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OPTIMAL INVESTMENT AND PRICE DEPENDENCE IN A SEMI-STATIC MARKET

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Abstract

This paper studies the problem of maximizing expected utility from terminal wealth in a semi-static market composed of derivative securities, which we assume can be traded only at time zero, and of stocks, which can be traded continuously in time and are modeled as locally-bounded semi-martingales. Using a general utility function defined on the positive real line, we first study existence and uniqueness of the solution, and then we consider the dependence of the outputs of the utility maximization problem on the price of the derivatives, investigating not only stability but also differentiability, monotonicity, convexity and limiting properties.

1 Introduction

A classical problem in financial economics is to understand the behavior of rational agents faced with an uncertain evolution of asset prices. In one of the most popular frameworks, one considers an investor who wants to maximize his expected utility from terminal wealth by investing in a frictionless market. This approach takes as inputs a utility function and a model for the future evolution of the stock prices; to implement this program in practice, typically one chooses a particular parametrized family of utility functions and of models for the stock price, and calibrates the value of the parameters to the available data.

Since the choice of the agent's utility and of the market model requires estimation, it is natural to ask how the agent's behavior is affected by misspecifications of the utility function and of the underlying market model. Indeed, following Hadamard's prescription, after investigating existence and uniqueness one should perform stability analysis, and only a problem whose solution exists, is unique and depends continuously on the initial data is worthy of the appellative 'well posed'.

In fact, much work has recently been done on the topic of sensitivity of the solution of the problem of expected utility maximization under perturbations of various initial conditions. Jouini and Napp [Joui 04], Carasus

and Rásonyi [Cara 07], and Larsen [Lars 09] consider misspecifications of the utility functions. Market perturbations are considered in Larsen and Žitković [Lars 07] who, working in a continuous time market, investigate the continuous dependence on the price of the stock (parametrized by the market-price of risk), and in Kardaras and Žitković [Kard 11], who perform a stability analysis of the problem under small misspecifications of the agent's preferences and of the market model. Hubalek and Schachermayer [Huba 98] study the convergence of prices of illiquid assets when the prices of the liquid assets converge, and stability of option pricing under market perturbations has been investigated by El Karoui et al. [Karo 98], while Kardaras [Kard 10] looks at the stability of the numéraire portfolio.

The previously mentioned results only deal with stability, and constitute a zeroth order approach to the problem. References which perform a first-order study are Henderson [Hend 02a], who studies, in a Brownian market, the expansion of the indifference price with respect to a small number of random endowments (see also Henderson and Hobson [Hend 02b]), and Kramkov and Sirbu [Kram 06], who generalize the first order approximation to semimartingale markets.

A setting which has been popular in recent years is the one of a semi-static market, where investors can trade continuously in time a number of stocks, as well as take static positions in some derivatives; for example see Campi [Camp 04, Camp 13], Ilhan et. al [Ilha 06, A Il 05] and Carr et al. [Carr 01], Schweizer and Wissel [Schw 08a, Schw 08b], Jacod and Protter [Jaco 10]. On of the advantages of this framework is that the price of a contingent claim which can be traded only at time zero is modeled simply as a vector in $\mathbb{R}^n$, instead of a general $\mathbb{R}^n$-valued semimartingale. Working in this simplified setting and using an exponential utility, Ilhan et al. [A Il 05] obtain *differentiability and strict-convexity* of the value function *as a function of the price* of the financial derivatives.

In the present paper we consider, as Ilhan et al. [A Il 05], the problem of maximizing expected utility from terminal wealth in a semi-static market framework; however, we use a general utility function defined on the positive real line. We study the existence and uniqueness of the solution, and the dependence of the value function, of its maximizer and of other quantities of interest on the (initial capital and on the) price p of the derivatives; we prove not only stability, but also differentiability, monotonicity, and convexity. Specifically, we reproduce in our model some economically sensible properties: the value function u has the expected monotonic behavior (as p increases u is initially decreasing, then constant, then increasing), and it diverges (along with the optimizer) when the price of the derivatives converge to an arbitrage price. We also show convexity in p of the largest feasible position, defined as the maximum number of shares of derivatives that the agent with given initial wealth can buy at price p and still be able to invest in the (liquid) stock market as to have a non-negative final wealth. This fact,

which is a consequence of the no-arbitrage assumption, is in our opinion not particularly intuitive (especially when there are multiple derivatives) and -to the best of our knowledge- was not noticed before. Unlike in the case of exponential utility, we show with an example that the maximal expected utility does not need to be a convex function of the derivative's prices; this makes proving differentiability a much trickier task, which we are able to carry on only under additional assumptions (most importantly in the case of power utility).

We emphasize that our problem does not fall under the general umbrella of utility maximization with convex constraints (we refer to Larsen and Žitković [Lars 13] for a survey), since it cannot be re-phrased asking that the portfolio and wealth process lie in some given convex set (possibly depending on t and ω); rather, we demand that the investor, after choosing his position at time zero *arbitrarily*, keep his position in derivatives unchanged for the rest of the time horizon, while freely investing in stocks. Moreover, Ilhan et al. [A Il 05], working in the framework of exponential utility, make essential use of relative entropy techniques and of some explicit representations of the maximal utility and of indifference prices; thus, as they point out, the 'extension to more general cases is not trivial', as a completely different approach is needed.

A simple but useful observation is that our problem can be decomposed in two steps: choosing the optimal amount of derivatives to buy at time zero, and then investing optimally in the continuous time stock market; the second step being the problem of optimal investment with random endowment, which is then closely related to our problem. To describe this relationship and make profitable use of it, we will need to slightly extend the main result of Hugonnier and Kramkov [Hugo 04] by considering endowments on the boundary of the domain of the utility; after personal communications with us, Mostovyi, who had generalized the results of [Hugo 04] to include the case of intermediate consumption, has analogously extended in [Most 11] his results to include endowments on the boundary.

The paper is organized as follows. In Section 2 we present the model of financial market and we define our problem. In Section 3 we state our main theorems, some of which we state in more detail in Sections 4 and 8. In Section 5 we prove existence and uniqueness of the solution. In Section 6 we generalize some results of Hugonnier and Kramkov [Hugo 04], and in Section 7 we study the relation between our problem and the problem of optimal investment with random endowment. In Section 8 we establish the continuity of the outputs of the utility maximization problem. In Section 9 we prove the convexity of the largest feasible position and we study the asymptotic behavior of the value function and the optimizer. In Section 10 we consider in more detail the one-dimensional case, and in Section 11 we provide an example of a value function which is not convex in p. Finally, in Section 12 we study the differentiability of the value function.

2 The model

Consider at first a model of a financial market composed of a savings account and d stocks which can be traded continuously in time. As is common in mathematical finance, we consider a finite deterministic[1] time horizon $[0, T]$, and we assume that the interest rate is 0; that is, the price process of the savings account is used as numéraire and is thus normalized to one. The price process $S = (S^i)_{i=1}^d$ of the stocks is assumed to be a locally-bounded[2] semi-martingale on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions.

Now enlarge the market by allowing also n contingent claims $f = (f_j)_{j=1}^n$ to be traded at price $p = (p_j)_{j=1}^n$; we assume that these contingent claims can be traded *only* at time zero, and that p is an arbitrage-free price for the European contingent claims f (in a sense which will be made precise later).

A self-financing portfolio is then defined as a triple (x, q, H), where $x \in \mathbb{R}$ represents the initial capital, $q_j \in \mathbb{R}$ represents the holding in the contingent claim f_j, and the random variable H_t^i specifies the number of shares of stock i held in the portfolio at time t.

An agent with portfolio (x, q, H) will invest his initial wealth x buying q European contingent claims at price p at time zero. This quantity is then held constant up to maturity, so the vector q represents the illiquid part of the portfolio and $qp := \sum_{j=1}^n q_j p_j$ represents the wealth invested in the European contingent claims (in this paper vw will always denote the dot product between two vectors v and w, and $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^{n+1}$).

He will then invest the remaining wealth $x - qp$ dynamically, buying H_t share of stocks at time $t \in [0, T]$, and put the rest (positive or negative) into the savings account. We will denote by X_t the value of the dynamic part of the portfolio, which will be called simply the wealth process; $x - qp$ will be called the initial value of the wealth process (which is different from the initial wealth x of the portfolio). The wealth process X evolves in time as the stochastic integral of H with respect to S:

$$X_t = x - qp + (H \cdot S)_t = x - qp + \int_0^t H_u dS_u, \qquad t \in [0, T],$$

where H is assumed to be a predictable S-integrable process.

For $x \geq 0$, we denote by $\mathcal{X}(x)$ the set of non-negative wealth processes

[1] In fact, one could take T to be a finite stopping time, as is the case in Hugonnier and Kramkov (2004), on which we rely.

[2] This assumption is not strictly speaking necessary, as the results in [Kram 99], [Kram 03], [Hugo 04], [Delb 97] on which our proofs hinge, although proved for a locally-bounded semi-martingale, are true also without the local boundedness assumption, if one replaces equivalent local-martingale measures with separating measures throughout (this fact is stated in [Hugo 05, Remark 3.4]).

whose initial value is equal to x, that is,

$$\mathcal{X}(x) := \{X \geq 0 : X_t = x + (H \cdot S)_t\}.$$

A probability measure Q is called an *equivalent local-martingale measure* if it is equivalent to P, and if S is a local-martingale under Q. We denote by $\mathcal{M}$ the family of equivalent local-martingale measures, and we assume that

$$\mathcal{M} \neq \emptyset. \tag{1}$$

This condition is essentially equivalent to the absence of arbitrage opportunities in the market without the European contingent claims: see [Delb 94] and [Delb 98] for precise statements as well as for further references.

In our model we consider an agent whose preferences are modeled via a utility function $U : (0, \infty) \to \mathbb{R}$, which is assumed to be strictly concave, strictly increasing and continuously differentiable and to satisfy the Inada conditions:

$$U'(0) := \lim_{x \to 0+} U(x) = \infty, \qquad U'(\infty) := \lim_{x \to \infty} U'(x) = 0. \tag{2}$$

It will be convenient to consider U defined on the whole real line. We want its extension to be concave and upper semi-continuous, and (2) implies that there is only one possible choice: we define $U(x)$ to be $-\infty$ for x in $(-\infty, 0)$, and to equal $U(0+) := \lim_{x \to 0} U(x)$ at $x = 0$.

We denote by $f = (f_j)_{j=1}^n$ the family of the $\mathcal{F}_T$-measurable payment functions of the European contingent claims with maturity T, and by $qf = \sum_{j=1}^n q_j f_j$ the payoff of the static part of the portfolio. The total payoff of the portfolio (x, q, H) is then $x - qp + (H \cdot S)_T + qf$.

A non-negative wealth process in $\mathcal{X}(x)$ is said to be maximal if its terminal value cannot be dominated by that of any other process in $\mathcal{X}(x)$. We assume that the European contingent claims can be sub- and super-replicated by trading in the stock; in other words, that there exists a maximal wealth process X' such that

$$|f| := \sqrt{\textstyle\sum_{j=1}^n f_j^2} \leq X'_T \,. \tag{3}$$

Since strictly positive maximal wealth processes are numéraires (see [Delb 95]), condition (3) can be interpreted as asking that $|f|$ be bounded with respect to some numéraire. If the contingent claims are uniformly bounded, as in [J Cv 01], then the optimization set is taken to be the set of wealth processes uniformly bounded from below. If the contingent claims are bounded just with respect to some numéraire, the optimization set has to be extended analogously. Following [Delb 97], we say that a wealth process X is *acceptable* if it admits a representation of the form $X = X' - X''$, where X' is a non-negative wealth process and X'' is a maximal wealth process. Since a wealth process X is *acceptable* if and only if it is bounded

below under some numéraire[3] N, the acceptable processes constitutes a natural optimization set for our optimal investment problem. Thus, following [Hugo 04], we define $\mathcal{X}(x,q)$ to be the set of acceptable wealth processes with initial value x whose terminal value dominates the random payoff $-qf$, i.e.,

$$\mathcal{X}(x,q) := \{X : X \text{ is acceptable}, X_0 = x \text{ and } X_T + qf \geq 0\}.$$

We will call $\bar{\mathcal{K}}$ the set of points (x,q) where $\mathcal{X}(x,q)$ is not empty, i.e.,

$$\bar{\mathcal{K}} := \{(x,q) \in \mathbb{R} \times \mathbb{R}^n : \mathcal{X}(x,q) \neq \emptyset\}. \tag{4}$$

As shown in [Hugo 04, Lemma 1 and 6], assumptions (1) and (3) imply that the convex cone $\bar{\mathcal{K}}$ defined in (4) is closed and its interior $\mathcal{K}$ contains $(x,0)$ for any $x > 0$, so $\bar{\mathcal{K}}$ is the closure of $\mathcal{K}$.

We will say that p is an *arbitrage-free price* for the European contingent claims f if any portfolio with zero initial capital and non-negative final wealth has identically zero final wealth, and we will denote by $\mathcal{P}$ the set of arbitrage-free prices; in other words,

$$\mathcal{P} := \{p \in \mathbb{R}^n : q \in \mathbb{R}^n, X \in \mathcal{X}(-pq,q) \text{ imply } X_T = -qf\}.$$

The objective of this paper is to study the problem of utility maximization in the enlarged market consisting of the bond, the stocks and the contingent claims, i.e., the following optimization problem, for $x > 0, p \in \mathcal{P}$,

$$\tilde{u}(x,p) := \sup\{\mathbb{E}[U(X_T + qf)] : X \text{ is acceptable }, q \in \mathbb{R}^n,\ X_0 = x - qp\}. \tag{5}$$

We follow the convention that $\mathbb{E}[U(X_T + qf)]$ equals $-\infty$ when $\mathbb{E}[U^-(X_T + qf)] = -\infty$ (whether or not $\mathbb{E}[U^+(X_T + qf)]$ is finite).

The problem of utility maximization with random endowment[4]

$$u(x,q) := \sup_{X \in\, \mathcal{X}(x,q)} \mathbb{E}[U(X_T + qf)], \quad (x,q) \in \mathbb{R}^{n+1}, \tag{6}$$

which was studied in [Hugo 04], is obviously closely related to (5); in fact it is easy to show that $\mathcal{K} \subseteq \{u > -\infty\} \subseteq \bar{\mathcal{K}}$, and trivially

$$\tilde{u}(x,p) = \sup_{q \in \mathbb{R}^n} u(x - qp, q) = \sup_{q \in \mathbb{R}^n : u(x-qp,q) > -\infty} u(x - qp, q), \quad x > 0, p \in \mathcal{P}. \tag{7}$$

By definition, p is a *marginal (utility-based) price* at (x,q) for f if the agent with initial endowment (x,q) given the opportunity to trade the contingent claims f at time zero at price p would neither buy nor sell any. We

[3] If $X = X' - X''$ take $N = 1 + X''$ to get $X/N \geq -1$; vice versa if $X/N \geq -1$ choose $X'' = N$ to get $X = X' - X''$.

[4] We use the convention that the sup (inf) over an empty set takes the value $-\infty$ ($+\infty$).

will denote by $\mathcal{P}(x,q)$ the set of marginal prices of f at $(x,q) \in \{u \in \mathbb{R}\}$; in other words, we set

$$\mathcal{P}(x,q) := \{p \in \mathbb{R}^n : u(x - q'p, q + q') \leq u(x,q) \text{ for all } q' \in \mathbb{R}^n\}.$$

The *largest feasible position* $m : (0,\infty) \times \mathcal{P} \longrightarrow [0,\infty]$ is defined as

$$m(x,p) := \sup\left\{\, |q| : q \in \mathbb{R}^n, (x - qp, q) \in \bar{\mathcal{K}} \,\right\}, \tag{8}$$

and it measures the maximum number of shares of derivatives that the agent with wealth x can buy at price p and still be able to invest in the stock market as to have a non-negative final wealth.

In this paper we study problem (5) and its relationship with problems (6) and (7). In particular, we investigate existence and uniqueness of the solution, and the dependence on the initial wealth x *and the initial price* p of the outputs of problem (5): the optimal position in derivatives $\tilde{q}$, the optimal final value of the dynamic part of the portfolio $\tilde{X}_T$, the maximal expected utility $\tilde{u}$ (and its derivatives), and the largest feasible position.

3 Statement of the main theorems

To state our main theorems we need to introduce some standard notation. Denote by $\mathcal{Y}(y)$ the family of non-negative processes Y with initial value y and such that for any non-negative wealth process X the product XY is a super-martingale, that is,

$$\mathcal{Y}(y) := \{Y \geq 0 : Y_0 = y, XY \text{ is a super-martingale for all } X \in \mathcal{X}(1)\}.$$

In particular, as $\mathcal{X}(1)$ contains the constant process 1, the elements of $\mathcal{Y}(y)$ are non-negative super-martingales. Note also that the set $\mathcal{Y}(1)$ contains the density processes of all $Q \in \mathcal{M}$.

The convex conjugate function V of the agent's utility function U is defined to be the Fenchel-Legendre transform of the function $-U(-\cdot)$; that is,

$$V(y) := \sup_{x\in\mathbb{R}}(U(x) - xy) = \sup_{x>0}(U(x) - xy), \quad y \in \mathbb{R}.$$

It is well known that, under the Inada conditions (2), the conjugate V of U is convex, lower semi-continuous, it is infinite on $(-\infty, 0)$ and, restricted to $(0,\infty)$, is a continuously differentiable, strictly decreasing and strictly convex function satisfying $V'(0) = -\infty$, $V'(\infty) = 0$, $V(0) = U(\infty)$, $V(\infty) = U(0)$, as well as the following bi-dual relation:

$$U(x) = \inf_{y\in\mathbb{R}}(V(y) + xy) = \inf_{y>0}(V(y) + xy), \quad x \in \mathbb{R}.$$

Following [Hugo 04], we will denote by w the value function of the problem of optimal investment without the European contingent claims, and by

$\tilde{w}$ its dual value function. In other words

$$w(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0\,; \qquad \tilde{w}(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)], \quad y > 0.$$

Recall that a random variable g is said to be *replicable* if there is an acceptable process X such that $-X$ is also acceptable and $X_T = g$ (if such a process exists, it is unique). In order to have uniqueness of the maximizer of (7) we will assume that

$$\text{for any non-zero } q \in \mathbb{R}^n \text{ the random variable } qf \text{ is not replicable.} \tag{9}$$

Note that, by discarding the redundant contingent claims, one can always reduce to the case where (9) holds (see [Hugo 04, Remark 6]), so assuming (9) does not comport a real loss of generality.

The following theorem is an analogue of the results found in [Kram 03], plus a description of the relationship between problems (5), (6), and (7); we state it here in an abridged format (for the full version see Theorems 5 and 6).

Theorem 1 *Assume that p is an arbitrage-free price for f, that conditions (1), (2), (3) and (9) hold, and that*

$$\tilde{w}(y) < \infty \text{ for all } y > 0.$$

Then the value function $\tilde{u}(\cdot, p)$ is finite, continuously differentiable, strictly increasing and strictly concave on $(0, \infty)$ and satisfies Inada conditions.

For any $x > 0$, the solution $(\tilde{X}, \tilde{q})$ to (5) and the solution $(x - \hat{q}p, \hat{q})$ to (7) exist, are unique and satisfy $\tilde{q} = \hat{q}$. Moreover, for every $(x, q) \in \{u > -\infty\}$ the solution $X(x, q)$ to (6) exist, is unique and satisfies $\tilde{X} = X(x - \hat{q}p, \hat{q})$.

In Theorem 1, the delicate point is that in general[5] problem (7) does not have a solution if we were to replace $\{u > -\infty\}$ (or $\bar{\mathcal{K}}$) with its interior $\mathcal{K}$ (see [Sior 12, Section 4]); however, in the existing literature problem (6) has been solved only in the case where (x, q) belongs to $\mathcal{K}$. As a consequence, to compare the problems (5) and (7) and establish Theorem 1, we need to finish carrying out an extension of the results of [Hugo 04] which was started in [Sior 12].

The following theorem does not have an analogue in [A Il 05] and, we believe, is very intuitive (given its economic interpretation). It shows that the dependence on p takes a particularly pleasing form in the case where q is one-dimensional.

[5] As we will show in Theorem 5, a sufficient condition for the solution of problem (7) to lie in $\mathcal{K}$ is that the final value of the optimal portfolio is bounded below by a strictly positive constant.

Theorem 2 *Under the assumptions of Theorem 1, if there is only one European contingent claim f, then there exists $\underline{p}, \overline{p} \in \mathbb{R}$ such that $(\underline{p}, \overline{p}) = \mathcal{P}$ and, for all $x > 0$, there exists a, b such that $\mathcal{P}(x, 0) = [a, b] \subset (\underline{p}, \overline{p})$. The function*

$$\begin{array}{rcl} (\underline{p}, \overline{p}) & \longrightarrow & \mathbb{R} \\ p & \mapsto & \tilde{q}(x, p) \end{array}$$

defined in Theorem 1 is continuous, it is strictly positive on $(\underline{p}, a)$, it equals zero on $[a, b]$, and it is strictly negative on $(b, \overline{p})$. The function

$$\begin{array}{rcl} (\underline{p}, \overline{p}) & \longrightarrow & \mathbb{R} \\ p & \mapsto & \tilde{u}(x, p) \end{array}$$

is continuous, it is strictly decreasing on $(\underline{p}, a)$, it is constant on $[a, b]$, and it is strictly increasing on $(b, \overline{p})$.

Moreover if $U(\infty) = \infty$ then $\tilde{q}(x, \underline{p}+) = \tilde{u}(x, \underline{p}+) = \tilde{u}(x, \overline{p}-) = \infty$ and $\tilde{q}(x, \overline{p}-) = -\infty$.

We recall that, in the general setting in which we work, $\mathcal{P}(x, 0)$ is not a singleton (see [Hugo 05, Theorem 3.1]), so in the above theorem it could actually happen that $a < b$.

Our next theorem is a stability result for the problem of optimal investment in the general multi-dimensional setting (for a more detailed statement see Theorem 11). It also shows that m is a convex function of the derivative's price and that, when the arbitrage-free prices converge to an arbitrage price, the corresponding utility and the optimal demand diverge.

Theorem 3 *Under the assumptions of Theorem 1, $\mathcal{P}$ is an open bounded convex set, the map $m(x, p)$ is finite valued, the map*

$$\begin{array}{rcl} (0, \infty) \times \mathcal{P} & \longrightarrow & \mathbb{R}^n \times L^0(P) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \\ (x, p) & \mapsto & \Big(\tilde{q},\, \tilde{X}_T,\, \tilde{u},\, \partial_x \tilde{u},\, m \Big) \end{array}$$

is continuous, $m(x, p)$ is positively homogeneous in x and is convex in p, and the supremum in (8) is attained. Moreover if $\mathcal{P} \ni p_n \to p \notin \mathcal{P}, x_n \to x > 0$ and $U(\infty) = \infty$ then, as $n \to \infty$,

$$\tilde{u}(x_n, p_n) \to \infty = \tilde{u}(x, p), \quad m(x_n, p_n) \to \infty \text{ and } \quad |\tilde{q}(x_n, p_n)| \to \infty.$$

We remark that in [A Il 05] the function $p \mapsto \tilde{u}(x, p)$ (corresponding to an exponential utility U) is proved to be strictly convex and differentiable. However, we will show that in our general framework convexity does not hold; this, and the fact that the maximizer of (7) may lay on the boundary of $\mathcal{K}$ make proving differentiability in p a very delicate task. The next theorem shows that these are the only impediments to differentiability, and that they can circumvented in some occasions.

Theorem 4 *Under the assumptions of Theorem 1, the function*

$$\begin{array}{rcl}(0,\infty)\times\mathcal{P} & \longrightarrow & \mathbb{R}\\ (x,p) & \mapsto & \tilde{u}(x,p)\end{array}$$

is continuously differentiable at all points if the function U is a power utility, and is continuously differentiable in a neighborhood of some point $(x^\star,p^\star)$ if either of the following conditions is satisfied:

1. *The optimizer $(x^\star-\hat{q}(x^\star,p^\star)p^\star,\hat{q}(x^\star,p^\star))$ of (7) belongs to $\mathcal{K}$.*
2. *The function $p\mapsto\tilde{u}(x^\star,p)$ is convex in a neighborhood of $p^\star$.*

Moreover, whenever the derivatives exist, they satisfy $\nabla_p\tilde{u}=-(\partial_x\tilde{u})\hat{q}$.

We remark that the equation $\nabla_p\tilde{u}=-(\partial_x\tilde{u})\hat{q}$ holds also for exponential utilities (see[6] [A Il 05, Theorem 3.1]), and that it can be derived heuristically in a simple fashion. Indeed, p is a marginal price at (x,q) iff the agent with endowment (x,q) could achieve no gains by trading in derivatives, so

$$p\in\mathcal{P}(x,q)\quad\text{iff}\quad\tilde{u}(x+qp,p)=u(x,q)=\min_{p'}\tilde{u}(x+qp',p'),\qquad(10)$$

which allows us to characterize marginal prices using $\tilde{u}$ instead of u. It follows from (10) that the function $g(p')=\tilde{u}(x+qp',p')$ has gradient zero at $p'=p$. Since $p\in\mathcal{P}(x,q)$ implies $q=\hat{q}(x+qp,p)$, we obtain $0=\nabla_{p'}g(p)=((\partial_x\tilde{u})\hat{q}+\nabla_p\tilde{u})(x+qp,p)$ as desired. Notice however how the above line of reasoning does not clarify at which points the equality is satisfied.

4 Results on existence and uniqueness

In this section we will state two theorems which subsume Theorem 1; the first deals with the existence and uniqueness of the solution of the problem (5), and the second solves problems (6) and (7) and describe their relationship with problem (5).

We first need to introduce the dual problems. If we define the set $\bar{\mathcal{L}}$ to be the polar of $-\bar{\mathcal{K}}$:

$$\bar{\mathcal{L}}:=-\bar{\mathcal{K}}^o:=\{v\in\mathbb{R}^{n+1}:vw\geq 0\text{ for all }w\in\bar{\mathcal{K}}\},$$

then clearly $\bar{\mathcal{L}}$ is a closed convex cone. We will denote by $\mathcal{L}$ its relative interior, so $\bar{\mathcal{L}}$ is the closure of $\mathcal{L}$. Given an arbitrary vector $(y,r)\in\mathbb{R}\times\mathbb{R}^n$,

[6] Actually, [A Il 05, Theorem 3.1] states that $\nabla_p\tilde{u}=\tilde{q}$; the missing minus sign in front of $\tilde{q}$ (which, in [A Il 05], is called $\lambda^\star$) is a typo, whereas the term $\partial_x\tilde{u}$ is missing because in this case $\partial_x\tilde{u}=1$, as it follows from [A Il 05, Theorem 4.1].

we denote by $\mathcal{Y}(y,r)$ the set of non-negative super-martingales $Y \in \mathcal{Y}(y)$ such that the inequality

$$\mathbb{E}[Y_T(X_T + qf)] \leq xy + qr$$

holds whenever $(x,q) \in \bar{\mathcal{K}}$ and $X \in \mathcal{X}(x,q)$; it is easy to show that $\mathcal{Y}(y,r)$ is non-empty if and only if $(y,r) \in \bar{\mathcal{L}}$ (see [Sior 12, Remark 5]).

We now define the problems dual[7] to (5) and to (6) to as follows[8]:

$$\tilde{v}(y,p) := \inf_{Y \in \mathcal{Y}(y,yp)} \mathbb{E}[V(Y_T)], \quad y \in \mathbb{R} \tag{11}$$

and

$$v(y,r) := \inf_{Y \in \mathcal{Y}(y,r)} \mathbb{E}[V(Y_T)], \quad (y,r) \in \mathbb{R} \times \mathbb{R}^n, \tag{12}$$

where $p \in \mathcal{P}$ is the vector of prices of the contingent claims f. We notice that trivially $\tilde{v}(y,p) = v(y,yp)$, and recall the following relationships between $w, \tilde{w}, u$ and v: $u(x,0) = w(x)$ (which follows from $\mathcal{X}(x,0) = \mathcal{X}(x)$) and $\tilde{w}(y) = \min_{p \in \mathcal{P}} v(y,yp)$ (see [Hugo 04, Lemma 2]). Moreover, recall that $\mathcal{L} = \{(y,yp) : y > 0 \text{ and } p \in \mathcal{P}\}$, or equivalently $\mathcal{P} = \{p : (1,p) \in \mathcal{L}\}$ (see [Sior 12, Lemma 3]). The following theorem is an analogue of the results found in [Kram 03].

Theorem 5 *Assume that p is an arbitrage-free price for f, that conditions (1), (2) and (3) hold, and that*

$$\tilde{w}(y) < \infty \text{ for all } y > 0. \tag{13}$$

Then one has:

1. *The value functions $\tilde{u}$ and $-\tilde{v}$ are finite, continuously differentiable, strictly increasing and strictly concave on $(0,\infty)$ and satisfy:*

$$\tilde{u}'(0) := \lim_{x \to 0} \tilde{u}'(x) = \infty, \qquad \tilde{v}'(\infty) := \lim_{y \to \infty} \tilde{v}'(y) = 0\,,$$

$$\tilde{u}'(\infty) := \lim_{x \to \infty} \tilde{u}'(x) = 0, \qquad \tilde{v}'(0) := \lim_{y \to 0} \tilde{v}'(y) = -\infty\,,$$

as well as the bi-conjugacy relationships:

$$\tilde{u}(x) = \min_{y>0}(\tilde{v}(y) + xy), \qquad x > 0,$$
$$\tilde{v}(y) = \max_{x>0}(\tilde{u}(x) - xy), \qquad y > 0$$

where the optimizers are unique and given by $\tilde{y} = \tilde{u}'(x)$ and $x = -\tilde{v}'(\tilde{y})$.

[7] The duality in (11) is with respect to the variable y only, with p playing the role of a parameter.

[8] We use the convention that the sup (inf) over an empty set takes the value $-\infty$ ($+\infty$).

2. *The solution* $(\tilde{X}(x), \tilde{q}(x))$ *to (5) exists for any* $x > 0$*, and* $-\tilde{X}(x)$ *is an acceptable wealth process. The final payoff* $\tilde{X}_T(x)+\tilde{q}(x)f$ *is unique.*

3. *For any* $y > 0$*,* $\mathcal{Y}(y, yp) \neq \emptyset$ *and the solution* $\tilde{Y}(y)$ *to (11) exists and is unique.*

4. *If* $\tilde{Y}_T(y)$ *is bounded (or equivalently if* $\tilde{X}_T(x) + \tilde{q}(x)f \geq \varepsilon > 0$*, for* x *given by* $\tilde{y} = \tilde{u}'(x)$*),* $\tilde{Y}(y)$ *equals the density process of an equivalent local-martingale measure and* $(x - \tilde{q}(x)p, \tilde{q}(x)) \in \mathcal{K}$*.*

5. *If* $x > 0$ *and* $\tilde{y} = \tilde{u}'(x)$ *the optimizers of (5) and (11) satisfy*

$$\tilde{Y}_T(\tilde{y}) = U'(\tilde{X}_T(x) + \tilde{q}(x)f)\,,$$
$$\mathbb{E}[\tilde{Y}_T(\tilde{y})(\tilde{X}_T(x) + \tilde{q}(x)f)] = x\tilde{y}\,.$$

To the best of our knowledge, the simple observation that the dual optimizer is a martingale if its terminal value is bounded is new; our proof also applies to the context of [Kram 03], mutatis mutandis.

It trivially follows from [Kram 99, Theorem 3.2], that a convenient sufficient condition for the validity of (13) is that the asymptotic elasticity of U is strictly less than one, and that $w(x) < \infty$ for some $x > 0$.

We now solve problems (6), and (7), and elucidate their relationship with problem (5).

Theorem 6 *Under the assumptions of Theorem 5, the following holds:*

1. *For any* $x > 0$ *the solutions* $(x - \hat{q}p, \hat{q})$ *to (7) exist, belong to* $dom(\partial u)$*, and are given by* $-\partial v(\tilde{y}, \tilde{y}p)$*, where* $\tilde{y} = \tilde{u}'(x)$*.*

2. *For every* $(x, q) \in \{u > -\infty\}$*, the solution* $X(x, q)$ *to (6) exists and is unique.*

3. *For any* $x > 0$ *the solutions* $(\tilde{X}(x, p), \tilde{q}(x, p))$ *to (5) are given by*

$$\{(X(x - qp, q), q) : (x - qp, q) \text{ solves (7)}\}.$$

4. *The following conditions are equivalent:*

   (a) *The solution to (5) is unique for all* $x > 0$*.*

   (b) *The solution to (7) is unique for all* $x > 0$*.*

   (c) *Condition (9) holds.*

   (d) *The function* v *is differentiable on* $\mathcal{L}$*.*

   If these conditions hold, the solution $(x - \hat{q}p, \hat{q})$ *to (7) is given by* $-\nabla v(\tilde{y}, \tilde{y}p)$*, where* $\tilde{y} = \tilde{u}'(x)$*.*

Notice that Theorem 6 allows to compute $\tilde{q}(x)$ explicitly, as long as one can compute v; results on how to approximate $\tilde{q}(x)$ are contained in [Kram 06].

Although in principle we could base our proof of Theorem 5 on the relationship between the different optimization problems stated in Theorem 6, we find it much more economical to reduce problem (5) to a setting where we can apply the abstract results of [Kram 99], [Kram 03]; this approach has the additional advantage of automatically providing an alternate version of Theorem 5 which holds under a different set of hypotheses. In fact, we can rely on [Kram 99, Theorem 3.1] to get a weaker version[9] of Theorem 5 in the case where we do not assume that $\tilde{w}$ is finite, but only that $\tilde{u}(x) < \infty$ for some $x > 0$. The next remark shows that this assumption is equivalent to the following more natural and at first sight weaker condition (14), and thus also to the stronger looking condition that $\tilde{u}$ is finite on $(0, \infty)$.

Remark 7 *Assume that conditions (1), (2), (3) hold, that $p \in \mathcal{P}$ and*

$$w(x) < \infty \text{ for some } x > 0. \tag{14}$$

Then $\tilde{u}(x) < \infty$ for all $x > 0$.

PROOF. As explained in [Hugo 04, Remark 6], we can assume without loss of generality that (9) holds. Notice that u is concave and $u > -\infty$ on $\mathcal{K}$ (since $u(x, 0) = w(x)$, this follows easily from (3): see [Hugo 04, Theorem 1]). It follows that u is a proper concave function, and so it is bounded above by an affine function. Since [Sior 12, Lemma 3] shows that, for any $x > 0$, the set $\{(x - qp, q) \in \bar{\mathcal{K}} : q \in \mathbb{R}^n\}$ is bounded, (7) implies $\tilde{u}(x) < \infty$ for all $x > 0$. □

5 Proofs of existence and uniqueness

In this section we prove Theorem 5 by making use of [Kram 03, Theorem 4]. We first need to introduce some notation: define

$$\mathcal{C}(x, q) := \{g \in L^0_+ : g \leq X_T + qf \text{ for some } X \in \mathcal{X}(x, q)\},$$

and

$$\tilde{\mathcal{C}}(x, p) := \bigcup_{q \in \mathbb{R}^n} \mathcal{C}(x - qp, q) = \bigcup_{q \in \mathbb{R}^n : (x-qp, q) \in \bar{\mathcal{K}}} \mathcal{C}(x - qp, q). \tag{15}$$

We will often write $\tilde{\mathcal{C}}(x)$ (resp. $\tilde{\mathcal{C}}$) as a shorthand for $\tilde{\mathcal{C}}(x, p)$ (resp. $\tilde{\mathcal{C}}(1, p)$), and we observe that $x\tilde{\mathcal{C}} = \tilde{\mathcal{C}}(x)$.

[9] The interested reader can easily write down the alternate statement after comparing [Kram 99, Theorem 3.1] with [Kram 99, Theorem 3.2].

Define $\mathcal{D}(y,r)$ to be the set of positive random variables dominated by the final value of some element of $\mathcal{Y}(y,r)$, i.e.,

$$\mathcal{D}(y,r) := \{h \in L^0_+ : h \leq Y_T \text{ for some } Y \in \mathcal{Y}(y,r)\} \tag{16}$$

We will write $\tilde{\mathcal{D}}$ as a shorthand for $\mathcal{D}(1,p)$, and we observe that $\mathcal{D}(y,r) \neq \emptyset$ if and only if $(y,r) \in \bar{\mathcal{L}}$ (see [Sior 12, Remark 5]), and $y\tilde{\mathcal{D}} = \mathcal{D}(y,yp)$.

We recall here two facts proved in [Hugo 04, Lemmas 8 and 9]. Let $\mathcal{M}'$ be the set of equivalent local-martingale measures Q such that the maximal process X' that appears in (3) is a uniformly integrable martingale under Q, and let $\mathcal{M}'(p)$ be the subset of measures $Q \in \mathcal{M}'$ such that $\mathbb{E}_Q[f] = p$. If $(1,p) \in \mathcal{L}$ and conditions (1) and (3) hold, then

$$\mathcal{M}'(p) \neq \emptyset, \text{ and if } Q \in \mathcal{M}'(p) \text{ then } dQ/dP \in \mathcal{D}(1,p). \tag{17}$$

We are now ready to prove the analogue of [Kram 99, Proposition 3.1]

Theorem 8 *Assume that $p \in \mathcal{P}$ and that conditions (1), (3), (9) hold. Then $\tilde{\mathcal{C}}$ is bounded in $L^0(\Omega,\mathcal{F},P)$ and it contains the constant function $g = 1$. The sets $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ satisfy the bipolar relations:*

$$g \in \tilde{\mathcal{C}} \iff g \in L^0_+ \text{ and } \mathbb{E}[gh] \leq 1 \quad \forall h \in \tilde{\mathcal{D}} \tag{18}$$

$$h \in \tilde{\mathcal{D}} \iff h \in L^0_+ \text{ and } \mathbb{E}[gh] \leq 1 \quad \forall g \in \tilde{\mathcal{C}}\,. \tag{19}$$

PROOF OF THEOREM 8. The implication $\Rightarrow$ in (18) and in (19), and the inclusion $1 \in \tilde{\mathcal{C}}$ follow directly from definitions. Now we use (17). If $g \in \tilde{\mathcal{C}}$ and $Q \in \mathcal{M}'(p)$ then $\mathbb{E}_Q[g] \leq 1$; it follows that $\tilde{\mathcal{C}}$ is bounded in $L^1(Q)$, and so also in $L^0(P)$, since Q is equivalent to P.

To finish the proof of (19) assume that h is a non-negative random variable such that $\mathbb{E}[gh] \leq 1 \quad \forall g \in \tilde{\mathcal{C}}$. Then, in particular, $\mathbb{E}[X_T h] \leq 1 \quad \forall X \in \mathcal{X}(1)$, so [Kram 99, Proposition 3.1] implies the existence of a process $Y \in \mathcal{Y}(1)$ such that $h \leq Y_T$. Define the process Z by setting

$$Z_t := \begin{cases} Y_t & \text{if } t < T \\ h & \text{if } t = T. \end{cases}$$

Then Z belongs to $\mathcal{Y}(1)$ and so, since $\mathcal{C}(x,q) \subseteq (x+qp)\tilde{\mathcal{C}}$, it belongs to $\mathcal{Y}(1,p)$. This proves $h \in \tilde{\mathcal{D}}$, i.e., the implication $\Leftarrow$ in (19).

To conclude, let us prove that $\tilde{\mathcal{C}}$ is closed with respect to the convergence in measure; the version of the bipolar theorem found in [Bran 99] and (19) then yield (18). So take $g_n \in \mathcal{C}(1-pq_n, q_n)$ and assume without loss of generality that g_n converges almost surely to g, and let's prove that $g \in \tilde{\mathcal{C}}$. Since [Sior 12, Lemma 3] implies that $(1-pq_n, q_n)$ is bounded, passing to a subsequence we can assume that q_n is converging to some q, so [Sior 12,

Lemma 4] shows that $g \in \tilde{\mathcal{C}}$ $\square$

We will need the following simple remark, which follows from the fact that a wealth process X is maximal iff there is a measure $Q \in \mathcal{M}$ such that X is a $Q-$uniformly integrable martingale (see [Delb 97, Theorem 2.5]).

Remark 9 *If conditions (1) and (3) hold, any acceptable wealth process with zero initial value and non-negative terminal wealth is indistinguishable from zero, i.e.,* $\mathcal{X}(0,0) = \{0\}$

PROOF OF THEOREM 5. As explained in [Hugo 04, Remark 6], we can assume without loss of generality that (9) holds. Since clearly we have

$$\tilde{u}(x) = \sup_{g \in \tilde{\mathcal{C}}} \mathbb{E}[U(xg)], \qquad \tilde{v}(y) = \inf_{h \in \tilde{\mathcal{D}}} \mathbb{E}[V(yh)],$$

Theorem 8 puts us in a position to apply [Kram 03, Theorem 4], as long as we prove that $\tilde{v}$ is finite at all points under the assumption that $\tilde{w}$ is; this follows from [Sior 12, Lemma 3] and [Hugo 04, Lemma 2].

Since problem (11) is a particular case of problem (12) (resp. because of item 3 of Theorem 6), it is enough to prove item 4 (resp. the fact minus the optimal wealth process is acceptable) in the context of optimal investment with random endowment; this will be done in Theorem 10, and involves of course no circular argument.

6 Optimal investment with random endowment

In this section we rely on the upper-semi-continuity of u (which was proved in [Sior 12]) to extend [Hugo 04, Theorem 2] by considering also the behavior on the boundary of $\mathcal{K}$ and $\mathcal{L}$. In particular, we show that the sub-differential of v is empty on the boundary of $\mathcal{L}$, that u and v are convex conjugates on $\mathbb{R}^n$ and that if the solution $Y_T(y,r)$ of the dual problem is a bounded random variable then $Y(y,r)$ equals the density process of an equivalent local-martingale measure and $-\partial v(y,r) \in \mathcal{K}$.

We will denote by $Im(\partial f)$ the image of the sub-differential of a concave function $f : \mathbb{R}^{n+1} \to (-\infty, \infty]$ (or of a convex and $[-\infty, \infty)$ valued function), by $dom(\partial f)$ its domain $\{z : \partial f(z) \neq \emptyset\}$, and by

$$\frac{\partial^+ f}{\partial w}(z) := \lim_{t \to 0+} \frac{f(z+tw) - f(z)}{t}$$

the right sided directional derivative of f at z in the direction of w.

Theorem 10 *Under the assumptions of Theorem 5 the following holds:*

1. *The functions u and $-v$ defined on $\mathbb{R}^{n+1}$ have values in $[-\infty, \infty)$, are concave and upper semi-continuous and satisfy the bi-conjugacy*

relationships:

$$u(x,r) = \inf_{(y,r)\in\mathbb{R}^{n+1}} \left(v(y,r) + xy + qr\right), \qquad (x,q) \in \mathbb{R}^{n+1}, \tag{20}$$

$$v(y,r) = \sup_{(x,q)\in\mathbb{R}^{n+1}} (u(x,r) - xy - qr), \qquad (y,r) \in \mathbb{R}^{n+1}. \tag{21}$$

2. *$\mathcal{K} \subseteq dom(\partial u) \subseteq \{u > -\infty\} \subseteq \bar{\mathcal{K}}$, and all inclusions can be strict.*

3. *$\mathcal{L} = dom(\partial v) \subseteq \{v < \infty\} \subseteq \bar{\mathcal{L}}$. In particular $\partial v(\mathcal{L}) = dom(\partial u)$ and $\mathcal{L} = \partial u(\bar{\mathcal{K}})$ and if (y,r) belongs to the relative boundary of $\mathcal{L}$ then $\partial v(y,r) = \emptyset$ and $\frac{\partial^+ v}{\partial w}(y,r) = -\infty$ for every $w \in \mathcal{L} - (y,r)$.*

4. *For all $(x,q) \in \{u > -\infty\}$ there exists a unique maximizer $X(x,q)$ of (6) and for all $(y,r) \in \{v < \infty\}$ there exists a unique minimizer $Y(y,r)$ of (12).*

5. *The process $-X(x,q)$ is acceptable, and if $Y_T(y,r)$ is bounded (or equivalently if $X_T(x,q) + qf \geq \varepsilon > 0$ for (x,q) such that $(y,r) \in \partial u(x,q)$) then $Y(y,r)$ equals the density process of an equivalent local-martingale measure and $-\partial v(y,r) \in \mathcal{K}$.*

6. *If $(x,q) \in dom(\partial u)$ and $(y,r) \in \partial u(x,q)$, the terminal values of the optimizers are related by*

$$Y_T(y,r) = U'(X_T(x,q) + qf), \tag{22}$$

$$\mathbb{E}[Y_T(y,r)(X_T(x,q) + qf)] = xy + qr. \tag{23}$$

PROOF. Since u is concave and takes real values on the open set $\mathcal{K}$ (see [Hugo 04, Lemma 2, Theorem 2]), u never takes the value ∞ and the inclusions in item 2 hold. Analogously, since v is convex and takes real values on the open set $\mathcal{L}$ (see [Hugo 04, Lemma 2]), v never takes the value $-\infty$, yielding the chain of inclusions in item 3 other than $dom(\partial v) \subseteq \mathcal{L}$, which will be proved later.

By [Sior 12, Theorem 6], u is upper-semi-continuous, and there exists a unique maximizer of (6) for any $(x,q) \in \{u > -\infty\}$. To prove that v is lower semi-continuous and that there exists a unique minimizer for any $(y,r) \in \{v < \infty\}$ let $h_n \in \mathcal{D}(y_n, r_n)$ for some converging sequence (y_n, r_n), and define $s := \sup_n y_n < \infty$. Then $h_n \in \mathcal{D}(s)$, so [Kram 99, Lemma 3.2] gives that the sequence $(V^-(h_n))_{n\geq 1}$ is uniformly integrable. The proof of [Sior 12, Theorem 3] then applies, mutatis mutandis, completing the proof of item 4.

To prove the bi-conjugacy relationships (20) and (21), call $\bar{u}$ the function defined by the right-hand side of (20). Since $v = \infty$ outside of $\bar{\mathcal{L}}$, the infimum defining $\bar{u}$ can equivalently be taken over $\bar{\mathcal{L}}$ instead of $\mathbb{R}^{n+1}$, and [Rock 70,

Theorem 7.5] implies that we can equivalently replace $\bar{\mathcal{L}}$ with $\mathcal{L}$ (analogously we can replace $\mathbb{R}^{n+1}$ with $\bar{\mathcal{K}}$ or $\mathcal{K}$ in (21)). Then the concave and upper semi-continuous functions u and $\bar{u}$, which are defined on $\mathbb{R}^{n+1}$, never take the value ∞ and coincide on $\mathcal{K}$ (as shown in [Hugo 04, Theorem 2]) and so on $\bar{\mathcal{K}}$ (by [Rock 70, Theorem 7.5])). By definition u is identically $-\infty$ outside $\bar{\mathcal{K}}$; let us show that this is also true of $\bar{u}$, so they coincide everywhere. If $(x,q) \notin \bar{\mathcal{K}}$ one can find $(y,r) \in \mathcal{L}$ such that $xy + qr < 0$, and so

$$\frac{\bar{u}(x,q)}{n} \leq \left(\frac{v(n(y,r)) + n(xy+qr)}{n} \right).$$

The thesis then follows taking limits for $n \to \infty$, using l'Hospital rule and the fact that, by Theorem 5, the function $\tilde{v}(\lambda) = v(\lambda(1, r/y))$ satisfies $\lim_{\lambda\to\infty} \tilde{v}'(\lambda) = 0$ (r/y is an arbitrage-free price: see [Sior 12, Lemma 3]). This concludes the proof of (20), and now (21) follows from the fact that v is convex, proper and lower semi-continuous, concluding the proof of item 1.

Let us prove item 6. If $(y,r) \in \partial u(x,q)$ then $(x,q) \in \partial v(y,r)$ and so, by definition of sub-differential, $(x,q) \in \{u > -\infty\}$ and $(y,r) \in \{v < \infty\}$. Item 4 implies the existence of the optimizers $X(x,q)$ and $Y(y,r)$, and we have

$$\mathbb{E}[\,\big|V(Y_T(y,r)) + X_T(x,q)Y_T(y,r) - U(X_T(x,q))\big|\,] =$$

$$\mathbb{E}[V(Y_T(y,r)) + X_T(x,q)Y_T(y,r) - U(X_T(x,q))]$$

$$\leq v(y,r) + xy + qr - u(x,q) = 0,$$

which implies (22) and (23).

Let us now prove item (5). Observe that there exists $X \in \mathcal{X}(x,q)$ such that $-X$ is acceptable and $X_T \geq X_T(x,q)$ (see [Sior 12, Lemma 2]); since $X(x,q)$ is an optimizer, this implies $X_T = X_T(x,q)$. Then $X(x,q) - X \in \mathcal{X}(0,0)$, so Remark 9 gives that $-X(x,q)$ is acceptable. Now, fix $(y,r) \in \partial u(x,q)$ and assume that $Y_T(y,r) \leq c < \infty$, which (22) implies to be equivalent to $X_T(x,q) + qf \geq \varepsilon > 0$ for $\varepsilon := (U')^{-1}(c)$; then

$$X(x,q) - \varepsilon =: X \in \mathcal{X}(x-\varepsilon, q), \text{ and in particular } (x-\varepsilon, q) \in \bar{\mathcal{K}}.$$

Since $(1,0) \in \mathcal{K}$ (see [Hugo 04, Lemma 6]) and $\bar{\mathcal{K}}$ is a convex cone, $(x,q) = (x-\varepsilon, q) + \varepsilon(1,0) \in \bar{\mathcal{K}} + \mathcal{K} \subseteq \mathcal{K}$, i.e., $-\partial v(y,r) \subseteq \mathcal{K}$. From the definition of $\mathcal{Y}(y,r)$ and $X \in \mathcal{X}(x-\varepsilon, q)$ it follows that

$$\mathbb{E}[Y_T(y,r)(X_T + qf)] \leq (x-\varepsilon)y + qr,$$

so (23) and the definition of X imply that $\mathbb{E}[-\varepsilon Y_T(y,r)] \leq -\varepsilon y$. Thus $\mathbb{E}[Y_T(y,r)] \geq y$, the super-martingale $Y(y,r)$ is actually a martingale, and so Q defined by $dQ = Y_T(y,r)dP$ is a probability measure whose density

process equals $Y(y,r) \in \mathcal{Y}(y)$; since S is locally-bounded, Q is an equivalent local-martingale measure.

Let us show with examples that the inclusions in item 2 can be strict; take U to be a power utility of exponent α and considering a market with no stocks and one contingent claim uniformly distributed in $[-1,1]$. Then one can easily compute explicitly $u(x,q)$ for all α's, and show the following. When α is smaller than -1, $dom(u) = \mathcal{K}$, so the last inclusion can be strict. When α is between -1 and 0, $dom(u) = \bar{\mathcal{K}} \setminus \{(0,0)\}$ and $dom(\partial u) = \mathcal{K}$, so the second inclusion can be strict. Finally, when α is between 0 and 1, $dom(\partial u) = \bar{\mathcal{K}} \setminus \{(0,0)\}$, so the first inclusion can be strict.

To conclude the proof of item 3, we only need to prove that $\partial u(\bar{\mathcal{K}}) \subseteq \mathcal{L}$, since item 1 implies that $\partial u(\mathbb{R}^n) = dom(\partial v)$ and $\partial v(\mathbb{R}^n) = dom(\partial u)$ (see [Rock 70, Theorem 23.5]), and the result on the partial derivatives of v follows then from [Rock 70, Theorem 23.3]. Notice first that

$$\partial u(\bar{\mathcal{K}}) = dom(\partial v) \subseteq \bar{\mathcal{L}}.$$

Now $\partial u(\bar{\mathcal{K}}) \subseteq \mathcal{L}$ follows exactly as in the proof of [Hugo 04, Theorem 2] once we notice that the following relationship holds for every $(x,q) \in \bar{\mathcal{K}}$ (and not just for $(x,q) \in \mathcal{K}$): for any non-negative measurable function g,

$$g \in \mathcal{C}(x,q) \iff \mathbb{E}[gh] \leq 1 \quad \forall h \in \tilde{\mathcal{D}}(x,q), \tag{24}$$

where by definition

$$A(x,q) := \{(y,r) \in \bar{\mathcal{L}} : xy + qr \leq 1\},$$

$$\tilde{\mathcal{D}}(x,q) := \bigcup_{(y,r)\in A(x,q)} \mathcal{D}(y,r).$$

Indeed, equivalence (24) corresponds to [Hugo 04, Proposition 1, Eq. (26)] which, although stated only for $(x,q) \in \mathcal{K}$, holds (with the same proof) for every $(x,q) \in \bar{\mathcal{K}}$ (this is true also for [Hugo 04, Lemma 10], on which [Hugo 04, Proposition 1] relies). □

Trivially there is a condition which is equivalent to the domain of u being the whole of $\bar{\mathcal{K}}$: that is, $U(0) := U(0+)$ needs to be real valued. Indeed

$$u(x,q) \geq u(0,0) \text{ at all } (x,q) \in \bar{\mathcal{K}}, \tag{25}$$

so $dom(u) = \bar{\mathcal{K}}$ iff $u(0,0) \in \mathbb{R}$, and Remark 9 implies that $u(0,0) = U(0)$.

7 Relation between the optimization problems

In this section we prove Theorem 6.

PROOF OF THEOREM 6. Item 2 is part of item 4 in Theorem 10, and item 3 is trivial. For the proof of item 1, fix x and p, and let f be the function

$f(q) := -u(x-pq, q)$, which is proper convex and lower semi-continuous (by Theorem 10); its minimizer is then given by $\partial f^*(0)$ (see [Hiri 01, Formula (1.4.6), Chapter E]), where f^* is the Fenchel-Legendre transform of f. To compute f^* we write $f = g \circ A$ with $g(a,b) := -u(-(a,b))$, $A(q) := (-x + pq, -q)$, and apply [Hiri 01, Chapter E, Theorem 2.2.3] to find

$$f^*(\mu) = min_{(y,r)}\{v(y,r) + xy : yp - r = \mu\}.$$

The previous expression for f^* allows us to compute $\partial f^*(0)$ using [Hiri 01, Chapter D, Theorem 4.5.1][10], and to find

$$\partial f^*(0) = \{q : (x - pq, q) \in -\partial v(y,r)\}, \tag{26}$$

where (y,r) is any solution of

$$min_{(y,r)}\{v(y,r) + xy : yp - r = 0\}. \tag{27}$$

Item 1 of Theorem 5 then implies that problem (27) has as unique solution, namely $(\tilde{y}, \tilde{y}p)$, with $\tilde{y} = \tilde{u}'(x)$. Stitching the pieces together, we obtain that the function $q \mapsto u(x - pq, q)$ is maximized at the points $(x - p\hat{q}, \hat{q})$ in $-\partial v(\tilde{u}'(x), \tilde{u}'(x)p) \subseteq -\partial v(\mathbb{R}^n) = dom(\partial u)$, which concludes the proof of item 1.

The equivalence of items 4c and 4d is part of [Hugo 04, Lemma 3], and the the equivalence of items 4a and 4b follows from items 2 and 3. If item 4d holds, item 1 shows that $-\nabla v(\tilde{y}, \tilde{y}p)$ is the unique solution to (7); in particular item 4b holds. To show that item 4b implies item 4d we have to proceed differently[11]; so, let us assume that item 4c does not hold, and let $q' \neq 0$ be such that $q'f$ is replicated by a process $X' \in \mathcal{X}(x')$. This implies that $x' = q'p$, since p is an arbitrage-free price. Now, let $(\tilde{X}, \tilde{q})$ be a solution to (5); then, $(\tilde{X} + x' - X', \tilde{q} + q')$ is also a solution to (5) (since it has the same final payoff), so item 4a does not hold. This concludes the proof of item 4. □

8 Continuity

In this section we state and prove a result that subsumes the part of Theorem 3 which deals with the continuous dependence on (x,p).

Theorem 11 *Under the assumptions of Theorem 5 the maps*

$$\begin{array}{rcl} (0,\infty) \times \mathcal{P} & \longrightarrow & L^0(P) \times L^1(P) \times \mathbb{R} \times \mathbb{R} \\ (x,p) & \mapsto & \left(\tilde{X}_T + \tilde{q}f, U(\tilde{X}_T + \tilde{q}f), \tilde{u}(x,p), \dfrac{\partial \tilde{u}(x,p)}{\partial x}\right) \end{array}$$

[10] Although stated only for finite convex functions, the theorem holds (with the same proof) for proper convex functions.

[11] Since item 4b implies that $-\nabla v(\tilde{y}, \tilde{y}p)$ exists at all $\tilde{y} > 0$, but only for the one fixed p which we used in problem (7), and not for all p such that $(1,p) \in \mathcal{L}$.

and

$$\begin{array}{rcl}(0,\infty)\times\mathcal{P} & \longrightarrow & L^0(P)\times L^1(P)\times\mathbb{R}\times\mathbb{R}\\ (y,p) & \mapsto & \left(\tilde{Y}_T, V(\tilde{Y}_T), \tilde{v}(y,p), \dfrac{\partial\tilde{v}(y,p)}{\partial y}\right)\end{array}$$

are continuous, and if we additionally assume (9) then the map

$$\begin{array}{rcl}(0,\infty)\times\mathcal{P} & \longrightarrow & \mathbb{R}^n\times L^0(P)\\ (x,p) & \mapsto & (\tilde{q}, \tilde{X}_T)\end{array}$$

is (well defined and) continuous.

PROOF: Step 1 To obtain the proof for $(\tilde{Y}_T, V(\tilde{Y}_T))$ apply the changes which we will now describe to the proof of [Kram 99, Lemma 3.6]. Replace y and y_n with (y, yp) and $(y_n, y_n p_n)$, $\hat{h}$ with $\tilde{Y}_T$ and the function that is there called v (which in this article we denote by $\tilde{w}$) with the function v defined in (12). Apply [Hugo 04, Lemma 2] to obtain the finiteness of v from our assumption that $\tilde{w}$ is finite, and then use [Sior 12, Lemma 4] to obtain $g \in \mathcal{D}(y, yp)$. Finally use $\mathcal{D}(y, yp) \subseteq \mathcal{D}(y)$ and the fact that v, being concave and finite on the open set $\mathcal{L} \ni (y, yp)$, is there continuous; this concludes the proof that $(\tilde{Y}_T, V(\tilde{Y}_T))$ is continuous. The continuity of $\tilde{v}$ is now trivial, and the continuity of its derivative in y follows from [Rock 70, Theorem 25.7].

Step 2 Here we prove the continuity of $\tilde{u}$. Since $\tilde{u}(\cdot, p)$ is concave, it is locally Lipschitz, so $\tilde{u}(\cdot,\cdot)$ is continuous if $\tilde{u}(x,\cdot)$ is continuous. Since, by Theorem 5, $\tilde{u}(x,\cdot)$ is the infimum over $y > 0$ of the continuous functions $p \mapsto v(y, yp) + xy$, it is upper-semi-continuous. To prove its lower semicontinuity define the open set

$$V_p := \{q \in \mathbb{R}^n : (x - qp, q) \in \mathcal{K}\},$$

and observe that, since u is upper-semi-continuous (see [Sior 12, Theorem 6]) and concave, [Rock 70, Theorem 7.5] yields

$$u(a,b) = \lim_{\varepsilon\to 0+} u((a,b)(1-\varepsilon) + (1,0)\varepsilon) \text{ for all } (a,b) \in \bar{\mathcal{K}},$$

which implies that for any $c \in \mathbb{R}$

$$\tilde{u}(x,p) \le c \quad \text{if and only if} \quad u(x - qp, q) \le c \text{ for all } q \in V_p. \tag{28}$$

Since u, being concave, is continuous when restricted to the interior $\mathcal{K}$ of its effective domain $\{u \in \mathbb{R}\}$, if $p_n \to p \in \mathcal{P}$ and $\tilde{u}(x, p_n) \le c$ then for all $q \in V_p$

$$u(x - qp, q) = \lim_n u(x - qp_n, q) \le c,$$

so (28) implies $\tilde{u}(x,p) \le c$, so $\{\tilde{u}(x,\cdot) \le c\}$ is closed, i.e., $\tilde{u}(x,\cdot)$ is lower-continuous and so $\tilde{u}(\cdot,\cdot)$ is continuous[12]. The continuity of $\partial_x \tilde{u}$ now follows from [Rock 70, Theorem 25.7]. As proved in Theorem 6, $\tilde{q}$ is uniquely defined iff (9) holds, and in this case $\tilde{q} = -\nabla v(\partial_x \tilde{u}, (\partial_x \tilde{u})\, p)$, so $\tilde{q}$ is continuous.

Step 3 Now we describe the changes one needs to apply to the proof of [Kram 99, Lemma 3.6] in order to obtain continuity of $(\tilde{X}_T + \tilde{q}f, U(\tilde{X}_T + \tilde{q}f))$. Replace y and y_n with (x,p) and (x_n,p_n), $\hat{h}$ with $\tilde{X}_T + \tilde{q}f$, V with $-U$ and the function that is there called v with the function $\tilde{u}$. Observe that we can assume without loss of generality that (9) holds, so if $\tilde{q}_n := \tilde{q}(x_n,p_n)$ and $\tilde{q} := \tilde{q}(x,p)$ then $\tilde{q}_n \to \tilde{q}$. Define

$$h_n := \frac{(\tilde{X}_T(x_n,p_n) + \tilde{X}_T(x,p)) + (\tilde{q}(x_n,p_n) + \tilde{q}(x,p))f}{2},$$

so that $h_n \in \mathcal{C}(a_n,b_n)$, where

$$(a_n,b_n) := \Big(\frac{x_n + x}{2} - \frac{(\tilde{q}_n p_n + \tilde{q}p)}{2}, \frac{\tilde{q}_n + \tilde{q}}{2}\Big).$$

Since (a_n,b_n) is a convergent sequence, (3) provides an $\bar{x} > 0$ such that $\mathcal{C}(a_n,b_n)$ is contained in $\mathcal{C}(\bar{x},0)$, which is bounded in $L^0(P)$. This yields [Kram 99, Formula (3.13)] in our case, and also allows to apply Kolmos' lemma to construct a sequence $(g_n)_{n\ge1}$ of forward convex combinations of $(h_n)_{n\ge1}$ that is converging almost surely to some random variable g, which [Sior 12, Lemma 4] shows to be in $\tilde{\mathcal{C}}(x,p)$. Using the continuity of $\tilde{u}$, since $(g_n)_{n\ge1} \subseteq \mathcal{C}(\bar{x},0)$ we can apply [Kram 03, Lemma 1] to prove the uniform integrability of $U^+(g_n)$. □

9 Asymptotic results and convexity of m

In this section we will conclude the proof of Theorem 3, addressing the convexity of m and the asymptotics for $\mathcal{P} \ni p_n \to p \notin \mathcal{P}$.

Lemma 12 *If (1) and (3) hold, $\mathcal{P}$ is bounded and convex, and it is open iff (9) holds.*

PROOF. The identity $\mathcal{P} = \{p : (1,p) \in \mathcal{L}\}$ (proved in [Sior 12, Lemma 3]) shows that $\mathcal{P}$ is convex, and is open if and only if condition (9) is satisfied (see [Hugo 04, Lemma 3]). Moreover, assume by contradiction that $\mathcal{P}$ is not

[12] We can also give a more elegant proof that $\tilde{u}(x,\cdot)$ is continuous, relying on a hard-to-prove theorem: since $\tilde{v}$ is continuous and $\tilde{u}(x) = \tilde{v}(\tilde{y}) + x\tilde{y}$, where $\tilde{y} = \partial_x \tilde{u}$, the continuity of $\tilde{u}$ follows from the one of $\tilde{y}$. To prove the latter, observe that the continuous bijection g of $(0,\infty) \times \mathcal{P}$ in itself given by $(y,p) \mapsto (-\partial_y \tilde{v}(y,p), p)$ has inverse $g^{-1}(x,p) = (\tilde{y},p)$, and the map g is open, by Brouwer's invariance domain theorem, so g^{-1} is continuous.

bounded, i.e., there exists $p_n \in \mathcal{P}$ such that $|p_n|$ is converging to infinity as $n \to \infty$, and define

$$q_n := -p_n/|p_n|^{3/2}. \quad \text{Then } 1 + q_n p_n = 1 - |p_n|^{1/2}$$

is converging to $-\infty$, and so it is negative for big enough n. Since $-\bar{\mathcal{K}}^o = \bar{\mathcal{L}} \ni (1, p_n)$, by definition of polar $(1, q_n) \notin \bar{\mathcal{K}}$, which by [Hugo 04, Lemma 1] contradicts (3), since $q_n \to 0$. □

To obtain our results on the function m, we will first study the following auxiliary function $d : \mathcal{L} \longrightarrow [0, \infty]$ defined as

$$d(w) := \sup\{|v| : v \in \bar{\mathcal{K}} \text{ and } vw \leq 1\}. \tag{29}$$

Lemma 13 *Assume conditions (1), (3), (9), then*

1. $xd(w) = d(\frac{w}{x})$ *for every* $x > 0, w \in \mathcal{L}$.
2. d *is finite and the supremum in* (29) *is attained.*
3. $d(w_1 + w_2) \leq \min(d(w_1), d(w_2))$ *for all* $w_1, w_2 \in \mathcal{L}$.
4. d *is locally bounded.*

PROOF. Item one is obvious, and then [Sior 12, Lemma 3] implies that d is finite at every point, so clearly by compactness the supremum defining d is attained. Item three follows from $\bar{\mathcal{L}} = -\bar{\mathcal{K}}^o$. Finally, let $k = 2^{n+1}$ and take $(h_i)_{i=1}^k \subset \mathcal{L}$ to be the vertices of a cube Q containing w in its interior, and $\lambda = (\lambda_i)_{i=1}^k$ be convex weights. Let $h_\lambda := \sum_{i=1}^k \lambda_i h_i$ be the generic point in Q, and choose j such that $\lambda_j \geq 1/k$; using item 1 and 3 we obtain

$$d(h_\lambda) \leq d(\lambda_j h_j) \leq k d(h_j) \leq k \max_{1 \leq i \leq k} d(h_i) =: C_Q < \infty.$$

Thus d is bounded by C_Q on Q, so d is locally bounded. □

PROOF OF THEOREM 3. The continuity properties follow from Theorem 11, and the properties of $\mathcal{P}$ from Lemma 12; let us prove the asymptotic properties, assuming $U(\infty) = \infty$. Since $\tilde{u}(\cdot, p)$ is increasing we can assume without loss of generality that $x_n = x$. Since p is not an arbitrage-free price, by definition there exists an $X \in \mathcal{X}(-qp, q)$ such that the inequality $X_T \geq 0$ holds and is strict with strictly positive probability. By monotone convergence

$$\lim_n \mathbb{E}[U(x + nX_T)] = P(X_T = 0)U(x) + P(X_T > 0)U(\infty) = \infty,$$

and so $\tilde{u}(x,p)=\infty$ for any $x>0$, and so given an arbitrary $M>0$ we can find a $q\in\mathbb{R}^n$ such that $u(\frac{x}{2}-qp,q)\geq M$. But then, if $\mathcal{P}\ni p_n\to p\notin\mathcal{P}$, for n big enough $q(p_n-p)<\frac{x}{2}$ and so

$$\tilde{u}(x,p_n)\geq\tilde{u}(\frac{x}{2}+q(p_n-p),p_n)\geq u(\frac{x}{2}+q(p_n-p)-qp_n,q)=u(\frac{x}{2}-qp,q)\geq M,$$

which proves $\lim_n\tilde{u}(x,p_n)=\infty$.

If the sequence $q_n:=\tilde{q}(x_n,p_n)$ was converging to some $q\in\mathbb{R}^n$, the upper-semicontinuity of u would imply

$$\infty=\lim_n\tilde{u}(x,p_n)=\lim_n u(x-q_np_n,q_n)\leq u(x-qp,q),$$

which is not possible since the concave function u is real valued on some open set. It follows analogously that q_n can not have any convergent subsequence, which implies $\lim_n|\tilde{q}(x_n,p_n)|=\infty$; since $|\tilde{q}|\leq m$, also $m(x_n,p_n)$ diverges. Let us now prove the results about the function m. The positive homogeneity in x is trivial and, since $\mathcal{P}=\{p:(1,p)\in\mathcal{L}\}$ (see [Sior 12, Lemma 3]), $m(1,p)$ is bounded above by $d(w)$ for $w:=(1,p)$. Thus m is finite, and by compactness this implies that the supremum in (8) is attained. To conclude, we only need to show that $m(1,\cdot)$ is convex, as this will also imply the continuity of $m(\cdot,\cdot)$. Note that

$$m(x,p)=\max\left\{\,|q|:(z,q)\in(\mathbb{R}\times\mathbb{R}^n)\cap\bar{\mathcal{K}}\text{ and }\ (z,q)(1,p)\leq x\right\},\qquad(30)$$

since trivially any maximizer $(\bar{z},\bar{q})$ of (30) satisfies $(\bar{z},\bar{q})(1,p)=x$. From Lemma 13 it follows that, for fixed $w\in\mathcal{L}$, there exists $\varepsilon>0$ s.t.

$$2\varepsilon\sup\left\{d(w'):w'\in B_\varepsilon(w)\right\}<1,\qquad(31)$$

where $B_\varepsilon(w)$ is the ball of radius ε centered at w. Let $p\in\mathcal{P}$, $p_0,p_1\in B_\varepsilon(p)$, and define $w:=(1,p)$ and $w_i:=(1,p_i)$ for $i=0,1$. Fix a generic $\lambda\in(0,1)$, let

$$p_\lambda:=\lambda p_1+(1-\lambda)p_0\text{ and }w_\lambda:=\lambda w_1+(1-\lambda)w_0=(1,p_\lambda),$$

and use (30) to choose a $v_\lambda=(z_\lambda,q_\lambda)\in(\mathbb{R}\times\mathbb{R}^n)\cap\bar{\mathcal{K}}$ that satisfies $|q_\lambda|=m(1,p_\lambda)$ and $v_\lambda w_\lambda\leq 1$, so that necessarily $v_\lambda w_\lambda=1$. Suppose that we can build $t_0,t_1\in\mathbb{R}$ such that

$$t_i>0,\quad t_iv_\lambda w_i\leq 1\text{ for }i=0,1\ \text{ and }\ \ 1=\lambda t_1+(1-\lambda)t_0.\qquad(32)$$

Then, if we define $v_i:=(z_i,q_i):=t_i(z_\lambda,q_\lambda)=t_iv_\lambda$ for $i=0,1$, we have that $v_i\in\bar{\mathcal{K}}$, $v_iw_i\leq 1$ and $v_\lambda=\lambda v_1+(1-\lambda)v_0$, and so (30) implies

$$m(1,p_\lambda)=|q_\lambda|\leq\lambda|q_1|+(1-\lambda)|q_0|\leq\lambda m(1,p_1)+(1-\lambda)m(1,p_0),$$

which shows that $m(1,\cdot)$ is locally convex and thus[13] convex; so, to conclude we just need to build t_0, t_1 that satisfy (32). To do so, define t_1 such that $t_1 v_\lambda w_1 = 1$, i.e.,

$$t_1 := \frac{1}{1 + (1-\lambda)v_\lambda(w_1 - w_0)}.$$

Note that t_1 is (well defined and) strictly positive, since (31) implies that

$$|v_\lambda(w_1 - w_0)| \leq 2|v_\lambda||(w_1 - w)| \leq 2d(1, w_\lambda)\varepsilon < 1. \tag{33}$$

Now, define t_0 such that $1 = \lambda t_1 + (1-\lambda)t_0$ holds. It is easy to show that (33) implies $t_0 > 0$, and somewhat lengthy but straightforward[14] algebraic manipulations show that $t_0 v_\lambda w_0 \leq 1$ is equivalent to $(\lambda - 1)(v_\lambda(w_1 - w_0))^2 \leq 0$, and so it is satisfied. □

10 The one dimensional case

In this section we prove Theorem 2; we will need the following lemma.

Lemma 14 *Under the assumptions of Theorem 1, if there is only one European contingent claim f, and $p_1, p_2 \in \mathcal{P}, x > 0$, then*

1. *If $p_2 < p_1$ and $\tilde{q}(x, p_1) > 0$ then $\tilde{q}(x, p_2) > 0$ and $\tilde{u}(x, p_1) < \tilde{u}(x, p_2)$.*
2. *If $p_2 > p_1$ and $\tilde{q}(x, p_1) < 0$ then $\tilde{q}(x, p_2) < 0$ and $\tilde{u}(x, p_1) < \tilde{u}(x, p_2)$.*

PROOF Assume that $p_2 < p_1$ and $q_1 := \tilde{q}(x, p_1) > 0$, and let $a_1 := x - q_1 p_1$, so that $u(a_1, q_1) = \tilde{u}(x, p_1) \geq w(x) > -\infty$ and $(a_1, q_1) \in \bar{\mathcal{K}}$. In particular

$$0 < (a_1, q_1)(1, p_2) = x - q_1(p_1 - p_2) < x,$$

and so there exists $t_1 > 1$ s.t. $t_1(a_1, q_1)(1, p_2) = x$. From [Sior 12, Lemma 7] it follows trivially that $t \mapsto u(tx, tq)$ is strictly increasing on $[1, \infty)$ if $(x, q) \in \{u > \infty\}$ and $(x, q) \neq 0$; thus

$$\tilde{u}(x, p_1) = u(a_1, q_1) < u(t_1(a_1, q_1)) \leq \tilde{u}(x, p_2). \tag{34}$$

Moreover if $q \leq 0$ then

$$(x - qp_2, q)(1, p_1) = x + q(p_1 - p_2) \leq x$$

and so $u(x - qp_2, q) \leq \tilde{u}(x, p_1)$; it follows that (34) implies that $q \neq \tilde{q}(x, p_2)$, concluding the proof of item one. Item two follows analogously. □

[13] It is enough to show this in dimension one, where a function is convex iff it is the integral of an increasing function; since a locally increasing functions is increasing, the thesis follows.

[14] Just remember to use the identities $w_0 = w_\lambda - \lambda(w_1 - w_0)$ and $v_\lambda w_\lambda = 1$.

PROOF OF THEOREM 2 As shown in [Sior 12, Corollary 8], $\mathcal{P}(x,0)$ is the image of the set $\partial u(x,0)$ through the 'perspective function' $(y,r) \mapsto r/y$. Since $\partial u(x,0)$ is compact (see [Bert 03, Proposition 4.4.2]) and convex, this implies that $\mathcal{P}(x,0)$ is also compact and convex[15] set; thus, there exist $a \leq b$ such that $[a,b] = \mathcal{P}(x,0)$. The inclusion $\mathcal{P}(x,0) \subset \mathcal{P}$ is proved in [Sior 12, Theorem 1]; from Lemma 12 it follows that there exist $\underline{p}, \overline{p}$ such that $\mathcal{P} = (\underline{p}, \overline{p})$. The continuity of $\tilde{q}$ and $\tilde{u}$ and the asymptotic results follow from Theorem 3. By definition $p \in \mathcal{P}(x,0)$ iff $\tilde{q}(x,p) = 0$, and so Lemma 14 implies that $\tilde{q}(x,\cdot)$ is strictly positive on $(\underline{p}, a)$, it equals zero on $[a,b]$, and it is strictly negative on $(b, \overline{p})$. Also, by definition $\tilde{u}(x,p) \geq u(x,0)$ with equality holding iff $p \in \mathcal{P}(x,0)$; so $\tilde{u}$ is constant on $\mathcal{P}(x,0)$, and Lemma 14 implies that it is strictly increasing on $(\underline{p}, a)$ and is strictly decreasing on $(b, \overline{p})$. $\square$

11 An example of a non-convex dependence on prices

In the case of an exponential utility U, the function $p \mapsto \tilde{u}(x,p)$ is strictly convex (see [A Il 05, Theorem 3.1]). However, in this section we will show that, in our general framework, convexity does not hold.

To do so, we will first build a counter-example using a 'utility function' U which is not differentiable at $x = 1, 3$, is convex but not strictly convex, and which satisfies otherwise all the properties we assume in Section 2 for a utility function. To obtain an example starting with a (differentiable, strictly convex) utility function, it is then enough to consider a sequence of (differentiable, strictly convex) utility functions U_n converging to U *uniformly*, and to note that trivially the corresponding maximal expected utilities $\tilde{u}_n$ converge uniformly to $\tilde{u}$; and so, since $\tilde{u}(x,\cdot)$ is not convex, for some n also $\tilde{u}_n(x,\cdot)$ must be not convex.

In our counter-example we will consider a function U which is affine in $[1/2, 1]$, $[1,3]$ and $[3,4]$, and which satisfies

$$U(1) = 0, \quad U'(1-) = 1000, \quad U'(1+) = 1, \quad U'(3+) = 1/1000,$$

so that in particular $U(3) = 2$. We then extend U to $(0,\infty)$ in a way that U is strictly increasing, convex, differentiable at all points other than $x = 1, 3$, and satisfies Inada conditions (2). We consider a market with no stocks, and one derivative f with distribution given by $P(f = 1) = 2/3$ and $P(f = -1) = 1/3$; the interval of its arbitrage-free prices is clearly $\mathcal{P} = (-1, 1)$.

[15] Because the perspective function is continuous, and sends convex sets to convex sets (as proved in [Boyd 04, Section 2.3.3]). Alternatively, convexity can also easily be proved directly from the definition of $\mathcal{P}(x,q)$

Then, the expected utility $u(x,q)$ equals $(2U(x+q)+U(x-q))/3$, and so the maximal expected utility $\tilde{u}(x,p)$ at $x=2$ is given by $\max_q g(q)$, where

$$g(q) := \tfrac{2}{3}U(2-qp+q) + \tfrac{1}{3}U(2-qp-q). \tag{35}$$

To find the maximizer $\tilde{q} = \tilde{q}(2,p)$ of g, notice that, because of the behavior of U outside $[1,3]$, it is clear that for small p we only need to consider the interval $[i,s]$ of values of q such that both $2-qp+q$ and $2-qp-q$ belong to $[1,3]$; and since g is affine on this interval, its maximum is attained on the boundary. Let us now consider the case of small $p>0$. Simple computations show that $-i = s = (1+p)^{-1}$, and since $U(x) = x-1$ on $[1,3]$ and $(1+p)^{-1} = 1-p+o(p)$ we get that

$$g(s) = \tfrac{2}{3}U(2+\tfrac{1-p}{1+p}) + \tfrac{1}{3}U(2-\tfrac{1+p}{1+p}) = \tfrac{2}{3}\Big(1+\tfrac{1-p}{1+p}\Big) = \tfrac{2}{3}\Big(2-2p+o(p)\Big).$$

and

$$g(i) = \tfrac{2}{3}U(2-\tfrac{1-p}{1+p}) + \tfrac{1}{3}U(2+\tfrac{1+p}{1+p}) = \tfrac{2}{3}\Big(1-\tfrac{1-p}{1+p}\Big) + \tfrac{2}{3} = \tfrac{2}{3}\Big(1+2p+o(p)\Big),$$

so that $g(s) > g(i)$. It follows that

$$\tilde{q}(2,p) = s = (1+p)^{-1} \quad \text{and} \quad \tilde{u}(2,p) = g(s) = 4/3 - 4p/3 + o(p).$$

Entirely analogous computations for the case of small $p<0$ give that $-i = s = (1-p)^{-1}$ and $g(s) > g(i)$, so that $\tilde{q}(2,p) = s = (1-p)^{-1} = 1+p+o(p)$ and

$$\tilde{u}(2,p) = g(s) = \tfrac{2}{3}U(2+\tfrac{1-p}{1-p}) + \tfrac{1}{3}U(2-\tfrac{1+p}{1-p}) = \tfrac{4}{3} + \tfrac{1}{3}\Big(1-\tfrac{1+p}{1-p}\Big) = \tfrac{2}{3}\Big(2-p+o(p)\Big).$$

Putting the pieces together, we obtain that

$$\partial_p \tilde{u}(2,0+) = -\frac{4}{3} < -\frac{2}{3} = \partial_p \tilde{u}(2,0-),$$

which shows that $\tilde{u}(x,\cdot)$ is not convex.

12 Differentiability

In this section we prove the differentiability in p of the value function. The following differentiability result is trivial, but useful.

Remark 15 *Under the assumptions of Theorem 1 the function*

$$\begin{array}{rcl} (0,\infty)\times\mathcal{P} & \longrightarrow & \mathbb{R} \\ (y,p) & \mapsto & \tilde{v}(y,p) \end{array}$$

is continuously differentiable, and its derivatives satisfy $\nabla_p \tilde{v}(y,p) = y\nabla_r v(y,yp)$ *and* $\partial_y \tilde{v}(y,p) = (\partial_y v + p\nabla_r v)(y,yp)$.

PROOF Since the function v defined in (12) is convex and differentiable (see [Hugo 04, Lemma 3]) and so continuously differentiable (see [Rock 70, Theorem 25.5]), the thesis follows from the identity $\tilde{v}(y,p) = v(y,yp)$. □

We also record that simple explicit examples show that the largest feasible position is not a differentiable function of the derivative's prices.

To facilitate the proof of Theorem 4 we state the following fact.

Lemma 16 *Let f, g, h be real valued functions defined on some open set $G \subseteq \mathbb{R}^n$, and assume that $f(z) \leq g(z) \leq h(z)$ for all $z \in G$, with equality at some $z = \bar{z} \in G$, and assume that h is differentiable at $z = \bar{z}$. Then, if g is differentiable at $z = \bar{z}$, it has gradient $\nabla g(\bar{z}) = \nabla h(\bar{z})$. Moreover, g is differentiable at $z = \bar{z}$ if either of the following additional conditions is satisfied:*

1. *g is convex.*
2. *f is differentiable at $z = \bar{z}$.*

We forego the proof of the previous real-analysis lemma, as it is a simple consequence of the squeeze theorem (a.k.a. sandwich theorem) and the following fact: if $L : \mathbb{R}^n \to \mathbb{R}$ is linear and $L(x) \geq o(x)$ for all x near zero then $L \equiv 0$ (as it follows from $\pm L(x) = L(\pm\varepsilon x)/\varepsilon \geq o(\pm\varepsilon x)/\varepsilon$, taking limits for $\varepsilon \to 0+$).

We will need the following lemma, a special case of which is the well known fact that the function $w = u(\cdot, 0)$ is differentiable (see [Kram 03, Theorem 2]). We recall that in general the function u is not differentiable on $\mathcal{K}$ (see [Hugo 05, Remark 3.1]).

Lemma 17 *Under the assumptions of Theorem 1, the function $t \mapsto u(t(x,q))$ is differentiable on $t > 0$ if $(x,q) \in \mathcal{K}$.*

PROOF Since $(x,q) \in \mathcal{K}$, the thesis follows from [Rock 70, Theorem 23.4] once we prove that, whenever $(y_i, r_i) \in \partial u(x,q)$, $i = 1, 2$, the equality $xy_1 + qr_1 = xy_2 + qr_2$ holds. The latter follows from the equality $xy_i + qr_i = \mathbb{E}[Y_T(y_i, r_i)(X_T(x,q) + qf)]$, since $Y_T(y_i, r_i) = U'(X_T(x,q) + qf)$ does not depend on i (for the last two identities, see [Hugo 04, Theorem 1]). □

We remark that the reason why the previous proof does not work when $(x,q) \in \partial\mathcal{K}$ is that the assumptions of [Rock 70, Theorem 23.4] are not satisfied. The assumptions of this theorem however cannot be weakened: in fact, it is easy to show[16] that the theorem fails for points not in the (relative) interior of the domain.

[16] Indeed, consider the convex function given by $g(x,y) := \max(1 - \sqrt{x}, |y|)$ for $x \geq 0$, and $g(x,y) = \infty$ if $x < 0$; then $\partial g(1,1) = (-\infty, 0] \times \{1\}$, yet the function $h(y) := g(0,y) = \max(1, |y|)$ is not differentiable at $y = 1$ even if $(a,b)(0,1)$ is constant over $(a,b) \in \partial g(1,1)$.

The previous lemma allows us to prove differentiability of $\tilde{u}$ when the solution $(x-\hat{q}p, \hat{q})$ of (7) belongs to $\mathcal{K}$; to show that it does, one can at times use that it equals $-\nabla v(\partial_x \tilde{u}, (\partial_x \tilde{u})p)$ (see Theorem 6) or that $\tilde{X}_T(x)+\tilde{q}(x)f \geq \varepsilon > 0$ (see Theorem 5) . Note that in the case of power utilities we can prove differentiability since somewhat explicit computations are possible; indeed, if $U(x) = x^\alpha/\alpha$ for some non-zero $\alpha \in (-\infty, 1)$, then $V(y) = -y^\beta/\beta$ for $\beta := \alpha/(\alpha-1)$. By homogeneity it easily follows that

$$\tilde{v}(y,p) = \frac{-y^\beta}{\beta}(-\beta\tilde{v}(1,p)),$$

and since $-\beta\tilde{v}(1,p) > 0$ the bi-conjugacy relationships yield

$$\tilde{u}(x,p) = \frac{x^\alpha}{\alpha}(-\beta\tilde{v}(1,p))^{1-\alpha}. \tag{36}$$

PROOF OF THEOREM 4. As explained in [Hugo 04, Remark 6], we can assume without loss of generality that (9) holds, so we can use Remark 15. If U is a power utility, the thesis follows from equation (36).

Denote by $y^\star := \tilde{y}(x^\star, p^\star), q^\star := \tilde{q}(x^\star, p^\star)$ the optimizers for $p = p^\star$. To prove that $p \mapsto \tilde{u}(x^\star, p)$ is differentiable if it is convex, simply apply Lemma 16 to the functions $g(p) := \tilde{u}(x^\star, p), \quad h(p) := \tilde{v}(y^\star, p) + x^\star y^\star$. To prove differentiability under the other hypothesis, for p close to $p^\star$ define

$$t(p) := \tfrac{x^\star}{x^\star + q^\star(p-p^\star)}, \text{ so that } \quad t(p) > 0 \text{ and } \; t(p)\big((x^\star - q^\star p^\star, q^\star)(1,p)\big) = x^\star.$$

Taking Lemma 17 into account, applying Lemma 16 to the functions

$$f(p) := u(t(p)(x^\star - q^\star p^\star, q^\star)), \quad g(p) := \tilde{u}(x^\star, p), \quad h(p) := \tilde{v}(y^\star, p) + x^\star y^\star$$

proves that $p \mapsto \tilde{u}(x,p)$ is differentiable at $p = p^\star$. Moreover, whenever $\tilde{u}$ is differentiable, Lemma 16 shows that

$$\nabla_p \tilde{u}(x^\star, p^\star) = \nabla_p \tilde{v}(y^\star, p^\star), = y^\star \nabla_r v(y^\star, y^\star p^\star),$$

which by Theorem 6 equals

$$\nabla_p \tilde{u}(x^\star, p^\star) = -q^\star \partial_x \tilde{u}(x^\star, p^\star).$$

Since $\tilde{q}$ and $\partial_x \tilde{u}$ are continuous function of (x,p) (by Theorem 3), the proof is finished. □

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