Normal Forms and Unfoldings of Linear Systems in Eigenspaces of (Anti)-Automorphisms of Order Two

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Abstract

In this article we classify normal forms and unfoldings of linear maps in eigenspaces of (anti)-automorphisms of order two. Our main motivation is provided by applications to linear systems of ordinary differential equations, general and Hamiltonian, which have both time-preserving and time-reversing symmetries. However the theory gives a uniform method to obtain normal forms and unfoldings for a wide variety of linear differential equations with additional structure. We give several examples and include a discussion of the phenomenon of orbit splitting.

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1 Introduction

Let V be a finite dimensional real vector space and $\mathbf{g}$ an eigenspace of an (anti)-automorphism of order two of the Lie algebra $\mathbf{gl}(V)$. Let $\mathbf{G}$ be a Lie subgroup of $\mathbf{Gl}(V)$ consisting of *structure preserving transformations* such that the action

$$L \mapsto gLg^{-1} \text{ with } L \in \mathbf{g} \text{ and } g \in \mathbf{G},$$

preserves $\mathbf{g}$. Then the $\mathbf{G}$-orbit of $L \in \mathbf{g}$ given by $\mathrm{Orb}_{\mathbf{G}}(L) = \{gLg^{-1} \mid g \in \mathbf{G}\}$ is again a subset of $\mathbf{g}$. In this paper we address the following two problems:

i) Classify all $\mathbf{G}$-orbits (normal forms) of elements L in $\mathbf{g}$;
ii) Find the unfoldings of L in $\mathbf{g}$.

We also briefly consider generalizations to abelian groups of (anti)-automorphisms of order two.

Setting and Motivation

Any local study of equilibrium points of vector fields starts with an analysis of their linearizations. These are in one-to-one correspondence with linear maps. This correspondence respects both the transformation properties of linear vector fields under linear coordinate changes and their Lie algebra structure. Moreover dynamical systems theory is often concerned with vector fields which preserve some structure. Well-known examples are equivariant, reversible and Hamiltonian vector fields. The linearizations of such vector fields preserve the same structure and the spaces of structure preserving linear maps can be identified with eigenspaces of (anti)-automorphisms of order two acting on the space of all linear maps.

Example 1.1 Consider reversible linear vector fields on $\mathbb{R}^n$. Such a vector field is determined by an infinitesimally reversible linear map L satisfying $RL = -LR$, where the linear *structure map* R satisfies $R^2 = I$ and $R \neq \pm I$. We can also write this condition as $\phi_R(L) = -L$ where the *automorphism* ϕ_R is defined as $\phi_R(A) := R^{-1}AR$ for all $A \in \mathbf{gl}(\mathbb{R}^n)$. The structure preserving transformation group $\mathbf{G}$ consists of R-equivariant maps $\mathbf{G} = \{g \in \mathbf{Gl}(\mathbb{R}^n) \mid gR = Rg\}$. ▷

Example 1.2 Similarly a Hamiltonian linear vector field is determined by an infinitesimally symplectic linear map L. Let ω be a symplectic form, ie a non-degenerate skew symmetric bilinear form, on $\mathbb{R}^{2n}$. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^{2n}$. Then there is a *structure map* J satisfying $J^* = -J$ and $J^2 = -I$ such that $\omega(x, y) = \langle x, Jy \rangle$ for all $x, y \in \mathbb{R}^{2n}$. An infinitesimally symplectic map L satisfies $\omega(Lx, y) = -\omega(x, Ly)$ or equivalently $\langle x, L^*Jy \rangle = -\langle x, JL \rangle$ for all $x, y \in \mathbb{R}^{2n}$. We write this condition as $\psi_J(L) = -L$, where ψ_J is the *anti-automorphism* defined by $\psi_J(A) := J^{-1}A^*J$, for all $A \in \mathbf{gl}(\mathbb{R}^{2n})$. In this case the structure preserving transformation group $\mathbf{G}$ consists of maps that preserve ω, that is $\mathbf{G} = \{g \in \mathbf{Gl}(\mathbb{R}^{2n}) \mid \omega(gx, gy) = \omega(x, y), \text{ for all } x, y \in \mathbb{R}^{2n}\}$, which we can rephrase using J as $\mathbf{G} = \{g \in \mathbf{Gl}(\mathbb{R}^{2n}) \mid g^*Jg = J\}$. ▷

Example 1.3 Combining the previous two examples, a reversible Hamiltonian linear vector field is determined by an infinitesimally reversible symplectic linear map L. Usually one requires that R is an anti-symplectic map, and then ϕ_R and ψ_J commute. Thus infinitesimally reversible symplectic maps on $\mathbb{R}^{2n}$ are elements of the intersection of two eigenspaces

$$\{A \in \mathbf{gl}(\mathbb{R}^{2n}) \mid \phi_R(A) = -A\} \cap \{A \in \mathbf{gl}(\mathbb{R}^{2n}) \mid \psi_J(A) = -A\},$$

which, by virtue of the fact that ϕ_R and ψ_J commute, is the simultaneous eigenspace of ϕ_R and ψ_J. The structure preserving transformation group for infinitesimally reversible symplectic maps is the intersection of the transformation groups of Examples 1.1 and 1.2: $\mathbf{G} = \{g \in \mathbf{Gl}(\mathbb{R}^{2n}) \mid gR = Rg,\ g^*Jg = J\}$. ▷

The main motivation for the theory developed in this paper is the normal form and unfolding problem for linear reversible equivariant vector fields in both the general and Hamiltonian cases. The spaces of such vector fields have been described by Lamb & Roberts [22] in the general case, and by Hoveijn, Lamb & Roberts [16] in the Hamiltonian case and can again be characterized as simultaneous eigenspaces of abelian groups of (anti)-automorphisms of order two. The theory developed in this article provides a uniform approach to all such problems and is applied to linear (Hamiltonian) reversible equivariant vector fields in Hoveijn, Lamb & Roberts [15, 17].

In some cases, including Hamiltonian and equivariant vector fields, the corresponding eigenspaces are Lie subalgebras of $\mathbf{gl}(V)$ and the normal form and unfolding theory for maps in $\mathbf{gl}(V)$ (see Section 2), carries over almost automatically. However reversible vector fields, for example, do not form a Lie subalgebra. This paper shows that, despite this, analogous normal form and unfolding theories can been developed.

Normal form and unfolding problems have a long history ranging from the classical Jordan normal form to the more modern unfolding theory of Arnol'd [1]. We give a brief overview without trying to be complete. Williamson [32] was the first to find normal forms for infinitesimally symplectic maps. Later a more constructive approach was given by Burgoyne & Cushman [4, 5]. Unfoldings of infinitesimally symplectic maps were independently given by Galin [12] and Koçak [21]. For extensive studies of particular systems also see van der Meer [23] and Cotter [7]. Normal forms and unfoldings of infinitesimally reversible maps were first studied by Palmer [26] and later by Sevryuk [28] and Shih [29]. A particular example where the linear part plays a crucial role can be found in Iooss [19]. Other contributions without a direct relation to dynamical systems are Dempwolff [9], Jacobson [20] for semi-linear maps and Djukovic et al. [10] and Patera & Rousseau [27] for subspaces of $\mathbf{gl}(V)$ which are not Lie algebras. Wiegman [33] considers normal forms for maps over the quaternions. Studies of mixed structures include Hoveijn [14] on infinitesimally reversible symplectic maps and Melbourne [24] and Melbourne & Dellnitz [25] on infinitesimally symplectic equivariant maps.

Remark 1.4 Note that the description using (anti)-automorphisms is not limited to linear vector fields. In fact the latter are just the 1-jets of C^∞-vector fields. The (anti)-automorphisms can equally well be defined on k-jet spaces of vector fields, where they are still Lie algebra (anti)-automorphisms of order two. The normalization procedures for C^∞-vector fields described in Broer et al. [3] can be combined with the ideas developed in this paper to give a corresponding nonlinear normal form theory. ▷

Remark 1.5 Another generalisation of the theory would be to consider a general (compact) group Γ of (anti)-automorphisms acting on $\mathbf{gl}(V)$ and classify normal forms and unfoldings of linear maps in an isotypic component $\mathbf{g}$ of the action of Γ on $\mathbf{gl}(V)$.

Main Results

The main results of the paper are the Reduction Lemma 3.8, the Unfolding Lemma 3.10 and the Orbit Splitting Theorem 3.12. A formal statement of the Reduction Lemma requires some technical notation, but it may be informally summarised as:

Reduction Lemma
The normal form of a linear map L in the eigenspace of an (anti)-automorphism is determined by the semi-simple part of L on a reduced space.

This lemma greatly simplifies the problem of finding normal forms, because the actual computations are limited to low dimensional spaces. It is essential for the Reduction Lemma that the Jordan-Chevalley decomposition holds in the eigenspaces of an (anti)-automorphism. Using the notation introduced above, the Unfolding Lemma reads as follows.

Unfolding Lemma
Let γ be an (anti)-automorphism of finite order and let $L \in \mathbf{g}$. Then the restriction of the $\mathbf{Gl}(V)$-centralizer unfolding of L in $\mathbf{g}$ is equivalent to the $\mathbf{G}$-centralizer unfolding in $\mathbf{g}$.

This means that we do not need to find a new way of computing unfoldings in a subset of $\mathbf{gl}(V)$ with a smaller structure preserving transformation group $\mathbf{G}$. We simply use a version of the existing Arnol'd or centralizer unfolding, see Lemma 2.5. As an alternative one might use the representation theory of $\mathbf{sl}(2)$ to find unfoldings, see Koçak [21] or Cushman & Sanders [8].

Orbit splitting is a well known phenomenon for infinitesimally symplectic maps. If such a map has a pair of double imaginary eigenvalues then there are two inequivalent normal forms. They may be distinguished by *signs*, sometimes called Krein signs, see Example 3.7. In general the $\mathbf{Gl}(V)$-orbit of a map $L \in \mathbf{g}$ may intersect $\mathbf{g}$ in several $\mathbf{G}$-orbits of L. The Orbit Splitting Theorem states that there are at most two such orbits.

Orbit Splitting Theorem
The intersection of $\mathbf{Gl}(V)$*-orbit of* L *in* $\mathbf{gl}(V)$ *and* $\mathbf{g}$ *consists of at most two* $\mathbf{G}$*-orbits.*

In general inequivalent $\mathbf{G}$-orbits have different unfoldings, which may give rise to *passing* and *splitting* of eigenvalues when parameters are varied, see Section 4.2 for details.

Organization

The remainder of this article is organized as follows. In Section 2 we review the theory for normal forms and unfoldings in $\mathbf{gl}(V)$. We use this as a starting point for finding normal forms and unfoldings in the eigenspace of an (anti)-automorphism in Section 3. In Section 4 we apply the results of Section 3 to present normal forms and unfoldings in eigenspaces of (anti)-automorphisms of order two. Finally in Section 5 we generalize our results to abelian groups of (anti)-automorphisms of order two. We also suggest some further possible generalizations.

2 Normal Forms and Unfoldings in $\mathbf{gl}(V)$

A linear differential equation is given by $\dot{x} = Ax$ where $A \in \mathbf{gl}(V)$. A coordinate change $y = gx$, with $g \in \mathbf{Gl}(V)$, transforms this to $\dot{y} = gAg^{-1}y$. Thus linear vector fields transform as linear maps. We therefore identify the space of linear vector fields on V with $\mathbf{gl}(V)$. Here we review the normal from and unfolding theory for linear maps in $\mathbf{gl}(V)$ in Sections 2.1 and 2.2 respectively.

2.1 Normal Forms

Let V be a finite dimensional real vector space. Then $\mathbf{gl}(V)$ is the Lie algebra of all linear maps from V to itself. The Lie group $\mathbf{Gl}(V)$ is the group of all invertible linear transformations from V to itself. The action of $\mathbf{Gl}(V)$ on $\mathbf{gl}(V)$ is given by the *adjoint action*, that is, by *similarity transformations*:

$$\mathrm{Ad}_g : L \mapsto gLg^{-1}$$

The $\mathbf{Gl}(V)$-orbits

$$\mathrm{Orb}_{\mathbf{Gl}(V)}(L) \;=\; \{gLg^{-1} \mid g \in \mathbf{Gl}(V)\}$$

of the adjoint action are precisely the equivalence classes we are interested in classifying. From now on we will use the word 'orbits' only. It is well known that for the adjoint action of $\mathbf{Gl}(V)$ the orbit of L in $\mathbf{gl}(V)$ is determined by two invariants: the eigenvalues and Jordan structure of L. The *Jordan-Chevalley decomposition*, *Reduction Lemma* and *Reconstruction Lemma* described below formalize this fact.

Jordan-Chevalley Decomposition

The Jordan-Chevalley decomposition splits a linear map L into the sum of its semi-simple and nilpotent parts. A map S is called *semi-simple* if the algebraic and geometric multiplicity of each of its eigenvalues are equal. A map N is called *nilpotent* if $N^n = 0$ for some integer n. The least such integer is called the *height* of N.

Theorem 2.1 (Jordan-Chevalley decomposition)
For each $L \in \mathbf{gl}(V)$ *there exist a unique semi-simple* $S \in \mathbf{gl}(V)$ *and a unique nilpotent* $N \in \mathbf{gl}(V)$ *such that* $[S, N] = 0$ *and* $L = S + N$.

The eigenvalues of L are determined by the semi-simple part S while the nilpotent part N determines its Jordan structure. The Jordan-Chevalley decomposition of a linear map is Ad_g-equivariant and so is a

property of the $\mathbf{Gl}(V)$-orbit rather than the individual map, see Humphreys [18]. The Jordan-Chevalley decomposition almost automatically extends to all the classical Lie algebras, see [18, 31]. We shall see in Section 3 that it also extends to eigenspaces of Lie algebra (anti)-automorphisms acting on $\mathbf{gl}(V)$.

Reduction

The Reduction Lemma, due to Burgoyne & Cushman [5, 6], exploits the Jordan-Chevalley decomposition to simplify the normal form problem for linear maps. It formalizes the classical Jordan normal form algorithm and works in all classical Lie algebras.

Let $L \in \mathbf{gl}(V)$ be a linear map with Jordan-Chevalley decomposition $L = S + N$. An L-invariant subspace of V is said to be *indecomposable* if it has no proper L-invariant subspaces. The restriction of L to an indecomposable L-invariant subspace has a unique eigenvalue and such a subspace is a generalized eigenspace for that eigenvalue. The space V decomposes as a direct sum of indecomposable L-invariant subspaces $V = \oplus_\lambda V_\lambda$ which is unique up to permutations unless there are two or more V_λ with equal eigenvalues *and* equal heights. Moreover the restrictions of S and N to the indecomposable subspaces are the semi-simple and nilpotent parts of the restrictions of L. The characteristic polynomial of L factors over the indecomposable L-invariant subspaces. The height n of the restriction of N to such a space can be computed from the characteristic polynomial of the restriction of L, see Burgoyne & Cushman [4].

Lemma 2.2 (Reduction Lemma)
Let V_λ be an indecomposable L-invariant space with eigenvalue λ. Assume that the restriction of N to V_λ has height n. Then there is an S-invariant complement W_λ of NV_λ in V_λ such that $V_\lambda = W_\lambda \oplus NV_\lambda = W_\lambda \oplus NW_\lambda \oplus \cdots \oplus N^{n-1}W_\lambda$. For each $j = 0 \ldots n-1$ we have $\dim N^j W_\lambda = \dim W_\lambda$. The restriction of L to V_λ is determined up to similarity by the restriction of S to W_λ.

By Lemma 2.2 we only have to classify the semi-simple part S.

Theorem 2.3 *Let $L = S+N$ be the Jordan-Chevalley decomposition of L and let V_λ be an indecomposable L-invariant subspace with eigenvalue λ. Assume that the restriction of N to V_λ has height n.*

a) **Real eigenvalues.** *If $\lambda = \alpha \in \mathbb{R}$ then $(L-\alpha)^n = 0$ for some n. In this case $\dim W_\alpha = 1$, $\dim V_\alpha = n$ and for all $e \in V_\alpha$ we have $Se = \alpha e$.*

b) **Complex eigenvalues.** *If $\lambda = \alpha \pm i\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$ then $((L-\alpha)^2 + \beta^2)^n = 0$ for some n. In this case $\dim W_{\alpha \pm i\beta} = 2$, $\dim(V_{\alpha \pm i\beta}) = 2n$ and for all $v \in V_{\alpha \pm i\beta}$ we have $(S-\alpha)^2 v = -\beta^2 v$. For any $e \in W_{\alpha \pm i\beta}$ let $f = \frac{1}{\beta}(S-\alpha)e$, then $\langle e, f \rangle$ is a basis of $W_{\alpha \pm i\beta}$.*

Thus the restriction of the semi-simple part S to a subspace W_λ always has normal form

$$\begin{pmatrix} \alpha \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

depending on whether λ is real or complex, respectively. Note that in case of complex eigenvalues we can always find a basis such that $\beta > 0$.

Reconstruction

Suppose we are given a linear map $L \in \mathbf{gl}(V)$ with Jordan-Chevalley decomposition $L = S + N$ and an indecomposable L-invariant space V_λ. From Lemma 2.2 we know that there is an S-invariant complement W_λ of NV_λ in V_λ such that $V_\lambda = W_\lambda \oplus NW_\lambda \oplus \cdots \oplus N^{n-1}W_\lambda$, where n is the height of N on V_λ and $\dim N^j W_\lambda = \dim W_\lambda$. In Theorem 2.3 we gave normal forms for the restriction of S to W_λ. To find normal forms for L on V_λ we start with the basis of W_λ used for the normal form of S on W_λ in Theorem 2.3 and apply N to this $n-1$ times to generate a basis for V_λ.

Lemma 2.4 (Reconstruction Lemma)
Let $L = S + N$ be the Jordan-Chevalley decomposition of L and V_λ be an indecomposable L-invariant subspace. Let W_λ be an S-invariant complement to NV_λ in V_λ and assume that N has height n on V_λ. Then, if $\langle e_1 \ldots, e_m \rangle$ is a basis for W_λ, the set $\langle e_1 \ldots, e_m, Ne_1 \ldots, Ne_m, N^{n-1}e_1 \ldots, N^{n-1}e_m \rangle$ is a basis for V_λ.

The corresponding normal forms for indecomposable L-invariant subspaces with real and complex eigenvalues are the familiar real Jordan blocks

$$\begin{pmatrix} \alpha & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \alpha \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha & -\beta & & & & & & \\ \beta & \alpha & & & & & & \\ 1 & 0 & \alpha & -\beta & & & & \\ 0 & 1 & \beta & \alpha & & & & \\ & & \ddots & & \ddots & & & \\ & & & \ddots & & \ddots & & \\ & & & & 1 & 0 & \alpha & -\beta \\ & & & & 0 & 1 & \beta & \alpha \end{pmatrix}$$

with respect to the bases $\langle e, Ne, \ldots, N^{n-1}e\rangle$ and $\langle e, f, Ne, Nf, \ldots, N^{n-1}e, N^{n-1}f\rangle$, respectively. Here e and f are vectors as in Theorem 2.3.

2.2 Unfoldings

A general theory for *unfoldings* or *deformations* of maps in $\mathbf{gl}(V)$ is given by Arnol'd [1]. Here we use the Reduction Lemma to describe this theory, see also [14]. Note that $\mathbf{Gl}(V)$-orbits are smooth submanifolds of $\mathbf{gl}(V)$, see Bredon [2].

Definition 2.1 *A smooth map $\mathcal{L} : \mathbb{R}^p \to \mathbf{gl}(V) : \mu \mapsto \mathcal{L}(\mu)$ with $\mathcal{L}(0) = L$ is called an* unfolding *or a* deformation *of L. If $\mathcal{L}$ is transverse to the $\mathbf{Gl}(V)$-orbit through L at L, then it is said to be* versal.

¿From now on we will only use the word 'unfolding'. We are especially interested in unfoldings of a map L having a minimum number of parameters but still parametrising a section transverse to the $\mathbf{Gl}(V)$-orbit through L_0. We therefore make the following definition, see Arnol'd [1].

Definition 2.2 *Two unfoldings $A(\mu)$ and $B(\mu)$ of L are called* equivalent *if they are similar as families of linear maps. This means that there is a smooth family of transformation $g(\mu) \in \mathbf{Gl}(V)$ such that $g(\mu)A(\mu)g(\mu)^{-1} = B(\mu)$ for all $\mu \in \mathbb{R}^p$. An unfolding $\mathcal{L}$ of L is called* miniversal *if (a) for every other unfolding $A : \mathbb{R}^q \to \mathbf{gl}(V)$ of L there exists a smooth map $\chi : \mathbb{R}^q \to \mathbb{R}^p$ such that A is equivalent to $\mathcal{L} \circ \chi$, and (b) $\mathcal{L}$ has the minimal number of parameters possible for unfoldings with this property.*

The number of parameters for a miniversal unfolding is equal to the codimension of the $\mathbf{Gl}(V)$-orbit through L and so is called the *codimension* of L. Geometrically, the image of a miniversal unfolding of L is a submanifold of $\mathbf{gl}(V)$ whose tangent space at L is a complement to the tangent space at L of the $\mathbf{Gl}(V)$-orbit through L.

Arnol'd [1] showed that miniversal unfoldings can be obtained by taking orthogonal complements to tangent spaces of $GL(V)$-orbits with respect to the inner product $\langle A, B\rangle = \mathrm{Trace}(A^*B)$ on $\mathbf{gl}(V)$. Some computations show that the *centralizer* or *Arnol'd unfolding* is:

$$\{L + M^* \mid M \in \mathrm{Ker}(\mathrm{ad}_L)\},$$

where $\mathrm{ad}_L(M) = [M, L] = ML - LM$. By applying the adjoint action of $\mathbf{Gl}(V)$ on $\mathbf{gl}(V)$ to this unfolding we can obtain an unfolding at any other point on the $GL(V)$ orbit through L. Transversality and miniversality are preserved by this transformation, but orthogonality will usually be lost.

We now give a more detailed description of the centralizer unfolding using the Reduction Lemma. The idea is to start with the centralizer unfolding of the semi-simple part of L and then reconstruct the unfolding of L in several steps. An advantage of this approach is that it depends only on the Jordan-Chevalley decomposition and not on a particular normal form for L. Furthermore we only have to compute matrices which commute with a semi-simple matrix on a low dimensional space.

Assume that $L = S + N$ has only one eigenvalue on V and let $V = V_1 \oplus \cdots \oplus V_p$, where each V_i is an indecomposable L-invariant subspace. For each V_i we have the decomposition $V_i = W_i \oplus NW_i \oplus \cdots \oplus N^{n_i-1}W_i$, where W_i is an S-invariant complement to NV_i in V_i. Assume that the heights n_i of N

restricted to V_i satisfy $n_1 \geq \cdots \geq n_p$. Let $W = W_1 \oplus \cdots \oplus W_p$ and let $\langle e_{i,1}, \ldots, e_{i,q} \rangle$ be a basis for W_i. Note that the W_i all have the same dimension $\dim W_i = q$, so we may assume that their bases have been chosen such that the matrices of $S_i = S|_{W_i}$ are equal.

To find the centralizer unfolding of L first choose a basis for $\mathbf{u}_{W_1} = \{M : W_1 \to W_1 \mid \mathrm{ad}_S(M) = 0\}$. Only in this step is it necessary to compute commuting matrices. The next step is to extend the basis for $\mathbf{u}_{W_1}$ to a basis for $\mathbf{u}_W = \{M : W \to W \mid \mathrm{ad}_S(M) = 0\}$. Then we extend this basis to the set of maps on V which commute with N. The final step is to construct a basis for $\mathbf{u}_V = \{M : V \to V \mid \mathrm{ad}_L(M) = 0\}$. We make this more precise in the following Unfolding Lemma, which is a corrected version of that in [14].

Lemma 2.5 (Unfolding Lemma)

1. *Construct a basis* $\langle B^{(1)}, \ldots, B^{(r)} \rangle$ *of* $\mathbf{u}_{W_1}$.
2. *Extend the basis of* $\mathbf{u}_{W_1}$ *to a basis of* $\mathbf{u}_W$ *by defining* $B_{ij}^{(k)} : W \to W$ *by:*

$$B_{ij}^{(k)} \;=\; \begin{cases} B^{(k)} & : \; W_j \to W_i \\ 0 & : \; W_{j'} \to W_{i'}, \; (i',j') \neq (i,j) \end{cases}$$

   Then $\{B_{ij}^{(k)} \mid i,j = 1, \ldots, p \text{ and } k = 1, \ldots, r\}$ *is a basis of* $\mathbf{u}_W$.
3. *Extend the basis of* $\mathbf{u}_W$ *to a basis of* $\mathbf{u}_V$ *by defining* $\tilde{B}_{ij}^{(k)} : V \to V$ *by:*

$$\tilde{B}_{ij}^{(k)} N^m e_{j,l} \;=\; \begin{cases} N^m B_{ij}^{(k)} e_{j,l} & j \geq i \\ N^{m+n_i-n_j} B_{ij}^{(k)} e_{j,l} & j < i \end{cases}$$

   for $l = 1, \ldots, q$ *and* $m = 0, \ldots, n_j - 1$. *Then the* $N^n \tilde{B}_{ij}^{(k)}$ *for* $0 \leq n < \min(n_i, n_j)$ *form a basis of* $\mathbf{u}_V$.
4. *Let* $\langle B_1, \ldots, B_d \rangle$ *be a basis of* $\mathbf{u}_V$, *then* $L(\mu) = L + \sum_{i=1}^{d} \mu_i B_i^*$ *is a miniversal deformation of* L.

Proof. The construction in the proposition is a formalization of the construction by Arnol'd [1], which in turn is based on Gantmacher's construction in [13]. We restrict ourselves to the case that L has a real eigenvalue. First we prove that the $B_1, \ldots, B_d$ are linearly independent, then we prove that they span $\mathbf{u}_V$. By construction the $B^{(1)}, \ldots, B^{(r)}$ are linearly independent. Then the $B_{ij}^{(k)}$ are also linearly independent since they map W_j to W_i. In step 3) the $B_{ij}^{(k)}$ are only extended to V so the $\tilde{B}_{ij}^{(k)}$ are still linearly independent. (Here we have to take into account that the height of N on W_j may be smaller than the height of N on W_i so that $N^{m+n_i-n_j} = 0$ as soon as $N^m = 0$.) The $N^n \tilde{B}_{ij}^{(k)}$ are linearly independent because they map W_j to $N^n W_i$. The number of $N^n \tilde{B}_{ij}^{(k)}$ is equal to $\sum_{i,j} \min(n_i, n_j) = \sum_i (2i-1) n_i$ which is equal to the number of parameters in the Arnol'd unfolding, so the $N^n \tilde{B}_{ij}^{(k)}$ span $\mathbf{u}_V$. Thus the $N^n \tilde{B}_{ij}^{(k)}$ form a basis of $\mathbf{u}_V$. □

3 Normal Forms and Unfoldings in an Eigenspace of an (Anti)-Automorphism

We outline a general theory for maps in eigenspaces of Lie algebra (anti)-automorphisms of order two acting on $\mathbf{gl}(V)$ and show that a $\mathbf{Gl}(V)$-orbit in $\mathbf{gl}(V)$ can split into at most two $\mathbf{G}$-orbits when intersected with such an eigenspace. Here $\mathbf{G}$ is the structure preserving transformation group.

To characterize normal forms in Section 2.1 and find their unfoldings in Section 2.2 we made essential use of the following facts. First, the equivalence classes are the orbits of the adjoint action of a transformation group. Second, the Jordan-Chevalley decomposition leads to a reduction of the normal form and unfolding problems to semi-simple maps. Third, the Lie algebra of the transformation group enables us to characterize tangent spaces to orbits and hence to find miniversal unfoldings as complements.

In this section we show that the eigenspaces of (anti)-automorphisms have all the Lie algebraic structure that is needed to generalize these facts. Appropriate *structure preserving* transformation Lie groups can be defined and the Jordan-Chevalley decomposition still holds. The latter leads to a Reduction Lemma which can be used to simplify the computation of normal forms and miniversal unfoldings.

3.1 (Anti)-Automorphisms

We begin by describing some properties of (anti)-automorphisms of $\mathbf{gl}(V)$. Where it is convenient we identify V with $\mathbb{R}^n$ and hence $\mathbf{gl}(V)$ with $\mathbf{gl}(n,\mathbb{R})$.

Definition 3.1 *A linear map $\gamma : \mathbf{gl}(V) \to \mathbf{gl}(V)$ is an* automorphism *of $\mathbf{gl}(V)$ if $\gamma(AB) = \gamma(A)\gamma(B)$ and an* anti-automorphism *if $\gamma(AB) = \gamma(B)\gamma(A)$.*

We will generally denote an automorphism by ϕ, an anti-automorphism by ψ and will write γ if we do not want to make a distinction. Let

$$\sigma(\gamma) \;=\; \begin{cases} 1, & \text{if } \gamma \text{ is an automorphism} \\ -1, & \text{if } \gamma \text{ is an anti-automorphism.} \end{cases}$$

Note that if γ is an automorphism or anti-automorphism then the Lie bracket $[A,B] = AB - BA$ on $\mathbf{gl}(V)$ satisfies

$$\gamma\left([A,B]\right) \;=\; \sigma(\gamma)\,[\gamma(A),\gamma(B)]\,,$$

so γ is a Lie algebra automorphism or anti-automorphism.

The next proposition describes the (anti)-automorphisms of $\mathbf{gl}(V) \cong \mathbf{gl}(n,\mathbb{R})$ and shows that they have associated *structure maps* $\mathbf{s} \in \mathbf{Gl}(n,\mathbb{R})$. A proof based on the symmetries of the Dynkin diagram can be found in Freudenthal & de Vries [11]. We will use $\gamma_{\mathbf{s}}$ to denote either $\phi_{\mathbf{s}}$ or $\psi_{\mathbf{s}}$.

Proposition 3.1 *Every automorphism of $\mathbf{gl}(n,\mathbb{R})$ has the form $\phi_{\mathbf{s}}(L) = \mathbf{s}^{-1}L\mathbf{s}$, $L \in \mathbf{gl}(n,\mathbb{R})$, for some $\mathbf{s} \in \mathbf{Gl}(n,\mathbb{R})$. The anti-automorphisms of $\mathbf{gl}(n,\mathbb{R})$ are all of the form $\psi_{\mathbf{s}} = \phi_{\mathbf{s}} \circ \psi_I$ where $\psi_I(L) = L^*$.*

Here L^* is defined as $\langle x, L^*y\rangle := \langle Lx, y\rangle$, for all $x, y \in \mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ is an inner product on $\mathbb{R}^n$. In the next lemma we collect some simple properties of an (anti)-automorphism of order two.

Lemma 3.2 *Let γ be an (anti)-automorphism of order two on $\mathbf{gl}(V)$ and $\mathbf{s}$ the structure map associated to γ. Then:*

a) $\gamma_{\alpha\mathbf{s}} = \gamma_{\mathbf{s}}$ *for all* $\alpha \in \mathbb{R}$*;*
b) The eigenvalues of γ *are* ± 1*;*
c) γ *is semi-simple.*

Note that a) holds for any (anti)-automorphism and c) is true for (anti)-automorphisms of finite order. The next proposition lists some properties of structure maps.

Proposition 3.3 *Let $\mathbf{s}$ be a structure map associated to an (anti)-automorphism γ of order two. Then we may assume that $\mathbf{s}$ has the following properties:*

a) $\det \mathbf{s} = \pm 1$*;*
b) $\mathbf{s}$ *is orthogonal;*
c) $\mathbf{s}^2 = \pm I$*;*
d) The matrix of $\mathbf{s}$ *is given by* $\mathbf{s} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ *or* $\mathbf{s} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.

Here a) is true in general because it is a consequence of property a) in Lemma 3.2 and b) holds for any (anti)-automorphism of finite order.

Proof of Proposition 3.3. Using $\gamma_{\alpha\mathbf{s}} = \gamma_{\mathbf{s}}$ for all $\alpha \in \mathbb{R}$, we can scale $\mathbf{s}$ such that $\det \mathbf{s} = \pm 1$. For the remaining parts we distinguish between automorphisms and anti-automorphisms. We use the fact that $\mathbf{s}$ transforms as $\mathbf{s} \mapsto g\mathbf{s}g^{-1}$ or $\mathbf{s} \mapsto g\mathbf{s}g^*$ under a coordinate transformation g depending on whether $\mathbf{s}$ is associated to an automorphism or an anti-automorphism, see Section 3.2. Let $\gamma_{\mathbf{s}} = \phi_{\mathbf{s}}$ be an automorphism. Then $\phi_{\mathbf{s}}^2 = I$ immediately implies $\mathbf{s}^2 = \pm I$. This in turn implies that $\mathbf{s}$ has eigenvalues ± 1 or $\pm i$ and that $\mathbf{s}$ is semi-simple, which yields the normal forms in d). Thus after a suitable transformation $\mathbf{s}$ is orthogonal. Let $\gamma_{\mathbf{s}} = \psi_{\mathbf{s}}$ be an anti-automorphism. Then $\psi_{\mathbf{s}}^2 = I$ implies $\mathbf{s}^* = \pm s$ and so $\mathbf{s}$ is semi-simple. Thus $\mathbf{s}$ has either real or purely imaginary eigenvalues. Restrict $\mathbf{s}$ to an indecomposable $\mathbf{s}$-invariant space, then by applying the scaling $g = \rho I$ the eigenvalues of $\mathbf{s}$ are scaled to ± 1 or $\pm i$. This transformed $\mathbf{s}$ is orthogonal and moreover satisfies $\mathbf{s}^2 = \pm I$. Thus it takes one of the forms in d) with respect to a suitable basis. □

3.2 Eigenspaces of (Anti)-Automorphisms

The eigenspace of an (anti)-automorphism γ corresponding to an eigenvalue μ is denoted by

$$\mathbf{gl}_\mu(V) \;=\; \{L \in \mathbf{gl}(V) \mid \gamma(L) = \mu L\}.$$

The next lemma gives some basic properties of the eigenspaces of γ.

Lemma 3.4 *Let γ be an (anti)-automorphism of order two on $\mathbf{gl}(V)$. Then:*

a) $\mathbf{gl}_\mu(V)$ is a Lie subalgebra of $\mathbf{gl}_\mu(V)$ if and only if $\mu = \sigma(\gamma)$;

b) $\mathbf{gl}_\mu(V)$ is a Lie submodule of $\mathbf{gl}(V)$ over the Lie subalgebra $\mathbf{gl}_{\sigma(\gamma)}(V)$;

c) $\mathbf{gl}(V)$ splits as a sum of eigenspaces, $\mathbf{gl}(V) = \mathbf{gl}_1(V) \oplus \mathbf{gl}_{-1}(V)$.

Note that a) and b) hold for any (anti)-automorphisms of finite order.

Proof of Lemma 3.4. Let $A \in \mathbf{gl}_\mu(V)$ and $B \in \mathbf{gl}_\nu(V)$. Then $\gamma([A,B]) = \sigma(\gamma)[\gamma(A),\gamma(B)] = \sigma(\gamma)\mu\nu[A,B]$. Since γ is of order two it is semi-simple and its eigenvalues are ± 1, and so $\mathbf{gl}(V)$ splits as in c). □

Jordan-Chevalley Decomposition

The Jordan-Chevalley Decomposition 2.1 holds in any Lie subalgebra $\mathbf{g}$ of $\mathbf{gl}(V)$, see [18]. We now show that it holds in any eigenspace of an (anti)-automorphism of $\mathbf{gl}(V)$.

Proposition 3.5 *If $L \in \mathbf{gl}_\mu(V)$ and $L = S + N$ with S semi-simple, N nilpotent and $[S,N] = 0$, then both S and N are elements of $\mathbf{gl}_\mu(V)$.*

Proof. Let $L \in \mathbf{gl}_\mu(V)$ have Jordan-Chevalley decomposition $L = S + N$ with $S, N \in \mathbf{gl}(V)$. Then $\gamma(L) = \mu L = \mu S + \mu N$ and so $\mu S + \mu N$ is the Jordan-Chevalley decomposition of $\gamma(L)$. We also have $\gamma(L) = \gamma(S) + \gamma(N)$. Since $\gamma(S) = \mathbf{s}^{-1}S\mathbf{s}$ or $\gamma(S) = \mathbf{s}^{-1}S^*\mathbf{s}$ we see that $\gamma(S)$ is semi-simple. Furthermore $\gamma(N)^n = \gamma(N^n) = 0$ and so $\gamma(N)$ is nilpotent. Finally $[\gamma(S),\gamma(N)] = \sigma(\gamma)\gamma([S,N]) = 0$. Thus $\gamma(S)+\gamma(N)$ is the Jordan-Chevalley decomposition of $\gamma(L)$. Since this decomposition is unique we have $\gamma(S) = \mu S$ and $\gamma(N) = \mu N$. □

Remark 3.1 Note that the subspaces of symmetric and skew-symmetric matrices in $\mathbf{gl}(n,\mathbb{R})$, the ± 1 eigenspaces of the anti-automorphism $L \mapsto L^*$, both consist entirely of semi-simple matrices. Thus for both these eigenspaces the Jordan-Chevalley decomposition is trivial. ▷

Coordinate Transformations

We next look at coordinate transformations. If $L \in \mathbf{gl}(V)$ and $g \in \mathbf{Gl}(V)$, then applying the coordinate change g transforms L to gLg^{-1}. An automorphism $\phi_{\mathbf{s}}$ of $\mathbf{gl}(V)$ transforms to $\phi_{g\mathbf{s}g^{-1}}$ and an anti-automorphism $\psi_{\mathbf{s}}$ to $\psi_{g\mathbf{s}g^*}$. We therefore consider the action of $\mathbf{Gl}(V)$ on pairs $(L,\mathbf{s})$ given by $(L,\mathbf{s}) \mapsto g.(L,\mathbf{s})$, where $g.(L,\mathbf{s})$ is a shorthand for $(gLg^{-1}, g\mathbf{s}g^{-1})$ if $\gamma_{\mathbf{s}}$ is an automorphism and $(gLg^{-1}, g\mathbf{s}g^*)$ if $\gamma_{\mathbf{s}}$ is an anti-automorphism. Classifying pairs with respect to this action is equivalent to first bringing $\mathbf{s}$ into normal form using any transformation from $\mathbf{Gl}(V)$, and then classifying maps L using only transformations which preserve $\mathbf{s}$. The group of such transformations is

$$\begin{aligned}\mathbf{Gl}_{\mathbf{s}}^{+1} &= \{g \in \mathbf{Gl}(V) \mid g\mathbf{s}g^{-1} = \mathbf{s}\}\\ \mathbf{Gl}_{\mathbf{s}}^{-1} &= \{g \in \mathbf{Gl}(V) \mid g\mathbf{s}g^{*} = \mathbf{s}\}\end{aligned}$$

for automorphism and anti-automorphisms, respectively. The group $\mathbf{Gl}(V)$ can be indentified with the set of invertible elements in $\mathbf{gl}(V)$ and so the (anti)-automorphism $\gamma_{\mathbf{s}}$ can also be regarded as an operator on $\mathbf{Gl}(V)$. Thus $\gamma_{\mathbf{s}}(g) = \mathbf{s}^{-1}g\mathbf{s}$ for automorphisms and $\gamma_{\mathbf{s}}(g) = \mathbf{s}^{-1}g^*\mathbf{s}$ for anti-automorphisms. As before

we set $\sigma(\gamma_{\mathbf{s}})$ equal to $+1$ for automorphisms to -1 for anti-automorphims. Then the group of *structure preserving* transformations can be characterized as

$$\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma_{\mathbf{s}})}(V) \;=\; \{g \in \mathbf{Gl}(V) \mid \gamma_{\mathbf{s}}(g) = g^{\sigma(\gamma_{\mathbf{s}})}\}.$$

With help of the structure preserving transformation group we summarize the discussion so far in the following lemma.

Lemma 3.6 (Orbit Lemma)
The $\mathbf{Gl}(V)$*-orbits of pairs* $(L, \mathbf{s})$ *are equivalent to* $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma_{\mathbf{s}})}(V)$*-orbits of* L*.*

There is, however, a slightly larger transformation group that also preserves the eigenspaces. This is the subgroup of $\mathbf{Gl}(V)$ consisting of transformations which preserve the (anti)-automorphism $\gamma = \gamma_{\mathbf{s}}$, rather than the structure map $\mathbf{s}$ itself. We denote this group by

$$\mathbf{Gl}_{\gamma}(V) = \left\{ g \in \mathbf{Gl}(V) \;\middle|\; \begin{array}{rcll} \phi_{g\mathbf{s}g^{-1}} & = & \phi_{\mathbf{s}}, & \text{if } \gamma = \phi_{\mathbf{s}} \text{ is an automorphism} \\ \psi_{g\mathbf{s}g^{*}} & = & \psi_{\mathbf{s}}, & \text{if } \gamma = \psi_{\mathbf{s}} \text{ is an anti-automorphism.} \end{array} \right\}$$

Equivalently, if $\phi_g(L) = g^{-1}Lg$ for $L \in \mathbf{gl}(V)$, then

$$\mathbf{Gl}_{\gamma}(V) \;=\; \{g \in \mathbf{Gl}(V) \mid \phi_g \circ \gamma = \gamma \circ \phi_g\}\,. \tag{1}$$

In this paper we will use the transformation groups $\mathbf{Gl}_{\mathbf{s}}^{\pm 1}(V)$ in the normal form and unfolding theories. However in Lemma 3.11 below we show that the $\mathbf{Gl}_{\gamma}(V)$ orbit through $L \in \mathbf{gl}_{\mu}(V)$ is precisely the intersection of the $\mathbf{Gl}(V)$ orbit in $\mathbf{gl}(V)$ with $\mathbf{gl}_{\mu}(V)$. Thus the difference between the two groups is closely related to the phenomenon of orbit splitting. The following proposition describes some of the elementary properties of these transformation groups.

Remark 3.2 With a slight abuse of notation we define $\mathbf{Sl}(V) := \{g \in \mathbf{Gl}(V) \mid \det g = \pm 1\}$. Note that the $\mathbf{Gl}(V)$ and $\mathbf{Sl}(V)$ orbits through any L are equal. We can therefore always work with the subgroups $\mathbf{Sl}_{\mathbf{s}}^{\sigma(\gamma_{\mathbf{s}})}(V) = \mathbf{Sl}(V) \cap \mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma_{\mathbf{s}})}(V)$ and $\mathbf{Sl}_{\gamma}(V) = \mathbf{Sl}(V) \cap \mathbf{Gl}_{\gamma}(V)$ rather than $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma_{\mathbf{s}})}(V)$ and $\mathbf{Gl}_{\gamma}(V)$ themselves.

Proposition 3.7 *Let* γ *be an (anti)-automorphism of order two on* $\mathbf{gl}(V)$ *with structure map* $\mathbf{s}$*. Let* $\mathbf{gl}_{\mu}(V)$ *be the eigenspace of* γ *with eigenvalue* μ*.*

a) The groups $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(V)$ *and* $\mathbf{Gl}_{\gamma}(V)$ *preserve the eigenspace* $\mathbf{gl}_{\mu}(V)$*.*

b) The Lie algebra of $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(V)$ *is the eigenspace* $\mathbf{gl}_{\sigma(\gamma)}(V)$*.*

c) The group $\mathbf{Sl}_{\mathbf{s}}^{\sigma(\gamma)}(V)$ *is equal to either the whole of* $\mathbf{Sl}_{\gamma}(V)$ *or to a normal subgroup of index two. If* $\dim V$ *is odd then* $\mathbf{Sl}_{\mathbf{s}}^{\sigma(\gamma)}(V) = \mathbf{Sl}_{\gamma}(V)$*. If* γ *is an automorphism then the same results hold with* $\mathbf{Gl}_{\mathbf{s}}^{-1}(V)$ *and* $\mathbf{Gl}_{\gamma}(V)$ *in place of* $\mathbf{Sl}_{\mathbf{s}}^{-1}(V)$ *and* $\mathbf{Sl}_{\gamma}(V)$*.*

Parts a) and b) of Proposition 3.7 also hold for (anti)-automorphisms of any finite order.

Example 3.3 Let ψ be the anti-automorphism $L \mapsto L^*$, for which the structure map $\mathbf{s}$ is the identity map $\mathbf{I}$. Then $\mathbf{Gl}_{\mathbf{s}}^{-1}(V) = \mathbf{Sl}_{\mathbf{s}}^{-1}(V) = \mathbf{Sl}_{\gamma}(V)$ is the group of orthogonal transformations. However $\mathbf{Gl}_{\gamma}(V)$ is the subgroup of $\mathbf{Gl}(V)$ consisting of elements g such that gg^* is a scalar multiple of the identity. Thus part c) of Proposition 3.7 does not hold with $\mathbf{Gl}_{\mathbf{s}}^{-1}(V)$ and $\mathbf{Gl}_{\gamma}(V)$ in place of $\mathbf{Sl}_{\mathbf{s}}^{-1}(V)$ and $\mathbf{Sl}_{\gamma}(V)$. $\triangleright$

Example 3.4 Consider the set of infinitesimally R-reversible maps on $\mathbb{R}^2$ with $R = \mathrm{diag}(1, -1)$ and let $\phi(L) = R^{-1}LR$. Then $\mathbf{Sl}_R^{+1}(V) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid ab = \pm 1 \right\}$ has two cosets in $\mathbf{Sl}_{\gamma}(V)$. One coset is $\mathbf{Sl}_R^{+1}(V)$, the other is $h\mathbf{Sl}_R^{+1}(V)$, where $h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In this case it is also true that $\mathbf{Gl}_{\gamma}(V) = \mathbf{Gl}_R^{+1}(V) \cup h\mathbf{Gl}_R^{+1}(V)$. $\triangleright$

Proof of Proposition 3.7. The proofs of parts *a)* and *b)* are straightforward calculations.

For part *c)* we use the homomorphism $\rho : g \mapsto \gamma(g)g^{-\sigma(\gamma)}$ of $\mathbf{Gl}(V)$. The characterization in equation (1) implies that for every element $g \in \mathbf{Gl}_\gamma(V)$ the element $\rho(g) = \gamma(g)g^{-\sigma(\gamma)}$ commutes with every linear map $L \in \mathbf{gl}(V)$ and so must be a real nonzero scalar multiple of the identity, $\gamma(g)g^{-\sigma(\gamma)} = \lambda I$, say. Let $n = \dim V$. Then taking determinants, and noting that $\det\gamma(g) = \det g$, implies that $\lambda^n = 1$ for automorphisms and $\lambda^n = (\det g)^2$ for anti-automorphisms. It follows that $\lambda = \pm 1$ for any $g \in \mathbf{Gl}_\gamma(V)$ if γ is an automorphisms, and for $g \in \mathbf{Sl}_\gamma(V)$ if γ is an anti-automorphism. Hence ρ takes values in $\mathbb{Z}_2 = \{\pm I\}$. If n is odd then $\lambda = 1$ in both cases and ρ is the constant mapping to the identity. The results now follow. □

3.3 Reduction

We will now reduce to the semi-simple case using a method analogous to that described in Section 2.1. The next lemma follows almost immediately from Lemma 2.2.

Lemma 3.8 (Reduction Lemma)
Let L be a map in $\mathbf{gl}_\mu(V) = \{L \in \mathbf{gl}(V) \mid \gamma_{\mathbf{s}}(L) = \mu A\}$, where $\gamma_{\mathbf{s}}$ is an (anti)-automorphism of order two with structure map $\mathbf{s}$. Furthermore let $L = S + N$ be the Jordan-Chevalley decomposition and let V_λ be the indecomposable L-invariant subspaces. Then there exists an indecomposable $\mathbf{s}$-invariant subspace $X_\lambda = V_\lambda + \mathbf{s}V_\lambda$. Furthermore for each X_λ there exists an S-invariant complement Y_λ of NX_λ in X_λ such that $X_\lambda = Y_\lambda \oplus NX_\lambda = Y_\lambda \oplus NY_\lambda \oplus \cdots \oplus N^{n-1}Y_\lambda$. If S is given on Y_λ, then L is determined on X_λ up to similarity.

On Y_λ we have a reduced (anti)-automorphism. For automorphisms it is easy to see that the reduction of $\phi_{\mathbf{s}}$ is simply $\phi_{\mathbf{s}}$ restricted to Y_λ so that $Y_\lambda = W_\lambda + \mathbf{s}W_\lambda$. Since Y_λ is $\mathbf{s}$-invariant, the normal form of $\mathbf{s}$ on X_λ follows from the normal form of $\mathbf{s}$ restricted to Y_λ.

Remark 3.5 Either $V_\lambda \cap \mathbf{s}V_\lambda = \{0\}$ or $V_\lambda \cap \mathbf{s}V_\lambda = V_\lambda$. For example for infinitesimally R-reversible maps we have $X_{\pm i\beta} = V_{\pm i\beta}$, but $X_\alpha = V_\alpha \oplus RV_\alpha = V_\alpha \oplus V_{-\alpha}$. ▷

For anti-automorphisms the situation is somewhat more complicated. Recall that an anti-automorphism comes from a non-degenerate bilinear form ω which is either symmetric or anti-symmetric, $\omega(y,x) = \varepsilon\omega(x,y)$ with $\varepsilon = \pm 1$. Then on X_λ the map L satisfies $\omega(Lx,y) = \mu\omega(x,Ly)$. On Y_λ we have a reduced form.

Lemma 3.9 *Let $L = S + N$ be the Jordan-Chevalley decomposition of L on the indecomposable $\mathbf{s}$-invariant subspace X_λ. Let Y_λ be an S-invariant complement to NX_λ in X_λ. Then $\tau(x,y) = \omega(x, N^{n-1}y)$ is non-degenerate bilinear form on Y_λ with $\tau(y,x) = \varepsilon\mu^{n-1}\rho(x,y)$. Furthermore $\tau(Sx,y) = \mu\tau(x,Sy)$.*

See Appendix A for a proof. The reduced anti-automorphism is $\psi_{\mathbf{t}}$ where the structure map $\mathbf{t}$ is defined by $\tau(x,y) = \langle x, \mathbf{t}y\rangle_Y$ for all $x, y \in Y = Y_\lambda$. Then $Y_\lambda = W_\lambda + \mathbf{t}W_\lambda$. Here Y_λ is *not* $\mathbf{s}$-invariant. However the freedom in choosing Y_λ can be used to put $\mathbf{s}$ into a normal form with respect to the splitting $X_\lambda = Y_\lambda \oplus NY_\lambda \oplus \cdots \oplus N^{n-1}Y_\lambda$, again see Appendix A.

Now we obtain an unfolding lemma which uses the $\mathbf{Gl}(V)$-unfolding from Lemma 2.5 as a starting point.

Lemma 3.10 (Unfolding Lemma)
Let γ be an (anti)-automorphism and let $L \in \mathbf{gl}_\mu(V)$. Then the restriction of the $\mathbf{Gl}(V)$ centralizer unfolding of L in $\mathbf{gl}(V)$ to $\mathbf{gl}_\mu(V)$ is equivalent to the $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(V)$ centralizer unfolding in $\mathbf{gl}_\mu(V)$.

Proof. Let $L \in \mathbf{gl}_\mu(V)$ and let $T_{\mathbf{Gl}(V)} = \{UL - LU \mid U \in \mathbf{gl}(V)\}$ be the tangent space at L to the $\mathbf{Gl}(V)$-orbit of L. Let $N_{\mathbf{Gl}(V)}$ be its orthogonal complement with respect to an inner product on $\mathbf{gl}(V)$. To simplify notation let $\mathbf{G} = \mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(V)$ and $\mathbf{g} = \mathbf{gl}_{\sigma(\gamma)}(V)$. Then we have $T_{\mathbf{G}} = \{UL - LU \mid U \in \mathbf{g}\}$ and its orthogonal complement $N_{\mathbf{G}}$ in $\mathbf{gl}_\mu(V)$. By shifting L to the origin in $\mathbf{gl}(V)$ we have $\mathbf{gl}(V) = T_{\mathbf{Gl}(V)} \oplus^\perp N_{\mathbf{Gl}(V)}$ and $\mathbf{gl}_\mu(V) = T_{\mathbf{G}} \oplus^\perp N_{\mathbf{G}}$. Now let Π be the orthogonal projection onto $\mathbf{gl}_\mu(V)$. Then

$\Pi(T_{\mathbf{Gl}(V)}) = \{UL - LU \mid U \in \mathbf{gl}(V), \gamma(UL-LU) = \mu(UL-LU)\} = \{UL - LU \mid U \in \mathbf{g}\} = T_{\mathbf{G}}$, since $\gamma(UL-LU) = \mu(UL-LU)$ iff $U \in \mathbf{g}$, and

$$\mathbf{gl}_\mu(V) = \Pi(\mathbf{gl}(V)) = \Pi(T_{\mathbf{Gl}(V)} \oplus^\perp N_{\mathbf{Gl}(V)}) = \Pi(T_{\mathbf{Gl}(V)}) \oplus^\perp \Pi(N_{\mathbf{Gl}(V)}) = T_{\mathbf{G}} \oplus^\perp \Pi(N_{\mathbf{Gl}(V)}).$$

Since orthogonal complements are unique, we have $\Pi(N_{\mathbf{Gl}(V)}) = N_{\mathbf{G}}$. □

3.4 Orbit Splitting

The transformation group $\mathbf{Gl}_\gamma(V)$ preserves $\mathbf{gl}_\mu(V)$ as a linear space and so maps $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(V)$-orbits into $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(V)$-orbits. The difference between the groups $\mathbf{Gl}_\gamma(V)$ and $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(V)$ gives rise to *splitting* of orbits. This means that a $\mathbf{Gl}(V)$-orbit in $\mathbf{gl}(V)$ splits into different $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(V)$-orbits when intersected with $\mathbf{gl}_\mu(V)$. In particular orbit splitting implies that eigenvalues and Jordan structure no longer suffice to characterize $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(V)$-orbits. Examples of additional invariants are *symplectic* and *reversible signs* [6, 14].

Lemma 3.11 (Intersection Lemma)
Let V be an indecomposable $\mathbf{s}$-invariant space and $\gamma_{\mathbf{s}}$ an (anti)-automorphism of order two with structure map $\mathbf{s}$. Let $L \in \mathbf{gl}_\mu(V)$. Then

$$\mathrm{Orb}_{\mathbf{Gl}(V)}(L) \;\cap\; \mathbf{gl}_\mu(V) \;=\; \mathrm{Orb}_{\mathbf{Gl}_\gamma(V)}(L).$$

Proof. If $M \in \mathrm{Orb}_{\mathbf{Gl}_\gamma(V)}(L)$ then clearly $M \in \mathrm{Orb}_{\mathbf{Gl}(V)}(L) \cap \mathbf{gl}_\mu(V)$. Next we assume $M \in \mathrm{Orb}_{\mathbf{Gl}(V)}(L) \cap \mathbf{gl}_\mu(V)$. The $\mathbf{Gl}(V)$-orbits of L and M are, of course, the same. From the Orbit Lemma 3.6 we know that the $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(V)$-orbits of L and M correspond to the $\mathbf{Gl}(V)$-orbits of the pairs $(L, \mathbf{s})$ and $(M, \mathbf{s})$. These orbits are not necessarily the same. From the $\mathbf{Gl}(V)$-classification of pairs in Section 4 we see that the normal form of $(L, \mathbf{s})$ is either $(L_0, \mathbf{s}_0)$ or $(L_0, \varepsilon\mathbf{s}_0)$ with $\varepsilon = \pm 1$. That is either eigenvalues and Jordan structure determine the orbit or there is an additional sign. In that case $(L_0, \mathbf{s}_0)$ and $(L_0, -\mathbf{s}_0)$ are *not* equivalent, there exists no $g \in \mathbf{Gl}(V)$ such that $g.(L_0, \mathbf{s}_0) = (L_0, -\mathbf{s}_0)$. The normal form of $(M, \mathbf{s})$ is $(L_0, \varepsilon\mathbf{s}_0)$ and in the absence of a sign we set $\varepsilon = 1$. If there is no sign then there are $g_1, g_2 \in \mathbf{Gl}(V)$ such that $g_1.(L, \mathbf{s}) = (L_0, \mathbf{s}_0)$ and $g_2.(M, \mathbf{s}) = (M_0, \mathbf{s}_0)$. But then we have $(M, \mathbf{s}) = g_2^{-1}g_1.(L, \mathbf{s})$ and thus $g_2^{-1}g_1 \in \mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(V) \subset \mathbf{Gl}_\gamma(V)$. If there is a sign we perform the same computation to find $g_1, g_2 \in \mathbf{Gl}(V)$ such that $(M, \mathbf{s}) = g_2^{-1}g_1.(L, \varepsilon\mathbf{s})$ and thus $g_2^{-1}g_1 \in \mathbf{Gl}_\gamma(V)$. Hence we may conclude that $M \in \mathrm{Orb}_{\mathbf{Gl}_\gamma(V)}(L)$. □

The following result is an immediate corollary of Proposition 3.7 and Lemma 3.11.

Theorem 3.12 (Orbit Splitting Theorem)
If $L \in \mathbf{gl}_\mu(V)$ then the $\mathbf{Gl}(V)$-orbit of L in $\mathbf{gl}(V)$ intersects $\mathbf{gl}_\mu(V)$ in at most two $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(V)$-orbits.

Example 3.6 Recall from Example 1.1 that the space of infinitesimally R-reversible maps is defined by $\mathbf{gl}_{-R}(V) := \{L \in \mathbf{gl}(V) \mid \phi_R(L) = -L\}$, where ϕ_R is an automorphism defined by $\phi_R(L) := R^{-1}LR$ with *structure map* R satisfying $R^2 = I$, $R \neq \pm I$. The structure map R has eigenvalues $+1$ and -1 with corresponding eigenspaces $E_+ = \{a \in V \mid Ra = a\}$ and $E_- = \{b \in V \mid Rb = -b\}$. For an infinitesimally R-reversible map L we have $LE_+ \subseteq E_-$ and $LE_- \subseteq E_+$. Thus eigenvectors of L for eigenvalue zero either belong to E_+ or to E_-. The new invariant, called the *reversible sign* indicates to which eigenspace such an eigenvector belongs. ▷

Example 3.7 Infinitesimally symplectic linear maps are defined by $\mathbf{sp}(V) := \{L \in \mathbf{gl}(V) \mid \omega(Lx, y) + \omega(x, Ly) = 0,\ \forall x, y \in V\}$, where ω is a non-degenerate skew symmetric bilinear form on V, see Example 1.2. With help of an inner product $\langle \cdot, \cdot \rangle$ on V we can find a map J satisfying $J^* = -J$ and $J^2 = -I$ such that $\omega(x, y) = \langle x, Jy \rangle$. Then $\mathbf{sp}(V) = \{L \in \mathbf{gl}(V) \mid \psi_J(L) = -L\}$, where ψ is the anti-automorphism defined by $\psi(L) := J^{-1}L^*J$. Here J is the *structure map*. Since $\mathbf{gl}_\sigma(V) = \mathbf{gl}_{-1}(V) = \mathbf{sp}(V)$, Proposition 3.7 says that $\mathbf{sp}(V)$ is a Lie algebra, as we already knew. The invertible transformations that preserve the symplectic structure are exactly those g for which $\omega(gx, gy) = \omega(x, y)$. Indeed according to Proposition 3.7 they are given by $\mathbf{Gl}_1(V) = \{g \in \mathbf{Gl}(V) \mid \psi_J(g) = g^{-1}\}$, which is equivalent to $g^*Jg = J$, reflecting

the transformation rule for bilinear forms. The order of ψ_J is two. So if splitting of an orbit occurs there are at most two inequivalent normal forms. A well known example is the distinction between 1:1 resonance and 1:-1 resonance. In both cases there are double eigenvalues $\pm\mathbf{i}$. But there is an additional invariant, namely a *symplectic* or *Krein* sign distinguishing the two cases. These signs are intimately related to the Morse index of the corresponding quadratic Hamilton functions. Computing the unfoldings in the 1:1 case one finds *passing* of imaginary eigenvalues (see Cotter [7], Galin [12] and Koçak [21]), which is a codimension three phenomenon. In the 1:-1 case one finds *splitting* of imaginary eigenvalues (see van der Meer [23]), which is a codimension one phenomenon. ▷

4 Normal Forms and Examples of Unfoldings

4.1 Normal Forms

In the present setting, where $\gamma_{\mathbf{s}}$ is an (anti)-automorphism of order two with structure map $\mathbf{s}$, we can classify maps satisfying $\gamma_{\mathbf{s}}(L) = \mu L$ into the eight different types listed in Table 1.

type	$\gamma_{\mathbf{s}}$	$\mathbf{s}^*$	μ	L
1	$\phi_{\mathbf{s}}$	$\mathbf{s}$	1	$\mathbf{s}$-equivariant of type $\mathbb{R}$
2	$\phi_{\mathbf{s}}$	$\mathbf{s}$	-1	$\mathbf{s}$-reversible of type $\mathbb{R}$
3	$\phi_{\mathbf{s}}$	$-\mathbf{s}$	1	$\mathbf{s}$-equivariant of type $\mathbb{C}$ or $\mathbb{C}$-linear
4	$\phi_{\mathbf{s}}$	$-\mathbf{s}$	-1	$\mathbf{s}$-reversible of type $\mathbb{C}$ or $\mathbb{C}$-semi-linear
5	$\psi_{\mathbf{s}}$	$\mathbf{s}$	1	symmetric
6	$\psi_{\mathbf{s}}$	$\mathbf{s}$	-1	anti-symmetric
7	$\psi_{\mathbf{s}}$	$-\mathbf{s}$	1	anti-symplectic
8	$\psi_{\mathbf{s}}$	$-\mathbf{s}$	-1	symplectic

Table 1: *Eight types of maps satisfying $\gamma_{\mathbf{s}}(L) = \mu L$, when $\gamma_{\mathbf{s}}$ is an (anti)-automorphism of order two.*

Applying the Reduction Lemma 3.8 it is straightforward to obtain normal forms for semi-simple maps of the types listed in Table 1. If $\gamma_{\mathbf{s}}$ is an anti-automorphism we apply the construction of Appendix A to put the structure map $\mathbf{s}$ into normal form. Finally we reduce to the smaller space Y on which we consider the semi-simple part S of L and the reduced structure map $\mathbf{t}$ (see Appendix A), such that $S \in \mathbf{gl}_\mu(Y) = \{A \in \mathbf{gl}(Y) \mid \gamma_{\mathbf{t}}(A) = \mu A\}$. Since we consider semi-simple maps in their own right we denote the reduced structure map again by $\mathbf{s}$. Note that in view of the Orbit Lemma 3.6 the normal forms in Tables 2 and 3 can be regarded as representatives of $\mathbf{Gl}_{\mathbf{s}}^{\sigma(\gamma)}(Y)$-orbits of S once a choice for $\mathbf{s}$ has been made, but they can also be seen as representatives of $\mathbf{Gl}(Y)$-orbits of the pair $(S, \mathbf{s})$.

Theorem 4.1 (Normal Form Theorem)
Let $\gamma_{\mathbf{s}}$ be an (anti)-automorphism of order two with structure map $\mathbf{s}$ and let $L \in \mathbf{gl}_\mu(X)$ be a linear map in one of the eigenspaces of $\gamma_{\mathbf{s}}$, where X is an indecomposable $(L, \mathbf{s})$-invariant space. Furthermore let $L = S + N$ be the Jordan-Chevalley decomposition of L. Then the normal form of L on X is determined by the normal form of S on Y, where $Y \subset X$ is a reduced space as in Proposition 3.8. The normal forms of S are listed in Tables 2 and 3.

The *signs* in the tables indicate that the normal forms for $+1$ and -1 are inequivalent. Since in each case there are at most two possibilities this shows that orbit splitting occurs as stated in Theorem 3.12. The relevance of the signs becomes most obvious when constructing unfoldings. We put the sign in the structure map $\mathbf{s}$, but for nonzero eigenvalues we can also put the sign in the map S.

Remarks 4.1 Let us make some remarks on the various cases in Tables 2 and 3.

1. There are only two cases where there are no signs at all, namely infinitesimally $\mathbf{s}$-reversible semi-simple maps of type $\mathbb{C}$ and anti-symplectic semi-simple maps.
2. In each of the other cases the sign has a geometric meaning. Let S be a semi-simple map in an eigenspace of an automorphism, see Table 1. For $\mathbf{s}$-equivariant maps of type $\mathbb{R}$ there are two isotypic components, labelled by the eigenvalues of $\mathbf{s}$. The sign indicates on which of the two isotypic components L acts. Similarly for $\mathbf{s}$-equivariant maps of type $\mathbb{C}$. Here the sign disappears for real eigenvalues of L. The latter are forced to be double and are non-generic. For infinitesimally $\mathbf{s}$-reversible maps of type $\mathbb{R}$ the sign for zero eigenvalues of L indicates to which eigenspace of $\mathbf{s}$ the corresponding eigenvector belongs. In case of symmetric and anti-symmetric maps the sign is related to the signature of the underlying bilinear form. If the signature is denoted by (p, q), that is
$$\langle x, y\rangle = \sum_{i=1}^{p} x_i\, y_i - \sum_{i=1}^{q} x_{p+i}\, y_{p+i},$$
then for signature $(n, 0)$ or $(0, n)$, symmetric maps only have real eigenvalues whereas anti-symmetric maps only have imaginary eigenvalues. In this case the sign distinguishes between $(n, 0)$ and $(0, n)$. For infinitesimally symplectic maps the structure map itself is symplectic. The dynamical interpretation is that the structure map defines a preferred direction of rotation. This gives a sign for imaginary eigenvalues of an infinitesimally symplectic map.
3. In case of types 3 and 4 the structure map $\mathbf{s}$ satisfies $\mathbf{s}^* = -\mathbf{s}$ and $\mathbf{s}^2 = -I$. Therefore $\mathbf{s}$ defines a complex structure on X. Maps of type 3 commute with $\mathbf{s}$, but maps of type 4 anti-commute with $\mathbf{s}$. Consequently maps of type 3 can be considered as $\mathbb{C}$-linear maps whereas maps of type 4 can be considered as $\mathbb{C}$-semi-linear maps.
4. In case of anti-automorphisms the reduced structure map $\mathbf{t}$ can be either symmetric or skew in eigenspaces with $\mu = -1$. This means that a reduced semi-simple infinitesimally symplectic map is either infinitesimally symplectic or anti-symmetric with respect to the reduced structure map. Similarly a reduced semi-simple anti-symmetric map is either anti-symmetric or infinitesimally symplectic. ▷

We conclude this section with a proof of the Normal Form Theorem.

Proof of Theorem 4.1. We will not give proofs for all the different cases for they are very similar to each other. Instead we give some representative proofs for automorphisms and anti-automorphisms with and without signs. Throughout the proof we use the following standard matrices
$$I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

1. Case 3 in Table 1; complex eigenvalues; $\mathbf{s}^2 = -I$. The maps S and $\mathbf{s}$ commute, both have complex eigenvalues, and so $Y_{\alpha\pm i\beta} = W_{\alpha\pm i\beta}$. Let e be any vector in $Y_{\alpha\pm i\beta}$ and define $f := \frac{1}{\beta}(S-\alpha)e$. Then $\langle e, f\rangle$ is a basis of $Y_{\alpha\pm i\beta}$ on which we have $S = \alpha I + \beta J$ and $\mathbf{s} = J$. Note that we can change the sign of β in S by applying the transformation R or T. Then the sign of $\mathbf{s}$ changes as well. Therefore we assume $\beta > 0$ and put the sign in $\mathbf{s} = \pm J$.
2. Case 4 in Table 1; real eigenvalues; $\mathbf{s}^2 = -I$. The maps S and $\mathbf{s}$ anti-commute. If $e \in W_\alpha$ then $\mathbf{s}e \in W_{-\alpha}$, so $Y_\alpha = W_\alpha \oplus W_{-\alpha}$. Then $\langle e, \mathbf{s}e\rangle$ is a basis of Y_α and we have $S = \alpha R$ and $\mathbf{s} = J$. Note that $\mathbf{s}S\mathbf{s}^{-1} = -S$, so we may assume that $\alpha \geq 0$. There is no sign in this case.
3. Case 5 in Table 1; complex eigenvalues; $\mathbf{s}^2 = I$. Here $Y_{\alpha\pm i\beta} = W_{\alpha\pm i\beta}$. Let e be any vector in $Y_{\alpha\pm i\beta}$ and define $f := \frac{1}{\beta}(S-\alpha)e$. Then $\langle e, f\rangle$ is a basis of $Y_{\alpha\pm i\beta}$. On this basis $S = \alpha I + \beta J$. Let τ be the reduced form on $Y_{\alpha\pm i\beta}$, that is $\tau(x, y) = \langle x, \mathbf{s}y\rangle$. Then indeed $\tau(e, f) = \frac{1}{\beta}(e, Se) = \frac{1}{\beta}(Se, e) = \tau(f, e)$. If u is a vector in $Y_{\alpha\pm i\beta}$ such that $\mathbf{s}u = u$, then $\mathbf{s}(S-\alpha)u = -(S-\alpha)\mathbf{s}u = -(S-\alpha)u$. So $\mathbf{s}$ is indeed symmetric and has eigenvalues ± 1 on $Y_{\alpha\pm i\beta}$. Thus there exists an orthogonal transformation such that $\langle u, \frac{1}{\beta}(S-\alpha)u\rangle$ is a new basis of $Y_{\alpha\pm i\beta}$ and $S = \alpha I + \beta J$, $\mathbf{s} = R$. Note that $\mathbf{s}S\mathbf{s}^{-1} = \alpha I - \beta J$, so we may assume $\beta \geq 0$ and there is no sign.
4. Case 6 in Table 1; imaginary eigenvalues; $\mathbf{s}^2 = I$. Again $Y_{\alpha\pm i\beta} = W_{\alpha\pm i\beta}$. Let e be any vector in $Y_{\pm i\beta}$ and define $f := \frac{1}{\beta}Se$. Then $\langle e, f\rangle$ is a basis of $Y_{\pm i\beta}$. On this basis $S = \beta J$. The reduced form τ

on $Y_{\alpha\pm i\beta}$ is symmetric, so $\tau(e,f)=\tau(f,e)$. On the other hand $\tau(e,f)=\frac{1}{\beta}\tau(e,Se)=-\frac{1}{\beta}\tau(Se,e)=-\tau(f,e)$, so $\tau(e,f)=0$. Furthermore $\tau(f,f)=\frac{1}{\beta^2}\tau(Se,Se)=-\frac{1}{\beta^2}\tau(S^2e,e)=\tau(e,e)$. Thus on the basis $\langle e,f\rangle$ we have $S=\beta J$, $\mathbf{s}=\pm I$ and we may assume that $\beta>0$. □

4.2 Examples of Unfoldings

Our aim is to present a list of low codimension unfoldings. We will concentrate on unfoldings of zero eigenvalues. Since such unfoldings give a parametrization of the full space of maps we also get information on the unfoldings of real, imaginary and complex eigenvalues. The classification is complete up to codimension two. Where appropriate we include some higher codimension cases to show the consequences of signs. In constructing unfoldings we use Lemmas 3.10 and 2.5 and we summarize the results in a theorem.

Theorem 4.2 (Unfolding Theorem)
Let $\gamma_{\mathbf{s}}$ be an (anti)-automorphism of order two with structure map $\mathbf{s}$ and let $L\in\mathbf{gl}_{\mu}(X)$ be a linear map, with X an indecomposable $(L,\mathbf{s})$-invariant space. Furthermore let $L=S+N$ be the Jordan-Chevalley decomposition of L. Then the unfoldings of L up to codimension two are listed in Tables 4 and 5.

Here we consider real maps with real parameters. In such maps simple real eigenvalues are forced to remain on the real axis when parameters are varied. Maps which are elements of an eigenspace corresponding to eigenvalue -1 of an (anti)-automorphism have eigenvalues which come in complex conjugate *and* opposite pairs. Thus a pair of simple opposite imaginary eigenvalues is forced to remain on the imaginary axis when parameters are varied. However there may also be collisions of such eigenvalues on the real or imaginary axis as the parameter varies. Generically eigenvalues *split* into the complex plane at collisions, but in the presence of signs generic *passing* also occurs.

There are several examples where we have passing or splitting depending on the signs. In $\mathbf{s}$-equivariant maps of type $\mathbb{R}$, type 1 in Table 1, passing of real eigenvalues with different signs and splitting of real eigenvalues with equal signs are both codimension one phenomena. This can be inferred from 1c and 1d in Table 4. In infinitesimally $\mathbf{s}$-reversible maps of type $\mathbb{R}$, type 2 in Table 1, only zero eigenvalues are signed. At collisions real and imaginary eigenvalues generically split. Maps of type 3 generically do not have eigenvalues on the real or imaginary axis. Maps of type 4 generically do have opposite pairs of real eigenvalues, but since there are no signs they split at collisions. Real eigenvalues of maps of type 5 with equal signs pass but split when the signs are different at collisions. This follows from 5b, 5c and 5d in Table 5. Similarly imaginary eigenvalues of anti-symmetric maps of type 6 split or pass when the signs are different or equal respectively. See 6e and 6f in Table 5. Note that passing is a codimension 3 phenomenon, but splitting is a codimension 1 phenomenon. This should come as no surprise because the anti-symmetric maps are closely related to infinitesimally symplectic maps of type 8. Here we have the same codimensions for splitting and passing, see 8c and 8d in Table 5. Maps of type 7 can generically have real eigenvalues. Since there are no signs they generically split at collisions.

5 Generalizations

Here we will generalize the results for a single (anti)-automorphism of order two to an abelian group Γ of (anti)-automorphisms of order two. In general the subset $\mathbf{g}$ in the Introduction will be an isotypic component of the action of Γ on $\mathbf{gl}(V)$, but for abelian groups these are equivalent to simultaneous eigenspaces. To make this more precise, let the V be a finite dimensional real vector space and let Γ be a abelian group of (anti)-automorphisms of order two acting on $\mathbf{gl}(V)$. Suppose Γ is generated by $\langle\gamma_1,\ldots,\gamma_p\rangle$ with $\gamma_i^2=I$ for $i=1,\ldots,p$. Then the simultaneous eigenspaces are given by

$$\mathbf{gl}_{\mu_1,\ldots,\mu_p}(V)=\{L\in\mathbf{gl}(V)\mid\gamma_1(L)=\mu_1L,\ldots,\gamma_p(L)=\mu_pL\},$$

where the eigenvalues μ_i are ±1. The structure map associated to γ_i is denoted by $\mathbf{s}_i$. Apart from Example 1.3 we encounter this situation with infinitesimally reversible equivariant and infinitesimally symplectic reversible equivariant maps. See [15, 17] for applications of the results of this article.

The theory developed for a single (anti)-automorphism almost immediately extends to an abelian group of (anti)-automorphisms. Let us review Section 3 and make some comments. The structure maps can again be taken orthogonal, but here we need to take a closer look at anti-automorphisms, see Appendix B. From the proof of Proposition 3.5 it follows immediately that the Jordan-Chevalley decomposition also holds in $\mathbf{gl}_{\mu_1,\ldots,\mu_p}(V)$. Moreover the *structure preserving* transformation group $\mathbf{G}$ is the intersection of the structure preserving transformations groups for each (anti)-automorphism γ_i. Once we have identified the transformation group we can classify its orbits in $\mathbf{gl}_{\mu_1,\ldots,\mu_p}(V)$. There is a Reduction Lemma similar to 3.8 where indecomposable $\mathbf{s}$-invariant subspaces are replaced by indecomposable $\mathbf{s}_1,\ldots,\mathbf{s}_p$-invariant subspaces. In the same way we have an Unfolding Lemma and an Orbit Splitting Theorem for Γ. But in the latter we now have at most 2^p inequivalent $\mathbf{G}$-orbits.

Remark 5.1 The indecomposable $\mathbf{s}_1,\ldots,\mathbf{s}_p$-invariant subspaces can be relatively large. Let us look at an $\mathbb{H}$-linear map on $\mathbb{R}^4$. The quaternionic structure on $\mathbb{R}^4$ is determined by two structure maps $\mathbf{c}$ and $\mathbf{q}$ with $\mathbf{c}^2 = -I$, $\mathbf{q}^2 = -I$ and $\mathbf{cq} = -\mathbf{qc}$, see [15]. Then $\mathbb{H}$-linear maps on $\mathbb{R}^4$ are defined as $\mathbf{gl}_{1,1}(\mathbb{R}^4) := \{A \in \mathbf{gl}(\mathbb{R}^4) \mid \phi_{\mathbf{c}}(A) = A,\ \phi_{\mathbf{q}}(A) = A\}$, where $\phi_{\mathbf{c}}(A) := \mathbf{c}^{-1}A\mathbf{c}$ and $\phi_{\mathbf{q}}(A) := \mathbf{q}^{-1}A\mathbf{q}$. Here $\Gamma = \{I, \phi_{\mathbf{c}}, \phi_{\mathbf{q}}, \phi_{\mathbf{c}} \circ \phi_{\mathbf{q}}\}$ since clearly $\phi_{\mathbf{c}}$ and $\phi_{\mathbf{q}}$ commute. Let $L \in \mathbf{gl}_{-1,-1}(\mathbb{R}^4)$ have real eigenvalues, then the indecomposable $L, \mathbf{c}, \mathbf{q}$-invariant space is $X_\alpha = V_\alpha \oplus \mathbf{c}V_\alpha \oplus \mathbf{q}V_\alpha \oplus \mathbf{cq}V_\alpha$. ▷

Example 5.2 The maximum number of 2^p inequivalent $\mathbf{G}$-orbits occurs in an example of an infinitesimally reversible symplectic linear map. We have already encountered such maps in Example 1.3. They are elements of the simultaneous eigenspace $\mathbf{gl}_{-1,-1}(\mathbb{R}^{2n}) := \{A \in \mathbf{gl}(\mathbb{R}^{2n}) \mid \phi_R(A) = -A,\ \psi_J(A) = -A\}$, where $\phi_R(A) := R^{-1}AR$ and $\psi_J(A) := J^{-1}A^*J$. Here $\Gamma = \{id, \phi_R, \psi_J, \psi_{RJ}\}$ is generated by ϕ_R and ψ_J, where $\psi_J \circ \phi_R = \psi_{RJ}$. Let L be a map in $\mathbf{gl}_{-1,-1}(\mathbb{R}^4)$ with two blocks of double zero eigenvalues and a nilpotent part of height two, see [14]. Then there are $4 = 2^2$ inequivalent $\mathbf{G}$-orbits in $\mathbf{gl}_{-1,-1}(\mathbb{R}^4)$. ▷

As mentioned in the Introduction our main motivation for studying eigenspaces of (anti)-automorphisms of order two comes from real ordinary differential equations. Other obvious generalizations apart from the one given in this section are not necessarily in this context. Such generalizations include single (anti)-automorphisms of finite order. Then the eigenspace $\mathbf{gl}_\mu(V)$ need not be real. This problem can be overcome by taking the real invariant space $\mathbf{gl}_{\mu,\bar{\mu}}(V)$ as the object of study, though one could also look at the complex space $\mathbf{gl}_\mu(V)$ in its own right. One could also look at abelian and non-abelian groups generated by (anti)-automorphisms of finite order, and more generally still one might consider general compact groups of (anti)-automorphisms. We will not pursue these matters here.

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A Standard Form of a Bilinear Form

Here we generalize a result of Burgoyne & Cushman [4, 5], which in turn is based on a theorem of Springer & Steinberg [30], to obtain a normal form for a skew or indefinite symmetric bilinear form on V which respects the splitting of V in the Reduction Lemma 3.8.

Let V be real vector space and let ω be a nondegenerate bilinear form on V which is either symmetric or skew, that is for each nonzero $x \in V$ there exists an $y \in V$ such that $\omega(x,y) \neq 0$ and for all $x, y \in V$, $\omega(y,x) = \varepsilon\omega(x,y)$ with $\varepsilon = \pm 1$. If $\langle\cdot,\cdot\rangle$ is the standard inner product on V then there is an invertible linear map $\mathbf{s}$ with $\mathbf{s}^* = \varepsilon\mathbf{s}$ such that $\omega(x,y) = \langle x, \mathbf{s}y\rangle$. We may assume that after scaling $\mathbf{s}^2 = \varepsilon I$, so $\mathbf{s}$ is orthogonal.

Let L be a linear map on V such that for all $x, y \in V$, $\omega(Lx, y) = \mu\omega(x, Ly)$ with $\mu = \pm 1$. This is equivalent to $\langle Lx, \mathbf{s}y\rangle = \mu\langle x, \mathbf{s}Ly\rangle$ or $L^*\mathbf{s} = \mu\mathbf{s}L$. Let $\psi_{\mathbf{s}}(L) = \mathbf{s}^{-1}L^*\mathbf{s}$. Then L satisfies $\psi_{\mathbf{s}}(L) = \mu L$.

Now we assume that V is an indecomposable L,$\mathbf{s}$-invariant space. If $L = S + N$ is the Jordan-Chevalley decomposition of L then there is an S-invariant complement W of NV in V such that $V = W \oplus NW \oplus \cdots \oplus N^{n-1}W$, where n is the height of N. Although $\mathbf{s}$ has a normal form as in Lemma 3.3, since $\mathbf{s}$ is orthogonal, we wish to find a normal form of $\mathbf{s}$ which respects the above splitting of V.

If ω is definite then it must be symmetric and thus $\mathbf{s} = \pm I$. Then every L satisfying $\psi_{\mathbf{s}}(L) = \mu L$ is semi-simple. Therefore in the present situation we need only consider indefinite forms. The main result of this section can now be stated.

Proposition A.1 *Let ω and L be defined as above. For every S-invariant complement W_1 of NV in V there is an invertible transformation g such that $W = gW_1$ is again S-invariant and on $W \oplus NW \oplus \cdots \oplus N^{n-1}W$ the matrix of $\mathbf{s}$ takes the form*

$$\begin{pmatrix} & & * \\ & \iddots & \\ * & & \end{pmatrix},$$

where $*$ is a $m \times m$ block and $m = \dim W$.

The procedure to transform W_1 runs as follows. Note that the matrix of $\mathbf{s}$ is upper triangular with respect to the main anti-diagonal. Furthermore blocks of $\mathbf{s}$ on an anti-diagonal differ only by a sign, see the proof of Proposition A.1 below. By adding a component of NW_1 to W_1 and setting $W_2 = W_1 + Ng_1W_1$ for a map g_1 to be specified later, we clear the first co-anti-diagonal. Then we set $W_3 = W_2 + N^2g_2W_2$, clearing the second co-anti-diagonal without affecting the first. This process stops after $n-1$ steps.

The following lemmas are useful in the proof of Proposition A.1. Although there is freedom in choosing an S-invariant complement W of NV in V the space $N^{n-1}W$ is unique.

Lemma A.2 *If W is an S-invariant complement W of NV in V, then $N^{n-1}W$ is unique.*

Proof. Let n be the height of N on V. For every $x \in N^{n-1}W$ we have $Nx = 0$. Since the eigenspaces of N are unique $N^{n-1}W$ is unique. Thus if U is also an S-invariant complement W of NV in V then $N^{n-1}U = N^{n-1}W$. □

The matrix of $\mathbf{s}$ with respect to a basis in $V = W \oplus NW \oplus \cdots \oplus N^{n-1}W$ has the following properties.

Lemma A.3 *Let $\omega(x, y) = \langle x, \mathbf{s}y\rangle_V$ and blocks of $\mathbf{s}$ are denoted by β_{ij}. Then $\beta_{i,j} = \mu\beta_{i-1,j+1}$ and $\beta_{ij} = 0$ if $i + j \geq n$.*

Proof. This follows immediately from $\omega(N^ix, N^jy) = \mu\omega(N^{i-1}x, N^{j+1}y) = \mu^i\omega(x, N^{i+j}y)$. □

In the proof of the proposition we need the bilinear forms $\tau_j(x, y) = \omega(N^jx, y)$ on W for $j = 1, \ldots, n-1$. Furthermore let T_j be defined as $\tau_j(x, y) = \langle x, T_jy\rangle_W$ for all $x, y \in W$.

Lemma A.4 *T_{n-1} is an invertible map.*

Proof. This follows from the fact that τ_{n-1} is nondegenerate on W. The form ω is nondegenerate on V, so for every nonzero $x \in V$ there is a $y \in V$ such that $\omega(x, y) \neq 0$. In particular for every $x \in W$ there is a $y \in V$ such that $\omega(N^{n-1}x, y) \neq 0$. Every such y has a unique decomposition $y = y_1 + y_2$ with $y_1 \in W$ and $y_2 \in NV$. Then $0 \neq \omega(N^{n-1}x, y) = \mu^{n-1}\omega(x, N^{n-1}y) = \mu^{n-1}\omega(x, N^{n-1}y_1 + N^{n-1}y_2) = \mu^{n-1}\omega(x, N^{n-1}y_1)$. Thus τ_{n-1} is nondegenerate on W and therefore T_{n-1} is invertible on W. □

Proof of Proposition A.1. Note that the blocks β_{ij} differ from the matrices of T_{j-1} by a sign only. Therefore if the bilinear forms τ_j are identically zero on W for $j = 1, n-2$, the matrix of $\mathbf{s}$ has the desired form.

Suppose $\tau_{n-2} \neq 0$ on W. Let $W_2 = W_1 + NgW_1$ where g is such that $\tau_{n-2}((I+Ng)x, (I+Ng)y) = 0$ for all $x, y \in W_1$. After a few computations we find $\tau_{n-2}(x+Ngx, y+Ngy) = \langle x, (T_{n-2}+\mu T_{n-1}g+g^*T_{n-1})y\rangle_W$. Assuming that $g^*T_{n-1} = \mu T_{n-1}g$ we set $g = -\frac{1}{2}\mu T_{n-1}^{-1}T_{n-2}$. Then $\tau_{n-2} = 0$ on W_2.

Now assume that $\tau_{n-2} = 0, \ldots, \tau_{n-j+1} = 0$ on W_{j-1}. Let $W_j = W_{j-1} + N^{j-1}gW_{j-1}$, where g is such that $\tau_{n-j}((I+N^{j-1}g)x, (I+N^{j-1}g)y) = 0$ for all $x, y \in W_{j-1}$. Again we find $\tau_{n-j}(x+N^{j-1}gx, y+N^{j-1}gy) =$

$\langle x, (T_{n-j} + \mu^{j-1} T_{n-1} g + g^* T_{n-1}) y \rangle_W$ and we set $g = -\frac{1}{2}\mu^{j-1} T_{n-1}^{-1} T_{n-j}$, so that $\tau_{n-j} = 0$ on W_j. It is easily checked that now $\tau_{n-2} = 0, \ldots, \tau_{n-j} = 0$ on W_j.

We still have to check that $g^* T_{n-1} = \mu^{j-1} T_{n-1} g$ in each step, but this follows from $T_{n-j}^* T_{n-1} = \mu^{j-1} T_{n-1} T_{n-j}$ because $N^* T_{n-1} = \mu T_{n-1} N$. Furthermore it is easy to see that each W_j is S-invariant. □

B Orthogonality of Structure Maps

In this appendix we give a precise statement of the properties of structure maps associated to the generators of an abelian group Γ of (anti)-automorphisms of order two. Their properties are essentially those of a structure map of a single (anti)-automorphism, but it is not a priori clear that we can transform them as in Proposition 3.3 so that they can all be assumed to be orthogonal. It might happen that a transformation which takes one structure map in good shape spoils another. The proposition below shows that this does not happen because of the commutation relations of the (anti)-automorphisms.

Proposition B.1 *Let Γ be a abelian group of (anti)-automorphisms of order two on $\mathbf{gl}(V)$ generated by $\langle \gamma_1, \ldots, \gamma_p \rangle$. Then we may assume that $\gamma_i = \gamma_{\mathbf{s}_i}$ where the structure maps $\mathbf{s}_i$ have the following properties:*

a) $\det \mathbf{s}_i = \pm 1$.
b) $\mathbf{s}_i$ *is orthogonal.*
c) $\mathbf{s}_i^2 = \pm I$.
d) $\mathbf{s}_i \mathbf{s}_j = \pm \mathbf{s}_j \mathbf{s}_i$.

Proof. Throughout the proof $\mathbf{s}$ and $\mathbf{t}$ will be any pair of structure maps from the set $\{\mathbf{s}_1 \ldots \mathbf{s}_p\}$.

Part a) follows from $\gamma_{\alpha \mathbf{s}} = \gamma_{\mathbf{s}}$ for all $\alpha \in \mathbb{R}$, so we can scale $\mathbf{s}$ such that $\det \mathbf{s} = \pm 1$. For every pair $\gamma_{\mathbf{s}}, \gamma_{\mathbf{t}} \in \Gamma$ we have $\gamma_{\mathbf{s}} \circ \gamma_{\mathbf{t}} = \gamma_{\mathbf{t}} \circ \gamma_{\mathbf{s}}$ and $\gamma_{\mathbf{s}}^2 = I$, $\gamma_{\mathbf{t}}^2 = I$. We distinguish three different cases.

1. $\gamma_{\mathbf{s}} = \phi_{\mathbf{s}}$ and $\gamma_{\mathbf{t}} = \phi_{\mathbf{t}}$ are automorphisms. From $\phi_{\mathbf{s}}^2 = I$ we have $\mathbf{s}^2 = \pm I$, so $\mathbf{s}$ is semi-simple and has eigenvalues ± 1 or $\pm i$. In order that $\phi_{\mathbf{s}} \circ \phi_{\mathbf{t}} = \phi_{\mathbf{t}} \circ \phi_{\mathbf{s}}$ we must have $\mathbf{st} = \pm \mathbf{ts}$. Then $\langle \mathbf{s}, \mathbf{t} \rangle$ generates a finite group. By a transformation, corresponding to averaging the inner product on V over this group, we obtain that $\mathbf{s}$ and $\mathbf{t}$ are orthogonal. We can do this at once for all structure maps associated to automorphisms in $\langle \gamma_1, \ldots, \gamma_p \rangle$.

2. $\gamma_{\mathbf{s}} = \phi_{\mathbf{s}}$ is an automorphism and $\gamma_{\mathbf{t}} = \psi_{\mathbf{t}}$ is an anti-automorphism. Because of 1) we assume that $\mathbf{s}$ has properties a), b) and c). Now $\psi_{\mathbf{t}}^2 = I$ implies $\mathbf{t}^* = \pm \mathbf{t}$ and so $\mathbf{t}$ is semi-simple, moreover $\mathbf{t}$ has either real or purely imaginary eigenvalues. Then it follows from $\phi_{\mathbf{s}} \circ \psi_{\mathbf{t}} = \psi_{\mathbf{t}} \circ \phi_{\mathbf{s}}$ that $\mathbf{st} = \pm \mathbf{ts}$. The latter implies that there exist indecomposable $\mathbf{s}, \mathbf{t}$-invariant subspaces on which $\mathbf{t}$ has either real eigenvalues in configurations α, $\alpha\alpha$ or $\pm\alpha$, or purely imaginary eigenvalues in configurations $\pm i\beta$ or $\pm i\beta \pm i\beta$. A scaling transformation acts on this subspace as $g = \rho I$, taking the eigenvalues of $\mathbf{t}$ to ± 1 or $\pm i$. Since $\mathbf{s}$ transforms as $\mathbf{s} \mapsto \mathbf{s} g^{-1}$ it is invariant under the scaling g. Thus $\mathbf{t}$ has properties a), b), c) and d).

3. $\gamma_{\mathbf{s}} = \psi_{\mathbf{s}}$ and $\gamma_{\mathbf{t}} = \psi_{\mathbf{t}}$ are anti-automorphisms. Because of 2) we assume that $\mathbf{s}$ has properties a), b) and c). Again $\psi_{\mathbf{t}}^2 = I$ implies $\mathbf{t}^* = \pm \mathbf{t}$ and so $\mathbf{t}$ is semi-simple, moreover $\mathbf{t}$ has either real or purely imaginary eigenvalues. From $\psi_{\mathbf{s}} \circ \psi_{\mathbf{t}} = \psi_{\mathbf{t}} \circ \psi_{\mathbf{s}}$ we infer that $\mathbf{ts} = \pm \mathbf{s} \mathbf{t}^{-1}$. Again we look for a transformation that takes eigenvalues of $\mathbf{t}$ to ± 1 or $\pm i$ but leaves $\mathbf{s}$ invariant. Summarising we have eight different cases $\mathbf{s}^2 = \varepsilon_1 I$, $\mathbf{s}^* = \varepsilon_1 \mathbf{s}$, $\mathbf{t}^* = \varepsilon_2 \mathbf{s}$, $\mathbf{ts} = \varepsilon_3 \mathbf{s} \mathbf{t}^{-1}$, where $\varepsilon_i = \pm 1$. Let us look at $\varepsilon_1 = \varepsilon_2 = -1$ and $\varepsilon_3 = \varepsilon = \pm 1$, the other cases being very similar. On any indecomposable $\mathbf{s}, \mathbf{t}$-invariant subspace $\mathbf{t}$ has eigenvalues $\pm i\beta$ and $\pm \frac{i}{\beta}$, $\beta > 0$. Suppose e is a vector such that $\mathbf{t}^2 e = -\beta^2 e$. Let f be defined as $f := \frac{1}{\beta} \mathbf{t} e$. Then $\mathbf{ts} e = \varepsilon \mathbf{s} \mathbf{t}^{-1} e = -\varepsilon \frac{1}{\beta} \mathbf{s} f$ and $\mathbf{ts} f = \varepsilon \mathbf{s} \mathbf{t}^{-1} f = \varepsilon \frac{1}{\beta} \mathbf{s} e$. Since $\mathbf{s}^2 = -I$, $\mathbf{s}^2 v = -v$ for each vector v. Thus on the basis $\langle e, f, \mathbf{s}e, \mathbf{s}f \rangle$, $\mathbf{t}$ and $\mathbf{s}$ have the following matrices

$$\mathbf{t} = \begin{pmatrix} 0 & -\beta & & \\ \beta & 0 & & \\ & & 0 & \varepsilon\beta^{-1} \\ & & -\varepsilon\beta^{-1} & 0 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} & & -1 & 0 \\ & & 0 & -1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix}.$$

Now let g be a transformation with blockdiagonal matrix $\mathrm{diag}(a^{-1}I_2, aI_2)$ with $a = \sqrt{\beta}$. Then $\mathbf{t}$ and $\mathbf{s}$ transform as

$$g\mathbf{t}g^* = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & \varepsilon \\ & & -\varepsilon & 0 \end{pmatrix}, \quad g\mathbf{s}g^* = \begin{pmatrix} & & -1 & 0 \\ & & 0 & -1 \\ 1 & 0 & & \\ 0 & 1 & & \end{pmatrix} = \mathbf{s}.$$

So we see that $\mathbf{t}$ has the properties listed in the lemma. □

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In the tables we use the following notation.

$$I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

type	λ	Y_λ	$\mathbf{s}$	S	remarks
1	α	W_α	± 1	α	sign
1	$\alpha \pm i\beta$	$W_{\alpha\pm i\beta}$	$\pm I_2$	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$	sign, $\beta > 0$
2	0	W_0	± 1	0	sign
2	$\pm\alpha$	$W_\alpha \oplus W_{-\alpha}$	T	$\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$	$\alpha > 0$
2	$\pm i\beta$	$W_{\pm i\beta}$	R	$\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}$	$\beta > 0$
2	$\pm\alpha \pm i\beta$	$W_{\alpha\pm i\beta} \oplus W_{-\alpha\pm i\beta}$	$\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha & -\beta & & \\ \beta & \alpha & & \\ & & -\alpha & \beta \\ & & -\beta & -\alpha \end{pmatrix}$	$\alpha > 0,\ \beta > 0$
3	$\alpha \pm i\beta$	$W_{\alpha\pm i\beta}$	$\pm J$	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$	sign, $\beta \geq 0$
4	$\pm\alpha$	$W_\alpha \oplus W_{-\alpha}$	J	$\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$	$\alpha \geq 0$
4	$\pm\alpha \pm i\beta$	$W_{\alpha\pm i\beta} \oplus W_{-\alpha\pm i\beta}$	$\begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha & -\beta & & \\ \beta & \alpha & & \\ & & -\alpha & \beta \\ & & -\beta & -\alpha \end{pmatrix}$	$\alpha \geq 0,\ \beta > 0$

Table 2: *Normal forms for semi-simple maps S in the eigenspace of an automorphism $\phi_{\mathbf{s}}$. W_λ is an indecomposable S-invariant space and $Y_\lambda = W_\lambda + \mathbf{s}W_\lambda$ is an indecomposable $S, \mathbf{s}$-invariant space.*

type	λ	Y_λ	$\mathbf{s}$	S	remarks
5	α	W_α	± 1	α	sign
5	$\alpha \pm i\beta$	$W_{\alpha\pm i\beta}$	R	$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$	$\beta > 0$
6	0	W_0	± 1	0	sign
6	$\pm\alpha$	$W_\alpha \oplus W_{-\alpha}$	T	$\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$	$\alpha > 0$
6	$\pm i\beta$	$W_{\pm i\beta}$	$\pm I_2$	$\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}$	sign, $\beta > 0$
6	$\pm\alpha \pm i\beta$	$W_{\alpha\pm i\beta} \oplus W_{-\alpha\pm i\beta}$	$\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha & -\beta & & \\ \beta & \alpha & & \\ & & -\alpha & -\beta \\ & & \beta & -\alpha \end{pmatrix}$	$\alpha > 0,\ \beta > 0$
7	$\alpha\alpha$	$W_\alpha \oplus \mathbf{s}W_\alpha$	J	$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$	
7	$\alpha \pm i\beta$	$W_{\alpha\pm i\beta} \oplus \mathbf{s}W_{\alpha\pm i\beta}$	$\begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha & -\beta & & \\ \beta & \alpha & & \\ & & \alpha & \beta \\ & & -\beta & \alpha \end{pmatrix}$	$\beta > 0$
8	$\pm\alpha$	$W_\alpha \oplus W_{-\alpha}$	J	$\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$	$\alpha \geq 0$
8	$\pm i\beta$	$W_{\pm i\beta}$	$\pm J$	$\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}$	sign, $\beta > 0$
8	$\pm\alpha \pm i\beta$	$W_{\alpha\pm i\beta} \oplus W_{-\alpha\pm i\beta}$	$\begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha & -\beta & & \\ \beta & \alpha & & \\ & & -\alpha & -\beta \\ & & \beta & -\alpha \end{pmatrix}$	$\alpha > 0,\ \beta > 0$

Table 3: *Normal forms for semi-simple maps in eigenspaces of anti-automorphisms.*

type	λ	X_λ	$\mathbf{s}$	$L(\nu)$	codim
$1a$	0^+	$\langle a\rangle$	1	ν	1
$1b$	$(0^+)^2$	$\langle a, Na\rangle$	I_2	$\begin{pmatrix} \nu_1 & \nu_2 \\ 1 & \nu_1 \end{pmatrix}$	2
$1c$	$(0^+)(0^-)$	$\langle a, b\rangle$	R	$\begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}$	2
$1d$	$(0^+)(0^+)$	$\langle a_1, a_2\rangle$	I_2	$\begin{pmatrix} \nu_1 & \nu_2 \\ \nu_3 & \nu_4 \end{pmatrix}$	4
$2a$	0^+	$\langle a\rangle$	1	0	0
$2b$	$(0^+)^2$	$\langle Nb, b\rangle$	R	$\begin{pmatrix} 0 & 1 \\ \nu & 0 \end{pmatrix}$	1
$2c$	$(0^+)(0^-)$	$\langle a, b\rangle$	R	$\begin{pmatrix} 0 & \nu_1 \\ \nu_2 & 0 \end{pmatrix}$	2
$2d$	$(0^+)^3$	$\langle a, N^2a, Na\rangle$	$\left(\begin{array}{c\|c} I_2 & \\ \hline & -1 \end{array}\right)$	$\left(\begin{array}{cc\|c} & & \nu \\ & & 1 \\ \hline 1 & \nu & \end{array}\right)$	1
$2e$	$(0^+)^2(0^+)$	$\langle a, Nb, b\rangle$	$\left(\begin{array}{c\|c} I_2 & \\ \hline & -1 \end{array}\right)$	$\left(\begin{array}{cc\|c} & & 0 \\ & & 1 \\ \hline \nu_2 & \nu_1 & \end{array}\right)$	2
$2f$	$(0^+)^2(0^-)$	$\langle Nb_1, b_1, b_2\rangle$	$\left(\begin{array}{c\|c} 1 & \\ \hline & -I_2 \end{array}\right)$	$\left(\begin{array}{c\|cc} & 1 & 0 \\ \hline \nu_1 & & \\ \nu_2 & & \end{array}\right)$	2
$3a$	00	$\langle e, \mathbf{s}e\rangle$	J	$\begin{pmatrix} \nu_1 & -\nu_2 \\ \nu_2 & \nu_1 \end{pmatrix}$	2
$3b$	$(\pm i\beta)^2$	$\langle e, \mathbf{s}e, Ne, N\mathbf{s}e\rangle$	$\begin{pmatrix} J & \\ & J \end{pmatrix}$	$\begin{pmatrix} \nu_1 & -\beta-\nu_2 & \nu_3 & -\nu_4 \\ \beta+\nu_2 & \nu_1 & \nu_4 & \nu_3 \\ 1 & 0 & \nu_1 & -\beta-\nu_2 \\ 0 & 1 & \beta+\nu_2 & \nu_1 \end{pmatrix}$	4
$3c$	$(\pm i\beta)(\pm i\beta)$	$\langle e, \mathbf{s}e, f, -\mathbf{s}f\rangle$	$\begin{pmatrix} J & \\ & -J \end{pmatrix}$	$\begin{pmatrix} \nu_1 & -\beta-\nu_2 & & \\ \beta+\nu_2 & \nu_1 & & \\ & & \nu_3 & -\beta-\nu_4 \\ & & \beta+\nu_4 & \nu_3 \end{pmatrix}$	4
4	0	$\langle e, \mathbf{s}e\rangle$	J	$\begin{pmatrix} \nu_1 & \nu_2 \\ \nu_2 & -\nu_1 \end{pmatrix}$	2

Table 4: *Unfoldings in eigenspaces of automorphisms.*

type	λ	X_λ	$\mathbf{s}$	$L(\nu)$	codim
$5a$	0	$\langle e\rangle$	1	ν	1
$5b$	0^2	$\langle e, Ne\rangle$	T	$\begin{pmatrix} \nu_1 & \nu_2 \\ 1 & \nu_1 \end{pmatrix}$	2
$5c$	00	$\langle a, b\rangle$	R	$\begin{pmatrix} \nu_1 & -\nu_3 \\ \nu_3 & \nu_2 \end{pmatrix}$	3
$5d$	00	$\langle a_1, a_2\rangle$	I	$\begin{pmatrix} \nu_1 & \nu_3 \\ \nu_3 & \nu_2 \end{pmatrix}$	3
$6a$	0	$\langle e\rangle$	1	0	0
$6b$	$(0^+)^3$	$\langle e, Ne, N^2e\rangle$	$\begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$	$\begin{pmatrix} 0 & \nu & 0 \\ 1 & 0 & \nu \\ 0 & 1 & 0 \end{pmatrix}$	1
$6c$	0^2	$\langle e, f, Ne, Nf\rangle$	$\begin{pmatrix} & J \\ -J & \end{pmatrix}$	$\begin{pmatrix} \nu_2 & \nu_3-\nu_4 & \nu_1 & 0 \\ \nu_3+\nu_4 & -\nu_2 & 0 & \nu_1 \\ 1 & 0 & \nu_2 & \nu_3-\nu_4 \\ 0 & 1 & \nu_3+\nu_4 & -\nu_2 \end{pmatrix}$	4
$6d$	$(\pm i\beta)^2$	$\langle e, f, Ne, Nf\rangle$	$\begin{pmatrix} & J \\ -J & \end{pmatrix}$	$\begin{pmatrix} 0 & -\beta-\nu_1 & \nu_2 & 0 \\ \beta+\nu_1 & 0 & 0 & \nu_2 \\ 1 & 0 & 0 & -\beta-\nu_1 \\ 0 & 1 & \beta+\nu_1 & 0 \end{pmatrix}$	2
$6e$	$(\pm i\beta)(\pm i\beta)$	$\langle e_1, f_1, e_2, f_2\rangle$	$\begin{pmatrix} I_2 & \\ & I_2 \end{pmatrix}$	$\begin{pmatrix} 0 & -\beta-\nu_1 & \nu_3 & -\nu_4 \\ \beta+\nu_1 & 0 & \nu_4 & \nu_3 \\ -\nu_3 & -\nu_4 & 0 & -\beta-\nu_2 \\ \nu_4 & -\nu_3 & \beta+\nu_2 & 0 \end{pmatrix}$	4
$7a$	0	$\langle e, \mathbf{s}e\rangle$	J	$\begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix}$	1
$7b$	0^2	$\langle e, \mathbf{s}e, Ne, N\mathbf{s}e\rangle$	$\begin{pmatrix} & J \\ J & \end{pmatrix}$	$\begin{pmatrix} \nu_1 & 0 & \nu_2 & \nu_3-\nu_4 \\ 0 & \nu_1 & \nu_3+\nu_4 & -\nu_2 \\ 1 & 0 & \nu_1 & 0 \\ 0 & 1 & 0 & \nu_1 \end{pmatrix}$	4
$8a$	0	$\langle e, \mathbf{s}e\rangle$	J	$\begin{pmatrix} \nu_1 & \nu_2-\nu_3 \\ \nu_2+\nu_3 & -\nu_1 \end{pmatrix}$	3
$8b$	0^2	$\langle e, Ne\rangle$	J	$\begin{pmatrix} 0 & \nu \\ 1 & 0 \end{pmatrix}$	1
$8c$	$(\pm i\beta)^2$	$\langle e, f, Ne, Nf\rangle$	$\begin{pmatrix} & I \\ -I & \end{pmatrix}$	$\begin{pmatrix} 0 & -\beta-\nu_1 & \nu_2 & 0 \\ \beta+\nu_1 & 0 & 0 & \nu_2 \\ 1 & 0 & 0 & -\beta-\nu_1 \\ 0 & 1 & \beta+\nu_1 & 0 \end{pmatrix}$	2
$8d$	$(\pm i\beta)(\pm i\beta)$	$\langle e_1, f_1, e_2, f_2\rangle$	$\begin{pmatrix} J & \\ & J \end{pmatrix}$	$\begin{pmatrix} 0 & -\beta-\nu_1 & \nu_3 & -\nu_4 \\ \beta+\nu_1 & 0 & \nu_4 & \nu_3 \\ -\nu_3 & -\nu_4 & 0 & -\beta-\nu_2 \\ \nu_4 & -\nu_3 & \beta+\nu_2 & 0 \end{pmatrix}$	4

Table 5: *Unfoldings in eigenspaces of anti-automorphisms.*