ZEROTH HOCHSCHILD HOMOLOGY OF PREPROJECTIVE ALGEBRAS OVER THE INTEGERS

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Abstract. We determine the $\mathbb{Z}$-module structure of the preprojective algebra and its zeroth Hochschild homology, for any non-Dynkin quiver (and hence the structure working over any base commutative ring, of any characteristic). This answers (and generalizes) a conjecture of Hesselholt and Rains, producing new p-torsion classes in degrees $2p^\ell, \ell \geq 1$. We relate these classes by p-th power maps and interpret them in terms of the kernel of Verschiebung maps from noncommutative Witt theory. An important tool is a generalization of the Diamond Lemma to modules over commutative rings, which we give in the appendix.

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1. Introduction and Main Results

This paper concerns the $\mathbb{Z}$-module $A_{\mathrm{cyc}} := A/[A,A] = A/\mathrm{Span}(\{ab-ba \mid a,b \in A\})$ (also known as the zeroth Hochschild homology, $HH_0(A)$) for certain graded algebras A over $\mathbb{Z}$ related to quivers and noncommutative Witt theory. Here, the span means the integral span. For these algebras, a curious phenomenon emerges:

(*) *Over a characteristic p field $\mathbf{k}$, the dimension of the $2p^\ell$-th graded part of $A_{\mathrm{cyc}} \otimes \mathbf{k}$ exceeds by one the dimension of the same part over a characteristic zero field, for all $\ell \geq 1$; in all other degrees, the two dimensions are the same.*

The simplest such algebra we study is

$$\Pi := \mathbb{Z}\langle x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\rangle / (\sum_{i=1}^{n} [x_i, y_i]), \quad n \geq 2, \tag{1.0.1}$$

where $\mathbb{Z}\langle t_1, \dots, t_m\rangle$ is by definition the free associative (noncommutative) algebra on m generators $t_1, \dots, t_m$. For this algebra $A = \Pi$, the phenomenon (*) was conjectured by Hesselholt and Rains, motivated by noncommutative Witt theory.

Moreover, Hesselholt and Rains produced specific homogeneous classes of Π_{cyc} that, if nonzero, are p-torsion, and conjectured that they are all nonzero and generate the p-torsion. These classes are given as follows: One has the commutative diagram of quotients, where $P := \mathbb{Z}\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle$

is the free algebra used above:

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & \Pi \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ P_{\mathrm{cyc}} & \xrightarrow{\overline{\alpha}} & \Pi_{\mathrm{cyc}}. \end{array} \tag{1.0.2}$$

Define

$$r := \sum_{i=1}^{n} [x_i, y_i]. \tag{1.0.3}$$

Then, the class $\pi(r^p)$ is a multiple of p in P_{cyc}. Hence, one may consider the class $r^{(p)} := \bar{\alpha}(\frac{1}{p}\pi(r^p)) \in \Pi_{\mathrm{cyc}}$, which evidently satisfies $pr^{(p)} = 0$. Similarly, $\pi(r^{p^\ell}) \in P_{\mathrm{cyc}}$ is a multiple of p for all $\ell \geq 1$, and one may thus define $r^{(p^\ell)} := \bar{\alpha}(\frac{1}{p}\pi(r^{p^\ell})) \in \Pi_{\mathrm{cyc}}$.

Conjecture 1.0.4 (Hesselholt-Rains)**.** For all primes $p \geq 2$, if $n \geq 2$, the classes $r^{(p^\ell)}$ are nonzero and generate the p-torsion of Π_{cyc}. There is no p^2-torsion in Π_{cyc}.

The conjecture implies (*). We give an elementary proof of this conjecture in §3 below, which is the first main result of this paper.

In terms of noncommutative Witt theory, the conjecture in particular implies that the noncommutative p-adic Verschiebung map is not, in general, injective, and the ghost components of its kernel are given by the nonvanishing p-torsion classes in Π_{cyc} stated in the conjecture. We will explain this in more detail in §2.

Remark 1.0.5. If we work over $\mathbb{F}_p$, then the classes $r^{(p^\ell)} := r^{(p^\ell)} \otimes 1 \in \Pi_{\mathrm{cyc}} \otimes_{\mathbb{Z}} \mathbb{F}_p$ are related by p-th power maps: generally, for every associative $\mathbb{F}_p$-algebra A, we have a well defined p-th power map $A_{\mathrm{cyc}} \to A_{\mathrm{cyc}}$, $[a]_{\mathrm{cyc}} \mapsto [a^p]_{\mathrm{cyc}}$, which is well-defined since $(a+b)^p - a^p - b^p \in [A, A]$ (modulo p).[1] Then, it is easy to verify explicitly that $r^{(p^\ell)} = (r^{(p^{\ell-1})})^p$. Hence, working over $\mathbb{Z}$, all the p-torsion of Π_{cyc} is generated from $r^{(p)} + p\Pi_{\mathrm{cyc}}$ by taking p-th powers (i.e., replacing a class $[f]$ by a class $[f^p]$, which is well defined modulo $p\Pi_{\mathrm{cyc}}$) and sums of resulting classes (since the images of the p-torsion modulo $p\Pi_{\mathrm{cyc}}$ are all generated from $r^{(p)}$ by p-th power maps and sums). Note that $p\Pi_{\mathrm{cyc}}$ itself contains no p-torsion since Π_{cyc} has no p^2-torsion.

1.1. **Quiver generalization.** A *quiver* is an oriented graph whose edges are called arrows. We will always assume the quiver to be connected, i.e., its underlying undirected graph is connected. We are interested in the quiver generalization of the preceding, in which P is replaced by the algebra of paths in a quiver, and the $n \geq 2$ condition is replaced by a certain non-Dynkin condition. The algebra Π is replaced by the *preprojective algebra of* Q, which was originally defined by Gelfand and Ponomarev [GP79] in the study of quiver representations.

In detail, for any quiver Q, let P_Q be the algebra over $\mathbb{Z}$ (i.e., the ring) generated by paths in Q, with concatenation as multiplication, called the path algebra (later on we will also work over a general commutative ring $\mathbf{k}$, in which case the path algebra over $\mathbf{k}$ is $P_Q \otimes_{\mathbb{Z}} \mathbf{k}$). Let Q_1 denote the set of arrows in the quiver Q and Q_0 the set of vertices. We will equip P_Q with the grading by length of paths, with $(P_Q)_m$ the subspace spanned by paths of length m; thus Q_0 forms a basis for $(P_Q)_0$ and Q_1 forms a basis for $(P_Q)_1$.

Define the *double quiver*, $\overline{Q}$, to be the quiver obtained from Q by adding a reverse arrow a^* for every $a \in Q_1$, with the same endpoints but the opposite orientation, and keeping the same set of

[1] In fact, as observed by Jacobson in the 1940's, $(a+b)^p \equiv a^p + b^p$ modulo the Lie algebra generated by a and b.

vertices. We replace P above with $P_{\overline{Q}}$, and recover the P of the previous subsection in the special case when Q has only one vertex and n arrows.

Let us assume that Q is finite (i.e., it has finitely many arrows, and hence, by connectivity, finitely many vertices). Define

$$r = \sum_{a \in Q_1} (aa^* - a^*a), \quad \Pi_Q := P_{\overline{Q}}/(r). \tag{1.1.1}$$

Our main object of study in this paper is $(\Pi_Q)_{\mathrm{cyc}}$, and as such we define notation for it:

$$\Lambda_Q := (\Pi_Q)_{\mathrm{cyc}}. \tag{1.1.2}$$

We may consider again the commutative diagram

$$\begin{array}{ccc} P_{\overline{Q}} & \xrightarrow{\alpha} & \Pi_Q \\ \downarrow \pi & & \downarrow \pi \\ (P_{\overline{Q}})_{\mathrm{cyc}} & \xrightarrow{\overline{\alpha}} & \Lambda_Q \end{array} \tag{1.1.3}$$

Then the Hesselholt-Rains conjecture generalizes as follows. We say that a quiver Q is (ADE) Dynkin if the underlying undirected graph is Dynkin of type A_n, D_n, or E_n (with $n = |Q_0|$ equal to the number of vertices); in particular this means there are no loops and at most one arrow between any pair of vertices. We say a quiver Q is extended Dynkin if its underlying undirected graph is the extended Dynkin diagram of a type ADE Dynkin diagram (in particular, this implies that it obtained from the latter by adding an additional vertex and one or two arrows). We consider the diagram with one vertex and one arrow (from the vertex to itself) to be extended Dynkin, and call it type $\tilde{A}_0$. Thus, the algebras P and Π from the previous subsection are path and preprojective algebras of an extended Dynkin quiver of type $\tilde{A}_0$ in the case $n = 1$ and of a Dynkin quiver of type A_1 in the case $n = 0$, whereas for $n \geq 2$ the corresponding quiver is neither Dynkin nor extended Dynkin.

Theorem 1.1.4. For all primes $p \geq 2$, if Q is non-Dynkin and non-extended Dynkin, the classes $r^{(p^\ell)} := \bar{\alpha}(\frac{1}{p}\pi(r^{(p^\ell)}))$ are nonzero and generate the p-torsion of Λ_Q. There is no p^2-torsion.

We also present a much more general question (Question 6.2.1), based on conversations with P. Etingof, that asks whether, for finitely-presented graded algebras over $\mathbb{Z}$ (or $\mathbb{Z}^{Q_0}$), for primes in which they are "asymptotic representation complete intersections (RCIs)" (see [EG06]) the new torsion is generated by classes of the above form, for relations r which lie in the integral span of commutators modulo p.

Remark 1.1.5. As in Remark 1.0.5, over $\mathbb{F}_p$ one has the statement $(r^{(p^\ell)}) = (r^{(p)})^{p^{\ell-1}}$ using the p-th power maps on $\Lambda_Q \otimes_{\mathbb{Z}} \mathbb{F}_p$, so all the p-torsion of Λ_Q is generated, modulo p, by $r^{(p)}$ using p-th power maps and sums.

1.2. **Strategy of proof of Theorem 1.1.4.** The main strategy for the proof of the theorem is to exploit an extended Dynkin subquiver $Q^0 \subsetneq Q$, i.e., a quiver such that $Q^0_0 := (Q^0)_0 \subseteq Q_0$ and $Q^0_1 := (Q^0)_1 \subseteq Q_1$. (It is well-known that such a quiver exists whenever Q is neither Dynkin or extended Dynkin, although there may be more than one choice of it.) We then use and develop facts about extended Dynkin quivers, and the relationship between Π_Q and Π_{Q^0}. Let $Q \setminus Q^0$ denote the quiver with the same vertex set as Q, and with arrows $Q_1 \setminus Q^0_1$ (thus, $(Q \setminus Q^0)_0 := Q_0$).

We prove that there is an isomorphism as $\mathbb{Z}$-modules, $\Pi_Q \cong \Pi_{Q^0} *_{\mathbb{Z}^{Q^0_0}} \Pi_{Q\setminus Q^0, Q^0_0}$, where $\Pi_{Q\setminus Q^0, Q^0_0}$ is a partial preprojective algebra (defined in [EE05]; see §4.1 below), an algebra over $\mathbb{Z}^{Q_0}$ (which

in turn we consider as a $\mathbb{Z}^{Q_0^0}$-algebra in the canonical way). The proof is then divided into three somewhat overlapping cases:

(1) The case of primes p which are good for the extended Dynkin quiver Q^0 (i.e., not a factor of the size of the corresponding finite group Γ under the McKay correspondence), given in Theorem 4.2.13. To prove this, we use the well-known Morita equivalence $\Pi_{Q^0} \otimes_{\mathbb{Z}} \mathbf{k} \simeq \mathbf{k}[x,y] \rtimes \Gamma$ for $\mathbf{k}$ is an algebraically closed field of characteristic p. This induces a Morita equivalence $\Pi_{Q,Q_0^0} \otimes_{\mathbb{Z}} \mathbf{k} \simeq (\mathbf{k}\langle x,y\rangle \rtimes \Gamma) *_{\mathbf{k}[\Gamma]} (\mathbf{k}[\Gamma] \otimes_{\mathbb{Z}^{Q_0^0}} \Pi_{Q\setminus Q^0,Q_0^0})$. We then prove the theorem in essentially the same manner as in the one-vertex case (where $\Gamma = \{1\}$). The latter case, which is essentially the original Hesselholt-Rains conjecture, is proved in §3 below in an elementary fashion using (a mild generalization of) the Diamond Lemma.

(2) The case when Q^0 is of type $\tilde{A}$ or $\tilde{D}$, given in Theorems 9.1.2 and 10.1.9. To prove these theorems, we use explicit integral bases for Π_{Q^0}, Π_Q and their zeroth Hochschild homology modulo torsion. We explicitly present these bases and verify that they are bases using the Diamond Lemma for modules over a commutative ring (which is discussed in Appendix A).

(3) The remaining cases where Q^0 is of type $\tilde{E}_n$ (for $n \in \{6,7,8\}$) and $p \leq 5$: these are proved in §13.2. Here we need to prove a more refined statement, given in Theorem 7.0.9, which relies on p-th power maps. For the cases at hand, the proof follows via Theorem 13.1.1, which computes Λ_Q in the type E Dynkin and extended Dynkin cases via straightforward computation using Gröbner generating sets (cf. Appendix A.1 and Proposition A.1.1 therein), and Proposition 12.5.8, which computes the zeroth Poisson homology of the necklace Lie algebra structure on Λ_Q [Gin01, BLB02, CBEG07] in these cases.

Above and below, we use the term "Gröbner generating set" since we work over arbitrary commutative rings (such as $\mathbb{Z}$) and it is slightly inconvenient for us to define the minimal such sets that, when working over fields, are customarily called Gröbner bases (and we do not need minimality here).

In the process, we obtain integral, rational, and characteristic p bases for Π_Q and Λ_Q modulo torsion. This relies on the $\mathbb{Z}$-module decomposition $\Pi_Q \cong \Pi_{Q^0} *_{\mathbb{Z}^{Q_0^0}} \Pi_{Q\setminus Q^0,Q_0^0}$. For Λ_Q, we can then (essentially) write classes as cyclic words in Λ_{Q^0} and $(\Pi_{Q\setminus Q^0,Q_0^0})_{\mathrm{cyc}}$. For details, see §9.4 for the case $Q^0 = \tilde{A}_n$, §10.4 for the case $Q^0 = \tilde{D}_n$, Propositions 7.0.5 and 7.0.7 for the case where p is good for Q^0 as above (i.e., not a factor of the size of the group given by the McKay correspondence); and for the general case see Theorem 7.0.9 and the preceding (note that only $Q^0 = \tilde{E}_n$ and $p \leq 5$ are not covered by the preceding cases). As observed above, these cases are overlapping, but note that the bases we obtain do *not* coincide in overlapping cases.

The bases should be interesting in their own right. As one simple application of the bases for Π_Q, we may deduce that Π_Q is torsion-free (which was proved in [EE05] for non-Dynkin quivers using Gelfand-Kirillov dimension). (This follows from much less work than is required for the proof of Theorem 1.1.4 itself, where the essential difficulty is in finding the $\mathbb{Z}$-module structure of Λ_Q, rather than merely Π_Q.)

In the Dynkin and extended Dynkin cases, we also compute explicit bases for Π_Q and Λ_Q modulo torsion, and give an explicit description of the torsion of Λ_Q. Here, it turns out that the torsion is finite, and the nonzero classes $r^{(p^\ell)}$ only occur in "stably bad primes": none for $\tilde{A}_n$; $p = 2$ for $\tilde{D}_n$, $p \in \{2,3\}$ for $\tilde{E}_6, \tilde{E}_7$, and $p \in \{2,3,5\}$ for $\tilde{E}_8$. The precise result is Theorem 13.1.1 (which refines a result of [MOV06], which established the cases in which $\Lambda_Q \otimes \mathbb{F}_p$ vanishes). Note that there is good reason why this torsion must be as described: indeed, our proof of Theorem 1.1.4 via the method of the proof of Theorem 7.0.9 in §13 could also be used backwards to deduce, assuming the statement of Theorem 1.1.4 (which does not single out any primes), the precise torsion structure of Λ_Q without computing it directly.

Remark 1.2.1. In fact, there are precise ways in which Π_Q is well-behaved in *all* primes, unlike $\mathbf{k}[x,y] \rtimes \Gamma$ when $\mathbf{k}$ is a field of characteristic dividing $|\Gamma|$: for instance, Π_Q is a Calabi-Yau algebra over the base ring $\mathbb{Z}^{Q_0}$, in the sense that Π_Q has a self-dual finitely-generated projective bimodule resolution of length two (see, e.g., [CBEG07, (9.2.2)]). In particular, this implies that $\Pi_Q \otimes_{\mathbb{Z}} \mathbf{k}$ has global dimension two for all fields $\mathbf{k}$, unlike $\mathbf{k}[x,y] \rtimes \Gamma$, which has infinite global dimension when $\mathbf{k}$ is a field of characteristic dividing $|\Gamma|$.

1.3. **Outline of the paper.** First, in §2, we explain the motivation and interpretation using noncommutative Witt theory.

Then, in §3, we prove Hesselholt and Rains's conjecture.

In §4 we prove its generalization, Theorem 1.1.4, in the case of good primes (not dividing $|\Gamma|$ where $\Gamma < SL_2(\mathbb{C})$ is associated to an extended Dynkin subquiver).

Next, §5 recalls the notion of NCCI algebras and proves a general result we will need about them.

In §6, we explain in detail the (suggestive) Hilbert series formulas resulting from Theorem 1.1.4. We also pose a more general question on asymptotic RCI algebras in positive characteristic (Question 6.2.1).

In the next crucial section, §7, we prove one direction of Theorem 1.1.4: that the classes $r^{(p^\ell)}$ are nonzero (Proposition 7.0.8). We then proceed to state a refinement of the main theorem (Theorem 7.0.9), using prime powers, which implies Theorem 1.1.4. The goal of the remainder of the paper will be to prove this refinement.

In §8, we study some algebraic structures related to the preprojective algebra that we will need. In particular, we define and generalize the Lie bialgebra structure on Λ_Q, obtained as a quotient of the necklace Lie algebra. We explain that it is actually a Poisson algebra in the extended Dynkin case, modulo torsion (by identifying Λ_Q modulo torsion with the center of Π_Q, which is a commutative algebra). We explain how $P_{\overline{Q}}$ is a "free product" deformation of Π_Q, and prove that in the extended Dynkin case it quantizes the Poisson bracket on the center coming from the McKay correspondence.

In §§9 and 10, we prove the main results in the cases where there exists a subquiver Q^0 which is extended Dynkin of type $\tilde{A}_n$ (Theorem 9.1.2) or $\tilde{D}_n$ (Theorem 10.1.9). These results imply Theorem 7.0.9 and hence the main Theorem 1.1.4 in these cases.

In preparation for the type $\tilde{E}_n$ cases, in §11, we prove some results on preprojective algebras of star-shaped quivers. Then, in §12, we give explicit bases (via Gröbner generating sets) for Π_Q in the case of (extended) Dynkin quivers of type E (Proposition 12.0.1). We also compute the Lie structure on Λ_Q (Proposition 12.5.1).

Finally, we complete the proof of Theorem 7.0.9 in §13.

In the appendix, we give a generalized version of the Diamond Lemma for modules over commutative rings (which we use to compute bases).

Remark 1.3.1. In this paper we make use of two structures on $(P_{\overline{Q}})_{\mathrm{cyc}} \otimes \mathbb{F}_p$ and $\Lambda_Q \otimes \mathbb{F}_p$: the p-th power maps $[f] \mapsto [f^p]$ discussed above, and the necklace Lie algebra structure of [Gin01, BLB02] discussed in §8 below. It is natural to ask if these structures are compatible in any way. In particular, one might ask if they form a restricted Lie algebra. This is, however, not true, because in the latter case an axiom of restricted Lie algebras would be $\mathrm{ad}[f^p] = (\mathrm{ad}\, f)^p$, but the LHS is an operator of degree $p|f|-2$ whereas the RHS is an operator of degree $p(|f|-2)$, which are not equal. We could not find any compatibility axiom which these structures enjoy.

1.4. **Notation and Definitions.** We will always let $\mathbf{k}$ denote a base commutative ring and will work with algebras and their modules over $\mathbf{k}$. When considering a quotient A/B for A an algebra,

we will usually only require B to be a graded $\mathbf{k}$-submodule rather than an ideal (so that the quotient is only a graded $\mathbf{k}$-module).

To avoid confusion with $\mathbf{k}$-submodules whose definition requires parentheses, the ideal generated by elements will henceforth be denoted with double parentheses: $(\!(r)\!) =$ the ideal generated by r.

Given a $\mathbf{k}$-module M and a subset $S \subseteq M$, we will let $\langle S \rangle$ denote the $\mathbf{k}$-linear span of the set S. This is a completely different use of $\langle - \rangle$ than that for the free algebra $\mathbf{k}\langle x_1, \ldots, x_n \rangle$ on indeterminates $x_1, \ldots, x_n$, and the usage will be clear from the context.

Given a set S, we let the free $\mathbf{k}$-module generated by S be denoted by $\mathbf{k} \cdot S$.

We will use square braces to indicate that the expression inside is taken up to cyclic permutations: i.e., we write cyclic words as $[a_1 a_2 \cdots a_m] = [a_2 a_3 \cdots a_m a_1]$. We will use this in many cases where it is not strictly necessary (i.e., where the context already guarantees that the expression is invariant under cyclic permutations).

As stated earlier, a quiver Q is a finite, directed, connected graph, allowing loops and multiple edges. The edges are called arrows and the set of arrows is denoted by Q_1, while the set of vertices is denoted by Q_0. Connected here means the underlying undirected graph is connected. We will maintain the definition of the path algebra P_Q and the preprojective algebra Π_Q above, as well as the double quiver $\overline{Q}$. Note that, in P_Q, the product $p_1 p_2$ of two paths is zero if p_1 does not terminate at the same vertex at which p_2 begins.

For an arrow $a \in Q_1$, the reverse arrow is denoted a^*, and we also use the notation $(a^*)^* := a$. If an arrow a goes from vertex $i \in Q_0$ to $j \in Q_0$, we say $a : i \to j$, and set $a_s = i; a_t = j$ ("s"="source", "t"="target").

Let $\mathbf{k}^{Q_0} = \bigoplus_{i \in Q_0} \mathbf{k}$ denote the ring which, as a $\mathbf{k}$-module, is the free $\mathbf{k}$-module $\mathbf{k} \cdot Q_0$ with basis Q_0, and which has product $ij = \delta_{ij} i$. Then, we consider $\mathbf{k} \cdot Q_1$ to be a $\mathbf{k}^{Q_0}$-bimodule with multiplication $iaj = \delta_{ia_s} \delta_{ja_t} e$. One has $P_Q \otimes_{\mathbb{Z}} \mathbf{k} = T_{\mathbf{k}^{Q_0}}(\mathbf{k} \cdot Q_1)$.

The "length" of a path in a quiver is the number of arrows in the path. Similarly, the length of a line segment of arrows is the number of arrows in the segment. As above, P_Q and Π_Q are graded by path length.

When A is a graded $\mathbf{k}$-module, we let A_m denote the degree-m component of A, so that $A = \bigoplus_m A_m$. The gradings we will need will be by path length (hence nonnegative).

We will let $\mathbb{Z}/n$ denote $\mathbb{Z}/n\mathbb{Z}$ throughout.

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2. Relations to Witt theory

In [Hes97] (see also [Hes05]), Hesselholt defined the abelian group of p-typical Witt vectors $W(A)$ for any noncommutative ring A, and the vectors of a given length ℓ, $W_\ell(A)$. In the case where A is commutative, this reduces to the usual Witt vectors [Wit37] (which form a commutative ring in this case). We recall briefly their definition.

First, one defines the $W_\ell(A)$ and the restriction maps $R : W_{\ell+1}(A) \twoheadrightarrow W_\ell(A)$, and then $W(A)$ will be the inverse limit of the $W_\ell(A)$. Each $W_\ell(A)$ is defined as a certain quotient of A^ℓ, such that there exist restriction maps R completing the commutative diagrams (where the left vertical arrow

is the projection to the first ℓ components):

$$\begin{array}{ccc} A^{\ell+1} & \twoheadrightarrow & W_{\ell+1}(A) \\ \downarrow & & \downarrow R \\ A^{\ell} & \twoheadrightarrow & W_{\ell}(A). \end{array} \tag{2.0.1}$$

We therefore represent elements of $W_\ell(A)$ (nonuniquely) by coordinates $(a_0, \dots, a_{\ell-1})$.

In terms of these coordinates, the sum and difference operations on $W_\ell(A)$ are expressed using noncommutative versions of the usual Witt polynomials for these operations. In more detail, define first the noncommutative *ghost map*, $w : A^\ell \to A^\ell_{\mathrm{cyc}}$, $w(a_0, \dots, a_{\ell-1}) = ([a_0], [a_0^p + pa_1], [a_0^{p^2} + pa_1^p + p^2 a_2], \dots)$. There exist (nonunique) noncommutative polynomials $s_i, d_i, i = 0, 1, \dots$ in infinitely many variables $(x_0, x_1, x_2, \dots)$ such that, for all ℓ, $s = (s_0, s_1, \dots, s_{\ell-1})$ and $d = (d_0, d_1, \dots, d_{\ell-1})$ are transported under w to the usual sum and difference in A^ℓ_{cyc}. Then, the $W_\ell(A)$ are the unique quotients (of sets) of the A^ℓ, compatible with (2.0.1), such that s and d descend to each $W_\ell(A)$, and each composition $(\{0\}^{\ell-1} \times A) \subset A^\ell \twoheadrightarrow W_\ell(A)$ has zero fiber $\{0\}^{\ell-1} \times [A, A]$ (i.e., one may inductively define a noncanonical bijection of sets $A^\ell_{\mathrm{cyc}} \cong W_\ell(A)$). Hesselholt showed that $W_\ell(A)$ then is an abelian group under s and d and the operations are independent of the choice of noncommutative polynomials s_i, d_i.

In addition to defining $W(A)$ for associative algebras A, Hesselholt generalized the Frobenius (F) and Verschiebung (V) operators on $W(A)$ to the noncommutative setting. These have the form $F : W_\ell(A) \to W_{\ell-1}(A)$ and $V : W_\ell(A) \to W_{\ell+1}(A)$.

In terms of coordinates, the Verschiebung operator continues to have the form

$$V(a_0, a_1, \dots, a_{\ell-1}) = (0, a_0, a_1, \dots, a_{\ell-1}), \tag{2.0.2}$$

and the Frobenius operator is the unique functorial operation whose expression in terms of ghost components $w(a_0, \dots, a_{\ell-1}) = (w_0, w_1, \dots, w_{\ell-1}) \in A^\ell_{\mathrm{cyc}}$ is

$$w(F(a_0, \dots, a_{\ell-1})) = (pw_1, pw_2, \dots, pw_{\ell-2}). \tag{2.0.3}$$

While, in the commutative case, the Verschiebung operator is always injective, it turns out this may not be the case in the noncommutative setting. Hesselholt defined functorial exact sequences,

$$HH_1(A) \xrightarrow{\delta_\ell} W_\ell(A) \xrightarrow{V} W_{\ell+1}(A), \tag{2.0.4}$$

using the polynomials s_i, d_i. In particular, if $\sum_{i=1}^n \bar{x}_i \otimes \bar{y}_i \in A \otimes A$ is a Hochschild one-cycle (i.e. $\sum_{i=1}^n [\bar{x}_i, \bar{y}_i] = 0$) for some $\bar{x}_i, \bar{y}_i \in A$, one may consider the map $\phi : \Pi \to A$ mapping $x_i \mapsto \bar{x}_i, y_i \mapsto \bar{y}_i$ (for Π as in the beginning of §1 with parameter n); then one has a commutative diagram with exact rows,

$$\begin{array}{ccccc} HH_1(\Pi) & \xrightarrow{\delta_\ell} & W_\ell(\Pi) & \xrightarrow{V} & W_{\ell+1}(\Pi) \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ HH_1(A) & \xrightarrow{\delta_\ell} & W_\ell(A) & \xrightarrow{V} & W_{\ell+1}(A), \end{array} \tag{2.0.5}$$

so that $\phi \circ \delta_\ell[\sum_i x_i \otimes y_i] = \delta_\ell[\sum_i \bar{x}_i \otimes \bar{y}_i]$. Hence, the kernel of the Verschiebung map is generated by all classes that can be obtained as the image of the classes $\delta_\ell[\sum_i x_i \otimes y_i]$ under algebra maps $\Pi \to A$ (for all choices of n).

Furthermore, one has $W_1(A) = A/[A, A]$, and Hesselholt computed that $\delta_1(\sum_i x_i \otimes y_i) = r^{(p)}$. More generally,

$$w(\delta_\ell(\sum_i x_i \otimes y_i)) = (r^{(p)}, r^{(p^2)}, \dots, r^{(p^\ell)}). \tag{2.0.6}$$

Hence, Conjecture 1.0.4 says that, not only is the kernel of V nonzero in general, but the universal elements of the kernel, $\delta_\infty(\sum_{i=1}^n x_i \otimes y_i) \in W(\Pi)$, have all nonzero ghost components (for $n \geq 2$).

Note that, by (2.0.2),(2.0.3), the ghost components of any element of $\ker(V)$ are p-torsion in A_{cyc}.[2] Conjecture 1.0.4 implies that, in fact, the ghost components of the universal element of $\ker(V)$ above form a $\mathbb{F}_p$-basis for the p-torsion in Π_{cyc}.

Remark 2.0.7. The quiver generalization of the above should be as follows: Given any set of vertices Q_0, one may consider noncommutative algebras over $\mathbb{Z}^{Q_0}$ instead of over $\mathbb{Z}$. The Witt group itself is unchanged, only the category of algebras is changed. Now, the universal elements of $\ker(V)$ should instead be written as $\delta_\infty(\sum_{a\in Q_1} a\otimes a^* - a^*\otimes a) \in W(\Pi_Q)$, replacing the parameter n by the quiver Q. We then see that, for non-Dynkin, non-extended Dynkin quivers, not only is the corresponding universal element nonzero, but all its ghost components are nonzero, and they form a $\mathbb{F}_p$-basis for the p-torsion of Λ_Q.

3. Hesselholt and Rains' conjecture

The purpose of this section is to prove the following main combinatorial result, which in particular implies Conjecture 1.0.4. The proof of this lemma will be generalized later, to compute bases of Π and Λ modulo torsion for quivers containing $\tilde{A}_n$ or $\tilde{D}_n$:

Definition 3.0.1. For a sequence $(x_1, \ldots, x_k) \in X^k$ for any set X, let $\mathrm{per}(x.)$ be the period (the least positive integer such that $x_i = x_{i+\mathrm{per}(x.)}$, with indices taken modulo k), and let $\mathrm{rep}(x.) := k/\mathrm{per}(x.)$ be the number of cyclic permutations which fix the sequence (the size of the stabilizer in $\mathbb{Z}/k$ of the sequence), which we can call the "number of times the sequence repeats itself", hence "rep".

Let $A := F/R$ where $F := \mathbb{Z}\langle x, y, r'\rangle$ is the free algebra and $R := (\!(r)\!)$ with $r := xy - yx + r'$. Let $V := A/[Ar'A, A]$. Let $\hat{V}$ be the $\mathbb{Z}$-module obtained from V by adjoining $[\frac{1}{p}(r')^{p^\ell}]$ for all p prime and $\ell \geq 1$, i.e., the quotient of $V \oplus \mathbb{Z}\cdot\{t_p\}_{p \text{ prime}, \ell\geq 1}$ by the relations $p \cdot t_p = [(r')^{p^\ell}]$ for all p prime and $\ell \geq 1$. Similarly let $\widehat{F/[F,F]}$ and $\widehat{A/[A,A]}$ be the modules obtained from $F/[F,F]$ and $A/[A,A]$, respectively, by adjoining $[\frac{1}{p}(r')^{p^\ell}]$ for all p prime and $\ell \geq 1$.

Henceforth, a "monomial" in X refers to a noncommutative monomial, i.e., an element of the form $x_1 \cdots x_k$ for $x_1, \ldots, x_k \in X$ (with any of the x_i allowed to be equal to each other), unless we say "cyclic monomial." A cyclic monomial $[x_1 \cdots x_k]$ is the same except where $[x_1 \cdots x_k] = [x_2 \cdots x_k x_1]$ by definition.

Lemma 3.0.2. (1) A basis for A is given by monomials in x, y, and r' such that the maximal submonomials in x, y are of the type

$$z_{a,b} := \begin{cases} (xy)^b x^{a-b}, & \text{if } a \geq b, \\ (yx)^a y^{b-a}, & \text{if } a < b, \end{cases} \tag{3.0.3}$$

i.e. of the form $z_{a_1,b_1} r' z_{a_2,b_2} \cdots r' z_{a_{m+1},b_{m+1}}$ for $m, a_i, b_i \geq 0$;

(2) A basis for $A/[Ar'A, A]$ is given by
   (a) $z_{a,b}$; and
   (b) classes of the form $[z_{a_1,b_1} r' z_{a_2,b_2} \cdots z_{a_m,b_m} r']$ (up to simultaneously cyclically permuting the a and b indices).

[2] In fact, $F \circ V$ is multiplication by p in $W(A)$, so $\ker(V)$ is p-torsion in $W(A)$ as well.

(3) The canonical quotient map $V = A/[Ar'A, A] \twoheadrightarrow A/[A, A]$ has kernel the free submodule $W \subset V$ with basis given by the classes (for $a > b \geq 1$):

$$W_{a,b} := \sum_{a.,b.} \frac{\gcd(a,b)}{\mathrm{rep}(a.,b.)} [\prod_{\ell=1}^{|\{a.\}|} (-r')z_{a_\ell,b_\ell}], \tag{3.0.4}$$

$$W_{b,a} := \sum_{a.,b.} \frac{\gcd(a,b)}{\mathrm{rep}(a.,b.)} [\prod_{\ell=1}^{|\{a.\}|} r' z_{b_\ell,a_\ell}], \tag{3.0.5}$$

$$W_{a,a} = [(xy + r')^a] - [(xy)^a], \tag{3.0.6}$$

where the sums $\sum_{a.,b.}$ are over all **distinct** pairs $(a., b.)$ of tuples of the same length, modulo simultaneous cyclic permutations of the indices, such that $W_{c,d}$ always has bidegree (c, d) in x, y (with r' having bidegree $(1, 1)$, so in the summation $a = \sum(a_i + 1)$ and similarly for b) and such that $a_\ell > b_\ell$ for all ℓ. The coefficient $\mathrm{rep}(i., j.)$ is as defined in (3.0.1) above, viewing $(i., j.)$ as a k-tuple of elements of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Finally, the product is taken in left-to-right order, i.e., in (3.0.4) it expands as $[(-r')z_{a_1,b_1}(-r')z_{a_2,b_2} \cdots (-r')z_{a_\ell,b_\ell}]$, and similarly for (3.0.5). Note that the coefficients of $W_{a,b}, W_{b,a}$, and $W_{a,a}$ above are integers.

(4) None of the classes $W_{a,b}$, considered as elements of V, are multiples of any non-unit in $\mathbb{Z}$. Under the quotient $V \twoheadrightarrow V/\langle [(r')^m] \rangle_{m \geq 1}$, the images of only the classes W_{p^ℓ,p^ℓ} are multiples of a non-unit in $\mathbb{Z}$, the greatest of which is p.

Then, the classes $\frac{1}{p} W_{p^\ell,p^\ell} \in \hat{V}$ and $\frac{1}{p}[r^{p^\ell}] \in \widehat{F/[F,F]}$ have the same image in $\widehat{A/[A,A]}$, are nonzero, and generate the torsion of $\widehat{A/[A,A]}$ ($\mathbb{Z}/p$ in degrees $2p^\ell$ and 0 otherwise).

We note that abstractly understanding $A/[A, A]$ is easy if we wanted to use cyclic words in x and y, but the point of finding bases of the above form is to allow one to obtain bases in a further quotient by a power of r', or in an extension of A (cf. Corollary 3.0.14). In particular, to prove Conjecture 1.0.4, we will view F as a subalgebra of the free algebra $P = \mathbb{Z}\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle$ by $x = x_1, y = y_1$, and $r' = \sum_{i=2}^n [x_i, y_i]$, in which case the image of $(r')^\ell$ in Λ is zero for all $\ell \geq 1$.

Proof. (1) We use the Diamond Lemma for modules as formulated in Appendix A. First, define the disorder $\mathrm{Dis}(M)$ of a monomial M in x, y to be the minimal number of swaps of adjacent letters in M needed to bring it to the form $z_{a,b}$. For a monomial $M = M_1 r' M_2 \cdots r' M_{n+1}$ for $n \geq 1$, define the disorder to be $\mathrm{Dis}(M) := \mathrm{Dis}(M_1) + \ldots + \mathrm{Dis}(M_{n+1})$: that is, $\mathrm{Dis}(M)$ is given by the sum of the above disorder over each maximal monomial in x's and y's.

Let $O(M)$ be the maximal nonnegative integer such that $M \in (r')^{O(M)}$. Then, we define the partial order on monomials such that $M_1 \prec M_2$ if and only if either $O(M_1) > O(M_2)$ or $O(M_1) = O(M_2)$ and $\mathrm{Dis}(M_1) < \mathrm{Dis}(M_2)$. Every relation $f(xy - yx + r')g$, where f and g are monomials in x, y, and r', then has leading term equal to either $fxyg$ or $fyxg$ (because of the O condition), and thus can be viewed as a reduction $fyxg \mapsto fxyg + fr'g$ or $fxyg \mapsto fyxg - fr'g$ which reduces the disorder of the leading term.

To prove that the reductions are confluent, we have to show that, if a monomial f in x, y, and r' can be reduced in two different ways, then their difference is integrally spanned by relations with leading monomials strictly less than f in the partial ordering. That is, we need to consider f in which there are two distinct instances where x and y are adjacent and swapping the pair of x and y results in a smaller monomial in the partial ordering.

We claim that these two adjacent instances of x and y do not overlap. If not, then there must be either an instance of xyx in f such that the disorder decreases if xyx is replaced by yxx as well as if xyx is replaced by xxy, or alternatively an instance of yxy in f such that the disorder decreases if yxy is replaced by yyx as well as if it is replaced by xyy. But this is impossible. Indeed, in the

case that the disorder of a monomial f_1yxxf_2 is less than that of f_1xyxf_2, then it easily follows that $\mathrm{Dis}(f_1xyxf_2) = \mathrm{Dis}(f_1yxxf_2) + 1$ and also $\mathrm{Dis}(f_1xxyf_2) = \mathrm{Dis}(f_1xyxf_2) + 1$.

So, the two adjacent instances of x and y in f in question do not overlap. Call the two reductions of f obtained by the two corresponding swaps f' and f''. Then f' and f'' both admit a common reduction, obtained by swapping the pair of x and y which was not yet swapped (i.e., reducing f' by swapping the pair of x and y which was swapped to obtain f'', and vice-versa). Since all monomials appearing in f' and f'' are strictly less than f, this implies that $f' - f''$ is a linear combination of relations (defining reductions) whose leading monomials are strictly less than f, which yields the confluence property in this case.

With either approach, all reductions yield relations with leading coefficient 1, so by Proposition A.2.5, the quotient F/R is a free bigraded $\mathbb{Z}$-module with the given basis (the elements not appearing as leading coefficients of relations).

(2) Let $F' \subset F$ be integrally spanned by monomials not containing r' and let $V' := F' \oplus Fr'F/[Fr'F, F]$. There is an obvious surjection $\beta : V' \twoheadrightarrow V := A/[Ar'A, A]$. V' has a basis consisting of monomials in x and y and cyclic monomials in x, y, and r', i.e., monomials in x, y, and r' up to cyclic permutation. On cyclic monomials, let $O(f)$ denote the number of times that r' occurs in f. Let us define a partial order on monomials in V': $f \prec g$ if either (1) $O(f) > O(g)$; or (2) $O(f) = O(g)$ and there are fewer swaps $yx \leftrightarrow xy$ needed to bring f to normal form than for g.

Note that, in V', monomials in x, y, and r' are invariant under cyclic permutations, which is why we place them in square braces (in accordance with §1.4).

Then, it is not difficult to see that the set of elements $f(xy-yx)g+[fr'g], [h(xy-yx+r')] \in V'$ (for f, g, h monomials, with $h \in (\!(r')\!)$ and $f, g \notin (\!(r')\!)$) form a confluent set W', so that $V'/W' \cong A/[Ar'A, A] = V$, as desired: the normal-form basis of the quotient is given by the classes listed in the statement of (2). One makes the same arguments as before: the main point is that cyclic monomials that contain r' behave similarly to regular monomials in that one can compute the minimal number of swaps $xy \leftrightarrow yx$ needed to reduce to a normal-form element, and any two reductions can be performed in either order with the same result.

In detail, to prove confluence, as before, it suffices to show that, if f is either a monomial in x and y, or a cyclic monomial in x, y, and r', and if there are two adjacent instances of x and y in f such that swapping either pair decreases the disorder of f, then the corresponding reductions f' and f'' admit a common reduction. In the case of monomials in x and y, this is exactly the same argument as before. In the case of cyclic monomials in x, y, and r', then as before, in this case the two adjacent instances of x and y must be disjoint, since $\mathrm{Dis}([f_1yxxf_2]) < \mathrm{Dis}([f_1xyxf_2])$ (for f_1 and f_2 monomials in x, y, and r' at least one of which contains r') implies $\mathrm{Dis}([f_1xyxf_2]) = \mathrm{Dis}([f_1yxxf_2]) + 1$ and $\mathrm{Dis}([f_1xxyf_2]) = \mathrm{Dis}([f_1xyxf_2]) + 1$. Therefore f' and f'' admit common reductions, obtained by swapping the other instance of x and y not yet swapped. Thus $f' - f''$ is a linear combination of relations with smaller leading (cyclic) monomial than that of f, proving the confluence property.

Again, the leading coefficients of the relations generating W' (corresponding to the reductions) all are one, so by Proposition A.2.5, the desired set (which consist of the generators of V' which are not leading coefficients of any relations) forms a basis of V.

(3) We claim that $[A, A] = [Ar'A, A] + [\langle z_{a,b-1}\rangle_{a,b\geq 1}, y] + [\langle z_{a-1,b}\rangle_{a,b\geq 1}, x]$. This follows immediately from the fact that the $z_{a,b}$ form a basis of A (Part (1)). This means that $[A, A] = V/W$ where W is integrally spanned by the relations

$$w_{a,b,y} := \eta([z_{a,b-1}, y]), \quad w_{a,b,x} := \eta([z_{a-1,b}, x] \in V, \tag{3.0.7}$$

where

$$\eta : A \twoheadrightarrow A/[Ar'A, A] \cong V \text{ is the quotient.} \tag{3.0.8}$$

It remains to show that

$$\langle w_{a,b,x}, w_{a,b,y}\rangle = \langle W_{a,b}\rangle. \tag{3.0.9}$$

This will complete the proof of (3).

We will prove the sharper result that

$$\frac{\gcd(a,b)}{b} w_{a,b,x} = -\frac{\gcd(a,b)}{a} w_{a,b,y} = \pm W_{a,b}, \quad a,b \geq 1, \tag{3.0.10}$$

using the positive choice of gcd, where $\pm$ is plus if $a \leq b$ and minus if $a > b$. Here the division by a or b makes sense since V is a free $\mathbb{Z}$-module.

First, we note that $aw_{a,b,x} + bw_{a,b,y} = 0$, since the left-hand side is equivalent in $A/[Ar'A, A]$ to

$$\sum_{i=1}^{a+b} \eta([h_{i+1}h_{i+2}\cdots h_{a+b}h_1h_2\cdots h_{i-1}, h_i]), \quad h_1\cdots h_{a+b} = z_{a,b}, \quad h_i \in \{x,y\}, \forall i. \tag{3.0.11}$$

Thus, it remains to compute only one of $w_{a,b,x}$ and $w_{a,b,y}$ for all positive integers a and b. Let us suppose that $a > b$. Then, it follows that $w_{a,b,x}$ is the result of successively commuting the x on the right of $(xy)^b x^{a-b}$ all the way to the left, and subtracting the resulting $x(xy)^b x^{a-b-1}$. So, $w_{a,b,x} = \sum_{0\leq b'\leq b-1}[(xy)^{b'}xr'(xy)^{b-b'-1}x^{a-b-1}]$. Each summand can then be reduced to a linear combination of monomials in r' and the monomials $z_{c,d}$ for all c,d. Specifically, this is the sum of all possible terms of the form $[r'z_{a_1,b_1}r'\cdots r'z_{a_k,b_k}]$ such that $b_1 \geq b - b' - 1$, and satisfying the conditions of (3.0.4): $a_\ell > b_\ell$ for all ℓ, and $a_1 + \cdots + a_k + k = a$, $b_1 + \cdots + b_k + k = b$. So, when we add up all the contributions to $w_{a,b,x}$, we get $-((b_1+1)+\cdots+(b_m+1) = b)/rep(a.,b.)$ copies of each $[(-r')z_{a_1,b_1}(-r')\cdots(-r')z_{a_m,b_m}]$, i.e., $-\frac{b}{\gcd(a,b)}W_{a,b}$.

For the same reason, when $b > a$ we get $w_{a,b,y} = -\frac{a}{\gcd(a,b)}W_{a,b}$. So it remains to consider $w_{b,b,x}$. Here, we get $\eta((yx)^b - (xy)^b) = [(xy+r')^b] - [(xy)^b]$, as desired.

It is clear that the classes $W_{a,b}$ are zero if either of a or b is zero. This completes the proof of part (3).

(4) Since V is a free $\mathbb{Z}$-module, and each relation (3.0.4)–(3.0.6) lives in a different bigraded degree, we find that the torsion part of V/W is a direct sum of its bigraded components, which are all cyclic (or trivial). In bidegree (a,b), this module is $\mathbb{Z}/g_{a,b}$, where $g_{a,b}$ is the gcd of all the coefficients $\dfrac{\gcd(a,b)}{\mathrm{rep}(a.,b.)}$ that appear in (3.0.4)–(3.0.6). To compute this, we claim first that the numbers $\mathrm{rep}(a.,b.)$ range over all positive factors of $\gcd(a,b)$: it's clear that any cyclic monomial of the form $[f^\ell]$ with bidegree (a,b) must have ℓ be a factor of a and b; on the other hand, for any such factor, we can form the cyclic monomial $[f^\ell]$ where $f = r'x^{\frac{a}{\ell}-1}y^{\frac{b}{\ell}-1}$. So, the gcd of the coefficients $\dfrac{\gcd(a,b)}{\mathrm{rep}(a.,b.)}$ is 1 for all a,b. When $a \neq b$, the same is true if we restrict to terms not of the form $[(r')^m]$, since such terms cannot have bidegree (a,b). On the other hand, in the case $a = b$, we see that, if we restrict to the terms other than $[(r')^m]$, then the gcd is equal to p in the case $m = p^\ell$ where p is prime (but still one in any other case). To see this, first note that the term $[(r')^m]$ is the unique one that can have coefficient equal to $1 = \frac{\gcd(a,a)}{a}$, since $\mathrm{rep}(a.,b.) = a = b$ if and only if $a. = b. = (0,0,\ldots,0)$ (tuples of length $a = b$ with all zeros). Next, the gcd of all positive factors of m other than one equals p when $m = p^\ell$ and equals one otherwise. This proves the claims of the first paragraph.

Finally, we need to show that the classes $\frac{1}{p}W_{p^\ell,p^\ell} \in \hat{V}$ and $\frac{1}{p}[r^{p^\ell}] \in \widehat{F/[F,F]}$ have the same image in $\widehat{A/[A,A]}$, are nonzero, and generate its torsion. First, we have already seen that the images of the classes $\frac{1}{p}W_{p^\ell,p^\ell} \in \hat{V}$ are nonzero and generate the torsion of $\widehat{A/[A,A]}$, since $\widehat{A/[A,A]}$ is obtained by

modding by the classes $W_{a,b}$, which now have greatest integer factor $= p$ if $a = b = p^\ell$ and greatest integer factor $= 1$ otherwise.

We can write the image of the class $\frac{1}{p}[r^{p^\ell}]$ in $A/[A,A]$ as follows:

$$\frac{1}{p}[r^{p^\ell} - (xy)^{p^\ell} - (-yx)^{p^\ell}] + \frac{1}{p}(1 + (-1)^{p^\ell})[(xy)^{p^\ell}], \tag{3.0.12}$$

and since the first term in square braces is actually p times an integral combination of cyclic words, we can replace terms (yx) by $(xy + r')$ (without destroying the ability to divide by p), and obtain

$$\frac{1}{p}[-(xy)^{p^\ell} - (-xy - r')^{p^\ell}] + \frac{1}{p}(1 + (-1)^{p^\ell})[(xy)^{p^\ell}] \equiv \frac{1}{p}\big([(xy + r')^{p^\ell}] - [(xy)^{p^\ell}]\big) \pmod{p}, \tag{3.0.13}$$

which is the image of $\frac{1}{p}W_{p^\ell,p^\ell}$ under $\widehat{F/[F,F]} \twoheadrightarrow \widehat{A/[A,A]}$, as desired. □

From the proof, we deduce the more general

Corollary 3.0.14. Let $B = B_+ \oplus \mathbf{k}$ be any (graded) augmented $\mathbf{k}$-algebra such that B_+ is a free $\mathbf{k}$-module. Let $r' \in B \setminus \mathbf{k}$ (of degree 2). Let $r := xy - yx + r'$, and set $F' = \mathbb{Z}\langle x, y\rangle * B$ (with $\mathbb{Z}\langle x, y\rangle$ the free algebra) and $A' = F'/(\!(r)\!)$. Then

$$A'_{\mathrm{cyc}} \xleftarrow{\sim} (\mathbb{Z}[x,y] * B)_{\mathrm{cyc}}/W, \tag{3.0.15}$$

where W is as defined in Lemma 3.0.2. In particular,

$$\mathrm{torsion}(A'_{\mathrm{cyc}}) \xleftarrow{\sim} \mathrm{torsion}(B_{\mathrm{cyc}}) \oplus \bigoplus_{p,\ell} \langle r^{(p^\ell)}\rangle, \tag{3.0.16}$$

where the sum is over all primes p and all $\ell \geq 1$ such that $(r')^{p^\ell} \in pB + [B,B]$, and $\langle r^{(p^\ell)}\rangle$ denotes the subgroup of A'_{cyc} generated by $r^{(p^\ell)}$. The isomorphisms are obtained from $F' \twoheadrightarrow A'$ by picking a $\mathbb{Z}$-module section $\mathbb{Z}[x,y] \hookrightarrow F'$ of the submodule $\mathbb{Z}[x,y] \subseteq A'$. The classes $r^{(p^\ell)}$ are all nonzero.

In the corollary, the class $r^{(p^\ell)}$ is defined as follows: Let $f(x,y,r')$ be a noncommutative polynomial in x, y, r' such that $f(x,y,r') + (r')^p \equiv (xy - yx + r')^p$ modulo commutators, and such that all the coefficients of $f(x,y,r')$ are divisible by p (such an f exists by the construction of $r^{(p^\ell)} \in \Lambda_Q$). Let $g \in B$ be an arbitrary element such that $pg - (r')^p \in [B,B]$. Then we can define $r^{(p^\ell)}$ as the image of $\frac{1}{p}f(x,y,r') + g$ in A'_{cyc}.

Proof. We apply the argument of the lemma. Namely, we apply the Diamond Lemma for free modules (as found in Appendix A) now to $F'/[F', F'B_+F']$ (note that our F' now replaces what was F before, and there is no notation for what was F' in the proof of the lemma). Since B_+ is free as a $\mathbf{k}$-module, we can do this. Our partial ordering only concerns, as before, the x and y variables. Namely, we can write $F'/[F', F'B_+F']$ as the direct sum of $(B_+)_{\mathrm{cyc}}$ and a free $\mathbf{k}$-module whose basis consists of noncommutative words in x and y together with cyclic noncommutative words in x, y, and a basis for B_+, with no two elements of B_+ appearing next to each other (a cyclic noncommutative word is, as before, a noncommutative word modulo cyclic permutations). Our partial ordering is based as before on the degree in x and y and on the disorder function, giving the number of swaps of x and y needed to yield an element which is a word in the $z_{a,b}$ and the basis of B. Then, the rest of the proof goes through in the same manner. □

Remark 3.0.17. The corollary immediately implies Conjecture 1.0.4, setting $B := \mathbb{Z}\langle x_2, y_2, \ldots, x_g, y_g\rangle$ to be the free algebra, and defining $r' = \sum_{j=2}^{g}[x_j, y_j]$. (In fact, it implies Theorem 1.1.4 for arbitrary quivers Q containing a loop, i.e., $Q \supsetneq Q^0 \cong \tilde{A}_0$. See Remark 4.2.21.)

Remark 3.0.18. The proof of parts (1)–(2) of Lemma 3.0.2 and some of (3) is slightly shorter if we used $x^a y^b$ instead of $z_{a,b}$; in this case we have

$$W_{a,b} = \sum_{a.,b.} \frac{\gcd(a,b)}{\mathrm{rep}(a.,b.)} [\prod_{\ell=1}^{k} (r' x^{a_\ell} y^{b_\ell})], \tag{3.0.19}$$

and the Diamond Lemma argument is a bit simpler. However, the disadvantage is that the given formula for $W_{a,a}$ does not follow explicitly from the computation; more importantly, the chosen convention resembles more closely the $Q \supsetneq \tilde{D}_n$ case (however, the elements $x^a y^b$ do have an analogue for the $Q \supsetneq \tilde{A}_{n-1}$ case).

In more detail, replacing $z_{a,b}$ by $x^a y^b$, we could avoid using our version of the Diamond Lemma altogether for part (1) above, using instead the Gröbner generating set $(xy - yx + r')$ with respect to the graded lexicographical ordering with $|r'| = |x| = |y| = 1$ and $r' < x < y$ (see Appendix A.1 and Proposition A.1.1 therein). To prove part (3) from parts (1) and (2), we could use that the normal form of $z_{a,b}$ with respect to the basis of monomials in x and y has leading term $x^a y^b$.

4. Proof of Theorem 1.1.4 for good primes: a Γ-equivariant version

In this section, we will prove a "Γ-equivariant" version of Lemma 3.0.2, which will allow us to prove the main Theorem 1.1.4 for good primes.

4.1. **Partial preprojective algebras.** We will need the notion of *partial preprojective algebra*, denoted $\Pi_{Q,J}$, which depends on a subset of vertices $J \subseteq Q_0$, and is defined as a quotient of $P_{\overline{Q}}$ by the relations *iri* only at vertices $i \in Q_0 \setminus J$, called *black* vertices. In particular, this includes $P_{\overline{Q}} = \Pi_{Q,Q_0}$ as well as $\Pi_Q = \Pi_{Q,\emptyset}$. We define this in detail below.

We remark that there is an analogy between Π_Q and closed Riemann surfaces, in which $\Pi_{Q,J}$ is obtained by adding *punctures* at the points corresponding to J: for more details, see §11.

Notation 4.1.1. If $Q_0' \subset Q_0$, and $\mathbf{k}$ is any commutative ring, then let $\mathbf{1}_{Q_0'} \in \mathbf{k}^{Q_0}$ denote the matrix which is 1 in entries $(i,i), \forall i \in Q_0'$, and zero elsewhere. In terms of vertex idempotents, $\mathbf{1}_{Q_0'} = \sum_{i \in Q_0'} i$.

Definition 4.1.2. [EE05] Given any quiver Q, and any subset $J \subset Q_0$ of vertices (called the **white** vertices, with $Q_0 \setminus J$ the set of **black** vertices), define the **partial preprojective algebra** $\Pi_{Q,J}$ by:

$$\Pi_{Q,J} := P_{\overline{Q}} / (\!(\mathbf{1}_{Q_0 \setminus J} r \mathbf{1}_{Q_0 \setminus J})\!). \tag{4.1.3}$$

Definition 4.1.4. $\Lambda_{Q,J} := (\Pi_{Q,J})_{\mathrm{cyc}}$.

4.2. **Proof of Theorem 1.1.4 for good primes.** Let Q^0 be an extended Dynkin quiver, and $\Gamma \subset SL_2(\mathbb{C})$ the corresponding finite subgroup under the McKay correspondence. Identify x and y with the standard basis of $\mathbb{C}^2$, and hence let $SL_2(\mathbb{C})$ and therefore Γ act on $\mathbb{C} \cdot \{x, y\}$. By [CBH98], §3, for $\mathbf{k} = \mathbb{C}$, we know that there are Morita equivalences

$$P_{\overline{Q^0}} \otimes \mathbf{k} \simeq \mathbf{k}\langle x, y \rangle \rtimes \Gamma, \tag{4.2.1}$$

$$\Pi_{Q^0} \otimes \mathbf{k} \simeq \mathbf{k}[x, y] \rtimes \Gamma, \tag{4.2.2}$$

where, as before, $\mathbf{k}\langle x, y \rangle$ is the free algebra, and for any algebra A with an action by a group Γ, $A \rtimes \Gamma$ denotes the skew group algebra. These Morita equivalences are given by

$$P_{\overline{Q^0}} \otimes \mathbf{k} \xrightarrow{\sim} \mathsf{f}(\mathbf{k}\langle x, y \rangle \rtimes \Gamma)\mathsf{f} \tag{4.2.3}$$

where $\mathsf{f} \in \mathbf{k}[\Gamma]$ is a sum of primitive idempotents, one for each irreducible representation of Γ. In more detail, $\mathsf{f} = \sum_{i \in Q_0^0} \mathsf{f}_i$, where the vertices Q_0^0 of the extended Dynkin quiver Q^0 also label the irreducible representations U_i of Γ, and f_i is chosen so that $\mathbf{k}[\Gamma]\mathsf{f}_i \cong U_i$. Then, one may choose elements $\tilde{a} \in \mathbf{k}\langle x, y\rangle \rtimes \Gamma$ for each arrow $a \in \overline{Q}_1$ so that the map $a \mapsto \mathsf{f}\tilde{a}\mathsf{f}, i \mapsto \mathsf{f}_i$ gives the isomorphism (4.2.3), and furthermore that the element $xy - yx \in \mathbf{k}\langle x, y\rangle \rtimes \Gamma$ maps to the element $r \in P_{\overline{Q^0}}$, yielding (4.2.2) (specifically, (4.2.3) descends to $\Pi_{Q^0} \otimes \mathbf{k} \xrightarrow{\sim} \mathsf{f}(\mathbf{k}[x, y] \rtimes \Gamma)\mathsf{f}$).

Then, the idea of the proof of Theorem 1.1.4 for good primes is to repeat the arguments of the previous section, generalizing to a "Γ-equivariant" version. This involves replacing $A = \mathbf{k}\langle x, y, r'\rangle/((xy - yx + r')) \cong \mathbf{k}\langle x, y\rangle$ by $A = (\mathbf{k}\langle x, y, r'\rangle/((xy - yx + r'))) \rtimes \Gamma \cong \mathbf{k}\langle x, y\rangle \rtimes \Gamma$. Here, we set the Γ-action on r' to be trivial.

There are a few problems with this. First, in the previous section, we worked over $\mathbb{Z}$; we cannot get any torsion information setting $\mathbf{k} = \mathbb{C}$ as above. We resolve this problem by restricting to "good characteristic": primes that do not divide $|\Gamma|$, where all of the above easily generalizes (as is well-known). As a slight modification, rather than working over an algebraically closed field in good characteristic, we will set $\mathbf{k} := \mathbb{Z}[\frac{1}{|\Gamma|}, e^{\frac{2\pi i}{|\Gamma|}}]$, since this allows us to see all of the torsion (except in bad primes) simultaneously. Moreover, for this choice of $\mathbf{k}$, up to conjugation, we can assume that $\Gamma < SL_2(\mathbf{k})$. (Note that $\mathbb{Z}[\frac{1}{|\Gamma|}, e^{\frac{2\pi i}{|\Gamma|}}]$ is not a principal ideal domain, but this will not cause us any difficulties, as we did not and will not need to assume that we obtain a direct sum of cyclic modules; in any case, the torsion information we obtain could equivalently be obtained by working over finite fields of all good characteristics, and the modules we are actually interested in are defined over $\mathbb{Z}$.)

Second, we will need a way to get from A above to actual preprojective algebras of quivers properly containing Q^0. This will follow from a generalization of Corollary 3.0.14 and some general arguments about (partial) preprojective algebras, further developing some of the ideas of [EE05, EG06].

Third, to understand (a presentation of) $A/[A, A]$, we first need to understand $F/[F, F]$ where $F = \mathbf{k}\langle x, y, r'\rangle \rtimes \Gamma$. For this, we use the general (known) Hochschild theory for skew group algebras, as follows:[3]

Proposition 4.2.4. Let A be an associative algebra over a commutative ring $\mathbf{k}$ and Γ a finite group acting on A. Assume that $\mathbf{k}$ contains $\frac{1}{|\Gamma|}$ (in particular, the characteristic of $\mathbf{k}$ does not divide Γ). For any $\gamma \in \Gamma$, let $\Gamma_\gamma := \{\gamma' \mid \gamma'\gamma = \gamma\gamma'\}$ denote the centralizer of γ. Then, for any $A \rtimes \Gamma$-bimodule M, one has

$$HH^\bullet(A \rtimes \Gamma, M) \cong HH^\bullet(A, M)^\Gamma, \quad HH^\bullet(A \rtimes \Gamma) \cong \bigoplus_\gamma HH^\bullet(A, A\gamma)^{\Gamma_\gamma}; \tag{4.2.5}$$

$$HH_\bullet(A \rtimes \Gamma, M) \cong HH_\bullet(A, M)_\Gamma, \quad HH_\bullet(A \rtimes \Gamma) \cong \bigoplus_\gamma HH_\bullet(A, A\gamma)_{\Gamma_\gamma}, \tag{4.2.6}$$

where we sum over a collection of representatives of the conjugacy classes of Γ (one for each conjugacy class).

Proof. We prove the second result (for Hochschild homology) since we will use that one more heavily; the first follows from the co-version of the proof.

We write

$$HH_\bullet(A \rtimes \Gamma, M) = \mathrm{Tor}_\bullet^{(A \otimes A^{\mathrm{op}}) \rtimes (\Gamma \times \Gamma^{\mathrm{op}})}(A \rtimes \Gamma, M). \tag{4.2.7}$$

For any algebra B, let LH^B denote the left-derived functor of $B - Bimod \to \mathbf{k} - mod$, given by $M \mapsto M/\langle mb - bm\rangle_{m \in M, b \in B}$: that is, the derived functor which yields the Hochschild homology of B

[3]Thanks to P. Etingof for explaining this (known) proposition and proof.

with coefficients in the given B-bimodule M. Let $\Gamma_\Delta := \{(g, g^{-1}) \in \Gamma \times \Gamma^{\mathrm{op}}\}$. By Shapiro's lemma, since $\mathrm{Ind}_{\Gamma_\Delta}^{\Gamma\times\Gamma^{\mathrm{op}}}(A) = A \otimes \mathbf{k}[\Gamma] \cong A \rtimes \Gamma$ as a $\Gamma \times \Gamma^{\mathrm{op}}$-module, where A (and M) are $\Gamma_\Delta \cong \Gamma$-modules by conjugation,

$$\begin{aligned}(4.2.8)\quad \mathrm{Tor}_\bullet^{(A\otimes A^{\mathrm{op}})\rtimes(\Gamma\times\Gamma^{\mathrm{op}})}(A \rtimes \Gamma, M) &= H_\bullet(LH^{A\otimes A^{\mathrm{op}}}(LH^{\mathbf{k}[\Gamma]\otimes\mathbf{k}[\Gamma]^{\mathrm{op}}}((A \rtimes \Gamma) \otimes M))) \\ &\cong H_\bullet(LH^{A\otimes A^{\mathrm{op}}}(LH^{\mathbf{k}[\Gamma_\Delta]}(A \otimes M))) = H_\bullet(LH^{\mathbf{k}[\Gamma_\Delta]}(LH^{A\otimes A^{\mathrm{op}}}(A \otimes M))).\end{aligned}$$

Now, by our assumption on $\mathbf{k}$, taking Γ-coinvariants (or invariants) is exact, so the RHS is

$$(4.2.9)\qquad \mathrm{Tor}_\bullet^{A\otimes A^{\mathrm{op}}}(A \otimes M)_\Gamma = HH_\bullet(A, M)_\Gamma.$$

Finally, specializing to $M := A \rtimes \Gamma$, we note that

$$(4.2.10)\qquad \mathbf{k}[\Gamma] = \bigoplus_C k[\Gamma, C], \quad \mathbf{k}[\Gamma, C] := \bigoplus_{\gamma\in C}\langle\gamma\rangle,$$

where C ranges over the conjugacy classes of Γ. Since $\mathbf{k}[\Gamma, C]$ is stable under the $\mathbf{k}[\Gamma]$-action (given by the conjugation action of Γ), we end up with the second formula in (4.2.6). □

Let us introduce the notation (for $a, b \in A$ and $\gamma \in \Gamma$)

$$(4.2.11)\qquad [a, b]_\gamma := a(\gamma \cdot b) - ba,$$

where $\cdot$ denotes the action of Γ on A (so $\gamma \cdot b = \gamma b\gamma^{-1} \in A \rtimes \Gamma$). Then, in the case of degree zero, we may rewrite (4.2.6) as

$$(4.2.12)\qquad (A \rtimes \Gamma)_{\mathrm{cyc}} \cong \bigoplus_C (A/[A, A]_{\gamma_C})_{\Gamma_{\gamma_C}},$$

where C ranges over the conjugacy classes of Γ, $\gamma_C \in C$ is a fixed choice of representative for each C, and $\Gamma_\gamma \subset \Gamma$ denotes the centralizer of γ in Γ. Note that this formula may also be obtained directly without using any homological algebra, but only the definition $B_{\mathrm{cyc}} := B/[B, B] = HH_0(B)$ and the decomposition of $\mathbf{k}[\Gamma]$ into a direct sum of conjugacy classes. However, we will use the above formulas later on, and felt it is better to explain the general result.

We now state our

Theorem 4.2.13. Let $\mathbf{k} := \mathbb{Z}[\frac{1}{|\Gamma|}, e^{\frac{2\pi i}{|\Gamma|}}]$ and $A = \mathbf{k}\langle x, y, r'\rangle/((xy - yx + r'))$, and let $\Gamma \subset SL_2(\mathbf{k})$ be a finite subgroup, which acts on x and y by the tautological action on $\mathbf{k} \cdot \{x, y\} \cong \mathbf{k}^2$ (the free $\mathbf{k}$-module generated by x and y), and fixes r'.

Then, the $\mathbf{k}$-module $(A \rtimes \Gamma)_{\mathrm{cyc}}$ has a canonical decomposition along conjugacy classes C of Γ,

$$(4.2.14)\qquad (A \rtimes \Gamma)_{\mathrm{cyc}} = \bigoplus_C (A \rtimes \Gamma)_{\mathrm{cyc},C},$$

presented as follows:

(i) For $C \neq \{1\}$ and any choice $\gamma_C \in C$, $(A \rtimes \Gamma)_{\mathrm{cyc},C}$ is torsion-free (and in fact $\mathbf{k}$-projective).

(ii) For $C = \{1\}$, there is an isomorphism $V_1/W_1 \xrightarrow{\sim} (A \rtimes \Gamma)_{\mathrm{cyc},\{1\}}$, where V_1 is the direct sum of $\mathbf{k}[x, y]^\Gamma$ and the $\mathbf{k}$-module of Γ-invariant classes of elements of the form

$$(4.2.15)\qquad \sum_{a\cdot,b\cdot} \alpha_{a\cdot,b\cdot} z_{a_1,b_1} r' \cdots z_{a_m,b_m} r',$$

modulo cyclic permutations, with $\alpha_{a\cdot,b\cdot} \in \mathbf{k}$, and W_1 is $\mathbf{k}$-linearly spanned by the Γ-invariant classes in W of Lemma 3.0.2, part (3) ((3.0.4)–(3.0.6)). This map is induced by the obvious morphism $V_1 \to (A \rtimes \Gamma)_{\mathrm{cyc},\{1\}}$.

(iii) The projection $W_1 \to [(\langle r' \rangle)]^\Gamma / [(\langle r' \rangle)^2]^\Gamma \cong (\langle r' \rangle \otimes \mathbf{k}[x,y]^\Gamma)$ is a monomorphism and becomes an isomorphism after tensoring by $\mathbb{C}$. The image is

$$\Big(\bigoplus_{a,b} \langle \gcd(a,b) r' \otimes x^{a-1} y^{b-1} \rangle\Big)^\Gamma. \tag{4.2.16}$$

(iv) $(A \rtimes \Gamma)_{\mathrm{cyc},\{1\}}$ is torsion-free (in fact, projective over $\mathbf{k}$), but its quotient by the image of $\langle [(r')^{\ell}] \rangle_{\ell \geq 1}$ has $\mathbf{k}/(\langle p \rangle)$-torsion in each degree $2p^\ell$ for p prime and $p \nmid |\Gamma|$. The torsion then appears in $V_1/(W_1 + \langle [(r')^\ell] \rangle_{\ell \geq 1}$ and is $\mathbf{k}$-linearly spanned by the image of $r^{(p^\ell)} = \frac{1}{p}[(xy - yx + r')^{p^\ell}]$ under the map $(\mathbf{k}\langle x, y, r' \rangle)_{\mathrm{cyc}} \twoheadrightarrow (A \rtimes \Gamma)_{\mathrm{cyc}} / \langle [(r')^\ell] \rangle_{\ell \geq 1}$.

The proof of the theorem will be given at the end of the section. First, we give consequences and comments.

Remark 4.2.17. In fact, one can deduce from the theorem that $(A \rtimes \Gamma)_{\mathrm{cyc}}$ itself is not merely $\mathbf{k}$-projective but free over $\mathbf{k}$. Namely, with f as in (4.2.3), we have $\mathsf{f}(A \rtimes \Gamma)\mathsf{f} = A' \otimes_{\mathbb{Z}} \mathbf{k}$, where $A := (P_{\overline{Q^0}} *_{\mathbb{Z}^{Q_0^0}} \mathbb{Z}[r'])/(\langle r_0 + r' \rangle)$, $r_0 \in P_{\overline{Q^0}}$ is the standard element $r_0 = \sum_{a \in Q_1^0} [a, a^*]$, the path algebra is defined over the integers, and the $\mathbb{Z}^{Q_0^0}$-bimodule structure on $\mathbb{Z}[r']$ is defined by the augmentation $\mathbb{Z}^{Q_0^0} \twoheadrightarrow \mathbb{Z}$. Therefore $(A \rtimes \Gamma)_{\mathrm{cyc}} = (\mathsf{f}(A \rtimes \Gamma)\mathsf{f})_{\mathrm{cyc}} = A'_{\mathrm{cyc}} \otimes_{\mathbb{Z}} \mathbf{k}$. On the other hand, $A'_{\mathrm{cyc}} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{|\Gamma|}]$ is torsion-free, and hence free (since $\mathbb{Z}[\frac{1}{|\Gamma|}]$ is a principal ideal domain). Thus $(A \rtimes \Gamma)_{\mathrm{cyc}}$ itself is free over $\mathbf{k}$. In the proof below, we will give a simple reason why the summands $(A \rtimes \Gamma)_{\mathrm{cyc},C}$ are projective, which does not immediately explain why their direct sum, $(A \rtimes \Gamma)_{\mathrm{cyc}}$, is free.

The analogue of Corollary 3.0.14 is then

Corollary 4.2.18. Let B be any nonnegatively graded $\mathbf{k} \oplus \mathbf{k}[\Gamma]$-algebra, whose graded components are finitely-generated free $\mathbf{k}$-modules. Let $r' \in B \setminus \mathbf{k}$, of degree two, be such that $(0 \oplus \mathbf{1}_{\mathbf{k}[\Gamma]}) r' (0 \oplus \mathbf{1}_{\mathbf{k}[\Gamma]}) = r'$. Set $F' = (\mathbf{k}\langle x, y \rangle \rtimes \Gamma) *_{\mathbf{k}[\Gamma]} B$ where $\mathbf{k}[\Gamma]$ acts on B by inclusion into the right summand (so does not contain the unit in $\mathbf{k} \oplus \mathbf{k}[\Gamma]$). Let $r := xy - yx + r'$, and set $A' := F'/(\langle r \rangle)$. Then one has isomorphisms

$$A'_{\mathrm{cyc}} \xleftarrow{\sim} ((\mathbf{k}[x,y] \rtimes \Gamma) *_{\mathbf{k}[\Gamma]} B)_{\mathrm{cyc}} / W_1, \tag{4.2.19}$$

where W_1 is as described in Theorem 4.2.13 (in terms of x, y, and r'). In particular,

$$\mathrm{torsion}(A'_{\mathrm{cyc}}) \xleftarrow{\sim} \mathrm{torsion}(B_{\mathrm{cyc}}) \oplus \bigoplus_{p,\ell} \langle r^{(p^\ell)} \rangle, \tag{4.2.20}$$

where p ranges over all primes and $\ell \geq 1$ ranges over positive integers such that $(r')^{p^\ell} \in pB + [B, B]$. The isomorphisms are obtained from $F' \twoheadrightarrow A'$ by picking a section of $(\mathbf{k}[x,y] \rtimes \Gamma)$ into F'. All of the classes $r^{(p^\ell)}$ are nonzero.

(We omit the proof, which is similar to that of Corollary 3.0.14. Note that $r^{(p^\ell)}$ is also defined in exactly the same way as is done there.) This corollary immediately gives us Theorem 1.1.4 for good primes:

Proof of Theorem 1.1.4 for good primes. Assume $Q \supsetneq Q^0$. We will prove the theorem for this quiver Q.

We will use the fact that $\Pi_{Q \setminus Q^0, Q_0^0}$ and $(\Pi_{Q \setminus Q^0, Q_0^0})_{\mathrm{cyc}}$ are torsion-free (see Proposition 7.0.1, whose proof does not use any other results from this paper aside from the Diamond Lemma as formulated in Appendix A). To prove the theorem, we claim that it suffices to work over $\mathbf{k} := \mathbb{Z}[\frac{1}{|\Gamma|}, e^{\frac{2\pi i}{|\Gamma|}}]$, i.e., to replace Π_Q and $\Pi_{Q \setminus Q^0, Q_0^0}$ by $\Pi_Q \otimes \mathbf{k}$ and $\Pi_{Q \setminus Q^0, Q_0^0} \otimes \mathbf{k}$. This is because Λ_Q is a direct sum of cyclic modules, and the number of copies of $\mathbb{Z}/p$, for $p \nmid |\Gamma|$, is still detected after tensoring by $\mathbf{k}$

(it becomes the number of copies of $\mathbf{k}/(\!(p)\!)$), as is the fact that $r^{(p^\ell)}$ is nonzero. So, it suffices to prove that $\Lambda_Q \otimes \mathbf{k}$ has torsion in degrees $2p^\ell$ for $p \nmid |\Gamma|$ and $\ell \geq 1$, and that that torsion is a copy of $\mathbf{k}/(\!(p)\!)$ which is $\mathbf{k}$-linearly spanned by $r^{(p^\ell)}$.

For readability, we will omit the $\otimes\mathbf{k}$ and work over $\mathbf{k}$ in the proof. We need to show that the torsion is $\mathbf{k}$-linearly spanned by nonzero classes $r^{(p^\ell)}$ for $p \nmid |\Gamma|$.

Viewing $\mathbf{k}^{Q_0^0}$ as the center of $\mathbf{k}[\Gamma]$, set $B = \mathbf{k}[\Gamma] \otimes_{\mathbf{k}^{Q_0^0}} \Pi_{Q\setminus Q^0, Q_0^0}$, which is a $\mathbf{k}[\Gamma] \oplus \mathbf{k}^{Q_0 \setminus Q_0^0}$-algebra. Using the map $\mathbf{k} \hookrightarrow \mathbf{k}^{Q_0\setminus Q_0^0}$ sending 1 to $\mathbf{1}_{Q_0\setminus Q_0^0}$, we view B as a $\mathbf{k} \oplus \mathbf{k}[\Gamma]$-algebra.

Let the element $r' \in B$ be given by $r' := \mathbf{1}_{Q_0^0} r_{Q\setminus Q^0} \mathbf{1}_{Q_0^0}$. Now, $F' = (\mathbf{k}\langle x, y\rangle \rtimes \mathbf{k}[\Gamma]) *_{\mathbf{k}[\Gamma]} B$. By definition and the precise version of Morita equivalence from [CBH98] outlined above, we see that $\mathsf{f}F'\mathsf{f} \xrightarrow{\sim} \Pi_{Q,Q_0^0}$, and that $\mathsf{f}r\mathsf{f} = \mathsf{f}(xy - yx + r')\mathsf{f}$ maps to the element r of Π_{Q,Q_0^0} under the isomorphism.

Then, by the corollary, Λ_Q, viewing Π_Q now as $\Pi_{Q,Q_0^0}/(\!(r)\!)$, has torsion $\mathbf{k}$-linearly spanned by the nonzero classes $r^{(p^\ell)}$ for $\ell \geq 1$ and primes $p \nmid |\Gamma|$. □

Remark 4.2.21. Using the above argument with $\Gamma = \{1\}$, we obtain Theorem 1.1.4 in the case of quivers containing a loop, $\tilde{A}_0$, for which all primes are good. This only requires Corollary 3.0.14, and not the Γ-generalization (Corollary 4.2.18).

Proof of Theorem 4.2.13. Specializing (4.2.12) to our case,

$$(A \rtimes \Gamma)_{\mathrm{cyc}} \cong \bigoplus_C \big(A/([A, Ar'A]_{\gamma_C} + \langle [z_{a,b}, x]_{\gamma_C}, [z_{a,b}, y]_{\gamma_C}\rangle_{a,b\geq 1})\big)_{\Gamma_{\gamma_C}}. \tag{4.2.22}$$

Next, fix C, and let $\lambda_C, \lambda_C^{-1} \in \{e^{2\pi i k/|\Gamma|}\}_{1\leq k\leq |\Gamma|}$ be the eigenvalues of γ_C (which has determinant one). Let us choose an eigenbasis x_C, y_C of γ_C acting on $\mathbf{k}\cdot\{x, y\} \cong \mathbf{k}^2$, which we may obtain using the projections $\frac{1}{\lambda_C^{\pm 1} - \lambda_C^{\mp 1}}(\gamma_C - \lambda_C^{\mp 1})$, unless $\gamma_C = \pm 1$ (which is equivalent to $\lambda_C = \pm 1$ since in the latter case $\lambda_C = \lambda_C^{-1}$). If $\gamma_C = \pm 1$, we set $x_C = x, y_C = y$. Let us assume $\gamma_C \cdot x_C = \lambda_C x_C$ and $\gamma_C \cdot y_C = \lambda_C^{-1} y_C$. Since $[x_C, y_C]$ must be a unit multiple of $[x, y] = -r'$, we may further assume that $[x_C, y_C] = [x, y]$, by rescaling. Then, for $g \in \langle x, y\rangle = \langle x_C, y_C\rangle$,

$$[g, x_C]_{\gamma_C} = [g, x_C] + (\lambda_C - 1)g x_C, \quad [g, y_C]_{\gamma_C} = [g, y_C] + (\lambda_C^{-1} - 1) g y_C. \tag{4.2.23}$$

Let us also define $z_{a,b}^C$ as in (3.0.3), but replacing x and y by x_C and y_C.

There now remain two steps: (1) to understand $(A/[A, Ar'A]_{\gamma_C})_{\Gamma_{\gamma_C}}$, and (2) to compute the needed relations analogous to (3.0.4)–(3.0.6). Both use the C-versions of x, y, and z and γ_C-commutators.

We begin with (1), which is the easier step. Since $|\Gamma|$ is invertible in $\mathbf{k}$, we may replace coinvariants by invariants in (4.2.22). Then, the RHS of (4.2.22) is isomorphic to

$$\big(A^{\langle\gamma_C\rangle}/(([A, Ar'A]_{\gamma_C})^{\langle\gamma_C\rangle} + (\langle [z_{a,b}, x]_{\gamma_C}, [z_{a,b}, y]_{\gamma_C}\rangle_{a,b\geq 1})^{\langle\gamma_C\rangle})\big)^{\Gamma_{\gamma_C}}, \tag{4.2.24}$$

where $\langle\gamma_C\rangle < \Gamma_{\gamma_C}$ is the cyclic subgroup generated by γ_C. Invariants under this subgroup are those polynomials in x_C and y_C such that the bidegree (a, b) satisfies $\lambda_C^{a-b} = 1$ (in other words, letting $|\gamma_C|$ denote the order of γ_C, $|\gamma_C| \mid (a - b)$).

Next, we generalize the argument of Lemma 3.0.2 to prove the following claim:

$$A^{\langle\gamma_C\rangle}/([A, Ar'A]_{\gamma_C})^{\langle\gamma_C\rangle} \tag{4.2.25}$$

is a free graded $\mathbf{k}$-module with basis the classes

(a) $z_{a,b}^C$ for $a, b \geq 0$ and $\lambda_C^{a-b} = 1$,

(b) $[z^C_{a_1,b_1} r' \cdots z^C_{a_m,b_m} r']$, for $m \geq 1$ and $a_i, b_i \geq 0$, **except** for those classes of the form

$$[f^\ell], \text{ where } f \text{ has bidegree } (a,b) \text{ with } \lambda_C^{a-b} \neq 1 \text{ (i.e., } \gamma_C \cdot f = \lambda_C^{a-b} f \neq f). \tag{4.2.26}$$

We now prove the claim. Set $F := \mathbf{k}\langle x, y, r' \rangle$. Then $F^{\langle \gamma_C \rangle}$ itself has a basis consisting of noncommutative monomials in x, y, and r' of bidegrees (a,b) satisfying λ_C^{a-b}. In particular, part (i) of Lemma 3.0.2 generalizes, with the same proof, to yield that $A^{\langle \gamma_c \rangle}$ itself has a basis consisting of elements of the form

$$z^C_{a_1,b_1} r' \cdots z^C_{a_m,b_m} r' z^C_{a_{m+1},b_{m+1}}, \quad \text{s.t. } \lambda_C^{\sum_i (a_i - b_i)} = 1.$$

To do so, we apply again the Diamond Lemma with the same partial ordering as before.

Next, note that $F^{\langle \gamma_C \rangle}/[F, Fr'F]^{\langle \gamma_C \rangle}$ has a basis consisting of (1) noncommutative monomials in x and y of bidegrees (a,b) such that $\lambda_C^{a-b} = 1$, and (2) cyclic words in x, y, and r', of bidegrees (a,b) such that $\lambda_C^{a-b} = 1$, and such that, if the cyclic word repeats itself m times (i.e., it is periodic of length $\frac{1}{m}$ times the length of the word), then it remains true that $\lambda_C^{(a-b)/m} = 1$. In particular, $F^{\langle \gamma_C \rangle}/[F, Fr'F]^{\langle \gamma_C \rangle}$ is a free $\mathbf{k}$-module.

Now, we can use the same partial ordering defined in the proof of Lemma 3.0.2 on $F^{\langle \gamma_C \rangle}/[F, Fr'F]^{\langle \gamma_C \rangle}$ to deduce the claim.

(2) Taking invariants under Γ_{γ_C} preserves the property of being torsion-free, and in fact yields a projective module (since it passes to a summand, namely the image of the symmetrizer idempotent $\frac{1}{|\Gamma_{\gamma_C}|} \sum_{g \in \Gamma_{\gamma_C}} g$). By (4.2.24), it remains only to compute the remaining relations $(\langle [z_{a,b}, x]_{\gamma_C}, [z_{a,b}, y]_{\gamma_C} \rangle_{a,b \geq 1})^{\langle \gamma_C \rangle}$, so that modding (4.2.25) by these, and taking Γ_{γ_C}-invariants, yields $(A \rtimes \Gamma)_{\mathrm{cyc}}$.

We claim that

$$\langle [z^C_{a,b}, x_C]_{\gamma_C}, [z^C_{a,b}, y_C]_{\gamma_C} \rangle_{a,b \geq 0} = \langle [z^C_{a,b}, x_C]_{\gamma_C}, [z^C_{a,b}, x_C y_C]_{\gamma_C}, [y^b_C, y_C] \rangle_{a,b \geq 0}. \tag{4.2.27}$$

This follows from the general formula $[f, gh]_{\gamma_C} = [f(\gamma_C \cdot g), h]_{\gamma_C} + [hf, g]_{\gamma_C}$.

Thus, we will consider commutators of the form

$$[z^C_{a-1,b}, x_C]_{\gamma_C} (a \geq 1), \quad [z^C_{a-1,b-1}, x_C y_C]_{\gamma_C} (a, b \geq 1), \quad [y_C^{b-1}, y_C]_{\gamma_C} (b \geq 1). \tag{4.2.28}$$

We would like to eliminate the second commutator above similarly to (3.0.10). There are two complications: the first is that we are now using γ_C-commutators; the second is that we are working in the $\mathbf{k}$-module of γ_C-twisted cyclic words $(A^{\langle \gamma \rangle}/([A, Ar'A]_{\gamma_C})^{\langle \gamma_C \rangle})$.

Let us assume $C \neq \{1\}$, since otherwise all our work is done in Lemma 3.0.2. For $a > b$, let us consider the equation

$$\sum_{i=0}^{a-b-1} [x_C^i (x_C y_C)^b x_C^{a-b-i-1}, x_C] + \sum_{i=0}^{b-1} [(x_C y_C)^i x_C^{a-b} (x_C y_C)^{b-i-1}, x_C y_C] = 0. \tag{4.2.29}$$

To turn this into an identity involving the γ_C-commutators in (4.2.28), we first note the following: if $f \in A$ is an eigenvector for the action of γ_C (e.g., if f is monomial in x_C, y_C) with eigenvalue λ_C^ℓ, and $g \in A$ is arbitrary, then

$$[fg, f] = f[g, f] \equiv \lambda_C^\ell [g, f] f = \lambda_C^\ell [gf, f] \pmod{([A, Ar'A]_{\gamma_C})^{\langle \gamma_C \rangle}}. \tag{4.2.30}$$

In particular, for $a > b$ (requiring $\lambda_C^{a-b} = 1$),

$$[x_C^i (x_C y_C)^b x_C^{a-b-i-1}, x_C] \equiv \lambda_C^i [z^C_{a-1,b}, x_C] \pmod{([A, Ar'A]_{\gamma_C})^{\langle \gamma_C \rangle}}, \tag{4.2.31}$$

$$[(x_C y_C)^i x_C^{a-b} (x_C y_C)^{b-i-1}, x_C y_C] \equiv [z^C_{a-1,b-1}, x_C y_C] \pmod{([A, Ar'A]_{\gamma_C})^{\langle \gamma_C \rangle}}. \tag{4.2.32}$$

Now, we combine (4.2.23) with (4.2.29),(4.2.31), (4.2.32) to obtain

$$\frac{1-\lambda_C^{a-b}}{1-\lambda_C}([z^C_{a-1,b}, x_C]_{\gamma_C} + (1-\lambda_C)z^C_{a,b}) + b[z^C_{a-1,b-1}, x_C y_C] \in ([A, Ar'A]_{\gamma_C})^{\langle\gamma_C\rangle}. \tag{4.2.33}$$

Since $\lambda_C^{a-b} = 1$ and $([A, Ar'A]_{\gamma_C})^{\langle\gamma_C\rangle}$ is saturated in $A^{\langle\gamma_C\rangle}$, this says that

$$[z^C_{a-1,b-1}, x_C y_C]_{\gamma_C} \in ([A, Ar'A]_{\gamma_C})^{\langle\gamma_C\rangle}. \tag{4.2.34}$$

It should be possible to verify this explicitly without using (4.2.29), but it seems more difficult.

Also,

$$[(x_C y_C)^{b-1}, x_C y_C]_{\gamma_C} = 0, \quad [y_C^{b-1}, y_C]_{\gamma_C} = (\lambda_C^{-1} - 1)y_C^b. \tag{4.2.35}$$

Thus, we conclude that, for $C \neq \{1\}$,

$$(\langle [z^C_{a-1,b}, x_C]_{\gamma_C}, [z^C_{a,b-1}, y_C]_{\gamma_C}\rangle_{a,b\geq 1})^{\langle\gamma_C\rangle} = \langle [z^C_{a-1,b}, x_C]_{\gamma_C}, (\lambda_C^{-1}-1)y_C^b\rangle_{a,b\geq 1;\lambda_C^{a-b}=1}. \tag{4.2.36}$$

Also, we easily see that

$$[z^C_{a-1,b}, x]_{\gamma_C} \equiv (\lambda_C - 1)z^C_{a,b} \pmod{(\!(r')\!)}. \tag{4.2.37}$$

Finally, note that $\lambda_C - 1$ is invertible in $\mathbf{k}$, as it is a factor of $|\gamma_C|$ (by plugging $t = 1$ into $\frac{1-t^{|\gamma_C|}}{1-t}$), which is a factor of $|\Gamma|$. Thus, for $C \neq \{1\}$, the image of the classes $(\langle [z_{a,b}, x]_{\gamma_C}, [z_{a,b}, y]_{\gamma_C}\rangle_{a,b\geq 1})^{\langle\gamma_C\rangle}$ in $A^{\langle\gamma_C\rangle}/([A, Ar'A]_{\gamma_C} + (\!(r')\!))^{\langle\gamma_C\rangle}$ has the same $\mathbf{k}$-linear span as the image of the elements $\{z^C_{a,b}\}_{a,b\geq 0}$.

We conclude that the summand of (4.2.24) corresponding to C is torsion-free. More precisely, this summand is the Γ_{γ_C}-invariant part of the free $\mathbf{k}$-module with basis given by the classes (b) from our list above (in part (1) of the proof), and hence $\mathbf{k}$-projective.

In the case $C = \{1\}$, on the other hand, we are back in the situation of Lemma 3.0.2, except that we have to take invariants at the end. So we get the summand $(V/W)^\Gamma \cong V^\Gamma/W^\Gamma$ where V, W are given in Lemma 3.0.2 (by construction, W is a $\mathbf{k}[\Gamma]$-module in this case). Furthermore, the projection of W modulo $(\!(r)\!)^2$ to $\bigoplus_{a,b}\langle [r' z_{a,b}]\rangle$ is an monomorphism of graded free $\mathbf{k}$-modules and a Γ-morphism, hence gives an embedding of $\mathbf{k}[\Gamma]$-modules (which is a $\mathbb{C}$-isomorphism). Thus, the quotient-mod-$(\!(r)\!)^2$ map gives a monomorphism

$$W^\Gamma \hookrightarrow (\langle r'\rangle \otimes \mathbf{k}[x, y]^\Gamma), \tag{4.2.38}$$

with image equal to (4.2.16). Explicitly, W^Γ can be obtained by the inverse of (4.2.16) by pulling back $\sum \alpha_{a,b} x^a y^b \in \mathbf{k}[x, y]^\Gamma$ with $\gcd(a, b) \mid \alpha_{a,b}$ to $\sum \frac{\alpha_{a,b}}{\gcd(a,b)} W_{a,b}$, which must be in W^Γ by the isomorphism.

Since V/W is torsion-free, so is $(V/W)^\Gamma$. If we considered instead $V/(W + \langle [(r')^\ell]\rangle_{\ell\geq 1})$, then V/W has a single copy of $\mathbf{k}/(\!(p)\!)$ in each degree $2p^\ell$ for p prime and $p \nmid |\Gamma|$, and $\ell \geq 1$. To conclude that $(V/W)^\Gamma$ has the same torsion, we need only check that each summand of $\mathbf{k}/(\!(p)\!)$ is Γ-invariant. This follows because it is generated by the image of $\frac{1}{p}[r^{p^\ell}] \in F/[F, F]$, which is Γ-invariant.

The rest of the theorem follows as in Lemma 3.0.2. For the statement about projectivity of $(A \rtimes \Gamma)_{\mathrm{cyc},\{1\}}$ in (iv), this follows because it is the Γ-invariant submodule of the free $\mathbf{k}$-module V/W constructed in Lemma 3.0.2, so it is a direct summand of a free module (as in step (2) above). □

5. Background on NCCI algebras

In order to state and prove many of our results, we will make essential use of the fact that the preprojective algebra of a non-Dynkin quiver is a noncommutative complete intersection (NCCI) (cf. e.g., [EG06, Definition 1.1.1]). This essentially follows from [EE05], although in the course of

our proof we will recover this fact without needing to refer to it (the arguments needed are similar, however).

In this section, we briefly recall the definitions of NCCI and formulate and prove properties of them that we will need. We begin with recollections from [EG06], then proceed to give somewhat different properties and characterizations (e.g., Proposition 5.2.1).

5.1. **Recollections.** Let A be a finitely-presented algebra $A = T_R V/J$. Assume for a moment R is a field. Then, in [EG06], A is called an NCCI if J/J^2 is a projective A-bimodule, or equivalently, A has a projective bimodule resolution of length 2. In the general case of R a commutative ring, we also require that A is a projective R-module.

We will additionally assume the following:

- $R = \mathbf{k}^I$ for some commutative ring $\mathbf{k}$ and some finite set I;
- V is a finitely-generated, $\mathbb{Z}_+$-graded (by "weight") R-bimodule, which is free over $\mathbf{k}$;
- $J = (\!(L)\!)$ where $L \subset T_R V$ is a finitely-generated, homogeneous (with respect to total degree), positive-weight R-subbimodule;
- L is a minimal generating R-subbimodule of $(\!(L)\!)$;
- For all $i, j \in I$ and all $m \geq 0$, $(iAj)_m$ is free over $\mathbf{k}$ (although this will be automatic in our usual context of $\mathbf{k} = \mathbb{Z}$ or a field).

Recall above that A_m denotes the m-th graded part of A. In particular, we can define the graded matrix Hilbert series of A, which is the following power series in t with coefficients in I-by-I matrices with nonnegative entries:

$$h(A;t) := \sum_{m \geq 0} [\mathrm{rk}\ (iAj)_m]_{i,j \in I} t^m \subset \mathbb{Z}_{\geq 0}^{I \times I}[[t]]; \tag{5.1.1}$$

similarly, we can define the same for J, or any other graded module M whose components $(iMj)_m$ are free over $\mathbf{k}$.

Recall the following characterizations of NCCI (the first of which is (6.1.2) discussed earlier).

Proposition 5.1.2. [Ani82], cf. [EG06, Theorem 3.2.4] The following are equivalent:

(1) A is an NCCI ($= (\!(L)\!)/(\!(L)\!)^2$ is a projective $T_R V$-bimodule);
(2) $h(A;t) = (1 - h(V;t) + h(L;t))^{-1}$;
(3) The Koszul complex of A-modules,

$$0 \to L \otimes_R A \to V \otimes_R A \to A \twoheadrightarrow R \to 0, \tag{5.1.3}$$

is exact, where the first nontrivial map is given by restriction of the map

$$T_R V \otimes_R A \to V \otimes_R A, \quad (v_1 \cdots v_n) \otimes a \mapsto v_1 \otimes (v_2 \cdots v_n a), \tag{5.1.4}$$

and the second nontrivial map is given by multiplication;
(4) Anick's A-bimodule complex [Ani86],

$$0 \to A \otimes_R L \otimes_R A \to A \otimes_R V \otimes_R A \to A \otimes_R A \twoheadrightarrow A \to 0, \tag{5.1.5}$$

is exact, where the first nontrivial map is given by restriction of the map

$$A \otimes_R T_R V \otimes_R A \to A \otimes_R V \otimes_R A, \quad a \otimes (v_1 \cdots v_n) \otimes b \mapsto \sum_{i=1}^{n} (a v_1 \cdots v_{i-1}) \otimes v_i \otimes (v_{i+1} \cdots v_n b), \tag{5.1.6}$$

and the second nontrivial map is given by

$$a \otimes v \otimes b \mapsto av \otimes b - a \otimes vb. \tag{5.1.7}$$

It is easy to prove the equivalence of parts (2), (3), and (4): this follows from the Euler-Poincaré principle (that the alternating sum of Hilbert series of an exact sequence is zero).

Remark 5.1.8. When $\mathbf{k}$ is a field and $L \subseteq V \otimes_R V$ is quadratic, it is easy to deduce from the above that, if A is an NCCI, it is also Koszul; cf. [EE05, Theorem 2.3.4].

In [EE05], it is proved that, in the special case where A is a preprojective algebra of a non-Dynkin quiver or A is a partial preprojective algebra with at least one white vertex (and $\mathbf{k}$ is any field), then A is Koszul; but actually the proof there shows A is an NCCI (the definition of NCCI came slightly later in [EG06]). Moreover, it is well-known that, when A is the preprojective algebra of a Dynkin quiver, then A is not NCCI; in fact, the minimal bimodule resolution of A is periodic (in particular infinite), as is the Koszul complex. Thus one concludes from [EE05] the following, over $\mathbb{Z}$ and hence over an arbitrary commutative ring $\mathbf{k}$:

Proposition 5.1.9. A (partial) preprojective algebra Π_{Q,Q'_0} is an NCCI if and only if either Q is not Dynkin or Q'_0 is nonempty.

As we mentioned at the beginning of the section, we will not need to use this proposition.

5.2. **General results on NCCIs.** We will use the following alternative characterization of NCCIs:

Proposition 5.2.1. The following are equivalent for A as above:

(1) A is an NCCI.
(2) Equip $T_R V$ with the descending filtration by powers of $(\!(L)\!)$. Then the canonical map $A *_R T_R L \to \operatorname{gr} T_R V$ is an isomorphism.
(3) For any section $A \hookrightarrow T_R V$ of the quotient map, the induced map $A *_R T_R L \to T_R V$ is an isomorphism.

Proof. It is clear that parts (2) and (3) are equivalent since A is a free $\mathbf{k}$-module. We show that (2) is equivalent to exactness of (5.1.3) ((3) of Proposition 5.1.2). It is easy to see that exactness is equivalent to injectivity of $L \otimes_R A \to V \otimes_R A$ (by definition of A). The latter is equivalent to the following formula in $T_R V$:

$$(L \cdot T_R V) \cap (V \cdot (\!(L)\!)) = L \cdot (\!(L)\!). \tag{5.2.2}$$

This last equation obviously follows from (2) or (3). Conversely, we may proceed by induction on L-degree of the filtered vector space $T_R V$. Let $\operatorname{gr}^L_i T_R V := (\!(L)\!)^i / (\!(L)\!)^{i-1}$. As the base case, it is clear that $A \cong \operatorname{gr}^L_0 T_R V$. Inductively, assume that $\operatorname{gr}^L_i T_R V = (A *_R T_R L)^L_i$. Then, (5.2.2) shows that

$$\operatorname{gr}^L_{i+1}(L T_R V) \cong L \cdot (\!(L)\!)^i / L \cdot (\!(L)\!)^{i+1}.$$

Since the multiplication $L \otimes_R T_R V \to T_R V$ is injective, the RHS identifies with $L \otimes_R \operatorname{gr}^L_i T_R V \cong L \otimes_R (A *_R T_R L)^L_i$. Similarly, for all $j \geq 1$,

$$\operatorname{gr}^L_{i+1}(V^j L (\!(L)\!)^i) = A_j \otimes_R L \otimes_R (A *_R T_R L)^L_i. \qquad \square$$

Note that, using the formula (Lemma 2.2.5 of [EE05])

$$h(A) = \frac{1}{1-\alpha}, h(B) = \frac{1}{1-\beta} \Rightarrow h(A *_R B) = \frac{1}{1-\alpha-\beta}, \quad \text{if } \alpha, \beta \in t\mathbb{Z}[[t]] \otimes \operatorname{End}(R), \tag{5.2.3}$$

one obtains an alternative proof of the fact that (2) of Proposition 5.2.1 is equivalent to (2) of Proposition 5.1.2. (Note that (5.2.3) is $T_R V *_R T_R W \cong T_R(V \oplus W)$ in the case that α, β are positive, or for arbitrary α, β if one allows V and W to be graded super-vector spaces.) We preferred a proof using tensors rather than Hilbert series.

We do not actually need to say much about NCCIs in general, but we mention these to explain the meaning of conditions (2), (3) of Proposition 5.2.1, which we will use heavily.

Proposition 5.2.1 easily implies the following analogue of Lemma 5.1.1 of [EG06] (which was for RCI algebras):

Proposition 5.2.4. Let A and B be finitely-presented as in Proposition 5.2.1.

(i) $A *_R B$ is an NCCI if and only if A and B are NCCIs. More generally, consider a commutative diagram

$$\begin{array}{ccccc} B & \hookrightarrow & C & \twoheadrightarrow & A \\ \| & & {\scriptstyle gr}\uparrow{\scriptstyle \sim,R} & & \| \\ B & \hookrightarrow & A *_R B & \twoheadrightarrow & A \end{array} \tag{5.2.5}$$

whose rows are graded R-algebra morphisms, and whose middle vertical arrow is an isomorphism of graded R-bimodules (but not necessarily an algebra homomorphism). Suppose that C is generated as a B-algebra by a positively-graded R-bimodule section $\tilde{A} \subset C$ of A intersecting B only in $\mathbf{k}$, and that the middle vertical arrow is obtained by taking the associated graded with respect to the descending filtration generated by the grading on B and by $|\tilde{A} \setminus \{0\}| = 0$. Then, C is an NCCI if and only if A and B are.

(ii) Assume $L \supset L'$ are finitely-generated, positively-graded homogeneous bimodules, which minimally generate the ideals $(\!(L)\!), (\!(L')\!)$. If $T_R V/(\!(L)\!)$ is an NCCI, then $T_R V/(\!(L')\!)$ is an NCCI.

Proof. (i) Set $A = T_R V/(\!(L)\!)$ and $B = T_R W/(\!(L')\!)$ for minimal, finitely-generated L, L', so that $A *_R B = T_R(V \oplus W)/(\!(L, L')\!)$, and $C = T_R(V \oplus W)/(\!(L', \tilde{L})\!)$ for some $\tilde{L}$ mapping isomorphically to L under the projection to $T_R V$. The result then follows from (2) of Proposition 5.2.1.

(ii) Assume $T_R V/(\!(L)\!)$ is an NCCI. Then,

$$T_R V \cong T_R V/(\!(L)\!) *_R T_R L \tag{5.2.6}$$

as graded R-bimodules (considering some section $T_R V/(\!(L)\!) \hookrightarrow T_R V$). Modding by $(\!(L')\!)$, we obtain

$$T_R V/(\!(L')\!) \cong T_R V/(\!(L)\!) *_R (T_R L/(\!(L')\!)), \tag{5.2.7}$$

By part (i), we may conclude the desired result if $T_R L/(\!(L')\!)$ is an NCCI. But, this is obvious since $L' \subset L$ is a subbimodule. □

Remark 5.2.8. More generally, part (ii) of Proposition 5.2.4 shows that, for any NCCI $T_R V/(\!(L)\!)$ (with L minimal and finitely-generated), if $L' \subset T_R L$, then letting $\bar{L}'$ be the image of L' in $T_R V$, one has that $T_R V/(\!(\bar{L}')\!)$ is an NCCI if and only if $T_R L/(\!(L')\!)$ is an NCCI.

Now, we note that all of the above results also hold for $\mathbf{k} = \mathbb{Z}$, dealing with free $\mathbb{Z}$-modules (since the definition, in Proposition 5.1.2.(i), of an NCCI includes being a free $\mathbb{Z}$-module).

We then deduce the following important result (needed for writing bases), now with $R = \mathbb{Z}^{Q_0}$ for a quiver Q:

Corollary 5.2.9. (i) If $\Pi_{Q,J}$ is an NCCI and $J \subset J' \subseteq Q_0$, then $\Pi_{Q,J'}$ is an NCCI, and

$$\mathrm{gr}^{((\mathbf{1}_{J'\setminus J} r \mathbf{1}_{J'\setminus J}))} \Pi_{Q,J'} \cong \Pi_{Q,J} *_R T_R\langle \mathbf{1}_{J'\setminus J} r \mathbf{1}_{J'\setminus J}\rangle, \tag{5.2.10}$$

by taking the associated graded algebra with respect to powers of the ideal $(\!(\mathbf{1}_{J'\setminus J} r \mathbf{1}_{J'\setminus J})\!)$. One also has a graded R-bimodule isomorphism $\xleftarrow{\sim}$ by choosing any graded R-bimodule section $\Pi_{Q,J} \hookrightarrow \Pi_{Q,J'}$;

(ii) If Q' and Q'' are subquivers of Q with vertex set Q_0 such that $Q_1 = Q'_1 \sqcup Q''_1$ (all arrows of Q are in either Q' or Q'' but not both), and $I \subset Q_0$ is the set of vertices incident to arrows from both Q'_1 and Q''_1, then if $\Pi_{Q',J}$ is an NCCI, one has

$$\mathrm{gr}^{((\overline{Q''_1}))} \Pi_{Q,J} \cong \Pi_{Q',J} *_R \Pi_{Q'',J\cup I}, \tag{5.2.11}$$

by taking the associated graded algebra with respect to the filtration by powers of the ideal $(\!(\bar{Q}_1'')\!)$. Similarly, one can obtain a graded R-bimodule isomorphism

$$\Pi_{Q,J} \xleftarrow{\sim} \Pi_{Q',J} *_R \Pi_{Q'',J\cup I} \tag{5.2.12}$$

by picking any weight-graded R-bimodule section $\Pi_{Q',J} \subset \Pi_{Q,J}$ of the quotient, and using the composition $\Pi_{Q'',J\cup I} \hookrightarrow \Pi_{Q,J\cup I} \twoheadrightarrow \Pi_{Q,J}$ of obvious maps.

(iii) In the situation of (ii), $\Pi_{Q,J}$ is an NCCI if and only if $\Pi_{Q'',J\cup I}$ is an NCCI.

Proof. (i) Let $V = \langle\, \overline{Q}_1 \,\rangle$. Let $\tilde{\Pi}_{Q,J} \subseteq T_R V$ be a graded R-bimodule section of $T_R V \twoheadrightarrow \Pi_{Q,J}$. Then,

$$T_R V = \tilde{\Pi}_{Q,J} *_R T_R\langle \mathbf{1}_{I\setminus J} r \mathbf{1}_{I\setminus J}\rangle = (\tilde{\Pi}_{Q,J} *_R T_R\langle \mathbf{1}_{J'\setminus J} r \mathbf{1}_{J'\setminus J}\rangle) *_R \langle \mathbf{1}_{I\setminus J'} r \mathbf{1}_{I\setminus J'}\rangle. \tag{5.2.13}$$

If we mod by $T_R \mathbf{1}_{I\setminus J'} r \mathbf{1}_{I\setminus J'}$, we obtain (5.2.10), which together with (5.2.13) proves the desired result.

(ii) It is easy to see that

$$\Pi_{Q,J\cup I} \cong \Pi_{Q',J\cup I} *_R \Pi_{Q'',J\cup I}. \tag{5.2.14}$$

Now, let $V_1 = \langle\, \overline{Q_1'} \,\rangle$ and $V_2 = \langle\, \overline{Q_1''} \,\rangle$. By our hypotheses and (5.2.10), we may rewrite (5.2.14) as

$$\mathrm{gr}^{(\!(\mathbf{1}_{I\setminus J} r \mathbf{1}_{I\setminus J})\!)}\, \Pi_{Q,J\cup I} \cong \Pi_{Q',J} *_R T_R\langle \mathbf{1}_{I\setminus J} r_{Q'} \mathbf{1}_{I\setminus J}\rangle *_R \Pi_{Q'',J\cup I}. \tag{5.2.15}$$

Now, modding by $(\!(\mathbf{1}_{I\setminus J} r_Q \mathbf{1}_{I\setminus J})\!)$, we replace all instances of $\mathbf{1}_{I\setminus J} r_{Q'} \mathbf{1}_{I\setminus J}$ with $-\mathbf{1}_{I\setminus J} r_{Q''} \mathbf{1}_{I\setminus J}$, and obtain (5.2.11) (upon further taking the associated graded algebra with respect to $(\!(\overline{Q_1''})\!)$).

(iii) This follows from (5.2.11) and Proposition 5.2.4.(i). □

The corollary implies the inductive result used in [EE05] to show that non-Dynkin quivers or partial preprojective algebras with at least one white vertex are Koszul: that is, it allows one to reduce to the case of extended Dynkin quivers and star-shaped quivers whose special vertex is white and whose branches are of unit length.

6. Hilbert series and a question about RCI algebras

In this short section, we explain the consequences of our results for Hilbert series in positive characteristic, and pose a question which would generalize our main theorem. This section is *not* needed for the proof of our main results.

Throughout this section, we will use the abusive notation $\Pi_Q := \Pi_Q \otimes \mathbf{k}$, where $\mathbf{k}$ will be always a field, since we are only interested in Hilbert series.

A main result of [EG06] is a formula for the Hilbert series $h(A;t)$ of certain $\mathbb{Z}_{\geq 0}$-graded algebras $A := \bigoplus_{m\geq 0} A_m$ over semisimple rings $R = \mathbb{C}^I$ which are noncommutative analogues of complete intersections, and also for $A/[A,A]$. For the latter, it turns out to be more natural to describe the graded vector space $\mathcal{O}(A) := \mathrm{Sym}(A/[A,A])_+$, where the $+$ means to pass to the positively-graded $\mathbb{C}$-linear subspace of $A/[A,A]$, which we need to do in order for the symmetric algebra to have finite dimension in each graded component. (The reason why $\mathcal{O}(A)$ is more natural to describe is because it is closely related to the $\mathbb{C}$-linear subspace of functions on the representation variety which are invariant under change of basis.)

Computing $h(\mathcal{O}(A);t)$ is tantamount to computing $h(A/[A,A];t)$: explicitly, if $h(A/[A,A];t) = \sum_m a_m t^m$, then

$$h(\mathrm{Sym}(A/[A,A])_+;t) = \prod_{m\geq 1} \frac{1}{(1-t^m)^{a_m}}. \tag{6.0.1}$$

6.1. **The non-Dynkin, non-extended Dynkin and partial preprojective cases.** First, let us work over the field $\mathbf{k} = \mathbb{C}$. Let (Q, J) be any pair where Q is a quiver, $J \subset Q_0$ is a subset of vertices, and either $J \neq \emptyset$ or Q is neither Dynkin nor extended Dynkin. Then, by [EG06]:

(6.1.1)

$$h(\Pi_{Q,J}; t) = (1 - t \cdot C + t^2 \cdot \mathbf{1}_{Q_0 \setminus J})^{-1}, \quad h(\mathcal{O}(\Pi_{Q,J}); t) = \big(\frac{1}{1-t^2}\big)^{\delta_{J,\emptyset}} \prod_{m \geq 1} \frac{1}{\det(1 - t^m \cdot C + t^{2m} \cdot \mathbf{1}_{Q_0 \setminus J})},$$

where C is the adjacency matrix of $\overline{Q}$, and $\delta_{J,\emptyset} = 1$ if $J = \emptyset$ and 0 otherwise.

The above formulas have the following representation-theoretic interpretation (which was used in their proof in [EG06]): Let $V = \langle \overline{Q}_1 \rangle$ and let $L = \langle \mathbf{1}_{Q_0 \setminus J} r \mathbf{1}_{Q_0 \setminus J} \rangle$: thus, $\Pi_Q = T_{\mathbf{k}^{Q_0}} V / (\!(L)\!)$. Let $L^\circ = L \cap [T_{\mathbf{k}^{Q_0}} V, T_{\mathbf{k}^{Q_0}} V]$. Then, (6.1.1) expresses as the following formula [EG06]:

(6.1.2)

$$h(\Pi_{Q,J}; t) = (1 - h(V; t) + h(L; t))^{-1}, \quad h(\mathcal{O}(\Pi_{Q,J})) = \frac{1}{1 - h(L^\circ; t)} \prod_{m \geq 1} \frac{1}{\det(1 - h(V; t^m) + h(L; t^m))}.$$

Since L is a minimal generating bimodule of $(\!(L)\!)$, the first formula above (for $h(\Pi; t)$) says that Π is a noncommutative complete intersection (NCCI) (cf. Proposition 5.1.2, taken from [EG06, Theorem 3.2.4], which was itself taken from, e.g., [Ani82]), and both formulas above together say that Π is an asymptotic representation-complete intersection (asymptotic RCI), by [EG06, Theorem 3.7.7 and Proposition 3.7.1]. Note also that, in fact, Π is Koszul (by [EE05, Theorem 2.3.4], any NCCI with quadratic relations (i.e., $L \subset T^2_{\mathbf{k}^{Q_0}} V$) is Koszul).

Furthermore, the expression $\prod_{m \geq 1} \frac{1}{1 - h(V; t^m) + h(L; t^m)}$ can be viewed as an analogue of the zeta function, so (following [EG06]) we define

$$\zeta(V, L; t) := \prod_{m \geq 1} \frac{1}{\det(1 - h(V; t^m) + h(L; t^m))}. \tag{6.1.3}$$

Finally, the expression $1 - t \cdot C + t^2 = t \cdot ((\frac{1}{t} + t) - C)$ can be viewed as the "t-analogue" of the Cartan matrix. Also, $\frac{1}{1 - tx + x^2}$ is the generating function for (suitably normalized) Chebyshev polynomials ϕ of the second type, so that $h(\Pi; t) = \phi(C; t) := 1 + \sum_{m \geq 1} \phi_m(C) t^m$.

Our main result, Theorem 1.1.4, together with the fact that Π_Q is torsion-free (which is a consequence of [EE05], or alternatively of the bases we construct), generalizes the above formulas to arbitrary characteristic:

Corollary 6.1.4. Take any pair (Q, J) where Q is a quiver and $J \subset Q_0$ is a subset of vertices, such that either $J \neq \emptyset$ or Q is neither Dynkin nor extended Dynkin. Over any field $\mathbf{k}$ of characteristic $p > 0$, the Hilbert series of $\Pi_{Q,J}$ and $\mathcal{O}(\Pi_{Q,J})$ are given as follows:

(6.1.5)

$$h(\Pi_{Q,J}) = (1 - t \cdot C + t^2 \cdot \mathbf{1}_{Q_0 \setminus J})^{-1}, \quad h(\mathcal{O}(\Pi_{Q,J}); t) = \big(\prod_{\ell \geq 0} \frac{1}{1 - t^{2p^\ell}}\big)^{\delta_{J,\emptyset}} \prod_{m \geq 1} \frac{1}{\det(1 - t^m \cdot C + t^{2m} \cdot \mathbf{1})},$$

Using the notation ζ, V, L, L° above, the formulas become

$$h(\Pi_{Q,J}) = \frac{1}{1 - h(V; t) + h(L; t)}, \quad h(\mathcal{O}(\Pi_{Q,J}); t) = \zeta(V, L; t) \prod_{\ell \geq 0} \frac{1}{1 - h(L^\circ; t^{p^\ell})}. \tag{6.1.6}$$

6.2. **A question on asymptotic RCI algebras in positive characteristic.** P. Etingof and the author have also done computer tests of some finitely presented algebras over $\mathbb{Z}$ which are asymptotic RCI over $\mathbb{Q}$, and in most sufficiently random cases, (6.1.6) has held. This motivates the following generalization of Theorem 1.1.4. Fix a prime $p \in \mathbb{Z}$, and let $\mathbf{k} := \mathbb{Z}_{(\!(p)\!)}$ be the localization of $\mathbb{Z}$ at the ideal $(\!(p)\!)$. Let I be a set, and $R := \mathbf{k}^I$.

Question 6.2.1. (P. Etingof and the author) Let $A = T_R V/(\!(L)\!)$ be a finitely-presented algebra over R, with L a minimal generating bimodule. Further suppose that L and $L \cap [T_R V, T_R V]$ are saturated, and A is an asymptotic RCI [EG06] (roughly, this says that the Koszul complex of the representation varieties over $\mathbf{k}$ are asymptotically exact). Is it then true that the p-torsion of A_{cyc} is isomorphic to a graded $\mathbb{F}_p$-vector space with basis the classes $r_j^{(p^\ell)}$, for $\ell \geq 1$? Here, (r_j) is a lift to L of an $\mathbb{F}_p$-basis of $(L \cap [T_R V, T_R V]) \otimes \mathbb{F}_p$, and $r_j^{(p^\ell)}$ is the image of $\frac{1}{p}[r_j^{p^\ell}]$.

As before, the classes $r_j^{(p^\ell)}$ must be p-torsion if they are nonzero. Beginning with an algebra A over $\mathbb{Z}$, the question asks whether these classes are always nonzero in primes such that $A \otimes_{\mathbb{Z}} \mathbb{Z}_{(\!(p)\!)}$ is an asymptotic RCI, and whether they generate all the p-torsion.

6.3. **The extended Dynkin case.** The formula for $h(\Pi_{Q,J})$ above still holds when Q is extended Dynkin and $J = \emptyset$, but the second must be modified (since Π_Q is still an NCCI but no longer an asymptotic RCI).

Suppose that $i_0 \in Q_0$ is an extending vertex of Q, i.e., removing i_0 and its incident arrows leaves one with the corresponding Dynkin quiver. Over $\mathbf{k} = \mathbb{C}$, one has the following isomorphisms of graded vector spaces:

$$(HH^0(\Pi_Q) \otimes \mathbf{k}) \xrightarrow{\sim} (i_0 \Pi_Q i_0 \otimes \mathbf{k}) \xrightarrow{\sim} (HH_0(\Pi_Q)_+ \otimes \mathbf{k}) \oplus \mathbf{k} = ((\Lambda_Q)_+ \otimes \mathbf{k}) \oplus \mathbf{k}, \tag{6.3.1}$$

where the final $\mathbf{k}$ is placed in degree zero. The first map is given by the projection $x \mapsto i_0 x i_0$, and the second is given by restriction of the obvious projection $(\Pi_Q)_+ \twoheadrightarrow HH_0(\Pi_Q)_+$, together with $(i_0(\Pi_Q)_0 i_0 \otimes \mathbf{k}) \cong \mathbf{k}$. The isomorphisms of (6.3.1) follow from the Morita equivalences recalled in §4 together with the computation of Hochschild (co)homology there, since $\Pi_Q \cong \mathsf{f}(\mathbf{k}[x,y] \rtimes \Gamma)\mathsf{f}$ induces $i_0 \Pi_Q i_0 \cong \mathbf{k}[x,y]^\Gamma = HH^0(\mathbf{k}[x,y] \rtimes \Gamma)$, and Hochschild (co)homology is invariant under Morita equivalence.

Note that the maps in (6.3.1) make sense also over $\mathbb{Q}$, and they must also be isomorphisms. Using the second isomorphism of (6.3.1), we obtain the following formula:

$$h(\mathcal{O}(\Pi_Q) \otimes \mathbb{Q}; t) = h(\mathrm{Sym}(i_0 \Pi_Q i_0 \otimes \mathbb{Q})_+; t) = \prod_{m \geq 1} \frac{1}{(1-t^m)^{a_m}}, \tag{6.3.2}$$

where

$$1 + \sum_{m \geq 1} a_m t^m = h(i_0 \Pi_Q i_0; t) = \Big(\frac{1}{1 - t \cdot C + t^2 \cdot \mathbf{1}}\Big)_{i_0 i_0} = \phi(C; t)_{i_0 i_0}. \tag{6.3.3}$$

Here, $_{i_0 i_0}$ denotes the entry of the matrix in the i_0, i_0 component. There is a general formula for NCCI algebras over a characteristic zero field [EG06]:

$$h(\mathcal{O}(A); t) = h(\mathrm{Sym}\, HH_2(A); t) \cdot \zeta(V, L; t). \tag{6.3.4}$$

Setting this equal to (6.3.2) in our case yields the following curious identity from [EG06], of which we will provide a more direct proof:

$$\prod_{m \geq 1} \frac{1}{(1-t^m)^{\varphi_m(C)_{i_0 i_0}}} = \frac{1}{1-t^2} \cdot \prod_{m \geq 1} \frac{1}{\det(\mathbf{1} - t^m \cdot C + t^{2m} \cdot \mathbf{1})}, \tag{6.3.5}$$

where $\varphi_m := \phi_m - \phi_{m-2}$ is the m-th Chebyshev polynomial of the first type. Specifically, we use $i_0 \Pi_Q i_0 \otimes \mathbb{Q} \cong \Lambda_Q \otimes \mathbb{Q}$, compute the Hilbert series of the former using bases, and compare it with the determinant of the t-analogue of the Cartan matrix, to obtain the RHS of (6.3.5). To compare it with the LHS, we use that Π_Q is an NCCI over $\mathbb{C}$, which could be verified explicitly using our bases, but it is easier to use the Morita equivalence.

From Theorem 13.1.1 below, together with the fact that Π_Q is torsion-free [EE05], we deduce the following generalization to arbitrary characteristic:

Proposition 6.3.6. Let Q be any extended Dynkin quiver, and $\mathbf{k}$ any field of characteristic $p > 0$. Then one has the following formulas:

$$h(\Pi_Q) = (1 - t \cdot C + t^2 \cdot \mathbf{1})^{-1} = \phi(C;t), \quad h(\Lambda_Q;t) = hT(Q) + \phi(C;t)_{i_0 i_0}, \tag{6.3.7}$$

$$hT(Q) = \begin{cases} \sum_{m=1}^{\lfloor \frac{n-2}{2} \rfloor} t^{4m}, & \text{if } p = 2 \text{ and } Q = \tilde{D}_n, \\ t^4, & \text{if } p = 2 \text{ and } Q = \tilde{E}_6, \\ t^4 + t^8 + t^{16}, & \text{if } p = 2 \text{ and } Q = \tilde{E}_7, \\ t^4 + t^8 + t^{16} + t^{28}, & \text{if } p = 2 \text{ and } Q = \tilde{E}_8, \\ t^6, & \text{if } p = 3 \text{ and } Q \in \{\tilde{E}_6, \tilde{E}_7\}, \\ t^6 + t^{18}, & \text{if } p = 3 \text{ and } Q = \tilde{E}_8, \\ t^{10}, & \text{if } p = 5 \text{ and } Q = \tilde{E}_8, \\ 0, & \text{otherwise.} \end{cases} \tag{6.3.8}$$

Here and elsewhere, we abuse notation and say that Q = some Dynkin or extended Dynkin quiver if, when orientations are discarded, one obtains the corresponding quiver (in other words, Q is given by choosing an orientation on each arrow of the Dynkin or extended Dynkin quiver).

Note that the torsion of Λ_Q in the extended Dynkin case only appears in what we call "stably bad primes": 2 for D_n, 2 and 3 for E_6 and E_7, and $2, 3$, and 5 for E_8 (this was observed also in [MOV06] where they were called merely "bad primes"). We will see that the same is true in the Dynkin case. These primes are a subset of what we call "bad primes": primes dividing the order of the group $\Gamma \subset SL_2(\mathbb{C})$ associated to the quiver under the McKay correspondence. We use the term "stably bad" because, for types A_n and D_n, these primes are bad independently of n.

6.4. **The Dynkin case.** In the case that Q is Dynkin and $J = \emptyset$, the formula for $h(\Pi_Q;t)$ is no longer valid: instead, $h(\Pi_Q;t)$ records the dimensions of irreducible representations of Q (because $\Pi_Q \otimes \mathbb{C}$ is a direct sum of one copy of each). One may easily show that Π_Q is torsion-free using Gröbner generating sets (cf. Appendix A.1 and Proposition A.1.1 therein).

Furthermore, as is proved in [MOV06], $\Lambda_Q \otimes \mathbb{Q} = 0$, and in fact $\Lambda_Q \otimes \mathbb{F}_p = 0$ if p is not a stably bad prime. As explained in Theorem 13.1.1, Λ_Q is isomorphic to the torsion of $\Lambda_{\widetilde{Q}}$ (for $\widetilde{Q}$ the extended Dynkin quiver which extends Q), so so we may compute the Hilbert series over any field:

Proposition 6.4.1. Let Q be a Dynkin quiver. Then $h(\Lambda_Q;t) = hT(\tilde{Q})$, given in (6.3.8) (for $\tilde{Q}$ = the extended Dynkin quiver associated to Q).

7. Refinement and partial proof of the main theorem ($r^{(p^\ell)} \neq 0$)

Recall that a *forest* is a (directed) graph without (undirected) cycles.

Proposition 7.0.1. Take any quiver Q together with a nonempty subset of white vertices, $J \subset Q_0$. Let $G \subset \overline{Q}_1$ be a forest such that the map $G \to Q_0$, $a \mapsto a_s$ (the source vertex) yields a bijection $G \xrightarrow{\sim} Q_0 \setminus J$. Then, a free $\mathbb{Z}$-basis of $\Pi_{Q,J}$ is given by monomials in the arrows Q that do not contain a subword aa^* for any $a \in G$. In particular, $\Pi_{Q,J}$ is an NCCI.

Furthermore, $\Lambda_{Q,J}$ is a free $\mathbb{Z}$-module with basis given by cyclic words not containing aa^* for any $a \in G$.

Proof. A forest satisfying the given condition can be constructed inductively as follows: To begin, for every vertex in $Q_0 \setminus J$ which is adjacent to J, add an arrow to G with source at that vertex

and target in J. Inductively, for each vertex of $Q_0 \setminus J$ which is not incident to G, but is adjacent to a vertex which is incident to G, add an arrow with source at that vertex and target at a vertex incident to G. When the process is completed, one obtains a forest satisfying the desired condition.

The first result follows immediately from the Diamond Lemma (Propositions A.2.3–A.2.5) if we let the partial order on monomials be given by the number of subwords aa^* with $a \in G$ that appear. To see that $\Pi_{Q,J}$ is an NCCI, we show (2) of Proposition 5.2.1. To do this, we adjoin generators $r_i (= iri)$ for all $i \in Q_0 \setminus J$, and apply the Diamond Lemma to reduce any path to a unique sum of monomials in $\overline{Q}_1$ and the r_i not containing aa^* for any $a \in G$.

For the final result, we note that one may still use the Diamond Lemma for $\Lambda_{Q,J}$ because the maximum number of swaps $aa^* \mapsto a^*a$ for $a \in G$ that may be performed in a cyclic word is still finite, since $a \in G$ cannot be a loop. This follows reverse-inductively on the distance of an arrow a from J, using that this distance is bounded. □

The fact that $\Pi_{Q,J}$ is an NCCI was first shown in [EE05] using Hilbert series, and in [EG06], $\Pi_{Q,J}$ was further shown to be a RCI. The fact that $\Lambda_{Q,J}$ is torsion-free over $\mathbb{Z}$ appears to be new.

As an application of the proposition, by comparing the above basis and the formula (6.1.2) for $h(\mathcal{O}(\Pi_{Q,J}); t)$ in the asymptotic RCI case (a generalization of RCI), one obtains a formula for computing the number of cyclic words in letters x_i, y_i, z_j not containing $x_i y_i$ for any i.

To handle the case where there are no white vertices, we need the

Proposition 7.0.2. [EE05] The algebra Π_Q is an NCCI over $\mathbb{Z}$ if Q is non-Dynkin (in particular, it is torsion-free).

Proof. By Corollary 5.2.9 (and the comments afterward), and Proposition 7.0.1, one may reduce to the case that Q is extended Dynkin. For the extended Dynkin case, the easiest proof is to use our bases to show that Π_Q is torsion free; then, after tensoring with $\mathbb{C}$, one may use the Morita equivalence of Π_Q with $\mathbb{C}[x, y] \rtimes \Gamma$ from §4. Alternatively, one could deduce the NCCI property from our computation of bases with some effort. □

For any non-Dynkin, non-extended Dynkin quiver Q, it is well-known (and easy to check) that $Q \supsetneq Q^0$ for some extended Dynkin quiver Q^0 with vertex set Q_0^0. The results of the previous section then allow us to write, as graded R-bimodules,

$$\Pi_Q = \tilde{\Pi}_{Q^0} *_R \Pi_{Q \setminus Q^0, Q_0^0}, \tag{7.0.3}$$

where $\tilde{\Pi}_{Q^0}$ is an arbitrary graded R-bimodule section of $\Pi_Q \twoheadrightarrow \Pi_{Q^0}$, and $\Pi_{Q \setminus Q^0, Q_0^0}$ embeds canonically into Π_Q via the sequence $\Pi_{Q \setminus Q^0, Q_0^0} \hookrightarrow \Pi_{Q, Q_0^0} \twoheadrightarrow \Pi_Q$.

Notation 7.0.4. In general, if $Q \supsetneq Q^0$ where Q^0 is extended Dynkin, then we will fix a graded R-bimodule section $\tilde{\Pi}_{Q^0} \subset \Pi_Q$ of $\Pi_Q \twoheadrightarrow \Pi_{Q^0}$, which exists because Π_{Q^0} is torsion-free. Then, for any subset $U \subset \Pi_{Q^0}$, we denote its image under the section by $\tilde{U}$.

Now, we proceed to one of our main goals: a description of Λ_Q when Q is non-Dynkin and non-extended Dynkin. We begin with

Proposition 7.0.5. Let $Q \supsetneq Q^0$ where Q^0 is non-Dynkin. Let $V := \Pi_Q / [(\langle \overline{Q}_1 \setminus \overline{Q_1^0} \rangle), \Pi_Q]$. Let B be the algebra $B := T_R((\Pi_{Q^0})_+ \otimes_R (\Pi_{Q \setminus Q^0, Q_0^0})_+)$. We have

(i)

$$V \xleftarrow{\sim} \Pi_{Q^0} \oplus (B/[B, B])_+ \oplus \Lambda_{Q \setminus Q^0, Q_0^0}, \tag{7.0.6}$$

as $\mathbb{Z}$-modules, where the map is given using (7.0.3).

(ii) $\Lambda_Q \cong V/W$ where W is the image in V of $[\tilde{\Pi}_{Q^0}, \langle \overline{Q_1^0} \rangle]$, using Notation 7.0.4.

Note that $(B/[B,B])_+$ in (7.0.6) has a basis consisting of alternating cyclic words in $(\Pi_{Q^0})_+$ and $(\Pi_{Q\setminus Q^0,Q_0^0})_+$. Here and below, an "alternating cyclic word" in sets $\mathcal{A}$ and $\mathcal{B}$ means a word of the form $a_1b_1\cdots a_mb_m$ (for $m\geq 1$) modulo simultaneous cyclic permutations of the indices, where $a_i\in\mathcal{A}$ and $b_i\in\mathcal{B}$.

Proof of Proposition 7.0.5. The first part follows from (7.0.3) and its proof, together with the Diamond Lemma argument from Lemma 3.0.2.(2). The second part follows from the observation (cf. Lemma 3.0.2.(3)) that $[\Pi_Q,\Pi_Q]=[\Pi_Q\langle\overline{Q}_1\setminus\overline{Q_1^0}\rangle\Pi_Q,\Pi_Q]+[\tilde{\Pi}_{Q^0},\langle\overline{Q_1^0}\rangle]$. □

From now on we will consider the case where Q^0 is extended Dynkin. Let $i_0\in Q_0^0$ be a fixed choice of extending vertex. Fix any quiver $Q\supsetneq Q^0$. Let $\Gamma\subseteq SL_2(\mathbb{C})$ be the group corresponding to Q^0.

First, we describe W (defined in Proposition 7.0.5.(ii) above) away from bad primes, i.e., we describe $W\otimes\mathbb{Z}[\frac{1}{|\Gamma|}]$. For simplicity, when we say "we work over S", for a commutative ring S, we mean that all $\mathbb{Z}$-modules should be tensored by S, e.g., W denotes $W\otimes_{\mathbb{Z}}S$, and we will omit the tensor product for ease of notation.

Proposition 7.0.7. We work over $\mathbb{Z}[\frac{1}{|\Gamma|}]$, with W the $\mathbb{Z}[\frac{1}{|\Gamma|}]$-module obtained by tensoring the one in Proposition 7.0.5.(ii) by $\mathbb{Z}[\frac{1}{|\Gamma|}]$. Let $W_0\subset W$ be the image of $[\Pi_{Q^0},\Pi_{Q^0}]$ under (7.0.6). Let $V':=V/W_0$ and $W':=W/W_0$. Then, these are graded modules with finitely-generated free homogeneous components, and

(i) $h(W';t)=t^2\cdot h(HH^0(\Pi_{Q^0})_+;t)=t^2\cdot h((i_0\Pi_{Q^0}i_0)_+;t)$.

(ii) The composition $W'\hookrightarrow V'\twoheadrightarrow V'/[(\!(r')\!)^2]$ is injective, giving an isomorphism $W'\xrightarrow{\sim}\widetilde{[r'HH^0(\Pi_{Q^0})_+]}$, using Notation 7.0.4.

Proof. Over $\mathbf{k}:=\mathbb{Z}[\frac{1}{|\Gamma|},e^{\frac{2\pi i}{|\Gamma|}}]$, this follows from Theorem 4.2.13.(iii) and the partial proof of Theorem 1.1.4 contained there, using the projection $\mathbf{k}[x,y]\rtimes\Gamma\twoheadrightarrow\mathsf{f}(\mathbf{k}[x,y]\rtimes\Gamma)\mathsf{f}\cong\Pi_{Q^0}$. Since $\mathbb{Z}[\frac{1}{|\Gamma|}]$ is a principal ideal domain, and the modules V' and W' are graded with finitely-generated homogeneous components, each homogeneous component of V' and W' is a direct sum of finitely many cyclic modules. On the other hand, the functor $M\mapsto M\otimes_{\mathbb{Z}[\frac{1}{|\Gamma|}]}\mathbf{k}$ does not annihilate any cyclic modules. Thus, since the target of this functor is free in each graded component, all of the cyclic modules occurring in V' and W' must be $\mathbb{Z}[\frac{1}{|\Gamma|}]$, i.e., V' and W' are free in each homogeneous component. Then, the result on Hilbert series follows because the aforementioned functor preserves Hilbert series of graded modules with finitely-generated free homogeneous components. □

As a corollary, we can deduce in full generality the easier direction of Theorem 1.1.4, that the classes $r^{(p^\ell)}$ are nonzero. We will use this in the proof of the other direction:

Proposition 7.0.8. For every prime $p>0$ and any $\ell\geq 1$, the class $r^{(p^\ell)}\in\Lambda_Q$ is nonzero.

Proof. For any quiver Q^0, we may perform the same procedure as in Proposition 7.0.5 to obtain a basis of $A':=P_{\overline{Q^0}}\langle r_i'\rangle_{i\in Q_0^0}/(\!(iri+r_i')\!)_{i\in Q_0^0}\cong P_{\overline{Q^0}}$. First, (7.0.3) becomes $A'\cong\Pi_{Q^0}*_{\mathbb{Z}^{Q_0^0}}\langle r_i'\rangle$ where we view $\langle r_i'\rangle$ as $\mathbb{Z}^{Q_0^0}$-modules by $jr_i'j'=\delta_{ij}\delta_{ij'}r_i'$. Let $r':=\sum_i r_i'$. Then, Proposition 7.0.5 presents $V_{A'}:=A'/[A',A'r'A']$ as the direct sum of $\tilde{\Pi}_{Q^0}$ and the free $\mathbb{Z}$-module with basis given by alternating cyclic words in a basis of Π_{Q^0} and $(r')^\ell,\ell\geq 1$. We may also compute the relations $W_{A'}$ as in the proposition. For any quiver $Q\supsetneq Q^0$, we have a canonical map $A'\to\Pi_Q$ which induces maps $V_{A'}\to V_Q$ and $W_{A'}\to W_Q$. It is easy to see that $W_{A'}\twoheadrightarrow W_Q$ is a surjection, since the relations are integrally spanned by commutators $[\tilde{\Pi}_{Q^0},\tilde{\Pi}_{Q^0}]$, which can be taken in A'.

Now, assume Q^0 is extended Dynkin. By Proposition 7.0.7, the rank of W_Q (which is free) does not depend on the choice of Q, but only on Q^0 (provided $Q \supsetneq Q^0$). Also, if $Q' \supsetneq Q^0$ is the quiver obtained from Q^0 by adjoining a loop to each vertex in Q_0^0, then the map $V_{A'} \to V_{Q'}$ has kernel equal to $\langle [r'] \rangle$. By Proposition 7.0.7, this shows that the map $W_{A'} \to W_{Q'}$ also has kernel equal to $\langle [r'] \rangle$, using Theorem 4.2.13.(iii). So, we obtain an isomorphism $W_{A'}/\langle [r'] \rangle \xrightarrow{\sim} W_{Q'}$. Since W_Q and $W_{Q'}$ have the same Hilbert series, one also must obtain an isomorphism $W_{A'}/\langle [r'] \rangle \xrightarrow{\sim} W_Q$. Finally, the kernel of $V_{A'} \to V_Q$ contains the kernel of $V_{A'} \to V_{Q'}$, since $[r']$ is also zero in V_Q.

Hence, in each graded degree m, one obtains an isomorphism $(W_{Q'})_m \xrightarrow{\sim} (W_Q)_m$, and a monomorphism of their saturations, $\mathrm{Sat}((W_{Q'})_m) \hookrightarrow \mathrm{Sat}((W_Q)_m)$. Here, the *saturation* of a $\mathbb{Z}$-submodule $M \subseteq V$ is the module $\mathrm{Sat}(M) := \{x \in V \mid \exists n \geq 1 \text{ s.t. } n \cdot x \in M\}$. We are taking the saturation of $(W_{Q'})_m$ inside $V_{Q'}[m]$ and of $(W_Q)_m$ inside $(V_Q)_m$.

In particular, if we lift the class $r^{(p^\ell)}$ in any way to $V_{Q'}$, it lies in $\mathrm{Sat}((W_{Q'})_m) \setminus (W_{Q'})_m$ by Conjecture 1.0.4 (i.e., the $\tilde{A}_0$ case of Theorem 1.1.4, proved in §3), and hence any lift to V_Q also has this property. □

To prove the other direction of Theorem 1.1.4 in full generality, we generalize the theorem by an analysis in each prime p using p-th powers, as follows.

Let us define $W' := W \cap [(\!(r')\!)]$ (similarly when we work over more general commutative rings than $\mathbb{Z}$). This coincides with $W \cap [(\!(\langle \overline{Q}_1 \setminus \overline{Q_1^0} \rangle)\!)]$, since W/W' is integrally spanned by $[\tilde{\Pi}_{Q^0}]$. Let $W'_p \subset V \otimes \mathbb{F}_p$ be the image of the map $W' \otimes \mathbb{F}_p \to V \otimes \mathbb{F}_p$ induced by inclusion. Then, for any $[w] \in W'_p$, we may consider $[w]^p = [w^p] \in (V \cap [(\!(r')\!)]) \otimes \mathbb{F}_p$. Note that $[w]^p \mapsto 0 \in \Lambda_Q$, since the same is true for $[w]$. Hence, $[w]^p \in W'_p$ as well. This observation allows us to state the following theorem, which will be proved in §13.2, and refines the main Theorem 1.1.4. We will use the canonical projection $V \twoheadrightarrow \Pi_{Q^0}$ whose kernel is the image of $(\!(r')\!)$. We will work below over $\mathbb{Z}_{(\!(p)\!)}$, and note that tensoring by $\mathbb{F}_p$ still makes sense, and in particular $W'_p = (W' \otimes \mathbb{Z}_{(\!(p)\!)}) \otimes_{Z(\!(p)\!)} \mathbb{F}_p$.

Theorem 7.0.9. We work over $\mathbb{Z}_{(\!(p)\!)}$ for any prime p. Then, W has the form

$$W = W_0 \oplus W', \tag{7.0.10}$$

where $W' = W \cap [(\!(r')\!)]$ and W_0 is a $\mathbb{Z}_{(\!(p)\!)}$-submodule such that the composition $W_0 \hookrightarrow V \twoheadrightarrow \Pi_{Q^0}$ is a monomorphism with image $[\Pi_{Q^0}, \Pi_{Q^0}]$. Moreover, these satisfy the following conditions:

(i) W_0 is saturated except in the cases that $(Q^0, p) \in \{(\tilde{D}_n, 2), (\tilde{E}_n, 2), (\tilde{E}_n, 3), (\tilde{E}_8, 5)\}$, when $W_0 = W_{0,s} \oplus W_{0,r}$ with $W_{0,s}$ saturated, and $W_{0,r}$ has finite rank and will be described in (iii).

(ii) $W' = W'_s \oplus W'_r$, where W'_s is saturated (see (iv)) and W'_r will be described in (iii).

(iii) $W_r := W_{0,r} \oplus W'_r$ has a basis of classes $\{f_\ell\}, \ell \geq 1$, with $|f_\ell| = 2p^\ell$, satisfying

$$\mathrm{ord}_p(f_\ell) = 1, \quad \frac{1}{p} f_{\ell+1} \equiv (\frac{1}{p} f_\ell)^p \pmod{p}, \tag{7.0.11}$$

where $\mathrm{ord}_p(f)$ denotes the greatest nonnegative integer m such that f is a multiple of p^m. One has $h(W_{0,r}; t) \leq hT(Q^0)$ from (6.3.8) (with the same p). The image of $\frac{1}{p} f_\ell$ in Λ_Q is $r^{(p^\ell)}$.

(iv) W'_s has a basis of classes $\{g_{i,\ell}\}$ as follows:

$$g_{i,\ell+1} \equiv g_{i,\ell}^p \pmod{p}. \tag{7.0.12}$$

Here, the $g_{i,0}$ $\mathbb{Z}_{(\!(p)\!)}$-linearly span a submodule which projects isomorphically mod $[(\!(\langle \overline{Q}_1 \setminus \overline{Q_1^0} \rangle)\!)^3] + (\!(p)\!)$ to $[r'U_p]$, and $U_p \subset HH^0(\Pi_{Q^0}) \otimes \mathbb{F}_p$ is a certain $\mathbb{F}_p$-vector subspace with Hilbert series $h(HH^0(\Pi_{Q^0})_+; t) - t^{2p-2} h(HH^0(\Pi_{Q^0}); t^p) + C_p(t)$. Here, $C_p(t) = 0$ unless

Λ_{Q^0} has p-torsion, in which case, letting m be the smallest positive integer such that $r^{(p^m)}$ is zero in Λ_{Q^0},

$$t^2 C_p(t) = t^{2p} - t^{2p^m} + \begin{cases} \sum_{\ell=1}^{\lfloor \frac{n}{4} \rfloor - 1} t^{4(2\ell+1)} - t^{2^{\lfloor \log_2 \frac{2(n-2)}{2\ell+1} \rfloor + 1} \cdot (2\ell+1)}, & \text{if } p = 2 \text{ and } Q^0 = \tilde{D}_n, \\ t^{28} - t^{56}, & \text{if } p = 2 \text{ and } Q^0 = \tilde{E}_8, \\ 0, & \text{otherwise.} \end{cases} \tag{7.0.13}$$

$i_0 U_p i_0$ contains the image of the Poisson bracket $\{,\} : (i_0 \Pi_{Q^0} i_0 \otimes \mathbb{F}_p)^{\otimes 2} \to (i_0 \Pi_{Q^0} i_0) \otimes \mathbb{F}_p$, to be defined in §§9.5, 10.5, and 12.5.

Here, the Poisson algebra $i_0 \Pi_{Q^0} i_0 \otimes \mathbb{F}_p$ is an analogue of $\mathbb{C}[x,y]^\Gamma$ (and, if p is a good prime, then $i_0 \Pi_{Q^0} i_0 \otimes \bar{\mathbb{F}}_p \cong \bar{\mathbb{F}}_p[x,y]^\Gamma$ by the McKay correspondence [CBH98]). For good primes, the above theorem (except for the final statement about Poisson bracket) is not difficult to deduce from Theorem 4.2.13.

Theorem 1.1.4 follows from Theorem 7.0.9, since we deduce that the torsion of $\Lambda_Q \otimes \mathbb{Z}_{(\!(p)\!)}$ for $Q \supsetneq Q^0$ is at most $\mathbb{Z}/p$ in each degree $2p^\ell$, and this is $\mathbb{Z}_{(\!(p)\!)}$-linearly spanned by the nonzero classes $r^{(p^\ell)}$.

Proposition 7.0.7 (together with Propositions 7.0.5 and 7.0.1) gives us explicit $\mathbb{Q}$-bases of Λ_Q. Fix a basis $\mathcal{P}$ of Π_{Q^0} by paths (which can be explicitly done using Theorems 9.1.2.(iii) and 10.1.9.(i), and Proposition (12.0.1).(ii)), such that, for every vertex $i \in Q_0^0$, the basis $\mathcal{P}$ includes a subset $\mathcal{P}_i$ of paths beginning (and ending) at i which maps to a basis of $\Lambda_{Q^0} \otimes \mathbb{Q}$: this is possible since $\Lambda_{Q^0 \setminus i} \otimes \mathbb{Q} = 0$ for all i, with $Q^0 \setminus i$ the quiver obtained by deleting i and all arrows incident to i. Fix also the basis $\mathcal{B}$ of $(\Pi_{Q \setminus Q^0, Q_0^0})_+$ given by Proposition 7.0.1.

Proposition 7.0.14. For every non-Dynkin, non-extended Dynkin quiver Q, a $\mathbb{Q}$-basis for $\Lambda_Q \otimes \mathbb{Q}$ is given as follows, depending on a choice of extended Dynkin subquiver $Q^0 \subsetneq Q$ on vertices $Q_0^0 \subsetneq Q_0$, and a **fixed** vertex $i' \in Q_0^0$ incident to an arrow of $Q \setminus Q^0$:

(1) The basis $\mathcal{P}$ above;
(2) Alternating cyclic words in $\mathcal{P}$ and $\mathcal{B}$ such that the source and target vertices of the basis elements match up (to give a nonzero product), **excluding** cyclic words of the form $[i' a^* a \pi]$ for $a \in G$ and $\pi \in \mathcal{P}_{i'}$, where $G \subset \overline{Q}_1 \setminus \overline{Q}_1^0$ is the forest chosen in Proposition 7.0.1 (so a must be the unique arrow of G incident to i');
(3) A basis of $\Lambda_{Q \setminus Q^0, Q_0^0} \otimes \mathbb{Q}$, given in Proposition 7.0.1.

Theorem 7.0.9 similarly gives explicit $\mathbb{F}_p$-bases of $\Lambda_Q \otimes \mathbb{F}_p$. One way to express this is as above, but replacing $a^* a \pi$ by elements $(a^* a \pi)^{p^\ell}$ where π is a leading path of a basis element of $\tilde{U}_p$; we also have to add in basis elements $r^{(p^\ell)}$ which have nonzero image in Λ_{Q^0}. The details are omitted.

8. The necklace Lie structure on Λ_Q and generalizations

In this section, we will recall the necklace Lie structure on $(P_{\overline{Q}})_{\mathrm{cyc}}$ and its quotient Λ_Q. In this section only, for convenience, we will let $P := P_{\overline{Q}}$, $L := (P_{\overline{Q}})_{\mathrm{cyc}}$, and $\Lambda := \Lambda_Q$. We will also let Q be an arbitrary quiver until §8.5, where we will have it contain an extended Dynkin quiver once again.

First, we will recall the fundamental necklace Lie bracket on L [BLB02, Gin01] as well as the fact that it descends to Λ_Q (by the comment after Lemma 8.3.2 in [CBEG07]). Then we will recall the related "double Poisson" bracket on P defined by [VdB08]. We will need to generalize these structures a bit, and in particular prove a noncommutative Leibniz identity (8.2.7). Then, we will deduce a fundamental formula relating commutators with the necklace bracket in the extended

Dynkin case (Proposition 8.5.9). As explained in §8.5, we interpret the latter as expressing $P = P_{\overline{Q}}$ as a "free-product" deformation of Π_Q in the sense of [GS12].

At the same time, we recall the necklace Lie bialgebra structure on L [Sch05] and prove that it also descends to $(\Lambda_Q)_+$, and derive some results such as a Batalin-Vilkovisky style identity (8.3.2), since they arise naturally in the above context and may be of independent interest. However, we will not require anything about the Lie cobracket in this paper, so the reader not interested in this can skip all results involving the Lie cobracket.

We will also prove in this section that the classes $r^{(p^\ell)}$ are in the kernel of the Lie bracket and cobracket (Proposition 8.1.9); in fact, as we mention in Remark 8.1.10 below, this is true for all of the torsion of Λ_Q. However, we will not need these results in this paper, so the reader may skip this as well if desired.

8.1. **The necklace Lie and double Poisson brackets.** We first recall the necklace Lie bracket [BLB02, Gin01]. Let $L := (P_{\overline{Q}})_{\mathrm{cyc}} = P_{\overline{Q}}/[P_{\overline{Q}}, P_{\overline{Q}}]$. Let ω be the natural symplectic form on the free $\mathbb{Z}$-module $\mathbb{Z}\cdot\overline{Q}_1$, of rank $|\overline{Q}_1| = 2|Q_1|$, with symplectic basis $(a, a^*)_{a\in Q_1}$. Also, for any arrow $a : i \to j$ in $\overline{Q}$, let $a_s := i$ and $a_t := j$. Then one defines the bracket $\{\,,\}$ by

$$\{[a_1a_2\cdots a_m], [b_1b_2\cdots b_n]\} = \sum_{i,j} \omega(a_i, b_j)[(a_i)_t a_{i+1}\cdots a_{i-1}b_{j+1}\cdots b_{j-1}], \tag{8.1.1}$$

$$\delta([a_1a_2\cdots a_m]) = \sum_{i<j} \omega(a_i, a_j)[(a_j)_t a_{j+1}\cdots a_{i-1}] \wedge [(a_i)_t a_{i+1}\cdots a_{j-1}]. \tag{8.1.2}$$

Here, we need the terms $(a_i)_t, (a_j)_t$ in case one has $n = m = 1$ in the first line, and in case $j = i+1$ or $j = m, i = 1$ in the second line.

Proposition 8.1.3. [Sch05] The above defines the structure of an involutive Lie bialgebra on L: that is, L is a Lie bialgebra satisfying $\mathrm{br}\circ\delta = 0$, where $\mathrm{br} : L\otimes L \to L$ is the bracket.

To define these more suggestively, we may define [Sch05] operators $\partial_a : L \to P, D_a : P \to P\otimes P$ by the formulas

$$\partial_a([a_1\cdots a_m]) := \sum_{i:\, a_i=a} (a_i)_t a_{i+1}\cdots a_{i-1}, \tag{8.1.4}$$

$$D_a(a_1\cdots a_m) := \sum_{i:\, a_i=a} a_1\cdots a_{i-1}(a_i)_s \otimes (a_i)_t a_{i+1}\cdots a_m, \tag{8.1.5}$$

so that $\{\,,\} : L\otimes L \to L, \delta : L \to L\otimes L$ are given by (letting $m : P\otimes P \to P$ be the multiplication)

$$\{\,,\} = \sum_{a\in Q_1} \mathrm{pr}\circ m\circ(\partial_a\otimes\partial_{a^*} - \partial_{a^*}\otimes\partial_a), \quad \delta = \sum_{a\in Q_1} (\mathrm{pr}\otimes\mathrm{pr})(D_a\partial_{a^*} - D_{a^*}\partial_a). \tag{8.1.6}$$

(Note that ∂_a was notated $\frac{\partial}{\partial a}$ in [Sch05] and D_a was notated $\tilde{D}_a$.)

Our first result is then

Proposition 8.1.7. The submodule $\mathbb{Z}\cdot Q_0 \oplus [Pr] \subset P/[P,P] = L$ is a Lie bi-ideal, where $\mathbb{Z}\cdot Q_0 = L_0$ is the $\mathbb{Z}$-linear span of vertices. Hence, $\Lambda_+ := \Lambda/\Lambda_0$ is a Lie bialgebra. Furthermore, $[Pr] \subset L$ is a Lie ideal, so that Λ itself is a Lie algebra.

(As mentioned above, the Lie part was proved in [CBEG07] in greater generality: see Proposition 4.4.3.(ii) and the comment after Lemma 8.3.2.)

Remark 8.1.8. It is an interesting question to explicitly quantize Λ in the sense of Drinfeld (cf. [EK96]): by [EK96] there exists a functorial quantization of Lie bialgebras; however, we were unable to find

any natural formulas or even an explicit algorithm for computing formulas which produces a quantization of Λ. In more detail, in [Sch05] we found an explicit quantization of L itself, but we were unable to find a Hopf ideal of that quantization corresponding to the Lie bi-ideal $\langle Q_0 \rangle \oplus [Pr]$.

We can also answer the natural question of how the classes $r^{(p^\ell)}$ behave under the Lie structure:

Proposition 8.1.9. In Λ_+, $r^{(p^\ell)}$ is in the kernel of both the bracket and cobracket. More generally, $[ir^m] \in L$ is in the kernel of the bracket for any $i \in Q_0, m \geq 1$, and is in the kernel of the cobracket on $L_+ := L/L_0$.

Remark 8.1.10. The complete kernel of the Lie bracket can be found in [Sch07, Theorem 9.2.2], and in [Sch07, Proposition 9.2.1] the first statement is also generalized to all of the torsion of Λ.

We will prove Propositions 8.1.7 and 8.1.9 in §8.4, after developing the noncommutative BV structure on L.

8.2. **Lifting brackets to P.** In [VdB08], lifts of the Lie bracket on L to P were defined as follows. Let τ denote the operator which permute components of tensor products. Namely, if $\sigma \in \Sigma_n$, then $\tau_\sigma : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to V_{\sigma^{-1}(1)} \otimes V_{\sigma^{-1}(2)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}$ is the result of applying the permutation σ.

Then, one has the double Poisson bracket $\{\!\{ , \}\!\} : P \otimes P \to P \otimes P$:

$$\{\!\{a_1 \cdots a_m, b_1 \cdots b_n\}\!\} := \sum_i \omega(a_i, b_j) b_1 \cdots b_{j-1}(a_i)_t a_{i+1} \cdots a_m \otimes a_1 \cdots a_{i-1}(a_i)_s b_{j+1} \cdots b_n. \tag{8.2.1}$$

In terms of the operator D_e of (8.1.5), one has

$$\{\!\{ , \}\!\} := \sum_{a \in Q_1} (m \otimes m) \circ \tau_{(13)} \circ (D_a \otimes D_{a^*} - D_{a^*} \otimes D_a). \tag{8.2.2}$$

As mentioned in [VdB08], the formula $m \circ \{\!\{ , \}\!\} : P \otimes P \to P$ defines a Loday bracket. Since this kills $[P, P] \otimes P$, for our purposes it will be more convenient to work with the induced map $\{ , \} : L \otimes P \to P$, which we henceforth call the Loday bracket:

$$\{[a_1 \cdots a_m], b_1 \cdots b_n\} := \sum_i \omega(a_i, b_j) b_1 \cdots b_{j-1}(a_i)_t a_{i+1} \cdots a_{i-1} b_{j+1} \cdots b_m. \tag{8.2.3}$$

In terms of (8.1.4), (8.1.5),

$$\{ , \}\Big|_{L \otimes P} := \sum_{a \in Q_1} m \circ (m \otimes 1) \circ \tau_{(12)} \circ (\partial_a \otimes D_{a^*} - \partial_{a^*} \otimes D_a). \tag{8.2.4}$$

Combining this with the Lie bracket $\{ , \} : L \otimes L \to L$, one can consider $\{ , \}$ to be a Lie bracket on $\tilde{P} := L \oplus P$, by defining

$$\{ , \}\Big|_{P \otimes L} := -\{ , \}\Big|_{L \otimes P} \circ \tau_{(12)}, \quad \{ , \}\Big|_{P \otimes P} = 0, \tag{8.2.5}$$

without losing any information: in fact, this encodes the fact that the bracket $\{ , \}$ is skew on L.

We may now compare the bracket $\{ , \}$ with the algebra multiplication on P, and it turns out that this satisfies a kind of Leibniz rule:

Proposition 8.2.6. Defining multiplication $m\big|_{(L \otimes P) \oplus (P \otimes L) \oplus (L \otimes L)} = 0$ by L to be zero, $(\tilde{P}, \{ , \}, m)$ satisfies

$$\{a, bc\} = b\{a, c\} + \{a, b\}c. \tag{8.2.7}$$

Proof. The fact that $\tilde{P}$ is Lie follows from a straightforward computation, and (8.2.7) follows immediately from the definitions. Alternatively, one may derive this from the double Poisson axioms that $\{\!\{,\}\!\}$ satisfies (see [VdB08]). □

Also, one has the rather obvious identity:

$$\mathrm{pr}\circ\{\,,\}\Big|_{P\otimes L} = \{\,,\}\circ(\mathrm{pr}\otimes 1),\ \text{and similarly}\ \mathrm{pr}\circ\{\,,\}\Big|_{L\otimes P} = \{\,,\}\circ(1\otimes\mathrm{pr}). \tag{8.2.8}$$

It turns out that the cobracket lifts to P as well:

$$\delta_\ell : P \to L\otimes P, \delta : \tilde{P}\to\tilde{P}\otimes\tilde{P},$$

$$\delta_\ell(a_1\cdots a_n) := \sum_{i<j} -\omega(a_i,a_j)[(a_i)_t a_{i+1}\cdots a_{j-1}]\otimes a_1\cdots a_{i-1}(a_i)_s a_{j+1}\cdots a_n, \tag{8.2.9}$$

$$\delta_\ell = \sum_{a\in Q_1}(\mathrm{pr}\otimes m)\tau_{(12)}\big((1\otimes D_a)D_{a^*} - (1\otimes D_{a^*})D_a\big), \tag{8.2.10}$$

$$\delta\Big|_P = \delta_\ell - \tau_{(12)}\delta_\ell, \tag{8.2.11}$$

with δ defined on L as usual. One then has

Proposition 8.2.12. The structure $(\tilde{P},\{\,,\},\delta)$ is an involutive Lie bialgebra.

Proof. This follows from the same proof as [Sch05, §2] (one simply needs to remember where the start and end of pieces that come from P are). □

As before, one may view δ on P as a lift of δ on L: If we define $\mathrm{pr}\Big|_L = \mathrm{id}$, one has

$$(\mathrm{pr}\otimes\mathrm{pr})\delta = \delta\circ\mathrm{pr}\,. \tag{8.2.13}$$

8.3. **Noncommutative BV structure.** One notices a BV-style connection encompassing the Poisson and Lie bialgebra structures, whose proof is straightforward and omitted:

Proposition 8.3.1. The following BV-style identity is satisfied by $\tilde{P}$: For any $a,b\in P$, one has

$$\delta_\ell(ab) = \delta_\ell(a)(1\otimes b) + (1\otimes a)\delta_\ell(b) + (\mathrm{pr}\otimes 1)\{\!\{a,b\}\!\}. \tag{8.3.2}$$

To get an equation involving δ, one can apply $(1-\tau_{(12)})$ to each side (which originally lives in $L\otimes P$).

What this says is that δ can be defined as the unique cobracket satisfying (8.3.2) which vanishes on $P_0\oplus P_1$. Since δ has total degree -2, it must be the unique **homogeneous** cobracket (or, indeed, homogeneous linear map $P\to L\otimes P$) of degree -2 satisfying (8.3.2).

8.3.1. **More general cobrackets.** The results of this subsection will not be needed elsewhere in the paper.

One could define more general inhomogeneous cobrackets which do not vanish on P_1. For example, one could start with $\delta'_\ell(a) = F_a\otimes a$, for elements $F_a\in L$ assigned to arrows $a\in\overline{Q}_1$, satisfying: $\delta(F_a)=0$ (using the old δ from (8.1.2)), $F_a = -F_{a^*}$, and $\{bb^*,F_a\}=0,\forall a,b\in\overline{Q}_1$. This extends to a unique δ'_ℓ on P satisfying (8.3.2), which induces a Lie coalgebra structure on L.

However, this does not necessarily yield an involutive Lie bialgebra: to obtain involutivity and the one-cocycle condition, we can set $\delta_\ell = \sum_{a\in Q_1} -F_a\otimes\mathrm{ad}[aa^*] + [aa^*]\otimes\mathrm{ad}\,F_a$.

More generally, any δ'_ℓ satisfying (8.3.2) must be of the form

$$\delta'_\ell = \delta_\ell + \sum_i F_i\otimes\theta_i, \tag{8.3.3}$$

where $F_i \in L$ and $\theta_i \in \mathrm{Der}(P)$. The condition that δ'_ℓ induce a cobracket on L (the co-Jacobi condition) is then

$$Skew \circ \big(\delta(\sum_i F_i \otimes \theta_i) + \frac{1}{2}\{\sum_i F_i \otimes \theta_i, \sum_j F_j \otimes \theta_j\}\big) = 0, \tag{8.3.4}$$

where

$$\delta(F \otimes \theta) := \delta(F) \otimes \theta - F \otimes [\delta_\ell, \theta], \tag{8.3.5}$$

$$[\delta_\ell, \theta] := \delta_\ell \circ \theta - (\theta \otimes 1 - \mathbf{1} \otimes \theta) \circ \delta_\ell, \tag{8.3.6}$$

$$\{F \otimes \theta, F' \otimes \theta'\} := (F \wedge \theta(F')) \otimes \theta' + (F' \wedge \theta'(F)) \otimes \theta - (F \wedge F') \otimes [\theta, \theta'], \text{ and} \tag{8.3.7}$$

$$Skew = \sum_{\sigma \in S_3} \mathrm{sign}(\sigma) \cdot \tau_\sigma. \tag{8.3.8}$$

In the case that the θ_i are inner derivations (equivalently, they kill r; in particular, they descend to Π), then we may consider maps of the form $\delta' = \delta + \sum(F_i \otimes \mathrm{ad}\, G_i - G_i \otimes \mathrm{ad}\, F_i)$. Then, one has $[\delta_\ell, \mathrm{ad}\, F] = (1 \otimes \mathrm{ad})\delta(F)$. Thus, (8.3.4) says

$$Skew\big((\delta \otimes 1)(\sum_i F_i \wedge G_i) + \frac{1}{2}\{\sum_i F_i \wedge G_i, \sum_j F_j \wedge G_j\}\big) = 0, \tag{8.3.9}$$

where now $\{F \otimes G, F' \otimes G\} = F \otimes \{G, F'\} \otimes G$, extended linearly.

This is a version of the Maurer-Cartan equation. The explanation is that it is the condition $(D')^2 = 0$, where D' is the operator on the exterior algebra $\Lambda^\bullet L$ induced by $\delta' := (\mathrm{pr} \otimes \mathrm{Id}) \circ \delta'_\ell$ and the bracket. Indeed, as observed in [Gin01, §2.10], involutive Lie (super)bialgebra structures on $\mathfrak{g}$ are equivalent to BV structures on the exterior algebra $\Lambda^\bullet \mathfrak{g}$ with a differential obtained from maps $\mathfrak{g} \to \wedge^2 \mathfrak{g}$ and $\wedge^2 \mathfrak{g} \to \mathfrak{g}$. As a consequence, solutions of (8.3.9) will give solutions of (8.3.2) which descend to involutive Lie bialgebra structures (not merely cobrackets). Furthermore, given that δ descends to Λ, so does the above δ'.

For example, for any $F_i, G_i \in [P_Q]$, one obtains the class $\delta'_\ell = \delta + \sum_i (F_i \otimes \mathrm{ad}\, G_i - G_i \otimes \mathrm{ad}\, F_i)$, which gives an involutive Lie bialgebra satisfying (8.3.2). Note that this still gives a graded Lie bialgebra using the "geometric" grading, given by setting $|Q^*| = 1$ and $|Q| = 0$: the bracket and cobracket then both have degree -1. (The total grading we are using everywhere else in this paper is the analogue of the Bernstein or additive grading. In [GS10], we use the geometric grading since it exists for much more general algebras (replacing $P_{\overline{Q}} = T_{P_Q} \mathrm{Der}(P_Q, P_Q \otimes P_Q)$ by $T_A \mathrm{Der}(A, A \otimes A)$ for more general A.)

For more details and a general construction of noncommutative BV structures, see [GS10]. For our purposes, we will only need the identity (8.3.2).

8.4. **Proof of Propositions 8.1.7 and 8.1.9.** We break Proposition 8.1.7 into two lemmas:

Lemma 8.4.1. The $\mathbb{Z}$-submodule $[Pr] \subset L = P/[P, P]$ is a Lie ideal.

Proof. Note that this is actually a special case of Proposition 4.4.3(ii) from [CBEG07]. We give an elementary proof. Pick $f = [a_1 \cdots a_m] \in L$ and let $g \in P$ be arbitrary. We make use of the Poisson

bracket $\{\,,\}$ on $\tilde{P} = L \oplus P$, to obtain

$$\begin{aligned}(8.4.2)\quad \{[a_1\cdots a_m], gr\} - \{[a_1\cdots a_m], g\}r &= g\{[a_1\cdots a_m], r\} \\ &= g\sum_j \omega(a_j, a_j^*)(-1)^{[a_j^*\in Q_1]}\,(a_{j+1}\cdots a_{j-1}a_j - a_j a_{j+1}\cdots a_{j-1}) \\ &= g\sum_j (a_{j+1}\cdots a_{j-1}a_j - a_j a_{j+1}\cdots a_{j-1}) = 0.\end{aligned}$$

Here [statement] is defined to equal one if "statement" is true and to equal zero if "statement" is false. Now, the result follows from (8.2.8). □

Lemma 8.4.3. The $\mathbb{Z}$-submodule $[Pr] + L_0 \subset L$ is a Lie bi-ideal.

Proof. It is obvious that L_0 is in the kernel of the Lie bracket, so we need only show that $[Pr] \oplus L_0$ is a Lie coideal. Let $f \in P$ be arbitrary; we compute $\delta_\ell(rf)$ by means of the BV identity (8.3.2):

$$(8.4.4)\qquad \delta_\ell(rf) = \delta_\ell(r)(1\otimes f) + (1\otimes r)\delta_\ell(f) + (\mathrm{pr}\otimes 1)\{\!\{r, f\}\!\}.$$

Now, $\delta_\ell(r) = \sum_{a\in Q_1}(a_t\otimes a_s - a_s\otimes a_t)$, so that the first term on the RHS is in $L_0\otimes P$. The second term is obviously in $L\otimes Pr$. Let us compute the last term on the RHS. Let us assume $f = a_1\cdots a_n$ is a single path:

$$\begin{aligned}(8.4.5)\quad \{\!\{r, a_1\cdots a_n\}\!\} = \sum_i \omega(a_i^*, a_i)(-1)^{[a_i\in Q_1]}\big(&a_1\cdots a_{i-1}a_i\otimes (a_i)_t a_{i+1}\cdots a_n \\ &- a_1\cdots a_{i-1}(a_i)_s\otimes a_i\cdots a_n\big) = a_1\cdots a_n\otimes (a_n)_t - (a_1)_s\otimes a_1\cdots a_n,\end{aligned}$$

since $\omega(a_i^*, a_i)(-1)^{[a_i^*\in Q_1]} = 1$ as before. This is in $P_0\otimes P + P\otimes P_0$. The lemma now follows from (8.2.13). □

Proof of Proposition 8.1.9. We prove the second assertion (which clearly implies the first, since L is a free module). To show that $[ir^m]$ is in the kernel of the Lie bracket on L for any $i \in Q_0$ and $m \geq 0$, one simply applies (8.4.2) multiple times, replacing r with iri (and hence limiting the sum to those arrows a_j which are adjacent to i).

Showing that $[ir^m]$ is in the kernel of the cobracket is a bit more difficult. We show more generally that ir^m is in the kernel of $\delta'_\ell := (q\otimes q')\circ\delta_\ell : P \to L_+\otimes P_+$, where $q : L \twoheadrightarrow L/L_0 = L_+, q' : P \twoheadrightarrow P/P_0 = P_+$ are the projections. Inductively, we need to show that

$$(8.4.6)\qquad 0 = \delta'_\ell(ir^{m+1}) = ir\delta'_\ell(ir^m) + \delta'_\ell(ir)r^m + (q\circ\mathrm{pr}\otimes q')\{\!\{ir, ir^m\}\!\} = (q\circ\mathrm{pr}\otimes q')\{\!\{ir, ir^m\}\!\}.$$

Now, considering (8.4.5), one verifies that most of the terms in $\{\!\{ir, ir^m\}\!\}$ cancel, leaving $ir^m\otimes i - i\otimes ir^m$. This is killed by $q\circ\mathrm{pr}\otimes q'$, verifying the desired result. □

8.5. **Commutators and Poisson brackets in the extended Dynkin case.** Throughout this subsection, let Q^0 be a (fixed) extended Dynkin quiver. Let $\Gamma < SL(2,\mathbb{C})$ be the corresponding finite group under the McKay correspondence.

8.5.1. **Preliminaries on $i_0\Pi_{Q^0}i_0$ and Λ_{Q^0}.** It will be useful to consider the following sequence of natural maps (cf. (6.3.1)):

$$(8.5.1)\qquad HH^0(\Pi_{Q^0}) \hookrightarrow \Pi_{Q^0} \twoheadrightarrow HH_0(\Pi_{Q^0}) = \Lambda_{Q^0}.$$

Tensoring over $\mathbf{k} := \mathbb{Z}[\frac{1}{|\Gamma|}, e^{\frac{2\pi i}{|\Gamma|}}]$, this composition is an isomorphism onto $((\Lambda_{Q^0})_+\otimes\mathbf{k})\oplus\mathbf{k}$ (where $1\in\mathbf{k}$ is the class of the unit in $P_{\overline{Q^0}}$). To see this, we can use the graded Morita equivalence $\Pi_{Q^0}\otimes\mathbf{k} \simeq \mathbf{k}[x,y]\rtimes\Gamma$ (cf. §4), which induces isomorphisms on HH^0 and HH_0. Thus, we can

replace $\Pi_{Q^0}\otimes\mathbf{k}$ by $\mathbf{k}[x,y]\rtimes\Gamma$, and the result follows from the fact that $HH^0(\mathbf{k}[x,y]\rtimes\Gamma)=\mathbf{k}[x,y]^\Gamma\to HH_0(\mathbf{k}[x,y]\rtimes\Gamma)=(\mathbf{k}[x,y]^\Gamma\oplus\mathbf{k}[\Gamma]^\Gamma)/\mathbf{k}$ is an isomorphism in positive degrees.

Note that, if we work over $\mathbb{Z}$ or a field of characteristic dividing $|\Gamma|$, then the map $HH^0(\Pi_{Q^0})\to\Lambda_{Q^0}$ is *not* an isomorphism in positive degrees, even for the cases of $\tilde{A}_{n-1},\tilde{D}_n$. For example, in the case $\tilde{A}_{n-1}$, a central element involves a sum over all vertices of cycles beginning and ending at that vertex; for each vertex, the corresponding summand has the same image in Λ_{Q^0} and hence the sum must be a multiple of n, which does not yield an isomorphism when we don't invert $|\Gamma|=n$.

Back to the situation above with $\mathbf{k}=\mathbb{Z}[\frac{1}{|\Gamma|},e^{\frac{2\pi i}{|\Gamma|}}]$, we may transplant the commutative multiplication on $HH^0(\Pi_{Q^0}\otimes\mathbf{k})\cong\mathbf{k}[x,y]^\Gamma\cong i_0\Pi_{Q^0}i_0\otimes\mathbf{k}$ to $((\Lambda_{Q^0})_+\otimes\mathbf{k})\oplus\mathbf{k}$ using the isomorphism $HH^0(\Pi_{Q^0}\otimes\mathbf{k})\tilde{\to}((\Lambda_{Q^0})_+\otimes\mathbf{k})\oplus\mathbf{k}$ above. It is easy to check that this multiplication is compatible with the necklace bracket, i.e., induces a graded Poisson algebra structure on $((\Lambda_{Q^0})_+\otimes\mathbf{k})\oplus\mathbf{k}$ (it essentially follows from Proposition 8.2.6). Since there is a unique generically nondegenerate Poisson bracket on $HH^0(\Pi_{Q^0}\otimes\mathbf{k})\cong\mathbf{k}[x,y]^\Gamma$ up rescaling, this shows that the isomorphism $HH^0(\Pi_{Q^0}\otimes\mathbf{k})\tilde{\to}((\Lambda_{Q^0})_+\otimes\mathbf{k})\oplus\mathbf{k}$ carries the standard Poisson bracket to the necklace bracket, up to scaling. This actually works over $\mathbb{Z}[\frac{1}{|\Gamma|}]$ since we do not need the roots of unity to express the center of $HH^0(\Pi_{Q^0}\otimes\mathbb{Z}[\frac{1}{|\Gamma|}])$.

It is useful to have the following result (still with $\mathbf{k}=\mathbb{Z}[\frac{1}{|\Gamma|},e^{\frac{2\pi i}{|\Gamma|}}]$):

Proposition 8.5.2. Let Q^0 be any extended Dynkin quiver with extending vertex $i_0\in Q_0^0$, and let $i\in Q_0^0$ be any vertex. Let $z\in HH^0(\Pi_{Q^0}\otimes\mathbf{k})$ be any central element. Then, taking image in $HH_0(\Pi_{Q^0}\otimes\mathbf{k})=\Lambda_{Q^0}\otimes\mathbf{k}$, we obtain

$$[iz]=\dim(\rho_i)[i_0z],\quad [z]=|\Gamma|[i_0z]. \tag{8.5.3}$$

Proof. Let us consider the sequence

$$HH^0(\Pi_{Q^0}\otimes\mathbf{k})\hookrightarrow\mathbf{k}[x,y]\rtimes\Gamma\twoheadrightarrow HH_0(\mathbf{k}[x,y]\rtimes\Gamma)\tilde{\to}\Lambda_{Q^0}\otimes\mathbf{k} \tag{8.5.4}$$

We know that the sequence is injective. By the analysis in §4, the image of $[iz]$ in $HH_0(\mathbf{k}[x,y]\rtimes\Gamma)$ consists of projection of $z\mathsf{f}_i$ to ze where $e\in\Gamma$ is the identity. But this is the projection of the idempotent f_i to e, which is taking the trace of the identity element in the representation ρ_i, which is the dimension. □

8.5.2. **P_{Q^0} as a free-product deformation of Π_{Q^0}.** Next, let $Q\supsetneq Q^0$ be any quiver, and as in §4 let $r':=\sum_{a\in Q_1\setminus Q_1^0}\mathbf{1}_{Q_0^0}(aa^*-a^*a)\mathbf{1}_{Q_0^0}$. For any $f\in\Pi_{Q^0}$ and $z\in HH^0(\Pi_{Q^0})$, choose lifts $\tilde{f},\tilde{z}\in P_{\overline{Q^0}}$. It follows using the Morita equivalence of §4 that

$$[\tilde{z},\tilde{f}]\equiv-\mu_{r'}\{\!\{\tilde{z},\tilde{f}\}\!\}\pmod{((r'))^2+[((r')),f]}, \tag{8.5.5}$$

where

$$\mu_{r'}(a\otimes b):=ar'b. \tag{8.5.6}$$

For example, in the case $\Gamma=\{1\}$, the above is an expansion of $[x^ay^b,x^cy^b]$ under the relation $[x,y]=r'$. In this case, Q^0 is the quiver with one vertex and one loop, and

$$[x^n,y]=-\sum_{i=0}^{n-1}x^ir'x^{n-i-1}\equiv-r'x^{n-1}\pmod{[((r'))^2]+[((r')),\Pi_Q]}. \tag{8.5.7}$$

Let us return to the case of general extended Dynkin Q^0.

Remark 8.5.8. Because of Proposition 11.2.5, or by the same argument, one deduces that (8.5.5) remains true after replacing Π_Q by $P_{\overline{Q^0}}$, or more generally by $\Pi_{Q,J}$, for $Q \supset Q^0$ and either $Q \neq Q^0$ or $J \neq \emptyset$; we then set $r' = -\sum_{a \in Q_1^0}(aa^* - a^*a) \in \Pi_{Q,J}$.

Now, replace Π_Q by $P_{\overline{Q^0}}$. By (7.0.3), as in the proof of Proposition 7.0.5.(i), one deduces that $P_{\overline{Q^0}}/[(\!(r')\!), f]$ and $P_{\overline{Q^0}}/([(\!(r')\!), f] + (\!(r')\!)^2)$ are torsion-free. Hence, (8.5.5) holds over $\mathbb{Z}$ (working in $P_{\overline{Q^0}}$).

We interpret (8.5.5), together with the NCCI property that $P_{\overline{Q^0}} \cong \Pi_{Q^0} *_R T_{\mathbb{Z}}\langle r\rangle$ (with $R = \mathbb{Z}^{Q_0}$), as saying that $P_{\overline{Q^0}}$ is a **noncommutative** or **free product** deformation of Π_{Q^0}, which "quantizes" the double bracket $\{\!\{\,,\}\!\}$ (more precisely, the Poisson bracket on $HH^0(\Pi_{Q^0})$, using the following propositions). One can also say that P_Q is a noncommutative deformation of Π_Q for any non-Dynkin, non-extended Dynkin quiver (the NCCI property yields "flatness"), but without a statement about Poisson bracket. For more details and the general theory of this type of "free product" deformation, see [GS12].

We may now deduce the following useful result, over $\mathbb{Z}$:

Proposition 8.5.9. Let $Q \supsetneq Q^0$ where Q^0 is extended Dynkin. For any $z \in HH^0(\Pi_{Q^0})$ and $x \in \Pi_{Q^0}$, and any lifts $\tilde{z}, \tilde{x}$ to Π_Q,

$$\tilde{z}\tilde{x} - \tilde{x}\tilde{z} \equiv [r'\psi(\{[i_0 z], [x]\})] \pmod{[(\!(\langle \overline{Q}_1 \setminus \overline{Q_1^0}\rangle)\!), \Pi_Q] + [(\!(\overline{Q}_1 \setminus \overline{Q_1^0})\!)^3]}, \tag{8.5.10}$$

where $\psi : \Lambda_{Q^0} = HH_0(\Pi_{Q^0}) \twoheadrightarrow HH^0(\Pi_{Q^0})$ is the composition

$$HH_0(\Pi_{Q^0}) \twoheadrightarrow HH_0(\Pi_{Q^0})/(\text{torsion} \oplus \bigoplus_{i \neq i_0}[i]) \cong i_0\Pi_{Q^0}i_0 \cong HH^0(\Pi_{Q^0}).$$

Proof. It follows from Proposition 7.0.5 that $\Pi_Q/([(\!(\langle \overline{Q}_1 \setminus \overline{Q_1^0}\rangle)\!), \Pi_Q] + (\!(\langle \overline{Q}_1 \setminus \overline{Q_1^0}\rangle)\!)^3)$ is torsion-free. Hence, it is enough to prove the above formula tensored over $\mathbf{k}$, where this follows from (8.5.5) and Proposition 8.5.2. □

Remark 8.5.11. Equation (8.5.5) also implies that, letting

$$\pi : P_{\overline{Q^0}} \twoheadrightarrow \Pi_{Q^0} \tag{8.5.12}$$

be the projection,

$$\mu\{\!\{P_{\overline{Q^0}}, \pi^{-1}(HH^0(\Pi_{Q^0}))\}\!\} \subset \pi^{-1}(HH^0(\Pi_{Q^0})), \tag{8.5.13}$$

where μ is the multiplication map. In other words, $\pi^{-1}(HH^0(\Pi_{Q^0}))$ is a Loday ideal with respect to the Loday bracket $L_{\overline{Q^0}} \otimes P_{\overline{Q^0}} \to P_{\overline{Q^0}}$ of §8.2.

9. Quivers containing $\tilde{A}_{n-1}$

9.1. **Bases of Π_Q for type A quivers and refinement of Theorem 1.1.4.** Here we describe bases of Π_Q for extended Dynkin quivers of type A and quivers which properly contain them. The resulting Theorem 9.1.2 proves Theorem 1.1.4 for all quivers containing a cycle. We work over $\mathbb{Z}$ throughout, i.e., with $\mathbb{Z}$-modules.

Let $Q^0 = \tilde{A}_{n-1}$, as depicted in Figure 1, forming a polygon which is oriented counter-clockwise (none of the results depend on this choice; in particular the choice of orientation does not affect the structure of Π_{Q^0}, Π_Q, or their zeroth Hochschild homology; we only make this choice for convenience). We define the following:

$$x = \sum_{a \in Q_1^0} a, \quad y = \sum_{a \in Q^0} a^*. \tag{9.1.1}$$

We call an arrow is *counter-clockwise-oriented* or *clockwise-oriented* depending on whether it forms part of the counter-clockwise or clockwise orientation of the polygon in Figure 1, i.e., whether it is in Q^0 or not. Let the vertex set Q_0^0 be given the natural structure of a $\mathbb{Z}/n$-torsor (i.e., affine space over $\mathbb{Z}/n$), where adding one means moving once counter-clockwise (if the reader prefers, one can let $Q_0^0 = \mathbb{Z}/n$, choosing a fixed vertex to be labeled by zero).

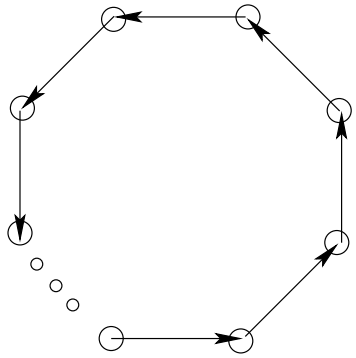

FIGURE 1. $Q^0 = \tilde{A}_{n-1}$ with counter-clockwise orientation

We then have the

Theorem 9.1.2. Let $Q^0 = \tilde{A}_{n-1}$ with the above notation and orientation.

(i) For any $i, j \in Q_0^0$, a basis of $i\Pi_{Q^0}j$ is given by $ix^a y^b j$ for $(a-b) \equiv (j-i) \pmod{n}$.
(ii) A basis of $i\Pi_{Q^0}j$ is also given by the nonzero elements $iz_{a,b}j$ (3.0.3), which are equal to the $ix^a y^b j$ above.
(iii) Λ_{Q^0} is a free $\mathbb{Z}$-module with basis given, for any fixed $i_0 \in Q_0^0$, by the classes $[i_0 x^a y^b] = [i_0 z_{a,b}]$ for $a, b \geq 0$ and $n \mid (b-a)$.
(iv) A basis of $HH^0(\Pi_{Q^0})$ is given by $z_{a,b}$ for $a, b \geq 0$ and $n \mid (b-a)$.
(v) For any $Q \supsetneq Q^0$, and any fixed vertex $i_0 \in Q_0^0$, W of Proposition 7.0.5.(ii) has the form $W = W' \oplus W_0$, where W_0 is the image of $[\Pi_{Q^0}, \Pi_{Q^0}]$ under (7.0.6) and and W' is a free $\mathbb{Z}$-module with basis the classes $W_{a,b}$ given by (3.0.4)–(3.0.6). As in the case of Lemma 3.0.2, $W_{a,b}$ is a generator of the $\mathbb{Z}$-submodule $\langle [iz_{a-1,b}, x], [iz_{a,b-1}, y] \rangle_{i \in Q_0^0}$.
(vi) The integral span of the classes $W_{a,b}$, for $(a,b) \neq (p^\ell, p^\ell)$ is a saturated $\mathbb{Z}$-submodule of V. The order of W_{p^ℓ, p^ℓ} in V is p.
(vii) The image of $\frac{1}{p} W_{p^\ell, p^\ell}$ in Λ_Q is nonzero, equal to $r^{(p^\ell)}$, and every p-torsion class in Λ_Q is an integral combination of these classes.

Parts (i)–(iv) are easy and their proofs are omitted. The remainder of the theorem will be proved in §9.3. As a corollary, one deduces Theorem 1.1.4 in this case, and moreover easily obtains a $\mathbb{Z}$-basis of Λ_Q modulo torsion together with $\mathbb{F}_p$-bases of all of the p-torsion for all p. Also, note that Conjecture 1.0.4 is a special case of $n = 1$ (the case of quivers with one vertex). We remark also that the above yields bases of $\Pi_{A_{n-1}}$ as well, by taking the images of all paths in the above basis of $\Pi_{\tilde{A}_{n-1}}$ and discarding those whose image is zero. This is true because, for each $i, j \in Q_0^0$ and $m \geq 1$, there is at most one basis element in $(i\Pi_{\tilde{A}_n} j)_m$ above that projects to a nonzero element of $(i\Pi_{A_n} j)_m$.

9.2. **The case of $\tilde{A}_0$.** Although Corollary 3.0.2 already implies Theorem 1.1.4 in the case $Q^0 = \tilde{A}_0$ (cf. Remark 4.2.21), and in fact Theorem 9.1.2 for $\tilde{A}_0$, we give a slightly different explanation using bases, as this will be generalized to $\tilde{A}_{n-1}$ for general n in the next subsection.

Let $Q \supsetneq Q^0 = \tilde{A}_0$, i.e., Q is a quiver containing an arrow which is a loop, say $x \in Q_1$, based at the vertex $i_0 \in Q_0$. Let Q^0 be the subquiver consisting of just the vertex i_0 and the loop x. Fix a maximal tree $G \subset \overline{Q}$ as in Proposition 7.0.1: here this means that all arrows of G are oriented towards the vertex i_0 (one can follow oriented arrows of G to arrive at i_0 from any vertex). We define $G^* := \{a^* \mid a \in G\}$, and one evidently has $G \cap G^* = \emptyset$.

Let the reverse of the arrow x be $y := x^*$, and take $z_{a,b}$ defined as in (3.0.3).

Let us make the isomorphism $\Lambda_Q \cong V/W$ from Proposition 7.0.6 more explicit in this case. Define $r_{Q_0^0} = \mathbf{1}_{Q_0^0} r \mathbf{1}_{Q_0^0} = i_0 r i_0$ as in the setup of Proposition 7.0.5. Let $F := \Pi_{Q,Q_0^0}$, and set $r' := r_{Q_0^0} - xy + yx \in \Pi_{Q\setminus Q^0,Q_0^0}$. Set $A = F/(\!(i_0 r i_0)\!) \cong \Pi_Q$.

Then, $V := A/[A(\overline{Q}_1 \setminus \overline{Q_1^0})A, A]$ has a basis consisting of:

(1) Cyclic words in $\overline{Q}_1$ containing an arrow from $\overline{Q \setminus Q^0}$, such that maximal subwords from $\overline{Q \setminus Q^0}$ are as dictated by Proposition 7.0.1, and maximal subwords from $\overline{Q^0}$ are of the form $z_{a,b}$;
(2) Monomials of the form $z_{a,b}$,

and is free (cf. Proposition 7.0.1). Then, $\Lambda_Q \cong A/[A,A] \cong V/W$ where $W = \langle W_{a,b}\rangle_{a,b\geq 1,(a,b)\neq 1}$ is as described in Lemma 3.0.2. We have assumed that $Q \setminus Q^0$ is nonempty, so that $r' \neq 0$. Also, since r' is a sum of commutators, $[(r')^{p^\ell}]$ is a multiple of p (as a class of $F/[F,F]$, and hence in V). Then, the rest of the result follows immediately from (the proof of) Lemma 3.0.2. We note that we could have alternatively proved this result by presenting Π_Q as a case of Corollary 3.0.14 (using that $\Lambda_{Q\setminus Q^0,Q_0^0}$ is torsion free).

This finishes the proof of Theorem 9.1.2 and hence Theorem 1.1.4 in the $\tilde{A}_0$ case, which includes Conjecture 1.0.4 as a special case.

9.3. **Proof of Theorem 9.1.2.** The proof generalizes the argument of the previous subsection. When there is any chance of confusion, if $i \in Q_0^0$ and $m \in \mathbb{Z}$, we use underlined notation, $\underline{i+m} \in Q_0^0$, for the result of adding m, i.e., moving m steps in the counterclockwise direction. We can describe the path algebra $P_{\overline{Q^0}}$ as generated by Q_0^0, x, and y, with relations/conditions: (1) Q_0^0 are idempotents of degree zero; (2) x and y have degree 1; and (3) $ix = x(i+1)$, $(i+1)y = yi$, and $ixj = jyi = 0$ if $j \neq i+1$. As before, we can define $z_{a,b}$ by (3.0.3). Here x and y are given in (9.1.1).

We define $r_{Q_0^0} := \sum_{i\in Q_0^0} iri$ and $r' := r_{Q_0^0} - \sum_{a\in Q_1}(aa^* - a^*a)$ analogously to the case $n = 0$. Let $F = \Pi_{Q,Q_0^0}$ and $A = F/(\!(r_{Q_0^0})\!) = \Pi_Q$. We will compute W and the quotient V/W, in the notation of Proposition 7.0.5. Clearly, $iWj = iVj$ when $i \neq j$, so it suffices to consider $\bar{W} := \bigoplus_{i\in Q_0^0} iWi$.

In this case, $\bar{W}$ is generated by reducing the elements $[z_{a-1,b}, ix], [z_{a,b-1}, iy] \in P_{\overline{Q}}$ for all $a, b \in \mathbb{Z}_{\geq 0}$ with $n \mid |b-a|$, and all $i \in Q_0^0$, modulo $r_{Q_0^0} = xy - yx + r'$ and commutators of elements which include a multiple of $\langle Q \setminus Q^0\rangle$ (including r').

One finds relations similar to (3.0.4)–(3.0.6), keeping track of the idempotents Q_0^0, and substituting $r_{Q_0^0} - xy + yx$ for r. We will now see why our choice of the $z_{a,b}$'s (intended originally to carry over well to the present setting) is convenient, stemming from the property that $ixy = xyi$ for all $i \in Q_0^0$. Let us assume that $a > b$, since essentially the same relations result in the other case (and the torsion must be the same). We may assume that $n \mid (b-a)$, or else the bidegree (a,b)-part is zero. Let $\eta : A \twoheadrightarrow V$ be the projection. Since, writing $iz_{a,b} = t_1 t_2 \cdots t_{a+b}$ where $t_j \in \overline{Q_1^0}$, one has

$$\sum_{m=1}^{a+b} \eta([t_m t_{m+1} \cdots t_{m-2}, t_{m-1}]) = 0, \tag{9.3.1}$$

it follows (using $\eta([Ar'A, A]) = 0$) that

$$b\eta([iz_{a,b-1}, y]) + b\eta([(\underline{i+1})z_{a-1,b}, x]) + \sum_{m=0}^{a-b-1} \eta([(\underline{i+m})z_{a-1,b}, x]) = 0. \tag{9.3.2}$$

So, we need only compute the $\eta([iz_{a-1,b}, x])$, or equivalently, the $\eta([z_{a-1,b}, ix])$: (recall that Q_0^0 is considered as a $\mathbb{Z}/n$-torsor and $\underline{i+m}$ is the operation of adding $m \in \mathbb{Z}$ to $i \in Q_0^0$):

$$\eta([z_{a-1,b}, ix]) = [(\underline{i+1} - i)z_{a,b}] + [\sum_{c=0}^{b-1} (\underline{i - (a-b-2)}) r'(xy)^c x^{a-b-1} (xy)^{b-1-c} x]. \tag{9.3.3}$$

Using only $n-1$ of the above n relations for each fixed a, b, this can be interpreted as eliminating the classes $[iz_{a,b}]$ for all $i \in Q_0^0$ except any fixed vertex $i_0 \in Q_0^0$. To add the last relation in, we need only consider the sum of all n relations, which together with (9.3.2) (allowing us to divide by $\dfrac{b}{\gcd(\frac{a-b}{n}, b)}$) gives (3.0.4) for $a > b$ (the coefficients of Q_0^0 disappear as we are summing over all translations around the cycle: every vertex of Q_0^0 becomes $\mathbf{1}_{Q_0^0}$). The same argument works for $a < b$, and so the proof from §9.2 shows that Λ_Q has no torsion in these cases. Note that the $W_{a,b}$ remain nonzero in this case since $Q \neq Q^0$—even if there is only an additional arrow at one particular vertex $i \in Q_0^0$, there are terms in $W_{a,b}$ where r only appears adjacent to this vertex i.

In the case $a = b = m$, again we see that the quotient V/W is the same as eliminating $[iz_{m,m}]$ for all $i \neq i_0$, and considering only the relation $W_{a,a}$ from (3.0.4)–(3.0.6). For the same reasons as in §9.2, it follows that V/W has torsion $\mathbb{Z}/p$ in exactly those bidegrees (p^ℓ, p^ℓ) for p prime and $\ell \geq 1$, and the torsion is generated by the class $[r^{p^\ell}]/p \in P_{\overline{Q}}/[P_{\overline{Q}}, P_{\overline{Q}}]$. This implies parts (v)–(vii) of Theorem 9.1.2, which were all that remained to be proved.

9.4. **Hilbert series and** (6.3.5) **in the $\tilde{A}_n$ case.** In this section, we verify the Hilbert series of $\Lambda_{\tilde{A}_n}$ using our bases, and give a direct proof of the curious identity (6.3.5) from [EG06] in this case. We provide this since it is an easy consequence of Theorem 9.1.2 (which we just proved), and gives a different proof from what is found elsewhere.

First, from Theorem 9.1.2 we deduce

Proposition 9.4.1. The Hilbert series of $\Lambda_{\tilde{A}_{n-1}}$ and $i\Pi_{\tilde{A}_{n-1}}i$, for any vertex i, are given by

$$h(i\Pi_{\tilde{A}_{n-1}}i; t) = h(\Lambda_{\tilde{A}_{n-1}}; t) = \frac{1+t^n}{(1-t^2)(1-t^n)} = \frac{1-t^{2n}}{(1-t^2)(1-t^n)^2}. \tag{9.4.2}$$

It immediately follows that one has the formula

$$h(\Lambda_{\tilde{A}_{n-1}}; t)(1-t^2) = 1 + \frac{2t^n}{1-t^n}, \tag{9.4.3}$$

which we can use to verify the following formula for Hilbert series from [EG06] (using (6.3.4)):

$$h(\mathcal{O}(\Pi); t) = \prod_{m \geq 1} \frac{1}{(1-t^m)^{a_m}}, \tag{9.4.4}$$

$$\prod_{m \geq 1} \frac{1}{(1-t^m)^{a_m - a_{m-2}}} = \frac{1}{1-t^2} \cdot \prod_{m \geq 1} \frac{1}{\det(1 - t^m \cdot C + t^{2m} \cdot 1)}, \tag{9.4.5}$$

where C is the adjacency matrix of $\overline{\tilde{Q}}$, and $a_{-1} = a_0 = 0$. The element $1 - t \cdot C + t^2 \cdot \mathbf{1}$ is $1/t$ times the so-called "t-analogue of the Cartan matrix", $(1 + \frac{1}{t}) \cdot \mathbf{1} - C$. For $\tilde{A}_{n-1}$, one has

$$\det(1 - t \cdot C + t^2 \cdot \mathbf{1}) = (1-t^n)^2. \tag{9.4.6}$$

To verify (9.4.5), set $h((\Lambda_{\tilde{A}_{n-1}})_+;t)=\sum_m a_m t^m$; one then has from (9.4.3)

$$a_m - a_{m-2} = 2[n \mid m], \quad m \geq 3, \tag{9.4.7}$$

which implies the desired identity.

Since $h(i_0\Pi_{\tilde{A}_{n-1}} i_0;t) = h(\Lambda_{\tilde{A}_{n-1}};t)$, by (6.1.1), the a_m's above satisfy

$$1+\sum a_m t^m = \big(\frac{1}{1-t\cdot C+t^2\cdot \mathbf{1}}\big)_{i_0 i_0}. \tag{9.4.8}$$

That is, $a_m = \phi_m(C)_{i_0 i_0}$, where ϕ_m is the m-th Chebyshev polynomial of the second type. Since $\phi_m - \phi_{m-2} = \varphi_m$, a Chebyshev polynomial of the first type, our work above explicitly verifies the identity (6.3.5) from [EG06].

So, from our point of view, this identity is the fact that $\Lambda_{\tilde{A}_{n-1}} \cong i_0\Pi_{\tilde{A}_{n-1}} i_0$, together with the similarity between the identity (9.4.3) and the formula for the determinant of the t-analogue of the Cartan matrix, (9.4.6).

9.5. **Poisson structure on $i_0\Pi_Q i_0$ for $Q=\tilde{A}_{n-1}$.** For Q extended Dynkin, there is an injection $i_0\Pi_Q i_0 \hookrightarrow \Lambda_Q$, whose cokernel is isomorphic to the torsion of Λ_Q. Hence, the necklace Lie bracket on Λ_Q induces a bracket on $i_0\Pi_Q i_0$, obtained by taking the image in Λ_Q, applying the bracket, and then using the isomorphism $i_0\Pi_Q i_0 \cong \Lambda_Q/\text{torsion}$. It is clear that the result is a Lie bracket. Moreover, $i_0\Pi_Q i_0$ is commutative. Then, it follows from (8.2.7) that $i_0\Pi_Q i_0$ is actually a Poisson algebra, i.e., that the Leibniz identity is satisfied.

In the case $Q=\tilde{A}_{n-1}$, by Theorem 13.1.1, Λ_Q is torsion-free, so in this case $i_0\Pi_Q i_0 \to \Lambda_Q$ is an isomorphism. Here we describe explicitly the resulting Poisson structure.

The simplest way to understand $\Lambda_{\tilde{A}_{n-1}}$ is in terms of the basis $[i_0 x^a y^b]$, where x denotes moving clockwise one arrow, and y denotes moving counterclockwise, and i_0 is a fixed vertex. One requires that $n \mid (a-b)$, and $a,b \in \mathbb{Z}_{\geq 0}$. To compute the bracket, we first compute the bracket in terms of the rational basis $[\mathbf{1}x^a y^b]$, where $\mathbf{1}$ is the identity (the sum of all vertices); since there is no torsion in $\Lambda_{\tilde{A}_{n-1}}$, this suffices. Then, it immediately follows that one can compute the bracket (and cobracket) by summing over ways to pair letters x,y that correspond to opposite arrows. One easily computes that δ is zero. The bracket is then

$$[[i_0 x^a y^b],[i_0 x^c y^d]] = i_0 \frac{ad-bc}{n} x^{a+c-1} y^{b+d-1}. \tag{9.5.1}$$

In other words, the Poisson structure on $i_0\Pi_Q i_0$ is given by $\{i_0 z_{a,b}, i_0 z_{c,d}\} = \frac{ad-bc}{n} i_0 z_{a+c-1,b+d-1}$.

In terms of the isomorphism $\Lambda_{\tilde{A}_{n-1}} \xrightarrow{\sim} \Lambda_{\tilde{A}_{n-1}} \otimes 1 \subset \Lambda_{\tilde{A}_{n-1}} \otimes \mathbb{C} \xrightarrow{\sim} \mathbb{C}[x,y]^{\mathbb{Z}/n}$, one has integral basis elements x^n, y^n, xy, with Poisson bracket which is $\frac{1}{n}$ times the usual Poisson bracket on $\mathbb{C}[x,y]$, restricted to $\mathbb{C}[x,y]^{\mathbb{Z}/n}$.

Summarizing, we can give an explicit presentation of $i_0\Pi_Q i_0$ as a graded Poisson algebra over $\mathbb{Z}$ as follows:

Proposition 9.5.2. The following is an explicit presentation of $i_0\Pi_Q i_0$ for $Q=\tilde{A}_{n-1}$:

$$X := [i_0 x^n], \quad Y := [i_0 y^n], \quad Z := [i_0 xy], \tag{9.5.3}$$

$$i_0\Pi_Q i_0 \xleftarrow{\sim} \mathbb{Z}[X,Y,Z]/((XY - Z^n)), \tag{9.5.4}$$

$$\{X,Y\} = nZ^{n-1}, \quad \{X,Z\} = X, \quad \{X,Y\} = -Y, \tag{9.5.5}$$

$$|X| = n, \quad |Y| = n, \quad |Z| = 2. \tag{9.5.6}$$

We remark that $i_0\Pi_Q i_0$ can be thought of as "$\mathbb{Z}[x,y]^\Gamma$" $= \mathbb{C}[x,y]^\Gamma \cap \mathbb{Z}[x,y]$ (using $\frac{1}{n}$ times the standard bracket), and this also allows one to make sense of "$\mathbb{F}_p[x,y]^\Gamma$" for primes $p \mid n$. The same

comment will apply for other extended Dynkin quivers Q, when we explicitly compute $i_0\Pi_Q i_0$ as a Poisson algebra in those cases as well.

10. Quivers containing $\tilde{D}_n$

10.1. **Bases of Π_Q for type D quivers and refinement of Theorem 1.1.4.** Here we describe bases of Π_Q for extended Dynkin quivers of type D and quivers properly containing them, leading to Theorem 10.1.9 which implies Theorem 1.1.4 in the case of quivers containing a $\tilde{D}_n$ quiver, i.e., containing either multiple nodes (vertices of valence ≥ 3) or a node of valence 4. Together with the type $\tilde{A}$ case in the previous section, this proves Theorem 1.1.4 in all cases except star-shaped quivers with three branches.

We will need the following notation. Suppose that $Q^0 = \tilde{D}_n$ is drawn and oriented as follows:

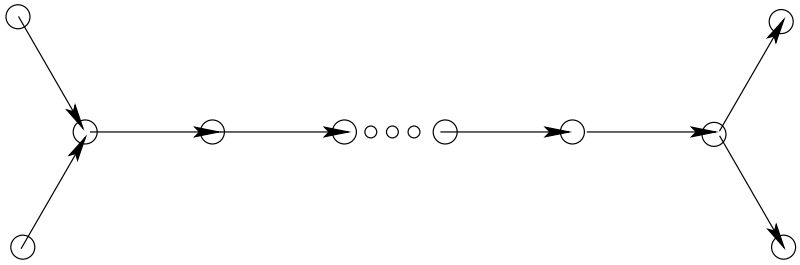

Figure 2. $Q^0 = \tilde{D}_n$ with the preferred orientation

As in the figure, we let $i_{LU}, i_{LD}, i_{RU}, i_{RD}$ denote the four external vertices (L, R, U, D stand for "left, right, up, down", respectively). Furthermore, we set $i_L := i_{LU} + i_{LD}$ and $i_R := i_{RU} + i_{RD}$, the sum of the leftmost and rightmost external vertices. We then define

$$\mathbf{1}_{\text{in}} := \sum_{i \text{ internal}} i, \tag{10.1.1}$$

the sum of internal vertices.

Next, we define

$$R := \sum_{a \in \overline{Q_1^0} | a \text{ is oriented rightward}} a, \quad L := \sum_{a \in \overline{Q_1^0} | a \text{ is oriented leftward}} a. \tag{10.1.2}$$

Here, "leftward" means an arrow which is either oriented from right to left in the diagram, or one whose terminal endpoint is one of the two leftmost vertices, i_{LU} and i_{LD}; similarly we define "rightward" to be all other arrows. One can think of R and L as analogous to x and y in the $\tilde{A}_{n-1}$ case (although the behavior is not exactly the same). The choice of orientation above is not essential, since none of the results (essentially) depend on it; in particular, the structures of Π_{Q^0} and Π_Q remain the same.

We will consider a slightly larger set of paths, called "generalized paths," $GP \subset P_{\overline{Q^0}}$ (we will also use the term for the image of such paths in Π_{Q^0} or Π_Q) as follows: Elements of GP_m are products of the form

$$i_1 X_1 i_2 X_2 \cdots i_m X_m i_{m+1}, \quad i_j \in Q_0^0 \cup \{i_L, i_R\}, \quad X_j \in \{L, R\}. \tag{10.1.3}$$

In other words, we allow products of not merely arrows, but also the sums of two external arrows which are pointed in the same left-right direction and share either the same initial vertex or share the same terminal vertex (note that the shared vertex is necessarily an internal vertex). Generalized paths have well-defined "generalized endpoints", which means either a vertex of Q_0^0 or one of the

elements i_L, i_R (the sum of the two left or right endpoints). In (10.1.3) these are given by i_1 and i_{m+1}.

It will be convenient to define "F," for "forward," as follows: if Y is any generalized path, then

$$YF := \begin{cases} YR, & \text{if } Y = Yi_L \text{ or } Y = Y'Ri \text{ where } i \in Q_0^0 \text{ is internal and } Y' \in GP, \\ YL, & \text{if } Y = Yi_R \text{ or } Y = Y'Li \text{ where } i \in Q_0^0 \text{ is internal and } Y' \in GP. \end{cases} \tag{10.1.4}$$

À priori, in order for symbols F to become elements of $P_{\overline{Q^0}}$, they must be multiplied on the left by a generalized path Y. It is easy to check, however, that one obtains a well-defined product $(P_{\overline{Q^0}})_+ \times \{F\} \to P_{\overline{Q^0}}$ by linearity (i.e., it is enough to multiply on the left by any positively-graded element of the path algebra).

We call an arrow "upward" if, in Figure 2, it goes diagonally up and to the left or diagonally up and to the right, i.e., it has either initiates at i_{LD} or i_{RD}, or else terminates at i_{LU} or i_{RU}. Similarly define "downward" to be an arrow going diagonally down and to the left or down and to the right.

Then, we define L_U and L_D by:

$$L_U := \sum_{a \in \overline{Q}_1 | a \text{ is leftward and not downward}} a, \quad L_D := \sum_{a \in \overline{Q}_1 | a \text{ is leftward and not upward}} a, \tag{10.1.5}$$

where the notation is chosen because, if P is a path, then PL_U, if nonzero, is a path of length one more which follows P by a leftward arrow which is not downward, i.e., either an arrow diagonally up and to the left, or a leftward arrow whose incident vertices are both internal; the similar statement holds for L_D.

We similarly define R_U and R_D to be the sum of all rightward arrows which are not downward and the sum of all rightward arrows which are not upward, respectively. Finally, define F_U by (10.1.4), replacing L and R by L_U and R_U, and similarly define F_D.

We now define our proposed basis elements, for any $c, C \geq 0$, and any choice of initial vertex i, terminal vertex j, and initial direction R or L:

$$z_{c,C,R,i,j} := \begin{cases} i(RL)^c j, & \text{if } C = 0, \\ i(RL)^c Rj, & \text{if } C = 1, \\ i(RL)^c R_U F_U^{C-2} F j, & \text{if } C \geq 2. \end{cases} \tag{10.1.6}$$

$$z_{c,C,L,i,j} := \begin{cases} i(LR)^c j, & \text{if } C = 0, \\ i(LR)^c Lj, & \text{if } C = 1, \\ i(LR)^c L_U F_U^{C-2} F j, & \text{if } C \geq 2. \end{cases} \tag{10.1.7}$$

Remark 10.1.8. The indices c, C in the $\tilde{D}_n$ case are analogous to the quantities $\min(a, b)$ and $|a-b|$ in the $\tilde{A}_{n-1}$ case: c gives the number of short cycles (RL or LR) and $\frac{1}{2(n-2)} \cdot C$ gives the number of long cycles (a power of f_U or f_D). We cannot keep the same notation in both cases because in the $\tilde{A}_{n-1}$ case, the winding number is a meaningful quantity (which is $\frac{1}{n} \cdot (a-b)$), whereas in the $\tilde{D}_n$ case, the only meaningful quantity is the number of short and long cycles: the starting direction R or L is only meaningful if beginning at an internal vertex, and the meaning is lost modulo commutators (or when passing to the center).

The main result of this section is:

Theorem 10.1.9. Let $Q^0 = \tilde{D}_n$. We use the notation of Proposition 7.0.5.

(i) A basis for $i\Pi_{Q^0} j$ for any $i, j \in Q_0^0$ consists of all nonzero elements having the form
   (a) $z_{c,C,R,i,j}$, for $c, C \geq 0$ and any i, j, or

(b) $z_{c,C,L,i,j}$, for $c, C \geq 0, i, j \in Q_0^0$ such that either $i \in \{i_{RU}, i_{RD}\}$ or $C \geq 1$.

In particular, for $i_0 := i_{LU}$, a basis of $i_0 \Pi_{Q^0} i_0$ is given by z_{c,C,R,i_0,i_0} for $c, C \geq 0$.

(ii) For any quiver $Q \supsetneq Q^0$, $W = W_1 \oplus W_2$, where $W_1 \subset W$ projects isomorphically under $V \twoheadrightarrow \Pi_{Q^0} \twoheadrightarrow \Pi_{Q^0}/\langle z_{c,C,R,i_{LU},i_{LU}}, z_{c,0,L,i_{RU},i_{RU}} \rangle_{c,C \geq 0}$ onto the (saturated) submodule

$$\bigoplus_{i \neq j} i\Pi_{Q^0} j \oplus \bigoplus_{i \in Q_0^0 | i \notin \{i_{LU}, i_{RU}\}} i\Pi_{Q^0} i \oplus \langle z_{c,C,L,i_{RU},i_{RU}} \rangle_{c \geq 0, C \geq 1}, \tag{10.1.10}$$

and W_2 has a basis of classes $W_{c,C}$, for $C = 2(n-2) \cdot C'$ and $c, C' \geq 0$, having the form

$$W_{c,C} := \sum_{c_\cdot, C_\cdot, i_\cdot, X_\cdot | (*)} \frac{\gcd(c, C')}{\mathrm{rep}(c_\cdot, C_\cdot)} [\prod_{m=1}^{|\{c_\cdot\}|} r' z_{c_m, C_m, X_m, i_m, i_{m+1}}] \quad (C > 0), \tag{10.1.11}$$

where $(*)$ is the condition that X_m always begin going forward:

$$\prod_{m'=1}^{m-1} z_{0,C_{m'},X_{m'},i_{m'},i_{m'+1}} F = \prod_{m'=1}^{m-1} z_{0,C_{m'},X_{m'},i_{m'},i_{m'+1}} X_m, \text{ and} \tag{10.1.12}$$

$$c = |\{c_\cdot\}| + \sum_m c_m, \quad C = \sum_m C_m; \tag{10.1.13}$$

finally,

$$W_{c,0} := (RL)^c - (LR)^c = i_L (RL)^c - i_R (LR)^c + \mathbf{1}_{\mathrm{in}}((RL)^c - (RL + r')^c)\mathbf{1}_{\mathrm{in}}. \tag{10.1.14}$$

(iii) The integral span of $W_{c,C}$ for $C > 0$ or $c \neq p^\ell$ for any prime p and $\ell > 1$ is saturated, and the order of $W_{p^\ell,0}$ is p (modulo the other $W_{c,C}$'s or otherwise).

(iv) The image of $\frac{1}{p} W_{p^\ell,0}$ is $r^{(p^\ell)}$ and these classes generate the torsion of Λ_Q (ranging over all p and ℓ).

Although we could have included a description of Λ_{Q^0} above, we relegate this to Theorem 13.1.1.

Note that, as in the A_n case, one way to obtain bases for Π_{D_n} is from the above basis. One can set $i_0 := i_{LD}$ and $D_n := \tilde{D}_n \setminus \{i_0\}$, and use those basis elements $z_{c,C,X,i,j}$ which are nonzero under the quotient $\Pi_{\tilde{D}_n} \twoheadrightarrow \Pi_{D_n}$, and which do not pass through i_{LU} (but may begin or end at i_{LU}).

10.2. **Proof of Theorem 10.1.9.(i).** Define a filtration on Π_{Q^0} by powers of the two-sided ideal $(\!(\mathbf{1}_{\mathrm{in}} RL \mathbf{1}_{\mathrm{in}}, \mathbf{1}_{\mathrm{in}} LR \mathbf{1}_{\mathrm{in}})\!)$. We will show that $\mathrm{gr}\, \Pi_{Q^0}$ is a free $\mathbb{Z}$-module with the given basis (more precisely, its image in $\mathrm{gr}\, \Pi_{Q^0}$), which shows that the same is true for Π_{Q^0}.

We begin by showing that the given elements integrally span Π_{Q^0} and $\mathrm{gr}\, \Pi_{Q^0}$. We first show that Π_{Q^0} and $\mathrm{gr}\, \Pi_{Q^0}$ are integrally spanned by a larger set: elements of the form $i(RL)^k Y$ or $i(LR)^k Y$ where Y is a path that satisfies the following property: The paths change from going to the left to right or vice-versa only at the four endpoints of $\tilde{D}_n$. This differs from the proposed basis only by allowing F_U's to change to F_D's, and replacing the first F_U or F by any of L_U, L_D, R_U,or R_D.

First, Π_{Q^0}, as well as $\mathrm{gr}\, \Pi_{Q^0}$, is obviously generated by elements $i(RL)^k Y$ or $i(LR)^k Y$ where Y is a generalized path. We can assume that Y does not begin with LR, RL, or any expression equal to one of these, since otherwise we could absorb this into the product $(LR)^k$ or $(RL)^k$: if Y begins at an internal vertex i, then $iLR = iRL$, whereas if Y begins at an external vertex, only one of iLR and iRL is nonzero. Next, we show that, if Y includes a change of left-right direction at a vertex other than an external one, then it is equivalent to an element that makes such a change of direction strictly earlier, up to elements of $(\!(\mathbf{1}_{\mathrm{in}} RL \mathbf{1}_{\mathrm{in}}, \mathbf{1}_{\mathrm{in}} LR \mathbf{1}_{\mathrm{in}})\!)^{k+1}$. Since we assume the change

of direction cannot happen at the beginning, this will prove the desired result (obviously, there are only finite number of possible expressions $(LR)^k Y, (RL)^k Y$ of a fixed length to consider).

Suppose that the first change of direction at an internal vertex (say j) is of the form $iLjR$ where i is also an internal vertex. Then, it must follow L, so that $LiLjR = LiLR = LiRL$, giving the desired element which changes direction at an internal vertex earlier. Similarly, we can handle $iRjL$ where i, j are internal.

Next, suppose that the first change of direction at an internal vertex (say j) is of the form $iLjR$ where i is external. Then it must be preceded by R_X for the appropriate $X \in \{U, D\}$, so we get $R_X LjR = R_X LR_{\overline{X}} = RLR_{\overline{X}}$ for $X \neq \overline{X} \in \{U, D\}$. If these three arrows are the first of Y, then we can absorb an additional RL (or LR) into the initial power of RL or LR. In this case, the element $i(RL)^k Y$ or $i(LR)^k Y$ lies in $\left((\mathbf{1}_{\text{in}} RL \mathbf{1}_{\text{in}}, \mathbf{1}_{\text{in}} LR \mathbf{1}_{\text{in}})\right)^{k+1}$, so we can discard it. Otherwise, these three arrows must be preceded by R, since we assumed this was the first change of direction. Then, we get $RiRLR_{\overline{X}} = RiLRR_{\overline{X}}$, again yielding an expression that changes direction at an internal vertex earlier.

This proves the desired claim. Now, to show that Π_{Q^0}, as well as $\operatorname{gr} \Pi_{Q^0}$, is integrally spanned by the elements stated in the theorem, we make two observations. First of all, if we have a nonzero expression $i(RL)^k Y$ for $k \geq 1$, where Y does not begin with R_U or R_D, then i must be an internal vertex and Y must begin with L_U or L_D. In this case, because $i(LR)^k = i(RL)^k$, this expression is already equal to $i(LR)^k Y$, so we don't need $i(RL)^k Y$ to integrally span Π_{Q^0}.

Next, take an expression of the form $i(RL)^k Y$ where $k \geq 1$ and Y begins with R_U or R_D. If Y includes L_D anywhere except at the very end, then it precedes R, and we may freely replace $L_D R$ with $LR - L_U R$. Since Y does not begin with L_D, the occurrence $L_D R$ in Y is preceded by L. Thus, we replace the resulting $LL_D R$ in Y by $LLR - LL_U R = LRL - LL_U R$. The resulting LRL term involves changing direction at an internal vertex, which we showed above must be in $\left((\mathbf{1}_{\text{in}} RL \mathbf{1}_{\text{in}}, \mathbf{1}_{\text{in}} LR \mathbf{1}_{\text{in}})\right)^{k+1}$. In other words, in $\operatorname{gr} \Pi_{Q^0}$, one can replace the L_D in Y by $-L_U$, and obtain the same element. Hence, the elements stated in the theorem integrally span $\operatorname{gr} \Pi_{Q^0}$, as well as Π_{Q^0}.

It remains to show linear independence of the given elements over $\mathbb{Z}$ (i.e., that they form a basis for the free $\mathbb{Z}$-module Π_{Q^0}). We do this by mirroring the above, but using the Diamond Lemma and a bit more careful analysis. For any generalized path $Y \in GP$, we can construct an alternating sequence

$$E(X) := (t_1, t_2, t_3, \ldots, t_m)_{t_\ell \in \{L,R\}}, \tag{10.2.1}$$

which we call "the ends of X", as follows: Start with the empty sequence, unless the initial vertex of X is an external one, in which case we start with the side L for left or R for right. Every time we hit the left end (a left external vertex or the superposition of both) we add an L to the sequence $E(X)$, unless $E(X)$ is already nonempty with last term equal to L. Every time we hit the right end, we add an R, unless the sequence is nonempty with last term R. That is, the sequence records the order in which the path hits left and right endpoints, throwing out multiple hits of one side before next hitting the opposite side. Put another way, it records which is the first side the path reaches, and the number of times it alternates from one side to another.

Now, for any $X \in GP \cap (P_{\overline{Q^0}})_k$ of length k, with $E(X) = (t_1, t_2, \ldots, t_m)$, we can write $X = X_1 i_{t_1} X_2 i_{t_2} \ldots i_{t_m} X_{m+1}$ in the unique way such that $X_1 \in P_{\overline{Q^0}}$ has minimal possible length, and for this value of X_1, X_2 has minimal possible length, etc.

Finally, let us consider the places $\ell \in \{1, 2, \ldots, k\}$ where X is "not going in the correct direction." By "not the correct direction at ℓ", we mean that the path is headed away from the next endpoint i_{t_m} (=towards the previous endpoint $i_{t_{m-1}}$) appearing in the path. Precisely, suppose that $X = a_1 a_2 \cdots a_k$ and X hits the endpoint i_{t_m}, at the time corresponding to when we add t_m to the sequence, between $a_{\ell_i - 1}$ and a_{ℓ_i} (i.e., if the last endpoint t_m is hit at the very end, set $\ell_m := k+1$).

Then, for all $\ell_i \leq \ell < \ell_{i+1}$, we say that ℓ is a place where X heads in the "wrong direction" if $a_\ell \neq t_{i+1}$. Now, for any $X \in GP \cap (P_{\overline{Q^0}})_k$ of the form $X = iXj$ for some $i, j \in Q_0^0$, let $WD(X) \subset \{1, 2, \ldots, k\}$ be the subset of places where X goes in the wrong direction. One easily sees that the size of $WD(X)$ is the difference between k and the shortest path with the same endpoints as X (this makes sense since we assumed X had definite endpoints).

Now, we may define $X \prec Y$ if either $|WD(X)| > |WD(Y)|$, or else $|WD(X)| = |WD(Y)|$ and there is an order-preserving bijection $\phi : WD(X) \to WD(Y)$ such that $\phi(c) \leq c$ for all $c \in C(X)$. That is, the places where X goes in the "wrong direction" occur strictly before the corresponding places in Y.

Now, we finally have the ordering we need such that the relations on $\langle GP \cap (P_{\overline{Q^0}})_k \rangle$ modulo which one gets $(\Pi_{Q^0})_k$ are confluent and give the desired elements as the normal form (one reduces as in the first part of this proof). So, the desired elements are linearly independent and form a basis of Π_{Q^0}. Also, since their images integrally span $\operatorname{gr} \Pi_{Q^0}$, they form a basis of $\operatorname{gr} \Pi_{Q^0}$ as well, which is therefore also a free $\mathbb{Z}$-module.

Similarly to the $\tilde{A}_{n-1}$ case, it is not difficult to deduce from the above procedure that Π_{Q^0} is an NCCI (that one has a unique reduction of $P_{\overline{Q^0}}$ to words in the above basis and the relations iri). Also, it is clear from the above proof that for any of the basis elements, we can feel free to replace any of the F_U's in the $z_{c,C,X,i,j}$ by F_D's, and we will still be left with a basis.

10.3. **Proof of the remainder of Theorem 10.1.9.** Suppose Q is any quiver with a proper subquiver $Q^0 \subsetneq Q$ with $\overline{Q^0} \cong \overline{\tilde{D}_n}$. Define $r_{Q_0^0} := \sum_{i \in Q_0^0} iri$. Fix a forest $G \subset \overline{Q}_1 \setminus \overline{Q_1^0}$ as in Proposition 7.0.1.

As in the $\tilde{A}$ case, and using the same notation V, W, and A, it suffices to compute $\bar{W} := \bigoplus_{i \in Q_0^0} iWi \subset V$, which is integrally spanned by $\eta([X, e])$ for X listed in Theorem 10.1.9.(i) and $a \in \overline{Q_1^0}$ such that $Xa \in \bar{\Pi}_Q := \sum_{i \in Q_0} i\Pi_Q i$ (cf. Proposition 7.0.5.(iii)). We can instead let a be one of L_U, L_D, R_U, or R_D.

First, we need to eliminate some of the commutators $\eta([X, a])$. We prefer to eliminate those $X = z_{c,C,X,i,j}$ with smaller c, so as to maximize the c of the remaining commutators. Essentially, this means continuing to use the filtration on A by powers of the ideal (LR, RL).

First note the basic

Lemma 10.3.1. If X is obtained from $z_{c,C,X,i,j}$ by changing α instances of $R_U L$ to $R_D L$ or $L_U R$ to $L_D R$, then $X \equiv (-1)^\alpha z_{c,C,X,i,j}$ modulo $\langle z_{c',C',X,i,j} \rangle_{c'>c}$.

Proof. This follows from the fact that $R_U L + R_D L = RL$, using the ideas in the proof of Theorem 10.1.9.(i) (in the last subsection). □

We use this to write some relations which are deduced similarly to (9.3.1), (9.3.2). These will allow us to express certain commutators in terms of commutators involving basis elements living in a smaller power of $(\!(\mathbf{1}_{\text{in}} RL \mathbf{1}_{\text{in}}, \mathbf{1}_{\text{in}} LR \mathbf{1}_{\text{in}})\!)$.

Notation 10.3.2. We say that an equality holds "modulo commutators with higher powers of $(\!(\mathbf{1}_{\text{in}} RL \mathbf{1}_{\text{in}}, \mathbf{1}_{\text{in}} LR \mathbf{1}_{\text{in}})\!)$" if the equation is true up to η of commutators with elements $z_{c',C',X',i',j'}$, where c' is greater than all the indices c which appear in the equation.

Take C such that $2(n-2) \mid C$ and $C > 0$. This is equivalent to the condition that a basis element of the form $z_{c,C,X,i,i}$ has either (1) i is internal and $z_{c,C,X,i,i} = z_{c,C-1,X,i,j} X$ for some j (i.e., it ends in X); or (2) i is external. Pick any choice of i, X such that $z_{c,C,X,i,i}$ is a basis element; let $c > 0$. Let $\tilde{X}$ be defined to be X when i is internal (first case), and $\overline{X}$ otherwise (second case), where here and below $\overline{X} \in \{L, R\}$ denotes the opposite direction from $X \in \{L, R\}$. That is, $\tilde{X}$ is the last

direction in $z_{c,C,X,i,i}$. One then computes, using (9.3.1), Lemma 10.3.1, and Notation 10.3.2, for $c \geq 1$,

$$\sum_j c\eta([z_{c-1,C+1,X,i,j}, j\overline{X}i]) + c\eta([z_{c,C-1,\tilde{X},j,i}, iXj]) + \frac{C}{2(n-2)} \sum_{i',j' \in Q_0^0 \setminus \{i_{rd}, i_{\ell d}\}, Y \in \{L,R\}} (-1)^{c\delta_{\overline{X},Y}} \eta([z_{c,C-1,Y,i',j'}, j'(L+R)i']) = 0 \tag{10.3.3}$$

modulo commutators with higher powers of $(\!(\mathbf{1}_{\text{in}}RL\mathbf{1}_{\text{in}}, \mathbf{1}_{\text{in}}LR\mathbf{1}_{\text{in}})\!)$,

which shows that we can eliminate the relations $\sum_j \eta([z_{c-1,C+1,X,i,j}, j\overline{X}i])$. In terms of commutators of basis elements with arrows, we do the following: In the case that i is not adjacent to external vertices by an arrow moving in the X direction (beginning with i), this sum is only over a single value of j, so we eliminate the relation $\eta([z_{c-1,C+1,X,i,j}, j\overline{X}i])$. Otherwise, the first sum is over two possible values of j, then we can choose to eliminate the relation $\eta([z_{c-1,C+1,X,i,i_{XD}}, i_{XD}\overline{X}i])$. Note in the above that the sums over all i', j' are effectively only over adjacent pairs i', j', and the $j'(L+R)i'$ is shorthand for the unique arrow in $\overline{Q^0}$ from j to i.

Also, the relation $\eta([z_{c-1,C+1,X,i,i_{XU}}, i_{XU}\overline{X}i])$ we left above can be interpreted simply as expressing $z_{c-1,C+2,X,i,i}$ as $z_{c,C,X,i_{XU},i_{XU}}$ plus some multiple of $r' = r_{Q_0^0} - \sum_{a \in Q_1^0}(aa^* - a^*a)$ in the quotient V/W. So we can also eliminate this relation if we also eliminate the generator $z_{c-1,C+2,X,i,i}$ from V. Actually, this paragraph is still true if $C = 0$, so we can also eliminate the relation $\eta([z_{c-1,1,X,i,i_{XU}}, i_{XU}\overline{X}i])$ and the generator $z_{c-1,2,X,i,i}$ (for i adjacent to external vertices by arrows of the form iX_Uj or iX_Dj).

Next, consider a basis element $z_{c,C,X,i,i}$ such that $2(n-2) \nmid C$ and $2(n-2) \nmid (C-2)$ (we dealt with the other cases in the last paragraph). In particular, i is internal and not adjacent to external vertices by moving in the X direction. Let $g = g(X,i)$ be defined to be the distance from i to the X-end. One has $g = \frac{C}{2} - (n-2)\lfloor \frac{C}{2(n-2)} \rfloor$ since the F_U^C-portion of the element $z_{c,C,X,i,i}$ goes forward g units, cycles $\lfloor \frac{C}{2(n-2)} \rfloor$ times around the long distance of $\tilde{D}_n$, and then moves g units forward back to i. We will show that (1) $\eta([z_{c,C-1,X,i,j}, j\overline{X}i]) \equiv -\eta([z_{c,C-1,\overline{X},j,i}, iXj])$ modulo commutators with higher powers (in particular, modulo terms that begin and end closer to the external vertices on the X-side than i), and that (2) either of these (sums of) commutators can be taken to express $z_{c,C,X,i,i}$ in terms of $z_{c+m,C-2m,X,j',j'}$ for $m > 0$ (and $g(X,j') < g(X,i)$); eventually this will reduce us to basis elements beginning and ending at an external vertex.

(1) follows from the identity

$$[z_{c,C-1,X,i,j}, j\overline{X}i] + [j(\overline{X}X)^c z_{0,C-1,X,j,i}, iXj] = [j\overline{X}Xj, z_{c,C-2,X,j,j}], \tag{10.3.4}$$

and (2) follows because $\eta([z_{c,C-1,X,i,j}, j\overline{X}i]) - (z_{c,C,X,i,i} - z_{c+1,C-2,X,j,j}) \in [r'A]$.

Similarly to (10.3.4), for $C > 2$ we can also show that $\eta([z_{c,C-1,\overline{X},i,i_{XD}}, i_{XD}\overline{X}i])$ is in the integral span of other commutators (note that $2(n-2) \mid C$ for this to be nonzero):

$$[i(X\overline{X})^c z_{0,C-1,\overline{X},i,i_{XD}}, i_{XD}\overline{X}i] + [i(X\overline{X})^c z_{0,C-1,\overline{X},i,i_{XU}}, i_{XU}\overline{X}i] + [z_{c,C-1,\overline{X},i_{XD},i}, iXi_{XD}] + [z_{c,C-1,\overline{X},i_{XU},i}, iXi_{XU}] = [(X\overline{X})^c z_{0,C-2,\overline{X},i,i}, X\overline{X}], \tag{10.3.5}$$

which shows that

$$\eta([z_{c,C-1,\overline{X},i,i_{XD}}, i_{XD}\overline{X}i]) + \eta([z_{c,C-1,\overline{X},i,i_{XU}}, i_{XU}\overline{X}i]) + \eta([z_{c,C-1,\overline{X},i_{XD},i}, iXi_{XD}]) + \eta([z_{c,C-1,\overline{X},i_{XU},i}, iXi_{XU}]) \in \eta([(\!(\mathbf{1}_{\text{in}}RL\mathbf{1}_{\text{in}}, \mathbf{1}_{\text{in}}LR\mathbf{1}_{\text{in}})\!)^{c+1}, \Pi_Q]), \tag{10.3.6}$$

so we can indeed throw out any one of these commutators: we choose $\eta([z_{c,C-1,\overline{X},i,i_{XD}}, i_{XD}\overline{X}i])$.

The only other commutators of the form $\eta([X,e])$, with X as in Theorem 10.1.9.(i) such that Xe is nonzero and in $\bar{\Pi}_Q$, that we have not yet mentioned, are those of the form $\eta([z_{c,1,X,i,j}, j\overline{X}i])$ where j is an internal vertex. Here we can make use of the identity, similar to the above: if i and j are internal, then

$$\eta([z_{c,1,X,i,j}, j\overline{X}i]) = \eta([z_{c,1,\overline{X},j,i}, iXj]), \tag{10.3.7}$$

and otherwise,

$$\sum_{Y\in\{U,D\}} \eta([z_{c,1,\overline{X},i_{XY},j}, jXi]) = \sum_{Y\in\{U,D\}} \eta([z_{c,1,X,j,i_{\overline{X}Y}}, i_{\overline{X}Y}\overline{X}j]). \tag{10.3.8}$$

Thus, we can eliminate the commutators $\eta([z_{c,1,L,i,j}, jRi])$ for $i \neq i_{RD}, j \neq i_{LD}$. Furthermore, in the cases $i = i_{RD}$ and $j = i_{LD}$, these commutators allow one to express $z_{c,0,L,i_{RD},i_{RD}}$ in terms of $z_{c,0,L,i_{RU},i_{RU}}$, and $z_{c,0,R,i_{LD},i_{LD}}$ in terms of $z_{c,0,R,i_{LU},i_{LU}}$, so we can eliminate all commutators $\eta([z_{c,1,L,i,j}, jRi])$ along with the generators $z_{c,0,\overline{X},i_{XD},i_{XD}}$ for $X \in \{L,R\}$. To summarize the result of the computation (with parenthetical English versions):

(1) We can eliminate $z_{c,C,X,i,i}$ from the basis if i is internal and $2(n-2) \nmid C$, thus passing to a submodule $V_0 \subset V$ (that is, we consider only cyclic paths that are a combination of length-two cycles at a vertex and long cycles around the length of $\tilde{D}_n$);
(2) The only commutators we then need to consider, in order to integrally span $W \cap V_0$ (up to dividing certain commutators by certain integers), and thus present $\Lambda_Q \cong V_0/(W \cap V_0)$, are
   (a) $\eta([z_{c,C-1,X,i,j}, j(L+R)i])$ for $2(n-2) \mid C$ and $j \notin \{i_{LD}, i_{RD}\}$ (we only need consider commutators obtained from the previous elements by separating the last arrow off and writing the commutator), and
   (b) $\eta([z_{c,1,R,i,j}, jLi])$.

In computing the remaining commutators, we use the following basic identities, which are similar to those for $\tilde{A}_0, \tilde{A}_n$: First, note that, by the choice of orientation of $\tilde{D}_n = Q^0$ in the previous subsection, in $A = \Pi_Q$, one has $\mathbf{1}_{\mathrm{in}}(LR-RL)\mathbf{1}_{\mathrm{in}} = \mathbf{1}_{\mathrm{in}}(LR-RL+r_{Q_0^0})\mathbf{1}_{\mathrm{in}} \in \langle \overline{Q}_1 \setminus \overline{Q_1^0}\rangle^2$. So, letting $r' := r_{Q_0^0} - \sum_{a\in Q_1^0}(aa^* - a^*a)$ (as before),

$$\begin{aligned} R_X(LR)i_R = (R_XLR)i_R = (RLR_{\overline{X}} + R_XLR_X - R_{\overline{X}}LR_{\overline{X}})i_R \\ = ((RL)R_{\overline{X}} - R_{\overline{X}}r')i_{R\overline{X}} + R_Xr'i_{RX}; \end{aligned} \tag{10.3.9}$$

$$R(RL)\mathbf{1}_{\mathrm{in}} = ((RL)R - Rr')\mathbf{1}_{\mathrm{in}}; \tag{10.3.10}$$

and similarly swapping left with right, and the element $r' \in \Pi_Q$ with $-r'$. We can use these to move an R (or L) past a power of RL (or LR), thus allowing one to compute $\eta([z_{c,C-1,X,i,j}, jRi])$ for $X = R$ and i internal, or for $X = L$ and i a right external vertex (and similarly swapping left with right). For i internal and $j \neq i_{LD}$ a vertex adjacent to i on the left, one obtains

$$\eta([z_{c,C-1,R,i,j}, jRi]) = z_{c,C,R,i,i} - z_{c,C,R,j,j} + \sum_{0\le c'<c} [r'z_{c',C-1,R,i,j}z_{c-c'-1,1,R,j,i}]. \tag{10.3.11}$$

(We already noted that we can discard the relation in the case $j = i_{LD}$.)

For $i \in \{i_{RU}, i_{RD}\}$ right external, one obtains the following. We use the simplified notation $i^{(c)} := \begin{cases} i, \text{if } c \text{ is even}, \\ \bar{i}, \text{if } c \text{ is odd}, \end{cases}$ where $\bar{i}$ is the other right external vertex from i. We'll need the same

definition for left external vertices later.

$$\eta([z_{c,C-1,L,i,j}, jRi]) = z_{c,C,L,i,i} - \eta(z_{c,1,R,j,i^{(c)}} z_{0,C-1,L,i^{(c)},j}) + \sum_{c'<c, \epsilon\in\{0,1\}} (-1)^{1+c'+\epsilon}[r' z_{c',C-1,L,i^{(\epsilon)},j} \cdot z_{c-c'-1,1,R,j,i^{(\epsilon)}}]; \tag{10.3.12}$$

We simplify the RHS by noting that

$$\eta(z_{c,1,R,j,i^{(c)}} z_{0,C-1,L,i^{(c)},j}) = \begin{cases} z_{c,C,R,j,j}, & \text{if } i^{(c)} = i_{RU}, \\ -z_{c,C,R,j,j} + Y, & \text{if } i^{(c)} = i_{RD}, \end{cases} \tag{10.3.13}$$

where $Y \in \langle z_{c+m(n-2),C-2m(n-2),R,j,j}\rangle_{m\geq 1}$ (that is, Y consists of "higher powers of $(\!(\mathbf{1}_{\text{in}}RL\mathbf{1}_{\text{in}}, \mathbf{1}_{\text{in}}LR\mathbf{1}_{\text{in}})\!)$").

We can thus consider the above relations (10.3.11),(10.3.12) as eliminating for each a, b the basis elements $z_{c,C,X,i,i}$ (where i is not lower external) for all but a single pair (X_0, i_0) where $i_0 \notin \{i_{RD}, i_{LD}\}$, leaving only a single relation. We can conveniently choose this relation to be the sum of (10.3.11) and (10.3.12) over all $i \notin \{i_{RD}, i_{LD}\}$ (with $j \neq i_{LD}$ for (10.3.11)), together with $(-1)^c$ times the sum of the commutators obtained from these by swapping left and right. This is zero if $c = 0$, so we can assume $c \geq 1$. By (10.3.3), to include all of the commutators we discarded, we need only divide the resulting relation by $\dfrac{c}{\gcd(c, \frac{C}{2(n-2)})}$.

We thus get the single relation (for all choices of $c, C \geq 1$ such that $2(n-2) \mid C$):

$$\frac{\gcd(c, \frac{C}{2(n-2)})}{c} \sum_{i,j\notin\{i_{LD},i_{RD}\}} \eta([z_{c,C-1,R,i,j}, jLi] + (-1)^c[z_{c,C-1,L,i,j}, jRi]) = \tag{10.3.14}$$

$$\frac{\gcd(c, \frac{C}{2(n-2)})}{c} \sum_{c',i,X;j\notin\{i_{LD},i_{RD}\}} (-1)^{c\delta_{X,L}+(c-c')(\delta_{i,i_{\overline{X}D}}+\delta_{i,i_{\overline{X}U}})+\delta_{i,i_{\overline{X}D}}} [r' z_{c',C-1,X,i,j} z_{c-c'-1,1,\overline{X},j,i}] + Y,$$

where Y is in the integral span of products of terms $z_{c',C',X,i',j'}$ and r', such that the sum of C'-degrees is strictly less than b. That is, Y is the image of a greater power of the ideal $(\!(\mathbf{1}_{\text{in}}RL\mathbf{1}_{\text{in}}, \mathbf{1}_{\text{in}}LR\mathbf{1}_{\text{in}})\!)$ than c (as is $z_{c,C,X,i,j}$ for any X, i, j). In other words, if we pass to the associated graded algebra of Π_Q with respect to the filtration by powers of the ideal $(\!(\mathbf{1}_{\text{in}}RL\mathbf{1}_{\text{in}}, \mathbf{1}_{\text{in}}LR\mathbf{1}_{\text{in}})\!)$ considered earlier, and the associated graded $\mathbb{Z}$-module with respect to the image of this filtration in $\bar{\Pi}_Q/([\bar{\Pi}_Q(\overline{Q}_1 \setminus \overline{Q}_1^0)\bar{\Pi}_Q, \bar{\Pi}_Q] \cap \bar{\Pi}_Q)$, then we can eliminate the Y term and have an equality above. In any case, by induction on filtration degree, we will see that one can neglect the Y portion.

We then expand the RHS in terms of the basis of V previously described: (1) the elements $z_{c,C,X,i,j}$ from Theorem 10.1.9.(i); and (2) cyclic alternating products of elements $z_{c,C,X,i,j}$ and monomials in $\overline{Q}_1 \setminus \overline{Q}_1^0$ not containing aa^* for any $a \in G$ (the forest we picked as in Proposition 7.0.1). This results in the expression

$$\frac{\gcd(c, \frac{C}{2(n-2)})}{c} \sum_{(c.,C.,X.,i.,j.)|(*)} (c_1+1) s(c., C., i., X.)[\prod_{m=1}^{|\{c.\}|} r' z_{c_m,C_m,X_m,i_m,i_{m+1}}] + Y, \tag{10.3.15}$$

$$s(c., C., i., X.) := (-1)^{c\delta_{X_1,L}} \prod_{m=1}^{|\{c.\}|} (-1)^{(\delta_{i_m,i_{RD}}+\delta_{i_m,i_{LD}})+c_m(\sum_{k\geq m} B_k)}, \tag{10.3.16}$$

where $B_k = \#$ of times that an arrow of the form $R_Z i_{RZ}$ or $L_Z i_{LZ}$ appears in the factor $z_{c_k,C_k,X_k,i_k,i_{k+1}}$, and $Y \in (\!(\mathbf{1}_{\text{in}}RL\mathbf{1}_{\text{in}}, \mathbf{1}_{\text{in}}LR\mathbf{1}_{\text{in}})\!)^{c+1}$.

where the $(*)$ in the first sum indicates that we sum only over distinct $|\{c.\}|$-tuples of elements of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \{L, R\} \times Q_0^0 \times Q_0^0$ such that $\sum c_m = c - |\{c.\}|, \sum C_m = C, C_m \geq 1, \forall m$, and X_{m+1} is always a forward direction that follows $z_{c_m,C_m,X_m,i_m,i_{m+1}}$ (which can be upward or downward). We take indices modulo the length of the tuples (e.g. $i_{|\{c.\}|+1} = i_1$).

Since $\sum_{m=1}^{|\{c.\}|} B_m$ is always even (because $2(n-m) \mid C$), when we pass to summing only over distinct **cyclic** $|\{c.\}|$-tuples, all of the contributing terms have the same sign, leaving us with

$$\sum_{(c.,C.,X.,i.,j.)|(*)} \pm \frac{\gcd(c, \frac{C}{2(n-2)})}{\mathrm{rep}(c., C., X., i., j.)} [\prod_{m \in \mathbb{Z}/|\{c.\}|} r' z_{c_m,C_m,X_m,i_m,i_{m+1}}] + Y, \tag{10.3.17}$$

which now so closely resembles (3.0.4)–(3.0.6) that it is straightforward to conclude that there is no torsion in the portion of Λ_Q corresponding to $C \geq 1$. To be precise, consider the grading on Π_Q and Λ_Q by length of paths, so that $(\Pi_Q)_m, (\Lambda_Q)_m$ denote the $\mathbb{Z}$-submodules integrally spanned by paths of length m. Consider the filtration by powers of the ideal $(\!(\mathbf{1}_{\mathrm{in}} RL \mathbf{1}_{\mathrm{in}}, \mathbf{1}_{\mathrm{in}} LR \mathbf{1}_{\mathrm{in}})\!)$ in Π_Q, and the filtration by the image of these powers in Λ_Q. Then we have the associated graded $\mathbb{Z}$-modules, $\mathrm{gr}\, \Pi_Q = \bigoplus_g \mathrm{gr}_g \Pi_Q$ and $\mathrm{gr}\, \Lambda_Q = \bigoplus_g \mathrm{gr}_g \Lambda_Q$. Thus, $\mathrm{gr}_g \Pi_Q = (\!(\mathbf{1}_{\mathrm{in}} RL \mathbf{1}_{\mathrm{in}}, \mathbf{1}_{\mathrm{in}} LR \mathbf{1}_{\mathrm{in}})\!)^g / (\!(\mathbf{1}_{\mathrm{in}} RL \mathbf{1}_{\mathrm{in}}, \mathbf{1}_{\mathrm{in}} LR \mathbf{1}_{\mathrm{in}})\!)^{g+1}$, and $\mathrm{gr}_g \Lambda_Q$ is its quotient by the integral span of commutators.

Then, the precise statement we infer from the above is that there is no torsion in $(\mathrm{gr}_g \Lambda_Q)_m$ except possibly when $2g = m$. This is the maximum possible value of g, so that $(\mathrm{gr}_g \Lambda_Q)_{2g} = [(\!(\mathbf{1}_{\mathrm{in}} LR \mathbf{1}_{\mathrm{in}}, \mathbf{1}_{\mathrm{in}} RL \mathbf{1}_{\mathrm{in}})\!)^g]_{2g}$ is actually a submodule of $(\Lambda_Q)_{2g}$. Therefore it has torsion if and only if $(\Lambda_Q)_{2g}$ itself does.

It remains to see if the module $[(\!(\mathbf{1}_{\mathrm{in}} LR \mathbf{1}_{\mathrm{in}}, \mathbf{1}_{\mathrm{in}} RL \mathbf{1}_{\mathrm{in}})\!)^g]_{2g}$ has torsion. We can present the module as follows. It is integrally spanned by $[z_{c,0,R,i,i}], [z_{c,0,L,i_{RY},i_{RY}}]$, and multiples of $\overline{Q}_1 \setminus \overline{Q_1^0}$ (with each $z_{c_t,0,X_t,i,i}$ having $X_t = R$ unless i is right external, in which case each $X_t = L$). The relations are spanned by $\eta([z_{c-1,1,R,i,j}, jLi])$ for $j \neq i_{RD}, i \neq i_{LD}$ (we can ignore other commutators by our initial arguments), and by $(\sum_{i \in Q_0 \setminus Q_0^0} iri)$. When i and j are internal vertices, we can view this as eliminating $z_{c,0,R,i,i}$ for all internal vertices except one, and expressing the other in terms of that one and multiples of r'. This leaves us with only two relations. Let j be the internal vertex adjacent to the left end. For convenience, let us replace $\eta([z_{c-1,1,R,i_{LU},j}, jLi_{LU}])$ by $\eta([z_{c-1,1,R,i_{LU},j}, jLi_{LU}]) + \eta([z_{c-1,1,R,i_{LD},j}, jLi_{LD}])$. (The added term is exactly what is used to eliminate the $z_{c,2,L,j,i_{LY}}$ we would otherwise need as a generator, cf. (10.3.3) and the following paragraphs.) Then, this relation can be viewed as expressing $z_{c,0,R,j,j}$ in terms of $z_{c,0,R,i_{LU},i_{LU}}$, since we already expressed $z_{c,0,R,i_{LD},i_{LD}}$ in terms of the latter. This eliminates the generator $z_{c,0,R,i,i}$ for the last remaining internal vertex i, since we may take $i = j$.

To express the final relation in terms of remaining basis elements, we take the sum over all adjacent i, j of $\eta([z_{c-1,1,R,i,j}, jLi])$. The result is

$$\sum_{i \in Q_0^0} \eta(i((RL)^c - (LR)^c)i) = \sum_{i \in Q_0^0 | i \text{ is internal}} [i((RL)^c - (RL + r')^c)i] + \sum_{X \in \{U,D\}} (i_{LX}(RL)^c i_{LX} - i_{RX}(LR)^c i_{RX}), \tag{10.3.18}$$

where we used the non-basis elements $z_{c,0,R,i_{LD},i_{LD}}$ and $z_{c,0,L,i_{RD},i_{RD}}$ for convenience, which can be replaced by basis elements with only a slight modification.

The result is nonzero, since there exists a vertex $i \in Q_0^0$ such that i is adjacent to an arrow of $\overline{Q \setminus Q^0}$; in this case, for every proper factor $1 \neq m \mid c$, one has the term $[i((RL)^{\frac{c}{m}-1} r')^m i]$, nonzero and independent of other terms in the expansion, with coefficient m. Hence, the gcd of all coefficients is one unless c is a prime power. We conclude that there is only torsion in Λ_Q in degrees $2p^k$ for p prime and $k \geq 1$, where it is generated by the single relation (10.3.18). For much

the same reason as in the $\tilde{A}_{n-1}$-case (by the similarity of (10.3.18) and (3.0.4)–(3.0.6); see the end of the proof for $\tilde{A}_{n-1}$), the torsion is a single copy of $\mathbb{Z}/p$, generated by $\frac{1}{p}[r^{p^k}]$.

This completes the proof of Theorem 10.1.9.

10.4. **Hilbert series for $\tilde{D}_n$ and** (6.3.5)**.** From Theorem 10.1.9, we deduce the

Proposition 10.4.1. The Hilbert series of $i_0\Pi_{\tilde{D}_n}i_0$ over any field, for i_0 the extending vertex (i.e. an external vertex) of $\tilde{D}_n$, and the Hilbert series of $\Lambda_{\tilde{D}_n}$ over characteristic zero, are given by

$$h(i_0\Pi_{\tilde{D}_{n-1}}i_0;t) = h(\Lambda_{\tilde{D}_n};t) = \frac{1}{(1-t^{2n-4})(1-t^2)} - \frac{t^2}{1-t^4} = \frac{1+t^{2n-2}}{(1-t^4)(1-t^{2n-4})}. \tag{10.4.2}$$

It immediately follows from the above that one has the formula

$$h(\Lambda_{\tilde{D}_n};t)(1-t^2) = 1 + \frac{t^{2n-4}}{1-t^{2n-4}} + \frac{2t^4}{1-t^4} - \frac{t^2}{1-t^2}, \tag{10.4.3}$$

which we can use to verify the formulas (9.4.4),(9.4.5). Namely, letting C be the adjacency matrix of $\overline{\tilde{D}_n}$, one has

$$\det(1 - t\cdot C + t^2\cdot \mathbf{1}) = \frac{(1-t^{2n-4})(1-t^4)^2}{1-t^2}. \tag{10.4.4}$$

One then has from (10.4.3)

$$a_m - a_{m-2} = [(2n-4) \mid m] + 2[4 \mid m] - [2 \mid m], \quad m \geq 3, \tag{10.4.5}$$

which implies the desired identity (9.4.5) and verifies (6.3.5) for $\tilde{D}_n$.

10.5. **The Poisson algebra $i_0\Pi_Q i_0$ for $Q = \tilde{D}_n$.** As in §9.5, the Lie structure on Λ_Q can be described in terms of a Poisson algebra structure on $i_0\Pi_Q i_0$. In particular, by Theorem 13.1.1 (which does not rely on the results of this subsection) we can define a Lie bracket on $i_0\Pi_Q i_0$ using the projection $q : (\Lambda_Q)_+ \twoheadrightarrow (i_0\Pi_Q i_0)_+$ and inclusion $j : i_0\Pi_Q i_0 \hookrightarrow \Lambda_Q$ by the formula

$$\{f,g\} = q\{j(f), j(g)\}. \tag{10.5.1}$$

It follows as before from (8.2.7) that this endows $i_0\Pi_Q i_0$ with the structure of a Poisson algebra. In view of Proposition 8.1.9 and Remark 8.1.10, in fact the Poisson structure on $i_0\Pi_Q i_0$ entirely captures the Lie algebra structure on Λ_Q.

We then compute the following:

Proposition 10.5.2. Let $Q = \tilde{D}_n$, and let $i_0 := i_{LU}$ be the upper-left external vertex. Set $X := i_0 z_{2,0} i_0, Y := i_0 z_{0,2(n-2)} i_0$, and $Z := i_0 z_{1,2(n-2)} i_0$. Then, one has

$$i_0\Pi_Q i_0 \xleftarrow{\sim} \mathbb{Z}[X,Y,Z]/((Z^2 + XY^2 - \delta_{2|n}X^{\frac{n}{2}}Y - \delta_{2\nmid n}X^{\frac{n-1}{2}}Z)), \tag{10.5.3}$$

$$\{X,Y\} = 2Z - \delta_{2\nmid n}X^{\frac{n-1}{2}}, \quad \{X,Z\} = 2XY - \delta_{2|n}X^{\frac{n}{2}}, \tag{10.5.4}$$

$$\{Y,Z\} = Y^2 - \delta_{2|n}\frac{n}{2}X^{\frac{n-2}{2}}Y - \delta_{2\nmid n}\frac{n-1}{2}X^{\frac{n-3}{2}}Z, \tag{10.5.5}$$

$$|X| = 2, \quad |Y| = 2(n-2), \quad |Z| = 2(n-1). \tag{10.5.6}$$

Notice that, over $\mathbb{Z}[\frac{1}{2}]$, $i_0\Pi_Q i_0 \otimes \mathbb{Z}[\frac{1}{2}]$ is generated as a Poisson algebra by X and Y.

Proof. This can all be computed using the results of the preceding sections. To see that the given relation is the only relation (i.e., the map in (10.5.3) is injective), we can compare Hilbert series. The rest can all be computed using the basis. To compute $\{Y,Z\}$, it is easiest to compute everything else first, and then compute $\{Y,Z^2\}$ two ways: either as $2Z\{Y,Z\}$, or using the formula (10.5.3)

for Z^2, and then use that $i_0\Pi_Q i_0$ is an integral domain (which is easy to see from the filtration and bases). □

11. (Partial) Preprojective algebras of star-shaped quivers

The purpose of this section is to establish some basic results about the structure of (partial) preprojective algebras of star-shaped quivers, and more generally to study what happens to preprojective algebras when one adds line segments or loops to the quiver. We will begin in §11.1 with geometric motivation, relating preprojective algebras to Riemann surfaces. The reader not interested in this purely motivational material may safely skip §11.1.

11.1. **A geometric analogy.** There is a known analogy between the preprojective algebra of star-shaped quivers with branches of lengths $d_1, \dots, d_n$ and the Riemann spheres with orbifold points of orders $d_1+1, \dots, d_n+1$, wherein the preprojective algebra is the "additive" version and the fundamental group of the associated orbifold is the "multiplicative" version. Namely, the fundamental group of the latter orbifold is the quotient of the free group on $x_1, \dots, x_n$ by the relations

$$x_1^{d_1+1}, \dots, x_n^{d_n+1}, x_1x_2\cdots x_n, \tag{11.1.1}$$

whereas, for i_s the special vertex of the preprojective algebra of the associated quiver Q, and now letting $x_i := e_i e_i^*$ where e_i is the arrow in $\overline{Q}$ in the i-th branch which begins at i_s (which we will assume for simplicity lies in Q),

$$i_s\Pi_Q i_s \cong \mathbb{Z}\langle x_1, \dots, x_n\rangle/(\!(x_1^{d_1+1}, \dots, x_n^{d_n+1}, x_1+x_2+\cdots+x_n)\!). \tag{11.1.2}$$

Next, suppose that $j \in Q_0$ is the endpoint of a branch, say the first branch. Then, if we take the partial preprojective algebra with respect to $\{j\}$, this changes the i_s-part by eliminating the relation $x_1^{d_1+1}$, i.e.:

$$i_s\Pi_{Q,\{j\}} i_s \cong \mathbb{Z}\langle x_1, \dots, x_n\rangle/(\!(x_2^{d_2+1}, \dots, x_n^{d_n+1}, x_1+x_2+\cdots+x_n)\!). \tag{11.1.3}$$

This is the same as letting $d_1 \to \infty$, and corresponds geometrically to replacing the first orbifold point by a puncture point.

11.2. **Preprojective algebras of quivers with line segments added.** The last comment suggests the following interpretation of the partial preprojective algebra: Take the underlying quiver and adjoin, at each white vertex $j \in J$, an infinite ray based at j, to form an extended quiver $\hat{Q}_{J,\infty}$. Then, the projection $\mathbf{1}_{Q_0}\Pi_{\hat{Q}_{J,\infty}}\mathbf{1}_{Q_0}$ of the preprojective algebra of this extended quiver to the span of paths beginning and ending at Q_0 (see Remark 11.2.3 below) is the partial preprojective algebra $\Pi_{Q,J}$. Precisely, for any quiver Q and any subset $J \subseteq Q_0$, we make the following definition:

Definition 11.2.1. For any map $f : J \to \mathbb{Z}_{\geq 0}$, let $\hat{Q}_{J,f}$ be the quiver obtained from Q by attaching a segment of length $f(j)$ to each $j \in J$ (i.e., the segment has $f(j)$ arrows and $f(j)$ vertices), oriented outward from j.

The choice of orientation in the definition is not important; we make it only for definiteness.

Of course, we could have taken $J = Q_0$ in the above definition without loss of generality, but it will be convenient for the limit we consider below to sometimes have J be a proper subset (i.e., to compare $\Pi_{Q,J}$ with a limit $\varprojlim_f \Pi_{\hat{Q}_{J,f}}$).

Definition 11.2.2. For any two functions $f, g : J \to \mathbb{Z}_{\geq 0}$, we say that $f \leq g$ if $f(j) \leq g(j)$ for all $j \in J$.

Note that, for any $f \geq g$, one has a surjection $\Pi_{\hat{Q}_{J,f}} \twoheadrightarrow \Pi_{\hat{Q}_{J,g}}$. One may consider the inverse limit of *graded algebras*, $\lim_{\underset{f}{\leftarrow}} \Pi_{\hat{Q}_{J,f}}$. Here, the fact that we are taking the limit as graded algebras means that the limit is defined separately in each degree, using sequences of homogeneous elements.

We may think of this limit as $\Pi_{\hat{Q}_{J,\infty}}$, where $\hat{Q}_{J,\infty}$ is the "infinite quiver" described above, using the following remark:

Remark 11.2.3. The definition of preprojective algebra makes sense even if the quiver has infinitely many arrows, as long as there are only finitely many arrows incident to each vertex. Call such a quiver a locally finite quiver. Then the preprojective algebra of a locally finite quiver Q on vertex set Q_0 is

$$\Pi_Q := P_{\overline{Q}}/(\sum_{a \in Q_1 : a_s = i} (aa^* - a^*a))_{i \in Q_0}, \tag{11.2.4}$$

i.e., even though the element r may no longer exist, we can still define the elements $r_i := \sum_{a \in Q_1 : a_s = i}(aa^* - a^*a)$, and take the ideal generated by these elements. In the case of a finite quiver, $r_i = iri$ and $(\!(r)\!) = (\!(iri)\!)_{i \in Q_0}$, so this indeed recovers the original definition.

Note that Hochschild homology does not commute with the above inverse limit. One has instead the following result:

Proposition 11.2.5. We have the following natural isomorphisms:

(i) $\Pi_{Q,J} \cong \lim_{\underset{f}{\leftarrow}} \mathbf{1}_{Q_0} \Pi_{\hat{Q}_{J,f}} \mathbf{1}_{Q_0}$.

(ii) $\lim_{\underset{f}{\leftarrow}} \Lambda_{\hat{Q}_{J,f}} \cong \Lambda_{Q,J}/\langle jr^\ell j \rangle_{j \in J, \ell \geq 1}$.

Proof. (i) This follows from the formula

$$\mathbf{1}_{Q_0} \Pi_{\hat{Q}_{J,f}} \mathbf{1}_{Q_0} \cong \Pi_{Q,J}/(\!(jr^{f(j)}j)\!)_{j \in J}. \tag{11.2.6}$$

Note that the limit is of graded algebras, and in each degree the RHS stabilizes to the LHS; in particular all classes in the RHS are represented by finite linear combinations of paths in $\hat{Q}_{J,\infty}$.

(ii) Let I be the vertex set of $\hat{Q}_{J,f}$. By the following Proposition 11.2.9, the inclusion $\mathbf{1}_{Q_0} \Pi_{\hat{Q}_{J,f}} \mathbf{1}_{Q_0} \hookrightarrow \Pi_{\hat{Q}_{J,f}}$ induces an isomorphism $(\mathbf{1}_{Q_0} \Pi_{\hat{Q}_{J,f}} \mathbf{1}_{Q_0})_{\mathrm{cyc}}/\langle [a_s^\ell], [i] \rangle_{s \in J, \ell \geq 1, i \in I \setminus Q_0} \xrightarrow{\sim} \Lambda_{\hat{Q}_{J,f}}$, where, for each $s \in J$, a_s is the cycle of length two obtained by beginning at s, traversing one arrow in $\overline{\hat{Q}_{J,f}} \setminus \overline{Q}$, and then its reverse back to i_s. If we now assume that f is such that $f(i) \geq N$ for all $i \in J$, then in degrees $m < 2N$, we evidently have $((\mathbf{1}_{Q_0} \Pi_{\hat{Q}_{J,f}} \mathbf{1}_{Q_0})_{\mathrm{cyc}})_m \cong (\Lambda_{Q,J})_m$, using the isomorphism (11.2.6). Under this isomorphism, the image of a_s^ℓ is $i_s r^\ell i_s$, for every $i_s \in J$. Now taking the limit $N \to \infty$ gives the result. □

In the above proposition, we used the following result for for (still more general) quivers $\hat{Q}$ obtained from Q by attaching any number of line segments to its vertices (not limiting to one line segment per vertex). In particular, this explains why adding infinite rays to a vertex is like adding "punctures" to the corresponding "surface."

Definition 11.2.7. Beginning with a quiver Q, a (not necessarily distinct) collection of vertices $i_1, \ldots, i_m \in Q_0$, and some positive integers $d_1, \ldots, d_m$, let $\hat{Q} := \hat{Q}_{i_1,\ldots,i_m,d_1,\ldots,d_m}$ be the quiver obtained from Q by attaching line segments $L_s, s \in \{1, \ldots, m\}$ of lengths d_s to vertices $i_s \in Q_0$, oriented outward from the vertex. That is, L_s is a quiver whose arrows are disjoint from Q, sharing only the vertex i_s. Then $(L_s)_1 \cap Q_1 = \emptyset$ and $(L_s)_0 \cap Q_0 = \{i_s\}$. Let $L_s^* \subseteq \hat{Q}^*$ be the oppositely oriented quiver and $\overline{L_s} := L_s \cup L_s^* \subseteq \overline{\hat{Q}}$ the double.

The choice of orientation in the definition of the segments L_s is not important; we made an arbitrary choice. We will also make use of the following notation:

Notation 11.2.8. For any branch L_s based at vertex i_s and any vertex $i \in (L_s)_0$, let $p_{i_s i}$ and $p_{i i_s}$ denote the straight-line paths in $\hat{Q}$ from i_s to i and from i to i_s inside $\overline{L_s}$, respectively. Let $d_{i_s,i} = d_{i,i_s}$ denote the lengths of these paths. For each line segment L_s, let C_s be the cycle in $\bar{\hat{Q}}$ of length two which begins at s, travels one arrow along the segment L_s, and then takes the reverse arrow back to s.

Proposition 11.2.9. (i) The inclusion $\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0} \hookrightarrow \Pi_{\hat{Q}}$ induces a surjection

$$(\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0})_{\mathrm{cyc}} \twoheadrightarrow \Lambda_{\hat{Q}} \tag{11.2.10}$$

with kernel $\langle [C_s^\ell] \rangle_{s\in\{1,\dots,m\},\ell\geq 1}$. As a consequence, the inclusion induces an isomorphism

$$\Lambda_{\hat{Q}} \xleftarrow{\sim} (\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0})_{\mathrm{cyc}}/\langle [C_s^\ell] \rangle_{s\in\{1,\dots,m\},\ell\geq 1}. \tag{11.2.11}$$

(ii) One obtains isomorphisms, for $J = \{i_1, \dots, i_m\}$,

$$\Lambda_{\hat{Q}} \xleftarrow{\sim} (\Pi_{Q,J} *_{\mathbb{Z}^{Q_0}} B)_{\mathrm{cyc}}/\langle [C_s^\ell] \rangle_{s\in\{1,\dots,m\},\ell\geq 1}, \text{ where} \tag{11.2.12}$$

$$B = \bigoplus_{j\in J} B_j, \quad B_j = \mathbb{Z}\langle C_s \mid i_s = j\rangle/(\!(C_s^{d_s}, \sum_{s'|i_{s'}=j} C_{s'})\!)_{s|i_s=j}. \tag{11.2.13}$$

Note that it is easy to combine Propositions 11.2.5 and 11.2.9 to describe the case of a quiver with some finite and some infinite line segments added. In this direction, we only state the promised

Corollary 11.2.14. Let Q be the quiver obtained by beginning with one vertex i_s, and attaching g loops, and m line segments of lengths $d_1, \dots, d_m$ (allowing for $d_j = \infty$). Then, defining $\Lambda_Q = (\Pi_Q)_{\mathrm{cyc}}$ where $\Pi_Q = P_{\overline{Q}}/(\!(iri)\!)_{i\in Q_0}$, one obtains formulas

$$i_s\Pi_Q i_s \cong A := \mathbb{Z}\langle x_1, \dots, x_g, y_1, \dots, y_g, p_1, \dots, p_m\rangle/(\!(\sum_{i=1}^{g}[x_i, y_i] + \sum_{i=1}^{m} p_i, p_j^{d_j+1})\!)_{j\in\{1,\dots,m\}}, \tag{11.2.15}$$

$$\Lambda_Q \cong A/([A, A] + \langle p_j^\ell \rangle_{j\in\{1,\dots,m\},\ell\geq 1}), \tag{11.2.16}$$

where we set by definition $p_j^\infty := 0$ (for any j). Thus, Λ_Q is the "additive analogue of the orbifold surface of genus g with orbifold/puncture points $p_1, \dots, p_m$ of orders $d_1 + 1, \dots, d_m + 1$."

The corollary easily follows from part (ii) of Proposition 11.2.9 using the argument of the proof of Proposition 11.2.5.

The rest of this section is devoted to the proof of Proposition 11.2.9.

Proof of Proposition 11.2.9. (i) First, we show surjectivity in (11.2.10). Any path in $P_{\overline{\hat{Q}}}$ which lies entirely in one of the line segments $\overline{L_s}$ (for any fixed s) maps to zero in $\Lambda_{\hat{Q}}$. Then, any other path projects to the same class as a path that begins and ends at a vertex in Q_0^0. This proves the surjectivity. Also, it is clear that $[a_s^\ell]$ is in the kernel of (11.2.10) for any $\ell \geq 1$ and $s \in \{1, \dots, m\}$, so it remains to show that the resulting map (11.2.11) is injective (and hence an isomorphism).

To do this, we construct an explicit inverse. For this, we need to characterize the $\mathbb{Z}$-module $j_1\Pi j_2$ in terms of $\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0}$ (in cases when at least one of j_1, j_2 are not in Q_0). We use the following lemma.

Lemma 11.2.17. In the situation of Proposition 11.2.9, let $\hat{Q}_0$ be the vertex set of $\hat{Q}$, and let $j_1, j_2 \in \hat{Q}_0 \setminus Q_0$, which are on line segments $\overline{L_{s_1}}$ and $\overline{L_{s_2}}$, respectively. Then, there is an exact

sequence of $\mathbb{Z}$-modules,

(11.2.18)

$$0 \to a_{s_1}^{d_{s_1}-d_{i_{s_1},j_1}+1}\Pi_{\hat{Q}} + \Pi_{\hat{Q}} a_{s_2}^{d_{s_2}-d_{i_{s_2},j_2}+1} \hookrightarrow i_{s_1}\Pi_{\hat{Q}} i_{s_2} \overset{x\mapsto p_{j_1 s_1} x p_{s_2 j_2}}{\twoheadrightarrow} j_1\Pi_{\hat{Q}} j_2 \to 0, \quad \text{if } s_1 \neq s_2,$$

(11.2.19)

$$0 \to a_{s_1}^{d_{s_1}-d_{i_{s_1},j_1}+1}\Pi_{\hat{Q}} + \Pi_{\hat{Q}} a_{s_2}^{d_{s_2}-d_{i_{s_2},j_2}+1} \hookrightarrow i_{s_1}\Pi_{\hat{Q}} i_{s_2} \overset{x\mapsto p_{j_1 s_1} x p_{s_2 j_2}}{\longrightarrow} j_1\Pi_{\hat{Q}} j_2 \to \sum_{\ell=0}^{\min(d_{i_{s_1} j_1}, d_{i_{s_2} j_2})-1} p_{j_1 j_2}(a')^\ell p_{j_2 j_1} \to 0, \quad \text{if } s_1 = s_2,$$

where a' is any cycle of length two inside $\overline{L_{s_2}}$, beginning and ending at j_2.

Furthermore, if one of j_1, j_2 is in Q_0 and the other is in $\hat{Q}_0 \setminus Q_0$, we have the exact sequence

(11.2.20)

$$0 \to a_{s_1}^{d_{s_1}-d_{i_{s_1},j_1}+1}\Pi_{\hat{Q}} \hookrightarrow i_{s_1}\Pi_{\hat{Q}} j_2 \overset{x\mapsto p_{j_1 s_1} x}{\twoheadrightarrow} j_1\Pi_{\hat{Q}} j_2 \to 0, \quad \text{if } j_1 \in (L_{s_1})_0, j_2 \in Q_0, \text{ or}$$

(11.2.21)

$$0 \to \Pi_{\hat{Q}} a_{s_2}^{d_{s_2}-d_{i_{s_2},j_2}+1} \hookrightarrow i_{s_1}\Pi_{\hat{Q}} i_{s_2} \overset{x\mapsto x p_{s_2 j_2}}{\twoheadrightarrow} j_1\Pi_{\hat{Q}} j_2 \to 0, \quad \text{if } j_2 \in (L_{s_2})_0, j_1 \in Q_0.$$

Proof. The image of any path in $\Pi_{\hat{Q}}$ which lies strictly inside a line segment $\overline{L_s}$ only depends on its endpoints and length. This and the fact that any path from j_1 to j_2 must pass through $\overline{Q}$ if $i_{s_1} \neq i_{s_2}$ shows the exactness at $j_1\Pi_{\hat{Q}} j_2$ above. Exactness (injectivity) at $a_{s_1}^{d_{s_1}-d_{i_{s_1},j_1}}\Pi_{\hat{Q}} + \Pi_{\hat{Q}} a_{s_2}^{d_{s_2}-d_{i_{s_2},j_2}}$ and at $\sum_{\ell=0}^{\min(d_{i_{s_1} j_1}, d_{i_{s_2} j_2})-1} p_{j_1 j_2}(a')^\ell p_{j_2 j_1}$ (surjectivity) is obvious. It remains to show exactness at $i_{s_1}\Pi_{\hat{Q}} i_{s_2}$. That is, it remains to compute the kernel of $x \mapsto p_{j_1 s_1} x p_{s_2 j_2}$. To do this, let us consider

(11.2.22)

$$j_1 P_{\bar{\hat{Q}}} j_2 \twoheadrightarrow j_1\Pi_{\hat{Q}} j_2.$$

The kernel of this is $j_1(\!(r)\!)j_2$, which we may rewrite as follows. For all s, let $\mathbf{1}_{L_s}$ be the sum of all vertices on L_s except i_s (so, the vertices from $\hat{Q}_0 \setminus Q_0$ on L_s). Then,

(11.2.23)

$$j_1(\!(r)\!)j_2 = j_1(\!((\mathbf{1}_{s_1} + \mathbf{1}_{s_2})r)\!)j_2 + p_{j_1 i_{s_1}}(\!(r)\!)p_{i_{s_2} j_2}.$$

We deduce (using the RHS and the observations at the beginning of the proof) that

(11.2.24)

$$p_{j_1 i_{s_1}} P_{\bar{\hat{Q}}} p_{i_{s_2} j_2} \cap j_1(\!(r)\!)j_2 = p_{j_1 i_{s_1}} a_{s_1}^{d_{s_1}-d_{i_{s_1},j_1}+1}\Pi_{\hat{Q}} p_{i_{s_2} j_2} + p_{j_1 i_{s_1}}\Pi_{\hat{Q}} a_{s_2}^{d_{s_2} p_{i_{s_2} j_2} - d_{i_{s_2},j_2}+1} + p_{j_1 i_{s_1}}(\!(r)\!)p_{i_{s_2} j_2}.$$

This yields the desired result. □

Now, we define the map $\Lambda_{\hat{Q}} \to (\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0})_{\mathrm{cyc}}/\langle [C_s^\ell]\rangle_{s\in\{1,\dots,m\},\ell\geq 1}$ as follows. First, let us define a map

(11.2.25)

$$\Pi_{\hat{Q}} \to (\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0})_{\mathrm{cyc}}.$$

First, the map sends ixj to zero if $i \neq j$ for vertices $i, j \in \hat{Q}_0$. Next, on $\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0}$, the map is the tautological one. Then, for any $j \in \hat{Q}_0 \setminus Q_0$, with j in the line segment L_s ($j \in (L_s)_0$), we set $p_{ji_s} x p_{i_s j} \mapsto [x(p_{i_s j} p_{ji_s})] = [xC_s^{d_{isj}}]$. To see that this is well-defined, by the lemma it suffices to show that if $x \in C_s^{d_s-d_{isj}+1}\Pi_{\hat{Q}} + \Pi_{\hat{Q}} C_s^{d_s-d_{isj}+1}$, then $[xC_s^{d_{isj}}] = 0 \in (\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0})_{\mathrm{cyc}}$. However, this follows from the fact that $[C_s^{d_s-d_{isj}+1}\Pi_{\hat{Q}} C_s^{d_{isj}}] + [\Pi_{\hat{Q}} C_s^{d_{isj}} C_s^{d_s-d_{isj}+1}] \subset [\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0}, \mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0}] + [C_s^{d_s+1}\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0}]$. Then, the only elements of $j\Pi_{\hat{Q}} j$ which can not be written in the form $p_{ji_s} x p_{i_s j}$ are (by the Lemma) those classes represented by paths lying entirely in L_s, of total length less than $2d_{ji_s}$. Let us define such paths to map to zero. We thus get a well-defined map (11.2.25).

Next, if we post-compose the map with the quotient

$$(\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0})_{\mathrm{cyc}} \twoheadrightarrow (\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0})_{\mathrm{cyc}}/\langle [C_s^\ell]\rangle_{s\in\{1,\dots,m\},\ell\geq 1},$$

then all elements of Π represented by paths which lie entirely in L_s (of any length) map to zero. It remains to show that the composite map kills $[\Pi_{\hat{Q}},\Pi_{\hat{Q}}]$. For this, we need to show that

(1) $[p_{ji_s}fp_{i_sj}, p_{ji_s}f'p_{i_sj}]$ maps to zero for any $f,f'\in i_s\Pi_{\hat{Q}}i_s$ and $j\in(L_s)_0$,
(2) $[p_{ji_s}fp_{i_sj},a']$ maps to zero for any $f\in i_s\Pi_{\hat{Q}}i_s$, $j\in(L_s)_0$ $(j\neq i_s)$, and where a' is a path of length two beginning and ending at j.

The image of the class in part (1) is $[fC_s^{d_{i_sj}}f'C_s^{d_{i_sj}}]-[f'C_s^{d_{i_sj}}fC_s^{d_{i_sj}}]=0$. The image of the class in part (2) is $[fC_s^{d_{i_sj}+1}]-[C_sfC_s^{d_{i_sj}}]=0$. This proves that our map descends to a map

$$\Lambda_{\hat{Q}} \to (\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0})_{\mathrm{cyc}}/\langle C_s^\ell\rangle_{s\in\{1,\dots,m\},\ell\geq 1}. \tag{11.2.26}$$

By definition, this map inverts the map in (11.2.11) induced by the inclusion.

(ii) This easily follows from (i) using the formula

$$\mathbf{1}_{Q_0}\Pi_{\hat{Q}}\mathbf{1}_{Q_0} \cong \Pi_{Q,Q_0^0} *_{\mathbb{Z}^{Q_0}} B. \tag{11.2.27}$$

□

11.3. **Presentation of Π_Q and Λ_Q for star-shaped quivers.** We will need the following notation, which makes sense not just for star-shaped quivers but for tree-shaped quivers (i.e., quivers such that, forgetting the orientations of arrows, one obtains a tree):

Notation 11.3.1. For two vertices i,j in a tree-shaped quiver Q, let $d_{i,j}$ denote their distance, i.e., the minimum number of arrows that must be traversed in $\overline{Q}$ to go from i to j. Let $p_{i,j}\in P_{\overline{Q}}$ be the unique path of length $d_{i,j}$ from i to j.

Note that, in the case that i and j lie in an external line segment of the tree Q, then the above notation is consistent with Notation 11.2.8.

From Corollary 11.2.14 and Lemma 11.2.17 we immediately deduce the following general result:

Proposition 11.3.2. Let Q be a star-shaped quiver with branches L_k of lengths d_k, for $k\in\{1,\dots,m\}$, and special vertex $i_s\in Q_0$. Then:

(i) $i_s\Pi_Q i_s\cong A:=\mathbb{Z}\langle x_1,\dots,x_m\rangle/((x_1+\cdots+x_m,x_1^{d_1+1},\dots,x_m^{d_m+1}))$.
(ii) For every pair of vertices j_1,j_2 in branches L_{k_1},L_{k_2}, respectively, $i\Pi_Q j\cong A/\langle x_1^{d_{k_1}-d_{i_s,j_1}+1}A+Ax_2^{d_{k_2}-d_{i_s,j_2}+1}\rangle$.
(iii) $\Lambda_Q\cong A/([A,A]+\langle x_k^q\rangle_{1\leq k\leq m,q\geq 1})$.

Note that the torsion structure of Λ_Q, as a special case of all quivers Q, is the main result of this article, but this cannot be obviously deduced for star-shaped quivers using only the above proposition.

12. Type $\tilde{E}_n$ quivers

Specializing Proposition 11.3.2 to the case of quivers of type $\tilde{E}_n$, and using noncommutative Gröbner generating sets (cf. Appendix A.1 and Proposition A.1.1 therein), we obtain the following bases of Π_Q:

Proposition 12.0.1. Let Q be an extended Dynkin quiver of type E. Let A be defined as in Proposition 11.3.2.(i). For readability, set $x:=x_1,y:=x_2,z:=x_3$, assume $d_1\geq d_2\geq d_3$, and set $d:=d_1$.

(i) A basis for $A \cong i_s \Pi_Q i_s$ (as a free $\mathbb{Z}$-module) is given by the elements, for all $0 \le \ell_1 < d$, $\ell_2 \ge 0$, and $\ell_3 \ge 0$,

$$x^{\ell_1}(yx^d)^{\ell_2}(yx^{d-1})^{\ell_3} \cdot Y, \quad Y \in \{yx^{d-2}y, yx^{\ell_4}, 1\}_{0 \le \ell_4 \le d-2}. \tag{12.0.2}$$

(i.e., Y is an initial subword of $yx^{d-2}y$).

(ii) A basis for $i_0 \Pi_Q i_0$, (as a free $\mathbb{Z}$-module), via Proposition 11.3.2.(ii), is given by

$$i_0, \tag{12.0.3}$$

$$p_{i_0 i_s}\big((yx^d)^{\ell} y\big) p_{i_s i_0} \quad (\ell \ge 0), \tag{12.0.4}$$

$$p_{i_0 i_s}\big((yx^d)^{\ell_1}(yx^{d-1})^{\ell_2} yx^{d-2} y\big) p_{i_s i_0} \quad (\ell_1, \ell_2 \ge 0). \tag{12.0.5}$$

This proposition follows from a computation of Gröbner generating sets that we performed with Magma for the $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$ cases separately. We give the details in the following four subsections.

12.1. **Type $\tilde{E}_6$.** We first compute a basis for the ring $A := \mathbb{Z}\langle x, y, z\rangle / (\!(x^3, y^3, z^3, x+y+z)\!) \cong i_s \Pi i_s$ (cf. Proposition 11.3.2). We do this by computing a Gröbner generating set for the ideal $(\!(x^3, y^3, z^3, x+y+z)\!) \subset \mathbb{Z}\langle x, y, z\rangle$, which we can do by computer using Buchberger's algorithm (we used Magma). It is also straightforward to verify by hand that the given elements form a Gröbner generating set.

Proposition 12.1.1. In the graded lexicographical order with $x \prec y \prec z$, the Gröbner generating set for the ideal $(\!(x^3, y^3, z^3, x+y+z)\!) \subset \mathbb{Z}\langle x, y, z\rangle$ is

$$yxyx^2 - xyxyx + x^2yxy + x^2yx^2, \tag{12.1.2}$$

$$y^3, \tag{12.1.3}$$

$$y^2x + yxy + yx^2 + xy^2 + xyx + x^2y, \tag{12.1.4}$$

$$x^3, \tag{12.1.5}$$

$$z + y + x. \tag{12.1.6}$$

By definition of Gröbner generating sets we immediately deduce Proposition 12.0.1.(i). We will prove Proposition 12.0.1.(ii) for E_6, E_7, and E_8 simultaneously in §12.4.

As a result of Proposition 12.0.1.(ii), we deduce, where $\Phi_m(t)$ denotes the cyclotomic polynomial whose roots are the primitive m-th roots of unity:

Corollary 12.1.7.

$$h(i_0 \Pi i_0; t) = \frac{\Phi_{24}(t)}{(1-t^4)(1-t^6)} = \frac{1-t^4+t^8}{(1-t^4)(1-t^6)} = \frac{(1-t^{24})}{(1-t^6)(1-t^8)(1-t^{12})}. \tag{12.1.8}$$

Furthermore, one has the following partial fraction decomposition:

$$h(i_0 \Pi i_0)(1-t^2) = 1 + \frac{2t^6}{1-t^6} + \frac{t^4}{1-t^4} - \frac{t^2}{1-t^2}. \tag{12.1.9}$$

Proof. For the first part, we note that our basis above shows that $h(i_0 \Pi i_0; t) = 1 + \frac{t^6}{(1-t^6)} + \frac{t^8}{(1-t^6)(1-t^4)}$. Putting this over the common denominator $(1-t^4)(1-t^6)$, we get a numerator of $1 - t^6 - t^4 + t^6 - t^{10} + t^{10} + t^8 = 1 - t^4 + t^8 = \Phi_{24}(t)$.

For the second part, one may explicitly verify the identity. Note that, since $(1-t^6)(1-t^4) = (1+t^2+t^4)(1+t^2)(1-t^2)^2$ is a decomposition into relatively prime factors, one sees that $h(i_0 \Pi i_0)(1-t^2)$ must have a partial fraction decomposition with denominators $1-t^6, 1-t^4, 1-t^2$, and the above is one such. □

The meaning of the partial-fraction decomposition (12.1.9) is again the identity (6.3.5) (cf. §9.4): setting $h(i_0\Pi i_0) = \sum a_m t^m$,

$$a_m - a_{m-2} = 2[6 \mid m] + [4 \mid m] - [2 \mid m], \quad m \geq 2. \tag{12.1.10}$$

This bears similarity to the determinant of the t-analogue of the Cartan matrix:

$$\det(\mathbf{1} - t \cdot C + t^2 \cdot \mathbf{1}) = \frac{(1-t^6)^2(1-t^4)}{1-t^2}. \tag{12.1.11}$$

Indeed, (12.1.10) says that

$$\prod_{m\geq 1} \frac{1}{(1-t^m)^{a_m - a_{m-2}}} = \prod_{m\geq 1} \frac{1-t^2}{(1-t^6)^2(1-t^4)}, \tag{12.1.12}$$

Then, as in §9.4, we verify (6.3.5) in this case.

12.2. **Type $\tilde{E}_7$.** We first compute a Gröbner generating set for the ideal $(\!(x^4, y^4, z^2, x+y+z)\!) \subset \mathbb{Z}\langle x, y, z\rangle$ (cf. Proposition 11.3.2), which we can do with Magma. It is also straightforward to verify by hand that the given elements form a Gröbner generating set.

Proposition 12.2.1. In the graded lexicographical order with $a \prec b \prec c$, the Gröbner generating set for the ideal $(\!(x^4, y^4, z^2, x+y+z)\!) \subset \mathbb{Z}\langle x, y, z\rangle$ is

$$yx^2yx^3 - xyx^2yx^2 + x^2yx^2yx - x^3yx^2y, \tag{12.2.2}$$

$$yxyx + yx^2y + yx^3 + xyxy + x^3y, \tag{12.2.3}$$

$$x^4, \tag{12.2.4}$$

$$y^2 + yx + xy + y^2, \tag{12.2.5}$$

$$z + y + x. \tag{12.2.6}$$

We immediately deduce Proposition 12.0.1.(i) for E_7. See the next subsection for the proof of (ii). As a result of (ii) we deduce the

Corollary 12.2.7. One has the formula

$$h(i_0\Pi_{\tilde{E}_7} i_0; t) = \frac{\Phi_{36}(t) = 1 - t^6 + t^{12}}{(1-t^6)(1-t^8)} = \frac{1-t^{36}}{(1-t^8)(1-t^{12})(1-t^{18})}. \tag{12.2.8}$$

Additionally, one has the partial fraction decomposition

$$h(i_0\Pi_{\tilde{E}_7} i_0; t)(1-t^2) = 1 + \frac{t^4}{1-t^4} + \frac{t^6}{1-t^6} + \frac{t^8}{1-t^8} - \frac{t^2}{1-t^2}. \tag{12.2.9}$$

Since, letting C be the adjacency matrix of $\overline{\tilde{E}_7}$, one has the formula

$$\det(1 - t \cdot C + t^2 \cdot \mathbf{1}) = \frac{(1-t^4)(1-t^6)(1-t^8)}{1-t^2}, \tag{12.2.10}$$

the identity (6.3.5) is verified.

12.3. **Type $\tilde{E}_8$.** We first compute a Gröbner generating set for the ideal $(\!(x^6, y^3, z^2, x+y+z)\!) \subset \mathbb{Z}\langle x, y, z\rangle$ (cf. Proposition 11.3.2), which we can do with Magma. It is also straightforward to verify by hand that the given elements form a Gröbner generating set.

Proposition 12.3.1. In the graded lexicographical order with $x \prec y \prec z$, the Gröbner generating set for the ideal $(\!(x^6, y^3, z^2, x+y+z)\!) \subset \mathbb{Z}\langle x, y, z\rangle$ is

$$yx^4yx^5 - xyx^4yx^4 + x^2yx^4yx^3 - x^3yx^4yx^2 + x^4yx^4yx - x^5yx^4y, \tag{12.3.2}$$

$$yx^3yx + yx^4y + yx^5 + xyx^3y - x^2yx^3 - x^3yx^2 + x^5y, \tag{12.3.3}$$

$$x^6, \tag{12.3.4}$$

$$yx^2y + yx^3 + xyx^2 + x^2yx + x^3y + 2x^4, \tag{12.3.5}$$

$$yxy - xyx - x^3, \tag{12.3.6}$$

$$y^2 + yx + xy + x^2, \tag{12.3.7}$$

$$z + y + x. \tag{12.3.8}$$

We immediately deduce Proposition 12.0.1.(i) for the E_8 case, which completes the proof of that part.

Corollary 12.3.9. The Hilbert series of $i_0\Pi_{\tilde{E}_8}i_0$ is given by

$$h(i_0\Pi_{\tilde{E}_8}i_0) = \frac{\Phi_{12}(t)\Phi_{60}(t) = 1 - t^{10} + t^{20}}{(1-t^{10})(1-t^{12})} = \frac{(1-t^{60})}{(1-t^{12})(1-t^{20})(1-t^{30})}. \tag{12.3.10}$$

Additionally, one has the partial fraction decomposition

$$h(i_0\Pi_{\tilde{E}_8}i_0)(1-t^2) = 1 + \frac{t^4}{1-t^4} + \frac{t^6}{1-t^6} + \frac{t^{10}}{1-t^{10}} - \frac{t^2}{1-t^2}. \tag{12.3.11}$$

Since, letting C be the adjacency matrix for $\overline{\tilde{E}_8}$, one has the formula

$$\det(1 - t\cdot C + t^2\cdot \mathbf{1}) = \frac{(1-t^4)(1-t^6)(1-t^{10})}{1-t^2}, \tag{12.3.12}$$

the identity (6.3.5) is verified.

12.4. **Proof of Proposition 12.0.1.(ii).** We prove this proposition simultaneously for all cases $\tilde{E}_6, \tilde{E}_7$, and $\tilde{E}_8$.

By Lemma 11.2.17, the map $i_s\Pi i_s \to i_0\Pi i_0, f \mapsto p_{i_0i_s} f p_{i_si_0}$ has image $(i_0\Pi i_0)_+$ and kernel integrally spanned by elements of the form xg, gx, $g \in i_s\Pi i_s$. First, the basis elements that begin or end in x are killed. Conversely, for any basis element g, expressing xg in terms of basis elements consists only of terms beginning with x. On the other hand, gx might not consist only of terms beginning or ending with x: this can happen in the case that $g = (yx^d)^{\ell_1}(yx^{d-1})^{\ell_2}yx^{d-2}y$. In this case, gx is a sum of basis elements beginning or ending in x, and the element $(yx^d)^{\ell_1}(yx^{d-1})^{\ell_2+1}$. Hence, we can simply eliminate the latter element from our basis, and we obtain the stated result and hence the proposition.

12.5. **Poisson structure on $i_0\Pi_Q i_0$.** As in §§9.5 and 10.5, Theorem 13.1.1 (which does not rely on any results of this subsection) implies that $i_0\Pi_Q i_0$ can be given a Poisson algebra structure for Q of types $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$, by (10.5.1). Here we give a description of this Poisson structure for $Q = \tilde{E}_n$ over $\mathbb{Z}$ using the bases of Proposition 12.0.1. Note that, by Remark 8.1.10, this also entirely describes the Lie structure on Λ_Q.

Proposition 12.5.1. The Poisson structure of $i_0\Pi_Q i_0$ is given as follows for $Q = \tilde{E}_n$. Let us use the notation of Proposition 12.0.1, and define

$$X := p_{i_0i_s} y p_{i_si_0}, \quad Y := p_{i_0i_s} yx^{d-2} y p_{i_si_0}, \quad Z := p_{i_0i_s} yx^{d-1} yx^{d-2} y p_{i_si_0}. \tag{12.5.2}$$

Finally, let us assume that all arrows of the quiver Q are oriented towards the special vertex.

(i) For $Q = \tilde{E}_6$, $i_0\Pi_Q i_0 \xleftarrow{\sim} \mathbb{Z}[X,Y,Z]/((Z^2 + Y^3 + ZX^2))$, and

$$\{X,Y\} = -2Z - X^2, \quad \{X,Z\} = 3Y^2, \quad \{Y,Z\} = -2XZ. \tag{12.5.3}$$

(ii) For $Q = \tilde{E}_7$, $i_0\Pi_Q i_0 \xleftarrow{\sim} \mathbb{Z}[X,Y,Z]/((Z^2 - X^3Y + Y^3))$, and

$$\{X,Y\} = -2Z, \quad \{X,Z\} = 3Y^2 - X^3, \quad \{Y,Z\} = 3X^2Y. \tag{12.5.4}$$

(iii) For $Q = \tilde{E}_8$, $i_0\Pi_Q i_0 \xleftarrow{\sim} \mathbb{Z}[X,Y,Z]/((Z^2 + X^5 + Y^3))$, and

$$\{X,Y\} = -2Z, \quad \{X,Z\} = 3Y^2, \quad \{Y,Z\} = -5X^4. \tag{12.5.5}$$

In particular, the homogeneous elements X, Y, Z generate $i_0\Pi_Q i_0$ as a graded algebra, and after tensoring over any commutative ring containing $\frac{1}{2}$, X, Y generate as a Poisson algebra. The degrees are $|X| = 2(d+1), |Y| = 4d$, and $|Z| = 6d$.

Proof. This can all be verified by an explicit computation. To see that the given relation (e.g., $Z^2 + Y^3 + ZX^2$ for $\tilde{E}_6$) is the only relation, we can also use the Hilbert series computed in [EG06] (cf. the first formula of (6.1.1), which holds for extended Dynkin quivers). To compute the relation and the necklace bracket formulas, we used Magma, but it is also tractable by hand (and only requires computations in fairly low degrees). □

We note that, in the above, the presentation in (i) of $\tilde{E}_6$ is somewhat more inconvenient than the presentations of $\tilde{E}_7, \tilde{E}_8$ since the latter have the form $Z^2 = P(X,Y)$ for polynomials P in X, Y (i.e., $i_0\Pi_Q i_0$ is a direct sum of the part with odd degree in Z and the part with even degree). Over $\mathbb{Z}[\frac{1}{2}]$, one can fix this:

$$Z' := Z + \frac{1}{2}X^2, \quad i_0\Pi_{\tilde{E}_6} i_0[\frac{1}{2}] \xleftarrow{\sim} \mathbb{Z}[\frac{1}{2}][X,Y,Z']/(((Z')^2 + Y^3 + \frac{1}{4}X^4)), \tag{12.5.6}$$

$$\{X,Z'\} = 3Y^2, \quad \{Y,Z'\} = X^3, \quad \{X,Y\} = -2Z'. \tag{12.5.7}$$

Finally, we explicitly compute the zeroth Poisson homology of $i_0\Pi_{Q^0} i_0 \otimes \mathbb{F}_p$ where p is a bad prime for Q^0, which will be needed to finish the proof of the main theorem. Let $M(m)$ denote the shift of a graded $\mathbb{Z}$-module M by degree m, i.e., $M(m)_n = M_{n+m}$.

Proposition 12.5.8. Let (Q^0, p) be one of the seven exceptional cases $(\tilde{E}_6, 2)$, $(\tilde{E}_6, 3)$, $(\tilde{E}_7, 2)$, $(\tilde{E}_7, 3)$, $(\tilde{E}_8, 2)$, $(\tilde{E}_8, 3)$, $(\tilde{E}_8, 5)$. Let $A := i_0\Pi_{Q^0} i_0$ be the Poisson algebra, and set $A_p := A \otimes \mathbb{F}_p$ and $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$. If $(Q^0, p) \neq (\tilde{E}_8, 2)$, then the zeroth Poisson homology $HP_0(A_p) = A_p/\{A_p, A_p\}$ is given, as a graded $A_p^{(p)} := \langle f^p \rangle_{f \in A_p}$-module, by

$$HP_0(A_p) \cong (A_p^{(p)})_+(2-2p) \oplus HP_0(A_p)', \tag{12.5.9}$$

where $HP_0(A_p)'$ is an A_p-module of the form $U \otimes \mathbb{F}_p$, where $\mathbb{F}_p$ is the augmentation module, and U is a graded $\mathbb{F}_p$-module with Hilbert series $\leq h(HP_0(A_{\mathbb{Q}}); t)$. Moreover, a basis of $HP_0(A_p)'$ can be taken to be the image of a subset of A which projects to linearly independent classes in $HP_0(A_{\mathbb{Q}})$.

If $(Q^0, p) = (\tilde{E}_8, 2)$, then we have a correction, related to the torsion class in $(\Lambda_{Q^0})_{28}$:

$$HP_0(A_2) \cong (A_2^{(2)})_+/\langle X^2 \rangle \oplus HP_0(A_2)', \quad X = p_{i_0 i_s} y p_{i_s i_0}. \tag{12.5.10}$$

Proof. This is done on a case-by-case basis. In all cases, we can assume that the quiver is oriented as in Proposition 12.5.1, because for any $a \in Q_1^0$, letting $(Q^0)_a$ be the quiver with the arrow a reversed, there is an isomorphism $\Pi_{Q^0} \xrightarrow{\sim} \Pi_{(Q^0)_a}$ sending a to $-a$ and fixing all other arrows and all vertices, which induces a Poisson isomorphism $i_0\Pi_{Q^0} i_0 \xrightarrow{\sim} i_0\Pi_{(Q^0)_a}$.

Now, we show how to prove the proposition for the case $(\tilde{E}_6, 2)$, and omit the other six cases (which are all similar). Note first that, by Proposition 12.5.1.(i), a basis of $i_0\Pi_{Q^0} i_0$ is given by

$\{X^aY^b, X^aY^bZ\}_{a,b\geq 0}$. We wish to compute the image of the Poisson bracket explicitly. As in the commutator case (in Lemma 3.0.2), we may use the formula

$$\{fg,h\} = f\{g,h\} + g\{f,h\} = \{f,gh\} + \{g,fh\} \tag{12.5.11}$$

to reduce to computing Poisson brackets of the form $\{X,f\}, \{Y,f\}$, and $\{Z,f\}$. Next, by (12.5.3), we may compute, now over $\mathbb{F}_2$:

$$\{x, x^ay^b\} = bx^{a+2}y^{b-1}, \quad \{y, x^ay^b\} = ax^{a+1}y^b, \quad \{z, x^ay^b\} = ax^{a-1}y^{b+2}, \tag{12.5.12}$$

(12.5.13)

$$\{x, x^ay^bz\} = bx^{a+2}y^{b-1}z + x^ay^{b+2}, \quad \{y, x^ay^bz\} = ax^{a+1}y^bz, \quad \{z, x^ay^bz\} = ax^{a-1}y^{b+2}z.$$

Now, it follows from this that the image of the Poisson bracket is integrally spanned by the elements

$$x^ay^b, x^ay^bz, \quad \text{where } a \text{ or } b \text{ is even, and } a \geq 2, b \geq 0, \tag{12.5.14}$$

$$x^{2a+1}y^{2b}(y^3+x^2z), \quad a,b \geq 0, xy^{2b+2}, y^{2b+2}, y^{2b+2}z, \quad a,b \geq 0. \tag{12.5.15}$$

In other words, the following gives a basis of $HP_0(A_2)$:

$$xy(x^{2a}y^{2b}), xyz(x^{2a}y^{2b}), xz(y^{2b}), yz, z, y, x, 1. \tag{12.5.16}$$

It is easy to see that the last five elements are killed by multiplication by A_2^2. The first three elements form a A_2^2-submodule isomorphic to $((A_2)_+)^2(-2)$, which can be seen by formally adjoining an element r' in degree two and setting $r'x^2 = xy, r'y^2 = xz$, and $r'z^2 = xyz$.

Finally, we note that $HP_0(A_{\mathbb{Q}}) \cong \langle xy, yz, z, y, x, 1\rangle$, which is known and can be computed directly.[4] □

13. Proof of Theorem 7.0.9 and hence Theorem 1.1.4

In this section we complete the proof of the main results. First we prove a result, Theorem 13.1.1, which explicitly describes the torsion of Λ_Q in the Dynkin and extended Dynkin cases, which is interesting in its own right. Note that, in the Dynkin case, Λ_Q is entirely torsion (and finite), so this actually describes all of Λ_Q in that case.

13.1. The torsion of Λ_Q in the extended Dynkin and Dynkin cases.

This subsection is devoted to the following result:

Theorem 13.1.1. For any Dynkin quiver Q with corresponding extended Dynkin quiver $\tilde{Q} \supset Q$, with extending vertex $i_0 \in \tilde{Q}_0$, one has the split exact sequence

$$i_0\Pi_{\tilde{Q}}i_0 \hookrightarrow \Lambda_{\tilde{Q}} \twoheadrightarrow \Lambda_Q, \quad \Lambda_{\tilde{Q}} \cong i_0\Pi_{\tilde{Q}}i_0 \oplus \Lambda_Q, \tag{13.1.2}$$

using the natural maps. Furthermore, $i_0\Pi_{\tilde{Q}}i_0$ is a free $\mathbb{Z}$-module, and Λ_Q is finite, given as follows:

1. $(\Lambda_{A_n})_+ = 0$.
2. $(\Lambda_{D_n})_m = 0$ for all $m \geq 1$ except when $4 \mid m$ and $m \leq 2(n-2)$, in which case $\Lambda_m \cong \mathbb{Z}/2$, and is integrally spanned by the class $[i_s(xy)^{m/4}]$, where i_s is the special vertex and x, y are the length-two cycles along the branches of length one (to the endpoint and back). This class equals $[r^{(m/2)}]$ when m is a power of 2.

[4] As pointed out by P. Etingof, by a spectral sequence argument, whenever V is a finite-dimensional complex symplectic vector space and G a finite subgroup of symplectic automorphisms, the dimension of $HP_0(\mathbb{C}[V]^G)$ is at least the dimension of the zeroth Hochschild homology of the Weyl algebra of V smashed with G, which by [AFLS00] is the number of conjugacy classes of G which do not have 1 as an eigenvalue. In our case, this is one less than the number of vertices of the quiver, and one in fact has equality. The equality does not hold in general, but in the appendix to [BEG04], $HP_0(\mathbb{C}[V]^G)$ is shown to always be finite-dimensional.

In $\Lambda_{\tilde{D}_n}$, the above class lifts to the torsion class $[i_{RU}(LR)^{m/2}] + [i_{LU}(RL)^{m/2}]$, which may also be described as $X^{m/4-1} \cup r^{(2)} := \left[X^{m/4-1}\widetilde{r^{(2)}}\right]$ for $X \in HH^0(\Pi_{\tilde{D}_n})$ as defined in Proposition 10.5.2, and $\widetilde{r^{(2)}} \in \Pi_{\tilde{D}_n}$ any class such that $\left[\widetilde{r^{(2)}}\right] = r^{(2)}$ (cf. Proposition 12.5.1).

(3) $(\Lambda_{E_6})_m = 0$ for all $m \geq 1$ except for $m \in \{4, 6\}$, where $(\Lambda_{E_6})_4 \cong \mathbb{Z}/2$ and $(\Lambda_{E_6})_6 \cong \mathbb{Z}/3$, generated by the classes $r^{(2)}$ and $r^{(3)}$.

(4) $(\Lambda)_{E_7})_m = 0$ for all $m \geq 1$ except when m is $4, 6, 8$, or 16, where we get $\mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/2$, and $\mathbb{Z}/2$, respectively, generated by the classes $r^{(2)}, r^{(3)}, r^{(4)}$, and $r^{(8)}$.

(5) $(\Lambda_{E_8})_m = 0$ for all $m \geq 1$ except when m is $4, 6, 8, 10$, 16, 18, or 28, where we get $\mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/2, \mathbb{Z}/5$, $\mathbb{Z}/2$, $\mathbb{Z}/3$, and $\mathbb{Z}/2$, respectively, generated by the classes $r^{(2)}, r^{(3)}$, $r^{(4)}, r^{(5)}$, $r^{(8)}$, $r^{(9)}$, and $[i_s x^4 y x^4 y x^3 y]$, where x is a length-two cycle in the direction of the longest branch, and y is a length-two cycle in the direction of the second-longest branch, and i_s is the special vertex.

In $(\Lambda_{\tilde{E}_8})_{28}$, the last class above lifts to the class $X^2 \cup r^{(2)} = \left[X^2\widetilde{r^{(2)}}\right]$, where X is defined in (12.5.2).

The classes above also lift to classes which generate the torsion of $\Lambda_{\tilde{Q}}$, which is isomorphic to Λ_Q under the second map of (13.1.2). Explicitly, these classes are the corresponding $r^{(p^\ell)}$, and in the $\Lambda_{\tilde{D}_n}$ and $(\Lambda_{\tilde{E}_8})_{28}$ cases, they are as noted above.

Proof of Theorem 13.1.1. Since the cokernel of (13.1.2), Λ_Q, is always torsion (i.e., $\Lambda_Q \otimes \mathbb{Q} = 0$ [MOV06], which is also easy to check using Gröbner generating sets), and the free rank of the first two terms in (13.1.2), $i_0\Pi_{\tilde{Q}}i_0$ and $\Lambda_{\tilde{Q}}$, are the same in each degree (6.3.1), the map $i_0\Pi_{\tilde{Q}}i_0 \to \Lambda_{\tilde{Q}}$ is always a monomorphism. It remains only to compute Λ_Q where Q is Dynkin, and check that the above descriptions indeed give a splitting $\Lambda_Q \hookrightarrow \Lambda_{\tilde{Q}}$. We first explain the second step (the splitting). We define the splitting on the torsion classes $r^{(p^\ell)}$ by $r^{(p^\ell)} \mapsto r^{(p^\ell)}$. This obviously is well-defined. Next, in the D_n case, it is easy to check that, in $\Lambda_{\tilde{Q}}$, and hence also in Λ_Q, $([i_{RU}(LR)^{m/2}] + [i_{LU}(RL)^{m/2}])$ is two-torsion using (10.1.14) for $c = \frac{m}{2}$. Thus we can define the splitting to send the corresponding class in Λ_Q to that in $\Lambda_{\tilde{Q}}$. For both the D_n and E_8 cases, the cup-product formula for the remaining elements of Λ_Q can be checked using Magma; the same formula defines two-torsion elements of $\Lambda_{\tilde{Q}}$, so we can define the splitting similarly to send the corresponding element of Λ_Q to that of $\Lambda_{\tilde{Q}}$.

It remains to verify the formulas for the torsion of Λ_Q where Q is Dynkin. This is a finite computation, since Π_Q is finite-dimensional. We do this directly as follows: (1) We note that here $i\Pi_{A_n}i$ is one-dimensional for each i, corresponding to the path of length zero. Thus Λ_{A_n} is torsion-free (and isomorphic to $i\Pi_{A_n}i$ as a graded $\mathbb{Z}$-module).

(2) Consider the quiver D_n obtained from $\tilde{D}_n$ by cutting the vertex i_{LD}. In this case, Λ_{D_n} is generated by closed paths that sit entirely on the two rightmost external arrows (we can apply relations of Π and cyclic rotations to move any cycle of length two there). Let i be the rightmost internal vertex. Then, such classes are zero if they are not a power of the length-four cycle of the form iR_ULR_DL. Furthermore, the additional relations that one obtains are: (i) any closed path that does not touch the rightmost external arrows is zero; (ii) $0 = [i(RL)^m] = [i((R_U + R_D)L)^m]$ for all i and m; and (iii) $i(RL)^{n-3} = 0$ (the latter is an algebra relation from Π_{D_n}). The last condition says that any closed cycle along the rightmost external arrows of length $\geq 2(n-2)$ is zero. The second condition says that $[(R_ULR_DL)^m] + [(R_DLR_UL)^m] = 0$, which combined with the fact that these two are equal (which was already true in $\Lambda_{\tilde{D}_n}$), shows that $[(L_URL_DR)^m]$ generates a copy of $\mathbb{Z}/2$. Finally, it is straightforward to check that the class $r^{(2m)}$ reduces to this class when m is a power of two.

(3),(4),(5) Using Proposition 11.3.2, we may give an explicit presentation of $i_s\Pi_{E_n}i_s$ and Λ_{E_n} for $n \in \{6,7,8\}$; this immediately cuts off the degree at ≤ 10 for $\tilde{E}_6$, ≤ 16 for $\tilde{E}_7$, and ≤ 28 for $\tilde{E}_8$ (these numbers may also be computed using the Coxeter numbers of the corresponding root systems: see e.g. [MOV06, EE07]). We completed the remaining finite computation by hand with the help of Magma: we computed the torsion abstractly by hand using Gröbner generating sets, and checked with Magma that the given classes generate the torsion. (We also double-checked the Hilbert series over finite fields including $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5$ in Magma.) □

Theorem 13.1.1, together with the explicit bases we already gave for $i_0\Pi i_0$ of an extended Dynkin quiver, gives an explicit basis of Λ for all extended Dynkin quivers.

13.2. **Proof of Theorem 7.0.9 (completing the proof of Theorem 1.1.4).**

Proof of Theorem 7.0.9. Recall that we are working over $\mathbb{Z}_{((p))}$; e.g., when we write Π_Q we mean $\Pi_Q \otimes \mathbb{Z}_{((p))}$.

First, in the cases that (Q^0, p) is a good pair ($p \nmid |\Gamma|$ where $\Gamma \subset SL_2(\mathbb{C})$ is the finite group associated to Q^0), the result follows from Theorem 4.2.13: it remains only to verify the statements concerning p-th powers and image of Poisson bracket. The statements about p-th powers follow from the form of the relations $W_{a,b}$ appearing in Theorem 4.2.13 (these elements should not be confused with the submodules of W appearing Theorem 7.0.9); in particular, it follows from (3.0.4)–(3.0.6) that:

$$W_{pa,pb} \equiv W_{a,b}^p \pmod{p}, \quad \frac{1}{p}W_{p^\ell,p^\ell} \equiv (\frac{1}{p}W_{p,p})^{p^{\ell-1}} \pmod{p}. \tag{13.2.1}$$

The final statement about Poisson bracket (iv) then follows immediately from Proposition 8.5.9.

Next, in the cases that $Q^0 = \tilde{A}_n$ or $Q^0 = \tilde{D}_n$, the results follow from Theorems 9.1.2 and 10.1.9, by a similar analysis of the given relations.

In the seven remaining cases $\{(\tilde{E}_6,2),(\tilde{E}_6,3),(\tilde{E}_7,2),(\tilde{E}_7,3),(\tilde{E}_8,2),(\tilde{E}_8,3),(\tilde{E}_8,5)\}$, we make use of the following claim, which follows from Proposition 12.5.8 (as we will explain):

Claim 13.2.2. If M_p is the image of the Poisson bracket on $i_0\Pi_Q i_0 \otimes \mathbb{F}_p$, then

$$t^2 h(i_0\Pi_Q i_0; t) = \sum_{\ell \geq 0} t^{2p^\ell}(h(M_p; t^{p^\ell}) + F(t^{p^\ell})), \tag{13.2.3}$$

where $F(t) = h(HP_0(i_0\Pi_Q i_0 \otimes \mathbb{F}_p)'; t) \leq h(HP_0(i_0\Pi_Q i_0 \otimes \mathbb{Q}); t)$.

Let us first use this claim. By Proposition 8.5.9, $r' \cdot M$ and $[HH^0(\Pi_{Q^0}), i_0\Pi_{Q^0}i_0] = [i_0\Pi_{Q^0}i_0, i_0\Pi_{Q^0}i_0]$ are equivalent mod $[(\langle \overline{Q}_1 \setminus \overline{Q_1^0}\rangle)^3] + [(\langle \overline{Q}_1 \setminus \overline{Q_1^0}\rangle), \Pi_Q] + (p)$. Moreover, this image lies in the image of $r'HH^0(\Pi_{Q^0})$, which we call $[r'HH^0(\Pi_{Q^0})] \otimes \mathbb{F}_p$. So, there exists a saturated free submodule $\tilde{M} \subset W'$ which projects to $[r'M]$. We will find a (saturated) $Z_{((p))}$-submodule $U_p' \subset HH^0(\Pi_{Q^0})$ with Hilbert series $h(U_p'; t) = F(t) - t^{2p^m-2}$ (where m is the smallest positive integer such that $r^{(p^m)}$ is zero in Π_{Q^0}), such that the image of U_p' and M in $HH^0(\Pi_{Q^0}) \otimes \mathbb{F}_p$ are linearly independent (over $\mathbb{F}_p$). We will then find a submodule $\tilde{U}_p' \subset W'$ which projects isomorphically mod $[(\langle \overline{Q}_1 \setminus \overline{Q_1^0}\rangle)^3] + [(\langle \overline{Q}_1 \setminus \overline{Q_1^0}\rangle), \Pi_Q]$ to $[r'U_p']$. In view of the observations before the statement of the theorem (given any element of $W \otimes \mathbb{F}_p$, its p-th power must also be in $W \otimes \mathbb{F}_p$), it then follows that $W' \otimes \mathbb{F}_p$ contains the direct sum of all p-th powers of $[r'U_p'] \otimes \mathbb{F}_p$ and $[r'M] \otimes \mathbb{F}_p$. We can set $W_s' := \tilde{M} \oplus \tilde{U}_p'$, which by the claim satisfies the conditions of (iv).

Once we also describe W_0 and W_r as claimed, the sum $U := W_0 \oplus (W_s' \oplus W_r')$ is a graded submodule $U \subseteq W$ which has the correct Hilbert series, i.e., such that W/U is torsion. Moreover, we obtain that V/U is torsion and generated by the classes $r^{(p^\ell)}$. Since we know that these classes

are nonzero by Proposition 7.0.8 (the easy direction of Theorem 1.1.4), we conclude that $U = W$, which proves the theorem.

It remains to find the submodules U'_p and W_r, and to describe W_0. We begin with the description of W_0. This is easy: using Theorem 13.1.1, it already has the desired form, except in the case of $(\tilde{E}_8, 2)$, where it is not immediately clear that $(W_0)_{28}$ is saturated. However, this can be verified explicitly (with the help of Magma): $[x^4y, x^4yx^3y]$ projects to $2x^4yx^4yx^3y - x^5yx^5y = 2x^4yx^4yx^3y - [p_{is}i_0r'p_{i_0i_s}yp_{i_si_0}r'p_{i_0i_s}y]$ in V, and this is not a multiple of 2 (working over $\mathbb{Z}$, the above relation is in fact not a multiple of any positive integer).

Next, we find U'_p. For Hilbert series reasons, U'_p should be $\mathbb{Z}_{((p))}$-linearly spanned by elements of $i_0\Pi_Q i_0$ which project to a subset of the generators of the zeroth Poisson homology of $i_0\Pi_Q i_0 \otimes \mathbb{Q}$ (note that $i_0\Pi_Q i_0 \otimes \mathbb{C} \cong \mathbb{C}[x,y]^\Gamma$). The degrees of elements of U'_p are ≤ 20 for $\tilde{E}_6$, ≤ 32 for $\tilde{E}_7$, and ≤ 56 for $\tilde{E}_8$. For these low degrees, the needed $\tilde{U}'_p \subset W'$ can be found (or its existence verified) using Magma.

Finally, let us find W_r. For each $\ell \geq 1$, define $f_\ell = p \cdot \bar{f}_\ell$, such that $\bar{f}_\ell$ is any noncommutative polynomial in x, y which projects, modulo commutators, to $\frac{1}{p}[(x+y)^{p^\ell} - x^{p^\ell} - y^{p^\ell}]$. Our choice of $f_\ell = p\bar{f}_\ell$ makes it clear that, in fact, $r^{(p^\ell)}$ and $\bar{f}_\ell$ have the same image in V/W. In particular, $p[\bar{f}_\ell] = [f_\ell] \in W$ and $[f_\ell] \in W'$ for $\ell \geq m$. Thus, it suffices to show that $\bar{f}_\ell$ is independent of $W'_s \otimes \mathbb{F}_p$ for all $\ell \geq m$; to do this it suffices to show it for $\ell = m$. In these cases, $W'_{|\bar{f}_m|} \otimes \mathbb{Q} \cong HH^0(\Pi_{Q^0} \otimes \mathbb{Q})(2)_{|\bar{f}_m|}$ is one-dimensional; thus, it suffices to show that $\bar{f}_m \notin pV + [((r'))^2]$. This we can explicitly verify with Magma.

It remains only to prove the claim. In general, $G(t) = \sum_{\ell \geq 0} H(t^{p^\ell})$ if and only if $H(t) = G(t) - G(t^p)$. This observation, together with Proposition 12.5.8, proves the desired result. □

Appendix A. The Diamond Lemma for modules

In this section, we prove a generalization of the Diamond Lemma that applies to free modules, rather than to free algebras (as in [Ber78]). As a consequence, we deduce results on Gröbner generating sets for free algebras (which are probably known). In the case of fields, one recovers usual Gröbner bases and the Diamond Lemma of [Ber78]. The arguments used are essentially the same.

We first recall Gröbner generating sets, since they are simpler, then proceed with the generalized Diamond Lemma, which implies the result on Gröbner generating sets.

A.1. **Gröbner generating sets.** First, suppose that $F = \mathbf{k}\langle x_1, \dots, x_n\rangle$ is the free noncommutative algebra over the commutative ring $\mathbf{k}$ generated by indeterminates $x_1, \dots, x_n$. Consider the graded lexicographical ordering, which means that $M_1 \prec M_2$ if either $|M_1| < |M_2|$ (where $|M|$ denotes the length of a monomial M), or $|M_1| = |M_2|$ and $M_1 \ll M_2$ with respect to the lexicographical ordering $\ll$ on $x_1, \dots, x_n$ where $x_1 < x_2 < \dots < x_n$.[5]

Given a set of elements $P_i \in F$, another polynomial P is said to be reducible with respect to the P_i if the leading monomial (with respect to $\prec$) $LM(P)$ of P contains as a subword the leading monomial of one of the P_i's. Otherwise, P is said to be irreducible with respect to the P_i's. If P is reducible, then a reduction of P is an element of the form $P - \lambda X P_i Y$ where X and Y are monomials, $\lambda \in \mathbf{k}$, and $\lambda X \cdot LM(P_i) \cdot Y$ is the leading term of P (i.e., $X \cdot LM(P_i) \cdot Y$ is the leading monomial of P, which appears with coefficient λ).

Let us call an **ideal** generating set $\{P_i\}$ for an ideal $I = (\!(P_i)\!)$ a **Gröbner generating set** if the leading monomial of each P_i has coefficient which is a unit in $\mathbf{k}$, and any polynomial has a

[5]We can generalize this to replace lexicographical ordering by any total ordering on monomials of a given degree, and generalize the degree $|\ |$ to a weighted degree where each x_i is assigned a positive integer, not necessarily 1.

unique reduction to an irreducible element with respect to the P_i. In other words, the irreducible monomials form a basis of the quotient F/I as a free $\mathbf{k}$-module.

The following criterion is well-known (and is the basis for the Buchberger algorithm for computing Gröbner generating sets):

Proposition A.1.1. A set (P_i) forms a Gröbner generating set for $I = (\!(P_i)\!)$ if and only if, for all elements P_i and P_j with leading terms $\lambda_i M_i$ and $\lambda_j M_j$, and all monomials M such that $M = M_i X = Y M_j$ for some monomials X and Y, one can reduce the element $\lambda_j P_i X - \lambda_i Y P_j$ to zero, meaning that there is a sequence of reductions, using the P_ℓ, taking this element to zero.

Proof. This follows from the Diamond Lemma in the next subsection. □

A.2. **The Diamond Lemma for modules.** Here, we formulate and prove a version of the Diamond Lemma for free modules over a arbitrary commutative ring $\mathbf{k}$, which we haven't seen in the literature. We then specialize to the free algebra case.

Let $\mathbf{k}$ be a commutative ring. Given any free module V over $\mathbf{k}$ with a fixed basis $(v_i)_{i\in I}$ labeled by a partially ordered set $(I, \prec)$ which satisfies the descending chain condition (meaning every strictly descending sequence in I is finite), and given any submodule $W \subseteq V$, the Diamond Lemma gives a criterion for a set $S = (w_j)_{j\in J} \subseteq W$ (for some index set J) to be a *confluent spanning set*, which is a generalization of the notion of Gröbner generating set, essentially meaning that applying reductions in any order yields the same result. In the case S is confluent, every element of V has a unique reduction to certain $\mathbf{k}$-linear combinations of the irreducible monomials $(v_i)_{i\in I'\subseteq I}$ for a certain subset $I' \subseteq I$ determined by S. More precisely, for each $i \in I'$, there exists a subset $R_i \subseteq \mathbf{k}$ so that every monomial has a unique reduction to a finite sum of the form $\sum_{i\in I'} \lambda_i v_i$, with $\lambda_i \in R_i$ for all i. (The subset $R_i \subseteq \mathbf{k}$ will be an arbitrary choice of representatives of the quotient $\mathbf{k}/Y_i$ for some ideal Y_i.)

Let $i \preceq i'$ mean $i = i'$ or $i \prec i'$. An element $w_j \in V$ defines a "reduction" if, writing $w_j = \sum_{i\in I} \lambda_{ji} v_i$ (all but finitely many λ_{ji} are nonzero for each j), there exists a unique $\psi(j) \in I$ such that $\lambda_{j\psi(j)} \neq 0$ and $\lambda_{ji} \neq 0$ implies $i \preceq \psi(j)$. Then, the "reduction" associated to w_j sends $\lambda_{j\psi(j)} v_{\psi(j)}$ to $-\sum_{i\prec\psi(j)} \lambda_{ji} v_i$.

For every collection $S = (w_j)_{j\in J} \subseteq W$ of elements defining reductions and every $i \in I$, let $S_i := \{w_j \mid j \in J, \psi(j) = i\}$ be the set of w_j which reduce scalar multiples of v_i (i.e., elements of the form λv_i for $\lambda \in \mathbf{k}$). Also, let $V_{\prec i} := \mathrm{Span}_{\ell\prec i}(v_\ell)$, where here and below span means the $\mathbf{k}$-linear span.

Definition A.2.1. A *confluent set* is a collection $S = (w_j)_{j\in J}$ of elements which define reductions such that, for all $i \in I$, $\mathrm{Span}(S_i) \cap V_{\prec i} \subseteq \mathrm{Span}(S_\ell)_{\ell\prec i}$. A *confluent spanning set* for W is a confluent set $S \subseteq W$ which $\mathbf{k}$-linearly spans W.

Remark A.2.2. Unlike the definition of Gröbner generating sets, we have included no minimality condition in the definition of confluent spanning sets. For example, if S is a confluent spanning set, then any superset of S obtained by adjoining scalar multiples of some elements of S with the same leading monomials (this last condition would be automatic if $\mathbf{k}$ were an integral domain) is still a confluent set. We have no need for minimality (this would be useful if we wanted minimal confluent spanning sets to be unique, but we do not need this).

Given $S = (w_j)_{j\in J}$, define for all $i \in I$ the ideal $Y_i \subseteq \mathbf{k}$ generated by all $\lambda \in \mathbf{k}$ such that there is an element $w_j \in S$ of the form $w_j = \lambda v_i + \sum_{i'\prec i} \mu_{i'} v_{i'}$ (in particular, when w_j defines a reduction, $\psi(j) = i$).

Proposition A.2.3. (The Diamond Lemma I) For any confluent set (w_j), and any choices of representatives $R_i \subseteq \mathbf{k}$ of $\mathbf{k}/Y_i$ (i.e. R_i maps bijectively to $\mathbf{k}/Y_i$) such that $0 \in R_i$ for all (but

finitely many) i, every class in V/W has a unique expression as a linear combination $\sum_i \mu_i v_i$ where $\mu_i \in R_i$.

Proposition A.2.4. (The Diamond Lemma II) For any confluent spanning set (w_j), one may obtain another confluent spanning set $(u_{i,r})$ for W by choosing, for each $i \in I$ such that $Y_i \neq (0)$, arbitrary elements $u_{i,r} \in \operatorname{Span} W_i$ such that $u_{i,r} = \lambda_{i,r} v_i + \sum_{\ell \prec i} \mu_{i,\ell,r} v_\ell$ such that the $\lambda_{i,r}$ satisfy $(\!(\lambda_{i,r})\!)_r = Y_i$.

Proposition A.2.5. (The Diamond Lemma III) Again suppose $S = (w_j)_{j \in J}$ is a confluent spanning set. Let $I_{\neq 1} \subseteq I$ be the subset such that $i \in I_{\neq 1}$ if and only if $Y_i \neq (1)$. Let $I_{\neq 1,0}$ be the subset such that $i \in I_{\neq 1,0}$ if and only if $Y_i \neq (1)$ and $Y_i \neq (0)$. Then V/W may be presented as $\operatorname{Span}_{i \in I_{\neq 1}}(v_i)/\operatorname{Span}_{i \in I_{\neq 1,0}}(T_i)$, where $T_i = \{u_{i,r}\}$ is a set of elements as in Proposition A.2.4.

We only prove the first version, since the other two follow fairly easily.

Proof of Proposition A.2.3. We claim that it is enough to assume that I is finite. In general, I is an inductive limit of finite subsets $I' \subseteq I$ which are downward-closed, i.e., if $i < i'$, then $i' \in I'$ implies $i \in I'$. For such $I' \subseteq I$, $W_{I'} := \operatorname{Span}_{i \in I'}(S_i) \subseteq V_{I'} := \operatorname{Span}_{i \in I'}(v_i)$, and then V is the union of such $V_{I'}$, and W is the union of such $W_{I'}$. Hence, it is enough to prove the statement for every $(V_{I'}, W_{I'})$.

So, assume I is finite. We prove the statement by induction on the size of I. If $|I| = 1$, the statement is clear. Inductively, suppose that $i_0 \in I$ is a maximal element and that the result is true for the submodules $V' \subsetneq V, W' \subseteq W$ where $V' = \operatorname{Span}_{i \neq i_0}(v_i)$ and $W' = \operatorname{Span}_{i \prec i_0}(S_i)$. Then, the result also holds for the pair (V, W'). Next, write $V/W = (V/W')/\operatorname{Span}(S_{i_0})$. It is evident that every element of V/W is the class of an element of the form $\sum_{i \in I} \mu_i v_i$ (for $\mu_i \in R_i$), since for every $y \in Y_{i_0}$, $y \cdot v_{i_0} \in (V' + W)$. It remains to show that every class is uniquely represented in this way. For this, it suffices to prove the following claim: If $\sum_{i \in I} \mu_i v_i \in W$ for some μ_i, then if $i_1 \in I$ is maximal such that $\mu_{i_1} \neq 0$, then $\mu_{i_1} \in Y_{i_1}$. Let us prove the claim by induction on $|I|$: that is, let us strengthen the inductive hypothesis to include this (it is clear when $|I| = 1$).

The claim is evident for $i_1 = i_0$, since for every element of W, the coefficient of v_{i_0} lies in Y_{i_0}. More generally, if $i_1 \not\prec i_0$, then this follows by the inductive hypothesis, since any $\mathbf{k}$-linear combination of elements of S_{i_0} has nonzero coefficients only on v_i with $i \preceq i_0$. It remains to prove the claim for $i_1 \prec i_0$. Given any element $\sum_{i \in I} \mu_i v_i$, we may subtract, for every maximal i' with $i' \not\prec i_0$, some element of $\operatorname{Span}(S_{i'})$, so that the new element $\sum_{i \in I} \mu'_i v_i$ has $\mu'_{i'} = 0$ for all such i', by the case $i_1 \not\prec i_0$ of the claim above. This will not change the set of maximal i_1 such that $\mu_{i_1} \neq 0$ which instead have the property $i_1 \prec i_0$. Thus, it suffices to assume that, for all maximal i_1 with $\mu_{i_1} \neq 0$, $i_1 \prec i_0$. In this case, $\sum_{i \in I} \mu_i v_i \in W \cap V_{\prec i_0}$, and by the confluence condition, it follows that $\sum_{i \in I} \mu_i v_i \in \operatorname{Span}(S_{\prec i_0})$. Now, the claim follows by induction. □

Note that we did **not** actually describe the abstract module structure of V/W above: for example, for $\mathbf{k} = \mathbb{Z}$, one could have $V = \langle v_1, v_2 \rangle$ and $W = \langle 3v_1 - v_2, 3v_2 \rangle$, where Proposition A.2.3 says that the quotient is set-theoretically the same as sums $\lambda_1 v_1 + \lambda_2 v_2$ for $\lambda_i \in \{0, 1, 2\}$. The abstract $\mathbb{Z}$-module structure, however, is $\mathbb{Z}/9$.

In the situation of Gröbner generating sets, as we will explain below, W as above could be taken to be multiples of a Gröbner generating set by monomials on either side; in particular, W is much larger than an actual Gröbner generating set.

Remark A.2.6. In the case that $\mathbf{k}$ is a field, then the Diamond Lemma says that a basis of V/W is given by $(\bar{v}_i)_{i \in I'}$ where $I' = I_{\neq 1}$ is the set of indices such that v_i does not appear as the leading term of any of the relations (w_j) in the confluent set.

Remark A.2.7. We could have instead taken $\prec$ to be a well-ordering (e.g. a labeling by $\mathbb{Z}_{\geq 0}$ if the module is countably generated) with no loss of generality. This is because we may convert

any partial ordering satisfying the descending chain condition into an arbitrary well ordering that preserves all relations $x \prec y$ from the partial order (i.e. such that there is a map of partially ordered sets from the original set to the totally ordered one), without changing any of the above objects. (Note that this requires the Axiom of Choice in general, and it works because the descending chain condition is equivalent to saying that any subset has a minimal element.) This would yield exactly the same results and proof. We state it in the generality of partially ordered sets because that setting is sometimes more convenient.

Now, let us specialize to the free algebra case. Let $A = \mathbf{k}\langle x_1, \ldots, x_m \rangle$ be a free noncommutative algebra generated by indeterminates x_i. Let $\prec$ be a partial order on the monomials in the x_i's satisfying the descending chain condition, such that $f \prec g$ implies $h_1 f h_2 \prec h_1 g h_2$ for any monomials h_1, h_2 (for instance, this is satisfied by the graded lexicographical ordering). Suppose $B \subseteq A$ is an ideal. Then we can define a set (b_i) to be a confluent ideal generating set of B if the elements $(f b_i g)$, for f, g ranging over all monomials in the x_i, form a confluent spanning set for B as an $\mathbf{k}$-module (with basis the monomials and partial order $\prec$). To understand what this means, call the "leading monomial" $LM(P)$ of an element $P \in A$ the highest monomial which appears with nonzero coefficient with respect to $\prec$, if such a monomial exists and exceeds all other monomials which have nonzero coefficient. In order to be confluent, we first require that each b_i have a leading monomial. Then, elements of the form $f b_i g$ define reductions, which reduce certain multiples of $f \cdot LM(b_i) \cdot g$ into linear combinations of smaller monomials. Then, the confluence condition says: if $h \in A$ admits multiple reductions, then every linear combination of elements defining such reductions which has a lower leading monomial than h is itself a linear combination of elements defining reductions whose leading monomials are all less than h.

Then, in this case, one concludes all of the Diamond Lemma versions. In the case $\mathbf{k}$ is a field, for example, one finds that A/B has a basis, as a vector space, given by those monomials in "normal form", which means that they do not contain the leading monomial of any of the b_i's as a subset (as a word).

Letting $\prec$ be the graded lexicographical ordering (or a variant as discussed in §A.1), we find that a minimal confluent ideal generating set (with leading monomials having coefficient one) is a Gröbner generating set for B, and recover Proposition A.1.1. In fact, at the cost of relinquishing uniqueness, we recover a version of Gröbner generating sets over arbitrary commutative rings (which should perhaps be called Gröbner generating sets).

References

[AFLS00] J. Alev, M. A. Farinati, T. Lambre, and A. L. Solotar, *Homologie des invariants d'une algèbre de Weyl sous l'action dun groupe fini*, J. Algebra **232** (2000), 564–577.

[Ani82] D. J. Anick, *Noncommutative graded algebras and their Hilbert series*, J. Algebra **78** (1982), 120–140.

[Ani86] ———, *On the homology of associative algebras*, Trans. Amer. Math. Soc. **296** (1986), 641–659.

[BEG04] Yuri Berest, Pavel Etingof, and Victor Ginzburg, *Morita equivalence of Cherednik algebras*, J. Reine Angew. Math. **568** (2004), 81–98, arXiv:math/0207295.

[Ber78] George M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), no. 2, 178–218.

[BLB02] Raf Bocklandt and Lieven Le Bruyn, *Necklace Lie algebras and noncommutative symplectic geometry*, Math. Z. **240** (2002), no. 1, 141–167.

[CBEG07] William Crawley-Boevey, Pavel Etingof, and Victor Ginzburg, *Noncommutative geometry and quiver algebras*, Adv. Math. **209** (2007), 274–336, math.AG/0502301.

[CBH98] William Crawley-Boevey and Martin P. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. J. **92** (1998), no. 3, 605–635.

[EE05] Pavel Etingof and Ching-Hwa Eu, *Koszulity and the Hilbert series of preprojective algebras*, math.RA/0512287, 2005.

[EE07] ———, *Hochschild and cyclic homology of preprojective algebras of ADE quivers*, arXiv:math/0609006, 2007.

[EG06] Pavel Etingof and Victor Ginzburg, *Noncommutative complete intersections and matrix integrals*, math.AG/0603272, 2006.

[EK96] Pavel Etingof and David Kazhdan, *Quantization of Lie bialgebras I*, Selecta Math (N.S.) **2** (1996), no. 1, 1–41.

[Gin01] Victor Ginzburg, *Non-commutative symplectic geometry, quiver varieties, and operads*, Math. Res. Lett. **8** (2001), no. 3, 377–400.

[GP79] I.M. Gelfand and V.A. Ponomarev, *Model algebras and representations of graphs*, Func. Anal. and Applic. **13** (1979), no. 3, 1–12.

[GS10] V. Ginzburg and T. Schedler, *Differential operators and BV structures in noncommutative geometry*, Selecta Math. (N.S.) **16** (2010), no. 4, 673–730, arXiv:0710.3392. MR 2734329 (2012k:18021)

[GS12] Victor Ginzburg and Travis Schedler, *Free products, cyclic homology, and the Gauss–Manin connection*, Adv. Math. **231** (2012), no. 3-4, 2352–2389, arXiv:0803.3655. MR 2964640

[Hes97] Lars Hesselholt, *Witt vectors of non-commutative rings and topological cyclic homology*, Acta Mathematica **178** (1997), 109–141.

[Hes05] ———, *Correction to "Witt vectors of non-commutative rings and topological cyclic homology"*, Acta Mathematica **195** (2005), 55–60.

[MOV06] Anton Malkin, Viktor Ostrik, and Maxim Vybornov, *Quiver varieties and Lusztig's algebra*, Adv. Math. **203** (2006), no. 2, 514–536, arXiv:math/0403222.

[Sch05] Travis Schedler, *A Hopf algebra quantizing a necklace Lie algebra canonically associated to a quiver*, Int. Math. Res. Not. (2005), no. 12, 725–760, IMRN/14217.

[Sch07] T. Schedler, *Hochschild homology of preprojective algebras over the integers*, arXiv:0704.3278v1, 2007.

[VdB08] M. Van den Bergh, *Double Poisson algebras*, Trans. Amer. Math Soc. **360** (2008), no. 11, 5711–5769, arXiv:math.QA/0410528.

[Wit37] E. Witt, *Zyklische Körper und Algebren der Charakteristik p vom grad p^n*, J. Reine Angew. Math. **176** (1937), 126–140.