The General Three-Nucleon Faddeev Equation

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by

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Abstract

A new formalism for the description of the non-relativistic three-nucleon systems is constructed from the Faddeev's equations and the Tensor Representation. This method takes into account higher orbital angular moments, non-separable potentials as well as spin and isospin dependent forces.

The Faddeev equation is represented in a basis that fully classifies the three-nucleon system, realizing an irreducible representation of the rotation and permutation groups in the three dimensional space of three particles. Higher orbital angular momentum harmonics as well as their symmetry properties are readily constructed from simple formulae.

The internal and external variables are separated in a moving system of reference such that the new coordinate set is invariant under any permutation of the particles. A generalized Omnes Angular Momentum reduction is applied.

The spin-isospin formalism is generalized for higher orbital angular momenta and for $S_z = 3/2$. The final model is set up with the Faddeev equation representing a set of coupled equations in one continuous variable in a generalized base of representation for arbitrary orbital angular momentum and with the spin-isospin formalism included.

In the memory of my beloved mother

For my dear father

To my devoted wife, Susie

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CHAPTER I

INTRODUCTION

The nuclear three-body problem has been under study for more than 40 years providing us with a steady and continuous progress in our knowledge about this basic interaction.

In the process of description of this physical problem it is possible to initiate the study of the three-nucleon problem from basic theoretical concepts and then develop them using abstract ideas in Mathematical-Physics to reach at the end numerical results that can be confirmed by experimental data - binding energies, scattering length, etc. It is readily possible to check any improvement in the theory by comparing the theoretical predictions with the laboratory observations. This fact has made this problem very attractive, being a subject of study of many Physicists and Mathematicians.

The main breakthrough for the exact calculations of the three-nucleon problems came with the formalism developed by Faddeev [1] who wrote the equation of motion of the three particles simultaneously in a coupled integral equation of the Fredholm type that is free of the ambiguities - disconnectedness of the kinematics - of the previous Lippmann-Schwinger equation. A brief discussion of the Faddeev equation is put forward in Chapter V.

The other widely used formalism developed by Lovelace [2] and Amado [3], make use of separable potentials in the interaction thus greatly reducing the complexity of the problem. However these potentials are not easily justifiable. One regards them as an approximation to non-separable ones [4].

In the present thesis the integral equation approach is used which has four very important advantages:

I - It offers in principle a complete and exact solution of the problem. The only approximations made are through the cut-off of the series in the angular momentum reduction, in the pole expansion of the t-matrices and in the numerical analysis. All of these approximations can be made as accurate as needed by calculating higher terms in the expansions.

II - It is a natural way of incorporating the boundary conditions of the complex three body problems.

III - The boundstate is studied on the same footing as the scattering problem giving the flexibility of solving both problems simultaneously.

IV - The formalism is valid for higher energies. The basic model of interaction can be retained when studying the effects of high energy scattering.

The previous attempts of solving the three-nucleon problem can be divided into six broad groups. For easy reference Table 1 shows the characteristic features of the first four groups.

The first group is characterized by the asymmetric angular momentum reduction of the variables in the Faddeev framework. Ahmadzadeh and Tjon [5] labelled the new states by (L^2, L_z, l^2, l_z) where L represents the relative orbital angular momentum of a pair of particles, L_z its projection on the z axis, l the relative angular momentum of the third particle relative to the other two and l_z the corresponding z projection. These states are not treated on equal footing. El Baz et al. [6] using spherical harmonics generalized this method to include the spin depence

Group	Group	Non-separable potentials	Uniqueness	Symmetric variables	One continuous variable	Higher orbital angular momentum	Spin-isospin formalism
		1	2	3	4	5	6
Ahmadzadeh and Tjon [5]	I	Yes	Yes	No	No	Yes	No
El Baz et al. [6]	I	Yes	Yes	No	Yes	Yes	Yes
Harper, Kim and Tubis [7]	I	Yes	Yes	No	Yes	Yes	Yes
Mitra [8]	II	No	Yes	No	Yes	Yes	Yes*
Sitenko [9], Kharchenko [9]	II	No	Yes	No	Yes	Yes	Yes
Bhakar [10]	II	No	Yes	No	Yes	Yes	Yes
Simonov, Badalyan [11], Pustovalov et al. [12]	III	Yes	No	Yes	Yes	Yes	Yes
Omnes [13]	IV	Yes	Yes	No	No	No	No
Osborn, Noyes [14]	IV	Yes	Yes	No	No	No	No
Lee [13]	IV	Yes	Yes	Yes	Yes	No	Yes
Present work	IV	Yes	Yes	Yes	Yes	Yes	Yes

Table 1

* A.N. Mitra did not include isospin dependence

and finally Harper, Kim and Tubis [7] incorporated the isospin formalism - with the interaction of tensor forces.

The second group represented by Mitra [8], Kharchenko, Sitentko [9] and Bhakar [10] solved the problem for separable potentials reducing the kernel of the integral to one integrable variable. This approach reduces the analytical solution of the integrals considerably.

The third group is represented by the work of Simonov, Badalyan, [11], Pustovalov et al. [12] who expanded the three nucleon wave function in generalized spherical harmonics. This procedure, called the k-harmonic method uses the Shroedinger equation directly, thus missing the benefits of the Faddeev approach.

The fourth group is characterized by a symmetric angular momentum reduction in the Faddeev formalism. Omnes [13] succeeded in reducing the number of continuous variables to three, the centre of mass momentum of the three particles. Osborn and Noyes [14] further reduced this equation to two continuous variables but introduced non-symmetric variables. To the fifth group belongs the work of Lovelace [2] and Amado [3] in whose formalism the t-matrices are of the separable type. Finally the sixth group is represented by the direct solution of the Faddeev integral equation through variational techniques [20]. The fifth and sixth groups are not shown in the table.

The columns in Table I are ordered in the following way:

Column I shows methods general enough to include non-separable potentials in the description of the system. According to Lovelace [2] the pole structure of the problem remains the same regardless of the choice of potential and so the separable potential can be thought of as being an approximation to the non-separable one.

Column 2 shows the formalisms with final unique solution. Uniqueness of description is the main advantage of the Faddeev equation over the usual Lippmann-Schwinger [15] one. The former contains all the transition operators simultaneously permitting a full description of all the possible channels in the same equations from the whole spectrum of multichannels in a three particle configuration while the Lippmann-Schwinger equations considers the various channels separately.

In column 3 the formalisms which use symmetrical variables are shown. Most of the authors have worked with unsymmetrical systems because of the natural way of writing the two body potentials or t-matrices in terms of two-body variables but this set of coordinates is not suitable for descriptions where the particles get mixed up, particularly in the boundstate problem, where the matrices become very cumbersome. Treating the three particles on the same footing gives wave functions with very simple properties.

Columns 4 shows those theories which are capable of reducing all the continuous variables to a set of coupled equations in discrete labels and only one continuous variable. This reduction makes possible the numerical solution of the problem by transforming multiple integrals with interlocking limits to discrete sums and only one single integral equation, thus simplifying the problem considerably.

Column 5 shows methods which have been generalized to higher orbital angular momentum. The solution for the S state ($L = 0$) is very simple since no angular dependence is present in the interaction leaving the equations of motion with only three internal variables. The extension to P and D states is very complex but essential for the exact

description of the three-nucleon bound and scattering states permitting to take into account higher angular momentum states and non-central forces.

Finally column 6 shows theories using the spin-isospin formalism included in the equations of motion of the system.

The present thesis is a generalization and extension of the previous work done by Hochberg, Lee and Sibbel [16] where the Faddeev equation was given a basis of representation according to the SU(3) group and solved for the binding energy of the triton in the $L = 0$ state.

However the SU(3) states were not used in the present work because of the difficulties encountered in solving the differential equation for polynomials of higher order and the lack of definite symmetry properties under permutations of all three particles for states with $L \neq 0$.

Instead, following the method proposed by Pustovalov and Simonov [17] a complete set of angular functions is constructed by the application of tensor differential operators to a scalar polynomial. Specifically the state for $L = 0$ obtained by Lee [18] is used as the initial polynomial upon which the whole representation is constructed. The surface harmonics for the S-state $S_{\lambda}^{\mu}(\hat{r})$ in Lee's work represents the scalar polynomial in the internal variables.

The presentation of this thesis is organised as follows:

In Chapter II a brief description of the coordinate system is given. The choice of the three internal and three external coordinates, the orientation of the body-fixed axis and the transformation properties of the set are explained.

Chapter III starts with a short account of the general formalism for the classification of three particles. The differential tensor operators as well as the symmetrisation and tracesless operators are introduced. The harmonic polynomials are then constructed over the $S_{\lambda}^{\mu\rho}(\hat{r})$ state for arbitrary orbital angular momentum. The final states are classified according to their permutational symmetries. A table with the symmetric, asymmetric and mixed states is produced with the explicit functions for $L = 0$, 1 and 2 and for λ up to 8.

Chapter IV describes a method for separating the internal and external variables from the higher orbital angular momentum states. A moving system of reference is introduced with the property of leaving invariant the new coordinate systems under the group of permutations of three particles. Angular harmonics are calculated for S, P and D states.

With the Euler angular dependence separated in the basis of representations, a generalized Omnes angular momentum reduction is applied in Chapter V, section (ii), where a short account of this method is described.

Chapter V, section (i), recalls the main results and properties of Faddeev's formalism. These equations with the kernel iterated are expressed in the Tensor Representation where the final equation is already a set of coupled integral equations.

Chapter V, section (iii), introduces the basic results of the spin-isospin formalism for a three-nucleon system developed by Verde [19]. The Faddeev equation is finally written in its most general form for arbitrary orbital angular momentum and with spin-isospin dependence.

Various Appendices with the more relevant mathematical derivations are given at the end of the last chapter.

CHAPTER II

THE COORDINATE SYSTEM

One of the main difficulties in solving a problem dealing with more than two particles is the complexity arising from the high number of degrees of freedom present in the problem.

The selection of an appropriate coordinate system is essential for a simple solution of the problem since the equations get very long and involved as soon as the various interactions are considered.

In the three body problem the number of variables needed to describe the configuration of the particles are twelve; nine for the position, two for the spin and isotopic spin and one for the hypercharge.

In selecting the right set of coordinates one has to have in mind the behaviour of the variables under the transformations of the operators and try to find a coordinate system which transforms in the same way as the state functions. Since the bases of representations are constructed from the rotation and permutation groups it is natural to seek a set of coordinates that behaves under the same transformations in the same manner. Even more would be gained if it were possible to find a coordinate system that treats the three particles on the same footing. Then information can be obtained on all the particles at the same time, simplifying the complete solution of the problem. Thus we know the spatial correlation between particles $|1|$ and $|2|$, $|2|$ and $|3|$ and $|1|$ and $|3|$ simultaneously whereas in the usual asymmetric system of reference, the correlation between $|1|$ and $|2|$ is directly known but that between $|2|$ and $|3|$ or $|1|$ and $|3|$ has to be evaluated separately with the help of recoupling coefficients. This is specially important

for the multichannel Faddeev equations where all the possible interactions are treated at the same time.

The system of coordinates used in this work is completely symmetric and has very simple properties under rotation and permutation for all the three particles. It was first used in this problem by Dragt [21] and originally described by Dalitz and Fabri [22].

A brief description of this coordinate system and its properties will be given below for further reference. A more detailed analysis is found in the original papers [22].

Starting with the reduced C.M. system of three point particles in configuration space, an orthogonal transformation is performed on the ray vector $(\underline{r}_1, \underline{r}_2, \underline{r}_3)$ to the new system $(\underline{r}^{(1)}, \underline{r}^{(2)})$ thus:

$$\underline{r}^{(1)} = \frac{1}{\sqrt{2}} (\underline{r}_2 - \underline{r}_1) \qquad \underline{r}^{(3)} = \frac{1}{\sqrt{3}} (\underline{r}_1 + \underline{r}_2 + \underline{r}_3) = 0$$

$$\underline{r}^{(2)} = \frac{1}{\sqrt{6}} (2\underline{r}_3 - \underline{r}_1 - \underline{r}_2) \tag{2.1}$$

The normalizations are chosen to satisfy the following relation

$$\underline{r}^{(1)2} + \underline{r}^{(2)2} = \underline{r}_1^{\,2} + \underline{r}_2^{\,2} + \underline{r}_3^{\,2} = r^2 \tag{2.2}$$

where $(\underline{r}^{(1)}, \underline{r}^{(2)}) = \underline{r}$ is a six dimensional vector.

The body fixed axis is chosen so as to satisfy the equation

$$\sum_i (\underline{r}_i \cdot \underline{u}) \cdot (\underline{r}_j \cdot \underline{v}) = 0 \tag{2.3}$$

This implies that the body fixed axis becomes the principle axis of inertia in a triangle formed by particles of equal mass.

Since $\underline{r}_1 + \underline{r}_2 + \underline{r}_3 = 0$

$$\underline{r}^{(2)} = \sqrt{3/2}(\underline{r}_3)$$

and

$$\underline{r}^{(1)} = \frac{1}{\sqrt{2}}(\underline{r}_2 - \underline{r}_1) \qquad (2.4)$$

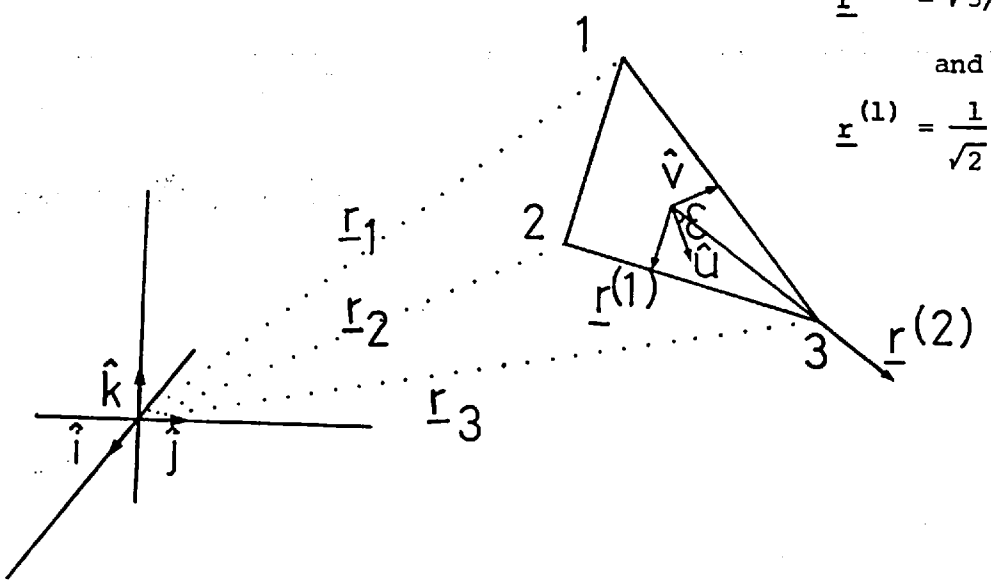


Figure 1

In the above figure 1 are shown the laboratory system of reference $(\hat{i}, \hat{j}, \hat{k})$, the body fixed system of reference $(\hat{u}, \hat{v}, \hat{w})$ and the angle ϵ between $\underline{r}^{(2)}$ and $\underline{u}$.

The rotation needed to bring the triangle formed by the particles in configuration space to the laboratory system is defined by the Euler angles (α, β, γ)

$$\begin{aligned} 0 \leq \alpha &\leq 2\pi \\ 0 \leq \beta &\leq \pi \\ 0 \leq \gamma &\leq 2\pi \end{aligned} \qquad (2.5)$$

These are called the external variables of the system.

The internal variables are parametrized bearing in mind the need to carry out a partial wave decomposition of harmonics in a 5 dimensional sphere. Consequently angular coordinates with finite ranges are required. This is accomplished by using a well known trigonometric property that says that the sum of the three distances from an interior point to the sides of an equilateral triangle of unit altitude is one.

The three particle system in the Dalitz plot, and the Fabri polar coordinates (ρ,ϕ) are defined by the relations

$$\underline{r}_i^{\,2} = \frac{1}{3}\underline{r}^2(1 + \rho\xi_i)$$

$$\underline{r}_{jk}^{\,2} = \underline{r}^2(1 - \rho\xi_i) = (\underline{r}_k - \underline{r}_j)^2 \qquad (2.6)$$

$$\xi_1 = \cos(\phi - 2/3\pi)$$

$$\xi_2 = \cos(\phi + 2/3\pi)$$

$$\xi_3 = \cos\phi$$

The Dalitz-Fabri system of coordinates, eqn. (2.6), satisfies in the C.M. the condition $\rho^2 \leq 1$. This allows us to define the angular variable ψ by $\rho = \cos 2\psi$ (2.7)

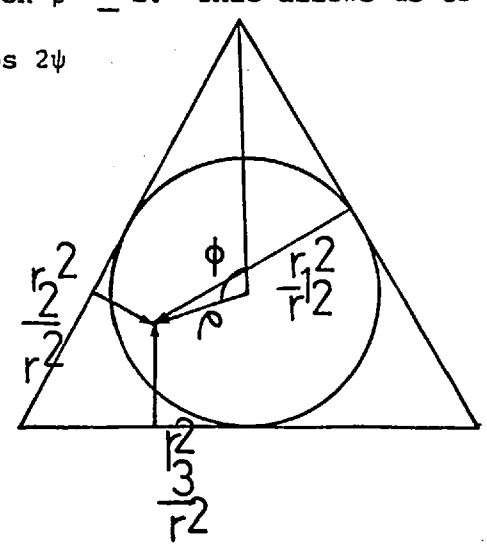


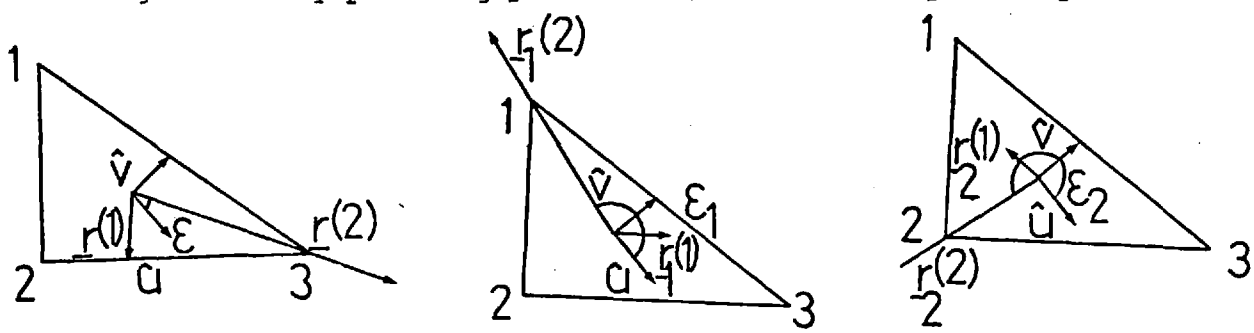
Figure 2. Kinematic triangle of unit altitude and O as centroid.

The circle inside this triangle (not to be confused with the configuration triangle) represents the kinematically feasible region of the three nucleon interactions.

To complete our description of coordinates we note that equation (2.3) does not define the body fixed axis uniquely. This arbitrariness is removed by defining the angle ε between $\underline{r}^{(2)}$ and $\underline{u}$ by the relations

$$\begin{aligned} \underline{r}^{(1)} &= |\underline{r}|(\cos\psi \sin\tfrac{\phi}{2}\,\underline{u} - \sin\psi\cos\phi\underline{v}) \\ \underline{r}^{(2)} &= |\underline{r}|(\cos\psi\cos\tfrac{\phi}{2}\,\underline{u} + \sin\psi\sin\tfrac{1}{2}\phi\underline{v}) \end{aligned} \tag{2.8}$$

and the angle ε is shown in fig. 1. In order to apply the generalized angular momentum reduction we need to know the transformation properties of the original system $(\underline{r}^{(1)},\underline{r}^{(2)},\varepsilon)$ and of the two complementary systems $(\underline{r}_1^{(1)}, \underline{r}_1^{(2)},\varepsilon_1)$ and $(\underline{r}_2^{(1)},\underline{r}_2^{(2)},\varepsilon_2)$ which are obtained from the original one by permuting particles 1, 2 and 2,3 respectively.



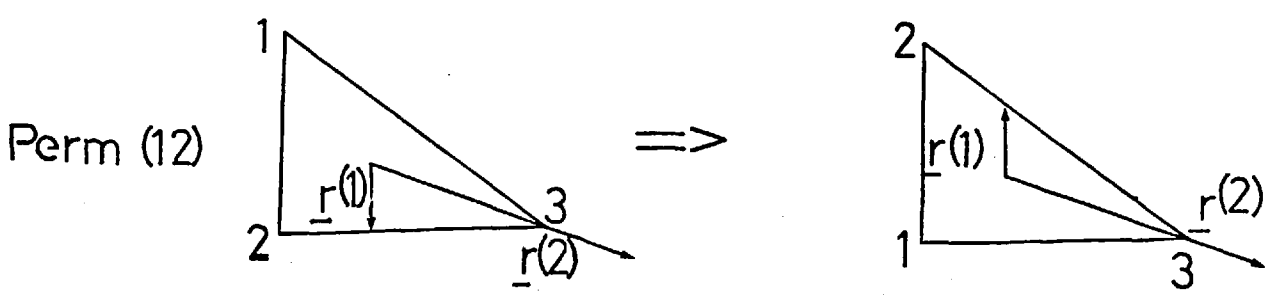


Figure 3

From the above figures can be easily seen that

$$(12) \quad \begin{pmatrix} \underline{r}^{(1)} \\ \underline{r}^{(2)} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \underline{r}^{(1)} \\ \underline{r}^{(2)} \end{pmatrix} \qquad \text{etc.}$$

Performing all the transformations indicated we obtain the following relations

$$\begin{pmatrix} \underline{r}_1^{(1)} \\ \underline{r}_1^{(2)} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \underline{r}^{(1)} \\ \underline{r}^{(2)} \end{pmatrix} \tag{2.9}$$

and

$$\begin{pmatrix} \underline{r}_2^{(1)} \\ \underline{r}_2^{(2)} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \underline{r}^{(1)} \\ \underline{r}^{(2)} \end{pmatrix}$$

where (12) = Permutation of 1 with 2.

The following figures illustrate the fact that a permutation of any two particles leaves the coordinates r and ψ invariant.

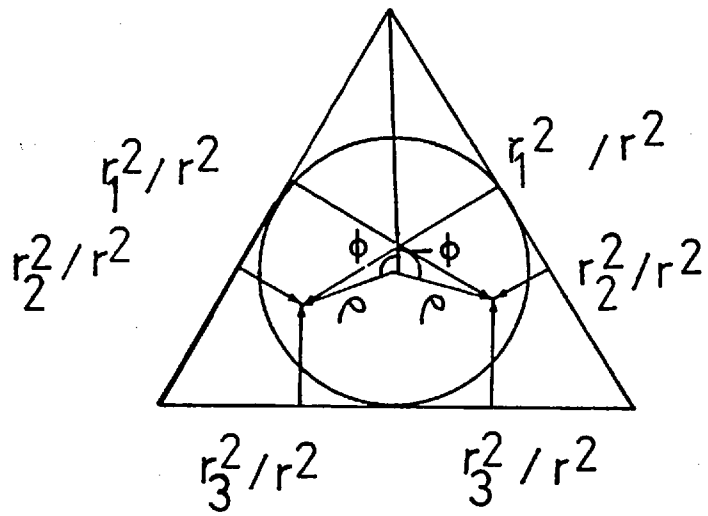


Figure 4.

We now express the transformation properties of ϕ such that for any permutation of the particles the Euler angles are invariant. Use is made of Appendix 2 in Lee's work [18].

Thus

$$
\begin{aligned}
(12)\phi &= -\phi & (12)\alpha &= (23)\alpha = (31)\alpha = \alpha \\
(23)\phi &= -\phi + \frac{2\pi}{3} & (12)\beta &= (23)\beta = (31)\beta = \beta - \pi \\
(31)\phi &= -\phi - \frac{2\pi}{3} & (12)\gamma &= (23)\gamma = (31)\gamma = \pi - \gamma \qquad (2.10) \\
(123)\phi &= \phi + \frac{2\pi}{3} \\
(132)\phi &= \phi + \frac{2\pi}{3}
\end{aligned}
$$

Finally the volume elements are given by

$$\underline{d\tau} = \frac{1}{8} r^5 \, dr \, \rho d\rho \, d\phi d\alpha \sin\beta d\beta d\gamma \qquad (2.11)$$

where

$$\underline{d\tau} = \prod_{i=1}^{6} dr_i$$

CHAPTER III

THE THREE NUCLEON STATES

i) The Classification of Three Particles

In order to solve the Faddeev equations for the three nucleon problem it is necessary to construct a complete set of wave functions and a maximal set of commuting operators which fully describes the system.

The choice of this classification is not unique. One uses the general principles of invariance to select an adequate set of operators that can characterize the observables in the systems. A natural way of doing this is to diagonalize those variables which are known to be constants of motion in a free system, that is, in a system without external or internal interacting forces. The eigenvalues of these maximal sets of operator will enumerate the bases of representation of all the physically plausible states.

Various successful attempts were made in the classification of nucleon states, the most useful being the group theory, the differential and the algebraic approaches. The group theoretical approach exploits the invariance of the free Hamiltonian of the system under the group O(6) or the subgroup SU(3) transformations. Dragt [21] succeeded in developing a full classification of the SU(3) group based on previous papers by Smith [23]. A brief description of this approach follows.

The most general group of transformations of the three ray vectors, eqn. (2.1), is given by the full orthogonal group O(9). This is reduced to O(6) when going to the C.M. system eliminating three variables. The

algebra of this group is given by 15 linearly independent 6 x 6 antisymmetric matrices. Dragt denotes this by L_0.

There are several ways of reducing this orthogonal group but in view of the need to include the spin and isotopic spin dependence in the formalism and since the Pauli principle requires that the wave functions must possess definite symmetry properties under permutations, Dragt carried this reduction to include the permutations group of three particles P(3). The basis of representations that possesses definite symmetry properties under permutation of all three particles are easily handled to construct states which are antisymmetric with respect to exchange of nucleons.

L_1 is constructed from L_0 such that all its elements treat the three particles symmetrically. Its Lie algebra is formed with nine elements that can be written as:

$$\underline{J} = \sum_{i=1}^{3} \underline{r}_i \times \underline{q}_i - (\Sigma \underline{r}_i) \times (\Sigma \underline{q}_i) \text{ for 3 objects and}$$

$$\sqrt{3}\,\underline{K} = (\underline{r}_1 - \underline{r}_2)\underline{q}_3 + (\underline{r}_2 - \underline{r}_3)\underline{q}_1 + (\underline{r}_3 - \underline{r}_1)\underline{q}_2 + (\underline{q} \leftrightarrow \underline{r}) \text{ for 6 objects} \tag{3.1}$$

where $\underline{J}$ is symmetric under interchange of two indices and $\underline{K}$ is antisymmetric under interchange of any two indices.

Extracting an operator $S = \frac{1}{2}\Sigma K_{ij}$ from K, a simple Lie Group of eight elements is obtained, the SU(3) group vid. Dragt [21].

In other words the problem of classifying the three nucleons reduces to finding a complete basis of representations of the group of rotations in ordinary three dimensional space and a representation of the group of permutations of three particles.

This classification of the SU(3) group on the six-dimensional sphere includes the O(3) group of rotations and the group P(3) of permutation. So here lies the powerful usefulness of this group for the three nucleon system.

According to the general theory of groups, there are required five labels to characterize an irreducible representation of the SU(3) group. Two are provided by the eigenvalues of the Casimir operator S and Λ^2 eqs. (3.4) and (3.5). Λ^2 is associated with the Grand Total Angular Momentum [23] and S with the permutation symmetries of the three particles. Two more labels are obtained by the eigenvalues of the three dimensional rotational generators J^2 and J_z, eqn (3.6) and (3.7). The fifth and last label w appears only when higher orbital angular momenta are taken into account, its interpretation does not have any direct physical meaning. This label w has been extensively discussed by various authors [24].

The final SU(3) states contains all the required properties for the full and adequate description of the present system but unfortunately this elegant method gets very involved and complicated for wave functions with λ greater than three, thus leaving the problem with an incomplete base of representation.

Faced with these shortcomings Lee in his Ph.D thesis [18] adopted a variation of the above mentioned method where the final states are obtained by the solution of a differential equation. This method was inspired by the original work of Zickendraht [25]. In a paper by Levy-Leblond and Leby-Nahas [26] the relation between the basis constructed from the SU(3) group approach and the differential approach by Zickendraht is revealed.

Following Zickendraht, Lee obtained on eigenstate for L = 0 in the SU(3) classifications on a five-dimensional sphere S_5 solving a reduced

Schroedinger equation where the internal and external variables had been previously separated. For a detailed presentation of this method the reader is strongly advised to refer to Lee's work [18].

The Casimir operators are expressed in terms of the Dalitz-Fabri coordinates. Thus:

$$\Lambda^2 = -\left(\frac{\partial^2}{\partial\psi^2} + \frac{4\cos 4\psi}{\sin 4\psi}\frac{\partial}{\partial\psi} + \frac{4}{\cos^2 2\psi}\frac{\partial^2}{\partial\phi^2}\right) + \left[\frac{2\underline{L}^2}{\sin^2 2\psi} + \left(\frac{1}{\cos^2 2\psi} - \frac{2}{\sin^2 2\psi}\right)\underline{L}_\omega^2 + \right.$$

$$\left. + 4i\hbar\,\frac{\sin 2\psi}{\cos^2 2\psi}\,\underline{L}_\omega\frac{\partial}{\partial\phi} + \frac{\cos 2\psi}{\sin^2 2\psi}\,(L_+^2 - L_-^2)\right] \tag{3.2}$$

and

$$S = 2i\hbar\frac{\partial}{\partial\phi} \tag{3.3}$$

where $\underline{L} = (L\underline{u}, L\underline{v}, L\underline{w})$ is the angular momentum operator with respect to $(\underline{u}, \underline{v}, \underline{w})$.

The surface harmonies $S^{\mu}_{\lambda LM}(\hat{r})$ are eigenfunctions of Λ^2, S, L^2 and L_z

1. $$\Lambda^2 S^{\mu}_{\lambda LM}(\hat{\underline{r}}) = \lambda(\lambda+4)\hbar^2 S^{\mu}_{\lambda LM}(\hat{\underline{r}}) \tag{3.4}$$

2. $$S\,S^{\mu}_{\lambda r}(\hat{\underline{r}}) = 2\mu\hbar S^{\mu}_{\lambda r}(\hat{\underline{r}}) \tag{3.5}$$

3. $$L^2 S^{\mu}_{\lambda LM}(\hat{\underline{r}}) = L(L+1)\hbar S^{\mu}_{\lambda LM}(\hat{\underline{r}}) \tag{3.6}$$

4. $$L_z S^{\mu}_{\lambda LM}(\hat{\underline{r}}) = M\hbar S^{\mu}_{\lambda LM}(\hat{\underline{r}}) \tag{3.7}$$

The final states are characterized by 5 quantum numbers (λ,μ,L,M,ω) where λ labels the six-dimensional angular momentum, μ the properties under permutations of all three particles, L the orbital angular momentums, M its projections on the z axis of the space-fixed reference frame and finally w the previously mentioned eigenvalue of the fifth operator, respectively.

The explicit derivations of the surface harmonies are obtained by expanding $S^{\mu}_{\lambda LM}(r)$ in terms of the rotation elements $D^{L}_{MK}(\alpha,\beta,\gamma)$ [27] of the SO(3) subgroup and substituting it in eqn. (3.4). Hence

$$S^{\mu LM}_{\lambda}(\hat{\underline{r}}) = \sum_{K} G^{LMK}_{\lambda,\mu}(\psi,\phi) D^{L}_{MK}(R) \tag{3.8}$$

The differential equation to be solved is

$$\left[\frac{\partial^2}{\partial\psi^2} + \frac{4\cos4\psi}{\sin4\psi}\frac{\partial}{\partial\psi} - \frac{4\mu^2}{\cos^2 2\psi} - \frac{2}{\sin^2\psi}(L(L+1) - K^2)\right.$$

$$\left. - 4\mu K\frac{\sin2\psi}{\cos^2 2\psi} + \lambda(\lambda+4)\right] g^{LM,K}_{\lambda\mu}(\psi) = \frac{\cos2\psi}{\sin^2 2\psi} x$$

$$(C^{L}_{K+1} g^{LM,K+2}_{\lambda\mu}(\psi) + C^{L}_{K-1} g^{LM,K-2}_{\lambda\mu}(\psi)) \tag{3.9}$$

with

$$C^{L}_{K} = [(L+K+1)(L+K)(L-K)(L-K+1)]$$

giving the solution for L = 0 equal to

$$S_{\lambda\mu}(\hat{\underline{r}}) = N_{\Omega}\, e^{-i\mu\phi}(\cos2\phi)^{|-\mu|} P^{|\mu|,0}_{\frac{1}{2}(\lambda/2-|\mu|)}(1 - 2\cos^2 2\psi) \tag{3.10}$$

Up to this point everything works very well and according to the original expectations. Unfortunately, for higher orbital angular momentum, $L \neq 0$, the solution of the above differential equation does not have definite symmetry properties under permutation of all three particles, curtailing the advantages of the present method. Furthermore for $L \neq 0$ the solutions get very involved and complicated, it being difficult to obtain explicit analytical expressions for the function $S^{\mu}_{\lambda,LM}(\hat{\underline{r}})$.

Thus continuing with this method one obtains functions $S^{\mu}_{\lambda,LM}(\hat{\underline{r}})$ expressed in terms of states with mixed symmetry properties under permutations and with a very complex angular dependence. Even if linear combinations were found with the appropriate symmetry properties, the result would not be useful since the final numerical computations would be excessive with present high speed electronic computers.

What is needed is a new way of constructing the states of the three nucleons, with the appropriate properties.

ii) Construction of the Tensor Representation

The intention is to construct a basis of representation that fully classifies the three nucleon states with the previously discussed properties.

A brief discussion of the role of the six-dimensional Laplacian in the classification of Zickendraht and Lee will lead to the development of the algebraic method employed in the present work.

The Grand total angular momentum operator can be expressed as a function of the six-dimensional Laplacian ∇^2_6 by [18]

$$\Lambda^2 = -\hbar r(\nabla_6^2 - \frac{1}{r^5}\frac{\partial}{\partial r}(r^5\frac{\partial}{\partial r})) \tag{3.11}$$

It can be shown that the Λ^2 operator corresponds to the angular part of the ∇_6^2 operator.

After separating the Laplacian into two parts, one of them a function of the internal variables and the other a function of the external variables, ∇_6^2 can be expressed in terms of the usual three-dimensional orbital angular momentum operator. In other words the decomposition of the Λ^2 operator in terms of ∇_6^2 enables one to specify the three-nucleon states as eigenfunctions of the orbital angular momentum operator, and hence in terms of elements of the group SO(3).

The surface harmonics $S^{\mu}_{\lambda LM}$ are related to the harmonic polynomials by the expression

$$S^{\nu}_{\lambda}(\hat{\underline{r}}) = \frac{1}{r\lambda} P^{\nu}_{\lambda}(\underline{r}) \tag{3.12}$$

where

$$\nabla_6^2 P^{\nu}_{\lambda}(\underline{r}) = 0 \tag{3.13}$$

From previous work of A.M. Baladyan and Yu. A. Simonov [11] it is known that a set of harmonic polynomials can be constructed to classify a three nucleon system. Their method of construction implies the solution of the following differential equation:

$$\frac{1}{\rho^5}\frac{d}{d\rho}(\rho^5\frac{d\chi_K}{d\rho}) + (\varepsilon - \frac{K(K+4)}{\rho^2})\chi_K = 0 \tag{3.14}$$

As can be easily seen this equation is similar to Lee's differential equation [18] eqn (4.14) having a solution with the same disadvantages. However those disadvantages can be avoided. What is needed is a complete set of functions which satisfy the equation

$$\nabla_6^2 \, r^\lambda \, S^{\mu\omega}_{\lambda,LM} = 0 \tag{3.15}$$

constructed in such a way that there is no need to solve the differential equation with its inherent complexities. This set should be classified by the previous "physical" quantum numbers (λ,μ,L,M,ω).

Following Simonov and Pustovalov [17] a complete set of harmonic polynomials in the six-dimensional real space is constructed performing tensor products of complex vectors.

Defining the complex variable $\underline{z}$ as

$$\begin{aligned} -\underline{z} &= \underline{r}^{(2)} + i\underline{r}^{(1)} \\ -\underline{z}^* &= \underline{r}^{(2)} - i\underline{r}^{(1)} \end{aligned} \tag{3.16}$$

A transformation from the six-dimensional real sphere to the three dimensional complex sphere is defined where

$$\underline{r}^{(1)^2} + \underline{r}^{(2)^2} = |\underline{z}|^2 = \underline{r}^2 \tag{3.17}$$

is an invariant of the group SU(3) $\supset$ O(6). From eqn (3.15) it can be seen that any operator applied to $r^\lambda S^{\mu\omega}_{\lambda,LM}$ will give a harmonic polynomial whenever such operator commutes with the three dimensional complex Laplacian.

Using general properties of the differential calculus equations (3.18) hold for a wide range of functions

$$[\Delta, \frac{\partial}{\partial z_i}] = [\Delta, \frac{\partial}{\partial z_j^*}] = 0 \tag{3.18}$$

where

$$\Delta = \frac{\partial^2}{\underline{\partial z}\,\underline{\partial z}^*}$$

Thus applying the differential operators $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial z_j^*}$ to the scalar polynomial $r^\lambda S^\mu_{\lambda,LM}$ for L = 0 equation (3.10) we obtain harmonic polynomials as functions of the complex vectors $\underline{z},\underline{z}^*$. Furthermore, since the differential operators are vectors, the geometric combination of L consecutive applications of these operators forms a tensor of rank L. So it is possible to construct harmonic polynomials for arbitrary L. We now define

$$Q^{\lambda\mu LM\omega}_{pq} = \frac{\partial}{\partial z_{i(1)}}\frac{\partial}{\partial z_{j(2)}} \cdots \frac{\partial}{\partial z_{k(P)}}\frac{\partial}{\partial z^*_{l(1)}}\frac{\partial}{\partial z^*_{m(2)}} \cdots \frac{\partial}{\partial z^*_{n(q)}} r^\lambda S^{\mu\omega}_{\lambda LM} \tag{3.19}$$

where P is the number of differential operator with respect to z_i

and q " " " " " " " " " " z_i^*

From equation (3.13) and equation (3.18) it follows that the function $Q^{\lambda\mu LM\omega}_{pq}$ are harmonic polynomials (H.P.). Thus for the H.P. corresponding to states L = 1,2,3 and n we obtain the following results.

The vector state L = 1 is given by putting p = 1 and q = 0. Thus

$$\Delta \frac{\partial}{\partial z_i} r^{\lambda} S_{\lambda}^{\nu} = \frac{\partial}{\partial z_i} r^{\lambda} S_{\lambda}^{\nu} = \Delta Q_{10}^{\lambda,\nu} = 0 \qquad k = \tfrac{1}{2}$$

and putting p = 0 and q = 1 we get

$$\Delta \frac{\partial}{\partial z_i^*} r^{\lambda} S_{\lambda}^{\nu} = \frac{\partial}{\partial z_i^*} \Delta r^{\lambda} S_{\lambda}^{\nu} = \Delta Q_{01}^{\lambda,\nu} = 0 \qquad k = -\tfrac{1}{2} \qquad (3.20)$$

The Tensor state L = 2, is obtained by putting p + q = 2, (p,q) = (2,0), (1,1), (0,2) giving

$$\Delta \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} r^{\lambda} S_{\lambda}^{\nu} = \Delta Q_{20}^{\lambda,\nu} = 0 \qquad k = 1,$$

$$\Delta \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j^*} r^{\lambda} S_{\lambda}^{\nu} = \Delta Q_{11}^{\lambda,\nu} = 0 \qquad k = 0, \qquad (3.21)$$

$$\Delta \frac{\partial}{\partial z_i^*} \frac{\partial}{\partial z_j^*} r^{\lambda} S_{\lambda}^{\nu} = \Delta Q_{02}^{\lambda,\nu} = 0 \qquad k = -1.$$

Similarly for Tensor state L = 3 expressions below are obtained.

$$\Delta \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_R} r^{\lambda} S_{\lambda}^{\nu} = \Delta Q_{30}^{\lambda,\nu} = 0 \qquad k = 3/2$$

$$\Delta \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_R^*} r^{\lambda} S_{\lambda}^{\nu} = \Delta Q_{21}^{\lambda,\nu} = 0 \qquad k = 1/2 \qquad (3.22)$$

$$\Delta \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j^*} \frac{\partial}{\partial z_R^*} r^{\lambda} S_{\lambda}^{\nu} = \Delta Q_{12}^{\lambda,\nu} = 0 \qquad k = -1/2$$

$$\Delta \frac{\partial}{\partial z_i^*} \frac{\partial}{\partial z_j^*} \frac{\partial}{\partial z_R^*} r^{\lambda} S_{\lambda}^{\nu} = \Delta Q_{03}^{\lambda,\nu} = 0 \qquad k = -3/2$$

Generalizing for a tensor of rank n we find putting L = n, or p + q = n, n + 1 states

$$\Delta \frac{\partial}{\partial z_{i(1)}} \frac{\partial}{\partial z_{j}(2)} \cdots \frac{\partial}{\partial z_{\ell(n)}} r^{\lambda} S_{\lambda}^{\nu} = \Delta Q_{n0}^{\lambda\nu} = 0 \qquad k = n/2 \qquad (3.23)$$

$$\Delta \frac{\partial}{\partial z_{i1}} \frac{\partial}{\partial z_{j2}} \cdots \frac{\partial}{\partial z_{(n-1)}} \frac{\partial}{\partial z^{*}_{m}(n)} r^{\lambda} S_{\lambda}^{\nu} = \Delta Q_{n-1,1}^{\lambda\nu} = 0 \qquad k = n/2 - 1$$

$$\Delta \frac{\partial}{\partial z^{*}_{i1}} \cdots \cdots \frac{\partial}{\partial z^{*}_{j}(n)} r^{\lambda} S_{\lambda}^{\nu} = \Delta Q_{0n}^{\lambda\nu} = 0 \qquad k = -n/2$$

where ν represents all the other indices in the polynomials.

The harmonic polynomials $Q_{pq}^{\lambda\nu}$ or order L + 1 is obtained from the H.P. of order L by applying the differential operator once. Performing L differentiations on $r^{\lambda} S_{\lambda}^{\nu}$ produces a H.P. of order L. The function S_{λ}^{ν} were obtained by Lee [18] being a representation of SU(3) with L = 0.

From the general theory of harmonic polynomials [28] it is known that our H.P.'s are labelled by five eigenvalues. So the correspondence between the SU(3) states and these harmonic polynomials is readily inferred.

In order to make the $Q_{pq}^{\lambda\mu LM\omega}$ eigenfunctions of the total orbital angular momentum operator it is necessary to symmetrize and make traceless the tensor differential operators $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial z^{*}_{j}}$.

This is done with the help of symmetrization and P_s^L is defined by trace vanishing operators [29], P_s^L and P_t^L respectively.

$$P_s^L T^L_{i_1 i_2 \cdots i_L} = (L!)\, \Sigma_P\, T^L_{i_1 i_2 \cdots i_L} \qquad (3.24)$$

where $T_{i_1 i_2 \cdots i_L}$ in a tensor of rank L and Σ_P is the summation over all the <u>different</u> permutations of the indices $i_1, i_2, \ldots, i_L$.

P_t^L is defined by

$$P_t^L T^L_{12..L} = T_{i_1 i_2 .. i_L} - (2L-1)^{-1} \Sigma_P \delta_{12} T_{i_n i_n i_3 \ldots i_L} +$$

$$+ (2L-1)^{-1}(2L-3)^{-1} \Sigma_P \delta_{12}\delta_{34} T^L_{i_n i_n i_R i_R i_S \ldots i_L} - \ldots . \qquad (3.25)$$

Therefore the operator $P^L = P_S^L P_t^L$ (3.26) when applied to an arbitrary tensor makes it symmetric and traceless. A detailed description of the working of this operator for a spherical tensor of rank two is given in equations (3.30), (3.31) and (3.32).

The third quantum number M, representing the projections of the total orbital angular momentum L onto the z axis can be obtained by using spherical coordinates. These are given in terms of their Cartesian components by

$$z_+ = \frac{1}{\sqrt{2}}(z_1 + iz_2) \qquad z_0 = z_3 \qquad z_- = -\frac{1}{\sqrt{2}}(z_1 - iz_2) \qquad (3.26)$$

Analogously for $\underline{z}^*$ we have

$$z_+^* = \frac{1}{\sqrt{2}}(z_1^* + iz_2^*) \qquad z_0^* = z_3^* \qquad z_-^* = -\frac{1}{\sqrt{2}}(z_1^* - iz_2^*) \qquad (3.27)$$

Where the space matrix g^{ij} is given by

$$g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The total orbital angular momentum operator L is symmetric with respect to $\underline{z}$ and $\underline{z}^*$, which is given by

$$L = -i\hbar([\underline{z}, \partial/\partial\underline{z}] + [\underline{z}^*, \partial/\partial\underline{z}^*])$$

Therefore the tensors $T^L_{i_1 i_2 \ldots i_L}$ can be expressed in spherical coordinates as $T^L_{lmn;l^*m^*n^*}$ with $(lmn;l^*m^*n^*)$ taking the components $(+,0,-)$. We take M the projection onto the z axis to be the sum of the projections of $\underline{z}$ and $\underline{z}^*$.

Therefore we have

$$M = (m - n) + (m^* - n^*) \qquad (3.28)$$

An illustration of the working of the present method for L = 2 is given.

Three spherical tensors of rank two can be constructed from the vectors $\underline{z}$ and $\underline{z}^*$, namely $z_i z_j$, $z_i z_j^*$ and $z_i^* z_j$ with $i,j = (+,0,-)$. Denoting by $l(r)$ the number of components $z_i (z_i^*)$ in the spherical tensor and defining k by $k = \frac{l-r}{2}$, (3.29), we can see that for L = 2 k takes one of the three possible values $(1,0,-1)$.

With the help of the operator P^L eqn. (3.26), we obtain the three symmetric and traceless spherical tensors T^{2M1}, T^{2M0}, T^{2M-1} which we write as matrix arrays of their components.

Thus,

L = 2, k = 1 l = 2 r = 0

$$P^2(z_i z_j)^M = T^{2M1} = \begin{bmatrix} z_+z_+ - 1/3z^2 & z_+z_0 & z_+z_- \\ z_0z_+ & z_0z_0 - 1/3z^2 & z_0z_- \\ z_-z_+ & z_-z_0 & z_-z_- - 1/3z^2 \end{bmatrix} \qquad (3.30)$$

$L = 2 \quad k = 0 \quad l = 1 \quad r = 1$

$$P^2(z_i z_j^*)^M = T^{2MO} = \begin{bmatrix} z_+ z_+^* - 1/3 \underline{z}\cdot\underline{z}^* & \tfrac{1}{2}(z_+ z_O^* + z_O z_+^*) & \tfrac{1}{2}(z_+ z_-^* + z_- z_+^*) \\ \tfrac{1}{2}(z_O z_+^* + z_+ z_O^* & z_O z_O^* - 1/3 \underline{z}\cdot\underline{z}^* & \tfrac{1}{2}(z_O z_-^* + z_- z_O^*) \\ \tfrac{1}{2}(z_- z_+^* + z_+ z_-^*) & \tfrac{1}{2}(z_- Z_O^* + z_O z_-^*) & \tfrac{1}{2}(z_- z_-^* - 1/2 \underline{z}\cdot\underline{z}^*) \end{bmatrix} \tag{3.31}$$

$L = 2 \quad k = -1 \quad l = 0 \quad r = 2$

$$P^2(z_i^* z_j^*)^M = T^{2M-1} = \begin{bmatrix} z_+^* z_+^* - 1/3 z^{*2} & z_+^* z_O^* & z_+^* z_-^* \\ z_O^* z_O^* & z_O^* z_O^* - 1/3 z^{*2} & z_O^* z_-^* \\ z_-^* z_+^* & z_-^* z_O^* & z_-^* z_-^* - 1/3 z^{*2} \end{bmatrix} \tag{3.32}$$

By the symmetry of the tensors $T^2_{lmn;l^*m^*n^*}$ we have

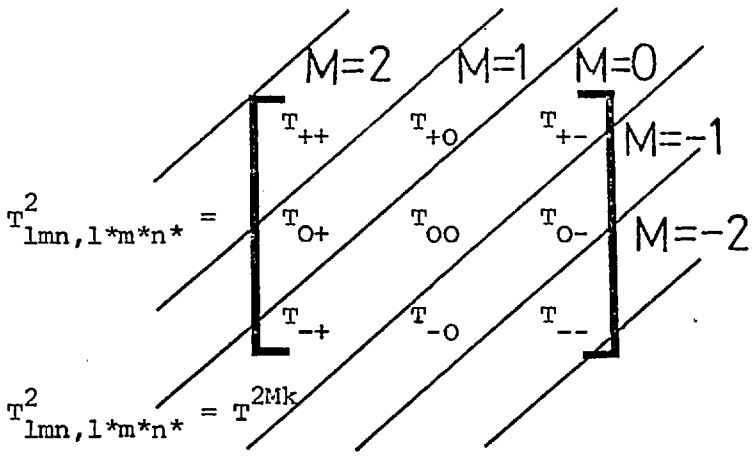
$$\begin{aligned} T_{+O} &= T_{O+} \\ T_{+-} &= T_{-+} \\ T_{-O} &= T_{-O} \end{aligned} \tag{3.33}$$

and due to the fact that the spherical tensors $T^2_{lmn;l^*m^*n^*}$ is traceless we have the following property which can be easily verified for $L = 2$; Thus

$$T_{-+} = \tfrac{1}{2} T_{OO} \tag{3.34}$$

Hence the tensor $T^2_{lmn,l^*m^*n^*}$ can be characterized by only two numbers $L = 2$ and M.

Thus for L = 2 we have



(3.35)

where for L = 2 M = 2,1,0,-1,-2

These properties of the spherical tensors for arbitrary L can be expressed as

$$T^{L}_{lmn,l*m*n*} = T^{LMk} \tag{3.36}$$

where M = -L, -L + 1,...L-1, L.

Defining

$$(\partial/\partial z,\partial/\partial z^*)^{L-M}_{pq} = \frac{\partial}{\partial z_{i(1)}}\frac{\partial}{\partial z_{j(2)}}\cdots\frac{\partial}{\partial z_{m(P)}}\frac{\partial}{\partial z^*_{n(1)}}\frac{\partial}{\partial z^*_{l(2)}}\cdots\frac{\partial}{\partial z^*_{n(q)}} \tag{3.37}$$

We can write the final wave function $\phi^{\mu\omega}_{\lambda LM}$ as

$$\phi^{\mu\omega}_{\lambda LM} = P^{L}Q^{\lambda\mu LM\omega}_{pq} = P^{L}(\partial/\partial z,\partial/\partial z^*)^{L-M}_{pq}\, r^{\lambda}S^{\mu}_{\lambda}(\hat{r}) \tag{3.38}$$

The action of the differentials $\partial/\partial z_i$ and $\partial/\partial z^*_j$ on $r^{\lambda}S^{\mu}_{\lambda}(\hat{r})$ is shown in Appendix 1 and the effect of the operator $P^{L}(\partial/\partial z_1\ \partial/\partial z^*)^{L-M}_{pq}$ on $r^{\lambda}S^{\mu}_{\lambda}$ for arbitrary L, M and (p,q) can be obtained by induction looking at the tensors for L = 1 and 2.

By eqns (A2.5), (A2.9) and (A2.12) we can write

$L = 1 \quad P = 1 \quad q = 0$

$$P^1 \frac{\partial}{\partial z_i} = T^{1M\frac{1}{2}}\Lambda_1 + T^{1M-\frac{1}{2}}\Lambda_2 \tag{3.39}$$

$L = 1 \quad P = 0 \quad q = 1$

$$P^1 \frac{\partial}{\partial z_i^*} = T^{1M-\frac{1}{2}}\Lambda_1^* + T^{1M\frac{1}{2}}\Lambda_2 \tag{3.40}$$

$L = 2 \quad P = 2 \quad q = 0$

$$P^2 \frac{\partial}{\partial z_{i(1)}} \frac{\partial}{\partial z_{j(2)}} = T^{2M1}\Lambda_1^2 + 2T^{2M0}\Lambda_1\Lambda_2 + T^{2M-1}\Lambda_2^2 \tag{3.41}$$

$$= (T^{1M\frac{1}{2}}\Lambda_1 + T^{1M-\frac{1}{2}}\Lambda_2)^2$$

$L = 2 \quad P = 1 \quad q = 1$

$$P^2 \frac{\partial}{\partial z_{i(1)}} \frac{\partial}{\partial z_{j(2)}^*} = T^{2M1}\Lambda_1\Lambda_2 + 2T^{2M0}(\Lambda_1\Lambda_2^* + \Lambda_2^2) + T^{2M-1}\Lambda^{*2}$$

$$= (T^{1M\frac{1}{2}}\Lambda_1 + T^{1M-\frac{1}{2}}\Lambda_2)(T^{1M\frac{1}{2}}\Lambda_2 + T^{1M-\frac{1}{2}}\Lambda_1^*) \tag{3.42}$$

$L = 2 \quad P = 0 \quad q = 2$

$$P^2 \frac{\partial}{\partial z_{i(1)}^*} \frac{\partial}{\partial z_{j(2)}^*} = T^{2M1}\Lambda_2^2 + 2\Lambda_1^*\Lambda_2 + T^{2M-1}\Lambda_1^*$$

$$= (T^{1M-\frac{1}{2}}\Lambda_1^* + T^{1M\frac{1}{2}}\Lambda_2)^2 \tag{3.43}$$

Thus for arbitrary L we have

$$P^L(\partial/\partial z, \partial/\partial z^*)_{pq}^{L-M} = [T^{1M\frac{1}{2}}\Lambda_1 + T^{1M-\frac{1}{2}}\Lambda_2]^P[T^{1M\frac{1}{2}}\Lambda_1^* + T^{1M-\frac{1}{2}}\Lambda_2]^q \tag{3.44}$$

Using the binomial expansions formula and expressions (p,q) in terms of L and ω the above relation can be written as

$$P^{L}(\partial/\partial z,\partial/\partial z^*)_{\omega}^{L-M} = \sum_{k=-L/2}^{L/2} T^{LkM}(z,z^*) \sum_{m=0}^{L+2\omega/2} \binom{L+2\omega}{2}{m} \binom{L-2\omega}{2}{k-\omega+m}$$

$$x\ \Lambda_1^{L/2+\omega-m}\ \Lambda_1^{*L/2-k-m}\ \Lambda_2^{2m+k-\omega} \qquad (3.45)$$

Where the effect of $\Lambda_1^{t}\Lambda_1^{*s}\Lambda_2^{a}$ on $r^{\lambda}S_{\mu}^{\lambda}(\hat{r})$ is given in Appendix 1.

The fourth quantum number μ representing the symmetric properties of the harmonic polynomials will be given by

$$\mu = \frac{1 - r}{2} \qquad (3.46)$$

The permutation properties of the $P^{L}Q_{pq}^{\lambda\mu LM\omega}$ are obtained from the behaviour of the complex variables z and z* by application of the generators of the group P(3).

Thus from eqns. (2.9) and (3.16) fthe following relations are easily obtained:

$$P_{12}\begin{pmatrix}\underline{z}\\ \underline{z}^*\end{pmatrix} = \begin{pmatrix}\underline{z}^*\\ \underline{z}\end{pmatrix} \qquad P_{13}\begin{pmatrix}\underline{z}\\ \underline{z}^*\end{pmatrix} = \begin{pmatrix}e^{i\pi/3}\,\underline{z}^*\\ e^{-i\pi/3}\,\underline{z}\end{pmatrix}$$

$$P_{23}\begin{pmatrix}\underline{z}\\ \underline{z}^*\end{pmatrix} = \begin{pmatrix}e^{-i\pi/3}\,\underline{z}^*\\ e^{i\pi/3}\,\underline{z}\end{pmatrix} \qquad (3.47)$$

The permutations of these variables are equivalent to rotations in the complex z-plane.

The construction of harmonic polynomials with definite symmetry properties under permutation are easily obtained by noting that the

action of P_{12} in $P^{I}Q^{\lambda\mu LM\omega}_{pq}$ is equivalent to interchanging z and z*. This implies that l and r (3.46) will also interchange thus,

$$(12)\phi^{\mu\omega}_{\lambda LM} = \phi^{-\mu-\omega}_{\lambda LM} \tag{3.48}$$

So we have

$$\mu' = \frac{l' - r'}{2} = -\frac{l - r}{2} = -\mu \tag{3.49}$$

Where the variables after the permutations are primed.

For (13) and (23) the harmonic polynomials transform as

$$(13)\ \phi^{\mu\omega}_{\lambda LM} = \phi^{-\mu-\omega}_{\lambda LM} e^{-4/3\pi i\mu} \tag{3.50}$$

$$(23)\ \phi^{\mu\omega}_{\lambda LM} = \phi^{-\mu-\omega}_{\lambda LM} e^{4/3\pi i\mu} \tag{3.51}$$

Using these properties it follows that

$${}_1\phi^{\mu\omega}_{\lambda LM} = \frac{1}{\sqrt{2}}(1 + (12))\phi^{\mu\omega}_{\lambda LM} \tag{3.52}$$

and

$${}_2\phi^{\mu\omega}_{\lambda LM} = \frac{1}{\sqrt{2}\,i}(1 - (12))\phi^{\mu\omega}_{\lambda LM} \begin{cases} 1 & \text{if } \mu = 3/2n \\ 2/3 \sin 4\pi/3\, n & \text{if } \mu \neq 3/2 \end{cases}$$

where for $\mu = 3/2n$, ${}_1\phi^{\mu\omega}_{\lambda LM}$ and ${}_2\phi^{\mu\omega}_{\lambda LM}$ are symmetric and antisymmetric respectively under any interchange of particles and for $\mu \neq 3/2$, ${}_1\phi^{\mu\omega}_{\lambda LM}$

and ${}_2\phi^{\mu\omega}_{\lambda LM}$ form the mixed representation of the group P(3).

Finally a word about orthogonality of this basis of representation. Since the four operators Λ^2, L^2, M and S commute with each other the eigenfunctions are orthogonal with respect to their indices. For L = 0, 1, 2 the harmonic polynomials constructed in Appendix 2 are also orthogonal with respect to ω. Since our polynomials are degenerate with respect to the quantum number ω, the orthogonalizations of the S.H. make it possible to fully and unambiguously classify the three nucleon states by the method developed above.

The tensor representations so far constructed in this chapter, together with the pseudotensor variant [17] obtained in a similar way but by performing vector products of $\underline{z}$ and $\underline{z}^*$, instead of the full tensor products, form a complete representation of the three body system.

In Appendix 2 an explicit derivation of the wave functions for L = 0, 1 and 2 is given together with a classification of their respective symmetries. Appendix 3 contains a table of wave functions for L = 0, 1, 2 and $\lambda \leq 8$.

CHAPTER IV

THE INTERNAL AND EXTERNAL COORDINATES

i) The Separation of the External Coordinates

A representation of the states of the three-nucleon systems was constructed by the Tensor polynomials $\phi^{\mu,\omega}_{\lambda LM}$ developed in Chapter III. These functions were constructed in the Euler-Dalitz-Fabri coordinate system specified by three internal variables (r,ψ,ϕ) and three external ones (α,β,γ), eqns. (2.5) and (2.6).

Even though these polynomials have all the required properties for the specification of all the possible states of three particles, its practical application proves difficult. This is apparent when substituting the $\phi^{\mu\omega}_{\lambda LM}$ states directly into the equation of motion, in our case the Faddeev equation. The difficulties are caused by the mixing of internal and external coordinates in the kernel of the integral equations and made worse by the large number of continuous variables. The powerful method of reducing the angular momentum - discussed in Chapter V - is inhibited by the mixing of the coordinates.

In its most general form the Tensor polynomials can be written using eqn. (3.38) as

$$\phi^{\mu\omega}_{\lambda LM} = P^{L}(\partial/\partial z,\ \partial/\partial z^{*})^{L-M}_{pq}\ r^{\lambda} S^{\mu\omega}_{\lambda LM} \tag{4.1}$$

In this expression the tensor $(\partial/\partial z,\ \partial/\partial z^{*})^{L-M}_{pq}$ is a function of the spherical components (z_0, z_-, z_+) which in turn are expressed as a mixed combination of all six coordinates. In other words, the polynomials

$\phi^{\mu\omega}_{\lambda LM}$ are parametrized by the six coordinates $(\alpha,\beta,\gamma,r,\psi,\phi)$ in such a way that the explicit expressions of these states contain all six coordinates in a mixed form.

In order to proceed to the generalized partial wave decomposition of the wave functions, $\phi^{\mu\omega}_{\lambda LM}$, it is advantageous to separate them into amplitudes containing only internal or external coordinates. We then associate the amplitudes of the external coordinates with the three dimensional rotation group; which is similar to the Omnes angular momentum reduction. (see Chapter V).

It must be pointed out that in separating the polynomials $\phi^{\mu\omega}_{\lambda LM}$ in terms of the internal (r,ψ,ϕ) and external coordinates (α,β,γ) the resulting functions should conserve its simple symmetry properties.

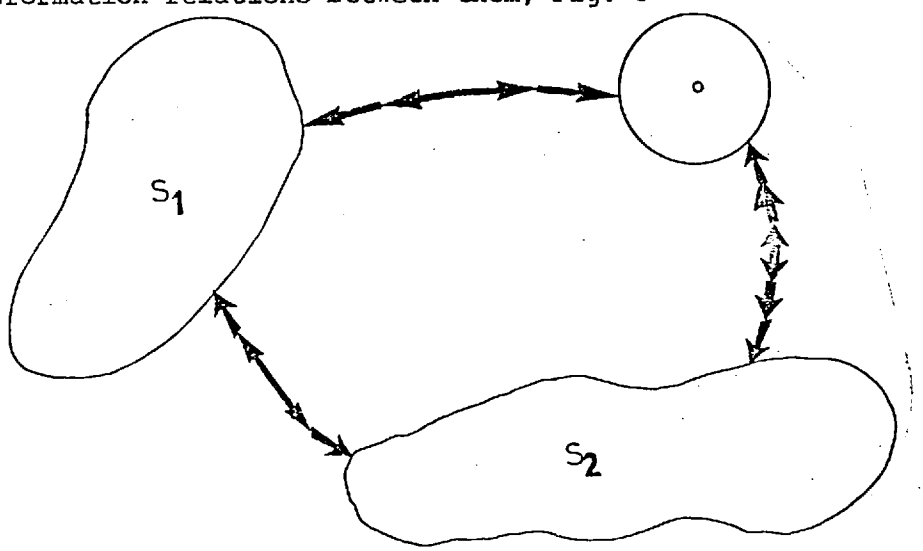
Thus our intention is to transform our previously found set of polynomials into a new set, that satisfies two constraints; firstly, adaptability to the angular momentum reduction formalism and secondly the requirement of being an irreducible representation of the SU(3) group. This is done in the present work with the help of the D-functions. Omnes [13] succeeded in reducing a classification of the three particles (different from our $\phi^{\mu\omega}_{\lambda LM}$) into a set of wave functions in only three continuous variables and Osborn [14] applied this "reduced" set to the Faddeev's equation obtaining a system of coupled integral equations in only two continuous variables.

In our formalism the D-functions play a key role. As these functions are invariant under permutations we can express our amplitude for the external coordinates in terms of D-functions and this enables us to relate the Omnes reduction method to ours. So we finally also obtain equations which are independent of the external coordinates.

In terms of the Euler angles (α,β,γ), the D-function can be expressed as [24]

$$D^{L}_{NL}(\alpha,\beta,\gamma) = (-1)^{L-N}\left[\binom{2L}{L-N}\right]^{\frac{1}{2}} e^{in\gamma}e^{iL\gamma}(\cos^2\frac{\beta}{2})^{L+N/2}(\sin^2\frac{\beta}{2})^{L-N/2} \tag{4.2}$$

Calling S_1 the set of functions in the Tensor Representation coordinate-system O) and S_2 our required set of polynomials, with the 'internal and external coordinates separated into two factors, the problem reduces to finding a projection of the set S_1 into S_2 and the transformation relations between them, Fig. 5



This is achieved by defining a moving set of coordinates such that the unit vectors (generating S_2) are invariant under any permutation of the particles. In this manner the transformation between S_1 and S_2 keeps the internal variables invariant so that all the changes occur in the external variables only. Thus it is possible to express these external changes as rotation with respect to the system, O, i.e. in terms of D-functions.

Performing the separation in this way the permutational properties of the functions are not affected by the permutational properties of the coordinate system.

Niri and Smorodinskii [30] and Pustovalov and Smorodinskii [24] used a different set of invariant moving vectors. The latter paper contains two mistakes (presumably typographical) which we found in trying to check

the invariance of two of the three defining vectors (equations 26 and 29 ibid.)

Let $(\underline{S}_{2x}, \underline{S}_{2y}, \underline{S}_{2z})$ be the vectors generating the set S_2. Invariance with respect to permutations of all three particles implies

$$\begin{array}{lll} (12)\ \underline{S}_{2x} = \underline{S}_{2x} & (12)\ \underline{S}_{2y} = \underline{S}_{2y} & (12)\ \underline{S}_{2z} = \underline{S}_{2z} \\ (23)\ \underline{S}_{2x} = \underline{S}_{2x} & (23)\ \underline{S}_{2y} = \underline{S}_{2y} & (23)\ \underline{S}_{2z} = \underline{S}_{2z} \\ (31)\ \underline{S}_{2x} = \underline{S}_{2x} & (31)\ \underline{S}_{2y} = \underline{S}_{2y} & (31)\ \underline{S}_{2z} = \underline{S}_{2z} \end{array} \qquad (4.3)$$

Where (ab) represents the permutation operator of particles a and b.

Using the permutational properties of the complex vectors $\underline{z}$ and $\underline{z}^*$, and of the internal coordinates (r,ψ,ϕ) given in Table 2, a collection of different sets of moving systems of reference that are invariant under any permutation can be constructed.

Table 2

$$\begin{array}{ll} (12)\ \underline{z} = \underline{z}^* & (12)\ \underline{z}^* = \underline{z} \\ (23)\ \underline{z} = \underline{z}^* e^{2\pi i/3} & (23)\ \underline{z}^* = \underline{z} e^{-2\pi i/3} \\ (31)\ \underline{z} = \underline{z}^* e^{-2\pi i/3} & (31)\ \underline{z}^* = \underline{z} e^{2\pi i/3} \end{array} \qquad (4.4)$$

$$\begin{array}{ll} (12)\ \phi = -\phi & (12)\ \sigma = \sigma^{-1} \\ (23)\ \phi = -\phi + 2\pi/3 & (23)\ \sigma = \sigma^{-1} e^{-2\pi i/3} \\ (31)\ \phi = -\phi - 2\pi/3 & (31)\ \sigma = \sigma^{-1} e^{2\pi i/3} \end{array} \qquad (4.5)$$

$$\begin{array}{lll} (12)\ \sigma^{-1} = \sigma & (12)\ \cos 2\psi = \cos 2\psi & (12)\ r = r \\ (23)\ \sigma^{-1} = \sigma e^{2\pi i/3} & (23)\ \cos 2\psi = \cos 2\psi & (23)\ r = r \\ (31)\ \sigma^{-1} = \sigma e^{-2\pi i/3} & (31)\ \cos 2\psi = \cos 2\psi & (31)\ r = r \end{array} \qquad (4.6)$$

where $\sigma = e^{-i\phi}$

The moving system of reference $(\underline{S}_{2x}, \underline{S}_{2y}, \underline{S}_{2z})$ is defined by,

$$\underline{S}_{2x} = \frac{\underline{z}}{z^{*2}} + \frac{\underline{z}^*}{z^2} \tag{4.7}$$

$$\underline{S}_{2z} = \sin 3\phi \left[z^{*2}\underline{z}^* \times z^2\underline{z}\right] \tag{4.8}$$

$$\underline{S}_{2y} = \underline{S}_{2y} \times \underline{S}_{2z} \tag{4.9}$$

where X is the vector product of two complex vectors.

By constructions the set is a dextral system of reference where $(\underline{S}_{2x}, \underline{S}_{2y}, \underline{S}_{2z})$ are linearly independent and satisfy the following orthogonality conditions:

$$\underline{S}_{2x} \cdot \underline{S}_{2y} = 0 \tag{4.10}$$

$$\underline{S}_{2x} \cdot \underline{S}_{2z} = 0 \tag{4.11}$$

$$\underline{S}_{2y} \cdot \underline{S}_{2z} = 0 \tag{4.12}$$

These relations are easily obtained with the use of the identity

$$(A \times B)\cdot C = (B \times C)\cdot A = (C \times A)\cdot B \tag{4.13}$$

In Appendix 4 the invariance of $(\underline{S}_{2x}, \underline{S}_{2y}, \underline{S}_{2z})$ under permutation is proved as well as the orthogonality conditions.

With the help of eqns. (4.9), (A5.7), (4.7) and (4.8) $\underline{S}_{2y}$ can be expressed in terms of $\underline{z}$ and $\underline{z}^*$ as follows:

$$\begin{aligned}\underline{S}_{2y} &= \sin 3\phi \left[\underline{z}^* \times \underline{z}\right] \times (z^2\underline{z} + z^{*2}\underline{z}^*) \\ &= \sin 3\phi \left[\underline{z}(z^2\underline{z}\cdot\underline{z}^* + z^{*4}) - \underline{z}^*(z^4 + z^{*2}\underline{z}^*\cdot\underline{z})\right]\end{aligned} \tag{4.14}$$

Using eqns. (3.17) the orthogonal vectors $(\underline{S}_{2x},\underline{S}_{2y},\underline{S}_{2z})$ can also be expressed in terms of the original vectors in configuration space $\underline{r}^{(1)}$ and $\underline{r}^{(2)}$, thus:

$$\underline{S}_{2x} = \underline{r}^{(1)}[1/4(r^{(2)^2} - r^{(1)^2}) \cdot (\underline{r}^{(1)}\cdot\underline{r}^{(2)})^{-1} + (r^{(1)}\cdot r^{(2)})]^{-1} -$$

$$- \underline{r}^{(2)}[1/2(r^{(2)^2} - r^{(1)^2}) + 2(\underline{r}^{(1)}\cdot\underline{r}^{(2)})^2(r^{(2)^2} - r^{(1)^2})^{-1}]^{-1} \tag{4.15}$$

$$\underline{S}_{2y} = [4 \sin 3\phi \ (r^{(2)^2} + r^{(1)^2})(\underline{r}^{(2)}\cdot\underline{r}^{(1)})(2 + i)](-\underline{r}^{(2)})$$

$$+ \underline{r}^{(1)}[4 \sin 3\phi \ i(r^{(2)^4} + r^{(1)^4} + 2(\underline{r}^{(1)}\cdot\underline{r}^{(2)})^2 - r^{(1)^2}\cdot r^{(2)^2})] \tag{4.16}$$

$$\underline{S}_{2z} = -2i \sin 3\phi \ [(r^{(2)^2} - r^{(1)^2})^2 + 4(\underline{r}^{(2)}\cdot\underline{r}^{(1)})^2][\underline{r}^{(1)} \times \underline{r}^{(2)}] \tag{4.17}$$

Appendix 5 contains the derivation of the above formulae.

From eqn. (4.15-17) the mixing of the internal and external variables can be easily appreciated. Using the permutational properties of the vectors $\underline{r}^{(1)}$ and $\underline{r}^{(2)}$ eqn. (2.9), the invariance of the vector $(\underline{S}_{2x},\underline{S}_{2y},\underline{S}_{2z})$ can be checked.

To obtain the transformation relations between the original at S_1 and the moving set S_2 it is necessary to project the vectors $\underline{z}$ and $\underline{z}^*$ onto S_2 (calling $\underline{x}$ and $\underline{x}^*$ to the vector $\underline{z}$ and $\underline{z}^*$ in the set S_2) and then find the rotation necessary to bring the set S_2 to coincide with $\underline{S}_1$, i.e. applying a rotation IR to the orthogonal system $(\underline{S}_{2x},\underline{S}_{2y},\underline{S}_{2z})$ such that

$$IR(\underline{S}_{2x},\underline{S}_{2y},\underline{S}_{2z}) = (\underline{S}_{1x},\underline{S}_{1y},\underline{S}_{1z}) \tag{4.18}$$

Since we are working in spherical coordinates and the vectors $(\underline{S}_{2x},\underline{S}_{2y},\underline{S}_{2z})$ form a cartesian set, it is necessary to evaluate the cartesian projections of the vectors $\underline{z}$ and $\underline{z}^*$ in S_2 and then construct the spherical components from their definition,

$$\begin{aligned} x_+ &= \frac{1}{\sqrt{2}}(x_1 + ix_2) & x_+^* &= \frac{1}{\sqrt{2}}(x_1^* + ix_2^*) \\ x_- &= \frac{-1}{\sqrt{2}}(x_1 - ix_2) & x_-^* &= \frac{-1}{\sqrt{2}}(x_1^* - ix_2^*) \\ x_o &= x_3 & x_o^* &= x_3^* \end{aligned} \tag{4.19}$$

The cartesian components (x_1,x_2,x_3) and (x_1^*,x_2^*,x_3^*) are given by

$$\begin{aligned} x_1 &= \underline{z}\cdot\frac{\underline{S}_{2x}}{|S_{2x}|} & x_1^* &= \underline{z}^*\cdot\frac{\underline{S}_{2x}}{|S_{2x}|} \\ x_2 &= \underline{z}\cdot\frac{\underline{S}_{2y}}{|S_{2y}|} & x_2^* &= \underline{z}^*\cdot\frac{\underline{S}_{2y}}{|S_{2y}|} \\ x_3 &= \underline{z}\cdot\frac{\underline{S}_{2z}}{|S_{2z}|} & x_3^* &= \underline{z}^*\cdot\frac{\underline{S}_{2z}}{|S_{2z}|} \end{aligned} \tag{4.20}$$

Hence we have

$$\begin{aligned} x_+ &= \frac{1}{\sqrt{2}}\left(\underline{z}\cdot\frac{\underline{S}_{2x}}{|S_{2x}|} + i\,\underline{z}\cdot\frac{\underline{S}_{2y}}{|S_{2y}|}\right) \\ x_- &= \frac{-1}{\sqrt{2}}\left(\underline{z}\cdot\frac{\underline{S}_{2x}}{|S_{2x}|} - i\,\underline{z}\cdot\frac{\underline{S}_{2y}}{|S_{2y}|}\right) \\ x_o &= \underline{z}\cdot\frac{\underline{S}_{2z}}{|S_{2z}|} \end{aligned} \tag{4.21}$$

$$x_+^* = \frac{1}{\sqrt{2}} \left(\underline{z}^* \frac{\underline{S}_{2x}}{|S_{2y}|} + i\, \underline{z}^* \frac{\underline{S}_{2y}}{|S_{2y}|} \right) \tag{4.22}$$

$$x_-^* = \frac{-1}{\sqrt{2}} \left(\underline{z}^* \cdot \frac{\underline{S}_{2x}}{|S_{2x}|} - i\, \underline{z}^* \cdot \frac{\underline{S}_{2y}}{|S_{2y}|} \right)$$

$$x_0^* = \underline{z}^* \cdot \frac{\underline{S}_{2z}}{|S_{2z}|}$$

The scalar combinations of the vectors $\underline{z}$ and $\underline{z}^*$ obtainable from eqns. (3.16) and (2.6-7) are given by

$$\begin{aligned} z^2 &= -\sigma r^2 \cos 2\psi \\ z^{*2} &= -\sigma^{-1} r^2 \cos 2\psi \\ z^2 z^{*2} &= r^4 \cos^2 2\psi \\ z \cdot z^* &= r^2 \end{aligned} \tag{4.23}$$

From eqns. (4.4-6) and (4.23) the following relations between the scalar combinations of the vectors $\underline{z}$ and $\underline{z}^*$ and their permutational properties are obtained,

$$\begin{aligned} z^2 &= \sigma^2 z^{*2} \\ z \cdot z^* &= -z^2 (\cos 2\psi)^{-1} e^{i\phi} = -e^{-i\phi} z^{*2} (\cos 2\psi)^{-1} \\ z^2 z^{*2} &= z^4 e^{2i\phi} = z^{*4} e^{-2i\phi} \end{aligned} \tag{4.24}$$

$$\begin{aligned} &(12)\ z^2 = z^{*2} && (12)\ z^{*2} = z^2 \\ &(23)\ z^2 = z^{*2} e^{-2\pi i/3} && (23)\ z^{*2} = z^2 e^{2\pi i/3} \\ &(31)\ z^2 = z^{*2} e^{2\pi i/3} && (31)\ z^{*2} = z^2 e^{-2\pi i/3} \end{aligned} \tag{4.25}$$

$$\begin{aligned} &(12)\ \underline{z} \cdot \underline{z}^* = (23)\ \underline{z} \cdot \underline{z}^* = (31)\ \underline{z} \cdot \underline{z}^* = \underline{z} \cdot \underline{z}^* \\ &(12)\ z^2 z^{*2} = (23)\ z^2 z^{*2} = (31)\ z^2 z^{*2} = z^{*2} z^2 \end{aligned} \tag{4.26}$$

Finally from eqns. (4.21-22), (4.7-8), (4.14) and (4.24) we obtain the projection of $\underline{z}$ and $\underline{z}^*$ onto S_2 in terms of the internal coordinates only, thus

$$x_{(\mp)}^{\pm} = (\bar{+})r\sigma^2[4-2\cos(2\psi)(\sigma^{-3}+\sigma^3)]^{-\frac{1}{2}}[\sigma^{-3}-\cos(2\psi)(\bar{+}) \frac{i(\sigma^{-6}-1)(1-\cos^2 2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^6)^{\frac{1}{2}}}] \tag{4.27}$$

$$x^*_{(\pm)} = (\bar{+})r\sigma^{-2}[4-2\cos(2\psi)(\sigma^{-3}+\sigma^3)]^{-\frac{1}{2}}[\sigma^{3}-\cos(2\psi)(\mp) \frac{i(1-\sigma^{6})(1-\cos^2 2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^6)^{\frac{1}{2}}}] \tag{4.28}$$

The permutational properties of the vectors $\underline{x}$ and $\underline{x}^*$ are given by

$$\begin{aligned}
(12)\ \underline{x} &= \underline{x}^* & (12)\ \underline{x}^* &= \underline{x} \\
(23)\ \underline{x} &= \underline{x}^* e^{2\pi i/3} & (23)\ \underline{x}^* &= \underline{x} e^{-2/3\pi i} \\
(31)\ \underline{x} &= \underline{x}^* e^{-2\pi i/3} & (31)\ \underline{x}^* &= \underline{x} e^{+2/3\pi i}
\end{aligned} \tag{4.29}$$

Hence as expected there are similar to the corresponding relations for $\underline{z}$ and $\underline{z}^*$ eqn. (4.4).

In Appendix 6 the derivation of the spherical components in the S_2 system is given in detail as well as the proof of the invariant properties of the vectors $\underline{x}$ and $\underline{x}^*$ under permutations.

Up to this point the set S_2 has been determined as a function of the required parameters (r,ϕ,ψ). We now seek the transformation relations between the two systems, i.e. between $(\underline{z},\underline{z}^*)$ and $(\underline{x},\underline{x}^*)$. To achieve this we denote by the Euler angles (α,β,γ) the orientation of the set $(\underline{S}_{2x},\underline{S}_{2y},\underline{S}_{2z})$ with respect to the coordinate system O.

From the general theory of rotations we know that the general rotation matrix in terms of the Euler angles is given by [27].

$$IR = \begin{Bmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma\ , & \sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\gamma, & -\sin\beta\cos\gamma \\ -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma, & -\sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma, & \sin\beta\sin\gamma \\ \cos\alpha\sin\beta\ , & \sin\alpha\sin\beta\ , & \cos\beta \end{Bmatrix} \tag{4.30}$$

Then the rotation needed to bring the system S_2 to coincide with the S_1 in terms of the cartesian coordinates is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = IR \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{4.31}$$

Using eqns. (4.30-31) and constructing the spherical components the following transformation relations are obtained [24].

$$\begin{bmatrix} z_+ \\ -z_- \end{bmatrix} = \begin{Bmatrix} e^{i(\gamma+\alpha)} \cos^{2\beta/2} & e^{-i(\gamma-\alpha)} \sin^{2\beta/2} \\ e^{i(\gamma+\alpha)} \sin^{2\beta/2} & e^{-i(\alpha+\gamma)} \cos^{2\beta/2} \end{Bmatrix} \begin{bmatrix} x_+ \\ -x_- \end{bmatrix} \tag{4.32}$$

In figure 6 the two different systems of reference ($\underline{S}_{2x}, \underline{S}_{2y}, \underline{S}_{2z}$) and ($\underline{S}_{1x}, \underline{S}_{1y}, \underline{S}_{1z}$) can be seen, as well as all the relevant parameters discussed in this chapter.

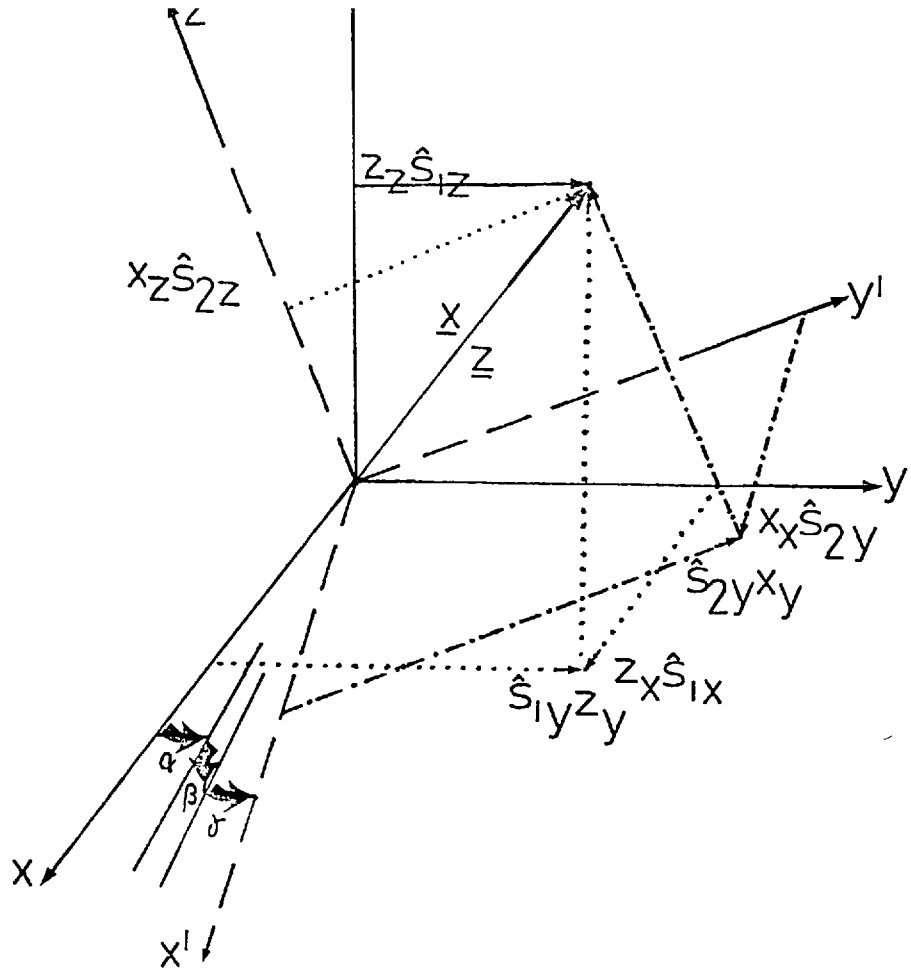


Figure 6

ii) The Tensor Representation in the S_2 System

Having developed the fundamental relations between S_1 and S_2 in subsection i), we now proceed to apply these results to our Tensor Representation.

The vectors $\underline{z}$ and $\underline{z}^*$ enter the harmonic polynomials as scalar products of their corresponding spherical components, as can be appreciated in eqns. (A2.9) and (A2.12). For given k we can write for a spherical tensor $T^k_{LL}(\underline{z},\underline{z}^*)$ without loss of generality

$$z_+^l z_+^{*r} = \left\{x_+ e^{i(\alpha+\gamma)} \cos^{2\beta/2} + (-x_-) e^{-i(\gamma-\alpha)} \sin^{2\beta/2}\right\}^l \times$$

$$\left\{x_+^* e^{-i(\alpha+\gamma)} \cos^{2\beta/2} + (-x_-^*) e^{i(\gamma-\alpha)} \sin^{2\beta/2}\right\}^r$$

$$= \sum_{m=0}^{l} \sum_{n=0}^{r} \binom{l}{m} \binom{r}{n} e^{i\gamma(l+r+2m-2n)} e^{i\alpha(l+r)} (\cos^{2\beta/2})^{l+r-m-n}$$

$$(\sin^{2\beta/2})^{m+n} (x_+)^{l-m} (-x_-)^{m} (x_+^*)^{r-n} (-x_-^*)^{n} \qquad (4.33)$$

Where $k = \frac{l-r}{2}$

After some manipulations Pustovalov and Smorodinskii [24] arrived at the expression

$$T^k_{LM}(z,z^*) = \sum_N D^L_{NM}(\alpha,\beta,\gamma) \left[\binom{2L}{L-N}\right]^{-\frac{1}{2}} \sum_n \binom{l}{\frac{(L-N)}{2} - n} \binom{r}{n} \times$$

$$(-x_-)^{\frac{(L-N)}{2}-n} (-x_-^*)^{n} (x_+)^{k+(N/2)+n} (x_+^*)^{\frac{L}{2}-k-n} \qquad (4.34)$$

where the summation variables m and n were replaced by $m + n = \frac{(L-N)}{2}$ and n.

Hence from eqns. (4.27-29) and (4.34) the spherical harmonic tensors can be expressed in terms of the external coordinates (α,β,γ) through the D-functions and in terms of the internal coordinates through the tensor $T^k_{LM}(r,\phi,\psi)$. Thus

$$T^k_{LM}(\underline{z},\underline{z}^*) = \sum_N D^L_{NM}(\alpha,\beta,\gamma)\ T^k_{LN}(r,\phi,\psi) \qquad (4.35)$$

Direct comparison of this equation with our initial $\phi^{\mu\omega}_{\chi LN}$ shows that the latter expression has the internal and external coordinates separated.

The tensors $T^k_{LN}(r,\phi,\psi)$ are obtained by substituting the values of $[x_+, x_+^*, (-x_-), (-x_-^*)]$ in terms of (r,ϕ,ψ) eqns. (4.27-28) into equation (4.34).

Thus

$$T^{k}_{LN}(r,\phi,\psi) = \left\{\binom{2L}{L-N}\right\}^{-\frac{1}{2}} \sum_{n}\binom{\frac{L+2k}{2}}{\frac{L-N}{2}-n}\binom{\frac{L-2k}{2}}{n} \quad x$$

$$x\left\{ - r\sigma^{2}\left[4-2\cos(2\psi)\,(\sigma^{-3}+\sigma^{3})\right]^{-\frac{1}{2}}\left[\sigma^{-3}-\cos 2\psi + i\frac{(\sigma^{-6}-1)}{(2-\sigma^{-6}-\sigma^{6})^{\frac{1}{2}}}\,(1-A^{2})^{\frac{1}{2}}\right]\right\}^{\frac{L-N}{2}-n}$$

$$x\left[- r\sigma^{-2}\left[4-2\cos(2\psi)\,(\sigma^{-3}+\sigma^{3})\right]^{-\frac{1}{2}}\left[\sigma^{3}-\cos 2\psi - i\,\frac{(1-\sigma^{6})}{(2-\sigma^{-6}-\sigma^{6})^{\frac{1}{2}}}\,(1-A^{2})^{\frac{1}{2}}\right]\right]^{n}$$

$$x\left[- r\sigma^{2}\left[4-2\cos(2\psi)\,(\sigma^{-3}+\sigma^{3})\right]^{-\frac{1}{2}}\left[\sigma^{-3}-\cos 2\psi - i\,\frac{(\sigma^{-6}-1)}{(2-\sigma^{-6}-\sigma^{6})^{\frac{1}{2}}}\,(1-A^{2})^{\frac{1}{2}}\right]\right]^{k+N/2+n}$$

$$x\left[- r\sigma^{-2}\left[4-2\cos(2\psi)\,(\sigma^{-3}+\sigma^{3})\right]^{-\frac{1}{2}}\left[\sigma^{3}-\cos 2\psi + i\,\frac{(1-\sigma^{6})}{(2-\sigma^{-6}-\sigma^{6})^{\frac{1}{2}}}\,(1-A^{2})^{\frac{1}{2}}\right]\right]^{L/2-k-n}$$

(4.36)

where $l + r = L$

$$\frac{l - r}{2} = k$$

Appendix 7 contains the explicit derivation of the tensors $T^{k}_{LM}(r,\phi,\psi)$. The final result can be expressed as

$$T^{k}_{LN}(r,\phi,\psi) = \left[\frac{(L+N)!\,(L-N)!}{2L!}\right]^{\frac{1}{2}} \left[\frac{(L/2+k)!}{(k+N/2)!}\right] (r\cos 2\psi)^{L} \sigma^{k-L/2} \times$$

$$P_1^{L/2} P_2^{N/2} P_3^{k} \times F(L/2-k-n+1,\ L/2-N/2-n+1,\ k+N/2-1;\ P_4) \qquad (4.37)$$

where

$$P_1 = \left[2 + (\sigma^{3}-\sigma^{-3})\left(\cos 2\psi - \frac{2i(1-\cos^{2}2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^{6})^{\frac{1}{2}}}\right)\right]$$

$$P_2 = \frac{(2-\sigma^{-6}-\sigma^{6})^{\frac{1}{2}}(\sigma^{-3}-\cos 2\psi) - i(\sigma^{-6}-1)(1-\cos^{2}2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^{6})^{\frac{1}{2}}(\sigma^{3}-\cos 2\psi) + i(\sigma^{-6}-1)(1-\cos^{2}2\psi)^{\frac{1}{2}}}$$

$$P_3 = \frac{(2-\sigma^{-6}-\sigma^{6})^{\frac{1}{2}}(\sigma^{-3}-\cos 2\psi) - i(\sigma^{-6}-1)(1-\cos^{2}2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^{6})^{\frac{1}{2}}(\sigma^{3}-\cos 2\psi) + i(1-\sigma^{6})(1-\cos^{2}2\psi)^{\frac{1}{2}}}$$

$$P_4 = \frac{2\cos^{2}2\psi + (\sigma^{3}-\sigma^{-3})\left(\cos 2\psi + \frac{2i(1-\cos^{2}2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^{6})^{\frac{1}{2}}}\right)}{2 + (\sigma^{3}-\sigma^{-3})\left(\cos 2\psi - \frac{2i(1-\cos^{2}2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^{6})^{\frac{1}{2}}}\right)}$$

for $-k-N/2+l \geq 0$

Where F(a,b,c;d) in the hypergeometric functions defined by

$$F(a,b,c;d) = \sum_n \frac{(a)_n (b)_n}{(c)_n} \frac{d^n}{n!} \tag{4.38}$$

and

$$(a)_n = a(a+1) \cdots (a+n-1) \tag{4.39}$$

Summarizing the results of this chapter we can say that the tensor wave functions $\phi^{\mu\omega}_{\lambda LM}$ can be expressed in its most general way as

$$\phi^{\mu\omega}_{\lambda LM} = \sum_{k=-L/2}^{L/2} \sum_N D^L_{NM}(\alpha,\beta,\gamma) A_{LN} \left[\frac{(L/2+k)!}{(k+N/2)!}\right] (r\cos 2\psi)^{L_\sigma k-L/2} P_1^{L/2} P_2^{N/2} P_3^k \times$$

$$F(L/2-k-n+1,\ L/2-n+1,\ k+N/2-1; P_4) \sum_{m=0}^{\frac{L+2\omega}{2}} \binom{L+2\omega}{2}{m} \binom{L-2\omega}{2}{k-\omega+m} \Lambda_1^{L/2+\omega-m} \times$$

$$\Lambda_1^{*L/2-k-m} \Lambda_2^{2m+k-\omega} r^\lambda S^{\mu\omega}_{\lambda LM} \tag{4.40}$$

where $A_{LN} = \left[\frac{(L+N)!\,(L-N)!}{2L!}\right]^{\frac{1}{2}}$

The explicit separation of the internal and external coordinates was achieved with the help of the D-function as was originally required. The tensors $\phi^{\mu\omega}_{\lambda LM}$ conserve their definite properties under rotation and permutations of all the particles.

CHAPTER V

THE GENERAL THREE-NUCLEON FADDEEV EQUATION

i) The Faddeev Equation in the Tensor Representation

Faddeev in his celebrated paper [1] succeeded in formulating the equations of motion of a non-relativistic system of three nucleons. The previous methods of solving this problem were based on the application of the Lippmann-Schwinger [15] (LS) equation which for the case of two particles gave successful results but presented difficulties for three or more particles.

One tried to overcome these difficulties by making extra assumptions or by arbitrarily modifying the Green function. However the solutions so obtained are not unique. In mathematical terms this difficulty appeared as δ-functions in the equations of motion impeding the usual method of solution by iteration.

The need to generalize the L-S formalisms is apparent when realising that the multichannel interactions between three particles contain special features which lead to singularities in the equation of motion. The two-body transition operator defined in a two particle Hilbert space were insufficient to describe adequately the additional interactions. Due to the third particle, Faddeev re-expressed the L-S equation in a more general way free of singularities, based on a suitable combination of two-body transition operators in a three-particle Hilbert space such that for any initial state there were four possible final states covering all the possible interactions without the presence of discontinuities.

We will not discuss the mathematical and physical justifications of the derivation of the Faddeev equation which was done at length by several authors [1] [14]. Instead we will make use of the final equation in terms of wave functions and give the necessary conditions which the kernel of the integrals equations must satisfy in order to use well defined methods of solution.

The Faddeev equation is given by

$$\begin{bmatrix} |\bar{\psi}^{(1)}\rangle \\ |\bar{\psi}^{(2)}\rangle \\ |\bar{\psi}^{(3)}\rangle \end{bmatrix} = \begin{bmatrix} \bar{\psi}_0^{(1)} \\ \bar{\psi}_0^{(2)} \\ \bar{\psi}_0^{(3)} \end{bmatrix} - \begin{bmatrix} 0 & G_0(z)T_1(z) & G_0(z)T_1(z) \\ G_0(z)T_2(z) & 0 & G_0(z)T_2(z) \\ G_0(z)T_3(z) & G_0(z)T_3(z) & 0 \end{bmatrix} \begin{bmatrix} |\bar{\psi}^{(1)}\rangle \\ |\bar{\psi}^{(2)}\rangle \\ |\bar{\psi}^{(2)}\rangle \end{bmatrix} \tag{5.1}$$

Where the interacting three-nucleon state $|\bar{\psi}\rangle$ is given by

$$|\bar{\psi}\rangle = |\bar{\psi}_0\rangle + |\bar{\psi}^{(1)}\rangle + |\bar{\psi}^{(2)}\rangle + |\bar{\psi}^{(3)}\rangle \tag{5.2}$$

In this equation $|\bar{\psi}_0\rangle$ and $|\bar{\psi}_0^{(i)}$ are known asymptotic states, $G_0(z)$ is the free Green's function for three nucleons, $T_i(z)$ is the two-nucleon transition operator in the three-particle Hilbert space and z is a complex variable.

It is customary to work in the momentum representation for simplicity's sake since the complicated asymptotic dependence which is present in configuration space is difficult to handle. The transformation to the momentum space leaves singularities, usually in the form of δ-functions, which are easier to work with.

The transition operator in the momentum representation is given by

$$\langle \underline{q} | T_i(z) | \underline{q}' \rangle = \delta(\underline{\xi}_i - \underline{\xi}_i') \langle \underline{\eta}_i | t_i(z - \underline{\xi}_i'^2) | \underline{\eta}_i' \rangle \tag{5.3}$$

where t_i is the off-shell transition amplitude between particles j and k and $(\underline{\xi}_i, \underline{\eta}_i)$ are the usual centre of mass relative momentum coordinates.

If the kernel of equation (5.1) is compact or completely continuous then it is possible to apply the Fredholm theory for the solution of integral equations. Compactness of the kernel means that it can be uniformly approximated by a sum of reparable kernels. (See for example Weinberg's Appendix [33]).

Due to the δ-function appearing in eqn. (5.3) the kernel is not completely continuous. Rubin, Sugar and Tiktopoulos [34] showed that the square of the Faddeev matrix kernel is of Fredholm type on the real axes for potentials analytic in the momentum variables. Thus the iterated Faddeev equation can be solved using the powerful Fredholm methods for integral equations using this type of potentials.

The iterated Faddeev equation is given by [18]

$$|\bar{\psi}^{(i)}\rangle = |\tilde{\bar{\psi}}_0^{(i)}\rangle + K^{(i,j)}(z) \; |\bar{\psi}^{(j)}\rangle \tag{5.4}$$

With the new asymptotic states $|\tilde{\bar{\psi}}_0^{(i)}\rangle$ being defined by

$$|\tilde{\bar{\psi}}_0^{(i)}\rangle = |\bar{\psi}_0^{(i)}\rangle - \sum_{j \neq i} G_0(z) T_j(z) |\bar{\psi}_0^{(j)}\rangle \tag{5.5}$$

and the iterated kernel $K^{(i,j)}(z)$ being given by

$$K(z) = \begin{pmatrix} G_oT_1G_oT_2+G_oT_1G_oT_3 & G_oT_1G_oT_3 & G_oT_1G_oT_2 \\ G_oT_2G_oT_3 & G_oT_2G_oT_1+G_oT_2G_oT_3 & G_oT_2G_oT_1 \\ G_oT_3G_oT_2 & G_oT_3G_oT_1 & G_oT_3G_oT_1+G_oT_3G_oT_2 \end{pmatrix} \tag{5.6}$$

Denoting our tensor polynomials in the bra and ket formulation byy

$$\phi^{\mu,\omega}_{\lambda LM} = |\phi\rangle \quad = |\mu,\omega,\lambda,LM\rangle \tag{5.7}$$

we can express the Faddeev equation in our representation by

$$\langle\phi|\bar{\psi}^{(i)}\rangle = \sum_{j=1}^{3} \sum_{\substack{\lambda LM \\ \mu\omega}} \int \langle\phi|K^{(i,j)}(z)|\phi'\rangle \, 2\underline{k}'d\underline{k}' \quad \langle\phi'|\bar{\psi}^{(j)}\rangle$$

$$i,j = 1,2,3 \tag{5.8}$$

where for simplicity only the homogeneous term in eqn. (5.4) is considered and the completeness relation in the momentum representation is given by

$$\sum_{\substack{\mu\omega\lambda \\ LM}} \int |q^2,\mu,\omega,L,\lambda M\rangle \, 2\underline{q}d\underline{q} \, \langle q^2,\mu,\omega,\lambda,L,M| \;=\; 1 \tag{5.9}$$

ii) The Symmetric Omnes Angular Momentum Reduction

Direct solution of eqn. (5.8) which is a coupled set of integral equations in one continuous variable, implies an enormous effort. Firstly due to the large number of variables involved in the intermediate states it is necessary to perform several many-dimensional integrations. Secondly it is necessary to construct the kernel from totally off-shell two-body amplitudes in all partial waves and thirdly combine both results into the final system of coupled equations solving for the desired physical magnitude, i.e. binding energy, scattering length, etc. These involve a lengthy and costly computational process.

Solutions along these lines, but using the Schrödinger equation instead of the Faddeev, as well as spherical harmonics similar to ours, were carried out by Dzyuba et al [35] but for higher orbital angular momentum the value of the binding energy of H^3 gave large errors.

In order to simplify eqn. (5.8) we use the symmetric Omnes angular momentum reduction [13] taking advantage of the previously discussed properties (Chapter IV) of both formalisms in relation to the D-function.

Omnes introduced the intermediate states $|\underline{E},L,M,N\rangle$ in which the total energy $\underline{E} = (q_1^2, q_2^2, q_3^2)$, the total angular momentum L, and its components M and N about the space-and-body-fixed axes respectively are diagonalized (which refer to the space and body fixed axes respectively).

The representation of the $|\underline{E},L,M,N\rangle$ states in momentum space donated by $|\underline{q}_i\rangle = |\underline{\xi}_i, \underline{n}_i\rangle$ where $(\underline{\xi}_i, \underline{n}_i)$ are the momentum variables corresponding to $(\underline{r}_i^{(1)}, \underline{r}_i^{(2)})$ eqn. (2.8) is given by [14]

$$\langle \underline{q}_i' | \underline{E}, L, M, N\rangle = \left\{ \frac{2L+1}{3\sqrt{3}\pi^2} \right\}^{\frac{1}{2}} \delta(\underline{E}'-\underline{E}) D^L_{MN}(R') \qquad (5.10)$$

where the normalizations factor was chosen to satisfy the orthogonality condition given by

$$\langle \underline{E},LMN|\underline{E}'L'M'N'\rangle = \delta(\underline{E}-\underline{E}')\ \delta_{LL'}\delta_{MM'}\delta_{NN'} \tag{5.11}$$

The transition operator defined in eqn. (5.3) can be expressed in terms of the Omnes states by projecting the $|\underline{k}\rangle$ representation into the complete $|\underline{E},LMN\rangle$ states.

With the help of eqn. (5.11) Lee [18] obtained the following expression:

$$\langle E,LMN|T_i(z)|\underline{E}',L'M'N'\rangle = 4\sqrt{2}\pi^2\delta(\underline{E}_i-\underline{E}_i')/E_i'^{\frac{1}{2}}\delta_{LL'}\delta_{MM'}\ x \tag{5.12}$$

$$\sum_{l'N_i} t_{i,l'}(\eta_i^2,\eta_i'^2;\ z-\xi_i'^2)D^{*L}_{NN_i''}(\pi/2,-\pi/2,-\varepsilon_i)D^{L}_{N'N_i''}(\pi/2,-\pi/2,-\varepsilon_i)\ x$$

$$Y_{l'N_i''}(\delta_i,0)Y_{l'N_i''}(\delta_i',0)$$

Where $Y_{lm}(\theta,\phi)$ are the spherical harmonics, δ_i,δ_i' refer to the angles between $\underline{\eta}_i$ and $\underline{\xi}_i$, $\underline{\eta}_i'$ and $\underline{\xi}_i'$ respectively, N_i the component of L along the body fixed z-axis such that $\underline{\xi}_i$ is collinear with it and ε_i is defined before eqn. (2.8).

In deriving eqn. (5.12) use was made of the following rotational property of the symmetric Omnes states given by

$$|\underline{E},LMN\rangle = \sum_{N_i''} D^{L}_{NN_i''}(\pi/2,-\pi/2,-\varepsilon_i)|\underline{E},LMN_i''\rangle \tag{5.13}$$

The kernel of the Faddeev eqn. in the tensor representation eqn. (5.8) can be expressed in terms of the intermediate Omnes states by

$$\langle\phi|K^{(i,j)}(z)|\phi'\rangle = \sum_{L^*M^*N^*} \sum_{L''M''N''} \int \langle\phi|\underline{E}^*L^*M^*N^*\rangle \, d\underline{E}^* \, x$$

$$x \, \langle \underline{E}^*,L^*M^*N^*|K^{(i,j)}(z)|\,\underline{E}'',L''M''N''\rangle \, dE'' \, \langle \underline{E}'',L''M''N''|\phi'\rangle \qquad (5.14)$$

Using eqn. (5.10) and the tensor polynomials in the momentum representations, we obtain for the transformation coefficient

$$\langle \underline{q}^2,\mu\omega\lambda LM|\underline{E}^*,L^*,M^*,N^*\rangle = \frac{3\sqrt{6\pi}}{2L+1}\left\{\frac{2L+1}{3\sqrt{3}}\right\}^{\frac{1}{2}} \delta_{LL^*}\delta_{MM^*}\delta_{NN^*} \quad x$$

$$\sum_{k=-L/2}^{L/2} \sum_{N} \Omega(\underline{q},\underline{q}^*) \, {}^{o}T^{k}_{LN}(\phi,\psi) \, C^{*\mu\omega}_{\lambda LM}(\phi,\psi) \qquad (5.15)$$

where

$$C^{\mu\omega}_{\lambda LM}(\phi,\psi) = \sum_{m=0}^{\frac{L+2\omega}{2}} \binom{L+2\omega}{2}{m} \binom{\frac{L-2\omega}{2}}{k-\omega+m} \Lambda_1^{L/2+\omega-m} \, \Lambda_1^{*L/2-k-m} \, \Lambda_2^{2m+k-\omega} \quad x$$

$$S^{\mu}_{\lambda}(\phi,\psi) \qquad (5.16)$$

$\Omega(\underline{q},\underline{q}^*)$ depends on the normalization chosen and ${}^{o}T^{k}_{LN}$ is given by

$${}^{o}T^{k}_{LN}(\phi,\psi) = r^{-L} \, {}^{k}T_{LN}(r,\phi,\psi) \qquad (5.17)$$

In obtaining eqn. (5.15) use was made of the following orthogonality property of the D-functions

$$\int D^{L}_{MN}(R')D^{L*}_{M*N*}(R')dR' = \frac{8\pi^2}{2L+1}\delta_{LL*}\delta_{MM*}\delta_{NN*} \qquad (5.18)$$

We are now in a position to express the matrix elements of the kernel in tensor representation as coupled integral equations in the internal variables only. Using eqn. (5.14) and (5.15) we have

$$\langle \underline{q}^2,\mu\omega\lambda LM|K^{(i,j)}(z)|\underline{q}'^2\mu'\omega'\lambda'L'M'\rangle =$$

$$= \frac{\pi^2}{6\sqrt{3}} \sum_{NN'} \int \langle \underline{E},LMN|K^{(i,j)}(z)|\underline{E}',LMN'\rangle\, C^{*\mu\omega}_{\lambda LM}(\phi*,\psi*)\, C^{\mu'\omega'}_{\lambda'LM}(\phi'',\psi'') \quad x$$

$$dqq^2q'^2\Omega(\underline{q},\underline{q})\Omega^{*}(\underline{q},\underline{q}')\cos2\psi*d(\cos2\psi*)\cos2\psi''d(\cos2\psi'')d\phi*d\phi''T^{o}_{LN}{}^{k}\,T^{*o}_{LN'}{}^{k} \qquad (5.19)$$

In obtaining eqn. (5.19) use was made of the rotational invariance of the kernel given by

$$\langle \underline{E}*,L*M*N*|K^{(i,j)}(z)|\underline{E}'',L''M''N''\rangle = \delta_{L*L''}\delta_{M*M''}\langle \underline{E}*L*M*N*|K^{(i,j)}(z)|\underline{E}''L''M''N''\rangle \qquad (5.20)$$

Usually $\Omega(\underline{q},\underline{q}*)$ can be expressed in terms of a delta function in the following form

$$\Omega(\underline{q},\underline{q}*) = \frac{\delta(q^2-q*^2)}{q^2}B(\underline{q}) \qquad (5.21)$$

where $B(\underline{q})$ is a function of $\underline{q}$.

In this case eqn. (5.19) reduces to a coupled set of integral equations in only ψ and ϕ.

Finally the Faddeev equation in the tensor representations can be expressed in its reduced form using eqns (5.12) and (5.19) where the number of integrations have been reduced to two.

The present method allows us to describe the nonrelativistic three-nucleon problem for arbitrary orbital angular momentum where the kernel has been simplified to permit a much shorter numerical calculation, even though it is still considerable.

To complete the description of the three-nucleon problem we need to include the spin and isospin dependence of the particles involved. This is carried out in the following section.

iii) Spin-Isospin Generalized Faddeev Equation

A complete set of three particle spin-isospin states was constructed by Verde [19] and applied to the Faddeev equation for $L = 0$, $S = S_z = \frac{1}{2}$, $I = -I_z = \frac{1}{2}$ by Lee [18] where S_z and I_z are the z-component of the spin and isospin operators respectively.

In this section we generalize this formalism to include higher orbital angular momentum and the other physically relevant states consisting of $S_z = 3/2$, $I_z = -\frac{1}{2}$. The normalized basis of spin-isospin states for $S_z = \frac{1}{2}$, $I_z = -\frac{1}{2}$, is given by

$$Z_s = Z(1) = \frac{1}{\sqrt{2}} (\Theta_1\Pi_1 + \Theta_2\Pi_2) \quad \text{Symmetric state}$$

$$Z_a = -Z(2) = \frac{1}{\sqrt{2}} (\Theta_1\Pi_2 - \Theta_2\Pi_1) \quad \text{Antisymmetric state}$$

$$Z_1 = -Z(3) = \frac{1}{\sqrt{2}} (\Theta_1\Pi_2 + \Theta_2\Pi_1)$$

Mixed state

$$Z_2 = Z(4) = \frac{1}{\sqrt{2}} (\Theta_1\Pi_1 - \Theta_2\Pi_2) \qquad (5.22)$$

where (Θ_1, Θ_2) and (Π_1, Π_2) are the mixed states in the spin and isospin spaces respectively.

For $S_z = 3/2$ and $I_z = -\frac{1}{2}$ the basis is given by

$$Z(5) = \Pi_2\Theta_s$$

$$Z(6) = -\Pi_1\Theta_s \qquad (5.23)$$

where Θ_s is the completely symmetric spin state.

In order to satisfy the Pauli Principle the total wave function must be fully antisymmetric in exchanges of the coordinates of any pair of nucleons in the spin, isospin and space states. Thus we can write the general ket for $S_z = \frac{1}{2}$, $I_z = -\frac{1}{2}$ as

$$|\bar{\psi}> = |\bar{\psi}_a> Z_s|\bar{\psi}_s> Z_a - |\bar{\psi}_2> Z_1 + |\bar{\psi}_1> Z_2$$

$$= \sum_{b=1}^{4} |\bar{\psi}(b)> Z(b) \qquad (5.24)$$

and for $S_z = 3/2$ $I_z = -\frac{1}{2}$ we have

$$|\bar{\psi}> = \sum_{b=3}^{4} |\bar{\psi}(b)> Z(b+2) \qquad (5.25)$$

Where $|\bar{\psi}_a\rangle = |\bar{\psi}_{(1)}\rangle$, $|\bar{\psi}_s\rangle = |\bar{\psi}_{(2)}\rangle$, $|\bar{\psi}_2\rangle = |\bar{\psi}_{(3)}\rangle$ and $|\bar{\psi}_1\rangle = |\bar{\psi}_{(4)}\rangle$ are the symmetric, antisymmetric and the mixed states respectively. These functions can be represented in terms of the Tensor Polynomials using eqn. (3.52) as

$$
\begin{aligned}
|\bar{\psi}_a\rangle &= \sum_{\substack{\mu_1\omega \\ \lambda LM}} \int |\, {}_2\phi^{\mu_1\omega}_{\lambda LM}\rangle \, dq^2 \, \langle {}_2\phi^{\mu_1\omega}_{\lambda LM} \, \bar{\psi}_a\rangle \\
|\bar{\psi}_s\rangle &= \sum_{\substack{\mu_1\omega \\ \lambda LM}} \int |\, {}_1\phi^{\mu_1\omega}_{\lambda LM}\rangle \, dq^2 \, \langle {}_1\phi^{\mu_1\omega}_{\lambda LM}| \, \bar{\psi}_s\rangle \qquad (5.26) \\
|\bar{\psi}_1\rangle &= \sum_{\substack{\mu_2\omega \\ \lambda LM}} \int |\, {}_1\phi^{\mu_2\omega}_{\lambda LM}\rangle \, dq^2 \, \langle {}_1\phi^{\mu_2\omega}_{\lambda LM}| \, \bar{\psi}_1\rangle \\
|\bar{\psi}_2\rangle &= \sum_{\substack{\mu_2\omega \\ \lambda LM}} \int |\, {}_2\phi^{\mu_2\omega}_{\lambda LM}\rangle \, dq^2 \, \langle {}_2\phi^{\mu_2\omega}_{\lambda LM}| \, \bar{\psi}_2\rangle
\end{aligned}
$$

where $\mu_1 = 3/2\ n \qquad n = 0,1,\ldots$

$\mu_2 \neq 3/2\ n$

Since the Faddeev equation expresses the total wave function as a sum of three terms, we decompose the states with definite symmetry properties as a sum of these components. Thus defining

$$
|\bar{\psi}(b)\rangle = \sum_{i=1}^{3} |\bar{\psi}^{(i)}(b)\rangle \qquad (5.27)
$$

the Faddeev equation operating in the spin-isospin and Hilbert spaces can be written as

$$|\bar{\psi}^{(i)}> = -G_0(z)T_i(|\bar{\psi}^{(j)}> + |\bar{\psi}^{(k)}>) \qquad i \neq j \neq k \tag{5.28}$$

where T_i is given by

$$T_i = \sum_{\alpha=1}^{4} P_{i,\alpha}(j,k)T_{i,\alpha} \tag{5.29}$$

$P_{i,\alpha}$ is the projection operator for the spin-isospin state of the two-nucleon subsystems (j.k) and $T_{i,\alpha}$ the transition operator of that state.

The $P_{i,\alpha}$ are given by

$$\begin{aligned} P_{i,1} &= P_\sigma^+ P_\tau^+ \\ P_{i,2} &= P_\sigma^+ P_\tau^- \\ P_{i,3} &= P_\sigma^- P_\tau^+ \\ P_{i,4} &= P_\sigma^- P_\tau^- \end{aligned} \tag{5.30}$$

where $P_{\sigma,\tau}^{\pm}(jk) = \frac{1}{2}[1 \pm (jk)_{\sigma,\tau}]$ and $(\ )_\sigma$ and $(\ )_\tau$ representing the permutations in the spin and isospin spaces respectively.

Using the orthogonality of the Z(b) states and eqn. (5.24) the spin-isospin kets are expressed as

$$|\bar{\psi}^{(i)}(b)> = -G_0(z)\left[\sum_{\alpha=1}^{4} Z(b)P_{i,\alpha}(jk)Z(b')T_{i,\alpha}\right](|\bar{\psi}^{(j)}(b') + |\bar{\psi}^{(k)}(b')>) \tag{5.31}$$

To evaluate eqn. (5.31) it is necessary to obtain the explicit matrix representations of the three permutation operators in the spin and isospin spaces.

For $S_z = \frac{1}{2}$ and $I_z = -\frac{1}{2}$ the matrices are given by

$$(23)_\sigma = \begin{pmatrix} 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix}$$

$$(23)_\tau = \begin{pmatrix} 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix} \qquad (5.32)$$

$$(31)_\tau = \begin{pmatrix} 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix}$$

$$(31)_\tau = \begin{pmatrix} 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \end{pmatrix}$$

$$(12)_\sigma = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$(12)_\tau = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

For $S_z = 3/2$ and $I_z = -\frac{1}{2}$ the matrices are given by

$$(12)_\sigma = (23)_\sigma = (31)_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(12)_\tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(5.33)

$$(23)_\tau = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$(31) \quad = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

With the help of eqn. (5.32-33) we obtain for the Faddeev equation a set of twelve coupled equations for $|\bar{\psi}^{(i)}(b)\rangle$ i = 1,2,3 and b = 1,2,3,4 for the case $S_z = \frac{1}{2}$, $I_z = -\frac{1}{2}$ and a set of six coupled equations for $|\bar{\psi}^{(i)}(b)\rangle$ i = 1,2,3 and b = 3,4 for the case $S_z = 3/2$ $I_z = -\frac{1}{2}$. After regrouping the Faddeev equation in terms of symmetric state we obtain for arbitrary orbital angular momentum and for $S_z = \frac{1}{2}$ $I_z = -\frac{1}{2}$ the following equations

$$\begin{bmatrix} |\bar{\psi}_s\rangle \\ |\bar{\psi}_a\rangle \\ |\bar{\psi}_2\rangle \\ |\bar{\psi}_1\rangle \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{bmatrix} |\bar{\psi}_s\rangle \\ |\bar{\psi}_a\rangle \\ |\bar{\psi}_2\rangle \\ |\bar{\psi}_1\rangle \end{bmatrix} \tag{5.34}$$

where

$$|\bar{\psi}(b)\rangle = \begin{pmatrix} |\bar{\psi}^1(b)\rangle \\ |\bar{\psi}^2(b)\rangle \\ |\bar{\psi}^3(b)\rangle \end{pmatrix} \qquad b = 1,2,3,4$$

and the S matrices are given in Appendix 8. The case for $S_z = 3/2$ $I_z = -\frac{1}{2}$ is obtained exactly in the same way.

Finally we iterate the kernel of eqn. (5.34) to obtain

$$\begin{Bmatrix} |\bar{\psi}_s\rangle \\ |\bar{\psi}_a\rangle \\ |\bar{\psi}_2\rangle \\ |\bar{\psi}_1\rangle \end{Bmatrix} = \begin{Bmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{Bmatrix} \begin{bmatrix} |\bar{\psi}_s\rangle \\ |\bar{\psi}_a\rangle \\ |\bar{\psi}_2\rangle \\ |\bar{\psi}_1\rangle \end{bmatrix} \tag{5.35}$$

where

$$\begin{aligned}
W_{11} &= S_{11}^2 + S_{13}^2 + S_{14}^2 \\
W_{12} &= W_{21} = S_{13}S_{32} + S_{14}S_{42} \\
W_{13} &= W_{31} = S_{11}S_{13} + S_{13}S_{33} + S_{14}S_{43} \\
W_{14} &= W_{41} = S_{11}S_{14} + S_{13}S_{34} + S_{14}S_{44} \\
W_{22} &= S_{22}^2 + S_{23}^2 + S_{24}^2 \\
W_{32} &= W_{23} = S_{22}S_{23} + S_{23}S_{33} + S_{24}S_{43} \\
W_{24} &= W_{42} = S_{22}S_{24} + S_{23}S_{34} + S_{24}S_{44} \\
W_{33} &= S_{31}^2 + S_{32}^2 + S_{33}^2 + S_{34}^2 \\
W_{34} &= W_{43} = S_{31}S_{14} + S_{32}S_{24} + S_{33}S_{34} + S_{34}S_{44} \\
W_{44} &= S_{14}^2 + S_{42}^2 + S_{34}^2 + S_{44}^2
\end{aligned} \tag{5.36}$$

The iterated spin-isospin Faddeev equation in the Tensor Representation is exact and general enough to describe the three body problem for arbitrary orbital angular momentum, a wide variety of potentials including non-separable ones and reduced to a manageable level of computation.

CHAPTER VI

CONCLUSION

A formalism has been developed capable of describing the non-relativistic three-nucleon problem in its most general form. We have taken into account higher orbital angular momenta, non-separable potentials and the spin-isospin dependence of the interactions.

The final equations can be approximated to any desired degree of accuracy by a finite set of coupled integral equations in one variable only.

The Faddeev equation was expressed in the Tensor Representation permitting the inclusion of higher orbital angular momenta in a very simple and practical form. This representation was parametrized in terms of internal and external coordinates to take full advantage of the symmetrical Omnes angular momentum reduction.

Finally, the spin-isospin space for any orbital angular momenta was introduced in the iterated Faddeev equation permitting the incorporation of more realistic spin-isospin dependent potentials in our description.

The next step in the present study would be a numerical computation based on our formalism applied to a given physical system, such as the calculation of the Triton binding energy.

The numerical results will provide a test for nucleon-nucleon potentials, since we have an exact formalism. Comparison with alternative approaches will serve to throw light on the working of other methods of solution and their accuracy.

Finally our general solution will permit taking a closer look at the

importance of higher orbital angular momenta in both the bound states and the scattering of three nucleons.

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APPENDICES

APPENDIX 1

The effect of the Λ_1, Λ_1^* and Λ_2 operators on the $r^\lambda S_\lambda^\mu$ function [17]. The evaluation of the differentials $\partial/\partial z_i$ and $\partial/\partial z_i^*$ is carried out by noting that the polynomials for L = 0 depend only on the internal variables (r,ϕ,ψ).

Thus

$$\frac{\partial}{\partial z_i} = \frac{\partial r}{\partial z_i}\frac{\partial}{\partial r} + \frac{\partial \cos 2\psi}{\partial z_i}\frac{\partial}{\partial \cos 2\psi} + \frac{\partial \sigma}{\partial z_i}\frac{\partial}{\partial \sigma} \tag{A1.1}$$

Performing the partial differentials using eqns. (4.23) the following relations are obtained

$$\frac{\partial}{\partial z_i} = g^{AB}(z_j\Lambda_1 + z_j^*\Lambda_2) \tag{A1.2}$$

and

$$\frac{\partial}{\partial z_i^*} = g^{AB}(z_j^*\Lambda_1^* + z_j\Lambda_2) \tag{A1.3}$$

where

$$\Lambda_1 = -r^{-2}(\sigma^{-1}\frac{\partial}{\partial \cos 2\psi} + \sec 2\psi\,\partial/\partial\sigma) \tag{A1.4}$$

$$\Lambda_2 = r^{-1}(2\partial/\partial r - \cos(2\psi)\sigma^{-1}\,\partial/\partial\cos 2\psi) \tag{A1.5}$$

$$\Lambda_1^* = r^{-2}\sigma(\sigma\,\sec 2\psi\,\partial/\partial\sigma - \partial/\partial\cos 2\psi) \tag{A1.6}$$

and the three operators Λ_1, Λ_1^* and Λ_2 commute with each other.

The action of these operators on the scalar polynomials $r^\lambda S_\mu^\lambda$ is obtained with the help of the following properties of the Jacobi polynomials (A1.7)[31].

For the evaluation of $\partial/\partial\cos 2\psi$ we have,

$$\frac{d^m}{dx^m} P_n^{(\alpha,\beta)}(x) = \frac{1}{2^m} \frac{\Gamma(n+m+\alpha+\beta+1)}{(n+\alpha+\beta+1)} P_{n-m}^{(\alpha+m,\beta+m)}(x) \qquad \text{(A1.7)}$$

$$m = 1,2\ldots n$$

where

$$\frac{\Gamma(\alpha+\beta+1+n+m)}{\Gamma(n+\alpha+\beta+1)} = (\alpha+\beta+1+n)_m \qquad \text{(A1.8)}$$

The functional relations between the Jacobi polynomials needed are given by

$$(1-x)P_n^{(\alpha+1,\beta)}(x) + (1+x)P_n^{(\alpha,\beta+1)}(x) = 2P_n^{(\alpha,\beta)}(x)$$

$$P_n^{(\alpha,\beta-1)}(x) - P_n^{(\alpha-1,\beta)}(x) = P_{n-1}^{(\alpha,\beta)}(x) \qquad \text{(A1.9)}$$

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$$

$$(2n+\alpha+\beta)P_n^{(\alpha-1,\beta)}(x) = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(x) - (n+\beta)P_{n-1}^{(\alpha,\beta)}(x)$$

Using these formulae the scalar polynomials $r^\lambda S_\mu^\lambda$ under the action of $\Lambda_1^t \Lambda_1^{*s} \Lambda_2^a$ given by

$$\Lambda_1{}^t \Lambda_1^{*s} \Lambda_2{}^a r^\lambda S_\lambda^\mu = (-1)^t \left(\frac{\lambda}{2} - \mu + 2s\right) \cdots \left(\frac{\lambda}{2} + \mu - 2t - a + \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}\right) \quad x$$

$$r^{\lambda - 2(s+t+a)} \sum_{P=0}^{\begin{Bmatrix} a/2 \\ \frac{a-1}{2} \end{Bmatrix}} \left(\frac{\lambda}{2} + \mu - 2t - a - \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\right) \cdots \left(\frac{\lambda}{2} + \mu - 2t - 2a - 2p + 2\right) \quad x$$

$$M^a_{a-p} \, (\sigma \cos 2\psi)^{\mu+s-t} \, P_{n-s-a+p}^{(\mu+s-t,\, s+t+2-p)} \, (1 - 2\cos^2 2\psi) \qquad \text{(A1.10)}$$

where $n = \dfrac{\lambda - 2\mu}{4}$.

for a $\begin{cases} \text{odd;} & \text{the upper line applies} \\ \text{even;} & \text{the lower line applies} \end{cases}$

and the coefficients M^a_{a-p} satisfy the equation

$$M^a_{a-p} = (a-2p+1) \, M^{a-1}_{a-p} + M^{a-1}_{a-p-1} \qquad \text{(A1.11)}$$

A transcription of the table of coefficients M^a_{a-p} for a = 1,2,...7 and a-p = 1,2,...6 is given below.

	S - P					
S	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	0	3	1	0	0	0
4	0	3	6	1	0	0
5	0	0	6	10	1	0
6	0	0	15	45	15	1
7	0	0	0	105	105	21

APPENDIX 2

Construction of the Tensor Polynomials $\phi^{\mu\omega}_{\lambda LM}$ for L = 0,1,2. [17].

The quantum number μ characterizing the permutational properties of the polynomials $\phi^{\lambda\omega}_{LKM}$ is separated in two parts, ε and k which respectively refer to the scalar and tensor parts of the spherical polynomials.

Thus

$$\mu = \varepsilon + k \tag{A2.1}$$

The polynomials obtained must have a definite quantum number μ since ε and k are not conserved separately.

From the application of the $(\partial/\partial z, \partial/\partial z^*)^{L-M}_{pq}$ operator to the scalar function $r^{\lambda} S^{\lambda\omega}_{L\mu M}$ the intervals of the numbers ω,μ and λ are obtained as follows

$$-\frac{L}{2} < \omega < L/2 \tag{A2.2}$$

$$-\frac{\lambda - L}{2} + \omega \leq \mu \leq \frac{\lambda - L}{2} + \omega \tag{A2.3}$$

$$\lambda = L, L+2, \ldots \tag{A2.4}$$

Since the effect of the operator (12) applied to the tensor polynomials is to change the sign of the quantum numbers ω and μ eqn. (3.49), it is possible to confine the value of ω to positive ones only.

Spherical harmonics for L = 0 are given by eqn. (3.10). For L = 1 the polynomials are given by

$$P^{l}\frac{\partial}{\partial z_i} = P^{l}g^{ij}(z_j\Lambda_1 + z_j^*\Lambda_2) = T^{lM\frac{1}{2}}\Lambda_1 + T^{lM-\frac{1}{2}}\Lambda_2 \tag{A2.5}$$

$$\phi_{\lambda}^{l\mu} = \Omega(K + 2\varepsilon + 3)r^{-l}[T^{lM\frac{1}{2}}(\sigma\cos 2\psi)^{\varepsilon}P_n^{(\varepsilon,1)}(1+\cos^2 2\psi)$$

$$- T^{lM-\frac{1}{2}}(\sigma\cos 2\psi)^{\varepsilon+1}P_n^{(\varepsilon+1,1)}(1+\cos^2 2\psi)] \tag{A2.6}$$

$$\varepsilon = -\frac{\lambda - 1}{2}\ ,\ -\frac{\lambda - 1}{2} + 2,\ \ldots\ \frac{\lambda - 1}{2} \tag{A2.7}$$

$$n = \frac{\lambda - L}{4} - \varepsilon/2 \tag{A2.8}$$

For $L = 2$ $\quad \omega = 1$

$$P^2\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} = g^{ik}g^{jl}\{(z_k z_l - 1/3g^{kl}z^2)\Lambda_1^2 + (z_k z_l^*$$

$$+ z_k^* z_l - 2/3g^{kl}\underline{z}\cdot\underline{z}^*)\Lambda_1\Lambda_2 + (z_k^* z_l^* - 1/3g^{kl}z^{*2})\Lambda_2^2\}$$

$$= T^{2Ml}\Lambda_1^2 + 2T^{2MO}\Lambda_1\Lambda_2 + T^{2M-1}\Lambda_2^2 \tag{A2.9}$$

$$\phi_{\lambda\mu}^{2Ml} = \Omega_2(\frac{\lambda-2}{2} + \varepsilon + 2)\ [T^{2Ml}(\sigma\cos 2\psi)^{\varepsilon}P_n^{(\varepsilon,2)} - 2T^{2MO}\ x$$

$$(\sigma\cos 2\psi)^{\varepsilon+1}P_{n-1}^{(\varepsilon+1,1)} + T^{2M-1}(\sigma\cos 2\psi)^{\varepsilon+2}P_{n-2}^{(\varepsilon+2,2)}] +$$

$$T^{2M-1}(\sigma\cos 2\psi)^{\varepsilon+2}P_{n-1}^{(\varepsilon+2,1)} \tag{A2.10}$$

$$\varepsilon = -\frac{\lambda - 2}{2}\ ,\quad -\frac{\lambda - 2}{2} + 2\ \cdots\ \frac{\lambda - 2}{2} \tag{A2.11}$$

$L = 2 \quad \omega = 0$

$$P^2 \frac{\partial}{\partial z_{i(1)}} \frac{\partial}{\partial z^*_{j(1)}} = g^{ik}g^{jl}\{(z_k z_l - 1/3 g^{kl} z^2)\Lambda_1\Lambda_2 + \tfrac{1}{2}(z_k z^*_l$$

$$+ z^*_k z_l - 2/3 g^{kl}(\underline{z}\cdot\underline{z}^*))(\Lambda_1\Lambda^*_1 + \Lambda_2^2) + (z^*_k z^*_l - 1/3 g^{kl} z^{*2})\Lambda^*_1\Lambda_2\}$$

$$= T^{2M1}\Lambda_1\Lambda_2 + 2T^{2MO}(\Lambda_1\Lambda^*_2 + \Lambda^2_2) + T^{2M-1}\Lambda^{*2} \qquad \text{(A2.12)}$$

$$\phi^{2\mu O}_{\lambda M_i} = \Omega_3 \left[\left(\frac{\lambda-2}{2} + \varepsilon + 4\right)T^{2M-1}(\sigma\cos 2\psi)^{\varepsilon+1} P^{(\varepsilon-1,2)}_{n-1}(1-2\cos^2 2\psi)\right]$$

$$T^{2MO}\left[(\sigma\cos 2\psi)^{\varepsilon} P^{(\varepsilon,1)}_n (1-2\cos^2 2\psi) + \left(\frac{\lambda-2}{2} + \varepsilon\right)(\sigma\cos 2\psi)^{\varepsilon} P^{(\varepsilon,2)}_{n-1}(1-2\cos^2 2\psi)\right.$$

$$\left. - \left(\frac{\lambda-2}{2} + \varepsilon + 4\right)(\sigma\cos 2\psi)^{\varepsilon} P^{(\varepsilon,2)}_n\right] - \left(\frac{\lambda-2}{2} + \varepsilon\right)T^{2M1}$$

$$(\sigma\cos 2\psi)^{\varepsilon-1} P^{(\varepsilon-1,2)}_n \qquad \text{(A2.13)}$$

$$\varepsilon = 0,2,\ldots \frac{\lambda-2}{2} \quad \text{for} \quad \frac{\lambda-2}{2} \quad \text{even} \qquad \text{(A2.14)}$$

$$\varepsilon = 1,3,\ldots \frac{\lambda-2}{2} \quad \text{for} \quad \frac{\lambda-2}{2} \quad \text{odd}$$

The symmetric and antisymmetric tensors for L = 0,1,2 and $\lambda \leq 8$ are given by the following values:

L	λ	μ	ω
0	0	0	
0	4	0	
0	6	3	
0	8	0	
1	3	3/2	1/2
1	5	-3/2	1/2
1	7	3/2	1/2
2	4	0	1
2	6	3	1
2	8	0	1
2	2	0	0
2	6	0	0
2	8	3	0

APPENDIX 3

Table of function $\phi^{L\omega}_{\lambda\mu M}$ [32]

L	λ	μ	ω	$\phi^{L\omega}_{\lambda\mu}$
0	0	0		1
	2	1		$\sigma\cos 2\psi$
	4	2		$\sigma^2\cos^2 2\psi$
		0		$1-2\cos^2 2\psi$
	6	3		$\sigma^3\cos^3 2\psi$
		1		$\sigma\cos 2\psi\,(2-3\cos^2 2\psi)$
	8	4		$\sigma^4\cos^4 2\psi$
		2		$\sigma^2\cos^2 2\psi\,(3-4\cos^2 2\psi)$
		0		$1-\sigma\cos^2 2\psi+\sigma\cos^4 2\psi$
1	1	1/2	1/2	$-T^{1,\frac{1}{2}}$
	3	3/2		$-T^{1,\frac{1}{2}}\sigma\cos 2\psi$
		-1/2		$2\cos^2 2\psi\sigma^{-1}T^{1,\frac{1}{2}} + T^{1,-\frac{1}{2}}$
	5	5/2		$-T^{1,\frac{1}{2}}\sigma^2\cos^2 2\psi$
		1/2		$-T^{1,\frac{1}{2}}(1-3\cos^2 2\psi) + T^{1,-\frac{1}{2}}\sigma\cos 2\psi$
		-3/2		$-3T^{1,\frac{1}{2}}\sigma^{-2}\cos^2 2 - 2T^{1,-\frac{1}{2}}\sigma^{-1}\cos 2\psi$
	7	7/2		$-T^{1,\frac{1}{2}}\sigma^3\cos^3 2\psi$
		3/2		$-2T^{1,\frac{1}{2}}\sigma\cos 2\psi\,(1-\cos^2 2\psi) + T^{1,-\frac{1}{2}}\sigma^2\cos^2 2\psi$
		-1/2		$3T^{1,\frac{1}{2}}\sigma^{-1}\cos 2\psi\,(1-2\cos^2 2\psi) + T^{1,-\frac{1}{2}}(1-3\cdot\cos^2 2\psi)$
		5/2		$4T^{1,\frac{1}{2}}\sigma^{-3}\cos^3 2\psi+3T^{1,-\frac{1}{2}}\sigma^{-2}\cos^2 2\psi.$
2	2	1	1	$T^{2,1}$
	4	2		$T^{2,1}\cos 2\psi$

L	λ	μ	ω	
2	4	0	1	$-\sigma T^{2,1}\sigma\cos 2\psi - 4T^{2,0} + T^{2,-1}\sigma\cos 2\psi$
	6	3		$T^{2,1}\sigma^2\cos^2 2\psi$
		1		$4T^{2,1}(1-4\cos^2 2\psi) - 8T^{2,0}\sigma\cos 2\psi + T^{2,-1}\sigma^2\cos^2 2\psi$
		-1		$12T^{2,1}\sigma^{-2}\cos^2 2\psi + 8T^{2,0}\sigma^{-1}\cos 2\psi + 3T^{2,-1}(1-\cos^2 2\psi)$
	8	4		$T^{2,1}\sigma^3\cos^3 2\psi$
		2		$T^{2,1}\cos 2\psi(2-5\cos^2 2\psi) - 12T^{2,0}\sigma^2\cos^2 2\psi + T^{2,-1}\cdot\sigma^3\cos^3 2\psi$
		0		$-4T^{2,1}\sigma^{-1}\cos 2\psi(2-5\cos^2 2\psi) - 4T^{2,0}(1-3\cos^2 2\psi) +$ $+ T^{2,-1}\sigma\cos 2\psi(3-\cos^2 2\psi)$
		-2		$-20T^{2,1}\sigma^{-3}\cos^3 2\psi - 12T^{2,0}\sigma^{-2}\cos^2 2\psi + 3T^{2,-1}\sigma^{-1}$ $\cos 2\psi(-3+2\cos^2 2\psi)$
	2	0	0	$-T^{2,0}$
	4	1		$-2T^{2,1} - 5T^{2,0}\sigma\cos 2\psi$
	6	2		$-4T^{2,1}\sigma\cos 2\psi - 7T^{2,0}\sigma^2\cos^2 2\psi$
		0		$2T^{2,1}\sigma^{-1}\cos 2\psi + T^{2,0}(-1+7\cos^2 2\psi) + 2T^{2,1}\cdot\sigma\cos 2\psi$
	8	3		$-2T^{2,1}\sigma^2\cos^2 2\psi - 3T^{2,0}\sigma^3\cos^3 2\psi$
		1		$-2T^{2,1}(1-4\cos^2 2\psi) + T^{2,0}\sigma\cos 2\psi \times$ $(-5+18\cos^2 2\psi) + 4T^{2,-1}\sigma^2\cos^2 2\psi$

APPENDIX 4

Properties of the $(\underline{S}_{2x}, \underline{S}_{2y}, \underline{S}_{2z})$ vectors

i) Proof of the invariance of the vectors $(\underline{S}_{2x}, \underline{S}_{2y}, \underline{S}_{2z})$ under permutations of all the particles:

In Chapter IV the moving system of reference S_2 was defined by

$$\underline{S}_{2x} = \frac{\underline{z}}{z^{*2}} + \frac{\underline{z}^*}{z^2} \tag{A4.1}$$

$$\underline{S}_{2y} = \sin 3\phi \{\underline{z}(z^2 \underline{z}\cdot\underline{z}^* + z^{*4}) - \underline{z}^*(z^4 + z^{*2}\underline{z}^*\cdot\underline{z})\} = \underline{S}_{2z} \times \underline{S}_{2x} \tag{A4.2}$$

$$\underline{S}_{2z} = \sin 3\phi \, (z^{*2}\underline{z}^* \times z^2 \underline{z}) \tag{A4.3}$$

Using eqns. (3.47) and (4.25-26) and applying the permutation operators (12), (23) and (13) to the above defined vectors, we obtain

For $\underline{S}_{2x}$

$$(12)\ \underline{S}_{2x} \doteq (12)\ \left(\frac{\underline{z}}{z^{*2}} + \frac{\underline{z}^*}{z^2}\right) = \frac{\underline{z}^*}{z^2} + \frac{\underline{z}}{z^{*2}} = \underline{S}_{2x} \tag{A4.4}$$

$$(23)\ \underline{S}_{2x} = \frac{\underline{z}^* e^{2\pi i/3}}{z^2 e^{2\pi i/3}} + \frac{\underline{z} e^{-2/3\pi i}}{z^{*2} e^{-2/3\pi i}} = \frac{\underline{z}^*}{z^2} + \frac{\underline{z}}{z^*} = \underline{S}_{2x} \tag{A4.5}$$

$$(13)\ \underline{S}_{2x} = \frac{\underline{z}^* e^{-2\pi i/3}}{z^2 e^{-2\pi i/3}} + \frac{\underline{z} e^{2/3\pi i}}{z^{*2} e^{2/3\pi i}} = \frac{\underline{z}^*}{z^2} + \frac{\underline{z}}{z^*} = \underline{S}_{2x} \tag{A4.6}$$

For $\underline{S}_{2z}$

$$(12)\ \underline{S}_{2z} = -\sin 3\phi[z^2\underline{z} \times z^{*2}\underline{z}^*] = \sin 3\phi[z^{*2}\underline{z}^* \times z^2\underline{z}] = \underline{S}_{2z} \qquad (A4.7)$$

$$(23)\ \underline{S}_{2z} = -\sin 3\phi[z^2 e^{2\pi i/3} e^{-2/3\pi i}\ \underline{z} \times z^{*2} e^{2\pi i/3} e^{-2\pi i/3}\underline{z}] =$$

$$= \sin 3\phi[z^{*2}\underline{z}^* \times z^2\underline{z}] = \underline{S}_{2z} \qquad (A4.8)$$

since (23) $\sin 3\phi = -\sin 3\phi$

$$(31)\ \underline{S}_{2z} = -\sin 3\phi\,(z^2 e^{-2/3\pi i} e^{2/3\pi i}\ \underline{z} \times z^{*2} e^{2/3\pi i} e^{-2/3\pi i}\underline{z}^*)$$

$$= \sin 3\phi\,(z^{*2}\underline{z}^* \times z^2\underline{z}) = \underline{S}_{2z} \qquad (A4.9)$$

since $\sin 3\phi = -\sin 3\phi$

$$(12)\ \underline{S}_{2y} = (12)(\underline{S}_{2z} \times \underline{S}_{2x}) = \underline{S}_{2y} \qquad (A4.10)$$

$$(23)\ \underline{S}_{2y} = (23)(\underline{S}_{2z} \times \underline{S}_{2x}) = \underline{S}_{2y} \qquad (A4.11)$$

$$(13)\ \underline{S}_{2y} = (31)(\underline{S}_{2z} \times \underline{S}_{2x}) = \underline{S}_{2y} \qquad (A4.12)$$

Thus the system S_2 is invariant under permutations.

ii) Proof of the orthogonality and dextrality of S_2.

Performing the scalar products of the vectors generating S_2 and using the relation of the scalar triple product eqn. (A5.7) we obtain,

$$\underline{S}_{2x}\cdot\underline{S}_{2z} = (\sin(3\phi)z^{*2}\underline{z}^{*})\cdot\left[\sin3\phi\ \underline{z}\ x\ \left(\frac{\underline{z}}{z^{*2}} + \frac{\underline{z}^{*}}{z^{2}}\right)\right]$$

$$= \left(\frac{\underline{z}^{*}}{z^{2}}\ x\ \sin3\phi\ z^{*2}\underline{z}^{*}\right)\ \ \sin3\phi\ z^{2}\underline{z} = 0 \qquad \text{(A4.13)}$$

$$\underline{S}_{2y}\cdot\underline{S}_{2z} = (\underline{S}_{2z}\ x\ \underline{S}_{2x})\cdot\underline{S}_{2z} = (\underline{S}_{2z}\ x\ \underline{S}_{2z})\ .\ \underline{S}_{2x} = 0 \qquad \text{(A4.14)}$$

$$\underline{S}_{2x}\cdot\underline{S}_{2y} = \underline{S}_{2x}\cdot(\underline{S}_{2z}\ x\ \underline{S}_{2x}) = \underline{S}_{2z}\cdot(\underline{S}_{2x}\ x\ \underline{S}_{2x}) = 0 \qquad \text{(A4.15)}$$

Thus the system S_2 is orthogonal.

Taking the scalar triple product of the three vectors $\underline{S}_{2x}$, $\underline{S}_{2y}$, $\underline{S}_{2z}$ we obtain

$$(\underline{S}_{2x}\ x\ \underline{S}_{2y})\cdot\underline{S}_{2z} = \underline{S}_{2y}\cdot(\underline{S}_{2z}\ x\ \underline{S}_{2x}) = (\underline{S}_{2z}\ x\ \underline{S}_{2x})\cdot(\underline{S}_{2z}\ x\ \underline{S}_{2x}) = 1 \qquad \text{(A4.16)}$$

Thus S_2 is a dextral system.

APPENDIX 5

The Set S_2 in Configuration Space

With the help of eqns. (3.16) and (4.7-9) it is possible to express the vectors of the moving system of coordinates ($\underline{S}_{2x}, \underline{S}_{2y}, \underline{S}_{2z}$) in terms of the vectors $\underline{r}^{(1)}$ and $\underline{r}^{(2)}$ in the configuration space defined by eqn. (2.1).

Thus

$$\underline{S}_{2z} = \sin 3\phi\,(z^{*2}\underline{z}^{*} \times z^{2}\underline{z}) = \sin 3\phi z^{*2}z^{2}(\underline{r}^{(2)} - i\underline{r}^{(1)}) \times$$

$$(\underline{r}^{(2)} + i\underline{r}^{(1)}) = \sin 3\phi z^{*2}z^{2}\{[\underline{r}^{(2)} \times \underline{r}^{(2)}] + [\underline{r}^{(2)} \times i\underline{r}^{(1)}] \times$$

$$+ [-i\underline{r}^{(1)} \times \underline{r}^{(2)}] + [-i\underline{r}^{(1)} \times i\underline{r}^{(1)}]\}$$

$$= \sin 3\phi z^{*2}z^{2}\{-2i(\underline{r}^{(1)} \times \underline{r}^{(2)})\} \qquad \text{(A5.1)}$$

From eqn. (3.16) the scalar products of z^{*2} and z^{2} can be constructed. Thus

$$z^{*2}z^{2} = (r^{(2)^2} - r^{(1)^2})^2 + 4(\underline{r}^{(2)} \cdot \underline{r}^{(1)})^2 \qquad \text{(A5.2)}$$

Substituting eqn. (A5.2) into eqn. (A5.11) we obtain

$$\underline{S}_{2z} = -2i \sin 3\phi[(r^{(2)^2} - r^{(1)^2})^2 + 4(\underline{r}^{(2)} \cdot \underline{r}^{(1)})^2](\underline{r}^{(1)} \times \underline{r}^{(2)}) \qquad \text{(A5.3)}$$

Similarly for $\underline{S}_{2x}$ we obtain

$$\underline{S}_{2x} = \frac{\underline{z}}{z^{*2}} + \frac{\underline{z}^*}{z^2} = \frac{4\underline{r}^{(1)}(\underline{r}^{(1)}\cdot\underline{r}^{(2)} - 2\underline{r}^{(2)}(r^{(2)^2} - r^{(1)^2}))}{(r^{(2)^2} - r^{(1)^2})^2 + 4(\underline{r}^{(2)}\cdot\underline{r}^{(1)})^2} \tag{A5.4}$$

Regrouping terms $\underline{S}_{2x}$ can be written as

$$\underline{S}_{2x} = \underline{r}^{(1)}[1/4(r^{(2)^2} - r^{(1)^2})^2(\underline{r}^{(2)}\underline{r}^{(1)})^{-1} + (\underline{r}^{(2)}\cdot\underline{r}^{(1)})]^{-1} -$$

$$\underline{r}^{(2)}[1/2(r^{(2)^2} - r^{(1)^2}) + 2(\underline{r}^{(2)}\cdot\underline{r}^{(1)})^2(r^{(2)^2} - r^{(1)^2})^{-1}]^{-1} \tag{A5.5}$$

Finally for the vector $\underline{S}_{2y}$ we obtain

$$\underline{S}_{2y} = \underline{S}_{2z} \times \underline{S}_{2x} = \sin 3\phi(z^{*2}\underline{z}^* \times z^2\underline{z}) \times (\frac{\underline{z}}{z^{*2}} + \frac{\underline{z}^*}{z^2})$$

$$= \sin 3\phi \frac{z^{*2}}{z^2}(\underline{z}^* \times \underline{z}) \times (z^2\underline{z} + z^{*2}\underline{z}^*) \tag{A5.6}$$

Using the well known triple vector product relation we obtain the explicit magnitude and direction of $\underline{S}_{2y}$. Thus

$$(B \times C) \times A = C(A\cdot B) - B(A\cdot C) \tag{A5.7}$$

$$\underline{S}_{2y} = \sin 3\phi\{\underline{z}[(z^2\underline{z} + z^{*2}\underline{z}^*)\cdot\underline{z}^*] - \underline{z}^*[(z^2\underline{z} + z^{*2}\underline{z}^*)\cdot\underline{z}]\}$$

$$= \sin 3\phi\ (\underline{z}\gamma - \underline{z}^*\delta)$$

$$= \sin 3\phi\{(\underline{r}^{(2)} - i\underline{r}^{(1)})\delta - (\underline{r}^{(2)} + \underline{r}^{(1)})\gamma\} \tag{A5.8}$$

Where $\gamma = z^2\underline{z}\cdot\underline{z}^* + z^{*4}$ $\delta = z^4 + z^{*2}\underline{z}^*\cdot\underline{z}$

Regrouping the vectors $\underline{r}^{(2)}$ and $\underline{r}^{(1)}$ we obtain

$$\underline{S}_{2y} = \sin 3\phi\{\underline{r}^{(2)}(\delta - \gamma) - \underline{r}^{(1)} i(\delta + \gamma)\} \tag{A5.9}$$

We evaluate $(\delta - \gamma)$ and $(\gamma + \delta)$ from eqn. (3.16). Thus:

$$(\delta - \gamma) = -(z^2\underline{z}\cdot\underline{z}^* + z^{*4} - z^4 - z^{*2}\underline{z}^*\cdot\underline{z}) \tag{A5.10}$$

$$= z^{*4} - z^4 - \underline{z}^*\cdot\underline{z}(z^2 - z^{*2})$$

$$= -(r^{(2)^2} + r^{(1)^2})4i(\underline{r}^{(2)}\cdot\underline{r}^{(1)}) + 8(r^{(2)^2} - r^{(1)^2})(\underline{r}^{(2)}\cdot\underline{r}^{(1)}) \tag{A5.11}$$

$$\gamma + \delta = z^2\underline{z}\cdot\underline{z}^* + z^{*2}(\underline{z}^*\cdot\underline{z}) + z^{*4} + z^4 = z^{*4} + z^4 + \underline{z}\cdot\underline{z}^*(z^2 + z^{*2})$$

$$= (r^{(2)^2} + r^{(1)^2})2(r^{(2)^2} - r^{(1)^2}) + 2[(r^{(2)^2} - r^{(1)^2}) - (2(\underline{r}^{(2)}\cdot\underline{r}^{(1)})^2]$$

$$= 4[r^{(2)^4} + r^{(1)^4} + 2(\underline{r}^{(2)}\cdot\underline{r}^{(1)})^2 - (\underline{r}^{(2)}\cdot\underline{r}^{(1)})] \tag{A5.12}$$

Substituting $(\delta - \gamma)$ and $\gamma + \delta$ in eqn. (A5.9) we obtain for $\underline{S}_{2y}$

$$\underline{S}_{2y} = \underline{r}^{(1)}[4i\sin 3\phi(r^{(2)^2} r^{(1)^2} - r^{(2)^4} - r^{(1)^4} + 2(\underline{r}^{(2)}\cdot\underline{r}^{(1)})^2]$$

$$- \underline{r}^{(2)}[(i + 2)4\sin 3\phi(r^{(2)^2} + r^{(1)^2})(\underline{r}^{(2)}\cdot\underline{r}^{(1)})] \tag{A5.13}$$

APPENDIX 6

The Spherical Components $x(\pm),x^*(\pm)$

i) Derivation of (x_+,x_-) and (x^*_+,x^*_-).

The projection of the vectors $\underline{z}$ and $\underline{z}^*$ onto the basis $(\underline{S}_{2x},\underline{S}_{2y},\underline{S}_{2z})$ defines the vectors $\underline{x}$ and $\underline{x}^*$ which in spherical coordinates are given by equations (4.21) and (4.22).

Thus

$$x_{(\pm)} = (\pm)\frac{1}{\sqrt{2}}\left(\underline{z}\cdot\frac{\underline{S}_{2x}}{|\underline{S}_{2x}|} \; (\pm)i\underline{z}\cdot\frac{\underline{S}_{2y}}{|\underline{S}_{2y}|}\right) \qquad x^*_{\pm} = (\pm)\frac{1}{2}\left(\underline{z}^*\cdot\frac{\underline{S}_{2x}}{|\underline{S}_{2y}|} \; (\pm)i\underline{z}^*\cdot\frac{\underline{S}_{2y}}{|\underline{S}_{2y}|}\right) \tag{A6.1}$$

We have omitted the components x_0 and x^*_0 since these are not needed for the construction of the tensors $T^k_{LL}(r,\phi,\psi)$.

Substituting eqns. (4.7) and (4.14) into (4.20) we get

$$x_1 = \underline{z}\cdot\frac{\underline{S}_{2x}}{|\underline{S}_{2x}|} = \left(\frac{z^2}{z^{*2}}+\frac{\underline{z}\cdot\underline{z}^*}{z^2}\right)\left(\frac{\underline{z}\cdot\underline{z}^*}{z^{*4}}+\frac{1}{z^{*2}}+\frac{1}{z^2}+\frac{\underline{z}\cdot\underline{z}^*}{z^4}\right)^{-\frac{1}{2}} \tag{A6.2}$$

$$x_2 = \underline{z}\cdot\frac{\underline{S}_{2y}}{|\underline{S}_{2y}|} = \sin 3\phi\left[\left(z^4\underline{z}\cdot\underline{z}^*+z^2z^{*4}\right)-\left(\underline{z}^*\cdot\underline{z}\,z^4+z^{*2}(\underline{z}^*\cdot\underline{z})^2\right)\right] \times$$

$$\left\{\sin^2 3\phi\left[\underline{z}\cdot\underline{z}^*(\gamma^2+\delta^2)-(z^2+z^{*2})\gamma\delta\right]\right\}^{-\frac{1}{2}} \tag{A6.3}$$

Where γ and δ are defined in eqn. (A5.8)

$$x_1^* = \underline{z}^* \cdot \frac{\underline{S}_{2x}}{|S_{2x}|} = \left(\frac{\underline{z}\cdot\underline{z}^*}{z^{*2}} + \frac{z^{*2}}{z^2}\right)\left(\frac{\underline{z}\cdot\underline{z}^*}{z^{*2}} + \frac{1}{z^{*2}} + \frac{1}{z^2} + \frac{\underline{z}\cdot\underline{z}^*}{z^4}\right)^{-\frac{1}{2}} \qquad \text{(A6.4)}$$

and

$$x_2^* = \underline{z}^* \cdot \frac{\underline{S}_{2y}}{|S_{2y}|} = \sin 3\phi\left[\left(z^2(\underline{z}\cdot\underline{z}^*)^2 + z^{*4}(\underline{z}\cdot\underline{z}^*)\right) - \left(z^4 z^{*2} + z^{*4}\underline{z}^*\cdot\underline{z}\right)\right] \times$$

$$\left\{\sin^2 3\phi\left[\underline{z}\cdot\underline{z}^*(\gamma^2 + \delta^2) - (z^2 + z^{*2})\gamma\delta\right]\right\}^{-\frac{1}{2}} \qquad \text{(A6.5)}$$

The above relations can be considerably simplified by expressing all the scalar products in terms of z^2. This is accomplished with the help of eqn. (4.24).

Thus

$$x_1|S_{2x}| = \sigma^2 - 1/\sigma\cos 2\psi$$

$$x_2|S_{2y}| = \sin 3\phi\ \frac{r^6\cos 2\psi}{\sigma}\left(\cos^2 2\psi - 1\right) \qquad \text{(A6.6)}$$

$$x_1^*|S_{2x}| = 1/\sigma^2 - \sigma/\cos 2\psi$$

$$x_2^*|S_{2y}| = -\sin 3\phi\ r^6\cos(2\psi)\sigma\left(\cos^2 2\psi - 1\right) \qquad \text{(A6.7)}$$

Where use was made of eqn. (4.23).

Following the same procedure the $|S_{2x}|$ and $|S_{2y}|$ are expressed in terms of the internal coordinates (r,ϕ,ψ). These are substituted into eqns. (A6.6) and (A6.1) giving the final result after factorizing $[4-2\cos(2\psi)(\sigma^{-3} + \sigma^3)]$.

Thus

$$x_{(\pm)} = (\mp) r\sigma^2 [4-2\cos(2\psi)(\sigma^{-3}+\sigma^3)]^{-\frac{1}{2}} [\sigma^{-3}-\cos 2\psi\,(\mp)\, \frac{i(\sigma^{-6}-1)(1-\cos^2 2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^6)^{\frac{1}{2}}}] \tag{A6.8}$$

$$x^*_{(\pm)} = (\mp) r\sigma^{-2} [4-2\cos(2\psi)(\sigma^{-3}+\sigma^3)]^{-\frac{1}{2}} [\sigma^{3}-\cos 2\psi\,(\pm)\, \frac{i(1-\sigma^6)(1-\cos^2 2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^6)^{\frac{1}{2}}}] \tag{A6.9}$$

ii) Permutational properties of $x_{(\pm)}$ and $x^*_{(\pm)}$.

In this section we show that $x_{(\pm)}$ and $x^*_{(\pm)}$o behave under permutations in the same way as $z_{(\pm)}$ and $z_{(\pm)}$ do, i.e.

$$\begin{aligned}
&(12)\ x_{(\pm)} = x^*_{(\pm)} && (12)\ x^*_{(\pm)} = x_{(\pm)} \\
&(23)\ x_{(\pm)} = x^*_{(\pm)} e^{2\pi i/3} && (23)\ x^*_{(\pm)} = x_{(\pm)} e^{-2\pi i/3} \\
&(31)\ x_{(\pm)} = x^*_{(\pm)} e^{-2\pi i/3} && (31)\ x^*_{(\pm)} = x_{(\pm)} e^{-2\pi i/3}
\end{aligned} \tag{A6.10}$$

Directly applying the permutation operators (12), (23) and (31) to eqns. (A6.8-9) we obtain

$$(12)\ x_{(\pm)} = (\mp)\ r\sigma^{-2} [4-2\cos(2\psi)(\sigma^{3}+\sigma^{-3})]^{-\frac{1}{2}} [\sigma^{3}-\cos 2\psi\,(\pm)\, \frac{i(1-\sigma^6)(1-\cos^2 2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^6)^{\frac{1}{2}}}] \tag{A6.11}$$

$$= x^*_{(\pm)}$$

$$(23)\quad x_{(\pm)} = (\mp)\, r\sigma^{-2} e^{-4\pi i/3}[4-2\cos 2\psi\,(\sigma^{3} e^{6\pi i/3}+\sigma^{-3} e^{-6\pi i/3})]^{-\frac{1}{2}}[\sigma^{3} e^{6\pi i/3} -$$

$$\cos 2\psi\,(\mp)\,\frac{i(1-\sigma^{6} e^{12\pi i/3})(1-\cos^{2}2\psi)^{\frac{1}{2}}}{(2-\sigma^{6} e^{12\pi i/3}-\sigma^{6} e^{-12\pi i/3})^{\frac{1}{2}}} = x^{*}_{(\pm)} e^{2\pi i/3} \qquad \text{(A6.12)}$$

since $e^{\pm 2n\pi i} = 1$ for $n = 0,1,2 \ldots$

$$e^{-4\pi i/3} = e^{2\pi i/3}\, e^{-2\pi i} = e^{2\pi i/3}$$

$$(31)\quad x_{(\pm)} = (\mp) r\sigma^{-2} e^{4\pi i/3}[4-2\cos 2\psi\,(\sigma^{3} e^{-6\pi i/3}+\sigma^{3} e^{6\pi i/3})]^{-\frac{1}{2}}[\sigma^{3} e^{-6\pi i/3} -$$

$$\cos 2\psi\,(\pm)\,\frac{i(1-\sigma^{6} e^{-12\pi i/3})(1-\cos^{2}2\psi)^{\frac{1}{2}}}{(2-\sigma^{6} e^{-12\pi i/3}-\sigma^{-6} e^{12\pi i/3})}] = x^{*}_{(\pm)} e^{-2\pi i/3} \qquad \text{(A6.13)}$$

Similarly for $x^{*}_{(\pm)}$ we have

$$(12)\quad x^{*}_{(\pm)} = (\mp) r\sigma^{-2}[4-2\cos(2\psi)\,(\sigma^{3}+\sigma^{-3})]^{-\frac{1}{2}}[\sigma^{-3} - \cos 2\psi\,(\mp)\,\frac{i(\sigma^{-6}-1)(1-\cos^{2}2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^{6})^{\frac{1}{2}}}]$$

$$= x_{(\pm)} \qquad \text{(A6.14)}$$

$$(23)\quad x^{*}_{(\pm)} = (\mp) r\sigma^{2} e^{4\pi i/3}[4-2\cos(2\psi)\,(\sigma^{3} e^{6\pi i/3}+\sigma^{-3} e^{-6\pi i/3})]^{-\frac{1}{2}}[\sigma^{-3} e^{-6\pi i/3}$$

$$-\cos(2\psi)\,(\mp)\,\frac{i(\sigma^{-6} e^{-12\pi i/3}-1)(1-\cos^{2}2\psi)^{\frac{1}{2}}}{(2-\sigma^{6} e^{12\pi i/3}-\sigma^{-6} e^{-12\pi i/3})^{\frac{1}{2}}}] = x_{\pm}\, e^{-2/3\pi i} \qquad \text{(A6.15)}$$

finally

$$(31)\quad x^{*}_{\pm} = (\mp)\sigma^{2}e^{-4\pi i/3}\left[(4-2\cos(2\psi))(\sigma^{3}e^{-6\pi i/3}+\sigma^{-3}e^{6\pi i/3})\right]^{-\frac{1}{2}}\left[\sigma^{-3}e^{6\pi i/3} - \cos(2\psi)(\mp)\frac{i(\sigma^{6}e^{12\pi i/3}-1)(1-\cos^{2}2\psi)^{\frac{1}{2}}}{(2-\sigma^{6}e^{-12\pi i/3}-\sigma^{-6}e^{12\pi i/3})^{\frac{1}{2}}}\right] = x_{\pm}\,e^{2/3\pi i} \qquad (A6.16)$$

Thus $x_{(\pm)}$ and $x^{*}_{(\pm)}$ satisfy eqn. (A6.10).

APPENDIX 7

Derivation of the $T^k_{LM}(r,\phi,\psi)$ Tensors

Using equation (4.36) we can write $T^k_{LM}(r,\phi,\psi)$ as

$$T^k_{LN}(r,\phi,\psi) = \left\{ \binom{2L}{L-N} \right\}^{-\frac{1}{2}} \sum_n \binom{1}{\frac{L-N}{2} - n} \binom{r}{n} r^L \left[A(C+D)\right]^{\frac{L-N}{2}-n} \times$$

$$[B(C' - E)]^n \, [A(C - D)]^{k+N/2+n} \, [B(C' + E)]^{L/2-k-n} \qquad \text{(A7.1)}$$

where
$$A = \sigma^2[4 - 2\cos(2\psi)(\sigma^{-3} + \sigma^3)]^{-\frac{1}{2}}$$

$$B = \sigma^{-2}[4 - 2\cos(2\psi)(\sigma^{-3} + \sigma^3)]^{-\frac{1}{2}}$$

$$C = \sigma^{-3} - \cos 2\psi$$

$$C' = \sigma^3 - \cos 2\psi$$

$$D = i(\sigma^{-6} - 1)\frac{(1-\cos^2 2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^6)^{\frac{1}{2}}} \qquad \text{(A7.2)}$$

$$E = i(1 - \sigma^6)\frac{(1-\cos^2 2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^6)^{\frac{1}{2}}}$$

Combining factors raised to the power n, the tensors $T^k_{LN}(r,\phi,\psi)$ can be written as

$$T^k_{LN}(r,\phi,\psi) = \left\{ \binom{2L}{L-N} \right\}^{-\frac{1}{2}} \sum_n \binom{1}{\frac{L-N}{2} - n} \binom{r}{n} (r\cos 2\psi)^L \sigma^{k-L/2} \times$$

$$(C+D)^{\frac{L-N}{2}} (C-D)^{k+\frac{N}{2}} (C'+E)^{\frac{L}{2}-k} \left[\frac{(C-D)(C'-E)}{(C+D)(C'+E)}\right]^n \qquad \text{(A7.3)}$$

Expressing the factor in the square brackets in terms of ψ and σ we get

$$CC'-CE+DE-DC' = 2\cos^2 2\psi+\cos(2\psi)(\sigma^3-\sigma^{-3}) - \frac{i(1-\cos^2 2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^6)^{\frac{1}{2}}}[(\sigma^{-3}-\cos 2\psi)(1-\sigma^6) +$$

$$(\sigma^3-\cos 2\psi)(\sigma^{-6}-1)] = 2\cos^2 2\psi+(\sigma^3-\sigma^{-3})(\cos 2\psi+ \frac{2i(1-\cos^2 2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^6)^{\frac{1}{2}}}) \qquad (A7.4)$$

$$= P_5$$

$$CC'+CE+DC'+DE = 2+\cos 2\psi(\sigma^3-\sigma^{-3}) + \frac{i(1-\cos^2 2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}+\sigma^6)^{\frac{1}{2}}}[(\sigma^{-3}-\cos 2\psi)(1-\sigma^6) +$$

$$+ (\sigma^3-\cos 2\psi)(\sigma^{-6}-1)] = 2+(\sigma^3-\sigma^{-3})(\cos 2\psi - \frac{2i(1-\cos^2 2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^6)^{\frac{1}{2}}}) = P_1 \qquad (A7.5)$$

Similarly the terms $(C+D)^{\frac{L-N}{2}}\ (C-D)^{k+\frac{N}{2}}\ (C'+E)^{L/2-k}$ in eqn. (A7.3) become after substitution

$$\left[(C+D)(C'+E)\right]^{L/2}\left[\frac{C-D}{C+D}\right]^{N/2}\left[\frac{C-D}{C'+E}\right]^{k} = P_1^{L/2}\left[\frac{(2-\sigma^{-6}-\sigma^6)^{\frac{1}{2}}(\sigma^{-3}-\cos 2\psi)-i(\sigma^{-6}-1)}{(2-\sigma^{-6}-\sigma^6)^{\frac{1}{2}}(\sigma^3-\cos 2\psi)+i(1-\sigma^6)}\right]$$

$$\left[\frac{(1-\cos^2 2\psi)^{\frac{1}{2}}}{(1-\cos^2 2\psi)^{\frac{1}{2}}}\right]^{k}\left[\frac{(2-\sigma^{-6}-\sigma^6)(\sigma^{-3}-\cos 2\psi)-i(\sigma^{-6}-1)(1-\cos^2 2\psi)^{\frac{1}{2}}}{(2-\sigma^{-6}-\sigma^6)(\sigma^3-\cos 2\psi)+i(\sigma^{-6}-1)(1-\cos^2 2\psi)^{\frac{1}{2}}}\right]^{N/2} = P_1^{L/2}P_2^{N/2}P_3^{k}$$

(A7.6)

Where P_1, P_2 and P_3 were defined in eqn. (4.37).

The binomial coefficients are given by

$$\left\{\binom{2L}{L-N}\right\}^{-\frac{1}{2}} = \left[\frac{(L+N)!(L-N)!}{2L!}\right]^{\frac{1}{2}} \qquad (A7.7)$$

$$\binom{l}{\frac{L-N}{2}-n}\binom{r}{n} = \frac{l!\ r!}{(l-\frac{L-N}{2}-n)!\,(\frac{L-N}{2}-n)!\,(r-n)!\,n!} \tag{A7.8}$$

Substituting into the above equation the relations

$$l = \frac{L+2k}{2}$$

$$r = \frac{L-2k}{2} \tag{A7.9}$$

and using eqns. (A7.3-10) we can write the tensor $T^{k}_{LN}(r,\phi,\psi)$ in terms of the hypergeometric functions F (a,b;c;d) as follows.

$$T^{k}_{LN}(r,\phi,\psi) = \left[\frac{(L+N)!\,(L-N)!}{2L!}\right]^{\frac{1}{2}}\left[\frac{L/2+k)!}{(k+n/2)!}\right](r\cos 2\psi)^{L}\sigma^{k-L/2}\ x$$

$$P_1^{L/2}P_2^{N/2}P_3^{k}\ F\ (L/2-k-n+1, L/2-N/2-n+1, k+N/2-1; P_4) \tag{A7.10}$$

for $-k-N/2+l \geq 0$

where use was made of the relation

$$\frac{a!}{(a-n)!} = (a-n+1)_n \tag{A7.11}$$

APPENDIX 8

The S Matrices

$$S_{11} = G_o(z) \begin{pmatrix} 0 & \frac{1}{2}(T_{1,1}+T_{1,4}) & \frac{1}{2}(T_{1,1}+T_{1,4}) \\ \frac{1}{2}(T_{2,1}+T_{2,4}) & 0 & \frac{1}{2}(T_{2,1}+T_{2,4}) \\ \frac{1}{2}(T_{3,1}+T_{3,4}) & \frac{1}{2}(T_{3,1}+T_{3,4}) & 0 \end{pmatrix}$$

$$S_{12} = S_{21} = G_o(z) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{13} = S_{31} = G_o(z) \begin{pmatrix} 0 & \frac{\sqrt{3}}{4}(T_{11}-T_{14}) & \frac{\sqrt{3}}{4}(T_{11}-T_{14}) \\ -\frac{\sqrt{3}}{4}(T_{21}-T_{24}) & 0 & -\frac{\sqrt{3}}{4}(T_{21}-T_{24}) \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{14} = S_{41} = G_o(z) \begin{pmatrix} 0 & 1/4(T_{1,1}-T_{1,4}) & 1/4(T_{1,1}-T_{1,4}) \\ 1/4(T_{2,1}-T_{2,4}) & 0 & 1/4(T_{2,1}-T_{2,4}) \\ -\frac{1}{2}(T_{3,1}+T_{3,4}) & -\frac{1}{2}(T_{3,1}+T_{3,4}) & 0 \end{pmatrix}$$

$$S_{22} = G_o(z) \begin{pmatrix} 0 & \frac{1}{2}(T_{1,2}+T_{1,3}) & \frac{1}{2}(T_{1,2}+T_{1,3}) \\ \frac{1}{2}(T_{2,2}+T_{2,3}) & 0 & \frac{1}{2}(T_{2,2}+T_{2,3}) \\ \frac{1}{2}(T_{3,2}+T_{3,3}) & \frac{1}{2}(T_{3,2}+T_{3,3}) & 0 \end{pmatrix}$$

$$S_{23} = S_{32} = G_o(z) \begin{pmatrix} & 1/4(T_{1,2}-T_{1,3}) & 1/4(T_{1,2}-T_{1,3}) \\ \frac{1}{2}(T_{2,2}+T_{2,3}) & 0 & \frac{1}{2}(T_{2,2}+T_{2,3}) \\ \frac{1}{2}(T_{3,2}+T_{3,3}) & \frac{1}{2}(T_{3,2}+T_{3,3}) & 0 \end{pmatrix}$$

$$S_{24} = S_{42} = G_o(z) \begin{pmatrix} 0 & -\frac{\sqrt{3}}{4}(T_{1,2}-T_{1,3}) & -\frac{\sqrt{3}}{4}(T_{1,2}-T_{1,3}) \\ \frac{\sqrt{3}}{4}(T_{2,2}-T_{2,3}) & 0 & \frac{\sqrt{3}}{4}(T_{2,1}-T_{2,3}) \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{33}=G_o(z) \begin{pmatrix} 0 & 1/8(3T_{1,1}+T_{1,2}+T_{1,3}+3T_{1,4}) & 1/8(3T_{1,1}+T_{1,2}+T_{1,3}+3T_{1,4}) \\ 1/8(3T_{2,1}+T_{2,2}+T_{2,3}+3T_{2,4} & 0 & 1/8(3T_{2,1}+T_{2,2}+T_{2,3}+3T_{2,4}) \\ \frac{1}{2}(T_{3,2}+T_{3,3}) & \frac{1}{2}(T_{3,2}+T_{3,3}) & 0 \end{pmatrix}$$

$$S_{34}=S_{43}=G_o(z) \begin{pmatrix} 0 & \frac{\sqrt{3}}{8}(T_{1,1}-T_{1,2}-T_{1,3}+T_{1,4}) & \frac{\sqrt{3}}{8}(T_{1,1}-T_{1,2}+T_{1,3}+T_{1,4}) \\ -\frac{\sqrt{3}}{8}(T_{2,1}-T_{2,2}-T_{2,3}+T_{2,4}) & 0 & \frac{\sqrt{3}}{8}(T_{2,1}-T_{2,2}-T_{2,3}+T_{2,4}) \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{44}=G_0(z)\begin{pmatrix} 0 & 1/8(T_{1,1}+3T_{1,2}+3T_{1,3}+T_{1,4}) & 1/8(T_{1,1}+3T_{1,2}+3T_{1,3}+T_{1,4}) \\ 1/8(T_{2,1}+3T_{2,2}+3T_{2,3}+T_{2,4} & 0 & 1/8(T_{2,1}+3T_{2,2}+3T_{2,3}+T_{2,4}) \\ \tfrac{1}{2}(T_{3,1}+T_{3,4}) & \tfrac{1}{2}(T_{3,1}+T_{3,4}) & 0 \end{pmatrix}$$