Approximations for optimal stopping and impulsive control of piecewise-deterministic processes

by

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Abstract :

Piecewise-deterministic processes (P.D.P.'s) are a class of Markov processes consisting of a mixture of deterministic motion and random jumps . This class of P.D.P.'s provides a general family of stochastic models covering virtually all non-diffusion applications . In this work we deal with the optimal stopping and impulsive control of P.D.P.'s . Optimal stopping is an essential ingredient for impulsive control , since the latter can be regarded as an implicit optimal stopping problem . First we present some technical improvements to some results currently in the literature of optimal stopping of P.D.P.'s , which are needed for the impulsive control problem . After that , we develop a computational method for solving the optimal stopping of P.D.P.'s . This method reduces the problem to a sequence of one dimensional minimizations . We then apply our results on optimal stopping to obtain some optimality conditions and a numerical technique for the impulsive control problem . Numerical examples to illustrate the methods are provided .

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to my parents

Chapter 1

Introduction

1 . Preliminaries

Piecewise-deterministic Markov processes (P.D.P.'s) have been introduced in Davis [9] as a general class of stochastic models suitable for formulating optimization problems in queueing , maintenance/replacement models , inventory systems , investment scheduling and many other areas of operations research . The control theory of P.D.P.'s has recently been developed both for optimal control ([10] , [36] , [38]) and optimal stopping ([15] , [23]) and specific problems have been solved (eg [11]) . However in many important applications , such like inventory systems , control takes place at discrete times in response to the random evolution of the system . This kind of control is known as impulsive control . The optimal stopping problem can be regarded as the simplest example of impulsive control where there is a singled admitted impulse (= stopping) . Presently there is already a good deal of work on several aspects concerning both problems . For the general theory of optimal stopping with applications we refer to [7] , [35] and [41] .

The optimal stopping and impulsive control problems were studied from an analytic point of view as application of variational and quasi-variational inequalities by Bensoussan and Lions [1] , [2] , [3] . Kushner [21] , [22] treats a computational method for both problems based on approximating the diffusion by controlled Markov chains . In [24] Lepeltier and Marchal consider the impulsive control of right processes ([14]) under very general assumptions and show the markovian form of the value function ; in particular they write the impulsive control problem as a sequence of optimal stopping problems , an approach due to Robin [32] .

As pointed out by Yushekevich in [39] , [40] the concept of impulsive control is essentially the same as that for interventions in Markov decision processes (MDP) , only in the latter the decision rules are formulated in terms of "policies" rather than stopping times . An early investigaton of MDP with interventions is due to De Leve [25] ; applications of the techniques developed there are presented in [26] . Afterwards Van der Duyn Schouten [37] admitted in his model , so-called continuous time Markov decision drift processes (CTMDP) both continuous (on the transition measure and jump rate) and impulsive control . CTMDP were analysed by discrete time approximations in [17] , [18] , [37] and by a direct deduction of optimality conditions in Yushekevich [39], [40] . In [17] a computational technique is proposed based on applying dynamic programming to a discrete time approximation of the original model .

In this work we develop an approach analogous to the classical theory of impulsive control for diffussion processes introduced by Bensoussan and Lions [1] but instead of studying the variational and quasi-variational inequalities associated to the optimal stopping and impulsive control problems we present some recursive methods (analogous to the discrete time case) to obtain a characterization of the value function for both problems . In the appendix of chapter 2 and chapter 3 we show a connection between the characterization results for the optimal stopping and impulsive control problems and the variational and quasi-variational inequalities . As mentioned earlier , impulsive control can be formulated in terms of a sequence of optimal stopping problems and thus the key to computing solutions of the former is an effective technique for the latter . Due to the special structure of P.D.P.'s there is a natural form of approximation which in conjuction with the recursive method mentioned above reduces the optimal stopping problem to a sequence of one dimensional searches . With this technique as a building block we are able to solve quite complicated problems of impulsive control of P.D.P.'s .

This work is organized in the following way . An overview of some of the general results on optimal stopping and impulsive control currently in the literature is given below . In chapter 2 we study the optimal stopping of P.D.P.'s from the point of view mentioned above . A numerical

technique with an example is presented . In chapter 3 we apply the results of chapter 2 to impulsive control of P.D.P.'s . Numerical applications to illustrate the computational method developed there are also presented . In chapter 4 we present some final conclusions .

2 . Some general results on optimal stopping

Let $X = (X_t , \mathcal{F}_t , P_x)$ be a standard Markov process (see [35] ch 1) on a state space E and let $\mathcal{M}_{\infty}$ denote the set of all $\mathcal{F}_t$-stopping times . The optimal stopping problem is defined by

$$\rho(x) = \inf_{\tau \in \mathcal{M}_{\infty}} E_x(e^{-\alpha\tau} g(X_\tau)) \qquad \alpha \geq 0$$

If g satisfies some regularities conditions (like boundedness and upper-semicontinuity on E) then from the general theory of optimal stopping (see theorems 1 , 3 ch 3 of [35]) the function $-\rho$ can be characterized as the smallest α-excessive function which majorizes -g on E (recall that a measurable function v is α-excessive if for all $x \in E$,

$e^{-\alpha t} E_x(v(X_t)) \leq v(x)$ for all $t \geq 0$ and

$\lim_{t \downarrow 0} e^{-\alpha t} E_x(v(X_t)) = v(x)$) .

Moreover for $\epsilon \geq 0$ define

$$\tau_\epsilon := \inf \{ t \geq 0 ; \ e^{-\alpha t} g(X_t) \leq e^{-\alpha t} \rho(X_t) + \epsilon \} \qquad (1)$$

Then τ_ϵ is an ϵ-optimal stopping time . Under more restrictive assumptions (like continuity of g on E) the stopping time τ_0 is optimal .

If however the process X is just a right process (cf Getoor [14]) and g a borel function then one needs to use the concept of strongly supermedian functions introduced by Mertens [28] to obtain a characterization for the function ρ similar to the one above (see [6]) .

Optimal stopping is also connected to some free boundary problems . Let $\mathbb{A}$ be the strong generator of the Markov process X and $\mathfrak{D}$ the set which defines the optimal stopping rule , i.e. , the stopping set . Then

$$\begin{cases} \mathbb{A}\rho(x) - \alpha\rho(x) \geq 0 & \forall\, x \in E \\ \rho(x) \leq g(x) & \forall\, x \in E \\ \mathbb{A}\rho(x) - \alpha\rho(x) = 0 & \text{for } x \in E - \mathfrak{D} \\ \rho(x) = g(x) & \text{for } x \in \mathfrak{D} \end{cases} \qquad (2)$$

provided that $\mathbb{A}\rho(x)$ is well defined . Consequently the optimal stopping problem is equivalent to finding a set $\mathfrak{D}$ and a function ρ which satisfies the conditions above . Problems of this type are called free boundary problems (or Stefan's problems) . Usually these sort of problems have a non-unique solution and to distinguish the unique solution coinciding with the payoff function $\rho(x)$ we need to find additional conditons which $\rho(x)$ must necessarily satisfy (for instance the "smooth pasting" condition given in [35] ch 3) .

Equations (2) above were studied for diffusions by Bensoussan and Lions [2] from an analytic point of view as an application of variational inequalities .

Another important approach to optimal stopping problems from theoretical and computational point of view is the so-called penalty method . It consists of finding a stopping intensity (μ_t) adapted to $\mathcal{F}_t$ satisfying the inequality $0 \leq \mu_t \leq \eta$, $t \geq 0$ which minimizes the functional

$$G^{\mu}(x) = E_x\Big(\int_0^{\infty} \mu_t\, g(X_t) \exp\Big(-\int_0^t (\mu_s + \alpha)\, ds \Big)\, dt \Big)$$

We denote by ρ^{η} the infimun value of G^{μ} . For each $\eta > 0$ the problem formulated above is a stochastic control problem and the associated Bellman equation has the following form :

$$\mathbb{A}\rho^{\eta} - \alpha\rho^{\eta} - \eta(\rho^{\eta} - g)^{+} = 0 \qquad (3)$$

Under some smoothness conditions we have the following result (cf [43] theorem 1) : ρ is a bounded continuous function and for arbitrary $\eta > 0$ there exists a unique continuous solution ρ^{η}

of the equation (3) above . Moreover for each $x \in E$, $\rho^{\eta}(x) \downarrow \rho(x)$ uniformly on compact sets of E as $\eta \uparrow \infty$ and τ_0 defined in (1) is an optimal stopping time . Sufficient conditions are X is a Feller process , E is a separable locally compact metric space , g is a bounded continuous function and $\alpha > 0$.

3 . Some general results on impulsive control

Impulsive control problems can be roughly described in the following way (for a more rigorous definition see chapter 3) . An strategy $\mathcal{S} = (\tau_i , R_i)$ is a sequence of non-anticipative intervention times $\tau_1 < \tau_2 < \ldots$ and non-anticipative E-valued random variables R_i , $i \geq 1$ defined on a measurable space $(\mathcal{W} , \mathcal{G})$. Between the intervention times τ_i , τ_{i+1} the motion of the system is determined by the process X starting from R_i . If an intervention takes place at $x \in E$ then the set of feasible points the decision maker can send the system is denoted by $\Gamma(x)$. The strategy $\mathcal{S}$ and the family of probability measures $(P_y)_{y \in E}$ will induce a probability measure $P_x^{\mathcal{S}}$ on $\mathcal{G}$. We say that $\mathcal{S}$ is an admissible strategy if $\tau_i \overset{i}{\rightarrow} \infty \infty$ $P_x^{\mathcal{S}}$ - a.s. .

Remark : More general formulations of the above problem can be found in the literature . For instance , some authors allow random rather than deterministic transitions after an intervention ; in this case R_i , $i \geq 1$ would be an $\mathcal{P}(E)$-valued random variables where $\mathcal{P}(E)$ denotes the space of all probability measures over E . However in this work and particularly in chapter 3 we will concentrate more on the "deterministic" case .

Let $(Y_t(w))_{t \geq 0}$, $w \in \mathcal{W}$ denote the path of the process with interventions . Associated to each admissible strategy $\mathcal{S}$ we have the following expected discounted cost

$$V^{\mathcal{S}}(x) = E_x^{\mathcal{S}}\Big(\int_0^{\infty} e^{-\alpha\tau_i} f(Y_s)ds \; + \; \sum_0^{\infty} e^{-\alpha\tau_i} c(Y_{\tau_i} , Y_{\tau_i +}) \Big)$$

We denote by $\hat{\rho}$ the infimun value of $V^{\mathcal{S}}$ over all admissible strategies .

Remark : Another performance criterion is the so-called long run average cost (see Robin [31]) but here we will consider the discounted cost case only .

As mentioned earlier optimal stopping and impulsive control are closely related . Define

$$Mv(x) = \inf_{y \in \Gamma(x)} (c(x,y) + v(y))$$

In [24] Lepeltier and Marchal show that under very general conditions (like X is a right process , random transitions after inerventions , etc) the impulsive control problem can be regarded as an implicit stopping time problem of the following form

$$\hat{\rho}(x) = \inf_{\tau \in \mathcal{M}_\infty} E_x(\int_0^\tau e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau} M\hat{\rho}(X_\tau)) \qquad (4.i)$$

This result is intuitively plausible since after an intervention we have essentially the same problem as when it started . Moreover it was shown that defining

$$h(x) = E_x(\int_0^\infty e^{-\alpha s} f(X_s) ds) \quad \text{and}$$

$$\mathcal{A}v(x) = \inf_{\tau \in \mathcal{M}_\infty} E_x(\int_0^\tau e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau} Mv(X_\tau)) \qquad (4.ii)$$

then $\mathcal{A}^n h(x) \downarrow \hat{\rho}(x)$ as $n \to \infty$ for all $x \in E$. So the impulsive control problem is the limit of the sequence of optimal stopping problems defined by (4.ii) .

Let $\mathfrak{D}$ be the set which defines the impulsive control rule . Then from (4.i) and the results for optimal stopping we have

$$\begin{cases} \mathbb{A}\hat{\rho}(x) - \alpha\hat{\rho}(x) \geq -f(x) & \forall\, x \in E \\ \hat{\rho}(x) \leq M\hat{\rho}(x) & \forall\, x \in E \\ \mathbb{A}\hat{\rho}(x) - \alpha\hat{\rho}(x) = -f(x) & \text{for } x \in E - \mathfrak{D} \\ \hat{\rho}(x) = M\hat{\rho}(x) & \text{for } x \in \mathfrak{D} \end{cases} \qquad (5)$$

provided that $\mathbb{A}\hat{\rho}(x)$ is well defined . Bensoussan and Lions [1] , [3] studied (5) above for diffusion processes using the theory of quasi-variational inequalities (the term "quasi" is due to the fact that in (5) , instead of g(x) as in (2) , we have the implicit function $M\hat{\rho}(x)$) .

The problem of finding optimal strategies has also been addressed by many authors (see [3] or [24] section 5) . Suppose there is a measurable transformation R from E into E such that for all $x \in E$

$$R(x) \in \Gamma(x) \quad \text{and} \quad M\hat{\rho}(x) = c(x,R(x)) + \hat{\rho}(R(x))$$

Suppose also that $\tau^* := \inf \{ t \geq 0 ; \hat{\rho}(X_t) = M\hat{\rho}(X_t) \}$ is optimal for the problem (4.i) . Then under some regularity conditions it can be shown that the stationary strategy

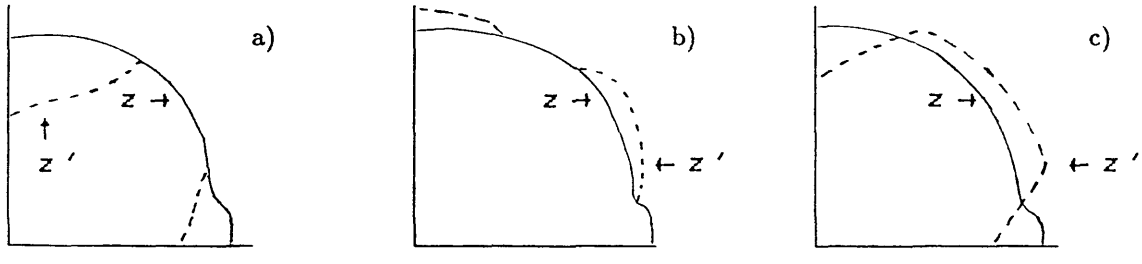
$$\tau_{i+1} := \tau_i + \tau^* \circ \theta_{\tau_i} \qquad \text{where } \theta_t \text{ is the translation operator on } \mathcal{W}$$

$$R_i(w) = R(Y_{\tau_i}(w)) \quad \text{for } w \in \mathcal{W}$$

is optimal (see [3]) .

Impulsive control problems have been studied from a different point of view for MDP . In [25] De Leve examined MDP with interventions using the long run average cost criterion . A simplified version of De Leve's ideas is presented in [27] . There they consider that for each $x \in E$ there is a finite set D(x) of feasible decisions in state x . A decision can either be an intervention which interrupts the natural process and causes an instantaneous (possibly random) change of

the state of the system or it can be a null-decision which leaves the natural process untouched . A policy z is a measurable function that adds to each state $x \in E$ a single decision $z(x) \in D(x)$. They propose a policy improvement algorithm to obtain a solution of the problem . An iteration step of the algorithm consists of three operations . The first part is the value-determination operation which consists of computing the average cost for the current policy . Next a policy improvement is performed which yields a new policy whose set of intervention states is at least as large as that of the previous policy . This implies that an additional operation which may cancel interventions in favour of the null decision is needed . This operation is called the cutting operation (see fig. below) . This algorithm has been applied to some numerical examples in [27] and [34] . In chapter 3 we will solve the latter example for the expected discounted cost case using the numerical technique presented in that chapter .



a) policy improvement operation, b) cutting operation, c) both policy improvement and cutting operation . z is the present strategy , z' is the improved one .

Van der Duyn Schouten [37] admitted in his model both continuous control affecting jump rate and jump distribution of the process and impulsive control. A CTMDP is a nine-tuple (E , A_1 , A_2 , λ , Q , p , f , c , ϕ) where E is the state space , A_1 is the set of continuous control , A_2 the set of impulsive control , $\lambda(\cdot , \cdot) : ExA_1 \rightarrow R_+$ the jump rate , Q , a transition probability from ExA_1 to E , is the jump distribution , p , a transition probability from ExA_2 to E , is the impulsive jump distribution , $f(\cdot , \cdot) : ExA_1 \rightarrow R$ is the cost per unit of time , $c(\cdot , \cdot) : ExA_2 \rightarrow R$ is the intervention cost and ϕ is the drift function which describes the

behaviour of the sample paths of the process between two sucessive jump epochs independently of the policy . Let $\{ \Delta \}$ denote a cemetery state and define

$$J[0,\infty) := \{ (t_j , x_j)_{j=1}^{\infty} : 0=t_1 \leq t_2 \leq \dots ; t_j \in [0,\infty] , x_j \in E \cup \{\Delta\} , j \geq 1 ; t_{j+2} - t_j > 0$$
$$\text{if } t_j < \infty ; x_j = \Delta \text{ iff } t_j = \infty ; \lim_{j \to \infty} t_j = \infty \}$$

According to Van der Duyn Schouten's concept a policy is a pair $R = (R_1 , R_2)$ with R_1 a transition probability from $J[0,\infty)x[0,\infty)$ to A_1 and R_2 a transition probability from $J[0,\infty)x[0,\infty)$ to $A_2 \cup \{\partial\}$ (where $\{\partial\}$ represents the "null-decision" seen above) such that R is non-anticipating , ie , it is independent of the "future" and for all $(w,t) \in J[0,\infty)x[0,\infty)$, $R_2(w,t)$ is either concentrated on A_2 or $\{\partial\}$. A policy is "conservative" if its induced probability measure P_R is concentrated on $J[0,\infty)$. Van der Duyn Schouten's approach is to approximate these CTMDP by discrete-time Markov decision processes (DTMDP) where the time between decisions is $\frac{1}{k}$. This discretization is called the $k^{\underline{th}}$ approximating DTMDP . Let $R^{(k)} = (R_1^{(k)} , R_2^{(k)})$ be a policy for the $k^{\underline{th}}$ approximating DTMDP and $P^{(k)}_{R^{(k)}}$ its induced probability measure . Let P_R denote the induced probability measure by a conservative policy R on $J[0,\infty)$. Theorem 5.2 of [17] provides sufficient conditions on model parameters and policies to obtain weak convergence of $P^{(k)}_{R^{(k)}}$ to P_R . Suppose now that $c_R(w)$ ($c^{(k)}_{R^{(k)}}(w)$ respectively) represents the total discounted cost incurred when the stochastic process induced by policy R ($R^{(k)}$) moves along the sample path $w \in J[0,\infty)$ ($w \in J^{(k)}[0,\infty)$ where $J^{(k)}[0,\infty)$ denotes a discretized subspace of $J[0,\infty)$, see [17] section 4) . Theorem 2.5 of [18] gives sufficient conditions to show

$$\int_{J^{(k)}[0,\infty)} c^{(k)}_{R^{(k)}}(w) dP^{(k)}_{R^{(k)}}(w) \quad k \to \infty \quad \int_{J[0,\infty)} c_R(w) dP_R(w)$$

With this theorem in hand Hordijk and Van der Duyn Schouten show in [18] how the above result can be used to identify the optimal policy of the CTMDP by means of the optimal policies of the approximating DTMDP's . If structural results of optimal policies for the DTMDP's are

known or obtainable via dynamic programming then Hordijk and Van der Duyn Schouten claim that the approximation results mentioned above provides the key to carry over these structural results to the continuous time case (see [17] , [18]) . The long run average cost criterion was considered in [16] . Applications of this technique to queueing , maintenance models and continuous review inventory models can be found in [16] , [17] , [18] , [19] .

Yushkevich [39] , [40] considers a slightly generalized and modified version of Van der Duyn Schouten's model and studies the problem by a direct deduction of optimality conditions . His aim is to examine to what extent inequalities analogous to (5) can be applied to CTMDP . Consider a CTMDP determined by (E , A_1 , A_2 , λ , Q , p , f , c , ϕ) and define for a universally measurable function $F : E \to R$ and all $x \in E$

$$LF(x,a) := \begin{cases} f(x,a) + \lambda(x,a)\int_E (F(y) - F(x))Q(dy,x,a) & , a \in A_1(x) \\ c(x,a) + \int_E F(y)p(dy,x,a) & , a \in A_2(x) \end{cases}$$

$$MF(x) := \inf_{a \in A_2(x)} LF(x,a) \quad , \quad \mathcal{U}F(x) := \inf_{a \in A_1(x)} LF(x,a)$$

where $A_i(x) \subset A_i$ is the set of actions admitted at state x . Then denoting by $\hat{\rho}(x)$ the value function of the minimization problem of the total expected (discounted) cost with discount factor $\alpha \geq 0$ Yushkevich [39] , [40] provides conditions under which $\hat{\rho}$ will satisfy the following systems of inequalities

$$\begin{aligned} &\hat{\rho}(x) \leq M\hat{\rho}(x) \\ &\mathcal{U}\hat{\rho}(\phi(t,x)) - \alpha\hat{\rho}(\phi(t,x)) \geq - \frac{d\hat{\rho}(\phi(t,x))}{dt} \end{aligned} \qquad (6.i)$$

where the second inequality is understood in the integral sense because the derivative may not exist . If for some $x \in E$ both functions $\hat{\rho}(\phi(t,x))$ and $M\hat{\rho}(\phi(t,x))$ are continuous in t then

theorem 4.14 of [40] shows that for each open interval $(s_1,s_2) \in \mathfrak{R}_X^c$, where

$\mathfrak{R}_X := \{ t : t \geq 0 , \hat{\rho}(\phi(t,x)) = M\hat{\rho}(\phi(t,x)) \}$,

the function $\hat{\rho}(\phi(t,x))$ is absolutely continuous on $[s_1,s_2]$ ($[s_1,\infty)$ if $s_2=\infty$) and

$$\mathcal{Q}\hat{\rho}(\phi(t,x)) - \alpha\hat{\rho}(\phi(t,x)) + \frac{d\hat{\rho}(\phi(t,x))}{dt} = 0 \qquad (6.ii)$$

Lebesgue-a.s. on $[s_1,s_2]$. These are according to Yushkevich the Bellman inequalities for CTMDP with the expected discounted cost criterion . In the appendix of chapter 3 we show a connection between the characterization result for the impulsive control of P.D.P.'s presented in that chapter and the Bellman inequalities for CTMDP when there is only impulsive control (that is , A_1 has only one element) . The long run average cost was also considered in [39] as well as some particular CTMDP (like models without drift or without impulse actions , cf [39] , [40]) .

Finally we conclude this chapter by mentioning that the optimal stopping problem with partial information has been analysed in [13] and [20] .

Chapter 2

Optimal stopping of piecewise-deterministic processes

1 . Preliminaries

In this chapter we deal with approximation techniques for the optimal stopping of a P.D.P. . The results obtained here are essential for the impulsive control problem which we will see in chapter 3 . P.D.P.'s are right processes (but not Feller processes) and therefore the general theory in [6] , [28] can be applied . However due to the special features of P.D.P.'s more specific results can be obtained by a direct approach .

This chapter is organized in the following way . In section 2 we give the main definitions and general assumptions . In section 3 we extend the results of [15] to the case when the gain function is just lower semi-analytic . In chapter 3 we will study the impulsive control problem of P.D.P.'s and we will need this extension . The main result of this section is the construction of ϵ-optimal stopping times. In section 4 we present a computational technique for solving the optimal stopping problem of a P.D.P. . We construct a discretized P.D.P. which retains the main characteristics of the original process and we show convergence of the payoff functions . Our computational technique reduces the optimal stopping problem to a sequence of one dimensional minimizations . The problem of finding ϵ-optimal stopping times of the original problem using the solution of the discretized problem is also considered . We conclude that section by presenting a numerical example . In the appendix we present some proofs which were omitted from the main text and a result showing a connection between the representation results of section 3 and the system (2) of chapter 1 .

2 . Notations and definitions

Let E be an open subset of R^d , ∂E the boundary of E , $\mathcal{E}$ the borel σ-field of E and $\mathfrak{U}(E)$ the universally measurable σ-field of E . Let Ω denote the space of E-valued functions on $[0,\infty)$ which are right continuous and have left hand limits at each $t < \infty$, $x_t(\omega) = \omega_t$ for every $\omega \in \Omega$, $\mathcal{F}_t^0 = \sigma\{x_s, s \leq t\}$ and $\mathcal{F}^0 = \mathcal{F}_\infty$. We denote by $B^*(E)$, $B(E)$ and $C(E)$ the space of real valued bounded functions on E which are respectively lower semi-analytic , borel and continuous . We will consider a P.D.P. taking values in E and determined by the following parameters :

a) the flow ϕ (t,x) in E , satisfying $\phi(t,\phi(s,x)) = \phi(t+s,x)$.

b) the jump rate $\lambda : E \to R_+$.

c) the transition measure $Q : \mathcal{E}x(E \cup \partial^* E) \to [0,1]$ where $\partial^* E := \{ z \in \partial E ; \phi(-t,z) \in E$ for $t \in (0,\epsilon)$ some $\epsilon > 0 \}$.

We define $t^*(x) := \inf \{ t > 0 ; \phi(t,x) \in \partial^* E \}$ for every $x \in E$. The general assumptions of the P.D.P. , which we shall denote by (X_t) , are as in Davis [9] . In addition to those assumptions we will also suppose that λ is bounded . The motion of the process (X_t) starting from x is constructed in the following way . Take a random variable T_1 such that $P(T_1 > t) = F(t)$ where :

$$F(t) = \begin{cases} \exp\left(- \int_0^t \lambda(\phi(s,x))ds \right) & , t < t^*(x) \\ 0 & , t \geq t^*(x) \end{cases}$$

Now select independently an E-valued random variable having distribution $Q(. ; \phi(T_1,x))$. The trajectory of X_t for $t \leq T_1$ is given by

$$X_t = \begin{cases} \phi(t,x) & , t < T_1 \\ Z_1 & , t = T_1 \end{cases}$$

Starting from $X_{T_1} = Z_1$ we now select the next inter-jump time T_2-T_1 and post-jump location $X_{T_2} = Z_2$ in a similar way . This gives a piecewise deterministic trajectory (X_t) with jump times T_1 , T_2 , ... and post-jump locations Z_1 , Z_2 , ... We will denote P_x the probability law on $(\Omega,\mathcal{F}^0)$ of the P.D.P. starting at x . We assume that $\lim_{n\to\infty} T_n = \infty$ P_x-a.s. . For $x \in E$ we define :

i) $Qv(x) := \int_E v(y)Q(dy;x)$ where $v \in B^*(E)$. Note that $Q(\ .\ ;x) : E \to [0,1]$ can be uniquely extended to $\mathcal{U}(E)$ (see [12] chap II) and thus the above integral is well defined on the measurable space $(E,\mathcal{U}(E))$.

ii) $\Lambda(t,x) := \int_0^t \lambda(\phi(s,x))ds$ where $0 \leq t \leq t^*(x)$

Finally we say that $g \in B^c(E)$ if $g \in B(E)$, $g(\phi(\ .\ ,x)) : [0,t^*(x)) \to R$ is continuous and $\lim_{t\to t^*(x)} g(\phi(t,x))$ exists whenever $\exp(-\Lambda(t^*(x),x)) > 0$.

3 . Optimal stopping of a P.D.P. with lower semi-analytic gain function

Let $\mathcal{F}$ be the P_x-completion of $\mathcal{F}^0$ and $\mathcal{F}_t$ the right continuous completion of $\mathcal{F}_t^0$ with respect to the P_x-null sets of $\mathcal{F}^0$. We denote by $\mathcal{M}$ the set of all P_x-a.s. finite $\mathcal{F}_t$-stopping times . We will consider the following optimal stopping problem for $x \in E$

$$\rho'(x) = \inf_{\tau\in\mathcal{M}} E_x(\ g\ (\ X_\tau)\ ) \qquad (1)$$

In [15] Gugerli studied (1) when $g \in B^c(E)$. In this section we will extend the results of [15] to the case when $g \in B^*(E)$. As mentioned before this extension is needed to study the impulsive control problem of P.D.P.'s . The main result of this section will be the construction of ϵ-optimal stopping times.

Let $\mathbb{B} := \{\ (t,x)\ ;\ x \in E$ and $0 \leq t < t^*(x)\ \} \subset R^{d+1}$. Since from our general assumptions $t^*(x)$ is a borel function in E we can easily show that $\mathbb{B}$ is a borel set of R^{d+1}. For

$(t,x) \in \mathbb{B}$ and v_1 , v_2 in $B^*(E)$ define

a) $J(v_1,v_2)(t,x) := E_x(v_1(\phi(t,x))1_{\{T_1>t\}} + v_2(Z_1)1_{\{T_1\leq t\}})$

$$= v_1(\phi(t,x))e^{-\Lambda(t,x)} + \int_0^t Qv_2(\phi(s,x))\lambda(\phi(s,x))e^{-\Lambda(s,x)}ds$$

b) $Kv_2(x) := E_x(v_2(Z_1)) = \int_0^{t^*(x)} Qv_2(\phi(s,x))\lambda(\phi(s,x))e^{-\Lambda(s,x)}ds + Qv_2(\phi(t^*(x),x))e^{-\Lambda(t^*(x),x)}$

c) $L(v_1,v_2)(x) := \{ \inf_{0\leq t<t^*(x)} J(v_1,v_2)(t,x) \} \wedge Kv_2(x)$

It follows from the semi-group property of the drift ϕ that

$$L(v_1,v_2)(\phi(t,x)) = e^{\Lambda(t,x)}\Big(\{ \inf_{t\leq s<t^*(x)} J(v_1,v_2)(s,x) \} \wedge Kv_2(x) - E_x(v_2(Z_1)1_{\{T_1\leq t\}}) \Big) \quad (2)$$

We can easily show that $L(v_1,v_2)(\cdot) : E \to R$ is in $B^*(E)$. Define the sequence of functions ρ_m , $m \in \mathbb{N} := \{ 0 , 1 , ... \}$ by $\rho_0 := g$, $\rho_{m+1} := L(g,\rho_m)$. Then $\rho_m \in B^*(E)$ for all $m \in \mathbb{N}$ and since $\rho_1 \leq g$, $\rho_{m+1} \leq \rho_m$ for all $m \in \mathbb{N}$. Therefore $\rho := \inf_m \rho_m = \lim_{m\to\infty} \rho_m$ exists and from lemma 7.30(2) of [4] , $\rho \in B^*(E)$. We shall show in Corollary 1 below that $\rho = \rho'$, the value function for the stopping problem . Corollary 1 of Gugerli [15] is readily modified to show the following results .

Proposition 1 : Let ρ and ρ_m be defined as above . Then

a) $\rho_{m+1} = L(\rho_m,\rho_m)$.

b) ρ is the biggest solution of

$$\begin{cases} v = L(g,v) \\ v \in B^*(E) \end{cases}$$

c) ρ is the biggest solution of

$$\begin{cases} v = L(v,v) \\ v \leq g \ , \ v \in B^*(E) \end{cases}$$

Function ρ can also be characterised in the following way :

Proposition 2 : ρ is the biggest solution of

$$\begin{cases} v(x) \leq E_x(v(X_{t\wedge T_1})) \quad \forall\, t \in [0,t^*(x)] \ , \ \forall\, x \in E \\ v \leq g \ , \ v \in B^*(E) \end{cases} \tag{3}$$

Proof : Noting that $L(\rho,\rho)(x) = \{ \inf_{0\leq t<t^*(x)} E_x(\rho(X_{t\wedge T_1})) \} \wedge E_x(\rho(Z_1))$ it follows from Prop. 1c) that for any $x \in E$

$\rho(x) \leq E_x(\rho(X_{t\wedge T_1}))$ for all $t \in [0,t^*(x))$ and $\rho(x) \leq E_x(\rho(Z_1))$

Hence ρ is a solution of (3) . Suppose v is another solution of (3) . We use induction in m to show that $\rho_m \geq v$ for all $m \in \mathbb{N}$. For $m = 1$, $\rho_1 = L(g,g) \geq L(v,v) \geq v$. Suppose now that $\rho_m \geq v$. It follows from Prop. 1a) that $\rho_{m+1} = L(\rho_m,\rho_m)(x) \geq L(v,v) \geq v$. Therefore $\rho = \lim_{m\to\infty} \rho_m \geq v$, proving the proposition . □

We shall now pass to the main part of this section which is the construction of ϵ-optimal stopping times . Let g and λ be bounded by a_1 and a_2 respectively and for $\epsilon > 0$ define

i) $\tilde{U}_\epsilon := \inf\{ t \geq 0 ; g(X_t) - \rho(X_t) < \epsilon/3 \}$. It follows from [12] (chapter IV-50) that $\tilde{U}_\epsilon$ is an $\mathcal{F}_t$-stopping time .

ii) $\zeta > 0$ is such that $(1-e^{-\zeta a_2})a_1 < \epsilon/3$.

iii) for every $k \in \mathbb{I} = \{ 0 , \pm 1 , \pm 2 , \dots \}$:

$A(k) := \{ (t,x) \in \mathbb{B} ; 0 \leq t \leq \zeta , g(\phi(t,x)) < (k+1)\epsilon/3 \}$.

$B(k) := \{ x \in E ; (k-1)\epsilon/3 \leq \rho(x) < k\epsilon/3 \}$.

iv) $S := \bigcup_{k\in\mathbb{I}} \{ (\mathrm{proj}_E A(k)) \cap B(k) \} \subset E$ where $\mathrm{proj}_E A(k) := \{ x \in E ; (t,x) \in A(k)$ for some $t \in R_+ \}$.

By using the representation of the function ρ given in Prop. 2 and the results of martingale theory in continuous time the proof of lemma 8 , chapter 2 of Shiryayev [35] is readily modified to show that :

Proposition 3 : $P_x(\tilde{U}_\epsilon < \infty) = 1$.

Therefore $\tilde{U}_\epsilon \in \mathcal{M}$. However we cannot use $\tilde{U}_\epsilon$ as an ϵ-optimal stopping time because $g(X_t)$ - $\rho(X_t)$ may not be right continuous P_x-a.s. and therefore we cannot say that $g(X_{U_\epsilon})$ - $\rho(X_{U_\epsilon}) \leq \epsilon/3$ P_x-a.s. . So we need to define a new stopping time . To do so we shall prove first :

Proposition 4 : There exists an analytically measurable function $\psi : S \to [0,\zeta]$ such that for all $x \in S$, $g(\phi(\psi(x),x)) < \rho(x) + 2\epsilon/3$.

Proof : Since g is lower semi-analytic (l.s.a.) and ϕ is continuous in $\mathbb{B}$ we have from lemma 7.30(3) of [4] that $g(\phi(\ .\ ,\ .\ )) : \mathbb{B} \to R$ is l.s.a. . Thus by definition of l.s.a. functions $\{ (t,x) \in \mathbb{B} ; g(\phi(t,x)) < (k+1)\epsilon/3 \}$ is analytic in $\mathbb{B}$. It is clear that $\{ (t,x) \in \mathbb{B} ; 0 \leq t \leq \zeta \}$ is borel in $\mathbb{B}$. By virtue of Prop 7.36 and corollary 7.35.2 of [4] (borel sets are analytic , countable intersections of analytic sets is analytic) A(k) is an analytic subset of $\mathbb{B}$. It follows from Jankov-von Neumann theorem (Prop. 7.49 of [4]) that for each $k \in \mathbb{I}$ there exists an analytically measurable function $\psi_k : \text{proj}_E A(k) \to R_+$ such that $Gr(\psi_k) := \{ (\psi_k(x),x) ; x \in \text{proj}_E A(k) \} \subset A(k)$. Note that $\text{proj}_E A(k)$ is an analytic subset of E (Prop 7.39 of [4]) . Since B(k) , $k \in \mathbb{I}$ are analytically measurable sets of E it is clear that S is also an analytically measurable set of E . Define the funtion $\psi : S \to [0,\zeta]$ by $\psi(x) := \psi_k(x)$ if $x \in (\text{proj}_E A(k)) \cap B(k)$. The sets B(k) are disjoint and therefore ψ is well defined . It is clear that ψ is analytically measurable . If $x \in B(k) \cap \text{proj}_E A(k)$ then $Gr(\psi_k) \subset A(k)$ and

$g(\phi(\psi(x),x)) = g(\phi(\psi_k(x),x)) < (k+1)\epsilon/3 \leq \rho(x) + 2\epsilon/3$

Hence ψ has the required properties and the proposition is proved . □

We next show that $X_{\tilde{U}_\epsilon}$ maps $\{\tilde{U}_\epsilon < \infty\}$ into S .

Proposition 5 : For every $\omega \in \{\tilde{U}_\epsilon < \infty\}$, $X_{\tilde{U}_\epsilon}(\omega) \in S$.

Proof : For $\omega \in \{\tilde{U}_\epsilon < \infty\}$ let us denote $x' = X_{\tilde{U}_\epsilon}(\omega)$. We want to show that $x' \in S$ or , in other words , there exists $k \in \mathbb{I}$ such that $(k-1)\epsilon/3 \leq \rho(x') < k\epsilon/3$ and for some $t' \in R_+$ we

have $0 \leq t' \leq \zeta$, $(t',x') \in \mathbb{B}$ and $g(\phi(t',x')) < (k+1)\epsilon/3$. We can define k in a unique way such that $(k-1)\epsilon/3 \leq \rho(x') < k\epsilon/3$. From Prop 1b) we have one of the following three possibilities : either

a) there exists $\beta > 0$ such that $\rho(x') = \{ \inf_{\beta \leq t < t^*(x')} J(g,\rho)(t,x') \} \wedge K\rho(x')$ or

b) $\rho(x') = \lim_{t_i \downarrow 0} \left(e^{-\Lambda(t_i,x')} g(\phi(t_i,x')) + \int_0^{t_i} Q\rho(\phi(s,x'))\lambda(\phi(s,x'))e^{-\Lambda(\phi(s,x'))}ds \right) = \lim_{t_i \downarrow 0} g(\phi(t_i,x'))$

for some positive sequence $t_i \downarrow 0$ or

c) $\rho(x') = g(x')$

For c) it is clear that $(0,x') \in A(k)$. For b) we can choose $t'=t_j$ where t_j is such that $(t_j,x') \in \mathbb{B}$, $0 \leq t_j \leq \zeta$ and

$| \rho(x') - g(\phi(t_j,x')) | < k\epsilon/3 - \rho(x')$ (since by definition of k , $k\epsilon/3 - \rho(x') > 0$)

Therefore $(t',x') \in \mathbb{B}$, $0 \leq t' \leq \zeta$ and $g(\phi(t',x')) < k\epsilon/3$ which implies that $(t',x') \in A(k)$. For a) we first note that $\rho(\phi(\,.\,,x')) : [0,\beta] \rightarrow R$ is continuous . Indeed from (2) and Prop. 1b) for every $0 \leq t \leq \beta$

$$\rho(\phi(t,x')) = e^{\Lambda(t,x')} \left(\rho(x') - \int_0^t Q\rho(\phi(s,x'))\lambda(\phi(s,x'))e^{-\Lambda(s,x')}ds \right)$$

which is continuous in t . From the definition of $\tilde{U}_\epsilon$ either there exists a sequence $t_i \downarrow 0$, $0 < t_i \leq \beta$ such that

$g(\phi(t_i,x')) < \rho(\phi(t_i,x')) + \epsilon/3$ or

$g(x') < \rho(x') + \epsilon/3$. For the last case it is clear that $(0,x') \in A(k)$. For the first case we can choose $t'=t_j$ where $(t_j,x') \in \mathbb{B}$, $0 \leq t_j \leq \zeta$ and

$| \rho(x') - \rho(\phi(t_j,x')) | < k\epsilon/3 - \rho(x')$

Therefore $(t',x') \in \mathbb{B}$, $0 \leq t' \leq \zeta$ and

$g(\phi(t',x')) < \rho(\phi(t',x')) + \epsilon/3 < k\epsilon/3 + \epsilon/3 = (k+1)\epsilon/3$

which proves the proposition . □

The mapping $X_{\tilde{U}_\epsilon} : (\{\tilde{U}_\epsilon < \infty\}, \mathcal{F}_{\tilde{U}_\epsilon}) \to (S,E)$ is $\mathcal{F}_{\tilde{U}_\epsilon}$-measurable (see chapter IV-64 of [12]) . Proposition 7.44 of [4] can be readily modified to show that $X_{\tilde{U}_\epsilon}$ is also $\mathcal{F}_{\tilde{U}_\epsilon}/\mathcal{U}(E)$-measurable and since $\psi : (S,\mathcal{U}(E)) \to ([0,\zeta],\mathcal{B}([0,\zeta]))$ is universally measurable (where $\mathcal{B}([0,\zeta])$ is the borel σ-fied of $[0,\zeta]$) then $\psi(X_{\tilde{U}_\epsilon})$ from $\{\tilde{U}_\epsilon<\infty\}$ into $[0,\zeta]$ is $\mathcal{F}_{\tilde{U}_\epsilon}$-measurable . We define :

$$U_\epsilon(\omega) = \begin{cases} \infty & \text{if } \tilde{U}_\epsilon(\omega)=\infty \text{ ; otherwise} \\ \tilde{U}_\epsilon(\omega) & \text{if } g(X_{\tilde{U}_\epsilon}(\omega)) \le \rho(X_{\tilde{U}_\epsilon}(\omega)) + \epsilon/3 \\ \tilde{U}_\epsilon(\omega) + \psi(X_{\tilde{U}_\epsilon}(\omega)) & \text{if } g(X_{\tilde{U}_\epsilon}(\omega)) > \rho(X_{\tilde{U}_\epsilon}(\omega)) + \epsilon/3 \end{cases}$$

It is clear that U_ϵ is $\mathcal{F}_{\tilde{U}_\epsilon}$-measurable , $\tilde{U}_\epsilon \le U_\epsilon$ and $P_x(U_\epsilon < \infty) = 1$. By virtue of [12] chapter IV-57-a) , U_ϵ is an $\mathcal{F}_t$-stopping time and therefore $U_\epsilon \in \mathcal{M}$. The next theorem is our main result in this section .

Theorem 1 : $E_x(g(X_{U_\epsilon})) < \rho(x) + \epsilon$.

In order to show this theorem we need the following lemmas

Lemma 1 : For $z \in E$ define $r := \inf\{ t\in[0,t^*(z)) ; g(\phi(t,z)) < \rho(\phi(t,z)) + \epsilon/3 \} \wedge t^*(z)$ where $\inf\{\emptyset\} := \infty$. Then $E_z(\rho(X_{r\wedge T_1})) = \rho(z)$.

Proof : If $r < t^*(z)$ then

$$E_z(\rho(X_{r\wedge T_1})) = \rho(\phi(r,z))e^{-\Lambda(r,z)} + E_z(\rho(Z_1)1_{\{T_1\le r\}})$$

But from Prop. 1b) , $\rho = L(g,\rho)$ and from (2)

$$L(g,\rho)(\phi(r,z))e^{-\Lambda(r,z)} = \{ \inf_{r\le s<t^*(z)} J(g,\rho)(s,z) \} \wedge K\rho(z) - E_z(\rho(Z_1)1_{\{T_1\le r\}})$$

It is clear then that

$$E_z(\rho(X_{r\wedge T_1})) = \begin{cases} \{ \inf_{r\le s<t^*(z)} J(g,\rho)(s,z) \} \wedge K\rho(z)) & \text{if } r < t^*(z) \\ E_z(\rho(Z_1)) = K\rho(z) & \text{if } r = t^*(z) \end{cases}$$

If $\rho(z) = K\rho(z) \le \inf_{0\le t<t^*(x)} J(g,\rho)(t,z)$ (see Prop.1b)) then obviously we

have $E_z(\rho(X_{r\wedge T_1})) = \rho(z)$. Now consider the other case , that is ,

$$K\rho(z) > \inf_{0\le t<t^*(x)} J(g,\rho)(t,z) = \rho(z)$$

We have at least one of the three possibilities below

a) $\rho(z) = J(g,\rho)(t',z)$ for some $0 \le t' < t^*(z)$

b) $\rho(z) = \lim_{t_i\uparrow t'} J(g,\rho)(t_i,z)$ for some $0 < t' \le t^*(z)$ and $t_i\uparrow t'$, $0 < t_i < t'$

c) $\rho(z) = \lim_{t_i\downarrow t'} J(g,\rho)(t_i,z)$ for some $0 \le t' < t^*(z)$ and $t_i\downarrow t'$, $t' < t_i < t^*(z)$

For a) it is clear from (2) that $\rho(\phi(t',z)) = g(\phi(t',z))$. Thus from the definition of r we immediately obtain $r \le t'$ which implies $E_z(\rho(X_{r\wedge T_1})) = \rho(z)$. For b) we first note that if $e^{-\Lambda(t',z)} = 0$ (so $t' = t^*(z) = \infty$) then

$$\lim_{t_i\uparrow\infty} J(g,\rho)(t_i,z) = \lim_{t_i\uparrow\infty} \Big(g(\phi(t_i,z))e^{-\Lambda(t_i,z)} + \int_0^{t_i} Q\rho(\phi(s,z))\lambda(\phi(s,z))e^{-\Lambda(s,z)}ds \Big)$$

$$= \int_0^{\infty} Q\rho(\phi(s,z))\lambda(\phi(s,z))e^{-\Lambda(s,z)}ds = E_z(\rho(Z_1))$$

which corresponds to the case $\rho(z) = K\rho(z)$. So we can concentrate on $e^{-\Lambda(t',z)} > 0$. We can find t_j such that

$$J(g,\rho)(t_j,z) < \rho(z) + \tfrac{\epsilon}{3}e^{-\Lambda(t',z)}$$

Therefore from (2)

$$\rho(\phi(t_j,z)) = e^{\Lambda(t_j,z)}\Big(\{ \inf_{t_j\le s<t^*(z)} J(g,\rho)(s,z) \} \wedge K\rho(z) - E_z(\rho(Z_1)1_{\{T_1\le t_j\}}) \Big)$$

$$= e^{\Lambda(t_j,z)}\Big(\rho(z) - E_z(\rho(Z_1)1_{\{T_1\le t_j\}}) \Big)$$

$$> e^{\Lambda(t_j,z)}\Big(J(g,\rho)(t_j,z) - \tfrac{\epsilon}{3}e^{-\Lambda(t',z)} - E_z(\rho(Z_1)1_{\{T_1\le t_j\}}) \Big)$$

$$= g(\phi(t_j,z)) - \tfrac{\epsilon}{3}\exp\{-\int_{t_j}^{t'}\lambda(\phi(s,z))ds\} \ge g(\phi(t_j,z)) - \tfrac{\epsilon}{3}$$

So $g(\phi(t_j,z)) < \rho(\phi(t_j,z)) + \frac{\epsilon}{3}$

and it follows immediately that $r \le t_j$. Thus $E_z(\rho(X_{r\wedge T_1})) = \rho(z)$.

For c) we first see that

$$\rho(z) = \lim_{t_i\downarrow t'} J(g,\rho)(t_i,z) = \lim_{t_i\downarrow t'} \Big(g(\phi(t_i,z))e^{-\Lambda(t_i,z)} + \int_0^{t_i} Q\rho(\phi(s,z))\lambda(\phi(s,z))e^{-\Lambda(s,z)}ds \Big)$$

$$= \Big(\lim_{t_i\downarrow t'} g(\phi(t_i,z)) \Big)e^{-\Lambda(t',z)} + E_z(\rho(Z_1)1_{\{T_1\le t'\}})$$

Define $\nu := \phi(t',z)$. From (2) and the above expression

$$\rho(\nu) = e^{\Lambda(t',z)}\Big(\{ \inf_{t'\le s<t^*(z)} J(g,\rho)(s,z) \} \wedge K\rho(z) - E_z(\rho(Z_1)1_{\{T_1\le t'\}}) \Big)$$

$$= e^{\Lambda(t',z)}\Big(\rho(z) - E_z(\rho(Z_1)1_{\{T_1\le t'\}}) \Big) = \lim_{t_i\downarrow t'} g(\phi(t_i,z))$$

For every $\gamma \in [0,t^*(\nu))$

$$\rho(\phi(\gamma,\nu)) = e^{\Lambda(\gamma,\nu)}\Big(\{ \inf_{\gamma\le s<t^*(\nu)} J(g,\rho)(s,\nu) \} \wedge K\rho(\nu) - E_\nu(\rho(Z_1)1_{\{T_1\le\gamma\}}) \Big)$$

$$\ge e^{\Lambda(\gamma,\nu)}\Big(\rho(\nu) - \int_0^{\gamma} Q\rho(\phi(s,\nu))\lambda(\phi(s,\nu))e^{-\Lambda(s,\nu)}ds \Big)$$

Then we can find $n_0 \in \mathbb{N}$ such that $| g(\phi(t_i,z)) - \rho(\nu) | < \epsilon/6$ and

$$\Big| \Big\{ e^{\Lambda(t_i-t',\nu)}\Big(\rho(\nu) - \int_0^{t_i-t'} Q\rho(\phi(s,\nu))\lambda(\phi(s,\nu))e^{-\Lambda(s,\nu)}ds \Big) \Big\} - \rho(\nu) \Big| < \epsilon/6$$

whenever $i \ge n_0$. Thus for all $i \ge n_0$

$$\rho(\phi(t_i,z)) = \rho(\phi(t_i-t',\phi(t',z))) = \rho(\phi(t_i-t',\nu)) > \rho(\nu) - \epsilon/6 > g(\phi(t_i,z)) - \epsilon/3$$

and so $g(\phi(t_i,z)) < \rho(\phi(t_i,z)) + \epsilon/3$ for all $i \ge n_0$. Since $t_i\downarrow t'$ as $i\to\infty$ it follows from the definition of r that $r \le t'$ and as before $E_z(\rho(X_{r\wedge T_1})) = \rho(z)$ proving the lemma . □

Lemma 2 : $E_x(\rho(X_{\tilde{U}_\epsilon \wedge T_m})) = \rho(x)$ for all $m \in \mathbb{N}$.

Proof : We will use induction in m to prove this lemma . The result is clear for $m = 0$ since $T_0 := 0$. Suppose now that $E_x(\rho(X_{\tilde{U}_\epsilon \wedge T_m})) = \rho(x)$ for some m . Then

$$E_x(\rho(X_{\tilde{U}_\epsilon \wedge T_{m+1}})) = E_x(\rho(X_{\tilde{U}_\epsilon \wedge T_m}) 1_{\{\tilde{U}_\epsilon < T_m\}} + \rho(X_{\tilde{U}_\epsilon \wedge T_{m+1}}) 1_{\{\tilde{U}_\epsilon \geq T_m\}}) \quad (4)$$

But from the definition of $\tilde{U}_\epsilon$

$$\tilde{U}_\epsilon \wedge T_{m+1} = (T_m + R(Z_m)) \wedge T_{m+1} \quad \text{on } \{ \tilde{U}_\epsilon \geq T_m \}$$

where $R(z) = \inf \{ t \in [0,t^*(z)) ; g(\phi(t,z)) < \rho(\phi(t,z)) + \epsilon/3 \} \wedge t^*(z)$. Thus from time homogeneity and the strong markov property

$$E_x(\rho(X_{\tilde{U}_\epsilon \wedge T_{m+1}}) 1_{\{\tilde{U}_\epsilon \geq T_m\}}) = E_x(E (\rho(X_{(T_m + R(Z_m)) \wedge T_{m+1}}) 1_{\{\tilde{U}_\epsilon \geq T_m\}} / \mathcal{F}_{T_m}))$$

$$= E_x(\chi(Z_m) 1_{\{\tilde{U}_\epsilon \geq T_m\}})$$

where $\chi(z) = E_z(\rho(X_{R(z) \wedge T_1}))$. From the previous lemma we get $\chi(z) = \rho(z)$. Replacing it in (4) we immediately obtain

$$E_x(\rho(X_{\tilde{U}_\epsilon \wedge T_{m+1}})) = E_x(\rho(X_{\tilde{U}_\epsilon \wedge T_m}) 1_{\{\tilde{U}_\epsilon < T_m\}} + \rho(Z_m) 1_{\{\tilde{U}_\epsilon \geq T_m\}})$$

$$= E_x(\rho(X_{\tilde{U}_\epsilon \wedge T_m})) = \rho(x)$$

where the last equality follows from the induction hypothesis for m . The lemma is proved . □

Lemma 3 : $E_x(\rho(X_{\tilde{U}_\epsilon})) = \rho(x)$

Proof : The proof follows immediately from lemma 2 using the bounded convergence theorem and the fact that $P_x(\tilde{U}_\epsilon < \infty) = 1$ and $T_m \to \infty$ P_x-a.s. □

We can now prove the theorem .

Proof of theorem 1 : Let

$$A_1 := \{ \tilde{U}_\epsilon < \infty , g(X_{\tilde{U}_\epsilon}) - \rho(X_{\tilde{U}_\epsilon}) \leq \epsilon/3 \}$$

$$A_2 := \{ \tilde{U}_\epsilon < \infty , g(X_{\tilde{U}_\epsilon}) - \rho(X_{\tilde{U}_\epsilon}) > \epsilon/3 , \text{ no jumps between } (\tilde{U}_\epsilon , \tilde{U}_\epsilon + \psi] \}$$

$$A_3 := \{ \tilde{U}_\epsilon < \infty , g(X_{\tilde{U}_\epsilon}) - \rho(X_{\tilde{U}_\epsilon}) > \epsilon/3 , \text{ at least one jump between } (\tilde{U}_\epsilon , \tilde{U}_\epsilon + \psi] \}$$

Then A_1 , A_2 , A_3 are in $\mathcal{F}$ and

$$E_x(g(X_{U_\epsilon})) = E_x(g(X_{\tilde{U}_\epsilon})1_{A_1} + g(\phi(\psi,X_{\tilde{U}_\epsilon}))1_{A_2} + g(X_{\tilde{U}_\epsilon+\psi})1_{A_3}) \qquad (5)$$

Now

$$E_x(g(X_{\tilde{U}_\epsilon})1_{A_1}) \leq E_x((\rho(X_{\tilde{U}_\epsilon}) + \epsilon/3)1_{A_1})$$

$$E_x(g(\phi(\psi,X_{\tilde{U}_\epsilon}))1_{A_2}) \leq E_x((\rho(X_{\tilde{U}_\epsilon}) + 2\epsilon/3)1_{A_2}) \text{ from Prop. 4}$$

$$E_x(g(X_{\tilde{U}_\epsilon+\psi})1_{A_3}) \leq a_1 E_x(E(1_{A_3} / \mathcal{F}_{\tilde{U}_\epsilon})) \quad (\text{since g is bounded by } a_1)$$

$$\leq a_1 E_x(P_{X_{\tilde{U}_\epsilon}} (\text{at least one jump in } (\tilde{U}_\epsilon , \tilde{U}_\epsilon + \psi]))$$

$$= a_1 E_x(1 - \exp \{ - \Lambda(\psi,X_{\tilde{U}_\epsilon}) \})$$

$$\leq a_1(1 - e^{-\zeta a_2}) < \epsilon/3$$

from the construction of ζ and boundedness of λ by a_2 . Replacing these inequalities in (5) we obtain

$$E_x(g(X_{U_\epsilon})) < E_x((\rho(X_{\tilde{U}_\epsilon}) + \epsilon/3)1_{A_1} + (\rho(X_{\tilde{U}_\epsilon}) + 2\epsilon/3)1_{A_2}) + \epsilon/3$$

$$\leq E_x(\rho(X_{\tilde{U}_\epsilon})) + 2\epsilon/3 + \epsilon/3 = \rho(x) + \epsilon$$

where the last equality is by virtue of lemma 3 .The theorem is proved . □

The final result of this section will show that ρ is the value function , i. e. , $\rho = \rho'$ (see (1)) . But first we need the following result .

Proposition 6 : For any $\tau \in \mathcal{M}$, $E_x(\rho(X_\tau)) \geq \rho(x)$.

Proof : We will show that for every $m \in \mathbb{N}$,

$$\rho(x) \leq E_x(\rho(X_{\tau \wedge T_m})) \qquad (6)$$

We use induction in m ; it is clear for $m = 0$. Suppose it holds for m . Then

$$E_x(\rho(X_{\tau \wedge T_{m+1}})) = E_x(\rho(X_{\tau \wedge T_m})1_{\{\tau < T_m\}} + \rho(X_{\tau \wedge T_{m+1}})1_{\{\tau \geq T_m\}})$$

$$= E_x(\rho(X_{\tau \wedge T_m})1_{\{\tau < T_m\}} + E(\rho(X_{(T_m+R_m)\wedge T_{m+1}})/\mathcal{F}_{T_m})1_{\{\tau \geq T_m\}}) \qquad (7)$$

since from the representation lemma for stopping times (see Davis [8])

$$\tau \wedge T_{m+1} = (T_m + R_m) \wedge T_{m+1} \qquad \text{on } \{ T_m \leq \tau \}$$

for some $\mathcal{F}_{T_m}$-measurable , non-negative function R_m . By virtue of time homogeneity and the strong Markov property we get

$$E(\rho(X_{(T_m+R_m)\wedge T_{m+1}}) / \mathcal{F}_{T_m}) = \chi(Z_m , R_m)$$

where

$$\chi(z , r) = E_z(\rho(X_{r \wedge T_1}))$$

It follows from Prop. 2 that $\chi(z , r) \geq \rho(z)$. Replacing it in (7) we obtain

$$E_x(\rho(X_{\tau \wedge T_{m+1}})) \geq E_x(\rho(X_{\tau \wedge T_m})1_{\{\tau < T_m\}} + \rho(Z_m)1_{\{\tau \geq T_m\}})$$

$$= E_x(\rho(X_{\tau \wedge T_m})) \geq \rho(x)$$

where the last inequality is by virtue of the induction hypothesis for m . Hence (6) is proved . The proposition follows from the same arguments as for lemma 3 . □

We can now show that $\rho = \rho'$.

Corollary 1 : $\rho(x) = \rho'(x)$ for all $x \in E$.

Proof : From theorem 1 and since $U_\epsilon \in \mathcal{M}$ we get

$$\rho'(x) = \inf_{\tau \in \mathcal{M}} E_x(g(X_\tau)) \leq E_x(g(X_{U_\epsilon})) < \rho(x) + \epsilon$$

Hence $\rho'(x) \leq \rho(x)$. On the other hand from Prop. 6 and the fact that $\rho \leq g$ we have

for any $\tau \in \mathcal{M}_b$

$$E_x(g(X_\tau)) \geq E_x(\rho(X_\tau)) \geq \rho(x)$$

So taking infimum over $\mathcal{M}_b$ we obtain

$$\rho'(x) = \inf_{\tau \in \mathcal{M}_b} E_x(g(X_\tau)) \geq \rho(x)$$

which shows the desired result . □

Remarks : 1) The results of this section also hold for the case when the state space E is the countable union of possibly different dimensional open sets .

2) We cannot characterise $-\rho$ as the smallest excessive function majorising -g . Proposition 2 is the best we can get in this sense . The following example shows this . Consider a P.D.P. determined by

$E = \{ \Delta \} \cup [0 , \infty)$,

$\phi(t,\Delta) = \Delta$, $\phi(t,x) = x + t$ for all x and $t \in [0 , \infty)$,

$\lambda(\Delta) = \lambda > 0$, $\lambda(x) = 0$ for all $x \in [0 , \infty)$ and

$Q(\{0\};\Delta) = 1$

Thus starting from Δ the process stays there until the first jump time T_1 (which is exponentially distibuted with parameter λ) when it jumps to $\{0\}$ and moves uniformly to the right . We define the gain function in the following way:

$g(\Delta) = 0$, $g(x) = 0$, $x \in (0 , \infty)$, $g(0) = - 1$.

It is clear that $\rho(\Delta) = \rho(0) = - 1$, $\rho(x) = 0$ for $x > 0$. However

$E_\Delta(g(X_t)) = 0 = g(\Delta)$ and

$E_x(g(X_t)) \geq g(x)$, for all x and $t \in [0 , \infty)$

Thus - g is the smallest excessive function majorising - g (and not - ρ) . Note that $E_\Delta(g(X_{T_1})) = - 1 < g(\Delta) = 0$ so that g is not a solution of (3) .

4 . Discretization for the optimal stopping of a P.D.P.

4.1. Introduction

In this section we will consider the discounted optimal stopping problem

$$\rho'(x) = \inf_{\tau \in \mathcal{M}_\infty} E_x(e^{-\alpha\tau} g(X_\tau))$$

where $\alpha > 0$ and $\mathcal{M}_\infty$ represents the set of all $\mathcal{F}_t$-stopping times (including ∞) . The results of section 3 can be readily modified for this case and , as we saw in corollary 1 , ρ (defined just before prop. 1) is equal to ρ' for all points in E . For reasons that will become clear later (see subsection 4.5) we will extend the definition of ρ to the points on the boundary ∂^*E . Note that the process jumps instantaneously when it starts from any point in ∂^*E . Suppose $g \in B^*(E)$ and is bounded and defined on ∂^*E . For $x \in \partial^*E$

$$\rho_{m+1}(x) = g(x) \wedge E_x(\rho_m(Z_1)) \quad , \quad \rho_0(x) = g(x)$$

$$\rho(x) = g(x) \wedge E_x(\rho(Z_1))$$

Remark : Note that for $x \in \partial^*E$, $\rho'(x) = E_x(\rho(Z_1))$ and thus may be different from $\rho(x)$.

Define $\tilde{E} := E \cup \partial^*E$. The following proposition can be easily proved (see proof in the appendix) .

Proposition 7 : For $x \in \tilde{E}$ and $m \in \mathbb{N}$

$$0 \leq \rho_m(x) - \rho(x) \leq E_x(e^{-\alpha T_m}(g(Z_m) - \rho(Z_m))) .$$

4.2. Assumptions

From now on we add the following assumptions to the previous ones (for notation see section 2) :

1) $t^* : E \to R_+ \cup \{ \infty \}$ is continuous

2) $\lambda(\phi(\cdot ,y)) \to \lambda(\phi(\cdot \, x))$ as $y \to x$ Lebesgue a. s. on $[0 , t^*(x))$ for all $x \in E$

3) For every $v \in C(E)$

3.i) $Qv(\cdot) : E \to R$ is continuous

3.ii) $Qv(\cdot) : \partial^* E \to R$ is continuous

4) g is continuous and bounded on $\tilde{E}$

5) $\Lambda(t^*(x),x) = \infty$ whenever $t^*(x) = \infty$

Remark : a) If λ is continuous on E then clearly assumption 2) will be satisfied

b) Assumptions 3.i) and 3.ii) mean that Q is weakly continuous on E and $\partial^* E$ respectively .

From the assumptions above we can easily show the following proposition (see proof in the appendix) .

Proposition 8 : $\Lambda(\cdot , \cdot) : \{ (t,x) ; x \in \tilde{E} , 0 \leq t \leq t^*(x) \} \to R_+ \cup \{ \infty \}$ is continuous (where $t^*(x) := 0$ if $x \in \partial^* E$) .

For $x \in \tilde{E}$ let us denote by μ_x the joint probability measure of (T_1^x , Z_1^x) on $(R^{d+1}, \mathcal{B}(R^{d+1}))$ (where $\mathcal{B}(R^{d+1})$ denotes the borel σ-field of R^{d+1}) . We have that for any $t \in R_+ \cup \{\infty\}$ and $A \in \mathcal{E}$

$$P_x(T_1 \leq t , Z_1 \in A) = \int_0^{t \wedge t^*(x)} Q(A ; \phi(s,x)) \lambda(\phi(s,x)) \, e^{-\Lambda(s,x)} ds +$$

$$Q(A ; \phi(t^*(x),x)) e^{-\Lambda(t^*(x),x)} 1_{\{ [t^*(x),\infty] \}}(t)$$

Let $\mathcal{P}(R^{d+1})$ be the space of all probability measures over R^{d+1} . We have the following proposition (see proof in the appendix) :

Proposition 9 : $\mu_{(\cdot)} : \tilde{E} \to \mathcal{P}(R^{d+1})$ is continuous with respect to the weak topology on $\mathcal{P}(R^{d+1})$.

4.3. Construction of the discretized process

Let $D := \{ z_1 , z_2 , \dots \}$ be a countable dense set in E and $D^N := \{ z_1 , \dots , z_N \}$. Define the sets $\{ A_i^N \}_{i=1}^N$ in the following form :

$B_i^N := \{ z \in E ; | z - z_i | \leq | z - z_j |$ for every $j = 1, \dots , N \}$, $i = 1, \dots ,N$

$A_1^N := B_1^N$

$A_i^N := B_i^N - \bigcup_{j=1}^{i-1} A_j^N \qquad i = 2 , \dots ,N$

Then these sets are clearly in E . The discretized P.D.P. , denoted by (X_t^N) , is determined by :

a) the flow $\phi(t,x)$

b) the jump rate λ

c) the transition probability Q^N defined as follows :

$$Q^N(\{ z_i \} ; x) := Q(A_i^N ; x) \text{ for } i = 1, \dots , N \text{ and all } x \in \tilde{E}$$

Therefore jumping from x the discretized process can go to a finite number of points $\{ z_1 , \dots , z_N \}$ only . From now on whenever necessary we will use the superscript N to distinguish the discretized process from the original one .

We will be concerned now with the construction of (T_1 , Z_1) and (T_1^N , Z_1^N) . For arbitrary $x \in \tilde{E}$ we have seen in Prop.9 that $\mu_y \rightarrow \mu_x$ weakly as $y \rightarrow x$. Let P denote the Lebesgue-[0,1] measure . Then using Skorohod's theorem (Billingsley [5] page 337) we can construct (T_1^y , Z_1^y) and (T_1^x , Z_1^x) measurable functions from the probability space $([0,1],\mathfrak{B}([0,1]),P)$ into $(R^{d+1},\mathfrak{B}(R^{d+1}))$ such that (T_1^y , Z_1^y) has distribution μ_y , (T_1^x , Z_1^x) has distribution μ_x and

$(T_1^y , Z_1^y) \rightarrow (T_1^x , Z_1^x)$ as $y \rightarrow x$ P-a.s.

For the discretized case the construction is the following : for every $\nu \in E$ and $u \in [0,1]$

$T_1^{N\nu}(u) = T_1^{\nu}(u)$

$Z_1^{N\nu}(u) = z_i$ if $Z_1^{\nu}(u) \in A_i^N$

It is easy to show that ($T_1^{N\nu}$, $Z_1^{N\nu}$) as defined above agrees with the definition of the discretized process (X_t^N) . For the next theorem we consider the construction seen above for some arbitrary fixed $x \in \tilde{E}$.

Proposition 10 : $Z_1^{Ny} \to Z_1^x$ as $y \to x$ and $N \to \infty$ P-a.s.

Proof : Define the set $U := \{ u \in [0,1] ; Z_1^y(u) \to Z_1^x(u) \text{ as } y \to x \}$. As we saw above , $P(U) = 1$. Take $u \in U$. Given $\epsilon > 0$ there exists $z_i \in D$ such that

$| Z_1^x(u) - z_i | < \epsilon/2$

because D is dense in E . Take $\delta > 0$ such that

$| Z_1^y(u) - Z_1^x(u) | < \epsilon/4$ whenever $| x-y | < \delta$

and $N_0 \in \mathbb{N}$ such that $z_i \in D^N$ whenever $N \geq N_0$. Thus for $N \geq N_0$ and $| x-y | < \delta$

$$\begin{aligned} | Z_1^x(u) - Z_1^{Ny}(u) | &\leq | Z_1^x(u) - Z_1^y(u) | + | Z_1^y(u) - Z_1^{Ny}(u) | \\ &\leq | Z_1^x(u) - Z_1^y(u) | + | Z_1^y(u) - z_i | \end{aligned}$$

where the last inequality follows from the definition of the sets $\{ A_i^N \}$. Hence

$$\begin{aligned} | Z_1^x(u) - Z_1^{Ny}(u) | &\leq | Z_1^x(u) - Z_1^y(u) | + | Z_1^y(u) - z_i | \\ &\leq | Z_1^x(u) - Z_1^y(u) | + | Z_1^y(u) - Z_1^x(u) | + | Z_1^x(u) - z_i | \\ &= 2| Z_1^x(u) - Z_1^y(u) | + | Z_1^x(u) - z_i | < \epsilon \end{aligned}$$

which proves the proposition . □

We can now show the following result :

Proposition 11 : For every $m \in \mathbb{N}$

a) $E_{(\cdot)}(\exp \{ -\alpha T_m \}) : \tilde{E} \to R_+$ is continuous .

b) $E_y^N (\exp \{ -\alpha T_m^N \}) \to E_x (\exp \{ -\alpha T_m \})$ as $y \to x$ and $N \to \infty$ for every $x \in \tilde{E}$.

Proof : We will show b) only since the proof of a) is similar . For arbitrary x in $\tilde{E}$ we use the construction seen in subsection 4.3 . We will use induction in m to prove the proposition . For $m = 0$ the result is clear . Suppose b) holds for m . Define for $z \in \tilde{E}$, $\chi(z) := E_z(\exp \{ -\alpha T_m \})$ and $\chi^N(z) := E_z^N (\exp \{ -\alpha T_m^N \})$. Then from time homogeneity and the strong Markov property

$$E_y^N (\exp \{ -\alpha T_{m+1}^N \}) = E_y^N (E^N(\exp \{ -\alpha T_{m+1}^N \} / \mathcal{F}_{T_1^N}))$$

$$= E_y^N (\exp \{ -\alpha T_1^N \} E^N(\exp \{ -\alpha (T_{m+1}^N - T_1^N) \} / \mathcal{F}_{T_1^N}))$$

$$= E_y^N (\exp \{ -\alpha T_1^N \} \chi^N (Z_1^N))$$

$$= E(\exp \{ -\alpha T_1^{Ny} \} \chi^N (Z_1^{Ny}))$$

where the last expectation is over [0,1] with measure P . Then from the construction of T_1^{Ny} ($= T_1^y$) , prop. 10 and the induction hypothesis for m

$\exp \{ -\alpha T_1^{Ny} \} \chi^N (Z_1^{Ny}) \rightarrow \exp \{ -\alpha T_1^x \} \chi(Z_1^x)$ as $y \rightarrow x$ and $N \rightarrow \infty$ P-a.s.

From the bounded convergence theorem

$$\lim_{\substack{y \rightarrow x \\ N \rightarrow \infty}} \left(E_y^N (\exp \{ -\alpha T_{m+1}^N \}) \right) = \lim_{\substack{y \rightarrow x \\ N \rightarrow \infty}} \left(E(\exp \{ -\alpha T_1^{Ny} \} \chi^N (Z_1^{Ny})) \right)$$

$$= E \left(\lim_{\substack{y \rightarrow x \\ N \rightarrow \infty}} \left(\exp \{ -\alpha T_1^{Ny} \} \chi^N (Z_1^{Ny}) \right) \right)$$

$$= E(\exp \{ -\alpha T_1^x \} \chi(Z_1^x))$$

$$= E_x(\exp \{ -\alpha T_{m+1} \})$$

Since x is arbritary the proposition is proved . □

4.4. Convergence results

We shall now consider the following discretized optimal stopping problem

$$\rho^{N,}(x) = \inf_{\tau \in \mathcal{M}_\infty^N} E_x^N(e^{-\alpha \tau} g^N(X_\tau^N))$$

where $g^N \in B^*(E)$, g^N is defined and bounded on $\partial^* E$, and $g^N \rightarrow g$ uniformly on compact sets of $\tilde{E}$ as $N \rightarrow \infty$. Of course we could take $g^N = g$ but we allow this extra generality to use the results we get here in the impulsive control problem . From the results of section 3 we have $\rho_m^N \downarrow \rho^N$ as $m \rightarrow \infty$.

The following proposition can be easily proved (see proof in the appendix) .

Proposition 12 : Suppose $v : \tilde{E} \to R$ is continuous . The following statements are equivalent :

a) $v^N \overset{N}{\to} v$ uniformly on compact sets of $\tilde{E}$.

b) $v^N(y) \to v(x)$ as $y \to x$ and $N \to \infty$ for all $x \in \tilde{E}$.

The next theorem will be used to show continuity of ρ and convergence of ρ^N to ρ .

Theorem 2 : For every $m \in \mathbb{N}$

a) $\rho_m(\cdot) : \tilde{E} \to R$ is continuous .

b) $\rho_m^N(y) \to \rho_m(x)$ as $y \to x$ and $N \to \infty$ for every $x \in \tilde{E}$.

Proof : Again we will show b) only . We will use induction in m . For $m = 0$ the result is immediate using proposition 12 ($\rho_0^N(y) = g^N(y) \to g(x) = \rho_0(x)$ as $y \to x$ and $N \to \infty$) . Suppose b) holds for m . We will show the result for $x \in E$ (the proof for the case $x \in \partial^* E$ is similar) . We will prove the following steps :

Step 1 : For any $t \in [0, t^*(x))$

i) $E_y^N(\exp \{ -\alpha T_1^N \} \rho_m^N(Z_1^N) 1_{\{T_1^N \le t\}}) \to E_x(\exp \{ -\alpha T_1 \} \rho_m(Z_1) 1_{\{T_1 \le t\}})$ as $y \to x$ and $N \to \infty$.

ii) $E_y^N(\exp \{ -\alpha T_1^N \} \rho_m^N(Z_1^N)) \to E_x(\exp \{ -\alpha T_1 \} \rho_m(Z_1))$ as $y \to x$ and $N \to \infty$.

Step 2 : For any sequence $y_k \to x$ as $k \to \infty$

$$\lim_{k \to \infty} P\left\{ T_1^x = t^*(x) , T_1^{y_k} < t^*(y_k) \right\} = 0 .$$

Step 3 : For any sequence $y_k \to x$, $N_k \to \infty$, $t_k \to t \le t^*(x)$ as $k \to \infty$ where $t_k < t^*(y_k)$ for all k

$$E_{y_k}^{N_k}(\exp\{ -\alpha T_1^{N_k} \} \rho_m^{N_k}(Z_1^{N_k}) 1_{\{T_1^{N_k} \le t_k\}}) \overset{k}{\to} \begin{cases} E_x(\exp \{ -\alpha T_1 \} \rho_m(Z_1) 1_{\{T_1 \le t\}}) & \text{if } t < t^*(x) \\ E_x(\exp \{ -\alpha T_1 \} \rho_m(Z_1) 1_{\{T_1 < t\}}) & \text{if } t = t^*(x) \end{cases}$$

Step 4 : $\inf_{0\le t<t^*(y)} J^N(g^N,\rho_m^N)(t,y) \to \inf_{0\le t<t^*(x)} J(g,\rho_m)(t,x)$ as $y\to x$ and $N\to\infty$.

Then from step 1.ii) and step 4 we get

$$\lim_{\substack{y\to x\\ N\to\infty}} \rho_{m+1}^N(y) = \lim_{\substack{y\to x\\ N\to\infty}} \Big(\Big\{ \inf_{0\le t<t^*(y)} J^N(g,\rho_m^N)(t,y) \Big\} \wedge K^N\rho_m^N(y) \Big)$$

$$= \Big(\lim_{\substack{y\to x\\ N\to\infty}} \Big\{ \inf_{0\le t<t^*(y)} J^N(g,\rho_m^N)(t,y) \Big\} \Big) \wedge \Big(\lim_{\substack{y\to x\\ N\to\infty}} K^N\rho_m^N(y) \Big)$$

$$= \Big\{ \inf_{0\le t<t^*(x)} J(g,\rho_m)(t,x) \Big\} \wedge K\rho_m(x) = \rho_{m+1}(x)$$

and since x is arbitrary in E the theorem will be proved .

Proof of step 1 : It is clear that

$$E_y^N(\exp\{ -\alpha T_1^N \} \rho_m^N(Z_1^N) 1_{\{T_1^N\le t\}}) = E(\exp\{ -\alpha T_1^{Ny} \} \rho_m^N(Z_1^{Ny}) 1_{\{T_1^{Ny}\le t\}})$$

where the last expectation is over [0,1] with measure P . From the construction of T_1^{Ny} , proposition 10 , the induction hypothesis for m and the fact that

$P(T_1^x = t) = P_x(T_1 = t) = 0$ we get

$$\exp\{ -\alpha T_1^{Ny} \} \rho_m^N(Z_1^{Ny}) 1_{\{T_1^{Ny}\le t\}} \to \exp\{ -\alpha T_1^x \} \rho_m(Z_1^x) 1_{\{T_1^x\le t\}} \text{ as } y\to x \text{ and } N\to\infty \text{ P-a.s.}$$

By using the bounded convergence theorem we immediately obtain part i) of step 1 . Part ii) is proved in a similar way .

Proof of step 2 : It is clear that

$$P\Big\{ T_1^{y_k}<t^*(y_k) , T_1^x=t^*(x) \Big\} = P\Big\{ T_1^x=t^*(x) \Big\} - P\Big\{ T_1^{y_k}=t^*(y_k) \Big\} + P\Big\{ T_1^{y_k}=t^*(y_k) , T_1^x<t^*(x) \Big\}$$

$$= e^{-\Lambda(t^*(x),x)} - e^{-\Lambda(t^*(y_k),y_k)} + P\Big\{ T_1^{y_k}=t^*(y_k) , T_1^x<t^*(x) \Big\}$$

Then from continuity of $\Lambda(t^*(\,.\,),\,.\,)$ on E we get

$$\limsup_{k\to\infty} P\Big\{ T_1^{y_k}<t^*(y_k) , T_1^x=t^*(x) \Big\} = \limsup_{k\to\infty} P\Big\{ T_1^{y_k}=t^*(y_k) , T_1^x<t^*(x) \Big\}$$

$$\le P\Big\{ \limsup_{k\to\infty} \{T_1^{y_k}=t^*(y_k)\} , T_1^x<t^*(x) \Big\} \qquad (8)$$

But since $T_1^{y_k} \to T_1^x$ as $k \to \infty$ P-a.s. and $t^*(y_k) \to t^*(x)$ as $k \to \infty$ it follows that

$\limsup_{k\to\infty} \left\{T_1^{y_k}=t^*(y_k)\right\} \subset \left\{T_1^x = t^*(x)\right\}$ P-a.s. ; thus (8) equals zero proving step 2 .

Proof of step 3 : If $t < t^*(x)$ then $P(T_1^x=t) = 0$ and it is clear that

$$1_{\{T_1^{y_k}\le t\}} \to 1_{\{T_1^x\le t\}} \text{ as } k \to \infty \text{ P-a.s.}$$

The rest of the proof follows from the proof of step 1 . Suppose now $t = t^*(x)$. Then we get

$$E \mid e^{-\alpha T_1^{y_k}} \rho_m^{N_k}(Z_1^{N_k y_k}) 1_{\{T_1^{y_k}\le t_k\}} - e^{-\alpha T_1^x} \rho_m(Z_1^x) 1_{\{T_1^x<t^*(x)\}} \mid =$$

$$E \mid \left(e^{-\alpha T_1^{y_k}} \rho_m^{N_k}(Z_1^{N_k y_k}) 1_{\{T_1^{y_k}\le t_k\}} - e^{-\alpha T_1^x} \rho_m(Z_1^x) \right) 1_{\{T_1^x<t^*(x)\}} \mid +$$

$$E \mid \left(e^{-\alpha T_1^{y_k}} \rho_m^{N_k}(Z_1^{N_k y_k}) 1_{\{T_1^{y_k}\le t_k\}} \right) 1_{\{T_1^x=t^*(x)\}} \mid \qquad (9)$$

Since $T_1^{y_k} \to T_1^x$ as $k \to \infty$ P-a.s. and $t_k \to t^*(x)$ as $k \to \infty$ it is clear that

$$1_{\{T_1^{y_k}\le t_k\}} \to 1 \text{ as } k \to \infty \text{ P-a.s. on } \{ T_1^x<t^*(x) \}$$

Therefore as in the proof of step 1 the first term of (9) goes to zero as $k\to\infty$. For the second term we obtain

$$E \mid \left(e^{-\alpha T_1^{y_k}} \rho_m^{N_k}(Z_1^{N_k y_k}) 1_{\{T_1^{y_k}\le t_k\}} \right) 1_{\{T_1^x=t^*(x)\}} \mid \le a_1 P\left\{ T_1^{y_k}\le t_k , T_1^x=t^*(x) \right\}$$

$$\le a_1 P\left\{T_1^{y_k}\le t^*(y_k), T_1^x=t^*(x)\right\} \to 0 \text{ as } k\to\infty$$

from step 2 proving step 3 .

Proof of step 4 : We will show that for any sequence $y_k \to x$ and $N_k \to \infty$ as $k \to \infty$ we have

$$\text{i) } \limsup_{k\to\infty} \left(\inf_{0\le t<t^*(y_k)} J^{N_k}(g^{N_k},\rho_m^{N_k})(t,y_k) \right) \le \inf_{0\le t<t^*(x)} J(g,\rho_m)(t,x) .$$

$$\text{ii) } \liminf_{k\to\infty} \left(\inf_{0\le t<t^*(y_k)} J^{N_k}(g^{N_k},\rho_m^{N_k})(t,y_k) \right) \ge \inf_{0\le t<t^*(x)} J(g,\rho_m)(t,x) .$$

Let us show i) first . Given $\epsilon > 0$ take $t_\epsilon < t^*(x)$ such that t_ϵ is ϵ-optimal for the right hand side

of i) . By virtue of continuity of $t^*(\ .\ )$ we can get $k_0 \in \mathbb{N}$ such that $t_\epsilon < t^*(y_k)$ for $k \geq k_0$. It is clear then that for $k \geq k_0$

$$\left\{ \inf_{0\leq t<t^*(y_k)} J^{N_k}(g^{N_k},\rho_m^{N_k})(t,y_k) \right\} - \left\{ \inf_{0\leq t<t^*(x)} J(g,\rho_m)(t,x) \right\}$$

$$\leq \quad J^{N_k}(g^{N_k},\rho_m^{N_k})(t_\epsilon,y_k) - J(g,\rho_m)(t_\epsilon,x) + \epsilon$$

$$\leq \quad \left(g^{N_k}(\phi(t_\epsilon,y_k))e^{-\Lambda(t_\epsilon,y_k)} - g(\phi(t_\epsilon,x))e^{-\Lambda(t_\epsilon,x)} \right) +$$

$$\left(E_{y_k}^{N_k}(e^{-\alpha T_1^{N_k}} \rho_m^{N_k}(Z_1^{N_k}) 1_{\{T_1^{N_k}\leq t_\epsilon\}}) - E_x(e^{-\alpha T_1} \rho_m(Z_1) 1_{\{T_1\leq t_\epsilon\}}) \right) + \epsilon \qquad (10)$$

By virtue of continuity of $\Lambda(\ .\ ,\ .\ )$ in $\mathbb{B}$ and the assumptions on g^N and g the first term of (10) goes to zero as $k \to \infty$. From step 1 we see that the second term of (10) also goes to zero as $k \to \infty$. Thus

$$\limsup_{k\to\infty} \left\{ \inf_{0\leq t<t^*(y_k)} J^{N_k}(g^{N_k},\rho_m^{N_k})(t,y_k) \right\} - \inf_{0\leq t<t^*(x)} J(g,\rho_m)(t,x) \quad \leq \quad \epsilon$$

and since it holds for every $\epsilon > 0$, part i) is proved . Let us show ii) now . Given $\epsilon > 0$ we can find $s_k < t^*(y_k)$ which is ϵ-optimal for $\inf_{0\leq t<t^*(y_k)} J^{N_k}(g^{N_k},\rho_m^{N_k})(t,y_k)$ for each $k \in \mathbb{N}$. Consider now any convergent subsequence

$$J^{N_{k_i}}(g^{N_{k_i}},\rho_m^{N_{k_i}})(s_{k_i},y_{k_i}) \quad \text{of} \quad J^{N_k}(g^{N_k},\rho_m^{N_k})(s_k,y_k)$$

There is no loss of generality in assuming that s_{k_i} converges to some s in $R_+ \cup \{\infty\}$. Since $s_{k_i} < t^*(y_{k_i}) \to t^*(x)$ as $i \to \infty$ it is clear that $s \leq t^*(x)$. Suppose $s < t^*(x)$. Then

$$J(g,\rho_m)(s,x) - J^{N_{k_i}}(g^{N_{k_i}},\rho_m^{N_{k_i}})(s_{k_i},y_{k_i}) = \left(g(\phi(s,x))e^{-\Lambda(s,x)} - g^{N_{k_i}}(\phi(s_{k_i},y_{k_i}))e^{-\Lambda(s_{k_i},y_{k_i})} \right)$$

$$+ \left(E_x(e^{-\alpha T_1} \rho_m(Z_1) 1_{\{T_1\leq s\}}) - E_{y_{k_i}}^{N_{k_i}}(e^{-\alpha T_1^{N_{k_i}}} \rho_m^{N_{k_i}}(Z_1^{N_{k_i}}) 1_{\{T_1^{N_{k_i}}\leq s_{k_i}\}}) \right) \to 0 \text{ as } i \to \infty$$

from the assumptions on g^N and g , continuity of Λ and step 3 . If $s = t^*(x)$ then from continuity of g on $\tilde{E}$ we have

$$\Big(\lim_{t \to t^*(x)} J(g,\rho_m)(t,x) \Big) - J^{N_{k_i}}(g^{N_{k_i}},\rho_m^{N_{k_i}})(s_{k_i},y_{k_i})$$

$$= \Big(g(\phi(t^*(x),x))e^{-\Lambda(t^*(x),x)} - g^{N_{k_i}}(\phi(s_{k_i},y_{k_i}))e^{-\Lambda(s_{k_i},y_{k_i})} \Big) +$$

$$\Big(E_x(e^{-\alpha T_1}\rho_m(Z_1)1_{\{T_1<t^*(x)\}}) - E_{y_{k_i}}^{N_{k_i}}(e^{-\alpha T_1^{N_{k_i}}}\rho_m^{N_{k_i}}(Z_1^{N_{k_i}})1_{\{T_1^{N_{k_i}} \le s_{k_i}\}}) \Big) \to 0 \text{ as } i \to \infty$$

from the assumptions on g^N and g , continuity of Λ and step 3 . Then we can conclude that

$$\inf_{0 \le t < t^*(x)} J(g,\rho_m)(t,x) \le \lim_{i \to \infty} J^{N_{k_i}}(g^{N_{k_i}},\rho_m^{N_{k_i}})(s_{k_i},y_{k_i})$$

and since it is true for any convergent subsequence of $J^{N_k}(g^{N_k},\rho_m^{N_k})(s_k,y_k)$ we get

$$\inf_{0 \le t < t^*(x)} J(g,\rho_m)(t,x) \le \liminf_{k \to \infty} \Big\{ J^{N_k}(g^{N_k},\rho_m^{N_k})(t,y_k) \Big\}$$

$$\le \liminf_{k \to \infty} \Big\{ \inf_{0 \le t < t^*(y_k)} J^{N_k}(g^{N_k},\rho_m^{N_k})(t,y_k) \Big\} + \epsilon$$

proving step 4 and the theorem . □

The next results show continuity of ρ and convergence of ρ^N to ρ .

Corollary 2 : ρ is continuous on $\tilde{E}$.

Proof : From prop. 7 we have for any $x \in \tilde{E}$

$$0 \le \rho_m(x) - \rho(x) \le E_x(e^{-\alpha T_m}(g(Z_m) - \rho(Z_m))) \le 2a_1 E_x(e^{-\alpha T_m})$$

The sequence $E_x(e^{-\alpha T_m})$ is obviously decreasing and converges to zero as $m \to \infty$. From prop. 11 $E_{(\,\cdot\,)}(e^{-\alpha T_m})$ is continuous on $\tilde{E}$. By virtue of Dini's theorem (see [30] page 135) $E_x(e^{-\alpha T_m}) \to 0$ as $m \to \infty$ uniformly on compact sets of $\tilde{E}$. So it is clear that $\rho_m \to \rho$ as $m \to \infty$ uniformly on compact sets of $\tilde{E}$. From theorem 2 ρ_m is continuous on $\tilde{E}$ which implies that ρ is continuous on $\tilde{E}$. □

Corollary 3 : $\rho^N \overset{N}{\to} \infty$ ρ uniformly on compact sets of $\tilde{E}$.

Proof : For any $x \in \tilde{E}$ we get from prop. 7 , prop. 11 and theorem 2

$$\begin{aligned} | \rho(x) - \rho^N(y) | &\leq | \rho(x) - \rho_m(x) | + | \rho_m(x) - \rho_m^N(y) | + | \rho_m^N(y) - \rho^N(x) | \\ &\leq 2a_1 E_x(e^{-\alpha T_m}) + | \rho_m(x) - \rho_m^N(y) | + 2a_1 E_y^N(e^{-\alpha T_m}) \\ &\to 4a_1 E_x(e^{-\alpha T_m}) \quad \text{as } y \to x \text{ and } N \to \infty \end{aligned}$$

Taking the limit as $m \to \infty$ we obtain that $\rho^N(y) \to \rho(x)$ as $y \to x$ and $N \to \infty$. The corollary follows from prop. 12 . □

Remark : The results of this section can be easily extended to the case when the state space E is the countable union of possibly different dimensional open sets .

4.5. Construction of ϵ-optimal stopping times using ρ^N

In this subsection we assume that $\tilde{E} := E \cup \partial E$ is a compact set of R^d and $g^N = g$. The next definitions are necessary in order to establish uniform convergence of ρ^N to ρ on $\tilde{E}$. For $x \in \partial E - \partial^* E$ we have one of the following three possibilities :

i) $\phi(\,.\,,x)$ will leave ∂E and enter E or

ii) $\phi(\,.\,,x)$ will leave $\tilde{E}$ or

iii) $\phi(\,.\,,x)$ will stay in $\partial E - \partial^* E$ forever

We define $\partial^* E' := \{ z \in \partial E - \partial^* E ; z = \phi(\gamma,x)$ where $\phi(\gamma^+,x) \notin \tilde{E}$ and $\phi(t,x) \in \partial E - \partial^* E$ for all $t \in [0,\gamma)$ for some $\gamma \in R_+$ and some x satisfying ii) above $\}$. We re-define $\partial^* E$ as the union of the old $\partial^* E$ with $\partial^* E'$ and define $\partial \hat{E} := \partial E - \partial^* E$. We extend the previous assumptions of this section replacing E by $E \cup \partial \hat{E}$. We can easily show then that all the previous results of this section still hold if we replace E by $E \cup \partial \hat{E}$. In particular corollaries 2 and 3 will say that ρ is continuous on $\tilde{E}$ and $\rho^N \to \rho$ as $N \to \infty$ uniformly on $\tilde{E}$. The purpose of the rest of this subsection is to construct an ϵ-optimal stopping time for the original problem using ρ^N . Given $\epsilon > 0$ consider $N \in \mathbb{N}$ such that $| \rho - \rho^N | < \epsilon/4$. Define

$$U_\epsilon^N := \inf\{ t \geq 0 ; e^{-\alpha t} g(X_t) - e^{-\alpha t} \rho^N(X_t) < \epsilon/2 \} \in \mathcal{M}_\infty$$

Note that from continuity of g on $\tilde{E}$ and the results in [15] we get that $\rho^N \in B^c(E)$ (see section 2 for definitions) . So we have right continuity of $\rho^N(X_t)$ and thus for all $\omega \in \Omega$

$$e^{-\alpha U_\epsilon^N(\omega)} g(X_{U_\epsilon^N(\omega)}) \; - \; e^{-\alpha U_\epsilon^N(\omega)} \rho^N(X_{U_\epsilon^N(\omega)}) \; \leq \; \epsilon/2$$

Suppose we can show the following result :

Theorem 3 : $E_x(e^{-\alpha U_\epsilon^N} \rho(X_{U_\epsilon^N})) \; = \; \rho(x)$.

Then noting that $\rho^N < \rho + \epsilon/4$ we obtain

$$E_x(e^{-\alpha U_\epsilon^N} g(X_{U_\epsilon^N})) \quad \leq E_x(e^{-\alpha U_\epsilon^N} \rho^N(X_{U_\epsilon^N})) \; + \; \epsilon/2 \; < \; E_x(e^{-\alpha U_\epsilon^N} \rho(X_{U_\epsilon^N})) \; + \; 3\epsilon/4$$

$$= \; \rho(x) + 3\epsilon/4 \; < \rho(x) + \epsilon$$

where the last equality is due to Theorem 3 . Therefore U_ϵ^N is an ϵ-optimal stopping time of the original problem . It remains to show Theorem 3 .

Proof of Theorem 3 : If we define for $(s,z) \in E \times R_+$

$$r := \inf\{ \; t \geq 0 \; ; \; e^{-\alpha(t+s)} g(\phi(t,z)) < e^{-\alpha(t+s)} \rho^N(\phi(t,z)) + \epsilon/2 \; \}$$

then all we have to show is that $E_z(e^{-\alpha(r \wedge T_1)} \rho(X_{r \wedge T_1})) \; = \; \rho(z)$. Lemmas 2 and 3 (see section 3) can be readily modified to complete the proof of the theorem . From the proof of lemma 1 we know that

$$E_z(e^{-\alpha(r \wedge T_1)} \rho(X_{r \wedge T_1})) = \begin{cases} \left\{ \inf_{r \leq t < t^*(z)} J(g,\rho)(t,z) \right\} \wedge K\rho(z) & \text{if } r < t^*(z) \\ K\rho(z) & \text{if } r \geq t^*(z) \end{cases}$$

The result is clear if $\rho(z) = K\rho(z)$. Suppose $\rho(z) = \inf_{0 \leq t < t^*(z)} J(g,\rho)(t,z) \; < \; K\rho(z)$. Since g and ρ are continuous we have at least one of the two possibilities :

a) $\rho(z) = J(g,\rho)(\gamma,z)$ for some $\gamma \in [0,t^*(z))$

b) $\rho(z) = \lim_{t \to t^*(z)} J(g,\rho)(t,z)$

By virtue of (2) we can easily show that for a) , $\rho(\phi(\gamma,z)) = g(\phi(\gamma,z))$ and for b) ,

$\lim_{t \to t^*(z)} J(g,\rho)(t,z) = g(\phi(t^*(z),z)$.

From this and continuity of g and ρ it follows that we can find $\gamma_\epsilon \in [0,\gamma]$ for a) and $\gamma_\epsilon \in [0,t^*(z))$ for b) such that

$g(\phi(\gamma_\epsilon,z)) < \rho(\phi(\gamma_\epsilon,z)) + \epsilon/4$

Note that $\rho(z) = \inf_{0 \le t < t^*(z)} J(g,\rho)(t,z) = \inf_{s \le t < t^*(z)} J(g,\rho)(t,z)$ for any $s \le \gamma_\epsilon$.

Since

$| \rho - \rho^N | < \epsilon/4$ we get

$g(\phi(\gamma_\epsilon,z)) < \rho(\phi(\gamma_\epsilon,z)) + \epsilon/4 < \rho^N(\phi(\gamma_\epsilon,z)) + \epsilon/2$

and thus

$e^{-\alpha(s+\gamma_\epsilon)} g(\phi(\gamma_\epsilon,z)) < e^{-\alpha(s+\gamma_\epsilon)} \rho^N(\phi(\gamma_\epsilon,z)) + \epsilon/2$

It follows from the definition of r that $r \le \gamma_\epsilon$ which implies that

$E_z(e^{-\alpha(r \wedge T_1)} \rho(X_{r \wedge T_1})) = \inf_{r \le t < t^*(z)} J(g,\rho)(t,z) = \rho(z)$

and the theorem is proved . □

4.6. Failure of continuity of ρ if $\alpha = 0$

We will show now that continuity of ρ may fail if we drop the condition $\alpha > 0$. Consider the following P.D.P. defined below (see fig. 1)

i) the state space $E \cup \partial\hat{E} = \{ (x,y) ; 0 \le x \le 1 , 0 \le y < e^{x-1} \}$

ii) the flow $\phi(t,(x,y)) = (x+t , ye^t)$

iii) $\lambda = 0$

iv) $Q(\{(0 , y)\} ; (1 , y)) = 1$ for $0 \le y \le e^{-1}$

$Q(\{(\ln y + 1 , y)\} ; (1 , y)) = 1$ for $e^{-1} \le y \le e^{-\epsilon}$

$Q(\{(1 - \epsilon , e^{-\epsilon})\} ; (1 , y)) = 1$ for $e^{-\epsilon} \le y \le 1$

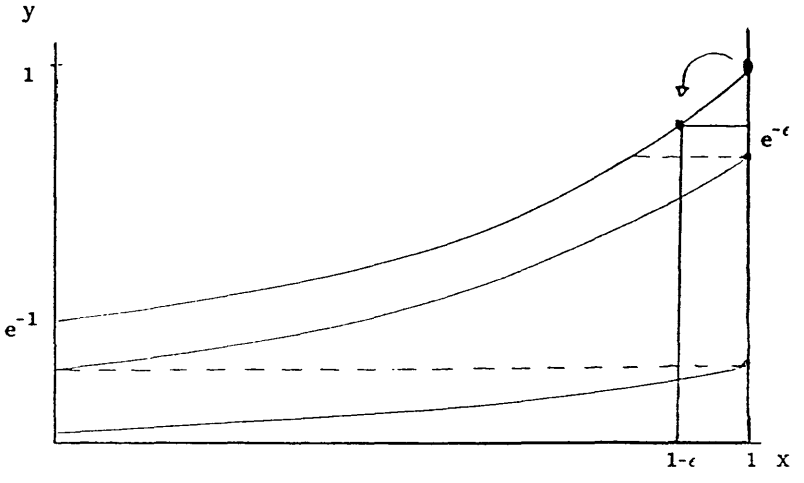


fig. 1

From above it is clear that $\partial^* E = 1x[0,1]$. We define the gain function $g(x,y) := y$. We want to find the solution of the optimal stopping problem

$$\rho(x,y) = \sup_{\tau \in \mathcal{M}_b} E_{(x,y)}(g(X_\tau))$$

All the assumptions of subsection 4.2 are satisfied . However it is easy to see that

$$\rho(x,y) = \begin{cases} 1 \text{ if } y > 0 \\ 0 \text{ if } y = 0 \end{cases}$$

and hence ρ is discontinuous . Consider now the discounted case . Then we get that for $\alpha < 1$

$$\rho(x,y) = \begin{cases} y^\alpha \text{ for } 0 \le y \le e^{x-1-\epsilon} \\ e^{-\alpha(1-x+\epsilon)} \text{ for } e^{x-1-\epsilon} \le y \le e^{x-1-\alpha\epsilon} \\ ye^{(1-\alpha)(1-x)} \text{ for } e^{x-1-\alpha\epsilon} \le y \le e^{x-1} \end{cases}$$

and for $\alpha \ge 1$, $\rho(x,y) = y$. Therefore ρ is continuous for this case .

4.7. Numerical technique and example

For the discretized process X_t^N note that

$$E_x^N(e^{-\alpha T_1^N} v(Z_1^N) 1_{\{T_1^N \le t\}}) = \sum_{i=1}^N v(z_i) \Big(\int_0^t e^{-\alpha s} Q(A_i^N;\phi(s,x))\lambda(\phi(s,x)) e^{-\Lambda(s,x)} ds \Big)$$

$$K^N v(x) = E_x^N(e^{-\alpha T_1^N} v(Z_1^N)) = \sum_{i=1}^N v(z_i) \Big(\int_0^{t^*(x)} e^{-\alpha s} Q(A_i^N;\phi(s,x))\lambda(\phi(s,x)) e^{-\Lambda(s,x)} ds \Big)$$

$$+ \sum_{i=1}^N v(z_i) e^{-\alpha t^*(x)} Q(A_i^N;\phi(t^*(x),x)) e^{-\Lambda(t^*(x),x)}$$

and hence the above expressions depend only on $v(z_i)$, i = 1 , ... , N . Thus considering the discretized problem with $g^N = g$ we start with $\rho_0^N(z_i) = g(z_i)$, i = 1 , ... , N and for m = 0 , 1 , ... we iterate

$$\rho_{m+1}^N(z_i) = \Big\{ \inf_{0 \le t < t^*(z_i)} J^N(g,\rho_m^N)(t,z_i) \Big\} \wedge K^N \rho_m^N(z_i) \qquad i = 1 , ... , N$$

Therefore each iteration consists of N one dimensional minimizations . The algorithm is consistent since as we saw above

$$J^N(g,\rho_m^N)(t,z_i) = g(\phi(t,z_i)) e^{-\Lambda(t,z_i)} + E_{z_i}^N(e^{-\alpha T_1} \rho_m^N(Z_1) 1_{\{T_1 \le t\}})$$

and $K^N \rho_m^N(z_i)$ depend only on $\rho_m^N(z_i)$, i = 1 , ... , N which were calculated in the previous iteration . In this way we get an approximation for $\rho^N(z_1)$, ... , $\rho^N(z_N)$ (recall that $\rho_m^N \downarrow \rho^N$ as $m \to \infty$) . If we want to calculate $\rho^N(x)$ for any $x \in E$ we have to do a further one dimensional minimization

$$\rho^N(x) = \{ \inf_{0 \le t < t^*(x)} J^N(g,\rho^N)(t,x) \} \wedge K^N \rho^N(x)$$

since $J^N(g,\rho^N)(t,x)$ and $K^N \rho^N(x)$ depend only on $\rho^N(z_1)$, ... , $\rho^N(z_N)$.

Let us consider the following example to illustrate our method . Customers arrive at a single-server queue according to a Poisson process with rate λ and fixed service time β . At time T the queue will be closed and only those who are on the queue at time T will be served . Our gain function is defined in the following way

$g(\xi,w) = w + c(T-\xi)^2 \quad , \quad \xi \leq T \, , \, w \geq 0$

where w is the waiting time of a customer arrivig at time ξ . The state space is $E = E_1 \cup E_2$ where $E_1 := [0,T]x(0,\infty)$, $E_2 := [0,T]$ and $\partial^* E_1 = [0,T]x\{0\}$. The motion in E_1 is given by $\phi(t,(\xi,w)) = (\xi+t,w-t)$ and in E_2 is $\phi(t,\xi) = \xi + t$. The transition measure is

$Q(\{(\, \xi \, , \, w+\beta) \};(\, \xi \, , \, w \,)) = 1 \quad , \xi \leq T \, , \quad w > 0$ (jumps in E_1)

$Q(\{\, \xi \,\};(\, \xi \, , \, 0 \,)) = 1 \, , \, \xi \leq T$ (jump from E_1 to E_2)

$Q(\{(\, \xi \, , \, \beta \,)\};(\, \xi \,)) = 1 \, , \, \xi \leq T$ (jump from E_2 to E_1)

and $\lambda(x) = \lambda$ for every $x \in E$. We want to find the solution of

$$\rho(x) = \inf_{\tau \in \mathcal{M}_\infty} E_x(g(X_\tau)) \quad , \, x \in E$$

This problem can be interpreted as someone who wants to minimize the waiting time on the queue and the free time not used . Although some of the assumptions are not satisfied ($\alpha=0$, g unbounded) we can show that for this specific example our method is valid . The discretization we consider is shown in fig. 2 (the hxh squares and h segment are the sets A_i^N) .

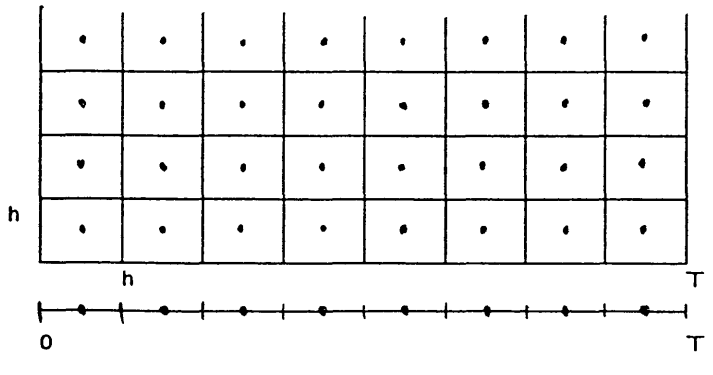


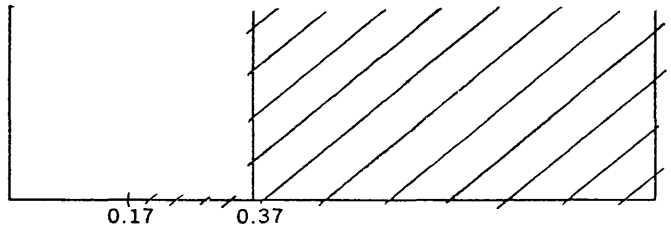
fig. 2

We make T = 1 , λ = 20 , β = 0.1 and vary c . For the discretization we fix h = 0.005 . Since the top points are not accurate we ingnored the in our results . We obtained the following results (the hatched area is the stopping set) :

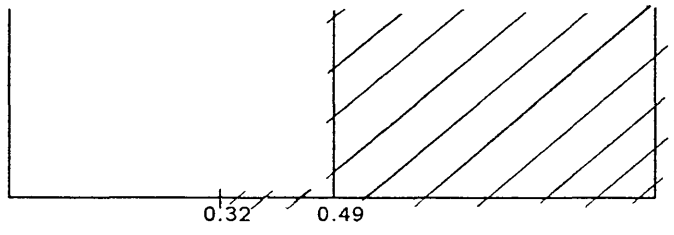
1) c = 0.1 . All the state space is stopping set

2) c = 0.5 . All the state space is stopping set

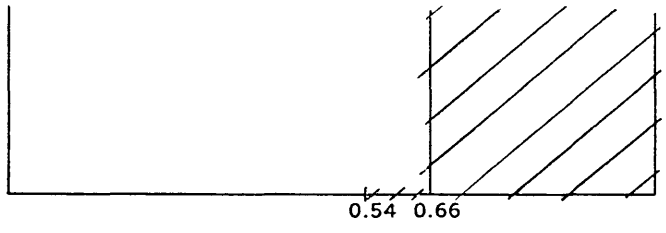
3) c = 0.8



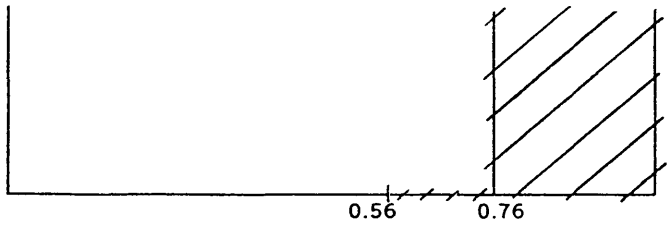
4) c = 1.0



5) c = 1.5



6) c = 2.0



Appendix of Chapter 2

In this appendix we present the proofs of propositions 7 , 8 , 9 , 12 and a connection between the solutions of the system $v = L(g,v)$ and system (2) of chapter 1 .

Proof of proposition 7 : First we show that for all $x \in \tilde{E}$ and $n \in \mathbb{N}$

$$0 \leq \rho_{n+1}(x) - \rho(x) \leq E_x(e^{-\alpha T_1}(\rho_n(Z_1) - \rho(Z_1))) \quad (A1)$$

Indeed if $\rho(x) = E_x(e^{-\alpha T_1}\rho(Z_1))$ then since $\rho_{n+1}(x) \leq K\rho_n(x) = E_x(e^{-\alpha T_1}\rho_n(Z_1))$ it is clear that (A1) holds . Now if $x \in \partial^* E$ and $\rho(x) = g(x)$ then it is immediate that $\rho_{n+1}(x) = g(x)$ and therefore (A1) is satisfied . Finally consider $x \in E$ and $\rho(x) = \inf_{0 \leq t < t^*(x)} J(g,\rho)(t,x)$. Given $\epsilon > 0$ there exists t_ϵ in $[0,t^*(x))$ such that

$J(g,\rho)(t_\epsilon,x) \leq \rho(x) + \epsilon$. Then since $\rho_{n+1}(x) \leq J(g,\rho_n)(t_\epsilon,x)$ we have

$$\begin{aligned} 0 \leq \rho_{n+1}(x) - \rho(x) &\leq J(g,\rho_n)(t_\epsilon,x) - J(g,\rho)(t_\epsilon,x) + \epsilon \\ &= E_x(e^{-\alpha T_1}(\rho_n(Z_1) - \rho(Z_1)) 1_{\{T_1 \leq t_\epsilon\}}) + \epsilon \\ &\leq E_x(e^{-\alpha T_1}(\rho_n(Z_1) - \rho(Z_1))) + \epsilon \end{aligned}$$

where the last inequality follows from the fact that $\rho_n \geq \rho$. Since the above inequality holds for every $\epsilon > 0$, (A1) is proved .

Now we are going to show that for all $m \in \mathbb{N}$

$$0 \leq \rho_m(x) - \rho(x) \leq E_x(e^{-\alpha T_m} (g(Z_m) - \rho(Z_m))) \quad \forall\, x \in \tilde{E} \qquad (A2)$$

We use induction in m . For $m = 1$ the result is immediate from (A1) with $n = 0$. Suppose (A2) holds for m . Define $\chi(z) := E_z(e^{-\alpha T_m} (g(Z_m) - \rho(Z_m))$. Then from (A1) , (A2) for m , time homogeneity and the strong Markov property we have

$$0 \le \rho_{m+1}(x) - \rho(x) \quad \le E_x(e^{-\alpha T_1}(\rho_m(Z_1) - \rho(Z_1)))$$

$$\le E_x(e^{-\alpha T_1} \chi(Z_1))$$

$$= E_x(e^{-\alpha T_1} E(e^{-\alpha(T_{m+1}-T_1)}(g(Z_{m+1}) - \rho(Z_{m+1})) / \mathcal{F}_{T_1}))$$

$$= E_x(e^{-\alpha T_{m+1}} (g(Z_{m+1}) - \rho(Z_{m+1})))$$

wich completes the proof . □

Proof of proposition 8 : Consider (t,x) , $x \in \tilde{E}$, $0 \le t \le t^*(x)$ and any sequence (t_i,y_i) , $y_i \in \tilde{E}$, $0 \le t_i \le t^*(y_i)$ such that $t_i \to t$, $y_i \to x$ as $i \to \infty$. If $t = 0$ then it is clear that $\Lambda(t_i,y_i) \to 0 = \Lambda(t,x)$ as $i \to \infty$. Suppose $0 < t < \infty$; choose $0 < \nu < t$ and take $i_0 \in \mathbb{N}$ such that $\nu < t_i$ for all $i \ge i_0$. From the fact that λ is bounded by a_2 , assumption 2 of subsection 4.2. and the bounded convergence theorem we obtain

$$| \Lambda(t,x) - \Lambda(t_i,y_i) | \quad \le | \Lambda(t,x) - \Lambda(\nu,x) | + | \Lambda(\nu,x) - \Lambda(\nu,y_i) | + | \Lambda(\nu,y_i) - \Lambda(t_i,y_i) |$$

$$\le \int_{\nu}^{t} \lambda(\phi(s,x)) \, ds + \int_{0}^{\nu} | \lambda(\phi(s,x)) - \lambda(\phi(s,y_i)) | \, ds + \int_{\nu}^{t_i} \lambda(\phi(s,y_i)) \, ds$$

$$\le a_2(t - \nu) + \int_{0}^{\nu} | \lambda(\phi(s,x)) - \lambda(\phi(s,y_i)) | \, ds + a_2(t_i - \nu)$$

$$\to 2a_2(t - \nu) \quad \text{as } i \to \infty$$

Taking the limit as $\nu \to t$ we conclude that $\Lambda(t_i,y_i) \overset{i}{\to} \Lambda(t,x)$. Finally suppose $t = \infty$. Then from assumption 5 of subsection 4.2 it is clear that $\Lambda(t,x) = \infty$. Given any number $b > 0$ we can find $0 < \nu < \infty$ such that $\Lambda(\nu,x) \ge b$. Take a sequence $\nu_i \to \nu$ as $i \to \infty$ such that $\nu_i \le t_i$ for all $i \in \mathbb{N}$. Then $\Lambda(t_i,y_i) \ge \Lambda(\nu_i,y_i)$ for all $i \in \mathbb{N}$ and using the result seen above we get

$$\liminf_{i \to \infty} \Lambda(t_i,y_i) \ge \lim_{i \to \infty} \Lambda(\nu_i,y_i) = \Lambda(\nu,x) \ge b$$

Since it holds for b arbitraryly large it follows that

$$\liminf_{i \to \infty} \Lambda(t_i,y_i) = \infty$$

which shows that $\lim_{i \to \infty} \Lambda(t_i,y_i) = \infty$ proving the proposition . □

Proof of proposition 9 : For arbitrary $x \in \tilde{E}$ we are going to show that

$$\lim_{y \to x} E_y (f(T_1 , Z_1)) = E_x(f(T_1 , Z_1))$$

for any real-valued bounded continuous function f on R_+xE . This means that $\mu_y \overset{y \to x}{\to} \mu_x$ weakly and since x is arbitrary the proposition will be proved . Define for $t \leq t^*(x)$

$$\ell(t,x) := \int_0^t \int_E f(s,z)\, Q(dz,\phi(s,x))\lambda(\phi(s,x))e^{-\Lambda(s,x)}ds = E_x(f(T_1 , Z_1)1_{\{T_1<t\}})$$

$$\mathcal{J}(x) := \Big(\int_E f(t^*(x),z)Q(dz;\phi(t^*(x),x)) \Big)e^{-\Lambda(t^*(x),x)} = E_x(f(T_1 , Z_1)1_{\{T_1 = t^*(x)\}})$$

and let $B = \sup_{(s,z) \in R_+xE} | f(s,z) |$. It is clear that

$$E_x(f(T_1 , Z_1)) = E_x(f(T_1 , Z_1)1_{\{T_1<t^*(x)\}}) + E_x(f(T_1 , Z_1)1_{\{T_1 = t^*(x)\}})$$

$$= \ell(t^*(x),x) + \mathcal{J}(x)$$

We will show the following steps :

i) $\ell(t^*(y),y) \to \ell(t^*(x),x)$ as $y \to x$

ii) $\mathcal{J}(y) \to \mathcal{J}(x)$ as $y \to x$

Proof of step i) : If $t^*(x) = 0$ (that is $x \in \partial^*E$) then i) is clear since $t^*(y) \to 0$ as $y \to x$. Suppose $t^*(x) > 0$. Consider $t < t^*(x)$ and y such that $t < t^*(y)$. Then from assumptions 2) and 3.i) of subsection 4.2 , proposition 8 and the bounded convergence theorem we have

$$| \ell(t^*(x),x) - \ell(t^*(y),y) | \leq | \ell(t^*(x),x) - \ell(t,x) | + | \ell(t,x) - \ell(t,y) | + | \ell(t,y) - \ell(t^*(y),y) |$$

$$\leq \int_t^{t^*(x)} \int_E |f(s,z)| Q(dz;\phi(s,x))\lambda(\phi(s,x))e^{-\Lambda(s,x)}ds + \int_t^{t^*(y)} \int_E |f(s,z)| Q(dz;\phi(s,y))\lambda(\phi(s,y))e^{-\Lambda(s,y)}ds$$

$$+ \int_0^t \Big(\Big| \int_E f(s,z)Q(dz;\phi(s,x))e^{-\Lambda(s,x)}\lambda(\phi(s,x)) - \int_E f(s,z)Q(dz;\phi(s,y))e^{-\Lambda(s,y)}\lambda(\phi(s,y)) \Big| \Big) ds$$

$$\leq B\left(e^{-\Lambda(t,x)} - e^{-\Lambda(t^*(x),x)} \right) + B\left(e^{-\Lambda(t,y)} - e^{-\Lambda(t^*(y),y)} \right)$$

$$+ \int_0^t \Big(\Big| \int_E f(s,z)Q(dz;\phi(s,x))e^{-\Lambda(s,x)}\lambda(\phi(s,x)) - \int_E f(s,z)Q(dz;\phi(s,y))e^{-\Lambda(s,y)}\lambda(\phi(s,y)) \Big| \Big) ds$$

$$\to 2B\left(e^{-\Lambda(t,x)} - e^{-\Lambda(t^*(x),x)} \right) \quad \text{as } y \to x$$

Taking the limit as $t \to t^*(x)$ we obtain the desired result . □

Proof of step ii) : If $t^*(x) = \infty$ then from assumption 5) , $\mathcal{J}(x) = 0$. By virtue of proposition 8 it is clear that $e^{-\Lambda(t^*(y),y)} \to 0$ as $y \to x$ which shows that $\mathcal{J}(y) \to 0$ as $y \to x$. Suppose $t^*(x) < \infty$. We need the folowing result (see proof in Theorem 2.5.3 of Hordijk and Van der Duyn Schouten [17]) :

Proposition A.1 : Let $(\mathcal{V}_n)_{n=1}^{\infty}$, $\mathcal{V}$ be bounded measurable functions from E to R such that if $\varphi_n \to \varphi$ then $\mathcal{V}_n(\varphi_n) \to \mathcal{V}(\varphi)$ as $n \to \infty$ for all $\varphi \in E$. Let $\mathbb{P}$, $\mathbb{P}_n$, $n \geq 1$ be probability measures on E . Suppose $\mathbb{P}_n \overset{n\to\infty}{\to} \mathbb{P}$ weakly . Then

$$\lim_{n\to\infty} \int_E \mathcal{V}_n(z)d\mathbb{P}_n(z) = \int_E \mathcal{V}(z)d\mathbb{P}(z)$$

In our case we have that for arbitrary sequence $y_n \overset{n\to\infty}{\to} x$ we define $\mathcal{V}_n(z) = f(t^*(y_n),z)$, $\mathcal{V}(z) = f(t^*(x),z)$, $\mathbb{P}_n(\,.\,) = Q(\,.\,;\phi(t^*(y_n),y_n))$ and $\mathbb{P}(\,.\,) = Q(\,.\,;\phi(t^*(x),x))$. From continuity of f on $R_+ xE$ it is clear that if $\varphi_n \to \varphi$ then $f(t^*(y_n),\varphi_n) \to f(t^*(x),\varphi)$ as $n \to \infty$ for all $\varphi \in E$

and from assumption 3.ii) of subsection 4.2 , $Q(\cdot ;\phi(t^*(y_n),y_n)) \overset{n\to\infty}{\to} Q(\cdot ; \phi(t^*(x),x))$ weakly . Therefore by virtue of proposition A.1 we obtain

$$\lim_{n\to\infty}\int_E f(t^*(y_n),z)Q(dz;\phi(t^*(y_n),y_n)) = \int_E f(t^*(x),z)Q(dz;\phi(t^*(x),x))$$

The proof of this step (and the proposition) is completed by noting that $e^{-\Lambda(t^*(y),y)} \to e^{-\Lambda(t^*(x),x)}$ as $y \to x$ from proposition 8 . □

Proof of proposition 12 : Let us show first that a) $\Rightarrow$ b) . For arbitrary $x \in \tilde{E}$ consider any sequence $y_i \to x$ and $N_i \to \infty$ as $i \to \infty$. Define the compact set $\mathcal{C} = \left(\bigcup_{i=1}^{\infty} \{ y_i \} \right) \cup \{ x \}$. Then we get

$$| v(x) - v^{N_i}(y_i) | \leq | v(x) - v(y_i) | + | v(y_i) - v^{N_i}(y_i) | \to 0 \quad \text{as } i \to \infty$$

from continuity of v and uniform convergence of v^N to v in $\mathcal{C}$. Since it is true for arbitrary $x \in \tilde{E}$ and every sequence $y_i \to x$ and $N_i \to \infty$ as $i \to \infty$ we get the desired result .

Now we prove that b) $\Rightarrow$ a) . Let $\mathcal{C}$ be a compact set of $\tilde{E}$. For any $x \in \mathcal{C}$ given $\epsilon > 0$ there exists N(x) and $\delta_1(x) > 0$ such that $| v(x) - v^N(y) | < \frac{\epsilon}{2}$ whenever $N \geq N(x)$ and $| x - y | < \delta_1(x)$. Due to continuity of v we can find $\delta_2(x) > 0$ such that $| v(x) - v(y) | < \frac{\epsilon}{2}$ whenever $| x - y | < \delta_2(x)$. Define $\delta(x) := \delta_1(x) \wedge \delta_2(x)$ and $\mathcal{U}(x) := \left\{ z \in R^d ; | x - y | < \delta(x) \right\}$. Then

$$| v(y) - v^N(y) | \leq | v(y) - v(x) | + | v(x) - v^N(y) | < \epsilon$$

for every $y \in \mathcal{U}(x) \cap \mathcal{C}$ and $N \geq N(x)$. Now let $\{ \mathcal{U}(x_1) , \dots , \mathcal{U}(x_r) \}$ be a finite subcover of $\{ \mathcal{U}(x) ; x \in \mathcal{C} \}$ for $\mathcal{C}$ and let $N_{max} := \max \{ N(x_1) , \dots , N(x_r) \}$. We get that for any $y \in \mathcal{C}$ there exists $i \in \{1 , \dots , r\}$ such that $y \in \mathcal{U}(x_i)$ and therefore $| v(y) - v^N(y) | < \epsilon$ whenever $N \geq N_{max} \geq N(x_i)$ proving the proposition . □

The remainder of this appendix is devoted to show a connection between the solutions of $v = L(g,v)$ and system (2) of chapter 1 associated to the optimal stopping problem . Let $B^{ac}(E)$ denote the space of functions g in $B^{c}(E)$ such that for each $x \in E$, $g(\phi(\,.\,,x)) : [0,t^*(x)) \to R$ is absolutely continuous . First we are going to show an auxiliary result . Define

$$L(g,v)(t,x) := \{ \inf_{t \le s < t^*(x)} J(g,v)(s,x) \} \wedge Kv(x) \qquad , t \in [0,t^*(x))$$

Proposition A.2 : Suppose that $g \in B^{ac}(E)$ and $v \in B(E)$. Then for every $x \in E$, $L(g,v)(\,.\,,x) : [0,t^*(x)) \to R$ ia absolutely continuous .

Proof : Since $g(\phi(\,.\,,x))$ is absolutely continuous it is clear that

$$J(g,v)(t,x) = e^{-\alpha t - \Lambda(t,x)} g(\phi(t,x)) + \int_0^t e^{-\alpha s - \Lambda(s,x)} \lambda(\phi(s,x)) Qv(\phi(s,x)) ds$$

is also absolutely continuous in t . Therefore for every compact set $\mathcal{C}$ of $[0,t^*(x))$ given $\epsilon > 0$ there exists $\delta > 0$ such that for each finite collection $[a_i , b_i]$, $i = 1 , \ldots , n$ of non-overlapping intervals of $\mathcal{C}$

$$\sum_{i=1}^{n} | J(g,v)(b_i,x) - J(g,v)(a_i,x) | < \epsilon \quad \text{if} \quad \sum_{i=1}^{n} (b_i - a_i) < \delta$$

For each $i = 1 , \ldots , n$ we have one of the possibilities below :

i) $L(g,v)(a_i,x) = J(g,v)(a_i',x)$ for some $a_i \le a_i' < b_i$ and in this case it is clear that

$L(g,v)(b_i,x) - L(g,v)(a_i,x) = L(g,v)(b_i,x) - L(g,v)(a_i',x) = L(g,v)(b_i,x) - J(g,v)(a_i',x) \le$

$J(g,v)(b_i,x) - J(g,v)(a_i',x)$. Note also that $b_i - a_i' \le b_i - a_i$.

ii) $L(g,v)(a_i,x) = L(g,v)(b_i,x)$ and in this case we define $a_i' = b_i$.

Then from the fact that $L(g,v)(t,x)$ is increasing in t we have

$$\sum_{i=1}^{n} | L(g,v)(b_i,x) - L(g,v)(a_i,x) | = \sum_{i=1}^{n} (L(g,v)(b_i,x) - L(g,v)(a_i,x))$$

$$= \sum_{i=1}^{n} (L(g,v)(b_i,x) - L(g,v)(a_i',x))$$

$$\leq \sum_{i=1}^{n} (J(g,v)(b_i,x) - J(g,v)(a_i',x))$$

$$\leq \sum_{i=1}^{n} | J(g,v)(b_i,x) - J(g,v)(a_i',x) | < \epsilon$$

since $\sum_{i=1}^{n} (b_i - a_i') \leq \sum_{i=1}^{n} (b_i - a_i) < \delta$. □

Consider $v \in B(E)$ such that $v(x) = L(g,v)(x)$, $g \in B^{ac}(E)$ for all $x \in E$. Since (cf (2) of chapter 2)

$$v(\phi(t,x)) = e^{\alpha t + \Lambda(t,x)} (L(g,v)(t,x) - \int_0^t e^{-\alpha s - \Lambda(s,x)} \lambda(\phi(s,x)) Q v(\phi(s,x)) ds)$$

it is clear from proposition A.2 that $v \in B^{ac}(E)$. Let $\mathfrak{X}$ and $\mathbb{A}$ be respectively the vector field and the extended generator of the P.D.P. (X_t) . The domain of $\mathbb{A}$, $\mathfrak{D}(\mathbb{A})$, has been characterized in Theorem 5.5 of Davis [9] . We have that if $v \in B^{ac}(E)$ then defining $v(x) := Qv(x)$ for $x \in \partial^* E$ we get $v \in \mathfrak{D}(\mathbb{A})$ and

$$\mathbb{A}v(x) = \mathfrak{X}v(x) + \lambda(x) \int_E (v(y) - v(x)) Q(dy,x) \quad , \quad \forall \ x \in \mathbb{E}$$

where

$\mathbb{E} := \{ x \in E ; \mathfrak{X}v(x) = D^+ v(x) := \frac{dv(\phi(t,x))}{dt}|_{t=0}$ exists at x $\}$ (recall that since $v(\phi(\cdot ,x))$ is absolutely continuous the derivative exists almost everywhere on $[0,t^*(x))$) .

Proposition A.3 : For $g \in B^{ac}(E)$ let v be a solution of

$$\begin{cases} v(x) = L(g,v)(x) \ , \ \forall\, x \in E \\ v \in B^{ac}(E) \end{cases}$$

Then

$$\begin{cases} \mathbb{A}v(x) - \alpha v(x) \ \geq \ 0 & , \ \forall\, x \in \mathbb{E} \\ v(x) \ \leq \ g(x) & , \ \forall\, x \in E \\ (\mathbb{A}v(x) - \alpha v(x)) (v(x) - g(x)) \ = \ 0 & , \ \forall\, x \in \mathbb{E} \end{cases}$$

Proof : Since $L(g.v)(\ . \ ,x) : [0,t^*(x)) \rightarrow R$ is absolutely continuous and increasing we have that $\frac{dL(g,v)(t,x)}{dt}$ exists almost everywhere and is positive . From

$$v(\phi(t,x)) = e^{\alpha t + \Lambda(t,x)} (\ L(g,v)(t,x) - \int_0^t e^{-\alpha s - \Lambda(s,x)} \lambda(\phi(s,x)) Q v(\phi(s,x)) ds \ )$$

we get that for $x \in \mathbb{E}$

$$\mathfrak{A}v(x) = D^+ v(x) = (\alpha + \lambda(x))v(x) + D^+ L(g,v)(0,x) - \lambda(x) Q v(x)$$

and thus

$$\mathfrak{A}v(x) + \lambda(x) \int_E (v(y) - v(x)) Q(dy;x) - \alpha v(x) = \mathbb{A}v(x) - \alpha v(x) \ = \ D^+ L(g,v)(0,x) \ \geq \ 0 \ .$$

From $v(x) = L(g,v)(x)$ it is immediate that $v(x) \leq g(x)$, $\forall\, x \in E$.

Now if for $x \in \mathbb{E}$, $L(g,v)(0,x) = v(x) < g(x) = J(g,v)(0,x)$ then from continuity of $L(g,v)(\ . \ ,x)$ and $J(g,v)(\ . \ ,x)$ we have that for some $\epsilon > 0$, $L(g,v)(t,x) < J(g,v)(t,x)$ for all $t \in [0,\epsilon)$ and therefore $L(g,v)(t,x)$ is constant on $[0,\epsilon)$. It means that if $v(x) < g(x)$ then $D^+ L(g,v)(0,x) = 0 = \mathbb{A}v(x) - \alpha v(x)$ for all $x \in \mathbb{E}$, proving the proposition . □

Chapter 3

Impulsive control of piecewise-deterministic processes

1 . Preliminaries

In the previous chapter we presented some characterization results and a numerical technique for the optimal stopping problem of a P.D.P. . In this chapter we will apply these results to the impulsive control problem . Our approach is analogous to the classical theory of impulsive control introduced by Bensoussan and Lions [1] but we charaterize the value function by some recursive methods rather than by variational inequalities . From the general theory of impulsive control of right processes (see [24]) such problems can be written as a sequence of optimal stopping and therefore by using the results of the previous chapter we can develop a numerical technique for computing optimal impulsive controls for P.D.P.'s .

This chapter is organized in the following way . We keep the same notations , defintions and general assumptions as in chapter 2 . In section 2 we give the formulation of the problem . In section 3 we study the impulsive control of P.D.P.'s under general conditions . We show that by iteration of the single-jump-or-intervention operator we obtain a sequence of functions converging to the value function of the problem . In section 4 we present a numerical technique for the impulsive control of P.D.P.'s . This technique consists of solving a sequence of one dimensional minimizations . We conclude that section by presenting some numerical examples . Finally in the appendix some proofs which were omitted from the main text are shown as well as a connection between the representation results of section 3 and the Bellman inequalities as defined by Yushkevich (see chapter 1 , equations (6)) .

2 . Problem formulation

2.1. Construction of the canonical space

In order to define the impulsive control problem in a proper way we need to use the idea of "renaissance de processus de Markov" (Meyer [29]) . Basically the idea is that we have a Markov process "running" until the first intervention time when the process is killed and another Markov process starts from some point $z \in E$; the procedure is then repeated . The reader is referred to Lepeltier and Marchal [24] for a more general and detailed explanation of this concepts for impulsive control . We shall now describe the construction of the canonical space which we shall use for the formulation of the impulsive control problem . Let $\{ \Delta \}$ be a cemetery state and let $\hat{\Omega}$ be the space of functions $\hat{\omega} : R_+ \to E \cup \{\Delta\}$ which satisfies one of the following properties :

- $\hat{\omega} \in \Omega$; in this case we define $\eta(\hat{\omega}) := \infty$.

- for some $a \in R_+$ $\hat{\omega}_t$ is right continuous with left limit in $[0,a]$ and $\hat{\omega}_t = \Delta$ for $t \in (a,\infty)$; we define $\eta(\hat{\omega}) := a$.

- $\hat{\omega}_t = \Delta$ for every $t \in R_+$; in this case we denote $\hat{\omega}$ by Δ and define $\eta(\Delta) := 0$.

For $\hat{\omega} \in \hat{\Omega}$ define $\hat{x}_t(\hat{\omega}) := \hat{\omega}_t$, $\hat{\mathcal{F}}^0_t := \sigma\{ \hat{x}_s ; s \leq t \}$, $\hat{\mathcal{F}}^0 := \hat{\mathcal{F}}_\infty$ and $\hat{\mathcal{F}}$ the universal completion of $\hat{\mathcal{F}}^0$. Let $(\hat{\Omega}_i,\hat{\mathcal{F}}_i)$ be a copy of the measurable space $(\hat{\Omega},\hat{\mathcal{F}})$. Now define $\mathcal{W} := \prod_{i=1}^{\infty} \hat{\Omega}_i$, $\mathcal{G}^0 := \sigma\{ \prod_{i=1}^{\infty} \hat{\mathcal{F}}_i \}$ and $\mathcal{G}$ the universal completion of $\mathcal{G}^0$. We will denote the elements of $\mathcal{W}$ by $w = (\hat{\omega}_1,\hat{\omega}_2,\dots)$. Let $\mathcal{W}_k := \prod_{i=1}^{k} \hat{\Omega}_i$, $\mathcal{G}^0_k := \sigma\{ \prod_{i=1}^{k} \hat{\mathcal{F}}_i \}$, $\mathcal{G}_k$ the universal completion of $\mathcal{G}^0_k$ and $w_k : \mathcal{W} \to \mathcal{W}_k$ the projection map $w_k(w) = (\hat{\omega}_1, \dots ,\hat{\omega}_k)$. We now set

$$\tau_k(w) = \tau_k(w_k) := \sum_{i=1}^{k} \eta(\hat{\omega}_i) \qquad (\infty + \infty := \infty)$$

$$\tau_\infty(w) := \lim_{k \to \infty} \tau_k(w)$$

and define the path $(Y_t(w))_{t\geq 0}$ by

$Y_0(w) := \hat{x}_0(\hat{\omega}_1)$

$$Y_t(w) := \begin{cases} \hat{x}_{t-\tau_i(w)}(\hat{\omega}_{i+1}) & \text{if } \tau_i(w) < t \leq \tau_{i+1}(w) \\ \Delta & \text{if } t > \tau_\infty(w) \end{cases}$$

2.2. Definition of the admissible stategies

Let Γ be an analytic set of ExE and let $\Gamma(x)$ denote the x-section of Γ , i.e. , $\Gamma(x) = \{ y \in E : (x,y) \in \Gamma \}$. We impose the following conditions :

i) $\{ \Gamma(x) \}_{x\in E}$ is a family of non-empty sets of E .

ii) for every $x \in E$ and $y \in \Gamma(x)$, $\Gamma(y) \subset \Gamma(x)$.

We define the class of admissible strategies $\mathbb{S}$ as the set of sequences $\mathcal{S} := (S_n , R_n)_{n=1}^{\infty}$ which verify :

a) for every $n \geq 1$, $R_n(\cdot) : \mathcal{W}_n \rightarrow E\cup\{\Delta\}$ is a universally measurable random variable such that

$$R_n(w_n) \begin{cases} = \Delta & \text{if } \tau_n(w_n) = \infty \\ \in \Gamma(\hat{x}_{\eta(\hat{\omega}_n)}(\hat{\omega}_n)) & \text{otherwise} \end{cases}$$

b) for every $n \geq 2$, $S_n(\cdot , \cdot) : \mathcal{W}_{n-1}x\Omega \rightarrow R_+\cup\{\infty\}$ is such that for $\omega \in \Omega$ fixed , $S_n(\cdot ,\omega)$ is a universally measurable function and for $w_{n-1} \in \mathcal{W}_{n-1}$ fixed , $S_n(w_{n-1}, \cdot)$ is an $\mathcal{F}_t$-stopping time of a P.D.P. (X_t) starting from $R_{n-1}(w_{n-1})$. For n=1 , S_1 is an $\mathcal{F}_t$-stopping time of a P.D.P. (X_t) starting from x .

c) for every $n \geq 2$, $S_n(w_{n-1}, \cdot)$ is zero if there is $k \leq n-1$ such that $\eta(\hat{\omega}_k) = \infty$.

d) for every $n \geq 2$, S_n is strictly positive except for case c) .

Remark : Note that given $w_{n-1} \in \mathcal{W}_{n-1}$, $(R_{n-1}(w_{n-1}) , S_{n-1}(w_{n-1}, \cdot))$ defines a mapping $\mathcal{Y}$ from Ω into $\hat{\Omega}$ in the following way . If $R_{n-1}(w_{n-1}) = \Delta$ then

$\mathcal{Y}(\omega) = \Delta \in \hat{\Omega}$ for all $\omega \in \Omega$.

Consider now $R_{n-1}(w_{n-1}) \neq \Delta$. If $S_{n-1}(w_{n-1},\omega) = \infty$ then $\mathcal{Y}(\omega) = \omega \in \hat{\Omega}$; otherwise

$$\hat{x}_t(\mathcal{Y}(\omega)) = \begin{cases} x_t(\omega) & \text{if } t \le S_n(w_{n-1},\omega) \\ \Delta & \text{if } t > S_n(w_{n-1},\omega) \end{cases}$$

and thus $\mathcal{Y}(\omega) \in \hat{\Omega}$. From the defintion of $\mathcal{Y}$ we have

$$\eta(\mathcal{Y}(\omega)) = S_n(w_{n-1},\omega) \qquad \forall\ \omega \in \Omega\ .$$

Suppose $\mathcal{S} = (S_n , R_n)_{n=1}^{\infty}$ satisfy the conditions a) , b) , c) and d) above . Then S_1 and P_x will induce a probability measure μ_x^1 on $(\hat{\Omega}_1 , \hat{\mathcal{F}}_1)$. Given w_{n-1} we define $\mu_x^n(\{\Delta\} ; w_{n-1}) = 1$ if $\tau_{n-1}(w_{n-1}) = \infty$; otherwise $S_n(w_{n-1}, \cdot)$ and $P_{R_{n-1}(w_{n-1})}$ will induce a probability measure $\mu_x^n(\cdot ,w_{n-1})$ on $(\hat{\Omega}_n , \hat{\mathcal{F}}_n)$. The family (μ_x^n) defines a probability measure $P_x^{\mathcal{S}}$ on $(\mathcal{W} , \mathcal{G})$. The last condition for $\mathcal{S}$ be admissible is the following :

e) $\tau_\infty = \infty$ $P_x^{\mathcal{S}}$ - a.s. .

2.3. Problem statement

Let $f : E \to R_+$ and $c : \Gamma \to R_+$ be bounded Borel functions satisfying the following conditions :

i) $c(x,y) \ge c_0 > 0$ for some $c_0 \in R_+$ and every $(x,y) \in \Gamma$; $c(x,\Delta) = 0$, $\forall\ x \in E$.

ii) for every $x \in E$, $y \in \Gamma(x)$ and $z \in \Gamma(y) \subset \Gamma(x)$, $c(x,y) + c(y,z) \ge c(x,z)$.

For each admissible strategy $\mathcal{S} \in \mathbb{S}$ we associate the following cost :

$$V^{\mathcal{S}}(x) = E_x^{\mathcal{S}}\Big(\int_0^{\tau_\infty} e^{-\alpha s} f(Y_s)ds \ + \ \sum_{i=1}^{\infty} e^{-\alpha \tau_i} c(Y_{\tau_i}, Y_{\tau_i}+) \ \Big)$$

where $\alpha > 0$ and the expectation is over $\mathcal{W}$ with probability measure $P_x^{\mathcal{S}}$. The value function of the impulsive control problem is :

$$\hat{\rho}'(x) = \inf_{\mathcal{S} \in \mathbb{S}} V^{\mathcal{S}}(x)$$

3 . Impulsive control problem of P.D.P.'s

3.1. General results on impulsive control of P.D.P.'s

For $(t,x) \in \mathbb{B}$ and v_1 , v_2 in $B^*(E)$ we re-define J , K and define M , $\mathcal{A}$ in the following way :

a) $$J(v_1,v_2)(t,x) := E_x\Big(\int_0^{T_1 \wedge t} e^{-\alpha s} f(\phi(s,x))ds \; + \; e^{-\alpha t} v_1(\phi(t,x)) 1_{\{T_1 > t\}} + e^{-\alpha T_1} v_2(Z_1) 1_{\{T_1 \le t\}} \Big)$$

b) $$Kv_2(x) := E_x\Big(\int_0^{T_1} e^{-\alpha s} f(\phi(s,x))ds + e^{-\alpha T_1} v_2(Z_1) \Big) \quad (= J(v_1,v_2)(\infty,x))$$

c) $$Mv_2(x) := \inf_{y \in \Gamma(x)} \{ c(x,y) + v_2(y) \}$$

d) $$\mathcal{A}v_2(x) := \inf_{\tau \in \mathcal{M}_\infty} E_x\Big(\int_0^{\tau} e^{-\alpha s} f(X_s)ds + e^{-\alpha \tau} Mv_2(X_\tau) \Big)$$

Recall that the definition of L is :

e) $$L(v_1,v_2)(x) := \{ \inf_{0 \le t < t^*(x)} J(v_1,v_2)(t,x) \} \wedge Kv_2(x)$$

Remark : Note that the above operators are monotone in $B^*(E)$.

Since M maps $B^*(E)$ into $B^*(E)$ (see Bertsekas-Shreve [4] prop. 7.47) we have from the results for the optimal stopping of P.D.P.'s (see chapter 2 proposition 1 and corollary 1) that $\mathcal{A}$ also maps $B^*(E)$ into $B^*(E)$. Define

$$h(x) := E_x\Big(\int_0^{\infty} e^{-\alpha s} f(X_s)ds \Big)$$

Remark : Note that h(x) corresponds to the cost of "no intervention" strategy .

The following proposition can be easily proved (see appendix) .

Proposition 1 : The function h is the smallest solution of the system

$$\begin{cases} v = Kv \\ v \geq 0 \quad , \quad v \in B(E) \end{cases}$$

Moreover if we define $h_{n+1} = Kh_n$, $h_0 = 0$ then $h_n \uparrow h$ as $n \to \infty$.

Since $\mathcal{A}$ is monotone , it is clear that $\mathcal{A}^n h$, $n = 0 , 1 , \ldots$ is a decreasing sequence of functions in $B^*(E)$ (note that for $n = 1$, $\mathcal{A}h \leq h$ by definition of the operator $\mathcal{A}$) . Therefore we have that $\lim_{n\to\infty} \mathcal{A}^n h$ exists and from lemma 7.30(2) of [4] it is in $B^*(E)$. From the general results of Lepeltier and Marchal [24] on impulsive control of right processes and the fact that P.D.P.'s are right processes we have the following Markovian form of the value function of the impulsive control problem :

Proposition 2 : $\hat{\rho}'$ is the biggest solution of the system

$$\begin{cases} v = \mathcal{A}v \\ v \in B^*(E) \end{cases}$$

Moreover $\hat{\rho}' = \lim_{n\to\infty} \mathcal{A}^n h$.

Proof : The second part of the statement follows from proposition 22 and 23 of Lepeltier-Marchal [24] . If we define

$$\mathcal{A}^+ v(x) := \inf_{\substack{\tau \in \mathcal{M}_{b\infty} \\ \tau > 0}} E_x\left(\int_0^\tau e^{-\alpha s} f(X_s)ds + e^{-\alpha\tau} M\rho(X_\tau)\right) \quad \text{and}$$

$$\hat{\rho}'^+(x) := \inf_{\substack{\vartheta \in \mathbf{S} \\ S_1 > 0}} V^\vartheta(x)$$

then it was shown in [24] that (proposition 22) $\hat{\rho}'^+(x) = \lim_{n\to\infty} (\mathcal{A}^+)^n h(x)$ and (proposition 23) $\hat{\rho}'(x) = M\hat{\rho}'^+(x) \wedge \hat{\rho}'^+(x) = \lim_{n\to\infty} \left(M(\mathcal{A}^+)^{n-1} h(x) \wedge (\mathcal{A}^+)^n h(x) \right)$; by virtue of condition ii) of subsection 2.2 and condition ii) of subsection 2.3 we can show (see appendix) that

$\mathcal{A}^n h(x) = M(\mathcal{A}^+)^{n-1}h(x) \wedge (\mathcal{A}^+)^n h(x)$. The first part is an immediate consequence of the second one . Indeed from the bounded convergence theorem it is clear that

$$\hat{\rho}'(x) = \inf_n \mathcal{A}^{n+1}h(x) = \inf_n \inf_{\tau \in \mathcal{M}_\infty} E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds + e^{-\alpha\tau} M\mathcal{A}^n h(X_\tau) \Big)$$

$$= \inf_{\tau \in \mathcal{M}_\infty} E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds + e^{-\alpha\tau} M(\inf_n \mathcal{A}^n h)(X_\tau) \Big) = \mathcal{A}\hat{\rho}'(x)$$

Moreover if $v = \mathcal{A}v$, $v \in B^*(E)$ then clearly $v = \mathcal{A}v \leq h$ and thus using monotonicity of $\mathcal{A}$ again we have $v = \mathcal{A}^n v \leq \mathcal{A}^n h \overset{n}{\rightarrow} \infty \hat{\rho}'$. □

Remark : The function $\mathcal{A}^n h$ can be regarded as the value function of the impulsive control problem which allows only n interventions . Note also that $\hat{\rho}'$ is the limit of a sequence of optimal stopping problems .

For $v \in B^*(E)$ define the operator $\hat{L}(v)(x) := L(Mv,v)(x)$. This operator when applied to v can be seen as the value function of the single-jump-or-intervention problem with gain function v . The next proposition links the operator $\hat{L}$ with $\hat{\rho}'$.

Proposition 3 : The cost function $\hat{\rho}'$ is the biggest solution of the system

$$\begin{cases} v = \hat{L}(v) \\ v \leq h \quad , \quad v \in B^*(E) \end{cases} \qquad (1)$$

Moreover if we define $\hat{\rho}'_{n+1} = \hat{L}(\hat{\rho}'_n)$, $\hat{\rho}'_0 = h$ then $\hat{\rho}'_n \downarrow \hat{\rho}'$ as $n \rightarrow \infty$.

Proof : By induction arguments and monotonicity of the operator $\hat{L}$, it is clear that the sequence of functions $\hat{\rho}'_n$ is decreasing in $B^*(E)$ (note that $\hat{\rho}'_1(x) = \hat{L}(h)(x) \leq Kh(x) = h(x)$ from prop. 1) . Therefore $\ell(x) := \lim_{n \rightarrow \infty} \hat{\rho}'_n(x)$ exists and is in $B^*(E)$. From the same arguments as for the proof of the first part of proposition 2 we get that ℓ is the biggest solution of system (1) . It

remains to show that $\ell = \hat{\rho}'$. From the optimal stopping results (see chapter 2 prop. 1 and corollary 1) and the fact that $\hat{\rho}'=\mathcal{A}\hat{\rho}'$ we obtain that $\hat{\rho}'= L(M\hat{\rho}',\hat{\rho}') = \hat{L}(\hat{\rho}')$ (from optimal stopping) , $\hat{\rho}' \leq h$ (from $\hat{\rho}'=\mathcal{A}\hat{\rho}'$ and definition of the operator $\mathcal{A}$) . So $\hat{\rho}'$ is a solution of (1) which implies that $\hat{\rho}' \leq \ell$ since ℓ is the biggest solution of (1) . If we show that $\ell = \mathcal{A}\ell$ then from proposition 2 , $\ell \leq \hat{\rho}'$ and the result will be proved . So all we need to show is that $\ell = \mathcal{A}\ell$. We will prove the following steps :

step 1 : $\ell \leq \mathcal{A}\ell$.

step 2 : $\ell \geq \mathcal{A}\ell$.

proof of step 1 : From the theoretical results for optimal stopping we know that $\mathcal{A}\ell = \inf_n \ell_n$ where $\ell_{n+1} := L(M\ell,\ell_n)$, $\ell_0 := M\ell$. We will use induction in n to prove that $\ell \leq \ell_n$ for all $n \in \mathbb{N}$. For $n = 0$ we have , by virtue of $\ell = \hat{L}(\ell) = L(M\ell,\ell)$, that $\ell \leq M\ell = \ell_0$. Suppose now that $\ell \leq \ell_n$. Then $\ell = \hat{L}(\ell) = L(M\ell,\ell) \leq L(M\ell,\ell_n) = \ell_{n+1}$ proving that $\ell \leq \ell_n$ for all $n \in \mathbb{N}$. Taking the infimum over n we obtain the desired result .

proof of step 2 : From the bounded convergence theorem we have

$$\mathcal{A}\ell(x) = \inf_{\tau \in \mathcal{M}_\infty} E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds + e^{-\alpha\tau} M(\inf_n \hat{\rho}_n')(X_\tau)\Big) =$$

$$= \inf_n \Big(\inf_{\tau \in \mathcal{M}_\infty} E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds + e^{-\alpha\tau} M\hat{\rho}_n'(X_\tau)\Big)\Big) = \inf_n \mathcal{A}\hat{\rho}_n'(x)$$

We will show by induction that $\mathcal{A}\hat{\rho}_n' \leq \hat{\rho}_{n+1}'$ for all $n \in \mathbb{N}$. For $n = 0$ we get $\mathcal{A}\hat{\rho}_0' = \mathcal{A}h \leq h$ (by definition of $\mathcal{A}$) and by virtue of the results for optimal stopping , $\mathcal{A}h = L(Mh,\mathcal{A}h)$. Thus $\mathcal{A}\hat{\rho}_0' = \mathcal{A}h = L(Mh,\mathcal{A}h) \leq L(Mh,h) = \hat{L}(h) = \hat{\rho}_1'$. Suppose $\mathcal{A}\hat{\rho}_{n-1}' \leq \hat{\rho}_n'$. Then $\mathcal{A}\hat{\rho}_n' \leq \mathcal{A}\hat{\rho}_{n-1}'$ (since $\hat{\rho}_m'$, $m \in \mathbb{N}$ is decreasing) $\leq \hat{\rho}_n'$ and $\mathcal{A}\hat{\rho}_n' = L(M\hat{\rho}_n',\mathcal{A}\hat{\rho}_n') \leq L(M\hat{\rho}_n',\hat{\rho}_n') = \hat{L}(\hat{\rho}_n') = \hat{\rho}_{n+1}'$ proving that $\mathcal{A}\hat{\rho}_n' \leq \hat{\rho}_{n+1}'$, $\forall\ n \in \mathbb{N}$. Hence $\mathcal{A}\ell = \inf_n \mathcal{A}\hat{\rho}_n' \leq \inf_n \hat{\rho}_{n+1}' = \ell$. □

Remark : The function $\hat{\rho}_n'$ can be understood as the value function of the impulsive control problem of a P.D.P. where only n jumps plus interventions are allowed and after that there are no further interventions .

Suppose now that $v_0 \in B^*(E)$ and $v_0 \geq h$. Define $v_{n+1} := \hat{L}(v_n)$.

Proposition 4 : $\lim_{n\to\infty} v_n(x) = \hat{\rho}'(x)$ for all $x \in E$.

Remark : This result shows that we only need to find an upper bound for h in order to get a sequence of functions converging to $\hat{\rho}'$.

Proof : There exists $\nu \in R_+$ such that $h \leq v_0 \leq h + \nu$. Define $q_0 := h + \nu$ and $q_{n+1} := \hat{L}(q_n)$. We will show the following steps :

Step 1 : $\hat{\rho}_n' \leq v_n \leq q_n$ for all $n \in \mathbb{N}$.

Step 2 : $q_n(x) \leq h(x) + E_x(e^{-\alpha T_n})\nu$ for all $x \in E$ and $n \in \mathbb{N}$.

Step 3 : $q_n \downarrow q \in B^*(E)$ and $q = \hat{L}(q)$.

From step 1 and step 3 we have $\hat{\rho}' = \lim_{n\to\infty} \hat{\rho}_n' \leq \lim_{n\to\infty} q_n = q$. From step 2 , the bounded convergence theorem and that $T_n \to \infty$ as $n \to \infty$ P_x-a.s. we obtain $q(x) = \lim_{n\to\infty} q_n(x) \leq h(x)$. So $q = \hat{L}(q)$, $q \in B^*(E)$ and $q \leq h$ which implies from prop. 3 that $q \leq \hat{\rho}'$. Thus $q = \hat{\rho}'$ and hence from step 1 , $\lim_{n\to\infty} v_n(x) = \hat{\rho}'(x)$. It remains to show steps 1 to 3 .

proof of step 1 : It is immediate since by definition $\hat{\rho}_0' = h \leq v_0 \leq h + \nu = q_0$ and $\hat{L}$ is monotone .

proof of step 2 : We use induction in n to show this result . For $n = 0$ the result is clear since $T_0 = 0$. Suppose it holds for n . Then defining $\chi(z) := E_z(e^{-\alpha T_n})$ we have

$$q_{n+1}(x) = \hat{L}(q_n)(x) \leq Kq_n(x) = E_x\Big(\int_0^{T_1} e^{-\alpha s} f(\phi(s,x))ds + e^{-\alpha T_1} q_n(Z_1)\Big)$$

$$\leq E_x\Big(\int_0^{T_1} e^{-\alpha s} f(\phi(s,x))ds + e^{-\alpha T_1}\big(h(Z_1) + \chi(Z_1)\nu\big)\Big)$$

$$= Kh(x) + E_x(e^{-\alpha T_1}(e^{-\alpha(T_{n+1}-T_1)}/\mathcal{F}_{T_1}))\nu = h(x) + E_x(e^{-\alpha T_{n+1}})\nu$$

where we have used above proposition 1 and the fact that

$$\chi(Z_1) = E(e^{-\alpha(T_{n+1}-T_1)} / \mathcal{F}_{T_1}) \qquad P_x \text{ -a.s.}$$

which follows from the strong Markov property and time homogeneity .

proof of step 3 : It is clear that $q_n \in B^*(E)$ and q_n , $n \in \mathbb{N}$ is a decreasing sequence of functions (note that

$$q_1(x) = \hat{L}(h+\nu)(x) \leq K(h+\nu)(x) = Kh(x) + E_x(e^{-\alpha T_1})\nu \leq h(x) + \nu = q_0(x)$$

where we have used prop. 1 to get $Kh = h$) . Therefore $q = \lim_{n\to\infty} q_n$ exists and is in $B^*(E)$. Using the same arguments as in the proof of the first part of prop. 2 we obtain that $q = \hat{L}(q)$ completing the proof of this step . □

3.2. Uniform convergence of $\mathcal{A}^n h$ to $\hat{\rho}'$

In [42] chapter 9 (see also [43] theorems 4 and 5) Zabczyk studies the properties of the operator $\mathcal{A}$ when the process is Feller and there is continuity of the parameters c , f and of the operator M . Theorem 9.2 of [42] can be readily modified to show the following proposition (see proof in appendix) . Let f be bounded by a_3 and recall that $c(x,y) \geq c_0$ for all $(x,y) \in \Gamma$. Define

$$\upsilon := \frac{1}{1 + \frac{\alpha c_0}{a_3}} .$$

Proposition 5 : For all $x \in E$ and $m \in \mathbb{N}$

$$0 \leq \mathcal{A}^{m-1}h(x) - \mathcal{A}^m h(x) \leq \upsilon^m \mathcal{A}^{m-1}h(x) .$$

Remark : It is immediate from proposition 5 that for all $x \in E$ and $m \in \mathbb{N}$

$$0 \leq \mathcal{A}^m h(x) - \hat{\rho}'(x) \leq \frac{\upsilon^{m+1}}{1-\upsilon} h(x) \leq \frac{\upsilon^{m+1}}{\alpha(1-\upsilon)} a_3 .$$

3.3. Construction of ϵ-optimal strategies

Let $\delta > 0$ and define the following functions :

i) $r_\delta(\cdot) : E \to E$ is an analytically measurable function such that $Gr(r_\delta) \subset \Gamma$ (where $Gr(r_\delta) := \{ (x,r_\delta(x)) ; x \in E \}$) and for all $x \in E$

$c(x,r_\delta(x)) + \hat{\rho}'(r_\delta(x)) \leq M\hat{\rho}'(x) + \frac{\delta}{2}$ (see prop. 7.50 of Bertsekas-Shreve [4] for selection theorem of lower-semianalytic functions)

ii) $U_\delta^x \in \mathcal{M}_\infty$ such that

$$E_x\left(\int_0^{U_\delta} e^{-\alpha s} f(X_s)ds + e^{-\alpha U_\delta} M\hat{\rho}'(X_{U_\delta}) \right) \leq \hat{\rho}'(x) + \frac{\delta}{2}$$

(see section 3 of chapter 2 for the construction of ϵ-optimal stopping times of P.D.P.'s with lower-semianalytic gain function) .

For n fixed we define the following strategy $\vartheta_{n\delta}$:

$S_1(\omega_1) := U_\delta^x(\omega_1)$, $\omega_1 \in \Omega$

$$R_1(\hat{\omega}_1) := \begin{cases} r_\delta(\hat{x}_{\eta(\hat{\omega}_1)}(\hat{\omega}_1)) & \text{if } \eta(\hat{\omega}_1) < \infty \\ \Delta & \text{otherwise} \end{cases}$$

and for $i = 2 , \ldots, n$ with $\kappa := R_{i-1}(w_{i-1})$:

$$S_i(w_{i-1},\omega_i) := \begin{cases} U_\delta^\kappa(\omega_i) & \text{if } \kappa \neq \Delta \\ 0 & \text{otherwise} \end{cases} \quad , \ \omega_i \in \Omega$$

$$R_i(w_i) := \begin{cases} r_\delta(\hat{x}_{\eta(\hat{\omega}_i)}(\hat{\omega}_i)) & \text{if } \eta(\hat{\omega}_i) < \infty \text{ and } \kappa \neq \Delta \\ \Delta & \text{otherwise} \end{cases}$$

$S_{n+1}(w_n,\omega_{n+1})$ is infinity if $R_n(w_n) \neq \Delta$, 0 otherwise . All the others R_j , $j \geq n + 1$ are equal to Δ and S_k , $k \geq n + 2$ are equal to zero .

Due to the Markovian form of this strategy (S_{i+1},R_{i+1} depends only on R_i) we get that for $n = 1,2, \ldots$

$$V^{\vartheta_{n\delta}}(x) = E_x^{\vartheta_{n\delta}}\Big(\int_0^{\tau_\infty} e^{-\alpha s} f(Y_s)ds + \sum_{i=1}^{\infty} e^{-\alpha \tau_i} c(Y_{\tau_i}, Y_{\tau_i}^+) \Big)$$

$$= E_X\Big(\int_0^{U_\delta} e^{-\alpha s} f(X_s)ds \; + e^{-\alpha U_\delta}\big(c(X_{U_\delta}, r_\delta(X_{U_\delta})) + V^{\vartheta_{n-1\delta}}(r_\delta(X_{U_\delta})) \big) \Big)$$

where we have defined $V^{\vartheta_{0\delta}} := h$, $X_\infty := \Delta$, $r_\delta(\Delta) := \Delta$ and $V^{\vartheta_{n\delta}}(\Delta) := 0$.

Remark : Note that we cannot guarantee $\vartheta_{n\delta}$ is in **S** because condition d) of subsection 2.2. may not be satisfied . We leave it open to be verified for each particular model .

From the remark of proposition 5 we have that

$0 \le h - \hat{\rho}' \le \frac{\upsilon}{\alpha(1-\upsilon)} a_3$. Let $b := \frac{\upsilon}{\alpha(1-\upsilon)} a_3$.

Proposition 6 : $V^{\vartheta_{n\delta}}(x) \le \hat{\rho}'(x) + n\delta + E_x^{\vartheta_{n\delta}}(e^{-\alpha \tau_n})b$.

Proof : We use induction in n . For n = 1 we have

$$V^{\vartheta_{1\delta}}(x) = E_X\Big(\int_0^{U_\delta} e^{-\alpha s} f(X_s)ds + e^{-\alpha U_\delta}\big(c(X_{U_\delta}, r_\delta(X_{U_\delta})) + h(r_\delta(X_{U_\delta})) \big) \Big)$$

$$\le E_X\Big(\int_0^{U_\delta} e^{-\alpha s} f(X_s)ds + e^{-\alpha U_\delta}\big(c(X_{U_\delta}, r_\delta(X_{U_\delta})) + \hat{\rho}'(r_\delta(X_{U_\delta})) + b \big) \Big)$$

$$\le E_X\Big(\int_0^{U_\delta} e^{-\alpha s} f(X_s)ds + e^{-\alpha U_\delta} M\hat{\rho}'(X_{U_\delta}) \Big) + \frac{\delta}{2} \; + E_X(e^{-\alpha U_\delta})b$$

$$\le \hat{\rho}'(x) + \delta + E_x^{\vartheta_{1\delta}}(e^{-\alpha \tau_1})b$$

Suppose now that the result holds for n . Then defining

$\chi(z) = E_z^{\vartheta_{n\delta}}(e^{-\alpha \tau_n})$

we have

$$V^{\vartheta_{(n+1)\delta}}(x) = E_x\Big(\int_0^{U_\delta} e^{-\alpha s} f(X_s)ds + e^{-\alpha U_\delta}\big(c(X_{U_\delta}, r_\delta(X_{U_\delta})) + V^{\vartheta_{n\delta}}(r_\delta(X_{U_\delta}))\big)\Big)$$

$$\leq E_x\Big(\int_0^{U_\delta} e^{-\alpha s} f(X_s)ds + e^{-\alpha U_\delta}\big(c(X_{U_\delta}, r_\delta(X_{U_\delta})) + \hat{\rho}'(r_\delta(X_{U_\delta}))$$

$$+ n\delta + \chi(r_\delta(X_{U_\delta}))b\big)\Big)$$

$$\leq E_x\Big(\int_0^{U_\delta} e^{-\alpha s} f(X_s)ds + e^{-\alpha U_\delta} M\hat{\rho}'(X_{U_\delta})\Big) + \frac{\delta}{2}$$

$$+ n\delta + E_x\big(e^{-\alpha U_\delta}\, \chi(r_\delta(X_{U_\delta}))b\big)$$

$$\leq \hat{\rho}'(x) + \frac{\delta}{2} + \frac{\delta}{2} + n\delta + E_x\big(e^{-\alpha U_\delta}\, \chi(r_\delta(X_{U_\delta}))b\big)$$

$$= \hat{\rho}'(x) + (n+1)\delta + E_x\big(e^{-\alpha U_\delta}\, \chi(r_\delta(X_{U_\delta}))b\big)$$

But since

$$E_x\big(e^{-\alpha U_\delta}\, \chi(r_\delta(X_{U_\delta}))\big) = E_x^{\vartheta_{(n+1)\delta}}\big(e^{-\alpha \tau_1}\, \chi(R_1)\big)$$

and

$$\chi(R_1) = E_x^{\vartheta_{(n+1)\delta}}\big(e^{-\alpha(\tau_{n+1} - \tau_1)} / R_1, \tau_1\big) \qquad P_x^{\vartheta_{(n+1)\delta}}\text{-a.s. on } \{\tau_1 < \infty\}$$

we get that

$$E_x^{\vartheta_{(n+1)\delta}}\big(e^{-\alpha \tau_1}\, \chi(R_1)\big) = E_x^{\vartheta_{(n+1)\delta}}\big(e^{-\alpha \tau_{n+1}}\big)$$

which completes the proof of the proposition . □

We can now show that given $\epsilon > 0$ we can find n and δ such that $\vartheta_{n\delta}$ is ϵ-optimal for the impulsive control problem .

Proposition 7 : Choose $n \in \mathbb{N}$ and $\delta > 0$ such that $n\delta + \frac{b}{c_0 n}\big(\frac{a_3}{\alpha} + n\delta + b\big) \leq \epsilon$. Then

$$V^{\vartheta_{n\delta}}(x) \leq \hat{\rho}'(x) + \epsilon .$$

Proof : From proposition 6 it is clear that

$V^{\vartheta_{n\delta}}(x) \leq \frac{a_3}{\alpha} + n\delta + b$ (recall that $\hat{\rho}'(x) \leq h(x) \leq \frac{a_3}{\alpha}$)

Note that

$$nc_0 E_x^{\vartheta_{n\delta}}(e^{-\alpha\tau_n}) \leq E_x^{\vartheta_{n\delta}} \left(\int_0^{\tau_\infty} e^{-\alpha s} f(Y_s) ds + \sum_{i=1}^{\infty} e^{-\alpha\tau_i} c(Y_{\tau_i}, Y_{\tau_i}^+) \right)$$

$$= V^{\vartheta_{n\delta}}(x) \leq \frac{a_3}{\alpha} + n\delta + b$$

So $E_x^{\vartheta_{n\delta}}(e^{-\alpha\tau_n}) \leq \frac{1}{nc_0}(\frac{a_3}{\alpha} + n\delta + b)$. Thus from proposition 6 again we get

$V^{\vartheta_{n\delta}}(x) \leq \hat{\rho}'(x) + n\delta + \frac{b}{nc_0}(\frac{a_3}{\alpha} + n\delta + b) \leq \hat{\rho}'(x) + \epsilon$,

proving the proposition . □

4 . Discretization for the impulsive control of a P.D.P.

4.1. Introduction

In this section we extend the definition of the sets $\Gamma(y)$ to the points y in $\partial^* E$ and assume that $\Gamma(y)$ is non-empty in $\tilde{E}$ for $y \in \partial^* E$. Conditions i) and ii) of subsection 2.2 are modified replacing E by $\tilde{E}$. Let $B^*(\tilde{E})$ ($B(\tilde{E})$, $C(\tilde{E})$) be the space of real valued bounded lower-semianalytic (Borel , continuous respectively) functions on $\tilde{E}$. We extend the domain of the operator $\mathcal{A}$ to $B^*(\tilde{E})$ in the following way : for $v \in B(\tilde{E})$ and $x \in E$, $\mathcal{A}v(x)$ is defined as before and for $x \in \partial^* E$, $\mathcal{A}v(x) = Mv(x) \wedge E_x(\mathcal{A}v(Z_1))$ where the process jumps instantaneously when it starts from any point in $\partial^* E$. Note that this definition is consistent since $Z_1 \in E$ and we first evaluate $\mathcal{A}v(x)$ for $x \in E$. We can easily show that proposition 1 and 5 still hold replacing E by $\tilde{E}$. Define for $x \in \tilde{E}$ $\hat{\rho}(x) := \lim_{n \to \infty} \mathcal{A}^n h(x)$. Note that $\hat{\rho}'(x) = \hat{\rho}(x)$ for $x \in E$ but for $x \in \partial^* E$, $\hat{\rho}(x)$ may be less than $\hat{\rho}'(x)$ ($= E_x(\hat{\rho}'(Z_1))$) .

4.2. Assumptions

Let $2^{\tilde{E}}$ denote the collection of all non-empty compact sets of $\tilde{E}$. The Hausdorff metric $d(\, . \, , \, . \,)$ in $2^{\tilde{E}}$ is defined in the following way (see Bertsekas-Shreve [4] appendix C) : for A , B , in $2^{\tilde{E}}$ and x in $\tilde{E}$

$$d(x,A) := \min_{a \in A} | x - a |$$

$$d(A,B) := \max \{ \max_{a \in A} d(a,B) \ , \ \max_{b \in B} d(b,A) \}$$

For $x \in \tilde{E}$ let $\tilde{\Gamma}(x)$ be the closure of $\Gamma(x)$ in $\tilde{E}$. In addition to assumptions 1 , 2 , 3 and 5 of subsection 4.2. , chapter 2 we impose the following ones :

1) $\tilde{\Gamma}(\, . \,)$ is a continuous set-valued mapping from $\tilde{E}$ into $2^{\tilde{E}}$ with respect to the Hausdorff metric in $2^{\tilde{E}}$.

2) $c(\, . \, , \, . \,) : \tilde{E} \times \tilde{E} \to R_+$ is bounded (by a_4) and continuous .

3) $f(\, . \,) : E \to R_+$ is bounded (by a_3) and continuous .

4.3. Auxiliary results

Proposition C.3 (appendix C) from Bertsekas-Shreve [4] can be readily modified to show the following result :

Proposition 8 : Statements a) and b) below are equivalent

statement a : $\tilde{\Gamma}(\, . \,) : \tilde{E} \to 2^{\tilde{E}}$ is continuous .

statement b : i) Consider any sequence $y_k \to x$ as $k \to \infty$, y_k and x in $\tilde{E}$. For every $\varphi_x \in \tilde{\Gamma}(x)$ there is a sequence $\varphi_k \in \tilde{\Gamma}(y_k)$ such that $\varphi_k \to \varphi_x$ as $k \to \infty$.

ii) Consider any sequence $y_k \to x$ as $k \to \infty$, y_k and x in $\tilde{E}$. For every sequence $\varphi_k \in \tilde{\Gamma}(y_k)$ there is a subsequence φ_{k_i} such that $\varphi_{k_i} \to \varphi_x \in \tilde{\Gamma}(x)$ as $i \to \infty$.

For $v \in C(\tilde{E})$ and $x \in \tilde{E}$ define $\tilde{M}v(x) := \min_{z \in \tilde{\Gamma}(x)} \{ c(x,z) + v(z) \}$. From our assumptions it is clear that $Mv(x) = \tilde{M}v(x)$. We have the following result :

Proposition 9 : M maps $C(\tilde{E})$ into $C(\tilde{E})$.

Proof : For arbitrary $x \in \tilde{E}$ consider any sequence $y_k \to x$ as $k \to \infty$. From compactness of $\tilde{\Gamma}(x)$, $\tilde{M}v(x) = c(x,\varphi_x) + v(\varphi_x)$ for some $\varphi_x \in \tilde{\Gamma}(x)$. From proposition 8 b.i) there exists a sequence $\varphi_k \to \varphi_x$ as $k \to \infty$ with $\varphi_k \in \tilde{\Gamma}(y_k)$. Since c and v are continuous on $\tilde{E}x\tilde{E}$ and $\tilde{E}$ respectively we get

$$\tilde{M}v(y_k) \leq c(y_k,\varphi_k) + v(\varphi_k) \to c(x,\varphi_x) + v(\varphi_x) = \tilde{M}v(x) \text{ as } k \to \infty$$

So $\limsup_{k \to \infty} \tilde{M}v(y_k) \leq \tilde{M}v(x)$.

On the other hand for each y_k we have from compactness of the set $\tilde{\Gamma}(y_k)$ that $\tilde{M}v(y_k) = c(y_k,\varphi_k) + v(\varphi_k)$ for some $\varphi_k \in \tilde{\Gamma}(y_k)$. From proposition 8 b.ii) we can find a subsequence y_{k_i} of y_k and φ_{k_i} of φ_k such that

$\varphi_{k_i} \to \varphi_x \in \tilde{\Gamma}(x)$ as $i \to \infty$ and

$$\liminf_{k \to \infty} \tilde{M}v(y_k) = \liminf_{k \to \infty} \left(c(y_k,\varphi_k) + v(\varphi_k) \right) = \lim_{i \to \infty} \left(c(y_{k_i},\varphi_{k_i}) + v(\varphi_{k_i}) \right)$$

Therefore from continuity of c and v on $\tilde{E}x\tilde{E}$ and $\tilde{E}$ respectively

$$\liminf_{k \to \infty} \tilde{M}v(y_k) = \lim_{i \to \infty} \left(c(y_{k_i},\varphi_{k_i}) + v(\varphi_{k_i}) \right) = c(x,\varphi_x) + v(\varphi_x) \geq \tilde{M}v(x)$$

and hence

$$\limsup_{k \to \infty} \tilde{M}v(y_k) \leq \tilde{M}v(x) \leq \liminf_{k \to \infty} \tilde{M}v(y_k)$$

which shows that

$$\lim_{k \to \infty} \tilde{M}v(y_k) = \tilde{M}v(x)$$

proving the proposition since it holds for every sequence $y_k \to x$, x is arbitrary in $\tilde{E}$ and $\tilde{M}v = Mv$. □

Let $D := \{ z_1 , z_2 , \dots \}$ be the countable dense set in E seen in chapter 2 and $D^N := \{ z_1 , \dots , z_N \}$. Define the sets $\{ \tilde{A}_i^N \}_{i=1}^N$ in the following way :

$$\tilde{B}_i^N := \{ z \in \tilde{E} ; | z - z_i | \leq | z - z_j | \text{ for every } j = 1 , \dots , N \} , i = 1 , \dots , N$$

$$\tilde{A}_1^N := \tilde{B}_1^N \quad , \quad \tilde{A}_i^N := \tilde{B}_i^N - \bigcup_{j=1}^{i-1} \tilde{A}_j^N \qquad i = 2 , \dots , N$$

So clearly $A_i^N = \tilde{A}_i^N - \partial^*E$. For each $N \in \mathbb{N}$ and $x \in \tilde{E}$ define

$$\Gamma^N(x) := \{ z_i \in D^N ; \tilde{A}_i^N \cap \tilde{\Gamma}(x) \neq \emptyset \} \qquad (2)$$

and for $v^N \in B^*(\tilde{E})$

$$M^N v^N(x) := \min_{y \in \Gamma^N(x)} \{ c(x,y) + v^N(y) \}$$

With these definitions we have the following proposition :

Proposition 10 : Suppose $v \in C(\tilde{E})$, $v^N \in B^*(\tilde{E})$ and $v^N \overset{N}{\underset{N}{\rightrightarrows}} v$ uniformly on compact sets of $\tilde{E}$. Then $M^N v^N \overset{N}{\underset{N}{\rightrightarrows}} Mv$ uniformly on compact sets of $\tilde{E}$.

Proof : For arbitrary $x \in \tilde{E}$ consider any sequence $y_k \to x$ and $N_k \to \infty$ as $k \to \infty$. As in the proof of the previous proposition for some $\varphi_x \in \tilde{\Gamma}(x)$

$$\tilde{M}v(x) = c(x,\varphi_x) + v(\varphi_x)$$

and we can find $\varphi_k \in \tilde{\Gamma}(y_k)$ such that $\varphi_k \to \varphi_x$ as $k \to \infty$. Define for each $k \in \mathbb{N}$

$\varphi_k^{N_k} := z_j$ where $j \in \{ 1 , \dots , N_k \}$ is such that $\varphi_k \in \tilde{A}_j^{N_k}$

From proposition 10 of chapter 2 it follows that

$$\varphi_k^{N_k} \to \varphi_x \text{ as } k \to \infty$$

From the above construction it is clear that $\varphi_k^{N_k} \in \Gamma^{N_k}(y_k)$. Thus

$$M^{N_k} v^{N_k}(y_k) \leq c(y_k,\varphi_k^{N_k}) + v^{N_k}(\varphi_k^{N_k}) \to c(x,\varphi_x) + v(\varphi_x) = \tilde{M}v(x) \text{ as } k \to \infty$$

where we have used above continuity of c on $\tilde{E}x\tilde{E}$, continuity of v on $\tilde{E}$ and uniform convergence of v^N to v on compact sets of $\tilde{E}$ (see proposition 12 of chapter 2) . Therefore

$$\limsup_{k \to \infty} M^{N_k} v^{N_k}(y_k) \leq \tilde{M}v(x)$$

Consider now $\varphi_k^{N_k} \in \Gamma^{N_k}(y_k)$ such that

$$M^{N_k} v^{N_k}(y_k) = c(y_k,\varphi_k^{N_k}) + v^{N_k}(\varphi_k^{N_k})$$

For each $\varphi_k^{N_k}$ we have from the definition of the set $\Gamma^{N_k}(y_k)$ (see (2) above) that

$\varphi_k^{N_k} = z_j$ where $j \in \{ 1 , \dots , N_k \}$ and $\tilde{\Gamma}(y_k) \cap \tilde{A}_j^{N_k} \neq \emptyset$

Thus associated to each $\varphi_k^{N_k}$ there is at least one $\varphi_k \in \tilde{\Gamma}(y_k)$ such that $\varphi_k \in \tilde{A}_j^{N_k}$. Using proposition 8 b.ii) we can find a subsequence y_{k_i} of y_k and φ_{k_i} of φ_k such that $\varphi_{k_i} \overset{i \to \infty}{\to} \varphi_x \in \tilde{\Gamma}(x)$ and

$$\liminf_{k \to \infty} M^{N_k} v^{N_k}(y_k) = \liminf_{k \to \infty} \left(c(y_k, \varphi_k^{N_k}) + v^{N_k}(\varphi_k^{N_k}) \right)$$

$$= \lim_{i \to \infty} \left(c(y_{k_i}, \varphi_{k_i}^{N_{k_i}}) + v^{N_{k_i}}(\varphi_{k_i}^{N_{k_i}}) \right)$$

Then again by virtue of proposition 10 of chapter 2 we get $\varphi_{k_i}^{N_{k_i}} \to \varphi_x$ as $i \to \infty$ and as before due to continuity of c on $\tilde{E}x\tilde{E}$, continuity of v on $\tilde{E}$ and uniform convergence of v^N to v on compact sets of $\tilde{E}$ we obtain that

$$\liminf_{k \to \infty} M^{N_k} v^{N_k}(y_k) = \lim_{i \to \infty} \left(c(y_{k_i}, \varphi_{k_i}^{N_{k_i}}) + v^{N_{k_i}}(\varphi_{k_i}^{N_{k_i}}) \right) = c(x, \varphi_x) + v(\varphi_x) \geq \tilde{M}v(x)$$

So $\limsup_{k \to \infty} M^{N_k} v^{N_k}(y_k) \leq \tilde{M}v(x) \leq \liminf_{k \to \infty} M^{N_k} v^{N_k}(y_k)$ and hence

$$M^{N_k} v^{N_k}(y_k) \overset{k \to \infty}{\to} \tilde{M}v(x) = Mv(x) .$$

Since it holds for every sequence $y_k \to x$, $N_k \to \infty$ as $k \to \infty$ and x is arbitrary in $\tilde{E}$ we get that $M^N v^N(y) \to Mv(x)$ as $y \to x$ and $N \to \infty$ for all $x \in \tilde{E}$. The result follows from proposition 12 of chapter 2 and continuity of Mv on $\tilde{E}$ (see proposition 9) . □

4.4. Convergence results

We will consider the impulsive control problem for the discretized P.D.P. (X_t^N) as defined in subsection 4.3. of chapter 2 with the sets $\{ \Gamma^N(y) \}_{y \in \tilde{E}}$ as defined in (2) above . From now on whenever necessary we will use the superscript N to distinguish the discretized problem from the original one . The following proposition can be easily proved (see appendix) :

Proposition 11 : $h \in C(\tilde{E})$ and $h^N \overset{N}{\to} h$ uniformly on compact sets of $\tilde{E}$ (recall definition of h in subsection 3.1.) .

The next proposition is important to show continuity of $\hat{\rho}$ on $\tilde{E}$.

Proposition 12 : For every $n \in \mathbb{N}$, $\mathcal{A}^n h \in C(\tilde{E})$.

Proof : We use induction in n to show the result . For $n = 0$ it is immediate from the previous proposition . Suppose it holds for n , that is , $\mathcal{A}^n h \in C(\tilde{E})$. Then from proposition 9 , $M\mathcal{A}^n h \in C(\tilde{E})$. Since $\mathcal{A}^{n+1} h$ is the value function of an optimal stopping problem with gain function $M\mathcal{A}^n h$ we obtain by virtue of corollary 2 of chapter 2 that $\mathcal{A}^{n+1} h \in C(\tilde{E})$ proving the proposition . □

From uniform convergence of $\mathcal{A}^n h$ to $\hat{\rho}$ (see proposition 5) and continuity of $\mathcal{A}^n h$ on $\tilde{E}$ the next corollary is immediate .

Corollary 1 : $\hat{\rho} \in C(\tilde{E})$.

The next proposition is needed to show convergence of $\hat{\rho}^N$ to $\hat{\rho}$.

Proposition 13 : For every $n \in \mathbb{N}$, $(\mathcal{A}^N)^n h^N \overset{N}{\to} \mathcal{A}^n h$ uniformly on compact sets of $\tilde{E}$.

Proof : We again use induction in n . For $n = 0$ the result is clear from prop. 11 . Suppose it holds for n , that is , $(\mathcal{A}^N)^n h^N \overset{N}{\to} \mathcal{A}^n h$ uniformly on compact sets of $\tilde{E}$. Since $\mathcal{A}^n h \in C(\tilde{E})$ (see prop. 12) and $(\mathcal{A}^N)^n h^N \in B^*(\tilde{E})$ we have from proposition 10 that $M^N(\mathcal{A}^N)^n h^N \overset{N}{\to} M\mathcal{A}^n h$ uniformly on compact sets of $\tilde{E}$. As mentioned before $\mathcal{A}^{n+1} h$ ($(\mathcal{A}^N)^{n+1} h^N$ respectively) is the value function of an optimal stopping problem of the original (discretized) process with gain function $M\mathcal{A}^n h$ ($M^N(\mathcal{A}^N)^n h^N$) . By virtue of corollary 3 of chapter 2

$(\mathcal{A}^N)^{n+1} h^N \overset{N}{\to} \mathcal{A}^{n+1} h$ uniformly on compact sets of $\tilde{E}$

proving the proposition . □

Noting that the constant υ seen in prop. 5 is the same for both the original problem and the discretized problem we obtain the following result :

Corollary 2 : $\hat{\rho}^N \overset{N\to\infty}{\to} \hat{\rho}$ uniformly on compact sets of $\tilde{E}$.

Proof : $| \hat{\rho} - \hat{\rho}^N | \leq (\mathcal{A}^n h - \hat{\rho}) + | \mathcal{A}^n h - (\mathcal{A}^N)^n h^N | + ((\mathcal{A}^N)^n h^N - \hat{\rho}^N)$

$$\leq 2 \frac{\upsilon^{n+1} a_3}{(1-\upsilon)\alpha} + | \mathcal{A}^n h - (\mathcal{A}^N)^n h^N | \overset{N\to\infty}{\to} 2 \frac{\upsilon^{n+1} a_3}{(1-\upsilon)\alpha}$$

uniformly on compact sets of $\tilde{E}$ by virtue of the previous proposition . Taking the limit as $n \to \infty$ we obtain the desired result . □

Remark : The results of this chapter can be extended to the case when the state space E is the countable union of possibly different dimensional open sets .

4.5. Construction of ϵ-optimal strategies using $M^N\hat{\rho}^N$ and $\hat{\rho}^N$

In this subsection we assume that $\tilde{E} := E \cup \partial E$ is a compact set of R^d . We redefine $\partial^* E$ and define $\partial\hat{E}$ as in the subsection 4.5. of chapter 2 . We can show that the previous results of this section still hold if we replace E by $E \cup \partial\hat{E}$; in particular corollaries 1 and 2 will say that $\hat{\rho}$ is continuous on $\tilde{E}$ and $\hat{\rho}^N \overset{N\to\infty}{\to} \hat{\rho}$ uniformly on $\tilde{E}$. Also , from propositon 11 , $M^N\hat{\rho}^N \overset{N\to\infty}{\to} M\hat{\rho}$ uniformly on $\tilde{E}$. The purpose of this subsection is to construct ϵ-optimal strategies of the original problem using $M^N\hat{\rho}^N$ and $\hat{\rho}^N$. Given $\epsilon > 0$ choose $n \in \mathbb{N}$ and $\delta > 0$ such that the inequality of prop. 7 is satisfied and take N such that

$$| \hat{\rho} - \hat{\rho}^N | < \frac{\delta}{24} \quad \text{and} \quad | M\hat{\rho} - M^N\hat{\rho}^N | < \frac{\delta}{24}$$

We will show first how to construct a stopping time U_δ^x satisfying ii) of subsection 3.3 using $\hat{\rho}^N$ and $M^N\hat{\rho}^N$. Define

i) $\tilde{U}_\delta^x := \inf \{ t \geq 0 ; e^{-\alpha t} M^N \hat{\rho}^N(X_t) < e^{-\alpha t} \hat{\rho}^N(X_t) + \frac{\delta}{8} \}$

ii) $\zeta > 0$ such that $(1 - e^{-a_2 \zeta})(\frac{a_3}{\alpha} + a_4) < \frac{\delta}{16}$ and $(1 - e^{-\alpha \zeta}) \frac{a_3}{\alpha} < \frac{\delta}{16}$

iii) For every $k \in \mathbb{N}$ define

$$A(k) := \{ (s,t,x) \in R_+ x \mathbb{B} ; 0 \leq t \leq \zeta , e^{-\alpha(s+t)} M^N \hat{\rho}^N(\phi(t,x)) < (k+1)\frac{\delta}{8} \}$$

$$B(k) := \{ (s,x) \in R_+ x E ; (k-1)\frac{\delta}{8} \leq e^{-\alpha s} \hat{\rho}^N(x) < k\frac{\delta}{8} \}$$

iv) $S := \bigcup_{k \in \mathbb{N}} (\text{proj}_{R_+ x E} A(k)) \cap B(k)$ where

$\text{proj}_{R_+ x E} A(k) = \{(s,x) \in R_+ x E : (s,t,x) \in A(k) \text{ for some } t \in R_+ \}$

Then as in proposition 4 and 5 of chapter 2 there is an analytically measurable function $\psi : S \to [0,\zeta]$ such that for every $(s,x) \in S$,

$$e^{-\alpha(s + \psi(s,x))} M^N \hat{\rho}^N(\phi(\psi(s,x),x)) < e^{-\alpha s} \hat{\rho}^N(x) + \frac{\delta}{4}$$

and for every $\omega \in \Omega$, $(\tilde{U}_\delta(\omega), X_{\tilde{U}_\delta}(\omega)) \in S$. We can now define the following $\mathcal{F}_t$-stopping time

$$U_\delta(\omega) := \begin{cases} \tilde{U}_\delta(\omega) \text{ if } e^{-\alpha \tilde{U}_\delta(\omega)} M^N \hat{\rho}^N(X_{\tilde{U}_\delta}(\omega)) \leq e^{-\alpha \tilde{U}_\delta(\omega)} \hat{\rho}^N(X_{\tilde{U}_\delta}(\omega)) + \frac{\delta}{8} \\ \tilde{U}_\delta(\omega) + \psi(\tilde{U}_\delta(\omega), X_{\tilde{U}_\delta}(\omega)) \text{ otherwise} \end{cases}$$

Theorem 3 of chapter 2 is readily modified to show the following result:

Proposition 14 : $E_x(\int_0^{\tilde{U}_\delta} e^{-\alpha s} f(X_s) ds + e^{-\alpha \tilde{U}_\delta} \hat{\rho}(X_{\tilde{U}_\delta})) = \hat{\rho}(x)$.

Then defining

$$A_1 := \{ \omega \in \Omega ; e^{-\alpha \tilde{U}_\delta(\omega)} M^N \hat{\rho}^N(X_{\tilde{U}_\delta}(\omega)) \leq e^{-\alpha \tilde{U}_\delta(\omega)} \hat{\rho}^N(X_{\tilde{U}_\delta}(\omega)) + \frac{\delta}{8} \}$$

$$A_2 := A_1^c \cap \{ \omega \in \Omega ; \text{no jumps between } [\tilde{U}_\delta(\omega) , \tilde{U}_\delta(\omega) + \psi(\tilde{U}_\delta(\omega), X_{\tilde{U}_\delta}(\omega))] \}$$

$$A_3 := A_1^c \cap \{ \omega \in \Omega ; \text{at least one jump in } [\tilde{U}_\delta(\omega) , \tilde{U}_\delta(\omega) + \psi(\tilde{U}_\delta(\omega), X_{\tilde{U}_\delta}(\omega))] \}$$

(A_1^c is the complement of A_1) and noting that

$M\hat{\rho} < M^N \hat{\rho}^N + \frac{\delta}{24}$, $\hat{\rho}^N < \hat{\rho} + \frac{\delta}{24}$ we obtain

$$E_x(\int_0^{U_\delta} e^{-\alpha s} f(X_s)ds + e^{-\alpha U_\delta} M\hat{\rho}(X_{U_\delta})) \quad \leq E_x(\int_0^{U_\delta} e^{-\alpha s} f(X_s)ds + e^{-\alpha U_\delta} M^N\hat{\rho}^N(X_{U_\delta})) + \frac{\delta}{24}$$

$$\leq E_x(\int_0^{\tilde{U}_\delta} e^{-\alpha s} f(X_s)ds + e^{-\alpha \tilde{U}_\delta}(1-e^{-\alpha\zeta})\frac{a_3}{\alpha} 1_{A_2 \cup A_3} +$$

$$e^{-\alpha \tilde{U}_\delta} M^N\hat{\rho}^N(X_{\tilde{U}_\delta}) 1_{A_1} + e^{-\alpha(\tilde{U}_\delta+\psi)} M^N\hat{\rho}^N(\phi(\psi, X_{\tilde{U}_\delta})) 1_{A_2} + e^{-\alpha U_\delta} M^N\hat{\rho}^N(X_{U_\delta}) 1_{A_3}) + \frac{\delta}{24}$$

$$\leq E_x(\int_0^{\tilde{U}_\delta} e^{-\alpha s} f(X_s)ds + (e^{-\alpha \tilde{U}_\delta}\hat{\rho}^N(X_{\tilde{U}_\delta}) + \frac{\delta}{8})1_{A_1} +$$

$$(e^{-\alpha \tilde{U}_\delta}\hat{\rho}^N(X_{\tilde{U}_\delta}) + \frac{\delta}{4})1_{A_2} + e^{-\alpha \tilde{U}_\delta}(1 - e^{-a_2\zeta})(\frac{a_3}{\alpha} + a_4) 1_{A_3}) + \frac{\delta}{24} + \frac{\delta}{16}$$

$$\leq E_x(\int_0^{\tilde{U}_\delta} e^{-\alpha s} f(X_s)ds + e^{-\alpha \tilde{U}_\delta}\hat{\rho}^N(X_{\tilde{U}_\delta})) + \frac{10}{24}\delta$$

$$\leq E_x(\int_0^{\tilde{U}_\delta} e^{-\alpha s} f(X_s)ds + e^{-\alpha \tilde{U}_\delta}\hat{\rho}(X_{\tilde{U}_\delta})) + \frac{11}{24}\delta$$

$$(\text{ from prop. 14 }) = \hat{\rho}(x) + \frac{11}{24}\delta < \hat{\rho}(x) + \frac{\delta}{2}$$

and thus U_δ^x satifies ii) of subsection 3.3 . Define $r_\delta(\,.\,) : E \rightarrow E$ such that $c(x,r_\delta(x)) + \hat{\rho}^N(r_\delta(x)) = M^N\hat{\rho}^N(x)$ (recall that the operator M^N minimizes over a finite number of points) . It is clear then that

$$c(x,r_\delta(x)) + \rho(r_\delta(x)) \quad \leq c(x,r_\delta(x)) + \rho^N(r_\delta(x)) + \frac{\delta}{24}$$

$$= M^N\hat{\rho}^N(x) + \frac{\delta}{24} \leq M\hat{\rho}(x) + \frac{\delta}{12} < M\hat{\rho}(x) + \frac{\delta}{2}$$

and hence r_δ satisfies i) of subsection 3.3. . Therefore we have constructed an ϵ-optimal strategy defined from U_δ and r_δ as in subsection 3.3. .

4.6. Numerical technique and examples

For the discretized impulsive control problem note that

$$M^N v(x) = \min_{y \in \Gamma^N(x)} \{ c(x,y) + v(y) \}$$

$$E_x^N(e^{-\alpha T_1^N} v(Z_1^N) 1_{\{T_1^N \le t\}}) = \sum_{i=1}^N v(z_i) \int_0^t e^{-\alpha s} Q(A_i^N; \phi(s,x)) \lambda(\phi(s,x)) e^{-\Lambda(s,x)} ds$$

$$K^N v(x) = E_x^N(e^{-\alpha T_1^N} v(Z_1^N)) = \sum_{i=1}^N v(z_i) \int_0^{t^*(x)} e^{-\alpha s} Q(A_i^N; \phi(s,x)) \lambda(\phi(s,x)) e^{-\Lambda(s,x)} ds +$$

$$\sum_{i=1}^N v(z_i) e^{-\alpha t^*(x)} Q(A_i^N; \phi(t^*(x),x)) e^{-\Lambda(t^*(x),x)}$$

and hence the above expressions depend only on $v(z_i)$, $i = 1 , \dots , N$. We start with $v_0^N = \frac{a_3}{\alpha}$ ($\ge h^N$ since f is bounded by a_3) and for $m = 0 , 1 , \dots$ we iterate

$$v_{m+1}^N(z_i) = (\inf_{0 \le t < t^*(z_i)} J^N(M^N v_m^N , v_m^N)(t,z_i)) \wedge K^N v_m^N(z_i) \qquad i = 1 , \dots , N$$

Therefore each iteration consists of N one dimensional minimizations . The algorithm is consistent since as we saw above

$$J^N(M^N v_m^N , v_m^N)(t,z_i) = E_{z_i}^N(\int_0^{T_1^N \wedge t} e^{-\alpha s} f(\phi(s,z_i)) ds + e^{-\alpha T_1^N} v_m^N(Z_1^N) 1_{\{T_1^N \le t\}})$$

$$+ M^N v_m^N(\phi(t,z_i)) e^{-\alpha t - \Lambda(t,z_i)}$$

and $K^N v_m^N(z_i)$ depend only on $v_m^N(z_i)$, $i = 1 , \dots , N$ which were calculated in the previous iteration . In this way we get an approximation for $\hat{\rho}^N(z_1) , \dots , \hat{\rho}^N(z_N)$ since by virtue of proposition 4 , $v_m^N \overset{m \to \infty}{\to} \hat{\rho}^N$. If we want to calculate $\hat{\rho}^N(x)$ for any $x \in E$ we have to do a further one dimensional minimization

$$\hat{\rho}^N(x) = (\inf_{0 \le t < t^*(x)} J^N(M^N \hat{\rho}^N , \hat{\rho}^N)(t,x)) \wedge K^N \hat{\rho}^N(x)$$

since $J^N(M^N \hat{\rho}^N , \hat{\rho}^N)(t,x)$ and $K^N \hat{\rho}^N(x)$ depend only on $\hat{\rho}^N(z_1) , \dots , \hat{\rho}^N(z_N)$.

Let us consider the following examples to illustrate our method .

Example 1 (maintenance-replacement) : This example is similar to the one presented in example 2.7 of [16] (see also [17]) . A device is subject to shocks which occur randomly in time according to a Poisson process with rate λ . Every shock causes independently of the others a certain amount of damage . The total damage incurred by a device accumulates additively . The amount of damage caused by a single shock is a random variable exponentially distributed with rate β . Besides the damage caused by shocks the device decays continuously at a constant rate $\sigma > 0$. When the total amount of damage exceeds B the device is out of order . The decision maker can reduce the total damage at every moment instantaneously . A cost of c(x,y) is incurred when the total damage is reduced from x to y , $x \geq y$. An operating cost f(x) per unit of time is incurred when the total damage is x . We want to find the optimal strategy which minimizes the total discounted cost with discount factor α .

The above problem can be modelled as a P.D.P. with $E = [0,B) \cup \{\Delta\}$ where Δ is a cemetery state and

i) $\phi(t,x) = x + \sigma t$ for $x \in [0,B)$

ii) $\lambda(x) = \lambda$, $x \in [0,B)$

iii) $Q([x,x+y) ; x) = 1 - e^{-\beta y}$ for $y \in [0,B-x)$ and $x \in [0,B)$

$Q(\{\Delta\} ; x) = e^{-\beta (B - x)}$ for $x \in [0,B)$

$Q(\{\Delta\} ; B) = 1$

and $\Gamma(x) = [0,x]$, $\Gamma(\Delta) = [0,B)$

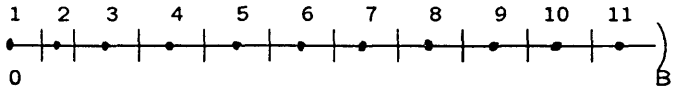
We will consider the following costs :

$c(x,y) = c_0 + c_1(x - y)$, $y \in \Gamma(x)$

$c(\Delta,y) = c_0' + c_1(B - y)$, $y \in [0,B)$

$f(x) = ax$, $f(\Delta) = a'$

The following fig. illustrate a discretization of the state space with N = 12



$\{ \Delta \}$ 12

For the simulations we used the following numerical values :

$c_0 = 0.1$, $c_0' = 1$, $a = 0.1$, $a' = 1$, $\lambda = 0.5$, $\alpha = 0.1$, $\beta = 10$, $B = 1$, $\sigma = 0.1$

and we varied c_1 . We started with $v_0(z_i) = 10$, $i = 1$, ... , N and the criterion to stop was ($m = 0 , 1 , \ldots$)

$$\sum_{i=1}^{N} | v_m^N(z_i) - v_{m+1}^N(z_i) | < p$$

where we varied p . We got that the optimal strategy is of the kind (s - S) . Below we give the numerical results obtained (the last column shows the number of iterations needed until the above criterion was satisfied) .

c	N	p	s	S	itera-tions
0.5	42	0.001	0.0	0.640404	82
1.0	42	0.001	0.0	0.735678	79
1.0	62	0.001	0.0	0.735043	82
1.5	42	0.001	0.0	0.800421	77
1.5	42	0.0005	0.0	0.800415	82
1.5	62	0.001	0.00833 (1^{st}pt)	0.800858	80
1.6	42	0.001	0.0875 (4^{th}pt)	0.811846	78
1.75	42	0.001	0.1625 (7^{th}pt)	0.824034	79
2.0	42	0.001	0.2625 (11^{th}pt)	0.843266	80

Fix now the same values as before except α and make $p = 0.001$, $N = 42$ and $c_1 = 0.5$. Varying α we obtained :

α	s	S	itera-tions
0.1	0.0	0.640404	82
0.2	0.0	0.748595	42
0.3	0.09	0.816139	30
0.6	0.4875	0.895656	19

Example 2 : (preventive maintenance of complex systems) : This example is similar to the one presented in [34] which was solved for the long run cost using the theory of generalized Markovian decision processes . We consider a system with two units which , once operating , are subject to random failures . These units operate separately but they share one repair facility . It means that a queueing situation for the repair facility may arise . We assume that each unit has a fixed maintenance/repair time r_i , a failure rate $\lambda_i(s)$ and a return rate $C_i(s)$ where i=1,2 and s is the age of unit i . Upon unit failure there is an immediate cost $B_i(s)$ plus the repair cost $M_i(s)$. If a unit is taken out of operation preventively and the repair facility is available , only repair costs $M_i(s)$ have to be paid ; but if the repair facility is engaged then a stopping cost $S_i(s)$ is incurred plus the repair cost $M_i(s)$ (where again i = 1,2 and s is the age of unit i) . We also assume that when the age of unit i reaches t_i^* then unit i is taken out of operation with no extra costs incurred but $M_i(s)$. If an unit is out of order and the repair facility is available then it is immediately sent to be repaired .

The above model can be seen as a P.D.P. with a 15-components state space . The three main state space components are :

$E_1 = [0,t_1^*)x[0,t_2^*)$ where units 1 and 2 are working

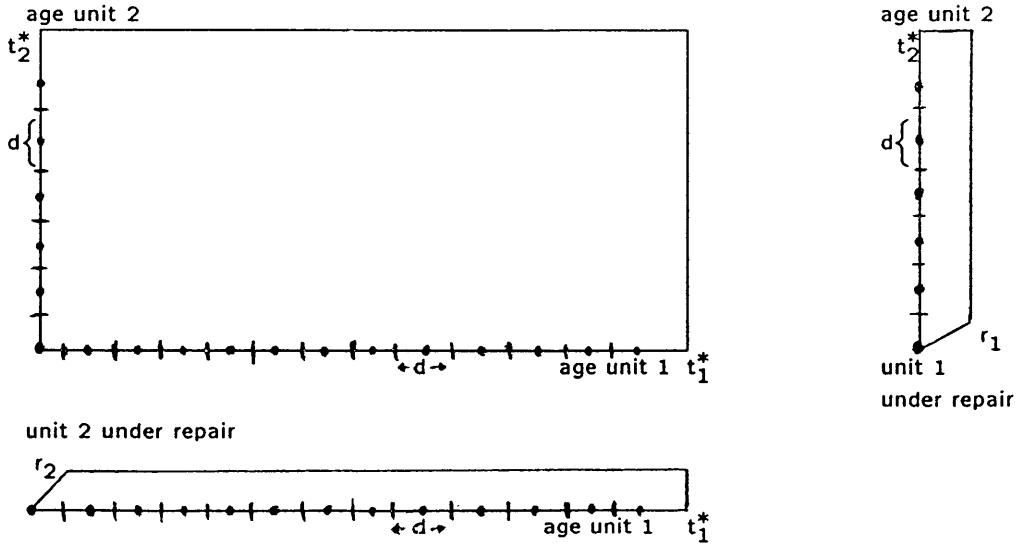
$E_2 = [0,t_1^*)x[0,r_2)$ where unit 1 is working and unit 2 is under repair

$E_3 = [0,r_1)x[0,t_2^*)$ where unit 1 is under repair and unit 2 is working

The other components are mainly cemetery states and are not important for the dynamic of the system . The decision maker can at any time send unit 1 or 2 to maintenance when the process is in E_1 and stop unit 1 (2 respectively) when the process is E_2 (E_3) . We want to minimize the total discounted cost . The following parameters were considered in the numerical simulations :

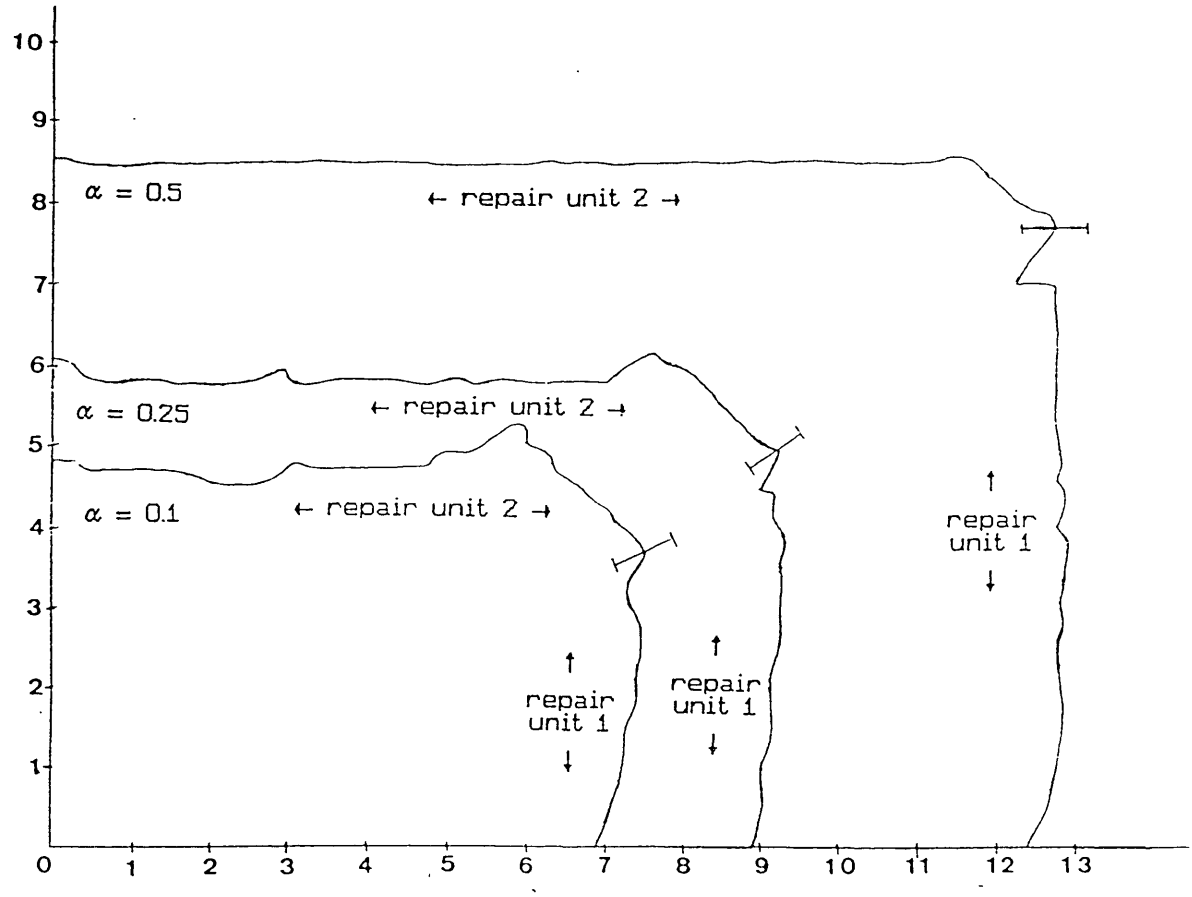
parameters	unit 1	unit 2
Barrier t_i^*	20	10
Maintenance/repair time r_i	2	1
Failure rate $\lambda_i(s)$	0.05	$0.13s^{0.3}$
Failure cost $B_i(s)$	9 + 0.55s	2.20
Maintenance/repair costs $M_i(s)$	$2 + 0.01s^2$	0.5 + 0.35s
Stopping costs $S_i(s)$	0.5	0.1
Rate of return $C_i(s)$	8.5 - 0.55s	8.1 - 0.58s

We used three values for the discount factor α (α = 0.5 , 0.25 and 0.1) and considered the discretization show below with d = 0.05



Our criterion to stop was $\left(\sum_{i=1}^{N} \frac{| v_{m+1}^{N}(z_i) - v_{m}^{N}(z_i) |}{v_{m}^{N}(z_i)} \right) / N \leq 0.0001$

and the number of iterations needed for convergence were 14 , 26 , 38 for α equal to 0.5 , 0.25 and 0.1 respectively . The next figure show the results obtained .



For E_2 the intervention region (stop unit 1) is $[s,t_i^*)x[0,r_2)$ where the value of s is 15.4 , 15.25 and 15 for α equal to 0.5 , 0.25 and 0.1 respectively (if we use the strategy of the fig. above it is clear that we will never intervene in E_2) . For E_3 the whole region is continuation , that is , no intervention (stop unit 2) should be taken .

Appendix of Chapter 3

The purpose of this appendix is to present the proofs of propositions 1 , 5 , 11 , the remainder of the proof of propositon 2 and a connection between the solutions of $v = \hat{L}(v)$ and the "Bellman inequalities" as defined by Yushkevich in [40] (see chapter 1 , equations (6)) .

Proof of proposition 1 : It is easy to see that the sequence $h_n \in B(E)$ and is increasing . From time homogeneity and strong Markov property we have

$$h_n(x) = E_x\Big(\int_0^{T_1} e^{-\alpha s} f(X_s)ds + \ldots + \int_{T_{n-1}}^{T_n} e^{-\alpha s} f(X_s)ds \Big)$$

Hence

$$0 \le h(x) - h_n(x) \le E_x\Big(\int_{T_n}^{\infty} e^{-\alpha s} f(X_x)ds \Big) \le a_3 E_x\Big(\int_{T_n}^{\infty} e^{-\alpha s} ds \Big) = \frac{a_3}{\alpha} E_x\big(e^{-\alpha T_n} \big)$$

where f is bounded by a_3 . Since $T_n^{\,n} \overset{}{\rightarrow} {}^{\infty}\infty$ P_x-a.s. we get $E_x(e^{-\alpha T_n}) \to 0$ as $n \to \infty$ and therefore $h_n \uparrow h \in B(E)$. From the bounded convergence theorem it is clear that

$$h(x) = \lim_{n\to\infty} h_{n+1}(x) = E_x\Big(\int_0^{T_1} e^{-\alpha s} f(X_s)ds + e^{-\alpha T_1} (\lim_{n\to\infty} h_n)(Z_1) \Big) = Kh(x) \quad .$$

Finally suppose $v \ge 0$, $v = Kv$. Then clearly $v = kv \ge Kh_n = h_{n+1} \uparrow h$. □

Remainder of the proof of proposition 2 : We want to prove that

$$\mathcal{A}^n h(x) = M(\mathcal{A}^+)^{n-1} h(x) \wedge (\mathcal{A}^+)^n h(x) \quad , \quad \forall\, x \in E \qquad (A1)$$

but first we are going to show the following result :

$$\mathcal{A}^n h(x) = M\mathcal{A}^{n-1} h(x) \wedge \mathcal{A}^+ \mathcal{A}^{n-1} h(x) \qquad (A2) .$$

Indeed the inequality $\mathcal{A}^n h(x) \leq M\mathcal{A}^{n-1}h(x) \wedge \mathcal{A}^+\mathcal{A}^{n-1}h(x)$ comes from the definition of the operators . Now for any $\tau \in \mathcal{M}_\infty$, $\{ \tau = 0 \} \in \mathcal{F}_0$ and so either $\tau = 0$ or $\tau > 0$ P_x-a.s. . Thus for any $\tau \in \mathcal{M}_\infty$

$$M\mathcal{A}^{n-1}h(x) \wedge \mathcal{A}^+\mathcal{A}^{n-1}h(x) \leq E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds + e^{-\alpha\tau} M\mathcal{A}^{n-1}h(X_\tau) \Big)$$

and taking infimum over $\tau \in \mathcal{M}_\infty$ we obtain (A2) .

Now we use induction to prove (A1) . For n = 1 (A1) follows from (A2) . Suppose that (A1) holds for n . From the definition of the operators $\mathcal{A}$ and $\mathcal{A}^+$

$$M\mathcal{A}^n h(x) \leq M(\mathcal{A}^+)^n h(x) \qquad \text{(A3)}$$

Consider now $\epsilon > 0$ and $y_\epsilon \in \Gamma(x)$ such that $c(x,y_\epsilon) + \mathcal{A}^n h(y_\epsilon) \leq M\mathcal{A}^n h(x) + \epsilon$. From the induction hypothesis for n , either $\mathcal{A}^n h(y_\epsilon) = M(\mathcal{A}^+)^{n-1}h(y_\epsilon)$ or $\mathcal{A}^n h(y_\epsilon) = (\mathcal{A}^+)^n h(y_\epsilon)$. For the latter it is clear that

$$M(\mathcal{A}^+)^n h(x) \leq c(x,y_\epsilon) + (\mathcal{A}^+)^n h(y_\epsilon) = c(x,y_\epsilon) + \mathcal{A}^n h(y_\epsilon) \leq M\mathcal{A}^n h(x) + \epsilon \qquad \text{(A4)}$$

while for the former we can find $z_\epsilon \in \Gamma(y_\epsilon) \subset \Gamma(x)$ (see condition ii) of subsection 2.2 , ch 3) such that $c(y_\epsilon,z_\epsilon) + (\mathcal{A}^+)^{n-1}h(z_\epsilon) \leq M(\mathcal{A}^+)^{n-1}h(y_\epsilon) + \epsilon$. Thus

$$\begin{aligned} c(x,y_\epsilon) + c(y_\epsilon,z_\epsilon) + (\mathcal{A}^+)^{n-1}h(z_\epsilon) - 2\epsilon &\leq c(x,y_\epsilon) + M(\mathcal{A}^+)^{n-1}h(y_\epsilon) - \epsilon \\ &= c(x,y_\epsilon) + \mathcal{A}^n h(y_\epsilon) - \epsilon \leq M\mathcal{A}^n h(x) \end{aligned}$$

Therefore

$$c(x,y_\epsilon) + c(y_\epsilon,z_\epsilon) + (\mathcal{A}^+)^{n-1}h(z_\epsilon) \leq M\mathcal{A}^n h(x) + 2\epsilon$$

However

$$\begin{aligned} M(\mathcal{A}^+)^n h(x) \leq c(x,z_\epsilon) + (\mathcal{A}^+)^n h(z_\epsilon) &\leq c(x,y_\epsilon) + c(y_\epsilon,z_\epsilon) + (\mathcal{A}^+)^{n-1}h(z_\epsilon) \\ &\leq M\mathcal{A}^n h(x) + 2\epsilon \qquad \text{(A5)} \end{aligned}$$

where the second inequality above is due to condition ii) of subsection 2.3 in ch 3 and the fact that $(\mathcal{A}^+)^m$ is a sequence of decreasing functions . Gathering (A4) and (A5) we obtain

$M(\mathcal{A}^+)^n h(x) \leq M\mathcal{A}^n h(x) + 2\epsilon \ , \ \forall \ \epsilon > 0$

and thus $M(\mathcal{A}^+)^n h(x) \leq M\mathcal{A}^n h(x)$. From (A3) it follows that

$$M(\mathcal{A}^+)^n h(x) = M\mathcal{A}^n h(x) \ , \ \forall \ x \in E \tag{A6}$$

and hence

$$\mathcal{A}^+\mathcal{A}^n h(x) = \inf_{\substack{\tau \in \mathcal{M}_\infty \\ \tau > 0}} E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds + e^{-\alpha\tau} M\mathcal{A}^n h(X_\tau) \Big)$$

$$= \inf_{\substack{\tau \in \mathcal{M}_\infty \\ \tau > 0}} E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds + e^{-\alpha\tau} M(\mathcal{A}^+)^n h(X_\tau) \Big) = (\mathcal{A}^+)^{n+1} h(x) \tag{A7}$$

From (A2) , (A6) and (A7)

$\mathcal{A}^{n+1} h(x) = M\mathcal{A}^n h(x) \wedge \mathcal{A}^+\mathcal{A}^n h(x) = M(\mathcal{A}^+)^n h(x) \wedge (\mathcal{A}^+)^{n+1} h(x)$

completing the proof . □

Proof of proposition 5 : In order to prove this proposition we need three auxiliary results .

Proposition A.1 : M and $\mathcal{A}$ are concave in $B^*(E)$; in other words , for any number $0 < \gamma < 1$ and functions v_1 , v_2 in $B^*(E)$ we have :

i) $M(\gamma v_1 + (1-\gamma)v_2) \geq \gamma M v_1 + (1-\gamma) M v_2$

ii) $\mathcal{A}(\gamma v_1 + (1-\gamma)v_2) \geq \gamma \mathcal{A} v_1 + (1-\gamma)\mathcal{A} v_2$

Proposition A.2 : $(1 - \upsilon)h(x) \leq \inf_{y \in \Gamma(x)} (c(x,y))$ (recall that $\upsilon = \dfrac{1}{1+\frac{\alpha c_0}{a_3}}$)

Proposition A.3 : For any $\tau \in \mathcal{M}_\infty$,

$$h(x) = E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds + e^{-\alpha\tau} h(X_\tau) \Big)$$

Proofs : Proposition A.1 follows immediately from the definitions of the operators M and $\mathcal{A}$. Proposition A.3 is an immediate consequence of the time homogeneity and strong Markov

property of the P.D.P. . Finally Proposition A.2 follows from $(1-\upsilon) \leq \frac{\alpha c_0}{a_3}$, $h(x) \leq \frac{a_3}{\alpha}$ and $c(x,y) \geq c_0$ for all $(x,y) \in \Gamma$.

Now we use induction in m to prove proposition 5 . We define the function $\tilde{0}(x) = 0$ for all x in E . We have from propositions A.2 and A.3 that for all $x \in E$

$$\mathcal{A}\tilde{0}(x) = \inf_{\tau \in \mathcal{M}_\infty} E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds + e^{-\alpha\tau} M\tilde{0}(X_\tau) \Big)$$

$$\geq \inf_{\tau \in \mathcal{M}_\infty} E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds + (1-\upsilon)e^{-\alpha\tau} h(X_\tau) \Big)$$

$$= \inf_{\tau \in \mathcal{M}_\infty} \Big((1-\upsilon)E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds + e^{-\alpha\tau} h(X_\tau) \Big) + \upsilon E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds \Big) \Big)$$

$$= \inf_{\tau \in \mathcal{M}_\infty} \Big((1-\upsilon)h(x) + \upsilon E_x\Big(\int_0^\tau e^{-\alpha s} f(X_s)ds \Big) \Big) = (1-\upsilon)h(x)$$

So $\mathcal{A}\tilde{0}(x) \geq (1-\upsilon)h(x)$ for all $x \in E$ and since $h \geq 0$ we get that $(1-\upsilon)h \leq \mathcal{A}\tilde{0} \leq \mathcal{A}h$. Thus $0 \leq h - \mathcal{A}h \leq \upsilon h$ which is the desired result for $m = 1$. Suppose now that the result holds for m , i.e. , $0 \leq \mathcal{A}^{m-1}h - \mathcal{A}^m h \leq \upsilon^m \mathcal{A}^{m-1}h$. Then

$\mathcal{A}^m h \geq (1-\upsilon^m)\mathcal{A}^{m-1}h = (1-\upsilon^m)\mathcal{A}^{m-1}h + \upsilon^m\tilde{0}$

and applying $\mathcal{A}$ to both sides we have by virtue of proposition A.1 that

$$\mathcal{A}^{m+1}h \geq \mathcal{A}\Big((1-\upsilon^m)\mathcal{A}^{m-1}h + \upsilon^m\tilde{0} \Big) \geq (1-\upsilon^m)\mathcal{A}^m h + \upsilon^m \mathcal{A}\tilde{0}$$

$$\geq (1-\upsilon^m)\mathcal{A}^m h + \upsilon^m(1-\upsilon)h = (1-\upsilon^{m+1})\mathcal{A}^m h - \upsilon^m(1-\upsilon)\mathcal{A}^m h + \upsilon^m(1-\upsilon)h$$

$$= (1-\upsilon^{m+1})\mathcal{A}^m h + \upsilon^m(1-\upsilon)(h-\mathcal{A}^m h)$$

and since $\mathcal{A}h \leq h$ it is clear that $\mathcal{A}^m h \leq h$. Thus

$\mathcal{A}^{m+1}h \geq (1-\upsilon^{m+1})\mathcal{A}^m h + \upsilon^m(1-\upsilon)(h-\mathcal{A}^m h) \geq (1-\upsilon^{m+1})\mathcal{A}^m h$

and re-arranging the terms we obtain $0 \leq \mathcal{A}^m h - \mathcal{A}^{m+1}h \leq \upsilon^{m+1}\mathcal{A}^m h$. □

Proof of proposition 11 : The proof of this proposition follows from the next results .

Proposition A.4 : For every $m \in \mathbb{N}$ we have

i) $h_m : \tilde{E} \to R_+$ is continous .

ii) $h_m^N(y) \to h_m(x)$ as $y \to x$ and $N \to \infty$ for every $x \in \tilde{E}$.

Proposition A.5 : $h : \tilde{E} \to R_+$ is continuous .

Proposition A.6 : $h^N \overset{N \to \infty}{\to} h$ uniformly on compact sets of $\tilde{E}$.

Proofs : The proof of propositions A.4 , A.5 and A.6 use respectively the same arguments as the proof of step 1 of theorem 2 , corollary 2 and corollary 3 in chapter 2 . □

The purpose of the remainder of this appendix is to show a connection between the solutions of $v = \hat{L}(v)$ and the equations (6) of chapter 1 when there is only impulsive control . Define for $v \in B^*(E)$,

$\mathcal{U}v(x) := f(x) + \lambda(x) \int_E (v(y) - v(x))Q(dy;x)$

We have the following result .

Proposition A.7 : Let v be a solution of

$$\begin{cases} v(x) = \hat{L}(v)(x) \ , \ v(x) \geq 0 & , \forall\ x \in E \\ v \in B^*(E) \end{cases}$$

Then for all $x \in E$,

$$\begin{cases} v(x) \leq Mv(x) & \text{(A8)} \\ v(\phi(\ .\ ,x)) : [0,t^*(x)) \to R \text{ is a Borel function , its derivative exists almost everywhere and} \\ \int_0^t (\mathcal{U}v(\phi(s,x)) - \alpha v(\phi(s,x)))ds \geq v(x) - v(\phi(t,x)) & , \forall\ t \in [0,t^*(x)) \quad \text{(A9)} \end{cases}$$

If moreover for some $x \in E$, $Mv(\phi(t,x))$ is continuous on $[0,t^*(x))$ then for each open interval $(s_1,s_2) \in \mathfrak{R}_x^c$, where $\mathfrak{R}_x := \{ 0 \leq t < t^*(x) ; v(\phi(t,x)) = Mv(\phi(t,x)) \}$, the function $v(\phi(t,x))$ is

absolutely continuous on $[s_1,s_2]$ ($[s_1,\infty)$ if $s_2 = \infty$) and

$$\mathcal{Q}v(\phi(t,x)) - \alpha v(\phi(t,x)) + \frac{dv(\phi(t,x))}{dt} = 0 \quad \text{almost everywhere on } [s_1,s_2] \tag{A10}$$

Proof : Equation (A8) is immediate from $v(x) = \hat{L}(v)(x)$. Equation (2) of chapter 2 (modified to include α and f) yields for any $t \in [0,t^*(x))$

$$v(\phi(t,x)) = e^{\alpha t+\Lambda(t,x)}\Big\{L(Mv,v)(t,x) - \int_0^t \Big(f(\phi(s,x)) + Qv(\phi(s,x))\lambda(\phi(s,x))\Big)e^{-\alpha s-\Lambda(s,x)}ds\Big\} \tag{A11}$$

where

$$L(Mv,v)(t,x) = \Big(\inf_{t\le s<t^*(x)} J(Mv,v)(s,x) \Big) \wedge Kv(x)$$

Since $L(Mv,v)(t,x)$ is increasing on $[0,t^*(x))$ it is clear that it is a Borel function and by virtue of theorem 2 on page 96 of Royden [33] , $\frac{dL(Mv,v)(t,x)}{dt}$ exists almost everywhere on $[0,t^*(x))$. From (A11) it follows that $v(\phi(t,x))$ is Borel measurable , has derivatives almost everywhere and

$$\frac{dv(\phi(t,x))}{dt} + \mathcal{Q}v(\phi(t,x)) - \alpha v(\phi(t,x)) = e^{\alpha t+\Lambda(t,x)}\frac{dL(Mv,v)(t,x)}{dt} \ge 0 \quad \text{a.e. on } [0.t^*(x)) \tag{A12}$$

where the last inequality is due to the fact that $L(Mv,v)(\cdot ,x)$ is increasing . It is clear from (A12) that for $t \in [0,t^*(x))$

$$\int_0^t \Big(\mathcal{Q}v(\phi(s,x)) - \alpha v(\phi(s,x))\Big)ds \ \ge\ -\int_0^t \frac{dv(\phi(s,x))}{ds}\,ds$$

and to show (A9) it remains to prove

$$\int_0^t \frac{dv(\phi(s,x))}{ds}\,ds \ \le\ v(\phi(t,x)) - v(x)$$

By applying again theorem 2 , [33] mentioned above on the increasing function

$$\Psi(t) := e^{\alpha t+\Lambda(t,x)}L(Mv,v)(t,x)$$

we get that it is differentiable a.e. on $[0.t^*(x))$, the derivative is Borel measurable and

$$\int_0^t \frac{d\mathcal{V}(s)}{ds}\, ds \;\le\; \mathcal{V}(t) - \mathcal{V}(0) \qquad \text{(A13)}$$

Since the function

$$\mathcal{R}(t) := -\, e^{\alpha t+\Lambda(t,x)} \int_0^t \Big(f(\phi(s,x)) + Qv(\phi(s,x))\lambda(\phi(s,x)) \Big) e^{-\alpha s-\Lambda(s,x)} ds$$

is absolutely continuous on $[0,t^*(x))$ we have for any $t \in [0,t^*(x))$

$$\int_0^t \frac{d\mathcal{R}(s)}{ds}\, ds = \mathcal{R}(t) - \mathcal{R}(0) \qquad \text{(A14)}$$

Equations (A11) , (A13) and (A14) yields

$$v(\phi(t,x)) \;= \mathcal{V}(t) + \mathcal{R}(t) \ge \mathcal{V}(0) + \int_0^t \frac{d\mathcal{V}(s)}{ds}\, ds + \mathcal{R}(0) + \int_0^t \frac{d\mathcal{R}(s)}{ds}\, ds$$

$$= \mathcal{V}(0) + \mathcal{R}(0) + \int_0^t \frac{d\,(\,\mathcal{V}(s) + \mathcal{R}(s)\,)}{ds}\, ds = \mathcal{V}(0) + \mathcal{R}(0) + \int_0^t \frac{dv(\phi(s,x))}{ds}\, ds$$

and since $\mathcal{V}(0) = L(Mv,v)(0,x) = v(x)$, $\mathcal{R}(0) = 0$, it is clear that

$$\int_0^t \frac{dv(\phi(s,x))}{ds}\, ds \;\le\; v(\phi(t,x)) - v(x)$$

proving (A9) .

Finally suppose $Mv(\phi(t,x))$ is continuous on $[0,t^*(x))$ for some $x \in E$. Then clearly $J(Mv,v)(t,x)$, $L(Mv,v)(t,x)$ and $v(\phi(t,x))$ are continuous on $[0,t^*(x))$. Also ,

$v(\phi(t,x)) < Mv(\phi(t,x))$ iff $L(Mv,v)(t,x) < J(Mv,v)(t,x)$.

For any open interval $(s_1,s_2) \in \mathfrak{R}_x^c$ and every $t \in (s_1,s_2)$, $v(\phi(t,x)) < Mv(\phi(t,x))$ and thus

$L(Mv,v)(t,x) < J(Mv,v)(t,x)$. Therefore $L(Mv,v)(t,x)$ must be constant on (s_1,s_2) and from continuity of $L(Mv,v)(t,x)$, it is constant on $[s_1,s_2]$ ($[s_1,\infty)$ if $s_2 = \infty$) . By virtue of (A11) and (A12) we get that $v(\phi(t,x))$ is absolutely continuous on $[s_1,s_2]$ and (A10) holds since $\frac{dL(Mv,v)(t,x)}{dt} = 0$. □

Chapter 4

Conclusions

We have considered in this work the optimal stopping and impulsive control problems of piecewise-deterministic processes (P.D.P.'s) . Optimal stopping is an important step in the theory of impulsive control . We considered the former in chapter 2 and presented some extensions to the results of Gurgeli [15] , which were needed for the application to impulsive control (chapter 3) . We also developed computational methods using the property that , if the transition measure is a probability mass function concentrated on a finite set , then the P.D.P. is always on one of the finite number of flow lines . In this case the otimal stopping problem can be reduced to a sequence of one dimensional minimizations . In chapter 3 we applied our results on optimal stopping to give conditions for optimality and computational methods for the impulsive control problem . Examples of applications to maintenance-replacement models were also given .

In our view the main advantage of the numerical technique developed here is its generality and explicit form . Example 2 of chapter 3 shows how general it can be . Another advantage is that boundaries are naturally taken into account under our framework . The main disadvantage is the computational cost that is needed in order to solve the sequence of one dimensional minimizations . In example 2 of chapter 3 the computational effort to solve the problem was considerable ; it took about 2:30 hours of C.P.U. in a VAX computer to obtain a solution for the case $\alpha = 0.1$ with around 600 points of discretization . Of course if we had taken less points in our discretization the computational effort would have been reduced at the expense of accuracy . However due to the complexity of the problem a high computational effort was expected . We noticed from the simulations some dependece between the number of iterations for convergence and the discount factor α ; the smaller α , the larger the number of iterations

(see example 1 of chapter 3) . This relation was also expected since α is killing the process and we are approximating the infinite horizon problem by a truncated one where at most m jumps plus interventions are allowed . So the smaller α , the larger m .

Van der Duyn Schouten [37] or Hordijk and Van der Duyn Schouten [17] proposed an approximation technique for CTMDP (see chapter 1) based on time-discretization . In their model they allow both continuous control (on the jump rate and transition measure) and impulsive control . Although they have not addressed themselves to the question of finding a numerical algorithm to solve the problem , the discrete time approximation (with possibly a state-space discretization) suggests a computational technique . We sketch below what this numerical method would look like for a CTMDP with :

a) "deterministic" transitions after interventions

b) no continuous control

Let $\hat{\rho}^k$ denote the value function of the impulsive control problem of the $k^{\underline{th}}$ approximating DTMDP (see chapter 1) . Then from standards results on dynamic programming we have the Bellman equation :

$$\hat{\rho}^k(x) = \min \{ M(\mathbb{Q}\hat{\rho}^k)(x) , \mathbb{Q}\hat{\rho}^k(x) \} \qquad , \forall \ x \in E$$

$$\alpha_k := \exp (-\tfrac{\alpha}{k})$$

$$\mathbb{Q}\hat{\rho}^k(x) := \tfrac{1}{k} f(x) + \alpha_k \Big((1 - \tfrac{\lambda(x)}{k}) \hat{\rho}^k(\phi(\tfrac{1}{k},x)) + \tfrac{\lambda(x)}{k} \int_E \hat{\rho}^k(\phi(\tfrac{1}{k},z)) \, Q(dz,x) \Big)$$

$$M(\mathbb{Q}\hat{\rho}^k)(x) = \inf_{y \in \Gamma(x)} \{ c(x,y) + \mathbb{Q}\hat{\rho}^k(y) \}$$

where we have used the following notation (which is slightly different from the one seen in the definition of CTMDP in ch 1) : $\Gamma \subset ExE$, $\Gamma(x)$, the x-section of Γ , is the set of feasible point to which the decision-maker can send the system after an intervention at x and $c : \Gamma \rightarrow R$ is the intervention cost .

The standard way in which these equations can be solved is by induction on the number of decisions epochs and then passing it to the limit . Note however that if the state space E is infinite , a discretization on E will also be needed in order to solve the problem numerically .

We make now a brief comparision between our method and the one sketched above . If structural results are known for the $k^{\underline{th}}$ approximating DTMDP (like , for instance , the optimality of (s-S) strategies in some inventory models) then these results can be carried over to the CTMDP and we should expect that a technique exploiting the knowledge of these properties would be more efficient than our method . However if no results are known for the discrete time approximations (like , for instance , in example 2 of ch 3) then we think that the method developed here should performe at least as well in terms of accuracy and speed of convergence as the one seen above since in the latter both time and space are discretized and therefore it is assumed that at most one jump may occur in the time interval $\frac{1}{k}$ (so the larger λ , the more points are needed to improve the accuracy) while in our method only state space is discretized and thus no restriction on the number of jumps in any interval is needed .

We have not considered in this work the impulsive control problem with the long run average criterion . It would be interesting to develop some optimality conditions and a computational method for this problem using the same approach as the one used here .

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