Morse Index Bound for Minimal Two Spheres

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ABSTRACT
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Colding and Minicozzi proved in their work [CM08] that a geometric invariant, that we call width, can be realized as the sum of areas of minimal 2-spheres. We prove that there exists a collection of such minimal 2-spheres with Morse index bound one.

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Chapter 1

Introduction

1.1 Motivation

There are two perspectives on the motivation of the main result. One from Morse theory and the other from the space of minimal surfaces. We describe both of them briefly below.

1.1.1 Morse Theory

Finite-dimensional Morse theory was developed by Morse [Mil63] to study geodesics. Index of a critical point of a proper nonnegative function on a manifold reflects its topology. The topology of the space of closed curves is well studied by Morse theory. For example, we know that the space of closed curves on S^n has the homotopy type of a CW-complex with one cell each in the dimensions 0, $n-1$, $2(n-1)$,... Since geodesics are critical points of energy functional for curves, a natural extension of Morse theory of closed geodesics would be a Morse theory of harmonic surfaces, which are critical points of energy functional for surfaces. However, the non-degeneracy of energy functional for surfaces can't be solved easily as in the case of closed curves. Sacks and Uhlenbeck introduced α-energy [SU81], which can be perturbed to be Morse functions. The α-energy approaches the usual energy as the parameter α in the perturbation goes to one, and the corresponding critical point of α-energy converges to a harmonic map. However, without curvature assumption [LLW17] or finite fundamental group [CT99] for the ambient manifold (M, g), the

harmonic sphere constructed by α-energy fails to realize the energy as α goes to one [LW15][Moo17, Remark 4.9.6]. Thus, we are motivated to prove the Morse index bound of the harmonic sphere produced by the min-max theory [CM08], which rules out the energy loss.

1.1.2 Space of Minimal Surfaces

On the other hand, harmonic maps from the sphere are conformal thus are minimal surfaces. In the thesis the *sphere* implies the two dimensional sphere, and *harmonic sphere* implies harmonic map with sphere as its domain unless stated otherwise. Almgren and Pitts' min-max theory proves the existence of embedded minimal hypersurfaces in closed manifold of dimension at least three at most seven. Using it Marques and Neves [MN16] proved the existence of an embedded minimal hypersurface with certain Morse index bound (the Morse index bound is equal to the dimension of the families used in the min-max process, see [MN16] for the details). This result plays an important role in proving Yau's conjecture [Son18], the conjecture states that for any closed three manifold there exist infinitely many embedded minimal surfaces. Moreover, Antoine Song [Son18] proved that there exist infinitely many embedded minimal hypersurfaces in any closed Riemannian manifold of dimension at least three at most seven using Almgren and Pitts' min-max theory. However, Almgren and Pitts' min-max theory doesn't say anything about minimal surfaces in higher codimension. While using the min-max theory of harmonic surfaces by Colding and Minicozzi [CM08], there's no restriction on codimension. So it motivates us to prove the Morse index bound of the harmonic sphere produced by the min-max theory of Colding and Minicozzi [CM08], which doesn't have the hypersurface restriction.

1.2 Min-Max Theory with Different Functional

Min-max theory is a way of finding critical points for variational problems. Let us recall the history of min-max theory. The idea of using min-max argument to show the existence of critical points for variational problems goes back to H.A. Schwarz in the 1870s in his work on harmonic functions. For finding closed geodesics on a sphere it was accomplished by Birkhoff [Bir17] in 1917. Since $\pi_1(S^2)$ is trivial, geodesic on a sphere cannot be obtained by simply minimizing the length of a closed curve. One needs to consider the *sweepouts*, which is the set $\tilde{\Lambda}$ of continuous maps

$$\sigma_1 : S^1 \times [0,1] \to S^2,$$

so that

1. for each t the map $\sigma_1(\cdot, t)$ is in $W^{1,2}$,
2. the map $t \mapsto \sigma_1(\cdot, t)$ is continuous from $[0, 1]$ to $W^{1,2}$,
3. σ_1 maps $S^1 \times \{0\}$ and $S^1 \times \{1\}$ to points.

We now consider the homotopy class $\tilde{\Lambda}_{\sigma_1}$, which is defined to be the set of map $\sigma' \in \tilde{\Lambda}$ that is homotopic to σ_1 through maps in $\tilde{\Lambda}$. The *width* in this case, which we denotes by W_c, is defined to be the following:

$$W_c := \inf_{\sigma' \in \tilde{\Lambda}_{\sigma_1}} \max_{t \in [0,1]} E(\sigma'(\cdot, t)),$$

here $E(\sigma'(\cdot, t))$ denotes the energy of the closed curve $\sigma'(\cdot, t)$. W_c is a geometric invariant. One can sweep S^2 out by the one parameter family $\sigma_1(\cdot, t)$ of closed curves and pull the sweepout tight using *curve shortening process*, which is considering the closed curves as piecewise linear maps with the chosen energy bound and replacing them with the unique shortest geodesic segments (the uniqueness is guaranteed by the energy bound). If the sweepout σ_1 induces a nontrivial class in $\pi_2(S^2)$, then we have

$$W_c > 0,$$

and each map in the tightened sweepout whose energy is close W_c must itself be close to a geodesic. Geodesics are critical points for energy of closed curves. That is how we prove the existence of closed geodesic on S^2 using min-max argument.

For finding minimal surfaces in any closed manifold M isometrically embedded into some Euclidean space $\mathbb{R}^{\mathbb{N}}$, with dimension at least three and nontrivial homotopy group $\pi_3(M)$, we have the min-max family for energy functional constructed by Colding and Minicozzi [CM08], which they used to prove the finite extinction time of Ricci flow. Colding and Minicozzi's min-max theory follows the same strategy as Birkhoff's: In this case, *sweepout* is defined to be the set Λ of continuous maps

$$\sigma : S^2 \times [0, 1] \to M,$$

so that

1. for each t the map $\sigma(\cdot, t)$ is in $C^0 \cap W^{1,2}$,
2. the map $t \mapsto \sigma(\cdot, t)$ is continuous from $[0, 1]$ to $W^{1,2}$ in a strong sense,
3. σ maps $S^2 \times \{0\}$ and $S^2 \times \{1\}$ to point maps.

We now consider the homotopy class Λ_σ, which is defined to be the set of map $\acute{\sigma} \in \Lambda$ that is homotopic to σ through maps in Λ. The *width* in this case, which we denotes by W, is defined to be the following:

$$W := \inf_{\acute{\sigma} \in \Lambda_\sigma} \max_{t \in [0,1]} E(\acute{\sigma}(\cdot, t)), \tag{1.1}$$

here $E(\acute{\sigma}(\cdot, t))$ denotes the energy of the map $\acute{\sigma}(\cdot, t)$. Given a map $u : \Sigma \to M$, the energy of it is defined to be:

$$E(u) = \int_\Sigma |\nabla u|^2,$$

energy functional E will be defined again precisely in section 2.1.1. Note that W is a geometric invariant depends on the manifold M only.

Now we pull the sweepout σ tight, in a continuous way, by harmonic replacement. Then if σ induces a nontrivial class in $\pi_3(M)$, we have

$$W > 0,$$

then each map in the tightened sweepout whose energy is close to W must itself be *close* to a finite collection of harmonic spheres. Close in both varifold distance and bubble tree sense, this will be defined precisely in section 4.2. Now we state Colding-Minicozzi's min-max theorem:

Theorem 1.2.1 (Colding-Minicozzi, [CM08]): Given a closed manifold (M^n, g) with dimension $n \geq 3$, and a map $\sigma \in \Lambda$ representing a nontrivial class in $\pi_3(M)$, there exists a sequence of sweepouts $\gamma_j(\cdot, t) \in \Lambda_\sigma$ with

$$\lim_{j \to \infty} \max_{t \in [0,1]} E(\gamma_j(\cdot, t)) = W,$$

and so that given $\epsilon > 0$, there exists $\bar{j}$ and $\delta > 0$ so that if $j > \bar{j}$ and

$$Area(\gamma_j(\cdot, s)) > W - \delta,$$

then there are finitely many harmonic maps $u_i : S^2 \to M$, $i = 0, ..., n$, with

$$d_V(\gamma_j(\cdot, s), \{u_i\}_{i=0}^n) < \epsilon.$$

Here $d_V(\cdot, \cdot)$ denotes the *varifold distance*, which will be defined in definition 3.3.1.

Corollary 1.2.2: Given a closed manifold (M^n, g) with dimension $n \geq 3$, and a map $\sigma \in \Lambda$ representing a nontrivial class in $\pi_3(M)$. If a sequence of sweepouts $\gamma_j(\cdot, t) \in \Lambda_\sigma$ has the property that

$$\lim_{j \to \infty} \max_{t \in [0,1]} E(\gamma_j(\cdot, t)) = W,$$

we define t_j to be $E(\gamma_j(\cdot, t_j)) = \max_{t\in[0,1]} E(\gamma_j(\cdot, t))$, then there exists a collection of finitely many harmonic spheres $\{u_i\}_{i=0}^m$ such that $\{\gamma_j(\cdot, t_j)\}_j$ "converges" to up to subsequence. Moreover,

$$\sum_{i=0}^{m} E(u_i) = W.$$

We will prove how theorem 1.2.1 implies corollary 1.2.2 in section 4.2 and define the "convergence" mentioned in corollary 1.2.2 again precisely.

Min-max theorem by Colding and Minicozzi shows that the width W is realized by the sum of energy of finitely many harmonic spheres. Harmonic spheres are critical points for energy functional. Since harmonic spheres are conformal thus are also minimal surfaces. That is how we prove the existence of minimal surfaces using min-max argument.

We can also do min-max to find critical points for α-energy. Let $(M, \sigma, \acute{\sigma}, \Lambda, \Lambda_\sigma)$ be given as before. α-energy E_α of a map $u' : \Sigma \to M$ is defined to be the following:

$$E_\alpha(u') := \int_\Sigma (1 + |\nabla u'|^2)^\alpha.$$

Clearly, when $\alpha = 1$, we have $E_1(u') = 4\pi + 2E(u)$. We can do the min-max argument using α-energy to define width. That is

$$W_\alpha := \inf_{\acute{\sigma}\in\Lambda_\sigma} \max_{t\in[0,1]} E_\alpha(\acute{\sigma}(\cdot, t)),$$

and if σ induces a nontrivial class in $\pi_3(M)$ then we have

$$W_\alpha > 4\pi,$$

for all $\alpha > 1$. The difference between doing min-max argument for energy and α-energy is that for $\alpha > 1$, E_α can be perturbed to be a *Morse function*, thus *Morse Theory* can apply and we know that there exists a map $u_\alpha : S^2 \to M$ which is a critical point for E_α, and

$$E_\alpha(u_\alpha) = W_\alpha.$$

Now given a sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ with $\alpha_i > 1$, and $\alpha_i \to 1$ as $i \to \infty$. As $\alpha_i \to 1$, u_{α_i} converges to a harmonic sphere. We refer to [MM88] for the details.

On the other hand, min-max theory for the area functional was first introduced by Almgren in the early 1960s [Alm62] and improved by Pitts in 1981. It provides the existence of an embedded minimal hypersurface in any closed manifold of dimension at least three at most seven.

1.3 Main Theorem and Related Theorems

1.3.1 Morse Index Bound for Embedded Minimal Hypersurfaces

If Σ is a smooth embedded closed minimal hypersurface, its *Morse index* is the maximal dimension of a subspace of normal variations restricted to which the second variation of area quadratic form is negative definite.

Using Almgren and Pitts' min-max theory Marques and Neves [MN16] proved the following theorem:

Theorem 1.3.1 ([MN16]): Let (M^n, g) be an n-dimensional closed Riemannian manifold with $3 \leq n \leq 7$. Given an integer k, there exists an embedded minimal hypersurface Σ^{n-1} with the following properties:

1. $\text{Area}(\Sigma) = w_k$,
2. $\text{Index}(\Sigma) \leq k$,

here w_k is the width for the k parameter sweepout. We refer to [MN16, theorem 1.2] for the precise definition.

In other words, there exists an embedded minimal hypersurface by Almgren Pitts' min-max theory with Morse index less than or equal to the number of parameters (dimension of the families used in the min-max process).

1.3.2 Main Theorem

If u is a harmonic sphere, its *Morse index* is the maximal dimension of a subspace of variations restricted to which the second variation of energy quadratic form is negative definite. We will define it again precisely in section 2.1.21. The main theorem of the thesis is the following:

Theorem 1.3.2 (Main Theorem): Let (M^n, g) be a closed Riemannian manifold with $n \geq 3$, g generic and a nontrivial homotopy group $\pi_3(M)$. Then there exists a collection of finitely many harmonic spheres $\{u_i\}_{i=0}^m$, $u_i : S^2 \to M$, which satisfies the following properties:

1. $\sum_{i=0}^{m} E(u_i) = W$,
2. $\sum_{i=0}^{m} Index(u_i) \leq 1$,

here W is the width. (See (1.1) or definition 4.2.3.)

The existence of finitely many harmonic spheres in theorem 1.3.2 is given by Colding and Minicozzi's min-max theory [CM08]. Theorem 1.3.2 is a variant of 1.3.1. We compare the difference between them here. Besides the obvious difference of harmonic spheres and embedded minimal hypersurfaces, energy functional and area functional, the embedded minimal hypersurface used in 1.3.1 is given by Almgren-Pitts' min-max theory, thus could have several components. When considering the variation of it, it means the variation of the whole configuration instead of each component. The index of theorem 1.3.1 is the maximal dimension on which the second variation of the energy functional of the whole configuration is negative definite. While in theorem 1.3.2, the index bound it obtains is the sum of each component, which is stronger than the bound for the whole configuration.

1.3.3 Harmonic Maps with Morse Index Bound

For obtaining a harmonic sphere u with index bound without the property of energy identity, i.e. $E(u) = W$, in theorem 1.3.2, there's another way by using α-energy to do min-max. This is done by Micallef and Moore in [MM88]. We now state it and sketch the idea of the proof here.

Theorem 1.3.3 ([MM88]): Let (M, g) be a closed Riemannian manifold with nontrivial homotopy group $\pi_k(M)$. Then there exists a harmonic sphere with Morse index bound $k - 2$.

If the metric g in theorem 1.3.3 is generic and the manifold has nontrivial $\pi_3(M)$ then theorem 1.3.3 gives the existence of a harmonic sphere with Morse index bound one.

So we consider the special case of theorem 1.3.3 such that $\pi_3(M)$ is nontrivial. The ideal of the proof is to do min-max using α-energy as we stated in section 1.2. Let $(E_\alpha, \{\alpha_i\}_{i\in\mathbb{N}}, u_{\alpha_i})$ be given as in section 1.2 so we know that by *Morse theory* we have $Index(u_{\alpha_i}) = 1$ for any $\alpha_i > 1$. Since u_{α_i} converges to a harmonic sphere

u as $i \to \infty$. Then by [Moo17, Lemma 4.10.7] we have

$$Index(u) \leq \liminf_{i\to\infty} Index(u_{\alpha_i}) = 1,$$

But the convergence doesn't preserve energy identity. i.e.,

$$\lim_{i\to\infty} E_{\alpha_i}(u_{\alpha_i}) \neq 4\pi + E(u).$$

1.3.4 Morse Index Conjecture

Marques and Neves proved that in the Almgren-Pitts setting, for one parameter case there is either an unstable embedded minimal surface with multiplicity one or one-sided minimal surface with even multiplicities and unstable double cover. They also proved in [MN18] under the assumption of multiplicity one, given any integer k, there exists a minimal hypersurface Σ with $Index(\Sigma) = k$, and area of Σ equals to the width. Thus they proposed the following Morse Index Conjecture.

Morse Index Conjecture *Given a manifold (M^n, g), $3 \leq n \leq 7$, g generic, and an integer k. There exists a smooth, embedded, two-sided and closed minimal hypersurface Σ^{n-1} such that $Index(\Sigma) = k$*

For harmonic maps, under the assumption that all the harmonic spheres obtained by min-max are of strong convergence, we are able to conclude that there exists a harmonic sphere which realizes the width and whose Morse index is exactly one. We recall that in section 1.2 we say that each map in the sweepout whose energy is close to W must itself be close to a finite collection of harmonic sphere, the assumption here is to assume that there exists a sweepout, so that for each map in the sweepout whose energy is close to the width, must itself be close in $W^{1,2}$-norm to *one* harmonic sphere whose energy realizes the width.

Definition 1.3.4 (Image set): Given a sweepout $\{\gamma_j(\cdot,t)\}_{j\in\mathbb{N}}$, we define the image set $\Psi(\{\gamma_j(\cdot,t)\}_{j\in\mathbb{N}})$ to be

$$\begin{aligned}\Psi(\{\gamma_j(\cdot,t)\}_{j\in\mathbb{N}}) := \Big\{ &\{u_i\}_{i=0}^m : \text{there exists a sequence } \{i_j\} \to \infty, \\ &t_{i_j} \in [0,1], \text{ such that } \gamma_{i_j}(\cdot,t_{i_j}) \text{ converges to } \{u_i\}_{i=0}^m, \\ &\text{and } \sum_{i=0}^m E(u_i) = W \Big\}.\end{aligned}$$

We will define image set again and what kind of convergence it refers to precisely in section 4.2.

Proposition 1.3.5: Let (M^n, g) be a closed Riemannian manifold with $n \geq 3$, g generic and a nontrivial homotopy group $\pi_3(M)$. If there exists a sweepout $\{\gamma_j(\cdot, t)\}_{j\in\mathbb{N}}$ which induces a non-trivial class in $\pi_3(M)$, and for each of $\{u_i\}_{i=0}^m \in \Psi(\{\gamma_j(\cdot, t)\}_{j\in\mathbb{N}})$, $\{u_i\}_{i=0}^m$ contains only one harmonic sphere. Then all the harmonic sphere in the image set has Morse index one.

By Proposition 1.3.5, Morse Index Conjecture is true for harmonic sphere when $k = 1$ under the assumption of strong convergence.

1.3.5 Countability for Harmonic Maps with Bounded Energy

Before proving the main theorem we need the result of countability for harmonic maps with bounded energy. We first define the equivalent classes of maps.

Definition 1.3.6: Given $u : S^2 \to M$, we define the equivalent class of u to be the following:

$$[u] := \{g : S^2 \to M | \text{ If } u = g \circ \phi \text{ for some } \phi \in PSL(2, \mathbb{C}).\}$$

Recall that $PSL(2, \mathbb{C})$ is the group of conformal automorphisms of S^2. So the equivalent class of a map $u : S^2 \to M$ is all the maps differ from u by a conformal automorphism of the domain S^2.

Theorem 1.3.7: Given a closed manifold (M^n, g), with $n \geq 3$, g generic, then the set of all harmonic spheres up to equivalent class with bounded energy is countable.

Theorem 1.3.7 is expected, given that we know a similar result holds for minimal embedded hypersurfaces. Namely:

Theorem 1.3.8 ([Sha17]): Given a closed manifold (M^n, g), $3 \leq n \leq 7$, with g generic, the set of embedded minimal hypersurfaces with bounded area and index is finite.

The idea to prove theorem 1.3.8 is the following: assume theorem 1.3.8 is not true, then there exists infinitely many embedded minimal hypersurfaces $\{\Sigma_j\}_{j\in\mathbb{N}}$

with bounded area and index. By the compactness theorem [Sha17], $\{\Sigma_j\}_{j\in\mathbb{N}}$ converges to a minimal embedded hypersurface Σ with the same area and index bound up to subsequence. We can use $\{\Sigma_j\}_{j\in\mathbb{N}}$ to construct a nontrivial Jacobi field on Σ. Then it contradicts the assumption of generic metric, which implies there's no nontrivial Jacobi field on minimal submanifolds [Whi17].

However, harmonic spheres with bounded energy are merely branched minimal immersions with bounded area. Theorem 1.3.8 doesn't imply theorem 1.3.7. The proof of theorem 1.3.7 brings together several important results like bumpy metric theorem for minimal surface [Whi17, theorem 2.1], bumpy metric theorem for harmonic map [Moo17] and comparison between second variation of area and energy of a minimal surface [EM08].

Before we prove theorem 1.3.7, the key step for theorem 1.3.7 is the following lemma 1.3.9, which is an analogy of theorem 1.3.8:

Lemma 1.3.9: Given (M, g), g generic, the dimension of M is at least three, and a harmonic sphere $u : S^2 \to M$. There exists $\epsilon = \epsilon(u) > 0$, so that if f is a harmonic sphere with the following property

$$\int_{S^2} |\nabla f - \nabla u|^2 < \epsilon,$$

then $[f] = [u]$.

In other words, Lemma 1.3.9 states that for generic metric harmonic spheres of same equivalent classes are locally isolated.

The idea of the proof is inspired by theorem 1.3.8. We argue by contradiction, if such $\epsilon > 0$ doesn't exist, then there exists a harmonic sphere $u : S^2 \to M$, a sequence of harmonic spheres $\{f_i\}_{i\in\mathbb{N}}$, $[f_i] \neq [u]$, such that

$$\int_{S^2} |\nabla f_i - \nabla u|^2 < 1/i,$$

We can use the sequence $\{f_i\}_{i\in\mathbb{N}}$ to construct a nontrivial Jacobi field for energy functional on u. We can't use bumpy metric theorem for harmonic map [Moo17] since it only applies to prime harmonic sphere. By [MM88, theorem 3.1] the nontrivial Jacobi field for energy functional on u induced a Jacobi field for area functional on u. We are left to check that u is an immersed minimal surface without branch points. By bumpy metric theorem for harmonic spheres [Moo17, theorem 5.1.1.], we know that the image of u is an immersed minimal surface free of branch points given that g is generic. Thus by [Whi17], which states that there's no nontrivial

Jacobi field on minimal immersed submanifold for g generic, we have the desired contradiction.

To prove theorem 1.3.7 , we pick $\{x_1, x_2, ..., x_{p_n}\} \subset S^2$ such that the union of each geodesic balls centered at x_i with radius $1/n$ covers S^2, i.e., $S^2 \subset \bigcup_{i=1}^{p_n} B_{1/n}(x_i)$. Then we consider the set

$$\mathcal{F}(n) = \Big\{[f] \mid \int_{B_{1/n}(x_i)} |\nabla f|^2 < \epsilon_{su}, \text{ for } i = 1, ..., p_n\Big\},$$

here ϵ_{su} is the constant given in [SU81, main estimate 3.2] , we need ϵ_{su} to obtain strong convergence in $W^{1,2}$ such that we can use lemma 1.3.9 to conclude that $\mathcal{F}(n)$ is finite for each $n \in \mathbb{N}$, so $\bigcup_{n\in\mathbb{N}} F(n)$ is countable. Thus the set of harmonic spheres with bounded energy is countable.

1.3.6 Idea of the Proof for Theorem 1.3.2

Given a manifold (M^n, g), $n \geq 3$, with g generic and a nontrivial $\pi_3(M)$, we consider a family of sweepouts $\{\gamma_j(\cdot, t)\}_{j\in\mathbb{N}}$, $\gamma_j : S^2 \times [0, 1] \to M$, which induces a nontrivial class in $\pi_3(M)$ with the following property:

$$\lim_{j\to\infty} \max_{t\in[0,1]} E(\gamma_j(\cdot, t)) = W.$$

We define t_j to be $E(\gamma_j(\cdot, t_j)) = \max_{t\in[0,1]} E(\gamma_j(\cdot, t))$. By corollary 1.2.2 there exists a collection of finitely many harmonic spheres $\{u_i\}_{i=0}^m$, $u_i : S^2 \to M$, and

$$\sum_{i=0}^{m} E(u_i) = W,$$

such that $\{\gamma_j(\cdot, t_j)\}_j$ converges to up to subsequence. More precisely, for every $u_i \in \{u_i\}_{i=0}^m$, there exists a sequence of conformal automorphisms $\{\psi_j^i\}_{j\in\mathbb{N}}$, $\psi_j^i : S^2 \to S^2$, such that $\gamma_j(\cdot, t_j) \circ \psi_j^i$ converges to u_i weakly in $W^{1,2}$ sense up to subsequence.

Recall that $\Psi(\{\gamma_j(\cdot, t)\}_j)$ is the image set of $\{\gamma_j(\cdot, t)\}_j$ (see definition 1.3.4). The idea is if $\{u_i\}_{i=0}^m \in \Psi(\{\gamma_j(\cdot.t)\}_j)$ have $\sum_{i=0}^m Index(u_i) > 1$, then we are able to perturb $\{\gamma_j(\cdot, t)\}_j$ to a new sweepout $\{\tilde{\gamma}_j(\cdot, t)\}_j$ such that

1. $\{\tilde{\gamma}_j(\cdot, t)\}_j$ is homotopic to $\{\gamma_j(\cdot, t)\}_j$,
2. $\lim_{j\to\infty} \max_{t\in[0,1]} E(\tilde{\gamma}_j(\cdot, t)) = W$,

3. $\{u_i\}_{i=0}^m \notin \Psi(\{\tilde{\gamma}_j(\cdot,t)\}_j)$,

then since $\lim_{j\to\infty}\max_{t\in[0,1]} E(\tilde{\gamma}_j(\cdot,t)) = W$, by corollary 1.2.2 we know that the image set $\Psi(\{\tilde{\gamma}_j(\cdot,t)\}_j)$ is nonempty. Let $\{v_i\}_{i=0}^{m_0} \in \Psi(\{\tilde{\gamma}_j(\cdot,t)\}_j)$. If $\sum_{i=0}^{m_0} Index(v_i) > 1$, we can perturb $\{\tilde{\gamma}_j(\cdot,t)\}_j$ again. By theorem 1.3.7, we know that the set of all harmonic spheres up to equivalent class with bounded energy W is countable, so we can perturb the sweepout inductively such that it is away from all harmonic spheres whose sum of Morse indexes is greater than 1. Since the perturbed sweepout is in the same homotopy class as $\{\gamma_j(\cdot,t)\}_j$ and has the property that the maximal slice of energy approaches W as $j\to\infty$, it converges to a collection of finitely many harmonic spheres whose sum of Morse indexes is bouned by 1.

Before we construct $\{\tilde{\gamma}_j(\cdot,t)\}_j$ we define the variation of a map. Suppose M is isometrically embedded in $\mathbb{R}^N$ and let $\Pi : \mathbb{R}^N \to M$ be the nearest point projection from $\mathbb{R}^N$ to M. Given a map $u : S^2 \to M$ and $X : S^2 \to \mathbb{R}^N$, with each $X^i \in C^\infty(S^2)$, we consider the variation of u with respect to X to be the following:

$$u_s := \Pi \circ (u + sX),$$

then it's clear that if another map $v : S^2 \to M$ is close to u in $W^{1,2}(S^2, M)$ sense, the variation v_s will be close to u_s as well.

Now we sketch how to construct $\{\tilde{\gamma}_j(\cdot,t)\}_j$. Assume $\{u_i\}_{i=0}^m \in \Psi(\{\gamma_j(\cdot,t)\}_j)$ such that $\{\gamma_j(\cdot,t_j)\}_j$ converges to up to subsequence, and $\sum_{i=0}^m Index(u_i) = k \geq 2$, then there exists k variations $\{X_l\}_{l=1}^k$, $X_l : S^2 \to \mathbb{R}^N$, with the following property: for each X_l, there exists at least one $u_l \in \{u_i\}_{i=0}^m$ so the second variation of energy of u_l with respect to X_l is negative. i.e.,

$$\frac{d^2}{ds^2}\Big|_{s=0} E(\Pi \circ (u_l + sX_l)) < 0, \quad l = 1, ..., k.$$

The idea is using $\{X_l\}_{l=1}^k$ to perturb $\gamma_j(\cdot,t)$. We first prove in lemma 6.1.1 that for $\gamma_j(\cdot,t)$ close to $\{u_i\}_{i=0}^m$, there exist corresponding cutoff functions $\eta_l^j : S^2 \to \mathbb{R}$. We can define $\tilde{X}_l := \eta_l^j \circ (\psi_l^j)^{-1} X_l \circ (\psi_l^j)^{-1}$, and define the variation of $\gamma_j(\cdot,t)$ with respect to $\tilde{X}_l$ to be:

$$\gamma_{j,s}(\cdot,t) := \Pi \circ \big(\gamma_j(\cdot,t) + \sum_{l=1}^k s_l \tilde{X}_l\big),$$

here $s = (s_1, ..., s_k) \in \bar{B}^k$, $\bar{B}^k$ is the k-dimensional unit ball, so that the energy of $\gamma_{j,s}(\cdot,t)$ is concave while changing $s \in \bar{B}^k$. That is, define $E_j^t(s) := E(\gamma_{j,s}(\cdot,t)), E_j^t : \bar{B}^k \to \mathbb{R}$, and we have

$$D^2 E_j^t(s) < 0, \quad \forall s \in \bar{B}^k.$$

If we can construct a continuous function $s_j : [0,1] \to \bar{B}^k$ so that $\gamma_{j,s_j(t)}(\cdot,t) = \gamma_j(\cdot,t)$ when $\gamma_j(\cdot,t)$ is away from $\{u_i\}_{i=0}^m$ (when t is away from t_j), and decrease the energy by certain amount when $\gamma_j(\cdot,t)$ is close to $\{u_i\}_{i=0}^m$. i.e.,

$$E(\gamma_{j,s_j(t)}(\cdot,t)) \leq E(\gamma_j(\cdot,t)) - c, \tag{1.2}$$

for some constant c. Since $\lim_{j\to\infty} \max_{t\in[0,1]} E(\gamma_j(\cdot,t)) = \lim_{j\to\infty} E(\gamma_j(\cdot,t_j)) = W$, equation 1.2 implies

$$\lim_{j\to\infty} E(\gamma_{j,s_j(t)}(\cdot,t_j)) \leq W - c,$$

then the sequence $\{\gamma_{j,s_j(t)}(\cdot,t_j)\}_j$ does not converge to $\{u_i\}_{i=0}^m$, and $\gamma_{j,s_j(t)}(\cdot,t)$ is the desired sweepout. In order to construct $s_j(t)$, we observe the following one parameter gradient flow $\{\phi_j^t(\cdot,x)\} \in \mathrm{Diff}(\bar{B}^k)$, with $x \in \bar{B}^k$ as starting point, generated by the vector field:

$$s \mapsto -(1-|s|^2)\nabla E_j^t(s), \quad s \in \bar{B}^k.$$

$\phi_j^t(\cdot,x)$ decreases the energy, except that $|s| = 1$ or $\nabla E_j^t(x) = 0$, that means the starting point x is the maximal point for the energy. The assumption of the lower bound of Morse index, $\sum_{i=0}^m Index(u_i) = k > 1$, now enables us to construct a continuous curve $y_j : [0,1] \to \bar{B}^k$ avoiding the maximal point of the function E_j^t as t varies. Namely, let $x_j : [0,1] \to \bar{B}^k$ be the maximal point for $E_j^t(s)$, i.e., $\nabla E_j^t(x(t)) = 0$, $x(t)$ is a continuous curve on $\bar{B}^k$, $k \geq 2$. Since the dimension of $\bar{B}^k$ is larger than 1, we can choose a continuous curve $y_j(t)$ on $\bar{B}^k$ which does not intersect with $x_j([0,1])$, then we can use $\{\phi_j^t(\cdot,y_j(t))\}$ to decrease energy, construct $s_j(t)$ and obtain the new sweepout $\tilde{\gamma}_j(\cdot,t) := \gamma_{j,s_j(t)}(\cdot,t)$. The new sweepout is in the same homotopy class as $\{\gamma_j(\cdot,t)\}_j$, and

$$E(\tilde{\gamma}_j(\cdot,t_j)) \leq W - c,$$

so $\tilde{\gamma}_j(\cdot,t_j)$ does not converge to $\{u_i\}_{i=0}^m$. We can repeat the procedure inductively and obtain a sweepout which is in the same homotopy class as $\{\gamma_j(\cdot,t)\}_j$, but away from all harmonic spheres whose sum of Morse indexes is larger than one. Then we can conclude that it converges to a collection of finitely many harmonic spheres whose sum of Morse indexes is bounded by 1.

Chapter 2

Preliminaries

2.1 Harmonic Maps

2.1.1 Definition of Harmonic Maps

Let M and N be Riemannian manifolds of dimension m and n, respectively. We write the metric tensor of M as

$$(h_{\alpha\beta})_{\alpha,\beta:1,\ldots,m},$$

and the one of N as

$$(g_{ij})_{i,j:1,\ldots,n},$$

and we use the following notations:

$$\begin{aligned}
(h^{\alpha\beta})_{\alpha,\beta=1,\ldots,m}, &\quad \text{inverse metric tensor}\\
h := \det(h_{\alpha\beta}), &\\
\Gamma^{\alpha}_{\beta\gamma} := \frac{1}{2}h^{\alpha\delta}(h_{\beta\delta,\gamma} + h_{\gamma\delta,\beta} - h_{\beta\gamma,\delta}), &\quad \text{Christoffel symbols of } M\\
(g^{ij})_{i,j=1,\ldots,n}, &\quad \text{inverse metric tensor}\\
g := \det(g_{ij}), &\\
\Gamma^{i}_{jk} = \frac{1}{2}g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l}), &\quad \text{Christoffel symbols of } N
\end{aligned}$$

Definition 2.1.1 (energy density): If $f : M \to N$ is a map of class C^1, we define its

energy density as

$$e(f)(x) := \frac{1}{2}h^{\alpha\beta}(x)g_{ij}(f(x))\frac{\partial f^i(x)}{\partial x^\alpha}\frac{\partial f^j(x)}{\partial x^\beta},$$

in local coordinates $(x^1, ..., x^m)$ on M, $(f^1, ..., f^n)$ on N.

The value of $e(f)(x)$ seems to depend on the choice of local coordinates. We are now going to interpret $e(f)$ intrinsically and see that it's not so. Recall that $f^{-1}TN$ is a bundle over M with metric $g_{ij}(f(x))$, while T^*M has metric $(h^{\alpha\beta}(x))$. Now we consider the the differential of f

$$df = \frac{\partial f^i}{\partial x^\alpha}dx^\alpha \otimes \frac{\partial}{\partial f^i},$$

a section of the bundle $T^*M \otimes f^{-1}TN$. Then, with $\frac{\partial f}{\partial x^\alpha} = \frac{\partial f^i}{\partial x^\alpha}\frac{\partial}{\partial f^i}$ we have

$$\begin{aligned} e(f) &= \frac{1}{2}h^{\alpha\beta}\langle\frac{\partial f}{\partial x^\alpha}, \frac{\partial f}{\partial x^\beta}\rangle_{f^{-1}TN} \\ &= \frac{1}{2}\langle df, df\rangle_{T*M\otimes f^{-1}TN}. \end{aligned}$$

Definition 2.1.2 (Energy): The energy of a C^1-map $f : M \to N$ is

$$E(f) := \int_M e(f)dM,$$

here dM is the volume form of M with respect to metric h. In local coordinates, we can write it as $dM = d\text{vol}_h = \sqrt{h}dx^1 \wedge ... \wedge dx^m$.

Definition 2.1.3 (Conformal Change of Riemannian Metric): Let M be a m-dimensional Riemannian manifold. Given two metrics h and h^* on M, if the angles between two vectors with respect to h and h^* are always equal to each other at each point of M, we say that h and h^* are conformal to each other. A necessary and sufficient condition that h and h^* of M be conformal to each other is that there exist a function $\rho : M \to \mathbb{R}$ such that

$$h^* = e^{2\rho}h.$$

We call such a change of metric h to h^* a conformal change of Riemannian metric

Theorem 2.1.4: If the dimension of M is two, then the energy of a C^1-map f is invariant under conformal change of Riemannian metric.

Proof. We compare the difference of energy density (definition 2.1.1) of f on the manifolds (M, h) and $(M, e^{2\rho}h)$, where ρ is a real-valued function on M. Let $e_h(f)$

denote the energy density of f with respect to metric h and $e_{e^{2\rho}h}(f)$ denote the energy density of f with respect to metric $e^{2\rho}h$, we have

$$e_{e^{2\rho}h}(f) = e^{-2\rho}e_h(f).$$

Moreover, we compare the volume form

$$d\mathrm{vol}_{e^{2\rho}h} = e^{m\rho}d\mathrm{vol}_h,$$

here m is the dimension of M. Thus,

$$\begin{aligned} E_{(M,e^{2\rho}h)}(f) &= \int_M e_{e^{2\rho}h}(f)d\mathrm{vol}_{e^{2\rho}h} \\ &= \int_M e^{(m-2)\rho}e_h(f)d\mathrm{vol}_h \\ &= E_{(M,h)}(f), \end{aligned} \tag{2.1}$$

the last equality holds because $m = 2$. □

Lemma 2.1.5: The Euler-Lagrange equation for E is:

$$\frac{1}{\sqrt{h}}\frac{\partial}{\partial x^\alpha}(\sqrt{h}h^{\alpha\beta}\frac{\partial f^i}{\partial x^\beta}) + h^{\alpha\beta}(x)\Gamma^i_{jk}(f(x))\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta} = 0. \tag{2.2}$$

Proof. Let f be a smooth critical point of E. We let φ be a vector field along f. It means that φ is a section of $f^{-1}TN$. In local coordinates

$$\varphi = \varphi^i(x)\frac{\partial}{\partial f^i}.$$

φ induces a variation of f by

$$f_t(x) := \exp_{f(x)}(t\varphi(x)), \tag{2.3}$$

so we have

$$\frac{\partial}{\partial t}\Big|_{t=0} f_t(x) = \varphi(x).$$

As f is a critical point of E, we have

$$\frac{d}{dt}\Big|_{t=0} E(f_t(x)) = 0,$$

thus

$$\begin{aligned}
0 =& \frac{d}{dt}\Big|_{t=0} \frac{1}{2}\int_M h^{\alpha\beta}(x) g_{ij}(f_t(x)) \frac{\partial f_t^i}{\partial x^\alpha}\frac{\partial f_t^j}{\partial x^\beta}\sqrt{h}dx^1\wedge ...\wedge dx^m \\
=& \int_M \frac{1}{2} h^{\alpha\beta}(x) g_{ij,k}(f(x)) \frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}\varphi^k\sqrt{h}dx^1\wedge ...\wedge dx^m \\
&+ \int_M h^{\alpha\beta}(x) g_{ij}(f(x)) \frac{\partial f^i}{\partial x^\alpha}\frac{\partial \varphi^j}{\partial x^\beta}\sqrt{h}dx^1\wedge ...\wedge dx^m \\
=& \int_M \frac{1}{2} h^{\alpha\beta}(x) g_{ij,k}(f(x)) \frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}\varphi^k\sqrt{h}dx^1\wedge ...\wedge dx^m \\
&- \int_M \frac{\partial}{\partial x^\beta}(\sqrt{h}h^{\alpha\beta}\frac{\partial f^i}{\partial x^\alpha}) g_{ij}(f(x))\varphi^j dx^1\wedge ...\wedge dx^m \\
&- \int_M h^{\alpha\beta}(x) g_{ij,k}(f(x)) \frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta}\varphi^j\sqrt{h}dx^1\wedge ...\wedge dx^m,
\end{aligned}$$

we can integrate by part assuming φ is a compactly supported bounded section of $f^{-1}TN$. Putting $\eta_i = g_{ij}\varphi^j$, and thus $\varphi^j = g^{ij}\eta_i$. We then obtain

$$\begin{aligned}
0 =& -\int_M \frac{1}{\sqrt{h}}\frac{\partial}{\partial x^\beta}(\sqrt{h}h^{\alpha\beta}\frac{\partial f^i}{\partial x^\alpha})\eta_i\sqrt{h}dx^1\wedge ...\wedge dx^m \\
&- \int_M \frac{1}{2} h^{\alpha\beta} g^{lj}(g_{ij,k}+g_{kj,i}-g_{ik,j})\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta}\eta_l\sqrt{h}dx^1\wedge ...\wedge dx^m \\
=& -\int_M \frac{1}{\sqrt{h}}\frac{\partial}{\partial x^\beta}(\sqrt{h}h^{\alpha\beta}\frac{\partial f^i}{\partial x^\alpha})\eta_i\sqrt{h}dx^1\wedge ...\wedge dx^m \\
&- \int_M \frac{1}{2} h^{\alpha\beta}\Gamma^l_{ik}\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta}\eta_l\sqrt{h}dx^1\wedge ...\wedge dx^m.
\end{aligned}$$

Then we have the desired equation 2.2. □

We are going to interpret equation 2.2 intrinsically now.

Definition 2.1.6 (tension field): We call the tension field of f to be the following

$$\tau(f) := \text{trace}\nabla df.$$

Lemma 2.1.7: Intrinsically, the Euler-Lagrange equation for E is

$$\tau(f) = 0. \tag{2.4}$$

Proof. Let f be a smooth critical point of E. We let φ be a vector field along f. It means that φ is a smooth section of $f^{-1}TN$. In local coordinates

$$\varphi = \varphi^i(x)\frac{\partial}{\partial f^i}.$$

φ induces a variation of f by

$$f_t(x) := \exp_{f(x)}(t\varphi(x)). \tag{2.5}$$

Let $\nabla^{f^{-1}TN}$ denote the Levi-Civita connection on $f^{-1}TN$, and we have

$$d\varphi = \nabla^{f^{-1}TN}_{\frac{\partial}{\partial x^\alpha}}(\varphi^i \frac{\partial}{\partial f^i}) \otimes dx^\alpha. \tag{2.6}$$

We also have to take derivative with respect to t. Here $\frac{\partial}{\partial t}$ is a vector tangent to $M \times \mathbb{R}$. The Levi-Civita connection on M and the trivial connection on $\mathbb{R}$ yield the Levi-Civita connection on $T^*(M \times \mathbb{R}) \otimes f^{-1}TN$. Moreover, instead of $\nabla^N_{df(\frac{\partial}{\partial x^\alpha})} = (f^*\nabla^N)_{\frac{\partial}{\partial x^\alpha}}$, we shall simply write $\nabla_{\frac{\partial}{\partial x^\alpha}}$.

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial t}} df_t|_{t=0} &= \nabla_{\frac{\partial}{\partial t}} \frac{\partial f_t^i}{\partial x^\alpha} \frac{\partial}{\partial f^i} \otimes dx^\alpha|_{t=0} \\
&= \nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial f_t^i}{\partial t} \frac{\partial}{\partial f^i} \otimes dx^\alpha|_{t=0} \\
&= \nabla_{\frac{\partial}{\partial x^\alpha}} \varphi^i \frac{\partial}{\partial f^i} \otimes dx^\alpha \\
&= d\varphi.
\end{aligned} \tag{2.7}$$

As f is a critical point of E, then

$$\begin{aligned}
0 = \frac{d}{dt}\Big|_{t=0} E(f_t) &= \frac{1}{2}\int_M \frac{d}{dt}\langle df_t, df_t\rangle dM|_{t=0} \\
&= \int_M \langle df_t, \nabla_{\frac{\partial}{\partial t}} df_t\rangle dM|_{t=0} \\
&= \int_M \langle df, d\varphi\rangle dM \\
&= \int_M \langle df, \nabla_{\frac{\partial}{\partial x^\alpha}} \varphi \otimes dx^\alpha\rangle dM \\
&= -\int_M \langle \nabla_{\frac{\partial}{\partial x^\alpha}} df, \varphi \otimes dx^\alpha\rangle dM \\
&= -\int_M \langle \mathrm{trace}\nabla df, \varphi\rangle dM.
\end{aligned}$$

The third equality follows from equation 2.7 and the forth equality follows from equation 2.6. Thus, the Euler-Lagrange equation for E is

$$\tau(f) = 0.$$

□

We now write $\tau(f)$ in local coordinatas and show that it is equivalent to equation 2.2. Let ∇ denote the Levi-Civita connection in $T^*M \otimes f^{-1}TN$ as before.

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x^\beta}}(df) =&\nabla_{\frac{\partial}{\partial x^\beta}}\big(\frac{\partial f^i}{\partial x^\alpha}dx^\alpha\frac{\partial}{\partial f^i}\big)\\ =&\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta}dx^\alpha\frac{\partial}{\partial f^i} + \big(\nabla_{\frac{\partial}{\partial x^\beta}}dx^\alpha\big)\frac{\partial f^i}{\partial x^\alpha}\frac{\partial}{\partial f^i} + \big(\nabla_{\frac{\partial f^j}{\partial x^\beta}\frac{\partial}{\partial f^i}}\frac{\partial}{\partial f^j}\big)\frac{\partial f^i}{\partial x^\alpha}dx^\alpha \quad (2.8)\\ =&\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta}dx^\alpha\frac{\partial}{\partial f^i} - \Gamma^\alpha_{\beta\gamma}dx^\gamma\frac{\partial f^i}{\partial x^\alpha}\frac{\partial}{\partial f^i} + \frac{\partial f^j}{\partial x^\beta}\Gamma^k_{ij}\frac{\partial}{\partial f^k}\frac{\partial f^i}{\partial x^\alpha}dx^\alpha,\end{aligned}$$

then we have

$$\tau^i(f) = h^{\alpha\beta}\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta} - h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\frac{\partial f^i}{\partial x^\gamma} + h^{\alpha\beta}\Gamma^i_{jk}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta}. \quad (2.9)$$

We compare equation 2.9 and equation 2.2. They are equivalent since one can easily compute that

$$\begin{aligned}\frac{1}{\sqrt{h}}\frac{\partial}{\partial x^\alpha}(\sqrt{h}h^{\alpha\beta}\frac{\partial f^i}{\partial x^\beta}) &= \frac{1}{\sqrt{h}}\big(\frac{\partial}{\partial x^\alpha}\sqrt{h}\big)h^{\alpha\beta}\frac{\partial f^i}{\partial x^\beta} + \big(\frac{\partial}{\partial x^\alpha}h^{\alpha\beta}\big)\frac{\partial f^i}{\partial x^\beta} + h^{\alpha\beta}\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta}\\ &= \frac{h}{2h}h^{\delta\gamma}h_{\delta\gamma,\alpha}h^{\alpha\beta}\frac{\partial f^i}{\partial x^\beta} - \big(h^{\delta\alpha}h^{\gamma\beta}h_{\delta\gamma,\alpha}\big)\frac{\partial f^i}{\partial x^\beta} + h^{\alpha\beta}\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta}\\ &= -h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\frac{\partial f^i}{\partial x^\gamma} + h^{\alpha\beta}\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta}.\end{aligned} \quad (2.10)$$

The operator

$$\Delta_M := \frac{1}{\sqrt{h}}\frac{\partial}{\partial x^\alpha}\big(\sqrt{h}h^{\alpha\beta}\frac{\partial}{\partial x^\beta}\big) \quad (2.11)$$

is called Laplace-Beltrami operator of M. By equation 2.9 and equation 2.10 we have the following equality

$$\tau(f) = \Delta_M f + h^{\alpha\beta}\Gamma^i_{jk}\frac{\partial f^j}{\partial x^\alpha}\frac{\partial f^k}{\partial x^\beta}. \quad (2.12)$$

Definition 2.1.8 (Harmonic Map): Given a C^2 function $f : M \to N$. It is a harmonic map if

$$\tau(f) = 0.$$

Let $\tilde{M}$ be a m-dimensional submanifold of the n-dimensional Riemannian manifold $\tilde{N}$. The metric on $\tilde{N}$ induces a metric on $\tilde{M}$, and for sections X Y of TM we have

$$\nabla^{\tilde{M}}_X Y = (\nabla^{\tilde{N}}_X Y)^\top,$$

where $(\cdot)^\top : T_x\tilde{N} \to T_x\tilde{M}$ denotes the orthogonal projection.

Definition 2.1.9 (Second Fundamental Form): For $X, Y \in T_x\tilde{M}$, we define the second fundamental form of the manifold $\tilde{M}$ in the ambient manifold $\tilde{N}$ to be the following

$$A_{\tilde{M}}(X, Y) := (\nabla^{\tilde{N}}_X Y)^{\perp},$$

here $(\cdot)^{\perp}$ denotes the orthogonal complement of $T_x\tilde{M}$ in $T_x\tilde{N}$.

If $f : M \to N$ is an isometric immersion. This means that for each $p \in M$, there exists a neighborhood U for which

$$f : U \to f(U)$$

is an isometry. We define

$$e_\alpha := f_*\big(\frac{\partial}{\partial x^\alpha}\big) = \frac{\partial f^i}{\partial x^\alpha}\frac{\partial}{\partial f^i}.$$

A submanifold $f(M)$ of N is call *minimal* if its *mean curvature* vanishes. i.e.,

$$h^{\alpha\beta}\big(\nabla^N_{e_\beta} e_\alpha\big)^{\perp} = 0, \tag{2.13}$$

here $(\cdot)^{\perp}$ denotes the orthogonal complement of T_xM in T_xN.

Theorem 2.1.10: An isometric immersion $f : M \to N$ is harmonic if and only if it represents a minimal submanifold of N.

Proof. If f is an isometric immersion, we can write mean curvature in local coordinates as the following:

$$\begin{aligned}
h^{\alpha\beta}(\nabla^N_{e_\beta} e_\alpha)^{\perp} &= h^{\alpha\beta}\nabla^N_{e_\beta} e_\alpha - h^{\alpha\beta}(\nabla^N_{e_\beta} e_\alpha)^{\top} \\
&= h^{\alpha\beta}\nabla^N_{e_\beta} e_\alpha - h^{\alpha\beta}\nabla^M_{\frac{\partial}{\partial x_\beta}}\frac{\partial}{\partial x_\alpha} \\
&= h^{\alpha\beta}\nabla^N_{\frac{\partial f^j}{\partial x^\beta}\frac{\partial}{\partial f^j}}\frac{\partial f^i}{\partial x^\alpha}\frac{\partial}{\partial f^i} - h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial x^\gamma} \\
&= h^{\alpha\beta}\frac{\partial^2 f^i}{\partial x_\alpha \partial x_\beta}\frac{\partial}{\partial f^i} + h^{\alpha\beta}\Gamma^k_{ij}\frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\beta}\frac{\partial}{\partial f^k} - h^{\alpha\beta}\Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial x^\gamma}.
\end{aligned} \tag{2.14}$$

We compare equation 2.14 and equation 2.9 and see that vanishing mean curvature implies the vanishing tension field:

$$\tau(f) = 0.$$

□

Lemma 2.1.11: For smooth maps $f : M \to N$, $p : N \to Q$ between Riemannian manifolds, the following chain rule holds

$$\tau(p \circ f) = h^{\alpha\beta}\nabla dp\big(\frac{\partial f}{\partial x^\alpha}, \frac{\partial f}{\partial x^\beta}\big) + (dp) \circ (\tau(f)).$$

Proof. By simple computation we have that

$$\begin{aligned}
\tau(p \circ f) &= \text{trace}\nabla d(p \circ f) \\
&= h^{\alpha\beta}\nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta}(p \circ f) \\
&= h^{\alpha\beta}\nabla_{\frac{\partial}{\partial x^\alpha}} \big(\frac{\partial p}{\partial f^i}\frac{\partial f^i}{\partial x^\beta}\big) \\
&= h^{\alpha\beta}\big(\nabla_{\frac{\partial f^j}{\partial x^\alpha}\frac{\partial}{\partial f^j}} \frac{\partial p}{\partial f^i}\big)\frac{\partial f^i}{\partial x^\beta} + h^{\alpha\beta}\frac{\partial p}{\partial f^i}\big(\nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial f}{\partial x^\beta}\big) \\
&= h^{\alpha\beta}\big(\nabla_{\frac{\partial}{\partial f^i}} \frac{\partial p}{\partial f^j}\big)\frac{\partial f}{\partial x^\alpha}\frac{\partial f}{\partial x^\beta} + (dp) \circ (\tau(f)) \\
&= h^{\alpha\beta}\nabla dp\big(\frac{\partial f}{\partial x^\alpha}, \frac{\partial f}{\partial x^\beta}\big) + (dp) \circ (\tau(f)).
\end{aligned} \tag{2.15}$$

□

Remark. In particular, if f is a harmonic map, then we have

$$\tau(p \circ f) = h^{\alpha\beta}\nabla dp\big(\frac{\partial f}{\partial x^\alpha}, \frac{\partial f}{\partial x^\beta}\big). \tag{2.16}$$

Remark. Given Riemannian manifolds (M, h) and (N, g), N is isometrically embedded in $\mathbb{R}^k$. We consider a harmonic map $f : M \to N \subseteq \mathbb{R}^k$, then by 2.12 and 2.16 we know the tension field of f becomes:

$$\Delta_M f - h^{\alpha\beta}A_N\big(\frac{\partial f}{\partial x^\alpha}, \frac{\partial f}{\partial x^\beta}\big) = 0, \tag{2.17}$$

here Δ_M is Laplace-Beltrami operator defined as 2.11 and $A_N(\cdot, \cdot)$ is the second fundamental form of N in $\mathbb{R}^k$.

Definition 2.1.12 (Curvature Tensor): The curvature tensor R^N of connection ∇ on manifold N is

$$R^N(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for vector fields X, Y, Z on N. We define the components R^N_{ijkl} by

$$\Big\langle R^N\big(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\big)\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\Big\rangle_{TN} = R^N_{ijkl},$$

and the Ricci curvature of N to be

$$R^N_{ij} = g^{kl}R^N_{ikjl}.$$

Now we state the Bochner Lemma, for the proof we refer to [Moo17, theorem 4.6.1].

Lemma 2.1.13 (Bochner Lemma): Let $f : M \to N$ be a harmonic map, then

$$\Delta(e(f)) = |\nabla df|^2 + h^{\alpha\gamma}h^{\beta\delta}R^M_{\alpha\beta}\frac{\partial f}{\partial x^\gamma}\frac{\partial f}{\partial x^\delta} - h^{\alpha\gamma}h^{\beta\delta}R^N_{ijkl}\frac{\partial f}{\partial x^\alpha}\frac{\partial f}{\partial x^\beta}\frac{\partial f}{\partial x^\gamma}\frac{\partial f}{\partial x^\delta}.$$

Now we give a few examples of harmonic maps.

Example 2.1.14: Constant maps and identity maps are harmonic maps.

Example 2.1.15 (Harmonic functions): In the case $N = \mathbb{R}$, harmonic maps are simply harmonic functions, since

$$\Gamma^i_{jk} = 0,$$

and the Euler-Lagrange equation 2.12 becomes:

$$\Delta_M f = 0.$$

Example 2.1.16 (Geodesics): If dimension of M is one, the Euler-Lagrange equation 2.2 can be written as

$$\frac{d^2 f^i}{dx^2} + \Gamma^i_{jk}\frac{df^j}{dx}\frac{df^k}{dx} = 0,$$

which is the equation for geodesics.

2.1.2 Second Variation for Energy

Let

$$\begin{aligned} &f_{st}(x) = f(x, s, t), \\ &f : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to N \end{aligned} \tag{2.18}$$

be a smooth family of maps between Riemannian manifolds of finite energy. M may have nonempty boundary, in which case we require $f(x, s, t) = f(x, 0, 0)$ for all $x \in \partial M$ and all s, t.

Theorem 2.1.17: For a smooth family of maps $f_{st} : M \to N$ defined as 2.18, with

$$\begin{aligned} V :=& \frac{\partial}{\partial s}\Big|_{s=t=0} f_{st} \\ W :=& \frac{\partial}{\partial t}\Big|_{s=t=0} f_{st}. \end{aligned}$$

To simplify the notation, we write f instead of f_{st}. The second variation of energy is:

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}\Big|_{s=t=0}E(f_{st}) = &\int_M \langle \nabla V, \nabla W\rangle_{f^{-1}TN}dM \\ &- \int_M trace_M\langle R^N(df,V)W,df\rangle_{f^{-1}TN}dM \\ &+ \int_M \langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial f}{\partial s}, trace_M\nabla df\rangle_{f^{-1}TN}dM,\end{aligned} \tag{2.19}$$

here ∇ denote the Levi-Civita connection in $f^{-1}TN$.

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial}{\partial s}\frac{1}{2}\Big\langle \frac{\partial f}{\partial x^\alpha}dx^\alpha, \frac{\partial f}{\partial x^\beta}dx^\beta\Big\rangle_{T^*M\otimes f^{-1}TN} =&\frac{\partial}{\partial t}\Big\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial f}{\partial x^\alpha}dx^\alpha, \frac{\partial f}{\partial x^\beta}dx^\beta\Big\rangle_{T^*M\otimes f^{-1}TN} \\ =&\frac{\partial}{\partial t}\Big\langle \nabla_{\frac{\partial}{\partial x^\alpha}}\frac{\partial f}{\partial s}dx^\alpha, \frac{\partial f}{\partial x^\beta}dx^\beta\Big\rangle_{T^*M\otimes f^{-1}TN} \\ =&\Big\langle \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial x^\alpha}}\frac{\partial f}{\partial s}dx^\alpha, \frac{\partial f}{\partial x^\beta}dx^\beta\Big\rangle_{T^*M\otimes f^{-1}TN} \\ &+\Big\langle \nabla_{\frac{\partial}{\partial x^\alpha}}\frac{\partial f}{\partial s}dx^\alpha, \nabla_{\frac{\partial}{\partial t}}\frac{\partial f}{\partial x^\beta}dx^\beta\Big\rangle_{T^*M\otimes f^{-1}TN} \\ =&\Big\langle \nabla_{\frac{\partial}{\partial x^\alpha}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial f}{\partial s}dx^\alpha, \frac{\partial f}{\partial x^\beta}dx^\beta\Big\rangle_{T^*M\otimes f^{-1}TN} \\ &+\Big\langle R^N(\frac{\partial f}{\partial t},\frac{\partial f}{\partial x^\alpha})\frac{\partial f}{\partial s}dx^\alpha, \frac{\partial f}{\partial x^\beta}dx^\beta\Big\rangle_{T^*M\otimes f^{-1}TN} \\ &+\Big\langle \nabla_{\frac{\partial}{\partial x^\alpha}}\frac{\partial f}{\partial s}dx^\alpha, \nabla_{\frac{\partial}{\partial x^\beta}}\frac{\partial f}{\partial t}dx^\beta\Big\rangle_{T^*M\otimes f^{-1}TN},\end{aligned}$$

then we evaluate everything at $s=t=0$ and get

$$\begin{aligned}\frac{\partial^2}{\partial s\partial t}\Big|_{s=t=0}E(f_{st}) = &\int_M \langle \nabla V, \nabla W\rangle_{f^{-1}TN}dM \\ &+ \int_M \Big\langle \nabla_{\frac{\partial}{\partial x^\alpha}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial f}{\partial s}dx^\alpha, \frac{\partial f}{\partial x^\beta}dx^\beta\Big\rangle_{T^*M\otimes f^{-1}TN}dM \\ &- \int_M trace\langle R^N(df,V)W,df\rangle_{f^{-1}TN}dM \\ = &\int_M \langle \nabla V, \nabla W\rangle_{f^{-1}TN}dM \\ &- \int_M \Big\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial f}{\partial s}, trace_M\nabla df\Big\rangle_{f^{-1}TN}dM \\ &- \int_M trace\langle R^N(df,V)W,df\rangle_{f^{-1}TN}dM.\end{aligned}$$

□

Corollary 2.1.18: If $f(x,0,0) : M \to N$ is harmonic we have

$$\begin{aligned}\frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} E(f_{st}) = &\int_M \langle \nabla V, \nabla W \rangle_{f^{-1}TN} dM \\ &- \int_M trace_M \langle R^N(df, V)W, df \rangle_{f^{-1}TN} dM.\end{aligned} \tag{2.20}$$

Proof. Since $f(x,0,0) : M \to N$ is harmonic we have

$$\tau(f(x,0,0)) = trace \nabla df(x,0,0) = 0.$$

Then it's obvious that 2.20 follows from 2.19. □

We write the second variation of energy as

$$\delta^2 E(f)(V,W) := \frac{\partial^2}{\partial s \partial t}\Big|_{s=t=0} E(f_{st}).$$

Definition 2.1.19 (Index Form): The index form of a harmonic map $f : M \to N$ is defined by

$$\begin{aligned}I(V,W) = &\int_M \langle \nabla V, \nabla W \rangle_{f^{-1}TN} dM \\ &- \int_M trace_M \langle R^N(df, V)W, df \rangle_{f^{-1}TN} dM.\end{aligned} \tag{2.21}$$

We define the Jacobe operator of f to be

$$\mathcal{J}_f(V) := -\Delta V - \mathrm{trace}_M R^N(V, df)df, \tag{2.22}$$

here Δ is the Laplacian on sections of $f^{-1}TN$ given in local coordinates on M by

$$\Delta = h^{\alpha\beta}(f^*\nabla^N)_{\frac{\partial}{\partial x^\alpha}}(f^*\nabla^N)_{\frac{\partial}{\partial x^\beta}},$$

so we have

$$I(V,W) = \int_M \langle \mathcal{J}_f(V), W \rangle dM.$$

Proposition 2.1.20 ([LW02]): For a smooth family of maps $f_{st} : M \to N$ defined as 2.18, with

$$\begin{aligned}V :=& \frac{\partial}{\partial s}\Big|_{s=t=0} f_{st} \\ W :=& \frac{\partial}{\partial t}\Big|_{s=t=0} f_{st}.\end{aligned}$$

Let $f_{00} = f$ be a smooth harmonic map. We have the Jacobi operator $\mathcal{J}_f$ defined as (2.22) to be the following:

$$\mathcal{J}_f(V) = -\frac{\partial}{\partial s}\Big|_{s=0}\tau(f_{s0}).$$

Proof. The proposition follows from the computation below:

$$\begin{aligned}
\frac{\partial^2}{\partial s\partial t}\Big|_{s=t=0}E(f_{st}) &= \frac{\partial}{\partial s}\Big|_{s=0}\int_M \langle df_{st}, \nabla_{\frac{\partial}{\partial t}} df_{st}\rangle|_{t=0} \\
&= -\frac{\partial}{\partial s}\Big|_{s=0}\int_M \langle \tau(f_{s0}), \frac{\partial}{\partial t}\Big|_{t=0} f_{st}\rangle \\
&= \int_M \langle -\frac{\partial}{\partial s}\Big|_{s=0}\tau(f_{s0}), \frac{\partial}{\partial t}\Big|_{s=t=0} f_{st}\rangle dM + \int_M \langle \tau(f_{00}), -\frac{\partial^2}{\partial s\partial t}\Big|_{s=t=0} f_{st}\rangle dM \\
&= \int_M \langle -\frac{\partial}{\partial s}\Big|_{s=0}\tau(f_{s0}), W\rangle dM \\
&= \int_M \langle \mathcal{J}_f(V), W\rangle dM.
\end{aligned}$$

The second to last equality follows by f_{00} is harmonic. Then it implies that

$$\mathcal{J}_f(V) = -\frac{\partial}{\partial s}\Big|_{s=0}\tau(f_{s0}).$$

□

Definition 2.1.21 (Index): The index of a harmonic map $f : M \to N$ is the maximal dimension of the subspace X of $\Gamma(f^{-1}TN)$ on which the index form is negative definite on X.

We define the nullity of a harmonic map f as the dimension of kernel of $\mathcal{J}_f$. The nullity of a harmonic map f is also finite when M is compact. Any

$$V \in \ker \mathcal{J}_f$$

is called a *Jacobi field* for the harmonic map f.

Definition 2.1.22: [Stable Harmonic Maps] We call a harmonic map *stable* if index of it is zero.

The second variation of energy of a harmonic map f derived in (2.19):

$$\begin{aligned}
\frac{\partial^2}{\partial s\partial t}\Big|_{s=t=0}E(f_{st}) = &\int_M \langle \nabla V, \nabla W\rangle_{f^{-1}TN} dM \\
&- \int_M trace_M \langle R^N(df, V)W, df\rangle_{f^{-1}TN} dM,
\end{aligned} \tag{2.23}$$

implies that if N has nonpositive sectional curvature, then

$$\int_M trace_M \langle R^N(df, V)V, df\rangle_{f^{-1}TN} dM < 0$$

thus any harmonic map into N is stable. This shows the difference between the stabilities of harmonic maps and minimal submanifolds. As we mentioned in the example before, when $f : M \to N$ is an isometric immersion, f is harmonic if and only if f is a minimal immersion. Suppose N is the Euclidean space. Then f as a harmonic map is always stable. On the other hand, the catenoid in $\mathbb{R}^3$ is a complete minimal surface which is not a stable minimal surface. Conversely, for the cylinder $S^3 \times \mathbb{R}$ the standard embedding

$$f : S^3 \to S^3 \times \mathbb{R}$$

is a stable minimal imbedding. But as a harmonic map it is unstable. The identity map which is obviously harmonic. But it might be unstable. R. T. Smith showed that the identity map in S^n is unstable and index of it is $n+1$. In general, we have the following result:

Theorem 2.1.23 ([Xin80]): When $n > 2$, any stable harmonic map from S^n into any Riemannian manifold must be a constant map.

Theorem 2.1.24 ([Xin80]): Let M be a compact manifold. When $n > 2$, any stable harmonic map from M into S^n must be a constant map.

When manifold M is two-dimensional. Instead of isometric immersion, it's known that the conformal parameterisation of a harmonic map is a minimal surface (see theorem 2.1.38). Let Σ_g be a Riemann surface with genus g and $u : \Sigma_g \to M$ be a (possibly branched) minimal immersion. We define i_A to be the index of u as a critical point for area functional, and let i_E be the index of u as a critical point for energy functional E. We have the following comparison result:

Theorem 2.1.25 ([EM08]): Let $u : \Sigma_g \to M$ be a (possibly branched) minimal immersion of a closed Riemann surface of genus g into a Riemannian manifold M. Then

$$i_E \leq i_A \leq i_E + r,$$

where, if $b =$ total number of branch points of u counted with multiplicity then

$$r = \begin{cases} 6g - 6 - 2b, & \text{for} \quad b \leq 2g - 3, \\ 4g - 2 + 2[\frac{-b}{2}], & \text{for} \quad 2g - 2 \leq b \leq 4g - 4, \\ 0, & \text{for} \quad 4g - 3 \leq b. \end{cases}$$

Remark. For a minimal immersion $u : S^2 \to M$, the index of it as a harmonic sphere and the index of it as a minimal immersion is the same.

2.1.3 Harmonic Maps from Riemann Surface

Let (Σ, h) be a two-dimensional Riemannian manifold and that the coordinates (x^1, x^2) are chosen on Σ for which $\frac{\partial}{\partial x^1}$ and $\frac{\partial}{\partial x^2}$ are always orthogonal and of the same length w.r.t. the metric $\langle \cdot, \cdot \rangle_h$ of Σ, that is

$$\left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1} \right\rangle_h = \left\langle \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2} \right\rangle_h,$$

$$\left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right\rangle_h = 0.$$

This is equivalent to the metric h being represented by

$$\lambda^2(x)(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$$

with a smooth function $\lambda : \Sigma \to \mathbb{R}$.

Definition 2.1.26 (Conformal Structure): A surface (Σ, h) with a conformal structure is a two-dimensional differentiable manifold with an atlas whose transition functions $y = \phi(x)$ satisfy

$$dy^1 \otimes dy^1 + dy^2 \otimes dy^2 = e^{\rho(x)}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$$

$y = (y^1, y^2)$, $x = (x^1, x^2)$, for some function $\rho : \Sigma \to \mathbb{R}$.

Definition 2.1.27 (Riemann Surface): A Riemann surface is a complex manifold of complex dimension 1.

Thus, given a Riemann surface (Σ, h), coordinate charts on a Σ are given by maps

$$\varphi_i : U_i \to \mathbb{C},$$

U_i open in Σ, for which the transition functions

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j),$$

are holomorphic maps between open subsets of $\mathbb{C}$.

We write the coordinates in $\mathbb{C}$ as

$$z = x + iy.$$

For a coordinate transformation $w(z) = u(x, y) + iv(x, y)$, we thus have the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

and in particular

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0.$$

We see that a Riemann surface has a conformal structure in the sense of definition 2.1.26.

We define operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big),$$

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big),$$

and 1-forms

$$dz = dx + idy,$$

$$d\bar{z} = dx - idy.$$

These satisfy

$$dz\Big(\frac{\partial}{\partial z}\Big) = 1 = d\bar{z}\Big(\frac{\partial}{\partial \bar{z}}\Big),$$

$$dz\Big(\frac{\partial}{\partial \bar{z}}\Big) = 0 = d\bar{z}\Big(\frac{\partial}{\partial z}\Big).$$

Definition 2.1.28 (Conformal Metric): A Riemannian metric $\langle \cdot, \cdot \rangle$ on a Riemann surface Σ is called conformal if in local coordinates it can be written as

$$\lambda^2(z) dz \otimes d\bar{z},$$

for some positive, real valued function $\lambda(z)$ on Σ. This means

$$\Big\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \Big\rangle = 0 = \Big\langle \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{z}} \Big\rangle,$$

$$\Big\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \Big\rangle = \lambda^2(z).$$

We refer to lemma 9.1.1. of [Jos11] for the proof of the following lemma:

Lemma 2.1.29: Every Riemann surface admits a conformal metric.

For the rest of the section, Σ will represent a Riemann surface with a conformal metric $\lambda^2(z)$, and (M, g) a Riemannian manifold. Then we have energy density (see definition 2.1.1) of a C^1 map $u : \Sigma \to M$ in local coordinates is

$$e(u) = \frac{1}{\lambda^2(z)} g_{ij}(u(z)) \frac{\partial u^i}{\partial z} \frac{\partial u^j}{\partial \bar{z}},$$

and the energy of u is

$$E(u) = \int_\Sigma g_{ij}(u(z)) \frac{\partial u^i}{\partial z} \frac{\partial u^j}{\partial \bar{z}} dz d\bar{z}. \tag{2.24}$$

Lemma 2.1.30: Given a Riemann surface (Σ, h) with $h = \lambda^2(z) dz \otimes d\bar{z}$. A map $u : \Sigma \to M$ of class C^2 is harmonic if it satisfies

$$\frac{\partial^2 u^i}{\partial z \partial \bar{z}} + \Gamma^i_{jk}(u(z)) \frac{\partial u^j}{\partial z} \frac{\partial u^k}{\partial \bar{z}} = 0. \tag{2.25}$$

Proof. Recall that the Euler-Lagrange equation for energy is

$$\frac{1}{\sqrt{h}} \frac{\partial}{\partial z^\alpha} (\sqrt{h} h^{\alpha\beta} \frac{\partial u^i}{\partial z^\beta}) + h^{\alpha\beta}(z) \Gamma^i_{jk}(u(z)) \frac{\partial u^j}{\partial z^\alpha} \frac{\partial u^k}{\partial z^\beta} = 0.$$

(see (2.2)), and

$$h_{\alpha\beta} = \begin{pmatrix} 0 & \lambda^2 \\ \lambda^2 & 0 \end{pmatrix}$$

substituting h then the Euler-Lagrange equation becomes

$$\frac{\partial^2 u^i}{\partial z \partial \bar{z}} + \Gamma^i_{jk}(u(z)) \frac{\partial u^j}{\partial z} \frac{\partial u^k}{\partial \bar{z}} = 0.$$

□

Remark. Tension field of u is

$$\tau(u) = \nabla_{\frac{\partial}{\partial \bar{z}}} \frac{\partial u}{\partial z}.$$

Moreover, let M be isometrically embedded in $\mathbb{R}^N$, consider equation 2.17 and we can write the Euler-Lagrange 2.25 as the following

$$\Big(\frac{\partial^2 u}{\partial z \partial \bar{z}} \Big)^\top = 0, \tag{2.26}$$

here $(\cdot)^\top$ denotes the orthogonal projection on $T_u M$.

Definition 2.1.31 (Weakly Conformal Map): A C^1-map $u : \Sigma \to M$ is called weakly conformal if

$$\Big\langle \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z} \Big\rangle_g = 0.$$

We are going to show that every harmonic map from Riemann sphere is weakly conformal.

Definition 2.1.32 (Hopf differential): Given a map $u : \Sigma \to M$, the Hopf differential of u is

$$\psi_u(z)dz^2 = \left\langle \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z} \right\rangle_g dz^2,$$

here $dz^2 := dz \otimes dz$.

We can see that $\psi_u \equiv 0$ if and only if u is weakly conformal.

Theorem 2.1.33: If $u : \Sigma \to M$ is a harmonic function. Then the Hopf differential ψ_u is a holomorphic function.

Proof. In local coordinates we have

$$\psi_u(z) = g_{ij}(u)\frac{\partial u^i}{\partial z}\frac{\partial u^j}{\partial z}.$$

We have to show that for u harmonic we have

$$\frac{\partial}{\partial \bar{z}}\Big(g_{ij}(u)\frac{\partial u^i}{\partial z}\frac{\partial u^j}{\partial z}\Big) = 0.$$

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}}\Big(g_{ij}(u)\frac{\partial u^i}{\partial z}\frac{\partial u^j}{\partial z}\Big) &= 2g_{ij}(u)\frac{\partial^2 u^i}{\partial z \partial \bar{z}}\frac{\partial u^j}{\partial z} + g_{ij,k}\frac{\partial u^i}{\partial z}\frac{\partial u^j}{\partial z}\frac{\partial u^k}{\partial \bar{z}} \\
&= 2g_{ij}(u)\frac{\partial^2 u^i}{\partial z \partial \bar{z}}\frac{\partial u^j}{\partial z} + (g_{lj,k} - g_{lk,j} - g_{jk,l})\frac{\partial u^k}{\partial \bar{z}}\frac{\partial u^l}{\partial z}\frac{\partial u^j}{\partial z} \\
&= 2g_{ij}(u)\frac{\partial u^j}{\partial z}\Big(\frac{\partial^2 u^i}{\partial z \partial \bar{z}} + \Gamma^i_{kl}(u(z))\frac{\partial u^k}{\partial z}\frac{\partial u^l}{\partial \bar{z}}\Big) \\
&= 0
\end{aligned}$$

□

Now we consider the case $\Sigma = S^2$, $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ with the following two coordinate charts:

$$\varphi_1 : S^2 \setminus \{(0,0,1)\} \to \mathbb{C}, \quad \varphi_1(x_1, x_2, x_3) = \frac{1}{1 - x_3}(x_1 + ix_2),$$

$$\varphi_2 : S^2 \setminus \{(0,0,-1)\} \to \mathbb{C}, \quad \varphi_2(x_1, x_2, x_3) = \frac{1}{1 + x_3}(x_1 - ix_2),$$

observe that the coordinate transformation is

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, \quad \varphi_2 \circ \varphi_1^{-1}(z) = \frac{1}{z},$$

and it's holomorphic. Thus we have verified that S^2 is a Riemann surface.

Lemma 2.1.34: Every holomorphic quadratic differential on S^2 vanishes identically.

Proof. We put $z = \varphi_1(x)$ and write the given holomorphic quadratic differential in the chart φ_1 as

$$\psi(z)dz^2, \quad \text{with } \psi : \mathbb{C}(= \varphi_1(S^2 \setminus \{(0,0,1)\})) \to \mathbb{C} \text{ holomorphic.}$$

Then with $\varphi_2(x) = w = \frac{1}{z}$ for $z \neq 0$,

$$\psi(z)dz^2 = \psi(z(w))\Big(\frac{\partial z}{\partial w}\Big)^2 dw^2 = \psi(z(w))\frac{1}{w^4}dw^2.$$

Since we have a holomorphic quadratic differential on S^2, it has to be bounded as $w \to 0$. We conclude that ψ is an holomorphic function on $\mathbb{C}$ with

$$\psi(z) \to 0 \text{ as } z \to \infty,$$

hence $\psi \equiv 0$ by Liouville's theorem. □

Corollary 2.1.35: Harmonic map $u : S^2 \to M$ is weakly conformal.

Proof. By theorem 2.1.33 and lemma 2.1.34. □

Here we also present a theorem by Almgren [Alm66, Lemma 1] which is an application of holomorphic differentials:

Theorem 2.1.36 ([Alm66]): Any nonconstant conformal harmonic map

$$u : S^2 \to S^3 \subseteq \mathbb{R}^4$$

is a parametrization of a totally geodesic two-sphere in S^3

Proof. Suppose that

$$u : S^2 \to S^3 \subseteq \mathbb{R}^4$$

is any parametrized minimal surface, let $\cdot$ denot the usual dot product in $\mathbb{R}^4$, then:

$$u \cdot u \equiv 1 \Rightarrow \frac{\partial u}{\partial z} \cdot u = 0 = \frac{\partial u}{\partial \bar{z}} \cdot u.$$

Since u is weakly conformal we have:

$$\frac{\partial u}{\partial z}\cdot\frac{\partial u}{\partial z}=0=\frac{\partial u}{\partial \bar{z}}\cdot\frac{\partial u}{\partial \bar{z}},$$

and

$$\frac{\partial u}{\partial z}\cdot\frac{\partial u}{\partial \bar{z}}=\lambda^2,$$

here λ^2 is a positive real function on S^2, and differentiation yields:

$$\frac{\partial^2 u}{\partial z^2}\cdot\frac{\partial u}{\partial z}=\frac{\partial^2 u}{\partial z\partial \bar{z}}\cdot\frac{\partial u}{\partial z}=0=\frac{\partial^2 u}{\partial \bar{z}^2}\cdot\frac{\partial u}{\partial \bar{z}}=\frac{\partial^2 u}{\partial z\partial \bar{z}}\cdot\frac{\partial u}{\partial \bar{z}}.$$

Since u is harmonic, by (2.26), we have

$$\Big(\frac{\partial^2 u}{\partial z\partial \bar{z}}\Big)^{\top}=0,$$

this implies

$$\frac{\partial^2 u}{\partial z\partial \bar{z}}=\mu u,$$

for some smooth function μ on S^2. Hence

$$\frac{\partial}{\partial \bar{z}}\Big(\frac{\partial^2 u}{\partial z^2}\cdot\frac{\partial^2 u}{\partial z^2}\Big)=2\Big(\frac{\partial}{\partial z}(\mu u)\Big)\cdot\frac{\partial^2 u}{\partial z^2}=0.$$

This implies that the quadratic differential

$$\Big(\frac{\partial^2 u}{\partial z^2}\cdot\frac{\partial^2 u}{\partial z^2}\Big)dz^4$$

vanishes by [Hop89, Lemma 2.6]. Then the space spanned by $u, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}}$ must be preserved by $\partial/\partial z$ and $\partial/\partial\bar{z}$ and hence parallel, and this implies that u maps into a three plane in $\mathbb{R}^4$. □

Definition 2.1.37 (Area Functional): Given a Riemann surface Σ with a conformal metric $\lambda^2(dx\otimes dx+dy\otimes dy)$, (M,g) a Riemannian manifold isometrically embedded in $\mathbb{R}^N$, and $u:\Sigma\to M$ of class C^1. The area functional $\mathcal{A}$ is defined to be:

$$\mathcal{A}:C^2(\Sigma,M)\to\mathbb{R},\quad \mathcal{A}(u)=\int_\Sigma\Big|\frac{\partial u}{\partial x}\wedge\frac{\partial u}{\partial y}\Big|dxdy.$$

Since

$$\Big|\frac{\partial u}{\partial x}\wedge\frac{\partial u}{\partial y}\Big|\le\frac{1}{2}\Big(\Big|\frac{\partial u}{\partial x}\Big|^2+\Big|\frac{\partial u}{\partial y}\Big|^2\Big).$$

We can see that $E(u)\ge\mathcal{A}(u)$ and the equality holds if and only if u is weakly conformal.

Remark. We mentioned minimal surface before (we defined minimal surface as a two dimensional submanifold whose mean curvature vanishes, see 2.13). Notice that the critical point of $\mathcal{A}$ defined above is a minimal surface.

Theorem 2.1.38: A weakly conformal harmonic map $u : \Sigma \to M$ is a critical point for the area function $\mathcal{A}$.

Proof. If u is weakly conformal and harmonic, and $t \mapsto u(t)$ is a smooth family of maps with $u(0) = u$, then

$$E(u(t)) \geq \mathcal{A}(u(t)) \quad \text{and} \quad E(u(0)) = \mathcal{A}(u(0)).$$

That implies

$$0 = \frac{d}{dt} E(u(t))\Big|_{t=0} = \frac{d}{dt} \mathcal{A}(u(t))\Big|_{t=0},$$

so u is critical for the area function $\mathcal{A}$. □

We have proved that any harmonic map on sphere is weakly conformal thus minimal.

2.1.4 Weakly Harmonic Surface

Throughout the section, (Σ, h) will represent a Riemann surface, and (M, g) will represent a closed Riemannian manifold which is isometrically embedded in $\mathbb{R}^N$. Up to now we have only considered smooth maps (or at least $C^2(\Sigma, M)$ maps), it's necessary to extend the definitions given to the case of Sobolev space $W^{1,2}(\Sigma, M)$ defined as follwoing:

Definition 2.1.39 (Sobolev Mappings):

$$W^{1,2}(\Sigma, M) := \{u \in W^{1,2}(\Sigma, \mathbb{R}^N) | \quad u(x) \in M\}. \tag{2.27}$$

Remark. The definition of $W^{1,2}(\Sigma, M)$ doesn't depend on the dimension of $\mathbb{R}^N$ in which the closed manifold (M, g) is isometrically embedded. (see [Hó2, lemma 1.4.3.]).

It's known that since dimension of Σ is two. The Sobolev space $W^{1,2}(\Sigma, M)$ defined as 2.27 is equilavent to the completion of $C^\infty(\Sigma, M)$ with respect to the norm $\|\cdot\|_{W^{1,2}}$:

$$\|u\|_{W^{1,2}} := \Big(\int_\Sigma |u|^2 + \int_\Sigma |du|^2 \Big)^{1/2},$$

for $u : \Sigma \to M \subseteq \mathbb{R}^N$. In other words, $C^\infty(\Sigma, M)$ is dense in $W^{1,2}(\Sigma, M)$ (see [Ha09, theorem 2.1] for the proof).

In order to define weakly harmonic surface, we introduce *nearest point projection* $\Pi : \mathbb{R}^N \to M$ which maps a point $x \in \mathbb{R}^N$ to the nearest point of M. Since we assume M is a closed manifold isometrically embedded in $\mathbb{R}^N$, there is a tubular neighborhood

$$M_\delta = \big\{x \in \mathbb{R}^N : \ \mathrm{dist}(x, M) < \delta\big\},$$

such that Π is well-defined and smooth. For a map $u : \Sigma \to M \subseteq \mathbb{R}^N$, $u = (u^1, u^2, .., u^N)$, and $u \in W^{1,2}(\Sigma, M)$, we write ∇u as the sum of *gradient* of u^i for $i = 1, ..., N$. That is, $\nabla u := \sum_{i=1}^N \nabla u^i$, and energy of u is simply

$$E(u) = \sum_{i=1}^N \int_\Sigma |\nabla u^i|^2 d\mathrm{vol}_h,$$

or equivalently, ∇ is the Levi-Civita connection of the bundle $u^{-1}T\mathbb{R}^N$. For a given $X \in C^\infty(\Sigma, \mathbb{R}^N)$, we consider the variation of u with respect to X defined as the follwoing

$$u_s = \Pi \circ (u + sX), \tag{2.28}$$

u_s is well-defined for s small enough such that the image of $u + sX$ is in the tubular neighborhood M_δ.

Definition 2.1.40 (Weakly Harmonic Map): We say that $u \in W^{1,2}(\Sigma, M)$ is weakly harmonic if and only if for any $X \in C^\infty(\Sigma, \mathbb{R}^N)$ we have

$$\lim_{s\to 0} \frac{E(u_s) - E(u)}{s} = 0. \tag{2.29}$$

Lemma 2.1.41: If $u \in W^{1,2}(\Sigma, M)$ is weakly harmonic. Then u satisfies the following equation

$$\Delta_\Sigma u \perp T_u M,$$

in the distrubution sense, i.e. $\forall X \in C^\infty(\Sigma, \mathbb{R}^N)$, if $X(x) \in T_{u(x)}M$ almost everywhere, then

$$\int_\Sigma h^{\alpha\beta} \Big\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial X}{\partial x^\beta} \Big\rangle d\mathrm{vol}_h = 0,$$

or equivalently u satisfies the following equation

$$\Delta_\Sigma u + h^{\alpha\beta} A_M\Big(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\Big) = 0, \tag{2.30}$$

in the distribution sense, i.e. $\forall X \in C^\infty(\Sigma, \mathbb{R}^N)$, we have

$$\int_M \Big\{ - h^{\alpha\beta} \Big\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial X}{\partial x^\beta} \Big\rangle + h^{\alpha\beta} \Big\langle A_M\Big(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\Big), X \Big\rangle \Big\} d\mathrm{vol}_h = 0.$$

Proof. The variation u_s of u can be written as the following:

$$u_s = \Pi \circ (u + sX) = u + sX_s,$$

where

$$X_s := \int_0^1 \frac{\partial \Pi}{\partial y^i}(u + stX)X^i dt.$$

Hence

$$E(u_s) = E(u) + s\int_\Sigma h^{\alpha\beta}\Big\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial X_s}{\partial x^\beta}\Big\rangle d\mathrm{vol}_h + o(s).$$

Thus (2.29) is equivalent to

$$\lim_{s\to 0}\int_\Sigma h^{\alpha\beta}\Big\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial X_s}{\partial x^\beta}\Big\rangle d\mathrm{vol}_h = 0.$$

Since Π is smooth, it follows that as $s \to 0$,

$$X_s \to d\Pi_u(X),$$

which is the orthogonal projection of X on T_uM. By assumption $X = d\Pi_u(X)$ almost everywhere. Therefore we have

$$\int_\Sigma h^{\alpha\beta}\Big\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial X}{\partial x^\beta}\Big\rangle d\mathrm{vol}_h = 0.$$

For the case $X(x) \notin T_{u(x)}M$ we have

$$\begin{aligned}
0 &= \lim_{s\to 0}\int_\Sigma h^{\alpha\beta}\Big\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial X_s}{\partial x^\beta}\Big\rangle d\mathrm{vol}_h \\
&= \int_\Sigma h^{\alpha\beta}\Big\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} d\Pi(X)\Big\rangle d\mathrm{vol}_h \\
&= \int_\Sigma h^{\alpha\beta}\Big\langle \frac{\partial u}{\partial x^\alpha}, d\Pi\big(\frac{\partial X}{\partial x^\beta}\big)\Big\rangle d\mathrm{vol}_h + \int_\Sigma h^{\alpha\beta}\Big\langle \frac{\partial u}{\partial x^\alpha}, \mathrm{Hess}\Pi_u\big(X, \frac{\partial u}{\partial x^\beta}\big)\Big\rangle d\mathrm{vol}_h \\
&= \int_\Sigma h^{\alpha\beta}\Big\langle \frac{\partial u}{\partial x^\alpha}, \frac{\partial X}{\partial x^\beta}\Big\rangle d\mathrm{vol}_h - \int_\Sigma h^{\alpha\beta}\Big\langle X, A_M\big(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\big)\Big\rangle d\mathrm{vol}_h.
\end{aligned}$$

The last equality follows from [Sim96, 2.12.3]. □

Given a map $u : \Sigma \to M$, $u \in W^{1,2}(\Sigma, M)$, and $X \in C^\infty(\Sigma, \mathbb{R}^N)$, by Taylor polynomial expansion of Π we have the following:

$$\begin{aligned}
u_s &= \Pi \circ (u + sX) \\
&= u + sd\Pi_u(X) + \frac{s^2}{2}\mathrm{Hess}\Pi_u(X, X) + o(s^2).
\end{aligned} \tag{2.31}$$

By applying ∇ to 2.31 we have

$$\begin{aligned}\nabla u_s =&\nabla u\\ &+ s(d\Pi_u(\nabla X) + \mathrm{Hess}\Pi_u(X, \nabla u))\\ &+ \frac{s^2}{2}\big(2\mathrm{Hess}\Pi_u(X, \nabla X) + \nabla\mathrm{Hess}\Pi_u(X, X, \nabla u)\big) + o(s^2),\end{aligned}$$

and the energy of u_s is

$$\begin{aligned}E(u_s) =& \int_\Sigma \langle\nabla u, \nabla u\rangle d\Sigma\\ &+s\int_\Sigma \langle\nabla u, d\Pi_u(\nabla X)\rangle + \langle\nabla u, \mathrm{Hess}\Pi_u(X, \nabla u)\rangle d\Sigma\\ &+s^2\Big\{\int_\Sigma \langle d\Pi_u(\nabla X), d\Pi_u(\nabla X)\rangle d\Sigma\\ &\quad+ 2\int_\Sigma \langle\mathrm{Hess}\Pi_u(X, \nabla u), d\Pi_u(\nabla X)\rangle d\Sigma\\ &\quad+ \int_\Sigma \langle\mathrm{Hess}\Pi_u(X, \nabla u), \mathrm{Hess}\Pi_u(X, \nabla u)\rangle d\Sigma\\ &\quad+ \frac{1}{2}\int_\Sigma \langle\nabla u, \big(2\mathrm{Hess}\Pi_u(X, \nabla X) + \nabla\mathrm{Hess}\Pi_u(X, X, \nabla u)\big)\rangle d\Sigma\Big\}\\ &+o(s^3).\end{aligned} \tag{2.32}$$

We now state the second variation of energy here:

$$\begin{aligned}\delta^2 E(u)(X, X) :=&\frac{d^2}{ds^2}\Big|_{s=0} E(u_s)\\ &= \int_\Sigma \langle d\Pi_u(\nabla X), d\Pi_u(\nabla X)\rangle d\Sigma\\ &\quad+ 2\int_\Sigma \langle\mathrm{Hess}\Pi_u(X, \nabla u), d\Pi_u(\nabla X)\rangle d\Sigma\\ &\quad+ \int_\Sigma \langle\mathrm{Hess}\Pi_u(X, \nabla u), \mathrm{Hess}\Pi_u(X, \nabla u)\rangle d\Sigma\\ &\quad+ \frac{1}{2}\int_\Sigma \langle\nabla u, \big(2\mathrm{Hess}\Pi_u(X, \nabla X) + \nabla\mathrm{Hess}\Pi_u(X, X, \nabla u)\big)\rangle d\Sigma.\end{aligned} \tag{2.33}$$

Consider a map $u \in W^{1,2}(\Sigma, M)$, and a sequence of maps $\{u_j\}_{j\in\mathbb{N}}$, $u_j \in W^{1,2}(\Sigma, M)$ for all j, such that $\|u_j - u\|_{W^{1,2}} \to 0$ as $j \to \infty$. Then for any $X \in C^\infty(\Sigma, \mathbb{R}^N)$, it's clear that from (2.33) we have

$$\delta^2 E(u)(X, X) - \delta^2 E(u_j)(X, X) \to 0, \quad \text{as } j \to \infty.$$

It's known that weakly harmonic map from Riemann surface Σ is smooth [H91]. Thus for a weakly harmonic map $u : \Sigma \to M \subseteq \mathbb{R}^N$, the variation of energy with respect to the variation defined below

$$u_s(x) := \Pi \circ (u(x) + sX(x)),$$

for some $X \in C^\infty(\Sigma, \mathbb{R}^N)$ is equivalent to the variation defined as (2.5) when $s \to 0$. That is, consider the smooth section $d\Pi_u(X)$ of $u^{-1}TM$ and the variation of u defined by

$$u_t(x) := \exp_{u(x)}(td\Pi_u(X)). \tag{2.34}$$

We have

$$\frac{d}{ds}\Big|_{s=0} E(u_s) = \frac{d}{dt}\Big|_{t=0} E(u_t).$$

So we have for a weakly harmonic map u, the second variation of energy (2.33) is equivalent to

$$\begin{aligned} \delta^2 E(u)(X, X) = &\int_\Sigma \langle \nabla V, \nabla V \rangle_{u^{-1}TM} d\Sigma \\ &- \int_\Sigma trace_\Sigma \langle R^M(du, V)V, du \rangle_{u^{-1}TM} d\Sigma, \end{aligned} \tag{2.35}$$

for $V = d\Pi_u(X)$.

2.1.5 A Priori Estimates and Removable Singularities

Let Σ be a Riemann surface, with conformal metric

$$\lambda^2(dx \otimes dx + dy \otimes dy),$$

here λ is a smooth real function on Σ, and let (M, g) be a compact Riemannian manifold. Given a harmonic map

$$f : \Sigma \to M.$$

Recall that the energy density takes the form

$$e(f) = \frac{1}{2\lambda^2}\Big(\Big|\frac{\partial f}{\partial x}\Big|^2 + \Big|\frac{\partial f}{\partial y}\Big|^2\Big),$$

and the area density

$$a(f) = \frac{1}{\lambda^2}\Big|\frac{\partial f}{\partial x} \wedge \frac{\partial f}{\partial y}\Big|.$$

It follows that

$$a(f) \leq e(f),$$

with equality holds if and only if f is conformal. We can rewrite the Bochner formula mentioned in lemma 2.1.13 in terms of $a(f)$ as

$$\Delta(e(f)) = |\nabla df|^2 + 2\tilde{K}e(f) - 2K(\sigma)a(f)^2, \tag{2.36}$$

here $\tilde{K}$ is the Gaussian curvature of Σ and $K(\sigma)$ is the sectional curvature of the two-plane generated by $\partial f/\partial x$ and $\partial f/\partial y$. If $K(\sigma) \leq 1$, our inequality between $a(f)$ and $e(f)$ implies that

$$\Delta(e(f)) \geq |\nabla df|^2 + 2\tilde{K}e(f) - 2e(f)^2 \geq 2\tilde{K}e(f) - 2e(f)^2. \tag{2.37}$$

When $\tilde{K} = 0$ and $K(\sigma) \leq 1$ we have

$$\Delta(e(f)) \geq -2e(f)^2. \tag{2.38}$$

Corollary 2.1.42: If M has nonpositive sectional curvatures, there are no harmonic map from S^2 into M and any harmonic map from T^2 into M must be totally geodesic. If M has negative sectional curvatures, there are no harmonic maps from S^2 or T^2 into M.

Proof. If $\Sigma = S^2$, we can choose the Riemannian metric on Σ so that $\tilde{K} \equiv 1$. By (2.36) we have

$$\Delta e(f) > 0.$$

But the continuous function $e(f)$ must obtain a maximum at some point. So there's no harmonic map from S^2 to M with nonpositive sectional curvature.

If $\Sigma = T^2$, we can choose the Riemannian metric on Σ so that $\tilde{K} \equiv 0$. In this case we have

$$\Delta e(f) = |\nabla df|^2 - 2K(\sigma)a(f)^2 \leq 0,$$

at the maximum point of $e(f)$. With the assumption $K(\sigma) \leq 0$ we know that

$$\nabla df = 0,$$

which implies that f is totally geodesic.

The case where M has negative sectional curvatures is proven similarly. □

We are going to state the ϵ-Regularity theorem and sketch the proof. The idea is to use the Bochner inequality (2.38):

$$\Delta|df|^2 \geq C|df|^2,$$

with the constant C depending only on M, Σ, and a standard result in PDE shows that

$$\sup_{\Omega} |df|^2 \leq C' \int_{\Sigma} |df|^2 d\Sigma,$$

for any open Ω compactly contained in Σ, where C' is a constant depends only on M, Σ, and $\text{dist}(\Omega, \partial\Sigma)$.

Theorem 2.1.43 ([SU81] Main Estimate 3.2): Let Σ be a compact Riemann surface and M a compact manifold. Then there exist constants $r_0 > 0$, $\epsilon_{su} > 0$, and $C > 0$ such that if D_r is a disk of radius $r \leq r_0$ in Σ and $f : D_r \to M$ is a harmonic map, then

$$\int_{D_r} e(f) d\Sigma < \epsilon_{su},$$

implies that

$$\sup_{D_{r/2}} e(f) < \frac{C}{\pi r^2} \int_{D_r} e(f) d\Sigma.$$

Once we know that $\nabla f \in L^{\infty}_{\text{loc}}$, it then follows from equation (2.30) that $\Delta f \in L^{\infty}_{\text{loc}}$, which implies by standard estimates on the inverse of the Laplacian that $f \in W^{2,p}_{\text{loc}}$ for all $p < \infty$. Hence we deduce that $\Delta f \in W^{1,p}_{\text{loc}}$ and hence $f \in W^{3,p}_{\text{loc}}$ for all $p > 0$. We can then repeat this argument to show that $f \in W^{k,p}_{\text{loc}}$, $\forall k$, and so the smoothness of the solution follows.

The ϵ-regularity theorem states that if the energy on a ball of small radius is sufficiently small, the maximum of the energy density on a ball of half the radius is bounded by a constant multiple of its average over the original ball. We start by proving this for harmonic maps on disks in the complex plane when the metric on the plane is flat:

Lemma 2.1.44: Suppose that $f : D_1 \to M$ is a harmonic map from a unit disk D_1 in complex plane, with the standard Euclidean metric $ds^2 = dx \otimes dx + dy \otimes dy$, into M a compact Riemannian manifold with sectional curvature $K(\tilde{\sigma}) \leq 1$. Then for any $r_0 \in (0, 1/2)$, if

$$\int_{D_1} e(f) d\Sigma < \pi(r_0^2 - 2r_0^4), \tag{2.39}$$

then we have the following inequality:

$$\max_{\sigma \in (0,1]} \sigma^2 \sup_{D_{1-\sigma}} e(f) < 4r_0^2. \tag{2.40}$$

Proof. Choose $\sigma_0 \in (0, 1]$ so that

$$\sigma_0^2 \sup_{D_{1-\sigma_0}} e(f) \geq \sigma^2 \sup_{D_{1-\sigma}} e(f), \quad \text{for all } \sigma \in (0, 1],$$

and choose $p_0 \in \overline{D_{1-\sigma_0}}$ such that

$$e_0 = e(f)(p_0) = \sup\{e(f)(p) : p \in \overline{D_{1-\sigma_0}}\}.$$

It suffices to show that if $\sigma_0^2 e_0 \geq 4r_0^2$ then (2.39) cannot hold. But

$$(\frac{\sigma_0}{2})^2 \sup_{D_{1-\sigma_0/2}} e(f) \leq \sigma_0^2 \sup_{D_{1-\sigma_0}} e(f) = \sigma_0^2 e_0,$$

implies that

$$\sup_{D_{1-\sigma_0/2}} e(f) \leq 4e_0.$$

and by the inequality $\sigma_0^2 e_0 \geq 4r_0^2$ we have that

$$\frac{r_0}{\sqrt{e_0}} \leq \frac{\sigma_0}{2},$$

so that the disk of radius $r_0/\sqrt{e_0}$ about p_0 is contained in D_1. If we rescale this disk to radius r_0, and define a new harmonic $g : D_{r_0} \to M$ by

$$g(q) = f\Big(p_0 + \frac{q}{\sqrt{e_0}}\Big),$$

so we have

$$e(g)(0) = 1, \quad \text{and } e(g) \leq 4 \text{ on } D_{r_0}.$$

It now follows from (2.38) that

$$\Delta e(g) \geq -2(e(g))^2 \geq -32.$$

If we define a smooth function $h : D_{r_0} \to \mathbb{R}$ in terms of the standard Euclidean coordinates (x, y) on the disk by

$$h = e(g) - [1 - 8(x^2 + y^2)],$$

then

$$\Delta h = \Delta(e(g)) + 32 \geq 0,$$

since h is subharmonic hence by mean value inequality we have

$$\begin{aligned} 0 = \pi r_0^2 h(0) &\leq \int_{D_{r_0}} h d\Sigma \\ &\leq \int_{D_{r_0}} \big(e(g) - (1 - 8(u^2 + v^2))\big) d\Sigma \\ &\leq \int_{D_{r_0}} e(g) d\Sigma - \int_0^{2\pi} \int_0^{r_0} (1 - 8r^2) r dr d\theta, \end{aligned}$$

and an integration yields the inequality

$$\int_{D_1} e(f) d\Sigma \geq \int_{D_{r_0}} e(g) d\Sigma \geq \pi(r_0^2 - 4r_0^4).$$

Thus we have proven the contrapositive of the assersion (2.39). □

proof for theorem 2.1.43. We only do the case for flat metric on D_r here. First note that $r_0^2 \leq \frac{1}{8}$ implies that

$$\frac{\pi}{2} r_0^2 \leq \pi(r_0^2 - 4r_0^4).$$

Thus, if

$$\int_{D_1} e(f) d\Sigma \leq \frac{\pi}{16} \text{ we can set } \int_{D_1} e(f) d\Sigma = \frac{\pi}{2} r_0^2,$$

and (2.39) implies

$$\begin{aligned} \int_{D_1} e(f) d\Sigma = \frac{\pi}{2} r_0^2 \leq \pi(r_0^2 - 4r_0^4), \\ \Longrightarrow \max_{\sigma \in (0,1]} \sigma^2 \sup_{D_{1-\sigma}} e(f) < 4r_0^2 = \frac{8}{\pi} \int_{D_1} e(f) d\Sigma. \end{aligned}$$

Under rescaling from D_1 to D_r, the energy integral remains unchanged, while

$$|df| \mapsto \frac{1}{r} |df|, \quad e(f) \mapsto \frac{1}{r^2} e(f),$$

and hence we have

$$\int_{D_r} e(f) d\Sigma < \frac{\pi}{16} \Longrightarrow \max_{\sigma \in (0,r]} \sigma^2 \sup_{D_{r-\sigma}} e(f) < \frac{8}{\pi} \int_{D_r} e(f) d\Sigma.$$

Now we set $\sigma = r/2$ to obtain:

$$\int_{D_r} e(f) d\Sigma < \frac{\pi}{16} \Longrightarrow \left(\frac{r}{2}\right)^2 \sup_{D_{r/2}} e(f) < \frac{8}{\pi} \int_{D_r} e(f) d\Sigma.$$

Finally, we set $\epsilon_{su} = \pi/16$ and $C = 32$ to finish the proof in our special case. □

Theorem 2.1.45 (Removable singularity, [SU81]): Let M be a compact manifold and D be the unit disk, if $u : D \backslash \{0\} \to M$ is a smooth harmonic map with finite energy, then u extends smoothly as a harmonic map from D to M.

Removable singularity theorem is originally proved by Sacks and Uhlenbeck in [SU81] Here we sketch the proof following the more conceptual proof by Parker [Par96]. The technique is also used to prove there's no loss in energy of the neck domain in later section. It shows that harmonic maps with small energy on long cylinder are almost radial then reaches the conclusion that harmonic maps with energy bounded away from 0 is far from being conformal, thus there's no energy loss on the long cylinder.

We focus on the case where the metric on D is flat metric and can be written as:

$$ds^2 = dr^2 + r^2 d\theta^2 = e^{-2t}(dt^2 + d\theta^2),$$

where $r = e^{-t}$. Given a compact manifold isometrically embedded in $\mathbb{R}^N$, consider smooth harmonic map $u : D \to M$. We use (r, θ) coordinates centered at p with $B_r(p) \subset D \setminus \{0\}$. Then by ϵ-regularity theorem (see Theorem 2.1.43) we know that there exists $\epsilon_{su} > 0$ so that whenever the energy on the punctured disk satisfies

$$E(u) < \epsilon_{su},$$

then

$$r^2 \sup_{x \in B_{r/2}(p)} |\nabla u(x)|^2 < CE(u).$$

This implies that when $E(u) < \epsilon_{su}$ the supremum of $r|\frac{\partial u}{\partial r}|$ and $|\frac{\partial u}{\partial \theta}|$ on the disk of radius $r/2$ about p is less than $\sqrt{2CE(u)}$. Thus implies

$$\text{supremum of } \Big|\frac{\partial u}{\partial t}\Big|, \Big|\frac{\partial u}{\partial \theta}\Big| \leq \sqrt{2CE(u)}, \tag{2.41}$$

in terms of the (u, θ) coordinates.

The (t, θ) coordinates establish an isomorphism from the punctured disk to a half-infinite cylinder $[0, \infty) \times S^1$ and the energy is simply:

$$E(u) = \frac{1}{2} \int_0^\infty \int_0^{2\pi} \Big[\Big|\frac{\partial u}{\partial t}\Big|^2 + \Big|\frac{\partial u}{\partial \theta}\Big|^2\Big] dt d\theta.$$

Let

$$P(t) = \frac{1}{2} \int_0^{2\pi} \Big|\frac{\partial u(t,\theta)}{\partial \theta}\Big|^2 d\theta, \text{ and } K(t) = \frac{1}{2} \int_0^{2\pi} \Big|\frac{\partial u(t,\theta)}{\partial t}\Big|^2 d\theta.$$

Then we have

$$E(u) = \int_0^\infty (P(t) + K(t)) dt.$$

Now we write ∂_θ for $\partial/\partial\theta$ and likewise ∂_t for $\partial/\partial t$. Recall that M is isometrically embedded in some Euclidean space $\mathbb{R}^N$, thus (2.17) gives the following relation

$$(\partial_t^2 u + \partial_\theta^2 u)^\top = 0, \tag{2.42}$$

which implies that

$$\begin{aligned} K'(t) - P'(t) &= \int_0^{2\pi} \langle \partial_t^2 u, \partial_t u \rangle d\theta - \int_0^{2\pi} \langle \partial_t \partial_\theta u, \partial_\theta u \rangle d\theta \\ &= \int_0^{2\pi} \langle \partial_t^2 u + \partial_\theta^2 u, \partial_t u \rangle d\theta \\ &= 0. \end{aligned}$$

So we know $K(t) = P(t) + c$ for some constant c. But then the boundedness of energy implies that $K(t) \equiv P(t)$. Our next goal is to calculate $P''(t)$. Instead of

writing $\nabla^M_{du(\frac{\partial}{\partial\theta})}$ as the Levi-Civita connection on $u^{-1}TM$ we should simply write ∇_θ and likewise for ∇_t.

$$\begin{aligned}
P''(t) &= \partial_t \int_0^{2\pi} \langle \nabla_t \partial_\theta u, \partial_\theta u\rangle d\theta \\
&= \int_0^{2\pi} |\nabla_t \partial_\theta u|^2 d\theta + \int_0^{2\pi} \langle \nabla_t^2 \partial_\theta u, \partial_\theta u\rangle d\theta \\
&= \int_0^{2\pi} |\nabla_t \partial_\theta u|^2 d\theta + \int_0^{2\pi} \langle R^M(\partial_t u, \partial_\theta u)\partial_t u, \partial_\theta u\rangle d\theta + \int_0^{2\pi} \langle \nabla_\theta \nabla_t \partial_t u, \partial_\theta u\rangle d\theta \\
&\geq \int_0^{2\pi} \langle R^M(\partial_t u, \partial_\theta u)\partial_t u, \partial_\theta u\rangle d\theta - \int_0^{2\pi} \langle \nabla_t \partial_t u, \nabla_\theta \partial_\theta u\rangle d\theta \\
&= \int_0^{2\pi} \langle R^M(\partial_t u, \partial_\theta u)\partial_t u, \partial_\theta u\rangle d\theta + \int_0^{2\pi} |\nabla_\theta \partial_\theta u|^2 d\theta.
\end{aligned} \tag{2.43}$$

The last equality is yielded by (2.42). Since $\int_0^{2\pi} \frac{\partial u}{\partial\theta} d\theta = 0$, by Wirtinger's inequality we have

$$2P(t) = \int_0^{2\pi} |\partial_\theta u|^2 d\theta \leq \int_0^{2\pi} |\partial_\theta^2 u|^2 d\theta,$$

and

$$\int_0^{2\pi} |\partial_\theta^2 u|^2 - |\nabla_\theta \partial_\theta u|^2 d\theta \leq C_1 \sup |\partial_\theta u| P(t),$$

where C_1 depends on the second fundamental form of M. Similarly

$$\int_0^{2\pi} |\langle R^M(\partial_t u, \partial_\theta u)\partial_t u, \partial_\theta u\rangle| d\theta \leq C_2 \sup |\partial_t u|^2 P(t),$$

where the constant C_2 now depends on the curvature of M. Putting it all together we have

$$P''(t) \geq 2P(t) - C_3 \sup\{|\partial_\theta u|, |\partial_t u|^2\} P(t),$$

by the boundedness of $E(u)$ and (2.41), we could choose $B_r(p) \subset D \setminus \{0\}$ so that $\sup\{|\partial_\theta u|, |\partial_t u|^2\} < 1/C_3$. Thus

$$P''(t) \geq P(t). \tag{2.44}$$

Given a sequence of real numbers $\{T_j\}_{j\in\mathbb{N}}$, $T_j \to \infty$, consider the function $f_j(t)$ such that it satisfies the following equation

$$f_j''(t) = f_j(t),$$

with $f_j(0) = P(0)$ and $f_j(T_j) = P(T_j)$. We know the solution of $f_j(t)$ is the following:

$$f_j(t) = \frac{1}{1-\alpha^2}(P(0) - \alpha P(T_j))e^t + \frac{\alpha}{1-\alpha^2}(P(T_j) - \alpha P(0))e^{-t}, \quad t \in [0, T_j], \tag{2.45}$$

where $\alpha = e^{T_j}$. Then by maximum principle we have

$$P(t) \leq f_j(t), \quad t \in [0, T_j].$$

As $j \to \infty$, $e^{T_j} \to \infty$. It implies that $P(t)$ decays exponentially:

$$P(t) \leq C_4 e^{-t},$$

In the (r, θ) coordinate it gives

$$e(u) \leq Cr,$$

which establishes Hölder continuity of the extension and shows that the limit is C^∞ by bootstrapping.

2.1.6 Existence of Minimal Sphere and the Sphere Theorem

Now we state the famous theorem by Sacks and Uhlenbeck and introduce the α-energy they used to overcome the noncompactness of conformal automorphisms of the domain.

Definition 2.1.46 (α-energy): Define the α-energy of a map $u : \Sigma \to M$ to be

$$E_\alpha(u) = \int_\Sigma \left(1 + |\nabla u|^2\right)^\alpha.$$

Remark. Unlike the energy functional, α-energy is not conformal invariant.

Theorem 2.1.47 (Sacks and Uhlenbeck, [SU81]): If M is a compact simply connected Riemannian manifold, then M contains at least one nonconstant minimal two sphere.

We sketch the idea of the proof here.

Since M is compact and simply connected there's a least integer $q \geq 2$ such that $H_q(M; \mathbb{Z}) \neq 0$, and it follows from the Hurewicz isomorphism that

$$\pi_i(M) = 0, \text{ for } 0 < i < q, \quad \text{and } \pi_q(M) \equiv H_q(M; \mathbb{Z})$$

Let $\mathcal{M}$ be the space of smooth maps from $S^2 \to M$ and let $\pi : \mathcal{M} \to M$ be defined by $\pi(f) = f(0)$, the evaluation of f at the base point $0 \in S^2 = \mathbb{C} \cup \{\infty\}$. It's a well-known fact from homotopy theorey that π is a fibration with fiber

$$\mathcal{M}_p = \pi^{-1}(p),$$

the set of basepoint preserving maps from S^2 to M. Moreover,

$$\pi_k(\mathcal{M}_p) \cong \pi_{k+2}(M),$$

and the fibration π induces a long exact sequence

$$\cdots \to \pi_k(\mathcal{M}_p) \to \pi_k(\mathcal{M}) \to \pi_{k-1}(\mathcal{M}_p) \to \cdots .$$

We note, moreover, that $\pi_* : \pi_k(\mathcal{M}) \to \pi_k(M)$ possesses a right inverse

$$i_* : \pi_k(M) \to \pi_k(\mathcal{M}),$$

induced by the inclusion $i : M \to \mathcal{M}$, which takes a point to the constant map at that point. Hence the long exact sequence splits and we conclude that

$$\pi_k(\mathcal{M}) \cong \pi_k(M) \oplus \pi_{k+2}(M).$$

In particular we have

$$\pi_{q-2}(\mathcal{M}) = \pi_q(M) \neq 0.$$

Now we identify the $\pi_{q-2}(\mathcal{M})$ with the space of free homotopy classes of maps from S^{q-2} into $\mathcal{M}$. We can perturb the α-energy for $\alpha > 1$ to make it nondegenerate and thus a Morse function. Choose a nonzero element $\beta \in \pi_{q-2}(\mathcal{M})$, let

$$\text{Minimax}(E_\alpha, \beta) := \inf_{u=\beta} \max_{x \in S^{q-2}} E_\alpha(u(x)).$$

Since $\beta \neq 0$, there exists $\epsilon > 0$ such that

$$\text{Minimax}(E_\alpha, \beta) > \epsilon, \quad \forall \alpha > 1.$$

Given a sequence of real numbers $\{\alpha_i\}_{i \in \mathbb{N}}$, $\alpha_i > 1$, such that $\alpha_i \to \infty$ as $i \to \infty$. Let $u_i : S^2 \to M$ be critical point of E_{α_i} with $E(u_i) = \text{Minimax}(E_{\alpha_i}, \beta)$ (the existence of such critical point u_i is guaranteed by Morse theory, see[Jos11, theorem 1.11.2.] for the precise statement and the proof.) There is a priori estimates for $\{u_i\}$ [SU81] (which is similar to theorem 2.1.43) so that if

$$\liminf_{i \to \infty} (\sup_{S^2} |\nabla u_i|) < \infty,$$

then u_i converges smoothly to a harmonic sphere $u : S^2 \to M$ as $i \to \infty$ up to subsequence. If

$$\sup_{S^2} |\nabla u_i| \to \infty, \quad i \to \infty,$$

then by rescaling the metric we can make $\liminf_{i \to \infty} (\sup_{S^2} |\nabla u_i|) < \infty$ and find the harmonic map. That proves the existence of minimal two sphere in a compact simply connected manifold.

We briefly state the sphere theorem as below and outline the idea of the proof, the same argument could be found at [MM88]. We'll also mention how it proves the existence of minimal surface with Morse index bound one on a closed manifold M with $\pi_3(M)$ nontrivial.

Theorem 2.1.48 (Sphere theorem, [MM88]): Suppose that M is a compact smooth simply connected Riemannian manifold of dimension at least four which has positive isotropic curvature. Then M is homeomorphic to a sphere.

The idea behind the proof is to show that M is a homotopy sphere and apply the solutions to the generalized Poincaré conjecture in dimensions no less than four to conclude that M is homeomorphic to a sphere.

By definition, a compact connected manifold M of dimension n is a homotopy sphere if $\pi_q(M) = 0$ for all integers such that $0 < q < n$. Thus if M isn't a homotopy sphere there must be an integer q with $0 < q < n$ such that $\pi_q(M) \neq 0$, and we choose the smallest such integer. By the Hurewicz isomorphism theorem, q is also the smallest positive integer such that $H_q(M; \mathbb{Z}) \neq 0$. By Poincaré duality, we can assume that $q \leq n/2$.

By similar argument as we just discussed in the proof of theorem 2.1.47, we obtain a sequence $u_i : S^2 \to M$ of critical points of E_{α_i} with $E(u_i) = \text{Minimax}(E_{\alpha_i}, \beta)$. Since $\beta \in \pi_{q-2}(\mathcal{M})$ is nontrivial, each u_i has Morse index no larger then $(q-2)$. Thus there is a subsequence of u_i converges to a harmonic map u with same Morse index bound $q-2$. So if $\pi_1(\mathcal{M}) = \pi_3(M) \neq 0$, then there exists a minimal two-sphere with Morse index at most one.

To complete the proof, we state the following lemma :

Lemma 2.1.49 ([MM88]): Suppose that M is a Riemannian manifold which has positive isotropic curvature. Then the Morse index of any harmonic two-sphere is at least $1/2(\dim M) - (3/2)$.

If $\pi_q(M)$ is nontrivial it implies that u has index bound $(q-2) \leq (1/2)(\dim M) - 2$ which contradicts Lemma 2.1.49. So $\pi_q(M) = 0$ for all $q < \dim(M)$ and thus M is a homotopy sphere.

Chapter 3

Compactness for Harmonic Spheres

3.1 Bubble Convergence

Recall that $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ is a Riemann surface. S^+ and S^- will denote the nothern and southern hemispheres of S^2, $p^+ = (0, 0, 1)$ and $p^- = (0, 0, -1)$ the north and south poles. We denote the stereographic projection by

$$\Pi_S : S^2 \subset \mathbb{R}^3 \to \mathbb{C} \cup \{\infty\}, \quad \Pi_S(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}.$$

Given a ball $B_r(x) \subset S^2$ the conformal dilation taking $B_r(x)$ to S^- is the composition of translation $x \to p^-$ followed by dilation of S^2 about p^- taking $B_r(p^-)$ to S^-.

Definition 3.1.1 (Möbius transformations): The group of automorphisms of the Riemann sphere is known as $PSL(2, \mathbb{C})$, it's also known as the group of *Möbius transformations*. Its elements are fractional linear transformations

$$\phi(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$

where $a, b, c, d \in \mathbb{C}$.

The group $PSL(2, \mathbb{C})$ is noncompact; A maximal compact subgroup of $PSL(2, \mathbb{C})$ is the group of isometries of S^2.

Definition 3.1.2 (Bubble Convergence): We say that a sequence $\{\gamma_j\}_{j\in\mathbb{N}}$, $\gamma_j : S^2 \to M$, of $W^{1,2}(S^2, M)$ maps *bubble converges* to a collection of $W^{1,2}(S^2, M)$ maps $u_0, ..., u_m : S^2 \to M$ if the following hold:

1. The γ_j converges weakly in $W^{1,2}(S^2, M)$ to u_0 and there's a finite set $\mathcal{S}_0 = \{x_0^1, ..., x_0^{k_0}\} \subset S^2$ so that the γ_j converge strongly to u_0 in $W^{1,2}(K, M)$ for any compact set $K \subset S^2 \setminus \mathcal{S}_0$.
2. For each $i > 0$, we get a point $x_{l_i} \in \mathcal{S}_0$ and a sequence of balls $B_{r_{i,j}}(y_{i,j})$ with $y_{i,j} \to x_{l_i}$. Further more, let $D_{i,j}$ be the dilation that takes the southern hemisphere to $B_{i,j}(y_{i,j})$. Then the map $\gamma_j \circ D_{i,j}$ converges to u_i as in 1.
3. if $i_1 \neq i_2$, then $\frac{r_{i_1,j}}{r_{i_2,j}} + \frac{r_{i_2,j}}{r_{i_1,j}} + \frac{|y_{i_1,j} - y_{i_2,j}|^2}{r_{i_1,j} r_{i_2,j}} \to \infty$.
4. $\sum_{i=0}^{m} E(u_i) = \lim_{j\to\infty} E(\gamma_j)$.

Now we give two examples of bubble convergence.

Example 3.1.3: Consider u as a fixed harmonic map and the sequence v_j to be:

$$v_j = u \circ \Psi_j,$$

Where $\Psi_j : S^2 \to S^2$ is a sequence of divergent dilations. Then the sequence $\{v_j\}_{j\in\mathbb{N}}$ bubble converges to u.

Example 3.1.4: Let Π_S be the stereographic projection

$$\Pi_S : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{R}^2,$$

and let $z = x + iy$ be complex coordinate on $\mathbb{R}^2 = \mathbb{C}$, if we set

$$f_j(z) = \frac{1}{jz} + z = \frac{z^2 + 1/j}{z}.$$

Then the map

$$v_j = \Pi_S^{-1} \circ f_j \circ \Pi_S : S^2 \to S^2$$

is conformal and harmonic. Since each v_j is a rational map of degree two, we have $E(v_j) = A(v_j) = 8\pi$. Moreover, v_j converges to identity map away from 0. The identity map has energy 4π. The other 4π of energy disappears at 0 but can be accounted for by composing with a divergent sequence of conformal dilations. In this case, the conformal dilation takes f_j to

$$\tilde{f}_j(z) = f_j(z/j) = \frac{1}{z} + \frac{z}{j},$$

which converges to the conformal inversion about the circle of radius one

We introduce $d_B(\cdot,\cdot)$ here to describe bubble convergence. Notice that $d_B(\cdot,\cdot)$ is not a distance function like $\|\cdot\|_{W^{1,2}}$ or $d_V(\cdot,\cdot)$ (see definition 3.3.1).

Definition 3.1.5: Given a collection of finitely many harmonic spheres $\{u_i\}_{i=0}^n$, and let $E=\sum_{i=0}^n E(u_i)$. For $\gamma: S^2\to M$, we say

$$d_B(\gamma,\{u_i\}_{i=0}^n)<\epsilon,$$

if the following holds:

$$\sum_{i=0}^{n}\Big(\int_{\Omega_i}|\nabla(\gamma-\nabla(u_i\circ D_i)|^2\Big)^{1/2}<\epsilon, \tag{3.1}$$

$$\int_{S^2\setminus\bigcup_{i=0}^n\Omega_i}|\nabla\gamma|^2<3\epsilon^2+2(n+1)\epsilon E, \tag{3.2}$$

$$\sum_{i=0}^{n}\Big(\int_{S^2\setminus\Omega_i}|\nabla(u_i\circ D_i)|^2\Big)^{1/2}<\epsilon, \tag{3.3}$$

for some conformal dilations $D_i: S^2\to S^2$, $i=0,...,n$, and pairwise disjoint domains $\Omega_0,...,\Omega_n$, $\bigcup_{i=0}^n\Omega_i\subset S^2$.

We write $d_B(\gamma,\{u_i\}_{i=0}^n)\geq\epsilon$ if there's no pairwise disjoint domains $\{\Omega_l\}_{i=0}^n$ and conformal dilations $\{D_l\}_{i=0}^l$ satisfying (3.1), (3.2), and (3.3).

3.2 Compactness for Harmonic Maps

Theorem 3.2.1 ([Par96]): Let $\{u_i\}_{i\in\mathbb{N}}$,$u_i:\Sigma\to M\ \forall i\in\mathbb{N}$, be a sequence of harmonic maps from a Riemann surface Σ to a compact Riemannian manifold M with bounded energy E_0. i.e.,

$$E(u_i)\leq E_0.$$

Then u_i bubble converges to a finte collection of harmonic maps $\{v_j\}_{j=0}^m$. Moreover,

$$\lim_{i\to\infty}E(u_i)=\sum_{j=0}^{m}E(v_j).$$

Remark. Given compact manifolds M,N, for general harmonic maps $f:M\to N$ (here we use the same notation as definition 2.1.1), it's proved by Takakuwa in [Tak98] such that if $m=\dim M$ is no less than 3 the set of harmonic maps with energy bounded by a constant is sequentially compact. That is, if a sequence $\{f_j\}$ of harmonic maps of M to N converges in the weak topology of $W^{1,m}$, than there exists a subsequence of $\{f_j\}$ which converges in the C^∞ topology.

Theorem 3.2.1 also holds for a sequence of *almost harmonic maps*.

Theorem 3.2.2 ([CM08]): Suppose that $0 < \epsilon \leq \epsilon_{su}$ (ϵ_{su} is given in theorem 2.1.43), $E_0 > 0$ are constants and $\{\gamma_j\}_{j\in\mathbb{N}}$, $\gamma_j : S^2 \to M$, is a sequence of $C^0 \cap W^{1,2}(S^2, M)$ maps with $E(\gamma_j) \leq E_0$ satisfying:

1. $A(\gamma_j) > E(\gamma_j) - 1/j$.
2. For any finite collection of disjoint closed balls B in S^2 with
$$\int_B |\nabla \gamma_j|^2 < \epsilon,$$
there is an energy minimizing map $v : B \to M$ that equals to γ_j on $\partial \frac{1}{8}B$ with
$$\int_{\frac{1}{8}B} |\nabla \gamma_j - \nabla v|^2 < \frac{1}{j}.$$

Then $\{\gamma_j\}_{j\in\mathbb{N}}$ bubble converges to a finite collection of harmonic spheres $u_0, ..., u_m$ up to subsequence.

Definition 3.2.3: Given a sequence of maps $\{u_i\}_{i\in\mathbb{N}}$, $u_i : \Sigma \to M$. We define $\mathcal{S}$ to be the set of points where energy concentrates. That is:
$$\mathcal{S} := \Big\{ x \in \Sigma \Big| \inf_{r>0} \Big(\lim_{j\to\infty} \int_{B_r(x)} |\nabla u_j|^2 \Big) \geq \epsilon_{su} \Big\},$$
here $\epsilon_{su} > 0$ is the constant given in theorem 2.1.43. Since $E(u_j) \leq E_0$, we know that $\mathcal{S}$ is a finite set.

Now we sketch the proof for Theorem 3.2.1. We refer to [Par96] for the details.

Sketch proof for Theorem 3.2.1. Let $u_j : \Sigma \to M$ be the sequence of harmonic maps with bounded energy E_0. Let $\mathcal{S}$ be given as definition 3.2.3. We know that $\mathcal{S}$ is a finite set since the energy is bounded. For $x \notin \mathcal{S}$, there exists $r_x > 0$ and $J_x \in \mathbb{N}$ so that:
$$\int_{B_{r_x}(x)} |\nabla u_j|^2 \leq \epsilon_{su}, \quad \text{for } j > J_x.$$
By theorem 2.1.43, we have
$$\sup_{B_{r_x/2}(x)} e(u_j) \leq C/r_x^2, \quad \forall i \in \mathbb{N}.$$

By Arzelà-Ascoli theorem there is a subsequence of u_i converges smoothly to a harmonic map $v_x : B_{r_x}(x) \to M$. Define

$$v_0 = \bigcup_{x \in \Sigma \setminus \mathcal{S}} v_x.$$

Then v_0 is harmonic on $\Sigma \setminus \mathcal{S}$. Since $\mathcal{S}$ is finite, by the removable singularity theorem 2.1.45, we know that v_0 extends to a smooth harmonic map on Σ. Therefore, u_j converges to v_0 smoothly on any compact set $K \subset \Sigma \setminus \mathcal{S}$.

We pick $x \in \mathcal{S}$ and fix a radius $\rho > 0$ so that x_0 is the only singular point in $B_{2\rho}(x)$ and choose a sequence $\epsilon_j \to 0$ by letting $\epsilon_j \leq \min\{\rho, 1/j\}$ be the largest number with

$$\int_{B_{\epsilon_j}(x)} |\nabla v_0|^2 \leq \frac{\epsilon_{su}}{j^2}.$$

Let $r_j > 0$ be the smallest radius such that

$$\inf_{y \in B_{\epsilon_j - r_j}(x)} \int_{B_{\epsilon_j}(x) \setminus B_{r_j}(y)} |\nabla u_j|^2 = \epsilon_{su}/2.$$

We choose a ball $B_{r_j}(y_j) \subset B_{\epsilon_j}(x)$ with

$$\int_{B_{\epsilon_j}(x) \setminus B_{r_j}(y_j)} |\nabla u_j|^2 = \epsilon_{su}/2.$$

Since $u_j \to v_0$ on compact subset of $B_\rho(x) \setminus \{x\}$ we get that

$$y_j \to x \quad \text{and} \quad r_j \to 0, \text{ as } j \to \infty.$$

After passing to a subsequence, we have $r_j < \epsilon_j / j^2$ [Par96, lemma 1.3]. For each j, let Ψ_j be the conformal dilation that takes $B_{r_j}(y_j)$ to S^-. Let $D_j := \Pi_S \circ \Psi_j \circ \Pi_S^{-1}$. The dilated map $\tilde{u}_j$ defined by:

$$\tilde{u}_j = u_j \circ D_j^{-1}$$

has the same energy as u_j, i.e., $E(\tilde{u}_j) = E(u_j)$. Hence we can consider the sequence $\tilde{u}_j$ and argue as before to have that there is a harmonic map v_1 and finite singular set $\mathcal{S}_1$ such that $\tilde{u}_j$ converges strongly to v_j on any compact set $K \subset \Sigma \setminus \mathcal{S}_1$.

Since the maximal energy concentration at any $y \in \mathcal{S}_1$ is at most:

$$\inf_{r>0} \Big[\lim_{j \to \infty} \int_{B_r(x)} |\nabla u_j|^2 \Big] - \epsilon_{su}/2.$$

We can repeat this procedure finitelcollection of harmonic maps $\{v_i\}$ which u_j bubble converges to. Now we're left to show that $\lim_{j\to\infty} E(u_j) = \sum_{i=0}^{m} E(v_i)$, which is equivalent to showing

$$\lim_{j\to\infty} \int_{B_{\epsilon_j}(x)\setminus B_{jr_j}(y_j)} |\nabla u_j|^2 = 0. \tag{3.4}$$

We call the following *neck maps* by restricting u_j to the intermediate annular region $B_{\epsilon_j}(x) \setminus B_{jr_j}(y_j)$. As we've shown in the proof of theorem 2.1.45, it suffices to consider the coordinate change

$$ds^2 = dr^2 + r^2 d\theta^2 = \epsilon_j^2 e^{-2t}(dt^2 + d\theta^2),$$

where $r = \epsilon_j e^{-t}$, and

$$P_j(t) = \frac{1}{2}\int_0^{2\pi} \Big|\frac{\partial u_j(t,\theta)}{\partial \theta}\Big|^2 d\theta, \quad t \in [0, T_j],$$

where $\epsilon_j e^{-T_j} = jr_j$. Recall that we have

$$P_j''(t) \geq P_j(t)$$

by (2.44). By (2.45) we get that

$$P_j(t) \leq \frac{1}{1-\alpha^2}(P_j(0) - \alpha P_j(T_j))e^t + \frac{\alpha}{1-\alpha^2}(P_j(T_j) - \alpha P_j(0))e^{-t}, \quad t \in [0, T_j], \tag{3.5}$$

where $\alpha = e^{T_j}$. The energy of u_j restricted on $B_{\epsilon_j}(x) \setminus B_{jr_j}(y_j)$ is

$$\begin{aligned}
\int_{B_{\epsilon_j}(x)\setminus B_{jr_j}(y_j)} |\nabla u_j|^2 =& 2\int_0^{T_j} P_j(t)dt \\
\leq& \int_0^{T_j} \Big\{ \frac{1}{1-\alpha^2}(P_j(0) - \alpha P_j(T_j))e^t \\
&+ \frac{\alpha}{1-\alpha^2}(P_j(T_j) - \alpha P_j(0))e^{-t} \Big\} dt \\
\leq& C(P_j(0) + P_j(T_j)).
\end{aligned}$$

By [Par96, lemma 1.7], $P_j(0) + P_j(T_j) \to 0$ as $j \to \infty$. So there's no energy loss on neck maps. □

3.3 Properties of Bubble Convergence

Now we introduce varifold distance and state the relation between bubble convergence and varifold convergence. The following definition of varifold distance $d_V(\cdot,\cdot)$ can be found at [CM11, Chapter 3].

Definition 3.3.1 (Varifold Distance): Fix a closed mainifold M, let

$$P_{\Pi} : \mathcal{G}_k M \to M$$

be the Grassmannian bundle of (unoriented) k-planes, that is, each fiber $P_{\Pi}^{-1}(p)$ is the set of all k-dimensional linear subspaces of the tangent space of M at p. Since $\mathcal{G}_k M$ is compact, we can choose a countable dense subset h_n of all continuous functions on $\mathcal{G}_k M$ with supremum norm at most one. Given a finite collection of maps

$$\begin{aligned} f_i : X_i \to M, \quad f_i \in W^{1,2}(X_i, M), i = 1, ..., k_1, \\ g_j : Y_j \to M, \quad g_j \in W^{1,2}(Y_j, M), j = 1, ..., k_2, \end{aligned}$$

here $\{X_i\}_{i=1}^{k_1}, \{Y_j\}_{j=1}^{k_2}$ are compact surfaces of dimension k. We consider the pairs $\{(X_i, F_i)\}_{i=1}^{k_1}$ and $\{(Y_j, G_j)\}_{j=1}^{k_2}$ with measurable maps

$$F_i : X_i \to \mathcal{G}_k M, \quad \text{and } G_j : Y_i \to \mathcal{G}_k M,$$

so that

$$f_i = P_{\Pi} \circ F_i, \quad \text{and } g_j = P_{\Pi} \circ G_j.$$

($F_i(x)$ is the linear subspace of $df_i(T_x M)$.) J_{f_i} is the Jacobian of f_i, then the varifold distance between them is defined by:

$$d_V(\{f_i\}_{i=1}^{k_1}, \{g_j\}_{j=1}^{k_2}) := \sum_{n=0}^{\infty} 2^{-n} \Big| \sum_{i=1}^{k_1} \int_{X_i} h_n \circ F_i J_{f_i} - \sum_{j=1}^{k_2} \int_{Y_j} h_n \circ G_j J_{g_j} \Big|. \tag{3.6}$$

Since M is a closed manifold isometrically embedded in $\mathbb{R}^N$. We can define varifold distance using $\mathbb{R}^N$ instead of M. Namely, let

$$P_{\tilde{\Pi}} : \mathcal{G}_k \mathbb{R}^N \to \mathbb{R}^N$$

be the Grassmanian bundle of k-planes. We can choose a countable dense subset $\{\tilde{h}_n\}$ of all continuous functions on $\mathcal{G}_k \mathbb{R}^N$ with supremum norm at most one. If the pairs $\{X_i, \tilde{F}_i\}$ and $\{Y_j, \tilde{G}_j\}$ represent compact surfaces $\{X_i, Y_j\}$ of dimension k with measurable maps $\tilde{F}_i : X_i \to \mathcal{G}_k \mathbb{R}^N$ and $\tilde{G}_j : Y_j \to \mathcal{G}_k \mathbb{R}^N$, so that

$$\begin{aligned} f_i = \tilde{\Pi} \circ \tilde{F}_i, \\ g_j = \tilde{\Pi} \circ \tilde{G}_j. \end{aligned}$$

Then the varifold distance between them is defined by

$$d_{V_R}(\{f_i\}_{i=1}^{k_1}, \{g_j\}_{j=1}^{k_2}) := \sum_{n=0}^{\infty} 2^{-n} \Big| \sum_{i=1}^{k_1} \int_{X_i} \tilde{h}_n \circ \tilde{F}_i J_{f_i} - \sum_{j=1}^{k_2} \int_{Y_j} \tilde{h}_n \circ \tilde{G}_j J_{g_j} \Big|. \tag{3.7}$$

Proposition 3.3.2 (Colding-Minicozzi, [CM08]): If a sequence $\{\gamma_j\}$ of $W^{1,2}(S^2, M)$ maps bubble converges to a collection of finitely many smooth maps $u_0, ..., u_n : S^2 \to M$ then it also varifold converges to $u_0, ..., u_n$.

Proof. Let $E = \sum_{i=0}^{n} E(u_i)$, since γ_j bubble converges to $\{u_i\}_{i=0}^{n}$, we assume without loss of generality that $d_B(\gamma_j, \{u_i\}_{i=0}^{n}) < 1/j$. So there exists conformal dilations $D_i^j \in PSL(2, \mathbb{C}), i = 0, ..., n$, and pairwise disjoint domains $\Omega_0^j, ..., \Omega_n^j, \bigcup_{i=0}^{n} \Omega_i^j \subset S^2$. such that the following holds:

$$\sum_{i=0}^{n} \Big(\int_{\Omega_i^j} |\nabla(\gamma_j - \nabla(u_i \circ D_i^j)|^2 \Big)^{1/2} < 1/j, \tag{3.8}$$

$$\int_{S^2 \setminus \bigcup_{i=0}^{n} \Omega_i^j} |\nabla \gamma_j|^2 < 3/j^2 + 2(n+1)E/j, \tag{3.9}$$

$$\sum_{i=0}^{n} \Big(\int_{S^2 \setminus \Omega_i^j} |\nabla(u_i \circ D_i^j)|^2 \Big)^{1/2} < 1/j. \tag{3.10}$$

For each u_i, let U_i denote the corresponding map to $\mathcal{G}_2 M$. Similarly, for each γ_j, let R_j denote the corresponding map to $\mathcal{G}_2 M$. We will also use that the map $\nabla u \to J_u$ is continuous as a map from L^2 to L^1 and thus area of u is continuous with respect to energy of u. (see [CM08, proposition A.3])

The proposition now follows by showing for each i and any $h \in C^0(\mathcal{G}_2 M)$ that

$$\begin{aligned}
\sum_{i=0}^{n} \int_{S^2} h \circ U_i J_{u_i} &= \sum_{i=0}^{n} \lim_{j \to \infty} \int_{\Omega_i^j} h \circ U_i \circ D_i^j J_{u_i \circ D_i^j} \\
&= \sum_{i=0}^{n} \lim_{j \to \infty} \int_{\Omega_i^j} h \circ R_j J_{\gamma_j} \\
&= \lim_{j \to \infty} \int_{\cup_i \Omega_i^j} h \circ R_j J_{\gamma_j} \\
&= \int_{S^2} h \circ R_j J_{\gamma_j},
\end{aligned}$$

where the first equality is simply the change of variables formula for integration, and the last equality follows from (3.9). □

Lemma 3.3.3: Given $\gamma_i : S^2 \to M$, $\gamma_i \in W^{1,2}(S^2, M)$, $i = 1, 2$. If $d_V(\gamma_1, \gamma_2) = 0$, then

$$E(\gamma_1) = E(\gamma_2).$$

Proof. Given $\epsilon > 0$, there exists a regularization $\tilde{\gamma}_i$ of γ_i with $\|\tilde{\gamma}_i - \gamma_i\|_{W^{1,2}} < \epsilon$, and $\tilde{\gamma}_i \in C^2(S^2, M)$ [SU81, Section 4]. The map $\tilde{\gamma}_i$ induces a C^1 pullback metric $\tilde{g}_i$ on S^2 (possibly degenerate, we can perturbe the metric to overcome this, see [CM08, Appendix D.2]) By the existence of isothermal coordinates [Jos91, Theorem 3.1.1], there is a unique orientation preserving $C^{1,1/2}$ conformal diffeomorphism τ_i such that

$$\tau_i : S^2_{g_0} \to S^2_{\tilde{g}_i},$$

here g_0 is the round metric on S^2 with constant curvature 1. We have

$$E(\tilde{\gamma}_i \circ \tau_i) = E(\tau_i : S^2_{g_0} \to S^2_{\tilde{g}_i}) = \text{Area}(S^2_{\tilde{g}_i}). \tag{3.11}$$

The condition $\|\tilde{\gamma}_i - \gamma_i\|_{W^{1,2}} < \epsilon$ implies that

$$E(\gamma_1) - E(\gamma_2) < E(\tilde{\gamma}_1) - E(\tilde{\gamma}_2) + 2\epsilon, \tag{3.12}$$

and

$$d_V(\tilde{\gamma}_1, \tilde{\gamma}_2) < C(\epsilon) + d_V(\gamma_1, \gamma_2), \tag{3.13}$$

here C is a continuous function such that $C(\epsilon) \to 0$ as $\epsilon \to 0$. Without loss of generality, let h_0 in definition 3.3.1 be the constant map 1, so that

$$\begin{aligned} d_v(\gamma_1, \gamma_2) &\geq \Big| \int_{S^2} J_{\gamma_1} - \int_{S^2} J_{\gamma_2} \Big| \\ &= |\text{Area}(\gamma_1) - \text{Area}(\gamma_2)|, \end{aligned} \tag{3.14}$$

and by (3.11), (3.12), (3.13), and (3.14) we have that

$$\begin{aligned} E(\gamma_1) - E(\gamma_2) &< E(\tilde{\gamma}_1) - E(\tilde{\gamma}_2) + 2\epsilon \\ &= \text{Area}(\tilde{\gamma}_1) - \text{Area}(\tilde{\gamma}_2) + 2\epsilon \\ &\leq d_V(\tilde{\gamma}_1, \tilde{\gamma}_2) + 2\epsilon \\ &< C(\epsilon) + 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ could be arbitrary small we thus have $E(\gamma_1) = E(\gamma_2)$. □

Given a collection of harmonic spheres $\{u_i\}_{i=0}^n$. Since $u_i : S^2 \to M$, $u_i \in W^{1,2}(S^2, M)$ for each i, and M is a closed Riemannian manifold isometrically embedded in $\mathbb{R}^N$, we have that

$$\sum_{i=0}^n u_i \circ \phi_i : S^2 \to \mathbb{R}^N, \quad \sum_{i=0}^n u_i \circ \phi_i \in W^{1,2}(S^2, \mathbb{R}^N),$$

for all $\phi_i \in PSL(S, \mathbb{C})$.

Claim 3.3.4: For a map $\gamma \in W^{1,2}(S^2, M)$ with $d_B(\gamma, \{u_i\}_{i=0}^n) < \epsilon$, $\epsilon < 1$, We have the following inequality

$$\inf\Big\{ \int_{S^2} |\nabla\gamma - \nabla\big(\sum_{i=0}^n u_i \circ \phi_i\big)|^2 \Big| \phi_i \in PSL(2,\mathbb{C})\Big\} < C(n)\epsilon,$$

for some constant $C(n)$ depends on $\{u_i\}_{i=0}^n$.

Proof. For any $\{\phi_i\}_{i=0}^n \in PSL(2,\mathbb{C})$ and pairewise disjoint domains $\{\Omega_i\}_{i=0}^n \subset S^2$ we have the following inequality:

$$\begin{aligned}\int_{S^2} |\nabla\gamma - \sum_{i=0}^n \nabla\big(u_i \circ \phi_i\big)|^2 \leq & \sum_{i=0}^n \Big(\int_{\Omega_i} |\nabla(\gamma - \nabla(u_i \circ \phi_i)|^2\Big) \\ & + \int_{S^2 \setminus \bigcup_{i=0}^n \Omega_i} |\nabla\gamma|^2 \\ & + \sum_{i=0}^n \Big(\int_{S^2\setminus\Omega_i} |\nabla(u_i \circ \phi_i)|^2\Big).\end{aligned}$$

By $d_B(\gamma, \{u_i\}_{i=0}^n) < \epsilon$, we know that there exist pairwise disjoint domains $\{\tilde{\Omega}_i\}_{i=0}^n \subset S^2$, and conformal transformations $\{\tilde{\phi}_i\}_{i=0}^n \in PSL(2,\mathbb{C})$ such that

$$\begin{aligned}\sum_{i=0}^n \Big(\int_{\tilde{\Omega}_i} |\nabla(\gamma - \nabla(u_i \circ \tilde{\phi}_i)|^2\Big) + \int_{S^2 \setminus \bigcup_{i=0}^n \tilde{\Omega}_i} |\nabla\gamma|^2 & \\ + \sum_{i=0}^n \Big(\int_{S^2\setminus\tilde{\Omega}_i} |\nabla(u_i \circ \tilde{\phi}_i)|^2\Big) & \\ < 2\epsilon^2 + 3\epsilon^2 + 2(n+1)\epsilon E, &\end{aligned}$$

here $E = \sum_{i=0}^n E(u_i)$. It implies the desired result:

$$\inf\Big\{ \int_{S^2} |\nabla\gamma - \nabla\big(\sum_{i=0}^n u_i \circ \phi_i\big)|^2 \Big| \phi_i \in PSL(2,\mathbb{C})\Big\} < C(n)\epsilon,$$

here $C(n) = 5 + 2(n+1)\sum_{i=0}^n E(u_i)$. □

Now we know that by proposition 3.3.2 bubble convergence implies varifold convergence, but the converse is not true, since varifold convergence simply implies that the images of two maps is close in measure sense. The following Proposition states that varifold convergence implies the term $d_B(\cdot,\cdot)$ goes to zero if the term:

$$\inf\Big\{ \int_{S^2} |\nabla\gamma - \nabla\big(\sum_{i=0}^n u_i \circ \phi_i\big)|^2 \Big| \phi_i \in PSL(2,\mathbb{C})\Big\}$$

also goes to zero.

Proposition 3.3.5: Given a collection of finitely many harmonic spheres $\{u_i\}_{i=0}^n$, for all $\epsilon > 0$, there exists $\delta > 0$ so that if $\gamma \in W^{1,2}(S^2, M)$ satisfies the following conditions:

$$\inf\Big\{ \int_{S^2} |\nabla\gamma - \nabla\big(\sum_{i=0}^{n} u_i \circ \phi_i\big)|^2 \Big| \phi_i \in PSL(2,\mathbb{C})\Big\} < \delta,$$

and

$$d_{V_R}(\gamma, \{u_i\}_{i=0}^n) < \delta,$$

we have

$$d_B(\gamma, \{u_i\}_{i=0}^n) < \epsilon$$

Proof. For the case that $\{u_i\}_{i=0}^n$ contains only one harmonic sphere say u_0, then given $\epsilon > 0$ and let $\delta = \epsilon/2$. The condition

$$\inf\Big\{ \int_{S^2} |\nabla\gamma - \nabla\big(u_0 \circ \phi\big)|^2 \Big| \phi \in PSL(2,\mathbb{C})\Big\} < \delta$$

implies that there exists $\phi_0 \in PSL(2,\mathbb{C})$ such that $\int_{S^2} |\nabla\gamma - \nabla\big(u_0 \circ \phi_0\big)|^2 < \epsilon$, thus we have $d_B(\gamma, u_0) < \epsilon$.

For the general case it suffices to consider that $\{u_i\}_{i=0}^n = \{u_0, u_1\}$. We argue by contradiction, suppose there exists $\epsilon > 0$ and a sequence $\{\gamma_j\}_{j\in\mathbb{N}}$, $\gamma_j \in W^{1,2}(S^2, M)$, such that

$$\inf\Big\{ \int_{S^2} |\nabla\gamma_j - \nabla\big(\sum_{i=0,1} u_i \circ \phi_i\big)|^2 \Big| \phi_i \in PSL(2,\mathbb{C})\Big\} < 1/j, \text{and } d_{V_R}(\gamma_j, \{u_i\}_{i=0,1}) < 1/j,$$

but does not imply $d_B(\gamma_j, \{u_0, u_1\}) < \epsilon$ for all $j \in \mathbb{N}$. That means for any parewise disjoint domains $\Omega_i \subset S^2$, $\Omega_0 \cap \Omega_1 = \{\emptyset\}$, and conformal automorphisms $\phi_i \in PSL(2,\mathbb{C})$, $i = 0, 1$ with $\sum_{i=0,1} \Big(\int_{S^2\setminus\Omega_i} |\nabla(u_i \circ \phi_i)|^2\Big)^{1/2} < \epsilon$, we have either

$$\int_{S^2\setminus\bigcup_i \Omega_i} |\nabla\gamma_j|^2 \geq 3\epsilon^2 + 4\epsilon E, \tag{3.15}$$

where $E = E(u_0) + E(u_1)$, or

$$\sum_{i=0,1} \Big(\int_{\Omega_i} |\nabla\gamma_j - \nabla(u_i \circ \phi_i)|^2\Big)^{1/2} \geq \epsilon. \tag{3.16}$$

(see definition 3.1.5). We first consider the case of (3.15) assuming the following condition $\sum_{i=0,1}\Big(\int_{\Omega_i}|\nabla\gamma_j-\nabla(u_i\circ\phi_i)|^2\Big)^{1/2}<\epsilon$. We consider the following inequality

$$\begin{aligned}\int_{S^2}|\nabla u_0|^2+\int_{S^2}|\nabla u_1|^2&=\sum_{i=0,1}\Big(\int_{\Omega_i}|\nabla(u_i\circ\phi)|^2+\int_{S^2\setminus\Omega_i}|\nabla(u_i\circ\phi)|^2\Big)\\&<\sum_{i=0,1}\Big(\int_{\Omega_i}|\nabla\gamma_j|^2\\&\qquad+2\big(\int_{\Omega_i}|\nabla\gamma_j|^2\big)^{1/2}\big(\int_{\Omega_i}|\nabla\gamma_j-\nabla(u_i\circ\phi)|^2\big)^{1/2}\\&\qquad+\int_{\Omega_i}|\nabla\gamma_j-\nabla(u_i\circ\phi)|^2+\int_{S^2\setminus\Omega_i}|\nabla(u_i\circ\phi)|^2\Big)\\&<\sum_{i=0,1}\int_{\Omega_i}|\nabla\gamma_j|^2+2\epsilon\big(\int_{\Omega_i}|\nabla\gamma_j|^2\big)^{1/2}+\epsilon^2+\epsilon^2.\end{aligned}$$

Since $d_{V_R}(\gamma_j,\{u_i\}_{i=0,1})<1/j$, by Lemma 3.3.3 we know that

$$\lim_{j\to\infty}E(\gamma_j)=E(u_0)+E(u_1).$$

For j sufficiently large, we have that

$$\begin{aligned}E(\gamma_j)&=\int_{\bigcup_i\Omega_i}|\nabla\gamma_j|^2+\int_{S^2\setminus\bigcup_i\Omega_i}|\nabla\gamma_j|^2\\&<\sum_{i=0,1}\int_{\Omega_i}|\nabla\gamma_j|^2+2\epsilon\big(\int_{\Omega_i}|\nabla\gamma_j|^2\big)^{1/2}+\epsilon^2+\epsilon^2\\&<\sum_{i=0,1}\int_{\Omega_i}|\nabla\gamma_j|^2+2\epsilon\big(E(u_0)+E(u_1)\big)^{1/2}+2\epsilon^2.\end{aligned}$$

which implies that

$$\int_{S^2\setminus\bigcup_i\Omega_i}|\nabla\gamma_j|^2<4\epsilon(E(u_0)+E(u_1))^{1/2}+2\epsilon^2,$$

contradicting our assumption (3.15).

Now we consider the case of (3.16). By our assumption of the sequence $\{\gamma_j\}_{j\in\mathbb{N}}$: $\inf\Big\{\int_{S^2}|\nabla\gamma_j-\nabla\big(\sum_{i=0,1}u_i\circ\phi_i\big)|^2\Big|\phi_i\in PSL(2,\mathbb{C})\Big\}<1/j$, we know that for each j there exist ϕ_0^j,ϕ_1^j such that

$$\int_{S^2}|\nabla\gamma_j-\nabla(u_0\circ\phi_0^j+u_1\circ\phi_1^j)|^2<2/j,$$

which implies that:

$$\begin{aligned} d_{V_R}\big((u_0\circ\phi_0^j+u_1\circ\phi_1^j),\{u_i\}_{i=0,1}\big) \leq & d_{V_R}(u_0\circ\phi_0^j+u_1\circ\phi_1^j,\gamma_j) \\ & + d_{V_R}(\gamma_j,\{u_i\}_{i=0,1}) \\ < & C(j)+1/j, \end{aligned} \tag{3.17}$$

for some $C(j)\to 0$ as $j\to\infty$. Since $(u_0\circ\phi_0^j+u_1\circ\phi_1^j)(S^2)=(u_0\circ(\phi_0^j\circ(\phi_1^j)^{-1})+u_1)(S^2)$, we can assume ϕ_1^j is identity for all $j\in\mathbb{N}$. Thus by (3.17) we know that $(u_0\circ\phi_0^j+u_1)(S^2)$ can be arbitrarily close to $\{u_0(S^2),u_1(S^2)\}$ as $j\to\infty$ in measure sense in $\mathbb{R}^N$ (see definition 3.3.1), this implies that $\{\phi_0^j\}_{j\in\mathbb{N}}$ is a divergent sequence in $PSL(2,\mathbb{C})$. Now we choose $R>0$ so that for $R'\geq R$ we have the following

$$\Big(\int_{S^2\setminus B_{R'}(p)}|\nabla u_1|^2\Big)^{1/2}<\epsilon/5,\ \forall p\in S^2.$$

Since $\{\phi_0^j\}_{j\in\mathbb{N}}$ is diverging, then for j sufficiently large we can find $q\in S^2$ such that

$$\int_{B_R(q)}|\nabla(u_0\circ\phi_0^j)|^2+\int_{S^2\setminus B_R(q)}|\nabla u_1|^2<\epsilon^2/10, \tag{3.18}$$

Then the assumption (3.16) implies

$$\Big(\int_{B_R(q)}|\nabla\gamma_j-\nabla u_1|^2\Big)^{1/2}+\Big(\int_{S^2\setminus B_R(q)}|\nabla\gamma_j-\nabla(u_0\circ\phi_0^j)|^2\Big)^{1/2}\geq\epsilon. \tag{3.19}$$

From the inequality

$$\begin{aligned} \int_{S^2}|\nabla\gamma_j-\nabla(u_0\circ\phi_0^j+u_1)|^2 \geq & \Big|\Big(\int_{B_R(q)}|\nabla\gamma_j-\nabla u_1|^2+\int_{S^2\setminus B_R(q)}|\nabla\gamma_j-\nabla(u_0\circ\phi_0^j)|^2\Big) \\ & -\Big(\int_{B_R(q)}|\nabla(u_0\circ\phi_0^j)|^2+\int_{S^2\setminus B_R(q)}|\nabla u_1|^2\Big)\Big|, \end{aligned}$$

with the assumption $\int_{S^2}|\nabla\gamma_j-\nabla(u_0\circ\phi_0^j+u_1)|^2<2/j$ and (3.19) we have the following:

$$\begin{aligned} \int_{B_R(q)}|\nabla(u_0\circ\phi_0^j)|^2+\int_{S^2\setminus B_R(q)}|\nabla u_1|^2 \geq & \int_{B_R(q)}|\nabla\gamma_j-\nabla u_1|^2 \\ & +\int_{S^2\setminus B_R(q)}|\nabla\gamma_j-\nabla(u_0\circ\phi_0^j)|^2 \\ & -\int_{S^2}|\nabla\gamma_j-\nabla(u_0\circ\phi_0^j+u_1)|^2 \\ & \geq\epsilon^2/5, \end{aligned}$$

for j sufficiently large. It contradicts with the assumption (3.18). Thus we have proved proposition 3.3.5. □

Remark. Combining claim 3.3.4 and proposition 3.3.5 we have for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_B(\gamma, \{u_i\}_{i=0}^n) < \epsilon,$$

if and only if

$$\inf \Big\{ \int_{S^2} |\nabla\gamma - \nabla\big(\sum_{i=0}^{n} u_i \circ \phi_i\big)|^2 \Big| \phi_i \in PSL(2,\mathbb{C}) \Big\} < \delta,$$

and

$$d_{V_R}(\gamma, \{u_i\}_{i=0}^n) < \delta.$$

Chapter 4

Min-Max Constructions

4.1 Sweepout by Curves: Birkhoff's Min-Max Theory

Min-max theory is a way of finding critical points for variational problems. Here we state Birkhoff's work on the curve shortening process, and the idea of using it to prove the existence of geodesics. Colding and Minicozzi use the same idea to maps from S^2 to the manifold to show the existence of harmonic maps.

Curve Shortening Process

Fix a large positive number L and define Λ as the space of all piecewise linear maps (geodesics) from S^1 to M with L breaks. Such that the length of each geodesic segment is at most 2π, parametrized by arclength proportionately and with Lipschitz bound. Let $G \subset \Lambda$ denote the set of immersed closed geodesics in M of length at most $2\pi L$. Note that Λ is finite dimensional.

Given a piecewise smooth curve γ with at most L-breaks. We fix a partition of S^1 by choosing $2L$ evenly spaced points

$$x_0, ..., x_{2L} = x_0 \in S^1,$$

so that $|x_j - x_{j+1}| = \pi/L$. The idea of curve shortening process is replacing γ with geodesics on each *even* interval, i.e., $[x_{2j}, x_{2j+1}]$ by the geodesic with the same

endpoints to get a piecewise linear curve $\gamma_e : S^1 \to M$, and again replacing γ_e on each *odd* interval, i.e., $[x_{2j+1}, x_{2j+2}]$ by the geodesics with the same endpoints to get the piecewise smooth curve γ_o. Now we reparametrize γ_0 so that it's parametrized by arclength proportionately. We call the above *replacing* and *reparametrization* curve shortening process and denote this procedure by the map Ψ. Ψ satisfies the following conditions, for the proof we refer to [CM11, Chapter 5].

1. $\Psi(\gamma)$ is homotopic to γ, and $\text{Length}(\Psi(\gamma)) \leq \text{Length}(\gamma)$,
2. $\Psi(\gamma)$ depends continuously on γ,
3. There's a continuous function $\phi : [0, \infty) \to [0, \infty)$, with $\phi(0) = 0$, so that
$$dist^2(\gamma, \Psi(\gamma)) \leq \phi\Big(\frac{\text{Length}^2(\gamma) - \text{Length}^2(\Psi(\gamma))}{\text{Length}^2(\Psi(\gamma))}\Big),$$
4. Given $\epsilon > 0$, there exists $\delta > 0$, so that if $\gamma \in \Lambda$ with $dist(\gamma, G) \geq \epsilon$ then
$$\text{Length}(\Psi(\gamma)) \leq \text{Length}(\gamma) - \delta.$$

Defining the Sweepout

Let Ω be the set of the maps
$$\sigma : S^1 \times [0, 1] \to M,$$
so that:

1. For each t the map $\sigma(\cdot, t)$ is in $W^{1,2}(S^1, M)$.
2. The map $t \to \sigma(\cdot, t)$ is continuous from $[0, 1]$ to $W^{1,2}(S^1, M)$ with respect to the norm $\| \cdot \|_{W^{1,2}}$.
3. σ maps $S^1 \times \{0\}$ and $S^1 \times \{1\}$ to point maps.

Given a map $\sigma \in \Omega_\sigma$, the homotopy class Ω_σ is defined to be the set of maps $\hat{\sigma} \in \Omega$ that are homotopic to σ through maps in Ω. The width $W = W(\sigma)$ associated to the homotopy class Ω_σ is defined by taking infimum of maximum of the energy of each slice $\hat{\sigma}(\cdot, t)$. That is
$$W = \inf_{\hat{\sigma} \in \Omega_\sigma} \max_{t \in [0,1]} E(\hat{\sigma}(\cdot, t)).$$

The width W is always nonnegative and is positive if $\hat{\sigma}$ is in a nontrivial homotopy class. For $W > 0$ we choose a sequence of sweepouts $\{\sigma_j\}$ in Λ such that:

$$\max_{t\in[0,1]} E(\sigma_j(\cdot,t)) < W + 1/j.$$

Since $\sigma_j \in \Lambda$, for all $j \in \mathbb{N}$, it's parametrized proportionately by arclength, thus we have $E(\gamma_j) = 2\pi L \cdot \text{Length}^2(\gamma_j)$. Now we *tighten* the sweepout by curve shortening procedure Ψ. Let $\gamma_j(\cdot,t) = \Psi(\sigma_j(\cdot,t))$. Since the curve shortening procedure will leave the map in the same homotopy class. Thus again we have:

$$\begin{aligned} W &\leq \max_{t\in[0,1]} E(\gamma_j(\cdot,t)) \\ &= E(\gamma_j(\cdot,t_j)) \qquad\qquad \text{for some } t_j \in [0,1] \\ &\leq E(\sigma_j(\cdot,t_j)) \\ &\leq \max_{t\in[0,1]} E(\sigma_j(\cdot,t)) < W + 1/j, \end{aligned}$$

By property 4 of the curve shortening procedure we know that if j is sufficiently large $\sigma_j(\cdot,t_j)$ is close to a geodesic. So we conclude the following: *Each curve in the tightened sweepout whose length is close to the length of the longest curve in the sweepout must itself be close to a closed geodesic. In particular, there are curves in the sweepout that are close to closed geodesics.*

4.2 Colding and Minicozzi's Min-Max Theory

Harmonic Replacement

Given a closed manifold M of dimension at least three, isometrically embedded in $\mathbb{R}^N$ for some N, we turn next to maps from S^2 to M. The rough idea is the same as for curves: we define an energy decreasing procedure and show that it strictly decreases the energy of those maps which are away from harmonic maps. The energy decreasing map will again be constructed by local replacement, but this time we use replacement by harmonic (energy-minimizing) maps on balls with small energy. This is often referred to as *harmonic replacement*. The first key property that makes harmonic replacement useful is that the energy functional is strictly convex on small energy maps. The precise statement is given by the following result of Cloding-Minicozzi, [CM08]:

Theorem 4.2.1 (Colding-Minicozzi, [CM08]): There exists a contant $\epsilon_1 > 0$ (depending on M) so that if u and v are $W^{1,2}$ maps from $B_1 \subset \mathbb{R}^2$ to M, u and v agree on

∂B_1, and v is weakly harmonic with energy at most ϵ_1, then

$$\int_{B_1} |\nabla u|^2 - \int_{B_1} |\nabla v|^2 \geq \frac{1}{2} \int_{B_1} |\nabla v - \nabla u|^2.$$

An immediate corollary is the uniqueness of solutions to the Dirichlet problem for small energy maps. The second consequence is that the harmonic replacement is continuous as map from $C^0(\bar{B}_1, M) \cap W^{1,2}(B_1, M)$ to itself if we restrict to small energy maps, the continuity of harmonic replacement is like the property 2 of curve shortening process, allowing us to *tighten* the sweepout by harmonic replacement and construct the new sweepout .

Corollary 4.2.2 (corollary 3.4 [CM08]): Let $\epsilon_1 > 0$ be as in theorem 4.2.1 and set

$$\mathcal{N} = \{u \in C^0(\bar{B}_1, M) \cap W^{1,2}(B_1, M) | E(u) \leq \epsilon_1\}.$$

Given $u \in \mathcal{N}$, there's a unique energy minimizing map w equal to u on ∂B_1 and w is in $\mathcal{N}$. Furthermore, there exists C depending on M so that if $u_1, u_2 \in \mathcal{N}$ with corresponding energy minimizing maps w_1, w_2, and we set $E = E(u_1) + E(u_2)$, then

$$|E(w_1) - E(w_2)| \leq C|u_1 - u_2|_{C^0} E + C|\nabla u_1 - \nabla u_2|_{L^2} E^{1/2}. \tag{4.1}$$

Finally, the map from u to w is continuous as a map from $C^0(\bar{B}_1) \cap W^{1,2}(B_1)$ to itself.

Proof. In the proof, we will use *nearest point projection* defined in section 2.1.4, and choose the tubular neighborhood M_δ of M in $\mathbb{R}^N$ so that Π is a smooth function defined on M_δ. For any $x \in M$, we have $d\Pi_x(V) \leq |V|$. Therefore, there's a constant C_Π depending on M so that for any $x \in M_\delta$, we have $|d\Pi_x(V)| \leq (1 + C_\Pi |x - \Pi(x)|)|V|$. In particular, we can choose $\delta > 0$ small enough so that $d\Pi_x(V) \leq 2|V|$ for all $x \in M_\delta$.

Define a map v_1 by

$$v_1 = \Pi \circ (w_2 + (u_1 - u_2)).$$

so that v_1 maps to M and agrees with u_1 on ∂B_1. We have

$$E(v_1) \leq \Big(1 + C_\Pi |u_1 - u_2|_{C^0}\Big)^2 \Big(E(w_2) + 2(E(w_2)E(u_1 - u_2))^{1/2} + E(u_1 - u_2)\Big). \tag{4.2}$$

Since $E(w_1) \leq E(v_1)$ and by symmetry we can assume $E(w_2) \leq E(w_1)$ so that (4.2) implies (4.1).

For showing the continuity, we suppose that u u_j are in $\mathcal{N}$ with $u_j \to u$ in $C^0(\bar{B}_1, \mathbb{R}^N) \cap W^{1,2}(B_1, \mathbb{R}^N)$ and w w_j are the corresponding energy minimizing maps. We will first show that $w_j \to w$ in $W^{1,2}(B_1, \mathbb{R}^N)$. To do this, we set

$$v_j = \Pi \circ (w + (u_j - u)),$$

so that v_j maps to M and agrees with u_j on ∂B_1. Since $u_j \to u$ in $C^0(\bar{B}_1, \mathbb{R}^N) \cap W^{1,2}(B_1, \mathbb{R}^N)$, by (4.1) we have $E(w_j) \to E(w)$, so we get that $E(v_j) - E(w_j) \to 0$. Therefore, applying theorem 4.2.1 to w_j and v_j gives that $\|v_j - w_j\|_{W^{1,2}} \to 0$. Since $u_j \to u$ in $C^0(\bar{B}_1, \mathbb{R}^N) \cap W^{1,2}(B_1, \mathbb{R}^N)$ and

$$w - v_j = \Pi \circ (w - w - (u_j - u)),$$

it follows that $\|w - v_j\|_{W^{1,2}} \to 0$. The triangle inequality gives $\|w - w_j\|_{W^{1,2}} \to 0$.

Finally, we will argue by contradiction to see that $w_j \to w$ in $C^0(\bar{B}_1)$. Suppose instead there's a subsequence (which we still denote w_j) with

$$|w_j - w|_{C^0} \geq \epsilon > 0. \tag{4.3}$$

Using the uniform energy bound for the w_j together with theorem 2.1.43 we can pass to a further subsequence so that the w_j converge uniformly in C^2 on any compact set in B_1. Finally, by [Qin93] we know that w_j are also equicontinuous near ∂B_1, so Arzelà-Ascoli theorem gives a further subsequence that converges uniformly on $\bar{B}_1$ to a harmonic map w_∞ agrees with w on the boundary. However (4.3) implies that $|w_\infty - w|_{C^0} \geq \epsilon > 0$ which contradicts the uniqueness of small energy harmonic maps. This completes the proof. □

Definition 4.2.3 (Width): Let Ω be the set of continuous maps $\sigma : S^2 \times [0,1] \to M$ so that for each $t \in [0,1]$ the map $\sigma(\cdot, t)$ is in $C^0(S^2, M) \cap W^{1,2}(S^2, M)$, the map $t \to \sigma(\cdot, t)$ is continuous from $[0,1]$ to $C^0(S^2, M) \cap W^{1,2}(S^2, M)$ in a strong sense. Given a map $\beta \in \Omega$, the homotopy class Ω_β is defined to be the set of maps $\sigma \in \Omega$ that is homotopic to β through maps in Ω. We'll call any such σ a *sweepout*. The width $W = W_E(\beta, M)$ associated to the homotopy class Ω_β is defined by:

$$W := \inf_{\sigma \in \Omega_\beta} \max_{t \in [0,1]} E(\sigma(\cdot, t)). \tag{4.4}$$

We could alternatively define the width using area rather than energy by setting

$$W_A := \inf_{\sigma \in \Omega_\beta} \max_{t \in [0,1]} \text{Area}(\sigma(\cdot, t)).$$

Remark. We're interested in the case where β induces a map in a nontrivial class in $\pi_3(M)$. In that case the width is positive and equals to the area of a non-empty finite collection of minimal 2-spheres. Corollary 4.2.2 gives a proof that the width is positive when the homotopy class is nontrivial or, equivalently, that

if $\max_{t\in[0,1]} E(\sigma(\cdot,t))$ is sufficiently small then σ is homotopically trivial. Namely, since $t \to \sigma(\cdot,t)$ is continuous from $[0,1]$ to $C^0(S^2,M)$, we can choose $r>0$ so that $B_r(p) \subset S^2$ and $\sigma(B_r(p),t) \subset B^t$ for some convex geodesic ball $B^t \subset M$ for every $t \in [0,1]$. If each $\sigma(\cdot,t)$ has energy less than ϵ_1 given by theorem 4.2.1, then replacing $\sigma(\cdot,t)$ outside of $B_r(p)$ by the energy minimizing map with the same boundary values gives a new sweepout $\tilde{\sigma}(\cdot,t)$. $\tilde{\sigma}$ is homotopic to σ by theorem 4.2.1. Moreover, the entire image of $\tilde{\sigma}(\cdot,t)$ is contained in B^t by [Jos11, lemma 4.1.3]. It follows that $\tilde{\sigma}$ is homotopically trivial by contracting each $\tilde{\sigma}(\cdot,t)$ to the point $\sigma(p,t)$ via a geodesic homotopy.

Now we will define an energy decreasing map from Ω_β, for $\beta \in \pi_3(M)$ nontrivial, to itself. This should be thought of as a generalization of Birkhoff's curve shortening process that plays a similar role when tightening a sweepout by curves.

Theorem 4.2.4: There's a constant $\epsilon_0 > 0$ and a continuous function $\Psi : [0,\infty) \to [0,\infty)$ with $\Psi(0) = 0$, both depending on M, so that given any $\tilde{\gamma} \in \Omega_\beta$ without nonconstant harmonic slices and $\beta \in \pi_3(M)$ nontrivial, there exists $\gamma \in \Omega_\beta$ so that

$$E(\gamma(\cdot,t)) \leq E(\tilde{\gamma}(\cdot,t)),$$

for each t and so for each t with $E(\tilde{\gamma}(\cdot,t)) \geq W/2$ we have the following: If $\mathcal{B}$ is any finite collection of disjoint closed balls in S^2 with $\int_{\mathcal{B}} |\nabla\gamma(\cdot,t)|^2 < \epsilon_0$ and $v : \cup_{B\in\mathcal{B}} \frac{1}{8}B \to M$ is an energy minimizing map equal to $\gamma(\cdot,t)$ on $\cup_{B\in\mathcal{B}} \frac{1}{8}\partial B$, then

$$\int_{\cup_{B\in\mathcal{B}} \frac{1}{8}B} |\nabla\gamma(\cdot,t) - \nabla v|^2 \leq \Psi(E(\tilde{\gamma}(\cdot,t)) - E(\gamma(\cdot,t))).$$

Theorem 4.2.4 is like properties 3 and 4 of curve shortening process. Note that if $E(\tilde{\gamma}(\cdot,t))$ and $E(\gamma(\cdot,t))$ are close then theorem 4.2.4 implies that $\gamma(\cdot,t)$ is close to harmonic maps. $\gamma(\cdot,t)$ is constructed from $\tilde{\gamma}(\cdot,t)$ by harmonic replacement on a family of balls in S^2 varying continuously in t, and then use theorem 4.2.1 and corollary 4.2.2 to argue that harmonic replacement does decrease the energy of $\tilde{\gamma}(\cdot,t)$ by certain amount bounded away from zero if $\tilde{\gamma}(\cdot,t)$ isn't harmonic. Finally, use corollary 4.2.4 to conclude that $\gamma(\cdot,t)$ varies continuously in t and it's a sweepout homotopic to $\tilde{\gamma}(\cdot,t)$. For the proof of theorem 4.2.4 and technical details of choosing such balls in S^2 we refer to [CM08].

Definition 4.2.5 (Minimizing sequence): Given a sweepout $\gamma_j(\cdot,t) : S^2 \times [0,1] \to M$, we call $\{\gamma_j(\cdot,\cdot)\}_{j\in\mathbb{N}}$ a *minimizing sequence* if

$$\lim_{j\to\infty} \max_{t\in[0,1]} E(\gamma_j(\cdot,t)) = W.$$

We call $\{\gamma_j(\cdot,t_j)\}_{j\in\mathbb{N}}$ a *min-max sequence* if

$$\lim_{j\to\infty} E(\gamma_j(\cdot,t_j)) = W.$$

Proposition 4.2.6 (Proposition 1.2 [CM08]): Given a closed manifold M with dimension $n \geq 3$, and a map $\beta \in \Omega$ representing a nontrivial class in $\pi_3(M)$. The width of energy W, and the width of area W_A associated to the homotopy class Ω_β are equal.

Remark. Let $\{\gamma_j\}_{j\in\mathbb{N}}$ be a minimizing sequence for W. Since $\text{Area}(\gamma_j(\cdot,t)) \leq E(\gamma_j(\cdot,t))$, and $W_A = W$ by Proposition 4.2.6, we have that

$$W_A \leq \lim_{j\to\infty} \max_{s\in[0,1]} \text{Area}(\gamma_j(\cdot,s)) \leq \lim_{j\to\infty} \max_{s\in[0,1]} E(\gamma_j(\cdot,s)) = W_A,$$

which implies that $\{\gamma_j\}_{j\in\mathbb{N}}$ is a minimizing sequence of area functional for W_A too.

Definition 4.2.7: We define the equivalent class of $u : S^2 \to M$ to be:

$$[u] := \Big\{ g : S^2 \to M \Big| \text{ if } u = g \circ \phi \text{ for some } \phi \in PSL(2,\mathbb{C}) \Big\},$$

Remark. Given maps $\gamma, \{u_i\}_{i=0}^n \in W^{1,2}(S^2, M)$ with $d_B(\gamma, \{u_i\}_{i=0}^n) < \epsilon$, we have

$$d_B(\gamma, \{g_i\}_{i=0}^n) < \epsilon, \text{ if } [g_i] = [u_i],\ i = 0, ..., n.$$

Definition 4.2.8 (Image set): The image set $\Lambda(\{\gamma_j(\cdot,t)\})$ of $\{\gamma_j(\cdot,t)\}_{j\in\mathbb{N}}$ is defined to be:

$$\Lambda(\{\gamma_j(\cdot,t)\}_{j\in\mathbb{N}}) := \Big\{ \{[u_i]\}_{i=0}^n : \text{there exists a sequence } \{i_j\} \to \infty, t_{i_j} \in [0,1], \text{ such that } \gamma_{i_j}(\cdot,t_{i_j}) \text{ bubble converges to } \{u_i\}_{i=0}^n \Big\},$$

Theorem 4.2.9 (Colding-Minicozzi, [CM08]): Given a closed manifold M with dimension $n \geq 3$, and a map $\beta \in \Omega$ representing a nontrivial class in $\pi_3(M)$, there exists a sequence of sweepouts $\gamma_j(\cdot,t) \in \Omega_\beta$ with $\max_{s\in[0,1]} \gamma_j(\cdot,s) \to W$, and so that given $\epsilon > 0$, there exists $\bar{j}$ and $\delta > 0$ so that if $j > \bar{j}$ and

$$Area(\gamma_j(\cdot,s)) > W - \delta,$$

then there are finitely many harmonic maps $u_i : S^2 \to M$, $i = 0, ..., n$, with

$$d_B(\gamma_j(\cdot, s), \{u_i\}_{i=0}^n) < \epsilon,$$

and

$$\sum_{i=0}^{n} E(u_i) = W.$$

Proof. Choose a sequence of maps $\tilde{\gamma} \in \Omega_\beta$ with

$$\max_{t\in[0,1]} E(\tilde{\gamma}_j(\cdot, t)) < W + \frac{1}{j},$$

and $\tilde{\gamma}_j(\cdot, t)$ is not harmonic unless it is a constant map.

Applying theorem 4.2.4 to $\tilde{\gamma}_j$ gives a sequence $\gamma_j \in \Omega_\beta$ with

$$E(\gamma_j(\cdot, t)) \leq E(\tilde{\gamma}_j(\cdot, t)).$$

We will argue by contradiction to show that γ_j have the desired property. Suppose that there exists $j_k \to \infty$ and $s_k \in [0, 1]$ with

$$Area(\gamma_{j_k}(\cdot, s_k)) > W - 1/k,$$

but doesn't imply the following

$$d_B(\gamma_{j_k}(\cdot, s_k), \{[u_i]\}_{i=0}^n) < \epsilon, \quad \forall \{[u_i]\}_{i=0}^n \in \mathcal{F}^{W+1}, \tag{4.5}$$

here $\mathcal{F}^{W+1}$ denote the set of collections of harmonic maps from S^2 to M so that the sum of the energies is at most $W + 1$. We denote $d_B(\gamma_{j_k}(\cdot, s_k), \mathcal{F}^{W+1}) > \epsilon$ for the above constraint. Now we have

$$\begin{aligned} W - 1/k < Area(\gamma_{j_k}(\cdot, s_k)) &\leq E(\gamma_{j_k}(\cdot, s_k)) \\ &\leq E(\tilde{\gamma}_{j_k}(\cdot, s_k)) \\ &\leq \max_{t\in[0,1]} E(\tilde{\gamma}_{j_k}(\cdot, t)) < W + 1/j_k, \end{aligned}$$

so we get

$$E(\tilde{\gamma}_{j_k}(\cdot, s_k)) - E(\gamma_{j_k}(\cdot, s_k)) \leq 1/k + 1/j_k \to 0, \tag{4.6}$$

and similarly

$$E(\gamma_{j_k}(\cdot, s_k)) - Area(\gamma_{j_k}(\cdot, s_k)) \to 0, \tag{4.7}$$

With (4.6) and recall the property of Ψ by theorem 4.2.4 and it gives the following:

If $\mathcal{B}$ is any finite collection of disjoint closed balls in S^2 with $\int_{\mathcal{B}} |\nabla \gamma_{j_k}(\cdot, s_k)|^2 < \epsilon_0$ and $v : \cup_{B\in\mathcal{B}} \frac{1}{8}B \to M$ is an energy minimizing map equal to $\gamma_{j_k}(\cdot, s_k)$ on $\cup_{B\in\mathcal{B}} \frac{1}{8}\partial B$, then

$$\begin{aligned}\int_{\cup_{B\in\mathcal{B}} \frac{1}{8}B} |\nabla \gamma_{j_k}(\cdot, s_k) - \nabla v|^2 &\leq \Psi(E(\tilde{\gamma}_{j_k}(\cdot, s_k)) - E(\gamma_{j_k}(\cdot, s_k))) \\ &\leq \Psi(1/k + 1/j_k) \to 0.\end{aligned}$$

Therefore, (4.7) and the above statement show that $\{\gamma_{j_k}(\cdot, s_k)\}$ is a sequence of almost harmonic maps, we can apply theorem 3.2.2 to get a subsequence of $\gamma_{j_k}(\cdot, s_k)$ which bubble converges to a collection of harmonic maps, contradicting the assumption $d_B(\gamma_{j_k}(\cdot, s_k), \mathcal{F}^{W+1}) > \epsilon$. We have proved the theorem. □

Corollary 4.2.10 (Colding-Minicozzi, [CM08]): Given a closed manifold M with dimension $n \geq 3$, and a map $\beta \in \Omega$ representing a nontrivial class in $\pi_3(M)$, there exists a sequence of sweepouts $\gamma_j(\cdot, t) \in \Omega_\beta$ with $\max_{s\in[0,1]} \gamma_j(\cdot, s) \to W$, so that given $\epsilon > 0$, there exists $\bar{j}$ and $\delta > 0$ so that if $j > \bar{j}$ and

$$Area(\gamma_j(\cdot, s)) > W - \delta,$$

then there are finitely many harmonic spheres $u_i : S^2 \to M$, $i = 0, ..., n$, with

$$d_V(\gamma_j(\cdot, s), \{u_i\}_{i=0}^n) < \epsilon.$$

Proof. By proposition 3.3.2 bubble convergence implies varifold convergence, thus theorem 4.2.9 implies corollary 4.2.10. □

Remark (Parameter spaces). Instead of using the unit interval [0,1] as the parameter space for the maps in the sweepout and assuming that the maps start and end in point maps, we could have used any compact finite dimensional topological space P and required that the maps are constant on ∂P (or $\partial P = \emptyset$).

Equivalent Statement

The goal of this section is to prove that *any min-max sequence bubble converges to a finite collection of harmonic spheres up to subsequence*, the statement doesn't follow immediately from theorem 4.2.10 since theorem 4.2.10 only states that there exists a sweepout, so that whenever the area of a slice of the sweepout is close to the width it must be close to a finite collection of harmonic spheres in bubble tree sense

itself. Theorem 4.2.10 is proven by showing that given any minimizing sequence $\{\tilde{\gamma}_j(\cdot,t)\}_{j\in\mathbb{N}}$, we can apply theorem 4.2.4, so that the *pulled-tight* sequence

$$\gamma_j(\cdot,t) = \Psi(\tilde{\gamma}_j(\cdot,t)),$$

(see theorem 4.2.4) contains a min-max sequence which bubble converges to a collection of finitely many harmonic spheres $\{u_i\}_{i=0}^n$ (whose sum of the energy realizes the width). That is,

$$\{[u_i]\}_{i=0}^n \in \Lambda(\{\gamma_j(\cdot,t)\}_{j\in\mathbb{N}}).$$

Our goal is showing that *any* minimizing sequence $\{\tilde{\gamma}_j(\cdot,t)\}_{j\in\mathbb{N}}$,

$$\lim_{j\to\infty}\max_{t\in[0,1]} E(\tilde{\gamma}_j(\cdot,t)) = W,$$

has nonempty image set $\Lambda(\{\tilde{\gamma}_j(\cdot,t)\}_{j\in\mathbb{N}})$. To do this, we observe that

$$E(\gamma_j(\cdot,t)) \leq E(\tilde{\gamma}_j(\cdot,t)),$$

for each j and for all $t\in[0,1]$. Since we already know that there exists a finite collection of harmonic spheres $\{[u_i]\}_{i=0}^n \in \Lambda(\{\gamma_j(\cdot,t)\}_{j\in\mathbb{N}})$. We can choose sequence $\{j_i\}\to\infty$, $t_{j_i}\in[0,1]$, such that $\{\gamma_{j_i}(\cdot,t_{j_i})\}$ bubble converges to $\{[u_i]\}_{i=0}^n$. Moreover, it's shown in the proof of theorem 4.2.4 (see [CM08, theorem 2.1]), $\Psi(\tilde{\gamma}_j(\cdot,t))$ is obtained from $\tilde{\gamma}_j(\cdot,t)$ by doing harmonic replacement on the disjoint closed balls on S^2 with energy at most $\epsilon_1>0$ (given in theorem 4.2.1), so by therorem 4.2.1 we have

$$E(\tilde{\gamma}_j(\cdot,t)) - E(\gamma_j(\cdot,t)) \geq \frac{1}{2}\int_{S^2} |\nabla\tilde{\gamma}_j(\cdot,t) - \nabla\gamma_j(\cdot,t)|^2.$$

Corollary 4.2.11: Given a sweepout $\{\gamma_j(\cdot,t)\}_{j\in\mathbb{N}}$, the pulled-tight of it: $\{\gamma_j(\cdot,t)\}_{j\in\mathbb{N}}$ given by [CM08, Theorem 2.1] has the following property:

$$E(\tilde{\gamma}_j(\cdot,t)) - E(\gamma_j(\cdot,t)) \geq \frac{1}{2}\int_{S^2} |\nabla\tilde{\gamma}_j(\cdot,t) - \nabla\gamma_j(\cdot,t)|^2.$$

Proof. The pulled-tight sweepout $\gamma_j(\cdot,t)$ is constructed by doing harmonic replacement on $\tilde{\gamma}_j(\cdot,t)$ over domain $\mathcal{B}\subset S^2$ with the energy of $\tilde{\gamma}_j(\cdot,t)$ on $\mathcal{B}$ at most $\epsilon_1>0$. Then by theorem 4.2.1 the corollary follows. □

With the above observation, we can state the following theorem:

Theorem 4.2.12: Given a closed manifold M with dimension $n\geq 3$, and a map $\beta\in\Omega$ representing a nontrivial class in $\pi_3(M)$, then for any sequence of sweepouts $\tilde{\gamma}_j\in\Omega_\beta$ with

$$\lim_{j\to\infty}\max_{s\in[0,1]} E(\tilde{\gamma}_j(\cdot,s)) = W,$$

there exists a subsequence $\{i_j\} \to \infty$, $t_{i_j} \in [0,1]$, and a collection of finitely many harmonic spheres $\{u_i\}_{i=0}^n$ such that

$$d_B(\tilde{\gamma}_{i_j}(\cdot, t_{i_j}), \{u_i\}_{i=0}^n) < 1/j,$$

and

$$\sum_{i=0}^{n} E(u_i) = W.$$

Proof. Given a minimizing sequence $\{\tilde{\gamma}_j\}_{j\in\mathbb{N}}$ such that

$$\max_{s\in[0,1]} E(\tilde{\gamma}_j(\cdot, s)) < W + 1/2j. \tag{4.8}$$

Applying theorem 4.2.4 to $\tilde{\gamma}_j$ gives a sequence $\gamma_j \in \Omega_\beta$ with

$$E(\gamma_j(\cdot, t)) \leq E(\tilde{\gamma}_j(\cdot, t)).$$

We choose $t_j \in [0,1]$ so that $E(\gamma_j(\cdot, t_j)) = \max_{s\in[0,1]} E(\gamma_j(\cdot, s))$. By similar argument in the proof of theorem 4.2.9 we know that $\{\gamma_j(\cdot, t_j)\}_{j\in\mathbb{N}}$ is an almost harmonic sequence, and by theorem 3.2.2, there exists a collection of finitely many harmonic spheres $\{u_i\}_{i=0}^n$ and subsequence $i_j \to \infty$ so that

$$d_B(\gamma_{i_j}(\cdot, t_{i_j}), \{u_i\}_{i=0}^n) < 1/2j.$$

By corollary 4.2.11 and (4.8) we have that

$$\frac{1}{2}\int_{S^2} |\nabla\tilde{\gamma}_j(\cdot, t_j) - \nabla\gamma_j(\cdot, t_j)|^2 \leq E(\tilde{\gamma}_j(\cdot, t_j)) - E(\gamma_j(\cdot, t_j)) < 1/j.$$

Thus we have

$$d_B(\tilde{\gamma}_{i_j}(\cdot, t_{i_j}), \{u_i\}_{i=0}^n) \to 0, \quad j \to \infty,$$

which is the desired result. □

Chapter 5

Countability for Harmonic Maps with Bounded Energy

5.1 Bumpy Metric

The world *bumpy metric* was introduced by Abraham in [Abr70]. Given a compact manifold M, let $\mathcal{H} := H^1(S^1, M)$ be the Hilbert manifold of absolutely continous maps $c : S^1 \to M$, and $E_g : \mathcal{H} \to \mathbb{R}$ the energy function for a Riemannian metric g on M. The a critical point $c \in \mathcal{H}$ of E_g is a closed geodesic parametrized proportionately to arclength. As the group S^1 acts continuously on $\mathcal{H}$ by

$$\begin{aligned} S^1 : \mathcal{H} &\to \mathcal{H}, \\ \theta(c)(t) = c(\theta + t), &\quad (\theta, c, t) \in S^1 \times \mathcal{H} \times S^1, \end{aligned}$$

and E_g is invariant under this action. The orbit $S^1(c)$ of a critical point $c \in \mathcal{H}$ consists entirely of critical points. In [Abr70], bumpy metric is defined as the the metric g, such that the Hessian of E_g has a one dimensional null space, the tangent space of $S^1(c)$ at c. A metric is called *generic* if it's a countable intersection of open dense subsets of the metric space, thus dense. We will talk about generic metric more later. It's showed in [Ano82] that bumpy metric is generic. We now state the bumpy metric theorem for geodesics:

Theorem 5.1.1 (Bumpy Metric Theorem for Geodesics, [Ano82]): If M is a compact manifold, then for a generic choice of metric on M, all nonconstant smooth closed geodesics have the property that their only Jacobi fields are those generated by the S^1 action on $\mathcal{H}$.

Now we define generic metric for a specific metric space. For $k \in \mathbb{N}$, the space of L^2_k Riemannian metrics on M simply denotes an open set of the Hilbert space of L^2_k-sections of the second symmetric power of T^*M.

Definition 5.1.2 (Generic Metrics, [Moo17]): By a generic Riemannian metric on a smooth manifold M we mean a Riemannian metric that belongs to a countable intersection of open dense subsets of the spaces of L^2_k Riemannian metrics on M with the L^2_k topology, for some choice of $k \in \mathbb{N}$, $k \geq 2$.

Remark. Notice that *generic metric* always implies a countable intersection of open dense subsets of the metric space. For geodesics and harmonic maps the metric space is L^2_k [Moo17], and for minimal submanifolds it's C^q Riemannian metric for $q \geq 3$, [Whi17].

Definition 5.1.3 (Bumpy Metric [Whi17]): A metric g on M is called *bumpy* if there is no smooth immersed minimal submanifold (minimal with respect to g) with a non-trivial Jacobi field.

Theorem 5.1.4 (Bumpy Metric Theorem for Minimal Submanifold, [Whi17]): If M is a compact manifold, then for a generic choice of metric of C^q g on M ($q \geq 3$), there are no minimal submanifolds with nonzero normal Jacobi fields. That is, each minimal submanifold has nullity 0.

We outline the idea of bumpy metric theorem here. Let $\mathcal{A}$ denote an open subset of C^q Riemannian metrics on M. Given a compact manifold N of dimension less than M, if u is a map from N to M let u' denote the set of all maps $u \circ \phi$ where $\phi : N \to N$ is a diffeomorphism. (Thus the element of u' are all parametrizations of the same surface $u(N)$.)

Bumpy metric theorem 5.1.4 is a study of the space $\mathcal{Z}$ of ordered pairs (g, u'), where $g \in \mathcal{A}$ and u is minimal with respect to g. First we define the projection map π defined as the following:

$$\pi : \mathcal{Z} \to \mathcal{A}$$
$$(g, u') \mapsto g.$$

It's shown in [Whi17, theorem 2.1] that π is a Fredholm map with Fredholm index 0. Moreover, the kernel of $D\pi(g, u')$ has dimension equal to the nullity of u' (i.e., the maximum number of linear independent normal Jacobi fields on u' with respect to area functional and metric g.) By the Sard-Smale theorem, this implies that the regular value of π is generic.

Bumpy Metric Theorem for Harmonic Maps

To state the bumpy metric theorem for harmonic maps, first we need to define prime harmonic map.

Definition 5.1.5 (Chapter 5 [Moo17]): Suppose that $f, h : S^2 \to M$ are harmonic spheres. f is called a *branched cover* of h if there exists a holomorphic map $g : S^2 \to S^2$ of degree $d \geq 2$ such that $f = h \circ g$. We call f a *prime* harmonic sphere if it's not a branched cover of another harmonic sphere.

Definition 5.1.6 (branch point): A point $p \in \Sigma$ is a *branch point* for the harmonic map $f : \Sigma \to M$ if $(\partial f/\partial z)(p) = 0$, where z is any complex coordinate near p.

Definition 5.1.7: If p is a branch point of f but there exists some neighborhood V containing p such that $f(V)$ is an immersed surface, then we say that p is a *false branch point*.

Definition 5.1.8 (Chapter 5 [Moo17]): If $f : \Sigma \to M$ is a parametrized minimal surface we say that $p \in \Sigma$ is an *injective point* for f if

$$df(p) \neq 0, \text{ and } \quad f^{-1}(f(p)) = 0.$$

If $f : \Sigma \to M$ is connected and has injective points, we say it is *somewhere injective*.

Lemma 5.1.9 (Chapter 5 [Moo17]): If a harmonic map $f : \Sigma \to M$ is somewhere injective, then it is prime.

Lemma 5.1.10 (Chapter 5 [Moo17]): If a conformal harmonic map $f : \Sigma \to M$ is prime, its injective points form an open dense subset of Σ.

Theorem 5.1.11 (Bumpy Metric Theorem for Harmonic Maps, theorem 5.1.1[Moo17]): If M is a compact smooth manifold of dimension at least three, then for a generic choice of Riemannina metric on M, all prime compact parametrized minimal surfaces $f : \Sigma \to M$ are free of branch points and lie on nondegenerate critical submanifolds, each such submanifold being an orbit of the group G of conformal automorphisms of Σ which are homotopic to the identity.

Remark. In Theorem 5.1.11, nondegeneracy of a prime harmonic map means that the Jacobi field of it are those generated by the automorphisms of Σ.

In our case $\Sigma = S^2$ and $G = PSL(2, \mathbb{C})$. Given a harmonic sphere $u : S^2 \to M$, if u is prime then it's free of branch points by Theorem 5.1.11. If u is not prime it can be written as a branched cover of a prime harmonic sphere, and all of its branched points are false. Theorem 5.1.11 implies that for all harmonic spheres $u : S^2 \to M$, $u(S^2)$ is an smooth immersed minimal submanifold.

We define the nullity $\mathcal{N}$ of a functional at a critical point u is the dimension of the space of Jacobi fields of the functional at u.

Theorem 5.1.12 (theorem 3.1[EM08]): Given a harmonic sphere $u : S^2 \to M$, let

$\mathcal{N}_A$ =nullity of u as a critical point of the area functional A,
$\mathcal{N}_E$ =nullity of u as a critical point of the energy functional E,
$\mathcal{N}_E^T$ =dimension of the space of purely tangential Jacobi fields of u, *as a critical point of* E.

Then

$$\mathcal{N}_A = \mathcal{N}_E - \mathcal{N}_E^T.$$

5.2 Countability of Harmonic Spheres with Bounded Energy

Lemma 5.2.1: Given (M, g), g bumpy, the dimension of M is at least three, and a harmonic sphere $u : S^2 \to M$. There exists $\epsilon = \epsilon(u) > 0$, so that if $\|f - u\|_{W^{1,2}} < \epsilon$, $f : S^2 \to M$ harmonic, then $[f] = [u]$. (See definition 4.2.7 for $[u]$.)

Proof. Given a harmonic sphere $u : S^2 \to M$, Theorem 5.1.11 implies that $u(S^2)$ is a smooth immersed minimal sphere. We argue by contradiction. If not, then there exists a sequence of harmonic spheres $\{f_i\}_{i\in\mathbb{N}}$, $f_i : S^2 \to M$, such that $\|f_i - u\|_{W^{1,2}} < 1/i$ and $[f_i] \neq [u]$ for all $i \in \mathbb{N}$. By strong convergence in $W^{1,2}$, Theorem 2.1.43, and Arzelà-Ascoli theorem we know that the convergence $f_i \to u$ is smooth and uniform. Thus we can choose subsequence $i(j) \to \infty$ as $j \to \infty$ such that

$$|f_{i(j)}(x) - u(x)| < 1/j, \ \forall x \in S^2, \tag{5.1}$$

and

$$\Big|\frac{\partial f_{i(j)}(x)}{\partial x^\alpha} - \frac{\partial u(x)}{\partial x^\alpha}\Big| < 1/j, \ \forall x \in S^2, \tag{5.2}$$

where (x^α) denotes the local coordinate system of S^2. Since $f_{i(j)}$ is harmonic we have that

$$\begin{aligned} 0 - 0 &= \tau(f_{i(j)}) - \tau(u) \\ &= \int_0^1 \frac{\partial}{\partial s}\Big|_{s=t} \tau(u + s(f_{i(j)} - u))dt. \end{aligned}$$

Thus there exists $t_j \in (0, 1)$ such that

$$0 = \frac{\partial}{\partial t}\Big|_{t=t_j} \tau(u + t(f_{i(j)} - u)).$$

Let

$$w := \lim_{j\to\infty} \frac{f_{i(j)} - u}{\max_{x\in S^2} |f_{i(j)}(x) - u(x)|}.$$

By proposition 2.1.20, we know that w is an nontrivial Jacobi field. i.e., $\mathcal{J}_u(w) = 0$. Assume that $f_{i(j)}(S^2) \neq u(S^2)$ for all $j \in \mathbb{N}$. Since $f_i(S^2) \neq u(S^2)$ implies that w is not purely tangential, Theorem 5.1.12 implies that u has a nontrivial Jacobi field for area functional as a smooth immersed minimal surface. This contradicts the bumpy metric assumption.

Now we consider the case that $f_{i(j)}(S^2) = u(S^2)$ for all $j \in \mathbb{N}$. If u is a prime harmonic sphere, then Theorem 5.1.11 implies that w is generated by $PSL(2, \mathbb{C})$, contradicting the assumption $[f_{i(j)}] \neq [u]$. Now we consider the case that u is not prime.

Claim 5.2.2: For j sufficiently large, we have

$$deg(f_{i(j)}) = deg(u).$$

Proof of the claim. It's known that if two maps are homotopic then they have the same degree. The stradegy of the proof is similar and can be found in [Hir94, Chapter 5].

For j sufficiently large, by equation 5.1 we know that $f_{i(j)}$ is homotopic to u. That is, there exists a continuous map $H_j : S^2 \times [0, 1] \to u(S^2)$:

$$H_j(x, t) := \gamma_x^j(t),\ x \in S^2,$$

such that $H_j(x, 0) = u(x)$ and $H_j(x, 1) = f_{i(j)}(x)$ for all $x \in S^2$, where $\gamma_x^j(t)$ denotes the unique geodesic with respect to the intrinsic metric starting from $u(x)$ with end point at $f_{i(j)}(x)$. Since $f_{i(j)}(S^2) = u(S^2)$, and by Lemma 5.1.10 we can choose $y \in u(S^2)$ so it satisfies the following conditions:

1. $\forall p \in u^{-1}(y)$, $du(p) \neq 0$,
2. $\forall p \in f_{i(j)}^{-1}(y)$, $df_i(p) \neq 0$,
3. y is a regular value for H_j and $H_j|_{\partial(S^2\times[0,1])}$.

$H_j^{-1}(y)$ is a compact 1-dimensional submanifold with boundary $(H_j|_{\partial(S^2\times[0,1])})^{-1}(y)$. In other words, it contains embedded arcs which are transverse to $\partial(S^2 \times [0, 1])$. By [Hir94, Chapter 5], given $p_1 \in H_j^{-1}(y)$, there is an unique $p_2 \in H_j^{-1}(y)$, $p_1 \neq p_2$, and

a component arc $\Gamma \in H_j^{-1}(y)$ with $\partial\Gamma = \{p_1, p_2\}$. By [Hir94, Chapter 5, Lemma 1.2], p_1 and p_2 are of opposite type for $H_j|_{\partial(S^2\times[0,1])}$. This implies that for any $p_1 \in u^{-1}(y)$, there is an unique corresponding $p_2 \in f_{i(j)}^{-1}(y)$. Then it follows that degree of u and degree of $f_{i(j)}$ are the same for j sufficiently large. □

By the smooth convergence of $f_{i(j)} \to u$, and $deg(f_{i(j)}) = deg(u)$. Let $u = \phi \circ \tilde{u}$ for some prime harmonic map $\tilde{u}$. We can write f_{i_j} as a branched cover of a prime harmonic map h_{i_j},i.e., $f_{i_j} = g_{i_j} \circ h_{i_j}$, with $deg(f_{i(j)}) = deg(g_{i_j}) = deg(u)$. By lemma 5.1.9 and lemma 5.1.10 we know that h_{i_j} is somewhere injective and the injective points of h_{i_j} form an open dense subset of S^2. So we obtain a sequence of prime harmonic spheres h_{i_j} that converges strongly to a prime harmonic sphere $\tilde{u}$ in $W^{1,2}$. It is the desired contradiction. □

Corollary 5.2.3: The $\epsilon(u) > 0$ given as lemma 5.2.1 is invariant under the equivalence relation. i.e., given harmonic spheres $f, g : S^2 \to M$, if $[f] = [g]$, then $\epsilon(f) = \epsilon(g)$.

Proof. We argue by contradiction. Given harmonic spheres $f, g : S^2 \to M$ with $[f] = [g]$, and assume $\epsilon(f) > \epsilon(g)$. Then there exists a harmonic map $u : S^2 \to M$ with

$$\epsilon(g) < \int_{S^2} |\nabla u - \nabla g|^2 < \epsilon(f),$$

such that $[u] \neq [g]$. Since $[g] = [f]$, there exists $\phi \in PSL(2, \mathbb{C})$ such that $f \circ \phi = g$. This implies

$$\begin{aligned}\epsilon(g) < \int_{S^2} |\nabla u - \nabla g|^2 &= \int_{S^2} |\nabla u - \nabla(f \circ \phi)|^2 \\ &= \int_{S^2} |\nabla(u \circ \phi^{-1}) - \nabla f|^2 < \epsilon(f),\end{aligned}$$

thus we have $[u] = [u \circ \phi^{-1}] = [f] = [g]$, which is the desired contradiction.

□

Definition 5.2.4: We define $\mathcal{F}^W$ to be:

$$\mathcal{F}^W := \Big\{ [f] \mid f : S^2 \to M \text{ harmonic }, E(f) \leq W \Big\}.$$

Corollary 5.2.5: Given a closed manifold (M, g), with generic g and the dimension of M is at least three. The set $\mathcal{F}^W$ is countable.

Proof. We pick a finite set $\{x_n^1, ..., x_n^{p^n}\}$ of S^2 so that $S^2 \subset \bigcup_{k=1}^{p^n} B_{1/n}(x_n^k)$. ($B_{1/n}(x_n^k)$ is the geodesic ball centered at the point x_n^k.) We define $\mathcal{F}^W(n)$ to be:

$$\mathcal{F}^W(n) = \Big\{ [f] \mid f \in \mathcal{F}^W, \int_{B_{1/n}(x_n^k)} |\nabla f|^2 < \epsilon_{su}, \text{ for } k = 1, ..., p^n \Big\},$$

where $\epsilon_{su} > 0$ is the constant given in Theorem 2.1.43. We can see that:

$$\mathcal{F}^W = \bigcup_{n \in \mathbb{N}} \mathcal{F}^W(n).$$

Now we prove that $\mathcal{F}^W(n)$ is finite. If not, then there exists a sequence $\{[f_i]\}_{i \in \mathbb{N}} \subset \mathcal{F}^W(n)$, and $[f_i] \neq [f_j]$ if $i \neq j$. By Theorem 3.2.1, f_i bubble converges to a harmonic map $f \in \mathcal{F}^W$ up to subsequence. Because of the assumption:

$$\int_{B_{1/n}(x_n^k)} |\nabla f_i|^2 < \epsilon_{su} \quad \forall i \in \mathbb{N}, k = 1, ..., p^n.$$

By Theorem 2.1.43, this implies that the convergence is strong in $W^{1,2}$ and $f \in \mathcal{F}^W(n)$. Then it contradicts Lemma 1.3.9. □

Chapter 6

Main Result

6.1 Unstable Lemma

The main focus of the section is lemma 6.1.1: proving the energy is concave for maps that are sufficiently close to a finite collection of harmonic spheres in bubble tree sense. We first consider the simplest example, for a given map $u \in W^{1,2}(S^2, M)$, and $X \in C^\infty(S^2, \mathbb{R}^N)$ with

$$\delta^2 E(u)(X, X) < 0.$$

By the form of second variation of energy (see (2.33)), clearly if $\epsilon > 0$ is sufficiently small, then for any $\gamma \in W^{1,2}(S^2, M)$ with $\|\gamma - u\|_{W^{1,2}} < \epsilon$, we have

$$\delta^2 E(\gamma)(X, X) < 0.$$

Now we consider the general case, given a finite collection of harmonic spheres $\{u_i\}_{i=0}^n$ with $\sum_{i=0}^n \text{Index}(u_i) = k > 0$. Index assumption implies there are k correspongding vector fields $\{X_l\}_{l=1}^k$, $X_l \in C^\infty(S^2, \mathbb{R}^N)$, positive constant $c_l > 0$ for each l, and the corresponding harmonic spheres $v_l \in \{u_i\}_{i=0}^n$, such that

$$\delta^2 E(v_l)(X_l, X_l) = -c_l < 0.$$

Now the goal is to choose $\epsilon > 0$ so that for any $\gamma \in W^{1,2}(S^2, M)$ with $d_B(\gamma, \{u_i\}_{i=0}^n) < \epsilon$, we can construct $\{\tilde{X}_l\}_{l=1}^k$, $\tilde{X}_l \in C^\infty(S^2, \mathbb{R}^N)$, then the variation $\gamma_s = \Pi \circ \big(\gamma + s\tilde{X}_l\big)$ satisfies

$$\frac{d^2}{ds^2}\Big|_{s=0} \int_{S^2} |\nabla \gamma_s|^2 < \frac{1}{2}\delta^2 E(v_l)(X_l, X_l),$$

for each l.

The proof of lemma 6.1.1 is long and detailed but the idea behind it is simple. It can be roughly spoken as the following: if $d_B(\gamma, \{u_i\}_{i=0}^n) < \epsilon$ for some $\gamma \in W^{1,2}(S^2, M)$, since $v_l \in \{u_i\}_{i=0}^n$, there exist $\Omega_l \subset S^2$ and $D_l \in PSL(2, \mathbb{C})$, so that

$$\int_{\Omega_l} |\nabla\gamma - \nabla(v_l \circ D_l)|^2 < \epsilon^2,$$

and

$$\int_{S^2 \setminus \Omega_l} |\nabla(v_l \circ D_l)|^2 < \epsilon^2, \tag{6.1}$$

see definition (3.1.5). If ϵ is sufficiently small the following term is small

$$\frac{d^2}{ds^2}\Big|_{s=0} \int_{\Omega_l} |\nabla\Pi \circ (v_l \circ D_l + sX_l \circ D_l)|^2 - \frac{d^2}{ds^2}\Big|_{s=0} \int_{\Omega_l} |\nabla\Pi \circ (\gamma + sX_l \circ D_l)|^2.$$

By choosing a suitable cutoff function η_l we can make the following term small

$$\frac{d^2}{ds^2}\Big|_{s=0} \int_{S^2} |\nabla\Pi \circ (\gamma + s\eta_l X_l \circ D_l)|^2 - \frac{d^2}{ds^2}\Big|_{s=0} \int_{\Omega_l} |\nabla\Pi \circ (\gamma + sX_l \circ D_l)|^2.$$

Let $\tilde{X}_l = \eta_l X_l \circ D_l$ and $\gamma_s = \Pi \circ \big(\gamma + s\tilde{X}_l\big)$, observe that

$$\begin{aligned} \delta^2 E(v_l)(X, X) =& \frac{d^2}{ds^2}\Big|_{s=0} \int_{\Omega_l} |\nabla\Pi \circ (v_l \circ D_l + sX \circ D_l)|^2 \\ &+ \frac{d^2}{ds^2}\Big|_{s=0} \int_{S^2 \setminus \Omega_l} |\nabla\Pi \circ (v_l \circ D_l + sX \circ D_l)|^2. \end{aligned} \tag{6.2}$$

The first term of the right hand side of (6.2) is close to $\frac{d^2}{ds^2}\Big|_{s=0} \int_{S^2} |\nabla\gamma_s|^2$, and the second term is small because of (6.1). We have the desired inequality

$$\frac{d^2}{ds^2}\Big|_{s=0} \int_{S^2} |\nabla\gamma_s|^2 < \frac{1}{2}\delta^2 E(v_l)(X_l, X_l).$$

We now state and prove the unstable lemma and specify how to choose $\epsilon > 0$ and η_l.

Lemma 6.1.1 (Unstable lemma): Let M be a closed manifold of dimension at least three, isometrically embedded in $\mathbb{R}^N$. Given a collection of finitely many harmonic spheres $\{u_i\}_{i=0}^n$ with $\sum_{i=0}^n Index(u_i) = k$. There exist $1 > c_0 > 0$ and $\epsilon > 0$, so that if $d_B(\gamma, \{u_i\}_{i=0}^n) < \epsilon$ for $\gamma \in W^{1,2}(S^2, M)$, then we can construct vector fields $\{\tilde{X}_l\}_{l=1}^k$, $\tilde{X}_l \in C^\infty(S^2, \mathbb{R}^N)$ for each l, define the variation γ_s of γ as

$$\gamma_s = \Pi \circ \big(\gamma + \sum_{l=1}^k s_l \tilde{X}_l\big), \quad s = (s_1, ..., s_k) \in \bar{B}^k,$$

and let $E_\gamma(s) := E(\gamma_s)$, so that the following hold:

1. $E_\gamma(s)$ has a unique maximum at $m_\gamma \in B^k_{\frac{c_0}{\sqrt{10}}}(0)$.

2. The map $\gamma \mapsto m_\gamma$ is continuous.

3. $\forall s \in \bar{B}^k$ we have
$$-\frac{1}{c_0} Id \leq D^2 E_\gamma(s) \leq -c_0 Id, \tag{6.3}$$
and
$$E_\gamma(m_\gamma) - \frac{1}{2c_0}|m_\gamma - s|^2 \leq E_\gamma(s) \leq E(m_\gamma) - \frac{c_0}{2}|m_\gamma - s|^2. \tag{6.4}$$

Proof. Index assumption implies there are k correspongding vector fields $\{X_l\}_{l=1}^k$, $X_l \in C^\infty(S^2, \mathbb{R}^N)$, positive constants $c_l > 0$ for each $l = 1, ..., k$, and the corresponding harmonic spheres $v_l \in \{u_i\}_{i=0}^n$, such that
$$\delta^2 E(v_l)(X_l, X_l) = -c_l < 0. \tag{6.5}$$
Let $\xi := c_l/C > 0$, C is a constant which will be chosen later. By (2.33), there exists $\delta(\xi) > 0$ depending on $\{M, X_l\}$ such that
$$|\delta^2 E(v_l)(X_l, X_l) - \delta^2 E(\gamma)(X_l, X_l)| < \xi, \tag{6.6}$$
for all γ with $\int_{S^2} |\nabla v_l - \nabla \gamma|^2 < \delta(\xi)$. We define $v_{l,s} := \Pi \circ (v_l + sX_l)$. There exists $\rho > 0$ such that for all $p \in S^2$ and $\varrho < \rho$ we have:
$$-\xi < \frac{d^2}{ds^2}\Big|_{s=0} \int_{B_\varrho(p)} |\nabla v_{l,s}|^2 < \xi, \quad \text{and} \int_{B_\varrho(p)} |\nabla X_l|^2 < \xi. \tag{6.7}$$
We choose $J \in \mathbb{N}$ so that
$$-\frac{1}{\log 1/J} < \xi, \tag{6.8}$$
and define ε_J to be $\min\left\{ \int_{B_{1/J}(p)} |\nabla v_l|^2 \,\middle|\, p \in S^2 \right\}$, $i \in \mathbb{N}$, note that ε_J is strictly positive. We now choose $\epsilon > 0$ to be the constant satisfying the following inequality
$$\max\{\epsilon^2, (3\epsilon^2 + 2(n+1)\epsilon E)\} < \min\{\varepsilon_J/2, \xi, \delta(\xi)\},$$
and consider $\gamma \in W^{1,2}(S^2, M)$ with $d_B(\gamma, \{u_i\}_{i=0}^n) < \epsilon$. Since $v_l \in \{u_i\}_{i=0}^n$, there is $\Omega_l \in S^2$ and a conformal dilation $D_l : S^2 \to S^2$ such that
$$\int_{\Omega_l} |\nabla \gamma - \nabla(v_l \circ D_l)|^2 < \epsilon^2, \tag{6.9}$$
and
$$\int_{S^2 \setminus \Omega_l} |\nabla(v_l \circ D_l)|^2 < \epsilon^2. \tag{6.10}$$

Moreover, we can choose $\tilde{\Omega}_l$ with $\Omega_l \subset \tilde{\Omega}_l$ so that

$$\int_{\tilde{\Omega}_l \setminus \Omega_l} |\nabla \gamma|^2 < 3\epsilon^2 + 2(n+1)\epsilon E,$$

here $E = \sum_{i=0}^{n} E(u_i)$. Assume that $S^2 \setminus (D_l \circ \tilde{\Omega}_l)$ and $S^2 \setminus (D_l \circ \Omega_l)$ are geodesic balls which center at some point $p \in S^2$, namely, $S^2 \setminus (D_l \circ \Omega_l) = B_r(p)$ and $S^2 \setminus (D_l \circ \tilde{\Omega}_l) = B_{r^k}(p)$ for some $1 < k \leq 2$. By equation 6.10 we know that

$$\int_{B_r(p)} |\nabla v_l|^2 < \varepsilon_J/2,$$

and it implies that r must be smaller than $1/J$.

Now we define the following piecewise smooth cutoff function, which was introduced by Choi and Schoen [CS85], $\eta : [0, \infty) \to [0, 1]$:

$$\eta(x) = \begin{cases} 0, & \text{for} \quad x < r^k, \\ (k+1) - (\log x)/(\log r), & \text{for} \quad r^k \leq x \leq r, \\ 1, & \text{for} \quad x > r, \end{cases}$$

so that

$$\frac{d\eta}{dx}(x) = \begin{cases} 0, & \text{for} \quad x < r^k, \\ -1/x(\log r), & \text{for} \quad r^k \leq x \leq r, \\ 0, & \text{for} \quad x > r, \end{cases}$$

and

$$\int_0^{2\pi} \int_{r^k}^{r} \Big(\frac{d\eta}{dx}(x)\Big)^2 x dx d\theta = -\frac{2\pi(k-1)}{\log r}.$$

Since we have $r < 1/J$, (6.8) implies that

$$-\frac{2\pi(k-1)}{\log r} < 2\pi(k-1)\xi.$$

Now we define $\eta_l = \eta \circ y_l : S^2 \to [0, 1]$, where $y_l : S^2 \to [0, \infty)$ and $y_l(q) = x$ for $q \in \partial B_x(p)$. η_l is compactly supported in $S^2 \setminus B_{r^k}(p)$ and has value 1 in $S^2 \setminus B_r(p)$, then

$$\int_{S^2} |\nabla \eta_l|^2 < 2\pi(k-1)\xi. \tag{6.11}$$

Let $\tilde{X}_l' := (\eta_l \circ D_l) X_l \circ D_l$, and define $v_{l,s} = \Pi \circ (v_l + sX_l)$, $\gamma_s := \Pi \circ (\gamma + s\tilde{X}_l')$.

Then we have:

$$\begin{aligned}\Big|\delta^2 E(\gamma)(\tilde{X}_l', \tilde{X}_l') - \delta^2 E(v_l)(X_l, X_l)\Big| &= \Big|\frac{d^2}{ds^2}\Big|_{s=0}\int_{S^2}|\nabla\gamma_s|^2 - \frac{d^2}{ds^2}\Big|_{s=0}\int_{S^2}|\nabla v_{l,s}|^2\Big| \\ &< \Big|\frac{d^2}{ds^2}\Big|_{s=0}\int_{\Omega_l}|\nabla\gamma_s|^2 - \frac{d^2}{ds^2}\Big|_{s=0}\int_{D_l\circ\Omega_l}|\nabla v_{l,s}|^2\Big| \\ &+ \Big|\frac{d^2}{ds^2}\Big|_{s=0}\int_{S^2\setminus(D_l\circ\Omega_l)}|\nabla v_{l,s}|^2\Big| \\ &+ \Big|\frac{d^2}{ds^2}\Big|_{s=0}\int_{S^2\setminus\Omega_l}|\nabla\gamma_s|^2\Big|.\end{aligned} \tag{6.12}$$

Since $\int_{\Omega_l}|\nabla v_l - \nabla\gamma|^2 < \epsilon^2 < \delta(\xi)$, (6.6) implies that

$$\Big|\frac{d^2}{ds^2}\Big|_{s=0}\Big(\int_{\Omega_l}|\nabla\gamma_s|^2 - |\nabla(v_{l,s}\circ D_l)|^2\Big)\Big| < \xi, \tag{6.13}$$

Since $S^2\setminus(D_l\circ\Omega_l) = B_r(p)$ by the assumption, the choice of ϵ implies that $r < \rho$ and equation 6.7 implies that

$$\Big|\frac{d^2}{ds^2}\Big|_{s=0}\int_{S^2\setminus(D_l\circ\Omega_l)}|\nabla v_{l,s}|^2\Big| < \xi. \tag{6.14}$$

Now we consider the last term of equation 6.12, namely:

$$\Big|\frac{d^2}{ds^2}\Big|_{s=0}\int_{S^2\setminus\Omega_l}|\nabla\gamma_s|^2\Big|,$$

by equation 2.33 we have:

$$\begin{aligned}\Big|\frac{d^2}{ds^2}\Big|_{s=0}\int_{S^2\setminus\Omega_l}|\nabla\gamma_s|^2\Big| &\le \int_{S^2\setminus\Omega_l}|d\Pi_\gamma(\nabla\tilde{X}_l')|^2 + C_1\int_{S^2\setminus\Omega_l}\big(|\nabla\tilde{X}_l'||\nabla\gamma| + |\tilde{X}_l'|^2|\nabla\gamma|^2\big) \\ &\le \int_{\tilde{\Omega}_l\setminus\Omega_l}|\nabla\tilde{X}_l'|^2 \\ &\quad + C_1\Big(\int_{\tilde{\Omega}_l\setminus\Omega_l}|\nabla\tilde{X}_l'|^2\Big)^{1/2}\Big(\int_{\tilde{\Omega}_l\setminus\Omega_l}|\nabla\gamma|^2\Big)^{1/2} \\ &\quad + C_1\sup_{p\in S^2}|X_l(p)|^2\int_{\tilde{\Omega}_l\setminus\Omega_l}|\nabla\gamma|^2,\end{aligned} \tag{6.15}$$

here C_1 is a constant which depends on M (since $\mathrm{Hess}\Pi_\gamma(\cdot,\cdot)$ is bounded by the second fundamental form of M [Sim96, Appendix 2.12] and $\nabla\mathrm{Hess}\Pi$ is bounded by

curvature of M). Then

$$\begin{aligned}\int_{\tilde{\Omega}_l\setminus\Omega_l}|\nabla\tilde{X}_l'|^2 =& \int_{B_r(p)\setminus B_{r^k}(p)}|\nabla(\eta_l X_l)|^2\\ =& \int_{B_r(p)\setminus B_{r^k}(p)}|(\nabla\eta_l)X_l+\eta_l\nabla X_l|^2\\ \leq& \int_{B_r(p)\setminus B_{r^k}(p)}|\nabla X_l|^2\\ &+2\sup_{p\in S^2}|X_l(p)|^2\Big(\int_{B_r(p)\setminus B_{r^k}(p)}|\nabla X_l|^2\Big)^{1/2}\Big(\int_{B_r(p)\setminus B_{r^k}(p)}|\nabla\eta_l|^2\Big)^{1/2}\\ &+\sup_{p\in S^2}|X_l(p)|^2\int_{B_r(p)\setminus B_{r^k}(p)}|\nabla\eta_l|^2\\ <& C_2\xi,\end{aligned} \tag{6.16}$$

for some constant $C_2(M,X_l)$, the last inequality follows from equation (6.11) and (6.7). By (6.16) we can bound (6.15) by:

$$\Big|\frac{d^2}{ds^2}\Big|_{s=0}\int_{S^2\setminus\Omega_l}|\nabla\gamma_s|^2\Big| < C_2\xi + C_1\sqrt{C_2}\xi + C_1\sup_{p\in S^2}|X_l(p)|^2\xi < C_3\xi. \tag{6.17}$$

Finally, combining the inequality (6.13), (6.14), and (6.17) we have that (6.12) is bounded by

$$\Big|\delta^2E(\gamma)(\tilde{X}_l',\tilde{X}_l') - \delta^2E(v_l)(X_l,X_l)\Big| < \xi+\xi+C_3\xi. \tag{6.18}$$

Since $\xi = c_l/C$, we now pick C to be a constant strictly larger than $5(C_3+2)$, then we have

$$-\frac{6}{5}c_l < \frac{d^2}{ds^2}\Big|_{s=0}E_\gamma(s) < -\frac{4}{5}c_l,$$

since $\frac{d^2}{ds^2}\Big|_{s=t}E_\gamma(s)$ is continuous with respect to t, there exists $\kappa_l(M,X_l,u_l,\epsilon)>0$ such that

$$-\frac{6}{5}c_l < \frac{d^2}{ds^2}\Big|_{s=t}E_\gamma(s) < -\frac{4}{5}c_l, \quad \text{for all } t\in[-\kappa_l,\kappa_l].$$

We can choose a constant $a(\kappa_l)>0$, let $\tilde{X}_l := a(\kappa_l)(\eta_l\circ D_l)X_l\circ D_l$, and redefine the variation of γ to be $\gamma_s := \Pi\circ(\gamma+s\tilde{X}_l)$ so that

$$-a^2(\kappa_l)\frac{3}{2}c_l < \frac{d^2}{ds^2}\Big|_{s=t}E_\gamma(s) < -a^2(\kappa_l)\frac{1}{2}c_l, \quad \text{for all } t\in[-1,1].$$

We finish the proof by choosing a constant $0<c_0<1$ such that $c_0 < \min_{l=1,..,k}a^2(\kappa_l)\frac{1}{2}c_l$ and $1/c_0 > \min_{l=1,\dots,k}a^2(\kappa_l)\frac{3}{2}c_l$.

□

Corollary 6.1.2: Let M be a closed manifold of dimension at least three, isometrically embedded in $\mathbb{R}^N$. Given a collection of finitely many harmonic spheres $\{u_i\}_{i=0}^n$ with $\sum_{i=0}^n Index(u_i) = k$. Let $1 > c_0 > 0$ and $\epsilon > 0$ be given as lemma 6.1.1. For a map $\gamma \in W^{1,2}(S^2, M)$ with $d_B(\gamma, \{u_i\}_{i=0}^n) < \epsilon$, let $\{\tilde{X}_l\}_{l=1}^k$ be vector fields given as lemma 6.1.1. For $\sigma \in W^{1,2}(S^2, M)$, we define

$$\sigma_s := \Pi \circ (\sigma + \sum_{l=1}^{k} s_l \tilde{X}_l) \quad \text{for } s = (s_1, ..., s_k) \in \bar{B}^k,$$

and let $E_\sigma(s) := E(\sigma_s)$, m_σ be the maximum of $E_\sigma(s)$. There exists $\delta_\gamma > 0$, such that for $\sigma \in W^{1,2}(S^2, M)$ satisfying

$$\int_{S^2} |\nabla\gamma - \nabla\sigma|^2 < \delta_\gamma,$$

the following properties hold

1. $E_\sigma(s)$ has a unique maximum at $m_\sigma \in B^k_{\frac{c_0}{\sqrt{10}}}(0)$.
2. The map $\sigma \mapsto m_\sigma$ is continuous.
3. $\forall s \in \bar{B}^k$ we have
$$-\frac{1}{c_0} Id \leq D^2 E_\sigma(s) \leq -c_0 Id, \tag{6.19}$$
and
$$E_\sigma(m_\sigma) - \frac{1}{2c_0}|m_\sigma - s|^2 \leq E_\sigma(s) \leq E(m_\sigma) - \frac{c_0}{2}|m_\sigma - s|^2. \tag{6.20}$$

Proof. By lemma 6.1.1, $\forall s \in \bar{B}^k$, we have

$$-\frac{1}{c_0} Id \leq D^2 E_\gamma(s) \leq -c_0 Id. \tag{6.21}$$

For each $s \in \bar{B}^k$, there exists $\delta(s) > 0$ so that for all $\sigma \in W^{1,2}(S^2, M)$ with

$$\int_{S^2} |\nabla\sigma - \nabla\gamma|^2 < \delta(s),$$

we have

$$-\frac{1}{c_0} Id \leq D^2 E_\sigma(s) \leq -c_0 Id. \tag{6.22}$$

Let $\delta_\gamma := \min_{s \in \bar{B}^k} \delta(s)$

Claim 6.1.3: $\delta_\gamma > 0$.

Proof of the claim. If not, there exists a sequence $\{s_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty}\delta(t_i) = 0$. Since $\bar{B}^k$ is compact, we have that $\lim_{i\to\infty} s_i = s' \in \bar{B}^k$, and $\delta(s') > 0$ implies the desired contradiction. □

Thus $\forall\sigma \in W^{1,2}(S^2, M)$ with

$$\int_{S^2} |\nabla\sigma - \nabla\gamma|^2 < \delta_\gamma,$$

we have

$$-\frac{1}{c_0} Id \leq D^2 E_\sigma(s) \leq -c_0 Id, \quad \forall s \in \bar{B}^k. \tag{6.23}$$

□

6.2 Deformation Theorem

Let M be a closed manifold with dimension at least three, isometrically embedded in $\mathbb{R}^N$. Consider a map $\beta \in \Omega$ representing a nontrivial class in $\pi_3(M)$, let W be the width associated to the homotopy class Ω_β (see definition 4.2.3, (4.4)), and given a sequence of sweepouts $\gamma_j(\cdot, t) \in \Omega_\beta$ which is minimizing, i.e.,

$$\lim_{j\to\infty} \max_{t\in[0,1]} E(\gamma_j(\cdot, t)) = W.$$

Moreover, let $K = \left\{ \{[k_i^1]\}_{i=0}^{m_1}, ..., \{[k_i^{N_k}]\}_{i=0}^{m_{N_k}} \right\}$ be a finite set of finite collection of equivalent classes of harmonic spheres, so there exist a constant $\epsilon_k > 0$ and $j_k \in \mathbb{N}$ such that

$$d_B(\gamma_j(\cdot, t), \{k_i^l\}_{i=0}^{m_l}) > \epsilon_k, \quad \forall t \in [0, 1],$$

for all $j > j_k$, $l = 1, ..., N_k$.

Theorem 6.2.1 (Deformation Theorem): As assumed above, given a collection of finitely many harmonic spheres $\{u_i\}_{i=0}^n$ with $\sum_{i=0}^n \text{Index}(u_i) = k > 1$ and $\sum_{i=0}^n E(u_i) = W$. There exists a sequence of sweepouts $\{\gamma_j'(\cdot, t)\}_{j\in\mathbb{N}}$ such that

1. $\gamma_j'(\cdot, t)$ is homotopic to $\gamma_j(\cdot, t)$,
2. $\{\gamma_j'(\cdot, t)\}_{j\in\mathbb{N}}$ is a minimizing sequence,

3. there exists $j'_k \in \mathbb{N}$ such that

$$d_B(\gamma'_j(\cdot,t), \{k_i^l\}_{i=0}^{m_l}) > \epsilon_k \quad \text{for } l = 1, ..., N_k, \forall t \in [0,1],$$

for all $j > j'_k$.

4. there exists $\epsilon_J > 0$ and $J \in \mathbb{N}$ such that

$$d_B(\gamma'_j(\cdot,t), \{u_i\}_{i=0}^{n}) > \epsilon_J, \quad \forall t \in [0,1],$$

for all $j > J$.

Proof. Assumption $\sum_{i=0}^{n} \text{Index}(u_i) = k$ implies there are k correspongding vector fields $\{X'_l\}_{l=1}^{k}$, $X'_l \in C^\infty(S^2, \mathbb{R}^N)$, such that $\delta^2 E(v_l)(X'_l, X'_l) < 0$, for $v_l \in \{u_i\}_{i=0}^{n}$. Let $\epsilon > 0$ be given as lemma 6.1.1. By proposition 3.3.5, there exists $\delta > 0$ such that $d_B(\gamma, \{u_i\}_{i=0}^{n}) < \epsilon$, if

$$\inf\Big\{ \int_{S^2} |\nabla\gamma - \nabla\big(\sum_{i=0}^{n} u_i \circ \phi_i\big)|^2 \Big| \phi_i \in PSL(2,\mathbb{C})\Big\} < \delta,$$

and

$$d_{V_R}(\gamma, \{u_i\}_{i=0}^{n}) < \delta,$$

for $\gamma \in W^{1,2}(S^2, M)$. We consider the following sets:

$$I_{j,\delta/2} := \Big\{ t \in [0,1] \Big| \inf\big\{ \int_{S^2} |\nabla\gamma_j(\cdot,t) - \sum_{i=0}^{n} \nabla(u_i \circ D_i)|^2 \Big| D_i \in PSL(2,\mathbb{C})\big\} \leq \delta/2 \Big\},$$

and

$$I'_{j,\delta/2} := \Big\{ t \in [0,1] \Big| d_{V_R}(\gamma_j(\cdot,t), \{u_i\}_{i=0}^{n}) \leq \delta/2 \Big\}.$$

Let $U_{j,\delta/2} := I_{j,\delta} \cap I'_{j,\delta/2}$. Proposition 3.3.5 implies that $d_B(\gamma_j(\cdot,t), \{u_i\}_{i=0}^{n}) < \epsilon$, for all $t \in U_{j,\delta/2}$. We define

$$E_j^t(s, \{Y_l\}_{l=1}^{k}) := \int_{S^2} |\nabla(\Pi \circ (\gamma_j(\cdot,t) + \sum_{l=1}^{k} s_l Y_l))|^2, \quad s = (s_1, ..., s_k) \in \bar{B}^k.$$

For $t_m \in U_{j,\delta/2}$, by lemma 6.1.1, we can construct vector fields $\{\tilde{X}_l(t_m)\}_{l=1}^{k}$, and the hessian of $E_j^{t_m}(s, \{\tilde{X}_l(t_m)\}_{l=1}^{k})$ with respect to $s \in \bar{B}^k$, which we denote by $D_s^2 E_j^{t_m}(s, \{\tilde{X}_l(t_m)\}$, satisfies

$$-\frac{1}{c_0} Id \leq D_s^2 E_j^{t_m}(s, \{\tilde{X}_l(t_m)\}_{l=1}^{k}) \leq -c_0 Id, \ \forall s \in \bar{B}^k \tag{6.24}$$

here c_0 is a constant given by lemma 6.1.1. By corollary 6.1.2 and the continuity of $\gamma_j(\cdot,t)$ in $W^{1,2}(S^2,M)$ with respect to t, we know there exists $\delta(t_m)>0$ so that for all $t\in(t_m-\delta(t_m),t_m+\delta(t_m))\cap U_{j,\delta/2}$ we have

$$-\frac{1}{c_0}Id\leq D_s^2E_j^t(s,\{\tilde{X}_l(t_m)\}_{l=1}^k)\leq -c_0Id,\ \forall s\in\bar{B}^k. \tag{6.25}$$

Let $I^{t_m}:=(t_m-\delta(t_m),t_m+\delta(t_m))\cap U_{j,\delta/2}$, since $U_{j,\delta/2}$ is compact we can cover $U_{j,\delta/2}$ by finitely many I^t, say $I^{t_1},...,I^{t_{N_1}}$. Moreover, after discarding some of the intervals, we can arrange that each t is in at least one closed interval $\bar{I}^{t_m}$, each $\bar{I}^{t_m}$ intersects at most two other $\bar{I}^{t_k}$'s, and the $\bar{I}^{t_k}$'s intersecting $\bar{I}^{t_m}$ do not intersect each other. For each $m=1,...,N_1$, choose a smooth function $\xi_m(t):[0,1]\to[0,1]$ which is supported in $\bar{I}^{t_m}$, and

$$\sum_{m=1}^{N_1}\xi_m(t)=1,\ \forall t\in[0,1].$$

We define $X_l(t)$ to be

$$X_l(t):=\sum_{m=1}^{N_1}\xi_m(t)\tilde{X}_l(t_m),\quad t\in[0,1],l=1,...,k. \tag{6.26}$$

Consider $E_j^t(s,\{X_l(t)\}_{l=1}^k)$, if $X_l(t)=\tilde{X}_l(t_m)$ for some $t\in U_{j,\delta/2}$, then obviously we have $-\frac{1}{c_0}Id\leq D_s^2E_j^t(s,\{X_l(t)\}_{l=1}^k)\leq -c_0Id$, $\forall s\in\bar{B}^k$. If $X_l(t)=\delta_a(t)\tilde{X}_l(t_a)+\delta_b(t)\tilde{X}_l(t_b)$, since $\delta_a(t)+\delta_b(t)=1$, we have for all $s\in\bar{B}^k$

$$\begin{aligned}-\frac{2}{c_0}Id&<D_s^2E_j^t(s,\{X_l(t)\}_{l=1}^k)\\&=D_s^2E_j^t(s,\{\delta_a(t)\tilde{X}_l(t_a)+\delta_b(t)\tilde{X}_l(t_b)\}_{l=1}^k)\\&<\delta_a^2(t)D_s^2E_j^t(s,\{\tilde{X}_l(t_a)\}_{l=1}^k)+\delta_b^2(t)D_s^2E_j^t(s,\{\tilde{X}_l(t_b)\}_{l=1}^k)\\&\leq-\frac{c_0}{2}Id.\end{aligned} \tag{6.27}$$

The last inequality follows from $\delta_a^2(t)+\delta_b^2(t)\geq 1/2$ By (6.27), we can choose $c=c_0/2$ such that $-\frac{1}{c}Id<D_s^2E_j^t(s,\{X_l(t)\}_{l=1}^k)<-cId$, $\forall s\in\bar{B}^k$. Now we define

$$\gamma_{j,s}(\cdot,t):=\Pi\circ\Big(\gamma_j(\cdot,t)+\sum_{l=1}^k s_lX_l(t)\Big),\quad s=(s_1,..,.s_k)\in\bar{B}^k,$$

and let $E_j^t(s):=E(\gamma_{j,s}(\cdot,t))$, then we have

1. $E_j^t(s)$ has a unique maximum at $m_j(t)\in B^k_{\frac{c}{\sqrt{10}}}(0)$.

2. The map $\gamma_j(\cdot,t)\mapsto m_j(t)$ is continuous.

3. $\forall s\in\bar{B}^k$ and $\forall t\in U_{j,\delta/2}$ we have

$$-\frac{1}{c}Id\le D^2E_j^t(s)\le -cId, \tag{6.28}$$

and

$$E_j^t(m_j(t))-\frac{1}{2c}|m_j(t)-s|^2\le E_j^t(s)\le E_j^t(m_j(t))-\frac{c}{2}|m_j(t)-s|^2. \tag{6.29}$$

Recall $\forall\{[k_i^l]\}_{i=0}^{m_l}\in K$ we have: $d_B(\gamma_j(\cdot,t),\{k_i^l\}_{i=0}^{m_l})>\epsilon_l$ for $l=1,...,N_k,\forall t\in[0,1]$. for all $j>j_k$. Without loss of generality (by rescaling $\{X_l(t)\}_{l=1}^k$ and c), we can assume that there exists $j_k'\in\mathbb{N}$ such that

$$d_B(\gamma_{j,s}(\cdot,t),\{k_i^l\}_{i=0}^{m_l})>\epsilon_l\quad\text{for } l=1,...,N_k,\forall s\in\bar{B}^k,\forall t\in[0,1], \tag{6.30}$$

for all $j>j_k'$.

Claim 6.2.2: For t belongs to the closure of $U_{j,\delta/2}\setminus U_{j,\delta/3}$, there exists a constant $\varepsilon=\varepsilon(\delta,\{u_i\}_{i=0}^n,\{X_l\}_{l=1}^k)>0$, so that for all $j\in\mathbb{N}$, $s\in\bar{B}^k$ satisfies

$$E(\gamma_{j,s}(\cdot,t))\le E(\gamma_j(\cdot,t))+\varepsilon,$$

then

$$d_B(\gamma_{j,s}(\cdot,t),\{u_i\}_{i=0}^n)>2\varepsilon.$$

Proof. We argue by contradiction. Assume that there are sequences $s_i'\in\bar{B}^k$ and $t_i'\in I$ satisfying

$$E(\gamma_{j,s_i'}(\cdot,t_i'))\le E(\gamma_j(\cdot,t_i'))+1/i, \tag{6.31}$$

and

$$d_B(\gamma_{j,s_i'}(\cdot,t_i'),\{u_i\}_{i=0}^n)\le 2/i.$$

Let $s_i'\to s=(s_1,...,s_k)\in\bar{B}^k$ and $t_i'\to t$. Hence we have $d_B(\gamma_{j,s}(\cdot,t),\{u_i\}_{i=0}^n)=0$, which implies that there are pairwise disjoint domains $\{\Omega_i\}_{i=0}^n$, $\cup_{i=0}^n\Omega_i\subset S^2$, and conformal dilations $\{D_i\}_{i=0}^n$ such that $\sum_{i=0}^n\int_{S^2\setminus\Omega_i}|\nabla(u_i\circ D_i)|^2=0$. For nontrivial harmonic sphere $u_i:S^2\to M$, $\int_{S^2\setminus\Omega_i}|\nabla(u_i\circ D_i)|^2=0$ is only possible when $\Omega_i=S^2$. That implies $\{u_i\}_{i=0}^n$ only contains one harmonic sphere, say u_0. The index assumption $\sum_{i=0}^n\text{Index}(u_i)=k$ now becomes $\text{Index}(u_0)=k$, and $\delta^2E(u_0)(X_l',X_l')<0$ for $l=1,...,k$. Moreover, the $X_l(t)$ we constructed in (6.26) is simply X_l'. By $\gamma_{j,s}(\cdot,t)\in C^0(S^2,M)\cap W^{1,2}(S^2,M)$, we have

$$\begin{aligned}u_0(x)&=\gamma_{j,s}(x,t)\\&=\Pi\circ\big(\gamma_j(x,t)+\sum_{l=1}^k s_lX_l'(x)\big),\end{aligned}$$

the equality holds for all $x \in S^2$. Moreover, because Π is the nearest point projection defined on a tubular neighborhood of M in $\mathbb{R}^N$, which implies the following

$$\sum_{l=1}^{k} s_l X_l'(x) = u_0(x) - \gamma_j(x,t) + \nu(x), \quad \nu(x) \in T^{\perp}_{u_0(x)} M.$$

Let $Y(x) := \gamma_j(x,t) - u_0(x)$, then

$$d\Pi_{u_0}(\sum_{l=1}^{k} s_l \nabla X_l') = d\Pi_{u_0}(-\nabla Y).$$

So we have

$$\delta^2 E(u_0)(Y,Y) < 0, \tag{6.32}$$

if $s \neq 0$. (6.32) implies that

$$E(\gamma_j(\cdot,t)) = E(\Pi \circ (u_0 + Y)) \leq E(u_0), \tag{6.33}$$

with the equality holds if and only if $s = 0$. On the other hand, by assumption (6.31)

$$E(u_0) = E(\gamma_{j,s}(\cdot,t)) \leq E(\gamma_j(\cdot,t)),$$

which forces s to be 0. Thus $\gamma_j(\cdot,t) = u_0$, which contradicts that t belongs to the closure of $U_{j,\delta/2} \setminus U_{j,\delta/3}$. □

Now we consider the one-parameter flow

$$\{\phi_j^t(\cdot,x)\}_{x\geq 0} \in \mathrm{Diff}(\bar{B}^k)$$
$$\phi_j^t(\cdot,\cdot) : \bar{B}^k \times [0,\infty) \to \bar{B}^k,$$

generated by the vector field:

$$s \mapsto -(1-|s|^2)\nabla E_j^t(s),\ s \in \bar{B}^k. \tag{6.34}$$

Claim 6.2.3: For all $\kappa < \frac{1}{4}$, there is T_j depending on $\left\{\{u_i\}_{i=0}^n, \{X_l\}_{l=1}^k, \kappa, \epsilon\right\}$ so that for any $t \in U_{j,\delta/2}$, and $v \in \bar{B}^k$ with $|v - m_j(t)| \geq \kappa$ we have:

$$E_j^t(\phi_j^t(v,T_j)) < E_j^t(0) - \frac{c}{10}. \tag{6.35}$$

Proof. By $m_j(t) \in B^k_{\frac{c}{\sqrt{10}}}(0)$ and (6.29) we know that for $\gamma_j(\cdot,t)$, $t \in U_{j,\delta/2}$, we have:

$$\sup_{s\in\bar{B}^k} E_j^t(s) = E_j^t(m_j(t)) \leq E_j^t(0) + \frac{c}{20}. \tag{6.36}$$

So, to prove (6.35), it suffices to show the existence of T_j such that

$$|v - m_j(t)| \geq \kappa \implies E_j^t(\phi_j^t(v, T_j)) < \sup_{s \in \bar{B}^k} E_j^t(s) - \frac{c}{5}.$$

We argue by contradiction and assume that there exists a constant $\frac{1}{4} > \kappa > 0$, a sequence $\{t_l\}_{l\in\mathbb{N}} \subset U_{j,\delta/2}$, and $\{s_l\}_{l\in\mathbb{N}} \subset \bar{B}^k$ with $|s_l - m_j(t_l)| \geq \kappa$ such that

$$E_j^{t_l}(\phi_j^{t_l}(s_l, l)) \geq E_j^{t_l}(0) - \frac{c}{10}. \tag{6.37}$$

Combining (6.37) with (6.36) we have $E_j^{t_l}(\phi_j^{t_l}(s_l, l)) \geq E_j^{t_l}(m_j(t_l)) - \frac{c}{5}$. Since $\phi_j^t(\cdot, \cdot)$ is an energy decreasing flow, we have

$$E_j^{t_l}(\phi_j^{t_l}(s_l, x)) \geq E_j^{t_l}(\phi_j^{t_l}(s_l, l)) \geq E_j^{t_l}(m_j(t_l)) - \frac{c}{5}, \ \forall 0 \leq x \leq l.$$

Since both $U_{j,\delta/2}$ and $\bar{B}^k$ are compact, we obtain subsequential limits $t \in U_{j,\delta/2}$ and $s \in \bar{B}^k$ with

$$E_j^t(\phi_j^t(s, x)) \geq \sup_{|v|\leq 1} E_j^t(v) - \frac{c_0}{5}, \quad \forall x \geq 0. \tag{6.38}$$

Since $\gamma_j(\cdot, t) \mapsto m_j(t)$ is a continuous map, $|s_l - m_j(t_l)| \geq \kappa$ implies $|s - m_j(t)| \geq \kappa$. Thus we have $\lim_{x\to\infty} |\phi_j^t(s, x)| = 1$ and thus we deduce from the equation (6.38) that

$$\sup_{|v|=1} E_j^t(v) \geq \sup_{|v|\leq 1} E_j^t(v) - \frac{c_0}{5}. \tag{6.39}$$

On the other hand, $m_j(t) \in B^k_{\frac{c}{\sqrt{10}}}(0)$ implies $|v - m_j(t)| > 2/3$ for all $v \in \bar{B}^k$ with $|v| = 1$. Hence, by equation 6.29 we have

$$\sup_{|v|=1} E_j^t(v) \leq \sup_{|v|\leq 1} E_j^t(v) - \frac{c_0}{2}\Big(\frac{2}{3}\Big)^2 < \sup_{|v|\leq 1} E_j^t(v) - \frac{c_0}{5},$$

which gives us the desired contradiction. □

We define a continuous homotopy:

$$H_j' : U_{j,\delta/2} \times [0, 1] \longrightarrow B^k_{1/2^j}(0),$$

so that

$$H_j'(t, 0) = 0, \text{ and } \inf_{t\in U_{j,\delta/2}} |H_j'(t, 1) - m_j(t)| \geq \kappa_j > 0.$$

We are able to define H_j' due to the assumption $\sum_i \text{Index}(u_i) = k \geq 2$. So we can choose a continuous path in $B^k_{1/2^j}(0)$ away from the curve of $m_j(t)$, $t \in U_{j,\delta/2}$. By claim 6.2.3, there exists T_j for $t \in U_{j,\delta/2}$ such that:

$$E_j^t\big(\phi_j^t(H_j'(t, 1), T_j)\big) < E_j^t(0) - \frac{c_0}{10}.$$

Let $c_j : [0,1] \longrightarrow [0,1]$ be a cutoff function which is supported in $U_{j,\delta/2}$, and has value one in $U_{j,\delta/3}$, value zero in $[0,1] \setminus U_{j,\delta/2}$. Define:

$$H_j(t,x) = H'_j(t, c_j(t)x),$$

and

$$H_j(t,x) = 0 \quad \forall t \in [0,1] \setminus U_{j,\delta/2}.$$

We now set $s_j(t) = (s_j^1(t), ..., s_j^k(t)) \in \bar{B}^k$ to be

$$s_j(t) = \phi_j^t(H_j(t,1), c_j(t)T_j), \quad \text{if } t \in U_{j,\delta/2},$$

and

$$s_j(t) = 0, \quad \text{if } t \in [0,1] \setminus U_{j,\delta/2}.$$

We define $\gamma'_j(\cdot,t)$ to be:

$$\gamma'_j(\cdot,t) := \Pi \circ \Big(\gamma_j(\cdot,t) + \sum_{l=1}^{k} s_j^l(t) X_l(t)\Big).$$

Since s_j is homotopic to the zero map in $\bar{B}^k$, so $\gamma'_j(\cdot,t)$ is homotopic to $\gamma_j(\cdot,t)$.

Claim 6.2.4: $\{\gamma'_j(\cdot,t)\}_{j\in\mathbb{N}}$ is a minimizing sequence.

Proof. From the energy non-increasing property of $\{\phi_j^t(\cdot,x)\} \in \mathrm{Diff}(\bar{B}^k)$ we have that for all $t \in [0,1]$

$$E(\gamma'_j(\cdot,t)) = E_j^t(\phi_j^t(H_j(t,1), c_j(t)T_j)) \leq E_j^t(H_j(t,1)). \tag{6.40}$$

Moreover, from (2.32), we know that there exists a continuous function $\Psi : [0,\infty) \to [0,\infty)$ with $\Psi(0) = 0$ such that

$$\Big|E(\gamma_{j,s}(\cdot,t)) - E(\gamma_j(\cdot,t))\Big| \leq \Psi\Big(\|\sum_{l=1}^{k} s_l X_l\|_{W^{1,2}}\Big),$$

and $H_j(t,1) \in B^k_{1/2^j}$ implies that

$$\begin{aligned} E(\gamma'_j(\cdot,t)) &= E_j^t(\phi_j^t(H_j(t,1), c_j(t)T_j)) \\ &\leq E_j^t(H_j(t,1)) \\ &\leq E(\gamma_j(\cdot,t)) + \Psi\Big(\frac{1}{2^j}\sum_{l=1}^{k}\|X_l\|_{W^{1,2}}\Big). \end{aligned} \tag{6.41}$$

By (6.41) and that $\gamma'_j(\cdot,t)$ is homotopic to $\gamma_j(\cdot,t)$ we have that

$$W \leq \lim_{j\to\infty}\max_{t\in[0,1]} E(\gamma'_j(\cdot,t)) = \lim_{j\to\infty}\max_{t\in[0,1]} E(\gamma_j(\cdot,t)) = W,$$

which finishes the proof □

Claim 6.2.5: There exists $j'_k \in \mathbb{N}$ such that

$$d_B(\gamma'_j(\cdot,t), \{k_i^l\}_{i=0}^{m_l}) > \epsilon_l \quad \text{for } l = 1, ..., N_k, \forall t \in [0,1],$$

for all $j > j_k$.

Proof. The claim follows from the assumption 6.30. □

Claim 6.2.6: there exists $\epsilon_J > 0$ and $J \in \mathbb{N}$ such that

$$d_B(\gamma'_j(\cdot,t), \{u_i\}_{i=0}^{n}) > \epsilon_J, \quad \forall t \in [0,1],$$

for all $j > J$.

Proof. There are three cases to consider.

Case 1: $t \in [0,1] \setminus U_{j,\delta/2}$

$\gamma'_j(\cdot,t) = \gamma_j(\cdot,t)$ for all j, so there exists $\epsilon_1 > 0$ such that $d_B(\gamma'_j(\cdot,t), \{u_i\}_{i=0}^{n}) > \epsilon_1$ for all $j \in \mathbb{N}$.

Case 2: $t \in U_{j,\delta/3}$

By claim 6.2.3 we have

$$\begin{aligned} E(\gamma'_j(\cdot,t)) &= E_j^t(\phi_j^t(H'_j(t,1), T_j)) \\ &< E_j^t(0) - \frac{c}{10} \\ &= E(\gamma_j(\cdot,t)) - \frac{c}{10}, \quad \forall j \in \mathbb{N}, \end{aligned}$$

so

$$\lim_{j\to\infty} \max_{t\in U_{j,\delta/3}} E(\gamma'_j(\cdot,t)) < \lim_{j\to\infty} \max_{t\in[0,1]} E(\gamma_j(\cdot,t)) - \frac{c}{10} = W - \frac{c}{10}.$$

It implies that there exists $\epsilon_2 > 0$ so $d_B(\gamma'_j(\cdot,t), \{u_i\}_{i=0}^{n}) > \epsilon_2$, or else by lemma 3.3.3 $\lim_{j\to\infty} \max_{t\in U_{j,\delta/3}} E(\gamma'_j(\cdot,t)) = \sum_{i=0}^{n} E(u_i) = W$.

Case 3: $t \in U_{j,\delta/2} \setminus U_{j,\delta/3}$

By (6.41) we have

$$E(\gamma'_j(\cdot,t)) \le E(\gamma_j(\cdot,t)) + \Psi\Big(\frac{1}{2^j} \sum_{l=1}^{k} \|X_l\|_{W^{1,2}}\Big),$$

for a continuous function $\Psi : [0,\infty) \to [0,\infty)$ with $\Psi(0) = 0$. and claim 6.2.2 implies that there exists $\epsilon_3 > 0$ so $d_B(\gamma_j'(\cdot,t), \{u_i\}_{i=0}^n) > \epsilon_3$ for j sufficiently large.

Let $\epsilon_J = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$, so we have

$$d_B(\gamma_j(\cdot,t), \{u_i\}_{i=0}^n) > \epsilon_J, \ \forall t \in [0,1],$$

for j sufficiently large. □

We have proved theorem 6.2.1. □

Theorem 1.3.2

Let (M,g) be a closed Riemannian manifold of dimension at least three, g generic and a nontrivial homotopy group $\pi_3(M)$, let W be the width associated to the homotopy class Ω_β (see definition 4.2.3, (4.4)). Then there exists a collection of finitely many harmonic spheres $\{u_i\}_{i=0}^m$, $u_i : S^2 \to M$, which satisfies the following properties:

1. $\sum_{i=0}^m E(u_i) = W$,
2. $\sum_{i=0}^m Index(u_i) \leq 1$.

Proof. Denote by $\mathcal{U}$ the collections of equivalent classes of harmonic spheres $\{[u_i]\}_{i=0}^n$ with $\sum_{i=0}^n \text{Index}(u_i) > 1$ and $\sum_{i=0}^n E(u_i) = W$. By corollary 5.2.5, $\mathcal{U}$ is countable and thus we can write $\mathcal{U} = \left\{\{[u_i^1]\}_{i=0}^{n_1}, \{[u_i^2]\}_{i=0}^{n_2}, ...\right\}$ with $\sum_{i=0}^{n_l} E(u_i^l) = W$ and $\sum_{i=0}^{n_l} \text{Index}(u_i^l) > 1$ for each $l \in \mathbb{N}$.

Given a minimizing sequence $\{\gamma_j(\cdot,t)\}_{j\in\mathbb{N}}$, we consider the collection of harmonic spheres: $\{[u_i^1]\}_{i=0}^{n_1}$, and by Theorem 6.2.1 there exists $\{\gamma_j^1(\cdot,t)\}_{j\in\mathbb{N}}$ so that

1. $\gamma_j^1(\cdot,t)$ is homotopic to $\gamma_j(\cdot,t)$,
2. $\{\gamma_j^1(\cdot,t)\}_{j\in\mathbb{N}}$ is a minimizing sequence,

3. there exists $\epsilon_1 > 0$ and $i_1 \in \mathbb{N}$ such that

$$d_B(\gamma_j^1(\cdot,t), \{u_i^1\}_{i=0}^{n_1}) > \epsilon_1, \quad \forall t \in [0,1],$$

for all $j > i_1$.

We can apply Theorem 6.2.1 again at the minimizing sequence $\{\gamma_j^1(\cdot,t)\}_{j\in\mathbb{N}}$, with $\{[u_i^2]\}_{i=0}^{n_2}$ the given collection of harmonic spheres, and set the compact set of harmonic spheres K to be $K^1 := \left\{\{[u_i^1]_{i=0}^{n_1}\}\right\}$, and obtain $\{\gamma_j^2(\cdot,t)\}_{j\in\mathbb{N}}$ so that

1. $\gamma_j^2(\cdot,t)$ is homotopic to $\gamma_j(\cdot,t)$,
2. $\{\gamma_j^2(\cdot,t)\}_{j\in\mathbb{N}}$ is a minimizing sequence,
3. there exist $\epsilon_1, \epsilon_2 > 0$ and $i_1, i_2 \in \mathbb{N}$ such that

$$d_B(\gamma_j^1(\cdot,t), \{u_i^l\}_{i=0}^{n_l}) > \epsilon_l, \quad j > i_l, \ \forall t \in [0,1],$$

$l = 1, 2$.

Proceeding inductively we can find $\{\gamma_j^m\}_{j\in\mathbb{N}}$ such that

1. $\gamma_j^m(\cdot,t)$ is homotopic to $\gamma_j(\cdot,t)$,
2. $\{\gamma_j^m(\cdot,t)\}_{j\in\mathbb{N}}$ is a minimizing sequence,
3. there exist $\epsilon_l > 0$ and $i_l \in \mathbb{N}$, $l = 1, ..., m$, such that

$$d_B(\gamma_j^1(\cdot,t), \{u_i^l\}_{i=0}^{n_l}) > \epsilon_l, \quad j > i_l, \ \forall t \in [0,1].$$

We can choose an increasing sequence $p_m > i_m$ such that

$$\max_{t\in[0,1]} E(\gamma_{p_m}^m(\cdot,t)) \leq W + \frac{1}{m}.$$

The sequence $\{\gamma_{p_m}^m(\cdot,t)\}_{m\in\mathbb{N}}$ is a minimizing sequence, thus by Theorem 4.2.12, there exists a sequence $\{t_m\}_{m\in\mathbb{N}} \subset [0,1]$ and a collection of finitely many harmonic spheres $\{v_i\}_{i=0}^m$ with $\sum_{i=0}^m E(v_i) = W$, such that $\{\gamma_{p_m}^m(\cdot,t_m)\}_{m\in\mathbb{N}}$ bubble converges to up to subsequence. i.e.,

$$d_B(\gamma_{p_m}^m(\cdot,t_m), \{v_i\}_{i=0}^m) \to 0, \ m \to \infty.$$

Since $\gamma_{p_m}^m(\cdot,t)$ is away from $\mathcal{U}$ as $m \to \infty$ so we have $\sum_{i=0}^m Index(v_i) \leq 1$, this is what we wanted to prove. □

Proposition 1.3.5

Let (M, g) be a closed Riemannian manifold of dimension at least three, g generic and a nontrivial homotopy group $\pi_3(M)$. Let $\{\gamma_j(\cdot, t)\}_{j\in\mathbb{N}}$ be a sequence of sweepouts which induces a non-trivial class in $\pi_3(M)$. If for each $\{u_i\}_{i=0}^n \in \Lambda(\{\gamma_j(\cdot, t)\}_{j\in\mathbb{N}})$, $\{u_i\}_{i=0}^n$ contains only one harmonic sphere. Then there $\{\gamma_j(\cdot, t)\}_{j\in\mathbb{N}}$ has Morse index one.

Proof. We argue by contradiction, if not, there exists a harmonic sphere $u : S^2 \to M$ such that $u \in \Lambda(\{\gamma_j(\cdot, t)\}_{j\in\mathbb{N}})$, and is stable (see definition 2.1.22).

Let $\epsilon(u) > 0$ be given by Lemma 1.3.9, and let $\epsilon_1 > 0$ be the constant such that if

$$\int_{S^2} |\nabla f - \nabla u|^2 < \epsilon_1,$$

for a map $f \in W^{1,2}(S^2, M)$, then the nearest point projection map on u with respect to $X(x) := f(x) - u(x)$ is well-defined in the sense that

$$E(f) = E(\Pi \circ (u + X)).$$

We define ϵ to be the minimum of $\{\epsilon_1, \epsilon(u)/10\}$. By Theorem 4.2.9, there exists $\delta > 0$ such that if

$$\text{Area}(\gamma_j(\cdot, s)) > W - \delta,$$

then

$$d_B(\gamma_j(\cdot, s), \{u_i\}_{i=0}^n) < \epsilon/2,$$

for a collection of finitely many harmonic maps. By the assumption $\{u_i\}_{i=0}^n$ contains only one harmonic sphere, say $u_s : S^2 \to M$, then $d_B(\gamma_j(\cdot, s), u_s) < \epsilon/2$ implies that

$$\int_{S^2} |\nabla\gamma_j(x, s) - \nabla u_s(x)|^2 < \epsilon/2.$$

We consider the following set:

$$I_{j,\delta} := \big\{t \in [0, 1] \big| E(\gamma_j(\cdot, t)) \geq W - \delta\big\}.$$

Since $u \in \Lambda(\{\gamma_j(\cdot, t)\}_{j\in\mathbb{N}})$. There exists $t_j \in I_{j,\delta}$ such that

$$\int_{S^2} |\nabla\gamma_j(x, t_j) - \nabla u(x)|^2 < \epsilon.$$

Claim 6.2.7: Let $\bar{I}_{j,\delta}$ be the connected component of $I_{j,\delta}$ which contains t_j. For all $t \in \bar{I}_{j,\delta}$, there exists $\phi(t) \in PSL(2,\mathbb{C})$ such that

$$\int_{S^2} |\nabla \gamma_j(x,t) - \nabla(u \circ \phi(t))|^2 < \epsilon$$

Proof. Let $\delta(t) > 0$ be the constant such that

$$\int_{S^2} |\nabla \gamma_j(x,s) - \nabla \gamma_j(x,t)|^2 < \epsilon, \quad \forall s \in (t - \delta(t), t + \delta(t)),$$

We can choose this $\delta(t) > 0$ since $\gamma_j(\cdot, t)$ is continuous strongly in $W^{1,2}(S^2, M)$ with respect to t. We know that for $s \in (t_j - \delta(t_j), t_j + \delta(t_j))$ we have

$$\begin{aligned}
\Big(\int_{S^2} |\nabla u_s(x) - \nabla u(x)|^2 \Big)^{1/2} <& \Big(\int_{S^2} |\nabla u_s(x) - \nabla \gamma_j(x,s)|^2 \Big)^{1/2} \\
&+ \Big(\int_{S^2} |\nabla \gamma_j(x,s) - \nabla \gamma_j(x,t_j)|^2 \Big)^{1/2} \\
&+ \Big(\int_{S^2} |\nabla u(x) - \nabla \gamma_j(x,t_j)|^2 \Big)^{1/2} \\
<& 3\sqrt{\epsilon} < \sqrt{\epsilon(u)}
\end{aligned}$$

By Lemma 1.3.9 we know that $[u] = [u_s]$. Since $\bar{I}_{j,\delta}$ is compact, $\bigcup_{t \in \bar{I}_{j,\delta}} (t - \delta(t), t + \delta(t))$ forms a finite subcovering of $\bar{I}_{j,\delta}$. We can do the previous argument finitely many times and get $[u_s] = [u]$, $\forall s \in \bar{I}_{j,\delta}$.

□

Let s be an end point of $\bar{I}_{j,\delta}$, i.e. $s \in \partial \bar{I}_{j,\delta}$. We know that $E(\gamma_j(\cdot, s)) = W - \delta$, and

$$\int_{S^2} |\nabla \gamma_j(x,s) - \nabla u_s(x)|^2 < \epsilon,$$

for a harmonic sphere $u_s : S^2 \to M$ such that $[u_s] = [u]$. Since u is strictly stable so is u_s. By the choice of $\epsilon > 0$ we know that

$$E(\gamma_j(\cdot, s)) = E(\Pi \circ (u_s + (\gamma_j(\cdot, s) - u_s))) > E(u_s) = W.$$

This is the desired contradiction. □

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