

FUNCTION SPACES ON LIE GROUPS AND APPLICATIONS

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Function spaces on Lie groups and applications

ABSTRACT

The overall goal of the thesis is to investigate hypoelliptic and subelliptic functional inequalities of different types and the corresponding function spaces on homogeneous nilpotent Lie groups, which include the cases of $\mathbb{R}^n$, Heisenberg, stratified and more general graded Lie groups. In such settings we derive a variety of functional inequalities including Hardy, Rellich, Hardy-Littlewood-Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg, uncertainty and other related inequalities, and their extensions.

The approach for obtaining hypoelliptic functional inequalities relies on establishing integral versions of Hardy inequalities on homogeneous Lie groups, for which we also find necessary and sufficient conditions for the weights for such inequalities to be true. Consequently, we link such integral Hardy inequalities to different hypoelliptic inequalities by using the Riesz and Bessel kernels associated to the described hypoelliptic operators.

Moreover, we describe Euler-Hilbert-Sobolev, Sobolev-Lorentz-Zygmund, Besov type, Morrey and generalised Morrey spaces on homogeneous Lie groups. We also study the boundedness of the Bessel-Riesz operators, generalised Bessel-Riesz operators, generalised fractional integral operators and Olsen type inequalities.

The obtained results on homogeneous Lie groups in this thesis give new statements in the Euclidean setting of $\mathbb{R}^n$ when we are working with anisotropic differential structures. Furthermore, even in the isotropic situation in $\mathbb{R}^n$, one novelty of all the obtained results is in the arbitrariness of the choice of any homogeneous quasi-norm, and some estimates are also new in the usual isotropic structure of $\mathbb{R}^n$ with the Euclidean norm.

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To my family

All analysts spend half their time hunting through the literature for inequalities which they want to use and cannot prove.

G.H. Hardy

1

INTRODUCTION

IN the modern literature, the classical L^p -Hardy inequality has the form

$$\left\| \frac{f}{|x|_E} \right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p} \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad n \geq 2, \quad 1 \leq p < n, \quad (1.1)$$

where $|x|_E = \sqrt{x_1^2 + \cdots + x_n^2}$ is the Euclidean norm and ∇ denotes the usual (Euclidean) gradient, $f \in C_0^\infty(\mathbb{R}^n)$, and the constant $\frac{p}{n-p}$ is known to be sharp. The one-dimensional version of (1.1) for $p = 2$ was obtained by Hardy in¹¹¹, and for more general p in¹¹². The multidimensional version of the Hardy inequality was shown by J. Leray¹³⁵.

The Hardy inequality has many applications in the analysis of linear and nonlinear PDEs. For example, the criteria on existence (or nonexistence) of solutions for the following second order partial differential equations

$$\left. \begin{array}{l} 0 \\ u_t \\ u_{tt} \end{array} \right\} - \Delta u = \lambda \frac{|u|^s}{|x|_E^2}$$

depends on a relation between the constant λ and the sharp constant $\frac{2}{n-2}$

from the Hardy inequality, see, e.g.,¹⁸ and²⁸. We can also refer to²⁸ and³¹ for a relationship between the Gelfand problem and the Hardy inequality. In the equation, instead of the Hardy potential $\frac{1}{|x|_E^2}$ one may have more general function which motivates to study weighted Hardy inequalities. Moreover, among many applications of the inequality we shall mention the uncertainty principle, which was presented by Heisenberg¹²⁰, then the mathematical details were provided by Hermann Weyl²¹⁸. Mathematically, on the Euclidean space $\mathbb{R}^n$, the uncertainty principle can be stated as the inequality

$$\left(\int_{\mathbb{R}^n} |\nabla \phi|^2 dx \right) \left(\int_{\mathbb{R}^n} |x|_E^2 \phi^2 dx \right) \geq \frac{n^2}{4} \left(\int_{\mathbb{R}^n} \phi^2 dx \right)^2, \quad (1.2)$$

for all $\phi \in C_0^\infty(\mathbb{R}^n)$ with the sharp constant, which can be shown as a consequence of (1.1).

The literature concerning Hardy inequalities and related analysis is extensive, and it would be impossible to do justice here to every contribution. Nevertheless, let us refer to the following extensive surveys and books for more information on this subject: Opic and Kufner¹⁵⁹, Davies and Hinz⁵⁶, Davies⁵⁵, Edmunds and Evans⁶³, Mazya^{142,143}, Ghoussoub and Moradifard^{92,93}, Balinsky, Evans and Lewis¹⁶, and the very recent book¹⁷⁷ by Ruzhansky and Suragan.

Among many, one of the related functional inequalities is the following inequality by Rellich¹⁶⁴, which now bears his name:

$$\left\| \frac{f}{|x|_E^2} \right\|_{L^2(\mathbb{R}^n)} \leq \frac{4}{n(n-4)} \|\Delta f\|_{L^2(\mathbb{R}^n)}, \quad n \geq 5, \quad (1.3)$$

where the constant is sharp. For history and details of (1.3) we can refer to, e.g., Davies and Hinz⁵⁶, and to Brézis and Vázquez³¹.

There is now a whole scope of related inequalities such as Sobolev, Rellich, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg, Hardy-Littlewood-Sobolev, Trudinger-Moser and others playing essential roles in many subjects, most importantly allowing one to obtain a priori estimates for the

well-posedness analysis of both linear and nonlinear PDEs.

The overall goal of this thesis is to study these inequalities in the setting of homogeneous Lie groups developed by Folland and Stein⁷⁸ with an idea of emphasising general results of harmonic analysis depending only on the group and dilation structures.

Such groups are necessarily nilpotent, and give almost the whole class of nilpotent Lie groups. For example, many well-known classes of nilpotent Lie groups such as *the Euclidean space, the Heisenberg group, H-type groups, polarizable Carnot groups, stratified groups, graded groups* are also examples of homogeneous Lie groups.

Actually, it can be shown that any (connected, simply connected) nilpotent Lie group is some $\mathbb{R}^n$ with a polynomial group law: for example, $\mathbb{R}^n$ with linear group law, Heisenberg group with quadratic group law in the last variable, etc. Note that in general, homogeneous Lie groups do not necessarily have homogeneous hypoelliptic left invariant differential operators, but if such an operator exists, then it must be in the class of *graded Lie groups* (see^{144, 212}). Such groups can also be characterised by dilations having rational weights, see⁶⁹ Section 4.1.1. An important subclass of graded Lie groups is the class of *stratified Lie groups*, where a homogeneous second-order sub-Laplacian $\mathcal{L}$ can be defined. Folland⁷¹ showed that the fundamental solution for the sub-Laplacian on stratified Lie groups behaves well. For more general hypoelliptic operators on graded Lie groups, we refer to Geller⁸⁶ and⁶⁹ Section 3.2.7. According to Montgomery¹⁴⁵, nilpotent graded Lie groups, or Carnot groups, are to sub-Riemannian geometry as Euclidean spaces are to Riemannian geometry. Especially, the importance of nilpotent Lie groups is widely acknowledged since they played an important role in obtaining sharp hypoelliptic estimates for differential operators on manifolds, starting from the path-breaking paper by Rothschild and Stein¹⁶⁹ (see also^{72,168}).

As for function spaces on homogeneous Lie groups, we can refer to Folland⁷¹ and Saka¹⁹⁷ for Sobolev and Besov spaces on stratified Lie groups, to Folland and Stein⁷⁸ for Hardy spaces on homogeneous Lie groups. On

graded Lie groups, we refer to^{70, 69} for Sobolev spaces, and to^{39, 40} for Besov spaces.

In the 1990s, a lot of work concerning Hardy inequalities was already developed in the context of the elliptic operators, but not very much has been done in the framework of (nonelliptic) hypoellipticity, in particular, for the Heisenberg sub-Laplacians. The study of hypoelliptic functional inequalities is important in many questions involving hypoelliptic PDEs, sub-Riemannian geometry, unique continuation, etc. Garofalo and Lanconelli have an important contribution to the development of Hardy inequalities on the Heisenberg group with their original approach, which is based on a far-reaching use of the fundamental solutions.

As for the setting of stratified Lie groups, in the literature, usually, there are three types of Hardy inequalities:

1. with the homogeneous quasi-norm associated to Folland's fundamental solution for $\mathfrak{L}$;
2. with the Carnot–Carathéodory distance, that is, the control distance associated to the sub-Laplacian;
3. with the Euclidean distance on the first stratum of the group.

In the above three cases, the best constants in the corresponding inequalities may depend on the quasi-norm that one is using. However, if one is not interested in the best constants in such inequalities, one can work with any of these equivalent quasi-norms. For example, Ciatti, Cowling and Ricci⁴⁸ established the Hardy inequality for the fractional order operator $\mathfrak{L}^{\gamma/2}$ with $0 < \gamma < Q/p$ and $1 < p < \infty$, namely they showed that the operator $T_\gamma f := |\cdot|^{-\gamma} \mathfrak{L}^{-\gamma/2} f$ extends uniquely to a bounded operator on $L^p(\mathbb{G})$. In general stratified groups, each power $\mathfrak{L}^{\gamma/2}$ of the sub-Laplacian involves a different homogeneous norm $|\cdot|_\gamma$. In⁴⁸, for the particular homogeneous norm $|\cdot|_0$, which is a limit of the norms $|\cdot|_\gamma$, it was shown that the operator norm satisfies

$$\|T_\gamma\|_{L^p(\mathbb{G}) \rightarrow L^p(\mathbb{G})} \lesssim 1 + C\gamma + O(\gamma^2). \quad (1.4)$$

Thus, in the first case, a typical Hardy inequality would take the form

$$\left\| \frac{f}{d(x)} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p} \|\nabla_H f\|_{L^p(\mathbb{G})}, \quad Q \geq 3, \quad 1 < p < Q, \quad (1.5)$$

where ∇_H is the horizontal gradient, and $d(x)$ is the so-called $\mathcal{L}$ -gauge. For such types of the inequalities we refer to, e.g. ^{85,94,98,156,51,53,129,222,48,95}.

As for the second case, the results on Hardy and other inequalities have been obtained mostly on the Heisenberg group. However, the last case has recently attracted considerable interest because of its geometrically clear nature and importance for questions in PDEs, see for example ¹³ and ⁵¹. In this case the horizontal Hardy inequality reads as follows

$$\left\| \frac{f}{|x'|} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{N-p} \|\nabla_H f\|_{L^p(\mathbb{G})}, \quad 1 < p < N, \quad (1.6)$$

where the constant is sharp, N is the dimension of the first stratum, and

$$|x'| = \sqrt{x_1'^2 + \cdots + x_N'^2}$$

is the Euclidean norm on the first stratum of $\mathbb{G}$.

The Hardy (1.6), Hardy-Rellich type and a special case of the Caffarelli-Kohn-Nirenberg inequalities on the general stratified Lie group were obtained in ¹⁷³. In Chapter 7 we discuss Sobolev type inequalities (improved versions of the Hardy inequality (1.6)), uncertainty inequalities and extended Caffarelli-Kohn-Nirenberg inequalities (see Section 7.2), which, in the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, extend the classical Caffarelli-Kohn-Nirenberg inequalities with respect to ranges of parameters, see Example 7.10. Moreover, in Section 7.3 using the factorization method of differential operators, inspired by the recent work of Gesztesy and Littlejohn⁸⁸ we obtain Hardy, Hardy-Rellich and refined Hardy inequalities on general stratified Lie groups. We also obtain analogues of Gesztesy and Littlejohn's 2-parameter version of the Rellich inequality on stratified Lie groups, and a new two-parameter estimate on $\mathbb{R}^n$ which can be regarded as a counterpart to the Gesztesy and Littlejohn's estimate. Furthermore, in Section 7.4

we prove horizontal weighted Rellich inequalities for the sub-Laplacian and for sub-Laplacians with drift on general stratified Lie groups. We discuss how the presence of a drift improves the known inequalities. Moreover, we show several versions of weighted Rellich inequalities for the sub-Laplacian with drift on the polarizable Carnot groups, with weights associated to the fundamental solution of the sub-Laplacian. The obtained results are new for the Laplacian in $\mathbb{R}^n$, embedding the classical Rellich inequality into a family of Rellich inequalities with parameter dependent drifts.

My Ph.D. research also focuses on obtaining analogues of functional inequalities on non-stratified graded Lie groups and non-graded homogeneous Lie groups. Beyond stratified Lie groups, the lack of the following properties make the topic itself even more interesting:

- non-stratified graded Lie groups $\mathbb{G}$ do not have homogeneous sub-Laplacians nor Laplacian but they always have the so-called Rockland operators, which are left-invariant homogeneous hypoelliptic differential operators on $\mathbb{G}$;
- no homogeneous (horizontal) gradient on non-stratified graded Lie groups means no horizontal estimates;
- no invariant hypoelliptic differential operator on non-graded homogeneous Lie groups to enter the right hand side of (1.3).

These prompt questions: how to formulate Hardy, Sobolev, Rellich and related functional inequalities on non-stratified graded Lie groups and non-graded homogeneous Lie groups?

Recall that, in general, homogeneous Lie groups do not have to be stratified or even graded, so the concept of horizontal gradients does not make sense. Nevertheless, there are still some useful properties, such as a good polar decomposition which, combined with dilations, allows us to investigate the radial structure of the group. Therefore, on general homogeneous Lie group, we work with the quasi-radial derivative operator $\mathcal{R}_{|\cdot|} := d/dr$,

which is the derivative with respect to a radial direction with respect to the quasi-norm $|\cdot|$, and the Euler operator $\mathbb{E} := |x|\mathcal{R}_{|x|}$. Note that $\mathcal{R}_{|\cdot|}$ is homogeneous of order -1 and is analogous to the radial derivative $\frac{x}{|x|} \cdot \nabla_E$ on $\mathbb{R}^n$, where ∇_E is the usual Euclidean gradient.

In¹⁷⁴, for the radial derivative operator $\mathcal{R}_{|\cdot|}$, the following L^p -Hardy inequality was obtained:

$$\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p} \|\mathcal{R}_{|x|} f\|_{L^p(\mathbb{G})}, \quad 1 < p < Q \quad (1.7)$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, where Q is the homogeneous dimension of $\mathbb{G}$. Then, in¹⁵², the following analogue of L^p -Rellich inequality was obtained

$$\left\| \frac{f}{|x|^2} \right\|_{L^p(\mathbb{G})} \leq \frac{p^2}{Q(p-1)(Q-2p)} \left\| \mathcal{R}_{|x|}^2 f + \frac{Q-1}{|x|} \mathcal{R}_{|x|} f \right\|_{L^p(\mathbb{G})}, \quad 1 < p < Q/2, \quad (1.8)$$

for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$. Moreover, it is known that the constants in (1.7) and (1.8) are sharp for any quasi-norm $|\cdot|$ and are attained if and only if $f = 0$. In the setting of general homogeneous Lie group, we refer to¹⁵³ and¹³² for the weighted Hardy and Rellich inequalities with Bessel pairs in the sense of Ghoussoub and Moradifam⁹², that is, the weights are related to the existence of solutions of certain ordinary differential equations. We also refer to¹⁷² and¹⁷⁰ for the local refined L^2 and L^p versions of Hardy's and Rellich's inequalities for sums of squares of vector fields on bounded sets of smooth manifolds under certain assumptions on the vector fields, respectively.

In Section 3.1.1 we show Hardy inequalities with more general weights using the factorization method. Moreover, in Chapter 3 we discuss Hardy inequalities with remainder terms, critical and subcritical Hardy inequalities, Heisenberg-Pauli-Weyl type uncertainty principles as well as Sobolev-Rellich, extended Caffarelli-Kohn-Nirenberg, Sobolev type (refined Hardy) inequalities for the radial derivative and Euler operators. Most inequalities are obtained with the best constants. Some of the obtained results are

new already in the anisotropic $\mathbb{R}^n$ as well as in the isotropic $\mathbb{R}^n$ due to the freedom in the choice of homogeneous quasi-norms.

It turns out that the Euler operator $\mathbb{E}$ on homogeneous Lie groups has many useful and interesting properties. For example, we show that the operator $\mathbb{A} := \mathbb{E}\mathbb{E}^*$ is a Komatsu-non-negative operator (see Lemma 2.14), hence one can define fractional powers of the operator $\mathbb{A} = \mathbb{E}\mathbb{E}^*$ as in¹⁴¹ Chapter 5, which allows us to define Euler-Hilbert-Sobolev spaces on general homogeneous Lie groups (see Section 5.0.1). Moreover, in Section 2.2 we define the operator semigroup $e^{-t\mathbb{E}^*\mathbb{E}}$ associated with the Euler operator on general homogeneous Lie groups, which will be useful in Section 5.0.4 to define Besov type spaces and obtain Gagliardo-Nirenberg type inequalities associated with the norm of the Besov type spaces. Furthermore, we discuss Sobolev-Lorentz-Zygmund and Morrey spaces on general homogeneous Lie groups and analyse their main properties in Chapter 5. There, we also discuss the boundedness of integral operators, namely, the Bessel–Riesz operators, generalised Bessel–Riesz operators, fractional integral operators and Olsen type inequalities in generalised Morrey spaces on homogeneous Lie groups.

The obtained results on homogeneous Lie groups in this dissertation give new statements already in the Euclidean setting of $\mathbb{R}^n$ when we are working with anisotropic differential structures. Furthermore, even in the isotropic situation in $\mathbb{R}^n$, one novelty of all the obtained results is in the arbitrariness of the choice of any homogeneous quasi-norm, and some estimates are also new in the usual isotropic structure of $\mathbb{R}^n$ with the Euclidean norm, which we will be indicating at relevant places. Thus, we could have worked directly in $\mathbb{R}^n$ with the anisotropic structure, but since the methods work equally well in the setting of Folland and Stein’s homogeneous Lie groups, we formulate all the results in such (greater) generality. In particular, it follows the general strategy initiated by their work, of distilling results of harmonic analysis depending only on the group and dilation structures: in this respect, Chapter 5 shows that the harmonic analysis on Morrey spaces largely falls into this category.

For example, let us consider the following Bessel–Riesz operators

$$I_{\alpha,\gamma}f(x) = \int_{\mathbb{R}^n} \frac{|x-y|_E^{\alpha-n}}{(1+|x-y|_E)^\gamma} f(y)dy, \quad (1.9)$$

where $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, $p \geq 1$, $\gamma \geq 0$ and $0 < \alpha < n$. The boundedness of the fractional integral operators $I_{\alpha,0} = I_\alpha$ on Lebesgue spaces was shown by Hardy and Littlewood in^{113, 114} and Sobolev in²⁰⁷. In the case of $\mathbb{R}^n$, the Hardy–Littlewood maximal operator, the generalised fractional integral operators, which are the generalised form of the Riesz potential I_α , Bessel–Riesz operators and Olsen type inequalities are widely analysed on Lebesgue spaces, Morrey spaces and generalised Morrey spaces (see e.g.^{3, 45, 147, 68, 67, 131, 148, 149, 106, 203, 123, 122}, and the recent survey³⁴). For some of their functional analytic properties, see also^{35,37} and references therein. We refer to^{101, 102, 100, 36, 163} for analysis in Morrey spaces and generalised Morrey spaces, and to^{9, 10} in anisotropic Morrey-type spaces. For the applications of Morrey spaces for non-Euclidean distances we refer to, e.g.,¹⁰⁴ and¹⁰⁵. In Chapter 5, we work with the following analogue of the Bessel–Riesz operator (1.9) on homogeneous Lie groups

$$I_{\alpha,\gamma}f(x) := \int_{\mathbb{G}} \frac{|xy^{-1}|^{\alpha-Q}}{(1+|xy^{-1}|)^\gamma} f(y)dy. \quad (1.10)$$

On graded Lie groups, in⁶⁹, Chapter 4, the authors use heat kernel methods to study the connection between these operators and Sobolev spaces. Here, in Chapter 5 we focus on the harmonic analysis aspects in the framework of Morrey spaces in the setting of general homogeneous Lie groups.

Thus, in most cases, we note that the functional inequalities for the radial derivative or Euler operators improve the ones with the usual Euclidean gradient. They are also useful in deriving functional inequalities for magnetic operators. For example, in Section 3.5.1 we show how such inequalities are useful in obtaining Hardy inequalities for Landau Hamiltonian and for Baouendi–Grushin operator with Aharonov–Bohm type magnetic field.

However, these functional inequalities associated to the operators $\mathcal{R}_{|\cdot|}$ and

$\mathbb{E}$ on homogeneous Lie groups do not always generalise the known inequalities on stratified Lie groups unless the homogeneous Lie groups are Abelian. Therefore, in order to keep consistency with the known inequalities, it is important to obtain analogues of these functional inequalities on graded Lie group, which is the largest class of homogeneous nilpotent Lie groups admitting homogeneous hypoelliptic left-invariant differential operators (^{144, 212}, see also a discussion in ⁶⁹ Section 4.1). Such operators are called *Rockland operators*. We note that, following the seminal paper by Rothschild and Stein¹⁶⁹, such Rockland operators can be considered as model “approximations” of general hypoelliptic partial differential operators on manifolds. In this thesis we will work with the Rockland differential operators which are positive in the sense of operators. Thus, our setting will include the higher order operators on $\mathbb{R}^n$ as well as higher order hypoelliptic invariant differential operators on the Heisenberg group, on general stratified Lie groups, and on general graded Lie groups.

From now on, we let $\mathcal{R}$ be a positive Rockland operator, that is, a positive left-invariant homogeneous hypoelliptic invariant differential operator on a graded Lie group $\mathbb{G}$ of homogeneous degree ν . Its powers $\mathcal{R}^a$ are understood through the functional calculus on the whole of $\mathbb{G}$, extensively analysed in ^{69,70}. In Chapter 6, we discuss the following analogue of (1.4), which we also call the Hardy-Sobolev inequalities since it contains the classical Hardy, Rellich and Sobolev inequalities: Let $1 < p \leq q < \infty$ and $0 < a < Q/p$. Let $0 \leq b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Then there exists a positive constant C for all $f \in \dot{L}_a^p(\mathbb{G})$ such that

$$\left\| \frac{f}{|x|^{\frac{b}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})}. \quad (1.11)$$

In the special case of $q = p$ we obtain the general hypoelliptic family of the Hardy inequalities:

$$\left\| \frac{f}{|x|^a} \right\|_{L^p(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \quad 0 < a < Q/p. \quad (1.12)$$

In particular, for $a = 1$ and $a = 2$ we obtain the hypoelliptic versions of Hardy and Rellich inequalities, respectively, which in this form are new already on the stratified Lie groups since the operator $\mathcal{R}$ does not have to be a sub-Laplacian and can be of any order. At the same time, for $b = 0$, (1.11) gives a simple proof of the Sobolev inequality obtained in⁷⁰:

$$\|f\|_{L^q(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{p}} f\|_{L^p(\mathbb{G})}, \quad 1 < p < q < \infty, \quad a = Q \left(\frac{1}{p} - \frac{1}{q} \right). \quad (1.13)$$

The homogeneous and inhomogeneous Sobolev spaces $\dot{L}_a^p(\mathbb{G})$ and $L_a^p(\mathbb{G})$ based on the positive left-invariant hypoelliptic differential Rockland operator $\mathcal{R}$ have been extensively investigated in⁷⁰ and⁶⁹ Section 4.4 to which we refer for the details of their properties. In these works, the authors generalised to graded Lie groups the Sobolev spaces based on the sub-Laplacian on stratified Lie groups analysed by Folland in⁷¹. Moreover, we discuss a critical version of (1.11), Heisenberg-Pauli-Weyl type uncertainty principle, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg, Hardy-Littlewood-Sobolev (HLS) and reversed HLS inequalities on general graded Lie group.

Actually, these hypoelliptic functional inequalities on graded Lie groups follow from the convolution Hardy inequalities that we will establish in the setting of general homogeneous Lie groups in Section 4.0.2. For example, let us state the following convolution Hardy inequality: Let $1 < p \leq q < \infty$ and $0 < a < Q/p$. Let $0 \leq b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Assume that $|T_a^{(1)}(x)| \leq C_2 |x|^{a-Q}$ for some positive $C_2 = C_2(a, Q)$. Then there exists a positive constant $C_1 = C_1(p, q, a, b)$ for all $f \in L^p(\mathbb{G})$ such that

$$\left\| \frac{f * T_a^{(1)}}{|x|^{\frac{b}{q}}} \right\|_{L^q(\mathbb{G})} \leq C_1 \|f\|_{L^p(\mathbb{G})}. \quad (1.14)$$

Indeed, we can obtain the Hardy-Sobolev inequalities (1.11) by taking $T_a^{(1)}$ in (1.14) to be the Riesz kernel (2.44) of a positive Rockland operator, since Riesz kernel satisfies the condition for $T_a^{(1)}$ by Lemma 2.20. Similarly, we obtain its critical versions by taking $T_a^{(2)}$ in (4.54) to be a combination of

Riesz and Bessel kernels. The proofs of these convolution Hardy inequalities are based on the integral Hardy inequalities that we will discuss in Section 4.0.1, for which we also find necessary and sufficient conditions for the weights for such inequalities to be true. Moreover, we will discuss other types of integral inequalities in Chapter 4.

As a by-product this gives a new proof of Sobolev inequalities, but most importantly, it provides a new way of using Sobolev and other differential structures in various weighted estimates, opening up wide perspectives of further applications to hypoelliptic partial differential equations of different types, both linear and nonlinear. For example, in ¹⁸⁵, ¹⁹⁰ and ¹⁸⁹ we clarified further links between the hypoelliptic functional inequalities and ground states for nonlinear Schrödinger type equations, as well as with extremisers of the corresponding variational problems, see Remark 6.14. We also refer to ¹⁸⁴ for the applications of the hypoelliptic Gagliardo-Nirenberg inequality to the nonlinear damped hypoelliptic wave equations, see Remark 6.14.

Incidentally, in ¹⁹¹ we showed that our proofs of the above hypoelliptic functional inequalities can be adapted to the setting of general connected Lie groups using the polar decompositions from Remark 4.5, which in particular extends Sobolev inequalities in ³³. We also refer to ¹⁹² where we investigated Hardy, weighted Trudinger-Moser and Caffarelli-Kohn-Nirenberg type inequalities on Riemannian manifolds with negative curvature.

I would like to honourably emphasise that most results of this Ph.D. thesis are included in the very recent open access book titled Hardy inequalities on homogeneous groups by Ruzhansky and Suragan ¹⁷⁷, and I am listed as the contributor of this book. There, they reviewed the most recent developments of the subelliptic functional inequalities on homogeneous Lie groups.

LIST OF PUBLICATIONS

We list here the publications written during this Ph.D. research. The thesis is based on the following articles, they are order by appearance in the text:

1. M. Ruzhansky, D. Suragan and N. Yessirkegenov, *Extended Caffarelli-Kohn-Nirenberg inequalities, and remainders, stability, and superweights for L_p -weighted Hardy inequalities*. Trans. Amer. Math. Soc. Ser. B, **5** (2018), 32–62.
2. M. Ruzhansky, D. Suragan, and N. Yessirkegenov. *Extended Caffarelli-Kohn-Nirenberg inequalities and superweights for L^p -weighted Hardy inequalities*. C. R. Math. Acad. Sci. Paris, **355** (2017), no. 6, 694–698.
3. M. Ruzhansky and N. Yessirkegenov. *Hypoelliptic functional inequalities*. arXiv:1805.01064 (2018).
4. M. Ruzhansky and N. Yessirkegenov. *Factorizations and Hardy-Rellich inequalities on stratified groups*. to appear in J. Spectr. Theory, arXiv:1706.05108 (2017).
5. M. Ruzhansky, D. Suragan and N. Yessirkegenov. *Euler semigroup, Hardy-Sobolev and Gagliardo-Nirenberg type inequalities on homogeneous groups*. arXiv:1805.01842 (2018).
6. M. Ruzhansky, D. Suragan and N. Yessirkegenov. *Sobolev type inequalities, Euler-Hilbert-Sobolev and Sobolev-Lorentz-Zygmund spaces on homogeneous groups*. Integral Equations and Operator Theory, (2018), 90:10.
7. A. Laptev, M. Ruzhansky and N. Yessirkegenov. *Hardy inequalities for Landau Hamiltonian and for Baouendi-Grushin operator with Aharonov-Bohm type magnetic field. Part I*. Math. Scand., **125** (2019), 239–269.
8. M. Ruzhansky, D. Suragan, and N. Yessirkegenov. *Hardy-Littlewood, Bessel-Riesz, and fractional integral operators in anisotropic Morrey and Campanato spaces*. Fract. Calc. Appl. Anal., **21** (2018), no. 3, 577–612.
9. M. Ruzhansky, D. Suragan, and N. Yessirkegenov. *Caffarelli-Kohn-Nirenberg and Sobolev type inequalities on stratified Lie groups*. NoDEA Nonlinear Differential Equations Appl., **24** (2017), no. 5, 24:56.

10. M. Ruzhansky and N. Yessirkegenov. *Rellich inequalities for sub-Laplacians with drift*. Proc. Amer. Math. Soc. **147** (2019), no. 3, 1335–1349.

Other publications by the author not included in this thesis:

11. M. Ruzhansky and N. Yessirkegenov. *Very weak solutions to hypoelliptic wave equations*. J. Differential Equations, **268** (2020), 2063–2088.

12. M. Ruzhansky and N. Yessirkegenov. *Limiting cases of Sobolev inequalities on stratified groups*. Proc. Japan. Acad., Ser. A, **95** (2019), no. 8, 83–87.

13. M. Ruzhansky and N. Yessirkegenov. *New progress on weighted Trudinger-Moser and Gagliardo-Nirenberg, and critical Hardy inequalities on stratified groups*. Landscapes of Time-Frequency Analysis, pp. 277–289. Birkhäuser, 2019.

14. M. Ruzhansky, N. Tokmagambetov and N. Yessirkegenov. *Best constants in Sobolev and Gagliardo-Nirenberg inequalities on graded groups and ground states for higher order nonlinear subelliptic equations*. arXiv:1704.01490 (2017).

15. M. Ruzhansky and N. Yessirkegenov. *Critical Sobolev, Gagliardo-Nirenberg, Trudinger and Brezis-Gallouet-Wainger inequalities, best constants, and ground states on graded groups*. arXiv:1709.08263 (2017).

16. M. Ruzhansky and N. Yessirkegenov. *Hardy, Hardy-Sobolev, Hardy-Littlewood-Sobolev and Caffarelli-Kohn-Nirenberg inequalities on general Lie groups*. arXiv:1810.08845 (2018).

17. M. Ruzhansky and N. Yessirkegenov. *Hardy, weighted Trudinger-Moser and Caffarelli-Kohn-Nirenberg type inequalities on Riemannian manifolds with negative curvature*. arXiv:1802.09072 (2018).

18. M. Ruzhansky and N. Yessirkegenov. *Existence and non-existence of global solutions for semilinear heat equations and inequalities on sub-Riemannian manifolds, and Fujita exponent on unimodular Lie groups*. arXiv:1812.01933 (2018).

19. A. Velicu and N. Yessirkegenov. *Rellich, Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg inequalities for Dunkl operators and applications*. arXiv:1904.08725 (2019).

2

Preliminaries

THE purpose of this chapter is to provide all the key notions to present our results, thereby introducing the reader to the analysis on homogeneous Lie groups and fixing notation for the rest of this thesis. Firstly we discuss relevant properties of general Lie groups and algebras, then focus on properties of homogeneous Lie groups that we will work with, and finally we introduce the notion of the Euler operator on homogeneous Lie groups and study some of its main properties.

2.1 HOMOGENEOUS LIE GROUPS

In this section we give the definitions and basic properties of nilpotent Lie algebras and groups following Folland and Stein's book⁷⁸ as well as the recent open access books⁶⁹ and¹⁷⁷. We can also refer to, e.g., Goodman⁹⁷, Corwin and Greanleaf⁵⁰, Ricci¹⁶⁵ and Folland's books^{73,74,75} for comprehensive treatments of nilpotent Lie groups.

Let $\mathbb{G}$ be a (connected and simply-connected) Lie group with a Lie algebra

(real and finite-dimensional) $\mathfrak{g}$. If the lower central series

$$\mathfrak{g}_{(1)} := \mathfrak{g}, \quad \mathfrak{g}_{(j)} := [\mathfrak{g}, \mathfrak{g}_{(j-1)}],$$

of a Lie algebra $\mathfrak{g}$ eventually vanishes in a finite number of steps, then we say that $\mathfrak{g}$ is nilpotent Lie algebra. Furthermore, $\mathfrak{g}$ is said to be nilpotent of step s if s is the smallest integer for which all Lie brackets of order $s + 1$ are zero, and corresponding $\mathbb{G}$ is called a nilpotent Lie group of step s . By the Baker-Campbell-Hausdorff formula for sufficiently small elements X and Y of $\mathfrak{g}$ one has

$$\exp X \exp Y = \exp H(X, Y) = X + Y + \frac{1}{2}[X, Y] + \cdots, \quad (2.1)$$

where $\exp : \mathfrak{g} \rightarrow G$ is the exponential map, $H(X, Y)$ is an infinite linear combination of X and Y and their iterated commutators, and the dots denote terms of order higher than two. In the case of nilpotent Lie algebra, the formula (2.1) is finite, and defines a polynomial map from $V \times V$ to V , where V is the underlying vector space of $\mathfrak{g}$.

Proposition 2.1 (¹⁷⁷ Proposition 1.1.1). *Let $\mathfrak{g}$ be a nilpotent Lie algebra, and let $\mathbb{G}$ be the corresponding connected and simply-connected nilpotent Lie group. Then we have*

- (i) *The exponential map $\exp : \mathfrak{g} \rightarrow \mathbb{G}$ is a diffeomorphism. Furthermore, the group law $(x, y) \mapsto xy$ is a polynomial map if $\mathbb{G}$ is identified with $\mathfrak{g}$ via $\exp$.*
- (ii) *If λ is a Lebesgue measure on $\mathfrak{g}$, then $\exp \lambda$ is a bi-invariant Haar measure on $\mathbb{G}$.*

Definition 2.2. A family of dilations of a Lie algebra $\mathfrak{g}$ is a family of linear mappings

$$\{\delta_r, r > 0\}$$

from $\mathfrak{g}$ to itself with the following properties:

- $\delta_r = \exp(A \log r)$, where A is a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues;

- every δ_r is a morphism of $\mathfrak{g}$, i.e., a linear mapping from $\mathfrak{g}$ to itself which respects the Lie bracket:

$$\forall X, Y \in \mathfrak{g}, r > 0 \quad [\delta_r X, \delta_r Y] = \delta_r [X, Y].$$

We note that $\delta_{rs} = \delta_r \delta_s$ for all $r, s > 0$. If $\{\delta_r\}$ is a family of dilations on $\mathfrak{g}$, then $\tilde{\delta}_r := \delta_{r^\alpha} = \exp(\alpha A \log r)$ gives also a new family of dilations $\{\tilde{\delta}_r\}$ for any $\alpha > 0$. One can always assume that the minimum eigenvalue of A is equal to 1 by adjusting α .

Note that if a Lie algebra $\mathfrak{g}$ is equipped with a family of dilations then it is nilpotent but the converse is not true in general. For example, Dyer⁶² constructed a (nine-dimensional) nilpotent Lie algebra that does not allow any compatible family of dilations.

Definition 2.3. We say that a Lie algebra $\mathfrak{g}$ is *graded* if it is endowed with a vector space decomposition

$$\mathfrak{g} = \bigoplus_{j=1}^{\infty} V_j$$

with the property $[V_i, V_j] \subset V_{i+j}$, where all but finitely many of the V_j 's are 0.

A Lie group is said to be *graded* when it is a connected simply-connected Lie group and its Lie algebra is graded.

Definition 2.4. We say that a graded Lie algebra $\mathfrak{g}$ is *stratified* if V_1 generates the whole of the Lie algebra $\mathfrak{g}$ by taking iterated commutators, that is, if $\mathfrak{g}$ is nilpotent of step m then

$$\mathfrak{g} = \bigoplus_{j=1}^m V_j, \quad [V_j, V_1] = V_{j+1}, \quad \forall j = 1, \dots, m,$$

holds. The natural dilations on $\mathfrak{g}$ are defined as follows

$$\delta_r \left(\sum_{k=1}^m X_k \right) = \sum_{k=1}^m r^k X_k, \quad (X_k \in V_k).$$

A Lie group is said to be *stratified* when it is a connected simply-connected Lie group and its Lie algebra is stratified.

Definition 2.5. A Lie group $\mathbb{G}$ is called homogeneous if it is a connected and simply-connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ admits a family of dilations $\{\delta_r\}$.

By Proposition 2.1, (i), the exponential mapping $\exp : \mathfrak{g} \rightarrow \mathbb{G}$ is a global diffeomorphism, hence it induces the corresponding family on $\mathbb{G}$ which one can still call the dilations on $\mathbb{G}$ and denote by δ_r . So, we write $\delta_r(x)$ or simply rx for $x \in \mathbb{G}$.

Here are some examples of homogeneous groups:

Example 2.6. The Abelian group $(\mathbb{R}^n, +)$ is the simplest example of a homogeneous group with the usual dilations $\delta_r x = rx$, $r > 0$, $x \in \mathbb{R}^n$.

Example 2.7. Let n be a positive integer. Then, the Heisenberg group $\mathbb{H}_n$, whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$, is homogeneous with the dilations

$$\delta_r(z_1, \dots, z_n, t) = (rz_1, \dots, rz_n, r^2t).$$

If $\nu_1, \dots, \nu_n$ are the eigenvalues of A , listed in nondecreasing order and with each value listed as many times as its multiplicity, then the mappings $\{\delta_r = \exp(A \log r)\}$ give the dilation structure to an n -dimensional homogeneous group $\mathbb{G}$, with

$$1 = \nu_1 \leq \nu_2 \leq \dots \leq \nu_n. \quad (2.2)$$

Now, we fix a basis $\{X_k\}_{k=1}^n$ of the Lie algebra $\mathfrak{g}$ such that for each k we have

$$AX_k = \nu_k X_k.$$

Then the *homogeneous dimension* of the corresponding Lie group $\mathbb{G}$ is defined by

$$Q := \sum_{k=1}^n \nu_k = \text{Tr}(A). \quad (2.3)$$

Letting these X_k 's be orthonormal we can define a standard Euclidean norm $\|\cdot\|$ on $\mathfrak{g}$, which can be also considered as a function on $\mathbb{G}$ by the following formula

$$\|x\| = \|\exp^{-1} x\|. \quad (2.4)$$

A *homogeneous quasi-norm* on $\mathbb{G}$ can be defined as a continuous function $x \mapsto |x|$ from $\mathbb{G}$ to $[0, \infty)$ such that

(a)

$$|x^{-1}| = |x| \quad \text{and} \quad |rx| = r|x|,$$

for all $x \in \mathbb{G}$ and $r > 0$,

(b)

$$|x| = 0 \iff x = 0.$$

Note that quasi-norms that are C^∞ -smooth on $\mathbb{G} \setminus \{0\}$ always exist.

In the sequel, we use the notation $rx = \delta_r x$ for the dilation of x induced by the dilations on the Lie algebra via the exponential mapping.

The quasi-ball centred at $x \in \mathbb{G}$ with radius $r > 0$ is defined by

$$B(x, r) := \{y \in \mathbb{G} : |x^{-1}y| < r\}.$$

Note that the quasi-ball $B(x, r)$ is the left translate by x of $B(0, r)$ and $B(0, r) = \delta_r B(0, 1)$. We also recall that $\overline{B(x, r)}$ is compact for any $x \in \mathbb{G}$ and $r > 0$.

Proposition 2.8 (⁶⁹ Proposition 3.1.35 (ii)). *Let $\mathbb{G}$ be a homogeneous group. Then any two homogeneous quasi-norms on $\mathbb{G}$ are equivalent, i.e., if we fix two homogeneous quasi-norms $|\cdot|$ and $|\cdot|'$ on $\mathbb{G}$, then there are positive constants $C_1, C_2 > 0$ such that*

$$C_1|x|' \leq |x| \leq C_2|x|'.$$

The following proposition shows that every homogeneous quasi-norm satisfies the triangle inequality up to a constant:

Proposition 2.9 (¹⁷⁷ Proposition 1.2.4). *Let $\mathbb{G}$ be a homogeneous group.*

(1) For any homogeneous quasi-norm on $\mathbb{G}$ there is a positive constant $C > 0$ such that

$$|xy| \leq C(|x| + |y|), \quad \forall x, y \in \mathbb{G}. \quad (2.5)$$

(2) Moreover, on $\mathbb{G}$, there is always a homogeneous quasi-norm $|\cdot|$ with the property

$$|xy| \leq |x| + |y| \quad (2.6)$$

for all $x, y \in \mathbb{G}$.

By Proposition 2.1 we see that the bi-invariant Haar measure on $\mathbb{G}$ comes from the Lebesgue measure on $\mathfrak{g}$. Let us require that the Haar measure of $B(0, 1)$ is 1 by fixing the normalisation of the Haar measure on $\mathbb{G}$. Then, in the case $\mathbb{G} = \mathbb{R}^n$ with the usual Lebesgue measure, the Haar measure is $\Gamma((n+2)/2)/\pi^{n/2}$ times the Lebesgue measure. The notation $|E|$ denotes the measure of any measurable set $E \subset \mathbb{G}$. We also use the notation $\int_{\mathbb{G}} f dx$ or $\int_{\mathbb{G}} f(x) dx$ for the integral of a function f with respect to this measure.

Note that

$$|\delta_r(E)| = r^Q |E|, \quad d(rx) = r^Q dx,$$

and $|B(x, r)| = r^Q$ for all $r > 0$ and $x \in \mathbb{G}$, where Q is the homogeneous dimension of $\mathbb{G}$.

Definition 2.10. If a function f on $\mathbb{G} \setminus \{0\}$ satisfies

$$f \circ \delta_r = r^\lambda f \text{ for all } r > 0,$$

then we say that the function is homogeneous of degree λ .

Since we have for f and g that

$$\int_{\mathbb{G}} f(x) (g \circ \delta_r)(x) dx = r^{-Q} \int_{\mathbb{G}} (f \circ \delta_{1/r})(x) g(x) dx,$$

the mapping $f \mapsto f \circ \delta_r$ can be extended to distributions by defining

$$\langle f \circ \delta_r, \phi \rangle = r^{-Q} \langle f, \phi \circ \delta_{1/r} \rangle,$$

for any distribution f and any test function ϕ with the usual duality $\langle \cdot, \cdot \rangle$ between functions and distributions. If the distribution f satisfies

$$f \circ \delta_r = r^\lambda f \text{ for all } r > 0,$$

then we say that the distribution is homogeneous of degree λ . If a linear operator D on $\mathbb{G}$ satisfies

$$D(f \circ \delta_r) = r^\lambda (Df) \circ \delta_r \text{ for all } r > 0,$$

for any f , then we say that D is homogeneous of degree λ . Note that Df is homogeneous of degree $\mu - \lambda$ if f is a homogeneous function of degree μ .

The following polar decomposition is very important for our analysis:

Proposition 2.11 (¹⁷⁷ Proposition 1.2.10). *If $\mathbb{G}$ is a homogeneous Lie group equipped with a homogeneous quasi-norm $|\cdot|$, then there always exists a (unique) positive Borel measure σ on the unit sphere*

$$\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\} \tag{2.7}$$

such that

$$\int_{\mathbb{G}} f(x) dx = \int_0^\infty \int_{\mathfrak{S}} f(ry) r^{Q-1} d\sigma(y) dr \tag{2.8}$$

holds for all $f \in L^1(\mathbb{G})$.

The convolution of two integrable functions on $\mathbb{G}$ is defined by

$$(f * g)(x) := \int_{\mathbb{G}} f(y) g(y^{-1}x) dy = \int_{\mathbb{G}} f(xy^{-1}) g(y) dy.$$

Note that the convolution is not commutative on the non-Abelian $\mathbb{G}$.

We will also often use the following Young inequality in our analysis:

Proposition 2.12 (¹⁷⁷ Proposition 1.2.13). *Assume that $1 \leq p, q, r \leq \infty$ and*

$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. If $f \in L^p$ and $g \in L^q$, then we have $f * g \in L^r$ and

$$\|f * g\|_{L^r(\mathbb{G})} \leq \|f\|_{L^p(\mathbb{G})} \|g\|_{L^q(\mathbb{G})}.$$

2.1.1 RADIAL AND EULER OPERATORS

In this section we discuss radial and Euler operators and study a number of properties that will be used throughout the thesis.

Let us begin by introducing a radial derivative operator that acts on a differentiable function f on a homogeneous group $\mathbb{G}$ defined by

$$\mathcal{R}_{|\cdot|} f(ry) := \frac{d}{dr} f(ry) \quad \text{if } |y| = 1, \quad (2.9)$$

for any $r > 0$.

It can be also noticed that

$$\mathcal{R}_{|\cdot|} f = \frac{d}{d|x|} f(x) = \frac{d}{d|x|} f\left(\frac{x}{|x|}|x|\right) = \frac{x \cdot \nabla_\delta}{|x|} f(x), \quad (2.10)$$

for any differentiable function f , where $\nabla_\delta = (\nu_1 \partial / \partial x_1, \dots, \nu_n \partial / \partial x_n)$ and the quasi-norm $|\cdot|$ can be arbitrary.

Let us now define the Euler operator on $\mathbb{G}$ as follows

$$\mathbb{E} := |x| \mathcal{R}_{|x|}, \quad (2.11)$$

which is homogeneous of degree 0.

Actually, one can note that the radial derivative operator does not depend on the group structure, instead only depends on the dilations and the homogeneous quasi-norm. In the same way, the Euler operator only depends on the dilations. Nevertheless, as discussed in the introduction, since there is no invariant hypoelliptic differential operator on general (non-graded) homogeneous Lie groups, we work there with the radial derivative and Euler operators.

Let us now show the following useful property of the Euler operators on taking derivatives with respect to points that are not on the quasi-sphere $\mathfrak{S}$: denoting $\rho := e^t r$ for $t \in \mathbb{R}$ we have for $x = ry \in \mathbb{G}$ with $y \in \mathfrak{S}$ that

$$\frac{d}{dt} (f(e^t x)) = \frac{d}{dt} (f(e^t r y)) = \frac{d}{dt} (f(\rho y)) = \rho \frac{d}{d\rho} (f(\rho y)) = \mathbb{E} f(\rho y) = \mathbb{E} f(e^t x),$$

which is

$$\frac{d}{dt} f(e^t x) = \mathbb{E} f(e^t x). \quad (2.12)$$

We will use this identity in Section 2.2. One can also note that $\mathbb{E} f(rx) = r \frac{d}{dr} f(rx)$.

There are also other useful properties of the Euler operator:

Proposition 2.13 (¹⁸² Ruzhansky-Suragan-Y., Lemmas 2.1, 2.2 and 2.4). *The followings hold true:*

(i) *For the differentiable function $f : \mathbb{G} \setminus \{0\} \rightarrow \mathbb{R}$ we have*

$$\mathbb{E}(f) = \nu f \iff f(rx) = r^\nu f(x) \quad (\forall r > 0, x \neq 0),$$

where $\nu \in \mathbb{R}$.

(ii) *Let $\mathbb{I}$ be the identity operator. Then the formal adjoint of the Euler operator $\mathbb{E}$ has the form*

$$\mathbb{E}^* = -Q\mathbb{I} - \mathbb{E}. \quad (2.13)$$

(iii) *Moreover, we have*

$$\|\mathbb{E} f\|_{L^2(\mathbb{G})} = \|\mathbb{E}^* f\|_{L^2(\mathbb{G})} \quad (2.14)$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$.

Proof of Proposition 2.13. Part (i). Using (2.9) for the function

$$f(rx) = r^\nu f(x), \quad \forall r > 0$$

with $x := \rho y \neq 0, y \in \mathfrak{S}$, we obtain $\mathbb{E} f = \nu f(x)$.

In the converse case, if we fix $x \neq 0$ and denote $g(r) := f(rx)$, then the equality $\mathbb{E}f(rx) = \nu f(rx)$ with (2.9) implies that

$$g'(r) = \frac{d}{dr}f(rx) = \frac{1}{r}\mathbb{E}f(rx) = \frac{\nu}{r}f(rx) = \frac{\nu}{r}g(r),$$

which gives, in particular, $g(r) = g(1)r^\nu$, that is, $f(rx) = r^\nu f(x)$.

Part (ii). Taking into account the polar decomposition in Proposition 2.11 and (2.9), and using integration by parts we have for the formal adjoint operator of $\mathbb{E}$ on $C_0^\infty(\mathbb{G} \setminus \{0\})$ that

$$\begin{aligned} \int_{\mathbb{G}} \mathbb{E}f(x)\overline{g(x)}dx &= \int_0^\infty \int_{\mathfrak{S}} \frac{d}{dr}f(ry)\overline{g(ry)}r^Q d\sigma(y)dr \\ &= - \int_0^\infty \int_{\mathfrak{S}} f(ry) \left(Qr^{Q-1}\overline{g(ry)} + r^Q \frac{d}{dr}g(ry) \right) d\sigma(y)dr \\ &= - \int_{\mathbb{G}} f(x)(Q + \mathbb{E})\overline{g(x)}dx. \end{aligned}$$

Part (iii). The identity (2.14) follows directly from the fact that $\mathbb{E}$ is normal, that is, $\mathbb{E}\mathbb{E}^* = \mathbb{E}^*\mathbb{E}$. $\square$

We will also work with the following operator

$$\mathbb{A} := \mathbb{E}\mathbb{E}^*,$$

which is formally self-adjoint, namely

$$\mathbb{A} = \mathbb{A}^* = -Q\mathbb{E} - \mathbb{E}^2.$$

Replacing f by $\mathbb{E}f$ in (2.14), for $\mathbb{A}$ we have

$$\|\mathbb{A}f\|_{L^2(\mathbb{G})} = \|\mathbb{E}^2 f\|_{L^2(\mathbb{G})} \tag{2.15}$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$. Another useful property of the operator $\mathbb{A}$ is that it is Komatsu-non-negative in $L^2(\mathbb{G})$, that is,

$(-\infty, 0)$ is included in the resolvent set $\rho(\mathbb{A})$ of $\mathbb{A}$ and

$$\exists M > 0, \quad \forall \lambda > 0, \quad \|(\lambda + \mathbb{A})^{-1}\|_{L^2(\mathbb{G}) \rightarrow L^2(\mathbb{G})} \leq M\lambda^{-1}.$$

For more details on Komatsu-non-negative operators and their applications in the theory of fractional powers of operators we can refer to⁶⁹ Section A.3.

Let us now discuss this property for the operator $\mathbb{A}$ in the following lemma:

Lemma 2.14 (¹⁸² Ruzhansky-Suragan-Y., Lemma 2.3). *The operator $\mathbb{A} = \mathbb{E}\mathbb{E}^*$ is Komatsu-non-negative in $L^2(\mathbb{G})$:*

$$\|(\lambda + \mathbb{A})^{-1}\|_{L^2(\mathbb{G}) \rightarrow L^2(\mathbb{G})} \leq \left(\lambda^2 + \frac{\lambda Q^2}{2} + \frac{Q^4}{16} \right)^{-2} \quad \text{for all } \lambda > 0. \quad (2.16)$$

Proof of Lemma 2.14. By Proposition 2.13, Part (ii), for $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ we have

$$\begin{aligned} \|(\lambda \mathbb{I} + \mathbb{A})f\|_{L^2(\mathbb{G})}^2 &= \|(\lambda \mathbb{I} - \mathbb{E}(Q\mathbb{I} + \mathbb{E}))f\|_{L^2(\mathbb{G})}^2 \\ &= \lambda^2 \|f\|_{L^2(\mathbb{G})}^2 + \|\mathbb{E}(Q\mathbb{I} + \mathbb{E})f\|_{L^2(\mathbb{G})}^2 - 2\lambda \langle f, (Q\mathbb{E} + \mathbb{E}^2)f \rangle \\ &= \lambda^2 \|f\|_{L^2(\mathbb{G})}^2 + \|\mathbb{E}(Q\mathbb{I} + \mathbb{E})f\|_{L^2(\mathbb{G})}^2 + 2\lambda \langle f, \mathbb{E}^*\mathbb{E}f \rangle \\ &= \lambda^2 \|f\|_{L^2(\mathbb{G})}^2 + \|\mathbb{E}(Q\mathbb{I} + \mathbb{E})f\|_{L^2(\mathbb{G})}^2 + 2\lambda \|\mathbb{E}f\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (2.17)$$

On the other hand, using the polar decomposition (2.8) and Hölder's inequality, we observe that

$$\begin{aligned} \int_{\mathbb{G}} |f(x)|^2 dx &= \int_0^\infty \int_{\mathbb{S}} |f(ry)|^2 r^{Q-1} d\sigma(y) dr \\ &= -\frac{2}{Q} \int_0^\infty r^Q \operatorname{Re} \int_{\mathbb{S}} f(ry) \frac{\overline{df(ry)}}{dr} d\sigma(y) dr = -\frac{2}{Q} \operatorname{Re} \int_{\mathbb{G}} f(x) \overline{\mathbb{E}f(x)} dx \\ &\leq \frac{2}{Q} \left(\int_{\mathbb{G}} |f(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{G}} |\mathbb{E}f(x)|^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

which implies

$$\frac{Q}{2} \|f\|_{L^2(\mathbb{G})} \leq \|\mathbb{E}f\|_{L^2(\mathbb{G})}. \quad (2.18)$$

Using this and taking into account (2.15), from (2.17) we obtain

$$\begin{aligned} \|(\lambda \mathbb{I} - \mathbb{E}(Q\mathbb{I} + \mathbb{E}))f\|_{L^2(\mathbb{G})}^2 &= \lambda^2 \|f\|_{L^2(\mathbb{G})}^2 + 2\lambda \|\mathbb{E}f\|_{L^2(\mathbb{G})}^2 + \|\mathbb{E}(Q\mathbb{I} + \mathbb{E})f\|_{L^2(\mathbb{G})}^2 \\ &\geq \left(\lambda^2 + 2\lambda \frac{Q^2}{4} + \frac{Q^4}{16} \right) \|f\|_{L^2(\mathbb{G})}^2, \end{aligned} \quad (2.19)$$

yielding (2.16). $\square$

In the next proposition we want to show another useful property of the Euler operator. Sometimes, in order to prove certain inequalities on homogeneous group $\mathbb{G}$ it is sufficient to prove them only for radially symmetric functions. For instance, we will use such ideas in Theorems 3.10 and 3.38. As usual, a function $f = f(x)$ on $\mathbb{G}$ is called radial (or radially symmetric) if it depends only on $|x|$.

Proposition 2.15. [¹⁸³ Ruzhansky-Suragan-Y., Lemma 2.2] *Let $1 < p < \infty$ and let $\{\phi_i\}_{i=1}^3 \in L_{loc}^1(\mathbb{G})$ be any radial functions. If we denote*

$$\tilde{f}(r) := \left(\frac{1}{|\mathbb{G}|} \int_{\mathbb{G}} |f(ry)|^p d\sigma(y) \right)^{1/p}, \quad (2.20)$$

then we have

$$\int_{\mathbb{G}} \phi_1(x) \left| \tilde{f}(|x|) \right|^p dx = \int_{\mathbb{G}} \phi_1(x) |f(x)|^p dx \quad (2.21)$$

for any $f \in L_{loc}^p(\mathbb{G})$. Moreover, we have

$$\int_{\mathbb{G}} \phi_2(x) \left| \mathbb{E}^k \tilde{f}(|x|) \right|^p dx \leq \int_{\mathbb{G}} \phi_2(x) |\mathbb{E}^k f(x)|^p dx \quad (2.22)$$

and

$$\int_{\mathbb{G}} \phi_3(x) \left| \mathcal{R}_{|x|}^k \tilde{f}(|x|) \right|^p dx \leq \int_{\mathbb{G}} \phi_3(x) |\mathcal{R}_{|x|}^k f(x)|^p dx \quad (2.23)$$

for any $k \in \mathbb{N}$ and all $f \in L_{loc}^p(\mathbb{G})$ such that $\mathbb{E}^k f \in L_{loc}^p(\mathbb{G})$ with sharp constants. The constants are attained when $f = \tilde{f}$.

Proof of Proposition 2.15. Using the representation (2.20), we obtain

$$\begin{aligned} \int_{\mathbb{G}} |\tilde{f}(|x|)|^p \phi_1(x) dx &= |\mathfrak{S}| \int_0^\infty |\tilde{f}(r)|^p \phi_1(r) r^{Q-1} dr \\ &= |\mathfrak{S}| \int_0^\infty \frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |f(ry)|^p d\sigma(y) \phi_1(r) r^{Q-1} dr = \int_{\mathbb{G}} |f(x)|^p \phi_1(x) dx, \end{aligned} \quad (2.24)$$

which is (2.21).

Now we show (2.22). For this, let us first prove the inequality

$$|\mathbb{E}^k \tilde{f}| \leq \left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |\mathbb{E}^k f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}} \quad (2.25)$$

for any $k \in \mathbb{N}$ by the induction. So, we need to check this for $k = 1$. Using Hölder's inequality, we get

$$\begin{aligned} |\mathbb{E} \tilde{f}| &= r \left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}-1} \frac{1}{|\mathfrak{S}|} \left| \int_{\mathfrak{S}} |f(ry)|^{p-2} f(ry) \overline{\frac{d}{dr} f(ry)} d\sigma(y) \right| \\ &\leq \left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}-1} \frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |f(ry)|^{p-1} \left| r \frac{d}{dr} f(ry) \right| d\sigma(y) \\ &\leq \left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}-1} \frac{1}{|\mathfrak{S}|} \left(\int_{\mathfrak{S}} \left| r \frac{d}{dr} f(ry) \right|^p d\sigma(y) \right)^{\frac{1}{p}} \\ &\quad \times \left(\int_{\mathfrak{S}} |f(ry)|^p d\sigma(y) \right)^{\frac{p-1}{p}} = \left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |\mathbb{E} f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}}. \end{aligned}$$

Now assuming that

$$|\mathbb{E}^\ell \tilde{f}| \leq \left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |\mathbb{E}^\ell f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}} \quad (2.26)$$

holds for $\ell \in \mathbb{N}$, we prove

$$|\mathbb{E}^{\ell+1} \tilde{f}| \leq \left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |\mathbb{E}^{\ell+1} f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}}.$$

Then, using (2.26) we calculate

$$\begin{aligned}
|\mathbb{E}^{\ell+1} \tilde{f}(r)| &= |\mathbb{E}(\mathbb{E}^\ell \tilde{f}(r))| \leq \left| \mathbb{E} \left(\left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |\mathbb{E}^\ell f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}} \right) \right| \\
&= r \left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |\mathbb{E}^\ell f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}-1} \\
&\quad \times \left| \frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |\mathbb{E}^\ell f(ry)|^{p-2} \mathbb{E} f(ry) \overline{\frac{d}{dr}(\mathbb{E}^\ell f(ry))} d\sigma(y) \right| \\
&\leq \left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |\mathbb{E}^\ell f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}-1} \left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |\mathbb{E}^\ell f(ry)|^{p-1} |\mathbb{E}^{\ell+1} f(ry)| d\sigma(y) \right) \\
&\leq \left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |\mathbb{E}^\ell f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}-1} \frac{1}{|\mathfrak{S}|} \left(\int_{\mathfrak{S}} |\mathbb{E}^{\ell+1} f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{\mathfrak{S}} |\mathbb{E}^\ell f(ry)|^p d\sigma(y) \right)^{\frac{p-1}{p}} = \left(\frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |\mathbb{E}^{\ell+1} f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}},
\end{aligned}$$

where we have used Hölder's inequality in the last line. Thus, we have proved (2.25).

Now using (2.25), we obtain

$$\begin{aligned}
\int_{\mathbb{G}} |\mathbb{E}^k \tilde{f}(x)|^p \phi_2(x) dx &= |\mathfrak{S}| \int_0^\infty |\mathbb{E}^k \tilde{f}(r)|^p \phi_2(r) r^{Q-1} dr \\
&\leq |\mathfrak{S}| \int_0^\infty \frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} |\mathbb{E}^k f(ry)|^p \phi_2(r) r^{Q-1} d\sigma(y) dr \\
&= \int_{\mathbb{G}} |\mathbb{E}^k f(x)|^p \phi_2(x) dx.
\end{aligned}$$

Thus, we have proved (2.22) for any $k \in \mathbb{N}$. Similarly, one can obtain (2.23). $\square$

2.2 EULER SEMIGROUP

In this section we introduce the operator semigroup $\{e^{-t\mathbb{E}^*\mathbb{E}}\}_{t>0}$ associated with the Euler operator, which extends the results of¹⁵ from the Euclidean setting $\mathbb{R}^n$ to general homogeneous groups.

First, let us prepare some preliminary results. We define the map $F : L^2(\mathbb{G}) \rightarrow L^2(\mathbb{R} \times \mathfrak{S})$ as

$$(Ff)(s, y) := e^{sQ/2}f(e^s y) \quad (2.27)$$

for $y \in \mathfrak{S}$ and $s \in \mathbb{R}$, and its inverse $F^{-1} : L^2(\mathbb{R} \times \mathfrak{S}) \rightarrow L^2(\mathbb{G})$ as

$$(F^{-1}g)(x) := r^{-Q/2}g(\ln r, y). \quad (2.28)$$

We note that F preserves the L^2 norm.

We will also use the dilations $U(t) : L^2(\mathbb{G}) \rightarrow L^2(\mathbb{G})$ given by

$$U(t)f(x) := e^{tQ/2}f(e^t x). \quad (2.29)$$

These form a group of unitary operators with generator $U(t) = e^{i\mathbb{B}t}$, where $\mathbb{B}$ is given by

$$\mathbb{B}f = \frac{1}{i} \frac{d}{dt} U(t)f|_{t=0} = \frac{1}{i} \left(\mathbb{E} + \frac{Q}{2} \right) f = -i\mathbb{E}f - i\frac{Q}{2}f, \quad (2.30)$$

where we have used (2.12). Using Proposition 2.13, Part (ii), we obtain from (2.30) that

$$\mathbb{B} = \mathbb{B}^* = -i\mathbb{E} - i\frac{Q}{2}, \quad (2.31)$$

which implies

$$\mathbb{E}^*\mathbb{E} = \left(-i\mathbb{B} - \frac{Q}{2} \right) \left(i\mathbb{B} - \frac{Q}{2} \right) = \mathbb{B}^2 + \frac{Q^2}{4}. \quad (2.32)$$

By (2.27) and (2.29), we have $(Ff)(s, y) = (U(s)f)(y)$ for $y \in \mathfrak{S}$ and $s \in \mathbb{R}$,

then this with the group property of the dilations $U(\cdot)$ gives that

$$(F(U(t)f))(s, y) = (U(s)(U(t)f))(y) = (U(s+t)f)(y) = (Ff)(s+t, y). \quad (2.33)$$

If we define $M : L^2(\mathbb{G}) \rightarrow L^2(\mathbb{R} \times \mathfrak{S})$ as

$$(Mf)(\tau, y) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-is\tau} (Ff)(s, y) ds, \quad (2.34)$$

then it follows from (2.33) and the change of variables that

$$\begin{aligned} (MU(t)f)(\tau, y) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-is\tau} (Ff)(s+t, y) ds \\ &= \frac{e^{it\tau}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-is\tau} (Ff)(s, y) ds \\ &= e^{it\tau} (Mf)(\tau, y). \end{aligned} \quad (2.35)$$

We note that the map $M = \mathcal{F} \circ F$ is the Mellin transformation, where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}$. The map M has an explicit representation using the group structure of $\mathbb{R}^+$ under multiplication: it is the Fourier transform on this group.

Now we are ready to give the representation of the operator semigroup $\{e^{-t\mathbb{E}^*\mathbb{E}}\}_{t>0}$ on homogeneous groups.

Theorem 2.16. [¹⁸³ Ruzhansky-Suragan-Y., Theorem 3.1] *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $x = ry$ and $r = |x|$ with $y \in \mathfrak{S}$. Then the semigroup $e^{-t\mathbb{E}^*\mathbb{E}}$ is given by*

$$\begin{aligned} (e^{-t\mathbb{E}^*\mathbb{E}}f)(x) &= \frac{e^{-tQ^2/4}}{\sqrt{4\pi t}} r^{-Q/2} \int_0^\infty e^{-\frac{(\ln r - \ln s)^2}{4t}} s^{-Q/2} f(sy) s^{Q-1} ds \\ &= \frac{e^{-tQ^2/4}}{\sqrt{4\pi t}} |x|^{-Q/2} \int_0^\infty e^{-\frac{(\ln |x| - \ln s)^2}{4t}} s^{-Q/2} f(sy) s^{Q-1} ds. \end{aligned} \quad (2.36)$$

In section 5.0.4 we use the Euler semigroup $\{e^{-t\mathbb{E}^*\mathbb{E}}\}_{t>0}$ to define Besov type spaces, and to prove Gagliardo-Nirenberg type inequality in these spaces.

Proof of Theorem 2.16. Let us first show that

$$(M\mathbb{B}f)(\tau, y) = \tau(Mf)(\tau, y) \quad (2.37)$$

for f in the domain $\mathcal{D}(\mathbb{B})$, which implies the fact that $f \in \mathcal{D}(\mathbb{B}) \Leftrightarrow (\tau, y) \mapsto \tau(Mf)(\tau, y) \in L^2(\mathbb{R} \times \mathfrak{S})$. Noting $i\mathbb{B}e^{it\mathbb{B}} = \partial_t U(t)$ we calculate

$$(Mi\mathbb{B}e^{i\mathbb{B}t}f)(\tau, y) = (M\partial_t U(t)f)(\tau, y) = \partial_t(MU(t)f)(\tau, y).$$

Now using (2.35) we get from above that

$$(Mi\mathbb{B}e^{i\mathbb{B}t}f)(\tau, y) = \partial_t e^{it\tau}(Mf)(\tau, y) = i\tau e^{it\tau}(Mf)(\tau, y), \quad (2.38)$$

which implies (2.37) after setting $t = 0$.

Now we prove that

$$(Me^{-t\mathbb{B}^2}f)(\tau, y) = e^{-t\tau^2}(Mf)(\tau, y). \quad (2.39)$$

To obtain (2.39), we write

$$(Me^{-t\mathbb{B}^2}f)(\tau, y) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} (M\mathbb{B}^{2k}f)(\tau, y). \quad (2.40)$$

On the other hand, by iteration we have from (2.37) that

$$(M\mathbb{B}^{2k}f)(\tau, y) = \tau^{2k}(Mf)(\tau, y), \quad k = 0, 1, 2, \dots$$

Putting this in (2.40), we obtain

$$(Me^{-t\mathbb{B}^2}f)(\tau, y) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \tau^{2k}(Mf)(\tau, y) = e^{-t\tau^2}(Mf)(\tau, y).$$

Thus, we have obtained (2.39). Then, it implies that

$$e^{-t\mathbb{B}^2} = M^{-1}e^{-t\tau^2}M.$$

Here using $M = \mathcal{F} \circ F$, one has

$$e^{-t\mathbb{B}^2} = F^{-1} \circ \mathcal{F}^{-1}(e^{-t\tau^2} \mathcal{F} \circ F). \quad (2.41)$$

Then, a direct calculation gives that

$$\begin{aligned} \mathcal{F}^{-1}(e^{-t\tau^2} Mf)(\lambda, y) &= \mathcal{F}^{-1}(e^{-t\tau^2} \mathcal{F} \circ F)(\lambda, y) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda\tau} e^{-t\tau^2} e^{-is\tau} (Ff)(s, y) ds d\tau \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-t\tau^2 + i(\lambda-s)\tau} d\tau \right) (Ff)(s, y) ds \\ &= \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(\lambda-s)^2}{4t}} (Ff)(s, y) ds =: \varphi_t(\lambda, y), \end{aligned}$$

where we have used the property of the Gaussian integral in the last line

$$\int_{\mathbb{R}} e^{-t\tau^2 + i(\lambda-s)\tau} d\tau = \sqrt{\frac{\pi}{t}} e^{-\frac{(\lambda-s)^2}{4t}}.$$

From this and (2.41), using (2.27), (2.28) and $M = \mathcal{F} \circ F$ with $x = ry$ we compute

$$\begin{aligned} (e^{-t\mathbb{B}^2} f)(ry) &= (F^{-1} \varphi_t)(ry) \\ &= r^{-Q/2} \varphi_t(\ln r, y) \\ &= \frac{1}{\sqrt{4\pi t}} r^{-Q/2} \int_{\mathbb{R}} e^{-\frac{(\ln r - s)^2}{4t}} (Ff)(s, y) ds \\ &= \frac{1}{\sqrt{4\pi t}} r^{-Q/2} \int_0^\infty e^{-\frac{(\ln r - \ln z)^2}{4t}} z^{\frac{Q}{2}-1} f(z) dz, \end{aligned}$$

where we have used the change of variables $z = e^s$ in the last line.

Since we have $e^{-t\mathbb{E}^*\mathbb{E}} = e^{-tQ^2/4} e^{-t\mathbb{B}^2}$ by (2.32), we obtain from above that

$$\begin{aligned} (e^{-t\mathbb{E}^*\mathbb{E}} f)(ry) &= e^{-tQ^2/4} (e^{-t\mathbb{B}^2} f)(ry) \\ &= \frac{1}{\sqrt{4\pi t}} r^{-Q/2} e^{-tQ^2/4} \int_0^\infty e^{-\frac{(\ln r - \ln z)^2}{4t}} z^{\frac{Q}{2}-1} f(z) dz \\ &= \frac{1}{\sqrt{4\pi t}} r^{-Q/2} e^{-tQ^2/4} \int_0^\infty e^{-\frac{(\ln r - \ln z)^2}{4t}} z^{-\frac{Q}{2}} f(z) z^{Q-1} dz, \end{aligned}$$

yielding (2.36). □

Remark 2.17. One can also define F as a unitary operator from $L^2(\mathbb{G})$ to $L^2(\mathbb{R} \times \mathfrak{S})$ that intertwines $-i(\mathbb{E} + Q/2)$ on $\mathbb{G}$ and $-id/dt$ in the $\mathbb{R}$ -variable on $\mathbb{R} \times \mathfrak{S}$, which would be essentially self-adjoint on $C_0^\infty(\mathbb{R} \times \mathfrak{S})$ and $C_0^\infty(\mathbb{G} \setminus \{0\})$, respectively.

2.3 GRADED LIE GROUPS

Graded Lie groups form the subclass of homogeneous nilpotent Lie groups admitting homogeneous hypoelliptic left-invariant differential operators (^{144, 212}, see also a discussion in ⁶⁹ Section 4.1). These operators are called Rockland operators from the Rockland conjecture, solved by Helffer and Nourrigat¹²¹.

We also note that the Rockland operators on graded Lie groups appear naturally in the analysis of subelliptic operators on manifolds, starting with the seminal paper of Rothschild and Stein¹⁶⁹.

Thus, let $\mathbb{G}$ be a graded Lie group, i.e. a connected simply connected Lie group such that its Lie algebra $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \bigoplus_{\ell=1}^{\infty} \mathfrak{g}_\ell,$$

where the $\mathfrak{g}_\ell$, $\ell = 1, 2, \dots$, are vector subspaces of $\mathfrak{g}$, all but finitely many equal to $\{0\}$, and satisfying

$$[\mathfrak{g}_\ell, \mathfrak{g}_{\ell'}] \subset \mathfrak{g}_{\ell+\ell'} \quad \forall \ell, \ell' \in \mathbb{N}.$$

Such groups are then necessarily nilpotent and homogeneous, they can be identified with $\mathbb{R}^n$ through the exponential mapping, with n being the topological dimension of $\mathbb{G}$, with the Haar measure on $\mathbb{G}$ given by the Lebesgue measure on $\mathbb{R}^n$ through this identification.

Let $\widehat{\mathbb{G}}$ be a unitary dual of $\mathbb{G}$ and let $\mathcal{H}_\pi^\infty$ be the space of smooth vectors for a representation $\pi \in \widehat{\mathbb{G}}$. A Rockland operator $\mathcal{R}$ on $\mathbb{G}$ is a left-invariant dif-

ferential operator which is homogeneous of positive degree and satisfies the condition:

(Rockland condition) for each representation $\pi \in \widehat{\mathbb{G}}$, except for the trivial representation, the operator $\pi(\mathcal{R})$ is injective on $\mathcal{H}_\pi^\infty$, i.e.,

$$\forall v \in \mathcal{H}_\pi^\infty, \quad \pi(\mathcal{R})v = 0 \Rightarrow v = 0.$$

Here $\pi(\mathcal{R}) := d\pi(\mathcal{R})$ is the infinitesimal representation of the Rockland operator $\mathcal{R}$ as of an element of the universal enveloping algebra of $\mathbb{G}$.

Different characterisations of such operators have been obtained by Rockland¹⁶⁶ and Beals²¹. For an extensive presentation about Rockland operators and for the theory of Sobolev spaces on graded groups we refer to⁷⁰ and⁶⁹ Chapter 4, and for the Besov spaces on graded Lie groups we refer to⁴⁰.

Since we will not be using the representation theoretic interpretation of these operators in this thesis, we define *Rockland operators as left-invariant homogeneous hypoelliptic differential operators on $\mathbb{G}$* . This is equivalent to the Rockland condition as it was shown by Helffer and Nourrigat in¹²¹.

The Rockland operators operators always exist. For example, for the Heisenberg group, the sub-Laplacian and its powers are Rockland operators. If $\mathbb{G}$ is a stratified Lie group with a basis $X_1, \dots, X_k \in \mathfrak{g}_1$ of $\mathfrak{g}_1$, the operators

$$\mathcal{R} = (-1)^m \sum_{j=1}^k a_j X_j^{2m}, \quad a_j > 0,$$

are positive Rockland operators for any $m \in \mathbb{N}$, yielding the sub-Laplacian for $m = 1$. More generally, for any graded Lie group $\mathbb{G} \sim \mathbb{R}^n$ with dilation weights $\nu_1, \dots, \nu_n$ and a basis $X_1, \dots, X_n$ of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ satisfying

$$D_r X_j = r^{\nu_j} X_j, \quad j = 1, \dots, n, \quad r > 0,$$

the operator

$$\mathcal{R} = \sum_{j=1}^n (-1)^{\frac{\nu_0}{\nu_j}} a_j X_j^{\frac{2\nu_0}{\nu_j}}, \quad a_j > 0, \quad (2.42)$$

is a Rockland operator of homogeneous degree $2\nu_0$, if ν_0 is any common multiple of $\nu_1, \dots, \nu_n$. There are other examples of Rockland operators that can be adapted to special selections of vector fields generating the Lie algebra in special ways, such as for example the vector fields from the first stratum on the stratified Lie groups. We refer to⁶⁹ Section 4.1.2 for other examples and a detailed discussion of Rockland operators.

Sobolev spaces associated to positive Rockland operators on graded Lie groups have been analysed in⁷⁰ and in⁶⁹ Section 4.4. In particular, for a positive Rockland operator of homogeneous degree $\nu \in \mathbb{N}$, for $a > 0$ and $1 < p < \infty$, we can define the homogeneous and inhomogeneous Sobolev spaces, respectively, by the norms

$$\|u\|_{\dot{L}_{a,\mathcal{R}}^p(\mathbb{G})} := \|\mathcal{R}^{\frac{a}{\nu}} u\|_{L^p(\mathbb{G})} \quad \text{and} \quad \|u\|_{L_{a,\mathcal{R}}^p(\mathbb{G})} := (\|\mathcal{R}^{\frac{a}{\nu}} u\|_{L^p(\mathbb{G})}^p + \|u\|_{L^p(\mathbb{G})}^p)^{\frac{1}{p}}. \quad (2.43)$$

If $\mathbb{G} = \mathbb{R}^n$ has isotropic dilations, and $\mathcal{R}$ is a homogeneous elliptic operator with constant coefficients, these spaces coincide with the usual Sobolev spaces on $\mathbb{R}^n$. If $\mathbb{G}$ is a stratified Lie group and $\mathcal{R}$ is a positive sub-Laplacian, then these Sobolev spaces have been analysed by Folland in⁷¹. For an extensive analysis of these Sobolev spaces on general graded Lie group we refer to⁷⁰ or⁶⁹ Section 4.4. In particular, it was shown that these spaces are independent of the choice of a positive operator $\mathcal{R}$, so that we can drop the subscript $\mathcal{R}$ in their notation, abbreviating them to $\dot{L}_a^p$ and L_a^p , respectively. While these spaces are independent of $\mathcal{R}$, the particular norms in (2.43) depend on it.

The operators $\mathcal{R}^{-a/\nu}$ for $\{a \in \mathbb{R}, 0 < a < Q\}$ and $(I + \mathcal{R})^{-a/\nu}$ for $a \in \mathbb{R}_+$ are called Riesz and Bessel potentials, respectively. If we denote their kernels by $\mathcal{I}_a$ and $\mathcal{B}_a$, then we have

$$\mathcal{I}_a(x) := \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^\infty t^{\frac{a}{\nu}-1} h_t(x) dt \quad (2.44)$$

for $0 < a < Q$ with $a \in \mathbb{R}$, and

$$\mathcal{B}_a(x) := \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^\infty t^{\frac{a}{\nu}-1} e^{-t} h_t(x) dt \quad (2.45)$$

for $a > 0$, where Γ denotes the Gamma function, and h_t is the heat kernel associated to the positive Rockland operator $\mathcal{R}$. We refer to⁶⁹ Section 4.3.4 for more details.

Before using $\mathcal{I}_a(x)$ and $\mathcal{B}_a(x)$, we recall the following results:

Theorem 2.18 (⁶⁹ Ruzhansky-Fischer, Theorem 4.2.7). *Let $\mathcal{R}$ be a positive Rockland operator on a graded Lie group $\mathbb{G}$. Let $|\cdot|$ be a fixed homogeneous quasi-norm. Let h_t be a heat kernel associated with the Rockland operator. Then each h_t is Schwartz and we have*

$$\forall s, t > 0 \quad h_t * h_s = h_{t+s}, \quad (2.46)$$

$$\forall x \in \mathbb{G}, r, t > 0 \quad h_{r^\nu t}(rx) = r^{-Q} h_t(x), \quad (2.47)$$

$$\forall x \in \mathbb{G} \quad h_t(x) = \overline{h_t(x^{-1})}, \quad (2.48)$$

$$\int_{\mathbb{G}} h_t(x) dx = 1. \quad (2.49)$$

Moreover, we have

$$\exists C = C_{\alpha, N, \ell} > 0 \quad \forall t \in (0, 1] \quad \sup_{|x|=1} |\partial_t^\ell X^\alpha h_t(x)| \leq C_{\alpha, N} t^N \quad (2.50)$$

for any $N \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^n$ and $\ell \in \mathbb{N}_0$.

Lemma 2.19 (⁶⁹ Ruzhansky-Fischer, Lemma 4.3.8). *Let $\mathcal{R}$ be a positive Rockland operator on graded group $\mathbb{G}$ and let h_t be its heat kernel as in Theorem 2.18. Let $|\cdot|$ be a homogeneous quasi-norm and $\alpha \in \mathbb{N}_0^n$ be a multi-index. Then for any real number a with $0 < a < (Q + [\alpha])/\nu$ there exists a positive constant C such that*

$$\int_0^\infty t^{a-1} |X^\alpha h_t(x)| dt \leq C |x|^{-Q-[\alpha]+\nu a}. \quad (2.51)$$

Replacing a by a/ν and putting $\alpha = 0$ in Lemma 2.19, and using the representation (2.44) for $\mathcal{I}_a(x)$, we obtain

Lemma 2.20 (¹⁹³ Ruzhansky-Y., Lemma 2.4). *Let $|\cdot|$ be a homogeneous quasi-norm. Let $0 < a < Q$ and $a \in \mathbb{R}$. Then there exists a positive constant $C = C(Q, a)$ such that*

$$|\mathcal{I}_a(x)| \leq C|x|^{-(Q-a)}. \quad (2.52)$$

Now let us prove the following useful lemma for $\mathcal{B}_a$, which may be not optimal, but sufficient for our purposes.

Lemma 2.21 (¹⁹³ Ruzhansky-Y., Lemma 2.5). *Let $|\cdot|$ be a homogeneous quasi-norm. Let $0 < a < Q$ and $a \in \mathbb{R}$. Then there exists a positive constant $C = C(Q, a)$ such that*

$$|\mathcal{B}_a(x)| \leq \begin{cases} C|x|^{-(Q-a)}, & \text{for } x \in \mathbb{G} \setminus \{0\}, \\ C|x|^{-Q}, & \text{for } x \in \mathbb{G} \text{ with } |x| \geq 1. \end{cases} \quad (2.53)$$

Proof of Lemma 2.21. We split the integral in (2.53) as follows

$$\begin{aligned} \mathcal{B}_a(x) &= \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^\infty t^{\frac{a}{\nu}-1} e^{-t} h_t(x) dt \\ &= \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^{|x|^\nu} t^{\frac{a}{\nu}-1} e^{-t} h_t(x) dt + \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_{|x|^\nu}^\infty t^{\frac{a}{\nu}-1} e^{-t} h_t(x) dt \\ &=: J_1 + J_2. \end{aligned} \quad (2.54)$$

To estimate J_1 using the property of homogeneity of h_t in (2.47), we calculate

$$\begin{aligned} |J_1| &= \left| \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^{|x|^\nu} t^{\frac{a}{\nu}-1} e^{-t} |x|^{-Q} h_{|x|^{-\nu}t} \left(\frac{x}{|x|} \right) dt \right| \\ &\leq \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} |x|^{-Q} \left(\sup_{|y|=1, 0 \leq t_1 \leq 1} |h_{t_1}(y)| \right) \int_0^{|x|^\nu} t^{\frac{a}{\nu}-1} dt \\ &= \frac{\nu}{a\Gamma\left(\frac{a}{\nu}\right)} |x|^{a-Q} \left(\sup_{|y|=1, 0 \leq t_1 \leq 1} |h_{t_1}(y)| \right) \\ &\leq C|x|^{a-Q}, \end{aligned} \quad (2.55)$$

where we have used that $\sup_{|y|=1, 0 \leq t_1 \leq 1} |h_{t_1}(y)|$ is finite by (2.50).

Now we estimate J_2 . A direct calculation gives that

$$\begin{aligned}
|J_2| &= \left| \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_{|x|^\nu}^\infty t^{\frac{a}{\nu}-1} e^{-t} h_t(x) dt \right| \\
&\leq \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_{|x|^\nu}^\infty t^{\frac{a}{\nu}-1} t^{-\frac{Q}{\nu}} |h_1(t^{-\frac{1}{\nu}}x)| dt \\
&\leq \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \|h_1\|_{L^\infty(\mathbb{G})} \int_{|x|^\nu}^\infty t^{\frac{a}{\nu}-1-\frac{Q}{\nu}} dt \\
&\leq C|x|^{a-Q},
\end{aligned} \tag{2.56}$$

where we have used that $\|h_1\|_{L^\infty(\mathbb{G})}$ is finite since h_1 is Schwartz. Combining (2.54), (2.55) and (2.56), we obtain (2.53).

On the other hand, when $|x| \geq 1$, one has for J_1

$$\begin{aligned}
|J_1| &= \left| \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_0^{|x|^\nu} t^{\frac{a}{\nu}-1} e^{-t} |x|^{-Q} h_{|x|^{-\nu}t}\left(\frac{x}{|x|}\right) dt \right| \\
&\leq \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} |x|^{-Q} \left(\sup_{|y|=1, 0 \leq t_1 \leq 1} |h_{t_1}(y)| \right) \int_0^{|x|^\nu} t^{\frac{a}{\nu}-1} e^{-t} dt \\
&\leq |x|^{-Q} \left(\sup_{|y|=1, 0 \leq t_1 \leq 1} |h_{t_1}(y)| \right) \\
&\leq C|x|^{-Q}.
\end{aligned} \tag{2.57}$$

It remains to estimate J_2 for $|x| \geq 1$. By a direct calculation, we obtain

$$\begin{aligned}
|J_2| &= \left| \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_{|x|^\nu}^\infty t^{\frac{a}{\nu}-1} e^{-t} h_t(x) dt \right| \\
&= \left| \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} \int_{|x|^\nu}^\infty t^{\frac{a}{\nu}-1-\frac{Q}{\nu}} e^{-t} h_1(t^{-\frac{1}{\nu}}x) dt \right| \\
&\leq \frac{1}{\Gamma\left(\frac{a}{\nu}\right)} |x|^{-Q} \|h_1\|_{L^\infty(\mathbb{G})} \Gamma\left(\frac{a}{\nu}\right) \\
&\leq C|x|^{-Q}.
\end{aligned} \tag{2.58}$$

Combining (2.54), (2.57) and (2.58), we obtain (2.53) for $|x| \geq 1$. $\square$

2.4 STRATIFIED LIE GROUPS

Let us begin with the definition of stratified Lie groups.

Definition 2.22. The triple $\mathbb{G} = (\mathbb{R}^n, \circ, D_\lambda)$ is called a *stratified Lie group* if it satisfies the conditions:

- For some natural numbers $N = N_1, N_2, \dots, N_r$ with $N + N_2 + \dots + N_r = n$, the following decomposition $\mathbb{R}^n = \mathbb{R}^N \times \dots \times \mathbb{R}^{N_r}$ is valid, and for each $\lambda > 0$ the dilation $D_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by

$$D_\lambda(x) = D_\lambda(x', x^{(2)}, \dots, x^{(r)}) := (\lambda x', \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)})$$

is an automorphism of $\mathbb{G}$. Here $x' \equiv x^{(1)} \in \mathbb{R}^N$ and $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 2, \dots, r$.

- Let N be as in above and let $X_1, \dots, X_N$ be the left invariant vector fields on $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}|_0$ for $k = 1, \dots, N$. Then

$$\text{rank}(\text{Lie}\{X_1, \dots, X_N\}) = n,$$

for each $x \in \mathbb{R}^n$, that is, the iterated commutators of $X_1, \dots, X_N$ span the Lie algebra of $\mathbb{G}$.

Note that the left invariant vector fields $X_1, \dots, X_N$ are called the (Jacobian) generators of the stratified group $\mathbb{G}$ and r is called a step of this stratified group $\mathbb{G}$. For the expressions for left invariant vector fields on $\mathbb{G}$ in terms of the usual (Euclidean) derivatives and further properties see e.g.⁶⁹ Section 3.1.5.

The homogeneous dimension Q of the stratified group $\mathbb{G}$ is then given by

$$Q = \sum_{k=1}^r k N_k, \quad N_1 = N.$$

The differential operator

$$\mathfrak{L} = \sum_{k=1}^N X_k^2, \tag{2.59}$$

is called a sub-Laplacian on the stratified group $\mathbb{G}$. This sub-Laplacian $\mathcal{L}$ is a left invariant homogeneous of order 2 hypoelliptic differential operator and it is known that $\mathcal{L}$ is elliptic if and only if $r = 1$. The left invariant vector fields X_j have an explicit form and satisfy the divergence theorem, which can be found in⁶⁹ Section 3.1.5 and¹⁷⁵:

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{\ell=2}^r \sum_{m=1}^{N_1} a_{k,m}^{(\ell)}(x', \dots, x^{\ell-1}) \frac{\partial}{\partial x_m^{(\ell)}}, \quad (2.60)$$

where $a_{k,m}^{(\ell)}$ is a homogeneous (with respect to δ_λ) polynomial function of degree $\ell - 1$. We denote the horizontal gradient and horizontal divergence by

$$\nabla_H := (X_1, \dots, X_N), \quad (2.61)$$

and

$$\operatorname{div}_H v := \nabla_H \cdot v,$$

respectively. We will also use the notation

$$|x'| = \sqrt{x_1'^2 + \dots + x_N'^2}$$

for the Euclidean norm on $\mathbb{R}^N$.

Since we have the explicit representation of the left invariant vector fields X_j in (2.60), one can readily obtain the identities

$$|\nabla_H |x'|^\gamma| = \gamma |x'|^{\gamma-1}, \quad (2.62)$$

and

$$\operatorname{div}_H \left(\frac{x'}{|x'|^\gamma} \right) = \frac{\sum_{j=1}^N |x'|^\gamma X_j x'_j - \sum_{j=1}^N x'_j \gamma |x'|^{\gamma-1} X_j |x'|}{|x'|^{2\gamma}} = \frac{N - \gamma}{|x'|^\gamma}, \quad (2.63)$$

for all $\gamma \in \mathbb{R}$, $x' \neq 0$.

On a general stratified Lie group $\mathbb{G}$, it is known that the sub-Laplacian $\mathcal{L}$

has the unique fundamental solution u (see⁷¹), that is,

$$-\mathcal{L}u = \delta,$$

where δ stands for the Dirac distribution with singularity at the identity element 0 of $\mathbb{G}$. The homogeneous fundamental solution is strictly positive by the maximum principle proved by Bony²⁶. Then there exists a homogeneous quasi-norm d ($\mathfrak{L}$ -gauge) such that

$$u = d^{2-Q}.$$

So we set

$$d(x) := \begin{cases} (u(x))^{\frac{1}{2-Q}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases} \quad (2.64)$$

Then, a stratified Lie group (or Carnot group) $\mathbb{G}$ is *polarizable* if the homogeneous norm $d = u^{1/(2-Q)}$ satisfies the following ∞ -sub-Laplace equation (see, e.g.,¹⁷)

$$\mathcal{L}_{\mathbb{G},\infty}d := \frac{1}{2} \langle \nabla_H(|\nabla_H d|^2), \nabla_H d \rangle = 0 \quad \text{in } \mathbb{G} \setminus \{0\}.$$

It follows that (see, e.g.,¹⁷ Proposition 2.20)

$$\mathcal{L}_{\mathbb{G},\infty}u = \frac{Q-1}{Q-2} \frac{|\nabla_H u|^4}{u},$$

that is,

$$\sum_{j=1}^N \frac{u X_j u X_j |\nabla_H u|}{|\nabla_H u|^3} = \frac{Q-1}{Q-2}. \quad (2.65)$$

For example, the Euclidean space, the Heisenberg group and the H-type groups are polarizable Carnot groups.

2.4.1 SUB-LAPLACIANS WITH DRIFT

In this section we discuss some properties of sub-Laplacians with drift. Although such operators can be considered on more general groups, in Chapter 7 we will use the results of this section only in the setting of the stratified groups $\mathbb{G}$.

Let us consider the operator

$$\mathcal{L}_X := - \sum_{i,j=1}^N a_{i,j} X_i X_j - \gamma X, \gamma \in \mathbb{R}, \quad (2.66)$$

on $C_0^\infty(\mathbb{G})$, where the matrix $(a_{i,j})_{i,j=1}^N$ is real, symmetric, positive definite, and $X \in \mathfrak{g}$ is a left-invariant vector field on $\mathbb{G}$. One can reduce (2.66) to the form (see e.g.¹¹⁷ or recent work¹⁴⁰)

$$\mathcal{L}_X = - \sum_{j=1}^N X_j^2 - \gamma X := \mathcal{L}_0 - \gamma X, \quad (2.67)$$

where

$$\mathcal{L}_0 = - \sum_{j=1}^N X_j^2. \quad (2.68)$$

For $X = \sum_{j=1}^N a_j X_j$ with $a_j \in \mathbb{R}$ let us denote $\|X\| := \left(\sum_{j=1}^N a_j^2 \right)^{1/2}$ and

$$b_X := \frac{\|X\|}{2}. \quad (2.69)$$

Then, $\mathcal{L}_X$ with domain $C_0^\infty(\mathbb{G})$ has the following properties:

Proposition 2.23 (¹⁹⁴ Ruzhansky-Y., Proposition 2.1). *Let $\mathbb{G}$ be a connected Lie group, and let $X_1, \dots, X_N$ be an algebraic basis of the corresponding Lie algebra $\mathfrak{g}$. Let $\gamma \in \mathbb{R}$ and $X \in \mathfrak{g} \setminus \{0\}$. Then*

- *The operator $\mathcal{L}_X$ is symmetric on $L^2(\mathbb{G}, \mu)$ for some positive measure μ on $\mathbb{G}$ if and only if there are positive character χ of $\mathbb{G}$ and constant C such that $\mu = C\mu_X$ and $\nabla_H \chi|_e = \gamma X|_e$, where μ_X is the measure absolutely*

continuous with respect to the Haar measure μ with density χ ;

- If we have $\nabla_H \chi|_e = \gamma X|_e$ for some positive character χ of $\mathbb{G}$, then $\mathcal{L}_X$ is essentially self-adjoint on $L^2(\mathbb{G}, \mu_X)$ and its spectrum is contained in $[\gamma^2 b_X^2, \infty)$.

Proof of Proposition 2.23. This proposition can be proven in the exact same way as the case $\gamma = 1$, see¹¹⁷ Proposition 3.1. So, let us briefly give a sketch of the proof. Let μ be a positive measure on $\mathbb{G}$. Then, we have for all test functions $\phi, \psi \in C_0^\infty(\mathbb{G})$ that

$$\begin{aligned}
\int_{\mathbb{G}} (\mathfrak{L}_X \phi) \psi d\mu &= - \sum_{j=1}^N \left(\int_{\mathbb{G}} (X_j^2 \phi) \psi d\mu \right) - \gamma \int_{\mathbb{G}} \psi X \phi d\mu \\
&= \sum_{j=1}^N \left(\int_{\mathbb{G}} X_j \phi X_j \psi d\mu + \int_{\mathbb{G}} \psi X_j \phi X_j \mu \right) + \gamma \int_{\mathbb{G}} \phi X \psi d\mu + \gamma \int_{\mathbb{G}} \phi \psi X \mu \\
&= \sum_{j=1}^N \left(- \int_{\mathbb{G}} \phi X_j^2 \psi d\mu - 2 \int_{\mathbb{G}} \phi X_j \psi X_j \mu - \int_{\mathbb{G}} \phi \psi X_j^2 \mu \right) \\
&\quad + \gamma \int_{\mathbb{G}} \phi X \psi d\mu + \gamma \int_{\mathbb{G}} \phi \psi X \mu \\
&= \int_{\mathbb{G}} \phi (\mathfrak{L}_X \psi) d\mu + 2\gamma \int_{\mathbb{G}} \phi X \psi d\mu - 2 \int_{\mathbb{G}} \phi \nabla_H \psi \nabla_H \mu + \int_{\mathbb{G}} \phi \psi (\mathfrak{L}_0 + \gamma X) \mu \\
&= \int_{\mathbb{G}} \phi (\mathfrak{L}_X \psi) d\mu + \langle \phi, 2\gamma(X\psi)\mu - 2\nabla_H \psi \nabla_H \mu + \psi(\mathfrak{L}_0 + \gamma X)\mu \rangle \\
&:= \int_{\mathbb{G}} \phi (\mathfrak{L}_X \psi) d\mu + \mathfrak{I}(\phi, \psi, \mu), \tag{2.70}
\end{aligned}$$

where $\nabla_H = (X_1, \dots, X_N)$ and $\langle \dots, \dots \rangle$ is the pairing between distributions and test functions on $\mathbb{G}$. From (2.70) we note that $\mathfrak{L}_X$ is symmetric on $L^2(\mathbb{G}, \mu)$ if and only if $\mathfrak{I}(\phi, \psi, \mu) = 0$ for all ϕ and ψ , that is,

$$(\mathfrak{L}_0 + \gamma X)\mu = 0, \quad \gamma(X\psi)\mu - \nabla_H \psi \nabla_H \mu = 0, \quad \forall \psi \in C_0^\infty(\mathbb{G}). \tag{2.71}$$

We know that the vector fields $X_1, \dots, X_N$ satisfy Hörmander's condition, so that $\mathfrak{L}_0 + \gamma X$ is hypoelliptic, which implies with the first equation of

(2.71) that μ has a smooth density ω with respect to the Haar measure. Then, as in¹¹⁷ Proof of the Proposition 3.1, one can show that

$$X = \sum_{j=1}^N a_j X_j, \quad (2.72)$$

for some coefficients $a_1, \dots, a_N$. Since $X_1, \dots, X_N$ are linearly independent, then from the second equation of (2.71) we derive that

$$X_k \omega = \gamma a_k \omega, \quad (2.73)$$

where $k = 1, \dots, N$. Similarly to¹¹⁷ Proof of the Proposition 3.1, we find the solution of (2.73)

$$\omega(x) = \omega(e) \exp \left(\gamma \int_0^1 \sum_{k=1}^N a_k \vartheta_k(t) dt \right), \quad (2.74)$$

which is a positive and uniquely determined by its value at the identity, where $\vartheta_k(t)$ is the piecewise C^1 path. We normalise ω so that $\omega(e) = 1$. Then ω is a character of $\mathbb{G}$.

Indeed, it is sufficient to note that for all $y \in \mathbb{G}$ the function $x \mapsto \omega(xy)/\omega(y)$ is a solution of (2.73), and its value at e is 1. Then for all $x, y \in \mathbb{G}$ we have $\omega(xy) = \omega(x)\omega(y)$, so that ω is a character of $\mathbb{G}$. The formulae (2.72) and (2.73) imply $\nabla_H \chi|_e = \gamma X|_e$ with $\chi = \omega$. Thus, we have proved the first part of Proposition 2.23.

As in the case $\gamma = 1$ (see¹¹⁷ Proposition 3.1), we consider the isometry $\mathcal{U}_2 f = \chi^{-1/2} f$ of $L^2(\mathbb{G}, \mu)$ onto $L^2(\mathbb{G}, \mu_X)$ to get

$$\chi^{\frac{1}{2}} \mathfrak{L}_X (\chi^{-\frac{1}{2}} f) = (\mathfrak{L}_0 + \gamma^2 b_X^2) f. \quad (2.75)$$

Since the operator on the right-hand side of (2.75) is essentially a self-adjoint on $L^2(\mathbb{G}, \mu)$, whose spectrum is contained in $[\gamma^2 b_X^2, \infty)$, then by (2.75) we conclude that $\mathfrak{L}_X$ is essentially self-adjoint operator on $L^2(\mathbb{G}, \mu_X)$ with the same spectrum.

□

In Section 7.4, we assume that there is a positive character χ of $\mathbb{G}$ such that

$$\nabla_H \chi|_e = \gamma X|_e.$$

By Proposition 2.23 it means that the operator $\mathcal{L}_X$ is formally self-adjoint with respect to the positive measure $\mu_X = \chi\mu$, where μ is the right Haar measure of $\mathbb{G}$. Thus, in this case we see that $\mathcal{L}_X$ is self-adjoint on $L^2(\mathbb{G}, \mu_X)$. Moreover,

$$L^2(\mathbb{G}, \mu) \ni f \mapsto \chi^{-1/2} f \in L^2(\mathbb{G}, \mu_X) \quad (2.76)$$

is an isometric isomorphism. We refer to^{116, 117} and¹⁴⁰ for a detailed discussion about properties of the sub-Laplacians with drift.

2.4.2 HEISENBERG GROUP

Recall that the Heisenberg group $\mathbb{H}_n$ is a manifold $\mathbb{R}^{2n+1}$ with the group law given by

$$\begin{aligned} (x^{(1)}, y^{(1)}, t^{(1)})(x^{(2)}, y^{(2)}, t^{(2)}) \\ := (x^{(1)} + x^{(2)}, y^{(1)} + y^{(2)}, t^{(1)} + t^{(2)} + \frac{1}{2}(x^{(1)}y^{(2)} - x^{(2)}y^{(1)})), \end{aligned}$$

for $(x^{(1)}, y^{(1)}, t^{(1)}), (x^{(2)}, y^{(2)}, t^{(2)}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \sim \mathbb{H}_n$, where we denote by $x^{(1)}y^{(2)}$ and $x^{(2)}y^{(1)}$ their usual scalar products on $\mathbb{R}^n$. The canonical basis of its Lie algebra $\mathfrak{h}_n$ is given by the left-invariant vector fields

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \quad j = 1, \dots, n, \quad T = \partial_t. \quad (2.77)$$

The canonical commutation relations of the basis $\{X_j, Y_j, T\}$ for $j = 1, \dots, n$ are given by

$$[X_j, Y_j] = T, \quad j = 1, \dots, n,$$

with all the other commutators being zero. We also note that the Heisenberg Lie algebra $\mathfrak{h}_n$ is stratified via $\mathfrak{h}_n = V_1 \oplus V_2$, where V_1 is linearly

spanned by the X_j 's and Y_j 's, and $V_2 = \mathbb{R}T$. Then, the natural dilations on the Heisenberg Lie algebra $\mathfrak{h}_n$ are given by

$$D_r(X_j) = rX_j, \quad D_r(Y_j) = rY_j, \quad D_r(T) = r^2T,$$

and on $\mathbb{H}_n$ by

$$D_r(x, y, t) = r(x, y, t) = (rx, ry, r^2t), \quad (x, y, t) \in \mathbb{H}_n, \quad r > 0.$$

Consequently, $Q = 2n + 2$ is the homogeneous degree of the Lebesgue measure $dx dy dt$ and the homogeneous dimension of the Heisenberg group $\mathbb{H}_n$ as well. The sub-Laplacian is given by

$$\mathfrak{L} := \sum_{j=1}^n (X_j^2 + Y_j^2) = \sum_{j=1}^n \left(\partial_{x_j} - \frac{y_j}{2} \partial_t \right)^2 + \left(\partial_{y_j} + \frac{x_j}{2} \partial_t \right)^2.$$

Let us now simplify and unify the following formulations, which will be useful in Section 7.3.2: we denote by $\tilde{x}'$ the variables of the first stratum

$$\tilde{x}'_j = \begin{cases} x_j, & 1 \leq j \leq n; \\ y_{j-n}, & n+1 \leq j \leq 2n, \end{cases} \quad (2.78)$$

with its dimension being $N = 2n$. So, its norm is

$$|\tilde{x}'| = \sum_{j=1}^N \sqrt{(\tilde{x}'_j)^2} = \sqrt{\sum_{j=1}^n (x_j^2 + y_j^2)}. \quad (2.79)$$

We will also use the notation

$$\nabla_H := (X_1, \dots, X_n, Y_1, \dots, Y_n) \quad (2.80)$$

for the horizontal gradient. For $\mathfrak{L}$ we can also write

$$\mathfrak{L} = \Delta_{\tilde{x}'} + \frac{|\tilde{x}'|^2}{4} \partial_t^2 + Z \partial_t, \quad \text{with } Z = \sum_{j=1}^n (x_j \partial_{y_j} - y_j \partial_{x_j}), \quad (2.81)$$

where $\Delta_{\tilde{x}'}$ is the Euclidean Laplacian with respect to $\tilde{x}'$, and Z is the angular derivative in the $\tilde{x}'$ -variables.

3

Hardy, Rellich, Caffarelli-Kohn-Nirenberg, Sobolev type inequalities on homogeneous Lie groups

THIS chapter is devoted to Hardy and Sobolev type inequalities and the analysis of their remainders in different forms as well as to other related functional inequalities such as Sobolev-Rellich, Caffarelli-Kohn-Nirenberg inequalities. Moreover, we discuss different aspects of this analysis such as their stability, higher-order inequalities, their weighted and extended versions. In addition, Heisenberg-Pauli-Weyl type and logarithmic uncertainty principles are established on homogeneous Lie groups. The analysis in this chapter is based on the joint works with Ruzhansky and Suragan^{180, 178, 182, 183}, and with Laptev and Ruzhansky¹³³, and with Ruzhansky^{193, 187}. In this chapter we adopt the notations from Chapter 2 concerning the homogeneous Lie groups and the operators defined on it.

3.1 HARDY INEQUALITIES

In this section we discuss several weighted versions of the Hardy inequality (1.7) on homogeneous Lie groups. We first establish Hardy type inequalities in L^p , $1 < p < \infty$, with the weights of the form $\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m}$ allowing for different choices of α and β , where $|\cdot|$ can be any homogeneous quasi-norm. Then, using the factorization method we show the Hardy inequalities with more general weights. Moreover, we discuss remainder estimates for L^p -weighted Hardy inequalities on homogeneous Lie groups, which are new in the Euclidean setting of $\mathbb{R}^n$. In addition, we investigate another improved version of L^p -weighted Hardy inequalities involving a distance and stability estimates. Furthermore, we show critical Hardy inequalities of logarithmic type and uncertainty type principles on homogeneous Lie groups. Lastly, we study the relation between critical and the subcritical Hardy inequalities. The obtained estimates will include both the isotropic and anisotropic settings of $\mathbb{R}^n$.

3.1.1 WEIGHTED HARDY INEQUALITIES

The classical Hardy inequalities and their extensions, such as the Caffarelli-Kohn-Nirenberg inequalities, usually involve the weights of the form $\frac{1}{|x|^m}$.

In this section we consider the weights of the form $\frac{(a+b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m}$ allowing for different choices of α and β . If $\alpha = 0$ or $\beta = 0$, this reduces to traditional weights. So, we are interested in the case when $\alpha\beta \neq 0$ and, in fact, we obtain two families of inequalities depending on whether $\alpha\beta > 0$ or $\alpha\beta < 0$:

Theorem 3.1 (¹⁸⁰ Ruzhansky-Suragan-Y., Theorem 8.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $a, b > 0$ and $1 < p < \infty$.*

(i) *If $\alpha\beta > 0$ and $m \leq \frac{Q-p}{p}$, then for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, we have*

$$\frac{Q - pm - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}} f \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} \mathcal{R}_{|x|} f \right\|_{L^p(\mathbb{G})}. \quad (3.1)$$

(ii) If $\alpha\beta < 0$ and $m \leq \frac{Q-p+\alpha\beta}{p}$, then for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, we have

$$\frac{Q - pm + \alpha\beta - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}} f \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} \mathcal{R}_{|x|} f \right\|_{L^p(\mathbb{G})}. \quad (3.2)$$

Directly from the inequalities (3.1) and (3.2), choosing different α, β, a, b, m and Q , one can obtain a number of Hardy type inequalities which have various consequences and applications. For instance, in the Abelian (isotropic or anisotropic) case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $Q = n$, so for any quasi-norm $|\cdot|$ on $\mathbb{R}^n$, all $a, b > 0$ and $1 < p < \infty$ these imply new inequalities. Thus, if $\alpha\beta > 0$ and $pm \leq n - p$, then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, we have

$$\frac{n - pm - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}} f \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} \frac{df}{d|x|} \right\|_{L^p(\mathbb{R}^n)}. \quad (3.3)$$

If $\alpha\beta < 0$ and $pm - \alpha\beta \leq n - p$, then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, we have

$$\frac{n - pm + \alpha\beta - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+1}} f \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} \frac{df}{d|x|} \right\|_{L^p(\mathbb{R}^n)}. \quad (3.4)$$

In the case of the standard Euclidean distance $|x|_E = \sqrt{x_1^2 + \dots + x_n^2}$ by using the Schwartz inequality from the inequalities (3.3) and (3.4) we obtain that if $\alpha\beta > 0$ and $pm \leq n - p$, then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$

$$\frac{n - pm - p}{p} \left\| \frac{(a + b|x|_E^\alpha)^{\frac{\beta}{p}}}{|x|_E^{m+1}} f \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{(a + b|x|_E^\alpha)^{\frac{\beta}{p}}}{|x|_E^m} \nabla f \right\|_{L^p(\mathbb{R}^n)}. \quad (3.5)$$

If $\alpha\beta < 0$ and $pm - \alpha\beta \leq n - p$, then for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, we have

$$\frac{n - pm + \alpha\beta - p}{p} \left\| \frac{(a + b|x|_E^\alpha)^{\frac{\beta}{p}}}{|x|_E^{m+1}} f \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{(a + b|x|_E^\alpha)^{\frac{\beta}{p}}}{|x|_E^m} \nabla f \right\|_{L^p(\mathbb{R}^n)}. \quad (3.6)$$

Note that (3.5) and (3.6) extend the inequalities that have been known for

$p = 2$ for some range of parameters from⁹² by Ghoussoub and Moradifam to the full range of $1 < p < \infty$. We also shall note that these inequalities have interesting applications in theory of ODE (see⁹² Theorem 2.1).

Proof of Theorem 3.1. We may assume that $Q \neq pm + p$ since for $Q = pm + p$ there is nothing to prove. Introducing polar coordinates $(r, y) = (|x|, \frac{x}{|x|}) \in (0, \infty) \times \mathfrak{S}$ on $\mathbb{G}$, where $\mathfrak{S}$ is the unit quasi-sphere (2.7), and using the polar decomposition (2.8), we have

$$\int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^\beta}{|x|^{pm+p}} |f(x)|^p dx = \int_0^\infty \int_{\mathfrak{S}} \frac{(a + br^\alpha)^\beta}{r^{pm+p}} |f(ry)|^p r^{Q-1} d\sigma(y) dr. \quad (3.7)$$

(i) Since $a, b > 0$, $\alpha\beta > 0$ and $m < \frac{Q-p}{p}$ we have

$$\begin{aligned} & \int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^\beta}{|x|^{pm+p}} |f(x)|^p dx \\ & \leq \int_0^\infty \int_{\mathfrak{S}} (a + br^\alpha)^\beta r^{Q-1-pm-p} \left(\frac{\alpha\beta br^\alpha}{(a + br^\alpha)(Q - pm - p)} + 1 \right) |f(ry)|^p d\sigma(y) dr \\ & = \int_0^\infty \int_{\mathfrak{S}} \frac{d}{dr} \left(\frac{(a + br^\alpha)^\beta r^{Q-pm-p}}{Q - pm - p} \right) |f(ry)|^p d\sigma(y) dr \\ & = -\frac{p}{Q - pm - p} \int_0^\infty (a + br^\alpha)^\beta r^{Q-pm-p} \operatorname{Re} \int_{\mathfrak{S}} |f(ry)|^{p-2} f(ry) \overline{\frac{df(ry)}{dr}} d\sigma(y) dr \\ & \leq \left| \frac{p}{Q - pm - p} \right| \int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^\beta |\mathcal{R}_{|x|} f(x)| |f(x)|^{p-1}}{|x|^{pm+p-1}} dx \\ & = \frac{p}{Q - pm - p} \int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^{\frac{\beta(p-1)}{p}} |f(x)|^{p-1}}{|x|^{(m+1)(p-1)}} \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} |\mathcal{R}_{|x|} f(x)| dx. \end{aligned}$$

By Hölder's inequality, it follows that

$$\begin{aligned} & \int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^\beta}{|x|^{pm+p}} |f(x)|^p dx \leq \frac{p}{Q - pm - p} \\ & \times \left(\int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^\beta}{|x|^{pm+p}} |f(x)|^p dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^\beta}{|x|^{pm}} |\mathcal{R}_{|x|} f(x)|^p dx \right)^{\frac{1}{p}}, \end{aligned}$$

yielding (3.1).

Let us now prove Part (ii). In the case $Q = pm + p - \alpha\beta$, there is nothing to prove, so we assume that $Q \neq pm + p - \alpha\beta$. By (3.7), and noting $\alpha\beta < 0$ and $pm - \alpha\beta < Q - p$ we calculate

$$\begin{aligned}
& \int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^\beta}{|x|^{pm+p}} |f(x)|^p dx \\
& \leq \int_0^\infty \int_{\mathfrak{S}} (a + br^\alpha)^\beta r^{Q-1-pm-p} \left(\frac{br^\alpha}{a + br^\alpha} + \frac{a}{a + br^\alpha} \cdot \frac{Q - pm - p}{Q - pm - p + \alpha\beta} \right) \\
& \quad \times |f(ry)|^p d\sigma(y) dr \\
& = \int_0^\infty \int_{\mathfrak{S}} \frac{(a + br^\alpha)^\beta r^{Q-1-pm-p}}{Q - pm - p + \alpha\beta} \left(\frac{\alpha\beta br^\alpha}{a + br^\alpha} + Q - pm - p \right) |f(ry)|^p d\sigma(y) dr \\
& = \int_0^\infty \int_{\mathfrak{S}} \frac{d}{dr} \left(\frac{(a + br^\alpha)^\beta r^{Q-pm-p}}{Q - pm - p + \alpha\beta} \right) |f(ry)|^p d\sigma(y) dr = \frac{-p}{Q - pm - p + \alpha\beta} \\
& \quad \times \int_0^\infty (a + br^\alpha)^\beta r^{Q-pm-p} \operatorname{Re} \int_{\mathfrak{S}} |f(ry)|^{p-2} f(ry) \frac{\overline{df(ry)}}{dr} d\sigma(y) dr \\
& \leq \left| \frac{p}{Q - pm - p + \alpha\beta} \right| \int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^\beta |\mathcal{R}_{|x}| f(x)| |f(x)|^{p-1}}{|x|^{pm+p-1}} dx \\
& = \frac{p}{Q - pm - p + \alpha\beta} \int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^{\frac{\beta(p-1)}{p}} |f(x)|^{p-1}}{|x|^{(m+1)(p-1)}} \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} |\mathcal{R}_{|x}| f(x)| dx.
\end{aligned}$$

Now we use the Hölder inequality to get

$$\begin{aligned}
& \int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^\beta}{|x|^{pm+p}} |f(x)|^p dx \leq \frac{p}{Q - pm - p + \alpha\beta} \\
& \times \left(\int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^\beta}{|x|^{pm+p}} |f(x)|^p dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{G}} \frac{(a + b|x|^\alpha)^\beta}{|x|^{pm}} |\mathcal{R}_{|x}| f(x)|^p dx \right)^{\frac{1}{p}},
\end{aligned}$$

which implies (3.2), as desired. $\square$

Now let us show higher order versions of (3.1) and (3.2).

Theorem 3.2 (¹⁸⁰ Ruzhansky-Suragan-Y., Theorem 9.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $a, b > 0$ and $1 < p < \infty, k \in \mathbb{N}$.*

(i) If $\alpha\beta > 0$ and $pm \leq Q - p$, then for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, we have

$$\prod_{j=0}^{k-1} \left(\frac{Q - p - p(m+j)}{p} \right) \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}} f}{|x|^{m+k}} \right\|_p \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}} \mathcal{R}_{|x|}^k f}{|x|^m} \right\|_p. \quad (3.8)$$

(ii) If $\alpha\beta < 0$ and $pm - \alpha\beta \leq Q - p$, then for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, we have

$$\prod_{j=0}^{k-1} \left(\frac{Q - p + \alpha\beta - p(m+j)}{p} \right) \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}} f}{|x|^{m+k}} \right\|_p \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}} \mathcal{R}_{|x|}^k f}{|x|^m} \right\|_p. \quad (3.9)$$

Note that when $k = 1$, (3.8) and (3.9) implies (3.1) and (3.2), respectively.

Proof of Theorem 3.2. Before starting the iteration process, let us recall (3.1), that is,

$$\frac{Q - pm - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}} f}{|x|^{m+1}} \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}} \mathcal{R}_{|x|} f}{|x|^m} \right\|_{L^p(\mathbb{G})}. \quad (3.10)$$

Now we replace f by $\mathcal{R}_{|x|} f$ to have

$$\frac{Q - pm - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}} \mathcal{R}_{|x|} f}{|x|^{m+1}} \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}} \mathcal{R}_{|x|}^2 f}{|x|^m} \right\|_{L^p(\mathbb{G})}. \quad (3.11)$$

If we replace m by $m + 1$ in (3.10), then we obtain

$$\frac{Q - p(m+1) - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}} f}{|x|^{m+2}} \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}} \mathcal{R}_{|x|} f}{|x|^{m+1}} \right\|_{L^p(\mathbb{G})}. \quad (3.12)$$

From this and (3.11) we derive that

$$\begin{aligned} \frac{Q - pm - p}{p} \cdot \frac{Q - p(m+1) - p}{p} \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+2}} f \right\|_{L^p(\mathbb{G})} \\ \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} \mathcal{R}_{|x|}^2 f \right\|_{L^p(\mathbb{G})}. \end{aligned}$$

Repeating this iteration process one can note that

$$\prod_{j=0}^{k-1} \left(\frac{Q - p}{p} - (m + j) \right) \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^{m+k}} f \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{(a + b|x|^\alpha)^{\frac{\beta}{p}}}{|x|^m} \mathcal{R}_{|x|}^k f \right\|_{L^p(\mathbb{G})}.$$

Similarly, but iterating (3.2) one can obtain (3.9) for $\alpha\beta < 0$, $pm - \alpha\beta \leq Q - 2$ and $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$. $\square$

FACTORIZATIONS AND WEIGHTED HARDY INEQUALITIES

Here we discuss weighted Hardy inequalities on general homogeneous groups using the factorization method of differential operators, inspired by the recent work of Gesztesy and Littlejohn⁸⁸. The approach is based on factorizing even-order differential equations. We refer to⁸⁸ and⁸⁹ for a thorough discussion of the factorization method, its history and different features. We also refer to⁹⁰ for obtaining the Hardy inequality and to⁸⁷ for logarithmic refinements by this factorization method.

In⁸⁸ the authors used the non-negativity of $T_{\alpha,\beta}^+ T_{\alpha,\beta}$ on $C_0^\infty(\mathbb{R}^n \setminus \{0\})$, for two-parameter differential expressions

$$T_{\alpha,\beta} := -\Delta + \alpha|x|^{-2}x \cdot \nabla + \beta|x|^{-2}$$

and its formal adjoint

$$T_{\alpha,\beta}^+ := -\Delta - \alpha|x|^{-2}x \cdot \nabla + (\beta - \alpha(n-2))|x|^{-2},$$

for $\alpha, \beta \in \mathbb{R}$, $x \in \mathbb{R}^n \setminus \{0\}$ and $n \geq 2$. As a result, they obtained the follow-

ing inequality for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$:

$$\begin{aligned} \int_{\mathbb{R}^n} |(\Delta f)(x)|^2 dx &\geq ((n-4)\alpha - 2\beta) \int_{\mathbb{R}^n} |x|^{-2} |(\nabla f)(x)|^2 dx \\ &\quad - \alpha(\alpha-4) \int_{\mathbb{R}^n} |x|^{-4} |x \cdot (\nabla f)(x)|^2 dx \\ &\quad + \beta((n-4)(\alpha-2) - \beta) \int_{\mathbb{R}^n} |x|^{-4} |f(x)|^2 dx, \end{aligned} \quad (3.13)$$

which implies classical Rellich and Hardy-Rellich type inequalities after choosing special values of parameters α and β .

We will discuss the factorization method also on stratified groups in Section 7.3. In particular, in Remark 7.20 we show the counterpart of the Gesztesy and Littlejohn estimate (3.13) which follows from the non-negativity of the operator $T_{\alpha,\beta} T_{\alpha,\beta}^+$.

Thus, by the factorization of differential expressions we obtain Hardy type inequalities with general weights $\phi(x)$ and $\psi(x)$, which are real-valued functions in $L_{loc}^2(\mathbb{G} \setminus \{0\})$, with their radial derivatives also in $L_{loc}^2(\mathbb{G} \setminus \{0\})$. The obtained results are new already in the standard setting of $\mathbb{R}^n$, but the proof below works for general homogeneous groups, in particular including $\mathbb{R}^n$ with both isotropic and anisotropic structures, as well as general stratified and graded Lie groups.

Theorem 3.3 (¹⁸⁷ Ruzhansky-Y., Theorem 3.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $\phi, \psi \in L_{loc}^2(\mathbb{G} \setminus \{0\})$ be any real-valued functions such that $\mathcal{R}_{|\cdot|}\phi, \mathcal{R}_{|\cdot|}\psi \in L_{loc}^2(\mathbb{G} \setminus \{0\})$. Let $\alpha \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ we have*

$$\begin{aligned} \int_{\mathbb{G}} (\phi(x))^2 |\mathcal{R}_{|x|} f(x)|^2 dx &\geq \alpha \int_{\mathbb{G}} (\phi(x) \mathcal{R}_{|x|} \psi(x) + \psi(x) \mathcal{R}_{|x|} \phi(x)) |f(x)|^2 dx \\ &\quad + \alpha(Q-1) \int_{\mathbb{G}} \frac{\phi(x) \psi(x)}{|x|} |f(x)|^2 dx - \alpha^2 \int_{\mathbb{G}} (\psi(x))^2 |f(x)|^2 dx \end{aligned} \quad (3.14)$$

and

$$\begin{aligned}
\int_{\mathbb{G}} (\phi(x))^2 |\mathcal{R}_{|x|} f(x)|^2 dx &\geq \alpha \int_{\mathbb{G}} (\psi(x) \mathcal{R}_{|x|} \phi(x) - \phi(x) \mathcal{R}_{|x|} \psi(x)) |f(x)|^2 dx \\
&+ \alpha(Q-1) \int_{\mathbb{G}} \frac{\phi(x) \psi(x)}{|x|} |f(x)|^2 dx - \alpha^2 \int_{\mathbb{G}} (\psi(x))^2 |f(x)|^2 dx \\
&- (Q-1) \int_{\mathbb{G}} \frac{(\phi(x))^2}{|x|^2} |f(x)|^2 dx + (Q-1) \int_{\mathbb{G}} \frac{\phi(x) \mathcal{R}_{|x|} \phi(x)}{|x|} |f(x)|^2 dx \\
&+ \int_{\mathbb{G}} \phi(x) \mathcal{R}_{|x|}^2 \phi(x) |f(x)|^2 dx. \tag{3.15}
\end{aligned}$$

From these we can get different weighted Hardy inequalities. For example, in the most physical situation with $\phi(x) \equiv 1$ we obtain for $\alpha \in \mathbb{R}$ and for any $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ the inequalities

$$\int_{\mathbb{G}} |\mathcal{R}_{|x|} f(x)|^2 dx \geq \int_{\mathbb{G}} \left(\alpha(Q-1) \frac{\psi(x)}{|x|} + \alpha \mathcal{R}_{|x|} \phi(x) - \alpha^2 (\psi(x))^2 \right) |f(x)|^2 dx$$

and

$$\begin{aligned}
\int_{\mathbb{G}} |\mathcal{R}_{|x|} f(x)|^2 dx &\geq \int_{\mathbb{G}} \left(\alpha(Q-1) \frac{\psi(x)}{|x|} - \alpha \mathcal{R}_{|x|} \phi(x) - \alpha^2 (\psi(x))^2 - \frac{Q-1}{|x|^2} \right) \\
&\times |f(x)|^2 dx.
\end{aligned}$$

Remark 3.4. If $\phi(x) = |x|^{-a}$ and $\psi(x) = |x|^{-b}$ for $a, b \in \mathbb{R}$, then (3.14) implies that

$$\int_{\mathbb{G}} \frac{|\mathcal{R}_{|x|} f(x)|^2}{|x|^{2a}} dx \geq \alpha(Q-a-b-1) \int_{\mathbb{G}} \frac{|f(x)|^2}{|x|^{a+b+1}} dx - \alpha^2 \int_{\mathbb{G}} \frac{|f(x)|^2}{|x|^{2b}} dx. \tag{3.16}$$

In the case $b = a + 1$, we have

$$\int_{\mathbb{G}} \frac{|\mathcal{R}_{|x|} f(x)|^2}{|x|^{2a}} dx \geq (\alpha(Q-2a-2) - \alpha^2) \int_{\mathbb{G}} \frac{|f(x)|^2}{|x|^{2a+2}} dx.$$

Then, by maximising the constant $(\alpha(Q-2a-2) - \alpha^2)$ with respect to α

we obtain the weighted Hardy inequality on homogeneous groups

$$\int_{\mathbb{G}} \frac{|\mathcal{R}_{|x|}f(x)|^2}{|x|^{2a}} dx \geq \frac{(Q - 2a - 2)^2}{4} \int_{\mathbb{G}} \frac{|f(x)|^2}{|x|^{2a+2}} dx. \quad (3.17)$$

It is known that the constant in (3.17) is sharp (see¹⁷⁴ Corollary 4.2). We also discuss L^p version of (3.17) as well as its critical version in Theorem 3.29, and prove the sharpness of the constant.

Thus, in the case $b = a + 1$ and $\alpha = \frac{Q-2a-2}{2}$, we see that (3.16) gives (3.17).

Remark 3.5. If $\phi(x) = |x|^{-a}(\log|x|)^c$ and $\psi(x) = |x|^{-b}(\log|x|)^d$ for $a, b, c, d \in \mathbb{R}$, then we derive from (3.14) the inequality

$$\begin{aligned} & \int_{\mathbb{G}} \frac{(\log|x|)^{2c}}{|x|^{2a}} |\mathcal{R}_{|x|}f(x)|^2 dx \\ & \geq \alpha \int_{\mathbb{G}} \left(\frac{(c+d)(\log|x|)^{c+d-1} + (Q-1-a-b)(\log|x|)^{c+d}}{|x|^{a+b+1}} \right) |f(x)|^2 dx \\ & \quad - \alpha^2 \int_{\mathbb{G}} \frac{(\log|x|)^{2d}}{|x|^{2b}} |f(x)|^2 dx. \end{aligned}$$

In the case when $a = \frac{Q-2}{2}$, $b = \frac{Q}{2}$, $c = 1$ and $d = 0$, it follows that

$$\int_{\mathbb{G}} \frac{(\log|x|)^2}{|x|^{Q-2}} |\mathcal{R}_{|x|}f(x)|^2 dx \geq (\alpha - \alpha^2) \int_{\mathbb{G}} \frac{|f(x)|^2}{|x|^Q} dx, \quad (3.18)$$

which after maximising the above constant with respect to α again, we obtain the critical Hardy inequality

$$\int_{\mathbb{G}} \frac{(\log|x|)^2}{|x|^{Q-2}} |\mathcal{R}_{|x|}f(x)|^2 dx \geq \frac{1}{4} \int_{\mathbb{G}} \frac{|f(x)|^2}{|x|^Q} dx. \quad (3.19)$$

In Theorem 3.29 we will show L^p version of (3.19) and prove the sharpness of the constant. So, we note that (3.18) gives (3.19) when $\alpha = \frac{1}{2}$.

Remark 3.6. On Carnot groups, we can refer to the recent work²²³ on Hardy inequalities for the horizontal gradient with a pair of non-negative weight functions.

Proof of Theorem 3.3. Let us consider the following one-parameter differential expression

$$T_\alpha := \phi(x)\mathcal{R}_{|x|} + \alpha\psi(x),$$

and its formal adjoint operator on $C_0^\infty(\mathbb{G} \setminus \{0\})$

$$T_\alpha^+ := -\phi(x)\mathcal{R}_{|x|} - \frac{Q-1}{|x|}\phi(x) - \mathcal{R}_{|x|}\phi(x) + \alpha\psi(x),$$

where $x \neq 0$. Using this and the representation of T_α we obtain

$$\begin{aligned} (T_\alpha^+ T_\alpha f)(x) &= -\phi(x)\mathcal{R}_{|x|}(\phi(x)\mathcal{R}_{|x|}f(x)) - \alpha\phi(x)\mathcal{R}_{|x|}(f(x)\psi(x)) \\ &\quad - \frac{Q-1}{|x|}(\phi(x))^2\mathcal{R}_{|x|}f(x) - \frac{\alpha(Q-1)}{|x|}\phi(x)\psi(x)f(x) - \phi(x)\mathcal{R}_{|x|}\phi(x)\mathcal{R}_{|x|}f(x) \\ &\quad - \alpha\psi(x)f(x)\mathcal{R}_{|x|}\phi(x) + \alpha\phi(x)\psi(x)\mathcal{R}_{|x|}f(x) + \alpha^2(\psi(x))^2f(x). \end{aligned}$$

Taking into account the non-negativity of $T_\alpha^+ T_\alpha$, we use the polar decomposition (2.8) to obtain

$$\begin{aligned} 0 &\leq \int_{\mathbb{G}} |(T_\alpha f)(x)|^2 dx = \int_{\mathbb{G}} f(x) \overline{(T_\alpha^+ T_\alpha f)(x)} dx \\ &= \operatorname{Re} \int_{\mathbb{G}} f(x) \overline{(T_\alpha^+ T_\alpha f)(x)} dx = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \end{aligned} \quad (3.20)$$

where

$$\begin{aligned} I_1 &= -\operatorname{Re} \int_0^\infty \int_{\mathfrak{S}} f(ry)\phi(ry) \overline{\frac{d}{dr} \left(\phi(ry) \frac{d}{dr}(f(ry)) \right)} r^{Q-1} d\sigma(y) dr, \\ I_2 &= -\alpha \operatorname{Re} \int_0^\infty \int_{\mathfrak{S}} f(ry)\phi(ry) \overline{\frac{d}{dr}(f(ry)\psi(ry))} r^{Q-1} d\sigma(y) dr, \\ I_3 &= -(Q-1) \operatorname{Re} \int_0^\infty \int_{\mathfrak{S}} f(ry) \frac{(\phi(ry))^2 \frac{d}{dr}(f(ry))}{r} r^{Q-1} d\sigma(y) dr, \\ I_4 &= -\alpha(Q-1) \int_{\mathbb{G}} \frac{\phi(x)\psi(x)}{|x|} |f(x)|^2 dx - \alpha \int_{\mathbb{G}} \mathcal{R}_{|x|}\phi(x)\psi(x) |f(x)|^2 dx \\ &\quad + \alpha^2 \int_{\mathbb{G}} (\psi(x))^2 |f(x)|^2 dx, \end{aligned} \quad (3.21)$$

$$I_5 = -\operatorname{Re} \int_0^\infty \int_{\mathfrak{S}} f(ry) \frac{d}{dr}(\phi(ry)) \phi(ry) \overline{\frac{d}{dr}(f(ry))} r^{Q-1} d\sigma(y) dr$$

and

$$I_6 = \alpha \operatorname{Re} \int_0^\infty \int_{\mathfrak{S}} f(ry) \phi(ry) \psi(ry) \overline{\frac{d}{dr}(f(ry))} r^{Q-1} d\sigma(y) dr.$$

Let us now simplify I_1 , I_2 , I_3 , I_5 and I_6 . For I_1 , by using the integration by parts we obtain

$$\begin{aligned} I_1 &= \frac{Q-1}{2} \int_0^\infty \int_{\mathfrak{S}} (\phi(ry))^2 \frac{d}{dr} |f(ry)|^2 r^{Q-2} d\sigma(y) dr \\ &\quad + \int_0^\infty \int_{\mathfrak{S}} (\phi(ry))^2 \left| \frac{d}{dr}(f(ry)) \right|^2 r^{Q-1} d\sigma(y) dr \\ &\quad + \frac{1}{2} \int_0^\infty \int_{\mathfrak{S}} \phi(ry) \frac{d}{dr}(\phi(ry)) \frac{d}{dr} |f(ry)|^2 r^{Q-1} d\sigma(y) dr. \end{aligned}$$

Using again the integration by parts to the first and third terms, we get

$$\begin{aligned} I_1 &= \int_{\mathbb{G}} (\phi(x))^2 |\mathcal{R}_{|x}| f(x)|^2 dx - \frac{(Q-1)(Q-2)}{2} \int_{\mathbb{G}} \frac{(\phi(x))^2}{|x|^2} |f(x)|^2 dx \\ &\quad - (Q-1) \int_{\mathbb{G}} \frac{\phi(x) \mathcal{R}_{|x}| \phi(x)}{|x|} |f(x)|^2 dx - \frac{1}{2} \int_{\mathbb{G}} (\mathcal{R}_{|x}| \phi(x))^2 |f(x)|^2 dx \\ &\quad - \frac{1}{2} \int_{\mathbb{G}} \mathcal{R}_{|x}|^2 \phi(x) \phi(x) |f(x)|^2 dx - \frac{Q-1}{2} \int_{\mathbb{G}} \frac{\phi(x) \mathcal{R}_{|x}| \phi(x)}{|x|} |f(x)|^2 dx. \quad (3.22) \end{aligned}$$

For I_2 , let us open the bracket and write as follows

$$\begin{aligned} I_2 &= -\frac{\alpha}{2} \int_0^\infty \int_{\mathfrak{S}} \phi(ry) \psi(ry) \frac{d}{dr} |f(ry)|^2 r^{Q-1} d\sigma(y) dr \\ &\quad - \alpha \int_0^\infty \int_{\mathfrak{S}} \phi(ry) \frac{d}{dr}(\psi(ry)) |f(ry)|^2 r^{Q-1} d\sigma(y) dr. \end{aligned}$$

By the integration by parts it implies

$$I_2 = \frac{\alpha}{2} \int_{\mathbb{G}} \psi(x) \mathcal{R}_{|x}| \phi(x) |f(x)|^2 dx + \frac{\alpha}{2} \int_{\mathbb{G}} \phi(x) \mathcal{R}_{|x}| \psi(x) |f(x)|^2 dx$$

$$+ \frac{\alpha(Q-1)}{2} \int_{\mathbb{G}} \frac{\phi(x)\psi(x)}{|x|} |f(x)|^2 dx - \alpha \int_{\mathbb{G}} \phi(x) \mathcal{R}_{|x|} \psi(x) |f(x)|^2 dx. \quad (3.23)$$

For I_3 , we have

$$\begin{aligned} I_3 &= -\frac{Q-1}{2} \int_0^\infty \int_{\mathfrak{S}} (\phi(ry))^2 \frac{d}{dr} |f(ry)|^2 r^{Q-2} d\sigma(y) dr \\ &= (Q-1) \int_{\mathbb{G}} \frac{\phi(x) \mathcal{R}_{|x|} \phi(x)}{|x|} |f(x)|^2 dx + \frac{(Q-1)(Q-2)}{2} \int_{\mathbb{G}} \frac{(\phi(x))^2}{|x|^2} |f(x)|^2 dx. \end{aligned} \quad (3.24)$$

For I_5 , one has

$$\begin{aligned} I_5 &= -\frac{1}{2} \int_0^\infty \int_{\mathfrak{S}} \frac{d}{dr} (\phi(ry)) \phi(ry) \frac{d}{dr} |f(ry)|^2 r^{Q-1} d\sigma(y) dr \\ &= \frac{1}{2} \int_{\mathbb{G}} \mathcal{R}_{|x|}^2 \phi(x) \phi(x) |f(x)|^2 dx + \frac{1}{2} \int_{\mathbb{G}} (\mathcal{R}_{|x|} \phi(x))^2 |f(x)|^2 dx \\ &\quad + \frac{Q-1}{2} \int_{\mathbb{G}} \frac{\mathcal{R}_{|x|} \phi(x) \phi(x)}{|x|} |f(x)|^2 dx. \end{aligned} \quad (3.25)$$

Finally, we calculate I_6 as follows,

$$\begin{aligned} I_6 &= \frac{\alpha}{2} \int_0^\infty \int_{\mathfrak{S}} \phi(ry) \psi(ry) \frac{d}{dr} |f(ry)|^2 r^{Q-1} d\sigma(y) dr \\ &= -\frac{\alpha}{2} \int_{\mathbb{G}} \mathcal{R}_{|x|} \phi(x) \psi(x) |f(x)|^2 dx - \frac{\alpha}{2} \int_{\mathbb{G}} \phi(x) \mathcal{R}_{|x|} \psi(x) |f(x)|^2 dx \\ &\quad - \frac{\alpha(Q-1)}{2} \int_{\mathbb{G}} \frac{\phi(x) \psi(x)}{|x|} |f(x)|^2 dx. \end{aligned} \quad (3.26)$$

Now we plug (3.21)-(3.26) into (3.20) to obtain

$$\begin{aligned} &\int_{\mathbb{G}} (\phi(x))^2 |\mathcal{R}_{|x|} f(x)|^2 dx \\ &- \alpha \int_{\mathbb{G}} \left(\phi(x) \mathcal{R}_{|x|} \psi(x) + \psi(x) \mathcal{R}_{|x|} \phi(x) + (Q-1) \frac{\phi(x) \psi(x)}{|x|} \right) |f(x)|^2 dx \\ &+ \alpha^2 \int_{\mathbb{G}} (\psi(x))^2 |f(x)|^2 dx \geq 0, \end{aligned}$$

yielding (3.14).

To obtain (3.15), we use the non-negativity of $T_\alpha T_\alpha^+$:

$$\begin{aligned}
(T_\alpha T_\alpha^+ f)(x) &= -\phi(x)\mathcal{R}_{|x|}(\phi(x)\mathcal{R}_{|x|}f(x)) + \alpha\phi(x)\mathcal{R}_{|x|}(f(x)\psi(x)) \\
&\quad - \frac{Q-1}{|x|}(\phi(x))^2\mathcal{R}_{|x|}f(x) - \frac{\alpha(Q-1)}{|x|}\phi(x)\psi(x)f(x) - \phi(x)\mathcal{R}_{|x|}\phi(x)\mathcal{R}_{|x|}f(x) \\
&\quad - \alpha\psi(x)f(x)\mathcal{R}_{|x|}\phi(x) - \alpha\phi(x)\psi(x)\mathcal{R}_{|x|}f(x) + \alpha^2(\psi(x))^2f(x) \\
&\quad - (Q-1)\phi(x)f(x)\mathcal{R}_{|x|}\left(\frac{\phi(x)}{|x|}\right) - \phi(x)f(x)\mathcal{R}_{|x|}^2\phi(x).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
0 &\leq \int_{\mathbb{G}} |(T_\alpha f)(x)|^2 dx = \int_{\mathbb{G}} f(x) \overline{(T_\alpha T_\alpha^+ f)(x)} dx \\
&= \operatorname{Re} \int_{\mathbb{G}} f(x) \overline{(T_\alpha T_\alpha^+ f)(x)} dx = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5 + \tilde{I}_6, \tag{3.27}
\end{aligned}$$

where

$$\tilde{I}_1 = I_1, \tilde{I}_2 = -I_2, \tilde{I}_3 = I_3, \tilde{I}_4 = I_4 + A, \tilde{I}_5 = I_5, \tilde{I}_6 = -I_6$$

with

$$A = -(Q-1) \int_{\mathbb{G}} \phi(x)|f(x)|^2 \mathcal{R}_{|x|}\left(\frac{\phi(x)}{|x|}\right) dx - \int_{\mathbb{G}} \phi(x)\mathcal{R}_{|x|}^2\phi(x)|f(x)|^2 dx.$$

By these and (3.21)-(3.26), one can obtain (3.15). $\square$

Remark 3.7. Note that proof of Theorem 3.3 can be reduced to one dimension (in fact to $\mathbb{R}^+$) by fixing y in the unit quasi-sphere and taking $f_y(r) = f(ry)$, $\phi_y(r) = \phi(ry)r^{\frac{Q-1}{2}}$ and $\psi_y(r) = \psi(ry)r^{\frac{Q-1}{2}}$.

WEIGHTED $L^p - L^q$ HARDY INEQUALITIES

Now we show another version of weighted Hardy inequalities with the necessary and sufficient condition on weights for such inequalities to be true.

Theorem 3.8 (¹⁹³ Ruzhansky-Y., Theorem 3.4). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let ϕ, ψ be positive weight functions on $\mathbb{G}$ and let $1 < p \leq q < \infty$. Then there exists a positive constant C such that*

$$\left(\int_{\mathbb{G}} \phi(x) |f(x)|^q dx \right)^{1/q} \leq C \left(\int_{\mathbb{G}} \psi(x) |\mathcal{R}_{|x|} f(x)|^p dx \right)^{1/p} \quad (3.28)$$

holds for all radial functions f with $f(0) = 0$ if and only if

$$A := \sup_{R>0} \left(\int_{|x| \geq R} \phi(x) dx \right)^{1/q} \left(\int_0^R \left(\int_{\mathfrak{S}} r^{Q-1} \psi(ry) d\sigma(y) \right)^{1-p'} dr \right)^{1/p'} < \infty. \quad (3.29)$$

Remark 3.9. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$, (3.28) was investigated in⁶¹ and²⁰⁴.

Proof of Theorem 3.8. The proof is based on Theorem 4.1 from Section 4.0.1. Let us introduce the following notations used in this proof: let $\tilde{f}(r) = f(x)$ for $r = |x|$ and

$$\Phi(r) = \int_{\mathfrak{S}} r^{Q-1} \phi(ry) d\sigma(y), \quad \Psi(r) = \int_{\mathfrak{S}} r^{Q-1} \psi(ry) d\sigma(y).$$

Taking into account $\tilde{f}(0) = 0$ and applying the inequality (4.1) in Theorem 4.1 we obtain

$$\begin{aligned} \left(\int_{\mathbb{G}} \phi(x) |f(x)|^q dx \right)^{1/q} &= \left(\int_{\mathfrak{S}} \int_0^\infty r^{Q-1} \phi(ry) |\tilde{f}(r)|^q dr d\sigma(y) \right)^{1/q} \\ &= \left(\int_0^\infty \Phi(r) |\tilde{f}(r)|^q dr \right)^{1/q} = \left(\int_0^\infty \Phi(r) \left| \int_0^r \mathcal{R}_r \tilde{f}(r) dr \right|^q dr \right)^{1/q} \\ &\leq C \left(\int_0^\infty \Psi(r) \left| \mathcal{R}_r \tilde{f}(r) \right|^p dr \right)^{1/p} = C \left(\int_{\mathbb{G}} \psi(x) |\mathcal{R}_{|x|} f(x)|^p dx \right)^{1/p} \end{aligned}$$

if and only if the condition (3.29) holds. □

3.1.2 REMAINDER ESTIMATES FOR L^p -WEIGHTED HARDY INEQUALITIES

In this section we discuss a family of remainder estimates in the weighted L^p -Hardy inequalities, with a freedom of choosing the parameter $b \in \mathbb{R}$ on general homogeneous groups. The obtained remainder estimates are new already in the standard setting of $\mathbb{R}^n$. The study of remainders in Hardy and other related inequalities is a classical topic going back to^{29,30,31}. Moreover, we establish Hardy-Sobolev type inequalities with an additional term and related uncertainty type inequalities.

Theorem 3.10 (¹⁸⁰ Ruzhansky-Suragan-Y., Theorem 3.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Let $2 \leq p < \infty$, $-\infty < \alpha < \frac{Q-p}{p}$ and $\delta_1 = Q - p - \alpha p - \frac{Q+pb}{p}$, $\delta_2 = Q - p - \alpha p - \frac{bp}{p-1}$ for any $b \in \mathbb{R}$. Then we have*

$$\begin{aligned} \int_{\mathbb{G}} \frac{|\mathcal{R}_{|x|} f(x)|^p}{|x|^{\alpha p}} dx - \left(\frac{Q - p - \alpha p}{p} \right)^p \int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{(\alpha+1)p}} dx \\ \geq C_p \frac{\left(\int_{\mathbb{G}} |f(x)|^p |x|^{\delta_1} dx \right)^p}{\left(\int_{\mathbb{G}} |f(x)|^p |x|^{\delta_2} dx \right)^{p-1}}, \end{aligned} \quad (3.30)$$

for all functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, where $C_p = c_p \left| \frac{Q(p-1)-pb}{p^2} \right|^p$ and

$$c_p = \min_{0 < t \leq 1/2} ((1-t)^p - t^p + pt^{p-1}).$$

Remark 3.11. Since the inequality (3.30) holds for any $b \in \mathbb{R}$, choosing $b = \frac{Q(p-1)}{p}$ so that $C_p = 0$, we obtain the L^p -weighted Hardy inequalities on homogeneous groups

$$\begin{aligned} \int_{\mathbb{G}} \frac{|\mathcal{R}_{|x|} f(x)|^p}{|x|^{\alpha p}} dx \geq \left(\frac{Q - p - \alpha p}{p} \right)^p \int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{(\alpha+1)p}} dx, \\ -\infty < \alpha < \frac{Q-p}{p}, \quad 2 \leq p < Q, \end{aligned} \quad (3.31)$$

for all functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$. We refer to¹⁷⁴ for the cases $p = 2$ and $\alpha = 1$ of (3.31). In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ with $Q = n$, the inequality

(3.31) gives the L^p -weighted Hardy inequalities for any quasi-norm on $\mathbb{R}^n$: For any function $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ we have

$$\int_{\mathbb{R}^n} \left| \frac{df(x)}{d|x|} \right|^p |x|^{-\alpha p} dx \geq \left(\frac{n-p-\alpha p}{p} \right)^p \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{p(\alpha+1)}} dx,$$

where $-\infty < \alpha < \frac{n-p}{p}$ and $2 \leq p < \infty$. By Schwarz's inequality with the standard Euclidean distance $|x|_E = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$, we obtain the Euclidean form of the L^p -weighted Hardy inequalities on $\mathbb{R}^n$

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^p}{|x|_E^{\alpha p}} dx &\geq \left(\frac{n-p-\alpha p}{p} \right)^p \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|_E^{p(\alpha+1)}} dx, \\ -\infty < \alpha &< \frac{n-p}{p}, \quad 2 \leq p < n, \end{aligned} \quad (3.32)$$

for any function $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, where ∇ is the standard gradient in $\mathbb{R}^n$.

Remark 3.12. We also note that in the Abelian case, (3.30) implies a new remainder estimate for any quasi-norm on $\mathbb{R}^n$: For any $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and for any $b \in \mathbb{R}$, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| \frac{df(x)}{d|x|} \right|^p |x|^{-\alpha p} dx - \left(\frac{n-p-\alpha p}{p} \right)^p \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{p(\alpha+1)}} dx \\ &\geq C_p \frac{\left(\int_{\mathbb{R}^n} |f(x)|^p |x|^{\delta_1} dx \right)^p}{\left(\int_{\mathbb{R}^n} |f(x)|^p |x|^{\delta_2} dx \right)^{p-1}}, \quad 2 \leq p < \infty, \quad -\infty < \alpha < \frac{n-p}{p}. \end{aligned} \quad (3.33)$$

As in Remark 3.11, by Schwarz's inequality with the standard Euclidean distance, we obtain the Euclidean version of the remainder estimate for L^p -weighted Hardy inequalities

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{|\nabla f(x)|^p}{|x|_E^{\alpha p}} dx - \left(\frac{n-p-\alpha p}{p} \right)^p \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|_E^{p(\alpha+1)}} dx \\ &\geq C_p \frac{\left(\int_{\mathbb{R}^n} |f(x)|^p |x|_E^{\delta_1} dx \right)^p}{\left(\int_{\mathbb{R}^n} |f(x)|^p |x|_E^{\delta_2} dx \right)^{p-1}}, \quad 2 \leq p < \infty, \quad -\infty < \alpha < \frac{n-p}{p}, \end{aligned} \quad (3.34)$$

for all function $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and for any $b \in \mathbb{R}$.

Thus, we note that the remainder estimate (3.34) is new already in the standard setting of $\mathbb{R}^n$.

Ghoussoub and Moradifam⁹¹ proved that there exists no strictly positive function $V \in C^1(0, \infty)$ such that the inequality

$$\int_{\mathbb{R}^n} |\nabla f|^2 dx \geq \left(\frac{n-2}{2} \right)^2 \int_{\mathbb{R}^n} \frac{|f|^2}{|x|^2} dx + \int_{\mathbb{R}^n} V(|x|) |f|^2 dx$$

holds for any $f \in W^{1,2}(\mathbb{R}^n)$. Cianchi and Ferone⁴⁷ showed that for all $1 < p < n$ there exists a constant $C = C(p, n)$ such that

$$\int_{\mathbb{R}^n} |\nabla f|^p dx \geq \left(\frac{n-p}{p} \right)^p \int_{\mathbb{R}^n} \frac{|f|^p}{|x|^p} dx (1 + C d_p(f)^{2p^*})$$

holds for all real-valued weakly differentiable functions f in $\mathbb{R}^n$ such that f and $|\nabla f| \in L^p(\mathbb{R}^n)$ go to zero at infinity. Here

$$d_p f = \inf_{c \in \mathbb{R}} \frac{\|f - c|x|^{-\frac{n-p}{p}}\|_{L^{p^*, \infty}(\mathbb{R}^n)}}{\|f\|_{L^{p^*, p}(\mathbb{R}^n)}}$$

with $p^* = \frac{np}{n-p}$, and $L^{\tau, \sigma}(\mathbb{R}^n)$ is the Lorentz space for $0 < \tau \leq \infty$ and $1 \leq \sigma \leq \infty$. In the case of a bounded domain Ω , Wang and Willem²¹⁶ for $p = 2$ and Abdellaoui, Colorado and Peral¹ for $1 < p < \infty$ investigated the improved versions of (3.32) (see also^{200, 201, 198} and¹⁹⁹ for more details).

We also note that Sano and Takahashi obtained the improved version of (3.32) in²⁰⁰ for $\Omega = \mathbb{R}^n$ and $\alpha = 0$ and then in²⁰¹ for any $-\infty < \alpha < \frac{n-p}{p}$: Let $n \geq 3$, $2 \leq p < n$ and $-\infty < \alpha < \frac{n-p}{p}$. Let $N \in \mathbb{N}$, $t \in (0, 1)$, $\gamma < \min \left\{ 1 - t, \frac{p-N}{p} \right\}$ and $\delta = N - n + \frac{N}{1-t-\gamma} \left(\gamma + \frac{n-p-\alpha p}{p} \right)$. Then there exists a constant $C > 0$ such that the inequality

$$\int_{\mathbb{R}^n} \frac{|\nabla f|^p}{|x|^{\alpha p}} dx - \left(\frac{n-p-\alpha p}{p} \right)^p \int_{\mathbb{R}^n} \frac{|f|^p}{|x|^{p(\alpha+1)}} dx \geq C \frac{\left(\int_{\mathbb{R}^n} |f|^{\frac{N}{1-t-\gamma}} |x|^\delta dx \right)^{\frac{p(1-t-\gamma)}{Nt}}}{\left(\int_{\mathbb{R}^n} |f|^p |x|^{-\alpha p} dx \right)^{\frac{1-t}{t}}}$$

holds for any radially symmetric function $f \in W_{0, \alpha}^{1, p}(\mathbb{R}^n)$, $f \neq 0$.

Now let us state the following lemma which will be useful in the proof of Theorem 3.10:

Lemma 3.13 (⁸² Frank and Seiringer). *Let $p \geq 2$ and let a, b be real numbers. Then there exists $c_p > 0$ such that*

$$|a - b|^p \geq |a|^p - p|a|^{p-2}ab + c_p|b|^p$$

holds, where $c_p = \min_{0 < t \leq 1/2} ((1-t)^p - t^p + pt^{p-1})$ is sharp in this inequality.

We will also use the following Caffarelli-Kohn-Nirenberg inequality:

Theorem 3.14 (¹⁶¹ Ozawa-Ruzhansky-Suragan, Theorem 3.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Let $1 < p < \infty$, $a, b \in \mathbb{R}$, and $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$. Then we have*

$$\left| \frac{Q - (a + b + 1)}{p} \right| \int_{\mathbb{G}} \frac{|f|^p}{|x|^{a+b+1}} dx \leq \left(\int_{\mathbb{G}} \frac{|\mathcal{R}_{|x}|f|^p}{|x|^{ap}} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} \frac{|f|^p}{|x|^{\frac{bp}{p-1}}} dx \right)^{\frac{p-1}{p}}, \quad (3.35)$$

where the constant $\left| \frac{Q - (a+b+1)}{p} \right|$ is sharp.

Proof of Theorem 3.10. First let us show the statement of Theorem 3.10 for a radial function $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$. Since f is radial, it can be represented as $f(x) = \tilde{f}(|x|)$. By Brezis-Vázquez's idea³¹, let us define

$$\tilde{g}(r) := r^{\frac{Q-p-\alpha p}{p}} \tilde{f}(r). \quad (3.36)$$

Then, taking into account $\tilde{f} = \tilde{f}(r) \in C_0^\infty(0, \infty)$ and $\alpha < \frac{Q-p}{p}$, we see that $\tilde{g}(0) = 0$ and $\tilde{g}(\infty) = 0$. Introducing polar coordinates $(r, y) = (|x|, \frac{x}{|x|}) \in (0, \infty) \times \mathfrak{S}$ on $\mathbb{G}$ and using (2.8), we write

$$\begin{aligned} J &:= \int_{\mathbb{G}} |\mathcal{R}_{|x}|f|^p |x|^{-\alpha p} dx - \left(\frac{Q - p - \alpha p}{p} \right)^p \int_{\mathbb{G}} \frac{|f|^p}{|x|^{p(\alpha+1)}} dx \\ &= |\mathfrak{S}| \int_0^\infty \left| \frac{d}{dr} \tilde{f}(r) \right|^p r^{Q-1-\alpha p} dr - |\mathfrak{S}| \left(\frac{Q - p - \alpha p}{p} \right)^p \int_0^\infty |\tilde{f}(r)|^p r^{Q-1-p(\alpha+1)} dr \end{aligned}$$

$$\begin{aligned}
&= |\mathfrak{S}| \int_0^\infty \left| \left(\frac{Q-p-\alpha p}{p} \right) r^{-\frac{Q-\alpha p}{p}} \tilde{g}(r) - r^{-\frac{Q-p-\alpha p}{p}} \frac{d}{dr} \tilde{g}(r) \right|^p r^{Q-1-\alpha p} dr \\
&\quad - |\mathfrak{S}| \left(\frac{Q-p-\alpha p}{p} \right)^p \int_0^\infty |\tilde{g}(r)|^p r^{-1} dr,
\end{aligned}$$

where $|\mathfrak{S}|$ is the $Q-1$ dimensional surface measure of the unit quasi-sphere. Let us now apply Lemma 3.13 to the integrand of the first term in the last expression above to get

$$\begin{aligned}
&\left| \left(\frac{Q-p-\alpha p}{p} \right) r^{-\frac{Q-\alpha p}{p}} \tilde{g}(r) - r^{-\frac{Q-p-\alpha p}{p}} \frac{d}{dr} \tilde{g}(r) \right|^p r^{Q-1-\alpha p} \\
&\geq \left(\left(\frac{Q-p-\alpha p}{p} \right)^p r^{-Q+\alpha p} |\tilde{g}(r)|^p \right) r^{Q-1-\alpha p} \\
&- p \left(\frac{Q-p-\alpha p}{p} \right)^{p-1} |\tilde{g}(r)|^{p-2} \tilde{g}(r) \frac{d}{dr} \tilde{g}(r) r^{-(\frac{Q-\alpha p}{p})(p-1)} r^{-(\frac{Q-p-\alpha p}{p})} r^{Q-1-\alpha p} \\
&\quad + c_p \left| \frac{d}{dr} \tilde{g}(r) \right|^p r^{-Q+p+\alpha p} r^{Q-1-\alpha p} \\
&= \left(\frac{Q-p-\alpha p}{p} \right)^p r^{-1} |\tilde{g}(r)|^p - p \left(\frac{Q-p-\alpha p}{p} \right)^{p-1} |\tilde{g}(r)|^{p-2} \tilde{g}(r) \frac{d}{dr} \tilde{g}(r) \\
&\quad + c_p \left| \frac{d}{dr} \tilde{g}(r) \right|^p r^{p-1}.
\end{aligned}$$

Here, we remark that

$$p \int_0^\infty |\tilde{g}(r)|^{p-2} \tilde{g}(r) \frac{d}{dr} \tilde{g}(r) dr = \int_0^\infty \frac{d}{dr} (|\tilde{g}(r)|^p) dr = 0,$$

since $\tilde{g}(0) = \tilde{g}(\infty) = 0$ and $p \geq 2$. Thus, we obtain a “ground state representation” of the Hardy difference J

$$J \geq c_p |\mathfrak{S}| \int_0^\infty \left| \frac{d}{dr} \tilde{g}(r) \right|^p r^{p-1} dr = c_p \int_{\mathbb{G}} |\mathcal{R}_{|x|} g(x)|^p |x|^{p-Q} dx, \quad (3.37)$$

where $g(x) := \tilde{g}(|x|)$ for $x \in \mathbb{G}$. Now, we need the case $a = \frac{Q-p}{p}$ of (3.35),

that is,

$$\begin{aligned} & \left| \frac{Q(p-1) - pb}{p^2} \right| \int_{\mathbb{G}} |g|^p |x|^{-\frac{Q+pb}{p}} dx \\ & \leq \left(\int_{\mathbb{G}} |\mathcal{R}_{|x}| g |x|^p |x|^{p-Q} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} |g|^p |x|^{-\frac{bp}{p-1}} dx \right)^{\frac{p-1}{p}} \end{aligned} \quad (3.38)$$

for any $b \in \mathbb{R}$. Then, combining this with (3.37) we arrive at

$$J \geq c_p \int_{\mathbb{G}} |\mathcal{R}_{|x}| g(x)^p |x|^{p-Q} dx \geq c_p \left| \frac{Q(p-1) - pb}{p^2} \right|^p \frac{\left(\int_{\mathbb{G}} |g|^p |x|^{-\frac{Q+pb}{p}} dx \right)^p}{\left(\int_{\mathbb{G}} |g|^p |x|^{-\frac{bp}{p-1}} dx \right)^{p-1}}. \quad (3.39)$$

By $g(x) = \tilde{g}(|x|)$, $x \in \mathbb{G}$, and (3.36), we note that

$$\begin{aligned} \int_{\mathbb{G}} |x|^{-\frac{Q+pb}{p}} |g(x)|^p dx &= |\mathfrak{S}| \int_0^\infty r^{Q-p-\alpha p} |\tilde{f}(r)|^p r^{-\frac{Q+pb}{p}} r^{Q-1} dr \\ &= \int_{\mathbb{G}} |f(x)|^p |x|^{Q-p-\alpha p - \frac{Q+pb}{p}} dx = \int_{\mathbb{G}} |f(x)|^p |x|^{\delta_1} dx. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} \int_{\mathbb{G}} |x|^{-\frac{bp}{p-1}} |g(x)|^p dx &= |\mathfrak{S}| \int_0^\infty r^{Q-p-\alpha p} |\tilde{f}(r)|^p r^{-\frac{bp}{p-1}} r^{Q-1} dr \\ &= \int_{\mathbb{G}} |f(x)|^p |x|^{Q-p-\alpha p - \frac{bp}{p-1}} dx = \int_{\mathbb{G}} |f(x)|^p |x|^{\delta_2} dx. \end{aligned}$$

We plug these equalities into (3.39) to obtain

$$J \geq c_p \left| \frac{Q(p-1) - pb}{p^2} \right|^p \frac{\left(\int_{\mathbb{G}} |f(x)|^p |x|^{\delta_1} dx \right)^p}{\left(\int_{\mathbb{G}} |f(x)|^p |x|^{\delta_2} dx \right)^{p-1}}. \quad (3.40)$$

Now, in order to show it for non-radial functions, let us consider the following radial function for a non-radial function f

$$U(r) = \left(\frac{1}{|\mathfrak{S}|} \int_{\mathbb{S}} |f(ry)|^p d\sigma(y) \right)^{\frac{1}{p}}. \quad (3.41)$$

Then, by Proposition 2.15 we have

$$\int_{\mathbb{G}} |\mathcal{R}_{|x|} U|^p |x|^{-\alpha p} dx \leq \int_{\mathbb{G}} |\mathcal{R}_{|x|} f|^p |x|^{-\alpha p} dx \quad (3.42)$$

and

$$\int_{\mathbb{G}} |U(|x|)|^p |x|^\theta dx = \int_{\mathbb{G}} |f(x)|^p |x|^\theta dx \quad (3.43)$$

for any $\theta \in \mathbb{R}$. Then, since we have proved (3.30) for radial functions, then by (3.42) and (3.43) we conclude that it also holds for all non-radial functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$. $\square$

Remark 3.15. The argument we used in the proof of Theorem 3.10, namely, proving first the inequalities for radial functions, then by Proposition 2.15 passing from radial to non-radial functions, is useful in proving many inequalities of the thesis. For example, this argument can be also applied to prove Theorems 3.1, 3.3, 3.16, 3.19, 3.20 and 3.29.

Let us now discuss the following Hardy-Sobolev type inequalities with an additional term on the right-hand side and uncertainty type inequality:

Theorem 3.16 (¹⁸³ Ruzhansky-Suragan-Y., Theorem 6.1). *For any quasi-norm $|\cdot|$, all differentiable $|\cdot|$ -radial functions ϕ , all $p > 1$, $Q \geq 2$, and all $f \in C_0^1(\mathbb{G})$ we have*

$$\int_{\mathbb{G}} \frac{\phi'(|x|)}{|x|^{Q-1}} |f(x)|^p dx \leq \int_{\mathbb{G}} |\mathcal{R}_{|x|} f(x)|^p dx + (p-1) \int_{\mathbb{G}} \frac{|\phi(|x|)|^{\frac{p}{p-1}}}{|x|^{\frac{p(Q-1)}{p-1}}} |f(x)|^p dx, \quad (3.44)$$

and

$$\int_{\mathbb{G}} \frac{\phi'(|x|)}{|x|^{Q-1}} |f(x)|^p dx \leq p \left(\int_{\mathbb{G}} |\mathcal{R}_{|x|} f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} \frac{|\phi(|x|)|^{\frac{p}{p-1}}}{|x|^{\frac{p(Q-1)}{p-1}}} |f(x)|^p dx \right)^{\frac{p-1}{p}}. \quad (3.45)$$

Remark 3.17. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, hence $Q = n$, note that the inequality (3.44) for $\phi = \log |x|$ and $p = n \geq 2$ implies the following new

inequality for any quasi-norm $|\cdot|$ on $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \frac{|f|^n}{|x|^n} dx \leq \int_{\mathbb{R}^n} |\mathcal{R}_{|\cdot|} f|^n dx + (n-1) \int_{\mathbb{R}^n} \frac{\left| \log \frac{1}{|x|} \right|^{\frac{n}{n-1}}}{|x|^n} |f|^n dx.$$

Here, taking the Euclidean distance $|x|_E = \sqrt{x_1^2 + \dots + x_n^2}$ if we use the Schwarz inequality, we obtain the critical Hardy inequality

$$\int_{\mathbb{R}^n} \frac{|f|^n}{|x|_E^n} dx \leq \int_{\mathbb{R}^n} |\nabla f|^n dx + (n-1) \int_{\mathbb{R}^n} \frac{\left| \log \frac{1}{|x|_E} \right|^{\frac{n}{n-1}}}{|x|_E^n} |f|^n dx, \quad (3.46)$$

where ∇ is the usual Euclidean gradient on $\mathbb{R}^n$. Recall that there is no positive constant C (see, e.g.¹⁶ Section 1.2.5) such that

$$\int_{\mathbb{R}^n} \frac{|f|^n}{|x|_E^n} dx \leq C \int_{\mathbb{R}^n} |\nabla f|^n dx$$

for all $f \in C_0^1(\mathbb{R}^n)$. Therefore, it seems that it is natural to have the inequality (3.46) with a positive additional term. Such type of inequalities can be useful, for instance in $\mathbb{R}^n$, for the study of the existence and nonexistence of positive exponentially bounded weak solutions to a parabolic type operator perturbed by a critical singular potential, see²⁰².

Remark 3.18. If we take $\phi = |x|^Q$ in the inequality (3.45), then we obtain

$$\int_{\mathbb{G}} |f|^p dx \leq \frac{p}{Q} \left(\int_{\mathbb{G}} |\mathcal{R}_{|\cdot|} f|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} |x|^{\frac{p}{p-1}} |f|^p dx \right)^{\frac{p-1}{p}}. \quad (3.47)$$

In the case $p = 2$ and $Q \geq 3$, since $\frac{2}{Q-2} \geq \frac{2}{Q}$ the inequality (3.47) improves the uncertainty inequality (3.47) with the constant $\frac{2}{Q-2}$ obtained in¹⁷⁴ and¹⁷⁶. Moreover, when $p = Q = 2$ the inequality (3.47) gives the critical version of their inequality. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, $n \geq 2$ for any quasi-norm $|\cdot|$ on $\mathbb{R}^n$ the inequality (3.47) becomes

$$\int_{\mathbb{R}^n} |f|^p dx \leq \frac{p}{n} \left(\int_{\mathbb{R}^n} |\mathcal{R}_{|\cdot|} f|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |x|^{\frac{p}{p-1}} |f|^p dx \right)^{\frac{p-1}{p}}.$$

If we take the Euclidean distance $|x|_E$, then by the Schwarz inequality it follows that

$$\int_{\mathbb{R}^n} |f|^p dx \leq \frac{p}{n} \left(\int_{\mathbb{R}^n} |\nabla f|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |x|^{\frac{p}{p-1}} |f|^p dx \right)^{\frac{p-1}{p}}, \quad (3.48)$$

which implies the classical Heisenberg-Pauli-Weyl uncertainty principle (1.2) when $p = 2$. Recall that equality in (3.48) for $p = 2$ holds for $f = C \exp(-b|x|_E)$, $b > 0$.

For the discussions on more general Lie groups and manifolds, we can refer to the survey⁷⁶ as well as to⁴⁹ and²¹¹.

We will discuss logarithmic uncertainty principles on homogeneous groups in Corollary 3.22, then Heisenberg-Pauli-Weyl type uncertainty principles on graded Lie groups for general homogeneous invariant hypoelliptic differential operators in Corollary 6.8, and then on stratified Lie groups for the horizontal gradient with the horizontal weight in Corollary 7.5.

Let us now prove Theorem 3.16.

Proof of Theorem 3.16. By the integration by parts, we have

$$\int_{\mathbb{G}} \frac{\phi'(|x|)}{|x|^{Q-1}} |f|^p dx = -p \operatorname{Re} \int_{\mathbb{G}} \frac{\phi(|x|) |f|^{p-2} f}{|x|^{Q-1}} \overline{\mathcal{R}_{|x|} f} dx.$$

By the Young inequality for $p > 1$, it follows that

$$\begin{aligned} \int_{\mathbb{G}} \frac{\phi'(|x|)}{|x|^{Q-1}} |f|^p dx &\leq p \int_{\mathbb{G}} \frac{|\phi| |f|^{p-1}}{|x|^{Q-1}} |\mathcal{R}_{|x|} f| dx \\ &\leq \int_{\mathbb{G}} |\mathcal{R}_{|x|} f|^p dx + (p-1) \int_{\mathbb{G}} \frac{|\phi(|x|)|^{\frac{p}{p-1}}}{|x|^{\frac{p(Q-1)}{p-1}}} |f|^p dx, \end{aligned} \quad (3.49)$$

which is (3.44). To obtain (3.45) we use Hölder's inequality for $p > 1$ in

(3.49):

$$\begin{aligned} \int_{\mathbb{G}} \frac{\phi'(|x|)}{|x|^{Q-1}} |f|^p dx &\leq p \int_{\mathbb{G}} \frac{|\phi| |f|^{p-1}}{|x|^{Q-1}} |\mathcal{R}_{|x|} f| dx \\ &\leq p \left(\int_{\mathbb{G}} |\mathcal{R}_{|x|} f|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} \frac{|\phi(|x|)|^{\frac{p}{p-1}}}{|x|^{\frac{p(Q-1)}{p-1}}} |f|^p dx \right)^{\frac{p-1}{p}}. \end{aligned}$$

□

3.1.3 STABILITY OF L^p -WEIGHTED HARDY INEQUALITIES

Sometimes the improved versions of different inequalities, or remainder estimates, are called stability of the inequality if the estimates depend on certain distances: see, e.g.,²² for stability of trace theorems,⁴³ for stability of Sobolev inequalities, etc. In this section we establish a remainder estimate in the L^p -weighted Hardy inequality involving the distance to the set of extremisers.

In order to state the theorem, let us introduce the following notations

$$f_{\alpha}(x) := |x|^{-\frac{Q-p-\alpha p}{p}} \quad (3.50)$$

for $-\infty < \alpha < \frac{Q-p}{p}$, and

$$d_R(f, g) := \left(\int_{\mathbb{G}} \frac{|f(x) - g(x)|^p}{\left| \log \frac{R}{|x|} \right|^p |x|^{(\alpha+1)p}} dx \right)^{\frac{1}{p}} \quad (3.51)$$

for functions f and g for which the integral is finite.

Theorem 3.19 (¹⁸⁰ Ruzhansky-Suragan-Y., Theorem 4.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Let $2 \leq p < \infty$ and $-\infty < \alpha < \frac{Q-p}{p}$. Then we have*

$$\int_{\mathbb{G}} \frac{|\mathcal{R}_{|x|} f|^p}{|x|^{\alpha p}} dx - \left(\frac{Q-p-\alpha p}{p} \right)^p \int_{\mathbb{G}} \frac{|f|^p}{|x|^{(\alpha+1)p}} dx$$

$$\geq c_p \left(\frac{p-1}{p} \right)^p \sup_{R>0} d_R(f, c_f(R)f_\alpha)^p, \quad (3.52)$$

for all radial functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, where $c_f(R) = R^{\frac{Q-p-\alpha p}{p}} \tilde{f}(R)$ with $f(x) = \tilde{f}(r)$, $|x| = r$, c_p is defined in Lemma 3.13, f_α and $d_R(\cdot, \cdot)$ are defined in (3.50) and (3.51), respectively.

Proof of Theorem 3.19. Since we have $p \geq 2$, then as in (3.37) in the proof of Theorem 3.10 we can obtain that

$$\begin{aligned} J(f) &= \int_{\mathbb{G}} |\mathcal{R}_{|x|} f|^p |x|^{-\alpha p} dx - \left(\frac{Q-p-\alpha p}{p} \right)^p \int_{\mathbb{G}} \frac{|f|^p}{|x|^{(\alpha+1)p}} dx \\ &\geq c_p |\mathfrak{S}| \int_0^\infty \left| \frac{d}{dr} \tilde{g} \right|^p r^{p-1} dr = c_p \int_{\mathbb{G}} \left| \frac{d}{dr} g \right|^p |x|^{p-Q} dx. \end{aligned} \quad (3.53)$$

Now we use¹⁷¹ Theorem 3.1 or Remark 3.21 with $\gamma = p$ to get

$$\begin{aligned} J(f) &\geq c_p \left(\frac{p-1}{p} \right)^p \int_{\mathbb{G}} \frac{\left| g(x) - g\left(\frac{Rx}{|x|}\right) \right|^p}{\left| \log \frac{R}{|x|} \right|^p |x|^Q} dx \\ &= c_p \left(\frac{p-1}{p} \right)^p \int_{\mathbb{G}} \frac{\left| |x|^{\frac{Q-p-\alpha p}{p}} f(x) - R^{\frac{Q-p-\alpha p}{p}} f\left(\frac{Rx}{|x|}\right) \right|^p}{\left| \log \frac{R}{|x|} \right|^p |x|^Q} dx \end{aligned}$$

for any $R > 0$. Then, noting $f(x) = \tilde{f}(r)$, $r = |x|$ it follows that

$$\begin{aligned} J(f) &\geq c_p \left(\frac{p-1}{p} \right)^p \int_{\mathbb{G}} \frac{\left| f(x) - R^{\frac{Q-p-\alpha p}{p}} \tilde{f}(R) |x|^{-\frac{Q-p-\alpha p}{p}} \right|^p}{\left| \log \frac{R}{|x|} \right|^p |x|^{(\alpha+1)p}} dx \\ &= c_p \left(\frac{p-1}{p} \right)^p \int_{\mathbb{G}} \frac{\left| f(x) - c_f(R) |x|^{-\frac{Q-p-\alpha p}{p}} \right|^p}{\left| \log \frac{R}{|x|} \right|^p |x|^{(\alpha+1)p}} dx, \end{aligned}$$

□

yielding (3.52).

3.1.4 CRITICAL HARDY INEQUALITIES

In this section we discuss critical Hardy and uncertainty type inequalities on homogeneous groups. In the Euclidean case of $\mathbb{R}^n$ it is known that the Hardy inequality (1.1) fails for any constant when $p = n$, see, e.g.,⁶⁵ and¹²⁴, and references therein. In such cases, it is natural to expect the appearance of the logarithmic terms. In Section 3.2.1 and 3.4 we also discuss more general critical L^p -Sobolev type inequalities (see Theorem 3.29) and critical Caffarelli-Kohn-Nirenberg inequalities (see Theorem 3.44), respectively.

Thus, we have the following critical Hardy inequalities of logarithmic type:

Theorem 3.20 (¹⁸⁰ Ruzhansky-Suragan-Y., Theorem 5.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Let $1 < \gamma < \infty$ and $\max\{1, \gamma - 1\} < p < \infty$. Then for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ and all $R > 0$ we have*

$$\left\| \frac{f - f_R}{|x|^{\frac{Q}{p}} \left(\log \frac{R}{|x|} \right)^{\frac{\gamma}{p}}} \right\|_{L^p(\mathbb{G})} \leq \frac{p}{\gamma - 1} \left\| \frac{\mathcal{R}_{|x|} f}{|x|^{\frac{Q-p}{p}} \left(\log \frac{R}{|x|} \right)^{\frac{\gamma-p}{p}}} \right\|_{L^p(\mathbb{G})}, \quad (3.54)$$

where $f_R(x) = f\left(R \frac{x}{|x|}\right)$, and the constant $\frac{p}{\gamma-1}$ is optimal.

Remark 3.21. In the Abelian isotropic case, this theorem implies¹³⁹ Theorem 1.1. On homogeneous group, we refer to¹⁷¹ Theorem 3.1 for the case $\gamma = p$.

Proof of Theorem 3.20. First, let us consider the integrals in (3.54) restricted to $B(0, R)$. Introducing polar coordinates $(r, y) = \left(|x|, \frac{x}{|x|}\right) \in (0, \infty) \times \mathfrak{S}$

on $\mathbb{G}$, where $\mathfrak{S}$ is the quasi-sphere as in (2.7), and using (2.8), we have

$$\begin{aligned}
\int_{B(0,R)} \frac{|f(x) - f_R(x)|^p}{|x|^Q \left| \log \frac{R}{|x|} \right|^\gamma} dx &= \int_0^R \int_{\mathfrak{S}} \frac{|f(ry) - f(Ry)|^p}{r^Q \left(\log \frac{R}{r} \right)^\gamma} r^{Q-1} d\sigma(y) dr \\
&= \int_0^R \frac{d}{dr} \left(\frac{1}{(\gamma-1) \left(\log \frac{R}{r} \right)^{\gamma-1}} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^p d\sigma(y) \right) dr - \frac{p}{\gamma-1} \\
&\times \operatorname{Re} \int_0^R \left(\log \frac{R}{r} \right)^{1-\gamma} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^{p-2} (f(ry) - f(Ry)) \frac{\overline{df(ry)}}{dr} d\sigma(y) dr \\
&= \frac{-p}{\gamma-1} \operatorname{Re} \int_0^R \left(\log \frac{R}{r} \right)^{1-\gamma} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^{p-2} (f(ry) - f(Ry)) \\
&\quad \times \frac{\overline{df(ry)}}{dr} d\sigma(y) dr,
\end{aligned}$$

where $p - \gamma + 1 > 0$, so that the boundary term at $r = R$ vanishes because of the following inequalities

$$|f(ry) - f(Ry)| \leq C(R - r)$$

and

$$\log \frac{R}{r} \geq \frac{R - r}{R}.$$

Now we use Hölder's inequality to get

$$\begin{aligned}
&\int_0^R \int_{\mathfrak{S}} \frac{|f(ry) - f(Ry)|^p}{r \left(\log \frac{R}{r} \right)^\gamma} d\sigma(y) dr = -\frac{p}{\gamma-1} \\
&\times \operatorname{Re} \int_0^R \left(\log \frac{R}{r} \right)^{-\gamma+1} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^{p-2} (f(ry) - f(Ry)) \frac{\overline{df(ry)}}{dr} d\sigma(y) dr \\
&\leq \frac{p}{\gamma-1} \int_0^R \left(\log \frac{R}{r} \right)^{-\gamma+1} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^{p-1} \left| \frac{df(ry)}{dr} \right| d\sigma(y) dr \\
&\leq \frac{p}{\gamma-1} \left(\int_0^R \int_{\mathfrak{S}} \frac{|f(ry) - f(Ry)|^p}{r \left(\log \frac{R}{r} \right)^\gamma} d\sigma(y) dr \right)^{\frac{p-1}{p}} \\
&\times \left(\int_0^R \int_{\mathfrak{S}} r^{p-1} \left(\log \frac{R}{r} \right)^{p-\gamma} \left| \frac{df(ry)}{dr} \right|^p d\sigma(y) dr \right)^{\frac{1}{p}}.
\end{aligned}$$

It follows that

$$\begin{aligned} & \left(\int_{B(0,R)} \frac{|f(x) - f_R(x)|^p}{|x|^Q \left| \log \frac{R}{|x|} \right|^\gamma} dx \right)^{\frac{1}{p}} \\ & \leq \frac{p}{\gamma - 1} \left(\int_{B(0,R)} |x|^{p-Q} \left| \log \frac{R}{|x|} \right|^{p-\gamma} |\mathcal{R}_{|x|} f(x)|^p dx \right)^{\frac{1}{p}}. \end{aligned} \quad (3.55)$$

Similarly, we obtain

$$\begin{aligned} & \left(\int_{B^c(0,R)} \frac{|f(x) - f_R(x)|^p}{|x|^Q \left| \log \frac{R}{|x|} \right|^\gamma} dx \right)^{\frac{1}{p}} \\ & \leq \frac{p}{\gamma - 1} \left(\int_{B^c(0,R)} |x|^{p-Q} \left| \log \frac{R}{|x|} \right|^{p-\gamma} |\mathcal{R}_{|x|} f(x)|^p dx \right)^{\frac{1}{p}}. \end{aligned} \quad (3.56)$$

Thus, combining (3.55) and (3.56) we obtain (3.54).

To finish the proof, it remains to prove the optimality of the constant $\frac{p}{\gamma-1}$ in (3.54). The inequality (3.54) implies that

$$\left(\int_{B(0,R)} \frac{|f|^p}{|x|^Q \left| \log \frac{R}{|x|} \right|^\gamma} dx \right)^{\frac{1}{p}} \leq \frac{p}{\gamma - 1} \left(\int_{B(0,R)} |x|^{p-Q} \left| \log \frac{R}{|x|} \right|^{p-\gamma} |\mathcal{R}_{|x|} f|^p dx \right)^{\frac{1}{p}}. \quad (3.57)$$

It is enough to prove the optimality of the constant $\frac{p}{\gamma-1}$ in (3.57). As in the Abelian case (see¹³⁹ Section 3), we define the following sequence of functions

$$f_k(x) := \begin{cases} (\log(kR))^{\frac{\gamma-1}{p}}, & \text{when } |x| \leq \frac{1}{k}, \\ \left(\log \frac{R}{|x|} \right)^{\frac{\gamma-1}{p}}, & \text{when } \frac{1}{k} \leq |x| \leq \frac{R}{2}, \\ \frac{2}{R} (\log 2)^{\frac{\gamma-1}{p}} (R - |x|), & \text{when } \frac{R}{2} \leq |x| \leq R \end{cases}$$

for large $k \in \mathbb{N}$. Letting $\tilde{f}_k(r) := f_k(x)$ with $r = |x| \geq 0$, we observe that

$$\frac{d}{dr} \tilde{f}_k(r) = \begin{cases} 0, & \text{if } r < \frac{1}{k}, \\ -\frac{\gamma-1}{p} r^{-1} \left(\log \frac{R}{r} \right)^{\frac{\gamma-1}{p}-1}, & \text{if } \frac{1}{k} < r < \frac{R}{2}, \\ -\frac{2}{R} (\log 2)^{\frac{\gamma-1}{p}}, & \text{if } \frac{R}{2} < r < R. \end{cases}$$

Denoting by $|\mathfrak{S}|$ the $Q - 1$ dimensional surface measure of the unit quasi-sphere, and by a direct calculation one has

$$\begin{aligned} \int_{B(0,R)} |x|^{p-Q} \left| \log \frac{R}{|x|} \right|^{p-\gamma} |\mathcal{R}_{|x|} f_k(x)|^p dx \\ &= |\mathfrak{S}| \int_0^R r^{p-1} \left| \log \frac{R}{r} \right|^{p-\gamma} \left| \frac{d}{dr} \tilde{f}_k(r) \right|^p dr \\ &= |\mathfrak{S}| \left(\frac{\gamma-1}{p} \right)^p \int_{\frac{1}{k}}^{\frac{R}{2}} r^{-1} \left(\log \frac{R}{r} \right)^{-1} dr \\ &\quad + |\mathfrak{S}| (\log 2)^{\gamma-1} \left(\frac{2}{R} \right)^p \int_{\frac{R}{2}}^R r^{p-1} \left(\log \frac{R}{r} \right)^{p-\gamma} dr \\ &= |\mathfrak{S}| \left(\frac{\gamma-1}{p} \right)^p ((\log(\log kR)) - \log(\log 2)) + C_{\gamma,p}, \end{aligned} \quad (3.58)$$

where

$$C_{\gamma,p} := 2^p (\log 2)^{\gamma-1} |\mathfrak{S}| \int_0^{\log 2} s^{p-\gamma} e^{-ps} ds.$$

Since $p - \gamma + 1 > 0$, we get $C_{\gamma,p} < \infty$. On the other hand, we note that

$$\begin{aligned} \int_{B(0,R)} \frac{|f_k(x)|^p}{|x|^Q \left| \log \frac{R}{|x|} \right|^\gamma} dx &= |\mathfrak{S}| \int_0^R \frac{|\tilde{f}_k(r)|^p}{r \left| \log \frac{R}{r} \right|^\gamma} dr \\ &= |\mathfrak{S}| (\log(kR))^{\gamma-1} \int_0^{\frac{1}{k}} r^{-1} \left(\log \frac{R}{r} \right)^{-\gamma} dr + |\mathfrak{S}| \int_{\frac{1}{k}}^{\frac{R}{2}} r^{-1} \left(\log \frac{R}{r} \right)^{-1} dr \\ &\quad + |\mathfrak{S}| (\log 2)^{\gamma-1} \left(\frac{2}{R} \right)^p \int_{\frac{R}{2}}^R r^{-1} (R-r)^p \left(\log \frac{R}{r} \right)^{-\gamma} dr \end{aligned}$$

$$= \frac{|\mathfrak{S}|}{\gamma - 1} + |\mathfrak{S}|(\log(\log(kR)) - \log(\log(2))) + C_{R,\gamma,p}, \quad (3.59)$$

where

$$C_{R,\gamma,p} := (\log 2)^{\gamma-1} \left(\frac{2}{R}\right)^p |\mathfrak{S}| \int_{\frac{R}{2}}^R r^{-1} (R-r)^p \left(\log \frac{R}{r}\right)^{-\gamma} dr.$$

The inequality $\log \frac{R}{r} \geq \frac{R-r}{R}$ for all $r \leq R$ and the assumption $p - \gamma > -1$ imply $C_{R,\gamma,p} < \infty$. Then, by (3.58) and (3.59), we obtain

$$\begin{aligned} & \left(\int_{B(0,R)} |x|^{p-Q} \left| \log \frac{R}{|x|} \right|^{p-\gamma} |\mathcal{R}_{|x|} f_k(x)|^p dx \right) \\ & \quad \times \left(\int_{B(0,R)} \frac{|f_k(x)|^p}{|x|^Q \left| \log \frac{R}{|x|} \right|^\gamma} dx \right)^{-1} \rightarrow \left(\frac{\gamma-1}{p} \right)^p \end{aligned}$$

as $k \rightarrow \infty$, which implies that the constant $\frac{p}{\gamma-1}$ in (3.57) is optimal. $\square$

The Hardy inequality (3.54) implies the following logarithmic uncertainty type principles:

Corollary 3.22 (¹⁸⁰ Ruzhansky-Suragan-Y., Corollary 5.2). *Let $1 < p < \infty$ and $q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Let $1 < \gamma < \infty$ and $\max\{1, \gamma-1\} < p < \infty$. Then for any $R > 0$ and $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ we have*

$$\left\| \frac{\mathcal{R}_{|x|} f}{|x|^{\frac{Q-p}{p}} \left(\log \frac{R}{|x|} \right)^{\frac{\gamma-p}{p}}} \right\|_{L^p} \|f\|_{L^q} \geq \frac{\gamma-1}{p} \left\| \frac{f(f-f_R)}{|x|^{\frac{Q}{p}} \left(\log \frac{R}{|x|} \right)^{\frac{\gamma}{p}}} \right\|_{L^2}. \quad (3.60)$$

Moreover, we have

$$\left\| \frac{\mathcal{R}_{|x|} f}{|x|^{\frac{Q-p}{p}} \left(\log \frac{R}{|x|} \right)^{\frac{\gamma-p}{p}}} \right\|_{L^p} \left\| \frac{f-f_R}{|x|^{\frac{Q}{p'}} \left(\log \frac{R}{|x|} \right)^{2-\frac{\gamma}{p}}} \right\|_{L^{p'}} \geq \frac{\gamma-1}{p} \left\| \frac{f-f_R}{|x|^{\frac{Q}{2}} \log \frac{R}{|x|}} \right\|_{L^2}^2, \quad (3.61)$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Remark 3.23. We refer to¹⁷¹ Corollary 3.2 for the case $\gamma = p$ of (3.60) and (3.61). For other types of uncertainty principles on general homogeneous group, we can also refer to¹⁷⁴ and¹⁷⁶.

Proof of Corollary 3.22. Using (3.54) we compute that

$$\begin{aligned} \left\| \frac{\mathcal{R}_{|x|}f}{|x|^{\frac{Q-p}{p}} \left(\log \frac{R}{|x|}\right)^{\frac{\gamma-p}{p}}} \right\|_{L^p} \|f\|_{L^q} &\geq \frac{\gamma-1}{p} \left\| \frac{f-f_R}{|x|^{\frac{Q}{p}} \left(\log \frac{R}{|x|}\right)^{\frac{\gamma}{p}}} \right\|_{L^p} \|f\|_{L^q} \\ &= \frac{\gamma-1}{p} \left(\int_{\mathbb{G}} \left| \frac{f(x)-f_R(x)}{|x|^{\frac{Q}{p}} \left(\log \frac{R}{|x|}\right)^{\frac{\gamma}{p}}} \right|^{2\frac{p}{2}} dx \right)^{\frac{1}{2} \frac{2}{p}} \left(\int_{\mathbb{G}} |f(x)|^{2\frac{q}{2}} dx \right)^{\frac{1}{2} \frac{2}{q}}. \end{aligned}$$

Now, we use the Hölder inequality to obtain

$$\left\| |x|^{\frac{p-Q}{p}} \left(\log \frac{R}{|x|}\right)^{\frac{p-\gamma}{p}} \mathcal{R}_{|x|}f \right\|_{L^p} \|f\|_{L^q} = \frac{\gamma-1}{p} \left\| \frac{f(f-f_R)}{|x|^{\frac{Q}{p}} \left(\log \frac{R}{|x|}\right)^{\frac{\gamma}{p}}} \right\|_{L^2}.$$

Similarly, one can prove (3.61). □

3.1.5 CRITICAL AND SUBCRITICAL HARDY INEQUALITIES

In this section, we study the relation between the critical and the subcritical Hardy inequalities on homogeneous groups.

Let $B^m(0, R)$ and $B^Q(0, R)$ be the quasi-balls of radius R around 0 with respect to the quasi-norm $|\cdot|$ on homogeneous groups of dimensions m and Q , respectively.

Proposition 3.24 (¹⁸⁰ Ruzhansky-Suragan-Y., Proposition 6.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q \geq m+1 \geq 3$ with $m \in \mathbb{N}$. Let $|\cdot|$ be a homogeneous quasi-norm. Then for any non-negative radial function $g \in C_0^1(B^m(0, R) \setminus \{0\})$, there exists a non-negative radial function $f \in$*

$C_0^1(B^Q(0, 1) \setminus \{0\})$ such that

$$\begin{aligned} & \int_{B^Q(0,1)} |\mathcal{R}_{|x|} f|^m dx - \left(\frac{Q-m}{m} \right)^m \int_{B^Q(0,1)} \frac{|f|^m}{|x|^m} dx = \frac{|\mathfrak{S}|}{|\tilde{\mathfrak{S}}|} \left(\frac{Q-m}{m-1} \right)^{m-1} \\ & \times \left(\int_{B^m(0,R)} |\mathcal{R}_{|z|} g|^m dz - \left(\frac{m-1}{m} \right)^m \int_{B^m(0,R)} \frac{|g|^m}{|z|^m \left(\log \frac{Re}{|z|} \right)^m} dz \right) \end{aligned} \quad (3.62)$$

holds true, where $|\mathfrak{S}|$ and $|\tilde{\mathfrak{S}}|$ are $Q-1$ and $m-1$ dimensional surface measure of the unit quasi-sphere, respectively.

Remark 3.25. Note that the right-hand side of (3.62) is critical Hardy inequality, while the left is subcritical Hardy inequality ($m \leq Q-1$). Therefore, it shows the relation between critical and subcritical Hardy inequalities.

Proof of Proposition 3.24. Let $r = |x|$, $x \in \mathbb{G}$ and $s = |z|$, $z \in \tilde{\mathbb{G}}$, where $\tilde{\mathbb{G}}$ is a homogeneous group of homogeneous dimension m . We define a radial function $f = f(x) \in C_0^1(B^Q(0, 1) \setminus \{0\})$ as

$$f(r) = g(s(r)) \quad (3.63)$$

for a non-negative radial function $g = g(z) \in C_0^1(B^m(0, R) \setminus \{0\})$, where $s(r) = R \exp \left(1 - r^{-\frac{Q-m}{m-1}} \right)$. Since

$$r^{-\frac{Q-m}{m-1}} = \log \frac{Re}{s}, \quad s'(r) = \frac{Q-m}{m-1} r^{-\frac{Q-m}{m-1}-1} s(r),$$

we remark that $s'(r) > 0$ for $r \in [0, 1]$ and $s(0) = 0$, $s(1) = R$. Noting $g(s) \equiv 0$ near $s = R$, we also see that $f \equiv 0$ near $r = 1$. Taking into account these properties, we compute that

$$\begin{aligned} & \int_{B^Q(0,1)} |\mathcal{R}_{|x|} f|^m dx - \left(\frac{Q-m}{m} \right)^m \int_{B^Q(0,1)} \frac{|f|^m}{|x|^m} dx \\ & = |\mathfrak{S}| \int_0^1 |f'(r)|^m r^{Q-1} dr - \left(\frac{Q-m}{m} \right)^m |\mathfrak{S}| \int_0^1 f^m(r) r^{Q-m-1} dr \end{aligned}$$

$$\begin{aligned}
&= |\mathfrak{S}| \int_0^R |g'(s)s'(r(s))|^m r^{Q-1}(s) \frac{ds}{s'(r(s))} - \left(\frac{Q-m}{m}\right)^m |\mathfrak{S}| \\
&\times \int_0^R g^m(s) r^{Q-m-1}(s) \frac{ds}{s'(r(s))} = |\mathfrak{S}| \left(\frac{Q-m}{m-1}\right)^{m-1} \int_0^R |g'(s)|^m s^{m-1} ds \\
&- \left(\frac{Q-m}{m}\right)^m \frac{m-1}{Q-m} |\mathfrak{S}| \int_0^R \frac{g^m(s)}{s \left(\log \frac{Re}{s}\right)^m} ds = \frac{|\mathfrak{S}|}{|\tilde{\mathfrak{S}}|} \left(\frac{Q-m}{m-1}\right)^{m-1} \\
&\times \left(\int_{B^m(0,R)} |\mathcal{R}_{|z}| g^m dz - \left(\frac{m-1}{m}\right)^m \int_{B^m(0,R)} \frac{|g|^m}{|z|^m \left(\log \frac{Re}{|z|}\right)^m} dz \right),
\end{aligned}$$

which is (3.62), as desired. $\square$

3.2 SOBOLEV TYPE AND SOBOLEV-RELLICH INEQUALITIES

In this section we discuss sharp remainder terms of L^p and weighted Sobolev type and Sobolev-Rellich inequalities on homogeneous Lie groups. Most inequalities are obtained with best constants. As consequences, we obtain analogues of the generalised classical Sobolev type and Sobolev-Rellich inequalities. A number of identities are derived including weighted and higher order types. In Section 7.1 we will discuss an analogue of the Sobolev type inequality but with the horizontal gradient on stratified Lie groups.

3.2.1 SOBOLEV TYPE INEQUALITIES

If $1 < p, p^* < \infty$ and

$$\frac{1}{p} = \frac{1}{p^*} - \frac{1}{n}, \quad (3.64)$$

then the (Euclidean) Sobolev inequality has the form

$$\|g\|_{L^p(\mathbb{R}^n)} \leq C(p) \|\nabla g\|_{L^{p^*}(\mathbb{R}^n)}, \quad (3.65)$$

where ∇ is the standard gradient in $\mathbb{R}^n$.

Ozawa and Sasaki¹⁶² (see also¹⁵) considered the so-called Sobolev type

inequality with respect to the operator $x \cdot \nabla$ instead of ∇ , that is,

$$\|g\|_{L^p(\mathbb{R}^n)} \leq C'(p) \|x \cdot \nabla g\|_{L^q(\mathbb{R}^n)}. \quad (3.66)$$

For any $\lambda > 0$, by substituting $g(x) = h(\lambda x)$ into (3.66), one readily observes that $p = q$ is a necessary condition to obtain (3.66).

In this section we discuss the L^p -Sobolev type inequality and its remainder formula on homogeneous Lie groups $\mathbb{G}$.

Theorem 3.26 (¹⁸² Ruzhansky-Suragan-Y., Theorem 3.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{G}$.*

(i) *Then we have*

$$\|f\|_{L^p(\mathbb{G})} \leq \frac{p}{Q} \|\mathbb{E}f\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \quad (3.67)$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, where the constant $\frac{p}{Q}$ is sharp and the equality is attained if and only if $f = 0$.

(ii) *Denoting by*

$$u := u(x) = -\frac{p}{Q} \mathbb{E}f(x), \quad v := v(x) = f(x),$$

we have the identity

$$\|u\|_{L^p(\mathbb{G})}^p - \|v\|_{L^p(\mathbb{G})}^p = p \int_{\mathbb{G}} I_p(v, u) |v - u|^2 dx, \quad 1 < p < \infty, \quad (3.68)$$

for all real-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, where

$$I_p(h, g) = (p-1) \int_0^1 |\xi h + (1-\xi)g|^{p-2} \xi d\xi.$$

(iii) *In the case $p = 2$, the identity (3.68) holds for all complex-valued functions*

$f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ and has the following form

$$\|\mathbb{E}f\|_{L^2(\mathbb{G})}^2 = \left(\frac{Q}{2}\right)^2 \|f\|_{L^2(\mathbb{G})}^2 + \left\|\mathbb{E}f + \frac{Q}{2}f\right\|_{L^2(\mathbb{G})}^2. \quad (3.69)$$

(iv) In the case $p = 2$ and $Q > 2$ the inequality (3.67) is equivalent to Hardy's inequality for any $g \in C_0^\infty(\mathbb{G} \setminus \{0\})$:

$$\left\|\frac{g}{|x|}\right\|_{L^2(\mathbb{G})} \leq \frac{2}{Q-2} \|\mathcal{R}_{|x|}g\|_{L^2(\mathbb{G})} = \frac{2}{Q-2} \left\|\frac{1}{|x|}\mathbb{E}g\right\|_{L^2(\mathbb{G})}. \quad (3.70)$$

(v) In the case $1 < p < Q$ the Sobolev type inequality (3.67) implies Hardy's inequality for any $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$:

$$\left\|\frac{f}{|x|}\right\|_{L^p(\mathbb{G})} \leq \frac{p}{Q-p} \|\mathcal{R}_{|x|}f\|_{L^p(\mathbb{G})}. \quad (3.71)$$

Remark 3.27. As mentioned above in the case of $\mathbb{R}^n$, one can observe that for any $\lambda > 0$ by setting $g(x) = h(\lambda x)$ into the inequality

$$\|g\|_{L^p(\mathbb{G})} \leq C(p) \|\mathbb{E}g\|_{L^q(\mathbb{G})}, \quad 1 < p, q < \infty \quad (3.72)$$

and using the fact the Euler operator is a homogeneous operator of order zero, we note that $p = q$ is a necessary condition to obtain (3.72).

Remark 3.28. We note that Part (ii) implies Part (i). Indeed, if we drop the non-negative term in the right-hand side of (3.68), then we obtain

$$\|f\|_{L^p(\mathbb{G})} \leq \frac{p}{Q} \|\mathbb{E}f\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \quad (3.73)$$

for all real-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$. Consequently, to have the inequality (3.73) holds for all complex-valued functions we use the identity (cf. Davies⁵⁴ p. 176)

$$\forall z \in \mathbb{C} : |z|^p = \left(\int_{-\pi}^{\pi} |\cos \theta|^p d\theta\right)^{-1} \int_{-\pi}^{\pi} |\operatorname{Re}(z) \cos \theta + \operatorname{Im}(z) \sin \theta|^p d\theta, \quad (3.74)$$

which follows from the representation $z = r(\cos \phi + i \sin \phi)$ by some manipulations.

So, the inequality (3.67) is valid with the sharp constant $\frac{p}{Q}$. We now claim that this constant is attained only for $f = 0$. By (3.74), it is enough to check this only for real-valued functions f . If the right-hand side of (3.68) is zero, then we must obtain

$$-\frac{p}{Q}\mathbb{E}f(x) = f(x),$$

which implies that $\mathbb{E}(f) = -\frac{Q}{p}f$. Taking into account the property of the Euler operator in Proposition 2.13, Part (i), it means that f is positively homogeneous of order $-\frac{Q}{p}$, that is, there exists a function $h : \mathfrak{S} \rightarrow \mathbb{C}$ such that

$$f(x) = |x|^{-\frac{Q}{p}} h\left(\frac{x}{|x|}\right), \quad (3.75)$$

where $\mathfrak{S}$ is defined by (2.7). In particular, (3.75) means that f cannot be compactly supported unless it is zero.

So, Remark 3.28 shows that Part (ii), namely (3.68) implies Part (i) of Theorem 3.26. Thus, for the part on Sobolev type inequality, we only need to prove Part (ii). Nevertheless, we also give an independent proof of (3.67) for complex-valued functions without using the formula (3.74), especially since this calculation will be also useful in proving Part (ii) of Theorem 3.26.

Proof of Theorem 3.26. Introducing polar coordinates $(r, y) = (|x|, \frac{x}{|x|}) \in (0, \infty) \times \mathfrak{S}$ on $\mathbb{G}$, where the quasi-sphere $\mathfrak{S}$ is defined in (2.7), and applying (2.8) and integrating by parts, we obtain

$$\begin{aligned} \int_{\mathbb{G}} |f(x)|^p dx &= \int_0^\infty \int_{\mathfrak{S}} |f(ry)|^p r^{Q-1} d\sigma(y) dr \\ &= -\frac{p}{Q} \int_0^\infty r^Q \operatorname{Re} \int_{\mathfrak{S}} |f(ry)|^{p-2} f(ry) \frac{\overline{df(ry)}}{dr} d\sigma(y) dr \\ &= -\frac{p}{Q} \operatorname{Re} \int_{\mathbb{G}} |f(x)|^{p-2} f(x) \overline{\mathbb{E}f(x)} dx. \end{aligned} \quad (3.76)$$

Here, we use Hölder's inequality for $\frac{1}{p} + \frac{1}{q} = 1$ to get

$$\begin{aligned} \int_{\mathbb{G}} |f(x)|^p dx &= -\frac{p}{Q} \operatorname{Re} \int_{\mathbb{G}} |f(x)|^{p-2} f(x) \overline{\mathbb{E}f(x)} dx \\ &\leq \frac{p}{Q} \left(\int_{\mathbb{G}} ||f(x)|^{p-2} f(x)|^q dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{G}} |\mathbb{E}f(x)|^p dx \right)^{\frac{1}{p}} \\ &= \frac{p}{Q} \left(\int_{\mathbb{G}} |f(x)|^p dx \right)^{1-\frac{1}{p}} \|\mathbb{E}f\|_{L^p(\mathbb{G})}, \end{aligned}$$

yielding (3.67) in Part (i).

We now prove Part (ii). If we denote

$$u := u(x) = -\frac{p}{Q} \mathbb{E}f, \quad v := v(x) = f(x),$$

then (3.76) can be rewritten as

$$\|v\|_{L^p(\mathbb{G})}^p = \operatorname{Re} \int_{\mathbb{G}} |v|^{p-2} v \bar{u} dx. \quad (3.77)$$

For any real-valued functions f we rewrite (3.76) and (3.77)

$$\int_{\mathbb{G}} |f(x)|^p dx = -\frac{p}{Q} \int_{\mathbb{G}} |f(x)|^{p-2} f(x) \mathbb{E}f(x) dx$$

and

$$\|v\|_{L^p(\mathbb{G})}^p = \int_{\mathbb{G}} |v|^{p-2} v u dx, \quad (3.78)$$

respectively. On the other hand, for all L^p -integrable real-valued functions u and v we have

$$\begin{aligned} \|u\|_{L^p(\mathbb{G})}^p - \|v\|_{L^p(\mathbb{G})}^p &+ p \int_{\mathbb{G}} (|v|^p - |v|^{p-2} v u) dx \\ &= \int_{\mathbb{G}} (|u|^p + (p-1)|v|^p - p|v|^{p-2} v u) dx \\ &= p \int_{\mathbb{G}} I_p(v, u) |v - u|^2 dx, \quad 1 < p < \infty, \end{aligned}$$

where

$$I_p(v, u) = (p-1) \int_0^1 |\xi v + (1-\xi)u|^{p-2} \xi d\xi.$$

Combining this with (3.78) one obtains (3.68).

Now let us show Part (iii). If $p = 2$, the identity (3.77) becomes

$$\|v\|_{L^2(\mathbb{G})}^2 = \operatorname{Re} \int_{\mathbb{G}} v \bar{u} dx. \quad (3.79)$$

Then, a direct calculation gives that

$$\begin{aligned} \|u\|_{L^2(\mathbb{G})}^2 - \|v\|_{L^2(\mathbb{G})}^2 &= \|u\|_{L^2(\mathbb{G})}^2 - \|v\|_{L^2(\mathbb{G})}^2 + 2 \int_{\mathbb{G}} (|v|^2 - \operatorname{Re} v \bar{u}) dx \\ &= \int_{\mathbb{G}} (|u|^2 + |v|^2 - 2 \operatorname{Re} v \bar{u}) dx \\ &= \int_{\mathbb{G}} |u - v|^2 dx, \end{aligned}$$

which is (3.69).

Now we prove Part (iv). First we show that the inequality (3.67) implies (3.70). Let $g = |x|f$. Then we have

$$\begin{aligned} \|\mathbb{E}f(x)\|_{L^2(\mathbb{G})}^2 &= \left\| \mathbb{E} \frac{g}{|x|} \right\|_{L^2(\mathbb{G})}^2 = \int_0^\infty \int_{\mathfrak{S}} \left| \left(r \frac{d}{dr} \right) \frac{g(ry)}{r} \right|^2 r^{Q-1} d\sigma(y) dr \\ &= \left\| -\frac{g}{|x|} + \frac{d}{d|x|} g \right\|_{L^2(\mathbb{G})}^2 \\ &= \left\| \frac{g}{|x|} \right\|_{L^2(\mathbb{G})}^2 - 2 \operatorname{Re} \int_{\mathbb{G}} \frac{g}{|x|} \overline{\frac{d}{d|x|} g} dx + \left\| \frac{d}{d|x|} g \right\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (3.80)$$

Since

$$\begin{aligned} -2 \operatorname{Re} \int_{\mathbb{G}} \frac{g}{|x|} \overline{\frac{d}{d|x|} g} dx &= -2 \operatorname{Re} \int_0^\infty \int_{\mathfrak{S}} \frac{g(ry)}{r} \overline{\frac{d}{dr} g(ry)} r^{Q-1} d\sigma(y) dr \\ &= -\operatorname{Re} \int_0^\infty \int_{\mathfrak{S}} \frac{d}{dr} (|\bar{g}|^2) r^{Q-2} d\sigma(y) dr = (Q-2) \operatorname{Re} \int_0^\infty \int_{\mathfrak{S}} |\bar{g}|^2 r^{Q-3} d\sigma(y) dr \end{aligned}$$

$$= (Q - 2) \left\| \frac{g}{|x|} \right\|_{L^2(\mathbb{G})}^2,$$

it follows that

$$\|\mathbb{E}f(x)\|_{L^2(\mathbb{G})}^2 = (Q - 1) \left\| \frac{g}{|x|} \right\|_{L^2(\mathbb{G})}^2 + \left\| \frac{d}{d|x|} g \right\|_{L^2(\mathbb{G})}^2. \quad (3.81)$$

We see from (3.67) and (3.81) that

$$\left\| \frac{g}{|x|} \right\|_{L^2(\mathbb{G})}^2 \leq \frac{4}{Q^2} \left((Q - 1) \left\| \frac{g}{|x|} \right\|_{L^2(\mathbb{G})}^2 + \left\| \frac{d}{d|x|} g \right\|_{L^2(\mathbb{G})}^2 \right),$$

which implies (3.70).

Conversely, we assume that (3.70) holds. For $f = g/|x|$ we have

$$\begin{aligned} \left\| \frac{d}{d|x|} (|x|f) \right\|_{L^2(\mathbb{G})}^2 &= \|f + \mathbb{E}f\|_{L^2(\mathbb{G})}^2 = \|f\|_{L^2(\mathbb{G})}^2 \\ &+ 2\operatorname{Re} \int_{\mathbb{G}} f(x) \overline{\mathbb{E}f(x)} dx + \|\mathbb{E}f\|_{L^2(\mathbb{G})}^2. \end{aligned}$$

Then, since we have

$$\begin{aligned} 2\operatorname{Re} \int_{\mathbb{G}} f(x) \overline{\mathbb{E}f(x)} dx &= 2\operatorname{Re} \int_0^\infty \int_{\mathfrak{S}} f(ry) r^Q \frac{d}{dr} \overline{f(ry)} d\sigma(y) dr \\ &= \int_0^\infty r^Q \int_{\mathfrak{S}} \frac{d}{dr} (|f(ry)|^2) d\sigma(y) dr \\ &= -Q \int_0^\infty \int_{\mathfrak{S}} |f(ry)|^2 r^{Q-1} d\sigma(y) dr \\ &= -Q \|f\|_{L^2(\mathbb{G})}^2, \end{aligned}$$

and (3.70), we obtain that

$$\|f\|_{L^2(\mathbb{G})}^2 \leq \frac{4}{(Q - 2)^2} \left(\|\mathbb{E}f\|_{L^2(\mathbb{G})}^2 - (Q - 1) \|f\|_{L^2(\mathbb{G})}^2 \right),$$

which means

$$\|f\|_{L^2(\mathbb{G})} \leq \frac{2}{Q} \|\mathbb{E}f\|_{L^2(\mathbb{G})}.$$

Let us now prove Part (v). Since

$$\|\mathcal{R}_{|x|}(|x|f)\|_{L^p(\mathbb{G})} = \|\mathbb{E}f + f\|_{L^p(\mathbb{G})} \geq \|\mathbb{E}f\|_{L^p(\mathbb{G})} - \|f\|_{L^p(\mathbb{G})},$$

then here we use (3.67) to obtain

$$\|\mathcal{R}_{|x|}(|x|f)\|_{L^p(\mathbb{G})} \geq \frac{Q-p}{p} \|f\|_{L^p(\mathbb{G})},$$

which concludes (3.71). $\square$

Now let us state weighted L^p -Sobolev type inequalities on $\mathbb{G}$.

Theorem 3.29 (¹⁸² Ruzhansky-Suragan-Y., Theorem 3.4). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $\alpha \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, $1 < p < \infty$, and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ for $\alpha p \neq Q$ we have*

$$\left\| \frac{f}{|x|^\alpha} \right\|_{L^p(\mathbb{G})} \leq \left| \frac{p}{Q - \alpha p} \right| \left\| \frac{1}{|x|^\alpha} \mathbb{E}f \right\|_{L^p(\mathbb{G})}. \quad (3.82)$$

If $\alpha p \neq Q$ then the constant $\left| \frac{p}{Q - \alpha p} \right|$ is sharp.

For $\alpha p = Q$ we have

$$\left\| \frac{f}{|x|^{\frac{Q}{p}}} \right\|_{L^p(\mathbb{G})} \leq p \left\| \frac{\log |x|}{|x|^{\frac{Q}{p}}} \mathbb{E}f \right\|_{L^p(\mathbb{G})}, \quad (3.83)$$

where the constant p is sharp.

Remark 3.30. In the case when $p = Q$ the inequality (3.83) was obtained in¹⁷¹. In Section 3.4, using these inequalities (3.82) and (3.83) we will obtain an extended version of classical Caffarelli-Kohn-Nirenberg inequalities with respect to ranges of parameters (see Theorem 3.44), which are new in the Euclidean setting of $\mathbb{R}^n$, as well as on homogeneous groups. Moreover, our methods give an improvement by replacing the full gradient by the radial derivative. In Section 3.5, we will show that Theorem 3.29

plays a key role in obtaining Hardy inequalities for the Landau Hamiltonian (or the twisted Laplacian).

The inequalities (3.82) and (3.83) imply the following uncertainty type principles:

Corollary 3.31. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $\alpha \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, and any homogeneous quasi-norm $|\cdot|$ on $\mathbb{G}$ for $\alpha p \neq Q$ we have*

$$\left\| \frac{1}{|x|^\alpha} \mathbb{E}f \right\|_{L^p(\mathbb{G})} \| |x|^\alpha f \|_{L^{p'}(\mathbb{G})} \geq \left| \frac{Q - \alpha p}{p} \right| \int_{\mathbb{G}} |f(x)|^2 dx, \quad (3.84)$$

where $1/p + 1/p' = 1$. For $\alpha p = Q$ we have

$$\left\| \frac{\log |x|}{|x|^{\frac{Q}{p}}} \mathbb{E}f \right\|_{L^p(\mathbb{G})} \left\| |x|^{\frac{Q}{p}} f \right\|_{L^{p'}(\mathbb{G})} \geq \frac{1}{p} \int_{\mathbb{G}} |f(x)|^2 dx. \quad (3.85)$$

Remark 3.32. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, hence $Q = n$, taking $|x|_E = \sqrt{x_1^2 + \dots + x_n^2}$ and $\alpha = 1$, using the Schwarz inequality we obtain from (3.84) the classical Heisenberg-Pauli-Weyl uncertainty principle. We refer to¹⁷¹ Corollary 4.3 for the case $p = Q$ of (3.85).

Proof of Corollary 3.31. Using (3.82) we obtain

$$\begin{aligned} \left\| \frac{1}{|x|^\alpha} \mathbb{E}f \right\|_{L^p(\mathbb{G})} \| |x|^\alpha f \|_{L^{p'}(\mathbb{G})} &\geq \left| \frac{Q - \alpha p}{p} \right| \left\| \frac{f}{|x|^\alpha} \right\|_{L^p(\mathbb{G})} \| |x|^\alpha f \|_{L^{p'}(\mathbb{G})} \\ &\geq \left| \frac{Q - \alpha p}{p} \right| \int_{\mathbb{G}} |f(x)|^2 dx, \end{aligned}$$

where we have used Hölder's inequality in the last line.

Similarly, using (3.83) one can obtain (3.85). □

Proof of Theorem 3.29. Using the integration by parts, for $\alpha p \neq Q$ we have

$$\begin{aligned} \int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\alpha p}} dx &= \int_0^\infty \int_{\mathfrak{S}} |f(ry)|^p r^{Q-1-\alpha p} d\sigma(y) dr \\ &= -\frac{p}{Q-\alpha p} \int_0^\infty r^{Q-\alpha p} \operatorname{Re} \int_{\mathfrak{S}} |f(ry)|^{p-2} f(ry) \frac{\overline{df(ry)}}{dr} d\sigma(y) dr \\ &\leq \left| \frac{p}{Q-\alpha p} \right| \int_{\mathbb{G}} \frac{|\mathbb{E}f(x)| |f(x)|^{p-1}}{|x|^{\alpha p}} dx = \left| \frac{p}{Q-\alpha p} \right| \int_{\mathbb{G}} \frac{|\mathbb{E}f(x)| |f(x)|^{p-1}}{|x|^{\alpha+\alpha(p-1)}} dx. \end{aligned}$$

Now we use the Hölder inequality to obtain

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\alpha p}} dx \leq \left| \frac{p}{Q-\alpha p} \right| \left(\int_{\mathbb{G}} \frac{|\mathbb{E}f(x)|^p}{|x|^{\alpha p}} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{\alpha p}} dx \right)^{\frac{p-1}{p}},$$

yielding (3.82).

To show the sharpness of the constant we check the equality condition in above Hölder's inequality. For this, we consider the function

$$g(x) = \frac{1}{|x|^C}, \quad (3.86)$$

where $C \in \mathbb{R}, C \neq 0$ and $\alpha p \neq Q$. We observe that

$$\left| \frac{1}{C} \right|^p \left(\frac{|\mathbb{E}g(x)|}{|x|^\alpha} \right)^p = \left(\frac{|g(x)|^{p-1}}{|x|^{\alpha(p-1)}} \right)^{\frac{p}{p-1}}, \quad (3.87)$$

which satisfies the equality condition in the Hölder inequality. Therefore, the constant $\left| \frac{p}{Q-\alpha p} \right|$ in (3.82) is sharp.

Now it remains to show (3.83). For this, by using the integration by parts we obtain

$$\begin{aligned} \int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q} dx &= \int_0^\infty \int_{\mathfrak{S}} |f(ry)|^p r^{Q-1-Q} d\sigma(y) dr \\ &= -p \int_0^\infty \log r \operatorname{Re} \int_{\mathfrak{S}} |f(ry)|^{p-2} f(ry) \frac{\overline{df(ry)}}{dr} d\sigma(y) dr \end{aligned}$$

$$\leq p \int_{\mathbb{G}} \frac{|\mathbb{E}f(x)||f(x)|^{p-1}}{|x|^Q} |\log|x|| dx = p \int_{\mathbb{G}} \frac{|\mathbb{E}f(x)| \log|x||}{|x|^{\frac{Q}{p}}} \frac{|f(x)|^{p-1}}{|x|^{\frac{Q(p-1)}{p}}} dx.$$

Here, we use the Hölder inequality to have

$$\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q} dx \leq p \left(\int_{\mathbb{G}} \frac{|\mathbb{E}f(x)|^p |\log|x||^p}{|x|^Q} dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^Q} dx \right)^{\frac{p-1}{p}},$$

which implies (3.83).

In order to show the sharpness of the constant, we check the equality condition in the above Hölder inequality. Consider

$$h(x) = (\log|x|)^C,$$

where $C \in \mathbb{R}$ and $C \neq 0$. Then we note that

$$\left| \frac{1}{C} \right|^p \left(\frac{|\mathbb{E}h(x)| |\log|x||}{|x|^{\frac{Q}{p}}} \right)^p = \left(\frac{|h(x)|^{p-1}}{|x|^{\frac{Q(p-1)}{p}}} \right)^{\frac{p}{p-1}}, \quad (3.88)$$

which satisfies the equality condition in the Hölder inequality. Thus, the constant p in (3.83) is sharp. $\square$

Let us discuss separately the case $p = 2$.

Theorem 3.33 (¹⁸² Ruzhansky-Suragan-Y., Theorem 3.6). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{G}$. Then for every complex-valued function $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ we have*

$$\left\| \frac{1}{|x|^\alpha} \mathbb{E}f \right\|_{L^2(\mathbb{G})}^2 = \left(\frac{Q}{2} - \alpha \right)^2 \left\| \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2 + \left\| \frac{1}{|x|^\alpha} \mathbb{E}f + \frac{Q - 2\alpha}{2|x|^\alpha} f \right\|_{L^2(\mathbb{G})}^2 \quad (3.89)$$

for any $\alpha \in \mathbb{R}$.

From (3.89) one can get different equalities and inequalities. For example,

for $\alpha = 1$, we obtain the equality

$$\left\| \frac{1}{|x|} \mathbb{E}f \right\|_{L^2(\mathbb{G})}^2 = \left(\frac{Q-2}{2} \right)^2 \left\| \frac{f}{|x|} \right\|_{L^2(\mathbb{G})}^2 + \left\| \frac{1}{|x|} \mathbb{E}f + \frac{Q-2}{2|x|} f \right\|_{L^2(\mathbb{G})}^2. \quad (3.90)$$

In (3.89), by dropping the non-negative last term we obtain the following statement:

Corollary 3.34. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{G}$. Let $\alpha \in \mathbb{R}$ and $Q - 2\alpha \neq 0$. Then we have*

$$\left\| \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})} \leq \frac{2}{|Q - 2\alpha|} \left\| \frac{1}{|x|^\alpha} \mathbb{E}f \right\|_{L^2(\mathbb{G})}, \quad (3.91)$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, where the constant in (3.91) is sharp and the equality is attained if and only if $f = 0$.

This statement on the constant and the equality follows by the same argument as that in Remark 3.28. We note a special case of $\alpha = 1$, then (3.90) gives the L^2 -Hardy inequality

$$\left\| \frac{f}{|x|} \right\|_{L^2(\mathbb{G})} \leq \frac{2}{Q-2} \left\| \frac{1}{|x|} \mathbb{E}f \right\|_{L^2(\mathbb{G})}, \quad Q \geq 3, \quad (3.92)$$

with sharp constant.

In the case $\alpha = 0$, the identity (3.89) recovers Part (iii) of Theorem 3.26. However, in the proof of Theorem 3.33 we will use Part (iii) of Theorem 3.26.

Proof of Theorem 3.33. Taking into account (2.9) and (2.11) for any $\alpha \in \mathbb{R}$ we have

$$\mathbb{E} \frac{1}{|x|^\alpha} = r \frac{d}{dr} \frac{1}{r^\alpha} = -\alpha \frac{1}{r^\alpha} = -\alpha \frac{1}{|x|^\alpha}, \quad r = |x|.$$

On the other hand, we also have

$$\mathbb{E} \frac{f}{|x|^\alpha} = \frac{1}{|x|^\alpha} \mathbb{E}f + f \mathbb{E} \frac{1}{|x|^\alpha}.$$

Combining these last two equalities, we observe that

$$\frac{1}{|x|^\alpha} \mathbb{E}f = \mathbb{E} \frac{f}{|x|^\alpha} + \alpha \frac{f}{|x|^\alpha}. \quad (3.93)$$

Then we obtain

$$\begin{aligned} \left\| \frac{1}{|x|^\alpha} \mathbb{E}f \right\|_{L^2(\mathbb{G})}^2 &= \left\| \mathbb{E} \frac{f}{|x|^\alpha} + \frac{\alpha f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2 \\ &= \left\| \mathbb{E} \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2 + 2\alpha \operatorname{Re} \int_{\mathbb{G}} \mathbb{E} \left(\frac{f}{|x|^\alpha} \right) \overline{\frac{f}{|x|^\alpha}} dx + \left\| \frac{\alpha f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (3.94)$$

In (3.69), replacing f by $\frac{f}{|x|^\alpha}$ and using (3.93) we have that

$$\left\| \mathbb{E} \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2 = \left(\frac{Q}{2} \right)^2 \left\| \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2 + \left\| \frac{1}{|x|^\alpha} \mathbb{E}f + \frac{Q-2\alpha}{2|x|^\alpha} f \right\|_{L^2(\mathbb{G})}^2. \quad (3.95)$$

Using formula (2.8) for polar coordinates, one obtains

$$\begin{aligned} 2\alpha \operatorname{Re} \int_{\mathbb{G}} \mathbb{E} \left(\frac{f}{|x|^\alpha} \right) \overline{\frac{f}{|x|^\alpha}} dx &= 2\alpha \operatorname{Re} \int_0^\infty r^{Q-1} \int_{\mathbb{S}} r \frac{d}{dr} \left(\frac{f(ry)}{r^\alpha} \right) \overline{\frac{f(ry)}{r^\alpha}} d\sigma(y) dr \\ &= \alpha \int_0^\infty r^Q \int_{\mathbb{S}} \frac{d}{dr} \left(\frac{|f(ry)|^2}{r^{2\alpha}} \right) d\sigma(y) dr = -\alpha Q \left\| \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (3.96)$$

Summing up all above we arrive at

$$\left\| \frac{1}{|x|^\alpha} \mathbb{E}f \right\|_{L^2(\mathbb{G})}^2 = \left(\frac{Q}{2} - \alpha \right)^2 \left\| \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2 + \left\| \frac{1}{|x|^\alpha} \mathbb{E}f + \frac{Q-2\alpha}{2|x|^\alpha} f \right\|_{L^2(\mathbb{G})}^2,$$

which implies (3.89). □

3.2.2 HIGHER ORDER SOBOLEV-RELICH INEQUALITIES

Here we show higher order version of the weighted Sobolev inequality (3.82) by the iteration process.

Corollary 3.35. *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and*

let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{G}$. Let $1 < p < \infty$, $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ be such that $Q \neq \alpha p$. Then for any complex-valued function $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ we have

$$\left\| \frac{f}{|x|^\alpha} \right\|_{L^p(\mathbb{G})} \leq \left| \frac{p}{Q - \alpha p} \right|^k \left\| \frac{\mathbb{E}^k f}{|x|^\alpha} \right\|_{L^p(\mathbb{G})}. \quad (3.97)$$

Remark 3.36. In the case when $k = 2$ the inequality (3.97) can be thought of as a (weighted) Sobolev-Rellich type inequality.

In the case $p = 2$ an interesting feature is that we also have the exact formula for the remainder which provides the sharpness of the constants as well.

Theorem 3.37 (¹⁸² Ruzhansky-Suragan-Y., Theorem 3.10). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{G}$. Let $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ be such that $Q \neq 2\alpha$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ the following inequality holds:*

$$\left\| \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})} \leq \left(\frac{2}{|Q - 2\alpha|} \right)^k \left\| \frac{1}{|x|^\alpha} \mathbb{E}^k f \right\|_{L^2(\mathbb{G})}. \quad (3.98)$$

The constant in (3.98) is sharp and the equality is attained if and only if $f = 0$.

Furthermore, for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \left\| \frac{1}{|x|^\alpha} \mathbb{E}^k f \right\|_{L^2(\mathbb{G})}^2 &= \left(\frac{Q - 2\alpha}{2} \right)^{2k} \left\| \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2 \\ &+ \sum_{m=1}^k \left(\frac{Q - 2\alpha}{2} \right)^{2k-2m} \left\| \frac{1}{|x|^\alpha} \mathbb{E}^m f + \frac{Q - 2\alpha}{2|x|^\alpha} \mathbb{E}^{m-1} f \right\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (3.99)$$

Although we often do not get sharp constants by iterative methods, since we have the formula (3.99), we can apply it to prove that the iterative constant is sharp. This may be a general feature of iterating Sobolev-Rellich type inequalities as the same phenomena was also investigated in $\mathbb{R}^n$ by Davies and Hinz⁵⁶, but they have used very different methods for their analysis (see also²⁹ and³⁰).

Proof of Theorem 3.37. Replacing f by $\mathbb{E}f$ in (3.89) we obtain for any $\alpha \in \mathbb{R}$ that

$$\left\| \frac{1}{|x|^\alpha} \mathbb{E}^2 f \right\|_{L^2(\mathbb{G})}^2 = \left(\frac{Q}{2} - \alpha \right)^2 \left\| \frac{\mathbb{E}f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2 + \left\| \frac{1}{|x|^\alpha} \mathbb{E}^2 f + \frac{Q - 2\alpha}{2|x|^\alpha} \mathbb{E}f \right\|_{L^2(\mathbb{G})}^2. \quad (3.100)$$

Combining (3.100) with (3.89) we get

$$\begin{aligned} \left\| \frac{1}{|x|^\alpha} \mathbb{E}^2 f \right\|_{L^2(\mathbb{G})}^2 &= \left(\frac{Q}{2} - \alpha \right)^4 \left\| \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2 \\ &\quad + \left(\frac{Q}{2} - \alpha \right)^2 \left\| \frac{1}{|x|^\alpha} \mathbb{E}f + \frac{Q - 2\alpha}{2|x|^\alpha} f \right\|_{L^2(\mathbb{G})}^2 \\ &\quad + \left\| \frac{1}{|x|^\alpha} \mathbb{E}^2 f + \frac{Q - 2\alpha}{2|x|^\alpha} \mathbb{E}f \right\|_{L^2(\mathbb{G})}^2. \end{aligned}$$

This iteration process implies

$$\begin{aligned} \left\| \frac{1}{|x|^\alpha} \mathbb{E}^k f \right\|_{L^2(\mathbb{G})}^2 &= \left(\frac{Q - 2\alpha}{2} \right)^{2k} \left\| \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2 \\ &\quad + \sum_{m=1}^k \left(\frac{Q - 2\alpha}{2} \right)^{2k-2m} \left\| \frac{1}{|x|^\alpha} \mathbb{E}^m f + \frac{Q - 2\alpha}{2|x|^\alpha} \mathbb{E}^{m-1} f \right\|_{L^2(\mathbb{G})}^2, \quad k = 1, 2, \dots \end{aligned}$$

By dropping non-negative terms, we obtain

$$\left\| \frac{1}{|x|^\alpha} \mathbb{E}^k f \right\|_{L^2(\mathbb{G})}^2 \geq \left(\frac{Q - 2\alpha}{2} \right)^{2k} \left\| \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2. \quad (3.101)$$

If $Q \neq 2\alpha$, this means

$$\left\| \frac{f}{|x|^\alpha} \right\|_{L^2(\mathbb{G})}^2 \leq \left(\frac{2}{Q - 2\alpha} \right)^{2k} \left\| \frac{1}{|x|^\alpha} \mathbb{E}^k f \right\|_{L^2(\mathbb{G})}^2, \quad (3.102)$$

which implies (3.98). Now let us show the sharpness of the constant in (3.98). The equality

$$\frac{1}{|x|^\alpha} \mathbb{E}^m f + \frac{Q - 2\alpha}{2|x|^\alpha} \mathbb{E}^{m-1} f = 0$$

can be restated as

$$\mathbb{E}(\mathbb{E}^{m-1}f) + \frac{Q-2\alpha}{2}\mathbb{E}^{m-1}f = 0,$$

and by Proposition 2.13, Part (i), it follows that $\mathbb{E}^{m-1}f$ is positively homogeneous of degree $-\frac{Q}{2} + \alpha$. Thus, if f is positively homogeneous of degree $m-1 - \frac{Q}{2} + \alpha$, then all the remainder terms vanish. Since this can be approximated by functions in $C_0^\infty(\mathbb{G} \setminus \{0\})$, the constant $\left(\frac{2}{Q-2\alpha}\right)^{2k}$ is sharp. Even if it were attained, then it would be on functions f which are positively homogeneous of degree $m-1 - \frac{Q}{2} + \alpha$. In this case $\frac{f}{|x|^{\alpha+m-1}}$ would be positively homogeneous of degree $-\frac{Q}{2}$, and these are in L^2 if and only if they are zero. $\square$

3.3 REMAINDER ESTIMATES FOR SOBOLEV TYPE INEQUALITIES

In this section we discuss L^2 -Sobolev type inequality (see (3.67) with $p = 2$) with remainder terms, which can be viewed as an analogue of Stubbe's inequality²⁰⁹ on homogeneous groups.

Theorem 3.38 (¹⁸³ Ruzhansky-Suragan-Y., Theorem 4.12). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q > 2$. Let $|\cdot|$ be a homogeneous quasi-norm. Then we have for all $f \in C_0^\infty(\mathbb{G})$ and $0 \leq \delta < \frac{Q^2}{4}$, the inequality*

$$\int_{\mathbb{G}} |\mathbb{E}f(x)|^2 dx - \delta \int_{\mathbb{G}} |f(x)|^2 dx \geq \frac{\left(\frac{Q^2}{4} - \delta\right)^{\frac{Q-1}{Q}}}{\left(\frac{(Q-2)^2}{4}\right)^{\frac{Q-1}{Q}}} S_Q \left(\int_{\mathbb{G}} |x|^{2^*} |g(|x|)|^{2^*} dx \right)^{\frac{2}{2^*}} \quad (3.103)$$

with sharp constant, where

$$g(|x|) = \mathcal{M}(f)(|x|) := \frac{1}{|\mathfrak{S}|} \int_{\mathfrak{S}} f(|x|, y) d\sigma(y)$$

and

$$S_Q := |\mathfrak{S}|^{\frac{2}{Q}} Q^{\frac{Q-2}{Q}} (Q-2) \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)} \right)^{\frac{2}{Q}}. \quad (3.104)$$

Remark 3.39. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$, the inequality (3.103) gives that

$$\begin{aligned} \int_{\mathbb{R}^n} |(x \cdot \nabla)f(x)|^2 dx - \delta \int_{\mathbb{R}^n} |f(x)|^2 dx &\geq \frac{\left(\frac{n^2}{4} - \delta\right)^{\frac{n-1}{n}}}{\left(\frac{(n-2)^2}{4}\right)^{\frac{n-1}{n}}} \\ &\times S_n \left(\int_{\mathbb{G}} |x|^{2^*} |g(|x|)|^{2^*} dx \right)^{\frac{2}{2^*}}. \end{aligned} \quad (3.105)$$

An interesting feature of the above inequality is that its constant is sharp for any quasi-norm $|\cdot|$, that is, it does not depend on the quasi-norm $|\cdot|$. Therefore, this inequality is new already in $\mathbb{R}^n$. For the case of the Euclidean distance $|x|_E = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$, we refer to¹⁵ Corollary 4.4 and²¹⁹ Theorem 1.1.

Before starting the proof of this result, let us recall the following result²⁴:

Lemma 3.40 (²⁴). *Let p, q be constants, such that $q > p > 1$. Let f be a non-negative function. Then for all $0 \leq s < \infty$ we have*

$$\left(\int_0^\infty \left| \int_0^s f(r) dr \right|^q r^{q/p-q-1} dr \right)^{p/q} \leq C_{p,q} \int_0^\infty |f(r)|^p dr, \quad (3.106)$$

where

$$C_{p,q} = (q - q/p)^{-p/q} \left(\frac{(q/p - 1) \Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right) \Gamma\left(\frac{p(q-1)}{q-p}\right)} \right)^{(q-p)/q}$$

is sharp. Moreover, the equality in (3.106) is attained for functions of the form

$$f(r) = c_1 (c_2 r^{q/p-1} + 1)^{\frac{q}{p-q}}, \quad c_1 > 0, \quad c_2 > 0. \quad (3.107)$$

Proof of Theorem 3.38. In this proof we will use the useful property of the Euler operator from Proposition 2.15. In other words, we first prove the theorem for radial functions, then by Proposition 2.15 we extend it to non-radial case. Thus, let us assume for now that $f(x) = \tilde{f}(|x|)$. Then, by the definition of the function g we obtain $g(r) = \tilde{f}(r)$. By a direct calculation

we have

$$\begin{aligned}
& \int_{\mathbb{G}} |\mathbb{E}\tilde{f}(|x|)|^2 dx - \beta(Q - \beta) \int_{\mathbb{G}} |\tilde{f}(|x|)|^2 dx \\
&= |\mathfrak{S}| \left(\int_0^\infty |\tilde{f}'(r)|^2 r^{Q+1} dr - \beta(Q - \beta) \int_0^\infty |\tilde{f}(r)|^2 r^{Q-1} dr \right),
\end{aligned} \tag{3.108}$$

where $0 \leq \beta < Q/2$. If we set $h(|x|) := |x|^\beta \tilde{f}(|x|)$, then the integration by parts gives that

$$\begin{aligned}
& \int_0^\infty |h'(r)|^2 r^{Q+1-2\beta} dr \\
&= \int_0^\infty |\beta r^{\beta-1} \tilde{f}(r) + r^\beta \tilde{f}'(r)|^2 r^{Q+1-2\beta} dr \\
&= \beta^2 \int_0^\infty |\tilde{f}(r)|^2 r^{Q-1} dr + \int_0^\infty |\tilde{f}'(r)|^2 r^{Q+1} dr + \beta \int_0^\infty \frac{d}{dr} |\tilde{f}(r)|^2 r^Q dr \\
&= \int_0^\infty |\tilde{f}'(r)|^2 r^{Q+1} dr - \beta(Q - \beta) \int_0^\infty |\tilde{f}(r)|^2 r^{Q-1} dr.
\end{aligned} \tag{3.109}$$

On the other hand, by changing the variables $s = r^{Q-2\beta}$ we also have that

$$\begin{aligned}
\int_0^\infty |h'(r)|^2 r^{Q+1-2\beta} dr &= \int_0^\infty |(Q - 2\beta) s^{\frac{Q-2\beta-1}{Q-2\beta}} h'(s)|^2 s^{\frac{Q-2\beta+1}{Q-2\beta}} \frac{s^{\frac{1}{Q-2\beta}-1} ds}{Q - 2\beta} \\
&= (Q - 2\beta) \int_0^\infty s^2 |h'(s)|^2 ds.
\end{aligned} \tag{3.110}$$

Taking into account (3.109) and (3.110), we rewrite (3.108) as

$$\begin{aligned}
& \int_{\mathbb{G}} |\mathbb{E}\tilde{f}(|x|)|^2 dx - \beta(Q - \beta) \int_{\mathbb{G}} |\tilde{f}(|x|)|^2 dx \\
&= |\mathfrak{S}| (Q - 2\beta) \left(\int_0^\infty s^2 |h'(s)|^2 ds \right).
\end{aligned} \tag{3.111}$$

Now denoting $\phi(s) := h'(s)$ and $\psi(s) := s^{-2}\phi(s^{-1})$, and using Lemma 3.40

with $p = 2$ and $q = \frac{2Q}{Q-2} =: 2^*$ we have

$$\begin{aligned}
& \int_0^\infty s^2 |h'(s)|^2 ds = \int_0^\infty s^2 |\phi(s)|^2 ds = \int_0^\infty |\psi(s)|^2 ds \\
& \geq \left(\frac{Q}{Q-2} \right)^{\frac{Q-2}{Q}} \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)} \right)^{\frac{2}{Q}} \left(\int_0^\infty \left| \int_0^s |\psi(t)| dt \right|^{2^*} s^{\frac{2-2Q}{Q-2}} ds \right)^{\frac{2}{2^*}} \\
& = \left(\frac{Q}{Q-2} \right)^{\frac{Q-2}{Q}} \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)} \right)^{\frac{2}{Q}} \left(\int_0^\infty \left| \int_{s^{-1}}^\infty |\phi(t)| dt \right|^{2^*} s^{\frac{2-2Q}{Q-2}} ds \right)^{\frac{2}{2^*}} \\
& \geq \left(\frac{Q}{Q-2} \right)^{\frac{Q-2}{Q}} \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)} \right)^{\frac{2}{Q}} \left(\int_0^\infty |h(s^{-1})|^{2^*} s^{\frac{2-2Q}{Q-2}} ds \right)^{\frac{2}{2^*}} \\
& = \left(\frac{Q}{Q-2} \right)^{\frac{Q-2}{Q}} \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)} \right)^{\frac{2}{Q}} \left(\int_0^\infty |h(s)|^{2^*} s^{\frac{2}{Q-2}} ds \right)^{\frac{2}{2^*}} \\
& = (Q-2\beta)^{\frac{2}{2^*}} \left(\frac{Q}{Q-2} \right)^{\frac{Q-2}{Q}} \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)} \right)^{\frac{2}{Q}} \\
& \quad \times \left(\int_0^\infty |r\tilde{f}(r)|^{2^*} r^{Q-1} dr \right)^{\frac{2}{2^*}},
\end{aligned}$$

where we have used $s = r^{Q-2\beta}$ and $h(r) = r^\beta \tilde{f}(r)$ in the last line. Combining this with (3.111), we obtain that

$$\begin{aligned}
& \int_{\mathbb{G}} |\mathbb{E}\tilde{f}(|x|)|^2 dx - \beta(Q-\beta) \int_{\mathbb{G}} |\tilde{f}(|x|)|^2 dx \\
& \geq |\mathfrak{S}|(Q-2\beta)^{\frac{2Q-2}{Q}} \left(\frac{Q}{Q-2} \right)^{\frac{Q-2}{Q}} \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)} \right)^{\frac{2}{Q}} \\
& \quad \times \left(\int_0^\infty |r\tilde{f}(r)|^{2^*} r^{Q-1} dr \right)^{\frac{2}{2^*}} = |\mathfrak{S}|^{\frac{2}{Q}} (Q-2\beta)^{\frac{2Q-2}{Q}} \left(\frac{Q}{Q-2} \right)^{\frac{Q-2}{Q}} \\
& \quad \times \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)} \right)^{\frac{2}{Q}} \left(\int_{\mathbb{G}} |x|^{2^*} |\tilde{f}(|x|)|^{2^*} dx \right)^{\frac{2}{2^*}}. \quad (3.112)
\end{aligned}$$

Here, if we set $\beta = (Q - \sqrt{Q^2 - 4\delta})/2$ for $0 \leq \delta < Q^2/4$, then recalling

$g(|x|) = \tilde{f}(|x|)$ we see that (3.112) implies that

$$\begin{aligned} \int_{\mathbb{G}} |\mathbb{E}\tilde{f}(|x|)|^2 dx - \delta \int_{\mathbb{G}} |\tilde{f}(|x|)|^2 dx &\geq |\mathfrak{S}|^{\frac{2}{Q}} (Q^2 - 4\delta)^{\frac{Q-1}{Q}} \left(\frac{Q}{Q-2} \right)^{\frac{Q-2}{Q}} \\ &\times \left(\frac{\Gamma(Q/2)\Gamma(1+Q/2)}{\Gamma(Q)} \right)^{\frac{2}{Q}} \left(\int_{\mathbb{G}} |x|^{2^*} |g(|x|)|^{2^*} dx \right)^{\frac{2}{2^*}} \\ &= \left(\frac{Q^2 - 4\delta}{(Q-2)^2} \right)^{\frac{Q-1}{Q}} S_Q \left(\int_{\mathbb{G}} |x|^{2^*} |g(|x|)|^{2^*} dx \right)^{\frac{2}{2^*}}, \end{aligned} \quad (3.113)$$

since (3.104). Thus, we have obtained (3.103) with sharp constant for all radial functions $f \in C_0^\infty(\mathbb{G})$.

Finally, using Proposition 2.15, and taking into account $g(|x|) = \tilde{f}(|x|)$ in (3.113), we obtain (3.103) for non-radial functions. The constant in (3.103) is sharp, since this constant is sharp for radial functions by Lemma 3.40. $\square$

3.4 EXTENDED CAFFARELLI-KOHN-NIRENBERG INEQUALITIES

In this section, we present new Caffarelli-Kohn-Nirenberg type inequalities in the Euclidean setting of $\mathbb{R}^n$ as well as on homogeneous groups. We will discuss their hypoelliptic and horizontal versions in Section 6.1 and 7.2, respectively.

Let us recall the classical Caffarelli-Kohn-Nirenberg inequality³⁸:

Theorem 3.41. *Let $n \in \mathbb{N}$ and let $p, q, r, a, b, d, \delta \in \mathbb{R}$ such that $p, q \geq 1, r > 0, 0 \leq \delta \leq 1$, and*

$$\frac{1}{p} + \frac{a}{n}, \frac{1}{q} + \frac{b}{n}, \frac{1}{r} + \frac{c}{n} > 0, \quad (3.114)$$

where $c = \delta d + (1 - \delta)b$. Then there exists a positive constant C such that

$$\| |x|^c f \|_{L^r(\mathbb{R}^n)} \leq C \| |x|^a |\nabla f| \|_{L^p(\mathbb{R}^n)}^\delta \| |x|^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta} \quad (3.115)$$

holds for all $f \in C_0^\infty(\mathbb{R}^n)$, if and only if the following conditions hold:

$$\frac{1}{r} + \frac{c}{n} = \delta \left(\frac{1}{p} + \frac{a-1}{n} \right) + (1-\delta) \left(\frac{1}{q} + \frac{b}{n} \right), \quad (3.116)$$

$$a-d \geq 0 \quad \text{if} \quad \delta > 0, \quad (3.117)$$

$$a-d \leq 1 \quad \text{if} \quad \delta > 0 \quad \text{and} \quad \frac{1}{r} + \frac{c}{n} = \frac{1}{p} + \frac{a-1}{n}. \quad (3.118)$$

The aim of this section is to extend the CKN-inequalities for general functions with respect to widening the range of indices (3.114). Moreover, another improvement will be achieved by replacing the full gradient ∇f in (3.115) by the radial derivative $\mathcal{R}_\parallel f = \frac{\partial f}{\partial r}$. It turns out that such improved versions can be established with sharp constants, and to hold both in the isotropic and anisotropic settings.

To compare with Theorem 3.41 let us first formulate the isotropic version of our extension in the usual setting of $\mathbb{R}^n$.

Theorem 3.42 (¹⁸⁰ Ruzhansky-Suragan-Y., Theorem 1.2). *Let $n \in \mathbb{N}$, $1 < p, q < \infty$, $0 < r < \infty$, with $p+q \geq r$, $\delta \in [0, 1] \cap [\frac{r-q}{r}, \frac{p}{r}]$ and $a, b, c \in \mathbb{R}$. Assume that $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$, $c = \delta(a-1) + b(1-\delta)$. If $n \neq p(1-a)$, then for any function $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ we have*

$$\| |x|^c f \|_{L^r(\mathbb{R}^n)} \leq \left| \frac{p}{n-p(1-a)} \right|^\delta \left\| |x|^a \left(\frac{x}{|x|} \cdot \nabla f \right) \right\|_{L^p(\mathbb{R}^n)}^\delta \| |x|^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta}. \quad (3.119)$$

In the critical case $n = p(1-a)$ for any function $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ we have

$$\| |x|^c f \|_{L^r(\mathbb{R}^n)} \leq p^\delta \left\| |x|^a \log |x| \left(\frac{x}{|x|} \cdot \nabla f \right) \right\|_{L^p(\mathbb{R}^n)}^\delta \| |x|^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta}, \quad (3.120)$$

for any homogeneous quasi-norm $|\cdot|$. If $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$, inequalities (3.119) and (3.120) imply, respectively,

$$\| |x|_E^c f \|_{L^r(\mathbb{R}^n)} \leq \left| \frac{p}{n-p(1-a)} \right|^\delta \| |x|_E^a \nabla f \|_{L^p(\mathbb{R}^n)}^\delta \| |x|_E^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta} \quad (3.121)$$

for $n \neq p(1-a)$, and

$$\| |x|_E^c f \|_{L^r(\mathbb{R}^n)} \leq p^\delta \| |x|_E^a \log |x|_E \nabla f \|_{L^p(\mathbb{R}^n)}^\delta \| |x|_E^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta}, \quad (3.122)$$

for $n = p(1-a)$. The inequality (3.119) holds for any homogeneous quasi-norm $|\cdot|$, and the constant $\left| \frac{p}{n-p(1-a)} \right|^\delta$ is sharp for $p = q$ with $a - b = 1$ or for $p \neq q$ with $p(1-a) + bq \neq 0$. Furthermore, the constants $\left| \frac{p}{n-p(1-a)} \right|^\delta$ and p^δ are sharp for $\delta = \overline{0, 1}$.

Note that if the conditions (3.114) hold, then the inequality (3.121) is contained in the family of Caffarelli-Kohn-Nirenberg inequalities in Theorem 3.41. However, already in this case, taking $p = q$ with $a - b = 1$ or $p \neq q$ with $p(1-a) + bq \neq 0$, we note that (3.121) implies (3.115) with sharp constant. In addition, the constants $\left| \frac{p}{n-p(1-a)} \right|^\delta$ and p^δ are sharp for $\delta = 0$ or $\delta = 1$. Our conditions $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ and $c = \delta(a-1) + b(1-\delta)$ imply the conditions (3.116)-(3.118) of Theorem 3.41 on the parameters, which are all necessary for having estimates of this type, at least under the conditions (3.114).

If the conditions (3.114) does not hold, then the inequality (3.121) is not covered by Theorem 3.41. Therefore, this implies an extension of Theorem 3.41 with respect to the range of parameters. In this sense, we call the obtained inequalities to be “extended Caffarelli-Kohn-Nirenberg”, from the point of view of an extension of the range of parameters (although Theorem 3.42 does not imply Theorem 3.41). The conditions (3.114) in Theorem 3.41 can be considered as “integrability” conditions, in particular ensuring the behaviour at the origin; however, both Theorem 3.42 and Theorem 3.41 apply to functions in $C_0^\infty(\mathbb{R}^n \setminus \{0\})$.

Let us give an example of an extension of the range of parameters:

Example 3.43. Let us take $1 < p = q = r < \infty$, $a = -\frac{n-2p}{p}$, $b = -\frac{n}{p}$ and

$c = -\frac{n-\delta p}{p}$. Then (3.121) becomes

$$\left\| \frac{f}{|x|_E^{\frac{n-\delta p}{p}}} \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{\nabla f}{|x|_E^{\frac{n-2p}{p}}} \right\|_{L^p(\mathbb{R}^n)}^\delta \left\| \frac{f}{|x|_E^{\frac{n}{p}}} \right\|_{L^p(\mathbb{R}^n)}^{1-\delta}, \quad 0 \leq \delta \leq 1, \quad (3.123)$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, where ∇ is the Euclidean gradient in $\mathbb{R}^n$. In this case, we have

$$\frac{1}{q} + \frac{b}{n} = \frac{1}{p} + \frac{1}{n} \left(-\frac{n}{p} \right) = 0,$$

which means that (3.114) fails. Thus, in this case, the inequality (3.123) is not covered by Theorem 3.41.

The improved versions of the Caffarelli-Kohn-Nirenberg (CKN) inequality for radially symmetric functions with respect to the range of parameters was investigated in⁵⁷. In²²⁷ and¹¹⁰, weighted Hardy type inequalities were obtained for the generalised Baouendi-Grushin vector fields, which is when $\gamma = 0$ gives the standard gradient in $\mathbb{R}^n$. We also refer to^{108, 107} for weighted Hardy inequalities on the Heisenberg group, to¹⁰⁹ and²²⁶ on the H-type groups, to¹⁶¹ for the special case of CKN on homogeneous group, and to a recent paper²²⁰ on Lie groups of polynomial growth as well as to references therein. For other extended CKN inequalities we refer to¹⁹¹ on general Lie groups, and to¹⁹² on Riemannian manifolds with negative curvature.

Theorem 3.42 is the special case of the following theorem:

Theorem 3.44 (¹⁸⁰ Ruzhansky-Suragan-Y., Theorem 7.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Let $1 < p, q < \infty$, $0 < r < \infty$ with $p + q \geq r$ and $\delta \in [0, 1] \cap \left[\frac{r-q}{r}, \frac{p}{r} \right]$ and $a, b, c \in \mathbb{R}$. Assume that $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ and $c = \delta(a-1) + b(1-\delta)$. Then we have the following Caffarelli-Kohn-Nirenberg type inequalities for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$:*

If $Q \neq p(1-a)$, then

$$\| |x|^c f \|_{L^r(\mathbb{G})} \leq \left| \frac{p}{Q - p(1-a)} \right|^\delta \| |x|^a \mathcal{R}_{|x|} f \|_{L^p(\mathbb{G})}^\delta \| |x|^b f \|_{L^q(\mathbb{G})}^{1-\delta}. \quad (3.124)$$

If $Q = p(1 - a)$, then

$$\| |x|^c f \|_{L^r(\mathbb{G})} \leq p^\delta \| |x|^a \log |x| \mathcal{R}_{|x|} f \|_{L^p(\mathbb{G})}^\delta \| |x|^b f \|_{L^q(\mathbb{G})}^{1-\delta}. \quad (3.125)$$

The constant in the inequality (3.124) is sharp for $p = q$ with $a - b = 1$ or $p \neq q$ with $p(1 - a) + bq \neq 0$. Moreover, the constants in (3.124) and (3.125) are sharp for $\delta = 0$ or $\delta = 1$.

Remark 3.45. Our conditions $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ and $c = \delta(a - 1) + b(1 - \delta)$ imply the condition (3.116) of Theorem 3.41, and in our case $a - d = 1$.

Let us now explain how Theorem 3.44 implies Theorem 3.42 in the following remark:

Remark 3.46. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$, (3.125) implies a new type of Caffarelli-Kohn-Nirenberg inequality for any quasi-norm on $\mathbb{R}^n$: Let $1 < p, q < \infty$, $0 < r < \infty$ with $p + q \geq r$ and $\delta \in [0, 1] \cap [\frac{r-q}{r}, \frac{p}{r}]$ and $a, b, c \in \mathbb{R}$. Assume that $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$, $n = p(1 - a)$ and $c = \delta(a - 1) + b(1 - \delta)$. Then we have the Caffarelli-Kohn-Nirenberg type inequality for any function $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and for any homogeneous quasi-norm $|\cdot|$:

$$\| |x|^c f \|_{L^r(\mathbb{R}^n)} \leq p^\delta \| |x|^a \log |x| \left(\frac{df}{d|x|} \right) \|_{L^p(\mathbb{R}^n)}^\delta \| |x|^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta}. \quad (3.126)$$

By the Schwarz inequality with the standard Euclidean distance given by $|x|_E = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$, we obtain the Euclidean form of the Caffarelli-Kohn-Nirenberg type inequality:

$$\| |x|_E^c f \|_{L^r(\mathbb{R}^n)} \leq p^\delta \| |x|_E^a \log |x|_E \nabla f \|_{L^p(\mathbb{R}^n)}^\delta \| |x|_E^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta}, \quad (3.127)$$

where ∇ is the standard gradient in $\mathbb{R}^n$. Similarly, we write the inequality (3.124) in the Abelian case: Let $1 < p, q < \infty$, $0 < r < \infty$ with $p + q \geq r$ and $\delta \in [0, 1] \cap [\frac{r-q}{r}, \frac{p}{r}]$ and $a, b, c \in \mathbb{R}$. Assume that $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$, $n \neq p(1 - a)$ and $c = \delta(a - 1) + b(1 - \delta)$. Then we have Caffarelli-Kohn-Nirenberg type inequality for any function $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and for any homogeneous

quasi-norm $|\cdot|$:

$$\| |x|^c f \|_{L^r(\mathbb{R}^n)} \leq \left| \frac{p}{n - p(1 - a)} \right|^\delta \left\| |x|^a \left(\frac{df}{d|x|} \right) \right\|_{L^p(\mathbb{R}^n)}^\delta \| |x|^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta}. \quad (3.128)$$

Then, using the Schwarz inequality with the standard Euclidean distance given by $|x|_E = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$, we obtain the Euclidean form of the Caffarelli-Kohn-Nirenberg type inequality:

$$\| |x|_E^c f \|_{L^r(\mathbb{R}^n)} \leq \left| \frac{p}{n - p(1 - a)} \right|^\delta \| |x|_E^a \nabla f \|_{L^p(\mathbb{R}^n)}^\delta \| |x|_E^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta}. \quad (3.129)$$

Thus, the inequalities (3.127) and (3.129) are new already in the Abelian case and, moreover, (3.124) and (3.125) hold for any choice of homogeneous quasi-norm.

The proof of Theorem 3.44 is based on Theorem 3.29:

Proof of Theorem 3.44. Case $\delta = 0$. In this case, since we have $q = r$ and $b = c$ the inequalities (3.124) and (3.125) become the trivial estimate

$$\| |x|^b f \|_{L^q(\mathbb{G})} \leq \| |x|^b f \|_{L^q(\mathbb{G})}.$$

Case $\delta = 1$. Here, we have $p = r$ and $a - 1 = c$. By Theorem 3.29 and $\mathbb{E} = |x| \mathcal{R}_{|x|}$, for $Q + pc = Q + p(a - 1) \neq 0$ we have

$$\begin{aligned} \| |x|^c f \|_{L^r(\mathbb{G})} &\leq \left| \frac{p}{Q + pc} \right| \| |x|^c \mathbb{E} f \|_{L^r(\mathbb{G})} \\ &= \left| \frac{p}{Q + pc} \right| \| |x|^{c+1} \mathcal{R}_{|x|} f \|_{L^r(\mathbb{G})} \\ &= \left| \frac{p}{Q - p(1 - a)} \right| \| |x|^a \mathcal{R}_{|x|} f \|_{L^p(\mathbb{G})}, \end{aligned}$$

yielding (3.124). Now, for the case $Q + pc = Q + p(a - 1) = 0$ we obtain

from Theorem 3.29 that

$$\begin{aligned}\| |x|^c f \|_{L^r(\mathbb{G})} &\leq p \| |x|^c \log |x| \mathbb{E} f \|_{L^r(\mathbb{G})} \\ &= p \| |x|^{c+1} \log |x| \mathcal{R}_{|x|} f \|_{L^r(\mathbb{G})} \\ &= p \| |x|^a \log |x| \mathcal{R}_{|x|} f \|_{L^p(\mathbb{G})},\end{aligned}$$

which is (3.125) when $\delta = 1$. Since the constants in Theorem 3.29 are sharp, so the constants in (3.124) and (3.125) are sharp.

Case $\delta \in (0, 1) \cap [\frac{r-q}{r}, \frac{p}{r}]$. Taking into account $c = \delta(a-1) + b(1-\delta)$, we write as follows

$$\| |x|^c f \|_{L^r(\mathbb{G})} = \left(\int_{\mathbb{G}} |x|^{cr} |f(x)|^r dx \right)^{\frac{1}{r}} = \left(\int_{\mathbb{G}} \frac{|f(x)|^{\delta r}}{|x|^{\delta r(1-a)}} \cdot \frac{|f(x)|^{(1-\delta)r}}{|x|^{-br(1-\delta)}} dx \right)^{\frac{1}{r}}.$$

Since we have $\delta \in (0, 1) \cap [\frac{r-q}{r}, \frac{p}{r}]$ and $p+q \geq r$, then we use the Hölder inequality for $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ to get

$$\begin{aligned}\| |x|^c f \|_{L^r(\mathbb{G})} &\leq \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x|^{p(1-a)}} dx \right)^{\frac{\delta}{p}} \left(\int_{\mathbb{G}} \frac{|f(x)|^q}{|x|^{-bq}} dx \right)^{\frac{1-\delta}{q}} \\ &= \left\| \frac{f}{|x|^{1-a}} \right\|_{L^p(\mathbb{G})}^{\delta} \left\| \frac{f}{|x|^{-b}} \right\|_{L^q(\mathbb{G})}^{1-\delta}.\end{aligned}\tag{3.130}$$

Here one can remark that when $p = q$ and $a - b = 1$, the equality condition in the Hölder equality holds for any function. In the case $p \neq q$, we observe that the function

$$h(x) = |x|^{\frac{1}{(p-q)}(p(1-a)+bq)}\tag{3.131}$$

satisfies the Hölder equality condition

$$\frac{|h|^p}{|x|^{p(1-a)}} = \frac{|h|^q}{|x|^{-bq}}.$$

If $Q \neq p(1-a)$, then using Theorem 3.29 we obtain

$$\left\| \frac{f}{|x|^{1-a}} \right\|_{L^p(\mathbb{G})}^{\delta} \leq \left| \frac{p}{Q - p(1-a)} \right|^{\delta} \left\| \frac{\mathbb{E} f}{|x|^{1-a}} \right\|_{L^p(\mathbb{G})}^{\delta}$$

$$= \left| \frac{p}{Q - p(1-a)} \right|^\delta \left\| \frac{\mathcal{R}_{|x|} f}{|x|^{-a}} \right\|_{L^p(\mathbb{G})}^\delta, \quad 1 < p < \infty. \quad (3.132)$$

Now we plug this into (3.130) to get

$$\| |x|^c f \|_{L^r(\mathbb{G})} \leq \left| \frac{p}{Q - p(1-a)} \right|^\delta \left\| \frac{\mathcal{R}_{|x|} f}{|x|^{-a}} \right\|_{L^p(\mathbb{G})}^\delta \left\| \frac{f}{|x|^{-b}} \right\|_{L^q(\mathbb{G})}^{1-\delta}.$$

In the case $p = q$ and $a - b = 1$, since Hölder's equality conditions of (3.130) and (3.132), respectively, hold for any function and for $g(x)$ from (3.86), then we can take the function $g(x)$ as a function that satisfies Hölder's equality condition of both (3.130) and (3.132). As for the case $p \neq q$ and $p(1-a) + bq \neq 0$, similarly, one can observe that $h(x)$ from (3.131) can be chosen as a function that satisfies Hölder's equality condition of both (3.130) and (3.132). Thus, we can conclude that the constant in (3.124) is sharp when $p = q$ and $a - b = 1$ or $p \neq q$ and $p(1-a) + bq \neq 0$.

To finish the proof, it remains to consider the case $Q = p(1-a)$. By Theorem 3.29, we have

$$\left\| \frac{f}{|x|^{1-a}} \right\|_{L^p(\mathbb{G})}^\delta \leq p^\delta \left\| \frac{\log |x|}{|x|^{1-a}} \mathbb{E} f \right\|_{L^p(\mathbb{G})}^\delta, \quad 1 < p < \infty.$$

Then, we put this in (3.130) to obtain

$$\begin{aligned} \| |x|^c f \|_{L^r(\mathbb{G})} &\leq p^\delta \left\| \frac{\log |x|}{|x|^{1-a}} \mathbb{E} f \right\|_{L^p(\mathbb{G})}^\delta \left\| \frac{f}{|x|^{-b}} \right\|_{L^q(\mathbb{G})}^{1-\delta} \\ &= p^\delta \left\| \frac{\log |x|}{|x|^{-a}} \mathcal{R}_{|x|} f \right\|_{L^p(\mathbb{G})}^\delta \left\| \frac{f}{|x|^{-b}} \right\|_{L^q(\mathbb{G})}^{1-\delta}. \end{aligned}$$

□

3.5 HARDY INEQUALITIES FOR LANDAU HAMILTONIAN AND FOR BAOUENDI-GRUSHIN OPERATOR WITH AHARONOV-BOHM TYPE MAGNETIC FIELD

In this section we establish Hardy inequalities for the Landau Hamiltonian (or the twisted Laplacian). Moreover, we discuss Poincaré type inequality and inequalities with more general families of weights, all with estimates for the remainder terms of these inequalities. In addition, we obtain weighted Hardy inequalities for the quadratic form of the magnetic Baouendi-Grushin operator for the magnetic field of Aharonov-Bohm type. For these, we show refinements of the known Hardy inequalities for the Baouendi-Grushin operator involving radial derivatives in some of the variables. In general, our approach in this section is mainly based on the inequalities for the radial derivative and Euler operators from the previous sections. For example, to prove the Hardy inequalities for the Landau Hamiltonian we use Theorems 3.1, 3.29, 5.5.

3.5.1 WEIGHTED HARDY INEQUALITIES FOR MAGNETIC BAOUENDI-GRUSHIN OPERATOR WITH AHARONOV-BOHM TYPE MAGNETIC FIELD

In this section we discuss weighted Hardy inequalities for the quadratic form of the magnetic Baouendi-Grushin operator with Aharonov-Bohm type magnetic field.

The classical L^2 -Hardy inequality has been generalised for Baouendi-Grushin vector fields by Garofalo⁸⁴,

$$\begin{aligned} \int_{\mathbb{R}^n} (|\nabla_x f|^2 + |x|^{2\gamma} |\nabla_y f|^2) dx dy \\ \geq \left(\frac{Q-2}{2} \right)^2 \int_{\mathbb{R}^n} \left(\frac{|x|^{2\gamma}}{|x|^{2+2\gamma} + (1+\gamma)^2 |y|^2} \right) |f|^2 dx dy, \end{aligned} \quad (3.133)$$

where $x \in \mathbb{R}^m, y \in \mathbb{R}^k$ with $n = m + k, m, k \geq 1, \gamma \geq 0, Q = m + (1 + \gamma)k$ and $f \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^k \setminus \{(0, 0)\})$. Here, $\nabla_x f$ and $\nabla_y f$ are the gradients of f

in the variables x and y , respectively. In the case when $\gamma = 0$ we derive the classical L^2 -Hardy inequality from (3.133).

Let us put this result in perspective. Let $z = (x_1, \dots, x_m, y_1, \dots, y_k) = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$ with $k, m \geq 1$, $k + m = n$ and $\gamma \geq 0$. Let us consider the vector fields

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \dots, m, \quad Y_j = |x|^\gamma \frac{\partial}{\partial y_j}, \quad j = 1, \dots, k.$$

Then, the corresponding subelliptic gradient and Baouendi-Grushin operator on $\mathbb{R}^{m+k}$ are defined by

$$\nabla_\gamma := (X_1, \dots, X_m, Y_1, \dots, Y_k) = (\nabla_x, |x|^\gamma \nabla_y) \quad (3.134)$$

and

$$\Delta_\gamma = \sum_{i=1}^m X_i^2 + \sum_{j=1}^k Y_j^2 = \Delta_x + |x|^{2\gamma} \Delta_y = \nabla_\gamma \cdot \nabla_\gamma, \quad (3.135)$$

respectively. Here, Δ_x and Δ_y are the Laplace operators in the variables $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, respectively. The Baouendi-Grushin operator for an even positive integer γ is a sum of squares of C^∞ vector fields satisfying Hörmander condition

$$\text{rank Lie}[X_1, \dots, X_m, Y_1, \dots, Y_k] = n.$$

We can define on $\mathbb{R}^{m+k}$ the anisotropic dilation attached to Δ_γ as

$$\delta_\lambda(x, y) = (\lambda x, \lambda^{1+\gamma} y)$$

for $\lambda > 0$, and the homogeneous dimension with respect to this dilation is

$$Q = m + (1 + \gamma)k. \quad (3.136)$$

A change of variables formula for the Lebesgue measure implies that

$$d \circ \delta_\lambda(x, y) = \lambda^Q dx dy.$$

It is easy to check that

$$X_i \circ \delta_\lambda = \lambda \delta_\lambda(X_i), \quad Y_i \circ \delta_\lambda = \lambda \delta_\lambda(Y_i),$$

and hence

$$\nabla_\gamma \circ \delta_\lambda = \lambda \delta_\lambda \nabla_\gamma.$$

Let $\rho(z)$ be the corresponding distance function from the origin for $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$ such that

$$\rho = \rho(z) := (|x|^{2(1+\gamma)} + (1+\gamma)^2 |y|^2)^{\frac{1}{2(1+\gamma)}}. \quad (3.137)$$

By a direct calculation one obtains

$$|\nabla_\gamma \rho| = \frac{|x|^\gamma}{\rho^\gamma}. \quad (3.138)$$

The described setup is non-group setting that presents similarities.

Weighted L^p -versions of (3.133) were obtained by D'Ambrosio⁵²: Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $p > 1, k, m \geq 1, \alpha, \beta \in \mathbb{R}$ be such that $m + (1+\gamma)k > \alpha - \beta$ and $m > \gamma p - \beta$. Then for every $f \in D_\gamma^{1,p}(\Omega, |x|^{\beta-\gamma p} \rho^{(1+\gamma)p-\alpha})$ we have

$$\int_\Omega |\nabla_\gamma f|^p |x|^{\beta-\gamma p} \rho^{(1+\gamma)p-\alpha} dx dy \geq \left(\frac{Q + \beta - \alpha}{p} \right)^p \int_\Omega |f|^p \frac{|x|^\beta}{\rho^\alpha} dx dy, \quad (3.139)$$

where $D_\gamma^{1,p}(\Omega, \omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the norm $(\int_\Omega |\nabla_\gamma f|^p \omega dz dy)^{1/p}$ for $\omega \in L_{loc}^1(\Omega)$ with $\omega > 0$ a.e. on Ω .

If no point $(0, y)$ is in Ω , then the constant $\left(\frac{Q + \beta - \alpha}{p} \right)^p$ in (3.139) is sharp. The inequality (3.139) has also been established in¹³⁰, and in²⁰⁶ for $\Omega = \mathbb{R}^n$ with sharp constant. Moreover, in²⁰⁶, the authors obtained the following Hardy-Rellich type inequality

$$\left(\frac{Q - \alpha - 2}{2} \right)^2 \int_{\mathbb{R}^n} |\nabla_\gamma f|^2 \rho^\alpha \leq \int_{\mathbb{R}^n} |\Delta_\gamma f|^2 \rho^{\alpha+2} |\nabla_\gamma \rho|^{-2},$$

where the constant is sharp, $p > 1, \frac{2-Q}{3} \leq \alpha \leq Q - 2, f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$.

The inequalities of this type have been also investigated for some subelliptic operators of different types, see e.g.^{84, 85, 51, 52, 155, 128} and⁵⁹.

Here we discuss the following refinement of the Hardy inequality for the Baouendi-Grushin operator:

Theorem 3.47 (¹³³ Laptev-Ruzhansky-Y., Theorem 2.5). *Let $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_k) \in \mathbb{R}^m \times \mathbb{R}^k$ with $k, m \geq 1, k + m = n$. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be such that*

$$Q + \alpha_1 - 2 > 0 \text{ and } m + \gamma\alpha_2 > 0.$$

Then we have

$$\begin{aligned} \int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \left(\left| \frac{d}{d|x|} f \right|^2 + |x|^{2\gamma} |\nabla_y f|^2 \right) dx dy \\ \geq \left(\frac{Q + \alpha_1 - 2}{2} \right)^2 \int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2 dx dy, \end{aligned} \quad (3.140)$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, where the constant $\left(\frac{Q + \alpha_1 - 2}{2}\right)^2$ is sharp.

Remark 3.48. Already in the absence of weights, i.e. for $\alpha_1 = \alpha_2 = 0$, the estimate (3.140) is new. The obtained family of inequalities extends the known case of $k = 0$ when one has the classical Hardy inequality, and the case of $m = 0$ when one has the radial version established in¹²⁵ (always for $\gamma = 0$). We note that since we can estimate $\left| \frac{d}{d|x|} f \right| \leq |\nabla_x f|$, the inequality (3.140) also gives a refinement to the inequality (3.139) for $p = 2$. Moreover, Theorem 3.47 implies the following inequality

$$\begin{aligned} \int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} (|\nabla_x f|^2 + |x|^{2\gamma} |\nabla_y f|^2) dx dy \\ \geq \left(\frac{Q + \alpha_1 - 2}{2} \right)^2 \int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2 dx dy, \end{aligned} \quad (3.141)$$

with sharp constant, which gives the result of D'Ambrosio (3.139) with $p = 2$ and $\Omega = \mathbb{R}^n$. We also refer to¹³⁰ and²⁰⁶ for (3.141).

The estimate (3.140), in addition to its own interest, will play an important

role in the derivation of estimates for magnetic operators.

Before proving Theorem 3.47 let us show the following lemma:

Lemma 3.49 (¹³³ Laptev-Ruzhansky-Y., Lemma 2.1). *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_k) \in \mathbb{R}^m \times \mathbb{R}^k$ with $k, m \geq 1$, $k + m = n$. Let $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ be such that*

$$Q + \alpha_1 - 2 > 0 \text{ and } m + \alpha_2\gamma > 0.$$

Then for any real-valued function $f \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} \int_{\Omega} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} |(\nabla_{\gamma} + i\beta \mathcal{A})f|^2 dx dy &\geq \left(\left(\frac{Q + \alpha_1 - 2}{2} \right)^2 + \beta^2 \right) \\ &\times \int_{\Omega} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} \frac{|x|^{2\gamma}}{\rho^{2\gamma+2}} |f|^2 dx dy, \end{aligned} \quad (3.142)$$

where

$$\mathcal{A} = \frac{\nabla_{\gamma} \rho}{\rho} = \left(\frac{\nabla_x \rho}{\rho}, |x|^{\gamma} \frac{\nabla_y \rho}{\rho} \right) \in \mathbb{R}^m \times \mathbb{R}^k. \quad (3.143)$$

Moreover, if $0 \in \Omega$, then the constant in (3.142) is sharp.

Proof of Lemma 3.49. By opening brackets we have

$$\begin{aligned} \int_{\Omega} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} |(\nabla_{\gamma} + i\beta \mathcal{A})f|^2 dx dy &= \int_{\Omega} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} |\nabla_{\gamma} f|^2 dx dy \\ &+ \beta^2 \int_{\Omega} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} |\mathcal{A}f|^2 dx dy. \end{aligned} \quad (3.144)$$

By (3.138) and (3.139) with $p = 2$, $\alpha = \gamma(\alpha_2 + 2) + 2 - \alpha_1$, $\beta = \gamma(\alpha_2 + 2)$, we have for any $Q + \alpha_1 - 2 > 0$ and $m + \alpha_2\gamma > 0$ that

$$\int_{\Omega} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} |\nabla_{\gamma} f|^2 dx dy \geq \left(\frac{Q + \alpha_1 - 2}{2} \right)^2 \int_{\Omega} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} \frac{|\nabla_{\gamma} \rho|^2}{\rho^2} |f|^2 dx dy. \quad (3.145)$$

Taking into account the form of the magnetic field

$$\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2) = \left(\frac{\nabla_x \rho}{\rho}, |x|^\gamma \frac{\nabla_y \rho}{\rho} \right)$$

in (3.143), and by a direct calculation we observe that

$$\begin{aligned} \beta^2 \int_{\Omega} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} |\mathcal{A}f|^2 dx dy &= \beta^2 \int_{\Omega} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} \frac{|\nabla_x \rho|^2 + |x|^{2\gamma} |\nabla_y \rho|^2}{\rho^2} |f|^2 dx dy \\ &= \beta^2 \int_{\Omega} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} \frac{|\nabla_{\gamma} \rho|^2}{\rho^2} |f|^2 dx dy. \end{aligned} \quad (3.146)$$

Now we plug (3.145) and (3.146) into (3.144) to obtain

$$\begin{aligned} \int_{\Omega} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} |(\nabla_{\gamma} + i\beta \mathcal{A})f|^2 dz &\geq \left(\left(\frac{Q + \alpha_1 - 2}{2} \right)^2 + \beta^2 \right) \\ &\times \int_{\Omega} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} \frac{|\nabla_{\gamma} \rho|^2}{\rho^2} |f|^2 dx dy, \end{aligned} \quad (3.147)$$

which gives (3.142) by (3.138). Since the constant in (3.145) is sharp when $0 \in \Omega$ by (3.139), then the constant in the obtained inequality is sharp too when $0 \in \Omega$. $\square$

We obtain the following corollary in $\mathbb{R}^{2+k}$ for the Aharonov-Bohm potential of the type considered in⁸.

Corollary 3.50. *Let $\Omega \subset \mathbb{R}^{2+k}$ be an open set. Let $(x, y) = (x_1, x_2, y) \in \mathbb{R}^2 \times \mathbb{R}^k$. Let $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ be such that $\alpha_1 + k(\gamma + 1) > 0$ and $\alpha_2\gamma + 2 > 0$. Then for any real-valued function $f \in C_0^\infty(\Omega)$ and for the following Aharonov-Bohm type magnetic field*

$$\tilde{\mathcal{A}} = \left(-\frac{\partial_{x_2} \rho}{\rho}, \frac{\partial_{x_1} \rho}{\rho}, -\frac{|x|^\gamma}{\sqrt{2}} \frac{\nabla_y \rho}{\rho}, \frac{|x|^\gamma}{\sqrt{2}} \frac{\nabla_y \rho}{\rho} \right), \quad (3.148)$$

we have

$$\int_{\Omega} \rho^{\alpha_1} |\tilde{\nabla}_{\gamma} \rho|^{\alpha_2} |(\tilde{\nabla}_{\gamma} + i\beta \tilde{\mathcal{A}})f|^2 dx dy \geq \left(\left(\frac{\alpha_1 + k(\gamma + 1)}{2} \right)^2 + \beta^2 \right)$$

$$\times \int_{\Omega} \rho^{\alpha_1} |\widetilde{\nabla}_{\gamma} \rho|^{\alpha_2} \frac{|x|^{2\gamma}}{\rho^{2\gamma+2}} |f|^2 dx dy, \quad (3.149)$$

where

$$\widetilde{\nabla}_{\gamma} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{|x|^{\gamma}}{\sqrt{2}} \nabla_y, \frac{|x|^{\gamma}}{\sqrt{2}} \nabla_y \right). \quad (3.150)$$

Moreover, if $0 \in \Omega$, then the constant $\left(\left(\frac{\alpha_1 + k(\gamma+1)}{2} \right)^2 + \beta^2 \right)$ in (3.149) is sharp.

Proof of Corollary 3.50. Since $m = 2$, we have $Q = 2 + k(1 + \gamma)$. Taking into account $|\widetilde{\mathcal{A}}f|^2 = |\mathcal{A}f|^2$, $|\widetilde{\nabla}_{\gamma} \rho| = |\nabla_{\gamma} \rho|$ and $|\widetilde{\nabla}_{\gamma} f|^2 = |\nabla_{\gamma} f|^2$, one can obtain (3.149) in the exact same way as in the proof of Lemma 3.49. $\square$

We are now in position to prove Theorem 3.47.

Proof of Theorem 3.47. Denote $r := |x|$ and $H(r, y) := \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2}$. Then, by (3.137) and (3.138) we rewrite $H(r, y)$ as follows

$$H(r, y) = \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} = r^{\alpha_2 \gamma} \rho^{\alpha_1 - \alpha_2 \gamma} = r^{\alpha_2 \gamma} (r^{2(1+\gamma)} + (1 + \gamma)^2 |y|^2)^{\frac{\alpha_1 - \alpha_2 \gamma}{2(1+\gamma)}}. \quad (3.151)$$

To obtain (3.140), it is enough to prove the following equality

$$\begin{aligned} & \int_{\mathbb{R}^k} \int_0^{\infty} \left(\left| \left(\partial_r + \alpha \frac{\partial_r \rho}{\rho} \right) f \right|^2 + r^{2\gamma} \left| \left(\nabla_y + \alpha \frac{\nabla_y \rho}{\rho} \right) f \right|^2 \right) r^{m-1} H(r, y) dr dy \\ &= \int_{\mathbb{R}^k} \int_0^{\infty} (|\partial_r f|^2 + r^{2\gamma} |\nabla_y f|^2) r^{m-1} H(r, y) dr dy \\ &\quad - ((Q + \alpha_1 - 2)\alpha - \alpha^2) \int_{\mathbb{R}^k} \int_0^{\infty} \frac{|\nabla_{\gamma} \rho|^2}{\rho^2} |f|^2 r^{m-1} H(r, y) dr dy. \end{aligned} \quad (3.152)$$

Indeed, then noting (3.151) and taking $\alpha = \frac{Q + \alpha_1 - 2}{2}$ we obtain (3.140).

Thus, to show (3.152) we compute

$$\begin{aligned} & \int_{\mathbb{R}^k} \int_0^{\infty} \left(\left| \left(\partial_r + \alpha \frac{\partial_r \rho}{\rho} \right) f \right|^2 + r^{2\gamma} \left| \left(\nabla_y + \alpha \frac{\nabla_y \rho}{\rho} \right) f \right|^2 \right) r^{m-1} H(r, y) dr dy \\ &= \int_{\mathbb{R}^k} \int_0^{\infty} (|\partial_r f|^2 + r^{2\gamma} |\nabla_y f|^2) r^{m-1} H(r, y) dr dy \end{aligned}$$

$$\begin{aligned}
& +\alpha^2 \int_{\mathbb{R}^k} \int_0^\infty \left(\left| \frac{\partial_r \rho}{\rho} \right|^2 + r^{2\gamma} \left| \frac{\nabla_y \rho}{\rho} \right|^2 \right) |f|^2 r^{m-1} H(r, y) dr dy \\
& + 2\alpha \operatorname{Re} \int_{\mathbb{R}^k} \int_0^\infty \frac{\partial_r \rho}{\rho} r^{m-1} H(r, y) \overline{\partial_r f} \cdot f dr dy \\
& + 2\alpha \operatorname{Re} \int_{\mathbb{R}^k} \int_0^\infty \frac{\nabla_y \rho}{\rho} \cdot \overline{\nabla_y f} r^{2\gamma+m-1} H(r, y) f dr dy \\
& =: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.153}$$

Let us now calculate I_2 , I_3 and I_4 separately. Since

$$\frac{\partial_r \rho}{\rho} = \frac{r^{2\gamma+1}}{\rho^{2\gamma+2}} \quad \text{and} \quad \frac{\nabla_y \rho}{\rho} = \frac{(\gamma+1)y}{\rho^{2\gamma+2}},$$

we have

$$\left| \frac{\partial_r \rho}{\rho} \right|^2 + r^{2\gamma} \left| \frac{\nabla_y \rho}{\rho} \right|^2 = \frac{r^{4\gamma+2} + r^{2\gamma}(\gamma+1)^2 |y|^2}{\rho^{4\gamma+4}} = \frac{r^{2\gamma}}{\rho^{2\gamma+2}} = \frac{|\nabla_\gamma \rho|^2}{\rho^2}. \tag{3.154}$$

Hence, for I_2 we get

$$I_2 = \alpha^2 \int_{-\infty}^\infty \int_0^\infty \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2 r^{m-1} H(r, y) dr dy. \tag{3.155}$$

For I_3 using the integration by parts we obtain

$$\begin{aligned}
I_3 &= -\alpha \int_{\mathbb{R}^k} \int_0^\infty (2\gamma + m + \gamma\alpha_2) \rho^{\alpha_1 - \alpha_2\gamma - 2\gamma - 2} r^{2\gamma+m-1 + \gamma\alpha_2} |f|^2 dr dy \\
& - \alpha \int_{\mathbb{R}^k} \int_0^\infty (\alpha_1 - \alpha_2\gamma - 2\gamma - 2) \rho^{\alpha_1 - \alpha_2\gamma - 4\gamma - 4} r^{4\gamma+m + \gamma\alpha_2 + 1} |f|^2 dr dy.
\end{aligned}$$

Taking into account $H(r, y) = r^{\alpha_2\gamma} \rho^{\alpha_1 - \alpha_2\gamma}$ by (3.151), it follows that

$$\begin{aligned}
I_3 &= -\alpha \int_{\mathbb{R}^k} \int_0^\infty \left((2\gamma + m + \gamma\alpha_2) \frac{r^{2\gamma}}{\rho^{2\gamma+2}} + (\alpha_1 - \alpha_2\gamma - 2\gamma - 2) \frac{r^{4\gamma+2}}{\rho^{4\gamma+4}} \right) \\
& \quad \times r^{m-1} H(r, y) |f|^2 dr dy
\end{aligned}$$

$$\begin{aligned}
&= -\alpha \int_{\mathbb{R}^k} \int_0^\infty \left(2\gamma + m + \gamma\alpha_2 + (\alpha_1 - \alpha_2\gamma - 2\gamma - 2) \frac{r^{2\gamma+2}}{\rho^{2\gamma+2}} \right) \frac{|\nabla_\gamma \rho|^2}{\rho^2} \\
&\quad \times |f|^2 r^{m-1} H(r, y) dr dy.
\end{aligned}$$

Similarly, for I_4 we have

$$\begin{aligned}
I_4 &= -\alpha \int_{\mathbb{R}^k} \int_0^\infty \operatorname{div}_y \left(H(r, y) \frac{\nabla_y \rho}{\rho} \right) r^{2\gamma+m-1} |f|^2 dr dy \\
&= -\alpha(\gamma + 1) \int_{\mathbb{R}^k} \int_0^\infty \operatorname{div}_y (\rho^{\alpha_1 - \alpha_2\gamma - 2\gamma - 2} y) r^{\alpha_2\gamma + 2\gamma + m - 1} |f|^2 dr dy \\
&= -\alpha \int_{\mathbb{R}^k} \int_0^\infty \left((\alpha_1 - \alpha_2\gamma - 2\gamma - 2) \rho^{\alpha_1 - \alpha_2\gamma - 2\gamma - 3} \frac{(\gamma + 1)^2 |y|^2}{\rho^{2\gamma+1}} \right) \\
&\quad \times r^{\alpha_2\gamma + 2\gamma + m - 1} |f|^2 dr dy \\
&\quad - \alpha \int_{\mathbb{R}^k} \int_0^\infty k(\gamma + 1) \rho^{\alpha_1 - \alpha_2\gamma - 2\gamma - 2} r^{\alpha_2\gamma + 2\gamma + m - 1} |f|^2 dr dy.
\end{aligned}$$

Since $\frac{r^{2\gamma}}{\rho^{2\gamma+2}} = \frac{|\nabla_\gamma \rho|^2}{\rho^2}$ by (3.154), then using $H(r, y) = r^{\alpha_2\gamma} \rho^{\alpha_1 - \alpha_2\gamma}$ we obtain

$$\begin{aligned}
I_4 &= -\alpha \int_{\mathbb{R}^k} \int_0^\infty \left((\alpha_1 - \alpha_2\gamma - 2\gamma - 2) \frac{(\gamma + 1)^2 |y|^2}{\rho^{2\gamma+2}} + k(\gamma + 1) \right) \\
&\quad \times \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2 r^{m-1} H(r, y) dr dy.
\end{aligned}$$

Then, recalling (3.137) we calculate $I_3 + I_4$ as follows

$$\begin{aligned}
I_3 + I_4 &= -\alpha \int_{\mathbb{R}^k} \int_0^\infty (\alpha_1 - \alpha_2\gamma - 2\gamma - 2 + 2\gamma + m + \gamma\alpha_2 + k(\gamma + 1)) \\
&\quad \times \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2 r^{m-1} H(r, y) dr dy \\
&= -\alpha \int_{\mathbb{R}^k} \int_0^\infty (Q + \alpha_1 - 2) \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2 r^{m-1} H(r, y) dr dy, \quad (3.156)
\end{aligned}$$

where we have used $Q = m + (1 + \gamma)k$. Now we put (3.155) and (3.156) into (3.153) to obtain (3.152).

Now, to show the sharpness of the constant in (3.140) we observe that

$$\begin{aligned}
& \int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} (|\nabla_x f|^2 + |x|^{2\gamma} |\nabla_y f|^2) dx dy \\
& \geq \int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \left(\left| \frac{d}{d|x|} f \right|^2 + |x|^{2\gamma} |\nabla_y f|^2 \right) dx dy \\
& \geq \left(\frac{Q + \alpha_1 - 2}{2} \right)^2 \int_{\mathbb{R}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|\nabla_\gamma \rho|^2}{\rho^2} |f|^2 dx dy.
\end{aligned}$$

Therefore, the constant $\left(\frac{Q+\alpha_1-2}{2}\right)^2$ in (3.140) is sharp, since we know that the same constant is sharp in (3.139). $\square$

We now state the main theorem of this section for complex-valued functions f .

Theorem 3.51 (¹³³ Laptev-Ruzhansky-Y., Theorem 2.4). *Let $(x, y) = (x_1, x_2, y) \in \mathbb{R}^2 \times \mathbb{R}^k$. Let $\alpha_1, \alpha_2, \beta \in \mathbb{R}$ be such that $\alpha_1 + k(\gamma + 1) > 0$, $\alpha_2 + 2\gamma > 0$ and $-1/2 \leq \beta \leq 1/2$. Recall $\widetilde{\nabla}_\gamma$, $\widetilde{\mathcal{A}}$ and ρ defined in (3.150), (3.148), (3.137), respectively. Then for any complex-valued function $f \in C_0^\infty(\mathbb{R}^{2+k} \setminus \{0\})$ we have*

$$\begin{aligned}
& \int_{\mathbb{R}^{2+k}} \rho^{\alpha_1} |\widetilde{\nabla}_\gamma \rho|^{\alpha_2} |(\widetilde{\nabla}_\gamma + i\beta \widetilde{\mathcal{A}})f|^2 dx dy \\
& \geq \left(\left(\frac{\alpha_1 + k(\gamma + 1)}{2} \right)^2 + \beta^2 \right) \int_{\mathbb{R}^{2+k}} \rho^{\alpha_1} |\widetilde{\nabla}_\gamma \rho|^{\alpha_2} \frac{|x|^{2\gamma}}{\rho^{2\gamma+2}} |f|^2 dx dy, \quad (3.157)
\end{aligned}$$

and the constant $\left(\frac{\alpha_1+k(\gamma+1)}{2}\right)^2 + \beta^2$ is sharp.

Proof of Theorem 3.51. By polar coordinates for x -plane $x_1 = r \cos \phi$, $x_2 = r \sin \phi$, we have $r = |x|$ and

$$\begin{aligned}
\frac{\partial_{x_1} \rho}{\rho} &= \frac{r^{2\gamma+1} \cos \phi}{r^{2(1+\gamma)} + (1+\gamma)^2 |y|^2}, \quad \frac{\partial_{x_2} \rho}{\rho} = \frac{r^{2\gamma+1} \sin \phi}{r^{2(1+\gamma)} + (1+\gamma)^2 |y|^2}, \\
\frac{\nabla_y \rho}{\rho} &= \frac{(1+\gamma)y}{r^{2(1+\gamma)} + (1+\gamma)^2 |y|^2}.
\end{aligned}$$

Thus, we can write

$$\int_{\mathbb{R}^{2+k}} \rho^{\alpha_1} |\widetilde{\nabla}_\gamma \rho|^{\alpha_2} |(\widetilde{\nabla}_\gamma + i\beta \widetilde{\mathcal{A}})f|^2 dx_1 dx_2 dy = I_1 + I_2, \quad (3.158)$$

where

$$\begin{aligned} I_1 = & \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \left| \left(\cos \phi \partial_r - \frac{\sin \phi}{r} \partial_\phi - i\beta \frac{r^{2\gamma+1} \sin \phi}{r^{2(1+\gamma)} + (1+\gamma)^2 |y|^2} \right) f \right|^2 \\ & \times r H(r, y) dr d\phi dy \\ & + \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \left| \left(\sin \phi \partial_r + \frac{\cos \phi}{r} \partial_\phi + i\beta \frac{r^{2\gamma+1} \cos \phi}{r^{2(1+\gamma)} + (1+\gamma)^2 |y|^2} \right) f \right|^2 \\ & \times r H(r, y) dr d\phi dy \end{aligned}$$

and

$$\begin{aligned} I_2 = & \frac{1}{2} \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \left| \left(\nabla_y - i\beta \frac{(1+\gamma)y}{r^{2(1+\gamma)} + (1+\gamma)^2 |y|^2} \right) f \right|^2 r^{2\gamma+1} H(r, y) dr d\phi dy \\ & + \frac{1}{2} \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \left| \left(\nabla_y + i\beta \frac{(1+\gamma)y}{r^{2(1+\gamma)} + (1+\gamma)^2 |y|^2} \right) f \right|^2 r^{2\gamma+1} H(r, y) dr d\phi dy. \end{aligned}$$

By opening the brackets we obtain

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \left(|\partial_r f|^2 + \frac{|\partial_\phi f|^2}{r^2} + \frac{\beta^2 r^{4\gamma+2} |f|^2}{(r^{2(1+\gamma)} + (1+\gamma)^2 |y|^2)^2} \right) r H(r, y) dr d\phi dy \\ &+ 2\operatorname{Re} \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \left(\frac{\partial_\phi f}{r} \cdot \frac{i\beta r^{2\gamma+1} f}{r^{2(1+\gamma)} + (1+\gamma)^2 |y|^2} \right) r H(r, y) dr d\phi dy \\ &= \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \left(|\partial_r f|^2 + \frac{1}{r^2} \left| \partial_\phi f + i\beta \frac{r^{2\gamma+2}}{r^{2(1+\gamma)} + (1+\gamma)^2 |y|^2} f \right|^2 \right) r H(r, y) dr d\phi dy \end{aligned}$$

and

$$I_2 = \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty |\nabla_y f|^2 r^{2\gamma+1} H(r, y) dr d\phi dy$$

$$+\beta^2(1+\gamma)^2 \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \frac{|y|^2|f|^2}{(r^{2(1+\gamma)} + (1+\gamma)^2|y|^2)^2} r^{2\gamma+1} H(r, y) dr d\phi dy.$$

Using the Fourier series for f we can expand

$$f(r, \phi, y) = \sum_{k=-\infty}^{\infty} f_k(r, y) e^{ik\phi},$$

and by a direct calculation, we get

$$\begin{aligned} & \frac{1}{r^2} \int_0^{2\pi} \left| \partial_\phi f + i\beta \frac{r^{2\gamma+2}}{r^{2(1+\gamma)} + (1+\gamma)^2|y|^2} f \right|^2 d\phi \\ &= \frac{2\pi}{r^2} \sum_k \left(k + \beta \frac{r^{2\gamma+2}}{r^{2(1+\gamma)} + (1+\gamma)^2|y|^2} \right)^2 |f_k(r, y)|^2 \\ &\geq \frac{2\pi}{r^2} \min_k \left(k + \beta \frac{r^{2\gamma+2}}{r^{2(1+\gamma)} + (1+\gamma)^2|y|^2} \right)^2 \sum_k |f_k(r, y)|^2 \\ &= \frac{1}{r^2} \min_k \left(k + \beta \frac{r^{2\gamma+2}}{r^{2(1+\gamma)} + (1+\gamma)^2|y|^2} \right)^2 \int_0^{2\pi} |f|^2 d\phi \\ &= \beta^2 \frac{r^{4\gamma+2}}{(r^{2(1+\gamma)} + (1+\gamma)^2|y|^2)^2} \int_0^{2\pi} |f|^2 d\phi. \end{aligned}$$

Now putting the obtained estimates for I_1 and I_2 in (3.158) we have

$$\begin{aligned} & \int_{\mathbb{R}^{2+k}} \rho^{\alpha_1} |\widetilde{\nabla}_\gamma \rho|^{\alpha_2} |(\widetilde{\nabla}_\gamma + i\beta \widetilde{\mathcal{A}})f|^2 dx_1 dx_2 dy \\ &\geq \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty (|\partial_r f|^2 + r^{2\gamma} |\nabla_y f|^2) r H(r, y) dr d\phi dy \\ &+ \beta^2 \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \left(\frac{r^{4\gamma+2} + (1+\gamma)^2|y|^2 r^{2\gamma}}{(r^{2\gamma+2} + (1+\gamma)^2|y|^2)^2} \right) |f|^2 r H(r, y) dr d\phi dy \\ &= \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty (|\partial_r f|^2 + r^{2\gamma} |\nabla_y f|^2) r H(r, y) dr d\phi dy \\ &+ \beta^2 \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \frac{r^{2\gamma}}{r^{2\gamma+2} + (1+\gamma)^2|y|^2} |f|^2 r H(r, y) dr d\phi dy. \quad (3.159) \end{aligned}$$

Since $|\widetilde{\nabla}_\gamma \rho| = |\nabla_\gamma \rho|$, then putting $m = 2$ in (3.145) and taking into account

(3.151) we obtain the following estimate for the first integral in (3.159)

$$\begin{aligned}
& \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty (|\partial_r f|^2 + r^{2\gamma} |\nabla_y f|^2) r H(r, y) dr d\phi dy \\
& \geq \left(\frac{\alpha_1 + k(\gamma + 1)}{2} \right)^2 \int_{\mathbb{R}^{2+k}} \rho^{\alpha_1} |\widetilde{\nabla}_\gamma \rho|^{\alpha_2} \frac{|\widetilde{\nabla}_\gamma \rho|^2}{\rho^2} |f|^2 dx dy. \tag{3.160}
\end{aligned}$$

Let us now calculate the second integral in (3.159) as follows

$$\begin{aligned}
& \beta^2 \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \frac{r^{2\gamma}}{r^{2\gamma+2} + (1 + \gamma)^2 |y|^2} |f|^2 r H(r, y) dr d\phi dy \\
& = \beta^2 \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \frac{r^{4\gamma+2} \cos^2 \phi}{(r^{2\gamma+2} + (1 + \gamma)^2 |y|^2)^2} |f|^2 r H(r, y) dr d\phi dy \\
& + \beta^2 \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \frac{r^{4\gamma+2} \sin^2 \phi}{(r^{2\gamma+2} + (1 + \gamma)^2 |y|^2)^2} |f|^2 r H(r, y) dr d\phi dy \\
& + \beta^2 \int_{\mathbb{R}^k} \int_0^{2\pi} \int_0^\infty \frac{(1 + \gamma)^2 r^{2\gamma} |y|^2}{(r^{2\gamma+2} + (1 + \gamma)^2 |y|^2)^2} |f|^2 r H(r, y) dr d\phi dy \\
& = \beta^2 \int_{\mathbb{R}^{2+k}} \rho^{\alpha_1} |\widetilde{\nabla}_\gamma \rho|^{\alpha_2} \left| \left(\frac{\partial_{x_1} \rho}{\rho}, \frac{\partial_{x_2} \rho}{\rho}, \frac{|x|^\gamma}{\sqrt{2}} \frac{\nabla_y \rho}{\rho}, \frac{|x|^\gamma}{\sqrt{2}} \frac{\nabla_y \rho}{\rho} \right) \right|^2 |f|^2 dx dy \\
& = \beta^2 \int_{\mathbb{R}^{2+k}} \rho^{\alpha_1} |\widetilde{\nabla}_\gamma \rho|^{\alpha_2} \frac{|\widetilde{\nabla}_\gamma \rho|^2}{\rho^2} |f|^2 dx dy. \tag{3.161}
\end{aligned}$$

Putting the estimates (3.160) and (3.161) in (3.159), we obtain (3.157). Since we have (3.157) with sharp constant for all real-valued functions by Corollary 3.50, then this constant is sharp also in the class of complex-valued functions in (3.157). $\square$

3.5.2 HARDY INEQUALITIES FOR LANDAU HAMILTONIAN

In this section we discuss Hardy inequalities for the Landau Hamiltonian (or the twisted Laplacian) using the inequalities (3.2), (3.82), (3.83) and (5.11) associated to the radial derivative operator.

Let us recall that the Landau Hamiltonian on $\mathbb{C}^n$ is defined as

$$\mathcal{L} = \sum_{j=1}^n \left[\left(i\partial_{x_j} + \frac{y_j}{2} \right)^2 + \left(i\partial_{y_j} - \frac{x_j}{2} \right)^2 \right]. \quad (3.162)$$

Setting

$$\tilde{X}_j = \partial_{x_j} - \frac{1}{2}iy_j \quad \text{and} \quad \tilde{Y}_j = \partial_{y_j} + \frac{1}{2}ix_j,$$

we have

$$\mathcal{L} = - \sum_{j=1}^n (\tilde{X}_j^2 + \tilde{Y}_j^2). \quad (3.163)$$

The twisted Laplacian can be also written as $\mathcal{L} = -\Delta + \frac{1}{4}(|x|^2 + |y|^2) + iN$, where

$$N = \sum_{j=1}^n (y_j \partial_{x_j} - x_j \partial_{y_j}) \quad (3.164)$$

is the rotation field. Let $\nabla_{\mathcal{L}}$ be the gradient operator associated with $\mathcal{L}$:

$$\nabla_{\mathcal{L}} f = (\tilde{X}_1 f, \dots, \tilde{X}_n f, \tilde{Y}_1 f, \dots, \tilde{Y}_n f). \quad (3.165)$$

Let $W_{\mathcal{L}}^{1,2}(\mathbb{C}^n)$ be the Sobolev space defined by

$$W_{\mathcal{L}}^{1,2}(\mathbb{C}^n) = \{f \in L^2(\mathbb{C}^n) : \tilde{X}_j f, \tilde{Y}_j f \in L^2(\mathbb{C}^n), 1 \leq j \leq n\}. \quad (3.166)$$

In the paper⁶, a version of the Hardy inequality for the Landau Hamiltonian (or the twisted Laplacian) was established for *real-valued* functions $f \in W_{\mathcal{L}}^{1,2}(\mathbb{C}^n)$, namely, the inequality

$$\frac{1}{4} \int_{\mathbb{C}^n} |f|^2 \omega(z) dz \leq \int_{\mathbb{C}^n} |\nabla_{\mathcal{L}} f|^2 dz,$$

with the weight

$$\omega(z) = \left(\frac{|\nabla_{\mathcal{L}} \varepsilon|^2}{\varepsilon^2} + \frac{|z|^2}{4} \right),$$

where ε is a fundamental solution to the twisted Laplacian on $\mathbb{C}^n$. Moreover, it is shown there that the weight $\omega(z)$ is optimal and never achieved, and the constant $1/4$ is the best for this weight, and never achieved.

In this section we obtain the versions of Hardy inequalities for the Landau Hamiltonian for both *complex-valued* and *real-valued functions*.

In fact, we obtain results for operators $\tilde{\mathcal{L}}_\psi$ of the form

$$\tilde{\mathcal{L}}_\psi = \sum_{j=1}^n \left[(i\partial_{x_j} + \psi(|z|)y_j)^2 + (i\partial_{y_j} - \psi(|z|)x_j)^2 \right],$$

where $\psi(|z|)$ is a radial real-valued differentiable function satisfying some conditions, $z = (x, y)$. Setting

$$\check{X}_j = \partial_{x_j} - i\psi(|z|)y_j \quad \text{and} \quad \check{Y}_j = \partial_{y_j} + i\psi(|z|)x_j,$$

we write

$$\widetilde{\nabla_{\mathcal{L}_\psi}} f = (\check{X}_1 f, \dots, \check{X}_n f, \check{Y}_1 f, \dots, \check{Y}_n f). \quad (3.167)$$

As usual, we will identify $\mathbb{C} \cong \mathbb{R}^2$.

Let $\Omega \subset \mathbb{R}^2$ be an open set and let $\widehat{\mathfrak{L}}_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega \setminus \{0\})$ with respect to

$$\|f\|_{\widehat{\mathfrak{L}}_0^{1,2}(\Omega)} = \|f\|_{L^2(\Omega)} + \|\mathbb{E}f\|_{L^2(\Omega)}. \quad (3.168)$$

We then have the following result for complex-valued functions f .

Theorem 3.52. *Let $\theta_1, \theta_2, \theta_3, \theta_4, a, b \in \mathbb{R}$ with $a, b > 0$, $\theta_1 \neq 0$, $\theta_2\theta_3 < 0$ and $2\theta_4 \leq \theta_2\theta_3$. Let $\psi = \psi(|z|)$ be a radial real-valued function such that $\psi \in L_{loc}^2(\mathbb{R}_+)$ with*

$$|\psi(r)|r^2 \leq \frac{1}{2}, \quad \forall r \in (0, \infty). \quad (3.169)$$

Let Ω be a bounded domain in $\mathbb{C}$ and $R = \sup_{z \in \Omega} \{|z|\}$.

Then we have the following inequalities:

(i) the weighted Hardy-Sobolev inequality

$$\int_{\mathbb{C}} \frac{|\widetilde{\nabla_{\mathcal{L}} f}|^2}{|z|^{2\theta_1}} dz - \theta_1^2 \int_{\mathbb{C}} \frac{|f|^2}{|z|^{2\theta_1+2}} dz \geq \int_{\mathbb{C}} \frac{(\psi(|z|))^2}{|z|^{2\theta_1-2}} |f|^2 dz, \quad (3.170)$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$;

(ii) the logarithmic Hardy inequality

$$\int_{\mathbb{C}} |\widetilde{\nabla}_{\mathcal{L}} f|^2 \log |z|^2 dz - \frac{1}{4} \int_{\mathbb{C}} |f|^2 dz \geq \int_{\mathbb{C}} (\psi(|z|))^2 |z|^2 \log |z|^2 |f|^2 dz, \quad (3.171)$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$;

(iii) the Poincaré inequality

$$\int_{\Omega} |\widetilde{\nabla}_{\mathcal{L}} f|^2 dz - \frac{1}{R^2} \int_{\Omega} |f|^2 dz \geq \int_{\Omega} (\psi(|z|))^2 |z|^2 |f|^2 dz, \quad (3.172)$$

for all complex-valued functions $f \in \widehat{\mathfrak{L}}_0^{1,2}(\Omega)$ satisfying $\frac{d}{d|z|} f \in L^2(\Omega)$, where the space $\widehat{\mathfrak{L}}_0^{1,2}(\Omega)$ is defined in (3.168);

(iv) the Hardy-Sobolev inequality with more general weights

$$\begin{aligned} \int_{\mathbb{C}} \frac{(a + b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4}} |\widetilde{\nabla}_{\mathcal{L}} f|^2 dz &\geq \frac{\theta_2 \theta_3 - 2\theta_4}{2} \int_{\mathbb{C}} \frac{(a + b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4+2}} |f|^2 dz \\ &+ \int_{\mathbb{C}} \frac{(\psi(|z|))^2 (a + b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4-2}} |f|^2 dz, \end{aligned} \quad (3.173)$$

for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$.

Proof of Theorem 3.52. Let $\kappa(|z|) \neq 0$ be a radial function. By polar coordinates for the z -plane $x = r \cos \phi$, $y = r \sin \phi$, we have

$$\begin{aligned} \int_{\mathbb{C}} \frac{|\widetilde{\nabla}_{\mathcal{L}} f|^2}{\kappa(|z|)} dz &= \int_{\mathbb{C}} (|(i\partial_x + y\psi(|z|)) f|^2 + |(i\partial_y - x\psi(|z|)) f|^2) \frac{dz}{\kappa(|z|)} \\ &= \int_0^\infty \int_0^{2\pi} \left(\left| \left(i \cos \phi \partial_r - \frac{i \sin \phi}{r} \partial_\phi + \psi(r) r \sin \phi \right) f \right|^2 \right) r d\phi \frac{dr}{\kappa(r)} \\ &+ \int_0^\infty \int_0^{2\pi} \left(\left| \left(i \sin \phi \partial_r + \frac{i \cos \phi}{r} \partial_\phi - \psi(r) r \cos \phi \right) f \right|^2 \right) r d\phi \frac{dr}{\kappa(r)} \\ &= \int_0^\infty \int_0^{2\pi} \left(|\partial_r f|^2 + \frac{|\partial_\phi f|^2}{r^2} + (\psi(r))^2 r^2 |f|^2 \right) r d\phi \frac{dr}{\kappa(r)} \end{aligned}$$

$$\begin{aligned}
& -2\operatorname{Re} \int_0^\infty \int_0^{2\pi} \left(\frac{i\partial_\phi f}{r} \cdot \overline{\psi(r)r f} \right) r d\phi \frac{dr}{\kappa(r)} \\
& = \int_0^\infty \int_0^{2\pi} \left(|\partial_r f|^2 + \frac{1}{r^2} |i\partial_\phi f - \psi(r)r^2 f|^2 \right) r d\phi \frac{dr}{\kappa(r)}.
\end{aligned}$$

Let us represent f via its Fourier series

$$f(r, \phi) = \sum_{k=-\infty}^{\infty} f_k(r) e^{ik\phi}.$$

Then from assumptions on ψ we obtain

$$\begin{aligned}
& \frac{1}{r^2} \int_0^{2\pi} |i\partial_\phi f - \psi(r)r^2 f|^2 d\phi = \frac{2\pi}{r^2} \sum_k (k + \psi(r)r^2)^2 |f_k(r)|^2 \\
& \geq \frac{2\pi}{r^2} \min_k (k + \psi(r)r^2)^2 \sum_k |f_k(r)|^2 = \frac{1}{r^2} \min_k (k + \psi(r)r^2)^2 \int_0^{2\pi} |f|^2 d\phi \\
& = r^2 (\psi(r))^2 \int_0^{2\pi} |f|^2 d\phi.
\end{aligned}$$

Thus, we arrive at

$$\int_{\mathbb{C}} \frac{|\widetilde{\nabla_{\mathcal{L}}} f|^2}{\kappa(|z|)} dz \geq \int_{\mathbb{C}} \frac{1}{\kappa(|z|)} \left| \frac{d}{d|z|} f \right|^2 dz + \int_{\mathbb{C}} \frac{|z|^2 (\psi(|z|))^2}{\kappa(|z|)} |f|^2 dz. \quad (3.174)$$

Now, we choose different $\kappa(|z|)$ in (3.174) so that we can use the inequalities (3.82), (3.83), (5.11) and (3.2) to obtain (3.170), (3.171), (3.172) and (3.173), respectively.

First, taking $\kappa(|z|) = |z|^{2\theta_1}$ we obtain

$$\int_{\mathbb{C}} \frac{|\widetilde{\nabla_{\mathcal{L}}} f|^2}{|z|^{2\theta_1}} dz \geq \int_{\mathbb{C}} \frac{1}{|z|^{2\theta_1}} \left| \frac{d}{d|z|} f \right|^2 dz + \int_{\mathbb{C}} \frac{(\psi(|z|))^2}{|z|^{2\theta_1-2}} |f|^2 dz. \quad (3.175)$$

In the Abelian case $\mathbb{G} = (\mathbb{R}^2, +)$, $p = 2$ and $\theta = \theta_1 + 1$ with $\theta_1 \neq 0$, (3.82) implies

$$\int_{\mathbb{C}} \frac{1}{|z|^{2\theta_1}} \left| \frac{d}{d|z|} f \right|^2 dz \geq \theta_1^2 \int_{\mathbb{C}} \frac{|f|^2}{|z|^{2\theta_1+2}} dz.$$

We use this in (3.175) to obtain (3.170).

Putting $\kappa(|z|) = |\log |z||^{-2}$ in (3.174), one has

$$\int_{\mathbb{C}} |\log |z||^2 |\widetilde{\nabla}_{\mathcal{L}} f|^2 dz \geq \int_{\mathbb{C}} |\log |z||^2 \left| \frac{d}{d|z|} f \right|^2 dz + \int_{\mathbb{C}} (\psi(|z|))^2 |z|^2 |\log |z||^2 |f|^2 dz. \quad (3.176)$$

In the Abelian case $\mathbb{G} = (\mathbb{R}^2, +)$ and $p = 2$, (3.83) implies

$$\int_{\mathbb{C}} |\log |z||^2 \left| \frac{d}{d|z|} f \right|^2 dz \geq \frac{1}{4} \int_{\mathbb{C}} |f|^2 dz.$$

Combining this with (3.176) we obtain (3.171).

We take $\kappa(|z|) = 1$ in (3.174) to get

$$\int_{\mathbb{C}} |\widetilde{\nabla}_{\mathcal{L}} f|^2 dz \geq \int_{\mathbb{C}} \left| \frac{d}{d|z|} f \right|^2 dz + \int_{\mathbb{C}} (\psi(|z|))^2 |z|^2 |f|^2 dz. \quad (3.177)$$

In the abelian case, when Ω is a bounded domain in $(\mathbb{R}^2, +)$, and $p = 2$, (5.11) implies

$$\int_{\Omega} \left| \frac{d}{d|z|} f \right|^2 dz \geq \frac{1}{R^2} \int_{\Omega} |f|^2 dz.$$

From this and (3.177) we derive (3.172).

Putting $\kappa(|z|) = \frac{(a+b|z|^{\theta_2})^{-\theta_3}}{|z|^{-2\theta_4}}$ in (3.174), we have

$$\begin{aligned} \int_{\mathbb{C}} \frac{(a+b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4}} |\widetilde{\nabla}_{\mathcal{L}} f|^2 dz &\geq \int_{\mathbb{C}} \frac{(a+b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4}} \left| \frac{d}{d|z|} f \right|^2 dz \\ &\quad + \int_{\mathbb{C}} \frac{\psi^2(|z|)(a+b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4-2}} |f|^2 dz. \end{aligned} \quad (3.178)$$

Again, in the Abelian case $\mathbb{G} = (\mathbb{R}^2, +)$ and $p = 2$, (3.2) gives

$$\frac{\theta_2\theta_3 - 2\theta_4}{2} \int_{\mathbb{C}} \frac{(a+b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4+2}} |f|^2 dz \leq \int_{\mathbb{C}} \frac{(a+b|z|^{\theta_2})^{\theta_3}}{|z|^{2\theta_4}} \left| \frac{d}{d|z|} f \right|^2 dz,$$

for $a, b > 0$, $\theta_2\theta_3 < 0$ and $2\theta_4 \leq \theta_2\theta_3$. Then, this and (3.178) imply (3.173).

□

Now we give some inequalities for real-valued functions to show the best estimates one can expect. While estimates for real-valued functions have less physical meaning than those for complex-valued functions in questions of the spectral theory, they also find their use in applications to the existence of real (or positive) solutions to some nonlinear equations.

Remark 3.53. Let $\psi = \psi(|z|)$ be a radial real-valued function such that $\psi \in L^2_{loc}(\mathbb{R}_+)$. By a direct calculation we have for all real-valued functions $f \in W^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ that

$$\begin{aligned} \int_{\mathbb{C}^n} |\widetilde{\nabla}_{\mathcal{L}} f|^2 dz &= \sum_{j=1}^n \int_{\mathbb{C}^n} \left(|(i\partial_{x_j} + y_j \psi(|z|)) f|^2 + |(i\partial_{y_j} - x_j \psi(|z|)) f|^2 \right) dz \\ &= \int_{\mathbb{C}^n} |\nabla f|^2 dz + \int_{\mathbb{C}^n} |z|^2 (\psi(|z|))^2 |f|^2 dz, \end{aligned} \quad (3.179)$$

where $W^{1,2}_{\mathcal{L}}$ is defined in (3.166). Using the well-known Hardy inequality for $n \geq 1$, we obtain

$$\int_{\mathbb{C}^n} |\widetilde{\nabla}_{\mathcal{L}} f|^2 dz \geq (n-1)^2 \int_{\mathbb{C}^n} \frac{|f|^2}{|z|^2} dz + \int_{\mathbb{C}^n} |z|^2 (\psi(|z|))^2 |f|^2 dz, \quad (3.180)$$

for all real-valued functions $f \in W^{1,2}_{\mathcal{L}}(\mathbb{C}^n)$ and $n \geq 1$. We note that for $\psi(|z|) = \frac{1}{2}$ we recover the classical Landau Hamiltonian, i.e.

$$\widetilde{\mathcal{L}}_{1/2} = -\mathcal{L} \text{ and } \widetilde{\nabla_{\mathcal{L}_{1/2}}} = \nabla_{\mathcal{L}}.$$

Let Ω is a bounded domain in $\mathbb{C}$ with $0 \in \Omega$ and $R \geq e \cdot \sup_{z \in \Omega} \{|z|\}$. Then, we use the critical Hardy inequality with sharp constant for $f \in W^{1,2}_{\mathcal{L}}(\Omega)$ (see for example^{4, 7)} to obtain

$$\int_{\Omega} |\widetilde{\nabla}_{\mathcal{L}} f|^2 dz \geq \frac{1}{4} \int_{\Omega} \frac{|f|^2}{|z|^2 \left(\log \frac{R}{|z|} \right)^2} dz + \int_{\Omega} |z|^2 (\psi(|z|))^2 |f|^2 dz, \quad (3.181)$$

for all real-valued functions $f \in W^{1,2}_{\mathcal{L}}(\Omega)$. Since the constants in the Hardy

and critical Hardy inequalities are sharp, then the constants in the obtained inequalities (3.180) and (3.181) are sharp too.

Remark 3.54. In the case $n = 1$ of the Remark 3.53, we can also use another type of the critical Hardy inequality (see, e.g., Solomyak²⁰⁸) in (3.179) to obtain

$$\int_{\mathbb{C}} |\widetilde{\nabla_{\mathcal{L}}} f|^2 dz \geq C \int_{\mathbb{C}} \frac{|f|^2}{|z|^2(1 + \log^2 |z|)} dz + \int_{\mathbb{C}} |z|^2 (\psi(|z|))^2 |f|^2 dz,$$

where C is a positive constant.

Remark 3.55. Let $\mathcal{L}_G$ be the magnetic Baouendi-Grushin operator on $\mathbb{C}^n$ with the constant magnetic field

$$\mathcal{L}_G = \sum_{j=1}^n ((i\partial_{x_j} + \psi_{1,j}(y_j))^2 + (i|x|^{\gamma}\partial_{y_j} + \psi_{2,j}(x_j))^2),$$

where $|x| = \sqrt{|x_1|^2 + \dots + |x_n|^2}$, $\psi_{1,j}, \psi_{2,j} \in L_{loc}^2(\mathbb{R} \setminus \{0\})$. Setting $\widehat{X}_j = i\partial_{x_j} + \psi_{1,j}(y_j)$ and $\widehat{Y}_j = i|x|^{\gamma}\partial_{y_j} + \psi_{2,j}(x_j)$, we write

$$\nabla_{\mathcal{L}_G} f = (\widehat{X}_1 f, \dots, \widehat{X}_n f, \widehat{Y}_1 f, \dots, \widehat{Y}_n f).$$

Then, by a direct calculation for all real-valued functions $f \in C_0^{\infty}(\mathbb{R}^{2n} \setminus \{0\})$ we have

$$\begin{aligned} & \int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} |\nabla_{\mathcal{L}_G} f|^2 dz \\ &= \sum_{j=1}^n \int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} \left(|(i\partial_{x_j} + \psi_{1,j}(y_j)) f|^2 + |(i|x|^{\gamma}\partial_{y_j} + \psi_{2,j}(x_j)) f|^2 \right) dz \\ &= \int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} |\nabla_{\gamma} f|^2 dz + \sum_{j=1}^n \int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} (|\psi_{2,j}(x_j)|^2 + |\psi_{1,j}(y_j)|^2) |f|^2 dz. \end{aligned}$$

Here, if we use the inequality (3.145) with $m = k = n$, hence $Q = n(2 + \gamma)$, then we obtain the following Hardy inequality for magnetic Baouendi-Grushin operator on $\mathbb{C}^n$ with the constant magnetic field

$$\int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_{\gamma} \rho|^{\alpha_2} |\nabla_{\mathcal{L}_G} f|^2 dz$$

$$\begin{aligned}
&\geq \left(\frac{n(2 + \gamma) + \alpha_1 - 2}{2} \right) \int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} \frac{|x|^{2\gamma}}{\rho^{2\gamma+2}} |f|^2 dz \\
&+ \sum_{j=1}^n \int_{\mathbb{C}^n} \rho^{\alpha_1} |\nabla_\gamma \rho|^{\alpha_2} (|\psi_{2,j}(x_j)|^2 + |\psi_{1,j}(y_j)|^2) |f|^2 dz,
\end{aligned}$$

for any real-valued function $f \in C_0^\infty(\mathbb{R}^{2n} \setminus \{0\})$ with sharp constant, where $n(2 + \gamma) + \alpha_1 - 2 > 0$ and $n + \alpha_2 \gamma > 0$.

4

Integral Hardy inequalities on homogeneous Lie groups

THIS chapter, whose content is based on joint work with Ruzhansky¹⁹³, is devoted to the various types of integral Hardy inequalities on homogeneous Lie groups, where instead of estimating a function by its gradient, we estimate the integral by the function itself. Such integral Hardy inequalities, namely, the convolution Hardy inequalities are useful in obtaining hypoelliptic functional inequalities that we will discuss in Chapter 6. In order to obtain the convolution Hardy inequalities, we first prove weighted integral Hardy type inequalities, which provide the characterisation of pairs of weights for the integral versions of Hardy inequalities to hold.

4.0.1 WEIGHTED INTEGRAL HARDY INEQUALITIES

In this section we introduce various types of weighted $L^p - L^q$ inequalities for the Hardy operator on homogeneous Lie groups for different ranges of indices $1 < p, q < \infty$. Moreover, we show necessary and sufficient

conditions on weights for such inequalities to be true.

Theorem 4.1 (¹⁹³ Ruzhansky-Y., Theorem 3.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $\{\phi_i\}_{i=1}^2$ and $\{\psi_i\}_{i=1}^2$ be positive functions on $\mathbb{G}$, and let $1 < p \leq q < \infty$. Then the inequalities*

$$\left(\int_{\mathbb{G}} \left(\int_{B(0,|x|)} f(z) dz \right)^q \phi_1(x) dx \right)^{\frac{1}{q}} \leq C_1 \left(\int_{\mathbb{G}} (f(x))^p \psi_1(x) dx \right)^{\frac{1}{p}} \quad (4.1)$$

and

$$\left(\int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,|x|)} f(z) dz \right)^q \phi_2(x) dx \right)^{\frac{1}{q}} \leq C_2 \left(\int_{\mathbb{G}} (f(x))^p \psi_2(x) dx \right)^{\frac{1}{p}} \quad (4.2)$$

hold for all $f \geq 0$ a.e. on $\mathbb{G}$ if and only if, respectively, we have

$$A_1 := \sup_{R>0} \left(\int_{\{|x| \geq R\}} \phi_1(x) dx \right)^{\frac{1}{q}} \left(\int_{\{|x| \leq R\}} (\psi_1(x))^{-(p'-1)} dx \right)^{\frac{1}{p'}} < \infty \quad (4.3)$$

and

$$A_2 := \sup_{R>0} \left(\int_{\{|x| \leq R\}} \phi_2(x) dx \right)^{\frac{1}{q}} \left(\int_{\{|x| \geq R\}} (\psi_2(x))^{-(p'-1)} dx \right)^{\frac{1}{p'}} < \infty. \quad (4.4)$$

Moreover, if $\{C_i\}_{i=1}^2$ are the smallest constants for which (4.1) and (4.2) hold, then

$$A_i \leq C_i \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_i, \quad i = 1, 2. \quad (4.5)$$

Remark 4.2. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$, we refer to¹⁴⁶ for $q = p$, to²⁷ and⁶¹ for $p \leq q$. Note that when $q = p > 1$ and $\phi_1(x) = |B(0, |x|)|^{-p}$ and $\psi_1(x) = 1$ in (4.1), we obtain $A_1 = (p-1)^{-1/p}$ and

$$\left(\int_{\mathbb{R}^n} \left| \frac{1}{|B(0, |x|)|} \int_{B(0, |x|)} f(z) dz \right|^p dx \right)^{\frac{1}{p}} \leq \frac{p}{p-1} \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}, \quad (4.6)$$

where $|B(0, |x|)|$ stands for the volume of the ball $B(0, |x|)$. We refer, e.g., to⁴⁶ for the inequality (4.6).

Proof of Theorem 4.1. Let us only prove (4.1) $\Leftrightarrow$ (4.3) since (4.2) $\Leftrightarrow$ (4.4) can be

proved similarly.

First, we prove (4.3) $\Rightarrow$ (4.1). Then, using the polar decomposition on $\mathbb{G}$ and denoting $r = |x|$, we can write

$$\begin{aligned} & \int_{\mathbb{G}} \phi_1(x) \left[\int_{B(0,r)} f(z) dz \right]^q dx \\ &= \int_0^\infty \int_{\mathfrak{S}} r^{Q-1} \phi_1(ry) \left[\int_0^r \int_{\mathfrak{S}} s^{Q-1} f(sy) d\sigma(y) ds \right]^q d\sigma(y) dr. \end{aligned} \quad (4.7)$$

Setting

$$g(r) = \left(\int_{\mathfrak{S}} \int_0^r s^{Q-1} (\psi_1(sy))^{1-p'} ds d\sigma(y) \right)^{1/(pp')}, \quad (4.8)$$

and using the Hölder inequality, we calculate

$$\begin{aligned} \int_0^r \int_{\mathfrak{S}} s^{Q-1} f(sy) d\sigma(y) ds &= \int_{\mathfrak{S}} \int_0^r s^{(Q-1)/p} f(sy) (\psi_1(sy))^{1/p} g(s) s^{(Q-1)/p'} \\ &\quad \times ((\psi_1(sy))^{1/p} g(s))^{-1} ds d\sigma(y) \\ &\leq \left(\int_{\mathfrak{S}} \int_0^r s^{Q-1} [f(sy) (\psi_1(sy))^{1/p} g(s)]^p ds d\sigma(y) \right)^{1/p} \\ &\quad \times \left(\int_{\mathfrak{S}} \int_0^r s^{Q-1} [(\psi_1(sy))^{1/p} g(s)]^{-p'} ds d\sigma(y) \right)^{1/p'}. \end{aligned} \quad (4.9)$$

We denote

$$U(s) := \int_{\mathfrak{S}} s^{Q-1} (f(sy) (\psi_1(sy))^{1/p} g(s))^p d\sigma(y), \quad (4.10)$$

$$V(r) := \int_0^r \int_{\mathfrak{S}} s^{Q-1} ((\psi_1(sy))^{1/p} g(s))^{-p'} d\sigma(y) ds, \quad (4.11)$$

$$W_1(r) := \int_{\mathfrak{S}} r^{Q-1} \phi_1(ry) d\sigma(y), \quad (4.12)$$

for $s, r > 0$. Now, we plug (4.9) into (4.7) to obtain

$$\int_{\mathbb{G}} \phi_1(x) \left(\int_{B(0,r)} f(z) dz \right)^q dx \leq \int_0^\infty W_1(r) \left(\int_0^r U(s) ds \right)^{q/p} (V(r))^{q/p'} dr. \quad (4.13)$$

Let us now recall the following continuous version of Minkowski's inequality (see, e.g.,⁶¹ Formula 2.1 or²³ Theorem 2.1): Let $\theta \geq 1$. Then for all $f_1(x), f_2(x) \geq 0$ on $(0, \infty)$ the inequality

$$\int_0^\infty f_1(x) \left(\int_0^x f_2(z) dz \right)^\theta dx \leq \left(\int_0^\infty f_2(z) \left(\int_z^\infty f_1(x) dx \right)^{1/\theta} dz \right)^\theta \quad (4.14)$$

holds. We use (4.14) with $\theta = q/p \geq 1$ in the right hand side of (4.13) to get

$$\begin{aligned} \int_{\mathbb{G}} \phi_1(x) \left(\int_{B(0,r)} f(z) dz \right)^q dx \\ \leq \left(\int_0^\infty U(s) \left(\int_s^\infty W_1(r) (V(r))^{q/p'} dr \right)^{p/q} ds \right)^{q/p}. \end{aligned} \quad (4.15)$$

Introducing

$$T(s) := \int_{\mathfrak{S}} s^{Q-1} (\psi_1(sy))^{1-p'} d\sigma(y),$$

and using (4.8), (4.11), the integration by parts, (4.3) and (4.12) we calculate $V(r)$ as follows

$$\begin{aligned} V(r) &= \int_{\mathfrak{S}} \int_0^r s^{Q-1} (\psi_1(sy))^{1-p'} \left(\int_0^s \int_{\mathfrak{S}} t^{Q-1} (\psi_1(tw))^{1-p'} d\sigma(w) dt \right)^{-1/p} ds d\sigma(y) \\ &= \int_0^r T(s) \left(\int_0^s T(t) dt \right)^{-1/p} ds = p' \int_0^r \frac{d}{ds} \left(\int_0^s T(t) dt \right)^{1/p'} ds \\ &= p' \left(\int_0^r T(s) ds \right)^{1/p'} = p' \left(\int_0^r \int_{\mathfrak{S}} s^{Q-1} (\psi_1(sy))^{1-p'} d\sigma(y) ds \right)^{1/p'} \\ &\leq p' A_1 \left(\int_r^\infty s^{Q-1} \int_{\mathfrak{S}} \phi_1(sw) d\sigma(w) ds \right)^{-1/q} = p' A_1 \left(\int_r^\infty W_1(s) ds \right)^{-1/q}. \end{aligned}$$

Similarly, using the integration by parts and (4.3), it follows that

$$\begin{aligned}
& \int_s^\infty W_1(r)(V(r))^{q/p'} dr \\
&= (p' A_1)^{q/p'} \int_s^\infty W_1(r) \left(\int_r^\infty W_1(s) ds \right)^{-1/p'} dr \\
&= (p' A_1)^{q/p'} p \left(\int_s^\infty W_1(r) dr \right)^{1/p} \\
&= (p' A_1)^{q/p'} p \left(\int_s^\infty \int_{\mathfrak{S}} r^{Q-1} \phi_1(ry) d\sigma(y) dr \right)^{1/p} \\
&\leq (p' A_1)^{q/p'} p A_1^{q/p} \left(\int_0^s r^{Q-1} \int_{\mathfrak{S}} (\psi_1(ry))^{1-p'} d\sigma(y) dr \right)^{-q/(p'p)} \\
&= A_1^q (p')^{q/p'} p (g(s))^{-q},
\end{aligned} \tag{4.16}$$

where we have used (4.8) in the last line. Finally, recalling (4.10) we plug (4.16) into (4.15) to obtain

$$\begin{aligned}
& \int_{\mathbb{G}} \phi_1(x) \left(\int_{B(0,r)} f(z) dz \right)^q dx \leq \left(\int_0^\infty U(s) A_1^p (p')^{p-1} p^{p/q} (g(s))^{-p} ds \right)^{q/p} \\
&= A_1^q (p')^{q/p'} p \left(\int_0^\infty U(s) (g(s))^{-p} ds \right)^{q/p} \\
&= A_1^q (p')^{q/p'} p \left(\int_0^\infty \int_{\mathfrak{S}} s^{Q-1} (f(sy))^p \psi_1(sy) d\sigma(y) ds \right)^{q/p} \\
&= A_1^q (p')^{q/p'} p \left(\int_{\mathbb{G}} \psi_1(x) (f(x))^p dx \right)^{q/p},
\end{aligned} \tag{4.17}$$

which implies (4.1) with $C_1 = A_1(p')^{1/p'} p^{1/q}$.

Now it remains to show (4.1) $\Rightarrow$ (4.3). If we take

$$f(x) := (\psi_1(x))^{1-p'} \chi_{(0,R)}(|x|)$$

with $R > 0$, then we have

$$\begin{aligned} & \left(\int_{\mathbb{G}} \psi_1(x) (f(x))^p dx \right)^{1/p} \left(\int_{|x| \leq R} (\psi_1(x))^{1-p'} dx \right)^{-1/p} \\ &= \left(\int_{|x| \leq R} (\psi_1(x))^{1-p'} dx \right)^{1/p} \left(\int_{|x| \leq R} (\psi_1(x))^{1-p'} dx \right)^{-1/p} = 1. \end{aligned} \quad (4.18)$$

Consequently, by (4.1) we also have

$$\begin{aligned} C &= C \left(\int_{\mathbb{G}} \psi_1(x) (f(x))^p dx \right)^{1/p} \left(\int_{|x| \leq R} (\psi_1(x))^{1-p'} dx \right)^{-1/p} \\ &\geq \left(\int_{\mathbb{G}} \phi_1(x) \left(\int_{|z| \leq |x|} f(z) dz \right)^q dx \right)^{1/q} \left(\int_{|x| \leq R} (\psi_1(x))^{1-p'} dx \right)^{-1/p} \\ &\geq \left(\int_{|x| \geq R} \phi_1(x) \left(\int_{|z| \leq |x|} f(z) dz \right)^q dx \right)^{1/q} \left(\int_{|x| \leq R} (\psi_1(x))^{1-p'} dx \right)^{-1/p} \\ &= \left(\int_{|x| \geq R} \phi_1(x) dx \right)^{1/q} \left(\int_{|z| \leq R} (\psi_1(z))^{1-p'} dz \right)^{1/p'}. \end{aligned} \quad (4.19)$$

Now, we combine (4.18) and (4.19) to obtain (4.3) with $C \geq A_1$. $\square$

Now let us discuss the case $q < p$ of Theorem 4.1:

Theorem 4.3 (¹⁹³ Ruzhansky-Y., Theorem 3.3). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $\{\phi_i\}_{i=3}^4$ and $\{\psi_i\}_{i=3}^4$ be positive functions on $\mathbb{G}$, and let $1 < q < p < \infty$ with $1/\delta = 1/q - 1/p$. Then the inequalities*

$$\left(\int_{\mathbb{G}} \left(\int_{B(0,|x|)} f(z) dz \right)^q \phi_3(x) dx \right)^{\frac{1}{q}} \leq C_1 \left(\int_{\mathbb{G}} (f(x))^p \psi_3(x) dx \right)^{\frac{1}{p}} \quad (4.20)$$

and

$$\left(\int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,|x|)} f(z) dz \right)^q \phi_4(x) dx \right)^{\frac{1}{q}} \leq C_2 \left(\int_{\mathbb{G}} (f(x))^p \psi_4(x) dx \right)^{\frac{1}{p}} \quad (4.21)$$

hold for all $f \geq 0$ if and only if, respectively, we have

$$A_3 := \int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0, |x|)} \phi_3(z) dz \right)^{\frac{\delta}{q}} \left(\int_{B(0, |x|)} (\psi_3(z))^{1-p'} dz \right)^{\frac{\delta}{q'}} (\psi_3(x))^{1-p'} dx < \infty \quad (4.22)$$

and

$$A_4 := \int_{\mathbb{G}} \left(\int_{B(0, |x|)} \phi_4(z) dz \right)^{\frac{\delta}{q}} \left(\int_{\mathbb{G} \setminus B(0, |x|)} (\psi_4(z))^{1-p'} dz \right)^{\frac{\delta}{q'}} (\psi_4(x))^{1-p'} dx < \infty. \quad (4.23)$$

Remark 4.4. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$, the inequalities (4.20) and (4.21) were studied in⁶¹.

Proof of Theorem 4.3. Let us only show (4.20) $\Leftrightarrow$ (4.22) since (4.21) $\Leftrightarrow$ (4.23) can be proved similarly.

First, we prove (4.22) $\Rightarrow$ (4.20). Let us introduce the notations

$$W_2(r) := \int_{\mathbb{G}} r^{Q-1} \phi_3(ry) d\sigma(y) \quad (4.24)$$

and

$$G(s) := \int_{\mathbb{G}} s^{Q-1} h(sy) (\psi_3(sy))^{1-p'} d\sigma(y) \quad (4.25)$$

for $h \geq 0$ on $\mathbb{G}$ to be chosen later. By the polar decomposition on $\mathbb{G}$ we

obtain

$$\begin{aligned}
& \int_{\mathbb{G}} \phi_3(x) \left(\int_{B(0,|x|)} h(z)(\psi_3(z))^{1-p'} dz \right)^q dx \\
&= \int_0^\infty \int_{\mathfrak{S}} r^{Q-1} \phi_3(rw) d\sigma(w) \left(\int_0^r \int_{\mathfrak{S}} s^{Q-1} h(sy)(\psi_3(sy))^{1-p'} d\sigma(y) ds \right)^q dr \\
&= \int_0^\infty W_2(r) \left(\int_0^r G(s) ds \right)^q dr \\
&= q \int_0^\infty G(s) \left(\int_0^s G(r) dr \right)^{q-1} \left(\int_s^\infty W_2(r) dr \right) ds \\
&= q \int_{\mathfrak{S}} \int_0^\infty s^{Q-1} h(sy)(\psi_3(sy))^{1-p'} \\
&\quad \times \left(\int_0^s \int_{\mathfrak{S}} r^{Q-1} h(rw)(\psi_3(rw))^{1-p'} d\sigma(w) dr \right)^{q-1} \left(\int_s^\infty W_2(r) dr \right) ds d\sigma(y) \\
&= q \int_{\mathfrak{S}} \int_0^\infty s^{Q-1} h(sy)(\psi_3(sy))^{(1-p')(\frac{1}{p} + \frac{q-1}{p} + \frac{p-q}{p})} \\
&\quad \times \left(\frac{\int_{\mathfrak{S}} \int_0^s r^{Q-1} h(rw)(\psi_3(rw))^{1-p'} dr d\sigma(w)}{\int_{\mathfrak{S}} \int_0^s r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w)} \right)^{q-1} \\
&\quad \times \left(\left(\int_{\mathfrak{S}} \int_0^s r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w) \right)^{q-1} \left(\int_s^\infty W_2(r) dr \right) \right) ds d\sigma(y).
\end{aligned}$$

Here, we use the Hölder inequality (with three factors) for $\frac{1}{p} + \frac{q-1}{p} + \frac{p-q}{p} = 1$ to get

$$\int_{\mathbb{G}} \phi_3(x) \left(\int_{B(0,|x|)} h(z)(\psi_3(z))^{1-p'} dz \right)^q dx \leq qK_1K_2K_3, \quad (4.26)$$

where

$$\begin{aligned}
K_1 &:= \left(\int_{\mathfrak{S}} \int_0^\infty s^{Q-1} (h(sy))^p (\psi_3(sy))^{1-p'} ds d\sigma(y) \right)^{1/p} \\
&= \left(\int_{\mathbb{G}} (h(x))^p (\psi_3(x))^{1-p'} dx \right)^{1/p}, \quad (4.27)
\end{aligned}$$

$$K_2 := \left(\int_{\mathfrak{S}} \int_0^\infty s^{Q-1} (\psi_3(sy))^{1-p'} \right. \\ \left. \times \left(\frac{\int_{\mathfrak{S}} \int_0^s r^{Q-1} h(rw) (\psi_3(rw))^{1-p'} dr d\sigma(w)}{\int_{\mathfrak{S}} \int_0^s r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w)} \right)^p ds d\sigma(y) \right)^{\frac{q-1}{p}} \quad (4.28)$$

and

$$K_3 := \left(\int_{\mathfrak{S}} \int_0^\infty s^{Q-1} (\psi_3(sy))^{1-p'} \left(\int_{\mathfrak{S}} \int_0^s r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w) \right)^{\frac{(q-1)p}{p-q}} \right. \\ \left. \times \left(\int_s^\infty W_2(r) dr \right)^{\frac{p}{p-q}} ds d\sigma(y) \right)^{\frac{p-q}{p}}. \quad (4.29)$$

Let us now estimate K_2 and K_3 . So, we restate K_2 as

$$K_2 = \left(\int_{\mathbb{G}} \frac{(\psi_3(x))^{1-p'}}{(\int_{B(0,|x|)} (\psi_3(z))^{1-p'} dz)^p} \left(\int_{B(0,|x|)} (\psi_3(z))^{1-p'} h(z) dz \right)^p dx \right)^{\frac{q-1}{p}}.$$

Here, in order to apply (4.1) with $p = q$, $f(x) = (\psi_3(x))^{1-p'} h(x)$ and

$$\phi_1(x) = \frac{(\psi_3(x))^{1-p'}}{(\int_{B(0,|x|)} (\psi_3(z))^{1-p'} dz)^p}, \quad \psi_1(x) = (\psi_3(x))^{(1-p')(1-p)},$$

let us check that

$$A_1(R) = \left(\int_{|x| \geq R} (\psi_3(x))^{1-p'} \left(\int_{B(0,|x|)} (\psi_3(z))^{1-p'} dz \right)^{-p} dx \right)^{1/p} \\ \times \left(\int_{|x| \leq R} (\psi_3(x))^{1-p'} dx \right)^{1/p'} < \infty \quad (4.30)$$

holds uniformly for all $R > 0$. Indeed, once (4.30) has been established,

the inequality (4.1) implies that

$$\begin{aligned} K_2 &\leq C \left(\int_{\mathbb{G}} (\psi_3(x))^{(1-p')(1-p+p)} (h(x))^p dx \right)^{\frac{q-1}{p}} \\ &= C \left(\int_{\mathbb{G}} (h(x))^p (\psi_3(x))^{1-p'} dx \right)^{\frac{q-1}{p}}. \quad (4.31) \end{aligned}$$

To check (4.30), denoting $S(s) = \int_{\mathfrak{S}} s^{Q-1} (\psi_3(sw))^{1-p'} d\sigma(w)$ and using the integration by parts we observe that

$$\begin{aligned} A_1(R) &= \left(\int_{\mathfrak{S}} \int_R^\infty r^{Q-1} (\psi_3(rw))^{1-p'} \left(\int_0^r S(s) ds \right)^{-p} dr d\sigma(w) \right)^{1/p} \\ &\times \left(\int_0^R S(s) ds \right)^{1/p'} = \left(\int_R^\infty \left(\int_0^r S(s) ds \right)^{-p} S(r) dr \right)^{1/p} \left(\int_0^R S(s) ds \right)^{1/p'} \\ &\leq \left(\frac{1}{p-1} \left(\int_0^R S(s) ds \right)^{1-p} \right)^{1/p} \left(\int_0^R S(s) ds \right)^{1/p'} = (p-1)^{-1/p} < \infty. \end{aligned}$$

For K_3 , since we have $\frac{1}{\delta} = \frac{1}{q} - \frac{1}{p} = \frac{p-q}{pq}$, then using (4.22) we note that

$$\begin{aligned} K_3 &= \left(\int_0^\infty \int_{\mathfrak{S}} \left(\int_s^\infty W_2(r) dr \right)^{\delta/q} \left(\int_{\mathfrak{S}} \int_0^r r^{Q-1} (\psi_3(rw))^{1-p'} dr d\sigma(w) \right)^{\delta/q'} \right. \\ &\quad \times \left. s^{Q-1} (\psi_3(sy))^{1-p'} d\sigma(y) ds \right)^{\frac{p-q}{p}} \\ &= \left(\int_{\mathbb{G}} \left(\int_{\mathbb{G} \setminus B(0,|x|)} \phi_3(z) dz \right)^{\delta/q} \left(\int_{B(0,|x|)} (\psi_3(z))^{1-p'} dz \right)^{\delta/q'} \right. \\ &\quad \times \left. (\psi_3(x))^{1-p'} dx \right)^{\frac{p-q}{p}} = A_3^{\frac{p-q}{p}} < \infty. \quad (4.32) \end{aligned}$$

Now, we plug (4.27), (4.31) and (4.32) into (4.26) to obtain

$$\begin{aligned} \int_{\mathbb{G}} \phi_3(x) \left(\int_{B(0,|x|)} h(z) (\psi_3(z))^{1-p'} dz \right)^q dx \\ \leq C A_3^{\frac{p-q}{p}} \left(\int_{\mathbb{G}} (h(x))^p (\psi_3(x))^{1-p'} dx \right)^{q/p}, \end{aligned}$$

which implies (4.20) after choosing $h := f\psi_3^{p'-1}$.

To show (4.20) $\Rightarrow$ (4.22), putting the functions

$$f_k(x) := \left(\int_{|y| \geq |x|} \phi_3(z) dz \right)^{\delta/(pq)} \left(\int_{\alpha_k \leq |z| \leq |x|} (\psi_3(z))^{1-p'} dz \right)^{\delta/(pq')} \\ \times (\psi_3(x))^{1-p'} \chi_{(\alpha_k, \beta_k)}(|x|), \quad k = 1, 2, \dots,$$

instead of $f(x)$ in (4.20), we get (4.22), where $0 < \alpha_k < \beta_k$ with $\alpha_k \searrow 0$ and $\beta_k \nearrow \infty$ for $k \rightarrow \infty$. $\square$

Remark 4.5. Actually, in¹⁹¹ we showed that the proof of Theorems 4.1 and 4.3 also work on general metric measure spaces possessing polar decompositions

$$\int_{\mathbb{X}} f(x) dx = \int_0^\infty \int_{\Sigma_r} f(r, \omega) \tau(r, \omega) d\omega_r dr \quad (4.33)$$

at $a \in \mathbb{X}$ for some set $\Sigma_r \subset \mathbb{X}$ (which may depend on r) with the measure $d\omega_r$ on Σ_r , and $(r, \omega) \rightarrow a$ as $r \rightarrow 0$, where $\mathbb{X}$ is the metric space with a Borel measure dx , and $\tau \in L^1_{loc}$, $f \in L^1(\mathbb{X})$. We used such integral inequalities to obtain Hardy, Hardy-Sobolev, Hardy-Littlewood-Sobolev, Caffarelli-Kohn-Nirenberg, Gagliardo-Nirenberg inequalities and their critical versions on general connected Lie groups, which include both unimodular and nonunimodular cases in compact and noncompact settings. We also refer to¹⁸⁶ for the integral Hardy inequalities (4.1) and (4.2) on $\mathbb{X}$.

In¹⁸³ we also established the following weighted exponential inequality:

Theorem 4.6 (¹⁸³ Ruzhansky-Suragan-Y., Theorem 5.1). *Let ϕ and ψ be positive functions defined on $\mathbb{G}$. Then there exists a positive constant C such that*

$$\int_{\mathbb{G}} \phi(x) \exp(\mathcal{M} \log f)(x) dx \leq C \int_{\mathbb{G}} \psi(x) f(x) dx \quad (4.34)$$

holds for all positive f if and only if

$$A := \sup_{R>0} R^Q \int_{|x| \geq R} \frac{\phi(x) \exp\left(\mathcal{M} \log \frac{1}{\psi}\right)(x)}{|x|^{2Q}} dx < \infty, \quad (4.35)$$

where $(\mathcal{M}f)(x) = \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} f(z)dz$.

Remark 4.7. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$, the inequality (4.34) was studied in¹¹⁹ for $n = 1$, and in⁶¹ for $n \geq 1$.

Since (4.34) involves Orlicz norm, we refer to²⁵ and references therein for Orlicz estimates for the Hardy integral operator.

4.0.2 CONVOLUTION HARDY INEQUALITIES

We now discuss integral Hardy inequalities in the convolution form. We will show in Chapter 6 that such inequalities are particularly useful to obtain hypoelliptic Hardy-Sobolev inequalities and their critical versions on graded groups if we take Riesz kernels of hypoelliptic differential operators instead of the convolution kernels.

Theorem 4.8 (¹⁹³ Ruzhansky-Y., Theorem 3.5). *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q . Let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p \leq q < \infty$ and $0 < a < Q/p$. Let $0 \leq b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Assume that $|T_a^{(1)}(x)| \leq C_2|x|^{a-Q}$ for some positive $C_2 = C_2(a, Q)$. Then there exists a positive constant $C_1 = C_1(p, q, a, b)$ such that*

$$\left\| \frac{f * T_a^{(1)}}{|x|^{\frac{b}{q}}} \right\|_{L^q(\mathbb{G})} \leq C_1 \|f\|_{L^p(\mathbb{G})} \quad (4.36)$$

holds for all $f \in L^p(\mathbb{G})$.

Proof of Theorem 4.8. Let us split the following integral into three parts as follows

$$\int_{\mathbb{G}} |(f * T_a^{(1)})(x)|^q \frac{dx}{|x|^b} \leq 3^q (M_1 + M_2 + M_3), \quad (4.37)$$

where

$$M_1 := \int_{\mathbb{G}} \left(\int_{\{2|y| < |x|\}} |T_a^{(1)}(y^{-1}x)f(y)| dy \right)^q \frac{dx}{|x|^b},$$

$$M_2 := \int_{\mathbb{G}} \left(\int_{\{|x| \leq 2|y| < 4|x|\}} |T_a^{(1)}(y^{-1}x)f(y)| dy \right)^q \frac{dx}{|x|^b}$$

and

$$M_3 := \int_{\mathbb{G}} \left(\int_{\{|y|>2|x|\}} |T_a^{(1)}(y^{-1}x)f(y)|dy \right)^q \frac{dx}{|x|^b}.$$

Let us first estimate M_1 . We can assume that $|\cdot|$ is a norm without loss of generality because of the existence such a norm (Proposition 2.9) and since replacing the seminorm by an equivalent one only changes the appearing constants. Although we could give a proof without this hypothesis, it simplifies the arguments below.

By the reverse triangle inequality and $2|y| < |x|$ we note that

$$|y^{-1}x| \geq |x| - |y| > |x| - \frac{|x|}{2} = \frac{|x|}{2}, \quad (4.38)$$

that is, $|x| < 2|y^{-1}x|$. Then, since $T_a^{(1)}(x)$ is bounded by a radial function which is non-increasing with respect to $|x|$, we have for M_1 that

$$\begin{aligned} M_1 &\leq \int_{\mathbb{G}} \left(\int_{\{2|y|<|x|\}} |f(y)|dy \right)^q \left(\sup_{\{|x|<2|z|\}} |T_a^{(1)}(z)| \right)^q \frac{dx}{|x|^b} \\ &\leq C \int_{\mathbb{G}} \left(\int_{\{2|y|<|x|\}} |f(y)|dy \right)^q \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b}. \end{aligned} \quad (4.39)$$

In order to apply (4.1) to M_1 , we need to check the condition (4.3), that is,

$$\left(\int_{\{2R<|x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} dx \right)^{\frac{1}{p'}} \leq A_1 \quad (4.40)$$

for all $R > 0$. For this, let us consider the cases $R \geq 1$ and $0 < R < 1$

separately. For $R \geq 1$ we have

$$\begin{aligned}
& \left(\int_{\{2R < |x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{|x| < R\}} dx \right)^{\frac{1}{p'}} \\
& \leq CR^{\frac{Q}{p'}} \left(\int_{\{2R < |x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \\
& \leq CR^{\frac{Q}{p'}} \left(\int_{\{2R < |x|\}} |x|^{(a-Q)q-b} dx \right)^{\frac{1}{q}} \\
& \leq CR^{\frac{Q}{p'}} R^{\frac{(a-Q)q-b+Q}{q}} \\
& \leq C,
\end{aligned} \tag{4.41}$$

since $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$ and $(a-Q)q - b + Q = -\frac{Qq}{p'} \neq 0$.

For $0 < R < 1$ noting $(a-Q)q - b + Q = -\frac{Qq}{p'} \neq 0$ we observe that

$$\int_{\{2R < |x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \leq CR^{(a-Q)q-b+Q}. \tag{4.42}$$

Here, taking into account $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$ we obtain

$$\begin{aligned}
& \left(\int_{\{2R < |x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)q} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{|x| < R\}} dx \right)^{\frac{1}{p'}} \\
& \leq CR^{a-Q-\frac{b}{q}+\frac{Q}{q}} R^{Q/p'} \leq C
\end{aligned} \tag{4.43}$$

for any $0 < R < 1$. Thus, the condition (4.40) holds. So, we can now apply (4.1) to M_1 to obtain

$$M_1^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_1 \|f\|_{L^p(\mathbb{G})}. \tag{4.44}$$

Now we estimate M_3 . Without loss of generality, we may assume again $|\cdot|$ is the norm. Then, similarly to (4.38) one can observe that $2|x| < |y|$

implies $|y| < 2|y^{-1}x|$. Taking into account this, we obtain for M_3 that

$$M_3 \leq \int_{\mathbb{G}} \left(\int_{\{|y|>2|x|\}} \left(\frac{|y|}{2} \right)^{(a-Q)} |f(y)| dy \right)^q \frac{dx}{|x|^b}.$$

Here, we want to apply (4.2) to M_3 . To do so, we need to verify the condition

$$\left(\int_{\{|x|<R\}} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)p'} dx \right)^{\frac{1}{p'}} \leq A_2. \quad (4.45)$$

For this, let us again consider the cases $R \geq 1$ and $0 < R < 1$ separately. For $R \geq 1$ using $|T_a^{(1)}(x)| \leq C|x|^{a-Q}$ and $Q \neq ap$, we note that

$$\left(\int_{\{2R<|x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)p'} dx \right)^{\frac{1}{p'}} \leq C \left(\int_{\{2R<|x|\}} |x|^{(a-Q)p'} dx \right)^{\frac{1}{p'}} \leq CR^{a-\frac{Q}{p}}. \quad (4.46)$$

It follows for $R \geq 1$ that

$$\left(\int_{\{|x|<R\}} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)p'} dx \right)^{\frac{1}{p'}} \leq CR^{a-\frac{Q}{p}+\frac{Q-b}{q}} \leq C,$$

since $b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$.

For $0 < R < 1$ noting $ap - Q < 0$ one has

$$\int_{\{2R<|x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)p'} dx \leq C \int_{\{2R<|x|\}} |x|^{(a-Q)p'} dx \leq CR^{(a-Q)p'+Q}. \quad (4.47)$$

Then, it gives with

$$\int_{\{|x|<R\}} \frac{dx}{|x|^b} \leq CR^{Q-b}$$

that

$$\begin{aligned}
& \left(\int_{\{|x| < R\}} \frac{dx}{|x|^b} \right)^{\frac{1}{q}} \left(\int_{\{2R < |x|\}} \left(\frac{|x|}{2} \right)^{(a-Q)p'} dx \right)^{\frac{1}{p'}} \\
& \leq CR^{\frac{Q-b}{q}} R^{\frac{(a-Q)p'+Q}{p'}} \\
& \leq C,
\end{aligned} \tag{4.48}$$

since $Q > b$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Thus, we have verified (4.45). So, by applying (4.2) to M_3 we obtain

$$M_3^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_2 \|f\|_{L^p(\mathbb{G})}. \tag{4.49}$$

Finally, it remains to estimate M_2 . Let us first rewrite M_2 as

$$M_2 = \sum_{k \in \mathbb{Z}} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |T_a^{(1)}(y^{-1}x)f(y)| dy \right)^q \frac{dx}{|x|^b}.$$

Taking into account $|x| \leq 2|y| \leq 4|x|$ and $2^k \leq |x| < 2^{k+1}$, we note that $2^{k-1} \leq |y| < 2^{k+2}$. As in (4.38), we assume $|\cdot|$ is the norm, and use the triangle inequality to obtain

$$3|x| = |x| + 2|x| \geq |x| + |y| \geq |y^{-1}x|, \tag{4.50}$$

which means $0 \leq |y^{-1}x| \leq 3|x| < 3 \cdot 2^{k+1}$. Setting $\tilde{I}_a(x) := C_2|x|^{a-Q}$, let us rewrite the assumption $|T_a^{(1)}(x)| \leq \tilde{I}_a(x)$. Then, we use the Young inequality in Proposition 2.12 for $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$ with $r \in [1, \infty]$ to estimate M_2 as follows

$$\begin{aligned}
M_2 & \leq \sum_{k \in \mathbb{Z}} 2^{-kb} \int_{\mathbb{G}} (([f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}]} * \tilde{I}_a)(x))^q dx \\
& = \sum_{k \in \mathbb{Z}} 2^{-kb} \|[f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}]} * \tilde{I}_a\|_{L^q(\mathbb{G})}^q \\
& \leq \sum_{k \in \mathbb{Z}} 2^{-kb} \|\tilde{I}_a \cdot \chi_{\{0 \leq |\cdot| < 3 \cdot 2^{k+1}\}}\|_{L^r(\mathbb{G})}^q \|f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}}\|_{L^p(\mathbb{G})}^q
\end{aligned}$$

$$\begin{aligned}
&= C_2 \sum_{k \in \mathbb{Z}} 2^{-kb} \left(\int_{|x| < 3 \cdot 2^{k+1}} |x|^{(a-Q)r} dx \right)^{\frac{q}{r}} \|f \cdot \chi_{\{2^{k-1} \leq |x| < 2^{k+2}\}}\|_{L^p(\mathbb{G})}^q \\
&\leq C \sum_{k \in \mathbb{Z}} 2^{-kb} (3 \cdot 2^{k+1})^{\left(\frac{(a-Q)pq}{pq+p-q} + Q\right) \frac{pq+p-q}{p}} \|f \cdot \chi_{\{2^{k-1} \leq |x| < 2^{k+2}\}}\|_{L^p(\mathbb{G})}^q \\
&= C \sum_{k \in \mathbb{Z}} 2^{-kb} (3 \cdot 2^{k+1})^b \|f \cdot \chi_{\{2^{k-1} \leq |x| < 2^{k+2}\}}\|_{L^p(\mathbb{G})}^q \\
&\leq C \sum_{k \in \mathbb{Z}} \|f \cdot \chi_{\{2^{k-1} \leq |x| < 2^{k+2}\}}\|_{L^p(\mathbb{G})}^q \\
&\leq C \|f\|_{L^p(\mathbb{G})}^q,
\end{aligned} \tag{4.51}$$

since $\frac{(a-Q)pq}{pq+p-q} + Q = \frac{bp}{pq+p-q} > 0$ and $q \geq p$.

Thus, combining (4.44), (4.49) and (4.51) we obtain (4.36), as desired. $\square$

Remark 4.9. In the case $q = p$, we can also prove Theorem 4.8 by using Schur's test⁷⁹. In the setting of stratified groups, we refer to⁴⁸ for such an argument for the Riesz kernels of the sub-Laplacian. Using the Schur test, let us give here the sketch of the proof of Theorem 4.8 when $q = p$.

In this case, we have $b = ap$ from $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Let $S_a f := |x|^{-b/p} (f * |x|^{a-Q})$, then $S_a^* g := (|x|^{-b/p} g) * |x|^{a-Q}$, where $(f, S_a^* g) = (S_a f, g)$. Since the integral kernel of S_a is positive, by Schur's test we see that instead of proving the estimate

$$\|S_a f\|_{L^p(\mathbb{G})} \leq A_{a,p}^{1/p'} B_{a,p}^{1/p} \|f\|_{L^p(\mathbb{G})}$$

for all $f \in L^p(\mathbb{G})$, it is enough to exhibit a positive function h and constants $A_{a,p}$ and $B_{a,p}$ such that

$$S_a(h^{p'})(x) \leq A_{a,p}(h(x))^{p'} \quad \text{and} \quad S_a^*(h^p)(x) \leq B_{a,p}(h(x))^p$$

for almost all $x \in \mathbb{G}$.

Let us take $h_c(x) := |x|^{c-Q}$ with $c > 0$ and consider the convolution integrals

$$h_c^{p'} * |x|^{a-Q} \quad \text{and} \quad (|x|^{-b/p} h_c^p) * |x|^{a-Q},$$

which arise in the computation of $S_a(h_c^{p'})$ and $S_a^*(h_c^p)$. We note that the

homogeneity orders of $h_c^{p'}$ and $|x|^{-b/p}h_c^p$ are $(c-Q)p'$ and $(c-Q)p-b/p$, respectively. It follows that the homogeneity orders of $h_c^{p'} * |x|^{a-Q}$ and $(|x|^{-b/p}h_c^p) * |x|^{a-Q}$ are $a-Q+(c-Q)p'$ and $a-Q+(c-Q)p-b/p$, respectively. Then, these convolution integrals converge absolutely in $\mathbb{G} \setminus \{0\}$ if and only if $0 < (c-Q)p' + Q < Q-a$ and $0 < (c-Q)p-b/p + Q < Q-a$, i.e. if

$$\max\left(\frac{Q}{p}, \frac{a}{p} + \frac{Q}{p'}\right) < c < Q - \frac{a}{p'}$$

since $b = ap$. One can observe that it is true if $0 < a < Q/p$.

Thus, recalling the Schur test we obtain

$$\| |x|^{-b/p} (f * |x|^{a-Q}) \|_{L^p(\mathbb{G})} \leq A_{a,p}^{1/p'} B_{a,p}^{1/p} \|f\|_{L^p(\mathbb{G})},$$

where $0 < a < Q/p$, $1 < p < \infty$, $f \in L^p(\mathbb{G})$ and $b = ap$. Finally, we use this and $|T_a^{(1)}(x)| \leq C|x|^{a-Q}$ to obtain

$$\begin{aligned} \left\| \frac{f * T_a^{(1)}}{|x|^{\frac{b}{p}}} \right\|_{L^p(\mathbb{G})} &\leq C \left\| \frac{|f| * |T_a^{(1)}|}{|x|^{\frac{b}{p}}} \right\|_{L^p(\mathbb{G})} \\ &\leq C \| |x|^{-b/p} (|f| * |x|^{a-Q}) \|_{L^p(\mathbb{G})} \leq C \|f\|_{L^p(\mathbb{G})}, \end{aligned} \quad (4.52)$$

as desired.

In the case $q > p$, the operator in (4.36) is dominated pointwise by

$$|x|^{-\frac{b}{q}} \left((|f| * K_{a-\frac{b}{q}}) * K_{\frac{b}{q}} \right),$$

where $K_a(x) = |x|^{-Q+a}$, and the above is the composition of $g \mapsto g * K_{a-\frac{b}{q}}$, which maps L^p into L^q , followed by $h \mapsto |x|^{-\frac{b}{q}} (h * K_{\frac{b}{q}})$ which falls in the $q = p$ case.

We now discuss the critical case $b = Q$ of Theorem 4.8.

Theorem 4.10 (¹⁹³ Ruzhansky-Y., Theorem 3.7). *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q . Let $|\cdot|$ be an arbitrary homogeneous quasi-norm and let $1 < p < r < \infty$ and $p < q < (r-1)p'$, where $1/p + 1/p' = 1$.*

Assume that for $a = Q/p$ we have

$$|T_a^{(2)}(x)| \leq C_2 \begin{cases} |x|^{a-Q}, & \text{for } x \in \mathbb{G} \setminus \{0\}, \\ |x|^{-Q}, & \text{for } x \in \mathbb{G} \text{ with } |x| \geq 1, \end{cases} \quad (4.53)$$

for some positive $C_2 = C_2(a, Q)$. Then there exists a positive constant $C_1 = C_1(p, q, r, Q)$ such that

$$\left\| \frac{f * T_{Q/p}^{(2)}}{\left(\log\left(e + \frac{1}{|x|}\right)\right)^{\frac{r}{q}} |x|^{\frac{Q}{q}}} \right\|_{L^q(\mathbb{G})} \leq C_1 \|f\|_{L^p(\mathbb{G})} \quad (4.54)$$

holds for all $f \in L^p(\mathbb{G})$.

Proof of Theorem 4.10. We split the following integral into three parts as follows

$$\int_{\mathbb{G}} |(f * T_{Q/p}^{(2)})(x)|^q \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q} \leq 3^q (N_1 + N_2 + N_3), \quad (4.55)$$

where

$$N_1 := \int_{\mathbb{G}} \left(\int_{\{2|y| < |x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)| dy \right)^q \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q},$$

$$N_2 := \int_{\mathbb{G}} \left(\int_{\{|x| \leq 2|y| < 4|x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)| dy \right)^q \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q}$$

and

$$N_3 := \int_{\mathbb{G}} \left(\int_{\{|y| > 2|x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)| dy \right)^q \frac{dx}{\left|\log\left(e + \frac{1}{|x|}\right)\right|^r |x|^Q}.$$

First, let us estimate N_1 . Similar to (4.38) from $2|y| < |x|$ we derive

$$|y^{-1}x| \geq |x| - |y| > |x| - \frac{|x|}{2} = \frac{|x|}{2}, \quad (4.56)$$

which is $|x| < 2|y^{-1}x|$. Setting

$$|T_a^{(2)}(x)| \leq \tilde{B}_a(x) := C_2 \begin{cases} |x|^{a-Q}, & \text{for } x \in \mathbb{G} \setminus \{0\}, \\ |x|^{-Q}, & \text{for } x \in \mathbb{G} \text{ with } |x| \geq 1, \end{cases} \quad (4.57)$$

we note that $T_{Q/p}^{(2)}(x)$ is bounded by $\tilde{B}_{Q/p}(x)$ which is non-increasing with respect to $|x|$. Hence, by (4.56) we can estimate N_1 as follows

$$\begin{aligned} N_1 &\leq \int_{\mathbb{G}} \left(\int_{\{2|y| < |x|\}} |f(y)| dy \right)^q \left(\sup_{\{|x| < 2|z|\}} |T_{Q/p}^{(2)}(z)| \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \\ &\leq \int_{\mathbb{G}} \left(\int_{\{2|y| < |x|\}} |f(y)| dy \right)^q \left(\tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q}. \end{aligned} \quad (4.58)$$

To apply (4.1) to N_1 , we need to check the condition (4.3), that is, that

$$\left(\int_{\{2R < |x|\}} \left(\tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \right)^{\frac{1}{q}} \left(\int_{\{|x| < R\}} dx \right)^{\frac{1}{p'}} \leq A_1 \quad (4.59)$$

holds for all $R > 0$. In order to check this, let us consider the cases $R \geq 1$ and $0 < R < 1$ separately. For $R \geq 1$ we use the second equality of (4.57) to obtain

$$\begin{aligned} &\left(\int_{\{2R < |x|\}} \left(\tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \right)^{\frac{1}{q}} \left(\int_{\{|x| < R\}} dx \right)^{\frac{1}{p'}} \\ &\leq CR^{\frac{Q}{p'}} \left(\int_{\{2R < |x|\}} \left(\tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right)^q \frac{dx}{|x|^Q} \right)^{\frac{1}{q}} \\ &= CR^{\frac{Q}{p'}} \left(\int_{\{2R < |x|\}} |x|^{-Qq-Q} dx \right)^{\frac{1}{q}} \\ &\leq CR^{-Q} R^{\frac{Q}{p'}} \\ &\leq C. \end{aligned} \quad (4.60)$$

For $0 < R < 1$ let us split the integral as follows

$$\begin{aligned}
& \int_{\{2R < |x|\}} \left(\tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \\
&= \int_{\{2R < |x| < 2\}} \left(\tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \\
&\quad + \int_{\{|x| \geq 2\}} \left(\tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q}. \quad (4.61)
\end{aligned}$$

Here, by the second equality in (4.57) it is easy to see that the second integral in the right hand side of (4.61) is finite. For the first integral, we use the first equality of (4.57) to obtain

$$\begin{aligned}
& \int_{\{2R < |x| < 2\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \\
&\leq \int_{\{2R < |x| < 2\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^q \frac{dx}{|x|^Q} \\
&\leq C \int_{\{2R < |x| < 2\}} |x|^{-Qq/p' - Q} dx \\
&\leq CR^{-Qq/p'}.
\end{aligned}$$

It implies with (4.61) that

$$\begin{aligned}
& \left(\int_{\{2R < |x|\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \right)^{\frac{1}{q}} \left(\int_{\{|x| < R\}} dx \right)^{\frac{1}{p'}} \\
&\leq C(R^{-Q/p'} + 1)R^{Q/p'} \leq C
\end{aligned}$$

for any $0 < R < 1$. Thus, we have checked (4.59). So, we can now apply (4.1) to N_1 to obtain

$$N_1^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_1 \|f\|_{L^p(\mathbb{G})}. \quad (4.62)$$

Now we estimate N_3 . Without loss of generality, let us assume again that $|\cdot|$ is the norm. By the similar argument used in (4.56), we derive $|y| < 2|y^{-1}x|$

from $2|x| < |y|$. Then, for N_3 we get

$$N_3 \leq \int_{\mathbb{G}} \left(\int_{\{|y|>2|x|\}} \left| \tilde{B}_{Q/p} \left(\frac{y}{2} \right) \right| |f(y)| dy \right)^q \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q}.$$

In order to apply (4.2) to N_3 , we need to check the condition

$$\left(\int_{\{|x|<R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \right)^{\frac{1}{p'}} \leq A_2. \quad (4.63)$$

For this, let us consider the cases $R \geq 1$ and $0 < R < 1$ separately. For $R \geq 1$ we use the second equality of (4.57) to obtain

$$\left(\int_{\{2R<|x|\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \right)^{\frac{1}{p'}} \leq C \left(\int_{\{2R<|x|\}} |x|^{-Qp'} dx \right)^{\frac{1}{p'}} \leq CR^{-\frac{Q}{p}}. \quad (4.64)$$

On the other hand, let us rewrite the term in the first bracket of (4.63) as follows

$$\begin{aligned} \int_{\{|x|<R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} &= \int_{\{|x|<\frac{1}{2}\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \\ &\quad + \int_{\{\frac{1}{2} \leq |x| < R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q}. \end{aligned}$$

Here, it is easy to see that the first summand in the right hand side of above is finite since $r > 1$. For the second term, we have

$$\int_{\{\frac{1}{2} \leq |x| < R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \leq \int_{\{\frac{1}{2} \leq |x| < R\}} \frac{dx}{|x|^Q} \leq C(1 + \log R). \quad (4.65)$$

Thus, for $R \geq 1$ the obtained estimates (4.64) and (4.65) imply

$$\left(\int_{\{|x| < R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \right)^{\frac{1}{q}} \left(\int_{\{2R < |x|\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \right)^{\frac{1}{p'}} \leq CR^{-\frac{Q}{p}} (1 + \log R)^{\frac{1}{q}} \leq C.$$

For $0 < R < 1$ in order to check the condition (4.63), let us first split the integral

$$\int_{\{2R < |x|\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx = \int_{\{2R < |x| < 2\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx + \int_{\{|x| \geq 2\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx. \quad (4.66)$$

Here, the second integral in the right hand side of above is finite because of the second equality in (4.57). For the first integral, we use the first equality of (4.57) to obtain

$$\int_{\{2R < |x| < 2\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \leq C \int_{\{2R < |x| < 2\}} |x|^{-Q} dx \leq C \log \left(\frac{1}{R} \right),$$

which implies with (4.66) that

$$\int_{\{2R < |x|\}} \left| \tilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \leq C \left(1 + \log \left(\frac{1}{R} \right) \right). \quad (4.67)$$

Then, by this inequality and

$$\int_{\{|x| < R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \leq C \left(\log \left(e + \frac{1}{R} \right) \right)^{-(r-1)},$$

noting $r > 1$ and $q < (r - 1)p'$ we obtain

$$\begin{aligned}
& \left(\int_{\{|x| < R\}} \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q} \right)^{\frac{1}{q}} \left(\int_{\{2R < |x|\}} \left| \widetilde{B}_{Q/p} \left(\frac{x}{2} \right) \right|^{p'} dx \right)^{\frac{1}{p'}} \\
& \leq C \left(\log \left(e + \frac{1}{R} \right) \right)^{-\frac{r-1}{q}} \left(1 + \left(\log \left(\frac{1}{R} \right) \right)^{\frac{1}{p'}} \right) \\
& \leq C.
\end{aligned} \tag{4.68}$$

Thus, we have checked (4.63). We can now apply (4.2) to N_3 to derive

$$N_3^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_2 \|f\|_{L^p(\mathbb{G})}. \tag{4.69}$$

Now, to complete the proof of the theorem we estimate N_2 . For this, we first write

$$\begin{aligned}
N_2 &= \sum_{k \in \mathbb{Z}} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)| dy \right)^q \\
&\quad \times \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q}.
\end{aligned}$$

Since the function $\left(\log \left(\frac{1}{|x|} \right) \right)^r |x|^Q$ is non-decreasing with respect to $|x|$ near the origin, there is an integer $k_0 \in \mathbb{Z}$ with $k_0 \leq -3$ such that this function is non-decreasing in $|x| \in (0, 2^{k_0+1})$. We decompose N_2 with k_0 as follows

$$N_2 = N_{21} + N_{22}, \tag{4.70}$$

where

$$\begin{aligned}
N_{21} &:= \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)| dy \right)^q \\
&\quad \times \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q}
\end{aligned}$$

and

$$N_{22} := \sum_{k=k_0+1}^{\infty} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)|dy \right)^q \times \frac{dx}{\left| \log \left(e + \frac{1}{|x|} \right) \right|^r |x|^Q}.$$

First, we estimate N_{22} . Since $|x| \leq 2|y| \leq 4|x|$ and $2^k \leq |x| < 2^{k+1}$, we also have $2^{k-1} \leq |y| < 2^{k+2}$. By (4.53) and $q > p$, for $\tilde{r} \in (1, \infty)$ with $1 + \frac{1}{q} = \frac{1}{\tilde{r}} + \frac{1}{p}$ we observe that

$$\begin{aligned} \int_{\mathbb{G}} |T_{Q/p}^{(2)}(x)|^{\tilde{r}} dx &= \int_{|x| < 1} |T_{Q/p}^{(2)}(x)|^{\tilde{r}} dx + \int_{|x| \geq 1} |T_{Q/p}^{(2)}(x)|^{\tilde{r}} dx \\ &\leq C_2 \left(\int_{|x| < 1} |x|^{-\frac{Qq(p-1)}{pq+p-q}} dx + \int_{|x| \geq 1} |x|^{-\frac{Qpq}{pq+p-q}} dx \right) < \infty. \end{aligned} \quad (4.71)$$

Now, we use this and Young's inequality (see Proposition 2.12) for $1 + \frac{1}{q} = \frac{1}{\tilde{r}} + \frac{1}{p}$ with $\tilde{r} \in (1, \infty)$ to estimate N_{22} as follows

$$\begin{aligned} N_{22} &\leq C \sum_{k=k_0+1}^{\infty} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |T_{Q/p}^{(2)}(y^{-1}x)f(y)|dy \right)^q dx \\ &\leq C \| [f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}}] * T_{Q/p}^{(2)} \|_{L^q(\mathbb{G})}^q \\ &\leq C \| T_{Q/p}^{(2)} \|_{L^{\tilde{r}}(\mathbb{G})}^q \sum_{k=k_0+1}^{\infty} \| f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}} \|_{L^p(\mathbb{G})}^q \\ &= C \sum_{k=k_0+1}^{\infty} \left(\int_{\{2^k \leq |x| < 2^{k+1}\}} |f(x)|^p dx \right)^{\frac{q}{p}} \\ &\leq C \left(\sum_{k \in \mathbb{Z}} \int_{\{2^k \leq |x| < 2^{k+1}\}} |f(x)|^p dx \right)^{\frac{q}{p}} \\ &= C \|f\|_{L^p(\mathbb{G})}^q. \end{aligned} \quad (4.72)$$

Now we have to estimate N_{21} . As in (4.56), we assume $|\cdot|$ is the norm, and

use the triangle inequality to obtain

$$3|x| = |x| + 2|x| \geq |x| + |y| \geq |y^{-1}x|, \quad (4.73)$$

where we have used $|y| \leq 2|x|$. We also have

$$\left(\log \left(\frac{1}{|x|} \right) \right)^r |x|^Q \geq \left(\log \left(\frac{1}{\left| \frac{y^{-1}x}{3} \right|} \right) \right)^r \left| \frac{y^{-1}x}{3} \right|^Q,$$

since $\left(\log \left(\frac{1}{|x|} \right) \right)^r |x|^Q$ is non-decreasing in $|x| \in (0, 2^{k_0+1})$ and $3|x| \geq |y^{-1}x|$. We use now this and (4.53) to obtain

$$\begin{aligned} & N_{21} \\ & \leq C \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} |y^{-1}x|^{-\frac{Q}{p'}} |f(y)| dy \right)^q \frac{dx}{\left(\log \left(\frac{1}{|x|} \right) \right)^r |x|^Q} \\ & = C \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} \frac{|y^{-1}x|^{-\frac{Q}{p'}} |f(y)|}{\left(\left(\log \left(\frac{1}{|x|} \right) \right)^r |x|^Q \right)^{\frac{1}{q}}} dy \right)^q dx \\ & \leq C \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} \frac{|y^{-1}x|^{-\frac{Q}{p'}} |f(y)| dy}{\left(\left(\log \left(\frac{3}{|y^{-1}x|} \right) \right)^r \left| \frac{y^{-1}x}{3} \right|^Q \right)^{\frac{1}{q}}} \right)^q dx. \end{aligned}$$

Since we have $|x| \leq 2|y| \leq 4|x|$ and $2^k \leq |x| < 2^{k+1}$ with $k \leq k_0$, then by (4.73) and $k_0 \leq -3$ we get $2^{k-1} \leq |y| < 2^{k+2}$ and $|y^{-1}x| \leq 3|x| < 3 \cdot 2^{k_0+1} \leq 3/4$. Then, introducing

$$g(x) := \frac{\chi_{B_{\frac{3}{4}}(0)}(x)}{\left(\log \left(\frac{1}{|x|} \right) \right)^{\frac{r}{q}} |x|^{\frac{Q}{q} + \frac{Q}{p'}}},$$

we obtain for N_{21} that

$$\begin{aligned} N_{21} &\leq C \sum_{k=-\infty}^{k_0} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{|x| \leq 2|y| \leq 4|x|\}} \frac{|f(y)| dy}{\left(\log\left(\frac{1}{|y^{-1}x|}\right)\right)^{\frac{r}{q}} |y^{-1}x|^{\frac{Q}{q} + \frac{Q}{p'}}} \right)^q dx \\ &\leq C \sum_{k=-\infty}^{k_0} \|[f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}}] * g\|_{L^q(\mathbb{G})}^q. \end{aligned}$$

Using $p < q < (r-1)p'$ and the Young inequality for $1 + \frac{1}{q} = \frac{1}{\tilde{r}} + \frac{1}{p}$ with $\tilde{r} \in (1, \infty)$, we observe that

$$N_{21} \leq C \|g\|_{L^{\tilde{r}}(\mathbb{G})}^q \sum_{k=-\infty}^{k_0} \|f \cdot \chi_{\{2^{k-1} \leq |\cdot| < 2^{k+2}\}}\|_{L^p(\mathbb{G})}^q \leq C \|f\|_{L^p(\mathbb{G})}^q, \quad (4.74)$$

since

$$\|g\|_{L^{\tilde{r}}(\mathbb{G})}^{\tilde{r}} = \int_{B(0,3/4)} \frac{dx}{\left(\log\left(\frac{1}{x}\right)\right)^{\frac{rp'}{p'+q}} |x|^Q} = C \int_{\log(\frac{4}{3})}^{\infty} \frac{dt}{t^{\frac{rp'}{p'+q}}} < \infty,$$

where we have used $\left(\frac{Q}{q} + \frac{Q}{p'}\right) \tilde{r} = Q$, $\frac{r\tilde{r}}{q} = \frac{rp'}{p'+q}$ and $q < (r-1)p'$ in the last line. Thus, (4.62), (4.69), (4.70), (4.72), (4.74) and (4.55) complete the proof of Theorem 4.10. $\square$

4.0.3 HARDY-LITTLEWOOD-SOBOLEV INEQUALITIES

In this section we establish Hardy-Littlewood-Sobolev inequality on homogeneous Lie groups as a consequence of the convolution Hardy inequality from the previous section. Moreover, we discuss the reversed Hardy-Littlewood-Sobolev inequalities on homogeneous Lie groups. In Section 6.1 we will discuss differential version of Hardy-Littlewood-Sobolev inequality for general homogeneous invariant hypoelliptic differential operators on graded Lie groups. Then, in Section 6.2 we will discuss the relation between the best constants of hypoelliptic Sobolev and Hardy-Littlewood-Sobolev inequalities for certain families of parameters.

Certainly, Hardy-Littlewood-Sobolev inequality is a very classical subject going back to Hardy-Littlewood^{113, 115} and Sobolev²⁰⁷. On homogeneous Lie groups, it was established by Folland and Stein⁷⁷ on the Heisenberg group, and its sharp constants were also investigated by Lieb¹³⁷ in the Euclidean setting, and then by Frank and Lieb⁸¹ on the Heisenberg group.

Theorem 4.11 (¹⁹³ Ruzhansky-Y., Theorem 4.1). *Let $\mathbb{G}$ be a homogeneous Lie group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $0 < \lambda < Q$ and $1 < p, q < \infty$ be such that $1/p + 1/q + (\alpha + \lambda)/Q = 2$ with $0 \leq \alpha < Q/p'$ and $\alpha + \lambda \leq Q$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(Q, \lambda, p, \alpha)$ such that*

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)}g(y)}{|x|^\alpha |y^{-1}x|^\lambda} dx dy \right| \leq C \|f\|_{L^p(\mathbb{G})} \|g\|_{L^q(\mathbb{G})} \quad (4.75)$$

holds for all $f \in L^p(\mathbb{G})$ and $g \in L^q(\mathbb{G})$.

Remark 4.12. Recently, in¹²⁷, using Theorem 4.1 the authors obtained the following Stein-Weiss inequality, which is a two-weight extension of the Hardy-Littlewood-Sobolev inequality: Let $0 < \lambda < Q, 1 < p < \infty, \alpha < \frac{Q}{p'}, \beta < \frac{Q}{q}, \alpha + \beta \geq 0, \frac{1}{q} = \frac{1}{p} + \frac{\alpha + \beta + \lambda}{Q} - 1$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then for $1 < p \leq q \leq \infty$ we have

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{u(y)h(x)}{|x|^\beta |y^{-1}x|^\lambda |y|^\alpha} dx dy \right| \leq C \|u\|_{L^p(\mathbb{G})} \|h\|_{L^{q'}(\mathbb{G})}, \quad (4.76)$$

holds for all $u \in L^p(\mathbb{G})$ and $h \in L^{q'}(\mathbb{G})$, where C is a positive constant independent of u and h .

Proof of Theorem 4.11. We denote $T_a^{(3)}(x) := |x|^{a-Q}$ with $0 < a < Q/r$ for some $1 < r < \infty$. Then, we use the Hölder inequality in left-hand side of

(4.75) to obtain

$$\begin{aligned} \left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)}g(y)}{|x|^\alpha |y^{-1}x|^\lambda} dx dy \right| &= \left| \int_{\mathbb{G}} \overline{f(x)} \frac{(g * T_{Q-\lambda}^{(3)})(x)}{|x|^\alpha} dx \right| \\ &\leq \|f\|_{L^p(\mathbb{G})} \left\| \frac{g * T_{Q-\lambda}^{(3)}}{|x|^\alpha} \right\|_{L^{p'}(\mathbb{G})}. \end{aligned} \quad (4.77)$$

Note that from the assumptions $\alpha + \lambda \leq Q$ and $1/p + 1/q + (\alpha + \lambda)/Q = 2$ we get $q \leq p'$. Moreover, we observe that $0 < \lambda < Q$, $\alpha < Q/p'$ and $1/p + 1/q + (\alpha + \lambda)/Q = 2$ imply

$$0 < Q - \lambda = Q - Q \left(2 - \frac{1}{p} - \frac{1}{q} \right) + \alpha < Q - Q \left(2 - \frac{1}{p} - \frac{1}{q} \right) + \frac{Q}{p'} = Q/q.$$

Taking into account $1 < q \leq p' < \infty$, $0 \leq \alpha p' < Q$, $0 < Q - \lambda < Q/q$ and $(Q - \lambda)/Q = 1/q - 1/p' + \alpha/Q$, we use Theorem 4.8 in (4.77) to obtain (4.75), as desired. $\square$

Remark 4.13 (Reversed Hardy–Littlewood–Sobolev inequality). Let us now discuss the validity of the following reversed Hardy–Littlewood–Sobolev inequality on homogeneous groups (see^{60, 151} and⁵⁸ for the recent Euclidean analysis of such inequalities):

$$\int_{\mathbb{G}} \int_{\mathbb{G}} f(x) |y^{-1}x|^\lambda f(y) dx dy \geq C_{Q,\lambda,p} \|f\|_{L^1(\mathbb{G})}^\theta \|f\|_{L^p(\mathbb{G})}^{2-\theta} \quad (4.78)$$

for any $0 \leq f \in L^1 \cap L^p(\mathbb{G})$ with $f \not\equiv 0$ and $0 < p < 1$, where $\lambda > 0$ and $\theta := (2Q - p(2Q + \lambda))/(Q(1 - p))$. In the case $\mathbb{G} = (\mathbb{R}^n, +)$, hence $Q = n$, we refer to⁶⁰ and¹⁵¹ for the case $p = 2n/(2n + \lambda)$, and to⁵⁸ for the case $p > n/(n + \lambda)$.

Here, in the case $0 < p \leq Q/(Q + \lambda)$, let us show that the inequality (4.78) is not valid, that is, we show that (4.78) fails for any $C_{Q,\lambda,p} > 0$. In the case $\mathbb{G} = (\mathbb{R}^n, +)$, this was shown in⁴² when $p < n/(n + \lambda)$ and in⁵⁸ when $p \leq n/(n + \lambda)$.

We consider

$$f_\varepsilon(x) := f(x) + A\varepsilon^{-Q}h(x/\varepsilon),$$

for a non-negative function f with compact support and for a non-negative smooth function h with the property $\int_{\mathbb{G}} h(x)dx = 1$, and for $A > 0$. Suppose (4.78) holds for some $C_{Q,\lambda,p} > 0$. We plug f_ε into the inequality (4.78) to obtain

$$\begin{aligned} C_{Q,\lambda,p} &\leq \frac{\int_{\mathbb{G}} \int_{\mathbb{G}} f_\varepsilon(x) |y^{-1}x|^\lambda f_\varepsilon(y) dx dy}{\|f_\varepsilon\|_{L^1(\mathbb{G})}^\theta \|f_\varepsilon\|_{L^p(\mathbb{G})}^{2-\theta}} \\ &\rightarrow \frac{\int_{\mathbb{G}} \int_{\mathbb{G}} f(x) |y^{-1}x|^\lambda f(y) dx dy + 2A \int_{\mathbb{G}} |x|^\lambda f(x) dx}{(\int_{\mathbb{G}} f(x) dx + A)^\theta (\int_{\mathbb{G}} (f(x))^p dx)^{(2-\theta)/p}} \end{aligned} \quad (4.79)$$

as $\varepsilon \rightarrow 0_+$, where we have used $\int_{\mathbb{G}} f_\varepsilon(x) dx = \int_{\mathbb{G}} f(x) dx + A$, and

$$\int_{\mathbb{G}} (f_\varepsilon(x))^p dx \rightarrow \int_{\mathbb{G}} (f(x))^p dx$$

and

$$\begin{aligned} \int_{\mathbb{G}} \int_{\mathbb{G}} f_\varepsilon(x) |y^{-1}x|^\lambda f_\varepsilon(y) dx dy &= \int_{\mathbb{G}} \int_{\mathbb{G}} f(x) |y^{-1}x|^\lambda f(y) dx dy \\ &+ 2A \int_{\mathbb{G}} \int_{\mathbb{G}} f(x) |(\varepsilon^{-1}y)^{-1}x|^\lambda h(y) dx dy + A^2 \varepsilon^{-2Q} \int_{\mathbb{G}} \int_{\mathbb{G}} h\left(\frac{x}{\varepsilon}\right) h\left(\frac{y}{\varepsilon}\right) dx dy \\ &\rightarrow \int_{\mathbb{G}} \int_{\mathbb{G}} f(x) |y^{-1}x|^\lambda f(y) dx dy + 2A \int_{\mathbb{G}} |x|^\lambda f(x) dx, \end{aligned}$$

since $\int_{\mathbb{G}} h(x)dx = 1$. Noting (4.79) is valid for all $A > 0$, we can take the limit as $A \rightarrow +\infty$ there for $\theta > 1$, that is, $p < Q/(Q + \lambda)$, so that we obtain $C_{Q,\lambda,p} = 0$. In the case $\theta = 1$, that is, $p = Q/(Q + \lambda)$, letting again $A \rightarrow +\infty$ in (4.79) we observe that

$$C_{Q,\lambda,p} \leq \frac{2 \int_{\mathbb{G}} |x|^\lambda f(x) dx}{(\int_{\mathbb{G}} (f(x))^p dx)^{1/p}}. \quad (4.80)$$

Now it is enough to show that the right-hand side of (4.80) goes to zero for some suitable f when $R \rightarrow \infty$. Indeed, in this case $p = Q/(Q + \lambda)$, if we

put the function

$$f_R(x) := \begin{cases} |x|^{-(Q+\lambda)}, & \text{for } 1 \leq |x| \leq R, \\ 0, & \text{otherwise,} \end{cases} \quad (4.81)$$

for any $R > 1$ in (4.80), then we note that

$$C_{Q,\lambda,p} \leq \frac{2 \int_{\mathbb{G}} |x|^\lambda f_R(x) dx}{(\int_{\mathbb{G}} (f_R(x))^p dx)^{1/p}} = 2(|\mathfrak{S}| \log R)^{-\lambda/Q} \rightarrow 0 \quad (4.82)$$

as $R \rightarrow \infty$, where $|\mathfrak{S}|$ is a $Q - 1$ dimensional surface measure of the unit quasi-sphere in $\mathbb{G}$.

Thus, we have proved that the reversed Hardy-Littlewood-Sobolev inequality (4.78) fails for any positive constant $C_{Q,\lambda,p}$ when $0 < p \leq Q/(Q + \lambda)$.

5

Function spaces on homogeneous Lie groups

IN this chapter, we introduce several function spaces on homogeneous Lie groups, with and without differentiability properties. The spaces involving radial derivatives are the Euler–Hilbert–Sobolev, which can be a version of Sobolev spaces associated to the Euler operator, and Sobolev–Lorentz–Zygmund spaces. The development of these function spaces is linked to relevant functional inequalities for the Euler operator from Chapter 3. As for function spaces without the differentiable structure, we define Morrey and generalised Morrey spaces on homogeneous Lie groups. Moreover, in these spaces, we study the boundedness of integral operators, namely, the Hardy–Littlewood maximal operator, Bessel–Riesz operators, generalised Bessel–Riesz operators, generalised fractional integral operators and Olsen type inequalities.

We refer to the monograph of Folland and Stein⁷⁸ for the Hardy spaces on homogeneous Lie groups. We also refer to the recent works developing function spaces on graded groups extending Folland and Saka’s constructions in the stratified case, see⁷⁰ and⁶⁹ for Sobolev spaces, and^{39, 40} for

Besov spaces. The constructions of this chapter are based on the analysis in the joint works with Ruzhansky and Suragan^{182, 181, 183}.

5.0.1 EULER-HILBERT-SOBOLEV SPACES

In this section we describe an Euler-Hilbert-Sobolev space on a homogeneous Lie group $\mathbb{G}$. First, recalling the definitions of the Euler operator $\mathbb{E}$ from Section 2.1.1, let us define more general *Euler-Sobolev spaces* by

$$\mathfrak{L}^{k,p}(\mathbb{G}) \equiv \overline{C_0^\infty(\mathbb{G} \setminus \{0\})}^{\|\cdot\|_{\mathfrak{L}^{k,p}(\mathbb{G})}}, \quad k \in \mathbb{Z}, \quad (5.1)$$

that is, as the completion of $C_0^\infty(\mathbb{G} \setminus \{0\})$ with respect to the semi-norm

$$\|f\|_{\mathfrak{L}^{k,p}(\mathbb{G})} := \|\mathbb{E}^k f\|_{L^p(\mathbb{G})}.$$

By (5.1), it is easy to see that the higher order Sobolev-Rellich inequality (3.97) with $\alpha = 0$, that is,

$$\|f\|_{L^p(\mathbb{G})} \leq \left(\frac{p}{Q}\right)^k \|\mathbb{E}^k f\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \quad k \in \mathbb{N}, \quad (5.2)$$

holds for all functions $f \in \mathfrak{L}^{k,p}(\mathbb{G})$. Thus, taking into account (5.1) and (5.2) we obtain

Corollary 5.1 (¹⁸² Ruzhansky-Suragan-Y., Proposition 4.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $1 < p < \infty$. Then the semi-normed space $(\mathfrak{L}^{k,p}(\mathbb{G}), \|\cdot\|_{\mathfrak{L}^{k,p}(\mathbb{G})})$, $k \in \mathbb{Z}$ is a complete space. The norm of the embedding operator $\iota : (\mathfrak{L}^{k,p}(\mathbb{G}), \|\cdot\|_{\mathfrak{L}^{k,p}(\mathbb{G})}) \hookrightarrow (L^p(\mathbb{G}), \|\cdot\|_{L^p(\mathbb{G})})$ satisfies*

$$\|\iota\|_{\mathfrak{L}^{k,p}(\mathbb{G}) \rightarrow L^p(\mathbb{G})} \leq \left(\frac{p}{Q}\right)^k, \quad k \in \mathbb{N}, \quad (5.3)$$

where we understand the embedding ι as an embedding of semi-normed subspace of $L^p(\mathbb{G})$.

By using Lemma 2.14 we can define fractional powers of the operator $\mathbb{A} =$

$\mathbb{E}\mathbb{E}^*$ as in¹⁴¹ Chapter 5 and we denote

$$|\mathbb{E}|^\beta := \mathbb{A}^{\frac{\beta}{2}}, \quad \beta \in \mathbb{R}.$$

For a brief account of the relevant theory of fractional powers we refer to⁶⁹ App.A.

Now, let us state the following fractional L^2 -Sobolev type inequality:

Theorem 5.2 (¹⁸² Ruzhansky-Suragan-Y., Theorem 4.2). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q , $\beta \in \mathbb{R}_+$ and let k be a positive integer with $k > \frac{\beta}{2}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ we have*

$$\|f\|_{L^2(\mathbb{G})} \leq C \left(k - \frac{\beta}{2}, k \right) \left(\frac{2}{Q} \right)^\beta \left\| |\mathbb{E}|^\beta f \right\|_{L^2(\mathbb{G})}, \quad (5.4)$$

where

$$C(\beta, k) = \frac{\Gamma(k+1)}{|\Gamma(\beta)\Gamma(k-\beta)|} \frac{2^{k-\beta}}{\beta(k-\beta)}. \quad (5.5)$$

Proof of Theorem 5.2. Recalling¹⁴¹ Proposition 7.2.1, p.176, we obtain the following interpolation inequality

$$\left\| |\mathbb{E}|^{-\beta} f \right\|_{L^2(\mathbb{G})} \leq C \left(k - \frac{\beta}{2}, k \right) \|f\|_{L^2(\mathbb{G})}^{1-\frac{\beta}{2k}} \left\| \mathbb{A}^{-k} f \right\|_{L^2(\mathbb{G})}^{\frac{\beta}{2k}}. \quad (5.6)$$

On the other hand, using (2.15) and (3.98) with $\alpha = 0$ we also have

$$\begin{aligned} & C \left(k - \frac{\beta}{2}, k \right) \|f\|_{L^2(\mathbb{G})}^{1-\frac{\beta}{2k}} \left\| \mathbb{A}^{-k} f \right\|_{L^2(\mathbb{G})}^{\frac{\beta}{2k}} \\ & \leq C \left(k - \frac{\beta}{2}, k \right) \|f\|_{L^2(\mathbb{G})}^{1-\frac{\beta}{2k}} \left(\frac{4}{Q^2} \right)^{\frac{\beta}{2}} \|f\|_{L^2(\mathbb{G})}^{\frac{\beta}{2k}} \\ & = C \left(k - \frac{\beta}{2}, k \right) \left(\frac{4}{Q^2} \right)^{\frac{\beta}{2}} \|f\|_{L^2(\mathbb{G})}. \end{aligned}$$

Finally, the combination of this and (5.6) implies (5.4), as desired. $\square$

Let us now define the *Euler-Hilbert-Sobolev space* by

$$\mathbb{H}^\beta(\mathbb{G}) \equiv \overline{C_0^\infty(\mathbb{G} \setminus \{0\})}^{\|\cdot\|_{\mathbb{H}^\beta(\mathbb{G})}}, \quad (5.7)$$

i.e., as the completion of $C_0^\infty(\mathbb{G} \setminus \{0\})$ with respect to

$$\|f\|_{\mathbb{H}^\beta(\mathbb{G})} := \| |\mathbb{E}|^\beta f \|_{L^2(\mathbb{G})}.$$

Taking into account (5.7) we obtain the inequality (5.4) for all $f \in \mathbb{H}^\beta(\mathbb{G})$:

$$\|f\|_{L^2(\mathbb{G})} \leq C \left(k - \frac{\beta}{2}, k \right) \left(\frac{2}{Q} \right)^\beta \| |\mathbb{E}|^\beta f \|_{L^2(\mathbb{G})}, \quad (5.8)$$

where $\beta \in \mathbb{R}_+$, $k > \frac{\beta}{2}$, $k \in \mathbb{N}$ and $C(k - \frac{\beta}{2}, k)$ is given by (5.5).

Thus, from the definition of the Euler-Hilbert-Sobolev space (5.7) and inequality (5.4) we conclude as follows:

Proposition 5.3 (¹⁸² Ruzhansky-Suragan-Y., Proposition 4.3). *The space $(\mathbb{H}^\beta, \|\cdot\|_{\mathbb{H}^\beta})$, $\beta \in \mathbb{R}$ is a complete space. Moreover, the norm of the embedding operator $\iota : (\mathbb{H}^\beta, \|\cdot\|_{\mathbb{H}^\beta}) \hookrightarrow (L^2, \|\cdot\|_{L^2})$ satisfies*

$$\|\iota\|_{\mathbb{H}^\beta(\mathbb{G}) \rightarrow L^2(\mathbb{G})} \leq C \left(k - \frac{\beta}{2}, k \right) \left(\frac{2}{Q} \right)^\beta, \quad \beta \in \mathbb{R}_+, k > \frac{\beta}{2}, k \in \mathbb{N}. \quad (5.9)$$

Remark 5.4. Note that the operator $|\mathbb{E}|^\beta$ has a bounded inverse, in particular it is injective and $\|\cdot\|_{\mathbb{H}^\beta(\mathbb{G})}$ is a norm.

POINCARÉ TYPE INEQUALITY

In this section we discuss Poincaré type inequality on a homogeneous Lie group $\mathbb{G}$.

Let $\Omega \subset \mathbb{G}$ be an open set and let $\widehat{\mathfrak{L}}_0^{1,p}(\Omega)$ be the completion of $C_0^\infty(\Omega \setminus \{0\})$ with respect to

$$\|f\|_{\widehat{\mathfrak{L}}_0^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\mathbb{E}f\|_{L^p(\Omega)}, \quad 1 < p < \infty. \quad (5.10)$$

Theorem 5.5 (¹⁸² Ruzhansky-Suragan-Y., Theorem 5.1). *Let Ω be a bounded open subset of $\mathbb{G}$. If $1 < p < \infty$, $f \in \widehat{\mathfrak{L}}_0^{1,p}(\Omega)$ and $\mathcal{R}_{|x|}f \equiv \frac{1}{|x|}\mathbb{E}f \in L^p(\Omega)$, then we have the following Poincaré type inequality on $\Omega \subset \mathbb{G}$:*

$$\|f\|_{L^p(\Omega)} \leq \frac{Rp}{Q} \|\mathcal{R}_{|x|}f\|_{L^p(\Omega)} = \frac{Rp}{Q} \left\| \frac{1}{|x|} \mathbb{E}f \right\|_{L^p(\Omega)}, \quad (5.11)$$

where $R = \sup_{x \in \Omega} |x|$.

The proof of Theorem 5.5 is based on the following proposition:

Proposition 5.6 (¹⁸² Ruzhansky-Suragan-Y., Proposition 5.2). *Let $\Omega \subset \mathbb{G}$ be an open set. If $1 < p < \infty$, $f \in \widehat{\mathfrak{L}}_0^{1,p}(\Omega)$ and $\mathbb{E}f \in L^p(\Omega)$, then we have*

$$\|f\|_{L^p(\Omega)} \leq \frac{p}{Q} \|\mathbb{E}f\|_{L^p(\Omega)}, \quad (5.12)$$

where the space $\widehat{\mathfrak{L}}_0^{1,p}(\Omega)$ is defined in (5.10).

Proof of Proposition 5.6. Let $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ be an even, smooth function such that

- $0 \leq \zeta \leq 1$,
- $\zeta(r) = 1$ if $|r| \leq 1$,
- $\zeta(r) = 0$ if $|r| \geq 2$.

For $\lambda > 0$, let us also set $\zeta_\lambda(x) := \zeta(\lambda|x|)$. Recall that we have the inequality (5.12) for $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ by (3.67). Taking into account the definition of $\widehat{\mathfrak{L}}_0^{1,p}(\Omega)$, we note that there is some $\{f_\ell\}_{\ell=1}^\infty \in C_0^\infty(\Omega \setminus \{0\})$ such that $f_\ell \rightarrow f$ in $\widehat{\mathfrak{L}}_0^{1,p}(\Omega)$ as $\ell \rightarrow \infty$. Let $\lambda > 0$. Then, from (3.67) we observe that

$$\|\zeta_\lambda f_\ell\|_{L^p(\Omega)} \leq \frac{p}{Q} (\|(\mathbb{E}\zeta_\lambda)f_\ell\|_{L^p(\Omega)} + \|\zeta_\lambda(\mathbb{E}f_\ell)\|_{L^p(\Omega)}) \quad (5.13)$$

holds for all $\ell \geq 1$. It follows that

$$\lim_{\ell \rightarrow \infty} \zeta_\lambda f_\ell = \zeta_\lambda f,$$

$$\begin{aligned}\lim_{\ell \rightarrow \infty} (\mathbb{E}\zeta_\lambda)f_\ell &= (\mathbb{E}\zeta_\lambda)f, \\ \lim_{\ell \rightarrow \infty} \zeta_\lambda(\mathbb{E}f_\ell) &= \zeta_\lambda(\mathbb{E}f)\end{aligned}$$

in $L^p(\Omega)$. We use these properties in (5.13) to obtain

$$\|\zeta_\lambda f\|_{L^p(\Omega)} \leq \frac{p}{Q} \left\{ \|(\mathbb{E}\zeta_\lambda)f\|_{L^p(\Omega)} + \|\zeta_\lambda(\mathbb{E}f)\|_{L^p(\Omega)} \right\}. \quad (5.14)$$

Since we also have

$$|(\mathbb{E}\zeta_\lambda)(x)| \leq \begin{cases} \sup |\mathbb{E}\zeta|, & \text{if } \lambda^{-1} < |x| < 2\lambda^{-1}; \\ 0, & \text{otherwise,} \end{cases}$$

then taking the limit as $\lambda \rightarrow 0$ in (5.14) we obtain (5.12). $\square$

Proof of Theorem 5.5. Recalling $R = \sup_{x \in \Omega} |x|$ in Proposition 5.6 we compute

$$\|f\|_{L^p(\Omega)} \leq \frac{p}{Q} \|\mathbb{E}f\|_{L^p(\Omega)} \leq \frac{Rp}{Q} \|\mathcal{R}_{|x|}f\|_{L^p(\Omega)} = \frac{Rp}{Q} \left\| \frac{1}{|x|} \mathbb{E}f \right\|_{L^p(\Omega)},$$

which is (5.11), as desired. $\square$

5.0.2 SOBOLEV-LORENTZ-ZYGMUND SPACES

Here we introduce several families of Lorentz type spaces and analyse their basic properties on a homogeneous Lie group $\mathbb{G}$.

We define the *Lorentz type spaces* by

$$L_{|\cdot|,Q,p,q}(\mathbb{G}) := \{f \in L_{loc}^1(\mathbb{G}) : \|f\|_{L_{|\cdot|,Q,p,q}(\mathbb{G})} < \infty\}, \quad 0 \leq p, q \leq \infty,$$

where

$$\|f\|_{L_{|\cdot|,Q,p,q}(\mathbb{G})} := \left(\int_{\mathbb{G}} (|x|^{\frac{Q}{p}} |f(x)|)^q \frac{1}{|x|^Q} dx \right)^{\frac{1}{q}}.$$

Here, Q and $|\cdot|$ are the homogeneous dimension of $\mathbb{G}$ and the homogeneous quasi-norm on $\mathbb{G}$, respectively. In the sequel, fixing them we will

often use the short notation $L_{p,q}(\mathbb{G}) := L_{|\cdot|,Q,p,q}(\mathbb{G})$.

In the Euclidean case $\mathbb{G} = (\mathbb{R}^n, +)$, for non-negative, radially symmetric and non-increasing (with respect to the radial direction) functions f we note that the norm $\|f\|_{L_{p,q}(\mathbb{R}^n)}$ coincides with the Lorentz norm in terms of the rearrangement, i.e.

$$\left(\int_{\mathbb{R}^n} \left(|x|^{\frac{n}{p}} |f(x)| \right)^q \frac{dx}{|x|^n} \right)^{\frac{1}{q}} = \omega_n^{\frac{1}{q} - \frac{1}{p}} \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$, and f^* denotes the symmetric decreasing rearrangement of f .

Let us also define the *Lorentz-Zygmund spaces* as follows

$$L_{p,q,\lambda}(\mathbb{G}) := \{f \in L_{loc}^1(\mathbb{G}) : \|f\|_{L_{p,q,\lambda}(\mathbb{G})} < \infty\}, \quad 0 \leq p, q \leq \infty, \lambda \in \mathbb{R},$$

where

$$\|f\|_{L_{p,q,\lambda}(\mathbb{G})} := \sup_{R>0} \left(\int_{\mathbb{G}} \left(|x|^{\frac{Q}{p}} \left| \log \frac{R}{|x|} \right|^\lambda |f(x)| \right)^q \frac{1}{|x|^Q} dx \right)^{\frac{1}{q}}.$$

Moreover, we define the *Sobolev-Lorentz-Zygmund spaces* by

$$W^1 L_{p,q,\lambda}(\mathbb{G}) := \left\{ f \in L_{p,q,\lambda}(\mathbb{G}) : \frac{1}{|x|} \mathbb{E} f \in L_{p,q,\lambda}(\mathbb{G}) \right\},$$

endowed with the norm

$$\|\cdot\|_{W^1 L_{p,q,\lambda}(\mathbb{G})} := \|\cdot\|_{L_{p,q,\lambda}(\mathbb{G})} + \left\| \frac{1}{|x|} \mathbb{E} \cdot \right\|_{L_{p,q,\lambda}(\mathbb{G})}.$$

Furthermore, we define $W_0^1 L_{p,q,\lambda}(\mathbb{G}) := \overline{C_0^\infty(\mathbb{G})}^{\|\cdot\|_{W^1 L_{p,q,\lambda}(\mathbb{G})}}$.

We also define the Lorentz-Zygmund spaces involving the double logarithmic weights by

$$L_{p,q,\lambda_1,\lambda_2}(\mathbb{G}) := \{f \in L_{loc}^1(\mathbb{G}) : \|f\|_{L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})} < \infty\},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and

$$\|f\|_{L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})} := \sup_{R>0} \left(\int_{\mathbb{G}} \left(|x|^{\frac{Q}{p}} \left| \log \frac{R}{|x|} \right|^{\lambda_1} \left| \log \left| \log \frac{R}{|x|} \right| \right|^{\lambda_2} |f(x)| \right)^q \frac{dx}{|x|^Q} \right)^{\frac{1}{q}}.$$

Remark 5.7. Note that the space $L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})$ extends the spaces $L_{p,q,\lambda}(\mathbb{G})$ and $L_{p,q}(\mathbb{G})$ in the sense that $L_{p,q,\lambda,0}(\mathbb{G}) = L_{p,q,\lambda}(\mathbb{G})$ and $L_{p,q,0,0}(\mathbb{G}) = L_{p,q}(\mathbb{G})$.

Similarly, we introduce the Sobolev-Lorentz-Zygmund spaces by

$$W^1 L_{p,q,\lambda_1,\lambda_2}(\mathbb{G}) := \left\{ f \in L_{p,q,\lambda_1,\lambda_2}(\mathbb{G}) : \frac{1}{|x|} \mathbb{E} f \in L_{p,q,\lambda_1,\lambda_2}(\mathbb{G}) \right\}, \quad (5.15)$$

endowed with the norm

$$\| \cdot \|_{W^1 L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})} := \| \cdot \|_{L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})} + \left\| \frac{1}{|x|} \mathbb{E} \cdot \right\|_{L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})},$$

and

$$W_0^1 L_{p,q,\lambda_1,\lambda_2}(\mathbb{G}) := \overline{C_0^\infty(\mathbb{G})}^{\| \cdot \|_{W^1 L_{p,q,\lambda_1,\lambda_2}(\mathbb{G})}}. \quad (5.16)$$

Using the special behaviour of the functions $f_R(x) := f(R \frac{x}{|x|})$, let us also define the Lorentz-Zygmund type spaces $\mathfrak{L}_{p,q,\lambda}(\mathbb{G})$ by

$$\mathfrak{L}_{p,q,\lambda}(\mathbb{G}) := \{ f \in L_{loc}^1(\mathbb{G}) : \|f\|_{\mathfrak{L}_{p,q,\lambda}(\mathbb{G})} < \infty \}, \quad \lambda \in \mathbb{R},$$

where

$$\|f\|_{\mathfrak{L}_{p,q,\lambda}(\mathbb{G})} := \sup_{R>0} \left(\int_{\mathbb{G}} \left(|x|^{\frac{Q}{p}} \left| \log \frac{R}{|x|} \right|^{\lambda} |f - f_R| \right)^q \frac{dx}{|x|^Q} \right)^{\frac{1}{q}}.$$

For $p = \infty$ we define

$$\|f\|_{\mathfrak{L}_{\infty,q,\lambda}(\mathbb{G})} := \sup_{R>0} \left(\int_{\mathbb{G}} \left(\left| \log \frac{R}{|x|} \right|^{\lambda} |f - f_R| \right)^q \frac{dx}{|x|^Q} \right)^{\frac{1}{q}}.$$

In addition, we define the Lorentz-Zygmund type spaces $\mathfrak{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{G})$ by

$$\mathfrak{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{G}) := \{f \in L_{loc}^1(\mathbb{G}) : \|f\|_{\mathfrak{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{G})} < \infty\}, \quad (5.17)$$

where

$$\begin{aligned} \|f\|_{\mathfrak{L}_{p,q,\lambda_1,\lambda_2}(\mathbb{G})} &:= \sup_{R>0} \left(\int_{\mathbb{G}} \left(|x|^{\frac{Q}{p}} \left| \log \frac{eR}{|x|} \right|^{\lambda_1} \left| \log \left| \log \frac{eR}{|x|} \right| \right|^{\lambda_2} \right. \right. \\ &\quad \times \left. \left. \left(\chi_{B(0,eR)}(x) |f - f_R| + \chi_{B^c(0,eR)}(x) |f - f_{e^2R}| \right) \right)^q \frac{dx}{|x|^Q} \right)^{\frac{1}{q}}, \\ \chi_{B(0,eR)}(x) &= \begin{cases} 1, & x \in B(0,eR); \\ 0, & x \notin B(0,eR). \end{cases} \end{aligned}$$

For $p = \infty$ we define

$$\begin{aligned} \|f\|_{\mathfrak{L}_{\infty,q,\lambda_1,\lambda_2}(\mathbb{G})} &:= \sup_{R>0} \left(\int_{\mathbb{G}} \left(\left| \log \frac{eR}{|x|} \right|^{\lambda_1} \left| \log \left| \log \frac{eR}{|x|} \right| \right|^{\lambda_2} \right. \right. \\ &\quad \times \left. \left. \left(\chi_{B(0,eR)}(x) |f - f_R| + \chi_{B^c(0,eR)}(x) |f - f_{e^2R}| \right) \right)^q \frac{dx}{|x|^Q} \right)^{\frac{1}{q}}. \end{aligned}$$

Now, let us show several embeddings between the Sobolev–Lorentz–Zygmund spaces.

Theorem 5.8 (¹⁸² Ruzhansky–Suragan–Y., Theorem 6.2). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{G}$. Let $1 < \gamma < \infty$ and $\max\{1, \gamma - 1\} < q < \infty$. Then the continuous embedding*

$$W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G}) \hookrightarrow \mathfrak{L}_{\infty,q,-\frac{1}{q},-\frac{\gamma}{q}}(\mathbb{G})$$

holds. In particular, for all $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$ and for any $R > 0$ the

following inequality holds

$$\begin{aligned} & \left(\int_{\mathbb{G}} \frac{\chi_{B(0,eR)}(x)|f - f_R|^q + \chi_{B^c(0,eR)}(x)|f - f_{e^2R}|^q}{\left| \log \left| \log \frac{eR}{|x|} \right| \right|^\gamma \left| \log \frac{eR}{|x|} \right|} \frac{dx}{|x|^Q} \right)^{\frac{1}{q}} \\ & \leq \frac{q}{\gamma - 1} \left(\int_{\mathbb{G}} |x|^{q-Q} \left| \log \frac{eR}{|x|} \right|^{q-1} \left| \log \left| \log \frac{eR}{|x|} \right| \right|^{q-\gamma} \left| \frac{1}{|x|} \mathbb{E}f \right|^q dx \right)^{\frac{1}{q}}, \quad (5.18) \end{aligned}$$

where the constant $\frac{q}{\gamma-1}$ is sharp and $f_R(x) := f(R\frac{x}{|x|})$.

Remark 5.9. Note that the integrand on the right-hand side of (5.18) has singularities for $|x| = R$, $|x| = eR$, and $|x| = e^2R$. Nevertheless, we do not need to subtract the boundary value of the functions on $|x| = eR$ on the left-hand side.

Remark 5.10. In the Abelian case of the Euclidean space $\mathbb{G} = \mathbb{R}^n$, we have $Q = n$ and taking $|\cdot|$ to be the Euclidean norm, Theorem 5.8 implies¹³⁹ Theorem 1.2. We also refer to²¹⁵ for the embedding of such Euclidean spaces for some indices and to¹³⁸ for the logarithmic type Hardy inequalities in Euclidean Sobolev–Zygmund spaces.

In the proof of Theorem 5.8, we will need the following proposition:

Proposition 5.11 (¹⁸² Ruzhansky–Suragan–Y., Proposition 6.5). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{G}$. Let $1 < \gamma < \infty$ and $\max\{1, \gamma - 1\} < q < \infty$. Then for all $f \in C_0^\infty(\mathbb{G})$ and any $R > 0$ the following inequality holds*

$$\begin{aligned} & \left(\int_{B(0,eR)} \frac{|f - f_R|^q}{\left| \log \left| \log \frac{eR}{|x|} \right| \right|^\gamma \left| \log \frac{eR}{|x|} \right|} \frac{dx}{|x|^Q} \right)^{\frac{1}{q}} \\ & \leq \frac{q}{\gamma - 1} \left(\int_{B(0,eR)} |x|^{q-Q} \left| \log \frac{eR}{|x|} \right|^{q-1} \left| \log \left| \log \frac{eR}{|x|} \right| \right|^{q-\gamma} \left| \frac{1}{|x|} \mathbb{E}f \right|^q dx \right)^{\frac{1}{q}}. \quad (5.19) \end{aligned}$$

Proof of Proposition 5.11. Let us first consider the integrals in (5.19) restricted to a quasi-ball $B(0, R)$. Using the polar decomposition and integration by

parts, we obtain

$$\begin{aligned}
& \int_{B(0,R)} \frac{|f - f_R|^q}{\left| \log \left| \log \frac{eR}{|x|} \right| \right|^\gamma \left| \log \frac{eR}{|x|} \right|} \frac{dx}{|x|^Q} \\
&= \int_0^R \frac{1}{r \left(\log \frac{eR}{r} \right) \left(\log \left(\log \frac{eR}{r} \right) \right)^\gamma} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^q d\sigma(y) dr \\
&= \frac{1}{\gamma - 1} \left[\left(\log \left(\log \frac{eR}{r} \right) \right)^{-\gamma+1} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^q d\sigma(y) \right]_{r=0}^{r=R} \\
&\quad - \frac{1}{\gamma - 1} \int_0^R \left(\log \left(\log \frac{eR}{r} \right) \right)^{-\gamma+1} \frac{d}{dr} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^q d\sigma(y) dr \\
&= -\frac{q}{\gamma - 1} \int_0^R \left(\log \left(\log \frac{eR}{r} \right) \right)^{-\gamma+1} \operatorname{Re} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^{q-2} \\
&\quad \times (f(ry) - f(Ry)) \frac{\overline{df(ry)}}{dr} d\sigma(y) dr.
\end{aligned}$$

In the above calculation, since we have $q - \gamma + 1 > 0$, the boundary term at $r = R$ vanishes because of the inequalities

$$\begin{aligned}
\log \left(\log \frac{eR}{r} \right) &= \int_1^{\log \frac{eR}{r}} \frac{dt}{t} \geq \frac{\log \frac{eR}{r} - 1}{\log \frac{eR}{r}} = \frac{\log \frac{R}{r}}{\log \frac{eR}{r}} \\
&= \frac{1}{\log \frac{eR}{r}} \int_1^{\frac{R}{r}} \frac{dt}{t} \geq \frac{1}{\log \frac{eR}{r}} \frac{\frac{R}{r} - 1}{\frac{R}{r}} = \frac{R - r}{R \log \frac{eR}{r}}
\end{aligned}$$

and

$$|f(ry) - f(Ry)| \leq C(R - r)$$

for $0 < r \leq R$. It follows that

$$\begin{aligned}
& \int_0^R \frac{1}{r(\log \frac{eR}{r})(\log(\log \frac{eR}{r}))^\gamma} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^q d\sigma(y) dr \\
& \leq \frac{q}{\gamma - 1} \int_0^R \frac{1}{(\log(\log \frac{eR}{r}))^{\gamma-1}} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^{q-1} \left| \frac{df(ry)}{dr} \right| d\sigma(y) dr \\
& = \frac{q}{\gamma - 1} \int_0^R \frac{1}{r^{\frac{q-1}{q}} (\log \frac{eR}{r})^{\frac{q-1}{q}} r^{-\frac{q-1}{q}} (\log \frac{eR}{r})^{-\frac{q-1}{q}}} \\
& \quad \times \frac{1}{(\log(\log \frac{eR}{r}))^{\frac{(q-1)\gamma}{q}} (\log(\log \frac{eR}{r}))^{\frac{\gamma-q}{q}}} \\
& \quad \times \int_{\mathfrak{S}} |f(ry) - f(Ry)|^{q-1} \left| \frac{df(ry)}{dr} \right| d\sigma(y) dr.
\end{aligned}$$

Now, we use Hölder's inequality to obtain

$$\begin{aligned}
& \int_0^R \frac{1}{r(\log \frac{eR}{r})(\log(\log \frac{eR}{r}))^\gamma} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^q d\sigma(y) dr \\
& \leq \frac{q}{\gamma - 1} \left(\int_0^R \int_{\mathfrak{S}} \frac{|f(ry) - f(Ry)|^q}{r(\log \frac{eR}{r})(\log(\log \frac{eR}{r}))^\gamma} d\sigma(y) dr \right)^{\frac{q-1}{q}} \\
& \times \left(\int_0^R \int_{\mathfrak{S}} r^{q-1} \left(\log \frac{eR}{r} \right)^{q-1} \left(\log \left(\log \frac{eR}{r} \right) \right)^{q-\gamma} \left| \frac{df(ry)}{dr} \right|^q d\sigma(y) dr \right)^{\frac{1}{q}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
& \left(\int_{B(0,R)} \frac{|f - f_R|^q}{\left| \log \left| \log \frac{eR}{|x|} \right| \right|^\gamma \left| \log \frac{eR}{|x|} \right| |x|^Q} dx \right)^{\frac{1}{q}} \\
& \leq \frac{q}{\gamma - 1} \left(\int_{B(0,R)} |x|^{q-Q} \left| \log \frac{eR}{|x|} \right|^{q-1} \left| \log \left| \log \frac{eR}{|x|} \right| \right|^{q-\gamma} \left| \frac{1}{|x|} \mathbb{E} f \right|^q dx \right)^{\frac{1}{q}}. \quad (5.20)
\end{aligned}$$

Now we consider the integrals in (5.19) restricted on $B(0, eR) \setminus B(0, R)$:

$$\begin{aligned}
& \int_{B(0, eR) \setminus B(0, R)} \frac{|f - f_R|^q}{\left| \log \left| \log \frac{eR}{|x|} \right| \right|^\gamma \left| \log \frac{eR}{|x|} \right|} \frac{dx}{|x|^Q} \\
&= \int_R^{eR} \frac{1}{r \left(\log \frac{eR}{r} \right) \left(\log \left(\left(\log \frac{eR}{r} \right)^{-1} \right) \right)^\gamma} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^q d\sigma(y) dr \\
&= -\frac{1}{\gamma - 1} \left[\left(\log \left(\left(\log \frac{eR}{r} \right)^{-1} \right) \right)^{-\gamma+1} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^q d\sigma(y) \right]_{r=R}^{r=eR} \\
&+ \frac{1}{\gamma - 1} \int_R^{eR} \left(\log \left(\left(\log \frac{eR}{r} \right)^{-1} \right) \right)^{-\gamma+1} \frac{d}{dr} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^q d\sigma(y) dr \\
&= \frac{q}{\gamma - 1} \int_R^{eR} \left(\log \left(\left(\log \frac{eR}{r} \right)^{-1} \right) \right)^{-\gamma+1} \operatorname{Re} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^{q-2} \\
&\quad \times (f(ry) - f(Ry)) \overline{\frac{df(ry)}{dr}} d\sigma(y) dr.
\end{aligned}$$

In the above calculation, since we have $q - \gamma + 1 > 0$, the boundary term at $r = R$ vanishes due to inequalities

$$\begin{aligned}
\log \left(\left(\log \frac{eR}{r} \right)^{-1} \right) &= \int_1^{\left(\log \frac{eR}{r} \right)^{-1}} \frac{dt}{t} \\
&\geq \left(\log \frac{eR}{r} \right) \left(\left(\log \frac{eR}{r} \right)^{-1} - 1 \right) = 1 - \log \frac{eR}{r} \geq \frac{r - R}{R}
\end{aligned}$$

and

$$|f(ry) - f(Ry)| \leq C(R - r)$$

for $R \leq r \leq eR$. Thus, we have

$$\begin{aligned}
& \int_R^{eR} \frac{1}{r(\log \frac{eR}{r})(\log((\log \frac{eR}{r})^{-1}))^\gamma} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^q d\sigma(y) dr \\
& \leq \frac{q}{\gamma - 1} \int_R^{eR} \left(\log \left(\left(\log \frac{eR}{r} \right)^{-1} \right) \right)^{-\gamma+1} \\
& \quad \times \int_{\mathfrak{S}} |f(ry) - f(Ry)|^{q-1} \left| \frac{df(ry)}{dr} \right| d\sigma(y) dr \\
& = \frac{q}{\gamma - 1} \int_R^{eR} \frac{1}{r^{\frac{q-1}{q}} (\log \frac{eR}{r})^{\frac{q-1}{q}} r^{-\frac{q-1}{q}} (\log \frac{eR}{r})^{-\frac{q-1}{q}}} \\
& \quad \times \frac{1}{(\log((\log \frac{eR}{r})^{-1}))^{\frac{(q-1)\gamma}{q}} (\log((\log \frac{eR}{r})^{-1}))^{\frac{\gamma-q}{q}}} \\
& \quad \times \int_{\mathfrak{S}} |f(ry) - f(Ry)|^{q-1} \left| \frac{df(ry)}{dr} \right| d\sigma(y) dr.
\end{aligned}$$

Here, we use Hölder's inequality to obtain

$$\begin{aligned}
& \int_R^{eR} \frac{1}{r(\log \frac{eR}{r})(\log((\log \frac{eR}{r})^{-1}))^\gamma} \int_{\mathfrak{S}} |f(ry) - f(Ry)|^q d\sigma(y) dr \\
& \leq \frac{q}{\gamma - 1} \left(\int_R^{eR} \int_{\mathfrak{S}} \frac{|f(ry) - f(Ry)|^q}{r(\log \frac{eR}{r})(\log((\log \frac{eR}{r})^{-1}))^\gamma} d\sigma(y) dr \right)^{\frac{q-1}{q}} \\
& \quad \times \left(\int_R^{eR} \int_{\mathfrak{S}} r^{q-1} \left(\log \frac{eR}{r} \right)^{q-1} \left(\log \left(\left(\log \frac{eR}{r} \right)^{-1} \right) \right)^{q-\gamma} \\
& \quad \times \left| \frac{df(ry)}{dr} \right|^q d\sigma(y) dr \right)^{\frac{1}{q}}.
\end{aligned}$$

This implies that

$$\begin{aligned} & \left(\int_{B(0,eR) \setminus B(0,R)} \frac{|f - f_R|^q}{\left| \log \left| \log \frac{eR}{|x|} \right| \right|^\gamma \left| \log \frac{eR}{|x|} \right|} \frac{dx}{|x|^Q} \right)^{\frac{1}{q}} \leq \frac{q}{\gamma - 1} \\ & \quad \times \left(\int_{B(0,eR) \setminus B(0,R)} |x|^{q-Q} \left| \log \frac{eR}{|x|} \right|^{q-1} \left| \log \left| \log \frac{eR}{|x|} \right| \right|^{q-\gamma} \left| \frac{1}{|x|} \mathbb{E}f \right|^q dx \right)^{\frac{1}{q}}. \end{aligned} \quad (5.21)$$

Thus, the inequalities (5.20) and (5.21) complete the proof of the proposition. $\square$

Similarly, one can show a dual inequality of (5.19) stated as follows:

Proposition 5.12 (¹⁸² Ruzhansky-Suragan-Y., Proposition 6.6). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q and let $|\cdot|$ be any homogeneous quasi-norm on $\mathbb{G}$. Let $1 < \gamma < \infty$ and $\max\{1, \gamma - 1\} < q < \infty$. Then for all $f \in C_0^\infty(\mathbb{G})$ and for any $R > 0$, the following inequality holds*

$$\begin{aligned} & \left(\int_{B^c(0,R)} \frac{|f - f_{eR}|^q}{\left| \log \left| \log \frac{R}{|x|} \right| \right|^\gamma \left| \log \frac{R}{|x|} \right|} \frac{dx}{|x|^Q} \right)^{\frac{1}{q}} \\ & \leq \frac{q}{\gamma - 1} \left(\int_{B^c(0,R)} |x|^{q-Q} \left| \log \frac{R}{|x|} \right|^{q-1} \left| \log \left| \log \frac{R}{|x|} \right| \right|^{q-\gamma} \left| \frac{1}{|x|} \mathbb{E}f \right|^q dx \right)^{\frac{1}{q}}. \end{aligned} \quad (5.22)$$

Now we are in position to prove Theorem 5.8.

Proof of Theorem 5.8. Replacing R by eR let us restate (5.22)

$$\begin{aligned} & \left(\int_{B^c(0,eR)} \frac{|f - f_{e^2R}|^q}{\left| \log \left| \log \frac{eR}{|x|} \right| \right|^\gamma \left| \log \frac{eR}{|x|} \right|} \frac{dx}{|x|^Q} \right)^{\frac{1}{q}} \\ & \leq \frac{q}{\gamma - 1} \left(\int_{B^c(0,eR)} |x|^{q-Q} \left| \log \frac{eR}{|x|} \right|^{q-1} \left| \log \left| \log \frac{eR}{|x|} \right| \right|^{q-\gamma} \left| \frac{1}{|x|} \mathbb{E} f \right|^q dx \right)^{\frac{1}{q}}. \end{aligned} \quad (5.23)$$

Then, the inequalities (5.19) and (5.23) imply (5.18) for $f \in C_0^\infty(\mathbb{G})$.

Now we prove (5.18) for $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$. We first show that (5.19) holds for $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$. Let $\{f_m\} \subset C_0^\infty(\mathbb{G})$ be a sequence such that $f_m \rightarrow f$ in $W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$ as $m \rightarrow \infty$ and almost everywhere by the definition (5.16). Setting

$$f_{R,m}(x) := \frac{f_m(x) - f_m\left(R \frac{x}{|x|}\right)}{\left| \log \left| \log \frac{eR}{|x|} \right| \right|^{\frac{\gamma}{q}} \left| \log \frac{eR}{|x|} \right|^{\frac{1}{q}}},$$

we observe that $\{f_{R,m}\}_{m \in \mathbb{N}}$ is a Cauchy sequence in $L^q(\mathbb{G}; dx/|x|^Q)$, since (5.19) holds for $f_m - f_k \in C_0^\infty(\mathbb{G})$. Consequently, there is $g_R \in L^q\left(\mathbb{G}; \frac{dx}{|x|^Q}\right)$ such that $f_{R,m} \rightarrow g_R$ in $L^q(\mathbb{G}; dx/|x|^Q)$ as $m \rightarrow \infty$. Taking into account

$$\begin{aligned} & \left\{ x \in \mathbb{G} \setminus \{0\} : f_m\left(R \frac{x}{|x|}\right) \not\rightarrow f\left(R \frac{x}{|x|}\right) \right\} \\ & \subset \bigcup_{r>0} \left\{ x \in \mathbb{G} \setminus \{0\} : f_m\left(r \frac{x}{|x|}\right) \not\rightarrow f\left(r \frac{x}{|x|}\right) \right\} \\ & = \{x \in \mathbb{G} \setminus \{0\} : f_m(x) \not\rightarrow f(x)\}, \end{aligned}$$

we obtain $f_m\left(R \frac{x}{|x|}\right) \rightarrow f\left(R \frac{x}{|x|}\right)$. It means that we have

$$\frac{f(x) - f\left(R \frac{x}{|x|}\right)}{\left| \log \left| \log \frac{eR}{|x|} \right| \right|^{\frac{\gamma}{q}} \left| \log \frac{eR}{|x|} \right|^{\frac{1}{q}}} = g_R(x)$$

almost everywhere. Thus, we have verified that (5.19) holds for all $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$. In a similar way, we can obtain the inequality (5.23) for all $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$. Since the inequalities (5.19) and (5.23) hold for all $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$, then we can conclude that the inequality (5.18) holds for $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(\mathbb{G})$ too.

Now let us prove the sharpness of the constant $\frac{q}{\gamma-1}$ in (5.18). By (5.18) we know that

$$\begin{aligned} & \left(\int_{B(0,R)} \frac{|f(x)|^q}{\left| \log \left| \log \frac{eR}{|x|} \right| \right|^\gamma \left| \log \frac{eR}{|x|} \right| |x|^Q} dx \right)^{\frac{1}{q}} \\ & \leq \frac{q}{\gamma-1} \left(\int_{B(0,R)} |x|^{q-Q} \left| \log \frac{eR}{|x|} \right|^{q-1} \left| \log \left| \log \frac{eR}{|x|} \right| \right|^{q-\gamma} \left| \frac{1}{|x|} \mathbb{E} f \right|^q dx \right)^{\frac{1}{q}} \quad (5.24) \end{aligned}$$

holds for all $f \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(B(0,R))$. Therefore, it is enough to show the sharpness of the constant $\frac{q}{\gamma-1}$ in (5.24). For this, as in the Abelian case (see¹³⁹ Section 3), for large $\ell \in \mathbb{N}$ let us consider the following sequence of functions

$$f_\ell(x) := \begin{cases} (\log(\log(\ell e R)))^{\frac{\gamma-1}{q}}, & \text{when } |x| \leq \frac{1}{\ell}, \\ \left(\log \left(\log \frac{eR}{|x|} \right) \right)^{\frac{\gamma-1}{q}}, & \text{when } \frac{1}{\ell} \leq |x| \leq \frac{R}{2}, \\ (\log(\log(2e)))^{\frac{\gamma-1}{q}} \frac{2}{R} (R - |x|), & \text{when } \frac{R}{2} \leq |x| \leq R. \end{cases}$$

It is easy to see that $f_\ell \in W_0^1 L_{Q,q,\frac{q-1}{q},\frac{q-\gamma}{q}}(B(0,R))$. Then, for $\tilde{f}_\ell(r) := f_\ell(x)$ with $r = |x| \geq 0$ we obtain

$$\frac{d}{dr} \tilde{f}_\ell(r) = \begin{cases} 0, & \text{when } r < \frac{1}{\ell}, \\ -\frac{\gamma-1}{q} r^{-1} \left(\log \left(\log \frac{eR}{r} \right) \right)^{\frac{\gamma-1}{q}-1} \left(\log \frac{eR}{r} \right)^{-1}, & \text{when } \frac{1}{\ell} < r < \frac{R}{2}, \\ -\frac{2}{R} (\log(\log(2e)))^{\frac{\gamma-1}{q}}, & \text{when } \frac{R}{2} < r < R. \end{cases}$$

It follows that

$$\begin{aligned}
& \int_{B(0,R)} |x|^{q-Q} \left| \log \frac{eR}{|x|} \right|^{q-1} \left| \log \left| \log \frac{eR}{|x|} \right| \right|^{q-\gamma} \left| \frac{1}{|x|} \mathbb{E} f_\ell(x) \right|^q dx \\
&= |\mathfrak{S}| \int_0^R r^{q-1} \left| \log \frac{eR}{r} \right|^{q-1} \left| \log \left| \log \frac{eR}{r} \right| \right|^{q-\gamma} \left| \frac{d}{dr} \tilde{f}_\ell(r) \right|^q dr \\
&= |\mathfrak{S}| \left(\frac{\gamma-1}{q} \right)^q \int_{\frac{1}{\ell}}^{\frac{R}{2}} r^{-1} \left(\log \frac{eR}{r} \right)^{-1} \left(\log \left(\log \frac{eR}{r} \right) \right)^{-1} dr \\
&+ (\log(\log(2e)))^{\gamma-1} \left(\frac{2}{R} \right)^q |\mathfrak{S}| \int_{\frac{R}{2}}^R r^{q-1} \left(\log \frac{eR}{r} \right)^{q-1} \left(\log \left(\log \frac{eR}{r} \right) \right)^{q-\gamma} dr \\
&= -|\mathfrak{S}| \left(\frac{\gamma-1}{q} \right)^q \int_{\frac{1}{\ell}}^{\frac{R}{2}} \frac{d}{dr} \left(\log \left(\log \left(\log \frac{eR}{r} \right) \right) \right) \\
&+ (\log(\log(2e)))^{\gamma-1} \left(\frac{2}{R} \right)^q |\mathfrak{S}| \int_{\frac{R}{2}}^R r^{q-1} \left(\log \frac{eR}{r} \right)^{q-1} \left(\log \left(\log \frac{eR}{r} \right) \right)^{q-\gamma} dr \\
&=: |\mathfrak{S}| \left(\frac{\gamma-1}{q} \right)^q (\log(\log(\log \ell e R)) - \log(\log(\log 2e))) + |\mathfrak{S}| C_{\gamma,q}, \quad (5.25)
\end{aligned}$$

where $|\mathfrak{S}|$ is the $Q-1$ dimensional surface measure of the unit sphere, and

$$C_{\gamma,q} := (2e)^q (\log(\log(2e)))^{\gamma-1} \int_0^{(\log(\log(2e)))} s^{q-\gamma} e^{q(s-e^s)} ds.$$

Since $q - \gamma + 1 > 0$ we note that $C_{\gamma,q} < +\infty$. On the other hand, we also have

$$\int_{B(0,R)} \frac{|f_\ell(x)|^q}{\left| \log \left| \log \frac{eR}{|x|} \right| \right|^\gamma \left| \log \frac{eR}{|x|} \right|} \frac{dx}{|x|^Q} = |\mathfrak{S}| \int_0^R \frac{|\tilde{f}_\ell(r)|^q}{\left| \log \left| \log \frac{eR}{r} \right| \right|^\gamma \left| \log \frac{eR}{r} \right|} \frac{dr}{r}$$

$$\begin{aligned}
&= |\mathfrak{S}|(\log(\log(\ell eR)))^{\gamma-1} \int_0^{\frac{1}{\ell}} r^{-1} \left(\log \frac{eR}{r} \right)^{-1} \left(\log \left(\log \frac{eR}{r} \right) \right)^{-\gamma} dr \\
&\quad + |\mathfrak{S}| \int_{\frac{1}{\ell}}^{\frac{R}{2}} r^{-1} \left(\log \frac{eR}{r} \right)^{-1} \left(\log \left(\log \frac{eR}{r} \right) \right)^{-1} dr \\
&\quad + |\mathfrak{S}|(\log(\log(2e)))^{\gamma-1} \left(\frac{2}{R} \right)^q \int_{\frac{R}{2}}^R r^{-1} (R-r)^q \left(\log \frac{eR}{r} \right)^{-1} \\
&\quad \quad \quad \times \left(\log \left(\log \frac{eR}{r} \right) \right)^{-\gamma} dr \\
&=: \frac{|\mathfrak{S}|}{\gamma-1} + |\mathfrak{S}|(\log(\log(\log(\ell eR)) - \log(\log(\log(2e)))) + |\mathfrak{S}|C_{R,\gamma,q}, \quad (5.26)
\end{aligned}$$

where

$$\begin{aligned}
C_{R,\gamma,q} &:= (\log(\log(2e)))^{\gamma-1} \left(\frac{2}{R} \right)^q \\
&\quad \times \int_{\frac{R}{2}}^R r^{-1} (R-r)^q \left(\log \frac{eR}{r} \right)^{-1} \left(\log \left(\log \frac{eR}{r} \right) \right)^{-\gamma} dr.
\end{aligned}$$

This constant is finite since $\log(\log \frac{eR}{r}) \geq \frac{R-r}{R}$ for all $r \leq R$ and $q - \gamma > -1$. Thus, (5.25) and (5.26) imply that

$$\begin{aligned}
&\int_{B(0,R)} |x|^{q-Q} \left| \log \frac{eR}{|x|} \right|^{q-1} \left| \log \left| \log \frac{eR}{|x|} \right| \right|^{q-\gamma} \left| \frac{1}{|x|} \mathbb{E}f_\ell \right|^q dx \\
&\quad \times \left(\int_{B(0,R)} \frac{|f_\ell(x)|^q}{\left| \log \left| \log \frac{eR}{|x|} \right| \right|^\gamma \left| \log \frac{eR}{|x|} \right|} \frac{dx}{|x|^Q} \right)^{-1} \rightarrow \left(\frac{\gamma-1}{q} \right)^q
\end{aligned}$$

as $\ell \rightarrow \infty$. Therefore, the constant $\frac{q}{\gamma-1}$ in (5.24) is sharp. $\square$

Remark 5.13. The argument from Remark 3.15 can be applied to obtain the inequalities (5.18), (5.19) and (5.22), which provides an alternative proof.

5.0.3 GENERALISED MORREY SPACES

In this section we analyse Morrey spaces and generalised Morrey spaces on homogeneous Lie groups. Moreover, we discuss the boundedness of

the Bessel-Riesz operators and fractional integral operators in generalised Morrey spaces on homogeneous Lie groups.

BESSEL-RIESZ KERNELS ON HOMOGENEOUS LIE GROUPS

We consider the following Bessel-Riesz operators on a homogeneous Lie group $\mathbb{G}$

$$I_{\alpha,\gamma}f(x) := \int_{\mathbb{G}} K_{\alpha,\gamma}(xy^{-1})f(y)dy = \int_{\mathbb{G}} \frac{|xy^{-1}|^{\alpha-Q}}{(1+|xy^{-1}|)^{\gamma}} f(y)dy \quad (5.27)$$

for any $f \in L^p_{loc}(\mathbb{G})$, $p \geq 1$ and $\gamma \geq 0$, $0 < \alpha < Q$ with the Bessel-Riesz kernel $K_{\alpha,\gamma}(x) = \frac{|x|^{\alpha-Q}}{(1+|x|)^{\gamma}}$, where $|\cdot|$ is any homogeneous quasi-norm on $\mathbb{G}$ of homogeneous dimension Q .

Let us first calculate the L^p -norms of the Bessel-Riesz kernel $K_{\alpha,\gamma}$.

Theorem 5.14 (¹⁸¹ Ruzhansky-Suragan-Y., Theorem 2.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. If $0 < \alpha < Q$ and $\gamma > 0$ then $K_{\alpha,\gamma} \in L^{p_1}(\mathbb{G})$ and*

$$\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \sim \left(\sum_{k \in \mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} \right)^{\frac{1}{p_1}},$$

for any $R > 0$ and $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$.

Let us recall the following result, which will be used in the sequel.

Lemma 5.15 (¹²³). *If $b > a > 0$ then $\sum_{k \in \mathbb{Z}} \frac{(u^k R)^a}{(1+u^k R)^b} < \infty$, for every $u > 1$ and $R > 0$.*

Proof of Theorem 5.14. Introducing polar coordinates $(r, y) = (|x|, \frac{x}{|x|}) \in (0, \infty) \times \mathfrak{S}$ on $\mathbb{G}$, where $\mathfrak{S}$ is the sphere as in (2.7), and using (2.8) for any $R > 0$, we have

$$\int_{\mathbb{G}} |K_{\alpha,\gamma}(x)|^{p_1} dx = \int_{\mathbb{G}} \frac{|x|^{(\alpha-Q)p_1}}{(1+|x|)^{\gamma p_1}} dx$$

$$= \int_0^\infty \int_{\mathbb{G}} \frac{r^{(\alpha-Q)p_1+Q-1}}{(1+r)^{\gamma p_1}} d\sigma(y) dr = |\sigma| \sum_{k \in \mathbb{Z}} \int_{2^k R \leq r < 2^{k+1} R} \frac{r^{(\alpha-Q)p_1+Q-1}}{(1+r)^{\gamma p_1}} dr,$$

where $|\sigma|$ is the $Q - 1$ dimensional surface measure of the unit sphere.

Then it follows that

$$\begin{aligned} \int_{\mathbb{G}} |K_{\alpha,\gamma}(x)|^{p_1} dx &\leq |\sigma| \sum_{k \in \mathbb{Z}} \frac{1}{(1+2^k R)^{\gamma p_1}} \int_{2^k R \leq r < 2^{k+1} R} r^{(\alpha-Q)p_1+Q-1} dr \\ &= \frac{|\sigma|(2^{(\alpha-Q)p_1+Q} - 1)}{(\alpha - Q)p_1 + Q} \sum_{k \in \mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}}. \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} \int_{\mathbb{G}} |K_{\alpha,\gamma}(x)|^{p_1} dx &\geq \frac{|\sigma|}{2^{\gamma p_1}} \sum_{k \in \mathbb{Z}} \frac{1}{(1+2^k R)^{\gamma p_1}} \int_{2^k R \leq r < 2^{k+1} R} r^{(\alpha-Q)p_1+Q-1} dr \\ &= \frac{|\sigma|(2^{(\alpha-Q)p_1+Q} - 1)}{2^{\gamma p_1}((\alpha - Q)p_1 + Q)} \sum_{k \in \mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}}. \end{aligned}$$

Therefore, for every $R > 0$ we arrive at

$$\int_{\mathbb{G}} |K_{\alpha,\gamma}(x)|^{p_1} dx \sim \sum_{k \in \mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}}.$$

For $p_1 \in \left(\frac{Q}{Q+\gamma-\alpha}, \frac{Q}{Q-\alpha}\right)$ using Lemma 5.15 with $u = 2, a = (\alpha-Q)p_1+Q, b = \gamma p_1$, we obtain $\sum_{k \in \mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} < \infty$ which implies $K_{\alpha,\gamma} \in L^{p_1}(\mathbb{G})$. $\square$

In view of Young's inequality (see Proposition 2.12) and taking into account the definition of the Bessel-Riesz operator (5.27), we immediately obtain:

Corollary 5.16 (¹⁸¹ Ruzhansky-Suragan-Y., Corollay 2.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Let $1 \leq p, q, p_1 \leq \infty$ with $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{p_1}$. Let $0 < \alpha < Q$ and $\gamma > 0$ with $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$. Then for all $f \in L^p(\mathbb{G})$ we have*

$$\|I_{\alpha,\gamma} f\|_{L^q(\mathbb{G})} \leq \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{L^p(\mathbb{G})}.$$

We conclude from Corollary 5.16 that $I_{\alpha,\gamma}$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ and

$$\|I_{\alpha,\gamma}\|_{L^p(\mathbb{G}) \rightarrow L^q(\mathbb{G})} \leq \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}.$$

HARDY-LITTLEWOOD MAXIMAL OPERATOR IN GENERALISED MORREY SPACES

In this section we define Morrey and generalised Morrey spaces on homogeneous Lie groups. Then we define the Hardy-Littlewood maximal operator in these spaces.

We define the Morrey spaces on a homogeneous group $\mathbb{G}$ of homogeneous dimension Q by

$$M^{p,q}(\mathbb{G}) := \{f \in L^p_{loc}(\mathbb{G}) : \|f\|_{M^{p,q}(\mathbb{G})} < \infty\}, \quad 1 \leq p \leq q, \quad (5.28)$$

where

$$\|f\|_{M^{p,q}(\mathbb{G})} := \sup_{r>0, a \in \mathbb{G}} r^{Q(1/q-1/p)} \left(\int_{B(a,r)} |f(x)|^p dx \right)^{1/p}.$$

Next, for a function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $1 \leq p < \infty$, let us define the generalised Morrey space $M^{p,\phi}(\mathbb{G})$ by

$$M^{p,\phi}(\mathbb{G}) := \{f \in L^p_{loc}(\mathbb{G}) : \|f\|_{M^{p,\phi}(\mathbb{G})} < \infty\}, \quad (5.29)$$

where

$$\|f\|_{M^{p,\phi}(\mathbb{G})} := \sup_{r>0, a \in \mathbb{G}} \frac{1}{\phi(r)} \left(\frac{1}{r^Q} \int_{B(a,r)} |f(x)|^p dx \right)^{1/p}.$$

In the sequel, we will assume that ϕ is nonincreasing and $t^{Q/p}\phi(t)$ is non-decreasing, so that ϕ satisfies the doubling condition, that is, there exists a positive constant $C_1 > 0$ such that

$$\frac{1}{2} \leq \frac{r}{s} \leq 2 \implies \frac{1}{C_1} \leq \frac{\phi(r)}{\phi(s)} \leq C_1. \quad (5.30)$$

By the definition of Morrey spaces (5.28), one can observe that

Lemma 5.17 (¹⁸¹ Ruzhansky-Suragan-Y., Lemma 3.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Then*

$$\|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})} \leq \|K_{\alpha,\gamma}\|_{M^{p_1,p_1}(\mathbb{G})} = \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}, \quad (5.31)$$

where $1 \leq p_2 \leq p_1$ and $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$.

Now, for every $f \in L^p_{loc}(\mathbb{G})$, we define the Hardy-Littlewood maximal operator M by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \quad (5.32)$$

where $|B(x,r)|$ stands for the Haar measure of the (quasi-)ball $B = B(x,r)$.

We now recall the result on the boundedness of the Hardy-Littlewood maximal operator in the generalised Morrey spaces from the work of Arai and Mizuhara¹¹ (see also Nakai¹⁵⁰ for extended versions), which was obtained on spaces of homogeneous type. Let us state this result on $\mathbb{G}$:

Theorem 5.18 (¹¹ Corollary 4.2). *Let $\mathbb{G}$ be a homogeneous group. Then for any $f \in M^{p,\phi}(\mathbb{G})$ and $1 < p < \infty$ there exists some positive constant $C_p > 0$ such that*

$$\|Mf\|_{M^{p,\phi}(\mathbb{G})} \leq C_p \|f\|_{M^{p,\phi}(\mathbb{G})}. \quad (5.33)$$

Remark 5.19. We refer to Nakai¹⁴⁷ for the isotropic Abelian case of (5.33), and to¹⁰³ Corollary 3.2 for the discussions on stratified groups.

Remark 5.20. The estimate (5.33) will be very useful in the next sections when we study boundedness of the Bessel-Riesz and more general operators in the generalised Morrey space, see e.g. proof of Theorems 5.21, 5.22, 5.24, 5.26 and 5.28.

BESSEL-RIESZ OPERATORS IN GENERALISED MORREY SPACES

Here we discuss the boundedness of the Bessel-Riesz operators in the generalised Morrey space (5.29). In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$ and $Q = n$

with the standard Euclidean distance the results of this section were obtained in¹²².

Theorem 5.21 (¹⁸¹ Ruzhansky-Suragan-Y., Theorem 4.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Let $\gamma > 0$ and $0 < \alpha < Q$. Let $\beta < -\alpha$, $1 < p < \infty$, and $\frac{Q}{Q+\gamma-\alpha} < p_1 < \frac{Q}{Q-\alpha}$. Assume that $0 < \phi(r) \leq Cr^\beta$ for all $r > 0$. Then for all $f \in M^{p,\phi}(\mathbb{G})$ we have*

$$\|I_{\alpha,\gamma}f\|_{M^{q,\psi}(\mathbb{G})} \leq C_{p,\phi,Q} \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}, \quad (5.34)$$

where $q = \frac{\beta p_1 p}{\beta p_1 + Q}$ and $\psi(r) = (\phi(r))^{p/q}$.

Proof of Theorem 5.21. For simplicity, let us take $a = 0$ when we use the definition of the generalised Morrey (5.29) space. For every $f \in M^{p,\phi}(\mathbb{G})$, we decompose $I_{\alpha,\gamma}f(x)$ as follows

$$I_{\alpha,\gamma}f(x) := I_1(x) + I_2(x),$$

where

$$I_1(x) := \int_{B(x,R)} \frac{|xy^{-1}|^{\alpha-Q} f(y)}{(1 + |xy^{-1}|)^\gamma} dy$$

and

$$I_2(x) := \int_{B^c(x,R)} \frac{|xy^{-1}|^{\alpha-Q} f(y)}{(1 + |xy^{-1}|)^\gamma} dy,$$

for some $R > 0$.

For I_1 , using the dyadic decomposition we have

$$\begin{aligned} |I_1(x)| &\leq \sum_{k=-\infty}^{-1} \int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} \frac{|xy^{-1}|^{\alpha-Q} |f(y)|}{(1 + |xy^{-1}|)^\gamma} dy \\ &\leq \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-Q}}{(1 + 2^k R)^\gamma} \int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} |f(y)| dy \\ &\leq CMf(x) \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-Q+Q/p_1} (2^k R)^{Q/p_1'}}{(1 + 2^k R)^\gamma}. \end{aligned}$$

Now we use the Hölder inequality for $\frac{1}{p_1} + \frac{1}{p_1'} = 1$ to obtain

$$|I_1(x)| \leq CMf(x) \left(\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} \right)^{1/p_1} \left(\sum_{k=-\infty}^{-1} (2^k R)^Q \right)^{1/p_1'}.$$

Using

$$\left(\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} \right)^{\frac{1}{p_1}} \leq \left(\sum_{k \in \mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} \right)^{\frac{1}{p_1}} \sim \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}, \quad (5.35)$$

it follows that

$$|I_1(x)| \leq C \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} Mf(x) R^{Q/p_1'}. \quad (5.36)$$

Using the Hölder inequality for $\frac{1}{p} + \frac{1}{p'} = 1$ we estimate I_2 as follows

$$\begin{aligned} |I_2(x)| &\leq \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q}}{(1+2^k R)^{\gamma}} \int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} |f(y)| dy \\ &\leq \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q}}{(1+2^k R)^{\gamma}} \left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} dy \right)^{\frac{1}{p'}} \left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} |f(y)|^p dy \right)^{\frac{1}{p}} \\ &= \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q}}{(1+2^k R)^{\gamma}} \left(\int_{2^k R}^{2^{k+1} R} \int_{\mathfrak{S}} r^{Q-1} d\sigma(y) dr \right)^{\frac{1}{p'}} \left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} |f(y)|^p dy \right)^{\frac{1}{p}} \\ &\leq C \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q}}{(1+2^k R)^{\gamma}} (2^k R)^{Q/p'} \left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} |f(y)|^p dy \right)^{\frac{1}{p}}. \end{aligned}$$

Here, taking into account $\phi(r) \leq Cr^{\beta}$ we obtain for I_2 that

$$\begin{aligned} |I_2(x)| &\leq C \|f\|_{M^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q+Q/p_1}}{(1+2^k R)^{\gamma}} \phi(2^k R) (2^k R)^{Q/p_1'} \\ &\leq C \|f\|_{M^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q+Q/p_1}}{(1+2^k R)^{\gamma}} (2^k R)^{\beta+Q/p_1'}. \end{aligned}$$

Using the Hölder inequality again, we arrive at

$$|I_2(x)| \leq C \|f\|_{M^{p,\phi}(\mathbb{G})} \left(\sum_{k=0}^{\infty} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} \right)^{1/p_1} \left(\sum_{k=0}^{\infty} (2^k R)^{\beta p'_1+Q} \right)^{1/p'_1}.$$

By the assumptions $p_1 < \frac{Q}{Q-\alpha}$ and $\beta < -\alpha$, we have $\beta p'_1 + Q < 0$. Then, Theorem 5.14 implies that

$$\left(\sum_{k=0}^{\infty} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} \right)^{1/p_1} \leq \left(\sum_{k \in \mathbb{Z}} \frac{(2^k R)^{(\alpha-Q)p_1+Q}}{(1+2^k R)^{\gamma p_1}} \right)^{1/p_1} \sim \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}.$$

Thus, we have for I_2 that

$$|I_2(x)| \leq C \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})} R^{Q/p'_1+\beta}. \quad (5.37)$$

Summing up the estimates (5.36) and (5.37), we obtain

$$|I_{\alpha,\gamma}f(x)| \leq C \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \left(Mf(x) R^{Q/p'_1} + \|f\|_{M^{p,\phi}(\mathbb{G})} R^{Q/p'_1+\beta} \right).$$

Here, assuming that f is not identically 0 and that Mf is finite everywhere, one can choose $R > 0$ such that $R^\beta = \frac{Mf(x)}{\|f\|_{M^{p,\phi}(\mathbb{G})}}$. Then, it follows that

$$|I_{\alpha,\gamma}f(x)| \leq C \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}^{-\frac{Q}{\beta p'_1}} (Mf(x))^{1+\frac{Q}{\beta p'_1}},$$

for every $x \in \mathbb{G}$. Taking $q = \frac{\beta p'_1 p}{\beta p'_1 + Q}$, for any $r > 0$ we obtain from the above that

$$\left(\int_{|x|<r} |I_{\alpha,\gamma}f(x)|^q dx \right)^{\frac{1}{q}} \leq C \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}^{1-\frac{p}{q}} \left(\int_{|x|<r} |Mf(x)|^p dx \right)^{\frac{1}{q}}.$$

Dividing both sides by $(\phi(r))^{p/q} r^{Q/q}$ we arrive at

$$\frac{\left(\int_{|x|<r} |I_{\alpha,\gamma}f(x)|^q dx \right)^{\frac{1}{q}}}{\psi(r) r^{Q/q}} \leq C \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}^{1-p/q} \frac{\left(\int_{|x|<r} |Mf(x)|^p dx \right)^{\frac{1}{q}}}{(\phi(r))^{p/q} r^{Q/q}},$$

where $\psi(r) = (\phi(r))^{p/q}$. Thus, we take the supremum over $r > 0$ to conclude that

$$\|I_{\alpha,\gamma}f\|_{M^{q,\psi}(\mathbb{G})} \leq C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{M^{p,\phi}(\mathbb{G})}^{1-p/q}\|Mf\|_{M^{p,\phi}(\mathbb{G})}^{p/q},$$

which implies (5.34), after applying (5.33). $\square$

By Lemma 5.17 we note that the Bessel-Riesz kernel belongs to the Morrey spaces. We use this property in the following statement refining the estimate (5.34):

Theorem 5.22 (¹⁸¹ Ruzhansky-Suragan-Y., Theorem 4.2). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Let $\gamma > 0$, $0 < \alpha < Q$ and $1 < p < \infty$. Let $\beta < -\alpha$, $\frac{Q}{Q+\gamma-\alpha} < p_2 \leq p_1 < \frac{Q}{Q-\alpha}$ with $p_2 \geq 1$. Assume that $0 < \phi(r) \leq Cr^\beta$ for all $r > 0$. Then for all $f \in M^{p,\phi}(\mathbb{G})$ we have*

$$\|I_{\alpha,\gamma}f\|_{M^{q,\psi}(\mathbb{G})} \leq C_{p,\phi,Q}\|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})}\|f\|_{M^{p,\phi}(\mathbb{G})}, \quad (5.38)$$

where $q = \frac{\beta p_1' p}{\beta p_1' + Q}$ and $\psi(r) = (\phi(r))^{p/q}$.

Proof of Theorem 5.22. We decompose $I_{\alpha,\gamma}f(x)$ as follows

$$I_{\alpha,\gamma}f(x) := I_1(x) + I_2(x),$$

where

$$I_1(x) := \int_{B(x,R)} \frac{|xy^{-1}|^{\alpha-Q}}{(1+|xy^{-1}|)^\gamma} f(y) dy$$

and

$$I_2(x) := \int_{B^c(x,R)} \frac{|xy^{-1}|^{\alpha-Q}}{(1+|xy^{-1}|)^\gamma} f(y) dy$$

for $R > 0$. Using the dyadic decomposition, we have for I_1 that

$$\begin{aligned}
|I_1(x)| &\leq \sum_{k=-\infty}^{-1} \int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} \frac{|xy^{-1}|^{\alpha-Q} |f(y)|}{(1 + |xy^{-1}|)^\gamma} dy \\
&\leq \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-Q}}{(1 + 2^k R)^\gamma} \int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} |f(y)| dy \\
&\leq CMf(x) \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-Q+Q/p_2} (2^k R)^{Q/p_2'}}{(1 + 2^k R)^\gamma},
\end{aligned}$$

where $1 \leq p_2 \leq p_1$. Here, Hölder's inequality for $\frac{1}{p_2} + \frac{1}{p_2'} = 1$ implies that

$$|I_1(x)| \leq CMf(x) \left(\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-Q)p_2+Q}}{(1 + 2^k R)^{\gamma p_2}} \right)^{1/p_2} \left(\sum_{k=-\infty}^{-1} (2^k R)^Q \right)^{1/p_2'}.$$

For simplicity, throughout this proof we take $a = 0$ when we use the definition of the generalised Morrey (5.29) space.

By making use (5.35), we obtain

$$\begin{aligned}
|I_1(x)| &\leq C_2 Mf(x) \left(\int_{0 < |x| < R} K_{\alpha, \gamma}^{p_2}(x) dx \right)^{\frac{1}{p_2}} R^{Q/p_2'} \\
&\leq C \|K_{\alpha, \gamma}\|_{M^{p_2, p_1}(\mathbb{G})} Mf(x) R^{Q/p_1'}.
\end{aligned} \tag{5.39}$$

For I_2 , we use the Hölder inequality for $\frac{1}{p} + \frac{1}{p'} = 1$ to get

$$|I_2(x)| \leq \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q}}{(1 + 2^k R)^\gamma} (2^k R)^{Q/p'} \left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} |f(y)|^p dy \right)^{1/p}.$$

Since

$$\begin{aligned}
&\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} K_{\alpha, \gamma}^{p_2}(xy^{-1}) dy \right)^{1/p_2} \\
&\sim \frac{(2^k R)^{(\alpha-Q)+Q/p_2}}{(1 + 2^k R)^\gamma} \geq C \frac{(2^k R)^{(\alpha-Q)}}{(1 + 2^k R)^\gamma} \left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} dy \right)^{1/p_2}, \tag{5.40}
\end{aligned}$$

it follows that

$$\begin{aligned}
|I_2(x)| &\leq C\|f\|_{M^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{(2^k R)^\alpha \phi(2^k R)}{(1+2^k R)^\gamma} \frac{\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} dy\right)^{1/p_2}}{(2^k R)^{Q/p_2}} \\
&\leq C\|f\|_{M^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \phi(2^k R) (2^k R)^{Q/p_1'} \frac{\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} K_{\alpha,\gamma}^{p_2}(xy^{-1}) dy\right)^{1/p_2}}{(2^k R)^{Q/p_2 - Q/p_1}}.
\end{aligned}$$

By the assumption $\phi(r) \leq Cr^\beta$ and the estimate

$$\frac{\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} K_{\alpha,\gamma}^{p_2}(xy^{-1}) dy\right)^{1/p_2}}{(2^k R)^{Q/p_2 - Q/p_1}} \lesssim \|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})}$$

for every $k = 0, 1, 2, \dots$, we obtain

$$|I_2(x)| \leq C\|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} (2^k R)^{\beta+Q/p_1'}.$$

Since $\beta + Q/p_1' < 0$ it follows that

$$|I_2(x)| \leq C\|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})} R^{\beta+Q/p_1'}. \quad (5.41)$$

Summing up the estimates (5.39) and (5.41), we obtain

$$|I_{\alpha,\gamma}f(x)| \leq C\|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})} (Mf(x)R^{Q/p_1'} + \|f\|_{M^{p,\phi}(\mathbb{G})} R^{\beta+Q/p_1'}).$$

Assume that f is not identically 0 and that Mf is finite everywhere, then we can choose $R > 0$ such that $R^\beta = \frac{Mf(x)}{\|f\|_{M^{p,\phi}(\mathbb{G})}}$. Inserting this in the above estimate we arrive at

$$|I_{\alpha,\gamma}f(x)| \leq C\|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}^{-\frac{Q}{\beta p_1'}} (Mf(x))^{1+\frac{Q}{\beta p_1'}}.$$

It follows for any $r > 0$ that

$$\begin{aligned} \left(\int_{|x|<r} |I_{\alpha,\gamma} f(x)|^q dx \right)^{\frac{1}{q}} \\ \leq C \|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}^{1-\frac{p}{q}} \left(\int_{|x|<r} |Mf(x)|^p dx \right)^{\frac{1}{q}}, \end{aligned}$$

where $q = \frac{\beta p_1' p}{\beta p_1' + Q}$. Here, dividing both sides by $(\phi(r))^{p/q} r^{Q/q}$ we obtain

$$\begin{aligned} \frac{\left(\int_{|x|<r} |I_{\alpha,\gamma} f(x)|^q dx \right)^{\frac{1}{q}}}{\psi(r) r^{Q/q}} &\leq C \|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}^{1-p/q} \\ &\quad \times \frac{\left(\int_{|x|<r} |Mf(x)|^p dx \right)^{1/q}}{(\phi(r))^{p/q} r^{Q/q}}, \end{aligned}$$

where $\psi(r) = (\phi(r))^{p/q}$. Now we take the supremum over $r > 0$, and then use (5.33) to obtain

$$\begin{aligned} \|I_{\alpha,\gamma} f\|_{M^{q,\psi}(\mathbb{G})} &\leq C \|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}^{1-p/q} \|Mf\|_{M^{p,\phi}(\mathbb{G})}^{p/q} \\ &\leq C_{p,\phi,Q} \|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}, \end{aligned}$$

as desired. $\square$

By Lemma 5.17, we note that Theorem 5.22 implies Theorem 5.21

$$\begin{aligned} \|I_{\alpha,\gamma} f\|_{M^{q,\psi}(\mathbb{G})} &\leq C \|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})} \\ &\leq C \|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}. \end{aligned}$$

In order to improve our results, we present the following lemma, which states that the kernel $K_{\alpha,\gamma}$ belongs to the generalised Morrey space $M^{p_2,\omega}(\mathbb{G})$ for some $p_2 \geq 1$ and some function ω .

Lemma 5.23 (¹⁸¹ Ruzhansky-Suragan-Y., Lemma 4.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $\gamma > 0$, $p_2 \geq 1$ and $Q - \frac{Q}{p_2} < \alpha < Q$. Assume that $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\omega(r) \geq Cr^{\alpha-Q}$ for all $r > 0$. Then we have*

$$K_{\alpha,\gamma} \in M^{p_2,\omega}(\mathbb{G}).$$

Proof of Lemma 5.23. Note that it is enough to evaluate the following integral around zero

$$\begin{aligned} \int_{|x| \leq R} K_{\alpha,\gamma}^{p_2}(x) dx &= \int_{|x| \leq R} \frac{|x|^{(\alpha-Q)p_2}}{(1+|x|)^{\gamma p_2}} dx \\ &\leq |\sigma| \int_{0 < r \leq R} r^{(\alpha-Q)p_2+Q-1} dr \leq C \omega^{p_2}(R) R^Q. \end{aligned}$$

Dividing both sides of the above inequality by $\omega^{p_2}(R) R^Q$ and taking p_2 th-root, it follows that

$$\frac{\left(\int_{|x| \leq R} K_{\alpha,\gamma}^{p_2}(x) dx \right)^{1/p_2}}{\omega(R) R^{Q/p_2}} \leq C^{1/p_2},$$

which implies

$$\sup_{R>0} \frac{\left(\int_{|x| \leq R} K_{\alpha,\gamma}^{p_2}(x) dx \right)^{1/p_2}}{\omega(R) R^{Q/p_2}} < \infty,$$

as desired. $\square$

Theorem 5.24 (¹⁸¹ Ruzhansky-Suragan-Y., Theorem 4.3). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Let $0 < \alpha < Q$, $\gamma > 0$ and $1 < p < \infty$. Let $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfy the doubling condition (5.30) and assume that $\omega(r) \leq Cr^{-\alpha}$ for all $r > 0$, so that $K_{\alpha,\gamma} \in M^{p_2,\omega}(\mathbb{G})$ for $\frac{Q}{Q+\gamma-\alpha} < p_2 < \frac{Q}{Q-\alpha}$ with $p_2 \geq 1$. Let $\beta < -\alpha < -Q - \beta$ and assume that $0 < \phi(r) \leq Cr^\beta$ for all $r > 0$. Then for all $f \in M^{p,\phi}(\mathbb{G})$ we have*

$$\|I_{\alpha,\gamma} f\|_{M^{q,\psi}(\mathbb{G})} \leq C_{p,\phi,Q} \|K_{\alpha,\gamma}\|_{M^{p_2,\omega}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}, \quad (5.42)$$

where $q = \frac{\beta p}{\beta + Q - \alpha}$ and $\psi(r) = (\phi(r))^{p/q}$.

Remark 5.25. Note that Theorem 5.24 gives the best estimate among the last three theorems. Indeed, if we take $\omega(R) := (1 + R^{Q/q_1}) R^{-Q/p_1}$ for some $q_1 > p_1$, then we obtain $\|K_{\alpha,\gamma}\|_{M^{p_2,\omega}(\mathbb{G})} \leq \|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})}$. Hence, Theorem

5.24 and Lemma 5.17 imply that

$$\begin{aligned}\|I_{\alpha,\gamma}f\|_{M^{q,\psi}(\mathbb{G})} &\leq C\|K_{\alpha,\gamma}\|_{M^{p_2,\omega}(\mathbb{G})}\|f\|_{M^{p,\phi}(\mathbb{G})} \\ &\leq C\|K_{\alpha,\gamma}\|_{M^{p_2,p_1}(\mathbb{G})}\|f\|_{M^{p,\phi}(\mathbb{G})} \\ &\leq C\|K_{\alpha,\gamma}\|_{L^{p_1}(\mathbb{G})}\|f\|_{M^{p,\phi}(\mathbb{G})}.\end{aligned}$$

Proof of Theorem 5.24. Let us write $I_{\alpha,\gamma}f(x)$ in the form

$$I_{\alpha,\gamma}f(x) := I_1(x) + I_2(x),$$

where

$$I_1(x) := \int_{B(x,R)} \frac{|xy^{-1}|^{\alpha-Q}f(y)}{(1+|xy^{-1}|)^\gamma} dy$$

and

$$I_2(x) := \int_{B^c(x,R)} \frac{|xy^{-1}|^{\alpha-Q}f(y)}{(1+|xy^{-1}|)^\gamma} dy$$

for some $R > 0$.

By using the dyadic decomposition, let us estimate I_1 as follows

$$\begin{aligned}|I_1(x)| &\leq \sum_{k=-\infty}^{-1} \int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} \frac{|xy^{-1}|^{\alpha-Q}|f(y)|}{(1+|xy^{-1}|)^\gamma} dy \\ &\leq \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-Q}}{(1+2^k R)^\gamma} \int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} |f(y)| dy \\ &\leq CMf(x) \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-Q+Q/p_2} (2^k R)^{Q/p_2'}}{(1+2^k R)^\gamma}.\end{aligned}$$

By the Hölder inequality for $\frac{1}{p_2} + \frac{1}{p_2'} = 1$, it follows that

$$|I_1(x)| \leq CMf(x) \left(\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-Q)p_2+Q}}{(1+2^k R)^{\gamma p_2}} \right)^{1/p_2} \left(\sum_{k=-\infty}^{-1} (2^k R)^Q \right)^{1/p_2'}.$$

For simplicity, throughout this proof let us take $a = 0$ when we use the definition of the generalised Morrey (5.29) space.

Recalling (5.35) we obtain

$$\begin{aligned}
|I_1(x)| &\leq CMf(x) \left(\int_{0 < |x| < R} K_{\alpha,\gamma}^{p_2}(x) dx \right)^{\frac{1}{p_2}} R^{Q/p_2'} \\
&\leq C \|K_{\alpha,\gamma}\|_{M^{p_2,\omega}(\mathbb{G})} Mf(x) \omega(R) R^Q \\
&\leq C \|K_{\alpha,\gamma}\|_{M^{p_2,\omega}(\mathbb{G})} Mf(x) R^{Q-\alpha}.
\end{aligned} \tag{5.43}$$

where we have used the assumption $\omega(r) \leq Cr^{-\alpha}$ in the last line.

Now, let us estimate I_2 as follows

$$\begin{aligned}
|I_2(x)| &\leq \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q}}{(1+2^k R)^\gamma} \int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} |f(y)| dy \\
&\leq C \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-Q}}{(1+2^k R)^\gamma} (2^k R)^{Q/p'} \left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} |f(y)|^p dy \right)^{1/p} \\
&\leq C \|f\|_{M^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{(2^k R)^\alpha \phi(2^k R)}{(1+2^k R)^\gamma} \frac{\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} dy \right)^{1/p_2}}{(2^k R)^{Q/p_2}},
\end{aligned}$$

where we have used that $\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} dy \right)^{1/p_2} \sim (2^k R)^{Q/p_2}$. By (5.40) it follows that

$$\begin{aligned}
|I_2(x)| &\leq C \|f\|_{M^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{(2^k R)^\alpha \phi(2^k R)}{(2^k R)^{\alpha-Q}} \\
&\quad \times \frac{\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} K_{\alpha,\gamma}^{p_2}(xy^{-1}) dy \right)^{1/p_2}}{(2^k R)^{Q/p_2}}.
\end{aligned}$$

Since we have $\phi(r) \leq Cr^\beta$ and $\omega(r) \leq Cr^{-\alpha}$ for all $r > 0$ by assumption, we arrive at

$$|I_2(x)| \leq C \|f\|_{M^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} (2^k R)^{Q-\alpha+\beta} \frac{\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} K_{\alpha,\gamma}^{p_2}(xy^{-1}) dy \right)^{1/p_2}}{\omega(2^k R) (2^k R)^{Q/p_2}}.$$

Here, using

$$\frac{\left(\int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} K_{\alpha, \gamma}^{p_2}(xy^{-1}) dy\right)^{1/p_2}}{\omega(2^k R)(2^k R)^{Q/p_2}} \lesssim \|K_{\alpha, \gamma}\|_{M^{p_2, \omega}(\mathbb{G})}$$

for every $k = 0, 1, 2, \dots$, we obtain for I_2 that

$$\begin{aligned} |I_2(x)| &\leq C \|K_{\alpha, \gamma}\|_{M^{p_2, \omega}(\mathbb{G})} \|f\|_{M^{p, \phi}(\mathbb{G})} \sum_{k=0}^{\infty} (2^k R)^{Q-\alpha+\beta} \\ &\leq C \|K_{\alpha, \gamma}\|_{M^{p_2, \omega}(\mathbb{G})} \|f\|_{M^{p, \phi}(\mathbb{G})} R^{Q-\alpha+\beta}, \end{aligned} \quad (5.44)$$

since $Q - \alpha + \beta < 0$. Summing up the estimates (5.43) and (5.44), we have

$$|I_{\alpha, \gamma} f(x)| \leq C \|K_{\alpha, \gamma}\|_{M^{p_2, \omega}(\mathbb{G})} (Mf(x) R^{Q-\alpha} + \|f\|_{M^{p, \phi}(\mathbb{G})} R^{Q-\alpha+\beta}).$$

Assume that f is not identically 0 and that Mf is finite everywhere. Then we can choose $R > 0$ such that $R^\beta = \frac{Mf(x)}{\|f\|_{M^{p, \phi}(\mathbb{G})}}$. Hence, we obtain

$$|I_{\alpha, \gamma} f(x)| \leq C \|K_{\alpha, \gamma}\|_{M^{p_2, \omega}(\mathbb{G})} \|f\|_{M^{p, \phi}(\mathbb{G})}^{(\alpha-Q)/\beta} (Mf(x))^{1+(Q-\alpha)/\beta}.$$

It follows for any $r > 0$ that

$$\begin{aligned} \left(\int_{|x| < r} |I_{\alpha, \gamma} f(x)|^q dx\right)^{\frac{1}{q}} \\ \leq C \|K_{\alpha, \gamma}\|_{M^{p_2, \omega}(\mathbb{G})} \|f\|_{M^{p, \phi}(\mathbb{G})}^{1-p/q} \left(\int_{|x| < r} |Mf(x)|^p dx\right)^{1/q}, \end{aligned}$$

where $q = \frac{\beta p}{\beta + Q - \alpha}$. Dividing both sides by $(\phi(r))^{p/q} r^{Q/q}$ we get

$$\begin{aligned} \frac{\left(\int_{|x| < r} |I_{\alpha, \gamma} f(x)|^q dx\right)^{\frac{1}{q}}}{\psi(r) r^{Q/q}} \\ \leq C \|K_{\alpha, \gamma}\|_{M^{p_2, \omega}(\mathbb{G})} \|f\|_{M^{p, \phi}(\mathbb{G})}^{1-p/q} \frac{\left(\int_{|x| < r} |Mf(x)|^p dx\right)^{1/q}}{(\phi(r))^{p/q} r^{Q/q}}, \end{aligned}$$

where $\psi(r) = (\phi(r))^{p/q}$. Finally, taking the supremum over $r > 0$ and using (5.33), we obtain

$$\begin{aligned} \|I_{\alpha,\gamma}f\|_{M^{q,\psi}(\mathbb{G})} &\leq C\|K_{\alpha,\gamma}\|_{M^{p_2,\omega}(\mathbb{G})}\|f\|_{M^{p,\phi}(\mathbb{G})}^{1-p/q}\|Mf\|_{M^{p,\phi}(\mathbb{G})}^{p/q} \\ &\leq C_{p,\phi,Q}\|K_{\alpha,\gamma}\|_{M^{p_2,\omega}(\mathbb{G})}\|f\|_{M^{p,\phi}(\mathbb{G})}, \end{aligned}$$

as desired. $\square$

GENERALISED BESSEL-RIESZ OPERATOR IN GENERALISED MORREY SPACES

In this section we discuss the boundedness of some generalisations of the Bessel-Riesz operators. Moreover, we establish Olsen type inequality for such operators in generalised Morrey spaces on homogeneous Lie groups.

We define the *generalised Bessel-Riesz operator* $I_{\tilde{\rho},\gamma}$ by

$$I_{\tilde{\rho},\gamma}f(x) := \int_{\mathbb{G}} \frac{\tilde{\rho}(|xy^{-1}|)}{(1 + |xy^{-1}|)^\gamma} f(y) dy, \quad (5.45)$$

where $\gamma \geq 0$, $\tilde{\rho} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the doubling condition (5.30) and the condition

$$\int_0^1 \frac{\tilde{\rho}(t)}{t^{\gamma-Q+1}} dt < \infty. \quad (5.46)$$

Note that if we take $\tilde{\rho}(t) = t^{\alpha-Q}$ for $\gamma < \alpha < Q$, then we obtain the Bessel-Riesz kernel (5.27)

$$K_{\tilde{\rho},\gamma} = K_{\alpha,\gamma} = \frac{|xy^{-1}|^{\alpha-Q}}{(1 + |xy^{-1}|)^\gamma}.$$

Theorem 5.26 (¹⁸¹ Ruzhansky-Suragan-Y., Theorem 5.1). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm and let $\gamma > 0$. Let $\tilde{\rho}$ and ϕ satisfy the doubling condition (5.30). Assume that ϕ is surjective and for some $1 < p < q < \infty$ satisfies the inequalities*

$$\int_r^\infty \frac{(\phi(t))^p}{t} dt \leq C_1(\phi(r))^p, \quad (5.47)$$

and

$$\phi(r) \int_0^r \frac{\tilde{\rho}(t)}{t^{\gamma-Q+1}} dt + \int_r^\infty \frac{\tilde{\rho}(t)\phi(t)}{t^{\gamma-Q+1}} dt \leq C_2(\phi(r))^{p/q}, \quad (5.48)$$

for all $r > 0$. Then we have

$$\|I_{\tilde{\rho},\gamma}f\|_{M^{q,\phi^{p/q}}(\mathbb{G})} \leq C_{p,q,\phi,Q}\|f\|_{M^{p,\phi}(\mathbb{G})}. \quad (5.49)$$

Proof of Theorem 5.26. For every $R > 0$, we decompose $I_{\tilde{\rho},\gamma}f(x)$ as follows

$$I_{\tilde{\rho},\gamma}f(x) = I_{1,\tilde{\rho}}(x) + I_{2,\tilde{\rho}}(x),$$

where

$$I_{1,\tilde{\rho}}(x) := \int_{B(x,R)} \frac{\tilde{\rho}(|xy^{-1}|)f(y)}{(1+|xy^{-1}|)^\gamma} dy$$

and

$$I_{2,\tilde{\rho}}(x) := \int_{B^c(x,R)} \frac{\tilde{\rho}(|xy^{-1}|)f(y)}{(1+|xy^{-1}|)^\gamma} dy.$$

For $I_{1,\tilde{\rho}}(x)$, we obtain

$$\begin{aligned} |I_{1,\tilde{\rho}}(x)| &\leq \int_{|xy^{-1}| < R} \frac{\tilde{\rho}(|xy^{-1}|)}{(1+|xy^{-1}|)^\gamma} |f(y)| dy \\ &\leq \int_{|xy^{-1}| < R} \frac{\tilde{\rho}(|xy^{-1}|)}{|xy^{-1}|^\gamma} |f(y)| dy \\ &= \sum_{k=-\infty}^{-1} \int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} \frac{\tilde{\rho}(|xy^{-1}|)}{|xy^{-1}|^\gamma} |f(y)| dy. \end{aligned}$$

By (5.30) and since

$$\int_{2^k R}^{2^{k+1} R} \frac{\tilde{\rho}(t)}{t^{\gamma-Q+1}} dt \geq C \frac{\tilde{\rho}(2^k R)}{(2^k R)^{\gamma-Q+1}} 2^k R \geq C \frac{\tilde{\rho}(2^k R)}{(2^k R)^{\gamma-Q}}, \quad (5.50)$$

it follows that

$$|I_{1,\tilde{\rho}}(x)| \leq C \sum_{k=-\infty}^{-1} \frac{\tilde{\rho}(2^k R)}{(2^k R)^\gamma} \int_{|xy^{-1}| < 2^{k+1} R} |f(y)| dy$$

$$\begin{aligned}
&\leq CMf(x) \sum_{k=-\infty}^{-1} \frac{\tilde{\rho}(2^k R)}{(2^k R)^{\gamma-Q}} \\
&\leq CMf(x) \sum_{k=-\infty}^{-1} \int_{2^k R}^{2^{k+1} R} \frac{\tilde{\rho}(t)}{t^{\gamma-Q+1}} dt \\
&= CMf(x) \int_0^R \frac{\tilde{\rho}(t)}{t^{\gamma-Q+1}} dt.
\end{aligned}$$

Here, we use (5.48) to obtain

$$|I_{1,\tilde{\rho}}(x)| \leq CMf(x)(\phi(R))^{(p-q)/q}. \quad (5.51)$$

For $I_{2,\tilde{\rho}}(x)$, applying (5.30) we have

$$\begin{aligned}
|I_{2,\tilde{\rho}}(x)| &\leq \int_{|xy^{-1}| \geq R} \frac{\tilde{\rho}(|xy^{-1}|)}{(1 + |xy^{-1}|)^\gamma} |f(y)| dy \\
&\leq \int_{|xy^{-1}| \geq R} \frac{\tilde{\rho}(|xy^{-1}|)}{|xy^{-1}|^\gamma} |f(y)| dy \\
&= \sum_{k=0}^{\infty} \int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} \frac{\tilde{\rho}(|xy^{-1}|)}{|xy^{-1}|^\gamma} |f(y)| dy \\
&\leq C \sum_{k=0}^{\infty} \frac{\tilde{\rho}(2^k R)}{(2^k R)^\gamma} \int_{|xy^{-1}| < 2^{k+1} R} |f(y)| dy.
\end{aligned}$$

Using Hölder's inequality, it follows that

$$\begin{aligned}
|I_{2,\tilde{\rho}}(x)| &\leq C \sum_{k=0}^{\infty} \frac{\tilde{\rho}(2^k R)}{(2^k R)^\gamma} \left(\int_{|xy^{-1}| < 2^{k+1} R} dy \right)^{1-\frac{1}{p}} \left(\int_{|xy^{-1}| < 2^{k+1} R} |f(y)| dy \right)^{\frac{1}{p}} \\
&\leq C \sum_{k=0}^{\infty} \frac{\tilde{\rho}(2^k R)}{(2^k R)^{\gamma-Q+\frac{Q}{p}}} \left(\int_{|xy^{-1}| < 2^{k+1} R} |f(y)| dy \right)^{\frac{1}{p}}.
\end{aligned}$$

For simplicity, throughout this proof let us take $a = 0$ when we use the definition of the generalised Morrey (5.29) space.

Taking into account this definition (5.29), we arrive at

$$\begin{aligned}
|I_{2,\tilde{\rho}}(x)| &\leq C\|f\|_{M^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \frac{\tilde{\rho}(2^{k+1}R)\phi(2^{k+1}R)}{(2^k R)^{\gamma-Q}} \\
&\leq C\|f\|_{M^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \int_{2^k R}^{2^{k+1}R} \frac{\tilde{\rho}(t)\phi(t)}{t^{\gamma-Q+1}} dt \\
&= C\|f\|_{M^{p,\phi}(\mathbb{G})} \int_R^{\infty} \frac{\tilde{\rho}(t)\phi(t)}{t^{\gamma-Q+1}} dt,
\end{aligned}$$

where we have used the fact that

$$\int_{2^k R}^{2^{k+1}R} \frac{\tilde{\rho}(t)\phi(t)}{t^{\gamma-Q+1}} dt \geq C \frac{\tilde{\rho}(2^{k+1}R)\phi(2^{k+1}R)}{(2^{k+1}R)^{\gamma-Q+1}} 2^k R \geq C \frac{\tilde{\rho}(2^{k+1}R)\phi(2^{k+1}R)}{(2^k R)^{\gamma-Q}}.$$

Now, we use (5.48) to obtain

$$|I_{2,\tilde{\rho}}(x)| \leq C\|f\|_{M^{p,\phi}(\mathbb{G})} (\phi(R))^{p/q}. \quad (5.52)$$

Summing up the estimates (5.51) and (5.52), we obtain

$$|I_{\tilde{\rho},\gamma}f(x)| \leq C(Mf(x)(\phi(R))^{(p-q)/q} + \|f\|_{M^{p,\phi}(\mathbb{G})}(\phi(R))^{p/q}).$$

Assume that f is not identically 0 and that Mf is finite everywhere. Then since ϕ is surjective we can choose $R > 0$ such that

$$\phi(R) = Mf(x) \|f\|_{M^{p,\phi}(\mathbb{G})}^{-1}.$$

For $x \in \mathbb{G}$, it follows that

$$|I_{\tilde{\rho},\gamma}f(x)| \leq C(Mf(x))^{\frac{p}{q}} \|f\|_{M^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}},$$

which implies

$$\left(\int_{B(0,r)} |I_{\tilde{\rho},\gamma}f(x)|^q \right)^{1/q} \leq C \left(\int_{B(0,r)} |Mf(x)|^p \right)^{1/q} \|f\|_{M^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}}.$$

Now, we divide both sides by $(\phi(r))^{p/q} r^{Q/q}$ to get

$$\begin{aligned} \frac{1}{(\phi(r))^{p/q}} \left(\frac{1}{r^Q} \int_{B(0,r)} |I_{\tilde{\rho},\gamma} f(x)|^q \right)^{1/q} \\ \leq C \frac{1}{(\phi(r))^{p/q}} \left(\frac{1}{r^Q} \int_{B(0,r)} |Mf(x)|^p \right)^{1/q} \|f\|_{M^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}}. \end{aligned}$$

Taking the supremum over $r > 0$ and using the boundedness of the maximal operator M on $M^{p,\phi}(\mathbb{G})$ shown in (5.33), we obtain (5.49), as desired. $\square$

Now let us show the following Olsen type inequality for the generalised Bessel-Riesz operator $I_{\rho,\gamma}$:

Theorem 5.27 (¹⁸¹ Ruzhansky-Suragan-Y., Theorem 5.2). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm and let $\gamma > 0$. Let $\tilde{\rho}$ and ϕ satisfy the doubling condition (5.30). Assume that ϕ is surjective and satisfies (5.47) and (5.48). Then for $1 < p < p_2 < \infty$ we have*

$$\|W I_{\tilde{\rho},\gamma} f\|_{M^{p,\phi}(\mathbb{G})} \leq C_{p,\phi,Q} \|W\|_{M^{p_2,\phi^{p/p_2}}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}, \quad (5.53)$$

provided that $W \in M^{p_2,\phi^{p/p_2}}(\mathbb{G})$.

Proof of Theorem 5.27. By Hölder's inequality, we have

$$\begin{aligned} \frac{1}{r^Q} \int_{B(0,r)} |W(x) I_{\tilde{\rho},\gamma} f(x)|^p dx \\ \leq \left(\frac{1}{r^Q} \int_{B(0,r)} |W(x)|^{p_2} dx \right)^{p/p_2} \left(\frac{1}{r^Q} \int_{B(0,r)} |I_{\tilde{\rho},\gamma} f(x)|^{\frac{pp_2}{p_2-p}} dx \right)^{\frac{p_2-p}{p_2}}. \end{aligned}$$

By taking the p -th roots and dividing both sides by $\phi(r)$, we obtain

$$\begin{aligned} \frac{1}{\phi(r)} \left(\frac{1}{r^Q} \int_{B(0,r)} |W(x) I_{\tilde{\rho},\gamma} f(x)|^p dx \right)^{1/p} \\ \leq \frac{1}{(\phi(r))^{p/p_2}} \left(\frac{1}{r^Q} \int_{B(0,r)} |W(x)|^{p_2} dx \right)^{1/p_2} \\ \times \frac{1}{(\phi(r))^{\frac{p_2-p}{p_2}}} \left(\frac{1}{r^Q} \int_{B(0,r)} |I_{\tilde{\rho},\gamma} f(x)|^{\frac{pp_2}{p_2-p}} dx \right)^{\frac{p_2-p}{pp_2}}. \end{aligned}$$

For simplicity, we take $a = 0$ in the definition of the generalised Morrey (5.29) space.

In the last inequality, taking the supremum over $r > 0$ and using the inequality (5.49), we arrive at

$$\|W I_{\tilde{\rho},\gamma} f\|_{M^{p,\phi}(\mathbb{G})} \leq C_{p,\phi,Q} \|W\|_{M^{p_2,\phi^{p/p_2}}(\mathbb{G})} \|I_{\tilde{\rho},\gamma} f\|_{L^{\frac{pp_2}{p_2-p},\phi^{\frac{p_2-p}{p_2}}}(\mathbb{G})}.$$

Since $1 < p < \frac{pp_2}{p_2-p} < \infty$, we put $q = \frac{pp_2}{p_2-p}$ into (5.49) to obtain (5.53). $\square$

FRACTIONAL INTEGRAL OPERATORS IN GENERALISED MORREY SPACES

In this section we investigate the boundedness of more general fractional integral operators. Moreover, we show Olsen type inequality for such operators in generalised Morrey spaces on homogeneous Lie groups.

We define the *generalised fractional integral operator* T_ρ by

$$T_\rho f(x) := \int_{\mathbb{G}} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} f(y) dy, \quad (5.54)$$

where $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfies the doubling condition (5.30) and the condition

$$\int_0^1 \frac{\rho(t)}{t} dt < \infty. \quad (5.55)$$

As in the usual Euclidean case, if we take $\rho(t) = t^\alpha$, $0 < \alpha < Q$, then we

obtain the classical Riesz transform

$$T_\rho f(x) = I_\alpha f(x) = \int_{\mathbb{G}} \frac{1}{|xy^{-1}|^{Q-\alpha}} f(y) dy.$$

Theorem 5.28 ⁽¹⁸¹⁾ Ruzhansky-Suragan-Y., Theorem 6.1). Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Let ρ and ϕ satisfy the doubling condition (5.30). Assume that ϕ is surjective and for some $1 < p < q < \infty$ satisfies the inequalities

$$\int_r^\infty \frac{\phi(t)^p}{t} dt \leq C_1(\phi(r))^p, \quad (5.56)$$

and

$$\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \leq C_2(\phi(r))^{p/q}, \quad (5.57)$$

for all $r > 0$. Then we have

$$\|T_\rho f\|_{M^{q,\phi^{p/q}}(\mathbb{G})} \leq C_{p,q,\phi,Q} \|f\|_{M^{p,\phi}(\mathbb{G})}. \quad (5.58)$$

Proof of Theorem 5.28. For every $R > 0$, we decompose $T_\rho f(x)$ as follows

$$T_\rho f(x) = T_1(x) + T_2(x),$$

where

$$T_1(x) := \int_{B(x,R)} \frac{\rho(|xy^{-1}|)}{(|xy^{-1}|)^Q} f(y) dy$$

and

$$T_2(x) := \int_{B^c(x,R)} \frac{\rho(|xy^{-1}|)}{(|xy^{-1}|)^Q} f(y) dy.$$

For $T_1(x)$, we have

$$\begin{aligned} |T_1(x)| &\leq \int_{|xy^{-1}| < R} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} |f(y)| dy \\ &= \sum_{k=-\infty}^{-1} \int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} |f(y)| dy. \end{aligned}$$

By (5.30), it follows that

$$\begin{aligned}
|T_1(x)| &\leq C \sum_{k=-\infty}^{-1} \frac{\rho(2^k R)}{(2^k R)^Q} \int_{|xy^{-1}| < 2^{k+1} R} |f(y)| dy \\
&\leq CM f(x) \sum_{k=-\infty}^{-1} \rho(2^k R) \\
&\leq CM f(x) \sum_{k=-\infty}^{-1} \int_{2^k R}^{2^{k+1} R} \frac{\rho(t)}{t} dt \\
&= CM f(x) \int_0^R \frac{\rho(t)}{t} dt,
\end{aligned}$$

where we have used the fact that

$$\int_{2^k R}^{2^{k+1} R} \frac{\rho(t)}{t} dt \geq C \rho(2^k R) \int_{2^k R}^{2^{k+1} R} \frac{1}{t} dt = C \rho(2^k R) \ln 2. \quad (5.59)$$

Thus, by (5.57) we obtain for $T_1(x)$ that

$$|T_1(x)| \leq CM f(x) (\phi(R))^{(p-q)/q}. \quad (5.60)$$

For $T_2(x)$, using (5.30) we have

$$\begin{aligned}
|T_2(x)| &\leq \int_{|xy^{-1}| \geq R} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} |f(y)| dy \\
&= \sum_{k=0}^{\infty} \int_{2^k R \leq |xy^{-1}| < 2^{k+1} R} \frac{\rho(|xy^{-1}|)}{|xy^{-1}|^Q} |f(y)| dy \\
&\leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)}{(2^k R)^Q} \int_{|xy^{-1}| < 2^{k+1} R} |f(y)| dy.
\end{aligned}$$

For simplicity, throughout this proof let us take $a = 0$ when we use the definition of the generalised Morrey (5.29) space.

In the last inequality, we use the Hölder inequality to obtain

$$|T_2(x)| \leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)}{(2^k R)^Q} \left(\int_{|xy^{-1}| < 2^{k+1} R} dy \right)^{1-1/p} \left(\int_{|xy^{-1}| < 2^{k+1} R} |f(y)| dy \right)^{1/p}$$

$$\begin{aligned}
&\leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)}{(2^k R)^{Q/p}} \left(\int_{|xy^{-1}| < 2^{k+1} R} |f(y)| dy \right)^{1/p} \\
&\leq C \|f\|_{M^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \rho(2^{k+1} R) \phi(2^{k+1} R) \\
&\leq C \|f\|_{M^{p,\phi}(\mathbb{G})} \sum_{k=0}^{\infty} \int_{2^k R}^{2^{k+1} R} \frac{\rho(t) \phi(t)}{t} dt = C \|f\|_{M^{p,\phi}(\mathbb{G})} \int_R^{\infty} \frac{\rho(t) \phi(t)}{t} dt,
\end{aligned}$$

where we have used the inequality

$$\begin{aligned}
\int_{2^k R}^{2^{k+1} R} \frac{\rho(t) \phi(t)}{t} dt &\geq C \rho(2^{k+1} R) \phi(2^{k+1} R) \int_{2^k R}^{2^{k+1} R} \frac{1}{t} dt \\
&= C \rho(2^{k+1} R) \phi(2^{k+1} R) \ln 2.
\end{aligned} \tag{5.61}$$

Thus, by (5.57) we obtain for $T_2(x)$ that

$$|T_2(x)| \leq C \|f\|_{M^{p,\phi}(\mathbb{G})} (\phi(R))^{p/q}. \tag{5.62}$$

Summing up the estimates (5.60) and (5.62), we arrive at

$$|T_\rho f(x)| \leq C (Mf(x) (\phi(R))^{(p-q)/q} + \|f\|_{M^{p,\phi}(\mathbb{G})} (\phi(R))^{p/q}).$$

Assume that f is not identically 0 and that Mf is finite everywhere. Then since ϕ is surjective, we can choose $R > 0$ such that

$$\phi(R) = Mf(x) \cdot \|f\|_{M^{p,\phi}(\mathbb{G})}^{-1}.$$

For $x \in \mathbb{G}$ it follows that

$$|T_\rho f(x)| \leq C (Mf(x))^{\frac{p}{q}} \|f\|_{M^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}},$$

which implies

$$\left(\int_{B(0,r)} |T_\rho f(x)|^q \right)^{1/q} \leq C \left(\int_{B(0,r)} |Mf(x)|^p \right)^{1/q} \|f\|_{M^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}}.$$

Here, we divide both sides by $(\phi(r))^{p/q} r^{Q/q}$ to get

$$\begin{aligned} \frac{1}{(\phi(r))^{p/q}} \left(\frac{1}{r^Q} \int_{B(0,r)} |T_\rho f(x)|^q \right)^{1/q} \\ \leq C \frac{1}{(\phi(r))^{p/q}} \left(\frac{1}{r^Q} \int_{B(0,r)} |Mf(x)|^p \right)^{1/q} \|f\|_{M^{p,\phi}(\mathbb{G})}^{\frac{q-p}{q}}. \end{aligned}$$

Finally, taking the supremum over $r > 0$ and using the boundedness of the maximal operator M on $M^{p,\phi}(\mathbb{G})$ shown in (5.33), we obtain (5.58), as desired. $\square$

Now let us turn to the Olsen type inequalities for the generalised fractional integral operator T_ρ and Bessel-Riesz operator $I_{\alpha,\gamma}$.

Theorem 5.29 (¹⁸¹ Ruzhansky-Suragan-Y., Theorem 6.2). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Let ρ and ϕ satisfy the doubling condition (5.30). Assume that ϕ is surjective and satisfy (5.56) and (5.57). Then we have*

$$\|WT_\rho f\|_{M^{p,\phi}(\mathbb{G})} \leq C_{p,\phi,Q} \|W\|_{M^{p_2,\phi^{p/p_2}}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}, \quad 1 < p < p_2 < \infty, \quad (5.63)$$

provided that $W \in M^{p_2,\phi^{p/p_2}}(\mathbb{G})$.

Proof of Theorem 5.29. By the Hölder inequality, we have

$$\begin{aligned} \frac{1}{r^Q} \int_{B(0,r)} |W(x)T_\rho f(x)|^p dx \\ \leq \left(\frac{1}{r^Q} \int_{B(0,r)} |W(x)|^{p_2} dx \right)^{p/p_2} \left(\frac{1}{r^Q} \int_{B(0,r)} |T_\rho f(x)|^{\frac{pp_2}{p_2-p}} dx \right)^{\frac{p_2-p}{p_2}}. \end{aligned}$$

Taking the p -th roots and dividing both sides by $\phi(r)$, we arrive at

$$\begin{aligned} \frac{1}{\phi(r)} \left(\frac{1}{r^Q} \int_{B(0,r)} |W(x)T_\rho f(x)|^p dx \right)^{1/p} \\ \leq \frac{1}{(\phi(r))^{p/p_2}} \left(\frac{1}{r^Q} \int_{B(0,r)} |W(x)|^{p_2} dx \right)^{1/p_2} \\ \times \frac{1}{(\phi(r))^{\frac{p_2-p}{p_2}}} \left(\frac{1}{r^Q} \int_{B(0,r)} |T_\rho f(x)|^{\frac{pp_2}{p_2-p}} dx \right)^{\frac{p_2-p}{pp_2}}. \end{aligned}$$

For simplicity, we take $a = 0$ when we use the definition of the generalised Morrey (5.29) space.

In the last inequality, we take the supremum over $r > 0$ to get

$$\|WT_\rho f\|_{M^{p,\phi}(\mathbb{G})} \leq C_{p,\phi,Q} \|W\|_{M^{p_2,\phi^{p/p_2}}(\mathbb{G})} \|T_\rho f\|_{L^{\frac{pp_2}{p_2-p},\phi^{\frac{p_2-p}{p_2}}}(\mathbb{G})}.$$

Since $1 < p < \frac{pp_2}{p_2-p} < \infty$, we put $q = \frac{pp_2}{p_2-p}$ in (5.58) to obtain (5.63), as desired. $\square$

Theorem 5.30 (¹⁸¹ Ruzhansky-Suragan-Y., Theorem 6.3). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $|\cdot|$ be a homogeneous quasi-norm. Let $0 < \alpha < Q$, $1 < p < \infty$ and $\gamma > 0$. Let $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfy the doubling condition and assume that $0 < \omega(r) \leq Cr^{-\alpha}$ for all $r > 0$, so that $K_{\alpha,\gamma} \in M^{p_2,\omega}(\mathbb{G})$ for $\frac{Q}{Q+\gamma-\alpha} < p_2 < \frac{Q}{Q-\alpha}$ with $p_2 \geq 1$, where $q = \frac{\beta p}{\beta+Q-\alpha}$. Assume that $0 < \phi(r) \leq Cr^\beta$ for all $r > 0$, where $\beta < -\alpha < -Q - \beta$. Then we have*

$$\|WI_{\alpha,\gamma} f\|_{M^{p,\phi}(\mathbb{G})} \leq C_{p,\phi,Q} \|W\|_{M^{p_2,\phi^{p/p_2}}(\mathbb{G})} \|f\|_{M^{p,\phi}(\mathbb{G})}, \quad (5.64)$$

provided that $W \in M^{p_2,\phi^{p/p_2}}(\mathbb{G})$, where $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{q}$.

Proof of Theorem 5.30. By the Hölder inequality for $\frac{p}{p_2} + \frac{p}{q} = 1$, we have

$$\begin{aligned} \frac{1}{r^Q} \int_{B(0,r)} |W(x)I_{\alpha,\gamma}f(x)|^p dx \\ \leq \left(\frac{1}{r^Q} \int_{B(0,r)} |W(x)|^{p_2} dx \right)^{p/p_2} \left(\frac{1}{r^Q} \int_{B(0,r)} |I_{\alpha,\gamma}f(x)|^q dx \right)^{p/q}. \end{aligned}$$

Taking the p -th roots and dividing both sides by $\phi(r)$, we obtain

$$\begin{aligned} \frac{1}{\phi(r)} \left(\frac{1}{r^Q} \int_{B(0,r)} |W(x)I_{\alpha,\gamma}f(x)|^p dx \right)^{1/p} \\ \leq \frac{1}{(\phi(r))^{p/p_2}} \left(\frac{1}{r^Q} \int_{B(0,r)} |W(x)|^{p_2} dx \right)^{1/p_2} \\ \times \frac{1}{(\phi(r))^{p/q}} \left(\frac{1}{r^Q} \int_{B(0,r)} |I_{\alpha,\gamma}f(x)|^q dx \right)^{1/q}. \end{aligned}$$

For simplicity, we take $a = 0$ when we use the definition of the generalised Morrey (5.29) space.

In the last inequality, we take the supremum over $r > 0$ to get

$$\|WI_{\alpha,\gamma}f\|_{M^{p,\phi}(\mathbb{G})} \leq C\|W\|_{M^{p_2,\phi^{p/p_2}}(\mathbb{G})}\|I_{\alpha,\gamma}f\|_{M^{q,\phi^{p/q}}(\mathbb{G})},$$

which implies (5.64) by Theorem 5.24 after putting $\psi(r) = (\phi(r))^{p/q}$. $\square$

5.0.4 BESOV TYPE SPACES

In this section we define Besov type spaces using the Euler semigroup $\{e^{-t\mathbb{E}^*\mathbb{E}}\}_{t>0}$ introduced in Section 2.2. Moreover, we establish another type of Gagliardo-Nirenberg inequality in these spaces.

We define the Besov type space $B^\alpha(\mathbb{R} \times \mathfrak{S})$ as the space of all tempered distributions f on $\mathbb{R} \times \mathfrak{S}$ with the norm

$$\|g\|_{B^\alpha(\mathbb{R} \times \mathfrak{S})} := \sup_{t>0} \{t^{-\alpha/2} \|Fe^{-t\mathbb{B}^2}F^{-1}f\|_{L^\infty(\mathbb{R} \times \mathfrak{S})}\} < \infty, \quad (5.65)$$

where the operators $\mathbb{B}$ and F are defined in Section 2.2, and $\mathfrak{S}$ is the quasi-sphere $\mathfrak{S} := \{x \in \mathbb{G} : |x| = 1\}$. Now let us state the following Gagliardo-Nirenberg type inequalities:

Theorem 5.31 (¹⁸³ Ruzhansky-Suragan-Y., Theorem 4.2). *Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension Q . Let $1 \leq p < q < \infty$. Then there exists a positive constant $C = C(p, q)$ such that*

$$\|f\|_{L^q(\mathbb{R} \times \mathfrak{S})} \leq C \|\mathcal{R}_{|\cdot|} f\|_{L^p(\mathbb{R} \times \mathfrak{S})}^{p/q} \|f\|_{B^{p/(p-q)}(\mathbb{R} \times \mathfrak{S})}^{1-p/q} \quad (5.66)$$

holds for all $f \in B^{p/(p-q)}(\mathbb{R} \times \mathfrak{S})$ such that $\mathcal{R}_{|\cdot|} f \in L^p(\mathbb{R} \times \mathfrak{S})$.

In the proof, we will use the following representation for $e^{-t\mathbb{B}^2}$, which is a corollary of Theorem 2.16:

Corollary 5.32. *Let F and F^{-1} be mappings as in (2.27) and (2.28), respectively. Then we have*

$$F e^{-t\mathbb{B}^2} F^{-1} f(r, y) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{(r-s)^2}{4t}\right) f(sy) ds. \quad (5.67)$$

Proof of Corollary 5.32. Plugging $e^{-t\mathbb{B}^2} = e^{tQ^2/4} e^{-t\mathbb{E}^*\mathbb{E}}$, (2.27) and (2.28) into the left hand side of (5.67), and using (2.36) we obtain

$$\begin{aligned} & F e^{-t\mathbb{B}^2} F^{-1} f(r, y) \\ &= F \left(e^{tQ^2/4} \frac{e^{-tQ^2/4} r^{-Q/2}}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(\ln r - \ln s)^2}{4t}} s^{Q/2-1} (s^{-Q/2} f(\ln s, y)) ds \right) \\ &= e^{rQ/2} \frac{e^{-rQ/2}}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(r - \ln s)^2}{4t}} \frac{f(\ln s, y)}{s} ds \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^\infty e^{-\frac{(r-s_1)^2}{4t}} f(s_1 y) ds_1, \end{aligned}$$

which is (5.67), where we have used the change of variables $s = e^{s_1}$ in the last line. $\square$

We will also use the one-dimensional case of the following result:

Theorem 5.33 (¹³⁴ Theorem 1). *Let $1 \leq p < q < \infty$. Then for all $f \in L_1^p(\mathbb{R}^n)$ there exists a positive constant $C = C(p, q, n)$ such that*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)}^{p/q} \|f\|_{B_{\infty,\infty}^{p/(p-q)}(\mathbb{R}^n)}^{1-p/q}, \quad (5.68)$$

where

$$\|f\|_{B_{\infty,\infty}^{\alpha}(\mathbb{R}^n)} := \sup_{t>0, x \in \mathbb{R}^n} \left\{ t^{-\alpha/2} \left| \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{-|x-y|^2/4t} dy \right| \right\}.$$

We are now ready to prove Theorem 5.31.

Proof of Theorem 5.31. By Theorem 5.33 with $n = 1$ and Corollary 5.32, for any $y \in \mathfrak{S}$ we have

$$\begin{aligned} & \int_{\mathbb{R}} |f(r, y)|^q dr \\ & \leq C^q \int_{\mathbb{R}} \left| \frac{\partial f(r, y)}{\partial r} \right|^p dr \left(\sup_{t>0, r \in \mathbb{R}} t^{p/(2(q-p))} \left| \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(r-s)^2/(4t)} f(s, y) ds \right| \right)^{q-p} \\ & = C^q \int_{\mathbb{R}} |\mathcal{R}_{|\cdot|} f(r, y)|^p dr \left(\sup_{t>0, r \in \mathbb{R}} t^{p/(2(q-p))} \left| F e^{-t\mathbb{B}^2} F^{-1} f(r, y) \right| \right)^{q-p} \\ & \leq C^q \int_{\mathbb{R}} |\mathcal{R}_{|\cdot|} f(r, y)|^p dr \left(\sup_{t>0} t^{p/(2(q-p))} \left\| F e^{-t\mathbb{B}^2} F^{-1} f \right\|_{L^\infty(\mathbb{R} \times \mathfrak{S})} \right)^{q-p} \\ & = C^q \int_{\mathbb{R}} |\mathcal{R}_{|\cdot|} f(r, y)|^p dr \|f\|_{B^{p/(p-q)}(\mathbb{R} \times \mathfrak{S})}^{q-p} \end{aligned}$$

since (5.65). Thus, we integrate the above inequality with respect to y over $\mathfrak{S}$ to obtain the desired Gagliardo-Nirenberg inequality (5.66). $\square$

6

Hypoelliptic functional inequalities

IN this chapter we derive a variety of functional inequalities for general homogeneous invariant hypoelliptic differential operators on nilpotent Lie groups. The obtained inequalities include Hardy, Rellich, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg, Hardy-Littlewood-Sobolev (HLS) and reversed HLS inequalities, and Heisenberg-Pauli-Weyl type uncertainty principle. Moreover, we also discuss the relation between the best constants of the hypoelliptic Sobolev and Hardy-Littlewood-Sobolev inequalities for certain families of parameters. In this chapter, we adopt the notations introduced in Section 2.3 concerning the graded Lie groups. Most of the results of this chapter are available in the joint work with Ruzhansky¹⁹³.

6.1 HYPOELLIPTIC HARDY, RELICH, CAFFARELLI-KOHN-NIRENBERG AND HARDY-LITTLEWOOD-SOBOLEV INEQUALITIES

Some of the hypoelliptic functional inequalities that we discuss in this section have been known in the case of the sub-Laplacians, however, for more general hypoelliptic operators almost all of them appear to be new as

no approaches for obtaining such estimates have been available. The approach developed in this section relies on the integral versions of Hardy inequalities on homogeneous groups from Chapter 4. Namely, we link such integral Hardy inequalities to different hypoelliptic inequalities by using the Riesz and Bessel kernels associated to the described hypoelliptic operators.

For example, since we have (2.52) for the Riesz kernel $\mathcal{I}_\alpha$ from (2.44), taking $T_a^{(1)}(x) = \mathcal{I}_a(x)$ in Theorem 4.8 and noting that $\mathcal{R}^{-\frac{a}{\nu}} f = f * \mathcal{I}_a$ by⁶⁹ Corollary 4.3.11, we obtain

Theorem 6.1 (¹⁹³ Ruzhansky-Y., Theorem 5.1). *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν . Let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p \leq q < \infty$ and $0 < a < Q/p$. Let $0 \leq b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Then there exists a positive constant C such that*

$$\left\| \frac{f}{|x|^{\frac{b}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})} \quad (6.1)$$

holds for all $f \in \dot{L}_a^p(\mathbb{G})$.

Remark 6.2. When $\mathbb{G}$ is a stratified group, if we let $\mathcal{R} = -\mathfrak{L}$ be the positive sub-Laplacian and $q = p$, hence $b = ap$, then (6.1) becomes

$$\left\| \frac{f}{|x|^a} \right\|_{L^p(\mathbb{G})} \leq C \|(-\mathfrak{L})^{\frac{a}{2}} f\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \quad 0 < a < Q/p, \quad (6.2)$$

which is the fractional Hardy inequality (1.4) obtained by Ciatti, Cowling and Ricci⁴⁸.

Moreover, note that the Hardy-Sobolev (6.1) inequalities contain the classical Hardy, Rellich and Sobolev inequalities, which we will discuss in the following remarks:

Remark 6.3. In the case $q = p$ and $a = 1$, the inequality (6.1) gives the Hardy

inequality on graded groups

$$\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{1}{\nu}} f\|_{L^p(\mathbb{G})}, \quad 1 < p < Q, \quad (6.3)$$

for all $f \in \dot{L}_1^p(\mathbb{G})$. It implies the classical Hardy inequality when $\mathbb{G} = (\mathbb{R}^n, +)$ and $\mathcal{R} = -\Delta$ is the positive Laplacian, hence $Q = n$ and $\nu = 2$,

$$\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{R}^n)} \leq C \|(-\Delta)^{\frac{1}{2}} f\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty, \quad (6.4)$$

where ∇ is the usual Euclidean gradient.

Remark 6.4. In the case $q = p$ and $a = 2$, the inequality (6.1) gives the Rellich inequality on graded groups

$$\left\| \frac{f}{|x|^2} \right\|_{L^p(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{2}{\nu}} f\|_{L^p(\mathbb{G})}, \quad 1 < p < \frac{Q}{2}, \quad (6.5)$$

for all $f \in \dot{L}_2^p(\mathbb{G})$. It implies the classical Rellich inequality when $\mathbb{G} = (\mathbb{R}^n, +)$ and $\mathcal{R} = -\Delta$ is the positive Laplacian, hence $Q = n$ and $\nu = 2$,

$$\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{R}^n)} \leq C \|\Delta f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \frac{n}{2}. \quad (6.6)$$

We note that the hypoelliptic versions of Hardy (6.3) and Rellich inequalities (6.5) are new already on stratified groups, since the operator $\mathcal{R}$ does not have to be a sub-Laplacian and can be of any order.

Remark 6.5. In the case $b = 0$, the inequality (6.1) gives the Sobolev inequality on graded groups, see⁶⁹ Proposition 4.4.13, (5): Let $1 < p \leq q < \infty$ and $0 < a < Q/p$ with $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q}$. Then there exists a positive constant C such that

$$\|f\|_{L^q(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})} \quad (6.7)$$

holds for all $f \in \dot{L}_a^p(\mathbb{G})$.

Remark 6.6. We can also prove Theorem 6.1 by using the Schur test argument from Remark 4.9 and the analysis on graded Lie groups developed

in⁶⁹. We first prove it for $p = q$. By (4.36), we have

$$\left\| \frac{f * T_{b/q}^{(1)}}{|x|^{\frac{b}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|f\|_{L^q(\mathbb{G})} \quad (6.8)$$

for all $f \in L^q(\mathbb{G})$, where $0 < b < Q$ and $1 < q < \infty$. Here, since we have (2.52), taking $T_{b/q}^{(1)}(x) = \mathcal{I}_{b/q}(x)$ and using $\mathcal{R}^{-\frac{b}{\nu q}} f = f * \mathcal{I}_{b/q}$ by⁶⁹ Corollary 4.3.11, we obtain from (6.8) that

$$\left\| \frac{f}{|x|^{\frac{b}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{b}{\nu q}} f\|_{L^q(\mathbb{G})} \quad (6.9)$$

holds for all $f \in \dot{L}_{b/q}^q(\mathbb{G})$, where $0 < b < Q$ and $1 < q < \infty$.

Now for $p \leq q$, we use the Sobolev inequality (6.9) to get

$$\left\| \frac{f}{|x|^{\frac{b}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{b}{\nu q}} f\|_{L^q(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})}, \quad (6.10)$$

where $0 < a < Q/p$ and $\frac{a}{\nu} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$.

Similarly, since we have (2.53), taking $T_a^{(2)}(x) = \mathcal{B}_a(x)$ in Theorem 4.10 and then using $(I + \mathcal{R})^{-\frac{a}{\nu}} f = f * \mathcal{B}_a$ by⁶⁹ Corollary 4.3.11, we obtain the critical case $b = Q$ of Theorem 6.1:

Theorem 6.7 (¹⁹³ Ruzhansky-Y., Theorem 5.6). *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν . Let $|\cdot|$ be an arbitrary homogeneous quasi-norm and let $1 < p < r < \infty$ and $p < q < (r-1)p'$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(p, q, r, Q)$ such that*

$$\left\| \frac{f}{\left(\log \left(e + \frac{1}{|x|} \right) \right)^{\frac{r}{q}} |x|^{\frac{Q}{q}}} \right\|_{L^q(\mathbb{G})} \leq C \|f\|_{L_{Q/p}^p(\mathbb{G})} \quad (6.11)$$

holds for all $f \in L_{Q/p}^p(\mathbb{G})$.

The Hardy inequality (6.1) implies the following Heisenberg-Pauli-Weyl uncertainty principle for general homogeneous invariant hypoelliptic differential operators:

Corollary 6.8 (¹⁹³ Ruzhansky-Y., Corollary 5.7). *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν . Let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p \leq q < \infty$ and $0 < a < Q/p$. Let $0 \leq b < Q$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{q} + \frac{b}{qQ}$. Then there exists a positive constant C such that*

$$\|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})} \| |x|^{\frac{b}{q}} f \|_{L^{q'}(\mathbb{G})} \geq C \int_{\mathbb{G}} |f(x)|^2 dx \quad (6.12)$$

holds for all $f \in \dot{L}_a^p(\mathbb{G})$, where $1/q + 1/q' = 1$.

Remark 6.9. Sometimes it is possible to obtain uncertainty principles without using Hardy inequalities, see, e.g., Ciatti, Ricci and Sundari⁴⁹. On stratified groups, we also refer to, e.g.,⁸⁵ and⁴⁸.

Proof of Theorem 6.8. Using Hölder's inequality and (6.1), we obtain

$$\|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})} \| |x|^{\frac{b}{q}} f \|_{L^{q'}(\mathbb{G})} \geq C \left\| \frac{f}{|x|^{b/q}} \right\|_{L^q(\mathbb{G})} \| |x|^{\frac{b}{q}} f \|_{L^{q'}(\mathbb{G})} \geq C \int_{\mathbb{G}} |f(x)|^2 dx.$$

□

As another consequence of Theorem 6.1, we also obtain a family of extended Caffarelli-Kohn-Nirenberg inequalities on graded groups.

Theorem 6.10 (¹⁹³ Ruzhansky-Y., Theorem 5.9). *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν . Let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p, q < \infty$, $\delta \in (0, 1]$ and $0 < r < \infty$ with $r \leq \frac{q}{1-\delta}$ for $\delta \neq 1$. Let $0 < a < Q/p$ and $\beta, \gamma \in \mathbb{R}$ with $\delta r(Q - ap - \beta p) \leq p(Q + r\gamma - r\beta)$ and $\beta(1 - \delta) - \delta a \leq \gamma \leq \beta(1 - \delta)$. Assume that $\frac{r(\delta Q + p(\beta(1-\delta) - \gamma - a\delta))}{pQ} + \frac{(1-\delta)r}{q} = 1$. Then there exists a positive constant C such that*

$$\| |x|^\gamma f \|_{L^r(\mathbb{G})} \leq C \|\mathcal{R}^{\frac{a}{\nu}} f\|_{L^p(\mathbb{G})}^\delta \| |x|^\beta f \|_{L^q(\mathbb{G})}^{1-\delta} \quad (6.13)$$

holds for all $f \in \dot{L}_a^p(\mathbb{G})$.

In the Euclidean case $\mathbb{G} = (\mathbb{R}^n, +)$ with $Q = n$, if the conditions (3.114) are not satisfied, then the inequality (6.13) is not covered by the classical Caffarelli-Kohn-Nirenberg inequality, see Theorem 3.41. Therefore, the inequality (6.13) actually also gives an extension of Theorem 3.41 with respect to the range of parameters. Let us give an example:

Example 6.11. If $1 < p = q = r < n$, $a = 1$, $\mathcal{R} = -\Delta$ and $\gamma = \beta(1 - \delta) - \delta$, then (6.13) takes the form

$$\| |x|^\gamma f \|_{L^p(\mathbb{R}^n)} \leq C \left\| (-\Delta)^{\frac{1}{2}} f \right\|_{L^p(\mathbb{R}^n)}^\delta \| |x|^\beta f \|_{L^p(\mathbb{R}^n)}^{1-\delta}. \quad (6.14)$$

Here, we can take e.g. $\beta \leq -n/p$ or $\gamma \leq -n/p$ so that the conditions (3.114) fail, then the inequality (6.14) is not covered by Theorem 3.41.

Remark 6.12. We note that the conditions $\beta = \gamma = 0$, $a > 0$, $1 < p < Q/a$, $1 < q \leq r \leq pQ/(Q - ap)$ and $\delta = (1/q - 1/r)(a/Q + 1/q - 1/p)^{-1}$ satisfy all the conditions of Theorem 6.10. Indeed, $\delta = (1/q - 1/r)(a/Q + 1/q - 1/p)^{-1}$, $r \geq q$ and $Q - ap > 0$ imply $r \leq \frac{q}{1-\delta}$, while $r \leq pQ/(Q - ap)$ gives $\delta r(Q - ap - \beta p) \leq p(Q + r\gamma - r\beta)$ since $\beta = \gamma = 0$ and $\delta \leq 1$. In this case, $\delta = (1/q - 1/r)(a/Q + 1/q - 1/p)^{-1}$ and $\beta(1 - \delta) - \delta a \leq \gamma \leq \beta(1 - \delta)$ are equivalent to $\frac{r(\delta Q + p(\beta(1-\delta) - \gamma - a\delta))}{pQ} + \frac{(1-\delta)r}{q} = 1$ and $a\delta \geq 0$, respectively. Thus, (6.13) recovers also the Gagliardo-Nirenberg inequality previously obtained in¹⁸⁴ and¹⁸⁵ on graded groups

$$\|f\|_{L^r(\mathbb{G})} \leq C \left\| \mathcal{R}^{\frac{a}{p}} f \right\|_{L^p(\mathbb{G})}^\delta \|f\|_{L^q(\mathbb{G})}^{1-\delta} \quad (6.15)$$

for all $f \in \dot{L}_a^p(\mathbb{G}) \cap L^q(\mathbb{G})$.

We also note that when $\mathbb{G} = (\mathbb{R}^n, +)$, $Q = n$ and $\mathcal{R} = -\Delta$, in the special case $p = q = 2$ and $a = 1$, the inequality (6.15) essentially gives the classical Gagliardo-Nirenberg inequality⁸³ and¹⁵⁴.

In¹⁹⁰ we also obtained the following critical case $a = Q/p$ of the Gagliardo-Nirenberg inequality (6.15):

Theorem 6.13 (¹⁹⁰ Ruzhanksy-Y., Theorem 3.3). *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν . Let $1 < p < \infty$. Then we have*

$$\|f\|_{L^r(\mathbb{G})} \leq Cr^{1-1/p} \|\mathcal{R}^{\frac{Q}{\nu p}} f\|_{L^p(\mathbb{G})}^{1-p/r} \|f\|_{L^p(\mathbb{G})}^{p/r} \quad (6.16)$$

for every r with $p \leq r < \infty$ and for every function $f \in L_{Q/p}^p(\mathbb{G})$, where the constant C depends only on p and Q .

Remark 6.14. Actually, there are already some applications: for example, in ¹⁸⁵ and ¹⁹⁰ we showed that the best constants of the Sobolev inequality (6.7) with inhomogeneous norm (replacing q by r), the Gagliardo-Nirenberg inequality (6.15) with $q = p$ and its critical version (6.16) can be related to ground states (least energy solutions) of the following Schrödinger-type equation

$$\mathcal{R}^{\frac{a}{\nu}} \left(|\mathcal{R}^{\frac{a}{\nu}} u(x)|^{p-2} \mathcal{R}^{\frac{a}{\nu}} u(x) \right) + |u(x)|^{p-2} u(x) = |u(x)|^{r-2} u(x).$$

Note that this is also a differential equation if a/ν is an integer. They are also related to the the variational problem

$$d = \inf_{u \in L_a^p(G) \setminus \{0\}} \mathfrak{L}(u),$$

for functionals

$$\mathfrak{L}(u) = \frac{1}{p} \int_{\mathbb{G}} |\mathcal{R}^{\frac{a}{\nu}} u(x)|^p dx + \frac{1}{p} \int_{\mathbb{G}} |u(x)|^p dx - \frac{1}{r} \int_{\mathbb{G}} |u(x)|^r dx$$

and

$$\mathfrak{J}(u) = \int_{\mathbb{G}} \left(|\mathcal{R}^{\frac{a}{\nu}} u(x)|^p + |u(x)|^p - |u(x)|^r \right) dx.$$

For such discussions in the setting of stratified groups we refer to ¹⁸⁹.

Furthermore, in ¹⁸⁴ the authors use the Gagliardo-Nirenberg inequality (6.15) to investigate global in time well-posedness for small data for the semilinear damped hypoelliptic wave equation.

We also refer to ²¹⁴ for the similar investigations but in the Dunkl settings.

Another type of Garliardo-Nirenberg inequality involving Besov norms on graded groups was obtained in¹⁴.

Moreover, in¹⁹⁰ using the critical Gagliardo-Nirenberg inequality (6.16) we obtained the following hypoelliptic Trudinger and Brezis-Gallouet-Wainger inequalities:

Theorem 6.15 (¹⁹⁰ Ruzhanksy-Y., Theorem 3.5). *Let $\mathbb{G}$ be a graded group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν . Let $1 < p < \infty$. Then there exist positive α and C such that*

$$\int_{\mathbb{G}} \left(\exp(\alpha |f(x)|^{p'}) - \sum_{0 \leq k < p-1, k \in \mathbb{N}} \frac{1}{k!} (\alpha |f(x)|^{p'})^k \right) dx \leq C \|f\|_{L^p(\mathbb{G})}^p \quad (6.17)$$

holds for all functions $f \in L_{Q/p}^p(\mathbb{G})$ with $\|\mathcal{R}^{\frac{Q}{\nu p}} f\|_{L^p(\mathbb{G})} \leq 1$, where $1/p + 1/p' = 1$.

Remark 6.16. In the case $\mathbb{G} = (\mathbb{R}^n, +)$ and $\mathcal{R} = -\Delta$ is the Laplacian, the inequality (6.17) was obtained in¹⁶⁰. In this Abelian case, we can also refer to¹⁵⁷ for $p = 2$, to¹⁵⁸ for $p = n = 2$, and to² for $p = n \geq 2$. We also refer to¹⁹³ for the weighted versions of (6.17).

Theorem 6.17 (¹⁹⁰ Ruzhanksy-Y., Theorem 4.4). *Let $\mathbb{G}$ be a graded group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν . Let $a, p, q \in \mathbb{R}$ with $1 < p, q < \infty$ and $a > Q/q$. Then there exists $C > 0$ such that we have*

$$\|f\|_{L^\infty(\mathbb{G})} \leq C(1 + \log(1 + \|\mathcal{R}^{a/\nu} f\|_{L^q(\mathbb{G})}))^{1/p'} \quad (6.18)$$

for all functions $f \in L_{Q/p}^p(\mathbb{G}) \cap L_a^q(\mathbb{G})$ with $\|f\|_{L_{Q/p}^p(\mathbb{G})} \leq 1$.

Remark 6.18. In the case $\mathbb{G} = (\mathbb{R}^n, +)$ and $\mathcal{R} = -\Delta$ is the Laplacian, the inequality (6.18) was obtained by Brezis and Wainger³² by employing Fourier transform methods. In this case, we also refer to Engler⁶⁶ for $n/p, m \in \mathbb{Z}$, and to Ozawa¹⁶⁰ for the general case without using the Fourier transform.

Theorem 6.19 (¹⁹⁰ Ruzhanksy-Y., Theorem 4.7). *Let $\mathbb{G}$ be a graded group of*

homogeneous dimension Q and let $1 < p < \infty$. Then we have

$$\int_{\Omega} |f(x)| dx \leq C_4 \|f\|_{L^p_{Q/p}(\mathbb{G})} |\Omega| (1 + |\log |\Omega||)^{1/p'} \quad (6.19)$$

for any function $f \in L^p_{Q/p}(\mathbb{G})$ and for any Lebesgue measurable set Ω with $|\Omega| < \infty$, where the constant C_4 depends only on p and Q , and $|\Omega|$ denotes the Lebesgue measure of Ω .

Remark 6.20. When $\mathbb{G} = (\mathbb{R}^n, +)$ and $\mathcal{R} = -\Delta$ is the Laplacian, the inequality (6.19) was obtained in³² Lemma 2 and in¹⁶⁰ Theorem 3. In³², using this estimate and Morrey's technique the authors proved the Brezis-Wainger inequality: for any function $f \in L^p_{n/p+1}(\mathbb{R}^n)$ and for each $x, y \in \mathbb{R}^n$, the inequality

$$|f(x) - f(y)| \leq C \|f\|_{L^p_{n/p+1}(\mathbb{R}^n)} |x - y| (1 + |\log |x - y||)^{1/p'}, \quad 1 < p < \infty,$$

holds true, where the constant C depends only on n and p .

Remark 6.21. For logarithmic Sobolev inequalities we can refer to, e.g., the lecture notes⁹⁹ by Guionnet and Zegarliński, and to¹¹⁸ for discussions on metric measure spaces.

Proof of Theorem 6.10. **Case $\delta = 1$.** In this case, note that $\frac{r(\delta Q + p(\beta(1-\delta) - \gamma - a\delta))}{pQ} + \frac{(1-\delta)r}{q} = 1$ gives $\frac{a}{Q} = \frac{1}{p} - \frac{1}{r} - \frac{\gamma}{Q}$. It means that the condition $\delta r(Q - ap - \beta p) \leq p(Q + r\gamma - r\beta)$ is equivalent to the trivial estimate $pQ \leq pQ$. The condition $\beta(1-\delta) - \delta a \leq \gamma \leq \beta(1-\delta)$ gives $-a \leq \gamma \leq 0$, which implies $r \geq p$ with $\frac{a}{Q} = \frac{1}{p} - \frac{1}{r} - \frac{\gamma}{Q}$. Thus, in this case, we observe that (6.13) is equivalent to (6.1).

Case $\delta \in (0, 1)$. We write

$$\| |x|^\gamma f \|_{L^r(\mathbb{G})} = \left(\int_{\mathbb{G}} |x|^{\gamma r} |f(x)|^r dx \right)^{\frac{1}{r}} = \left(\int_{\mathbb{G}} \frac{|f(x)|^{\delta r}}{|x|^{r(\beta(1-\delta) - \gamma)}} \cdot \frac{|f(x)|^{(1-\delta)r}}{|x|^{-\beta r(1-\delta)}} dx \right)^{\frac{1}{r}}.$$

Note that $\delta > 0$, $Q > ap$ and $\beta(1-\delta) - \gamma \geq 0$ imply $r(\delta Q + p(\beta(1-\delta) - \gamma - a\delta)) > 0$, while $\delta r(Q - ap - \beta p) \leq p(Q + r\gamma - r\beta)$, $\delta < 1$ and $r \leq \frac{q}{1-\delta}$ give $\frac{pQ}{r(\delta Q + p(\beta(1-\delta) - \gamma - a\delta))} \geq 1$ and $\frac{q}{(1-\delta)r} \geq 1$, respectively. Then by using

Hölder's inequality for $\frac{r(\delta Q + p(\beta(1-\delta) - \gamma - a\delta))}{pQ} + \frac{(1-\delta)r}{q} = 1$, we obtain

$$\begin{aligned}
& \| |x|^\gamma f \|_{L^r(\mathbb{G})} \\
& \leq \left(\int_{\mathbb{G}} \frac{|f(x)|^{\frac{\delta p Q}{\delta Q + p(\beta(1-\delta) - \gamma - a\delta)}}}{|x|^{\frac{p Q (\beta(1-\delta) - \gamma)}{\delta Q + p(\beta(1-\delta) - \gamma - a\delta)}}} dx \right)^{\frac{\delta Q + p(\beta(1-\delta) - \gamma - a\delta)}{p Q}} \left(\int_{\mathbb{G}} \frac{|f(x)|^q}{|x|^{-\beta q}} dx \right)^{\frac{1-\delta}{q}} \\
& = \left\| \frac{f}{|x|^{\frac{\beta(1-\delta) - \gamma}{\delta}}} \right\|_{L^{\frac{\delta p Q}{\delta Q + p(\beta(1-\delta) - \gamma - a\delta)}}(\mathbb{G})}^\delta \left\| \frac{f}{|x|^{-\beta}} \right\|_{L^q(\mathbb{G})}^{1-\delta}.
\end{aligned} \tag{6.20}$$

We also note that the conditions $\frac{\delta p Q}{\delta Q + p(\beta(1-\delta) - \gamma - a\delta)} \geq \delta r > 0$ and $\beta(1-\delta) - \gamma \geq 0$ imply

$$\frac{\delta p Q}{\delta Q + p(\beta(1-\delta) - \gamma - a\delta)} \cdot \frac{\beta(1-\delta) - \gamma}{\delta} \geq 0, \tag{6.21}$$

while $Q > ap$ and $\delta > 0$ give

$$\frac{\delta p Q}{\delta Q + p(\beta(1-\delta) - \gamma - a\delta)} \cdot \frac{\beta(1-\delta) - \gamma}{\delta} < Q. \tag{6.22}$$

Then, (6.21), (6.22) and

$$\frac{a}{Q} = \frac{1}{p} - \frac{1}{\frac{\delta p Q}{\delta Q + p(\beta(1-\delta) - \gamma - a\delta)}} + \frac{\frac{\beta(1-\delta) - \gamma}{\delta}}{Q}$$

with $\gamma \geq \beta(1-\delta) - \delta a$ imply $\frac{\delta p Q}{\delta Q + p(\beta(1-\delta) - \gamma - a\delta)} \geq p$, so that we can use Theorem 6.1 in (6.20) to obtain (6.13). $\square$

Consequently, by working with Riesz kernels of positive Rockland operators, we subsequently obtain the following differential version of Hardy-Littlewood-Sobolev inequalities:

Theorem 6.22 (¹⁹³ Ruzhansky-Y., Theorem 5.12). *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $|\cdot|$ be an arbitrary homogeneous quasi-norm. Let $1 < p, q < \infty$, $0 \leq a < Q/p$ and $0 \leq b < Q/q$. Let $0 < \lambda < Q$, $0 \leq \alpha < a + Q/p'$ and $0 \leq \beta \leq b$ be such that $(Q - ap)/(pQ) + (Q - q(b -$*

$\beta)/ (qQ) + (\alpha + \lambda)/Q = 2$ and $\alpha + \lambda \leq Q$, where $1/p + 1/p' = 1$. Then there exists a positive constant $C = C(Q, \lambda, p, \alpha, \beta, a, b)$ such that

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)}g(y)}{|x|^\alpha |y^{-1}x|^\lambda |y|^\beta} dx dy \right| \leq C \|f\|_{\dot{L}_a^p(\mathbb{G})} \|g\|_{\dot{L}_b^q(\mathbb{G})} \quad (6.23)$$

holds for all $f \in \dot{L}_a^p(\mathbb{G})$ and $g \in \dot{L}_b^q(\mathbb{G})$.

Proof of Theorem 6.22. We first prove it for $a \neq 0$ and $b \neq 0$. We want to use Theorem 4.11 in the left-hand side of (6.23) to get

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} \frac{\overline{f(x)}g(y)}{|x|^\alpha |y^{-1}x|^\lambda |y|^\beta} dx dy \right| \leq C \|f\|_{L^{p_1}(\mathbb{G})} \left\| \frac{g}{|y|^\beta} \right\|_{L^{q_1}(\mathbb{G})}, \quad (6.24)$$

where $p_1 := \frac{pQ}{Q-ap}$ and $q_1 := \frac{qQ}{Q-q(b-\beta)}$. For this, let us check the conditions of Theorem 4.11. Note that $0 < a < Q/p$ implies $1 < p_1 < \infty$, while $0 < b < Q/q$ and $0 \leq \beta \leq b$ imply $1 < q_1 < \infty$. We also note that $0 \leq \alpha < a + Q/p'$ implies $0 \leq \alpha < Q/p'_1$ with $p'_1 = p_1/(p_1 - 1)$ and

$$2 = \frac{Q - ap}{pQ} + \frac{Q - q(b - \beta)}{qQ} + \frac{\alpha + \lambda}{Q} = \frac{1}{p_1} + \frac{1}{q_1} + \frac{\alpha + \lambda}{Q}.$$

Thus, since we also have $0 < \lambda < Q$ and $\alpha + \lambda \leq Q$, by Theorem 4.11 we obtain (6.24).

Now we estimate the right-hand side of (6.24) by using the Sobolev (6.7) and Hardy-Sobolev inequalities (6.1).

Note that we have $1 < p < p_1 < \infty$, $0 < a < Q/p$ and $\frac{a}{Q} = \frac{1}{p} - \frac{1}{p_1}$ since $p_1 = \frac{pQ}{Q-ap}$. Then we can apply the Sobolev inequality (6.7) to obtain

$$\|f\|_{L^{p_1}(\mathbb{G})} \leq C \|f\|_{\dot{L}_a^p(\mathbb{G})}. \quad (6.25)$$

From $Q - q(b - \beta) > 0$ and $Q - qb > 0$ we get $0 \leq \frac{\beta q Q}{Q - q(b - \beta)} < Q$, that is, $0 \leq \beta q_1 < Q$. We also note that $b/Q = 1/q - 1/q_1 + \beta/Q$ since $1 < q \leq q_1 < \infty$.

Then, we can use the Hardy-Sobolev inequality (6.1) to obtain

$$\left\| \frac{g}{|y|^\beta} \right\|_{L^{q_1}(\mathbb{G})} \leq C \|g\|_{\dot{L}_b^q(\mathbb{G})}. \quad (6.26)$$

Finally, putting (6.25) and (6.26) in (6.24), we obtain (6.23).

In the case $a = 0$, the inequalities (6.26) and (6.24) give (6.23).

When $b = 0$, we have $\beta = 0$ since $0 \leq \beta \leq b$, then (6.24) and (6.25) imply (6.23). $\square$

6.2 ON THE BEST CONSTANTS IN HLS AND SOBOLEV INEQUALITIES

In this section we discuss the relation between the best constants of Sobolev and Hardy-Littlewood-Sobolev inequalities on graded groups for certain families of parameters.

Since the homogeneous order of the Riesz potential $I_{Q-\lambda}(y^{-1}x)$ and $|y^{-1}x|^{-\lambda}$ is equal to $-\lambda$, then taking $p = q$, $f = g$ and $\lambda = Q - 2a$ for $0 < a < Q/2$ in (4.75) with $\alpha = 0$ we obtain the following version of the Hardy-Littlewood-Sobolev inequality on graded groups

$$\left| \int_{\mathbb{G}} \int_{\mathbb{G}} I_{2a}(y^{-1}x) \overline{f(x)} f(y) dx dy \right| \leq C \|f\|_{L^{\frac{2Q}{Q+2a}}(\mathbb{G})}^2 \quad \text{for all } f \in L^{\frac{2Q}{Q+2a}}(\mathbb{G}). \quad (6.27)$$

Let C_{HLS} be the best constant in (6.27). We show the relation between this constant and the best constant C_S in the Sobolev inequality (6.7) with $p = 2Q/(Q + 2a)$, $0 < a < Q/2$, and $q = 2$, that is, C_S is the best constant in the inequality

$$\|f\|_{L^2(\mathbb{G})} \leq C \|f\|_{\dot{L}_a^{\frac{2Q}{Q+2a}}(\mathbb{G})} \quad \text{for all } f \in \dot{L}_a^{2Q/(Q+2a)}(\mathbb{G}). \quad (6.28)$$

We note here that the Riesz potential I_a as well as homogeneous Sobolev spaces norm in (6.28) above correspond to the particular fixed positive

Rockland operator $\mathcal{R}$ of homogeneous degree ν , and is defined by $\|f\|_{\dot{L}_a^p(\mathbb{G})} = \|\mathcal{R}_\nu^{\frac{a}{p}} f\|_{L^p(\mathbb{G})}$. While it is known^{69,70} that these Sobolev spaces are independent of the choice of a positive Rockland operator $\mathcal{R}$, the best constants clearly depend on the precise expressions of the norms.

Theorem 6.23. *Let $\mathbb{G}$ be a graded group of homogeneous dimension Q , and let $0 < a < Q/2$. Then we have*

$$C_S = C_{HLS}. \quad (6.29)$$

Remark 6.24. In the Euclidean case, we refer to^{137, 41, 80} for the best constant in Hardy-Littlewood-Sobolev inequality, and refer to^{167, 12, 210} and¹³⁷ for the best constant in Sobolev inequality. We also note that according to our knowledge the best constant in the Hardy-Littlewood-Sobolev inequality is not known yet on general stratified groups (beyond the Heisenberg group). Indeed, in⁸¹, Frank and Lieb found the value of the best constant in the Hardy-Littlewood-Sobolev inequality on the Heisenberg group, however, using $|y^{-1}x|^{-\lambda}$ instead of the Riesz potential in (6.27). Although the homogeneous functions $|\cdot|^{-\lambda}$ and $I_{Q-\lambda}(\cdot)$ are equivalent, the best constant in the Hardy-Littlewood-Sobolev inequality does depend on the choice of this weight. In our case, it is the use of the Riesz potential that implies the validity of Theorem 6.23.

Remark 6.25. In^{217, 44} and¹⁸⁵ the best constant in the Sobolev inequality with inhomogeneous norm for the parameters different than those in (6.28) is expressed in the variational form as well as in terms of the ground state solutions of the nonlinear Schrödinger equation when $\mathbb{G}$ is $(\mathbb{R}^n, +)$, the Heisenberg group, and a general graded group, respectively.

Proof of Theorem 6.23. Taking into account that $\mathcal{R}^{-\frac{a}{p}} f = f * \mathcal{I}_a$ (see⁶⁹ Corollary 4.3.11), we rewrite the left-hand side of (6.27) as

$$\begin{aligned} \left| \int_{\mathbb{G}} \int_{\mathbb{G}} I_{2a}(y^{-1}x) \overline{f(x)} f(y) dx dy \right| &= \left| \int_{\mathbb{G}} \overline{f} \mathcal{R}^{-\frac{2a}{p}} f dx \right| = |(\mathcal{R}^{-\frac{2a}{p}} f, f)| \\ &= |(\mathcal{R}^{-\frac{a}{p}} f, \mathcal{R}^{-\frac{a}{p}} f)| = \|\mathcal{R}^{-\frac{a}{p}} f\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (6.30)$$

Putting (6.30) in (6.27), we arrive at

$$\|f\|_{L^2(\mathbb{G})} \leq C_{\text{HLS}} \|f\|_{\dot{L}_a^{\frac{2Q}{Q+2a}}(\mathbb{G})}. \quad (6.31)$$

Since C_S is best constant in (6.28), that is, C_S is the best constant in (6.31), we have $C_S \leq C_{\text{HLS}}$. On the other hand, similarly, one can obtain (6.27) with the constant C_S from (6.28) using (6.30), which means $C_{\text{HLS}} \leq C_S$. $\square$

7

Functional inequalities on stratified Lie groups

IN this chapter we discuss versions of some of the inequalities from the previous chapters on stratified Lie groups using the horizontal gradient (2.61) and sub-Laplacian (2.59).

As already mentioned in the introduction, in the literature there are three types of estimates on stratified Lie groups available: with the Carnot–Carathéodory distance, the fundamental solution of the sub-Laplacian $\mathcal{L}$ or the Euclidean distance on the first stratum of the group. Here, in this chapter we concentrate on the last two cases.

First, we show Sobolev type and Caffarelli-Kohn-Nirenberg inequalities with weights from the first stratum of a stratified group $\mathbb{G}$. Such inequalities also give new insights (proofs) in the Euclidean setting. Moreover, the Caffarelli-Kohn-Nirenberg inequalities on $\mathbb{G}$ are new even in the Abelian case, that is, in the Euclidean case these extend the classical Caffarelli-Kohn-Nirenberg inequalities with respect to ranges of parameters, see Example 7.10.

Then, we use the factorization method, similar to the ideas explained in Section 3.1.1, but now in the setting of stratified groups. As a result we obtain two-parameter inequalities analogous to the Gesztesy–Littlejohn type inequalities (3.13).

Finally, in Section 7.4 we show an embedding of the classical Rellich inequality¹⁶⁴ for $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$

$$\int_{\mathbb{R}^n} |\Delta f(x)|^2 dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^4} dx, \quad n \neq 2, \quad (7.1)$$

into a family of inequalities associated to (sub-)Laplacians with drifts. Such operators have recently attracted considerable interest in the literature in the context of the spectral multiplier estimates (see^{116,117} and¹⁴⁰).

In particular, in $\mathbb{R}^n$ with $n \geq 5$, for all $a_i \in \mathbb{R}$ for $i = 1, \dots, n$ and all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ we obtain a new family of inequalities

$$\begin{aligned} \left\| \left(\Delta + \sum_{i=1}^n a_i \partial_{x_i} \right) f \right\|_{L^2(\mathbb{R}^n, \mu_X)}^2 &\geq \frac{n^2(n-4)^2}{16} \left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n, \mu_X)}^2 \\ &\quad + b_X^4 \|f\|_{L^2(\mathbb{R}^n, \mu_X)}^2 + b_X^2 \frac{(n-2)^2}{2} \left\| \frac{f}{|x|} \right\|_{L^2(\mathbb{R}^n, \mu_X)}^2, \end{aligned} \quad (7.2)$$

where $d\mu_X = e^{-\sum_{i=1}^n a_i x_i} dx$ and dx is the Lebesgue measure, and $b_X = \frac{1}{2} \left(\sum_{j=1}^n a_j^2 \right)^{1/2}$. Note that the inequality (7.2) reduces to the classical Rellich inequality (7.1) if all $a_i = 0$. If $a_i \neq 0$ for some $i = 1, \dots, n$, then the drift term $\sum_{i=1}^n a_i \partial_{x_i}$, although being of low order, brings an improvement to the Rellich inequality in the form of the addition of two positive terms in (7.2) compared to (7.1). Given the drift, the choice of the measure μ_X is essentially unique to make the operator self-adjoint. Moreover, the constants in (7.2) are sharp in the sense that there exists a sequence of functions such that the equality in (7.2) is attained in the limit of this sequence of functions.

In addition to (7.2), we obtain the weighted versions of such inequalities: Let μ_X and b_X be as in (7.2). Let $n \geq 3$, $a_i \in \mathbb{R}$ for $i = 1, \dots, n$ and let $\alpha \in \mathbb{R}$

with $n + 2\alpha - 4 > 0$. Then we have

$$\begin{aligned}
& \left\| |x|^\alpha \left(\Delta + \sum_{i=1}^n a_i \partial_{x_i} \right) f \right\|_{L^2(\mathbb{R}^n, \mu_X)}^2 \\
& \geq \frac{(n + 2\alpha - 4)^2 (n - 2\alpha)^2}{16} \left\| |x|^{\alpha-2} f \right\|_{L^2(\mathbb{R}^n, \mu_X)}^2 \\
& + b_X^2 \frac{(n + 2\alpha - 2)(n - 2\alpha - 2)}{2} \left\| |x|^{\alpha-1} f \right\|_{L^2(\mathbb{R}^n, \mu_X)}^2 + b_X^4 \left\| |x|^\alpha f \right\|_{L^2(\mathbb{R}^n, \mu_X)}^2,
\end{aligned} \tag{7.3}$$

for all functions $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. If $(n - 2\alpha)(n - 2\alpha - 2) \neq 0$ with $\alpha \geq 1$ and $n \geq 2\alpha$, then the constants in (7.3) are sharp, in the sense that there exists a sequence of functions such that the equality in (7.3) is attained in the limit of this sequence of functions.

In order to put in perspective the weighted Rellich inequalities, let us recall the following L^p -Rellich inequality that was obtained by Davies and Hinz⁵⁶: Let Ω be a proper, non-empty open subset of $\mathbb{R}^n$. Let $V \in C(\Omega)$ be a positive function such that $\Delta V < 0$ and $\Delta(V^a) \leq 0$ for some $a > 1$. Then for $p \in (1, \infty)$, and for all functions $f \in C_0^\infty(\Omega)$, we have

$$\int_{\Omega} |\Delta V(x)| |f(x)|^p dx \leq \left(\frac{p^2}{(p-1)a+1} \right)^p \int_{\Omega} \frac{(V(x))^p}{|\Delta V(x)|^{p-1}} |\Delta f(x)|^p dx. \tag{7.4}$$

If we take $\Omega = \mathbb{R}^n \setminus \{0\}$, $a = \frac{n-2}{2\alpha+2}$ and $V(x) = |x|^{-2\alpha-2}$ with $-1 < \alpha < \frac{n-4}{2}$, then (7.4) yields

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{2\alpha+4}} dx \leq \left(\frac{p^2}{(n-2\alpha-4)((p-1)n+2\alpha+4-2p)} \right)^p \int_{\mathbb{R}^n} \frac{|\Delta f(x)|^p}{|x|^{2\alpha+4-2p}} dx, \tag{7.5}$$

for any $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. For a Rellich-type inequality with the weight given by a mean distance function we refer to¹⁶ Corollary 6.2.7 and⁶⁴. We also refer to²²¹ for inequalities on the Heisenberg group, to²²⁴ on H-type groups, to¹⁷² for Rellich inequalities for Hörmander's sums of squares of vector fields. On stratified groups, we can refer to¹⁹⁶ and¹⁷³ for Hardy-Rellich inequalities, to^{129, 128, 126} and¹³⁶ for the inequalities with the weights in terms of the fundamental solution of the sub-Laplacian, and to¹⁷⁴ on

general homogeneous groups. See also ^{5, 19, 20} and ²¹³ as well as references therein.

Thus, in Section 7.4 we discuss the inequalities (7.2) and (7.3) on stratified Lie groups. For this, we first prove a new weighted Rellich inequality without drift with horizontal weights, see Theorem 7.26. Then, in the case of polarizable Carnot groups we show a family of weighted Rellich inequalities for the sub-Laplacians with drift, where the weights are given by the powers of the fundamental solution of the sub-Laplacian. In this chapter, we adopt the notations introduced in Section 2.4 concerning the stratified Lie groups.

The exposition of the present chapter is based on the joint works with Ruzhansky and Suragan ¹⁷⁹, and with Ruzhansky ^{187, 194}.

7.1 HORIZONTAL HARDY AND SOBOLEV TYPE INEQUALITIES

In this section we show an analogue of the Sobolev type inequality from Section 3.2 on a stratified Lie group $\mathbb{G}$ using the horizontal gradient (2.61). Moreover, we show the equivalence of the Sobolev type and Hardy inequalities (1.6) in L^2 , and that the Sobolev type inequality refines the Hardy inequality.

Theorem 7.1 (¹⁷⁹ Ruzhansky-Suragan-Y., Theorem 3.1). *Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum, and let $\alpha \in \mathbb{R}$. Then for any $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$ we have*

$$\frac{|N - \alpha p|}{p} \left\| \frac{f}{|x'|^\alpha} \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{x' \cdot \nabla_H f}{|x'|^\alpha} \right\|_{L^p(\mathbb{G})}, \quad 1 < p < \infty, \quad (7.6)$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. If $N \neq \alpha p$, then the constant $\frac{|N - \alpha p|}{p}$ is sharp.

Remark 7.2. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, \dots, \partial_{x_n})$, the inequality (7.6) implies the L^p -weighted Sobolev type

inequality obtained in¹⁶², that is,

$$\frac{|n - \alpha p|}{p} \left\| \frac{f}{|x|_E^\alpha} \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{x \cdot \nabla f}{|x|_E^\alpha} \right\|_{L^p(\mathbb{R}^n)}, \quad (7.7)$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, where $|x|_E = \sqrt{x_1^2 + \dots + x_n^2}$ and the constant is sharp.

Remark 7.3. In Section 7.3.1 we will obtain special case of (7.6) by the factorization method, see Theorem 7.16.

By using the Schwarz inequality in the right hand side of (7.6) we note that it can be a refinement of the L^p -weighted Hardy inequality on stratified groups from¹⁷³:

Corollary 7.4. *Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum, and let $\alpha \in \mathbb{R}$. Then for any $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$, and all $1 < p < \infty$, we have*

$$\frac{|N - \alpha p|}{p} \left\| \frac{f}{|x'|^\alpha} \right\|_{L^p(\mathbb{G})} \leq \left\| \frac{\nabla_H f}{|x'|^{\alpha-1}} \right\|_{L^p(\mathbb{G})}, \quad (7.8)$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. If $N \neq \alpha p$ then the constant $\frac{|N - \alpha p|}{p}$ is sharp.

Thus, the inequality (7.6) can be regarded as a refinement of (7.8).

We refer to²²⁵ for the Hardy type inequalities on crystallographic groups.

As in Corollary 3.31, using the Hölder inequality, from the inequality (7.6) one can derive the following horizontal version of the uncertainty inequality:

Corollary 7.5. *Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum. Let $1 < p < \infty$ and $\alpha \in \mathbb{R}$. Then for any $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$ we have*

$$\left\| \frac{x' \cdot \nabla_H f}{|x'|^\alpha} \right\|_{L^p(\mathbb{G})} \| |x'|^\alpha f \|_{L^{p'}(\mathbb{G})} \geq \frac{|N - \alpha p|}{p} \int_{\mathbb{G}} |f(x)|^2 dx, \quad (7.9)$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$ and $1/p + 1/p' = 1$.

Remark 7.6. We refer to¹⁷⁷ Corollary 6.1.3 for the case $\alpha = 1$ of (7.9) with ∇_H instead of $x \cdot \nabla_H$. Note that in the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, taking $N = n$ and $\alpha = 1$, using the Schwarz inequality from (7.9) we obtain the classical Heisenberg-Pauli-Weyl uncertainty principle.

Let us now prove Theorem 7.1.

Proof of Theorem 7.1. We may assume that $\alpha p \neq N$ since when $\alpha p = N$ the inequality (7.6) is trivial. Using (2.63) and the divergence theorem, we have

$$\begin{aligned} \int_{\mathbb{G}} \frac{|f(x)|^p}{|x'|^{\alpha p}} &= \frac{1}{N - \alpha p} \int_{\mathbb{G}} |f(x)|^p \operatorname{div}_H \left(\frac{x'}{|x'|^{\alpha p}} \right) dx \\ &= -\frac{p}{N - \alpha p} \operatorname{Re} \int_{\mathbb{G}} p f(x) |f(x)|^{p-2} \frac{x' \cdot \nabla_H f}{|x'|^{\alpha p}} dx \\ &\leq \left| \frac{p}{N - \alpha p} \right| \int_{\mathbb{G}} \frac{|f(x)|^{p-1}}{|x'|^{\alpha p}} |x' \cdot \nabla_H f| dx \\ &\leq \left| \frac{p}{N - \alpha p} \right| \int_{\mathbb{G}} \frac{|f(x)|^{p-1}}{|x'|^{\alpha(p-1)}} \frac{|x' \cdot \nabla_H f|}{|x'|^{\alpha}} dx \\ &\leq \left| \frac{p}{N - \alpha p} \right| \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x'|^{\alpha p}} dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{G}} \frac{|x' \cdot \nabla_H f|^p}{|x'|^{\alpha p}} dx \right)^{\frac{1}{p}}, \end{aligned}$$

yielding (7.6), where we have used Hölder's inequality in the last line. Now let us prove the sharpness of the constant. We note that the function

$$h_1(x) = \frac{1}{|x'|^{\frac{|N - \alpha p|}{p}}}, \quad N \neq \alpha p,$$

satisfies the following equality condition in Hölder's inequality

$$\left| \frac{p}{N - \alpha p} \right|^p \frac{|x' \cdot \nabla_H h_1(x)|^p}{|x'|^{\alpha p}} = \frac{|h_1(x)|^p}{|x'|^{\alpha p}}.$$

Approximation of this function by functions in $C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$ concludes that the constant $\frac{|N - \alpha p|}{p}$ is sharp. $\square$

The above also gives a simple proof of a part of⁵¹ Theorem 2.12.

We also note that the proof of Theorem 7.1 gives the following similar statement in $\mathbb{R}^n$: Let $x = (x', x'') \in \mathbb{R}^N \times \mathbb{R}^{n-N}$, $1 \leq N \leq n$ and $\alpha \in \mathbb{R}$. Then for any $f \in C_0^\infty(\mathbb{R}^n \setminus \{x' = 0\})$, and all $1 < p < \infty$, we have

$$\frac{|N - \alpha p|}{p} \left\| \frac{f}{|x'|^\alpha} \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{\nabla f}{|x'|^{\alpha-1}} \right\|_{L^p(\mathbb{R}^n)}, \quad (7.10)$$

where $|x'|$ is the Euclidean norm on $\mathbb{R}^N$. If $N \neq \alpha p$ then the constant $\frac{|N - \alpha p|}{p}$ is sharp.

The best constant in the Hardy inequality (7.10) when $\alpha = 1$ was conjectured by Badiale and Tarantello in¹³ Remark 2.3. Then, it was solved in²⁰⁵ and¹⁷³.

Now, let us show the equivalence of the Sobolev type and Hardy inequalities on stratified groups in L^2 :

Theorem 7.7 (¹⁷⁹ Ruzhansky-Suragan-Y., Theorem 3.4). *Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum with $N \geq 3$. Then the following two statements are equivalent:*

(a) *For any $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$, we have*

$$\|f\|_{L^2(\mathbb{G})} \leq \frac{2}{N} \|x' \cdot \nabla_H f\|_{L^2(\mathbb{G})}. \quad (7.11)$$

(b) *For any $g \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$, we have*

$$\left\| \frac{g}{|x'|} \right\|_{L^2(\mathbb{G})} \leq \frac{2}{N-2} \left\| \frac{x'}{|x'|} \cdot \nabla_H g \right\|_{L^2(\mathbb{G})}. \quad (7.12)$$

Proof. Let us first show that the statement (a) gives (b). For this, setting $g = |x'|f$ we have

$$\begin{aligned} \|x' \cdot \nabla_H f\|_{L^2(\mathbb{G})}^2 &= \left\| -\frac{g}{|x'|} + \frac{x'}{|x'|} \cdot \nabla_H g \right\|_{L^2(\mathbb{G})}^2 \\ &= \left\| \frac{g}{|x'|} \right\|_{L^2(\mathbb{G})}^2 - 2\operatorname{Re} \int_{\mathbb{G}} \frac{\overline{g(x)}}{|x'|} \frac{x'}{|x'|} \cdot \nabla_H g(x) dx + \left\| \frac{x'}{|x'|} \cdot \nabla_H g \right\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (7.13)$$

Using (2.63), we observe that

$$\begin{aligned} -2\operatorname{Re} \int_{\mathbb{G}} \frac{\overline{g(x)}}{|x'|} \frac{x'}{|x'|} \cdot \nabla_H g(x) dx &= - \int_{\mathbb{G}} \frac{x'}{|x'|^2} \nabla_H |g(x)|^2 dx \\ &= \int_{\mathbb{G}} \operatorname{div}_H \left(\frac{x'}{|x'|^2} \right) |g(x)|^2 dx = (N-2) \int_{\mathbb{G}} \frac{|g(x)|^2}{|x'|^2} dx. \end{aligned}$$

Then, from the statement (a) and (7.13) we obtain that

$$\left\| \frac{g}{|x'|} \right\|_{L^2(\mathbb{G})}^2 \leq \frac{4}{N^2} \left((N-1) \left\| \frac{g}{|x'|} \right\|_{L^2(\mathbb{G})}^2 + \left\| \frac{x'}{|x'|} \cdot \nabla_H g \right\|_{L^2(\mathbb{G})}^2 \right),$$

which implies (7.12).

Now, let us show the converse case. We assume that (b) holds. Then, setting $f = g/|x'|$ we get

$$\begin{aligned} \left\| \frac{x'}{|x'|} \cdot \nabla_H (|x'|f) \right\|_{L^2(\mathbb{G})}^2 &= \|f + x' \cdot \nabla_H f\|_{L^2(\mathbb{G})}^2 \\ &= \|f\|_{L^2(\mathbb{G})}^2 + 2\operatorname{Re} \int_{\mathbb{G}} x' f(x) \overline{\nabla_H f} dx + \|x' \cdot \nabla_H f\|_{L^2(\mathbb{G})}^2. \end{aligned}$$

Here, using (2.63) we note that

$$2\operatorname{Re} \int_{\mathbb{G}} x' f(x) \overline{\nabla_H f} dx = -N \|f\|_{L^2(\mathbb{G})}^2.$$

It follows with the statement (b) that

$$\|f\|_{L^2(\mathbb{G})}^2 \leq \frac{4}{(N-2)^2} (\|x' \cdot \nabla_H f\|_{L^2(\mathbb{G})}^2 - (N-1) \|f\|_{L^2(\mathbb{G})}^2),$$

yielding (7.11). □

7.2 HORIZONTAL CAFFARELLI-KOHN-NIRENBERG INEQUALITIES

In this section we introduce new Caffarelli-Kohn-Nirenberg type inequalities in the setting of stratified Lie groups. The proof is quite simple relying

on weighted Hardy inequalities and the Hölder inequality. However, we note that already in the Euclidean setting of $\mathbb{R}^n$ it also gives an extension of Theorem 3.41 from the point of view of indices.

Theorem 7.8 (¹⁷⁹ Ruzhansky-Suragan-Y., Theorem 4.1). *Let $\mathbb{G}$ be a stratified group with N being the dimension of the first stratum. Let $1 < p, q < \infty$, $0 < r < \infty$ with $p + q \geq r$ and $\delta \in [0, 1] \cap [\frac{r-q}{r}, \frac{p}{r}]$. Let $a, b, c \in \mathbb{R}$ and $N \neq p(1 - a)$. Assume that*

$$\frac{\delta r}{p} + \frac{(1 - \delta)r}{q} = 1 \text{ and } c = \delta(a - 1) + b(1 - \delta).$$

Then we have

$$\| |x'|^c f \|_{L^r(\mathbb{G})} \leq \left| \frac{p}{N + p(a - 1)} \right|^\delta \| |x'|^a \nabla_H f \|_{L^p(\mathbb{G})}^\delta \| |x'|^b f \|_{L^q(\mathbb{G})}^{1-\delta} \quad (7.14)$$

for all $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$. The constant in the inequality (7.14) is sharp when $p = q$ with $a - b = 1$ or $p \neq q$ with $p(1 - a) + bq \neq 0$, or when $\delta = 0, 1$.

Remark 7.9. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, \dots, \partial_{x_n})$, the inequality (7.14) implies the following Caffarelli-Kohn-Nirenberg type inequality: Let $1 < p, q < \infty$, $0 < r < \infty$ with $p + q \geq r$ and $\delta \in [0, 1] \cap [\frac{r-q}{r}, \frac{p}{r}]$. Let $a, b, c \in \mathbb{R}$ and $n \neq p(1 - a)$. Assume that $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ and $c = \delta(a - 1) + b(1 - \delta)$. Then we have

$$\| |x|_E^c f \|_{L^r(\mathbb{R}^n)} \leq \left| \frac{p}{n + p(a - 1)} \right|^\delta \| |x|_E^a \nabla f \|_{L^p(\mathbb{R}^n)}^\delta \| |x|_E^b f \|_{L^q(\mathbb{R}^n)}^{1-\delta}, \quad (7.15)$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, where $|x|_E = \sqrt{x_1^2 + \dots + x_n^2}$. The constant in the inequality (7.15) is sharp when $p = q$ with $a - b = 1$ or $p \neq q$ with $p(1 - a) + bq \neq 0$, or when $\delta = 0, 1$.

In the following example, let us show that the inequalities (7.15) give an extension of the classical Caffarelli-Kohn-Nirenberg inequality from Theorem 3.41 with respect to the range of indices:

Example 7.10. If we take $1 < p = q = r < \infty$, $a = -\frac{n-2p}{p}$, $b = -\frac{n}{p}$ and

$c = -\frac{n-\delta p}{p}$ in (7.15), then we obtain

$$\left\| \frac{f}{|x|^{\frac{n-\delta p}{p}}} \right\|_{L^p(\mathbb{R}^n)} \leq \left\| \frac{\nabla f}{|x|^{\frac{n-2p}{p}}} \right\|_{L^p(\mathbb{R}^n)}^\delta \left\| \frac{f}{|x|^{\frac{n}{p}}} \right\|_{L^p(\mathbb{R}^n)}^{1-\delta}, \quad 0 \leq \delta \leq 1, \quad (7.16)$$

for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$, where ∇ is the usual Euclidean gradient in $\mathbb{R}^n$. In this case, since

$$\frac{1}{q} + \frac{b}{n} = \frac{1}{p} + \frac{1}{n} \left(-\frac{n}{p} \right) = 0,$$

which means that (3.114) fails, so that the inequality (7.16) is not covered by Theorem 3.41.

Remark 7.11. We refer to¹⁹⁵ for the local and global weighted (singular) Trudinger–Moser inequalities with remainder terms, critical Hardy-type and weighted Gagliardo–Nirenberg inequalities on general stratified groups.

Proof of Theorem 7.8. Case $\delta = 0$. In this case, we have $q = r$ and $b = c$ by $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ and $c = \delta(a-1) + b(1-\delta)$, respectively. Then, the inequality (7.14) becomes the trivial estimate

$$\| |x'|^b f \|_{L^q(\mathbb{G})} \leq \| |x'|^b f \|_{L^q(\mathbb{G})}.$$

Case $\delta = 1$. Here we have $p = r$ and $a-1 = c$. From the inequality (7.8) for $N + cp \neq 0$ we obtain

$$\| |x'|^c f \|_{L^r(\mathbb{G})} \leq \left| \frac{p}{N + cp} \right| \| |x'|^{c+1} \nabla_H f \|_{L^r(\mathbb{G})}.$$

In this case, the constant in (7.14) is sharp, since the constant in (7.8) is sharp.

Case $\delta \in (0, 1) \cap [\frac{r-q}{r}, \frac{p}{r}]$. Taking into account $c = \delta(a-1) + b(1-\delta)$, we write

$$\| |x'|^c f \|_{L^r(\mathbb{G})} = \left(\int_{\mathbb{G}} |x'|^{cr} |f(x)|^r dx \right)^{\frac{1}{r}} = \left(\int_{\mathbb{G}} \frac{|f(x)|^{\delta r}}{|x'|^{\delta r(1-a)}} \frac{|f(x)|^{(1-\delta)r}}{|x'|^{-br(1-\delta)}} dx \right)^{\frac{1}{r}}.$$

Since we have $\delta \in (0, 1) \cap [\frac{r-q}{r}, \frac{p}{r}]$ and $p+q \geq r$, here we can use the Hölder inequality for $\frac{\delta r}{p} + \frac{(1-\delta)r}{q} = 1$ to obtain

$$\begin{aligned} \| |x'|^c f \|_{L^r(\mathbb{G})} &\leq \left(\int_{\mathbb{G}} \frac{|f(x)|^p}{|x'|^{p(1-a)}} dx \right)^{\frac{\delta}{p}} \left(\int_{\mathbb{G}} \frac{|f(x)|^q}{|x'|^{-bq}} dx \right)^{\frac{1-\delta}{q}} \\ &= \left\| \frac{f}{|x'|^{1-a}} \right\|_{L^p(\mathbb{G})}^{\delta} \left\| \frac{f}{|x'|^{-b}} \right\|_{L^q(\mathbb{G})}^{1-\delta}. \end{aligned} \quad (7.17)$$

In the case $p = q$ and $a - b = 1$, we see that Hölder's equality condition is satisfied for any $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. We also note that in the case $p \neq q$ the function

$$h_2(x) = |x'|^{\frac{1}{(p-q)}(p(1-a)+bq)} \quad (7.18)$$

satisfies Hölder's equality condition

$$\frac{|h_2|^p}{|x'|^{p(1-a)}} = \frac{|h_2|^q}{|x'|^{-bq}}.$$

Since $N \neq p(1-a)$, then by the inequality (7.8) we have

$$\left\| \frac{f}{|x'|^{1-a}} \right\|_{L^p(\mathbb{G})}^{\delta} \leq \left| \frac{p}{N + p(a-1)} \right|^{\delta} \left\| \frac{\nabla_H f}{|x'|^{-a}} \right\|_{L^p(\mathbb{G})}^{\delta}. \quad (7.19)$$

Now, we use this inequality in (7.17) to obtain

$$\| |x'|^c f \|_{L^r(\mathbb{G})} \leq \left| \frac{p}{N + p(a-1)} \right|^{\delta} \left\| \frac{\nabla_H f}{|x'|^{-a}} \right\|_{L^p(\mathbb{G})}^{\delta} \left\| \frac{f}{|x'|^{-b}} \right\|_{L^q(\mathbb{G})}^{1-\delta}.$$

When we prove (7.19), in the same way as in the proof of Theorem 7.1, one can observe that the function

$$h_3(x) = |x'|^C, \quad C \neq 0, \quad (7.20)$$

satisfies Hölder's equality condition. Hence, we see that in the case $p = q$ and $a - b = 1$ Hölder's equality condition of the inequalities (7.17) and (7.19) holds true for the same function $h_3(x)$ in (7.20). We also note that in the case $p \neq q$ and $p(1-a) + bq \neq 0$ Hölder's equality condition of

the inequalities (7.17) and (7.19) holds true for the same function $h_2(x)$ in (7.18). Thus, we have observed that the constant in (7.14) is sharp when $p = q, a - b = 1$ or $p \neq q, p(1 - a) + bq \neq 0$. $\square$

7.3 HORIZONTAL INEQUALITIES BY FACTORIZATION

In this section we obtain Hardy-Rellich and improved Hardy inequalities by the factorization method, similar to the ideas discussed in Section 3.1.1, but now on stratified Lie groups. In the isotropic Abelian case, we refer to^{87, 90} and to the recent paper⁸⁸ for obtaining Hardy and Hardy-Rellich inequalities using the factorization method.

7.3.1 TWO-PARAMETER HARDY-RELLICH INEQUALITIES BY FACTORIZATION

In this section we apply the factorization method on general stratified Lie groups.

Theorem 7.12 (¹⁸⁷ Ruzhansky-Y., Theorem 4.1). *Let $\mathbb{G}$ be a stratified group with $N \geq 2$ being the dimension of the first stratum, and let $\alpha, \beta \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$ we have*

$$\begin{aligned} \|\mathcal{L}f\|_{L^2(\mathbb{G})}^2 &\geq (\alpha(N-2) - 2\beta) \left\| \frac{\nabla_H f}{|x'|} \right\|_{L^2(\mathbb{G})}^2 - \alpha^2 \left\| \frac{x' \cdot \nabla_H f}{|x'|^2} \right\|_{L^2(\mathbb{G})}^2 \\ &\quad + (\alpha(N-4)(N-2) - \alpha^2(N-2) + 2\beta(4-N) - \beta^2 \\ &\quad + \alpha\beta(N-2)) \left\| \frac{f}{|x'|^2} \right\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (7.21)$$

We note that the term $x' \cdot \nabla_H f$ can be eliminated.

Remark 7.13. If we use the Cauchy-Schwarz inequality

$$\int_{\mathbb{G}} \frac{|x' \cdot (\nabla_H f)(x)|^2}{|x'|^4} dx \leq \int_{\mathbb{G}} \frac{|(\nabla_H f)(x)|^2}{|x'|^2} dx \quad (7.22)$$

in (7.21), we obtain

$$\begin{aligned} \|\mathcal{L}f\|_{L^2(\mathbb{G})}^2 &\geq (\alpha(N-2) - 2\beta - \alpha^2) \left\| \frac{\nabla_H f}{|x'|} \right\|_{L^2(\mathbb{G})}^2 \\ &\quad + (\alpha(N-4)(N-2) - \alpha^2(N-2) + 2\beta(4-N) - \beta^2 \\ &\quad + \alpha\beta(N-2)) \left\| \frac{f}{|x'|^2} \right\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (7.23)$$

Using the weighted Hardy inequality (7.8) for $\alpha(N-2) - 2\beta - \alpha^2 \geq 0$ we get

$$\begin{aligned} \|\mathcal{L}f\|_{L^2(\mathbb{G})}^2 &\geq \left(\frac{N(N-2)(N-4)}{4} \alpha - \frac{N(N-4)}{2} \beta - \frac{N^2 - 4N + 8}{4} \alpha^2 \right. \\ &\quad \left. - \beta^2 + (N-2)\alpha\beta \right) \left\| \frac{f}{|x'|^2} \right\|_{L^2(\mathbb{G})}^2. \end{aligned} \quad (7.24)$$

Here, one can note that except for the case $N = 4$ (which is the critical case of the Rellich inequality) one can always choose the parameters α and β so that the right-hand side of (7.24) is positive.

Remark 7.14. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, \dots, \partial_{x_n})$ is the usual Euclidean gradient, the inequality (7.21) implies for $\alpha, \beta \in \mathbb{R}$ and for any $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ with $n \geq 2$ the inequality

$$\begin{aligned} &\|\Delta f\|_{L^2(\mathbb{R}^n)}^2 \\ &\geq (\alpha(n-2) - 2\beta) \left\| \frac{\nabla f}{|x|} \right\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad - \alpha^2 \left\| \frac{x \cdot \nabla f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad + (\alpha(n-4)(n-2) - \alpha^2(n-2) + 2\beta(4-n) - \beta^2 + \alpha\beta(n-2)) \left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned} \quad (7.25)$$

Proof of Theorem 7.12. We will be applying the factorization method with

the following differential expressions for two real parameters $\alpha, \beta \in \mathbb{R}$:

$$T_{\alpha,\beta} := -\mathcal{L} + \alpha \frac{x' \cdot \nabla_H}{|x'|^2} + \frac{\beta}{|x'|^2} \quad (7.26)$$

and its formal adjoint

$$T_{\alpha,\beta}^+ := -\mathcal{L} - \alpha \frac{x' \cdot \nabla_H}{|x'|^2} - \frac{\alpha(N-2) - \beta}{|x'|^2}, \quad (7.27)$$

where $x' \neq 0$. For any function $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$ let us now calculate $(T_{\alpha,\beta}^+ T_{\alpha,\beta} f)(x)$ as follows

$$\begin{aligned} & (T_{\alpha,\beta}^+ T_{\alpha,\beta} f)(x) \\ &= \left(-\mathcal{L} - \alpha \frac{x' \cdot \nabla_H}{|x'|^2} - \frac{\alpha(N-2) - \beta}{|x'|^2} \right) \left(-(\mathcal{L}f)(x) + \alpha \frac{x' \cdot (\nabla_H f)}{|x'|^2} + \frac{\beta f(x)}{|x'|^2} \right) \\ &= (\mathfrak{L}^2 f)(x) + \alpha \left(-\mathfrak{L} \left(\frac{x' \cdot (\nabla_H f)}{|x'|^2} \right) (x) + \frac{x' \cdot \nabla_H}{|x'|^2} (\mathfrak{L}f)(x) + \frac{N-2}{|x'|^2} (\mathfrak{L}f)(x) \right) \\ &\quad + \beta \left(-\mathfrak{L} \left(\frac{f}{|x'|^2} \right) (x) - \frac{(\mathfrak{L}f)(x)}{|x'|^2} \right) \\ &\quad + \alpha\beta \left(-\frac{x' \cdot \nabla_H}{|x'|^2} \left(\frac{f}{|x'|^2} \right) (x) + \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} - \frac{(N-2)f(x)}{|x'|^4} \right) \\ &\quad + \alpha^2 \left(-\left(\frac{x' \cdot \nabla_H}{|x'|^2} \right) \left(\frac{x' \cdot (\nabla_H f)}{|x'|^2} \right) (x) - (N-2) \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} \right) + \beta^2 \frac{f(x)}{|x'|^4}. \end{aligned} \quad (7.28)$$

Now we calculate $(T_{\alpha,\beta} T_{\alpha,\beta}^+ f)(x)$ as follows

$$\begin{aligned} (T_{\alpha,\beta} T_{\alpha,\beta}^+ f)(x) &= \left(-\mathcal{L} + \alpha \frac{x' \cdot \nabla_H}{|x'|^2} + \frac{\beta}{|x'|^2} \right) \\ &\quad \times \left(-(\mathcal{L}f)(x) - \alpha \frac{x' \cdot (\nabla_H f)(x)}{|x'|^2} - \frac{(\alpha(N-2) - \beta)f(x)}{|x'|^2} \right) \end{aligned}$$

$$\begin{aligned}
&= (\mathfrak{L}^2 f)(x) + \alpha \left(\mathfrak{L} \left(\frac{x' \cdot (\nabla_H f)}{|x'|^2} \right) (x) - \frac{x' \cdot \nabla_H}{|x'|^2} (\mathfrak{L} f)(x) + (N-2) \mathfrak{L} \left(\frac{f}{|x'|^2} \right) (x) \right) \\
&\quad + \beta \left(-\mathfrak{L} \left(\frac{f}{|x'|^2} \right) (x) - \frac{(\mathfrak{L} f)(x)}{|x'|^2} \right) \\
&\quad + \alpha \beta \left(\frac{x' \cdot \nabla_H}{|x'|^2} \left(\frac{f}{|x'|^2} \right) (x) - \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} - \frac{(N-2)f(x)}{|x'|^4} \right) \\
&\quad + \alpha^2 \left(-\left(\frac{x' \cdot \nabla_H}{|x'|^2} \right) \left(\frac{x' \cdot (\nabla_H f)}{|x'|^2} \right) (x) - (N-2) \frac{x' \cdot \nabla_H}{|x'|^2} \left(\frac{f}{|x'|^2} \right) (x) \right) \\
&\quad + \beta^2 \frac{f(x)}{|x'|^4}. \quad (7.29)
\end{aligned}$$

Since we have

$$\begin{aligned}
\mathfrak{L} \left(\frac{f}{|x'|^2} \right) (x) &= \sum_{j=1}^N X_j^2 \left(\frac{f}{|x'|^2} \right) (x) \\
&= \sum_{j=1}^N X_j \left(\frac{X_j f}{|x'|^2} - \frac{2x'_j f}{|x'|^4} \right) (x) \\
&= \sum_{j=1}^N \left(\frac{(X_j^2 f)(x)}{|x'|^2} - \frac{4x'_j (X_j f)(x)}{|x'|^4} + \frac{8(x'_j)^2 f(x)}{|x'|^6} - \frac{2f(x)}{|x'|^4} \right) \\
&= \frac{(\mathfrak{L} f)(x)}{|x'|^2} - \frac{4x' \cdot (\nabla_H f)(x)}{|x'|^4} - (2N-8) \frac{f(x)}{|x'|^4} \quad (7.30)
\end{aligned}$$

and

$$\begin{aligned}
\frac{x' \cdot \nabla_H}{|x'|^2} \left(\frac{f}{|x'|^2} \right) (x) &= \frac{f(x)}{|x'|^2} \sum_{j=1}^N x'_j X_j (|x'|^{-2}) + \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} \\
&= -2 \sum_{j=1}^N \frac{(x'_j)^2 f(x)}{|x'|^6} + \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} \quad (7.31) \\
&= -2 \frac{f(x)}{|x'|^4} + \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4},
\end{aligned}$$

we obtain for $(T_{\alpha,\beta}^+ T_{\alpha,\beta} f)(x) + (T_{\alpha,\beta} T_{\alpha,\beta}^+ f)(x)$ that

$$(T_{\alpha,\beta}^+ T_{\alpha,\beta} f)(x) + (T_{\alpha,\beta} T_{\alpha,\beta}^+ f)(x)$$

$$\begin{aligned}
&= 2(\mathfrak{L}^2 f)(x) + 2\alpha(N-2) \left(\frac{(\mathfrak{L}f)(x)}{|x'|^2} - 2 \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} + (4-N) \frac{f(x)}{|x'|^4} \right) \\
&\quad + 2\beta \left(-\frac{2(\mathfrak{L}f)(x)}{|x'|^2} + \frac{4x' \cdot (\nabla_H f)(x)}{|x'|^4} + (2N-8) \frac{f(x)}{|x'|^4} \right) \\
&\quad - 2\alpha\beta(N-2) \frac{f(x)}{|x'|^4} + 2\alpha^2 \left(-\left(\frac{x' \cdot \nabla_H}{|x'|^2} \right) \left(\frac{x' \cdot (\nabla_H f)(x)}{|x'|^2} \right) (x) \right. \\
&\quad \left. - (N-2) \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} + (N-2) \frac{f(x)}{|x'|^4} \right) + 2\beta^2 \frac{f(x)}{|x'|^4}. \quad (7.32)
\end{aligned}$$

In order to simplify this, let us calculate the following expression

$$\begin{aligned}
&-2 \left(\frac{x' \cdot \nabla_H}{|x'|^2} \right) \left(\frac{x' \cdot (\nabla_H f)(x)}{|x'|^2} \right) (x) - 2(N-2) \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} + 2(N-2) \frac{f(x)}{|x'|^4} \\
&= -\frac{2 \sum_{j,k=1}^N (x'_j X_j)(x'_k (X_k f))(x)}{|x'|^4} - 2 \sum_{j,k=1}^N x'_j (-2) |x'|^{-3} X_j |x'| \frac{x'_k (X_k f)(x)}{|x'|^2} \\
&\quad - 2(N-2) \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} + 2(N-2) \frac{f(x)}{|x'|^4} \\
&= -\frac{2 \sum_{k=1}^N x'_k (X_k f)(x)}{|x'|^4} - \frac{2 \sum_{j,k=1}^N x'_j x'_k X_j (X_k f)(x)}{|x'|^4} + \frac{4 \sum_{k=1}^N x'_k (X_k f)(x)}{|x'|^4} \\
&\quad - 2(N-2) \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} + 2(N-2) \frac{f(x)}{|x'|^4} \\
&= -2(N-3) \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} - \frac{2 \sum_{j,k=1}^N x'_j x'_k (X_j X_k f)(x)}{|x'|^4} + 2(N-2) \frac{f(x)}{|x'|^4}. \quad (7.33)
\end{aligned}$$

Using this in (7.32), we arrive at

$$(T_{\alpha,\beta}^+ T_{\alpha,\beta} f)(x) + (T_{\alpha,\beta} T_{\alpha,\beta}^+ f)(x)$$

$$\begin{aligned}
&= 2(\mathfrak{L}^2 f)(x) + (2\alpha(N-2) - 4\beta) \frac{(\mathfrak{L}f)(x)}{|x'|^2} \\
&\quad + (-4\alpha(N-2) - 2\alpha^2(N-3) + 8\beta) \frac{x' \cdot (\nabla_H f)(x)}{|x'|^4} \\
&\quad + (2\alpha(N-2)(4-N) + 2\alpha^2(N-2) - 2\alpha\beta(N-2) \\
&\quad + (4N-16)\beta + 2\beta^2) \frac{f(x)}{|x'|^4} - 2\alpha^2 \frac{\sum_{j,k=1}^N x'_j x'_k (X_j X_k f)(x)}{|x'|^4}. \quad (7.34)
\end{aligned}$$

Since $T_{\alpha,\beta}^+ T_{\alpha,\beta} + T_{\alpha,\beta} T_{\alpha,\beta}^+$ is non-negative, using the integration by parts we obtain

$$\begin{aligned}
&\int_{\mathbb{G}} |(T_{\alpha,\beta} f)(x)|^2 dx + \int_{\mathbb{G}} |(T_{\alpha,\beta}^+ f)(x)|^2 dx \\
&= \int_{\mathbb{G}} \overline{f(x)} ((T_{\alpha,\beta}^+ T_{\alpha,\beta} + T_{\alpha,\beta} T_{\alpha,\beta}^+) f)(x) dx \geq 0.
\end{aligned}$$

Now, we put (7.34) into this inequality to get

$$\begin{aligned}
&2 \int_{\mathbb{G}} |(\mathcal{L}f)(x)|^2 dx + (2\alpha(N-2) - 4\beta) \int_{\mathbb{G}} \frac{\overline{f(x)} (\mathfrak{L}f)(x)}{|x'|^2} dx \\
&\quad + (-4\alpha(N-2) - 2\alpha^2(N-3) + 8\beta) \int_{\mathbb{G}} \frac{\overline{f(x)} (x' \cdot (\nabla_H f)(x))}{|x'|^4} dx \\
&\quad + (2\alpha(N-2)(4-N) + 2\alpha^2(N-2) - 2\alpha\beta(N-2) \\
&\quad + (4N-16)\beta + 2\beta^2) \int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^4} dx - 2\alpha^2 \sum_{j,k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)} x'_j x'_k (X_j X_k f)(x)}{|x'|^4} dx \geq 0. \quad (7.35)
\end{aligned}$$

Since we have

$$\begin{aligned}
\int_{\mathbb{G}} \frac{\overline{f(x)} (\mathfrak{L}f)(x)}{|x'|^2} dx &= - \sum_{j=1}^N \int_{\mathbb{G}} \frac{\overline{(X_j f)(x)} (X_j f)(x)}{|x'|^2} \\
&\quad - \sum_{j=1}^N \int_{\mathbb{G}} \overline{f(x)} (-2) |x'|^{-3} X_j |x'| (X_j f)(x) dx
\end{aligned}$$

$$= 2 \int_{\mathbb{G}} \frac{\overline{f(x)}(x' \cdot (\nabla_H f)(x))}{|x'|^4} dx - \int_{\mathbb{G}} \frac{|(\nabla_H f)(x)|^2}{|x'|^2} dx \quad (7.36)$$

and

$$\begin{aligned} & \sum_{j,k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)}x'_j x'_k (X_j X_k f)(x)}{|x'|^4} dx - (N-1) \sum_{k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)}x'_k (X_k f)(x)}{|x'|^4} dx \\ & - 2 \sum_{k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)}x'_k (X_k f)(x)}{|x'|^4} dx - \sum_{j,k=1}^N \int_{\mathbb{G}} \frac{x'_j x'_k \overline{(X_j f)(x)} (X_k f)(x)}{|x'|^4} dx \\ & - \sum_{j,k=1}^N \int_{\mathbb{G}} \overline{f(x)} x'_j x'_k (X_k f)(x) (-4) |x'|^{-5} X_j |x'| dx \\ & = -(N+1) \sum_{k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)}x'_k (X_k f)(x)}{|x'|^4} dx + 4 \sum_{j,k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)}(x'_j)^2 x'_k (X_k f)(x)}{|x'|^6} dx \\ & - \int_{\mathbb{G}} \frac{|x' \cdot (\nabla_H f)(x)|^2}{|x'|^4} dx = -(N-3) \int_{\mathbb{G}} \frac{\overline{f(x)}(x' \cdot (\nabla_H f)(x))}{|x'|^4} dx \\ & - \int_{\mathbb{G}} \frac{|x' \cdot (\nabla_H f)(x)|^2}{|x'|^4} dx, \end{aligned} \quad (7.37)$$

then the inequality (7.35) follows that

$$\begin{aligned} & 2 \int_{\mathbb{G}} |(\mathcal{L}f)(x)|^2 dx + (4\alpha(N-2) - 8\beta - 4\alpha(N-2) + 8\beta - 2\alpha^2(N-3) + 2\alpha^2(N-3)) \\ & \quad \times \int_{\mathbb{G}} \frac{\overline{f(x)}(x' \cdot (\nabla_H f)(x))}{|x'|^4} dx \\ & + (2\alpha(N-2)(4-N) + 2\alpha^2(N-2) - 2\alpha\beta(N-2) + (4N-16)\beta + 2\beta^2) \\ & \quad \times \int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^4} dx - (2\alpha(N-2) - 4\beta) \int_{\mathbb{G}} \frac{|(\nabla_H f)(x)|^2}{|x'|^2} dx \\ & \quad + 2\alpha^2 \int_{\mathbb{G}} \frac{|x' \cdot (\nabla_H f)(x)|^2}{|x'|^4} dx \geq 0, \end{aligned}$$

which implies (7.21), as desired. $\square$

Now let us show a very elementary proof of a version of the Hardy inequality (7.8) with $\alpha = 1$ on stratified Lie groups using the factorization

method.

Theorem 7.15. *Let $\mathbb{G}$ be a stratified group with $N \geq 3$ being the dimension of the first stratum. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$ we have*

$$\|\nabla_H f\|_{L^2(\mathbb{G})} \geq \frac{N-2}{2} \left\| \frac{f}{|x'|} \right\|_{L^2(\mathbb{G})}, \quad (7.38)$$

where the constant $\frac{N-2}{2}$ is sharp.

Proof of Theorem 7.15. Here we use the following one-parameter differential expression

$$\tilde{T}_\alpha := \nabla_H + \alpha \frac{x'}{|x'|^2},$$

and its formal adjoint

$$\tilde{T}_\alpha^+ := -\operatorname{div}_H(\cdot) + \alpha \frac{x'}{|x'|^2},$$

where $x' \neq 0$ and $\alpha \in \mathbb{R}$. Taking into account (2.63) we have

$$\begin{aligned} \tilde{T}_\alpha^+ \tilde{T}_\alpha f &= -(\mathfrak{L}f)(x) - \alpha \operatorname{div}_H \left(\frac{x'}{|x'|^2} f \right) (x) + \alpha \frac{x' \cdot (\nabla_H f)(x)}{|x'|^2} + \alpha^2 \frac{f(x)}{|x'|^2} \\ &= -(\mathfrak{L}f)(x) - \alpha \operatorname{div}_H \left(\frac{x'}{|x'|^2} \right) f(x) + \alpha^2 \frac{f(x)}{|x'|^2} \\ &= -(\mathfrak{L}f)(x) + \frac{\alpha(\alpha + 2 - N)}{|x'|^2} f(x). \end{aligned}$$

Using the integration by parts and the non-negativity of $\tilde{T}_\alpha^+ \tilde{T}_\alpha$, it follows that

$$\begin{aligned} 0 &\leq \int_{\mathbb{G}} |\tilde{T}_\alpha f|^2 dx = \int_{\mathbb{G}} \overline{f(x)} (\tilde{T}_\alpha^+ \tilde{T}_\alpha f)(x) dx \\ &= \int_{\mathbb{G}} |\nabla_H f|^2 dx + \alpha(\alpha + 2 - N) \int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^2} dx, \end{aligned}$$

that is,

$$\int_{\mathbb{G}} |\nabla_H f|^2 dx \geq \alpha(N - 2 - \alpha) \int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^2} dx.$$

From this we derive (7.38) after maximising the constant with respect to α . The sharpness of the constant in the obtained inequality was shown in¹⁷³. $\square$

The interesting result here is that by modifying the differential expression $\tilde{T}_\alpha$, the factorization method gives a refinement of the Hardy inequality (7.38):

Theorem 7.16 (¹⁸⁷ Ruzhansky-Y., Theorem 4.5). *Let $\mathbb{G}$ be a stratified group with $N \geq 3$ being the dimension of the first stratum. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$ we have*

$$\left\| \frac{x' \cdot \nabla_H f}{|x'|} \right\|_{L^2(\mathbb{G})} \geq \frac{N-2}{2} \left\| \frac{f}{|x'|} \right\|_{L^2(\mathbb{G})}, \quad (7.39)$$

where the constant $\frac{N-2}{2}$ is sharp.

Remark 7.17. By Theorem 7.7 we know that the L^2 -weighted Sobolev type inequality (7.39) implies L^2 -Hardy inequality (7.38).

Proof of Theorem 7.16. Here we work with the following one parameter differential expression

$$\hat{T}_\alpha := \frac{x' \cdot \nabla_H}{|x'|} + \frac{\alpha}{|x'|},$$

and its formal adjoint

$$\hat{T}_\alpha^+ := -\frac{x' \cdot \nabla_H}{|x'|} + \frac{\alpha - N + 1}{|x'|},$$

where $x' \neq 0$ and $\alpha \in \mathbb{R}$. Using (2.62) we observe that

$$\begin{aligned} & \left(\frac{x' \cdot \nabla_H}{|x'|} \right) \left(\frac{x' \cdot (\nabla_H f)(x)}{|x'|} \right) \\ &= \frac{\sum_{j,k=1}^N (x'_j X_j)(x'_k (X_k f)(x))}{|x'|^2} + \sum_{j,k=1}^N x'_j (-1) |x'|^{-2} X_j |x'| \frac{x'_k (X_k f)(x)}{|x'|} \end{aligned}$$

$$\begin{aligned}
&= \frac{\sum_{k=1}^N x'_k (X_k f)(x)}{|x'|^2} + \frac{\sum_{j,k=1}^N x'_j x'_k (X_j X_k f)(x)}{|x'|^2} - \frac{\sum_{k=1}^N x'_k (X_k f)(x)}{|x'|^2} \\
&= \frac{\sum_{j,k=1}^N x'_j x'_k (X_j X_k f)(x)}{|x'|^2}
\end{aligned}$$

and

$$\begin{aligned}
(x' \cdot \nabla_H) \left(\frac{f}{|x'|} \right) &= \frac{\sum_{k=1}^N x'_k (X_k f)(x)}{|x'|} + \sum_{k=1}^N x'_k (-1) |x'|^{-2} X_k |x'| f(x) \\
&= \frac{x' \cdot (\nabla_H f)(x)}{|x'|} - \frac{f(x)}{|x'|}.
\end{aligned}$$

Taking into account these identities, we obtain

$$\begin{aligned}
&\widehat{T}_\alpha^+ \widehat{T}_\alpha f(x) \\
&= - \frac{\sum_{j,k=1}^N x'_j x'_k (X_j X_k f)(x)}{|x'|^2} - (N-1) \frac{x' \cdot (\nabla_H f)(x)}{|x'|^2} + \frac{\alpha(\alpha+2-N)}{|x'|^2} f(x).
\end{aligned}$$

Using (2.62) we also note that

$$\begin{aligned}
&\sum_{j,k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)} x'_j x'_k (X_j X_k f)(x)}{|x'|^2} dx = -(N-1) \sum_{k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)} x'_k (X_k f)(x)}{|x'|^2} dx \\
&\quad - 2 \sum_{k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)} x'_k (X_k f)(x)}{|x'|^2} dx - \sum_{j,k=1}^N \int_{\mathbb{G}} \frac{x'_j x'_k \overline{(X_j f)(x)} (X_k f)(x)}{|x'|^2} dx \\
&\quad - \sum_{j,k=1}^N \int_{\mathbb{G}} \overline{f(x)} x'_j x'_k (X_k f)(x) (-2) |x'|^{-3} X_j |x'| dx \\
&= -(N+1) \sum_{k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)} x'_k (X_k f)(x)}{|x'|^2} dx + 2 \sum_{j,k=1}^N \int_{\mathbb{G}} \frac{\overline{f(x)} (x'_j)^2 x'_k (X_k f)(x)}{|x'|^4} dx \\
&\quad - \int_{\mathbb{G}} \frac{|x' \cdot (\nabla_H f)(x)|^2}{|x'|^2} dx \\
&= -(N-1) \int_{\mathbb{G}} \frac{\overline{f(x)} (x' \cdot (\nabla_H f)(x))}{|x'|^2} dx - \int_{\mathbb{G}} \frac{|x' \cdot (\nabla_H f)(x)|^2}{|x'|^2} dx. \quad (7.40)
\end{aligned}$$

Using the integration by parts and the non-negativity of the operator $\widehat{T}_\alpha^+ \widehat{T}_\alpha$,

we obtain

$$\begin{aligned}
0 &\leq \int_{\mathbb{G}} |\widehat{T}_\alpha f|^2 dx = \int_{\mathbb{G}} \overline{f(x)} (\widehat{T}_\alpha^+ \widehat{T}_\alpha f)(x) dx \\
&= - \int_{\mathbb{G}} \left(\frac{\sum_{j,k=1}^N x'_j x'_k \overline{f(x)} (X_j X_k f)(x)}{|x'|^2} + \frac{(N-1) \overline{f(x)} (x' \cdot (\nabla_H f)(x))}{|x'|^2} \right) dx \\
&\quad + \alpha(\alpha - N + 2) \int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^2} dx.
\end{aligned}$$

Here, we use (7.40) to get

$$\int_{\mathbb{G}} \left(\frac{|x' \cdot (\nabla_H f)(x)|^2}{|x'|^2} + \alpha(\alpha - N + 2) \frac{|f(x)|^2}{|x'|^2} \right) dx \geq 0,$$

which follows that

$$\int_{\mathbb{G}} \frac{|x' \cdot (\nabla_H f)(x)|^2}{|x'|^2} dx \geq \alpha((N-2) - \alpha) \int_{\mathbb{G}} \frac{|f(x)|^2}{|x'|^2} dx.$$

As before, we obtain (7.39) after maximising $\alpha((N-2) - \alpha)$ with respect to α . The sharpness of the constant follows from Theorem 7.1. $\square$

7.3.2 FACTORIZATIONS ON THE HEISENBERG GROUP

Now we discuss the Hardy-Rellich inequalities on the Heisenberg group. In this section we adopt the notations concerning the Heisenberg group from Section 2.4.2 as well as some new notation that will be useful in the computations:

$$\tilde{x}'_j = \begin{cases} x_j, & 1 \leq j \leq n; \\ y_{j-n}, & n+1 \leq j \leq 2n, \end{cases}, \quad \tilde{X}_j = \begin{cases} X_j, & 1 \leq j \leq n; \\ Y_{j-n}, & n+1 \leq j \leq 2n, \end{cases} \quad (7.41)$$

and

$$|\tilde{x}'| = \sqrt{\sum_{j=1}^N (\tilde{x}'_j)^2} = \sqrt{\sum_{j=1}^n (x_j^2 + y_j^2)}. \quad (7.42)$$

Taking into account the above notations, ∇_H and the sub-Laplacian are defined by

$$\nabla_H := (\tilde{X}_1, \dots, \tilde{X}_N) \quad (7.43)$$

and

$$\mathfrak{L} := \sum_{j=1}^N \tilde{X}_j^2 = \sum_{j=1}^n \left(\partial_{\tilde{x}'_j} - \frac{\tilde{x}'_{j+n}}{2} \partial_t \right)^2 + \left(\partial_{\tilde{x}'_{j+n}} + \frac{\tilde{x}'_j}{2} \partial_t \right)^2, \quad (7.44)$$

respectively. Therefore, we see that $N = 2n$ and the formulae (2.62) and (2.63) also hold on the Heisenberg group.

For the following formulation we recall the angular derivative operator $Z = \sum_{j=1}^n (x_j \partial_{y_j} - y_j \partial_{x_j})$ given in (2.81).

Theorem 7.18 (¹⁸⁷ Ruzhansky-Y., Theorem 5.1). *Let $\alpha, \beta \in \mathbb{R}$. Then for all complex-valued functions $f \in C_0^\infty(\mathbb{H}_n \setminus \{\tilde{x}' = 0\})$ we have*

$$\begin{aligned} \|\mathcal{L}f\|_{L^2(\mathbb{H}_n)}^2 &\geq ((N-4)\alpha - 2\beta) \left\| \frac{\nabla_H f}{|\tilde{x}'|} \right\|_{L^2(\mathbb{H}_n)}^2 \\ &\quad - \alpha(\alpha-4) \left\| \frac{\tilde{x}' \cdot \nabla_H f}{|\tilde{x}'|^2} \right\|_{L^2(\mathbb{H}_n)}^2 \\ &\quad + \beta((N-4)(\alpha-2) - \beta) \left\| \frac{f}{|\tilde{x}'|^2} \right\|_{L^2(\mathbb{H}_n)}^2 \\ &\quad - 2\alpha \int_{\mathbb{H}_n} \frac{\overline{f(\tilde{x}', t)} Z T f(\tilde{x}', t)}{|\tilde{x}'|^2} d\tilde{x}' dt + \alpha \|Tf\|_{L^2(\mathbb{H}_n)}^2 \end{aligned} \quad (7.45)$$

and

$$\begin{aligned}
\|\mathcal{L}f\|_{L^2(\mathbb{H}_n)}^2 &\geq (N\alpha - 2\beta) \left\| \frac{\nabla_H f}{|\tilde{x}'|} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad - \alpha(\alpha + 4) \left\| \frac{\tilde{x}' \cdot \nabla_H f}{|\tilde{x}'|^2} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad + (2(N - 4)(\alpha(N - 2) - \beta) - 2\alpha^2(N - 2) + \alpha\beta N - \beta^2) \left\| \frac{f}{|\tilde{x}'|^2} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad + 2\alpha \int_{\mathbb{H}_n} \frac{\overline{f(\tilde{x}', t)} ZTf(\tilde{x}', t)}{|\tilde{x}'|^2} d\tilde{x}' dt - \alpha \|Tf\|_{L^2(\mathbb{H}_n)}^2.
\end{aligned} \tag{7.46}$$

Moreover,

$$\begin{aligned}
\|\mathcal{L}f\|_{L^2(\mathbb{H}_n)}^2 &\geq ((N - \alpha)\alpha - 2\beta) \left\| \frac{\nabla_H f}{|\tilde{x}'|} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad + \beta((N - 4)(\alpha - 2) - \beta) \left\| \frac{f}{|\tilde{x}'|^2} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad - 2\alpha \int_{\mathbb{H}_n} \frac{\overline{f(\tilde{x}', t)} ZTf(\tilde{x}', t)}{|\tilde{x}'|^2} d\tilde{x}' dt + \alpha \|Tf\|_{L^2(\mathbb{H}_n)}^2
\end{aligned} \tag{7.47}$$

for $\alpha(\alpha - 4) \geq 0$, and

$$\begin{aligned}
\|\mathcal{L}f\|_{L^2(\mathbb{H}_n)}^2 &\geq (-2\beta + \alpha(N - \alpha) - 4\alpha) \left\| \frac{\nabla_H f}{|\tilde{x}'|} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad + (2(N - 4)(\alpha(N - 2) - \beta) - 2\alpha^2(N - 2) + \alpha\beta N - \beta^2) \left\| \frac{f}{|\tilde{x}'|^2} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad + 2\alpha \int_{\mathbb{H}_n} \frac{\overline{f(\tilde{x}', t)} ZTf(\tilde{x}', t)}{|\tilde{x}'|^2} d\tilde{x}' dt - \alpha \|Tf\|_{L^2(\mathbb{H}_n)}^2
\end{aligned} \tag{7.48}$$

for $\alpha(\alpha + 4) \geq 0$.

Remark 7.19. For functions $f(\tilde{x}', t) = f(\tilde{x}')$ the last two terms in the above inequalities vanish, while for functions $f(\tilde{x}', t) = f(|\tilde{x}'|, t)$ we have $Zf = 0$, so that the inner product term involving Zf vanishes. One can also note

that the estimates (7.45) and (7.47) for $\alpha > 0$, and (7.46) and (7.48) for $\alpha < 0$ show that $\|Tf\|_{L^2(\mathbb{H}_n)}$ is also controlled by $\|\mathcal{L}f\|_{L^2(\mathbb{H}_n)}$, although without the weight, which is natural in view of the homogeneity degrees.

Remark 7.20. In the case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $\nabla_H = \nabla = (\partial_{x_1}, \dots, \partial_{x_n})$, and we can note that $T = [X_j, Y_j] = 0$. So, the last two terms in (7.45) and (7.46) vanish. Therefore, in this case, (7.45) would coincide with the Gesztesy and Littlejohn's estimate (3.13), while (7.46) would give the following new estimate for $\alpha, \beta \in \mathbb{R}$ and $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ with $n \geq 2$:

$$\begin{aligned}
& \|\triangle f\|_{L^2(\mathbb{R}^n)}^2 \\
& \geq (n\alpha - 2\beta) \left\| \frac{\nabla f}{|x|} \right\|_{L^2(\mathbb{R}^n)}^2 \\
& - \alpha(\alpha + 4) \left\| \frac{x \cdot \nabla f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)}^2 \\
& + (2(n-4)(\alpha(n-2) - \beta) - 2\alpha^2(n-2) + \alpha\beta n - \beta^2) \left\| \frac{f}{|x|^2} \right\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned} \tag{7.49}$$

Remark 7.21. Let us show that the term

$$\int_{\mathbb{H}_n} \frac{\overline{f(\tilde{x}', t)} ZT f(\tilde{x}', t)}{|\tilde{x}'|^2} d\tilde{x}' dt$$

in Theorem 7.18 is actually real-valued. Indeed, using integration by parts we obtain

$$\begin{aligned}
& \int_{\mathbb{H}_n} \frac{\overline{f(\tilde{x}', t)} ZT f(\tilde{x}', t)}{|\tilde{x}'|^2} d\tilde{x}' dt \\
& = \sum_{j=1}^n \int_{\mathbb{H}_n} \frac{\overline{f(\tilde{x}', t)} (x_j \partial_{y_j} - y_j \partial_{x_j}) (\partial_t f(\tilde{x}', t))}{|\tilde{x}'|^2} d\tilde{x}' dt \\
& = -\frac{1}{2} \sum_{j=1}^n \int_{\mathbb{H}_n} \overline{\partial_t f(\tilde{x}', t)} (\partial_{y_j} (f(\tilde{x}', t)) \partial_{x_j} (\log |\tilde{x}'|^2) \\
& \quad - \partial_{x_j} (f(\tilde{x}', t)) \partial_{y_j} (\log |\tilde{x}'|^2)) d\tilde{x}' dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{j=1}^n \int_{\mathbb{H}_n} \log |\tilde{x}'|^2 (\partial_{y_j}(f(\tilde{x}', t)) \overline{\partial_{tx_j} f(\tilde{x}', t)} \\
&\quad - \partial_{x_j}(f(\tilde{x}', t)) \overline{\partial_{ty_j} f(\tilde{x}', t)}) d\tilde{x}' dt \\
&= \frac{1}{2} \sum_{j=1}^n \int_{\mathbb{H}_n} \log |\tilde{x}'|^2 (\partial_{y_j}(f(\tilde{x}', t)) \overline{\partial_{tx_j} f(\tilde{x}', t)} + \overline{\partial_{y_j}(f(\tilde{x}', t)) \partial_{tx_j} f(\tilde{x}', t)}) d\tilde{x}' dt \\
&= \sum_{j=1}^n \int_{\mathbb{H}_n} \log |\tilde{x}'|^2 \operatorname{Re}(\partial_{y_j}(f(\tilde{x}', t)) \overline{\partial_{tx_j} f(\tilde{x}', t)}) d\tilde{x}' dt.
\end{aligned}$$

Remark 7.22. Note that we can use that $ZT = TZ$ and integration by parts in t , so that the term

$$\int_{\mathbb{H}_n} \frac{\overline{f(\tilde{x}', t)} ZT f(\tilde{x}', t)}{|\tilde{x}'|^2} d\tilde{x}' dt = - \left(\frac{Zf}{|\tilde{x}'|}, \frac{Tf}{|\tilde{x}'|} \right)_{L^2(\mathbb{H}_n)}$$

can be seen as an interaction between the central and the angular derivatives.

Writing (2.81) in the form

$$ZT = \mathfrak{L} - \Delta_{\tilde{x}'} - \frac{|\tilde{x}'|^2}{4} T^2, \tag{7.50}$$

we have

$$\begin{aligned}
&\int_{\mathbb{H}_n} \frac{\overline{f(x)} ZT f(x)}{|\tilde{x}'|^2} d\tilde{x}' dt \\
&= \left(\frac{\mathfrak{L}f}{|\tilde{x}'|}, \frac{f}{|\tilde{x}'|} \right)_{L^2(\mathbb{H}_n)} - \left(\frac{\Delta_{\tilde{x}'} f}{|\tilde{x}'|}, \frac{f}{|\tilde{x}'|} \right)_{L^2(\mathbb{H}_n)} + \frac{1}{4} \|Tf\|_{L^2(\mathbb{H}_n)}^2. \tag{7.51}
\end{aligned}$$

Consequently, inequalities (7.45) and (7.46) can be also written as

$$\begin{aligned}
\|\mathcal{L}f\|_{L^2(\mathbb{H}_n)}^2 &\geq ((N-4)\alpha - 2\beta) \left\| \frac{\nabla_H f}{|\tilde{x}'|} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad - \alpha(\alpha - 4) \left\| \frac{\tilde{x}' \cdot \nabla_H f}{|\tilde{x}'|^2} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad + \beta((N-4)(\alpha - 2) - \beta) \left\| \frac{f}{|\tilde{x}'|^2} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad - 2\alpha \left(\frac{\mathfrak{L}f}{|\tilde{x}'|}, \frac{f}{|\tilde{x}'|} \right)_{L^2(\mathbb{H}_n)} + 2\alpha \left(\frac{\Delta_{\tilde{x}'} f}{|\tilde{x}'|}, \frac{f}{|\tilde{x}'|} \right)_{L^2(\mathbb{H}_n)} + \frac{\alpha}{2} \|Tf\|_{L^2(\mathbb{H}_n)}^2,
\end{aligned} \tag{7.52}$$

and

$$\begin{aligned}
\|\mathcal{L}f\|_{L^2(\mathbb{H}_n)}^2 &\geq (N\alpha - 2\beta) \left\| \frac{\nabla_H f}{|\tilde{x}'|} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad - \alpha(\alpha + 4) \left\| \frac{\tilde{x}' \cdot \nabla_H f}{|\tilde{x}'|^2} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad + (2(N-4)(\alpha(N-2) - \beta) - 2\alpha^2(N-2) + \alpha\beta N - \beta^2) \left\| \frac{f}{|\tilde{x}'|^2} \right\|_{L^2(\mathbb{H}_n)}^2 \\
&\quad + 2\alpha \left(\frac{\mathfrak{L}f}{|\tilde{x}'|}, \frac{f}{|\tilde{x}'|} \right)_{L^2(\mathbb{H}_n)} - 2\alpha \left(\frac{\Delta_{\tilde{x}'} f}{|\tilde{x}'|}, \frac{f}{|\tilde{x}'|} \right)_{L^2(\mathbb{H}_n)} - \frac{\alpha}{2} \|Tf\|_{L^2(\mathbb{H}_n)}^2,
\end{aligned} \tag{7.53}$$

respectively.

Proof of Theorem 7.18. Similarly as the proof of Theorem 7.12, we apply the factorization method with the differential expressions (7.26) and (7.27). We refer to¹⁸⁷ for the rest of the proof. $\square$

7.4 RELICH INEQUALITIES FOR SUB-LAPLACIANS WITH DRIFT

Here we discuss Rellich inequalities for sub-Laplacians with drift on stratified Lie groups. In this section we adopt the notations introduced in Sec-

tion 2.4.1 concerning the sub-Laplacians with drift.

7.4.1 HORIZONTAL RELlich INEQUALITIES FOR SUB-LAPLACIANS WITH DRIFT ON STRATIFIED LIE GROUPS

In this section we establish weighted Rellich inequalities for sub-Laplacians with drift on a stratified Lie group $\mathbb{G}$ expressing the weights in terms of the Euclidean norm on the first stratum of $\mathbb{G}$. In Section 7.4.2, we will show a version with weights in terms of the fundamental solution of the sub-Laplacian on polarizable Carnot groups.

Theorem 7.23 (¹⁹⁴ Ruzhansky-Y., Theorem 3.2). *Let $\mathbb{G}$ be a stratified group with $N \geq 3$ being the dimension of the first stratum.*

- *If $\delta \in \mathbb{R}$ and $-N/2 \leq \delta \leq -1$, then we have*

$$\begin{aligned} \left\| \frac{\mathcal{L}_X f}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 &\geq \left(\frac{(N - 2\delta - 4)(N + 2\delta)}{4} \right)^2 \left\| \frac{f}{|x'|^{\delta+2}} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 \\ &+ \gamma^2 b_X^2 \frac{(N - 2\delta - 2)(N + 2\delta - 2)}{2} \left\| \frac{f}{|x'|^{\delta+1}} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 + \gamma^4 b_X^4 \left\| \frac{f}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu_X)}^2, \end{aligned} \quad (7.54)$$

for all functions $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$, where $\mathcal{L}_X$, b_X and μ_X are defined in (2.67), (2.69) and (2.76), respectively. If $(N + 2\delta)(N + 2\delta - 2) \neq 0$, then the constants in (7.54) are sharp.

- *In the case $\delta = 0$ and $N > 4$, we have*

$$\begin{aligned} \|\mathcal{L}_X f\|_{L^2(\mathbb{G}, \mu_X)}^2 &\geq \left(\frac{N(N - 4)}{4} \right)^2 \left\| \frac{f}{|x'|^2} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 \\ &+ \gamma^2 b_X^2 \frac{(N - 2)^2}{2} \left\| \frac{f}{|x'|} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 + \gamma^4 b_X^4 \|f\|_{L^2(\mathbb{G}, \mu_X)}^2, \end{aligned} \quad (7.55)$$

for any function $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$ with sharp constants.

The constants in (7.54) and (7.55) are sharp in the sense that there exists a sequence of functions such that the equalities in (7.54) and (7.55) are attained in

the limit of this sequence of functions, respectively.

Remark 7.24. In the case $(N - 2\delta - 2)(N + 2\delta - 2) \geq 0$, by dropping positive terms in (7.54) we obtain the following “standard” Rellich type inequality

$$\left\| \frac{\mathcal{L}_X f}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 \geq \left(\frac{(N - 2\delta - 4)(N + 2\delta)}{4} \right)^2 \left\| \frac{f}{|x'|^{\delta+2}} \right\|_{L^2(\mathbb{G}, \mu_X)}^2, \quad (7.56)$$

for all functions $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$, where $\delta \in \mathbb{R}$ with $-N/2 \leq \delta \leq -1$ and $N \geq 3$.

Similarly, we derive from (7.55) that

$$\|\mathcal{L}_X f\|_{L^2(\mathbb{G}, \mu_X)}^2 \geq \left(\frac{N(N - 4)}{4} \right)^2 \left\| \frac{f}{|x'|^2} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 \quad (7.57)$$

for any function $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$ and $N > 4$.

Remark 7.25. In the Abelian case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, \dots, \partial_{x_n})$ is the usual Euclidean gradient, if we take $X = \sum_{i=1}^N a_i \partial_{x_i}$ with $a_i \in \mathbb{R}$ and $\delta = -\alpha$, then (7.54) gives (7.3) for $\alpha \geq 1$ and $n \geq 2\alpha$ with sharp constants, while (7.55) implies (7.2) again with sharp constants.

We will use the following Rellich inequality in the proof of Theorem 7.23:

Theorem 7.26 (¹⁹⁴ Ruzhansky-Y., Theorem 3.1). *Let $\mathbb{G}$ be a stratified group with $N \geq 3$ being the dimension of the first stratum. Let $\delta \in \mathbb{R}$ with $-N/2 \leq \delta \leq -1$. Then we have*

$$\left\| \frac{\mathcal{L}f}{|x'|^\delta} \right\|_{L^2(\mathbb{G})} \geq \left| \frac{(N - 2\delta - 4)(N + 2\delta)}{4} \right| \left\| \frac{f}{|x'|^{\delta+2}} \right\|_{L^2(\mathbb{G})} \quad (7.58)$$

for all functions $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$. If we assume $N + 2\delta \neq 0$, then the constant is sharp.

Proof of Theorem 7.26. Replacing α by $\delta + 2$ let us rewrite (7.8) for $p = 2$

$$\left\| \frac{\nabla_H f}{|x'|^{\delta+1}} \right\|_{L^2(\mathbb{G})} \geq \left| \frac{N - 2\delta - 4}{2} \right| \left\| \frac{f}{|x'|^{\delta+2}} \right\|_{L^2(\mathbb{G})}, \quad (7.59)$$

where the constant is sharp when $N - 2\delta - 4 \neq 0$. On the other hand,¹⁷³ Theorem 5.1 with $p = 2$ and $\gamma = 2\beta$ follows that

$$\left\| \frac{\mathcal{L}f}{|x'|^{\beta-1}} \right\|_{L^2(\mathbb{G})} \geq \left| \frac{N + 2\beta - 2}{2} \right| \left\| \frac{\nabla_H f}{|x'|^\beta} \right\|_{L^2(\mathbb{G})} \quad (7.60)$$

for $2 - N \leq 2\beta \leq 0$ and $N \geq 3$. Here, replacing β by $\delta + 1$ we get

$$\left\| \frac{\mathcal{L}f}{|x'|^\delta} \right\|_{L^2(\mathbb{G})} \geq \left| \frac{N + 2\delta}{2} \right| \left\| \frac{\nabla_H f}{|x'|^{\delta+1}} \right\|_{L^2(\mathbb{G})} \quad (7.61)$$

for $2 - N \leq 2\delta + 2 \leq 0$ and $N \geq 3$. Thus, from (7.59) and (7.61) we obtain (7.58).

Now let us show sharpness of the constant in (7.58). In¹⁷³ Proof of Theorem 3.1, the authors proved the sharpness of the constant in (7.59) by checking the equality condition in Hölder's inequality. Namely, the function $|x'|^{C_1}$ satisfies this equality condition for any real number $C_1 \neq 0$. Similarly, in¹⁷³ Formula 5.7 (i.e. in the proof of (7.61)), checking again the equality condition in Hölder's inequality, one can note that the same function $|x'|^{C_1}$ satisfies the equality condition. Therefore, we can conclude that the constant in (7.61) is sharp when $N + 2\delta \neq 0$, hence the constant in (7.58) is also sharp for $N + 2\delta \neq 0$. $\square$

Now we are ready to prove Theorem 7.23.

Proof of Theorem 7.23. Let χ be the positive character on $\mathbb{G}$ from Proposition 2.23, and let $g = g(x) \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$ be such that $f = \chi^{-1/2}g$. Then, noting (2.76) we write

$$\left\| \frac{\mathcal{L}_X f}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu_X)} = \left\| \frac{\chi^{1/2} \mathcal{L}_X f}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)} = \left\| \frac{\chi^{1/2} \mathcal{L}_X (\chi^{-1/2} g)}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)}.$$

Using (2.75) and integration by parts we obtain

$$\left\| \frac{\mathcal{L}_X f}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 = \left\| \frac{(\mathcal{L}_0 + \gamma^2 b_X^2)g}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)}^2$$

$$\begin{aligned}
&= \left\| \frac{\mathcal{L}_0 g}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)}^2 + 2\gamma^2 b_X^2 \operatorname{Re} \int_{\mathbb{G}} \frac{\mathcal{L}_0 g(x) \overline{g(x)}}{|x'|^{2\delta}} dx + \gamma^4 b_X^4 \left\| \frac{g}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)}^2 \\
&= \left\| \frac{\mathcal{L}_0 g}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)}^2 - 2\gamma^2 b_X^2 \operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} \frac{X_j^2 g(x) \overline{g(x)}}{|x'|^{2\delta}} dx + \gamma^4 b_X^4 \left\| \frac{g}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)}^2 \\
&= \left\| \frac{\mathcal{L}_0 g}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)}^2 + 2\gamma^2 b_X^2 \int_{\mathbb{G}} \frac{|\nabla_H g(x)|^2}{|x'|^{2\delta}} \\
&\quad - 4\delta \gamma^2 b_X^2 \operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} \frac{x'_j X_j g(x) \overline{g(x)}}{|x'|^{2\delta+2}} dx + \gamma^4 b_X^4 \left\| \frac{g}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)}^2, \quad (7.62)
\end{aligned}$$

where $\mathfrak{L}_0 := -\mathfrak{L}$ is the positive sub-Laplacian.

Here, the third term can be simplified as follows

$$\begin{aligned}
&\operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} \frac{x'_j X_j g(x) \overline{g(x)}}{|x'|^{2\delta+2}} dx \\
&= (2\delta + 2 - N) \int_{\mathbb{G}} \frac{|g(x)|^2}{|x'|^{2\delta+2}} dx - \operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} \frac{x'_j g(x) \overline{X_j g(x)}}{|x'|^{2\delta+2}} dx,
\end{aligned}$$

which is

$$\operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} \frac{x'_j X_j g(x) \overline{g(x)}}{|x'|^{2\delta+2}} dx = \frac{2\delta + 2 - N}{2} \int_{\mathbb{G}} \frac{|g(x)|^2}{|x'|^{2\delta+2}} dx.$$

Taking into account this, we rewrite (7.62) as follows

$$\begin{aligned}
&\left\| \frac{\mathcal{L}_X f}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 = \left\| \frac{\mathcal{L}_0 g}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)}^2 + 2\gamma^2 b_X^2 \left\| \frac{\nabla_H g}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)}^2 \\
&+ 2\delta(N - 2\delta - 2)\gamma^2 b_X^2 \left\| \frac{g}{|x'|^{\delta+1}} \right\|_{L^2(\mathbb{G}, \mu)}^2 + \gamma^4 b_X^4 \left\| \frac{g}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)}^2. \quad (7.63)
\end{aligned}$$

By the Rellich (7.58) and Hardy (7.59) inequalities, we obtain from (7.63)

that

$$\begin{aligned} \left\| \frac{\mathcal{L}_X f}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 &\geq \left(\frac{(N - 2\delta - 4)(N + 2\delta)}{4} \right)^2 \left\| \frac{g}{|x'|^{\delta+2}} \right\|_{L^2(\mathbb{G}, \mu)}^2 \\ &\quad + \gamma^4 b_X^4 \left\| \frac{g}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu)}^2 + 2\gamma^2 b_X^2 \left(\frac{N - 2\delta - 2}{2} \right)^2 \left\| \frac{g}{|x'|^{\delta+1}} \right\|_{L^2(\mathbb{G}, \mu)}^2 \\ &\quad + 2\delta(N - 2\delta - 2)\gamma^2 b_X^2 \left\| \frac{g}{|x'|^{\delta+1}} \right\|_{L^2(\mathbb{G}, \mu)}^2. \end{aligned}$$

It follows that

$$\begin{aligned} \left\| \frac{\mathcal{L}_X f}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 &\geq \left(\frac{(N - 2\delta - 4)(N + 2\delta)}{4} \right)^2 \left\| \frac{f}{|x'|^{\delta+2}} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 \\ &\quad + \gamma^4 b_X^4 \left\| \frac{f}{|x'|^\delta} \right\|_{L^2(\mathbb{G}, \mu_X)}^2 + \gamma^2 b_X^2 \frac{(N - 2\delta - 2)(N + 2\delta - 2)}{2} \left\| \frac{f}{|x'|^{\delta+1}} \right\|_{L^2(\mathbb{G}, \mu_X)}^2. \end{aligned} \quad (7.64)$$

As discussed in the proof of Theorem 7.26, one can note that the same function satisfies the equality conditions in Hölder's inequalities. Therefore, the constants in (7.54) are sharp.

Similarly, one can obtain (7.55), the unweighted case $\delta = 0$, but using the following Rellich inequality from¹⁹⁶ Formula 1.5 instead of (7.58):

$$\|\mathcal{L}f\|_{L^2(\mathbb{G})} \geq \frac{N(N-4)}{4} \left\| \frac{f}{|x'|^2} \right\|_{L^2(\mathbb{G})}, \quad N \geq 5, \quad (7.65)$$

where the constant is sharp and $f \in C_0^\infty(\mathbb{G} \setminus \{x' = 0\})$. The sharpness of the constants in (7.55) also follows from the same argument as for the constants in (7.54). $\square$

7.4.2 RELICH INEQUALITIES FOR SUB-LAPLACIANS WITH DRIFT ON POLARIZABLE CARNOT GROUPS

In this section we discuss the weighted Rellich inequality for sub-Laplacians with drift on a polarizable Carnot group $\mathbb{G}$, where the weights are given

by the powers of the fundamental solution of the sub-Laplacian.

Theorem 7.27 (¹⁹⁴ Ruzhansky-Y., Theorem 4.1). *Let $\mathbb{G}$ be a polarizable Carnot group of homogeneous dimension $Q \geq 3$ and let $\theta \in \mathbb{R}$ with $Q + 2\theta - 4 > 0$. Then we have*

$$\begin{aligned}
& \left\| \frac{d^\theta}{|\nabla_H d|} \mathcal{L}_X f \right\|_{L^2(\mathbb{G}, \mu_X)}^2 \geq \frac{(Q + 2\theta - 4)^2 (Q - 2\theta)^2}{16} \|d^{\theta-2} |\nabla_H d| f\|_{L^2(\mathbb{G}, \mu_X)}^2 \\
& + \gamma^4 b_X^4 \left\| \frac{d^\theta}{|\nabla_H d|} f \right\|_{L^2(\mathbb{G}, \mu_X)}^2 + \gamma^2 b_X^2 \left(\frac{(Q + 2\theta - 2)(Q - 2\theta - 2)}{2} \right) \|d^{\theta-1} f\|_{L^2(\mathbb{G}, \mu_X)}^2 \\
& + 2\gamma^2 b_X^2 (Q - 1)(3Q - 4) \|d^{\theta-1} f\|_{L^2(\mathbb{G}, \mu_X)}^2 \\
& + 2\gamma^2 b_X^2 \int_{\mathbb{G}} \frac{d^{2\theta+2Q-2}}{|\nabla_H d|^4} (d^{1-Q} |\nabla_H d| \mathcal{L}(d^{1-Q} |\nabla_H d|) - 3 |\nabla_H (d^{1-Q} |\nabla_H d|)|^2) \\
& \quad \times |f(x)|^2 d\mu_X(x), \quad (7.66)
\end{aligned}$$

for all functions $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, where $\mathcal{L}_X$ and b_X are defined in (2.67) and (2.69), respectively.

Remark 7.28. In the Euclidean case $\mathbb{G} = (\mathbb{R}^n, +)$, we have $N = n$, $\nabla_H = \nabla = (\partial_{x_1}, \dots, \partial_{x_n})$ is the usual Euclidean gradient, $d = |x|_E$ is the Euclidean distance. In this case, if we take $X = \sum_{i=1}^N a_i \partial_{x_i}$ with $a_i \in \mathbb{R}$, then noting $|\nabla_H d| = 1$ we see that the last two terms in (7.66) cancel each other, so that we get (7.3).

Remark 7.29. We refer to Kombe¹²⁹ Theorem 4.1 for the case $\gamma = 0$, hence $\mathcal{L}_X = \mathcal{L}_0$, of (7.66), where $\mathfrak{L}_0 := -\mathfrak{L}$ is the positive sub-Laplacian.

Proof of Theorem 7.27. Let $g = g(x) \in C_0^\infty(\mathbb{G} \setminus \{0\})$ be such that $f = \chi^{-1/2} g$. Since $L^2(\mathbb{G}, \mu) \ni g \mapsto \chi^{-1/2} g \in L^2(\mathbb{G}, \mu_X)$ by (2.76), we can write

$$\begin{aligned}
\left\| \frac{d^\theta}{|\nabla_H d|} \mathcal{L}_X f \right\|_{L^2(\mathbb{G}, \mu_X)} &= \left\| \frac{d^\theta}{|\nabla_H d|} \chi^{1/2} \mathcal{L}_X f \right\|_{L^2(\mathbb{G}, \mu)} \\
&= \left\| \frac{d^\theta}{|\nabla_H d|} \chi^{1/2} \mathcal{L}_X (\chi^{-1/2} g) \right\|_{L^2(\mathbb{G}, \mu)},
\end{aligned}$$

where μ is the Haar (Lebesgue) measure on $\mathbb{G}$. Here, using (2.75) and integration by parts we calculate

$$\begin{aligned}
& \left\| \frac{d^\theta}{|\nabla_H d|} \mathcal{L}_X f \right\|_{L^2(\mathbb{G}, \mu_X)}^2 = \left\| \frac{d^\theta}{|\nabla_H d|} (\mathcal{L}_0 + \gamma^2 b_X^2) g \right\|_{L^2(\mathbb{G}, \mu)}^2 = \left\| \frac{d^\theta}{|\nabla_H d|} \mathcal{L}_0 g \right\|_{L^2(\mathbb{G}, \mu)}^2 \\
& \quad + 2\gamma^2 b_X^2 \operatorname{Re} \int_{\mathbb{G}} \frac{d^{2\theta}}{|\nabla_H d|^2} \mathcal{L}_0 g(x) \overline{g(x)} dx + \gamma^4 b_X^4 \left\| \frac{d^\theta}{|\nabla_H d|} g \right\|_{L^2(\mathbb{G}, \mu)}^2 \\
& = \left\| \frac{d^\theta}{|\nabla_H d|} \mathcal{L}_0 g \right\|_{L^2(\mathbb{G}, \mu)}^2 - 2\gamma^2 b_X^2 \operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} \frac{d^{2\theta}}{|\nabla_H d|^2} X_j^2 g(x) \overline{g(x)} dx \\
& \quad + \gamma^4 b_X^4 \left\| \frac{d^\theta}{|\nabla_H d|} g \right\|_{L^2(\mathbb{G}, \mu)}^2 = \left\| \frac{d^\theta}{|\nabla_H d|} \mathcal{L}_0 g \right\|_{L^2(\mathbb{G}, \mu)}^2 + 2\gamma^2 b_X^2 \left\| \frac{d^\theta}{|\nabla_H d|} \nabla_H g \right\|_{L^2(\mathbb{G}, \mu)}^2 \\
& \quad + \gamma^4 b_X^4 \left\| \frac{d^\theta}{|\nabla_H d|} g \right\|_{L^2(\mathbb{G}, \mu)}^2 + 2\gamma^2 b_X^2 \operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} X_j g(x) \overline{g(x)} X_j \left(\frac{d^{2\theta}}{|\nabla_H d|^2} \right) dx \\
& = \left\| \frac{d^\theta}{|\nabla_H d|} \mathcal{L}_0 g \right\|_{L^2(\mathbb{G}, \mu)}^2 + 2\gamma^2 b_X^2 \left\| \frac{d^\theta}{|\nabla_H d|} \nabla_H g \right\|_{L^2(\mathbb{G}, \mu)}^2 + \gamma^4 b_X^4 \left\| \frac{d^\theta}{|\nabla_H d|} g \right\|_{L^2(\mathbb{G}, \mu)}^2 \\
& \quad + 4\gamma^2 b_X^2 \theta \operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} X_j g(x) \overline{g(x)} \frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} dx \\
& \quad - 4\gamma^2 b_X^2 \operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} X_j g(x) \overline{g(x)} \frac{d^{2\theta} X_j |\nabla_H d|}{|\nabla_H d|^3} dx \\
& := \left\| \frac{d^\theta}{|\nabla_H d|} \mathcal{L}_0 g \right\|_{L^2(\mathbb{G}, \mu)}^2 + 2\gamma^2 b_X^2 \left\| \frac{d^\theta}{|\nabla_H d|} \nabla_H g \right\|_{L^2(\mathbb{G}, \mu)}^2 + \gamma^4 b_X^4 \left\| \frac{d^\theta}{|\nabla_H d|} g \right\|_{L^2(\mathbb{G}, \mu)}^2 \\
& \quad + I_1 + I_2. \quad (7.67)
\end{aligned}$$

To continue the proof, we need the following Hardy and Rellich inequalities from¹²⁹:

Theorem 7.30 (¹²⁹ Theorem 3.1). *Let $\mathbb{G}$ be a polarizable Carnot group with homogeneous dimension $Q \geq 3$. Let $1 < p < Q$ and $\alpha \in \mathbb{R}$ be such that $\alpha > -Q$.*

Then we have

$$\int_{\mathbb{G}} d^{\alpha+p} \frac{|\nabla_H d \cdot \nabla_H f|^p}{|\nabla_H d|^{2p}} dx \geq \left(\frac{Q + \alpha}{p} \right)^p \int_{\mathbb{G}} d^{\alpha} |f|^p dx, \quad (7.68)$$

for $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ with sharp constant.

Theorem 7.31 (¹²⁹ Theorem 4.1 or ⁹⁶). *Let $\mathbb{G}$ be a Carnot group with homogeneous norm $d = u^{1/(2-Q)}$ and $Q \geq 3$. Let $\alpha \in \mathbb{R}$ be such that $Q + \alpha - 4 > 0$. Then we have*

$$\int_{\mathbb{G}} \frac{d^{\alpha}}{|\nabla_H d|^2} |\mathcal{L}_0 f|^2 dx \geq \frac{(Q + \alpha - 4)^2 (Q - \alpha)^2}{16} \int_{\mathbb{G}} d^{\alpha} \frac{|\nabla_H d|^2}{d^4} |f|^2 dx, \quad (7.69)$$

for $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$ with sharp constant.

Then, Theorems 7.30 and 7.31 imply that

$$2\gamma^2 b_X^2 \left\| \frac{d^{\theta}}{|\nabla_H d|} \nabla_H g \right\|_{L^2(\mathbb{G}, \mu)}^2 \geq 2\gamma^2 b_X^2 \left(\frac{Q + 2\theta - 2}{2} \right)^2 \|d^{\theta-1} g\|_{L^2(\mathbb{G}, \mu)}^2 \quad (7.70)$$

for $Q + 2\theta - 2 > 0$, and

$$\left\| \frac{d^{\theta}}{|\nabla_H d|} \mathcal{L}_0 g \right\|_{L^2(\mathbb{G}, \mu)}^2 \geq \frac{(Q + 2\theta - 4)^2 (Q - 2\theta)^2}{16} \|d^{\theta-2} |\nabla_H d| g\|_{L^2(\mathbb{G}, \mu)}^2 \quad (7.71)$$

for $Q + 2\theta - 4 > 0$, respectively. These inequalities with (7.67) imply for $Q + 2\theta - 4 > 0$ that

$$\begin{aligned} & \left\| \frac{d^{\theta}}{|\nabla_H d|} \mathcal{L}_X f \right\|_{L^2(\mathbb{G}, \mu_X)}^2 \geq \frac{(Q + 2\theta - 4)^2 (Q - 2\theta)^2}{16} \|d^{\theta-2} |\nabla_H d| g\|_{L^2(\mathbb{G}, \mu)}^2 \\ & + 2\gamma^2 b_X^2 \left(\frac{Q + 2\theta - 2}{2} \right)^2 \|d^{\theta-1} g\|_{L^2(\mathbb{G}, \mu)}^2 + \gamma^4 b_X^4 \left\| \frac{d^{\theta}}{|\nabla_H d|} g \right\|_{L^2(\mathbb{G}, \mu)}^2 + I_1 + I_2. \end{aligned} \quad (7.72)$$

Now let us simplify I_1 as follows

$$\begin{aligned}
I_1 &= 4\gamma^2 b_X^2 \theta \operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} X_j g(x) \overline{g(x)} \frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} dx \\
&= -4\gamma^2 b_X^2 \theta \operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} g(x) \overline{X_j g(x)} \frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} dx \\
&\quad - 4\gamma^2 b_X^2 \theta \operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} |g(x)|^2 X_j \left(\frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} \right) dx,
\end{aligned}$$

which implies

$$\begin{aligned}
I_1 &= 4\gamma^2 b_X^2 \theta \operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} X_j g(x) \overline{g(x)} \frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} dx \\
&= -2\gamma^2 b_X^2 \theta \sum_{j=1}^N \int_{\mathbb{G}} |g(x)|^2 X_j \left(\frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} \right) dx. \quad (7.73)
\end{aligned}$$

Taking into account $d = u^{\frac{1}{2-Q}}$ we write

$$\begin{aligned}
\sum_{j=1}^N X_j \left(\frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} \right) &= (2-Q) \sum_{j=1}^N X_j \left(\frac{u^{\frac{2\theta-Q}{2-Q}}}{|\nabla_H u|^2} X_j u \right) \\
&= (2-Q) \sum_{j=1}^N \frac{2\theta-Q}{2-Q} u^{\frac{2\theta-2}{2-Q}} \frac{(X_j u)^2}{|\nabla_H u|^2} \\
&\quad + (2-Q) \sum_{j=1}^N \frac{u^{\frac{2\theta-Q}{2-Q}} X_j^2 u}{|\nabla_H u|^2} - 2(2-Q) \sum_{j=1}^N \frac{u^{\frac{2\theta-Q}{2-Q}} X_j u}{|\nabla_H u|^3} X_j |\nabla_H u|.
\end{aligned}$$

Thus, by this and (2.65) we obtain for I_1 that

$$\begin{aligned}
I_1 &= -2\gamma^2 b_X^2 \theta \int_{\mathbb{G}} \left(\sum_{j=1}^N X_j \left(\frac{d^{2\theta-1} X_j d}{|\nabla_H d|^2} \right) \right) |g(x)|^2 dx \\
&= -2\gamma^2 b_X^2 \theta (2\theta + Q - 2) \int_{\mathbb{G}} u^{\frac{2\theta-2}{2-Q}} |g(x)|^2 dx
\end{aligned}$$

$$= -2\gamma^2 b_X^2 \theta(2\theta + Q - 2) \int_{\mathbb{G}} |g(x)|^2 d^{2\theta-2} dx. \quad (7.74)$$

Next, we calculate I_2 . Integrating by parts, we have

$$\begin{aligned} I_2 &= -4\gamma^2 b_X^2 \operatorname{Re} \sum_{j=1}^N \int_{\mathbb{G}} X_j g(x) \overline{g(x)} \frac{d^{2\theta} X_j |\nabla_H d|}{|\nabla_H d|^3} dx \\ &= 2\gamma^2 b_X^2 \sum_{j=1}^N \int_{\mathbb{G}} |g(x)|^2 X_j \left(\frac{d^{2\theta} X_j |\nabla_H d|}{|\nabla_H d|^3} \right) dx. \end{aligned} \quad (7.75)$$

Putting $d = u^{\frac{1}{2-Q}}$ we get

$$\begin{aligned} \sum_{j=1}^N X_j \left(\frac{d^{2\theta} X_j |\nabla_H d|}{|\nabla_H d|^3} \right) &= (2-Q)^2 \sum_{j=1}^N X_j \left(\frac{u^{\frac{2\theta-3Q+3}{2-Q}}}{|\nabla_H u|^3} X_j \left(u^{\frac{Q-1}{2-Q}} |\nabla_H u| \right) \right) \\ &= (2-Q)^2 \sum_{j=1}^N X_j \left(\frac{Q-1}{2-Q} u^{\frac{2\theta-Q}{2-Q}} \frac{X_j u}{|\nabla_H u|^2} + u^{\frac{2\theta-2Q+2}{2-Q}} \frac{X_j |\nabla_H u|}{|\nabla_H u|^3} \right) \\ &:= (2-Q)^2 J_1 + (2-Q)^2 J_2. \end{aligned} \quad (7.76)$$

By (2.65) one has for J_1 that

$$\begin{aligned} J_1 &= \sum_{j=1}^N X_j \left(\frac{Q-1}{2-Q} u^{\frac{2\theta-Q}{2-Q}} \frac{X_j u}{|\nabla_H u|^2} \right) = \frac{(Q-1)(2\theta-Q)}{(Q-2)^2} u^{\frac{2\theta-2}{2-Q}} \sum_{j=1}^N \frac{(X_j u)^2}{|\nabla_H u|^2} \\ &\quad + \frac{Q-1}{2-Q} u^{\frac{2\theta-Q}{2-Q}} \sum_{j=1}^N \frac{X_j^2 u}{|\nabla_H u|^2} - \frac{2(Q-1)}{(2-Q)} u^{\frac{2\theta-Q}{2-Q}} \sum_{j=1}^N \frac{X_j u X_j |\nabla_H u|}{|\nabla_H u|^3} \\ &= \frac{(Q-1)(2\theta-Q)}{(Q-2)^2} u^{\frac{2\theta-2}{2-Q}} + \frac{Q-1}{2-Q} u^{\frac{2\theta-Q}{2-Q}} \frac{\mathcal{L}u}{|\nabla_H u|^2} + \frac{2(Q-1)^2}{(Q-2)^2} u^{\frac{2\theta-2}{2-Q}}. \end{aligned} \quad (7.77)$$

For J_2 we have

$$J_2 = \sum_{j=1}^N X_j \left(u^{\frac{2\theta-2Q+2}{2-Q}} \frac{X_j |\nabla_H u|}{|\nabla_H u|^3} \right) = \frac{2\theta-2Q+2}{2-Q} u^{\frac{2\theta-Q}{2-Q}} \sum_{j=1}^N \frac{X_j u X_j |\nabla_H u|}{|\nabla_H u|^3}$$

$$\begin{aligned}
& + u^{\frac{2\theta-2Q+2}{2-Q}} \sum_{j=1}^N \frac{X_j^2 |\nabla_H u|}{|\nabla_H u|^3} - 3u^{\frac{2\theta-2Q+2}{2-Q}} \sum_{j=1}^N \frac{(X_j |\nabla_H u|)^2}{|\nabla_H u|^4} \\
& = \frac{2\theta - 2Q + 2}{2 - Q} \left(\frac{Q - 1}{Q - 2} \right) u^{\frac{2\theta-2}{2-Q}} + u^{\frac{2\theta-2Q+2}{2-Q}} \frac{\mathcal{L} |\nabla_H u|}{|\nabla_H u|^3} - 3u^{\frac{2\theta-2Q+2}{2-Q}} \frac{|\nabla_H |\nabla_H u||^2}{|\nabla_H u|^4},
\end{aligned} \tag{7.78}$$

where we have used (2.65) in the last equality.

Now, using the representations of J_1 and J_2 we rewrite (7.76) as follows

$$\begin{aligned}
& \sum_{j=1}^N X_j \left(\frac{d^{2\theta} X_j |\nabla_H d|}{|\nabla_H d|^3} \right) = (Q - 1)(3Q - 4) u^{\frac{2\theta-2}{2-Q}} + (2 - Q)(Q - 1) \frac{\mathcal{L} u}{|\nabla_H u|^2} \\
& + (Q - 2)^2 u^{\frac{2\theta-2Q+2}{2-Q}} |\nabla_H u|^{-4} (|\nabla_H u| \mathcal{L} |\nabla_H u| - 3 |\nabla_H |\nabla_H u||^2).
\end{aligned}$$

Plugging this into (7.75) we arrive at

$$\begin{aligned}
I_2 & = 2\gamma^2 b_X^2 \sum_{j=1}^N \int_{\mathbb{G}} |g(x)|^2 X_j \left(\frac{d^{2\theta} X_j |\nabla_H d|}{|\nabla_H d|^3} \right) dx \\
& = 2\gamma^2 b_X^2 (Q - 1)(3Q - 4) \int_{\mathbb{G}} u^{\frac{2\theta-2}{2-Q}} |g(x)|^2 dx \\
& + 2\gamma^2 b_X^2 (Q - 2)^2 \int_{\mathbb{G}} u^{\frac{2\theta-2Q+2}{2-Q}} |\nabla_H u|^{-4} (|\nabla_H u| \mathcal{L} |\nabla_H u| - 3 |\nabla_H |\nabla_H u||^2) |g(x)|^2 dx.
\end{aligned}$$

Noting $u = d^{2-Q}$ we obtain for I_2 that

$$\begin{aligned}
I_2 & = 2\gamma^2 b_X^2 (Q - 1)(3Q - 4) \int_{\mathbb{G}} d^{2\theta-2} |g(x)|^2 dx \\
& + 2\gamma^2 b_X^2 \int_{\mathbb{G}} \frac{d^{2\theta+2Q-2}}{|\nabla_H d|^4} (d^{1-Q} |\nabla_H d| \mathcal{L}(d^{1-Q} |\nabla_H d|) - 3 |\nabla_H (d^{1-Q} |\nabla_H d|)|^2) \\
& \quad \times |g(x)|^2 dx. \tag{7.79}
\end{aligned}$$

Finally, putting (7.74) and (7.79) in (7.72) and using (2.76) we obtain (7.27), as desired. $\square$

8

Conclusion and future work

In general, homogeneous Lie groups do not have to be stratified or even graded, so there may be no horizontal gradients nor homogeneous left-invariant hypoelliptic differential operators. Nevertheless, in order to obtain analogues of the classical functional inequalities on general homogeneous Lie groups, we can work with the radial derivative or the Euler operators, or integral inequalities, where instead of estimating a function by its gradient, we estimate the integral by the function itself. Moreover, it turns out that it is possible to obtain functional inequalities with the best constants in some cases, implying new results already in the anisotropic $\mathbb{R}^n$ case as well as the isotropic $\mathbb{R}^n$ case due to the freedom in the choice of any homogeneous quasi-norm. Consequently, using the obtained functional inequalities, we defined the Euler–Hilbert–Sobolev, Sobolev–Lorentz–Zygmund, Besov type, Morrey and generalised Morrey spaces on general homogeneous Lie groups, and provided their embedding properties.

The non-stratified graded Lie groups do not have a homogeneous sub-Laplacian nor Laplacian but they always have the so-called Rockland operators, which are left-invariant homogeneous hypoelliptic differential operators. Here, we managed to link the integral versions of Hardy inequal-

ities on homogeneous Lie groups to their hypoelliptic versions through the Riesz and Bessel kernels of the Rockland operators (hypoelliptic left-invariant homogeneous differential operators, following the Helffer-Nourrigat's resolution of the Rockland conjecture in the 80s). Consequently, this leads to general hypoelliptic versions of Hardy-Sobolev, Hardy-Littlewood-Sobolev, Caffarelli-Kohn-Nirenberg, Gagliardi-Nirenberg and other inequalities. Inequalities of such type are important by themselves but also play an important role in wider analysis, in particular in view of the seminal results of Rothschild and Stein¹⁶⁹ linking the analysis of hypoelliptic differential operators on nilpotent Lie groups to differential operators on manifolds. The approach developed in this thesis for obtaining hypoelliptic functional inequalities relies on establishing integral versions of Hardy inequalities on homogeneous Lie groups, for which we also found necessary and sufficient conditions for the weights for such inequalities to be true.

Furthermore, we discussed versions of some of the above inequalities in the setting of stratified Lie groups using the horizontal gradient and sub-Laplacian. There are three versions of Hardy type inequalities on stratified Lie groups available in the literature: with the Carnot–Carathéodory distance, and the fundamental solution of the sub-Laplacian $\mathfrak{L}$ and the Euclidean distance on the first stratum of the group. In this thesis, we discussed the last two cases. Moreover, using the factorization method of differential operators, inspired by the recent work of Gesztesy and Littlejohn⁸⁸ we obtained Hardy and Hardy-Rellich inequalities on general stratified Lie groups, which in particular, implies a new two-parameter estimate on $\mathbb{R}^n$ which can be regarded as a counterpart to the Gesztesy and Littlejohn estimate. Furthermore, we showed the Rellich inequalities with the horizontal weights for the sub-Laplacian and for sub-Laplacians with drift on stratified Lie groups, also with the weights associated to the fundamental solution of the sub-Laplacian. The obtained results are already new for the Laplacian in the usual Euclidean setting of $\mathbb{R}^n$, embedding the classical Rellich inequality into a family of Rellich inequalities with parameter dependent drifts.

In addition, using the obtained inequalities we also established Heisenberg-

Pauli-Weyl type uncertainty principles on stratified Lie groups for the horizontal gradient with the horizontal weight, and on graded Lie groups for general homogeneous invariant hypoelliptic differential (Rockland) operators, and on homogeneous Lie groups for the radial and Euler operators.

One possible future development of the subject is to extend the obtained results to more general settings. Since we know from the works¹⁸⁶ and¹⁹¹ that Theorems 4.1 and 4.3 hold on general metric measure spaces possessing polar decompositions, then by our method it might be possible to extend other functional inequalities to the settings possessing polar decompositions. For example, we have already succeeded to extend them to a general connected Lie group in¹⁹¹, and to Riemannian manifolds with negative curvature in¹⁹².

Another possible development is to work on applications of the obtained functional inequalities to hypoelliptic PDEs. In particular, they can be applied to higher order hypoelliptic equations, where the traditional heat kernel methods largely fail due to the heat kernel being no longer real-valued. For example, as we mentioned in Remark 6.14, the best constants of the Sobolev and Gagliardo-Nirenberg inequalities can be expressed in the variational form as well as in terms of the ground state solutions of the corresponding nonlinear hypoelliptic equations (see^{185, 190} and¹⁸⁹). Moreover, in¹⁸⁴ the authors use the Gagliardo-Nirenberg inequality to investigate global in time well-posedness for small data for the semilinear damped hypoelliptic wave equation. We also refer to¹⁸⁸ for the study of very weak solutions to hypoelliptic wave equations. Thus, since functional inequalities are usually useful in obtaining a priori estimates for the well-posedness analysis of PDEs, for future research it is natural direction to study hypoelliptic PDEs using the obtained hypoelliptic functional inequalities.

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