HIDDEN SYMMETRIES IN GENERAL RELATIVITY:
KILLING TENSORS AND ANISOTROPIC SPACE-TIMES

by

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To my parents

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ABSTRACT

The goal of this thesis is to study hidden symmetries in general relativity arising from an important class of tensors, namely Killing tensors, which give the simplest non-linear irreducible first integrals of the geodesic equations. An extensive study of some anisotropic static spherically symmetric space-times possessing Killing tensors is also made.

In the first chapter the formalism developed by Newman and Penrose is presented using the tetrad and spinor calculus. This formalism is widely used in the modern literature and in the present thesis; the most important theorems on Killing tensors, and the junction conditions for the Einstein tensor and the propagation of Killing vectors are written in terms of it. Killing spinors which arise naturally in this formalism gave new impulse to the study of Killing tensors in general relativity.

In the second chapter the theory of Killing vectors and tensors is presented as developed historically. Together with conformal Killing vectors and tensors and Killing-Yano tensors, which are also studied, they are related to first integrals of geodesics and solutions of the geodesic deviation equation. Their connection with the abelian separation of the Hamilton-Jacobi equation is emphasized. A new method of generating Killing tensors of high order is presented using the Poisson and Schouten brackets. The Lie algebra of Killing vectors is extended in this way to a Lie algebra of higher dimension.

In the third chapter the extension of the Lie algebra is generated in the case of Kimura space-times which possess quadratic Killing tensors. A major role in obtaining the results of this chapter is played by algebraic computing, so an introduction to it is presented and the packages used (SHEEP and STENSOR) are studied in detail. New programs for Killing's equations and for the generation of high order Killing tensors are given.

In the fourth chapter the anisotropic static spherically symmetric space-times of Kimura are studied. The Bowers-Liang extension of Bondi's analysis of static fluid spheres to the anisotropic case is applied to these space-times and their physical properties are listed and discussed. In particular, those metrics describing possible interiors for stars and their junction conditions are examined.

THESIS OUTLINE

This thesis deals with the problem of hidden symmetries in general relativity and, in particular, in anisotropic static spherically symmetric space-times, and is divided into four chapters. Each chapter has an introductory section of its own, while the results and conclusions are summarized in the final sections.

In the first chapter the Newman-Penrose formalism is studied and emphasis is put both on tetrad and spinor calculi which are presented in sections 1.2 and 1.3 respectively. The relationship between spinors and tensors is established and the geometric interpretation of a spinor is also given. In section 1.4 the Newman-Penrose equations are presented together with the Petrov classification and the theorem of Goldberg and Sachs. The various Newman-Penrose quantities are studied geometrically. In section 1.5 this formalism is used to define and discuss Killing spinors in relation to irreducible Killing tensors of order two.

The second chapter is devoted to the study of Killing vectors and tensors in general relativity. The basic definitions as well as the integrability conditions for Killing's equations are given in section 2.2. The significance of the existence of irreducible Killing tensors in space-times is discussed in section 2.3. Their importance arises from the fact that they give the simplest non-linear first integrals of the geodesic equations and identify solutions of the geodesic deviation equation. They can also be used to investigate the conditions under which the Hamilton-Jacobi equation is solved by separation of variables. The theoretical background required to understand the method of generating an extension of the Lie algebra of isometries is presented in section 2.4. The Schouten-Nijenhuis bracket, which is closely related to the Poisson bracket of functions in the cotangent bundle, is introduced. Kimura space-times and their Killing tensors of second order are presented in section 2.5, since they provide the specific examples for the generation of new Killing tensors of high order obtained by the above method.

The subject of algebraic computing, used to obtain important programs and results, is reviewed in the third chapter. The problems and requirements of constructing efficient computing packages are discussed in section 3.2. The computing language LISP and the important packages SHEEP, CLASSI and STENSOR, which are based on LISP and designed for research in general relativity, are studied in detail in sections 3.3 and 3.4. Two important applications are presented in section 3.5. In the first, specific programs are listed which can be used to write Killing's equations of first, second and third orders for any metric. In the second, a new program defining the Poisson bracket of two functions for the computer is given and also used to obtain the extension of the Lie algebra of Killing vectors in the case of Kimura metrics.

The fourth chapter discusses the anisotropic static spherically symmetric space-times with Killing tensors. A theoretical study of anisotropic fluid spheres in general relativity and their Bondi analysis are presented in section 4.2. The physical properties of Kimura fluid spheres are investigated in section 4.3 in relation to the existence of hidden symmetries. The stability conditions of those spheres describing possible interiors for stars are considered. Their junction conditions are examined in section 4.4 with the use of the Newman-Penrose formalism. And the reciprocal transformation of the static Kimura space-times as well as its result on their Killing tensors are studied in section 4.5.

Various subjects related to the text and listings of some computer programs and results are presented in the appendices.

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NOTATION

The notational conventions presented here are those used throughout the thesis, unless otherwise indicated. All symbols are explained in the text.

An n-dimensional manifold is denoted by M. If P is a point in M, T_PM and T^*_PM denote the tangent and cotangent spaces of M at P, while $TM = \bigcup_{P\in M} T_PM$ and $T^*M = \bigcup_{P\in M} T^*_PM$ are the tangent and cotangent bundles over M. In local coordinates a point $P\in M$ is represented by an ordered set of real numbers x^μ ($\mu = 1,\ldots,n$), while any vector $V\in T_P$ and any one-form $p\in T^*M$ are expressed as $V = V^\mu \frac{\partial}{\partial x^\mu}$ and $p = p_\mu dx^\mu$ respectively. The induced coordinates $\{z^A\}$ of TM are then defined by $\{z^A\} = \{x^\mu, V^\mu\}$ and the induced coordinates $\{z^A\}$ of T*M by $\{z^A\} = \{x^\mu, p_\mu\}$.

A metric with signature ---+ is used, although the metric signature +++- was found to be more convenient for the calculations in appendices A3 and A10.

Greek indices $\alpha,\beta,\gamma,\ldots$ take the values 1,2,3,4 and the values $1,\ldots,n$ in 4-dimensional and n-dimensional manifolds respectively, and are tensor indices. Latin indices a,b,c,... take the values 1,2,3,4 and are tetrad indices. Spinor indices are of two types. Capital Latin indices A,B,... are abstract spinor indices and do not take numerical values. Small Latin indices a,b,... are spinor dyad indices and take the values 0,1. The convention $\alpha \to AA'$, $\beta \to BB'$,... indicating the correspondence between tensor and spinor indices is adopted. The Einstein summation convention is assumed everywhere.

Symmetrization and antisymmetrization are denoted respectively by round and square brackets around indices.

Partial derivatives are denoted by ∂ or a comma in front of an index, while covariant derivatives are denoted by ∇ or a semicolon in front of an index. Lie derivatives with respect to a vector field $\underline{V}$ are denoted by $\mathcal{L}_{\underline{V}}$.

Geometrized units are used where the speed of light and the gravitational constant are set equal to one.

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CHAPTER 1

THE NEWMAN-PENROSE FORMALISM IN GENERAL RELATIVITY

1.1 INTRODUCTION

The formalism developed by Newman and Penrose [1], which is also called the method of spin coefficients, consists of the use of tetrad and spinor calculus for studying the Einstein equations and the equations of other fields [2]. The expanded system of equations connecting the spinor components of the curvature tensor with the spin coefficients (the components of spinor connection) has become known as the system of Newman-Penrose equations.

The language of spin coefficients has become generally accepted and widely used in the modern literature. The main reason for such popularity of the Newman-Penrose formalism consists in its adequacy and internal adaptability for studying the solutions of Einstein equations of one or another algebraic class, as well as the general properties of a gravitational field (in particular the asymptotic properties).

The property of a gravitational field to belong to one or another algebraic type is connected with the fulfilment of certain algebraic conditions on the spinor components of the Weyl tensor. The fact that such components appear linearly in the Newman-Penrose equations permits the successful use of these equations in the study of solutions of a certain algebraic type.

The Newman-Penrose equations have also turned out to be convenient for studying the asymptotic properties of gravitational fields. Using these equations together with the definition of asymptotic flatness [3], we can study the behaviour of Riemann and metric tensors at infinity.

The Newman-Penrose formalism has been particularly useful in the theoretical analysis of gravitational fields and in obtaining exact solutions of the field equations. It has found many applications in the study of shock waves, the study of black holes [4] or in solving the wave equations by separation of variables in the Kerr metric [5]. Recently, the junction conditions for Einstein equations as well as the conditions for the propagation of isometries in general relativity have been stated in terms of the Newman-Penrose variables [6].

1.2 TETRAD CALCULUS

1.2.1 The tetrad representation

General relativity is based on the concept of space-time, which is a 4-dimensional differentiable (C^∞ Hausdorff) manifold M endowed with a metric $g_{\mu\nu}$ which can be transformed to $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1,-1,-1,+1)$ at every point P of M.

In the standard tetrad formalism as appears in [5], one sets up, at each point of space-time, a basis of four contravariant vectors

$$e_a^\mu \quad (a = 1, 2, 3, 4) \tag{1.2.1}$$

where Latin and Greek indices are tetrad and tensor indices respectively. Associated with the contravariant vectors (1.2.1) we have the covariant vectors

$$e_{a\mu} = g_{\mu\nu}\, e_a^{\nu} \tag{1.2.2}$$

where $g_{\mu\nu}$ denotes the metric tensor. The inverse e_μ^a of the matrix e_a^μ is defined so that

$$e_a^\mu \; e_\mu^b = \delta_a^b \quad \text{and} \quad e_a^\mu \; e_\nu^a = \delta_\nu^\mu \tag{1.2.3}$$

where the summation convention is assumed. We also assume that

$$e_a^\mu \; e_{b\mu} = \eta_{ab} = \text{constant} \tag{1.2.4}$$

while η^{ab}, the inverse of the matrix η_{ab}, is defined so that

$$\eta^{ab}\; \eta_{bc} = \delta_c^a \; . \tag{1.2.5}$$

We have in addition the identities

$$\eta_{ab}\; e_\mu^a = e_{b\mu} \quad \text{and} \quad \eta^{ab}\; e_{a\mu} = e_\mu^b \tag{1.2.6}$$

and

$$e_{a\mu}\; e_\nu^a = g_{\mu\nu} \; . \tag{1.2.7}$$

Given any tensor, $T_{\mu\nu}$, we can project it onto the tetrad frame to obtain its tetrad components. Thus,

$$T_{ab} = e_a^\mu\, e_b^\nu\, T_{\mu\nu} \quad \text{and} \quad T_{\mu\nu} = e_\mu^a\, e_\nu^b\, T_{ab}. \tag{1.2.8}$$

1.2.2 Directional derivatives; Ricci rotation coefficients; Intrinsic derivative

The contravariant vectors e_a, considered as tangent vectors, define the directional derivatives

$$e_a = e_a^{\mu} \frac{\partial}{\partial x^{\mu}} \tag{1.2.9}$$

so for a scalar function φ we write

$$\varphi,a = e_a^{\mu} \frac{\partial \varphi}{\partial x^{\mu}} = e_a^{\mu} \varphi,_{\mu}. \tag{1.2.10}$$

More generally we write

$$T_{a,b} = e_b^{\mu} \frac{\partial}{\partial x^{\mu}} T_a = e_b^{\mu} \frac{\partial}{\partial x^{\mu}} (e_a^{\nu} T_{\nu}) = e_b^{\mu} (e_a^{\nu} T_{\nu}),_{\mu}$$

$$= e_b^{\mu} (e_a^{\nu} T_{\nu,\mu} + T_{\nu} e_{a,\mu}^{\nu}). \tag{1.2.11}$$

If we assume the coordinates are geodesic, partial derivatives can be replaced by covariant ones and we write

$$T_{a,b} = e_b^{\mu} (e_a^{\nu} T_{\nu;\mu} + T_{\nu} e_{a;\mu}^{\nu}) = e_b^{\mu} (e_a^{\nu} T_{\nu})_{;\mu}$$

$$= T_{\nu;\mu} e_a^{\nu} e_b^{\mu} + e_{a\nu;\mu} e_b^{\mu} e_c^{\nu} T^c. \tag{1.2.12}$$

Using the definition

$$\gamma_{cab} = e_c^{\nu}\, e_{a\nu;\mu} e_b^{\mu} \tag{1.2.13}$$

we rewrite (1.2.12) in the form

$$T_{a,b} = T_{\mu;\nu}\, e^{\mu}_{a}\, e^{\nu}_{b} + \gamma_{cab}\, T^{c} . \tag{1.2.14}$$

The quantities γ_{cab} are the so-called Ricci rotation-coefficients. They are antisymmetric in the first pair of indices, that is ,

$$\gamma_{bac} = - \gamma_{abc}. \tag{1.2.15}$$

Therefore, there are 24 independent rotation coefficients in our 4-dimensional space-time.

Defining

$$\lambda_{abc} = e_{b\mu,\nu}\, (e^{\mu}_{a}\, e^{\nu}_{c} - e^{\nu}_{a}\, e^{\mu}_{c}) \tag{1.2.16}$$

and rewriting it in the form

$$\lambda_{abc} = (e_{b\mu,\nu} - e_{b\nu,\mu})\, e^{\mu}_{a}\, e^{\nu}_{c} \tag{1.2.17}$$

we have, by replacing again the ordinary derivatives by covariant ones, that

$$\lambda_{abc} = (e_{b\mu;\nu} - e_{b\nu;\mu})\, e^{\mu}_{a}\, e^{\nu}_{c} = \gamma_{abc} - \gamma_{cba} . \tag{1.2.18}$$

Hence, we can write

$$\gamma_{abc} = \frac{1}{2}\, (\lambda_{abc} + \lambda_{cab} - \lambda_{bca}). \tag{1.2.19}$$

It can be seen from equations (1.2.16) and (1.2.19) that the evaluation of the rotation coefficients does not require the evaluation of covariant derivatives (and therefore of the Christoffel symbols). (1.2.14) can be rewritten as

$$T_{\mu;\nu}\, e^{\mu}_{a}\, e^{\nu}_{b} = T_{a,b} - \gamma_{cab} T^{c} = T_{a,b} - \eta^{nm}\, \gamma_{nab}\, T_{m}. \tag{1.2.20}$$

The quantity on the right-hand side of (1.2.20) is called the <u>intrinsic derivative</u> of T_a in the direction b and written $T_{a/b}$, thus

$$T_{a/b} = T_{\mu;\nu}\, e^{\mu}_{a}\, e^{\nu}_{b}. \tag{1.2.21}$$

The notion of intrinsic derivative can easily be extended. Thus, if $T_{\alpha\beta\gamma\delta}$ is a fourth-order tensor, we have

$$T_{abcd/m} = T_{\alpha\beta\gamma\delta;\mu}\, e^{\alpha}_{a}\, e^{\beta}_{b}\, e^{\gamma}_{c}\, e^{\delta}_{d}\, e^{\mu}_{m} . \tag{1.2.22}$$

Since

$$T_{abcd,m} = (T_{\alpha\beta\gamma\delta}\, e^{\alpha}_{a}\, e^{\beta}_{b}\, e^{\gamma}_{c}\, d^{\delta}_{d})_{;\mu}\, e^{\mu}_{m} \tag{1.2.23}$$

we also have

$$T_{abcd/m} = T_{abcd,m} - \eta^{pq}\, (\gamma_{pam}\, T_{qbcd} + \gamma_{pbm}\, T_{aqcd}$$

$$+ \gamma_{pcm}\, T_{abqd} + \gamma_{pdm}\, T_{abcq}) . \tag{1.2.24}$$

We can use the Ricci rotation coefficients when studying the <u>commutator</u> $[e_a, e_b]$, which, being a tangent vector itself, can be expanded in terms of the same basis $\{e_a\}$ as follows

$$[e_a, e_b] = C^c_{ab}\ e_c \tag{1.2.25}$$

functions

with C^c_{ab} the structure constants. Considering the effect of this commutator on a scalar function f and using (1.2.9) and (1.2.13), we find that

$$C^c_{ab} = \gamma^c_{ba} - \gamma^c_{ab} \ . \tag{1.2.26}$$

<u>The Ricci and Bianchi identities</u> can also be written in terms of the Ricci rotation-coefficients. Projecting the Ricci identity

$$e_{a\mu;\alpha\beta} - e_{a\mu;\beta\alpha} = R^{\nu}_{\mu\alpha\beta}\ e_{a\nu} \tag{1.2.27}$$

where $R_{\nu\mu\alpha\beta}$ is the Riemann tensor, onto the tetrad frame we have

$$R_{abcd} = R_{\nu\mu\alpha\beta}\ e^{\nu}_a\ e^{\mu}_b\ e^{\alpha}_c\ e^{\beta}_d \ . \tag{1.2.28}$$

Then

$$R_{abcd} = e^{\mu}_b\ e^{\alpha}_c\ e^{\beta}_d\ \ (e_{a\mu;\alpha\beta} - e_{a\mu;\beta\alpha}) \ . \tag{1.2.29}$$

From the definition of Ricci rotation-coefficients

$\gamma_{paq} = e^{\mu}_p e_{a\mu;\alpha} e^{\alpha}_q$ we have

$$e_{a\mu;\alpha} = \gamma_{paq}\ e^p_{\mu}\ e^q_{\alpha} \tag{1.2.30}$$

hence

$$e_{a\mu;\alpha\beta} = \gamma_{paq;\beta}\, e^p_\mu\, e^q_\alpha + \gamma_{paq}\, e^p_{\mu;\beta}\, e^q_\alpha$$

$$+ \gamma_{paq}\, e^p_\mu\, e^q_{\alpha;\beta} \,. \tag{1.2.31}$$

Using (1.2.31) we have from (1.2.29) that

$$R_{abcd} = - \gamma_{abc;d} + \gamma_{abd;c} + \gamma_{abp}(\gamma^p_{cd} - \gamma^p_{dc})$$

$$+ \gamma_{apc}\, \gamma^p_{bd} - \gamma_{apd}\, \gamma^p_{dc} \,. \tag{1.2.32}$$

The relationship between the Riemann tensor, Weyl tensor and Ricci tensor goes over in tetrad formalism unchanged, that is,

$$R_{abcd} = C_{abcd} + \frac{1}{2}\,(\eta_{ac}\, R_{bd} - \eta_{bc}\, R_{ad} - \eta_{ad}\, R_{bc} + \eta_{bd}\, R_{ac})$$

$$- \frac{R}{6}\,(\eta_{ac}\, \eta_{bd} - \eta_{ad}\, \eta_{bc}). \tag{1.2.33}$$

Finally, the Bianchi identities

$$R_{\alpha\beta[\gamma\delta;\mu]} = 0 \tag{1.2.34}$$

in terms of intrinsic derivatives take the form

$$R_{ab[cd/m]} = 0 \tag{1.2.35}$$

or

$$\sum_{[cdm]} \left\{ R_{abcd,m} - \eta^{pq}(\gamma_{pam} R_{qbcd} + \gamma_{pbm} R_{aqcd} + \gamma_{pcm} R_{abqd} + \gamma_{pdm} R_{abcq}) \right\} . \tag{1.2.36}$$

1.2.3 The complex null tetrads and the spin coefficients

One of the ways to obtain information on the structure of a gravitational field is by studying the behaviour of different test particles in the field. In particular, when massless particles are considered (photons, neutrinos) the trajectories corresponding to them are null curves. Thus it is important to study the geometric properties of congruences* of such null curves**. It turns out to be convenient to introduce a suitably chosen field of tetrads which will be adapted to the given congruence [7].

We construct at an arbitrary point P of space-time a null vector ℓ^{μ} which is tangent to a null curve of the congruence passing through P. Any other null vector n^{μ} distinct from ℓ^{μ} and directed into the future is linearly independent of ℓ^{μ} and $n^{\mu}\ell^{\mu} > 0$. The pair of the real null vectors ℓ^{μ}, n^{μ} determines

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* The term "congruence" is used in the present thesis to indicate a family of curves such that there exists an element of the family passing through an arbitrarily given nonsingular point of space.

** There exists an important property which differentiates null curves from space-like and time-like curves. The point is that although the null curves form a submanifold of a space possessing a metric, the notion of distance along them is not defined. The parallel transport of vectors along null curves is induced by imbedding them into space-time and thus, since they have no metric properties, the null curves are one-dimensional affine spaces.

a two-dimensional subspace orthogonal to them (Fig. 1.1). We introduce an orthonormal basis in this subspace consisting of the pair of space-like vectors a^μ, b^μ

$$a_\mu a^\mu = -1; \quad b_\mu b^\mu = -1; \quad a_\mu b^\mu = 0 \qquad (1.2.37)$$

and thereby obtain a basis (ℓ^μ, n^μ, a^μ, b^μ) in the tangent space.

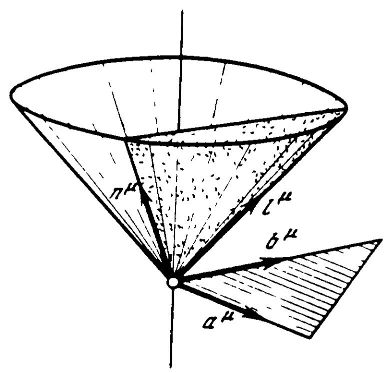

Figure 1.1. Construction of a null tetrad (Frolov, 1979).

But it is more convenient to work not with this basis but with a complex modification of it. So, putting

$$m^\mu = \frac{1}{\sqrt{2}}(a^\mu + i b^\mu) \quad \text{and} \quad \bar{m}^\mu = \frac{1}{\sqrt{2}}(a^\mu - i b^\mu) \qquad (1.2.38)$$

we define a tetrad

$$Z^\mu_m = (\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu) \qquad (m = 1,2,3,4) \qquad (1.2.39)$$

with ℓ^μ, n^μ being real null vectors and m^μ and its complex conjugate $\bar{m}^\mu$ being complex null vectors. The orthogonality properties of these vectors are

$$\left.\begin{aligned} \ell_\mu \ell^\mu &= m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = n_\mu n^\mu = 0 \\ \ell_\mu n^\mu &= - m_\mu \bar{m}^\mu = 1 \\ \ell_\mu m^\mu &= \ell_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0 \end{aligned}\right\} \quad . \tag{1.2.40}$$

This tetrad is normalized by the condition

$$Z_m^\mu \, Z_{n\mu} = \eta_{mn} \tag{1.2.41}$$

where η_{mn} is the flat-space metric

$$\eta_{mn} = \eta^{mn} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \tag{1.2.42}$$

Such tetrads are called <u>complex null tetrads</u>. The following relations are easily seen to be true

$$g_{\mu\nu} = Z_{m\mu} Z_{n\nu} \eta^{mn} = \ell_\mu n_\nu + n_\mu \ell_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu \, m_\nu \tag{1.2.43}$$

$$\eta_{mn} = Z_{m\mu} \, Z_{n\nu} \, g^{\mu\nu} . \tag{1.2.44}$$

Complex Ricci coefficients are now defined by

$$\gamma_m^{\ np} = Z_{m\mu;\nu} \quad Z^{n\mu} Z^{p\nu} \tag{1.2.45}$$

again with the symmetry $\gamma^{mnp} = - \gamma^{nmp}$.

The null basis, which underlies the Newman-Penrose formalism, can be derived from an orthonormal basis $(\underline{e}_t, \underline{e}_x, \underline{e}_y, \underline{e}_z)$, where $\underline{e}_t$ is time-like and $\underline{e}_x, \underline{e}_y, \underline{e}_z$ are space-like as follows

$$Z_1^{\mu} = \ell^{\mu} = \frac{1}{\sqrt{2}} (\underline{e}_t + \underline{e}_z)$$

$$Z_2^{\mu} = n^{\mu} = \frac{1}{\sqrt{2}} (\underline{e}_t - \underline{e}_z)$$

$$Z_3^{\mu} = m^{\mu} = \frac{1}{\sqrt{2}} (\underline{e}_x + i\underline{e}_y)$$

$$Z_4^{\mu} = \bar{m}^{\mu} = \frac{1}{\sqrt{2}} (\underline{e}_x - i\underline{e}_y) .$$

The basis vectors considered as directional derivatives are designated by special symbols

$$\left.\begin{aligned} \underline{D} &= \ell^{\mu}\partial_{\mu} = \ell^{\mu}\nabla_{\mu} \\ \underline{\Delta} &= n^{\mu}\partial_{\mu} = n^{\mu}\nabla_{\mu} \\ \underline{\delta} &= m^{\mu}\partial_{\mu} = m^{\mu}\nabla_{\mu} \\ \underline{\bar{\delta}} &= \bar{m}^{\mu}\partial_{\mu} = \bar{m}^{\mu}\nabla_{\mu} . \end{aligned}\right\} \qquad (1.2.46)$$

The various Ricci rotation coefficients, now called <u>spin coefficients</u>, are also designated by special symbols

$$\gamma_{131} = \varkappa \qquad (1.2.47a)$$

$$\gamma_{132} = \tau \qquad (1.2.47b)$$

$$\gamma_{133} = \sigma \qquad (1.2.47c)$$

$$\gamma_{134} = \rho \qquad (1.2.47d)$$

$$\gamma_{241} = -\pi \qquad (1.2.47e)$$

$$\gamma_{242} = -\nu \qquad (1.2.47f)$$

$$\gamma_{243} = -\mu \qquad (1.2.47g)$$

$$\gamma_{244} = -\lambda \qquad (1.2.47h)$$

$$\frac{1}{2}(\gamma_{121} - \gamma_{341}) = \varepsilon \qquad (1.2.47i)$$

$$\frac{1}{2}(\gamma_{122} - \gamma_{342}) = \gamma \qquad (1.2.47j)$$

$$\frac{1}{2}(\gamma_{123} - \gamma_{343}) = \beta \qquad (1.2.47k)$$

$$\frac{1}{2}(\gamma_{124} - \gamma_{344}) = \alpha \ . \qquad (1.2.47\ell)$$

The name "spin coefficients" indicates that these quantities arise in a natural way in the spinor formalism, as we shall see later.

1.3. SPINOR CALCULUS

1.3.1 Spinors and tensors

The fundamental fact, on which the spinor formalism is based, is the local isomorphism between the proper Lorentz group and the group SL(2,C) of unimodular complex 2 X 2 matrices (that is, matrices with determinant equal to one). More precisely, the group SL(2,C) is a twofold covering group of the Lorentz group. Unimodular matrices may also be viewed as linear operators in the two-dimensional complex linear space $\mathbf{C}^2$. We may therefore assume that an arbitrary Lorentz transformation is associated with a certain transformation in $\mathbf{C}^2$ (precisely, two transformations). The corresponding representation of the Lorentz group is called a spin representation, and the elements (complex vectors) of the linear space $\mathbf{C}^2$ (the spinor space) are called spinors [8].

Spinors are denoted, in what follows, by small Greek letters and spinor indices by capital Latin letters which do not take numerical values, for example ξ^A, η^B, κ^C etc.

As in tensor calculus, upper indices are called contravariant and lower indices are called covariant. Any tensor expression can be translated into a spinor form, each four-dimensional tensor index being replaced by a pair of two-dimensional spinor indices, one unprimed and one primed. Spinors with any number of indices, primed or unprimed, can be defined. A spinor with p indices of one kind is called a p-spinor, a spinor with p indices of one kind and q indices of the other kind is called a pq-spinor [9].

The connection between tensors and spinors is achieved by means of the quantities $\sigma^{\alpha}_{AA'}$, introduced by Infeld and Van der Waerden [10], satisfying

$$g_{\alpha\beta}\,\sigma^{\alpha}_{AA'}\,\sigma^{\beta}_{BB'} = \varepsilon_{AB}\,\varepsilon_{A'B'}\ . \tag{1.3.1}$$

The connecting quantities $\sigma^{\alpha}_{AA'}$ transform as space-time vectors on the index α and as spinors on the indices A,A'; they are Hermitian, that is,

$$\sigma_{\alpha}^{AA'} = \bar{\sigma}_{\alpha}^{\ AA'} \tag{1.3.2}$$

and they satisfy the relations

$$\sigma_{\alpha}^{AA'}\,\sigma^{\alpha}_{BB'} = \delta^{A}_{B}\,\delta^{A'}_{B'} \tag{1.3.3}$$

$$\sigma^{\alpha}_{AA'}\,\sigma_{\beta}^{AA'} = \delta^{\alpha}_{\beta} \tag{1.3.4}$$

$$\sigma_{\alpha}^{AA'}\,\sigma^{\alpha}_{BB'}\,\sigma_{\beta}^{BB'} = \sigma_{\beta}^{AA'}\quad * \ . \tag{1.3.5}$$

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* One set of σ's that satisfy these conditions is (apart from a factor) the set of Pauli matrices in a locally Cartesian coordinate frame

$$\sigma_1^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_2^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$\sigma_3^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \sigma_4^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ .$$

The 'ε' is the Levi-Civita alternating symbol in two dimensions defined by

$$\varepsilon^{AB} = \varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.3.6}$$

so that $\varepsilon_{01} = \varepsilon_{0'1'} = \varepsilon^{01} = \varepsilon^{0'1'} = 1$.

It is used for lowering and raising spinor indices as follows

$$\begin{aligned} \xi^A &= \varepsilon^{AB}\xi_B; \qquad & \xi_B &= \xi^A\varepsilon_{AB}; \\ \eta^{A'} &= \varepsilon^{A'B'}\eta_{B'}; \qquad & \eta_{B'} &= \eta^{A'}\varepsilon_{A'B'} . \end{aligned} \tag{1.3.7}$$

Because of the antisymmetry $\varepsilon_{AB} = -\varepsilon_{BA}$, $\varepsilon^{AB} = -\varepsilon^{BA}$ we must be very careful about orderings when we raise or lower indices, so note that

$$\xi_A\eta^A = \xi^B\varepsilon_{BA}\eta^A = -\eta^A\varepsilon_{AB}\xi^B = -\eta_A\xi^A . \tag{1.3.8}$$

The correspondence between spinors and tensors is established as follows: to every tensor of type $\binom{p}{q}$, $X^{\alpha\beta\cdots\cdot}_{\gamma\cdots\cdots}$ is associated a spinor of type $\binom{pp'}{qq'}$ defined by

$$X^{\alpha\beta}_{\gamma} \leftrightarrow X^{AA'BB'}_{CC'} = \sigma^{AA'}_{\alpha} \; \sigma^{BB'}_{\beta} \; X^{\alpha\beta}_{\gamma} \; \sigma^{\gamma}_{CC'} \tag{1.3.9}$$

and inversely

$$X^{\alpha\beta}_{\gamma} = \sigma^{\alpha}_{AA'}\,\sigma^{\beta}_{BB'}X^{AA'BB'}_{CC'}\,\sigma^{CC'}_{\gamma} \quad . \tag{1.3.10}$$

The spinor so formed is called the spinor equivalent of $X^{\alpha\beta}_{\gamma}$. So, in fact, equation (1.3.1) tells us that $\varepsilon_{AB}\varepsilon_{A'B'}$ is the spinor equivalent of $g_{\alpha\beta}$.

Note that when taking the complex conjugate of a spinor unprimed indices become primed and primed indices become unprimed. For example, the complex conjugate of $X^{AA'BB'}_{CC'}$ is $\bar{X}^{A'AB'B}_{C'C}$ whence the condition for $X^{\alpha\beta}_{\gamma}$ to be real is the Hermitian property

$$X^{AA'BB'}_{CC'} = \bar{X}^{A'AB'B}_{C'C} \quad . \tag{1.3.11}$$

Finally, it is worth mentioning that the first impression of complication, in spinor equations, is deceptive and it will turn out that the spinor description of various tensor quantities is extremely convenient for the study of many problems in the theory of general relativity.

1.3.2. Spinor expressions for certain tensors

In this section we consider the question of how various characteristics of vectors and tensors carry over to their spinor analogs [7].

Proposition 1: A spinor $\varkappa_A$ uniquely determines a real null vector by the relation

$$K_{\alpha} = \sigma^{AA'}_{\alpha}\,\varkappa_A\bar{\varkappa}_{A'} \quad . \tag{1.3.12}$$

Distinct spinors define the same null vector K_α if and only if they differ by a complex factor with the modulus equal to one. For example, the spinors $\varkappa_A$ and $\varkappa_A e^{i\vartheta}$, with ϑ real, define the same null vector.

<u>Proposition 2:</u> A real bivector, that is, a skew-symmetric two-index tensor, has the following spinor representation

$$F_{\alpha\beta} \leftrightarrow F_{AA'BB'} = \varepsilon_{AB}\bar{\Phi}_{A'B'} + \varepsilon_{A'B'}\Phi_{AB} \qquad (1.3.13)$$

where Φ_{AB} is a symmetric spinor $\Phi_{AB} = \Phi_{(AB)}$. Thus, a spinor $\varkappa_A$ also determines uniquely a bivector $F_{\alpha\beta}$ by the relation

$$F_{\alpha\beta} = \sigma_\alpha^{AA'}\sigma_\beta^{BB'}\ (\varepsilon_{AB}\bar{\varkappa}_{A'}\bar{\varkappa}_{B'} + \varepsilon_{A'B'}\varkappa_A\varkappa_B)\ . \qquad (1.3.14)$$

(1.3.14) follows from (1.3.13) if we put $\Phi_{AB} = \varkappa_A\varkappa_B$.

<u>Proposition 3:</u> The Levi-Civita tensor $\varepsilon_{\alpha\beta\gamma\delta}$ has the following spinor representation

$$\varepsilon_{\alpha\beta}^{\gamma\delta} \leftrightarrow \varepsilon_{AA'BB'}^{CC'DD'} = i(\delta_A^C\delta_B^D\delta_{A'}^{D'}\delta_{B'}^{C'} - \delta_A^D\delta_B^C\delta_{A'}^{C'}\delta_{B'}^{D'})\ . \qquad (1.3.15)$$

<u>Proposition 4:</u> A tensor $R_{\alpha\beta\gamma\delta}$ satisfying the symmetry properties

$$R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]} = R_{[\gamma\delta][\alpha\beta]} \text{ and } R_{\alpha[\beta\gamma\delta]} = 0$$

analogous to those of the curvature tensor, has the following spinor representation

$$R_{\alpha\beta\gamma\delta} \leftrightarrow R_{AA'BB'CC'DD'} = \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{A'B'C'D'}$$

$$+ \Phi_{ABC'D'}\varepsilon_{CD}\varepsilon_{A'B'} + \Phi_{CDA'B'}\varepsilon_{AB}\varepsilon_{C'D'}$$

$$+ 2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{AB}\varepsilon_{CD}\varepsilon_{A'D'}\varepsilon_{B'C'})$$

(1.3.16)

where Ψ_{ABCD} is a symmetric spinor $\Psi_{ABCD} = \Psi_{(ABCD)}$ and $\Phi_{ABC'D'}$ is a Hermitian spinor symmetric with respect to every pair of indices $\Phi_{ABC'D'} = \overline{\Phi}_{C'D'AB} = \Phi_{(AB)(C'D')}$.

Proposition 5: If the curvature tensor $R_{\alpha\beta\gamma\delta}$ has the spinor representation of proposition 4, then the Ricci tensor, the scalar curvature and the Weyl tensor have the following spinor representations respectively

$$R_{\alpha\beta} \leftrightarrow R_{AA'BB'} = - 2\Phi_{ABA'B'} + 6\Lambda\varepsilon_{AB}\varepsilon_{A'B'} \qquad (1.3.17)$$

$$R = 24\Lambda \qquad (1.3.18)$$

$$C_{\alpha\beta\gamma\delta} \leftrightarrow C_{AA'BB'CC'DD'} = \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{A'B'C'D'} \cdot$$

(1.3.19)

1.3.3 Geometric interpretation of a spinor

The question to be answered here is how one can visualize a spinor [9]. As we have already seen in section 1.3.2, a spinor $\varkappa^A$ uniquely determines a real null vector K^α by relation (1.3.12) and also a bivector $F_{\alpha\beta}$ by relation (1.3.14). In order to simplify

(1.3.14) we introduce a spinor η^A which forms a basis with $\varkappa^A$ by $\varkappa_A \eta^A = 1$; then it follows that $\varepsilon_{AB} = \varkappa_A \eta_B - \eta_A \varkappa_B$.
Now, if we substitute ε_{AB} in (1.3.14) we have

$$F_{\alpha\beta} = 2K_{[\alpha}W_{\beta]} \tag{1.3.20}$$

where K_α is given by (1.3.12) and W_β is a space-like vector, given by

$$W_\beta = \sigma_\beta^{BB'}\ (\eta_B \bar{\varkappa}_{B'} + \bar{\eta}_{B'} \varkappa_B) \tag{1.3.21}$$

and satisfying $W_\beta K^\beta = 0$.

Without violating the condition $\varkappa_A \eta^A = 1$, we may transform the spinor

$$\eta^A \to \eta^A + \lambda \varkappa^A \tag{1.3.22}$$

and then the vector W_β is determined up to a transformation

$$W_\beta \to \hat{W}_\beta = W_\beta + (\lambda + \bar{\lambda})K_\beta . \tag{1.3.23}$$

$F_{\alpha\beta}$ is unaffected by the transformation (1.3.22) since by definition it is independent of η^A.

It follows that an arbitrary spinor $\varkappa^A$ determines the 2-space which is spanned by K^α and any W^β . This space can be interpreted as a <u>flag</u>. The direction of the flagpole is the direction of the null vector K^α and the position of the flag (the two-dimensional area of the flag) is determined by $F_{\alpha\beta}$ (Fig. 1.2).

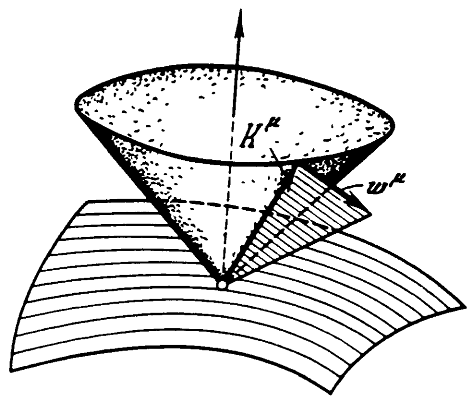

Figure 1.2. Geometric interpretation of a spinor (Frolov, 1979).

The change $\varkappa^A \to e^{i\vartheta}\varkappa^A$ of the spinor $\varkappa^A$ and the corresponding change $\eta^A \to e^{-i\vartheta}\eta^A$ of the spinor η^A produce the following change in W_α

$$W_\alpha \to W_\alpha \cos 2\vartheta + V_\alpha \sin 2\vartheta \; *$$

---

* Proof:

$W_\alpha = \sigma^{AA'}(\eta_A\bar{\varkappa}_{A'} + \bar{\eta}_{A'}\varkappa_A) =$

$\sigma_\alpha^{AA'} \{(e^{-i\vartheta}\eta_A)(e^{-i\vartheta}\bar{\varkappa}_{A'}) + (e^{i\vartheta}\bar{\eta}_{A'})(e^{i\vartheta}\varkappa_A)\} =$

$\sigma_\alpha^{AA'} (e^{-2i\vartheta}\eta_A\bar{\varkappa}_{A'} + e^{2i\vartheta}\bar{\eta}_{A'}\varkappa_A) =$

$\sigma_\alpha^{AA'} \{(\cos 2\vartheta - i \sin 2\vartheta)\eta_A\bar{\varkappa}_{A'} + (\cos 2\vartheta + i \sin 2\vartheta)\bar{\eta}_{A'}\varkappa_A\} =$

$\sigma_\alpha^{AA'} \{\cos 2\vartheta(\eta_A\bar{\varkappa}_{A'} + \bar{\eta}_{A'}\varkappa_A) + i \sin 2\vartheta(\bar{\eta}_{A'}\varkappa_A - \eta_A\bar{\varkappa}_{A'})\} =$

$W_\alpha \cos 2\vartheta + V_\alpha \sin 2\vartheta$,

where $V_\alpha = i(\varkappa_A\bar{\eta}_{A'} - \eta_A\bar{\varkappa}_{A'})$.

so that the change of phase ϑ turns the flag round the flagpole through an angle 2ϑ. Note that for $\vartheta = \pi$, the spinor κ^A changes sign, but flag and flagpole are unchanged, that is, κ^A and $-\kappa^A = e^{i\pi}\kappa^A$ define the same null flag. This ambivalence of sign is related to the fact that spin transformations yield a double-valued representation of Lorentz transformations.

1.3.4 Spinor analysis

The covariant derivative $\nabla_\mu \xi_A$ of the spinor ξ_A is defined by

$$\nabla_\mu \xi_A = \partial_\mu \xi_A - \xi_B \Gamma^B_{A\mu} \tag{1.3.24}$$

where $\Gamma^B_{A\mu}$ is the spinor affine connection [1] and [7]. The corresponding quantity $\bar{\Gamma}^{B'}_{A'\mu}$ deals with the primed indices.

The rules of covariant differentiation for spinor indices are exactly analogous to those for tensor indices. The choice of $\Gamma^B_{A\mu}$ is fixed by the requirement that the covariant derivatives of $\sigma^\mu_{AA'}$, ε_{AB}, $\varepsilon_{A'B'}$ shall vanish. It is important to notice here that, by (1.3.1), the four expressions

$$\sigma^\mu_{00'}, \quad \sigma^\mu_{11'}, \quad \sigma^\mu_{01'}, \quad \sigma^\mu_{10'} \tag{1.3.25}$$

satisfy the same orthogonality relations (1.2.40) as the four vectors ℓ^μ, n^μ, m^μ, $\bar{m}^\mu$. We would like, therefore, to identify the expressions (1.3.25) as a convenient tetrad. However, this would not be strictly accurate, since the expressions (1.3.25) do not really denote vectors as they stand and also although the covariant derivative of $\sigma^{\mu}_{AA'}$ is zero, this is not so for ℓ^{μ}, n^{μ}, m^{μ}, $\bar{m}^{\mu}$.

To get around this difficulty we introduce two basis spinors o^A, ι^A, a "dyad" [1], normalized thus

$$o_A \, \iota^A = \varepsilon_{AB} \, o^A \, \iota^B = - \iota_A \, o^A = 1 \tag{1.3.26}$$

and satisfying $\varepsilon^{AB} = o^A \iota^B - \iota^A o^B$.

A dyad in spin space is analogous to a tetrad in vector space. Moreover, given a dyad in spin space there corresponds a unique complex null tetrad to it defined by the relations

$$\left.\begin{aligned}
Z^{\mu}_1 &\equiv \ell^{\mu} = \sigma^{\mu}_{AA'} \, o^A \, \bar{o}^{A'} && (= \sigma^{\mu}_{00'}) \\
Z^{\mu}_2 &\equiv n^{\mu} = \sigma^{\mu}_{AA'} \, \iota^A \, \bar{\iota}^{A'} && (= \sigma^{\mu}_{11'}) \\
Z^{\mu}_3 &\equiv m^{\mu} = \sigma^{\mu}_{AA'} \, o^A \, \bar{\iota}^{A'} && (= \sigma^{\mu}_{01'}) \\
Z^{\mu}_4 &\equiv \bar{m}^{\mu} = \sigma^{\mu}_{AA'} \, \iota^A \, \bar{o}^{A'} && (= \sigma^{\mu}_{10'}) \ .
\end{aligned}\right\} \tag{1.3.27}$$

It is this correspondence that leads to the close connection between spinor methods and the complex null tetrad formalism, already presented.

If $(t^{\mu}, x^{\mu}, y^{\mu}, z^{\mu})$ is a standard restricted (that is, right-handed and isochronous) Minkowski tetrad, we can write [11]

$$t^{\mu} = \frac{1}{\sqrt{2}} (\ell^{\mu} + n^{\mu}) = \frac{1}{\sqrt{2}} (o^{A}\bar{o}^{A'} + \iota^{A}\bar{\iota}^{A'})$$

$$x^{\mu} = \frac{1}{\sqrt{2}} (m^{\mu} + \bar{m}^{\mu}) = \frac{1}{\sqrt{2}} (o^{A}\bar{\iota}^{A'} + \iota^{A}\bar{o}^{A'})$$

$$y^{\mu} = \frac{i}{\sqrt{2}} (m^{\mu} - \bar{m}^{\mu}) = \frac{i}{\sqrt{2}} (o^{A}\bar{\iota}^{A'} - \iota^{A}\bar{o}^{A'})$$

$$z^{\mu} = \frac{1}{\sqrt{2}} (\ell^{\mu} - n^{\mu}) = \frac{1}{\sqrt{2}} (o^{A}\bar{o}^{A'} - \iota^{A}\bar{\iota}^{A'}) .$$

The geometrical relation between (o^{A}, ι^{A}) and $(t^{\mu}, x^{\mu}, y^{\mu}, z^{\mu})$ can be seen in Figure 1.3.

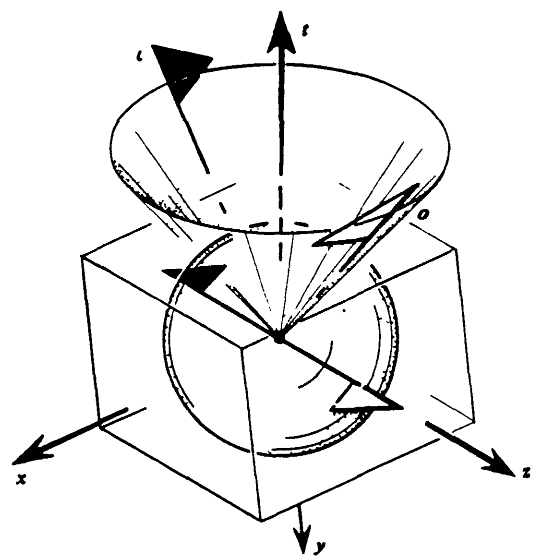

Figure 1.3. Relation between a spin frame (o, ι) and a restricted Minkowski tetrad (t,x,y,z) (Penrose, 1986).

As with the tetrads, it is convenient to have a generic symbol for both o^{A} and ι^{A}. Define ζ_{a}^{A}, $\bar{\zeta}_{a'}^{A'}$ by

$$\zeta_{0}^{A} = o^{A}; \quad \zeta_{1}^{A} = \iota^{A}; \quad \bar{\zeta}_{0'}^{A'} = \bar{o}^{A'}; \quad \bar{\zeta}_{1'}^{A'} = \bar{\iota}^{A'}. \qquad (1.3.28)$$

Then, for example, given a spinor $\psi_{AB'C}$ we can define its dyad components by

$$\psi_{ab'c} = \psi_{AB'C}\, \zeta^{A}_{a}\, \bar{\zeta}^{B'}_{b'}\, \zeta^{C}_{c} \, . \tag{1.3.29}$$

For ε_{AB} we have

$$\varepsilon_{ab} = \varepsilon_{AB}\, \zeta^{A}_{a}\, \zeta^{B}_{b} \, . \tag{1.3.30}$$

The lower-case indices behave in the same way algebraically as ordinary (capital) spinor indices, but when covariant differentiation is applied, no term involving an affine connection appears for the lower case indices.

Note that if we choose the components of ζ^{A}_{a} to be the Kronecker delta, then the dyad components of any spinor will be identical with its spinor components.

The expression

$$\sigma^{\mu}_{aa'} = \sigma^{\mu}_{AA'}\zeta^{A}_{a}\bar{\zeta}^{A'}_{a'} \tag{1.3.31}$$

now gives us ℓ^{μ}, n^{μ}, m^{μ}, $\bar{m}^{\mu}$ as aa' take the values 00', 11', 01', 10' respectively, by (1.3.27). With this interpretation the expressions (1.3.25) may indeed be thought of as giving the required tetrad. Also, (1.3.24) now gives

$$\nabla_{\mu}\zeta_{aA} = -\,\zeta_{aC}\Gamma^{C}{}_{A\mu} = \zeta_{a}{}^{C}\Gamma_{CA\mu} = \Gamma_{aA\mu} . \tag{1.3.32}$$

The covariant differentiation operator ∇_μ introduced above takes spinor quantities into geometric objects for which an additional tensor index μ appears along with the spinor indices. We can easily define instead of this operator a covariant differentiation operator $\nabla_{AA'}$ taking spinors of type $\begin{pmatrix} pp' \\ qq' \end{pmatrix}$ into spinors of type $\begin{pmatrix} pp' \\ q+1 \;\; q'+1 \end{pmatrix}$ by the following relation

$$\nabla_{AA'} = \sigma^{\mu}_{AA'} \nabla_\mu \tag{1.3.33}$$

or in the basis ζ^A_a

$$\nabla_{aa'} = \sigma^{\mu}_{aa'} \nabla_\mu \tag{1.3.34}$$

where

$$\nabla_{aa'} = \zeta^A_a \, \bar{\zeta}^{A'}_{a'} \, \nabla_{AA'} \tag{1.3.35}$$

and the operators $\nabla_{aa'}$ are covariant differentiation operators along the null directions defined by the vectors of the tetrad $(\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu)$.

The derivative operator $\nabla_{AA'}$ is linear, satisfies the Leibnitz rule, commutes with contraction and in addition satisfies two further conditions, namely [12]

i) for any spinor η we have $(\overline{\nabla\eta}) = \nabla\bar{\eta}$, that is $\nabla_{AA'}$ is real;

ii) $\nabla_{AA'} \, \varepsilon_{BC} = 0$ *. (1.3.36)

---

* Note that we can also define an ordinary differentiation operator $\partial_{AA'}$ by $\partial_{AA'} = \sigma^{\mu}_{AA'} \partial_\mu$ satisfying all the above properties.

Then we can introduce the following notation

$$D = \ell^{\mu}\nabla_{\mu} = \nabla_{00'}; \qquad \Delta = n^{\mu}\nabla_{\mu} = \nabla_{11'};$$
$$\delta = m^{\mu}\nabla_{\mu} = \nabla_{01'}; \qquad \bar{\delta} = \bar{m}^{\mu}\nabla_{\mu} = \nabla_{10'} . \tag{1.3.37}$$

For the analog of the Ricci rotation coefficients we introduce the following quantities possessing only spinor indices

$$\Gamma_{abcd'} = \zeta_{aA;\mu}\, \zeta^{A}_{b}\, \sigma^{\mu}_{cd'} \tag{1.3.38}$$

with the symmetry

$$\Gamma_{abcd'} = \Gamma_{bacd'} . \tag{1.3.39}$$

The quantities (1.3.38) can be expressed directly in terms of derivatives of $\sigma_{\mu ab'}$ as follows

$$\Gamma_{abcd'} = \frac{1}{2}\, \varepsilon^{p'q'}(\sigma_{cd'ap'bq'} - \sigma_{cd'bq'ap'} - \sigma_{ap'bq'cd'}) \tag{1.3.40}$$

where

$$\sigma_{ab'cd'ef'} = \sigma^{[\mu}_{ab'}\, \sigma^{\nu]}_{cd'}\, \sigma_{\mu ef',\nu} \tag{1.3.41}$$

or

$$\Gamma_{abcd'} = \frac{1}{2}\, \varepsilon^{p'q'}\, \sigma^{\mu}_{aq'}\, \sigma^{\nu}_{cd'}\, \sigma_{\mu bp';\nu} . \tag{1.3.42}$$

When applied to a scalar function φ the covariant differentiation operators ∇_μ and ∇_ν commute with one another, that is $(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)\varphi = 0$. Commutativity no longer holds for the operators $\nabla_{ab'}$ introduced in (1.3.34). It turns out that the spin coefficients $\Gamma_{abcd'}$ serve as a measure of the lack of commutativity of these operators. More precisely, the following equality holds for a scalar φ

$$(\nabla_{ab'}\nabla_{cd'} - \nabla_{cd'}\nabla_{ab'})\varphi = \{\varepsilon^{pq}(\Gamma_{pacd'}\nabla_{qb'} - \Gamma_{pcab'}\nabla_{qd'})$$

$$+ \varepsilon^{r's'}(\bar{\Gamma}_{r'b'd'c}\nabla_{as'} - \bar{\Gamma}_{r'd'b'a}\nabla_{cs'})\}\,\varphi\,. \qquad (1.3.43)$$

(1.3.43) permits one to write the commutation relations for the differential operators (1.3.37). By a slight extension of this calculation, when the derivatives act on a basis ζ^A_a, we obtain (for a proof see p. 139 Frolov [7])

$$\nabla_{fe'}\Gamma_{acdb'} - \nabla_{db'}\Gamma_{acfe'}$$

$$= \varepsilon^{pq}(\Gamma_{apdb'}\Gamma_{qcfe'} + \Gamma_{acpb'}\Gamma_{qdfe'} - \Gamma_{apfe'}\Gamma_{qcdb'} - \Gamma_{acpe'}\Gamma_{qfdb'})$$

$$+ \varepsilon^{r's'}(\Gamma_{acdr'}\bar{\Gamma}_{s'b'e'f} - \Gamma_{acfr'}\bar{\Gamma}_{s'e'b'd})$$

$$+ \Psi_{acdf}\varepsilon_{e'b'} + \Lambda\varepsilon_{e'b'}(\varepsilon_{cd}\varepsilon_{af} + \varepsilon_{ad}\varepsilon_{cf}) + \Phi_{acb'e'}\varepsilon_{fd}$$

$$(1.3.44)$$

where Ψ_{abcd}, $\Phi_{abc'd'}$ are the dyad components of the spinors Ψ_{ABCD}, $\Phi_{ABC'D'}$ which together with Λ appear in proposition 4 and correspond to the Weyl tensor, the trace-free part of the Ricci tensor and the scalar curvature respectively.

The following relations have been used to obtain (1.3.44)

$$\nabla^{A'}_{(A} \nabla_{B)A'} \xi_C = - \Psi_{ABCD} \xi^D + \Lambda \xi_{(A} \varepsilon_{B)C} \tag{1.3.45}$$

$$\nabla_{C(C'} \nabla^{C}_{D')} \xi_A = \Phi_{ABC'D'} \xi^B . \tag{1.3.46}$$

Relations (1.3.44) are the Newman-Penrose equations (NP equations for short). Their explicit form will be given in the next section but before that the spinor form of Bianchi identities and Maxwell equations will be presented here.

i) The Bianchi identities (1.2.34) in spinor form are

$$\left.\begin{aligned} &\nabla^{A}_{B'} \Psi_{ABCD} = \nabla^{A'}_{(B} \Phi_{CD)A'B'} \\ &\nabla^{CA'} \Phi_{CDA'B'} + 3\nabla_{DB'}\Lambda = 0. \end{aligned}\right\} \tag{1.3.47}$$

When Einstein's vacuum equations $\Phi_{ABC'D'} = 0$ hold, they reduce to $\nabla^{AA'} \Psi_{ABCD} = 0$.

ii) The Maxwell equations

$$\nabla_\beta F^{\alpha\beta} = 0; \quad F_{[\alpha\beta,\gamma]} = 0 \tag{1.3.48}$$

have the spinor form

$$\nabla^{BB'} \Phi_{AB} = 0 . \tag{1.3.49}$$

1.4. THE NEWMAN-PENROSE EQUATIONS AND RELATED QUANTITIES

1.4.1 Spin coefficients

In this section it will be shown how the formalisms developed in sections 1.2 and 1.3 can be put into a relatively concise form, despite the fact that all summations will be written out explicitly.

We start by presenting the twelve complex functions, called spin coefficients, which are defined in [1] in terms of either the Ricci rotation coefficients or the spinor affine connection

$$\kappa = \gamma_{131} = \ell_{\mu;\nu} m^{\mu} \ell^{\nu} \quad (1.4.1a)$$

$$\tau = \gamma_{132} = \ell_{\mu;\nu} m^{\mu} n^{\nu} \quad (1.4.1b)$$

$$\sigma = \gamma_{133} = \ell_{\mu;\nu} m^{\mu} m^{\nu} \quad (1.4.1c)$$

$$\rho = \gamma_{134} = \ell_{\mu;\nu} m^{\mu} \bar{m}^{\nu} \quad (1.4.1d)$$

$$\pi = -\gamma_{241} = -n_{\mu;\nu} \bar{m}^{\mu} \ell^{\nu} \quad (1.4.1e)$$

$$\nu = -\gamma_{242} = -n_{\mu;\nu} \bar{m}^{\mu} n^{\nu} \quad (1.4.1f)$$

$$\mu = -\gamma_{243} = -n_{\mu;\nu} \bar{m}^{\mu} m^{\nu} \quad (1.4.1g)$$

$$\lambda = -\gamma_{244} = -n_{\mu;\nu} \bar{m}^{\mu} \bar{m}^{\nu} \quad (1.4.1h)$$

$$\varepsilon = \frac{1}{2} (\gamma_{121} - \gamma_{341}) = \frac{1}{2} (\ell_{\mu;\nu} n^{\mu} \ell^{\nu} - m_{\mu;\nu} \bar{m}^{\mu} \ell^{\nu}) \quad (1.4.1i)$$

$$\gamma = \frac{1}{2}(\gamma_{122} - \gamma_{342}) = \frac{1}{2}(\ell_{\mu;\nu}n^{\mu}n^{\nu} - m_{\mu;\nu}\bar{m}^{\mu}n^{\nu}) \qquad (1.4.1j)$$

$$\beta = \frac{1}{2}(\gamma_{123} - \gamma_{343}) = \frac{1}{2}(\ell_{\mu;\nu}n^{\mu}m^{\nu} - m_{\mu;\nu}\bar{m}^{\mu}m^{\nu}) \qquad (1.4.1k)$$

$$\alpha = \frac{1}{2}(\gamma_{124} - \gamma_{344}) = \frac{1}{2}(\ell_{\mu;\nu}n^{\mu}\bar{m}^{\nu} - m_{\mu;\nu}\bar{m}^{\mu}\bar{m}^{\nu}) \qquad (1.4.1\ell)$$

or

$\Gamma_{abcd'}$ =

cd' \ ab	00	01 or 10	11
00'	κ	ε	π
10'	ρ	α	λ
01'	σ	β	μ
11'	τ	γ	ν

(1.4.2)

Thus it can easily be seen that the spin coefficients appear more naturally when dealing with spinors than with tetrad vectors. And now, equations (1.3.44) will be written down in explicit form using the notation (1.4.1)

$$D\rho - \bar{\delta}\kappa = (\rho^2+\sigma\bar{\sigma}) + (\varepsilon+\bar{\varepsilon})\rho - \bar{\kappa}\tau - \kappa(3\alpha+\bar{\beta}-\pi) + \Phi_{00} \qquad (1.4.3a)$$

$$D\sigma - \delta\kappa = (\rho+\bar{\rho})\sigma+(3\varepsilon-\bar{\varepsilon})\sigma-(\tau-\bar{\pi}+\bar{\alpha}+3\beta)\kappa + \Psi_0 \qquad (1.4.3b)$$

$$D\tau - \Delta\kappa = (\tau+\bar{\pi})\rho+(\bar{\tau}+\pi)\sigma+(\varepsilon-\bar{\varepsilon})\tau-(3\gamma+\bar{\gamma})\kappa+\Psi_1+\Phi_{01} \qquad (1.4.3c)$$

$$D\alpha - \bar{\delta}\varepsilon = (\rho+\bar{\varepsilon}-2\varepsilon)\alpha + \beta\bar{\sigma}-\bar{\beta}\varepsilon-\kappa\lambda-\bar{\kappa}\gamma+(\varepsilon+\rho)\pi+\Phi_{10} \qquad (1.4.3d)$$

$$D\beta - \delta\varepsilon = (\alpha+\pi)\sigma + (\bar{\rho}-\bar{\varepsilon})\beta - (\mu+\gamma)\kappa-(\bar{\alpha}-\bar{\pi})\varepsilon+\Psi_1 \qquad (1.4.3e)$$

$$D\gamma - \Delta\epsilon = (\tau+\bar{\pi})\alpha+(\bar{\tau}+\pi)\beta-(\epsilon+\bar{\epsilon})\gamma-(\gamma+\bar{\gamma})\epsilon+\tau\pi-\nu\kappa+\Psi_2-\Lambda+\Phi_{11} \qquad (1.4.3f)$$

$$D\lambda - \bar{\delta}\pi = (\rho\lambda+\bar{\sigma}\mu)+\pi^2+ (\alpha-\bar{\beta})\pi-\nu\bar{\kappa}-(3\epsilon-\bar{\epsilon})\lambda+\Phi_{20} \qquad (1.4.3g)$$

$$D\mu - \delta\pi = (\bar{\rho}\mu+\sigma\lambda)+\pi\bar{\pi}-(\epsilon+\bar{\epsilon})\mu-\pi(\bar{\alpha}-\beta)-\nu\kappa+\Psi_2+2\Lambda \qquad (1.4.3h)$$

$$D\nu - \Delta\pi = (\pi+\bar{\tau})\mu+(\bar{\pi}+\tau)\lambda+(\gamma-\bar{\gamma})\pi-(3\epsilon+\bar{\epsilon})\nu+\Psi_3+\Phi_{21} \qquad (1.4.3i)$$

$$\Delta\lambda - \bar{\delta}\nu = -(\mu+\bar{\mu})\lambda-(3\gamma-\bar{\gamma})\lambda+(3\alpha+\bar{\beta}+\pi-\bar{\tau})\nu-\Psi_4 \qquad (1.4.3j)$$

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha}+\beta)-\sigma(3\alpha-\bar{\beta})+(\rho-\bar{\rho})\tau-(\mu-\bar{\mu})\kappa-\Psi_1+\Phi_{01} \qquad (1.4.3k)$$

$$\delta\alpha - \bar{\delta}\beta = (\mu\rho-\lambda\sigma)+\alpha\bar{\alpha}+\beta\bar{\beta}-2\alpha\beta+\gamma(\rho-\bar{\rho})+\epsilon(\mu-\bar{\mu})-\Psi_2+\Lambda+\Phi_{11} \qquad (1.4.3\ell)$$

$$\delta\lambda - \bar{\delta}\mu = (\rho-\bar{\rho})\nu+(\mu-\bar{\mu})\pi+\mu(\alpha+\bar{\beta})+\lambda(\bar{\alpha}-3\beta)-\Psi_3+\Phi_{21} \qquad (1.4.3m)$$

$$\delta\nu - \Delta\mu = (\mu^2+\lambda\bar{\lambda})+(\gamma+\bar{\gamma})\mu-\bar{\nu}\pi+(\tau-3\beta-\bar{\alpha})\nu+\Phi_{22} \qquad (1.4.3n)$$

$$\delta\gamma - \Delta\beta = (\tau-\bar{\alpha}-\beta)\gamma+\mu\tau-\sigma\nu-\epsilon\bar{\nu}-\beta(\gamma-\bar{\gamma}-\mu)+\alpha\bar{\lambda}+\Phi_{12} \qquad (1.4.3o)$$

$$\delta\tau - \Delta\sigma = (\mu\sigma+\bar{\lambda}\rho)+(\tau+\beta-\bar{\alpha})\tau-(3\gamma-\bar{\gamma})\sigma-\kappa\bar{\nu}+\Phi_{02} \qquad (1.4.3p)$$

$$\Delta\rho - \bar{\delta}\tau = -(\rho\bar{\mu}+\sigma\lambda)+(\bar{\beta}-\alpha-\bar{\tau})\tau+(\gamma+\bar{\gamma})\rho+\nu\kappa-\Psi_2-2\Lambda \qquad (1.4.3q)$$

$$\Delta\alpha - \bar{\delta}\gamma = (\rho+\epsilon)\nu-(\tau+\beta)\lambda+(\bar{\gamma}-\bar{\mu})\alpha+(\bar{\beta}-\bar{\tau})\gamma-\Psi_3 \;. \qquad (1.4.3r)$$

Equations (1.4.3) are the Newman-Penrose equations as appear in Kramer et al. [13]. This version is believed to be the most accurate published version since it has been checked thoroughly by MacCallum and Aman using algebraic computing methods [14].

The quantities $\Psi_0, \Psi_1, \ldots\ldots, \Phi_{00}, \ldots$ and Λ are, respectively, related to components of the Weyl tensor, Ricci tensor and scalar curvature as follows [1]

$$\Psi_0 = - C_{1313} = - C_{\alpha\beta\gamma\delta} \ell^{\alpha} m^{\beta} \ell^{\gamma} m^{\delta} = \Psi_{0000}$$

$$\Psi_1 = - C_{1213} = - C_{\alpha\beta\gamma\delta} \ell^{\alpha} n^{\beta} \ell^{\gamma} m^{\delta} = \Psi_{0001}$$

$$\Psi_2 = - \frac{1}{2} (C_{1212} + C_{1234}) = - \frac{1}{2} C_{\alpha\beta\gamma\delta} (\ell^{\alpha} n^{\beta} \ell^{\gamma} n^{\delta} + \ell^{\alpha} n^{\beta} m^{\gamma} \bar{m}^{\delta}) = \Psi_{0011}$$

$$\Psi_3 = C_{1224} = C_{\alpha\beta\gamma\delta} \ell^{\alpha} n^{\beta} n^{\gamma} \bar{m}^{\delta} = \Psi_{0111}$$

$$\Psi_4 = - C_{2424} = - C_{\alpha\beta\gamma\delta} n^{\alpha} \bar{m}^{\beta} n^{\gamma} \bar{m}^{\delta} = \Psi_{1111}$$

$$\Phi_{00} = - \frac{1}{2} R_{11} = \Phi_{000'0'} = \bar{\Phi}_{00} \qquad (1.4.4)$$

$$\Phi_{11} = - \frac{1}{4} (R_{12} + R_{34}) = \Phi_{010'1'}$$

$$\Phi_{01} = - \frac{1}{2} R_{13} = \Phi_{000'1'} = \bar{\Phi}_{10}$$

$$\Phi_{12} = - \frac{1}{2} R_{23} = \Phi_{011'1'}$$

$$\Phi_{10} = - \frac{1}{2} R_{14} = \Phi_{010'0'} = \bar{\Phi}_{01}$$

$$\Phi_{21} = - \frac{1}{2} R_{24} = \Phi_{110'1'}$$

$$\Phi_{02} = -\frac{1}{2} R_{33} = \Phi_{001'1'} = \bar{\Phi}_{20}$$

$$\Phi_{22} = -\frac{1}{2} R_{22} = \Phi_{111'1'}$$

$$\Phi_{20} = -\frac{1}{2} R_{44} = \Phi_{110'0'}$$

$$\Lambda = R/24 .$$

The commutation relations for the operators D, Δ, δ, $\bar{\delta}$ are now obtained from (1.3.43) using the expressions for $\Gamma_{abcd'}$ in terms of the spin coefficients

$$(\Delta D - D\Delta) = [(\gamma+\bar{\gamma})D+(\varepsilon+\bar{\varepsilon})\Delta-(\tau+\bar{\pi})\bar{\delta}-(\bar{\tau}+\pi)\delta] \tag{1.4.5a}$$

$$(\delta D - D\delta) = [(\bar{\alpha}+\beta-\bar{\pi})D+\varkappa\Delta-\sigma\bar{\delta}-(\bar{\rho}+\varepsilon-\bar{\varepsilon})\delta] \tag{1.4.5b}$$

$$(\delta\Delta - \Delta\delta) = [-\bar{\nu}D+(\tau-\bar{\alpha}-\beta)\Delta+\bar{\lambda}\bar{\delta}+(\mu-\gamma+\bar{\gamma})\delta] \tag{1.4.5c}$$

$$(\bar{\delta}\delta - \delta\bar{\delta}) = [(\bar{\mu}-\mu)D+(\bar{\rho}-\rho)\Delta-(\bar{\alpha}-\beta)\bar{\delta}-(\bar{\beta}-\alpha)\delta] \quad . \tag{1.4.5d}$$

The full Bianchi identities (1.2.34) or (1.3.47), when written out in general, are very long and unwieldy. However, in empty space ($R_{\alpha\beta} = \Phi_{ABC'D'} = 0$), they have the simple form

$$D\Psi_1 - \bar{\delta}\Psi_0 = -3\varkappa\Psi_2+(2\varepsilon+4\rho)\Psi_1 - (-\pi+4\alpha)\Psi_0 \tag{1.4.6a}$$

$$D\Psi_2 - \bar{\delta}\Psi_1 = -2\varkappa\Psi_3+3\rho\Psi_2 - (-2\pi+2\alpha)\Psi_1-\lambda\Psi_0 \tag{1.4.6b}$$

$$D\Psi_3 - \bar{\delta}\Psi_2 = -\varkappa\Psi_4-(2\varepsilon-2\rho)\Psi_3+3\pi\Psi_2-2\lambda\Psi_1 \tag{1.4.6c}$$

$$D\Psi_4 - \bar{\delta}\Psi_3 = -(4\varepsilon-\rho)\Psi_4+(4\pi+2\alpha)\Psi_3-3\lambda\Psi_2 \tag{1.4.6d}$$

$$\Delta\Psi_0 - \delta\Psi_1 = (4\gamma-\mu)\Psi_0 - (4\tau+2\beta)\Psi_1+3\sigma\Psi_2 \tag{1.4.6e}$$

$$\Delta\Psi_1 - \delta\Psi_2 = \nu\Psi_0+(2\gamma-2\mu)\Psi_1-3\tau\Psi_2+2\sigma\Psi_3 \tag{1.4.6f}$$

$$\Delta\Psi_2 - \delta\Psi_3 = 2\nu\Psi_1-3\mu\Psi_2+(-2\tau+2\beta)\Psi_3+\sigma\Psi_4 \tag{1.4.6g}$$

$$\Delta\Psi_3 - \delta\Psi_4 = 3\nu\Psi_2-(2\gamma+4\mu)\Psi_3+(-\tau+4\beta)\Psi_4. \tag{1.4.6h}$$

Finally, in the Newman-Penrose formalism the Maxwell equations (1.3.49) take the form

$$D\Phi_1 - \bar{\delta}\Phi_0 = (\pi-2\alpha)\Phi_0+2\rho\Phi_1-\varkappa\Phi_2 \tag{1.4.7a}$$

$$D\Phi_2 - \bar{\delta}\Phi_1 = -\lambda\Phi_0+2\pi\Phi_1+(\rho-2\varepsilon)\Phi_2 \tag{1.4.7b}$$

$$\delta\Phi_1 - \Delta\Phi_0 = (\mu-2\gamma)\Phi_0+2\tau\Phi_1-\sigma\Phi_2 \tag{1.4.7c}$$

$$\delta\Phi_2 - \Delta\Phi_1 = -\nu\Phi_0+2\mu\Phi_1+(\tau-2\beta)\Phi_2 \tag{1.4.7d}$$

with

$$\left.\begin{aligned} \Phi_0 &= F_{\mu\nu}\, \ell^\mu\, m^\nu \\ \Phi_1 &= \tfrac{1}{2}\, F_{\mu\nu}(\ell^\mu\, n^\nu + \bar{m}^\mu\, m^\nu) \\ \Phi_2 &= F_{\mu\nu}\, \bar{m}^\mu\, n^\nu\ . \end{aligned}\right\} \tag{1.4.8}$$

The Newman-Penrose equations (1.4.3), the commutator relations (1.4.5) and the full Bianchi identities form a complete set of equations from which exact solutions of Einstein equations may be found. Despite the fact that we have to solve a considerable number of equations, the Newman-Penrose formalism* has great advantages. All differential equations are of first order. Gauge transformations of the tetrad can be used to simplify the field equations. One can extract invariant properties of the gravitational field without using a coordinate basis and the algebraic structure of the Weyl tensor can be easily specified [13].

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* The above presented Newman-Penrose formalism has been modified and extended by several authors. Geroch, Held and Penrose [15] developed a modified calculus adapted to physical situations in which a pair of real null directions is picked out at each space-time point. This new version of the spin coefficient method leads to even simpler formulae than the standard Newman-Penrose technique.

Allnut [16] modified the standard Newman-Penrose formalism so that it is more suited to the study of perfect-fluid space-times. And finally, Jogia and Griffiths [17] have extended the standard formalism to include the possibility of an anti-symmetric connection, so that it may be useful in Einstein-Cartan theory or in other theories which consider space-times with torsion.

1.4.2. Useful formulae containing the spin-coefficients

The definition of the spin coefficients allows us to obtain the following relations [7]

$$\ell_{\mu;\nu} = (\gamma+\bar{\gamma})\ell_\mu\ell_\nu + (\varepsilon+\bar{\varepsilon})\ell_\mu n_\nu - (\alpha+\bar{\beta})\ell_\mu m_\nu - (\bar{\alpha}+\beta)\ell_\mu\bar{m}_\nu$$

$$- \bar{\tau}m_\mu\ell_\nu - \tau\bar{m}_\mu\ell_\nu - \bar{\kappa}m_\mu n_\nu + \bar{\sigma}m_\mu m_\nu + \bar{\rho}m_\mu\bar{m}_\nu - \kappa\bar{m}_\mu n_\nu$$

$$+ \rho\bar{m}_\mu m_\nu + \sigma\bar{m}_\mu\bar{m}_\nu \qquad (1.4.9a)$$

$$n_{\mu;\nu} = -(\gamma+\bar{\gamma})n_\mu\ell_\nu - (\varepsilon+\bar{\varepsilon})n_\mu n_\nu + (\alpha+\bar{\beta})n_\mu m_\nu + (\bar{\alpha}+\beta)n_\mu\bar{m}_\nu$$

$$+ \nu m_\mu\ell_\nu + \pi m_\mu n_\nu - \lambda m_\mu m_\nu - \mu m_\mu\bar{m}_\nu + \bar{\nu}\bar{m}_\mu\ell_\nu + \bar{\pi}\bar{m}_\mu n_\nu$$

$$- \bar{\mu}\bar{m}_\mu m_\nu - \bar{\lambda}\bar{m}_\mu m_\nu \qquad (1.4.9b)$$

$$m_{\mu;\nu} = \bar{\nu}\ell_\mu\ell_\nu + \bar{\pi}\ell_\mu n_\nu - \bar{\mu}\ell_\mu m_\nu - \bar{\lambda}\ell_\mu\bar{m}_\nu - \tau n_\mu\ell_\nu - \kappa n_\mu n_\nu$$

$$+ \rho n_\mu m_\nu + \sigma n_\mu\bar{m}_\nu + (\gamma-\bar{\gamma})m_\mu\ell_\nu + (\varepsilon-\bar{\varepsilon})m_\mu n_\nu - (\alpha-\bar{\beta})m_\mu m_\nu$$

$$- (\beta-\bar{\alpha})m_\mu\bar{m}_\nu \qquad (1.4.9c)$$

$$\bar{m}_{\mu;\nu} = \nu\ell_\mu\ell_\nu + \pi\ell_\mu n_\nu - \mu\ell_\mu\bar{m}_\nu - \lambda\ell_\mu m_\nu - \bar{\tau}n_\mu\ell_\nu - \bar{\kappa}n_\mu n_\nu$$

$$+ \bar{\rho}n_\mu\bar{m}_\nu + \bar{\sigma}n_\mu m_\nu + (\bar{\gamma}-\gamma)\bar{m}_\mu\ell_\nu + (\bar{\varepsilon}-\varepsilon)\bar{m}_\mu n_\nu - (\bar{\alpha}-\beta)\bar{m}_\mu\bar{m}_\nu$$

$$- (\bar{\beta}-\alpha)\bar{m}_\mu m_\nu \qquad (1.4.9d)$$

These equations give after contraction

$$\ell^{\mu}{}_{;\mu} = -(\rho+\bar{\rho}) + \varepsilon + \bar{\varepsilon} \qquad (1.4.10a)$$

$$n^{\mu}{}_{;\mu} = -(\gamma+\bar{\gamma}) + \mu + \bar{\mu} \qquad (1.4.10b)$$

$$m^{\mu}{}_{;\mu} = -\bar{\alpha} + \bar{\pi} - \tau + \beta \qquad (1.4.10c)$$

$$\bar{m}^{\mu}{}_{;\mu} = -\alpha + \pi - \tau + \bar{\beta}. \qquad (1.4.10d)$$

Antisymmetrization of equations (3.9) leads to the following relations

$$\ell_{[\mu;\nu]} = -2\mathrm{Re}(\varepsilon)\ell_{[\nu}n_{\mu]} - (\bar{\tau}-\alpha-\bar{\beta})\ell_{[\nu}m_{\mu]} - (\tau-\bar{\alpha}-\beta)\ell_{[\nu}\bar{m}_{\mu]}$$

$$- \bar{\kappa}n_{[\nu}m_{\mu]} - \kappa n_{[\nu}\bar{m}_{\mu]} + 2i\mathrm{Im}(\rho)m_{[\nu}\bar{m}_{\mu]} \qquad (1.4.11a)$$

$$n_{[\mu;\nu]} = -2\mathrm{Re}(\gamma)\ell_{[\nu}n_{\mu]} + \nu\ell_{[\nu}m_{\mu]} + \bar{\nu}\ell_{[\nu}\bar{m}_{\mu]} + (\pi-\alpha-\bar{\beta})n_{[\nu}m_{\mu]}$$

$$+ (\bar{\pi}-\bar{\alpha}-\beta)n_{[\nu}\bar{m}_{\mu]} + 2i\mathrm{Im}(\mu)m_{[\nu}\bar{m}_{\mu]} \qquad (1.4.11b)$$

$$m_{[\mu;\nu]} = -(\bar{\pi}+\tau)\ell_{[\nu}n_{\mu]} + (2i\mathrm{Im}(\gamma)+\bar{\mu})\ell_{[\nu}m_{\mu]} + \bar{\lambda}\ell_{[\nu}\bar{m}_{\mu]}$$

$$+ (2i\mathrm{Im}(\varepsilon)-\rho)n_{[\nu}m_{\mu]} - \sigma n_{[\nu}\bar{m}_{\mu]} - (\bar{\alpha}-\beta)m_{[\nu}\bar{m}_{\mu]} \qquad (1.4.11c)$$

$$\bar{m}_{[\mu;\nu]} = -(\pi+\bar{\tau})\ell_{[\nu}n_{\mu]} + (-2i\mathrm{Im}(\gamma)+\mu)\ell_{[\nu}\bar{m}_{\mu]} + \lambda\ell_{[\nu}m_{\mu]}$$

$$+ (-2i\mathrm{Im}(\varepsilon)-\bar{\rho})n_{[\nu}\bar{m}_{\mu]} - \bar{\sigma}n_{[\nu}m_{\mu]} - (\alpha-\bar{\beta})\bar{m}_{[\nu}m_{\mu]} \qquad (1.4.11d)$$

where we use the notation $\gamma+\bar{\gamma} = 2\mathrm{Re}(\gamma)$ and $\gamma-\bar{\gamma} = 2i\mathrm{Im}(\gamma)$.

Using the covariant differentiation operators D, Δ, δ and $\bar{\delta}$, we can rewrite equations (1.4.9) in the following form

$$\left.\begin{aligned}
D\ell^\mu &= (\varepsilon+\bar{\varepsilon})\ell^\mu - \bar{\kappa}m^\mu - \kappa\bar{m}^\mu \\
\Delta\ell^\mu &= (\gamma+\bar{\gamma})\ell^\mu - \bar{\tau}m^\mu - \tau\bar{m}^\mu \\
\delta\ell^\mu &= (\bar{\alpha}+\beta)\ell^\mu - \bar{\rho}m^\mu - \sigma\bar{m}^\mu \\
\bar{\delta}\ell^\mu &= (\alpha+\bar{\beta})\ell^\mu - \bar{\sigma}m^\mu - \rho\bar{m}^\mu
\end{aligned}\right\} \qquad (1.4.12a)$$

$$\left.\begin{aligned}
Dn^\mu &= -(\varepsilon+\bar{\varepsilon})n^\mu + \pi m^\mu + \bar{\pi}\bar{m}^\mu \\
\Delta n^\mu &= -(\gamma+\bar{\gamma})n^\mu + \nu m^\mu + \bar{\nu}\bar{m}^\mu \\
\delta n^\mu &= -(\bar{\alpha}+\beta)n^\mu + \mu m^\mu + \bar{\lambda}\bar{m}^\mu \\
\bar{\delta}n^\mu &= -(\alpha+\bar{\beta})n^\mu + \lambda m^\mu + \bar{\mu}\bar{m}^\mu
\end{aligned}\right\} \qquad (1.4.12b)$$

$$\left.\begin{aligned}
Dm^\mu &= \bar{\pi}\ell^\mu - \kappa n^\mu + (\varepsilon-\bar{\varepsilon})m^\mu \\
\Delta m^\mu &= \bar{\nu}\ell^\mu - \tau n^\mu + (\gamma-\bar{\gamma})m^\mu \\
\delta m^\mu &= \bar{\lambda}\ell^\mu - \sigma n^\mu + (\beta-\bar{\alpha})m^\mu \\
\bar{\delta}m^\mu &= \bar{\mu}\ell^\mu - \rho n^\mu + (\alpha-\bar{\beta})m^\mu
\end{aligned}\right\} \qquad (1.4.12c)$$

$$\left.\begin{aligned}
D\bar{m}^{\mu} &= \pi\ell^{\mu} - \bar{\varkappa}n^{\mu} + (\bar{\varepsilon}-\varepsilon)\bar{m}^{\mu} \\
\Delta\bar{m}^{\mu} &= \nu\ell^{\mu} - \bar{\tau}n^{\mu} + (\bar{\gamma}-\gamma)\bar{m}^{\mu} \\
\delta\bar{m}^{\mu} &= \mu\ell^{\mu} - \bar{\rho}n^{\mu} + (\bar{\alpha}-\beta)\bar{m}^{\mu} \\
\bar{\delta}\bar{m}^{\mu} &= \lambda\ell^{\mu} - \bar{\sigma}n^{\mu} + (\bar{\beta}-\alpha)\bar{m}^{\mu} \ .
\end{aligned}\right\} \qquad (1.4.12d)$$

1.4.3 Transformation properties of the various Newman-Penrose quantities

Having chosen a tetrad frame - either an orthonormal frame as in the conventional tetrad formalism, or a complex null-frame as in the Newman-Penrose formalism - at each point in space-time, we can subject the frame to Lorentz transformations. In considering the effect of such Lorentz transformations on the various Newman-Penrose quantities, we shall find it convenient to regard a general Lorentz transformation of the basis (ℓ^{μ},n^{μ}, m^{μ},$\bar{m}^{\mu}$) as made up of the following three classes of rotations [5].

(a) Rotations of class I which leave the vector ℓ^{μ} unchanged;

(b) Rotations of class II which leave the vector n^{μ} unchanged;

(c) Rotations of class III which leave the directions of ℓ^{μ} and n^{μ} unchanged but rotate m^{μ} (and $\bar{m}^{\mu}$) by an angle ϑ in the (m^{μ},$\bar{m}^{\mu}$) plane.

Associated with these three classes of rotations, we have the following explicit transformations.

I. Null rotations around $\underline{\ell}$:

$$\ell^\mu \to \ell^\mu$$

$$n^\mu \to n^\mu + \bar{a}m^\mu + a\bar{m}^\mu + a\bar{a}\ell^\mu$$

$$m^\mu \to m^\mu + a\ell^\mu \tag{1.4.13a}$$

$$\bar{m}^\mu \to \bar{m}^\mu + \bar{a}\ell^\mu$$

II. Null rotations around $\underline{n}$:

$$\ell^\mu \to \ell^\mu + \bar{b}m^\mu + b\bar{m}^\mu + b\bar{b}n^\mu$$

$$n^\mu \to n^\mu$$

$$m^\mu \to m^\mu + bn^\mu \tag{1.4.13b}$$

$$\bar{m}^\mu \to \bar{m}^\mu + \bar{b}n^\mu$$

III. Spatial rotations in the $(\underline{m},\underline{\bar{m}})$-plane:

$$\ell^\mu \to \frac{1}{A}\,\ell^\mu$$

$$n^\mu \to An^\mu$$

$$m^\mu \to e^{i\vartheta}m^\mu \tag{1.4.13c}$$

$$\bar{m}^\mu \to e^{-i\vartheta}\bar{m}^\mu$$

where a and b are two complex functions and A and ϑ are two real functions.

The effect of a rotation of class I on the Weyl scalars is

$$\begin{aligned}
&\Psi_0 \to \Psi_0 \\
&\Psi_1 \to \Psi_1 + \bar{a}\Psi_0 \\
&\Psi_2 \to \Psi_2 + 2\bar{a}\Psi_1 + (\bar{a})^2\Psi_0 \qquad (1.4.14)\\
&\Psi_3 \to \Psi_3 + 3\bar{a}\Psi_2 + 3(\bar{a})^2\Psi_1 + (\bar{a})^3\Psi_0 \\
&\Psi_4 \to \Psi_4 + 4\bar{a}\Psi_3 + 6(\bar{a})^2\Psi_2 + 4(\bar{a})^3\Psi_1 + (\bar{a})^4\Psi_0 \,.
\end{aligned}$$

The spin-coefficients are transformed as follows

$$\begin{aligned}
&\varkappa \to \varkappa \\
&\tau \to \tau + a\rho + \bar{a}\sigma + a\bar{a}\varkappa \\
&\sigma \to \sigma + a\varkappa \\
&\rho \to \rho + \bar{a}\varkappa \\
&\varepsilon \to \varepsilon + \bar{a}\varkappa \\
&\alpha \to \alpha + \bar{a}(\rho + \varepsilon) + (\bar{a})^2\varkappa \qquad (1.4.15)\\
&\beta \to \beta + a\varepsilon + \bar{a}\sigma + a\bar{a}\varkappa \\
&\pi \to \pi + 2\bar{a}\varepsilon + (\bar{a})^2\varkappa + D\bar{a} \\
&\gamma \to \gamma + a\alpha + \bar{a}(\beta+\tau) + a\bar{a}(\rho+\varepsilon) + (\bar{a})^2\sigma + a(\bar{a})^2\varkappa \\
&\lambda \to \lambda + \bar{a}(2\alpha+\pi) + (\bar{a})^2(\rho+2\varepsilon) + (\bar{a})^3\varkappa + \bar{\delta}\bar{a} + \bar{a}D\bar{a} \\
&\mu \to \mu + a\pi + 2\bar{a}\beta + 2a\bar{a}\varepsilon + (\bar{a})^2\sigma + a(\bar{a})^2\varkappa + \delta\bar{a} + aD\bar{a}
\end{aligned}$$

$$\nu \to \nu + a\lambda + \bar{a}(\mu+2\gamma) + (\bar{a})^2(\tau+2\beta) + (\bar{a})^3\sigma + a\bar{a}(\pi+2\alpha)$$

$$+ a(\bar{a})^2(\rho+2\varepsilon) + a(\bar{a})^3\kappa + (\Delta+\bar{a}\delta+a\bar{\delta}+a\bar{a}D)\bar{a}\,.$$

The scalars representing the Maxwell field become

$$\begin{aligned} &\Phi_0 \to \Phi_0 \\ &\Phi_1 \to \Phi_1 + \bar{a}\Phi_0 \\ &\Phi_2 \to \Phi_2 + 2\bar{a}\Phi_1 + (\bar{a})^2\Phi_0\,. \end{aligned} \tag{1.4.16}$$

The corresponding effects of a rotation of class II on the various Newman-Penrose quantities can be readily written down from the foregoing formulae, since the effect of interchanging ℓ^μ and n^μ results in the transformation

$$\begin{aligned} &\Psi_0 \rightleftarrows \bar{\Psi}_4 \\ &\Psi_1 \rightleftarrows \bar{\Psi}_3 \\ &\Psi_2 \rightleftarrows \bar{\Psi}_2 \\ &\Phi_0 \rightleftarrows -\bar{\Phi}_2 \\ &\Phi_1 \rightleftarrows -\bar{\Phi}_1 \\ &\kappa \rightleftarrows -\bar{\nu} \\ &\rho \rightleftarrows -\bar{\mu} \\ &\sigma \rightleftarrows -\bar{\lambda} \end{aligned} \tag{1.4.17}$$

$$\alpha \rightleftarrows -\bar{\beta}$$

$$\varepsilon \rightleftarrows -\bar{\gamma}$$

$$\pi \rightleftarrows -\bar{\tau}.$$

In particular, the effect of a rotation of class II on the Weyl scalars is

$$\begin{aligned}
\Psi_0 &\to \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 \\
\Psi_1 &\to \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4 \\
\Psi_2 &\to \Psi_2 + 2b\Psi_3 + b^2\Psi_4 \\
\Psi_3 &\to \Psi_3 + b\Psi_4 \\
\Psi_4 &\to \Psi_4.
\end{aligned} \tag{1.4.18}$$

Finally, the effect of a rotation of class III on the Newman-Penrose quantities is

$$\begin{aligned}
\Psi_0 &\to A^{-2}e^{2i\vartheta}\Psi_0 \\
\Psi_1 &\to A^{-1}e^{i\vartheta}\Psi_1 \\
\Psi_2 &\to \Psi_2 \\
\Psi_3 &\to Ae^{-i\vartheta}\Psi_3 \\
\Psi_4 &\to A^2e^{-2i\vartheta}\Psi_4
\end{aligned}$$

$$
\begin{aligned}
&\Phi_0 \to A^{-1}e^{i\vartheta}\Phi_0 \\
&\Phi_1 \to \Phi_1 \\
&\Phi_2 \to Ae^{-i\vartheta}\Phi_2 \\
&\kappa \to A^{-2}e^{i\vartheta}\kappa \\
&\tau \to e^{i\vartheta}\tau \\
&\sigma \to A^{-1}e^{2i\vartheta}\sigma \\
&\rho \to A^{-1}\rho \\
&\pi \to e^{-i\vartheta}\pi \\
&\nu \to A^{2}e^{-i\vartheta}\nu \\
&\mu \to A\mu \\
&\lambda \to Ae^{-2i\vartheta}\lambda \\
&\varepsilon \to A^{-1}\varepsilon - \frac{1}{2}A^{2}DA + \frac{1}{2}iA^{-1}D\vartheta \\
&\gamma \to A\gamma - \frac{1}{2}\Delta A + \frac{1}{2}iA\Delta\vartheta \\
&\beta \to e^{i\vartheta}\beta + \frac{1}{2}ie^{i\vartheta}\delta\vartheta - \frac{1}{2}A^{-1}e^{i\vartheta}\delta A \\
&\alpha \to e^{-i\vartheta}\alpha + \frac{1}{2}ie^{-i\vartheta}\bar{\delta}\vartheta - \frac{1}{2}A^{-1}e^{-i\vartheta}\bar{\delta}A\ .
\end{aligned}
\tag{1.4.19}
$$

1.4.4 The Petrov classification

In section 1.4.3 we have seen how the various Newman-Penrose equations transform under Lorentz transformations. Choose now a tetrad frame such that $\Psi_4 \neq 0$. Consider a rotation of class II (as in section 1.4.3) with parameter b. The Weyl scalars $\Psi_0, \Psi_1, \ldots$ transform according to (1.4.18). It is clear that by a rotation of this class, we can make $\Psi_0 = 0$ if b is a root of the equation

$$\Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 = 0 . \qquad (1.4.20)$$

This equation has always four roots and the corresponding new directions of ℓ^μ, namely $\ell^\mu + \bar{b}m^\mu + b\bar{m}^\mu + b\bar{b}n^\mu$ are called the principal null-directions of the field. In general there are four principal null vectors. If some of the roots coincide, the field is said to be algebraically special. The various ways in which the roots can coincide lead to the Petrov classification [18].

(a) Petrov type I

All four roots are distinct, let them be b_1, b_2, b_3, b_4. Then by a rotation of class II, with parameter $b = b_1$ (say) we can make Ψ_0 vanish. Next, by a rotation of class I we can make $\Psi_4 = 0$ without affecting the vanishing of Ψ_0.

(b) Petrov type II

Two roots are equal, say $b_1 = b_2$ ($\neq b_3 \neq b_4$). In this case besides (1.4.20), its derivative, namely

$$\Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4 = 0 \qquad (1.4.21)$$

will also be satisfied when $b = b_1$ $(=b_2)$. Therefore, by a rotation of class II with parameter $b = b_1$ $(=b_2)$, Ψ_0 and Ψ_1 will vanish simultaneously. Then, by a rotation of class I we can also make Ψ_4 vanish.

(c) Petrov type D

Two pairs of roots coincide, say $b_1 = b_2$ and $b_3 = b_4$ ($\neq b_1$ or b_2). By a rotation of class II with parameter $b = b_1 (=b_2)$ we can make Ψ_0 and Ψ_1 vanish simultaneously. Then by a rotation of class I we can also make Ψ_4 and Ψ_3 vanish.

(d) Petrov type III

Three of the roots coincide, say $b_1 = b_2 = b_3 \neq b_4$. By a rotation of class II with parameter $b = b_1$ ($=b_2 = b_3$) we can make Ψ_0, Ψ_1,Ψ_2 vanish simultaneously. Then by a rotation of class I we can also make Ψ_4 vanish.

(e) Petrov type N

All four roots coincide, so there is only one distinct root, b (say). By a rotation of class II with parameter b we can make Ψ_0, Ψ_1, Ψ_2, Ψ_3 vanish simultaneously.

The classification scheme is the following.

[1111]	Petrov type I	algebraically general
[211]	Petrov type II	algebraically special
[22]	Petrov type D	algebraically special
[31]	Petrov type III	algebraically special
[4]	Petrov type N	algebraically special
[0]	Petrov type O	

The increasing specialization which appears as one moves down the table of Petrov types suggests a hierarchical arrangement, which may be set out in a Penrose diagram in which the arrows point in directions of increasing specialization.

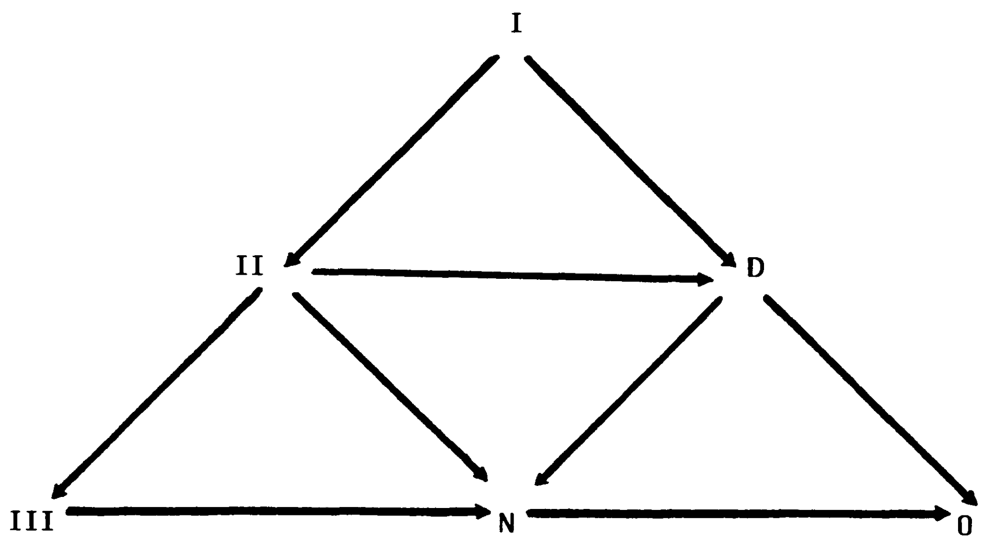


Figure 1.4. Penrose diagram

The hierarchy of types has a considerable physical significance, [9] and [19]. One can show that the curvature tensor of an isolated matter distribution allows, under certain assumptions about the sources, the following expansion, at large distance

$$R_{\alpha\beta\gamma\delta} = \frac{{}^{N}{}_{\alpha\beta\gamma\delta}}{r} + \frac{{}^{III}{}_{\alpha\beta\gamma\delta}}{r^2} + \frac{{}^{D}{}_{\alpha\beta\gamma\delta}}{r^3} + \frac{{}^{II}{}_{\alpha\beta\gamma\delta}}{r^4} + \frac{{}^{I}{}_{\alpha\beta\gamma\delta}}{r^5} + O(r^{-6}) \quad (1.4.22)$$

where the symbols N,III,D... refer to tensors of the respective algebraic types. The far field of every source of gravitational radiation (if such a field exists) is therefore a plane wave (type N) locally; as one approaches closer to the source, the four initially coincident directions of the null eigenvectors separate. Sachs [20] describes the phenomenon represented by equation (1.4.22) as the "peeling-off" of principal null directions. In Appendix A1, where we obtain the Petrov classification by studying the canonical decomposition of symmetric spinors, this peeling-off property is explained and represented geometrically.

1.4.5 Geometric meaning of spin coefficients

Before proceeding to an examination of the geometric meaning of spin coefficients, it is useful to write down some basic information about null geodesic congruences. The null geodesic congruences can be defined parametrically in the form $x^\mu = x^\mu(s, y^i)$ $(i = 1,2,3)$, an individual curve of the congruence being defined by the equation $y^i = c^i = \text{const}$, and s is an affine parameter along the curve. A field of tangent vectors ℓ^μ is defined by the equation $\ell^\mu = \frac{\partial x^\mu}{\partial s}$.

The key to understanding the geometric properties of a congruence of null geodesics is obtained by considering the following Gedanken experiment [7]. Assume that at a given moment of time an observer places an opaque object in the path of light rays. If another observer places a screen in the path of the rays, the shape, dimension and position of the shadow on the screen are determined not only by the characteristics of the object but also depend essentially on the geometric properties of the congruence of light rays.

Ehlers-Sachs theorem [21]: In the experiment considered above the following conditions hold.

i) All parts of the shadow reach the screen at the same time.

ii) The dimension, shape and orientation of the shadow depend only on the position of the screen and do not depend on the velocity of motion of the observer.

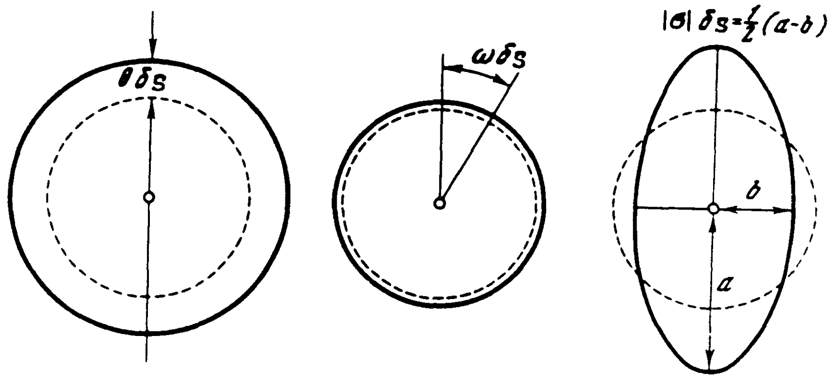
iii) If the screen is positioned at a small distance δs from the object, the shadow is magnified, rotated and deformed by the quantities $\vartheta\delta s$, $\omega\delta s$ and $|\sigma|\delta s$, where

$$\vartheta = \frac{1}{2}\,\ell^{\mu}{}_{;\mu} \qquad \underline{\text{expansion}} \qquad (1.4.23a)$$

$$\omega = \{\frac{1}{2}\,\ell_{[\mu;\nu]}\ell^{\mu;\nu}\}^{1/2} \qquad \underline{\text{twist}} \qquad (1.4.23b)$$

$$|\sigma| = \{\frac{1}{2}\,\ell_{(\mu;\nu)}\ell^{\mu;\nu} - \vartheta^2\}^{1/2} \qquad \underline{\text{shear}}\quad . \qquad (1.4.23c)$$

The effects of expansion, twist and shear have been represented diagrammatically by Sachs as follows



ϑ expansion $\qquad$ ω twist $\qquad$ $|\sigma|$ shear

Figure 1.5. Optical scalars.

The quantities ϑ, ω and $|\sigma|$ are called the <u>optical scalars.</u>

Now we are ready to examine briefly the geometric meaning of the spin coefficients. Using the relation (1.4.9a) we can find

$$\ell_{\mu;\nu}\ell^{\nu} = -\varkappa\bar{m}_{\mu} - \bar{\varkappa}m_{\mu} + (\varepsilon+\bar{\varepsilon})\,\ell_{\mu}. \tag{1.4.24}$$

From this equation we can see that if $\varkappa = 0$, ℓ^{μ} are tangent to null geodesics. In addition, by a change in scale $\ell^{\mu} \to G\ell^{\mu}$ which does not affect the directions of ℓ^{μ}, we can make $\varepsilon = 0$. Then, when $\varkappa = \varepsilon = 0$ we have from equation (1.4.9a)

$$\ell_{\mu;\nu} = (\gamma+\bar{\gamma})\ell_{\mu}\ell_{\nu} - (\bar{\alpha}+\beta)\ell_{\mu}\bar{m}_{\nu} - (\alpha+\bar{\beta})\ell_{\mu}m_{\nu} - \tau\bar{m}_{\mu}\ell_{\nu}$$

$$+ \sigma\bar{m}_{\mu}\bar{m}_{\nu} + \bar{\sigma}m_{\mu}m_{\nu} + \rho\bar{m}_{\mu}m_{\nu} + \bar{\rho}m_{\mu}\bar{m}_{\nu} - \bar{\tau}m_{\mu}\ell_{\nu} \tag{1.4.25}$$

and from equation (1.4.11a)

$$\ell_{[\mu;\nu]} = -(\bar{\alpha}+\beta-\tau)\ell_{[\mu}\bar{m}_{\nu]} - (\alpha+\bar{\beta}-\bar{\tau})\ell_{[\mu}m_{\nu]} + (\rho-\bar{\rho})\bar{m}_{[\mu}m_{\nu]} \tag{1.4.26}$$

from which we obtain

$$\ell_{[\mu;\nu}\ell_{\lambda]} = (\rho-\bar{\rho})\bar{m}_{[\mu}m_{\nu}\ell_{\lambda]}\,. \tag{1.4.27}$$

Now, from equations (1.4.25) and (1.4.26) we have

$$\frac{1}{2}\,\ell^{\mu}{}_{;\mu} = -\frac{1}{2}\,(\rho+\bar{\rho}) = \vartheta \tag{1.4.28a}$$

$$\ell_{[\mu;\nu]}\ell^{\mu;\nu} = -\frac{1}{2}\,(\rho-\bar{\rho})^{2} = 2\omega^{2} \tag{1.4.28b}$$

$$\frac{1}{2}\,[\ell_{(\mu;\nu)}\ell^{\mu;\nu} - \frac{1}{2}\,(\ell^{\mu}{}_{;\mu})^2] = |\sigma|^2. \tag{1.4.28c}$$

In this way, spin coefficients are related to the optical scalars of a congruence, with tangent vector ℓ^{μ} [5]. If ℓ^{μ} are taken to be tangent to a geodesic congruence and we wish to propagate the remainder of the tetrad system parallelly along this congruence, then we must take

$$\kappa = \varepsilon = \pi = 0 \tag{1.4.29}$$

since then by equations (1.4.12) we have

$$D\ell^{\mu} = Dn^{\mu} = Dm^{\mu} = D\bar{m}^{\mu} = 0.$$

If, in addition, ℓ^{μ} are taken to be hypersurface orthogonal, that is, proportional to a gradient field, then, as follows from (1.4.27), we must take

$$\rho = \bar{\rho} \tag{1.4.30}$$

and if ℓ^{μ} are equal to a gradient field we have

$$\rho = \bar{\rho} \quad \text{and } \tau = \bar{\alpha} + \beta \tag{1.4.31}$$

as a result of (1.4.26).

From (1.4.9a)

$$\ell_{\mu;\nu}n^{\nu} = -\,\tau\bar{m}_{\mu} - \bar{\tau}m_{\mu} + (\gamma+\bar{\gamma})\ell_{\mu}. \tag{1.4.32}$$

The quantity τ describes how the direction of ℓ^{μ} changes as we move in the direction n^{μ}, as can be seen from (1.4.32).

Again, we can make $\gamma = 0$ by the change $\ell^{\mu} \to G\ell^{\mu}$. The spin coefficients ν,μ,λ,π are, in this case, analogous to $\kappa,-\rho,-\sigma,\tau$, respectively, the difference being that the congruence now used is given by n^{μ} instead of ℓ^{μ}.

1.4.6 Goldberg-Sachs theorem

The study of null geodesic congruences has proved to be a powerful tool in the investigation of gravitational fields. In particular, it has been discovered that the properties of such congruences are closely related to the Petrov classification. A very well known theorem describing this connection is the Goldberg-Sachs theorem [22] which, in fact, relates algebraic speciality with the geodesic and shear-free character of the congruence formed by the principal null direction.

Statement of the theorem: In the empty space ($R_{\alpha\beta} = 0$) if there exists a shear-free null geodesic, then the Riemann tensor of the space must be algebraically specialized.

Proof: The theorem will be proved here using the Newman-Penrose formalism. First we shall prove that if the Riemann tensor is algebraically specialized, having $\Psi_0 = \Psi_1 = 0$, then $\sigma = \kappa = 0$. With these assumptions Bianchi identities become

$$3\sigma\Psi_2 = 0 \tag{1.4.33a}$$

$$-\delta\Psi_2 = -3\tau\Psi_2 + 2\sigma\Psi_3 \tag{1.4.33b}$$

$$\Delta\Psi_2 - \delta\Psi_3 = -3\mu\Psi_2 + (-2\tau + 2\beta)\Psi_3 + \sigma\Psi_4 \tag{1.4.33c}$$

$$-3\kappa\Psi_2 = 0 \tag{1.4.33d}$$

$$D\Psi_2 = -2\kappa\Psi_3 + 3\rho\Psi_2 \qquad (1.4.33e)$$

$$D\Psi_3 - \bar{\delta}\Psi_2 = -\kappa\Psi_4 - (2\varepsilon - 2\rho)\Psi_3 + 3\pi\Psi_2. \qquad (1.4.33f)$$

From (1.4.33a) either $\sigma = 0$ or $\Psi_2 = 0$. If $\sigma \neq 0$ then $\Psi_2 = 0$ and by (1.4.33b) and (1.4.33c) also $\Psi_3 = 0$ and $\Psi_4 = 0$. That is, $\sigma = 0$ unless the space is flat. Similarly, from (1.4.33d), (1.4.33e) and (1.4.33f) we have $\kappa = 0$ unless the space is flat.

The converse is more difficult to prove. We assume $\sigma = \kappa = 0$ and wish to prove $\Psi_0 = \Psi_1 = 0$. By transformation of the form $m^\mu \to e^{i\vartheta}m^\mu$ and by using a suitable scaling of ℓ^μ, we can set $\varepsilon = 0$. Equations (1.4.3) then become

$$D\rho = \rho^2 \qquad (1.4.34a)$$

$$\Psi_0 = 0 \qquad (1.4.34b)$$

$$D\tau = (\tau + \bar{\pi})\rho + \Psi_1 \qquad (1.4.34c)$$

$$D\beta = \beta\bar{\rho} + \Psi_1 \qquad (1.4.34d)$$

$$\delta\rho = \rho(\bar{\alpha} + \beta) + (\rho - \bar{\rho})\tau - \Psi_1 \ . \qquad (1.4.34e)$$

With the fact that $\Psi_0 = 0$ the needed Bianchi identities and commutators are

$$\delta\Psi_1 = (4\tau + 2\beta)\Psi_1 \qquad (1.4.35a)$$

$$D\Psi_1 = 4\rho\Psi_1 \qquad (1.4.35b)$$

$$(D\delta - \delta D)\varphi = (\bar{\pi} - \bar{\alpha} - \beta)D\varphi + \bar{\rho}\delta\varphi \ . \qquad (1.4.35c)$$

Consider now the null rotation (1.4.13a). This rotation of the tetrad does not change ℓ^{μ} or disturb the relation $\varepsilon = 0$. The complex function a can be chosen so that $\tau = 0$, then (1.4.35a) becomes

$$\delta\Psi_1 = 2\beta\Psi_1. \tag{1.4.36}$$

Case I: If $\rho = 0$, then from (1.4.34e) we have $\Psi_1 = 0$ and the converse is proved.

Case II: If $\rho \neq 0$, then (1.4.36) and (1.4.35b) are rewritten as follows

$$\delta \ln \Psi_1 = 2\beta \tag{1.4.37}$$

$$D \ln \Psi_1 = 4\rho. \tag{1.4.38}$$

From (1.4.37) and (1.4.38) we have

$$(D\delta - \delta D)\ln \Psi_1 = 2D\beta - 4\delta\rho = 2\beta\bar{\rho} - 4\rho(\bar{\alpha} + \beta) + 6\Psi_1 \tag{1.4.39}$$

after using (1.4.34d) and (1.4.34e).

The commutator (1.4.35c) with $\varphi = \ln \Psi_1$ and using (1.4.37) and (1.4.38) becomes

$$(D\delta - \delta D)\ln \Psi_1 = 2\beta\bar{\rho} - 4\rho(\bar{\alpha} + \beta) + 4\rho\bar{\pi}. \tag{1.4.40}$$

Subtracting (1.4.40) from (1.4.39) we have

$$\Psi_1 = \frac{2}{3}\,\bar{\pi}\rho. \tag{1.4.41}$$

From (1.4.34c), since $\tau = 0$, we have that

$$\Psi_1 = -\,\bar{\pi}\rho. \tag{1.4.42}$$

This is clearly incompatible with (1.4.41) since ρ is assumed different from zero. Hence, $\Psi_1 = 0$,which completes the proof.

Corrolary

If the field is algebraically special and of Petrov type D, then the congruences formed by the two principal null directions, ℓ^μ and n^μ, must both be geodesic and shear-free, that is $\kappa = \sigma = \nu = \lambda = 0$ when $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and conversely. Petrov type D is a particularly interesting class to study for several reasons. The Schwar zschild and Kerr metrics are familiar members of this class. Besides being free of line singularities these two metrics are also stationary in time.

But do all type D fields share this property? One can only prove a weak converse, namely, that an algebraically special field which is stationary, time reversible and with a finite source can only be of type D. All type D vacuum metrics are derived by Kinnersley [23] with the help of Newman-Penrose formalism.

1.5 KILLING SPINORS

The algebraic characterization of the curvature tensor is of great importance in the study of the properties of gravitational fields. The Kerr solution [24], in particular, which is axially symmetric and stationary, has an algebraically special vacuum curvature tensor of type D. The special interest in the Kerr solution is due to the fact that it describes the final stage of the gravitational collapse of a star, that of a black hole.

In a very well known paper, Carter [25] showed how to integrate the geodesic equations of the Kerr metric using the fact that the Hamilton-Jacobi equation for the geodesics can be solved, in this case, by separation of variables. Later, Walter and Penrose [26] presented an alternative procedure to obtain the same results and related Carter's fourth constant of motion to the existence of a Killing tensor of order two. This paper, in which they also introduced the notion of a Killing spinor, was the beginning of a new interest in Killing tensors and of a thorough study of their properties by many authors.

A symmetric spinor of second order χ_{BC} is called a Killing spinor if it satisfies the twistor equation

$$\nabla^{A'}_{(A}\,\chi_{BC)} = 0. \tag{1.5.1}$$

As we have already seen, if $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor of a vacuum solution, the Weyl spinor $\Psi_{ABCD} = \Psi_{(ABCD)}$ is defined by

$$C_{\alpha\beta\gamma\delta} \leftrightarrow \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{A'B'C'D'}.$$

The Weyl spinor is completely determined by the five complex scalars Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4 of Newman-Penrose formalism defined in terms of Ψ_{ABCD} and the spinor dyad $\{o^A, \iota^A\}$ normalized as $o_A\iota^A = 1$. In the case of a type D Weyl spinor, $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$, provided that o^A and ι^A are chosen to be repeated principal spinors of Ψ_{ABCD} so that the corresponding null vectors ℓ^μ and n^μ lie along the repeated principal null directions of $C_{\alpha\beta\gamma\delta}$.

If we put

$$\Psi = 6\Psi_2 \tag{1.5.2}$$

then the Weyl spinor of a type D vacuum solution is given by

$$\Psi_{ABCD} = \Psi o_{(A}o_B\iota_C\iota_{D)} . \tag{1.5.3}$$

Then, if we write

$$\chi_{BC} = \Psi^{-1/3}\ o_{(B}\iota_{C)} \tag{1.5.4}$$

the spinor χ_{BC} is a Killing spinor, that is, it satisfies $\nabla^{A'}_{(A}\ \chi_{BC)} = 0$. This spinor defines a real symmetric trace-free tensor

$$P_{\alpha\beta} \leftrightarrow \chi_{AB}\bar{\chi}_{A'B'} \tag{1.5.5}$$

which satisfies the <u>conformal Killing equation</u>

$$\nabla_{(\alpha}P_{\beta\gamma)} = P_{(\alpha}g_{\beta\gamma)} \tag{1.5.6}$$

with P^α given by

$$P^{\alpha} = \frac{1}{3} \nabla_{\beta} P^{\alpha\beta} \tag{1.5.7}$$

and $g_{\beta\gamma}$ the metric tensor.

Note that if

$$\varkappa^{A} \bar{\varkappa}^{A'} \nabla_{AA'} \varkappa_{B} = 0 \tag{1.5.8}$$

then

$$\varkappa^{A} \bar{\varkappa}^{A'} \nabla_{AA'} (\varkappa^{B} \varkappa^{C} \chi_{BC}) = 0 . \tag{1.5.9}$$

If equation (1.5.8) holds, the null vectors $K^{\alpha} \leftrightarrow \varkappa^{A} \bar{\varkappa}^{A'}$ are tangent to null geodesics, since it implies that $K^{\alpha} \nabla_{\alpha} K^{\beta} = 0$. Then equation (1.5.9) in fact tells us that the complex scalar

$$\chi = \varkappa^{B} \varkappa^{C} \chi_{BC} \tag{1.5.10}$$

is <u>constant</u> along the null geodesics.

In addition, arg χ is also constant along null geodesics and this enables us to determine the parallel propagation of "flag planes" or polarization planes along them, for equation (1.5.8) states, in addition to the fact that the K^{α} are tangent to null geodesics, that the flag plane of $\varkappa^{A}$ is also parallelly propagated along them. Knowledge of χ_{BC} defines this parallel propagation since, for example, if $\chi > 0$ at one point of a null geodesic, then $\chi > 0$ also at every other point of it. This fixes the flag plane of $\varkappa^{A}$ in relation to χ_{AB}. And so knowledge of any non-zero Killing spinor χ_{AB} gives information on the propagation of polarisation of a particle along any null geodesics.

The absolute value of χ, that is, the real scalar $|\chi|^2$, is also a constant of motion for null geodesics associated with the conformal Killing tensor $P_{\alpha\beta}$.

The equation defining a Killing spinor of second order can be generalized to the case of a Killing spinor with arbitrary (unequal) numbers of symmetric upper and lower indices and so we obtain

$$\nabla^{(A'_0}_{(A_0} \chi_{B_1 \ldots B_s)}^{A'_1 \ldots A'_r)} = 0 \qquad (1.5.11)$$

which in flat space-times represents in a conformally invariant way the general trace-free symmetric twistor equation of valence $\binom{r}{s}$.

In the particular case of the Killing spinor $\chi_{BC\ldots D}$ satisfying $\nabla^{A'}_{(A} \chi_{BC\ldots D)} = 0$ we have as before the complex scalar

$$\chi = \chi_{BC\ldots D}\, \varkappa^B \varkappa^C \ldots \varkappa^D \qquad (1.5.12)$$

constant along the null geodesics. The absolute value of χ, which is also a constant of motion for null geodesics, is in this case associated with the conformal Killing tensor

$$P_{\beta\gamma\ldots\delta} \leftrightarrow \chi_{BC\ldots D}\, \bar{\chi}_{B'C'\ldots D'}. \qquad (1.5.13)$$

To verify that this tensor is a conformal Killing tensor one needs only to imagine the decomposition of $\nabla_{(\alpha}P_{\beta\gamma\ldots\delta)}$ into symmetric spinors [27]. The term which is symmetric in all spinor indices of each type vanishes because $\chi_{BC\ldots D}$ is a Killing spinor. All other terms in the decomposition include factors of ε_{AB} and $\varepsilon_{A'B'}$. If ε_{AB} occurs as a factor, then $\varepsilon_{A'B'}$ must also occur in the same term in order that the expression be symmetric in tensor indices α and β. Then, because of the identity $g_{\alpha\beta} = \varepsilon_{AB}\varepsilon_{A'B'}$, it follows that each non-zero term in the decomposition of $\nabla_{(\alpha}P_{\beta\gamma\ldots\delta)}$ contains the metric as a factor and therefore the tensor $P_{\beta\gamma\ldots\delta}$ satisfies the conformal Killing tensor equation.

Going back to our initial case of the Killing spinor χ_{BC} and its associated conformal Killing tensor $P_{\beta\gamma}$ and writing

$$\Sigma = \Psi^{-1/3}\, \bar{\Psi}^{-1/3} \tag{1.5.14}$$

since $o^A\bar{o}^{A'} \leftrightarrow \ell^\mu$ and $\iota^A\bar{\iota}^{A'} \leftrightarrow n^\mu$, we have

$$P_{\beta\gamma} = \Sigma\,(\ell_{(\beta}n_{\gamma)} - \frac{1}{4}\, g_{\beta\gamma})\,. \tag{1.4.15}$$

Define

$$Q_{\beta\gamma} = \Sigma\, \ell_{(\beta}n_{\gamma)} \tag{1.5.16}$$

then from the equation $\nabla_{(\alpha}P_{\beta\gamma)} = P_{(\alpha}g_{\beta\gamma)}$ we have that

$$\nabla_{(\alpha}Q_{\beta\gamma)} = Q_{(\alpha}g_{\beta\gamma)} \tag{1.5.17}$$

where

$$Q_\alpha = P_\alpha + \frac{1}{4}\nabla_\alpha\Sigma \, . \tag{1.5.18}$$

Supposing that there exists a scalar Q such that Q_α is the gradient of this scalar, that is $Q_\alpha = \nabla_\alpha A$, and putting

$$K_{\beta\gamma} = Q_{\beta\gamma} - Qg_{\beta\gamma} \tag{1.5.19}$$

we end up with

$$\nabla_{(\alpha}K_{\beta\gamma)} = \nabla_{(\alpha}Q_{\beta\gamma)} - Q_{(\alpha}g_{\beta\gamma)} = 0 \tag{1.5.20}$$

and so $K_{\beta\gamma}$ is a Killing tensor of second order.

Walker and Penrose [26] proved that the Kerr solution admits $K_{\beta\gamma}$ as a Killing tensor and consequently that this tensor is irreducible provided the angular momentum parameter "a" of the Kerr metric is not zero.

It is also possible, in the case of the Kerr space-time, to give a physical meaning of the Killing spinor χ_{AB} [28]. Suppose that from the Killing spinor χ_{AB} we construct the bivector

$$M_{\alpha\beta} \leftrightarrow \chi_{AB}\varepsilon_{A'B'} + \bar{\chi}_{A'B'}\varepsilon_{AB} \, . \tag{1.5.21}$$

Then, in the linear approximation to Einstein's theory, this bivector can be shown to be related to the expression for the total angular momentum of the source of the gravitational field.

Finally, it should be mentioned that, although historically irreducible Killing tensors have been in use in differential geometry from the time of Jacobi in the 1840's [29], it was only after the work of Carter, Walker and Penrose on the Kerr metric that these tensors attracted the attention of several researchers and became important in general relativistic studies. And this leads naturally to Chapter 2, where Killing tensors, their properties and literature will be presented.

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CHAPTER 2

THE STUDY OF KILLING TENSORS IN GENERAL RELATIVITY

2.1 INTRODUCTION

Einstein's field equations $R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} = 8\pi T_{\mu\nu}$ are ten coupled partial differential equations for the unknown components $g_{\mu\nu}$ of the metric tensor and are so complicated that it is very hard to find exact solutions, even in the vacuum case, where the components $T_{\mu\nu}$ of the energy-momentum tensor vanish.

Exact solutions for different forms of $T_{\mu\nu}$ and different forms of boundary conditions can be found when a number of assumptions have been made about the form of the metric tensor or the Petrov type of the Weyl tensor and so on [1].

The commonest assumption made by people looking for exact solutions is that the metric admits a Killing vector or isometry. But not even the assumption of two commuting Killing vectors is sometimes enough to lead to the solution of field equations. So, the question arises as to what other kinds of assumptions on the metric must be made in order to help solve the field equations or lead to a better understanding of other areas in general relativity.

The existence of other types of symmetries rather than isometries is a very useful assumption that may lead to interesting results. Hidden symmetries associated with Killing tensors are the symmetries studied in the present thesis. Killing tensors, which are generalizations of Killing vectors, yield constants of motion just like Killing vectors do and enable the separation of the Hamilton-Jacobi equation in most type D vacuum solutions. Moreover, it is proved in this chapter that they give a Lie algebraic extension of the Lie algebra of isometries and generate new solutions of the geodesic deviation equation.

2.2 KILLING VECTORS AND TENSORS: DEFINITIONS AND PROPERTIES

2.2.1 Killing vectors and isometries

Let the manifold M be endowed with the metric $g_{\mu\nu}$ and let $\{x^\mu\}$ be a coordinate system for the manifold at its point P. An infinitesimal point transformation

$$\bar{x}^\mu = x^\mu + \varepsilon X^\mu(x) \qquad (\varepsilon << 1) \tag{2.2.1}$$

is said to define an infinitesimal motion or isometry [2] on M if the infinitesimal distance ds between two arbitrary points x^μ and $x^\mu + dx^\mu$ is equal to the infinitesimal distance $d\bar{s}$ between the two corresponding points $\bar{x}^\mu$ and $\bar{x}^\mu + d\bar{x}^\mu$ (except for higher terms in ε) or equivalently if it leaves the metric $g_{\mu\nu}$ invariant. Since

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \quad \text{and} \quad d\bar{s}^2 = g_{\mu\nu}(\bar{x})d\bar{x}^\mu d\bar{x}^\nu \tag{2.2.2}$$

we see that a necessary and sufficient condition for (2.2.1) to be an isometry on the manifold is that $d\bar{s}^2 = ds^2$, that is ,

$$g_{\mu\nu}(\bar{x})d\bar{x}^\mu d\bar{x}^\nu = g_{\mu\nu}(x)dx^\mu dx^\nu \tag{2.2.3}$$

or that

$$(g_{\mu\nu} + \varepsilon X^{\sigma} g_{\mu\nu,\sigma})(dx^{\mu} + \varepsilon X^{\mu}{}_{,\sigma} dx^{\sigma})(dx^{\nu} + \varepsilon X^{\nu}{}_{,\sigma} dx^{\sigma})$$

$$= g_{\mu\nu}(x) dx^{\mu} dx^{\nu} \tag{2.2.4}$$

be satisfied for any dx^{μ} (except for higher terms in ε), that is to say, if

$$X^{\sigma} g_{\mu\nu,\sigma} + X^{\sigma}{}_{,\mu} g_{\sigma\nu} + X^{\sigma}{}_{,\nu} g_{\mu\sigma} = 0\,. \tag{2.2.5}$$

(2.2.5) can also be written in the form

$$X_{\mu;\nu} + X_{\nu;\mu} = 0 \iff \nabla_{\mu} X_{\nu} + \nabla_{\nu} X_{\mu} = 0 \tag{2.2.6}$$

as well as in the form

$$X^{\sigma} g^{\mu\nu}{}_{,\sigma} - X^{\mu}{}_{,\sigma} g^{\sigma\nu} - X^{\nu}{}_{,\sigma} g^{\mu\sigma} = 0. \tag{2.2.7}$$

Equation (2.2.5) is called <u>Killing's equation</u> and any vector X^{μ} satisfying it is called a <u>Killing vector.</u>

Note that if we choose a coordinate system in which the vector X^{μ} has the components, say,

$$X^{\mu} = \delta^{\mu}_{1} \tag{2.2.8}$$

then from (2.2.5) we have that

$$g_{\mu\nu,1} = 0 \tag{2.2.9}$$

which shows that the components $g_{\mu\nu}$ of the metric tensor are independent of the variable x^1 in this particular coordinate system (see Fig. 2.1).

Some properties of Killing vectors which follow immediately from definition (2.2.5) are:

i) $$X^{\mu}{}_{;\mu} = 0. \tag{2.2.10}$$

ii) If $T^{\mu\nu} = T^{\nu\mu}$ and $\nabla_{\nu}T^{\mu\nu} = 0$ are the formal equations of the energy-momentum tensor, then

$$\nabla_{\mu}(T^{\mu\nu}X_{\nu}) = 0 \tag{2.2.11}$$

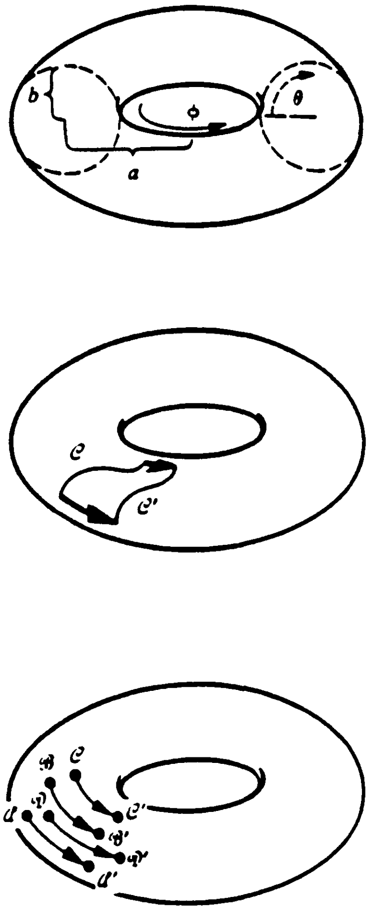
which means that the vector field $T^{\mu\nu}X_{\nu}$ is a conserved current.

iii) If $u^{\mu}u_{\mu} = \pm 1$ and $u^{\nu}\nabla_{\nu}u^{\mu} = 0$, that is, if u^{μ} is the unit tangent vector to a time-like/space-like geodesic, then $X_{\mu}u^{\mu}$ is a conserved quantity along the geodesic orbit.

This last property can be stated as a theorem.

Theorem: In any geometry endowed with a symmetry described by a Killing vector field X^{μ}, motion along any geodesic leaves constant the scalar product of the tangent vector with the Killing vector [3].

Killing vectors and isometries (illustrated here by a doughnut)



We suppose that on our manifold, here a doughnut with $ds^2 = b^2 d\vartheta^2 + (a - b\cos\vartheta)^2 d\varphi^2$, in a given coordinate system, the metric components are independent of a particular coordinate x^μ, here the coordinate $x^\mu = \varphi$.

We translate an arbitrary curve C through the infinitesimal displacement $\varepsilon \underline{X} = \varepsilon \frac{\partial}{\partial\varphi}$, $\varepsilon << 1$, to form a new curve C'. Because $\partial g_{\mu\nu}/\partial\varphi = 0$ the two curves have the same length * .

Since the length of every curve is preserved by the above translation, the distances between neighbouring points are also preserved. We say that $\underline{X} = \frac{\partial}{\partial\varphi}$ is the generator of an isometry on the manifold.

A vector field $\underline{X}$ generates an isometry if and only if it satisfies Killing's equation $X_{(\mu;\nu)} = 0$.

---

*The original curve has length $L_C = \int_{\lambda_1}^{\lambda_2} [g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}]^{1/2} d\lambda$ and

the displaced one has length $L_{C'} = \int_{\lambda_1}^{\lambda_2} [(g_{\mu\nu} + \varepsilon \frac{\partial g_{\mu\nu}}{\partial\varphi}) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}]^{1/2} d\lambda$.

Obviously $L_C = L_{C'}$.

Figure 2.1. Killing vectors and isometries.

(Misner - Thorne - Wheeler, 1970).

If, instead of (2.2.3), $d\bar{s}^2 = e^{2\varphi}ds^2$ is true, that is, if

$$g_{\mu\nu}(\bar{x})d\bar{x}^{\mu}d\bar{x}^{\nu} = e^{2\varphi}g_{\mu\nu}(x)dx^{\mu}dx^{\nu} \tag{2.2.12}$$

then the infinitesimal point transformation (2.2.1) is said to define a <u>conformal motion</u> in M. The vector X^{μ} satisfies the equation

$$\nabla_{\mu}X_{\nu} + \nabla_{\nu}X_{\mu} = 2\varphi g_{\mu\nu} \tag{2.2.13}$$

and is called a <u>conformal Killing vector</u>. It defines a conformal symmetry on M in the same way that a Killing vector defines an isometry on M.

Let us now turn to the manifold M and suppose that the metric $g_{\mu\nu}$ is invariant under several isometries. The set of isometries on M has the structure of a <u>group</u> [4]: an associative product is defined (the product of isometries A and B is A followed by B), an inverse exists for each element, and a unit transformation (the identity) exists. The <u>group of isometries</u> is the <u>symmetry group</u> of M.

The Killing vector X^{μ} has then a direct geometrical interpretation as the tangent to the orbit at some point of M under the action of a one-parameter group of isometries on $(M, g_{\mu\nu})$ which can be expressed mathematically by

$$\mathcal{L}_{\underline{X}}\, g_{\mu\nu} = 0 \tag{2.2.14}$$

with $\mathcal{L}_{\underline{X}}$ being the Lie derivative* with respect to X^{μ}.

---

* See Appendix A2.

Isometries are obtained from the Killing vectors by exponentiation in the same way that group elements are obtained from the infinitesimal generators which form the Lie algebra of the group. In an abstract group the commutators of the infinitesimal group elements define the structure constants of the group. Thus if Y^μ ($\mu = 1,\ldots,m$) are the basis elements of the Lie algebra, then the structure constants of G are defined by

$$[Y_\mu, Y_\nu] = C^\sigma_{\mu\nu}\, Y_\sigma. \tag{2.2.15}$$

The Jacobi identity* applied to Y^μ shows that the $C^\sigma_{\mu\nu}$ must satisfy

$$C^\sigma_{\mu\nu}\, C^\lambda_{\sigma\varkappa} + C^\sigma_{\nu\varkappa}\, C^\lambda_{\sigma\mu} + C^\sigma_{\varkappa\mu}\, C^\lambda_{\sigma\nu} = 0. \tag{2.2.16}$$

Further, any set of $C^\sigma_{\mu\nu}$ (antisymmetric in μ,ν) which satisfy (2.2.16) are the structure constants for some Lie group.

The group of isometries of the manifold M is isomorphic to some abstract group G. The Lie algebra commutator of (2.2.15) is replaced by the commutator [,] of the Killing vectors X^μ of the symmetry group. The m independent Killing vectors X^μ (m may or may not be equal to the dimension of M) obey

$$[X_\mu, X_\nu] = C^\sigma_{\mu\nu}\, X_\sigma. \tag{2.2.17}$$

---

* The Jacobi identity for vector fields X,Y,Z is $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$

The structure coefficients in (2.2.17) are constants and are equal to the structure constants of the Lie algebra of the abstract group of isometries. Given a Lie group G with structure constants $C^{\sigma}_{\mu\nu}$, a manifold M is said to be invariant under the group G if there are m (the dimension of G) Killing vector fields X^{μ} which obey the Lie algebra relation (2.2.17) [5].

2.2.2 Integrability conditions for Killing vector equations.

Killing vector equation (2.2.5) is much more restrictive than it looks, for it allows us to determine the whole function $X^{\mu}(x)$ from given values of X^{μ} and $X_{\mu;\nu}$ at some point P of the manifold M [6]. To see this, we only need to recall the Ricci identity

$$X_{\mu;\nu\varkappa} - X_{\mu;\varkappa\nu} = - R^{\sigma}_{\mu\nu\varkappa} X_{\sigma} \tag{2.2.18}$$

and the cyclic sum rule for the curvature tensor

$$R^{\sigma}_{\mu\nu\varkappa} + R^{\sigma}_{\nu\varkappa\mu} + R^{\sigma}_{\varkappa\mu\nu} = 0. \tag{2.2.19}$$

By adding (2.2.18) and its two cyclic permutations, we find that the vector X_{σ} must satisfy the relation

$$0 = X_{\mu;\nu\varkappa} - X_{\mu;\varkappa\nu} + X_{\nu;\varkappa\mu} - X_{\nu;\mu\varkappa} + X_{\varkappa;\mu\nu} - X_{\varkappa;\nu\mu}. \tag{2.2.20}$$

Killing vector equations (2.2.5) and (2.2.20) then give

$$X_{\mu;\nu\varkappa} - X_{\mu;\varkappa\nu} - X_{\varkappa;\nu\mu} = 0 \qquad (2.2.21)$$

and using (2.2.18) we finally have that

$$X_{\varkappa;\nu\mu} = - R^{\sigma}_{\mu\nu\varkappa}\, X_{\sigma}. \qquad (2.2.22)$$

And so Killing vector equations are, in fact, equivalent to the system of equations

$$X_{\mu;\nu} = \omega_{\mu\nu} = - \omega_{\nu\mu} \qquad (2.2.23a)$$

$$X_{\mu;\nu\varkappa} = \omega_{\mu\nu;\varkappa} = - R^{\sigma}_{\varkappa\nu\mu}\, X_{\sigma}. \qquad (2.2.23b)$$

Equations (2.2.23a) and (2.2.23b) may be regarded as a system of linear homogeneous first order differential equations for the quantities X^{μ} and $\omega_{\mu\nu}$. Hence, given X^{μ} and $X_{\mu;\nu}$ at some point P of the manifold, we can determine the second derivatives of $X^{\mu}(x)$ at P from equation (2.2.22) and we can successively find higher derivatives of X^{μ} at P by taking derivatives of this equation. All the derivatives of X^{μ} at P will thus be determined as linear combinations of $X^{\mu}(P)$ and $X_{\mu;\nu}(P) = \omega_{\mu\nu}(P)$. Whether (2.2.22) is soluble for a given set of initial data $X^{\mu}(P)$ and $X_{\mu;\nu}(P)$ depends on the integrability of this equation, which in turn depends on the metric. One integrability condition follows from the general formula for commutators of covariant derivatives of tensors

$$X_{\mu;\nu\varkappa\lambda} - X_{\mu;\nu\lambda\varkappa} = - R^{\sigma}_{\mu\varkappa\lambda}\, X_{\sigma;\nu} - R^{\sigma}_{\nu\varkappa\lambda}\, X_{\mu;\sigma}. \qquad (2.2.24)$$

Equation (2.2.22) will satisfy condition (2.2.24) if and only if

$$R^{\sigma}_{\varkappa\mu\nu} X_{\sigma;\lambda} - R^{\sigma}_{\lambda\mu\nu} X_{\sigma;\varkappa} + (R^{\sigma}_{\varkappa\mu\nu;\lambda} - R^{\sigma}_{\lambda\mu\nu;\varkappa})X_{\sigma}$$

$$= - R^{\sigma}_{\mu\varkappa\lambda} X_{\sigma;\nu} - R^{\sigma}_{\nu\varkappa\lambda} X_{\mu;\sigma}. \qquad (2.2.25)$$

This condition is of course empty for a flat space, but in general it will impose linear relations among the X^{μ} and $X_{\mu;\nu}$ at any given point. Alternatively, if we know something about the Killing vector of an unknown metric, then, using (2.2.25), we can learn something about its curvature tensor.

2.2.3 Killing and Killing-Yano tensors

A generalization of the notion of a Killing vector is that of a Killing tensor of order p, defined as follows [1]. A Killing tensor of order p is a symmetric tensor $K_{\mu_1 \ldots \mu_p}$ which satisfies the generalized Killing equation

$$K_{(\mu_1 \ldots \mu_p;\lambda)} = 0. \qquad (2.2.26)$$

It is obvious from (2.2.26) that a Killing vector is a first order Killing tensor (p = 1). Killing tensors of order two satisfy

$$K_{\mu\nu} = K_{\nu\mu} \text{ and } K_{(\mu\nu;\lambda)} = 0 \Leftrightarrow \nabla_{(\lambda} K_{\mu\nu)} = 0. \qquad (2.2.27)$$

Equation (2.2.27) can also be written in the form

$$K_{(\mu\nu,\lambda)} + K^{\sigma}_{(\lambda}g_{\mu\nu),\sigma} = 0 \qquad (2.2.28)$$

or in the form

$$K^{(\mu\nu}{}_{,\sigma}g^{\lambda)\sigma} - g^{(\mu\nu}{}_{,\sigma}K^{\lambda)\sigma} = 0. \qquad (2.2.29)$$

Trivial examples of Killing tensors of order two are the metric tensor $g_{\mu\nu}$ and all products $K_{\mu\nu} = \underset{1}{X}{}_{(\mu}\underset{2}{X}{}_{\nu)}$ of Killing vectors $\underset{1}{X}{}^{\mu}$, $\underset{2}{X}{}^{\nu}$ (not necessarily different) and linear combinations of these with constant coefficients α,β,γ

$$K_{\mu\nu} = \alpha\underset{1}{X}{}_{\mu}\underset{2}{X}{}_{\nu} + \beta\underset{2}{X}{}_{\mu}\underset{1}{X}{}_{\nu} + \gamma(\underset{1}{X}{}_{\mu}\underset{2}{X}{}_{\nu} + \underset{2}{X}{}_{\mu}\underset{1}{X}{}_{\nu}). \qquad (2.2.30)$$

Killing tensors which are constructible in this way are called reducible or trivial. In the study of the properties of gravitational fields, however, those Killing tensors with real importance are the irreducible ones which cannot be constructed from the already existing Killing vectors.

Some other examples of Killing tensors of order two are the following.

i) If two Riemannian spaces with metrics $g_{\mu\nu}$ and $g'_{\mu\nu}$ have the same geodesic paths, then they are projectively related. The affine connections have the relationship $\Gamma'^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} + 2\,\delta^{\mu}_{(\nu}\partial_{\lambda)}\psi$ for some scalar ψ. It follows that $K_{\mu\nu} := e^{-4\psi}g'_{\mu\nu}$ is a Killing tensor in the space of $g_{\mu\nu}$.

ii) An affine collineation is a vector field $\underline{X}$ satisfying $\mathcal{L}_{\underline{X}}\Gamma^{\mu}_{\nu\lambda} = 0$. Using the identity $\mathcal{L}_{\underline{X}}\,\Gamma^{\mu}_{\nu\lambda} = \nabla_{\nu}\nabla_{\lambda}X^{\mu} - R^{\mu}_{\lambda\nu\sigma}\,X^{\sigma}$ [7]

it can be seen that $\nabla_{(\nu}\nabla_{\lambda}X_{\mu)} = 0$, if X^{μ} is an affine collineation. In this case $\nabla_{(\lambda}X_{\mu)}$ is a Killing tensor.

iii) A projective collineation is a vector field $\underline{Y}$ satisfying $\mathcal{L}_{\underline{Y}}\,\Gamma^{\mu}_{\nu\lambda} = 2\delta^{\mu}_{(\nu}\partial_{\lambda)}\varphi$ for some scalar φ. Then the tensor $K_{\mu\nu} := \nabla_{(\mu}Y_{\nu)} - 2\varphi g_{\mu\nu}$ is a Killing tensor [8].

iv) Symmetric tensors $K_{\mu\nu}$ which satisfy conditions of the form $K_{\mu\nu;\lambda} = A_{\mu\lambda}B_{\nu} + A_{\nu\lambda}B_{\mu} - 2A_{\mu\nu}B_{\lambda}$ with $A_{\mu\nu} = A_{\nu\mu}$ are Killing tensors [9].

The Killing equation (2.2.27) splits naturally into two parts. Define $P_{\mu\nu}$ as the trace-free part of $K_{\mu\nu}$ and $K := K^{\mu}_{\mu}$ as its trace, so that we can write

$$K_{\mu\nu} = P_{\mu\nu} + \frac{1}{4}Kg_{\mu\nu}. \tag{2.2.31}$$

When the Killing equation is transvected with $g^{\mu\nu}$ we obtain the trace equation in the form

$$\nabla^{\mu}P_{\mu\nu} + \frac{3}{4}\nabla_{\nu}K = 0. \tag{2.2.32}$$

The remaining trace-free part of the Killing equation is

$$\nabla_{(\lambda}P_{\mu\nu)} - \frac{1}{3}g_{(\mu\nu}\nabla^{\sigma}P_{\lambda)\sigma} = 0. \tag{2.2.33}$$

A tensor $P_{\mu\nu}$ for which equation (2.2.33) holds is referred to as a conformal Killing tensor [10]. Conformal Killing tensors form a subalgebra of the much larger algebra of all totally symmetric tensors on a manifold, and so the sum, product and

Lie bracket of two conformal Killing tensors is again a conformal Killing tensor [11].

Walker and Penrose [12] and Hughston and Sommers [10] used the notion of a Killing spinor (presented in chapter 1) to prove the following theorems for Killing tensors of order two in type D vacuum solutions.

Theorem 1: Each vacuum solution of the Einstein field equations (with or without cosmological constant), for which the Weyl tensor is of type D, admits a conformal Killing tensor $P_{\mu\nu}$ satisfying equation (2.2.33), which is irreducible provided the space-time admits fewer than four independent Killing vectors.

Theorem 2: In every type D vacuum solution, with the exception of the C-metric and its rotating generalization due to Kinnersley [13], the divergence $\frac{1}{3}\nabla^{\sigma}P_{\lambda\sigma}$ is the gradient of a scalar. Then, the tensor $K_{\mu\nu}$ given by (1.5.19) is a Killing tensor of order two for this solution.

Finally, a Killing-Yano tensor of order p is a skew-symmetric tensor $a_{\mu_1\ldots\mu_p}$ which satisfies the equation

$$a_{\mu_1\ldots\mu_p;\nu} + a_{\mu_1\ldots\nu;\mu_p} = 0. \tag{2.2.34}$$

Equation (2.2.34) is also a generalization of Killing's equation [1]. A Killing-Yano tensor of order two satisfies*

* Killing-Yano tensors of order two have been studied in the context of general relativity by Penrose and Floyd [15], so Carter and McLenaghan [16] have introduced the name Penrose-Floyd tensors and the name Penrose-Floyd equation for Killing-Yano equation (2.2.35).

$$a_{\mu\nu} = -a_{\nu\mu} \quad \text{and} \quad a_{\mu\nu;\lambda} + a_{\mu\lambda;\nu} = 0 \tag{2.2.35}$$

and can be used to construct a symmetric Killing tensor $K_{\mu\nu}$ * by taking the contracted product with itself as follows

$$a_{\mu\sigma} a^{\sigma}_{\nu} = K_{\mu\nu}. \tag{2.2.36}$$

A theorem relating the existence of Killing-Yano tensors of order two to the existence of a symmetric Killing tensor in type D vacuum solutions is proved by Stephani [14] using the Newman-Penrose formalism. In terms of a null tetrad $(\ell^{\mu},n^{\mu},m^{\mu},\bar{m}^{\mu})$ satisfying the orthogonality conditions (1.2.40) this theorem is as follows.

Theorem. All type D vacuum solutions which admit a symmetric Killing tensor $K_{\mu\nu}$, also admit a Killing-Yano tensor of order two $a_{\mu\nu}$, which can be written in terms of the null vectors of the above-mentioned tetrad as

$$a_{\mu\nu} = A(\ell_{\mu}n_{\nu} - n_{\mu}\ell_{\nu}) + iB(m_{\mu}\bar{m}_{\nu} - m_{\nu}\bar{m}_{\mu}).$$

---

* The inverse problem is discussed in Collinson [17], who studied the conditions which must be satisfied in order that a given Killing tensor $K_{\mu\nu}$ can be written as the contracted product of a Killing-Yano tensor with itself.

The two tensors are connected by

$$K_{\mu\nu} = a_{\mu\sigma}a^{\sigma}_{\nu} = (A^2 + B^2)(\ell_{\mu}n_{\nu} + \ell_{\nu}n_{\mu}) + B^2 g_{\mu\nu}.$$

Conformal Killing-Yano tensors of order p, in general, are introduced by Yano and Bochner [2] as generalizations of Killing vectors. A conformal Killing-Yano tensor of order p is a skew-symmetric tensor $a_{\mu_1 \ldots \mu_p}$ which satisfies the equation

$$a_{\mu_1 \ldots \mu_{p-1}\mu_p;\nu} + a_{\mu_1 \ldots \mu_{p-1}\nu;\mu_p} = 2\varphi_{\mu_1 \ldots \mu_{p-1}} g_{\mu_p \nu} \qquad (2.2.37)$$

with

$$\varphi_{\mu_1 \ldots \mu_{p-1}} = \frac{1}{n} g^{\mu_p \nu} a_{\mu_1 \ldots \mu_{p-1}\mu_p;\nu} \qquad (2.2.38)$$

and n is the dimension of the manifold.

2.2.4 Integrability conditions for Killing tensor equations

We have seen in section 2.2.2 that Killing vector equations are equivalent to the system of equations (2.2.23a) and (2.2.23b). Hauser and Malhiot [18 and 19] have derived equations analogous to those for the case of a Killing tensor $K_{\mu\nu}$ which they called the structural equations for a Killing tensor of order two.

They introduced two new tensors playing roles analogous to that of the bivector $\omega_{\mu\nu}$, as follows

$$L_{\mu\nu\sigma} = K_{\nu\sigma;\mu} - K_{\mu\sigma;\nu} \qquad (2.2.39)$$

$$M_{\mu\nu\sigma\rho} = \frac{1}{2} \{L_{\mu\nu[\sigma;\rho]} + L_{\sigma\rho[\mu;\nu]}\}. \tag{2.2.40}$$

The symmetries of $L_{\mu\nu\sigma}$ and $M_{\mu\nu\sigma\rho}$ follow directly from their definitions and hold for any symmetric tensor $K_{\mu\nu}$ whether or not it is a Killing tensor. Equations (2.2.39) and (2.2.40) imply that

$$L_{\mu\nu\sigma} = L_{[\mu\nu]\sigma} \text{ and } L_{[\mu\nu\sigma]} = 0 \tag{2.2.41}$$

and therefore in a 4-dimensional space-time $L_{\mu\nu\sigma}$ has 20 independent components, and that $M_{\mu\nu\sigma\rho}$ has the same symmetries as the Riemann tensor.

The definition $K_{(\mu\nu;\sigma)} = 0$ of a Killing tensor of second order and equation (2.2.39) imply that

$$K_{\mu\nu;\sigma} = \frac{1}{3} (L_{\sigma\mu\nu} + L_{\sigma\nu\mu}) = \frac{2}{3} L_{\sigma(\mu\nu)}. \tag{2.2.42}$$

The expressions for the covariant derivatives of $L_{\mu\nu\sigma}$ and $M_{\mu\nu\sigma\rho}$ are the following

$$L_{\mu\nu\sigma;\varkappa} = -3R_{\mu\nu\lambda(\sigma}K^{\lambda}_{\varkappa)} - \frac{9}{4} K^{\lambda}_{[\varkappa}R_{\mu\nu]\sigma\lambda} - \frac{3}{4} K^{\lambda}_{[\sigma}R_{\mu\nu]\varkappa\lambda}$$

$$+ M_{\mu\nu\sigma\varkappa} \tag{2.2.43}$$

$$M_{\mu\nu\sigma\rho;\varkappa} = R_{\mu\nu\sigma\rho;\lambda}K^{\lambda}_{\varkappa}$$

$$+ (\delta^{\alpha\beta}_{\mu\nu}\delta^{\tau\pi}_{\sigma\rho} + \delta^{\tau\pi}_{\mu\nu}\delta^{\alpha\beta}_{\sigma\rho})\{\tfrac{1}{2}(R_{\alpha\beta\tau\varkappa;\lambda} - \tfrac{3}{4} R_{\alpha\beta\tau\lambda;\varkappa})K^{\lambda}_{\pi}$$

$$- \frac{1}{4} R_{\alpha\beta\varkappa}{}^{\lambda}L_{\tau\pi\lambda} + \frac{1}{3}(R_{\alpha\varkappa\tau}{}^{\lambda} + R_{\alpha\tau\varkappa}{}^{\lambda})L_{\pi\lambda\beta}$$

$$+ \frac{1}{24} R_{\alpha\beta\tau}{}^{\lambda}(5L_{\varkappa\lambda\pi} + 7L_{\pi\varkappa\lambda})\} . \qquad (2.2.44)$$

Equations (2.2.42), (2.2.43) and (2.2.44), which are analogous to (2.2.23a) and (2.2.23b) for Killing vectors, are the structural equations for a Killing tensor of order two. The first integrability condition for these equations is obtained by computing $M_{\mu\nu\sigma\rho;[\varkappa\lambda]}$ and it implies the following theorem.

Theorem. A 4-dimensional space-time admits at most 50 linearly independent Killing tensors of order two. The maximum number of 50 is attained if and only if the space-time is of constant curvature.

2.2 THE SIGNIFICANCE OF THE EXISTENCE OF KILLING TENSORS IN SPACE-TIMES

2.3.1 First integrals of the geodesic equations

Killing vectors and tensors are important for the study of geodesic and geodesic deviation equations. In particular, Killing tensors originated, historically, in their connection with first integrals of geodesic equations.

In a Riemannian space, the differential equations of geodesics

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\varkappa} u^\nu u^\varkappa = 0 \qquad (2.3.1)$$

with $u^\mu = \frac{dx^\mu}{ds}$, admit linear first integrals of the form

$$X_\mu u^\mu = L \qquad (2.3.2)$$

which are constant along the geodesics if and only if $X_{(\mu;\nu)} = 0$, that is, if and only if the X_μ's are Killing vectors.

Similarly, equations (2.3.1) admit quadratic first integrals of the form

$$K_{\mu\nu} u^\mu u^\nu = Q \qquad (2.3.3)$$

with $K_{\mu\nu} = K_{\nu\mu}$, which are constant along the geodesics if and only if $K_{(\mu\nu;\lambda)} = 0$, that is, if and only if the $K_{\mu\nu}$'s are Killing tensors of second order. Note that, since the contracted product of a Killing-Yano tensor with itself is a symmetric Killing tensor of second order, quadratic first integrals of the form (2.3.3) can also be generated by a Killing-Yano tensor as follows

$$a_{\mu\mu_1 \ldots \mu_p} \, a^{\mu_1 \ldots \mu_p}{}_{\nu} u^\mu u^\nu = Q \qquad (2.3.4)$$

where

$$a_{\mu\mu_1 \ldots \mu_p} \, a^{\mu_1 \ldots \mu_p}{}_{\nu} = K_{\mu\nu}.$$

A trivial example of a constant of motion, associated with a Killing tensor, is that of the norm of the tangent to a geodesic;

it is the constant of motion associated with the metric tensor $g_{\mu\nu}$ which trivially satisfies Killing's equation of second order.

In general, every Killing tensor of order p $K_{\mu_1 \cdots \mu_p}$ identifies a first integral of degree p for the geodesic equations of motion, defined by

$$K_{\mu_1 \cdots \mu_p} u^{\mu_1} \cdots u^{\mu_p} = K \qquad (2.3.5)$$

which is constant along the geodesics [12]. If $K_{\mu_1 \cdots \mu_p}$ is irreducible, the corresponding first integral is called proper, while improper first integrals correspond to reducible Killing tensors * .

If we restrict ourselves to the case p = 2, then the problem of finding all space-times of a certain type which admit proper quadratic first integrals of the geodesic motion can be thought of as an extension of the classical problem of determining those central force fields in which a nonrelativistic particle has a constant of motion which is a second degree polynomial in the momentum components and which is functionally independent

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* Apart from the above stated (usual) first integrals of geodesic equations, rational first integrals of geodesic equations of the form

$$I = \frac{a_\mu u^\mu}{b_\nu u^\nu}$$

and their integrability conditions have been recently studied by Collinson [20].

of the Hamiltonian and the angular momentum [9]. This correspondence illustrates how <u>hidden symmetry</u> conditions [21], such as the existence of irreducible Killing tensors, can represent dynamically interesting fields, and justifies in this way the importance of the study of Killing tensors.

The Kerr metric [22] gives additional support for such a study in general relativity, since it provides a physically important example of the presence of a quadratic constant of motion which arises from a second-order Killing tensor.

Moreover, in order to interpret the physical significance of a certain solution of Einstein's equations, we often need to understand the global nature of space-time. The global analysis of a solution, however, is usually very difficult since in most cases the metric $g_{\mu\nu}$ is given only locally. It is then the knowledge of first integrals of geodesic equations that greatly facilitates this analysis and helps us to extend a locally given space-time to a maximal inextendible space-time. Theorems on first integrals are, therefore, of great interest in general relativity.

Now, in certain cases of the global analysis, it is sufficient to know only the conformal structure. In this case, a knowledge of first integrals of <u>null</u> geodesics will suffice, and this is equivalent to the requirement that

$$P_{\mu_1 \ldots \mu_p} u^{\mu_1} \ldots u^{\mu_p} = P \qquad (2.3.6)$$

be constant along null geodesics with parallelly propagated tangent vector u^{μ}, where now $P_{\mu_1 \dots \mu_p}$ is a conformal Killing tensor of order p.

Before concluding the present section, it is worth mentioning some important results proved by many authors concerning the number of linearly independent first integrals of geodesics in an n-dimensional Riemannian space V_n.

For an n-dimensional Riemannian space V_n, the condition that the geodesic equations admit linear first integrals is equivalent to the condition that the V_n admits a group of isometries. A V_n admits a group of isometries of maximum dimension $N_{LI} = \frac{1}{2} n(n+1)$ if and only if it is a space of constant curvature [23].

It was also proved by Egorov [24] that for a V_n $(n \geqslant 3,\ n \neq 4)$ there is no group of isometries of dimension r such that $\frac{1}{2} n(n+1) > r > \frac{1}{2} n(n-1)+1$. This is an example of a "gap" theorem.

Thomas [25] showed that in a V_n the maximum number of linearly independent quadratic first integrals of geodesics is $N_{QI} = \frac{1}{12} n(n+1)^2(n+2)$. Then Katzin and Levine [26] showed that this maximum number N_{QI} occurs if the space V_n is of constant curvature.

Kimura [27] generalized Egorov's result and showed that in a V_n $(n \geqslant 3)$ a second maximum number of linearly independent quadratic first integrals of geodesics is admitted which is $\frac{1}{12}(n^4+5n^2+54n-120)$. And this is Kimura's gap theorem for quadratic Killing tensors.

Weir [28] proved that the maximum number of linearly independent conformal quadratic first integrals of the geodesics is $N_{QCO} = \frac{1}{12}(n-1)(n+2)(n+3)(n+4)$ and that this maximum number occurs if V_n is of constant curvature, but no gap theorem exists in this case.

Finally, Katzin, Levine and Halsey [29] showed that the maximum number of linearly independent cubic first integrals of geodesics is $N_{CI} = \frac{1}{144} n(n+1)^2(n+2)^2(n+3)$ and that this maximum number, again, occurs if V_n is a space of constant curvature. No gap theorem exists in this case either.

2.3.2 Killing tensors and geodesic deviation equation

It is well known that in a Riemannian space with metric $g_{\mu\nu}$, if X^μ is a Killing vector defined by equation (2.2.6) and u^μ is the unit tangent vector to a time-like/space-like geodesic, then $X_\mu u^\mu$ is a constant of motion <u>and</u> the equation of geodesic deviation*

$$u^\nu u^\sigma \nabla_\nu \nabla_\sigma V^\mu + R^\mu{}_{\nu\varkappa\sigma} u^\nu V^\varkappa u^\sigma = 0 \qquad (2.3.7)$$

is satisfied by such a Killing vector.

A similar result for Killing tensors is also true in a Riemannian space if $K_{\mu_1 \ldots \mu_p}$ is a Killing tensor of order p, then $K_{\mu_1 \ldots \mu_p} u^{\mu_1} \ldots u^{\mu_p}$ is a constant of motion <u>and</u> it identifies a solution for the equation of geodesic deviation (2.3.7), as

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* The fundamental role of the equation of geodesic deviation in general relativity is connected with the analysis of tidal forces, the focusing effect of gravity.

is proved by Caviglia, Zordan and Salmistraro [30] in the following two theorems.

<u>Theorem 1:</u> Let $K_{\mu\mu_1\ldots\mu_p}$ be a Killing tensor of order $(p+1)$. Then the field V^μ defined by

$$V^\mu = K^\mu{}_{\mu_1\ldots\mu_p} u^{\mu_1}\ldots u^{\mu_p} \tag{2.3.8}$$

is a solution of the equation of geodesic deviation (u^μ is the tangent vector to a geodesic).

<u>Proof:</u> We write first the definition $\nabla_{(\nu}K_{\mu\mu_1\ldots\mu_p)} = 0$ of the Killing tensor $K_{\mu\mu_1\ldots\mu_p}$ in the form

$$\nabla_\nu K_{\mu\mu_1\ldots\mu_p} = -\{\nabla_\mu K_{\mu_1\ldots\mu_p\nu} + \ldots + \nabla_{\mu_p} K_{\nu\mu\mu_1\ldots\mu_{p-1}}\} \,. \tag{2.3.9}$$

Evaluating the $u^\sigma\nabla_\sigma$ derivative of both sides of this equation and transvecting with $u^\nu u^{\mu_1}\ldots u^{\mu_p}$, we obtain

$$\begin{aligned}&(p+1)u^\nu u^\sigma \nabla_\nu\nabla_\sigma(K_{\mu\mu_1\ldots\mu_p} u^{\mu_1}\ldots u^{\mu_p})\\ &= -u^\nu u^\sigma u^{\mu_1}\ldots u^{\mu_p}\nabla_\sigma\nabla_\mu K_{\mu_1\ldots\mu_p\nu} \,.\end{aligned} \tag{2.3.10}$$

If we now apply the commutation rule for covariant differentiation and the definition of a Killing tensor in the right-hand side of equation (2.3.10), we can easily prove that the vector V^μ satisfies

$$u^\nu u^\sigma \nabla_\nu\nabla_\sigma V^\mu + R^\mu{}_{\nu\varkappa\sigma}u^\nu V^\varkappa u^\sigma = 0 \tag{2.3.11}$$

which is the equation of geodesic deviation. □

A weak converse of theorem 1 exists and can be stated as follows.

Theorem 2: Let $K_{\mu\mu_1\ldots\mu_p}$ be a totally symmetric tensor field of order (p+1) such that every field V^μ defined by equation (2.3.8) is a solution of the equation of geodesic deviation. Then $K_{\mu\mu_1\ldots\mu_p}$ satisfies the relation

$$\nabla_\sigma \nabla_{(\nu} K_{\mu\mu_1\ldots\mu_p)} = 0. \tag{2.3.12}$$

In general, relation (2.3.15) is not equivalent to the definition of a Killing tensor $\nabla_{(\nu} K_{\mu\mu_1\ldots\mu_p)} = 0$. However, every solution of (2.3.15) will give a Killing tensor of order (p+1) if the metric of the space-time admits only null parallel totally symmetric tensor fields of order (p+2).

Proof: If we use the Ricci identities and the symmetries of the tensor field $K_{\mu\mu_1\ldots\mu_p}$, we can write the equation of geodesic deviation in the form

$$u^\sigma u^{\mu_1}\ldots u^{\mu_p} u^\nu \nabla_\nu \{(p+1)\nabla_\sigma K_{\mu\mu_1\ldots\mu_p} + \nabla_\mu K_{\sigma\mu_1\ldots\mu_p}$$

$$= u^\sigma u^{\mu_1}\ldots u^{\mu_p} u^\nu \nabla_\mu \nabla_\nu K_{\sigma\mu_1\ldots\mu_p}. \tag{2.3.13}$$

Since equation (2.3.13) is true for arbitrary u^μ, it follows that

$$\nabla_\sigma \nabla_{(\nu} K_{\mu\mu_1 \ldots \mu_p)} + \nabla_\nu \nabla_{(\mu_1} K_{\mu\mu_2 \ldots \mu_p \sigma)} + \ldots$$

$$+ \nabla_{\mu_p} \nabla_{(\sigma} K_{\mu\nu\mu_1 \ldots \mu_{p-1})} = \nabla_\mu \nabla_{(\sigma} K_{\nu\mu_1 \ldots \mu_p)} . \tag{2.3.14}$$

Subtracting the term $(p+2)\nabla_\mu \nabla_{(\sigma} K_{\nu\mu_1 \ldots \mu_p)}$ from both sides of equation (2.3.14), we obtain

$$\nabla_\sigma \nabla_{(\nu} K_{\mu\mu_1 \ldots \mu_p)} - \nabla_\mu \nabla_{(\sigma} K_{\nu\mu_1 \ldots \mu_p)} + \ldots$$

$$+ \nabla_{\mu_p} \nabla_{(\sigma} K_{\mu\nu\mu_1 \ldots \mu_{p-1})} - \nabla_\mu \nabla_{(\sigma} K_{\nu\mu_1 \ldots \mu_p)}$$

$$= - (p+1)\nabla_\mu \nabla_{(\sigma} K_{\nu\mu_1 \ldots \mu_p)} . \tag{2.3.15}$$

Now, interchanging the indices μ and ν and subtracting the new equation thus obtained from equation (2.3.15), we can easily prove that the expression in the first square brackets of equation (2.3.15) vanishes. By repeating this procedure we end up with the left-hand side of equation (2.3.15) equal to zero, which proves that equation (2.3.12) holds. □

2.3.3 The theory of separability of the Hamilton-Jacobi equation in General Relativity

During the century or so following the work of Jacobi, much research in Hamiltonian mechanics was directed towards understanding the separability conditions for the Hamilton-Jacobi equation. In the problem of finding the motion of a classical particle one has the choice of dealing with a system of second order ordinary differential equations (Lagrange's equations) or a single first order partial differential equation (the Hamilton-Jacobi equation). In practice the latter method is less frequently used because of the difficulty in finding complete integrals. However, when these integrals are obtainable, the method leads rapidly to the equations of the trajectories. Furthermore, it is of fundamental theoretical importance since it is more easily assimilated into quantum theory [31].

The classical research, initiated by Liouville [32] and Stäckel [33], was aimed at finding the most general coordinate form of a metric and potential with a separable Hamilton-Jacobi equation. A lot of questions, however, are still unanswered, such as, for example, whether the constants of motion which arise from the local separation of the Hamilton-Jacobi equation are well behaved in the large [34].

More recently, a great deal of work has been carried out on the subject of the separability of the Hamilton-Jacobi equation in Riemannian spaces and, in particular, in the curved space-times of general relativity. Amongst the motivations for this work has been the need to study the behaviour of geodesics in order to find both local and global properties of the space-time.

A newer motivation for relativists to investigate separation problems is the work done by Carter [35 and 36] concerning the separability of Hamilton-Jacobi and Schrödinger equations of type D space-times. Carter has shown how to integrate the geodesic equations in type D vacuum solutions of Einstein's equations which include the (charged) Kerr solution, a solution with a great significance in astrophysical research. His method is based on the fact that the Hamilton-Jacobi equation for the geodesic problem in these space-times can be solved by separation of variables.

Matravers [37] also proved that the separability of the Hamilton-Jacobi equation implies the existence of Killing tensors in the case of Carter metrics.

The appearance of these papers renewed attention on Killing tensors in connection with their role in the problem of the separation of variables in the Hamilton-Jacobi equation for geodesics.

In particular, Woodhouse [34] studied the relationship between Killing tensors and separable systems for the Hamilton-Jacobi equation; he found that there exist essentially two types of separable systems for the (massless) Hamilton-Jacobi equation, whose separable coordinate is adapted either to a (conformal) Killing vector or to an eigenvector of a (conformal) Killing tensor.

Because of its importance in connection with the existence of Killing vectors and tensors, the theory of the separability of the Hamilton-Jacobi equation will be presented here, following the way of Benenti and Francaviglia [38].

Let M be any n-dimensional Riemannian manifold. A chart $U \subseteq M$ with local coordinates x^μ ($\mu = 1,\ldots,n$) is called a <u>separable chart</u> and the x^μ are called <u>separable coordinates</u> if the Hamilton-Jacobi equation for geodesics

$$\frac{1}{2} g^{\mu\nu}\partial_\mu S \partial_\nu S - h = 0 \qquad (2.3.16)$$

with S the Jacobi action and h a real constant, is integrable by separation of the variables x^μ in U, that is, if a complete integral S exists in U such that S decomposes into the sum of n functions S_μ, each one depending only on the corresponding coordinate x^μ *.

At a given point $P \in M$ several separable charts may exist. Integrals with the above properties, corresponding to two different separable charts, can be strictly different, that is they can represent different n-parameter families of real functions on M. We can collect the separable charts at a point $P \in M$ in classes, each class containing all the charts to which the same family of solutions corresponds. Such a class will be called a (local) separability structure at $P \in M$. For each separability structure a family of separable coordinates exists such that each coordinate system in the family admits a maximum number r of ignorable coordinates ($0 \leqslant r \leqslant n$). Each system in this family is called a normal separable system of coordinates and the corresponding separability structure will be denoted by S_r. A separability structure S_r is regular [39] if, in some separable coordinate system associated with it, none of the metric components $g^{\sigma\sigma}$ ($\sigma = r+1,\dots,n$) is zero.

In the theory of separability structures an important role

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* A separable coordinate x^μ is called a trivial separable coordinate for the Hamilton-Jacobi equation with $h \neq 0$, if and only if it is adapted to a Killing vector.

is played by Killing vectors and tensors. Benenti and Francaviglia [40] have studied further the separability problem and proved the following theorem which relates this problem with the existence of Killing vectors and tensors in the manifold.

<u>Theorem</u> : A manifold $(M,g_{\mu\nu})$ admits a local regular separability structure S_r if and only if it admits r <u>commuting</u> Killing vectors X_α $(0 \leqslant \alpha \leqslant r-1)$ and n-r Killing tensors $\underset{a}{K}$ $(r \leqslant a \leqslant n)$ which are independent * and satisfy the following conditions .

i) In the Lie algebra of Killing vectors and tensors the following commutation relations hold

$$[X_\alpha,X_\beta] = 0; \quad [\underset{a}{K},\underset{b}{K}] = 0; \quad [\underset{a}{K},X_\alpha] = 0 \quad \forall\ a,b,\alpha,\beta \tag{2.3.17}$$

ii) The Killing tensors $\underset{a}{K}$ have n-r common eigenvectors $\underset{a}{X}$ such that

$$[\underset{a}{X},X_\alpha] = 0; \quad [\underset{a}{X},\underset{b}{X}] = 0 \qquad \forall a,b,\alpha \tag{2.3.18}$$

$$g(\underset{a}{X},X_\alpha) = 0 \qquad \forall a,\alpha\ . \tag{2.3.19}$$

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* From the mechanical point of view this stated independence means that the linear first integrals associated with the Killing vectors X_α and the quadratic first integrals associated with the Killing tensors $\underset{a}{K}$ are functionally independent. As a consequence both Killing vectors and tensors can be adjusted to be linearly independent at each point of the manifold.

The brackets appearing in i) are the Schouten-Nijenhuis brackets, which will be studied in the next section.

As an application of the theory of separability of the Hamilton-Jacobi equation in general relativity, Carter's work on Kerr space-time is presented in Appendix A3, where with the use of the above theory a fourth constant of the motion, related to the existence of a second order Killing tensor, is obtained.

2.4 EXTENSION OF THE LIE ALGEBRA OF ISOMETRIES

2.4.1 The Schouten-Nijenhuis bracket

The Lie derivative is one of the best known types of bilinear differential concomitants of tensor fields that does not involve a connection.

The Lie derivative operation $\mathcal{L}_{\underline{V}}$ with respect to any vector field $\underline{V} = V^{\mu}\partial_{\mu}$ maps vector fields $\underline{U}$ into vectors fields $\mathcal{L}_{\underline{V}}\underline{U}$ and tensor fields T into tensor fields $\mathcal{L}_{\underline{V}}T$ of the same valence. If $\underline{U} = U^{\mu}\partial_{\mu}$ and $T = T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_r}$ then

$$(\mathcal{L}_{\underline{V}}U)^{\mu} \equiv [V,U]^{\mu} = V^{\sigma}\partial_{\sigma}U^{\mu} - U^{\sigma}\partial_{\sigma}V^{\mu} = - [U,V]^{\mu} \qquad (2.4.1)$$

$$(\mathcal{L}_{\underline{V}}T)^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_r} \equiv [X,T]^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_r}$$

$$= V^{\sigma}\partial_{\sigma}T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_r} - \sum_{h=1}^{p} T^{\mu_1 \dots \sigma \dots \mu_p}{}_{\nu_1 \dots \nu_r}\partial_{\sigma} V^{\mu_h}$$

$$+ \sum_{k=1}^{r} T^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \sigma \dots \nu_r}\partial_{\nu_k} V^{\sigma} \, . \qquad (2.4.2)$$

The consequences of these formulae are:

(i) the Jacobi identity for vector fields

$$[\underline{V},[\underline{U},\underline{W}]] + [\underline{U},[\underline{W},\underline{V}]] + [\underline{W},[\underline{V},\underline{U}]] = 0 \qquad (2.4.3)$$

(ii) a "Jacobi-type" relation [41]

$$[\underline{V},[\underline{U},T]] - [\underline{U},[\underline{V},T]] - [[\underline{V},\underline{U}],T] = 0 \qquad (2.4.4)$$

which can also be written as

$$(\mathcal{L}_{\underline{V}}\mathcal{L}_{\underline{U}} - \mathcal{L}_{\underline{U}}\mathcal{L}_{\underline{V}})T = \mathcal{L}_{[\underline{V},\underline{U}]}T. \qquad (2.4.5)$$

Let $F^{\mu_0\ldots\mu_p}$ and $G^{\mu_0\ldots\mu_r}$ be two contravariant tensor fields of order $(p+1)$ and $(r+1)$ respectively. Then, as Schouten [42] proved, there is a concomitant $[F,G]^{p+r+1}$, called the Schouten bracket of F and G, with components

$$[F,G]^{\mu_0\ldots\mu_{p+r}} = \sum_{a=0}^{p} F^{(\mu_0\ldots\mu_{a-1}|\sigma|\mu_a\ldots\mu_{p-1}}\partial_\sigma G^{\mu_p\ldots\mu_{p+r})}$$

$$+ \sum_{a=0}^{p} (-1)^a F^{[\mu_0\ldots\mu_{a-1}|\sigma|\mu_a\ldots\mu_{p-1}}\partial_\sigma G^{\mu_p\ldots\mu_{p+r}]}$$

$$- \sum_{b=0}^{r} G^{(\mu_0\ldots\mu_{b-1}|\sigma|\mu_b\ldots\mu_{r-1}}\partial_\sigma F^{\mu_r\ldots\mu_{p+r})}$$

$$- \sum_{b=0}^{r} (-1)^{pr+p+r+b} G^{[\mu_0\ldots\mu_{b-1}|\sigma|\mu_b\ldots\mu_{r-1}}\partial_\sigma F^{\mu_r\ldots\mu_{p+r}]}. \qquad (2.4.6)$$

The proof of the tensor character of equation (2.4.6) is given by remarking that (2.4.6) does not change if ∂_σ is everywhere replaced by the covariant differentiation operator ∇_σ.

The vertical bars around the summation index σ make it immune to both symmetry and antisymmetry operations on indices.

For p = 0 the equation (2.4.6) reduces to the Lie derivative with respect to F^μ of the sum of the symmetric and skew-symmetric parts of G, $G^{(\mu_0 \ldots \mu_r)}$ and $G^{[\mu_0 \ldots \mu_r]}$ respectively [43].

As follows from equation (2.4.6), the Schouten bracket [F,G] can be thought of as a type of Lie derivative of the tensor field G with respect to the tensor field F, $\mathcal{L}_F G$.

In particular, if F and G are totally symmetric tensor fields, that is $SF \equiv F^{(\mu_0 \ldots \mu_p)}$ and $SG \equiv G^{(\mu_0 \ldots \mu_r)}$, then their Schouten bracket is given by

$$[SF,SG]^{\mu_1 \ldots \mu_p \varkappa \nu_1 \ldots \nu_r}$$

$$= (p+1) F^{\sigma(\mu_1 \ldots \mu_p} \partial_\sigma G^{\varkappa \nu_1 \ldots \nu_r)}$$

$$- (r+1)\, G^{\sigma(\nu_1 \ldots \nu_r} \partial_\sigma F^{\varkappa \mu_1 \ldots \mu_p)} \tag{2.4.7}$$

which is again a symmetric tensor field [43].

The bracket is antisymmetric in F and G, that is $[SF,SG] = -[SG,SF]$, is linear in each slot, is unchanged if ordinary partial derivatives are replaced by covariant derivatives and again reduces to the Lie derivative of G along F if F is a vector field [8].

Nijenhuis has proved in his paper that the Schouten bracket satisfies the Jacobi identity

$$[SF,[SG,SH]] + [SG,[SH,SF]] + [SH,[SF,SG]] = 0 \qquad (2.4.8)$$

where F,G,H are contravariant tensor fields of order (p+1), (r+s), (s+1) respectively.

Finally, if the cup product $F \cap G$ is defined to be the symmetrized outer product of the symmetric tensors F and G, by

$$SF \cap SG = F^{(\mu_0 \ldots \mu_p}G^{\nu_0 \ldots \nu_r)} = SG \cap SF \qquad (2.4.9)$$

then the Schouten bracket satisfies the following Leibnitz rule

$$[SF, SG \cap SH] = [SF,SG] \cap SH + [SF,SH] \cap SG \qquad (2.4.10)$$

where F,G,H are the same as before.

Thus, it is shown in the above analysis that, with respect to the Schouten bracket, the totally symmetric contravariant tensor fields on a space have a Lie algebraic structure* which

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* Corresponding relations to (2.4.7) and (2.4.8) exist for totally antisymmetric tensor fields F,G,H with $AF = F^{[\mu_0 \ldots \mu_p]}$, $AG = G^{[\mu_0 \ldots \mu_r]}$ and $AH = H^{[\mu_0 \ldots \mu_s]}$. The Schouten bracket of two such tensors is given by the equation

$$[AF,AG]^{\mu_1 \ldots \mu_p \varkappa \nu_1 \ldots \nu_r} = (p+1)F^{\sigma[\mu_1 \ldots \mu_p}\partial_\sigma G^{\varkappa \nu_1 \ldots \nu_r]} - (r+1)G^{\sigma[\nu_1 \ldots \nu_r}\partial_\sigma F^{\varkappa \mu_1 \ldots \mu_p]}.$$

This bracket satisfies $[AF,AG] = -(-1)^{(p+1)(r+1)}[AG,AF]$ and a "Jacobi-type" relation of the form is a direct generalization of the vector field equations (2.4.1) and (2.4.2). It then becomes obvious that the set of Killing tensor fields on a space satisfy a Lie algebra, which, as Geroch [11] pointed out, can be defined as the subalgebra of symmetric tensor fields, consisting of those tensors $K^{\mu\nu\ldots\lambda}$ that commute with the metric tensor $g^{\mu\nu}$, that is, those that satisfy

$$[K,g] = 0. \tag{2.4.11}$$

Since this equation is, in fact, Killing's equation (when covariant differentiation is used in the product definition), it follows that, indeed, the Lie algebra of Killing tensors on a space is a direct generalization of the Lie algebra of Killing vectors. If now K and L denote two Killing tensors (L may well be a Killing vector), then a simple consequence of Jacobi identity [11] (pointed out by Collinson)

$$[g,[K,L]] + [K,[L,g]] + [L,[g,K]] = 0 \tag{2.4.12}$$

is that $[[K,L],g] = 0$, which means that $[K,L]$ is also a Killing

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$$(-1)^{s(p+1)}[AF,[AG,AH]] + (-1)^{p(r+1)}[AG,[AH,AF]]$$
$$+ (-1)^{r(s+1)}[AH,[AF,AG]] = 0. \quad [43]$$

It is clear from the above relations that the totally antisymmetric tensor fields have a graded Lie algebraic structure [41].

tensor (or a Killing vector if L is to be taken as a Killing vector and K is of a suitable order). And in this way, Killing tensors give a Lie algebraic extension of the Lie algebra of Killing vectors, which may be finite or infinite, a matter that will be discussed in the next chapter.

2.4.2 Interpretation of the Schouten-Nijenhuis bracket

It is well known that if a dynamical system has n coordinates $\{q^\mu\}$ corresponding to its n degree of freedom, then these coordinates define a manifold M called the configuration space [44] and the evolution of the system in time is described by a curve $q^\mu(t)$ in M. The Lagrangian $\mathcal{L}$ of the system is a function of q^μ and $\dot{q}^\mu = dq^\mu/dt$ and so it is a function on TM, the tangent bundle of M. Then, the momentum, defined by $p_\mu = \partial\mathcal{L}/\partial\dot{q}^\mu$, is a one-form field on M, that is a cross-section of the cotangent bundle T*M.

It follows that the phase space which is the manifold with coordinates $\{q^\mu, p_\mu\}$ is nothing but the cotangent bundle T*M. Thus, any function which depends both on the position of a particular point in a manifold M and on the vector $\underline{p}$ of the cotangent space at that point of M, should be regarded as a function on the cotangent bundle T*M of M. And this is exactly what we need for the interpretation of the Schouten bracket.

Let the space-time cotangent bundle T*M have local coordinates (x^μ, p_μ). Then the Poisson bracket of two real analytic functions f,g defined on T*M is

$$(f,g) = \frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial x^\mu} \tag{2.4.13}$$

which is clearly independent of the local coordinates. It depends only on the symplectic form [44] * .

Now, a function f on T*M may be associated with any contravariant symmetric tensor $F^{\mu_0\ldots\mu_p}$ on M by $f = F^{\mu_0\ldots\mu_p} p_{\mu_0}\ldots p_{\mu_p}$ that is, it can be written as a homogeneous polynomial in p_μ. Thus, if f and g are functions associated with the symmetric tensors $F^{\mu_0\ldots\mu_p}$, $G^{\mu_0\ldots\mu_r}$ respectively by

$$f = F^{\mu_0\ldots\mu_p} p_{\mu_0}\ldots p_{\mu_p}$$

$$g = G^{\mu_0\ldots\mu_r} p_{\mu_0}\ldots p_{\mu_r}$$

then the tensor bracket operation,known as the Schouten bracket, yields a new symmetric tensor whose associated scalar function is the Poisson bracket of f and g, that is ,

$$(f,g) = [SF,SG]^{\mu_0\ldots\mu_{p+r}} p_{\mu_0}\ldots p_{\mu_r} . \tag{2.4.14}$$

If we also write

$$h \equiv H^{\mu_0\ldots\mu_s} p_{\mu_0}\ldots p_{\mu_s}$$

then equation (2.4.8) is nothing but the Jacobi identity of

* For more about the symplectic form see Appendix A4.

classical mechanics $(f,(g,h)) + (g,(h,f)) + (h,(f,g)) = 0$.
And thus the Poisson bracket of functions on the cotangent bundle and the symmetric Schouten-Nijenhuis bracket are directly related, a fact which is the starting point for my work.

As we have already seen, we can have a Lie algebraic extension of the Lie algebra of Killing vectors if we consider Schouten-Nijenhuis brackets of Killing vectors and Killing tensors with themselves. But recalling the definition of the Schouten bracket, it is very easy to realize how complicated the calculations in terms of it may be. However, the Poisson bracket, which is directly related to the Schouten bracket by (2.4.14), provides one with a very simple way to do the same calculations.

For example, as an application of the Poisson bracket formalism, which I will use later, I present here Killing's equations in terms of this formalism.

Let X^μ, Y^μ be two Killing vectors and $K^{\mu\nu}$ a Killing tensor of order two of a space-time with metric $g_{\mu\nu}$; then the following functions on the cotangent bundle are associated with them

$$\alpha = X^\mu p_\mu \ ,$$

$$\beta = Y^\mu p_\mu \ ,$$

$$\varkappa = K^{\mu\nu} p_\mu p_\nu \ .$$

The energy function E is defined as the function associated with the metric tensor $g_{\mu\nu}$ by

$$E = g^{\mu\nu}p_\mu p_\nu. \tag{2.4.15}$$

Then the Poisson bracket of the functions α and β is given by

$$(\alpha,\beta) = (Y^\sigma X^\mu{}_{,\sigma} - X^\sigma Y^\mu{}_{,\sigma})p_\mu$$

$$= (Y^\sigma X^\mu{}_{;\sigma} - X^\sigma Y^\mu{}_{;\sigma})p_\mu.$$

The <u>general covariance</u> of Poisson bracket formalism must be emphasized here and as a result of it all equations written in terms of this formalism remain unchanged if partial derivatives are replaced by covariant ones.

Killing's equation for the vector X^μ is written in terms of the Poisson bracket formalism as

$$(\alpha,E) = 0$$

where $(\alpha,E) = (X^\mu{}_{,\sigma}g^{\sigma\nu} + X^\nu{}_{,\sigma}g^{\sigma\mu} - X^\sigma g^{\mu\nu}{}_{,\sigma})p_\mu p_\nu.$

Similarly, the Killing tensor equation for the tensor $K^{\mu\nu}$ is written as

$$(\varkappa,E) = 0$$

where $(\varkappa,E) = K^{(\mu\nu}{}_{,\sigma}g^{\rho)\sigma} - g^{(\mu\nu}{}_{,\sigma}\, K^{\rho)\sigma}p_\mu p_\nu p_\rho$.

In this formalism, constants of motion are defined as the functions which commute with the energy function E.

As can be seen from the above equations, the Poisson bracket formalism is of a straightforward algorithmic character and it is this character that makes it very convenient for my work. In particular, I used it to write computer programs (which will be presented in the next chapter) for the evaluation of high order Killing tensors and thus the generation of the extension of the Lie algebra of Killing vectors in the case of a class of metrics, namely the Kimura metrics.

2.5 KIMURA SPACE-TIMES AND CORRESPONDING KILLING TENSORS

Since the objective of studying Killing tensors is not only to find all Killing tensors of a given space-time, but also to classify space-times with respect to the non-trivial Killing tensors they admit, any such classification is of considerable interest.

The problem of determining and classifying all isotropic static spherically symmetric space-times which admit non-trivial Killing tensors of second order has been discussed by Hauser and Malhiot [9], under the assumption that the quadratic first integrals of geodesics associated with such tensors are stationary, that is, independent of time.

Kimura [45, 46 and 47] generalized the above work and discussed the problem without making such an assumption. He obtained a whole class of static spherically symmetric space-times which admit Killing tensors of second order and are, in addition, anisotropic. It must be noted here that, since Hauser and Malhiot space-times are a subclass of those of Kimura, one must not assume that Killing tensors are always associated with imperfect fluids.

The static spherically symmetric space-time has a line element of the form

$$ds^2 = - Adr^2 - r^2 d\vartheta^2 - r^2 \sin^2\vartheta d\varphi^2 + Cdt^2 \qquad (2.5.1)$$

where A and C are arbitrary positive-valued functions of r. The quadratic first integrals obtained by Kimura are the integrals given by

$$Q = H_{\mu\nu}\, u^{\mu}\, u^{\nu} \qquad (2.5.2)$$

where $u^{\mu} = dx^{\mu}/ds$ with s denoting arc length along the geodesics and $H_{\mu\nu}$ is a symmetric tensor satisfying

$$H_{(\mu\nu;\varkappa)} = 0. \qquad (2.5.3)$$

For the line element (2.5.1), equation (2.5.3) may be written explicitly as follows

$$\partial_1 H_{11} - A'A^{-1} H_{11} = 0$$

$$\partial_2 H_{11} + 2\partial_1 H_{12} - (4r^{-1} + A'A^{-1})H_{12} = 0$$

$$\partial_3 H_{11} + 2\partial_1 H_{13} - (4r^{-1} + A'A^{-1})H_{13} = 0$$

$$\partial_4 H_{11} + 2\partial_1 H_{14} - (A'A^{-1} + 2C'C^{-1})H_{14} = 0$$

$$\partial_1 H_{22} + 2\partial_2 H_{12} + 2rA^{-1} H_{11} - 4r^{-1} H_{22} = 0$$

$$\partial_1 H_{23} + \partial_2 H_{13} + \partial_3 H_{12} - 2 \cot \vartheta \, H_{13} - 4r^{-1} H_{23} = 0$$

$$\partial_1 H_{24} + \partial_2 H_{14} + \partial_4 H_{12} - (C'C^{-1} + 2r^{-1}) H_{24} = 0$$

$$\partial_1 H_{33} + 2\partial_3 H_{13} + 2rA^{-1} \sin^2 \vartheta \, H_{11} + 2 \sin \vartheta \cos \vartheta \, H_{12}$$

$$- 4r^{-1} H_{33} = 0$$

$$\partial_3 H_{14} + \partial_4 H_{13} + \partial_1 H_{34} - (C'C^{-1} + 2r^{-1}) H_{34} = 0$$

$$\partial_1 H_{44} + 2\partial_4 H_{14} - C'A^{-1} H_{11} - 2C'C^{-1} H_{44} = 0$$

$$\partial_2 H_{22} + 2rA^{-1} H_{12} = 0$$

$$\partial_3 H_{22} + 2\partial_2 H_{23} + 2rA^{-1} H_{13} - 4 \cot \vartheta \, H_{23} = 0$$

$$\partial_2 H_{33} + 2\partial_3 H_{23} + 2rA^{-1} \sin^2 \vartheta \, H_{12} + 2 \sin \vartheta \cos \vartheta \, H_{22}$$

$$- 4 \cot \vartheta \, H_{33} = 0$$

$$\partial_3 H_{33} + 2rA^{-1} \sin^2 \vartheta \, H_{13} + 2 \sin \vartheta \cos \vartheta \, H_{23} = 0$$

$$\partial_4 H_{22} + 2\partial_2 H_{24} + 2rA^{-1} H_{14} = 0$$

$$\partial_2 H_{34} + \partial_3 H_{24} + \partial_4 H_{23} - 2 \cot \vartheta \, H_{34} = 0$$

$$\partial_4 H_{33} + 2\partial_3 H_{34} + 2rA^{-1} \sin^2 \vartheta \; H_{14} + 2 \sin \vartheta \cos \vartheta \; H_{24} = 0$$

$$\partial_2 H_{44} + 2\partial_4 H_{24} - C'A^{-1} H_{12} = 0$$

$$\partial_3 H_{44} + 2\partial_4 H_{34} - C'A^{-1} H_{13} = 0$$

$$\partial_4 H_{44} - C'A^{-1} H_{14} = 0.$$

The above system of partial differential equations can be solved if several assumptions on constants of integration are made, and then the whole class of Kimura metrics as seen in Table 2.1 is determined, together with the corresponding Killing tensors of second order, presented in Table 2.2.

Kimura types are subdivided by names such as IIA, IIIC and so on, but for my purposes 1,...,17 seems a better labelling. Note that the metrics numbers 9, 10, 11, 12 and 13 are the same as in Hauser and Malhiot, and thus the corresponding Killing tensors are stationary.

Finally, it must be emphasized here that every work on Killing tensors, that can be found in the literature, is done under the assumption that Killing tensors commute. In the case of Kimura metrics, the Killing tensors do <u>not</u> share this property, so that the Lie algebraic extension of the Lie algebra of Killing vectors based on them is <u>non-central</u>* and the separability of

---

* Suppose that X_α are the original Killing vectors of a space-time. A possible extension of the Lie algebra of Killing vectors by another generator Y (a Killing vector or a Killing tensor) is called <u>central</u> if $[X_\alpha, Y] = 0$, otherwise <u>non-central</u> [49].

of the Hamilton-Jacobi equation is lost. This particular character of the Killing tensors of Kimura metrics is of great interest and it is, in fact, the motivation for my work on them.

Table 2.1

KIMURA METRICS

The form of the metric components A and C

Case IA	1.	$A = a_1 C^2(1 - J_2 r^2)^{-3}$
		$C = 2j_2 r^2(1-J_2 r^2)[-a_1 l_1 J_2 r^4 + 2l_5(1-J_2 r^2) + a_2(1-J_2 r^2)^2]^{-1}$
		where a_1, a_2, l_1, l_5, J_2 are constants.
Case IB	2.	$A = C^2(a_1 + a_2 r^2)^{-3}$
		$C = r^2(a_3 + a_4 r^2)^{-1}$
		where $a_1 \neq 0$, $a_3 \neq 0$ and a_4 are constants.
Case IIA	3.	$A = C'^2 C^{-1}(a_1 C + a_2)^{-1}$
		$C = [-a_1 a_2^{-1} + r^2(a_3 + a_4 r^2)^{-1}]^{-1}$
		where $a_2 \neq 0$, $a_3 \neq 0$ and a_1, a_4 are constants.
Case IIA	4.	$A = C'^2 C^{-1}(a_1 C + a_2)^{-1}$
		$C = a_2 a_4 r^4[(a_3^2 - a_1 a_4) r^4 + a_2 a_3 r^2 + \frac{1}{4} a_2^2]^{-1}$
		where $a_2 \neq 0$ and a_1, a_3, a_4 are constants.

Case IIC 5. $A = r^{-2}(a_1r^2+a_2)^{-2}$

$C = r^2(a_1r^2+a_2)^{-1}$

where a_1 and a_2 are constants and $a_1a_2\neq 0$.

Case IIC 6. $A = a_2^{-2}r^{-2}$

$C = a_2^{-1}r^2$

where a_2 is a constant.

Case III 7. $A = (a_1r^2+a_2)^{-3}$

$C = 1$

where $a_2\neq 0$ and a_1 are constants.

Case III 8. $A = a_1^{-3}r^{-6}$

$C = 1$

where a_1 is a constant.

---

9. $A = 2[1-a_1r^2\pm\{(1-a_1r^2)^2-a_2^2r^4\}^{1/2}][1-a_1r^2)^2-a_2^2r^4]^{-1}$

(9a and 9b)

$C = r^2[r^2+a_3\{1-a_1r^2\mp((1-a_1r^2)^2-a_2^2r^4)^{1/2}\}]^{-1}$

where a_1,a_2,a_3 are constants.

---

10. $A = 4(1-a_1r^2)^{-1}$

$C = 1$

where $a_1\neq 0$ is a constant.

---

11. $A = (b_0 r^2+4)^{-1}$

$C = 1$

where $b_0 \neq 0$ is a constant.

---

12. $A = (b_0 r^2+4)^{-1}$

$C = r[(b_0 r^2+4)^{1/2}+b_5 r]^{-1}$

where b_0, b_5 are constants.

---

13. $A = \frac{1}{2}[1-a_1 r^2 \pm \{(1-a_1 r^2)^2-4a_2^2 r^4\}^{1/2}][(1-a_1 r^2)^2-4a_2^2 r^4]^{-1}$

(13a and 13b)

$C = r^2[r^2+a_3\{1-a_1 r^2 \mp ((1-a_1 r^2)^2-4a_2^2 r^4)^{1/2}\}]^{-1}$

where $a_2 \neq 0$ and a_1, a_3 are constants.

---

14. $A = (b_0 r^2+4)^{-1}$

$C = b_0 r^2+4$

where $b_0 \neq 0$ is a constant.

---

15. $A = \frac{1}{4}$

$C = 1$

---

16. $A = 4(1-a_1 r^2)^{-1}$

$C = 1-a_1 r^2$

where $a_1 \neq 0$ is a constant.

---

17. $A = 4$

$C = 1$

Table 2.2

The form of Killing tensors of second order

---

1. $$K_{\mu\nu} = e^{\omega t}[A\delta^1_\mu\delta^1_\nu - A^{\frac{1}{2}}Ca_1^{\frac{1}{2}}\omega(1-j_2r^2)^{-\frac{1}{2}}r\delta^1_{(\mu}\delta^4_{\nu)} + r^2(1-j_2r^2)\delta^2_\mu\delta^2_\nu$$

$$+ r^2(1-j_2r^2)\sin^2\vartheta\delta^3_\mu\delta^3_\nu - \{C+[a_1i_1r^2(1-j_2r^2)^{-1}-i_5]C^2\}\delta^4_\mu\delta^4_\nu]$$

$$I_{\mu\nu} = e^{-\omega t}[A\delta^1_\mu\delta^1_\nu + A^{\frac{1}{2}}Ca_1^{\frac{1}{2}}\omega(1-j_2r^2)^{-\frac{1}{2}}r\delta^1_{(\mu}\delta^4_{\nu)} + r^2(1-j_2r^2)\delta^2_\mu\delta^2_\nu$$

$$+ r^2(1-j_2r^2)\sin^2\vartheta\delta^3_\mu\delta^3_\nu - \{C+[a_1i_1r^2(1-j_2r^2)^{-1}-i_5]C^2\}\delta^4_\mu\delta^4_\nu]$$

---

2. $$K_{\mu\nu} = [A\delta^1_\mu\delta^1_\nu + r^2(1+a_1^{-1}a_2r^2)(\delta^2_\mu\delta^2_\nu + \sin^2\vartheta\delta^3_\mu\delta^3_\nu)$$

$$- \{C-(a_1a_4-a_2a_3)a_1^{-1}C^2\}\delta^4_\mu\delta^4_\nu]t^2$$

$$- 2a_1^{-1}(a_1+a_2r^2)^{-\frac{1}{2}}A^{\frac{1}{2}}Crt\delta^1_{(\mu}\delta^4_{\nu)} + a_1^{-2}(a_1+a_2r^2)^{-1}r^2C^2\delta^4_\mu\delta^4_\nu$$

$$I_{\mu\nu} = [A\delta^1_\mu\delta^1_\nu + r^2(1+a_1^{-1}a_2r^2)(\delta^2_\mu\delta^2_\nu + \sin^2\vartheta\delta^3_\mu\delta^3_\nu)$$

$$- \{C-(a_1a_4-a_2a_3)a_1^{-1}C^2\}\delta^4_\mu\delta^4_\nu]t - a_1^{-1}(a_1+a_2r^2)^{-\frac{1}{2}}A^{\frac{1}{2}}Cr\delta^1_{(\mu}\delta^4_{\nu)}$$

---

3. $$K_{\mu\nu} = e^{\mu t}[A\delta^1_\mu\delta^1_\nu + 2A^{\frac{1}{2}}Cr(a_3+a_4r^2)^{-\frac{1}{2}}\delta^1_{(\mu}\delta^4_{\nu)}$$

$$+ r^2(1+a_4a_3^{-1}r^2)(\delta^2_\mu\delta^2_\nu + \sin^2\vartheta\delta^3_\mu\delta^3_\nu) + C(1+a_1a_2^{-1}C)\delta^4_\mu\delta^4_\nu]$$

where $\mu = a_2^{\frac{1}{2}}$

$$I_{\mu\nu} = e^{-\mu t}[A\delta^1_\mu\delta^1_\nu - 2A^{\frac{1}{2}}Cr(a_3+a_4r^2)^{-\frac{1}{2}}\delta^1_{(\mu}\delta^4_{\nu)}$$

$$+ r^2(1+a_4a_3^{-1}r^2)(\delta^2_\mu\delta^2_\nu + \sin^2\vartheta\delta^3_\mu\delta^3_\nu) + C(1+a_1a_2^{-1}C)\delta^4_\mu\delta^4_\nu]$$

---

4. $$K_{\mu\nu} = e^{\nu t}[2A^{\frac{1}{2}}C\delta^1_{(\mu}\delta^4_{\nu)} - 2a_4^{\frac{1}{2}}a_2^{-\frac{1}{2}}r^4(\delta^2_\mu\delta^2_\nu + \sin^2\vartheta\,\delta^3_\mu\delta^3_\nu)$$

$$+ a_2^{-\frac{1}{2}}a_4^{-\frac{1}{2}}C^2(r^{-2} + 2a_3a_2^{-1})\delta^4_\mu\delta^4_\nu]$$

where $\nu = \frac{1}{2}\, a_2^{\frac{1}{2}}$

$$I_{\mu\nu} = e^{-\nu t}[2A^{\frac{1}{2}}C\delta^1_{(\mu}\delta^4_{\nu)} + 2a_4^{\frac{1}{2}}a_2^{-\frac{1}{2}}r^4(\delta^2_\mu\delta^2_\nu + \sin^2\vartheta\,\delta^3_\mu\delta^3_\nu)$$

$$- a_2^{-\frac{1}{2}}a_4^{-\frac{1}{2}}C^2(r^{-2} + 2a_3a_2^{-1})\delta^4_\mu\delta^4_\nu]$$

---

5. and 6.

$$K_{\mu\nu} = 2A^{\frac{1}{2}}Ct\delta^1_{(\mu}\delta^4_{\nu)} - t^2r^4(\delta^2_\mu\delta^2_\nu + \sin^2\vartheta\,\delta^3_\mu\delta^3_\nu) + C^2(r^{-2} + a_2t^2)\delta^4_\mu\delta^4_\nu$$

$$I_{\mu\nu} = 2A^{\frac{1}{2}}C\delta^1_{(\mu}\delta^4_{\nu)} - 2tr^4(\delta^2_\mu\delta^2_\nu + \sin^2\vartheta\,\delta^3_\mu\delta^3_\nu) + 2a_2C^2t\delta^4_\mu\delta^4_\nu$$

---

7. $$K_{\mu\nu} = [A\delta^1_\mu\delta^1_\nu + a_2^{-2}r^2(a_1r^2 + a_2)(\delta^2_\mu\delta^2_\nu + \sin^2\vartheta\,\delta^3_\mu\delta^3_\nu)]t^2$$

$$- 4a_2^{-1}(a_1r^2 + a_2)^{-2}rt\delta^1_{(\mu}\delta^4_{\nu)} + a_2^{-2}(a_1r^2 + a_2)^{-1}r^2\delta^4_\mu\delta^4_\nu$$

$$I_{\mu\nu} = [A\delta^1_\mu\delta^1_\nu + a_2^{-1}r^2(a_1r^2 + a_2)(\delta^2_\mu\delta^2_\nu + \sin^2\vartheta\,\delta^3_\mu\delta^3_\nu)]t$$

$$- 2a_2^{-1}(a_1r^2 + a_2)^{-2}r\delta^1_{(\mu}\delta^4_{\nu)}$$

---

8. $$K_{\mu\nu} = -a_1^{3/2}r^4t^2(\delta^2_\mu\delta^2_\nu + \sin^2\vartheta\,\delta^3_\mu\delta^3_\nu) + 4A^{\frac{1}{2}}t\delta^1_{(\mu}\delta^4_{\nu)} + a_1^{-3/2}r^{-2}\delta^4_\mu\delta^4_\nu$$

$$I_{\mu\nu} = -2a_1^{3/2}r^4t(\delta^2_\mu\delta^2_\nu + \sin^2\vartheta\,\delta^3_\mu\delta^3_\nu) + 4A^{\frac{1}{2}}\delta^1_{(\mu}\delta^4_{\nu)}$$

In the remaining cases more than two linearly independent proper quadratic integrals are admitted which, in addition, have a very complicated form and so only the number of these integrals will be presented here.

9.	3 Killing tensors of second order, which are stationary
10.	3 Killing tensors of second order, which are stationary
11.	3 Killing tensors of second order, which are stationary
12.	3 Killing tensors of second order, which are stationary
13.	5 Killing tensors of second order, which are stationary
14.	8 Killing tensors of second order
15.	8 Killing tensors of second order
16.	11 Killing tensors of second order
17.	11 Killing tensors of second order

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CHAPTER 3

ALGEBRAIC COMPUTING IN GENERAL RELATIVITY AND ITS USE IN THE GENERATION OF HIGH ORDER KILLING TENSORS

3.1 INTRODUCTION

Since Einstein first put forward his theory of gravitation [1] a great deal of research has been carried out in relativity, hundreds of exact solutions of Einstein field equations have been found [2] and still relativists are aware of the enormous amount of time and work they must spend in order to carry out the extended calculations that this theory involves. Fortunately, though, general relativistic calculations are mostly of an algebraic nature and this is what makes the use of computers possible and invaluable for research in this area [3]. In the words of Jean Sammet [4] "... it has become obvious that there are a large number of problems requiring very tedious, time-consuming, error-prone and straightforward algebraic manipulation and these characteristics make computer solution both necessary and desirable..."

Almost every relativist must have embarked on problems in which an algebraic computing system would have made life easier, since it is always more precise and efficient to solve a problem algebraically, and only substitute numbers in the final formula, than to work through a series of numerical approximations.

The main reasons for using an electronic computer to carry out algebraic calculations [5] are the following.

First, it removes the tediousness of repeating routine but lengthy tasks and saves a great deal of time. A famous example is the well-known Bondi metric [6]. The Riemann tensor of this metric took about 6 months to calculate by hand and SHEEP (see section 3.4.1) does it in about 30 seconds!

A second reason is that once the program one writes is correct and the input is correct (both, of course, involve some non-trivial work to ensure accuracy) the answer will also be correct. In the example of the Bondi metric mentioned above, computer results showed that the original calculations contained mistakes. The corrections are given in d'Inverno [7].

Finally, the use of algebraic computing leads to new and unforeseen directions in research [5], not only because, with the power of a computer, projects formerly unthinkable can be carried out, but also because the necessity of stating procedures for use by the machine in a very precise algorithmic form helps the researcher to focus his/her mind on aspects of a problem which might not have been adequately considered before.

Algebraic computing plays a major role in obtaining the results of this chapter and so an introduction to its requirements and problems is presented, and the computer algebra packages used, namely SHEEP and STENSOR, are studied in detail.

3.2 PROBLEMS AND REQUIREMENTS OF ALGEBRAIC COMPUTING

3.2.1 Computer languages

To use a computer one has to formulate a problem in a language understandable by that machine. There are three types of languages in which one can write a program [8].

i) Machine code: This consists of detailed instructions for the machine in a number-coded form for the step-by-step solution of a problem. Machine code varies from one make of computer to another.

ii) Assembly language: This is very similar in nature to machine code but also possesses a compiler which translates the assembly language into machine code. These languages are again machine dependent.

iii) High-level language: A high-level language enables the programmer to write in either English or mathematical-like notation and it also possesses a compiler which translates the language into machine code. High-level languages are, in principle, machine independent, that is, a program that works on one make of computer will also work on another.

High-level programming languages are divided into two main groups, "procedural" and "functional" languages [9].

Languages of a "procedural" nature are designed mainly for "number-crunching" computations. Clearly then, procedural languages are not adequate for algebraic calculations ("symbol-crunching") where symbols are needed for most formatting.

The present interest in computer science is concentrated on "functional" languages which play a fundamental role in developing "artificial intelligence". Functional languages provide much greater liberty in defining a multitude of data-types which include both numerical and symbolic data. Hence, it is, in general, beneficial to start with a functional language as a base for an efficient algebraic computing package.

3.2.2 Garbage collector

When algebra systems are presented with specific problems it soon becomes clear that algebraic calculations are several orders of magnitude more difficult for a computer than comparable numerical ones. It is a fact that in an iterative calculation involving numbers the complexity of the calculation grows linearly or at worse powerwise, whereas in an iterative calculation involving algebra the complexity grows exponentially [8]. This obviously would lead to a memory "blow-up", that is, to a halt of the program because the computer storage is choked up with algebraic expressions.

Let us consider as a simple example the calculation of the third Legendre polynomial p_3 using the recurrence relation

$$p_n = x\ p_{n-1} + \frac{x^2-1}{n}\left(\frac{d}{dx}\ p_{n-1}\right) \tag{3.2.1}$$

where $p_1 = x$. Three of the intermediate expressions that are obtained are

$$\begin{aligned} p_3 &= x[x^2 + \tfrac{1}{2}(x^2-1)] + \tfrac{1}{3}(x^2-1)(3x) \\ &= x^3 + \tfrac{1}{2}x^3 - \tfrac{1}{2}x + x^3 - x \\ &= \tfrac{5}{2}x^3 - \tfrac{3}{2}x. \end{aligned}$$

This example illustrates what happens in general, namely, that intermediate expressions are usually significantly longer than the final answer. So, suppose now that we ask a certain computer with a given amount of store to compute as many Legendre polynomials as possible. This will lead to a "blow-up" because the computer storage will be choked up with intermediate expressions which are no longer needed.

Thus, what we really require is to have our chosen language equipped with a special device to round up the discarded information, reclaim the store occupied by this unwanted information and give a new start to the program. Such an important device is technically called a "garbage collector".

3.2.3 Simplification

The most fundamental problem in the design of an algebraic system concerns the question of simplification of expressions; this is because it is not always clear which is the simpler form for writing an expression. For example, there are some cases in which a^2-b^2 is a desirable answer and others in which $(a+b)(a-b)$ is more desirable if, say, this result is subsequently divided by $(a+b)$.

Fitch [10] gives the following three reasons for the need to simplify algebraic expressions.

i) To make expressions smaller in store and hence speed the calculations (the compactness problem).

ii) To put an expression presented to the user into an intelligible form (the intelligibility problem).

iii) To check if an expression is identically zero (the identity problem.

These problems are obviously dependent on one another, and they are also very difficult to solve since simplification is not an algorithmic procedure.

The best way to solve simplification problems is to provide an interactive environment for the user to do the calculations, interfere with the work of the computer at suitable times and tell it how to expand an expression. In other words, the human-machine interaction is the best method for overcoming such problems. Consequently, it is always of great help to choose a computer language that can run interactive programs as well as "Batch jobs"*.

3.2.4 Substitutions

Substitutions are needed in algebraic computing for dividing the calculation into stages, and thus increase the speed and storage efficiency of the manipulation [9].

There are three types of substitutions.

i) Expansion: one term is replaced by its full value (which might also contain expandable terms).

ii) Algorithmic substitutions: these substitutions are carried out in all basic computer systems. A simple example of algorithmic substitution is a "gathering substitution", which is used to gather some terms of an expression in one term.

iii) Sum substitutions: these are the most difficult substitutions since the relation given will be of the form $a+b+c = 0$ with a,b,c possibly complicated expressions. An important example of sum substitutions is the trigonometric relation

---

* "Batch job" is a technical term used for non-interactive jobs that are submitted to the computer [9].

$$\cos^2 x + \sin^2 x - 1 = 0. \qquad (3.2.2)$$

The difficulty in such a substitution is that there is no standard way of deciding which term should be removed and replaced by the other two.

Although there exist some very successful trigonometric simplifiers for expression (3.2.2), the best way to handle sum substitutions is again the user-machine interaction.

3.2.5 Calculus and polynomial division

Since calculus is a very important mathematical tool, one naturally expects that all algebraic computing packages would provide calculus facilities.

Differentiation of a given expression involving standard functions is usually a straightforward process and it is of an algorithmic nature; thus, it is available in almost all algebraic computing packages. However, the inverse process of integration would not appear to be algorithmic and that is why the integration programs had severe restrictions in the past.

The situation looks more hopeful at present, since an integration program exists in MACSYMA and a formalism developed by Risch, Norman and Davenport [11] is now implemented in REDUCE.

Another problem, intimately related to integration, is the "common factor problem", that is, the problem of factoring polynomials. Although this is an algorithmic problem, it involves much testing and thus consumes a great deal of computer time. Again, in recent years, special programs have been developed possessing algorithms for polynomial factorization, and systems such as REDUCE and MACSYMA possess factorization facilities.

3.2.6 Interface

A final problem that must be examined is that of the interface between user and machine. There are two aspects of this problem.

i) One is that of the format for the input of algebraic expressions and the form of their output. The interface must allow the checking of the input expressions in order to reduce errors. It must also allow the user to read any stage of the calculation and interfere when he/she thinks it is necessary to give commands about the way in which the work should be carried out. This, of course, can help to reduce the possibility of an error even further.

ii) The other is the matter of usability of the system, which covers the question of how easy it is for a new user to learn to use the system, or what level of programming is needed and so on.

Both aspects are clearly related to the interactibility of the language, which is a desirable quality of any computing language if we want it to be used in constructing an efficient algebraic computing package. This is because algebraic computing involves intelligent decision-making which is not algorithmic and not programmable and makes the interference of the user with the work of the computer indispensable.

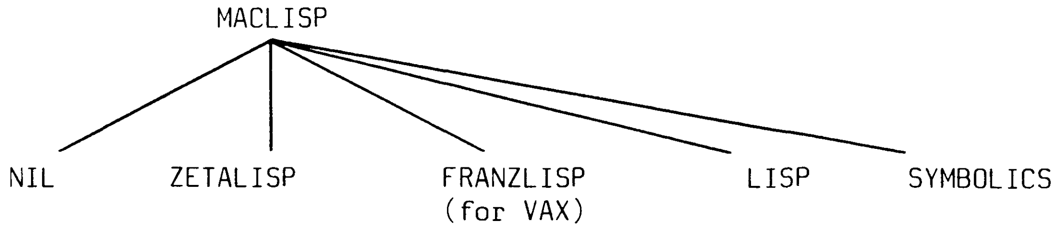
Understanding the requirements for algebraic computing, as well as the difficulties involved, can help one to choose the most appropriate language for building a system useful in a specific research area. Most algebraic systems and packages designed for use in general relativity are LISP-based systems, that is, systems written in the language LISP which is presented in the next section.

3.3 THE COMPUTING LANGUAGE LISP

LISP, an acronym for LISt Processing, was designed by John McCarthy [12] around 1960 and other features of the language were later added by the artificial intelligence group at MIT. It is a high level, functional, interactive programming language which fulfils all the requirements needed for constructing an algebraic computing system. It has, for example, a garbage collector, it provides good facilities for trapping errors, tracing faulty programs and testing code as it is written.

LISP is a language of contrasts. It is one of the oldest computer languages but it is still forward-looking. Many large programs have been written in LISP, including artificial intelligence programs, and thus the language provides good support for the professional programmer while, on the other hand, it is an excellent language for beginners.

Most makes of mainframe and an increasing number of micros support LISP. As a result, there exist now many LISP dialects. The most widely used version of LISP is the MACLISP (MIT version), which itself has the following dialects



FRANZLISP, in particular, which relates to SHEEP, is a dialect for the VAX machine, the most used machine in artificial intelligence. The research work presented in this chapter was carried out on such a VAX machine.

The building blocks of LISP are quantities called <u>atoms</u>. An atom can be an integer, a floating point number of a symbol composed of one or more characters (but not starting with a numeric character*). For example,

4 , -7 , 2.8 , A , TODAY ...

are atoms. Atoms can be grouped together to form <u>lists</u>, which are the basic data structures of LISP. A list may be thought of, mathematically, as a row vector. Lists can also be grouped together to form higher-level lists, and thus the elements in a list are either atoms or lists themselves. Parentheses are used to denote the beginning and end of a list, and spaces to separate the items. For example,

```
(A  B  C)
(A  (A  B)  (C  D)  )  ...
```

---

* since LISP will use the first character to discriminate between a variable and a number.

are lists. Atoms and lists collectively are called symbolic expressions.

LISP is an interactive language which (as opposed to other computing languages) has an interpreter. Thus, a running LISP constitutes an infinite READ-EVAL loop, that is, the interpreter reads the statement typed from the terminal, evaluates it, prints the answer and then waits for more input.

The syntax of LISP is extremely simple and thus LISP is a very easy language to learn. No previous knowledge of any other programming language is needed; on the contrary, having experience of another language may be a disadvantage. Thus, it is possible to learn to program quite well in LISP in three weeks or so, unless one has learnt FORTRAN first, in which case it will take about six weeks! [5]

The basic LISP functions are also very simple and to understand them is all that is needed in order to be able to write LISP programs. A function is a kind of procedure for which it is convenient to think in terms of an output called the value and inputs called the arguments. In LISP, the function to be performed is always given first, followed by the arguments, the things that the function is to work with.

The basic LISP functions are the following (note that some differences may exist in different dialects) [9].

1. QUOTE or symbol ' returns its single argument unevaluated. For example, if you type (QUOTE (ARBITRARY LIST)), LISP will respond (ARBITRARY LIST). Note that the form 'X is just a shorthand form for (QUOTE X).

2. ATOM returns T (for true) if its argument is any atom otherwise, that is, if its argument is a list, it returns NIL. For example, if you type (ATOM 'LIST), you have as an answer T, while (ATOM '(LIST)) gives NIL.
3. EQ applies to two arguments and compares them for equality. For example, if you type (EQ 4 4), you get T*.
4. CAR returns the first member of the argument list and it has no meaning if applied to an atom. For example, (CAR '(A B C)) will give you A, that is, CAR returns an atom.
5. CDR returns the tail of the argument list. For example, (CDR '(A B C)) will give you (B C), that is, CDR returns a list.
6. CONS constructs a new list out of its arguments. For example, (CONS 'A '(B C D)) will give you (A B C D). The second argument must always be a list.
7. COND is the main structure provided in LISP for testing and acting on conditions. COND can be followed by any number of lists, each of which contains a test and something to return if the test succeeds. Thus, the syntax is as follows:

```
( COND ( < test 1 > ... < result 1 > )
       ( < test 2 > ... < result 2 > )
         :
       ( < test n > ... < result n > ))
```

---

* There is also a command EQUAL which compares list structures to see whether they have the same shape and atoms and also whether their values are represented by the same memory cells.

Each list is called a clause. The idea is to search through the clauses, checking first if the CAR of the first list is empty; if not, its CDR is evaluated and this value is returned as the value for the COND expression and the searching stops. Otherwise, the same work is done on the next list and so on. If no successful clause is found, COND returns NIL.

8. DE is a convenient way of defining new functions. It takes three arguments, none of which is evaluated; the first is the name of the function that has been defined, the second is the list of variables that the function uses and the third is the algorithm of the function. For example, (DE ADD2 (X) (PLUS x 2)) defines the function ADD2.

Two examples will now be given where the above LISP functions have been used; they were carried out at Imperial College as the first applications of LISP.

Example 1: Intersection of two sets X and Y

```
(DE   INTERSECTION   (X  Y )
  (COND  ((NULL X)  NIL)
   ((MEMQ*  (CAR X)  Y )
    (CONS  (CAR X)  (INTERSECTION (CDR X)  Y )))
     (T  (INTERSECTION  (CDR X)  Y )))).
```

---

* MEMQ takes two arguments and checks if the first argument is an element of the second.

<u>Example 2</u>: Union of two sets X and Y

```
(DE UNION (X Y)
   (COND ((NULL X) Y)
    ((MEMQ (CAR X) Y) (UNION (CDR X) Y))
  (T (CONS (CAR X) (UNION (CDR X) Y))))).
```

These two small programs can be applied to any sets X and Y, and LISP will then return their intersection and union as an answer.

The significance of LISP in algebra is that it offers a natural way of representing algebraic expressions since, in addition to the functions mentioned above, LISP knows all the traditional functions of one variable (log, exp, cos,...), the arithmetic operations (+, x, ...) and the Boolean operations (AND, NOT,...).

Many more built-in functions exist in LISP. For details on the various LISP functions, see [13] and [14].

LISP also allows one to define recursive operations. A <u>recursive operation</u> is one that is defined in terms of itself. Although LISP possesses a function LOOP for repeating a sequence of operations, the use of its additional powerful tool of recursion gives the user enormously greater flexibility in the control of a program, and leads to both shorter and clearer programs [14].

Finally, it must be mentioned that, although LISP is very powerful for algebraic computing, it has, however, two main disadvantages:

i) According to M. Wand [15], LISP stands for "Lots of Irritating Single Parentheses"! which appears as a fair accusation. In fact, the parentheses problem goes away as soon as a person learns, after a little experience, to put things down on the page properly.

ii) A more significant disadvantage of LISP is the problem of "dialects", the lack of a definitive LISP standard, a problem common to all computer languages, which is due to the continuous contributions and improvements of LISP at different research centres.

3.4 IMPORTANT PACKAGES BASED ON LISP

Algebraic computing packages* had two different sources: computer scientists and theoretical physicists. While computer scientists were interested in designing packages for general algebraic procedures, theoretical physicists had specific problems to solve, and so they built less sophisticated packages which became very successful. R. d'Inverno in [8] gives an extensive review of the historical development of algebraic computing, in which he presents all algebraic packages, even those now out of date.

The main LISP-based packages are MASCYMA [16], REDUCE [17] and SHEEP [18]. SHEEP is the most suitable for research in

---

* The word "package" usually refers to a collection of programs, while a collection of packages is referred to as a "system". Systems have wider applications but they also require larger computers.

general relativity, and its storage and running is less expensive, in terms of computer time, than those of the other two systems. Both MASCYMA and REDUCE, however, have been used in gravitational calculations and have produced very important results [19 and 20].

SHEEP and STENSOR, a package implemented in SHEEP, will be presented in what follows, since they were used for carrying out the work of this chapter. A few comments on CLASSI, a package for classifying geometries, will also be made.

3.4.1 The package SHEEP

SHEEP is the latest member of the LAM family of computer systems (Lisp Algebraic Manipulator) which are intended mainly for calculations in general relativity and were developed by R. d'Inverno. SHEEP was designed by I. Frick [18] for the DEC PDP 10 machine (and SHEEP stands for LAM(B) grown up'). The most important new feature of SHEEP is that it can be used interactively [21]. In addition, many of the old facilities and algorithms of the systems of the LAM family have been redesigned and significantly improved.

SHEEP consists of three machine-coded packages, which are loaded on top of each other into the LISP system [22].

i) The basic package contains routines that read, write, simplify, substitute into and differentiate algebraic expressions.

ii) On top of this, there is a package of more user-oriented commands that manipulate tensor components.

iii) On top of this, there is one of the two packages CORD or FRAME that computes the Einstein tensor. The CORD version of SHEEP computes the Einstein tensor in a coordinate frame from a given metric. The FRAME version is similar to the CORD one but the computations are done with respect to a moving frame with constant frame metric $H^{\alpha\beta}$. This includes orthonormal and null tetrad frames. The frame (tetrad) components of the Einstein tensor are computed from $H^{\alpha\beta}$ and from the components of the frame (tetrad) vectors.

A system can be applied more efficiently if the user knows its capacity and limitations. Thus, the main advantages and disadvantages of SHEEP will be presented here in order to understand what SHEEP can and cannot do.

The main advantages of SHEEP are:

i) It has a comparatively small size; quite a lot of work can be done in 30K of words [9].

ii) SHEEP is very fast and probably faster than any other system for tensor manipulations. The very well known example of calculating the Einstein tensor for the Bondi radiating metric, presented in the introduction 3.1, was worked out by SHEEP in about 30 seconds using about 35K of memory.

iii) It has a general tensor manipulation package, which can be used in any dimension and allows for the easy definition of new tensors.

iv) It possesses facilities for expanding in terms of a small parameter using power series algorithms.

v) Interactiveness is the most important property of SHEEP, which can prevent a lot of mistakes.

vi) SHEEP has the most readable output compared to all other systems [8] and its notation for the input is straightforward. Moreover, it accepts input from files and gives output to files.

vii) SHEEP is very easy to learn so that any potential user can be converted into an actual one in a very short time and become able to tackle interesting research problems.

The main disadvantages of SHEEP are:

i) SHEEP was written in SLISP dialect, the Standard LISP and therefore has the same disadvantages as LISP. It can only run, at present, on PDP 10 and VAX machines of DEC computers [9].

ii) SHEEP has no symbolic integration program, a price that had to be paid for compactness since all known integration programs are extensively large.

iii) Polynomial division and factorization are also missing from SHEEP.

The last twc disadvantages can be overcome when REDUCE (which includes advanced integration or polynomial division algorithms) and SHEEP live together in the same machine, since SHEEP has instructions for communicating with REDUCE.

SHEEP is the most heavily employed system for solving problems in general relativity. Two very important modules have been added to SHEEP, namely, CLASSI written by J. Aman [23] which is a classification program for geometrics in general relativity and STENSOR written by L. Hornfeldt [24] which is a system specialized on formulas with tensors or spinors with symbolic indices.

3.4.2 CLASSI: classifying geometries

CLASSI contains classification programs for geometries in general relativity written for SHEEP. The most important classification program is that for invariant classification of geometries, related to the "equivalence problem" in exact solutions of Einstein's equations [25], that is, the problem of the elimination of duplications arising from the possibility of writing the same metric in different coordinate systems.

CLASSI is an extension of SHEEP, and thus with CLASSI it is possible to calculate more tensors and quantities than originally with SHEEP. In particular, using CLASSI one can calculate the Weyl tensor, contravariant and mixed components of the Einstein tensor as well as its divergence, that is, the equations of motion. Several covariant derivatives (of high order) of the Riemann tensor, and squares (scalars) of covariant derivatives of the Riemann tensor, the Ricci tensor and the scalar curvature can also be evaluated. Electrodynamical calculations can be performed in CLASSI and the electromagnetic field tensor can be found if both the potential A_μ and the metric are given. CLASSI also performs the Petrov classification of metrics, calculates Newman-Penrose quantities and can handle spinors.

3.4.3 STENSOR: tensor manipulator

STENSOR was developed by L. Hornfeldt and it is based on SHEEP. It is a system for indicial tensor calculus and automatic generation of tensor algorithms. More explicitly, STENSOR is designed to handle any type of indicial quantities, that is, quantities with symbolic indices, such as tensors or spinors.

It can perform all known tensor-calculus operations, such as addition, multiplication, partial and covariant differentiation of tensors and so on.

It can use the symmetries of the different indices during computation. For example, if $A_{\mu\nu}$ and $S^{\mu\nu}$ are defined to be antisymmetric and symmetric tensors respectively, STENSOR will automatically give the value zero to the contracted quantity $A_{\mu\nu}S^{\mu\nu}$ (where a repeated index is understood as a dummy index).

STENSOR accepts and handles anticommuting or non-commuting objects and it also possesses a very sophisticated substitution program.

It has the best trigonometric simplifier known which exploits optimally the trigonometric summation relation
sin^2 (x) + cos^2 (x) = 1.

But the most important feature of STENSOR is the automatic generation of tensor algorithms: there is a compiler, which from any given tensor formula generates a program for computing individual tensor components by performing contractions, derivations etc. This means that SHEEP is no longer restricted to calculating the usual tensors of general relativity using the metric, but new tensors and scalars can be defined and computed without having any relation to the metric!

Finally, the input notation of STENSOR is extremely easy to learn since it is very similar to ordinary textbook conventions and so a user can start using it in various research problems after very little time. In fact, STENSOR has been very successfully used in various problems of classical relativity, quantum gravity or supergravity [19].

3.5 APPLICATIONS

In what follows, two of the applications that were carried out at Imperial College are presented. The first is selected because it demonstrates very clearly the most important features of SHEEP and STENSOR and also serves as a tutorial for the system.

In the second application, the computer is used to generate the algebraic extension of the Lie algebra of Killing vectors in some particular cases of Kimura metrics. Higher order Killing tensors are produced and mistakes in the enormous calculations involved are avoided. The whole procedure, which is very simple and straightforward, verified the theoretical results discussed in Chapter 2.

3.5.1 Killing's equations

Killing's equations for a symmetric tensor $K_{\mu_1 \ldots \mu_p}$ of order p are given by $K_{(\mu_1 \ldots \mu_p ; \nu)} = 0$. To write down explicitly all the components of Killing's equations for a Killing tensor of arbitrary order is extremely hard, especially for complicated metrics. The computer, however, can do this job very quickly and without errors.

The basic idea is to think of Killing's equations as equivalent to demanding the vanishing of the tensor

$$KT_{\mu_1 \ldots \mu_p \nu} = K_{(\mu_1 \ldots \mu_p ; \nu)} = 0. \tag{3.5.1}$$

A very simple program can then be written, introducing the new tensor to SHEEP. After loading this program to STENSOR, its command TCOMPOUT will use the Tensor COMPiler to generate a program for computing the individual components of the desired tensor via SHEEP for any metric [24]. The TCOMPOUT command must be of the form (TCOMPOUT file name tsr1 ...) and will save the building properties of the tensor on a file named "FILENAME.BLF" ("Bulf" is a technical term which refers to "automatic program generation"). The BLF-file contains everything needed for computing the components of the tensor, that is, it constitutes an "application package". This file can be loaded onto a naked SHEEP-version (that is, SHEEP on its own, where it is not necessary to have STENSOR running as well, so that we can save memory) and after entering the components of a particular metric, the command WMAKE applied to the tensor will give the exact values of its components.

The files "KILLING.ONE", "KILLING.TWO" and "KILLING.THREE", containing programs for the evaluation of Killing's equations for a Killing tensor of first, second and third order respectively, are presented in Tables 3.1, 3.2 and 3.3.

Note that STENSOR does not accept Greek letters or small Latin letters and that vector, tensor and tetrad indices run from 0 to 3 in 4-dimensional manifolds.

Listing of files with programs for Killing's equations

Table 3.1

The listing of the file "KILLING.ONE"

```
%  *** KILLING.ONE***    (Killing vector)
(SETQ  AFFIN  "GAMUC)
(PDEF  KT1  S12)
< M  (I  ;(J > $
(PPDEF  KT1  '(  )
(NOASK  (EDEF  KT1))
(TCOMPOUT  KILLING1  KT1)
(SYMBOLIC  M)
(FUNS  (M  T  R  H  F ) )
```

Table 3.2

The listing of the file "KILLING.TWO"

```
% *** KILLING.TWO***   (Killing tensor of second order)
(SETQ  AFFIN  "GAMUC )
(PDEF  KT2  S123)
< N  (I  (J  ;(K > $
(PPDEF  KT2  '(  )
(NOASK (EDEF  KT2) )
(TCOMPOUT  KILLING2  KT2 )
(SYMBOLIC N)
(FUNS  (N  T  R  H  F ) )
```

Table 3.3

The listing of the file "KILLING.THREE"

```
% *** KILLING.THREE *** (Killing tensor of third order)
(SETQ  AFFIN  "GAMUC)
(PDEF  KT3  (1  (1  2))  (1  (1  3) )  (1  (2  3) )
   (1  (2  4) )  (1  (3  4) )  (1  (1  4 ) ) )
< Z   (I   (J   (K  ;(L  > $
(PPDEF   KT3  '(   )
(NOASK  (EDEF   KT3)  )
(TCOMPOUT  KILLING3  KT3 )
(SYMBOLIC  Z )
(FUNS  (Z  T  R  H  F ) )
```

Before explaining how one can use these programs to write Killing's equations for a particular metric, I should say a few words about the commands involved.

Note first that the symbol % means that the rest of the line is a comment. It is there to clarify the program to the reader and will be overlooked by the computer.

The command SETQ takes two arguments and is the normal LISP assignment operator. SETQ returns the second argument as the LISP value of the first. For example, if you type (SETQ X '(A B)), any subsequent use of X will be understood by LISP to be (A B), and if you type only X, LISP will respond by printing (A B). Note also that if you type (SETQ X (PLUS 25 -20)), LISP will evaluate the second argument and in any subsequent use of X it will understand X as having the value 5. Thus, SETQ evaluates its second argument when unquoted. In our case, this will cause the AFFINity value to be GAMU, which we must inhibit with an extra quote so that the AFFINity value will be 'GAMU. This is because 'GAMU is the affinity used by STENSOR for the covariant differentiation of tensors. GAMUC corresponds to the coordinate version, while for the frame version we must type GAMUF [24].

The command PDEF is used to define the tensors KT1, KT2, KT3. The PDEF command must be of the form (PDEF tsr P1 P2 ...), P1, P2 ... being the Properties of the defined tensor. On this command the system responds with " = " and halts, waiting for an indicial formula, the "definition" of the tensor, to be typed in. When the user has done this, he/she ends the input with the symbol $ (escape).

The symbol S12 means symmetry between the first and second indices, while S123 means that the defined tensor is symmetric in all three indices. In the case of the third order Killing tensor, the symmetries of the indices have been written explicitly, in order to show that there is also another way to present them to the computer.

The command PPDEF means Perform the Permutation notations indicated in the DEFinition of the tensor.

EDEF is the function for Evaluating DEFinitions.

EDEFASK is a switch which, when on, causes EDEF to ask for interactive evaluation orders. Thus, NOASK, which can take any number of commands as arguments, performs these commands with the switch EDEFASK temporarily turned off. This command is useful in a file when the user is sure that he/she wants the indicated evaluations.

TCOMPOUT produces, after each file has been loaded to STENSOR, the BLF files "KILLING1.BLF", "KILLING2.BLF", "KILLING3.BLF".

SYMBOLIC M,N,Z declares M,N,Z as the symbols for the component of the tensors KT1, KT2, KT3 respectively.

(FUNS (...)) declares that the tensor components are functions of the variables (T,R,H,F) ≡ (t,r,ϑ,φ), that is, the programs deal with the general case of time-dependent Killing tensors.

To use these programs, the following should be done.

1. Log-on the computer and start STENSOR by typing 'stensor'. (LOAD CORD) for the coordinate version.
2. Load the file containing the necessary definitions, using the LISP command (LOAD "KILLING.ONE"). The computer will then produce the file "KILLING1.BLF", which must be loaded to STENSOR using the command (RELOAD "KILLING1.BLF"). The file "KILLING1.BLF" is presented in Appendix A5. A similar procedure must be followed in the cases of the "KILLING.TWO" and "KILLING.THREE" files.
3. To apply the program to a given metric you can do either of the following.

3a. Enter the components of the particular metric using the command (RPL GDD) with which you define the covariant components of the metric tensor.

3b. LOAD a file that you have already created with all the details of the particular metric (this is usually a very useful and convenient way to proceed, especially if you want to perform many calculations for this metric). Then use the command (WMAKE KT1) and leave the rest to the computer; similarly for KT2 and KT3. The SHEEP command

WMAKE computes the individual components of a tensor and also writes them explicitly on the screen.

For my purposes, a file containing all the information about variables and metric components of the spherically symmetric Kimura space-times is needed. This file, with the name "KIMURA.BASIC", is presented in Table 3.4.

Table 3.4

The listing of the file "KIMURA.BASIC"

```
% *** KIMURA.BASIC ***
(VARS   T   R   H   F)
(FUNS   (A   T   R)   (C   T   R) )
(ON   NOZERO   DIAGONAL)
(RPL   GDD)
C $
-A $
-R^2 $
-R^2 * SIN(H)^2 $
```

The line element described in this file is of the form

$$ds^2 = Cdt^2 - Adr^2 - r^2 d\vartheta^2 - r^2 \sin^2\vartheta \, d\varphi^2 \qquad (3.5.2)$$

where, for the sake of generality, the metric components A and C are denoted as functions of t and r. And, in fact, this file can be thought of as representing any spherically symmetric

metric, when specific forms of A and C are considered. In the case of <u>static</u> spherically symmetric metrics, we must, of course, think of A and C as functions of r only.

Killing vector equations for the above metric can be written by the computer in fractions of seconds, when the files "KILLING.ONE", "KILLING1.BLF" (produced by the tensor compiler) and "KIMURA.BASIC" are loaded to STENSOR. These equations, as produced by the computer, are presented in Table 3.5, where their right-hand sides must be put equal to zero and M_0, M_1, M_2, M_3 denote the components of the Killing vector.

The corresponding results for Killing tensor equations of second and third order are not presented here because of the large number of equations involved.

Table 3.5

The listing of Killing vector equations

*** "KILLING1.LAM" ***

$KT1_{00} = 2M_{0,T} - A^{-1}C_{,R}M_1 - C^{-1}C_{,T}M_0$

$KT1_{01} = M_{0,R} + M_{1,T} - A^{-1}A_{,T}M_1 - C^{-1}C_{,R}M_0$

$KT1_{02} = M_{0,H} + M_{2,T}$

$KT1_{03} = M_{0,F} + M_{3,T}$

$KT1_{11} = 2M_{1,R} - A^{-1}A_{,R}M_1 - C^{-1}A_{,T}M_0$

$KT1_{12} = M_{1,H} + M_{2,R} - 2R^{-1}M_2$

$KT1_{13} = M_{1,F} + M_{3,R} - 2R^{-1}M_3$

$KT1_{22} = 2M_{2,H} + 2A^{-1}RM_1$

$KT1_{23} = - 2\cos(H)M_3 \sin^{-1}(H) + M_{2,F} + M_{3,H}$

$KT1_{33} = 2\cos(H)M_2 \sin(H) + 2M_{3,F} + 2A^{-1}RM_1 \sin^2(H)$

3.5.2 The generation of high order Killing tensors

As we have seen in Chapter 2, by using the Schouten-Nijenhuis bracket for totally symmetric tensors we can have an algebraic extension of the Lie algebra of Killing vectors. It has also been proved that the Poisson bracket of the functions, on the cotangent bundle, associated with these symmetric tensors is directly related to their Schouten-Nijenhuis bracket. Thus I shall use below the Poisson bracket of functions associated with the Killing tensors of Kimura space-times, to give specific examples of the Lie algebra extension and to obtain new Killing tensors of high order.

The Poisson bracket of two functions $\varkappa$ and ι with respect to the variables (x^μ, p_μ) is defined by

$$(\varkappa,\iota) = \frac{\partial\varkappa}{\partial x^\mu}\frac{\partial\iota}{\partial p_\mu} - \frac{\partial\varkappa}{\partial p_\mu}\frac{\partial\iota}{\partial x^\mu} . \tag{3.5.3}$$

We need first to write a program where the Poisson bracket of two functions is defined in a way understandable to the computer. This program, written in the basic file "POI.DEF", is presented in Table 3.6.

It is worth mentioning that very little knowledge of STENSOR is needed to write down this program, since one does not even need to know how to formulate the covariant derivative in STENSOR, and yet the program is very <u>general</u>; "POI.DEF" can be used with <u>any</u> metric' a fact that makes this program very useful.

The symbol DEFAULTDIMENSION'* must be explained. Since we have passed now to the cotangent bundle which has coordinates

Table 3.6

The listing of the program defining the Poisson bracket of two functions

```
    % *** POI.DEF ***
% The first file on Poisson brackets, using explicit  evaluation.
(SETQ  DIM  (SETQ  CODIM  (SETQ  DEFAULTDIMENSION!*  8)))
% Poi Sson of Kappa and Iota:
(PDEF   PSKI)
< KAPPA   ,0 > < IOTA   ,4  >  +
< KAPPA   ,1 > < IOTA   ,5  >  +
< KAPPA   ,2 > < IOTA   ,6  >  +
< KAPPA   ,3 > < IOTA   ,7  >  -
< KAPPA   ,4 > < IOTA   ,0  >  -
< KAPPA   ,5 > < IOTA   ,1  >  -
< KAPPA   ,6 > < IOTA   ,2  >  -
< KAPPA   ,7 > < IOTA   ,3  >   $

(TCOMPOUT   POI   PSKI  )
```

(x^{μ}, p_{μ}) we are, in fact, in a manifold with different dimension from the original one. In particular, while the original manifold is 4-dimensional, the new one is 8-dimensional. We must therefore indicate this to the computer. This is done by the command

(SETQ DIM (SETQ CODIM (SETQ DEFAULTDIMENSION!* 8))).

The symbol DEFAULTDIMENSION!* is a SHEEP symbol that should have a positive value. Note that when one writes down the command

TCOMP for a tensor, TCOMP declares arrays for this tensor as well as for the tensors occurring in its definition. Therefore, always set the desired dimension before TCOMPing! [24] .

The numbers 0,1,2,3 in the program correspond to the x^{μ} coordinates, while the numbers 4,5,6,7 correspond to the p_{μ} coordinates. When we load this file into STENSOR, the tensor compiler produces the file "POI.BLF" which will help the computer to evaluate Poisson brackets in the future. The file "POI.BLF" is presented in Appendix A5. We are now ready to discuss some particular examples.

The first case worked out in detail was case IIC of the Kimura metrics

$$ds^2 = Cdt^2 - Adr^2 - r^2d\vartheta^2 - r^2\sin^2\vartheta d\varphi^2$$

with $C = a_2^{-1}r^2$ and $A = a_2^{-2}r^{-2}$, where a_2 is a constant. The two Killing tensors of second order for this case are given in Table 2.2, number 6, and are, after substitution of the above values of A and C, the following

$$K_{\mu\nu} = \frac{2rt}{a_2^2}\,\delta^4_{(\mu}\delta^1_{\nu)} - t^2r^4(\delta^2_{\mu}\delta^2_{\nu}+\sin^2\vartheta\delta^3_{\mu}\delta^3_{\nu}) + \frac{r^4}{a_2^2}\,(r^{-2}+a_2t^2)\,\delta^4_{\mu}\delta^4_{\nu} \tag{3.5.4}$$

$$I_{\mu\nu} = \frac{2r}{a_2^2}\,\delta^4_{(\mu}\delta^1_{\nu)} - 2tr^4(\delta^2_{\mu}\delta^2_{\nu} + \sin^2\vartheta\delta^3_{\mu}\delta^3_{\nu}) + \frac{2r^4t}{a_2}\,\delta^4_{\mu}\delta^4_{\nu}. \tag{3.5.5}$$

These two Killing tensors are associated with the functions $\varkappa = K^{\mu\nu}p_{\mu}p_{\nu}$ and $\iota = I^{\mu\nu}p_{\mu}p_{\nu}$ on the cotangent bundle. The functions

$\varkappa$ and ι have the explicit forms

$$\varkappa = K^{\mu\nu}p_\mu p_\nu = -\,2a_2 rtp_r p_t - t^2p_\vartheta^2 - \frac{t^2}{\sin^2\vartheta}\,p_\varphi^2 + (r^{-2}+a_2t^2)p_t^2 \tag{3.5.6}$$

$$\iota = I^{\mu\nu}p_\mu p_\nu = -\,2a_2 rp_r p_t - 2tp_\vartheta^2 - \frac{2t}{\sin^2\vartheta}\,p_\varphi^2 + 2a_2tp_t^2 \tag{3.5.7}$$

and they must be written in a way understandable to the computer if we want to command it to perform the required calculations.

First of all, since STENSOR accepts only capital letters, particular symbols are needed for the eight coordinates (x^μ, p_μ) and the constant a_2. I have chosen the following symbols

$$(t,r,\vartheta,\varphi,p_t,p_r,p_\vartheta,p_\varphi) \equiv (T,R,H,F,V,P,Q,S)$$

while the constant a_2 is denoted by B. All the above information on case IIC can be put into a new file which is listed in Table 3.7 under the name "POIKM6.DEF" (the number 6 is the number of the metric appearing in Table 2.1).

"Reading" the files "POI.BLF" and "POIKIM6.DEF" is all that the computer needs in order to start the evaluation of Poisson brackets! Thus, after "being" in STENSOR, we have to type the following commands

```
STE > (LOAD  "POI.BLF")
STE > (LOAD  "POIKIM6.DEF")
STE > (WMAKE   PSKI)
```

Table 3.7

The listing of the file "POIKIM6.DEF"

```
% *** POIKIM6.DEF ***
```

% The Poisson for Kimura case IIC with $C = a_2^{-1}r^2$ and $A = a_2^{-2}r^{-2}$.

```
(SETQ  DIM  (SETQ  CODIM  (SETQ  DEFAULTDIMENSION!* 8 ) ) )
(VARS   T  R  H  F  V  P  Q  S  )
(FUNS  (B) )
(RPL   KAPPA)
-2  B  R  T  P  V  -  T^2  Q^2 - T^2  S^2  SIN(H)^(-2)
     + V^2  R^(-2) + B  T^2  V^2   $
(RPL   IOTA )
-2  B  R  P  V  -  2  T  Q^2  - 2  T  S^2  SIN(H)^(-2)
    + 2  B  T  V^2  $
```

---

and the computer will then evaluate the Poisson bracket of $\varkappa$ and ι, namely, $(\varkappa,\iota) = \lambda$. The answer that the computer gives is

$$PSKI = - 4BQ^2T^2V - 4BS^2T^2V \sin^{-2}(H) - 8B^2PRTV^2 + 4B^2P^2R^2V$$
$$+ 4B^2T^2V^3 + 4Q^2R^{-2} + 4R^{-2}S^2V \sin^{-2}(H)$$

that is,

$$(\varkappa,\iota) = \lambda = -\ 4a_2t^2p_tp_\vartheta^2 - 4a_2t^2\sin^{-2}\vartheta p_tp_\varphi^2 - 8a_2^2rtp_rp_t^2$$

$$+ 4a_2^2r^2p_r^2p_t + 4a_2^2t^2p_t^3 + \frac{4}{r^2}\ p_\vartheta^2p_t + \frac{4}{r^2}\ \sin^{-2}\vartheta p_\varphi^2p_t. \qquad (3.5.8)$$

The function λ can be thought of as the function, on the cotangent bundle, associated with the symmetric tensor of third order $L^{\mu\nu\varkappa}$, that is, $\lambda = L^{\mu\nu\varkappa}p_\mu p_\nu p_\varkappa$. The tensor $L^{\mu\nu\varkappa}$ is a Killing tensor, since both $K^{\mu\nu}$ and $I^{\mu\nu}$ are, and it has the form

$$L^{\mu\nu\varkappa} = -4a_2t^2\delta_4^\mu\delta_2^\nu\delta_2^\varkappa - 4a_2\ \frac{t^2}{\sin^2\vartheta}\ \delta_4^\mu\delta_3^\nu\delta_3^\varkappa - 8a_2^2rt\delta_1^\mu\delta_4^\nu\delta_4^\varkappa$$

$$+ 4a_2^2r^2\delta_1^\mu\delta_1^\nu\delta_4^\varkappa + 4a_2^2t^2\delta_4^\mu\delta_4^\nu\delta_4^\varkappa + \frac{4}{r^2}\ \delta_2^\mu\delta_2^\nu\delta_4^\varkappa$$

$$+ \frac{4}{r^2\sin^2\vartheta}\ \delta_3^\mu\delta_3^\nu\delta_4^\varkappa. \qquad (3.5.9)$$

So far, only one Poisson bracket has been evaluated, which gives rise to a new Killing tensor of third order. This process, however, can be continued and Killing tensors of higher and higher order can be obtained from Poisson brackets in the following way

$$(\varkappa,\iota) = \lambda = L^{\mu\nu\varkappa}p_\mu p_\nu p_\varkappa$$

$$(\varkappa,\lambda) = (\varkappa,(\varkappa,\iota)) = \mu = M^{\mu\nu\varkappa\lambda}p_\mu p_\nu p_\varkappa p_\lambda$$

$$(\varkappa,\mu) = (\varkappa,(\varkappa,(\varkappa,\iota))) = \nu = N^{\mu\nu\varkappa\lambda\alpha}p_\mu p_\nu p_\varkappa p_\lambda p_\alpha$$

$$(\varkappa,\nu) = (\varkappa,(\varkappa,(\varkappa,(\varkappa,\iota)))) = \pi = P^{\mu\nu\varkappa\lambda\alpha\beta}p_\mu p_\nu p_\varkappa p_\lambda p_\alpha p_\beta$$

$$(\varkappa,\pi) = (\varkappa,(\varkappa,(\varkappa,(\varkappa,(\varkappa,\iota))))) = \rho = R^{\mu\nu\varkappa\lambda\alpha\beta\gamma}p_\mu p_\nu p_\varkappa p_\lambda p_\alpha p_\beta p_\gamma$$

$$(\varkappa,\rho) = (\varkappa,(\varkappa,(\varkappa,(\varkappa,(\varkappa,(\varkappa,\iota)))))) = \sigma = S^{\mu\nu\varkappa\lambda\alpha\beta\gamma\delta}p_\mu p_\nu p_\varkappa p_\lambda p_\alpha p_\beta p_\gamma p_\delta .$$

$\vdots$

It is obvious that to do all these calculations by hand is extremely hard. There is, however, a very easy way of performing the above calculations on the computer. After STENSOR has finished the job ordered by the command (WMAKE PSKI), we simply ask it to RePLace IOTA by PSKI, that is, by λ, by typing

```
STE > (RPL  IOTA)
IOTA = : PSKI  $
```

and ask it again to evaluate the new Poisson bracket $(\varkappa,\lambda)$ using the command

```
STE > (WREMAKE PSKI).
```

And this process can continue for as long as we want! In practice, though, there are some restrictions imposed by the large storage space required at very high stages of the calculation.

In this first case of Kimura metrics the following Poisson brackets were evaluated:

1. $(\varkappa,\iota) = \lambda$
2. $(\varkappa,\lambda) = (\varkappa,(\varkappa,\iota)) = \mu$

3. $(\varkappa,\mu) = (\varkappa,(\varkappa,(\varkappa,\iota))) = \nu$

4. $(\varkappa,\nu) = (\varkappa,(\varkappa,(\varkappa,(\varkappa,\iota)))) = \pi$ (60 terms)

5. $(\varkappa,\pi) = (\varkappa,(\varkappa,(\varkappa,(\varkappa,(\varkappa,\iota))))) = \rho$ (70 terms) and

6. $(\varkappa,\rho) = (\varkappa,(\varkappa,(\varkappa,(\varkappa,(\varkappa,(\varkappa,\iota)))))) = \sigma$ (140 terms!)

The first five of them are presented in Appendix A6.

Three more cases of Kimura metrics have been worked out as above.

* The case IIC with $C = r^2(a_1r^2 + a_2)^{-1}$ and $A = r^{-2}(a_1r^2 + a_2)^{-2}$, where a_1 and a_2 are constants. For this case the file "POIKIM5.DEF", listed in Table 3.8, was created where the constants a_1 and a_2 are denoted by D and B respectively.

The following Poisson brackets were evaluated:

1. $(\varkappa,\iota) = \lambda$

2. $(\varkappa,\lambda) = (\varkappa,(\varkappa,\iota)) = \mu$ (39 terms)

3. $(\varkappa,\mu) = (\varkappa,(\varkappa,(\varkappa,\iota))) = \nu$ (64 terms) and

4. $(\varkappa,\nu) = (\varkappa,(\varkappa,(\varkappa,(\varkappa,\iota)))) = \pi$ (168 terms!)

The first three of them are presented in Appendix A7.

Table 3.8

The listing of the file "POIKIM5.DEF"

```
% *** POIKIM5.DEF ***
% The Poisson for Kimura case IIC with C = r^2(a_1 r^2 + a_2)^-1 and
%    A = r^-2(a_1 r^2 + a_2)^-2.

(SETQ  DIM  (SETQ  CODIM (SETQ  DEFAULTDIMENSION!*  8)))

(VARS    T  R  H  F  V  P  Q  S )

(FUNS   (B) )

(FUNS   (D) )

(RPL   KAPPA)

- 2  (D  R^2  +  B)  R  T  P  V  - T^2  Q^2 - T^2  S^2  SIN(H)^(-2)

+  R^(-2)   V^2   +   B   T^2   V^2   $

(RPL   IOTA)

- 2  (D  R^2  +  B)  R P V  - 2 T Q^2 - 2 T  S^2  SIN(H)^(-2)

+  2  B  T  V^2   $
```

---

*The case III with C = 1 and $A = (a_1 r^2 + a_2)^{-3}$, where a_1 and a_2 are constants. For this case the file "POIKIM7.DEF", listed in Table 3.9, was created where again the constants a_1 and a_2 are denoted by D and B respectively.

Table 3.9

The listing of the file "POIKIM7.DEF"

% *** POIKIM7.DEF***

% The Poisson for Kimura case III with C = 1 and A = $(a_1r^2+a_2)^{-3}$.

```
(SETQ  DIM  (SETQ  CODIM  (SETQ  DEFAULTDIMENSION'* 8)))

(VARS   T  R  H  F  V  P  Q  S)

(FUNS   (B))

(FUNS   (D))

(RPL   KAPPA)

T^2  (D  R^2  +  B)^3  P^2  +  4  R  T  (D  R^2 + B)  P V B^(-1)

+ T^2 (D  R^2  +  B)  B^(-1)  R^(-2)  Q^2

+ T^2 (D  R^2  +  B)  B^(-1)  R^(-2)  SIN(H)^(-2)  S^2

+ R^2  B^(-2)  (D  R^2  +  B)^(-1)  V^2  $

(RPL  IOTA)

T  (D  R^2  +  B)^3  P^2  + 2 R (D  R^2 + B)  B^(-1)  P  V

+ T (D  R^2  +  B)  B^(-1)  R^(-2)  Q^2

+ T (D  R^2  +  B)  B^(-1)  R^(-2)  SIN(H)^(-2)  S^2  $
```

The Poisson brackets evaluated, in this case, are only two:

1. $(\varkappa,\iota) = \lambda$ (31 terms)
2. $(\varkappa,\lambda) = (\varkappa,(\varkappa,\iota)) = \mu$ (233 terms!!)

for the obvious reason that a large number of terms is involved. The first of them is presented in Appendix A8.

*The case III with C = 1 and $A = a_1^{-3} r^{-6}$, where a_1 is a constant. For this case the file POIKIM8.DEF", listed in Table 3.10, was created, where a_1 is denoted by D.

Table 3.10

The listing of the file "POIKIM8.DEF"

```
%***POIKIM8.DEF***
% The Poisson for Kimura III with C = 1 and A = a1^-3 r^-6.

(SETQ  DIM  (SETQ  CODIM  (SETQ  DEFAULTDIMENSION'* 8)))
(VARS   T  R  H  F  V  P  Q  S)
(FUNS   (D))
(RPL   KAPPA)
-4 T  D^(3/2)  R^3  P  V  -  D^(3/2)  T^2  Q^2
-  D^(3/2)  T^2  SIN(H)^(-2)  S^2  +  D^(-3/2)  R^(-2)  V^2  $
(RPL   IOTA)
- 4  D^(3/2)  R^3  P  V  -  2  D^(3/2)  T  Q^2
- 2  T  D^(3/2)  SIN(H)^(-2)  S^2  $
```

The following Poisson brackets were evaluated:

1. $(\varkappa,\iota) = \lambda$
2. $(\varkappa,\lambda) = (\varkappa,(\varkappa,\iota)) = \mu$
3. $(\varkappa,\mu) = (\varkappa,(\varkappa,(\varkappa,\iota))) = \nu$
4. $(\varkappa,\nu) = (\varkappa,(\varkappa,(\varkappa,(\varkappa,\iota)))) = \pi$ (33 terms)
5. $(\varkappa,\pi) = (\varkappa,(\varkappa,(\varkappa,(\varkappa,(\varkappa,\iota))))) = \rho$ (37 terms)
6. $(\varkappa,\rho) = (\varkappa,(\varkappa,(\varkappa,(\varkappa,(\varkappa,(\varkappa,\iota)))))) = \sigma$ (73 terms)

and they are presented in Appendix A9.

3.6 CONCLUSION

The solutions of the equation $[K,g] = 0$, where K and g denote a Killing and the metric tensors respectively, are defined as hidden symmetries [26]. This definition, together with the interpretation of the Schouten bracket as the Poisson bracket of functions, on the cotangent bundle, associated with symmetric Killing tensors, make possible the systematic search for hidden symmetries in many physical situations.

Killing tensors also identify solutions of the equation of geodesic deviation. In spite of the importance of this equation in the study of space-times, very few exact solutions of this equation can be found in the literature because of the complexity of the differential equations involved.

With the method proposed in this chapter, new Killing tensors of high order can be evaluated, which lead both to an extension of the Lie algebra of isometries and to new solutions of the equation of geodesic deviation. Moreover, the proposed method is very simple and very general and helps enormously to overcome the problems of the complicated calculations involved.

It must also be noted that the Lie algebraic extension of the Lie algebra of Killing vectors may be finite or indeed infinite. Since Killing tensors are symmetric tensors, there is no upper bound on the number of their indices, that is, of their order, and so the extended Lie algebra does not have to be finite-dimensional. The situation, however, is totally different with antisymmetric tensors. Since in any n-dimensional manifold, the upper bound in the order of an antisymmetric tensor is n, the graded Lie algebraic extension [27] must always be finite.

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CHAPTER 4

ANISOTROPIC STATIC SPHERICALLY SYMMETRIC SPACE-TIMES WITH KILLING TENSORS

4.1 INTRODUCTION

The concept of a spherically symmetric space-time has played an important role in the theory of general relativity [1]. In the first four decades of research in general relativity, the overwhelming majority of exact solutions of Einstein's equations were obtained under the assumption of spherical symmetry [2]. The exterior Schwarzschild solution [3], which provided the decisive experimental verification of general relativity, is a well known example. We may also say that relativistic cosmology and astrophysics were developed following the advance of investigations of spherically symmetric space-times.

The physical as well as the mathematical properties of these space-times have been thoroughly studied by many authors and from various points of view. In particular, spherically symmetric distributions of matter have been discussed as relativistic models of stellar configurations, such as white dwarfs, neutron stars (and their connection to pulsars), supermassive stars (and their connection to quasars), black holes or relativistic star clusters [4].

Most problems in astrophysical studies of such massive objects have been investigated under the assumption of local isotropy. The existence of local anisotropy has not usually been considered, mainly because of the complexity of the equations involved. Recent theoretical works, however, on more realistic stellar models, suggest that superdense matter may be locally anisotropic, at least in certain density ranges [5].

A complete theoretical understanding of the structure of massive objects does, therefore, require a knowledge of the possible anisotropic properties of dense matter, as a result of which local anisotropy can no longer be neglected.

The theory of anisotropic spherically symmetric distributions of matter will be presented in this chapter and the physical properties of Kimura anisotropic static fluid spheres will be studied.

4.2 ANISOTROPIC FLUID SPHERES IN GENERAL RELATIVITY

4.2.1 Anisotropic matter and Einstein's equations

Einstein's field equations connect the distribution of matter and energy, described by the components of the energy-momentum tensor $T_{\mu\nu}$, with the resulting gravitational field, described by the potential $g_{\mu\nu}$, and they have the form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} = 8\pi T_{\mu\nu} \; * \tag{4.2.1}$$

where $G_{\mu\nu}$ is the Einstein tensor.

---

* Einstein's equations are, in this sense, analogous to Maxwell's equations $\partial^{\nu}\partial_{\nu}A_{\mu} = - 4\pi j_{\mu}$, with the energy-momentum tensor $T_{\mu\nu}$ serving as the source of the gravitational field in much the same way as the current j_{μ} serves as the source of the electromagnetic field [6].

It is always very hard to find exact solutions of equations (4.2.1) and the problem is made particularly difficult by the nonlinear character of these equations *. Considerable simplifications can be introduced, however, if one is searching for solutions of equations (4.2.1) that would correspond to an equilibrium distribution of a fluid.

Since the condition of gravitational equilibrium for a fluid will, on physical grounds, be a static and spherically symmetric distribution of matter [8], we can choose Schwarzschild-type or radiation coordinates (r,ϑ,φ,t) in which the solution of equations (4.2.1) will be described by the line element

$$ds^2 = - e^{\lambda}dr^2 - r^2d\vartheta^2 - r^2\sin^2\vartheta d\varphi^2 + e^{\nu}dt^2 \qquad (4.2.2)$$

where λ and ν are functions of r alone. (4.2.1) and (4.2.2) then lead to the following expressions for the only surviving components of the energy-momentum tensor

---

* Maxwell's equations are linear because the electromagnetic field does not itself carry charge, whereas gravitational fields do carry energy and momentum, and do therefore contribute to their own source. That is, the nonlinearity of Einstein's equations represents the effect of gravitation on itself [7].

$$\left.\begin{aligned} 8\pi T^r_r &= - e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} \\ 8\pi T^\vartheta_\vartheta = 8\pi T^\varphi_\varphi &= - e^{-\lambda} \left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu'-\lambda'}{2r}\right) \\ 8\pi T^t_t &= e^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \end{aligned}\right\} \qquad (4.2.3)$$

where a dash denotes differentiation with respect to r.

If the matter involved in the distribution is by hypothesis an anisotropic fluid, we can obtain a direct connection with the properties of the fluid by making use of the general expression [9]

$$T^{\mu\nu} = (p_\perp + \rho)u^\mu u^\nu - p_\perp g^{\mu\nu} + (p_r - p_\perp)\chi^\mu\chi^\nu \qquad (4.2.4)$$

where u^μ is the velocity of the matter and χ^μ is a space-like vector satisfying $\chi^\mu\chi_\mu = -1$. Then the spherical symmetry implies that (in canonical coordinates) the energy-momentum tensor T^μ_ν is diagonal*, that is, $T^\mu_\nu = \mathrm{diag}(-p_r, -p_\vartheta, -p_\varphi, \rho)$, and, moreover, that $p_\vartheta = p_\varphi$ which will be denoted by $p_\perp$. The only surviving components of the energy-momentum tensor are therefore

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* In general, a fourth nonvanishing component would appear written in covariant notation as $T_{14} = T_{41}$, but with the choice of canonical coordinates we can make it vanish. Since T_{14} is the density of momentum for the observer involved in the choice of coordinates , the requirement $T_{14} = 0$ means that the motion of the observer is the mean motion of the matter [10].

$$T^r_r = -p_r, \quad T^\vartheta_\vartheta = T^\varphi_\varphi = -p_\perp \text{ and } T^t_t = \rho. \tag{4.2.5}$$

Without further specifications about the relations between p_r, $p_\perp$ and ρ this form is completely general. The quantities p_r and $p_\perp$ can contain contributions from fluid pressures as well as other stresses. In the following we shall always call p_r <u>the radial pressure</u> and $p_\perp$ <u>the tangential pressure</u>; ρ is the <u>density</u> of the matter.

For the line element (4.2.2), Einstein's equations (4.2.1) become, after substitution of the expressions (4.2.5)

$$\left.\begin{aligned}
8\pi p_r &= e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \\
8\pi p_\perp &= e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu'-\lambda'}{2r}\right) \\
8\pi\rho &= e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}.
\end{aligned}\right\} \tag{4.2.6}$$

The differential equations (4.2.6)* relate the unknown functions λ and ν, appearing in the line element (4.2.2), with the pressures p_r and $p_\perp$ and the density ρ of the fluid.

Thus, we only have three structure equations for the five unknown quantities λ, ν, p_r, $p_\perp$ and ρ (as functions of r). The problem will hence become determinate as soon as we introduce

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* Note that the only difference between equations (4.2.6) and those describing isotropic spherically symmetric matter is that $p_r \neq p_\perp$.

two further independent equations corresponding to some additional hypotheses as to the nature of the fluid. From a physical point of view, it seems most natural to introduce these additional hypotheses in the form of equations of state, $p_r = p_r(\rho)$ and $p_\perp = p_\perp(\rho)$ describing the relation between the pressures p_r and $p_\perp$ and the density ρ of the fluid under consideration*. When solved, these five equations (that is, equations (4.2.6) plus the two equations of state) will give us a solution of (4.2.1) which, however, may not be physically interesting or even physically possible. Nevertheless, having obtained an explicit solution, it becomes relatively easy to examine the implied physics and check if it has a character that could be expected for actual fluids [8].

Equations (4.2.6) can be used to construct an equivalent set of structure equations which are easier to handle and more useful for applications.

Integration of the third of equations (4.2.6) gives

$$e^{-\lambda} = 1 - \frac{2m}{r} \tag{4.2.7}$$

where

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr \tag{4.2.8}$$

---

* In general, p_r and $p_\perp$ will depend on additional variables (such as entropy, magnetic fields, etc.) in which case further equations determining these will be needed.

is the mass inside a sphere of radius r as seen by a distant observer.

From (4.2.6) the following equation for the gradient of the radial pressure p_r can also be obtained

$$\frac{dp_r}{dr} = -\frac{1}{2}\nu'(p_r + \rho) + \frac{2}{r}(p_\perp - p_r). \tag{4.2.9}$$

It must be noted, however, that there is <u>no</u> equation governing the gradient of the tangential pressure. The second term in (4.2.9) is the only term in the structure equations which contains the tangential pressure explicitly.

The first and third of equations (4.2.6) can be used to show that

$$\frac{1}{2}\nu' = \frac{m + 4\pi r^3 p_r}{r(r - 2m)} . \tag{4.2.10}$$

This expression reduces to the familiar formula

$$\frac{1}{2}\nu' = \frac{m}{r^2} \tag{4.2.10N}$$

in the Newtonian limit.

In most studies of stellar structure, one replaces equation (4.2.10) by the equivalent equation

$$\frac{dp_r}{dr} = -\frac{(p_r+\rho)(m+4\pi r^3 p_r)}{r(r-2m)} + \frac{2}{r}(p_\perp - p_r)\ * \ . \tag{4.2.11}$$

---

* In the isotropic case with $p_r = p_\perp = p$, equation (4.2.11) reduces to the Oppenheimer-Volkoff equation of hydrostatic equilibrium.

In the Newtonian limit, equation (4.2.11) has the form

$$\frac{dp_r}{dr} = - \frac{\rho m}{r^2} + \frac{2}{r} (p_\perp - p_r) \quad * \; . \tag{4.2.11N}$$

The anisotropy term is, therefore, of Newtonian origin, at least in the particular case of spherical symmetry.

Thus, finally (4.2.7), (4.2.9), (4.2.10) and the equations of state can be used to solve the problem of determining λ, ν, p_r, $p_\perp$ and ρ as functions of r. A solution of these equations, however, and the consequent construction of a stellar model are possible only when certain boundary conditions have been imposed, as follows.

B_1: We require that the interior of any mass distribution be free of singularities, which imposes the condition $m(r) \to 0$ as $r \to 0$. Equivalently, we require $\lambda \to 0$ as $r \to 0$ to ensure that the metric is nonsingular at the centre.

---

* Compare two stellar models, one relativistic and one Newtonian, which at a given radius r have the same values of p_r, $p_\perp$, ρ and m. Then, from equations (4.2.11) and(4.2.11N) we can see that general relativity predicts stronger gravitational forces in a stationary body than does Newtonian theory. These forces, among their other important effects, can pull certain white-dwarf and supermassive stars into gravitational collapse under circumstances where Newtonian theory would have predicted stable hydrostatic equilibrium [4].

B_2: The pressures p_r and $p_\perp$ and the density ρ must be everywhere defined inside.

B_3: Assuming that p_r is finite at $r = 0$, then from (4.2.6) we have $\nu' \to 0$ as $r \to 0$. Therefore, the gradient dp_r/dr will be finite at $r = 0$, only if $(p_\perp - p_r)$ vanishes at least as rapidly as r when $r \to 0$, as can be seen from (4.2.9).

B_4: The radial pressure vanishes on the outer boundary. Thus the radius R of a stellar model will be determined by the condition $p_r(R) = 0$* (note that the density and the tangential pressure do not necessarily vanish on the outer boundary).

B_5: Outside our spherical distribution of matter, space is empty so that it is described by the exterior Schwarzschild metric, that is, an exterior vacuum Schwarzschild metric is always matchable to our interior solution across $r = R$ as long as $p_r(R) = 0$, even though $p_\perp(R)$ and $\rho(R)$ may be discontinuous.

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* As in isotropic models, the inclusion of an "atmosphere" will result in complications since, if an atmosphere is included, we should require $p_r(R) = P_{ph}$ the "photospheric" pressure.

4.2.2 Bondi analysis for anisotropic matter

Bondi's method used in his analysis of isotropic gravitating spheres [11] has been generalized by Bowers and Liang [12] to include the case of locally anisotropic matter. New variables have been introduced as follows

$$u(r) = \frac{m(r)}{r} \tag{4.2.12}$$

$$v(r) = 4\pi r^2 p_r \tag{4.2.13}$$

$$x = \frac{p_\perp - p_r}{p_r} \tag{4.2.14}$$

where x, the fractional anisotropy, is in general a function of u, v and r, and again $m(r) = \int_0^r 4\pi r^2 \rho(r)dr$. Then, equations (4.2.7) and (4.2.10) become respectively

$$e^{-\lambda} = 1 - 2u \tag{4.2.15}$$

$$\frac{1}{2}\, r\, \frac{d\nu}{dr} = \frac{u + v}{1 - 2u} \tag{4.2.16}$$

and equation (4.2.9) reduces to the form

$$\frac{1}{r}\frac{dr}{du} = \frac{(1 - 2u)(\frac{dv}{du} - \alpha)}{H(u,v,x)} \; . \tag{4.2.17}$$

Also, equations (4.2.8) and (4.2.12)-(4.2.14) imply that

$$4\pi r^2 \rho = u \, \frac{(\frac{dv}{du} - \beta)}{(\frac{dv}{du} - \alpha)} \; . \tag{4.2.18}$$

The functions α, β and H are defined by

$$H = 2v(1 - 2u)(1 + x) - (u + v)^2 \tag{4.2.19}$$

$$\alpha = - \frac{u + v}{1 - 2u} \tag{4.2.20}$$

$$\beta = - \frac{v}{u} \frac{(5 - 2u - v)}{(1 - 2u)} - \frac{2vx}{u} . \tag{4.2.21}$$

Note that α is independent of x and that the above equations reduce to Bondi's equations for isotropic matter, when $x = 0$.

The expressions defined above will be used to divide the (u,v)-plane into several regions of interest. The construction of curves in such a plane was first discussed by Bondi for the case $x = 0$. The introduction of anisotropy results only in quantitative shifts of these curves.

When H vanishes, that is, when $2v(1-2u)(1+x)-(u+v)^2 = 0$, we obtain a hyperbola H in the (u,v)-plane (see Fig. 4.1). Inside the hyperbola H is positive and outside it H is negative, that is, the curves $H(u,v,x) = 0$ divide a given model into <u>a core $H > 0$</u> and <u>an envelope $H < 0$</u>. For $x = 0$, the hyperbola H touches $u = 0$ at $v = 0$, $u = \frac{1}{4}$ at $v = \frac{1}{4}$ and $u = \frac{1}{2}$ at $v = -\frac{1}{2}$.

When the denominator of (4.2.18) vanishes, that is, when $(1-2u)\frac{dv}{du} + u + v = 0$, we obtain, on integration, the family of parabolae

$$v = \sqrt{A(1 - 2u)} - (1 - u). \tag{4.2.22}$$

They all pass through $u = \frac{1}{2}$, $v = -\frac{1}{2}$ and apart from the line corresponding to $A = 0$, they all touch the line $u = \frac{1}{2}$ at this point. These parabolae correspond to surfaces of constant v and infinite density and thus represent thin mass shells. Their position in the (u,v)-plane is unaffected by anisotropy. The position of the H-curves, however, in the (u,v)-plane is clearly determined by the amount and the nature of anisotropy [12].

For $x > 0 \Leftrightarrow p_\perp > p_r$ such a curve is shifted away from the v-axis and intersects the curves $(1-2u)\frac{dv}{du} + u + v = 0$ corresponding to larger values of the constant A. For $x < 0 \Leftrightarrow p_\perp < p_r$ we have the opposite result. Note that the boundary condition $x = 0$ at $u = v = 0$ guarantees that $H = 0$ always passes through the origin, independent of the form of x.

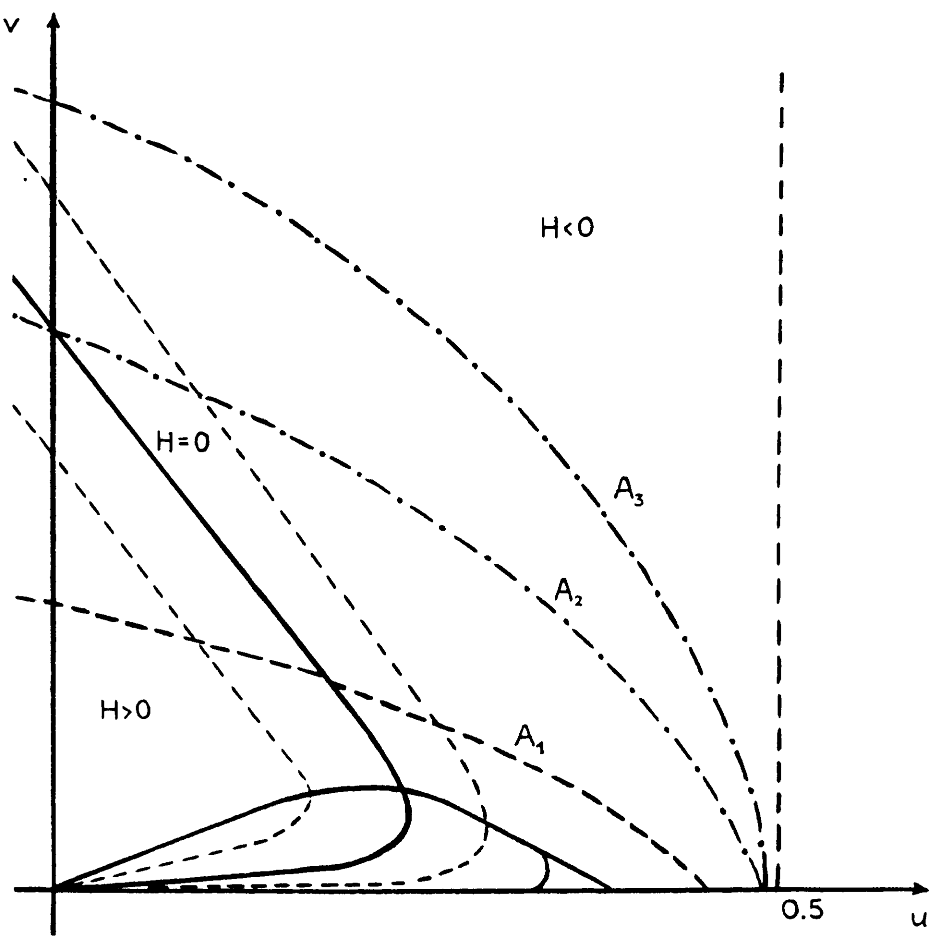

Figure 4.1. Bondi diagram of the (u,v)-plane. Anisotropic case. (Bowers-Liang, 1974).

What Bondi does in his analysis is to draw and study the permissible curves which start at the centre and end on the surface of the sphere corresponding, for example, to a star. Every equilibrium configuration is thus represented, for a given equation of state and, in the anisotropic case, for given x, by a particular permissible curve in the (u,v)-plane which starts in the core and finishes in the envelope.

Moreover, making the assumption that the quantities p_r, $p_\perp$ and ρ are finite at the origin and non-negative everywhere, $u \geqslant 0$ and $v \geqslant 0$ and thus every such curve lies in the first quadrant of the Bondi diagram. The surface condition $p_r = 0$ implies that the curve returns to $v = 0$, except for infinite-radius models, and if $\rho = 0$ at the surface, it also meets the u-axis tangentially.

Every permissible curve moves towards non-decreasing values of $\dot{A}$ in the core ($H > 0$), crosses $H = 0$ tangentially at a A = constant curve, and moves towards non-increasing values of A in the envelope ($H < 0$).

Finally, the largest possible value of u, on the assumptions $p_r > 0$, $\rho > 0$, is obtained for a model whose curves rise as rapidly as possible in the core and descend as gradually as possible in the envelope. This corresponds to a model with as high a pressure and as low a density as possible in the core and a thin infinite-density mass shell for an envelope. The maximum value of u_s at the surface has been evaluated by Bondi [13] in the isotropic case to be $u_s \sim 0.485$.

And with this we conclude our discussion on the theory of the anisotropic fluid spheres.

Before we go on to the next section, however, it must be mentioned here that the static anisotropic fluid spheres are not widely studied in general relativity, while the non-static ones have more often been discussed. Herrera, Ruggeri and Witten [14], for example, studied such non-static anisotropic spherically symmetric distributions of matter. They managed to find solutions corresponding to some strange anisotropic spheres sustained only by transverse stresses ($p_r = 0$, $p_\vartheta = p_\varphi \neq 0$), which have already been considered by Lemaître [15]. In their analysis, they followed the above presented method of Bowers and Liang, which, however, is restricted to static solutions only; although their results are presented as physical, they are dependent on the highly idealized structure of the used model.

Among the few examples of anisotropic static spherically symmetric solutions are the interior Schwarzschild solutions presented by Florides [16] and their charged generalizations given by Mehra [17]. In both cases, the component T_r^r of the energy-momentum tensor vanishes, while $T_\vartheta^\vartheta = T_\varphi^\varphi \neq 0$.

Letelier [18] also studied anisotropic static fluids as sources of the gravitational field under the assumption that their energy-momentum tensor is the sum of two tensors representing the energy-momentum tensors of an isotropic or a null fluid.

The lack of specific examples of anisotropic static spherically symmetric solutions of Einstein's equations in the literature, together with the fact that the massive spheres of Kimura are both static and anisotropic, was the motivation for the study of their physical properties, which are presented in the next section.

4.3 THE STUDY OF KIMURA ANISOTROPIC STATIC FLUID SPHERES

4.3.1 Physical properties and stability conditions

We now turn to the Kimura space-times where the metric is written as

$$ds^2 = - A(r)dr^2 - r^2 d\vartheta^2 - r^2 \sin^2\vartheta d\varphi^2 + C(r)dt^2 \qquad (4.3.1)$$

and where we may have to examine the functions

$$\lambda = \ln A \quad \text{and } \nu = \ln C$$

for singular behaviour. The physical properties of these space-times will be investigated in what follows.

The components of the Einstein tensor in this case are given by

$$\left.\begin{aligned}
G^r_r &= \frac{1}{r^2} - \frac{1}{rAC}\frac{\partial C}{\partial r} - \frac{1}{r^2 A} \\
G^\vartheta_\vartheta &= G^\varphi_\varphi = \frac{1}{2rA^2}\frac{\partial A}{\partial r} + \frac{1}{4A^2 C}\frac{\partial A}{\partial r}\frac{\partial C}{\partial r} + \frac{1}{4AC^2}\left(\frac{\partial C}{\partial r}\right)^2 \\
&\quad - \frac{1}{2rAC}\frac{\partial C}{\partial r} - \frac{1}{2AC}\frac{\partial^2 C}{\partial r^2} \\
G^t_t &= \frac{1}{r^2} + \frac{1}{rA^2}\frac{\partial A}{\partial r} - \frac{1}{r^2 A}
\end{aligned}\right\} \qquad (4.3.2)$$

and so, clearly, the Kimura metrics represent static spherically symmetric distributions of matter which are <u>anisotropic</u>.

One of the most important features of Kimura space-times is that they all possess irreducible Killing tensors of second order. It is worth looking, therefore, to see if the Killing tensor criterion can function as a constraint on solutions that correspond to interesting matter tensors, and so relate the problem of hidden symmetries to physical characteristics of space-times. Thus, some features of the matter tensors involved must be described. Using Einstein's equations $G^{\mu}_{\nu} = 8\pi T^{\mu}_{\nu}$ with $T^{\mu}_{\nu} = \mathrm{diag}(-p_r, -p_{\perp}, -p_{\perp}, \rho)$ the radial and tangential pressures and the density of the fluids can be found as functions of r. In some cases the possible existence of a cosmological constant is examined. Obviously then, Einstein's equations must have the form

$$R_{\mu\nu} - \frac{1}{2} Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \qquad (4.3.3)$$

where Λ denotes the cosmological constant*.

The Bondi variables u, v, x and H, related to p_r, $p_{\perp}$ and ρ, are also evaluated and discussed.

The expressions for the pressures and the density of the fluids are used to produce the equations of state $p_r = p_r(\rho)$ and $p_{\perp} = p_{\perp}(\rho)$.

The radius R of each fluid sphere is evaluated from the condition $p_r(R) = 0$, while the mass inside the sphere of radius r is found as a function of r using the expression (4.2.8).

The stability of the spheres is also a problem of importance. The gravitational mass M of every sphere, as determined in the Newtonian far field, is evaluated by the condition

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* The cosmological constant was first introduced by Einstein in 1917 [19].

$$m(R) = M. \tag{4.3.4}$$

Then, the following criterion of stability must be satisfied, if the fluid sphere under consideration is to be thought of as representing the interior of a star (for a proof see p. 220, Stephani [20]).

A spherically symmetric star can only exist in a state of stable equilibrium (that is, can only compensate its own gravitational attraction with a finite pressure) if its mass M and its radius R satisfy the inequality

$$\frac{2M}{R} > \frac{8}{9} \quad . \tag{4.3.5}$$

Note that this criterion is valid for any equation of state and that its proof covers the anisotropic case as well as the isotropic one. In both cases, the expression (4.2.8) defining the mass m(r) as a function of r, and the condition of a vanishing radial pressure at the surface, from which the radius R is obtained, are the same.

The inequality (4.3.5) expresses the fact that a star of fixed surface area is only stable as long as its mass lies below a critical mass. A star whose mass transgresses this limit must inevitably collapse into itself as a consequence of its gravitational attraction which will then be too strong. Thus, we can comment that while in the linear Newtonian gravitational theory a predominance of the gravitational force can be compensated by a contraction and the associated finite increase in pressure, or by additional forces, in the non-linear Einstein theory a pressure increase or an extra force acts, via the energy-momentum tensor, to increase further the strength of the gravitational field above the critical mass (4.3.5).

Moreover, the inequality (4.3.5) can be immediately translated into a statement about the redshift of spectral lines from the surface of any star

$$z = \left(1 - \frac{2M}{R}\right)^{-1/2} - 1 \tag{4.3.6}$$

and thus it imposes on z the upper limit $z < 2$, that is, the redshift of a light signal coming from the surface of a stable star has the maximum value $z_{max} = 2$.

Bowers and Liang have shown in their paper [12] that anisotropy with $x > 0$ ($x < 0$) results in an increase (decrease) of the maximum equilibrium mass and surface redshift in comparison with the corresponding isotropic values. They have also proved that in realistic situations the increase in the surface redshift will probably be of the same magnitude as the amount of anisotropy incorporated, independent of the details of the model, and in particular that an infinite anisotropy permits an infinite value of the surface redshift (that is, $x \to \infty$, then $z \to \infty$), independent of the equation of state or form of anisotropy.

Finally, it should be mentioned that the well known Tolman-Ehrenfest formula for the thermostatic equilibrium of isotropic matter [21]

$$T\sqrt{g_{44}} = \text{constant} \tag{4.3.7}$$

where T is the proper temperature of the matter, can also be applied in the case of anisotropic matter, and thus in Kimura fluid spheres, to describe the temperature distribution at equilibrium. A proof of this result is given in Appendix A 10.

In what follows, selected Kimura cases, including Einstein-like, de Sitter-like and anti-de Sitter-like line elements, are presented and the physical properties of the fluids involved are examined. Most of these models are proved unphysical. Some of them, however, have been found stable and thus they can very well represent interiors of stars.

---

6. $A = a_2^{-2} r^{-2}$ and $C = a_2^{-1} r^2$, where a_2 is a constant.

The pressures and density of the fluid are

$$p_r = -\frac{1}{8\pi r^2} + \frac{3a_2^2}{8\pi}$$

$$p_\perp = \frac{3a_2^2}{8\pi} = \text{constant}$$

$$\rho = \frac{1}{8\pi r^2} - \frac{3a_2^2}{8\pi}.$$

Clearly, the constant $3a_2^2$ can be interpreted as a cosmological constant, that is, $\Lambda = 3a_2^2 > 0$.

The corresponding Bondi variables are given by

$$u = \frac{1}{2}(1 - a_2^2 r^2)$$

$$v = -\frac{1}{2}(1 - 3a_2^2 r^2)$$

$$x = \frac{1}{3a_2^2 r^2 - 1}, \quad x \gtrless 0 \quad \text{for} \quad r \gtrless \frac{1}{\sqrt{3}a_2}$$

$H = 2a_2^4 r^4 > 0$ for all r, so the model consists of a core. The equations of state are

$$p_r + \rho = 0$$

$$p_\perp + \rho = \frac{1}{8\pi r^2} \ .$$

The radius of the sphere is $R = \frac{1}{\sqrt{3}a_2}$, $a_2 \neq 0$.

The mass inside the sphere is given, as a function of r, by $m(r) = \frac{r}{2}(1 - a_2^2 r^2)$ and the mass M of the sphere is $M = \frac{1}{3\sqrt{3}a_2}$. Since $\frac{2M}{R} = \frac{2}{3} < \frac{8}{9}$ it follows that this model is <u>stable</u>, and thus it can represent the interior of a star. The apparent singularity of the metric at $r = 0$ is only a coordinate one since $g = -a_2^3 r^4 \sin^2\vartheta$ is not singular at $r = 0$. Moreover, the surface redshift was evaluated to be $z = 0.73 < 2$, as expected for a stable model.

---

7. $A = (a_1 r^2 + a_2)^{-3}$ and $C = 1$, where $a_2 \neq 0$ and a_1 are constants. The pressures and density of the fluid are

$$p_r = -\frac{1}{8\pi r^2} + \frac{1}{8\pi r^2}(a_1r^2 + a_2)^3$$

$$p_\perp = \frac{3a_1}{8\pi}(a_1r^2 + a_2)^2$$

$$\rho = \frac{1}{8\pi r^2} - \frac{1}{8\pi r^2}(a_1r^2 + a_2)^3 - \frac{6a_1}{8\pi}(a_1r^2 + a_2)^2.$$

The corresponding Bondi variables are given by

$$u = \frac{1}{2}[1 - (a_1r^2 + a_2)^3]$$

$$v = -\frac{1}{2}[1 - (a_1r^2 + a_2)^3]$$

$$x = \frac{3a_1r^2(a_1r^2 + a_2)^2 - (a_1r^2 + a_2)^3 + 1}{(a_1r^2 + a_2)^3 - 1}$$

$$H = 3a_1r^2(a_1r^2 + a_2)^5.$$

The equations of state are

$$p_r + \rho = -\frac{6a_1}{8\pi}(a_1r^2 + a_2)^2$$

$$p_\perp + \rho = \frac{1}{8\pi r^2} - \frac{1}{8\pi r^2}(a_1r^2 + a_2)^3 - \frac{3a_1}{8\pi}(a_1r^2 + a_2)^2.$$

The radius of the sphere is $R = \sqrt{\frac{1-a_2}{a_1}}$, $a_1 \neq 0$ and $a_2 \neq 1$. The mass inside the sphere is given, as a function of r, by $m(r) = \frac{r}{2}[1-(a_1r^2+a_2)^3]$ and the mass M of the sphere is M = 0! Thus, this model cannot represent the interior of a star. Note, also, that there is a genuine singularity of the metric at $r = \sqrt{-\frac{a_2}{a_1}}$.

8. $A = a_1^{-3} r^{-6}$ and $C = 1$, where a_1 is a constant. Since this model is a special case of 7, for $a_2 = 0$, it is not expected to represent the interior of a star. And this is in fact the case, since again $M = 0$! Note, also, that there is a genuine singularity of the metric at the origin.

---

10. $A = 4(1 - a_1 r^2)^{-1}$ and $C = 1$, where $a_1 \neq 0$ is a constant. The pressures and density of the fluid are

$$p_r = -\frac{3}{32\pi r^2} - \frac{a_1}{32\pi}$$

$$p_\perp = -\frac{a_1}{32\pi} = \text{constant}$$

$$\rho = \frac{3}{32\pi r^2} + \frac{3a_1}{32\pi} .$$

The corresponding Bondi variables are given by

$$u = \frac{1}{8}(3 + a_1 r^2)$$

$$v = -\frac{1}{8}(3 + a_1 r^2)$$

$$x = -\frac{3}{3 + a_1 r^2} , \quad x \gtrless 0 \quad \text{for} \quad r \lessgtr \sqrt{-\frac{3}{a_1}}$$

$$H = \frac{1}{16} a_1 r^2 (a_1 r^2 - 1).$$

The equations of state are

$$p_r + \rho = \frac{a_1}{16\pi}$$

$$p_\perp + \rho = \frac{3}{32\pi r^2} + \frac{a_1}{16\pi} \; .$$

The radius of the sphere is $R = \sqrt{-\frac{3}{a_1}}$, $a_1 < 0$.

The mass inside the sphere is given, as a function of r, by $m(r) = \frac{r}{8}(3 + a_1 r^2)$ and the mass M of the sphere is $M = 0$!

Thus, this model cannot represent the interior of a star.

---

11. $A = (b_0 r^2 + 4)^{-1}$ and $C = 1$, where $b_0 \neq 0$ is a constant. (Einstein-like).

The pressures and density of the fluid are

$$p_r = \frac{3}{8\pi r^2} + \frac{b_0}{8\pi}$$

$$p_\perp = \frac{b_0}{8\pi} = \text{constant}$$

$$\rho = -\frac{3}{8\pi r^2} - \frac{3b_0}{8\pi} \; .$$

The corresponding Bondi variables are given by

$$u = -\frac{1}{2}(3 + b_0 r^2)$$

$$v = \frac{1}{2}(3 + b_0 r^2)$$

$$x = -\frac{3}{3 + b_0 r^2}$$

$$H = b_0 r^2 (4 + b_0 r^2).$$

The equations of state are

$$p_r + \rho = -\frac{b_0}{4\pi}$$

$$p_\perp + \rho = -\frac{3}{8\pi r^2} - \frac{b_0}{4\pi}.$$

The radius of the sphere is $R = \sqrt{-\frac{3}{b_0}}$, $b_0 < 0$.

The mass inside the sphere is given, as a function of r, by $m(r) = \frac{r}{2}(-3 - b_0 r^2)$ and the mass M of the sphere is $M = 0$!

The model cannot represent the interior of a star. Note, also, that there is a genuine singularity of the metric at $r = \sqrt{-\frac{4}{b_0}}$.

---

14. $A = (b_0 r^2 + 4)^{-1}$ and $C = b_0 r^2 + 4$, where $b_0 \neq 0$ is a constant. (Anti-de Sitter-like).

The pressures and density of the fluid are

$$p_r = \frac{3}{8\pi r^2} + \frac{3b_0}{8\pi}$$

$$p_\perp = \frac{3b_0}{8\pi} = \text{constant}$$

$$\rho = -\frac{3}{8\pi r^2} - \frac{3b_0}{8\pi} .$$

The similarity of these expressions to the corresponding ones of case 11 is obvious and thus the model is expected to be unphysical. And indeed it was found to be so, since in this case the radius of the sphere is $R = \sqrt{-\frac{1}{b_0}}$ and its mass is $M = -\sqrt{-\frac{1}{b_0}}$. For R to be a real number we must require $b_0 < 0$ and then $M < 0$, which is a very unphysical result. The model cannot therefore represent the interior of a star. Note, however, that the apparent singularity of the metric at $r = \sqrt{-\frac{4}{b_0}}$ is only a coordinate one, since $g = -r^4 \sin^2\vartheta$ is not singular for this value of r.

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15. $A = \frac{1}{4}$ and $C = 1$

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17. $A = 4$ and $C = 1$

These two cases are treated together because of their apparent similarity. In both cases, the radial pressure p_r of the fluid involved vanishes as $\sim \frac{1}{r^2}$, and thus the condition $p_r(R) = 0$ is not true. Obviously, these models cannot represent interiors of stars.

---

16. $A = 4(1 - a_1 r^2)^{-1}$ and $C = 1 - a_1 r^2$, where $a_1 \neq 0$ is a constant. (de Sitter-like).

The pressures and density of the fluid are

$$p_r = - \frac{3}{32\pi r^2} - \frac{3a_1}{32\pi}$$

$$p_\perp = - \frac{3a_1}{32\pi} = \text{constant}$$

$$\rho = \frac{3}{32\pi r^2} + \frac{3a_1}{32\pi} \quad .$$

Clearly, the constant $- 3a_1/4$ can be interpreted as a cosmological constant, that is, $\Lambda = - 3a_1/4$.

The corresponding Bondi variables are given by

$$u = \frac{1}{8} (3 + a_1 r^2)$$

$$v = - \frac{3}{8} (1 + a_1 r^2)$$

$$x = - \frac{1}{1 + a_1 r^2} , \quad x \gtrless 0 \quad \text{for} \quad r \lessgtr \sqrt{- \frac{1}{a_1}}$$

$$H = \frac{1}{16} a_1 r^2 (2a_1 r^2 - 3).$$

The equations of state are

$$p_r + \rho = 0$$

$$p_\perp + \rho = \frac{3}{32\pi r^2} \ .$$

The radius of the sphere is $R = \sqrt{-\frac{1}{a_1}}$, $a_1 < 0$. With the requirement $a_1 < 0$, the cosmological constant is positive and also $H > 0$ for all values of r, so the model consists of a core.

The mass inside the sphere is given, as a function of r, by $m(r) = \frac{r}{8}(3 + a_1 r^2)$ and the mass M of the sphere is $M = \frac{1}{4}\sqrt{-\frac{1}{a_1}}$. Since $\frac{2M}{R} = \frac{1}{2} < \frac{8}{9}$ it follows that this model is stable, and thus it can represent the interior of a star. Moreover, the surface redshift was evaluated to be $z = 0.41 < 2$, as expected for a stable model.

4.3.2 Optical characteristics

When studying the optical characteristics of a static medium, one usually starts with the form of the metric

$$ds^2 = -d\sigma^2 + f^2 dt^2 \tag{4.3.8}$$

with

$$d\sigma^2 = g_{ab} dx^a dx^b \qquad (a,b = 1,2,3) \tag{4.3.9}$$

where g_{ab} and f are independent of time * .

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* Only in these circumstances is there a close formal resemblance between the equations which follow and those of the familiar optics, mainly because then one has to deal with a theory which is only three-dimensional.

Then, Fermat's Principle [22] can be written as*

$$\delta \int_{A}^{A'} f^{-1}\, d\sigma = 0 \qquad (4.3.10)$$

for variations with fixed end points A and A'.

If x^a and x'^a are the coordinates of the points A and A' respectively, the value of the integral appearing in (4.3.10), taken along the extremal which joins A and A', is the point characteristic $V(x'^1,x'^2,x'^3;\ x^1,x^2,x^3)$ of the medium, that is,

$$V(x'^1,x'^2,x'^3;x^1,x^2,x^3) = \int_{A}^{A'} f^{-1}\, d\sigma. \qquad (4.3.11)$$

The point characteristic V of a medium is thus defined to be the optical length of the ray (appearing in Fermat's principle) between two points considered as a function of their coordinates. It is important to note that this function V is defined by the medium.

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* Fermat's Principle, known also as the principle of the shortest optical path, asserts that the optical length of an actual ray between any two points A and A' is shorter than the optical length of any other curve which joins these points and which lies in a certain regular neighbourhood of it. (By a regular neighbourhood we mean one that may be covered by rays in such a way that one and only one ray passes through each point of it.)

From equations (4.3.10) it follows that V satisfies the pair of simultaneous differential equations

$$g^{ab}(x)\, V_{,a}V_{,b} = f^{-2}(x); \quad g^{ab}(x')\, V_{,a'}V_{,b'} = f^{-2}(x') \qquad (4.3.12)$$

where subscripts a,a' following a comma denote partial differentiation with respect to x^a and x'^a. Once V is known, the covariant unit tangent vectors $u_a(x)$ and $u_a(x')$ to the ray at A and A' are given by

$$u_a(x) = f(x)V_{,a}; \quad u_a(x') = f(x')V_{,a'}. \qquad (4.3.13)$$

To solve equations (4.3.12) for the case of spherically symmetric space-times, following Buchdahl [23], we write the line element as

$$ds^2 = -\, p^2dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) + f^2dt^2 \qquad (4.3.14)$$

with f and p being functions of r alone.

For any pair of terminal points A and A' we further imagine the coordinate system so arranged that both points lie in the equatorial plane. Then, the ray joining these points will also lie entirely in this plane, and we may set $d\varphi = 0$ in (4.3.14). A and A' have spatial coordinates (r,ϑ) and (r',ϑ') respectively.

Equations (4.3.12) now become, in an obvious notation,

$$\begin{aligned} p^{-2}(r)V^2_{,r} + r^{-2}V^2_{,\vartheta} &= f^{-2}(r) \\ p^{-2}(r')V^2_{,r'} + (r')^{-2}V^2_{,\vartheta'} &= f^{-2}(r'). \end{aligned} \qquad (4.3.15)$$

The complete integral of (4.3.15) may be obtained by separating variables. After some calculations we end up with

$$V = \int_r^{r'} p(f^{-2} - a^2 r^{-2})^{1/2} dr + a(\vartheta' - \vartheta) \tag{4.3.16}$$

where a is an arbitrary constant satisfying the equation $\frac{\partial V}{\partial a} = 0$, that is,

$$0 = - a \int_r^{r'} r^{-2} p(f^{-2} - a^2 r^{-2})^{-1/2} dr + (\vartheta' - \vartheta). \tag{4.3.17}$$

Using (4.3.17) we can rewrite (4.3.16) as follows

$$\underline{V = \int_r^{r'} p f^{-2}(f^{-2} - a^2 r^{-2})^{-1/2}\, dr.} \tag{4.3.18}$$

It is convenient to put $\underline{\vartheta' - \vartheta = \psi}$, so that

$$\underline{\psi = a\int_r^{r'} r^{-2} p(f^{-2} - a^2 r^{-2})^{-1/2} dr.} \tag{4.3.19}$$

The point characteristic V is thus given by equation (4.3.18), the constant $a(r, r', \psi)$, which occurs in it, being determined by equation (4.3.19).

If now the 3-dimensional metric $d\sigma^2$ is conformally flat, that is, if in suitable coordinates there exists a function q^2 such that we can write

$$d\sigma^2 = q^2\{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\} \tag{4.3.20}$$

then (4.3.8) becomes

$$ds^2 = - q^2\{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\} + f^2dt^2 \tag{4.3.21}$$

and the refractive index of the medium can be defined by

$$N(x^1,x^2,x^3) = \frac{q}{f} . \tag{4.3.22}$$

When $q = f$, $N = 1$ and ds^2 as a whole is conformally flat and the light cone is conformally invariant.

In the case of a spherically symmetric space-time, (4.3.21) has the form

$$ds^2 = - q^2\{dr^2 + r^2d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2\} + f^2dt^2. \tag{4.3.23}$$

The form (4.3.23) of a spherically symmetric metric is the so-called isotropic form. Historically, the isotropic form of the metric was first studied by Einstein and Strauss[24] for the general, not necessarily static, spherically symmetric case. Tolman [21] introduced a new variable r' in accordance with the equation

$$\frac{dr'}{r'} = p\,\frac{dr}{r} \tag{4.3.24}$$

which, when substituted in (4.3.14) and after dropping primes, gives the metric in the form (4.3.23). He also worked with the general non-static case. His proof, however, is not correct because, as Wyman [25] pointed out, the right-hand side of (4.3.24) is not a perfect differential when p is a function of t also. Kustaanheimo [26] attempted a new proof for the transformation leading to the isotropic form of the metric, which for the static case reduces to

$$\frac{d\bar{r}}{\bar{r}} = p\,\frac{dr}{r} \text{ and } d\bar{t} = dt. \qquad (4.3.25)$$

The transformation (4.3.25) will be used in what follows to obtain the isotropic form of certain Kimura metrics. In this case, (4.3.14) and (4.3.1) give

$$p^2 = A \text{ and } f^2 = C$$

and thus the transformation needed is

$$\frac{d\bar{r}}{\bar{r}} = \sqrt{A}\,\frac{dr}{r} \text{ and } d\bar{t} = dt. \qquad (4.3.26)$$

The refractive index of the corresponding media will be evaluated using (4.3.22). But before that, the point characteristics of those media will be evaluated using (4.3.18) and (4.3.19).

6. $A = a_2^{-2}r^{-2}$ and $C = a_2^{-1}r^2$, where a_2 is a constant. In this case $p = a_2^{-1}r^{-1}$ and $f^2 = a_2^{-1}r^2$, and thus

$$V = \int_r^{r'} pf^{-2}(f^{-2} - a^2r^{-2})^{-1/2}\,dr = \int_r^{r'} \frac{dr}{r^2(a_2 - a^2)^{1/2}} \Rightarrow$$

$$V = (a_2 - a^2)^{-1/2}\left\{\frac{r' - r}{r'r}\right\}$$

the constant a being determined by

$$\psi = a\int_r^{r'} r^{-2}p(f^{-2} - a^2r^{-2})^{-1/2}\,dr = \int_r^{r'} \frac{a\,dr}{a_2r^2(a_2 - a^2)^{1/2}} \Rightarrow$$

$$\psi = aa_2^{-1}(a_2 - a^2)^{-1/2}\left\{\frac{r' - r}{r'r}\right\}.$$

8. $A = a_1^{-3}r^{-6}$ and $C = 1$, where a_1 is a constant. In this case $p = a_1^{-3/2}r^{-3}$ and $f^2 = 1$, and thus

$$V = \int_r^{r'} pf^{-2}(f^{-2} - a^2r^{-2})^{-1/2}\,dr = \int_r^{r'} \frac{dr}{a_1^{3/2}r^2(r^2 - a^2)^{1/2}}.$$

Put $r = a\cosh y \rightarrow y = \cosh^{-1}\frac{r}{a}$, then $dr = a\sinh y\,dy$ and $r^2 - a^2 = a^2\sinh^2 y$, so that

$$V = \int_r^{r'} \frac{dy}{a_1^{3/2} a^2 \cosh^2 y} = \frac{1}{a_1^{3/2} a^2} [\tanh y]_r^{r'}$$

$$= \frac{1}{a_1^{3/2} a^2} [\tanh(\cosh^{-1} \frac{r}{a})]_r^{r'} \Rightarrow$$

$$V = a_1^{-3/2} a^{-2} \{\tanh(\cosh^{-1} \frac{r'}{a}) - \tanh(\cosh^{-1} \frac{r}{a})\}$$

---

the constant a being determined by

$$\psi = a \int_r^{r'} r^{-2} p(f^{-2} - a^2 r^{-2})^{-1/2} dr = a \int_r^{r'} \frac{dr}{a_1^{3/2} r^4 (r^2 - a^2)^{1/2}} ;$$

working as above we have

$$\psi = \frac{1}{a_1^{3/2} a^3} \int_r^{r'} \frac{dy}{\cosh^4 y} = \frac{1}{3a_1^{3/2} a^3} [\tanh y(\operatorname{sech}^2 y + 2)]_r^{r'} \Rightarrow$$

$$\psi = 3a_1^{-3/2} a^{-3} \{[\tanh(\cosh^{-1} \frac{r'}{a})][\operatorname{sech}^2(\cosh^{-1} \frac{r'}{a}) + 2]$$

$$- [\tanh(\cosh^{-1} \frac{r}{a})][\operatorname{sech}^2(\cosh^{-1} \frac{r}{a}) + 2]\} .$$

---

15. $A = \frac{1}{4}$ and $C = 1$.

In this case $p = \frac{1}{2}$ and $f^2 = 1$, and thus

$$V = \int_r^{r'} pf^{-2}(f^{-2} - a^2 r^{-2})^{-1/2} \, dr = \int_r^{r'} \frac{1}{2} \frac{dr}{(1 - \frac{a^2}{r^2})^{1/2}}$$

$$= \frac{1}{2} [(r^2 - a^2)^{1/2}]_r^{r'} \Rightarrow$$

$$V = \frac{1}{2} \{(r'^2 - a^2)^{1/2} - (r^2 - a^2)^{1/2}\}$$

---

the constant a being determined by

$$\psi = a\int_r^{r'} r^{-2}p(f^{-2} - a^2r^{-2})^{-1/2}\,dr = \frac{a}{2}\int_r^{r'} \frac{dr}{r(r^2 - a^2)^{1/2}} \;.$$

Put r = a cosh y, so that

$$\psi = \frac{1}{2}\int_r^{r'} \frac{dy}{\cosh y} \;.$$

Put also $t = \tanh\frac{y}{2}$, then $dy = \dfrac{2dt}{1 - t^2}$ and $\cosh y = \dfrac{1 + t^2}{1 - t^2}$,
so finally

$$\psi = \int_r^{r'} \frac{dt}{1 + t^2} = [\tan^{-1}t]_r^{r'} = [\tanh^{-1}(\tanh\tfrac{y}{2})]_r^{r'} \Rightarrow$$

$$\psi = \{\tan^{-1}[\tanh(\tfrac{1}{2}\cosh^{-1}\tfrac{r'}{a})] - \tan^{-1}[\tanh(\tfrac{1}{2}\cosh^{-1}\tfrac{r}{a})]\}.$$

---

17. A = 4 and C = 1.

In this case p = 2 and $f^2 = 1$, and thus

$$V = \int_r^{r'} pf^{-2}(f^{-2} - a^2r^{-2})^{-1/2}\,dr = \int_r^{r'} \frac{2rdr}{(r^2 - a^2)^{1/2}} = [(r^2 - a^2)^{1/2}]_r^{r'} \Rightarrow$$

$$V = \{(r'^2 - a^2)^{1/2} - (r^2 - a^2)^{1/2}\}$$

---

the constant a being determined by

$$\psi = a\int_r^{r'} r^{-2}p(f^{-2} - a^2r^{-2})^{-1/2}dr = 2a\int_r^{r'} \frac{dr}{r(r^2 - a^2)^{1/2}} \;.$$

Put again r = a cosh y, so that

$$\psi = 2\int_{r}^{r'} \frac{dy}{\cosh y}$$

and also $t = \tanh \frac{y}{2}$, so finally

$$\psi = 4\int_{r}^{r'} \frac{dt}{1+t^2} = 4[\tan^{-1} y]_{r}^{r'} = 4[\tan^{-1}(\tanh \tfrac{y}{2})]_{r}^{r'} \Rightarrow$$

$$\psi = 4\{\tan^{-1}[\tanh(\tfrac{1}{2}\cosh^{-1}\tfrac{r'}{a})] - \tan^{-1}[\tanh(\tfrac{1}{2}\cosh^{-1}\tfrac{r}{a})]\}.$$

Isotropic coordinates and refractive index

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6. $A = a_2^{-2} r^{-2}$ and $C = a_2^{-1} r^2$, where a_2 is a constant.

In this case the transformation (4.3.26) becomes

$$\left.\begin{array}{l} \dfrac{d\bar{r}}{d\bar{r}} = \dfrac{1}{a_2 r}\dfrac{dr}{r} \\[2em] d\bar{t} = dt \end{array}\right\} \Rightarrow \begin{array}{l} \bar{r} = \exp[-\dfrac{1}{a_2 r}] \\[2em] \bar{t} = t. \end{array}$$

The initial metric

$$ds^2 = -\frac{1}{a_2^2 r^2}\, dr^2 - r^2 d\vartheta^2 - r^2 \sin^2\vartheta d\varphi^2 + \frac{r^2}{a_2}\, dt^2$$

can be written in the isotropic form

$$ds^2 = -\frac{1}{a_2^2 \bar{r}^2 (\ln \bar{r})^2}\,[d\bar{r}^2 + r^{-2} d\vartheta^2 + \bar{r}^2 \sin^2\vartheta d\varphi^2] + \frac{1}{a_2^3 (\ln \bar{r})^2}\, dt^2.$$

The refractive index of the medium is then given by

$$N = \frac{a_2^{1/2}}{\bar{r}}.$$

---

8. $A = a_1^{-3} r^{-6}$ and $C = 1$, where a_1 is a constant.

In this case the transformation (4.3.26) becomes

$$\left.\begin{aligned} \frac{d\bar{r}}{\bar{r}} &= \frac{1}{a_1^{3/2} r^3}\frac{dr}{r} \\ d\bar{t} &= dt \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \bar{r} &= \exp\left[-\frac{1}{3a_1^{3/2} r^3}\right] \\ \bar{t} &= t. \end{aligned}$$

The initial metric

$$ds^2 = -\frac{1}{a_1^3 r^6}\, dr^2 - r^2 d\vartheta^2 - r^2 \sin^2\vartheta d\varphi^2 + dt^2$$

can be written in the isotropic form

$$ds^2 = -\frac{1}{\bar{r}^2 (3a_1^{3/2} \ln \bar{r})^{2/3}} [d\bar{r}^2 + \bar{r}^2\, d\vartheta^2 + \bar{r}^2 \sin^2\vartheta d\varphi^2] + dt^2.$$

The refractive index of the medium is then given by

$$N = \frac{1}{\bar{r}(3a_1^{3/2} \ln \bar{r})^{1/3}} .$$

---

15. $A = \frac{1}{4}$ and $C = 1$.

The transformation (4.3.26) now becomes

$$\left.\begin{aligned} \frac{d\bar{r}}{\bar{r}} &= \frac{1}{2}\frac{dr}{r} \\ d\bar{t} &= dt \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \bar{r} &= r^{1/2} \\ \bar{t} &= t. \end{aligned}$$

The initial metric

$$ds^2 = -\frac{dr^2}{4} - r^2 d\vartheta^2 - r^2 \sin^2\vartheta d\varphi^2 + dt^2$$

can be written in the isotropic form

$$ds^2 = -\bar{r}^{-2}[d\bar{r}^2 + \bar{r}^2 d\vartheta^2 + \bar{r}^2 \sin^2\vartheta d\varphi^2] + dt^2 .$$

The refractive index of the medium is then given by

$N = \bar{r}$.

---

17. $A = 4$ and $C = 1$.

The transformation (4.3.26) becomes

$$\left.\begin{aligned} \frac{d\bar{r}}{\bar{r}} &= 2\frac{dr}{r} \\ d\bar{t} &= dt \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \bar{r} &= r^2 \\ \bar{t} &= t. \end{aligned}$$

The initial metric

$$ds^2 = -4dr^2 - r^2 d\vartheta^2 - r^2 \sin^2\vartheta d\varphi^2 + dt^2$$

can be written in the isotropic form

$$ds^2 = -\frac{1}{\bar{r}}[d\bar{r}^2 + \bar{r}^2 d\vartheta^2 + \bar{r}^2 \sin^2\vartheta d\varphi^2] + dt^2.$$

The refractive index of the medium is then given by

$$N = \frac{1}{\bar{r}^{1/2}} .$$

Static spherically symmetric space-times are expected to have a refractive index function N(r) with the form of Maxwell's fish-eye*. Buchdahl [23] proved this in the cases of the Schwarzschild exterior field, the de Sitter and the Einstein space-times, under the assumption of isotropy of matter. In every examined Kimura case, however, as a possible result of the anisotropy of the matter involved, the refractive index function does <u>not</u> have the form of Maxwell's fish-eye.

4.4 JUNCTION CONDITIONS IN TERMS OF NEWMAN-PENROSE FORMALISM

4.4.1 Junction conditions and the propagation of isometries.

A new set of junction conditions stated in terms of the Newman-Penrose variables has been proposed by Herrera and Jiménez [28], who also studied the propagation of Killing vectors under the assumption of the fulfilment of these conditions.

The problem of matching two solutions of Einstein's equations belonging to two regions M, $\bar{M}$ of the space-time separated by a non-null hypersurface S, is of great importance in general relativity.

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* The "fish-eye" refractive index function has the form $N(r) = \dfrac{1}{1 + (\frac{r}{k})^2}$, where k is a constant, and was first investigated by Maxwell [27].

A solution of Einstein's equations in one of the regions, say M, can be considered as representing the source of the metric in the other, say $\bar{M}$, only if the two metrics can be matched on the boundary S of the source. Thus the search for appropriate interior solutions is a problem closely related to the study of the junction conditions.

Recently, Bonnor and Vickers [29] studied in detail the three most used sets of junction conditions in general relativistic problems, namely, those of Darmois (hereafter denoted by D) [30], O'Brien and Synge (hereafter denoted by OS) [31] and Lichnerowicz (hereafter denoted by L) [32]. They showed that the conditions of D and L are equivalent, and that the conditions of OS are more restrictive than the others; the conditions of OS may rule out, however, some physically interesting possibilities. They also pointed out that, since the conditions of OS are not covariantly stated, they are equivalent to the L (and therefore to the D) conditions in some coordinate systems and not in others.

Herrera and Jiménez proved in their paper that their set of junction conditions (hereafter denoted by HJ) are equivalent to the L (and therefore to the D) conditions.

The HJ junction conditions are stated as follows. Two regions M and $\bar{M}$ of the space-time separated by a non-null hypersurface S are said to match across S if

i) there exists a coordinate system such that the tetrad vectors are continuous across S

ii) the spin coefficients, taken with respect to the continuous tetrad whose existence is being asserted, are continuous across S.

This choice of junction conditions presents two main advantages:

i) With respect to the L conditions: The HJ junction conditions are partially stated in covariant form (the conditions on the spin coefficients).

ii) With respect to the D conditions: The quantities on which the HJ junction conditions are imposed (tetrad vectors and spin coefficients) are the dynamical variables of the Newman-Penrose formalism (besides the Riemann tensor components). Thus no additional calculations are needed in order to verify that the junction conditions are satisfied (for example, there is no need to calculate the second fundamental form as in the case of D conditions).

It must also be noted that the HJ junction conditions can be successfully used with metrics representing both isotropic and anisotropic distributions of matter.

Introducing the null tetrad $(\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ satisfying the orthogonality properties (1.2.40) and using (1.4.12) we find

$$Dn^\mu - \Delta\ell^\mu = -(\gamma+\bar{\gamma})\ell^\mu + (\pi+\bar{\tau})m^\mu + (\tau+\bar{\pi})\bar{m}^\mu - (\varepsilon+\bar{\varepsilon})n^\mu \qquad (4.4.1a)$$

$$\delta\ell^\mu - Dm^\mu = (\bar{\alpha}+\beta-\bar{\pi})\ell^\mu - (\bar{\rho}+\varepsilon-\bar{\varepsilon})m^\mu - \sigma\bar{m}^\mu + \kappa n^\mu \qquad (4.4.1b)$$

$$\delta n^\mu - \Delta m^\mu = -\bar{\nu}\ell^\mu + (\mu-\gamma+\bar{\gamma})m^\mu + \lambda\bar{m}^\mu + (\tau-\bar{\alpha}-\beta)n^\mu \qquad (4.4.1c)$$

$$\delta\bar{m}^{\mu}-\bar{\delta}m^{\mu} = (\mu-\bar{\mu})\ell^{\mu}+(\bar{\beta}-\alpha)m^{\mu}+(\bar{\alpha}-\beta)\bar{m}^{\mu}+(\rho-\bar{\rho})n^{\mu}. \qquad (4.4.1d)$$

Then we can show that the L (and therefore the D) conditions imply the HJ conditions.

Recall that the L conditions demand the existence of a coordinate system such that the metric components and their first derivatives are continuous across the hypersurface S which separates the two regions M and $\bar{M}$ of the space-time. The continuity of the metric components is implied by the continuity of the tetrad vectors, as follows from the expression (1.2.43).

Conversely, the continuity of tetrad vectors is implied by the continuity of the metric components since, given the metric tensor, the tetrad field is determined up to

(a) a null rotation around $\underline{\ell}$;

(b) a boost in the $(\underline{\ell},\underline{n})$-plane and a spatial rotation in the $(\underline{m},\underline{\bar{m}})$-plane ;

(c) a null rotation around $\underline{n}$.

Thus, if there exists a coordinate system for which the metric components are continuous, for this coordinate system we can demand the tetrad vectors to be continuous; this corresponds to the first of the HJ conditions.

From equations (4.4.1) we obtain the following set of equations

$$\ell_{\mu}(Dn^{\mu} - \Delta\ell^{\mu}) = -(\varepsilon + \bar{\varepsilon}) \qquad (4.4.2a)$$

$$n_{\mu}(Dn^{\mu} - \Delta\ell^{\mu}) = -(\gamma + \bar{\gamma}) \qquad (4.4.2b)$$

$$m_{\mu}(Dn^{\mu} - \Delta\ell^{\mu}) = -(\tau + \bar{\pi}) \qquad (4.4.2c)$$

$$\bar{m}_\mu(Dn^\mu - \Delta\ell^\mu) = -\,(\pi + \bar{\tau}) \qquad (4.4.2d)$$

$$\ell_\mu(\delta\ell^\mu - Dm^\mu) = \varkappa \qquad (4.4.2e)$$

$$n_\mu(\delta\ell^\mu - Dm^\mu) = (\bar{\alpha} + \beta - \bar{\pi}) \qquad (4.4.2f)$$

$$m_\mu(\delta -^\mu - Dm^\mu) = \sigma \qquad (4.4.2g)$$

$$\bar{m}_\mu(\delta\ell^\mu - Dm^\mu) = (\bar{\rho} + \varepsilon - \bar{\varepsilon}) \qquad (4.4.2h)$$

$$\ell_\mu(\delta n^\mu - \Delta m^\mu) = (\tau - \bar{\alpha} - \beta) \qquad (4.4.2i)$$

$$n_\mu(\delta n^\mu - \Delta m^\mu) = -\,\bar{\nu} \qquad (4.4.2j)$$

$$m_\mu(\delta n^\mu - \Delta m^\mu) = -\,\bar{\lambda} \qquad (4.4.2k)$$

$$\bar{m}_\mu(\delta n^\mu - \Delta m^\mu) = -\,(\mu + \gamma + \bar{\gamma}) \qquad (4.4.2\ell)$$

$$\ell_\mu(\delta\bar{m}^\mu - \bar{\delta}m^\mu) = (\rho - \bar{\rho}) \qquad (4.4.2m)$$

$$n_\mu(\delta\bar{m}^\mu - \bar{\delta}m^\mu) = (\mu - \bar{\mu}) \qquad (4.4.2n)$$

$$m_\mu(\delta\bar{m}^\mu - \bar{\delta}m^\mu) = -\,(\bar{\alpha} - \beta) \qquad (4.4.2o)$$

$$\bar{m}_\mu(\delta\bar{m}^\mu - \bar{\delta}m^\mu) = -\,(\alpha - \bar{\beta}). \qquad (4.4.2p)$$

Thus, if the first derivatives of the tetrad vectors are continuous across S, using equations (4.4.2), the spin coefficients can be proved continuous across S; this corresponds to the second of the HJ conditions. And this completes the proof that the L (and therefore the D) conditions imply the HJ conditions.

The converse is also true, namely that the HJ conditions imply the L (and therefore the D) conditions.

Recall that the D conditions demand that both the first fundamental form $(g_{\mu\nu}dx^\mu dx^\nu)_S$ and the second $(P_{\mu;\nu}dx^\mu dx^\nu)_S$ be continuous across S. P_μ is a unit vector normal to S. From (1.2.43) it is obvious that the continuity of the tetrad vectors across S implies the continuity of the first fundamental form; this corresponds to the first of the D conditions.

The unit vector P^μ can be written as $P^\mu = f\ell^\mu\, gn^\mu + qm^\mu + \bar{q}\bar{m}^\mu$

with $fg - q\bar{q} = \frac{1}{2}$. The metric will remain unchanged if, without changing the coordinate system, we perform a boost in the $(\underline{\ell},\underline{n})$-plane

$$\ell^{\mu} \rightarrow \ell^{\mu}$$

$$n^{\mu} \rightarrow \frac{1}{\lambda}\, n^{\mu}$$

$$m^{\mu} \rightarrow m^{\mu}$$

and a null rotation (1.4.13b) around $\underline{n}$.

Since λ and b are arbitrary functions, we may choose $q = \bar{b}$ and $f = \lambda$ and then we get that $P^{\mu} = \ell^{\mu} + \frac{1}{2}\, n^{\mu}$, where $P^{\mu}P_{\mu} = 1$.

Thus, since the covariant derivatives of the tetrad vectors are expressed in terms of the spin coefficients, the continuity of the spin coefficients across S implies the continuity of the second fundamental form; this corresponds to the second of the D conditions. And this completes the proof that the HJ conditions imply the L (and therefore the D) conditions.

In order to use the HJ junction conditions to examine the problem of the propagation of Killing vectors across S, we must first write Killing's equations in terms of the Newman-Penrose formalism.

In general, any vector X^{μ} can be written in terms of the null tetrad $(\ell^{\mu},n^{\mu},m^{\mu},\bar{m}^{\mu})$ as

$$X^{\mu} = A\ell^{\mu} + Bn^{\mu} + Cm^{\mu} + \bar{C}\bar{m}^{\mu} \qquad (4.4.3)$$

where A, B, C are functions of the coordinates.

Let X^{μ} be a Killing vector. Then, using the expressions (1.4.9), Killing's equations $X_{(\mu;\nu)} = 0$ can be written as

$$2A\mathrm{Re}(\gamma)+2\mathrm{Re}(C\bar{\nu})+\Delta A = 0 \tag{4.4.4a}$$

$$2A\mathrm{Re}(\varepsilon)-2B\mathrm{Re}(\gamma)+2\mathrm{Re}(C\bar{\pi})-2\mathrm{Re}(C\tau)+DA+\Delta B = 0 \tag{4.4.4b}$$

$$B\bar{\nu}-A(\bar{\alpha}+\beta+\tau)-C\bar{\lambda}-\bar{C}\mu-2iC\mathrm{Im}(\gamma)-\delta A+\Delta\bar{C} = 0 \tag{4.4.4c}$$

$$-2B\mathrm{Re}(\varepsilon)-C\varkappa-\bar{C}\bar{\varkappa}+DB = 0 \tag{4.4.4d}$$

$$-A\varkappa+B(\bar{\alpha}+\beta+\bar{\pi})+C\sigma+\bar{C}\{\bar{\rho}-2i\mathrm{Im}(\varepsilon)\}-\delta B+D\bar{C} = 0 \tag{4.4.4e}$$

$$2A\mathrm{Re}(\rho)-2B\mathrm{Re}(\mu)+2\mathrm{Re}\{C(\bar{\alpha}-\beta)\}-\delta C-\bar{\delta}\bar{C} = 0 \tag{4.4.4f}$$

$$A\sigma-B\bar{\lambda}-\delta\bar{C}-\bar{C}(\bar{\alpha}-\beta) = 0. \tag{4.4.4g}$$

The problem of the propagation of Killing vectors can then be stated as follows.

Let us assume that there exists a Killing vector X^{μ}_{M} in M, such that, on the separating hypersurface S, it coincides with one of the Killing vectors of $\bar{M}$, say $X^{\mu}_{\bar{M}}$. We want to know under which conditions X^{μ}_{M} define inside M the same kind of symmetry as the one defined by $X^{\mu}_{\bar{M}}$ in $\bar{M}$.

Suppose that we take the metric of the outer region $\bar{M}$ to be the Schwarzschild metric, then, in radiation coordinates, the separating hypersurface S is defined by $r = r_1$. We take the metric of the inner region M to be any metric (written in radiation coordinates) which is proved to represent the source of the metric in $\bar{M}$, that is, one that matches with the Schwarzschild metric on the separating hypersurface S.

If we start by considering one of the Killing vectors of the Schwarzschild metric generating the spherical symmetry, namely

$$X_S = -\frac{r}{\sqrt{2}}(\cos\varphi + i\cos\vartheta\sin\varphi)m_S^{\mu} - \frac{r}{\sqrt{2}}(\cos\varphi - i\cos\vartheta\sin\varphi)\bar{m}_S^{\mu}$$

and demanding that $X^{\mu} = A\ell^{\mu} + Bn^{\mu} + Cm^{\mu} + \bar{C}\bar{m}^{\mu}$ is a Killing vector in M that coincides with X_S^{μ} on the hypersurface $r = r_1$, that is,

$$\left.\begin{aligned} A(t,r_1,\vartheta,\varphi) &= 0 \\ B(t,r_1,\vartheta,\varphi) &= 0 \\ C(t,r_1,\vartheta,\varphi) &= -\frac{r_1}{\sqrt{2}}(\cos\varphi + i\cos\vartheta\sin\varphi) \end{aligned}\right\} \qquad (4.4.5)$$

then, using equations (4.4.4) and (4.4.5), we can easily show that the violation of any of the junction conditions automatically destroys the propagation of the Killing vector inside M. Thus, the propagation of the symmetry of the region $\bar{M}$ into the region M is related to the fulfilment of the HJ junction conditions.

4.4.2 Applications

In this section, the HJ junction conditions will be used to match the static spherically symmetric Kimura metrics (4.3.1) with the Schwarzschild metric

$$ds^2 = -\frac{1}{1-\frac{2m}{r}}\,dr^2 - r^2 d\vartheta^2 - r^2\sin^2\vartheta d\varphi^2 + (1-\frac{2m}{r})dt^2 \tag{4.4.6}$$

on a hypersurface S defined by the equation

$$r = r_1. \tag{4.4.7}$$

The appropriate tetrad for doing calculations in the case of spherical symmetry, with the metric given by (4.3.1), is, according to Davis [33],

$$\left.\begin{aligned} \ell_\mu &= \frac{1}{\sqrt{2}}\,(\sqrt{C}\delta^4_\mu + \sqrt{A}\delta^1_\mu) \\ n_\mu &= \frac{1}{\sqrt{2}}\,(\sqrt{C}\delta^4_\mu - \sqrt{A}\delta^1_\mu) \\ m_\mu &= \frac{r}{\sqrt{2}}\,(\delta^2_\mu + i\sin\vartheta\delta^3_\mu) \\ \bar{m}_\mu &= \frac{r}{\sqrt{2}}\,(\delta^2_\mu - i\sin\vartheta\delta^3_\mu). \end{aligned}\right\} \tag{4.4.8}$$

The spin coefficients for the tetrad (4.4.8) are given by

$$\left.\begin{aligned} \varkappa = \sigma &= \tau = \nu = \lambda = \pi = 0 \\ \rho = \mu &= \frac{1}{r\sqrt{2}}\frac{1}{\sqrt{A}} \\ \varepsilon = \gamma &= -\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{A}}\frac{1}{2C}\frac{\partial C}{\partial r}. \end{aligned}\right\} \tag{4.4.9}$$

For the selected Kimura cases studied in section 4.3.1, the spin coefficients (4.4.9) are the following.

---

6. $A = a_2^{-2} r^{-2}$ and $C = a_2^{-1} r^2$, where a_2 is a constant.

$\varkappa = \sigma = \tau = \nu = \lambda = \pi = 0$

$\rho = \mu = \dfrac{a_2}{\sqrt{2}}$

$\varepsilon = \gamma = -\dfrac{a_2}{\sqrt{2}}$

---

7. $A = (a_1 r^2 + a_2)^{-3}$ and $C = 1$, where $a_2 \neq 0$ and a_1 are constants.

$\varkappa = \sigma = \tau = \nu = \lambda = \pi = 0$

$\rho = \mu = \dfrac{1}{r\sqrt{2}} (a_1 r^2 + a_2)^{3/2}$

$\varepsilon = \gamma = 0$

---

8. $A = a_1^{-3} r^{-6}$ and $C = 1$, where a_1 is a constant.

$\varkappa = \sigma = \tau = \nu = \lambda = \pi = 0$

$\rho = \mu = \dfrac{1}{\sqrt{2}} a_1^{3/2} r^2$

$\varepsilon = \gamma = 0$

---

10. $A = 4(1 - a_1 r^2)^{-1}$ and $C = 1$, where $a_1 \neq 0$ is a constant.

$\varkappa = \sigma = \tau = \nu = \lambda = \pi = 0$

$\rho = \mu = \dfrac{1}{2r\sqrt{2}} (1 - a_1 r^2)^{1/2}$

$\varepsilon = \gamma = 0$

---

11. $A = (b_0 r^2 + 4)^{-1}$ and $C = 1$, where $b_0 \neq 0$ is a constant.

$\varkappa = \sigma = \tau = \nu = \lambda = \pi = 0$

$\rho = \mu = \dfrac{1}{r\sqrt{2}} (b_0 r^2 + 4)^{1/2}$

$\varepsilon = \gamma = 0$

---

14. $A = (b_0 r^2+4)^{-1}$ and $C = b_0 r^2+4$, where $b_0 \neq 0$ is a constant.

$\varkappa = \sigma = \tau = \nu = \lambda = \pi = 0$

$$\rho = \mu = \frac{1}{r\sqrt{2}} (b_0 r^2+4)^{1/2}$$

$$\varepsilon = \gamma = -\frac{1}{2\sqrt{2}} b_0 r(b_0 r^2+4)^{-1/2}$$

---

15. $A = \frac{1}{4}$ and $C = 1$.

$\varkappa = \sigma = \tau = \nu = \lambda = \pi = 0$

$$\rho = \mu = \frac{2}{r\sqrt{2}}$$

$\varepsilon = \gamma = 0$

---

16. $A = 4(1-a_1 r^2)^{-1}$ and $C = 1-a_1 r^2$, where $a_1 \neq 0$ is a constant.

$\varkappa = \sigma = \tau = \nu = \lambda = \pi = 0$

$$\rho = \mu = \frac{1}{2r\sqrt{2}} (1-a_1 r^2)^{1/2}$$

$$\varepsilon = \gamma = \frac{1}{4\sqrt{2}} a_1 r(1-a_1 r^2)^{-1/2}$$

---

17. $A = 4$ and $C = 1$.

$\varkappa = \sigma = \tau = \nu = \lambda = \pi = 0$

$$\rho = \mu = \frac{1}{2r\sqrt{2}}$$

$$\varepsilon = \gamma = 0.$$

The corresponding expressions for the spin coefficients of the Schwarzschild metric are

$$\left.\begin{aligned} &\kappa_S = \sigma_S = \tau_S = \nu_S = \lambda_S = \pi_S = 0 \\ &\rho_S = \mu_S = \frac{1}{r\sqrt{2}} \frac{\sqrt{r-2m}}{\sqrt{r}} \\ &\varepsilon_S = \gamma_S = -\frac{m}{2r\sqrt{2}} \frac{1}{\sqrt{r(r-2m)}} \end{aligned}\right\} \qquad (4.4.10)$$

where the subscript S stands for Schwarzschild.

We start by demanding the continuity of all spin coefficients across the hypersurface $r = r_1$ separating the two regions M and $\bar{M}$ of space-time. We let the metric of the <u>inner</u> region M be any <u>Kimura metric</u>, and the metric of the outer region $\bar{M}$ be the Schwarzschild metric.

Then, from the continuity of ρ across $r = r_1$ we find the mass inside the sphere of radius $r = r_1$, in complete agreement with the results of section 4.3.1. The continuity of ε across $r = r_1$ gives $r_1 = R$, where R is the radius of the sphere obtained in section 4.3.1 by the condition $p_r(R) = 0$.

As an example, consider the particular case 6 of Kimura metrics. Then, the continuity of ρ across $r = r_1$, that is $[\rho_S=\rho]_{r=r_1}$, implies that $[\frac{1}{r\sqrt{2}}\frac{\sqrt{r-2m}}{\sqrt{r}} = \frac{a_2}{\sqrt{2}}]_{r=r_1}$ or equivalently $[m = \frac{r}{2}(1-a_2^2r^2)]_{r=r_1}$, which is the expression for the mass inside the sphere of radius r_1. The continuity of ε across $r=r_1$, that is $[\varepsilon_S=\varepsilon]_{r=r_1}$, implies that $[\frac{m}{2r\sqrt{2}}\frac{1}{\sqrt{r(r-2m)}} = \frac{a_2}{2\sqrt{2}}]_{r=r_1}$. From this expression and the expression for the mass we obtain the equation $[3a_2^4r^4+2a_2^2r^2-1=0]_{r=r_1}$ which, when solved, gives us $r_1 = \frac{1}{\sqrt{3}a_2}$, that is, r_1 is equal to the radius R of the sphere.

In all cases, the continuity of the metric components across the hypersurface $r=r_1$ appears automatically as a consequence of the continuity of the spin coefficients.

4.5 RECIPROCAL TRANSFORMATION AND KILLING'S EQUATIONS

In an n-dimensional Riemannian space a line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{4.5.1}$$

is called <u>static</u> if there is a coordinate x^a such that the $g_{\mu\nu}$ satisfy the conditions

$$g_{ia} = 0 \quad \text{and} \quad g_{\mu\nu,a} = 0 \tag{4.5.2}$$

that is, if the line element (4.5.1) has the form

$$ds^2 = g_{ik}(x^j)dx^i dx^k + g_{aa}(x^j)(dx^a)^2 \tag{4.5.3}$$

where Greek indices take the values 1,...,n and Latin indices run over the same range but with a omitted. The index a being fixed, no summation over it is to be carried out, even when it is repeated. Buchdahl [34] proved the following theorem.

Theorem: If (4.5.3) is a given static solution of Einstein's equations in empty space $R_{\mu\nu} = 0$, then another static solution is immediately given by

$$ds^2 = (g_{aa})^2 g_{ik} dx^i dx^k + \frac{1}{g_{aa}} (dx^a)^2. \tag{4.5.4}$$

Any two solutions related in this way will be called reciprocal to one another.

In all cases with $g_{aa} = 1$ this is trivially true and these cases correspond to self-reciprocal solutions.

Note that, if g and $\bar{g}$ denote the determinants of the metric tensors of (4.5.3) and (4.5.4) respectively, they are related by

$$\sqrt{(-\bar{g})} = (g_{aa})^2 \sqrt{(-g)}. \tag{4.5.5}$$

It is convenient to refer to the process of formation of a reciprocal solution as a reciprocal transformation. Then it is obvious that in certain cases a solution reciprocal to a given one will also be obtainable from the latter by means of a transformation of coordinates. This is the case, for example, if one applies to the general spherically symmetric solution of $R_{\mu\nu} = 0$ a reciprocal transformation which does not destroy spherical symmetry.

As a simple example consider the Schwarzschild solution with line element given by (4.4.6). Its reciprocal solution, according to (4.5.4), is

$$ds^2 = -(1 - \frac{2m}{r})^2[\frac{1}{(1 - \frac{2m}{r})} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2] + \frac{1}{(1 - \frac{2m}{r})} dt^2.$$

Then, the coordinate transformation $r = -\bar{r} + 2m$ reproduces the usual Schwarzschild solution.

As an example of a non-trivial reciprocal transformation consider the general Kimura solution with line element given by (4.3.1). Its reciprocal solution, according to (4.5.4), is

$$ds^2 = -C^2(Adr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2) + \frac{1}{C} dt^2. \qquad (4.5.6)$$

It must be noted here that the reciprocal transformation for one form of the metric is different from that for another form of the same metric; thus, for example, the standard and the isotropic forms of a static spherically symmetric metric will give different reciprocal solutions.

Buchdahl also studied reciprocal solutions of Einstein's equations in the presence of matter [35] and proved that his theorem can again be applied. Assuming that the x^a coordinate is the time coordinate x^4, he also proved that the energy-momentum tensors T^μ_ν and $\bar{T}^\mu_\nu$ of a given metric and its reciprocal are related by

$$\bar{T}^j_i = \frac{1}{(g_{44})^2} T^j_i \quad \text{and} \quad \bar{T}^4_4 = -\frac{1}{(g_{44})^2}(T^4_4 - 2T^i_i). \tag{4.5.7}$$

If the components of the energy-momentum tensor, as measured by an observer at rest with respect to the coordinates x^μ, are given by (4.2.21), then the results of measurements carried out by such an observer on distributions represented by a metric $g_{\mu\nu}$ and its reciprocal $\bar{g}_{\mu\nu}$ are related by

$$\left.\begin{aligned} \bar{p}_r &= \frac{1}{(g_{44})^2} p_r \\ \bar{p}_\perp &= \frac{1}{(g_{44})^2} p_\perp \\ \bar{\rho} &= -\frac{1}{(g_{44})^2}(\rho + 2p_r + 4p_\perp) . \end{aligned}\right\} \tag{4.5.8}$$

Equations (4.5.8) are the <u>extension</u> of Buchdahl's results for the case of an <u>anisotropic</u> distribution of matter.

A very interesting equation, which will be answered here, is whether or not Killing's equations remain unchanged under a reciprocal transformation of the form

$$\bar{g}_{ij} = (g_{44})^2 g_{ij} \quad \text{and} \quad \bar{g}_{44} = \frac{1}{g_{44}} . \tag{4.5.9}$$

The problem will be examined, in what follows, for the case of the static spherically symmetric Kimura space-times.

We start by examining Killing vector equations (2.2.5). It is well known that the static symmetric Kimura space-times admit the following four Killing vectors

$$\left.\begin{aligned}
X_1 &= \sin\varphi \frac{\partial}{\partial\vartheta} + \cot\vartheta\cos\varphi \frac{\partial}{\partial\varphi} \\
X_2 &= -\cos\varphi \frac{\partial}{\partial\vartheta} + \cot\vartheta\sin\varphi \frac{\partial}{\partial\varphi} \\
X_3 &= -\frac{\partial}{\partial\varphi} \\
X_4 &= \frac{\partial}{\partial t}.
\end{aligned}\right\} \qquad (4.5.10)$$

Spherical symmetry also implies that

$$X^{\mu}\,\partial_{\mu}(f(r)) = 0 \qquad (4.5.11)$$

where X^{μ} denotes a Killing vector and $f(r)$ is any arbitrary function of r.

Three separate cases of (2.2.5) need to be considered.

i) $\underline{\mu = \nu = 4}$. Then, from (2.2.5) we have

$$X^1 g_{44,1} + 2X^4_{\ ,4}\, g_{44} = 0. \qquad (4.5.12)$$

Using (4.5.9), the left-hand side of this equation becomes $X^1\bar{g}_{44,1} + 2X^4_{\ ,4}\bar{g}_{44} = -\dfrac{1}{(g_{44})^2} X^1 g_{44,1}$ since $X^4_{\ ,4} = 0$. Because of (4.5.11), $X^1 g_{44,1} = 0$ and so the form of (4.5.12) remains unchanged under the reciprocal transformation (4.5.9).

ii) $\underline{\mu = 4\ ;\ \nu = 1}$. Then, from (2.2.5) we have

$$X^j_{\ ,4}\, g_{1j} + X^4_{\ ,1}\, g_{44} = 0. \qquad (4.5.13)$$

Using (4.5.9), the left-hand side of this equation becomes $X^j_{,4}\bar{g}_{ij} + X^4_{,i}\bar{g}_{44}$, which is obviously equal to zero since $X^j_{,4} = 0$ and $X^4_{,i} = 0$. Thus the form of (4.5.13) remains unchanged under the reciprocal transformation (4.5.9).

iii) $\underline{\mu = i}$; $\underline{\nu = j}$. Then, from (2.2.5) we have

$$X^s g_{ij,s} + X^s_{,i} g_{sj} + X^s_{,j} g_{si} = 0. \qquad (4.5.14)$$

Using (4.5.9), the left-hand side of this equation becomes

$$X^s \bar{g}_{ij,s} + X^s_{,i}\bar{g}_{sj} + X^s_{,j}\bar{g}_{si}$$
$$= (g_{44})^2[X^s g_{ij,s} + X^s_{,i} g_{sj} + X^s_{,j} g_{si}] + 2g_{44}g_{44,s}X^s g_{ij} \ .$$

Because of (4.5.11), $X^s g_{44,s} = 0$. Clearly, the form of (4.5.14) again remains unchanged under the reciprocal transformation (4.5.9).

Next we examine Killing tensor equations (2.2.29), by considering the particular case 6 of Kimura metrics, with $g_{11} = -a_2^{-2}r^{-2}$, $g_{22} = -r^2$, $g_{33} = -r^2\sin^2\vartheta$ and $g_{44} = a_2^{-1}r^2$ and its Killing tensor of second order with components $K^{14} = K^{41} = -a_2 rt$, $K^{22} = -t^2$, $K^{33} = -\dfrac{t^2}{\sin^2\vartheta}$ and $K^{44} = \dfrac{1}{r^2} + a_2 t^2$.

Four separate cases of (2.2.29) need to be considered.

i) $\mu = \nu = 4$; $\lambda = 4$

ii) $\mu = \nu = 4$; $\lambda = i$

iii) $\mu = 4$; $\nu = i$; $\lambda = j$

iv) $\mu = i$; $\nu = j$; $\lambda = k$.

In each case, using (4.5.9) in (2.2.29) and after substitution of the above values of the metric and Killing tensor components, the form of (2.2.29) does <u>not</u> remain the same.

As an example, consider the case i). Then, from (2.2.29) we have $K^{44}_{,4}g^{44} - g^{44}_{,s}K^{4s} = 0$. Using (4.5.9), the left-hand side of this equation becomes

$$K^{44}_{,4}\bar{g}^{44} - \bar{g}^{44}_{,s}K^{4s} = K^{44}_{,4}\,\frac{1}{g_{44}} + \frac{1}{(g_{44})^2}\,g^{44}_{,s}K^{4s}$$

which after substitution of the above values of the metric and Killing tensor components is found different from zero. Thus, this particular component of (2.2.29) is not preserved under the reciprocal transformation (4.5.9).

All the above results can be summarized in an important theorem.

<u>Theorem</u>: A given static solution of Einstein's equations and its reciprocal possess the same isometry group, and consequently the Killing vector equations remain unchanged under a reciprocal transformation. Killing tensor equations, however, are not preserved by such a transformation.

4.6 CONCLUSION

The main features of the examined Kimura models can now be summarized. From a mathematical point of view, they constitute a complete class of static spherically symmetric space-times possessing irreducible Killing tensors of second order. They all match with the exterior Schwarzschild solution as expected, since every static spherically symmetric space-time has to obey

Birkhoff's theorem [36].

Their main interest from a physical point of view, however, is the anisotropic character of their energy-momentum tensors which probably results in the instability of the matter observed in most cases. Only a few cases correspond to physically realistic distributions of matter and can thus represent interiors of stars. The remainder, and in particular those which admit a large number of Killing tensors, are proved to be unphysical and consequently these Killing tensors do not correspond to any physical source of the metric. It seems, therefore, that the existence of Killing tensors cannot function as an independent and useful constraint on solutions corresponding to interesting matter tensors.

Finally, the reciprocal solutions obtained from the Kimura ones admit the same isometry group, but since Killing tensors do change under a reciprocal transformation, the Lie algebra extension discussed in Chapter 2 will also change under such a transformation, and different Killing tensors of high order will then be obtained.

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APPENDIX A1

CANONICAL DECOMPOSITION OF SYMMETRIC SPINORS

THE PETROV-PIRANI CLASSIFICATION

Every symmetric spinor with n indices $\varphi_{AB\ldots K} = \varphi_{(AB\ldots K)}$ can be written as a symmetrized product of 1-spinors [1]

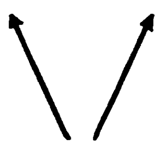
$$\varphi_{AB\ldots K} = \alpha_{(A}\beta_{B}\cdots\pi_{K)}. \tag{A1.1}$$

Each one of the spinors $\alpha_A,\ldots,\pi_K$, which are called principal spinors, is determined up to a complex scalar factor and in turn determines a real null direction. They need not be all distinct. Therefore, the decomposition (A1.1), called the canonical decomposition of $\varphi_{AB\ldots K}$, determines at least one and at most n real null directions which are called the principal null directions. Two principal null directions coincide if and only if the principal spinors corresponding to them are parallel. The mutual position of the principal null directions is basic to the algebraic classification of symmetric spinors and the tensors defined by them. If the spinor $\varphi_{AB\ldots K}$ has p distinct principal directions, each of which is m_i-fold degenerate, we say that it has the type $[m_1,m_2,\ldots,m_p]$. The spinor $\varphi_{AB\ldots K}$ is called algebraically general if it has the type $[1,1,\ldots,1]$. If at least one principal light direction is degenerate, it is said to be algebraically special.

The simplest case is the symmetric 2-spinor Φ_{AB} which can be written in the form $\Phi_{AB} = \alpha_{(A}\beta_{B)}$. The algebraic classification of the spinor Φ_{AB}, which determines the electromagnetic field tensor $F_{\alpha\beta}$, is as follows

Partition	Name of corresponding bivector	Form of Φ_{AB}
[11]	algebraically general	$\Phi_{AB} = \alpha_{(A}\beta_{B)}$ α_A not proportional to β_B
[2]	algebraically special	$\Phi_{AB} = \alpha_{(A}\beta_{B)}$ α_A proportional to β_B
0		$\Phi_{AB} = 0$

The principal null directions for each case are as follows



[11]

[2]

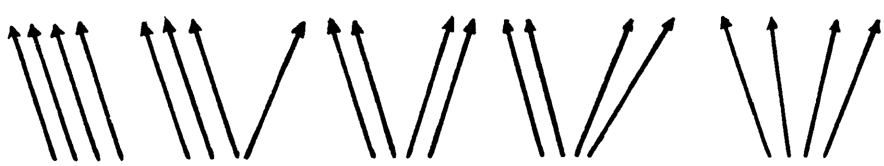
The gravitational principal null directions are undefined if $\Phi_{AB} = 0$.

Applying the classification process to the Weyl tensor we find Penrose's form of the Petrov classification.

The Weyl tensor $C_{\alpha\beta\gamma\delta}$ is determined by the symmetric 4-spinor Ψ_{ABCD} which can be written in the form $\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$. α_A, β_B, γ_C, δ_D are <u>not</u> proportional to each other. The classification by principal null directions is as follows

Partition	Symbol for Petrov type	Form of Ψ_{ABCD}
[1111]	I algebraically general	$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$
[211]	II algebraically special	$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$
[22]	D algebraically special	$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$
[31]	III algebraically special	$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$
[4]	N algebraically special	$\Psi_{ABCD} = \alpha_A\beta_B\gamma_C\delta_D$
[0]	0	$\Psi_{ABCD} = 0$

The principal null vectors for each case are as follows



[4] N   [31] III   [22] D   [211] II   [1111] I

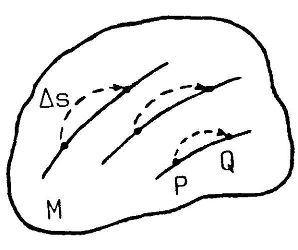
The gravitational principal null directions are undefined if $\Psi_{ABCD} = 0$.

APPENDIX A2

LIE DERIVATIVES

If physical realities are represented by tensor fields on the space-time manifold, physical laws must say how different fields vary together from one point of the manifold to another. This involves a comparison between the values of each field at neighbouring points. Several methods of comparison exist, one of them being the Lie derivative along the congruence defined by a vector field.

A congruence is a set of curves that fill the manifold (or part of it) smoothly and continuously without intersecting. Each point of the manifold is on one and only one curve. The key point from which everything else follows is that the congruence provides a natural mapping of the manifold into itself. If the parameter on the curve is s, then any sufficiently small number Δs defines a mapping in which each point is wrapped into the one, a parameter distance Δs further along the same curve of the congruence (see figure below). If such a 1-1 mapping exists for all Δs we call this a "dragging" along the congruence,



The mapping of the manifold M to itself defined by mapping each point to another point on the same curve of the congruence whose parameter is Δs larger.

or a Lie dragging [2]. For each point $P \in M$, a vector field $\underline{V}$ on M determines a unique curve $\gamma_P(s)$ such that $\gamma_P(0) = P$ and $\underline{V}$ is the tangent vector to the curve. The family of such curves is the congruence associated with the vector field. The concept of dragging permits the definition of a derivative along the congruence, which will be useful in any problem where a congruence plays a central role. To introduce this new type of derivative we consider the map φ_s dragging each point P with coordinates x^μ along the curve $\gamma_P(s)$ through P into another point $Q = \varphi_s(P)$ with coordinates y^μ. For sufficiently small values of s the map φ_s is a 1-1 map which induces a map $\varphi_s^* T$ of any tensor T.

We define the Lie derivative [3] of T with respect to $\underline{V}$ by

$$\mathcal{L}_{\underline{V}} T = \lim_{s \to 0} \frac{1}{s} (\varphi_s^* T - T) . \qquad \text{(A2.1)}$$

The tensors T and $\varphi_s^* T$ have the same type and are both evaluated at the same point P. Therefore the Lie derivative (A2.1) is also a tensor of the same type at P.

The Lie derivative is an operation defined on a differentiable manifold without imposing additional structures. It depends on the vector $\underline{V}$ not only at P but also at neighbouring points, so that given a congruence we can define a unique Lie derivative which, however, does depend on the congruence.

In a coordinate basis, we have the following components for the Lie derivative [4] of

i) a function f: $\mathcal{L}_{\underline{V}} f = V^\mu f_{,\mu}$

ii) a one-form ω: $\mathcal{L}_{\underline{V}} \omega = (V^\sigma \omega_{\mu,\sigma} + \omega_\sigma V^\sigma{}_{,\mu}) dx^\mu$

iii) a vector field $\underline{U}$: $\mathcal{L}_{\underline{V}} \underline{U} = [\underline{V}, \underline{U}] = (V^\sigma U^\mu{}_{,\sigma} - U^\sigma V^\mu{}_{,\sigma}) \frac{\partial}{\partial x^\mu}$

iv) an arbitrary tensor T:

$$(\mathcal{L}_{\underline{V}} T)^{\mu\nu\ldots}{}_{\varkappa\lambda\ldots} = V^\sigma T^{\mu\nu\ldots}{}_{\varkappa\lambda\ldots,\sigma} - T^{\sigma\nu\ldots}{}_{\varkappa\lambda\ldots} V^\mu{}_{,\sigma}$$

$$- T^{\mu\sigma\ldots}{}_{\varkappa\lambda\ldots} V^\nu{}_{,\sigma} + T^{\mu\nu\ldots}{}_{\sigma\lambda\ldots} V^\sigma{}_{,\varkappa} + T^{\mu\nu\ldots}{}_{\varkappa\sigma\ldots} V^\sigma{}_{,\lambda} + \ldots$$

v) an arbitrary spinor χ [5]:

$$(\mathcal{L}_{\underline{V}} \chi)^{A\ldots A'\ldots}_{D\ldots D'\ldots} = V^\sigma \nabla_\sigma \chi^{A\ldots A'\ldots}_{D\ldots D'\ldots} - h^A_{A_0} \chi^{A_0\ldots A'\ldots}_{D\ldots D'\ldots} - \ldots$$

$$- \bar{h}^{A'}_{A'_0} \chi^{A\ldots A'_0\ldots}_{D\ldots D'\ldots} + h^{D_0}_{D} \chi^{A\ldots A'\ldots}_{D_0\ldots D'\ldots} + \ldots$$

$$+ \bar{h}^{D_0}_{D'_0} \chi^{A\ldots A'}_{D\ldots D'_0} \ldots + \ldots$$

where $h^A_D = \frac{1}{2}(\nabla_{DD'} V^{AD'} + \bar{\lambda}\varepsilon^A_D)$ and λ is defined, $\mathcal{L}\varepsilon^{AB}$ being skew in AB, by $\mathcal{L}\varepsilon^{AB} = \lambda\varepsilon^{AB}$.

The Lie derivative of tensors, in general, has the following properties (stated here without proof) [6] :

a) It satisfies the Leibniz product rule.

b) It satisfies the Jacobi identity.

c) It commutes with the operation of contraction.

d) It commutes with the partial derivative.

e) It commutes with the exterior derivative.

f) It can be applied to arbitrary linear geometrical objects, to the Christoffel symbols for example.*

One of the principal uses of Lie derivatives in physics is to express the notion that a tensor field is invariant under some transformation. We say that a tensor field T is <u>invariant</u> under a vector field $\underline{V}$ if

$$\mathcal{L}_{\underline{V}} T = 0. \qquad \text{(A2.2)}$$

A vector field $\underline{V}$ for which the metric tensor satisfies (A2.2) is a Killing vector field. The metric tensor is invariant with respect to a Killing vector field, that is ,

$$\mathcal{L}_{\underline{V}} g_{\mu\nu} := V^{\sigma} g_{\mu\nu,\sigma} + g_{\mu\sigma} V^{\sigma}{}_{,\nu} + g_{\sigma\nu} V^{\sigma}{}_{,\mu} = 0. \qquad \text{(A2.3)}$$

Equation (A2.3) is Killing's equation and defines a motion or isometry on M.

---

* For more about Lie derivatives of geometrical objects see K. Yano, The Theory of Lie Derivatives and its Applications [4].

APPENDIX A3

FIRST INTEGRALS OF GEODESICS IN KERR SPACE-TIME

The Kerr family of solutions of Einstein and Einstein-Maxwell equations is the most general class of solutions known at present which could represent the field of a rotating neutral or electrically charged body in asymptotically flat space, and it is, therefore, especially important for understanding the gravitational collapse of a rotating star.

The original and most useful form of the Kerr family of solutions of the source-free Einstein-Maxwell equations is given in terms of Kerr coordinates u, r, ϑ, φ as follows.

The covariant form of the metric is

$$\begin{aligned} ds^2 &= \rho^2 d\vartheta^2 - 2a \sin^2\vartheta \, dr \, d\varphi + 2 \, dr \, du \\ &+ \rho^{-2}\{(r^2+a^2)^2 - \Delta a^2 \sin^2\vartheta\} \sin^2\vartheta \, d\varphi^2 \qquad \text{(A3.1)} \\ &- 2a\rho^{-2}(2mr-e^2) \sin^2\vartheta \, d\varphi \, du - \{1 - \rho^{-2}(2mr-e^2)\}du^2 \end{aligned}$$

and the corresponding covariant form of the electromagnetic field is

$$\begin{aligned} F = 2e\rho^{-4}\{&(r^2-a^2\cos^2\vartheta)dr \wedge du - 2a^2 r \cos\vartheta \sin\vartheta \, d\vartheta \wedge du \\ &- a \sin^2\vartheta \, (r^2-a^2\cos^2\vartheta)dr \wedge d\varphi + 2ar(r^2+a^2)\cos\vartheta \sin\vartheta \, d\vartheta \wedge d\varphi\} \end{aligned} \qquad \text{(A3.2)}$$

where we use the abbreviations $\rho^2 = r^2 + a^2 \cos^2 \vartheta$ and $\Delta = r^2 - 2mr + a^2 + e^2$; the symbol "$\wedge$" denotes the operation of taking the antisymmetrized tensor product [7].

The parameters a, m, e appearing in the above equations are the angular momentum per unit mass, the mass and the charge of the rotating star represented by the Kerr solution, respectively. The form (A3.1) of the metric is the Kerr-Newman solution which reduces to the Kerr solution when $e = 0$. When $a = 0$, equations (A3.1) and (A3.2) represent the charged Reissner-Nordström solution which reduces to the Schwarzschild solution when also $e = 0$.

The Kerr solutions, being stationary and axisymmetric, have an abelian two-parameter group of isometries, generated by the commuting Killing vectors $\partial/\partial u$ and $\partial/\partial\varphi$. Also, in all Kerr spaces, the Weyl tensor is of type D in the Petrov classification.

The equations of motion of a freely falling test particle of mass μ and charge ε in the Kerr field are given by

$$\frac{d^2x^\mu}{d\tau^2} = \frac{\varepsilon}{\mu} F^\mu{}_\nu \frac{dx^\nu}{d\tau} \tag{A3.3}$$

where τ denotes proper time.

These equations may be derived from the Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \varepsilon A_\mu \dot{x}^\mu \tag{A3.4}$$

where the vector potential A satisfies $F = 2dA$ and where a dot over a symbol denotes ordinary differentiation with respect to the proper time by $\tau = \mu\lambda$, which is equivalent to imposing the normalising condition

$$g_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu} = -\,\mu^{2}. \tag{A3.5}$$

Next, we introduce the momenta

$$p_{\mu} = g_{\mu\nu}\,\dot{x}^{\nu} + \varepsilon A_{\mu}. \tag{A3.6}$$

By substituting these in the Lagrangian we obtain the Hamiltonian

$$H = \frac{1}{2}\,g^{\mu\nu}(p_{\mu} - \varepsilon A_{\mu})(p_{\nu} - \varepsilon A_{\nu}). \tag{A3.7}$$

Since it does not depend explicitly on λ, the Hamiltonian is automatically a constant of the motion and using (A3.5) we have

$$H = -\,\frac{1}{2}\,\mu^{2}. \tag{A3.8}$$

Now, the simplest vector potential giving rise to the electromagnetic field (A3.2) is

$$A = e\rho^{-2}r\,(du - a\,\sin^{2}\vartheta\,d\varphi\,). \tag{A3.9}$$

From (A3.1) we then obtain the momenta

$$p_{u} = -\,\{1 - \rho^{-2}(2mr-e^{2})\}\dot{u} - a\rho^{-2}(2mr-e^{2})\sin^{2}\vartheta\dot{\varphi}$$

$$+\,\dot{r} + e\rho^{-2}r \tag{A3.10}$$

$$p_\varphi = -a\rho^{-2}(2mr-e^2)\sin^2\vartheta\dot{u} + \rho^{-2}\{(r^2+a^2)^2-\Delta a\sin^2\vartheta\}\sin^2\vartheta\dot{\varphi}$$

$$- a\sin^2\vartheta\dot{r} - e\rho^{-2}ar\sin^2\vartheta \qquad (A3.11)$$

$$p_r = \dot{u} - a\sin^2\vartheta\dot{\varphi} \qquad (A3.12)$$

$$p_\vartheta = \rho^2\dot{\vartheta} \qquad (A3.13)$$

and the Hamiltonian

$$H = \frac{1}{2}\rho^{-2}[\Delta p_r^2 + 2\{(r^2+a^2)p_u+ap_\varphi-\varepsilon er\}p_r+p_\vartheta^2+(a\sin\vartheta p_u+\sin^{-1}\vartheta p_\varphi)^2]. \qquad (A3.14)$$

Hamilton's equation $dp_\mu/d\lambda = -\partial H/\partial x^\mu$ guarantees that p_u and p_φ are constants of motion, corresponding to the conservation of energy and angular momentum about the symmetry axis, respectively. We write

$$p_u = -E \qquad (A3.15)$$

$$p_\varphi = \Phi. \qquad (A3.16)$$

In addition, we also have the constant of motion given by (A3.8), corresponding to the conservation of rest mass.

These three constants of motion are sufficient to determine the orbit of the particle only when some restriction is imposed which reduces the problem to three or fewer dimensions. In order to determine the orbit of the particle through 4-dimensional space-time, a fourth constant of motion is needed which cannot come from the obvious symmetries of the metric. It turns out, however, that such a constant can be obtained by taking advantage of the fact that the Hamilton-Jacobi equation can be solved by separation of variables in the coordinate system (A3.1) and with the choice of gauge (A3.9).

From (A3.7) we obtain the general form of the Hamilton-Jacobi equation as

$$\frac{\partial S}{\partial \delta} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial S}{\partial x^{\mu}} - \varepsilon A_{\mu}\right)\left(\frac{\partial S}{\partial x^{\nu}} - \varepsilon A_{\nu}\right) \tag{A.3.17}$$

where S is the Jacobi action.

If there is a separable solution, in terms of the already known constants of motion, it must take the form

$$S = -\frac{1}{2}\mu^{2}\lambda - Eu + \Phi\varphi + S_{\vartheta} + S_{r} \tag{A3.18}$$

with S_{ϑ} and S_{r} functions of only ϑ and r respectively. Substituting (A3.18) in (A3.14), we see that the equation can be separated as follows

$$\left(\frac{dS_{\vartheta}}{d\vartheta}\right)^{2} + a^{2}\mu^{2}\cos^{2}\vartheta + (aE\sin\vartheta - \Phi\sin^{-1}\vartheta)^{2}$$

$$= -\Delta\left(\frac{dS_{r}}{dr}\right)^{2} + 2\{(r^{2}+a^{2})E - a\Phi + \varepsilon er\}\frac{dS_{r}}{dr} - \mu^{2}r^{2}. \tag{A3.19}$$

Both sides must be equal to a new constant of motion, which we denote by K and which is related to the momenta by

$$p_\vartheta^2 + (aE \sin\vartheta - \Phi \sin^{-1}\vartheta)^2 + a^2\mu^2\cos^2\vartheta = K \qquad \text{(A3.20a)}$$

$$\Delta p_r^2 - 2\{(r^2+a^2)E - a\Phi + \varepsilon e r\}p_r + \mu^2 r^2 = -K. \qquad \text{(A3.20b)}$$

Equation (A3.19) splits up to give two ordinary differential equations

$$\frac{dS_\vartheta}{d\vartheta} = \sqrt{\Theta} \qquad \text{(A3.21)}$$

$$\frac{dS_r}{dr} = \Delta^{-1}(P + \sqrt{R}) \qquad \text{(A3.22)}$$

where the functions $\Theta(\vartheta)$, $P(r)$, $R(r)$ are defined by

$$\Theta = Q - \cos^2\vartheta\{a^2(\mu^2 - E^2) + \Phi^2 \sin^{-2}\vartheta\}$$

$$P = E(r^2 + a^2) - \Phi a + \varepsilon e r \qquad \text{(A3.23)}$$

$$R = P^2 - \Delta(\mu^2 r^2 + K)$$

and

$$Q: = K - (\Phi - aE)^2. \qquad \text{(A3.24)}$$

The constant K, or equivalently the constant Q, together with the three other constants given by (A3.8), (A3.15) and (A3.16) form a complete set of first integrals of the motion. The final form of the Jacobi action is

$$S = -\frac{1}{2}\mu^2\lambda - Eu + \Phi\varphi + \int^{\vartheta}\sqrt{\Theta}d\vartheta + \int^{r}\Delta^{-1}Pdr$$

$$+\int^{r}\Delta^{-1}\sqrt{R}\,dr \qquad . \tag{A3.25}$$

It is worth giving here another form of the Kerr metric, namely its form in the Boyer-Lindquist coordinates [8]

$$ds^2 = -(1-2Mr/\Sigma)dt^2-2(2Mr/\Sigma)a\sin^2\vartheta dtd\varphi+(\Sigma/\Delta)dr^2+\Sigma d\vartheta^2$$

$$+(A/\Sigma)\sin^2\vartheta d\varphi^2 \tag{A3.26}$$

where $\Sigma = r^2+a^2\cos^2\vartheta$, $\Delta = r^2+a^2-2Mr$ and $A = (r^2+a^2)^2-\Delta a^2\sin^2\vartheta$.

Introducing the complex null tetrad $(\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ by

$$\ell^\mu = \frac{1}{2}\Sigma^{-1}(r^2+a^2,\Delta,0,a)$$

$$n^\mu = \Delta^{-1}(r^2+a^2,-\Delta,0,a) \tag{A3.27}$$

$$m^\mu = (2\Sigma)^{-1/2}(ia\sin\vartheta,0,1,i/\sin\vartheta)$$

normalized so that $g_{\mu\nu} = -2\ell_{(\mu}n_{\nu)}+2m_{(\mu}\bar{m}_{\nu)}$ [9], the Killing tensor $K_{\mu\nu}$ associated with the fourth constant K of the Kerr metric is written as [10]

$$K_{\mu\nu} = 2\Sigma\ell_{(\mu}n_{\nu)} + r^2g_{\mu\nu}. \tag{A3.28}$$

APPENDIX A4

SYMPLECTIC FORM AND SYMMETRIES OF DYNAMICAL SYSTEMS

The Hamiltonian version of a dynamical system of equations begins with the Lagrangian $\mathcal{L}(q,\dot{q})$. The momentum p is defined by $p_\mu = \partial\mathcal{L}/\partial\dot{q}^\mu$ and then the Hamiltonian H of the system is given by

$$H = \sum_{\mu=1}^{n} p_\mu \dot{q}^\mu - \mathcal{L} = H(p,q) \qquad \text{(A4.1)}$$

where n are the degrees of freedom of the dynamical system.

The definition of the Hamiltonian, given by(A4.1) suggests the expression [11]

$$\vartheta = \sum_{\mu=1}^{n} p_\mu dq^\mu . \qquad \text{(A4.2)}$$

Since p_μ and q^μ are quantities on T*M, each dq^μ is a one-form on T*M and so is each scalar multiple $p_\mu dx^\mu$ and hence ϑ itself. Any change in the local coordinates q^μ followed by the corresponding change in the momenta p_μ leaves the one-form ϑ unchanged. Thus, on every cotangent bundle there is an invariant one-form ϑ.

Moreover, on every cotangent bundle T*M, our invariant one-form ϑ yields by exterior differentiation a two-form $\Omega = d\vartheta$ with

$$\Omega = \sum_{=1}^{n} dp_\mu \wedge dq^\mu . \qquad \text{(A4.3)}$$

Ω is a closed two-form ($d\Omega = 0$ since $d\Omega = dd\vartheta$) and is of maximal rank, which means that its exterior power $\Omega^n = \Omega \wedge \ldots \wedge \Omega$ to n factors is nonzero. The two-form defined by (A4.3) is called a <u>symplectic form</u> and every manifold equipped with such a form is called a <u>symplectic manifold</u>. Thus, we have just seen that every cotangent bundle ($\Leftrightarrow$ every phase space) T*M is a symplectic manifold. Any vector field $\underline{X}$ that satisfies

$$\mathcal{L}_{\underline{X}}\Omega = 0 \tag{A4.4}$$

is called a <u>Hamiltonian vector field</u>.

If such a vector field satisfies also the relation

$$\mathcal{L}_{\underline{X}} H = \underline{X}(H) = 0 \tag{A4.5}$$

then the dynamical system described by the Hamiltonian H is conservative, that is, in Hamiltonian mechanics, constants of motion correspond to symmetries of the dynamical system via the symplectic structure of the theory. This means that a symmetry is a one-parameter group of transformations of the phase space * whose corresponding vector field $\underline{X}$ satisfies $\mathcal{L}_{\underline{X}}\Omega = 0$ and $\mathcal{L}_{\underline{X}} H = 0$ [12].

---

* And this is an "obvious" symmetry. A hidden symmetry is one which cannot be derived from such a transformation because it mixes up coordinates and momenta in a non-trivial way.

APPENDIX A5

BLF-FILES PRODUCED BY TENSOR COMPILER

This appendix contains the files "KILLING1.BLF" and "POI.BLF", produced by Tensor COMPiler [13], which are used by the computer to write Killing vector equations and evaluate the Poisson bracket of functions, respectively.

```
% KILLING1.BLF Produced by Tensor COMPiler v.84-04    Sun Mar  3 13:41:56 1985
(OR (TENP (QUOTE M)) (DECLTEN M D))
(DPUT GAMUC UPINDX (1))
(DPUT GAMUC NAM (GAM))

(DE CBKT1 (INDX1 INDX2)
 (PROG (ANS)
    (ADANZ (SIMP (SDIFF (M INDX2) INDX1)))
    (ADANZ (SIMP (SDIFF (M INDX1) INDX2)))
    (SUM!*1 DEFAULTDIMENSION!*
     '(SMMINS (SIMP (DOMULC (GAMUC !*1 INDX2 INDX1) (M !*1)))))
    (SUM!*1 DEFAULTDIMENSION!*
     '(SMMINS (SIMP (DOMULC (GAMUC !*1 INDX1 INDX2) (M !*1)))))
    (RETURN ANS)))

(DPUT KT1 BULF CBKT1)
(DPUT KT1 PREQ (GAMUC M))
(OR (TENP (QUOTE KT1)) (DECLTEN KT1 S))
```

```
% POI.BLF Produced by Tensor COMPiler v.84-04    Sun Mar  3 12:24:18 1985
(OR (TENP (QUOTE KAPPA)) (DECLTEN KAPPA SCL))
(OR (TENP (QUOTE IOTA)) (DECLTEN IOTA SCL))

(DE CBPSKI NIL
 (XMPLUS (SIMP (MKTIMS (SDIFF (KAPPA) 0) (SDIFF (IOTA) 4)))
  (SIMP (MKTIMS (SDIFF (KAPPA) 1) (SDIFF (IOTA) 5)))
  (SIMP (MKTIMS (SDIFF (KAPPA) 2) (SDIFF (IOTA) 6)))
  (SIMP (MKTIMS (SDIFF (KAPPA) 3) (SDIFF (IOTA) 7)))
  (SMMINS (SIMP (MKTIMS (SDIFF (KAPPA) 4) (SDIFF (IOTA) 0))))
  (SMMINS (SIMP (MKTIMS (SDIFF (KAPPA) 5) (SDIFF (IOTA) 1))))
  (SMMINS (SIMP (MKTIMS (SDIFF (KAPPA) 6) (SDIFF (IOTA) 2))))
  (SMMINS (SIMP (MKTIMS (SDIFF (KAPPA) 7) (SDIFF (IOTA) 3))))))

(DPUT PSKI BULF CBPSKI)
(DPUT PSKI PREQ (IOTA KAPPA))
(OR (TENP (QUOTE PSKI)) (DECLTEN PSKI SCL))
```

APPENDIX A6

THE POISSON BRACKETS FOR KIMURA METRIC NUMBER 6

This appendix contains the computer output for the Poisson brackets, in case IIC of Kimura metrics with $C = a_2^{-1} r^2$ and $A = a_2^{-2} r^{-2}$.

Mar 24 12:00 1985 pski1.lam Page 1

$PSKI = -4BQ^2T^2V - 4BS^2T^2V\sin^{-2}(H) - 8B^2PRTV^2 + 4B^2P^2R^2V + 4B^2T^2V^3 + 4Q^2R^{-2}V$
$+4R^{-2}S^2V\sin^{-2}(H)$

Mar 24 11:49 1985 pski2.lam Page 1

$PSKI = -8BPQ^2R^{-1}V - 8BPR^{-1}S^2V\sin^{-2}(H) + 8BQ^2R^{-2}TV^2 + 16BQ^2S^2T^3\sin^{-2}(H) + 8BQ^4T^3$
$+8BR^{-2}S^2TV^2\sin^{-2}(H) + 8BS^4T^3\sin^{-4}(H) + 24B^2PRS^2T^2V\sin^{-2}(H) + 24B^2PQ^2RT^2V$
$-8B^2P^2Q^2R^2T - 8B^2P^2R^2S^2T\sin^{-2}(H) - 16B^2Q^2T^3V^2 - 16B^2S^2T^3V^2\sin^{-2}(H) - 24B^3PRT^2V^3$
$+24B^3P^2R^2TV^2 - 8B^3P^3R^3V + 8B^3T^3V^4 - 16Q^2R^{-2}S^2T\sin^{-2}(H) - 8Q^4R^{-2}T - 8R^{-2}S^4T\sin^{-4}(H)$

Mar 9 10:56 1985 pski3.lam Page 1

$$
\begin{aligned}
PSKI = {} & -64BQ^{2}R^{-2}S^{2}T^{2}V\sin^{-2}(H) - 32BQ^{4}R^{-2}T^{2}V - 32BR^{-2}S^{4}T^{2}V\sin^{-4}(H) \\
& -64B^{2}PQ^{2}R^{-1}TV^{2} - 64B^{2}PR^{-1}S^{2}TV^{2}\sin^{-2}(H) + 32B^{2}P^{2}Q^{2}V + 32B^{2}P^{2}S^{2}V\sin^{-2}(H) \\
& +32B^{2}Q^{2}R^{-2}T^{2}V^{3} + 32B^{2}Q^{2}S^{2}T^{4}V\sin^{-2}(H) + 16B^{2}Q^{4}T^{4}V + 32B^{2}R^{-2}S^{2}T^{2}V^{3}\sin^{-2}(H) \\
& +16B^{2}S^{4}T^{4}V\sin^{-4}(H) + 64B^{3}PRS^{2}T^{3}V^{2}\sin^{-2}(H) + 64B^{3}PQ^{2}RT^{3}V^{2} - 32B^{3}P^{2}Q^{2}R^{2}T^{2}V \\
& -32B^{3}P^{2}R^{2}S^{2}T^{2}V\sin^{-2}(H) - 32B^{3}Q^{2}T^{4}V^{3} - 32B^{3}S^{2}T^{4}V^{3}\sin^{-2}(H) - 64B^{4}PRT^{3}V^{4} \\
& +96B^{4}P^{2}R^{2}T^{2}V^{3} - 64B^{4}P^{3}R^{3}TV^{2} + 16B^{4}P^{4}R^{4}V + 16B^{4}T^{4}V^{5} + 32Q^{2}R^{-4}S^{2}V\sin^{-2}(H) + 16Q^{4}R^{-4}V \\
& +16R^{-4}S^{4}V\sin^{-4}(H)
\end{aligned}
$$

```
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gc:[*list:1620{23%}; fixnum:4{23%}; ut':67%]

PSKI has 60 terms.

                   2 -3 2     -2          4 -3          -3 4     -4
PSKI =  -64BPQ R  S Vsin  (H) -32BPQ R  V -32BPR  S Vsin  (H)

     2 -4 2  2   -2            2 -2 4 3   -4          4 -4  2
+64BQ R  S TV sin  (H) +192BQ R  S T sin  (H) +32BQ R  TV

      4 -2 2 3   -2          6 -2 3        -4 4  2   -4          -2 6 3   -6
+192BQ R  S T sin  (H) +64BQ R  T  +32BR  S TV sin  (H) +64BR  S T sin  (H)

     2  2 -1 2 2    -2          2  4 -1 2      2  -1 4 2    -4
+384B PQ R  S T Vsin  (H) +192B PQ R  T V +192B PR  S T Vsin  (H)

     2 2 2 2    -2          2 2 4      2 2 4    -4          2 2 -2 2 3 2   -2
-128B P Q S Tsin  (H) -64B P Q T -64B P S Tsin  (H) -256B Q R  S T V sin  (H)

    2 2 4 5   -4          2 4 -2 3 2     2 4 2 5   -2          2 6 5
-96B Q S T sin  (H) -128B Q R  T V  -96B Q S T sin  (H) -32B Q T

     2 -2 4 3 2   -4          2 6 5   -6          3   4 4    -4
-128B R  S T V sin  (H) -32B S T sin  (H) -160B PRS T Vsin  (H)

     3  2  2 4    -2           3  2 -1 2 3      3  4  4
-320B PQ RS T Vsin  (H) -192B PQ R  T V  -160B PQ RT V

     3  -1 2 2 3   -2          3 2 2  2      3 2 2 2 2 3   -2          3 2 4 2 3
-192B PR  S T V sin  (H) +192B P Q TV  +128B P Q R S T sin  (H) +64B P Q R T

    3 2 2 4 3   -4          3 2 2  2   -2          3 3  2    -2          3 3 2
+64B P R S T sin  (H) +192B P S TV sin  (H) -64B P RS Vsin  (H) -64B P Q RV

    3 2 -2 3 4       3 2 2 5 2   -2         3 4 5 2      3 -2 2 3 4   -2
+64B Q R  T V  +192B Q S T V sin  (H) +96B Q T V  +64B R  S T V sin  (H)

    3 4 5 2   -4           4   2 4 3   -2          4  2  4 3      4 2 2 2 3 2
+96B S T V sin  (H) +320B PRS T V sin  (H) +320B PQ RT V  -384B P Q R T V

     4 2 2 2 3 2   -2          4 3 2 3 2      4 3 3 2 2    -2          4 4 2 4
-384B P R S T V sin  (H) +192B P Q R T V +192B P R S T Vsin  (H) -32B P Q R T

    4 4 4 2    -2          4 2 5 4     4 2 5 4   -2          5   4 5
-32B P R S Tsin  (H) -96B Q T V  -96B S T V sin  (H) -160B PRT V

     5 2 2 3 4      5 3 3 2 3      5 4 4  2      5 5 5      5 5 6
+320B P R T V  -320B P R T V  +160B P R TV  -32B P R V +32B T V

    2 -4 4    -4         4 -4 2    -2         6 -4       -4 6    -6
-96Q R  S Tsin  (H) -96Q R  S Tsin  (H) -32Q R  T -32R  S Tsin  (H)
```

```
Mar  9 11:17 1985  pski5.lam Page 1

gc:[*list:1620{24%}; fixnum:4{32%}; ut':93%]

PSKI has 70 terms.

                  2 -4 4 2      -4                4 -4 2 2      -2                6 -4 2
PSKI =  -5768Q R   S T Vsin   (H) -5768Q R   S T Vsin   (H) -1928Q R   T V

       -4 6 2      -6             2  2 -3 2  2      -2              2  4 -3  2
-1928R   S T Vsin   (H) -7688 PQ R   S TV sin   (H) -3848 PQ R   TV

      2  -3 4  2     -4             2 2 2 -2 2      -2              2 2 4 -2
-3848 PR   S TV sin   (H) +3848 P Q R   S Vsin   (H) +1928 P Q R   V

      2 2 -2 4      -4              2 2 -4 2 2 3      -2              2 2 -2 4 4      -4
+1928 P R   S Vsin   (H) +3848 Q R   S T V sin   (H) +5768 Q R   S T Vsin   (H)

      2 4 -4 2 3          2 4 -2 2 4      -2              2 6 -2 4
+1928 Q R   T V   +5768 Q R   S T Vsin   (H) +1928 Q R   T V

      2 -4 4 2 3      -4              2 -2 6 4      -6              3  2 -1 2 3 2      -2
+1928 R   S T V sin   (H) +1928 R   S T Vsin   (H) +15368 PQ R   S T V sin   (H)

      3  4 -1 3 2          3  -1 4 3 2      -4              3 2 2 2 2      -2
+7688 PQ R   T V   +7688 PR   S T V sin   (H) -7688 P Q S T Vsin   (H)

      3 2 4 2          3 2 4 2      -4              3 2 -2 2 4 3      -2
-3848 P Q T V -3848 P S T Vsin   (H) -7688 Q R   S T V sin   (H)

      3 2 4 6      -4              3 4 -2 4 3          3 4 2 6      -2              3 6 6
-1928 Q S T Vsin   (H) -3848 Q R   T V   -1928 Q S T Vsin   (H) -648 Q T V

      3 -2 4 4 3      -4              3 6 6      -6              4   4 5 2      -4
-3848 R   S T V sin   (H) -648 S T Vsin   (H) -3848 PRS T V sin   (H)

      4  2  2 5 2      -2              4  2 -1 3 4          4  4  5 2
-7688 PQ RS T V sin   (H) -7688 PQ R   T V   -3848 PQ RT V

      4  -1 2 3 4      -2              4 2 2 2 2 4      -2              4 2 2 2 3
-7688 PR   S T V sin   (H) +3848 P Q R S T Vsin   (H) +11528 P Q T V

      4 2 4 2 4          4 2 2 4 4      -4              4 2 2 2 3      -2
+1928 P Q R T V +1928 P R S T Vsin   (H) +11528 P S T V sin   (H)

      4 3  2  2      -2              4 3 2   2          4 4 2 2          4 4 2 2      -2
-7688 P RS TV sin   (H) -7688 P Q RTV   +1928 P Q R V +1928 P R S Vsin   (H)

      4 2 -2 4 5          4 2 2 6 3      -2              4 4 6 3          4 -2 2 4 5      -2
+1928 Q R   T V   +3848 Q S T V sin   (H) +1928 Q T V   +1928 R   S T V sin   (H)

      4 4 6 3      -4              5   2 5 4      -2              5  2  5 4          5 2 2 2 4 3
+1928 S T V sin   (H) +7688 PRS T V sin   (H) +7688 PQ RT V   -11528 P Q R T V

       5 2 2 2 4 3      -2              5 3 2 3 3 2          5 3 3 2 3 2      -2
-11528 P R S T V sin   (H) +7688 P Q R T V   +7688 P R S T V sin   (H)

      5 4 2 4 2          5 4 4 2 2      -2              5 2 6 5          5 2 6 5      -2
-1928 P Q R T V -1928 P R S T Vsin   (H) -1728 Q T V   -1928 S T V sin   (H)

      6   5 6          6 2 2 4 5          6 3 3 3 4          6 4 4 2 3          6 5 5  2
-3848 PRT V   +9608 P R T V   -12808 P R T V   +9608 P R T V   -3848 P R TV

     6 6 6          6 6 7          2 -6 4      -4              4 -6 2      -2              6 -6
+648 P R V +648 T V   +1920 R   S Vsin   (H) +1920 R   S Vsin   (H) +640 R   V

    -6 6      -6
+64R   S Vsin   (H)
```

APPENDIX A7

THE POISSON BRACKETS FOR KIMURA METRIC NUMBER 5

This appendix contains the computer output for the Poisson brackets, in case IIC of Kimura metrics with $C = r^2(a_1r^2 + a_2)^{-1}$ and $A = r^{-2}(a_1r^2 + a_2)^{-2}$.

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```
                 3  2        2 4       2 2       2 2    -2        3    2    2
PSKI =   -8BDPR  TV  +8BDP  R  V -4BQ  T  V -4BS  T  Vsin  (H) +4DV  -8B PRTV

   2 2 2      2 2 3     2 2 6     2 -2     -2 2    -2
+4B  P  R  V +4B  T V  +4D  P  R  V +4Q  R  V +4R  S Vsin  (H)
```

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PSKI has 39 terms.

```
                  3         2 3 2         3 2 2    -2              4        2 2 4
PSKI =  -40BDPRV  +24BDPQ R T V +24BDPR S T Vsin  (H) +40BDTV  -16BDP Q R T

      2 4 2    -2          2 -1        -1 2    -2          2 2 6  2      2 3 7
-16BDP R S Tsin  (H) -8BPQ R   V -8BPR  S Vsin  (H) +24BD P R TV  -24BD P R V

   2 -2  2        2 2 3   -2         4 3      -2 2  2   -2          4 3   -4
+8BQ R   TV  +16BQ S T sin  (H) +8BQ T   +8BR  S TV sin  (H) +8BS T sin  (H)

      2    -2          2          2  2       2  2   -2         2   3 2 3
-8DPRS Vsin  (H) -8DPQ RV -40DQ TV  -40DS TV sin  (H) -24B DPR T V

   2  2 4  2      2  3 5      2   2 2     -2        2  2  2      2 2 2 2
+48B  DP R TV  -24B DP R V +24B PRS T Vsin  (H) +24B PQ RT V -8B P Q R T

  2 2 2 2     -2        2 2 3 2      2 2 3 2    -2        3   2 3      3 2 2  2
-8B P R S Tsin  (H) -16B Q T V  -16B S T V sin  (H) -24B PRT V  +24B P R TV

   3 3 3     3 3 4     2  3 3     2 2 2 6     2 2 6 2    -2      3 3 9
-8B P R V +8B T V  -40D PR V  -8D P Q R T -8D P R S Tsin  (H) -8D P R V

    2 -2 2    -2        4 -2     -2 4    -4
-16Q R   S Tsin  (H) -8Q R   T -8R  S Tsin  (H)
```

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PSKI has 64 terms.

$$
\begin{aligned}
PSKI = {} & 320 B D P R S^{2} T V^{2} \sin^{-2}(H) + 320 B D P Q^{2} R T V^{2} + 64 B D P^{2} Q^{2} R^{2} V + 64 B D P^{2} R^{2} S^{2} V \sin^{-2}(H) \\
& - 480 B D Q^{2} T^{2} V^{3} - 480 B D S^{2} T^{2} V^{3} \sin^{-2}(H) - 576 B D^{2} P R^{3} T V^{4} - 32 B D^{2} P^{2} Q^{2} R^{6} T^{2} V + 576 B D^{2} P^{2} R^{4} V^{3} \\
& - 32 B D^{2} P^{2} R^{6} S^{2} T^{2} V \sin^{-2}(H) - 64 B D^{3} P^{3} R^{9} T V^{2} + 64 B D^{3} P^{4} R^{10} V - 64 B Q^{2} R^{-2} S^{2} T^{2} V \sin^{-2}(H) \\
& - 32 B Q^{4} R^{-2} T^{2} V - 32 B R^{-2} S^{4} T^{2} V \sin^{-4}(H) + 96 D Q^{2} R^{-2} V^{3} + 384 D Q^{2} S^{2} T^{2} V \sin^{-2}(H) + 192 D Q^{4} T^{2} V \\
& + 96 D R^{-2} S^{2} V^{3} \sin^{-2}(H) + 192 D S^{4} T^{2} V \sin^{-4}(H) - 576 B^{2} D P R T V^{4} + 64 B^{2} D P Q^{2} R^{3} T^{3} V^{2} \\
& + 64 B^{2} D P R^{3} S^{2} T^{3} V^{2} \sin^{-2}(H) - 64 B^{2} D P^{2} Q^{2} R^{4} T^{2} V + 288 B^{2} D P^{2} R^{2} V^{3} - 64 B^{2} D P^{2} R^{4} S^{2} T^{2} V \sin^{-2}(H) \\
& + 288 B^{2} D T^{2} V^{5} - 64 B^{2} P Q^{2} R^{-1} T V^{2} - 64 B^{2} P R^{-1} S^{2} T V^{2} \sin^{-2}(H) + 96 B^{2} D^{2} P^{2} R^{6} T^{2} V^{3} \\
& - 192 B^{2} D^{2} P^{3} R^{7} T V^{2} + 96 B^{2} D^{2} P^{4} R^{8} V + 32 B^{2} P^{2} Q^{2} V + 32 B^{2} P^{2} S^{2} V \sin^{-2}(H) + 32 B^{2} Q^{2} R^{-2} T^{2} V^{3} \\
& + 32 B^{2} Q^{2} S^{2} T^{4} V \sin^{-2}(H) + 16 B^{2} Q^{4} T^{4} V + 32 B^{2} R^{-2} S^{2} T^{2} V^{3} \sin^{-2}(H) + 16 B^{2} S^{4} T^{4} V \sin^{-4}(H) \\
& - 64 B^{3} D P R^{3} T^{3} V^{4} + 192 B^{3} D P^{2} R^{4} T^{2} V^{3} - 192 B^{3} D P^{3} R^{5} T V^{2} + 64 B^{3} D P^{4} R^{6} V \\
& + 64 B^{3} P R S^{2} T^{3} V^{2} \sin^{-2}(H) + 64 B^{3} P Q^{2} R T^{3} V^{2} - 32 B^{3} P^{2} Q^{2} R^{2} T^{2} V - 32 B^{3} P^{2} R^{2} S^{2} T^{2} V \sin^{-2}(H) \\
& - 32 B^{3} Q^{2} T^{4} V^{3} - 32 B^{3} S^{2} T^{4} V^{3} \sin^{-2}(H) - 64 B^{4} P R T^{3} V^{4} + 96 B^{4} P^{2} R^{2} T^{2} V^{3} - 64 B^{4} P^{3} R^{3} T V^{2} \\
& + 16 B^{4} P^{4} R^{4} V + 16 B^{4} T^{4} V^{5} + 384 D^{2} P Q^{2} R^{3} T V^{2} + 384 D^{2} P R^{3} S^{2} T V^{2} \sin^{-2}(H) + 32 D^{2} P^{2} Q^{2} R^{4} V \\
& + 32 D^{2} P^{2} R^{4} S^{2} V \sin^{-2}(H) + 80 D^{2} V^{5} + 288 D^{3} P^{2} R^{6} V^{3} + 16 D^{4} P^{4} R^{12} V + 32 Q^{2} R^{-4} S^{2} V \sin^{-2}(H) \\
& + 16 Q^{4} R^{-4} V + 16 R^{-4} S^{4} V \sin^{-4}(H)
\end{aligned}
$$

APPENDIX A8

THE POISSON BRACKET FOR KIMURA METRIC NUMBER 7

This appendix contains the computer output for the Poisson bracket $(\kappa,\iota) = \lambda$, in case III of Kimura metrics with $C = 1$ and $A = (a_1 r^2 + a_2)^{-3}$.

```
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PSKI has 31 terms.

              3  2     2 -2        2     2 -1     2 2 4 2      2 2      2 -1
PSKI =  -4BDPR TV (B+DR )    +4BPRTV (B+DR )   +12BD P R T V -2BP R V(B+DR )

      3  2     2 -1     2 4      2 -1   -3  2 2      2 -1
+12DPR TV (B+DR )   -6DP R V(B+DR )  -2B  DQ R V(B+DR )

   -3  2 2      2 -1   -2       -3  4 3     2 -1    -3 2 6 3     2 -2
-2B  DR S V(B+DR )  sin  (H) +4B  DR V (B+DR )   -4B  D R V (B+DR )

   -2  4 3     2 -2    -2 2 2 6     -2 3  7  2     2 -1    -2 3 2 8      2 -1
-4B  DR V (B+DR )   +8B  D P R V +4B  D PR TV (B+DR )   -2B  D P R V(B+DR )

   -2 4  9  2     2 -2    -2 2      2 -1    -2 2 3     2 -1
-4B  D PR TV (B+DR )   -2B  Q V(B+DR )   +4B  R V (B+DR )

   -2 2      2 -1   -2        -1  2 4     -1  2 2     -1  2 2   -2
-2B  S V(B+DR )  sin  (H) +16B  DP R V +4B  DQ T V +4B  DS T Vsin  (H)

    -1 2  5  2     2 -1    -1 2 2 6      2 -1     -1 3  7  2     2 -2
+12B  D PR TV (B+DR )   -6B  D P R V(B+DR )   -12B  D PR TV (B+DR )

    2  2 2 2     3 2 2      2  5  2     2 -2    3 2 6 2     2 2     2 -2 2
+12B DP R T V +4B P T V -12D PR TV (B+DR )   +4D P R T V +8P R V +4Q R  T V

   -2 2 2    -2
+4R  S T Vsin  (H)
```

APPENDIX A9

THE POISSON BRACKETS FOR KIMURA METRIC NUMBER 8

This appendix contains the computer output for the Poisson brackets, in case III of Kimura metrics with C = 1 and $A = a_1^{-3} r^{-6}$.

```
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           3 2 6    2 -2     -2 2    -2      3
PSKI = 16D P R V +4Q R  V +4R  S Vsin  (H) +8V
```

```
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              3/2   2    -2          3/2  2          3/2  3 3     3/2 2  2
PSKI =  -16D   PRS Vsin  (H) -16D    PQ RV -160D    PR V  -80D    Q TV

    3/2 2 -2 2    -2        3/2 4 -2      3/2 -2 4     -4
-16D   Q R  S Tsin  (H) -8D   Q R  T -8D   R  S Tsin  (H)

    3/2 2  2   -2        9/2 2 2 6      9/2 2 6 2     -2        9/2 3 9
-80D   S TV sin  (H) -32D   P Q R T -32D   P R S Tsin  (H) -64D   P R V
```

```
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              3  2 3  2          3  3 2  2   -2          3 2 2 4
PSKI = 1536D PQ R TV  +1536D PR S TV sin  (H) +128D P Q R V

     3 2 4 2    -2          3 2 6 3     3 2 2 2    -2          3 4 2
+128D P R S Vsin  (H) +2304D P R V  +768D Q S T Vsin  (H) +384D Q T V

     3 4 2    -4        6 4 12      2 -4 2    -2           2 -2 3    4 -4
+334D S T Vsin  (H) +256D P R  V +32Q R  S Vsin  (H) +192Q R  V  +16Q R  V

   -4 4    -4       -2 2 3   -2        5
+16R  S Vsin  (H) +192R  S V sin  (H) +320V
```

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PSKI has 33 **terms.**

```
                  3/2   2 3    -2              3/2  2  3        3/2  2 -1 2     -2
PSKI =  -5888D   PRS V sin  (H) -5888D   PQ RV  -128D   PQ R  S Vsin  (H)

     3/2  4 -1       3/2  -1 4     -4             3/2  3 5      3/2 2  4
-64D   PQ R  V -64D   PR  S Vsin  (H) -15616D  PR V  -7808D   Q TV

     3/2 2 -4 4     -4       3/2 2 -2 2  2    -2       3/2 4 -4 2     -2
-96D   Q R  S Tsin  (H) -5888D   Q R  S TV sin  (H) -96D   Q R  S Tsin  (H)

       3/2 4 -2  2     3/2 6 -4      3/2 -4 6    -6
-2944D   Q R  TV  -32D   Q R  T -32D   R  S Tsin  (H) -

       3/2 -2 4  2    -4        3/2 2  4    -2           9/2  2 3 2 2    -2
-2944D   R  S TV sin  (H) -7808D   S TV sin  (H) -9216D   PQ R S T Vsin  (H)

       9/2  4 3 2         9/2  3 4 2     -4        9/2 2 2 4 2    -2
-4608D   PQ R T V -4608D   PR S T Vsin  (H) -512D   P Q R S Tsin  (H)

         9/2 2 2 6  2       9/2 2 4 4       9/2 2 4 4     -4
-20992D   P Q R TV  -256D   P Q R T -256D   P R S Tsin  (H)

         9/2 2 6 2  2    -2          9/2 3 2 7        9/2 3 7 2     -2
-20992D   P R S TV sin  (H) -512D   P Q R V -512D   P R S Vsin  (H)

         9/2 3 9 3        9/2 2 4 3    -4          9/2 4 2 3    -2         9/2 6 3
-29696D   P R V  -2304D   Q S T sin  (H) -2304D   Q S T sin  (H) -768D   Q T

       9/2 6 3    -6         15/2 4 2 12        15/2 4 12 2     -2
-768D   S T sin  (H) -512D     P Q R  T -512D     P R  S Tsin  (H)

        15/2 5 15
-1024D    P R  V
```

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gc:[list:1620(22%); *fixnum:14(12%); ut!:0%]

PSKI has 37 terms.

```
                3   4  2    -4                3  2  2  2    -2                3  2 3  4
PSKI = 67584D PRS TV sin  (H) +135168D PQ RS TV sin  (H) +380928D PQ R TV

        3  4    2            3  3 2  4   -2             3 2 2 2 2    -2
+67584D PQ RTV  +380928D PR S TV sin  (H) +1536D P Q R S Vsin  (H)

         3 2 2 4 3        3 2 4 2         3 2 2 4    -4
+115712D P Q R V  +768D P Q R V +768D P R S Vsin  (H)

         3 2 4 2 3    -2             3 2 6 5         3 2 -2 4 2    -4
+115712D P R S V sin  (H) +4904960 P R V  +506880 Q R  S T Vsin  (H)

         3 2 2 2 3    -2            3 4 -2 2 2     -2            3 4 2 3
+1204640 Q S T V sin  (H) +506880 Q R  S T Vsin  (H) +95232D Q T V

        3 6 -2 2          3 -2 6 2    -6            3 4 2 3    -4
+168960 Q R  T V +168960 R  S T Vsin  (H) +95232D S T V sin  (H)

         6 2 2 6 2 2    -2            6 2 4 6 2          6 2 6 4 2    -4
+135168D P Q R S T Vsin  (H) +67584D P Q R T V +67584D P R S T Vsin  (H)

         6 3 2 9  2           6 3 9 2  2    -2            6 4 2 10
+270336D P Q R TV  +270336D P R S TV sin  (H) +3072D P Q R  V

       6 4 10 2    -2             6 4 12 3        9 6 18       2 -6 4    -4
+3072D P R  S Vsin  (H) +3665920 P R  V  +4096D P R  V +192Q R  S Vsin  (H)

        2 -4 2 3    -2           2 -2 5       4 -6 2    -2           4 -4 3
+12032Q R  S V sin  (H) +27392Q R  V  +192Q R  S Vsin  (H) +6016Q R  V

      6 -6      -6 6    -6           -4 4 3    -4            -2 2 5    -2
+64Q R  V +64R  S Vsin  (H) +6016R  S V sin  (H) +27392R  S V sin  (H)

       7
+31232V
```

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PSKI has 73 terms.

```
                  3/2   2 5   -2                 3/2  2  5
PSKI =  -1772544D   PRS V sin  (H) -1772544D   PQ RV

     3/2  2 -3 4    -4                3/2  2 -1 2 3   -2
-768D   PQ R  S Vsin  (H) -420864D   PQ R  S V sin  (H)

     3/2  4 -3 2    -2                3/2  4 -1 3      3/2  6 -3
-768D   PQ R  S Vsin  (H) -210432D   PQ R  V  -256D   PQ R  V

     3/2  -3 6    -6                3/2  -1 4 3   -4             3/2  3 7
-256D   PR  S Vsin  (H) -210432D   PR  S V sin  (H) -2836480D   PR V

         3/2 2  6       3/2 2 -6 6    -6             3/2 2 -4 4  2   -4
-1418240D   Q TV  -512D   Q R  S Tsin  (H) -3156480   Q R  S TV sin  (H)

         3/2 2 -2 2  4   -2          3/2 4 -6 4    -4
-1772544D   Q R  S TV sin  (H) -768D   Q R  S Tsin  (H)

        3/2 4 -4 2  2   -2             3/2 4 -2  4       3/2 6 -6 2    -2
-3156480   Q R  S TV sin  (H) -8862720   Q R  TV  -512D   Q R  S Tsin  (H)

        3/2 6 -4  2      3/2 8 -6      3/2 -6 8    -8
-1052160   Q R  TV  -128D   Q R  T -128D   R  S Tsin  (H)

        3/2 -4 6  2   -6             3/2 -2 4  4   -4
-1052160   R  S TV sin  (H) -8862720   R  S TV sin  (H)

         3/2 2  6   -2          9/2    6 2    -6
-1418240D   S TV sin  (H) -202752D   PRS T Vsin  (H)

        9/2  2  4 2    -4              9/2  2 3 2 2 3   -2
-6082560   PQ RS T Vsin  (H) -8478720D   PQ R S T V sin  (H)

        9/2  4  2 2    -2              9/2  4 3 2 3          9/2  6  2
-6082560   PQ RS T Vsin  (H) -4239360D   PQ R T V  -202752D   PQ RT V

         9/2  3 4 2 3   -4          9/2 2 2 2 4    -4
-4239360D   PR S T V sin  (H) -4608D   P Q R S Tsin  (H)

         9/2 2 2 4 2  2   -2              9/2 2 2 6  4
-24944640   P Q R S TV sin  (H) -120238080   P Q R TV

      9/2 2 4 2 2    -2             9/2 2 4 4  2        9/2 2 6 2
-4608D   P Q R S Tsin  (H) -12472320   P Q R TV  -1536D   P Q R T

      9/2 2 2 6    -6               9/2 2 4 4  2   -4
-1536D   P R S Tsin  (H) -12472320   P R S TV sin  (H)

          9/2 2 6 2  4   -2          9/2 3 2 5 2    -2
-120238080   P R S TV sin  (H) -6144D   P Q R S Vsin  (H)

         9/2 3 2 7 3       9/2 3 4 5       9/2 3 5 4    -4
-1953792D   P Q R V  -3072D   P Q R V -3072D   P R S Vsin  (H)
```

```
         9/2 3 7 2 3   -2                9/2 3 9 5         9/2 2 -2 6 3   -6
-1953792D   P R S V sin  (H) -12742656D   P R V  -135168D   Q R  S T sin  (H)

         9/2 2 4 3 2   -4               9/2 4 -2 4 3   -4
-2119680D   Q S T V sin  (H) -202752D   Q R  S T sin  (H)

         9/2 4 2 3 2   -2               9/2 6 -2 2 3   -2             9/2 6 3 2
-2119680D   Q S T V sin  (H) -135168D   Q R  S T sin  (H) -706560D   Q T V

       9/2 8 -2 3        9/2 -2 8 3   -8             9/2 6 3 2   -6
-33792D   Q R  T  -33792D   R  S T sin  (H) -706560D   S T V sin  (H)

        15/2 2 2 6 4 3   -4             15/2 2 4 6 2 3   -2
-405504D    P Q R S T sin  (H) -405504D    P Q R S T sin  (H)

        15/2 2 6 6 3        15/2 2 6 6 3   -6
-135168D    P Q R T  -135168D    P R S T sin  (H)

         15/2 3 2 9 2 2    -2             15/2 3 4 9 2
-1622016D    P Q R S T Vsin  (H) -811008D    P Q R T V

        15/2 3 9 4 2    -4             15/2 4 2 10 2    -2
-811008D    P R S T Vsin  (H) -12288D    P Q R  S Tsin  (H)

         15/2 4 2 12  2      15/2 4 4 10        15/2 4 10 4    -4
-3305472D    P Q R  TV  -6144D    P Q R  T -6144D    P R  S Tsin  (H)

         15/2 4 12 2  2   -2            15/2 5 2 13
-3305472D    P R  S TV sin  (H) -12288D    P Q R  V

       15/2 5 13 2    -2              15/2 5 15 3       21/2 6 2 18
-12288D    P R  S Vsin  (H) -4448256D    P R  V  -8192D    P Q R  T

      21/2 6 18 2    -2            21/2 7 21
-8192D    P R  S Tsin  (H) -16384D    P R  V
```

APPENDIX 10

TOLMAN-EHRENFEST FORMULA FOR THE THERMOSTATIC EQUILIBRIUM OF GRAVITATING MATTER.

A well known principle of classical thermodynamics is that a state of thermal equilibrium of a system would necessarily be characterized by uniformity of temperature. This result has been extended by Tolman and Ehrenfest [14] in the general theory of relativity. They have proved that the distribution of proper temperature T of matter in thermodynamic equilibrium situated in a static gravitational field whose metric is

$$ds^2 = g_{ik}dx^i dx^k - g_{44}(dx^4)^2 \qquad \text{(A10.1)}$$

with i,k taking the values 1,2,3, x^4 being the time-coordinate and $\partial g_{ik}/\partial t = \partial g_{44}/\partial t = 0$, is described by the relation

$$T\sqrt{-g_{44}} = \text{constant} \qquad \text{(A10.2)}$$

the constant being the same in all parts of the matter distribution.

Thus a system is in thermostatic equilibrium in general relativity not when the temperature gradient vanishes but rather when the gradient of $T\sqrt{-g_{44}}$ becomes zero.

A new proof of the Tolman-Ehrenfest formula (A10.2) will be given in what follows. Starting by considering the metric as static and spherically symmetric, we assume the existence of a unique vector field u^μ representing the average velocity of matter at each point of space-time, which satisfies the normalization condition $u^\mu u_\mu = -1$ (a time-like vector) and is given, in the comoving frame, by

$$u^\mu = (0,\ 0,\ 0,\ \frac{1}{\sqrt{-g_{44}}}). \tag{A10.3}$$

We define the projection tensor [15] by

$$h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu. \tag{A10.4}$$

It determines at each point a {3+1} splitting of space-time into {space + time} and satisfies $h_{\mu\nu}u^\nu = 0$. We also define the acceleration vector a^ν by

$$a^\nu = \dot{u}^\nu := u^\nu_{\ ;\mu}\, u^\mu. \tag{A10.5}$$

It is a space-like vector and satisfies $\dot{u}^\mu u_\mu = 0$.

Using the velocity vector field u^μ we can split the energy-momentum tensor as follows

$$T^{\mu\nu} = \rho u^\mu u^\nu + S^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu) \tag{A10.6}$$

where ρ is the density of matter, $S^{\mu\nu}$ is the stress tensor and q^μ is the energy flux relative to u^μ, which represents processes such as diffusion and/or heat conduction and satisfies $q^\mu u_\mu = 0$.

We may obtain the fluid temperature distribution from the heat flux q^μ if we use the following equation due to Eckart [16]

$$q^{\mu} = - \chi(T,x^{i},t)\, h^{\mu\nu}\, (T_{,\nu} + Ta_{\nu}) \tag{A10.7}$$

where $T(x^{i},t)$ is the temperature, $\chi > 0$ is the thermal conductivity of the fluid, $h^{\mu\nu}$ and a^{μ} the projection tensor and the acceleration vector respectively. The term $h^{\mu\nu}T_{,\nu}$ is obviously the relativistic temperature gradient. The term $- Ta_{\nu}$ has no classical analog and implies a flow of heat in accelerated matter in a direction opposite to the acceleration.

In cases with vanishing heat flow, $q^{\mu} = 0$, we have from equation (A10.7) that

$$T_{,\nu} + Ta_{\nu} = 0. \tag{A10.8}$$

Then, for a static spherically symmetric metric and with the use of the velocity vector (A10.3) the only non-vanishing component of the acceleration vector is

$$a_{1} = \frac{1}{2}\, (-g^{44})(-g_{44})_{,1} \quad . \tag{A10.9}$$

Going back into (A10.8), we then have

$$T_{,1} + \frac{1}{2}\, \left(\frac{T}{-g_{44}}\right)\, (-g_{44})_{,1} = 0 \tag{A10.10}$$

from which, after integration, it follows that

$$T\, \sqrt{-g_{44}} = \text{constant.}$$

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CORRECTIONS

CHAPTER 1

page 26 , line 14 : $n_\mu l^\mu > 0$ (instead of $n^\mu l^\mu > 0$)
page 78 , line 14 : (1.5.15) (instead of (1.4.15))
page 79 , line 3 : $Q_\alpha = \nabla_\alpha Q$ (instead of $Q_\alpha = \nabla_\alpha A$)

CHAPTER 2

page 107, line 11 : (2.3.12) (instead of (2.3.15))
page 121, line 6 : emphasized and as a result of it all equations written here
page 121, line 16 : (..... - .....) $p_\mu p_\nu p_\rho$
page 125, line 25 : 48 (instead of 49)

CHAPTER 3

page 145, line 6 : or a symbol (instead of of a symbol)
page 148, line 13 : (PLUS X 2) (instead of (PLUS x 2))

CHAPTER 4

page 195, line 10 : $\frac{2M}{R} < \frac{8}{9}$ (instead of $\frac{2M}{R} > \frac{8}{9}$)
page 232, line 12 : question (instead of equation)

APPENDIX A1

page 242, table :

	Form of Ψ_{ABCD}
211	$\Psi_{ABCD} = \alpha_{(A}\alpha_B \gamma_C \delta_{D)}$
22	$\Psi_{ABCD} = \alpha_{(A}\alpha_B \gamma_C \gamma_{D)}$
31	$\Psi_{ABCD} = \alpha_{(A}\alpha_B \alpha_C \delta_{D)}$
4	$\Psi_{ABCD} = \alpha_{(A}\alpha_B \alpha_C \alpha_{D)}$

APPENDIX A2

page 243, line 16 : mapped (instead of wrapped)

APPENDIX A3

page 251, line 10 : $\frac{\partial S}{\partial \lambda}$ (instead of $\frac{\partial S}{\partial \delta}$)