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THEORETICAL STUDIES OF WAVE PROPAGATION
IN
DISPERSIVE MEDIA

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A thesis submitted to the
University of London
for the degree of Doctor of Philosophy.
March 1969

ABSTRACT

The theory of dispersive waves in linear and nonlinear systems is discussed in Chapter I. Particular attention is given to the classical theory of geometrical optics and its recent extension to finite amplitude waves by Whitham. Karpman's method for dealing with moderate amplitude systems is also presented and discussed in relation to the Whitham theory.

Chapter II deals with a detailed application of Whitham's theory to open-channel steady flow of deep water past a slowly modulated wavy wall. The calculation is based on the method of imaginary characteristics, the appearance of a "phase jump" in the solution being compared with previous theoretical and experimental results. Such phase jumps can occur in a wide range of similar problems. Chapter III may be regarded as a lead-in to the more general problem of phase jumps, and uses the principle of conservation of energy to predict the possible position of the discontinuity encountered in the previous chapter.

Chapter IV contributes to the theory of the intense heating of the Solar atmosphere which occurs above a height of about 1000 km. A mechanism is examined which involves the concept of "tongues of turbulence" which intermittently protrude above the convective zone and generate gravity waves in the stratified atmosphere. In regions of strong magnetic field disturbances created by these tongues may be dragged clear of the convective zone and propagated in the form of Alfvén waves. The distortion of the vortex loops in such a propagating wave is described in the case where the initial disturbance is a Hill's spherical vortex. Estimates of the Joule heating of the atmosphere in this case are used to discuss the relation of the above mechanism to the small scale granules and the spicules.

Acknowledgements

It is a great pleasure to record here my indebtedness to Professor M.J.Lighthill, Sec.R.S., who has supervised the research presented in this thesis and has been a constant source of encouragement and stimulating ideas. I would also like to thank Dr.C.I.Devanathan of the Indian Institute of Science, Bangalore, for many valuable discussions on the theory of wave propagation.

Financial support was provided by a grant from the Central Electricity Generating Board, which is gratefully acknowledged.

M.S.H.

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INTRODUCTION

Summary

This chapter is primarily concerned with a brief review and discussion of those notions in the theory of wave propagation which are used and developed in the present thesis. In §1 the linear theory of wave propagation in dispersive media is considered. This is relevant to the discussion of Karpman's (1968) theory of moderate amplitude slowly varying wave trains, given in §3, and also to the calculations of Chapters II and IV. The major part of this thesis deals with a detailed application of Whitham's theory of finite amplitude wave trains, and the essentials of this theory constitute the discussion of §2. Karpman's method is a modification applicable when the waves are of moderate amplitude. A short note on "wave action" (Bretherton & Garrett 1968) is also included in §2.

1.1 Linear theory of wave dispersion

The theory of wave propagation governed by linear systems of equations is now a well established part of the literature. This section is devoted to a brief review of the fundamental ideas concerning waves which propagate in a dispersive medium.

Consider the wave field produced by a localized disturbance. If the disturbance is analyzed into components of different wavelength, then the medium is said to be dispersive if these components propagate at different velocities. This occurs when the equation connecting the frequency and wavenumber (the dispersion relation) is not linear. Ultimately the disturbance spreads out to fill a large region of space in such a manner that locally it may be regarded as an almost periodic wave train. Let $(\underline{x}, t)$ be the rectangular Cartesian coordinates of a point in space-time. Then in a region V_0 of space-time whose dimensions are large compared to the wavelength and wave period, but small compared to the region of space-time occupied by the wave train, the waveform may be represented by

$$\psi = a(\underline{x}, t) e^{i(\underline{k}_0 \underline{x} - \omega_0 t + \phi)} \quad (1)$$

Now a uniform wave train has constant amplitude a , and its phase is equal to $\underline{k}_0 \underline{x} - \omega_0 t$, where $\underline{k}_0$ and ω_0 are respectively the wavenumber and frequency. The solution represented by (1) may therefore be regarded as a monochromatic wave subject to

both amplitude and phase modulation, the scales of the modulating effects being large compared to the wavelength and wave period. At any point in the wave train the wavenumber differs from its unmodulated value by an amount equal to $\partial\phi/\partial\underline{x}$, and the frequency by an amount equal to $-\partial\phi/\partial t$, where ϕ is the phase modulation term in the solution.

Let the linear equation governing the wave propagation have the form

$$F\left(-i\frac{\partial}{\partial\underline{x}}, i\frac{\partial}{\partial t}, \underline{x}, t\right)\psi = 0, \quad (2)$$

where the space and time scales of the differential operator F are large compared to a typical wavelength and wave period. Substitute a solution of the form (1) in equation (2) to obtain

$$F\left(\underline{k}_0 - i\frac{\partial}{\partial\underline{x}}, \omega_0 + i\frac{\partial}{\partial t}, \underline{x}, t\right)ae^{i\phi} = 0. \quad (3)$$

Then in a region of space-time where the wave train may be regarded as a weak modulation of a wave of wavenumber $\underline{k}_0$ and frequency ω_0 , the phase ϕ , as well as the amplitude a , will be a slowly varying function. The operator F may then be expanded in powers of $\partial/\partial\underline{x}$ and $\partial/\partial t$. If only the first small term in this expansion is retained, and the real and imaginary parts equated to zero, one obtains (cf. de Broglie, 1930)

$$F(\underline{k}, \omega, \underline{x}, t) = 0$$

(4)

$$\frac{\partial a^2}{\partial t} + \frac{\partial \omega}{\partial \underline{k}_0} \frac{\partial a^2}{\partial \underline{x}} = 0$$

where the group velocity

$$\frac{\partial \omega}{\partial \underline{k}_0} = - \frac{\partial F / \partial \underline{k}}{\partial F / \partial \omega}$$

is evaluated at $\underline{k} = \underline{k}_0$, $\omega = \omega_0$.

In the case where t does not appear explicitly in the differential operator F , a^2 may be regarded as a measure of the wave energy density.* The second of equations (4) states that changes in a^2 propagate at the group velocity. The first equation is the dispersion relation, already mentioned. If the total phase in the solution (1) is denoted by θ , then the wavenumber and frequency are given by

$$\underline{k} = \frac{\partial \theta}{\partial \underline{x}}, \quad \omega = - \frac{\partial \theta}{\partial t}. \quad (5)$$

Hence, if the dispersion equation is solved as an equation for ω (each eigenfrequency corresponding to a separate species of wave), then the frequency is given by an expression of the form

$$\frac{\partial \theta}{\partial t} + H\left(\frac{\partial \theta}{\partial \underline{x}}, \underline{x}, t\right) = 0. \quad (6)$$

This is the Hamilton-Jacobi partial differential equation familiar in the theory of classical mechanics. Along the

*provided that the medium is at rest.

characteristics of equation (6), viz., those curves which satisfy

$$\frac{dx}{dt} = \frac{\partial H}{\partial k}, \quad (7)$$

the wavenumber and frequency vary in accordance with Hamilton's equations

$$\frac{dk}{dt} = - \frac{\partial H}{\partial x}, \quad \frac{d\omega}{dt} = \frac{\partial H}{\partial t}. \quad (8)$$

Equations (7) and (8) may be regarded as the equations of motion of a "wave packet" of frequency ω and wavenumber k .

The present approximation, which is valid at large distances from the initial disturbance, therefore corresponds to the approximation of geometrical optics (or W.K.B. approximation; Heading (1962), Frömen & Frömen (1965)), and the procedure is the reverse of that often adopted in deriving the equations of wave mechanics (in which the wave operator F is deduced from the Hamilton-Jacobi partial differential equation).

1.2 Damped waves

For waves propagating in a dissipative medium the frequency determined by the dispersion equation is no longer real, but possesses a negative imaginary component. Consider, for example, damped waves propagating in the direction of the x -axis in a homogeneous, time independent

medium. The complex frequency Ω has the form

$$\Omega = \omega(k) - i\gamma(k), \quad (9)$$

and the damping rate will be small provided that

$$\gamma(k) \ll \omega(k).$$

The characteristics of the Hamilton-Jacobi equation are now complex, so that setting

$$x = X + iY,$$

where X and Y are both real, the process of dissipation may be imagined as one in which energy is steadily radiated out of the real (X, t) -plane into the three dimensional (X, Y, t) -space.

Along a characteristic

$$\frac{dx}{dt} = \frac{dX}{dt} + i\frac{dY}{dt} = \omega'(k) - i\gamma'(k);$$

hence the characteristic makes an angle with the real plane whose tangent is given by

$$\frac{dY}{dX} = - \frac{\gamma'(k)}{\omega'(k)}. \quad (10)$$

Provided that this angle is sufficiently small one may continue to regard changes in the wave field as being propagated along the rays

$$\frac{dX}{dt} = \omega'(k) \quad (11)$$

in the real plane. (Details are given by Lighthill, 1965a.)

The half life of a wave packet which experiences such light damping, that is, the time in which its energy is diminished to half of its initial value, is equal to

$$\frac{\log_e 2}{2\gamma(k)} .$$

It follows that the distance travelled by the wave packet during this time - the 'median dissipation length' - is given by

$$\Lambda = \frac{\log_e 2}{2\gamma(k)} \cdot \omega'(k), \quad (12)$$

and is large if the damping is light. This notion will be given an approximate generalization in Chapter IV, in which the problem of Alfvén wave propagation in a compressible, stratified atmosphere is considered.

In the notation of Chapter IV, for large but finite conductivity σ , the dispersion relation for Alfvén waves propagating along the x-axis (which is taken parallel to the magnetic field) has the form (Alfvén & Fälthammar 1963)

$$\Omega = V k - \frac{ic^2 k^2}{8\pi\mu\sigma},$$

where V is the Alfvén velocity. The median dissipation length is therefore given by

$$\Lambda = \frac{4\pi\mu\sigma V \log_e 2}{c^2 k^2} = \frac{\mu\sigma \lambda_V^2 \log_e 2}{\pi c^2} \quad (13)$$

where the wavelength $\lambda = 2\pi/k$.

2.1 Whitham's theory of almost periodic finite amplitude wave trains

Whitham (1965) has given what is essentially an extension to finite amplitude wave trains of the geometrical optics approximation discussed in §1. It requires that, when the various coefficients determining the slowly varying properties of the medium are held constant, the set of nonlinear equations which describe the wave field admit of plane periodic solutions depending only on the phase $\theta = \underline{k} \cdot \underline{x} - \omega t$, where the wavenumber and frequency are constant. In general, a given problem will require the introduction of further wave parameters such as the amplitude, as well as certain 'pseudo-frequencies' which arise whenever the wave field is described by means of one or more potential functions. All of these extra parameters - or adiabatic invariants - are, of course, constant for the plane periodic solution.

For example, in the case of large amplitude waves propagating on water of finite depth, the plane periodic solution (Stokes's wave) may be expressed in the form (Whitham, 1967)

$$\left. \begin{aligned} \theta &= k_1 x_1 - \omega t \\ \phi &= \beta_1 x_1 - \gamma t + \Phi(\theta, x_3) \\ h &= H(\theta) \end{aligned} \right\} \quad (14)$$

(where the repeated index implies summation over $i = 1, 2$ only, i.e., over horizontal coordinates). Here ϕ is the velocity potential, h the elevation of the free surface, and x_3 the vertical space coordinate. The wavenumber $\underline{k}$ and frequency ω are constant, as are the pseudo-frequencies β_i and γ . The mean stream velocity is equal to β_i , while $-\gamma$ may be interpreted as the constant which appears on the right hand side of Bernoulli's equation (Lamb 1932, page 19):

$$\frac{p}{\rho} + \frac{1}{2} \left(\frac{\partial \phi}{\partial \underline{x}} \right)^2 - \frac{\partial \phi}{\partial t} = -\gamma. \quad (15)$$

Two further constants must be introduced to complete the specification of the wave profile. These consist of a measure of the wave amplitude a , say, and the mean elevation of the free surface above its undisturbed level, denoted by b .

Whitham's procedure is to generalize this solution to deal with situations in which the wave parameters vary slowly in space and time; more precisely, to those situations in which their proportional change is small over distances of the order of a wavelength and times of the order of a wave period. That is, the waveform is assumed to be locally a close approximation to the exactly periodic case, and the definitions of k_i , ω , β_i , γ are generalized to

$$\left. \begin{aligned} k_i &= \frac{\partial \theta}{\partial x_i} , & \omega &= - \frac{\partial \theta}{\partial t} \\ \beta_i &= \frac{\partial \psi}{\partial x_i} , & \gamma &= - \frac{\partial \psi}{\partial t} \end{aligned} \right\} \quad (16)$$

the potential ϕ now being defined in terms of θ and ψ by

$$\phi = \psi + \Phi(\theta, x_3).$$

The adiabatic invariants are slowly varying functions of x_i and t , and the theory derives the equation governing their evolution by an application of Hamilton's principle in an averaged form:

$$\oint \mathcal{L}(k_i, \omega, a; \beta_i, \delta, b) dx_1 dx_2 dt = 0. \quad (17)$$

The average Lagrangian density, $\mathcal{L}$, is equal to the mean value of the of the exact Lagrangian per unit horizontal surface area of water over one wavelength of the plane periodic solution. In (17) admissible variations vanish on the boundaries of the region of integration and satisfy the constraints imposed by (16).

It is customary to adopt an Eulerian description of fluid motion, and it is then not sufficient to take the Lagrangian density to be equal to the difference between the kinetic and potential energies per unit area. The difficulty arises because it is also necessary to take account of the boundary conditions on the free surface and on the bottom of the stream, which would involve the introduction of Lagrange multipliers. Luke (1967) has given a more convenient 'Lagrangian' which automatically incorporates the boundary conditions into the variational principle, so that they follow as a consequence of the Euler equations. This is given by

$$L = \int_0^{h(x_1, t)} \left\{ \frac{\partial \phi}{\partial t} + \frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 + g x_3 \right\} dx_3 . \quad (18)$$

The difficulties mentioned above have been discussed at length by Seliger & Whitham (1968), who make extensive use of Clebsch potentials.

Whitham's averaging technique may be justified by the following simple argument:

For an almost periodic wave train two space-time scales are important. The slow variations in the adiabatic invariants are governed by a scale of order $1/\varepsilon$ ($\varepsilon \ll 1$), while the rapid local fluctuations result from the variation of the phase θ over successive intervals of width 2π , say, and are associated with scales of order 1 . The averaging procedure essentially filters out these high frequency 'ripples' from the Lagrangian, so that the resulting Euler equations describe the evolution of the wave parameters over distances and times of the order $1/\varepsilon$.

When the wave train is locally a good approximation to an exactly periodic solution, the exact Lagrangian density (18) may be expressed as a function of the slowly varying adiabatic invariants and of the phase θ . The exact variational principle is then

$$\delta \iiint L(k_1, \omega, a; \beta_1, \gamma, b; \theta) dx_1 dx_2 dt = 0 , \quad (19)$$

and if the proportional change in the adiabatic invariants over a wavelength is small, the space integration may be

performed by writing the integral as a sum of integrals each of which is taken over a small rectangular surface element of the water having two adjacent wavecrests as opposite sides, and over which the wave parameters may be held fixed at their local values. In other words, the space integration in (19) is approximated by the space integral of the average value of L over a wavelength. This idea has been developed further by Bretherton (1968), who gives a careful discussion of the approximation and shows that the error involved is $O(\epsilon^2)$.

Luke (1966) has given a formal justification of Whitham's approximation for exact variational principles of the form

$$\oint \int \int L(\phi, \phi_t, \phi_x, \underline{x}, t, \lambda) d\underline{x} dt = 0, \quad (20)$$

where L is the exact Lagrangian density and λ denotes one or more external parameters which may be required to specify slowly varying properties of the medium. His method is a generalization of the well known procedure often adopted in the derivation of the equations of geometrical optics (see, for example, Frömen & Frömen 1965). Thus the solution of the wave problem is assumed to possess an asymptotic expansion of the form

$$\phi(\theta, X_1, T) = \sum_{n=0}^N \epsilon^n \phi_n(\theta, X_1, T),$$

where X_1, T are stretched coordinates defined by $\epsilon x_1 = X_1$, $\epsilon t = T$. The wave parameters are assumed to be functions of

the stretched coordinates alone, i.e., to have a length scale $O(1/\epsilon)$, as before, while the rapid local variations of the solution are determined by the variation of the phase θ , whose definition is unchanged. The zeroth order solution ϕ_0 corresponds to the locally periodic solution obtained by Whitham's method, and, for fixed X_1 and T , is just the exactly periodic function of θ which is used in the evaluation of the average Lagrangian density.

Luke shows how the higher order corrections may be obtained for waves described by the nonlinear Klein-Gordon equation

$$\phi_{tt} - \phi_{xx} + V'(\phi) = 0.$$

The iteration procedure is rather effective for this equation because the solution ϕ_0 is symmetrical about the zeroes of $\partial\phi_0/\partial\theta$, enabling the higher order equations to be simplified. However, the zeroth order solution of the more general case described by (20) does not usually possess this property, so that no such simplification of the higher order expressions would normally be expected.

A systematic expansion procedure has also been given by Bisshopp (1968). This depends on the fact that the variational principle

$$\delta \iint L(\phi, \epsilon \phi_{X_1}, \epsilon \phi_T, X_1, T, \epsilon) dX dT = 0,$$

which is the form assumed by (20) when written in terms of the stretched coordinates X_1, T , is equivalent to the variational principle

$$\delta \iint dX_1 dT \int d\theta L(\phi, P_{X_1} \phi_{\theta} + \epsilon \phi_{X_1}, P_T \phi_{\theta} + \epsilon \phi_T, X_1, T, \epsilon) = 0.$$

The potential ϕ and phase θ are defined by

$$\phi = \phi(\theta, X_1, T, \epsilon), \quad \theta = \frac{1}{\epsilon} P(X_1, T, \epsilon).$$

They are to be determined by means of an expansion procedure similar to Luke's, and Bisshopp defines a sequence of successive approximations which derives from the work of Gardner (1959) on adiabatic invariants.

The difficulty with these expansion procedures, however, is that the analysis tends to become rather involved, and doubly so since the zeroth order solution itself is generally obtained in the form of an expansion about linear theory (see Whitham 1967). This means that it will not normally be feasible to calculate higher order terms in the asymptotic expansions. It is therefore desirable to check in as many cases as possible the region of validity of Whitham's (zeroth order) solution. It is suggested that the calculations presented in Chapters II and III of this thesis provide a sound basis for such an experimental evaluation of the theory. It may be that, as in the W.K.B. approximation, the zeroth order solution attains an accuracy far greater than may reasonably be expected.

Finally it is interesting to note that Hoogstraten (1968) has applied perturbation expansions, similar to Luke's, directly to the governing equations of (i) cnoidal waves, (ii) Boussinesq waves, (iii) Stokes waves. In (i) and (ii) he derives in the second approximation to the lowest order equation of the asymptotic expansion, the axisymmetric wave equation first discussed by Lighthill (1965b) for more general systems. In the same approximation the equations derived in (iii) are shown to agree with those obtained by Whitham (1967).

2.2 Conservation equations and wave action

The possibility of initially smooth finite amplitude wave trains developing shock-like discontinuities in one or more of the wave parameters has been discussed by Whitham (1965). Particular attention was paid to those cases where the Euler equations constitute a hyperbolic system. Later Benjamin & Feir (1967) presented experimental and theoretical evidence to the effect that the finite amplitude Stokes's wave on deep water is unstable. At the same time this instability was predicted by Lighthill (1965b) in a pioneering paper on the application of Whitham's theory to moderate amplitude systems. A variant of this problem is discussed at length in the present thesis (Chapters II and III), where the contention is that, although the governing equations now turn out to be elliptic, the unstable Stokes's wave develops

a dissipation free discontinuity in phase and wavenumber (a "phase jump") which can be discussed in terms of the physical conservation equations.

It is convenient to be able to derive the relevant conservation equations directly from the invariance properties of the Lagrangian integral. This is done in Chapter III, and the present discussion is limited to a statement of the important theorem involved:

NOETHER'S THEOREM If the functional

$$S(u) = \int_R L(\underline{x}, u, \frac{\partial u}{\partial \underline{x}}) d\underline{x}$$

is invariant under the family of transformations

$$\underline{x}' = \underline{x} + \varepsilon \phi(\underline{x}, u, \frac{\partial u}{\partial \underline{x}}) + o(\varepsilon)$$

$$u' = u + \varepsilon \psi(\underline{x}, u, \frac{\partial u}{\partial \underline{x}}) + o(\varepsilon)$$

for an arbitrary region R of $\underline{x}$ -space, then

$$\frac{\partial}{\partial x_i} \left(\bar{\psi} \frac{\partial L}{\partial \left(\frac{\partial u}{\partial x_i} \right)} + \phi_i L \right) = 0$$

on each extremal surface of $S(u)$, where

$$\bar{\psi} = \psi - \phi_i \frac{\partial u}{\partial x_i}.$$

The proof is long but not difficult (see, for example, Gel'fand & Fomin 1963).

The conservation equation for a variational principle of the form

$$\delta \iint \mathcal{L}(\underline{k}, \omega) \, d\underline{x} dt = 0$$

which corresponds to integral invariance under phase translation, is identical with the Euler equation

$$\frac{\partial \mathcal{L}}{\partial t} - \frac{\partial \mathcal{L}}{\partial x_i} k_i = 0. \quad (21)$$

Bretherton & Garrett (1968) (see also Garrett, Thesis 1968) have used this result to obtain the equation describing the conservation of "wave action" in linear conservative systems. If E is the average perturbation wave energy per wavelength, then

$$\begin{aligned} E &= \bar{\omega} \mathcal{L} - \mathcal{L} \\ &= \bar{\omega} \mathcal{L}, \end{aligned}$$

since $\mathcal{L} \equiv 0$ for linear systems, and the wave action is defined to be equal to the ratio $E/\bar{\omega}$. Here $\bar{\omega}$ denotes the frequency with respect to a frame at rest relative to the local medium. Substituting in (21) gives the equation of conservation of wave action, viz.,

$$\frac{\partial}{\partial t} \left(\frac{E}{\bar{\omega}} \right) + \frac{\partial}{\partial x_i} \left(\frac{\partial \omega}{\partial k_i} \frac{E}{\bar{\omega}} \right) = 0 \quad (22)$$

where $\partial \omega / \partial k_i$ is the group velocity with respect to a fixed system of axes.

Actually, equation (22) is a particular form of the

Boltzmann equation used to describe, for example, phonon interactions in crystals (Peierls, 1955) and plasmon interactions in ionized media (Kadomtsev, 1965). If $n(\underline{k}, \underline{x}, t)$ is the density of wave packets in the element $d\underline{k}d\underline{x}$ of phase space, then for a collisionless system subject to negligible damping, n satisfies

$$\frac{\partial n}{\partial t} + \frac{\partial \omega}{\partial \underline{k}} \cdot \frac{\partial n}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial n}{\partial \underline{k}} = 0, \quad (23)$$

where ω is the frequency of the wave packet with respect to fixed axes. Now if the total wave energy in $d\underline{k}d\underline{x}$ is equal to E , then, by Planck's formula, n is proportional to E/ω , and this has a delta function dependence on $\underline{k}$ in the case of a wave train of almost constant wavenumber. Substituting in (23) and integrating over $\underline{k}$ then leads to the wave action conservation equation (22).

2.3 The average Lagrangian for waves on deep water

In order to obtain the average Lagrangian density it is necessary to know the form of the plane periodic solution which satisfies a nonlinear ordinary differential equation, the phase θ being the independent variable. In general it is difficult to find a closed expression for such a solution, and various expansion procedures have been adopted. For example, Whitham (1967) has used a Fourier expansion of the uniform wave solution of the Stokes's wave problem on water of finite depth to calculate the average Lagrangian (from

equation (18)) correct to terms in the fourth power of the wave slope. However, such a procedure has the disadvantage of limiting the nonlinear discussion to moderate amplitude waves, and it often happens that important nonlinear effects can be overlooked (see Lighthill 1967, and Chapters II and III).

The difficulty mentioned in § 2.1 concerning the appropriate Eulerian representation of the Lagrangian density function no longer arises in the case of Stokes's waves propagating on deep water. For infinite depth a suitable Lagrangian coordinate of the motion is the elevation of the free surface, so that, in the usual notation, the classical form of the Lagrangian density, $L = T - V$, may be used when expressed in terms of the surface elevation. Lighthill (1967) has succeeded in using this representation to obtain an approximate expression for the average Lagrangian density valid for all possible periodic wave slopes. This was accomplished by interpolating between the known form of the Lagrangian at small wave slopes (also calculated by Lighthill 1965b) and the calculated form for waves of maximum height. The maximum amplitude periodic wave is that for which the surface makes Stokes's 120° angle at the crest, and was first studied in detail by Michell (1893).

For infinite depth the pseudo-frequencies are absent from the description of the wave motion, the average form of which may now be expressed in terms of the wavenumber k and frequency ω alone — the remaining wave parameters, a

and b , being determined by the free surface conditions.

Lighthill's average Lagrangian is therefore a function of $\underline{k}$ and ω only, and the precise form is given in Chapter II.

3. Karpman's method

Karpman (1968) has given a method of calculation for moderate amplitude dispersive waves which is capable of allowing for more rapid variations in the wave parameters. A short account of his theory is presented below for the purpose of comparison with the Whitham theory.

Consider waves governed by a nonlinear, but isotropic dispersion relation of the form

$$\omega = \omega(k^2, a^2) , \quad (24)$$

where a is a measure of the amplitude, and for convenience the wavenumber vector $\underline{k}$ is taken to be parallel to the x -axis. The wave slope ka is assumed to be small, so that to a first approximation beyond linear theory (24) becomes

$$\omega = \omega_0(k^2) + \left(\frac{\partial \omega}{\partial a^2} \right)_{a=0} a^2 , \quad (25)$$

where $\omega = \omega_0(k^2)$ is the linear theory dispersion relation.

As in the Whitham theory, modulation processes are considered for nonlinear non-stationary waves in which the wave parameters vary slowly on a scale of wavelength and

wave period. In the notation of §1.1 set:

$$\begin{aligned}\theta(\underline{x}, t) &= k_0 x - \omega_0 t + \phi(\underline{x}, t) \\ \omega &= \omega_0 - \phi_t \\ \underline{k} &= \underline{k}_0 + \frac{\partial \phi}{\partial \underline{x}},\end{aligned}\quad (26)$$

and let a_0 be the constant amplitude of an initially unmodulated wave.

In general, as the modulation of an almost periodic wave train proceeds, the direction of the wavenumber vector will vary in accordance with the phase modulation term ϕ . Expand the first term on the right hand side of equation (25) correct to terms which are second order in these changes, i.e.,

$$\begin{aligned}\omega_0(\underline{k}^2) &= \omega_0(k_0^2) + \frac{\partial \omega_0}{\partial k_j} \frac{\partial \phi}{\partial x_j} + \frac{1}{2} \cdot \frac{\partial^2 \omega_0}{\partial k_i \partial k_j} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \\ &= \omega_0(k_0^2) + U_0 \phi_x + \frac{1}{2} U_0' \phi_x^2 + \frac{U_0}{2k_0} (\phi_y^2 + \phi_z^2)\end{aligned}$$

where the group velocity $U_0 = \partial \omega_0 / \partial k$, and $U_0' = \partial^2 \omega_0 / \partial k^2$ are evaluated at $\underline{k} = \underline{k}_0$. By substituting in (25) and subtracting out the uniform wave one obtains:

$$\begin{aligned}\phi_t + U_0 \phi_x + \frac{1}{2} \left\{ U_0' \phi_x^2 + \frac{U_0}{k_0} (\phi_y^2 + \phi_z^2) \right\} \\ + \left(\frac{\partial \omega}{\partial a^2} \right)_{a=0} (a^2 - a_0^2) = 0.\end{aligned}\quad (26)$$

Equation (26), although correct when the Whitham approximation is expected to be valid, cannot describe more rapid variations in the wave parameters. To overcome this weakness Karpman advocates a procedure which is similar to that used in §1.1, in which the linearized wave operator F is expanded to terms of the second order in the partial derivatives of slowly varying quantities. Consider, for example, the simple case

$$F(\underline{k}, \omega) = \omega^2 - \Omega(k^2). \quad (27)$$

Expanding as in §1.1 one finds that to second order the wave equation becomes

$$\left[i \left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x} \right) + \frac{1}{2\omega} \left(U_0^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) + \frac{1}{2} U_0' \frac{\partial^2}{\partial x^2} + \frac{U_0}{2k_0} \Delta_1 \right] a e^{i\phi} = 0, \quad (28)$$

where $\Delta_1 = \partial^2 / \partial z^2 + \partial^2 / \partial y^2$.

Karpman now introduces the approximation

$$\frac{1}{2\omega} \left(U_0^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) = 0,$$

although it is not clear why this term should be small compared to the first term on the left of (28). Then equating real and imaginary parts to zero it follows that

$$\begin{aligned} \phi_t + U_0 \phi_x + \frac{1}{2}(U_0' \phi_x^2 + \frac{U_0}{k_0} (\nabla_{\perp} \phi)^2) \\ - \frac{U_0'}{2a} a_{xx} - \frac{U_0}{2k_0} \frac{\Delta_{\perp} a}{a} = 0 \quad (29) \end{aligned}$$

$$\frac{\partial a^2}{\partial t} + U_0 \frac{\partial a^2}{\partial x} + U_0' \frac{\partial (\phi_x a^2)}{\partial x} + \frac{U_0}{k_0} \nabla_{\perp} \cdot (a^2 \nabla_{\perp} \phi) = 0 \quad (30)$$

where $\nabla_{\perp} = (0, \partial/\partial y, \partial/\partial z)$.

Equation (30) is the modified energy propagation equation (cf. equation (4) above). Equation (29) differs from equation (26) in two ways: firstly it does not contain terms proportional to a^2 , since it is derived from the linear equation (28). Secondly it contains second order derivatives in the amplitude with respect to the space coordinates. Equation (29) is the first correction, neglecting nonlinear terms, to the geometrical optics approximation, and equation (26) is the first nonlinear correction to the geometrical optics approximation. Hence, the correct equation for ϕ , which includes terms of order a^2 , and derivatives up to and including the second order, must be

$$\begin{aligned} \phi_t + U_0 \phi_x + \frac{1}{2}(U_0' \phi_x^2 + \frac{U_0}{k_0} (\nabla_{\perp} \phi)^2) \\ + \left(\frac{\partial \omega}{\partial a^2} \right)_{a=0} (a^2 - a_0^2) - \frac{U_0}{2a} a_{xx} - \frac{U_0 \Delta_{\perp} a}{2k_0 a} = 0. \quad (31) \end{aligned}$$

Equations (30,31) constitute a complete system containing

correction terms which are not included in Whitham's adiabatic approach. They are valid for dispersion relations of the form (24) which arise, for example, in the theories of nonlinear optics, plasma waves and acoustic waves. Karpman (1968) has used them to study the disintegration of a weakly modulated wave train into "solitons".

The main disadvantage of Karpman's method is that the equations are considerably more complicated than those encountered in the application of Whitham's theory to the moderate amplitude case. It may also be remarked that Karpman's equations are constructed to deal with situations in which variations in the wave parameters have a length scale which is about an order of magnitude smaller than those which arise in problems which can be treated by the Whitham theory. In practice, however, such length scales are likely to be of importance in cases where it is not in general permissible to represent the wave by a locally periodic solution. It seems doubtful, therefore, whether the extra effort required to obtain such a perturbation solution will be justified by the amount of extra information derived from that solution. In the problem considered in Chapters II and III it is found that a good description of the wave field can be derived from the Whitham theory if the region where the wave parameters vary rapidly is treated as a discontinuity, across which almost periodic solutions are matched by means of certain jump conditions.

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- II -

Nonlinear theory of open-channel steady flow past a
solid surface of finite-wave-group shape

Nonlinear theory of open-channel steady flow past a solid surface of finite-wave-group shape

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(Received 17 June 1967)

This paper deals with a detailed application of the Whitham theory of finite amplitude wave propagation to open-channel steady flow of water of infinite depth past a slowly modulated wavy wall. A numerical procedure first used by Garabedian & Lieberstein (1958) for the solution of 'stably-posed' elliptic Cauchy problems is employed to obtain a map of the wave pattern on the free surface of the water. The appearance of a 'shock' in this solution is discussed in terms of previous analytical and experimental results.

1. Introduction

A general theory governing the dispersion of slowly varying nonlinear wave trains has been proposed by Whitham (1965*a, b*). It consists in first supposing the wave train to be locally a uniform solution of the equations of motion from which an average Lagrangian is calculated in terms of the wave parameters. The equations describing the slow variation of these parameters are then derived by an application of Hamilton's Principle, that the time integral of the Lagrangian of the whole system is stationary.

Lighthill (1965, 1967) has considered in some detail applications of the theory to moderate amplitude waves on deep water (where certain 'pseudo-frequencies' are absent); he discusses in particular the dispersive effects of: (i) a large frequency spread, and (ii) substantial amplitude variation but almost constant wave-number, on a finite amplitude wave-group. In his second paper a polynomial approximation to the average Lagrangian for deep water waves is derived, valid for all amplitudes. Whitham (1967) considers the one-dimensional propagation of finite amplitude waves in the case of finite depth. Both authors report substantial agreement between their analytical results and the results of Benjamin & Feir (1967) obtained in their experimental and analytical study of the instability of deep water waves.

A general account of the whole field was given in a recent Royal Society Discussion Meeting (Lighthill 1967). On page 33, there is a preliminary discussion of the application of the Whitham theory to waves that are stationary on a uniform stream. This is followed up in more detail in the present paper, which derives from an attempt to apply the Whitham theory to the ship-wave problem. Relative to a ship in uniform motion the wave pattern is stationary, and it should be

correspondingly easier to check theoretical predictions of nonlinear effects experimentally than in the non-steady problems previously considered.

The theory of ship-waves was first considered by Kelvin who showed that in general the wave pattern contains two families of waves: (i) the transverse waves which follow in the wake of the ship, and which tend to predominate at low Froude numbers, and (ii) the lateral or diverging wave system important at higher Froude numbers. On the whole more attention has been given to the transverse waves and it might be of more interest now to study the development of the lateral part of Kelvin's solution produced by a speedboat, say. Although a non-linear treatment of waves generated by a ship moving at a high Froude number would be very valuable, it appears still too difficult to carry out in practice, and a somewhat simpler problem is treated here, chosen as a possible introduction to the ship-wave case. The difficulty in the latter case is that the energy of waves generated by a ship tends to be spread over a rather large region of wave-number space, so that near the ship it is in general impossible to have a gradually varying wave pattern. One cannot, therefore, specify suitably smooth boundary conditions such as are necessary if the theory is to predict the future development of the waves. An easier problem is to consider open-channel steady flow past a slowly modulated wavy wall of constant wavelength. In this case the energy induced in the wave motion occupies a relatively narrow band of wave-numbers.

This is the problem treated below. The water is assumed to have infinite depth so that a suitably modified form of Lighthill's formula for the average Lagrangian can be used. Although time dependence is now removed, the dispersion equation proves to be so complex that in the present context only a numerical solution can be contemplated.

2. Derivation of the dispersion equation

In terms of a Cartesian frame (x, y, z) with associated unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ the wall is given by $y = f(x)$, and in the undisturbed state the water will be supposed to occupy the region $-\infty < x < \infty$, $f(x) < y < \infty$, $-\infty < z < 0$. The mean velocity of the water is taken to be $U\mathbf{i}$.

Lighthill (1967) has given an approximation to the Lagrangian density function, $\mathcal{L}$, for waves on deep water in terms of the frequency ω and the wave-number κ . This says that $\mathcal{L}\kappa^2/\rho g$ (where ρ is the density of the water) is a function of $\Omega = \omega^2/g\kappa$ alone, and that the formula

$$\mathcal{L} = \frac{\rho g}{8\kappa^2} \{(\Omega - 1)^2 - (\Omega - 1)^3 - (\Omega - 1)^4\} \quad (1)$$

holds to a good approximation throughout the interval from $\Omega = 1$ (infinitesimal amplitude waves) to $\Omega = 1.20$ (waves of maximum height).

Equation (1) is written in terms of axes fixed relative to the undisturbed motion of the water. To obtain a form appropriate to steady flow at speed U past the wavy wall note that, if $\mathbf{v}$ is the phase velocity at a point where the wave-number is $\kappa = (l, m)$, and the time frequency ω , then

$$\omega = \mathbf{v} \cdot \kappa.$$

In terms of the frame fixed relative to the wall the frequency is given by $(U\mathbf{i} + \mathbf{v}) \cdot \boldsymbol{\kappa}$, and if this vanishes then

$$\omega_{\pm} = -Ul. \quad (2)$$

The required form of the Lagrangian is derived from (1) by replacing ω by $-Ul$ and the wave-number magnitude κ by $\sqrt{l^2 + m^2}$.

To obtain the dispersion equation for the wave-field a phase function $\theta(x, y)$ is introduced by means of the definition

$$l = \theta_x, \quad m = \theta_y, \quad (3)$$

where suffixes denote partial differentiations. The level curves $\theta(x, y) = \text{constant}$, map out the wave crests and troughs in the (x, y) -plane. The average Lagrangian $\mathcal{L}$ is now written in terms of the partial derivatives of θ and Hamilton's Principle applied in the simplified form

$$\delta \iint \mathcal{L}(\theta_x, \theta_y) dx dy = 0.$$

The calculus of variations gives the Euler equation

$$\frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \theta_x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \mathcal{L}}{\partial \theta_y} \right) = 0,$$

$$\text{or} \quad \frac{\partial^2 \mathcal{L}}{\partial \theta_x^2} \theta_{xx} + 2 \frac{\partial^2 \mathcal{L}}{\partial \theta_x \partial \theta_y} \theta_{xy} + \frac{\partial^2 \mathcal{L}}{\partial \theta_y^2} \theta_{yy} = 0; \quad (4)$$

this is a quasi-linear partial differential equation in θ , the coefficients depending only on the first derivatives θ_x and θ_y .

It is convenient to work in terms of dimensionless variables X, Y, L, M defined by

$$\left. \begin{aligned} X &= \frac{xg}{U^2}, & Y &= \frac{yg}{U^2}, \\ L &= \frac{U^2 l}{g}, & M &= \frac{U^2 m}{g}, \end{aligned} \right\} \quad (5)$$

in which case (4) takes the form

$$a(L, M) \theta_{XX} + 2b(L, M) \theta_{XY} + c(L, M) \theta_{YY} = 0, \quad (6)$$

where the coefficients are obtained from (1) to be:

$$\begin{aligned} a(L, M) &= \{(-2L^{10} - 10L^8 M^2 + 24L^6 M^2 - 56L^6 M^4 + 6L^6 \\ &\quad + 10L^4 M^2 - 24L^2 M^6 + 2L^2 M^4 - 2M^6) \\ &\quad + \sqrt{(L^2 + M^2)}(-2L^8 - 15L^6 M^2 + 9L^4 M^2 + 90L^4 M^4 + 9L^2 M^4 - 2M^6)\}, \end{aligned} \quad (7)$$

$$\begin{aligned} b(L, M) &= \{(-16L^7 M + 48L^7 M^3 + 8L^5 M + 16L^5 M^3 + 16L^3 M^3 + 32L^3 M^5 + 8LM^5) \\ &\quad + \sqrt{(L^2 + M^2)}(15L^7 M - 9L^5 M - 90L^5 M^3 - 3L^3 M^3 + 6LM^5)\}, \end{aligned} \quad (8)$$

$$\begin{aligned} c(L, M) &= \{(6L^{10} + 8L^8 - 42L^8 M^2 - 2L^6 - 32L^6 M^2 + 2L^4 M^2 - 40L^4 M^4 \\ &\quad + 10L^2 M^4 + 6M^6) + \sqrt{(L^2 + M^2)}(-15L^8 + 3L^6 + 90L^6 M^2 - 9L^4 M^2 - 12L^2 M^4)\}. \end{aligned} \quad (9)$$

The boundary conditions determining the solution of the dispersion equation are such that L, M and θ are specified on some curve Γ in the (X, Y) -plane, and

it is important to know the form of (6) in the neighbourhood of this initial curve. This is governed by the sign of the discriminant

$$\Delta = b^2 - ac \quad (10)$$

the equation being elliptic if $\Delta < 0$, parabolic if $\Delta = 0$ and hyperbolic if $\Delta > 0$. In figure 1 the continuous curves are the loci of points for which Δ vanishes. The

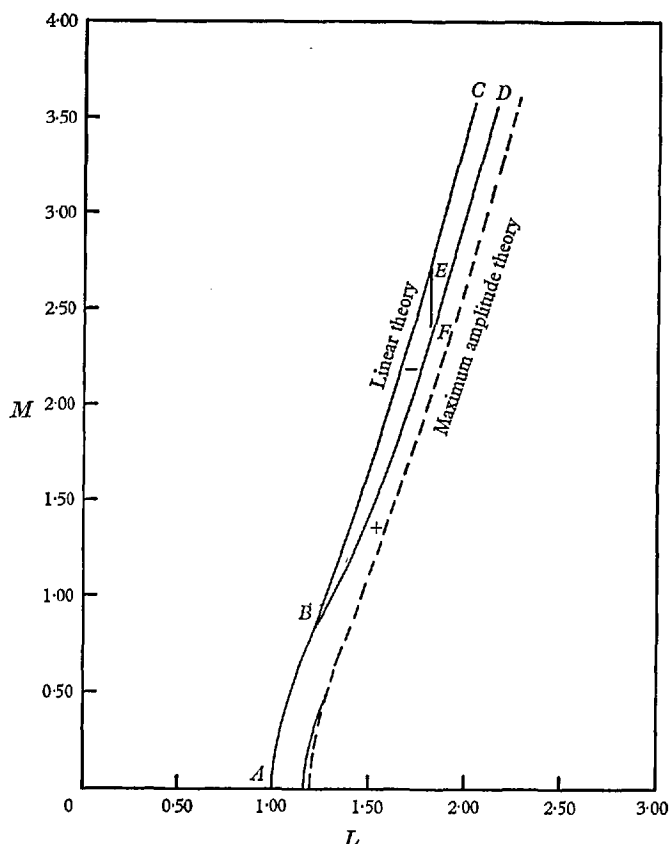


FIGURE 1. The mode of dispersion of a wave-group is determined by the spread of its energy in (L, M) -space between the linear theory and maximum amplitude theory curves. In the regions marked $-$, $+$ the dispersion is governed respectively by an elliptic equation and by a hyperbolic equation. The segment EF denotes the wave-number band occupied by the initial conditions used in the calculation.

magnitude of M is plotted against that of L , and on ABC they are related by the formula

$$L^2 = \sqrt{(L^2 + M^2)}, \quad (11)$$

which is that obtaining under the linear theory approximation. The portion BD separates the two regions marked $+$ and $-$ where respectively Δ is positive and negative. In the positive region the characteristics of the dispersion equation are real and distinct, and solutions θ would be expected to exhibit the characteristic splitting of the group velocity noted by Whitham (1965*a*). In the negative

area there are no real characteristics. Maximum amplitude waves are represented by the broken curve, which was shown by Lighthill (1967) to be given by

$$L^2 = 1.20\sqrt{(L^2 + M^2)}. \quad (12)$$

The region included between the linear theory curve and the maximum amplitude curve is that actually occupied by real waves.

The point *B* in figure 1 corresponds to $L = \sqrt{3/2}$, $M = \frac{1}{2}\sqrt{3}$, and its significance is illustrated by noting that, for example, on the linear theory of ship-waves, this is the point in wave-number space corresponding to the familiar cusp in the Kelvin wave pattern; the transverse waves occupy the segment *AB* and the lateral waves *BC*. Thus for moderate amplitude ship-waves the lateral waves are governed by an elliptic equation and the transverse waves by a hyperbolic equation. The present calculation is concerned with the lateral system, and replaces the ship by a wavy wall which results in the preferential generation of a wave pattern for which, near the wall, (L, M) lies well within the elliptic region.

3. The boundary conditions

The underlying idea of the Whitham theory is that the average Lagrangian is first calculated for a perfectly periodic wave-form which is afterwards allowed to vary slowly on a scale of wavelength. In an exactly parallel manner the boundary conditions for the present problem can be obtained by first supposing the wavy wall to have a constant amplitude. The case of a wave-group surface is then deduced by allowing the amplitude factor in the conditions so obtained to vary slowly on a scale of the wall wavelength.

Apart from requiring the slow variation of the wave parameters, the Whitham theory demands also that they be single-valued, or at least that, if two or more families of waves do overlap, the energy associated with one should completely swamp that of the others. Hence it is not feasible to attempt to obtain suitable boundary conditions too close to the wall. One must be content to prescribe the conditions near the wall, yet far enough away for spatial transients to be unimportant. In the case of the moderate amplitude cosine wall to be considered it will be seen that this can be satisfactorily achieved by taking the initial curve Γ to be the straight line $Y = 1$, parallel to the wall.

It is known (Lighthill 1967) that, for moderate amplitudes at least, the amplitude of a wave of given longitudinal wave-number L varies with distance Y from the wall as in the linear theory, although its position satisfies

$$-\left(\frac{\partial X}{\partial Y}\right)_L = \left(\frac{\partial M}{\partial L}\right)_Y. \quad (13)$$

If Y is not larger than one or two wavelengths, substantial departures from the geometric optics approximation will not have occurred and energy will be received from the wall along the linear theory group velocity lines. These are given by

$$X - \chi(L)Y = \text{constant}, \quad (14)$$

where $\chi(L) = \partial M / \partial L$, and L and M are related by (11). Specifically, if the wall is given by

$$f(X) = A_0 \cos WX, \quad (15)$$

where A_0 and W are respectively the dimensionless amplitude and wave-number

of the wall, it will be shown that on a linear theory approximation the wave elevation ζ near the wall may be taken as

$$\zeta = \frac{-2A_0 W}{\sqrt{(W^2-1)}} \sin [WX - WY\sqrt{(W^2-1)}]. \quad (16)$$

The factor $2A_0 W/\sqrt{(W^2-1)}$ is the wave amplitude, and is assumed to be the first Fourier coefficient in the expansion of the waveform.

If A_0 is now allowed to vary slowly with X , $A_0 = A_0(X)$, near the wall the effect of this variation would be expected to be felt along the lines

$$X - \chi(W)Y = \text{constant}. \quad (17)$$

Thus on the initial line Γ , near enough to the wall for the accumulative effects of nonlinearity to be ignored, the appropriate wave amplitude would be

$$A = \frac{2WA_0(X - \chi(W)Y)}{\sqrt{(W^2-1)}}. \quad (18)$$

Here $A = \delta g/U^2$, where δ is the true amplitude.

To determine completely the wave-number (L, M) and phase θ on Γ for the real, finite amplitude waves, the compatibility relation connecting l, m and δ :

$$\omega^2 = g\kappa(1 + \kappa^2\delta^2 + \frac{5}{4}\kappa^4\delta^4 + \dots) \quad (19)$$

given by Lamb (1932, p. 420) must be used. As before $\omega^2 = U^2 l^2$, $\kappa = \sqrt{(l^2 + m^2)}$. The longitudinal wave-number L is taken to be W , that of the forcing mechanism, and the amplitude δ is given by (18). M is now computed from (19) by reverting the series and expanding M in terms of A and L . It is easily shown that

$$M = M(L, A)$$

$$= -L\sqrt{(L^2-1)} \left\{ 1 - \left(\frac{L^2}{L^2-1} \right) (AL^2)^2 + \left(\frac{L^2}{L^2-1} \right) \left(\frac{9}{4} - \frac{L^2}{2(L^2-1)} \right) (AL^2)^4 + \dots \right\}, \quad (20)$$

where the minus sign anticipates backward-facing waves.

Hence, since (20) gives M as a slowly varying function of X , the boundary conditions for the dispersion equation can be set in the form,

$$L = W, \quad M = M(W, A), \quad \theta = LX + MY, \quad (21)$$

on $Y = 1$.

In the next section the linear theory solution will be obtained.

4. Linear theory solution

The wave pattern generated by a wall

$$Y = f(X), \quad (22)$$

where $f(X)$ is small, is analysed by introducing a potential $\phi(X, Y, Z)$ such that the dimensionless velocity $\mathbf{v}$ of the flow is given by

$$\mathbf{v} = \mathbf{i} + \nabla\phi. \quad (23)$$

On infinitesimal amplitude theory ϕ satisfies the equation

$$\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial Y^2} + \frac{\partial^2 \phi}{\partial Z^2} = 0, \quad (24)$$

where $Z = zg/U^2$, in the region $-\infty < X < \infty$, $0 < Y < \infty$, $-\infty < Z < 0$. Apart from the usual condition at infinity, ϕ further satisfies

$$\frac{\partial \phi}{\partial Y} = \frac{df(X)}{dX}, \quad (25)$$

as $Y \rightarrow +0$, and

$$\left. \begin{aligned} \zeta + \frac{\partial \phi}{\partial X} &= 0, \\ \frac{\partial \phi}{\partial Z} - \frac{\partial \zeta}{\partial X} &= 0, \end{aligned} \right\} \quad (26)$$

or

$$\frac{\partial \phi}{\partial Z} + \frac{\partial^2 \phi}{\partial X^2} = 0, \quad (27)$$

at $Z = 0$, where $\zeta(X, Y)$ is the dimensionless surface elevation.

The problem is readily solved by considering the double Fourier transform $\psi(L, M; Z)$:

$$\psi(L, M; Z) = \frac{1}{\pi\sqrt{(2\pi)}} \int_{-\infty}^{\infty} dX \int_0^{\infty} \phi(X, Y, Z) \cos(MY) e^{-iLX} dY. \quad (28)$$

ψ is found to satisfy

$$\frac{\partial^2 \psi}{\partial Z^2} - (L^2 + M^2) \psi = \sqrt{\frac{2}{\pi}} iL \bar{f}(L), \quad (29)$$

where

$$\bar{f}(L) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(X) e^{-iLX} dX \quad (30)$$

and

$$\frac{\partial \psi}{\partial Z} = L^2 \psi, \quad (31)$$

as $Z \rightarrow -0$. The solution which is bounded as $Z \rightarrow -\infty$ is easily determined, and at $Z = 0$ reduces to

$$\psi = \sqrt{\frac{2}{\pi}} \frac{iL \bar{f}(L)}{[L^2 - \sqrt{(L^2 + M^2)}] \sqrt{(L^2 + M^2)}}. \quad (32)$$

The potential ϕ now follows from the inversion theorem for Fourier transforms, and at $Z = 0$ is,

$$\phi = \frac{2}{\pi} \int_{-\infty}^{\infty} dL \int_0^{\infty} \frac{iL \bar{f}(L) e^{iLX} \cos(MY) dM}{[L^2 - \sqrt{(L^2 + M^2)}] \sqrt{(L^2 + M^2)}}. \quad (33)$$

In evaluating this double integral account must be taken of the radiation condition that no waves enter or leave at $X = -\infty$. To do this consider the portion

$$\begin{aligned} I &= \int_0^{\infty} \frac{\cos(MY) dM}{[L^2 - \sqrt{(L^2 + M^2)}] \sqrt{(L^2 + M^2)}} \\ &= \frac{1}{2} \int_0^{\infty} \frac{e^{iMY} dM}{[L^2 - \sqrt{(L^2 + M^2)}] \sqrt{(L^2 + M^2)}} + \frac{1}{2} \int_0^{\infty} \frac{\bar{e}^{iMY} dM}{[L^2 - \sqrt{(L^2 + M^2)}] \sqrt{(L^2 + M^2)}} \\ &= I_1 + I_2, \text{ say.} \end{aligned}$$

The radiation condition will be automatically satisfied if L is replaced by $L - i\epsilon$,

where ϵ is a small positive constant, I_1 and I_2 evaluated, and ϵ then allowed to tend to zero.

First note that both integrands possess simple poles at

$$M = \pm L\sqrt{(L^2 - 1)},$$

and branch points at $M = \pm iL$. To evaluate I_1 consider the integral about the contour shown in figure 2, consisting of the real axis from $M = 0$ to R , the circular

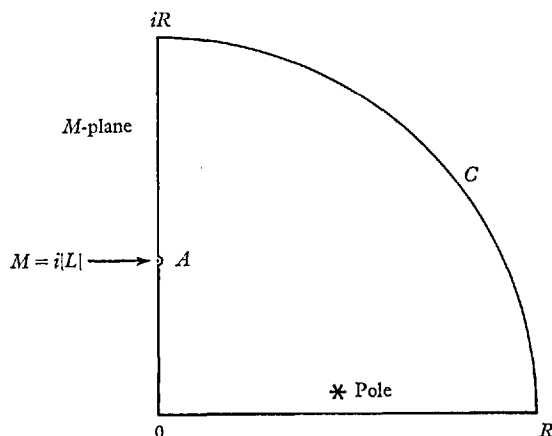


FIGURE 2. The contour used in evaluating I_1 . The pole lies within the contour if $L < 0$ and outside if $L > 0$.

arc C from $M = R$ to iR , and the imaginary axis indented at the branch point A , from $M = iR$ to 0 . When L is replaced by $L - i\epsilon$ in the integrand, the pole on the positive real axis is shifted into the interior of the contour if $L < 0$, and into the lower half-plane if $L > 0$. Hence, after letting $\epsilon \rightarrow 0$, its contribution to the integral may be written in the form

$$\frac{\pi i H(-L)}{L\sqrt{(L^2 - 1)}} \exp(-iLY\sqrt{(L^2 - 1)}) = H(-L)P(L), \text{ say,}$$

where H is the Heaviside unit function. Thus, as $R \rightarrow \infty$,

$$I_1 = H(-L)P(L) + \int_0^A + \int_A^{i\infty},$$

$$\text{i.e. } I_1 = H(-L)P(L) + \frac{1}{2} \int_0^{|L|} \frac{i e^{-\lambda Y} d\lambda}{[L^2 - \sqrt{(L^2 - \lambda^2)}]\sqrt{(L^2 - \lambda^2)}} + \frac{1}{2} \int_{|L|}^{\infty} \frac{e^{-\lambda Y} d\lambda}{[L^2 - i\sqrt{(\lambda^2 - L^2)}]\sqrt{(\lambda^2 - L^2)}}. \quad (34)$$

In a similar manner it is shown that,

$$I_2 = H(L)P(L) - \frac{1}{2} \int_0^{|L|} \frac{i e^{-\lambda Y} d\lambda}{[L^2 - \sqrt{(L^2 - \lambda^2)}]\sqrt{(L^2 - \lambda^2)}} + \frac{1}{2} \int_{|L|}^{\infty} \frac{e^{-\lambda Y} d\lambda}{[L^2 + i\sqrt{(\lambda^2 - L^2)}]\sqrt{(\lambda^2 - L^2)}}. \quad (35)$$

Combining I_1 and I_2 one eventually obtains

$$\phi = -2 \int_{-\infty}^{\infty} \frac{\bar{f}(L) \exp\{iLX - iLY\sqrt{(L^2-1)}\} dL}{\sqrt{(L^2-1)}} + 2i \int_{-\infty}^{\infty} dL \int_{|L|}^{\infty} \frac{L^3 \bar{f}(L) e^{-\lambda Y + iLX} d\lambda}{[L^4 + (\lambda^2 - L^2)] \sqrt{(\lambda^2 - L^2)}} \quad (36)$$

at $Z = 0$.

In the first instance $f(X)$ is the uniform wall

$$f(X) = A_0 \cos(WX), \quad (37)$$

for which
$$\bar{f}(L) = \frac{1}{2} A_0 [\delta(L+W) + \delta(L-W)], \quad (38)$$

where $\delta(x)$ is the Dirac function. Substitution in (36) and use of (26) leads to the expression

$$\zeta = \frac{-2A_0 W}{\sqrt{(W^2-1)}} \sin(WX - WY\sqrt{(W^2-1)}) + \frac{2A_0 W^4}{\pi} \int_W^{\infty} \frac{\cos(WX) e^{-\lambda Y} d\lambda}{[W^4 + (\lambda^2 - W^2)] \sqrt{(\lambda^2 - W^2)}} \quad (39)$$

for the elevation of the free surface. The second term here represents a disturbance confined to the immediate neighbourhood of the wall, and the proportional error involved in its neglect is easily shown to be less than

$$\frac{\sqrt{(W^2-1)} e^{-WY}}{W \sqrt{(2\pi WY)}}.$$

Γ is taken to be the straight line $Y = 1$. The error along Γ is therefore less than 4% and is rapidly decreasing further as the solution progresses (figure 4).

The dimensionless amplitude A thus takes the form anticipated in (18) on Γ , and since the group velocity lines are

$$X - \frac{(2W^2-1)Y}{\sqrt{(W^2-1)}} = \text{constant}, \quad (40)$$

the geometric optics approximation to the amplitude at (X, Y) finally takes the form

$$A = \frac{2W}{\sqrt{(W^2-1)}} A_0 \left(X - Y \frac{(2W^2-1)}{\sqrt{(W^2-1)}} \right). \quad (41)$$

This is the value to be used in the conditions (21).

5. Solution of the dispersion equation

The problem of solving the dispersion equation,

$$a(L, M) \theta_{XX} + 2b(L, M) \theta_{XY} + c(L, M) \theta_{YY} = 0, \quad (42)$$

must now be considered. The solution is required in the half-plane $Y > 1$, and the boundary conditions to be imposed on $Y = 1$ have been shown to be

$$L = W, \quad M = M(W, A), \quad \theta = LX + MY, \quad (43)$$

where A is given by (41). Since (42) should be elliptic near the wall, W must be

chosen to lie in the elliptic region of wave-number space, and in the calculation presented below $W = 1.8$. The values of L in the solution must be expected to be of the same order of magnitude, while, by (11), $M \sim 2.7$. Under these circumstances it is difficult to construct an efficient approximation to the coefficients a, b, c which might enable some form of analytic solution to be obtained. It is more satisfactory to retain all the terms in the expressions for the coefficients and to adopt the numerical method of solution to be described.

The problem, then, is to solve the elliptic equation (42) under the Cauchy-type boundary conditions (43). In the classical sense this is 'incorrectly set'. However, the Hadamard concept of a well-posed initial value problem, namely that the solution be stable under arbitrary perturbations of the boundary conditions, is not applicable in this case. Here the boundary conditions are all functionally related, and arbitrary, independent perturbations therefore not permitted. Lieberstein (1959) has discussed such problems in relation to second order, linear elliptic equations. He defines the initial value problem to be 'stably-posed' if the perturbations can be confined to a restricted class of functions, S , for which the solution remains stable. It will be seen later that this definition may be extended to cover the present quasi-linear equation by requiring the class S , and so also the boundary conditions, to consist of those real functions for which there exists a sufficiently large region of the complex X -plane into which they may be analytically continued. This implies that the initial values of the wave parameters and their analytic continuations be smoothly varying functions.

Usually Cauchy-type boundary conditions are associated with hyperbolic equations, and the natural way is to solve by the method of characteristics. Recently techniques have been developed which permit elliptic type problems to be solved in the same way. Garabedian & Lieberstein (1958) have described in detail such a method, and used as an example the problem of subsonic flow behind a bow shock wave. A brief résumé will be given here.

Let (α, β) denote the characteristic co-ordinates of (42). When the equation is elliptic these will in general be complex. One can define a new pair of co-ordinates (ξ, η) , however, by

$$\xi = \frac{\alpha + \beta}{2}, \quad \eta = \frac{\alpha - \beta}{2i}, \quad (44)$$

which have the property that real solutions of (42) in the (ξ, η) -plane correspond to real solutions in the (X, Y) -plane. Of course the relationship between (X, Y) and (ξ, η) is not known in advance, and must be calculated in a step-by-step manner simultaneously with L, M and θ .

In terms of these new co-ordinates the solution of the dispersion equation subject to the conditions (43) is equivalent to the solution of the canonical elliptic system

$$\begin{aligned} X_\xi &= (bX_\eta - aY_\eta)/\sqrt{(ac - b^2)}, \\ Y_\xi &= (cX_\eta - bY_\eta)/\sqrt{(ac - b^2)}, \\ L_\xi &= (bL_\eta + cM_\eta)/\sqrt{(ac - b^2)}, \\ M_\xi &= -(aL_\eta + bM_\eta)/\sqrt{(ac - b^2)}, \\ \theta_\xi &= [(bL + cM)X_\eta - (aL + bM)Y_\eta]/\sqrt{(ac - b^2)}, \end{aligned}$$

under the same boundary conditions. This set of equations is conveniently denoted by

$$R_{\xi} = BR_{\eta}, \quad (45)$$

where R is the column vector of unknowns and B the matrix of coefficients. Since any function of a characteristic co-ordinate is still a characteristic co-ordinate, there is a freedom of choice of the initial curve γ in the (ξ, η) -plane along which the conditions (43) are to be imposed. This may be taken as the η -axis. Further, it may be supposed that along γ , $X = F(\eta)$, where F is an arbitrary real analytic function of η ; in the present application $F(\eta) \equiv \eta$.

The important step now is to notice that, if (X, Y, L, M, θ) can all be considered as analytic functions of the complex variable $\eta = \lambda + i\sigma$, say, then, by the Cauchy-Riemann equations, the derivative R_{η} ($= R_{\lambda}$) on the right-hand side of (45) may be replaced by R_{σ}/i . If λ is now held fixed at λ_0 , the canonical system which is elliptic in the real (ξ, η) -plane becomes the hyperbolic system

$$R_{\xi} = (B/i)R_{\sigma}, \quad (46)$$

with the real characteristics

$$\xi \pm \sigma = \text{constant}$$

in the (ξ, σ) -plane. Hence, provided that for each λ_0 the data given on the (real) η -axis can be analytically continued into the complex η -plane to become data on the σ -axis, the hyperbolic system can be solved by a stable step-by-step procedure in the (ξ, σ) -plane. At $\sigma = 0$ the solution so obtained reduces to the solution in the real (ξ, η) -plane on the line $\eta = \lambda_0$. A complete covering of any portion of the real (ξ, η) -plane is obtained by repeating this procedure over a whole range of λ_0 .

The philosophical difficulties involved in requiring that the data be continuable into the complex η -plane have already been discussed (Lighthill 1967). The Whitham theory assumes at the outset that the wave parameters are in some sense 'smooth' functions of position. It is now clear that their variation must be smooth enough to permit their being continued sufficiently far into the complex X -plane without meeting any singularities. This is precisely the case in the present problem.

The system (46) is solved numerically by adopting a finite difference approximation. That used here is the scheme originally proposed by Garabedian & Lieberstein in their paper. Let $\Delta\xi$, $\Delta\sigma$ denote positive increments in ξ and σ , and for a given λ_0 set

$$\left. \begin{aligned} R_{m,n} &= R[(\tfrac{1}{2}m\Delta\xi), (\tfrac{1}{2}n\Delta\sigma)], \\ B_{m,n} &= B[(\tfrac{1}{2}m\Delta\xi), (\tfrac{1}{2}n\Delta\sigma)], \end{aligned} \right\} \quad (47)$$

where $R(\xi, \sigma)$, $B(\xi, \sigma)$ are the vector and matrix of (46). The latter is now approximated by

$$R_{m+1,n} = R_{m-1,n} + s[(B_{m,n+1} + B_{m,n-1})/2i](R_{m,n+1} - R_{m,n-1}), \quad (48)$$

where $s = \Delta\xi/\Delta\sigma$. The characteristic roots of B/i are easily determined to be 1, -1, 0, so that the solution of the initial value problem for (48) would be expected to be stable provided $s \leq 1$. The truncation error in using (48) is of order $(\Delta\sigma)^2$.

It is convenient to take $\Delta\xi = \Delta\sigma$ and to adopt the mesh of points shown in figure 3, where only values of m and n whose sum is even are required. The only mesh points used in the calculation for which σ is negative occur at $n = -1$, and since both $R(\xi, 0)$, $B(\xi, 0)$ are real, their values at these points are given by the reflexion principle formulae:

$$R_{m,-1} = R_{m,1}^*, \quad B_{m,-1} = B_{m,1}^*, \quad (49)$$

where $R_{m,1}^*$, $B_{m,1}^*$ denote the complex conjugates of $R_{m,1}$, $B_{m,1}$.

The calculation may be summarized as follows.

I. Calculate $R_{0,n}$ for $n = 0, 2, 4, \dots, 2N$, say, from the initial conditions.

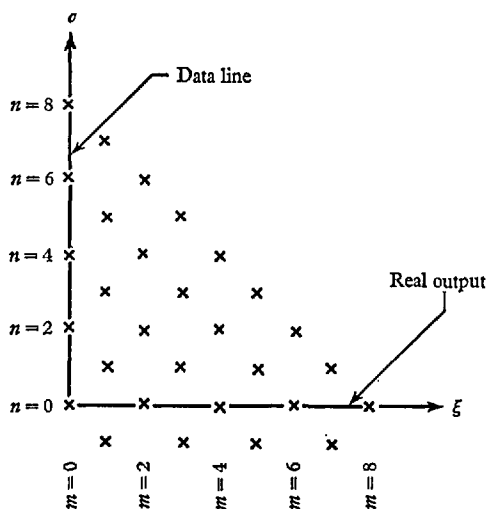


FIGURE 3. The system of mesh points used in the finite difference calculation. Complex data 'read-in' along the σ -axis produces real output along the ξ -axis.

II. Calculate $R_{1,n}$ for $n = 1, 3, 5, \dots, 2N-1$, by means of (48) with $R_{-1,n}$ replaced by $\frac{1}{2}(R_{0,n+1} + R_{0,n-1})$ and s replaced by $\frac{1}{2}s$.

III. Calculate through repeated application of (48) the remaining values of $R_{m,n}$ such that $m+n$ is even and does not exceed $2N$.

The 'real output' of the calculation occurs along the ξ -axis in figure 3 at the N points $m = 2, 4, 6, \dots, 2N$. This output will be the numerical value of the vector $R = (X, Y, L, M, \theta)$, so that each line $\eta = \lambda_0$ corresponds to a curve C , say, in the (X, Y) -plane along which L, M, θ take values given by the output R .

6. Numerical solution for the wavy wall

In the present calculation the slowly modulated wall is given by

$$f(X) = (S/W) \exp\{-(2X/L_G)^2\} \cos(WX), \quad (50)$$

where, relative to the length scale U^2/g , S/W is the maximum amplitude and L_G is the effective length of the wave-group surface. The wall is characterized by the

three dimensionless parameters S , L_G and W . S represents the maximum slope of the wall, L_G its length, and W the 'natural' frequency. The values used here are:

$$S = 0.07, \quad L_G = 6\pi, \quad W = 1.8.$$

The wavelength of the wall is $2\pi/1.8 \sim 3.5$, so that the wave-group surface consists effectively of seven humps, three on either side of the central maximum. The maximum slope is small, but reference to figure 1 shows that the region of wave-number space occupied by the initial conditions for this value of S is represented by the line EF , and the tip F of this line, corresponding to the maximum amplitude of the wall, is close to the boundary BD of the elliptic region. This would seem to indicate that quite large waves may be generated by moderate amplitude walls, and is a consequence of our considering the large Froude number case.

The object of the numerical calculation is to obtain a map of the curves of constant phase, θ , near the wall. This has been accomplished by solving the dispersion equation, in the manner described, above forty-one times at equally spaced points on the initial line $Y = 1$, from $X = -10$ to $X = +10$. In each case N , the number of output points was taken to be 30, and the mesh size $\Delta\xi = \Delta\sigma$ was never greater than 0.17. By interpolation between the output points on C , co-ordinates (X, Y) of the points of intersection of 100 level curves of θ with C (from $\theta = -21.0$ to 18.6) were obtained, and the curves of constant phase subsequently plotted.

The calculation was performed on an IBM 7090 which requires about 3.7 min to obtain the output for each curve C , including the interpolation. The complete calculation requires a minimum of $2\frac{1}{2}$ h machine time. Except in the shock-like region to be discussed below the numerical procedure proved to be very stable, differences being noted only in the sixth significant figure on trebling the mesh size.

7. Discussion of results

The phase plot is shown in figure 4. It can be seen that the numerical calculation has produced a regular pattern of wave crests which on the left are almost rectilinear, but which tend to exhibit a bulging towards the left on moving in the positive X -direction. Eventually this bulging becomes so pronounced that a genuine discontinuity develops, resulting in the formation of a type of 'shock'. The dispersion equation remains elliptic except at the shock, where the numerical procedure becomes violently unstable. The interpretation of these results is that the wave amplitude is greatest where the bulging occurs. The latter is thus a consequence of amplitude dispersion, where larger amplitude waves have the larger phase velocity. Indeed, it is easy to see that the development and beginning of the shock occur approximately along that linear theory group velocity line which passes through the point marked P in the figure where the wall amplitude first becomes significant. Moreover, inspection of the numerical results reveals that below this line the wave-number lies further into the elliptic region than above, so that the waves here have a correspondingly larger amplitude. The

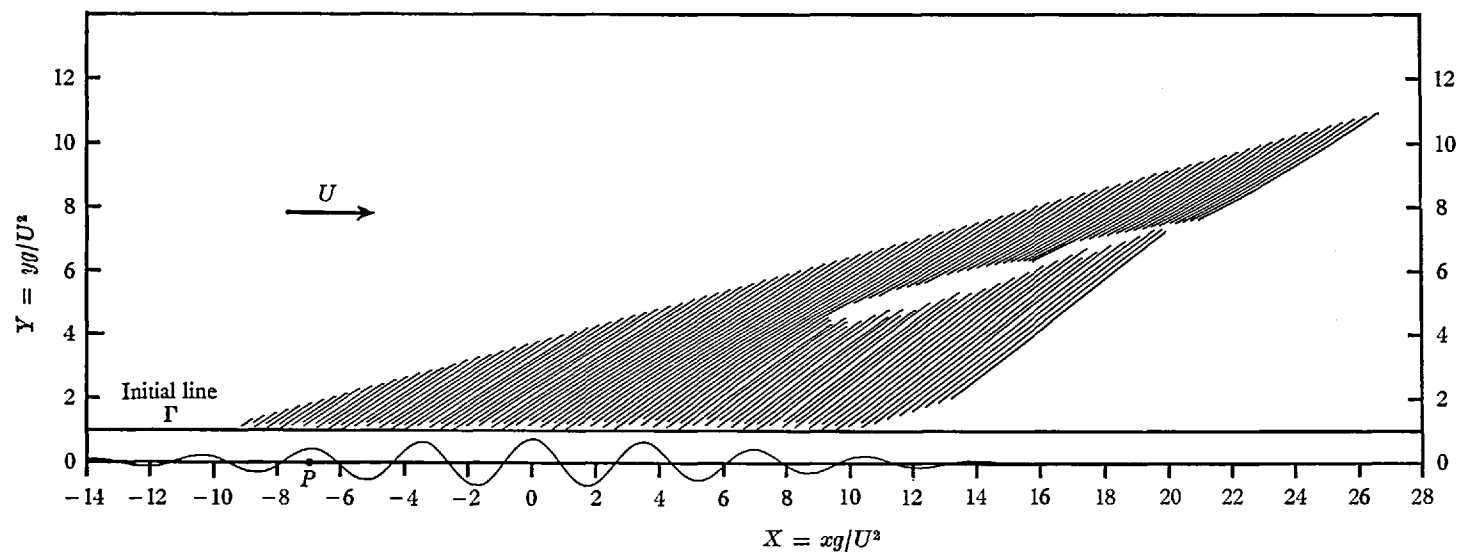


FIGURE 4. The wave pattern produced by flow past the wavy wall. The 'shock' develops approximately along the linear theory group velocity line from P . The wall is drawn to an exaggerated scale.

result is a tendency for the outer wave crests to lag behind because of their smaller phase velocities. This is clearly visible in the figure, and strikingly so after the shock has developed.

Across the shock there is an abrupt change in the direction of the wave crests and the magnitude of the wave-number, the waves being more closely packed on the side remote from the wall. Feir's experiments on one-dimensional, non-steady wave-trains indicate that in the region of the shock there is no pronounced turbulent dissipation or humping-up of the water: it is a confused region in which the wave crests execute a curious 'wiggle' and emerge on the other side changed in direction and spacing. It is possible, however, that the solution obtained here is not valid behind the shock. In the case of gas dynamics the simple wave approach to the solution behind a caustic gives misleading results for strong shocks, and the solution there must be derived from jump conditions across the shock.

Whitham (1965*a*) has suggested that it might be possible for shocks to develop in the hyperbolic case, and that they could probably be treated in the context of his theory by introducing suitable jump conditions. Lighthill (1965, 1967) has already anticipated the appearance of shocks in the elliptic case. In his treatment of the theory he studies the development of a slowly modulated wave-group of nearly constant wave-number, and predicts the appearance of a singularity in the wave-number and amplitude after a certain time, t_c , say. He shows further that there is an apparent 'incubation period' in which very little change in the wave-form occurs before time $t = 0.7t_c$. This may be compared with the present case where the shock appears to develop quite rapidly from a fairly smooth wave pattern. Also, as in Lighthill's case, the wave-number magnitude is increased ahead of the shock and decreased behind.

Experimental support for the Whitham theory has so far been confined to the study of the development in time and distance of a one-dimensional wave group on deep water (Benjamin & Feir 1967), where a shock-like instability has been observed. Since this interesting phenomenon also occurs in the present case, where, moreover, the wave pattern is stationary, permitting relative ease of measurement, it seems desirable to do experiments with a wavy wall to see what happens in the neighbourhood of the discontinuity. Information obtained in this way would also be of considerable value to the further study of high Froude number ship-waves.

The author takes great pleasure in acknowledging his indebtedness to Prof. M. J. Lighthill, F.R.S., for his constant help and guidance throughout this investigation. He would also like to thank Dr C. I. Devanathan of the Indian Institute of Science, Bangalore, for many useful discussions concerning the material of this paper. Finally, the financial support of the Central Electricity Generating Board is gratefully acknowledged.

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APPENDIX

It was pointed out in § 2.6 that the waves generated by the wavy wall have a fairly substantial amplitude, especially in the vicinity of the "shock", even though the wall has only a moderate maximum slope $S = 0.07$. For any fixed S the wave steepness near the wall is determined by the value of the Froude number W , equal to the natural frequency of the wall, and taken to be 1.8. As already mentioned in Chapter II, this value of W was chosen in order to ensure that the initial conditions along Γ were well within the elliptic region of figure 2.1.

The initial values of the wavenumber vector on Γ near the centre of the wavy wall, and represented by points near F on the segment EF of figure 2.1, lie in the vicinity of the boundary BD of the elliptic and hyperbolic regions. For a fixed, sufficiently large transverse wavenumber component L , say, there will be a critical wave steepness above which the dispersion equation (2.4) becomes hyperbolic.

Wave steepness $\mathcal{J}$ is defined by

$$\mathcal{J} = \frac{H}{\lambda},$$

where H is the vertical distance between the wave crest and wave trough, and λ is the wavelength. To a good approximation,

$\mathcal{J}$ may be expressed in terms of (L, M) by

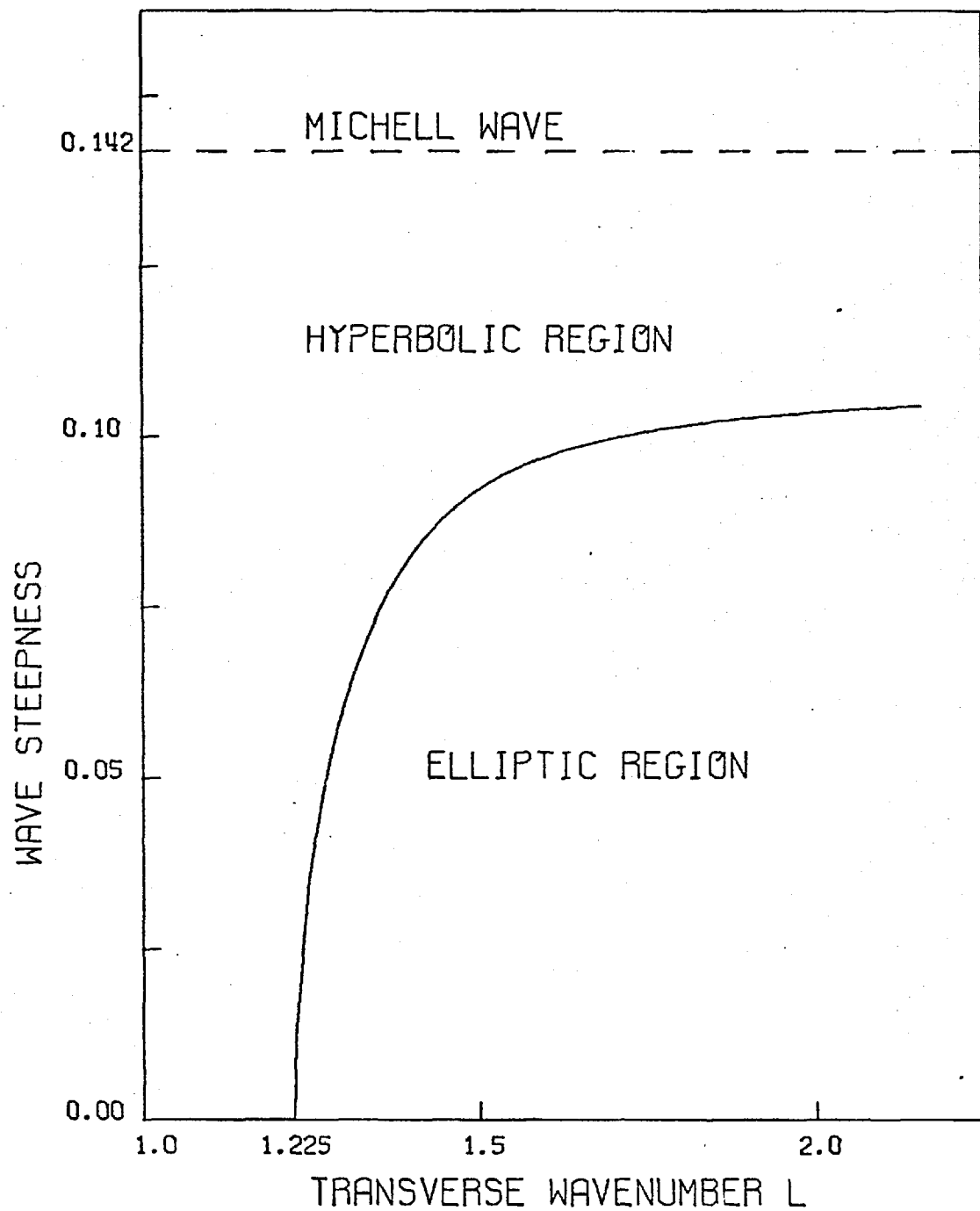


Figure A1.

$$\mathcal{S} = \frac{1}{\pi} \left\{ \frac{L^2}{(L^2 + M^2)^{\frac{1}{2}}} - 1 \right\}.$$

The maximum possible wave steepness occurs for the Michell wave, and is equal to 0.142. Figure A.1 is a plot of critical wave steepness against L . The broken line denotes the Michell wave (whose wavenumber vector lies on the broken-line curve of figure 2.1). It is clear that for $L = W = 1.8$, the waves in the vicinity of the point F (figure 2.1) have a steepness exceeding two-thirds that of a wave of maximum height.

Inspection of the numerical results indicates that in the "gap" in the solution represented by figure 2.4, the dispersion equation describing the wave field has indeed crossed over into the hyperbolic region, which, in some manner associated with the appearance of real characteristics, accounts for the rapid transition in the values of the wave parameters. Actually this conclusion is rather speculative, since in regions of rapid variability Whitham's fundamental approximation can no longer be relied on to give a good representation of the wave field, and other arguments which treat the discontinuity as a weak shock must be applied.

Bearing this in mind, it is perhaps of interest to record the following speculations on the effect of transition from the elliptic to the hyperbolic region. Consider, for example, a moderate amplitude uniform wave train propagating on deep

water. In order that such a wave train maintain a permanent form a balance must be established between the effects of amplitude dispersion and frequency dispersion. The wave profile of large amplitude waves is associated with the presence of a large number of shorter wavelength components, which have progressively slower (group) velocities of propagation. The two effects of amplitude dispersion and frequency dispersion therefore tend to act in opposite directions and must be delicately balanced in a wave of permanent form. Lighthill (1965, 67) has shown that such one-dimensional wave trains satisfy an elliptic dispersion equation, provided the wave steepness is not too large. When such a wave train receives a small modulation, this modulation can only deepen with time (since the amplitude cannot pass through a maximum or a minimum). However, in those regions where the amplitude steadily increases, the dispersion equation will ultimately cross over into the hyperbolic region as the critical wave steepness is approached and exceeded. This results in the appearance of two characteristic velocities (Whitham's "splitting" of the group velocity). Conceivably this could trigger-off a splitting of the wave group into two parts. Each part, being of reduced amplitude owing to the sharing out of the available wave energy, subsequently falls back into the elliptic region. The separation into two groups is completed by further contraction of each group as the original process is repeated.

Feir (1967) has conducted experiments on deep water waves which tend to exhibit just this phenomenon. In his case, however, it might be more appropriate to describe the process as one in which the instability results in the emission of a small amplitude wave packet ahead of the group.

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- III -

Phase jumps

Phase jumps

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(Received 1 December 1967)

An earlier paper (Howe 1967) considered a non-linear theory of open-channel steady flow of deep water past a slowly modulated wavy wall. The wave pattern on the free surface of the water was obtained as the solution of a stably posed elliptic Cauchy problem, the main feature of the solution being the appearance of a 'shock' across which there is an abrupt change of phase. Such phase jumps can occur in a wide range of similar problems, but the advantage of the present case is that it is rather well suited to experimental investigation. This paper is therefore a lead-in to the more general problem of phase jumps, and uses the principle of conservation of energy in conjunction with the earlier solution to predict the possible position of the discontinuity on the free surface of the water. The possible nature of the free surface in the vicinity of the phase jump is also discussed (figure 4). This is a region where the width of the wave troughs becomes dramatically shorter than that of the neighbouring troughs. An approximate method of determining the line along which the phase jump occurs, not depending on a knowledge of the solution of the Cauchy problem, is also presented.

1. Introduction

In an earlier paper (Howe 1967) a non-linear theory of steady flow of deep water past a slowly modulated wavy wall was considered. The discussion was based on a theory proposed by Whitham (1965*a, b*) which describes the dispersion of slowly varying wave trains of large amplitude. Briefly, the method consists in first supposing the wave train to be locally a close approximation to an exactly periodic solution of the full non-linear equations of motion, from which an average Lagrangian is calculated in terms of the wave parameters. The dispersion equations governing the slow variation of these parameters are then obtained by an application of Hamilton's Principle.

Previous calculations of the dispersion of large amplitude wave-groups (Lighthill 1965, 1967; Whitham 1967*a*) had shown that in the particular case in which the dispersion equations formed an elliptic system a certain instability in the solutions of the initial value problems arose. Experimental confirmation of such instability, in the case of the development in time and space of a one-dimensional finite amplitude wave-group on deep water, has been reported by Benjamin & Feir (1967). However, because of the experimental difficulties involved in making accurate observations of moving wave trains, it appeared desirable to do a non-linear calculation for a case in which the wave pattern is steady. Such steady wave patterns could in principle be observed from a ship

moving at constant velocity relative to still water but, because of difficulties mentioned by Howe (1967), it is not immediately clear how to apply the Whitham theory to the general ship-wave problem. The simpler problem of steady flow of deep water past a wavy wall of locally sinusoidal shape, whose amplitude decays slowly from a central maximum, was therefore considered, as a possible introduction to the ship-wave case. As explained in some detail in the earlier paper, the dispersion equation describing the wave-field on the free surface of the water is of elliptic type for moderate wall amplitudes, provided the wavelength of the wall is not too large.

The result of the earlier calculation is shown in figure 1, which is a map of the curves of constant phase on the free surface of the water (corresponding to the wave-crests and troughs). The x -axis is taken from left to right along the mean surface of the wall, the origin being at the point of maximum wall amplitude, and the positive direction of the y -axis is outwards from the wall. All distances and wavelengths are quoted with respect to U^2/g as the unit of length, where U is the undisturbed free stream speed which is in the positive x -direction. The amplitude of the wall is exaggerated by a factor of 8 in the figure. The wave pattern was obtained as the solution of an initial value problem, the wave-number κ and the phase θ being prescribed along the line $y = 1$.

It can be seen from the figure that there is a region of the (x, y) -plane within which the solution is indeterminate. This will be referred to as the 'gap region'. It is argued in the earlier paper that this is a manifestation of a shock-like phenomenon, across which there are discontinuous jumps in the wave-number and phase, the direction and spacing of the wave-crests changing abruptly through the shock. The shock, or 'phase jump', forms largely as a result of amplitude dispersion, whereby the larger amplitude waves have the larger phase velocity. The present paper aims to complete the solution of the problem by predicting the position of the discontinuity using the solution obtained and shown in figure 1 on either side of the shock. This is described in §3, and involves an application of the principle of conservation of energy. Out of all possible positions of the shock only one ensures that this is not violated.

Comparison with the analogous case in gas dynamics suggests the possibility of error in the solution which is being used behind (i.e. to the right of) the gap region. Such errors are third order effects in gas dynamics, and can often be neglected in the case of weak shocks; however, misleading results may be obtained in the strong shock case. Hence, the possible need for the solution behind the gap region to be modified, with possible consequential changes in shock position, must not be overlooked.

The present calculation assumes therefore that the shock is sufficiently weak for the method to be valid. Support for this view comes from the observations of Benjamin & Feir (1967), which indicate that the instability of finite amplitude waves does not result in turbulent dissipation of energy, but rather a redistribution of the energy in the spectrum of the wave-group.

The relevant conservation equations are derived in the next section. In §3 the predicted position of the shock is found, and the predicted amount of phase jump across the shock is used to estimate the difference in the number of wave-

crests entering and leaving the shock. The possible form of the free surface in the neighbourhood of the phase jump is obtained by taking cross-sections of the free surface at right-angles to the shock. Observations on this would form a rather precise test of the theory. Finally, in §4 a crude physical argument is given which interestingly enough enables one to predict rather well the line along which the phase jump lies, although the method does not give its location on the line correctly.

2. Conservation equations and the dispersion equation

Let $\mathcal{L}(\kappa, \omega)$ denote the average Lagrangian per unit horizontal area for finite amplitude waves on deep water, where $\kappa = (l, m)$ is the wave-number and ω the time frequency. The system is assumed to be non-dissipative, since in an almost uniform wave train velocity gradients are smoothly varying functions of position and so the effect of viscosity may be ignored. In this case Lighthill (1967) has given an explicit expression for $\mathcal{L}$ approximately valid for all possible amplitudes, and this has been used in the computations of Howe (1967).

The wave-field is described in terms of a phase function $\theta(x_i, t)$ (where t is the time), which is smoothly varying and takes successive integral multiple values of 2π on successive wave-crests. In terms of θ the wave-number and frequency are given by

$$\kappa_i = \partial\theta/\partial x_i, \quad \omega = -\partial\theta/\partial t. \quad (2.1)$$

$$\text{Hamilton's Principle,} \quad \delta \iint \mathcal{L}(\kappa_i, \omega) d\mathbf{x} dt = 0, \quad (2.2)$$

then leads to the Euler equation

$$\frac{\partial}{\partial t}(\mathcal{L}_\omega) = \frac{\partial}{\partial x_i}(\mathcal{L}_{\kappa_i}) \quad (2.3)$$

which, as pointed out by Whitham (1967*b*), is the conservation equation representing the balance between changes in the space-like adiabatic invariant $\mathcal{L}_{\kappa_i}$ and the time-like adiabatic invariant $\mathcal{L}_\omega$. Other conservation equations may be derived from (2.2) by applying Noether's Theorem (Bogoliubov & Shirkov 1957, p. 20). In particular the invariance of $\mathcal{L}$ with respect to arbitrary time and space translations lead respectively to the energy equation

$$\frac{\partial}{\partial t}(\omega \mathcal{L}_\omega - \mathcal{L}) - \frac{\partial}{\partial x_i}(\omega \mathcal{L}_{\kappa_i}) = 0 \quad (2.4)$$

and the momentum equation,

$$\frac{\partial}{\partial t}(\kappa_j \mathcal{L}_\omega) - \frac{\partial}{\partial x_i}(\kappa_j \mathcal{L}_{\kappa_i} - \mathcal{L} \delta_{ij}) = 0. \quad (2.5)$$

The definitions (2.1) give the two further conservation equations

$$\frac{\partial \omega}{\partial x_i} + \frac{\partial \kappa_i}{\partial t} = 0, \quad \text{curl } \kappa = 0. \quad (2.6)$$

The first of (2.6) represents the conservation of wave-crests, and the second says that wave-crests originate and terminate only on the boundaries of the wave-

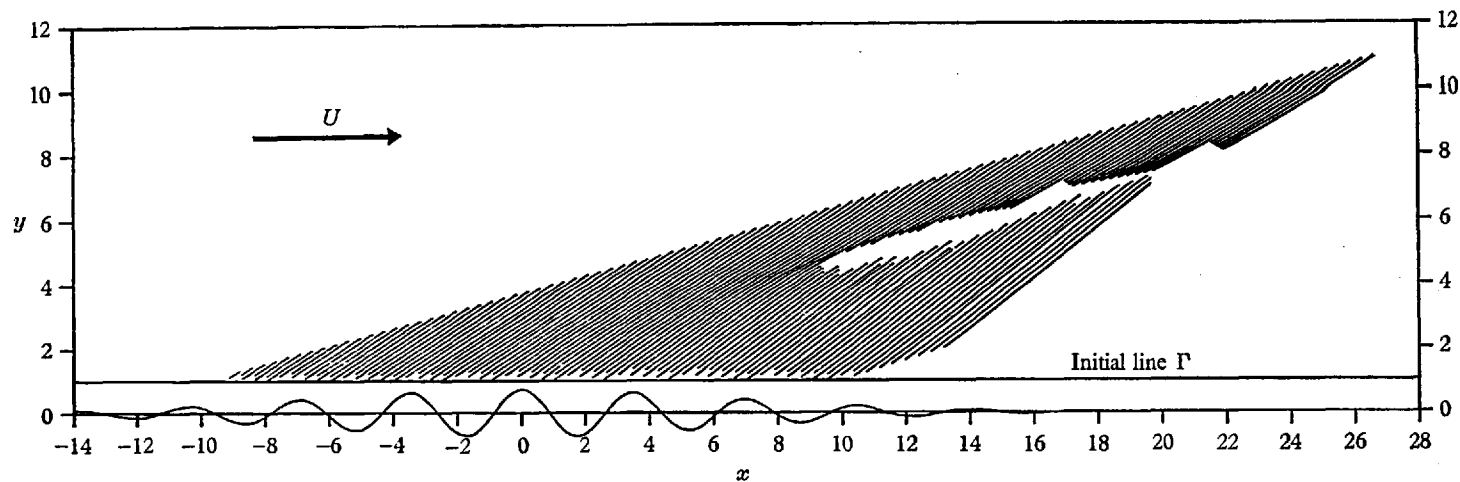


FIGURE 1. The surface wave pattern produced by steady flow at speed U past the wavy wall

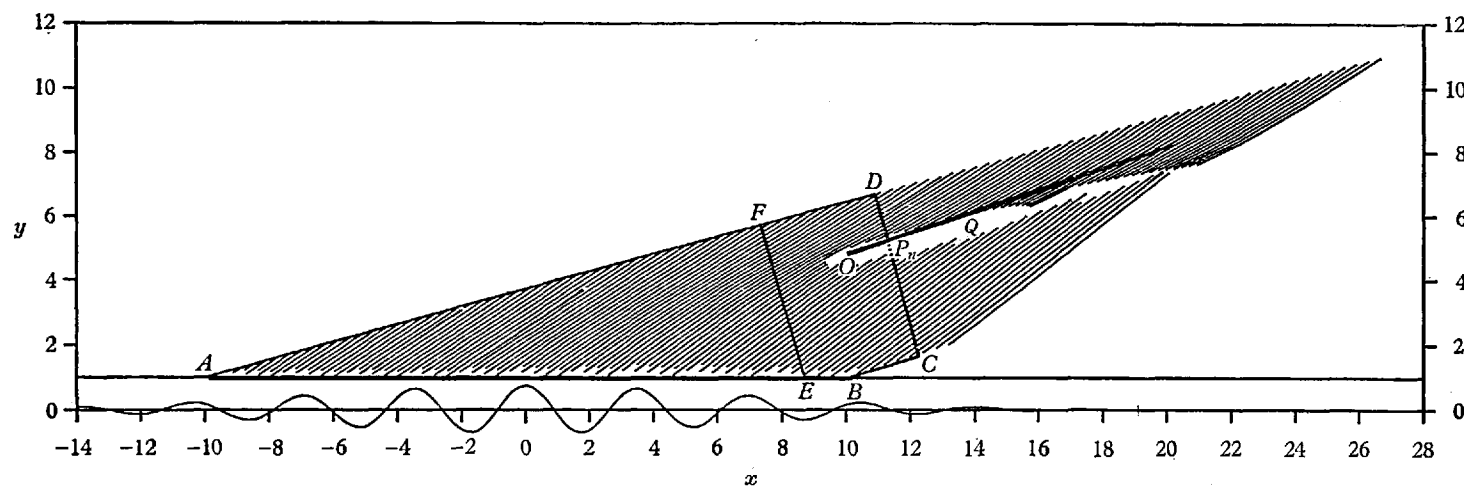


FIGURE 2. The position of the phase jump OQ as determined by the principle of conservation of energy. $ABCD A$ is a typical contour used to predict the P_n , and $AEFA$ is a typical test-integral contour.

field. However, (2.3) and (2.6) are valid only in those regions where the wave-field varies slowly on a scale of wavelength, whereas the energy and momentum equations (2.4), (2.5) are valid under all circumstances when the average Lagrangian $\mathcal{L}$ is replaced by the exact Lagrangian density.

The wavy wall problem concerns the steady-state distribution of waves on the free surface, and it is shown in the earlier paper how Lighthill's Lagrangian is expressed in terms of axes fixed relative to the wall, with respect to which the wave pattern is steady, so that in (2.1) and (2.6) $\omega = 0$ and $\partial/\partial t = 0$. The dispersion equation obtained from (2.1) and (2.3) becomes in this case

$$a(l, m) \theta_{xx} + 2b(l, m) \theta_{xy} + c(l, m) \theta_{yy} = 0, \quad (2.7)$$

where $a(l, m) = \mathcal{L}_{ll}$, $b(l, m) = \mathcal{L}_{lm}$, $c(l, m) = \mathcal{L}_{mm}$.

For the particular wall considered (2.7) is a quasi-linear elliptic equation, and the associated initial value problem was solved by the method of imaginary characteristics, the data l, m, θ being specified along the line $y = 1$.

3. The phase jump in the wavy wall problem

In this section the position of the phase jump in the wavy wall problem is predicted using the data obtained in the earlier paper on either side of the gap region. Discontinuous jumps across a shock are usually treated in terms of the physical conservation equations of the problem. Whitham (1967*b*) has pointed out that in non-linear problems of the present type there are always more conservation equations than the number of required shock conditions. Essentially one must distinguish between those conservation equations which remain valid in an un-averaged form, i.e. through the shock, and those which are true only for slowly varying wave trains. Thus, for example, it would be wrong to choose the wave conservation equations (2.6). However, the energy and momentum equations (2.4) and (2.5) may be used.

Consider in particular the energy conservation equation

$$\frac{\partial}{\partial t}(\omega \mathcal{L}_\omega - \mathcal{L}) - \frac{\partial}{\partial x_i}(\omega \mathcal{L}_{\kappa_i}) = 0. \quad (3.1)$$

Here $\mathcal{L} = \mathcal{L}(\kappa_i, \omega)$ is the Lighthill average Lagrangian, and is in terms of axes fixed relative to the mean motion of the water. In order that (3.1) may be written in terms of axes fixed relative to the wall, for which the wave pattern is steady, one first computes the derivatives $\mathcal{L}_\omega$ and $\mathcal{L}_{\kappa_i}$ in (3.1) and then replaces ω by $-Ul$ and $\partial/\partial t$ by $U_i(\partial/\partial x_i)$ where $U_i = (U, 0)$ is the undisturbed stream velocity, to obtain,

$$U_i \frac{\partial}{\partial x_i} [Ul \mathcal{L}_\omega + \mathcal{L}] - \frac{\partial}{\partial x_i} [Ul \mathcal{L}_{\kappa_i}] = 0. \quad (3.2)$$

Thus about any contour S in the (x, y) -plane

$$\oint_S [(Ul \mathcal{L}_\omega + \mathcal{L}) U_i - Ul \mathcal{L}_{\kappa_i}] n_i ds = 0, \quad (3.3)$$

where n_i is the outward normal of S and ds the curvilinear length element. Equation (3.3) represents the balance of the energy fluxes into and out of S .

The position of the phase jump is determined by choosing a suitable sequence of contours S_n which pass through the gap region of the wavy wall solution. Since the solution is known on either side of this region one then extrapolates the integrand along the contour from either side of the gap region to a point P_n , say. At P_n there will be a discontinuous jump in the value of the integrand, and the precise location of P_n on the contour is adjusted to make the integral about the whole contour vanish. One may note that an integral relation of the form (3.3) can also be derived from the momentum equation (2.4). The same procedure applied in this case would lead to the same prediction of the position of P_n .

Figure 2 reproduces the solution of the wavy wall problem and shows also the position of the phase jump as determined by the above method. A typical integration contour is also shown, consisting of a segment AB from $x = -10$ to $+10$ of the initial line $y = 1$, a portion BC of the 'output curve' from the point $(10, 1)$ (see §4), a transversal CD cutting the shock approximately at right-angles, and the segment DA of the output curve from $(-10, 1)$. Before attempting to find the position of the jump, the reliability of the numerical solution obtained by Howe (1967) in satisfying (3.3) was examined by integrating around several closed contours not passing through the gap region, e.g., $AEFA$. The data available from the solution gave the wave-number κ to four decimal places, and when used in the test integrals the latter were found to differ from zero only in the fifth decimal place (the contributions along AE and EF separately were of order 1); in no case was the error in excess of ± 0.00005 .

The sequence of contours S_n was generated by moving the component CD of the contour shown in figure 2 along AD and BC . The points P_n determined in this way were found to lie closely along a straight line, and the location of the phase jump in figure 2 is on a segment of the line

$$y = 0.3372x + 1.4875, \quad (3.4)$$

which is the least squares fit to the points P_n . The standard deviation of the fit is 0.012. The tip of the shock is rather ill-defined, but is approximately at the point O where $x = 9.17$, and data for the line (3.4) extends as far as $x = 18.61$.

The difference in the number of wave-crests which enter the segment OQ of the shock from the left (1) and which leave it on the right (2) is given by

$$\delta = \frac{1}{2\pi} \int_0^Q [\kappa] \cdot ds. \quad (3.5)$$

The integration is taken along the discontinuity, and the square bracket denotes the jump in wave-number. Since $\kappa = \text{grad } \theta$, and $\theta_1 = \theta_2$ at O ,

$$\delta = \frac{1}{2\pi} \int_0^Q [d\theta] = \frac{1}{2\pi} (\theta_2 - \theta_1)_Q. \quad (3.6)$$

The variation of $2\pi\delta$ with distance s of Q along the jump from O is shown in figure 3, for possible future comparison with experiment.

One may illustrate the possible nature of the free surface of the water in the neighbourhood of the phase jump by plotting the surface elevation along a line

such as CD in figure 2, cutting the jump at right angles. The free surface elevation is obtained from the solution of the earlier paper by substituting the amplitude and phase into the Stokes expansion for finite amplitude surface waves. Possible forms for the surface in the gap region may then be inferred by graphical interpolation between the curves of the known elevation on either side. By taking a large number of such cross-sections and plotting them on the same diagram a picture of the surface which seems almost three-dimensional is obtained.

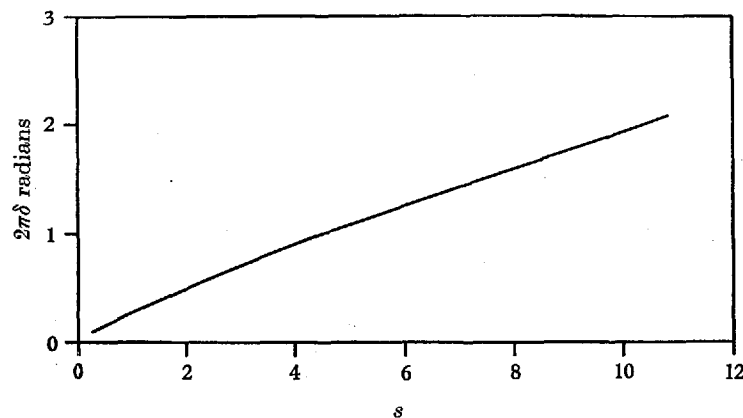


FIGURE 3. $2\pi\delta$ radians is the increase in phase in passing across the discontinuity in the positive x -direction at a distance s from the tip O of the phase jump.

This has been done in figure 4. Each wavy curve represents a cross-section of the free surface (magnified by a factor of 5) along a line such as CD in figure 2. Such a line is taken as the abscissa and the positive direction of the ordinate is in the sense OQ of figure 2. Thus figure 4 should be viewed by looking from the left along the direction of the phase jump line, which is also shown. With respect to this orientation the left-hand tip of each curve occurs where the wave energy is very small and so lies approximately at the undisturbed level of the free surface. The broken-line segments indicate a possible form for the free surface in the gap region.

The effect of the phase jump is rather dramatically emphasized by the shortening of the widths of those troughs shown entering the jump from below. Comment may also be made on the change in form of the crest which passes just to the left of the jump origin. The wave is of characteristic trochoidal section which becomes more peaked as it nears the tip of the jump. However, this tendency towards breaking is apparently arrested by the appearance of the jump which effectively acts as a means of transfer of wave energy from that crest, which moves on above the jump, to the waves in the region beneath the jump, so that the wave crest emerges on the other side much reduced in amplitude.

4. An approximate method for locating the line along which the jump lies

This discussion of the wavy wall problem is conveniently concluded by a consideration of an approximate method of predicting the line along which the

phase jump is formed, which gives rather more physical insight than the purely numerical approach.

Consider the dispersion equation (2.7)

$$a\theta_{xx} + 2b\theta_{xy} + c\theta_{yy} = 0.$$

The characteristics of this equation are given by

$$ady^2 - 2bdxdy + cdx^2 = 0,$$

or, setting $\Delta = \sqrt{(ac - b^2)}$, and noting that the equation is elliptic, by

$$\frac{dx}{dy} = \frac{b}{c} \pm i \frac{\Delta}{c}. \quad (4.1)$$

Remembering that the dispersion equation was originally solved by analytic continuation into the complex x -plane, (4.1) states that, if y is real, the solution at any point P , say, in the real (x, y) -plane is determined by the two conjugate characteristics through P originating at two conjugate points in that complex x -plane which meets the real (x, y) -plane along $y = 1$. Data at these points is obtained by keeping the real part of x fixed and analytically continuing the data from the initial line Γ (i.e. $y = 1$) into the complex x -plane. At P these two characteristics define a plane element cutting the real (x, y) -plane along the curve

$$dx/dy = b/c. \quad (4.2)$$

As a first approximation (4.2) may be regarded as defining the rays along which energy is propagated. Indeed in the case of infinitesimal amplitude they do degenerate into the steady-state analogue of the group velocity lines, since

$$\frac{dx}{dy} = \frac{b}{c} = \frac{\mathcal{L}_{lm}}{\mathcal{L}_{mm}} = \frac{\partial(\mathcal{L}_m, m)}{\partial(l, m)} \cdot \frac{\partial(m, l)}{\partial(\mathcal{L}_m, l)} = - \left(\frac{\partial m}{\partial l} \right)_{\mathcal{L}_m}.$$

For finite amplitudes the neglect of the imaginary part of (4.1) implies that the propagation of changes along the approximate 'rays' (4.2) would be to some extent blurred.

Suppose now that the wall has constant amplitude. The solution of the dispersion equation is then trivially a plane wave, and the energy rays (4.2) become the family of parallel straight lines

$$x = \frac{b}{c}y + \text{const.} \quad (4.3)$$

Here b/c is a function of κ which is constant. But when the amplitude of the wall varies, κ will vary along Γ . Provided that this variation is slow enough, in the sense of Whitham's approximation, the energy rays would be expected to remain straight for some distance from Γ , but the slope b/c would vary in accordance with the variation of κ along the initial line. It is conceivable that under these circumstances the family (4.3) would envelope a caustic along which the solution of the dispersion equation would be indeterminate. Physically one then expects to find a shock in the solution.

These ideas may be illustrated by referring to the results of the wavy wall calculation. The dispersion equation was solved by reducing it to a set of five

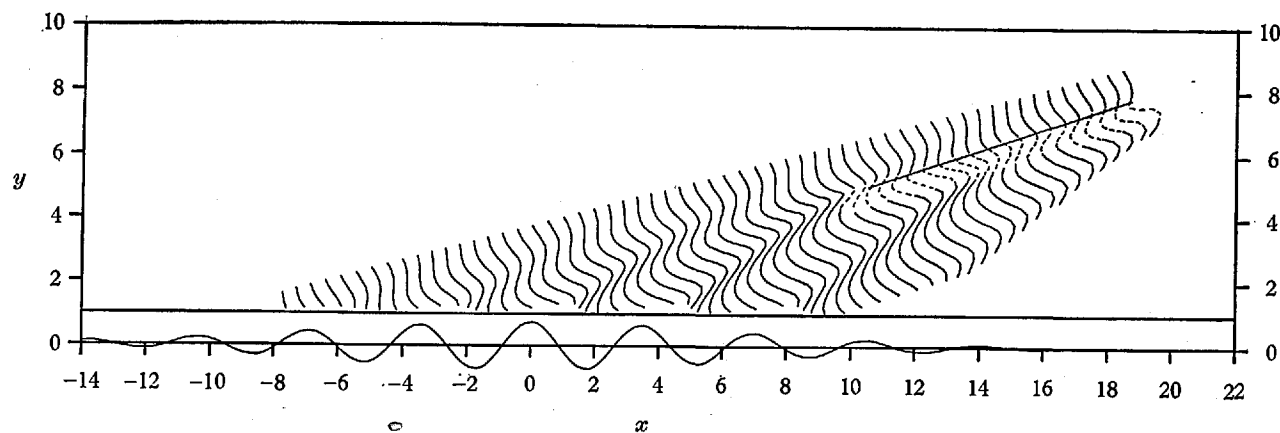


FIGURE 4. Each wavy curve is a cross-section of the free surface along a direction perpendicular to the phase jump. The figure should be viewed by looking from the left along the direction of the discontinuity. The broken-line segments indicate a possible form for the free surface in the neighbourhood of the phase jump.

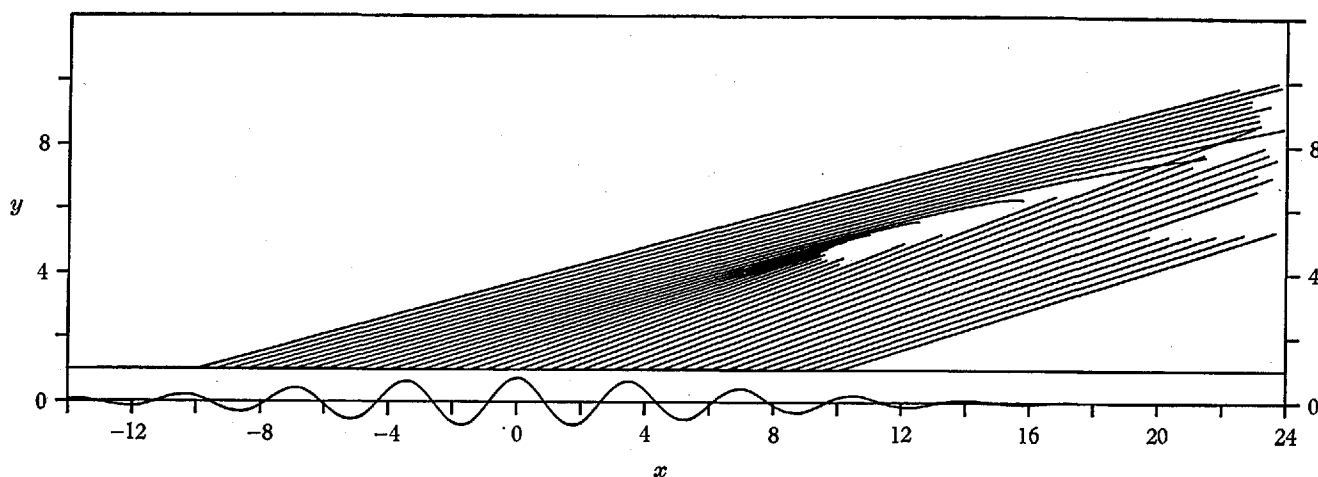


FIGURE 5. The 'output' curves obtained by Howe (1967). The method of imaginary characteristics used in solving the dispersion equation gives the values of l , m and θ along these rays.

first order partial differential equations in which the unknowns (x, y, l, m, θ) were to be determined in terms of co-ordinates (ξ, η) defined by

$$\xi = \frac{\alpha + \beta}{2}, \quad \eta = \frac{\alpha - \beta}{2i},$$

where α, β are the complex characteristic co-ordinates of the dispersion equation. The method involves continuing the initial data into the complex η -plane, where if $\eta = \lambda + i\sigma$, λ is held fixed at λ_0 , say. In the (ξ, σ) -plane the system of equations is hyperbolic and is soluble by the method of characteristics. At $\sigma = 0$ the solution so obtained reduces to the solution in the real (ξ, η) -plane along the straight line $\eta = \lambda_0$. A complete covering of any portion of the real (ξ, η) -plane is obtained by repeating this procedure over a complete interval of λ_0 .

Since the solution of the system of equations corresponds to a constant real value of η , it follows that the image curve in the (x, y) -plane of the straight line $\eta = \lambda_0$ satisfies

$$d\alpha - d\beta = 0, \quad (4.4)$$

where, as described in the earlier paper,

$$x = \lambda_0, \quad y = 1, \quad \text{on} \quad \alpha + \beta = 0. \quad (4.5)$$

This set of 'output curves' obtained in the wavy wall calculation is shown in figure 5. They are straight except near the region where breakdown occurs. Because of this it is valid to approximate to each by giving to κ the value it has where it cuts the boundary $y = 1$, and to solve (4.4) on the assumption that κ is constant. This approximation readily yields the solution

$$x = (b/c)y + \text{const.}, \quad (4.6)$$

the constant being determined by the starting-point on $y = 1$. Thus under these circumstances the output curves correspond to the energy rays (4.3).

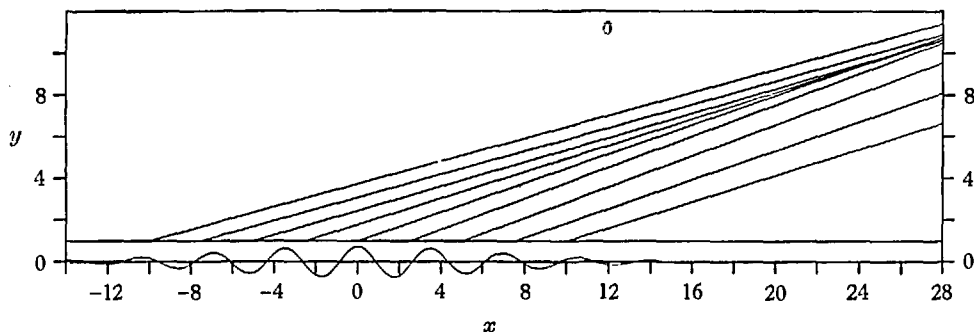


FIGURE 6. The family of 'energy rays' given by equation (4.3). The effect of finite amplitude is to blur the propagation of changes along these rays.

Figure 5 reveals that the output curves of the wavy wall calculation *do* in fact tend to form an envelope along the front of the gap region. This may be compared with figure 6 in which are drawn the energy rays (4.3) originating from the same points on $y = 1$ as the output curves. The energy rays form a cusp

at (23.37, 9.21). This is quite a lot farther from the initial line than the start of the envelope in figure 5.

However, the line (3.4) along which the phase jump occurs is rather well determined by that along which the cusp forms in the case of the energy rays. Taking our definition of the latter line as that ray which touches the cusp at its tip, one obtains for its equation,

$$y = 0.3235x + 1.6609. \quad (4.7)$$

This is negligibly different from (3.4) within the range $x = 10, 20$ of validity of the phase-jump calculation. Thus the blurring of the energy rays may alter the distance to formation of the discontinuity but apparently does not alter the direction in which the phase-jump is formed.

It is again a great pleasure to acknowledge the generous help and guidance of Professor M. J. Lighthill, Sec. R.S., throughout the preparation of this work. Support for this research was provided by a grant from the Central Electricity Generating Board, which is gratefully acknowledged.

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- IV -

On gravity-coupled magnetohydrodynamic waves in the
Sun's atmosphere

Summary

This chapter contributes to the theory of the intense heating of the Solar chromosphere and corona which occurs above a height of about 1000 km. The efficiency of the sources of acoustic radiation within the convective zone appears hardly sufficient to account for all of this heating, and alternative mechanisms of mechanical wave generation, made more powerful by the presence of a magnetic field, are needed to explain chromospheric brightness in magnetically active regions. Lighthill (1967) proposes a mechanism involving the concept of "tongues of turbulence" which intermittently protrude above the convective zone and generate gravity waves in the stratified atmosphere, which at higher altitudes may be converted into magnetohydrodynamic waves. Furthermore, in regions of strong magnetic field (30 gauss, say) disturbances created by these tongues may be dragged clear of the convective zone and propagated along the magnetic field lines. A property of such an Alfvén wave in a stratified

medium is that the vertical component of the vorticity field is propagated without change along the lines of force, whereas horizontal components tend to be spread out. Under these circumstances it is desirable to investigate the manner in which the initially closed vortex loops are affected by the motion, in order to ensure that basic principles, which require the loops to remain closed, are not violated. Such a verification in terms of a simple model of a tongue of turbulence (Hill's spherical vortex) is presented in § 5. The effect of dissipation due to Joule heating of the atmosphere on the motion of the vortex wave is discussed in § 6, where it is estimated that a tongue of turbulence should have a diameter of the order of the granulation size if it is to escape into the upper chromosphere and contribute to the heating process.

1. Introduction

It is known that the temperature in the chromosphere and corona increases outwards from the surface of the Sun. The possible explanation of such heating in terms of the dissipation of wave motions generated in and above the convective zone was discussed in some detail at the 28th. Symposium of the I.A.U., and in particular by Lighthill (1967) who gave a comprehensive review of current theories and suggested possible alternatives.

A satisfactory analysis of wave propagation in the atmosphere of the Sun involves a consideration of the effects of gravity and of a magnetic field on a partially ionized compressible medium. A general, nonlinear analysis of this nature, although very desirable, seems still to be difficult to carry out in practice, however. In general, previous authors have confined themselves to linearized theories of propagation in which the influence of compressibility, or gravity, or the magnetic field, has been supposed small. For example, Lighthill (1960) discusses the case of a compressible conducting atmosphere in the presence of a magnetic field, and Moore & Spiegel (1964) that of a compressible atmosphere subject only to a gravitational field.

Lighthill (1967) reviews the various mechanisms that exist for the generation of waves in and above the convective zone. He concludes that inside the convective zone sound wave

generation alone is possible, since gravity waves can only occur in a stably stratified medium, and magnetohydrodynamic waves appear to be excluded because for moderate magnetic field intensities (e.g., 30 gauss) the gas speed is in excess of the wave propagation speed (Alfvén speed) by more than an order of magnitude, so that such waves would be multiply scattered by nonlinear interactions with the turbulence before they could escape from the convective zone. However, measurements of the Solar magnetic field are tending to indicate that there may be regions where the magnetic field takes on much higher values, so that the question of the possible generation of Alfvén waves in the convective zone, and their conversion to acoustic modes at higher levels, is not yet completely settled. At the same time, the available evidence is not strong enough to give confidence that the radiation of sound and of magnetohydrodynamic waves from the convective zone (and their subsequent dissipation in shocks) is sufficient to account for all of the chromospheric heating.

Because of this possible discrepancy Lighthill proposes an additional mechanism of wave generation in which "tongues of turbulence" intermittently protrude from the convective zone into the stably stratified layers above. Provided that the excitation period of these tongues is greater than about 250 sec. their energy can be radiated by means of

gravity waves, which at higher altitudes may be converted into magnetohydrodynamic waves. The experimental data of Noyes (1967) tends to support this mechanism of wave generation.

If, further, the magnetic field is strong enough, at least of the order of several gauss, it is argued that the tendency for the vertical component of vorticity in these tongues of turbulence to propagate along the magnetic lines of force may well exceed any tendency of the nonlinear convective forces to retain that vorticity within the turbulent region. The result is the propagation of a torsional oscillation along the magnetic lines of force. Lighthill suggests that field strengths at the supergranulation boundaries are adequate for this purpose; for example, a field strength of about 30 gauss where the density, $\rho = 10^{-7} \text{ gm cm}^{-3}$ would probably suffice. A mechanism of this type would usefully help to explain chromospheric brightness in magnetically active regions, including plages and the lines of the chromospheric network. The propagation along the magnetic lines of force might also explain the "rosettes" of spicules of Noyes (1967).

Since the question of what happens to the total vorticity of this propagating Alfvén wave, specifically the manner in which the closed loops of vortex lines are deformed, was raised at the I.A.U. Symposium (Lighthill 1967, page 468), it seemed desirable to consider in detail a highly simplified model for a tongue of turbulence in which

the vortex loops are rather well defined, and to follow its motion along the magnetic field lines. Such a model is provided by Hill's spherical vortex. A first attempt at such an analysis is presented in this chapter. In § 5 the vortex is shown to propagate as a wave along the magnetic lines of force in which the vertical component of vorticity remains unchanged, but the horizontal parts of the vorticity become spread out according to a different law of propagation. A brief discussion of energy losses caused by Joule heating of the atmosphere is also included, in which an estimate is given of the median dissipation level (cf. Chapter I, § 1.2), i.e., the vertical distance moved by the vortex wave from its point of inception just above the convective zone during which one half of its initial energy is transformed into heat. A formula given by Devanathan, Uberoi & Bhatnagar (1965) for a multi-component assembly is applied to estimate the conductivity of the atmosphere for use in this calculation.

In order to simplify the analysis it will be supposed that a linearized approach will suffice, at least as far as the present calculation may be regarded as a preliminary investigation, intended to exhibit the main features of the motion. Further, it will be assumed that the scale of the motions are such that the atmosphere is effectively incompressible (§ 2), except in so far as a Boussinesq type of approximation is applicable. This assumption is reasonable when the Alfvén speed is small compared to the speed of sound

V_s , say. Table 1 is taken from Osterbrock (1961) (with the third column calculated from Osterbrock's Table 2), and shows the relation between V_s and the Alfvén velocity for different heights z , when the magnetic field is 30 gauss. The zero level for z is that at which the optical depth is 0.003.

z km.	V_s km.sec ⁻¹	Alfvén speed km.sec ⁻¹
-300	6.7	0.15
-200	6.5	0.23
-100	6.4	0.37
0	6.3	0.6
250	6.4	1.38
500	6.5	3.0
750	6.6	6.1
1000	6.7	12
1500	7.0	39
2000	7.4	109

TABLE 1

Thus for moderate values of the magnetic field strength the proposed approximation should give reasonable predictions below about 700 km., but above this level the neglect of compressibility would not be justified.

It is concluded (§ 5) that the propagation without change of the vertical component of vorticity does not lead to

a contradiction of basic principles, which require the vortex lines to be closed. Secondly, the estimate of the median dissipation level ($\delta 6$), although only approximate, should be a reasonable order of magnitude result, and indicates that tongues of turbulence effective in the heating process can be associated with the small scale photospheric granules.

2. Equations of motion of a stratified atmosphere

For the relatively small scale motions of the present discussion the effect of curvature of the Sun's surface is negligible, and it will be sufficient to adopt a rectangular system of coordinates with the z-axis vertical. Take the origin in the surface of the Sun (the zero level of Table 1), and denote the three fundamental unit vectors by $\underline{i}, \underline{j}, \underline{k}$.

In the usual notation, the field equations, written in terms of Gaussian units with displacement current neglected, take the form

$$\begin{aligned} \text{curl } \underline{B} &= \frac{4\pi}{c} \underline{j} \quad , \quad \text{curl } \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t} \\ \text{div } \underline{B} &= 0 \quad , \quad \underline{j} = \sigma (\underline{E} + \frac{\underline{v} \wedge \underline{B}}{c}) \end{aligned} \quad (2.1)$$

and the fluid motion equations are,

$$\begin{aligned} \rho \frac{d\underline{v}}{dt} &= \rho \underline{g} - \nabla p + \frac{\underline{j} \wedge \underline{B}}{c} \\ \frac{\partial \rho}{\partial t} + \text{div}(\rho \underline{v}) &= 0 \end{aligned} \quad (2.2)$$

It will also be necessary to note the integral form of the energy equation:

$$\frac{\partial}{\partial t} \int \left\{ \frac{1}{2} \rho \underline{v}^2 + \frac{B^2}{8\pi} + \frac{E^2}{8\pi} + \rho \underline{g} \cdot \underline{\xi} + e \right\} dV = 0, \quad (2.3)$$

the volume integral being taken throughout all space. In (2.2), (2.3), $\underline{\xi}$ denotes the displacement of a fluid particle from its initial undisturbed position, $\underline{v}$ its velocity, $\underline{g}$ the gravitational acceleration ($\sim 274 \text{ m sec}^{-2}$), ρ the density, p the pressure, and e the internal energy per unit volume of fluid.

Let $p_0(z)$, $\rho_0(z)$ be the static distributions of pressure and density respectively, which are related by

$$\frac{\partial p_0}{\partial z} + \rho_0 g = 0. \quad (2.4)$$

In the following analysis it will be supposed that the effect of compressibility may be neglected, except in so far as the buoyancy force on an element of fluid which moves to a new level is influenced by the density change due to the undisturbed pressure p_0 taking different values at the old and new levels. The conditions for this to be valid in the linearized approximation may be set in the form

$$\frac{U^2}{v_s^2} \ll 1, \quad (2.5)$$

where U is the Alfvén speed, and

$$\frac{\ell}{\rho_0} \cdot \frac{\partial \rho_0}{\partial z} \ll 1 \quad (2.6)$$

where ℓ is the length scale of the velocity field. The above mentioned buoyancy condition is expressed by

$$p' \ll \eta \frac{\partial p_0}{\partial z} \quad (2.7)$$

where $p' = p - p_0$ is the fluctuating pressure, and η the vertical displacement of a fluid particle from its equilibrium position. Reference to Table 1 shows that (2.5) should be satisfied in the photosphere and lower chromosphere.

Under these circumstances the second of (2.2) reduces to

$$\text{div } \underline{v} = 0, \quad (2.8)$$

and the linearized approximation then permits one to set

$$\underline{v} = \frac{\partial \underline{\xi}}{\partial t}, \quad \text{div } \underline{\xi} = 0. \quad (2.9)$$

Since $\text{div } \underline{\xi}$ is known it suffices in order to completely determine $\underline{\xi}$ to know one component of $\underline{\xi}$ and the corresponding component of $\text{curl } \underline{\xi}$. This fact will be used below.

First note that the equation giving the magnetic field perturbation $\underline{b}$, say, in terms of the velocity is

$$\frac{\partial \underline{b}}{\partial t} = (\underline{B} \cdot \nabla) \underline{v} + \frac{c^2}{4\pi\sigma} \nabla^2 \underline{b} \quad (2.10)$$

where $\underline{B}$ now denotes the undisturbed field. For a medium of

infinite conductivity this becomes

$$\underline{b} = (\underline{B} \cdot \nabla) \underline{\xi} . \quad (2.11)$$

Next, assuming that each fluid element moves adiabatically,

$$\rho' = \frac{p'}{v_s^2} + \frac{\rho_0}{g} N^2 \underline{\xi} \cdot \underline{k} , \quad (2.12)$$

where $\rho' = \rho - \rho_0$ is the density perturbation, and

$$N = \left[-\frac{g^2}{v_s^2} - \frac{g}{\rho_0} \frac{\partial \rho_0}{\partial z} \right]^{\frac{1}{2}}$$

is the Väisälä-Brunt resonance frequency. In view of (2.7) the first term on the right of (2.12) may be dropped, the equation of motion then becomes

$$\rho_0 \frac{\partial \underline{v}}{\partial t} = -\rho_0 N^2 \underline{\xi} \cdot \underline{k} - \nabla p' + \frac{1}{4\pi} \text{curl}(\underline{b} \wedge \underline{B}) .$$

The pressure is eliminated from this equation by taking the curl, so that after substituting for $\underline{b}$ from (2.11), and taking account of (2.6), one obtains

$$\frac{\partial^2}{\partial t^2} \text{curl} \underline{\xi} = N^2 \underline{k} \wedge \nabla (\underline{\xi} \cdot \underline{k}) + \frac{(\underline{B} \cdot \nabla)^2 \text{curl} \underline{\xi}}{4\pi \rho_0} . \quad (2.13)$$

Hence, if $\zeta = \underline{k} \cdot \text{curl} \underline{\xi}$, then

$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{(\underline{B} \cdot \nabla)^2 \zeta}{4\pi \rho_0} . \quad (2.14)$$

This determines the vertical component of $\text{curl} \underline{\xi}$; the whole motion will be completely determined if one also knows the vertical component of $\underline{\xi}$. Let $\eta = \underline{\xi} \cdot \underline{k}$, then the equation for

η is obtained by taking the vertical component of the curl of equation (2.13), taking note of (2.9). It is

$$\left(\frac{\partial^2}{\partial t^2} + N^2 \right) \nabla^2 \eta = N^2 \frac{\partial^2 \eta}{\partial z^2} + \frac{(\underline{B} \cdot \nabla)^2 \nabla^2 \eta}{4\pi\rho_0}. \quad (2.15)$$

Equations (2.14), (2.15) are those used by Lighthill (1967). Together they determine ξ ; $\underline{b}$ is then given by (2.11).

3. Formal solution of the equations

In this section equations (2.14), (2.15) will be solved in the case where the coefficients may be assumed constant, i.e., when the size of the initial disturbance is small compared to the scale height of the atmosphere. At time $t = 0$ it will be supposed that

$$\begin{aligned} \eta &= \eta_0(\underline{x}) \quad , \quad \frac{\partial \eta}{\partial t} = \eta_1(\underline{x}) \\ \xi &= \xi_0(\underline{x}) \quad , \quad \frac{\partial \xi}{\partial t} = \xi_1(\underline{x}). \end{aligned} \quad (3.1)$$

Consider first equation (2.15). After resolution into Fourier transforms it is readily deduced that the particular solution satisfying (3.1) is given by

$$\eta = \lim_{\varepsilon \rightarrow +0} \iint_{-\infty}^{\infty} \frac{[\omega \bar{\eta}_0(\underline{k}) - i \bar{\eta}_1(\underline{k})] e^{i(\underline{k} \cdot \underline{x} + \omega t)}}{2\pi i [(\omega - i\varepsilon)^2 - \mu^2]} d\underline{k} d\omega \quad (3.2)$$

$$\text{where } \mu^2 = \frac{N^2(1^2 + m^2)}{k^2} + \frac{(\underline{B} \cdot \underline{k})^2}{4\pi\rho_0}, \quad (3.3)$$

$\underline{k} = (l, m, n)$ is the wavenumber vector, and for a function $f(\underline{x})$,

$$\bar{f}(\underline{k}) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} f(\underline{x}) e^{-i\underline{k} \cdot \underline{x}} d\underline{x}.$$

The introduction into (3.2) of $\varepsilon > 0$ ensures that $\eta(\underline{x}, t) = 0$ for $t < 0$.

Suppose the initial disturbance (the "tongue of turbulence") is confined to a neighbourhood of the origin, then it may be shown from (3.2) that at a great distance R from the origin

$$\eta(\underline{x}, t) \sim O(1/R^{3/2}). \quad (3.4)$$

See Lighthill (1960, 1965) for a thorough discussion of results of this nature. By equation (2.11) the same result is seen to hold for the vertical component of the magnetic field perturbation b_z .

Next consider the equation for $\zeta(\underline{x}, t)$. Equation (2.14) is the one dimensional wave equation, and says that (provided ρ_0 is assumed constant) ζ is propagated without change along the magnetic field lines $\underline{B}$ at the Alfvén speed $B/(4\pi\rho_0)^{1/2}$. To simplify the discussion of this case define the scaled distance X measured along the magnetic field lines by

$$dX = (4\pi\rho_0)^{1/2} \frac{\underline{B} \cdot d\underline{x}}{B^2},$$

then equation (2.14) becomes

$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial^2 \zeta}{\partial X^2}. \quad (3.5)$$

Thus if one now supposes that the initial conditions for

ξ refer to the initial distributions of $\xi_0(X)$, $\xi_1(X)$, say, along a magnetic field line, then the solution along that line may be set in the form

$$\xi(X, t) = \frac{1}{2}(\xi_0(X-t) + \xi_0(X+t)) + \frac{1}{2} \int_{X-t}^{X+t} \xi_1(x) dx \quad (3.6)$$

Our concern is with that portion of the solution which is propagated upwards into the photosphere, and it will be supposed that this is along the characteristics $X - t = \text{const.}$ Particular interest is attached to the vorticity propagation, which in this case is given by

$$\frac{\partial \xi}{\partial t} = \frac{1}{2} \left[\frac{\partial \xi_0(X-t)}{\partial t} + \xi_1(X-t) \right]. \quad (3.7)$$

If now $\beta = \underline{k} \cdot \text{curl } \underline{b}$, then by (2.11)

$$\beta = (\underline{B} \cdot \nabla) \xi = -(4\pi\rho_0)^{\frac{1}{2}} \frac{\partial \xi}{\partial t}. \quad (3.8)$$

The general situation is thus seen to be that, at large distances from the initial disturbance the vertical components of $\underline{\xi}$ and $\underline{b}$ suffer spherical attenuation. However, the corresponding components of $\text{curl } \underline{\xi}$ and $\text{curl } \underline{b}$ are propagated without change along the magnetic lines of force. The effect of gravity has been to damp out any vertical disturbances, whereas horizontal motions can propagate without disturbing the gravitational potential energy of the fluid particles.

When the Alfvén wave has travelled a distance equal to several length scales of the initial disturbance, it may be supposed that the vertical perturbations have suffered

complete attenuation, and that the varying part of the field is determined by ξ and β alone.

Let $\underline{v} = (u, v, w)$, (where $w \equiv 0$), then since $\text{div } \underline{v} = 0$, and

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial \xi}{\partial t},$$

it follows that the equation determining $\underline{v}$ in terms of ξ is

$$\nabla_{\perp}^2 \underline{v} = - \text{curl} \left[\frac{\partial \xi}{\partial t} \underline{k} \right] \quad (3.9)$$

where $\nabla_{\perp}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In a similar manner

$$\nabla_{\perp}^2 \underline{b} = - \text{curl}(\beta \underline{k}),$$

hence for the upward propagating wave which satisfies (3.8)

$$\nabla_{\perp}^2 \underline{b} = (4\pi\rho_0)^{\frac{1}{2}} \text{curl} \left(\frac{\partial \xi}{\partial t} \underline{k} \right),$$

i.e.,

$$\underline{b} = -(4\pi\rho_0)^{\frac{1}{2}} \underline{v}. \quad (3.10)$$

Similarly it may be shown that for the downward propagating wave (i.e., that which moves along the characteristics $x+t = \text{const.}$), $\underline{b} = +(4\pi\rho_0)^{\frac{1}{2}} \underline{v}$, so that since the vertical vorticity, and therefore by (3.9) the associated velocity $\underline{v}$, is propagated symmetrically along the magnetic lines of force from the initial disturbance, it follows that the propagation of $\underline{b}$ is asymmetric.

Turn attention now to the energy equation (2.3). Following the Alfvén wave disturbance it has been seen that eventually the motion of the fluid is wholly horizontal, enabling the energy equation to be set in the form

$$\frac{\partial}{\partial t} \int \left\{ \frac{1}{2} \rho_0 \underline{v}^2 + \frac{(\underline{B} + \underline{b})^2}{8\pi} + \frac{E^2}{8\pi} \right\} dV = - \frac{\partial}{\partial t} \int e dV \quad (3.11)$$

the integrals being taken over all space. Now internal energy increases through Joule heating and viscous dissipation, and this will be confined to regions where $\underline{v} \neq 0$. However, radiation and conduction also serve to transmit energy. It will be supposed that in (3.11) the integrals may be taken over those regions of space where $\underline{v} \neq 0$, and that it is permissible to neglect the contribution to the energy balance due to radiation and conduction in comparison with the Joule heating. The small contribution of viscous dissipation will also be neglected. Under these circumstances the right hand member of (3.11) may be replaced by

$$- \int \frac{j^2}{\sigma} dV = - \int \frac{c^2 \rho_0}{4\pi \sigma} (\text{curl } \underline{v})^2 dV$$

where the relations (2.1), (3.10) have been used.

Next consider the left hand side of (3.11). For a highly conducting atmosphere E is of the same order as vB/c , so that energy of the electric field may be neglected in comparison with the magnetic energy. Thus substituting for $\underline{b}$, and noting that the term in $\underline{B} \cdot \underline{b}$ necessarily integrates to zero over the whole of space, one obtains for the final form of the wave-energy equation

$$\frac{\partial}{\partial t} \int \rho_0 \underline{v}^2 dV = - \int \frac{c^2 \rho_0}{4\pi \sigma} (\text{curl } \underline{v})^2 dV. \quad (3.12)$$

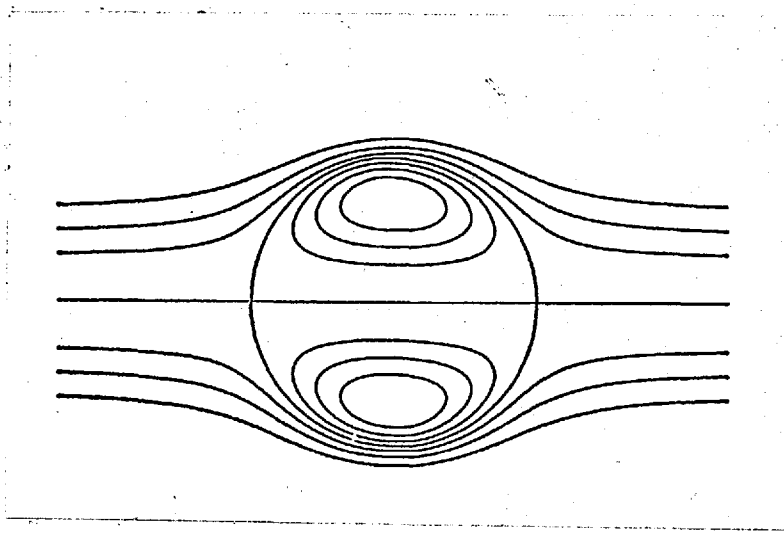
As expected there is equipartition of energy between the magnetic and velocity fields.

4. Hill's spherical vortex

As mentioned in the introduction it is the intention here to apply the analytical results of the previous section to the case of a "tongue of turbulence". Because of the peculiarity of the motion, namely that the vertical component of vorticity propagates without change along the magnetic field lines, whereas the effect of gravity is to extract all the energy from any vertical motions that may have been initially introduced, it would seem to be of considerable interest to consider a case where, initially at least, the system of vortex lines is well defined, and to see how these closed loops are deformed by the motion.

A reasonable model suitable for a consideration of the evolution of the vortex lines, and also as a model of a tongue of turbulence, is Hill's spherical vortex (Lamb 1932, Hill 1894). This is an exact solution of the inviscid, incompressible equations of motion. It combines the classical solution for irrotational steady flow of a stream with general velocity $-U$ along the x -axis, say, past a rigid sphere of radius a , with a rotational solution for all points within that sphere, conditions being matched on its surface. The streamlines of the motion, which are rotationally symmetric about the x -axis, are shown in figure 1.

Impress on everything a velocity $\underline{U_i}$ to obtain the case of a spherical vortex advancing with constant velocity U



Hill's Spherical Vortex

Figure 1.

through an atmosphere which is at rest at infinity. Then inside such a moving vortex the velocity is given by

$$\underline{v} = \left((5a^2 - 6(y^2 + z^2) - 3x^2) \frac{U}{2a^2}, \frac{3Uxy}{2a^2}, \frac{3Uxz}{2a^2} \right) \quad (4.1)$$

and the vorticity inside is $\underline{\omega}$, say, given by

$$\underline{\omega} = (0, -15Uz/2a^2, 15Uy/2a^2), \quad (4.2)$$

the vortex lines being the circles

$$y^2 + z^2 = \text{constant} \quad (4.3)$$

perpendicular to the x-axis.

Outside the sphere the flow is irrotational with

$$\underline{v} = ((2x^2 - y^2 - z^2)a, 3axy, 3axz)Ua^2/2(x^2 + y^2 + z^2)^{5/2} \quad (4.4)$$

The advantage in using Hill's spherical vortex as a model results essentially from the simplicity of the form of the vortex lines (4.3). Further, the dissipation length scale of the resultant Alfvén wave in the atmosphere of the Sun would not be expected to depend critically on the fine scale motions within the vortex, so that an estimate of the median dissipation level should be of value in explaining the subsequent motion of that wave.

5. Solution of the spherical vortex problem

As in §3 it will be supposed here that the coefficients in the equations of motion may be considered to be constant,

modifications for the variable case will be discussed in the next section.

Suppose that initially the centre of the vortex is located at the origin, and that the sphere is in a state of uniform translation at speed U along the x -axis in an atmosphere which is at rest at infinity. Assume further that this velocity distribution is instantaneously created, so that each fluid particle is imparted with velocity, but is not displaced from its equilibrium position. This would model the case of a tongue of turbulence which has arched over at the top of its orbit, and so be a region of horizontal momentum.

It has been seen in § 3 that when interest is centred only on the motion of the Alfvén wave it may eventually be supposed that all the velocities are horizontal. These horizontal velocities are given in terms of the vertical component of vorticity by equation (3.9), so that the motion is determined once the distribution of $\partial \zeta / \partial t$ in all horizontal planes is known. But from (3.7), with $\zeta_0(X) \equiv 0$ at time $t = 0$, it is seen that, if one follows the motion of the upward moving Alfvén wave along the lines of force (assumed parallel), then in any horizontal plane which moves with the vortex this distribution remains constant and equal to one half of that distribution given (at $t = 0$) by (4.2). Thus if the coordinate system is allowed to move with the vortex sphere along the lines of force, the centre being the moving origin, then the functional form of $\partial \zeta / \partial t$ is

independent of the time. By (3.9) the same is true for the velocity distribution.

Now the solution of the equation

$$\nabla^2 \phi = F(x, y)$$

$(-\infty < x < \infty, -\infty < y < \infty)$ which is bounded at infinity is given by

$$\phi = \frac{1}{4\pi} \iint_{-\infty}^{\infty} F(\xi, \eta) \log_e \left[(x-\xi)^2 + (y-\eta)^2 \right] d\xi d\eta.$$

Apply this result to equation (3.9) with $\partial \psi / \partial t$ taking one half the value given by (4.2) for the upward propagating wave. Then it may be shown that $\underline{v} = (u, v, 0)$, where, if $r^2 = x^2 + y^2 + z^2$,

$$u = \begin{cases} \frac{-15U(x^2 + 3y^2 + 2z^2 - 2a^2)}{32a^2}, & r < a \\ \frac{-15U(a^2 - z^2)^2}{32a^2(x^2 + y^2)}(y^2 - x^2) & r > a, |z| < a \\ 0, & |z| > a \end{cases}; \quad (5.1)$$

$$v = \begin{cases} \frac{15Uxy}{16a^2} & r < a \\ \frac{15U(a^2 - z^2)^2}{16a^2(x^2 + y^2)}xy & r > a, |z| < a \\ 0 & |z| > a. \end{cases} \quad (5.2)$$

It follows that

$$\underline{\omega} = \begin{cases} (0, -15Uz/8a^2, 15Uy/4a^2), & r < a \\ \left(\frac{15Uxyz(a^2-z^2)}{4a^2(x^2+y^2)^2}, \frac{15Uz(a^2-z^2)(y^2-x^2)}{8a^2(x^2+y^2)^2}, 0 \right) & r > a, \\ & |z| < a \\ 0, & |z| > a. \end{cases} \quad (5.3)$$

Note that the vertical component of vorticity inside the sphere as calculated from the solutions (5.1), (5.2) and given in (5.3) is the same as in the initial conditions (one half of (4.2)), and so provides a check of the calculations.

Thus even though initially the velocity is non-zero for $|z| > a$, in its final form all the fluid particles move in horizontal orbits inside a horizontal slab of thickness $2a$ which is moving through the atmosphere at the Alfvén velocity. In other words only those horizontal velocities associated with vertical spins are preserved by the motion. Vertical motions have been damped out by gravity.

Further, initially the vorticity was confined to the region within the sphere, but the action of gravity in creating purely horizontal motion is to produce horizontal shearing outside the sphere resulting in the creation there of horizontal vorticity. The velocity lines lie in planes of constant z , and if $\rho^2 = a^2 - z^2$, they are given by

$$\begin{aligned} y^3 - 2\rho^2y + x^2y &= \text{constant} & (x^2 + y^2 < \rho^2) \\ x^2 + y^2 &= 2Cy & (x^2 + y^2 > \rho^2) \end{aligned} \quad (5.4)$$

(C being constant). The vortex lines are given by:

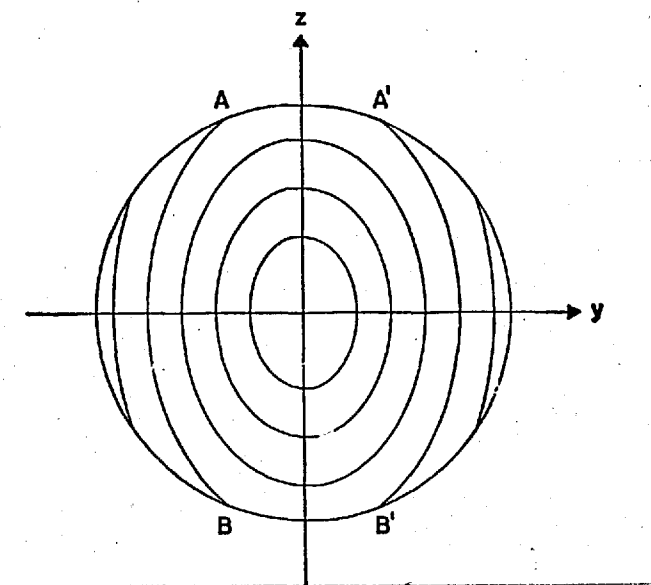


Figure 2.

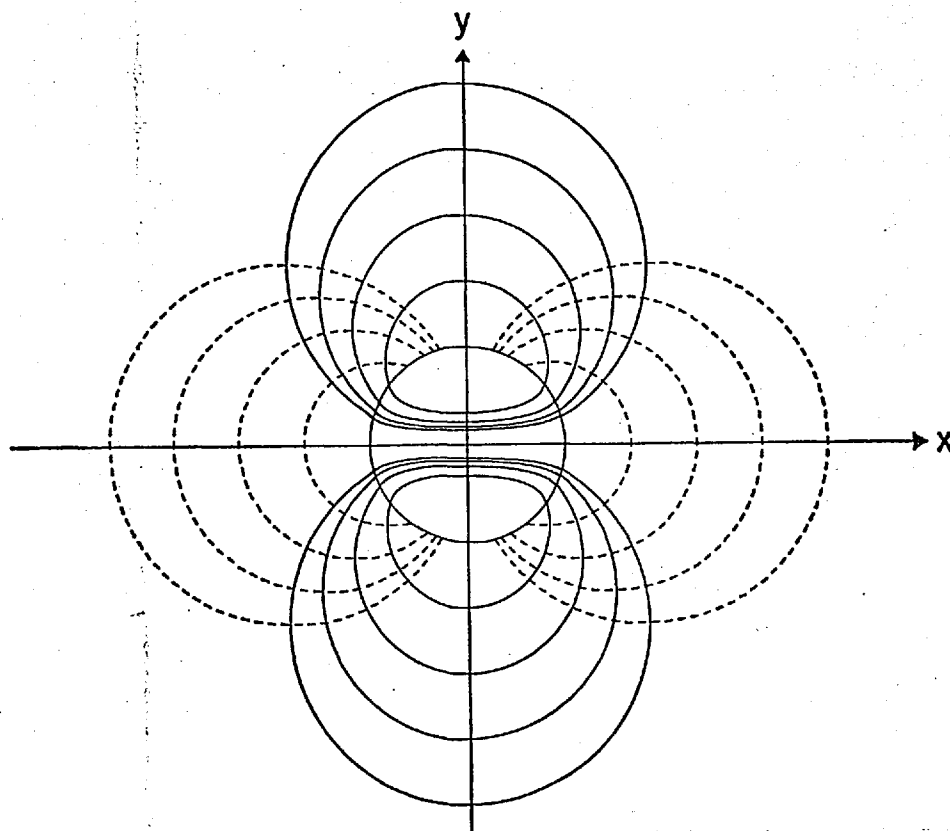


Figure 3.

$$\left. \begin{aligned}
 y^2 + \frac{1}{2}z^2 &= \text{constant} & (r < a) \\
 x^2 + y^2 &= 2Cx & (r > a)
 \end{aligned} \right\} \quad (5.5)$$

in planes perpendicular to the x-axis, and by

in planes of constant z .

From (5.3), (5.5) it is clear that the neglect of vertical motions in the calculation of the solutions has reduced the horizontal component of vorticity within the sphere to one half of its value at $t = 0$. Hence vortex loops which were initially circular, (4.3), have contracted to the elliptical form (5.5). Because of this it is now impossible for all the vortex loops to close within the sphere. Figure 2 is a cross-section of the sphere in a plane perpendicular to the x-axis, and shows closed elliptical vortex loops and elliptical segments of vortex loops which are unable to close within the sphere. When the latter pass through the surface of the sphere they are bent over to become horizontal and segments of the circle (5.5). These circles are the broken line curves of figure 3, and join up elliptic vortex lines (whose semi-minor axis exceeds $((a^2 - x^2)/2)^{\frac{1}{2}}$) which are symmetrically placed with respect to the zx -plane. For example, referring again to figure 2, the vortex sections AB and A'B' are joined A to A', B to B' by such horizontal circles. Thus it is now clear that even though it is only the vertical component of vorticity which has been propagated without change, horizontal components tend to spread out from the propagating sphere

in such a manner that all the vortex loops remain closed. In this final stage when η has become zero it is obvious that the vortex continues to propagate without change in its velocity field, i.e., all components of vorticity propagate without change along the field lines. This also follows from equation (2.13), since there can be no effect of stratification as soon as $\eta = 0$. Similarly any disturbance which initially had no vertical motions associated with it would propagate without change along the field lines.

The solid curves in figure 3 represent the horizontal velocity lines in a section of the vortex system made by a plane of constant z .

6. The median dissipation level

The discussion of this section will be in terms of the fixed coordinate system of §2.

In the solutions given above it has been assumed that the density $\rho_0(z)$ is constant, i.e., that the scale height of the atmosphere is large compared to the radius of the vortex sphere, so large, in fact, that the solutions (5.1), (5.2), (5.3), in which it is assumed that $w \equiv 0$, are attained before any appreciable change has occurred in the density. However, if it is desired to follow the motion of the vortex wave over a considerable distance it is necessary to take account of the slow variations of the density. The next approximation assumes

that each component disturbance of the wave propagates at the local Alfvén velocity. Because of this, and the fact that the atmospheric density decreases with height, one is led to expect a gradual thickening (as calculated below in (6.1)) of the horizontal slab in which the fluid motions are confined as it propagates through the atmosphere. Similarly the original vortex sphere would be sheared and stretched. Since the density variation is with respect to z alone, any disturbance confined initially to a horizontal plane remains in a horizontal plane.

It has also been assumed that the atmosphere has infinite conductivity. When this is not the case there will be finite conversion of kinetic and magnetic energy into heat in accordance with equation (3.12). Now it is well known (see Chapter I, § 1.2, or Alfvén & Färlthammar, § 3.4) that the dissipation length, over which the amplitude of a mode of wavelength λ is reduced by a factor $1/e$, is proportional to λ^2 . The corresponding length for the spherical vortex would therefore be expected to be proportional to a^2 .

It will be assumed that the effect of dissipation is sufficiently gradual to enable one to continue to suppose that a disturbance moves at the local Alfvén velocity. The change in size of a small volume element ΔV , say, of the Alfvén wave can therefore be calculated from the fact that each point of its bounding surface is moving at the local Alfvén velocity. Hence if d/dt denotes the time rate of change

following the wave, i.e., $d/dt = \partial/\partial t + (4\pi\rho_0)^{-\frac{1}{2}}(\underline{B} \cdot \nabla)$, then

$$\frac{1}{\Delta V} \frac{d\Delta V}{dt} = \text{div} \left\{ \frac{\underline{B}}{(4\pi\rho_0)^{\frac{1}{2}}} \right\}.$$

From this it is deduced that, since $\text{div } \underline{B} = 0$, if dV_0, dV represent corresponding volume elements of the travelling disturbance when at heights z_0, z respectively, then

$$dV = \left\{ \frac{\rho_0(z_0)}{\rho_0(z)} \right\}^{\frac{1}{2}} dV_0. \quad (6.1)$$

In the absence of dissipation, (6.1) leads directly to the result that the fluctuating velocity amplitudes within the vortex vary as $\rho_0^{-\frac{1}{4}}$ (Walen 1944, Cowling 1957).

From (3.12) the total wave energy $\mathcal{E}$, say, is given by

$$\mathcal{E} = \int \rho_0 v^2 dV \quad (6.2)$$

the integration being taken over the volume of the moving slab. In order to estimate the velocity on the right hand side of this equation one may proceed as follows.

In the absence of dissipation the vertical component of vorticity propagates without change when the density is constant, and increases in proportion to $\rho_0^{-\frac{1}{4}}$ when the effects of slow variations in the density are included. In the latter case at any given instant, the vertical vorticity distribution in a given horizontal plane, considered to be moving with the vortex, is obtained from the distribution at $t = 0$, say, by multiplying the latter by a factor which depends only on

the vertical distance moved by that plane from its initial position. It is assumed that this approximation may be carried over to the dissipative case, in other words, that there exists a multiplying factor $\theta(z, z_0)$ such that the amplitude of the propagating vorticity when at height z is obtained from its value when at height z_0 by multiplying the latter by this factor, i.e.,

$$\frac{\partial \xi(z)}{\partial t} = \theta(z, z_0) \frac{\partial \xi(z_0)}{\partial t}.$$

Further, it will be supposed that θ may be taken to be constant throughout the vertical extent of the vortex, so that $z - z_0$ may be thought of as the vertical distance moved by the centre of the vortex. The assumptions are equivalent to supposing that the energy of the moving disturbance is confined to a fairly narrow band of wavenumbers, and that the thickness of the moving slab is much smaller than the scale height of the atmosphere. Appealing to equation (3.9) one obtains the equivalent result $\underline{v}(z) = \theta(z, z_0) \underline{v}(z_0)$.

The function $\theta(z, z_0)$ may be determined from (3.12). First note that the operator $\partial / \partial t$ in that equation may be replaced by $(4\pi\rho_0)^{\frac{1}{2}} \underline{B} \cdot \underline{k} \partial / \partial z$, i.e., differentiation following the vertical motion of the vortex. Then θ satisfies the equation

$$\frac{\underline{B} \cdot \underline{k}}{(4\pi\rho_0)^{\frac{1}{2}}} \frac{\partial}{\partial z} \int \rho_0(z) \theta(z, z_0)^2 \underline{v}(z_0)^2 dV = - \int \frac{c^2 \rho_0(z) \theta(z, z_0)^2 \omega(z_0)^2}{4\pi\sigma(z)} dV. \quad (6.3)$$

The integrals here (in which $\rho_0(z)$, $\theta(z, z_0)$, $\sigma(z)$ may be

regarded as constant) are taken over the volume of the moving slab when its centre is at height z , but may be evaluated in terms of the known initial conditions by means of the transformation (6.1).

In order to calculate the variation of $\mathcal{E}$ with z one has

$$\begin{aligned}\mathcal{E} &= \int \rho_0 \underline{v}^2 dv \\ &= \int_{V_0} \rho_0(z) \theta(z, z_0)^2 \underline{v}(z_0)^2 \left[\frac{\rho_0(z_0)}{\rho_0(z)} \right]^{\frac{1}{2}} dv_0 ,\end{aligned}$$

the integration being over the volume V_0 of the slab when at height z_0 , i.e.,

$$\mathcal{E} = \theta(z, z_0)^2 (\rho_0(z) \rho_0(z_0))^{\frac{1}{2}} \int_{V_0} \underline{v}(z_0)^2 dv_0 . \quad (6.4)$$

Hence dividing (6.3) by (6.4),

$$\frac{1}{\mathcal{E}} \frac{\partial \mathcal{E}}{\partial z} = \frac{-c^2}{B(z) \sigma(z)} \left[\frac{\rho_0(z)}{4\pi} \right]^{\frac{1}{2}} \frac{\int_{V_0} \underline{\omega}(z_0)^2 dv_0}{\int_{V_0} \underline{v}(z_0)^2 dv_0} \quad ((6.5))$$

where $B = \underline{B} \cdot \underline{k}$.

Next assume that the solutions of the vortex problem given by (5.1), (5.2), (5.3) are attained before the vortex wave has moved a distance large compared to the scale height of the atmosphere, and before it has suffered any appreciable dissipation, so that they may be regarded as initial conditions to be applied at z_0 . Then the ratio of the two

integrals in (6.5), is found to be equal to $2l/2a^2$. Hence the median dissipation level z , i.e., the level attained by the vortex wave from z_0 during which one half of its initial energy is transformed into heat, is the root of the equation

$$\log_e 2 = \frac{2lc^2}{4\pi^{\frac{1}{2}}a^2} \int_{z_0}^z \frac{\sqrt{\rho_0(\xi)}^7}{B(\xi)\sigma(\xi)} d\xi \quad (6.6)$$

In evaluating this integral numerically use has been made of data concerning the number densities of electrons, H^+ , H and M^+ , where M is an easily ionized metal (taken here to be typically Na), given by Osterbrock (1961). The conductivity was calculated from a formula given by Devanathan, Uberoi & Bhatnagar (1965) for a multi-component assembly (their equation (4.13)), in which the particle collisions were assumed to be perfectly elastic, and the average kinetic temperature in the photosphere and lower chromosphere was taken to be $6000^\circ K$. The calculation was carried out for four values of B : 30, 100, 200 and 300 gauss.

The variation of σ and ρ_0 with height is shown in Table 2, where $z_0 = -400$ km.

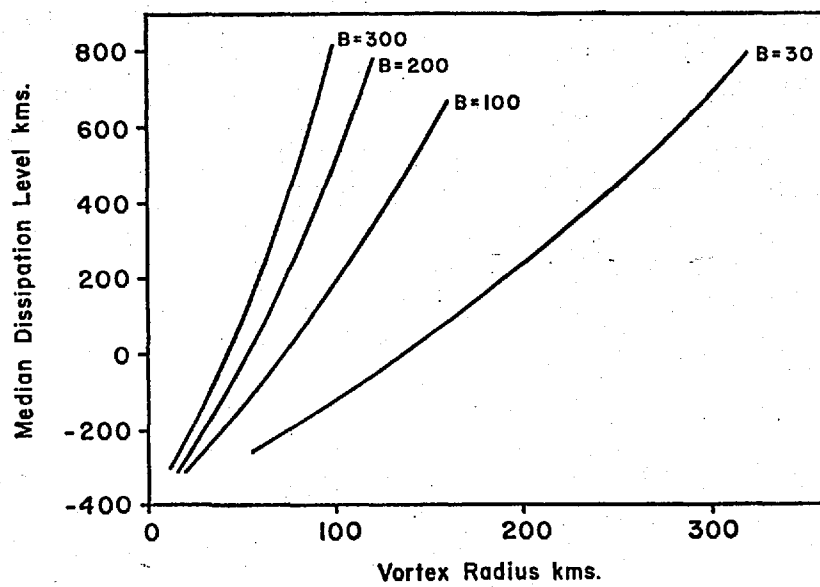


Figure 4.

z km.	$\log_{10} \rho_0$ (gm cm ⁻³)	$\log_{10} \sigma$ (sec ⁻¹)
-400	-6.6	11.6
-300	-6.8	10.9
-200	-7.1	9.8
-100	-7.6	9.4
0	-8.1	9.3
250	-8.6	8.8
500	-9.3	8.3
750	-9.9	8.6

TABLE 2

Figure 4 shows the median dissipation level z plotted against the vortex radius a for the four values of the magnetic field strength considered. It appears that in the least favourable case of B equal to 30 gauss a vortex whose radius exceeds about 300 km may escape from the lower chromosphere retaining over half of its initial energy. Before leaving it will be stretched vertically and sheared horizontally, but would otherwise remain a fairly compact unit. However, at heights greater than about 700 km the effect of compressibility can no longer be neglected. This means that the dispersion relation governing the propagation of small disturbances is no longer linear, so that the vortex system, while in the main continuing to move parallel to the lines of force, would be influenced by the appearance of acoustic modes, and

in particular would constitute a source of shock waves which would give additional means of dissipation. For the higher values of B (which recent measurements of the Solar magnetic field have revealed between the supergranules) the effect is to increase the height of penetration of a vortex, and consequently the heating efficiency of the vortex mechanism.

The results indicate that in general the smallest observable disturbance of the present type would have a diameter of about 600 km. This is significant if these waves are to be interpreted as the spicules of Noyes (1967), since this minimum size of a tongue of turbulence is of the same order as the diameter of a small scale photospheric granule. Now recent evidence suggests that the spicules, whose properties are quite well known from observations at the limb, occur in clusters (the "rosettes of mottles" of Beckers (1963)) located above supergranulation boundaries, which are positively correlated with the chromospheric network of strong magnetic field (up to a hundred times the quiet Sun field intensity). The birth rate of the spicules in these clusters has been estimated by Beckers (1964) to be of the same order as that of the small scale granules of the underlying photosphere. Hence these small scale granules would appear to play the rôle of Lighthill's protruding tongues of turbulence which, in these regions of strong magnetic field, produce disturbances which are dragged clear of the convective zone and propagated along the field lines into

the upper chromosphere.

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