Multi-beam Energy Moments of Multi-beam Particle Velocity Distributions

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Key Points:

Standard velocity moments of effectively-disjoint measured multibeam particle velocity distributions, f(v) can be misleading.

So-called multibeam moments of multibeam f(v) are found by taking a standard moment of each beam and then summing over beams.

Standard thermal moments (pressure, thermal flux, etc.) of multibeam f(v) contain pseudothermal parts which are absent in multibeam moments.

Abstract

High resolution electron and ion velocity distributions, f(**v**), which consist of N effectively disjoint beams, have been measured by NASA's Magnetospheric Multi-Scale Mission (MMS) and in reconnection simulations. Commonly used *standard* velocity moments assume a single *mean-flow-velocity* for the entire distribution. This can lead to counterintuitive results for a multi-beam f(**v**). An example is the standard thermal energy density moment (at a given space-time point) of a pair of equal and opposite cold particle beams. This standard moment is nonzero even though each beam has zero thermal energy density. By contrast, a *multibeam* moment of two or more cold beams at a given position and time has no thermal energy. A multibeam moment is obtained by taking a standard moment of each beam and then summing over beams. In this paper we will generalize these notions, explore their consequences and apply them to an f(**v**) which is sum of tri-Maxwellians.

Both standard and multibeam energy moments have coherent and incoherent pieces. Examples of *incoherent* moments are the thermal energy density, the pressure and the thermal energy flux (enthalpy flux plus heat flux). Corresponding *coherent* moments are the bulk kinetic energy density, the RAM pressure and the bulk kinetic energy flux. The difference between a standard incoherent moment and its multi-beam counterpart will be defined as the "*pseudothermal part"* of the standard moment. The *sum* of a pair of corresponding coherent and incoherent moments is the *undecomposed* moment. Undecomposed standard moments are always equal to the corresponding undecomposed multi-beam moments.

1 Introduction

1.1 VELOCITY MOMENTS OF MULTIBEAM VELOCITY DISTRIBUTIONS, f(**v**)

In space plasma physics is it very common to discuss the properties of a plasma in terms of its velocity moments – density, velocity, pressure, etc. and to use a fluid theory framework to describe and predict the behavior of many space plasma phenomena (Cravens, 2004; Bellan, 2006; Kulsrud, 2005). Two different kinds of space plasmas are commonly treated as fluids: collisional and collisionless.

- In collisional space plasmas, such as the chromosphere and lower ionosphere, the collisional mean free path is short and the particle velocity distributions, f(**v**), are at or close to Maxwellian, locally (i.e., f(**v**) is close to local thermal equilibrium). In this case the fluid equations for a given species can be derived from the Boltzmann equation for that species by the Chapman-Enskog procedure (Braginskii, 1963) and include collisional transport coefficients such as electron-ion equilibration rates and thermal conductivity.
- However, in Earth's upper ionosphere, magnetosphere and solar wind, the collisional mean free path is large, so the plasma is *collisionless* and often *far* from thermal equilibrium [Pashmann, et al, 1998]. *Standard* collisionless fluid equations and energy transport equations can be derived for constituent species by taking velocity moments of the Vlasov eqn. for each species (Bellan, 2006; Aunai, et al, 2011; Goldman, et al, 2015). The transport equations are expressed in terms of (standard) velocity moments of the particle distribution, f(**v**). In collisionless space-plasma fluid equations there are no electron-ion collisions and there is no collisional thermal conductivity. Energy transport can only occur by *convection*, described in terms of particle energy fluxes or by *radiation* (Poynting flux).

In practical space plasma applications, collisionless velocity distributions, f(**v**), are measured by electrostatic analyser instruments (Fazakerley, et al., 1998), and (standard) moments can be constructed on the ground or onboard (so reducing the telemetry) (Reme et al, 2001; McFadden et al. 2007) The launch of the Fast Plasma Instrument (FPI) (Pollock et al., 2016) on board the Magnetosphere MultiScale (MMS) Mission (Burch, et al, 2016) now enables the full 3D electron and ion distribution to be measured over unprecedented time-intervals of 30 ms and 150 ms cadence respectively (or 7.5 ms and 37.5 ms in certain operating modes (Rager, et al, 2018).

In many regions of interest, not least the electron dissipation region of magnetic reconnection (e.g., Burch, et al, 2015), the ion or electron velocity distribution, f(**v**), consists of a number of effectively disjoint pieces or "*beams,*" in velocity space. Ambiguities may arise from the use of standard moments because, experimentally, the distribution is not contiguous; nor is it single peaked. F

Consequently, for such multi-beam f(**v**) it may be useful to take moments beam by beam and then add them together. We will call such sums *multi-beam* moments. The purpose of this paper is to lay out a systematic framework for understanding multi-beam moments taken at a single space-time point (or an average over a small space-time neighborhood of such a point).

The *standard moments* of a given f(**v**) include only *one* flow velocity, **u** and only *one* mean density, n. This is the case both for multi-beam f(**v**) and for single-peaked f(**v**). Higher-order moments are either coherent (involving **u**) or "incoherent," corresponding to moments

which are correlations of velocity fluctuations, such as $\int d^3\mathbf{v} f(\mathbf{v})|\mathbf{v} - \mathbf{u}|^2$ or $\int d^3\mathbf{v} f(\mathbf{v})\,(\mathbf{v} - \mathbf{u})\,|\mathbf{v} - \mathbf{u}|^2$. For example, the energy density , U, of electrons or ions is the sum of a bulk kinetic energy density, $U_{bulk} = nmu^2/2$, and a "thermal" energy density moment, $U_{therm} = m\int d^3\mathbf{v} f(\mathbf{v})|\mathbf{v} - \mathbf{u}|^2/2$; the energy density *flux*, **Q** is a sum of a coherent bulk kinetic energy flux, $\mathbf{Q}_{bulk} = \mathbf{u}nmu^2/2$, and a "thermal" energy flux moment, $\mathbf{Q}_{therm}$ (enthalpy plus heat flux). The term *"incoherent"* is used here to refer to velocity moments which are correlation functions of velocity *fluctuations*, $(\mathbf{v} - \mathbf{u})$. We use another term, *thermal,* interchangeably with *incoherent. Thermal* does *not* imply *thermal equilibrium* or *Maxwellian* velocity distributions. It only means *incoherent*.

Standard moments of multi-beam (effectively disjoint) $f(\mathbf{v})$ can give rise to ambiguities. A classic example is the standard thermal energy-density moment of a pair of equal and opposite cold particle beams. As shown explicitly in the next section, the *standard* thermal energy-density moment, U_{therm} of this two-beam system is non-zero even though each beam has zero thermal energy density.

The definition $U_{therm} = m\int d^3\mathbf{v} f(\mathbf{v})|\mathbf{v} - \mathbf{u}|^2/2$ guarantees that U_{therm} is incoherent. In this example, the flow velocity, **u**, is zero because, there is zero number density flux due to symmetry. However, U_{therm}/n cannot readily be interpreted as the "temperature" of the two-beam system because the fluctuations, $(\mathbf{v} - \mathbf{u})$ are not taken with respect to the flow velocities, $\mathbf{u}_j$ of the beams, but rather with respect to $\mathbf{u} = 0$. For this reason, we will call U_{therm} a *pseudothermal* energy density. More generally, for beams that are not cold, only a *part, ΔU_{therm},* of the standard moment, U_{therm} is pseudothermal. The remainder is equal to the *multi-beam* thermal moment, U^{MB}_{therm}.

In general, a two-cold-beam system may develop in time into *one* beam possessing a standard thermal energy density equal to ΔU_{therm}. Such time evolution might be due either to collisions or to the nonlinear evolution of the two-stream instability. This interpretation could be valid and useful for some applications. However, in reconnecting magnetospheric plasmas and in other collisionless plasmas, collisional relaxation to thermal equilibrium is too slow to be relevant. Furthermore, multimodal distributions (especially for ions) are not necessarily two-stream unstable. This appears to be the case for the example presented below (Fig. 3) of a multi-beam ion distribution from a simulation of magnetic reconnection in the magnetosphere.

In this paper an alternate method is developed for taking velocity moments, at a single space-time point, of an N-beam velocity distribution, $f(\mathbf{v})$, of form $f(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v}) + \ldots + f_N(\mathbf{v})$. Such *multi-beam moments* are created by taking standard coherent moments or standard thermal moments of each beam, $f_j(\mathbf{v})$ and then summing over beams j = 1 to N. This is especially relevant to an assembly of beams because fluctuations in each beam are then automatically taken with respet to the to the flow velocity, $\mathbf{u}_j$ of that beam, rather than to the centroid velocity of the assembly of beams. This procedure eliminates *pseudothermal parts* of standard moments of $f(\mathbf{v})$ whose interpretations are ambiguous or counter-intuitive such as the standard "thermal" energy density and related "temperature," which are *system* properties, rather than individual *beam* properties. We do not consider, in the multi-beam moment analysis in this paper, the time-dependence of moments, which might eventually transform them (via nonlinear saturation or collisions) into a state in which their temperature is unambiguous.

It is important to note that the sum of the coherent (bulk) and "thermal" parts of an energy density or energy density flux moment remains the same whether the moments are standard or multibeam. Such a sum will be called an *undecomposed* moment. An example is the

undecomposed particle bulk kinetic energy density, $m\int d^3\mathbf{v} f(\mathbf{v})|\mathbf{v}|^2/2$, which can be decomposed by writing $\mathbf{v} = \mathbf{u} + (\mathbf{v} - \mathbf{u})$ or by writing $\mathbf{v} = \mathbf{u}_j + (\mathbf{v} - \mathbf{u}_j)$ for each beam in the sum of beams.

1.2 PSEUDOTHERMAL ENERGY MOMENT OF A PAIR OF COLD BEAMS

Fig. 1 shows a distribution, $f(\mathbf{v})$, consisting of two equal and opposite cold electron beams, with velocities $\mathbf{u}_0$ and $-\mathbf{u}_0$ and equal densities n_0. According to standard moment theory the effective velocity distribution, $f(\mathbf{v})$ is one entity, with one flow velocity, $\mathbf{u}$. In this example, $\mathbf{u} = 0$, so the *bulk kinetic energy moment, U_{bulk}* is zero and the (single) density is $n = 2n_0$. The *incoherent* energy density moment found from *standard* moment theory is $mn(u_0)^2/2$. This incoherent part of the energy density is often called the *thermal energy density* and written as $U_{therm} = n$"T." This yields an effective *temperature*, "T" $= m(u_0)^2/2$. A difficulty with such a *standard* moment is that the pair of *cold moving* beams appears to have a temperature, which we call here a *pseudotemperature*, but *no* bulk kinetic energy density.

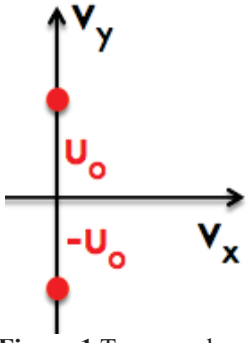


Figure 1 Two equal and opposite cold beams

This is customarily remedied by simply considering $f(\mathbf{v})$ to be a *two-beam* system, $f(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v})$. The density and velocity moments of f_1 and f_2 are $\{n_1, \mathbf{u}_1\}$ and $\{n_2, \mathbf{u}_2\}$, where, $n_1 = n_2 = n_0$, $\mathbf{u}_1 = \mathbf{u}_0$ and $\mathbf{u}_2 = -\mathbf{u}_0$. The energy moments of *each* beam are now $U_{bulk1,2} = n_0 m(u_0)^2/2$ and $U_{therm1,2} = 0$. For the *system* of two beams, $(U_{bulk})^{2\text{-beam}} = U_{bulk1} + U_{bulk2} = nm(u_0)^2/2$, where $n = n_1 + n_2 = 2n_0$ and $(U_{therm})^{2\text{-beam}} = 0 + 0$, which is more intuitive than the results of the standard moment analysis, $U_{bulk} = 0$ and $U_{therm} = nm(u_0)^2/2$.

We summarize the results for the two different ways of taking energy moments of $f(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v})$ in the table below:

Table 1 *Energy density moments for two system of equal and opposite cold (e or i) beams*

Standard energy moments: 2 cold beams		**Multibeam energy moments: 2 cold beams**	
U_{bulk}	0	$(U_{bulk})^{2\text{-beam}}$	$nm(u_0)^2/2$
U_{therm}	$nm(u_0)^2/2$	$(U_{therm})^{2\text{-beam}}$	0
$U = U_{bulk} + U_{therm}$	$nm(u_0)^2/2$	$U = (U_{bulk})^{2\text{-beam}} + (U_{therm})^{2\text{-beam}}$	$nm(u_0)^2/2$

The bottom (shaded) row demonstrates an important general result: that the *sum* of the bulk and thermal energy terms has the same value, $U = nm(u_0)^2/2$, for both the *standard* and the *multibeam* methods of taking moments. In this paper, the *generalized* U will be referred to as the *undecomposed* energy moment and the bulk and thermal energies as the decomposed energy moments. It is clear from Table 1 that the values of the decomposed moments differ, depending on the method used for taking moments. Since U is independent of the approach used, it can be employed reliably to quantify the properties of the plasma without ambiguity. This offers a path

forward in studying the properties of complex distribution functions such as those observed by MMS.

1.3. ORGANIZATION AND OUTLINE OF THIS PAPER

In the present paper the concepts discussed above in Fig. 1 and Table 1 for two cold beams will be generalized to N-beams and the full set of multibeam pressure, energy density and energy density flux moments. Standard and multibeam moments then will be evaluated and compared for an analytic particle velocity distribution, f(**v**), consisting of a sum of N tri-Maxwellian beams.

The content of this paper is indicated below in terms of questions and issues to be addressed:

1. Examples of measured and simulated f(v) *which are effectively disjoint.*

   What are examples of effectively disjoint velocity distributions, $f(\mathbf{v}) \equiv f(\mathbf{v}, \mathbf{r}_0, t_0)$, found at certain locations and times during magnetic reconnection in the magnetopause and found in PIC simulations of reconnection in the magnetotail? Examples treated in this paper (Fig. 2 and 3) focus on disjoint *ion* velocity distributions. However, the theory applies equally well to disjoint *electron* velocity distributions, such as electron velocity distributions containing "crescents." One physical mechanism for producing a pair of velocity-space *ion* beams at $\{\mathbf{r}_0, t_0\}$ has been identified in Fig. (3) in terms of Lagrangian beam particle trajectories through $\{\mathbf{r}_0, t_0\}$ and other points in space and time (Eastwood, et al, 2014).

2. Standard energy transport theory

   What are the various energy moments and transport equations associated with a given f(**r**,**v**,t)? We derive and discuss moments and standard energy transport equations in terms of kinetic theory (i.e., the Vlasov equation) in a collisionless plasma

3. Multibeam moments.

   What is the difference between *undecomposed* energy density, pressure and energy flux moments and *decomposed* energy density moments (bulk, thermal), pressure moments (RAM, thermal) and energy flux moments (bulk, enthalpy, heat flux)? What is the relationship between *standard* and *multibeam* decomposed moments? How are pseudothermal parts of standard decomposed moments determined? These questions are addressed in terms of the example of the *standard* and *multibeam* moments of a set of N tri-Maxwellian beams.

2. Measurements and simulations exhibiting multibeam velocity distributions

2.1 MULTIPLE ION BEAMS NEAR X-LINE DURING DAYSIDE RECONNECTION

In a pioneering paper (Burch, et al, 2016), high resolution data from the Fast Plasma Instrument (FPI) onboard MMS satellites was used to study electron and ion velocity distributions in the electron diffusion region of Earth's magnetopause during magnetic reconnection. The left image in Figure 2 shows the magnetic field and standard particle and field energy fluxes as functions of time along the trajectory of MMS4. The reconnection x-line is crossed at time 1307, as indicated by the vertical dashed line. The right image (here constructed from the data) shows 3D contour maps of the ion distribution, f(**v**), measured over a time interval of 150ms a few seconds before and a few seconds after an x-line crossing. Details are given in the Figure caption. It is clear that both ion distributions, f(**v**), are far from equilibrium and disjoint. Each f(**v**) is therefore an ion multibeam distribution amenable to the multibeam moment treatment described in this paper. Such a treatment will generally give ion energy moments *different* from the ones on the left in Figure (2).

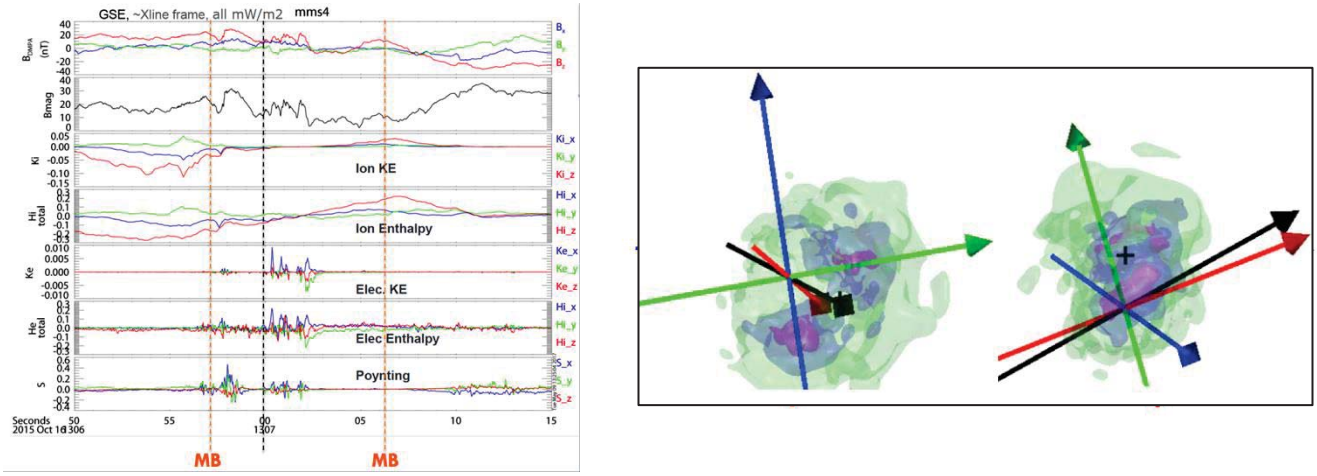


Figure 2: Data and 3-D ion velocity distribution measured by MMS-4 during dayside EDR event [Burch, et al, 2016]. Left panel: MMS4-measured electron and ion energy density fluxes (bulk kinetic energy density flux, enthalpy flux and Poynting fluxes) showing x-point around time 1307 (vertical black dashed line). The two vertical orange dotted lines labelled MB are at times ~3s before 1307 and ~6s after 1307. Multiple-beam-like features are found in the ion velocity distribution, f(**v**), at both of those times. The right two panels show f(**v**) determined by the Fast Plasma Instrument (FPI) at the later time from two different perspectives. In those panels the black arrow is in the direction of **B** and the blue, green and red arrows show the directions of GSE coordinates in x, y, z. The tip of each colored arrow is at a velocity corresponding to an energy of 1.6 keV. The values of f(**v**) on successive surface contours differ by factors of 5.

2.2 TAIL RECONNECTION SIMULATION SHOWS MULTIPLE ION BEAMS

Multibeam (disjoint) ion distributions, f(**v**), have also been found in *simulations* of magnetic reconnection in Earth's magnetotail. An example can be found in (Eastwood, et al, 2014) in which PIC simulations are carried out in support of THEMIS observations of magnetic reconnection on February 27, 2009 and complex ion distributions were observed. The THEMIS magnetic field data were used to establish appropriate comparison cuts through a particle-in-cell simulation of reconnection, and very good agreement was found between the observed and simulated ion distributions on both sides of the dipolarization front.

The principal feature of interest in the simulation is the dynamics of a pair of counterpropagating ion beams found in the dipolarization front about five ion inertial lengths in front of the unperturbed plasma just below the neutral sheet at time t = 30. They are visible in the 2D reduced distribution in the v_x-v_y (reconnection) plane shown in Figure 3a. This reduced distribution is integrated over v_z. The two beams are represented by two small cubes in velocity space labeled A and B. The other two orthogonal reduced distribution functions are shown in Figures 3b and 3c.

Figures 3d–3g show the real space self-consistent *trajectories* of the bunches of ions that pass through these two phase-space cubes at $t = 30$ over the time interval $23 < t < 35$. Trajectories are of particular interest because they illustrate *how* multibeam particle distributions, f(**v**), might arise.

The trajectories are superposed on plots of $E_{z\mathrm{SIM}}$ (approximately the reconnection electric field) and $E_{y\mathrm{SIM}}$ (approximately the Hall electric field), respectively. The fields and circled groups of ions in Figures 3d and 3e correspond to the start time for the trajectories ($t = 23.28$). The fields and circled bunches of ions in Figures 3f and 3g correspond to the time when the trajectories cross, thereby giving rise to the disjoint reduced distribution function show in Figures 3a-c. The ions composing the two beams are clearly distinguishable, with beam A moving primarily downward and to the right, while beam B moves primarily upward and to the right (as labeled in Figure 3d).

At the start time, the ion bunches are widely separated in space, with each bunch near a different separatrix. The ion bunch trajectories later in time are determined by the electric and magnetic forces. The ions are not frozen-in (this is the ion-diffusion region). For details concerning the forces and their influence on the ion trajectories refer to Eastwood, et al, (2014).

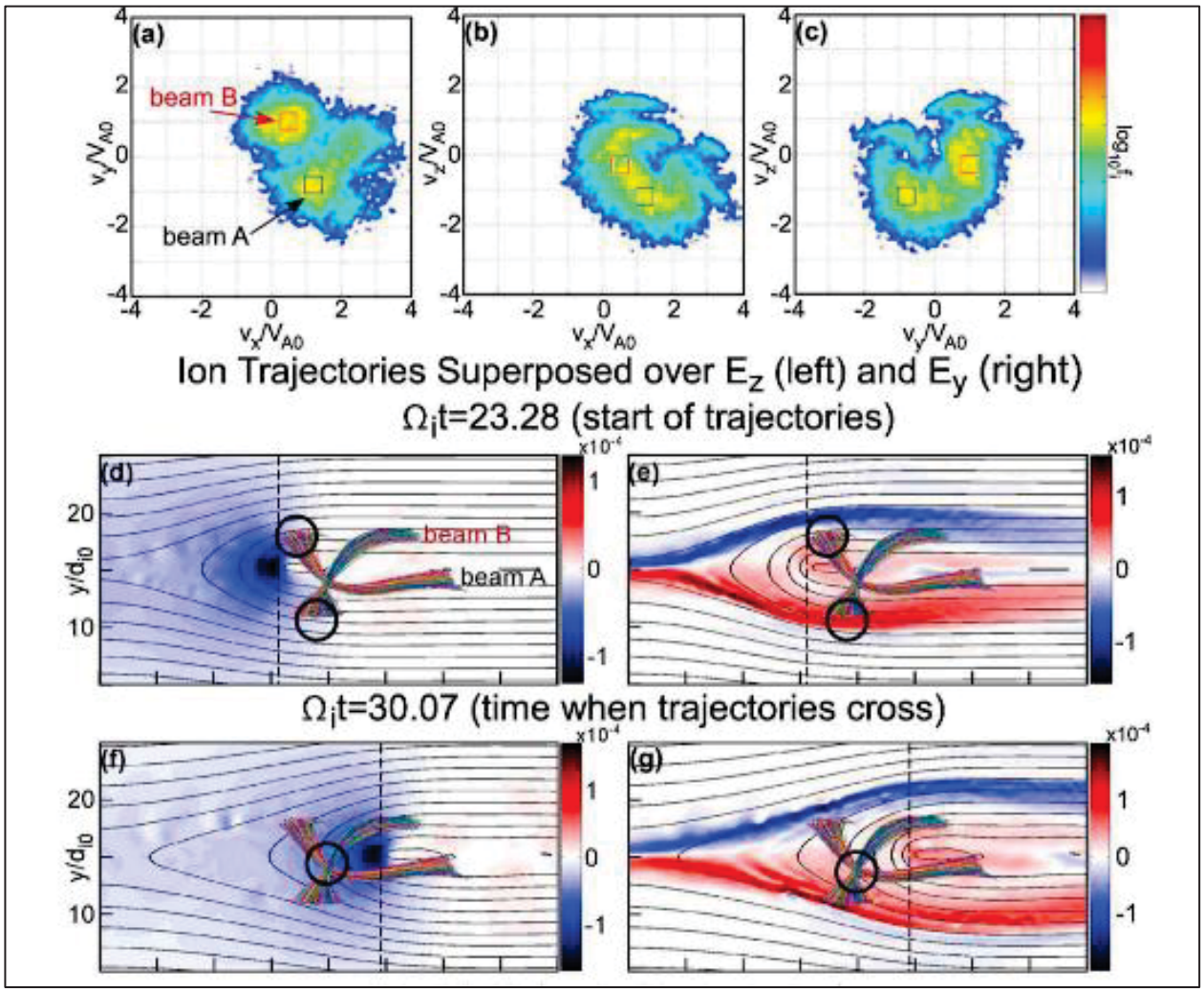


Figure 3 Ion beams from PIC simulation from (Eastwood, et al, 2014). (a–c) Ion beams observed in the ion distribution at *x*, *y*, and *t* = 120, 14, and 30.07 are shown. (d–h) Back and forward tracing analysis of the ion trajectories in space between *t* = 23 and *t* = 35, superimposed on snapshots of E_{zSIM} (d and f) and E_{ySIM} (a and g) at the times shown in the panel captions. For each snapshot, the physical locations of the ions in each beam along the overall trajectory are approximately indicated by the black circles.

3 "Standard" energy-transport theory

3.1 REVIEW OF STANDARD ENERGY-TRANSPORT THEORY

There are a number of different ways to take energy moments and derive corresponding energy transport equations for a given particle species. A common strategy, the so-called standard approach, is based on the assumption that there is a single mean-flow-velocity, **u** of a distribution f(**v**), for electrons or for ions. For a given particle species the well-known equations (Birn & Hesse, 2005; Bellan, 2006; Aunai, et al, 2011) for the transport of the bulk coherent energy density, U_{bulk}, and the "thermal" (incoherent) energy density, U_{therm} are given (for electrons or ions) by,

$$(1)\quad \begin{aligned} &a)\ \ \partial_t U_{bulk} + \nabla\cdot\mathbf{Q}_{bulk} = \mathbf{J}_s\cdot\mathbf{E} - \mathbf{u}\cdot\nabla\cdot\boldsymbol{P} \\ &b)\ \ \partial_t U_{thermal} + \nabla\cdot\mathbf{Q}_{therm} = \mathbf{u}\cdot\nabla\cdot\boldsymbol{P},\quad \mathbf{Q}_{therm} \equiv \mathbf{Q}_{heatflux} + \mathbf{Q}_{enthalpy} \end{aligned} \qquad \square$$

In Eqn. (1a), **u** is the flow-velocity of the species, $U_{bulk} = mnu^2/2$, $\mathbf{Q}_{bulk}$ is the bulk kinetic energy density flux, $\mathbf{u}U_{bulk}$, and $\mathbf{J_s}\cdot\mathbf{E}$ is the work per unit voume performed by the electric field, **E** against the current density, $\mathbf{J}_s = \mathbf{J}_e$ or $\mathbf{J}_s = \mathbf{J}_i$. The term $-\mathbf{u}\cdot\nabla\cdot\boldsymbol{P}$ is the work per unit volume performed on the flow velocity, **u**, by the pressure force per unit volume, $-\nabla\cdot\boldsymbol{P}$.

In Eqn. (1b), $\mathbf{Q}_{therm}$ is the thermal (incoherent) energy flux, $\mathbf{Q}_{therm} = \mathbf{Q}_{heatflux} + \mathbf{Q}_{enthalpy}$, consisting of a part, $\mathbf{Q}_{heatflux}$, which is invariant under Galilean velocity-frame transformations and a part, $\mathbf{Q}_{enthalpy}$, proportional to **u**. The term $+\mathbf{u}\cdot\nabla\cdot\boldsymbol{P}$ is the work/volume done *by* **u** against the pressure force/volume. Hence, the work associated with pressure governs the transfer of energy between coherent and incoherent energy densities.

Eqs. (1) may be derived from MHD with an appropriate equation of state (Birn & Hesse, 2005) or, more fundamentally, from kinetic theory (Bellan, 2006; Aunai, et al, 2011), as will be demonstrated in Sec.3.2.

Another (electron or ion) energy transport equation results from adding Eqn. (1a) to Eqn. (1b). The work terms associated with pressure forces then drop out entirely, resulting in:

$$(2a)\quad \partial_t U + \nabla\cdot\mathbf{Q} = \mathbf{J}_s\cdot\mathbf{E}$$

$$(2b)\quad U \,..\, U_{bulk} + U_{therm},\quad \mathbf{Q} \,..\, \mathbf{Q}_{bulk} + \mathbf{Q}_{therm}$$

Eqn. (2) describes particle energy transport (for ions or electrons) without regard to flow velocity or pressure (Goldman, et al, 2016). We shall refer to Eqn. (2) as the *undecomposed* particle energy transport equation, and U as the undecomposed particle energy density. U_{bulk} and U_{therm} can be understood as resulting from the decomposition of U into a coherent and an incoherent part. In a similar manner, **Q** is the undecomposed energy density flux whose decomposed pieces, $\mathbf{Q}_{bulk}$ and $\mathbf{Q}_{therm}$ are its coherent and incoherent parts.

At first glance energy transport Eqn. (2) appears less useful than energy transport Eqns. (1) because U and **Q** do not distinguish between coherent and incoherent (thermal) energies and fluxes. However, energy transport in terms of U and **Q** can be very useful because U and **Q** are

independent of whether their decomposed parts are standard (single-beam) moments or multibeam moments of a given velocity distributions, f(**v**). Note also that U and **Q** enter into the electromagnetic energy transport equation in the same way as the electromagnetic energy density and Poynitng flux when $-(\mathbf{J}_e+\mathbf{J}_i)\cdot\mathbf{E}$ is eliminated by using Eqn. (2a) for each species:

$$(3)\qquad \partial_t\left(U_{EM}+U_e+U_i\right)+\nabla\cdot\left(\mathbf{S}_{Poynting}+\mathbf{Q}_e+\mathbf{Q}_i\right)=0$$

In Eqn. (3) the c.g.s. electromagnetic energy, $U_{EM} = (E^2+B^2)/8\pi$, and Poynting flux, $\boldsymbol{S} = c(\boldsymbol{E}\times\boldsymbol{B})/4\pi$, are on the same footing as $U_{e,i}$ and $\mathbf{Q}_{e,i}$. From Eqn. (3) it is evident that when electromagnetic energy is converted locally it can be transformed into local particle energy densities, $U_{e,i}$, and/or into Poynting flux and particle fluxes, $\mathbf{Q}_e$ and $\mathbf{Q}_i$.

At the level of description of Eqn (3) one cannot determine whether lost magnetic energy goes mainly into local particle heating, into particle acceleration or is spatially transported by electromagnetic fields or particle energy density fluxes. To determine in standard transport theory whether coherent or incoherent processes dominate changes in particle energy (or energy flux) one must employ the standard decomposed energy transport Eqns. (1).

3.2 KINETIC THEORY OF STANDARD ENERGY TRANSPORT

Further insight into the meaning of Eqn. (2) is gained by deriving it from kinetic theory (Aunai, et al, 2011; Bellan, 2006; Goldman, et al, 2016). In kinetic theory, undecomposed energy transport is described by taking a kinetic energy moment of the Vlasov equation for either species:

$$\int d^3\mathbf{v}\left[\left(\frac{m_s v^2}{2}\right)\left\{\partial_t+\mathbf{v}\cdot\nabla+\frac{q_s}{m_s}\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\partial_{\mathbf{v}}\right\}f_s\right]=0,\quad \mathrm{s}=\mathrm{e},\mathrm{i}$$

The result is Eqn. (2) with the undecomposed U and **Q** now explicitly defined as velocity moments of f(**v**),

$$4a)\quad U(\mathbf{r},t)=\ \textit{undecomposed}\ \text{energy (e or i)}\ \equiv\int d^3\mathbf{v}f(\mathbf{v},\ \mathbf{r},\ t)\left[\frac{mv^2}{2}\right]$$

$$4b)\quad \mathbf{Q}(\mathbf{r},t)=\textit{undecomposed}\ \text{energy flux (e or i)}\ \equiv\int d^3\mathbf{v}f(\mathbf{v},\ \mathbf{r},\ t)\cdot\left[\mathbf{v}\frac{mv^2}{2}\right]$$

In the usual standard approach employing the mean-flow-velocity decomposition of U and **Q** moments, one substitutes $\mathbf{v} = \mathbf{u} + \delta\mathbf{v}$ into $mv^2/2$ and $m\mathbf{v}v^2/2$ in the velocity integrals in Eqns (4a) and (4b). Integrals whose integrands are odd powers $\delta\mathbf{v} = (\mathbf{v}-\mathbf{u})$ vanish — hence the term "incoherent" for moments proportional to $(\delta\mathbf{v})$.[2] This kind of decomposition is most meaningful for a single electron or ion beam, f(**v**), which is single-peaked and contiguous. In physical units, the mean-flow-velocity decomposition of U and **Q** moments for either electrons or ion is

$$5a)\ \ \mathrm{U}_{\mathrm{bulk}}=\frac{\mathrm{n}mu^2}{2},\ \ \mathrm{U}_{therm}=\frac{m}{2}\int d^3\mathbf{v}f(\mathbf{v})(\delta v)^2$$

$$5b)\quad \mathbf{Q}_{bulk} = n\mathbf{u}\frac{mu^2}{2}$$

$$5c)\quad \mathbf{Q}_{heatflux} = \int d^3\mathbf{v} f(\mathbf{v})\frac{m}{2}\left[\delta\mathbf{v}(\delta\mathbf{v})^2\right]$$

$$5d)\quad \mathbf{Q}_{enthalpy} = \int d^3\mathbf{v} f(\mathbf{v})\frac{m}{2}\left[\mathbf{u}(\delta\mathbf{v})^2 + 2\mathbf{u}\cdot\delta\mathbf{v}\delta\mathbf{v}\right]$$

The enthalpy flux is second-order in δv so it may be defined in terms of the pressure tensor, $\mathbf{P}$. In dimensional units,

$$5e)\quad \mathbf{Q}_{enthalpy} = \frac{\mathbf{u}\,\mathrm{Tr}(\mathbf{P})}{2} + \mathbf{u}\cdot\mathbf{P}, \quad \mathbf{P} \equiv m\int d^3\mathbf{v} f(\mathbf{v})\delta\mathbf{v}\,\delta\mathbf{v}, \quad (\text{for electrons or ions})$$

The collisionless fluxes in Eqs. (5) describe energy densities carried by coherent and incoherent velocities. Coherent and incoherent energy densities are manifestly carried in $\mathbf{Q}_{\mathrm{bulk}}$ and $\mathbf{Q}_{\mathrm{enthalpy}}$ by the flow velocity, **u**. The second term in $\mathbf{Q}_{\mathrm{enthalpy}}$ allows for orthogonal components of the flow velocity to carry different thermal energy densities in the three orthogonal directions for a diagonal pressure tensor.

The moments in Eqn. (5) enter into the two independent energy transport Eqns. (1a,b) for U_{bulk} and U_{therm}. They can be derived from the Vlasov equation for electrons or ions (Bellan, 2006; Goldman, et al, 2016) by using

- the transport Eqn (2a) for $U = U_{\mathrm{bulk}} + U_{\mathrm{therm}}$ (as found above from the Vlasov Eqn.) and
- a transport equation for U_{bulk} obtained by dotting u into the *momentum* eqn (*first* moment of the Vlasov eqn.).

The distinction between undecomposed and decomposed moments may be applied to the pressure tensor moment as well as to the energy and energy flux moments. The *stress tensor*, $\mathbf{T}$, is the *undecomposed* moment which decomposes into a *RAM pressure* tensor (dyadic) and a *"thermal" pressure*, $\mathbf{P}$ when the mean velocity, **u** is introduced:

$$6)\quad \mathbf{T} \equiv m\int d^3\mathbf{v} f(\mathbf{v})\mathbf{v}\mathbf{v} = \mathbf{P}_{RAM} + \mathbf{P}, \qquad \mathbf{P}_{RAM} = \mathrm{nm}\mathbf{u}\mathbf{u}, \quad (\text{for electrons or ions})$$

Note that the pressure tensor, $\mathbf{P}$, is Galilean-frame-invariant (like U_{therm} and $\mathbf{Q}_{\mathrm{heatflux}}$), whereas $\mathbf{T}$ and $\mathbf{P}_{\mathrm{RAM}}$ are not. In *dimensional units,*

$$6a)\quad \mathrm{Tr}(\mathbf{P})/2 = U_{\mathrm{therm}}$$

This helps explain the enthalpy flux in Eqn. (5e), since the first term on the right is now $\mathbf{u}U_{\mathrm{therm}}$. The undecomposed stress tensor enters directly into the force-momentum equation, which is the *first* moment of the Vlasov equation for electrons or ions and may be written in the following form:

$$6b)\quad \partial_t(mn\mathbf{u}) + \nabla\cdot\mathbf{T} = q\mathbf{E} + q\frac{\mathbf{u}}{c}\times\mathbf{B}, \quad (\text{dimensional units}) \quad \square$$

When $\mathbf{T}$ is decomposed the "thermal" pressure force appears explicitly and the RAM pressure leads (together with the continuity eqn.) to the convective derivative, $\mathbf{u}\cdot\nabla\mathbf{u}$.

A space-time transport eqn for $\mathbf{T}(\mathbf{r},t)$ is generated from kinetic theory by multiplying the Vlasov eqn by m**vv** and integrating over **v**. However, space-time transport will not be studied in this paper.

*3.3 STANDARD MOMENTS OF ONE TRI-MAXWELLIAN BEAM, f(**v**)*

To give explicit examples of standard and multibeam energy moments, we first consider a single tri-Maxwellian distribution function, f(**v**), for electrons or ions,

$$(7)\quad f(\mathbf{v}) = \frac{n}{w_x w_y w_z (2\pi)^{3/2}} Exp\left\{ -\frac{1}{2}\left(\frac{(v_x - u_x)^2}{w_x^2} + \frac{(v_y - u_y)^2}{w_y^2} + \frac{(v_z - u_z)^2}{w_z^2} \right) \right\}$$

Here, f(**v**) is normalized to the density, n, and centered at the velocity, $\mathbf{u} = \{u_x, u_y, u_z\}$. The three thermal velocities are $w_k = \sqrt{(T_k/m_s)}$, where k = (x, y, z), s = e or i, and T_j is the temperature (in energy units) associated with direction j. We define a thermal velocity vector, $\mathbf{w} = \{w_x, w_y, w_z\}$.

All of the standard moment integrals in Eqns. (5) and (6) can be performed analytically, with the following results (in dimensional units) :

$$(8a)\quad \int d^3\mathbf{v} f(\mathbf{v}) = n = \text{density}$$

$$(8b)\quad \int d^3\mathbf{v} f(\mathbf{v})\mathbf{v} = n\mathbf{u} = \text{particle flux of electrons or ions}$$

$$(8c)\quad U_{bulk} = \frac{nmu^2}{2}, \quad U_{therm} = \frac{nmw^2}{2}$$

$$(8d)\quad \mathbf{P}_{RAM} = \text{nm}\mathbf{uu}, \quad \mathbf{P} = \text{nm}\begin{bmatrix} w_x^2 & 0 & 0 \\ 0 & w_y^2 & 0 \\ 0 & 0 & w_z^2 \end{bmatrix}, \text{ or, } P_{kn} = \text{nm}\delta_{kn} w_k^2 \text{ (no sum over k)}$$

$$(8e)\quad \mathbf{Q}_{bulk} = \frac{n\mathbf{u}mu^2}{2}, \quad \mathbf{Q}_{enthalpy-k} = \frac{mn}{2}\left(u_k w^2 + 2u_k w_k^2\right) \text{ (no sum over k)}$$

$$(8f)\quad \mathbf{Q}_{heatflux} = 0$$

Note that the pressure tensor is diagonal in this coordinate system. Such a coordinate system may be found for a non-diagonal pressure tensor in terms of its eigenvalues and eigenvectors.

The enthalpy flux vector has been calculated in terms of the pressure, as in Eqn. (5e), $\mathbf{Q}_{enthalpy} = \mathbf{u}\,\text{Tr}(\mathbf{P})/2 + \mathbf{u}\cdot\mathbf{P}$

The standard heat flux, $\mathbf{Q}_{\text{heatflux}}$, is zero for a tri-Maxwellian beam of the form in Eqn. (7) due to reflection symmetries which makes the heat flux integral vanish.

4. Multibeam moments

4.1 MULTIBEAM MOMENTS FOR $f(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v}) + ... + f_N(\mathbf{v})$

Consider a 3-D electron or ion velocity distribution, f(**v**), consisting of N pieces, which will be referred to as *beams*:

(9) $f(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v}) + ... + f_N(\mathbf{v})$.

In the *standard* method for taking moments, a single mean flow velocity, **u**, is used, as in Eqns. (5) and (6) for the set of N beams embodied in f(**v**). Such an approach is used in the typical calculation of on-board moments on a satellite, such as the Fast Plasma Instruments on board the MMS satellites as well as instruments on Cluster, THEMIS, etc. It is usually applied without regard to whether the measured f(**v**) consists of one or more than one effectively disjoint "beams".

In the *multibeam* decomposition the *moments of each* beam are taken first (using flow velocities $\mathbf{u}_j$ of each *beam*, j). One then sums the moments over all beams. In general, the multibeam and standard decomposition methods give *different* results for the *decomposed* moments.

By contrast, the *undecomposed* (electron or ion moments), U, **Q**, and **T**, which are independent of flow velocities, are the *same* whether one uses f*(v)* or its equivalent, $f_1(\mathbf{v}) + f_2(\mathbf{v}) + ... + f_N(\mathbf{v})$. In dimensional units,

$$(10a)\quad U \equiv \int d^3\mathbf{v} f(\mathbf{v}) \left[\frac{mv^2}{2}\right] = \sum_j U_j, \qquad U_j \equiv \int d^3v f_j(\mathbf{v}) \left[\frac{mv^2}{2}\right]$$

$$(10b)\quad \mathbf{Q} \equiv \int d^3\mathbf{v} f(\mathbf{v}) \left[\mathbf{v}\frac{mv^2}{2}\right] = \sum_j \mathbf{Q}_j, \quad \mathbf{Q}_j \equiv \int d^3\mathbf{v} f_j(\mathbf{v}) \left[\mathbf{v}\frac{mv^2}{2}\right]$$

$$(10c)\quad \mathbf{T} \equiv \int d^3v f(v) [m\mathbf{vv}] = \sum_j \mathbf{T}_j, \quad \mathbf{T}_j \equiv \int d^3\mathbf{v} f_j(\mathbf{v}) [m\mathbf{vv}]$$

The sum in Eqn. (10a) is a sum over the N beams of *undecomposed* beam energy densities, U_j; the sum in Eqn. (10b) is a sum over (electron or ion) beams of undecomposed beam energy fluxes, $\mathbf{Q}_j$; and the sum in Eqn. (10c) is a sum over beams of undecomposed individual beam stress tensors. It should be clear from Eqns. (10) that the *sum over undecomposed* moments *of each beam* gives the *same* result as the undecomposed moments of f(**v**). The undecomposed moments are therefore a robust and unambiguous characterization of the plasma, whether it consists of one or more beams.

The "*standard*" decomposition of U, **Q** and **T** based on the mean flow velocity **u**, is correctly given by Eqns. (5) and (6). The standard decomposed moments U_{bulk}, U_{therm}, $\mathbf{Q}_{bulk}$, $\mathbf{Q}_{heatflux}$, and $\mathbf{Q}_{enthalpy}$, $\mathbf{T}_{RAM}$ and **T** are found, for example, in the moments file of the Fast Plasma Instrument (FPI) on each of the MMS spacecraft.

By contrast, in the alternative, *multibeam* decomposition one first decomposes the moments of each beam using that beam's mean velocity, $\mathbf{u}_j$ and then substitutes, $\mathbf{v} = \mathbf{u}_j + (\mathbf{v}_j - \mathbf{u}_j)$ in $mv^2/2$ and $m\mathbf{v}v^2/2$ in the velocity integrals for U_j and $\mathbf{Q}_j$ and $\mathbf{T_j}$ in Eqn. (10).

A key point in this paper is that each *multibeam decomposed* moment of the *assembly* of (electron or ion) beams is given by the sum over beams, j, of the corresponding *standard moment* of each beam j,

$$(11a)\quad U^{MB}_{bulk} = \sum_{j=1}^{N} U_{j-bulk} \neq U_{bulk}, \quad U^{MB}_{therm} = \sum_{j=1}^{N} U_{j-therm} \neq U_{therm},$$

$$\text{where, } U_{j-therm} = \frac{m}{2}\int d^3\mathbf{v} f_j(\mathbf{v})\left(\mathbf{v}-\mathbf{u}_j\right)^2$$

$$(11b)\quad \mathbf{Q}^{MB}_{bulk} = \sum_{j=1}^{N} \mathbf{Q}_{j-bulk} \neq \mathbf{Q}_{bulk}, \quad \mathbf{Q}^{MB}_{heatflux} = \sum_{j=1}^{N} \mathbf{Q}_{j-heatflux} \neq \mathbf{Q}_{heatflux}, \quad \mathbf{Q}^{MB}_{enthalpy} = \sum_{j=1}^{N} \mathbf{Q}_{j-enthalpy} \neq \mathbf{Q}_{enthalpy}$$

$$(11c)\quad \mathbf{P}^{MB} = \sum_{j=1}^{N} \mathbf{P}_j \neq \mathbf{P}, \quad \mathbf{P}^{MB}_{RAM} = \sum_{j=1}^{N} \mathbf{P}_{j-RAM} \neq \mathbf{P}_{RAM}$$

In general, none of the *multibeam* decomposed moments are equal to the corresponding *standard* decomposed moment.

Based on Eqns. (10) and (11) the *undecomposed* (electron or ion) moments U, **Q** and **T** may therefore be decomposed into either a sum of standard moments or a sum of multibeam moments:

$$(12a)\quad U = U^{MB}_{bulk} + U^{MB}_{therm} = U_{bulk} + U_{therm}$$

$$(12b)\quad \mathbf{T} = \mathbf{P}^{MB}_{RAM} + \mathbf{P}^{MB} = \mathbf{P}_{RAM} + \mathbf{P}$$

$$(12c)\quad \mathbf{Q} = \mathbf{Q}^{MB}_{bulk} + \mathbf{Q}^{MB}_{therm} = \mathbf{Q}_{bulk} + \mathbf{Q}_{therm}, \text{ where,}$$

$$(12d)\quad \mathbf{Q}^{MB}_{therm} \equiv \mathbf{Q}^{MB}_{heatflux} + \mathbf{Q}^{MB}_{enthalpy} \text{ and } \mathbf{Q}_{therm} \equiv \mathbf{Q}_{heatflux} + \mathbf{Q}_{enthalpy}$$

The *thermal* energy density fluxes $\mathbf{Q}_{therm}$ and $(\mathbf{Q}^{MB})_{therm}$ have been defined in Eqn (12d) as sums of enthalpy flux and heat flux. Section (4.4) will show how Eqns. (12) determine the *pseudothermal* parts of standard moments. First, we consider lower-order moments.

4.2 DENSITIES AND PARTICLE FLUXES IN STANDARD AND MULTI-BEAM PICTURES

Taking the zero-order-moment of both sides in $f(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v}) + ... + f_N(\mathbf{v})$ yields,

$$13a)\quad \int d^3\mathbf{v} f(\mathbf{v})/n = 1 = \eta_1 + \eta_2 + ... + \eta_N, \quad \eta_j \equiv n_j/n$$

Here, n_j is the density of beam j, n is the density of the ensemble of beams and $\eta_j = n_j/n$ is the fractional density of each beam. Taking the first-order moment of both sides of $f(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v}) + ... + f_N(\mathbf{v})$ yields the particle flux. When divided by n, this becomes

$$(13b)\quad \int d^3\mathbf{v}\mathbf{v}f(\mathbf{v})/n = \mathbf{u} = \eta_1\mathbf{u}_1 + \eta_2\mathbf{u}_2 + ... + \eta_N\mathbf{u}_N,$$

where $\mathbf{u}_j$ is the velocity of the j-th beam and $\mathbf{u}$ is the mean velocity of the ensemble of beams. Each beam velocity, $\mathbf{u}_j$, in the weighted mean, $\mathbf{u}$, is multiplied by the fractional beam density, η_j.

Eqns. (13) illustrate that the beam velocities, $\mathbf{u}_j$, and the beam densities, n_j, are not independent of each other. Their relationship depends on the standard density, n and flow velocity, $\mathbf{u}$. We next consider a sum of tri-Maxwellian beams.

4.3 MULTIBEAM MOMENTS OF A SUM OF TRI-MAXWELLIANS

Consider an *analytic* velocity distribution consisting of a sum of N electron or N ion beams, $f(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v}) + ... f_N(\mathbf{v})$, in which each beam, $f_j(\mathbf{v})$, is tri-Maxwellian:

$$(14)\quad f_j(\mathbf{v}) = \frac{n_j}{w_{jx}w_{jy}w_{jz}(2\pi)^{3/2}} Exp\left\{-\frac{1}{2}\left(\frac{(v_x-u_{jx})^2}{w_{jx}^2} + \frac{(v_y-u_{jy})^2}{w_{jy}^2} + \frac{(v_z-u_{jz})^2}{w_{jz}^2}\right)\right\},\quad j = 1, 2, ..., N$$

Here n_j is the density of beam j, $\mathbf{u}_j$ is its mean velocity, and $mw_{jx}^2 = T_{jx}$, $mw_{jy}^2 = T_{jy}$, $mw_{jz}^2 = T_{jz}$ are the three temperatures of beam j. We have taken all of the tri-Maxwellians to be co-aligned, with temperatures along the x, y and z axes. Consequently, all beam pressure tensor moments will here be diagonal. This of course, is not the most general possible sum of tri-Maxwellians but it will suffice to illustrate the concepts considered in this paper. More commonly treated is the even less general limit of a sum of *spherically symmetric* Maxwellians, each of temperature T_j, arrived at by setting $w_{jx} = w_{jy} = w_{jz} = \sqrt{(T_j/m)}$.

As in Eqns. (11), the *decomposed multibeam* moments of *any* multibeam $f(\mathbf{v})$ are given simply by the sum over beams, j, of the corresponding standard moments of each beam j. Since each beam is a tri-Maxwellian and the standard moments of a tri-Maxwellian are given by Eqns. (8), the *decomposed multibeam* moments of the sum of tri-Maxwellians can be expressed as,

$$(15a)\quad U_{bulk}^{MB} = \frac{mn}{2}\sum_{j=1}^{N}\eta_j\left|\mathbf{u}_j\right|^2,\quad U_{therm}^{MB} = \frac{mn}{2}\sum_{j=1}^{N}\eta_j\left|\mathbf{w}_j\right|^2$$

$$(15b)\quad P_{kn}^{MB} = mn\delta_{kn}\sum_{j=1}^{N}\eta_j w_{jk}^2,\quad \text{(no sum over k or n = x, y, z)}$$

$$(15c)\quad \mathbf{Q}_{bulk}^{MB} = \frac{mn}{2}\sum_{j=1}^{N}\eta_j\mathbf{u}_j\left|\mathbf{u}_j\right|^2$$

$$(15d)\quad \mathbf{Q}_{enthalpy}^{MB} = \frac{mn}{2}\sum_{j=1}^{N}\eta_j\left[\mathbf{u}_j\left|\mathbf{w}_j\right|^2 + 2\mathbf{u}_j\cdot\mathbf{w}_j\mathbf{w}_j\right]$$

$$(15e)\quad \mathbf{Q}_{heatflux}^{MB} = 0$$

Here the integer j labels *beams* while the integers k and n label vector and tensor *components* in the three Cartesian coordinates, {x, y, z}. The *multibeam* heat flux is zero because of

velocity reflection symmetry in each tri-Maxwellian beam. We have defined a thermal *vector*, $\mathbf{w}_j \equiv \{w_{jx}, w_{jy}, w_{jz}\}$. In the *spherically symmetric limit*, $|\mathbf{w}_j|^2 = 3T_j/m$. In the *cold beam limit,* $w_j \rightarrow 0$, for all beams so the ensemble thermal energy, enthalpy flux and pressure go to zero.

4.4 HOW TO FIND *STANDARD* MOMENTS FROM *MULTIBEAM* MOMENTS

The decomposed moments obtained using the *standard* (e.g., FPI) mean-flow-velocity, **u**, and associated density, n, are given by the (electron or ion) f(v)-integrals in Eqns. (5) and (6).

First substitute $f(\mathbf{v}) = f_1(\mathbf{v}) + ... + f_N(\mathbf{v})$ into Eqns. (5) and (6). Then, substitute $[(\mathbf{u}_j\text{-}\mathbf{u})+(\mathbf{v}\text{-}\mathbf{u}_j)]$ for $\delta\mathbf{v} \equiv (\mathbf{v} - \mathbf{u})$ in each $f_j(\mathbf{v})$ integral in each moment. Note that $f_j(\mathbf{v})$ integrals proportional to $(\mathbf{v}\text{-}\mathbf{u}_j)$ vanish because they are fluctuation moments. In physical units, the standard energy density Eqns. (5a) then become,

$$(16a)\quad U_{bullk} = \frac{mn}{2}u^2, \quad U_{therm} = U_{therm}^{MB} + \frac{mn}{2}\sum_{j=1}^{N}\eta_j\left(\mathbf{u}-\mathbf{u}_j\right)^2 ,$$

the standard pressure and standard enthalpy flux Eqns. (5e) become,

$$(16b)\quad \mathbf{P} = \mathbf{P}^{MB} + mn\sum_{j=1}^{N}\eta_j\left(\mathbf{u}-\mathbf{u}_j\right)\left(\mathbf{u}-\mathbf{u}_j\right), \quad \mathbf{Q}_{enthalpy} = \frac{\mathbf{u}\,\mathrm{Tr}(\mathbf{P})}{2} + \mathbf{u}\cdot\mathbf{P} ,$$

and the standard bulk energy flux remains equal to Eqn. (5b):

$$(16c)\quad \mathbf{Q}_{bulk} = \frac{mn}{2}\mathbf{u}u^2$$

The standard heat flux in Eqn. (5c) requires a bit more algebra to relate to the multibeam heat flux:

$$(16d)\quad \mathbf{Q}_{heatflux} = \mathbf{Q}_{heatflux}^{MB} - \frac{nm}{2}\left\{\sum_{j=1}^{N}\eta_j\left[\mathbf{u}-\mathbf{u}_j\right]\left|\mathbf{u}-\mathbf{u}_j\right|^2 - 2\sum_{j=1}^{N}\eta_j\left[\left(\mathbf{u}-\mathbf{u}_j\right)\cdot\mathbf{P}_j + \left(\mathbf{u}-\mathbf{u}_j\right)\frac{Tr\mathbf{P}_j}{2}\right]\right\}$$

4.4.1 STANDARD MOMENTS OF AN ASSEMBLY OF TRI-MAXWELLIANS

For the assembly of N co-aligned tri-Maxwellians, $(U_{therm})^{MB}$ and $\mathbf{P}^{MB}$ are known, so the *standard thermal* moments for the assembly are:

$$(17a)\quad U_{therm} = U_{therm}^{MB} + \frac{mn}{2}\sum_{j=1}^{N}\eta_j\left|\delta\mathbf{u}_j\right|^2 = \frac{mn}{2}\left[\sum_{j=1}^{N}\eta_j\left(\left|\mathbf{w}_j\right|^2 + \left|\delta\mathbf{u}_j\right|^2\right)\right],\quad \delta\mathbf{u}_j \equiv \mathbf{u}_j - \mathbf{u}$$

$$(17b)\quad \mathbf{P} = \mathbf{P}^{MB} + mn\sum_{j=1}^{N}\eta_j\delta\mathbf{u}_j\delta\mathbf{u}_j = \sum_{j=1}^{N}\mathbf{P}_j,\ \ \mathbf{P}_j \equiv \mathrm{mn}\eta_j\begin{bmatrix} w_{jx}^2 + \left|\delta u_{jx}\right|^2 & \delta u_{jx}\delta u_{jy} & \delta u_{jx}\delta u_{jz} \\ \delta u_{jx}\delta u_{jy} & w_{jy}^2 + \left|\delta u_{jy}\right|^2 & \delta u_{jz}\delta u_{jy} \\ \delta u_{jx}\delta u_{jz} & \delta u_{jz}\delta u_{jy} & w_{jz}^2 + \left|\delta u_{jz}\right|^2 \end{bmatrix}$$

$$(17c)\quad \mathbf{Q}_{enthalpy} = \frac{\mathbf{u}\,\mathrm{Tr}(\mathbf{P})}{2} + \mathbf{u}\cdot\mathbf{P} = \mathbf{u}U_{therm} + \mathbf{u}\cdot\mathbf{P}$$

$$(17d)\quad \mathbf{Q}_{heatflux} = 0 + \frac{nm}{2}\left\{\sum_{j=1}^{N}\eta_j\delta\mathbf{u}_j\left|\delta\mathbf{u}_j\right|^2\right\} + 2\sum_{j=1}^{N}\left[\delta\mathbf{u}_j\cdot\mathbf{P}_j + \delta\mathbf{u}_j\frac{Tr\mathbf{P}_j}{2}\right]$$

There are both *real* and *pseudo* thermal parts contained in the *standard* thermal energy density, heat flux and enthalpy flux of the assembly of N co-aligned tri-Maxwellian beams. Although the *multibeam* heat flux, $(\mathbf{Q}_{heatflux})^{MB}$ of the assembly is zero, owing to the symmetry of each tri-Maxwellian beam, the *standard* heat flux of the assembly, $\mathbf{Q}_{heatflux}$, does *not* vanish. It is composed entirely of *pseudothermal* heat flux. Note that the dependence of $\mathbf{Q}_{heatflux}$ on $\mathbf{P}_j$ means that the pseudothermal heat flux contains thermal energy $|w_{jk}|^2$, where the index k is the Cartesian coordinate, k = x, y or z. However, in the cold beam limit $w_{jk} \to 0$, The standard heat flux, $\mathbf{Q}_{heatflux}$, of the assembly of tri-Maxwellians does *not* go to zero.

4.4.2 MOMENTS OF EQUAL AND OPPOSITE MAXWELLIAN DISTRIBUTIONS

The above case is a generalization of the "pseudothermal" standard energy density moment of two equal and opposite *cold* beams described in the introduction. For the two equal and opposite cold beams in Fig. (1), the mean flow velocity, u = 0, and, from Eqns. (15 and 16),

$$n_1 = n_2 = \frac{\mathrm{n}}{2},\ \ \mathbf{u}_1 = \mathbf{u}_0,\ \mathbf{u}_2 = -\mathbf{u}_0,\ \mathbf{u} = 0$$

$$U_{bulk} = 0,\ \ U_{therm} = \frac{m}{2}\left(n_1u_0^2 + n_1u_0^2\right) = mn_1u_0^2$$

$$U_{bulk}^{MB} = \frac{m}{2}\left(n_1u_0^2 + n_1u_0^2\right) = mn_1u_0^2,\ U_{therm}^{MB} = 0$$

Thus, the standard (FPI) moments identify the energy of an assembly of cold beams as totally "thermal," whereas multibeam moments correctly identify the system energy as composed of bulk kinetic energy only. In general, U_{therm} will be a mixture of real and pseudothermal energy densities. If the two cold beams in Fig. 1 and Table 1 are replaced by two Maxwellian beams,

each with isotropic temperature, $T = mw^2$, Eqns. (17) still hold, except for the "thermal" energy densities expressions which are changed to:

$$U_{therm} = \frac{mn\left\{w^2 + u_0^2\right\}}{2} \qquad U_{therm}^{2-beam} = \frac{mnw^2}{2} = U_{therm} - \left(\frac{mnu_0^2}{2}\right)$$

Table (2), below, compares the standard energy density moments to the multibeam energy density moments for this example of two beams.

Table 2: *STANDARD VS. MULTIBEAM* energy density moments for two equal (density), oppositely-directed *warm* beams (e or i). (Dimensional units)

***Standard* energy moments**		***Multibeam* energy moments**	
U_{bulk}	0	$(U_{bulk})^{2\text{-beam}}$	$\frac{mnu_0^2}{2}$
U_{therm}	$\frac{mn\left\{w^2 + u_0^2\right\}}{2}$	$(U_{therm})^{2\text{-beam}}$	$\frac{mnw^2}{2}$
$U = U_{bulk} + U_{therm}$	$\frac{mn\left\{w^2 + u_0^2\right\}}{2}$	$U = (U_{bulk})^{2\text{-beam}} + (U_{therm})^{2\text{-beam}}$	$\frac{mn\left\{w^2 + u_0^2\right\}}{2}$

4.5 *PSEUDOTHERMAL* PARTS OF DIMENSIONLESS STANDARD MOMENTS

Thus, *standard* thermal energy moments may contain *pseudothermal* parts, whereas the corresponding *multibeam* thermal energy moments are entirely *true* thermal.

In general, pseudothermal *parts* of standard moments are obtained by subtracting from each standard moment the corresponding multibeam moment. Hence, the *pseudothermal* moments, ΔU, $\Delta \mathbf{P}$, and $\Delta \mathbf{Q}$ are defined as:

$$(18a)\quad \Delta U \equiv U_{therm} - U_{therm}^{MB}$$

$$(18b)\quad \Delta \mathbf{P} \equiv \mathbf{P} - \mathbf{P}^{MB}$$

$$(18c)\quad \Delta \mathbf{Q} \equiv \mathbf{Q}_{therm} - \mathbf{Q}_{therm}^{MB}$$

Since Eqn.(12a), for example, may be rewritten as $U_{therm} - U_{therm}^{MB} = U_{bulk}^{MB} - U_{bulk}$, Eqns. (12) allow us to replace the difference moments in Eqn. (18) by differences of the much simpler *bulk* moments:

$$(19a)\quad \Delta U \equiv U_{bulk}^{MB} - U_{bulk}$$

$$(19b)\quad \Delta \mathbf{P} \equiv \mathbf{P}_{RAM}^{MB} - \mathbf{P}_{RAM}$$

$$(19c)\quad \Delta \mathbf{Q} \equiv \mathbf{Q}_{bulk}^{MB} - \mathbf{Q}_{bulk}$$

First consider energy density moments. Define <u>*dimensionless*</u> *energy units* by dividing *all* energy densities by a *normalization* standard bulk energy density, $U_{norm} = mnu_n^2/2$. We allow for the possibility that the normalization flow velocity moment, $\mathbf{u}_n$ may (or may not) differ from the standard flow velocity moment, $\mathbf{u}$, of the given $f(\mathbf{v})$.

The *standard and multibeam bulk energy density moments* are independent of the exact shape of the (electron or ion) velocity distribution, $f(\mathbf{v}) = f_1(\mathbf{v}) + ... + f_N(\mathbf{v})$ and of the shape of the individual beam distributions, $f_j(\mathbf{v})$. These moments depend only on densities and mean velocities, $(n, \mathbf{u})$, of $f(\mathbf{v})$, and $(n_j, \mathbf{u}_j)$, of $f_j(\mathbf{v})$, so that the *pseudothermal energy density moment* in dimensionless units is given by,

$$20a)\quad \Delta U \equiv U_{bulk}^{MB} - U_{bulk} \equiv \frac{1}{u_n^2}\left[\sum_{j=1}^{N} \eta_j u_j^2 - u^2\right], \quad \eta_j \equiv \frac{n_j}{n} \quad \text{(dimensionless)}$$

Two equivalent alternate expressions for the *dimensionless* ΔU are

$$(20b)\quad \Delta U = \frac{1}{u_n^2}\sum_{j=1}^{N} \eta_j \left(\mathbf{u}_j - \mathbf{u}\right)^2$$

$$(20c)\quad \Delta U = \frac{1}{u_n^2}\sum_{\substack{i,j=1 \\ i<j}}^{N} \eta_i \eta_j \left(\mathbf{u}_i - \mathbf{u}_j\right)^2$$

Eqn. (20b) has N terms. It is equivalent to Eqn. (20a) as one can see by explicitly squaring $(\mathbf{u}\text{-}\mathbf{u}_j)^2$ in each term and summing. In this form, ΔU is manifestly invariant under Galilean frame changes since the beam velocities, $\mathbf{u}_j$ are all measured from the mean flow

velocity, **u**. The right side of Eqn. (20c) depends on *differences* of beam velocities, ($\mathbf{u}_i$-$\mathbf{u}_j$) so it too is explicitly Galilean frame-independent. In this format there are N(N-1)/2 terms.

Any of the Eqns. (20) may be used to *calculate* the pseudothermal energy density, ΔU, in terms of zero-order and first-order moments of f(**v**) and of zero-order and first-order moments of each of the beams, $f_j(\mathbf{v})$. However, there is another possible use of ΔU which will be demonstrated in a later paper dealing with methods for finding approximate multibeam moments of *measured* f(**v**). From Eqn. (19a), the multibeam thermal energy density $(U_{therm})^{MB}$ is,

$$(21)\quad U_{therm}^{MB} = U_{therm} - \Delta U$$

Thus, for a given (i.e., measured) f(v), and a known (i.e., measured) U_{therm} the decomposed multi-beam energy density, $(U_{therm})^{MB}$ can be found by subtracting ΔU. This is usually easier than calculating $(U_{therm})^{MB}$ by summing all of the beam thermal energy densities (as in Eqn. (11a))

Consistent with the treatment leading to the multibeam energy density, we next treat the vector, *pseudoenergy flux,* Δ**Q**. Here, all (electron or ion) energy *fluxes* are rendered dimensionless by dividing by $mnu_n^3/2$:

$$(22)\quad \mathbf{Q}_{bulk}^{MB} = \mathbf{Q}_{bulk} + \Delta\mathbf{Q}, \quad \Delta\mathbf{Q} \equiv \frac{1}{u_n^3}\left[\sum_{j=1}^{N} \eta_j \mathbf{u}_j u_j^2 - \mathbf{u}u^2\right], \quad \text{(dimensionless)}$$

By the same reasoning that led to the "*thermal*" multibeam *energy* moment in Eqn. (21), we can express the dimensionless "*thermal*" multibeam *energy flux* vector moment as,

$$(23)\quad \mathbf{Q}_{therm}^{MB} = \mathbf{Q}_{therm} - \Delta\mathbf{Q}$$

Hence, if we know the standard moment $\mathbf{Q}_{therm}$ for f(**v**), we can find $(\mathbf{Q}_{therm})^{MB}$ by subtracting the "pseudothermal" part, Δ**Q**. Adding Eqns. (22) and (23) verifies that either way the moments are taken (standard method or multibeam method) one obtains the same undecomposed **Q**. The same is true of the undecomposed energy density, *U*, as is evident when *ΔU* is eliminated from Eqns. (18a) and (21).

One can use a similar strategy to find the (electron or ion) multibeam pressure tensor, $\mathbf{P}^{MB}$, from the known standard pressure tensor, **P** by introducing the RAM pressure tensor moment, $\mathbf{P}_{RAM}$ and stress tensor moment, **T**, as in Eqn. (12). Here, $\mathbf{T} = \mathbf{P} + \mathbf{P}_{RAM}$, has the same value for standard or multibeam pressure moments. Dimensionless pressures have been created by dividing all pressures by nmu_n^2. If Δ**P** is defined as, $\Delta\mathbf{P} \equiv \mathbf{P}_{RAM} - \mathbf{P}_{RAM}^{MB}$, then, by the same arguments as above,

$$24)\quad \mathbf{P}^{MB} = \mathbf{P} - \Delta\mathbf{P}, \quad \Delta\mathbf{P} \equiv \frac{1}{u_n^2}\left[\sum_{j=1}^{N} \eta_j \mathbf{u}_j \mathbf{u}_j - \mathbf{u}\mathbf{u}\right] \quad \text{(dimensionless, for electrons or ions)}$$

Note that the multibeam pressure moment tensor, $\mathbf{P}^{MB}$, contains the scalar multibeam energy density, U_{therm}^{MB}, since, in *dimensionless units*, the *trace* of $\mathbf{P}^{MB}$, $\mathrm{Tr}[\mathbf{P}^{MB}] = U_{therm}^{MB}$.

When **u**, **P**, and **Q** are *known* standard moments of a known or measured electron/ion f(**v**), one need only find the density and velocity (n_j, u_j) of each beam, $f_j(\mathbf{v})$, in order to evaluate

ΔU, $\Delta\mathbf{P}$ and $\Delta\mathbf{Q}$, and, from them, $\mathbf{P}^{MB}$, $(U_{therm})^{MB}$ and $(\mathbf{Q}_{therm})^{MB}$. For the visual method, this is a more effective strategy than calculating a sum of beam thermal moments, as in the definitions of $\mathbf{P}^{MB}$, $(U_{therm})^{MB}$ and $(\mathbf{Q}_{itherm})^{MB}$ given in Eqns. (11).

In Table (3), below, we have summarized the formulas for finding the pseudothermal parts, ΔU, $\Delta\mathbf{P}$ and $\Delta\mathbf{Q}$, of known standard moments, U_{therm}, $\mathbf{P}$, and $\mathbf{Q}_{therm}$ from the standard moments and the N beam centroid-velocities, $\mathbf{u}_j$ and N densities, n_j. The multibeam moments are then determined by subtracting ΔU, $\Delta\mathbf{P}$, and $\Delta\mathbf{Q}$ from U_{therm}, $\mathbf{P}$, and $\mathbf{Q}_{therm}$. All quantities are in dimensionless units, constructed by dividing each moment in *physical units* by the appropriate normalization factor in the third column. *The normalization factors are a kinetic energy density,* $mnu_n^2/2$, *an effective scalar pressure,* mnu_n^2, *and a kinetic energy density flux magnitude,* $mnu_n^3/2$. The density, n and flow velocity u_n can conveniently be taken as the *standard* density and flow velocity of the given (e or i) f(**v**), so that the normalizations become magnitudes of the standard U_{bulk} and $|\mathbf{Q}_{bulk}|$. In other words, $mnu_n^2/2 = U_{bulk}$, $mnu_n^3/2 = |\mathbf{Q}_{bulk}|$ and $mnu_n^2 = 2U_{bulk}$.

Table 3: ***Formulas for dimensionless multibeam thermal moments. (normalization in third column).***

Moment	**Pseudothermal part of standard moment**	**Normalization**	
Energy Density	$\Delta U \equiv \frac{1}{u_n^2}\left[\sum_{j=1}^{N}\eta_j u_j^2 - u^2\right]$	$\frac{mnu_n^2}{2}$	$U_{therm}^{MB} = U_{therm} - \Delta U_{therm}$ □
Pressure tensor	$\Delta\mathbf{P} \equiv \frac{1}{u_n^2}\left[\sum_{j=1}^{N}\eta_j \mathbf{u}_j\mathbf{u}_j - \mathbf{uu}\right]$	mnu_n^2	$\boxed{\mathbf{P}^{MB} = \mathbf{P} - \Delta\mathbf{P}}$ $Tr\mathbf{P} = U_{therm}$, $Tr\mathbf{P}^{MB} = U_{therm}^{MB}$
Energy Flux vector	$\Delta\mathbf{Q} = \frac{1}{u_n^2}\left[\sum_{j=1}^{N}\eta_j \mathbf{u}_j u_j^2 - \mathbf{u}u^2\right]$	$\frac{mnu_n^3}{2}$	$\boxed{\mathbf{Q}_{therm}^{MB} = \mathbf{Q}_{therm} - \Delta\mathbf{Q}}$ $\mathbf{Q}_{therm} \equiv \mathbf{Q}_{enthalpy} + \mathbf{Q}_{htflux}$ $\mathbf{Q}_{enthalpy} \equiv \hat{\mathbf{u}}Tr\mathbf{P} + 2\hat{\mathbf{u}}\cdot\mathbf{P}$

5. Summary and conclusions

5.1 MULTIBEAM VELOCITY DISTRIBUTIONS AND PSEUDOTHERMAL MOMENTS

In this paper we have addressed issues that arise when taking energy moments of effectively disjoint particle velocity distributions, f(**v**), which can be expressed as a sum of N beams, $f(\mathbf{v}) = f_1(\mathbf{v}) + \ldots + f_N(\mathbf{v})$.

The *standard* moments of any f(**v**) have only *one* flow velocity, **u**, associated with f(**v**). Examples of standard *coherent* moments are the bulk kinetic energy density, $U_{bulk} = nmu^2/2$, of the system and the bulk kinetic energy density flux vector, $\mathbf{Q}_{bulk} = n\mathbf{u}mu^2/2$ of the system. Moments which depend on correlations of fluctuations (**v** – **u)** are said to be *incoherent or "thermal"*. Examples are the standard thermal energy density, $U_{therm} = \int d^3\mathbf{v} f(\mathbf{v})\, m|\mathbf{v} - \mathbf{u}|^2/2$, and the standard thermal energy flux vector, $\mathbf{Q}_{therm}$, defined, in our convention, to be the sum of the enthalpy and heat flux vectors. There are separate space-time transport equations for the coherent and the incoherent energy moments (Eqns. (1)). In this paper we do not address the space-time evolution of energy moments of electrons or ions but rather consider their properties averaged over short time periods and small spatial regions.

Standard moments of multibeam velocity distributions $f(\mathbf{v}) = f_1(\mathbf{v}) + \ldots + f_N(\mathbf{v})$ tend to inflate U_{therm} and change the direction and magnitude of $\mathbf{Q}_{therm}$ by amounts we call the *pseudo-thermal parts* of these standard thermal moments, ΔU_{therm} and $\Delta\mathbf{Q}_{therm}$. A simple example is the pseudothermal energy, ΔU_{therm}, of a pair of cold beams. In this case ΔU_{therm} arises from the standard-moment misidentification of their center-of-mass-frame kinetic energy as thermal energy.

There is an alternate way of taking moments well-suited for multibeam velocity distributions of form $f(\mathbf{v}) = f_1(\mathbf{v}) + \ldots + f_N(\mathbf{v})$, which we have called *multibeam* moments. *Multibeam* thermal energy density moments, U^{MB}_{therm} and energy density flux moments, $\mathbf{Q}^{MB}_{therm}$, do not have pseudothermal parts. They are calculated by taking a *standard* thermal moment of *each beam* and summing over beams. For example, the multibeam thermal energy density moment is given by,

$$U^{MB}_{therm} = \sum_{j=1}^{N} f_j(\mathbf{v}) m\,|\,\mathbf{v} - \mathbf{u}_j\,|^2/2$$

The pseudothermal part of a thermal moment is the difference between the standard and multibeam thermal moments. For example, $\Delta U_{therm} = U_{therm} - U^{MB}_{therm}$, and, $\Delta\mathbf{Q}_{therm} = \mathbf{Q}_{therm} - \mathbf{Q}^{MB}_{therm}$. Coherent multibeam moments are calculated in the same way. For example, the multibeam kinetic energy density is given by,

$$U^{MB}_{bulk} = \sum_{j=1}^{N} mn_j\,|\,\mathbf{u}_j\,|^2/2$$

The time-evolution of the *multibeam* coherent and thermal energy densities is governed by a pair of space-time energy equations (Eqns. (26)) similar to those for the standard bulk and thermal energy densities. Energy transfer between coherent and thermal multibeam energy densities is

now governed by the sum over beams of the work/volume done by the multibeam pressure density force of each beam on or against the beam's flow velocity, $\mathbf{u}_j$.

To illustrate the advantages of taking multibeam energy moments the method was applied to a system of N co-axial tri-Maxwellians. Due to symmetry, none of the N tri-Maxwellians had a standard heat flux moment. However, the standard heat flux moment of the sum of beams, $f(\mathbf{v}) = f_1 + \ldots + f_N$ was non-zero. Hence, for this example, *all* of the standard heat flux of $f(\mathbf{v})$ is *pseudothermal and* the multibeam heat flux moment of $f(\mathbf{v})$ is zero.

For the special case of a pair of equal and opposite cold beams the standard thermal energy density is nonzero, while the multibeam thermal energy density is zero.

5.2 UNDECOMPOSED ENERGY MOMENTS

There is still another way of taking energy moments of $f(\mathbf{v})$, this time with no reference to either a single beam flow velocity or to multiple beam flow velocities and hence no distinction between coherence or incoherence of energy moments.

An example is the overall energy moment, $U = \int d^3\mathbf{v} f(\mathbf{v}) mv^2/2$. If v^2 is replaced by $[(\mathbf{v}\text{-}\mathbf{u}) + \mathbf{u}]^2$, U decomposes into a sum of the *standard* coherent (bulk kinetic) energy density and incoherent (thermal) energy density, $U = U_{bulk} + U_{therm}$. For that reason, we have referred to U as the *undecomposed* energy density. For a system of N beams the substitution of $f(\mathbf{v}) = f_1(\mathbf{v}) + \ldots + f_N(\mathbf{v})$ in the integral for U, followed by a replacement of v^2 by $[(\mathbf{v}\text{-}\mathbf{u}_j) + \mathbf{u}_j]^2$ in each beam integral, j, leads to the decomposition of U into a sum of coherent and incoherent multibeam energy densities, $U = U_{bulk}^{MB} + U_{therm}^{MB}$.

Therefore, regardless of whether $f(\mathbf{v})$ consists of one or more beams, the undecomposed energy moments are meaningful and invariant under which method of decomposition is employed. This is true for the undecomposed energy flux vector, **Q**, as well as for U. The space-time energy transport equation for U is given by Eqn. (2a).

Analysis of *measured* velocity distributions, $f(\mathbf{v})$, using undecomposed moments such as U and **Q** has the advantage that undecomposed energy moments are well-defined and meaningful for single-peaked, multi-peaked, disjoint or arbitrarily-shaped $f(\mathbf{v})$, far from thermal equilibrium. The disadvantage is that the energy and energy flux cannot be interpreted as "coherent" or "incoherent" (i.e., *bulk* or *thermal*).

One interesting and important consequence of the invariance of U and **Q** is that pseudothermal parts of standard thermal moments which are absent in multibeam *thermal* moments appear in the corresponding multibeam *bulk* moments. As an example, standard thermal energy density in an assembly of beams will appear as an increase in (multi-beam) bulk kinetic energy density. Equivalently, pseudotemperature absent in the multibeam moment shows up as increased coherent kinetic energy per particle in the assembly of beams.

We have shown that the invariance of undecomposed energy moments leads to a simplification in the determination of the pseudothermal parts of the standard thermal energy density and energy density flux moments. For example, we may use the invariance of U and **Q** to rewrite the pseudothermal part of the standard thermal energy density moment and the standard thermal energy flux moment in terms of $\Delta U_{therm} = = U^{MB}_{bulk} - U_{bulk}$ and $\Delta \mathbf{Q}_{therm} = \mathbf{Q}^{MB}_{bulk} - \mathbf{Q}_{bulk}$.

Hence, the pseudothermal parts of these standard thermal moments can be expressed entirely in terms of coherent beam velocities and beam densities. This will form the basis of the so-called "visual method" for finding pseudothermal moments of measured f(**v**) to be described in a future paper.

5.3 PRESSURE MOMENTS

In addition to *U* and **Q**, another undecomposed moment we have treated is the particle stress tensor, $\boldsymbol{T} = \int d^3\mathbf{v} f(\mathbf{v}) m\, \mathbf{vv}$, whose divergence appears in the particle momentum equation (Eqn. (6b)). $\boldsymbol{T}$ decomposes into the standard "thermal" pressure tensor moment, $\boldsymbol{P}$ and the RAM pressure tensor, $\mathbf{P}_{\mathrm{RAM}} = m\mathbf{uu}$ when both **v**'s in the dyadic **vv** are replaced by $\mathbf{v} = (\mathbf{v} - \mathbf{u}) + \mathbf{u}$. It also decomposes into the multibeam pressure tensor, $\boldsymbol{P}^{MB}$, and the RAM pressure tensor, $\boldsymbol{P}^{MB}_{RAM} = \sum_{j=1}^{N} mn_j \left(\mathbf{u}_j \mathbf{u}_j \right)$, when f(**v**) is replaced by $(f_1(\mathbf{v}) + \ldots + f_N(\mathbf{v}))$ and **v** is expressed as $\mathbf{v} = (\mathbf{v} - \mathbf{u_j}) + \mathbf{u_j}$ in the RAM pressure of each of the N beams. Once again, there will be a (tensor) *pseudothermal* part of the standard pressure tensor moment, $\boldsymbol{P}$, in a multibeam f(**v**), given by the difference, $\Delta \boldsymbol{P} = \boldsymbol{P} - \boldsymbol{P}^{MB}$. Pseudothermal pressure will show up in the multibeam RAM pressure tensor due to the invariance of $\boldsymbol{T}$.

5.4 SIGNIFICANCE

Recent high-resolution simulations and measurements of electron and ion velocity distributions, f(**v**), during magnetic reconnection have revealed that f(**v**) can be very far from thermal equilibrium, quite complex, and often effectively disjoint. Hence the interpretation of particle energetics sometimes requires new tools although we are still wedded to using long-standing familiar ones such as fluid concepts and fluid models. Kinetic theory modeling of processes in collisionless plasmas is increasingly used to understand space physics measurements, especially through particle-in-cell simulations. Vlasov equations contain particle and field physics which is not present in fluid equations. Velocity moments of the Vlasov equation are often the basis for deriving fluid variables and equations.

In this paper we retain this framework but introduce a *multibeam* method of taking velocity moments which is sometimes more appropriate than the "*standard*" method of taking velocity moments for interpreting distributions, f(**v**), which are far from equilibrium and effectively disjoint. In the *standard* method of taking moments it is assumed that there is only one overall flow velocity associated with f(**v**) and the energy moments of f(**v**) are coherent (bulk kinetic energy, bulk energy flux, RAM pressure, etc.) and incoherent (pressure, thermal energy density, enthalpy flux, heat flux, etc.). This can lead to counterintuitive results, such as the *pseudo*thermal

energy in a system, f(**v**) of two cold equal and opposite cold electron beams. The remedy is well known – consider each electron beam to be a separate *species*. When the system thermal energy is calculated for each species separately and then summed, the system energy is found to be entirely kinetic rather than entirely thermal.

The underlying approximation in the multibeam method is to express f(**v**) as a sum of distributions, which we call beams, so that for electrons or for ions, $f(\mathbf{v}) = f_1(\mathbf{v}) + \ldots + f_N(\mathbf{v})$. We can think of each beam as a separate subspecies of the electron or ion f(**v**). Each beam has a centroid velocity, $\mathbf{u}_j$, a density, n_j, etc. Multibeam energy density moments of f(**v**) are constructed by taking standard energy density moments of each beam and then summing over beams. Thus, multibeam energy moments are again coherent (multibeam bulk kinetic energy, multibeam bulk energy flux, multibeam RAM pressure, etc.) and incoherent (multibeam pressure, multibeam thermal energy density, multibeam enthalpy flux, multibeam heat flux, etc.). *Multibeam* incoherent moments do not contain pseudothermal parts. However, any of the *standard* incoherent energy density moments can contain pseudothermal parts which actually should be reinterpreted (whether scalar, vector or tensor) as additions present in the corresponding multibeam coherent moment.

This has practical significance for *standard* fluid interpretations of *multibeam* f(**v**) measured by spacecraft during magnetospheric reconnection or found in PIC simulations. A hypothetical scenario illustrating the differences in interpretation when taking multibeam moments of a multibeam electron distribution, f(**v**), rather than taking standard moments is:

- If an elevated standard electron *temperature* $T = (U_{therm}/n)$ measured by an MMS spacecraft over a 30 ms time interval is interpreted as due to a *heating process* and the electron velocity distribution, f(**v**) measured by the FPI instrument is found to consist of disjoint electron beams, the multibeam electron temperature will be lower, and the multibeam bulk kinetic energy per particle higher by the same amount. Although the standard temperature suggests electron heating has occurred, the multibeam temperature reveals that some of the higher electron energy is coherent rather than incoherent, arising from electron acceleration to a higher bulk kinetic energy. The relative magnitudes of thermal energy density vs bulk kinetic energy remain to be studied for specific measurements.

It is worth restating and discussing the assumptions and limitations of the approach developed here for taking energy moments of non-equilibrium particle distributions, f(**v**). A number of these issues will be addressed by work currently in progress.

The multibeam moment method described here applies to non-equilibrium velocity distributions, f(**v**), at essentially one time and place, which can be approximated as a sum of sub-distributions, $f_1(\mathbf{v}) + \ldots + f_N(\mathbf{v})$, referred to as "beams." While an analytic example has been given of a sum of tri-Maxwellians, it has not been demonstrated in this paper when or if an arbitrary *measured* non-equilibrium f(**v**) can be expressed in this manner. One strategy would be to approximate f(**v**) as a sum of parametrized analytic sub-distributions (not necessarily tri-Maxwellians) and to determine the parameters by a least-squares fit to the measured f(**v**). There are preliminary indications that this method can be useful and can be adapted to treat high-

energy distributed velocity-space "clouds" or "haloes" as particular sub-distributions (beams) in the sum.

Aside from the question of uniqueness of a beam decomposition for a given N it is not always clear how many beams, N, should be included in the sum. In the theoretical limit in which N equals the number of *particles* in f(**v**), one enters the microscopic kinetic regime, where there is not even a concept of fluid variables such as temperature. However, published work on machine learning suggests a single digit-number of beams, N, is reasonable (DuPuis, et al, 2020).

A *least-squares fitting* method and other strategies for approximating velocity distributions measured during magnetic reconnection in the magnetosphere (e.g., Fig. 2) as sums of beams have been presented at various conferences. A sequel to the present paper, which develops this and other methods for construction of multi-beam moments, is about to be submitted for publication.

The present paper does not consider multibeam particle energy transport in space and time. The multibeam fluid moments taken here are of a Eulerian multibeam distribution, f(**v**), at one space-time point (or an average over at most a small range of such points). The study of *processes* such as multibeam particle energization and space-time energy transfer are beyond the scope of the moment methods discussed here. Nevertheless, *undecomposed* energy moments discussed in this paper may be useful in following spacecraft-measured Eulerian f(**v**) in time along the spacecraft orbit, especially when multiple beams appear to merge and split and when the distinction between coherent and incoherent moments is not essential.

Considerable insight is gained into the *origins and energization* of beams from *Lagrangian* studies based on kinetic theory (i.e., simulations), such as the particle trajectories traced in Fig. (3). For example, a pair of beams may be two-stream unstable only so long as the space-time particle trajectories remain crossed at the same spatial point. Even if instability does occur, the spacecraft must remain at that crossing point long enough for it to be observed.

In conclusion, care must be taken when applying fluid concepts to highly non-Maxwellian particle velocity distributions, especially when they are disjoint.

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Figure 1.

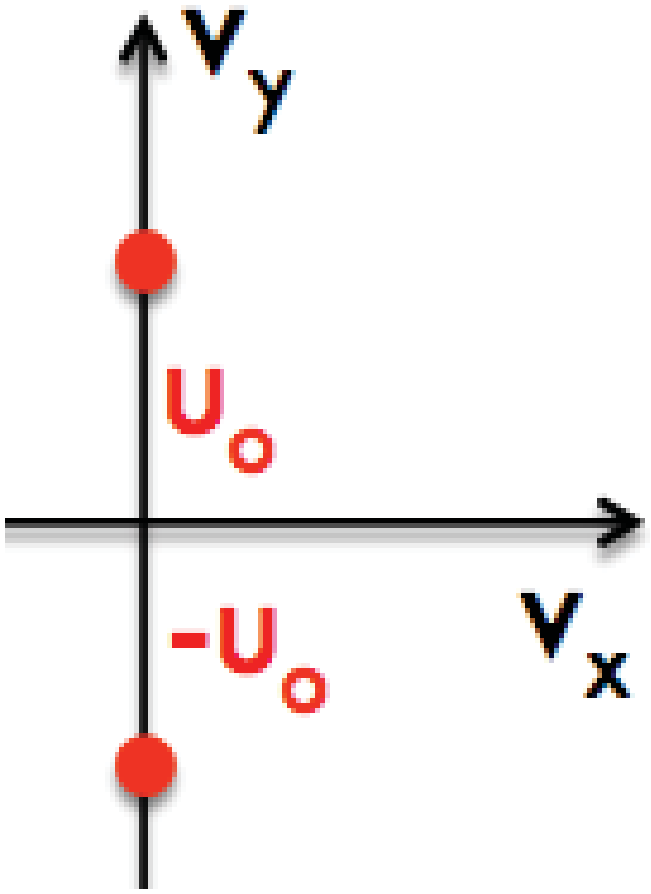
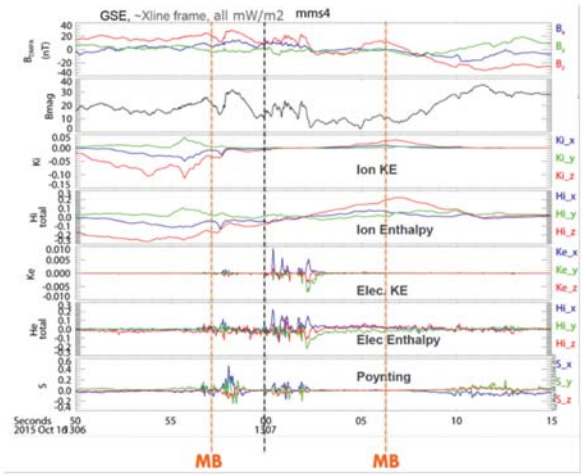

v_y
U_o
$-U_o$
v_x

Figure 2.


GSE, ~Xline frame, all mW/m2 mms4
Ion KE
Ion Enthalpy
Elec. KE
Elec Enthalpy
Poynting
MB
MB

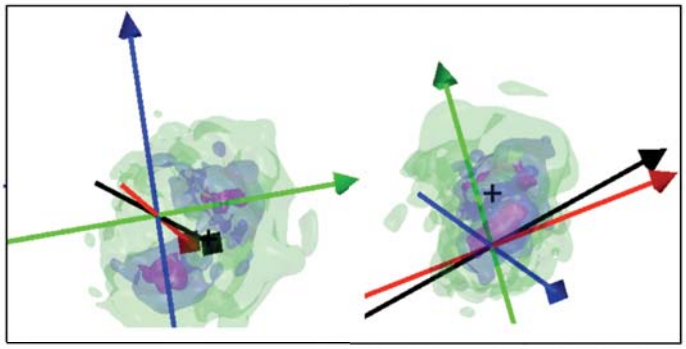
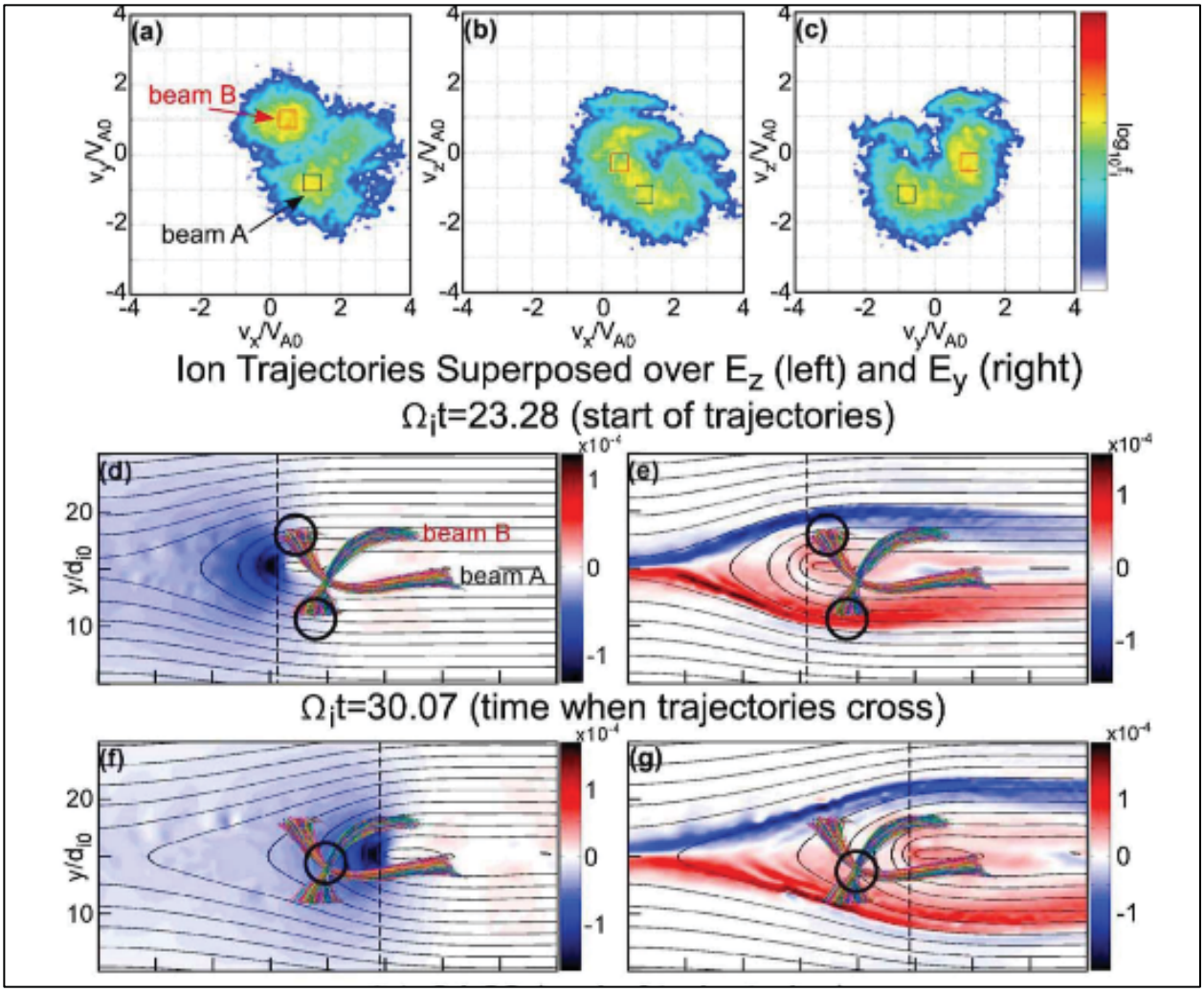


Figure 3.


(a)
(b)
(c)
beam B
beam A
v_y/V_{A0}
v_z/V_{A0}
v_x/V_{A0}
$\log_{10} f_i$
Ion Trajectories Superposed over E_z (left) and E_y (right)
$\Omega_i t$=23.28 (start of trajectories)
(d)
(e)
(f)
(g)
y/d_{i0}
x10^{-4}
$\Omega_i t$=30.07 (time when trajectories cross)