SOME LIMIT THEOREMS FOR SHOT-NOISE PROCESSES

by

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ABSTRACT

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Shot-noise processes consist of an underlying process of points, each of which triggers off a contribution, which may itself be a stochastic process. They form a flexible and widely used class of model for stochastic processes. While moments are easily calculated using Campbell's Theorem, the distribution is rarely available in closed form, so that approximations are of great interest. This thesis is concerned with the central limit theorem, when the process is considered over a long time interval.

A detailed discussion of the case when the primary point process is Poisson and the secondary processes independent, is contained in Chapter I. The characteristic function is expressed in terms of its Feller canonical measure, leading to necessary and sufficient conditions for convergence to any infinitely divisible distribution. Both normal and non-normal limits are discussed in the examples which include two important special cases, the cluster point process and the integrated stationary shot-noise. A Berry-Essèen bound for the error in using a normal approximation, is given.

The extension to the generalized shot-noise, with non-Poisson point process and dependent secondary processes, requires an extensive development of point process theory. In Chapter II, the properties of Palm probabilities are discussed and the various definitions of the conditional intensity measure of a point process, compared. The generalization of known results on this measure, culminates in a conditional version of Campbell's Theorem for shot-noise.

In Chapter III, it is shown that the characteristic function of the generalized shot-noise, while not available in closed form, may be expressed in terms of a random canonical measure. By considering convergence of processes of such measures, sufficient conditions for convergence to infinitely divisible distributions are derived. Criteria for convergence to a normal limit are deduced as a special case.

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INTRODUCTION

As a model for stochastic phenomena in a wide variety of contexts shot-noise processes have a strong intuitive appeal. Each member of a primary, random process of points $\{\tau_j\}$, $j = 1,2,\ldots$ is assumed to trigger off a secondary stochastic process, $H(t,\tau_j)$. The shot-noise process is the sum of these individual contributions, $\sum H(t,\tau_j)$, over an appropriate subset of $\{\tau_j\}$. This is also sometimes referred to as a filtered point process, see for example Parzen (1962). Since the τ_j and the $H(t,\tau_j)$ can take values in quite general spaces, the model can embrace a wide variety of applications, a few of which are mentioned now as illustrations.

The classical example is the shot effect in thermionic vacuum tubes - S.O. Rice (1944-5), Davenport and Root (1958) - where the point events are electrons striking the anode and the secondary process or "noise" is the response of an output circuit. In hydrology - Weiss (1974), Lawrence and Kattegoda (1977) - the point events are rain showers or storms and the noise corresponds to the rate of flow in rivers. In astronomy - Chandrasekhar (1943) - the points may be stars or galaxies in space-time, while the noise is the light intensity, gravitational field etc. at distant points. Marcus (1972) has proposed a shot-noise model for the lunar surface, the primary process here consisting of the points of impact of meteorites and the noise being the craters they throw up. The noise and chemical pollution caused by motor vehicles has also been modelled in this way by Marcus (1973, 74), with the position of vehicles on the road being the point events. In bacteriology, the sequences of births of microbes is a point process giving rise to a flow of toxins into the bloodstream of the host, as in Jagers (1967). Other applications may be found in the works referred to by Daley (1971) or J. Rice (1977).

Throughout this thesis, I will assume that the primary process consists of point events ordered in time and that the secondary processes are real-valued. In addition, the point τ_j will be the starting point of the process $H(t,\tau_j)$, so that for $t < \tau_j$, $H(t,\tau_j)$ is zero. While many of my results carry over easily to the case where the noise processes are two-sided, the ordering properties of time will become important in Chapters II and III.

The objective is to find the limiting distribution, for large t, of

$$Y(t) = \sum_{\tau_j \in [0,t]} H(t,\tau_j)$$

and to describe this distribution in terms of two functions, a centering function μ_t and a norming function σ_t, and a limiting distribution for

$$Z(t) = \frac{Y(t) - \mu_t}{\sigma_t} .$$

Typically, μ_t and σ_t might be $EY(t)$ and $\sqrt{\text{Var } Y(t)}$, respectively, with the limit distribution being $N(0,1)$. While the characteristic function and moments of $Y(t)$ are easy to derive, as shown in Section 2 of Chapter I, the distribution function is not usually available in closed form. It follows that approximations to the distribution are of considerable importance.

For this type of central limit theorem to make sense, it is clear that the variation of $Y(t)$ about its mean must tend to increase unboundedly with time. For instance, in the case of finite variance, $\text{Var } Y(t) \to \infty$, as $t \to \infty$. Now the shot-noise process usually considered in applications is

$$Y(t)^* = \sum_{\tau_j \in (-\infty, t]} H(t, \tau_j)$$

i.e. the sum of all contributions triggered off prior to time t. To make sense for phenomena in equilibrium, it is usual to consider stationary shot-noise processes for which Var $Y(t)^*$ clearly cannot explode to infinity. However, there are three contexts in which "exploding" shot-noise is of interest.

Firstly, cluster point processes are of this type. Here the primary process is a sequence of cluster centres each of which triggers off a further point process. Then $H(t, \tau_j)$ is the number of points in the secondary process starting at τ_j and occurring up to time t. The total number of points in [0,t] is given by Y(t). Models of this type have been proposed by, among others, Barlett (1963) for bunching in traffic, Lewis (1964 a,b) for computer failures and Vere-Jones (1970) for earthquake aftershocks. The total number of points is clearly an exploding shot-noise in the sense we are considering. In fact, central limit theorems for these processes have been proved, with varying degrees of generality and elegance, by Lewis (1964a, 1969, 1970), Smith (1964), Daley (1972) and Westcott (1973, 77). In fact, one motivation for this research was to see to what extent Westcott's results could be extended to more general shot-noise processes.

Secondly, the integral of a stationary shot-noise is also a shot-noise and it has this exploding nature, as will be shown in more depth in Chapter I, Sections 7 and 9. In view of the importance of the sum (and arithmetic mean) in statistical analyses of sequences of random variables, it seems likely that the integral (and Cesaro mean) will prove important

in the analysis of shot-noise. The distribution of the integral, therefore, becomes of great interest.

The third area where exploding shot noises appear naturally is closely related to the second. Often it is the integral, the area under the trajectory, of the shot-noise which is actually the function of greatest interest. Thus the production of toxins by bacteria - Jagers (1967), Pakes (1972) - can be expressed as the integral of a shot-noise. Similarly, if the cost of an illness is proportional to the time the disease lasts then the total cost of an infectious disease will be given by the integral of a shot-noise. Further similar applications, to dam storage, inventory and queuing problems, are referred to by McNeil (1970). The cumulative processes of Smith (1955) and the renewal-reward process of Ross (1970) may also be considered in this way.

The thesis falls into three parts. Chapter I consists of a fairly extensive study of the central limit theorem for the case where the primary process is Poisson and the secondary processes are independent. The principal tool used is the characteristic function and its expression in terms of the Feller canonical measure (Feller, 1971). In fact, this turns out to be a rather neat illustration of the power of Feller's approach. The results obtained are applied to several examples including the cluster point process and the integrated, stationary shot-noise.

Chapter III is devoted to proving a central limit theorem for a broad class of generalised shot-noise processes where the primary point process may be non-Poisson and the secondary process may depend on past events. This result is derived by extending the Feller canonical measure to allow for conditioning on the past of the process. However, the application of

this limit theorem to particular examples remains a matter for further work.

The arguments of Chapter III depend heavily on some preliminary results on point processes established in Chapter II, in particular on the use of the conditional intensity measure. The development in Chapters II and III is more difficult than in Chapter I, so that a more rigorous treatment is necessary. While some fundamental questions, connected with the definition of shot-noise stochastic processes, have been discussed, I have not tackled the difficult question of finding useful criteria for their existence.

As for the reference system, all works to which reference is made are listed in alphabetical order of author's name and are referred to by name and year, e.g. Lewis (1964 a). Equations and expressions are numbered within sections, within chapters. Thus (I.2.1) means equation (1) of section 2 of Chapter I. However, if the reference were made in Chapter I itself, the "I" would be dropped, as would the "I.2" if the reference occurred in Section 2. Similar conventions apply to Theorems, Lemmas and Corollaries, which have their own numbering system, and also to Examples.

CHAPTER I

THE POISSON SHOT-NOISE

§1 Introduction

In this chapter we consider the case where the primary process is a Poisson process with time-dependent intensity $\lambda(t)$. We assume that $\lambda(t)$ is non-negative and write $\Lambda(t) = \int_0^t \lambda(s)\, ds$.

The sequence of events is denoted by $\{\tau_i\}, i \in \mathbb{N}$ where the τ_i are ranked in increasing order. The number in an interval, $[t_1, t_2]$ is denoted by $N[t_1, t_2]$, with $N[0,t]$ abbreviated to $N(t)$.

Following Vere-Jones (1970) we have that:

(a) $N[a,b]$ has a Poisson distribution with parameter $\Lambda(b) - \Lambda(a)$;

(b) if I_1 and I_2 are disjoint intervals, $N(I_1)$ and $N(I_2)$ are independent random variables.

Corresponding to each possible starting point $s \in \mathbb{R}$, we have a real-valued stochastic process $\{H(t,s)\}, t \in \mathbb{R}$, for which $H(t,s) = 0$, if $t < s$. The processes, $H(\cdot,s)$, are all assumed independent of each other and of the point process. The shot-noise process we are interested in is

$$Y(t) = \sum_{i=1}^{N(t)} H(t,\tau_i) \qquad \ldots \qquad \ldots \qquad (1)$$

which can also be written as an integral with respect to the counting measure N introduced above, viz.

$$Y(t) = \int_0^t H(t,s)\, dN(s) \qquad \ldots \qquad \ldots \qquad (2)$$

Note that a consequence of our independence assumptions is that the distribution of $H(t,\tau_i)$ cannot depend on i, the rank of τ_i.

Of previous authors, Takacs (1954) and Daley (1971), have considered the conceptually restrictive form $f(t - \tau_i, \chi_i)$, for a deterministic function f and $\{\chi_i\}$ a random sample from some distribution. Westcott (1976) and Rice (1977) use stochastic processes, $H(t - \tau_i)$, while Parzen (1962, P.152) calls (1) a filtered Poisson process (extended sense).

The significance of the form we use is best illustrated by an example.

Example 1

Consider a Poisson cluster process, cf. Vere-Jones (1970), in which $H(t - s)$ is the number of points of a cluster, starting at s, which occur in (s,t]. An important (random) parameter is the total cluster-size $K = \lim_{u\to\infty} H(u)$. Now suppose $K(\tau_i)$ is thought to depend on some explanatory variable $X(\tau_i)$ whose changing value we can chart but not control. We can no longer assume that the $H(t - \tau_i)$ are realizations of the same stochastic process. In the definition above, however, we do allow their distributions to vary according to their starting point, τ_i.

We are interested in the convergence in distribution of

$$Z(t) = \frac{Y(t) - \mu_t}{\sigma_t}, \text{ as } t \to \infty,$$

for appropriate centering and norming functions, μ_t and σ_t, respectively.

From Levy's continuity theorem, Loève (1963, p.191), this is equivalent to considering convergence of the characteristic function of $Z(t)$ to a continuous limit. In deriving conditions for the central limit theorem, our principal tools will therefore, be the characteristic function and the corresponding Feller canonical measure.

In Section 2, the characteristic function of $Y(t)$ is derived. Feller's canonical measure makes its appearance in Section 3 where the measure corresponding to a Poisson shot-noise is derived and $Z(t)$ is shown to be infinitely divisible. The main theorem of Chapter I is derived in Section 4. This gives necessary and sufficient conditions for convergence to a particular limit distribution. We also include a discussion of the choice of centering, which is rather fuller than Feller's. The usefulness of the necessary and sufficient conditions is illustrated in Section 5, which contains a thorough discussion of a particular shot-noise process, whose strikingly non-normal limiting behaviour was surprising to this writer, at least.

The remainder of the chapter deals exclusively with convergence to the standard normal distribution, the case of greatest practical interest. The necessary and sufficient conditions then reduce, in Section 6, to an analogue of the celebrated Lindeberg condition for the convergence of a sum of independent random variables. From this, we can easily derive a number of sufficient conditions which are simple enough to be of considerable practical use. These are discussed in Section 7. In Section 8, we find a bound for the error in approximating the true distribution of $Z(t)$ by its $N(0,1)$ limit. This is a straightforward analogue of the Berry-Esseen bound for the sum of independent random variables case. Finally, in Section 9, we turn to the equilibrium shot-noise process and extend our central limit theorem to cover the

useful example of the integrated shot-noise process.

§2 The Characteristic Function

$\mathbb{C}_{Y(t)}[\theta]$, the characteristic function of $Y(t)$, can be derived simply using a well-established method due to Takacs (1954). He uses the following Lemma.

Lemma 1

Conditionally on $N(t) = n$, the events $\{\tau_i\}$, $i = 1,\ldots,n$ are distributed as the order statistics from a random sample, size n, with common distribution $F(s) = \Lambda(s)/\Lambda(t)$, $s \in [0,t]$.

Theorem 2

$$\mathbb{C}_{Y(t)}[\theta] = \exp \int_0^t \{\mathbb{C}_H[\theta;t,s] - 1\}\lambda(s)\, ds$$

$$\text{where } \mathbb{C}_H[\theta;t,s] = \mathbb{E}e^{i\theta H(t,s)} \qquad \ldots \quad \ldots \quad (1)$$

Proof

Since $e^{i\theta Y(t)} = \prod_{j=1}^{N(t)} e^{i\theta H(t,\tau_j)}$, we have

$$\mathbb{E}[e^{i\theta Y(t)} | N(t), \{\tau_j\}] = \prod_{j=1}^{N(t)} \mathbb{C}_H[\theta;t,\tau_j]$$

using the independence of the $H(t,\tau_j)$. Now, conditionally on $N(t)$, the unordered τ_j are a random sample on $[0,t]$, with density $\lambda(s)/\Lambda(t)$.

$$\therefore \quad \mathbb{E}[e^{i\theta Y(t)} | N(t)] = \left[\int_0^t \mathbb{C}_H[\theta;t,s] \, \frac{\lambda(s)\, ds}{\Lambda(t)} \right]^{N(t)}$$

Finally, $N(t)$ has a Poisson distribution whose probability generating function is

$$\mathbb{E}[z^{N(t)}] = \exp[\Lambda(t)\ (z-1)]$$

$$\therefore \quad \mathbb{C}_{Y(t)}[\theta] = \exp[\Lambda(t)\{\int_0^t \mathbb{C}_H[\theta;t,s]\frac{\lambda(s)ds}{\Lambda(t)} - 1\}]$$

$$= \exp \int_0^t \{\mathbb{C}_H[\theta;t,s] - 1\}\lambda(s)\ ds$$

It follows immediately that

$$\mathbb{E}Y(t) = \int_0^t \mathbb{E}H(t,s)\lambda(s)\ ds,\ \text{Var}\ Y(t) = \int_0^t \mathbb{E}H^2(t,s)\lambda(s)\ ds \quad \ldots \quad (2)$$

provided these quantities are finite. This is usually known as Campbell's Theorem, cf. Campbell (1909 a, b).

§3 The Canonical Measure

We will be interested in the normed and centered process

$$Z(t) = \frac{Y(t) - \mu_t}{\sigma_t} \qquad \ldots \qquad \ldots \quad (1)$$

Later in this chapter $Y(t)$ will be assumed to have a finite variance so that we would hope to be able to take $\mu_t = \mathbb{E}Y(t)$ and $\sigma_t^2 = \text{Var}\ Y(t)$.

We first derive the canonical measure assuming $\mu_t = \mathbb{E}Y(t)$ but leaving σ_t unspecified.

From Theorem 2, we see that

$$\log \mathbb{C}_{Z(t)}[\theta] = \int_0^t \{\mathbb{C}_H[\theta\sigma_t^{-1};t,s] - 1\}\lambda(s)\,ds - i\theta\sigma_t^{-1}\int_0^t \mathbb{E}H(t,s)\lambda(s)\,ds$$

Introducing the distribution function of $H(t,s)$,

$$F_H[x;t,s] = \Pr\{H(t,s) \leqslant x\},$$

and writing $F_H[dx;t,s]$ for the differential form, we have that

$$\log \mathbb{C}_{Z(t)}[\theta] = \int_0^t\int_{-\infty}^{\infty}\left\{e^{i\theta\sigma_t^{-1}x} - 1 - i\theta\sigma_t^{-1}x\right\}F_H[dx;t,s]\lambda(s)\,ds$$

$$= \int_{-\infty}^{\infty}\frac{e^{i\theta y} - 1 - i\theta y}{y^2}\cdot y^2\int_0^t F_H(\sigma_t dy;t,s)\lambda(s)\,ds \qquad \ldots \quad (2)$$

Define $M_t(dy) = y^2\int_0^t F_H[\sigma_t dy;t,s]\lambda(s)\,ds$

Now Feller (1971, P.560) defines a measure M to be <u>canonical</u> when $M[x_1,x_2]$ is finite, $\forall\, x_1,x_2 \in \mathbb{R}$, and the integrals $M^+(x) = \int_x^{\infty} y^{-2}M(dy)$, $M^-(-x) = \int_{-\infty}^{-x} y^{-2}M(dy)$ both converge, $\forall\, x > 0$.

Clearly $M_t[x_1,x_2]$, $M_t^+(x)$ and $M_t^-(-x)$ are all finite provided $\Lambda(t) < \infty$.

A distribution, F, is said to be <u>infinitely divisible</u> if for every positive integer n, there exists a distribution, F_n, such that $F = F_n^{n*}$, the n-fold convolution of F_n. Equivalently, the characteristic function ω is infinitely divisible if $\forall n$, $\omega = (\omega_n)^n$, for some characteristic function ω_n. Feller (1971, Chapter XVII.1) shows that all infinitely divisible characteristic functions can be written as

$$\exp\left[\int_{-\infty}^{\infty}\frac{e^{i\theta y} - 1 - i\theta\tau(y)}{y^2}\,M(dy) + ib\theta\right] \qquad \ldots \quad (3)$$

for a canonical measure M, a centering function $\tau(y)$ and a real constant b, which depends on the particular choice of $\tau(y)$. We must choose $\tau(y)$ so that the integral in (3) exists.

From (2)

$$\mathbb{E}_{Z(t)}[\theta] = \exp \int_{-\infty}^{\infty} \frac{e^{i\theta y} - 1 - i\theta y}{y^2} M_t(dy)$$

When $EY(t) = \int_0^t EH(t,s)\lambda(s)ds < \infty$, $\forall t$, we can make the obvious choice of $\tau(y) = y$, while when $EY(t)$ is infinite we can follow Feller in choosing $\tau(y) = \sin y$. Then

$$\frac{e^{i\theta y} - 1 - i\theta \sin y}{y^2} \to \tfrac{1}{2}\theta^2, \qquad y \to 0$$

just as with $\tau(y) = y$, but now $\sin y$ itself is bounded and the integral in (3) exists. It follows that $Z(t)$ has an infinitely divisible distribution.

§4 Necessary and Sufficient Conditions for Convergence

A sequence of canonical measures $\{M_t\}$ is said to converge properly to the canonical measure M, if

$$M_t(I) \to M(I) \text{ for all intervals } I \text{ of continuity for } M,$$

and if

$$M_t^+(x) \to M^+(x),\ M_t^-(-x) \to M^-(-x), \text{ at all } x > 0 \text{ which}$$

are points of continuity for M.

The fundamental result we use to investigate the convergence in distribution of $Z(t)$ is the following theorem of Feller (1971, P.564.)

Theorem 1

Let M_t be a canonical measure and

$$\psi_t(\theta) = \exp\left[\int_{-\infty}^{\infty} \frac{e^{i\theta y} - 1 - i\theta\tau(y)}{y^2} M_t(dy) + i\, b_t\, \theta \right]$$

a sequence of characteristic functions. In order that ψ_t tends to a continuous limit ψ, it is necessary and sufficient that there exist

a canonical measure M and real constant b such that $M_t \to M$ and $b_t \to b$ as $t \to \infty$.

Then $\psi(\theta) = \exp\left[\int_{-\infty}^{\infty} \frac{e^{i\theta y} - 1 - i\theta\tau(y)}{y^2} M(dy) + ib\theta\right]$.

Remarks 1. Feller proves the theorem for a sequence M_n, $n \in \mathbb{N}$, but the proof for real t is a straightforward extension in which we consider all sequences $t_n \to \infty$.

2. As we remarked above, Feller uses the centering $\tau(y) = \sin y$ and shows the following useful property. Suppose $\psi_t(\theta) = \exp[\alpha_t\{\phi_t(\theta) - 1 - i\beta_t\theta\}]$ where $\alpha_t > 0$, β_t are real constants and $\phi_t(\theta)$ a sequence of characteristic functions with distribution functions $F_t(x)$. Then, if there exist centering constants β_t such that $\psi_t(\theta)$ has a continuous limit, $\beta_t = \int_{-\infty}^{\infty} \sin x \, F_t(dx)$ will suffice. This allows us to separate the question of the appropriate centering from the more fundamental question of the limiting measure.

Combining Theorem 1 with the continuity theorem for characteristic functions, Loève (1963, P.191), necessary and sufficient conditions for convergence of Z(t) are:

$$\int_{-x}^{x} y^2 \int_0^t F_H[\sigma_t dy; t, s]\lambda(s)ds \to M[-x, x], \text{ for some } x > 0 \quad \ldots (1)$$

and

$$\int_x^{\infty} \int_0^t F_H[\sigma_t dy; t, s]\lambda(s)ds \to M^+(x)$$

$$\int_{-\infty}^{-x} \int_0^t F_H[\sigma_t dy; t, s]\lambda(s)ds \to M^-(-x), \ \forall x > 0 \quad \ldots \quad \ldots (2)$$

We can say for some $x > 0$ in (1) because, for all intervals not covering the origin, $M_t[x_1, x_2] \to M[x_1, x_2]$, from (2) and the Helly-Bray Lemma, Loève (1963, P.180).

We prefer to rewrite these in terms of the tails of the distribution F_H, namely,

$$\bar{F}_H([-x,x];t,s) = F_H[-x;t,s] + \bar{F}_H[x;t,s]$$

$$= \Pr\{H(t,s) \leqslant -x\} + \Pr\{H(t,s) \geqslant x\}$$

Then (2) is equivalent to

$$\int_o^t \bar{F}_H[\sigma_t x;t,s]\lambda(s)ds \to M^+(x)$$

$$\int_o^t F_H[-\sigma_t x;t,s]\lambda(s)ds \to M^-[-x] \qquad \ldots \quad \ldots \quad \ldots \quad (3)$$

while, integrating by parts in (1), we have

$$\int_x^x y^2 \int_o^t F_H[\sigma_t dy;t,s]\lambda(s)ds$$

$$= -x^2 \int_o^t \bar{F}_H([-\sigma_t x,\sigma_t x];t,s)\lambda(s)ds$$

$$+ 2\int_o^x y \int_o^t \bar{F}([-\sigma_t y,\sigma_t y];t,s)\lambda(s)ds\,dy .$$

Since the first term on the right converges to

$$-x^2[M^+(x) + M^-(-x)] \text{ by (3)},$$

(1) and (2) can be replaced by (3) and

$$2\int_o^x y \int_o^t \bar{F}_H([-\sigma_t y,\sigma_t y];t,s)\lambda(s)ds\,dy$$

$$\to \bar{M}(x) = M[-x,x] + x^2[M^+(x) + M^-(-x)], \text{ for some } x > 0. \qquad \ldots \quad \ldots \quad \ldots \quad (4)$$

Finally, when $Y(t)$ has a finite variance which increases without bound, we can take

$$\sigma_t^2 \sim \text{Var } Y(t) = \int_o^t \mathbb{E}H^2(t,s)\lambda(s)ds$$

$$= 2\int_o^t \int_o^\infty x\bar{F}_H([-x,x];t,s)dx\,\lambda(s)\,ds$$

so that (4) is equivalent to

$$\frac{2}{\sigma_t^2} \int_{x\sigma_t}^{\infty} y \int_0^t \bar{F}_H([-y,y];t,s)\lambda(s)\, ds\, dy \to 1 - \bar{M}(x), \text{ for some } x > 0 \quad \ldots \quad (5)$$

For brevity we will sometimes denote the expression

$$\int_0^t \bar{F}([-y,y];t,s)\ \lambda(s)\ ds \text{ by } F_t(y).$$

Remark 3. In the finite variance case, M is a finite measure, so that (1) $\Rightarrow$ (2) by the Helly-Bray Theorem, Loève (1963, p.181), and, if we take $\sigma_t^2 \sim \text{Var } Y(t)$, we have only one condition, namely (1). However the pair of conditions (3) and (5) often seem easier to work with, especially as calculating the expression in (5) involves calculating that in (3) anyway.

There remains one detail to be settled. Recalling the expression (3.2) and Remark 2 following Theorem 1 it is clear that, if $\mathbb{E}Y(t) < \infty$, we would like to use the centering function $\tau(y) = y$, which is equivalent to taking $Z(t) = \frac{Y(t) - \mathbb{E}Y(t)}{\sigma_t}$. When can we deduce that, as $t \to \infty$,

$$\log \mathbb{C}_{Z(t)}[\theta] = \int_{-\infty}^{\infty} \frac{e^{i\theta y} - 1 - i\theta y}{y^2}\, d M_t(y) \quad \ldots \quad (6)$$

$$\to \int_{-\infty}^{\infty} \frac{e^{i\theta y} - 1 - i\theta y}{y^2}\, d M(y) \quad ?$$

Corollary 2 a)

Suppose that (1) is true and that, for $\forall\ x > 0$ which are continuity points of M,

$$\int_x^\infty y \int_0^t F_H[\sigma_t\, dy; t,s]\ \lambda(s)\ ds \to \int_x^\infty \frac{1}{y}\, dM(y)$$

$$\int_{-\infty}^{-x} y \int_0^t F_H[\sigma_t\, dy; t,s]\ \lambda(s)\ ds \to \int_{-\infty}^{-x} \frac{1}{y}\, dM(y), \qquad \ldots \quad (7)$$

where the left sides are finite.

Then (6) holds.

b) In particular, (7) is true if $\int_0^t \mathbb{E}|H(t,s)|^{1+\delta}\ \lambda(s)\ ds < \infty$, some $\delta > 0$.

Proof a) Note that on any interval $[x,\infty)$, $x > 0$, $y\int_0^t F_H[\sigma_t dy; t,s]\lambda(s)\, ds$ defines a finite measure, $\frac{1}{y} M_t(dy)$, which by (7) is completely convergent to $\frac{1}{y}M(dy)$. From Loève (1963, p.182) we see that

$$M_t^+(X) = \int_x^\infty \frac{1}{y} \cdot \frac{1}{y}\ M_t(dy) \to M^+(x)$$

Treating $(-\infty, -x]$ similarly we see that (7) $\Rightarrow$ (2). To apply Theorem 1, we need only prove that

$$b_t = \int_{-\infty}^{\infty} \frac{\sin y - y}{y^2}\, dM_t(y) \to \int_{-\infty}^{\infty} \frac{\sin y - y}{y^2}\, dM(y) = b$$

Now $\frac{\sin y - y}{y^2}$ is continuous so the contribution to the integral from any bounded interval, converges, by Loève (1963, P.180). Again taking $y\varepsilon[x,\infty)$, $x > 0$ and noting that $\frac{\sin y - y}{y}$ is bounded

$$\int_x^\infty \frac{\sin y - y}{y} \cdot \frac{1}{y}\, dM_t(y) \to \int_x^\infty \frac{\sin y - y}{y^2}\, dM(y) \quad \text{by}$$

Loève (1963, P.182). A similar argument for $(-\infty, x]$ concludes the proof.

b) Here, taking $y \in [x,\infty)$, $x > 0$, we note that $\frac{1}{y^{1-\delta}}\, d M_t(y)$ is a finite measure which is certainly weakly convergent by (1). Since $\frac{\sin y - y}{y^{1+\delta}} \to 0$, $y \to \infty$, we can use Loève (1963, P.181) to deduce (7).

Note 1. In the special case where $Y(t)$ has finite variance, we can use part (b) with $\delta = 1$ so that centering to $EY(t)$ is viable without any further investigation.

2. $\frac{1}{\sigma_t} \int_0^t E|H(t,s)|\, \lambda(s)\, ds$ must be finite by (7) but if it tends to a limit this need not be equal to $\int_{-\infty}^{\infty} \frac{1}{y} M(y)$ since there is the possibility of an atom at the origin in $\lim_{t\to\infty} \int_{-x}^{x} \int_0^t F_H[\sigma_t\, dy; t,s]\lambda(s)ds$.

§5 Examples

Example 1

To illustrate the use of the canonical measure we investigate the convergence of $Z(t)$ when $\lambda(t) = 1, \forall\, t$ and $H(s) = +\sqrt{1 + s^2 e^{-Xs}}$, $s \geq 0$, where X is a random variable, exponentially distributed with unit mean.

For $s > 0$, $1 < H^2(s) < 1 + s^2$

and $\lim_{s\to\infty} H^2(s) = 1$, a.s.

For $1 < u < (1 + s^2)^{\frac{1}{2}}$,

$$F(u,s) = \Pr\{H(s) \leq u\}$$

$$= \Pr\{e^{-X} \leq [\frac{u^2 - 1}{s^2}]^{\frac{1}{s}}\}$$

$$= [\frac{u^2 - 1}{s^2}]^{\frac{1}{s}} .$$

Now $$\mathbb{E}H^2(s) = 1 + \int_0^\infty s^2 e^{-Xs} e^{-X} dx$$

$$= 1 + \frac{s^2}{s + 1}$$

Therefore $\operatorname{Var} Y(t) = \int_0^t (1 + \frac{s^2}{s + 1})\, ds$

$$= \frac{t^2}{2} + \log(t+1) \sim \frac{t^2}{2} , \quad t \to \infty,$$

so we take $\sigma_t^2 = \frac{t^2}{2}$ and, for $\sigma_t x > 1$,

$$\overline{F}(\sigma_t x, s) = 1 - \left[\frac{\frac{1}{2}t^2x^2 - 1}{s^2}\right]^{\frac{1}{s}}, \quad s \geq \sqrt{\tfrac{1}{2}t^2x^2 - 1} \qquad \ldots \quad \ldots \quad (1)$$

$$= 0 \quad , \text{ otherwise}$$

For $x > \sqrt{2}$, $t \leq \sqrt{\tfrac{1}{2}t^2x^2 - 1}$ for large t so that $\int_0^t \overline{F}(\sigma_t x; s)\,ds$ will be identically zero giving us $M^+(x) = 0$ by (4.3).

For $x \leq \sqrt{2}$, we want

$$\lim_{t\to\infty} \int_{\sqrt{\frac{1}{2}t^2x^2-1}}^{t} \{1 - \left(\frac{\frac{1}{2}t^2x^2 - 1}{s^2}\right)^{\frac{1}{s}}\}ds$$

The integrand is $1 - \exp\left[-\frac{2}{s}\log\frac{s}{\sqrt{\frac{1}{2}t^2x^2 - 1}}\right]$

$$= \frac{2}{s}\log\frac{s}{\sqrt{\frac{1}{2}t^2x^2 - 1}} + O(\tfrac{1}{t^2}) ,$$

Now

$$\int_{\sqrt{\frac{1}{2}t^2x^2-1}}^{t} \frac{2}{s}\log\frac{s}{\sqrt{\frac{1}{2}t^2x^2 - 1}}\,ds$$

$$= \log^2\frac{t}{\sqrt{\frac{1}{2}t^2x^2 - 1}} \to \log^2\frac{x}{\sqrt{2}} , \text{ as } t \to \infty .$$

Since the range of integration $[\sqrt{\tfrac{1}{2}t^2x^2 - 1}, t]$ has length $O(t)$ the remainder term has integral $O(\frac{1}{t})$ and we deduce that

$$M^+(x) = \log^2\frac{x}{\sqrt{2}} , \text{ for } 0 \leq x \leq \sqrt{2} .$$

Next from (4.5) we have

$$1 - \overline{M}(x) = \lim_{t\to\infty} 2\int_x^\infty y \int_0^t \overline{F}(\sigma_t y, s)\,ds\,dy .$$

Again, when $x > \sqrt{2}$, $\sqrt{\frac{1}{2}t^2y^2 - 1} > t$, $\forall\, y \geq x$, provided t is large enough, so that from (1) the inner integral above is zero and $\overline{M}(x) = 1$.

For $x \leq \sqrt{2}$,

$$1 - \overline{M}_t(x) = 2\int_x^{\sqrt{2(1+\frac{1}{t^2})}} y \int_{\sqrt{\frac{1}{2}t^2y^2-1}}^{t} \{1 - [\frac{\frac{1}{2}t^2y^2 - 1}{s^2}]^{\frac{1}{s}}\}\,ds\,dy$$

since, if $y > \sqrt{2(1+\frac{1}{t^2})}$ then, for all $s \leq t$, $s < \sqrt{\frac{1}{2}t^2y^2 - 1}$ and so, again from (1), the inner integral is zero.

The substitution $z^2 = \frac{1}{2}y^2 - t^{-2}$ yields

$$1 - \bar{M}_t(x) = 4 \int_{(\frac{1}{2}x^2 - t^{-2})^{\frac{1}{2}}}^{1} z \int_{tz}^{z} \{1 - [\frac{t^2 z^2}{s^2}]^{\frac{1}{s}}\} \, ds \, dz \ .$$

Consider the function on $[0,1]$

$$g_t(z) = z \int_{tz}^{t} \{1 - [\frac{t^2 z^2}{s^2}]^{\frac{1}{s}}\} \, ds \, I_{[(\frac{1}{2}x^2 - t^{-2})^{\frac{1}{2}},\ 1]}(z)$$

where $I_{[a,b]}(z)$ is an indicator function.

Now $$g_t(z) \leq z \int_{tz}^{t} \{1 - \exp(-\frac{1}{s} \log \frac{s^2}{t^2 z^2})\} ds$$

$$\leq z \int_{tz}^{t} \frac{2}{s} \log \frac{s}{tz} \, ds$$

$$= z \log^2 z \ ,$$

which is integrable on $[0,1]$.

From the calculation for $M^+(x)$ we know that $\lim_{t\to\infty} g_t(z) = z \log^2 z$ on $[\sqrt{\frac{1}{2}x^2}, 1]$.

Dominated convergence shows that

$$1 - \bar{M}(x) = 4 \int_{\sqrt{\frac{1}{2}x^2}}^{1} z \log^2 z \, dz$$

$$= 1 - x^2 [\log^2 \frac{x}{\sqrt{2}} - \log \frac{x}{\sqrt{2}} + \tfrac{1}{2}] ,$$

$$0 \leq x \leq \sqrt{2} \ .$$

In terms of the original canonical measure,

$$M[0,x] = \bar{M}(x) - x^2 M^+(x)$$

$$= \left(\frac{x}{\sqrt{2}}\right)^2 + x^2 \left|\log \frac{x}{\sqrt{2}}\right|, x \leq \sqrt{2}$$

so that $M(dx) = 2x \log \frac{\sqrt{2}}{x}\, dx$, $x \in [0, \sqrt{2}]$

$= 0$, otherwise

I have not been able to evaluate the corresponding distribution function but its cumulants are easy to determine.

The characteristic function is $\exp \psi(\theta)$

where

$$\psi(\theta) = \int_0^{\sqrt{2}} \frac{e^{i\theta x} - 1 - i\theta x}{x^2} \cdot 2x \log \frac{\sqrt{2}}{x}\, dx$$

$$= \int_0^{\sqrt{2}} \sum_{r=2}^{\infty} \frac{(i\theta x)^r}{r!} \cdot \frac{2}{x} \log \frac{\sqrt{2}}{x}\, dx$$

$$= 2 \sum_{r=2}^{\infty} \frac{(i\theta)^r}{r!} \int_0^{\sqrt{2}} x^{r-1} \log \frac{\sqrt{2}}{x}\, dx$$

$$= \sum_{r=2}^{\infty} \frac{(i\theta)^r}{r!} \frac{2^{\frac{r}{2}+1}}{r^2} ,$$

so that the cumulants are: $\gamma_1 = 0$

$$\gamma_k = \frac{2^{\frac{k}{2}+1}}{k^2}, \quad k = 2, 3, \ldots.$$

and, in particular, $\gamma_3 = 0.943$, $\gamma_4 = 0.33\dot{3}$.

Note: The same results remain true if, instead of taking $H^2(s) = 1+s^2e^{-Xs}$ with X distributed exponentially, we take $H^2(s) = 1 + s^2(F[X])^S$ where F is any continuous distribution function and X is a random variable with precisely that distribution.

In some ways this is a disturbing example. H(s) is, from a naive point of view, "well-behaved" being very smooth and bounded in s, having finite moments of all orders for each s and a positive limit which also has moments of all orders. Yet we have a strikingly non-normal limit.

To try to establish more clearly what a "well-behaved" shot-noise process looks like we take two approaches: in §6 we investigate sufficient conditions for a normal limit; here, we look at the dependence of the type of model we considered in Example 2 on various parameters. We write

$$H(s) = +\sqrt{(\alpha + \beta s^\gamma e^{-\delta X s})},$$

with X distributed exponentially with mean λ^{-1} and all the parameters ~~non-negative.~~ positive except α which is non-negative.

Then $$\mathbb{E}H^2(s) = \alpha + \frac{\beta\lambda s^\gamma}{\delta s+\lambda}$$

$$= \begin{cases} \alpha + \frac{\beta\lambda}{\delta} s^{\gamma-1} + o(s^{\gamma-1}), & \gamma \geq 1 \\ \alpha + o(1), & \gamma < 1 \end{cases}$$

and so $\quad \text{Var } Y(t) = \int_0^t \mathbb{E}H^2(s)\,ds \sim \frac{\beta\lambda}{\gamma\delta} t^{\gamma}, \quad \gamma > 1$

$$\sim [\alpha + \frac{\beta\lambda}{\delta}]t, \gamma = 1$$

$$\sim \alpha t \quad , \quad \gamma < 1$$

For $x \geq \sqrt{\alpha}$,

$$F(x,s) = \left[\frac{x^2 - \alpha}{\beta s^{\gamma}}\right]^{\frac{\lambda}{\delta s}} \quad , \quad \alpha \leq x^2 \leq \alpha + \beta s^{\gamma} \qquad \ldots \quad \ldots \quad (2)$$

$$= 1 \quad , \text{ otherwise}$$

Example 2

Consider first the case, $\gamma < 1$. Introducing new parameters $\kappa = \frac{\alpha}{\beta}$, $\omega = \frac{\delta}{\lambda}$ we take $\sigma_t^2 = \alpha t$ and, for $\sigma_t x \geq \sqrt{\alpha}$,

$$\overline{F}[\sigma_t x, s] = 1 - \left[\frac{\kappa(tx^2 - 1)}{s^{\gamma}}\right]^{\frac{1}{\omega s}} \quad , \; s \geq (\kappa[tx^2 - 1])^{\frac{1}{\gamma}}$$

$$= 0 \quad , \text{ otherwise}$$

Unlike Example 1, we have a linear σ_t^2.

Condition (4.3) is then

$$\int_{[\kappa(tx^2-1)]^{\frac{1}{\gamma}}}^{t} \left\{1 - \left[\frac{\kappa(tx^2 - 1)}{s^{\gamma}}\right]^{\frac{1}{\omega s}}\right\} ds \to M^{+}(x), \; \forall\, x > 0 \,,$$

Since $\gamma < 1$, $\kappa(tx^2 - 1) > t^{\gamma}$, for all t large enough, so that $M^{+}(x) = 0$, $\forall x$. Turning to condition (4.5), we see that

$$2\int_{x}^{\left[\frac{1}{t}\left(\frac{t^{\gamma}}{\kappa}+1\right)\right]^{\frac{1}{\gamma}}} y \int_{\left[\kappa(tx^2-1)\right]^{\frac{1}{\gamma}}}^{t} \left\{1 - \left[\frac{\kappa(tx^2-1)}{s^{\gamma}}\right]^{\frac{1}{\omega s}}\right\} ds\, dx$$

ultimately vanishes for each x, for similar reasons.

So $\bar{M}(x) = 1$, $M^{+}(x) = 0$, $M^{-}(-x) = 0, \forall\, x > 0$ and so $M[-x,x] = 1, \forall\, x > 0$. Clearly M(dx) has an atom at the origin and is zero elsewhere, which is the canonical measure corresponding to the normal distribution; see, for example, Feller (1971, P.566). So for $\gamma < 1$, we have a normal limit.

Example 3

When $\gamma \geq 1$ we define

$$\sigma^2 = \begin{cases} \frac{1}{\gamma\omega} = \frac{\lambda}{\gamma\delta} & , \quad \gamma > 1 \\ (\kappa + \frac{1}{\omega}) = (\frac{\alpha}{\beta} + \frac{\lambda}{\delta}) & , \quad \gamma = 1 \end{cases}$$

and take $\sigma_t^{\,2} = \beta\sigma^2 t^{\gamma}$.

From (2), we see that for $x \geq \left(\frac{\kappa}{\sigma^2 t^{\gamma}}\right)^{\frac{1}{2}}$

$$\bar{F}(\sigma_t x, s) = 1 - \left[\frac{\sigma^2 t^{\gamma} x^2 - \kappa}{s^{\gamma}}\right]^{\frac{1}{\omega s}}, \quad s \geq [\sigma^2 t^{\gamma} x^2 - \kappa]^{\frac{1}{\gamma}}$$

$$= 0 \qquad , \quad \text{otherwise}$$

Abbreviate $[\sigma^2 t^{\gamma} x^2 - \kappa]^{\frac{1}{\gamma}}$ to $\phi_t(x)$. Then, by the same argument as in Example 1,

$$\int_{\phi_t(x)}^{t} \left\{1 - \left[\frac{\phi_t(x)}{s}\right]^{\frac{\gamma}{\omega s}}\right\} ds = \frac{\gamma}{2\omega} \log^2 \frac{t}{\phi_t(x)} + O(\tfrac{1}{t}), \quad t \to \infty$$

so that $M^{+}(x) = \lim_{t\to\infty} \frac{\gamma}{2\omega} \log^2 [\sigma^2 x^2 - \frac{\kappa}{t^{\gamma}}]^{\frac{1}{\gamma}}$

$$= \begin{cases} \frac{1}{2\gamma\omega} \log^2 \sigma^2 x^2 & , \quad x \leq \frac{1}{\sigma} \\ 0 & , \quad x \geq \frac{1}{\sigma} \end{cases}$$

To evaluate $\bar{M}(x)$, we make the transformation $z^2 = \sigma^2 y^2 - \kappa/t^\gamma$. Just as in Example 1, a dominated convergence argument produces

$$1 - \bar{M}(x) = \frac{4}{\omega\gamma\sigma^2} \int_{\sigma x}^{1} z \log^2 z \, dz$$

$$= \frac{1}{\omega\gamma\sigma^2} \left[1 - \frac{\sigma^2 x^2}{2} \left[4 \log^2 \sigma x - 4 \log \sigma x + 2 \right] \right]$$

$$= \frac{1}{2\gamma\omega\sigma^2} \left[2 - \sigma^2 x^2 \{ \log^2 \sigma^2 x^2 - 2 \log \sigma^2 x^2 + 2 \} \right], \quad x \leqslant \frac{1}{\sigma}$$

The canonical measure can be calculated as before:

$$\gamma > 1 \quad : \quad M(dx) = \begin{cases} 2\sigma^2 x \left| \log \sigma^2 x^2 \right|, & x \leqslant \frac{1}{\sigma} \\ 0, & x \geqslant \frac{1}{\sigma} \end{cases} \qquad \ldots \quad (3)$$

$$\gamma = 1 \quad : \quad M(dx) = \begin{cases} \frac{\kappa}{\sigma^2} \delta(x) + (1 - \frac{\kappa}{\sigma^2}) \, 2\sigma^2 x \left| \log \sigma^2 x^2 \right|, & x \leqslant \frac{1}{\sigma} \\ 0, & x \geqslant \frac{1}{\sigma} \end{cases} \qquad \ldots (4)$$

The latter distribution has a normal component at the origin with variance $\frac{\kappa}{\sigma^2}$. Here also, σ_t^2 was linear so that this does not seem to be the decisive factor in producing normality in the limit, contrary to a conjecture of Westcott (1973).

Example 4

When $\alpha = 0$, we can take

$$\sigma_t^2 = \beta\sigma^2 t^\gamma, \quad \text{with} \quad \sigma^2 = \frac{\lambda}{\gamma\delta} = \frac{1}{\gamma\omega}$$

$$F[\sigma_t x; s] = \left[\frac{\sigma^2 x^2 t^\gamma}{s^\gamma} \right]^{\frac{1}{\omega s}}, \text{ for } 0 \leqslant x^2 \leqslant \frac{s^\gamma}{\sigma^2 t^\gamma}$$

$$= \left[\frac{\phi_t(x)}{s} \right]^{\frac{\gamma}{\omega s}}, \text{ as before, and 0 elsewhere.}$$

Again, $\int_0^t \overline{F}(\sigma_t x; s)\, ds \sim \frac{\gamma}{\omega}\int_{\phi_t(x)}^t \frac{1}{s}\log\left(\frac{s}{\phi_t(x)}\right) ds$

$$= \frac{\sigma^2}{2}\log^2[\sigma^2x^2] \quad , \quad x \leqslant \frac{1}{\sigma}$$

and vanishes outside $[0,\frac{1}{\sigma}]$ while

$$2\int_x^\infty y\int_0^t \overline{F}(\sigma_t y, s)\, ds\, dy$$

$$\to \sigma^2\int_x^{\frac{1}{\sigma}} y\log^2\sigma^2y^2\, dy$$

$$= 1 - \frac{\sigma^2x^2}{2}[\log^2\sigma^2x^2 - 2\log\sigma^2x^2 + 2] \ , \ x \leqslant \frac{1}{\sigma}$$

and is zero otherwise, so that we have as canonical measure

$$\begin{aligned} M(dx) &= 2\sigma^2 x\left|\log\sigma^2x^2\right| , \quad x \leqslant \frac{1}{\sigma} \\ &= 0 \qquad\qquad , \quad x \geqslant \frac{1}{\sigma} \end{aligned} \qquad \ldots \quad (5)$$

just as in Example 3 for $\gamma > 1$.

As in Example 1, we can easily calculate the cumulants using (3) – (5):

$$\gamma_1 = 0$$

$$\gamma_r = \frac{4}{(\sigma^2)^{r/2-1}r^2} \quad , \quad r \geqslant 2$$

with the exceptional case, $\gamma = 1$, $\alpha > 0$ when $\gamma_1 = 0$, $\gamma_2 = 1$,

$\gamma_r = [1 - \frac{\kappa}{\sigma^2}]\frac{4}{(\sigma^2)^{r/2-1}r^2}$.

§6 · Convergence to Normal

Because of the constant use of the normal distribution in statistical analyses, there is particular interest in identifying those processes H, for which $Z(t)$ can be expected to become approximately normal for large t.

As we noted in Example 5.2 the normal distribution has the canonical measure

$$M[-x,x] = 1 \qquad \forall x > 0$$

$$M^{+}(x) = M^{-}(-x) = 0$$

The necessary and sufficient conditions with $\sigma_t^2 = \text{Var } Y(t)$, ie. (4.5) and (4.3), become

$$\frac{2}{\sigma_t^2} \int_{x\sigma_t}^{\infty} y F_t(y)\, dy \to 0 \qquad \ldots \quad \ldots \quad (1)$$

and $$F_t(\sigma_t x) \to 0 \qquad \ldots \quad \ldots \quad (2)$$

where $$F_t(y) = \int_0^t \bar{F}_H([-y,y]; t,s)\lambda(s)ds.$$

If (1) holds for $x_o > 0$ we can truncate the LS at $x > x_o$. We have

$$\frac{2}{\sigma_t^2} \int_{x_o\sigma_t}^{\infty} y F_t(y)\, dy \geq 2 \int_{x_o}^{x} y F_t(\sigma_t y)\, dy$$

$$\geq (x^2 - x_o^2) F_t(\sigma_t x)$$

implying that (2) holds, $\forall\, x > x_o$ and hence for $\forall x > 0$. We therefore have the following theorem analogous to the Lindeberg condition for sums of independent random variables , Feller (1971, P.262).

Theorem 1

If $\int_0^t \mathbb{E}H^2(t,s)\lambda(s)\,ds < \infty$, $\forall t$, and increases without bound as $t \to \infty$

and if we choose $\sigma_t^2 \sim \int_0^t \mathbb{E}H^2(t,s)\lambda(s)\,ds$, then

$$\frac{1}{\sigma_t^2} \int_{x\sigma_t}^{\infty} y \int_0^t \bar{F}([-y,y];t,s)\lambda(s)\,ds\,dy \to 0, \quad \forall x > 0 \qquad \ldots \qquad \ldots \quad (3)$$

is necessary and sufficient for the complete convergence of $Z(t)$ to a $N(0,1)$ random variable.

Note: If the primary Poisson process is stationary with rate λ, we have

$$\sigma_t^2 = \lambda \int_0^t \mathbb{E}H^2(t,s)\,ds = \lambda h_t, \text{ say,}$$

and criterion (3) reduces to

$$\frac{1}{h_t} \int_{x(\lambda h_t)^{\frac{1}{2}}}^{\infty} y \int_0^t \bar{F}([-y,y];t,s)\,ds\,dy \to 0, \quad \forall x > 0 \qquad \ldots \qquad \ldots \quad (4)$$

Now, instead of considering large t, it is equally legitimate to let $\lambda \to \infty$, in which case (4) is automatically satisfied.

This generalizes the central limit theorems for large λ, which customarily appear in the literature, e.g. Parzen (1962, P.157). It is arguable that this assumption of large λ is often inadequate. As Rice (1977) has shown, shot noise processes converge weakly to Gaussian processes, for high rates λ, under quite weak conditions. Often, however, it is precisely because the realisations of the process do not look Gaussian that a shot-noise model is chosen. For instance, the river-flow data

considered by Weiss (1974) was not time-reversible, and therefore not Gaussian, since it had sharp increases and slow declines.

Theorem 1 is, then, much more flexible than the usual central limit theorems for shot-noise processes.

§7 Sufficient Conditions for Asymptotic Normality

We restrict ourselves to the situation where $\lambda(t)$ is bounded while $\Lambda(t) = \int_0^t \lambda(s)\, ds$ is unbounded. This includes most cases of practical interest including step-functions with known change points, bounded periodic functions of known period and asymptotically constant functions.

Corollary 1

If (a) the random variable $H_M = \max_{t,s} |H(t,s)|$ has a finite 2nd moment, (b) $\liminf \frac{\sigma_t^2}{\Lambda(t)} > 0$, then $Y(t)$ is asymptotically normal.

Proof

$\forall t \geq s \geq 0$, $\bar{F}([-y,y];t,s) \leq \bar{F}_M(y)$, where F_M is the distribution function of H_M and so

$$F_t[y] \leq \Lambda(t)\, \bar{F}_M(y)$$

Therefore

$$\frac{1}{\sigma_t^2} \int_{x\sigma_t}^{\infty} y F_t[y]\, dy$$

$$\leq \frac{\Lambda(t)}{\sigma_t^2} \int_{x\sigma_t}^{\infty} y \bar{F}_M(y)\, dy .$$

The first term is bounded by (b) while the second tends to zero by (a), thus satisfying (6.3).

Note: If in fact $\lambda(t)$ were to be unbounded we might nevertheless have condition (b) holding if σ_t^2, in a sense, "automatically" kept pace with the increasing rate of primary events. The proof would then remain valid.

Corollary 2

Suppose H is of the form $H(t-s)$ and possesses a limiting r.v. $H_\infty = p \lim_{s\to\infty} H(s)$, not identically zero. Then condition (a) of Corollary 1 suffices for a normal limit.

Proof

We know, from Loève (1963, P. 157,164), that $\mathbb{E}H^2(s) \to \mathbb{E}H^2_\infty$, if (a) holds, while

$$\sigma_t^2 \sim \int_o^t \mathbb{E}H^2(t-s)\lambda(s)\,ds = \int_o^t \mathbb{E}H^2(s)\lambda(t-s)\,ds$$

We can choose t_o s.t.

$$\frac{1}{\Lambda(t)} \left| \int_o^t (\mathbb{E}H^2(s) - \mathbb{E}H^2_\infty)\,\lambda(t-s)\,ds \right|$$

$$\leq \frac{2\mathbb{E}H_M^2}{\Lambda(t)} \int_o^{t_o} \lambda(t-s)\,ds + \frac{\Lambda(t-t_o)}{\Lambda(t)} \cdot \varepsilon$$

The first term can be made as small as we please since $\lambda(s)$ is bounded while the second is less than ε.

Therefore $\dfrac{\sigma_t^2}{\Lambda(t)} \to \mathbb{E}H^2_\infty > 0$ and condition (b) of Corollary 1 is also valid.

Note: Again bounded $\lambda(t)$ is not strictly necessary. The proof remains valid for $\lambda(t) \sim t^\gamma$, $\gamma > 0$; $\lambda(t) \sim \log t$; but not $\lambda(t) \sim e^t$.

Example 1

Consider a Poisson cluster point process with cluster size K and unspecified cluster structure. Clearly $H(s)$ is just the number of points, in a cluster starting at 0, occurring in $(0,s]$. $H_\infty = H_M = K$, while $Y(t)$ is the total number of points in $[0,t]$.

If $\mathbb{E}K^2 < \infty$, Corollary 2 applies with $\sigma_t^2 \sim \Lambda(t)\mathbb{E}K^2$.

Example 2

The extension to the situation of Example 1.1 is straightforward. As a simple, but plausible example, suppose $K(X(s))$ has a multiplicative form $X(s)R$ where $X(s)$ are the observed variables and R is a random variable which can be considered as standardised cluster size. Suppose $X(s) \in [x_o, x_1]$, an interval of $(0,\infty)$, while $0 < \mathbb{E}R^2 < \infty$.

Then $H(t,s) \uparrow X(s)R$, as $t \to \infty$, while $H_M \leq x_1 R$. By the argument of Corollary 2

$$\sigma_t^2 \sim \int_0^t \mathbb{E}H^2(t,s)\lambda(s)\,ds \sim \mathbb{E}R^2 \int_o^t X^2(s)\lambda(s)\,ds \qquad \ldots \quad \ldots \quad (1)$$

$$\geq x_o^2 \mathbb{E}R^2 \Lambda(t)$$

Conditions (a) and (b) of Corollary 1 are satisfied provided we take σ_t^2 from (1).

Note: When $\lambda(s) = \lambda$ in Example 1, i.e. the primary Poisson process is homogeneous, we have Theorem 2 of Westcott (1973). When $\lambda(s) \to \lambda$, we have result (C) of Westcott (1977).

Example 3

The process of §5 had

$$H(s) = +\sqrt{(\alpha + \beta s^\gamma e^{-\delta X s})} \quad \text{and} \quad H^2_\infty = \alpha > 0, \text{ a.s.}$$

The maxima of these curves occur at $s = \frac{\gamma}{\delta X}$ with $H_M^2 = \alpha + \beta e^{-\gamma}\left(\frac{\gamma}{\delta X}\right)^\gamma$,

so that $EH_M^2 < \infty$ iff $EX^{-\gamma} < \infty$. Choosing X exponentially distributed, we require $\gamma < 1$ to prove asymptotic normality using Corollary 2 and this is borne out by the direct analysis of Examples 5.2, 5.3. On the other hand, if X has a Gamma distribution with density $\frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}$, Corollary 2 applies when $\gamma < k$. It is interesting that there is a family of examples for which these fairly crude criteria separate out those with a normal limit from the rest.

The case $\gamma = 1$, X ~ exponential, of Example 5.3 illustrates the fact that linearity of Var Y(t) does not suffice to produce a normal limit. The exponentially distributed X produces too many big peaks (corresponding to small X values). The result is a highly skewed non-normal limit.

Example 4

Shot-noise processes are most commonly used to model stationary phenomena. Westcott (1976)shows that (for instance) if G(s) is non-negative and almost surely bounded, while the primary Poisson process is stationary, then $Y(t) = \sum_{i=-\infty}^{N(t)} G(t-\tau_i)$ exists if and only if

$$\int_0^\infty EG(s)\,ds < \infty \qquad \ldots \quad \ldots \quad \ldots \quad (2)$$

This still gives a sufficient condition if we drop a.s. boundedness.

Now $\int_0^t (Y(u) - Y(0))\,du$ is clearly also a shot-noise process, namely,

$$\int_0^t\int_s^t G(u-s)\,du\,dN(s) + \int_{-\infty}^0\int_0^t G(u-s)\,du\,dN(s) \qquad \ldots \quad \ldots \quad (3)$$

As a first step we investigate here the first term, which is the integral of a transient shot-noise. Note that we can write it as $\int_0^t H(t-s)\,dN(s)$ where $H(t-s) = \int_0^{t-s} G(u)\,du$. Its mean is then $\lambda\int_0^t\int_0^s EG(u)\,du\,ds$ and variance $\lambda\int_0^t E\left[\int_0^s G(u)\,du\right]^2 ds$ by Campbell's Theorem, (2.1).

Now $\int_0^s G(u)\,du$ increases to a maximum limit H_∞ and we strengthen (2) by insisting that $EH_\infty^2 < \infty$. Then Corollary 2 gives us a normal limit taking $\sigma_t^2 = \lambda t EH_\infty^2$.

Corollaries 1 and 2 deal with a fairly restricted situation. For even if no limiting r.v. H_∞ exists, σ_t^2 must behave roughly like $\Lambda(t)$ for large t, since, for some $k > 0$,

$$k \leq \frac{\sigma_t^2}{\Lambda(t)} \leq EH_M^2$$

However, when $EH^2(s) \to 0$, $\frac{\text{Var } Y(t)}{\Lambda(t)}$ also vanishes and the methods of Corollaries 1 and 2 appear to fail. Fortunately, we can make stronger demands on H_M to get

Corollary 3

If (a) $EH_M^{2\alpha} < \infty$, $\alpha > 1$ and (b) $\liminf \frac{\text{Var } Y(t)}{\Lambda(t)^{\frac{1}{\alpha}}} > 0$, then Z(t) is asymptotically normal.

Proof

$$\frac{\Lambda(t)}{\sigma_t^2}\int_{y\sigma_t}^{\infty} x\bar{F}_M(x)\,dx \leq \frac{\Lambda(t)}{y^{2(\alpha-1)}\sigma_t^{2\alpha}}\int_{y\sigma_t}^{\infty} x^{2\alpha-1}\bar{F}_M(x)\,dx \to 0$$

under the above conditions. The argument in the proof of Corollary 1 now yields the result.

Example 5

Suppose $H(s) = \frac{X}{(s+1)^{\gamma/2}}$, $0 < \gamma < 1$ with the primary process stationary and $EX^{\frac{2}{1-\gamma}} < \infty$. Then $\sigma_t^2 \sim \frac{t^{1-\gamma}}{1-\gamma}$ and $\frac{\sigma_t^2}{t^{1-\gamma}}$ is bounded away from zero. Taking $\alpha = \frac{1}{1-\gamma}$ in Corollary 3 we see that the limit will be normal.

Note: It is interesting that in Example 5.4 where $\sigma_t^2 \sim t^\gamma$, $\mathbb{E}H_M^{2/\gamma} < \infty \iff \mathbb{E}X^{-1} < \infty$ as we saw in Example 3. For X exponentially distributed this just fails to be the case.

The interplay between the strength of the bound H_M and the rate of growth of σ_t^2 continues in the next Corollary.

Corollary 4

If $|H(t,s)|$ is bounded a.s. in each interval $[0,t]$, say by M_t, then $M_t/\sigma_t \to 0$ implies asymptotic normality of $Z(t)$.

Proof

$$\forall x > M_t/\sigma_t, \quad F_t[\sigma_t x] = 0$$

Since $\int_x^{M_t/\sigma_t} y\, F_t[\sigma_t y]\, dy$ becomes identically zero for large t, Theorem 6.1 applies.

Next we derive a criterion which parallels Liapunov's Condition in the sums of independent random variables situation , Loève (1963, P.275).

Corollary 5

$$\int_0^t \mathbb{E}|H(t,s)|^{2+\delta}\lambda(s)\, ds = o(\sigma_t^{2+\delta})$$

$$\Rightarrow Z(t) \text{ asymptotically normal.}$$

Proof

Applying Tchebyshev's Inequality for $(2+\delta)^{th}$ moments we have

$$\frac{1}{\sigma_t^2} \int_{x\sigma_t}^\infty y\, F_t[y]\, dy$$

$$\leq \frac{1}{\sigma_t^2} \int_0^t \mathbb{E}|H(t,s)|^{2+\delta}\lambda(s)\, ds \int_{x\sigma_t}^\infty \frac{1}{y^{1+\delta}}\, dy$$

$$= \frac{1}{\delta x^\delta} \cdot \frac{\int_0^t \mathbb{E}|H(t,s)|^{2+\delta}\lambda(s)\, ds}{\sigma_t^{2+\delta}} \to 0, \quad t \to \infty$$

We use this first of all to consider cases where H is unbounded.

Corollary 6

Consider the sequence of bounding r.v.'s $H_{M_t} = \max_{s\in[0,t]} |H(t,s)|$

Then $\dfrac{\Lambda(t)\mathbb{E}H_{M_t}^{2+\delta}}{\sigma_t^{2+\delta}} \to 0 \Rightarrow Z(t)$ asymptotically normal.

Proof

This is immediate from Corollary 5.

Example 6

Suppose $H(s) \sim Xh(s)$, where X has distribution function F_X with finite variance and $h(s)$ is a deterministic function, ultimately increasing.

$$\frac{\int_0^t \mathbb{E}|H(t,s)|^{2+\delta}\lambda(s)\,ds}{\sigma_t^{2+\delta}}$$

$$\leq \frac{\mathbb{E}|X|^{\delta} h(t)^{\delta}\ \sigma_t^{2}}{\sigma_t^{2+\delta}}, \quad \text{for large enough } t.$$

So if $h(t)/\sigma_t \to 0$, we can apply Corollary 5. This would be true for t^{ν} or $\log t$, for instance, but not for e^t. In the latter case

$$\int_0^t \overline{F}(\sigma_t x, s)\,ds = \int_0^t \overline{F}_X(xe^{t-s}/\sqrt{2})\,ds$$

$$= \int_0^t \overline{F}_X(xe^{s}/\sqrt{2})\,ds \neq 0$$

The bounding conditions of Corollaries 1 and 2 can be replaced by others serving the same purpose.

Corollary 7

(a) Suppose $H(s) \xrightarrow{p} H_\infty$, a limiting r.v. not concentrated at 0.

Then $\mathbb{E}|H(s)|^{2+\delta}$ bounded is sufficient for asymptotic normality.

(b) If $\dfrac{\sigma_t^2}{\Lambda(t)^{1/\alpha}} \to \infty$, $\alpha > 1$, $\mathbb{E}|H(s)|^{2\alpha}$ bounded is sufficient.

Proof

(a) The bounded moments condition ensures that $\mathbb{E}H^2(s) \to \mathbb{E}H^2_\infty$, Loève (1963, P.164). So $\sigma_t^2/\Lambda(t) \to \mathbb{E}H^2_\infty$, also. But, from Corollary 5, we need only that $\Lambda(t)/\sigma_t^{2+\delta} \to 0$

(b) Using Tchebyshev's Inequality for $2\alpha^{th}$ moments we see that

$$\frac{1}{\sigma_t^2}\int_{\sigma_t x}^{\infty} y \int_0^t \bar{F}\{[-y,y];s\}\,\lambda(t-s)\,ds\,dy$$

$$\leq \frac{1}{\sigma_t^2}\int_0^t \mathbb{E}|H(s)|^{2\alpha}\lambda(t-s)\,ds \int_{\sigma_t x}^{\infty} \frac{1}{y^{2\alpha-1}}\,dy$$

$$= O\left(\frac{\Lambda(t)}{\sigma_t^{2\alpha}}\right) \to 0$$

It has already been remarked that $\mathbb{E}H^2(s) \to \mathbb{E}H^2_\infty$, which implies the asymptotic linearity of Var $Y(t)$, is not sufficient for a normal limit. If however $H(s) \uparrow H_\infty$, a.s. Corollary 2 applies. Our final criterion gives another such situation (c.f. Examples 5.2, 5.3).

Corollary 8

If $H(s)$ is non-negative, $H(s) \xrightarrow{p} H_\infty$, not concentrated at 0, and $H(s) \geq H_\infty$ a.s., then $\mathbb{E}H^2(s) \to \mathbb{E}H^2_\infty < \infty \Rightarrow Z(t)$ asymptotically normal.

Proof

$$\sigma_t^2 = \Lambda(t)\mathbb{E}H^2_\infty \quad \text{and so}$$

$$\frac{\Lambda(t)}{\sigma_t^2}\int_{y\sigma_t}^{\infty} y\overline{F}(y;\infty)\,dx \to 0 \qquad \ldots \quad \ldots \quad \ldots \quad (4)$$

since $\mathbb{E}H^2_\infty < \infty$.

Moreover, $\overline{F}(y,s) - \overline{F}(y,\infty)$ is always non-negative so

$$\left|\frac{1}{\Lambda(t)}\int_0^t\int_{x\cdot\Lambda(t)^{\frac{1}{2}}}^{\infty} y\,\{\overline{F}(y,s) - \overline{F}(y,\infty)\}\,dy\,\lambda(t-s)\,ds\right|$$

$$\leq \frac{1}{\Lambda(t)}\int_0^t\left|\int_0^{\infty} y\,\{\overline{F}(y,s) - \overline{F}(y,\infty)\}dy\right|\lambda(t-s)\,ds$$

$$= \frac{1}{\Lambda(t)}\int_0^t |\mathbb{E}H^2(s) - \mathbb{E}H^2_\infty|\lambda(t-s)\,ds$$

$$\to 0\,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

(4) and (5) combine to prove the result.

§8 Berry-Esseen Theory for Shot-Noise

The usefulness of the normal approximation is considerably strengthened if we have some idea of the size of the error as observations accumulate. In the sums of independent r.v.'s case, one important result is the Berry-Esseen Theorem which asserts that the error is uniformly $O(n^{-\frac{1}{2}})$. A related result can be proved for the Poisson shot-noise. In the special case when $H(t,s)$ is a deterministic function, our result may be deduced from a theorem of Papoulis (1971). However, even here, we have an improved bound.

In what follows $Z(t) = \frac{Y(t) - EY(t)}{\sqrt{\text{Var } Y(t)}}$ with distribution function $F_{Z(t)}(x)$ and characteristic function $\mathbb{C}_{Z(t)}(\theta)$. $\Phi(x)$ denotes the standard normal distribution function and $\rho_t = \int_0^t \mathbb{E}|H(t,s)|^3 \lambda(s)ds$

We base our result on the following theorem from Feller (1971, pp.538, 543).

Lemma 1

$$\Delta = \pi|F_{Z(t)}(x) - \Phi(x)|$$

$$\leq \int_{-T}^{T} |\mathbb{C}_{Z(t)}(\theta) - e^{-\frac{1}{2}\theta^2}| \frac{d\theta}{|\theta|} + \frac{24}{T\sqrt{2\pi}}$$

for all x and $T > 0$.

Theorem 2

If $\rho_t < \infty$ then

$$|F_{Z(t)}(x) - \Phi(x)| \leq \frac{C\rho_t}{\sigma_t^3}$$

with the constant $C < 3.292$.

Proof

We use the inequality

$$|e^{\alpha} - e^{-\beta}| = e^{-\beta}|e^{\alpha+\beta} - 1|$$

$$\leq |\alpha + \beta| e^{|\alpha+\beta|-\beta}, \quad \beta \text{ real}$$

with $\alpha = \log \mathbb{C}_{Z(t)}(\theta)$

$$= \int_0^t [\mathbb{C}_H(\theta\sigma_t^{-1}; t,s) - 1 - i\theta\sigma_t^{-1}\mathbb{E}H(t,s)]\lambda(s)ds$$

and $\beta = \frac{1}{2}\theta^2$.

Recalling that $\sigma_t^2 = \int_0^t \mathbb{E}H^2(t,s)\lambda(s)\,ds$, we see that

$$|\alpha + \beta|$$

$$= \left|\int_0^t \{Ee^{i\theta\sigma_t^{-1}H(t,s)} - 1 - i\theta\sigma_t^{-1}EH(t,s) + \tfrac{1}{2}\theta^2\sigma_t^{-2}EH^2(t,s)\}\lambda(s)\,ds\right|$$

$$\leqslant \frac{1}{6}\frac{|\theta|^3}{\sigma_t^3}\int_0^t E|H(t,s)|^3\lambda(s)\,ds = \frac{1}{6}|\theta|^3\frac{\rho_t}{\sigma_t^3}$$

Take $T = \sigma_t^3/\rho_t$ in Lemma 1

$$\Delta \leqslant \frac{1}{6T}\int_{-T}^{T}\theta^2\exp\left[-\tfrac{1}{2}\theta^2 + \frac{1}{6T}|\theta|^3\right]d\theta + \frac{24}{T\sqrt{2\pi}}$$

$$\leqslant \frac{1}{6T}\int_{-T}^{T}\theta^2 e^{-1/3\theta^2}\,d\theta + \frac{24}{T\sqrt{2\pi}}$$

$$\leqslant \frac{1}{T}\left[\frac{1}{6}\left(\frac{3}{2}\right)^{3/2}\sqrt{2\pi} + \frac{24}{\sqrt{2\pi}}\right]$$

The constant C is just that on the right divided by π and so the proof is complete.

As it stands Theorem 2 has a severe drawback in that it assumes that we have centred Y(t) exactly by its mean and normed it exactly by the square root of its variance. In practice these may be unnecessarily complicated and we would like to replace them by asymptotic equivalents.

Suppose then that $\sigma_t^2 = \text{Var } Y(t)$ while $\tilde{\sigma}_t^2 \sim \sigma_t^2$, $t \to \infty$.

Similarly $\mu_t = EY(t)$ while $\dfrac{\tilde{\mu}_t - \mu_t}{\tilde{\sigma}_t} \to 0$.

Let $\tilde{Z}(t) = \dfrac{\sigma_t}{\tilde{\sigma}_t}Z(t)$. Then

$$F_{\tilde{Z}(t)}(x) = F_{Z(t)}\left(\frac{\tilde{\sigma}_t}{\sigma_t}x\right) \text{ and}$$

$$|F_{\tilde{Z}(t)}(x) - \Phi(x)| \leqslant \left|\Phi\left(\frac{\tilde{\sigma}_t}{\sigma_t}x\right) - \Phi(x)\right| + C\frac{\rho_t}{\sigma_t^3}$$

Without loss of generality take $x > 0$. The 1st term is

$$\frac{1}{\sqrt{2\pi}} \left| \int_{x}^{\tilde{\sigma}_t/\sigma_t x} e^{-\frac{1}{2}z^2} dz \right|$$

$$= \frac{1}{\sqrt{2\pi}} \left| \int_{1}^{\tilde{\sigma}_t/\sigma_t x} e^{-\frac{1}{2}x^2z^2} dz \right|$$

$$\leq \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \left| \int_{1}^{\tilde{\sigma}_t/\sigma_t} \frac{1}{z} dz \right| = \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \left| \log \frac{\tilde{\sigma}_t}{\sigma_t} \right|$$

$$\left.\begin{aligned} &\leq \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \left(\frac{\tilde{\sigma}_t}{\sigma_t} - 1 \right), && \text{if } \tilde{\sigma}_t \geq \sigma_t \\ \text{and} \quad &\leq \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \left(\frac{\sigma_t}{\tilde{\sigma}_t} - 1 \right), && \text{if } \sigma_t \geq \tilde{\sigma}_t \end{aligned}\right\}$$

Furthermore, if $\hat{Z}(t) = \tilde{Z}(t) + \dfrac{\mu_t - \tilde{\mu}_t}{\tilde{\sigma}_t}$

then $\left| F_{\hat{Z}(t)}(x) - \Phi(x) \right|$

$$\leq \frac{C\rho_t}{\sigma_t^3} + \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \left| \log \frac{\tilde{\sigma}_t}{\sigma_t} \right| + \left| \Phi\left[x - \frac{(\mu_t - \tilde{\mu}_t)}{\tilde{\sigma}_t}\right] - \Phi(x) \right|$$

$$\leq \frac{C\rho_t}{\sigma_t^3} + \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}} \left| \log \frac{\tilde{\sigma}_t}{\sigma_t} \right| + \frac{1}{\sqrt{2\pi}} \left| \frac{\mu_t - \tilde{\mu}_t}{\tilde{\sigma}_t} \right|$$

Unfortunately, while these inequalities give some advance on Theorem 2, they are still unsatisfactory. For example, consider the general

Poisson cluster process. Using equation (9) of Westcott (1973), it is easily shown that

$$\mathbb{E}Y(t) = \lambda t \mathbb{E}S - \lambda \sum_{j=1}^{\infty} j R_j \int_0^t \bar{F}_j(u)\, du$$

$$\mathrm{Var}\, Y(t) = \lambda t \mathbb{E}S^2 - \lambda \sum_{j=1}^{\infty} (2j-1) R_j \int_0^t \bar{F}_j(u)\, du$$

$$\text{while} \qquad \rho_t = \lambda t \mathbb{E}S^3 - \lambda \sum_{j=1}^{\infty} (3j^2 - 3j + 1) R_j \int_0^t \bar{F}_j(u)\, du$$

where R_j and $F_j(u)$ are as defined in that paper.
So while we can deduce, for instance, that

$$\frac{t^{\frac{1}{2}} \rho_t}{\sigma_t^{\,3}} \sim \frac{\mathbb{E}S^3}{(\mathbb{E}S^2)^{3/2}} ,$$

any upper bound will depend on the fine structure of the clusters i.e. on R_j, $F_j(u)$.

§9 An Equilibrium Shot-noise Process

The integrated shot-noise process of Example 7.4 is a special case of the following. We assume for simplicity that our primary Poisson process is stationary with rate 1. $H(s)$ is a non-negative increasing process, $s \geq 0$ with limit H_∞ where $0 < \mathbb{E}H^2_\infty < \infty$. We want the distribution of

$$Y(t)^* = \int_0^t H(t-s)\, dN(s) + \int_{-\infty}^{0} [H(t-s) - H(-s)]\, dN(s)$$

$$= Y_1(t) + Y_2(t), \text{ say} \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

First we must ensure that $Y_2(t)$ is a.s finite. Consider $Y_2(t)$ for fixed t as the value of a stationary shot-noise process,

$$\int_{-\infty}^{u} [H(t+u-s) - H(u-s)]\, dN(s) , \text{ at } u = 0.$$

Since the integrand is non-negative, condition (27) of Westcott (1976)

gives a necessary and sufficient condition. In fact, the condition

$$\int_0^\infty \mathbb{E}[H(t+s) - H(s)]\, ds < \infty \qquad \ldots \qquad \ldots \qquad (2)$$

is more than adequate.

Now $\quad EY^*(t) = \int_0^t \mathbb{E}H(s)\, ds + \int_0^\infty \mathbb{E}[H(t+s) - H(s)]\, ds$

If $\quad \int_0^\infty \mathbb{E}(H_\infty - H(s))\, ds < \infty \qquad \ldots \qquad \ldots \qquad (3)$

we can decompose the second term into

$$\int_t^\infty \mathbb{E}(H_\infty - H(s))\, ds - \int_0^\infty \mathbb{E}(H_\infty - H(s))\, ds$$

leaving $\quad EY^*(t) = t\mathbb{E}H_\infty$

Note that (3) $\Rightarrow$ (2).

Var $Y^*(t)$ = Var $Y_1(t)$ + Var $Y_2(t)$ because of our inbuilt independence assumptions.

$$\text{Var } Y_1(t) = \int_0^t \mathbb{E}H^2(s)\, ds \sim t\mathbb{E}H^2_\infty \qquad \text{while}$$

$$\text{Var } Y_2(t) = \int_0^\infty \mathbb{E}[H(t+s) - H(s)]^2 ds$$

$$= \int_t^\infty \mathbb{E}[H(s) - H_\infty]^2 ds + \int_0^\infty \mathbb{E}[H(s) - H_\infty]^2 ds$$

$$- 2\int_0^\infty \mathbb{E}[H_\infty - H(t+s)][H_\infty - H(s)]\, ds$$

provided $\quad \int_0^\infty \mathbb{E}[H_\infty - H(s)]^2 ds < \infty \qquad \ldots \qquad \ldots \qquad (4)$

The first term tends to zero while the second is constant. For each s, as $t \to \infty$, $[H_\infty - H(t+s)][H_\infty - H(s)] \to 0$, a.s, and it is smaller than

$[H_\infty - H(s)]^2$, for $t > 0$. By (4) and the dominated convergence theorem the 3rd term also vanishes as $t \to \infty$.

Var $Y^*(t)$, therefore, behaves asymptotically as a constant + $t\mathbb{E}H^2_\infty$.

Finally, put $Z^*(t) = \dfrac{Y^*(t) - t\mathbb{E}H_\infty}{t^{\frac{1}{2}}(\mathbb{E}H^2_\infty)^{\frac{1}{2}}}$

$$= \frac{Y_1(t) - \mathbb{E}Y_1(t)}{(t\mathbb{E}H^2_\infty)^{\frac{1}{2}}} + \frac{Y_2(t) - \mathbb{E}Y_2(t)}{(t\mathbb{E}H^2_\infty)^{\frac{1}{2}}}$$

The first expression has a N(0,1) limit by Corollary 7.2 while the second tends in probability to zero since Var $Y_2(t) = o(t)$ as $t \to \infty$.

Theorem 1

Under conditions (3) and (4)

$$Z^*(t) = \frac{Y^*(t) - t\mathbb{E}H_\infty}{(t\mathbb{E}H^2_\infty)^{\frac{1}{2}}} \text{ is asymptotically } N(0,1).$$

Example 1

When the process is an integrated shot-noise then $H(s) = \int_0^s G(u)\,du$, for some process $G(u)$, and $H_\infty = \int_0^\infty G(u)\,du$.

Condition (3) becomes

$$\int_0^\infty \int_s^\infty \mathbb{E}G(u)\,du\,ds < \infty$$

$$\text{i.e.} \quad \int_0^\infty u\mathbb{E}G(u)\,du < \infty \quad \text{which, in the case when}$$

$\mathbb{E}G(u)$ is bounded, implies $\mathbb{E}H_\infty < \infty$. Condition (4) becomes

$$\int_0^\infty \mathbb{E}\left[\int_s^\infty G(u)\,du\right]^2 ds < \infty.$$

CHAPTER II

THE CONDITIONAL INTENSITY OF A POINT PROCESS

§1. Introduction

As soon as the independence assumptions of the last chapter are abandoned, a much more careful treatment of the shot-noise process becomes essential. The primary purpose of this chapter is to give detailed definitions and preliminary results which may then be taken for granted when we return to the central limit theorem in Chapter III.

The reader may find it helpful to consider first the much simpler case of a shot-noise process on the integers. We have a probability space, $(\Omega,\mathcal{H},P)$, carrying a binary r.v., N_j, and a real r.v., H_{ij}, for each pair of integers $(i,j) \in \mathbb{Z}^2$. The "history" of the process prior to r is the σ-algebra,

$$\mathcal{H}_r = \sigma\{N_j, H_{ij}N_j;\ \forall i < r, j < r\},$$

i.e. the σ-algebra generated by the "points" and the H-processes they "trigger off" prior to r.

Let us define

$$\hat{\lambda}_j = E\{N_j|\mathcal{H}_j\} = P\{N_j = 1|\mathcal{H}_j\}$$

and

$$E_j[H_{ij}|\mathcal{H}_j] = E[H_{ij}|\mathcal{H}_j, N_j = 1]$$

Then

$$E[H_{ij}N_j|\mathcal{H}_j] = E_j[H_{ij}|\mathcal{H}_j]\hat{\lambda}_j \ .$$

Now, considering the shot-noise process

$$Y_n = \sum_{j=1}^{n} H_{nj}N_j \ ,$$

we see that

$$\mathbb{E}Y_n = \mathbb{E}\sum_{j=1}^{n} \mathbb{E}_j[H_{nj}|\mathcal{H}_j]\hat{\lambda}_j \qquad \ldots \qquad (1.1)$$

On the other hand

$$\mathbb{E}\sum_{j=1}^{n} H_{nj}\hat{\lambda}_j = \mathbb{E}\sum_{j=1}^{n} \mathbb{E}[H_{nj}|\mathcal{H}_j]\hat{\lambda}_j \qquad \ldots \qquad (1.2)$$

The thread running through this chapter is the search for continuous time analogues of $\mathcal{H}_r$ in Section 2, of $\mathbb{E}_j[H_{ij}|\mathcal{H}_j]$ in Section 3, of $\hat{\lambda}_j$ in Section 4 and of (1.1) and (1.2) in Section 5.

At the same time, we take the opportunity to prove some results about the conditional intensity function of the point process, $\hat{\lambda}(s)$, which is the analogue of $\hat{\lambda}_j$. This was introduced by McFadden (1965) as (in effect) the right derivative

$$\lim_{h\to 0} \frac{1}{h} \Pr\{N(s,s+h] = 1|\mathcal{Y}_{s+}\} \qquad \ldots \qquad (1.3)$$

where $N(s,s+h]$ is the number of events in $(s,s+h]$ and $\mathcal{Y}_{s+}$ is the "history", of the point process only, up to and including s. McFadden pointed out that this could be seen as representing a point process as a generalised birth process. The related derivative

$$\lim_{h\to 0} \frac{1}{h} \Pr\{N(s,s+h] \geqslant 1|\mathcal{Y}_{s+}\} \qquad \ldots \qquad (1.4)$$

was suggested by Cox and Lewis (1972), under the name complete intensity function, as an aid to the statistical analysis of multivariate point processes. In neither case were the existence of $\hat{\lambda}$, or the rules for manipulating it, considered. A particular form for (1.3) was proposed by Hawkes (1971a,b) (1972), Hawkes and Oakes (1974) and Oakes (1975) in their definitions of self-exciting and

mutually-exciting point processes. Here $\hat{\lambda}(s)$ was taken to have the linear form of a shot-noise

$$\nu + \int_{-\infty}^{t} \gamma(t - u)dN(u)$$

where ν was a constant and γ a deterministic function, such as a mixture of negative exponentials. The use of (1.3) and (1.4) in modelling point processes was also discussed by Snyder (1972),(1975). The main concern there was the detection or filtering of a signal corrupted by "noise" in the sense that only point events are observable. In this approach (1.3) and (1.4) were both assumed equal to a given functional form.

The existence of $\hat{\lambda}$ has been studied extensively by Papangelou, initially in the case of a stationary point process (1972a,b). Consider the interval function $\mathbb{E}[N[u,v) \mid \mathcal{F}_u]$, where $\mathcal{F}_u$ is the σ-algebra generated by point process events strictly prior to u. Papangelou defines $\hat{\Lambda}$ as the integral of this set function, usually referred to as a Burkill integral, c.f. Burkill (1923), and shows that it exists a.s., and also as an L^1limit, if $\mathbb{E}N[u,v) < \infty$. The derivative, where this exists is $\hat{\lambda}$. Now Papangelou's principal concern was a characterisation of processes of lines in two dimensions, the Davidson problem. He, therefore, went on to consider, (1974b), point processes on very general spaces where the total ordering property of the real line is not available. Unfortunately, neither paper considers the case of a time-ordered, non-stationary point process.

The existence problem for a general space, without ordering, has been considered further by Kallenberg (1978a). The conditional

intensity measure is shown to exist as the solution of a certain integral equation, related to those in my Theorem 5.3, below. This measure is shown, in turn, to be equal to the integral of the interval function $P[N[u,v) = 1 | \mathcal{J}_u]$ and, provided $\mathbb{E}N[u,v) < \infty$, to be equal to that of $E[N[u,v) | \mathcal{J}_u]$, also. Note that I have given here a time-ordered version of Kallenberg's Theorem 4.1 so that it may be compared more easily with the other results.

Now, in the course of proving the existence of $\hat{\Lambda}$ in (1972a), Papangelou showed that, if $F_0(x)$ is the conditional distribution function of the interval following a point, given the past, and $\{t_i\}$ are the ordered points of a stationary process, then

$$\hat{\Lambda}(t_i, t_{i+1}] = \int_{(0, t_{i+1}-t_i]} \frac{dF_0(x)}{1 - F_0(x)} .$$

Furthermore, if F_0 is continuous, the transformation $\tau_i = \hat{\Lambda}[0, t_i]$ applied to the positive points, yields a Poisson process which has points $\{\tau_i\}$ on $(0,\infty)$ and which is independent of $\mathcal{J}_0$. Rudemo (1973) gives a short proof of this result, working directly with $F_0(x)$, and extends the result to non-stationary point processes with $\mathbb{E}N[u,v)$ possibly infinite. He therefore defines the conditional intensity, $\hat{\lambda}(t)$, at least implicitly, as

$$\frac{f_{t_i}(t - t_i)}{1 - F_{t_i}(t - t_i)}, \text{ if } t \in (t_i, t_{i+1}], \quad \ldots \qquad (1.5)$$

where $F_{t_i}(x)$ is the distribution function of an interval starting at t_i, conditional on the past, $\mathcal{J}_{t_i}$.

Unquestionably, the most sustained attack on this problem has utilised the theory of continuous time martingales, in particular the Doob-Meyer decomposition theorem. This is the approach taken by, for instance, Bremaud (1974), Jacod (1975) and Brown (1978), by Aalen (1976), whose interest is primarily in the properties of certain statistical procedures, and by a considerable body of researchers in control theory, e.g. Boel, Varaiya and Wong (1975), Segall and Kailath (1975a,b), Davis (1976) and Segall, Davis and Kailath (1975). There is also an extensive related literature in French. The key observation is that, for all $s < t$,

$$\mathbb{E}[N(t) \mid \mathcal{F}_{s+}] \geq \mathbb{E}[N(s) \mid \mathcal{F}_{s+}] = N(s), \qquad \ldots \qquad (1.6)$$

so that $N(t)$ is a positive, right-continuous submartingale adapted to the σ-algebras, $\{\mathcal{F}_{t+}\}$ for $t \geq 0$. By Meyer (1966), VII T3) there is a unique, "natural" increasing process $A(t)$, such that $N(t) - A(t)$ is a right-continuous martingale. The significance of "natural" will be explained in Section 5. This process, $A(t)$, can be shown to define the distribution of the point process, Jacod (1975), and to possess the main properties of a conditional intensity measure. Thus, Aalen (1976) shows that, when they exist, the derivatives (1.3) and (1.4) equal that of $A(t)$, as does the derivative of $\hat{\Lambda}$. Brown (1978) shows that A is the integral of (1.6) and that it is "L-calculable", a property closely related to the existence of the Burkill integral, $\hat{\Lambda}$, defined in Papangelou (1972a). Brown calls $A(t)$ the "compensator" while Segall and Kailath (1975a) call it the "integrated (random) rate".

The martingale approach has a number of attractions which stem from its foundations in an established body of theory. It can easily be extended to the case where $EN[u,v)$ may be infinite by the use of "stopping times", essentially a random truncation technique. Furthermore, it is also possible to define conditional intensity measures relative to larger σ-algebras than $\{\mathcal{H}_{t+}\}$, since the equivalent of (1.7) will still be valid. For instance, it may be very useful to consider the σ-algebras $\{\mathcal{H}_{t+}\}$ which contain the past events of both point and secondary processes. On the other hand, it involves the introduction of a considerable amount of terminology which does not appear relevant to the shot-noise problem. The basic definition of the conditional intensity measure as the integral of a set function seems conceptually simpler, perhaps because it is more concrete.

The approach adopted here is, therefore, essentially that of Papangelou and the main proofs in Sections 3 and 4 are inspired by his. The development has four aims: firstly, to prove the existence of $\hat{\Lambda}$ in the non-stationary case; secondly, to shed light on some uncertainties concerning the proof of almost sure convergence in Theorem 1 of Papangelou (1972a); thirdly, to show that $\hat{\Lambda}$ can be defined as the Burkill integral of any one of the three set functions, $P[N[u,v) = 1|\mathcal{H}_u]$, $P[N[u,v) \geq 1|\mathcal{H}_u]$ and $E[N[u,v)|\mathcal{H}_u]$; and fourthly, to relate this definition to the derivatives, (1.3) - (1.5), to Rudemo's definition, (1.6) and to the Doob-Meyer decomposition. Finally, the results of Section 5 are substantial generalisations of Theorems 3 and 4 of Papangelou (1972a).

§2. Definitions and Preliminary Results

The purpose of this section is to define the primary and secondary processes, which comprise the shot-noise process, and to construct the σ-algebras, $\mathcal{H}_t$, which describe its history. We denote our overall sample space by Ω. If $Q \subset \mathbb{R}$, then $\mathcal{B}(Q)$ is the σ-algebra of Borel sets on Q. We abbreviate $\mathcal{B}(\mathbb{R})$ to $\mathcal{B}$ and $\mathcal{B}(-\infty,s)$ to $\mathcal{B}_s$ for each $s \in \mathbb{R}$. Before proceeding, we outline some elementary properties of generated σ-algebras, for ease of reference.

Suppose $\mathcal{C}$ is a collection of Ω sets. In general $\sigma(\cdot)$ denotes the smallest σ-algebra containing the sets, or collections of sets, in the brackets; or, if real functions are mentioned inside the brackets, the smallest σ-algebra which makes them Borel measurable.

Suppose $\mathcal{C} \subset \mathcal{D}$, where $\mathcal{D}$ is a σ-algebra. Then

$$\sigma(\mathcal{C}) \subset \mathcal{D} \qquad \ldots \qquad \ldots \qquad (2.1)$$

Suppose $A \subset \Omega$. Then

$$\sigma(\mathcal{C}) \wedge A = \sigma(\mathcal{C} \wedge A) \qquad \ldots \qquad \ldots \qquad (2.2)$$

Suppose f is a map from E into Ω. Then,

$$f^{-1}[\sigma(\mathcal{C})] = \sigma[f^{-1}(\mathcal{C})] \qquad \ldots \qquad \ldots \qquad (2.3)$$

Suppose $\mathcal{C}_i$ is a collection of sets in Ω_i and $\bigvee_i \sigma(\mathcal{C}_i)$ is the σ-algebra, on $\bigcup_i \Omega_i$, $\sigma\{\sigma(\mathcal{C}_i), \forall_i\}$. Then,

$$\bigvee_i \sigma(\mathcal{C}_i) = \sigma\{\bigcup_i \mathcal{C}_i\} \qquad \ldots \qquad \ldots \qquad (2.4)$$

Consider, now, the primary point process. Let Ψ_N denote the collection of unbounded countable subsets of $\mathbb{R}$, which have no cluster points and let $\omega \to \psi(\omega)$ be a map from Ω on to Ψ_N. We

define the counting measure N on $\mathcal{B}(\mathbb{R})$, s.t. $N(Q;\omega)$ is the cardinality of $Q \cap \psi(\omega)$, for $Q \in \mathcal{B}$. For brevity, we write $N([0,t];\omega)$ as $N(t,\omega)$ or even $N(t)$, if the meaning is clear. We make N into a random measure by defining the generated σ-algebra on Ω

$$\mathcal{N} = \sigma\{N(Q);\ Q \in \mathcal{B}\} .$$

We also require σ-algebras which describe the past of the point process. Let

$$\mathcal{N}_t = \sigma\{N(Q);\ Q \in \mathcal{B}_t\}$$

Of course $\mathcal{N} = \bigvee_{t\in\mathbb{R}} \mathcal{N}_t$. Note that we have skirted the issue of defining a σ-algebra on Ψ_N and making $\omega \to \psi(\omega)$ measurable. This is discussed in Daley and Vere-Jones (1972) and Papangelou (1974a).

It will prove very useful to have the points of each $\psi(\omega)$ labelled in order from the origin. We define

$$\tau_j(\omega) = \begin{cases} \inf_{[0,\infty)} \{t : N([0,t];\omega) \geq j\}; & j=1,2,\ldots. \\ \sup_{(-\infty,0)} \{t : N([t,0);\omega) > |j|+1\}; & j=0,-1,-2,\ldots. \end{cases}$$

Note that the random variables, $\tau_j I_{\{\tau_j<t\}}$, are $\mathcal{N}_t$-measurable only for $t \geq 0$. In the case $s \leq t < 0$ when $j \leq 0$,

$$\begin{aligned} \{\tau_j < s\} &= \{N[s,0) < |j|+1\} \\ &= \bigcup_{j\leq r\leq 0} \{N[t,0) < |r|+1\} \cap \{N[s,t) \leq r-j\} \\ &= \bigcup_{j\leq r\leq 0} \{\tau_r < t\} \cap \{N[s,t) \leq r-j\} . \end{aligned}$$

This set is in $\sigma\{\mathcal{N}_t \cap \{\tau_r < t\},\ j \leq r \leq 0\}$ which is a σ-algebra on

$\{\tau_j < t\}$. Because of this, we define σ-algebras, $\mathcal{F}_t^j$ on $\{\tau_j < t\}$, as follows:-

if $j \geq 1$, $t > 0$: $\mathcal{F}_t^j = \mathcal{F}_t \wedge \{\tau_j < t\}$;

if $j \leq 0$, $t \geq 0$: $\mathcal{F}_t^j = \mathcal{F}_t$ since $\{\tau_j < t\} = \Omega$;

if $j \geq 1$, $t \leq 0$: $\mathcal{F}_t^j = \{\phi\}$ since $\{\tau_j < t\} = \phi$;

if $j \leq 0$, $t \leq 0$: $\mathcal{F}_t^j = \sigma\{\mathcal{F}_t \wedge \{\tau_r < t\},\ j \leq r \leq 0\}$.

With these definitions, the restriction of τ_j to $\{\tau_j < t\}$ is $\mathcal{F}_t^j$ measurable.

Lemma 2.1(a) The σ-algebras $\mathcal{F}$, $\mathcal{F}_t$, and $\mathcal{F}_t^j$, for $t \in \mathbb{R}$ and $j \in \mathbb{Z}$, are all countably generated.

(b) The family $\{\mathcal{F}_t\}$ is increasing and left-continuous.

Proof (a) Since the Borel sets, $\mathcal{B}$, are generated by the rational intervals, $\mathcal{F}$ is generated by the events, $\{N[u,v) \geq r\}$ for $u,v \in \mathbb{Q}$ and $r \in \mathbb{N}$. By considering only those rational intervals which generate $\mathcal{B}_t$, $\mathcal{F}_t$ is seen to be countably generated, as also is $\mathcal{F}_t^j$, from its definition.

(b) $\{\mathcal{F}_t\}$ is increasing by definition so that $\bigvee_{s<t} \mathcal{F}_s \subset \mathcal{F}_t$. The reverse conclusion follows, from (2.1), since $\{N[u,v \geq r\} = \bigcup_{\substack{s \in \mathbb{Q} \\ s \in (u,v)}} \{N[u,s) \geq r\}$, which is in $\bigvee_{s<t} \mathcal{F}_s$, if $v \leq t$.

For use in Section 3, we also prove:

Lemma 2.2 For each $u < s$,

$$\mathcal{F}_s \wedge \{N(u,s) = 0\} \wedge \{N\{u\} = 1\} = \mathcal{F}_u \wedge \{N(u,s) = 0\} \wedge \{N\{u\} = 1\}.$$

Proof Suppose $Q \in \mathcal{B}_s$ and $r \in \mathbb{N}$. Then

$$\{N(Q) = r\} \wedge \{N(u,s) = 0\} \wedge \{N\{u\} = 1\}$$

$$= \{N(Q\backslash[u,s)) = \varepsilon_r\} \wedge \{N(u,s) = 0\} \wedge \{N\{u\} = 1\}$$

where $\varepsilon_r = r-1$ if $u \in Q$

$= r$ otherwise

Either way, by 2.2 and 2.1, the left-hand σ-algebra is in the right-hand one. The reverse inclusion is immediate since $\mathcal{F}_u \subset \mathcal{F}_s$.

In Section 3, we will need notation for the points immediately prior to s and at, or immediately after, s. Let

$$t_0(s) = u \text{ if } u < s,\ N(u,s) = 0 \text{ and } N\{u\} = 0$$

while

$$t_1(s) = v \text{ if } v \geq s,\ N[s,v] = 1 \text{ and } N\{v\} = 1$$

Using this notation, Lemma 2.2 says that

$$\mathcal{F}_s \wedge \{t_0(s) = u\} = \mathcal{F}_u \wedge \{t_0(s) = u\}.$$

The second step in our construction is to introduce a two-parameter random function, $H(t,s)$. Let Ψ_H be the collection of real-valued functions on $\mathbb{R}^2$ which are right-continuous, with left limits in both parameters. In Chapter III, we make the restriction $H(t,s) = 0$, $\forall t < s$, and it is helpful to think of s as the "starting" point of H. This restriction is not necessary for the construction, though many of the methods below, which depend on an idea of time-ordering, would make little sense without it.

Let H be a map $\omega \to H(\cdot,\cdot;\omega)$ from Ω on to Ψ_H. Then $\omega \to H(t,s;\omega)$ is the composition of the $(s,t)^{th}$ coordinate projection and H. We make each of these measurable by defining an increasing family of generated σ-algebras,

$$\mathcal{G}_{t,s} = \sigma\{H(u,v);\ u < t,\ v < s\}$$

Let $\mathcal{G} = \bigvee_{\substack{t\in\mathbb{R}\\ s\in\mathbb{R}}} \mathcal{G}_{t,s}$. If, also, $\mathcal{H}=\mathcal{F}\vee\mathcal{G}$, $\mathcal{H}$ will contain all the events of both primary and secondary processes.

Lemma 2.3(a) The σ-algebras $\mathcal{G}_{t,s}$, $\mathcal{G}$ and $\mathcal{H}$ are all countably generated.

(b) For each $t \in \mathbb{R}$ and $u < t$, the random function $H(u,s)$, $s < t$, is progressively measurable w.r.t. $\{\mathcal{G}_{t,s}\}_{s<t}$ in the following sense. For each $u < t$, the map

$$H_u : (v,\omega) \to H(u,v;\omega)$$

is measurable from $((-\infty,s) \times \Omega, \mathcal{B}_s \otimes \mathcal{G}_{t,s})$ into $(\mathbb{R},\mathcal{B})$, for any $s \leqslant t$.

Proof (a) By the right-continuity of H(t,s)

$$\{H(u,v) \leqslant x\} = \bigcap_{r>x} \bigcup_{\delta>0} \bigcap_{y\in(u,u+\delta)} \bigcap_{z\in(v,v+\delta)} \{H(y,z) < r\}$$

where $r, \delta, y, z \in \mathbb{Q}$. The events $\{H(y,z) < r\}$, for appropriately chosen rational y, z, r form a countable generating set for each $\mathcal{G}_{t,s}$ and also for $\mathcal{G}$. $\mathcal{H}$ is countably generated since $\mathcal{F}$ and $\mathcal{G}$ are, by (2.4).

(b) This follows by a simple adaptation of T.47 of Meyer (1966, p. 70). Suppose that, for each $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ s.t. $\frac{k}{n} < s$, we define on $(-\infty,s) \times \Omega$,

$$H_n(v,\omega) = H(u, \tfrac{k}{n}; \omega), \text{ if } v \in \left[\tfrac{k-1}{n}, \tfrac{k}{n}\right] \text{ for some } k$$

$$= 0 \qquad , \text{otherwise}$$

Then $H_n(v,\omega)$ is $\mathcal{B}_s \otimes \mathcal{G}_{t,s}$-measurable and converges on $(-\infty,s) \times \Omega$ to $H(u,v;\omega)$, which must also be $\mathcal{B}_s \otimes \mathcal{G}_{t,s}$-measurable.

Now let us construct the σ-algebras, $\mathcal{H}_t$, which contain the "history", i.e. past events of the process, prior to t. Let $\mathcal{G}^j_{t,s} = \mathcal{G}_{t,s} \cap \{\tau_j < s\}$. Now, for each $j \in \mathbb{Z}$, $t \in \mathbb{R}$ and $u < t$, consider the restriction of the map

$$\omega \to H(u, \tau_j(\omega); \omega)$$

to the set $\{\tau_j < s\}$, for a given $s \leq t$. This is the composition of two measurable maps, namely,

$$T_j : \omega \to (\tau_j(\omega), \omega)$$

from $(\{\tau_j < s\}, \mathcal{J}^j_s)$ into $((-\infty,s) \times \{\tau_j < s\}, \mathcal{B}_s \otimes \mathcal{J}^j_s)$ and, using Lemma 2.3(b) (progressive measurability),

$$H_u : (v,\omega) \to H(u,v;\omega)$$

from $((-\infty,s) \times \{\tau_j < s\}, \mathcal{B}_s \otimes \mathcal{G}^j_{t,s})$ into $(\mathbb{R}, \mathcal{B})$.

If $Q \times A$ is a rectangle in $\mathcal{B}_s \otimes \mathcal{G}^j_{t,s}$, then $T_j^{-1}(Q \times A) = \{\tau_j \in Q\} \cap A$, which is in $\mathcal{J}^j_s \vee \mathcal{G}^j_{t,s}$, since $Q \subset (-\infty,s)$. Thus, from (2.3), if $B \in \mathcal{B}$ then

$$(H_u \circ T_j)^{-1}(B) \in \mathcal{J}^j_s \vee \mathcal{G}^j_{t,s} \qquad \ldots \qquad \ldots \qquad (2.5)$$

Now this remains true for all $s \leqslant t$, the restrictions to $\{\tau_j < s\}$ being trivial in the anomalous cases $\{j \leqslant 0, s \geqslant 0$ when $\{\tau_j < s\} = \Omega$, and $\{j \geqslant 1, s \leqslant 0\}$ when $\{\tau_j < s\} = \phi$.

Motivated by the foregoing argument, let us define

$$\mathcal{H}_{t,j} = \{A \in \mathcal{G}_t^j \vee \mathcal{G}_{t,t}^j : A \cap \{\tau_j < s\} \in \mathcal{G}_s^j \vee \mathcal{G}_{t,s}^j, \forall s \leqslant t\} .$$

When $j \leqslant 0$ and $t \geqslant 0$, the restrictions for $0 \leqslant s \leqslant t$ merely ensure that $A \in \bigcap_{0 \leqslant s \leqslant t} \mathcal{F}_t \vee \mathcal{G}_{t,s} = \mathcal{G}_0 \vee \mathcal{G}_{t,0}$, while, when $j \geqslant 1$, the restrictions for $s \leqslant 0$ are trivial. To extend $\mathcal{H}_{t,j}$ from $\{\tau_j < t\}$ to Ω, define $\overline{\mathcal{H}}_{t,j} = \sigma\{\mathcal{H}_{t,j}, \{\tau_j \geqslant t\}\}$. This construction ensures that $\{\tau_j \geqslant t\}$ is the only event of the point process at, or after, t which is included. Finally, let

$$\mathcal{H}_t = \bigvee_{j \in \mathbb{Z}} \overline{\mathcal{H}}_{t,j} .$$

We will show that $\mathcal{H}_t$ has indeed got all the properties appropriate to the "history" of the process.

Firstly, the "observable" r.v's occurring before t are $\mathcal{H}_t$-measurable. We have already shown that $H(u,\tau_j) I_{\{\tau_j < t\}}$ is $\mathcal{H}_t$-measurable for all $j \in \mathbb{Z}$ and $u < t$. Moreover, for all $u < v \leqslant t$ and $r \in \mathbb{N}$,

$$\{N[u,v) \geqslant r\} = \bigcup_{n \in \mathbb{Z}} \bigcap_{j=n+1}^{n+r} \{\tau_j \in [u,v)\} \qquad \ldots \qquad (2.6)$$

which is in $\mathcal{H}_t$, since $\{\tau_j \in [u,v)\} \in \mathcal{H}_{t,j}$. Therefore, the point process events prior to t are also $\mathcal{H}_t$-measurable.

On the other hand, the "non-observable" events are not $\mathcal{H}_t$-measurable. Thus,

$$\{H(u,v) \leqslant x\} \cap \{\tau_j < s\} \in \mathcal{G}_s^j \vee \mathcal{G}_{t,s}^j$$

if and only if $v < s$, by the definition of $\mathcal{G}_{t,s}$. Therefore, $H(u,v)$ and, *a fortiori*, $H(u,v)I_{\{N\{s\}=0\}}$ cannot be $\mathcal{H}_t$-measurable for any t. Furthermore, the only sets in $\overline{\mathcal{H}}_{t,j}$ not in $\{\tau_j < t\}$, have the form $\{\tau_j \geq t\} \cap A$ for $A \in \mathcal{H}_{t,j}$. This excludes $\{\tau_j \geq v\}$, $\forall v > t$ and $H(u,\tau_j)I_{\{\tau_j \geq t\}}$ from $\mathcal{H}_t$-measurability.

The main properties of $\mathcal{H}_t$ are detailed in the following lemma.

Lemma 2.4(a) $\mathcal{G}_t^j \subset \mathcal{H}_t$, $\forall t \in \mathbb{R}$

(b) $\mathcal{H}_t$ is countably generated, $\forall t \in \mathbb{R}$

(c) $\{\mathcal{H}_t\}_{t\in\mathbb{R}}$ is a left-continuous, increasing family of σ-algebras.

Proof (a) This follows from (2.6) by using (2.1)

(b) Suppose, first, that $A \in \mathcal{H}_{t,\tau_j}$. Then $A \subset \{\tau_j < t\}$ and we can write

$$A = \bigcup_{\substack{r\in\mathbb{Q} \\ r\leq t}} A \cap \{\tau_j < r\} \qquad \ldots \qquad (2.7)$$

Now each set $A \cap \{\tau_j < r\} \in \mathcal{G}_r^j \vee \mathcal{G}_{t,r}^j$ and thus can be expressed as a countable union of its generating sets, using (2.2) and Lemmas 2.1(a) and 2.3(a). Since we also have

$$\{\tau_j \geq t\} = \bigcap_{\substack{r\in\mathbb{Q} \\ r<t}} \{\tau_j \geq r\} \qquad \ldots \qquad (2.8)$$

$$= \left[\bigcup_{\substack{r\in\mathbb{Q} \\ r<t}} \{\tau_j < r\}\right]^c, \text{ (where } A^c \text{ denotes the complement of } A),$$

the union of these generating sets over all rational $r \leq t$ and $j \in \mathbb{Z}$ forms a countable generating set for $\mathcal{H}_t$.

(c) It is enough to show that, for all j,

$$\mathcal{H}_{t,j} = \bigvee_{u<t} \mathcal{H}_{u,j}$$

Suppose $w < s$, $u < t$. Since $\{\tau_j < w\} \subset \{\tau_j < s\}$, we have

$$\mathcal{G}^j_w \vee \mathcal{G}^j_{u,w} \subset \mathcal{G}^j_s \vee \mathcal{G}^j_{t,s} \qquad \ldots \qquad (2.9)$$

For $A \in \mathcal{H}_{u,j}$, $A \subset \{\tau_j < u\}$ so that, for $s > u$,

$A \cap \{\tau_j < s\} = A \cap \{\tau_j < u\}$ which is in $\mathcal{G}^j_s \vee \mathcal{G}^j_{t,s}$ from (2.9) and the definition of $\mathcal{H}_{u,j}$. Thus, $\mathcal{H}_{u,j} \subset \mathcal{H}_{t,j}$ for all $u < t$. Now

$$\{\tau_j \geq u\} = \{\tau_j \geq t\} \cup \{\tau_j \in [u,t)\}.$$

The second set on the right is in $\mathcal{H}_{t,j}$ so that $\{\tau_j \geq u\} \in \mathcal{H}_{t,j}$. From (2.1), it follows that $\mathcal{H}_{u,j}$ is an increasing family and $\bigvee_{u<t} \mathcal{H}_{u,j} \subset \mathcal{H}_{t,j}$. The reverse inequality is an immediate consequence of (2.7) and (2.8).

The construction of the underlying stochastic processes, from which the shot-noise process is formed, is completed by specifying a probability measure, P, on $(\Omega, \mathcal{H})$. It is usual to start by defining the finite dimensional distributions. In this case, we would ascribe probabilities to the events

$$\{H(u_i, v_i) \leq x_i, \quad N[s_j, t_j) \leq r_j\}$$

where u_i, v_i, x_i range over finite collections of real points, $[s_j, t_j)$ over finite collections of real intervals and r_j over finite collections of integers. The set function, P, induced on Ω, will only be a measure under certain conditions. The "existence problem"

for the stochastic processes, i.e. the problem of finding these conditions, is beyond the scope of my investigations so far. However, in the case of a point process alone, $(\Omega,\mathcal{H})$, the conditions can be found in Daley and Vere-Jones (1972) or Westcott (1972). In the case of the secondary processes, the well-known Daniell-Kolmogorov conditions, see for example Kingman and Taylor (1966, p. 381), will be necessary but do not guarantee the existence of a version which is right-continuous with left limits. We require an additional assumption of continuity in probability, Doob (1953, p. 61). Suppose that at each point (s,t), $H(u,v)$ possesses a limit as $u \to s$ and $v \to t$ where the limit may be either from above or below. Then, for almost all sample paths, H is continuous, except for a countable number of points of discontinuity at which we may choose to make H right-continuous with left limits. This follows from Doob's Theorem 11.1, (1953, p. 356), extended to 2 dimensions.

Note, however, that most models for shot-noise processes would be constructed by specifying conditional rather than finite-dimensional joint distributions. For instance, it may be more natural to specify the probability of secondary process events conditionally on the past of point and secondary processes. Clearly the conditions involved will be much more complex and I have not investigated them.

We make the assumption, henceforth, that

$$\mu(Q) = \mathbb{E}\,N(Q) < \infty \qquad \ldots \qquad \ldots \qquad (2.10)$$

for every bounded $Q \in \mathcal{B}$ i.e. the point process has a σ-finite first moment measure, μ. Results obtained in Kallenberg (1978) and Brown (1978) suggest that this condition might be weakened but we do not do so here.

Conditional expectations can be defined in the usual way. Suppose X is an integrable, real random variable. If $\mathbb{E}[X|\mathcal{H}_s](\omega)$ is an $\mathcal{H}_s$-measurable r.v. such that

$$\int_A \mathbb{E}[X|\mathcal{H}_s](\omega)\ P(d\omega) = \int_A X(\omega)\ P(d\omega)$$

for every $A \in \mathcal{H}_s$, then it is (a version of) the conditional expectation of X given $\mathcal{H}_s$. In particular, for $A \in \mathcal{H}$, we can define $P[A|\mathcal{H}_s](\omega)$ up to a P-null set. Suppose we can go further and choose versions of $P[A|\mathcal{H}_s](\omega)$ for each $A \in \mathcal{H}$ such that:

$$\text{for all } \omega \in \Omega,\ P[\cdot|\mathcal{H}_s](\omega) \text{ is a}$$

$$\text{probability measure on } (\Omega,\mathcal{H}); \qquad \ldots \qquad \ldots \qquad (2.11)$$

then $P[\cdot|\mathcal{H}_s](\omega)$ is called a regular conditional probability measure. The conditional expectation of X can now be written as:

$$\mathbb{E}[X|\mathcal{H}_s](\omega) = \int_\Omega X(\psi)\ P[d\psi|\mathcal{H}_s](\omega)$$

$$= \int_{\mathbb{R}} x\, P[X^{-1}(dx)|\mathcal{H}_s](\omega)$$

To prove the existence of a regular version, we could use the Corollary on p. 193 of Neveu (1965). Since $\mathcal{H}$ has been shown to be countably generated, we would need to prove that $\mathcal{H}$ contains a compact class $\mathcal{C}$ having the "approximation property"

$$P(F) = \sup\{P(C);\ C \in \mathcal{C},\ C \subset \mathcal{F}\},\ \forall F \in \mathcal{F} \qquad \ldots \qquad (2.12)$$

Note that in Neveu's Definition I.6.2, $\mathcal{C}$ is compact if every countable subset of $\mathcal{C}$ which has an empty intersection, possesses a

finite subset with an empty intersection. Equivalently, every countable family in $\mathcal{C}$ with the finite intersection property, has a non-empty intersection, which is Meyer's definition of a semi-compact paving (1966, III D3). Meyer refers to (2.12) as "regularity" while Dellacherie and Meyer (1978, III46) talk about "inner regularity" or "tightness". For the connection with the more usual definition of tightness, see Billingsley (1968, p. 11, Problem 7).

The obvious way of introducing the compact class $\mathcal{C}$, is to consider the compact sets under some topology on Ω. One way of introducing such a topology is by defining a metric on Ω, hopefully one for which the open sets generate the σ-algebra $\mathcal{H}$, i.e. for which $\mathcal{H}$ is the Borel σ-algebra. Then, provided P satisfies (2.12), regular conditional probabilities exist w r t any subalgebra of $\mathcal{H}$, Dellacherie and Meyer (1978, III 72). In fact, if Ω can be shown to be a complete, separable, metric space (i.e. Polish space) with Borel σ-algebra $\mathcal{H}$, every probability defined on it automatically satisfies (2.12).

This approach of setting up a Polish space, is successfully followed by Papangelou (1972a, 1974a) for point processes. I have not investigated this possibility for the more complicated space defined here, so I will merely make the more general assumption that P satisfies (2.12) for a suitable compact class $\mathcal{C}$. This guarantees the existence of a regular conditional probability measure w r t $\mathcal{H}_s$, for each s. From now on, $P[\cdot|\mathcal{H}_s]$ denotes this regular version.

Throughout the remainder of this thesis, we will constantly be considering stochastic processes formed from sequences of conditional expectations such as $E[f(s)|\ \mathcal{H}_s]$, $s \in \mathbb{R}$. We will require to take limits and integrals of these processes without

detailed comment especially in Chapter III. We conclude this section on fundamentals by gathering together some results which make this possible.

In considering bounds and limits of a process $X(s)$, we are often faced with the following situation. We know that for some interval $A \subset \mathbb{R}$, $X(s) \in A$, a.s. i.e. except for a P-null exceptional set of realizations, E_s, depending on s. For any countable collection $\{s_j\}$, we can deduce that $X(s_j) \in A$, $\forall_j$ a.s. since $\bigcup_j E_{s_j}$ is measurable and P-null. However, without further argument we cannot deduce that $X(s) \in A$, $\forall s$, a.s., since $\bigcup_s E_s$ need not be measurable. To overcome this we introduce Doob's concept of separability (with respect to closed intervals).

We say that $X(s)$ is separable if there is a sequence $\{s_j\}$ of parameter values and a P-null set E such that, if A is any closed interval and I any open interval,

$$\{X(s_j) \in A, \quad s_j \in I\} \setminus \{X(s) \quad \in A, \; s \quad \in I\} \quad \subset \quad E.$$

As Doob points out (1953, p.52), this is equivalent to demanding that for $\omega \notin E$

$$\sup_{s \in I} X(s) = \sup_{s_j \in I} X(s_j)$$

and

$$\inf_{s \in I} X(s) = \inf_{s_j \in I} X(s_j),$$

for every open interval I. Clearly, if almost all the sample paths of a process possess both right and left limits at each s, then it is separable. Similarly, any continuous or monotone function of a separable process will also be separable. Further, Doob shows (1953, p.57 Theorem 2.4) that any real-valued process $X(s)$, $s \in \mathbb{R}$ has a separable modification $\tilde{X}(s)$ such that

$$P(X(s) = \tilde{X}(s)) = 1, \quad \forall s.$$

Now when $X(s)$ is a version of a conditional expectation then $\tilde{X}(s)$ is, also. Therefore, whenever we consider a sequence of conditional expectations, we can and will choose versions which give us a separable stochastic process.

For integration of processes like $E[f(s)|\mathcal{H}_s](\omega)$, we require $\mathcal{B}$ - measurability in s, but so that we can manipulate integrals by Fubini's Theorem, we need joint $\mathcal{B} \otimes \mathcal{H}$-measurability in (s,ω). In fact, we will prove progressive measurability, a stronger property which, as we saw following Lemma 2.3 allows substitution of suitable random variables for s. We approach this by investigating the continuity properties of $E[\cdot \mid \mathcal{H}_s]$.

Lemma 2.5 For each s, $f(s,\omega)$ is a positive, integrable r.v. while, for each ω, the sample path is left-continuous and increasing (decreasing). In either case, we can choose $E[f(s)\,|\,\mathcal{H}_s](\omega)$ to be left-continuous, for almost all ω.

Proof Consider the increasing case first. Let

$Y_s = E\,[f(s)\,|\,\mathcal{H}_s]$. If $s > t$, then

$$E[Y_t\,|\,\mathcal{H}_s] \quad = E[f(t)\,|\,\mathcal{H}_s] \;\geq\; E[f(s)\,|\,\mathcal{H}_s\,] \quad = Y_s.$$

Y_s is therefore a positive submartingale, adapted to the left-continuous family $\{\mathcal{H}_s\}$ and, by Proposition IV. 5.1 of Neveu (1965), uniformly integrable..

Now we choose a rational sequence $s_n \uparrow t$ and apply the submartingale convergence theorem, Doob (1953, p.324). There is an integrable r.v., Z_t say, for which $Y_{s_n} \to Z_t$, a.s., $n \to \infty$. Let $A \in \mathcal{H}_u$, for some $u < t$. Then

$$\int_A Z_t \, dP = \lim_{n\to\infty} \int_A Y_{s_n} \, dP$$

by the uniform integrability of Y_{s_n}, while by monotone convergence,

$$\int_A f(t) \, dP = \lim_{n\to\infty} \int_A f(s_n) dP.$$

Now, provided $s_n \geq u$,

$$\int_A Y_{s_n}\, dP = \int_A E[f(s_n) \mid \mathcal{H}_{s_n}]\, dP = \int_A f(s_n)\, dP.$$

Combining the last three equations,

$$\int_A Z_t\, dP = \int_A f(t)\, dP \qquad ---- \qquad (2.13)$$

for all $A \in \bigcup_{u<t} \mathcal{H}_u$. Since each side of (2.13) is an integral, it defines a measure on the field $\bigcup_{u<t} \mathcal{H}_u$ which, by Theorem 4.2 of Kingman and Taylor (1966), has a unique extension to the generated σ-algebra $\mathcal{H}_{t-} = \bigvee_{u<t} \mathcal{H}_u$. By left continuity, $\mathcal{H}_{t-} = \mathcal{H}_t$, so that (2.13) is true for all $A \in \mathcal{H}_t$. Therefore, Z_t is the required version of $E[f(t) \mid \mathcal{H}_t]$.

In the case when $f(s)$ is decreasing, Y_{s_n} is a positive supermartingale. Now $Y_{s_n} \leq E[f(s_1) \mid \mathcal{H}_{s_n}]$ which, by V T 18 of Meyer (1966), is a uniformly integrable martingale. So Y_{s_n} is also uniformly integrable and the above argument remains valid.

We have proved that, for each $t \in \mathbb{R}$, and sequence $s_n \uparrow t$,

$$E[f(s_n) \mid \mathcal{H}_{s_n}] \to E[f(t) \mid \mathcal{H}_t]$$

except for a P-null set, E_t say, of realizations. Since we have chosen $E[f(s)\,|\,\mathcal{H}_s]$ to be a separable stochastic process, we may apply Theorems 2.1 (ii) and 2.3 of Doob (1953, pp.53,55) and the definition of separability to deduce that, for all t, $E[f(s)\,|\,\mathcal{H}_s] \to E[f(t)\,|\,\mathcal{H}_t]$ as $s \uparrow t$, except on a single null set of realizations, as required.

Note This lemma applies, in particular, to the conditional probabilities, $P(A|\mathcal{H}_s), A \in \mathcal{H}$. On the one hand, we have chosen regular versions of these so that $P(\cdot\,|\,\mathcal{H}_s)$ is a true probability; on the other hand, in Lemma 2.5, we chose versions so that $P(A|\mathcal{H}_t) = \lim_{s_n \uparrow t} P(A|\mathcal{H}_{s_n})$. Are these compatible? Suppose A is a countable union of disjoint subsets, A_r, and that the $P(\cdot\,|\,\mathcal{H}_{s_n})$ are all regular conditional probability measures. Then

$$\begin{aligned} P(A|\mathcal{H}_t) &= \lim_{s_n \uparrow t} P\Big(\bigcup_r A_r\,\Big|\,\mathcal{H}_{s_n}\Big) \\ &= \lim_{s_n \uparrow t} \sum_r P(A_r|\,\mathcal{H}_{s_n}) \\ &= \sum_r P(A_r|\,\mathcal{H}_t) \end{aligned}$$

Clearly $P(\cdot\,|\,\mathcal{H}_t)$ is also a probability measure so that the modification in Lemma 2.5 preserves regularity.

Corollary 2.6 If $f(s,\omega)$ is an integrable $\mathcal{B}\otimes\mathcal{H}$-measurable function then $E[f(s)\,|\,\mathcal{H}_s]\,(\omega)$ is progressively measurable.

Proof We first show that $P(A|\mathcal{H}_s)(\omega)$ is progressively measurable, for any $A \in \mathcal{H}$. For each $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ for which $\frac{k+1}{n} \leq s$,

define on $(-\infty,s) \times \Omega$,

$$X_n(u,\omega) = P(A \mid \mathcal{H}_{\frac{k}{n}})(\omega), \text{ if } u \in [\tfrac{k}{n}, \tfrac{k+1}{n}) \text{ for some } k,$$

$$= 0 \quad , \quad \text{otherwise}$$

Then X_n is $\mathcal{B}_s \otimes \mathcal{H}_s$ measurable and, by Lemma 2.5, converges to $P(A \mid \mathcal{H}_u)(\omega)$ which is therefore also $\mathcal{B}_s \otimes \mathcal{H}_s$ measurable on $(-\infty,s) \times \Omega$, as required.

Now, for any $Q \in \mathcal{B}_s$, $A \in \mathcal{H}$ and $u < s$,

$$E[I_{Q \times A}(u,\psi) \mid \mathcal{H}_u](\omega) = I_Q(u)\, P(A \mid \mathcal{H}_u)(\omega).$$

The right side is clearly $\mathcal{B}_s \otimes \mathcal{H}_s$-measurable, so that the corollary is true for the indicator functions, $f(s,\omega) = I_{Q \times A}(s,\omega)$. By monotone convergence, it is true for positive measurable f and, hence, for integrable f, by considering positive, negative and, if necessary, real and imaginary parts.

Remark The key properties of the family $\{\mathcal{H}_t\}$ are contained in Lemmas 2.4 and 2.3a. The results which follow, from Lemma 2.5 on, apply to any family on Ω, which possesses these properties. In particular, we can make the choice $\mathcal{H}_t = \mathcal{F}_t$, $\forall\, t$.

Corollary 2.7 Each of the processes, $P[N[u,v) = 1 | \mathcal{F}_u]$ $P[N[u,v) \geq 1 | \mathcal{F}_u]$ and $E[N[u,v) | \mathcal{F}_u]$, is progressively measurable wrt. $\{\mathcal{F}_s\}_{s \in \mathbb{R}}$. In particular, they are each $\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{F}$-measurable on $\mathbb{R} \times \mathbb{R} \times \Omega$.

Proof Each process is left-continuous in both u and v by Lemma 2.5 and the remark above. Let $X(u,v;\omega)$ denote any of the three. We must prove that the map $(u,v;\omega) \to X(u,v;\omega)$ is measurable from $((-\infty,s) \times (-\infty,s) \times \Omega, \mathcal{B}_s \otimes \mathcal{B}_s \otimes \mathcal{F}_s)$ into $(\mathbb{R}, \mathcal{B})$, for any $s \geq v > u$.

For all $n,m \in \mathbb{N}$ and $k,j \in \mathbb{Z}$ s.t. $\frac{k+1}{n} < \frac{j+1}{m} \leq s$, we define on $(-\infty,s) \times (-\infty,s) \times \Omega$,

$$X_{n,m}(u,v;\omega) = X(\tfrac{k}{n},\tfrac{j}{m};\omega), \text{ if } u \in [\tfrac{k}{n}, \tfrac{k+1}{n}) \text{ and } v \in [\tfrac{j}{m}, \tfrac{j+1}{m}) \text{ for some } k \text{ and } j,$$
$$= 0, \quad \text{otherwise.}$$

Then $X_{n,m}$ is $\mathcal{B}_s \otimes \mathcal{B}_s \otimes \mathcal{F}_s$-measurable and converges on $(-\infty,s) \times (-\infty,s) \times \Omega$ to $X(u,v;\omega)$, as $n, m \to \infty$. X is therefore $\mathcal{B}_s \otimes \mathcal{B}_s \otimes \mathcal{F}_s$-measurable, as required.

§3 Some Results on Palm Distributions

Much of the argument of the next chapter hinges on taking expectations conditionally on the event $\{N\{s\} = 1\}$ and $\mathcal{H}_s$, the σ-algebra of events of the shot-noise process prior to s. We will denote such a conditional expectation by $E_s[\cdot|\mathcal{H}_s]$; its definition and properties are the subject of this section.

The probability of A, conditional on the occurrence of a point at a specified point s, will be denoted $P_s(A)$ and is called a Palm probability of A. In defining these, we use the approach of Ryll-Nardzewski (1960) and Papangelou (1972b), (1974a) but extend the domain of definition from $\mathcal{Y}_1$ to $\mathcal{H}$, to achieve a simple but useful increase in generality. Alternative definitions are reviewed in Daley and Vere-Jones (1972). We begin, however, by outlining an argument which justifies the extension of Propositions I.6.1 and III. 2.1 of Neveu (1965) from probabilities to σ-finite measures. This will prove useful several times in this chapter.

Let $(S,\mathcal{S})$ and $(T,\mathcal{Y})$ be measurable spaces. Our starting point is a set function $m(Q\times A)$, where $Q \in \mathcal{S}$ and $A \in \mathcal{Y}$. The collection of rectangles $\{Q \times A\}$ is a semi-ring generating $\mathcal{S}\otimes\mathcal{Y}$. If m can be proved σ-additive and σ-finite on $\{Q \times A\}$

then, by Theorems 3.5 and 4.2 of Kingman and Taylor (1966), it can be extended to a unique σ- finite measure, m, on $\mathcal{S} \otimes \mathcal{T}$. Clearly, if $m = m_1$ on $\{Q \times A\}$, for some other set function m_1, then m_1 is also a σ-finite measure and equals m on $\mathcal{S} \otimes \mathcal{T}$.

Next we define a σ-finite transition measure, M, as a mapping from $\mathcal{S} \times T$ into $\mathbb{R}$, such that:

a) for each $t \in T$, $M(\cdot, t)$ is a σ-finite measure on $(S, \mathcal{S})$;

b) for each $Q \in \mathcal{S}$, $M(Q, \cdot)$ is measurable on $(T, \mathcal{T})$.

In the case, where $(T, \mathcal{T}, Pr)$ is a probability space then M is just a random measure while when $M(\cdot, \omega)$ is a probability, Neveu calls it a transition probability.

If we have, in addition, a σ-finite measure, ν, defined on $(T, \mathcal{T})$ and we put

$$m(Q \times A) = \int_A M(Q,t)\ \nu\, dt), \quad - \ - \ - \ - \qquad (3.1)$$

it is easily proved that m is σ-additive, as in Neveu (1965, pp.74-75). If it can be proved σ-finite, we can deduce the existence of a unique σ-finite measure on $\mathcal{S} \otimes \mathcal{T}$ for which (3.1) holds. Furthermore, the remainder of Neveu's Proposition III.2.1,

i.e. Fubini's Theorem, is valid. Thus, for every m-integrable, complex function X, measurable on $(S \times T, \mathcal{S} \otimes \mathcal{T})$, the function

$$Y(t) = \int_S X(u,t) \; M(du,t) \quad - \; - \; - \; - \qquad (3.2)$$

is $\mathcal{T}$ - measurable on T and integrable wrt. ν. Moreover,

$$\int_{S\times T} X \, dm = \int_{T\times S} \int X(u,t) \; M(du,t) \; \nu \; (dt) \; - \; - \; - \; - \qquad (3.3)$$

Due to the linearity of the integral, the extension of Neveu's proof to complex X is a straightforward consideration of real and imaginary parts.

As our first application of this extension argument, let us define for each $Q \in \mathcal{B}$, $A \in \mathcal{H}$,

$$n(Q \times A) = \int_A N(Q,\omega) \; P(d\omega) \quad - \; - \; - \; - \qquad (3.4)$$

Now, $N(Q,\omega)$ is clearly a random measure and n is σ-finite, since $n(Q \times A) \leq \mu(Q) < \infty$, for bounded Q. Thus n has a unique extension to $\mathcal{B} \otimes \mathcal{H}$; the function

$$Y(\omega) = \int_{\mathbb{R}} X(s,\omega) \; d \; N(s,\omega) \quad - \; - \; - \; - \qquad (3.5)$$

is $\mathcal{H}$-measurable and integrable wrt. P, while

$$\int_{\mathbb{R}\times\Omega} X \, dn = \int_{\Omega\times\mathbb{R}} \int X(s,\omega) \, dN(s,\omega) \, P(d\omega) \quad - - - - \quad (3.6)$$

The measure, n, is called the master measure by Papangelou (1974a) and the Campbell measure by Kallenberg (1976), following Matthes (1972).

Example 3.1 Taking $X(s,\omega) = H(t,s,\omega) \, I_{[o,t]}(s)$ in (3.5), we see that the shot-noise is defined as the stochastic integral

$$Y(t,\omega) = \int_o^t H(t,s,\omega) \, dN(s,\omega)$$

For fixed $A \in \mathcal{H}$, the random measure, $n(\cdot \times A)$, is clearly absolutely continuous wrt. $\mu(\cdot) = \int_\Omega N(\cdot,\omega) \, P(d\omega)$ and so by the Radon-Nikodym theorem, there is a unique, finite and non-negative, $\mathcal{B}$ - measurable function $P_\cdot(A)$ s.t.

$$n(Q \times A) = \int_Q P_s(A) \, \mu(ds) \quad - - - - \quad (3.7)$$

We may choose a regular version of $P_s(A)$, for $A \in \mathcal{H}$, such that for each $s \in \mathbb{R}$, $P_s(\cdot)$ is a probability measure on $(\Omega,\mathcal{H})$. This possibility stems from the same assumption (2.12) which justified the choice of a regular version of $P(\cdot|\mathcal{H}_s)$ in Section 2. It follows that $P_s(A)$ is a transition measure (probability) from

$E \times \mathcal{H}$ into $[0,1]$ and, by Fubini's Theorem and (3.7),

$$\int_{\Omega \times E} X \, dn = \int_{E} \int_{\Omega} X(s,\omega) \, P_s(d\omega) \, \mu(ds) \qquad (3.8)$$

The interpretation of $P_s(A)$ as the probability of A given $\{N\{s\} = 1\}$ is confirmed by the following lemma.

Lemma 3.1

a) For $A \in \mathcal{H}$, $Q \in \mathcal{B}$

$$\int_{Q} P_s(A)\mu(ds) = \int_{Q} P_s(A \cap \{N\{s\} = 1\})\mu(ds)$$

b) $P_s\{N\{s\} = 1\} = 1$, for μ almost all s.

Proof This is contained in Papangelou (1974a).

We can define conditional expectations wrt. $P_s(\cdot)$ in the usual way, for each s. Suppose X is an integrable r.v. If $E_s[X|\mathcal{H}_s](\omega)$ is an $\mathcal{H}_s$- measurable function s.t. for all $A \in \mathcal{H}_s$,

$$\int_A E_s[X|\mathcal{H}_s](\omega)\ P_s(d\omega) = \int_A X(\omega)\ P_s(d\omega) \quad - - - - \quad (3.9)$$

then it is (a version of) the conditional expectation of X given $N\{s\} = 1$ and $\mathcal{H}_s$. Once again, with assumption (2.12), we can choose a regular version s.t. $P_s(\cdot|\mathcal{H}_s)$ is a true probability.

In view of Lemma 2.5, it seems plausible that these conditional probabilities can also be chosen to be left-continuous with right-limits in s. However, as I have not been able to prove this, I must introduce an assumption of progressive measurability, which would, of course, have followed from left-continuity. I assume that:

for any real r.v. X, the stochastic process, $E_s[X|\mathcal{H}_s]$, $s \in \mathbb{R}$, is progressively measurable wrt $\{\mathcal{H}_s\}$. . . . (3.10)

Remark All the above definitions, along with (3.10), apply to the special case where $\mathcal{H}_s = \mathcal{F}_s$, $\forall$ s.

If D is a $\mathcal{B} \otimes \mathcal{H}$ set its u-section is

$$D_u = \{\omega \in \Omega : (u,\omega) \in D\}$$

Let $\mathcal{D}_s = \{D \in \mathcal{B}\otimes\mathcal{H} : D_u \in \mathcal{H}_u, \forall u < s\}$ and $\mathcal{D} = \vee\mathcal{D}_s = \{D \in \mathcal{B}\otimes\mathcal{H} : D_u \in \mathcal{H}_u, \forall u\}$. $\mathcal{D}_s$ is called the σ-algebra of sets adapted to $\{\mathcal{H}_u\}_{u<s}$. Similarly, we let $\mathcal{K}_s$ be the σ-algebra of $\mathcal{B}\otimes\mathcal{J}$ sets adapted to $\{\mathcal{J}_u\}_{u<s}$.

We next show that $E_s[X|\mathcal{H}_s](\omega)$ could equally well have been defined as a "conditional expectation" given $\mathcal{D}$ wrt. the measure, n, on $\mathcal{B}\otimes\mathcal{H}$:

Lemma 3.2 For any n-integrable, $\mathcal{B}\otimes\mathcal{H}$ measurable function $X(s,\omega)$ and each set $D \in \mathcal{D}$

$$\int_D X(s,\omega)\, dn(s,\omega) = \int_D E_s[X(s)|\mathcal{H}_s](\omega)\, dn(s,\omega)$$

Proof

Consider first the indicator function $I_{Q \times A}$ where Q is a bounded Borel set and $A \in \mathcal{H}$.

$$\int_D I_{Q \times A} \, dn$$

$$= \int_Q \int_{D_s} I_A(\omega) \, P_s(d\omega) \mu(ds)$$

$$= \int_Q \int_{D_s} P_s[A|\mathcal{H}_s](\omega) \, P_s(d\omega) \mu(ds), \text{ since } D_s \in \mathcal{H}_s$$

$$= \int_D \mathbb{E}_s[I_{Q \times A}|\mathcal{H}_s](\omega) \, dn(s,\omega)$$

since $P_s[A|\mathcal{H}_s](\omega)$ is $\mathcal{D}$-measurable by (3.10).

Using linearity of the integral and monotone convergence the Lemma follows by extension.

Note that, in particular,

$$\int_\Omega \int_{\mathbb{R}} X(s,\omega) \, dN(s,\omega) \, P(d\omega)$$

$$= \int_\Omega \int_{\mathbb{R}} \mathbb{E}_s[X(s)|\mathcal{H}_s](\omega) \, dN(s,\omega) P(d\omega) \quad - - - - \qquad (3.11)$$

Recall from Section 2 that $t_o(s,\omega)$ is the point immediately prior to s and $t_1(s,\omega)$ that at, or immediately after, s. We also define for each $s \in \mathbb{R}$, the map

$$T_s : \omega \to (t_o(s), \omega)$$

Then $T_s^{-1}(Q \times A) = A \cap \{t_o(s) \in Q\} \in \mathcal{H}$, for all $Q \in \mathcal{B}$ and $A \in \mathcal{H}$,

since $\omega \to t_o(s,\omega)$ is $\mathcal{F}_s$ - measurable and $\mathcal{F}_s \subset \mathcal{H}_s \subset \mathcal{H}$. Thus T_s is a measurable map from $(\Omega, \mathcal{H})$ into $((-\infty, s) \times \Omega, \mathcal{B}_s \otimes \mathcal{H})$ and

$$P[T_s^{-1}(Q \times A)] = \int_A I_{\{t_o(s) \in Q\}}(\omega)\, P(d\omega)$$

$$= \int_A \int_Q I_{\{N(v,s) = 0\}}(v,\omega)\, dN(v,\omega)\, P(d\omega)$$

$$= \int_{Q \times A} I_{\{N(v,s) = 0\}}(v,\omega)\, dn(v,\omega) \quad .$$

If $X(v,\omega)$ is n-integrable and $\mathcal{B} \otimes \mathcal{H}$-measurable, then an extension to monotone limits of simple functions shows that

$$\int_\Omega X(t_o(s),\omega)\, P(d\omega) = \int_{(-\infty,s) \times \Omega} X(v,\omega) I_{\{N(v,s) = 0\}}\, dn(v,\omega) \qquad (3.12)$$

We also recall, from the assumption (3.10) and the remark following it, that $E_u[X(u,\cdot) \mid \mathcal{F}_u](\omega)$ is $\mathcal{K}_s$ - measurable on $(-\infty, s) \times \Omega$ where $\mathcal{K}_s$ is the σ-algebra

$$\{K \in \mathcal{B}_s \otimes \mathcal{F}_s : K_u \in \mathcal{F}_u,\ \forall u < s\}$$

If we define $\mathcal{F}_{t_o(s)} = T_s^{-1}(\mathcal{K}_s)$ then

$\mathbb{E}_{t_o(s)}[X(t_o(s),\cdot)|\mathcal{F}_{t_o(s)}](\omega)$ will be an $\mathcal{F}_{t_o(s)}$ - measurable random variable. Note that since T_s is measurable from $(\Omega,\mathcal{F}_s)$ into $((-\infty,s) \times \Omega, \mathcal{B}_s \otimes \mathcal{F}_s)$, we must have $\mathcal{F}_{t_o(s)} \subset \mathcal{F}_s$. The inclusion is strict, since $t_o(s)$ is $\mathcal{F}_s$ but not $\mathcal{F}_{t_o(s)}$ - measurable, and so we can interpret $\mathcal{F}_{t_o(s)}$ as the σ- algebra of events of the point process prior to $t_o(s)$, the point before s. We proceed by investigating the relationship between $\mathbb{E}[\cdot|\mathcal{F}_s]$ and $\mathbb{E}_{t_o(s)}[\cdot|\mathcal{F}_{t_o(s)}]$.

Suppose $T_s^{-1}(K) \in \mathcal{F}_{t_o(s)}$ and $\mathbb{E}|Y| < \infty$,

Then $\int_{T_s^{-1}(K)} \mathbb{E}_{t_o(s)}[Y \mid \mathcal{F}_{t_o(s)}](\omega)\ P(d\omega)$

$$= \int_{(-\infty,s)} \int_{K_V} \mathbb{E}_V[Y|\mathcal{F}_V](\omega)\ I_{\{N(V,s)=0\}}\ dn(V,\omega), \quad ---- \qquad 3.13)$$

by (3.12). On the other hand,

$$\int_{T_s^{-1}(K)} Y(\omega)\ P(d\omega)$$

$$= \int_{(-\infty,s)} \int_{K_V} Y(\omega)\ I_{\{N(V,s)=0\}}\ dn(V,\omega)$$

$$= \int_{(-\infty,s)} \int_{K_V} \mathbb{E}_V[YI_{\{N(V,s)=0\}}|\mathcal{F}_V](\omega)\ dn(V,\omega)$$

which is not equal to (3.13). The usual defining relationship of a conditional expectation does not, therefore, hold and we

must prove certain properties we would normally take for granted.

Lemma 3.3 If $\mathbb{E}|Y| < \infty$, then for each $s \in \mathbb{R}$,

$$\mathbb{E}_{t_o(s)}\left[Y\, I_{\{N(t_o(s),s) = 0\}} \,|\, \mathcal{F}_{t_o(s)}\right](\omega)$$

$$= \mathbb{E}[Y|\mathcal{F}_s]P_{t_o(s)}\left[N(t_o(s),s) = 0 \,|\, \mathcal{F}_{t_o(s)}\right](\omega), \text{ a.s.}$$

Proof Let $A \in \mathcal{F}_s$. Then from Lemma 2.2. for $\forall\, u < s$, there is a set $A(u) \in \mathcal{F}_u$ s.t. $A \cap \{t_o(s) = u\} = A(u) \cap \{t_o(s) = u\}$. Now, from (3.12),

$$\int_A \mathbb{E}_{t_o(s)}\left[Y I_{\{N(t_o(s),s) = 0\}} \,|\, \mathcal{F}_{t_o(s)}\right](\omega)\, P(d\omega)$$

$$= \int\int_{(-\infty,s)A(u)} \mathbb{E}_u\left[Y I_{\{N(u,s) = 0\}} \,|\, \mathcal{F}_u\right](\omega)\, I_{\{N(u,s) = 0\}} P_u(d\omega)\,\mu(du), - - - \quad (3.14)$$

where we have used the properties that P_u has support $\{N\{u\} = 1\}$ by Lemma 3.1(b), and that $t_o(s) = u \Leftrightarrow N(u,s) = 0$ and $N\{u\} = 1$.

Since $A(u) \in \mathcal{F}_u$, (3.14) equals

$$\int\int_{(-\infty,s)A(u)} Y(\omega) I_{\{N(u,s) = 0\}} P_u[N(u,s) = 0 \,|\, \mathcal{F}_u](\omega) P_u(d\omega)\, \mu(du)$$

$$= \int_A Y(\omega)\ P_{t_o(s)}\ [N(t_o(s),s) = 0 | \mathcal{F}_{t_o(s)}](\omega) P(d\omega) \quad ----(3.15)$$

$$= \int_A \mathbb{E}[Y|\mathcal{F}_s](\omega)\ P_{t_o(s)}[N(t_o(s),s) = 0 | \mathcal{F}_{t_o(s)}](\omega) P(d\omega),$$

since $A \in \mathcal{F}_s$ and $\mathcal{F}_{t_o(s)} \subset \mathcal{F}_s$.

Corollary 3.4 If Y is a positive, integrable r.v., then for each $s \in \mathbb{R}$,

$$\mathbb{E}_{t_o(s)}\ [Y I_{\{N(t_o(s),s) = 0\}} | \mathcal{F}_{t_o(s)}](\omega) < \infty\ ,\ \text{a.s.}$$

Proof By (3.15),

$$\int_\Omega \mathbb{E}_{t_o(s)}[Y\ I_{\{N(t_o(s),s)=0\}} | \mathcal{F}_{t_o(s)}](\omega)\ P(d\omega)$$

$$\leq \int_\Omega Y(\omega)\ P(d\omega)$$

$$< \infty\ ,\ \text{hence the result.}$$

Lemma 3.5 For each $s \in \mathbb{R}$,

$$P_{t_o(s)}[N(t_o(s), t_1(s)) = 0 | \mathcal{F}_{t_o(s)}](\omega) > 0,\ \text{a.s.}$$

Proof Let $E = \{(u,\omega) : u<s,\ P_u[N(u,t_1(s)=0 | \mathcal{F}_u](\omega) = 0\}$

which is $\mathcal{H}_s$ - measurable. By (3.12),

$$P[T_s^{-1}(E)] = \int_{(-\infty,s)} \int_{E_u} I_{\{N(u,s) = 0\}} P_u(d\omega)\ \mu(du)$$

Since $N(u,s) = 0 \iff N(u,t_1(s)) = 0$, by the definition of $t_1(s)$, the right side equals

$$\int_{(-\infty,s)} \int_{E_u} I_{\{N(u,t_1(s)) = 0\}} P_u(d\omega)\,\mu(du)$$

$$= \int_{(-\infty,s)} \int_{E_u} P_u[N(u,t_1(s)) = 0 | \mathcal{F}_u]\ P_u(d\omega)\ \mu(du)$$

$= 0$, by the definition of E_u.

$\therefore$ Provided $\omega \notin T_s^{-1}(E)$,

$$P_{t_o(s)}[N(t_o(s),t_1(s)) = 0 | \mathcal{F}_{t_o(s)}](\omega) > 0.$$

Finally, note that $t_o(s)$ is left-continuous in s. Indeed, for given ω, the realisation of the point process $\Psi(\omega)$ has no cluster points, so that there is some interval $(s-\delta, s]$, $\delta > 0$, in which $t_o(s)$ is constant. Similarly, $t_1(s)$ is left-continuous and indeed constant on some interval to the left of s. We, therefore, can deduce

Lemma 3.6 For each $s \in \mathbb{R}$ and integrable, $\mathcal{B} \otimes \mathcal{H}$- measurable function, $X(u,\omega)$,

$$E_{t_o(s)}\left[X(t_o(s)) \mid \mathcal{F}_{t_o(s)}\right] \quad \text{is left-continuous in } s.$$

§4 The Conditional Intensity

Consider an interval $[u,v)$ in $\mathbb{R}$. Let $\{\Delta^n[u,v)\}$, $n=1,2,\text{----}$ be a sequence of dissections, where Δ^n has points

$$u = c_o < c_1 < c_2 < - - - - < c_{k_n+1} = v,$$

Δ^{n+1} is a refinement of Δ^n and

$$||\Delta^n|| = \sup_{0 \le r \le k_n} (c_{r+1} - c_r) \to 0, \text{ as } n \to \infty,$$

If we need to indicate the dependence of c_r on n, we write c_r^n.

We can define three random interval functions on $\{\Delta^n\}$,

$$Z_r^{(1)}(\omega) = P[N(c_r, c_{r+1}) = 1 \mid \mathcal{F}_{c_r}](\omega),$$

$$Z_r^{(2)}(\omega) = P[N(c_r, c_{r+1}) \geq 1 \mid \mathcal{F}_{c_r}](\omega),$$

$$Z_r^{(3)}(\omega) = E[N(c_r, c_{r+1}) \mid \mathcal{F}_{c_r}](\omega)$$

and, corresponding to them, three dissection sums,

$$S_n^{(i)}([u,v);\omega) = \sum_{r=0}^{k_n} Z_r^{(i)}(\omega), \quad i = 1, 2, 3.$$

Note that

$$0 \leq S_n^{(1)} \leq S_n^{(2)} \leq S_n^{(3)}.$$

We will prove that, as $n \to \infty$, all three dissection sums for the same limit in both the a.s. and mean senses. This result can be considered as a conditional version of Korolyuk's Theorem which, in the above notation, states that

$$\mu[u,v) = \lim_{n\to\infty} \sum_{r=1}^{k_n} P[N(c_r, c_{r+1}) \geq 1]$$

An exhaustive discussion of this and related equalities is given by Daley (1974).

In proving the a.s. limit theorem, it is important to show that neither the limit nor the P-null set of exceptional realizations depends on the choice of dissection points. This, in part, motivates the careful treatment of null sets in Lemmas 4.1 - 4.4. It should be noted that, in the proof of the a.s. limit theorem, Theorem 1 of Papangelou (1972a) we find the assertion that, for each realization ω, the interval $[u,v)$ can be chosen, without loss of generality, so that a point occurs at u and none in (u,v). This is only true if a dissection point is placed at each point of the realization. In itself, this is a perfectly legitimate manoeuvre ; if this limit exists, it is the unique limit. However, in proving the theorem, Papangelou implicitly discards a large collection of null ω-sets which arise in the applications of his Lemma 7 (equivalent to my Lemmas 3.3 and 3.5) These are indexed by the countable collection, $\{c_r^n\}$, for any choice of dissections. There is an apparent circularity in this argument; null ω-sets are discarded which depend on the choice of dissection, which depends on the realization, ω. The proof below avoids this circularity at the expense of being rather longer. The arguments of Papangelou (1974b) are much more careful but unfortunately inapplicable here.

Lemma 4.1 Suppose, for each $t \in \mathbb{R}$, we have a family of intervals, $[x,y) \downarrow t$ i.e. $t \in [x,y)$ and $x \uparrow t$, $y \downarrow t$. Then, except for a P-null set of realizations,

$$P[N[x,y) = 1 | \mathcal{F}_x](\omega) \to P[N\{t\} = 1 | \mathcal{F}_t](\omega),$$

for each t.

Proof By Lemma 2.5

$$P[N[x,y) = 1 | \mathcal{F}_x] \to P[N[t,y) = 1 | \mathcal{F}_t], \text{ as } x \uparrow t.$$

For any rational sequence $y_n \downarrow t$,

$$P[N[t,y_n) = 1 | \mathcal{F}_t] \to P[N\{t\} = 1 | \mathcal{F}_t], \text{ a.s.},$$

by monotone convergence, Neveu (1965, P.122). Since our choice of conditional expectations makes $P[N[t,y) = 1 | \mathcal{F}_t]$ separable, this convergence holds except for a single P-null set and the lemma is proved.

Lemma 4.2 For P-almost all ω and all real intervals, (s,t), the event $\omega \in \{N(s,t) = 0\}$ implies that $P[N[x,z) = 1 | \mathcal{F}_x](\omega)$ is a subadditive interval function on the half-open intervals in (s,t).

Proof Let [x,z) be an arbitrary interval in (s,t), and $y \in [x,z)$, a point of subdivision. It is sufficient to show that, for almost all $\omega \in \{N(s,t) = 0\}$,

$$P[N[x,z) = 1 | \mathcal{F}_x]$$

$$\leq P[N[x,y) = 1 \mid \mathcal{F}_x] \quad + \quad P[N[y,z) = 1 \mid \mathcal{F}_y] \quad \text{- - - - (4.1)}$$

From Lemmas 3.3 and 3.5, the left side is a.s equal to

$$\frac{P_{t_o(x)}\left[N[x,z) = 1 \text{ and } N(t_o(x),\ x) = 0 \mid \mathcal{F}_{t_o(x)}\right]}{P_{t_o(x)}\left[N(t_o(x),\ x) = 0 \mid \mathcal{F}_{t_o(x)}\right]},$$

while the first term on the right is a.s.

$$\frac{P_{t_o(x)}\left[N[x,y) = 1 \text{ and } N(t_o(x),x) = 0 \mid \mathcal{F}_{t_o(x)}\right]}{P_{t_o(x)}\left[N(t_o(x),x) = 0 \mid \mathcal{F}_{t_o(x)}\right]}.$$

Now, for $\omega \in \{N[x,z) = 0\}$, $t_o(y,\omega) = t_o(x,\omega)$ so that the second term on the right of (4.1) is a.s.

$$\frac{P_{t_o(x)}\left[N[y,z) = 1 \text{ and } N(t_o(x),y) = 0 \mid \mathcal{F}_{t_o(x)}\right]}{P_{t_o(x)}\left[N(t_o(x),y) = 0 \mid \mathcal{F}_{t_o(x)}\right]}.$$

Since

$$P_{t_o(x)}\left[N(t_o(x),y) = 0 \mid \mathcal{F}_{t_o(x)}\right] \leq P_{t_o(x)}\left[N(t_o(x),x) = 0 \mid \mathcal{F}_{t_o(x)}\right]$$

and since the event $\{N(t_o(x),x) = 0 \text{ and } N[x,z) = 1\}$ implies that either $\{N(t_o(x),y) = 0 \text{ and } N[y,z) = 1\}$ or $\{N(t_o(x),x) = 0 \text{ and } N[x,y) = 1\}$, the inequality (4.1) now follows.

The exceptional null sets in the applications of Lemmas 3.3 and 3.5 are indexed by x,y,z, an uncountable collection. However, the half-open intervals are countably generated by the rational intervals. Now, we can exclude a single null ω-set, outside which (4.1) holds for all rational intervals. Using the continuity properties of Lemma 2.5, for any x, y and z we may choose rational sequences, $x_n \uparrow x$, $z_n \uparrow z$ and $y_n \uparrow y$ s.t.

$$P[N[x,z) = 1 \mid \mathcal{F}_x] = \lim_{n \to \infty} P[N[x_n,z_n) = 1 \mid \mathcal{F}_{x_n}]$$

with a similar approximation for the right side of (4.1). The inequality (4.1) holds term-by-term for non-exceptional ω, and so also in the limit, i.e. (4.1) holds for all x,y z except for a single P-null set, as required.

Lemma 4.3 For P-almost all ω, and all real intervals $[x,y)$,

$$E\left[N[x,y)\, I_{\{N[x,y) \geq 2\}} \mid \mathcal{F}_x\right]$$

$$\leq \frac{E_{t_o(x)}\left[N[x,y)\, I_{\{N[x,y)\geq 2\}} I_{\{N(t_o(x),x)\,=\,0\}} \mid \mathcal{F}_{t_o(x)}\right]}{P_{t_o(x)}\left[N(t_o(x),t_1(x)) = 0 \mid \mathcal{F}_{t_o(x)}\right]} \qquad (4.2)$$

Proof From Lemmas 3.3 and 3.5 the left side is a.s. equal to

$$\frac{E_{t_o(x)}\left[N[x,y) I_{\{N[x,y)\geq 2\}} I_{\{N(t_o(x),x)\,=\,0\}} \mid \mathcal{F}_{t_o(x)}\right]}{P_{t_o(x)}\left[N(t_o(x),x) = 0 \mid \mathcal{F}_{t_o(x)}\right]}.$$

Then (4.2) follows as $P_{t_o(x)}\left[N(t_o(x),x) = 0 \mid \mathcal{F}_{t_o(x)}\right]$ $\geq P_{t_o(x)}\left[N(t_o(x),t_1(x)) \mid \mathcal{F}_{t_o(x)}\right]$, a.s. Since the half-open intervals are countably generated by the rational intervals, we may use the continuity properties of Lemmas 2.5 and 3.6 to approximate each side of (4.2) by a suitable sequence. Then (4.2) can be seen to hold simultaneously for all $[x,y)$ except for a countable number of null sets indexed by the rational end-points. Hence (4.2) is valid for all $[x,y)$ except for a single P-null set,

as required.

Lemma 4.4. For P-almost all ω and all bounded real intervals, $[x,y)$,

$$E_{t_o(x)}\left[N[x,y)I_{\{N(t_o(x),x) = 0\}} \mid \mathcal{F}_{t_o(x)}\right](\omega) < \infty$$

Proof Since $EN[x,y) < \infty$, we can apply Corollary 3.4. We need only consider the countable collection of null sets, indexed by the rationals, due to the following argument.

Fix ω and let $t_o(x,\omega) = z$. Now choose a rational interval $[x_1,y_1)$ s.t. $x_1 \in (z,x)$ and $y_1 > y$. Then $t_o(x_1,\omega) = z$ also, and

$$E_z\left[N[x,y)\ I_{\{N(z,x) = 0\}} \mid \mathcal{F}_z\right](\omega)$$

$$\leq E_z\left[N[x_1,y_1)\ I_{\{N(z,x_1) = 0\}} \mid \mathcal{F}_z\right](\omega).$$

This is finite by Corollary 3.4 except for a P-null set indexed by $[x_1,y_1)$. The lemma follows, as required, except for a single P-null set.

Let us denote the exceptional null sets in Lemmas 4.1 - 4.4 by E_i, $i = 1, \ldots, 4$, respectively. Now, since $\mu(\Omega) = EN(\Omega) < \infty$,

for all bounded $Q \in \mathcal{B}$, $N(Q)$ is a.s. finite. Moreoever, since any such Q can be put into a rational interval, there is a single null set E_5, say, s.t. for $\omega \notin E_5$, $N(Q,\omega) < \infty$, for all bounded $Q \in \mathcal{B}$.

Theorem 4.5 Suppose our point process is without multiple points and has a σ-finite first moment measure, μ. Then, for P-almost all ω, and for all bounded real intervals $[u,v)$, the dissection sums, $S_n^{(i)}([u,v);\omega)$, $i = 1,2,3$, converge to the same finite limit, as $n \to \infty$. This limit and the exceptional P-null set are independent of the choice of dissection sequences, $\{\Delta^n\}$.

Proof If we exclude ω in the null sets E_i, $i = 1, - - - 5$, then $N[u,v) < \infty$ for all intervals $[u,v)$ and Lemmas 4.1 - 4.4 are valid.

Since the points of the process are distinct, we can, for a fixed non-exceptional ω, choose n sufficiently large that each interval $[c_r,c_{r+1})$ contains at most one point. $S_n^{(1)}$ can then be decomposed into two sums over subintervals covering, respectively, one and no points. In fact, for each dissection Δ^n, we let

$$f_n(t) = \sum_{r=0}^{k_n} Z_r^{(1)} I_{[c_r,c_{r+1})}(t)$$

and

$$g_n(s,t) = \sum_{r \in \mathcal{C}_{s,t}} Z_r^{(1)}$$

where $\mathcal{C}_{s,t} = \{r : [c_r, c_{r+1}) \subset (s,t)\}$. Then

$$S_n^{(1)}[u,v) = \int_{[u,v)} f_n(t)dN(t) + \iint_J g_n(s,t) I_{\{N(s,t)=0\}}^{(s,t)} dN(s)dN(t) - - - \quad (4.3)$$

where

$$J = \{(s,t) : s < \min(t,v),\ t > u\} \qquad - - - - \quad (4.4)$$

Now, both terms on the right of (4.3) are finite sums so that we need only show that $f_n(t)$ and $g_n(s,t)I_{\{N(s,t)=0\}}$ converge for non-exceptional ω. By Lemma 4.1,

$$f_n(t) \to P[N\{t\} = 1 \mid \mathcal{F}_t] , \ n \to \infty ,$$

while $g_n(s,t)\ I_{\{N(s,t)\ =\ 0\}}$ is increasing, both due to its subadditivity, shown in Lemma 4.2, and due to the absorption of new intervals, $[c_r, c_{r+1})$, approaching s and t as the dissections become finer. Hence $S_n^{(1)}[u,v)$ is a.s. convergent.

Next, we let $Z_r^{(4)} = \mathbb{E}\left[N[c_r, c_{r+1})I_{\{N[c_r,c_{r+1}) \geq 2\}} \mid \mathcal{F}_{c_r} \right]$.

Note that

$$S_n^{(3)} - S_n^{(1)} = \sum_{r=0}^{k_n} Z_r^{(4)} \qquad ---- \quad (4.5)$$

Defining

$$h_n(s,t) = \sum_{c_r \in (s,t]} Z_r^{(4)}$$

we have

$$S_n^{(3)} - S_n^{(1)} = \iint_J h_n(s,t)\, I_{\{N(s,t)=0\}}\, dN(s)dN(t)$$

where J is defined at (4.4). This is a finite sum so, again, it is sufficient to consider the convergence of $h_n(s,t)$ for realizations in which $t_0(c_r) = s$ and $t_1(c_r) = t$, for all $c_r \in (s,t]$. By Lemma 4.3, we then have

$$h_n(s,t) \leq \frac{E_s\left[\sum_{c_r \in (s,t]} N[c_r, c_{r+1})I_{\{N[c_r,c_{r+1}) \geq 2\}} I_{\{N(s,c_r)=0\}} \mid \mathcal{F}_s \right]}{P_s[N(s,t) = 0 \mid \mathcal{F}_s]} \qquad ----(4.6)$$

Consider the sum in the numerator of (4.6). Note that

$I_{\{N[c_r, c_{r+1}) \geq 2\}} \to 0$, as $n \to \infty$. Now for a fixed realization, at most one term of this sum can be non-zero for each n, and that term will be decreasing to zero, as $n \to \infty$. To show that (4.5) tends to zero, it is therefore enough to check that the numerator of (4.6) is finite for each n. Let

$$b_n = \min \{c_r^n : c_r^n > s\}.$$

Then the sum in (4.6) is dominated by

$$N[b_n, v)\ I_{\{N(s,b_n) = 0\}}$$

Since $t_o(b_n, \omega) = s$,

$$E_s\left[N[b_n, v)\ I_{\{N(s,b_n) = 0\}} \mid \mathcal{F}_s\right](\omega) < \infty$$

from Lemma 4.4. Thus $S_n^{(3)} - S_n^{(1)} \to 0$, so that $S_n^{(3)}$ and, a fortiori, $S_n^{(2)}$ converge to the same a.s. limit as $S_n^{(1)}$.

Finally, $E\, S_n^{(3)}[u,v) = E\, N[u,v) = \mu[u,v) < \infty$. By Fatou's Lemma,

$$E \lim_{n \to \infty} S_n^{(3)}[u,v) \leq \mu[u,v) < \infty \quad - - - - \qquad (4.7)$$

implying that $\lim_{n \to \infty} S_n^{(i)}[u,v)$ is a.s. finite for $i = 1,2,3$. This remains true for P-almost all ω, simultaneously for all bounded intervals, since we can insert any such interval into a rational one.

We will denote a.s. $\lim_{n \to \infty} S_n^{(i)}([u,v);\omega)$ for $i = 1,2,3$ by $\hat{\Lambda}([u,v);\omega)$. As Papangelou has noted, $\hat{\Lambda}$ is the Burkill integral of $Z^{(i)}$. Now, considering $E[N[u,v) \mid \mathcal{F}_u]$ as a function of u, we see from Lemma 2.5 that it tends to zero as $u \uparrow v$. By a well-known result on integrals of set functions, see for example Riesz and Sz. Nagy (1955, p.22),

$$\lim_{u \uparrow v} \hat{\Lambda}[u,v) = \lim_{u \uparrow v} E[N[u,v) \mid \mathcal{F}_v] = 0 \quad - - - - \qquad (4.8)$$

For any fixed non-exceptional ω, $\hat{\Lambda}(\cdot,\omega)$ is clearly a finitely additive interval function defined on the semi-ring of half-open intervals, $\{[u,v)\}$. Now suppose we have a sequence, v_j, s.t. $[u,v) = \bigcup_j [u,v_j)$. Since

$$\hat{\Lambda}[u,v_j) = \hat{\Lambda}[u,v) - \hat{\Lambda}[v_j,v)$$

we see, from (4.8), that $\hat{\Lambda}[u,v_j) \to \hat{\Lambda}[u,v)$, as $j \to \infty$,

so that $\hat{\Lambda}$ is σ-additive. By (4.7), it is also σ- finite, so it possesses a unique extension to a σ-finite measure on $(\mathbb{R}, \mathcal{B})$. On the exceptional null set, we may set $\hat{\Lambda}(\cdot, \omega)$ equal to zero. Then $\hat{\Lambda}(Q, \omega)$ is a σ-finite random measure on $(\mathbb{R}, \mathcal{B})$, as defined in Section 3.

Next, for $Q \in \mathcal{B}$, $A \in \mathcal{H}$, let

$$\hat{n}(Q \times A) = \int_A \hat{\Lambda}(Q; \omega)\, P(d\omega)$$

Now $\hat{n}$ is σ-finite since, by (4.7),

$$\hat{n}(Q \times A) \leq \mathbb{E}(Q) \leq \mu(Q) < \infty$$

Using the extension argument outlined in Section 3, $\hat{n}$ has a unique extension to a σ-finite measure on $\mathcal{B} \otimes \mathcal{H}$. Moreover, for any integrable, $\mathbb{C}$-valued, $\mathcal{B} \otimes \mathcal{H}$ measurable function, $f(s, \omega)$, we have

$$\int_{\mathbb{R} \times \Omega} f \, d\hat{n} = \int_{\Omega} \int_{\mathbb{R}} f(s, \omega)\, d\hat{\Lambda}(s, \,)\, P(d\omega) \qquad (4.9)$$

In the remainder of this section, we point out the connection between this construction of the conditional intensity and two alternatives mentioned in Section 1. The left and right derivatives of the set function $\hat{\Lambda}(\cdot, \omega)$, if they exist at s, may be denoted by $\hat{\lambda}(s-, \omega)$, $\hat{\lambda}(s+, \omega)$ and, if they are equal,

$\hat{\lambda}(s,\omega)$. Since $\hat{\Lambda}$ is the integral of each of the set functions $Z^{(i)}$ defined earlier, and since a set function and its integral possess the same derived numbers, Riesz and Sz. Nagy (1955, P.23), we have relationships like

$$\hat{\lambda}(s+,\omega) = \lim_{h \to 0} \frac{P[N[s,s+h) \geq 1 | \mathcal{F}_s](\omega)}{h},$$

whenever either side exists. This is essentially the definition (1.4) given by Cox and Lewis (1972). Under the hypotheses of Theorem 4.5, we have the following rather weak result. We denote the derivative of $Z^{(i)}$ at s by $D Z^{(i)}(s;\omega)$. Thus

$$DZ^{(2)}(s;\omega) = \lim_{u \uparrow s, v \downarrow s} \frac{P[N[u,v) \geq 1 | \mathcal{F}_u](\omega)}{v-u}.$$

Corollary 4.6 For almost every s, the derivative $D Z^{(i)}(s;\omega)$ exists and almost surely equals $\hat{\lambda}(s;\omega)$ for $i = 1,2,3$.

Proof For fixed ω, $\hat{\lambda}(s,\omega)$ exists for almost all s since $\hat{\Lambda}$ is additive and therefore monotone. For such s, by our earlier argument, we have

$$\hat{\lambda}(s,\omega) = D Z^{(i)}(s,\omega),$$

Now, define A to be the complement of the set

$\{(s,\omega) \in \mathbb{R} \times \Omega : DZ^{(i)}(s,\omega)$ exists and equals $\hat{\lambda}(s,\omega)\}$

This is a $\mathcal{B} \otimes \mathcal{Y}$ - measurable set by Corollary 2.7. We have shown that A^{ω} has Lebesgue measure zero. By Fubini's Theorem

$$\int\int_{\mathbb{R}\Omega} I_{A_s}(\omega)\, P(d\omega)\, ds = \int\int_{\Omega\,\mathbb{R}} I_{A^{\omega}}(s)\, ds\, P(d\omega)$$

$$= 0$$

Thus, for almost all s, A_s is P-null as required.

The second alternative definition is that of Rudemo (1973), cf. (1.5), which expresses $\hat{\Lambda}$ in terms of $F_s(x)$, the conditional distribution of the interval, X_s, which follows a point at s, given the past, $\mathcal{Y}_s$.

$$\text{Let } F_s(x) = P_s[N(s, s+x] \geq 1 \mid \mathcal{Y}_s](\omega)$$

$$= 1 - P_s[N(s, s+x) = 0 \mid \mathcal{Y}_s](\omega)$$

$$= P_s[X_s \geq x \mid \mathcal{Y}_s](\omega). \text{ - - - -} \qquad (4.10)$$

$F_s(x)$ is left continuous in x whereas a distribution function is usually right continuous. On the other hand, the definition (4.10), using the survivor function, or tail, of X_s, is quite common in some applications of point processes.

Corollary 4.7 Under the hypotheses of Theorem 4.5, for $Q \in \mathcal{B}$

$$\hat{\Lambda}(Q) = \int_Q \frac{dF_{t_o(y)}(y-t_o(y))}{1-F_{t_o(y)}(y-t_o(y))}$$

Proof It is most convenient to work with $Z^{(2)}$ as, if $t_o(c_r) = s$,

$$Z_r^{(2)} = \frac{P_s[N[c_r,c_{r+1}), N(s,c_r) = 0 \mid \mathcal{F}_s]}{P_s[N(s,c_r) = 0 \mid \mathcal{F}_s]}$$

$$= \frac{F_s(c_{r+1}-s) - F_s(c_r-s)}{1 - F_s(c_r-s)} \quad - - - - \qquad (4.11)$$

by Lemmas 3.3 and 3.5.

We can divide up $S_n^{(2)}[u,v)$ into a finite sum as in the proof of Theorem 4.5. Let

$$k_n(s,t) = \sum_{c_r \in (s,t]} Z_r^{(2)} ;$$

then

$$S_n^{(2)}[u,v) = \int_J k_n(s,t)\, I_{\{N(s,t)=0\}} dN(s)\,dN(t)$$

where J is defined at (4.4). It is enough to consider the convergence of $k_n(s,t)$ for realizations in which $t_o(c_r) = s$ and $t_1(c_r) = t$, for all $c_r \in (s,t]$. Without loss of generality, we may consider an interval $[u,v)$, containing one point at t, and a dissection such that $s < u = c_o < c_1 < c_2 \ldots < c_{j_n} \leq t < c_{j_n+1} \leq v$, where $c_{j_n+1} \downarrow t$, as $n \to \infty$. Then, by (4.11),

$$k_n(s,t) = \int_{[u-s,t-s]} f_n(x)\, dF_s(x)$$

$$+ \int_{(t-s,\, c_{j_n+1}-s)} \frac{dF_s(x)}{1-F_s(c_{j_n}-s)} \quad , \qquad \ldots\ldots (4.12)$$

where $f_n(x) = \dfrac{1}{1-F_s(c_r-s)}$, for $s+x \in [c_r, c_{r+1})$,

$r = 0, \ldots, j_n-1$,

$$= \frac{1}{1-F_s(t-s)} \quad , \quad \text{for } s+x \in [c_{j_n}, t).$$

Since $f_n(x) \uparrow \dfrac{1}{1-F_s(x)}$ for $x+s \in [u,t]$, the first term in (4.12) converges to

$$\int_{[u-s,t-s]} \frac{dF_s(x)}{1-F_s(x)} = \int_{[u,t]} \frac{dF_{t_o(y)}(y-t_o(y))}{1-F_{t_o(y)}(y-t_o(y))} \, .$$

The second term tends to zero as $c_{j_{n+1}} \downarrow t$. The result for a general Borel set Q now follows since $\hat{\Lambda}$ is a measure.

Corollary 4.8 If F_t is a continuous distribution function for all t and $\{t_i\}$, $i \in \mathbb{Z}$, denote the ordered points of a realization of the process, then

$$\hat{\Lambda}(t_i, y] = -\log[1-F_{t_i}(y-t_i)] \text{ for } y \in (t_i, t_{i+1}] - - - - \quad (4.13)$$

Proof This is immediate by change of variable in the integral.

In the special case of a renewal process, $\hat{\Lambda}$ is just the integral of the hazard function, Z(t), and (4.13) expresses the well-known fact, Cox and Lewis (1966, p 141) that

$$Z(t) = -\frac{d}{dt}\{\log[1-F(t)]\}$$

$$= \frac{f(t)}{1-F(t)}$$

where F and f are, respectively, the distribution and density function of the intervals between events.

§5 The Conditional Form of Campbell's Theorem

In this section, we prove that $\hat{\Lambda}$ can also be considered as an L', or mean, limit and deduce some important consequences. In particular, we can prove an analogue of Campbell's Theorem i.e. (2.2) of Chapter I.

Lemma 5.1 The sequences of dissection sums, $S_n^{(i)}[u,v)$ are uniformly integrable as $n \to \infty$, for $i = 1, 2, 3$.

Proof
$$S_n^{(1)} = \sum_{r=0}^{k_n} Z_r^{(1)} I_{\{N[c_r, c_{r+1}) = 0\}}$$
$$+ \sum_{r=0}^{k_n} Z_r^{(1)} I_{\{N[c_r, c_{r+1}) \geq 1\}}$$

The first sum is increasing to a limit smaller than $\hat{\Lambda}[u,v)$ by Lemma 4.2 and the proof of Theorem 4.5. The second is dominated by $N[u,v)$. Since both these have finite expectations, $S_n^{(1)}$ is uniformly integrable.

Now,

$$\mathbf{E}\left[\sum_{r=0}^{k_n} Z_r^{(4)}\right] = E\left[\sum_{r=0}^{k_n} N[c_r, c_{r+1}) I_{\{N[c_r, c_{r+1}) \geq 2\}}\right]$$

which tends to zero by dominated convergence and so is trivially uniformly integrable. It follows from (4.5) that $S_n^{(3)}$ and, a fortiori, $S_n^{(2)}$ are uniformly integrable.

An immediate consequence is that $\hat{\Lambda}$ exists in the mean sense also i.e.

$$\mathbb{E}\,|\,S_n^{(i)}[u,v) - \hat{\Lambda}[u,v)\,| \to 0\ ,\ n \to \infty.$$

We require a stronger result which allows us to manipulate integrals wrt. $\hat{\Lambda}$.

Lemma 5.2 Suppose that for P-almost all ω and for every $s \in \mathbb{R}$ and $Q \in \mathcal{B}[s,\infty)$, we have

$$\mathbb{E}[N(Q)\,|\,\mathcal{H}_s] = \mathbb{E}[N(Q)\,|\,\mathcal{G}_s] \quad - - - - \qquad (5.1)$$

Then, for any $A \in \mathcal{H}$ and real interval, $[u,v)$

$$\hat{n}([u,v) \times A) = \int_A \hat{\Lambda}([u,v);\omega)P(d\omega) = \int_{[u,v)\times\Omega} P[A|\mathcal{H}_s](\omega)\ dn(s,\omega).$$

Proof By the previous lemma,

$$\int_A \hat{\Lambda}[u,v)dP = \lim_{n\to\infty} \sum_{r=0}^{k_n} \int_A E[N[c_r, c_{r+1}) \mid \mathcal{H}_{c_r}] \, dP. \quad - - - - \quad (5.2)$$

Under condition (5.1), each integral on the right of (5.2) is

$$\int_\Omega I_A E[N[c_r, c_{r+1}) \mid \mathcal{H}_{c_r}] \, dP$$

$$= \int_\Omega P[A \mid \mathcal{H}_{c_r}] \, N[c_r, c_{r+1}) dP$$

so that (5.2) equals

$$\lim_{n\to\infty} \int\int_{\Omega\,[u,v)} g_n(s) \, d\,N(s) \, dP \quad - - - - \quad (5.3)$$

where

$$g_n(s) = P[A \mid \mathcal{H}_{c_r}], \quad \text{if} \quad s \in [c_r, c_{r+1}).$$

Now, as $c_r^n \uparrow s$, we have that a.s.

$$P[A \mid \mathcal{H}_{c_r^n}] \to P[A \mid \mathcal{H}_s]$$

by Lemma 2.5. Since $E\ N[u,v] < \infty$, the dominated convergence theorem shows that (5.3) equals

$$\iint_{\Omega\times[u,v)} P[A \mid \mathcal{H}_s](\omega)\, dN(s,\omega)\, P(d\omega).$$

Theorem 5.3 Suppose that $f(s,\omega)$ is an integrable, $\mathbb{C}$ - valued, $\mathcal{B}\otimes\mathcal{H}$ - measurable function and condition (5.1) holds. Then

$$\text{(a)}\quad E\int_{\mathbb{R}} f(s)\, d\hat{\Lambda}(s) = E\int_{\mathbb{R}} E[f(s)\mid \mathcal{H}_s]\, d\hat{\Lambda}(s).$$

$$\text{(b)}\quad E\int_{\mathbb{R}} f(s)\, dN(s) = E\int_{\mathbb{R}} E_s[f(s)\mid \mathcal{H}_s]\, d\hat{\Lambda}(s).$$

Proof a) Consider the semi-ring of rectangles $\{Q\times A\}$, for $Q\in\mathcal{B}$ and $A\in\mathcal{H}$. By Lemma 5.2, we have

$$\hat{n}(Q\times A) = \iint_{\Omega\ \mathbb{R}} E[I_{Q\times A}\mid \mathcal{H}_s]\, dN(s)\, P(d\omega).$$

The usual extension argument using monotone limits of simple functions yields

$$\iint_{\Omega\ \mathbb{R}} f(s,\omega)\, d\hat{\Lambda}(s,\omega)\, P(d\omega)$$

$$= \iint_{\Omega\ \mathbb{R}} E[f(s)\mid \mathcal{H}_s](\omega)\, dN(s,\omega)\, P(d\omega). \qquad (5.4)$$

In particular,

$$\iint_{\Omega \mathbb{E}} E[f(s) \mid \mathcal{H}_s](\omega)\, d\hat{\Lambda}(s,\omega)\, P(d\omega)$$

$$= \iint_{\Omega \mathbb{E}} E[f(s) \mid \mathcal{H}_s](\omega)\, d\, N(s,\omega)\, P(d\omega). \quad - - - - \qquad (5.5)$$

Part (a) follows by combining (5.4) and 5.5).

(b) From (3.11), we know that

$$\iint_{\Omega \mathbb{E}} f(s,\omega)\, dN(s,\omega)\, P(d\omega)$$

$$= \iint_{\Omega \mathbb{E}} E_s[f(s) \mid \mathcal{H}_s](\omega)\, d\, N(s,\omega)\, P(d\omega)$$

Since $E_s[f(s) \mid \mathcal{H}_s](\omega)$ is an integrable function, $\mathcal{B} \otimes \mathcal{H}$ - measurable and, indeed $\mathcal{H}_s$ - measurable for fixed s, part (b) follows by applying (5.4).

Example 5.1 We can take

$$f(s,\omega) = H(t, s;\omega) I_{[o,t]}(s)$$

in part (b) of the theorem to deduce that for a shot-noise process

$$E\,Y(t) = E\int_0^t E\,[H(t,s)\,|\,\mathcal{H}_s]\,d\hat{\Lambda}(s)$$

if $E\int_0^t E[\,|H(t,s)|\,|\mathcal{H}_s]\,d\hat{\Lambda}(s) < \infty$. This is the conditional analogue of Campbell's Theorem referred to above. It is the continuous time analogue of (1.1). The analogue of (1.2) follows from Theorem 5.3(b).

Next, note that condition (5.1) of Lemma 5.2 excludes "feedback" from the model, in that we cannot have a situation where, for example, a high value of $Y(t)$ has either an inhibiting or exciting effect on the primary point process. However, an examination of the proof of the lemma shows that (5.2) equals (5.3) under the weaker condition that

$$\int_A \sum_{r=0}^{k_n} \{E\,[N[c_r,c_{r+1})\,|\,\mathcal{F}_{c_r}] - E[N[c_r,c_{r+1})\,|\,\mathcal{H}_{c_r}]\}dP$$

$$\to 0\,, \qquad \text{- - - -} \qquad (5.6)$$

for all $A \in \mathcal{H}_u$, as $n \to \infty$.

More generally, we could define

$$\hat{\Lambda}_H[u,v) = \text{a.s.} \lim_{n\to\infty} \sum_{r=0}^{k_n} E[N[c_r, c_{r+1}) \,|\, \mathcal{H}_{c_r}]$$

provided this limit exists. Assuming also that the sequence of dissection sums is uniformly integrable, we can follow the reasoning leading to Theorem 5.3 to deduce that

$$E \int_{\mathbb{R}} f(s)dN(s) = E \int_{\mathbb{R}} E_s[f(s) \,|\, \mathcal{H}_s] \, d\hat{\Lambda}(s).$$

Now (5.6) implies that $\hat{\Lambda} = \hat{\Lambda}_H$ and so is a condition of local independence of the point process from past H-process events. One situation when (5.6) holds is as follows:

Corollary 5.4 Suppose that μ has no atoms and that for P-almost all ω and for each interval $[u,v)$, there is a constant M s.t. for all $x \in [u,v)$

$$E[N[x, x+h) \,|\, \mathcal{H}_x] \leq M \, E[N[x, x+h) \,|\, \mathcal{H}_x] \quad - - - - \qquad (5.8)$$

for h sufficiently small. Then (5.6) holds and Theorem 5.3 remains valid.

Proof $\int_\Omega E[N[c_r, c_{r+1}) \,|\, \mathcal{H}_{c_r}] dP \leq \int_\Omega N[c_r, c_{r+1}) \, dP$

$\to \mu\{c_r\} = 0$, as $n \to \infty$.

By (5.8), $E[N[c_r, c_{r+1}) \mid \mathcal{F}_{c_r}] - E[N[c_r, c_{r+1}) \mid \mathcal{H}_{c_r}]$

$\to 0$, a.s. as $n \to \infty$.

Now, for sufficiently large n,

$$\left| \sum_{r=0}^{k_n} \{E[N[c_r, c_{r+1}) \mid \mathcal{F}_{c_r}] - E[N[c_r, c_{r+1}) \mid \mathcal{H}_{c_r}]\} \right| \quad ---- \quad (5.9)$$

$$\leq (M+1) \sum_{r=0}^{k_n} E[N[c_r, c_{r+1}) \mid \mathcal{F}_{c_r}] ,$$

which converges a.s. Thus (5.9) tends a.s. to zero by dominated convergence. Furthermore, (5.9) is uniformly integrable, so (5.6) is valid, as required.

Theorem 5.3 allows us to clarify the relationships between the integral definition of $\hat{\Lambda}$ and two, further definitions outlined in Section 1. First, suppose $D \in \mathcal{D}$, the σ-algebra of sets adapted to $\{\mathcal{H}_s\}$. Then

$$\int_D d\hat{\Lambda}(s,\omega) \, P(d\omega) = \int_D dN(s,\omega) \, P(d\omega).$$

This can be considered as an integral equation for $\hat{\Lambda}$;

the existence of a solution follows from the Radon-Nikodym theorem. Allowing for changes due to the absence of time ordering, this is a close relative of the equation investigated by Kallenberg (1978, 3.1) in solving the existence problem on a general time set.

Secondly, suppose $Y(s)$ is a positive, bounded, right-continuous martingale adapted to the right-continuous family of σ-algebras, $\{\mathcal{H}_{s+}\}$. Let $\mathcal{K}_s = \mathcal{H}_{s+}$. Then $\mathcal{K}_{s-} = \bigvee_{u<s} \bigwedge_{v>u} \mathcal{H}_v = \mathcal{H}_{s-}$, as can easily be seen by comparing generating sets of each side. Furthermore, by left continuity $\mathcal{K}_{s-} = \mathcal{H}_s$. Now, from Theorem 4.3 of Doob (1953, p.355).

$$\lim_{u \uparrow s} \mathbf{E}[Y(s) \mid \mathcal{K}_u] = \mathbf{E}[Y(s) \mid \mathcal{K}_{s-}]$$

$$= \mathbf{E}[Y(s) \mid \mathcal{H}_s].$$

Since, by the martingale property, $\mathbf{E}[Y(s) \mid \mathcal{K}_u] = Y(u)$,

we have shown that

$$\mathbf{E}[Y(s) \mid \mathcal{H}_{s+}] = \lim_{u \uparrow s} Y(u) = Y(s-)$$

Applying Theorem 5.3(a) we have, for each t,

$$E\int_0^t Y(s)d\hat{\Lambda}(s) \quad = \quad E\int_0^t Y(s-)d\hat{\Lambda}(s).$$

Comparing this with VII.D.18 of Meyer (1966), we have shown that $\hat{\Lambda}$ is "natural", so that $\hat{\Lambda}[0,t]$ is a natural, increasing process in t. Now the Doob-Meyer decomposition, Meyer (1966, VII T3) yields a unique such process for given σ-algebras, $\{\mathcal{H}_{s+}\}$. Thus $\hat{\Lambda}[0,t]$ must be the "compensator" of Brown (1978) and the "integrated rate" of Segall and Kailath, (1975a).

Moreoever, a comparison of our results with these approaches and with Kallenberg's, shows room for improvement on two fronts. It should be possible to weaken the condition that μ is σ-finite by considering the interval function, $P[N[u,v) = 1|\mathcal{H}_u]$, only. Also, it should be possible to show the existence of conditional intensities with respect to any suitable family, $\{\mathcal{H}_s\}$. In particular $\hat{\Lambda}_H$, which was defined at (5.7) should exist under very general conditions. I have only discussed the very special case where $\hat{\Lambda}_H = \Lambda$.

CHAPTER III

A Central Limit Theorem for the Generalised Shot-Noise Process

§1. Introduction

The aim of this chapter is to prove a very general limit theorem, discarding most of the independence restrictions of Chapter I. The point process is allowed to be non-Poisson, while $H(t,s)$ can depend both on point process and H-process events. We must, however, retain some of the structure. The model is still additive and τ_i is still to be interpreted as the starting point of $H(\cdot,\tau_i)$. Conditionally on their pasts, the different $H(\cdot,\tau_i)$ evolve independently, while dependence of the point process on past H-process events, is restricted to allow the exploitation of Corollary II.5.4 and its consequences. These conditions are detailed in §2, along with others of a technical, rather than structural, nature.

We outline three possible approaches to the problem. Firstly, we could adopt the method used by Daley (1972) for the special case of stationary cluster point processes. We would then suppose that each $H(t,\tau_j)$ has a limit $H(\infty,\tau_j)$, and that these are identically and independently distributed with mean $\mathbb{E}H_\infty < \infty$. Recall that $\mu[0,t] = \mathbb{E}N(t)$, so that by II.3.8, $\mathbb{E}Y(t) = \int_0^t \mathbb{E}_s[H(t,s)]\mu(ds)$. We can write

$$Y(t) - \mathbb{E}Y(t) = \int_0^t [H(\infty,s) - \mathbb{E}H_\infty]dN(s)$$

$$+ [N(t) - \mu[0,t]]\mathbb{E}H_\infty$$

$$+ \Delta(t) \qquad \ldots \qquad (1.1)$$

where

$$\Delta(t) = \int_0^t [H(t,s) - H(\infty,s)]dN(s) - \int_0^t [\mathbb{E}_s H(t,s) - \mathbb{E}H_\infty]\mu(ds).$$

Since the first term is a random sum of i.i.d. random variables, we can use well-known theorems, see, for instance, Wittenberg (1964), to give conditions for its convergence, suitably normed, to the normal distribution, or to stable laws with index $0 < \alpha < 2$. Of course, if the mean is infinite, then another centering will be necessary.

The second term in (1.1), after suitable norming may also have a limiting distribution, while under sufficiently stringent conditions the remainder, Δ_t, will tend in probability to zero.

The limit distribution for Z(t) will depend on which of the first two terms becomes dominant as $t \to \infty$, with the possibility of a mixture of the two limit distributions occurring when neither dominates. Note that Wittenberg's results would allow us to treat even the cluster process case in greater generality than Daley, since N(t) need not be independent of the $H(\infty,\tau_j)$'s.

Nonetheless, there are several grounds for dissatisfaction. The conditions derived compare unfavourably with the compactness of those in Chapter I. While it is possible to generalise them to cover $H(s) \sim s^\alpha H_\infty$ or $H(\infty,\tau_j)$ non-identically distributed, the results become increasingly clumsy. Finally, expression (1) suggests that $\mathbb{E}H_\infty\mu[0,t]$ should be the natural centering term; for then the second term in $\Delta(t)$ would be unnecessary. Yet we know from Chapter I, that centering by EY(t) gives the simpler conditions.

A second line of attack would be to attempt to find an expression for the characteristic function of Z(t) in the spirit of (I.3.2).

Suppose that, conditionally on $\mathcal{F}[0,t]$, the $H(t,\tau_j)$, $\tau_j \in [0,t]$ are independent. Then

$$\mathbb{E}e^{i\theta Y(t)} = \mathbb{E}\prod_{j=1}^{N(t)} \mathbb{E}\left[e^{i\theta H(t,\tau_j)} \middle| \mathcal{F}[0,t]\right]$$

$$= \mathbb{E} \exp \int_0^t \log \mathbb{E}\left[e^{i\theta H(t,s)} \middle| \mathcal{F}[0,t]\right] dN(s) \qquad \ldots \quad (1.2)$$

This leads to an expression in terms of what Westcott (1972) has called the KLM measure, c.f. Matthes (1972), Daley and Vere-Jones (1972). However, this does not appear to yield the neat and unified solution we seek.

The approach which we do adopt below, is motivated by three considerations. Firstly, the basic idea of the central limit theorem of Chapter I, is to summarise the observed process, $Y(t)$, in terms of two "smooth" functions, μ_t and σ_t, and an additional "noise" component, usually a $N(0,1)$ limiting random variable. In the most common case, where we take $\mu_t = \mathbb{E}Y(t)$, this noise component arises as a form of Cesaro limit of the increments

$$H(t,s)dN(s) - \mathbb{E}H(t,s)\lambda(s)ds,$$

each of which has expectation zero. A natural generalisation in the dependent case, is to consider the increments

$$H(t,s)dN(s) - \mathbb{E}_s[H(t,s)\,|\,\mathcal{H}_s]\,\hat{\lambda}(s)ds,$$

which have conditional mean zero. We might reasonably expect that the limit in distribution of

$$Z(t) = \frac{1}{\sigma_t}\left[Y(t) - \int_0^t \mathbb{E}_s[H(t,s)\,|\,\mathcal{H}_s]\hat{\lambda}(s)ds\right] \qquad \ldots \quad (1.3)$$

would yield, for suitable σ_t, a non-degenerate variable, say N(0,1). Of course, a crucial property of μ_t and σ_t is that, given some model, we should be able to calculate them and then examine the distribution of the deviations, $\frac{Y(t) - \mu_t}{\sigma_t}$. Now, it may be that in simple cases $\int_0^t \mathbb{E}_s[H(t,s)\,|\mathcal{H}_s]\hat{\lambda}(s)ds$ can be calculated from <u>observable</u> random variables occurring prior to s. In this case, it will make a suitable centering function. Otherwise, we would hope to approximate it by a more practical centering, perhaps even $\mathbb{E}Y(t)$. We would, then, also be interested in the limiting behaviour of

$$\frac{1}{C_t}\left[\int_0^t \mathbb{E}_s[H(t,s)\,|\mathcal{H}_s]\hat{\lambda}(s)ds - \mathbb{E}Y(t)\right] \qquad \ldots \quad (1.4)$$

for suitable norming functions C_t. The ultimate limit theorem might sometimes involve a convolution of non-degenerate limiting distributions arising from (1.3) and (1.4). However, we will not investigate this possibility below.

The second line of thought which inspired this approach stemmed from the observation that the results of Chapter I are similar in spirit to those for sums of independent, non-identically distributed random variables. It would, therefore, not be surprising if, in the dependent case, we paralleled known central limit theorems for discrete-time martingale sequences. The methods used were, in fact, suggested by those of Brown and Eagleson (1971) for triangular arrays of martingale differences. Subsequent work by Eagleson (1975) and Rootzen (1977a,b) has indicated the further possibility of using a random norming function, $\sigma_t(\omega)$, but this will not be attempted here.

Our final motivation is to try to emulate the successful use of Feller's canonical measure in Chapter I. The crucial step is to develop a tractable analogue of expression (I.3.2) for the log characteristic function. The main result of Section 2 provides this by showing that a sufficient condition for convergence of the characteristic function of $Z(t)$, is a.s convergence of $\mathbb{C}_t(\theta;\omega)$ where

$$\log \mathbb{C}_t(\theta;\omega) = \int_{-\infty}^{\infty} \frac{(e^{i\theta y} - 1 - i\theta\tau(y))}{y^2} \cdot y^2 \int_0^t \hat{F}_H[\sigma_t dy; t,s;\omega] d\hat{\Lambda}(s,\omega)$$

with $\hat{F}_H[x;t,s;\omega] = P_s\{H(t,s) \leq x | \mathcal{H}_s\}(\omega)$ and $\tau(y)$ an appropriate centering function.

Section 3 is then devoted to a theory of random canonical measures, which extends Feller's results to deal with convergence of $\mathbb{C}_t$. These results are used in Section 4 to derive sufficient conditions for the convergence in distribution of $Z(t)$ to infinitely divisible laws and, in particular, to the normal distribution.

§2. A Representation Formula for the Characteristic Function

In this section we will derive a manageable formula for the characteristic function of

$$Z(t;\omega) = \frac{Y(t;\omega) - \mu_t(\omega)}{\sigma_t}, \quad t \geq 0$$

where $\mu_t(\omega)$ and σ_t are suitable random centering and non-random norming functions, respectively. We first list the conditions under which the formula will be derived. These are additional to the, possibly redundant, conditions II.2.12 and II.3.10 and to the definition of the underlying probability space $(\Omega,\mathcal{H},P)$, given in Chapter II.

Firstly, concerning the point process, we assume that:-

it has no multiple occurrences and the 1st moment measure, μ, is σ-finite and non-atomic; ... ... (2.1)

for each interval, $[u,v)$, there is a constant M, such that, for each realization ω,

$$\mathbb{E}\left[N[x,x+h)\,|\mathcal{H}_x\right](\omega) \leqslant M\mathbb{E}\left[N[x,x+h)\,|\mathcal{G}_x\right](\omega) \quad \ldots \quad \ldots \quad (2.2)$$

for h sufficiently small, uniformly in $x \in [u,v)$.

From (2.1) we can conclude that, almost surely, $\hat{\Lambda}$ exists and is finite. Furthermore, it is non-atomic, since $\mathbb{E}\hat{\Lambda}(Q) = \mu(Q)$, $Q \in \mathcal{B}$. Condition (2.2) is the "limited feedback" condition which ensures the validity of Corollary II.5.4 and, *a fortiori*, of Theorem II.5.3. The important case when a.s.

$$\mathbb{E}\left[N[x,x+h)\,|\mathcal{H}_x\right] = \mathbb{E}\left[N[x,x+h)\,|\mathcal{G}_x\right],$$

is clearly included.

Strictly speaking, for any specific model, condition (2.2) can be verified only up to a collection of P-null exceptional ω-sets indexed by the intervals $[u,v)$. However, the separability arguments familiar from Chapter II, show that the union of these sets is P-null. We may, therefore, choose versions of our conditional expectations such that (2.2) is trivially true on the exceptional set. For instance,

$$\mathbb{E}\left[N[x,x+h)\,|\mathcal{H}_x\right](\omega) = \mathbb{E}\left[N[x,x+h)|\,\mathcal{G}_x\right](\omega) = 0,$$

for exceptional ω, will do.

Concerning the secondary processes, H, we assume that:-

$H(t,s)$ is right-continuous with left limits in t and s simultaneously and that

$$H(t,s) = 0, \quad \forall t < s; \quad \ldots \quad \ldots \quad (2.3)$$

for each s, conditionally on $\mathcal{H}_s$, H(t,s) is independent of H(t,u) wrt the Palm distribution P_s, for every $u < s$ and $t \geqslant s$. ... ... (2.4)

Condition (2.4) ensures that, once started, each H-process evolves independently of its precursors.

A similar comment applies to (2.4) as did to (2.2). For each s, the condition may be verified up to a P_s-null set. We then choose versions for which (2.3) holds trivially.

Thirdly, concerning the centering and norming functions, we assume that:-

σ_t is deterministic and right-continuous with left-limits; (2.5)

$$\mu_t(\omega) = \int_0^t E_s[\tau\{H(t,s)\}|\mathcal{H}_s](\omega)\, d\hat{\Lambda}(s,\omega) \quad \ldots \quad \ldots \quad (2.6)$$

where τ is a real-valued, right-continuous function with left limits.

The final condition, we introduce largely for its temporary convenience. It will be removed in Section 4 and so is not as restrictive as it may at first seem. We assume that:-

for each θ, there exists a constant $C \in (0,\infty)$, not depending on t, such that

$$\int_0^t E_s[1 - \cos\{\theta\sigma_t^{-1} H(t,s)\}|\mathcal{H}_s]\, d\hat{\Lambda}(s) \leqslant C, \text{ a.s.} \quad \ldots \quad \ldots \quad (2.7)$$

Notice that, since $1 - \cos x \leqslant \frac{1}{2}x^2$, a special case of (2.7) is when

$$\frac{1}{\sigma_t^2} \int_0^t E_s[H^2(t,s)|\mathcal{H}_s]\, d\hat{\Lambda}(s) \leqslant C, \text{ a.s.}.$$

If, now, we seek an expression for $E[e^{i\theta Z(t)}]$, a major difficulty is that $e^{i\theta Y(t)}$ is not an additive function of the $H(t,\tau_i)$ so that the

methods of Chapter II are not immediately available, as they were in Example II.5.1. This is exemplified by the invalidity of (1.2) in the absence of independence. However, we can adapt a standard trick, used in proving central limit theorems for discrete time martingale sequences, to introduce an additive structure into the problem, c.f. Brown and Eagleson (1971).

We define

$$g_\theta(x) = e^{i\theta x} - 1 - i\theta\tau(x)$$

and

$$\mathbb{C}_t[\theta\ \omega] = \exp\{\int_0^t \mathbb{E}_s[g_{\theta\sigma_t^{-1}}\{H(t,s)\}|\mathcal{H}_s](\omega)\, d\hat{\Lambda}(s,\omega)\} \quad \ldots \quad (2.8)$$

Although ideally we would like to prove that

$$\mathbb{E}e^{i\theta Z(t)} = \mathbb{E}\mathbb{C}_t[\theta]$$

we have to be content with the weaker statement that for each θ and t,

$$\mathbb{E}\left[e^{i\theta Z(t)}\, \mathbb{C}_t[\theta]^{-1}\right] = 1 \ .$$

We will require a formula for integration by parts, when the measures concerned need not be absolutely continuous. The following Lemma is a straightforward extension of the result given by Meyer (1966), p.114. In both Lemmas 2.1 and 2.2 we consider a fixed realization ω and a fixed time t.

Lemma 2.1 Suppose $e(s)$ and $g(s)$ are $\mathbb{C}$-valued functions, integrable with respect to N and $\hat{\Lambda}$ respectively.

Let $$E(s) = \int_{-\infty}^{s} e(u)\ dN(u)$$

and $$G(s) = \int_{-\infty}^{s} g(u)\ d\hat{\Lambda}(u) \ .$$

Then $$\int_0^t E(s-)\, g(s)\, d\hat{\Lambda}(s) + \int_0^t G(s)\, e(s)\, dN(s) = E(t)\, G(t) - E(0-)\, G(0-) \ .$$

Proof Since $e(s)\,g(u)$ is integrable with respect to $dN(s)\,d\hat{\Lambda}(u)$, we may apply Fubini's Theorem.

$$\text{Thus } \int_0^t \int_u^t e(s)\,g(u)\,dN(s)\,d\hat{\Lambda}(u) = \int_0^t [E(t) - E(u-)]\,g(u)\,d\hat{\Lambda}(u) \quad \ldots \quad (2.9)$$

while

$$\int_0^t \int_0^{u-} e(s)\,g(u)\,dN(s)\,d\hat{\Lambda}(u) = \int_0^t \int_{s+}^t g(u)\,d\hat{\Lambda}(u)\,e(s)\,dN(s)$$

$$= \int_0^t [G(t) - G(s)]\,e(s)\,dN(s) \quad \ldots \quad (2.10)$$

$$\text{Also } \int_0^t \int_0^t e(s)\,g(u)\,dN(s)\,d\hat{\Lambda}(u) = [E(t) - E(0-)][G(t) - G(0-)] \quad \ldots \quad (2.11)$$

Since (2.9) plus (2.10) equals (2.11) the lemma follows easily.

Lemma 2.2 Let $f(s)$ be $\mathbb{C}$-valued and integrable with respect to $\hat{\Lambda}$. Then

$$W_t[\theta] = \exp\{i\theta Y(t) - \int_0^t f(s)\,d\hat{\Lambda}(s)\} - 1$$

$$= \int_0^t \exp\{i\theta Y(s-) - \int_0^s f(u)\,d\hat{\Lambda}(u)\} \cdot$$

$$\times [(e^{i\theta H(t,s)} - 1)dN(s) - f(s)\,d\hat{\Lambda}(s)] \quad \ldots \quad (2.12)$$

Proof We adopt the convention that $\int_0^s$ is zero if $s < 0$.

Let $E(t) = e^{i\theta Y(t)} = \exp\{i\theta \int_0^t H(t,s)\,dN(s)\}$, $t \geq 0$,

and $G(t) = \exp\{-\int_0^t f(s)\,d\hat{\Lambda}(s)\}$, $t \geq 0$. For all $t < 0$, define $E(t) = G(t) = 1$.

Thus

$$W_t[\theta] = E(t)\,G(t) - E(0-)\,G(0-).$$

To apply Lemma 2.1, we must find suitable "derivatives" for these functions.

Let $e(s) = \exp\{i\theta\int_0^{s-} H(t,u)\,dN(u)\}\,[e^{i\theta H(t,s)} - 1]$, $s \geq 0$

and, for $s < 0$, choose $e(s)$ such that $\int_{-\infty}^{0-} e(s)\,dN(s) = 1$. Then, considering $\int_0^s e(u)\,dN(u)$ as a finite sum, we see that it equals $\exp\{i\theta\int_0^s H(t,u)\,dN(u)\} - 1$ so that

$$\int_{-\infty}^{s} e(u)\,dN(u) = E(s)\ .$$

Similarly, define $g(s) = -\exp\{-\int_0^s f(u)\,d\hat{\Lambda}(u)\}\,f(s)$, $s > 0$, and, for $s < 0$, choose $g(s)$ such that $\int_{-\infty}^{0-} g(s)\,d\hat{\Lambda}(s) = 1$. If we let $J(s) = \int_0^s f(u)\,d\hat{\Lambda}(u)$, then

$$\int_0^s g(u)\,d\hat{\Lambda}(u) = -\int_0^s e^{-J(u)}\,dJ(u) = \int_C [-e^{-z}]\,dz\ ,$$

where the curve $C = \{z = J(u);\ 0 \leq u \leq s\}$ is rectifiable. Therefore,

$$\int_0^s g(u)\,d\hat{\Lambda}(u) = e^{-J(s)} - 1$$

so that

$$\int_{-\infty}^{s} g(u)\,d\hat{\Lambda}(u) = G(s)\ .$$

Now e is clearly N-integrable, while g is $\hat{\Lambda}$-integrable provided f is. An integration by parts now establishes (2.12).

If, next, we take

$$f(s) = E_s[e^{i\theta\sigma_t^{-1}H(t,s)} - 1|\mathcal{H}_s]$$

and introduce the abbreviation

$$U(s) = i\theta\sigma_t^{-1}\int_0^{s-} H(t,u)\,dN(u) - \int_0^s E_u[e^{i\theta\sigma_t^{-1}H(t,u)} - 1|\mathcal{H}_u]d\hat{\Lambda}(u)$$

then, by Lemma 2.2, we have

$$W_t[\theta\sigma_t^{-1}]$$

$$= \int_0^t e^{U(s)}\{(e^{i\theta\sigma_t^{-1}H(t,s)} - 1)dN(s) - E_s[e^{i\theta\sigma_t^{-1}H(t,s)} - 1|\mathcal{H}_s]d\hat{\Lambda}(s)\} \quad \ldots \quad (2.13)$$

Note that this integral has expected "increments" of zero, conditionally on the past, $\mathcal{H}_s$. Indeed, we will prove that, under assumptions (2.1) to (2.7),

$$EW_t[\theta\sigma_t^{-1}] = 0 .$$

Lemma 2.3 For each $t > 0$,

$$E\int_0^t e^{U(s)}(e^{i\theta\sigma_t^{-1}H(t,s)} - 1)\,dN(s)$$

$$= E\int_0^t E_s[e^{U(s)}|\mathcal{H}_s]\,E_s[e^{i\theta\sigma_t^{-1}H(t,s)} - 1|\mathcal{H}_s]\,d\hat{\Lambda}(s) \quad \ldots \quad \ldots \quad (2.14)$$

Proof Note that, for $Q \in \mathcal{P}[0,t)$, $N(Q)$ is $\mathcal{G}_s$ and hence $\mathcal{H}_s$-measurable. By assumption (2.4), conditionally on $\mathcal{H}_s$, $H(t,s)$ is a.s. independent of $\int_0^{s-} H(t,u)\,dN(u)$. Since $\int_0^s E_u[e^{i\theta\sigma_t^{-1}H(t,u)} - 1|\mathcal{H}_u]\,d\hat{\Lambda}(u)$ is also $\mathcal{H}_s$-measurable, the random variable $U(s)$ is conditionally independent of $H(t,s)$. It follows that

$$E_s[e^{U(s)}(e^{i\theta\sigma_t^{-1}H(t,s)} - 1)|\mathcal{H}_s]$$

$$= E_s[e^{U(s)}|\mathcal{H}_s]E_s[e^{i\theta\sigma_t^{-1}H(t,s)} - 1)|\mathcal{H}_s].$$

Now $\quad \mathrm{Re}[e^{i\theta\sigma_t^{-1}H(t,s)} - 1] = \cos\theta\sigma_t^{-1}H(t,s) - 1 .$

Thus, by assumption (2.7),

$$|e^{U(s)}| = \exp\int_0^s E_u[1 - \cos\{\theta\sigma_t^{-1}H(t,u)\}|\mathcal{H}_u]\, d\hat{\Lambda}(u)$$

$$\leq e^C , \text{ a.s.}$$

$$\therefore \quad \int_0^t |E_s[e^{U(s)}|\mathcal{H}_s]\, E_s[e^{i\theta\sigma_t^{-1}H(t,s)} - 1|\mathcal{H}_s]|\, d\hat{\Lambda}(s)$$

$$\leq 2e^C\hat{\Lambda}[0,t] , \text{ a.s.} \qquad \ldots \quad \ldots \qquad (2.15)$$

and so has finite expectation.

We may apply Theorem II.5.3(b) to deduce the lemma.

Lemma 2.4 For all $t > 0$,

$$E\int_0^t e^{U(s)}\, E_s[e^{i\theta\sigma_t^{-1}H(t,s)} - 1|\mathcal{H}_s]d\hat{\Lambda}(s)$$

$$= E\int_0^t E_s[e^{U(s)}|\mathcal{H}_s]\, E_s[e^{i\theta\sigma_t^{-1}H(t,s)} - 1|\mathcal{H}_s]\, d\hat{\Lambda}(s).$$

Proof This follows immediately from Theorem II.5.3(a) and (2.15).

Theorem 2.5 Under conditions (2.1) - (2.7),

$$E\left[e^{i\theta Z(t)}\, \mathbb{L}_t[\theta]^{-1}\right] = 1 , \text{ for each } t > 0.$$

Proof Recall that $Z(t) = \sigma_t^{-1}\left[Y(t) - \int_0^t \mathbb{E}_s[\tau\{H(t,s)\}|\mathcal{H}_s]\, d\hat{\Lambda}(s)\right]$

Now, $\mathbb{E}_s[\sigma_t^{-1}\tau\{H(t,s)\}|\mathcal{H}_s] = \sigma_t^{-1}\mathbb{E}_s[\tau\{H(t,s)\}|\mathcal{H}_s]$, since σ_t is deterministic.

$\therefore$ From (2.8)

$$\log \mathbb{C}_t[\theta] = \int_0^t \mathbb{E}_s[e^{i\theta\sigma_t^{-1}H(t,s)} - 1|\mathcal{H}_s]\, d\hat{\Lambda}(s) - i\theta\sigma_t^{-1}\int_0^t \mathbb{E}_s[\tau\{H(t,s)\}|\mathcal{H}_s]\, d\hat{\Lambda}(s)$$

$\therefore$ $e^{i\theta Z(t)}\mathbb{C}_t[\theta]^{-1}$

$$= \exp\left\{i\theta\sigma_t^{-1}\int_0^t H(t,s)dN(s) - \int_0^t \mathbb{E}_s[e^{i\theta\sigma_t^{-1}H(t,s)} - 1|\mathcal{H}_s]d\hat{\Lambda}(s)\right\}$$

$$= W_t[\theta\sigma_t^{-1}] + 1,$$

if we take $f(s) = \mathbb{E}_s[e^{i\theta\sigma_t^{-1}H(t,s)} - 1|\mathcal{H}_s]$ in the definition of W_t.

From (2.13), $W_t[\theta\sigma_t^{-1}]$ consists of two terms. By Lemmas 2.3 and 2.4 each term has the same expectation, given by

$$\mathbb{E}\int_0^t \mathbb{E}_s[e^{U(s)}|\mathcal{H}_s]\,\mathbb{E}_s[e^{i\theta\sigma_t^{-1}H(t,s)} - 1|\mathcal{H}_s]\, d\hat{\Lambda}(s),$$

so that $\mathbb{E}W_t[\theta\sigma_t^{-1}] = 0$ and the theorem is proved.

Theorem 2.6 Suppose conditions (2.1) to (2.7) hold and that $\mathbb{C}[\theta]$ is a (non-random) function, continuous at the origin. If

$$\mathbb{C}_t[\theta] \xrightarrow{a.s.} \mathbb{C}[\theta], \text{ as } t \to \infty. \quad \ldots \quad (2.16)$$

then $$\mathbb{E}[e^{i\theta Z(t)}] \longrightarrow \mathbb{C}[\theta], \text{ as } t \to \infty. \quad \ldots \quad (2.17)$$

$\mathbb{C}$ is the characteristic function of the limit law of $Z(t)$ as $t \to \infty$.

Proof Notice that $|\mathbb{C}_t[\theta]| \leq 1$, a.s. for all θ,t so that $|\mathbb{C}[\theta]| \leq 1$, $\forall\theta$, also. Now

$$|\mathbb{E}[e^{i\theta Z(t)}] - \mathbb{C}[\theta]|$$

$$\leq |\mathbb{E}[e^{i\theta Z(t)}\mathbb{C}_t[\theta]^{-1}] - 1|.|\mathbb{C}[\theta]|$$

$$+ \mathbb{E}\left[|e^{i\theta Z(t)}\mathbb{C}[\theta]|.|\mathbb{C}[\theta]^{-1} - \mathbb{C}_t[\theta]^{-1}|\right].$$

The first term is zero by the previous theorem, while $|e^{i\theta Z(t)}\mathbb{C}[\theta]| \leq 1$. By condition (2.7), just as in the derivation of (2.15), $|\mathbb{C}_t[\theta]^{-1}| \leq e^{C}$, a.s., this condition holding simultaneously for all $t \geq 0$. Dominated convergence yields

$$\mathbb{E}|\mathbb{C}[\theta]^{-1} - \mathbb{C}_t[\theta]^{-1}| \to 0$$

and (2.17) is proved. The last part of Theorem 2.6 follows from Levy's continuity theorem.

Theorem 2.6 focusses our attention on the convergence of $\mathbb{C}_t[\theta]$. If we define

$$\hat{F}_H(x;t,s;\omega) = P_s\{H(t,s) \leq x|\mathcal{H}_s\}(\omega)$$

and

$$M_t(dx;\omega) = x^2\int_0^t \hat{F}_H(\sigma_t dx;t,s;\omega)\,d\hat{\Lambda}(s,\omega) \quad \ldots \quad (2.18)$$

then

$$\log \mathbb{C}_t[\theta;\omega] = \int_{-\infty}^{\infty} \frac{e^{i\theta x} - 1 - i\theta\tau(x)}{x^2} M_t(dx;\omega). \quad \ldots \quad (2.19)$$

It follows that we are interested in the convergence properties of random measures like M_t.

§3. Random Measures

Let us call M a random measure of order k, or r.m.(k), if:-

(a) for P - almost all ω, $M(\cdot,\omega)$ is a measure on $(\mathbb{R},\mathcal{B})$ such that $M(I,\omega) < \infty$, for all finite intervals, while, for all $x > 0$,

$$M^+(x;\omega) = \int_x^\infty y^{-k} M(dy;\omega) < \infty \text{ and } M^-(-x;\omega) = \int_{-\infty}^{-x} |y|^{-k} M(dy;\omega) < \infty;$$

(b) for each $Q \in \mathcal{B}$, $M(Q,\cdot)$ is $\mathcal{H}$-measurable.

Two special cases deserve mention. An r.m.(2) is a random version of Feller's canonical measure while for an r.m.(0), $M(\mathbb{R}) < \infty$ a.s, so that we usually refer to it as an a.s. finite random measure.

Let C_M be the set in $\mathbb{R}$:

{x s.t. $M(\cdot;\omega)$ is continuous at x for each ω}

Then $\mathcal{R}_k$ will denote the class of r.m.(k)'s for which C_M is dense in $\mathbb{R}$.

A process M_t, $t \in \mathbb{R}$, is in the class $\mathcal{M}_k$ if:-

(a) M_t is an r.m.(k) for each t;

(b) for each bounded I and $x > 0$, $M_t(I)$, $M_t^+(x)$ and $M_t^-(-x)$ are separable stochastic processes.

Now suppose $M_t \in \mathcal{M}_k$ and $M \in \mathcal{R}_k$. Then M_t converges almost surely to M in the class $\mathcal{M}_k$, written $M_t \xrightarrow{\mathcal{M}_k} M$ a.s. if:-

$$M_t(I) \xrightarrow{a.s.} M(I), \text{ as } t \to \infty, \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

for all bounded intervals of continuity for $M(\cdot,\omega)$ for all ω, while

$$M_t^+(x) \xrightarrow{a.s.} M(x) \text{ and } M_t^-(-x) \xrightarrow{a.s.} M(-x), \ t \to \infty \quad \ldots \quad (3.2)$$

for every $x \in C_M$.

In the special case of a.s. finite r.m.'s, $\mathfrak{M}_0$ convergence reduces to a.s. complete convergence. We write $M_t \overset{C}{\to} M$ a.s., if (3.1) holds for all I bounded or unbounded. However, if (3.1) holds for bounded I only, we have a.s. weak convergence written $M_t \overset{W}{\to} M$ a.s.

For convenience, we point out here an obvious property of any a.s. convergent r.m. process which we will call a.s. stochastic boundedness. By analogy with Feller's usage, (1971, p.254) a separable stochastic process, X_t, is a.s. stochastically bounded if, for each $\varepsilon > 0$, there exist constants C and t_0 s.t.

$$P\{ \sup_{t > t_0} |X_t| > C\} < \varepsilon .$$

Clearly if M_t converges a.s. in $\mathfrak{M}_k$, then it must be a.s. stochastically bounded in the sense that $M_t(I)$, $M_t^{+}(x)$ and $M_t^{-}(-x)$ must all be a.s. stochastically bounded for all appropriate I and x.

Our first requirement is for analogues of the Helly-Bray Lemmas, Loève (1963, p.180-182) for r.m.(k) processes. We prove these first for a.s. finite r.m.'s and then extend them to order k. Since the essence of the proofs is in Loève, we merely state the additional points. Brown and Eagleson (1971) have proved them, by implication, in the course of their Theorem 1 but a more systematic approach seems worthwhile.

Lemma 3.1 Let $a, b \in C_M$, $g(x)$ be a continuous function and $M_t \overset{W}{\to} M$ a.s., $t \to \infty$. Then

$$\int_a^b g(x)\, M_t(dx) \xrightarrow{\text{a.s.}} \int_a^b g(x)\, M(dx) .$$

Proof For each $m \in \mathbb{N}$, choose a partition

$$a < x_0 < x_1 < \text{- - - -} < x_n = b$$

s.t. $\sup_{1\leq j\leq m} (x_j - x_{j-1}) \to 0$ as $m \to \infty$ and all the $x_j \in C_M$.

Let $g_m(x) = \sum_{j=1}^{m} g(x_{j-1}) I_{[x_{j-1},x_j)}(x)$ and put $k = \sup_{[a,b]} |g(x)|$.

Now, by a.s. stochastic boundedness, given $\delta, \eta > 0$, we can choose ε, t_0 s.t.

$$P\{ \sup_{t \geq t_0} M_t[a,b] + M[a,b] > \delta/2\varepsilon \} < \eta$$

Choose m s.t. $\sup_{[a,b]} |g(x) - g_m(x)| < \varepsilon$ and $t_1 \geq t_0$ s.t.

$$P\{ \sup_{1\leq j\leq m} |M_t[x_{j-1},x_j) - M[x_{j-1},x_j)| > \delta/2mk, \text{ for any } t \geq t_1 \} < \eta .$$

Just as in Loève (1963, p.180-1),

$$\left| \int_a^b g \, dM_t - \int_a^b g \, dM \right|$$

$$\leq \varepsilon \{ M_t[a,b] + M[a,b] \}$$

$$+ mk \sup_{j\leq m} |M_t[x_j,x_j) - M[x_{j-1},x_j)|$$

$$\therefore \quad P\{ \sup_{t\geq t_1} \left| \int_a^b g \, dM_t - \int_a^b g \, dM \right| > \delta \} < 2\eta .$$

Remark The restriction that the x_j be in C_M, leads me to consider only limits $M \in \mathcal{R}_k$. This may well be too narrow a class. For instance, in a different context, we might want to consider a.s. convergence to the Poisson counting measure, which clearly is not in $\mathcal{R}$.

Lemma 3.2 Suppose M_t is a.s. finite and $M_t \overset{w}{\to} M$ a.s. If $g(x) \to 0$, as $|x| \to \infty$, and is continuous then

$$\int_{-\infty}^{\infty} g \, dM_t \xrightarrow{a.s.} \int_{-\infty}^{\infty} g \, dM$$

Proof Again, as in Loève, p.181

$$\left| \int_{-\infty}^{\infty} g \, dM_t - \int_{-\infty}^{\infty} g \, dM \right|$$

$$\leq \sup_{x \notin [-b,a]} |g(x)| \{M_t(\mathbb{R}) + M(\mathbb{R})\} + \left| \int_{-b}^{a} g \, dM_t - \int_{-b}^{a} g \, dM \right|$$

Now, by Lemma 3.1, the 2nd term tends a.s. to zero. Given $\delta, \eta > 0$ choose ε, t_0 s.t.

$$P\{\sup_{t \geq t_0} M_t(\mathbb{R}) + M(\mathbb{R}) > \delta/2\varepsilon\} < \eta$$

Then choose $[-b,a]$ s.t. $\sup_{x \notin [-b,a]} |g(x)| < \varepsilon$ and $t_1 \geq t_0$ s.t.

$$P\left\{\sup_{t \geq t_1} \left| \int_{-b}^{a} g \, dM_t - \int_{-b}^{a} g \, dM \right| > \delta/2 \right\} < \eta .$$

Then

$$P\left\{\sup_{t \geq t_1} \left| \int_{-\infty}^{\infty} g \, dM_t - \int_{-\infty}^{\infty} g \, dM \right| > \delta \right\} < 2\eta .$$

Lemma 3.3 Suppose M_t is a.s. finite and $M_t \overset{c}{\to} M$, a.s. If g is bounded and continuous then

$$\int_{-\infty}^{\infty} g \, dM_t \xrightarrow{a.s.} \int_{-\infty}^{\infty} g \, dM .$$

Proof Since $M_t \xrightarrow{a.s.} M$ completely, given ε, $\delta > 0$, $\exists$ t_0, A s.t.

$$P\{\sup_{t \geq t_0} M_t[a,\infty) > \delta\} < \varepsilon, \text{ for } a \geq A,$$

with similar relationships holding for $M_t(-\infty,-b]$, $M[a,\infty)$ and $M(-\infty,-b]$. The proof then follows as in Loève, p.181-2.

Lemma 3.4 If $M_t \xrightarrow{\mathfrak{M}_k} M$ and g(x) is continuous and $O(x^{-k})$, as $|x| \to \infty$, then

$$\int_{-\infty}^{\infty} g(x) \, dM_t \longrightarrow \int_{-\infty}^{\infty} g(x) \, dM, \quad t \to \infty .$$

Proof Choose $a \in C_M$ s.t. $a > 0$. The measure $x^{-k} M_t(dx)$ on $[a,\infty)$ is finite and a.s. completely convergent to $x^{-k} M(dx)$. Since $x^k g(x)$ is bounded and continuous, we deduce from Lemma 3.3 that

$$\int_{a}^{\infty} g \, dM_t \xrightarrow{a.s.} \int_{a}^{\infty} g \, dM .$$

The convergence for $(-\infty,-b]$, for $b > 0$, $-b \in C_M$ follows similarly, while Lemma 3.1 yields convergence over $[-b,a]$, immediately.

Remark An immediate consequence of Lemma 3.4 is that $\mathfrak{M}_k$ convergence implies $\mathfrak{M}_\ell$ convergence for any $\ell > k$, and, in particular, a.s. complete convergence of an a.s. finite r.m. process implies its $\mathfrak{M}_\ell$ convergence, for any $\ell > 0$.

We now investigate convergence of expressions like (2.19) by considering the special case of random canonical measures, i.e. r.m.(2)'s.

Theorem 3.5 Suppose $M_t \in \mathcal{M}_2$, $M \in \mathcal{R}_2$

Define

$$\mathbb{C}_t[\theta] = \int_{-\infty}^{\infty} \frac{e^{i\theta x} - 1 - i\theta \sin x}{x^2} M_t(dx) + i\theta C_t$$

and

$$\mathbb{C}[\theta] = \int_{-\infty}^{\infty} \frac{e^{i\theta x} - 1 - i\theta \sin x}{x^2} M(dx) + i\theta C$$

for some separable process, C_t, and r.v. C. If

$$M_t \xrightarrow{\mathcal{M}_2} M, \text{ a.s. and } C_t \xrightarrow{\text{a.s.}} C$$

then

$$\mathbb{C}_t[\theta] \xrightarrow{\text{a.s.}} \mathbb{C}[\theta]$$

Proof Since $e^{i\theta x} - 1 - i\theta \sin x$ is bounded and $O(x^2)$, as $|x| \to 0$, $\frac{e^{i\theta x} - 1 - i\theta \sin x}{x^2}$ satisfies the conditions of Lemma 3.4 with $k = 2$.

The theorem follows immediately.

Corollary 3.6(a) If, in Theorem 3.5, $M_t \xrightarrow{\mathcal{M}_1} M$ a.s., we may replace the centering function $\sin x$ by x and omit the term $i\theta C$.

(b) In particular, this is valid if $M_t \in \mathcal{M}_{1-\delta}$ and $M_t \xrightarrow{\mathcal{M}_2} M$, a.s.

Proof We require to prove that

$$\int_{-\infty}^{\infty} \frac{\sin x - x}{x^2} M_t(dx) \xrightarrow{a.s.} \int_{-\infty}^{\infty} \frac{\sin x - x}{x^2} M(dx)$$

(a) $\frac{\sin x - x}{x^2}$ is continuous and $O(x^{-1})$, as $|x| \to \infty$ so we can apply Lemma 3.4 with $k = 1$ to give the a.s. convergence required.

(b) The measure $x^{-(1-\delta)} M_t(dx)$, defined away from the origin, is almost surely finite, by assumption, and weakly convergent to $x^{-(1-\delta)} M(dx)$, by Lemma 3.1. Writing $x^{-1} M_t(dx) = x^{-\delta} x^{-(1-\delta)} M_t(dx)$, we can apply Lemma 3.2 to prove that, for $a > 0$,

$$\int_a^{\infty} x^{-1} M_t(dx) \xrightarrow{a.s.} \int_a^{\infty} x^{-1} M(dx), \text{ as } t \to \infty .$$

A similar convergence holds for the lower tail. Since, by assumption, $M_t(I) \xrightarrow{a.s.} M(I)$, for any bounded I, $M_t \xrightarrow{\mathcal{M}_1} M$. a.s. and we may apply part (a).

§4. The Limit Theorem

We now return to the special case of the random measure of (2.18), namely,

$$M_t(dx) = x^2 \int_0^t \hat{F}_H(\sigma_t dx; t, s)\, d\hat{\Lambda}(s)$$

which, by assumptions (2.1) and (2.3), is in $\mathcal{M}_2$.

Suppose that $M_t \xrightarrow{\mathcal{M}_2} M$, a.s. for some non-random canonical measure, M. Since M_t must be a.s. stochastically bounded, we have, for each θ and for any $\varepsilon > 0$ and $0 < a < \infty$, a constant $0 < C < \infty$ and a time t_0, s.t.:-

$$P\{\sup_{t\geq t_0} M_t[-a,a] > \frac{2C}{3\theta^2}\} < \varepsilon/3 ,$$

$$P\{\sup_{t\geq t_0} M_t^{+}(a) > \frac{C}{6}\} < \varepsilon/3 \text{ and } P\{\sup_{t\geq t_0} M_t^{-}(-a) > \frac{C}{6}\} < \varepsilon/3 .$$

Now, let $K(t,s) = \int_0^s E_u[1 - \cos\{\theta\sigma_t^{-1}H(t,u)\} \mid \mathcal{H}_u] \, d\hat{\Lambda}(u)$.

Since $1 - \cos x \leq \min(\frac{x^2}{2}, 2)$, we have

$$K(t,t) = \int_{-\infty}^{\infty} (1 - \cos\theta x) \int_0^t \hat{F}_H(\sigma_t dx; t, s) d\hat{\Lambda}(s)$$

$$\leq \frac{\theta^2}{2} \int_{-a}^{a} M_t(dx) + 2\{M_t^{+}(a) + M_t^{-}(-a)\}$$

for arbitrary $a > 0$. Therefore,

$$P\{\sup_{t\geq t_0} K(t,t) > C\} < \varepsilon .$$

Next, introduce the truncated variables

$$H(t,s)^T = H(t,s) \quad , \text{ if } K(t,s) \leq C$$

$$= 0 \quad , \text{ otherwise}$$

Note that condition (2.7) is automatically satisfied for the truncated process since the integrand becomes zero if $H(t,s)^T = 0$. (2.3) is satisfied, since $K(t,s)$ is increasing with s, while (2.4) is also valid, because $K(t,s)$ is $\mathcal{H}_s$-measurable. The remaining conditions, 2.1, 2, 5, 6, are clearly unaffected by the truncation.

From now on, we use the superscript 'T' to denote variables from the truncated process. It is not difficult to show that $M_t^T \xrightarrow{m_2} M$, a.s.

For we can find a time, $t_1 \geq t_0$, s.t.

$$P\{\sup_{t\geq t_1} |M_t[a,b]^T - M[a,b]| > \delta\}$$

$$\leq P\{\sup_{t\geq t_1} |M_t[a,b] - M[a,b]| > \delta\}$$

$$+ P\{M_t[a,b]^T \neq M[a,b], \text{ for some } t \geq t_1\}$$

$$< \varepsilon + P\{\sup_{t\geq t_1} K(t,t) > C\}$$

$$< 2\varepsilon . \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4.1)$$

Similarly, $M_t^+(a)^T$ and $M_t^-(-a)^T$ converge a.s. to $M^+(a)$ and $M^-(-a)$, respectively.

It follows that M_t^T satisfies all the conditions of Theorems 2.6 and 3.5, and of Corollary 3.6. Therefore

$$\mathbb{E}\, e^{i\theta Z(t)^T} \to \mathbb{C}[\theta], \; t \to \infty .$$

Now

$$|\mathbb{E}\, e^{i\theta Z(t)} - \mathbb{C}[\theta]|$$

$$\leq |\mathbb{E}\, e^{i\theta Z(t)^T} - \mathbb{C}[\theta]| + 2P\{Z(t) \neq Z(t)^T\}$$

so that, arguing as at (4.1), we must also have

$$\mathbb{E}\, e^{i\theta Z(t)} \to \mathbb{C}[\theta], \; t \to \infty .$$

We have, therefore, proved the following result.

Theorem 4.1 Under conditions (2.1) - (2.6), the sequence of measures, M_t, defined at (2.18) is in $\mathfrak{M}_2$.

Suppose that

$$M_t \xrightarrow{\mathfrak{M}_2} M, \text{ a.s.}$$

for some non-random canonical measure, M. Then Z(t) converges in distribution to an infinitely divisible random variable, whose log characteristic function is

$$\log \mathbb{E}[\theta] = \int_{-\infty}^{\infty} \frac{e^{i\theta x} - 1 - i\theta \sin x}{x^2} \, dM(x) + iC\theta .$$

The constant C depends on the centering τ, and is given by

$$C = \lim_{t\to\infty} \int_{-\infty}^{\infty} \frac{\sin x - \tau(x)}{x^2} \, dM_t(x) .$$

The natural centering, $\tau(x) = x$, may be used if

$$M_t \xrightarrow{\mathfrak{M}_1} M, \text{ a.s.}$$

and, in particular, when $M_t \in \mathfrak{M}_{1-\delta}$, for some $\delta > 0$.

To illustrate the conditions on M_t, we derive a Lindeberg-type condition for convergence to a N(0,1) variable. As we observed in Chapter I, Section 6. , the appropriate limiting normal canonical measure has

$$M[-a,a] = 1$$

$$\text{and } M^+(a) = M^-(-a) = 0$$

for all $a > 0$.

Theorem 4.2 Suppose that conditions (2.1) - (2.6) hold, that

$$\int_0^t \mathbb{E}_s[H^2(t,s)\,|\mathcal{H}_s]\,d\hat{\Lambda}(s) < \infty, \text{ a.s.} \quad \ldots \quad (4.2)$$

and that we can choose a norming function σ_t s.t.

$$\frac{1}{\sigma_t^2}\int_0^t \mathbb{E}_s[H^2(t,s)\,|\mathcal{H}_s]\,d\hat{\Lambda}(s) \to 1, \text{ a.s.} \quad \ldots \quad (4.3)$$

If, as $t \to \infty$,

$$\frac{1}{\sigma_t^2}\int_0^t \int_{|x|>\sigma_t a} x^2\,\hat{F}_H(dx;t,s)\,d\hat{\Lambda}(s) \to 0, \text{ a.s.} \quad \ldots \quad (4.4)$$

for $a > 0$, then

$$Z(t) = \sigma_t^{-1}\,[Y(t) - \int_0^t \mathbb{E}_s[H(t,s)\,|\mathcal{H}_s]\,d\hat{\Lambda}(s)]$$

converges in distribution to a $N(0,1)$ limit.

Proof The random measure $M_t(dx) = x^2\int_0^t \hat{F}_H(\sigma_t dx;t,s)\,d\hat{\Lambda}(s)$ is a.s. finite by (4.2). Since, for any $\delta \leq 1$, $M_t \in \mathcal{M}_{1-\delta}$, the centering to $\tau(x) = x$ is justified.

Now

$$M_t[-a,a] = \int_{-a}^{a} x^2 \int_0^t \hat{F}_H(\sigma_t dx;t,s)\,d\hat{\Lambda}(s)$$

$$= \frac{1}{\sigma_t^2}\int_0^t \mathbb{E}_s[H^2(t,s)\,|\mathcal{H}_s]\,d\hat{\Lambda}(s)$$

$$- \frac{1}{\sigma_t^2}\int_0^t \int_{|x|>\sigma_t a} x^2\,\hat{F}_H(dx;t,s)\,d\hat{\Lambda}(s).$$

Combining (4.3) and (4.4), $M_t[-a,a] \to 1$, a.s., and, in fact, (4.3) states that $M_t(\mathbb{R}) \to 1$, a.s., also. We have, therefore, proved a.s. complete convergence to the normal canonical measure. Since this clearly implies $\mathfrak{M}_2$-convergence also, the proof may be completed by an application of Theorem 4.1.

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