ANALYTIC FUNCTIONS AND ITERATION THEORY

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ABSTRACT

The first Chapter of this thesis deals with the fractional iteration of analytic functions near a fix point of multiplier 1, as outlined in its introduction. In particular we have produced new results on the embedding problem.

In Chapter 2, we have looked at iteration of analytic functions on the lines initiated by Fatou, Julia and Radström and derived results for entire functions with infinite domains of normality.

We have studied in Chapter 3, Picard sets for entire transcendental functions and extended some results in this theory and in the fourth chapter, we have looked at factorisation of entire functions (in the form of composition) and derived criteria for prime functions, producing at the same time prime functions of very small growth.

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CONTENTS

CHAPTER I

FRACTIONAL ITERATION NEAR A FIX POINT OF MULTIPLIER 1

INTRODUCTION:-

An analytic function $f(z)$ is said to have a fix point $\zeta \neq \infty$ of multiplier 1, if $f(\zeta) = \zeta$, $f'(\zeta) = 1$. The function then has an expansion,

$$(1)\quad f(z) = \zeta + (z - \zeta) + \sum_{k=m+1}^{\infty} a_k(z - \zeta)^k,\ a_{m+1} \neq 0,\ m \geq 1.$$

It has been shown (e.g. in Baker [1]) that there is for every complex λ, a unique formal iterate,

$$(2)\quad f_\lambda(z) = \zeta + (z - \zeta) + \sum_{k=m+1}^{\infty} a_k(\lambda)(z - \zeta)^k,\ a_{m+1}(\lambda) = \lambda a_{m+1},$$

where the $a_k(\lambda)$ are well defined polynomials in λ determined by comparing coefficients in the formal identity $f_\lambda \circ f(z) = f \circ f_\lambda(z)$. For positive integral values $\lambda = n$ the series (2) is the same as that of the n^{th} iterate of $f(z) = f_1(z)$ and by analogy the $f_\lambda(z)$ are in general called the fractional iterates.

Without loss of generality, we choose our fix point at the origin. For simplicity, we shall work with the case $m = 1$, when (1) and (2) reduce to

$$(3)\quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad , \quad a_2 \neq 0,$$

$$(4)\quad f_\lambda(z) = z + \sum_{k=2}^{\infty} a_k(\lambda) z^k, \quad a_2(\lambda) = \lambda a_2, \quad \text{respectively.}$$

We note that the series (4) does not necessarily have a positive radius of convergence for each λ; in fact it is shown in Baker [1] that the values of λ corresponding to a positive radius of convergence either fill out the whole complex plane, or form a discrete one or two dimensional lattice. When the values fill out the whole plane we shall call f <u>EMBEDDABLE</u>.

In the following, we shall be deriving specific classes of non-embeddable functions; in the course of the discussion we shall give a new proof of a lemma of Szekeres using methods based on the work of Baker [I, secn 7].

We shall also establish the fact that when the values of λ corresponding to a positive radius of convergence include a two dimensional lattice, f is in fact embeddable, i.e. the case of a <u>discrete</u> two dimensional lattice cannot occur.

It is sometimes convenient to transfer the fix point to ∞. We shall employ the substitutions $z = \frac{k}{t}$, $z_1 = \frac{k}{t_1}$, choosing $a_2 k = -1$. Applying these to the transformation $z_1 = f(z)$ of (3) we get,

$$(5) \quad t_1 = t + 1 + \sum_{k'=1}^{\infty} b_{k'} t^{-k'} = g(t)$$

with fix point at ∞ and with b_1 satisfying

$$(6) \quad b_1 = (a_2^2 - a_3)/a_2^2.$$

Similarly (4) transforms to (7)

$$(7) \quad t_\lambda = t + \lambda + \sum_{k'=1}^{\infty} b_{k'}(\lambda) t^{-k'} = g_\lambda(t)$$

where $g_\lambda(t)$ form the unique family of formal series (7) commuting with (5). The series (4) is embeddable precisely when the series (7) converges for some $t \neq \infty$, for every λ. We shall from now on assume $g = g_1$ convergent for $|t| > R$.

1.2 A CRITERION FOR EMBEDDABLE FUNCTIONS.

We quote the following results from Baker [1]

LEMMA 1 [1, p 272]

If $\mathcal{D}(k) = \bigcup_{-\pi/4 \leq \alpha \leq \pi/4} \mathcal{C}(\alpha, k)$,

where $\mathcal{C}(\alpha, k)$ is the half plane $\{z \mid \mathrm{Re}(ze^{-i\alpha}) > k\}$, then for all sufficiently large $k (>R)$, $g_n(z)$ is regular,

(8) $g_n(z) \in \mathcal{D}(k) \qquad n = 1, 2, \ldots\ldots,$

and

(9) $\mathrm{Re}\, g_n(z) \to \infty \qquad \text{as } n \to \infty$

for all z in the closure $\overline{\mathcal{D}}(k)$ of $\mathcal{D}(k)$. Moreover (9) holds locally uniformly in $\mathcal{D}(k)$. (cf. [1, p273 (21)])

LEMMA 2 [1, p 273]

For all sufficiently large k the domain $\mathcal{D}(k)$ of Lemma 1 has the properties

(10) $A(t) = \lim_{n\to\infty} \{g_n(t) - n - b_1 \log n\}$

(where b_1 is as defined in (6)) exists uniformly for $t \in \mathcal{D}(k)$; moreover $A(t)$ is regular and univalent in $\mathcal{D}(k)$ and $A'(t) \to 1$ uniformly as $t \to \infty$ in $\mathcal{D}(k)$. Also

(11) $A\{g_n(t)\} = A(t) + n \quad$ for $t \in \mathcal{D}(k)$

LEMMA 3 [1, p 279]

Let $f_\lambda(z) = z + \lambda a_{m+1} z^{m+1} + \sum_{n=m+2}^{\infty} a_n z^n$, $a_{m+1} \neq 0$ $m \geq 1$

be a commuting family of formal power series. For $p > 0$, let Ωp be the class of complex λ with $|\lambda| \leq p$ for which $f_\lambda(z)$ has positive radius of convergence. Then there exists constants $\rho>0$ and $M>0$ such that

(i) $f_\lambda(z)$ converges in $|z| \leq \rho$ for all $\lambda \in \Omega p$

and

(ii) $|f_\lambda(z)| < M$ uniformly for all $|z| \leq \rho$ and all $\lambda \in \Omega p$.

We next state and prove Lemma 4.

LEMMA 4

If the series $t_\lambda = g_\lambda(t) = t + \lambda + \sum_{k'=1}^{\infty} b_{k'}(\lambda) t^{-k'}$ has a positive radius of convergence for every λ, i.e. g is embeddable, then $D(t) = \Lambda'(t)$ is regular, in a full neighbourhood of $t = \infty$ and has an expansion

(12) $D(t) = 1 - \dfrac{b_1}{t} + \sum_{k'=2}^{\infty} s_{k'} t^{-k'}$

which may be calculated from

(13) $D(g(t)) = D(t)/g'(t)$

This lemma is due to Szekeres [1]. The following proof, however, will be based on Baker's work in [1, §7].

Proof:

We shall show the existence of an infinitesimal transformation and work on the resulting differential equation.

Since $g(t)$ is embeddable, we have a local continuous group of functions $g_\lambda(t)$ with complex parameter λ, and $g_\lambda(g_\mu(t)) = g_\mu(g_\lambda(t))$. We shall transfer our fix point to zero, employing the usual transformation $z = k/t$, $z_1 = k/t_1$; Thus we get

(14) $f(z) = z + a_2 z^2 + \sum_{k'=3}^{\infty} a_{k'} z^{k'}$,

and in general,

(15) $f_\lambda(z) = z + \lambda a_2 z^2 + \sum_{n=3}^{\infty} b_n(\lambda) z^n.$

Now we restrict λ to $|\lambda| \le 1$ and by Lemma 3 there exists $\delta > 0$ such that all $f_\lambda(z)$ converge in $|z| \le \delta$ and satisfy there also $|f_\lambda(z)| < M$ for suitable M. The coefficients in (15) satisfy

$$|b_n(\lambda)| \le M/\delta^n \text{ for } |\lambda| \le 1$$

By Baker, [I p 267] $b_n(\lambda)$ is a polynomial of degree at most $n - 1$ with $b_n(0) = 0$. Applying Schwarz's lemma,

$$|b_n(\lambda)| \le \frac{M|\lambda|}{\delta^n} \quad \text{for } |\lambda| \le 1,$$

and $|b_n'(o)| \le M/\delta^n$.

Also $p_n(\lambda) = \dfrac{b_n(\lambda)}{\lambda} - b_n'(o)$ is a polynomial with vanishing constant term.

$$|p_n(\lambda)| \le |b_n(\lambda)| + |b_n'(o)| \le \frac{2M}{\delta^n} \text{ for } |\lambda| = 1,$$

and since $p_n(o) = o$, one has

(16) $|p_n(\lambda)| \le 2M/\delta^n \, |\lambda| \quad$ for $|\lambda| \le 1$.

The series

(17) $I(z) = a_2 z^2 + \sum_3^{\infty} b_n'(o) z^n$ converges for $|z| < \delta$,

where moreover,

$$\left|I(z) - \left(\frac{f_\lambda(z) - z}{\lambda}\right)\right| = \left|\sum_{n=3}^{\infty} p_n(\lambda) z^n\right|$$

$$< |\lambda| \sum_{n=3}^{\infty} 2M|z|^n \delta^{-n}$$

$$= \frac{|\lambda|}{\delta^2} \frac{2M|z|^3}{(\delta - |z|)}$$

Thus $I(z) = \lim_{\lambda\to o} \dfrac{f_\lambda(z) - z}{\lambda}$ holds for $|z|<\delta$.

Now consider $I(z) = \dfrac{\partial f_\lambda(z)}{\partial\lambda}\Big|_{\lambda\to o}$

$$I(f(z)) = \frac{\partial}{\partial\lambda} f_\lambda(f(z))\Big|_{\lambda\to o} = \frac{\partial f}{\partial\lambda}(f_\lambda(z))\Big|_{\lambda\to o}$$

Near $z = 0$, we get

$$\frac{\partial}{\partial\lambda} f_\lambda(f(z))\Big|_{\lambda\to o} = f'(f_\lambda(z)).\frac{\partial}{\partial\lambda} f_\lambda(z)\Big|_{\lambda\to o} ,$$

$$= f'(z)\frac{\partial}{\partial\lambda} f_\lambda(z)$$

Hence

(18) $I(f(z)) = f'(z)I(z)$.

We introduce $B = \frac{1}{I}$ and deduce

(19) $B(f(z)) = B(z)/f'(z)$

Suppose now, we transfer our fix point to ∞ once more. Using $z = k/t$, $z_1 = k/t_1$ we note that $f(z) = \dfrac{k}{g(t)}$ so that $f'(z) = \dfrac{t^2}{g^2(t)} g'(t)$

Hence by (19) $B(k/g(t)) = \dfrac{B(k/t)}{f'(z)}$

and so $\dfrac{1}{g^2} B(k/g(t)) = \dfrac{B(k/t)}{t^2 g'(t)}$.

Hence if μ is a constant,

(20) $D(t) = \dfrac{\mu}{t^2} B(k/t)$, defined in a full neighbourhood of ∞,

satisfies

(21) $D(g(t)) = D(t)/g'(t)$ near ∞, which is identical with (13).

It remains now to show that $D(t)$ as defined in (20) satisfies the condition $D(t) = A'(t)$ and has the correct expansion near ∞. We determine $b_3'(o)$ from $f_\lambda \circ f = f \circ f_\lambda$ and find by comparison of coefficients $b_3(\lambda) = \lambda(a_3 - a_2^2) + \lambda^2 a_2^3$

$$b_3'(o) = a_3 - a_2^2$$

Thus (17) becomes

$$I(z) = a_2 z^2 + (a_3 - a_2^2) z^3 + \text{----}$$

and so

$$B = 1/I(z) = \frac{1}{a_2 z^2}\left(1 - \frac{(a_3 - a_2{}^2)}{a_2} z + \text{----}\right)$$

and

$$D(t) = \mu/t^2\, B(k/t) = \mu/t^2\ \frac{t^2}{k^2 a_2}\left(1 - \frac{a_3 - a_2{}^2}{a_2}\frac{k}{t} + \text{----}\right)$$

$$= \mu a_2\left(1 + \frac{a_3 - a_2{}^2}{a_2{}^2}\frac{1}{t} + \text{----}\right) \text{ since } ka_2 = -1.$$

Choosing $\mu a_2 = 1$, we have

$$D(t) = 1 - \frac{a_2{}^2 - a_3}{a_2{}^2} \cdot \frac{1}{t} + \text{-----},$$

$$= 1 - \frac{b_1}{t} + \sum_{l=2}^{\infty} S_l t^{-l}, \text{ regular in a full neighbourhood of } \infty$$

Finally we show $D(t) = A'(t)$; Let t_o be a fixed point in $\mathfrak{R}(k)$ and set

$$(22)\quad E(t) = \int_{t_o}^{t} D(t)dt = (t - t_o) - b_1 \log t/t_o + \sum_{n=2}^{\infty} \frac{s_n}{1-n}(t^{1-n} - t_o^{1-n}),$$

$$= k_1 + t - b_1 \log t + \text{-----}$$

defined for $t \varepsilon \mathfrak{R}(k)$, where the integral is taken along an arbitrary path in the simply-connected region $\mathfrak{R}(k)$. Any fixed branch of log t may be taken.

$$\frac{d}{dt} E(g(t)) \;=\; D(g(t))g'(t) = D(t) = E'(t), \text{ using (21)}$$

Hence in $\mathfrak{R}(k)$ we have

$$(23)\quad E(g(t)) = E(t) + k_2, \qquad k_2 \text{ constant}$$

Also by Lemmas 1,2 for $t \varepsilon \mathfrak{R}(k)$ and $n \to \infty$ we have $g_n(t) \varepsilon \mathfrak{R}(k)$ and

$$E\{g_n(t)\} = E\{n + b_1 \log n + A(t) + o(1)\}$$

$$\begin{aligned} = k_1 + n &+ b_1 \log n + A(t) + o(1) \\ &- b_1 \log n \\ &- \log\left(1 + b_1 \frac{\log n}{n} + \frac{A(t)}{n}\right) \\ &+ O(1/n), \end{aligned}$$

i.e.

$$(24)\quad E\{g_n(t)\} = k_1 + n + A(t) + o(1) \text{ as } n \to \infty$$

By (23) however

$$E\{g_n(t)\} = k_2 n + E(t)$$

Therefore $k_2 = 1$ and $E(t) = A(t) + k_1$. Thus $D(t) = A'(t)$ concluding the proof of the lemma.

1.3 ON NON-EMBEDDABLE FUNCTIONS

We now quote the following result from Baker [2]

LEMMA 5

If the series (7) has a positive radius of convergence for each λ (i.e. $g(t)$ is embeddable) and if there exists a fixed $R_1 > R > 0$ and arbitrarily large integers n such that the total number of branches obtained by analytic continuation of $g_n(t) = t + n + \Sigma b_k(n)t^{-k}$ within $|t| > R_1$ is finite, then the coefficient b_1 in (6) is zero.

The proof of this Lemma depends mainly on the result of Lemma 4. We shall not reproduce it here, but it is rather similar to the arguments employed in proving Theorem 3.

Next we state and prove Lemma 6.

LEMMA 6

Let D be a domain bounded by a finite number of non-intersecting analytic curves, denoted by δ. We suppose D is bounded and contains the origin. If (i) f is regular and single-valued in D,

(ii) f has the expansion $f(z) = z + a_2z^2 + ----$,

and (iii) the set of analytic curves δ is a natural boundary for f, Then

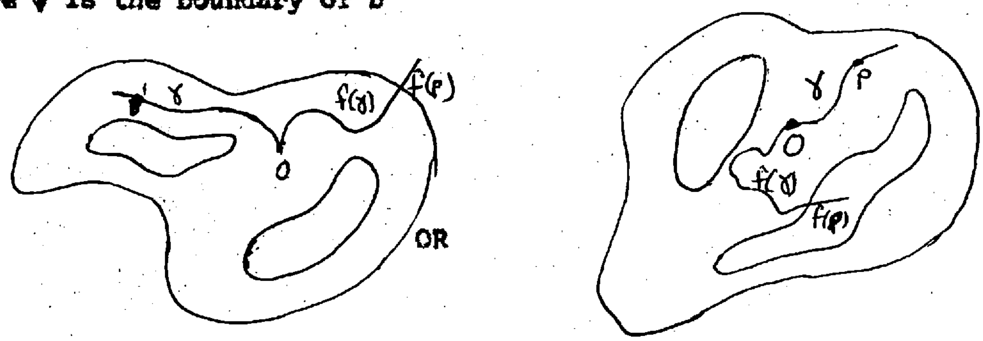
(a) If $f_n(z)$ is analytically continuable with at most algebraic singularities from its expansion $f_n(z) = z + na_2z^2 + ----$, at the origin, along a path γ of D, so are all $f_j(z)$ $j<n$ and $f_j(\gamma) \subset D$, $j<n$.

(b) $f_n(z)$, $n \geq 1$ is single valued as far as continuable analytically from $f_n(z) = z + na_2z^2 + ----$, by paths lying completely in D.

<u>Proof:</u> The proof that follows is by induction;

<u>Case</u> $n = 2$.

Given $f(z) = z + a_2z^2 + \text{-----}$, with (i) and (iii) satisfied we consider the analytic continuation of $f_2(z)$ from $f_2(z) = z + 2a_2z^2 + \text{---}$, at 0. If we can analytically continue $f_2(z)$ along a path γ in D, starting at 0, then $f(\gamma) \subset D$; for otherwise there exists $p \in D \cap \gamma$ such that $f(p) \in \delta$, where δ is the boundary of D

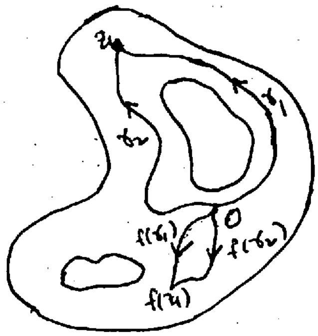


We suppose p to be the first such point on γ starting from 0. Then

$$(25) \quad f(z) = f_2(f_{-1}(z)),$$

where $f_2(z) = z + 2a_2z^2 + \text{----}$, and $f_{-1}(z) = z - a_2z^2 + \text{----}$, near 0; If we consider (25) as z runs along $f(\gamma)$ from 0 to $f(p)$, $f_{-1}(z)$ traverses γ from 0 to p and at $f(p)$ there is a branch of $f_{-1}(z)$ having at most an algebraic singularity such that $f_{-1}(f(p)) = p$. (25) thus in fact gives a continuation of $f(z)$ having at most an algebraic singularity over its natural boundary at $f(p)$. This is impossible, hence $f(\gamma) \subset D$ for any path γ in D, on which $f_2(z)$ is regular.

Suppose now $f_2(z)$ can be analytically continued from 0 to z_1 by two paths γ_1, γ_2 in D. Then $\gamma_1 \circ \gamma_2^{-1}$ is a path from 0 to 0 lying completely in D. As z traverses this closed path, $f(z)$ traverses another closed path $\bar{\gamma} = f(\gamma_1)\cdot f(\gamma_2)^{-1}$ which lies completely in D. (by preceding argument).



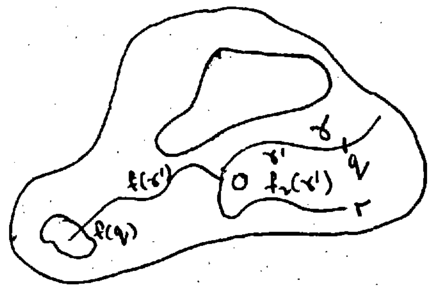
As w traces $\bar{\gamma}$, $f(w)$ is regular and continues from the branch $f(w) = w + a_2w^2 + ----$, at 0 to the same branch at the end of the path. Thus $f_2(z) = f(f(z))$ continues from the initial branch $f_2(z) = z + 2a_2z^2 + ----$, at $z = 0$, back to the same branch at the end of the path as z traces $\gamma_1\cdot\gamma_2^{-1}$. Hence continuation of $f_2(z)$ from 0 along γ_1 or γ_2 to z_1 yields identical results.

<u>Case n = 3</u>

Suppose $f_3(z)$ can be analytically continued from 0, along γ, in D and suppose further that continuation of $f_2(z)$ first breaks down at q on $\gamma \cap$ D where $f_3(z)$ is regular. Let γ' be the section of γ between 0 and q. (exclusive of q) Now f_2 is regular on γ'; By induction (case n = 2) $f(\gamma') \subset$ D. If $f(q)$ were also in D then $f_2(z)$ would be regular at q. Therefore $f(q) \in$ boundary δ.

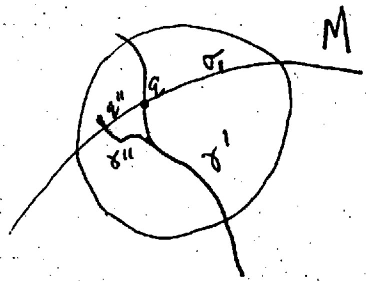
Now on γ', $f_3(z)$ is regular. Hence on $f(\gamma')$ which is in D, $f_2(z)$

is regular; for it is analytically continuable. (with perhaps algebraic singularities) But if $f_2(z)$ had an algebraic singularity on $f(\gamma')$ it would be many valued on some paths in D, which is impossible. $f_2(\gamma') = f(f(\gamma')) \subset D$ by the first part.

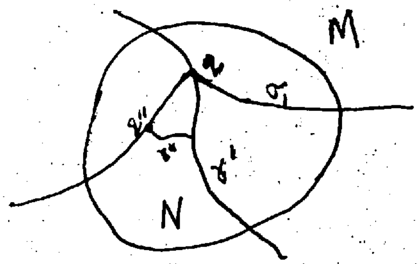


Now if for some sequence $\{z_j\} \in \gamma'$ with $z_j \to q$ $w_j = f_2(z_j) \to r$ and $r \in D$; then since $f_3(z) = f(f_2(z))$ along γ'. $f_3(z_j) = f(w_j)$. But $f(w)$ is regular at $w = r$ and $f(r) = f_3(q) = t$ say. If we consider the branch $w = p(u)$ of the inverse $f_{-1}(u)$ of $u = f(w)$ obtained by inverting the Taylor expansion $u = f(w) = t + f'(r)(w - r) + \cdots$ so that $r = p(t)$, then $p(f_3(z))$ where $f_3(z) = t + f_3'(q)(z - q) + \text{----}$, is an expansion involving at most an algebraic singularity at q, and at z_j taking for at least one branch, the values $p(f_3(z_j)) = w_j$. i.e. agreeing with f_2. We thus have a continuation of f_2 over q along γ'. This gives a contradiction. Thus as $z \to q$ on γ', $f_2(z)$ tends uniformly to the boundary δ.

We now look at an arc σ of δ surrounding $f(q)$. We claim that there exists an analytic arc σ_1 contained in D, passing through q and such that $f(\sigma_1) = \sigma$ of boundary δ. σ_1 does not meet γ' except at q since $f(\gamma') \subset D$. We can thus assume that σ_1 divides a neighbourhood N of q into two components one of which, say N contains the part of γ' near q and satisfies $f(N) \subset D$. N is chosen so small, so that it lies in the region of regularity of $f_3(z)$.



OR



Points in σ_1 near q lie in the region of regularity of $f_3(z)$ and we may modify γ' near q to γ'' to meet a point q'' of σ_1 rather then q, and to lie in N, apart from the end point at q''. We may assume that q'' is any point of σ_1 sufficiently near q, and γ'' meets σ_1 only in its end point q''. Then f_3 is regular on γ''.

Now $f_2(z)$ is regular on $\gamma'' - (q'')$ because $f(\gamma'' - \{q''\}) \subset D$ but cannot be continued along γ'' over q'' since $f(q'') \varepsilon$ boundary δ. Hence replacing q by q'' in the argument of the first part of case $n = 3$, we see that f maps σ_1 to σ on boundary δ in the neighbourhood of f(q) while $f_2(z) \to \delta$ as $z \to q''$.

For a fixed point w_o on σ, let w_n be a sequence in D with $w_n \to w_o$. Since σ is analytic we can assume that the w_n lie on a curve $\tilde{\gamma}$ in D which ends in w_o. (if e.g. σ were a line segment we could take a polygon joining successive w_n and this extends to the general case as an analytic curve is a conformal map of a line segment). γ is the image $f(\gamma'')$ of a curve γ'' of the type described above, with the end point q'' of γ'' mapping onto $f(q'') = w_o$, and with $f(z_n) = w_n$ for suitable points z_n in γ''. Since $f_2(z)$ tends to δ as z tends to q'' on γ'', we have $f(w_n) = f_2(z_n)$ which tends to δ as n tends to infinity. i.e. every boundary value of f on σ is in δ.

Let σ be an analytic arc $z = g(t)$, $\alpha \leq t \leq \beta$, $g'(t) \neq 0$, g analytic. We may assume that g maps the rectangle $\alpha \leq t \leq \beta$, $o < u \leq \eta$ of the $s = t + iu$ plane into part of D. If σ' is the subarc of σ corresponding to $[\alpha_1, \beta_1]$, where $\alpha < \alpha_2 < \alpha_1 < \beta_1 < \beta_2 < \beta$ and if R_n are the rectangles $\alpha_2 \leq t \leq \beta_2$, $0 < u < \eta/n$ $n = 1,2,3,$--- then $f(R_n) = D_n$ is a region $D_n \subset D$ whose boundary consists of points of (including at least σ') and otherwise of interior points of D. The set of boundary values of f on σ' is contained in $B = \bigcap_1^\infty \overline{\{f(D_n)\}}$ which is in turn contained in the set of boundary values of f on σ (to any $\beta \varepsilon B$ there is a $z_n \varepsilon D_n$ with $|f(z_n) = \beta| < 1/n$. One can pick a subsequence of z_n which converges to z_o on σ since $z_n = g(s_n)$ say, $s_n \varepsilon R_n$ and in the nested rectangles R_n, one can choose s_n convergent to say t_o) Now the $\overline{f(D_n)}$ are continua (compact connected sets), compact since $f(D_n) \subset D$ which is bounded, and these sets decrease with increasing n. Hence the intersection B is a continuum and so connected.

Therefore B, which lies on the boundary δ lies on just one of the analytic curves δ_1 which make up δ. Since the component of the complement of δ_1 containing D may be mapped on a unit disc and since σ' may be mapped on a circular arc, we may apply the Schwarz reflection principle [Carathéodory II], We see that f is regular on σ which lies on δ;

But this contradicts the hypothesis of the theorem. Hence of $f_3(z)$ is regular on γ in D then so is $f_2(z)$.

We now show that if f_3 is continuable along a path γ of D, then $f_2(\gamma) \subset D$. Consider a path γ in D along which f_3 is continuable. By earlier arguments we know f_2 is also continuable on γ. Since f_2 is continuable on γ, $f(\gamma) \subset D$ (first part), but f_3 is regular on γ implies $f_2(z)$ is regular on $f(\gamma)$ which by above is in D. Hence $f_2(\gamma) = f(f(\gamma)) \subset D$. The single valuedness of $f_3(z)$ now follows by almost exactly the same argument described for $f_2(z)$.

<u>General inductive step</u>

We observe that the proof of the Lemma hinges essentially on the establishment of (a).

We suppose $f_m(z)$, $m > 2$ is continuable from O, along γ in D and that analytic continuation of $f_{m-1}(z)$ breaks down for the first time at q in $\gamma \cap D$. Let γ' be the part of γ along which $f_{m-1}(z)$ is continuable. By hypothesis, $f_j(\gamma') \subset D$, $j < m - 1$; Further since f_m, f_{m-1} are regular on γ', it is easy to see that $f_{m-1}(\gamma') \subset D$. We consider t, $2 \leq t \leq m - 1$, the smallest integer such that along γ, continuation of f_t breaks down at q. then $f_{t-1}(\gamma') \subset D$, and since $f_{t-1}(q) \in D$ implies $f_t(z)$ regular at q we see that $f_{t-1}(q)$ boundary δ.

If we now consider a sequence $\{z_j\}$, $z_j \in D \cap \gamma'$, such that $z_j \to q$, and suppose $w_j = f_{m-1}(z_j) \to r$, $r \in D$, (we recollect that $f_{m-1}(\gamma') \subset D$)

Then since $f(w)$ is regular at r, $f(r) = f_m(q) = p$ say. If we denote by $f_{-1}(w)$ the branch of the inverse of $f(z)$ obtained by inverting the Taylor series $w - p = f'(r)(z - r) + \ldots$ such that $f(r) = p$, $f_{-1}(p) = r$, then as in the case $m = 3$ $f_{m-1}(z_j) = f_{-1}(f_m(z_j)) = w_j$. We thus have a continuation of $f_{m-1}(z)$ over q which contradicts our supposition. Hence, $f_{m-1}(z)$ tends uniformly to boundary δ as z tends to q on γ'.

We note that since $f_m(z)$ is regular on γ', on $f(\gamma')$ which is in D, $f_{m-1}(z)$ is also regular; Hence $f_{m-1}(\gamma') = f_{m-2}(f(\gamma')) \subset D$, by the induction hypothesis. We look at an arc σ of δ surrounding $f_{t-1}(q)$. There exists an arc σ_1 through q such that $f_{t-1}(\sigma_1) = \sigma$. σ_1 does not meet γ' except at q, since $f_{t-1}(\gamma') \subset D$. We can assume that σ_1 divides a neighbourhood M of q into two components, one of which, say N contains the part of γ' near p, and satisfies $f_{t-1}(N) \subset D$. M is chosen so small, as to lie in the region of regularity of $f_m(z)$ and of $f_{t-1}(z)$. Points of σ_1 near q lie in the region of regularity of f_m and we may modify γ' to γ'' near q to meet a point q'' of σ_1 instead of q and to lie in N, apart from the end point at q''. Now f_{t-1} is regular on γ'' (including q'') and $f_{t-1}(\gamma'' - (q'')) \subset D$, so f_t can be continued along $\gamma'' - (q'')$. However, since $f_{t-1}(q'') \in f_{t-1}(\sigma_1) \subset \sigma \subset \delta$ we see that f_t cannot be continued over q'' along γ''.

We have thus established: $f_{t-1}(\sigma_1) = \sigma \subset \delta$. On any of the paths γ'' described above, $\bar{\gamma} = f_{t-1}(\gamma'' - (q'')) \subset D$ and f_m is continuable along γ'', so f_{m-t+1} is continuable along $\bar{\gamma}$, and so by the induction hypothesis f_{m-t} is continuable along $\bar{\gamma}$; that is f_{m-1} is continuable along $\gamma'' - q''$. However, f_{m-1} is not continuable along γ'' over q'' for in this case the induction hypothesis would show f_t to be continuable over q''.

Thus we may replace q by q'' throughout the preceding argument and find that in particular, $f_{m-1}(z) \to \delta$, as $z \to q''$ from inside N. Thus since $f_{m-1}(z) = f_{m-t}(f_{t-1}(z))$ we see that as $w \to \sigma$ from inside D, $f_{m-t}(w) \to \delta$. By the Schwarz reflection principle, just as in the case $m = 3$ (when $t = 2$), f_{m-t} is then continuable across $\sigma \ni f_{t-1}(q)$ and so $f_{m-1}(z)$ is continuable along γ'' over q. This contradicts the assumptions and so we have proved that all f_j, $j < m$ can be continued along γ.

To show $f_j(\gamma) \subset D$ $j < m$; we have just shown that $f_{m-1}(z)$ is continuable along γ in D; Then by the induction hypothesis, $f_j(\gamma) \subset D$ $j < m-1$. In particular $f_{m-2}(\gamma) \subset D$. Now if f_m is continuable along γ in D, then so are all f_j $j < m$; in particular f_{m-1} is, and by above argument $f_{m-2}(\gamma) \subset D$. Now f_2 is regular on $f_{m-2}(\gamma)$, since f_m is regular on γ. Hence $f(f_{m-2}(\gamma)) \subset D$ which implies $f_{m-1}(\gamma) \subset D$ as required. The question of single-valuedness now follows by exactly the same argument earlier outlined.

<u>We now prove</u>

<u>Theorem I</u>: <u>If</u> f <u>and</u> D <u>are as in Lemma 6 and if</u> $a_2{}^2 - a_3 \neq 0$, <u>then</u> f(z) <u>is not embeddable</u>.

<u>Proof</u>: By Lemma 6 if f(z) is as defined, then $f_n(z)$, $n \geq 1$ is single

valued as far as continuable by paths in D. Considering Lemma 5 and by applying our usual transformation $z = k/t$ to bring our fix point to infinity, then if $f(z)$ is embeddable, since it is single-valued in D, the coefficient b_1 in (6) is zero. Hence our result follows.

We shall proceed to show that there exist non-embeddable functions which are single valued in their whole domain of existence D. (i.e. the boundary of D is a natural boundary).

We shall then remove the restriction on the analyticity of the boundary and derive the corresponding conditions for a non-embeddable class of functions. Further in Theorem 3 we shall remove the restriction on the coefficients, a_2, a_3 in the case where we have analytic boundaries.

<u>Examples and Counter examples</u>

<u>EXAMPLE 1</u> Given $f(z) = z + z^2 + \sum_{n=2}^{\infty} z^{2^n}$ which has the unit disc as its exact domain of existence, since $a_2^2 - a_3 = 1$, $f(z)$ is non-embeddable (by Theorem 1)

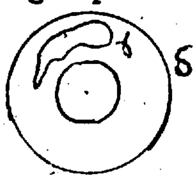
<u>Counter example 2</u>

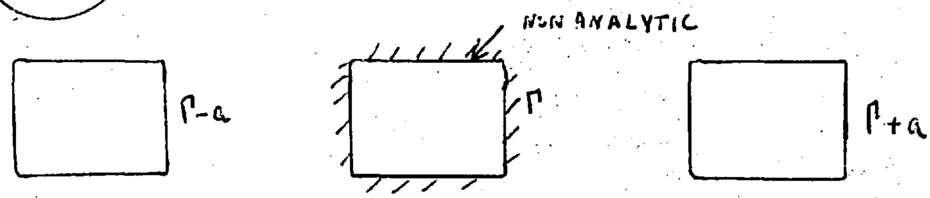
We shall give an example of a single valued function $f(z)$ with "NATURAL BOUNDARIES" which is embeddable. More exactly there exists an embeddable $f(z) = z + b_2z^2 + ---$, $|1/z| < \rho$, $\rho > 0$, $b_2 \neq 0$ whose exact domain of existence D, is the region between $|z| = 1$ and a non-analytic Jordan curve γ.

Construction:

Let δ be the circumference $|z| = 1$, Δ the domain $|z| < 1$

Let $w = A(z) = \frac{\lambda}{z} + a_1 z + a_2 z^2 + ---$, map Δ univalently onto the exterior of a non analytic Jordan curve Γ. (A is the Abel function of our group [Baker 1])





Then if a is large enough, $a > d$ say, where d = diameter of Γ (in the usual sense $d = \max_{x,y \in \Gamma} |x - y|$,) $\Gamma - a$, $\Gamma + a$ are disjoint. So $\Gamma \pm a$ lie in the exterior of Γ. Let the image of $\Gamma - a$ in Δ under $w = A(z)$ be γ [γ will be a non analytic curve with 0 outside it]. Let D be the region between γ and δ. Consider now

$$f(z) = A_{-1}(A(z) + a) = z + b_2 z^2 + ----, \text{ near } z = o$$

In fact $A_{-1}(w) = z = \frac{\lambda}{w} + \frac{a_1 \lambda^2}{w^3} + ----$,

and

$$f(z) = A_{-1}(A(z) + a) = z - \frac{a}{\lambda} z^2 + \frac{a^2}{\lambda^2} z^3 + ----,$$

(By construction, a is non zero and so $b_2 = a/\lambda$).

Now $f(z)$ is defined in D and univalent there. We can continue $f(z)$ along any curve in D surrounding γ. Moreover as z tends to the analytic curve δ from within D, $A(z) + a \to \Gamma + a$ and $f(z) \to$ non analytic $A_{-1}(\Gamma + a)$

Also as z tends to the non analytic curve γ in Δ, $A(z) + a \to \Gamma$ and $f(z) \to$ analytic δ; Hence f is not continuable over δ or γ. Hence f has the asserted properties and is embeddable in the family of iterates

$$f_\mu(z) = A_{-1}(A(z) + \mu a) = z - \frac{\mu a}{\lambda} z^2 + \text{----},$$

which has $f_1(z) = f(z)$ and a positive radius of convergence for each μ. Such a family satisfies $f_\mu \circ f = f \circ f_\mu = f_{\mu+1}$ and is thus the unique family associated with f.

We note that the assumptions of theorem I are violated both in that the boundary of D is not wholly analytic and in that $a_2{}^2 - a_3 = 0$.

One may obtain the conclusions of Lemma 6 under various modified sets of assumptions. As an example we give:-

LEMMA 7: If (i) D is a bounded domain which is bounded by a finite number of disjoint Jordan curves and which contains the origin.

(ii) f is regular in D and the boundary δ of D is a natural boundary for f

(iii) The boundary values of f on δ all lie outside $\overline{D}$

Then (i) If $f_n(z)$ is continuable along a path $\gamma \subset D$, so are all $f_j(z)$, $j < n$ and $f_j(\gamma) \subset D$, $j < n$,

(ii) $f_n(z)$, $n \geq 1$ is single valued as far as continuable from the origin along paths completely lying in D. We note that δ is no longer assumed to consist of analytic curves.

Proof: Case $n = 2$, follows immediately by the argument of Lemma 6.

Hence if f_2 is continuable from O, along $\gamma \subset D$, $f(\gamma) \subset D$ and f_2 is single valued as far as continuable along paths completely lying in D.

Case n = 3; Suppose $f_3(z)$ is continuable from O, along $\gamma \subset D$ and suppose continuation of $f_2(z)$ first breaks down at q on $\gamma \cap D$, then by our argument in Lemma 6, there exists an arc σ_1 in the neighbourhood of q, such that $f(\sigma_1)$ is an arc of boundary δ. Also as $z \to \sigma_1$ from a region N as in Lemma 6 containing γ', $f_2(z)$ takes boundary values on δ, contradicting (iii) of our hypothesis. Hence $f_2(z)$ is continuable along γ and the single valuedness follows as in Lemma 6.

In the general inductive step, if f_m is continuable along a curve γ in D, suppose continuation of $f_{m-1}(z)$ first breaks down at q on $\gamma \cap D$. Then let t be the least integer such that continuation of $f_t(z)$ breaks down at q; $(2 \leq t \leq m-1)$ Then $f_{t-1}(q) \in$ boundary δ. We consider γ', the part of γ ending in q, along which $f_{m-1}(z)$ is continuable. By hypothesis, $f_j(\gamma') \subset D$ $j < m-1$

Consider now a sequence of points $\{z_j\} \in D \cap \gamma'$ such that $z_j \to q$ where continuation of f_t breaks down for the first time. Since f_m, f_{m-1} are regular on γ', $f_{m-1}(\gamma') \subset D$ we consider an arc σ of δ surrounding $f_{t-1}(q)$. There is an arc σ_1 through q such that $f_{t-1}(\sigma_1) = \sigma$. σ_1 meets γ' only at q, since $f_{t-1}(\gamma') \subset D$, we see that σ_1 divides a neighbourhood M of q into two components, one of which N say, contains the part of γ' near q and satisfies $f_{t-1}(N) \subset D$; M is chosen so as to be in the region of regularity of $f_m(z)$ and $f_{t-1}(z)$. By modifying γ' to γ'' near q to meet σ_1 at q'', we get $f_{t-1}(z)$ regular on γ'' and $f_{t-1}(\gamma'' - (q'')) \subset D$.

So $f_t(z)$ can be continued along $\gamma'' - (q'')$. However since $f_{t-1}(q'')$ $f_{t-1}(\sigma_1) \subset \sigma \subset \delta$, $f_t(z)$ cannot be continued over q'' along γ''.

Introducing $\bar{\gamma} = f_{t-1}(\gamma'' - (q'')) \subset D$, since $f_m(z)$ is continuable along γ'', $f_{m-t+1}(z)$ is continuable along $\bar{\gamma}$ and by the induction hypothesis, f_{m-t} is continuable along $\bar{\gamma}$; that is f_{m-1} is continuable along $\gamma'' - (q'')$. But $f_{m-1}(z)$ is not continuable along γ'' over q'' for in this case, the induction hypothesis would show $f_t(z)$ continuable over q''. Thus replacing q by q'' in the argument, we get $f_{m-1}(z) \to \delta$ as $z \to q''$ from inside N. Since $f_{m-1}(z) = f_{m-t}(f_{t-1}(z))$ we see that $f_{m-t}(z)$ takes on some arc of δ boundary values which lie on δ.

There are two cases:- First if $t = m-1$ then $m - t = 1$ and we see that f takes boundary values on $\delta \subset \bar{D}$ on a certain arc of δ, which contradicts the assumptions. Secondly if $t < m - 1$, $m - t \geq 2$, then we see that f_{m-t} can be continued in D along paths leading up to the boundary arc σ and so by the induction hypothesis can f_j $j \leq m - t$, in particular f_2. But as z approaches δ, $f(z)$ lies outside D and so $f(f(z))$ has a singularity on any path $\gamma \subset D$ approaching δ. In either case we have a contradiction and the lemma is proved.

<u>Theorem 2</u>

<u>If</u> $f(z)$ <u>is defined as in Lemma</u> 7 <u>with the expansion</u>

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$$ <u>then if</u> $a_2^2 - a_3 \neq 0$, $f(z)$ <u>is non embeddable.</u>

<u>Proof:</u> This follows immediately by applying Lemma 4 to the above established Lemma 7.

Remarks: In the light of the above results, we briefly discuss a further example of a non embeddable function.

Example 3.

Let δ be any Jordan curve whose interior D, is a region containing the origin; let λ, μ, be the minimum and maximum distances of δ from the origin respectively. Now consider $f(z) = a_o + a_1 z + ---$, analytic in D, such that $a_o^2 \neq a_1$: and f(z) is not continuable across δ. We suppose further that $w = f(z)$ has boundary values all in $|w| > \theta > 0$, say, on δ. Then $g(z) = z + kz^2 f(z)$ is also analytic in D, with natural boundary δ. For suitable choice of k, the boundary values of g on δ satisfy

$$|g(z)| > k\lambda^2\theta - \mu > 2\mu$$

and are therefore outside $\bar{D}$ Also $g = z + a_o z^2 + a_1 z^3 + \ldots$, $a_o^2 - a_1 \neq 0$, so by theorem 2 g(z) is non-embeddable.

We now remove the earlier restrictions made on the coefficients in Theorem 1.

Theorem 3 If $f(z) = z + a_2 z^2 + ---$, $a_2 \neq 0$ is single valued in a domain D containing the origin, such that boundary D is a natural boundary for f and is made up of a finite number of disjoint analytic Jordan curves then f(z) is non-embeddable

Remark Counter example 2 shows we cannot drop the assumption that the boundaries are analytic.

Proof: We consider f(z) as defined above; suppose f(z) is embeddable.

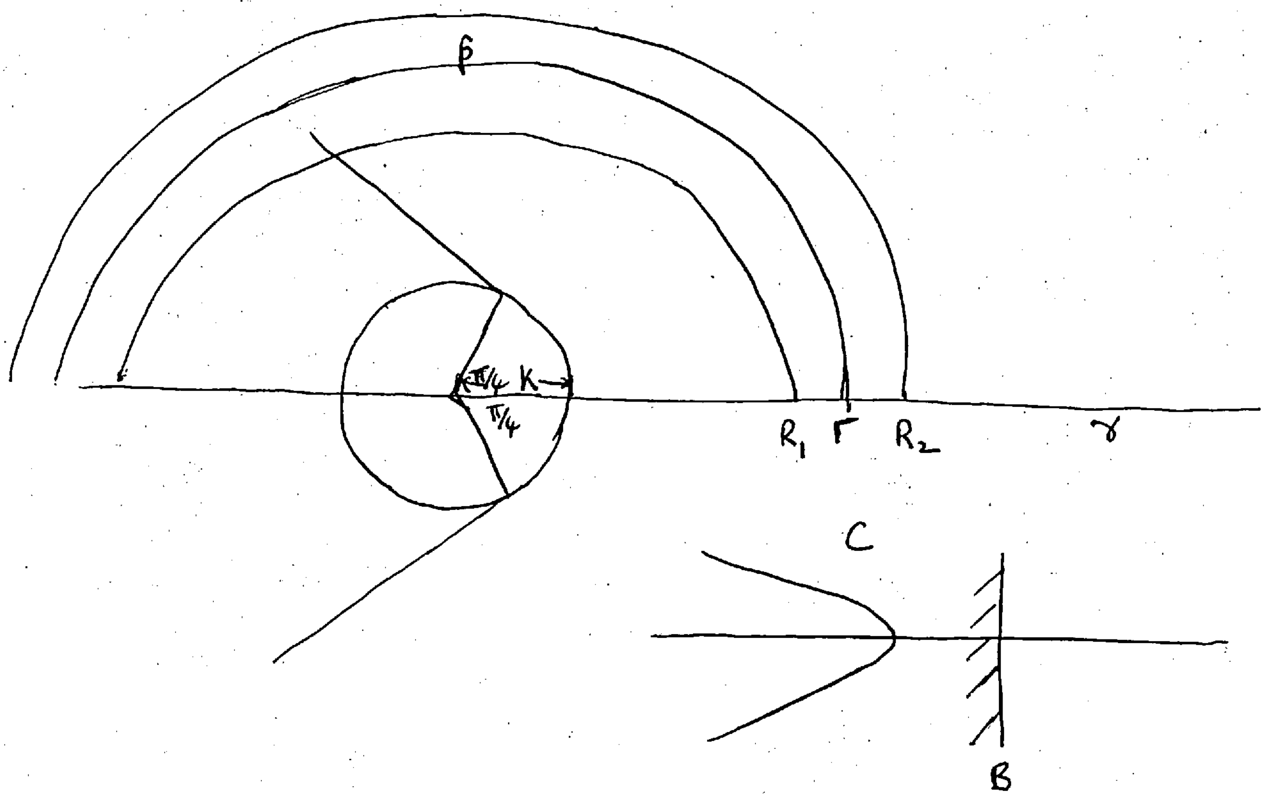
We move our fix point to infinity applying the usual transformation $z = \frac{k}{t}$, $z_1 = \frac{k}{t_1}$. We get

$$(26) \quad g(t) = t + 1 + \sum_{k=1}^{\infty} b_k t^{-k}$$

and for $f_n(z)$ we get

$$(27) \quad g_n(t) = t + n + \sum_{k=1}^{\infty} b_k(n) t^{-k}$$

with the usual properties. Then $g(t)$ is embeddable. Also by Lemma 6 all $g_n(t)$ are single valued as far as analytically continuable from an expansion about infinity and with a certain $|t| \geq T$ independent of n. We now consider $\mathfrak{D}(k)$ as defined in Lemma 1.



We choose k so large such that Lemma 2 holds and such that $A'(t)$ is regular in $|t| > k$ (cf Lemma 4). By choosing k large enough we may suppose $A'(t)$ uniformly close to 1 in $\Pi(k)$. Then $w = A(t)$ maps $\Pi(k)$ univalently and conformally to a region C of the w plane lying to the right of a curve which approaches ∞ in directions $\arg w = \pm \frac{3\pi}{4}$. C contains a half plane $R_e w > B$. We now take $R_o > k$, $R_o \ll R_1 \ll R_2$. Let $R_1 < r < R_2$ and γ be the segment $t > r$ of the real axis, β the semi-circle $t = re^{i\theta}$, $o \le \theta \le \pi$; Now $A'(t)$ is regular on $\beta u\gamma$ and so $A(t)$ may be continued regularly along $\beta u\gamma$ to $t = -r$ with the values of $A(\beta)$ being bounded. For large enough n, $A(\beta) + n$ lies in $R_e w > B$ while for all positive n, $t \in \gamma \subset \Pi(k)$ implies $A(t) + n = A(g_n(t)) \in C$. Thus $A(\beta u\gamma) + n$ is a curve in C.

Consider now $h(t) = A_{-1}\{A(t) + n\}$ on $\beta u\gamma$. On γ, $h(t) = g_n(t)$ while as t describes $\beta u\gamma$ $A(t) + n$ describes $A(\beta u\gamma) + n$ in C and the inverse of the univalent map $A: \Pi(k) \to C$ gives a regular continuation $h(t)$ of $g_n(t)$ along β to $-r$. Moreover for $t = re^{i\theta}$ $o \le \theta \le \pi$ and $R_1 \le r \le R_2$ $g_n(t)$ lies in a compact subset of $\Pi(k)$. Similarly considering a path $\beta' + \gamma$, $\beta' = re^{i\theta}$, $o \ge \theta \ge -\pi$, we conclude we can get a regular continuation of $g_n(t)$ along β' to $-r$. Now since $g_n(t)$ is single valued as far as continuable within $|t| \ge T$ and we have chosen $R_1 > T$, the upper and lower continuations yield identical results, for $g_n(t)$.

Thus $g_n(t)$ is regular in the annulus $R_1 \le t \le R_2$ and it maps the annulus to a compact subset of $\Pi(k)$ and further by Lemma 1 $g_n(t) \to \infty$ uniformly as n tends to infinity, in $\Pi(k)$ and hence in $R_1 \le t \le R_2$.

Now by Lemma 6, if $g_n(t)$ is regular for all $n \geq N_o$ say in annulus in the neighbourhood of infinity, then so is it for all $n < N_o$. Hence for all n, there exists R_1 so that $|t| > R_1$,

(28) $g_n(t)$ is regular for all n

(29) $g_n(t) \to \infty$ uniformly as $n \to \infty$

Transferring our fix point to the origin once more, we see that there exists a $\rho_o > 0$, such that in some $|z| < \rho_o$

(30) $f_n(z)$ is regular for all n, and

(31) $f_n(z) \to 0$ uniformly as $n \to \infty$

Now, considering our expansion $f(z) = z + a_2z^2 + ---$, we claim $\{f_n(z)\}$ cannot form a normal family in any $|z| < \rho_o$. If it could, there would exist a subsequence $\{f_{n_k}(z)\}$ uniformly convergent in $|z| \leq \rho < \rho_o$; that is $f_{n_k}(z) \to f(z) \equiv 0$ by (31) This implies $f'_{n_k}(0) \to 0$, contradicting the fact that $f'_{n_k}(0) = 1$. Hence f(z) cannot be embeddable.

In Baker [2] it was shown that if $f = z + a_2z^2 + ----$, is meromorphic in the plane, then it is not embeddable, except in the case:

$$f(z) = z/1-a_2z$$

The question was raised whether the result extends to any f which is single valued and our counter example 2 shows this is not universally true; there are embeddable single valued functions other than $\frac{z}{1-a_2z}$.

However theorems 1, 2, 3 give new classes of non-embeddable f, with natural boundaries given by certain curves. Examples where the boundaries form a discrete set are given by the following theorem 4, which slightly extends the work of Baker [2]

LEMMA 8

If the series (32) $f(z) = z + a_2z^2 + \sum_{k=3}^{\infty} a_kz^k \quad a_2 \neq 0$ is embeddable and if $f_n(z)$ are single valued in their whole domain of existence, $n = 1, 2, 3, ---$. Then there exists $\rho_o > 0$, such that for z in any annulus,

$$0 < \rho_2 \leq |z| \leq \rho_1 < \rho_o \text{ one has for all large enough } n$$

(i) $f_n(z)$ regular and

(ii) $f_n(z) \to 0$ uniformly as $n \to \infty$

Remark: The proof of this Lemma is given in Baker [2]]and is almost identical to that of Theorem 3. It will therefore not be reproduced here.

Theorem 4

If $f(z)$ is single valued and meromorphic in the whole complex plane except for a countable number of essential singularities, each isolated from each other, and if near the origin $f(z)$ has an expansion of the form

$$(33) \quad f(z) = z + a_2z^2 + \sum_{k=3}^{\infty} a_kz^k \quad a_2 \neq 0$$

then $f(z)$ is not embeddable, except in the case

$$f(z) = z/(1 - a_2z)$$

<u>Proof:</u> The case when f is meromorphic in the plane has been dealt with in [2] and covers the rational case in particular. We may therefore assume f has an essential singularity at say a. If $T(z) = -az/(z-a)$, then $g = T \circ f \circ T^{-1}(z) = z + a_2 z^2 + \text{----}$, has an essential singularity at ∞. Moreover f is embeddable in a family $f_\lambda = z + \lambda a_2 z^2 + \text{----}$, if and only if gis embeddable in a family $g_\lambda = T \circ f_\lambda \circ T^{-1} = z + \lambda a_2 z^2 + \text{----}$. Clearly g is single valued and has essential singularities at points $T(z')$ where z' are the essential singularities of f.

The iterates g_n are single valued and regular except at the countable set of points formed by

$\bigcup_{m=o}^{n-1} g_{-m}(s)$, where s is the set of singularities (including poles) of g. Once g_k has an essential singularity or pole at z', then so does g_n for $n > k$.

Suppose g is embeddable, then Lemma 8 shows that there exists ρ_o such that for any $o < \rho_1 \leq |z| \leq \rho_2 < \rho_o$ and all large enough n, g_n is regular in $\rho_1 \leq |z| \leq \rho_2$. Hence all g_n are regular in $o < |z| < \rho_o$. The lemma also states that $g_n(z) \to 0$ uniformly in $\rho_1 \leq |z| \leq \rho_2$ and hence on $|z| \leq \rho_2$, by the maximum modulus theorem. But this implies $g_n'(o) \to 0$ which contradicts $g_n'(o) = 1$. Hence the theorem is proved.

1.4 ON CONVERGENCE OF THE FAMILY f_λ

We now turn our attention to the values of λ corresponding to a positive radius of convergence for $f_\lambda(z)$. We shall in fact show that when these values of λ form a two dimensional lattice, $f(z)$ is embeddable.

Theorem 5. If the set of values of λ corresponding to a positive radius of convergence for $f_\lambda(z)$ includes a two dimensional lattice, then $f(z)$ is embeddable. (that is the set is in actual fact the whole plane).

Thus if f is not embeddable, f_λ converges only for $\lambda = n\ \lambda_o$, λ_o fixed, n integer.

Proof: For simplicity we shall consider $f_\lambda(z)$ in the form

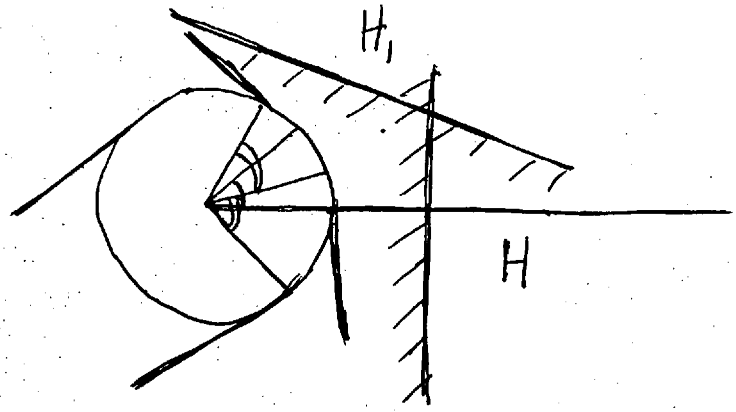
(33) $f_\lambda(z) = z + \lambda a_2 z^2 + \sum_{n=3}^{\infty} b_n(\lambda) z^n$ convergent for some positive R_1. For convenience we shift our fix point to infinity using $z = \frac{k}{t}$, $z_1 = \frac{k}{t_1}$, ---$z_\lambda = \frac{k}{t_\lambda}$, where $a_2 k = -1$ to get

(34) $g_\lambda(t) = t + \lambda + \sum_{k'=1}^{\infty} b_{k'}(\lambda) t^{-k'}$,

From Lemma 2 we recall,

(35) $A(t) = \lim_{n\to\infty} \{g_n(t) = n - b_1 \log n\}$ exists uniformly for $t \epsilon \Pi(k)$ as defined in Lemma 1. Also $A(t)$ is regular and univalent in $\Pi(k)$ with $A'(t) \to 1$ uniformly as $t \to \infty$ in $\Pi(k)$. Further since $t \epsilon \Pi(k)$ implies $g(t) \epsilon \Pi(k)$ we derive Abels equation.

(36) $A(g_n(t)) = A(t) + n$, $t \epsilon \Pi(k)$



Thus without loss of generality if $\lambda_o = 1$ and λ_1, $o < \arg \lambda_1 < \frac{\pi}{2}$ belong to the lattice (where our 2-D lattice is described by $m\lambda_o + n\lambda_1$, $\frac{\lambda_o}{\lambda_1} \neq$ real $\lambda_o \neq 0$, $\lambda_1 \neq 0$) then we can choose a right half plane H such that $H \subset \sqcap(k)$ as illustrated above and $t \in H \to g_{\lambda_1}(t) \in \sqcap(k)$ and hence Abels equation is satisfied.

$$(37) \quad A(g_{\lambda_1}(t)) = A(t) + \lambda_1$$

Differentiating (37) we get

$$(38) \quad A'(g_{\lambda_1}(t))\, g'_{\lambda_1}(t) = A'(t)$$

Thus

$$A'(t) = \{\prod_{n=o}^{N-1} g_{\lambda_1}'(g_{n\lambda_1}(t))\}\, A'(g_{N\lambda}(t))$$

Noting $g'_\lambda(t) \sim 1 + O(1/t^2)$, $g_{n\lambda_1} \sim t + n\lambda$, and $A'(t) \to 1$ as $t \to \infty$ in H we see that

$$(39) \quad A'(t) = \prod_{n=o}^{\infty} g_{\lambda_1}'(g_{n\lambda_1}(t)) \qquad \text{holds in } H .$$

Now using the fact that $g_{\lambda_1}(t)$ is convergent we similarly define

$$(40) \quad A_1(t) = \lim_{n \to \infty} \{g_{n\lambda_1}(t) - n_{\lambda_1} - \frac{b_1}{\lambda_1} \log n\}$$

This is regular in $\Pi_1(k)$ a domain similar to $\Pi(k)$ rotated through arg λ_1. In a similar manner we take a half plane H_1 Re $ze^{-i\lambda} > K$. Then $t \varepsilon$ H_1 implies $g_{\lambda_1}(t) \varepsilon \Pi_1(k)$ and Abels equation yields after differentiation,

$$(41) \quad A_1'(t) = \prod_{n=o}^{\infty} g_{\lambda_1}'(g_{n\lambda_1}(t)) \quad \text{regular in } H_1$$

Clearly $H \cap H_1$ is non empty and in this intersection the right hand side of (39) and (41) are equal. Hence $A'(t) = A_1'(t)$ hence $A_1'(t)$ can be considered as an analytic continuation of $A'(t)$.

Employing g_{-1}, $g_{-\lambda}$ and g_1 in a similar argument we get a full continuation of $A'(t)$ into planes Re $z < K'$, then Re $ze^{-i\lambda} < K''$ and finally Re $z > K'''$.

Thus $A'(t)$ has been continued right round the origin in a punctured neighbourhood of ∞ and the value obtained ~~by~~ on Re $z > K'''$ at the last step is (as in (41)) given by $\prod_{o}^{\infty} g'(g_n(t))$ which agrees with the original value in H. Thus $A'(t)$ is regular in a punctured neighbourhood of ∞.

We recall $A'(t) \to 1$ as $t \to \infty$ hence ∞ is at most a removable singularity for the $A'(t)$ function. Hence $A'(t)$ is regular in a full neighbourhood of ∞. Thus there is $A'(t)$ regular at ∞ and satisfying

$$(42) \quad A'(g(t)) = A'(t)/g'(t), \quad A'(t) = 1 - \frac{b_1}{t} \text{-----},$$

$$g(t) = g_1(t) = t + 1 + \frac{b_1}{t} + \text{----}.$$

Now in fact the converse of Lemma 4 holds, i.e. we can conclude in this case that g (and hence f) is embeddable. This has been proved independently by Erdös and Jabotinsky [1, P361-76] and by Baker [1 p289-290]

In fact if

$$A'(t) = 1 - \frac{b_1}{t} + \frac{c_1}{t^2} + \text{----}, \text{ set}$$

$$(43) \quad A(t) = t - b_1 \log t - \frac{c_1}{t}\text{---}$$

Since if we put $\frac{w}{t} = (1 + \nu)$ in the equation

$$(44) \quad A(w) = A(t) + \lambda, \qquad \text{we get}$$

$$\nu - \frac{b_1}{t} \log (1 + \nu) - \frac{c_1}{t^2}(1 + \nu) + \text{----} = \frac{\lambda}{t} - \frac{c_1}{t^2}\text{------}$$

which is satisfied for $1/t = 0$, $\nu = 0$, is analytic near $1/t = 0$, $\nu = 0$ (with $\log (1 + \nu) = \nu - \frac{\nu^2}{2} + \text{-----}$) and the derivative of the left hand side for $\nu = 1$ at $1/t = 0$, $\nu = 0$, there is a regular solution $\nu = \frac{\lambda}{t} + \text{----}$ at $t = \infty$.

$$(45) \quad w = t + \lambda + \text{-----} \qquad = h_\lambda(t) \text{ say}$$

The functions $h_\lambda(t)$ form a commuting family and are the only functions of form (45) at ∞ which satisfy (44). But g satisfies (42). Thus if t and so $g(t) \sim t + 1$ are large and choosing the logs so that $\log g(t) - \log t = \log \frac{g(t)}{t} \sim 0$ we do have

$A(g) - A(t) =$ constant δ in a whole neighbourhood of ∞ and by (42), (43) $\delta = 1$. Thus our assertion is established (incidentally by a proof rather different from those quoted which depend on differential equations existence theorems.).

1.5 HISTORICAL REMARKS

The theory of natural iteration and of embedding functions $f(z) = a_1z + a_2z^2 + \text{----}$, near the fix point 0 was treated for $|a_1| < 1$ in the late nineteenth century and early twentieth, e.g. by Koenigs [1]

The somewhat more difficult case $a_1 = 1$ was treated as regards the behaviour of f_n, n integer by Leau [1] and Fatou [1,2].

The case where a_1 is a root of unity is essentially equivalent to $a_1 = 1$. When $a_1 = \exp(\pi i \theta)$, θ irrational the situation becomes extremely complicated, c.f. Cremer [1], Siegel [1] and Russmann [1].

The embedding problem for $a_1 = 1$ received some attention after Baker [3] showed that if $f(z) = e^z - 1$ then f_σ has positive radius of convergence for real σ only if σ is an integer. There followed more or less simultaneous papers of Baker [1] and Erdős-Jabotinsky [1] including e.g. the result that the σ corresponding to positive radius of convergence form a one or two dimensional lattice or the whole plane (embeddable case).

M. Lewin [1], a student of Jabotinsky showed that $z + z^2$, $\dfrac{z}{(1-z)^2}$ are non-embeddable and Szekeres [1] and Baker [2] showed respectively that entire and meromorphic functions are non-embeddable except for $z/1-\alpha z$.

Later A. Ran [1] showed the algebraic function $z/\sqrt{1+z}$ is non-embeddable and Baker [4] extended this establishing a more general result and proving that a large class of functions is non-embeddable.

In a recent note J. Écalle [1] has announced an extension of the above results which would yield in our case when $a_2 \neq 0$: If $f = f_1 + f_2 = z + a_2 z^2 + ----$, where f_1 is a branch of an algebraic function and f_2 is meromorphic in the plane and if $a_2^2 - a_3 \neq 0$ then f is not embeddable.

Our contribution to the theory contains:-

(1) disproof of the suggestion that all single valued f are non-embeddable (except for trivial cases) by construction of counter example 2.

(2) Construction of new classes of non-embeddable functions especially examples with natural boundaries (Theorems 1, 2, 3, especially 3).

(3) A proof that either f is embeddable or else f_λ converges only for λ of the form $n\lambda_o$, $\lambda_o \neq 0$ fixed, n integer (Theorem 5).

In addition a number of new proofs and discussions of established Lemmas are given.

CHAPTER II - GLOBAL ITERATION THEORY

2.1 INTRODUCTION:- The theory of the iteration of a rational or entire function $f(z)$ of the complex variable z deals with the sequence of natural iterates $f_n(z)$ defined by $f_1(z) = f(z)$, $f_{n+1}(z) = f_1(f_n(z))$, $n = o, 1,2,$-----

In the theory developed by Fatou [1,2] and Julia [1] an important part is played by the set $F = F(f)$ of those points of the complex plane where $\{f_n(z)\}$ is not a normal family. Unless $f(z)$ is a rational function of order 0 or 1, the set $F(f)$ has the following properties, proved by Fatou for rational functions in [1] and for entire functions in [2]

I $F(f)$ is a non empty perfect set.

II $F(f_n) = F(f)$ for any integer $n \geq 1$

III $F(f)$ is completely invariant under the mapping $z \to f(z)$, i.e. if α belongs to $F(f)$ then so do $f(\alpha)$ and every solution β of $f(\beta) = \alpha$

IV For every $\alpha \in F(f)$ and for every complex β (excluding at most two exceptional β-values) there exists a sequence of positive integers n_k $k = 1, 2,$--- and a sequence of complex numbers α_k such that

$$f_{n_k}(\alpha_k) = \beta \quad \text{and} \lim_{k\to\infty} \alpha_k = \alpha$$

A fix point α of order n of $f(z)$ is a solution of $f_n(z) = z$. It has exact order n if $f_k(\alpha) = \alpha$ for $k = n$ but not for any $k < n$ and in this case $f_n'(\alpha)$ is called the multiplier of α provided $\alpha \neq \infty$. The fix point is called attractive indifferent or repulsive according as the modulus of its multiplier is less than, equal to or greater than 1, respectively. With this nomenclature we have:

V $F(f)$ contains every repulsive fix point of f of any order. Attractive fix points belong to the complement of $F(f)$

VI $F(f)$ is the derivative of the set of fix points (of all orders) of f; indeed by Baker [5] it is the derivative of the set of repulsive fix points.

The complement $\mathcal{C} = \mathcal{C}(f)$ of $F(f)$ is an open set, whose components of connectivity are the maximal domains of normality of $\{f_n(z)\}$. By III $\mathcal{C}(f)$ is completely invariant. One of the main problems of the theory is to investigate the various possibilities for the structure of $F(f)$ and the way in which $F(f)$ divides the plane. Results on this problem have been obtained in ths original papers of Fatou and Julia and later by Töpfer [1], Myrberg [1-3] and Baker [5-8].

It follows from III and IV that if $F(f)$ has a non empty interior then it is the whole plane. In the case of rational f, this can occur as was shown by Lattes [1] in the case where f is the rational function such that $P(2u) = f(P(u))$, $P(u)$ being the Weierstrass elliptic function. Baker [9] has now solved the long standing open problem proposed by Fatou in [2] on finding an entire transcendental f, for which $F(f)$ is the whole plane.

P. Bhattacharyya [1] in a recent study, has looked at the growth of entire functions with infinite domains of normality. Under some conditions, one can, given components of $\mathcal{C}(f)$, say something about the growth of f in certain regions.

He also discussed the nature of $F(f)$ and $\mathcal{C}(f)$ when $f(z) = e^z - 1$. He showed that for this f, the half plane $H \equiv \text{Re } z < 0$ is an invariant domain $f(H) \subset H$ belonging to $\mathcal{C}(f)$ and is contained in a completely invariant domain $G^* \subset \mathcal{C}(f)$. From the complete invariance of G^* it is clear that G^* includes also the strips

$s_n i \{(2n + 1/2)\pi < y < (2n + 3/2)\pi, \ \ x \geq 0 \}$

This suggests an investigation of the restrictions placed on f by assuming that its F and $\mathcal{C}$ have properties similar to the above. We shall show that the existence of an invariant left half plane and of a horizontal strip extending to infinity on the right, such as s_n in $\mathcal{C}(f)$ give restrictions on the growth of f in the strip. We also show that for a function of order ≤ 1 there cannot be a sequence of such strips of arbitrarily large width in addition to an invariant left half-plane.

2.2 ON ENTIRE FUNCTIONS WITH INFINITE DOMAINS OF NORMALITY

First of all we quote the following Lemmas;

LEMMA 2.1 Landau and Valiron [1]

Let $\phi(z)$ be regular and $\text{Re } \phi(z) \geq 0$ for $\zeta = \text{Re } z > 0$

Then we can find $c \geq 0$ so that writing $\phi(z) - cz = \psi(z))$

(i) $\text{Re } \psi(z) \geq 0$ for $\zeta > 0$

(ii) for every $p > 0$ $\frac{\psi(z)}{z} \to 0$ uniformly in $|z| \leq p\zeta$ as $|z| \to \infty$

LEMMA 2.2 P. Bhattacharayya [1, Theorem 2.7]

Let D be an infinite simply connected invariant domain of an entire function. Let D contain an angle A

$\alpha < \arg(z - z_o) < \alpha + \beta$, $\beta > 0$, Then for arbitrary $\epsilon > o$ $|f(z)| = O(z^k)$ for some $k = k(\beta,\epsilon)$ as $|z| \to \infty$ in $\alpha + \epsilon < \arg(z - z_o) < \alpha + \beta - \epsilon$

We now state and prove our first theorem which will be applied in our later work on iteration.

Theorem 2.1

Let $f(z)$ be entire of order ≤ 1 and zero exponential type and suppose further that on the negative real axis, there exists k such that $f(-r) = O(r^k)$ as $r \to \infty$ and suppose that all the zeros of $f(z)$ lie in the right half plane $\operatorname{Re} z \geq 0$. Then $f(z)$ is a polynomial.

Proof: We may assume $f(o) \neq 0$ (otherwise consider $f(z + \delta)$ for a > suitable $\delta > 0$. Suppose f is of zero exponential type and not a polynomial, so it has an infinity of zeros, all in $\operatorname{Re} z \geq 0$

In the first part, we shall be employing Carleman's theorem, Titchmarsh [1, p 130]

We evaluate

$\operatorname{Re} \frac{1}{2\pi i} \int_c \log f(z) \left(\frac{1}{z^2} + \frac{1}{R^2}\right) dz$ by contour integration round c, where c is a contour consisting of the semi circles $|z| = \rho$, $|z| = R$ $-\pi/2 \leq \arg z \leq \pi/2$ and the parts of the imaginary axis joining them, and $f(z)$ has zeros $a_j = r_j e^{i\theta_j}$, $j = o, 1, \ldots$ inside c. Thus we get

$$\sum \left(\frac{1}{r_j} - \frac{r_j}{R^2}\right) \cos \theta_j = \frac{1}{\pi R} \int_{-\pi/2}^{\pi/2} \log \; f(Re^{i\theta})| \cos \theta \, d\theta$$

$$+ \frac{1}{2\pi} \int_\rho^R \left(\frac{1}{y^2} - \frac{1}{R^2}\right) \log \; |f(iy)| dy - \operatorname{Re} \frac{1}{2\pi i} \int_{\substack{|z|=\rho \\ -\pi/2 \leq \arg z \leq \pi/2}} \log f(z)\left(\frac{1}{z^2} + \frac{1}{R^2}\right) dz$$

Similar considerations with a contour c_1 of some type as c with $\pi/2 \leq \arg z \leq 3\pi/2$ i.e. in the left half plane and noting f(z) has no zeros in this region, yields

$$0 = \frac{1}{R}\int_{\pi/2}^{3\pi/2} \log|f(Re^{i\theta})| \cos \ d\theta - \operatorname{Re}\frac{1}{2\pi}\int_{\substack{|z|=\rho \\ \pi/2\leq\arg z\leq 3\pi/2}} \log f(z)\left(\frac{1}{z^2}+\frac{1}{R^2}\right) dz$$

$$-\frac{1}{2\pi}\int_{\rho}^{R}\left(\frac{1}{y^2}-\frac{1}{R^2}\right)\log|f(-iy)|dy$$

$$-\frac{1}{2\pi}\int_{\rho}^{R}\left(\frac{1}{y^2}-\frac{1}{R^2}\right)\log|f(iy)|dy$$

Adding we get

$$\sum_{r_j\leq R}\left(\frac{1}{r_j}-\frac{r_j}{R^2}\right)\cos\theta_j = \frac{1}{\pi R}\int_0^{2\pi}\log|f(Re^{i\theta})| \cos\theta\, d\theta$$

$$-\operatorname{Re}\frac{1}{2\pi i}\int_{|z|=\rho}\log f(z)\left(\frac{1}{z^2}+\frac{1}{R^2}\right)dz$$

If we assume no zeros of f(z) in $|z| \leq \rho$, then we can take log f(z) regular and by residue theorem,

$$\operatorname{Re}\frac{1}{2\pi i}\int_{|z|=\rho}\log f(z)\left(\frac{1}{z^2}+\frac{1}{R^2}\right)dz = \operatorname{Re}\left\{\frac{f'(o)}{f(o)}\right\}$$

Thus $$\sum_{r_j\leq R}\left(\frac{1}{r_j}-\frac{r_j}{R^2}\right)\cos\theta_j \leq -\operatorname{Re}\left\{\frac{f'(o)}{f(o)}\right\}+\frac{1}{\pi R}\int_0^{2\pi}\log|f\ Re^{i\theta}|\cos\theta\, d\theta$$

$$< -\operatorname{Re}\left\{\frac{f'(o)}{f(o)}\right\}+\frac{1}{\pi R}\int_E\log|f(Re^{i\theta})|\cos\theta\, d\theta$$

E is the set of θ where $\log|f(Re^{i\theta})| \cos\theta \geq 0$

$= E_1 \cup E_2$ where E_1 = set where $\log|f(Re^{i\theta})| > 0$, $\cos\theta > 0$

and E_2 = set where $\log |f(Re^{i\theta})| < 0$, $\cos\theta < 0$

$$\leq -\mathrm{Re}\{\frac{f'(o)}{f(o)}\} + \frac{1}{\pi R}\{m(R,f) + m(R,\frac{1}{f})\}$$

$$\leq -\mathrm{Re}\{\frac{f'(o)}{f(o)}\} + \frac{1}{\pi R}\{T(R,f) + T(R,\frac{1}{f})\}$$

Since $T(R,f) - T(R,1/f) = O(1)$, $T(R,f) < \log m(R,f) < \quad \epsilon R$

$$\frac{1}{\pi R}\{T(R,f) + T(R,1/f)\} \to 0 \quad \text{as } R \to \infty$$

hence for all sufficiently large R,

$$\sum_{|r_j| \leq R} (\frac{1}{r_j} - \frac{r_j}{R^2}) \cos\theta_j \leq -\mathrm{Re}\{\frac{f'(o)}{f(o)}\} + 1 \qquad - \quad (1)$$

We now consider the series $\sum_{j=1}^{\infty} \frac{\cos\theta_j}{r_j}$

For any N, we choose R so large that

$\sum_1^N \frac{\cos\theta_j}{r_j} < \sum_1^N (\frac{1}{r_j} - \frac{r_j}{R^2}) \cos\theta_j + 1$, thus by argument above

$$\sum_1^N \frac{\cos\theta_j}{r_j} < -\mathrm{Re}\{\frac{f'(o)}{f(o)}\} + 2$$

But $\frac{\cos\theta_j}{r_j} \geq 0$ for all j, so that $\sum_{j=1}^{\infty} \frac{\cos\theta_j}{r_j}$ converges and has a sum λ, say $\lambda \geq 0$

By the hypothesis of the theorem, f(z) is of genus at most 1 and so we can write,

$$f(z) = e^{cz} \prod_1^{\infty} (1 - z/a_n)e^{z/a_n}, \text{ where } a_n = r_n e^{i\theta_n}$$

are the zeros of f. We also introduce

$$g(z) = \overline{f(\bar{z})} = e^{\bar{c}z} \prod_{1}^{\infty} (1 - z/\bar{a}_n)e^{z/\bar{a}_n}$$

If there exists k so that

$$f(-r) = O(r^k)$$

then $\quad g(-r) = \overline{f(-r)} = O(r^k)$

and $\quad F(z) = f(z)g(z)$ satisfies

$$F(-r) = O(r^{2k})$$

But

$$F(z) = e^{(c+\bar{c})z} \prod_{1}^{\infty} (1 - z/a_n)(1 - z/\bar{a}_n)e^{(z/a_n + z/\bar{a}_n)}$$

since $\dfrac{1}{a_n} + \dfrac{1}{\bar{a}_n} = \dfrac{2\cos\theta_n}{r_n}$, the product $\prod_{1}^{\infty} e^{(z/a_n + z/\bar{a}_n)} = e^{\lambda z}$

is convergent and

$$F(z) = e^{(c+\bar{c}+\lambda)z} \prod_{n=1}^{\infty} (1 - z/a_n)(1 - z/\bar{a}_n)$$

Now for $z = -r$, because a_n and $\bar{a}_n$ are in the right half plane, Re $z \geq 0$ we have

$$\left|\frac{a_n + r}{a_n}\right| \geq 1 \qquad \text{and} \qquad \left|\frac{\bar{a}_n + r}{a_n}\right| \geq 1$$

and so

$$|F(-r)| \geq e^{-dr}$$

where $d = c + \bar{c} + \lambda$ is real

Since f and hence g and F are of order 1, and type zero, we see that $d \geq 0$

$$F(z) = e^{dz}\,\Pi\{1 - (z/a_n + z/\bar{a}_n) + \frac{z^2}{|a_n|^2}\}$$

$$F(-r) = e^{-dr}\Pi\{1 + 2c_n r + r^2/r_n{}^2\}, \text{ where } c_n = \frac{\cos\theta_n}{r_n} \geq 0$$

$$\text{and } r_n = |a_n|$$

$$= e^{-dr} P, \text{ say}$$

$$P = \prod_{n=1}^{n_o-1} \{1 + 2c_n r + r^2/r_n{}^2\} \prod_{n=n_o}^{\infty} \{1 + 2c_n r + r^2/r_n{}^2\}$$

We choose n_o so that given $\epsilon > 0$

$$0 < 2\sum_{n_o}^{\infty} \frac{\cos\theta_n}{r_n} < \epsilon \qquad (2)$$

Then

$$\prod_{n_o}^{\infty} \{1 + 2c_n r + r^2/r_n{}^2\} \leq \prod_{n=n_o}^{\infty} (1 + 2c_n r)(1 + r^2/r_n{}^2)$$

Now, $\Pi(1 + 2c_n r) < \exp\{2c_n r\} < \exp(\epsilon r)$ by (2)

We also have

$$\log \prod_{n_o}^{\infty} (1 + r^2/r_n{}^2) = \sum_{n=n_o}^{\infty} \log(1 + r^2/r_n{}^2)$$

$$\leq \int_{\delta}^{\infty} \log(1 + r^2/t^2)\,dn(t)$$

$$= n(t)\log(1 + r^2/t^2)\Big|_{\delta}^{\infty} + 2\int_{\delta}^{\infty} \frac{n(t)r^2\,dt}{t(t^2 + r^2)}$$

where $n(t)$ counts the a_n, $\bar{a}_n$ and where $\delta > 0$ is so small that $n(\delta) = 0$. Then the integrated part vanishes in the above inequality.

Since F is of zero exponential type we have T(r) and hence N(r) and n(r) less than r for larger. Then

$$\log \prod_{n=n_0}^{\infty} (1 + r^2/r_n^{\,2}) \le 2\int_{\delta}^{\infty} \frac{\epsilon \cancel{t} r^2 dt}{\cancel{t}(t^2+r^2)} \le \pi\epsilon r$$

Thus $P \le p(r) \exp(\epsilon r + \pi\epsilon r)$ where p(r) is a polynomial in r, and if $d > 0$, we may assume $d > (2 + \pi)\epsilon$ so that as $r \to \infty$,

$$0 < F(-r) < e^{-\epsilon r}, \quad \epsilon > 0 \tag{3}$$

Next we appeal to a theorem in Valiron [1, p 89] which says that if q(z) is an entire function of finite order and $k > 1$, there is a corresponding constant $H = H(k)$ such that for large R, the annulus $R \le |z| \le kR$ contains at least one circle $|z| = r$ on which

$$|q(z)| \ge \{M(q,kr)\}^{-H}$$

Thus we can take $k = 2$ say, $H = H(2)$ and then in any interval $s \le r \le 2s$, there is at least one value r with

$$|q(-r)| \ge \min_{|z|=r} |q(z)| \ge \{M(q,2r)\}^{-H(2)}$$
$$\ge \{\exp(2\eta r)\}^{-H}$$
$$= \exp(-2\eta Hr)$$

In this estimate η may be taken arbitrarily small. Applying this to F(z) with η chosen so that $2\eta H < \epsilon$, we see that the assumption $d > 0$, on which (3) depends, leads to a contradiction.

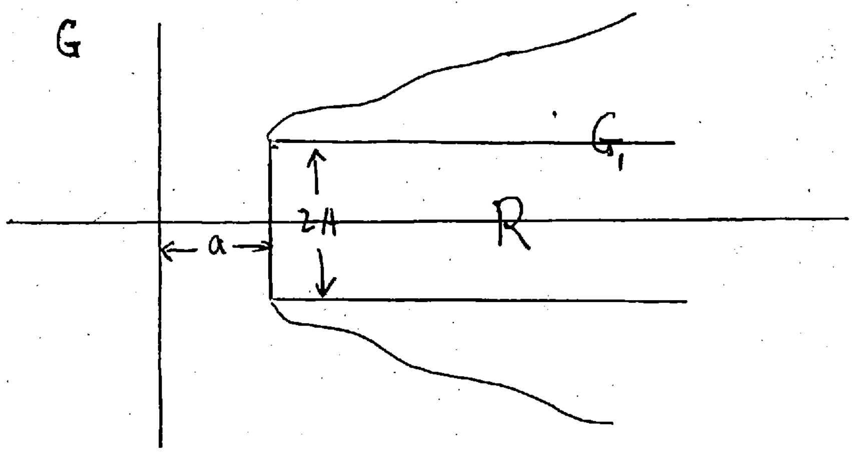
Thus $d = 0$ and

$$F(z) = \prod_{n=1}^{\infty} (1 - z/a_n)(1 - z/\bar{a}_n)$$

But since $|(1 - z/a_n)(1 - z/\bar{a}_n)| > 1$ for all $z = -r$, $r > 0$ we see that $F(-r) \to \infty$ faster than any polynomial as $r \to \infty$. Thus the assumptions of the theorem and the assumption, made at the beginning of the proof that f is not a polynomial are mutually inconsistent. The theorem follows.

We now proceed to state and prove a theorem which tells us about the growth of our function $f(z)$ once $\mathcal{C}(f)$ has been characterised.

Theorem 2.2. Suppose $f(z)$ is an entire transcendental function of order ≤ 1 and $\mathcal{C}(f)$ has a simply connected component G which contains the left half plane. Suppose also that the region R: $\{x > a > 0,\ |y| < A\}$ belongs to $\mathcal{C}(f)$. Then $f(z) = O(e^{k|z|/A})$ in every R_1, where R_1: $\{x > a > 0,\ |y| < A - \delta'\}$ for arbitrary δ'. k an absolute constant.



Proof: Let k denote the right half plane from which the unit semi-circle has been removed. We consider the transformation $z = a + \mu \log t$ $\mu = \frac{2A}{\pi}$ which maps k to R. R belongs to some component, say G_1 of $\mathcal{C}(f)$. Since $F(f)$ and $\mathcal{C}(f)$ are invariant (property III of the introduction) $f(G_1)$ belongs to a component of $\mathcal{C}(f)$. Töpfer 1 has proved that if

$\mathcal{C}(f)$ has an infinite component G_o say, then every other component of $\mathcal{C}(f)$ is simply connected. If we take G_o to be the component of $\mathcal{C}(f)$ which contains the left half plane (and by our assumptions is simply connected as well as infinite) we see that in our case every component of $\mathcal{C}(f)$ is simply connected.

Since $R \subset G_1$, $f(R)$ lies in a simply connected component G_2 of $\mathcal{C}(f)$.

Let v_o be a point of $F(f)$. Then the ray $v = v_o - \tau$, $\tau \geq o$ lies in the left half plane and so in G for all large τ, and so meets $F(f)$ a last time at say, v_1 on the frontier of G $(v_1 \varepsilon F)$ The function $w = (v - v_1)^{1/4}$ maps the simply connected region G into four disjoint regions $G^{(1)}$, $G^{(2)}$, $G^{(3)}$, $G^{(4)}$ one containing each of the rays $\arg w = \frac{\pm\pi}{4}, \pm \frac{3\pi}{4}$ except for the end point $w = o$ of these rays, which is a boundary point of each $G^{(i)}$.

We now take any one $\hat{G}$ say, of the four images of G_2 under the same transformation $W = (v - v_1)^{1/4}$. $\hat{G}$ is either one of the $G^{(1)} \ldots G^{(4)}$ or is disjoint from all of them. In either case $\hat{G}$ lies in a half plane $\theta < \arg z < \theta + \pi$

Then $w = (f(\zeta) - v_1)^{1/4} = \{f(a + \mu \log t) - v_1\}^{1/4} = g(t)$ say maps K into a half plane. For some α,

$r = e^{i\alpha} \{f[a + \mu \log(t + 1)] - v_1\}^{1/4}$ maps half plane $\operatorname{Re} t > 0$ into $\operatorname{Re} r > 0$.

By Lemma 2.1 $\quad |f(a + \mu \log (t + 1)) - v_1|^{1/4} < k_1|t|$

$|f(a + \mu \log (t + 1)| < K|t|^4$ as $t \to \infty$ in an angle $|\arg t| < \pi/2 - \delta$

Considering

$z = a + \mu \log (t + 1)$ we see that

$$|f(z)| < k\left|\frac{z-a}{\mu} - 1\right|^4 = k\left|\frac{\pi(z-a)}{2A} - 1\right|^4$$

$$< k(1 + e^{\frac{\pi|z|}{2A}} \cdot e^{\frac{-\pi a}{2A}})^4$$

$$< k' e^{2\pi|z|/A} \quad \text{as } z \to \infty \text{ in a slightly smaller strip } R_1$$

as defined in the statement of the theorem. This concludes the proof:

<u>Further discussion of</u> f <u>in Theorem 2.2</u>

We now go on to look at the indicator function. Since we are looking at entire functions of order ≤ 1, the indicator function $h(\theta)$ is defined by

$$h(\theta) = \limsup_{r\to\infty} \log |f(re^{i\theta})| r^{-1}$$

clearly $h(o) \leq 2\pi/A$

In the left half plane ($\subset G$) $f:G$ either $= G$ or a component disjoint from G. If $f:G = G$, that is G is an invariant domain, Lemma 2.2 shows that $f(z) = O(z^k)$ in a smaller half plane. If however the values of $f:G$ lie in the right half plane ($-f$ has an invariant half plane) and the same conclusion holds. In either case $h(\theta) = 0$ for $\pi/2 < \theta < 3\pi/2$. By the continuity of $h(\theta)$ Boas [1] $h(\pm \pi/2) = 0$.

Also by applying a result in Titchmarsh [1, 5.71] we put $H(\theta) = \frac{2\pi}{A} \cos \theta$, i.e. $H(\theta)$ is the unique sinosoid which takes the values $\frac{2\pi}{A}$, 0 at 0, $\pm \pi/2$ respectively. Then

$$h(\theta) \leq \frac{2\pi}{A} \cos \qquad |\theta| \leq \pi/2$$

$$= 0 \quad \text{otherwise}$$

In particular f is of exponential type $\leq \frac{2\pi}{A}$. For $\epsilon > 0$, $M(f,r) < ke^{(2\pi A+\epsilon)r}$ for all large enough r.

Remark: Consideration of $g(z) = \frac{1}{\epsilon}(e^{\epsilon z} - 1)$ with $\mathcal{C}(g)$ containing strips $\pi/2\epsilon < y < 3\pi/2\epsilon$ of width π/ϵ shows that our result is fairly sharp.

Further deductions from Theorems 2.1 and 2.2

Let us suppose there exists a succession of parallel strips of increasing width $A_k \to \infty$ in the right half plane, each belonging to $\mathcal{C}(f)$, and assume as before a simply connected domain containing the left half plane in $\mathcal{C}(f)$.

We first consider the indicator function $h(\theta)$ earlier defined. We show that for a function of exponential type, $h(\theta)$ is the same for $f(z)$ and $f(z + c)$ for any c. Since $f(z)$ is of exponential type,

$$\log |f(z)| < k|z| \qquad \text{say} \quad \text{as } z \to \infty$$

$h(\theta)$ is finite for all θ, in fact $h(\theta) < k$. Since $h(\theta)$ is continuous for all θ, given $\epsilon > 0$, if $|\theta| < \delta = \delta(\epsilon)$, $h(\theta) < h(o) + \epsilon$. Take a small η, $o < \eta < \delta$. Consider

$$g(z) = f(z)\exp\{-\frac{(h(o) + 3\epsilon)z}{\cos \eta}\}$$

On the rays $|\theta| = |\arg z| = \eta$ we have

$$|g(z)| < |f(re^{i\eta})| \exp\{-\frac{(h(o) + 3\epsilon)r \cos \eta}{\cos \eta}\}$$

$$< \exp(h(\eta) + \epsilon)r \;\; \exp\{-(h(o) + 3\epsilon)\, r\}$$

$$< \exp\{h(o) + 2\epsilon - h(o) - 3\epsilon\}\, r = \exp(-\epsilon r) \text{ for large } r$$

g is bounded on $\theta = \pm\eta$ and is of order 1. Therefore by a Phragmen-Lindelof result (c.f. Titchmarsh [1, p 177]) g is bounded in $|\arg z| \leq \eta$

Therefore $|f(z)| < M \left|\exp\left(\frac{h(o) + 3\epsilon}{\cos \eta}\right) z\right|$ in $|\arg z| \leq \eta$

We can take η so small that

$$\frac{h(o) + 3\epsilon}{\cos \eta} < h(o) + 4\epsilon \text{ and then}$$

$$\log |f(z)| < (h(o) + 4\epsilon)|z| + O(1) \text{ in } |\arg z| \leq \eta$$

Therefore in $|\arg z| < \eta$, $\log |f(z)| < \{h(o) + 4\epsilon\}|z| + O(1)$ for all large enough z. For any c, let $h_c(\theta) = h(\theta)$ for $f(z+c)$. Hence

$$h_c(o) = \limsup_{x\to\infty} \frac{\log |f(x+c)|}{x}$$

For large x, x + c is in $|\arg z| < \eta$ and so

$$\log |f(x+c)| < \{h(o) + 4\epsilon\} |x + c| + O(1)$$

$$\frac{\log |f(x+c)|}{x} < (h(o) + 4\epsilon)\frac{|x + c|}{x} + O(1/x)$$

hence $h_c(o) \leq h(o)$

But $f(z) = f(z+c-c)$ implies $h(o) \leq h_c(o)$. Thus $h_c(o) = h(o)$ and similarly for any θ, $h_c(\theta) = h(\theta)$. With this property of the indicator function established, by looking at the strips of increasing width A_k, we conclude that $h(o) = 0$ for our function f and since $h = o$ for $\pi/2 \leq \theta \leq 3\pi/2$ it is therefore a function of zero exponential type by the argument used above.

But by theorem 2.1 f(z) must be a polynomial. Hence if there are strips with arbitrarily large A_n, f is of zero exponential type. The fact that the left half plane belongs to $\mathcal{C}$ implies F belongs to Re z $\geq$ o. Take any α F, then all solutions of $f(z) = \alpha$ are in F and so belong to the right half plane. As pointed out above f(z) is of polynomial growth in the left half plane which belongs to $\mathcal{C}(f)$. Thus theorem 2.1 applied to $f(z) - \alpha$ would show $f(z) - \alpha$ is a polynomial.

We state our result in the form:

<u>Theorem 2.3</u> <u>If</u> f <u>is an entire transcendental function of order</u> ≤ 1 <u>with the left half plane belonging to some component of</u> $\mathcal{C}(f)$ <u>then there exists a constant</u> c <u>such that every strip of the form</u> $|y - y_o| < c$, $x > B$ <u>meets</u> F(f).

CHAPTER III - PICARD SETS FOR ENTIRE TRANSCENDENTAL FUNCTIONS

3.1 INTRODUCTION: Following Lehto [1] we state:-

Definition 3.1: A PICARD set E is a totally disconnected closed set in the extended z plane in whose complement CE, say, each single valued meromorphic function f(z) with at least one essential singularity in E takes every value infinitely often except for at most two values.

By Picards theorem a transcendental meromorphic function assumes infinitely often all values in the plane except at most two. Hence given any set with a finite number of points, it is automatically a Picard set. The obvious question is: Are there infinite point sets which are Picard sets?

Lehto [1] has answered this question by showing that all sufficiently thin countable sets with one limit point are also Picard sets. More precisely

Theorem 3.1

Let $\{a_n\}$, n = 1,2,3,---- be a point set whose points converge to infinity. Let E denote the union of $\{a_n\}$ n = 1,2,3,---- and the point at infinity. Then E is a Picard set if the points $\{a_n\}$ satisfy the condition

(1) $(\log |a_n|)^{2+\alpha} = O(\log |a_{n+1}|), \quad \alpha > 0.$

Matsumoto [1] then established the same result with

(2) $|a_n|^3 = O(|a_{n+1}|)$

Toppila [1] has now shown that the inequality (2) can be improved to

(3) $|a_n|^2 = O(|a_{n+1}|)$ and that this is best possible.

One can also define the idea of a Picard set for entire transcendental functions.

Definition 3.2. A set E in the complex plane is a Picard set for entire functions if for every entire (non-rational) f, f takes all values in the complement CE of E with at most one exception.

Remarks: (1) E need not be totally disconnected in this definition

(2) We note that if E is not a Picard set for entire functions, then there exists an entire transcendental function f(z) which takes 0 and 1 only in the set E. For entire functions, with which we are more concerned, Lehto established the following result.

Theorem 3.2. A set $E = \{a_n\}$ $n = 1,2,3,\text{---}$ is a Picard set for entire functions if

$$(4)\quad \left|\frac{a_n}{a_{n+1}}\right| = O(1/n^2)$$

By restricting the points of E to lie on a ray, the condition (4) can be weakened to yield

$$(5)\quad \left|\frac{a_{n+1}}{a_n}\right| \geq q > 1$$

Remark: We note that the entire function $f(z) = 1/2(1 + \cos\sqrt{z})$ takes 0 and 1 at the points $z = n^2\pi^2$ $n = 0,1,2,\text{----}$ with

$$(6)\quad \left|\frac{a_{n+1}}{a_n}\right| > 1 + 2/n$$

The condition (6) does not imply E to be a Picard set.

We shall in the following pages give a new proof of the fact that (5) is sufficient to ensure a Picard set for entire functions. We shall then weaken the separation between the $\{a_n\}$'s and derive conditions for Picard sets for entire functions. Finally we shall give some new theorems which improve the so far known conditions to ensure for certain sets of small discs to be Picard sets for entire transcendental functions.

We shall in a later chapter apply our results on Picard sets to the factorisation of an entire function F(z) in the form F(z) = f(g(z)) f, g entire. i.e. to a problem related to iteration theory.

3.2 Conditions for a sequence $\{a_n\}$ to be a Picard set

First of all we quote Schottky's theorem. (Hayman [1]).

LEMMA 3.1 *If $f(z)$ is regular and satisfies $f(z) \neq 0, 1$ in $|z| < 1$, then*

(7) $M(f,r) \leq \Omega(a_o,r)$ *where* $r = |z| < 1$, *and* $\Omega(a_o r)$ *satisfies*

(8) $\Omega(a_o,r) < \exp \frac{1}{1-r} \{(1+r) \log \max (1,|a_o|) + 2Cr\}$,

C *an absolute constant*, $a_o = f(o)$.

We now state and prove our second Lemma.

LEMMA 3.2 *Let $f(z)$ be an entire function and $\{a_n\}$ be a sequence with $|a_n| < |a_{n+1}|$, $|a_n| \to \infty$. If all the solutions of $f(z) = 0$ and $f(z) = 1$ occur at the points a_n, then there exists arbitrarily large n and points z_n with $|z_n| = \sqrt{a_n a_{n+1}}$ such that*

$$|f(z_n)| < 2$$

<u>Proof:</u> <u>If</u> $\alpha = 0$ or 1 is taken at most a finite number of times by $f(z)$, then there is a path Γ leading to ∞ on which $f(z) \to \alpha$. The points at which Γ cuts $|z| = \sqrt{|a_n a_{n+1}|}$ for large enough n satisfy the assertion of the lemma.

So we can assume f takes 0,1 infinitely often. We consider an arbitrarily large z_o such that $f(z_o) = 1$. Then z_o is an a_n. Then there is a curve γ through z_o on which $|f(z)| = 1$. There are two possibilities. Either γ is not closed but extends to infinity in which case argument above applies or γ is closed, enclosing a region D say in which $|f(z)| < 1$ and which contains a zero, a_n of $f(z)$. We let $z_o = a_m$ $a_m \neq a_n$. Then there is a path σ (in D apart from one end point at z_o) joining a_n, z_o and this meets either $|z| = \sqrt{|a_m a_{m-1}|}$ or $\sqrt{|a_m a_{m+1}|}$ at a point where $|f(z)| < 1$. Since $|z_o| = |a_m|$ may be chosen arbitrarily large the lemma is proved.

<u>Theorem 3.3</u> If $\left|\frac{a_{n+1}}{a_n}\right| \geq q > 1$ <u>then</u> $E = \{a_n\}$ $n = 1,2,3,$----
<u>is a Picard set for entire transcendental functions.</u> <u>Moreover we can obtain Picard sets</u> $E = \{a_n\}$ <u>in which</u> $|a_n| < |a_{n+1}| \to \infty$, $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ <u>as</u> $n \to \infty$
<u>In particular any set with</u>

(9) $|a_n| = \exp\left(\frac{\delta n}{\log n}\right)$ $\delta > k\pi^2$, $k > 1$ <u>is such a set.</u>

<u>Proof:</u> Under our hypothesis, $E = \{a_n\}$ with $|a_n| < |a_{n+1}| \to \infty$. Suppose E is not a Picard set. Then there exists an entire transcendental function f, whose zeros and ones lie entirely in E. Then by Lemma 3.2, there are arbitrarily large n such that the annulus A: $|a_n| < |z| < |a_{n+1}|$ contains no zeros or ones of f and there is a z_n with $|z_n| = \sqrt{|a_n a_{n+1}|}$, $|f(z_n)| < 2$. We may without loss of generality suppose z_n is on the

real axis. We map A by the transformation $z = e^t$ to a strip,

$$\log \alpha = \log |a_n| < \mathrm{Re}\,t < \log |a_{n+1}| = \log \beta$$

using $u = c + dt$, $d = \pi/2\log\beta/\alpha$ the strip is mapped into a second strip,

$\frac{-\pi}{4} < \mathrm{Re}\,u < \pi/4$ which is mapped by $w = \tan u$ onto the unit disc $|w| < 1$. The circle $|z| = \rho e^{it}$, $\rho = \sqrt{|a_n a_{n+1}|}$, $-\pi \leq t \leq \pi$, maps into a segment $w = i\tau$, $|\tau| \leq \tan h \frac{\pi^2}{2\log \beta/\alpha} = 1 - \frac{2}{\{e^{\pi^2/\log\beta/\alpha} + 1\}}$

By use of these transformations, the function $f(z)$ can be regarded as a function $g(w)$ in the disc, $g(w) \neq 0, 1$ in the disc

$|g(o)| = |f(z_n)| < 2$ and so by Lemma 3.1

$$(10) \quad M(r,g) < \exp \frac{1}{1-r} \{(1 + r) \log \max (1, a_o) + 2cr\}$$

$$< \exp \frac{k'}{1-r} \quad \text{for } r = |w| < 1, \quad |a_o| = |g(o)| < 2$$

where k' is an absolute constant.

For values of $f(z)$ on $|z| = \rho$ we have

$$(11) \quad |f(z)| < \exp \frac{k'}{1-\tau} = \exp\{ \frac{k'}{2} (e^{\pi^2/\log\beta/\alpha} + 1)\}$$

Going back to our coefficient condition in the statement of our theorem

$\left|\frac{a_{n+1}}{a_n}\right| > q$ implies $\beta/\alpha > q$ and hence (11) gives

$(12) \quad |f(z)| < k$, a bound independent of n, for z on $|z| = \rho = \sqrt{|a_n a_{n+1}|}$ for certain arbitrarily large ρ. This contradicts Liouvilles theorem and hence E must be a Picard set thus establishing our theorem.

If we now take $|a_n| = \exp (\frac{\delta n}{\log n})\delta > K\pi^2$, $k > 1$

$\log\left|\frac{a_{n+1}}{a_n}\right|, \sim \frac{\delta}{\log n} + 0 \qquad$ Thus $\log \beta/\alpha > \frac{1}{k} \frac{\delta}{\log n}$, and $\frac{\pi^2}{\log\rho/\alpha} < \frac{k\pi^2 \log n}{\delta} < \sigma \log n$ for some constant $\sigma < 1$.

Using (11) and this inequality we get for large n

(13) $|f(z)| < \exp\{\frac{k'}{2}(e^{\sigma\log n} + 1)\} < A \exp \frac{k'}{2} n^{\sigma} < \exp(\frac{\delta n}{\log n})$,

where $A = \exp \frac{k'}{2}$.

Thus on $|z| = \rho$, $|f(z)| < |a_n| < \rho$

This again contradicts Liouvilles theorem since f(z) is entire transcendental.

Further Remarks:- We shall now quote a result of Matsumoto 2 which throws some more light on our theorem.

Theorem 3.4 Let f(z) be an entire transcendental function. Then f(z) takes every finite value outside E infinitely often except for at most one, if the points of E satisfy the condition

$$\log \left|\frac{a_{v+1}}{a_v}\right| \geq m(v)$$

where $m(v)$, $v = 1,2,\text{---}$ are positive numbers such that

$$\limsup_{n\to\infty} \frac{k^{1/m(n)}}{\sum_{v=1}^{n} m(v)} < \infty, \qquad k \text{ constant}.$$

Here k is a certain constant related to Schottky's theorem.

In the considerations of theorem 3.3 we used an explicit choice of a_n, but in general our method works provided on $|z| = \rho$, $\rho = \sqrt{|a_n a_{n+1}|}$

$$\log |f(z)| < \frac{k'}{2}\left(e^{\pi^2/\log\left|\frac{a_{n+1}}{a_n}\right|} + 1\right)$$

implies

$\log |f(z)| < k_1 \log\rho$ and hence f(z) a polynomial.

This is the case if

$$1 + e^{(\pi^2/\log\left|\frac{a_{n+1}}{a_n}\right|)} < k'' \quad \log |a_n| \quad , \text{ that is when}$$

$$e^{(\pi^2/\log\left|\frac{a_{n+1}}{a_n}\right|)} < k_2 \log |a_n|$$

If we now put $m_n = \log \left|\frac{a_{n+1}}{a_n}\right|$

$$\log |a_n| = \log \left|\frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \text{-----------} a_1\right|$$

$$= \sum_{v=1}^{n-1} m_v + \log |a_1|$$

We see that our method works if $\dfrac{e^{(\pi^2/m_n)}}{\sum_{v=1}^{n-1} m_v}$ is bounded for all

arbitrarily large n.

i.e. if $\limsup_{n\to\infty} \dfrac{e^{\pi^2/m_n}}{\sum_{v=1}^{n-1} m_v} < \infty$

Hence our result of Theorem 3.3 is a simpler proof of a result of the type of Theorem 3.4. Matsumoto's proof is in fact very similar to the one given here.

Returning to the condition

(13') $\quad \left|\dfrac{a_{n+1}}{a_n}\right| \geq q > 1$

a set $E = \{a_n\}$ which satisfies this condition has $\overline{n}(r) = O(\log r)$, where $\overline{n}(r)$ is the number of a_n in $|a_n| \leq r$, each counted once. It follows from Nevanlinna's second fundamental theorem that an entire function f(z) whose 0's, 1's belong to E, satisfies

$$(14)\quad \begin{cases} T(r,f) = O(\log r)^2 \\ \log M\,(r,f) = O(\log r)^2 \end{cases}$$

where $T(r,f)$ is the Nevanlinna characteristic and $M(r,f)$ the maximum modulus. Indeed (14) depends only on $\overline{n}(r) = O(\log r)$ and not on the explicit separation condition (13'). This helps us to generalise theorem 3.3 and obtain a theorem of Winkler [1]

Theorem 3.5 If there is a constant $q > 1$ and a positive constant $(k > 1)$ such that

(i) for all sufficiently large r the annulus $\frac{r}{q} < |z| < qr$ contains at most two members of a set E,

(ii) the only point of E contained in $|z - a_n| < |a_n|^{-k}$ is a_n,

then E is a Picard set for entire transcendental functions.

Winkler's result is a little more general in that he considers the union of a finite number of sequences each satisfying (13') and such that the union satisfies our condition (ii)

Proof: Suppose E is not a Picard set. Hence there is an entire transcendental function $f(z)$ whose zeros and ones are in E. The counting function of E satisfies $\overline{n}(r) = O(\log r)$ and so f satisfies (14). Then (c.f. e.g. Boas [1]) the minimum modulus function $m(f,r)$ is, for any $\epsilon > 0$, greater than $M(f,r)^{1-\epsilon}$ on a set of unit density, i.e. for any $q > \lambda^1 > \lambda > 1$ there exist for all large enough r numbers r', r'' such that

$$\frac{r}{\lambda^1} < r' < \frac{r}{\lambda} < r < \lambda r < r'' < \lambda' r$$

and

$$m(f,r') > M(f,r')^{1-\epsilon} > 1$$

$$m(f,r'') > M(f,r'')^{1-\epsilon} > 1$$

We choose $r = |b_n|$ where b_n is a large zero of $f(z)$ and define r', r'' as above.

By Rouche's theorem f, and $f-1$ have the same number of zeros in the annulus

$$A: \quad r' < |z| < r''$$

and since there are at most two members of E in the annulus it follows that there are exactly two, say a_n, b_n and b_n, a_n are a zero and one respectively, of the same multiplicity m_n. Each b_n is embedded in such an annulus A.

Now we suppose all large b_n have multiplicity $m_n \geq k$; then all large a_n have the same multiplicity. Since f is entire, by Nevanlinna's theory on deficient values (c.f. Hayman [1])

$\Theta(\infty) = 1$ and $\Sigma\, \Theta(a) \leq 1$ where the sum is over finite values a. Considering the sum over 0, and 1, we see that $\Theta(a) \geq 1 - 1/k$ and hence $k \leq 2$. We now consider $\frac{f(z)}{(z-b_n)^k}$ which is regular and non-zero in the annulus A, between $c_1: |z| = r'$ and $c_2: |z| = r''$ and on the boundaries.

Since we may suppose b_n and hence r', r'' so large that for any constant $\bar{A}$, chosen $>2k$ in (ii) of the theorem.

$$M(f,r') > \{4\lambda'^{4+2\bar{A}}\, r'^{(\bar{A}+2)}\}^{1/1-\epsilon}$$

we have on c_1

$$\left|\frac{f(z)}{(z-b_n)^k}\right| \geq \frac{m(f,r')}{(z-b_n)^k} \geq \frac{M(f,r')^{1-\epsilon}}{(2r'')^k}$$

$$\geq \frac{\{4\lambda'^{4+2\bar{A}} r'^{(\bar{A}+2)}\}}{(2r'')^k} \geq \lambda'^{2\bar{A}} r'^{\bar{A}}$$

and similarly on c_2

$$\left|\frac{f(z)}{(z-b_n)^k}\right| \geq \frac{M(f,r'')^{1-\epsilon}}{(2r'')^k} - \frac{M(f,r')^{1-\epsilon}}{(2r'')^k} \geq (r'')^{\bar{A}}$$

Then we have throughout A

$$\left|\frac{f(z)}{(z-b_n)^k}\right| \geq (r'')^{\bar{A}}$$

In particular at $z = a_n$ where $f(z) = 1$, we have

$$\left|\frac{1}{(a_n-b_n)^k}\right| \geq (r'')^{\bar{A}} \geq |a_n|^{\bar{A}}$$

$|a_n - b_n| < 1/|a_n|$ contradicting hypothesis (ii) of our theorem since $\bar{A} > 2k$

We now state and prove a new sufficient condition for a Picard set

<u>Theorem 3.6:</u> <u>If</u> $a_n = \lambda_n e^{n^{1/\rho}}$, $n = 1,2,3,$----, where $\lambda_n > 0$, λ_n <u>is non-decreasing, and if</u> $\rho \leq 2$, <u>then</u> $\{a_n\}$ <u>is a Picard set for entire transcendental functions.</u>

<u>Remarks</u>: <u>Without loss of generality we can assume</u> $\lambda_1 \geq 1$; <u>It suffices to discuss the case</u> $p > 1$ <u>since</u> $p \leq 1$ <u>is already covered by the earlier theorem 3.3.</u>

<u>Proof</u>: If $\{a_n\}$ is not a Picard set then there exists an entire function $f(z)$ with all its zeros and ones in $\{a_n\}$. Calculating the characteristic function and employing Nevanlinna's 2nd fundamental theorem, we see that

$T(r,f) = O(\log r)^{p+1}$ and deduce that $f(z)$ is of order zero.

We now derive a factorisation for $f(z)$. If $\{\alpha_n\}$ denotes the zeros of f, then $\{\alpha_n\} \subset \{a_n\}$ and we can write $f(z) = k \prod_{n=1}^{\infty}(1 - z/\alpha_n)^{\mu_n'}$ since there exists $\{d_n\} \subset \{a_n\}$ such that $f(d_n) = 1$, k is real. We now show $\mu_n' = 1$ or 2.

We take a component γ of $|f(z)| < 1$ containing α_n. f is real on the real axis, so γ must be symmetric about it, simply connected and bounded by a simple closed curve Γ on which $|f(z)| = 1$. Γ meets R only in two points. If $\mu_n' > 2$, f must take 1 $\mu_n' > 2$ times on Γ, that is at some nonreal point as well as at the two points where Γ meets R, contradicting the fact that all "ones" are in $\{a_n\}$. In general using $\{a_n\}$ in our factorisation, we have

$$f(z) = k \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)^{\mu_n} \text{ where } \mu_n = 0, 1 \text{ or } 2$$

(when $a_n \neq$ an α_n, $\mu_n = 0$.)

Now we calculate

$$P_n = \prod_{k=n+1}^{\infty} \left(1 - \frac{a_n}{a_k}\right)$$

$$0 > \log P_n = \sum_{k=n+1}^{\infty} \log\left(1 - \frac{a_n}{a_k}\right) = \sum_{k=n+1}^{\infty} \log\left(1 - \frac{\lambda_n}{\lambda_k}\, e^{n^{1/P} - K^{1/P}}\right)$$

$$\geq \sum_{k=n+1}^{\infty} \log\left(1 - e^{n^{1/P} - k^{1/P}}\right) = T_n \qquad (\lambda_n \leq \lambda_k)$$

Since $\phi(x) = \log(1 - e^{n^{1/P} - x^{1/P}})$ is increasing to 0 we have

$$\sum_{k=n+1}^{\infty} \phi(k) = T_n > \int_n^{\infty} \phi(x)dx$$

$$= \int_n^{\infty} \log\left(1 - e^{n^{1/P} - x^{1/P}}\right)dx = J(\text{say}).$$

using $n^{1/P} - x^{1/P} = -t$

$$J = p\int_0^{\infty} \{\log(1 - e^{-t})\}(n^{1/P} + t)^{P-1}dt$$

This integral clearly converges.

$$-J = p\int_0^{n^{1/P}} \{-\log(1 - e^{-t})\}(n^{1/P} + t)^{P-1}dt$$

$$+ p\int_{n^{1/P}}^{\infty} \{-\log(1 - e^{-t})\}(n^{1/P} + t)^{P-1}dt$$

$$< p\int_0^{n^{1/P}} \{-\log(1 - e^{-t})\}(2n^{1/P})^{P-1}dt$$

$$+ p\int_{n^{1/P}}^{\infty} \{-\log(1 - e^{-t})(2t)^{P-1}dt$$

$$< 2^{P-1}pn^{\frac{P-1}{P}} \int_0^{\infty} -\log(1 - e^{-t})dt + 2p\int_0^{\infty} -t^{P-1}\log(1 - e^{-t})dt$$

$$= A_n n^{P-1/P} + B$$

$$< Cn^{(p-1)/p} \quad \text{for a suitable constant C.}$$

We choose a C for which this holds and assume from now on that $C > 16$.

Thus $0 > \log P_n > J > -Cn^{(p-1)/p}$

$$1 > P_n > \exp(-Cn^{(p-1)/p}),$$

Since $1 < p \leq 2$, $(p-1)/p < 1/p$ and so

$$1 > P_n > \exp(-Cn^{1/p}) \geq \frac{1}{\lambda_n^{\,c}} \exp(-Cn^{1/p})$$

since $\lambda_n \geq \lambda_1 \geq 1$.

Thus $1 > P_n > 1/a_n{}^C$

We return to the consideration of $f(z)$ which is of zero order. For any $\varepsilon > 0$ and all sufficiently large R there exists in $(R, R^{1+\varepsilon})$ values r where the minimum modulus function $m(f,r)$ satisfies $m(f,r) > r^{3C}$ (see e.g. Baker [3]) Hence taking an arbitrarily large a_m which is a zero of $f(z)$, there is such an r with $a_m > r > a_m^{3/4}$ and if a_n is the first zero of $f(z)$ larger than r, we have

$$\left.\begin{aligned} m(f,r) &> r^{3C} \\ a_n^{3/4} \leq a_m^{3/4} &< r < a_n \\ f(a_n) &= 0 \end{aligned}\right\} .$$

We assume a_m and thus r chosen so large that f has $>3C$ zeros in $(0,r)$.

We put
$$Q(z) = k \prod_{a_m \leq r} (1 - z/a_m)^{\mu_m}$$

$$R(z) = \prod_{m=n+2}^{\infty} (1 - z/a_m)^{\mu_m}$$

so that
$$f(z) = Q(z)(1 - z/a_n)^{\mu_n}(1 - z/a_{n+1})^{\mu_{n+1}} R(z)$$

$$\mu_n = 1 \text{ or } 2$$

$$\mu_{n+1} = 0,\ 1, \text{ or } 2$$

Now
$$m(f,r) = f(r) = Q(r)(1 - r/a_n)^{\mu_n}(1 - r/a_{n+1})^{\mu_{n+1}} R(r) > r^{3C}$$

whence $\quad Q(r) = m(Q,r) > r^{3C}$

Since $Q(z)$ has more than $3C$ zeros, we see that

$\left|\frac{Q(z)}{z^{3C}}\right| \not\to 0$ as $|z| \to \infty$ and so by maximum modulus theorem

$$|Q(z)| > |z|^{3C} \quad \text{for all } |z| \geq r.$$

Now if $g(z) = Q(z)\,R(z)$ we have on $|z| = r$

$$|g(z)| \geq g(r) > f(r) > r^{3C} > a_n^{\,C}$$

while on $|z| = a_{n+1}$ we have

$$|g(z)| > |Q(a_{n+1})R(a_{n+1})|$$
$$\geq |a_{n+1}|^{3C} \prod_{m=n+2}^{\infty} \left(1 - \frac{a_{n+1}}{a_m}\right)^{\mu_m} > |a_{n+1}|^{3C} p_{n+1}^2$$

$$\geq |a_{n+1}|^{3C} 1/|a_{n+1}|^{2C} = |a_{n+1}|^{C}$$

Thus on the boundary of the annulus A: $r \leq |z| \leq |a_{n+1}|$ and hence throughout A, the regular non-vanishing function g(z) satisfies

$$|g(z)| > |a_n|^C$$

and

$$|f(z)| > |a_n|^C\left(1 - \frac{z}{a_n}\right)^{\mu_n}\left(1 - \frac{z}{a_{n+1}}\right)^{\mu_{n+1}}$$

Now by Laguerre's theorem, (Titchmarsh [1]) an entire function F(z) of order less than two, real on the real axis and having all its zeros real is such that the zeros of F'(z) are all real and separated from each other by the zeros of F(z). Applied to f(z) and f(z) - 1 in our case, we see that the zeros of f'(z) are real and separated by the zeros of f(z) and also by the "ones" of f(z). Thus f(z) cannot be negative immediately on both sides of a zero, a_n of f(z), for if not we would have two zeros of f'(z) not separated by a "one" of f. Now from the preceding estimate for f(z) in A we have

$$\left|f\left(\frac{a_n + a_{n+1}}{2}\right)\right| > |a_n|^C \left|\frac{a_{n+1} - a_n}{2a_n}\right|^{\mu_n} \left|\frac{a_{n+1} - a_n}{2a_{n+1}}\right|^{\mu_{n+1}}$$

$$> 1/16\ |a_n|^C\ (a_{n+1} - a_n)^{\mu_n+\mu_{n+1}} \frac{1}{a_n^2 a_{n+1}^2}$$

Thus if $|a_{n+1}| \leq 2|a_n|$

$$\left|f\left(\frac{a_n + a_{n+1}}{2}\right)\right| > 1/16\ |a_n|^C \frac{1}{4|a_n|^4} (a_{n+1} - a_n)^4$$

$$\geq 1/64\ |a_n|^{12}(a_{n+1} - a_n)^4,$$

while if $a_{n+1} > 2a_n$

$$|f(\frac{a_n + a_{n+1}}{2})| > \frac{|a_n|^C}{16} \left|\frac{a_{n+1} - a_n}{2a_n}\right|^{\mu_n}, \qquad \mu_n = 1 \text{ or } 2$$

$$> \frac{|a_n|^{14}}{64} (a_{n+1} - a_n)^2 \qquad \text{or } \frac{|a_n|^{15}}{32} (a_{n+1} - a_n)$$

Thus in all three cases (using $a_n > 8$)

$$|f(\frac{a_n + a_{n+1}}{2})| > a_n^{10}(a_{n+1} - a_n) > 1 \quad \text{for all large enough n.}$$

Since for large n $a_{n+1} - a_n > \lambda_n(e^{(n+1)^{1/p} - n^{1/p}}) > e^{n^{1/p}}(1/p\, n^{1/p-1}) > 1$

We note that $|a_n - a_{n-1}| > 1$ and $|f(r)| > 1$.

Thus one of $f(r)$, $f(\frac{a_n+a_{n+1}}{2})$ is positive and greater than 1, since the only zero in $(r, \frac{a_n+a_{n+1}}{2})$ is a_n and f is positive in at least one of (r,a_n), $(a_n a_{n+1})$. But the "ones" are at a_k so we cannot have any in (a_n,a_{n+1}). Hence there must be a "one" in (r,a_n).

Now in $r \leq |z| \leq a_n$

$$|f(z)| = |Q(z)(1 - z/a_n)^{\mu_n} \prod_{m \geq n+1}^{\infty} (1 - z/a_m)^{\mu_m}|$$

$$> |Q(z)(1 - z/a_n)^{\mu_n} p_n^{\ 2}|$$

$$\geq \frac{r^{3C}}{|a_n|^{2C}}(1 - z/a_n)^{\mu_n}$$

If for some s, $r < s < a_n$, $f(s) = 1$

then $1 > \frac{s^{3C}}{|a_n|^{2C}} (1 - s/a_n)^{\mu n}$

$$\text{thus } (1 - s/a_n)^{\mu_n} < \frac{|a_n|^{2C}}{r^{3C}} < \frac{a_n^{\ 2C}}{a_n^{\ 9/4C}} \; \frac{1}{|a_n|^{C/4}} < \frac{1}{|a_n|^4} \qquad (C > 16)$$

$$\text{Hence } \left|\frac{a_n - s}{a_n}\right|^{\mu_n} < \frac{1}{|a_n|^4} \qquad \mu_n = 1 \text{ or } 2$$

$$\text{Thus } |a_n - s| < \frac{1}{a_n^{\ 3}} \text{ or } \frac{1}{a_n}, \text{ so } (a_n - s) < \frac{1}{a_n} < 1.$$

However $|a_n - s| < \frac{1}{a_n} < 1$ contradicts the fact that $|a_n - a_{n-1}| > 1$. Hence $\{a_n\}$ as defined is a Picard set.

__Remarks:__ This gives an improvement on the 'density' of $\{a_n\}$ so far achieved. In the particular case when all $\lambda_n = 1$, $p = 2$ we have $a_n = e^{\sqrt{n}}$. A function with all its zeros and ones at these points would have

$T(r) = 1/3(\log r)^3$ compared with $T(r) = 0\,(\log r)^2$ in the case $\frac{a_{n+1}}{a_n} > q > 1$. In general using our assumptions,

$T(r) = 0(\log r)^{p+1}$ and for $2 \geq p > 1$

we have an improvement on earlier results.

3.3 ON LARGER PICARD SETS

We shall in this section be considering conditions for Picard sets which contain a sequence of discs.

We first of all quote a lemma of Matsumoto [2] used also by Toppila [2].

LEMMA 3.3 *If $\mu \geq 3$, and f is analytic in the annulus A: $1 < |z| < e^{\mu}$ and if f omits 0 and 1 in A, then there exists a constant K, independent of f,μ and such that the spherical length and hence the diameter of the image of $|z| = e^{\mu/2}$ under f is $\leq ke^{-\mu/2}$.*

Remark: By considering $f(z/\rho)$ instead of $f(z)$ we see that A may be replaced by $\rho < |z| < \rho e^{\mu} = \rho_1$ if $|z| = e^{\mu/2}$ is replaced by $|z| = \rho e^{\mu/2} = \sqrt{\rho\rho_1}$.

We shall for completeness, include the proof of this lemma.

Proof: By applying Schottky's theorem to an annulus as in Theorem 3.3, we see that if g is analytic in $\rho < |z| < \rho e^{\sigma}$ and does not take the values $0,1$ and if

$$m(g,\rho e^{\sigma/2}) = \min_{|z|=\rho e^{\sigma/2}} |g(z)| < a,\ a > 0,$$ then there is a bound $b = b(a,\sigma)$ such that the maximum modulus of g on $|z| = \rho e^{\sigma/2}$ satisfies

$$(15)\quad M(g,\rho e^{\sigma/2}) \leq b$$

We denote by $C(\alpha,\delta)$ the spherical disc with centre α and (spherical) radius δ, i.e. the set of points on the sphere whose spherical distance from α is less than δ. We choose $\delta > 0$, so that $C(0,\delta)$, $C(1,\delta)$ and $C(\infty,\delta)$ are mutually disjoint.

For a fixed σ, $0 < \sigma < \mu$ ($\sigma = 1$ say) and points z_1 on $|z| = e^{\sigma/2}$,

z_2 on $|z| = e^{\mu-\sigma/2}$. Then there is at least one of the discs $C(\alpha,\delta)$, $\alpha = 0$, 1 or ∞ which does not contain $f(z_1)$, $f(z_2)$. Suppose $C(\infty,\delta)$ has this property. Then applying the remark at the beginning of the proof to the annuli $1 < |z| < e^{\sigma}$, and $e^{\mu-\sigma} < |z| < e^{\mu}$, and noting that $f(z_i) \notin C(\infty,\delta)$ implies $|f(z_i)| < a$, $a = a(\delta)$, we see that there is a bound $b = b(a,\delta) = b(\delta,\sigma)$ such that the images under f of $|z| = e^{\sigma/2}$, $z = e^{\mu-\sigma/2}$ lie in $|z| \leq b$ since δ is independent of μ,f this bound depends only on σ and so there is a $\delta_\infty > 0$ depending only on σ such that the images in question lie outside $C(\infty,\delta_\infty)$.

Next suppose it is $C(1,\delta)$ which does not contain $f(z_1)$, $f(z_2)$. By using the bilinear transformation L which maps $0,1,\infty$ to $1,\infty,0$ respectively we see that $L(f)(z_i)$ lies outside $C(\infty,\delta)$ and so the images of $|z| = e^{\sigma/2}$, $|z| = e^{\mu-\sigma/2}$ by $L(f)$ lie outside $C(\infty,\delta)$. Thus on transforming back by L^{-1}, there is a $\delta_1 > 0$ such that the images of $|z| = e^{\sigma/2}$, $|z| = e^{\mu-\sigma/2}$ by f lie outside $C(1,\delta_1)$. Similarly if $(f(z_i)$ are outside $C(0,\delta)$ there is a $\delta_0 > 0$ such that the images in question lie outside $C(0,\delta_0)$.

Take $\delta' = \min(\delta_0, \delta_1, \delta_\infty)$, so that δ' depends only on σ. Then there is at least one $C(\alpha,\delta')$ $\alpha = 0,1,\infty$ such that the image of the boundary of

$$B: e^{\sigma/2} \leq |z| \leq e^{\mu-\sigma/2}$$

by f lies outside $C(\alpha,\delta')$. By the maximum principle applied to f, $\frac{1}{f}$ or $\frac{1}{f-1}$ as appropriate, it follows that $f(\overline{B})$ then lies outside $C(\alpha,\delta')$. Without loss of generality we may assume $\alpha = \infty$, since without altering the spherical length of the image of $|z| = e^{\mu/2}$ by $f(z)$ we may apply a rotation of the sphere of w-values to bring 0 or 1 to ∞. Assume then $f(\overline{B})$ lies outside $C(\infty,\delta')$. Then

$$(16) \quad |f(z)| \leq M = \sqrt{1-\delta'^2}/\delta' \quad \text{on } \overline{B}$$

Applying Cauchy's theorem to B

$$(17) \quad f'(z) = \frac{1}{2\pi i} \int_{|t|=r_1} \frac{f(t)}{(t-z)^2}\,dt - \frac{1}{2\pi i} \int_{|t|= r_2} \frac{f(t)}{(t-z)^2}\,dt$$

where $r_1 = \exp(\mu - \sigma/2)$, $r_2 = \exp \sigma/2$

If $\mu \geq 2 + \sigma$ (16) and (17) give on $|z| = e^{\mu/2}$

$$|f'(z)| \leq \frac{Me^{\mu-\sigma/2}}{(e^{\mu-\sigma/2}-e^{\mu})^2} + \frac{Me^{\sigma/2}}{(e^{\mu/2}-e^{\sigma/2})^2}$$

$$= \frac{2Me^{-\sigma/2}}{(e^{\mu-\sigma/2}-1)^2}$$

$$= \frac{2Me^{2-\sigma/2-\mu+\sigma}}{(e-e^{1-\mu/2-+\sigma/2})^2}$$

$$\leq \frac{2Me^{2+\sigma/2}}{(e-1)^2} \cdot e^{-\mu} = N \text{ (say)}$$

since $1 + \sigma/2 - \mu/2 \leq 0$.

Hence

$$(18) \quad \int_{|z|=e^{\mu/2}} |f'(z)|\,|dz| \leq N.2\pi e^{\mu/2} = \frac{4\pi Me^{2+\sigma/2}e^{-\mu/2}}{(e-1)^2}$$

Since the length of the image curve $f(|z| = e^{\mu/2})$ with respect to the euclidean metric $|dw|$ is not less than the chordal length calculated with respect to

$ds = \frac{|dw|}{1+|w|^2}$ we see that the spherical length is bounded by (18) which is indeed of form $ke^{-\mu/2}$, since M depends only on σ. The result then holds for all $\mu > \sigma = 1$ with $\mu \geq 2 + \sigma$, so for all $\mu \geq 3$.

This completes the proof of the Lemma. Toppila [2] has proved the following theorem:

<u>Theorem 3.7</u> <u>Let</u> D_n, $n = 1,2,3,$--- <u>be a sequence of discs with centre</u> z_n, $\mathrm{Re}z_n > 1$, <u>radius</u> ρ_n. <u>If</u>

(19) $$\left|\frac{z_{n+1}}{z_n}\right| > q > 1$$

<u>and</u>

(20) $$|z_n| = o(-\log \rho_n) \text{ as } n \to \infty$$

<u>Then</u> $E = \cup D_n$ <u>is a Picard set for entire transcendental functions</u>.

Our next theorem will be arranged somewhat differently from Toppila's and will yield a slightly more general result.

<u>Theorem 3.8</u>. <u>Let</u> A <u>be the angle</u> $|\arg z| < \frac{\alpha\pi}{2}$, $0 < \alpha < 2$. <u>Let</u> D_n <u>be a sequence of discs with centre</u> z_n, <u>radius</u> ρ_n <u>such that</u> $D_n \subset A$ <u>and</u> (19) <u>holds</u>. <u>Also</u> $\rho_n \to 0$ <u>so fast as</u> $n \to \infty$ <u>that</u>

(21) $$|z_n|^{1/2-\alpha} = o(-\log \rho_n)$$

<u>then</u> $E = \cup D_n$ <u>is a Picard set for entire transcendental functions</u>.

This theorem is essentially the same as the preceding one, in the case when $\alpha = 1$. The proof will make use of the following Lemma.

<u>LEMMA 3.4 Bieberbach [1]</u>

<u>If</u> $f(z)$ <u>is regular in the angle</u> W: $|\arg z| \leq \pi\beta/2$ <u>and omits</u> 0,1 <u>there</u>, <u>then in every smaller angle</u>

$|\arg z| \leq \frac{\pi\beta'}{2} < \pi\beta/2$, <u>we have</u>

$$|f(z)| < \exp (k|z|^{1/\beta}),$$

The proof is to consider $t = \frac{w-1}{w+1}$ which maps $|t| < 1$ to $\operatorname{Re} w > 0$ and then put $W^{\beta} = z$, so that $z = (\frac{1+t}{1-t})^{\beta}$ maps $|t| < 1$ on to W. Application of Schottky's theorem to $\phi(t) = f(z(t))$ in $|t| < 1$ gives the result. The constant K depends on β' and on $f(1)$.

<u>Proof of Theorem 3.8</u>

We assume $E = UD_n$ satisfies the conditions of the theorem but is not a Picard set. Then there is an entire transcendental function $f(z)$ such that $f(z)$ is not 0 or 1 outside E. We apply Lemma 3.4 to $f(z)$ which has no 0's or 1's in the complementary angle W to A, which has aperture $(2 - \alpha)\pi = \beta\pi$ say. Thus there is a constant K such that on the negative real axis $f(z)$ satisfies

(22) $|f(z)| < \exp (K|z|^{1/\beta})$

For any constant λ satisfying $1 < \lambda < \lambda^3 < q$ and for all sufficiently large n, the annulus

A_n: $\lambda|z_n| < |z| < \lambda^3|z_n|$

is free of discs D_n and so of zeros and ones of $f(z)$. Applying Schottky's theorem in A_n, as in proof of theorem 3.3 we obtain for $|z| = \sigma_n = \lambda^2|z_n|$

$$(23) \quad M(f,\sigma_n) \leq \exp[\{\exp(\pi^2/\log \lambda^2) + 1\}\{\log |f(-\lambda^2|z_n|) + c\}]$$

$$\leq E \exp \{F|z_n|^{1/\beta}\}$$

where E,F are constants independent of n.

We note by a similar argument, that if for some constant k there are arbitrarily large n such that $|z| = \sigma_n$ contains points with $|f(z)| \leq k$, the application of Schottky's theorem in A_n shows that there is a constant M (independent of n) such that $M(f,\sigma_n) < M$ for these values of n and f is identically constant.

If f has only finitely many zeros or ones, then 0 or 1 is an asymptotic value taken along a path which must meet infinitely many $|z| = \sigma_n$ and the above argument shows this to be impossible. Similarly if there is a D_n which contains zeros but no ones of f, then a component of $\{z; |f(z)| < 1\}$ which meets D_n must meet some σ_m, $m = n + 1$.

Thus f must have an infinity of 0's, 1's and there is an infinity of discs D_n such that D_n contains both 0's and 1's of f. We take such a D_n; For large n f has no zeros or ones in

$$B_n: \ \rho_n < |z - z_n| < k|z_n|,$$

where k is a fixed constant such that $0 < k < (q - 1)/q)$.

We apply Matsumoto's lemma to the annulus B_n for which

$$e^\mu = k|z_n|/\rho_n$$

Then the circle C_n: $|z - z_n| = \rho_n e^{\mu/2} = \sqrt{k\rho_n|z_n|} = t_n$ say, maps onto an image γ_n whose spherical length L_n satisfies

$$(24) \quad L_n < k'\rho_n^{1/2}/(|z_n|k)^{1/2}$$

where k' is a certain absolute constant. Since $\rho_n \to 0$, $|z_n| \to \infty$ as $n \to \infty$, $L_n \to 0$.

Let $\hat{D}_n$ denote the interior of C_n, so that $\hat{D}_n \supset D_n$.
Now $f(\hat{D}_n)$ is a domain whose boundary is contained in $f(c_n) = \gamma_n$.
For a point p on γ_n, γ_n belongs to the spherical disc $D' = D(p,L_n)$.
At least one of $0,1$ lies outside D' and so, since $f(\hat{D}_n) \supset f(D_n)$ and so contains both $0,1$, we see that $f(D_n)$ contains the exterior of D'.
Since $f(D_n)$ does not contain ∞, it follows that $\infty \in D'$ i.e. the distance of p from ∞ is less than L_n.
Thus $[p,\infty] = \dfrac{1}{\sqrt{1+|p|^2}} < L_n$ and hence

$$(25) \qquad |p| > \frac{\sqrt{1 - L_n^2}}{L_n} > \frac{1}{2L_n}$$

for sufficiently large n.

By the assumptions of the theorem, ρ_n and $|z_n|$ are so related that $t_n \to 0$ as $n \to \infty$ and hence

c_n: $|z - z_n| = \sqrt{k\rho_n|z_n|} = t_n$ lies inside the circle $|z| = \sigma_n = \lambda^2|z_n|$, $\lambda > 1$, for large n. Hence for large n we must have

$$(26) \quad M' = \max_{p\in\gamma_n = f(c_n)} |p| \leq M(f,\sigma_n) \leq E \exp\{F|z_n| 1/\beta\} \quad \text{by (23).}$$

Also for arbitrarily large n such that D_n contains $0,1$ we have

$$(27) \quad M' \geq \frac{1}{2L_n} = 2\,(\,z_n\,k)^{1/2}/k'\rho_n^{\,1/2} \qquad \text{by (25).}$$

Thus there must be arbitrarily large n for which

$$(28) \quad \frac{1}{(\rho_n)^{\frac{1}{2}}} \le 2(|z_n|k)^{1/2}/k'\rho_n^{\frac{1}{2}} \le E \exp\{F|z_n|^{1/\beta}\},$$

$$(29) \quad \log(1/\rho_n) < 2F|z_n|^{1/\beta} + O(1).$$

If we suppose, as in the statement of the theorem, that $\log(1/\rho_n)/|z_n|^{1/\beta} \to \infty$ as $n \to \infty$, then there cannot exist a constant F for which (29) holds and so the assumption that there exists entire transcendental f whose 0's, 1's lie in UD_n leads to a contradiction. This completes the proof of the theorem.

The restriction of the discs D_n to lie in the angle A is an essential feature of the preceding theorem which yields no statement in the case $\alpha = 2$, i.e. when the discs are not subject to any angular restriction. In this case we can still obtain a result, although the ρ_n must be taken smaller. Toppila [2] has proved the following:-

Theorem 3.9 Let D_n be a sequence of discs of centre z_n, radius ρ_n such that (19) holds and $\rho_n \to 0$ so fast as $n \to \infty$ that

$$(30) \quad |z_n|^n = O \log(1/\rho_n)$$

then $E = UD_n$ is a Picard set for entire transcendental functions.

Remark: It is noteworthy that in none of the three preceding theorems does q enter the relationship between z_n and ρ.

By our methods one can prove the following

Theorem 3.10 Let D_n be a sequence of discs of centre z_n, radius ρ_n, such that (19) holds. Then there exists a finite constant $k = k(q)$

such that if

(31) $|z_n|^k = O(\log(1/\rho_n))$ *as* $n \to \infty$

then $E = \cup D_n$ *is a Picard set for entire transcendental functions:*

Remark: (i) In this theorem we may in particular take K to be $k(q) + 1$ say, where $k(q)$ is the explicit function given in Lemma (3.5) below.

(ii) Condition (30) of Toppila's theorem implies

$|z_n|^k = o(\log 1/\rho_n)$ for each constant k so our result is stronger. Since moreover $\limsup_{q\to\infty} k(q) < \infty$, this last theorem actually improves theorem 3.8 if is close to 2 and q large.

The idea of the proof is essentially the same as that of theorem 3.8 except that Bieberbach's lemma is replaced by the following.

Lemma 3.5 *If the entire function* f *takes the values* 0,1 *at most in the discs* D_n *of centre* z_n, *radius* $\rho_n < 1$, *where* (19) *holds*, *then* f *is of finite order* $\leq k = k(q)$. *In this result we may take*

$$k(q) = 6 \log \{4 \exp(\frac{3\pi^2}{\log\sigma}) + 5\}/\log \sigma$$

where

$$\sigma = (3q + 1)/(2q + 2)$$

The proof depends on Lemma 3.6 which is quoted and proved below

Lemma 3.6 *If for all sufficiently large* r *the maximum modulus* $M(f,r)$ *of the entire function* $f(z)$ *satisfies*

(32) $M(f,\lambda r) < M(f,r)^K$

where λ, k *are constants* (>1), *then the order of* f *is at most*

$(\log k)/(\log \lambda)$.

Proof of Lemma 3.6: If f is non constant we may suppose (32) holds and also $M(f,r) > 1$ for $r \geq r_o > 1$. For such r it follows that

$$M(f,\lambda^n r) < M(f,\lambda^{n-1} r)^k < \text{---} < M(f,r)^{k^n}, \; r \geq r_o$$

For all large enough R, there is a positive integer m such that $R = \lambda^m r$ for some r in $r_o \leq r \leq \lambda r_o$. For such m

$$M(f,R) = M(f,\lambda^m r) < M(f,r)^{k^m} < A^{k^m}$$

where $A = M(f,\lambda r_o)$. Thus

$$\log\log M(f,R) < m \log k + \log\log A$$

since

$$\log R = m \log\lambda + \log r \geq m \log\lambda + \log r_o \geq m \log \lambda,$$ we get

$$\frac{\log\log M(f,R)}{\log R} \quad \frac{m \log k + \log\log A}{m \log\lambda}$$

whence the assertion of the lemma follows.

We now embark on the proof of Lemma 3.5.

Proof: We assume f is non constant; for any β with

(33) $1 < \beta < q$

the annulus

$A(r)$: $r/\beta < |z| < r$ for large r ($\geq r_o$) meets at most one D_n.

This is also true for

$B(r)$: $r < |z| < \beta r$

There is therefore a point z_o with $|z_o| = r$, such that the disc $|z - z_o| = r - r/\beta = t$ meets none of the discs D_n and so contains neither 0's nor 1's of $f(z)$.

To see this we suppose $A(r)$ meets a D_{n_o}. Without loss of generality we may assume there is a point, p, on the positive real axis belonging to $A(r) \cap D_{n_o}$ and the D_{n_o} will be contained in $|z - p| < 2$. For z_o st. $z_o = \pm ir$, the distance of z_o from the real segment $[r/\beta, r]$ is at least $> r + 2$ for large r. Thus the discs $|z \pm ir| < r$ do not meet D_{n_o}. The discs $|z \pm ir| < t = r(1 - 1/\beta)$ have this property and belong to $A(r) \cup B(r)$. At most one of them can contain a disc D_{n_1} (which must meet $B(r)$) and so the other satisfies our assertion.

Applying Schottky's theorem to $f(z_o + tw)$ in $|w| < 1$, we see that on $|w| < 1/2$

$$|f(z_o + \rho w)| < \exp[2\{3/2 \log \max(1, |f/z_o|) + C\}]$$

$$< \exp\{3 \log M(f,r) + 2C\}$$ where C is the universal constant entering Schottky's theorem. For all large enough r then

$$|f(z_o + \rho w)| < M(f,r)^4 \text{ on } |w| = 1/2.$$

Thus each circumference $|z| = R$, $r \leq R \leq r + t/2 = \sigma r$ where $\sigma = \dfrac{3\beta-1}{2\beta}$ contains a point where

$$(34) \qquad |f(z)| < M(f,r)^4$$

(Note $1 < \sigma < \beta$).

Of the annuli

$$A_j \quad \sigma^{j/3} r \leq |z| \leq \sigma^{(j+1)/3} r, \quad j = 0,1,2$$

for larger, the union $A_o \cup A_1 \cup A_2 \subset B(r)$ meets at most one D_n and so there is at least one A_j which meets no D_n and thus contains no 0's or ones of f. If A_j is such an annulus, applying Schottky's theorem in A_j gives

$$M(f,\sigma^{j/3+1/6}r) < \exp[\{\exp(\frac{\pi^2}{\log\sigma 1/3}) + 1\}\{\log\max(1,|f(z_1)|) + c\}]$$

where we have chosen z_1 such that (34) holds and $|z_1| = \sigma^{j/3+1/6}r$.

Thus

$$M(f,\sigma^{1/6}r) \leq M(f,\sigma^{j/3+1/6}r) < M(f,r)^{4L}e^{cL}$$

$$\text{where } L = \exp(\frac{3\pi^2}{\log\sigma}) + 1$$

Thus for all large r

$$M(f,\lambda r) \quad M(f,r)^k$$

with $\lambda = \sigma^{1/6}$, $\quad k = 4\exp(\frac{3\pi^2}{\log\sigma}) + 5$

We may for example choose $\beta = \dfrac{1+q}{2}$ which satisfies (33). One then obtains

$\sigma = (3q+1)/(2q+2)$ so that the order k of f is at most ~~log~~ log k/log λ

We now give the proof of theorem 3.10.

Proof: This now follows on the same lines as in the proof of theorem 3.8. The only difference is that the estimate (23) is replaced by

$$(35) \quad M(f,r) < \exp r^k = \exp r^{k+1},\ k = k+1.$$

for all sufficiently large r since the order of f is $\leq$ k and so < k + 1.

The estimate (28) is then replaced by

$$(36) \quad \frac{1}{\rho_n^{\frac{1}{2}}} \leq \frac{1}{2L_n} \leq M' = M(f,\sigma_n) = M(f,\lambda^2|z_n|) \leq \exp(\lambda^2|z_n|)^k$$

Thus

$$(37) \quad \log(1/\rho_n) \leq \cancel{\log\sqrt{2\pi}2}\,\lambda^{2k}|z_n|^k$$

If we then suppose that

$\log (1/\rho_n)/|z_n|^k \to \infty$ as in the statement of the theorem, then (37) cannot be satisfied for any sequence of integers n leading to infinity and the proof finishes exactly as in the earlier theorem.

3.4 COUNTER EXAMPLE

It is fairly obvious that any set of discs forming a Picard set as in the preceding theorems, must have very small radii. We give an example of a set of fairly small discs which is not a Picard set.

Counter Example: Let α be a constant, $\alpha > 4$

$$f(z) = \prod_{m=1}^{\infty} (1 + z/\alpha^m)$$

Then in $|z| < \alpha^{N+1/2}$, $g = \prod_{N+1}^{\infty} (1 + z/\alpha^m)$ satisfies

$m(g,r) = \min\limits_{|z|=r} |g(z)| = g(-r)$ since $g(z)$ has no zeros in $|z| < \alpha^{N+1/2}$ and so takes its minimum on the circumference and since g is a canonical product of genus zero, with negative zeros the minimum modulus on each circle is taken on the negative axis. Thus

$$|g(z)| > (1 - 1/\alpha 1/2)(1 - 1/\alpha 3/2)(1 - 1/\alpha 5/2) \text{----} = C \text{ say}$$

Also

$$h(z) = \prod_{m=1}^{N-1} (1 + z/\alpha^m) \quad \text{satisfies on } |z| \geq \alpha^{N-1/2}$$

$$|h(z)| > |h(-|z|)| \geq (\alpha^{\frac{1}{2}} - 1)(\alpha^{3/2} - 1) \text{ -- } (\alpha^{N-\frac{1}{2}} - 1)$$

Since $\alpha > 4 > 1$, we see that

$$|h(z)| > 1.\alpha.\alpha^2 \text{--} \alpha^{N-2} = \alpha^{(N-1)(N-2)/2}$$

Thus in

$$A_N: \quad \alpha^{N-\frac{1}{2}} \le |z| \le \alpha^{N+\frac{1}{2}}$$

$$|f(z)| > C\alpha^{(N-1)(N-2)/2}|(1 + z/\alpha^N)|$$

Thus any 1-point of $f(z)$ in the annulus A_N satisfies

$$1 = |f(z)| > C\alpha^{(N-1)(N-2)/2}\,|1 + \frac{z}{\alpha^N}|$$

$$|z + \alpha^N| < \frac{\alpha^N}{c\alpha^{(N-1)(N-2)/2}} = \frac{1}{c\alpha^{\frac{N^2-5N+2}{2}}} = \rho_N$$

Thus the discs

D_N: $|z + \alpha^N| < \rho_N = 1/c\alpha^{\frac{N^2-5N+2}{2}}$ do not form a Picard set since there is a function, viz f, whose 0's and 1's lie in these discs. It is clear that since on $|z + \alpha^N| = \rho_N$ $|f(z)| > 1$, by applying Rouche's theorem to, f, f-1, we have the same number of zeros and ones in D_N viz one.

In the light of theorem 3.3 and 3.10 we see that if we have $z_n = -\alpha^N$ as the centre of our discs,

$$\left|\frac{z_{n+1}}{z_n}\right| \ge \alpha > 1$$

$$\log(1/\rho_N) \sim \frac{N^2}{2}\log\alpha$$

Hence

$\alpha^N = |z_n|$ is not $o\theta(\log(1/\rho_N))$ z_n's in a half plane.

It is interesting to consider the preceding counter example in the light of the methods of §3. The zeros of the function f, form a

set $\{a_n\}$, $a_n = -\alpha^n$ and the ones form a set $\{b_n\}$, with
$|b_n - a_n| = \rho_n < k\alpha^{-(\frac{n^2-5n+2}{2})}$

Thus

$$(38)\quad \log 1/\rho_n \sim \frac{n^2}{2} \log \alpha$$

We show that our counter example could not have been much improved without changing its nature considerably. We prove the following, which in contrast with the theorem 3.5 which states that two reasonably "separated" sequences of type $\{\alpha^n\}$ form a Picard set, states

Theorem 3.11 *If* $\{a_n\}$ *satisfies*

$$(39)\quad \left|\frac{a_{n+1}}{a_n}\right| > q > 1, \quad n = 1,2,3,\text{----}$$

and if the sequence $\{b_n\}$ *satisfies* $0 < |b_n - a_n| = \rho_n$, *where*

$$(40)\quad (\log|a_n|)^2 = o(\log(1/\rho_n))$$

then the union $E = \{a_n\} \cup \{b_n\}$ *is a Picard set for entire transcendental functions.*

Proof: Suppose E is not a Picard set and let f be a function whose zeros and ones lie in E. Since the counting function of E has $\bar{n}(r) = O(\log r)$ we have

$$\log M(r,f) = O(\log r)^2$$

It follows that there is an r_n in each

$$\text{Max}(|a_{n-1}|,|b_{n-1}|) < r_n < \text{Min}(|a_n||b_n|), \quad n \geq n_0$$

with

$$m(f,r_n) > 1$$

and so if a_n, $n \geq n_o$ is a zero or one of f then b_n is a one or zero respectively. One then notes that for such n the disc D_n: $|z-a_n| < 2\rho_n$ contains a zero and one of f while the annulus

$$2\rho_n < |z - a_n| < k|a_n|, \qquad k = 1 - 1/q,$$

is free of zeros and ones.

Using the method of theorem 3.8 we see that by a precisely similar argument (28) is replaced by the conclusion that for suitable arbitrarily large n

$$\frac{1}{(2\rho_n)^{\frac{1}{2}}} = \frac{2(|a_n|k)^{\frac{1}{2}}}{k'(2\rho)^{\frac{1}{2}}} \leq M(f,q|a_n|) < \exp\{A(\log|a_n|)^2\}$$

so that $\log(1/\rho_n) \leq A_1(\log|a_n|)^2$ for a suitable constant A_1 independent of n.

This contradicts the assumption of the theorem, namely

$$(\log|a_n|)^2 = o \log(1/\rho_n)$$

Hence E is not a Picard set.

Applied to our example where $a_n = -\alpha^n$ we see that the condition on ρ_n becomes $n^2 = o(\log(1/\rho_n))$. On the other hand the above counter example shows that in the case $a_n = \alpha^n$ the condition $n^2 \sim \log(1/\rho_n)$ or in the general case

$$(\log|a_n|)^2 \sim \log(1/\rho_n)$$

does not ensure that E is a Picard set in theorem 3.11.

CHAPTER IV - COMPOSITION OF ENTIRE FUNCTIONS

4.1 INTRODUCTION

Related to the study of iteration is the question of expressing an entire (meromorphic) function $F(z)$ in the form

(1) $F(z) = f(g(z))$

f,g entire (g meromorphic if F meromorphic). Such an expression is a factorisation of $F(z)$. If we now restrict our discussion to entire functions, we shall call F pseudo-prime [PRIME] if a factorisation of F has at least one factor as a polynomial [LINEAR].

Polya [1] and Clunie [1] have considered the rate of growth of F in terms of f and g. In particular, Polya proved the following:

LEMMA 4.1 *If f and g are entire functions with*

$g(o) = 0$, *then for* $R > 0$, $0 < \rho < 1$

(2) $M(R, f\circ g) \geq M(C(\rho)M(\rho R, g), f)$

where

(3) $C(\rho) = (1-\rho)^2/4\rho$

Cor. *If $f\circ g$ is of finite order and g is not a polynomial then f is of zero order:*

Remark: The condition on $g(o)$ is not of much significance when we are employing the Lemma for large R only.

Rosenbloom [1,2] considered the factorisation of functions F with at most a finite number of fix points, i.e. such that $F(z) - z = 0$ has at most finitely many solutions. Factorisation of periodic entire functions was considered by A. Renyi and C. Renyi [1] and Baker [9].

Many special problems of factorisation theory were also considered by F. Gross [1 - 6], M. Ozawa [1] and R. Goldstein [1].

Typically these results prove that the assumption of some property for F implies some inadmissible property for f and or g and so shows F to be prime or pseudo prime.

For instance if F is entire of finite order and has a Picard value $a \neq \infty$, then F is pseudo-prime. For if $F = f(g)$, f,g transcendental, then by Polya's result, $f(w)$ has zero order and so takes the value a at an infinity of points $w = w_1, w_2, ----w_n, ----$. Then $g = w_n$ has an infinity of roots for all but perhaps one w_n and thus $f(g) = a$ has an infinity of roots. Thus any factorisation $F = f(g)$ must have at least one factor polynomial. Thus for example e^z is pseudo-prime, although not prime because we may take

$$f = w^k, \; g = e^{z/k}, \quad \text{for any integer } k$$

Many authors have noticed the above example which forms the starting point for generalisations. e.g. Ozawa [1] and Goldstein [1]. Goldstein went on to find cases when F has a deficient value a in the sense of Nevanlinna and proved for example that $\psi(z) + \phi(z)e^z$ is prime if ψ and ϕ are non-zero functions of order <1.

This chapter contains two special criteria for F to be prime. These are of a rather different kind from those previously considered and incidentally provide examples of prime functions of arbitrarily slow growth.

4.2 A Condition Involving Picard Sets

We now state and prove our first result.

<u>Theorem 4.1.</u> <u>Let</u> E <u>be the union of a set of discs</u> D_n <u>of centre</u> z_n, <u>radius</u> ρ_n <u>where</u>

$$(4)\quad \left|\frac{z_{n+1}}{z_n}\right| > q > 1 \quad , \quad n = 1,2,---,$$

<u>and</u> $\rho_n \to 0$ <u>as</u> $n \to \infty$, <u>such that</u> E <u>is a Picard set</u> (<u>in particular the sets of theorem</u> 3.8 <u>and</u> 3.10 <u>satisfy these conditions</u>). <u>Let</u> F(z) <u>be an entire transcendental function of finite order whose zeros all lie in</u> E. <u>Then</u> F <u>is pseudo prime and if</u> F <u>has an infinity of zeros at least one of which is simple then</u> F <u>is prime</u>.

<u>Proof:</u> We consider an entire function F(z) of finite order whose zeros lie within E. We suppose $F(z) = f(g(z))$ has order $A < \infty$. Thus

$$(5)\quad M(f(g),r) < e^{r^{A+\epsilon}} \quad \text{for large } r$$

If g(z) is not a polynomial,

$$(6)\quad cM(g,r/2) > r^k \text{ for any given } k, \text{ and large } r$$

Hence applying Lemma 4.1

$$M(f,r^k) < M(f, cM(g,r/2)) < M(f(g),r) < e^{r^{A+\epsilon}}$$

putting $s = r^k$

$$(7)\quad M(f,s) < e^{s^{(A+\epsilon)/k}}$$

For large r, s, $(A+\epsilon)/k$ is as small as we like and hence f is of order zero. Thus $f(t) = 0$ has infinitely many solutions $t_1, t_2, ---$. Since g is not a polynomial $g(z) = t_i$ has solutions outside the set E for some t_i as E defines a Picard set for entire transcendental functions. Hence there is a z' which is not in E such that

$F(z') = f(g(z')) = f(t_k) = 0$ against our hypothesis. Hence F is at least pseudo prime. We now go on to show that F is indeed prime if it has an infinity of zeros.

Suppose $F(z)$ as defined can be factorised such that $F(z) = f(g(z))$, $g(z)$ a polynomial. $F(z) = f(g(z)) = 0$ at an infinity of points z_k in E. For large z_k, we have $t_k = g(z_k)$ large and also any other solution of

$$t_k = g(z')$$

satisfies

$$F(z') = f(g(z')) = f(t_k) = 0$$

If g is a polynomial of degree $m > 1$, then for large t_k, $g(z') = t_k$ has m solutions (asymptotically) equally spaced on the circle $|z| = \lambda|t_k|^{1/m}$ (λ constant). However each disc D_n of E lies in an annulus

$$A_n: \quad q^{-\frac{1}{2}} \, |z_n| < |z| < q^{\frac{1}{2}} \, |z_n|$$

and the A_n are disjoint for large n. Since $\rho_n \to 0$ each disc D_n and therefore the annulus A_n can contain at most one of the roots of $g(z') = t_k$ for large t_k. This gives a contradiction.

It remains to consider the case when f is a polynomial and g transcendental. Suppose $f = 0$ at α and β, $\alpha \neq \beta$. Then g takes either α or β outside E and so F has zeros outside E. This is impossible, so $f = 0$ at only one point, say α, and has the form $f = \mu(z - \alpha)^k$. If $k > 1$ then $F = \mu(g - \alpha)^k$ and every zero of F has multiplicity at least k. Since F has at least one simple zero we conclude $k = 1$ and F is prime.

Remark: We can obtain examples of any finite order to which the theorem applies. For example, by taking canonical products. The conditions on the existence of an infinity of zeros, some simple, cannot be dropped as shown by the examples

$p(z)e^{p(z)}$ where $p(z)$ is a polynomial with zeros in E and $\{g(z)\}^k$, $k > 1$ where g is a function of finite order with zeros in E, which are pseudo prime but not prime.

4.3 A PRIME GAP SERIES

Before going to our next theorem we shall recall some results of Erdős and Macintyre [11] on entire functions with gap power series, on which our method of proof depends.

Theorem 4.2 *Let* $f(z) = \Sigma a_n z^{\lambda_n}$, λ_n *a strictly increasing sequence of non negative integers and*

$$\max_{|z|=r} |f(z)| = M(r)$$

$$\min_{|z|=r} |f(z)| = m(r)$$

$$\max_{n} |a_n| r^n = \mu(r)$$

By improving a result of Polya, they showed that if $f(z)$ *is entire and*

$$(8) \quad \sum_{n=0}^{\infty} \frac{1}{\lambda_{n+1}-\lambda_n} < \infty,$$

Then,

$$(9) \quad \limsup_{r\to\infty} \frac{\mu(r)}{M(r)} = \limsup_{r\to\infty} \frac{m(r)}{M(r)} = 1$$

If $f(z)$ *is of finite order, the condition* (8) *can be replaced by*

$$(10) \quad \sum_{k=0}^{n} \frac{1}{\lambda_{k+1} - \lambda_k} = o(\log \lambda_n)$$

<u>and if</u> $f(z)$ <u>is of zero order</u> (9) <u>is replaced by</u>

$$(11)\quad \sum_{k=o}^{n} \frac{1}{\lambda_{k+1} - \lambda_k} = O(\log \lambda_n)$$

We now give a proof of the following theorem

<u>Theorem 4.3</u> <u>Let</u> $\{p_n\}$ <u>be a sequence of primes such that</u>

$$(12)\quad \sum_{n=o}^{\infty} \frac{1}{p_{n+1} - p_n} < \infty$$

<u>Let</u> $F(z) = \sum_{n=o}^{\infty} a_n z^{p_n}$ <u>be an entire function, then</u> $F(z)$ <u>is prime.</u>

<u>Remark:</u> <u>If we assume</u> $F(z)$ <u>to be of finite order</u> (12) <u>may be replaced by</u>

$$(13)\quad \sum_{k=o}^{n} \frac{1}{p_{k+1} - p_k} = {}_{o}O(\log p_n)$$

<u>If</u> $F(z)$ <u>is of zero order we can replace</u> (12) <u>by</u>

$$(14)\quad \sum_{k=o}^{n} \frac{1}{p_{k+1} - p_k} = O(\log p_n)$$

<u>Proof:</u> Under the assumptions of our theorem, the theorem of Erdös and Macintyre gives us

$$(15)\quad \limsup_{r\to\infty} \frac{m(r)}{M(r)} = \limsup_{r\to\infty} \frac{\mu(r)}{M(r)} = 1$$

In fact they show that there is a sequence $A_n \leq A_{n+1} \leq$ ---- such that in $A_n \leq |z| \leq A_{n+1}$ the maximum term $\mu(r) = a_k z^{p_k}$, $k = k(n)$ tends to ∞ with n, and there exists certain arbitrarily large n, such that on $|z| = \sqrt{A_n A_{n+1}} = r_n$ we have

$$(16)\quad |F(z) - a_k z^{p_k}| = \underset{c}{\phi}(|a_k| r_n^{p_k})$$

It is enough for us to replace the right hand side of (16) by $< \frac{1}{4}|a_k|r_n^{p_k}$. We choose λ, $0 < \lambda < 1$ so that $\lambda^{p_k} = 1/2$. Then on $|z| = \lambda\, r_n$ we have

$$|F(z)| \leq |F(z) - a_k z^{p_k}| + |a_k z^{p_n}|$$
$$\leq \frac{1}{4}|a_k|r_n^{p_k} + \frac{1}{2}|a_k|r_n^{p_k}$$

that is $|F(z)| < \frac{3}{4}|a_k|r_n^{p_k}$ on $|z| = \lambda r_n$,

whilst

$$|F(z)| > \frac{3}{4}|a_k|r_n^{p_k} \quad \text{on } |z| = r_n$$

There is therefore a single simply connected component G of

$\{z|\ |F(z)| < M = \frac{3}{4}|a_k|r_n^{p_k}\}$ which lies wholly in $|z| < r_n$ and whose boundary is in the annulus $\lambda r_n < |z| < r_n$. By Rouche's theorem, there are p_k zeros of $F(z)$ in $\{|z| < \lambda r_n\} \subset G$. The boundary of G is a level curve Γ of $|F(z)| = M$.

We now suppose $F(z) = f(g(z))$, f,g entire. $u = g(z)$ maps Γ on to a level curve γ: $|f(\gamma)| = M$ in the u plane. γ has no self intersections and bounds $g(G)$. Finally $w = f(u)$ maps $g(G)$ onto $|w| \leq M$, $\gamma \to |w| = M$. If g is α-valent, f β valent (as z traverses Γ, g traverses γ, α times, f traverses $|w| = M$ $\alpha\beta$ times) then $\alpha\beta = p_k$.

Since p_k is prime we have either $\alpha = 1$ or $\beta = 1$. Now in the above argument n and hence M and the regions G, g(G) can be taken arbitrarily large. It follows that either for g is linear and F is prime.

Remark: Proof of the theorem relating to functions of finite order is exactly analogous. The construction of Erdös and Macintyre shows the existence of a single dominant term in $F(z)$ for certain $|z| = R_n$ and the above argument goes through.

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