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Hamiltonian structure of passive defect dynamics in two-dimensional nematic liquid crystals in unbounded domains

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Hamiltonian structures for the passive dynamics of topological defects in nematic liquid crystals in two-dimensional systems are presented. Following the work by Miyoshi *et al.* (Miyoshi *et al.* 2024 *Proc. R. Soc. A* **480**, 20240405. ([doi:10.1098/rspa.2024.0405](https://doi.org/10.1098/rspa.2024.0405))), which demonstrated that the regularized Frank free energy associated with topological defects is equivalent to the Kirchhoff–Routh path function, the harmonic conjugate of the regularized Frank energy is investigated to elucidate the defect dynamics. Since defect motion is driven by the gradient of the Frank energy, the application of the Cauchy–Riemann equations shows that the harmonic conjugate of the regularized Frank energy serves as a Hamiltonian governing the defect dynamics. This Hamiltonian is expressed as a weighted sum of arguments between defect positions. Several conserved quantities, as well as the conditions for self-similar defect dynamics, are identified based on this Hamiltonian. The Hamiltonian structure is also applicable for estimating defect motion and defect–defect collisions within a limited field of view of the alignment angles.

1. Introduction

Topological defects are singularities of alignment fields, where the alignment angle is undefined at the singular point. These singularities, also known as disclinations,

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have been extensively studied in liquid crystals because they fundamentally determine their structure and physical properties [1]. In recent years, it has been demonstrated that topological defects strongly influence the dynamics and organization of biological systems composed of elongated cells, particularly in the context of active nematics [2,3]. Active nematics containing multiple topological defects can exhibit a variety of remarkable phenomena, such as topological chaos [4,5] and morphogenesis in *Hydra* [6]. The study of such phenomena remains an important and active area of research in the field of active nematics.

Mathematical modelling of active nematic dynamics is crucial for understanding and controlling the behaviour of topological defects. Giomi *et al.* provided a theoretical framework for defect dynamics in active nematics by coupling nematic liquid crystal theory with incompressible fluid dynamics [7]. Building on this foundation, numerous studies have investigated defect dynamics in active fluids, including the regulation of turbulent behaviour through modifications of confining geometries [8,9]. These examples highlight both the importance and the continued development of mathematical approaches to defect analysis in active nematics.

Most active nematics are realized as two-dimensional films [10]. Within the framework of liquid crystal theory and under the one-constant approximation [1], harmonic function theory in two dimensions has become a central tool for representing alignment angles in the presence of topological defects, particularly in confined geometries [11–14]. The equation of motion described by equation (2.6) arises in a wide range of applications. For example, see [15] for the dynamics of Janus particles in harmonic alignment fields. By contrast, in point vortex dynamics, the Kirchhoff–Routh function not only determines equilibrium positions but also serves as the Hamiltonian governing vortex motion. For instance, Chandler & Spagnolie [16] derived the force exerted on boundaries with tangential anchoring, where the nematic liquid crystal aligns along the boundary. They obtained a formula for the force on immersed bodies that is mathematically analogous to the classical ‘Blasius formula’ from fluid dynamics [17]. Using similar assumptions, they also analysed the motion of such bodies induced by topological defects in two-dimensional domains [14]. More recently, Čopar & Kos [18] derived equations of motion for defects under prescribed far-field orientations based on harmonic function theory and a quasi-static force approximation. They showed that $\pm 1/2$ topological defects in unbounded domains can undergo annihilation and braiding phenomena owing to constraints on local orientation fields [18]. Recent advances have further provided explicit formulas for the Frank free energy in simply connected [12,19] and doubly connected domains [20,21], where analytic expressions for alignment angles can be obtained [12,20]. This line of work has been extended by Miyoshi *et al.* [22] to multiply connected domains using the Schottky–Klein prime functions developed by Crowdy [23].

Topological defects in nematic liquid crystals play a central role in understanding both static configurations and dynamic behaviour. While traditional approaches typically focus on minimizing the Frank free energy to identify equilibrium states [12,24], recent progress has introduced more advanced analytical techniques inspired by vortex dynamics in two-dimensional fluids. In particular, Miyoshi *et al.* proposed a regularized Frank energy functionally analogous to the Kirchhoff–Routh path function in point vortex dynamics [19]. This formulation allows equilibrium configurations of topological defects to be characterized as stationary points of the regularized energy in bounded domains. However, their analysis was limited to static equilibria and did not address time-dependent defect dynamics.

Subsequently, Miyoshi *et al.* [21] demonstrated that the same equilibrium states can be obtained by imposing equivalent local force balance conditions. Although this approach offers computational advantages, it remains restricted to equilibrium analysis and does not provide insight into defect dynamics. In classical vortex dynamics, the Hamiltonian structure persists even in multiply connected domains, as established by Lin [25]. By contrast, for nematic liquid crystals, it has been shown that the motion of $+1/2$ and $-1/2$ defects follows a gradient flow that decreases the Frank free energy, indicating that the regularized Frank energy proposed in [19] cannot serve as a Hamiltonian for such dynamics.

The primary aim of this paper is to derive a Hamiltonian formulation for passive defect dynamics in unbounded domains. Here, passive defect dynamics refers to defect motion in the absence of active forces, governed solely by the gradient of the Frank free energy. This assumption is consistent with frameworks adopted in several previous studies [14,18,26]. By analysing the equations of motion under the dilute approximation for liquid crystals, we show that defect dynamics can be reformulated as a Hamiltonian system. The corresponding Hamiltonian is identified as the harmonic conjugate of the regularized Frank energy, constructed explicitly through analogy with the Kirchhoff–Routh path function from point vortex dynamics.

This formulation provides a new framework for describing the conservative dynamics of topological defects in nematic systems, thereby establishing a connection between liquid crystal theory and the broader mathematical framework of Hamiltonian fluid dynamics. Within this perspective, we derive conditions for the existence of homothetic configurations—self-similar dynamical arrangements of defects. Such configurations were originally studied by Borisov & Mamaev [27] and Lacomba [28] in the context of two-dimensional fluids with point sources and sinks; our Hamiltonian formulation reveals analogous behaviour in nematic defect dynamics.

Another application of the Hamiltonian formulation developed here is the analysis of defect motion in regions with limited observational access. In experimental studies of cellular alignment, the field of view is restricted by the imaging area of the camera [29]. By combining measured alignment angles within the observable region with Poisson’s integral formula applied on the boundary, one can estimate the alignment field outside the observed domain. The Hamiltonian can then be constructed from the harmonic conjugate of this extended field, enabling the estimation of defect trajectories even in limited-view regions. Defect trajectories have been widely studied; for example, Mitchell *et al.* [30] analysed defect paths using the nematic locking principle. The present framework is also applicable to the analysis of defect interactions and collision (annihilation) processes.

The paper is organized as follows. Section 2 presents the mathematical formulation of topological defects in unbounded two-dimensional domains. Section 3 derives the Hamiltonian structure and associated conserved quantities, and provides conditions for the integrability of passive defect motion. Section 4 examines homothetic configurations and conditions for self-similar defect dynamics. Section 5 explores applications to regions with limited observational access. Finally, Section 6 presents concluding remarks and discusses future applications of the Hamiltonian framework to active nematic systems.

2. Mathematical formulation

(a) Problem setting

We consider N topological defects located at $z_k = x_k + iy_k \in \mathbb{C}$, $k = 1, \dots, N$, in the complex plane $z = x + iy$. Assuming that the size of the nematic liquid crystal molecules is negligible, we focus on the alignment angle $\phi(x, y)$ of the director field $\mathbf{n}(x, y) = (\cos \phi(x, y), \sin \phi(x, y))^T$. Under the one-constant approximation [11], the angle $\phi(x, y)$ is harmonic and can therefore be expressed as the real part of an analytic function $f(z)$, where

$$f(z) := \sum_{k=1}^N (-iq_k + \tau_k) \log(z - z_k). \quad (2.1)$$

Here, q_k denotes the charge of the defect and τ_k denotes the torque, which characterizes the local orientation around the defect [13]. Topological charges and torques are also referred to as winding numbers and spiral charges (see Čopar & Kos [18]). In this paper, we adopt the terminology ‘charge’ and ‘torque’, following Tang *et al.* [13]. The reader should note that these terms are equivalent.

Since the alignment angle represents a director field, $\phi(x, y)$ is defined up to congruence modulo π , i.e. ϕ is equivalent to $\hat{\phi} \in [0, \pi)$. Accordingly, the charge q_k must be an integer or a half-integer (positive or negative), whereas the torque τ_k may take arbitrary real values. An example of a director field with three topological defects is shown in the left-hand panel of figure 1.

The nematic elastic energy, known as the Frank free energy, is defined as the integral of the squared gradient of the alignment angle ϕ over a two-dimensional domain [1]:

$$F := \frac{K}{2} \int |\nabla \phi(x, y)|^2 dx dy, \quad (2.2)$$

where we have used the one-constant approximation $K_1 = K_3 = K$, and K is the Frank elastic constant [1]. From equation (2.2), the Frank energy diverges as $|\nabla \phi(x, y)|^2 \rightarrow \infty$ when $z \rightarrow z_k$ or $z \rightarrow \infty$. To regularize F , we follow Čopar & Kos [18] by introducing a defect core of radius ϵ for each defect (a short-range cut-off of the order of the isotropic core [3]) and by confining the system within a large radius R [18]. That is, we assume $\epsilon \ll |z_k - z_l| \ll R$ for $k \neq l$. This is known as the *dilute* approximation, in which defects are well separated from each other and far from the boundaries [1,31]. Using Green's second identity, the Frank energy F in equation (2.2) can be expressed explicitly as

$$F = \pi K \left[\left(\sum_{k=1}^N q_k \sum_{l=1}^N q_l \right) + \left(\sum_{k=1}^N \tau_k \sum_{l=1}^N \tau_l \right) \right] \log \frac{R}{\epsilon} + \pi K \mathcal{F}_d + c_0(\epsilon), \quad (2.3)$$

where $\mathcal{F}_d$ is the regularized Frank energy [19], defined by

$$\mathcal{F}_d := - \sum_{k=1}^N \sum_{l \neq k}^N (q_k q_l + \tau_k \tau_l) \log |z_k - z_l|, \quad (2.4)$$

and $c_0(\epsilon) := \pi K \log \epsilon \cdot \sum_{k=1}^N \sum_{l \neq k}^N (q_k q_l + \tau_k \tau_l)$ is a constant independent of z_k . The first term in equation (2.3) accounts for far-field contributions and local core effects, whereas the second term represents defect–defect interactions. Note that both the first and the last terms in equation (2.3) are independent of the defect positions z_k .

It follows that, for fixed ϵ , q_k and τ_k , the Frank energy is minimized when the second term in equation (2.3) is locally minimized [11]. Miyoshi *et al.* [21] used an analytical expression for the Frank energy in simply connected domains, similar to equation (2.3), to determine equilibrium defect positions. These equilibrium configurations are equivalently obtained by enforcing local force balance conditions derived from the energy gradient.

(b) Motion of defects

The motion of defects can be described under the quasi-static approximation of the nematic field [18], meaning that the relaxation time of the alignment field is much shorter than that of defect motion. Under this assumption, the alignment angle remains harmonic during defect evolution. The force $(f_{k,x}, f_{k,y})$ acting on the defect at z_k is given by the gradient of the Frank free energy and can be written as

$$f_{k,x} - i f_{k,y} = -2 \frac{\partial F}{\partial z_k} = -2\pi K \frac{\partial \mathcal{F}_d}{\partial z_k}, \quad (2.5)$$

where expression (2.3) has been used to obtain the final equality. This force is equivalent to the Peach–Koehler force acting between disclination lines in crystalline solids [32,33].

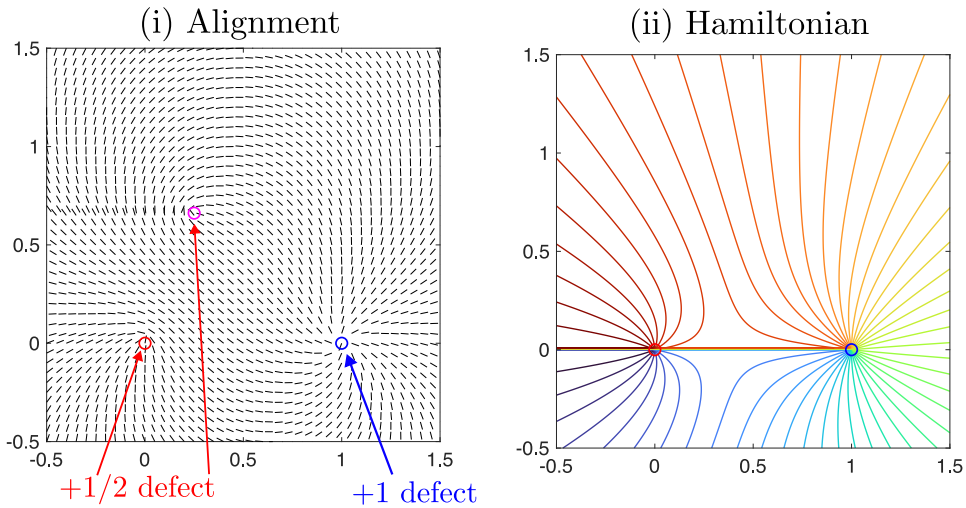


Figure 1. Topological defects with charges of $+1/2$ located at $z_1 = 0$ and $z_3 = 0.25 + 0.6614i$, and a defect with charge $+1$ located at $z_2 = 1$. The director field generated by these three defects is shown in panel (i). Assuming that the defects at $z = z_1$ and $z = z_2$ are stationary, the $+1/2$ defect at $z = z_3$ moves under the gradient of the regularized Frank free energy. The trajectory is shown in panel (ii) as contour lines of the Hamiltonian, computed as the harmonic conjugate of the regularized Frank energy. The branch cut of the Hamiltonian arising from the logarithmic terms is chosen between the defects in this configuration.

Topological defects are assumed to move passively under this force. Neglecting active contributions and using the quasi-static approximation [18], we obtain [34]

$$\hat{\xi} \frac{dz_k}{dt} = f_{k,x} - if_{k,y} = -2\pi K \frac{\partial F_d}{\partial z_k}, \quad (2.6)$$

where $\hat{\xi}$ is an effective friction coefficient [18], and the final equality follows from equation (2.5).

The quasi-static approximation commonly employed for passive nematic systems [34,35] assumes that the nematic field relaxes on a much shorter time scale than defect motion, so that the director field can be regarded as being in a sequence of instantaneous equilibrium states. In general, for active nematics, activity can continuously inject energy into the system and generate flows and self-driven defect dynamics [7]. Although the quasi-static approximation may not always hold in active nematics, the recent work by Miyazako *et al.* [34] shows that the active force on the defects derived by the quasi-static approximation predicts the defect dynamics in dry active nematics. This result suggests that the quasi-static approximation is still applicable to active nematics in certain regimes, especially for dry active nematics.

The equation of motion (equation (2.6)) is governed by the gradient of the energy and is therefore not a Hamiltonian system. However, in an unbounded domain, the regularized Frank energy F_d in equation (2.4) can be expressed as the imaginary part of an analytic function $\mathcal{G}$ defined in equation (3.1), which admits a harmonic conjugate that is unique up to an additive constant, and provides the basis for a Hamiltonian description of the dynamics.

3. Hamiltonian formulation and conserved quantities

(a) Hamiltonian representation of defect dynamics

Henceforth, each defect location z_k is taken to be a function of time, although this dependence is suppressed in our notation, except where needed for emphasis. We consider an analytic extension

of the reduced Frank energy $\mathcal{F}_d$, defined by

$$\mathcal{G}(z_1, z_2, \dots, z_N) := -i \sum_{k=1}^N \sum_{l \neq k}^N (q_k q_l + \tau_k \tau_l) \log(z_k - z_l). \quad (3.1)$$

It is straightforward to verify that $\mathcal{G}$ is analytic with respect to z_k , $k = 1, \dots, N$, and satisfies $\mathcal{F}_d = \text{Im}[\mathcal{G}]$. We then consider the real part of $\mathcal{G}$, namely,

$$\mathcal{H}(z_1, z_2, \dots, z_N) := \text{Re}[\mathcal{G}] = \sum_{k=1}^N \sum_{l \neq k}^N (q_k q_l + \tau_k \tau_l) \arg(z_k - z_l). \quad (3.2)$$

Since $\mathcal{G}$ is analytic with respect to z_k , the equation of motion in [equation \(2.6\)](#), after division by $\hat{\xi}$, can be rewritten as

$$\frac{d\bar{z}_k}{dt} = -\frac{2\pi K}{\hat{\xi}} \frac{\partial}{\partial z_k} \left(\frac{\mathcal{G} - \bar{\mathcal{G}}}{2i} \right) = 2i \frac{\pi K}{\hat{\xi}} \frac{\partial}{\partial z_k} \left(\frac{\mathcal{G}}{2} \right) = 2i \frac{\pi K}{\hat{\xi}} \frac{\partial}{\partial z_k} \left(\frac{\mathcal{G} + \bar{\mathcal{G}}}{2} \right) = 2i \frac{\partial \mathcal{H}}{\partial z_k}, \quad (3.3)$$

where we used $\text{Im}[\mathcal{G}] = (\mathcal{G} - \bar{\mathcal{G}})/(2i)$ and set the scaling factor $\pi K/\hat{\xi}$ to 1. [Equation \(3.3\)](#) implies that

$$\frac{d\bar{z}_k}{dt} = 2i \frac{\partial \mathcal{H}}{\partial z_k} = \sum_{l \neq k}^N \frac{q_k q_l + \tau_k \tau_l}{z_k - z_l}. \quad (3.4)$$

For the canonical form, writing $z_k = x_k + iy_k$, one obtains from [equation \(3.3\)](#) that

$$\frac{dx_k}{dt} = \frac{\partial \mathcal{H}}{\partial y_k}, \quad \frac{dy_k}{dt} = -\frac{\partial \mathcal{H}}{\partial x_k}, \quad (3.5)$$

for $k = 1, \dots, N$. This shows that $\mathcal{H}(z_1, z_2, \dots, z_N)$ serves as a Hamiltonian for the dynamics of the defect locations z_k , $k = 1, \dots, N$, and that the defect dynamics in unbounded domains, driven solely by the Frank free energy, is a Hamiltonian system.

Moreover, a direct calculation shows that the total time derivative of $\mathcal{H}$ is

$$\frac{d\mathcal{H}}{dt} = \sum_{k=1}^N \left(\frac{\partial \mathcal{H}}{\partial z_k} \frac{dz_k}{dt} + \frac{\partial \mathcal{H}}{\partial \bar{z}_k} \frac{d\bar{z}_k}{dt} \right) = 0. \quad (3.6)$$

Hence, $\mathcal{H}$ is conserved by the dynamics. It should be noted that the Hamiltonian in [equation \(3.2\)](#) is only locally defined, since each polar angle $\arg(z_k - z_l)$ is defined modulo 2π . However, its exterior differential is globally defined on $\mathbb{C} \setminus \Delta$, where $\Delta := \{z_1, \dots, z_N\}$ [\[28\]](#).

This formulation enables the identification of several conserved quantities associated with defect dynamics, which have not been reported in previous works. In addition, it reveals that the dynamical evolution of topological defects is more closely analogous to the free motion of point sources [\[27\]](#) than to the classical dynamics of point vortices [\[36\]](#).

In the context of point source dynamics in ideal fluids, Lacombe [\[28\]](#) derived a Hamiltonian formulation for N point sources based on an analogy with point vortices. Binary collisions between two sources were analysed and subsequently generalized to systems with multiple point sources. In the setting of topological defects, such binary collisions correspond to defect annihilation. The annihilation time and trajectories can therefore be examined using the Hamiltonian system given by [equation \(3.5\)](#).

Note that this Hamiltonian structure does not generally extend in any obvious way to bounded domains. Indeed, such a structure is not even established for point sources in bounded fluid domains [\[27,28\]](#). This limitation arises from the non-harmonicity of the regularized Frank energy in [equation \(2.4\)](#) with respect to z_k , as shown in [\[19\]](#).

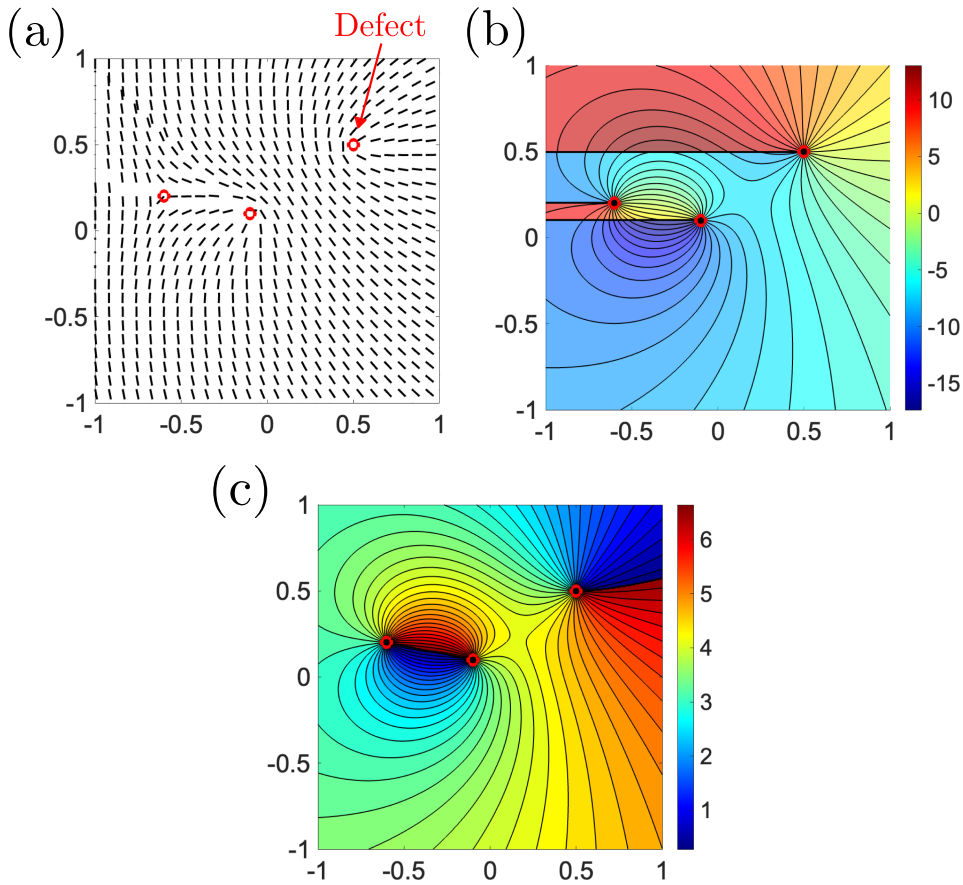


Figure 2. Topological defects with charges of $+1/2$ located at $z_1 = -0.1 + 0.1i$ and $z_2 = 0.5 + 0.5i$, and a defect with charge $-1/2$ located at $z_3 = -0.6 + 0.2i$. All defects are torque-free, i.e. $\tau_k = 0$ for $k = 1, 2, 3$. (a) Alignment field generated by the three defects. (b) Contours of the Hamiltonian $\mathcal{H}(z)$ with branch cuts chosen parallel to the real axis. (c) Contours of $\mathcal{H}(z)$ with branch cuts aligned along its contour lines. The contour lines shown in black in panel (c) are continuous.

To illustrate this, consider the alignment angle in the unit disc with tangential anchoring boundary conditions [12]. Tangential anchoring requires that the alignment angle be tangent to the boundary, i.e. $\phi = \theta - \pi/2$ for $\zeta = e^{i\theta}$ on the boundary. Miyazako & Nara [12] derived explicit expressions for the alignment angle $\phi(x, y)$ with multiple defects, involving both $\log(z - z_k)$ and $\log(1 - \bar{z}_k z)$ terms. Thus, in the presence of boundaries, the dependence involves not only z_k but also $1/\bar{z}_k$, which prevents the construction of a harmonic conjugate of the energy.

(b) Branch cut of the Hamiltonian

The Hamiltonian $\mathcal{H}$ is defined as the real part of the analytic function in $\mathcal{G}$ in equation (3.1), which has branch points at z_k , $k = 1, \dots, N$. Assuming that the defects z_k , $k = 1, \dots, N$, are fixed, consider an additional defect with charge q and torque τ located at a point z and which, in contrast, is free to move. This defect moves along the contour lines of the Hamiltonian $\mathcal{H}(z) = \mathcal{H}(z, z_1, \dots, z_N)$, which are determined by the level sets of

$$\mathcal{H}(z) = 2 \sum_{k=1}^N (qq_k + \tau\tau_k) \arg(z - z_k). \quad (3.7)$$

The right-hand panel of [figure 1](#) shows the contour lines of the Hamiltonian with two fixed defects located at $z = 0$ and $z = 1$.

However, an arbitrary choice of branch cuts renders the contour lines of $\mathcal{H}$ discontinuous. To avoid this issue, the branch cuts are chosen to lie along the contour lines of $\mathcal{H}(z)$. [Figure 2a](#) shows the director field with three topological defects, with charges $+1/2$ at $z_1 = -0.1 + 0.1i$ and $z_2 = 0.5 + 0.5i$, and a defect with charge $-1/2$ at $z_3 = -0.6 + 0.2i$, indicated by red dots. [Figure 2b](#) shows the contours of $\mathcal{H}(z)$ when the branch cuts of $\log(z - z_k)$ are taken along the negative real axis. In this case, the contour lines of $\mathcal{H}(z)$ are clearly discontinuous. By contrast, [figure 2c](#) shows the contours when the branch cuts are aligned with the contour lines of $\mathcal{H}(z)$. With this choice of branch cuts, the contour lines of $\mathcal{H}(z)$ become continuous.

There are multiple possible choices for the branch cuts, since infinitely many contour lines emanate from each z_k , as shown in [figure 2c](#). However, for any choice of branch cuts that lie along contour lines of the Hamiltonian, the resulting contours remain continuous, because distinct contour lines do not intersect. A more rigorous mathematical justification of this property remains a topic for future work.

(c) Several conserved quantities

The Hamiltonian formulation yields several conserved quantities, analogous to those in point vortex dynamics [36]. These invariants can be used to study first integrals and collision (blow-up) behaviour of topological defects. First, consider the centre position of z_k multiplied by N defined by

$$Z = Q + iP := \sum_{k=1}^N z_k. \quad (3.8)$$

A direct calculation shows that the quantity Z is conserved in motion, that is,

$$\frac{d\bar{Z}}{dt} = \sum_{k=1}^N \frac{d\bar{z}_k}{dt} = \sum_{k=1}^N \sum_{l \neq k}^N \frac{q_k q_l + \tau_k \tau_l}{z_k - z_l} = 0, \quad (3.9)$$

where [equation \(3.2\)](#) and [equation \(3.3\)](#) have been used. In addition, it can be shown that the quantity $\mathcal{A} := \sum_k \left(z_k \times \frac{dz_k}{dt} \right)$ is also invariant for the system. By a direct calculation,

$$\mathcal{A} = \sum_{k=1}^N \left(z_k \times \frac{dz_k}{dt} \right) = \text{Im} \left[\sum_{k=1}^N \bar{z}_k \frac{dz_k}{dt} \right] = \text{Im} \left[\sum_{k=1}^N \sum_{l \neq k}^N \frac{\bar{z}_k (z_k - z_l)}{|z_k - z_l|^2} (q_k q_l + \tau_k \tau_l) \right] = 0. \quad (3.10)$$

However, unlike in point source dynamics [27,28], the quantity $I := \sum_{k=1}^N |z_k|^2$, is not generally conserved. A direct computation yields

$$\begin{aligned} \frac{dI}{dt} &= \sum_{k=1}^N \left(z_k \frac{d\bar{z}_k}{dt} + \bar{z}_k \frac{dz_k}{dt} \right) = \sum_{k=1}^N \sum_{l \neq k}^N \left(\frac{z_k}{z_k - z_l} + \frac{\bar{z}_k}{\bar{z}_k - \bar{z}_l} \right) (q_k q_l + \tau_k \tau_l) \\ &= \sum_{k=1}^N \sum_{l \neq k}^N (q_k q_l + \tau_k \tau_l) \left(\frac{z_k (\bar{z}_k - \bar{z}_l)}{|z_k - z_l|^2} + \frac{\bar{z}_l (z_k - z_l)}{|z_l - z_k|^2} \right) = \sum_{k=1}^N \sum_{l \neq k}^N (q_k q_l + \tau_l \tau_k). \end{aligned} \quad (3.11)$$

This indicates that when the topological charges q_k and the torques τ_k , $k = 1, \dots, N$ satisfy the condition $\sum_{k=1}^N \sum_{l \neq k}^N (q_k q_l + \tau_l \tau_k) = 0$, then the quantity I is conserved by the dynamics. Otherwise, [equation \(3.11\)](#) means that the sum of $|z_k|^2$ increases linearly with time.

(d) Integrability

Since the motion of topological defects is governed by the Hamiltonian system in equation (3.5), it is natural to investigate its integrability within the framework of classical Hamiltonian theory. Hamiltonian formulations for vortex dynamics are well established [36], including results on vortex equilibria and chaotic behaviour [37]. Classically, the motion of three vortices is integrable, and that of four vortices is integrable when the total vorticity vanishes. By contrast, the dynamics of point sources and sinks have been less extensively studied, but with notable contributions by Lacombe [28] and Borisov & Mamaev [27]. Although the present system resembles point source dynamics, differences arise in the properties of I and $\mathcal{H}$.

To analyse integrability, we define the Poisson bracket as

$$\{f, g\} := \sum_{k=1}^N \left(\frac{\partial f}{\partial x_k} \frac{\partial g}{\partial y_k} - \frac{\partial g}{\partial x_k} \frac{\partial f}{\partial y_k} \right). \quad (3.12)$$

For any observable w , the Hamiltonian dynamics satisfy $dw/dt = \{w, \mathcal{H}\}$. In particular, $dx_k/dt = \{x_k, \mathcal{H}\}$ and $dy_k/dt = \{y_k, \mathcal{H}\}$. It can be shown that $\{Q, \mathcal{H}\} = \{P, \mathcal{H}\} = 0$, where Q and P are defined in equation (3.8). However, a direct calculation gives

$$\{Q, P\} = N \neq 0, \quad (3.13)$$

indicating that Q and P are not in involution. This contrasts with the case of point source dynamics [27, 28].

Assume that the charges and torques satisfy $\{I, \mathcal{H}\} = \sum_{k=1}^N \sum_{l \neq k} (q_k q_l + \tau_k \tau_l) = 0$. Then it can be shown that

$$\{Q^2 + P^2, \mathcal{H}\} = \{Q^2 + P^2, I\} = 0. \quad (3.14)$$

The first equality follows from the conservation of Q and P , while the second follows from

$$\{Q^2 + P^2, I\} = \sum_{k=1}^N \left(2Q \frac{\partial I}{\partial y_k} - 2P \frac{\partial I}{\partial x_k} \right) = 4 \sum_{k=1}^N \sum_{l=1}^N (x_l y_k - x_k y_l) = 0. \quad (3.15)$$

These results indicate that if $\sum_{k \neq l} (q_k q_l + \tau_k \tau_l) = 0$, then a three-defect system in the plane is Liouville integrable [36].

For $N = 3$ and torque-free defects, the integrability condition reduces to $q_1 q_2 + q_2 q_3 + q_1 q_3 = 0$. Examples of charge configurations satisfying this condition include $(q_1, q_2, q_3) = \pm(1/2, -1, -1)$ and $(q_1, q_2, q_3) = \pm(-1, 3/2, 3)$, subject to the constraint that q_k are integers or half-integers. Thus, infinitely many defect configurations yield integrable dynamics for three-defect systems.

Note that the lowest-energy configurations typically involve charges $q_k = \pm 1/2$, making such pairs the most commonly observed. However, $+1$ defect-like configurations can arise when small inclusions are present in domains with tangential or normal anchoring conditions [14]. This highlights the relevance of considering a broader range of charge values in practical settings involving nematic liquid crystals with boundaries.

4. Numerical examples for two and three topological defects

In point source dynamics, binary collision (or collapse) is a well-known phenomenon in which a point source merges with a point sink [28]. A similar behaviour is observed in classical defect dynamics, referred to as defect annihilation [7], where defects of opposite charge merge. Motivated by classical results in point source dynamics, this section presents several numerical examples of topological defect dynamics.

(a) Two topological defects

The first case is that two topological defects are located at $z = z_1$ and $z = z_2$, respectively. We assume that the defects are torque-free, i.e. $\tau_1 = \tau_2 = 0$. Without loss of generality, the coordinate system can be rotated so that both z_1 and z_2 are real. The equations of motion for $z_1(t)$ and $z_2(t)$ are then given by

$$\frac{d\bar{z}_1}{dt} = \frac{q_1 q_2}{z_1 - z_2}, \quad \frac{d\bar{z}_2}{dt} = \frac{q_1 q_2}{z_2 - z_1}. \quad (4.1)$$

Note that if initial conditions are such that $z_1(0)$ and $z_2(0)$ are both real, this property is preserved by the dynamics. It is straightforward to verify that the sum $z_1 + z_2$ is conserved. Defining the relative distance $\xi := z_1 - z_2$, we obtain

$$\frac{d\xi}{dt} = \frac{2q_1 q_2}{\xi}. \quad (4.2)$$

Equation (4.2) implies that if $q_1 q_2 > 0$, the distance ξ increases, whereas if $q_1 q_2 < 0$, ξ decreases and the two defects eventually merge. In point source terminology, this merging is referred to as binary collapse [28], while in defect dynamics it is known as annihilation [7]. The equation of motion in equation (4.2) is solved explicitly by

$$\xi(t) = \sqrt{4q_1 q_2 t + \xi_0^2}, \quad (4.3)$$

where $\xi_0 = |z_1 - z_2|$ is the initial separation between the defects. Equation (4.3) shows that annihilation occurs at $t = -\xi_0^2/(4q_1 q_2)$, consistent with the results of Giomi *et al.* [7], Miyazako *et al.* [34] and Denniston [35].

(b) Three topological defects: homothetic configurations

As discussed in §3, the dynamics of three topological defects are integrable when $\{I, \mathcal{H}\} = \sum_{k \neq l}^3 q_k q_l = 0$, where I was defined earlier and $\mathcal{H}$ is the Hamiltonian. In this case, the system is Liouville integrable and the motion evolves on a three-dimensional torus [36].

Beyond integrability, another notable feature of three-defect dynamics is the existence of homothetic configurations [27,28]. A homothetic configuration is one in which the geometric configuration remains self-similar at all times. Equivalently, the ratios of distances between defects remain constant during the motion [38]. Mathematically, this is expressed as

$$\frac{d}{dt} \left(\frac{D_{jk}}{D_{lm}} \right) = 0, \quad (4.4)$$

where $D_{jk} := |z_j - z_k|^2$, is the squared distance between z_j and z_k for $j \neq k$. The dynamics of z_k , $k = 1, 2, 3$, is given by

$$\begin{cases} \frac{d\bar{z}_1}{dt} = \frac{q_1 q_2}{z_1 - z_2} + \frac{q_1 q_3}{z_1 - z_3}, \\ \frac{d\bar{z}_2}{dt} = \frac{q_2 q_1}{z_2 - z_1} + \frac{q_2 q_3}{z_2 - z_3}, \\ \frac{d\bar{z}_3}{dt} = \frac{q_3 q_1}{z_3 - z_1} + \frac{q_3 q_2}{z_3 - z_2}, \end{cases} \quad (4.5)$$

from which we obtain

$$\begin{aligned}\frac{dD_{23}}{dt} &= \overline{(z_2 - z_3)} \left(\frac{dz_2}{dt} - \frac{dz_3}{dt} \right) + (z_2 - z_3) \overline{\left(\frac{dz_2}{dt} - \frac{dz_3}{dt} \right)} \\ &= 4q_2q_3 + q_1q_2 \frac{D_{23} - D_{13} + D_{12}}{D_{12}} + q_1q_3 \frac{D_{23} - D_{12} + D_{13}}{D_{13}},\end{aligned}\quad (4.6)$$

where equation (3.4) has been used. Similar expressions can be derived for dD_{13}/dt and dD_{12}/dt .

To analyse homothetic configurations, we introduce the variables (X, Y) defined by

$$D_{23} = XD_{12}, \quad D_{13} = YD_{12}, \quad dt = D_{12}XYd\tau. \quad (4.7)$$

Using these parameterizations, it is possible to recast the differential equations as follows:

$$\frac{dX}{d\tau} = XY \left[q_1q_2(1 - 3X - Y) + q_2q_3(3 - X + Y) + q_3q_1 \frac{(1 - X)(Y - 1 - X)}{Y} \right] \quad (4.8)$$

$$\text{and } \frac{dY}{d\tau} = XY \left[q_1q_2(1 - X - 3Y) + q_2q_3 \frac{(1 - Y)(X - 1 - Y)}{X} + q_3q_1(3 + X - Y) \right]. \quad (4.9)$$

Thus, the qualitative behaviour of the system can be analysed in the (X, Y) -plane. Homothetic configurations correspond to fixed points where both derivatives vanish. After simplification, one finds that such configurations occur, in particular, when

$$X = q_3/q_1 \quad \text{and} \quad Y = q_3/q_2. \quad (4.10)$$

There also exists a trivial homothetic configuration with negative and positive defects. For example, it is possible to find $z_1 = -z_3 = h \in \mathbb{R}$, $z_2 = 0$ with $q_1 = q_3 = +1/2$ and $q_2 = -1/2$. In that case, the two defects approach the defect with a charge of $-1/2$.

Two numerical examples of homothetic configurations are shown in figure 3. The simulations were performed using a standard time integration method based on the MATLAB ODE solver (ode45). Case (i) corresponds to $q_1 = q_2 = q_3 = 1/2$, while case (ii) corresponds to $q_1 = q_2 = 1/2$ and $q_3 = 1$. The circle dots correspond to the initial positions of the defects, and the cross dots correspond to the final positions of the defects after some time. The arrows indicate the direction of the motion of the defects. The black dots are the similarity centre of the three defects. In both cases, the self-similar motion of the three defects is observed.

It can be shown, by using equation (4.6) and symmetry of the system, that

$$\sum_{j < k}^3 D_{jk} - 4 \left(\sum_{k \neq l}^3 q_k q_l \right) t = \text{const.} \quad (4.11)$$

In other words, the sum of squared distances between each defect changes at a rate of $\sum_{k \neq l}^3 q_k q_l$. This indicates that the sum of squared distances is a constant of the motion when the motion of the three defects is integrable.

It is also useful to consider the angle $\Theta := \arg(z_2 - z_1)$ [28]. By a direct calculation, we have

$$\begin{aligned}\frac{d\Theta}{dt} &= \frac{d}{dt} \text{Im} [\log(z_2 - z_1)] = -\text{Im} \left[\frac{z_2 - z_1}{D_{12}} \left(\frac{2q_1q_2}{z_2 - z_1} + \frac{q_2q_3}{z_2 - z_3} - \frac{q_1q_3}{z_1 - z_3} \right) \right] \\ &= -\frac{2q_3(q_2D_{13} - q_1D_{23})}{D_{12}D_{23}D_{13}} \Delta,\end{aligned}\quad (4.12)$$

where $\Delta := \text{Im}[\overline{(z_2 - z_1)}(z_3 - z_1)]/2$ is the area of the triangle spanned by z_1, z_2 and z_3 . Since $q_2D_{13} = q_1D_{23}$ owing to the homothetic configuration according to equation (4.10), we can derive that $\dot{\Theta} = 0$, which means that three defects move along fixed straight lines passing through defect locations.

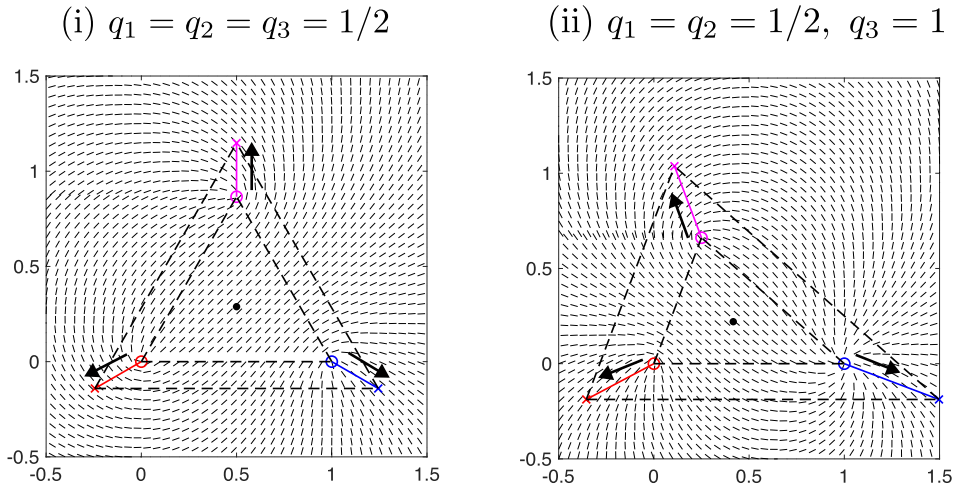


Figure 3. Homothetic configurations with (i) $q_1 = q_2 = q_3 = 1/2$ and (ii) $q_1 = q_2 = 1/2, q_3 = 1$. The three defects are indicated by coloured dots. The alignment fields are computed from the initial defect positions. The corresponding (X, Y) values lie at fixed points of the system in equations (4.8) and (4.9), and the defects move along trajectories that preserve a self-similar configuration. The circle dots correspond to the initial positions of the defects, and the cross dots correspond to the final positions of the defects after some time. The arrows indicate the direction of the motion of the defects. The black dots are the similarity centre of the three defects.

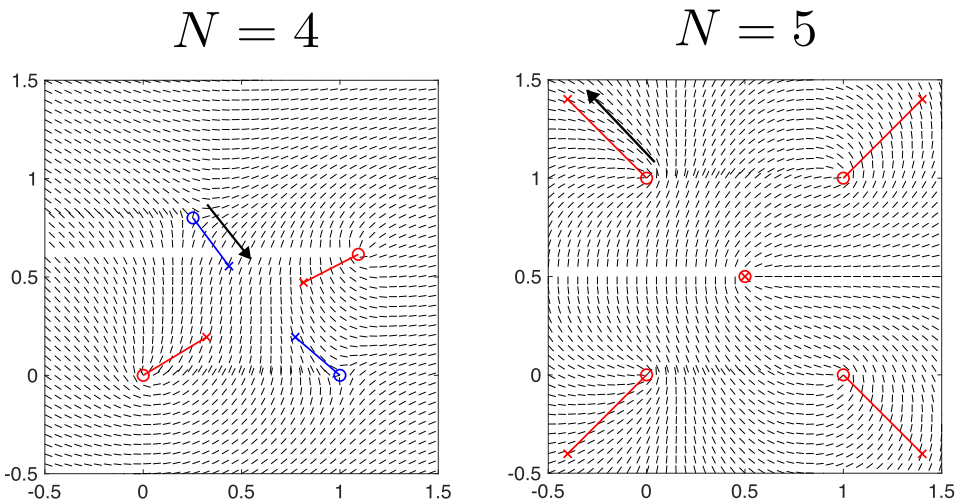


Figure 4. Homothetic configurations with $N = 4$ is shown on the left. The circle dots correspond to the initial positions of the defects, and the cross dots correspond to the final positions of the defects after some time. There are two topological defects with a charge of $+1/2$ and two topological defects with a charge of $-1/2$. Homothetic configuration with $N = 5$ is shown on the right. The charges of all topological defects are $+1/2$. For $N = 5$, the point defect located at the origin is stationary.

(c) Homothetic configuration of N topological defects

The argument presented in §4b can be extended to the dynamics of N topological defects. We define the distance between z_j and z_k as $D_{jk} := |z_j - z_k|^2$. By direct calculation,

$$\dot{D}_{jk} = 4q_j q_k + \sum_{n \neq j, n \neq k} \left(q_j q_n \frac{D_{jk} - D_{kn} + D_{jn}}{D_{jn}} + q_k q_n \frac{D_{jk} - D_{jn} + D_{kn}}{D_{kn}} \right), \quad (4.13)$$

where we used the following equation:

$$\frac{dz_j}{dt} = \sum_{l \neq j} \frac{q_j q_l}{z_j - z_l} = \frac{q_j q_k}{z_j - z_k} + \sum_{l \neq j, l \neq k} \frac{q_j q_l}{z_j - z_l}. \quad (4.14)$$

As in §4b, we define $D_{jk} = X_{jk,lm} D_{lm}$ for $k = 1, \dots, N$. A homothetic configuration is characterized by the condition that the time derivative of $X_{jk,lm}$ vanishes, that is,

$$\frac{d}{dt} \left(\frac{D_{jk}}{D_{lm}} \right) = \frac{\dot{D}_{jk} D_{lm} - D_{jk} \dot{D}_{lm}}{D_{lm}^2} = 0, \quad (4.15)$$

for any combination of (j, k, l, m) .

It is generally difficult to determine defect configurations that satisfy the condition in equation (4.15). One practical approach is to use nonlinear solvers starting from arbitrary initial configurations of z_k .

Figure 4 illustrates homothetic configurations for $N = 4$ and $N = 5$. The left-hand panel shows a configuration with $N = 4$, consisting of two defects with charge $+1/2$ and two with charge $-1/2$. The right-hand panel shows a configuration with $N = 5$, where all defects have charge $+1/2$. The circle dots correspond to the initial positions of the defects, and the cross dots correspond to the final positions of the defects after some time. The arrows indicate the direction of the motion of the defects. In both cases, the defect motion exhibits self-similar behaviour.

As noted earlier, homothetic configurations represent a special class of self-similar dynamical configurations. Although several such configurations were identified for $N = 4$ and $N = 5$, a general construction method for $N > 3$ remains unknown. It is also unclear whether homothetic configurations always exist for arbitrary charges and torques when $N > 2$.

Another challenge is that, in real experiments, defect dynamics is influenced not only by the Frank free energy but also by active forces and boundary effects. These perturbations can drive the system away from homothetic configurations. To address this issue, perturbative approaches may be useful for constructing approximate homothetic solutions.

5. Infinite region with torque-free defects with limited view

In §3, we examined the dynamics of topological defects in an unbounded region. The Hamiltonian formulation can be derived when complete information about all defects is available. In many experiments, however, observations are limited by the finite field of view of the imaging system. It is therefore natural to consider defect dynamics in an effectively infinite domain with torque-free defects, observed only within a limited region. Here, we assume that the observable region is a disc of radius 1.

In this section, we make two assumptions: the defects are torque-free, and the observation region is a unit disc. All defects evolve according to the gradient of the Frank free energy.

Let N denote the number of defects within the observation region, with charges and positions $q_1, \dots, q_N$ and $z_1, \dots, z_N$. Let M denote the number of defects outside the observation region, with charges and positions $q_{N+1}, \dots, q_{N+M}$ and $z_{N+1}, \dots, z_{N+M}$. The total number of defects is therefore $N + M$.

The alignment angle ϕ observed within the region is given by the real part of the function $f(z)$:

$$f(z) = -i \sum_{j=1}^N q_j \log(z - z_j) - i \sum_{j=N+1}^{N+M} q_j \log(z - z_j) =: f_{\text{inner}}(z) + f_{\text{outer}}(z). \quad (5.1)$$

However, the values of q_j and z_j for $j = N + 1, \dots, N + M$, as well as the number of external defects M , are unknown owing to the limited field of view. Therefore, it is necessary to estimate the contribution of the external field from these quantities from the observed data.

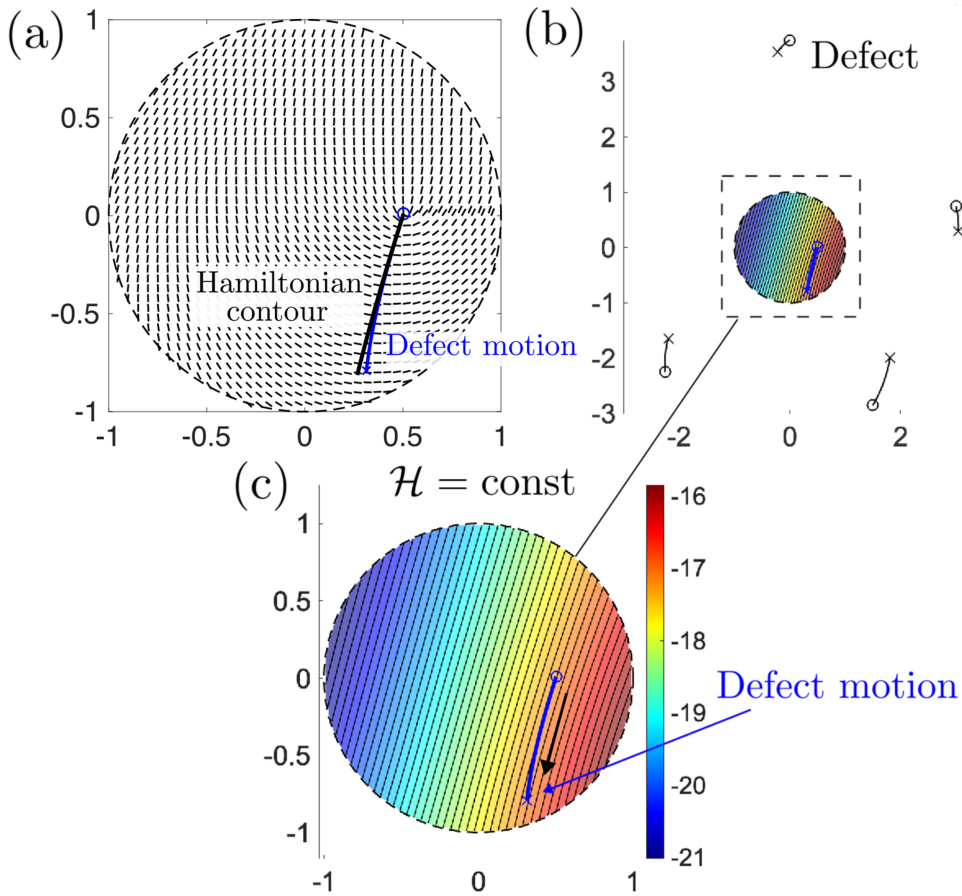


Figure 5. (a), (b) Defect dynamics in the limited view region, and (c) the corresponding Hamiltonian contour plots. We assume that a single defect with charge $+1/2$ exists within the limited view region, while the remaining defects are located far away. The initial locations of the four external defects are given by $(z_2, z_3, z_4, z_5) = \frac{3}{2}(2.0 + 0.5i, -1.5 - 1.5i, 1.0 - 1.9i, 2.5i)$ with charges $(q_2, q_3, q_4, q_5) = (1/2, -1, -1, +1)$. The initial location of the internal defect is $z_1 = 0.503 + 0.01i$. The circle dots correspond to the initial positions of the defects, and the cross dots correspond to the final positions of the defects at $t = 2.0$. (a) Alignment angle ϕ in the limited view region induced by the five defects (four external and one internal). (b) Trajectories of all defects in the full domain driven by the gradient of the Frank free energy over the time interval $t \in [0, 2.0]$. (c) Motion of the defect within the limited view region over the same time interval. The black dashed lines in panels (a) and (c) show the contours of the Hamiltonian $\mathcal{H}$ computed from $\text{Re}[f_{\text{outer}}(z)]$. The function $\text{Re}[f_{\text{outer}}(z)]$ is reconstructed using the Poisson integral defined in equation (5.2). The slight discrepancy between the Hamiltonian contours and the actual defect trajectory in panel (a) arises from the motion of defects initially located outside the limited view region.

To reconstruct $f_{\text{outer}}(z)$ from the observed data $\phi(z)$ within the limited view region, we exploit the fact that $f_{\text{outer}}(z)$ is harmonic in this region and contains no logarithmic singularities. Thus, the Poisson integral formula can be used to solve the Dirichlet problem for the two-dimensional Laplace equation with prescribed boundary data [39]:

$$f_{\text{outer}}(z) = \frac{1}{2\pi i} \oint_{C_0} \frac{\phi_0(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} d\zeta + ic_0, \quad c_0 \in \mathbb{R}, \quad (5.2)$$

with the boundary data on the circle C_0 :

$$\phi_0(z) = \phi(z) - \text{Re}[f_{\text{inner}}(z)]. \quad (5.3)$$

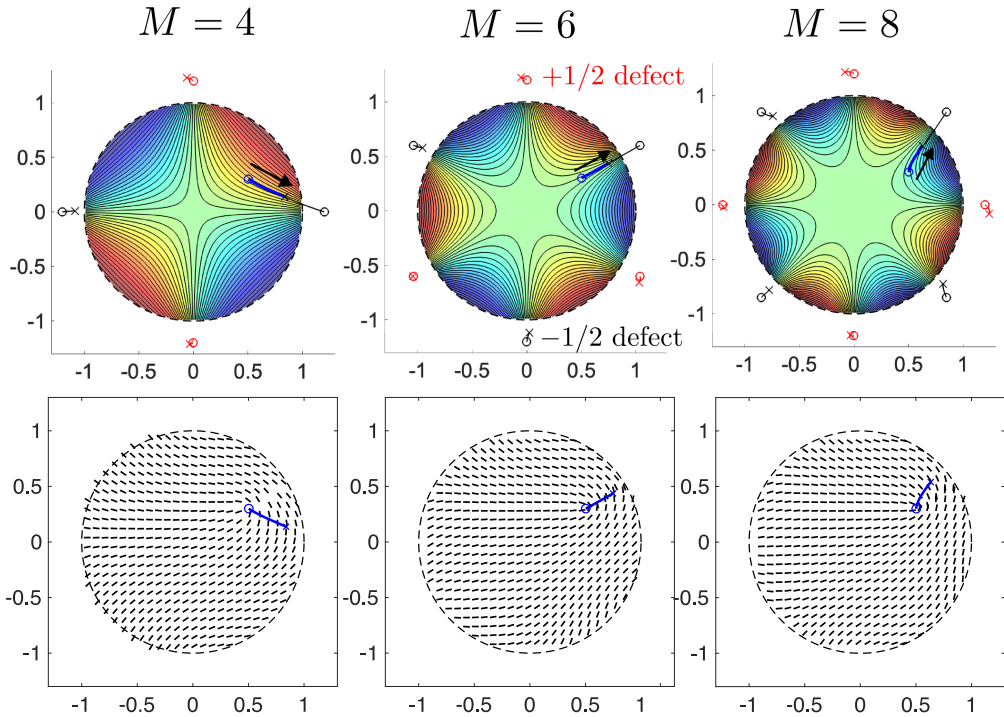


Figure 6. Symmetric configurations of external defects with charges $+1/2$ and $-1/2$. The internal defect with charge $+1/2$ is initially located at $z_1 = 0.503 + 0.3i$ at $t = 0$. Defects with charge $+1/2$ are located at $z_{2m+1} = \exp(2\pi i(2m-1)/M)$, whereas defects with charge $-1/2$ are located at $z_{2m} = \exp(2\pi i(2m-2)/M)$, $m = 1, \dots, M/2$. The circle dots correspond to the initial positions of the defects, and the cross dots correspond to the final positions of the defects at some time. Defect annihilation occurs, and the Hamiltonian contour plots approximate the collision trajectories.

Using this expression for $f_{\text{outer}}(z)$, we can determine the influence of external defects on the dynamics. Consider the regularized Frank free energy. The term $\mathcal{F}_d$ in equation (2.4) can be written as

$$\begin{aligned}\mathcal{F}_d &= - \sum_{k=1}^{N+M} \sum_{l \neq k}^{N+M} q_k q_l \log |z_k - z_l| \\ &= - \sum_{k=1}^N \sum_{l \neq k}^N q_k q_l \log |z_k - z_l| + 2 \sum_{k=1}^N q_k \text{Im}[f_{\text{outer}}(z_k)] + \mathcal{F}_{\text{outer}},\end{aligned}\quad (5.4)$$

where the energy $\mathcal{F}_{\text{outer}}$ is the (unknown) energy of the defects outside the limited view region, calculated by the presence of the defects located at z_k , $k = N+1, \dots, N+M$.

The defect dynamics, governed by the gradient of the Frank free energy, is given by

$$\frac{d\bar{z}_k}{dt} = -2 \frac{\partial \mathcal{F}_d}{\partial z_k} = -2 \left(- \sum_{l \neq k}^N \frac{q_k q_l}{z_k - z_l} - i q_k f'_{\text{outer}}(z_k) \right) = 2i \frac{\partial \mathcal{H}}{\partial z_k}, \quad (5.5)$$

where the Hamiltonian of the system is, by the analogy of the Kirchhoff–Routh path function equation (3.2), given by

$$\mathcal{H} = \sum_{k=1}^N \sum_{l \neq k}^N q_k q_l \arg(z_k - z_l) + 2 \sum_{k=1}^N q_k \text{Re}[f_{\text{outer}}(z_k)]. \quad (5.6)$$

This result is significant because it shows that defect motion can be inferred solely from the alignment field $\phi(z)$ within the limited view region. The expression in equation (5.6) is analogous to the Hamiltonian for vortex dynamics in a background flow with complex potential $f_{\text{outer}}(z)$ [40]. Notably, the Hamiltonian here involves the real parts of the logarithmic terms and $f_{\text{outer}}(z)$, rather than the imaginary parts.

For numerical simulations, we assume that a single defect with charge $+1/2$ lies within the limited view region, while the remaining defects are far away. The initial positions of the external defects are $(z_2, z_3, z_4, z_5) = \frac{3}{2}(2.0 + 0.5i, -1.5 - 1.5i, 1.0 - 1.9i, 2.5i)$ with charges of $(q_2, q_3, q_4, q_5) = (1/2, -1, -1, +1)$. The initial location of the internal defect is given by $z_1 = 0.503 + 0.01i$. Figure 5 shows the motion of the defects in the limited view region. Figure 5a shows the alignment angle ϕ at the initial time. Figure 5b shows the motion of defects in the whole region owing to the gradient of the Frank energy in the interval $t \in [0, 2.0]$. Circles correspond to the initial locations of defects, and the crosses correspond to the locations of defects at $t = 2.0$. Figure 5c shows the motion of defects in the limited view region owing to the gradient of the Frank energy in the interval $t \in [0, 2.0]$. The black line in figure 5a,c shows the contour of the Hamiltonian $\mathcal{H}$ calculated by the $\text{Re}[f_{\text{outer}}(z)]$. The function $\text{Re}[f_{\text{outer}}(z)]$ is recovered by Poisson's integral defined in equation (5.2). The slight difference between the contour of the Hamiltonian $\mathcal{H}$ and the actual defect motion in panel (a) is due to the motion of the defects outside the limited view region.

In experiments, the observed alignment angle ϕ is defined modulo π . To construct a continuous boundary function $\phi_0(z)$ on the circle, the data must be unwrapped. Specifically, after sampling $\phi(z)$, we define $\phi_0(z)$ as in equation (5.3) and apply

$$\phi_0(z) \mapsto \phi_0(z) + n\pi, \quad n \in \mathbb{Z}, \quad (5.7)$$

to have a continuous function $\phi_0(z)$ on the circle. The generated function $\phi_0(z)$ differs from $\text{Re}[f_{\text{outer}}(z)]$ by $m\pi$, $m \in \mathbb{Z}$, but it is easy to see that this constant does not affect the Hamiltonian structure.

The contour lines of the Hamiltonian shown in the black lines in figure 6 estimate the motion of the defects in the limited view region. Figure 6 shows the motion of the defects in the limited view region. The initial location of the defect with a charge of $+1/2$ in the limited view region is given by $z_1 = 0.503 + 0.3i$. The defects with a charge of $+1/2$ are located at $z_{2m+1} = \exp(2\pi i(2m-1)/M)$, whereas the defects with a charge of $-1/2$ are located at $z_{2m} = \exp(2\pi i(2m-2)/M)$, $m = 1, \dots, M/2$. It can be seen that the annihilation of two defects happens, and the Hamiltonian contour plots exhibit how the defects collide. The Hamiltonian contour plots shown in black lines in figure 6 estimate well the annihilation of the defects. This suggests that the Hamiltonian framework can be used to predict defect trajectories from partial observations and may also enable control of defect motion through engineered background fields.

6. Discussion

In this paper, we have derived a Hamiltonian formulation for the dynamics of topological defects in nematic liquid crystals, inspired by analogies with vortex dynamics and Kirchhoff–Routh theory. Our approach also demonstrates how the motion of defects within a limited observation region can be described in terms of observable alignment angles, by accounting for contributions from defects both inside and outside the observation domain. This has several important implications and opens new directions for research and applications.

First, extending the Hamiltonian formulation to bounded domains may provide new research opportunities. Geometrical confinement of defects is known to offer potential control over defect motion in nematic liquid crystals [5]. It has been shown that alignment angles in the unit disc can be represented by logarithmic singularities located at $z = z_k$ and $z = 1/\bar{z}_k$ [20]. While the present theoretical framework applies only to unbounded domains, it may be possible to extend the formalism by treating boundary effects as perturbations. Such perturbations

to the Hamiltonian could be analysed using Melnikov theory [36]. In this context, analytical tools such as Melnikov theory may help characterize the persistence of defect trajectories and the influence of domain boundaries. Analysis on the boundary effects and the perturbation of the Hamiltonian system by the active force is an important direction for future work.

This paper assumes that the torques are constant in motion. If the torques are allowed to become dynamical variables, torques are changed by the gradient of the Frank free energy as follows:

$$\frac{d\tau_k}{dt} = -b \frac{\partial F}{\partial \tau_k}, \quad (6.1)$$

where b is a relaxation parameter for torques. However, Melnikov theory would be applicable if the relaxation parameter b is small, so that the torque dynamics can be seen as a perturbation of the Hamiltonian dynamics.

The Hamiltonian structure also provides a new numerical framework for simulating defect dynamics. Structure-preserving numerical schemes, such as symplectic integrators, are well-suited for Hamiltonian systems [41]. Developing efficient algorithms to compute quantities such as collision times and periodic defect trajectories is an important direction for future work.

Chaotic advection and nonlinear dynamics represent another promising avenue of study. The Hamiltonian structure suggests the possibility of complex, and even chaotic, defect motion, particularly in systems with many defects or in multiply connected domains. Chaotic behaviour of singularities has been observed in both vortex dynamics and nematic systems [42]. It is therefore plausible that defect trajectories may exhibit sensitive dependence on initial conditions and intricate phase-space structures.

Finally, periodic and collective motions, such as leapfrogging orbits, have recently been observed in numerical studies [43]. The present formulation provides a theoretical basis for analysing such collective behaviours, including the leapfrogging dynamics reported for nematic defects [44]. These periodic motions arise from the interplay between defect interactions and background fields, and may be strongly influenced by domain geometry and boundary conditions [43,45]. Future work will focus on extending the present results to more complex geometries and investigating their effects on periodic defect motion.

Data accessibility. Data are available at [46].

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

Authors' contributions. H.M.: conceptualization, funding acquisition, resources, software, writing—original draft; D.G.C.: conceptualization, writing—review and editing.

Both authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Conflict of interest declaration. We declare we have no competing interests.

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