

TURBULENT NATURAL AND FORCED CONVECTION
IN PLUMES AND CAVITIES

by

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ABSTRACT

The partial-differential equations governing steady, turbulent, two-dimensional natural and forced convection in plumes and cavities are solved by means of finite-difference calculation methods. Closure of the governing equations is obtained from two turbulence properties, namely, turbulence kinetic energy and its dissipation rate, which are derived from their transport equations.

The flows considered are axi-symmetric forced plumes in calm, uniform and stably-stratified surroundings, and flow in a square cavity driven by both buoyancy and shear (with buoyancy due to heating of one of the walls of the cavity, and shear due either to an adjoining forced flow or the motion of a sliding wall). The plumes studied also include radioactive ones, in which heat is generated by radioactive fission products and transported by convection, diffusion and radiation.

Two numerical procedures are employed, one of which is the SIMPLE algorithm of Patankar and Spalding, and the other a new algorithm which is devised in this study and herein named EMIT. The former algorithm is applied to all flows dealt with in the study, while the latter has been tested for only forced plumes. For these test cases, the EMIT algorithm appears to be more efficient than the SIMPLE algorithm.

The closure of the governing equations is based on an existing non-buoyant version of turbulence model, which is extended to encompass buoyancy effects. An additional constant introduced in the modelling of turbulence dissipation rate in buoyant flows is found to be of order unity.

A method incorporating a "multi-group flux model", is developed for predicting radioactive heating effects in buoyant plumes. The method has also been applied to the plume in stably-stratified surroundings, and the results look promising.

For the forced plumes, there is generally a good agreement between present predictions and available data, with the predictions falling mainly within less than about 10% of the data. The predictions also reveal additional aspects of the flow (e.g. velocity, buoyancy and turbulence profiles under stable stratification) for which no data or previous analytical solutions are available. It is also found that the plume becomes self-similar as and when the integrated flux Richardson number becomes constant, which condition appears to coincide with the local Richardson number reaching a constant value of about 0.6.

For the cavity flows, the agreement between the predictions and available data is fairly satisfactory for forced convection, with errors varying between about 1 and 25%. However, the agreement is poor for mixed convection, with errors varying between about 10 and 50%; this is attributed to the inadequacy of the turbulence model employed. Also, for the mixed convection, stable advancement of the solution procedure is obtained only through the use of an "inertial relaxation method" which is devised in this study. Further, for the cavity flow driven by a sliding wall, the effective velocity impelling the main vortex appears not to be the wall velocity, but some smaller effective velocity of translation which is about 40% of the wall value. When this difference is accounted for, the characteristics of the flow within most of the cavity appears to be independent of the driving mechanism, although appreciable differences are observed in the immediate vicinity of the "downstream wall".

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NOMENCLATURE

<u>Symbol</u>	<u>Meaning</u>
A	area
A_j, A'_j	coefficients in finite-difference equation for line solution procedure
Ar	Archimedes number ($=Gr/Re^2 = g\beta\Delta TL/U^2$)
a	constant; coefficient in finite-difference equation
B	buoyancy parameter
B_j	coefficients in finite-difference equation for line solution procedure
B_μ	build-up factor
b	local width of plume; coefficient of linearised source term
C_1, C_2, C_3	constants in modelled ϵ -equation
C_D	constant in dissipation term of K-equation
C_μ	constant in effective viscosity expression
C_j, C'_j	coefficients in finite-difference equation for line solution procedure
C_p	specific heat
c	constant; coefficient of linearised source term
d	diameter of nozzle
E	integration constant in wall region; energy of γ -photons
e_{ij}	rate of strain tensor ($= \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}$)
F	buoyancy flux; total (convective and diffusive) flux across a surface; sum of directional photon flux { $= (I + J)/2$ }
F_w	total weight-deficiency produced per unit time at the source ($=\rho_{\infty,0} F_0$)
Fr	densimetric Froude number $= U^2 / \{g(\rho_\infty - \rho)d/\rho_{\infty,0}\}$
f	weighting factor; relaxation factor; general function/variable
G	generation of turbulence energy; gradient parameter ($= \frac{g}{\rho_{\infty,0}} \frac{d\rho_\infty}{dx}$); mass flux ($=\rho U$)

Gr	Grashof number ($= \beta g \Delta T L^3 / \nu^2$)
g_i	gravitational vector $\{ = (-g, 0, 0) \}$
H	height above vertical nozzle; height of cavity
$\tilde{h}$	total specific energy ($= h + \frac{1}{2} U_j U_j$)
h	specific enthalpy
I	positive-direction photon flux
i	turbulence intensity
J	negative-direction photon flux
$\dot{J}''$	scalar flux
K	turbulence energy
L	linear dimension
l_ϵ	dissipation length scale
M	residual mass source
m_j	mass fraction of species j
$\dot{m}$	mass flow rate ($= \rho U A$)
$\dot{m}'''$	mass source per unit volume
n	constant
n_r	heating parameter
Nu	Nusselt number ($= \dot{q} L / \gamma A \Delta T$)
P, p	pressure (mean and fluctuating values)
Pe	Peclet number ($= \rho U L / \Gamma$)
Pr	Prandtl number
Q	resultant directional photon flux ($= I - J$)
$\dot{Q}$	mass flow rates within cavity; heat per unit time per unit volume
$\dot{q}$	total heat transfer rate across a surface; specific heating rate at nozzle exit
R	gas constant; radial scale; residual source
R_f	flux Richardson number $\{ = -\beta g_i \overline{u_i \theta'} / (-\overline{u_i u_j} \frac{\partial U_i}{\partial x_j}) \}$

Re	Reynolds number ($=UL/\nu$)
Ri	Richardson number ($=g\Delta\rho L/\rho U^2$)
r	radial distance from axis of plume
r_0	radius of nozzle
S	source term in conservation equations
T	fluid temperature
t	time
U^+	non-dimensional velocity in wall region ($=U/U_\tau$)
U_i	mean velocity components in Cartesian tensor notation
U_τ	friction velocity ($=\sqrt{\tau_w/\rho}$)
u_i	fluctuating velocity components in Cartesian tensor notation
$-\overline{\rho u_i u_j}$	Reynolds stresses
$-\overline{\rho u_i \theta'}$, $-\overline{\rho u_i \tilde{h}'}$	turbulent heat flux
U, V, W	mean velocity components
u, v, w	fluctuating velocity components
W	width of cavity
x	distance measured in streamwise direction
x_i	general cartesian co-ordinates
y	distance measured in cross-stream direction; distance from wall
y^+	non-dimensional distance from wall ($=\rho U_\tau y/\mu$)
z	constant

Greek symbols

α	constant
β	coefficient of cubical expansion; constant
δ_{ij}	Kronecker delta ($= 1$ if $i = j$; otherwise $= 0$)
δV	volume of control volume or cell

∇	gradient operator ($= \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$)
$\Delta\phi$	ϕ difference (where ϕ is any variable)
ϵ	dissipation rate of turbulence energy
γ	constant; thermal conductivity of fluid
τ	shear stress
η	profile variable
κ	von Karman constant
θ	excess temperature ($= T - T_{\infty}$)
ϕ	dependent variable; photon flux spectrum
μ	laminar viscosity
ν	kinematic viscosity ($= \mu/\rho$)
ρ	fluid density
σ	Prandtl/Schmidt number; transfer cross-section
σ_k	constant in modelled K-equation
σ_{ϵ}	constant in modelled ϵ -equation
λ	spreading ratio; empirical function
Γ	exchange coefficient ($= \frac{\mu}{\sigma}$)
Ψ	stream function
ω	vorticity
Ω	eddy strength; spatial direction vector

Subscripts

1, 2, 3	directions of Cartesian co-ordinate system
a	relating to absorption
e	relating to entrainment
eff	"effective" or turbulent value
f	false value; relating to fission products
g	energy group interval of photon flux
h	relating to thermal energy

I	relating to inertial relaxation method, or inertial sublayer
m	relating to species concentration; centre-line value
max	maximum value
min	minimum value
0	value at nozzle exit; value at entrance to channel; origin of photon beam
r	relating to some ratio
rad	relating to radiation
ref	reference value
s	relating to scattering
t	turbulent value
u	"upwind" value
w, wall	wall value
β	relating to β -radiation
γ	relating to γ -radiation
ϕ	relating to variable ϕ
∞	ambient fluid
ζ	value at centre-line
N, E, S, W, P	} grid points
n, e, s, w	
ew, ns	

Superscripts

'	fluctuating quantity; correction; energy or direction other than that considered
*	guessed value (or value of previous iteration/pass); non-dimensional value
o	previous iteration/pass

PART I: INTRODUCTION

CHAPTER 1

INTRODUCTION

1.1 THE PROBLEMS CONSIDERED, THEIR PRACTICAL RELEVANCE, AND OBJECTIVES OF THE INVESTIGATION

1.11 Preamble

In recent years there has been considerable progress in the application of numerical procedures and turbulence models to problems of fluid flow, heat and mass transfer. Confidence has since steadily been increasing in the capabilities of such methods, based on reasonably successful applications to a wide range of problems in engineering and the physical sciences. This study is undertaken as a contribution to the fast growing horizon of these methods.

One of the major current interests in the computer prediction methods, and to which this work addresses itself, is their extension to turbulent flows which may be under the combined or separate influences of natural and forced convection. There is a wide range of turbulent flows of this type, but the particular ones dealt with in this study are buoyant plumes and flow in cavities; further, the cases considered are those where the flows are assumed to be "steady", in the commonly-accepted sense.

1.12 The Plume Problem

(a) Background and practical relevance

With the increasing dependence of modern society on power plants which dispose waste products into the atmosphere or rivers, there has been an increasing awareness of the detrimental effects (or "pollution") caused by this activity. Typical examples are the discharge into the atmosphere of smoke and other effluents from chimneys, and the disposal into rivers of wastes from chemical plants.

In most cases, the disposal of the wastes into natural bodies, (i.e. the atmosphere, rivers, etc.) is in the form of a continuous source of buoyant flow which in turn produces "buoyant plumes", which are maintained largely by gravity effects. A study of the ecological consequences of pollution therefore centres around an understanding of the mechanics of the flow in buoyant plumes.

The recent emphasis in the United Kingdom and other countries on seeking sources of energy other than from traditional fossil fuels has led to increased interest in nuclear power. As more nuclear power plants are constructed, the possibility increases of atmospheric pollution of a more serious and far-reaching kind, in the form of escape of radioactive gases from the power plants. In the extreme case a reactor accident could occur in which large amounts of radioactive fission products in gaseous or particulate form are released to the atmosphere. Since the fluid thus released will have high sensible heat augmented by radioactive heating, there results what may be called a "radioactive buoyant plume". These possibilities have led to a considerable interest in, and support of, part of this work by the United Kingdom Atomic Energy Authority.

Included in the study are buoyant plumes discharged into a stably-stratified environment. These have been observed to have a definite height limit. A common example is the limited height attained by smoke plumes from factory chimneys on a windless day when a temperature inversion exists. In this case, the maximum height influences air pollution, as the concentration of effluent at the ground level is increased when the maximum height is reduced.

Apart from the question of pollution, the study of buoyant plumes contributes to the understanding of the influence of buoyancy on free turbulence phenomena.

(b) The particular problems considered

The turbulent buoyant plumes considered in this study are of two-dimensional, axi-symmetric type, although the possibility exists of extending the procedures employed to encompass three-dimensional situations. The present restriction to two-dimensional situations dictate that there must be no cross wind, and the axis of the plume must be vertically orientated.

Both uniform and stably-stratified environments are considered, and in Fig. 1.1 are shown schematic representations of the associated flow patterns. The specific case-studies are as follows:

1. The momentum jet;
2. Conventional (i.e. non-radioactive) buoyant plumes in uniform surroundings;
3. Conventional buoyant plumes in stably-stratified surroundings;
- and 4. Radioactive buoyant plumes in stably-stratified surroundings;

1.13 The Cavity Flow Problem

(a) Background and practical relevance

In many engineering systems in which heat is removed by a coolant flow, it is necessary to understand the flow and thermal characteristics of the system. This is especially important in fast breeder reactors, often submerged in a large pool of sodium, a typical example of which is given in the schematic representation shown in Fig. 1.2 (see also Singer et al, 1976). Similar fast breeder reactors are used in the United Kingdom, France and elsewhere (see, for example, Grand, 1975). As typified by Fig.1,2. the reactor contains large volumes of liquid Sodium which undergo forced convection as a result of the circulation caused by the pumps, as well as natural convection caused by the resultant temperature differences. It will be observed in Fig. 1.2 that the system comprises several regions that can exhibit cavity-type flow.

Interest in most of the work on cavity flows in this study was stimulated by the discovery of mixed convection data (of which there are few) obtained by Grand (1975) owing to the interest of the French nuclear group (D.T.C.E., Centre d'Etudes Nucleaires, Grenoble) in understanding the flow characteristics of the cavity-type flows in the reactor system. Knowledge of the flow and temperature fields in the system is important, since these determine the stresses to which the system components are subjected.

In addition to the application to flow in nuclear reactor systems, cavity flows are of considerable interest in many branches of fluid mechanics. For example, in fluid dynamics, there is the problem of additional drag due to changes in pressures and velocities in and near a step, cavity, or other surface irregularities (e.g. wheel bays). Further, studies of the stationary, captured vortex in a cavity provide an aid to the understanding of recirculating flows in various circumstances. In hydraulics systems, cavity-type flow occurs in flows past sudden enlargements and contractions. In lubrication, a lubricating groove between sliding plates can often be simulated as cavity flow driven by a sliding wall. In heat transfer, interest in cavity flows arises in connection with the thermal insulating properties of cavities (e.g. double-glazed windows) and in bluff-body flame stabilization, etc.

(b) The particular problems considered

Turbulent, square-cavity flows driven by buoyancy and/or shear are considered in this study, and schematic representations are given in Fig. 1.3(a), (b), and (c). The specific cases dealt with are:

- (i) Cavity flow driven by shear due to an outer channel flow (Fig. 1.3a)
- (ii) Cavity flow driven by buoyancy and shear, with shear due to an outer channel flow and buoyancy due to a heated top wall (Fig. 1.3b).

(iii) Cavity flow driven by shear due to a sliding wall (Fig 1,3c),

1.14 Objective of the Investigation

The aim of the present investigation is to adapt, test by comparison with experimental data, and improve a combination of an existing turbulence model and an existing numerical method, to produce computer-based predictions of the flow categories listed in sub-sections 1.12 and 1.13.

1.2 BRIEF OUTLINE OF STATUS OF PREDICTION PROCEDURES

1.21 Numerical Methods

Numerical methods for the solution of the equations governing fluid flow in either two- or three- dimensions have been largely of the finite-difference type, owing to the complexity of the equations. One of the early successful numerical methods for the solution of these equations is the two-dimensional boundary-layer procedure of Patankar and Spalding (1967; revised 1970). This was quickly followed by another two-dimensional procedure, but for "general flows" (i.e. flows in which recirculation may occur), devised by Gosman et al (1969). Other two- and three-dimensional methods for steady and unsteady flows have also been reported by Harlow and Welch (1965), Chorin (1968), Amsden and Harlow (1970), and many others.

Another method for the calculation of three-dimensional boundary-layer flows was reported by Gosman and Spalding (1971). However, this was later superseded by a more efficient procedure called SIMPLE (see Chapter 3), which was devised by Patankar and Spalding (1972), and applicable to two- and three-dimensional general flows. SIMPLE has so far been the standard procedure at Imperial College for the calculation of general flows, and has been applied in various flow circumstances. (e.g. Caretto et al, 1972; McGuirk, 1975; Gosman et al, 1977). Recently, modified versions of SIMPLE have been applied to flows which are boundary-layer in nature except for the pressure (e.g. Pratap, 1975, Bergeles, 1976).

1.22 Turbulence Models

There has been a vast amount of research on the nature of turbulence and the modelling (i.e. approximate mathematical representation) of turbulence stresses. Since the early work of Prandtl (1925), which relates the turbulence stresses to velocity gradients through a prescribed length-scale distribution (i.e. scales representing typical sizes of the turbulent eddies), research on turbulence phenomena has led to methods of obtaining the turbulence stresses from one-, two-, or multi-equations of characteristic variables of the turbulence field.

The present status of such models has been reviewed by Launder and Spalding (1972), Bradshaw (1972), Harlow (1973), Mellor and Herring (1973), Launder and Spalding (1973), Hossain and Rodi (1974), Reynolds (1976), Launder (1976), and others. The work at Imperial College has established the suitability and economy for a wide range of flows of a two-equation model of turbulence in which the turbulence stresses are derived from the solution of two transport equations for the turbulence energy, K , and its dissipation rate, ϵ (e.g. Rodi, 1972; Launder and Spalding 1974, Pratap, 1975; McGuirk, 1975).

The Imperial College Group have also employed higher levels of closure in which transport equations are solved for each of the Reynolds stresses (e.g. Launder et al, 1972; Gibson and Launder, 1974; Launder, 1975, 1976; Bryant, 1977; Morse, 1977). Similar high-level closures have also been devised by Lumley (1972) and others. For some classes of flows (e.g. asymmetric channel flows, Hanjalic and Launder, 1972) models employing transport equations for K and ϵ are inadequate while the higher-order models show great promise; for others (e.g. recirculating flows) insufficient testing has been done.

1.3 OUTLINE OF PRESENT CONTRIBUTIONS

(a) Numerical Methods

Two numerical methods are employed in this study:

- Because the SIMPLE procedure has been previously well-tested for general flows, a version of the TEACH-2E computer program (Gosman and Ideriah, 1977) incorporating it is adapted for the computation of all flows studied here, thus providing further experience of application of this procedure.
- A new procedure, called EMIT (see Chapter 3), is devised and tested for the circumstances of the plume in both uniform and stably-stratified surroundings. For the flows tested, EMIT appears to be more efficient than SIMPLE.

(b) Turbulence Models

An existing model of turbulence for non-buoyant flows (see, Launder and Spalding, 1974), in which the turbulence fluxes are derived from the solution of two transport equations for the turbulence energy and its dissipation rate, is employed and extended to encompass buoyant-flow situations. Further, the work throws light on the suitability of the model employed, for the buoyant flows studied.

(c) Radioactive Heating Model

A "multi-group flux model" is developed for the evaluation of radioactive heating in buoyant plumes containing radioactive fission products. The model is further tested by way of application to radioactive plumes in stably-stratified surroundings.

(d) Flow Characteristics

The classes of flows listed in sub-sections 1.12(b) and 1.13(b) have been predicted, and a modest collection has also been made of previously available information on most of these flows. In addition, the present predictions provide additional information on some features

for which no data are available,

1.4 LAYOUT OF THE THESIS

(a) Main body

The main body of the thesis is made up of five parts, comprising a total of seven chapters. Part I is the introductory section of the thesis and embraces this chapter.

Part II lays the general mathematical foundation on which the work is based. It consists of Chapters 2 and 3. Chapter 2 deals with the mathematical modelling of natural and forced convection: there the exact forms of the governing equations of motion are presented, the closure problem is discussed briefly, and the turbulence model employed for closure is described; further, the final equations are presented in the form in which they are solved. In Chapter 3, the numerical methods employed are discussed: the finite-difference forms of the relevant equations are given, and the two solution algorithms (SIMPLE and EMIT) are presented; also discussed there are matters relating to the insertion of boundary conditions, accuracy, numerical stability, etc., of the solution procedures.

Part III presents the investigation of buoyant plumes, and consists of Chapters 4 and 5. Chapter 4 describes the work on conventional buoyant plumes. There a review is made of previous studies, the details are given of the present computations, and the results are discussed. Chapter 5 deals with radioactive buoyant plumes, commencing with a survey on previous work. Then the mathematical modelling of the heating due to radioactive fission products is described, the details of the computations are given, and the results are discussed.

Part IV covers the investigation undertaken of cavity flows, and embraces Chapter 6. After a review of the previous work are given details of present computations, followed by a presentation and discussion of the results.

Part V is the closing section of the thesis, and embodies Chapter 7. It summarizes the main results of the study and provides some recommendations for future work.

The manner in which the main body of the thesis is written renders it possible to read Parts III and IV independently (and for that matter also Chapters 4 and 5 of Part III) in conjunction with Part II.

(b) Appendices

The thesis incorporates 6 appendices to which the letters A to F are ascribed. Appendix A presents the various modes of turbulence models for natural and forced convection. Appendix B contains the derivations of some expressions of the turbulence model for wall regions. Appendix C describes the origin of extra expressions in the momentum and turbulence equations, and in Appendix D a one-dimensional analysis is performed for the source of buoyancy-induced numerical instability. In Appendix E a derivation is given of an "inertial relaxation" method employed in the study, and, finally, Appendix F outlines the derivation of a formula for the estimation of dissipation length scale at the exit of a nozzle.

PART II:

GENERAL MATHEMATICAL FOUNDATION

CHAPTER 2

MATHEMATICAL MODELLING OF NATURAL AND FORCED CONVECTION

2.1 INTRODUCTION

This chapter and the next lay the complete mathematical foundation on which the present study is based: the present chapter gives the mathematical statement of the physical laws which govern the flows, while the next one describes the manner in which the governing equations are solved.

The prediction of all turbulent flows requires formulation of the governing equations in the form of a closed set which is amenable to economical solution. The manner in which a closed set is arrived at in the present study is detailed in this chapter.

Section 2.2 presents the starting point, in the form of the time-dependent equations of motion, energy and conservation of a chemical species, while Section 2.3 presents them in their time-averaged form. Then, in Section 2.4 the closure problem is dealt with, and it is shown how the unknown turbulent fluxes may be determined by means of a turbulence model. Section 2.5 summarises the complete set of equations in the final form employed in this study, and Section 2.6 gives the boundary conditions which apply to the flows here considered.

2.2 THE TIME-DEPENDENT EQUATIONS OF NATURAL AND FORCED CONVECTION

2.21 The Equations: The basic conservation equations which govern natural and forced convection are those for mass (continuity), momentum, energy, and chemical species. They may be expressed in Cartesian tensor notation as follows (Batchelor, 1970; Bird et al, 1960):

Mass Conservation:
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho u_j) = 0 \quad (2.1)$$

Momentum Conservation: for direction i ($i = 1, 2, 3$)

$$\frac{\partial}{\partial t}(\rho U_i) + \frac{\partial}{\partial x_j}(\rho U_j U_i) = \frac{\partial}{\partial x_j} \{-p \delta_{ij} + \mu(e_{ij} - 1/3 e_{ii} \delta_{ij})\} + \rho g_i \quad (2.2)$$

Energy Conservation:

$$\frac{\partial}{\partial t}(\rho \tilde{h}) + \frac{\partial}{\partial x_j}(\rho U_j \tilde{h}) = \frac{\partial}{\partial x_j}(\Gamma_h C_p \frac{\partial T}{\partial x_j}) + \frac{\partial}{\partial x_j} \{U_i \mu (e_{ij} - 1/3 e_{ii} \delta_{ij})\} + S_h \quad (2.3)$$

Species Conservation:

$$\frac{\partial}{\partial t}(\rho m_j) + \frac{\partial}{\partial x_j}(\rho U_j m_j) = \frac{\partial}{\partial x_j}(\Gamma_m \frac{\partial m_j}{\partial x_j}) + S_{mj} \quad (2.4)$$

where U_i = velocity in the Cartesian direction x_i ($i = 1, 2, 3$)

p = local pressure

$\tilde{h} = h + \frac{1}{2} U_j U_j$ = total specific energy (2.5)

$h = h_{ref} + \int_{T_{ref}}^T C_p dT$ = specific enthalpy (2.6)

T = local temperature

C_p = specific heat of fluid at constant pressure

m_j = mass fraction of species j

g_i = gravitational vector

$e_{ij} = \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}$ = rate of strain tensor (2.7)

δ_{ij} = Kronecker delta ($\delta_{ij} = 1$ if $i = j$; $\delta_{ij} = 0$ if $i \neq j$)

S_h = energy source due to radiation, electrical heating, etc.

S_{mj} = species source due to chemical reaction, etc.

ρ = fluid density

μ = fluid viscosity

Γ = laminar exchange coefficient

and the symbols U_j , p , $\tilde{h}$, etc. stand for the instantaneous values of these variables.

In natural convection fluid motion arises because ρ in equation (2.2) is variable owing to variable T and/or m_j , giving rise to nett buoyant forces.

2.22 Laminar and Turbulent Flows

The foregoing equations govern both laminar and turbulent flows: they are six in number, with nine unknowns, namely U_i ($i = 1, 2, 3$), p , T , m_j , ρ , μ , and Γ . The remaining equations are in the form of auxiliary relations for the density ρ as well as μ and Γ of the form*:

$$\rho = \rho(m_j, T, p) \quad (2.8)$$

$$\mu = \mu(m_j, T, p) \quad (2.9)$$

$$\Gamma = \Gamma(m_j, T, p) \quad (2.10)$$

For most laminar flows these equations can be solved with little or no difficulty by currently available analytical or numerical methods to yield accurate results.

On the other hand, for turbulent flows this situation does not obtain, since such flows are essentially three-dimensional and unsteady. In addition, fine-scale effects play a dominant role in turbulent flows; for example, the size of the small eddies for turbulent natural convection from a heat source (e.g. a radiator) in a room of characteristic length L is of order $10^{-3}L$, while these eddies are very effective agents, for example, for smearing out temperature gradients (Tennekes and Lumley, 1973). Numerical computation involves dividing the flow domain into non-overlapping sub-domains called "grids", and the computing times increase with the size of the grid. Thus the grid size and computing times required for detailed numerical computation of such fine-scale effects will be uneconomical.

Fortunately, in most flows of interest to the engineer, it is the time-averaged properties which are of major interest; these vary much

* For a perfect gas, $\rho = P/RT$; where R is the gas constant.

Generally, for gases both μ and Γ obey relations of the form:

$$\mu \propto T^{3/2}/(T + C) \quad (\text{Sutherland's formula}), \text{ where } C \approx 114^\circ\text{K for air}$$

and $\Gamma = \frac{\mu}{\sigma}$, where σ = Prandtl/Schmidt number

more gradually in space, and therefore do not require excessive fine grids for their prediction.

The approach employed here for turbulent flows is therefore based on the time-averaged equations, which are presented in the next section.

2.3 THE TIME-AVERAGED EQUATIONS

The Reynolds method of decomposing the instantaneous value $\hat{\phi}$ of any variable into a time-averaged (or mean) component $\bar{\phi}$ and fluctuating component ϕ' gives the mean value $\bar{\phi}$ as:

$$\bar{\phi} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \hat{\phi} dt \quad (2.11)$$

$$\hat{\phi} = \bar{\phi} + \phi' \quad (2.12)$$

In the present context, ϕ may stand for any of the variables ρ , U_i , P , $\tilde{h}$, T , and m_j . For brevity the overbars indicating mean values will hereafter be omitted for all variables, and lower case letters will be used to indicate the fluctuating components of U_i and P .

Introduction of relations like (2.12) for the individual variables in equations (2.1) to (2.4) and then time-averaging yields for steady⁺⁺ flows (Hinze, 1959):

Continuity: $\frac{\partial}{\partial x_j} (\rho U_j) = 0 \quad (2.13)$

Momentum: $\frac{\partial}{\partial x_j} (\rho U_j U_i) = \frac{\partial}{\partial x_j} \{ -\overline{\rho u_i u_j} - P \delta_{ij} + \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{1}{3} \bar{e}_{ii} \delta_{ij} \right) \} + \rho g_i \quad (2.14)$

Energy: $\frac{\partial}{\partial x_j} (\rho U_j \tilde{h}) = \frac{\partial}{\partial x_j} (-\overline{\rho u_j \tilde{h}} + \Gamma_h C_p \frac{\partial T}{\partial x_j}) + S_h \quad (2.15)$

Species: $\frac{\partial}{\partial x_j} (\rho U_j m_j) = \frac{\partial}{\partial x_j} (-\overline{\rho u_j m_j} + \frac{\Gamma}{m} \frac{\partial m_j}{\partial x_j}) + S_m \quad (2.16)$

The above equations have been written for "nearly-incompressible"

+ Here the limit $t \rightarrow \infty$ represents a time t large enough such that the results obtained can be reproduced; for example $t \approx 200$ seconds, Kotsovinos (1976)

++ In this work, the term "steady" flows will be applied to flows with no well defined time-dependent structure.

flows, and for which, in line with the Boussinesq approximation (Turner, 1973), the influence of density variations may be neglected except in terms contributing to buoyancy effects, (i.e. terms expressing the influence of buoyancy). This simplification⁺, which will be upheld in this study, is not required by the numerical methods employed in the work, but it does enhance economy; hence it is used.

2.4 MATHEMATICAL MODELLING OF TURBULENCE IN NATURAL AND FORCED CONVECTION

2.41 The Closure Problem

The main problem in solving the foregoing time-averaged equations is that they contain new unknown terms, namely the Reynolds stresses $-\overline{\rho u_i u_j}$, the turbulent heat flux $-\overline{\rho u_j h'}$, and the turbulent mass flux $-\overline{\rho u_j m'_j}$. These play a decisive role in determining the flow behaviour because the fine-scale effects are primarily expressed through them.

It is possible to derive additional transport equations for these unknown correlations (Rotta 1969, 1971; Hanjalic and Launder 1972; Launder, 1975). Unfortunately, these new equations contain further unknown higher order terms like $\overline{u_k u_i u_j}$, $\overline{u_k u_j h'}$, etc. At some stage, therefore, it is necessary to make assumptions about the unknown correlations such that a closed set of equations is obtained. The path to the determination of the unknown correlations is often called "turbulence modelling".

A summary of the various classes of turbulence model which have been proposed for natural and forced convection is given in Appendix A, and more extensive information is provided by Launder and Spalding (1972), Lumley (1972), Gibson and Launder (1974), Launder (1975) and Hossain

⁺ For flows where these approximations are not made additional terms appear in the equations, involving correlations of fluctuating density and velocity or scalar quantities (see, for example, Lin, 1959), and the energy equation also contains further terms representing heating due to compression and frictional resistance.

and Rodi (1974). In this study, for the sake of economy and by way of providing further development and validation, a two-equation model often called the "K-ε effective-viscosity model", is employed and this forms the subject of the next section.

2.42 The K-ε Effective-Viscosity Model

Basic Concept:

The central theme in all effective viscosity models is the assumption that the diffusion of momentum, heat and matter in turbulent flows may be expressed by laws or 'constitutive relations' similar to those valid for laminar flows*, with the turbulent diffusional fluxes expressed in terms of "effective" diffusivities (or 'exchange coefficients') multiplying gradients of mean flow properties. These take the form (Hinze, 1959):

$$-\overline{\rho u_i u_j} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho K \delta_{ij} \quad (2.17)$$

$$-\overline{\rho \phi' u_j} = \Gamma_{\phi, t} \frac{\partial \Phi}{\partial x_j} \quad (2.18)$$

where μ_t = 'turbulent' or 'eddy' viscosity

K = turbulence kinetic energy ($\equiv \frac{1}{2} \overline{u_i u_i}$)

ϕ = h, T, or m_j

δ_{ij} = Kronecker delta

It is then necessary to seek a way of determining the turbulent viscosity, μ_t , and the turbulent diffusivity, $\Gamma_{\phi, t}$. The manner in which this is done is shown in the following paragraphs.

The turbulent viscosity, μ_t :

The local value of μ_t depends on the fluid density and the local properties of turbulence. Following Prandtl (1925), the general relation may be expressed as:

* For laminar flows, the transport laws referred to are: Newton's law of viscosity, Fourier's law of conduction, and Fick's law of diffusion.

$$\mu_t = C_\mu \rho V' l \quad (2.19)$$

where C_μ = an empirical function/constant

V' = characteristic velocity of the velocity fluctuation field

l = characteristic length scale of the turbulence eddies.

Applying the proposal of Kolmogorov (1942) and Prandtl (1945) that V' is equivalent to the square root of the kinetic energy of turbulence, K , expression (2.19) becomes

$$\mu_t = C_\mu \rho K^{\frac{1}{2}} l \quad (2.20)$$

In effective viscosity models, the variables K and l (or any combination thereof of the form $z = K^m l^n$, where m and n are constants) are determined from their respective transport equations. In the present study the variables employed are K and the time-mean rate of dissipation of turbulence energy, ε^+ , which is related to K and l through (Chou, 1945; Davidov, 1961; Harlow and Nakayama, 1968; Jones and Launder, 1972):

$$\varepsilon = K^{3/2} l^{-1} \quad (2.21)$$

Thus, in terms of K and ε , μ_t becomes

$$\mu_t = C_\mu \rho K^2 / \varepsilon \quad (2.22)$$

+ The rate of dissipation of turbulence energy, ε , is usually defined in the literature as (Corrsin, 1953):

$$\varepsilon = \nu \left(\frac{\partial u_i}{\partial x_j} \right) \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \quad (2.21a)$$

where ν is the laminar kinematic viscosity.

However, when the Reynolds number is high enough, the viscous dissipation takes place in the smallest eddies in which distribution of energy is "isotropic" so that equation (2.21a) reduces to

$$\varepsilon = \nu \left(\frac{\partial u_i}{\partial x_j} \right)^2 \quad (2.21b)$$

since because of "local isotropy" there cannot exist a correlation between $\frac{\partial u_i}{\partial x_j}$ and $\frac{\partial u_j}{\partial x_i}$ for the smallest eddies (Rotta 1951; Batchelor, 1960; Launder and Spalding, 1972).

Note that the small-scale structure of turbulence tends to be independent of any orientation effects introduced by the mean shear, so that all averages relating to the small eddies do not change under rotations or reflections of the co-ordinate system. If this is the case, the small-scale structure is called isotropic. Isotropy at small scales is called local isotropy (Tennekes and Lumley, 1973).

and the task of determining μ_t is now transformed to that of calculating K and ϵ .

The turbulent diffusivity, $\Gamma_{\phi,t}$:

For non-buoyant flows, the turbulent diffusivity $\Gamma_{\phi,t}$ is found to be of the same order of magnitude as the turbulent viscosity μ_t (eg Kestin and Richardson, 1963). The approach in effective-viscosity models is therefore to focus attention on the ratio $\mu_t/\Gamma_{\phi,t}$, often termed the turbulent Prandtl/Schmidt number $\sigma_{\phi,t}$. From expressions (2.17) and (2.18),

$$\sigma_{\phi,t} = \frac{\mu_t}{\Gamma_{\phi,t}} = \frac{(\overline{\rho u_i u_j} - 2/3 \rho K \delta_{ij}) / (\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i})}{\overline{\rho u_j \phi'} / \frac{\partial \phi}{\partial x_j}} \quad (2.23)$$

Generally, in non-buoyant free shear flows most workers choose uniform values of $\sigma_{\phi,t}$ ranging from 0.5 to 0.7 (Launder and Spalding, 1972) while for near-wall flows a value close to unity (most commonly 0.9) appears to be suitable possibly as a consequence of wall-affected pressure fluctuations modifying the Reynolds stress structure (Kestin and Richardson, 1963; Bakewell and Lumley, 1967; Launder, 1976).

The K- ϵ equations

The formula for μ_t and $\Gamma_{\phi,t}$ having been established through expressions (2.22) and (2.23), the problem reduces to one of obtaining the quantities (K and ϵ) on which they are based. Here both the exact and modelled forms of the K and ϵ transport equations will be given, and the details of these equations are given in Appendix A.

(i) The exact equations: For steady flows, the exact equations (within the framework of the Boussinesq approximation) for K and ϵ take the following forms⁺ in Cartesian tensor notation:

$$\underbrace{\frac{\partial}{\partial x_j} (\rho U_j K)}_A = - \underbrace{\rho \overline{u_i u_j} \frac{\partial U_i}{\partial x_j}}_G - \underbrace{\frac{\partial}{\partial x_j} \{ \rho \overline{u_j k'} + \overline{p u_j} - \mu (\frac{\partial K}{\partial x_j} + \frac{\partial}{\partial x_i} \overline{u_i u_j}) \}}_E$$

Convection of K by mean motion	generation of K by turbulent shear stress	diffusion of K by: (a) velocity fluctuations (b) pressure fluctuations, and (c) viscous action

⁺ See Appendix A

$$\underbrace{-\rho\epsilon}_D + \underbrace{g_i \overline{\rho' u_i}}_F \quad (2.24)$$

viscous dissipation
of K to "heat"

Generation/abstraction of K by
the working of velocity fluctu-
ations against buoyancy effects.

$$\underbrace{\frac{\partial}{\partial x_j}(\rho U_j \epsilon)}_A = -2\mu \underbrace{\left(\frac{\partial u_i}{\partial x_1} \frac{\partial u_j}{\partial x_1} + \frac{\partial u_i}{\partial x_i} \frac{\partial u_1}{\partial x_j} \right) \frac{\partial u_i}{\partial x_j}}_{G_1} - 2\mu u_j \underbrace{\frac{\partial u_i}{\partial x_1} \frac{\partial^2 u_i}{\partial x_j \partial x_1}}_{G_2}$$

Convection of ϵ by mean motion Generation of ϵ by mean flow: G_1 = production due to vortex stretching, G_2 = secondary generation

$$\underbrace{-2\mu \frac{\partial u_i}{\partial x_1} \frac{\partial u_j}{\partial x_1} \frac{\partial u_i}{\partial x_j}}_{G_3} - \underbrace{2\rho \left(v \frac{\partial^2 u_i}{\partial x_j \partial x_1} \right)^2}_D$$

generation due to "vortex stretching" by turbulence

viscous destruction of ϵ

$$\underbrace{-\frac{\partial}{\partial x_j} \left\{ \rho \overline{u_j \epsilon} + v \frac{\partial p}{\partial x_1} \frac{\partial u_i}{\partial x_1} - \mu \frac{\partial \epsilon}{\partial x_j} \right\}}_E + \underbrace{2v g_i \frac{\partial u_i}{\partial x_1} \frac{\partial \rho'}{\partial x_1}}_F \quad (2.25)$$

diffusive transport of ϵ due to (a) velocity fluctuations, (b) pressure fluctuations, (c) molecular action

buoyancy effects on ϵ

In the foregoing equations, $K' (= \frac{u_i^2}{2})$ is the instantaneous turbulence energy, ϵ is as defined by (2.21b), $\epsilon' (= v (\frac{\partial u_i}{\partial x_j})^2)$ is the instantaneous value of the dissipation rate, and g_i is the gravitational vector.

Terms F of equations (2.24) and (2.25), which express the effects of fluctuating buoyancy forces $\rho' g_i$, are of particular interest in flows where buoyancy is the main source of energy and therefore require some further elucidation:.. These terms express the generation or abstraction of turbulence energy and its dissipation rate from the velocity fluctuation field by the working of motions which are in the selective direction of gravity against fluctuating buoyancy forces[†].

[†] Apparently, the form of equation (2.24) indicates that there exists situations in which turbulence need not be maintained by shear stresses because it can be maintained by fluctuating buoyancy forces.

This selective generation/destruction of turbulence in the direction of gravity results in greater anisotropy of the large scale turbulence structure in flows where buoyancy plays a dominant role.

(ii) The modelled equations: In this study, the non-buoyant terms of the K-ε equations have been modelled in the form indicated by Launder and Spalding (1974). The buoyancy term F in the K-equation is treated as "exact". Following Launder (1973), the buoyancy term F in the ε-equation is assumed proportional to the rate of selective destruction of turbulence energy by buoyancy. For steady flows the modelled equations take the following forms in Cartesian tensor notation:

$$\begin{array}{ccccccc} & \text{I} & & \text{II} & & \text{III} & \text{IV} & \text{V}^+ \\ \circ & \frac{\partial}{\partial x_j} (\rho U_j K) & = & -\rho \overline{u_i u_j} \frac{\partial U_i}{\partial x_j} & + & \frac{\partial}{\partial x_j} \left\{ \frac{\mu_t}{\sigma_K} \frac{\partial K}{\partial x_j} \right\} & - & \rho \epsilon - \rho \beta g_i \overline{u_i \theta} \end{array} \quad (2.26)$$

+ The Boussinesq approximation of replacing density differences by temperature differences has been invoked in these terms. Since, for gases, the local density is a function of the local pressure and temperature, that is

$$\rho = \rho(P, T) \quad (2.27a)$$

differences in density may be written as a double expansion in terms of the temperature and pressure differences; thus

$$\rho_\infty - \rho = \left(\frac{\partial \rho}{\partial T} \right)_P (T_\infty - T) + \dots + \left(\frac{\partial \rho}{\partial P} \right)_T (P_\infty - P) + \dots + \frac{\partial^2 \rho}{\partial T \partial P} (T_\infty - T)(P_\infty - P) + \dots \quad (2.27b)$$

Hence taking only the first term of the pure temperature effects, there results:

$$\begin{aligned} \rho_\infty - \rho &= \rho \beta (T - T_\infty) \\ &= \rho \beta \theta \end{aligned} \quad (2.27c)$$

$$\text{where } \beta = - \frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_P = \text{coefficient of cubical} \quad (2.27d)$$

expansion.

In reality this approximation applies to both gases and liquids, and is even better for liquids than gases (Gebhart, 1971).

In the Navier-Stokes equation, the buoyancy term may be approximated by

$$-\beta \theta g_i, \text{ with } \theta = T - T_\infty \quad (2.27e)$$

This approximation is true in both laminar and turbulent natural convection. But, in turbulent flow θ can be decomposed into mean and fluctuating parts:

$$\theta = \bar{\theta} + \theta' \quad (2.27f)$$

that is, in turbulent flow, the body force becomes

$$-\beta \bar{\theta} g_i - \beta g_i \theta' \quad (2.27g)$$

In deriving the equation for K (see Appendix A) these terms are multiplied by the fluctuating velocity u_i and then time-averaged to yield

$$-\beta g_i \overline{u_i \theta'} \quad (2.27h)$$

which expresses the rate at which the fluctuating body force performs work,

$$\begin{array}{cccccc} \text{I} & & \text{II} & & \text{III} & & \text{IV} & & \text{V}^+ \\ \circ \frac{\partial}{\partial x_j} (\rho U_j \epsilon) & = & -C_1 \frac{\epsilon}{K} \rho u_i u_j \frac{\partial U_i}{\partial x_j} & + & \frac{\partial}{\partial x_j} \left\{ \frac{\mu_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_j} \right\} & - & C_2 \rho \frac{\epsilon^2}{K} & - & C_3 \rho \frac{\epsilon}{K} g_i \overline{u_i \theta'} \end{array} \quad (2.27)$$

where I = convection by mean motion

II = generation by turbulence stresses acting on mean flow

III = diffusive transport

IV = viscous dissipation

V = generation/destruction by buoyant forces.

β = coefficient of cubical expansion (see equation 2.27d in footnote).

$C_1, C_2, C_3, \sigma_K, \sigma_\epsilon \equiv$ empirical functions (assumed to be constants at high Reynolds numbers)

The foregoing modelled equations are based on the following fundamental assumptions:

(a) The Reynolds number of turbulence, Re_T , defined as

$$Re_T = \rho V' l / \mu = \rho K^2 / \mu \epsilon \quad (2.28)$$

is assumed to be sufficiently large that molecular transport may be neglected.

(b) Based on (a), the small scales of turbulence which are responsible for destruction of turbulence are assumed to be isotropic (see footnote on page 18).

Note that the buoyancy effects in the K- ϵ equations are now expressed in terms of fluctuating temperature difference, since the instantaneous buoyancy force depends on the instantaneous temperature difference⁺ θ' (mean value = $\theta = T - T_\infty$). In this study, $\overline{u_i \theta'}$ is further expressed in line with equation (2.18).

2.43 Some Special Features of Turbulence in Stratified Flows

As has been indicated in the foregoing section, the selective influence of buoyancy introduces a preferred direction (anisotropy) in turbulent, buoyant flows. In a stably-stratified fluid, (i.e. one in which density decreases with height in the direction of gravity),

+ See footnote on previous page.

motion away from the equilibrium position along the direction of the buoyancy force vector leads to an opposing force (Turner, 1973): hence, large excursions in the direction of gravity are prohibited, leading to strong anisotropy in the partitioning of turbulence energy and in the transport by turbulent motion. Thus horizontal motions are preferred over, and may survive, vertical motion; and, the trend of turbulence in such instances is towards a two-dimensional structure as its final state (Stuhmiller, 1976).

On the other hand, in an unstable fluid buoyancy introduces a completely opposite effect: it augments the motion, and the turbulent transport, in the direction of gravity.

Generally, buoyancy damps the turbulence if the flux Richardson number, R_f , defined as

$$R_f = - \frac{\text{Buoyant production of turbulence energy}}{\text{Shear production of turbulence energy}} \\ = - \beta g_i \overline{u_i \theta'} / (- \overline{u_i u_j} \frac{\partial U_i}{\partial x_j}) \quad (2.29)$$

is positive (as it is for stable flow), and augments it if R_f is negative (unstable flow).

In highly stably-stratified flows, the turbulent Prandtl number $\sigma_{\phi,t}$ (see equation (2.23)) is no longer uniform but becomes a strong function of the flux Richardson number: measurements by Webster (1964) (see also Launder, 1975) in a free shear flow indicate that, based on fluxes in the vertical direction, $\sigma_{h,t}$ rises steadily and fairly steeply with R_f , reaching a value of about twice that for non-buoyant flows at R_f of only 0.08. This implies that in stably stratified flows heat fluxes are damped much more rapidly than momentum transport in the direction of gravity; and the higher the stratification the more $\sigma_{h,t}$ departs from "normal" values.

The data of Webster are well correlated by the analysis of Launder (1975) who arrived at the following expression for $\sigma_{h,t}$:

$$\sigma_{h,t} = \sigma_0 \{ (1 + a_1 (0.8 - 0.51a_1)B) / (1 + a_1 a_2 B) \} \quad (2.30)$$

where σ_0 = turbulent Prandtl number for non-stratified flows

$$B = \beta g_i \frac{K^2}{\epsilon^2} \frac{\partial T}{\partial x_i} = \text{non-dimensional buoyancy parameter}$$

$$a_1 = 1/3.2$$

$$a_2 = 0.2$$

However, equation (2.30) strictly applies to flows remote from walls.

2.44 Treatment of Near-Wall Regions

(i) The Purpose

The wall regions have two major characteristics that distinguish them from the central region of the flow: (a) steep non-linear gradients in the flow properties, and (b) levels of turbulence Reynolds number Re_T (see equation (2.28)) sufficiently low for molecular viscosity to influence the production, destruction and diffusion of turbulence energy, with the eddy structure strongly influenced by the presence of the wall such that local isotropy of small scale turbulence no longer prevails (Laufer, 1954). Hence the basic turbulence model becomes inapplicable in these regions, and an alternative representation must be found. Two possibilities are outlined below.

(ii) The Available Approaches

There are two commonly-employed ways of accounting for the flow phenomena in wall regions, namely, the wall-function method and the method of modelling the low- Re_T effect. Wall-function methods are based on the fact that for most wall flows both mean and turbulence quantities are nearly-universal functions* of the dimensionless distance along a normal to the wall, y^+ , defined as:

$$y^+ = \rho U_\tau y / \mu \quad (2.31)$$

where y = normal distance from the wall

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}} = \text{friction velocity} \quad (2.32)$$

τ_w = wall shear stress

* See footnote on page 25

Experiments (see, for example, Tennekes and Lumley, 1973), show that in all "equilibrium"* wall flows, the thin fluid layer close to the wall can be sub-divided into three main regions based on y^+ : the viscous sublayer ($y^+ \leq 5$ to 6.7) where viscous forces dominate; followed by the buffer layer ($5 < y^+ < 30$) which is a "transition" region in which the flow is neither completely dominated by viscous effects nor completely turbulent; and the inertial sublayer ($y^+ > 30$) where the flow is assumed to be completely turbulent, but sufficiently close to the wall so that the shear stress is approximately uniform***.

* Here reference is made to wall flows which are in "local equilibrium", i.e. those where the life of an eddy is small compared with the time taken for the mean flow to change its form appreciably. Thus local rates of production and dissipation are so large that aspects of turbulent motion concerned with these processes are determined solely by the distribution of shear stress within the region and are independent of conditions outside it (Townsend, 1961).

Exceptions from this are cases where stream wise pressure gradients surface mass fluxes, very steep property gradients, and buoyancy, centrifugal or coriolis forces play a major role. For these cases, the "universal" functions may be modified to give fruitful results. See for example, Powell and Strong (1970), Cebeci et al (1970), Jones and Launder (1969), and Koosinlin and Lockwood (1974).

** In many engineering calculations, the buffer layer is disposed of by assuming that the viscous sublayer extends to $y^+ \approx 11.63$, beyond which point the flow is taken to be "fully turbulent". (The point $y^+ \approx 11.63$ is defined by the intersection, on a semi-logarithmic plot of $U^+ (= U/U_\tau)$ versus y^+ , of the velocity distributions for the viscous and inertial sublayers). This implies that the "low- Re_τ region" covers the range $0 < y^+ < 11.63$.

This approximation usually causes an abrupt change from purely viscous to purely turbulent stress at $y^+ \approx 11.63$. However, for the sake of economy, it is also employed in this study.

*** See footnote on page 27 for definition of the region of uniform shear stress.

Moreover, the characteristics of these three regions are fairly well established both experimentally and theoretically (see, for example, Tennekes and Lumley, 1973 for a review). Wall-function methods therefore exploit this "universal" nature of flows close to walls by avoiding the low- Re_T regions** (the viscous sublayer and the buffer layer) and taking boundary conditions at the beginning of the fully turbulent region**: these boundary conditions are specified by "wall functions" which are derived from the well established data, or one-dimensional Couette flow analyses or functions fitted to finite-difference solutions of the low- Re_T versions of equations (2.26) and (2.27) for a selection of standard cases. This approach has been proposed and used earlier by several workers including Patankar and Spalding (1967), Jones and Launder (1969), Powell and Strong (1970), Cebeci et al (1970), Ng and Spalding (1972), and Koosinlin and Lockwood (1974).

The method of modelling the low- Re_T phenomena involves modifying the basic high- Re_T assumptions to include both the effects of molecular viscosity and anisotropy of small-scale turbulence, with the resultant equations assumed to apply right up to the wall. However, owing to the steep gradients in flow properties in the wall regions finite-difference calculations through the viscous sublayer require very fine grids and consequently excessive computer storage and time. For example, Jones and Launder (1972), while using this method for wall boundary-layer flows, employed about 50 grid nodes in the region where $y^+ < 50$.

For the sake of economy in the use of computer time and storage the wall-function method is employed in this study. The main features of this method are given in the following paragraphs.

(iii) The Wall-Function Method

In this study it will be assumed that the thin fluid layer close to the wall is a zone of "local equilibrium" (see footnote on page 25)

** See footnote on page 25

with buoyancy effects assumed negligible in the thin layer and the wall taken to be both smooth and impermeable. Experiments (Klebanoff, 1954; Gardner and Kestin, 1963) show that the shear stress in the layer is uniform*, and a one-dimensional Couette flow analysis yields the following universal logarithmic law of the wall ('log-law') in the "fully-turbulent" region of the layer (e.g. Patankar and Spalding, 1970):

$$U^+ = \frac{1}{\kappa} \log_e(Ey^+) \quad (2.33)$$

where $U^+ = U/U_\tau$

κ, E = log-law constants determined from experiments

$$(\kappa = 0.4187; E = 9.793 \text{ from Patel, 1965}) \quad (2.34)$$

Here y^+ and U_τ are defined by (2.31) and (2.32).

The implications of (2.33) are (see also Appendix B):

$$(a) \quad \frac{\partial U}{\partial y} = U_\tau / \kappa y \quad (2.35)$$

and, since the shear stress is uniform in the layer,

$$(b) \quad \mu_t = \rho \kappa U_\tau y \quad (2.36)$$

The manner of evaluation within the thin fluid layer of the shear stress as well as K , ϵ , and the scalar quantity $\phi(= \tilde{h}, m_j)$ is given below:

The Shear Stress, τ :

In the uniform shear layer, the production and dissipation of turbulence energy are in balance; it can then be shown that**

$$\tau = \tau_w = \rho C_\mu^{1/2} K \quad (2.37)$$

Using this expression together with (2.33), the shear stress can be related to the turbulence energy K through the relation**

$$\tau = \rho \kappa C_\mu^{1/4} K U / \{\log_e(Ey^+)\} \quad (2.38)$$

* Strictly, $\tau/\tau_w \approx 1$ for $0 < y^+ < 2 \times 10^2$, and the ratio falls to about 0.8 at $y^+ \approx 8 \times 10^2$ (see, for example, Gardner and Kestin, 1963)

** See Appendix B

The dissipation rate, ϵ :

Employing the notion that the production and dissipation of K are equal, and using expressions (2.35) and (2.36), the expression for ϵ becomes*

$$\epsilon = \frac{U^3 \tau}{\kappa Y} \quad (2.39)$$

And, substituting for $U_\tau (= \sqrt{\frac{\tau_w}{\rho}})$ from (2.37), yields the following result:

$$\epsilon = C_\mu^{3/4} K^{3/2} / \kappa Y \quad (2.40)$$

The turbulence energy, K:

K may be obtained from direct solution of the regular energy equation, but with the diffusion of K to the wall set to zero i.e.

$$\left(\frac{\partial K}{\partial Y} \right)_{\text{wall}} = 0 \quad (2.41)$$

and appropriate alterations made to the generation and dissipation expressions to allow for the presence of the Couette layer. Details will be given in Chapter 6.

Scalar property, ϕ :

A one-dimensional Couette flow analysis of the variation of any scalar conserved property ϕ yields a relationship similar to (2.33); thus (Spalding, 1964):

$$\phi^+ = \sigma_{\phi, t} \left[U^+ + \rho t \frac{\sigma_\phi}{\sigma_{\phi, t}} \right] \quad (2.42)$$

where $\phi = h, m_j, \text{ etc}$

$$\phi^+ = \frac{\rho U_\tau (\phi - \phi_w)}{\dot{J}_\phi''} \quad (2.43)$$

$\dot{J}_\phi'' = \phi\text{-flux from the wall}$

Here ϕ^+ is the dimensionless value of the conserved property ϕ , and has the significance of a non-dimensional effective "resistance"

(ϕ - difference per unit ϕ -flux from the wall) posed by the wall layer

* See Appendix B

to the transfer of ϕ . The P-function expresses the contribution of the laminar sublayer to the effective resistance of the layer.

For the transport of heat, Jayatillaka (1967) derived the following P-function from experimental data for equilibrium flow over smooth surfaces:

$$P\left(\frac{\sigma_h}{\sigma_{h,t}}\right) = 9.24 \left[\left(\frac{\sigma_h}{\sigma_{h,t}}\right)^{3/4} - 1 \right] \left[1 + 0.28 \exp\left(-0.007 \frac{\sigma_h}{\sigma_{h,t}}\right) \right] \quad (2.44)$$

and this relationship is used in this study.

2.45 Assessment of the turbulence model

The Reynolds stresses $-\overline{\rho u_i u_j}$ and turbulent heat fluxes $-\overline{\rho u_j \theta'}$, which are estimated through the K- ϵ viscosity model, arise principally from the larger eddies since the latter are best able to interact with, and thus constitute the main agency for the extraction of energy from, the mean flow. These eddies are of size comparable to that of the width of the flow and are anisotropic (Bradshaw 1971; Tennekes and Lumley, 1973). The anisotropy of the large scales becomes more pronounced when there is a strong directional preference imposed by an external force (e.g. due to buoyancy, as indicated in Section 2.43, or swirl).

In expressing the Reynolds stresses and turbulent heat fluxes by mean gradient of transport quantities in the manner of laminar transport, it is implied that the large scales of turbulence are isotropic, since it is implicit in the model that at any given location of the turbulence field the turbulent diffusivity* $\Gamma_{\phi,t} (= \frac{\mu_t}{\sigma_{\phi,t}})$ is independent of direction. However, anisotropy of the large scales demands that

$$(\Gamma_{\phi,t})_1 \neq (\Gamma_{\phi,t})_2 \neq (\Gamma_{\phi,t})_3 \quad (2.45)$$

where subscripts 1, 2, 3, stand for the three spacial directions. For example, measurements made by Quarmby and Quirk (1972,1974), and also followed up by Launder (1976), on asymmetric gaseous diffusion in a

* for momentum transfer, $\Gamma_{\phi,t} = \mu_t$

pipe show that the Γ 's for turbulent heat fluxes are anisotropic. Consequently, for flows where the large scales are highly anisotropic the basic K- ϵ model may be inadequate.

The model also assumes isotropy of small scale eddies which are responsible for the viscous dissipation of turbulence energy into thermal internal energy. The value of the dissipation rate ϵ is consequently based on the prevalence of this local isotropy⁺. This aspect of the model is founded on the classical Kolmogorov's universal equilibrium theory: through the cascade of turbulent energy, vortex stretching eradicates the original anisotropy of the large eddies and produces even smaller scales which tend to lose all sense of direction and assume a universally isotropic⁺ structure that is independent of both the mean flow and the large scales that created them (Batchelor 1960; Bradshaw, 1971).

However, it is worth mentioning that some researchers have put forward theories that in a stratified medium buoyancy affects the small scales of turbulence. Bolgiano (1959, 1962) and Monin (1967) have suggested that in such medium buoyancy affects the small-scale turbulence structure appreciably so that local isotropy no longer prevails. Nevertheless, this contradicts the observations of Grigg and Stewart (1963) that the small scales of turbulence within buoyant "puffs" of fluid in motion are little affected by ambient stratification whereas the large scales are strongly affected.

For flows where both the mean velocity gradient and the shear stress do not fall to zero at the same location, or where both have opposite signs, the K- ϵ model will prove inadequate. However, experiments show that this difficulty does not arise for axi-symmetric jets/plumes. Nevertheless, in cavity flows this situation can arise.

+ See footnote on page 18

A further assumption which is implicit in the form of the K- ϵ model employed here, that the turbulence phenomena has no well defined time-dependent process throughout the fluid, may be invalid in some parts of the turbulence field. Signals from anamometers reveal that in some parts of a trubulence field the flow may not be continuously turbulent, but that it may exhibit intermittency⁺ (Bradshaw, 1971; Tennekes and Lumley, 1973). For example, the free edge of a turbulent flow (e.g. a jet or wake) is quite well defined, but highly irregular⁺⁺ and unsteady (Bradshaw, 1971).

A rather important and fundamental aspect of the K- ϵ model, which has already been indicated earlier, is that it is only valid for flows where the Reynolds number of turbulence (see expression (2.28)) is sufficiently large that molecular transport may be neglected. The model is therefore most likely to be inadequate in highly stably-stratified flows in which buoyancy dampens the turbulence to a level where both molecular and turbulence transports are of comparable strengths.

Also, the neglect of the influence of buoyancy in the wall functions employed in this study may be inadequate in wall regions where buoyancy plays a dominant role.

Despite the limitations indicated here of the K- ϵ turbulence model the non-buoyant version of it has been employed with considerable success by numerous workers in predicting various high Reynolds number two- and three-dimensional flows (See, for example, Launder and Spalding (1972),

+ A fixed point in some parts of a turbulence field may spend only a fraction of its time in the turbulent flow. This fraction is called the intermittency (1 for a point in the fully turbulent flow, 0 for one that is always outside the turbulence).

++ Indeed, something can be learnt about the large eddies that produce this irregularity, by studying the intermittency. For example, the region over which the intermittency decreases from 1 to 0 is much narrower in axisymmetric jets than in wakes, thus implying that the large eddies which are responsible for contorting the free edge are relatively smaller in the jets (Tennekes and Lumley, 1973)

Laundry et al (1972) for a review). Recently, McGuirk (1975) employed the K-ε model for the study of three-dimensional turbulent buoyant jets in co-flowing streams, but he omitted the buoyancy terms in the K and ε equations.

This work will shed further light on the influence, if any, of the various limitations indicated here on the version of K-ε model employed for the different flows studied here.

2.5 THE EQUATIONS TO BE SOLVED

In this section the governing equations are assembled in the manner they are to be solved for steady, axi-symmetric flows:

Hydrodynamic equations:

$$\circ \quad \frac{\partial}{\partial x}(r\rho U) + \frac{\partial}{\partial r}(r\rho V) = 0 \quad (2.46)$$

$$\circ \quad \frac{1}{r}\left\{\frac{\partial}{\partial x}(r\rho UU) + \frac{\partial}{\partial r}(r\rho VU)\right\} = \frac{1}{r}\left\{\frac{\partial}{\partial x}(r\mu_{eff}\frac{\partial U}{\partial x}) + \frac{\partial}{\partial r}(r\mu_{eff}\frac{\partial U}{\partial r})\right\} - \frac{\partial P}{\partial x} + g(\rho_{\infty} - \rho) + S_u \quad (2.47)$$

$$\circ \quad \frac{1}{r}\left\{\frac{\partial}{\partial x}(r\rho UV) + \frac{\partial}{\partial r}(r\rho VV)\right\} = \frac{1}{r}\left\{\frac{\partial}{\partial x}(r\mu_{eff}\frac{\partial V}{\partial x}) + \frac{\partial}{\partial r}(r\mu_{eff}\frac{\partial V}{\partial r})\right\} - \frac{\partial P}{\partial r} - \mu_{eff}\frac{V}{r^2} + S_v \quad (2.48)$$

Scalar transport equations:

$$\circ \quad \frac{1}{r}\left\{\frac{\partial}{\partial x}(r\rho U\tilde{h}) + \frac{\partial}{\partial r}(r\rho V\tilde{h})\right\} = \frac{1}{r}\left\{\frac{\partial}{\partial x}(rC_p\Gamma_{h,eff}\frac{\partial T}{\partial x}) + \frac{\partial}{\partial r}(rC_p\Gamma_{h,eff}\frac{\partial T}{\partial r})\right\} + S_h \quad (2.49)$$

$$\circ \quad \frac{1}{r}\left\{\frac{\partial}{\partial x}(r\rho U m_j) + \frac{\partial}{\partial r}(r\rho V m_j)\right\} = \frac{1}{r}\left\{\frac{\partial}{\partial x}(r\Gamma_{m,eff}\frac{\partial m_j}{\partial x}) + \frac{\partial}{\partial r}(r\Gamma_{m,eff}\frac{\partial m_j}{\partial r})\right\} + S_{mj} \quad (2.50)$$

Turbulence model equations:

$$\circ \quad \frac{1}{r}\left\{\frac{\partial}{\partial x}(r\rho UK) + \frac{\partial}{\partial r}(r\rho VK)\right\} = \frac{1}{r}\left\{\frac{\partial}{\partial x}(r\frac{\mu_{eff}}{\sigma_K}\frac{\partial K}{\partial x}) + \frac{\partial}{\partial r}(r\frac{\mu_{eff}}{\sigma_K}\frac{\partial K}{\partial r})\right\} + G - C_D\rho\varepsilon + \rho\sqrt{gu\theta'} \quad (2.51)$$

$$\begin{aligned} \circ \quad \frac{1}{r} \left\{ \frac{\partial}{\partial x} (r \rho U \epsilon) + \frac{\partial}{\partial r} (r \rho V \epsilon) \right\} &= \frac{1}{r} \left\{ \frac{\partial}{\partial x} \left(r \frac{\mu_{\text{eff}}}{\sigma_{\epsilon}} \frac{\partial \epsilon}{\partial x} \right) + \frac{\partial}{\partial r} \left(r \frac{\mu_{\text{eff}}}{\sigma_{\epsilon}} \frac{\partial \epsilon}{\partial r} \right) \right. \\ &\quad \left. + C_1 \frac{\epsilon}{K} G - C_2 \rho \frac{\epsilon^2}{K} + C_3 \rho \frac{\epsilon}{K} \beta g \overline{u \theta'} \right\} \end{aligned} \quad (2.52)$$

$$\begin{aligned} \circ \quad G \equiv \mu_t \left\{ 2 \left[\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial r} \right)^2 + \left(\frac{V}{r} \right)^2 \right] + \left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x} \right)^2 \right. \\ \left. - \frac{2}{3} \left[\frac{1}{r} \frac{\partial}{\partial r} (r V) + \frac{\partial U}{\partial x} \right]^2 \right\} \end{aligned} \quad (2.53)$$

Auxiliary:

$$\circ \quad S_u \equiv \frac{\partial}{\partial x} (\mu_{\text{eff}} \frac{\partial U}{\partial x}) + \frac{1}{r} \frac{\partial}{\partial r} (r \mu_{\text{eff}} \frac{\partial V}{\partial r}) \quad (2.54)$$

$$S_v \equiv \frac{\partial}{\partial x} (\mu_{\text{eff}} \frac{\partial U}{\partial r}) + \frac{1}{r} \frac{\partial}{\partial r} (r \mu_{\text{eff}} \frac{\partial V}{\partial r}) - \mu_{\text{eff}} \frac{V}{r^2} \quad (2.55)$$

$$\circ \quad \mu_{\text{eff}} = \mu + \mu_t \quad (2.56)$$

$$\Gamma_{\phi, \text{eff}} = \frac{\mu}{\sigma_{\phi}} + \frac{\mu_t}{\sigma_{\phi, t}} \quad \text{where } \phi = h, m_j \quad (2.57)$$

$$\begin{aligned} \circ \quad \overline{\rho u \theta'} &= - \frac{\mu_t}{\sigma_{h, t}} \frac{\partial \theta}{\partial x} \quad (\text{for case where gravity is in x-direction}) \\ &= - \frac{\mu_t}{\sigma_{h, t}} \left\{ \frac{\partial T}{\partial x} - \frac{\partial T_{\infty}}{\partial x} \right\} \end{aligned} \quad (2.58)$$

$$\circ \quad \mu_t = C_{\mu} \rho K^2 / \epsilon \quad (2.59)$$

The origin of the expressions for S_u , S_v , and G is given in Appendix C. And the appropriate values of $\sigma_{h, t}$ used in this study are given in the relevant sections (See Chapters 4 - 6).

General form of equations:

The equations to be solved can be summarised by a single equation of the form:

$$\underbrace{\frac{1}{r} \left\{ \frac{\partial}{\partial x} (r \rho U \phi) + \frac{\partial}{\partial r} (r \rho V \phi) \right\}}_{\text{convection terms}} - \underbrace{\frac{1}{r} \left\{ \frac{\partial}{\partial x} \left(r \Gamma \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial r} \left(r \Gamma \frac{\partial \phi}{\partial r} \right) \right\}}_{\text{diffusion terms}}$$

$$- S_{\phi} = 0 \quad (2.60)$$

source terms

where ϕ is a general variable defined in conjunction with the expressions for Γ and S_{ϕ} in Table 1.

Table 1

Definition of ϕ , Γ , and S_ϕ of equation (2.60)

Conserved Property	ϕ	Γ	S_ϕ
Mass	1	0	
Streamwise Momentum	U	μ_{eff}	$-\frac{\partial P}{\partial x} + g(\rho_\infty - \rho) + \frac{\partial}{\partial x} (\mu_{eff} \frac{\partial U}{\partial x}) + \frac{1}{r} \frac{\partial}{\partial r} (r \mu_{eff} \frac{\partial V}{\partial r})$
Cross-stream Momentum	V	μ_{eff}	$-\frac{\partial P}{\partial r} - 2\mu_{eff} \frac{V}{r^2} + \frac{\partial}{\partial x} (\mu_{eff} \frac{\partial U}{\partial r}) + \frac{1}{r} \frac{\partial}{\partial r} (r \mu_{eff} \frac{\partial V}{\partial r})$
Thermal Energy	$\tilde{h}$	$\Gamma_{h,eff}$	S_h
Species Conservation	m_j	$\Gamma_{m,eff}$	S_{mj}
Turbulence Energy	K	$\frac{\mu_{eff}}{\sigma_K}$	$G - C_D \rho \epsilon + \rho \beta g \overline{u\theta'}$
Dissipation rate	ϵ	$\frac{\mu_{eff}}{\sigma_\epsilon}$	$C_1 \frac{\epsilon}{K} G - C_2 \rho \frac{\epsilon^2}{K} + C_3 \rho \frac{\epsilon}{K} \beta g \overline{u\theta'}$

2.6 BOUNDARY CONDITIONS

The flows examined in this study are of an elliptic⁺ nature; it is therefore necessary to supply conditions on the variables at all the boundaries of the flow domain. Generally, these conditions are of three kinds: specification of the value of the dependent variable at the boundary, or the value of the associated flux, or a relation between the variable and the flux.

⁺ Elliptic flows are those where perturbations of conditions at any point in the flow can influence conditions at any other point. Such flows are governed by partial differential equations of the type known as "elliptic" - such as those given by equation (2.60).

Since velocities and pressures are inter-dependent, where velocities are prescribed pressures need not be. Thus the variables for which boundary conditions are specified are U , V , $\tilde{h}$, m_j , K , and ϵ ; and the general conditions at the various boundaries of the different flows studied here are as follows:

(1) Axi-symmetric Jet/Plume

With reference to the co-ordinate system shown in Fig 4.1,

$$\frac{\partial \phi}{\partial r} = 0, \text{ at } r = 0 \quad (2.61)$$

$$\phi = \phi_{\infty}, \text{ as } r \rightarrow \infty \quad (2.62)$$

$$\text{where } \phi = U, V, \tilde{h}, m_j, K, \epsilon \quad (2.63)$$

and subscript ∞ (infinity) stands for values in the stagnant ambient fluid.

At the nozzle exit, ($x = 0, 0 \leq \frac{r}{R} \leq 1$),

$$\begin{aligned} U &= U_0; \quad V = V_0 \\ \tilde{h} &= \tilde{h}_0, \quad m_j = m_{j,0} \\ K &= K_0 \\ \epsilon &= \epsilon_0 \end{aligned} \quad (2.64)$$

where subscript 0 stands for values at the exit (values usually prescribed).

(ii) Cavity-flows

At a wall, ($y = 0$),

$$U = 0, \text{ for stationary wall}$$

$$U = U_w, \text{ for moving wall}$$

$$V = 0$$

$$\phi = \phi_w, \text{ for wall with constant } \phi (= \tilde{h}, m_j)$$

$$J''_{\phi} = J''_{\phi,w}, \text{ for wall with prescribed } \phi\text{-flux}$$

$$K = 0$$

$$\epsilon = 0$$

where subscript w denotes the value at the wall ($y = 0$); y is the

normal distance measured from the wall; U is here taken to be the velocity parallel to the wall, and V that perpendicular to the wall; and $\dot{J}_\phi$ is the ϕ -flux ($\phi = h, m_j$).

At the entrance of a flow-driven cavity, boundary conditions are specified as given by equation (2.64).

2.7 CONCLUDING REMARKS

This Chapter has developed a mathematical representation of turbulent natural and forced convection as applied in this study. The main features of the Chapter may be summarised as follows:

- (i) The basic equations governing the transport of heat, mass, and momentum have been presented in both their time-dependent and time-averaged forms.
- (ii) The magnitude of the problem of forming a closed set of equations for turbulent flows has been indicated, and the essential features of the particular closure (the $K-\epsilon$ scalar viscosity turbulence model) employed in this study have been given.
- (iii) Some important aspects of turbulence in stratified flows have also been indicated as well as the special treatment involved in near-wall flows. Further, an indication has been given of the limitations of the turbulence model employed in this study.
- (iv) The complete set of equations for both mean flow variables and turbulence quantities has been finally presented in the form they are to be solved, and the similarity between the various equations has been shown.
- (v) Lastly, the boundary conditions required for the solution of the equations have also been given.

The stage is now set for the consideration of the calculation

procedure employed in solving the foregoing equations, and this is the subject of the next chapter.

CHAPTER 3

THE SOLUTION PROCEDURE

3.1 INTRODUCTION

The governing differential equations given in Chapter 2 are complex and inter-linked, and therefore require numerical methods of solution. In this chapter the particular numerical procedures employed in this study will be described.

In Section 3.2 the main features of the governing equations are identified, and the general route to their solution is outlined. The particular approach adopted involves the transformation of the differential equations into their finite difference equivalents, and this forms the subject of Section 3.3.

The solution of the resultant difference equations is obtained by two "guess-and-correct" procedures, one being an existing algorithm called SIMPLE and the other a new algorithm developed in this study and herein called EMIT. The essential features of both algorithms are discussed in Section 3.4. Then in Section 3.5 are given some special aspects of the solution techniques, including convergence, numerical stability, and accuracy of solutions to be obtained.

Finally, in Section 3.6 a summary is given of the major aspects of the chapter.

3.2 APPRECIATION OF THE PROBLEM

3.21 Nature of equations to be solved

The set of equations (2.60) which are to be solved have the following important features that govern the method of solution:

- (i) They are numerous*, and mainly of elliptic, non-linear, partial-

* For two-dimensional flows, there are seven partial-differential equations for continuity, U , V , h , m_j , K , and e .

differential type.

(ii) They are complex and inter-linked: the velocities U and V appear in all equations; the pressure P appears in the momentum equations; the turbulence quantities K and ϵ are strongly coupled, while the turbulence viscosity μ_t depends on both; buoyancy links the momentum and energy equations, etc.

(iii) There is no obvious equation from which the pressure P may be extracted, for while each of the variables U , V , $\tilde{h}$, m_j , K and ϵ appears as the dominant variable in its own transport equation, the sole remaining (continuity) equation does not contain P .

These features render the equations unsuitable for direct analytical solution, and necessitate the use of numerical methods.

3.22 Path to numerical solution of the equations

Various decisions are involved in devising numerical methods to solve the set of equations (2.60). Detailed discussion on this aspect has been given by Roache (1972), Gosman (1976), and others. Here only a brief outline will be given:

(i) Choice of variables

The type of variables (dependent and independent) employed will influence the nature and flexibility of the method of solution.

For the independent variables, the practice of working in Eulerian co-ordinates is adopted in this study, in common with many others (see, for example, Harlow, 1969; Gosman et al 1969). Other possibilities exist (e.g. Lagrangian (Hirt et al, 1970; Chan and Street, 1970) and hybrid Eulerian - Lagrangian (Hirt et al, 1974) methods) but these are often less suitable for recirculating flows owing to excessive distortion of the computational mesh.

There are two major alternatives for the dependent variables; these are the stream function - vorticity set which has been widely exploited

by Gosman et al (1969), and the velocity-pressure set (e.g. Harlow and Welch, 1965; Patankar and Spalding, 1972). The former set eliminates pressure and entails the solution of only two hydrodynamic equations for two-dimensional flows; however, for three-dimensional flows the equivalent set comprises six equations; moreover it is unsuitable for compressible flows (i.e. when $\frac{\partial \rho}{\partial t} \neq 0$ and/or the density is a function of pressure).

On the other hand, the velocity-pressure set involves the more familiar "primitive" variables (i.e. velocities and pressure), and it is straight forwardly applicable to three-dimensional flows as well as being suitable for compressible flows. The drawback of this set is that there is no direct equation for pressure; nevertheless this drawback has been overcome by methods which derive "pressure equations" by combining momentum and continuity. (e.g. Chorin, 1968; Amsden and Harlow, 1970; Patankar and Spalding, 1972). This set is adopted here as it has much more general applicability as well as greater ease in physical understanding.

(ii) Method of deriving algebraic equations

Numerical methods of solving the partial-differential equations entail reducing them to a set of algebraic equations. Two main approaches may be employed in doing this, namely, "finite-element" in which coefficients in an assumed algebraic distributions of variables are obtained from the partial-differential equations by application of minimisation principles (e.g. Baker, 1973; Oden and Wellford, 1972), and "finite difference" in which differential coefficients are replaced by finite-difference ones through the use of Taylor series or "micro-integral" approaches (e.g. Gosman et al, 1969; Roache, 1972).

The finite-element method is now widely used in stress analysis, and is claimed to be more flexible by way of applicability to arbitrary

boundary conditions, and also more accurate. Nevertheless, there is little supporting evidence for these claims, especially for large Reynolds number flows. On the other hand, finite-difference methods are well-established and proven for flows at all Reynolds numbers; moreover, they are more flexible than is commonly supposed, as indicated by the Los Alamos non-orthogonal mesh method (Hirt et al, 1974).

The finite-difference method is employed in this study.

(iii) Differencing schemes for convective and diffusive fluxes

In order to formulate the finite-difference equations, the manner in which convective and diffusive fluxes are to be evaluated needs to be defined. This is best illustrated by reference to a one-dimensional convection/diffusion example. The problem is then to replace

$$G \frac{d\phi}{dX} - \Gamma \frac{d^2 \phi}{dX^2} = 0 \quad (3.1)$$

by an algebraic equivalent; where ϕ is a scalar variable, G is the mass flux, Γ the diffusivity, and X the spatial co-ordinate. In doing this, equation (3.1) is generally assumed to be satisfied at discrete spatial locations, for example, W, P, and E shown in Fig. 3.1 (c).

There are two common practices used (Courant, Isaacson and Rees, 1952; Gosman et al, 1969; Roache, 1972): the "central difference" in which the variable ϕ is assumed to vary linearly; thus for point P in Fig. 3.1(c), and for the uniform spacing shown there, the algebraic equivalent of equation (3.1) becomes

$$\frac{G(\phi_E - \phi_W)}{2\delta X} - \Gamma \left(\frac{\phi_W + \phi_E - 2\phi_P}{\delta X^2} \right) = 0 ; \quad (3.2)$$

and the "upwind" difference in which, with respect to convection, ϕ is assumed to vary in a piece-wise manner and takes upstream value; thus for point P in Fig. 3.1 (c) the algebraic equivalent of equation (3.1) becomes

$$\frac{1}{2} \left\{ (G + |G|) \frac{(\phi_P - \phi_W)}{\delta X} + (G - |G|) \frac{(\phi_E - \phi_P)}{\delta X} \right\} - \Gamma \left(\frac{\phi_W + \phi_E - 2\phi_P}{\delta X^2} \right) = 0 \quad (3.3)$$

By comparison with the exact one-dimensional solution for given ϕ_E and ϕ_W , the central difference scheme gives very accurate results when $\left| \frac{G\delta X}{\Gamma} \right| < 2$, where $\frac{G\delta X}{\Gamma}$ is the Peclet number; outside this range it gives unrealistic results and numerical instability when the equations are solved iteratively (Gosman et al, 1969; Spalding, 1972). On the other hand, the "upwind" difference gives realistic and stable behaviour for all values of the Peclet number, and is therefore used in boundary layer procedures (e.g. Patankar and Spalding, 1970).

However, hybrids of the two schemes exist, which take full advantage of the merits of both (e.g. Spalding, 1972; Runchal, 1972). This method is therefore employed in this study.

It may be noted that the accuracy of all the existing schemes may be impaired by both truncation and round-off errors. The truncation error is the difference between the solution of the partial-differential equations and that of the finite difference equations, and it arises because of the approximations involved in the replacement of differential coefficients by finite-difference ones. Some detailed consideration of the truncation error has been made by Wolfshtein (1967) and Gosman et al (1969); in essence, it introduces an additional "false" diffusion of ϕ into the equations, which effect is primarily due to the fact that the calculated ϕ at any point is based on values at a finite number of neighbouring points through which the streamline through the central point may or may not pass; if the latter happens, the smearing occurs. All differencing schemes suffer from this "false diffusion", and its effect is small when the flow is parallel to the mesh or the cell Peclet number is small. This is therefore a major source of error in the computation of recirculating flows, where the flow will generally be inclined to the mesh.

Round-off errors are associated with iterative methods, and arise because computers only carry out calculations to a limited number of significant figures. This error is, however, unimportant as calculations of the same problem with 6, 8, and 16 significant figures make negligible difference in the results (Gosman et al, 1969), and most computers use at least 6 significant figures.

(iv) Solution of resultant algebraic equations

The finite-difference formulations result in a set of algebraic equations which are often non-linear and inter-linked. There are two rather connected classes of options for the solution of these algebraic equations (e.g. Ames, 1965):

(a) For a single dependent variable, ϕ : direct (i.e. matrix inversion) method, or iterative, point or block (e.g. line) methods. The direct method is generally unsuitable for non-linear equations.

(b) For a complete set of variables, ϕ 's: successive iteration method in which the equation set for each variable is solved in turn; or simultaneous method (e.g. matrix inversion) in which the equation sets for all variables are solved simultaneously either locally or over the whole field.

In this study, the successive iteration method is adopted as it is very suitable for the non-linear and inter-linked equations which are involved in the study.

3.3 FORMULATION OF THE FINITE-DIFFERENCE EQUATIONS

The transformation of the partial-differential equations (2.60) into finite-difference forms can be accomplished through various approaches, including Taylor series expansion, polynomial fitting, (e.g. Roache, 1972) and "micro-integration" (e.g. Gosman, 1976). In the present study the "micro-integration" approach is employed, as is described below.

3.31 The Grid

The first step in deriving the finite-difference equations is the establishment of a conventional grid and the definition of the points at which various flow variables are located. The grid here employed, when viewed in a plane passing through the axis or plane of symmetry, is rectangular and divides the domain of interest completely into a set of non-overlapping sub-domains as illustrated in Fig. 3.1 (a). The intersections of the grid lines, whose spacing is arbitrary, are called "grid nodes".

As shown by the dashed lines in Figs 3.1(a) and (b), the boundaries of the flow domain as well as those of imaginary control volumes or "cells" surrounding each grid node (e.g. P, N, S, E, W) lie mid-way between grid lines. These control volumes represent regions over which the partial-differential equations are integrated to obtain their finite-difference form.

In general, the control volumes surround the storage locations at which the discrete values of the flow variables, which values represent averages over the respective control volumes, are located. These storage locations, and hence the control volumes which surround them, are different for the various flow variables. As shown in Fig. 3.2, the pressure P and the other scalar variables ($\bar{h}$, T , m_j , K , ϵ) are stored at the main grid nodes, while the velocities U and V are stored mid-way between the main grid nodes at locations denoted by $(\rightarrow)$ and $(\uparrow)$ respectively.

This "staggered grid" system, which was originated by Harlow and Welch (1965) and subsequently adopted by Patankar and Spalding (1972), has the advantage that the variables U and V are located between the pressures which drive them: moreover, the velocities are located where they are needed for the calculation of the convective fluxes of scalar flow variables.

3.32 The derivation of the finite-difference equation for scalar variables ($\tilde{h}$, m_j , K , ϵ)

(i) Integration of the partial-differential equations

Following the "micro-intergral" procedure used by Gosman (1976), Gosman and Ideriah (1977), and others, the partial-differential equation given by (2.60) is integrated over the shaded control volume shown in Fig. 3.3(a), thus:

$$\iiint_V \left\{ \frac{1}{r} \left[\frac{\partial}{\partial x} (r \rho U \phi) + \frac{\partial}{\partial r} (r \rho V \phi) \right] - \frac{1}{r} \left[\frac{\partial}{\partial x} (r \Gamma \frac{\partial \phi}{\partial x}) + \frac{\partial}{\partial r} (r \Gamma \frac{\partial \phi}{\partial r}) \right] - S_\phi \right\} dV = 0 \quad (3.4)$$

After application of Gauss's theorem* to replace volume integral by surface ones, there results:

$$\begin{aligned} & \iint_{ns} \left\{ \underbrace{(r \rho U \phi - r \Gamma \frac{\partial \phi}{\partial x})_e}_{F_e} - \underbrace{(r \rho U \phi - r \Gamma \frac{\partial \phi}{\partial x})_w}_{F_w} \right\} dA_{ns} \\ & + \iint_{ew} \left\{ \underbrace{(r \rho V \phi - r \Gamma \frac{\partial \phi}{\partial r})_n}_{F_n} - \underbrace{(r \rho V \phi - r \Gamma \frac{\partial \phi}{\partial r})_s}_{F_s} \right\} dA_{ew} - \iiint_V S_\phi dV = 0 \quad (3.5) \end{aligned}$$

Here the subscripts n, s, w, e denote the control volume boundaries shown in fig. 3.3(a), $\iint_{kq}$ and dA_{kq} represent the integral and elemental area respectively along a surface bounded by planes k and q , and

$$F_i = \iint_A (r \rho U_j \phi - r \Gamma \frac{\partial \phi}{\partial x_j})_i dA, \quad \begin{aligned} i &= n, s, e, w \\ j &= 1, 2 \text{ (for 2D flows)} \end{aligned} \quad (3.6)$$

is the total (corrective and diffusive) flux across surface i .

It is to be noted that in equation (3.5) the flux terms, F_i , appear as surface integrals around the control volume boundaries, thus aiding

* The Gauss (or Divergence) theorem states that in a region R bounded by a surface S , within which the vector $\vec{f}$ as well as $\nabla \cdot \vec{f}$ is continuous, the volume integral of $\nabla \cdot \vec{f}$ equals the surface integral of $\vec{f}$. Thus (Lass, 1950)

$$\iiint_V (\nabla \cdot \vec{f}) dV = \iint_S \vec{f} \cdot \vec{n} ds$$

or in terms of Cartesian tensor notation,

$$\iiint_V \left(\frac{\partial}{\partial x_j} f_j \right) dV = \iint_S f_j dA_j, \quad j = 1, 2, 3 \quad (3.5a)$$

where $\vec{n}$ = unit vector at surface element ds ; f_j = component of function in direction j ; A_j = area perpendicular to direction j .

In application of (3.5a) to equation (3.4), the f_j 's represent the convective and diffusive fluxes; thus

for convective terms, $f_j = r \rho U_j \phi$; U_j = velocity in direction j ($j = 1, 2, 3$)
for diffusive terms, $f_j = r \Gamma \frac{\partial \phi}{\partial x_j}$; $j = 1, 2, 3$

where ϕ = scalar value ($U_i, \tilde{h}, m_j, K, \epsilon$, or unity for mass)

physical understanding and drawing attention to the need to preserve the conservation laws on which the partial differential equation is based.

(ii) Expressions for the total flux terms, F_i

The next stage in deriving the finite-difference equations is to express the total flux terms F_i in terms of the nodal values of the variables. This may be done by reference to the special case of one-dimensional transport across the cell boundaries, since it is reasonable to require that, in the one-dimensional limit, the fluxes should be accurately calculated.

For this limiting case, the "exact" solution, e.g. for the west boundary (see fig. 3.3(b)), yields (Zuber, 1972); see also Section 3.22)

$$F_w \approx \rho_w U_w A_w \{f_w \phi_w + (1 - f_w) \phi_p\} \quad (3.7)$$

$$\text{where } f_w \equiv \exp(Pe_w) / (\exp(Pe_w) - 1) \quad (3.8)$$

$$Pe_w \equiv \rho_w U_w \delta x_{pw} / \Gamma_w, \text{ cell Peclet number} \quad (3.9)$$

$$\rho_w \equiv (\rho_w + \rho_p) / 2; \quad \Gamma_w \equiv (\Gamma_w + \Gamma_p) / 2 \quad (3.10)$$

$$A_w \equiv r_p \delta r_{ns} \quad (3.11)$$

Expression (3.7) gives the convected ϕ as a weighted mean of the values ϕ_w and ϕ_p at the nodes bracketing the cell boundary in question, with weighting factor f depending on the local Peclet number Pe formed from values of U , ρ , and Γ at the cell boundary and δx_{pw} . The ρ 's and Γ 's at the cell boundaries are taken as averages of values at the main grid nodes, thus ensuring continuity of flux.

In equation (3.8),

$$\text{as } Pe_w \rightarrow 0, \quad f_w \rightarrow (1 + 2Pe_w^{-1}) / 2 \quad (3.12)$$

Substitution of (3.12) into expression (3.7) yields

$$F_w \approx \frac{\rho_w U_w A_w}{2} \{ (1 + 2Pe_w^{-1}) \phi_w + (1 - 2Pe_w^{-1}) \phi_p \}$$

$$\text{or, } F_w \approx \rho_w U_w A_w \frac{(\phi_p + \phi_w)}{2} - \frac{\Gamma_w A_w}{\delta x_{pw}} (\phi_p - \phi_w) \quad (3.13)$$

Expression (3.13) is the familiar central difference representation of the total-flux F_w through the west face of the control volume shown in Fig. 3.3(b).

Again, in equation (3.8),
as $Pe_w \rightarrow \begin{matrix} +\infty \\ -\infty \end{matrix}$, $f_w \rightarrow \begin{matrix} 1 \\ 0 \end{matrix}$ (3.14)

Substitution of (3.14) into (3.7) yields the "upwind" convection terms; however, the full "upwind" formula also derives diffusion fluxes from central difference as has been indicated earlier; thus the complete "upwind" formula for the total flux F_w through the west boundary shown in Fig. 3.3(b) is:

$$F_w \approx \begin{cases} \rho_w U_w A_w \phi_w - \frac{\Gamma_w A_w}{\delta x_{pw}} (\phi_p - \phi_w), & \text{if } Pe_w > 0 \\ \rho_w U_w A_w \phi_p - \frac{\Gamma_w A_w}{\delta x_{pw}} (\phi_p - \phi_w), & \text{if } Pe_w < 0 \end{cases} \quad (3.15)$$

Expressions (3.7), (3.13), and (3.15) therefore give the total flux derived from the "exact" solution, the central difference scheme, and "upwind" difference schemes respectively. In Fig. 3.4(c) is shown a plot against Pe_w of the coefficient of ϕ_w derived from these three expressions. It will be observed there that the central difference is preferable for small Pe_w , and the "upwind" difference for large values of Pe_w ; on the other hand, neither is entirely satisfactory for the complete range of Pe_w (i.e. $-\infty \leq Pe_w \leq \infty$).

However, shown in dotted lines in Fig. 3.4(c) is a compromise between the two schemes, and satisfactorily approximates the "exact" solution. This is the "hybrid" scheme, and consists of the asymptotes of the exact solution; it coincides with the central difference for $|Pe_w| < 2$, and the "upwind difference" outside this range. The $F_w \sim Pe_w$ relation for this scheme, which is employed in this study, is (Patankar and Spalding, 1972b):

$$\frac{F_w}{\rho_w U_w A_w} = \begin{cases} \frac{1}{2} \{ (1 + 2Pe_w^{-1}) \phi_w + (1 - 2Pe_w^{-1}) \phi_p \}, & \text{for } -2 < Pe_w < 2 \\ \phi_w, & \text{for } Pe_w \geq 2 \\ \phi_p, & \text{for } Pe_w \leq -2 \end{cases} \quad (3.16)$$

Re-writing expression (3.16) in the manner of (3.7), then the hybrid form of the weighting factor f_w becomes

$$f_w = \begin{cases} \frac{1}{2} (1 + 2Pe_w^{-1}), & \text{for } -2 < Pe_w < 2 \\ 1, & \text{for } Pe_w \geq 2 \\ 0, & \text{for } Pe_w \leq -2 \end{cases} \quad (3.17)$$

Similarly, for an east boundary the central difference formulation of the total-flux F_e is again in the manner of (3.16), but the "upwind" formulation changes; thus

$$\frac{F_e}{\rho_e U_e A_e} = \begin{cases} \frac{1}{2} \{ (1 + 2Pe_e^{-1}) \phi_E + (1 - 2Pe_e^{-1}) \phi_p \}, & \text{for } -2 < Pe_e < 2 \\ \phi_p, & \text{for } Pe_e \geq 2 \\ \phi_E, & \text{for } Pe_e \leq -2 \end{cases} \quad (3.18)$$

And, re-writing (3.18) in the same form as (3.7), the hybrid form of the weighting factor f_e becomes

$$f_e = \begin{cases} \frac{1}{2} (1 + 2Pe_e^{-1}), & \text{for } -2 < Pe_e < 2 \\ 0, & \text{for } Pe_e \geq 2 \\ 1, & \text{for } Pe_e \leq -2 \end{cases} \quad (3.19)$$

The remaining total-flux expressions F_n and F_s for the north and south boundaries of the control volume are derived in a similar manner.

(iii) Expression for Source term

Whatever the form of the particular expression for the source S_ϕ , the integrated value over the control volume can be reduced to a linearised form; thus

$$\iiint_V S_\phi dV = b\phi_p + C \quad (3.20)$$

where b and C are generally functions of ϕ , and are deduced from the integrated and linearised form of the source. Expressions for particular variables will be given later.

(iv) Assembly of final equation

Substitution of the flux and source expressions into the integrated equation (3.5) yields, with the aid of continuity:

$$(a_p - b)\phi_p = \sum_n a_n \phi_n + C \quad (3.21)$$

where $a_p = \sum_n a_n$; $\sum_n$ = summation over neighbouring ϕ -locations (N,S,E,W)

$$a_W = \rho_W U_W A_W f_W, \quad a_S = \rho_S V_S A_S f_S \quad (3.22)$$

and f_W, f_S etc, defined by expressions (3.17), (3.19), etc.

Here the a_n 's are combined convection/diffusion coefficients and are always positive. It may be noted that when $b = C = 0$, ϕ_p becomes the weighted mean of neighbouring ϕ -values, which is both physically plausible and important from the numerical stability point of view, as will be seen later.

3.33 Derivation of the finite-difference equations for velocity components U and V

The finite-difference equations for the velocity components are derived in a manner similar to those for the scalar variables, but with the following slight modifications:

- (i) The control volumes are displaced from those of the scalar variables (see Fig. 3.3c) because the storage locations of the velocities are displaced. As illustrated in Fig. 3.3(c), the control volumes pass through the locations of the pressures that drive the velocities.
- (ii) The pressure gradients are singled out from the other source terms for later attention in the derivation of the "pressure correction" equation. This is a consequence of the special solution procedures adopted, which are dealt with in Section 3.4.

For example, the finite-difference U-momentum equation for the shaded volume shown in Fig. 3.3(c) is:

$$(a_p - b)U_p = \sum_n a_n U_n + A_{ew}(P_W - P_p) + C \quad (3.23)$$

where $a_p = \sum_n a_n$; $\sum_n$ = summation over neighbouring U-locations (N,S,E,W)

$$a_W = \rho_W U_W A_{e_W} f_W, \text{ etc.} \quad (3.24)$$

$$f_W = f(\text{Re}_W), \text{ defined by (3.17) with } \text{Pe}_W \equiv \text{Re}_W \quad (3.25)$$

$$\text{Re}_W = \rho_W U_W \delta X_{PW} / \mu_W \quad \text{cell Reynolds number} \quad (3.26)$$

$$\rho_W U_W = (\rho_W U_W + \rho_P U_P) / 2 \quad (3.27)$$

Here the cell-boundary velocities, densities, etc., are again interpolated so as to satisfy continuity of total flux. The similarity between expressions (3.23) to (3.27) on the one hand and (3.21) to (3.22) on the other may be noted. The weighting factor, f_W , is now function of the cell Reynolds number.

3.34 Insertion of boundary conditions

At the boundaries of the calculation domain, the regular finite-difference equations are no longer applicable since they refer to points lying outside the domain. As shown in Fig. 3.3(d), the grid arrangement is such that the boundaries coincide with the cell walls. This is advantageous for ensuring conservation and for flux calculation.

For the particular example shown in Fig. 3.3(d), the link between ϕ_P and ϕ_W in the regular finite-difference equation must be suppressed, as the general F_W expression is now inappropriate. Thus F_W is suppressed by setting the appropriate difference coefficient to zero: i.e.,

$$a_W = 0 \quad (3.28)$$

and this breaks the normal $\phi_P \sim \phi_W$ link.

However, the correct expression for F_W still needs to be inserted. The appropriate value will depend on the particular conditions imposed at the boundary. For example, the boundary may be one of a prescribed flux (e.g. $F_W = F_B$, where subscript B stands for boundary), a prescribed flow variable ϕ_B {e.g. $F_W = a'_W(\phi_P - \phi_B)$, where a'_W is some coefficient}, or, indeed, for an internal node, a fixed value of ϕ . There are several ways of inserting the appropriate value of F_W , but in all cases the method adopted is a "false" source treatment in the manner of expression

(3.20); thus

$$F_w = b\phi_p + C \quad (3.29)$$

where b and C will generally be functions of the boundary value of ϕ .

This approach is applicable to all forms of conditions that may prevail at any type of boundary (e.g. wall, axis of symmetry, free-boundary, etc.), and it is particularly convenient for programming.

3.4 THE SOLUTION OF THE FINITE-DIFFERENCE EQUATIONS

3.41 Main Features

The solution procedures employed in this study solve the set of finite-difference equations for all variables simultaneously. Since the equations are non-linear and inter-linked, it becomes necessary to employ iterative methods. The particular methods employed in this study involve "guess-and-correct" procedures whereby initial guesses of all variables are made over the calculation domain, and these are used to calculate the coefficients in the finite-difference equations. The latter are then solved successively for each variable, using a "line method" of "sweeping" through the grid. The procedures are then employed repeatedly, using values of the previous iteration as new "guesses", until the desired solution is achieved.

Of particular interest in the methods are the special treatments given to the pressure-linked momentum and continuity equations, and these form the basis of two solution procedures employed in the study: they are the SIMPLE algorithm of Patankar and Spalding (1972), and the new EMIT algorithm originated in this study. Both algorithms solve the fully elliptic forms (see footnote on page 34) of the momentum equations, but the SIMPLE algorithm solves them for all grid nodes in the computation field simultaneously, while the EMIT algorithm solves them for grid nodes along only one plane after the other,

The major aspects of the line method and both the SIMPLE and EMIT procedures are covered in the remaining parts of this section.

3.42 The Line Solution Procedure

Before the SIMPLE and EMIT algorithms are discussed, it may be useful to introduce the line method which features in both algorithms. Fig. 3.4(a) illustrates the manner in which the grid nodes are generally scanned in the line iteration procedure. For each grid node on the line considered in Fig. 3.4(a), e.g. node j , the finite-difference equations (e.g. equation 3.21) can be expressed as

$$\phi_j = A_j \phi_{j+1} + B_j \phi_{j-1} + C_j \quad (3.30)$$

where, with respect to equation (3.21),

$$\phi_j = \phi_P; \phi_{j+1} = \phi_N; \phi_{j-1} = \phi_S \quad (3.31)$$

and the coefficients A_j , B_j , and C_j take the forms

$$A_j = \frac{a_n}{a_p - b}; \quad B_j = \frac{a_s}{a_p - b}; \quad C_j = \frac{a_W \phi_W + a_E \phi_E + C}{a_p - b} \quad (3.32)$$

Thus, C_j contains ϕ 's of points on the neighbouring lines, and these are temporarily supposed to be known, and A_j and B_j are known coefficients.

A set of equations of the type given by (3.30) can be solved directly with the use of the tri-diagonal matrix algorithm (TDMA). The TDMA is based on an elimination procedure derived by successive subtraction of a suitable multiple of each equation from the next, yielding a general recurrence relation for ϕ_j (Ames, 1965):

$$\phi_j = A'_j \phi_{j+1} + C'_j, \quad j = n-1, n-2, \dots, 1 \quad (3.33)$$

for a set of n equations; where

$$A'_j = \frac{A_j}{1 - A'_{j-1} B_j}; \quad C'_j = \frac{C_j + C'_{j-1} B_j}{1 - A'_{j-1} B_j} \quad (3.34)$$

and A_j , B_j and C_j are as defined by (3.32).

For each ϕ , the TDMA is used to solve simultaneously for all the values of ϕ at all the grid nodes on the line considered, taking the values at the nodes of the neighbouring lines as known. This is the line method employed in this study for deriving the ϕ 's of grid nodes

along any given line. The modes in which this method is employed in both the SIMPLE and EMIT algorithms are indicated in the appropriate sub-sections.

3.43 The SIMPLE algorithm

The pressure-linked momentum equations require some special attention: these are solved by first estimating the pressures at all nodes of the computation field, then obtaining preliminary values simultaneously for all grid nodes of U and V by solving the momentum equations, and finally correcting the pressure field through a "pressure-correction" equation so as to bring the velocity field into conformity with the continuity equation. The new values of U, V, and P are next used as new "guesses", and the process repeated (starting from fresh solution of the momentum equations) until the desired solution is obtained. This procedure is known as SIMPLE (=Semi Implicit Method for Pressure Linked Equations) and was developed by Patankar and Spalding (1972) along the lines of the earlier methods of Chorin (1968), Harlow and Welch (1965), and Amsden and Harlow (1970).

The procedure involves the following basic steps:

1. Initial guess of variable fields

Initially, the pressure and velocities are guessed at all grid nodes of the domain of computation.

2. First calculation of velocity field

A first estimate of the velocities is obtained by assembling the coefficients of the momentum equations and solving for U* and V* by means of the line procedure, using the guessed pressure field; e.g.

$$(a_p - b)U_p^* = \sum_n a_n U_n^* + A_{ew} (P_W^* - P_P^*) + C \quad (3.35)$$

where the superscript * given to U and V denotes that these are based on an estimated pressure field P*. In applying the line procedure to solve equation (3.35), for example, it is applied to one grid line after

the other in an ordered sequence, and then repeatedly employed to "sweep" through the grid. The velocities thus obtained will generally not satisfy continuity, and the correct values of the variables may thus be expressed as:

$$\begin{aligned} U &= U^* + U' \\ V &= V^* + V' \\ P &= P^* + P' \end{aligned} \quad (3.36)$$

where the primed values are the corrections.

3. The "pressure-correction" equation

(i) Integration of the continuity equation in the manner of equation (3.5) for the typical control volume shown in Fig. 3.3(a) yields

$$(\rho UA)_e - (\rho UA)_w + (\rho VA)_n - (\rho VA)_s = 0 \quad (3.37)$$

(ii) Now, advantage is taken of the "staggered" grid system by expressing the velocity corrections (U' , V') in terms of the pressure corrections P' ; e.g.

$$U_w = U_w^* + U'_w \quad (3.38)$$

and U'_w is expressed as

$$U'_w \approx \frac{\partial U_w^*}{\partial (P_w^* - P_p^*)} \cdot (P'_w - P'_p) \quad (3.39)$$

Further, from (3.35) it follows that

$$D_w \equiv \frac{\partial U_w^*}{\partial (P_w^* - P_p^*)} = a_{ew} / (a_p - b) \quad (3.40)$$

Thus (3.38) may finally be expressed as

$$U_w = U_w^* + D_w (P'_w - P'_p)$$

similarly,

$$\begin{aligned} U_e &= U_e^* + D_e (P'_p - P'_e) \\ V_s &= V_s^* + D_s (P'_s - P'_p) \\ V_n &= V_n^* + D_n (P'_p - P'_n) \end{aligned} \quad (3.41)$$

(iii) Substitution of (3.41) into (3.37) yields the following form of the continuity equation which has the pressure correction P' as its

potential, and is therefore called "pressure-correction" equation:

$$(a_p - b)P'_p = \sum_n a_n P'_n + M_p + C \quad (3.42)$$

where $a_p = \sum_n a_n$, $\sum_n$ = summation over neighbours (N, S, E, W)

$$a_W = \rho_w D_w A_w, \quad a_E = \rho_e D_e A_e, \quad \text{etc.} \quad (3.43)$$

$$M_p = (\rho U^* A)_w - (\rho U^* A)_e + (\rho V^* A)_s - (\rho V^* A)_n \quad (3.44)$$

Here M_p is the residual mass source associated with the first-stage velocities U^* , V^* . It should be noted that equation (3.42) is the finite difference form of a Poisson equation for pressure.

4. Correction of velocities and pressures, and final step

The next step is to correct the pressures and velocities so as to annihilate the residual mass source M_p defined by (3.44). The corrections are obtained by:

- (i) Solving equation (3.42) by the line procedure in the manner indicated in step 2 to obtain the entire field of pressure corrections, P' ;
- (ii) obtaining the velocity corrections U' and V' from the pressure corrections as defined by the second terms on the right-hand-side of (3.41).

Final Step

Finally, the new values of P , U , V are evaluated from expressions (3.36). This completes one "outer iteration" or simply "iteration" of the SIMPLE procedure.

The results thus obtained are tested for convergence⁺. If convergence is not attained the new values of U , V , and P are used as new "guesses", and the entire procedure is repeated from Step 2.

+ A converged solution is one in which each new "outer iteration" produces negligible difference in the values of the calculated variables, and each of the "residual sources" (see Section 3.51) is of the order of 10^{-3} times some reference value.

6. Solution of Other Scalar Variables

The new values of U and V produced at each iteration of the SIMPLE procedure are then used in the solution of equation (3.21) for each required scalar variable ($\tilde{h}$, m_j , K , and ϵ) for that iteration, using the line method.

3.44 The EMIT algorithm

(a) Introduction

The work on the plume in stably-stratified surroundings provided the motivation for seeking a new solution procedure for flows that may be elliptic in nature. As will be shown in Chapters 4 and 5, the flow in the initial 40% (approximately) of the full length of that plume is mainly parabolic⁺ in nature, although the remaining portion is fully elliptic. When the SIMPLE procedure is applied to such flows that have well-defined separate regions that are parabolic and elliptic in nature, it becomes uneconomical as it does not take advantage of the parabolic nature of part of the flow.

Nevertheless, owing to the elliptic nature of part of the flow, boundary layer procedures, such as that of Patankar and Spalding (1970), cannot yield satisfactory prediction of the entire flow, either. Similarly, although research has led to the adaptation of SIMPLE to semi-elliptic⁺⁺ flows (e.g. Pratap, 1975; Bergeles, 1976), the resultant is again inapplicable to the fully elliptic region of the plume in stably-

+ A flow field is classified as "parabolic" if (i) it has a single predominant direction of motion, (ii) shear stresses, heat fluxes, and mass-diffusion fluxes are significant only in directions perpendicular to the predominant direction, and (iii) downstream pressure field has little influence on upstream flow conditions. Such flows are governed by partial differential equations of the type known as "parabolic".

++ Semi-elliptic flows are those which are "parabolic" in nature except that, as in elliptic flows (see footnote on page 34), downstream pressure field significantly influences upstream flow conditions.

stratified surroundings.

Against this background efforts were made in the present study to devise a solution procedure that would cope with all regions of the plume in stably-stratified surroundings (i.e. one that will be valid for both parabolic and elliptic flows), but would be more economical than the SIMPLE procedure. The basic difference between SIMPLE and the new method is that it employs a "marching procedure" whereby the elliptic flow equations are solved completely along only one grid line or "plane" at a time, and then the entire grid lines are visited one after the other in an ordered sequence, which is repeated until a solution is obtained.

Although the method is presented here for two-dimensional flows in accordance with the flow situations pertaining to this study, it is also applicable to three-dimensional flow situations. The main features of the method are as follows:

- (i) It solves, along one plane after the other, the elliptic pressure-linked momentum equations by first estimating the velocities from a guessed pressure field, and then correcting both velocities and pressures with a view to satisfying continuity via the solution of a Poisson equation for "pressure-correction".
- (ii) The new pressures thus obtained along a given plane are then re-adjusted so that they are correctly linked to "upstream"* values, based on the requirement that the upstream velocities remain unchanged.
- (iii) Steps (i) and (ii) are applied at each plane of the flow domain in an ordered sequence until a complete "pass" is made through the

* Here the word "upstream" is used with respect to the direction of marching, and not the direction of flow. This special notion (including the counter-part term "downstream") will be upheld in the discussion of the EMIT algorithm.

domain; because the procedure does not incorporate boundary layer features, it is applied repeatedly (i.e. several "passes" are made through the flow domain) until the desired solution is obtained.

This procedure is here designated EMIT (= Elliptic Marching Integration Technique). A detailed outline of the method as well as various ways of obtaining faster convergence rate (see footnote on page 55) in its application is given under sub-sections (b) and (c) below.

(b) Description of the EMIT algorithm

1. The grid: A "staggered" grid system (see Fig. 3.5) similar to that of Fig. 3.2 is employed. However, in contrast to Fig. 3.2, the boomerang-shaped cluster of storage locations for variables solved for at each grid node N, S, E, W, or P is forward oriented with respect to the direction of marching. This does not alter the forms of the derived finite-difference equations.

2. Initial guess of variable fields:

Fig. 3.6 shows the main control volume and the triads of variables for the calculation plane II, the velocities and pressures of the triads of the adjacent upstream plane I are assumed to be "known", either from boundary conditions, or from the most recent calculation for that plane. Values associated with triads of grid nodes on the adjacent downstream plane III and the calculation plane II are guessed.

3. First improvement of velocity field:

A first estimate of the velocities of the calculation plane II is then obtained by assembling the coefficients of the momentum equations and solving for new values of U^* and V^* along the plane by means of the TDMA, using the initial guessed pressures and velocities;

e.g.
$$(a_p - b)U_E^* = \sum_n a_n U_n^* + A_{ew} (P_P^* - P_E^*) + C \quad (3.45)$$

4. The "pressure-correction" equation:

The continuity equation is integrated in the manner of equation (3.37) for each main cell along the calculation plane. The velocity corrections (U' , V') are connected to pressure corrections P' as shown in expressions (3.38) to (3.41), but now with the following important modifications:

- o The pressures on the adjacent upstream plane are temporarily taken as known, thus:

$$P'_W = 0 \quad (3.46)$$

- o Further, it is also assumed that the pressures on the adjacent downstream plane are also temporarily "known". Thus

$$P'_E = 0 \quad (3.47)$$

This assumption evidently introduces some error until convergence (see footnote on page 55) is nearly or fully attained; but it is the price to be paid for adopting a marching procedure, since conditions at the downstream plane are unknown. The effect of this approximation is to produce incorrect pressures and U_E 's along the calculation plane, but this is largely off-set by measures described below (see equations (3.49) and (3.51)).

- o The corrected velocities along the calculation plane now become

$$\begin{aligned} U_e &= U_e^* + D_e \cdot P_P' \\ V_s &= V_s^* + D_s \cdot (P'_s - P'_P) \\ V_n &= V_n^* + D_n \cdot (P'_P - P'_N) \end{aligned} \quad (3.48)$$

Note: U'_W is zero since U_W is taken to be correctly defined by boundary conditions or from most recent calculation for the upstream plane.

Introduction of expressions (3.48) into (3.37) yields again the "pressure-correction" equation as given by equation (3.42), with due

regard to the modifications introduced by (3.46) and (3.47),

Although U_w is not corrected, a_w (see equation (3.43)) is not set to zero in the pressure-correction equation. Thus,

$$a_w \neq 0 \quad (3.49)$$

This has a stabilizing effect by reducing the level of pressure-corrections P'_p given by equation (3.42); and thus helps to counteract errors introduced by assumption (3.47). Physically, a change in pressure at P transmits itself in all directions; however, U_w is not adjusted because it is desired not to destroy the momentum and continuity balance existing at the upstream plane.

5. Correction of velocities and pressures:

The TDMA is employed to solve equation (3.42) to give the set of P' for the calculation plane. The velocity corrections are then obtained from the P' , thus

$$\begin{aligned} U'_e &= D_e \cdot P'_p \\ V'_s &= D_s (P'_s - P'_p) \\ V'_n &= D_n (P'_p - P'_N) \end{aligned} \quad (3.50)$$

Subsequently, the corrected velocities and pressures (U , V , and P) for the calculation plane are derived in the manner of expressions (3.36).

6. Linkage of pressure to values on upstream plane:

In view of expressions (3.46) and (3.47), the calculated pressures will not in general accord with the known velocities and pressures (U_w , P_w) of the upstream plane. The P'_p 's are therefore re-adjusted with reference to values on the upstream plane through consideration of the finite-difference momentum equation for U_w .

Thus, for a specific node P on the calculation plane, the adjustment δp which will yield correct pressure level P_p with respect to U_w and P_w will be:

$$\delta p \equiv P_p - P_w = - \left\{ \frac{(a_p - b) U_w - \sum_n a_n U_n - C}{A_{ew}} \right\} \quad (3.51)$$

This block adjustment δ_p is made to all pressures on the calculation plane. This completes the EMIT procedure for that plane.

7. Solution of other scalar variables:

When the EMIT procedure is completed at each plane of calculation, the new values of U and V are then used for the solution of equation (3.21) for that plane for each required scalar variable ($\tilde{h}$, m_j , K, and ϵ), using the TDMA.

8. Forward Step

On completion of the above steps, the entire process starting from step 2 is next applied to the adjacent and subsequent downstream planes in an ordered sequence until the end of the solution domain is reached. This is termed a "pass" through the solution domain, and more than one such "pass" is generally required to give a converged solution.

At the end of each "pass" the new field of U, V, P and other scalar variables is tested for convergence. If this is not attained, the new field is used as initial guess for the next "pass".

(c) Some improvements to the EMIT procedure

While the stability and convergence of the SIMPLE and EMIT algorithms are discussed in the next section, it is useful to describe here some measures that can be used to promote faster convergence of the EMIT procedure:

1. First "pass" through the solution domain

During the first marching integration, the following measures may be fruitfully employed:

(i) To provide a better initial guess for the first "pass", of the variables to be computed at each plane, values obtained at the adjacent upstream plane may be projected forward one or two planes, depending on the variable.

(ii) With the good guesses of the velocities and pressures thus provided,

advantage may further be taken by not under-relaxing the calculated variables within the first "pass".

2. Block adjustment of Stream-wise velocities

The calculated U velocities at each plane may be further corrected by applying a uniform increment δU to them, with reference to the integral mass-balance across the plane. Thus,

$$U_j = U_j^0 + \delta U \quad (3.52)$$

$$\text{and,} \quad \delta U = (\bar{m} - \sum_{\text{all } j} \rho U_j^0 \delta A_j) / \sum_{\text{all } j} \rho \delta A_j \quad (3.53)$$

where U^0 is the velocity before application of the block adjustment, δA is the cell area perpendicular to the axial direction, the subscript j refers to the j -th cell on the calculation plane, and $\bar{m}$ is the total mass flow rate across the plane. This procedure can be applied to both free and confined flows. If applied, this modification is made immediately after the starred velocities (see equation 3.45) are obtained.

When modification 1. above is applied, the present modification is best started from the second "pass" through the solution domain.

3. Adjustment of downstream pressures:

After the pressures and velocities are obtained for a given plane a block adjustment of downstream pressures may be made to mirror the influence of the newly calculated velocities and pressures. In instances where the pressure increases downstream, this may procure a faster rate of build up of the pressures.

4. Plane iteration

The procedure applied at a plane may be successively applied more than once at a plane, before advancing to the next plane. This is termed "plane iteration".

3.5 OTHER SPECIAL FEATURES OF THE SOLUTION PROCEDURES

3.51 Convergence

Equation (3.21) may be re-written as

$$\phi_p = \sum_n \frac{a_n}{(a_p - b)} \cdot \phi_n + \frac{c}{a_p - b} \quad (3.54)$$

A general rule which ensures the convergence of a successive substitution method when applied to a set of equations of this type is the "diagonal dominance rule", sometimes called the "Scarborough criterion" (Scarborough, 1930; Forsythe and Wasow, 1960), which states that provided the equations are linear (i.e. a,b,c are constants) convergence is ensured if

$$\frac{\sum_n |a_n|}{|a_p - b|} \leq 1 \quad (3.55)$$

All the equations to be solved satisfy this condition since:

- a_n 's > 0
 - $a_p = \sum_n a_n$
 - b usually ≤ 0
- (3.56)

However, the coefficients are not constant in the present case: indeed they may sometimes depend directly on ϕ . Fortunately condition (3.55) is known to be over-stringent (it is a 'sufficient' rather than "necessary" condition) and when it is obeyed it is usually possible to successfully solve even the non-linear equations.

A test is necessary at the end of each "iteration" or "pass" to assess how nearly the current solution approximates to the exact solution of the finite-difference equations for each dependent variable. The method employed in this study is based on the "residual sources" (R's) of the difference equations, which for equation (3.21), for example, is defined by:

$$R_\phi = (a_p - b)\phi_p - \sum_n a_n \phi_n - c \quad (3.57)$$

If the current solution exactly satisfies the difference equation, R_ϕ will of course be zero. For an inexact solution, R_ϕ is a measure of the local error and it is required that:

$$\frac{\sum_{\text{all nodes}} |R_\phi|}{R_{\phi, \text{ref}}} < \lambda \quad (3.58)$$

where λ is a constant of order 10^{-3} , and $R_{\phi, \text{ref}}$ is an appropriately chosen reference flow quantity (typically the value of the relevant

extensive property at the entrance of the flow domain). This normalised residual-source test ensures that the finite-difference equations have been solved.

For the "pressure-correction" equation (3.42), the residual source R_ϕ is taken to be the mass source M_p defined by equation (3.44).

3.52 Numerical Stability

(a) Source and general effect of instability

Since successive substitution procedures do not effect direct simultaneous solution of the difference equations, they are generally prone to numerical instability. However, since it is known that iterative solution of a set of linear equations which satisfies the "diagonal dominance rule" will converge, instability must be a consequence of interlinkage between the equations.

Generally, the interlinkage between the difference equations give rise to numerical instabilities which result in the non-satisfaction of mass conservation on a local and over-all basis, and the oscillation rather than the convergence to constant values of the calculate values of the flow variables. Specifically, for buoyant flows there are three major causes:

- (i) the non-linearities and velocity and pressure-linkages of the momentum and continuity equations
- (ii) the buoyancy-linkage of the momentum and energy equations as well as the K and ϵ equations, and,
- (iii) the buoyancy-linkage of the turbulence viscosity μ_t through the buoyancy terms in the K - ϵ equations.

These three sources of instability occur independently, and their combined effect presents a major obstacle to obtaining solutions for conditions of buoyancy-influenced/flows at high Grashof numbers.

(b) Illustration of buoyancy-induced numerical instability

It is useful to examine the mechanism of numerical instability

in calculations of a buoyant flow. This may be done by perturbing a hydrodynamic and temperature field which satisfies the conservation equations exactly. One such field is that of a stably-stratified fluid confined between parallel, horizontal plates which are maintained at uniform and different temperatures, the cooler one being located at the bottom.

In order to illuminate the nature of the buoyancy-induced numerical instability we consider a one-dimensional analysis of the effects of a perturbation of the temperature at a single node P shown in Fig. 3.7(a) which also shows the unperturbed temperature distribution. Employing central differencing for the diffusive terms and "upwind" differencing for the convective terms, if the temperature T_p is perturbed by a small increment δT and all variables outside the perturbed cell are assumed to have their equilibrium (i.e. unperturbed) values, there results the following simplified forms of the energy and 'flow' (i.e. combined momentum and continuity) equations* (Caretto, Gosman and Spalding, 1971):

Energy:

$$\frac{T - T_{ref}}{T_u - T_{ref}} = \frac{|Pe|}{2 + |Pe|} \quad (3.59)$$

Flow:

$$\frac{T - T_{ref}}{T_u - T_{ref}} = \frac{4 + |Re|}{Pr \cdot Gr} \cdot Pe \quad (3.60)$$

and

$$Gr = \frac{\beta g h^3}{\nu^2} (T_u - T_{ref}) \quad (3.61)$$

where Pe , Re , Gr are the cell Peclet, Reynolds, and Grashof numbers respectively, T_{ref} is the unperturbed value of T_p , T_u is the "upwind" difference value of T_p (i.e. T_N or T_S), and h is the cell dimension.

The exact analytical solution to the above equations gives zero

* See also Appendix D.

flow with $T = T_{\text{ref}}$. The behaviour of the successive iteration procedure is illustrated graphically in Fig. 3.7(b). The dashed lines marked "iteration path" represent successive relaxation of the flow equation (horizontal lines) and the energy equation (vertical lines): a convergent procedure should approach the intersection of two curves representing the flow and energy equations and thereby recover the physically-stable conditions.

However, as shown in Fig. 3.7(c) which illustrates the iteration paths for different levels of Grashof number, the physically-stable condition can generally be recovered for only small Grashof numbers, for which cases the angle between the two curves is less than 90° , while for very large Grashof numbers the procedure diverges and may ultimately oscillate between T_N and T_S in a 'flip-flop' manner.

Further, Fig. 3.7(d) illustrates the extreme case in which the fluid thermal conductivity is zero (i.e. $Pe = Pr = \infty$): there is then no diffusion and the smallest motion of the fluid caused by the perturbation of T_p results straight away in the adjusted value oscillating between T_S and T_N owing to the "upwind" differencing.

Such instabilities occur in all flows arising from body forces (e.g. Gosman and Spalding, 1970; Gosman et al, 1975; Crossley, 1975). In particular, Crossley developed a point iteration method which cures the instability at even very high Grashof numbers (up to 10^{25} for the stably stratified case illustrated in Fig. 3.7a). However, the extension of this idea to line-iteration procedures is far from straightforward and could not be undertaken in the present study.

(c) Remedy for numerical instabilities

Whatever the nature of the numerical instability, experience shows that one or more of the following measures provides at least a partial remedy, although they provide a rather limited success in flows dominated by buoyancy:

(i) Use of appropriate method of under-relaxation; this has the effect of reducing the magnitude of current error levels in the field during the course of solution.

A relaxation factor f is employed, which, with reference to equation (3.21), is defined by

$$\phi_p^n = f\phi_p + (1 - f)\phi_p^o \quad (3.62)$$

where ϕ_p^o = value at the previous iteration

ϕ_p = new value for the current iteration but without relaxation.

ϕ_p^n = new value for the current iteration with relaxation.

$f > 1$ implies over-relaxation, and $0 < f < 1$ gives under-relaxation.

$f = 1$ indicates that the values of ϕ are not relaxed.

(ii) Increasing the number of "sweeps" of the tri-diagonal matrix algorithm: this has the effect of obtaining more complete solution of the equations during each iteration; the "pressure-correction" equation is the most sensitive in this respect (these comments apply only to the SIMPLE algorithm). For the EMIT algorithm, applying more than one iteration may be useful.

(iii) Specifying realistic initial fields of all variables, e.g. by using results of previous computations.

Further comments on these remedies as they apply to each study-case are made in the relevant section on computational details in Chapters 4 to 6.

An additional stability-procuring measure:

An additional precautionary measure is necessary to avoid the instability that may arise during the course of iteration on account of the mass flows not satisfying continuity as shown in Fig. 3.4(b). In such a case, all the finite-difference coefficients a_n 's become zero when the flows are strongly outwards-directed. The finite-difference

equations may then become singular, since $a_p = \sum_n a_n$ ($n = N, S, E, W$).

The remedy is to introduce a false mass source in the manner of (3.20); thus with respect to equation (3.21),

$$-\iiint_V S_{\text{false}} dV = b_f \phi_p + C_f \quad (3.63)$$

where

$$b_f = -|\dot{m}_{\text{net}}|$$

$$C_f = |\dot{m}_{\text{net}}| \phi_p^0$$

$$\dot{m}_{\text{net}} = \sum_n \dot{m}_n \quad (n = n, s, e, w) \quad (3.64)$$

$$\dot{m}_w = \rho_w U_w A_w, \text{ etc.}$$

$$\phi_p^0 = \phi_p \text{ of previous iteration}$$

Then, for example, equation (3.21) becomes

$$(a_p - b - b_f) \phi_p = \sum_n a_n \phi_n + C + C_f \quad (3.65)$$

and the possibility of a zero coefficient multiplying ϕ_p no longer arises. This false source has no effect on the final solution, for then $\phi_p^0 = \phi_p$ and $\dot{m}_{\text{net}} = 0$.

3.53 Accuracy of the Solution

The accuracy of the solution obtained depends on the following factors:

- (i) the degree to which the solution satisfies the finite-difference equations: this is measured by both the level of the residual sources defined by (3.46) and the relative change in magnitude of computed variables between successive iterations;
- (ii) the degree to which the solution satisfies the partial-differential equations: this is assessed by 'refining' the grid; that is for each flow configuration a grid-independent solution must be sought by increasing the number of grid lines until no significant changes are observed in the solution with further refinement.
- (iii) the location of, and conditions imposed at, boundaries: this is assessed by adjusting the boundary conditions as well as the location

of the boundaries, when the latter are not fixed by the physical constraints of the problem considered.

(iv) adequacy of the turbulence model: given a fully converged, grid-independent solution with satisfactory treatment of boundaries, the degree to which the predictions reflect reality depends on the adequacy of the turbulence model; this is assessed by comparison with experimental data.

It may be noted that while errors due to factors (i), (ii) and (iii) are mainly or solely computational in origin and can usually be eliminated, those due to factor (iv) are solely outside the realm of the numerical method and their elimination requires development of better turbulence models.

3.54 The Computer Program

The finite-difference equations and SIMPLE algorithm form the basis of the computer program called TEACH-2E(= Teaching Elliptic Axi-symmetric Characteristics Heuristically), which is described by Gosman and Ideriah (1977). Versions of TEACH-2E were developed for the various flows considered in this study, and the same program was restructured to enable the EMIT algorithm to be developed and tested.

3.6 SUMMARY

This chapter has described with the main features of the mathematical problem and solution procedures employed in this study. They may be summarized as follows:

(i) In the study, Eulerian co-ordinates are employed for the dependent variables, while "primitive" variables (i.e. U , V , P) are used as the hydrodynamic dependent ones. Further, the partial differential equations governing the fluid flow are reduced to their algebraic equivalents by finite-difference formulations: in doing this, the differential equations are discretized to obtain their finite difference forms

through a "micro-integral" approach, together with a "hybrid" difference scheme (combination of central and "upwind" differencing) for the convective and diffusive fluxes.

(ii) The difference equations are solved by two iteration methods involving guess-and-correct procedures and a line method. The special features of the procedures are the SIMPLE and EMIT algorithms which are employed to solve the pressure-linked momentum equations.

Both the SIMPLE and EMIT procedures solve the elliptic forms of the governing equations, but SIMPLE solves at each time the difference equations for all grid nodes in the computation field simultaneously (an "outer iteration"), while EMIT solves the equations only for grid nodes along one plane at a time and then "marches" to the next plane in an ordered sequence until all the planes are visited in the computation field (a "pass"). Both procedures require repeated "iterations"/"passes" to obtain the desired solution.

(iii) Some sources and effects of the numerical instability involved in solving the algebraic equations by successive substitution methods have been indicated, and the general manner of partly over-coming such instabilities has also been indicated as well as the way in which accurate solutions may be obtained.

PART III:

THE INVESTIGATION OF BUOYANT PLUMES

CHAPTER 4

CONVENTIONAL BUOYANT PLUMES

4.1 INTRODUCTION

This chapter presents the work carried out in this study on conventional plumes (i.e. plumes without radioactive heating). The starting point is Section 4.2; there are discussed the previous theoretical and experimental studies on plumes in both uniform and stably stratified surroundings. Section 4.3 gives the details of the present computations of plumes, including the method of application of both the solution procedures (SIMPLE and EMIT) and the turbulence model. Also given here is an assessment of which of the two solution procedures is more efficient in the plume application.

The presentation and discussion of the results is made in Section 4.4; results obtained for the axisymmetric momentum jet, and buoyant plumes in both uniform and stably stratified surroundings are there presented. Finally, conclusions are drawn in Section 4.5.

4.2 PREVIOUS WORK

2.1 Previous Theoretical Work

(i) Analytical Studies

Previous prediction methods for axisymmetric turbulent buoyant plumes discharged into calm surroundings have been based mainly on the integral approach. Using the independent variable η ($= r/b$), and assuming the following Gaussian profiles for the dependent variables U and $\Delta\rho$

$$U/U_m = e^{-\alpha\eta^2} \quad (4.1)$$

$$\Delta\rho/\Delta\rho_m = e^{-\gamma\eta^2} \quad (4.2)$$

and a mixing length closure for the turbulent shear stress $\overline{uv}$ of the form

$$\overline{uv} = nU_m^2 f(\eta) , \quad (4.3)$$

previous investigators reduced the boundary layer forms of the equations of mean motion to their following integral equivalents:

Conservation of mass:

$$\frac{d}{dx} (b^2 U_m) = 2\alpha E \quad (4.4)$$

Conservation of vertical momentum:

$$\frac{d}{dx} (b^2 U_m^2) = \frac{-2\alpha}{\gamma} b^2 g \frac{\Delta \rho_m}{\rho_0} \quad (4.5)$$

Conservation of thermal energy:

$$\frac{d}{dx} (b^2 U_m \Delta \rho_m) = -\frac{\alpha+\gamma}{\alpha} b^2 U_m \frac{d\rho_\infty}{dx} \quad (4.6)$$

Conservation of Kinetic energy of mean motion:

$$\frac{d}{dx} (b^2 U_m^3) = -cbU_m^3 - \frac{6\alpha}{\alpha+\gamma} b^2 g U_m \frac{\Delta \rho_m}{\rho_0} \quad (4.7)$$

where suffices 0, m, ∞ stand for values at the inlet, centre-line, and infinity respectively, r is the radial distance, b the local width of the plume, U and $\Delta \rho$ the vertical velocity and excess density, E is the entrainment rate, α and γ are shape constants, n is a constant, and c is the plume spreading coefficient given by

$$C = 24n\alpha^2 \int_0^\infty \eta^2 f e^{-\alpha\eta^2} d\eta \quad (4.8)$$

(Equation (4.6) assumes a linear relationship between ρ and T .)

The previous investigators used different combinations of equations (4.4) to (4.7), but their approaches can be grouped into three main categories.

The first category which includes work by Rouse et al (1952), Priestley and Ball (1955), and Schmidt (1957) uses equations (4.5), (4.6), and (4.7) for the conservation of momentum, thermal energy, and kinetic energy of mean motion respectively. The use of this set of conservation equations requires closure through equation (4.3), which introduces the unknown plume spreading coefficient C that needs to be specified.

Generally, a linear spread is assumed, with C taken as a known constant in all forms of stratification. While investigators in this group omitted equation (4.4) and thereby did not need to specify the entrainment rate E , their specification of plume spread is equivalent to assumptions about entrainment.

Investigators in the second category, including Morton et al (1956), Morton (1959a, 1959b), Lee and Emmons (1961), Hino (1962), and Abraham and Eysink (1969), employed equations (4.4), (4.5), and (4.6) for the conservation of mass, momentum, and thermal energy. In the use of equation (4.4), an assumption is necessary about the entrainment rate E (i.e. the rate of inflow per unit height of ambient fluid into the plume).

The various investigators named above specified E as

$$E = aV_e b \quad (4.9)$$

where a is a constant and V_e is the entrainment velocity which, based on the now-classical entrainment assumption of Taylor (1958), is given by

$$V_e = \alpha_e U_m \quad (4.10)$$

with U_m as the centre-line velocity and α_e as an entrainment constant. α_e is assumed to be a 'universal'* constant and values for which have been obtained from experiments by various workers on momentum jets & buoyant plumes as given in Table 2. These results indicate that α_e is higher for buoyant plumes than for momentum jets. Morton (1959a, 1959b) used a value of α_e of 0.116 for plumes, and the other investigators used similar values.

* in reality α_e is not truly a universal constant.

	TURBULENT ROUND JET		TURBULENT ROUND PLUME				
	Ricou & Spalding	Hill	Rouse et al	Morton et al	Schmidt	Nakagome & Hirata	George et al
	(1961)	(1972)	(1952)	(1956)	(1941)	(1976)	(1976)
<u>Calculated by:</u>							
Turner(1973)	0.08	-	0.12	-	-	-	-
Morton et al (1956)	-	-	0.085	0.093	0.125	-	-
Middleton (1975)	0.057	-	-	-	-	-	-
Present author ⁺⁺	0.08	0.08	0.0833	-	-	0.11	0.10

Table 2: Values of entrainment constant α_e as calculated by Turner (1973), Morton et al (1956), Middleton (1975), and present author using the experimental data of the authors indicated

⁺⁺ Calculations based on equation (4.48).

Investigators in the third category, pioneered by Fox (1970) and subsequently followed by Hirst (1971), used a procedure which combined the approaches of the first two categories: they solved all the integrated equations (4.4) to (4.7) for the conservation of mass, momentum, thermal energy, and kinetic energy of mean motion. Essentially, their method involves satisfying the continuity equation with a closure that requires the conservation of kinetic energy of mean motion, resulting in an entrainment rate E that is a function of the Reynolds stress, the form of the similarity profiles, and the local densimetric Froude number, i.e.:

$$E = (\beta_1 + \beta_2 Fr^2) U_m b \quad (4.11)$$

Or, with reference to equations (4.9) and (4.10), the entrainment velocity V_e is

$$V_e = (\beta_1 + \beta_2 Fr^{-2}) U_m \quad (4.12)$$

where β_1 and β_2 are parameters based on the assumed velocity and shear stress distributions (equations (4.1) and (4.3)) and Fr is the local densimetric Froude number, defined as:

$$Fr = U^2 / \left\{ \frac{g(\rho_\infty - \rho)}{\rho_\infty} d \right\} \quad (4.13)$$

with d being the diameter of the nozzle, g the gravitational constant, ρ the density and subscripts ∞ and 0 stand for values at infinity and nozzle-exit level.

The form of equation (4.12) indicates that the entrainment coefficient $(\beta_1 + \beta_2 Fr^{-2})$ increases as the Froude number decreases, and therefore helps to elucidate the need for different values of α_e in equation (4.10) for high and low Froude number flows (momentum jets and plumes respectively) as is indicated by Table 2.

A major defect of all the integral methods is the need for prescription of velocity and buoyancy profiles which are, indeed, major unknowns and are aspects of the flow for which predictions are required. Hence the integral analysis carried out by all the previous workers have suffered from the unrealistic assumptions of a Gaussian profile at all heights of a plume in stable ambient stratification.

Furthermore, the use of integral methods also requires the prescription of either the entrainment rate or the rate of spread of the plume, and these are additional important unknowns. The assumption of universal entrainment constants (or, indeed, the variable entrainment coefficient of Fox, 1970) has inherent errors especially in the region of reversed buoyancy for the plume in stably-stratified surroundings. The actual entrainment behaviour in turbulent plumes is a very complex phenomena, driven by the intermittency of the turbulence in the plume (Townsend, 1966) and cannot be prescribed with sufficient accuracy by simple relationships. Indeed, Telford (1966) has suggested a form of V_e (see equation (4.10))

that depends on the level of turbulence:

$$v_e = a_1 \left(\frac{\rho_\infty}{\rho} \overline{u^2} - \overline{u_\infty^2} \right) \quad (4.14)$$

where $\overline{u^2}$ and $\overline{u_\infty^2}$ are the rms turbulent velocities inside and outside the plume, and a_1 is a mixing constant.

Moreover, the assumption by some investigators of a linear spread of the plume with distance from the orifice in the zone where the plume fluid spreads out horizontally for the stably-stratified case is also unrealistic.

Despite the various short-comings of the integral approaches, they yield results that, in most cases, are in reasonable agreement with experimental data: in some cases the predicted height of the plume in stably-stratified surroundings was within 2% of the measured height. Nevertheless integral methods are valid for a restricted set of conditions; for example, a change in the boundary conditions often needs a change in the coefficients. Further, information produced by such methods is rather limited on matters like the turbulence structure of the flow. However, the design engineer needs a general method of solving the full governing equations, which will be applicable to arbitrary initial and boundary conditions and also capable of yielding more detailed information on the structure of the flow.

(ii) Numerical Studies

Recently, in two independent and parallel studies, Chen and Rodi (1975) and Madni and Pletcher (1975) employed finite-difference schemes for the prediction of plumes in both uniform and stably-stratified surroundings.

Chen and Rodi used the finite difference scheme of Patankar and Spalding (1970), and, following Launder (1975), they obtained closure from simplified forms of the transport equations (neglecting convective and diffusive transport) for the turbulent kinematic shear stress $\overline{uv}$ and heat fluxes $\overline{vT}$ and $\overline{uT}$:

$$\overline{uv} = -\frac{k}{\epsilon} (a_1 \overline{v^2} \frac{\partial U}{\partial r} + a_2 g \beta \overline{v T'}) \quad (4.15)$$

$$\overline{v T'} = -a_3 \frac{k}{3\epsilon} \overline{v^2} \frac{\partial T}{\partial r} \quad (4.16)$$

$$\overline{u T'} = -a_3 \frac{k}{3\epsilon} (\overline{uv} \frac{\partial T}{\partial r} + a_4 \frac{\partial U}{\partial r} - a_5 g \beta \overline{T'^2}) \quad (4.17)$$

where the a's are constants, g is the gravitational constant, β the coefficient of cubical expansion, and T' is the temperature fluctuation. $\overline{v^2}$ is taken as 0.53K, and the variables k, ϵ , and $\overline{T'^2}$ are obtained from the modelled forms of their respective transport equations. (See Appendix A).

However, the solution procedure employed by Chen and Rodi is of the boundary-layer type, and is therefore strictly inapplicable in the region of reversed flow for the plume in stably-stratified surroundings, nor is it applicable when axial diffusion is significant. Their procedure therefore fails within some 60% way to the maximum height of the plume in stably-stratified surroundings, beyond which point the flow becomes fully elliptic. Their predictions will be shown later.

Madni and Pletcher (1975) used an explicit finite-difference scheme of the DuFort-Frankel type to solve the boundary layer equations, together with Prandtl's mixing length hypothesis employed for the turbulent fluxes. Again, their procedure being of a boundary-layer type suffers from the same defects as does that of Chen and Rodi for the plume in stably-stratified surroundings. However, for the plume in uniform surroundings, their predictions were generally in satisfactory agreement with experimental data.

4.22 Previous Experimental Work

(i) Plumes in Uniform Surroundings

Early experimental work on plumes in uniform surroundings have been reported by Schmidt (1941), Yih (1951), Rouse et al (1952), Railstone (1954), Abraham, (1960) and Lee and Emmons (1961). The data of these investigators established that the profiles of velocity and buoyancy are approximately Gaussian.

Despite advances of the last two decades in flow measurement techniques, the data of Rouse et al remained until recently the basis for testing theoretical models (see, for example, Turner 1973, Tennekes and Lumley 1973) as they provided clearer measurements of velocity and buoyancy profiles. Their measurements made at a variety of heights for a variety of source strengths are well fitted by the following Gaussian functions for the radial velocity and buoyancy (effectively temperature) profiles:

$$U/(F_w/\rho_0 X)^{1/3} = 4.7 \exp(-96r^2/X^2) \quad (4.18)$$

$$-\Delta\gamma/(\rho_0 F_w^2/X^5)^{1/3} = 11.0 \exp(-71r^2/X^2) \quad (4.19)$$

where, subscript 0 stands for the source

X = distance from the source

r = radial distance from the axis

$\Delta\gamma = -g(\rho_\infty - \rho)$

= local change in weight density (4.20)

$$F_w = \rho_{\infty,0} F_0 \quad (4.21)$$

= total weight deficiency per unit time at the source

and

$$F_0 = 2\pi \int_0^R \frac{U_0 g (\rho_\infty - \rho)}{\rho_{\infty,0}} r dr \quad (4.22)$$

= buoyancy "flux" at the source.

It is noteworthy that the measurement of Schmidt (1941), among the earlier works, gave at a given axial location a wider spread of velocity than of buoyancy; but the measurement of Rouse et al indicate the converse.

Recently, the measurement of George et al (1976) and Nakagome and Hirata (1976) have provided a clearer picture than had hitherto been available of axi-symmetric buoyant plumes. Both have provided well-documented data on mean and fluctuating quantities in a turbulent plume of rising hot air. The measurements of Nakagome, made for the circumstances of a pure plume ($Fr_0 = 0$), show that the flow is accelerated

by buoyancy forces from the source to a point about 4 diameters downstream, and then decelerated by the actions of turbulent shear. On the other hand, the temperature decreases rapidly and continuously from the source.

Beyond some 4 diameters downstream, the measured centre-line velocities and excess temperatures of Nakagome and Hirata are well correlated by:

$$\frac{U_c}{U_{\max}} = 3.29 (x + C)^{-1/3} \quad (4.23)$$

$$\frac{\theta_c}{\theta_{\max}} = 13.5 (x + C)^{-5/3} \quad (4.24)$$

where θ is the excess temperature ($T - T_\infty$), x is the distance from the source, C is a constant, the suffix c stands for value at centre-line, and suffix $\max$ stands for maximum value. Their measurements also show similarity in profiles of both mean and fluctuating quantities beyond some 4 diameters downstream, while the work of George et al on slightly forced plumes ($Fr_0 = 1.6$) indicates similarity in profiles which are obtained beyond some 8 diameters downstream of the source.

The measured mean buoyancy profiles of George et al show good agreement with the Gaussian profiles obtained by Rouse et al (1952). However, contrary to the results of Rouse et al, their velocity profile is broader than the buoyancy profile. Similarly, the work of Nakagome and Hirata establishes broader profiles of momentum than of buoyancy.

(ii) Plumes in Stably-Stratified Surroundings

All the published experiments on plumes in a linearly, stably-stratified environment have been concerned mainly with determination of the maximum heights attained by the plumes.

Crawford and Leonard (1962) are apparently the only investigators who reported measurements on nearly pure plumes ($Fr_0 \approx 0$) in stably-stratified surroundings. Their rise-height data, obtained for the rise

of air above a heated source in nearly linearly stably-stratified environment, are satisfactorily correlated by the relation⁺:

$$H_{\max} = n F_0^{1/4} G^{-3/8} \quad (4.25)$$

where

$$G = \frac{g}{\rho_{\infty,0}} \frac{d\rho_{\infty}}{dx} \quad (4.26)$$

$H_{\max}$ is the maximum height of the plume, and n is a constant having a value of about 4.0.

Experimental studies on forced plumes ($Fr_0 > 0$) in linearly stably-stratified surroundings have been reported by Morton et al (1956), Fan (1967), Abraham and Eysink (1969), and Fox (1970). These experiments were all conducted by injecting water jets into brine solutions. The measured heights are all satisfactorily given by relation (4.25), with however the 'constant' n varying from one investigation to the other. The data of Morton et al are fitted satisfactorily by $n \approx 3.2$, while those of Abraham and Eysink are fitted by $n \approx 4.5$. The variation in the value of n may be due to the departure from the initial value during the various experiments of the ambient stratification.

More recent work on forced plumes in stably-stratified surroundings has been reported by Sneek and Brown (1974) who experimented with hot air jets. Again, the measured height conforms with expression (4.25), with $n \approx 3.6$. Measurements of the centre-line temperature are also provided, and indicate that within the region of reversed flow the temperature distribution is not uniform but nearly linear with a slightly smaller slope than that of the surroundings. However, as was the case with the experiment of Abraham and Eysink, the ambient temperature stratification changed during the course of the measurements.

⁺ Equation (4.25) can also be derived from dimensional analysis as shown by Briggs (1965)

4.23 Status of Existing Information

The status of existing information, both theoretical and experimental, on turbulent axi-symmetric buoyant plumes which are discharged into calm, uniform or stably-stratified surroundings may be summarised as follows:

- 1) While turbulent axi-symmetric plumes have been the subject of numerous theoretical investigations in the past, the analytical or numerical studies have been rather restrictive in one form or the other. There is therefore the need to establish a prediction method which is less restrictive in nature.
- 2) For the plume in uniform surroundings, recently-available experimental data on both mean flow and turbulence fields have enabled testing of both numerical procedures and turbulence models. However, more complete data on the turbulence field are required for a fuller understanding of the turbulence structure in buoyant plumes.
- 3) For the plume in stably stratified surroundings, experimental data on both the maximum height and the decay of centre-line temperatures provide some valuable information for the testing of numerical procedures and turbulence models. But more complete measurements of the mean flow and the turbulence field (of which no data exists at present) are required.

4.3 COMPUTATIONAL DETAILS

4.31 Application of the Solution Procedure

(i) Inlet Conditions: The flow is assumed to enter the domain of computation with uniform streamwise velocity U_0 , and zero cross-stream velocity V_0 . The distributions of the other mean flow variables ρ_0 , h_0 , and $m_{j,0}$ are also assumed to be uniform at the inlet. These conditions are believed to be consistent with those of the experimental situations which are modelled in the computations.

The levels of turbulence energy K_0 and its dissipation rate ϵ_0 at the nozzle exit are rather important parameters which, for a given specification of mean flow variables at the exit, may strongly influence the flow characteristics in the initial region of the flow. This situation arises because these two parameters determine the effective viscosity $\mu_{t,0}$ at the exit through the relation.

$$\mu_{t,0} = C_{\mu} \rho_0 K_0^{3/2} / \epsilon_0 \quad (4.27)$$

and, for a given nozzle configuration, different values of $\mu_{t,0}$ will initiate different rates of mixing of the jet fluid with the ambient fluid. Generally, experiments do not provide estimates of $\mu_{t,0}$: the best information available (e.g. Bradshaw et al, 1964; Bradbury, 1965; George et al, 1976) is simply the indication that at the nozzle exits of turbulent jets and plumes there exists a potential core, thus implying that $\mu_{t,0}$ may be far less than the laminar viscosity μ .

However, it is not sufficient to know that $\mu_{t,0}$ is small; its precise value is required as this indirectly fixes ϵ_0 for any given value of K_0 : since consequent values of ϵ in the initial region will there determine the level of K , the resultant development of the turbulence (and thus the mixing rate) of the jet depends on this value of ϵ_0 .

Faced with this problem, Launder et al (1972) in their work on free-shear flows sought detailed initial profiles of all dependent variables, but had only sufficient information on the mean variables and, in a few of the study cases, K as well. They therefore adopted a trial-and-error method whereby, for a given value of K_0 , trial calculations were made based on estimating ϵ_0 from a guessed constant effective viscosity $\mu_{t,0}$, and then adjusting the value of the constant until the predicted development in the initial region of the jet agreed with the measurements. Similarly, in his work on buoyant jets, McGuirk

(1975) used a method whereby, for a given value of K_0 , interest was centred rather on ϵ_0 than on the overall value of $\mu_{t,0}$: ϵ_0 was expressed as $K_0^{3/2}/cd$, where d is the diameter of the nozzle, and c is a constant; various values of c were then prescribed and trial calculations made until the predictions accorded with experiment.

In this study an approach is adopted that is akin to the methods used by Launder et al and McGuirk. K_0 and ϵ_0 are expressed as

$$K_0 = 3/2 i^2 U_0^2 \quad (4.28)$$

$$\epsilon_0 = \frac{C_D K_0^{3/2}}{l_\epsilon} \quad (4.29)$$

where U_0 is the mean streamwise velocity, i is the nozzle exit turbulence intensity, l_ϵ is the dissipation length scale, and C_D (≈ 1.0) is a constant. It is then necessary to prescribe values for i and l_ϵ .

Here attention will be concentrated on nozzles of smooth contraction, with uniform-velocity low-turbulence flow at the exit. Based on well-documented experimental data on such nozzle exit conditions (see, for example, Crow and Champagne, 1971, and Planchon, 1974, for momentum jets; and George et al, 1976, for buoyant plumes) the intensity i is taken as 0.1%.

Attention is then focused on the dissipation length scale l_ϵ . At the exit of a nozzle l_ϵ may be governed by the following important factors, among others (see also Appendix F):

- i the geometry (i.e. shape and size) of the nozzle and upstream grids, setting screens, etc.
- ii the turbulence intensity, i
- iii the nature of the boundary layers on the jet nozzle: these are controlled mainly by the Reynolds number and temperature of the flow (Bradshaw, 1966)

It is assumed that l_ϵ may be expressed as

$$l_e = \lambda L \quad (4.30)$$

where L is a linear dimension of the exit (i.e. the diameter of the nozzle), and λ is an empirical function which is assumed to have the following functional dependence:

$$\lambda = \lambda(i, Re, \theta_0/\theta_r, \dots) \quad (4.31)$$

where $Re (= \rho_0 U_0 d / \mu)$ is the Reynolds number of the flow, θ^* is the temperature, and θ_r is some reference value. From computer optimisation with respect to the development of the mean and turbulence quantities based on the data of Bradshaw et al (1964), Sneck and Brown (1974), and George et al (1976), the following relationship for λ is found to give a satisfactory development of the initial region of both jets and plumes covered in this study (see also Appendix F):

$$\lambda = A_e Re^{1/8} i^3 \{ (\theta_0/\theta_r)^n \} \quad (4.32)$$

where $n = 3.104$, $\theta_r = 1.97 \times 10^2$, and $A_e = 1.765$ (for momentum jets) or 6.425×10^2 (for plumes). The higher value of A_e for plumes is necessary in order to initiate turbulence much closer to the nozzle as is revealed by experiments (e.g. Sneck and Brown, 1974; George et al, 1976). The influence of θ_0 on λ , as given by the formula, is negligible for values of $\theta_0 < 150^\circ\text{C}$.

Expression (4.32) is employed in all computation of turbulent jets and plumes in this study. Its use eliminates the uneconomical method of trial-and-error runs for the simulation of each new set of experimental data.

(ii) Other Boundary Conditions: Fig. 4.1 (a) and (b) show the boundaries of the domain of computation for the plume in both uniform and stratified surroundings.

At the axis of symmetry, the desired zero normal gradient condition

* Taken, for convenience, relative to the ambient value.

given by expression (2.61) is obtained by simply setting to zero the relevant finite-difference coefficient, as explained Section 3.34. This applies to all the variables except V which is zero at the axis.

At the 'free' boundaries A and C (Fig. 4.1) where the ambient fluid is little disturbed by the plume, U and V are obtained from a complete momentum balance, with the external velocities set to zero, and the pressure as well as the temperature and species concentration taken to be those of the ambient fluid; further, K and ϵ are there assumed to be negligible. Tests on the influence of the location of boundary A shows that it does not affect the predictions provided it is located at distances greater than 9 diameters from the axis of symmetry for Fig. 4.1(a) and 1 diameter from the top of the plume for Fig. 4.1(b).

At the exit plane B (Fig. 4.1), the velocities are obtained from consideration of mass balance within the flow domain; in doing this the exit cross-stream velocities are set to zero, while the exit streamwise velocities are based on uniform increments on upstream values, with the total mass flow rate into the domain equated to that at the exit. For Fig. 4.1(b) the other variables are set at the ambient values at this plane, while for Fig. 4.1(a) their values there do not affect the computations owing to the boundary-layer nature of the flow.

4.32 Application of the Turbulence Model

For flows at high Reynolds number, the empirical functions in equations (2.26) and (2.27) tend to become constants (see for example, Launder and Spalding, 1974). Some previous workers, notably Rodi (1972), Launder et al (1972), Lockwood and Naguib (1974), and Morse (1977) have applied the non-buoyant version of the K - ϵ turbulence model to the particular circumstances of the free, turbulent, round jet. Table 3 shows a comparison of the values of the constants

used in the present study with those of these workers, together with the range of jet lengths covered.

The values used in the present study have been shown to give good predictions for plane flows (see, for example, Launder and Spalding, 1974). Rodi (1972) and Launder et al (1972) further modified C_μ and C_2 for the axi-symmetric jet; however, for the range of jet length covered in this investigation it is found, as did Lockwood and Naguib (1974), that better predictions are obtained without these additional modifications. Morse (1977) found that a value of C_1 of 1.44 is more appropriate in the initial region ($x/d < 15$) of the jet, while further downstream slightly higher values (ranging to a maximum of 1.56) are desirable, with the other constants left unaltered.

Table 3

The values of the constants of the K- ϵ model used by various workers.

	C_μ	C_D	C_1	C_2	σ_K	σ_ϵ	x/d (approx.)
1. Rodi (1972)	0.09+	1.0	1.43	1.92 ⁺	1.0	1.3	0-80
2. Launder et al (1972)	0.09 ⁺	1.0	1.43	1.92 ⁺	1.0	1.3	0-80
3. Lockwood and Naguib (1974)	0.09	1.0	1.44	1.89	1.0	1.3	0-75
4. Morse (1977)	0.09	1.0	1.44 to 1.56	1.92	1.0	1.3	0-80
5. Present study	0.09	1.0	1.44	1.92	1.0	1.3	0-20

⁺ Values further modified (see text)

Consideration of the characteristics of the initial region of the axi-symmetric jet issuing from a nozzle explains why constants (e.g. as established by Launder and Spalding, 1974) for plane flows are more appropriate for the correct prediction of that region. The eddies generated at the lips of the nozzle due to velocity discontinuity

between the jet and the surrounding fluid result in a region of intensive mixing, often called the mixing layer. This layer is nearly cylindrical, but it behaves very much like a plane mixing layer (Townsend 1956; Bradshaw et al 1964). The flow properties remain roughly the same as those of a plane mixing layer up to about 6.5 nozzle diameters from the exit plane; beyond this point there is a sudden departure from this behaviour as the influence of curvature becomes more pronounced, and the transition to "axi-symmetric jet" becomes rather rapid (Bradshaw et al 1964).

However, although owing to this rapid transition the mean flow variables become reasonably self-similar beyond some 8 to 10 nozzle diameters (Townsend, 1956), the influence of the quasi-plane mixing layer on the entire flow (i.e. mean and turbulence quantities) dies down very slowly as it actually requires a very long distance (up to about 70 diameters as determined by Wygnanski and Fielder, 1969) before it becomes truly fully-developed or self-preserving. In contrast, for a plane jet, Bradbury (1965) established that self-preservation of the entire flow is attained some 30 jet widths downstream of the nozzle.

Of particular interest in this study is the constant C_3 of equation (2.27), since there are appreciable temperature (or density) gradients in the direction of gravity in the flows considered and the buoyancy terms in the K- ϵ equations therefore play a significant role. In their study of horizontal free shear flows under significant buoyant influence, Gibson and Launder (1974) neglected the buoyancy term in the ϵ -equation, thereby implying $C_3 = 0$. The same approach was employed by Lockwood and Naguib (1974) for free, round, diffusion flames. On the other hand, Chen and Rodi (1975) used $C_3 = 1.45$. In this study C_3 is taken as 1.0 as this value does not only enhance numerical stability but also gives predictions that are in best agreement with experimental data (e.g. Sneck and Brown, 1974).

Finally, a uniform value of 0.7 is used as the turbulent Prandtl number for both jets and plumes, as this is a suitable average value for most free shear flows (e.g. Launder, 1976).

4.33 Other Computational Details

In all the plume calculations, a non-uniform grid is employed in the radial direction while a uniform grid is used in the axial direction. For most of the calculations, the overall length of the domain of computation is 18 nozzle diameters while the radius is 10 diameters. For this typical domain, a grid having 21 nodes in the axial direction and 19 in the radial direction (i.e. a 21 x 19 grid) is found to give a grid-independent solution, as higher grid numbers make negligible change in the predictions. Thus the 21 x 19 grid is used in the computations. Figs. 4.4(c) and 4.25 show the influence of the grid for the flow in uniform and stratified surroundings respectively. While Fig. 4.4(c) is for the momentum jet, similar influence is obtained for the plume in uniform surroundings.

Both the SIMPLE and EMIT algorithms are employed in the computation of the jets and plumes. For both algorithms, numerical stability is enhanced by the choice of appropriate relaxation factors f defined by equation (3.62). The following values of f are employed for both algorithms: 0.3 for the velocities U and V ; 1.0 for the pressure correction p' , temperature T , and concentration m_j ; and 0.5 for the turbulence energy K and dissipation rate ϵ as well as the turbulent viscosity μ_t .

It is found that for some of the flow variables the SIMPLE procedure also needed more than one "sweep" of the tri-diagonal matrix algorithm (TDMA) in order to ensure stable advancement of the calculations. A single sweep is found adequate for the velocities U and V , while 6 sweeps are necessary to ensure that reasonably correct values of P' are calculated and, thereby, continuity is satisfied; further, 6 sweeps are also necessary to speed convergence of the solu-

solution of all other scalar variables (T , m_j , K , and ϵ).

While for the EMIT algorithm the afore-mentioned choices of the relaxation factors f are found to be all that is basically necessary for stable advancement of the calculation, the application in the first "pass" of $f = 1$ for all variables (see Section 3.44C) as well as the block adjustment of stream-wise velocities defined by equations (3.52) and (3.53) is found to promote convergence for the case of momentum jet and plume in uniform surroundings. On the other hand, for the plume in stably-stratified surroundings, convergence is best attained without such additional modifications.

In all computations, a converged solution is defined as being reached when each new iteration (SIMPLE) or pass (EMIT) produces negligible difference from the previous one, with the residual sources (see equation 3.58) at least 10^{-3} times their respective reference values (for each variable, the reference value is taken to be the nozzle exit value of the corresponding extensive property). The SIMPLE algorithm takes about 150 iterations to give converged solutions for the momentum jet, about 250 for the plume in uniform surroundings, and about 380 for the plume in stably-stratified surroundings. The number of passes taken by the EMIT algorithm to give converged solution is some 15% less than the number of iterations required for SIMPLE.

All computations were carried out on a CDC 6600, and Table 4 shows a comparison of the machine time taken by each iteration/pass for both the SIMPLE and EMIT algorithms for the 21×19 grid. Overall, the EMIT algorithm procures some 22% savings in time,

Table 4

CDC 6600 Machine time for each iteration/pass of SIMPLE and EMIT algorithms

	SIMPLE (Seconds)	EMIT (Seconds)	Time ratio: EMIT/SIMPLE
1. Momentum Jet	0.83	0.647	.779
2. Plume in uniform surroundings	0.868	0.679	.782
3. Plume in stably-stratified surroundings	0.87	0.67	.77

4.4 PRESENTATION AND DISCUSSION OF RESULTS

4.41 The Plume in uniform surroundings

(i) The Momentum Jet

Although the K- ϵ turbulence model has been extensively tested by previous workers for the prediction of the axi-symmetric non-buoyant jets with the aid of boundary-layer solution procedures (e.g. Rodi, 1972; Launder et al, 1972; Lockwood and Naguib, 1974), it was decided as a preliminary phase of this study to also predict the momentum jet, since this flow has much in common with buoyant plumes, and the methods of solution differed from those used earlier.

Profiles:

Figs. 4.2, 4.3, and 4.4 show results of the application of the procedures⁺ to the momentum jet. Fig. 4.2 shows the radial distribution of the axial velocity within the mixing layer region, at 2 diameters downstream of the nozzle. The predictions are there compared with

⁺ In all flows where both the SIMPLE and EMIT procedures are employed, identical results are obtained.

the data of Bradshaw et al (1964), The agreement is very good, and both results indicate that the eddies of the mixing layer have not penetrated to the axis of the jet at 2 diameters downstream of the nozzle. Here the potential core extends to a radial distance of about $0.6r_0$, where r_0 is the radius of the nozzle.

Fig. 4.3 shows a comparison of the predicted axial (Fig. 4.3a) and radial (Fig. 4.3b) distributions of mean velocity and temperature with the data of Corrsin and Uberoi (1949) for both hot ($T_0/T_\infty = 2.03$, with $T_\infty = 290^\circ\text{K}$) and cold isothermal ($T_0/T_\infty = 1.05$) air jets issuing into calm surroundings. Also shown in Fig. 4.3(a) are the predictions of Lockwood and Naguib (1974) who used the boundary-layer procedure of Patankar and Spalding (1970). Generally, the agreement between the present predictions and experiment is very good. The over-all slightly higher trend of the predictions of Lockwood and Naguib may be due to initial conditions.

Fig. 4.4(a) & (b) show a comparison of the predicted axial and radial distributions respectively of the turbulence energy. Agreement between predictions and data is generally satisfactory; however, in the outer region of Fig. 4.4(b), and especially for the profile of $x/d = 4$, there is rather poor agreement with the data. Along the axis (Fig. 4.4a), both predictions and data show that the turbulence energy remains negligible up to some three diameters downstream, indicating the extension of the potential core up to that extent; indeed, it is not until some five diameters downstream that the turbulence energy increases considerably. On the other hand, and as would be expected, Fig. 4.4(b) shows that in the initial region of the jet a high level of turbulence energy is produced by shear within the mixing layer, and most of the energy resides quite near the level of the lips of the nozzle, $y/r_0 = 1$ (about two-thirds of the area under the curve lies between $y/r_0 = 0.5$ and 1.5). Moreover, the maximum turbulence energy occurs at $y/r_0 = 1$.

Rate of spread

Shown for the momentum jet in part of Table 5 are the predicted rates of spread of velocity and temperature, $dy_{0.5}/dx$ and $dy_{0.5T}/dx$ respectively, as well as the spreading ratio λ_r defined as

$$\lambda_r = \frac{dy_{0.5T}/dx}{dy_{0.5}/dx} \quad (4.33)$$

Here, $y_{0.5}$ and $y_{0.5T}$ are respectively the distances from the symmetry axis at which the velocity and temperature differences assume half their maximum values. Also shown for momentum jets in the table are the experimental measurements of $dy_{0.5}/dx$ of Rodi (1972) and $dy_{0.5T}/dx$ of Corrsin and Uberoi (1950), giving a value of λ_r of 1.244 and thus indicating a greater spread of energy than of momentum. The predictions of Chen and Rodi (1975) for $dy_{0.5}/dx$ is also shown. The latter and present predictions are both in very good agreement with the data.

Table 5

Rates of Spread: Momentum Jet and Plume in Uniform Surroundings

	MOMENTUM JET			BUOYANT PLUME		
	$\frac{dy_{0.5}}{dx}$	$\frac{dy_{0.5T}}{dx}$	λ_r	$\frac{dy_{0.5}}{dx}$	$\frac{dy_{0.5T}}{dx}$	λ_r
Experiment (Nakagome and Hirata, 1976, for plume)	0.086	0.107	1.244	0.133	0.105	0.789
Prediction (Chen and Rodi, 1975)	0.082	-	-	0.111	0.099	0.892
Present Prediction	0.085	0.107	1.26	0.12	0.105	0.875

(ii) The Buoyant Plume in Calm Uniform Surroundings

Computations for the plume in uniform surroundings were carried out for initial Froude numbers Fr_0 (see equation 4.13) ranging from 0.03 to 40. Typical characteristics of the plume are provided by the

well-documented experimental data on mean and turbulence quantities of George et al (1976) for an axi-symmetric plume generated by a hot air jet; of particular interest, therefore, are the computations made to simulate their experiment. The following parameters of the experiment were used for the computations; $d = 0.063\text{m}$, $U_0 = 0.67\text{ m/s}$, $T_0 = 573^\circ\text{K}$, $T_\infty = 302^\circ\text{K}$. These nozzle conditions correspond to a Froude number Fr_0 of 1.6 and a Reynolds number of 870 (for these conditions George et al reported turbulent behaviour for tests carried out beyond 2 diameters downstream). Generally, the predictions obtained are compared not only with the data of George et al but also with another recent set of well-documented data by Nakagome and Hirata⁺ (1976) for $Fr_0 = 0$.

Radial Profiles

Figs. 4.5 and 4.6 show comparisons of the predicted mean velocity profiles with the data of George et al and Nakagome and Hirata respectively. Figs 4.7 and 4.8 show similar comparison for the temperature difference (or buoyancy). There is a good agreement between the predictions and both sets of data. The measurements of George et al were made at $x/d \geq 8$, while those of Nakagome and Hirata were taken at $x/d \geq 4.1$, and both sets of data exhibit self-similarity. Similarly the predicted profiles become self-similar beyond some seven diameters downstream. Thus the profiles of mean variables become self-similar much closer to the nozzle for plumes than for plumes than for momentum jets in which self-similarity is attained at $x/d \geq 8$.

Both the predictions and the two sets of data also show that the velocity profiles are wider than the corresponding temperature or

+ It is shown later (see pages to) that, whatever the initial Froude number, the ultimate behaviour of the plume depends on the value of the integrated flux Richardson number (equations 4.43, 4.44 and 4.45). Profiles of variables for various Fr_0 become self-similar, and independent of Fr_0 when $\bar{R}_f$ becomes constant.

buoyancy profiles. This is shown more clearly in Fig. 4.9, which contains a comparison of the predicted radial distributions of mean velocity and temperature of the plume with those of the heated momentum jet. It is clear that in the momentum jet the temperature profile is wider than the velocity profile, while the converse is true for the plume. That is, unlike the case of the momentum jet, the turbulent diffusion of momentum is greater than the turbulent diffusion of energy in the plume.

Rates of spread

The present results confirm that in the buoyancy-driven plume, the process of mixing with the ambient fluid is much different from that in the shear-driven momentum jet. This is further highlighted by Fig. 4.10 which shows the predicted growth (i.e. the distributions of the half-widths $y_{0.5}$ and $y_{0.5T}$) of both the plume and the momentum jet. Also shown there are the measured growths by Nakagome and Hirata for $Fr_0 = 0$. The predictions and the data show several interesting features.

First, although the predictions for the plume are for $Fr_0 = 1.6$ they are surprisingly in rather good agreement with the data for $Fr_0 = 0$ beyond some 2.5 diameters downstream. This is because, although at $Fr_0 = 1.6$ the plume is given some initial momentum at the source, as shown by the predictions in Fig. 4.10, it undergoes a rather pronounced contraction close to the nozzle, thus indicating that the influence of the initial momentum flux is small compared with the initial buoyancy flux.

The second interesting feature shown by Fig. 4.10 is that the velocity half-width of the plume is much wider than that of the temperature, while the converse is true for the momentum jet. Thirdly, because the plume starts spreading much closer to the nozzle exit, the half-widths of both velocity and temperature are wider than their

corresponding values in the momentum jet.

Finally, the rate of spread of momentum, $dy_{0.5}/dx$, is much higher for the plume than for the momentum jet, while that of energy, $dy_{0.5T}/dx$, is about the same for both cases. The values of the rates of spread and their ratio λ_r (see equation 4.33) are shown in Table 5 where the data of Nakagome and Hirata as well as the predictions of Chen and Rodi (1975) are also shown. The predicted rate of spread of energy is in good agreement with the data; but that of momentum is slightly lower than the data, resulting in a higher value of the spreading rate λ_r of the prediction than of the data. It may also be noted in Table 5 that while the spreading ratio λ_r is greater than unity for the momentum jet, it is less than unity for the plume.

Axial profiles

Fig. 4.11 shows the axial distribution of the mean velocity. Experience with the plume predictions shows that, when normalised by the maximum velocity U_{max} along the axis, the distributions for various initial Froude numbers follow a fairly well-defined curve beyond $x/d \approx 3.5$. In Fig. 4.11 the predictions are compared with the data of Nakagome and Hirata; also shown there for further comparison are the predictions for the momentum jet. For $5 < x/d < 8$ the plume predictions are lower than the data by some 5%, however, beyond eight diameters the agreement with the data is much more satisfactory. The flow is accelerated by buoyancy effects from the nozzle exit to a point about 3.2 diameters downstream, and then decelerated by turbulent shear. By employing dimensional analysis in the manner of Abraham (1960), the predicted distribution beyond some five diameters may be well correlated by the relationship

$$\frac{U_z}{U_{max}} = 2.5 Fr_0^{-2/3} (x/d + a)^{-1/3} \quad (4.34)$$

where $a (= 2.0)$ is the value of x/d defining the virtual origin of the plume (i.e., the point at which the line for $y_{0.5}/d$ meets the abscissa in Fig. 4.10). The approximate analysis of Abraham (1960) yields a value of the multiplying factor of 4.0.

In Fig. 4.12 is shown a comparison of the predicted axial distribution of mean temperature with the data of George et al, both prediction and data being at $Fr_0 = 1.6$. Also shown there are the distributions for the pure plume ($Fr_0 = 0$) and the momentum jet ($Fr_0 = \infty$). The measurements of George et al are about coincident with the prediction, although they were made at just two locations. As would be expected, the axial decay of the temperature is a strong function of the Froude number Fr_0 : the predicted distribution for $Fr_0 = 1.6$ is higher than the data for $Fr_0 = 0$, while being much lower than that for the momentum jet. The prediction also indicates that a potential core, which normally would carry fluid at the discharge temperature up to some distance above the nozzle exit, does not exist, as the axial temperature decreases monotonically along the plume's path. This indicates that the plume fluid mixes far more with the ambient fluid than does a momentum jet. Again, from dimensional analysis, the predicted distribution of temperature beyond some five diameters downstream is well correlated by the relationship

$$\frac{\theta}{\theta_0} = 4.2 Fr_0^{2/3} (x/d + a)^{-5/3} \quad (4.35)$$

where $a = 2$, as defined for equation (4.34). The approximate analysis of Abraham yields a value of the multiplying factor of 7.8.

Turbulence Parameters

The distributions of the predicted turbulence quantities are given in Figs. 4.13, 4.15, 4.16, and 4.17. Figs. 4.13 and 4.15 show comparisons of the predicted turbulence energy with the data of both George et al and Nakagome and Hirata. Their experiments do not provide

data for all the three components of the Reynolds normal stresses, but only for the component $\overline{u_1^2}$ in the direction of gravity. However, Launder (1975) has shown that in a very unstable horizontal buoyant flow where the flux Richardson number R_f (see equation 2.29) is large and negative (as is the case for a buoyant plume in uniform surroundings), the ratios of the normal stresses are as follows:

$$\overline{u_1^2} : \overline{u_2^2} : \overline{u_3^2} = 0.94 : 0.53 : 0.53 \quad (4.36)$$

where subscript 1 stands for the vertical direction, and 2 and 3 for the other two directions. For a vertical buoyant plume in uniform surroundings, this ratio may be expected to be even more unequal in favour of the vertical direction since the stream wise direction is now vertical. However, in the absence of further information, the following ratios are assumed in the calculation of the turbulence energy K from the available experimental data:

$$\overline{u_1^2} : \overline{u_2^2} : \overline{u_3^2} = 1.0 : 0.5 : 0.5$$

That is, $\sqrt{\overline{u_1^2}} : \sqrt{\overline{u_2^2}} : \sqrt{\overline{u_3^2}} \approx 1.0 : 0.7 : 0.7 \quad (4.37)$

Fig. 4.13 shows a comparison of the predicted axial distribution of turbulence energy (normalised by U_g^2) with both sets of experimental data. Close to the source the predictions lie below the data of Nakagome and Hirata; however, for x/d greater than about 6, the agreement is more satisfactory. The most interesting feature about the results is that beyond some six diameters downstream both the predictions and the data show an almost constant level of K/U_g^2 . This implies that, rather unexpectedly, the turbulence becomes fully-developed (i.e. self-similar) as close to the nozzle as about seven diameters downstream. Indeed, Nakagome and Hirata contend, and their detailed measurements confirm, that the fully-developed turbulence structure starts from about 4.1 diameters downstream. In contrast, as already indicated in Section 4.32, the measurements of Wygnanski and Fielder

(1969) show that the turbulence structure in the axis-symmetric jet becomes fully-developed only beyond some 70 diameters downstream. This constitutes another major difference between the buoyancy-driven plume and the momentum jet which is driven mainly by shear.

It is worthwhile seeking a possible explanation for this rather surprising result. In the momentum jet, energy is imparted by the mean motion directly to the streamwise fluctuating component $\overline{u_1^2}$, and only the pressure-strain correlations $\frac{P'}{\rho} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^*$ further redistribute the energy to the other components of the turbulence motion. Self-similarity of the entire flow is therefore necessarily attained over a long distance: first, the mean flow takes a while to become similar and produces some $\overline{u_1^2}$ in that process; next, a balance has to be sort between the mean flow and $\overline{u_1^2}$; finally, after the achievement of this balance, isotropy of large scale turbulence (i.e. $\overline{u_1^2} \approx \overline{u_2^2} \approx \overline{u_3^2}$) has to be reached.

For the dynamic turbulence field of the buoyant plume, energy is imparted mainly by buoyancy forces also directly to the vertical component $\overline{u_1^2}$, and again redistributed by the pressure-strain correlations to the other components. But the buoyancy forces affect in full both the mean flow and the fluctuating component $\overline{u_1^2}$ (as well as $\overline{u_1\theta'}$) simultaneously rather than in stages, with the main interaction between the mean and turbulence fields provided only through the production of fluctuating excess temperature $\overline{\theta'}$ (i.e. $-2\overline{u_j\theta'} \frac{\partial \theta}{\partial x_j}$)*. Further, the final state of the large scales of turbulence is anisotropic (i.e. $\overline{u_1^2} \neq \overline{u_2^2} \neq \overline{u_3^2}$), indicating that redistribution of energy from $\overline{u_1^2}$

* See Appendix A

to the other components may be comparatively less influential. The possibility therefore exists of the mean and turbulence fields reaching self-similarity rather quickly and nearly in step.

This self-similarity and how quickly it starts is, indeed, better illustrated by examining the equation for the transport of turbulence energy (equation 2.24 or 2.26) along similar lines as Kotsovinos (1976). This may be re-written as

$$\begin{aligned} \frac{\partial}{\partial x_j} (\rho U_j K) &= -\overline{\rho u_i u_j} \frac{\partial U_i}{\partial x_j} \left\{ 1 + \frac{(+g_i \overline{\rho' u_i})}{-\overline{\rho u_i u_j} \frac{\partial U_i}{\partial x_j}} \right\} + \text{other terms} \\ &= -\overline{\rho u_i u_j} \frac{\partial U_i}{\partial x_j} \{ 1 + R_f \} + \text{other terms} \end{aligned} \quad (4.38)$$

$$\text{where } R_f = +g_i \overline{\rho' u_i} / (-\overline{\rho u_i u_j} \frac{\partial U_i}{\partial x_j}) \quad (4.39)$$

Here, R_f is the flux Richardson number (defined in equation 2.29 in terms of fluctuating excess temperature, θ'), ρ' is the fluctuating excess density, and g_i is the gravitational vector. Since the shear production is zero along the plume axis, and maximum in the region of maximum shear ($y/r_0 \approx 1$), R_f varies across the plume. However, the dependence on radial distance may be eliminated by integrating equation (4.38) across the plume, yielding

$$\int DK \cdot dy = -(\int \overline{\rho u v} \frac{\partial U}{\partial y} \cdot dy) (1 + \overline{R_f}) + \text{other terms} \quad (4.40)$$

where $D (= \frac{\partial \rho U_i}{\partial x_j})$ is a general operator for the convection term, $\int_{\text{Plume}}$ stands for integration across the entire plume, and the overall relative importance of the buoyant and shear productions is given by the integrated flux Richardson number $\overline{R_f}$ defined as:

$$\overline{R_f} = \frac{\int_{\text{Plume}} \overline{g \rho' u} \cdot dy}{\int_{\text{Plume}} \overline{\rho u v} \frac{\partial U}{\partial y} \cdot dy} \quad (4.41)$$

where g is the gravitational constant. The magnitude of $\overline{R_f}$ may be estimated by employing an order of magnitude analysis, wherein it may be assumed

$$\begin{aligned} Q(U) &= U_{\underline{c}}^2, & Q(\rho') &= c\Delta\rho, \\ Q\left(\int_{\text{plume}} dy\right) &= b, & Q(uv) &= Q(u)Q(v) = c^2 U_{\underline{c}}^2, \\ Q(u) &= Q(v) = cU_{\underline{c}}, & Q(\rho'u) &= Q(\rho')Q(u) = c^2 \Delta\rho U_{\underline{c}} \end{aligned} \quad (4.42)$$

where $U_{\underline{c}}$ is the centre-line velocity, b is the local width (i.e. the local diameter) of the plume, $\Delta\rho (= \rho_{\infty} - \rho_{\underline{c}})$ is the mean density difference, and c is a constant. Thus expression (4.41) becomes

$$\overline{R}_f = O\left(\frac{g\Delta\rho b}{\rho_0 U_{\underline{c}}^2}\right) = O(Ri(x)) \quad (4.43)$$

$$\text{where } Ri(x) = \frac{g\Delta\rho b}{\rho_0 U_{\underline{c}}^2} \quad (4.44)$$

Here, Ri is the local Richardson number and is generally a function of the distance x from the source. Note that the Richardson number Ri is simply the inverse of the local densimetric Froude number (see equation 4.13, with ρ_0 used in place of $\rho_{\infty,0}$). Expression (4.43) indicates that the overall production of turbulence energy by buoyancy forces relative to shear production is of the order of, and therefore varies as, the local Richardson number.

In Fig. 4.14 is shown the predicted variation of the local Richardson number $Ri(x)$. There, the local diameter of the plume b is defined as $2y_{0.9}$, where $y_{0.9}$ is the radial distance at which the streamwise velocity is 90% less than the centre-line value; as shown by the profile of $U/U_{\underline{c}}$ in Fig. 4.16 this occurs at $y = 2.1y_{0.5}$ in the region ($x/d > 1.5$) where that profile holds. Fig. 4.14 shows that Ri decays sharply from the nozzle exit, but beyond about 6.5 diameters downstream it becomes uniform at

$$Ri = 0.62 \quad (4.45)$$

In relation to expression (4.43), this constant value of Ri therefore indicates that the overall buoyant production of turbulence energy relative to the shear production is a constant for the plume beyond some 6.5 diameters downstream. This constant ratio implies that the

plume must be self-preserving, with the self-preservation starting from the location where the ratio becomes constant,

The value of 0.62 obtained in this study is most gratifying and interesting, for the recent experimental data of Kotsovinos (1975, 1976) for a plane vertical buoyant plume also gives a constant value of

$$Ri = 0.63 \pm 0.05 \quad (4.46)$$

for all measured locations $x/d \geq 6$ (where, here, d is the width of the slot), and for which locations both mean and turbulence quantities are self-similar.

It must also be emphasized that the region $x/d < 6.5$ (Fig. 4.14) in which the value of Ri is not constant forms the zone where the plume is not self-similar. Thus a plume becomes self-preserving only when the local integrated flux Richardson number $\overline{R}_f$, which is of the same order of magnitude as the local Richardson number Ri , becomes uniform and the latter of value $Ri \approx 0.6$. For a pure plume this condition is likely to be attained much closer to the source (owing to the absence of initial momentum) and this explains why the measurements of Nakagome and Hirata show self-preservation as close to the source as $x/d = 4$. On the other hand, for a buoyant jet ($Fr_0 \neq 0$) self-preservation cannot be attained within a given distance if the initial conditions are such that the uniform value of $Ri \approx 0.6$ is never reached.

Fig. 4.15 shows a comparison of the predicted radial profiles of turbulence energy, K/U^2 , with the two sets of experimental data, within the fully-developed region. The data of Nakagome and Hirata are for various axial locations beyond x/d of 4, while those of George et al are for $x/d \geq 8$. Although there is a fairly large scatter in both sets of data, each set shows self-similarity of the turbulence profile. The predicted profile is attained as from about seven diameters downstream, thus confirming self-similarity of the turbulence very close to the nozzle. In view of the scatter in the data, the prediction may

be said to be generally in satisfactory agreement with the latter; however the prediction shows a slightly lower trend close to the axis.

Another interesting feature about Fig. 4.15 is that, from the ratios given in equation (4.37), it follows that the stream-wise component of the normal stress along the plume axis in the fully-developed region has an average value of

$$\frac{\sqrt{\overline{u_1^2}}}{U_\infty} \approx 0.25 \quad (4.47)$$

which clearly indicates that the relative turbulence fluctuations in the buoyant plume are large. Further, this value of 0.25 compares favourably with the approximate value of 0.28 established by the measurements of Wygnanski and Fielder (1969) in the fully-developed region of the axi-symmetric momentum jet. This rather surprising result may be explained in part by the fact that although buoyancy augments the turbulence considerably, the mean velocity is also relatively increased and it decays as $x^{-1/3}$ (see equation 4.34) while in the momentum jet it decays as x^{-1} . On the other hand, as would be expected from the broader radial profile of mean velocity shown in Fig. 4.9, the complete radial profile of $\sqrt{\overline{u_1^2}}/U_\infty$ is broader in the plume than in the momentum jet.

In Fig. 4.16 is shown the predicted development of the radial profiles of turbulence energy which is now plotted against $y/y_{0.5}$, for axial locations $x/d = 2, 4.5$, and 6.96 . The profile for $x/d = 6.96$ is the self-similar profile that exists for axial locations here and further downstream. Also shown in Fig. 4.16 are the profiles of mean velocity and temperature which, now plotted against $y/y_{0.5}$ ($y/y_{0.5T}$ for temperature) are attained beyond $x/d \approx 1.5$. As shown in the plot, the predicted mean profiles about coincide with the predictions of Chen and Rodi (1975).

Fig. 4.17 shows a comparison of the predicted axial distributions

of the three turbulence quantities, the turbulence energy K , the turbulence dissipation rate ϵ , and the dissipation length scale l_ϵ , for both the plume and the momentum jet. The distribution of ϵ is qualitatively similar to that of K in both flows, being also very high close to the nozzle and nearly uniform beyond some seven diameters downstream in the case of the plume, while being negligible up to some five diameters downstream and still developing within the range of computation in the case of the momentum jet.

Of particular interest in Fig. 4.17 is the distribution of the dissipation length scale l_ϵ which characterizes the size of the turbulent eddies. It should be recalled that l_ϵ is normalized by the velocity half-width $y_{0.5}$ and the latter has been shown to be much larger for the plume than for the momentum jet at values of $x/d > 2$ (see Fig. 4.10). The well defined peak in the distribution for the plume is due to the fact the $y_{0.5}$ first decreases slightly close to the nozzle and then starts to increase sharply at $x/d \approx 2.5$, at about which point the peak occurs. The prediction indicates that, as compared with the momentum jet, a source of turbulent plume generates much larger eddies and this explains much of the characteristics of the plume. First, it is consistent with the non-prevailence of the potential core which characterizes the comparatively smaller size eddies generated by a source of non-buoyant jet since the larger eddies will allow much greater penetration of, and mixing with, the ambient fluid right from the nozzle exit.

Secondly, this increased mixing implies a higher rate of spread of momentum $dy_{0.5}/dx$ in the plume than in the momentum jet, as is testified by both Table 5 and Fig. 4.10. Incidentally, if Taylor's entrainment principle were assumed to hold, and since the measured and predicted mean velocity profiles are approximately Gaussian, substitution in equation (4.4) of a Gaussian similarity expression for the velocity yields the following expression for the entrainment constant (see, for

example, Turner, 1973; George et al, 1976);

$$\alpha_e = \frac{5}{6} (y_{0.5}/X) \quad (4.48)$$

where α_e is the entrainment constant defined by equation (4.10), and $y_{0.5}/X$ is here the rate of spread of the momentum. The present predictions therefore give an entrainment constant of 0.10 for the plume (in comparison Morton, 1959a, 1959b, used a value of 0.116) and a value of 0.072 for the momentum jet (in comparison, Turner, 1973, and the present author calculate a value of 0.08 from the data of Ricou and Spalding, 1961, see Table 2). Thus even the crude entrainment method confirms much greater mixing of the plume fluid with the ambient fluid than obtains in the momentum jet.

Thirdly, the increased mixing results invariably in greater dilution of the plume fluid right from the nozzle exit, and thus explains why the half-width of the temperature $y_{0.5T}$, decays far more rapidly than that of velocity, $y_{0.5}$, in the region close to the nozzle (see Fig. 4.10). This effective swamping in that region of the plume fluid by the ambient fluid and the attendant pronounced decay of $y_{0.5T}$ explains why, contrary to the case of the momentum jet, $y_{0.5T}$ is less than $y_{0.5}$ at all axial locations (Fig. 4.10), with the implication also that the temperature profiles will be narrower than the velocity profiles (Fig. 4.9).

4.42 The Plume in Calm, Stably-Stratified Surroundings

For the nearly pure plume ($Fr \rightarrow 0$) in stably, linearly-stratified surroundings the only data available are the measurements of maximum height made by Crawford and Leonard (1962), and in Fig. 4.18 predictions for this case (Fr_0 of order 0.01 and less) are compared with their data. There, F_0 is the initial buoyancy flux (equation 4.22) and G is the stratification parameter defined by equation (4.26). Also shown in Fig. 4.18 are the analytical solutions for the pure plume as given by Priestley and Ball (1955), Morton et al (1956), Morton (1959), and

Abraham and Eysink (1969). The data exhibit considerable scatter caused in part by fluctuations in the level of the plume. However, the present predictions agree satisfactorily and also coincide with the analysis given by Abraham and Eysink. The other theoretical solutions also show satisfactory agreement with the data, except that of Morton et al.

For the plume at moderate Froude numbers in stably, linearly-stratified surroundings computations were made to simulate the more recent experiment of Sneek and Brown (1974). The computations were carried out with the following parameters of their experiment: $d = 0.178\text{m}$, $U_0 = 0.5\text{m/s}$, $T_0 = 315^\circ\text{K}$, $T_{\infty,0} = 306.7^\circ\text{K}$, and $dT_{\infty}/dx = 1.8^\circ\text{K/m}$. These conditions give an initial Froude number, Fr_0 of 10.58.

Fig. 4.19(a), (b), (c) and (d) give an overall picture of the predicted flow field for the simulation of the data of Sneek and Brown. The predictions for other values of Fr_0 and dT_{∞}/dx show similar patterns. The vector plot of the mean velocity within the domain of computation is shown in Fig. 4.19(a). There, the arrows ($\uparrow$) are generally drawn to scale, but those marked with N ($\uparrow$) indicate vectors that are larger than indicated. Marked on the plot as dotted lines are the approximate boundaries of the plume: the lower boundary is defined as the approximate points at which the velocity vector (resultant of U and V) is $90\% \overset{\text{less than}}{\wedge}$ the centre-line vector, while the upper boundary gives the approximate points where U (or V for the horizontal part) is 1% of the nozzle exit velocity. The flow field indicates large entrainment of ambient fluid into the lower portions of the plume, and comparatively small entrainment at the uppermost boundary.

The plume boundaries given in Fig. 4.19(a) are also shown in Figs. 4.19(b), (c) and (d). Fig. 4.19(b) shows the predicted temperature contour within the flow field. The values at the outer boundary of the flow domain are those fixed for the ambient fluid in the computations. Outside the plume boundary, the contours tend to have positive

slope with distance from the axis of symmetry in the region where at the same height the centre-line temperature is greater than that of the ambient fluid, and negative slope where the former is less than the latter. At about 60% of the maximum height of the plume the contour is essentially horizontal from the axis to the outer boundary, indicating the level of zero buoyancy. Within the main plume boundaries, and in regions greater than about 60% way to the maximum height, the contour is mainly horizontal.

Fig. 4.19(c) shows the contour of the turbulence energy (K/U_0^2) within the flow field. In the area below about 60% of the maximum height of the plume, very high levels of turbulence occur in the region immediately surrounding $y/r_0 = 1$. Much of the turbulence generated in this zone is transferred radially outwards and inwards. Nearer to the top of the plume the turbulence energy is essentially constant. Fig. 4.19(d) shows the contour of dissipation length scale ($l_\epsilon/d \times 10^{-1}$). Within the plume boundary, and especially in the region beyond about 60% way to the top of the plume, the magnitude of l_ϵ is large.

Fig. 4.20 shows a comparison of the predicted axial distribution of temperature with the experimental data. Also shown there are the predictions of Chen and Rodi (1975) for the same calculations, but, as already indicated in Section 4.21 (ii), with a boundary-layer procedure. The present predictions agree well with those of Chen and Rodi up to some three diameters downstream of the source, but thereafter their predictions are much higher than those of the present study. Moreover, the Chen predictions stop at about nine diameters downstream (about 65% of the height of the plume). The present predictions may be said to be in good agreement with the data, since during the course of the experiment the ambient temperature gradient departed from its initial linear value of 1.8°K/m .

In Fig. 4.21 is shown the axial variation of the mean velocity, U_z , for both the present predictions and those of Chen and Rodi (1975). Again both sets of predictions agree well up to some three diameters downstream, within which region the plume accelerates, indicating that it is little affected by stratification in that region. However, although beyond that point both predictions show a decay, those of Chen and Rodi show much slower rate of decay as though the plume were in an ambient fluid of much smaller stratification (as is also the case for their predicted temperature in Fig. 4.20). Further, up to about nine diameters downstream, which is well within the region where the flow deflects horizontally, their prediction gives U_z/U_0 of about 1.0 while the present predictions give about 0.4 at the same level.

The reasons for the discrepancy between the present results and those of Chen and Rodi for $3 < x/d < 9$ are not clear, as the flow in the plume is of the boundary layer type within this region. However, it is felt that the cause may be due to the neglect of the influence of ambient stratification in their modelling of the rather important term $\overline{uT'}$ (see equation 4.17) which they have used to reflect the effect of buoyancy in the turbulence field. Physically, the correct buoyancy component is $\overline{u\theta'}$ as used in this study, where θ' is the fluctuating excess temperature, and its modelling must reflect the ambient stratification when the latter is non-zero. The form of equation (4.17) is only physically plausible under zero ambient stratification. Thus their prediction tends to behave as though the ambient stratification is nearly zero.

The height, H_{max} , of the plume is taken to be where U_z falls to zero. The experimental height is marked in Fig. 4.21, and it occurs at x/d of 13.85 (or 2.465m). The predicted height, as shown by the point where U_z goes to zero in Fig. 4.21, falls within some 0.2% of the experimental height.

Also shown in Fig. 4.21 is the predicted axial distribution of the excess temperature θ_z . The latter shows a monotonic decrease, and the level of zero buoyancy ($\theta_z = 0$) occurs at $X/H_{\max}$ of 0.61 beyond which θ_z decays very slowly owing to reduced entrainment in the region where the plume fluid spreads out horizontally.

In Fig. 4.21 are also shown the axial distributions of the turbulence energy, K/U_0^2 , the dissipation length scale, l_ϵ/d , and the plume half-width, $y_{0.5}$. These show that close to the nozzle exit ($X/d < 2$) the flow characteristics are mainly determined by momentum, resulting there in near-zero values of K/U_0^2 and l_ϵ/d as has been shown to be the case for momentum jet in Fig. 4.17. Also it should be noted that in the approximate region $4 < X/d < 9$, and also for $10 < X/d < 12$, both $y_{0.5}/d$ and l_ϵ/d vary almost linearly with distance.

Further, the existence of relatively small turbulent eddies in the exit region, as is implied by the low values of l_ϵ indicates a comparatively smaller dilution within that region of the plume fluid by the ambient fluid than is the case for the plume in uniform surroundings ($Fr_0 = 1.6$) discussed earlier. Little wonder that while for the latter case θ_z/θ_0 decays rapidly to a value of 0.6 within $X/d < 2$ (see Fig. 4.12), within the same region of the present case ($Fr_0 = 10.58$) it decays to only about 0.9 and actually stays very close to unity for $x/d < 1$. In this region the ambient stratification has earlier been indicated to be of little influence. Indeed, any influence there of ambient stratification would lead to a more rapid decay of θ_z/θ_0 than would be expected for a plume in uniform surroundings with $Fr_0 = 10.58$.

Both the turbulence energy and the dissipation length scale in Fig. 4.21 increase considerably beyond some three diameters downstream and K/U_0^2 becomes nearly constant in the region $6 < x/d < 11$. However, the turbulence energy exhibits a rather steep decline, owing to the destruction by buoyancy effects, close to the top of the plume.

Another interesting feature in Fig. 4.21 is that both $y_{0.5}$ and l_e increase considerably beyond the level of zero buoyancy, indicating the horizontal spread there of the plume fluid.

Fig. 4.22 shows the predicted radial distribution of mean velocity. The interesting feature there is that the velocity profile is nearly self-similar up to $X/H_{\max}$ of about 0.68 which is well within the initial part of the region where the plume deflects horizontally. Beyond that level, the profiles cease to be self-similar as indicated by those at $X/H_{\max}$ of 0.84 and 0.99.

The predicted radial distribution of excess temperature is shown in Fig. 4.23. For the lower region of the plume, the plot is made against $y/y_{0.5T}$, where $y_{0.5T}$ is the temperature half-width. Except in the immediate vicinity of the nozzle, the profile is self-similar in the major part of the lower region of the plume as is indicated by the profile for $0.1 < X/H_{\max} < 0.5$. In the upper region where the plume spreads out horizontally, $y_{0.5T}$ occurs far from the axis, and indeed, in the uppermost region it is not attained. Thus in that region the plot is made against y/d . The plots for $X/H_{\max} = 0.686$ and 0.76 indicate that in that region the temperature decays very little radially, and it becomes nearly uniform as the top of the plume is approached. This is a consequence of very little entrainment of ambient fluid in the upper region of the plume, as is shown by the vector plot in Fig. 4.19(a).

Fig. 4.24 shows the radial profiles of turbulence energy at various axial locations. Here no self-similarity exists, and, similar to the temperature profiles, the distributions tend to flatten out as the top of the plume is approached.

4.5 CONCLUDING REMARKS

This chapter has presented the work undertaken in this study on conventional buoyant plumes discharged into calm, uniform or stably-stratified surroundings. The main features of the chapter may be summarized as follows:

- (i) A review of both previous theoretical and experimental work on axisymmetric plumes in uniform and stably stratified surroundings have been made. The previous theoretical studies have employed practices that have been severely restrictive in one form or the other. The need therefore arises for establishment of a prediction method that will be rather general in nature, but without being excessively complicated for the engineer; it is on this basis the study covered in this chapter is undertaken.
- (ii) Recent data on plumes have rendered possible a satisfactory testing of both numerical procedures and turbulence models. However, more data are required on the mean flow and turbulence quantities for the plume in stably-stratified surroundings, and on the turbulence field for the plume in uniform surroundings.
- (iii) In computation of the initial region of turbulent jets and plumes, an uncertainty exists about the correct prescription of the initial dissipation rate. The formula derived in this study for the estimation of the initial dissipation length scale appears to clear the uncertainty.
- (iv) The turbulence constants which have been well proven for plane flows do not appear to need any modifications for the prediction of the initial region of the axisymmetric jets and plumes. The reason is attributed to the influence on the initial region of the quasi-plane mixing layer generated from the nozzle lips.
- (v) Both the SIMPLE procedure of Patankar and Spalding (1972) and the EMIT procedure developed in this study have been applied to the circumstances of the plume in both uniform and stably stratified

surroundings. Apart from the fact that the EMIT procedure basically involves considerable savings in computer storage facilities, it also procures some 22% savings in computing time.

(vi) The work on the plume in uniform surroundings has yielded several important results:

- o The plume flow can be divided into two regions: the developing and fully turbulent regions. In the developing region the flow is accelerated by buoyancy, and then becomes fully turbulent. In the fully turbulent region, similarity of both mean and turbulence quantities is attained; and this occurs very close to the source. The contrast between this behaviour and that of the momentum jet is attributed to the difference in the methods of generation of the turbulence.
- o An analysis has been given that reveals that self-similarity can generally occur of turbulence in the plume if, and only if, the integrated flux Richardson number becomes constant. The relevant constant is shown to be attained only when the local Richardson number becomes constant and has a value of about 0.60.
- o The rate of decay of velocity along the axis of the plume is smaller than that of temperature. Further, the rate of decay of the velocity is much smaller for the plume than for the momentum jet, owing to the fact that the former is continuously being supplied with momentum by buoyancy.
- o The spread of momentum in the plume is wider than that of thermal energy. Further, the spread of momentum in the plume is also wider than that in the momentum jet.
- o The rate of spread of momentum, $dy_{0.5}/dx$, in the plume is higher than that of thermal energy, $dy_{0.5T}/dx$. In addition, the rate of spread of momentum in the plume is greater than that in the momentum jet, implying greater mixing of plume fluid with the ambient fluid.

These are attributed to the generation of larger turbulent eddies by a source of buoyant plume.

(Vii) The work on the plume in stably stratified surroundings have also revealed some interesting features;

- As earlier indicated by previous workers, the height attained by the plume obeys a simple linear relationship based on the initial buoyancy flux and the stratification parameter.
- The level of zero buoyancy occurs at about 61% way to the maximum height of the plume.
- In the lower region of the plume, profiles of mean velocity and temperature are similar, but the turbulence is nowhere similar.
- Further, the plume fluid accelerates in parts of the lower region close to the source.
- In the top region of the plume, entrainment of ambient fluid is very small, leading to rather little change with height of the temperature.

CHAPTER 5

RADIOACTIVE BUOYANT PLUMES

5.1 INTRODUCTION

The work to be reported in this chapter is closely related to that of the previous chapter, in that it considers the rise in a stably-stratified atmosphere of a plume that contains radioactive fission products. The objective is to provide a means of assessing the consequences of a nuclear reactor accident in which fission products in gaseous or particulate form are released into the atmosphere.

If an element of a plume contains radioactive fission products, heat will be supplied continuously to it as a result of the process of radioactive decay. Indeed, the same is true if the fluid were influenced by chemical reaction, thermal radiation, or latent heat exchange. The effect of such heating is the continuous addition of buoyancy to the plume, through its influence on the density. This contrasts with the conventional plumes studied in the previous chapter, in which the source of heat is only at the plume stack. Further, plumes discharged from a reactor accident involving the rupture of a pressurized container may be ejected with a high velocity and, owing to steam from the reactor coolant, they may have a higher moisture content than conventional plumes, in addition to having a sensible heat release from the stack.

It is the absorption of radioactivity within a given element of the plume that produces heat in it. The radioactivity generally consists of β -radiation, α -particle emission, and γ -radiation. It is estimated (see Gifford, 1967) that about half the radioactivity consists of β -radiation and α -particle emission, and is effectively absorbed locally. Thus, the local production of heat due to the β - and α -effects will depend mainly on the amount of fission products

at the location in question. However, the remaining half of the radioactivity, consisting of γ -radiation, can penetrate air to the depth of about 100 metres (e.g. Gifford, 1967). Thus there will be transfer of heat from one location within the plume to another, owing to the transport of γ -radiation. A detailed and precise account of this effect can therefore be obtained only from complete knowledge of the γ -radiation transport.

With reference to equation (2.49) for the transport of thermal energy within the plume, there will now be a finite heat source S_h due to the radioactivity, and this may be decomposed into the two major parts described above:

$$S_{\text{rad}} = \dot{Q}_\beta + \dot{Q}_\gamma \quad (5.1)$$

where S_{rad} is the total heat generated by radioactive fission products per unit time per unit volume, $\dot{Q}_\beta$ is the directly deposited heat per unit time per unit volume due to β -radiation and α -particle emission from fission products, and $\dot{Q}_\gamma$ is the heat per unit time per unit volume due to γ -radiation from fission products at all locations within the plume. A complete assessment of the radioactive effects in the plume therefore requires determining both $\dot{Q}_\beta$ and $\dot{Q}_\gamma$. Some methods of doing this have been initiated in this study, and they form the basis of the radioactive plume work reported in this chapter.

Section 5.2 provides the starting point; there, previous studies of radioactive plumes are reviewed. Then in Section 5.3 is presented the mathematical modelling of the heating due to radioactivity; an analysis is given there of the models employed for the evaluation of both $\dot{Q}_\beta$ and $\dot{Q}_\gamma$. The details are then set out in Section 5.4 of the specific computations made, and the presentation and discussion of the results are given in Section 5.5.

5.2 PREVIOUS WORK

Previous studies on radioactive buoyant plumes have invariably been theoretical in nature (due to the obvious constraints on experimentation), and only the works of Gifford (1967) and Russo (1976) are known to the present author. The former authors studied the case of plume in a cross-wind, and also considered the rise under stable stratification. Gifford neglected the initial and bending-over stages of the plume, while Russo's work included both stages. However, both authors employed integral approaches for mass, momentum, and thermal energy, the essence of which have been outlined in Section 4.21, with inclusion of the angular dependence in the integral conservation equations for the case where initial and bending-over stages are also studied. Both sets of authors also assumed the familiar Gaussian (or "top hat") profiles of velocity and buoyancy, and their models generally suffer from the same deficiencies as have been outlined for integral approaches in Section 4.21.

As regards the radioactive heating defined by equation (5.1), both authors only considered the effects of directly-deposited heat within each element of the plume. In particular they assumed that all of the alpha and beta decay particles and one-half of the gamma emissions are absorbed; further, the rate of heat generation in each element of the plume is assumed to be determined by the total initial radioactive power "flux"⁺ Q^* (watts/sec) released by the source. In addition, Gifford argued that, since the fission products generated in the reactor decay fairly slowly as compared with the time t of the plume rise, Q^* is a constant all through the plume. However, Russo assumed an exponential decay rate; thus

⁺ The term "power flux" is here used to mean the "power emission rate" (units:watts/sec)

$$S = \frac{Q^* e^{-at}}{U} \quad (5.2)$$

where t is the time from release, U is the local plume velocity, and a is the effective radioactive decay rate which is estimated by taking the log of the ratio of activities at shutdown and one day later and dividing by the number of seconds in a day. Russo also took into account the effect of plume moisture content, but found that the effect on the plume rise is generally negligible.

The work of these authors give estimates of the rise height of plumes under different atmospheric conditions. It is concluded that rise due to radioactivity becomes significant for large release rates.

5.3 MATHEMATICAL MODELLING OF HEATING DUE TO RADIOACTIVITY

5.31 Heating due to β -radiation and α -particle emission

The directly deposited heat $\dot{Q}_\beta$ due to the β -radiation and α -particle emission is taken to be a function of the mass concentration m_j of the fission products in unit mass of the plume; that is

$$\begin{aligned} \dot{Q}_\beta &= \dot{Q}_\beta(m_j) \\ \text{or, } \dot{Q}_\beta &= \alpha m_j \end{aligned} \quad (5.3)$$

$$\text{where } m_j = \frac{\text{mass of fission products in unit volume}}{\text{mass of plume fluid in unit volume}} \quad (5.4)$$

and α is a factor which depends on the radioactive heat emission rate at the stack exit. The task of determining $\dot{Q}_\beta$ then reduces to one of determining the appropriate expression for α .

In doing this, some basic assumptions will be made. Firstly, it will be assumed that the entire fission products exiting from the stack release heat at a rate $\ddot{Q}_f$ equal to a given fraction of the sensible heat per unit time. That is,

$$\ddot{Q}_f = n_r \dot{Q}_{sen} \quad \text{per sec} \quad (J/s^2) \quad (5.5)$$

$$\text{and } \dot{Q}_{sen} = \rho_0 A_0 U_0 C_p \Delta T_0 \quad (J/S) \quad (5.6)$$

where $\ddot{Q}_f$ is the total rate of release of heating rate (i.e. power emission rate) at the stack exit due to fission products, $\dot{Q}_{sen}$ is

the total sensible heat flow from the nozzle, n_r is a release parameter, C_p the specific heat of the plume fluid; $\rho, U, \Delta T$ are the density, velocity, and temperature difference respectively, A the stack cross-sectional area, and subscript 0 stands for values at the stack exit.

Consequently, the heat emission rate per unit mass of fission products ($\dot{q}_f$) at the nozzle exit becomes

$$\dot{q}_f = \frac{\dot{Q}_f}{\dot{M}_f} \quad (\text{J/Kgs}) \quad (5.7)$$

where $\dot{M}_f (= m_{j,0} \rho_0 A_0 U_0)$ is the mass flow rate of fission products at the stack exit. Substituting for $\dot{Q}_f$ from (5.5), then

$$\dot{q}_f = \frac{n_r \dot{Q}_{sen}}{m_{j,0} \rho_0 A_0 U_0} \quad (\text{J/Kgs}) \quad (5.8)$$

Secondly, following Gifford (1967), it will be assumed that the fission products decay fairly slowly compared to the time of plume rise; thus it will be assumed that the specific heat emission rate $\dot{q}_f$ at the stack exit remains constant throughout the plume. This, however, is not an essential assumption.

For a unit volume of gas of density ρ , and containing mass fraction m_j , of radioactive material, we have

$$\text{mass of fission products in unit volume} = \rho m_j \quad (\text{Kg}) \quad (5.9)$$

$\therefore$ Heat, $\dot{Q}_\beta$, produced by fission products in unit volume is

$$\dot{Q}_\beta = \dot{q}_f \rho m_j \quad (\text{J/S}) \quad (5.10)$$

Expression (5.10) is employed in this study in evaluating the heating due to β -radiation and α -particle emission. It should be noted that for a given stack configuration and exit flow conditions, the value of $\dot{Q}_\beta$ may be varied by selecting different values of the parameter n_r in the expression for $\dot{q}_f$ (equation 5.8).

The use of equation (5.10) also requires the determination of the concentration m_j within the plume element. This is obtained from the solution of the species transport equation (2.50).

5.32 Heating due to γ -radiation

(i) Background

Like thermal radiation, a beam of γ -radiation consists of photons of a rather wide range of energy band. The energy, E , is often measured in millions of electron volts, MeV ($1 \text{ MeV} = 1.6 \times 10^{-13} \text{ Joules}$). Generally, the total heating for the entire energy range of γ -photons in unit volume of a beam at any spatial position, r , may be expressed as (Claiborne, 1968):

$$\dot{Q}_{\gamma} = c \int_{E_{\min}}^{E_{\max}} \sigma_a E \phi dE \quad (5.11)$$

where, $\dot{Q}_{\gamma} = \dot{Q}_{\gamma}(r)$, heat generation rate due to γ -rays at spatial position r (W/m^3)

E = energy of γ - photon (MeV)

$\phi = \phi(E, r)$, photon flux spectrum (number of photons per unit area per second at energy E in the beam) ($\text{photons} \cdot \text{m}^{-2} \cdot \text{s}^{-1} \cdot \text{MeV}^{-1}$)

$\sigma_a = \sigma_a(E, r)$ a kind of energy deposition coefficient or macroscopic cross section⁺ for energy deposition (m^{-1})

c = conversion factor, 1.6×10^{-13} (Ws/MeV)

In using equation (5.11), the problem lies in determining the photon flux spectrum ϕ . This is generally made up of two parts:

- unscattered part (obeys simple exponential law)
- scattered part (provides major contribution to heating, but cannot be expressed simply).

Theoretically, for any spatial point, there is no problem in determining ϕ , as the phenomenon of γ -ray transport in a phase space

+ The cross section σ of a stationary 'target' for a beam of photons of energy E , with respect to a given single process (e.g. absorption, scattering, etc) may be thought of as the effective area presented by the "target" for interception by one photon of the beam. (Or, σ is the probability per unit length for one photon of the beam to be caused to undergo the specific process by the "target").

of position, angle, energy, and time is governed accurately by the linear intergro-differential Boltzmann equation which balances the photon loss from an infinitesimal small volume element of phase space against the photon gain.

For steady-state (i.e. photon production rate = photon loss rate), the Boltzmann transport equation per unit volume of the elemental control volume shown in Fig. 5.1(a) is given by (e.g. Glasstone and Bell 1970):

$$\begin{array}{ccccccc}
 \text{I} & & \text{II} & & \text{III} & & \text{IV} \\
 \Omega \cdot \nabla \phi & + & \sigma_T \phi & = & \int_{\Omega'} \int_E \sigma \phi' dE' d\Omega' & + & S_0 \quad (5.12) \\
 \underbrace{\hspace{1cm}} & & \underbrace{\hspace{1cm}} & & \underbrace{\hspace{1cm}} & & \\
 \text{rate of} & & \text{absorption plus} & & \text{scattering in} & & \\
 \text{change} & & \text{self-scattering} & & & & \\
 & & \text{and out-scattering} & & & &
 \end{array}$$

where r = spatial co-ordinate

Ω = spatial direction vector

$\phi = \phi(r, E, \Omega)$, the number angular flux spectrum (i.e.

number of photons per unit area per second at position r travelling in the direction Ω with energy E)

$\phi' = \phi'(r, E', \Omega')$, the number angular flux spectrum for direction Ω' with energy E'

$\sigma_T = \sigma_T(r, E)$, total cross section (i.e. absorption cross section plus self-scattering and out-scattering cross section).

$\sigma = \sigma(r, E' \rightarrow E, \Omega' \rightarrow \Omega)$, transfer cross section (i.e. cross section for transfer from energy E' to E , and from direction Ω' to Ω)

$S_0 = S_0(r, E, \Omega)$, the source of photons of energy E in direction Ω at position r

$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$, gradient operator

The terms of equation (5.12) have the following meanings:

Term I = Rate of change: net change of ϕ of energy E in direction Ω ;

Term II = Absorption plus scattering-out: loss rate of ϕ of energy E travelling in direction Ω due to

a) absorption, and

b) scattering collisions resulting in change in direction or energy (i.e. self-scattering);

Term III = Scattering-in: production rate of ϕ of energy E travelling in direction Ω due to scattering collisions in two ways:

a) photons travelling in direction Ω but of other energy E' scattered into energy E in the same direction, and

b) photons travelling in other directions Ω' and of any energy scattered into direction Ω with energy E;

Term IV = Source: production rate of ϕ of energy E travelling in direction Ω due to sources of photons in unit volume.

It may be noted that the Boltzmann equation (5.12) has a wide interdisciplinary application in many branches of physical science and mathematics. For example, although employed here in the context of γ -ray transport, the same equation is used in kinetic theory of gases to describe the dynamics of molecular interactions; the astrophysicists employ it for energy transfer in stars, planets, etc.; heat transfer engineers employ it for the transport of thermal radiation in furnaces, etc.; and some chemical engineers (e.g. Dankwerts, 1957, Zwietering, 1959, and Curl, 1963) have attempted to employ it for turbulent mixing in which the turbulent flow is considered to be composed of macroscopic "parcels" of fluid whose transport, like the molecules of a gas, is governed by the Boltzmann equation (however, this particular approach of turbulent mixing has not yet attained general applicability owing to the fact that it requires assumptions regarding the life of and interaction between "parcels").

Apart from the Boltzmann equation, γ -ray transport, being basically

a statistical phenomenon, is also amenable to such statistical methods of analysis as Monte Carlo techniques (e.g. Cashwell and Everett, 1959) in which a number of photons are released from the source and their subsequent life histories traced: when a large number of case histories are obtained, the average result approaches the exact solution to the Boltzmann equation.

However, practical considerations limit the direct numerical integration of the Boltzmann equation or the use of Monte Carlo method. Further, radiation is like turbulence: complete knowledge would overburden the engineer; the art is to gain a satisfactory estimate for practical application. Various approximate methods of analysis have therefore been devised (see, for example, Clairborne, 1968) two of which are the build-up factor method and the multi-group flux model, both described below.

(ii) The Build-up Factor Method

Considering one-dimensional transport of uncollided photons (i.e. photons undisturbed by the intervening medium), the Boltzmann equation reduces to

$$\frac{d\phi}{dr}(r) = -\sigma_T \phi(r) + S \quad (5.13)$$

For a beam travelling in a medium containing no photon sources

(i.e. $S = 0$), integration of equation (5.13) yields

$$\phi(r) = \phi(r_0) \exp(-\sigma_T r) \quad (5.14)$$

$$\text{or,} \quad \phi = \phi_0 \exp(-\sigma_T r)$$

where subscript 0 stands for the source of the beam.

Substitution of expression (5.14) into equation (5.11) yields the following value of γ -heating for each energy considered:

$$\dot{Q}_\gamma = \phi_0 E_0 \sigma_T \exp(-\sigma_T r) \quad (5.15)$$

where E_0 is the energy of the γ -photon emitted by the source. Thus,

$$\begin{aligned} \text{effective transfer cross section} &= \sigma_T \exp(-\sigma_T r) \\ \text{for } \underline{\text{uncollided}} \text{ photons} \end{aligned} \quad (5.16)$$

In the build-up factor method, this effective transfer cross section representing the transfer of energy from "uncollided" photons from unit volume of a point source (K-cell, Fig 5.1b) which is at a distance r from a receiving point (j-cell, Fig 5.1b) is multiplied by an appropriate energy-deposition build-up factor B_μ to give the combined effect of both uncollided and scattered photons from the point source. Thus,

$$\text{effective transfer cross section for} \quad = B_\mu \sigma_T \exp(-\sigma_T r) \quad (5.17)$$

$$\text{uncollided and scattered photons}$$

For a medium of distributed source, this effective cross section must then be integrated over the source volume for each energy considered. In the manner of expression (5.15), the γ -heating per unit volume, $\dot{Q}_\gamma$, at any point for an isotropic source emitting ϕ_0 photons at energy E_0 per unit volume then becomes (Clairborne, 1968):

$$\dot{Q}_\gamma = E_0 \sigma_T \int_V \frac{\phi_0 B_\mu \exp(-\sigma_T r)}{4\pi r^2} \quad (5.18)$$

where r = distance between source and point of energy deposition

V = source volume

and, generally, ϕ_0 may be assumed proportional to the concentration of fission products, m_k ; that is,

$$\phi_0 = \hat{C} m_k \quad (5.19)$$

where $\hat{C}$ is an empirical constant relating concentration of fission products m_k to energy source rate.

Applying equation (5.18) to the j- and k-cells in Fig. 5.1(b), the energy deposition per unit volume of the j-cell from all other k-cells and for all energy bands becomes

$$\dot{Q}_\gamma = \sum_i \sum_k \hat{C}_i m_k E_{k,i} \hat{P}_{jki} \quad (5.20)$$

where, subscript j denotes j-cell

" k " k-cell (a typical one of all other cells)

" i " ith energy band

and
$$\hat{P}_{jki} = \frac{\sigma_{T,i} B_{\mu} e^{-\sigma_{T,i} r} \Delta V_k}{4\pi r^2} \quad (5.21)$$

= transfer probability of energy from k-cell to j-cell

within the ith energy band

ΔV_k = volume of k-cell

Equation (5.20) represents the final build-up factor expression for estimating the γ -heating. In its application, the values of $\hat{C}_i$, $\sigma_{a,i}$, and B_{μ} are required, in addition to the concentration of fission products m_k . It is similar to the equations which arise in the Hottel's "zone" method (Hottel and Sarofim, 1967) for thermal radiation, and like the zone method it is very uneconomical and unsuitable for combination with finite-difference fluid-flow computations: thus, if the flow field is divided into n cells (see Fig. 5.1b), n non-linear algebraic equations must be solved, each possessing n terms, for each ith energy band considered. For the plume calculations, for example, the flow field may be divided typically into 20 x 20 cells; therefore the application of the build-up factor method requires iteratively solving 400-term equations at some 400 cells, for each energy band. This would involve a most uneconomical usage of computer facilities.

This method is therefore not adopted in this study; rather a "flux model", which is more suitable for combination with the flow calculation method is developed in this study, and this forms the subject of the remaining parts of this section.

(iii) The Multi-group⁺ Flux Model

A very useful method of approximating the Boltzmann equation is to reduce it to a set of closed, first order, ordinary differential equations (sometimes termed "flux" equations), by integration with respect to solid angles. The early applications of such flux models were

+ The term "multi-group" refers to the division of the total photon energy into various intervals or "groups"

by the astro-physicists, and were all one-dimensional (the approach was originally proposed by Schuster, 1905, and later extended by Hamaker, 1947). The main advantage is that the flux equations can be recast into second order ones of the diffusion type and as such are easily and rapidly solved using the same numerical algorithms as for the flow computations.

Recently, as a result of progress in numerical computation of fluid flow, there has been a revived interest in flux models within the Mechanical Engineering Department of Imperial College: Spalding (1971) extended the original one-dimensional approach of Schuster to the analysis of thermal radiation in two and three dimensions; Gosman and Lockwood (1973) demonstrated the potential of a combined flux-model/fluid-flow numerical method by application of these equations to calculate thermal radiation in a two-dimensional furnace; and De Marco and Lockwood (1975) have presented a more rigorously derived flux model for thermal radiation, but this is not yet completely developed for axisymmetric cases with sources. The flux model which is developed in this study for γ -radiation transport has benefitted from the experiences of these previous workers.

The model developed here is also of a multi-group type. The multi-group approach is generally one of the most popular and efficient methods for solving the Boltzmann transport equations in which the energy dependence of photon fluxes and interaction cross sections are averaged over relatively broad energy intervals, coupled with approximations for the flux behaviour in each group interval.

(a) Assumptions of the model

A part of the basic assumptions to be made here is one that is often called the straightahead scattering method (Clairborne, 1968), and it involves the following assumptions:

- all γ -photons make Compton collisions⁺;
- energy degradation takes place after collision in accordance with the correct Klein-Nishina formula (Klein and Nishina, 1929);
- but, the γ -photons are not deflected from their original direction.

These assumptions lead to a considerable simplification of the Boltzmann equation, and the subsequent calculations can be made by the multi-group approach.

The following additional assumptions will be made, relating specifically to the flux model:

- the whole γ -radiation field at any point P (see Fig. 5.2a) can be represented by ascribing values to four fluxes, namely, pairs of positive γ -photon fluxes I and J in the positive and negative coordinate directions; (if we wish we could use a larger number of fluxes; but four are most convenient for a two-dimensional flow field).;
- each photon flux (I or J) undergoes no deflection from its original direction; (this is also in line with the straightened scattering method);
- the manner in which each I or J flux varies with position under the influences of absorption, scattering and production is governed by the Boltzmann equation.

(b) Derivation of multigroup equations

In the multi-group formulation, the quantity calculated is the integrated photon flux ϕ_g for each energy group, g , of width ΔE_g (see illustration in Fig. 5.2c). This requires the transport equation to be

+ In Compton scattering, a photon collides with an electron, loses some of its energy and is deflected from its original direction of travel. The basic theory of this effect, assuming the electron to be free and at rest, is that of Klein and Nishina (1929) and has been well confirmed experimentally.

integrated over energy group ΔE_g . The integrated group flux ϕ_g is defined as

$$\phi_g(r, \Omega) = \int_{\Delta E_g} \phi(r, E, \Omega) dE \quad (5.22)$$

where $\phi(r, E, \Omega)$ is the flux per unit solid angle per unit energy, E is the energy and Ω the angular direction.

The integration of the Boltzmann transport equation (5.12) over the energy band ΔE_g , assuming that energy is angular variation independent, gives (e.g. Chin et al, 1976; Goddard, 1976):

$$\begin{aligned} \Omega \cdot \nabla \phi_g(r, \Omega) + \bar{\sigma}_{T,g}(r, \Omega) \phi_g(r, \Omega) \\ = \sum_{g'=1}^G \int_{\Omega'} \sigma_{g' \rightarrow g}(r, \Omega' \rightarrow \Omega) \phi_{g'}(r, \Omega') d\Omega' + S_{0,g}(r, \Omega) \end{aligned} \quad (5.23)$$

where g refers to the particular group of interest and g' any of the various groups, G is the total number of groups (See Fig. 5.2c), Ω the particular angular direction considered and Ω' any of the various directions, and $S_{0,g}$ the group source. Here, the group-averaged cross-section coefficients are represented by the integral expressions:

$$\bar{\sigma}_{T,g}(r, \Omega) = \frac{\int_{\Delta E_g} \sigma_T(r, E, \Omega) \phi(r, E, \Omega) dE}{\int_{\Delta E_g} \phi(r, E, \Omega) dE} \quad (5.24)$$

$$\text{and } \sigma_{g' \rightarrow g}(r, \Omega' \rightarrow \Omega) = \frac{\int_{\Delta E_g} \int_{\Delta E_g} \sigma(r, E' \rightarrow E, \Omega' \rightarrow \Omega) \phi'(r, E' \rightarrow E) dE dE'}{\int_{\Delta E_g} \phi'(r, E') dE'} \quad (5.25)$$

where ϕ' = a weighted spectrum

$$\left. \begin{array}{l} E_g = \text{final energy interval} \\ E_{g'} = \text{initial energy interval} \end{array} \right\} \begin{array}{l} \text{defined by group structure} \\ \text{(see illustration in Fig. 5.2d)} \end{array}$$

In order to represent the photon transport by the "flux model", equation (5.23) needs to be integrated over each of the required number of solid angle intervals (four intervals in this case, as shown in Fig. 5.2e). Here the approach to integration follows the lines indicated by Goddard (1976). Integration of (5.23) over the solid angle Ω_1 of Fig. 5.2(d), for example, gives

$$\begin{aligned}
 & \int_{\Omega_1} \Omega \cdot \nabla \phi_g(r, \Omega) d\Omega + \int_{\Omega_1} \bar{\sigma}_{T,g}(r, \Omega) \phi_g(r, \Omega) d\Omega \\
 &= \underbrace{\sum_{g'=1}^G \int_{\Omega_1} \int_{\Omega'=\Omega} \sigma_{g' \rightarrow g}(r, \Omega' \rightarrow \Omega) \phi_{g'}(r, \Omega') d\Omega' d\Omega}_{\text{Scattering within interval } \Omega_1} \\
 &+ \underbrace{\sum_{g'=1}^G \int_{\Omega_1} \int_{\Omega' \neq \Omega_1} \sigma_{g' \rightarrow g}(r, \Omega' \rightarrow \Omega) \phi_{g'}(r, \Omega') d\Omega' d\Omega}_{\text{Scattering from outside interval } \Omega_1 \text{ with interval } \Omega_1} + \int_{\Omega_1} S_{0,g}(r, \Omega) d\Omega \quad (5.26)
 \end{aligned}$$

If, as earlier assumed for the flux model, the total (scalar) flux is to be represented as the sum of four angular components,

$$\begin{aligned}
 & \phi_1(r) + \phi_2(r) + \phi_3(r) + \phi_4(r) \\
 \text{where} \quad & \phi_1(r) = \int_{\Omega_1} \phi(r, \Omega) d\Omega \\
 & \phi_2(r) = \int_{\Omega_1} \phi(r, \Omega) d\Omega \quad (5.27)
 \end{aligned}$$

etc.

then, for component $\phi_1(r)$, the term in equation (5.26) for scattering from outside intervals becomes

$$\sum_{g'=1}^G \sum_{k=2}^4 \sigma_{g' \rightarrow g} \phi_{g',k}(r) \quad (5.28)$$

Similarly, the term for scattering within interval Ω_1 becomes

$$\sum_{g'=1}^G \sigma_{g' \rightarrow g} \phi_{g',1}(r) \quad (5.29)$$

where,

$$\sigma_{g' \rightarrow g} = \frac{\int_{\Omega_1} \int_{\Omega_k} \sigma_{g' \rightarrow g}(r, \Omega' \rightarrow \Omega) \phi'_{g'}(r, \Omega') d\Omega' d\Omega}{\int_{\Omega_k} \phi'_{g'}(r, \Omega') d\Omega} \quad (5.30)$$

$$\phi_{g,k}(r) = \int_{\Omega_k} \phi_g(r, \Omega) d\Omega \quad (5.31)$$

and, ϕ' = a weighted spectrum.

Substitution of expressions (5.27), (5.28), and (5.29) into equation (5.26), and writing $S_{g,1}$ in place of $\int_{\Omega_1} S_{0,g}(r, \Omega) d\Omega$, then yields the final form of the multi-group flux equation for direction 1 of Fig.

5.2(e):

$$\Omega \cdot \nabla \phi_{g,1}(r) + \bar{\sigma}_T \phi_{g,1}(r) = \sum_{g'=1}^G \sigma_{g' \rightarrow g} \phi_{g',1}(r) + \sum_{g'=1}^G \sum_{k=2}^4 \sigma_{g' \rightarrow g} \phi_{g',k}(r) + S_{g,1} \quad (5.32)$$

$\Omega_1 \rightarrow \Omega_1$ $\Omega_k \rightarrow \Omega_1$

c) The four-flux equations in cylindrical polar co-ordinates

For the plume flows considered in this study, interest lies mainly in axi-symmetric flow cases. The complete equations of the flux model will therefore next be written in cylindrical polar coordinates. In doing this subscript g of equation (5.32) will be dropped, and the ϕ_g 's will be written as I 's and J 's (i.e. I_x , J_x and I_r , J_r) as shown in Fig. 5.2(a). It may be noted that for our particular purpose the transfer probabilities in equation (5.32) now have the following significance:

$$\sigma_{g' \rightarrow g} = \sigma_s, \text{ the transfer probability per unit energy} \quad (5.33)$$

$\Omega_1 \rightarrow \Omega_1$ (for transfer within same angle)

$$\sigma_{g' \rightarrow g} = \frac{\sigma_s}{4\pi}, \text{ the transfer probability per unit energy per unit} \quad (5.34)$$

$\Omega_k \rightarrow \Omega_1$ solid angle (for inter-angular transfer)

For the sake of simplicity, the summation over various energies is temporarily suppressed, and the scattering is assumed to be isotropic. The scattering terms in equation (5.32) may then be written as

$$\sum_{g'=1}^G \sigma_{g' \rightarrow g} \phi_{g',1}(r) = \sigma_s \phi_1 \quad (5.35)$$

$$\text{and, } \sum_{g'=1}^G \sum_{k=2}^4 \sigma_{g' \rightarrow g} \phi_{g',k}(r) = \frac{\sigma_s}{4\pi} \sum_{k=2}^4 \phi_k \Delta\Omega_{k,1} \quad (5.36)$$

$\Omega_k \rightarrow \Omega_1$

where $\Delta\Omega_{k,1}$ is the angle between direction k and direction 1.

And, allowing for some overlap in the angular summations as shown in Fig. 5.2(f), the one-dimensional transport equations for the fluxes I and J in Fig. 5.2(a) become:

Axial direction

$$\frac{\partial I_x}{\partial x} = -\sigma_T I_x + \sigma_s I'_x + \frac{\sigma_s}{4} (2J_x + I_r + J_r) + S_0 \quad (5.37)$$

$$\frac{\partial J_x}{\partial r} = \sigma_T J_x - \sigma_s J'_x - \frac{\sigma_s}{4} (2I_x + I_r + J_r) - S_0 \quad (5.38)$$

Radial direction

$$\frac{\partial}{\partial r}(rI_r) = r\{-\sigma_T I_r + \frac{J_r}{r} + \sigma_s I'_r + \frac{\sigma_s}{4}(2J_r + I'_x + J_x) + S_0\} \quad (5.39)$$

$$\frac{\partial}{\partial r}(rJ_r) = r\{\sigma_T J_r + \frac{J_r}{r} - \sigma_s J'_r - \frac{\sigma_s}{4}(2I_r + I'_x + J_x) - S_0\} \quad (5.40)$$

In equations (5.39) and (5.40), the term J_r/r enters as a result of the cylindrical geometry illustrated in Fig. 5.2(b): a fraction dr/r of J which enters the cylinder of radius $r + dr$ does not enter the cylinder of radius r . Here the particular form of the term J_r/r arises because of assumption of isotropic distribution of the spectral energy density (ϕ or J) with angle. Other assumptions about the directional dependence of the spectral energy density would result in a non-unity factor multiplying the term J_r/r .

Each pair of first-order flux equations in each direction may be combined to yield a single second-order equation. To do this, the following quantities may be defined:

$$\text{Resultant directional photon flux, } Q \equiv I - J \quad (5.41)$$

$$\text{Sum of directional photon fluxes, } F \equiv (I + J)/2 \quad (5.42)$$

Both Q and F assume the subscripts x or r , depending on the direction of fluxes considered.

Addition of equations (5.37) and (5.38), or of equations (5.39) and (5.40) results, for each pair, in an expression for the resultant directional photon flux

$$Q = \Gamma\{-2\frac{\partial E_n}{\partial n} + \sigma_s Q'_n\} \quad (5.43)$$

where n stands for x or r , and

$$Q' = I' - J'$$

$$\Gamma_x \equiv \frac{1}{\sigma_T + \sigma_s/2} \quad (5.44)$$

$$\Gamma_r \equiv \frac{r}{1 + r(\sigma_T + \sigma_s/2)}$$

Subtraction of the two respective pairs of equations and combination with equation (5.43) yields:

Axial direction

$$\frac{\partial}{\partial x} \left\{ \frac{1}{(\sigma_T + \sigma_S/2)} \left(2 \frac{\partial F_x}{\partial x} - \sigma_S Q'_x \right) \right\} = 2(\sigma_T F_x - S_0 - \sigma_S F'_x) - \sigma_S (F_x + F_r) \quad (5.45)$$

Radial direction

$$\frac{1}{r} \frac{\partial}{\partial r} \left\{ \frac{r^2}{1 + r(\sigma_T + \sigma_S/2)} \left(2 \frac{\partial F_r}{\partial r} - \sigma_S Q'_r \right) \right\} = 2(\sigma_T F_r - S_0 - \sigma_S F'_r) - \sigma_S (F_x + F_r) \quad (5.46)$$

These are the final equations, but with summation over various energies suppressed.

d) The complete energy dependent flux equations

Introducing the complete summation over the various energies in order to account for the simplified forms introduced by expressions (5.35) and (5.36), equations (5.45) and (5.46) become:

Axial direction

$$\begin{aligned} \frac{\partial}{\partial x} \left\{ \Gamma_x \left(2 \frac{\partial F_x}{\partial x} - \sum_{g'=1}^G \sigma_S Q'_{x,g'} \right) \right\} = 2(\sigma_T F_x - S_0 - \sum_{g'=1}^G \sigma_S F'_{x,g'}) \\ - \sum_{g'=1}^G \sigma_S (F_x + F_r)_{g'} \end{aligned} \quad (5.47)$$

Radial direction

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left\{ \Gamma_r \cdot r \left(2 \frac{\partial F_r}{\partial r} - \sum_{g'=1}^G \sigma_S Q'_{r,g'} \right) \right\} = 2(\sigma_T F_r - S_0 - \sum_{g'=1}^G \sigma_S F'_{r,g'}) \\ - \sum_{g'=1}^G \sigma_S (F_x + F_r)_{g'} \end{aligned} \quad (5.48)$$

$$\text{And, } Q_n = \Gamma_n \left\{ -2 \frac{\partial F_n}{\partial n} + \sum_{g'=1}^G \sigma_S Q'_{n,g'} \right\} \quad (5.49)$$

These are the full working equations. Note that Q' and F' represent values for energy bands other than that considered.

e) The flux equations for a single energy band

The flux equations for a single energy band may often be required. Then, writing the total cross-section σ_T as

$$\begin{aligned} \sigma_T &= \text{absorption cross-section} + \text{self-scattering cross-section} \\ &= \sigma_a + \sigma_s \end{aligned} \quad (5.50)$$

equations (5.47) and (5.48) become

Axial direction

$$\frac{\partial}{\partial x} \left\{ \frac{2}{(\sigma_a + 3/2\sigma_s)} \frac{\partial F_x}{\partial x} \right\} = 2(\sigma_a F_x - S_0) - \sigma_s (F_r - F_x) \quad (5.51)$$

Radial direction

$$\frac{1}{r} \frac{\partial}{\partial r} \left\{ \frac{2r^2}{1 + r(\sigma_a + 3/2\sigma_s)} \frac{\partial F_r}{\partial r} \right\} = 2(\sigma_a F_r - S_0) - \sigma_s (F_x - F_r) \quad (5.52)$$

And,

$$Q_n = \Gamma_n \left\{ -2 \frac{\partial F_n}{\partial n} \right\} \quad (5.53)$$

Here, again, isotropic scattering is assumed.

f) Treatment of the boundaries

Generally, the boundaries of the domain of computation may absorb part of the incident radiation as well as emitting fresh radiation owing to finite concentration of fission products. If at a particular boundary,

I = outgoing photon flux

and J = incoming photon flux

and if the boundary absorbs a fraction α_B of the outgoing radiation, and also has a finite emission ($\frac{S_0}{\sigma_T}$, where $S_0 = S_B$), then

$$J = (1 - \alpha_B)I + e_B \frac{S_B}{\sigma_T} \quad (5.54)$$

where e_B = emissivity of the boundary (5.55)

= 1, if the boundary is emitting radiation

= 0, if the boundary is not emitting radiation

From the definition of F and Q , it follows that

$$I = F + Q/2 \quad (5.56)$$

$$F = F - Q/2$$

Introducing these into equation (5.54), and substituting for Q from equation (5.49), the following general boundary condition for equation (5.47) and (5.48) is obtained:

$$\alpha_B F_n \pm \frac{(2-\alpha_B)}{2} \Gamma_n \left\{ 2 \frac{dF_n}{dn} - \sum_{g'=1}^G \frac{\sigma_s}{g' \rightarrow g} Q'_{n,g'} \right\} = e_B \frac{S_B}{\sigma_T} \quad (5.57)$$

where the +ve sign is used at boundaries where +ve co-ordinate flux is the outgoing flux, and -ve sign is used at boundaries where -ve coordinate flux is the outgoing flux.

g) Auxiliary information

For "closure" of the differential equations (5.47) and (5.48) and the related boundary conditions, information needs to be provided of the total cross-section $\sigma_T (= \sigma_a + \sigma_s)$, the energy scattering cross-section σ_s , and the source rate S_0 .

$g' \rightarrow g$

Absorption cross-section, σ_a

Table 6 gives the values of the absorption cross-section σ_a for 12 energy groups interpolated from data given by Hubbel and Berger (1968) for air. The 12 groups are postulated according to the WEERIE program (Clarke, 1973). They may be further condensed into smaller groups (e.g. 3, 4, or 5 groups) as required.

Self-scattering cross-section, σ_s

For any given energy group, σ_s measures both energy scattering effects and inter-angular scattering effects. Here it may be assumed, though strictly incorrect, that for a single energy group the self-scattering brings about mainly inter-angular re-distribution, with the radiation energy remaining mainly within the group. Various values of σ_s may therefore be used, depending on the level of inter-angular re-distribution required.

Energy Scattering Cross-section, $\sigma_{s_{g' \rightarrow g}}$

The energy scattering cross-section $\sigma_{s_{g' \rightarrow g}}$ determines the inter-group transfers when more than one energy group is employed. σ_s is obtained by combining the Klein-Nishina formula and the Compton equation relating photon energy and scattering angle (Cashwell and Everett, 1959):

$$\sigma_E(E') = \frac{\pi r_0^2}{E} \left\{ \frac{E'}{E} + \left(\frac{2}{E} + \frac{1}{E^2} \right) + \left(E - \frac{2}{E} - 2 \right) \left(\frac{1}{E'} \right) + \left(\frac{1}{E'} \right)^2 \right\} \quad (5.58)$$

$$\sigma_E(E) = 2\pi r_0^2 \left\{ \frac{1+E}{(1+2E)^2} + \frac{2}{E^2} + \frac{E^2 - 2E - 2}{2E^2} \log_e (1+2E) \right\} \quad (5.59)$$

And the scattering kernel $\sigma_{s_{g' \rightarrow g}}$ is given by

$$\sigma_{s_{g' \rightarrow g}}(E' \rightarrow E) = \frac{\sigma_E(E')}{\sigma_E(E)} \quad (5.60)$$

where E = initial energy (the maximum)

E' = transferable energy (i.e. energy groups lower than E)

$r_0 = e^2/m_0c^2 = 0.28183 \times 10^{-12}$ cm, the classical Thompson radius of an electron

For the particular usage of E and E' in equations (5.58), (5.59), and (5.60) each is made dimensionless as follows:

$$E \text{ or } E' = \frac{\text{Energy in MeV}}{0.51083 \text{ MeV}} \quad (5.61)$$

That is, they are measured in numbers of photon energies (units, mc^2 ; and $1mc^2 = 0.51083 \text{ MeV}$).

Source rate, S_0

The source per unit volume, S_0 , of γ -heating in a plume element depends on the concentration of the fission products within the element. It will be assumed that the source rate is proportional to the β heating within the element; that is,

$$S_0 = c_0 \dot{Q}_\beta \quad (5.62)$$

where c_0 is a constant and $\dot{Q}_\beta$ is as defined by equation (5.10).

$$\text{Hence } S_0 = c_0 (\dot{q}_f \rho m_j) \quad (5.63)$$

where m_j is the mass concentration of fission products, and $\dot{q}_f$ is the specific heating rate of fission products as defined by equation (5.7).

h) Solution of the flux equations

For the solution of the differential equations (5.47) and (5.48), the equations and their related boundary conditions are cast into finite-difference forms in accordance with the practices given in Chapter 3. The calculation of the radiation fluxes is included in the iteration cycle used for the other variables.

At each cycle of iteration, the local source rates S_0 are deduced from the prevailing field of concentration of fission products. Then the F 's are calculated and employed to evaluate the γ -ray heating source, per unit volume $\dot{Q}_\gamma$ (see equation 5.1) of the thermal energy equation from

Table 6: Values of absorption coefficients, σ_a

Energy group g	Mean energy of photon (or γ -ray), E (MeV)	σ_a
		Density of air ⁺ (cm ² /gm)
1	0.27	0.11
2	0.51	0.086
3	0.76	0.072
4	1.0	0.0635
5	1.55	0.051
6	2.2	0.044
7	2.5	0.040
8	2.9	0.036
9	3.5	0.0337
10	4.5	0.0297
11	5.4	0.0266
12	6.2	0.0255

+ Density of air, $\rho_{\text{air}} = 1.293 \times 10^{-3} \text{ gm/cm}^3$

$$\begin{aligned} \dot{Q}_\gamma &= \frac{dQ_x}{dx} + \frac{1}{r} \frac{d(rQ_r)}{dr} \quad (\text{MeV/sec.m}^3) \\ &= 2\{\sigma_a(F_x + Fr) - 2S_0\} \end{aligned} \quad (5.64)$$

Here, $\dot{Q}_\gamma$ can be converted to J/sec.m³ by multiplying by 1.6×10^{-13} .

(i) Assessment of the flux model

There are some aspects of the flux model that may limit its accuracy. For example, γ -radiation flux is anisotropic, but the flux model is based on isotropic assumptions; further, the assumed angular coupling of the fluxes is a gross simplification of the actual phenomena.

The accuracy of the model also depends on the ratio

$$\tilde{\kappa} = \frac{\text{absorption path length } (=1/\sigma_a)}{\text{size of the flow domain}}$$

The model becomes less accurate as $\tilde{\kappa}$ becomes very large.

5.4 COMPUTATIONAL DETAILS

The practices employed at the inlet and other boundaries, as well as for the turbulence constants, are as given in Section 4.3. In the computations, again a non-uniform grid with exponential expansion is employed in the radial direction while a uniform grid is used in the axial direction. With radioactivity introduced in the calculations, the overall length required of the domain of computation depends on the level of radioactivity in the modelled plume, and the grid size is chosen to match the size of the domain of computation.

For the particular case of the full-scale plume stack calculation to be discussed in the next section, the overall length of the domain of computation is 32 stack diameters while the width is 12 diameters. For this case, a grid size of 32 nodes in the axial direction and 19 in the radial direction (i.e. a 32 x 19 grid) is used, as experience with the non-radioactive calculations indicated that such a grid will suffice for this case.

The SIMPLE algorithm is employed in the calculations, as this was used to initiate the radioactive plume study and it was decided that to repeat the calculations using the EMIT algorithm could serve no further purpose. Values employed of the under-relaxation factors f defined by equation (3.62), the number of "sweeps" of the tri-diagonal matrix algorithm (TDMA), and the manner of assessment of convergence are as given in Section 4.33.

The radiation flux equations are not solved at every iteration; rather they are visited after every 40 to 50 iterations of the fluid flow equations, with the radiation heat sources in the thermal energy equation taken to be those due to the prevailing radiation fluxes. The

equation for concentration of fission products is however, solved at each iteration.

When solved, the radiation flux equations require some 20 to 50 iterations to give converged solutions, and these iterations are completed before the solution of the fluid flow equations is continued. A total of about 800 iterations of the fluid flow equations are required to give converged solutions when γ -heating is calculated, and about 500 iterations when only β -heating is calculated.

All computations were carried out on a CDC 6600 computer, and the time required for each iteration (32 x 19 grid) is about 1.79 seconds.

5.5 PRESENTATION AND DISCUSSION OF RESULTS

5.51 Tests of the flux model equations

Before the flux model equations are linked up with the fluid flow equations, it is necessary to check that the finite difference forms of the flux equations yield the correct solutions for simple test cases. To this end, first each flux equation is decoupled and tested separately for conditions where analytical solutions are known; second, both equations are solved simultaneously for a uniform concentration field. The equations are tested for a single energy band.

(i) The decoupled equations (no scattering)

If the scattering cross-section σ_s is set to zero in equations (5.51) and (5.52), each of the equations becomes independent of the other.

Axial direction

If the flux J_x is set to zero equation (5.51) for the flux sum $F_x \{=(I_x + J_x)/2\}$ becomes identical with the equation

$$\frac{dI_x}{dx} + \sigma_a I_x - S_0 = 0 \quad (5.65)$$

where σ_a is the absorption cross-section. The general analytical solution of equation (5.65) is

$$I = \frac{1}{\sigma_a} \{ c e^{-\sigma x} + S_0 \} \quad (5.66)$$

where c is a constant to be determined from boundary conditions.

For example, if at $x = 0$, $I_B = \sigma_a I_B - S_B$, where subscript B stands for the boundary, then $c = \sigma_a I_B - S_B$. That is,

$$I = \frac{1}{\sigma_a} \{ (\sigma_a I_B - S_B) e^{-\sigma_a x} + S_0 \} \quad (5.67)$$

Thus, when equation (5.51) for the flux sum $F_x (= I_x/2$ in this case) is solved with boundary conditions defined by equation (5.67), the F-equation should yield identical results with equation (5.67) throughout the domain of computation. This was found to be the case when this test was applied to a uniform rectangular concentration field.

Radial direction

Again, if the flux F_r is set to zero, equation (5.52) for the flux sum $F_r \{ = (I_r + J_r)/2 \}$ becomes identical with the equation

$$\frac{1}{r} \frac{d}{dr} (r I_r) = -\sigma_a I_r + S_0 \quad (5.68)$$

The general analytical solution of this equation is

$$I = S_0 \cdot \frac{1}{\sigma_a} \left(1 - \frac{1}{r \sigma_a} \right) + \frac{C}{r} e^{-\sigma_a r} \quad (5.69)$$

where C is an integration constant. If at $r = R$, $I = I_R$, say, then

$$C = e^{\sigma_a R} \left\{ R I_R - \frac{1}{\sigma_a} S_R \left(R - \frac{1}{\sigma_a} \right) \right\}$$

Thus,

$$I = S_0 \cdot \frac{1}{\sigma_a} \left(1 - \frac{1}{r \sigma_a} \right) + e^{\sigma_a (R - r)} \left\{ \frac{R}{r} I_R - \frac{1}{\sigma_a} S_R \left(\frac{R}{r} - \frac{1}{r \sigma_a} \right) \right\} \quad (5.70)$$

As will be expected, when $R \rightarrow \infty$ equation (5.70) reduces to equation (5.67) for the plane case. Again, if equation (5.52) for the flux sum $F_r (= I_r/2)$ is solved for a cylindrical shell with boundary conditions as defined by equation (5.70), the F-equation should yield identical results with equation (5.70) throughout the computational field. Again, this was found to be the case when this test was applied to a cylindrical shell of uniform concentration.

(ii) The coupled flux equations

The coupled flux equations, that is those with scattering present, are tested for uniform concentration fields, with boundary conditions as defined by equation (5.57). For a single energy band, scattering will tend to make the radiation field more isotropic (i.e. it will make the fluxes become independent of direction), since as demanded by equations (5.51) and (5.52), $|F_r - F_x|$ tends to zero as σ_s becomes very large.

a) With the field of F_x fixed at the value for zero scattering, F_r is solved with varying values of the scattering cross-section σ_s . The axial distribution, at a fixed radius, of F_r for varying values of σ_s is shown in Fig. 5.3. It will be observed there that as σ_s becomes very large, F_r approaches the same distribution as F_x as demanded by the isotropic relations.

b) Next, with the field of F_r fixed at the value for zero scattering, F_x is solved with varying values of the scattering cross-section σ_s . For a fixed radius, the axial distribution of F_x for varying values of σ_s is shown in Fig. 5.4. Again, the distribution shows that F_x attains the same distribution as F_r when σ_s becomes very large.

c) Both F_x and F_r are next solved simultaneously. The axial and radial profiles of both quantities shown in Figs 5.5 and 5.6 indicate that the distributions for both quantities become identical as σ_s becomes very large.

5.52 The Radioactive Plume Predictions

a) β -heating effect for conditions of Sneek and Brown (1974) experiment

The first part of the radioactive plume predictions was carried out by studying the effect of β - heating on the rise height of a hypothetical plume corresponding to the conditions of the Sneek and Brown (1974) experiment which has been discussed in Chapter 4 (Section 4.42). The nozzle and other boundary conditions are as discussed in Section 4.42.

The sensible heat per unit mass of plume fluid released from the nozzle is 9×10^3 J/s. Various values, ranging from 0 to $0.2S^{-1}$, were employed for the heating parameter n_r which is defined by equation (5.4). The concentration m_j is zero at the outer boundaries, and the boundary radiation fluxes are as defined by equation (5.57).

The predicted axial distributions of the mean velocity u_z/u_0 , excess temperature θ_z/θ_0 , and concentration of fission products m_z/m_0 are shown in Fig. 5.7. The top of the plume occurs where u_z/u_0 falls to zero. Two major features are discernible in the predictions:

- Changes in the parameter n_r appear to produce a non-linear change in the rise height: for n_r less than 0.1 there is hardly a recognisable difference in the rise height while for a value of 0.2 there is a dramatic increase in the height of about 60%;
- For the conditions specified by the model of Sneck and Brown, additional rise due to radioactivity is negligible except when fission product activity (as measured by the value of n_r) is considerably large.

In addition, the level of zero buoyancy (i.e. where θ_z falls to zero) also occurs at about 60% way to the top of the plume in all cases shown in Fig. 5.7.

b) Full-scale plume stack

The main radioactive plume predictions made in this study were made for the conditions of a full scale plume stack, with dimensions as given by Robins (1973) for some power stations of the Central Electricity Generating Board. The following are the full conditions for which the predictions are made: $d = 5\text{m}$, $U_0 = 1.5\text{ m/s}$, $T_0 = 302.8^\circ\text{K}$, $T_{\infty,0} = 290^\circ\text{K}$, and $dT_{\infty}/dx = 0.0243^\circ\text{K/m}$. These conditions give an initial Froude number Fr_0 of 2.1, and a sensible heat release of 1.5×10^4 J/S per unit mass

of plume fluid from the stack. Thus, as compared with the laboratory model of Sneek and Brown, the specific sensible heat release is nearly doubled, while the atmospheric stratification is reduced by some 99%.

For the calculation of γ -heating effect, a single energy band of 1MeV is used; from Table 6 the corresponding absorption coefficient σ_a is 0.0082m^{-1} . Further, the source constant C_0 defined by equation (5.62) is taken to be unity; and the self-scattering coefficient σ_s (see equation 5.50) is chosen to be 0.246 (or $30\sigma_a$) as the value gives adequate inter-angular re-distribution of the fluxes.

The Flow Field

Fig. 5.8(a), (b), (c), (d), and (e) give an overall picture of the predicted flow field for both non-radioactive and radioactive ($\beta + \gamma$ heating; $n_r = 4 \times 10^{-3}$) cases of the full-scale plume stack study. Fig. 5.8(a) shows the vector plots of the mean velocity within the domain of computation. Shown there are arrows ($\uparrow$) and ($\uparrow$) which are as defined in chapter 4. Again, marked on the plots in dotted lines are the approximate boundaries of the plume: the lower boundaries are defined as the approximate points at which the velocity vector (resultant of U and V) is 90% less than the corresponding centre-line vector, while the upper boundaries give the approximate points where U (or V for the horizontal part) is about 98% less than the stack exit velocity.

It should be noted that, owing to the much greater height of the radioactive plume, the entrainment flow in the lower portion of that plume is nearly horizontal for a greater part of the domain of computation. Comparatively, for the non-radioactive plume, the flow in the lower region of the ambient fluid is inclined to the horizontal line. Thus the ambient flow characteristics differ in both cases.

The plume boundaries identified in Fig. 5.8(a) are also shown in Fig. 5.8(b), (c), (d), and (e). Fig. 5.8(b) shows the predicted

temperature contours (isotherms) within the flow field. The values at the outer boundaries of the flow domains are those fixed for the ambient fluid in the computations. Owing to effect of radioactive heating, at any given distance from the stack the temperature is higher within the radioactive plume than in the non-radioactive one. It may be noted that in the lower portion of the former plume, where the ambient fluid is little disturbed and the entrainment flow is horizontal, the contours are essentially horizontal outside the plume; however, in the upper region of the lower plume boundary, where the entrainment flow is not horizontal (compare with Fig 5.8(a)), the contour tends to have positive slope outside the plume as is also the case for the lower region of the non-radioactive plume here the entrainment flow is not horizontal. Further, the contours in either plume tend to be horizontal again in the region where the plume fluid spreads out horizontally and results in a near horizontal flow (compare with Fig. 5.8(a)).

Fig. 5.8(c) shows the contours of the concentration of fission products m_j . For the non-radioactive plume calculation, fission products are assumed to be present in the plume but with zero heating; this is done in order to effect a complete comparison between the two flow cases. At corresponding axial locations within the plumes, the concentration of fission products is much higher for the radioactive plume.

Fig. 5.8(d) shows the contours of turbulence energy ($k^* = K/U_0^2$) within the flow field. In both cases, as has been shown also in Fig. 4.20, very high level of turbulence is concentrated in the region immediately surrounding $y/r_0=1$. Again, much of the turbulence generated in this zone is transferred radially outwards and towards the axis of symmetry in addition to being convected axially. However, owing to the difference in the ambient flow in the lower portions of both plumes, the turbulence spreads out much more in this zone for

the non-radioactive case. It will be observed that for the radioactive case, the turbulence spreads out into the ambient fluid only in the upper region where flow curvature and/or horizontal spreading of the plume fluid occurs (compare with Fig. 5.8a).

The contours of the dissipation length scale ($l_\epsilon^* = l_\epsilon/d$) within the flow field are shown in Fig. 5.8(c). In both cases, within the plume boundary, and especially in the region beyond about 60% way to the top of the plume, the magnitude of the length scale is large. Owing to the difference in the flow structure within the lower portion of the ambient fluid, the distributions within this zone of l_ϵ are quite different.

Axial distribution of variables

Fig. 5.9 shows, for β -radiation heating only, the influence of the heating parameter n_r (see equation 5.4) on the axial distribution of the mean velocity U_z/U_0 , the excess temperature θ_z/θ_0 , and the concentration of fission products m_z/m_0 . For this case, the influence of radioactive heating becomes appreciable for a value of n_r of about 4×10^{-3} . As compared with similar tests earlier reported for the model of Sneck and Brown, a much smaller value of n_r is now required to produce some appreciable change in the height of the plume owing to the fact that, as indicated earlier, the specific heat release at the stack is now about doubled and the ambient stratification reduced by some 99%.

Again, it will be noted in Fig. 5.9 that, as was the case with the study on the model of Sneck and Brown, changes in the value of the heating parameter n_r do not produce a linear increase in the height of the plume; for n_r of 4×10^{-3} the increase in the height is only about 4%, while for n_r of 6×10^{-3} the increase is about 36%. Here again the level of zero buoyancy occurs about 60% way to the top of the plume in all cases.

Fig. 5.10 shows a similar plot as for Fig. 5.9, but now for a fixed value of the heating parameter n_r ($= 4 \times 10^{-3}$) and for both γ -heating and $\beta + \gamma$ heating. The value of the source rate constant C_0 (see equation 5.62) of 1.0 used in the calculation produces in each plume element a γ -heating rate more than double the β -heating. This is reflected in the considerable increase in the rise height (about 70% increase) when γ -heating is included for n_r of 4×10^{-3} (compare with Fig. 5.9 for the β -heating at the same value of n_r). Addition of β -heating to the γ -heating (i.e. combined $\beta + \gamma$ heating) at this value of n_r produces a comparatively small change in the rise height. The level of zero buoyancy again occurs in all cases at about 60% way to the top of the plume.

Fig. 5.11 shows the axial distribution of the turbulence energy k_ϵ/U_0^2 , the dissipation length scale l_ϵ/d , and the plume half-width $y_{0.5}/d$ for both non-radioactive and radioactive ($\beta + \gamma$ heating) cases. The distributions of l_ϵ/d and $y_{0.5}/d$ confirm that in the lower portions of the plumes the non-radioactive case spreads much more. The turbulence level k_ϵ/U_0^2 in the initial 30 meters of the plume is lower for the radioactive case, but becomes higher thereafter.

Radial distribution of variables

Fig. 5.12 shows the radial distributions of the mean velocity at various non-dimensional heights $x/H_{\max}$ of the plume, where $H_{\max}$ is the maximum height attained by the plume. While for the non-radioactive plume the profiles are similar for values of $x/H_{\max}$ up to about 0.55, for the radioactive case they show varying trends within the same height limit in the outer region of the plume ($y/y_{0.5} > 1.6$), although they are similar in the inner region.

In Fig. 5.13 are shown the profiles of excess temperature, the plots are made against $y/y_{0.5T}$ and y/d in the lower and upper regions of the plume, respectively. For the non-radioactive plume, the profiles

are self similar for $x/H_{\max} < 0.45$, but they are not similar in the upper region. For the radioactive plume, the profiles are similar only in the inner region ($y/y_{0.5T} < 1.6$) of the flow for $x/H_{\max} < 0.4$; again, in the upper region of the plume no self-similarity exists.

Fig. 5.14 shows the profiles of the concentration of fission products; the plots are made against $y/y_{0.5m}$, where $y_{0.5m}$ is the half-width of the fission products. For the non-radioactive plume, again similarity of the profiles exists for $x/H_{\max}$ less than 0.45, although thereafter they become non-similar. For the radioactive plume, the profiles for $x/H_{\max} < 0.4$ are similar except in the outer region of the plume ($y/y_{0.5m} > 1.6$); in the upper region of the plume the profiles are not similar.

Finally, the profiles of the turbulence energy are shown in Fig. 5.15. Here no self-similarity exists in either plume. It may be noted that owing to the fact that the entrainment flow is inclined to the horizontal in the lower region of the non-radioactive plume (see Fig. 5.8a), the turbulence has higher values in the ambient fluid in this case than it does within the lower region of the radioactive plume. This is due to the consequent difference in the two cases of the production of k by shear, defined by the term $G = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j}$ (see Appendix A). In the entrainment flow of the lower region of the radioactive plume, the vertical component of mean velocity, U , is approximately zero; thus $G \approx \mu_t \left\{ 2 \left(\frac{\partial v}{\partial r} \right)^2 + \left(\frac{v}{r} \right)^2 \right\}$ (see also Appendix C). However, for the same region of the non-radioactive plume U and V are of comparable value and the complete expression for G (see Appendix C) holds; thus the higher values of k in this case.

5.6 CONCLUDING REMARKS

This chapter has presented the work on radioactive plumes as was undertaken in this study. The main features of the chapter may be summarized as follows:

(i) Little work has been done previously on radioactive plumes. The few available works done previously are theoretical in nature, and are based on integral approaches, with the over-riding question avoided of gamma radiation transport. The need therefore arises for numerical studies that do not suffer from the many defects of integral approaches, and for establishment of ways in which γ -radiation transport may be accounted for. This chapter has attempted to satisfy these needs.

(ii) Models have here been initiated for the evaluation of heating due to β -radiation and α -particle emission, and, most important, for the effect of γ -radiation. These models have further been tested here, and the results look promising.

(iii) The results confirm that γ -radiation is an important factor in radioactive heating in a plume. They also indicate that the effect of radioactivity on the rise of the plume in calm, stably-stratified surroundings depends on both the ambient stratification as well as the specific heat release of the fission products. When the ambient stratification is severe, much higher heat release of fission products appears to be necessary in order to make appreciable difference in the rise height of the plume.

PART IV:

THE INVESTIGATION OF CAVITY FLOWS

CHAPTER 6

RECTANGULAR CAVITY FLOWS

6.1 INTRODUCTION

The term "cavity flow", first introduced by Squire (1956), is generally employed to describe separation regions consisting of a central recirculating core surrounded by a continuous boundary layer, illustrations of which are given in Fig.6.1(a) and (b). Such flow regions result from the separation at a sharp salient edge of a solid surface of an oncoming streamline motion. A thin region of high shear stress develops from the separation edge, and this provides the requisite energy to maintain the cavity flow.

The driving streamline motion may be provided by an oncoming free stream or channel flow on the one hand or by a sliding wall on the other. The resultant cavity flow may be steady or unsteady, two- or three- dimensional, and laminar or turbulent. The study reported in this chapter embraces all three driving mechanisms for turbulent flows and highlights the similarity between the various cases, although computations are made only for the channel-flow and sliding-wall cases.

Section 6.2 provides the starting point; there both the previous theoretical and experimental work on flows in rectangular cavities are presented. To achieve a more unified treatment, the previous work are reviewed for all three driving mechanisms, and the close similarity between them are there indicated. Further, in order to provide a better understanding of rectangular cavities reference is also made to cavities other than square ones, although computations are made only for the latter.

Then, in Section 6.3 the details are given of the present computations, starting with the method of application of the solution procedures

and the turbulence model. Then there follows the presentation and discussion of the results obtained, in Section 6.4. Finally, in Section 6.5 are presented some concluding remarks.

6.2 PREVIOUS WORK

6.21 Previous Theoretical Work

(i) Analytical Studies

All analytical methods for turbulent cavity flows have had two main elements : firstly, an identification of separate flow regimes (a central core region, and an inner boundary layer) within the cavity as shown in Fig. 6.1; secondly, the application of different theories to the separate regimes, with the gross features of the flow obtained by matching the solutions for the different regimes.

The core, which occupies the bulk of the cavity, is generally assumed to be a region in which viscous effects are small. Thus, even if the boundary layer surrounding the core is fully turbulent, it is implied in the treatment that the core itself contains negligible stresses due to turbulence. On this premise evolves a hypothesis to which strong mathematical support is lent by Batchelor (1956) and physical backing by Squire (1956) that the vorticity throughout the core is uniform. Further, following similar lines, Burggraf (1965) and Grimshaw (1969) showed that, like the vorticity, the temperature is uniform in non-isothermal flows even when buoyancy effects are present.

Thus, throughout the core,

$$\begin{aligned}\omega &= \omega_c \quad (\text{constant}) \\ T &= T_c \quad (\text{constant})\end{aligned}\tag{6.1}$$

where ω and T are the vorticity and temperature respectively, and subscript c stands for the value within the core. This result is argued to apply to any given region of closed streamlines in a cylinder⁺, rectangular cavity, etc.

⁺ See footnote on page 150.

By definition, (for 2-D flows)

$$w = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \quad (6.2)$$

and the stream function ψ is defined by

$$U = \frac{\partial \psi}{\partial y}, \quad V = -\frac{\partial \psi}{\partial x} \quad (6.3)$$

Substitution of (6.3) in (6.2) yields

$$\nabla^2 \psi = -\omega \quad (6.4)$$

Hence, within the core, from (6.1) and (6.4),

$$\nabla^2 \psi = -\omega_c \quad (6.5)$$

$$T = T_c$$

The values of ω_c and T_c require to be determined, and are obtained from consideration of the boundary streamline that separates the core from the inner boundary layer.

The inner boundary layer is assumed by the various workers to be a viscous area, and the Reynolds number of the flow is taken to be very high so that this region is very thin. Further, the motion in the region is assumed to be steady, incompressible, and two dimensional, and also governed by the usual-form of the boundary layer equations which, assuming zero pressure gradient, can be linearised in the Oseen manner to yield the following forms which are analogous to one-dimensional heat conduction equation:

$$\begin{aligned} U \frac{\partial U}{c \partial x} &= \nu \frac{\partial^2 U}{t \partial y^2} \\ U \frac{\partial \theta}{c \partial x} &= \Gamma \frac{\partial^2 \theta}{t \partial y^2} \end{aligned} \quad (6.6)$$

+ For a circular cavity, expression (6.1) for the vorticity implies that the core must rotate as a solid body and the central velocity profiles will then be linear. In rectangular cavities, uniform vorticity does not imply a solid body rotation of the core and linearity of the velocity profiles throughout the core; but near the centre of the core in a square cavity the profiles are linear (Mills, 1961).

where U_c is the velocity at the outer edge of the boundary layer, and θ is any linear function of the total enthalpy of the flow.

Equations (6.5) and (6.6) have been employed by the various workers⁺ to obtain the gross features of flows with closed streamlines by matching the solutions for the two regions. Batchelor (1956), Squire (1956) and Grimshaw (1969) applied the treatment to a circular cavity, while Burggraf (1965), Mills (1961, 1965a) applied it to rectangular cavities. Recently, Grand (1975) extended the approach to include the effect of pressure gradients for rectangular cavities.

The analyses of these workers gave satisfactory results on such matters as the length of corner eddies and strength of the recirculating flow. In particular, the analysis of Grand for the laminar square cavity flow gave a value of the eddy strength that was close to the turbulent value. However, the previous analyses are limited by the inbuilt and rather over-riding assumptions, and the fluids engineer needs to have a fairly more general tool for the calculation of the mean flow and turbulence fields within the cavity in order to assess more accurately the drag and heat transfer associated with such cavities.

(ii) Numerical Studies

Cavity flows have formed the basis for the testing of various numerical techniques in the past, but these have been mainly for laminar flows. The works in this field include those of Burggraf (1966) Pan and Acrivos (1967), Gosman et al (1969), Mills (1965b), Mehta and Lavan (1969), Donovan (1970), and Torrance et al (1972), in all of which the convergence and accuracy of the numerical solutions have also been discussed.

There is comparatively little published work on the application of

⁺ Some authors (e.g. Mills, 1965a) introduced the stream function into the momentum equation for the boundary layer, using von Mises method (Schlichting, 1968, p.143).

numerical methods to turbulent cavity flow problems. Those available include the work of Greenspan (1969), Gosman et al (1969) and Chin et al (1973), and their methods employ the stream function and vorticity as the primary dependent variables together with a uniform eddy viscosity (except Gosman et al who employed a two-equation turbulence model). Moreover, only the non-buoyant case is studied by these workers.

6.22 Previous Experimental Work on Turbulent Rectangular Cavity Flows

(i) Free-Stream Drive

Previous experimental work on flow-driven cavity indicates that, depending on the depth to width ratio H/W of the cavity, the boundary layer separating at the leading edge subsequently reattaches itself either at the roof or along the downstream wall of the cavity.

The work of Mills (1961), Charawat et al (1961), Tani et al (1961), Moss, Baker, and Bradbury (1977) and others shows that for the range $0 < H/W < 0.1$ the flow reattaches itself on the roof of the cavity as shown in Figs. 6.1(b) and 6.2(a). For subsonic flow, Charawat's work and that of Tani as well as Bradbury indicate that a negative pressure exists on the upstream cavity-wall, followed initially by a further slight drop in pressure on the roof, and then a rather rapid rise indicating the reattachment of the separated flow. Tani's work, confirmed by the recent measurements of Bradbury also show that the turbulence and shear stress increase downstream of the leading edge in the mixing region, and the positions of their maxima coincide at the outset with the mean dividing streamline, but deviate outward as the reattachment point is approached. Further, the base pressure on the walls as well as the distributions across the mixing region of the mean velocity, turbulence intensity and shear stress are relatively insensitive to changes in the thickness of the boundary layer or in the height of the upstream cavity wall.

Within the same broad range of H/W illustrated in Fig. 6.2(a), Mills' work also showed that for extremely shallow cavities, $0 < H/W \leq 0.025$, the pressure at the outer edge of the downstream wall increase sharply with increasing H/W owing to direct impingement of the boundary layer flow. However, on further increase of H/W beyond the value of 0.025, there is a sudden reduction at this edge in pressure which reaches a minimum at $H/W = 0.1$ as a result of the reattachment point moving along the roof towards the downstream wall.

The experiments of Mills also established that a main vortex is formed in the cavity at the critical value of $H/W \approx 0.1$, and in the range $0.1 \leq H/W \leq 0.2$ the stagnation point moves down the downstream wall towards its outer edge (see Fig. 6.2b), resulting in decrease in curvature of the dividing streamline so that the pressure at this edge rises again. The work of Maull and East (1963) indicates that within this range, and indeed for $H/W < 0.45$, the cavity flow is two-dimensional.

The investigations of Roshko (1955) and Mills established that while the flow processes for $H/W < 0.25$ are steady, the vortex in the cavity is unsteady in the range $0.25 \leq H/W \leq 0.8$ (see Fig. 6.2c), and the pressure, while varying with time, can take any value between two limits. Similarly, Fox (1964, 1965) observed that the pressure coefficient C_p along the cavity walls is not uniquely specified by H/W alone within the range $0.57 \leq H/W \leq 0.8$, and values obtained depend also on time of measurement. There is no consensus of opinion as to the cause of the unsteadiness. Roshko explains it as being a consequence of switching back and forth between distinct equilibrium flow regimes, while Mills takes the view that Roshko's explanation is not likely but that the unsteadiness may in fact take the form of an alternate up and down movement of the eddy (transverse to the mainstream direction) in order to explain the nature of the unsteadiness in the recorded pressure.

However, Maull and East offer a different interpretation. Within the range $0.45 \leq H/W \leq 0.8$, their investigations as well as those of Kistler and Tan (1967) depict a three-dimensional flow pattern embodied in each of a series of what may be called a "unit cell" transverse to the main stream direction, each cell being a mirror image of those adjacent to it. Both the number of cells and their strength (as measured by some boundary pressure change across the cavity) depend on the geometry of the cavity. For a given cavity several cell configurations are sometimes possible depending apparently on the starting process or the type and location of any disturbances to the flow. Maull and East indicate that the unsteadiness in this range is a function of the geometry of the cavity, and resulted only when the latter is such that no regular cellular configurations can be produced.

The results of most previous experimental work show that "square-cavity" type flow, which is of primary concern in this study, occurs in the range $0.8 \leq H/W \leq 1.4$ covering slightly "undersquare" and slightly "oversquare" cavities. As observed by most previous workers (e.g. Roshko, 1955; Mills, 1961; Fox, 1964, 1965; Haugen and Dhanak, 1966), this regime is one of a stable flow with a steady main vortex and secondary corner eddies as shown in Fig. 6.2(d), with the main vortex in almost solid-body rotation for $H/W = 1$. The work of Maull and East also established that the cavity flow is again 2-D within the approximate "square cavity" range $0.8 \leq H/W \leq 1.3$; they explain the sharp change from intermittent to steady pressure, observed by Roshko as well as Mills around H/W of 0.8, as being caused by the sudden collapse of the 3-D cellular structure of the preceding flow regime.

For the special case of $H/W = 1$, the observations of Mills, complemented in part by those of Fox, indicate that near the centre of the cavity both streamlines and lines of constant velocity are almost circular, but on proceeding away from the centre they become nearly square.

Further, the streamline pattern is asymmetrical with respect to the cartesian co-ordinate axes of the cavity, and displaced slightly from the geometrical centre of the cavity. Mills' measurements also show that the velocity is uniform and of value $0.55U_{\infty}$ along the line joining the outer edges of the cavity, while the tangential velocity of the core is approximately $0.3U_{\infty}$, and the strength of the vortex, Ω , is about 0.10, where Ω is defined by

$$\Omega = \frac{\dot{Q}}{\rho U_{\infty} W} \quad (6.7)$$

with $\dot{Q}$ representing the rate of mass circulation in the vortex, and U_{∞} the free stream velocity. In addition Mills' work indicated that the dividing streamline meets the downstream wall at about $0.015H$ from the outer edge and it curves outside the cavity in the central region; also the width δ_0 of the outer boundary layer (see Fig. 6.1(a)) is given by

$$\delta_0 = 0.10X \quad (6.8)$$

where X is the downstream distance from the leading edge and δ_0 is the distance between $U = 0.55U_{\infty}$ and $U = 0.99U_{\infty}$.

Again for the square cavity at $H/W = 1$, Fox's measurements showed that the temperature within the vortex is low as compared with the temperature of the walls which are heated for heat transfer measurements. As regards turbulence quantities, the measurements within the free shear layer of Mills, and Haugen and Dhanak, as well as Majumdar and Maji (1973) show that both normal and shear stress exhibit maxima in the neighbourhood of the dividing streamline at the outset, but for the whole length of the mixing region these maxima are clearly associated with a line or ray lying just outside the cavity. The maxima themselves are approximately uniform along this line. These observations of maximum stresses and their loci are similar to those of Tani et al (1961) for shallow cavities of the type shown in Fig. 6.1(b). Tani's measurements of the turbulence

and shear stress for deep cavities show that they are less strong for a square cavity than for a shallower one (e.g. $H/W = 0.4$).

The work of Roshko, Mills, and Haugen and Dhanak shows that the range $1.4 \leq H/W \leq 2.4$ (see Fig. 6.2e) represents another regime of cavity flow, with two vortices of smaller strength than the vortex in a square cavity. Specifically, for $1.4 \leq H/W \leq 1.75$ there appear secondary vortices in transition, and at $H/W = 2$ two well-defined vortices of approximately equal strength are formed. An interesting feature established by the studies of both Roshko and Mills is that within the approximate range $1.4 < H/W < 2.4$ a "hysteresis" effect develops, with the pressure taking either of two discrete values but cannot wander in between. The observations of Maull and East as well as Kistler and Tan confirm that within this range the flow is again 3-D with the usual cellular structure. Maull and East suggest that for each value of H/W within this range there are two steady cellular configurations possible which give the two discrete pressure distributions.

The observations of Haugen and Dhanak further indicate that transition again takes place around $H/W = 2.5$ when the number of vortices oscillates between 2 and 3. Subsequently, for H/W of approximately 3, three vortices in a state of counter rotation to, and on top of, one another are formed. The studies of Maull and East and Kistler and Tan establish that for $H/W > 2.2$ the flow takes a permanent two-dimensional structure.

(ii) Channel-Flow Drive

By channel flow it is here implied a flow of the type studied by Laufer (1951), which is bounded by closely spaced walls in the cross-stream direction so that the velocity varies continuously (except in the vicinity of the centre of the flow) in that direction, and the mixing region of the driven "cavity flow" may consequently interact with the boundary layer of the outer wall of the channel.

The only experimental work known to the author on turbulent cavity flow driven by such mechanism is by Grand (1975) whose work covers both forced and mixed convection in the cavity. For both cases the channel flow is fully-developed upstream of the cavity, and measured velocities and turbulence intensities in the cavity are based on the mean velocity U_0 of the channel flow defined by (Laufer, 1951):

$$U_0 = U_{\max}/1.23 \quad (6.9)$$

where $U_{\max}$ is the maximum velocity through the channel.

Grand measured the mean velocities, turbulence intensities, mean temperatures and heat transfer rates for Reynolds numbers $Re (=U_0 W/\nu)$ of 5×10^4 to 4×10^5 for cavities of $H/W = 1, 2$ and a span-width ratio $S/W = 1$ (Span, S = dimension normal to plane of flow), with relative sizes of channel and cavity as shown in Fig.1.3 for the square cavity.

Grand's measurement of the hydrodynamic field for forced convection in the cavity shows that, as in the case of free-stream drive (e.g. Mills, 1961, Tani et al, 1961) the cavity flow is maintained by a mixing region of high turbulence intensity, with severe mean velocity gradient. Inside the cavity, as again for free stream drive, the mean velocity tends to be linear with gradient approximately 20 times less than the maximum gradient in the mixing layer. In the wall region profiles of velocity and turbulence intensities have their maxima at about the same locations, with the velocities being in good agreement with measurements of Roshko (1955) for free stream drive. Both profiles, measured across mid-planes, are not anti-symmetric about the geometric centre of the cavity, being greater in the ascending flow (downstream wall zone) than in the descending flow (upstream wall zone).

The strength of the vortex Ω (see also equation 6.7) for the square cavity in forced convection is given by Grand as

$$\Omega = \frac{\dot{Q}}{\rho U_0 W} = 0.105 \pm .003 \quad (6.10)$$

which is in good agreement with the value obtained by Mills for a free-stream driven flow.

Buoyancy effects are introduced in Grand's experiments by either of the two ways:

- a) heating the top wall, with the two side walls insulated
- or b) heating either of the side walls, with the two remaining sides insulated.

In both cases, buoyancy effects produce a stagnant upper layer with a stable temperature stratification resulting in the dampening there of transport processes, and a lower recirculating region where the temperature is uniform or quasi-uniform with value close to that of the driving channel flow. The relative extents of the two regions depend on both the heated side of the cavity walls and the Archimedes number⁺, Ar , defined as

$$Ar = \frac{Gr}{Re^2} = \frac{g\beta\Delta TW}{U_0^2} \quad (6.11)$$

where $Gr = \frac{\beta g \Delta T W^2}{\nu^2} = \text{Grashof number} \quad (6.12)$

$$Re = \frac{U_0 W}{\nu} = \text{Reynolds number} \quad (6.13)$$

$$\Delta T = T_H - T_0 \quad (6.14)$$

with T_H as the uniform temperature of the heated wall, T_0 the uniform temperature of the upstream channel flow, and U_0 is defined by expression (6.9).

For a heated upper wall, Grand's experimental work indicates that for a given Re an increase of buoyancy effects (i.e. of Ar) increases the extent of the upper stratified layer together with a reduction of both the extent of the uniform-temperature recirculating zone and the magnitude of its uniform temperature. The same effect is true even for the case of a heated downstream wall: this is at first sight surprising

+ A modified Froude number. For a given Re , the greater the Archimedes number the more pronounced are the buoyancy effects.

since natural convection along the downstream wall aids the flow; nevertheless, owing to the thermal insulation of the other walls of the cavity, the fluid heated along the downstream wall is trapped at the top of the cavity and results in the development there of the stagnant stratified layer.

The heat transfer across the cavity for forced convection in the range $3 \times 10^4 \leq Re \leq 3 \times 10^5$ covered by the work is correlated by the following expression for the Nusselt number, Nu:

$$\frac{Nu}{Re^{0.5}} = 1.16 \quad (6.15)$$

For the square cavity, this holds for both heated top wall and heated downstream wall. With introduction of buoyancy effects for the heated top wall, the ratio $Nu/Re^{0.5}$ departs (in a decreasing sense) from the constant value of 1.16, the departure being more the lower the Re, with a monotonic decrease as the Archimedes number increases.

In contrast, for a heated downstream wall, the Nusselt number initially increases with the Archimedes number owing to the favourable influence of natural convection, with the experimental data about parallel to the line

$$\frac{Nu}{Gr^{1/3}} = \text{const.} \quad (6.16)$$

which characterizes turbulent natural convection along a vertical wall. But on further increase of the Archimedes number, the top stable layer develops and decreases the length of the downstream wall that is favourably influenced by natural convection, thus resulting thereon in a monotonic decrease of Nu with increasing Ar.

(iii) Sliding-Wall Drive

For the turbulent, rectangular cavity flow driven by a sliding wall, only the experimental work of Mills (1965a) and Girard and Curtet (1975) are known to the author. The measurements of Mills were for only the square cavity, $H/W = 1$, while those of Girard and Curtet covered the

range $1 \leq H/W \leq 5$. Both sets of experiments were for forced convection only. Further, Mills measured only the mean velocities, while Girard and Curtet measured both mean and fluctuating quantities.

For the case of $H/W = 1$, which is common to both sets of experiments, there is agreement between the two sets in respect of the fact that, when normalised on the basis of the velocity of the sliding wall U_{wall} as reference, the measured vortex strength, mean velocities, and turbulence quantities in either case are found to be much lower than the corresponding values for the free-stream and channel-flow drives in which the quantities are normalised with reference velocities U_{∞} and U_0 respectively. However, if the effective velocity of translation for the sliding wall drive is taken to be U_r , define as

$$U_r = U_{wall}/n \quad (6.17)$$

where n (>1) is a constant, and if U_r is used as the basis for normalization of both mean and turbulence quantities, the non-dimensional values become identical with those for free-stream and channel-flow drives.

Unfortunately, the factor n turns out to be different for both sets of measurements: Mills' work yields a value of n of 3.3 while that of Girard and Curtet gives 2.4. The different values of n may be due to air leakage from the cavity, reported by Mills in his work. Nevertheless, the fact that n is much greater than unity in either case implies that the vortex in a wall driven cavity is maintained by some effective velocity, U_r , which is much lower than the wall velocity, U_{wall} .

The measurements of Girard and Curtet at two values of the Reynolds number Re ($= U_{wall}W/\nu$) for the square cavity give the following values of vortex strength Ω :

$$\begin{aligned} \Omega = \frac{\dot{Q}}{\rho U_r W} &= 0.117 \text{ for } Re = 5 \times 10^{14} \\ &= 0.112 \text{ for } Re = 2 \times 10^5 \end{aligned} \quad (6.18)$$

where Ω is as defined by equations (6.7) and (6.10). This compares

favourably with the values of Ω obtained by Mills (1961) and Grand (1975) for the free-stream and channel-flow drives respectively.

For the entire range $1 \leq H/W \leq 5$ studied by Girard and Curtet, their measurements indicate that the vortex strength defined by (6.18) decreases with increasing values of H/W , with the value at $H/W = 5$ being approximately half that for the square cavity $H/W = 1$.

The work of Girard and Curtet also showed three-dimensional unsteady effects. It has been indicated earlier for the free-stream drive that the work of Maull and East (1963) established that two-dimensional flow in a rectangular cavity occurs only in the following ranges:

$$0 < H/W < 0.45 \quad 0.8 < H/W < 1.3 \quad H/W > 2.2 \quad (6.19)$$

Outside these limits Maull and East as well as others observed unsteady three-dimensional cellular effects. The work of Girard and Curtet for $H/W > 1$ indicates that, on the basis of the level of fluctuations at different locations of measured values of Ω for each H/W , similar unsteady effects occur in the wall-driven cavity within the second unsteady range ($1.3 < H/W < 2.2$) identified by Maull and East.

6.21 Status of Existing Information

The status of available information, both theoretical and experimental, on turbulent rectangular cavity flows may be summarized as follows:

1) Although cavity flows have been the subject of series of theoretical investigations in the past, the methods employed have been rather restrictive: previous analytical studies have been based mainly on the application of non-unified theories to the various regions of the flow, while available numerical work have mainly been based on stream function and vorticity as the primary dependent variables with application primarily to the non-buoyant case; further, such numerical studies have largely employed a uniform eddy viscosity concept. There is therefore the need to employ a fairly more general numerical method together

with a less restrictive turbulence model, with application also to the buoyant case.

2) There are extensive experimental data, mainly on the mean flow quantities in forced convection, for the square cavity driven by a free stream. Recent work has provided sufficient data on both the mean flow and turbulence quantities in both forced and natural convection for the square cavity driven by a channel flow, and in forced convection for a sliding-wall drive. But more complete data are required on the turbulence field. However, the available data have rendered possible the adequate testing of both numerical procedures and turbulence models.

6.3 COMPUTATIONAL DETAILS

6.31 Application of the Solution Procedure and Turbulence Model

(i) Inlet Conditions: for the channel flow drive, entrance conditions need to be specified upstream of the cavity. In line with the experimental conditions which are simulated in the computations, fully-developed channel flow profiles of U , V , T , K , and ϵ are prescribed. Generally, the mean profiles and turbulence energy are close to, and therefore can be obtained from, the measurements in fully-developed channel flows of Laufer (1951) and others. The dissipation rate ϵ may then be obtained in the manner of equations (4.29) and (4.30), with L as the width of the channel, and λ taken to be some suitable constant.

However, a much more direct and also accurate manner of obtaining the fully-developed profiles of all mean and turbulence quantities is by making a separate and rather simple computation for a channel flow, starting with uniform profiles U_0 , V_0 , T_0 , K_0 , and ϵ_0 , and carrying out the computations to the fully-developed region. This is the method employed in this study. The initial uniform profiles are as provided by the relevant experimental situation that is simulated, with K_0 and ϵ_0

as given in equations (4.28), (4.29), and (4.30). The value of λ employed in this case is unimportant, as the final fully-developed profiles in the channel will be independent of the initial condition.

The fully developed profiles thus obtained are then specified upstream of the cavity.

(ii) Outlet Conditions: For the cavity driven by channel flow, the velocities at the exit of the channel are obtained from consideration of mass balance within the flow domain. There the values of the other flow variables do not affect the computations owing to the boundary-layer nature of the exiting channel flow.

(iii) Wall Boundaries: In the wall regions, the following practices are employed for the incorporation of the appropriate boundary conditions for the various flow variables:

Mean velocities

A tangential velocity U_p for a node next to a wall is obtained from the usual momentum balance, but with the usual shear force expression F_i (see equation 3.6) along the wall boundary suppressed in the finite-difference equation. For the configuration shown in Fig. 6.3 (a), the usual shear force F_s is suppressed by setting

$$a_s = 0 \quad (6.20)$$

where a is as defined by equation (3.24) and subscript s stands for the south boundary of the relevant cell.

For a point P (see fig. 6.3a) within the turbulent region ($y^+ > 11.63$ where y^+ is as defined by equation 2.31), the correct shear force expression is then inserted through expression (2.38);

$$\begin{aligned} \text{thus } F_s &= \tau_s \delta x_{ew} \\ &= -\rho C_\mu^{1/4} K_{pw} (U_p - U_s) \kappa \delta x_{ew} / \log_e (E y_p^+) \end{aligned} \quad (6.21)$$

$$\text{where, } K_{pw} = (k_p + K_w) / 2 \quad (6.22)$$

$$y_p^+ = \rho C_\mu^{1/4} K_{pw} y_p / \mu \quad (6.23)$$

and y_p is the distance between the node P and the wall, and U_s is the velocity of the south boundary of the cell (i.e. the wall velocity).

If the point P falls within the viscous sublayer ($y^+ \leq 11.63$), the correct shear force expression becomes

$$\begin{aligned} F_s &= \tau_s \delta x_{ew} \\ &= -\mu (U_p - U_s) \delta x_{ew} / y_p \end{aligned} \quad (6.24)$$

The appropriate form of F_s is then incorporated via the source treatment, using coefficients b and c, as illustrated by equation (3.29).

For velocities normal to a wall, no special treatment is necessary.

Scalar property (e.g. T)

The incorporation of scalar wall boundary condition follows similar lines as that for velocities. With respect to Fig. 6.3(b), the usual flux expression a_s is suppressed in the manner of expression (6.20). For a constant-temperature wall condition, if the point P lies within the turbulent region ($y^+ > 11.63$), the correct scalar flux $\dot{Q}_s$ contributed by the wall is inserted through expression (2.42); thus

$$\begin{aligned} \dot{Q}_s &= \dot{q}'' \delta x_{ew} \\ &= \frac{-\rho C_\mu^{1/4} K_p^{1/2} (T_p - T_s) \delta x_{ew}}{T^+} \end{aligned} \quad (6.25)$$

$$\text{where } T^+ = \sigma_{h,t} \{ U^+ + P \left(\frac{\sigma_h}{\sigma_{h,t}} \right) \}$$

with U^+ and the P-function as defined by equations (2.33) and (2.44) respectively, and T_s is the temperature of the south boundary (i.e. the wall temperature).

If P lies within the viscous sublayer ($y^+ < 11.63$), the correct scalar flux becomes

$$\begin{aligned} \dot{Q}_s &= \dot{q}'' x_{ew} \\ &= \frac{\mu}{\sigma_h} (T_p - T_s) \delta x_{ew} / y_p \end{aligned} \quad (6.27)$$

Again, the appropriate form of $\dot{Q}_s$ is incorporated through the source treatment illustrated by equation (3.29).

For a wall of prescribed heat flux $\dot{q}_B$ (e.g. for an adiabatic wall, $\dot{q}_B = 0$), again a_s is suppressed and the correct condition is incorporated through the source treatment of equation (3.29), where now

$$b = 0; \text{ and } C = \dot{q}_B \quad (6.28)$$

Turbulence energy

With reference to Fig. 6.3(b), the turbulence energy for node P (i.e. K_p) is obtained from the usual K-balance, but with the diffusion of K to the wall set to zero as indicated in equation (2.41). This is achieved by suppressing a_s as shown in (6.20).

In addition, the source terms $G - C_D \rho \epsilon$ (see equation 2.51) are adjusted in order to reflect the influence of the wall:

- the generation term, G, is modified to account for the wall shear stress τ_s as given by (2.38), by noting that

$$\int_V \mu_t \left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x} \right)^2 dV \approx \tau_s (U_p - U_s) \delta V / y_p \quad (6.29)$$

where τ_s , U_p , and U_s are averages of nearby values, and δV is the volume of the cell;

- the dissipation term, $C_D \rho \epsilon$, is also modified to reflect expression (2.40), and is assigned an average value over the control volume, thus⁺

$$\begin{aligned} C_D \rho \bar{\epsilon} &= C_D \rho \int_V \epsilon dV \\ &= C_D \rho C_\mu^{3/4} (K_p^*{}^{1/2} K_p - K_s^{3/2}) U^+ \delta V / y_p \end{aligned} \quad (6.30)$$

$$\text{where } U^+ = \begin{cases} 1/\kappa \log_e(Ey^+) & \text{for } y^+ > 11.63 \\ y^+ & \text{for } y^+ \leq 11.63 \end{cases} \quad (6.31)$$

and K_p^* is the value of K_p of the previous iteration, and $K_s (= 0)$ the turbulence energy at the wall.

Again expressions (6.29) and (6.30) are incorporated through the source treatment of equation (3.29).

⁺ See Appendix B

Energy dissipation

With reference to Fig. 6.3(b), the dissipation rate ϵ_p is not obtained from the regular ϵ -balance, but it is assigned a value which is determined from equilibrium relations as shown by expressions (2.39) and (2.40). Thus,

$$\epsilon_p = C_\mu^{3/4} K_p^{3/2} / \kappa Y_p \quad (6.32)$$

This value is incorporated by overwriting the usual ϵ -balance through the source treatment of equation (3.29), where now

$$b = -\alpha; \quad C = \epsilon_p \cdot \alpha \quad (6.33)$$

and α is a very large number (e.g., 10^{30}).

(iv) Constants of the Turbulence Model: The constants C_μ , C_D , C_1 , C_2 , σ_K , and σ_ϵ used in the computation of all cavity flows are as given in Table 3. Further, for the particular case of buoyant flows a value of 1.0 is again used for the constant C_3 (see Section 4.32).

For flows of negligible buoyancy (Archimedes number, $Ar \approx 0$), a uniform value of 0.9 is used as the turbulent Prandtl number. However, when buoyancy is influential, with severe stable stratification, the turbulent Prandtl number needs to be modified to reflect effects of buoyancy and variation with shear stress. In this case recourse is made to expression (2.30) at all locations.

6.32 Special Practice for Relaxation of Velocity (Inertial-Relaxation Method)

In some instances where the flow is strongly influenced by external forces (e.g. buoyancy or centrifugal forces), the attendant numerical instabilities may not be cured only by the remedies given in Section 3.52(c). In such cases, the introduction of a special under-relaxation method of an "inertial" type may yield very fruitful results.

The form of the relaxation may best be explained by re-writing equation (3.23) in the equivalent form:

$$\phi_p = \frac{\sum_n a_n \phi_n + C + I\phi_p^*}{a_p - b + I} \quad (6.34)$$

where ϕ stands for the velocity (U or V) in the direction of gravity, C is here taken to include the usual pressure and buoyancy source terms of the momentum equation, ϕ_p^* stands for ϕ_p of the previous iteration, and I is some relaxation parameter. Clearly, if I is very large only a small change can occur in the value of ϕ .

The problem is to determine what value should be ascribed to I. Often, I can be derived from appropriately modified forms of the inertia term of the relevant momentum equation. Hence I is here called an "inertial-relaxation" parameter.

In this study, the following expression for I was devised (see Appendix E):

$$I = \frac{\alpha_I \rho g \beta (T_p - T_r)}{\sqrt{g \beta T_r L}} \quad (6.35)$$

where $\rho g \beta (T_p - T_r)$ is the usual buoyancy term (written in terms of temperature) in the momentum equation, T_r is the reference temperature, and L is a characteristic length (typically the dimension of the flow field), and α_I is a relaxation constant. Generally, α_I may be positive or negative.

For stability, we require that

$$I \geq 0 \quad (6.36)$$

Consequently, for

- $T_p > T_r$ (positive buoyancy), $\alpha_I > 0$
- $T_p < T_r$ (negative buoyancy), $\alpha_I < 0$

For the computation of cavity flows of moderately high Archimedes number Ar ($0.4 < Ar < 1.0$, for heated top wall), numerical stability is procured only when this "inertial relaxation method" is employed. A value of α_I of 0.2 is used. However, for very high Archimedes numbers ($Ar > 1$, for heated top wall) the method does not ensure stability.

6.33 Other Computational Details

In the computation of all cavity flows, a non-uniform grid is employed within the cavity in both X- and Y- directions (see Fig.1.3). The grid is expanded in an exponential manner from each wall to the centre of the cavity, in either direction, so that the grid nodes are concentrated in the wall region. For the flow driven cavity, the nodes are also concentrated in the mixing region.

For the sliding wall drive, a grid size of 31 nodes in either direction (i.e. a 31 x 31 grid) is found to give a grid-independent solution, as higher grid sizes make negligible change in the predictions, (see Fig. 6.31). For the channel flow drive, a 37 x 37 grid gives a grid-independent solution (see Fig 6.32), the higher grid size being necessary owing to a few additional grid employed for the channel.

Only the SIMPLE algorithm was used in the computation of the cavity flows, as this part of the study was completed much earlier than the origination of the EMIT algorithm. For flows of low Archimedes number, Ar , numerical instabilities are cured in part by the choice of appropriate relaxation factors f defined by equation (3.62). The values of f which are employed for both the channel-flow and sliding-wall wall drives are given in Table 6. For the case of moderately high Archimedes number the inertial relaxation method is employed in addition to the relaxation factors given in Table 7.

Again, more than one "sweep" of the tri-diagonal matrix algorithm (TDMA) is employed for most of the variables in order to ensure stable advancement of the calculations. Generally, a single sweep is sufficient for the velocities U and V , while 6 sweeps are necessary for p' in order to satisfy continuity; further, 6 sweeps are also necessary to obtain faster convergence of the solution of all other scalar variables (T , K , and ϵ).

Table 7

Under-relaxation factors, f , (equation 3.62) employed in cavity flows

Variable, ϕ	Relaxation Factor, f	
	Channel Flow	Sliding Wall
U	0.5	0.5
V	0.5	0.5
P'	1.0	1.0
T	1.0	-
K	0.5	0.3
ϵ	0.5	0.3
μ_t	0.5	0.3

As already indicated in Section 4.33, a converged solution is reached when each new iteration produces negligible difference from the previous one, with the residual sources (see equation (3.58) at least 10^{-3} times their respective reference values. Generally, about 350 iterations give converged solutions for non-buoyant cases, and about 500 for buoyant ones.

All computations were carried out on a CDC 6600, and each iteration of the sliding-wall drive (31x31 grid) takes about 2.8 seconds, while that of the channel flow (37x37) takes about 4.5 seconds.

6.4 PRESENTATION AND DISCUSSION OF RESULTS

6.41 Flow Driven Cavity

For the flow driven cavity, the computations are designed to simulate the experimental set-up of Grand(1975) for the square cavity, in which the latter is driven by a channel flow (see Fig 1.3). Both forced and mixed convection cases are studied. For the latter cases, buoyancy is introduced by heating the top wall. The fluid simulated is water (Prandtl number, $\sigma_h = 10$).

The study is made for various Reynolds numbers Re (see equation 6.13) ranging from 10^4 to 2×10^5 , and Archimedes numbers Ar (see equation 6.11) from 0 to 0.4. The predictions are generally compared not only with the data of Grand but also with those for free-stream drive where such data are available. Except otherwise stated, U_0 and T_0 refer to the average values of the relevant variable at the entrance to the cavity.

(i) Predicted Flow, Scalar, and Turbulence Fields

Flow Field

Fig. 6.4(b), (c), and (d) show the predicted flow fields for the channel-flow drive, for various Re and Ar . In all cases, the prediction depicts a single primary vortex with secondary corner eddies.

Fig. 6.4(b), for $Re = 2 \times 10^5$ and $Ar = 0.004$, effectively represents a case at high Reynolds number dominated by forced convection. There it will be observed that the motion in the main vortex is parallel to the cavity walls over much of its length. Thus the entire main vortex does not rotate as a solid body, as is the case for cavity flows in a circular cylinder. However, in the central region of the cavity (approximately the inner 40% of the main vortex), the flow is nearly circular, indicating a close similarity there to a solid-body rotation. The main vortex can therefore be divided into two parts: an outer region that is strongly influenced by the square walls (wall region), and a central region of almost solid-body rotation. The implication of a region of near solid-body rotation is that the velocity profiles will there be expected to be nearly linear, and the turbulence nearly homogeneous.

Generally, the nature of the main vortex and the secondary corner eddies depend on the Reynolds number and the Archimedes number. For the various cases shown in Fig. 6.4, the strength of the primary eddy (equation 6.10) and the location of its centre are given in Table 8. Also given in the table are the experimentally determined eddy strengths, where these are available.

A decrease in the Reynolds number (compare Fig. 6.4b and c) produces the following effects:

- the strength of the primary vortex decreases, accompanied by an increase in the size of the corner eddies,
- the vortex centre shifts both downstream and vertically downwards.

Table 8

Strength of primary eddy and location of its centre (for Fig. 6.4)

Fig.	$Re^* = \frac{UW}{\nu}$	$Ar = Gr/Re^2$	Driving Mechanism	Eddy Strength $\Omega^* = \dot{Q}/(\rho UW)$		Predicted coords. of eddy centre	
				Prediction	Experiment	X/W	y/H
6.4(a)	2×10^5	.000	Sliding Wall	.075	.112 (Girard & Curtet, 1975)	.51	.5
6.4(b)	2×10^5	.004	Channel Flow	.069	$.105 \pm .003$ (Grand 1975)	.53	.5
6.4(c)	10^4	.000	"	.0643	-	.57	.495
6.4(d)	10^4	.370	"	.0571	-	.62	.455
-	5×10^4 to 4×10^5	.000	Free Stream	-	.10 (Mills, 1961)	-	-

* $U = U_0, U_\infty, \frac{U_{wall}}{2.4}$ for channel flow, free stream, and sliding wall respectively.

Further, in addition to producing the same effects as for a decrease in Re , an increase in the Archimedes number (compare Fig. 6.4 c and d) increases the upstream corner eddy size at the expense of the downstream eddy.

Temperature Field

Fig. 6.5 shows the temperature contours ($T^* = T - T_0/\Delta T$, where ΔT is as defined by equation 6.14). Generally, the various plots reveal that the upper region of the cavity is one of a stable stratification, with the stratified zone being more extensive in the region identified as the upstream corner eddy in Fig. 6.4.

For the particular case of predominantly forced convection (Fig. 6.5a and b), the temperature within the cavity is much more uniform and the stratified zone almost negligible for the higher Re . This is due to increase in strength of the recirculating eddy, resulting in higher mixing of the fluid, for the higher Re .

When buoyancy effects become appreciable (compare Fig. 6.5b and c) the region of stable stratification increases, with a tendency of the warm buoyant fluid to remain nearer the top of the cavity. Generally the increased extent of the stratified region behaves like an adverse pressure gradient, moving the separation point of the upstream corner eddy further downstream along the top wall as is depicted in Fig. 6.4.

Turbulence Energy Field

Fig. 6.6(b), (c), and (d) show the contours of turbulence energy ($K^* = K/U_0^2$) for the channel flow drive. All three plots strongly exhibit a non-symmetrical turbulence field; the turbulence is much higher in the ascending flow (downstream wall region) than in the descending flow (upstream wall region). This is due to the fact that much of the turbulence produced in the thin shear layer is convected upwards directly along the downstream wall, where it is partly diffused in the cross-stream direction and partly further convected in the stream-wise direction.

For forced convection (compare Fig. 6.6b and c), the K^* -field has higher values within the main part of the cavity for the lower Re . This is due to the fact that, as is shown by the two turbulence fields, the production of K^* due to shear in the mixing region increases as the main recirculating eddy in the cavity becomes weaker as is the case for the lower Re .

When buoyancy becomes appreciable, its effect in the upper region of the cavity is to reduce the turbulence energy there owing to stable stratification (compare Fig. 6.6 c and d). However, in the thin region of the shear layer and also in the region where stratification is small, production of K^* due to shear still plays the dominant role, leading to a tendency there for higher K^* for the weaker main recirculating eddy (Fig. 6.6d).

Length Scale Field

Fig. 6.7(b), (c), and (d) show the contour of length scale l_ϵ^* ($= l_\epsilon/W$) for the channel flow. In the wall regions, owing to effects of local equilibrium, the dissipation length scales vary almost linearly with distance from the wall, and l_ϵ^* are highest in the central portions of the cavity. In addition, l_ϵ^* tends to be higher for the flow with primary vortex of lower strength.

(ii) Detailed Distributions of Variables, and Comparison with Experimental Data

Mean Velocities

Fig. 6.8 shows the distribution along the vertical axis at $x/W = 0.5$ of the predicted horizontal component of the mean velocity, U/U_0 , for the various Reynolds and Archimedes numbers. The data of Grand (1975) for channel-flow drive and those of Mills (1961) for free-stream drive (U/U_∞) are also shown. Both sets of data are in good agreement, and the predictions for $Re = 5 \times 10^4$ and 2×10^5 (predominantly forced convection) compare fairly favourably with them. However, in the range $1.4 \leq y/H \leq .32$ and $.68 \leq y/H \leq .86$ the agreement between prediction and data is not so good, with the difference varying between some 10 and 30%.

At the channel level, where buoyancy effects are negligible in all cases, U/U_0 depends on the Reynolds number. At $Re = 2 \times 10^5$ it attains a maximum value of

$$\frac{U}{U_0} \approx 1.1 \quad (6.37)$$

and this is in good agreement with the value of 1.2 given by the measurement of Grand, while at $Re = 10^4$ it falls to a value of about 0.75. At the top of the cavity there is also a marked difference between the profiles for high and low Reynolds number forced flows, with the wall boundary layer being much thinner for the higher Re (i.e. $\geq 5 \times 10^4$).

In the region close to the interface between the channel and the cavity, the velocity gradient is very large, indicating the presence there of a mixing layer. Within the central region of the cavity, the velocity profile is nearly linear for the case of $Re = 5 \times 10^4$ and 2×10^5 , thus indicating that the fluid rotates nearly like a solid body there as has also been indicated by the nearly circular flow field in that zone in Fig. 6.4(b). However, the gradient of this linear profile is approximately 20 times less than that in the mixing layer.

For the case of mixed convection at $Re = 10^4$, buoyancy forces are effective in reducing the velocity mainly near the top of the cavity. Generally, the boundary layer at the top of the cavity becomes broader the lower the strength of the primary eddy, as would be expected.

Fig. 6.9 shows the distribution along the horizontal axis at $y/H = 0.5$ of the predicted vertical component of mean velocity, V/U_0 , for the various Re and Ar . Also shown there for the effectively forced convection at high Re ($> 5 \times 10^4$) are the data of Grand (channel flow) and Mills (Free stream). The two sets of data are again in fairly good agreement except very close to the walls, and the corresponding predictions compare fairly favourably with them. However, the agreement between the prediction and data is rather poor, with differences up to some 30% for $.2 \leq x/W \leq .45$ and $.62 \leq x/W \leq .85$. It may be noted that both the data and predictions for channel flow at $Re = 2 \times 10^5$ show thinner boundary layers on both walls than do the data for free stream.

Nevertheless, it is clear that for the high Reynolds numbers the primary vortex is separated from the walls by rather very thin boundary layers irrespective of the driving mechanism. But the vortex is not perfectly symmetrical, so that the velocity maximum on the downstream side is higher than that on the upstream side. This asymmetry becomes pronounced for lower Re , and even more so when buoyancy becomes appreciable.

Generally, the boundary layers on the side walls become much wider and the magnitude of V/U_0 much reduced as the eddy strength decreases. For the particular case of predominantly forced convection at high Re , the profile is again linear in the region close to $X/W = 0.5$, and, for the same radial distance from the centre, its magnitude is also close to that in the zone around $y/H = 0.5$ in Fig. 6.8. This confirms that the flow rotates almost like a solid body in the central portion.

Mean Temperature

Fig. 6.10 shows temperature distributions ($T^* = (T - T_0)/\Delta T$) along the vertical axis at $X/W = 0.5$, for the various values of Re and Ar . Fig. 6.10(a) shows the effect of Reynolds number at moderately low (i.e. approximately fixed) Archimedes number. Also shown there are the data of Grand for $Re = 5 \times 10^4$. The prediction for $Re = 5 \times 10^4$ is in fairly satisfactory agreement with the data.

Fig. 6.10(a) shows that the temperature within the cavity is uniform and of value close to that at the entrance of the cavity. Comparison with the appropriate flow fields in Fig. 6.4 indicates that this uniform temperature exists within the main recirculating eddy, thus indicating that the fluid is mixed thoroughly within the main vortex. The uniform temperature meets the maximum temperature at the top wall through a zone of nearly uniform temperature gradient (stable stratification). At very high Reynolds number, typified by the profile for $Re = 2 \times 10^5$, this zone is almost negligible, and the temperature assumes the uniform, near-entrance value very close to the top wall. As the Reynolds number decreases the stratified zone increases, and, with a warmer layer of fluid settling at the top (typified by the profile for $Re = 10^4$), the resultant uniform temperature of the recirculating eddy is increased. For $Re = 5 \times 10^4$, this increase is noticeable only in the top 10% of the cavity.

Generally, the thermal boundary layer under the top wall presents an effective resistance to the heat transfer across the cavity. Thus, the increase of the boundary layer with decreasing Re indicates, as would be expected, smaller heat transfer across the cavity for the lower Re .

Fig. 6.10(b) shows the influence of buoyancy at a fixed Reynolds number ($Re = 10^4$). The predictions show that, with increased buoyancy effects (i.e. of Ar), both the value and extent of the uniform temperature region is reduced. The increased buoyancy has the effect not only of increasing the thickness of the boundary layer, but, as has been indicated earlier, also of moving the separation point of the upstream corner eddy further downstream along the top wall, with the reattachment point also extended further down along the upstream wall (compare Fig. 6.4 c and d). This much greater extent of the upper region of near stagnant flow, with its attendant increase in resistance to heat transfer, explains the reduction in the value and extent of the uniform temperature of the central region.

Also shown in Fig. 6.10(b) are the data of Grand for $Re = 10^4$, and $Ar = 0.37$. There is a rather poor agreement between the data and the corresponding prediction. This is due to the inadequacy of the turbulence model in general, and in particular in yielding the correct turbulent Prandtl number for this rather low Reynolds number flow.

Fig. 6.11(a) and (b) show the temperature distribution along the horizontal axis at $y/H = 0.5$ for fixed Reynolds number and low Archimedes numbers respectively. These plots show the same trends as have already been indicated for the corresponding case in Fig. 6.10. In particular, for $Re = 5 \times 10^4$ and 2×10^5 (Fig 6.11b) the uniform temperature profile again occurs nearly across the entire cavity. On the other hand, the starting point of the uniform temperature region moves downstream for both lower Re (Fig.6.11b) and higher Ar (Fig.6.11a), thus indicating the shifting downstream for these cases of the central region of the

recirculating eddy.

The temperature level in plot (b) for $Ar = 0.37$ is again underpredicted.

Turbulence quantities

Fig. 6.12 shows the predicted distribution of turbulence energy K^* ($=K/U_0^2$) along the vertical axis at $X/W = 0.5$, for the various values of Re and Ar . Also shown there are the experimental data of Grand for $Re = 2 \times 10^5$, and $Ar = 0.004$, the 'data' points were deduced from the measurements of $\overline{u_1^2}$ on the assumption of isotropy of turbulence (i.e. $\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}$). The corresponding prediction is in fairly satisfactory agreement with the data.

A rather prominent feature that is exhibited in Fig. 6.12 is the production within the shear layer of a higher value of K^* at lower Re (for approximately fixed Ar) or at higher Ar (for fixed Re). This result is rather surprising at first sight. However, it is known (e.g. Watt, 1967) that, in a plane mixing layer produced between a primary stream U_0 and a secondary stream U_c , the turbulence quantities (e.g. K , $\overline{uv}$, etc.) attain a constant value when normalized by $U_0 - U_c$, irrespective of the values of U_0 and U_c . Thus,

$$\frac{K}{(U_0 - U_c)^2} = z \quad (6.38)$$

where z is a constant. For the plane mixing layer of the cavity flow, the relevant secondary velocity U_c may be taken as the characteristic velocity of the recirculating eddy. The latter velocity may be defined as that which when imposed at the periphery of the eddy produces the same vortex strength.

From equation (6.10), it follows that

$$U_c = 2U_0\Omega \quad (6.39)$$

where Ω is the strength of the vortex. Equation (6.38) may be recast in the form of K^* ($=K/U_0^2$); thus

$$\frac{K}{U_0^2} = \frac{z(U_0 - U_c)^2}{U_0^2} \quad (6.40)$$

Hence K^* depends on the relative values of U_0 and U_c ; K^* is increased if the difference between U_0 and U_c is large, and vice versa.

At approximately fixed Archimedes number, what determines the flow characteristics is the Reynolds number $Re (=U_0 W/\nu)$ which in this case depends only on U_0 , since W and ν are fixed. For two flow cases of different Re (e.g. cases identified as "a" and "b"), the ratio of the characteristic velocities is

$$\frac{U_{c,a}}{U_{c,b}} = \frac{U_{0,a} \Omega_a}{U_{0,b} \Omega_b} \quad (6.41)$$

where subscripts a and b stand for the different cases. For $Re_a > Re_b$, we have that $U_{0,a} > U_{0,b}$; further it has been shown in Table 8 that for such a condition $\Omega_a > \Omega_b$; hence

$$\frac{U_{c,a}}{U_{c,b}} > \frac{U_{0,a}}{U_{0,b}} \quad (6.42)$$

Consequently, for increased Re , U_0 and U_c do not increase at the same rate, rather, U_c increases comparatively more for an increase in U_0 , thus yielding a lower value of K^* as demanded by expression (6.40).

On the other hand, at fixed Re (i.e. fixed U_0), U_c is directly proportional to Ω as shown by expression (6.39). In this case, an increase in buoyancy effects (i.e. of Ar) produces smaller Ω , and thus also smaller U_c , yielding a higher value of K^* .

The varying production of turbulence shown by the predictions in Fig. 6.12 is thus in line with the characteristics of the plane mixing layer: a higher value of K^* for a comparatively weaker recirculating eddy.

When buoyancy becomes appreciable, its effect is again seen to be predominant in the top region of the cavity. There, as shown by a comparison in Fig. 6.12 of the predictions for $Ar = 0$ and 0.37 at $Re = 10^4$, the prevalent stable stratification reduces the turbulence for the high Archimedes number.

Fig. 6.13 shows the predicted distribution of turbulence energy K^* along the horizontal axis, for the various values of Re and Ar . Also shown there are the data of Grand, again derived from assumption of isotropic turbulence, for $Re = 2 \times 10^5$ and $Ar = 0.004$. The agreement between the data and the corresponding prediction is fairly satisfactory in the upstream region but not however, further downstream, perhaps due to the assumption of local isotropy.

The general trend of higher K^* for a weaker main recirculating eddy is again depicted by the predictions in Fig. 6.13. It is also shown there (compare profiles for $Ar = 0$ and 0.37 , at $Re = 10^4$) that, owing to stable stratification, buoyancy reduces to some considerable degree the turbulence energy in the upstream wall region and to some smaller extent in the downstream wall region.

Shown in Fig. 6.14 are the predicted distributions at axial location $X/W = 0.5$ of the turbulent kinematic shear stress $\overline{uv}^*$ ($=\overline{uv}/\frac{1}{2}U_0^2$) across the shear layer, for the various values of Re and Ar . Also shown there are the data of Haugen and Dhanak (1966) for free stream drive at $Re = 1.3 \times 10^5$. The comparison with the data of the channel-flow prediction at $Re = 2 \times 10^5$ shows a rather interesting point: although outside the cavity ($y/H < 0$) the distribution of the turbulent shear stress depends on the driving mechanism, inside the cavity ($y/H > 0$) the distribution is independent of the driving mechanism.

As has already been shown for the distributions of K^* , those for $\overline{uv}^*$ in Fig. 6.14 show correspondingly higher values for the cavity flows of smaller eddy strength. Further, for high Re (2×10^5) both turbulence energy K^* (Fig. 6.12) and turbulent shear stress $\overline{uv}^*$ (Fig. 6.14) attain their maxima at a point very near $y = 0$. On the other hand, for the lower Re (10^4), the maxima move well into the cavity at $y/H \approx 0.03$. These facts suggest that the streamline dividing

the channel flow from the cavity flow moves higher up towards the cavity for the lower Re .

Fig. 6.15 shows a similar plot to Fig. 6.14, but now across the entire cavity. As is also the case with the turbulence energy K^* (Figs. 6.12 and 6.13), within the core of the cavity, and especially for $Re = 2 \times 10^5$, the shear stress $\overline{uv}^*$ remains fairly uniform. Further, this uniform value is nearly zero for the shear stress (as would be expected for a fluid in near solid-body rotation). Fig. 6.16 shows the predicted distribution of the shear stress along the horizontal axis at $y/H = 0.5$. Again, for $Re = 2 \times 10^5$ the stress is uniform and nearly zero within the core. However, for the high Archimedes number the uniform zone is greatly reduced.

Fig. 6.17 and 6.18 show the distribution of the effective viscosity (μ_{eff}/μ_{lam}) across the mid-planes of the cavity. They highlight the fact that the effective viscosity is particularly low for the case of $Re = 10^4$, with μ_{eff} being only about 20 times the laminar viscosity. Thus the high Reynolds number turbulence model employed in this study is basically unsuitable for the very low Re case studied here. Experience shows that the turbulent Prandtl number, among other aspects of the flow, particularly needs improvement: this explains why the predicted temperature profiles in Fig. 6.10(b) and 6.11(a) are much lower than the data. Note that in Fig. 6.17 the trough at the lower end of the cavity indicates the approximate upper limit of the shear layer for each case.

Pressure distribution

In Fig. 6.19 are shown the predicted pressure distributions along the cavity walls, for the various Re and Ar . Also shown there are the experimental data of Roshko (1955) for free stream drive at high Re (4.25×10^5). The pressure coefficient C_p is based on a reference value P_1 which is the pressure at the entrance lip. For the free stream

drive P_1 has the same value as the free stream pressure P_∞ . Agreement between the experimental data and the prediction for high Re (2×10^5) is fairly satisfactory, indicating that the pressure distribution is fairly independent of the driving mechanism. The distribution is however very sensitive to the Reynolds number when the latter is not high enough, and it also changes considerably at fixed Re when buoyancy becomes influential. Note that for the high Re , C_p is not symmetrical along the cavity walls, thus indicating again that the main vortex is not perfectly symmetrical.

Figs. 6.20 and 6.21 show the longitudinal pressure distribution in the shear layer. In fig. 6.20, C_p is calculated relative to the local centre-line pressure in the channel, P_m , and the predictions are there compared with the measurements of Haugen and Dhanak (1966) for the free stream drive ($P_m = P_\infty$, the free stream pressure). Both the predictions and the data show that C_p remains uniform along the major part of the shear layer, the only exception being at the entrance region (sudden expansion region) and the exit region (impingement region).

However, when C_p is based on the fixed pressure P_1 at the entrance lip (as shown in Fig. 6.21 for the prediction only) it undergoes dramatic changes, becoming much higher in the major part of the shear layer as the strength of the primary eddy decreases, but, as would be expected in the impingement region the values are higher for higher Re .

Friction on cavity walls

Fig. 6.22 shows the distribution on the cavity walls of the skin-friction coefficient, $C_F = \bar{\tau}_{wall} / \frac{1}{2} \rho U_0^2$ where $\bar{\tau}_{wall}$ is the wall shear stress:

$$\tau_{wall} = \mu \left(\frac{\partial U}{\partial y} \right)_{wall} \quad (6.43)$$

Here, U is the velocity parallel to the wall, and y is the normal distance measured from the wall.

Also shown there are the data of Roshko (1955) at the mid-point of each wall, for free stream drive at high Re ($\geq 1.5 \times 10^5$). The prediction for high Re (2×10^5) is in satisfactory agreement with the data. For this case, the prediction shows that along the top wall the zone of zero or reversed skin-friction is very small, thus indicating the small extent of the secondary corner eddies. Further, the skin-friction distribution on all three walls is not symmetrical, thus confirming that the primary vortex is not perfectly symmetrical at the high Re. The asymmetry is again seen to be more pronounced at the lower Re (10^4), and even more so when buoyancy is appreciable.

For the various computed cases, the relative sizes of the upstream and downstream corner eddies are well defined by the region along both the top wall and the adjoining part of the side walls, where the skin-friction changes sign. It will be noted that, as has already been shown by the flow field in Fig. 6.4,

- at zero or moderately low Ar, both corner eddies increase in content as Re decreases, with the upstream one being larger than that at the downstream corner; for the case of $Re = 10^4$, the upstream corner eddy is larger than the downstream one by some 26%;
- at fixed Re (10^4), when buoyancy becomes appreciable the upstream corner eddy increases considerably in size while that of the downstream corner actually decreases: for the cases shown, with increased buoyancy the upstream corner eddy increases by some 80% while the downstream one decreases by some 28%.

It should also be noted that the distributions show separation in the region close to the lips of both the upstream and downstream walls.

Heat Transfer

The heat transfer coefficient across the cavity is measured by the Nusselt number which is defined as

$$Nu = \frac{qL}{\gamma A(T_{\text{wall}} - T_0)} \quad (6.47)$$

where $\dot{q} = -\gamma A \left(\frac{\partial T}{\partial y} \right)_{\text{wall}}$ = heat transfer across the heated wall, (6.48)

L is a reference length ($=W$), γ is the thermal conductivity of the fluid, and A is the area of the wall.

Fig. 6.23 shows a plot of the predicted Nusselt number, Nu , for the heated top wall case at various Reynolds numbers and low Archimedes numbers ($Ar < 0.09$). Also shown in the plot are Grands' experimental data for heated top, downstream, and upstream walls, for which almost identical values of Nu are obtained at low Archimedes numbers. The data are correlated by the relation

$$Nu = cRe^{0.5} \quad (6.49)$$

where C is a constant (≈ 1.6). The prediction for very high Re ($=2 \times 10^5$) almost coincides with the line defined by equation (6.49); however, the lower Re predictions are in rather poor agreement with the data.

In Fig. 6.24 is shown the influence of buoyancy on the heat transfer across the cavity. There, the ratio $Nu/Re^{0.5}$ is plotted against the Archimedes number, and both the predictions and the data of Grand are shown. Also shown is the limiting line for forced convection for which $Nu/Re^{0.5} \approx 1.16$, and the prediction for $Re = 2 \times 10^5$ is seen to fall in well with it. At fixed Reynolds number (e.g. $Re = 10^4$ for the predictions), both the predictions and the data show a monotonic decrease of the heat transfer as the Archimedes number increases. However, for the higher Archimedes number ($Ar = 0.37$) the predictions is some 15% higher than the data.

6.42 WALL DRIVEN CAVITY

For the wall driven cavity, computations are made for the case of forced convection at a Reynolds number Re of 2×10^5 , where $Re = U_{\text{wall}} W / \nu$.

(i) Predicted Flow and Turbulence Fields

The plots of the flow and turbulence fields have generally been based on a reference velocity U_r as defined by equation (6.17) with

the constant $n = 2.4$,

Fig. 6.4(a) shows the vector plot of the predicted velocity field. It exhibits much the same features as the case of the channel flow drive at high Re (Fig. 6.4b): both the main vortex and corner eddies of each case are very similar to those of the other. This close similarity between the two cases suggest that the flow structure in the cavity is independent of the driving mechanism, as was remarked earlier.

The predicted strength of the main vortex and the location of its centre are given in Table 8. Also shown there is the data of Girard and Curtet (1975) for the sliding wall drive at $Re = 2 \times 10^5$. The predicted vortex strength is lower than that of their experiment, and also those obtained by Grand (1975) and Mills (1961) for channel-flow and free-stream drives respectively.

Figs. 6.6(a) and 6.7(a) show the predicted turbulence energy $K^* (=K/U_r^2)$ and length scale $l^* (=l_e/W)$ fields respectively. Again, each is comparable with the corresponding case for channel flow at $Re = 2 \times 10^5$ (Figs. 6.6 b and 6.7b). This also shows that the turbulence in the cavity is fairly independent of the driving mechanism.

(ii) Detailed Distribution of Variables, and Comparison with Experimental Data

The predictions are here compared not only with the experimental data for sliding wall drive, but also with those for channel flow and free stream drives. In doing so, the graphs are constructed on two scales for the variables: one based on the velocity U_0 (or U_∞) for the channel-flow and free-stream drives, and the other based on the wall velocity U_{wall} for the sliding-wall drive. The two scales differ by the factor n defined by equation (6.17), with n taken as 2.4.

Mean Velocities

Figs. 6.25 and 6.26 show the distributions along the mid-planes of the cavity of the vertical and horizontal components, respectively, of

the mean velocity. There are shown the predictions for both the sliding-wall and channel-flow drives; also shown are the experimental data of Girard and Curtet (Fig. 6.26), Grand, and Mills for the sliding wall, channel-flow, and free-stream drives respectively. There is a fairly satisfactory agreement between the two predictions. However, the agreement between the predictions and the data is not very good in the central regions of the cavity. Nevertheless, both sets of predictions and the various sets of data indicate that, when plotted as shown, the velocity field is fairly independent of the driving mechanism, except in the regions very close to the walls.

It should be noted that at $y/H = 0$ in Fig. 6.25, the prediction for the sliding wall extends outside the graph, as would be expected, to a value of $U/U_{\text{wall}} = 1$ which occurs at $U/U_0 = 2.4$. This accounts for the slightly higher velocities for the sliding wall drive in the region of the downstream wall ($x/W \approx 1$) in Fig. 6.26 and the top wall ($y/H \approx 1$) in Fig. 6.25. The measurements of Girard and Curtet also show the same trend in the region of the downstream wall in Fig. 6.26.

Turbulence Energy

Figs. 6.27 and 6.28 show the distribution across the mid-planes of the turbulence energy. The predictions for both sliding-wall and channel-flow drives are shown there, as well as the data of Girard and Curtet, and Grand for both drives. The experimental 'data' points are derived from the measurements on the assumption of isotropic turbulence (i.e. $\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}$). On the whole there is a fairly satisfactory agreement between both predictions, and also between them and the data (except in the downstream wall region in Fig. 6.28). It should be noted that in Fig. 6.28 the prediction for the sliding-wall drive agrees well with the data of Girard and Curtet across the entire cavity,

Both the predictions and the data confirm that the turbulence structure within a large part of the cavity is fairly independent of the driving mechanism when the effective velocity of translation is taken as $U_{\text{wall}}/2.4$ for the wall drive; the differences exist mainly in the region close to the downstream wall.

Pressure distribution

Fig. 6.29 shows the pressure distribution along the cavity walls. The coefficient C_p is based on the pressure P_1 at the lip of the upstream wall. Again, the predictions for both sliding-wall and channel-flow drives are shown, also shown there are the experimental data of Roshko (1955) for the free-stream drive. On the whole there is fairly satisfactory agreement between the two predictions, and also between both predictions and the data. However, C_p is much higher near the lip of the downstream wall for the sliding wall drive, since, as has been shown in connection with Fig. 6.25, the effective velocity at $y/H = 0$ is about 2.4 times higher for the sliding wall drive than for the flow drives, and all the fluid along the sliding wall region impinges on the downstream wall region without some parts of it escaping, unlike the case for the flow drives. This higher trend for the sliding wall drive is repeated in the region of the corner eddies.

Friction on cavity walls

Fig. 6.30 shows the distribution on the cavity walls of the skin-friction coefficient, C_F . There again are shown both predictions for the sliding-wall and channel-flow drives, as well as the experimental data of Roshko. The friction coefficient C_F is much higher along the downstream wall for the wall drive. On the other walls, however, the agreement between the two sets of predictions, and between them and the data is rather more satisfactory.

Note that in the region of the lip of the downstream wall, C_F does

not become negative for the sliding wall drive, thus testifying the rather obvious fact that no fluid escapes from the cavity in the region of the lip of that wall (i.e. no negative V -velocity in the lip region), unlike the case for the channel flow drive.

6.5. CONCLUDING REMARKS

The major findings of this chapter may be summarized as follows;

(i) A fairly detailed review has been made of both the previous theoretical and experimental work on rectangular, turbulent cavity flows.

Previous analytical studies have been based mainly on non-unified theories for various regions of the flow, while available numerical work have been based largely on stream function - vorticity methods.

(ii) Taking the three driving mechanisms (free stream, channel flow, and sliding wall) together for the square cavity, there exists sufficient data on both mean and turbulence quantities for the adequate testing of both numerical methods and turbulence models.

(iii) At high Archimedes numbers in cavity flows with stable stratification, there results considerable numerical instabilities owing to buoyancy coupling of momentum, thermal energy, and turbulence fields, which generally cannot be cured by commonly known remedial practices. However, the inertial-relaxation parameter devised in this study achieves some considerable success. Without its application, predictions could only be made for very low Archimedes numbers ($Ar \approx .04$). However, the method fails to cure the instability for very high values of Ar ($Ar > 1$ for the case of the heated top wall).

(iv) The following main features are borne out by the results of the computations:

- Forced convection in a square cavity involves a single, primary vortex with secondary, corner eddies. The structure of these is independent of the mechanism (free stream, channel flow, or sliding wall) driving the vortex.

- However, based on the wall velocity, the vortex strength for a sliding-wall drive is found to be about 42% (or $1/2.4$) of that measured for the other two drives. This suggests, and, in line with earlier findings of Girard and Curtet (1975), it has here been shown to be the case that when this difference is taken into account the measured mean velocity, pressure, skin-friction, and turbulence fields are fairly independent of the driving mechanism. It is therefore the case that, for the sliding-wall drive, it is not the velocity of the sliding wall that actually drives the main vortex, but some smaller effective velocity of translation which is about 42% of the wall velocity.
- The main vortex in the cavity is not perfectly symmetrical about the centre of the cavity, for forced convection. This is attributed to asymmetry in friction along the cavity walls.
- Turbulent, forced convection in a square cavity consists of three distinguishable regions:
 - a) A mixing layer which provides the requisite energy to drive the cavity flow;
 - b) a central region where the fluid rotates almost like a solid body, and in which the distribution of temperature is uniform and the turbulence nearly homogeneous;
 - c) a wall region in which the motion is nearly "square", being parallel to the cavity walls over much of its length, and which controls the heat transfer across the cavity.
- The turbulence generated by the mixing layer, and hence also the the turbulence field in the cavity, has been shown to increase (when measured as $K^* = K/U_0^2$) with decreasing strength of the main recirculating eddy inside the cavity. This behaviour is shown to be in line with the general characteristics of plane mixing layers.

- The relative extents of the central and wall regions, the strength and location of the accompanying main vortex and relative sizes of the corner eddies, the distribution of mean flow variables (velocities, temperature, pressure, skin-friction) and turbulence quantities, and the heat transfer across the cavity all depend on both the Reynolds number (when this is not very high) and the buoyancy effects measured by the Archimedes number.
- While proving fairly adequate for the high Reynolds number cavity flows, the turbulence model employed is inadequate for the low Reynolds number flows especially with respect to the turbulent Prandtl number.

PART V:

CLOSURE

CHAPTER 7

CLOSURE

7.1 CONCLUSIONS

Specific conclusions relating to the main aspects of this study have been given in Sections 2.7, 3.6, 4.5, 5.6, and 6.5; here considerations will be made on a more general basis.

1) Numerical Procedures

- The SIMPLE procedure of Patankar and Spalding (1972) has been successfully applied to the circumstances of all flows (buoyant-plumes, and cavity flows) dealt with in this study. The work has therefore provided further experience of application of this procedure.

For the cavity flows at moderately high Archimedes numbers (i.e. $Ar < 1$, for heated top wall), the procedure worked only through the use of the "inertial relaxation method" which was devised in this study. However, for very high values of the Archimedes number ($Ar > 1$ for the heated top wall), even this method fails to procure stable advancement of the solution procedure.

- A new numerical procedure, called EMIT, has been devised in this study and successfully applied to the calculation of the forced plumes in both uniform and stably-stratified surroundings. For these hot cases, the procedure appears to be more efficient than the SIMPLE procedure.

2) Turbulence Model

- The $k-\epsilon$ turbulence model has been extended to buoyant flows, with the effects of buoyancy in the ϵ -equation modelled in the form earlier suggested by Launder (1973). The additional constant c_3 introduced in the modelling is found to be of order unity.

- While proving fairly adequate for momentum jets, and buoyant plumes as well as for forced convection in the cavities, the $k-\epsilon$ model proves inadequate for predicting mixed convection in cavities at moderately high Archimedes numbers.
- More light has also been shed on the validity of the $k-\epsilon$ model for axi-symmetric jets, with the finding that the constants c_μ and c_2 applicable to plane flows appear to be satisfactory for the prediction of the initial region ($x/d < 15$); the reasons are attributed to the quasi-plane nature of the initial region.

3) Radioactive Heating Model

- A method has been developed for predicting radioactive heating effects in buoyant plumes. The major feature of the method is the use of a "multi-group flux model" for heating due to gamma-radiation. The method has been applied to the plume in stably-stratified surroundings, and the results look promising.
- It is, however, not yet certain how accurately the model represents the actual phenomena of radioactive heating.

4) Flow Studies

a) Buoyant plumes

- For the plume in uniform surroundings, there are now sufficient experimental data for testing both numerical methods and turbulence models, and the agreement between present predictions and the data is fairly satisfactory.
- It has been shown that the plume becomes self-similar as and when the integrated flux Richardson number $\overline{R_f}$ becomes constant, which appears to coincide with the local Richardson number Ri reaching a constant value of about 0.6 (which value also appears to be the same for both axi-symmetric and plane buoyant plumes). Also, if the self-similarity is attained, the resulting flow characteristics are independent of the initial Froude number. It is further shown

that the self-similarity can be attained much closer to the nozzle for plumes than for momentum jets; the reasons are attributed to the marked differences in the origins of turbulence generation for the two cases.

- For the plume in stably-stratified surroundings, more experimental data are required; however, the present predictions are in satisfactory agreement with the few available measurements of the rise-height of the plumes.

Further, some interesting aspects of the flow (regarding the velocity, buoyancy and turbulence profiles) for which no data or previous analytical solutions are available, have been revealed by the predictions.

- The predictions for the radioactive plume in stably-stratified surroundings provide further information on the flow characteristics, for which no data are available.

(b) Cavity flows

- Taking the three driving mechanisms (i.e. free stream, channel flow, and sliding wall) together, there exist some reliable experimental data for the testing of both numerical methods and turbulence models for the square cavity; but more detailed measurements using laser doppler anemometer, and focusing on the turbulence structure (e.g. stress distributions, etc.), are needed. For forced convection, the agreement between present predictions and the data is fair on the whole, with errors varying between about 1 and 25%. However, for mixed convection the agreement is poor, with errors varying between about 10 and 50%; this is attributed to the inadequacy of the $k-\epsilon$ turbulence model.
- For the sliding-wall drive, the effective velocity impelling the main vortex seems not to be the wall velocity, but some smaller

effective velocity of translation which is about 42% of the wall value. When this difference is accounted for, the characteristics of the flow in the cavity becomes fairly independent of the driving mechanism, within most of the cavity although appreciable differences appear in the immediate vicinity of the downstream wall.

7.2 RECOMMENDATIONS FOR FUTURE WORK

The following are suggested as possible areas for future study in connection with this work:

1) Numerical Procedures: The EMIT algorithm should be tested for various other flow circumstances; in particular, its suitability for strongly elliptic flows (e.g. cavity flows) needs to be established.

The "inertial relaxation method" needs to be improved or some other method devised for the procurement of numerical stability in the application of SIMPLE (and, perhaps, also EMIT) to cavity flows of very high Archimedes numbers.

2) Turbulence Model: For the buoyant plumes, a comparison is needed of higher order levels of turbulence closures on the one hand and the $k-\epsilon$ model on the other.

For the cavity flows, the inadequacy shown in this study of the $k-\epsilon$ model for mixed convection indicates an urgent need for application of improved forms of the $k-\epsilon$ model and, perhaps, also higher order closures. Specific attention needs to be given to the improvement of the turbulent Prandtl number and the wall functions (with respect to the inclusion of buoyancy effects).

3) Radioactive Heating Model: There is an urgent need for assessment of the accuracy of the "multi-group flux model" in representing gamma-ray transport.

4) Flow Studies: More extensive experimental measurements are needed for both the buoyant plumes (especially under stable stratification) and the cavity flows (mainly with respect to the turbulence field).

For the conventional buoyant plumes, the approach employed in this study should be extended to cover three-dimensional cases with cross-wind, and for the near-, mid-, and far-field regions.

For the radioactive buoyant plume, applications should be made to cases with initial and boundary conditions which are more akin to most probable circumstances of a nuclear reactor accident, and the method should be extended to include effects of cross-winds.

Finally, further calculations of cavity flows should be made for depth-to-width ratios other than unity, and for cases where the side walls are heated.

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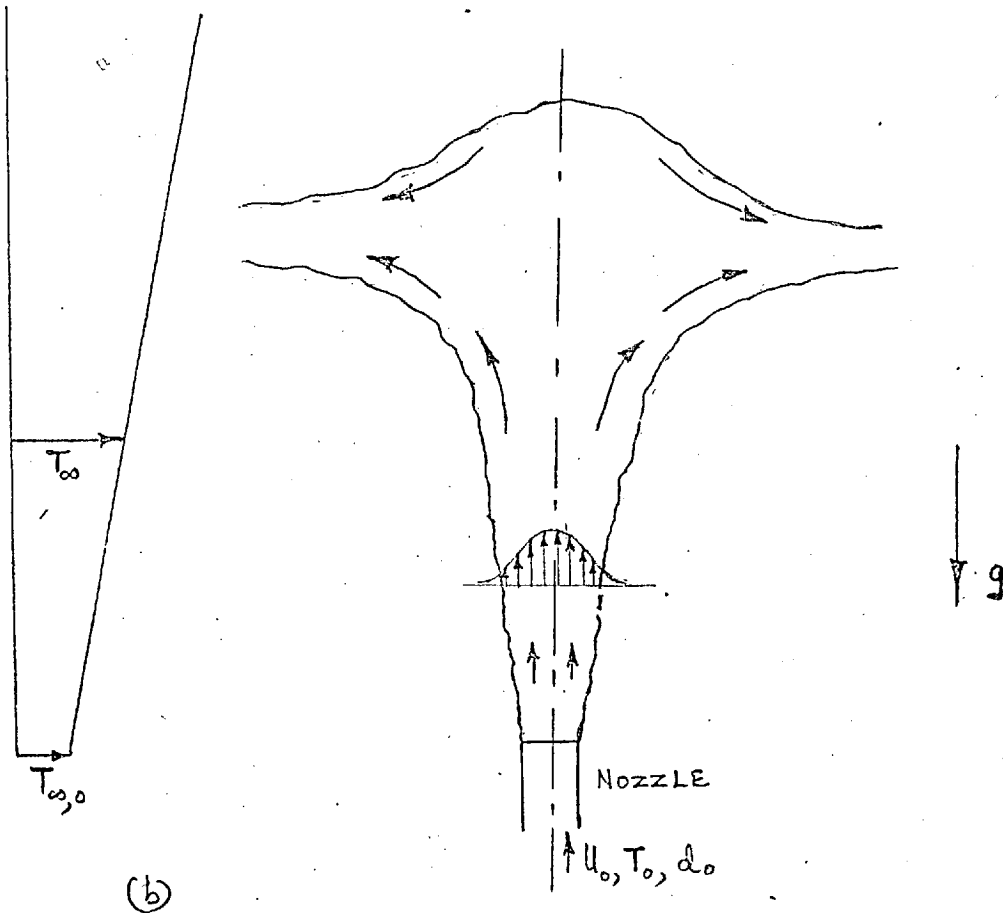
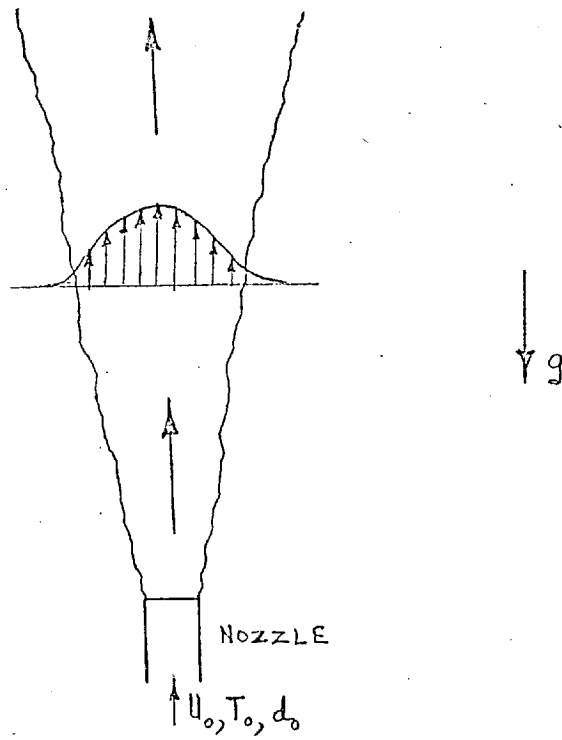
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(a)



(b)

Fig.1.1 Schematic representation of buoyant plumes in different ambient conditions:
 (a) uniform surroundings; (b) stably-stratified surroundings

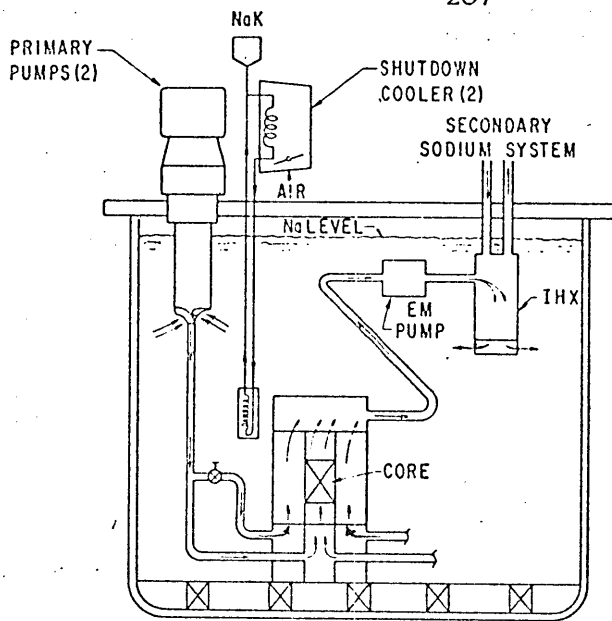


Figure 1.2 Schematic of the EBR-II Primary System (from Singer et al, 1976)

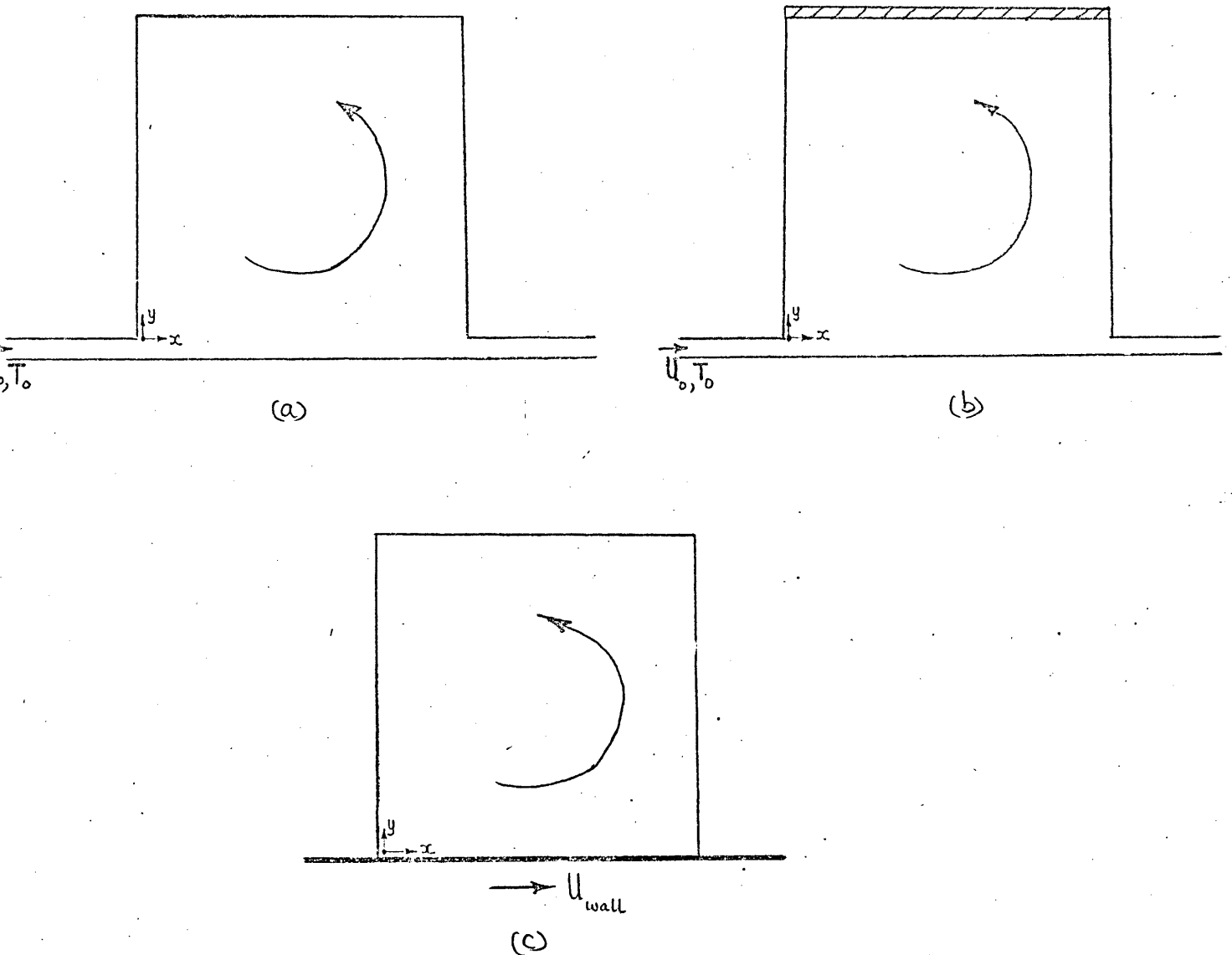
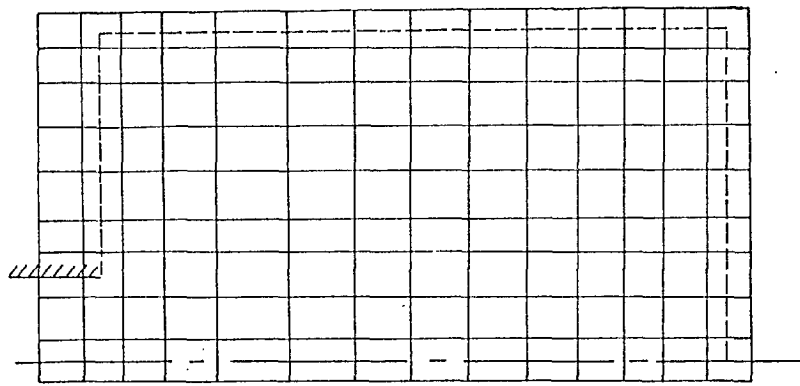


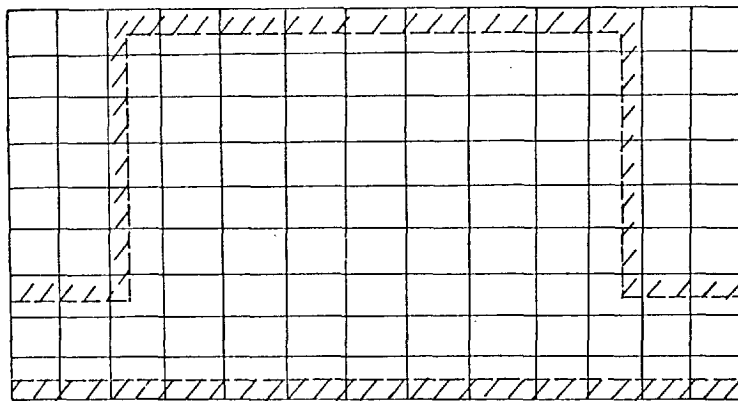
Fig. 1.3 Schematic representation of cavity flows : (a) Channel-flow drive; forced convection
(b) channel-flow drive; heated top wall (c) Sliding-wall drive



(a)

(i) Plume flow

Boundaries of flow domain:



(ii) Cavity flow

----- Axis of symmetry
 ----- Free boundary
 // // // // // Wall boundary

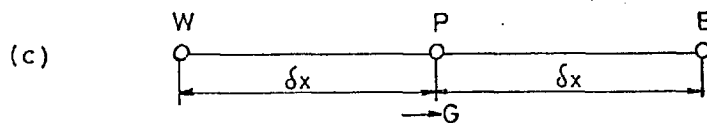
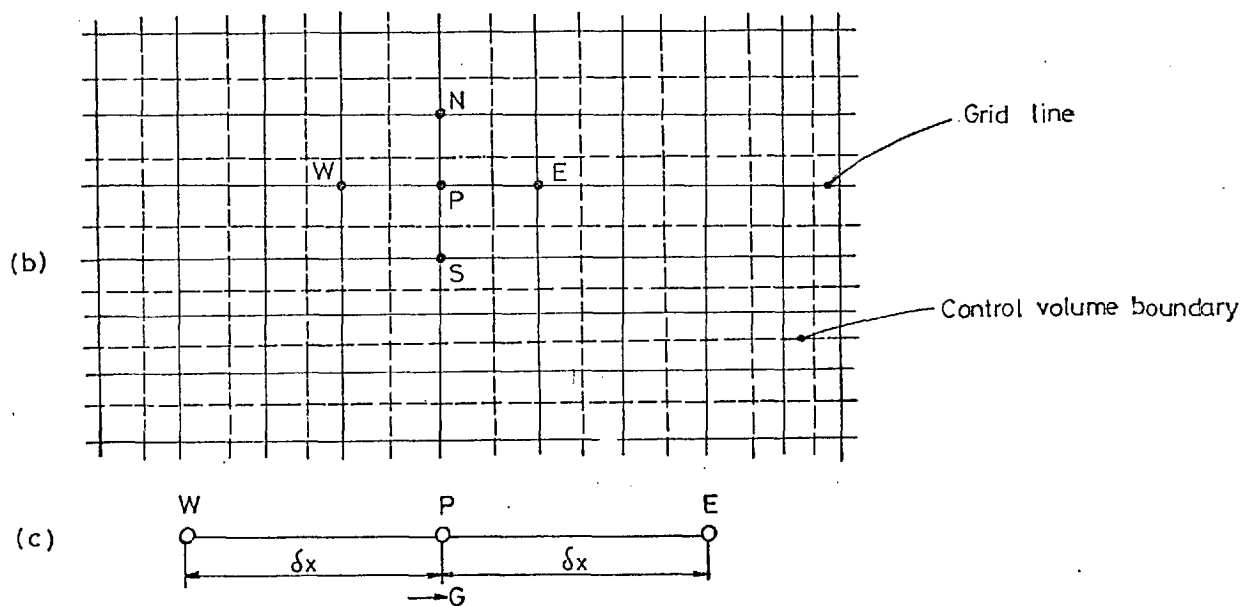


Fig. 3.1 Definition of grid and control volume boundaries:

a) Grid and boundaries of flow domain b) grid and control volumes c) illustration of one-dimensional flow grid

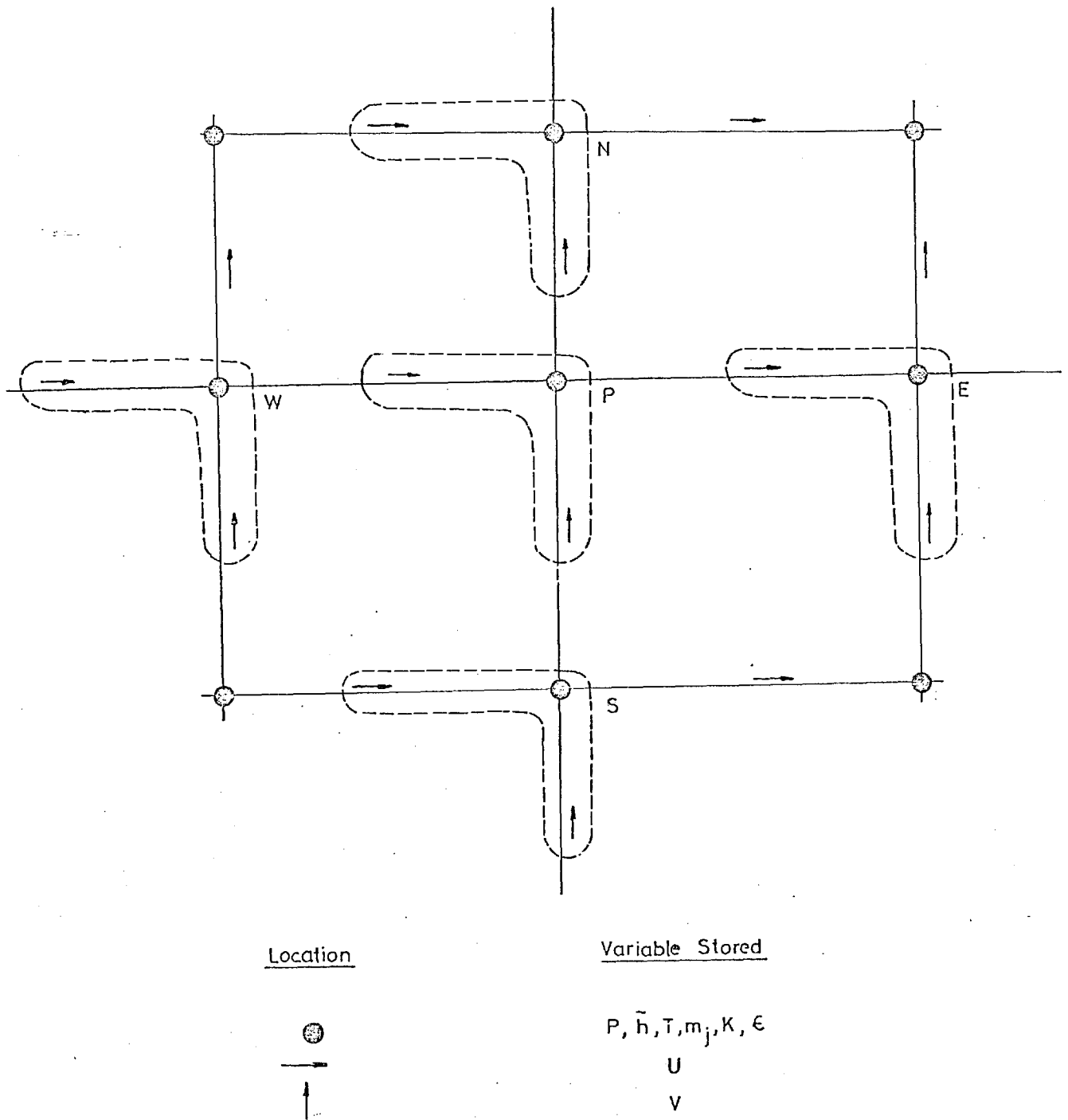
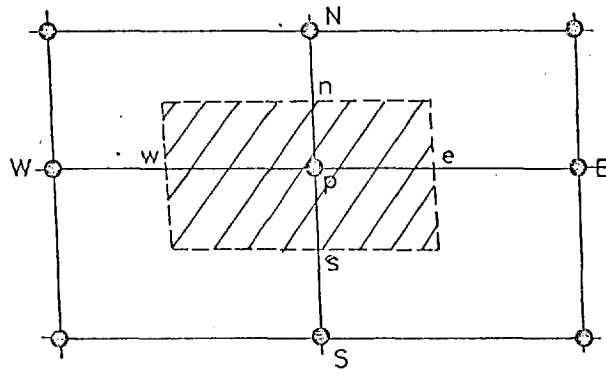
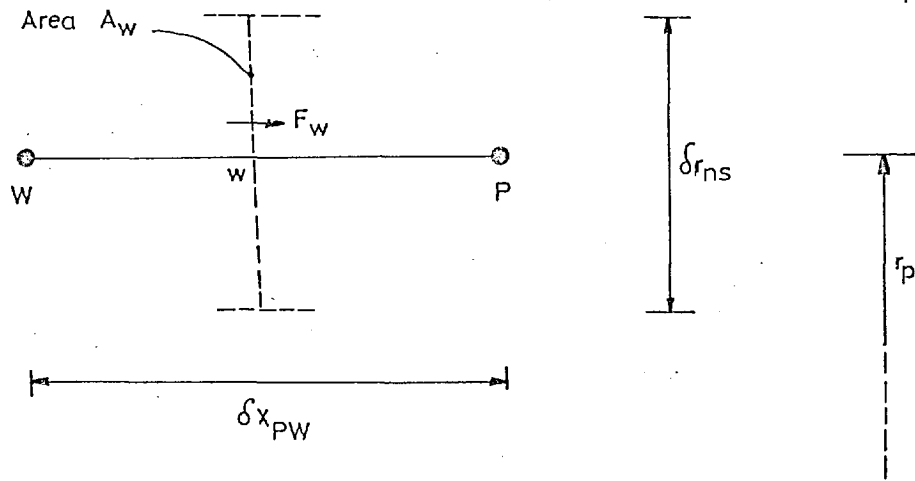


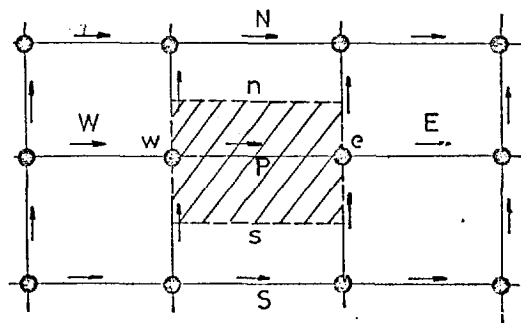
Fig. 3.2 The "staggered grid" (Solution triads for SIMPLE procedure)



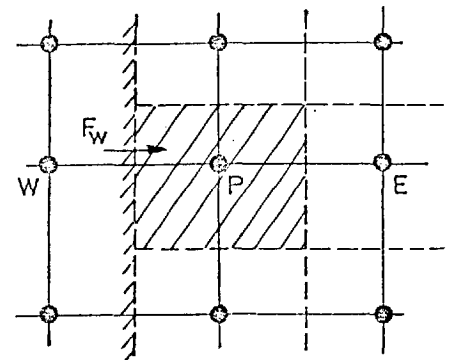
(a) Typical Scalar Cell



(b) Dimensions of, and flux through, west boundary of Cell

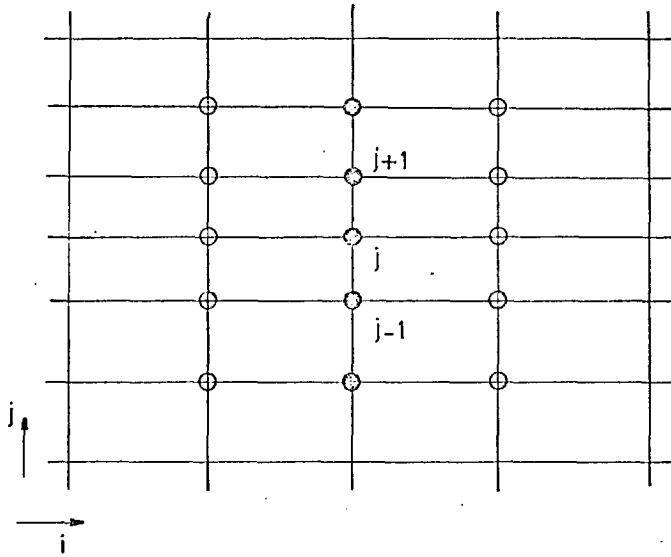


(c) Typical U-Cell



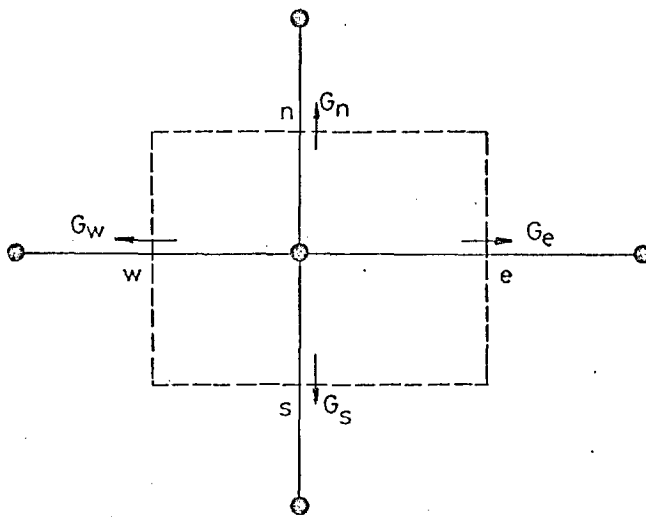
(d) Typical boundary Cell.

Fig. 3.3 Illustration of typical control volumes for formulation of finite-difference equations (n, s, w, e denote cell boundaries)



⊙ = points on line considered
○ = points on neighbouring lines

3.4(a) Illustration of line procedure



$$G_w = \rho_w U_w A_w$$

$$G_e = \rho_e U_e A_e$$

$$G_n = \rho_n V_n A_n$$

$$G_s = \rho_s V_s A_s$$

3.4(b) Illustration of possible condition where mass flows do not satisfy continuity

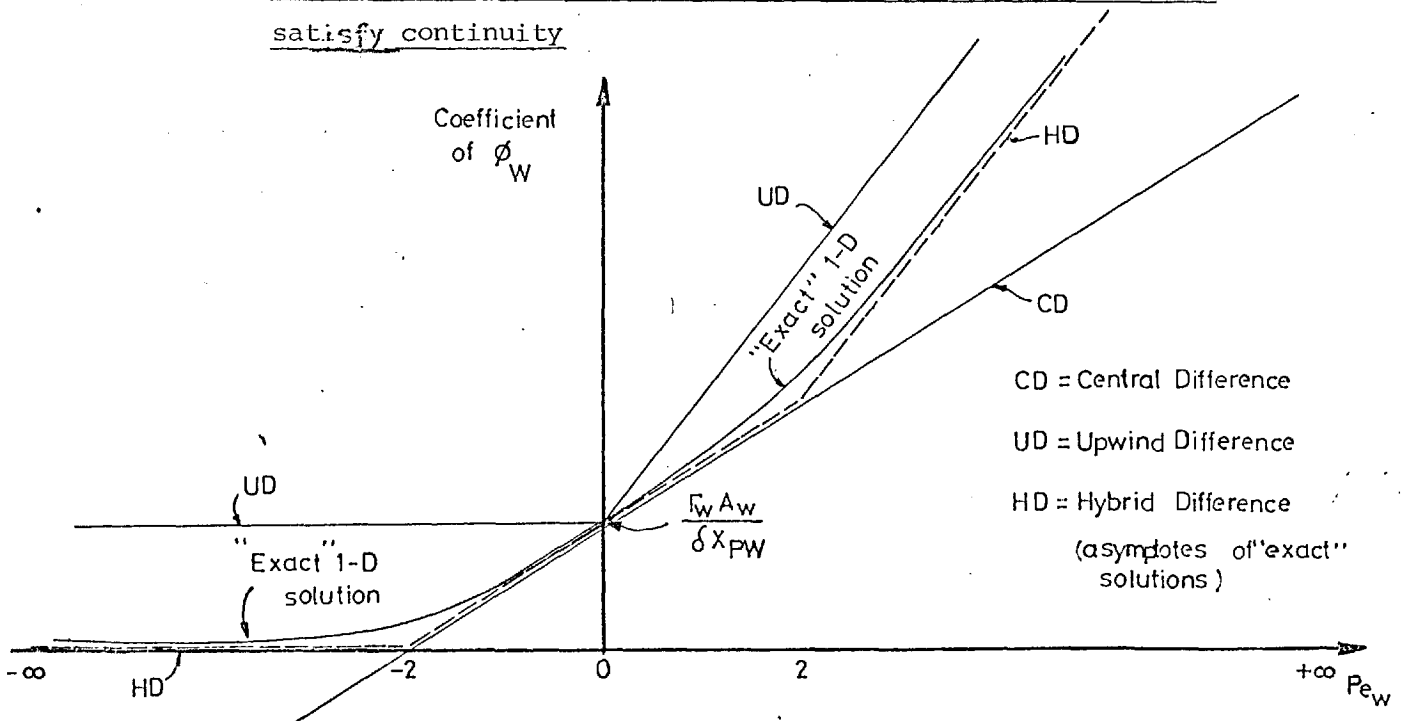


Fig. 3.4(c) Comparison of total-flux expressions for various difference schemes

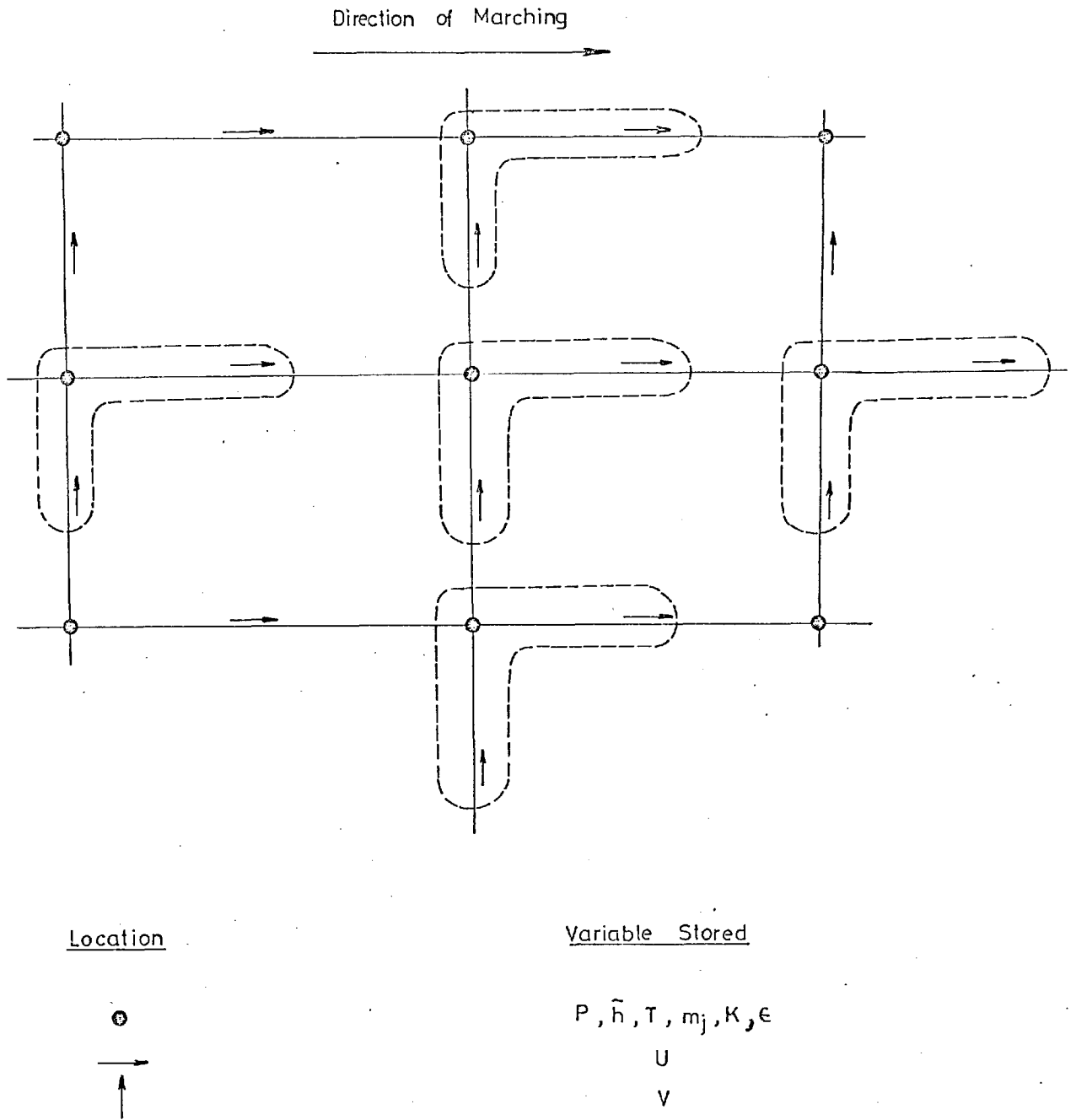


Fig. 3.5 The "Staggered grid" (Solution triads for EMIT procedure)

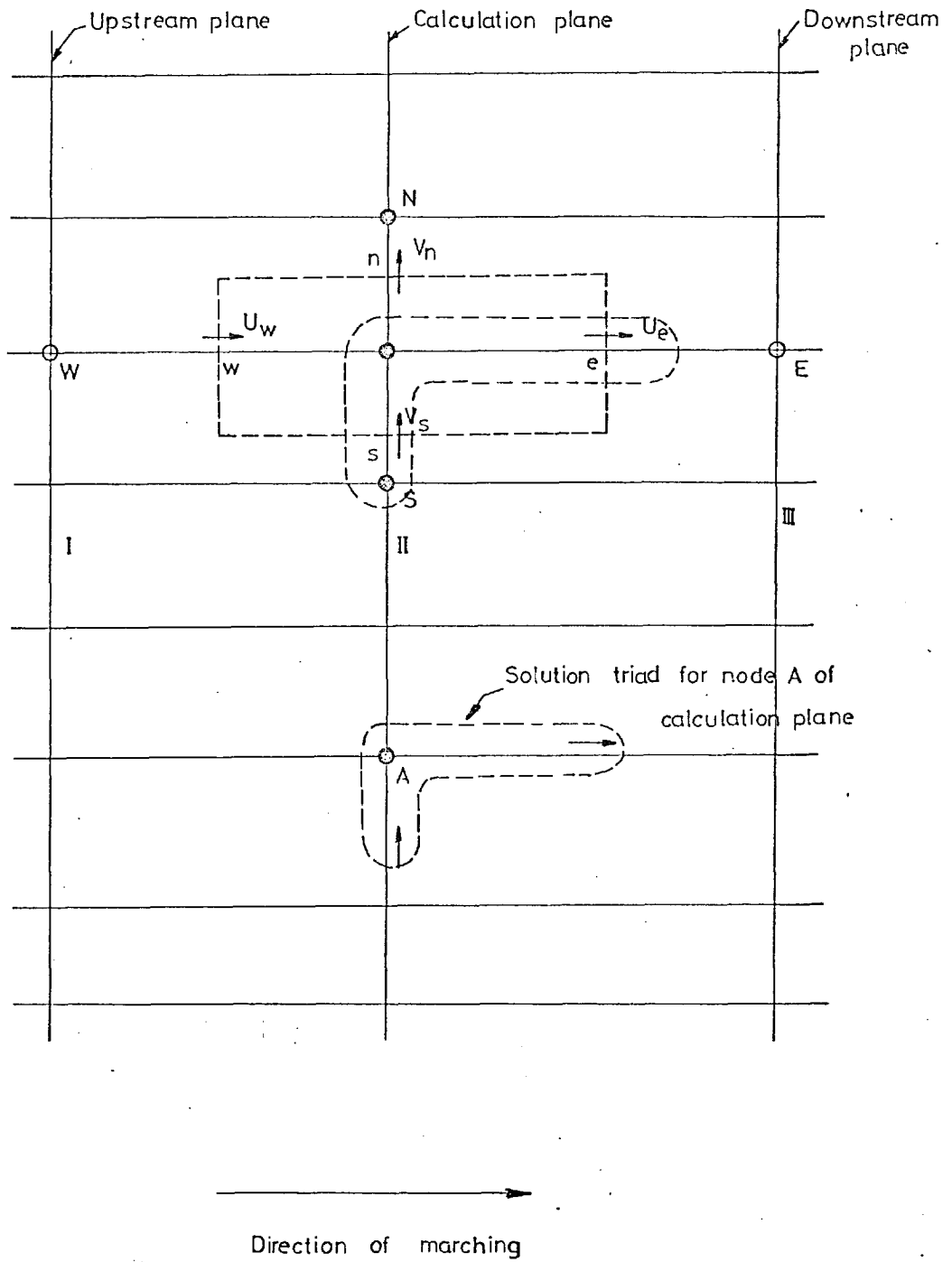
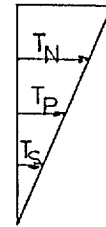
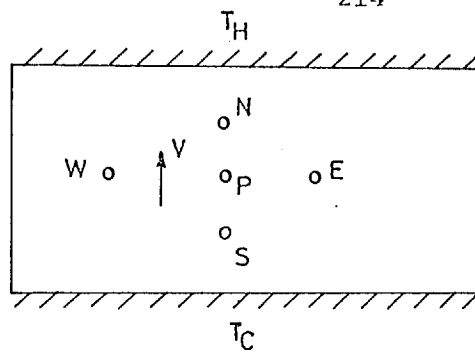
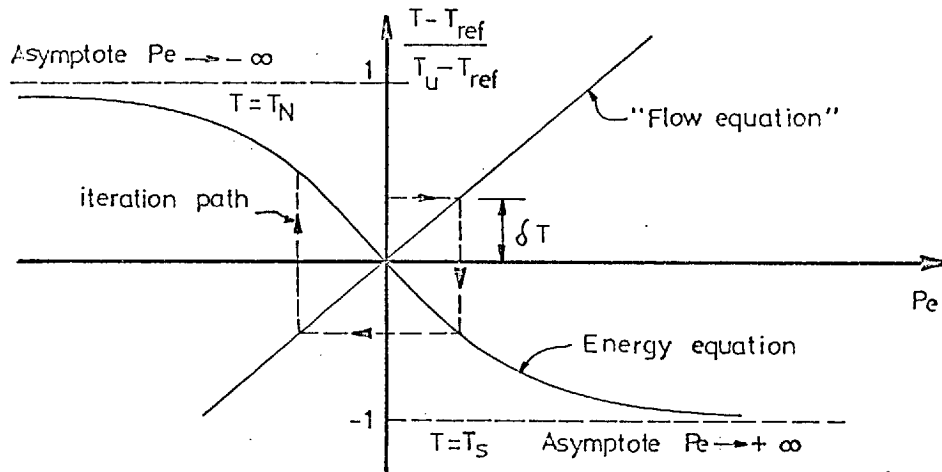


Fig. 3.6 Illustration of calculation plane for EMIT procedure

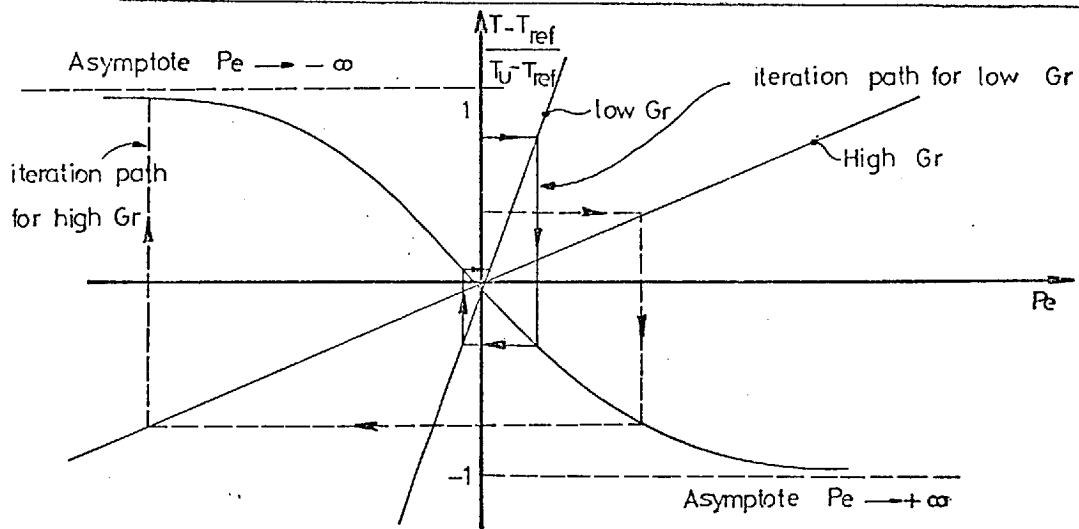


unperturbed temperature distribution.

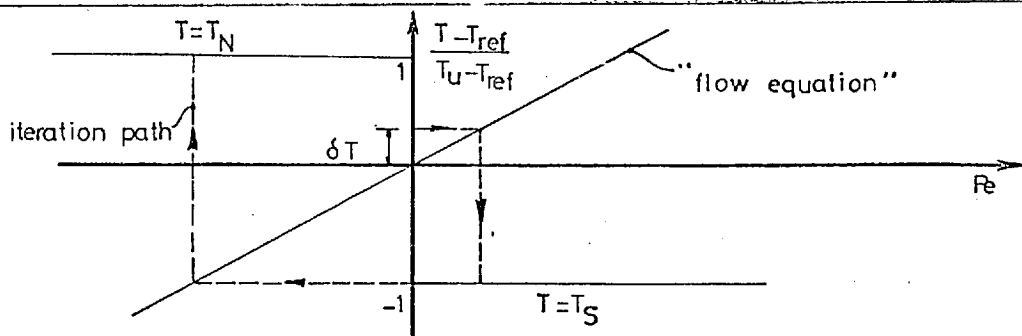
(a) One dimensional Stably-Stratified flow field.



(b) Graphical representation of successive relaxation procedure.



(c) Influence of Grashof number on Stability.



(d) Effect of Small perturbation when fluid thermal conductivity is zero ($Pe = Pr = \infty$).

Fig. 3.7 Illustration of buoyancy-induced numerical instability
($T_u = T_w$ or T_s , $T_{ref} = T_p$)

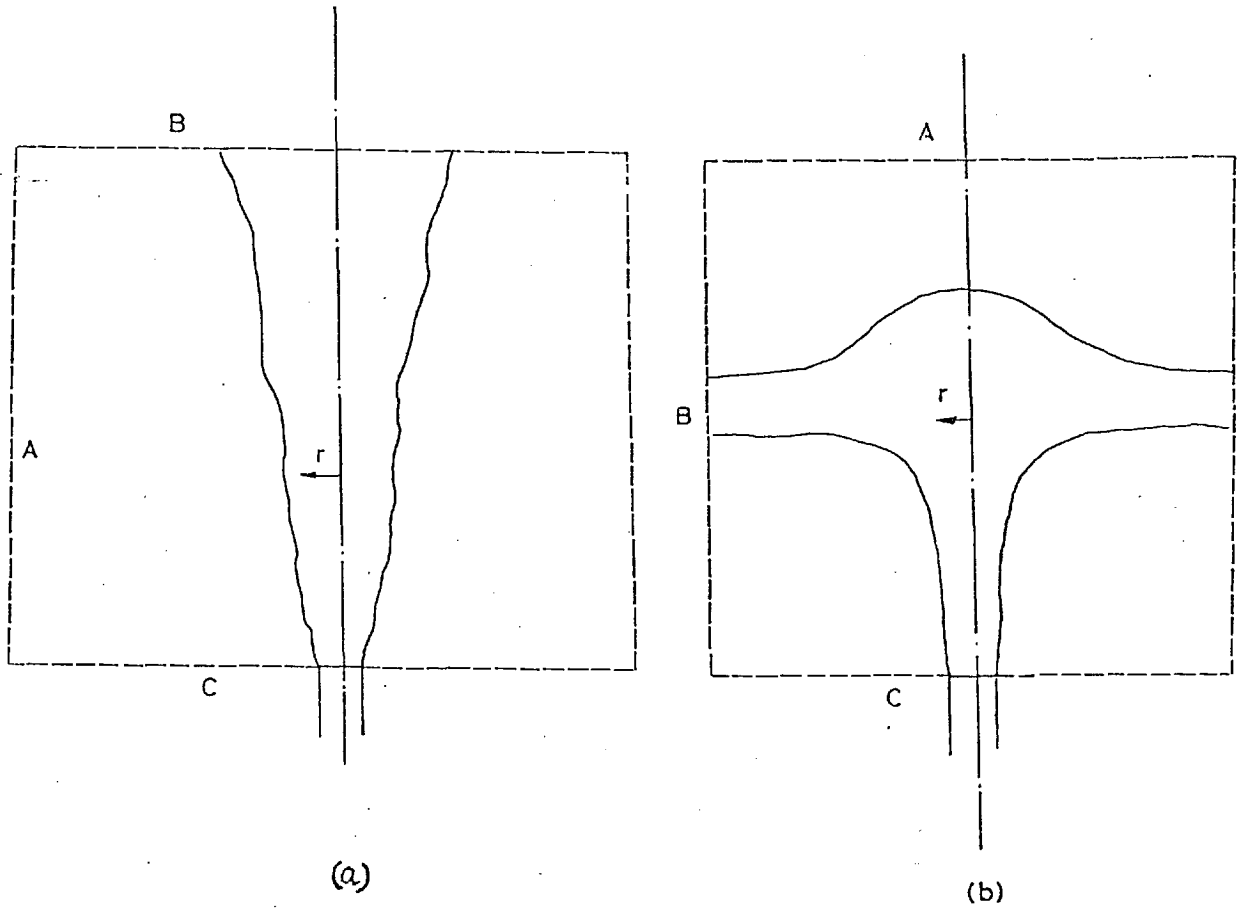


Fig. 4.1 Definition of domain of computation for plumes in both uniform and stably-stratified surroundings. Computations made for only one-half of flow field owing to symmetry.

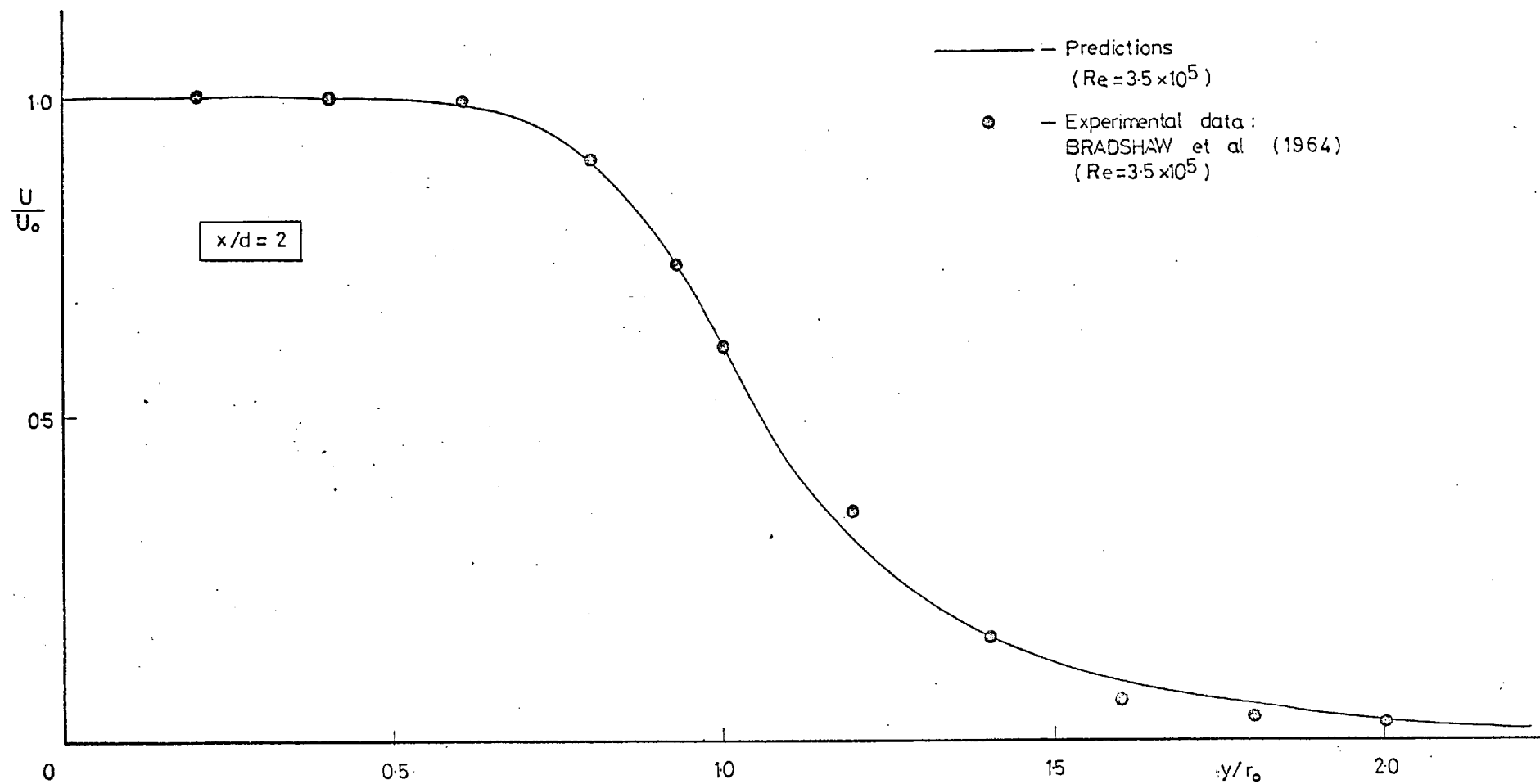


Fig. 4.2 MOMENTUM JET: Radial distribution of mean velocity in the mixing-layer zone

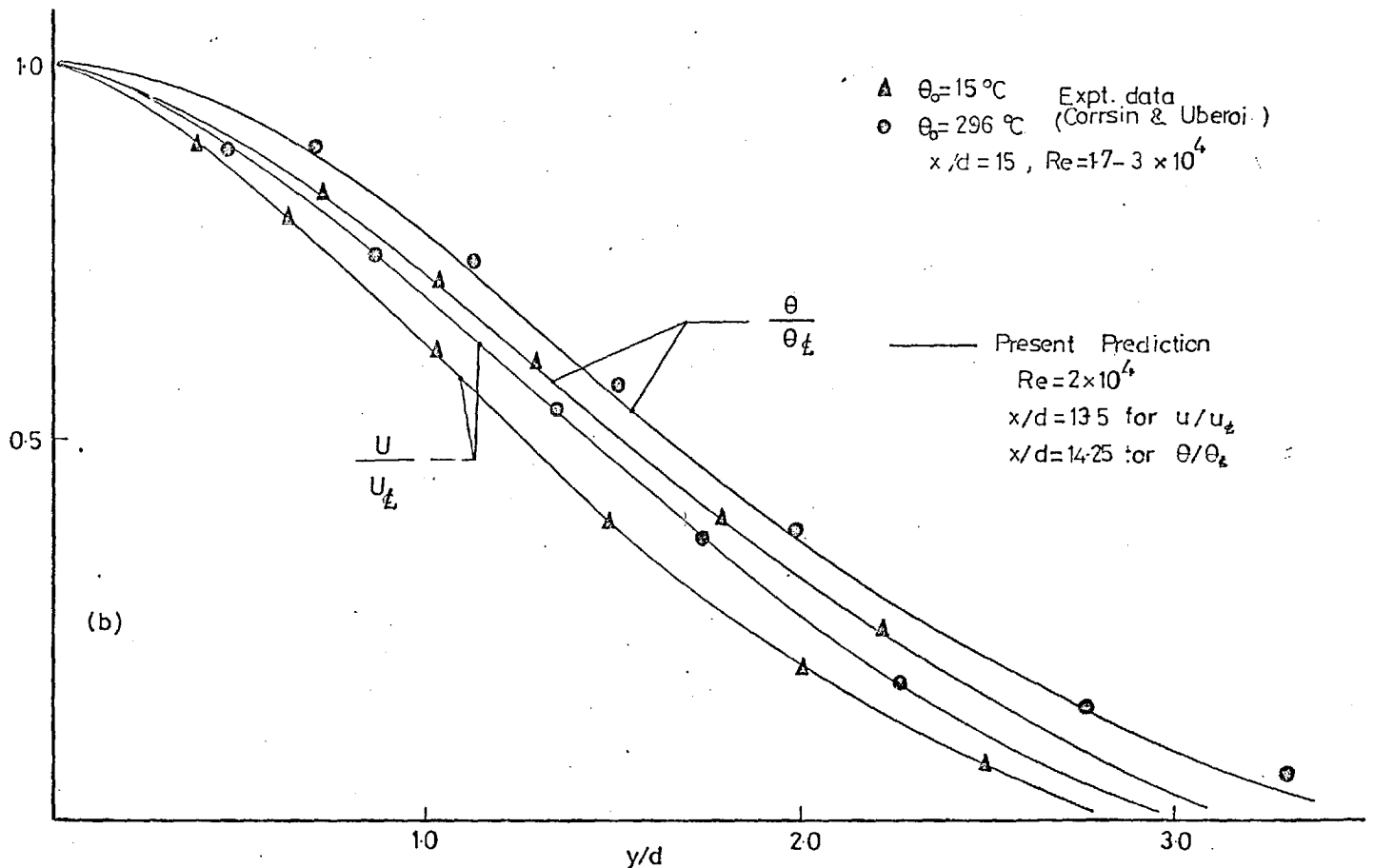
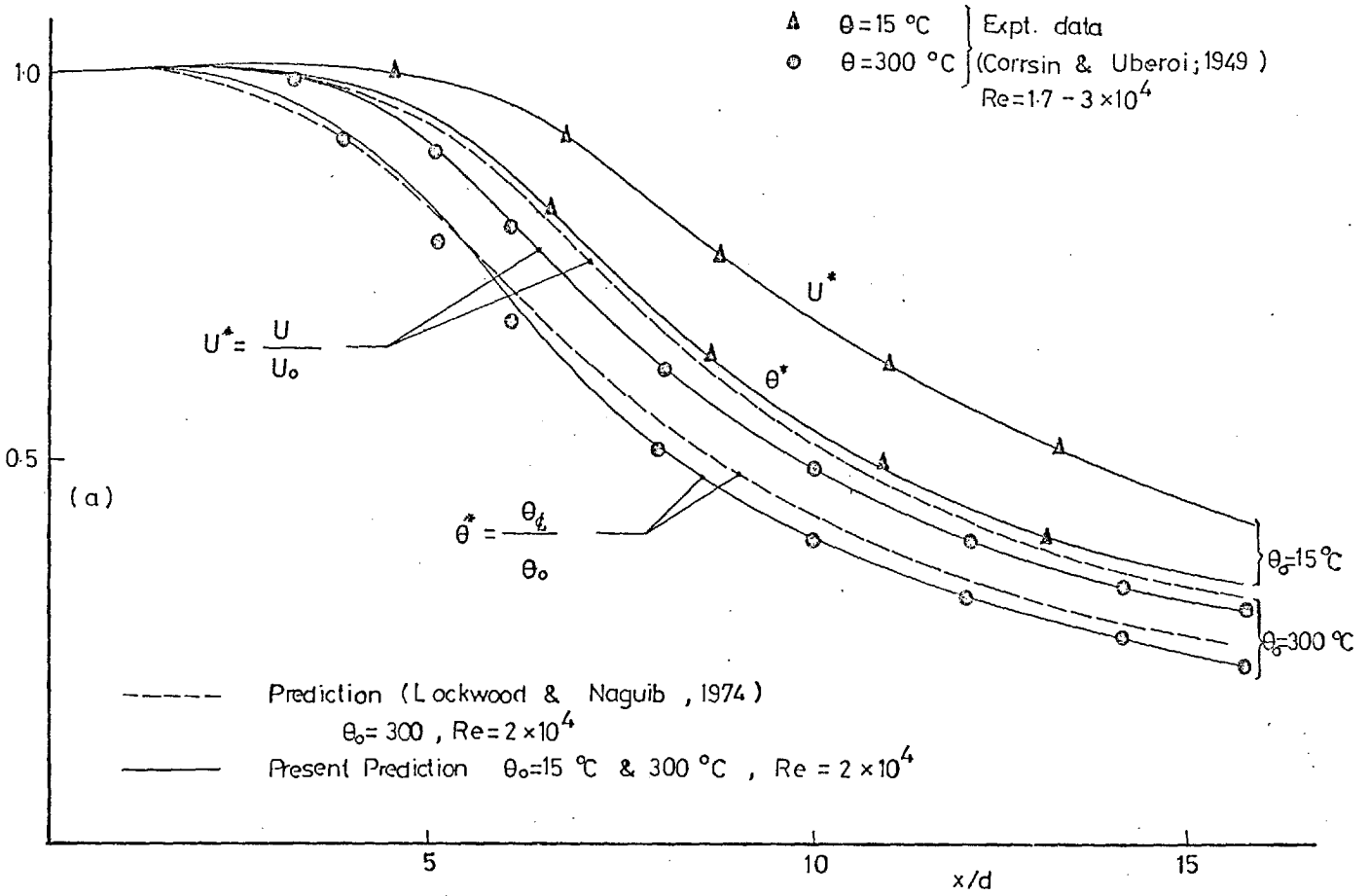


Fig. 4.3 Momentum Jet: Velocity and Temperature Distributions

(a) Axial Dist. (b) Radial Dist. in the "fully-developed" region

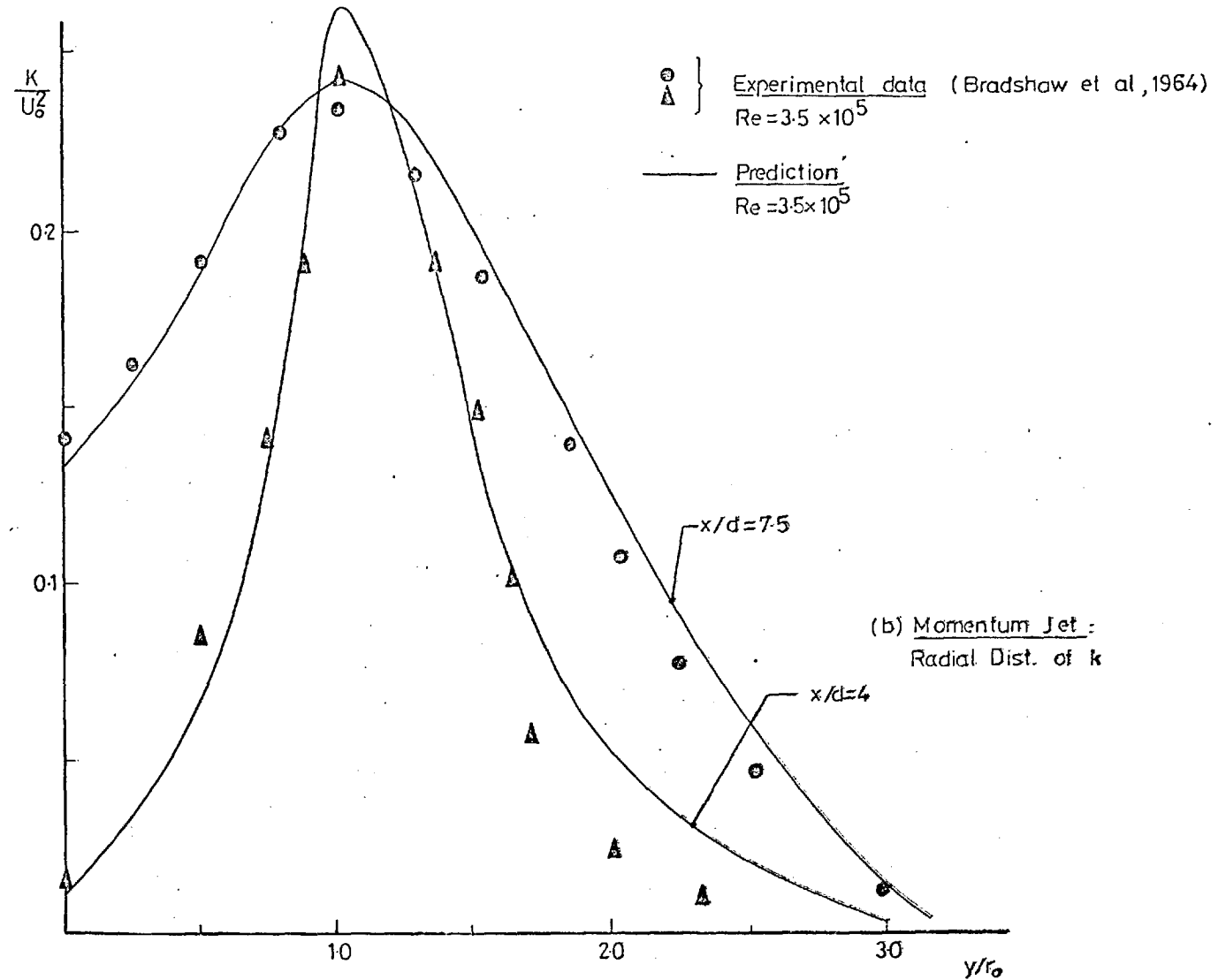
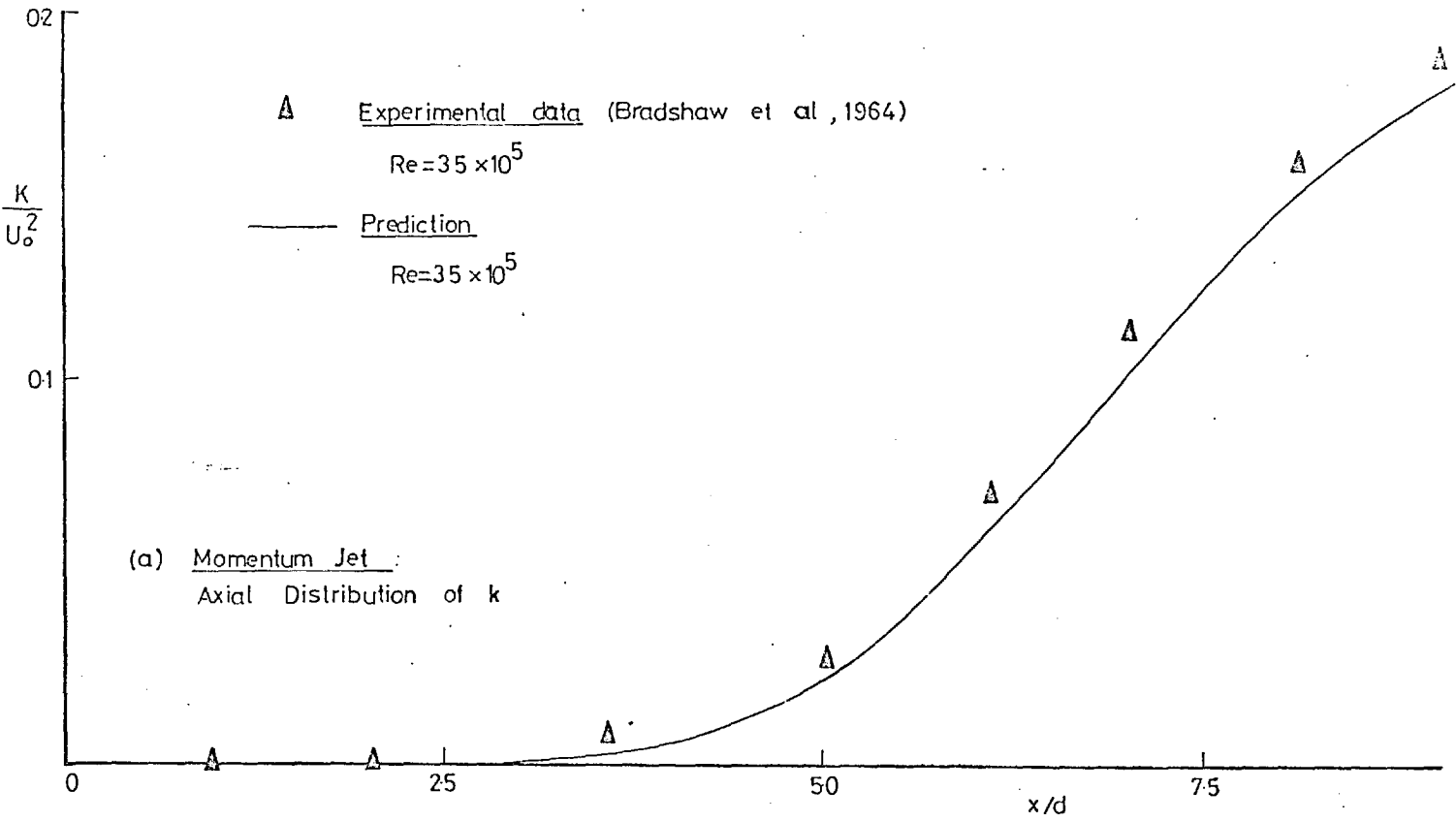
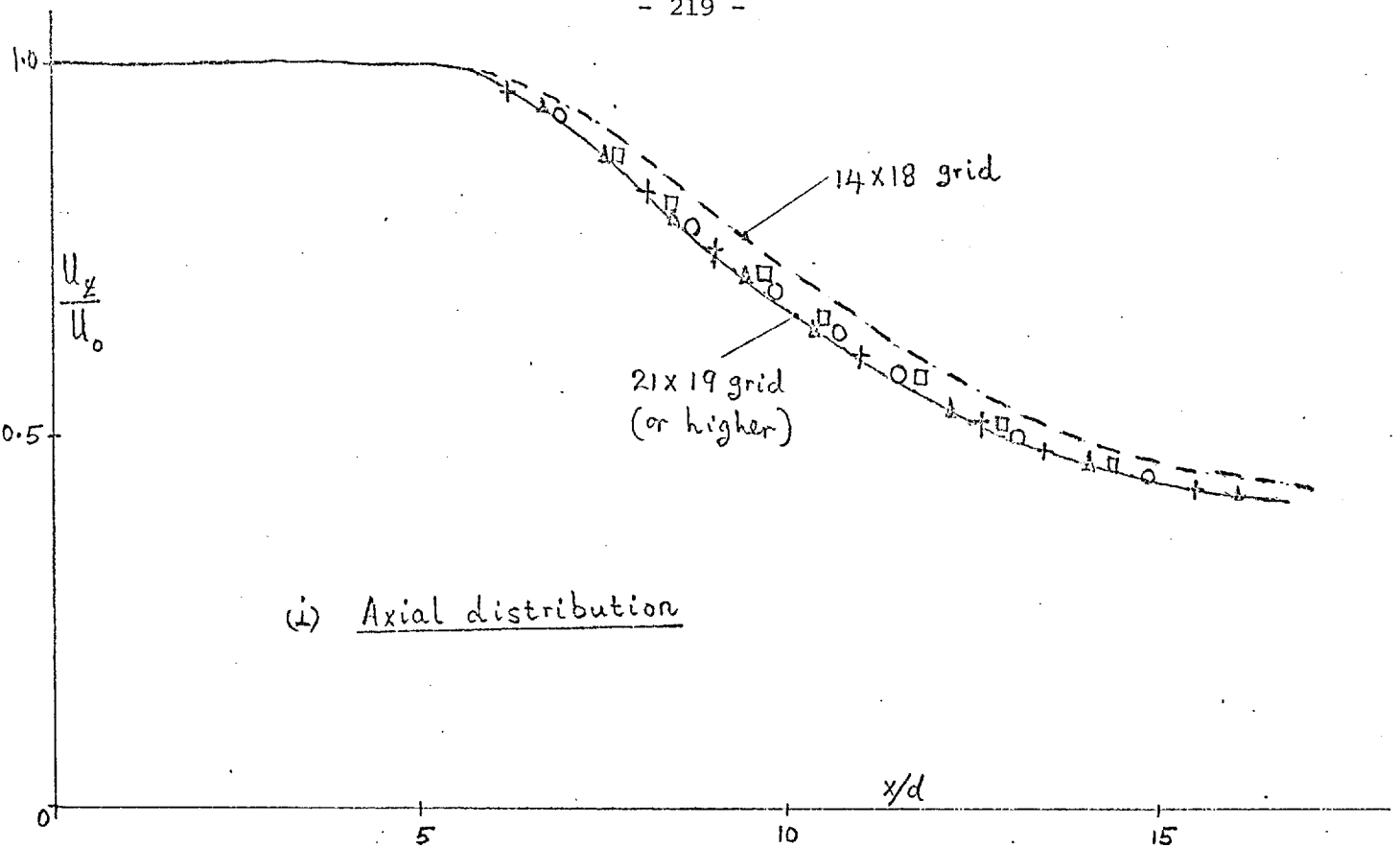


Fig. 4.4 Momentum Jet: Distribution of turbulence energy



▲ Expt. Data (Corrsin & Uberoi, 1949)
 $Re = 2 \times 10^4$, $\theta_0 = 15^\circ C$

Predictions ($Re = 2 \times 10^4$, $\theta_0 = 15^\circ C$)

+	21 x 19, 25 x 19	} grid
O	18 x 21	
□	17 x 19	

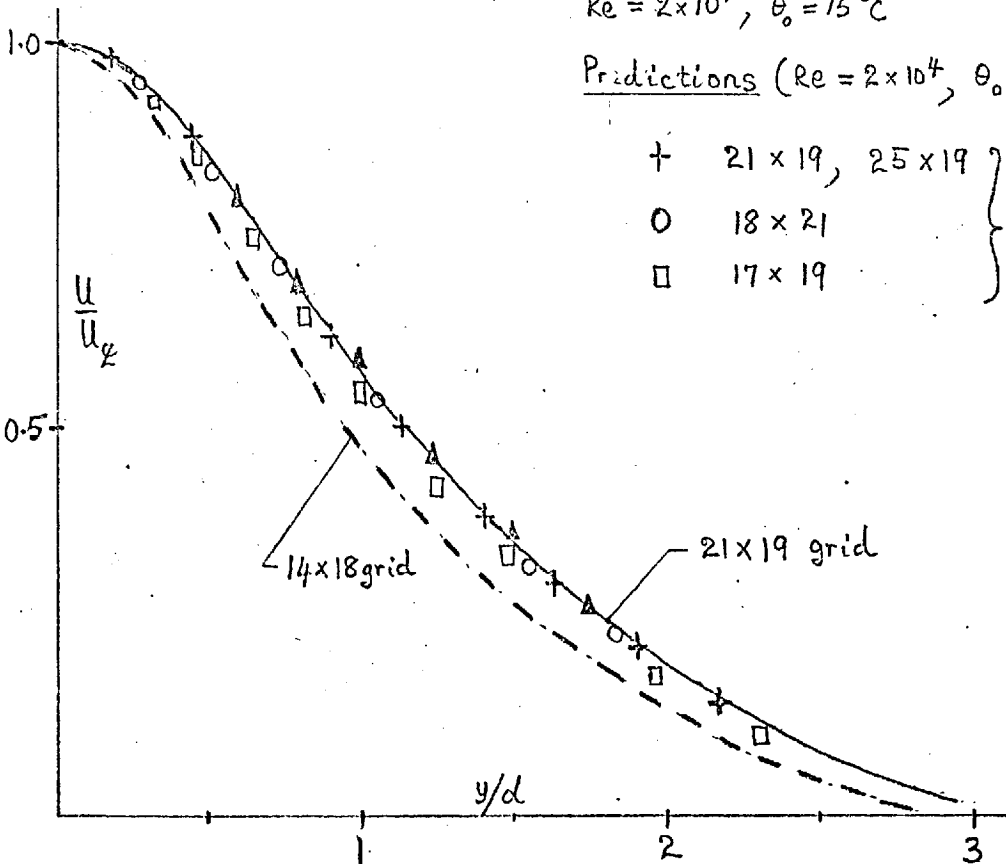


Fig. 4.4 (c) Forced plume in uniform surroundings (Momentum Jet):
Grid test

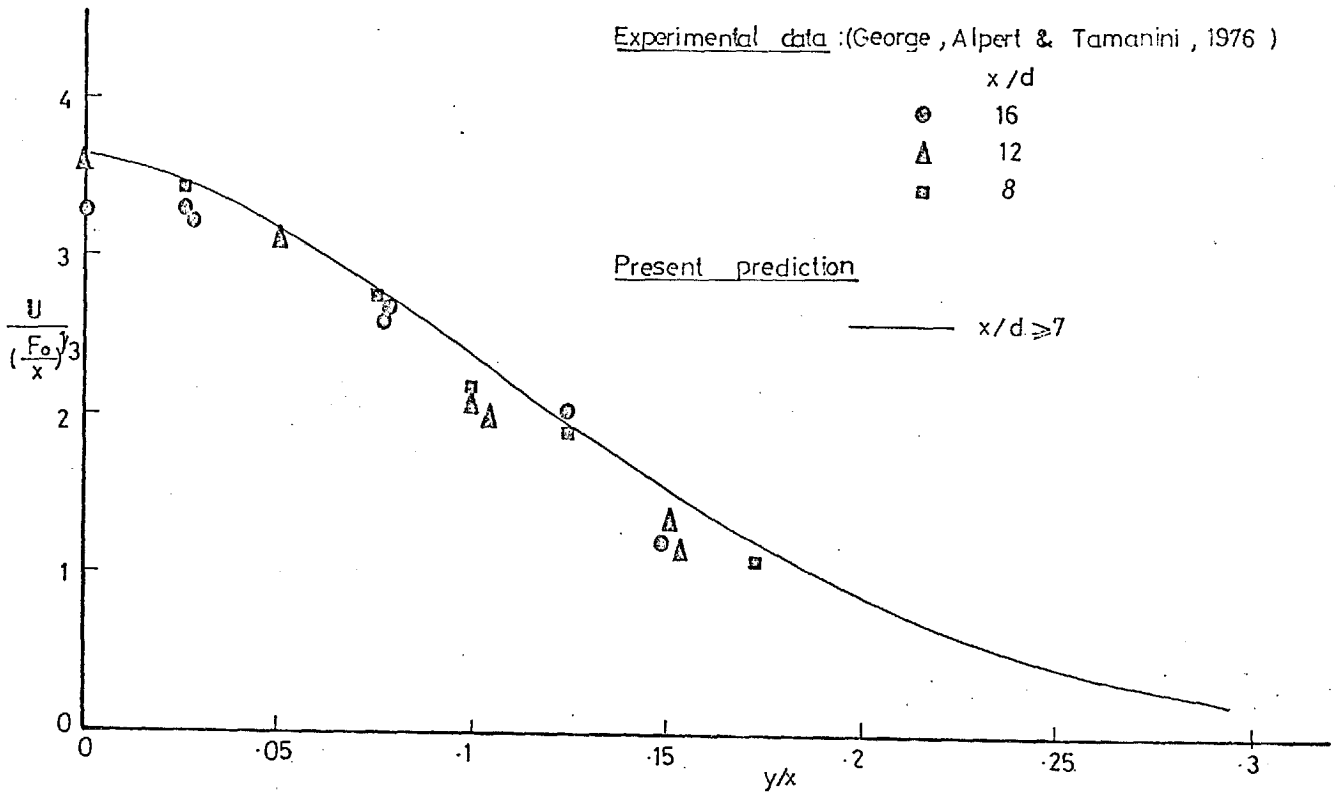


Fig. 4.5 Plume in Uniform Surroundings: Radial distribution of mean velocity

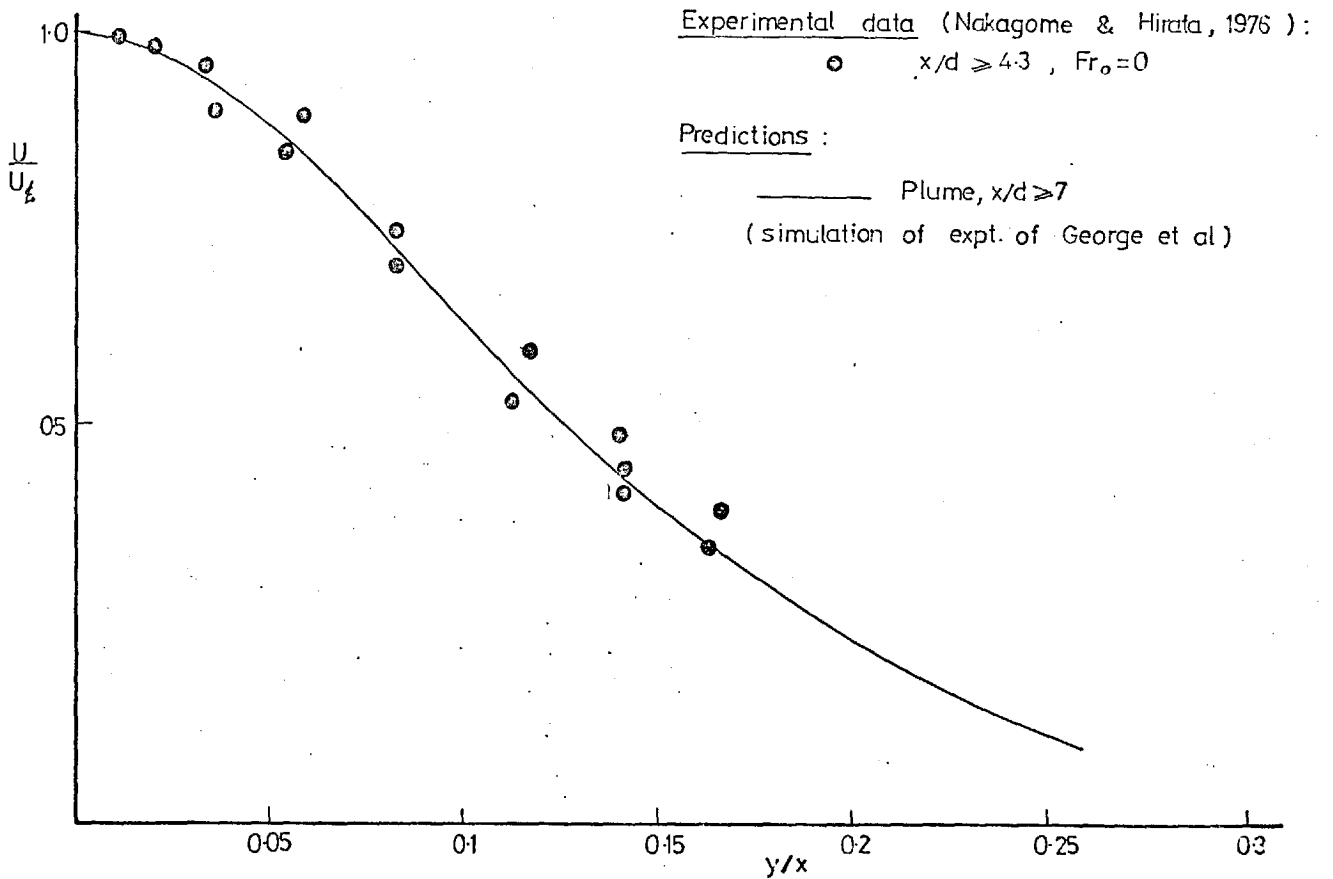


Fig. 4.6 Plume in Uniform Surroundings: Radial distribution of mean velocity

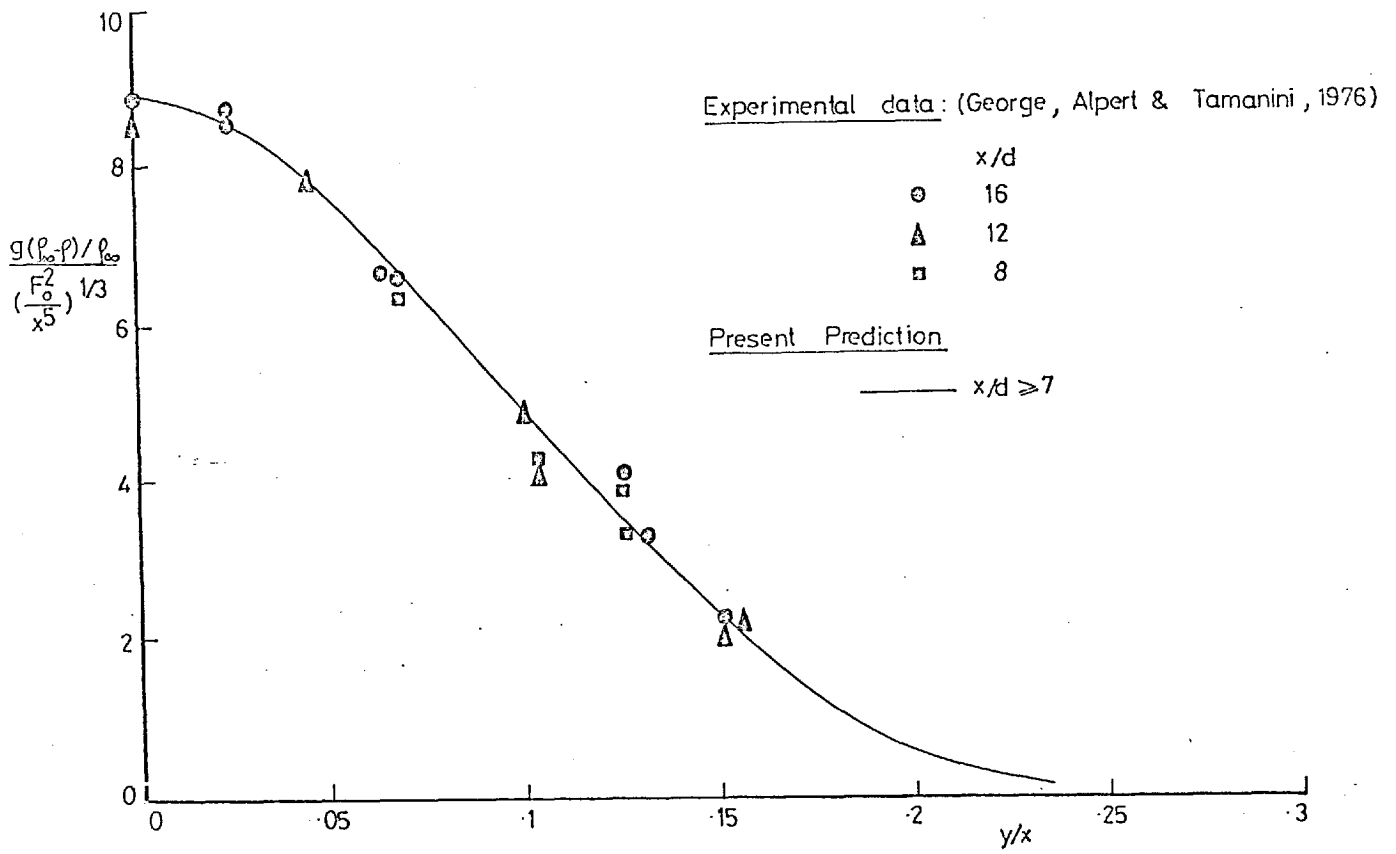


Fig. 4.7 Plume in Uniform Surroundings: Radial distribution of mean density (or buoyancy)

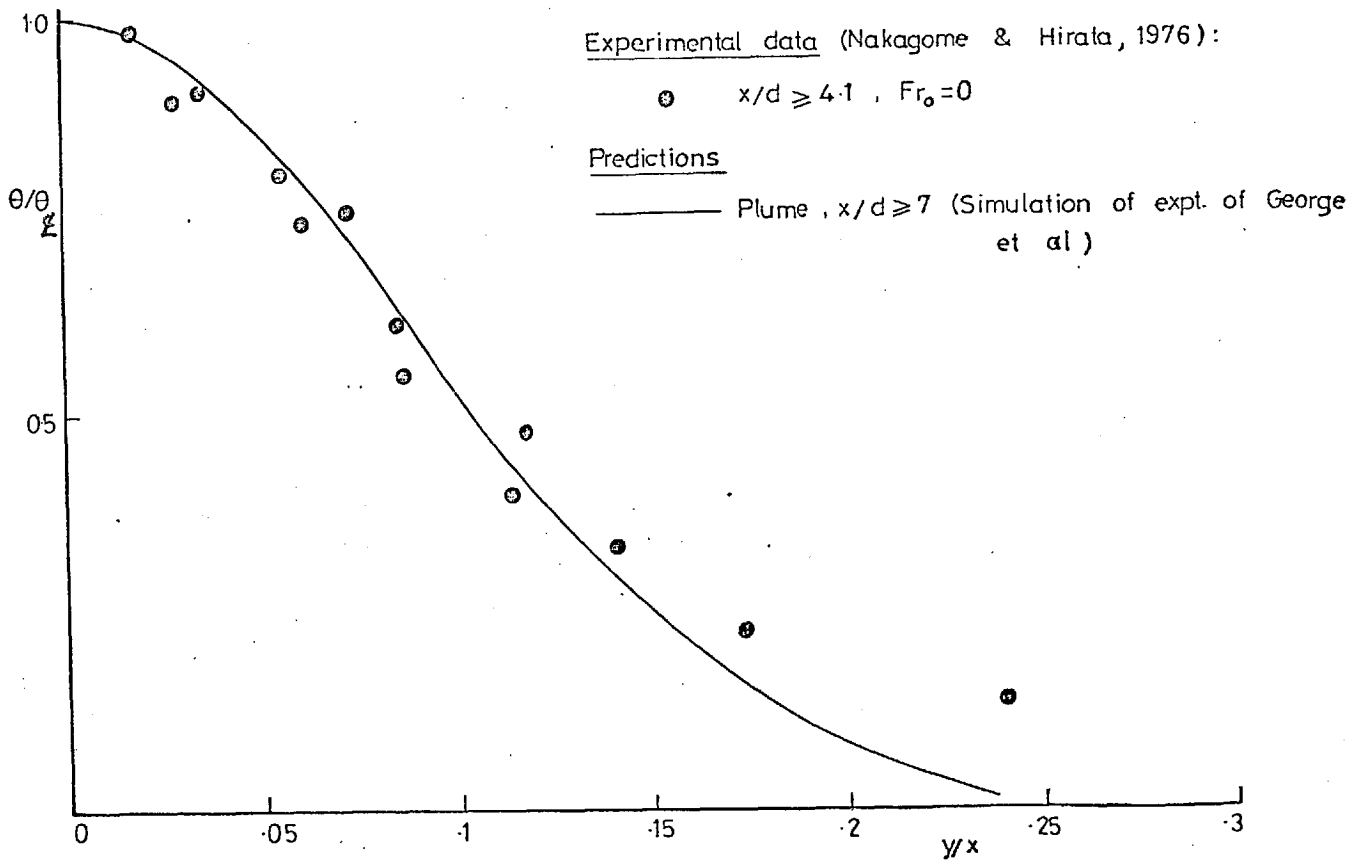


Fig. 4.8 Plume in uniform surroundings: Radial distribution of temperature

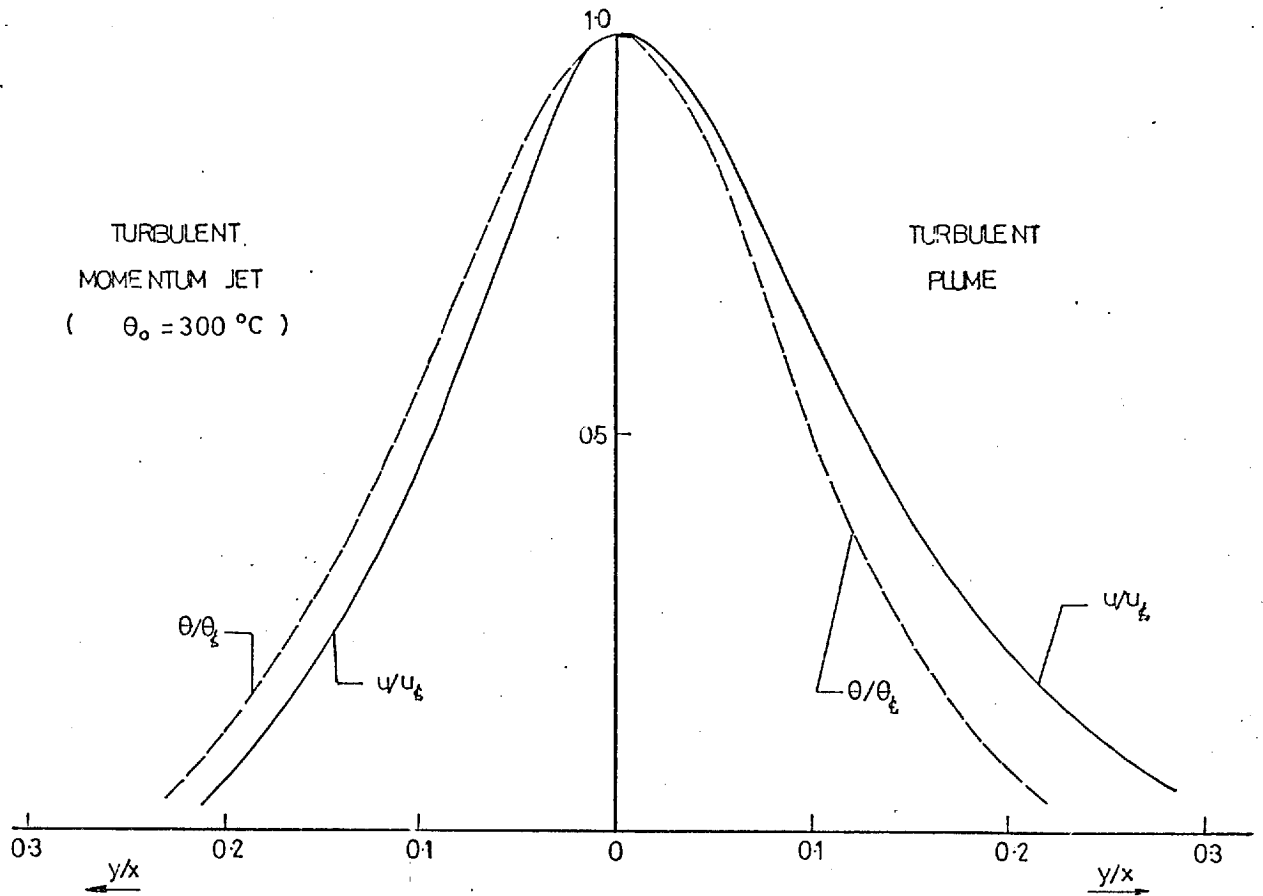


Fig. 4.9 Plume in uniform surroundings: Predicted radial distributions of mean velocity and temperature of the plume and the heated momentum jet

Experimental data (Nakagome & Hirata, 1976):

$\circ$ $y_{0.5}/d$ Plume
 Δ $y_{0.5T}/d$ ($Fr_0=0$)

Predictions

— Plume (Simulation of expt. of George et al, 1976)
 - - - Momentum Jet

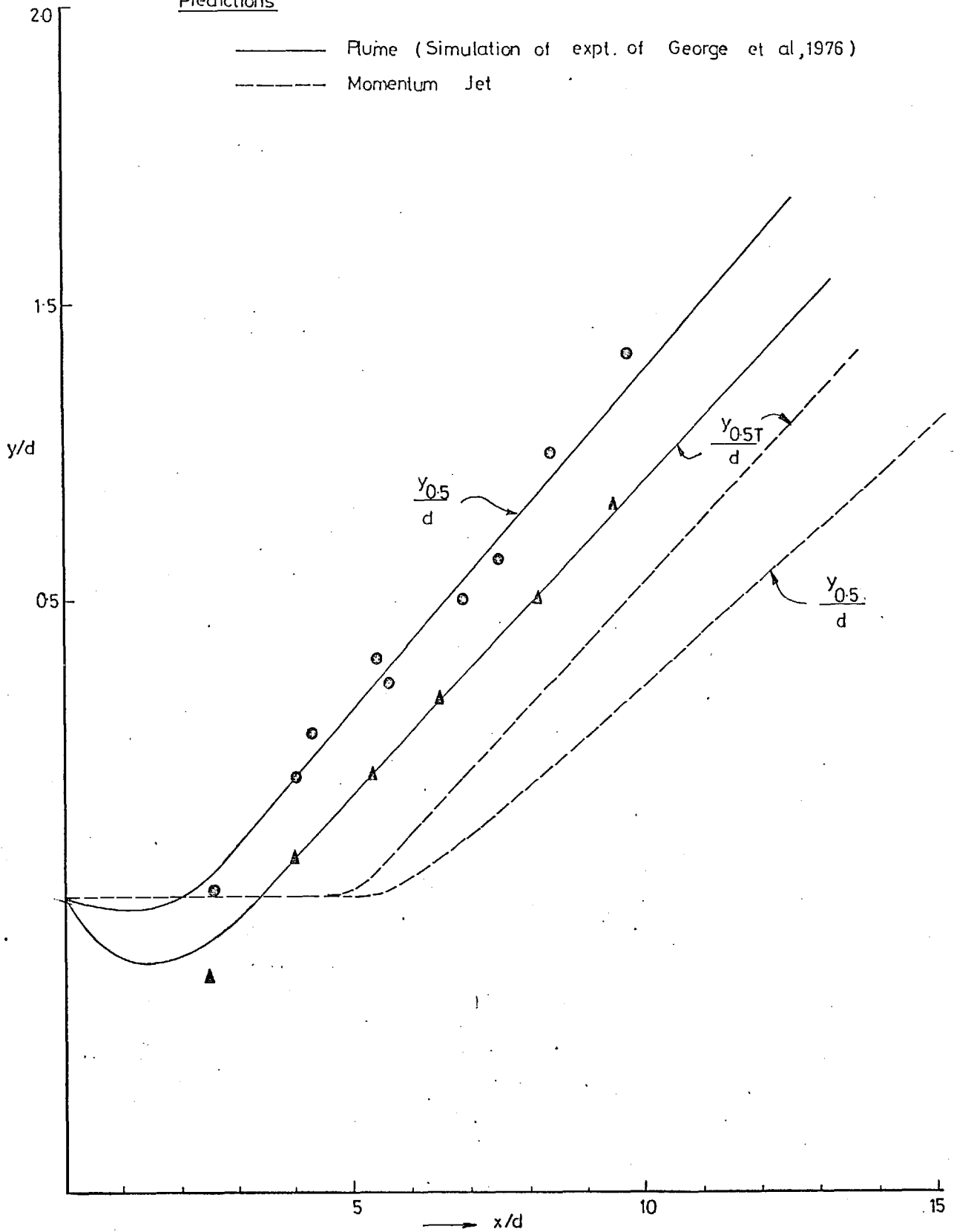


Fig. 4.10 Plume in uniform surroundings: Growth rate (distribution of half width) of the plume and momentum jet

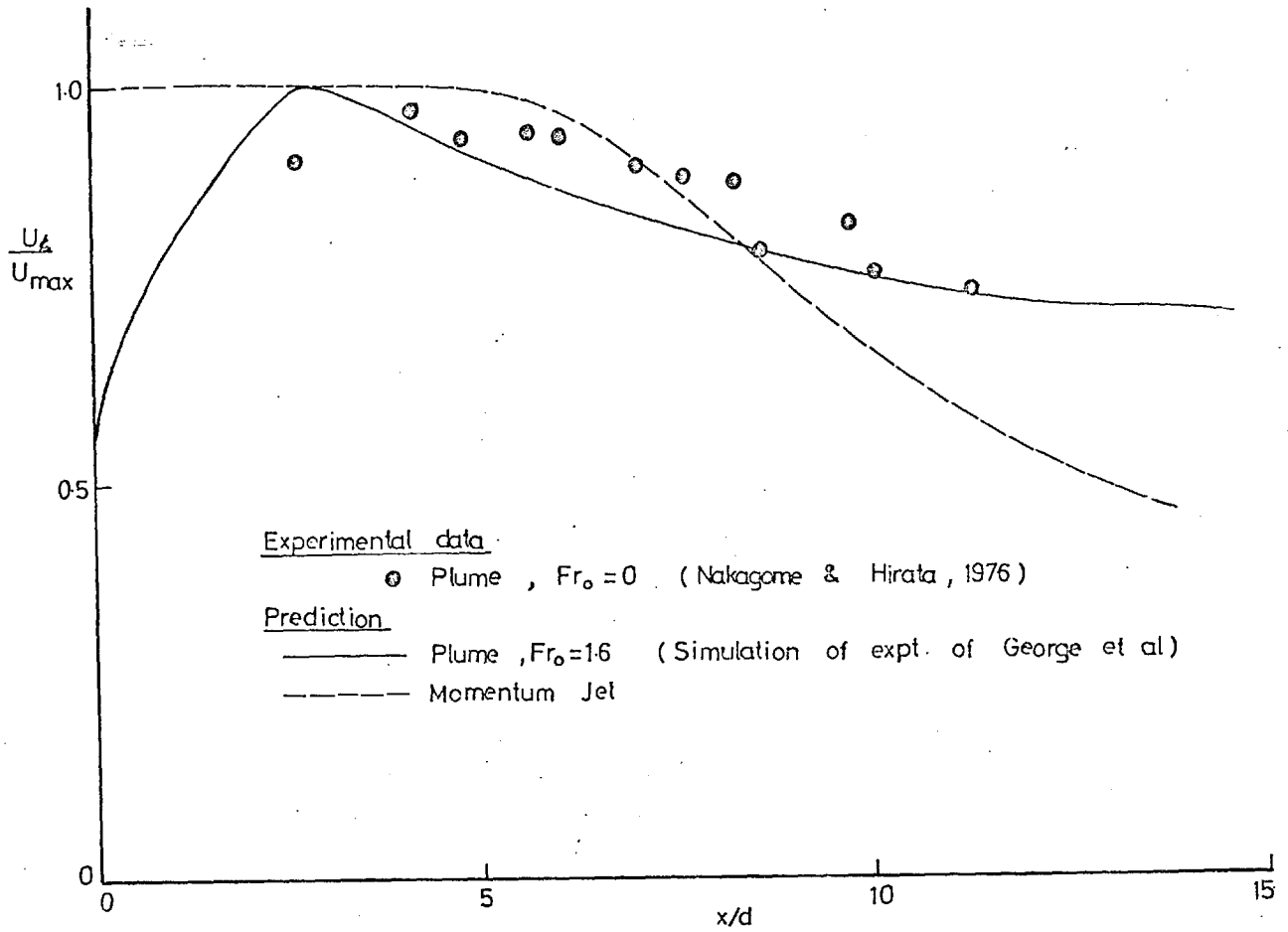


Fig. 4.11 Plume in uniform surroundings: Axial distribution of mean velocity

Experimental data

- ▲ Plume, $Fr_0=0$ (Nakagome & Hirata, 1976)
- Plume, $Fr_0=1.6$ (George et al, 1976)

Prediction

- Plume, $Fr_0=1.6$ (Simulation of expt. of George et al)
- - - Momentum Jet

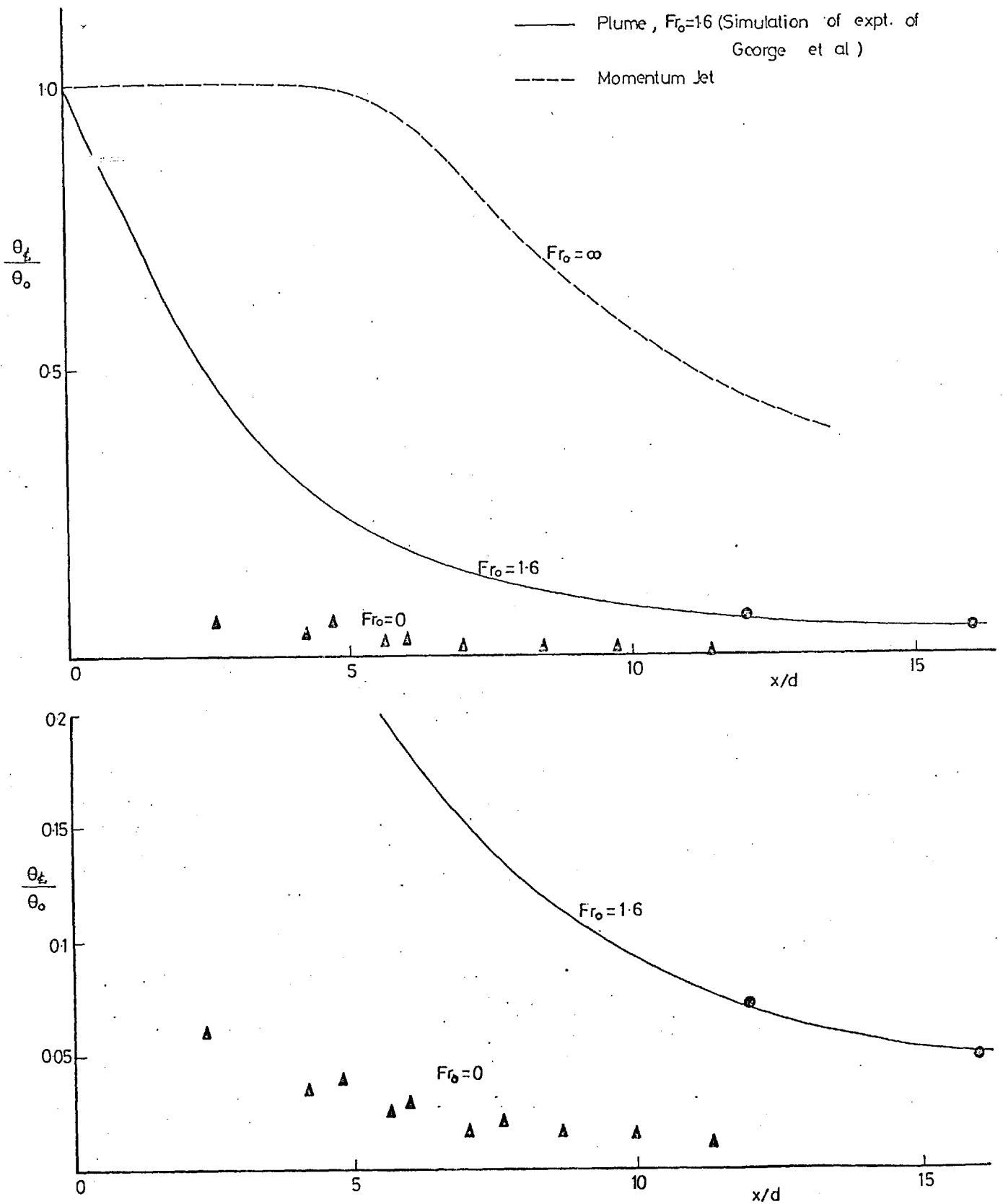


Fig. 4.12 Plume in uniform surroundings: Axial distribution of mean temperature

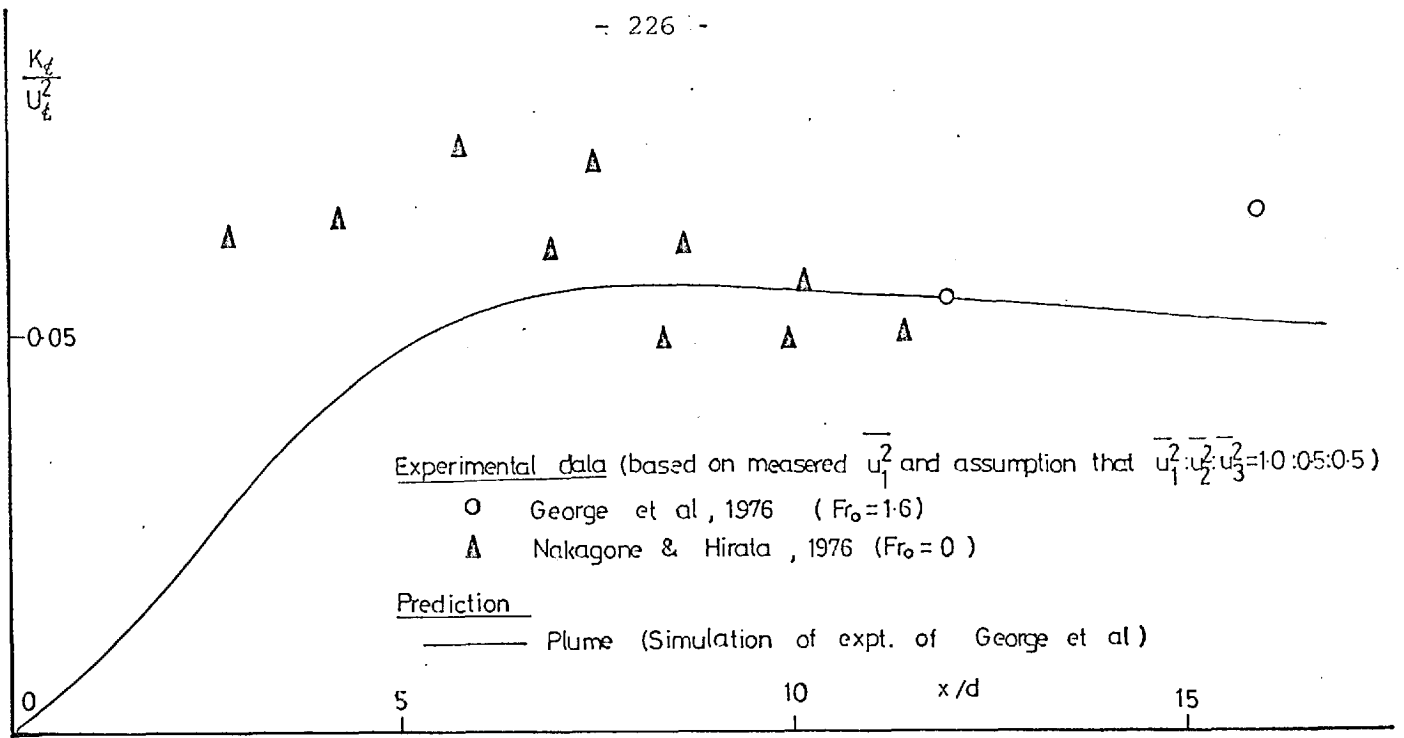
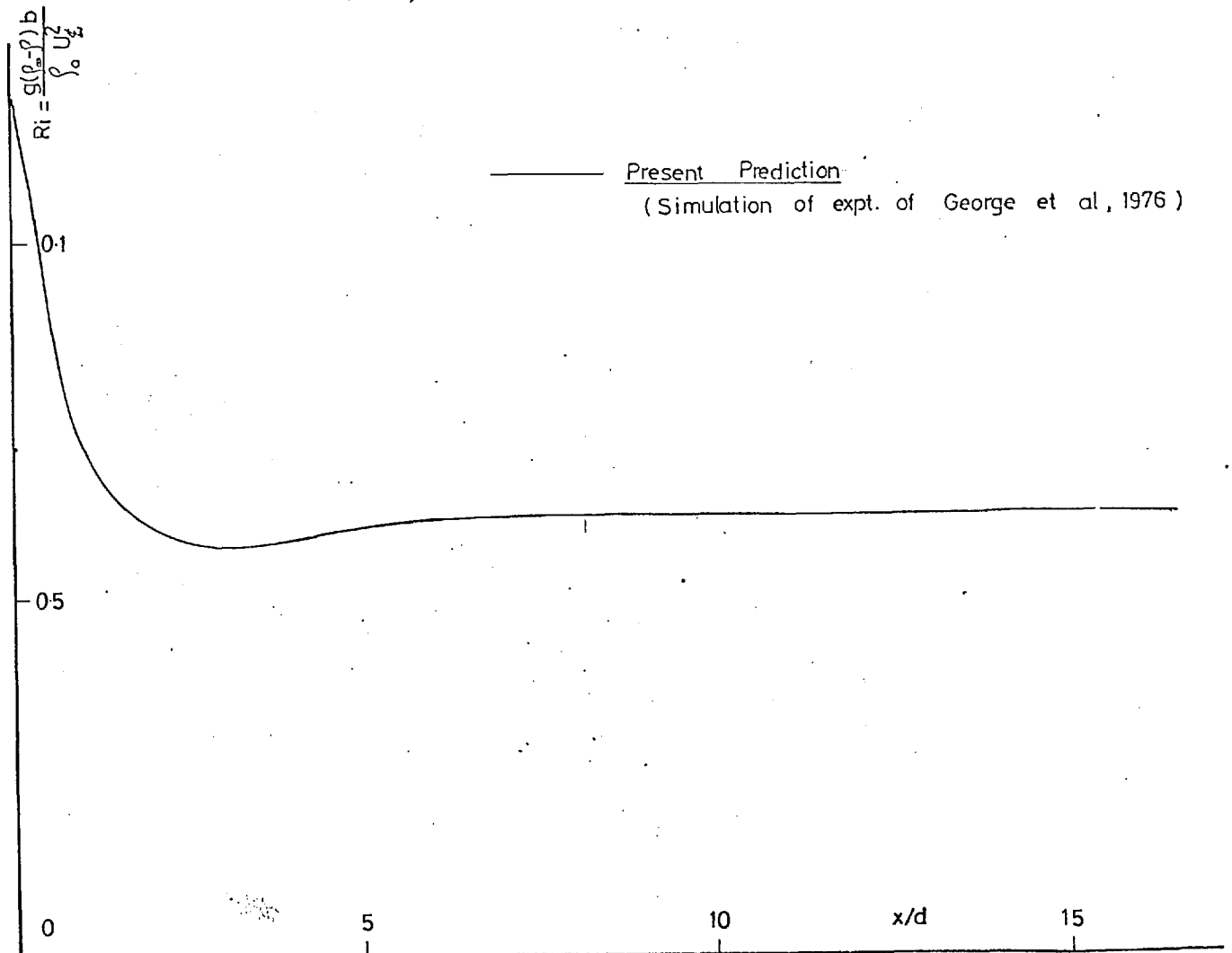


Fig. 4.13 Plume in uniform surroundings: Axial distribution of turbulence energy

Fig. 4.14 Plume in uniform surroundings: Predicted variation of the Richardson number along the plume



Experimental data (based on $\overline{u_1^2}$; $\overline{u_2^2}$; $\overline{u_3^2} = 1.0; 0.5; 0.5$)

(i) George et al (1976):

	x/d	
○	16	
Δ	12	$Fr_0 = 16$
■	8	

(ii) Nakagome & Hirata (1976):

Δ	$4.1 \leq x/d \leq 11.5$	$Fr_0 = 0$

Prediction

———— Plume, $x/d \geq 7$ (simulation of expt. of George et al)

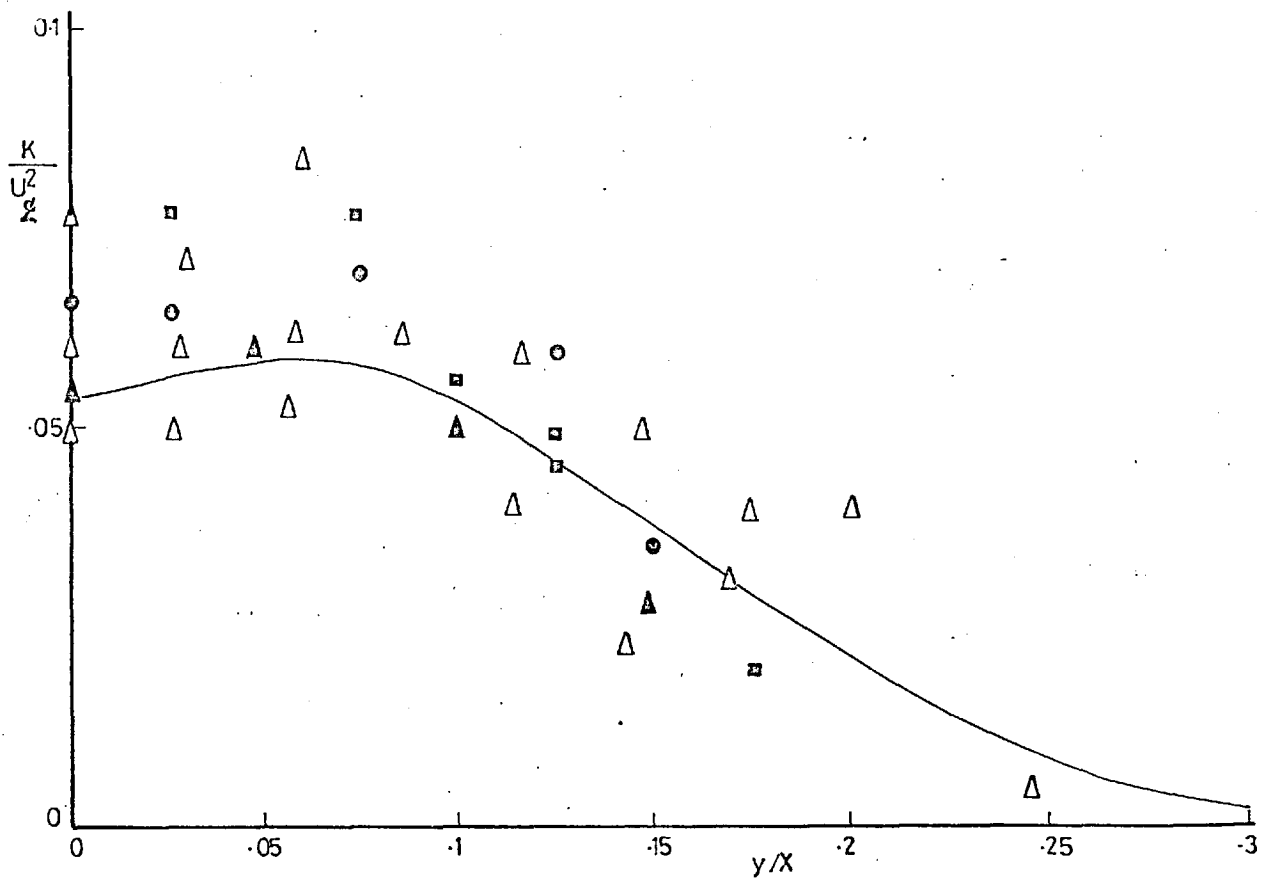


Fig. 4.15 Plume in uniform surroundings: Radial distribution of turbulence energy

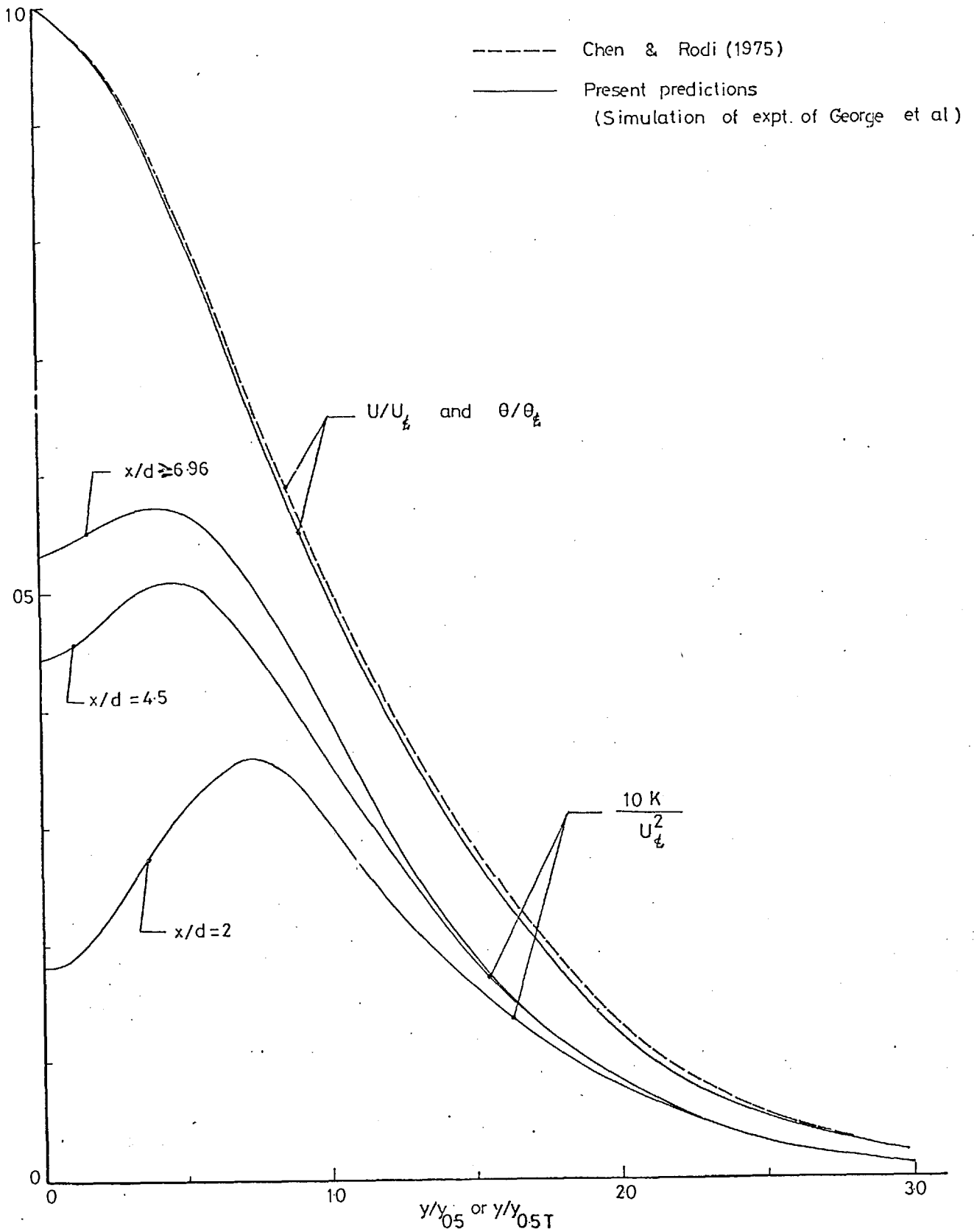


Fig. 4.16 Plume in uniform surroundings: Radial distribution of turbulence energy at various axial locations, and also self-similar profiles of mean velocity and temperature

— Plume (Simulation of expt. George et al)
 - - - Momentum Jet

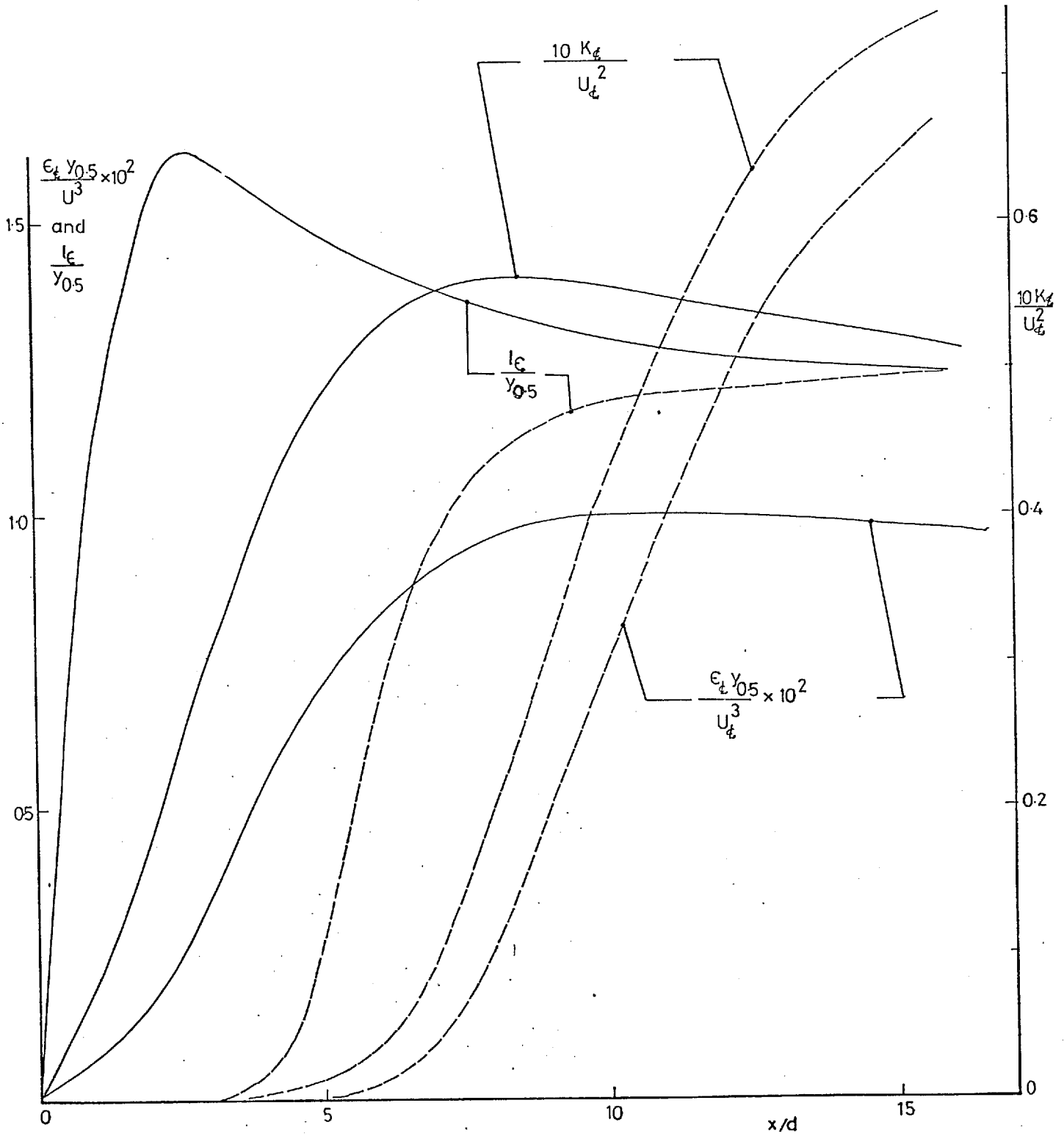


Fig. 4.17 Plume in uniform surroundings: Axial distribution of turbulence energy K , turbulence dissipation rate ϵ , and dissipation length scale l_ϵ .

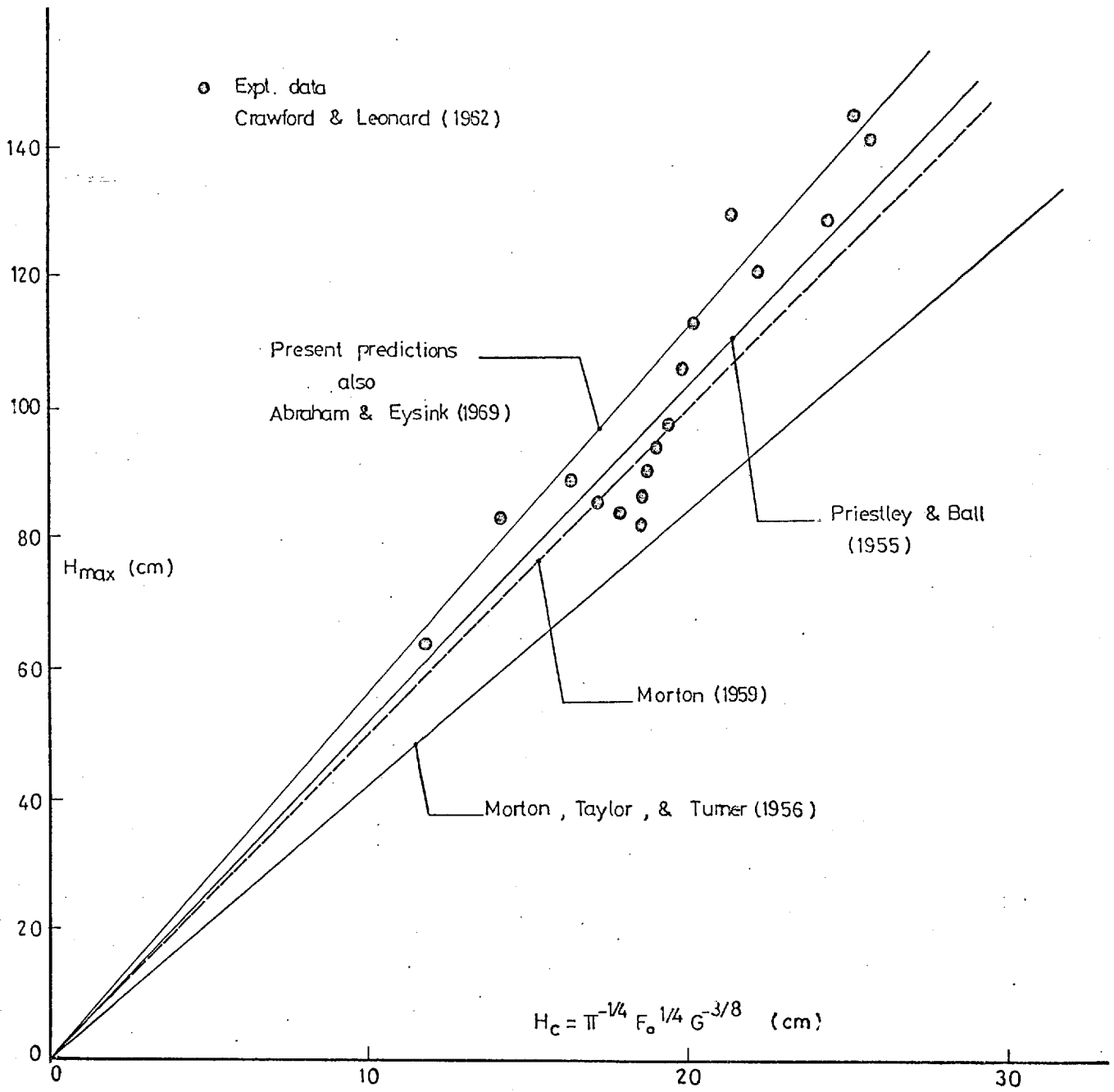
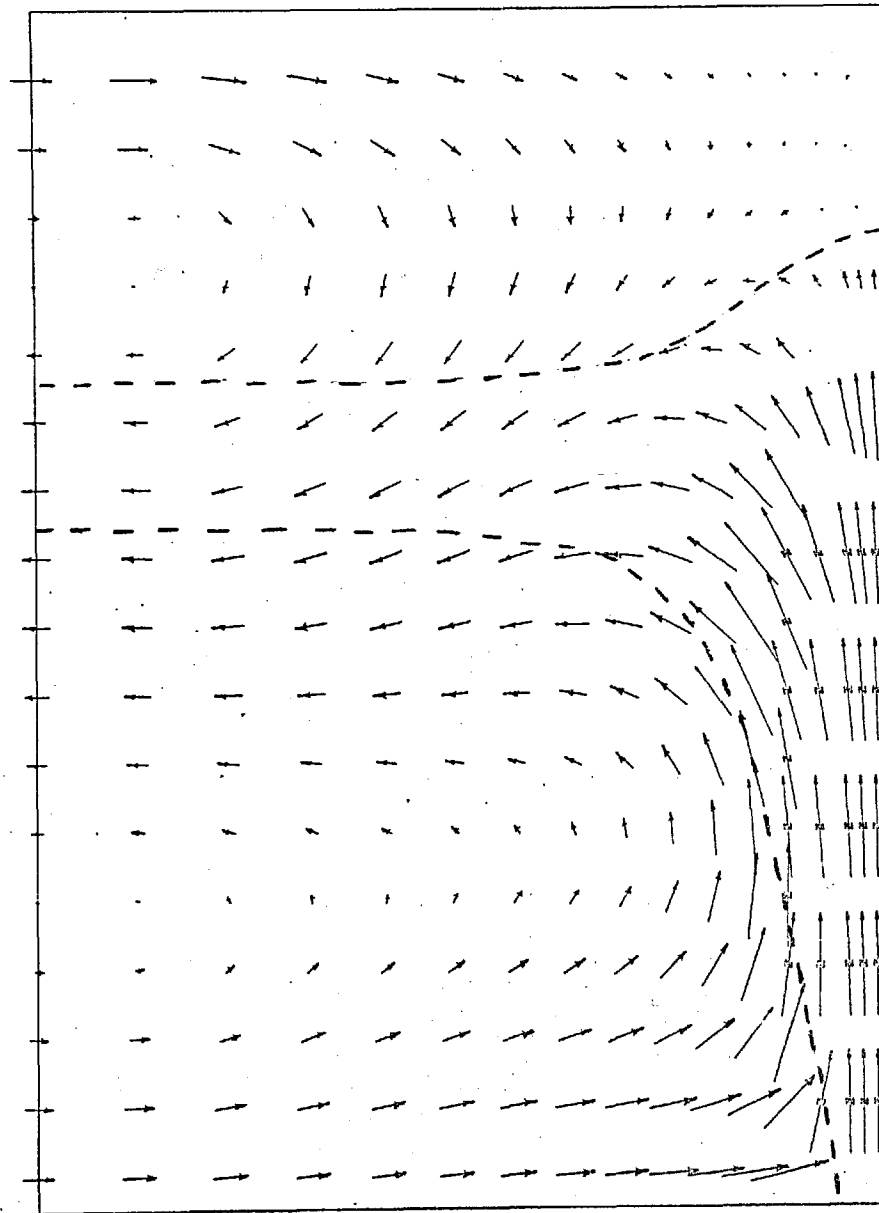
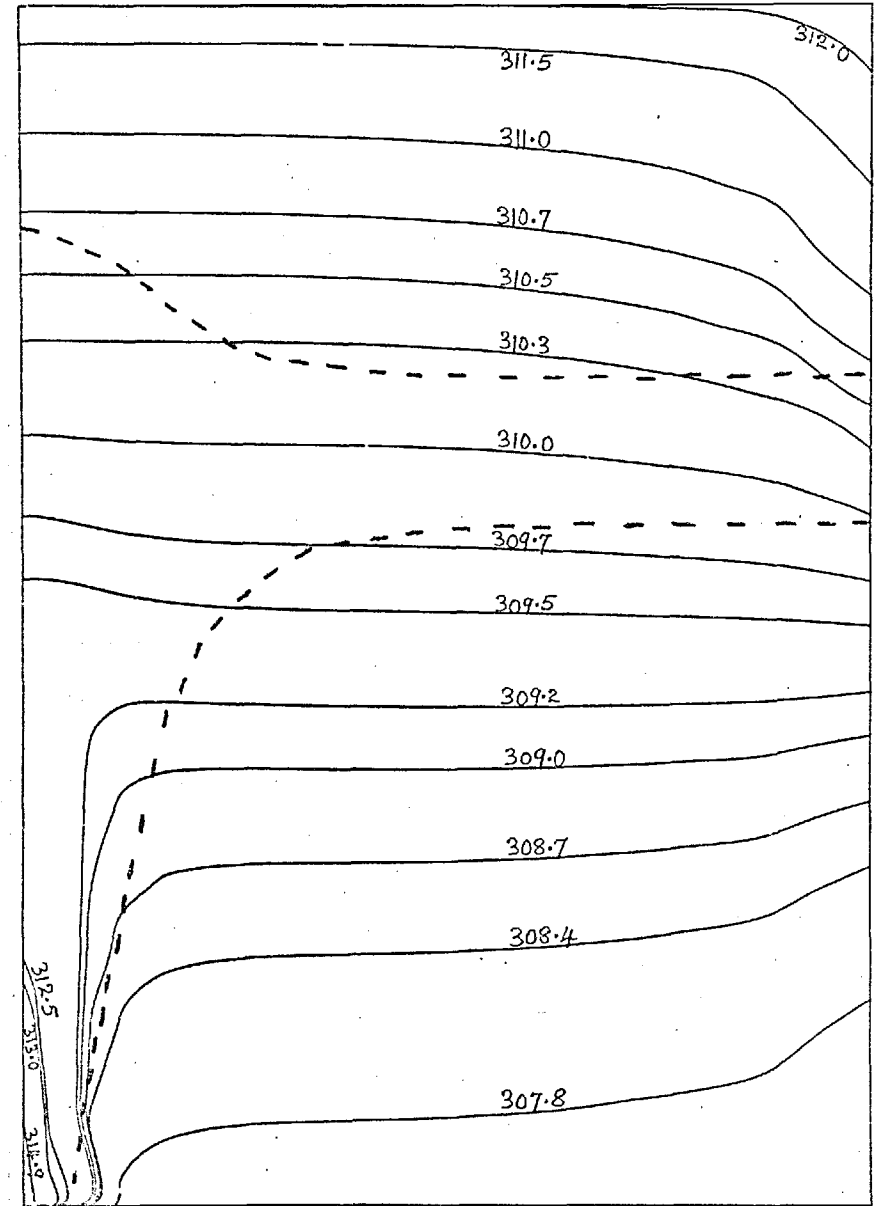


Fig. 4.18 Nearly Pure Plume in ambient fluid whose density decreases linearly with height: Maximum height of plume

Scale
1 cm = 0.2 m



(a) Vector plot of flow field

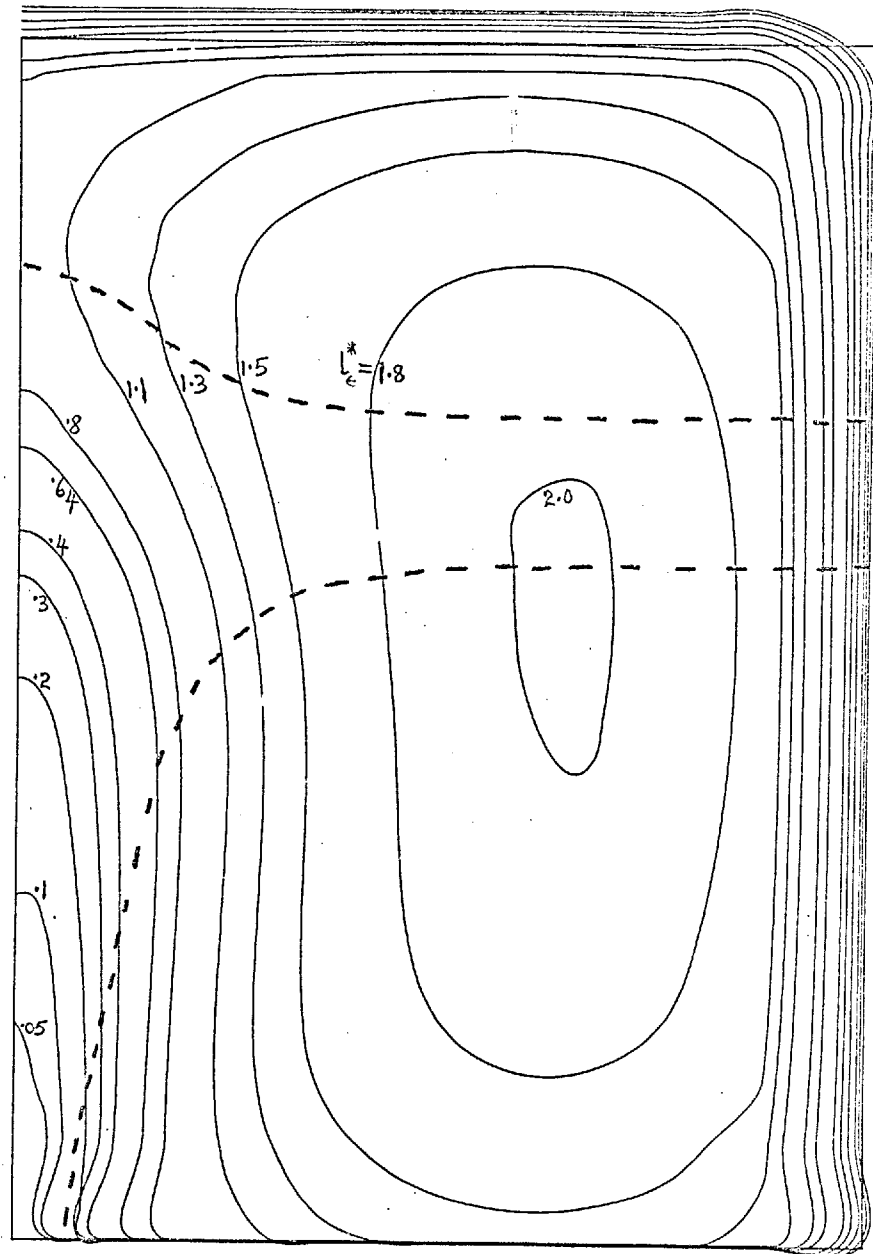
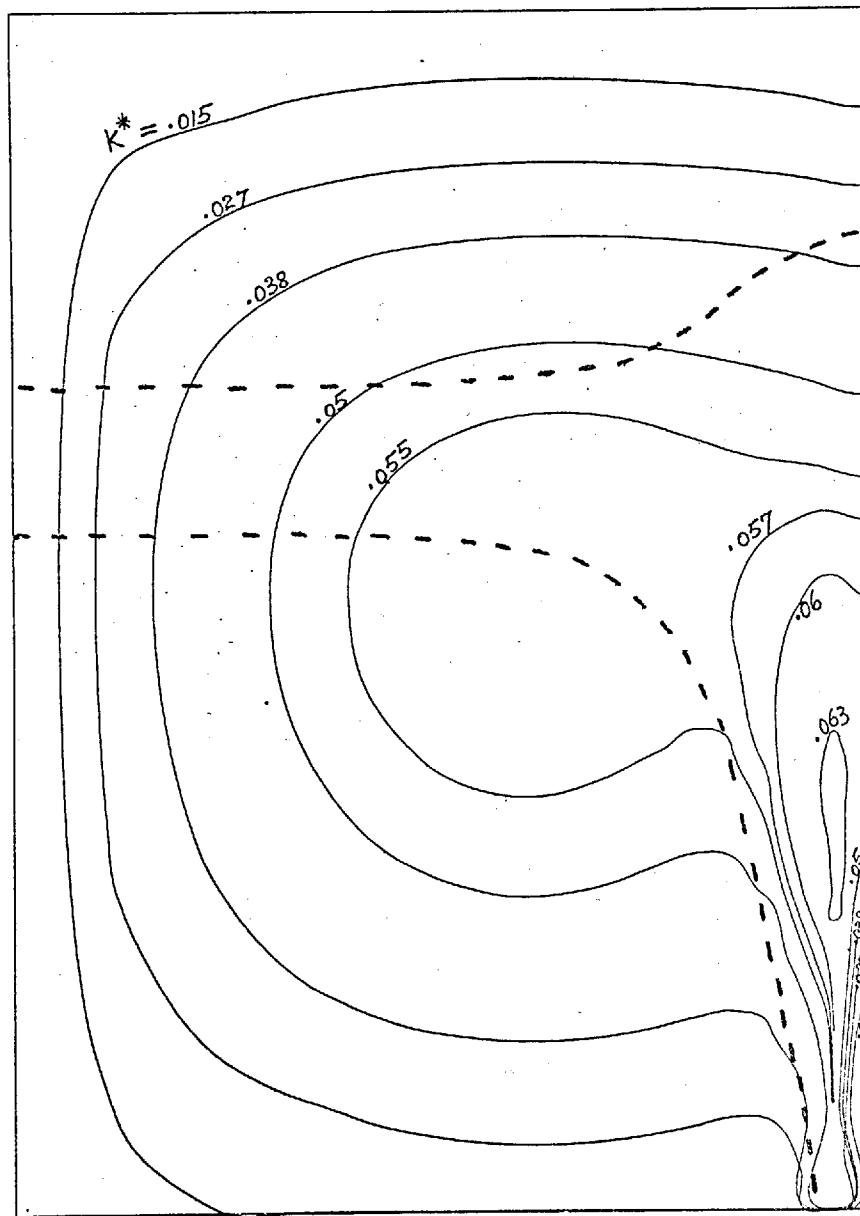


(b) Isotherms

Fig. 4.19 : Plume in stably-stratified surroundings

Scale

$1\text{cm} = 0.2\text{m}$



(c) Contour of turbulence energy ($K^* = K/U_0^2$)

(d) Contour of dissipation length scale ($l_\epsilon^* = l_\epsilon/d \times 10^{-1}$)

Fig. 4.19 : Plume in stably-stratified surroundings

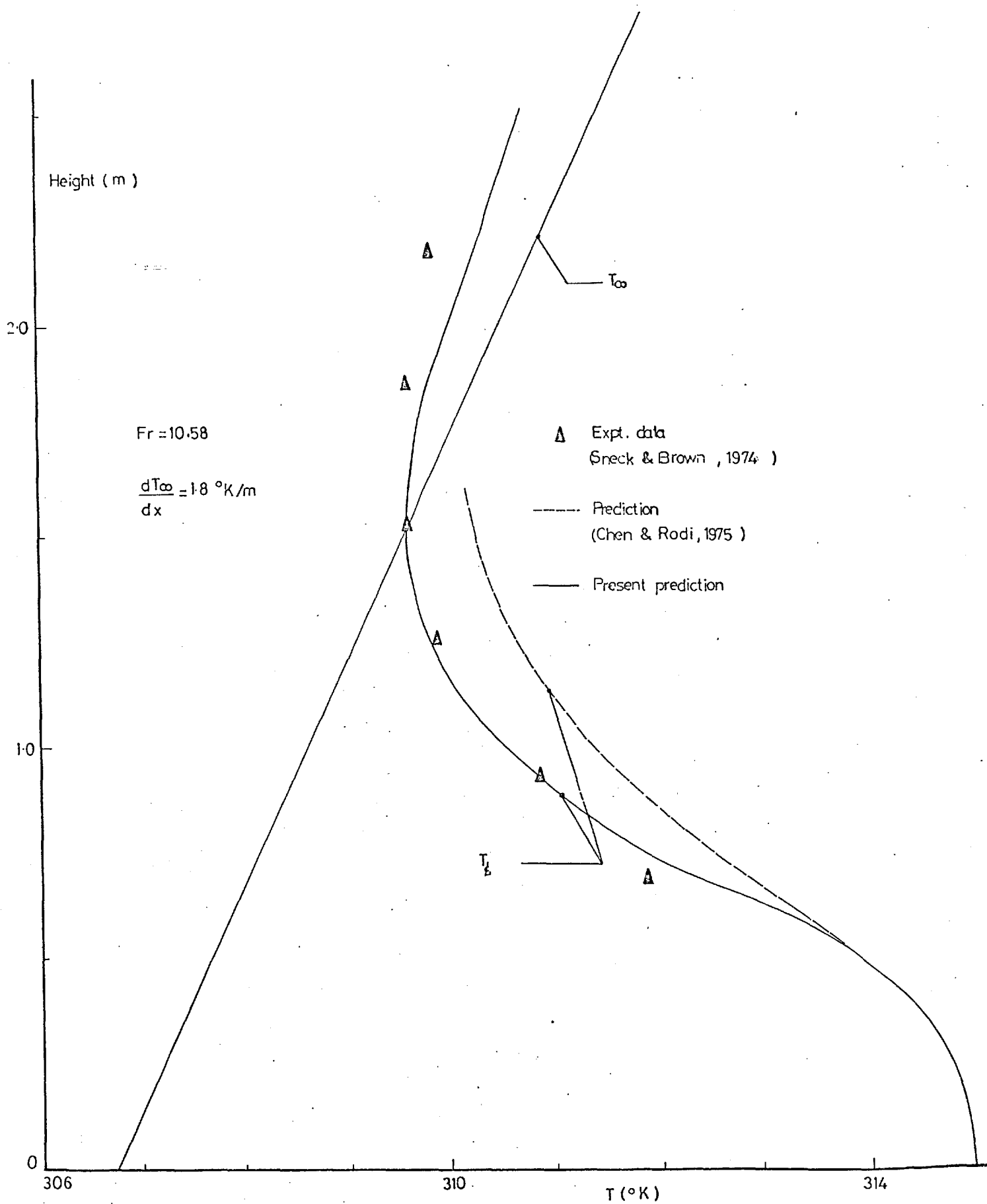
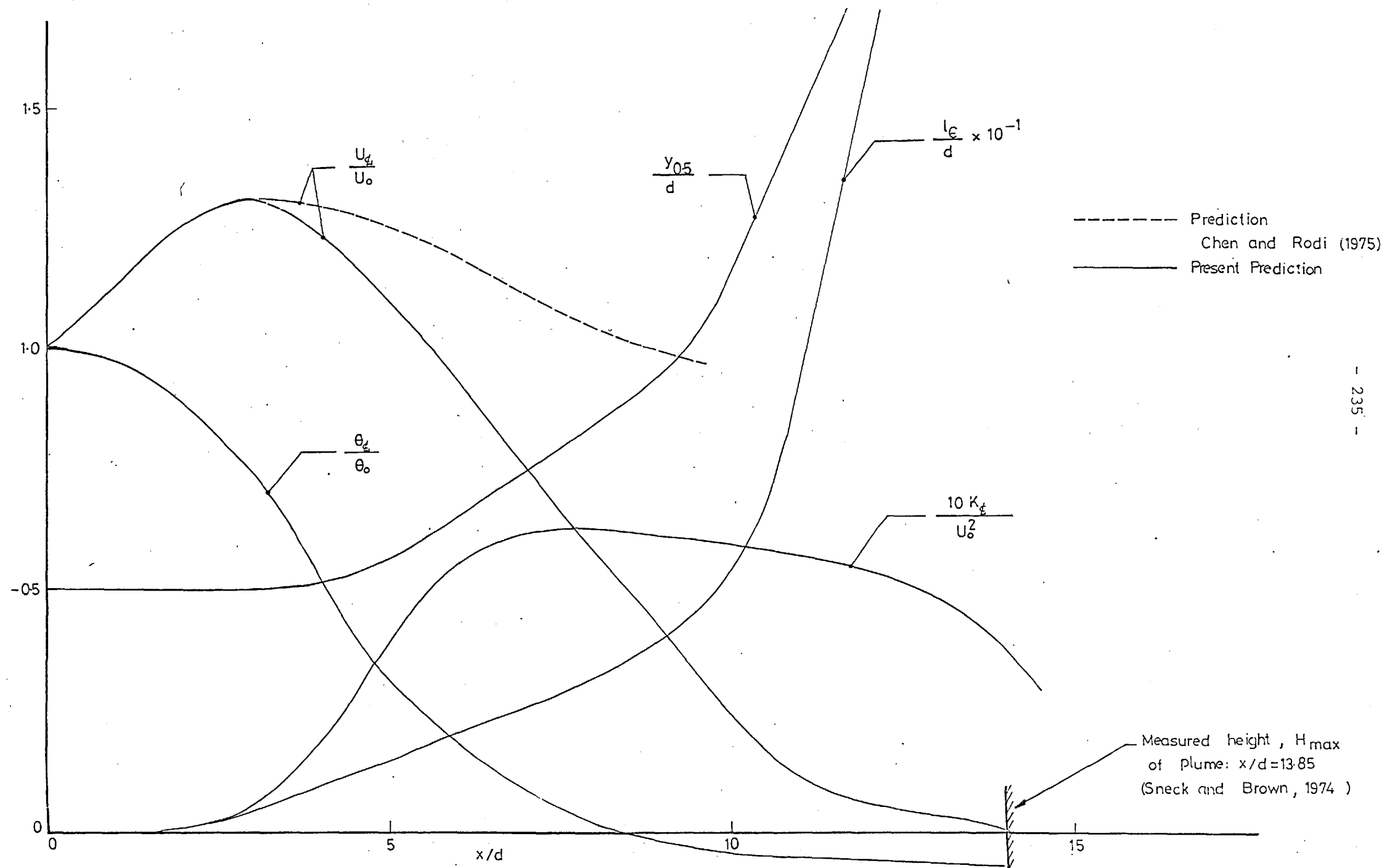


Fig. 4.20 Plume in Stably-Stratified Surroundings: Distribution of T

Fig. 4.21 Plume in stably-stratified surroundings: Growth rate ($y_{0.5}$), and axial distribution of U, θ, K and L_ϵ



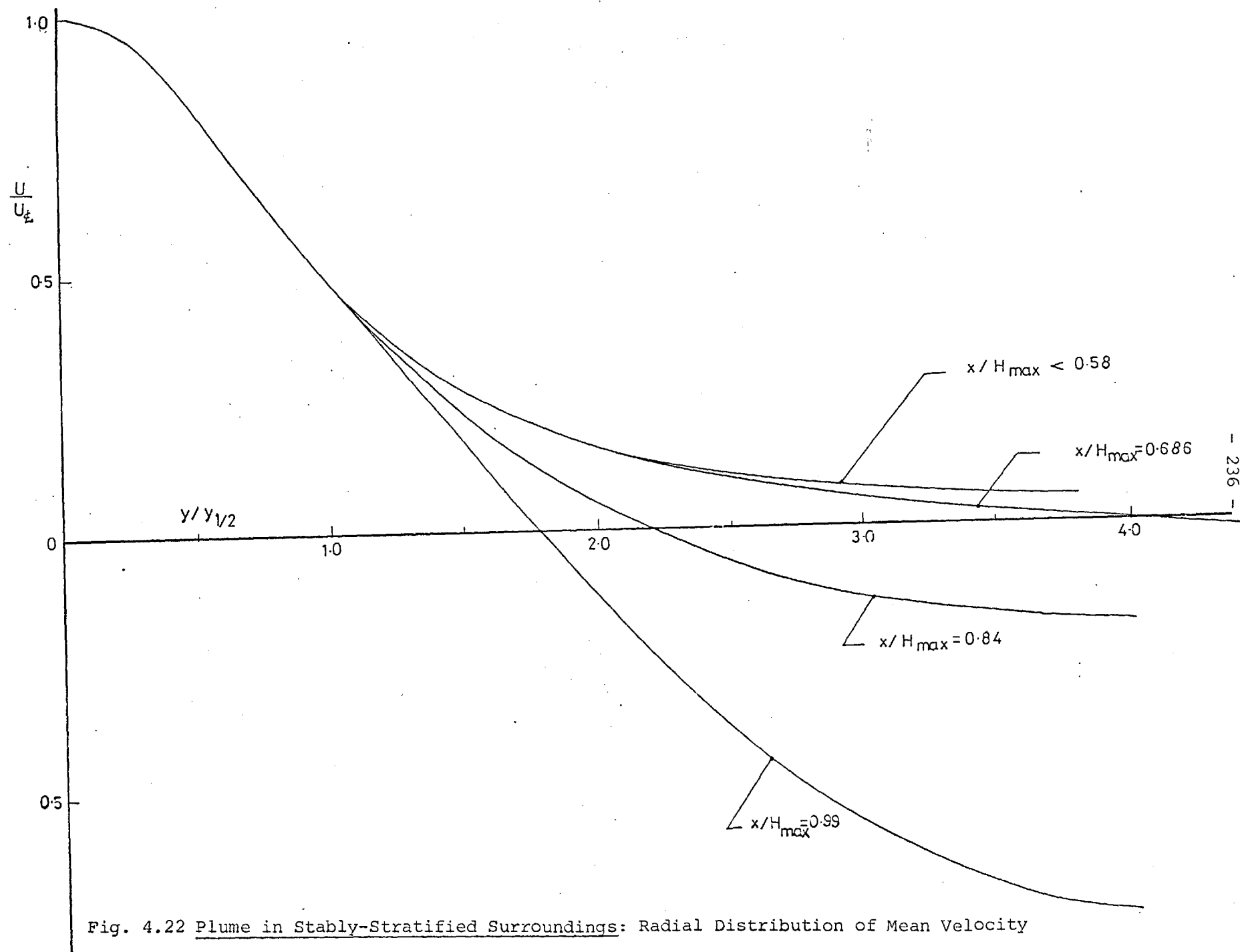


Fig. 4.22 Plume in Stably-Stratified Surroundings: Radial Distribution of Mean Velocity

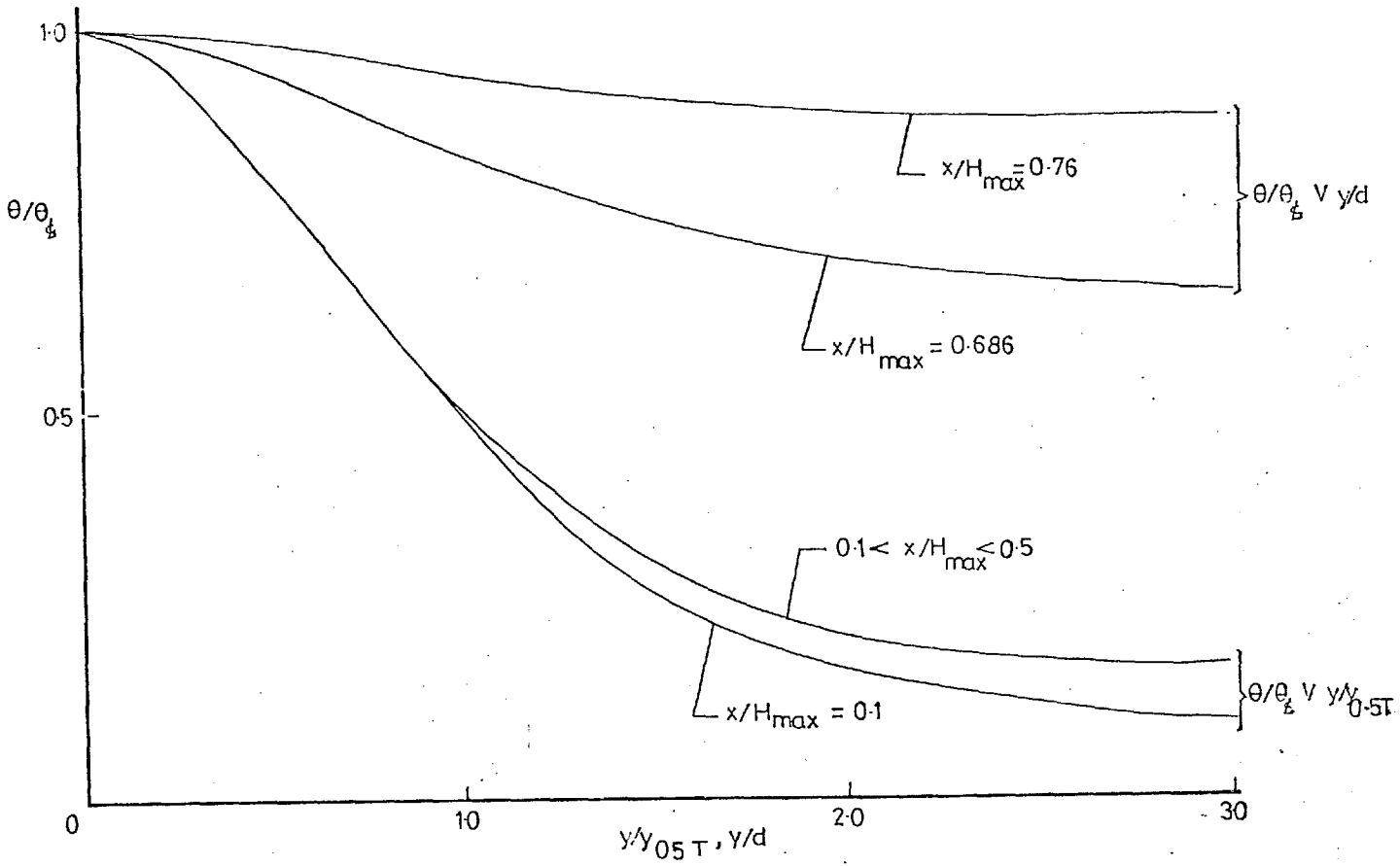


Fig. 4.23 Plume in stratified surroundings: Radial distribution of mean temperature

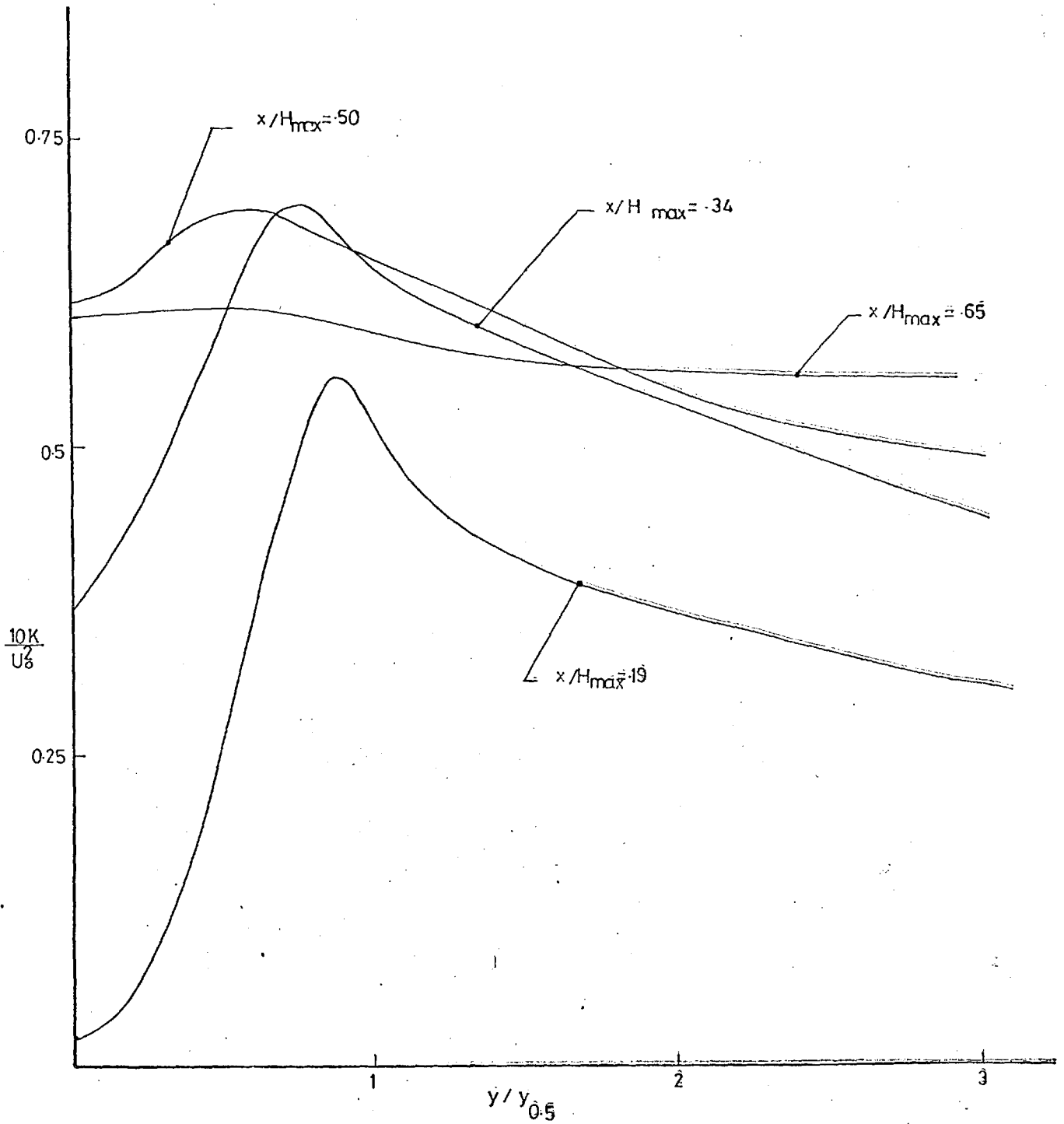


Fig. 4.24 Plume in Stably-Stratified Surroundings: Radial distribution of turbulence energy

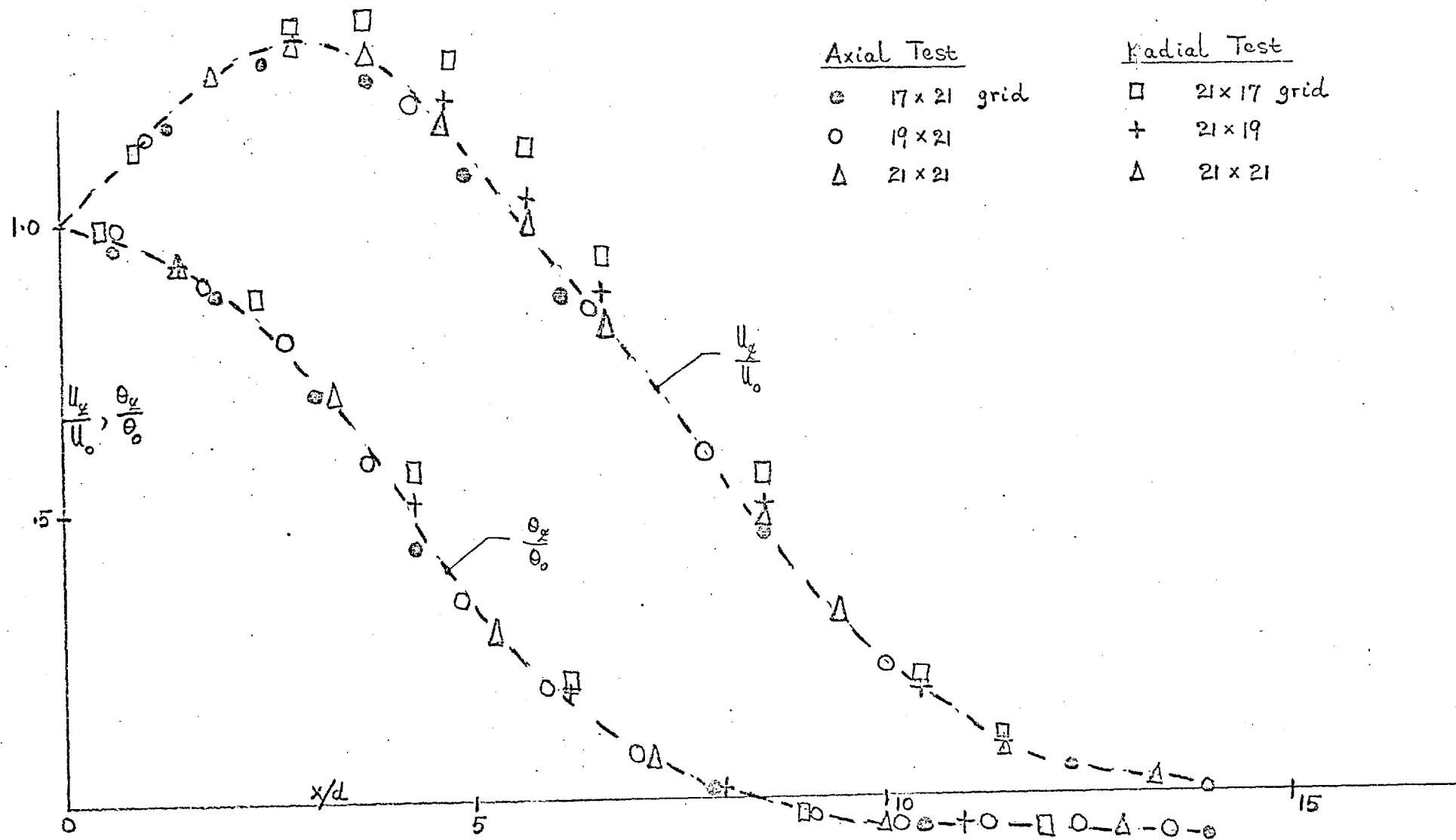
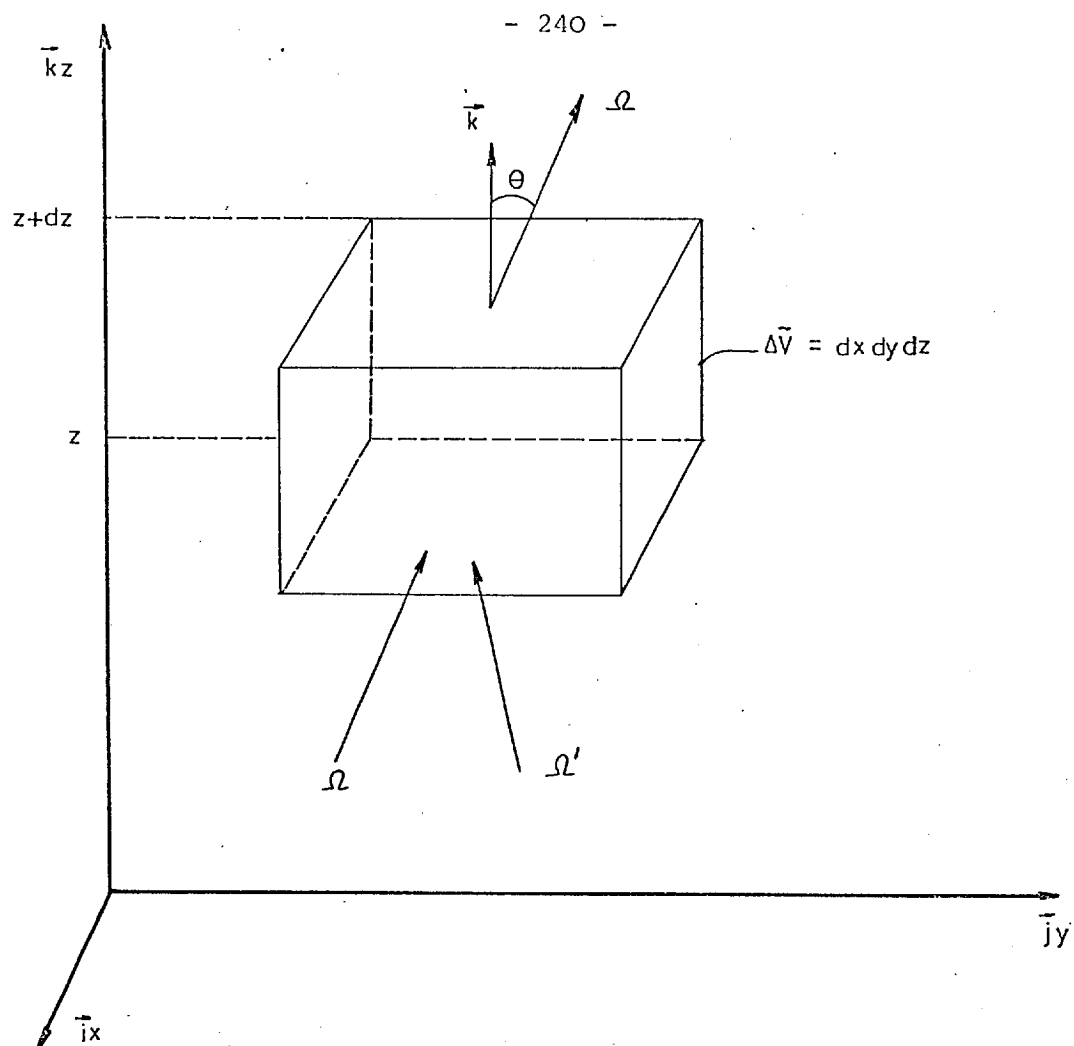
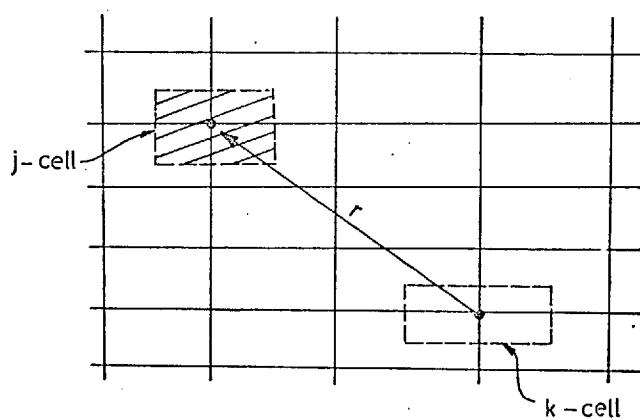


Fig. 4.25 : Grid Test : Simulation of expt. of Sneek & Brown

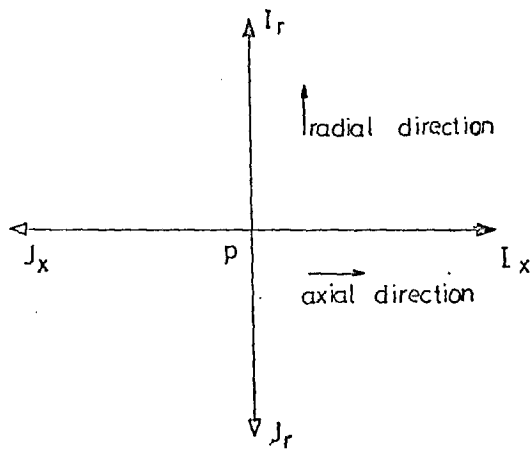


- (a) Photons in the volume element $\Delta \tilde{V}$, moving in direction Ω .
 (Note $\vec{i}, \vec{j}, \vec{k}, \Omega$ and Ω' are unit vectors in the directions shown)

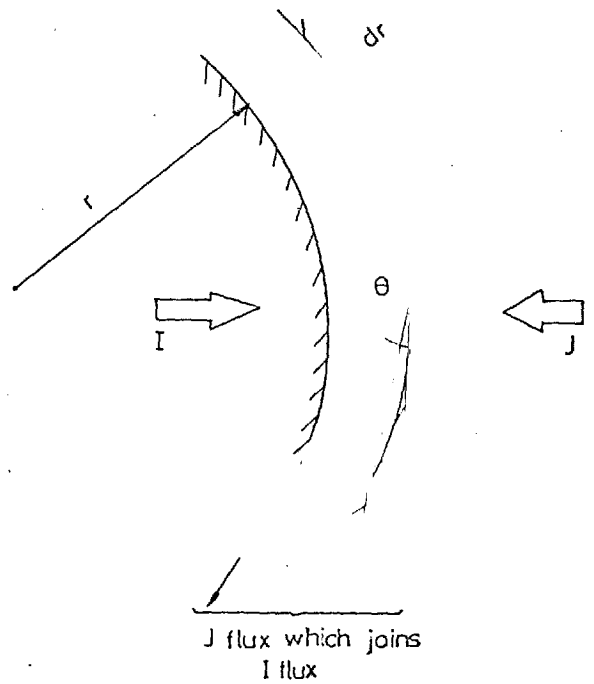


- (b) Illustration of j- and k-cells.

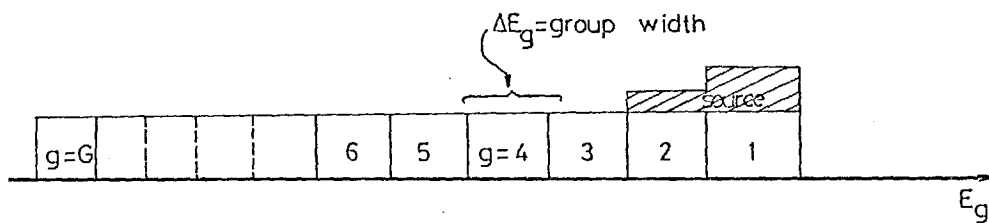
Fig. 5.1 Illustration of Control Volumes



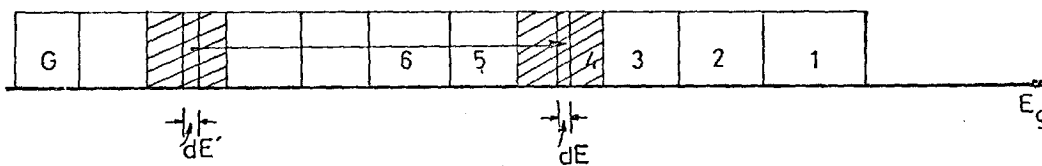
(a) Four fluxes at a point



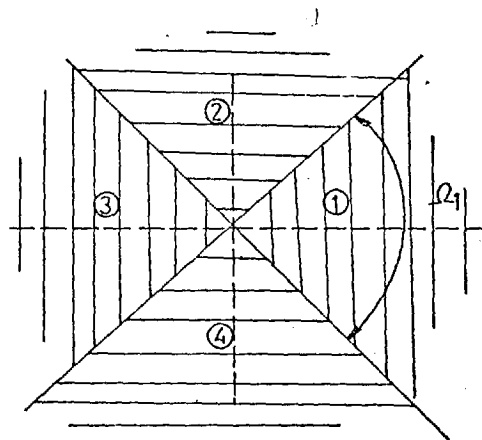
(b) Sketch of radial fluxes in a cylindrical geometry.



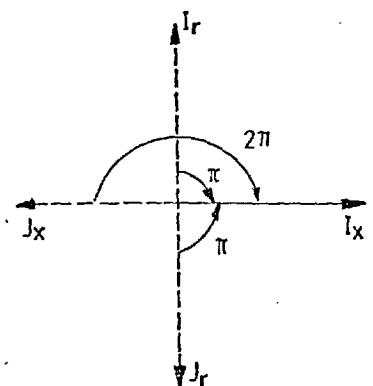
(c) Illustration of multi-group structure.



(d) Illustration of initial and final energy intervals.



(e) Illustration of solid angles.



(f) Solid angles between I_r and the other three fluxes J_x , I_x , J_r .

Fig. 5.2 Illustrations for development of multi-group model

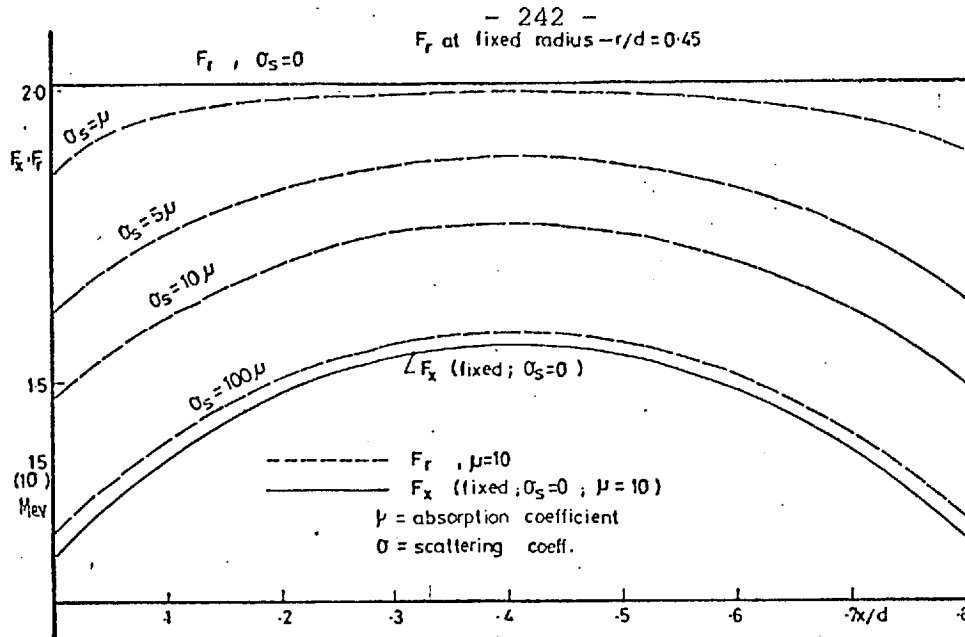


Fig. 5.3 Uniform Concentration Field: F_r at fixed radius $-r/d = 0.45$

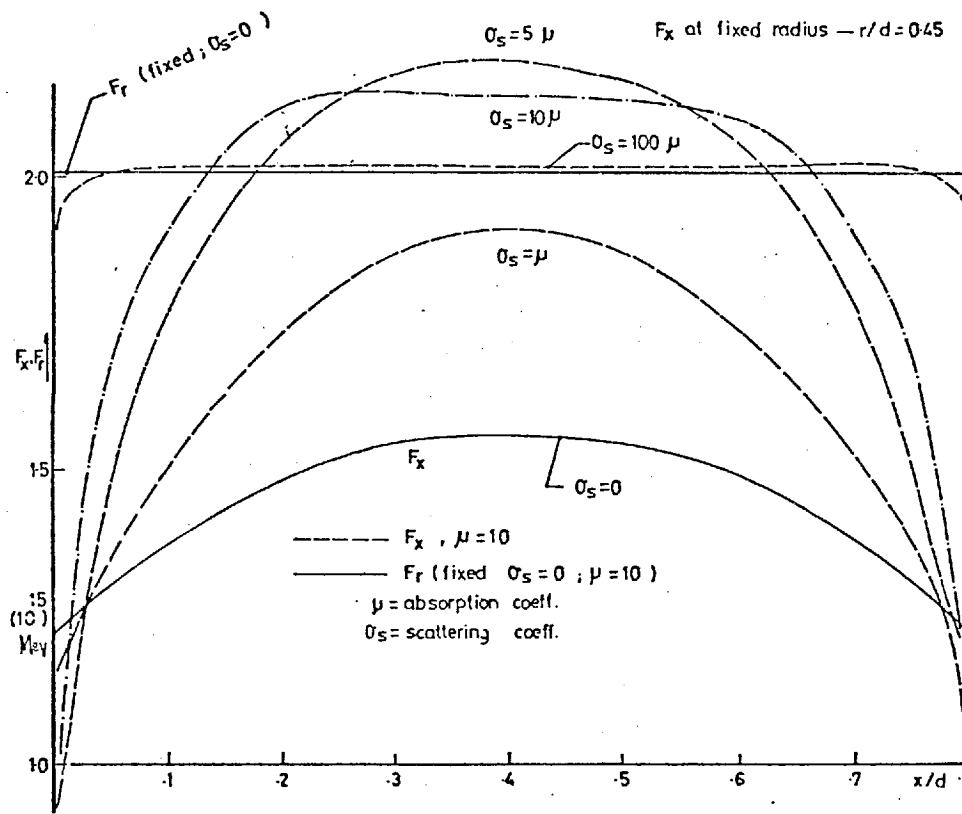


Fig. 5.4 Uniform Concentration Field: F_x at fixed radius $-r/d = 0.45$

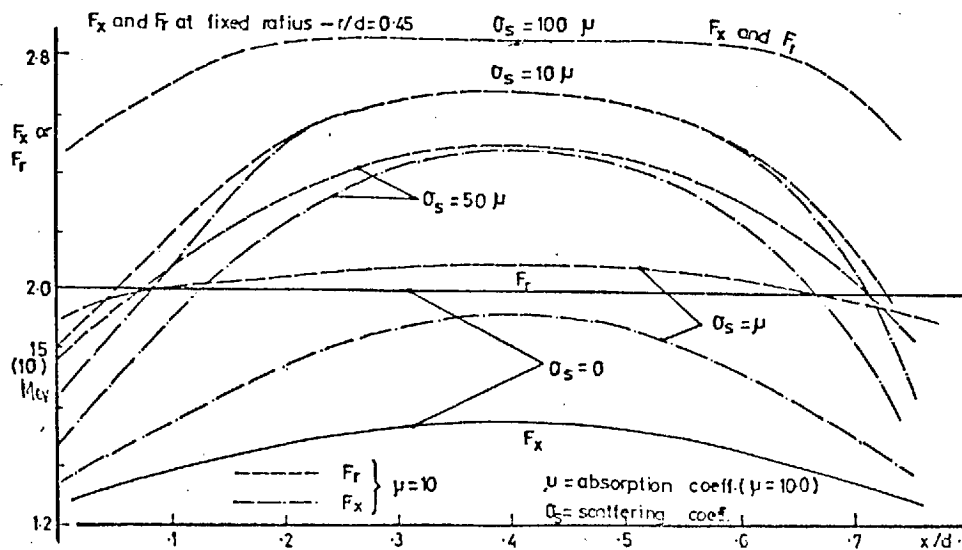


Fig. 5.5 Uniform Concentration field: F_x and F_r at fixed radius $-r/d = 0.45$

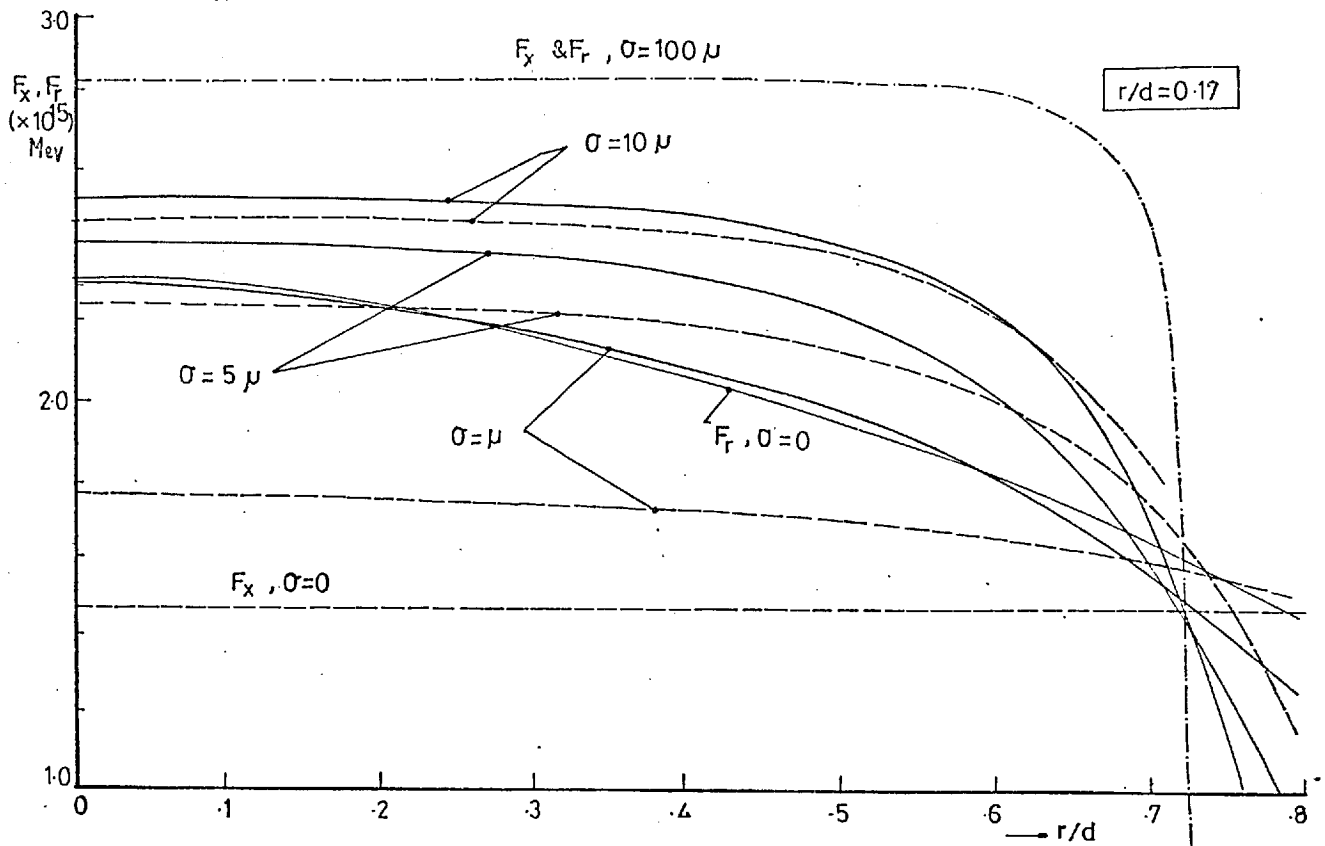
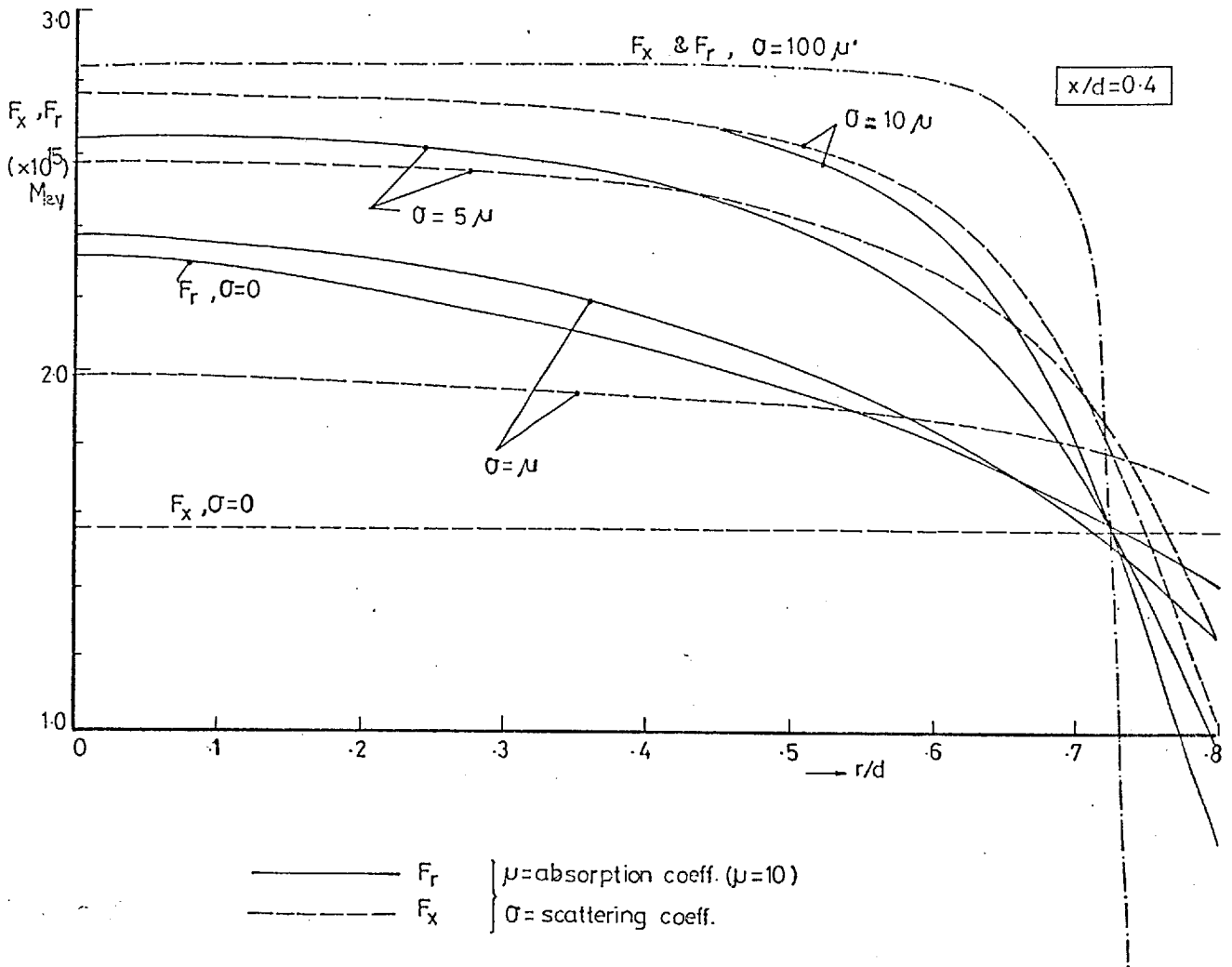


Fig. 5.6 Uniform concentration Field: Radial profiles of F_x and F_r at fixed values of x/d

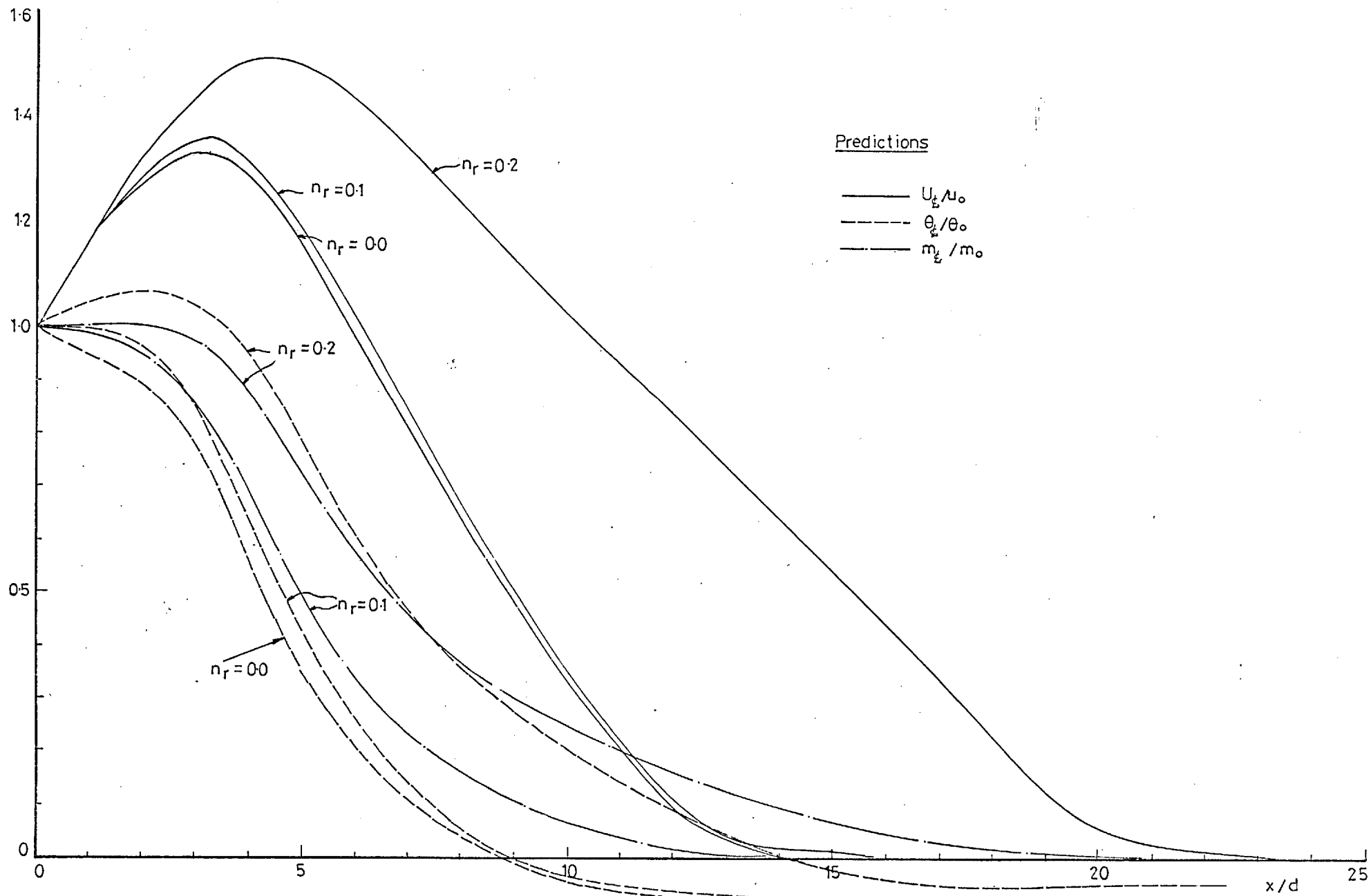
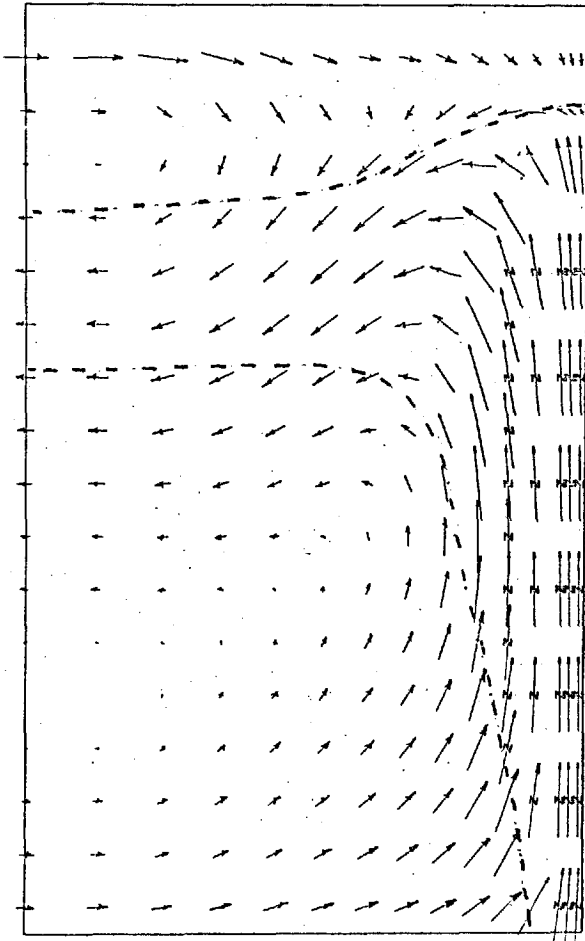


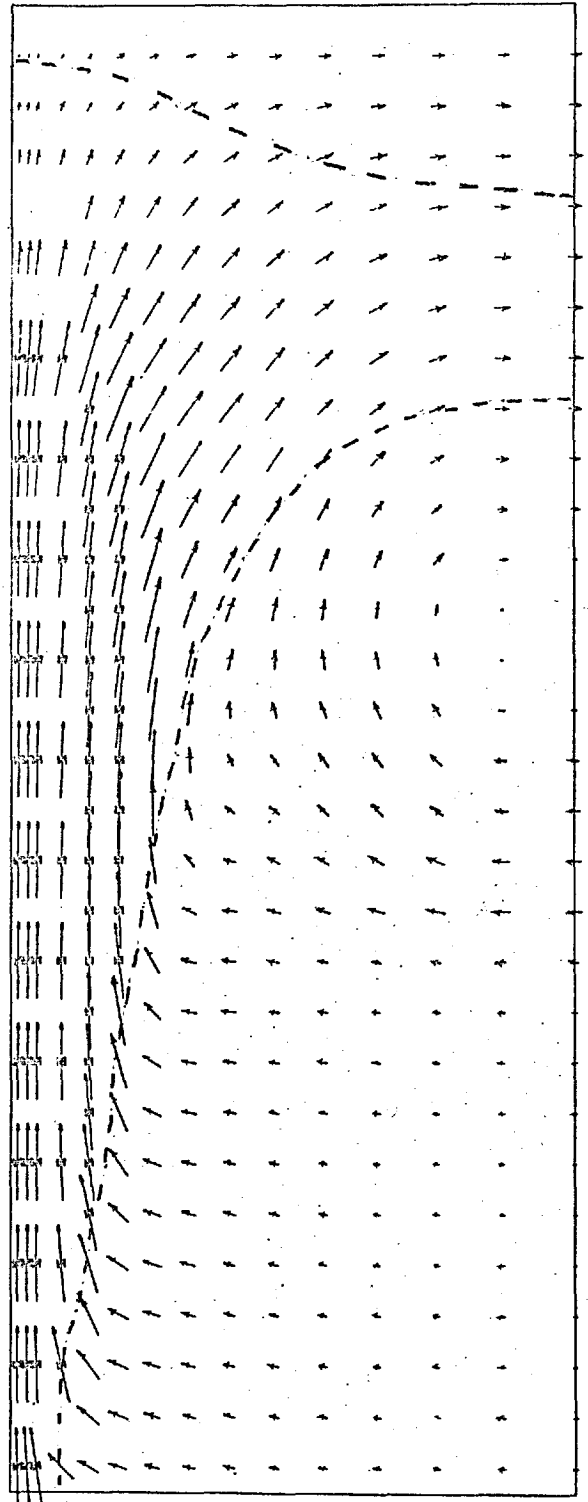
Fig. 5.7 Beta heating effect on model of Sneck and Brown (1974): Axial distribution of mean variables

Scale

1cm = 8.4m



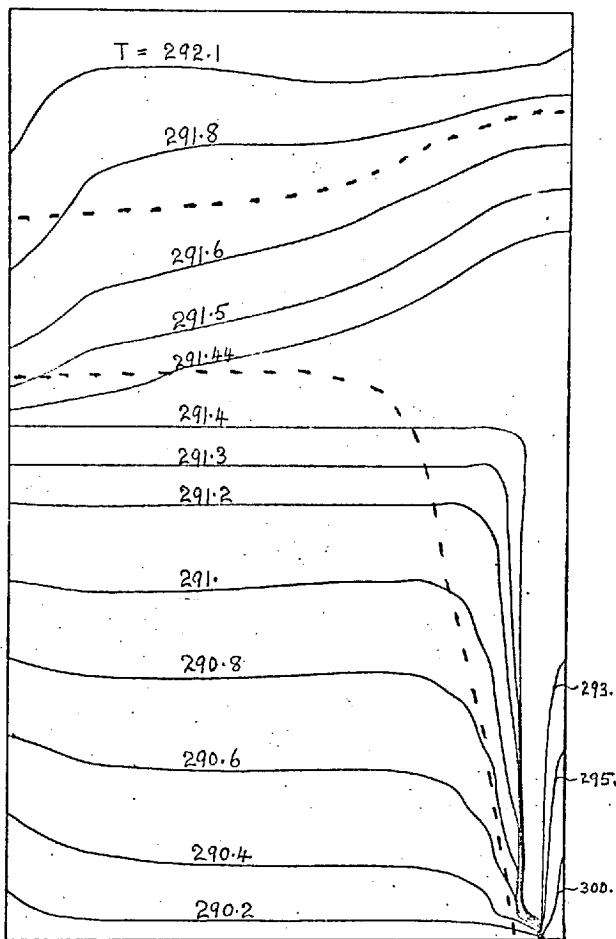
(i) Non-radioactive plume



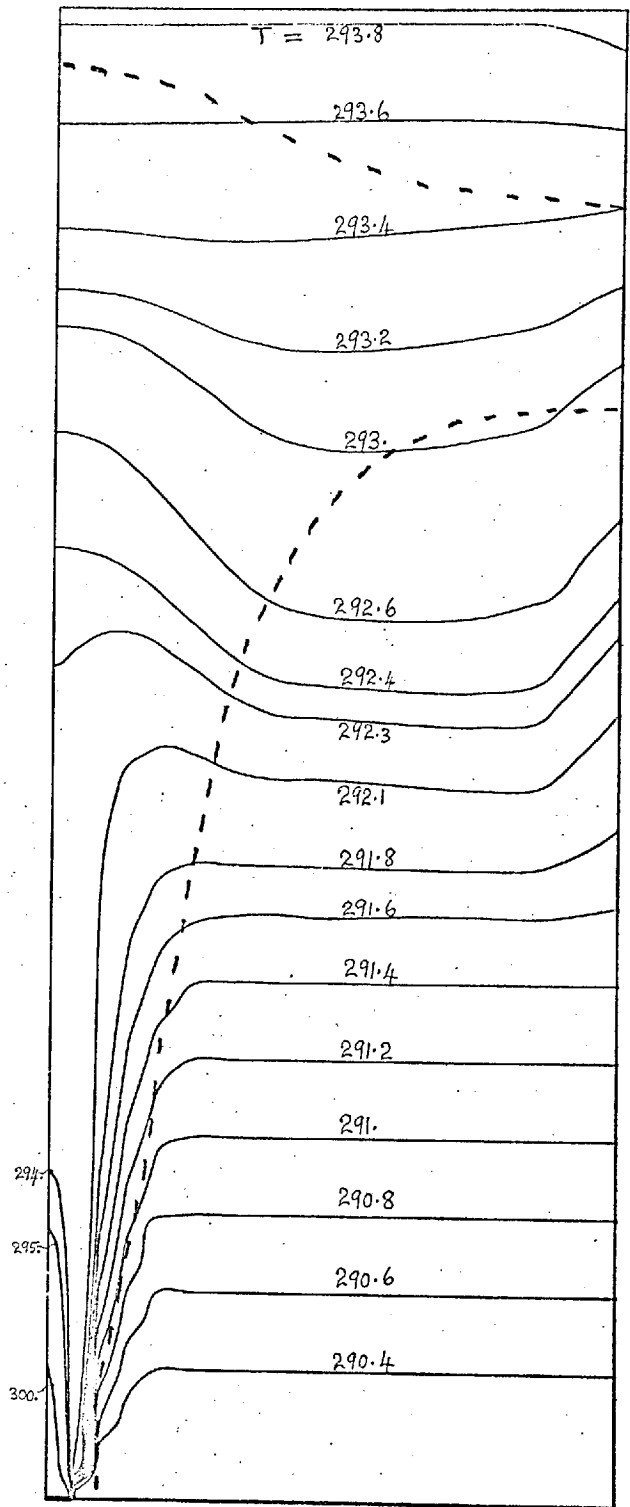
(ii) Radioactive plume (E+ γ heating)

Fig. 5.8 (a) Plume in stably-stratified surroundings: Vector plot of flow field

scale
1 cm = 8.4 m



(i) Non-radioactive plume

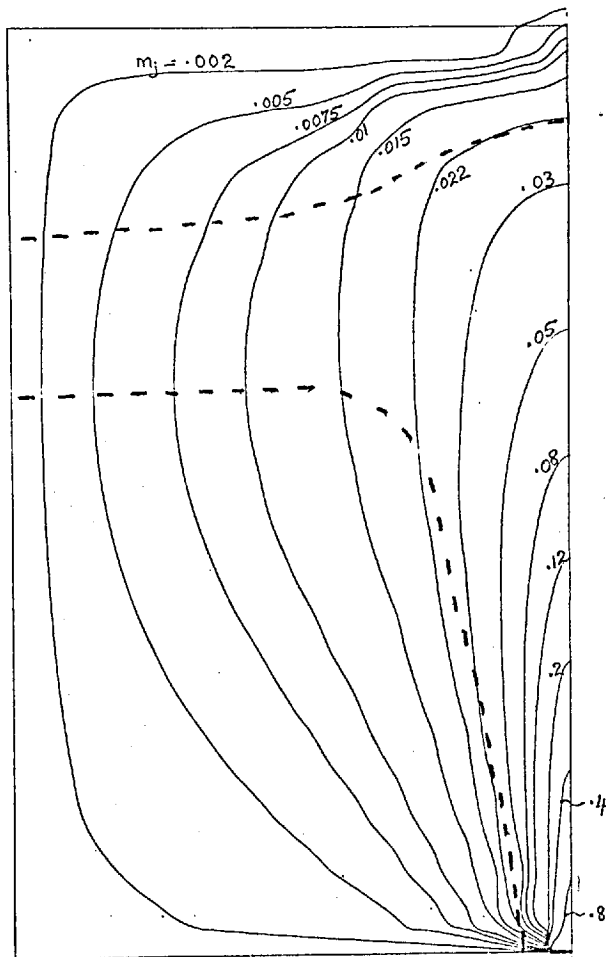


(ii) Radioactive plume (with heating)

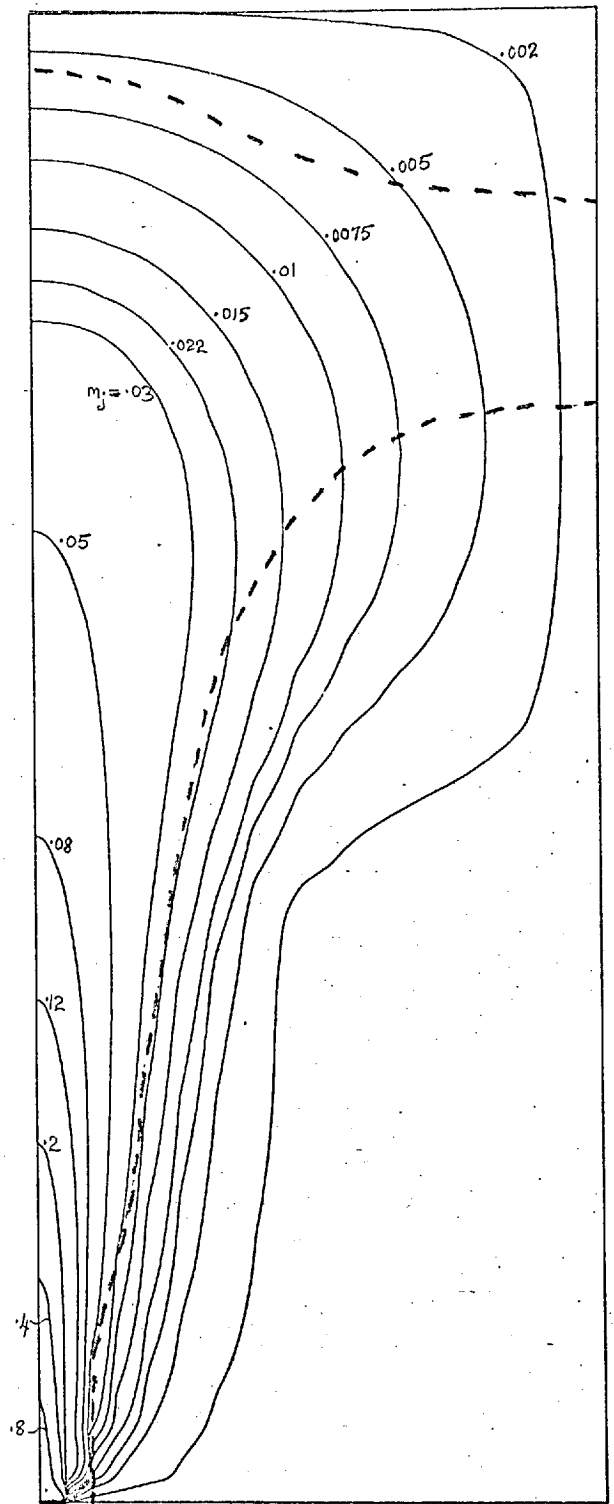
Fig. 5.8(b) Plume in stably-stratified surroundings: Isotherms

Scale

1 cm = 8.4 m



(i) Non-radioactive plume

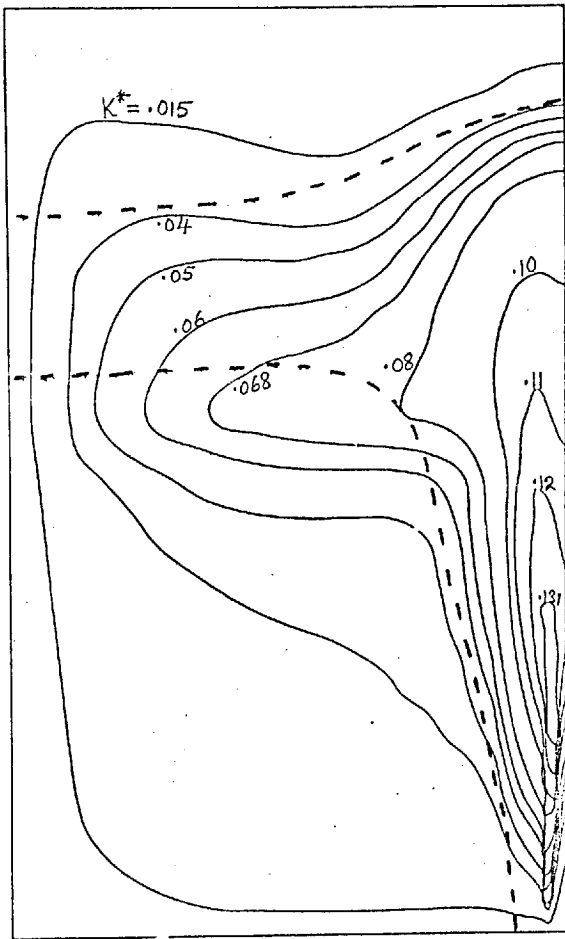


(ii) Radioactive plume ($\beta + \gamma$ heating)

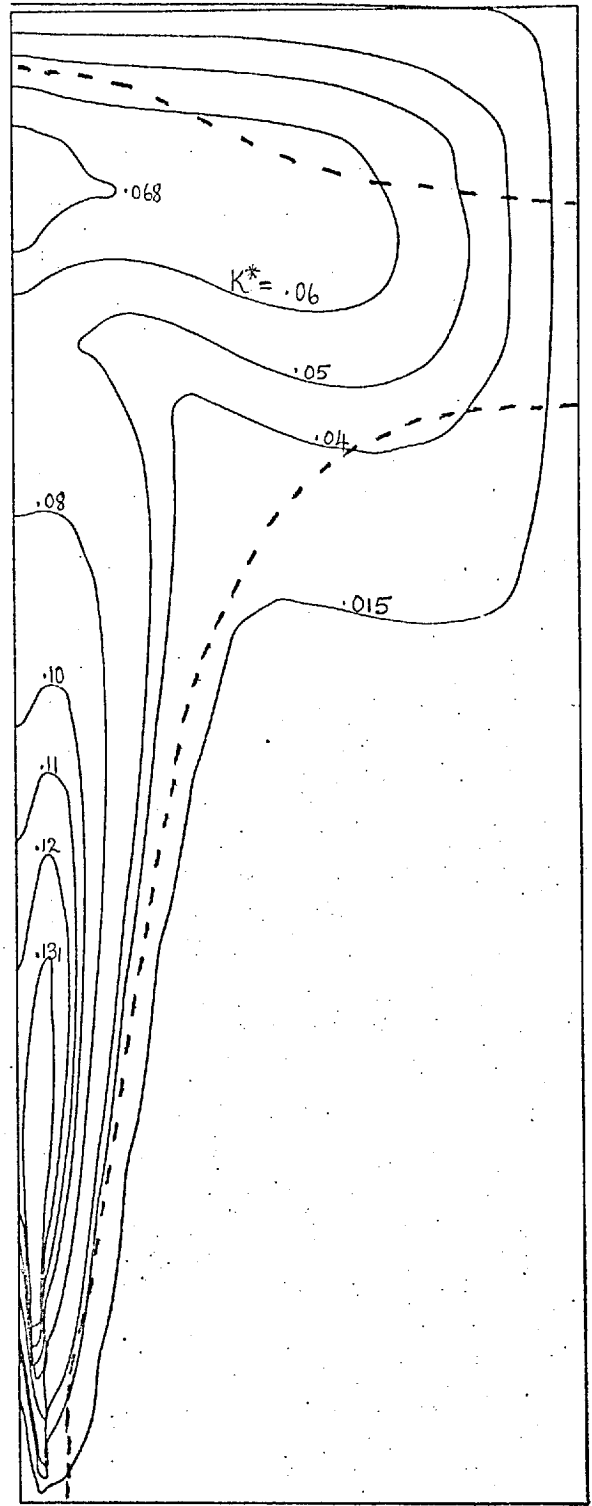
Fig. 5.8(c) Plume in stably-stratified surroundings: contour of concentration of fission products, m_j

scale

1cm = 8.4m



(i) Non-radioactive plume

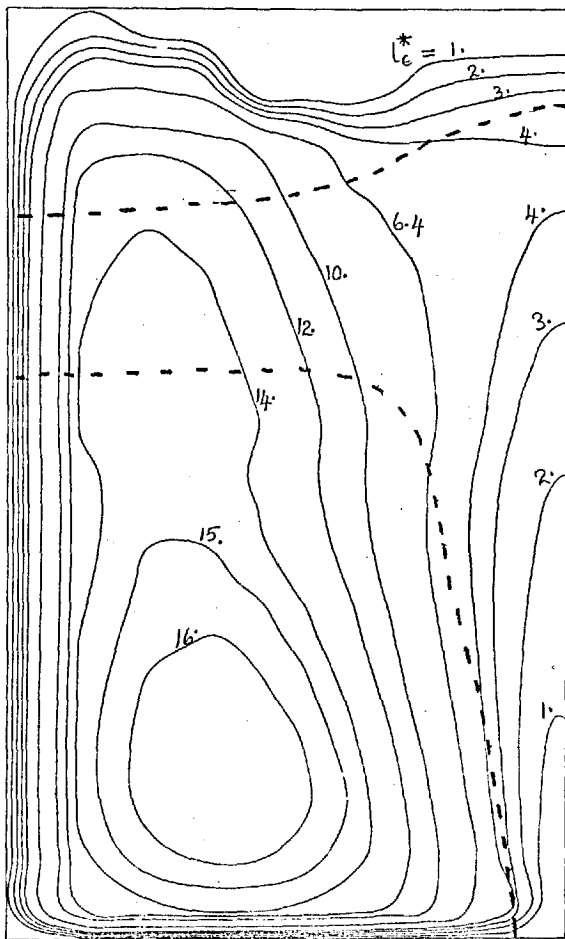


(ii) Radioactive plume ($E + \gamma$ heating)

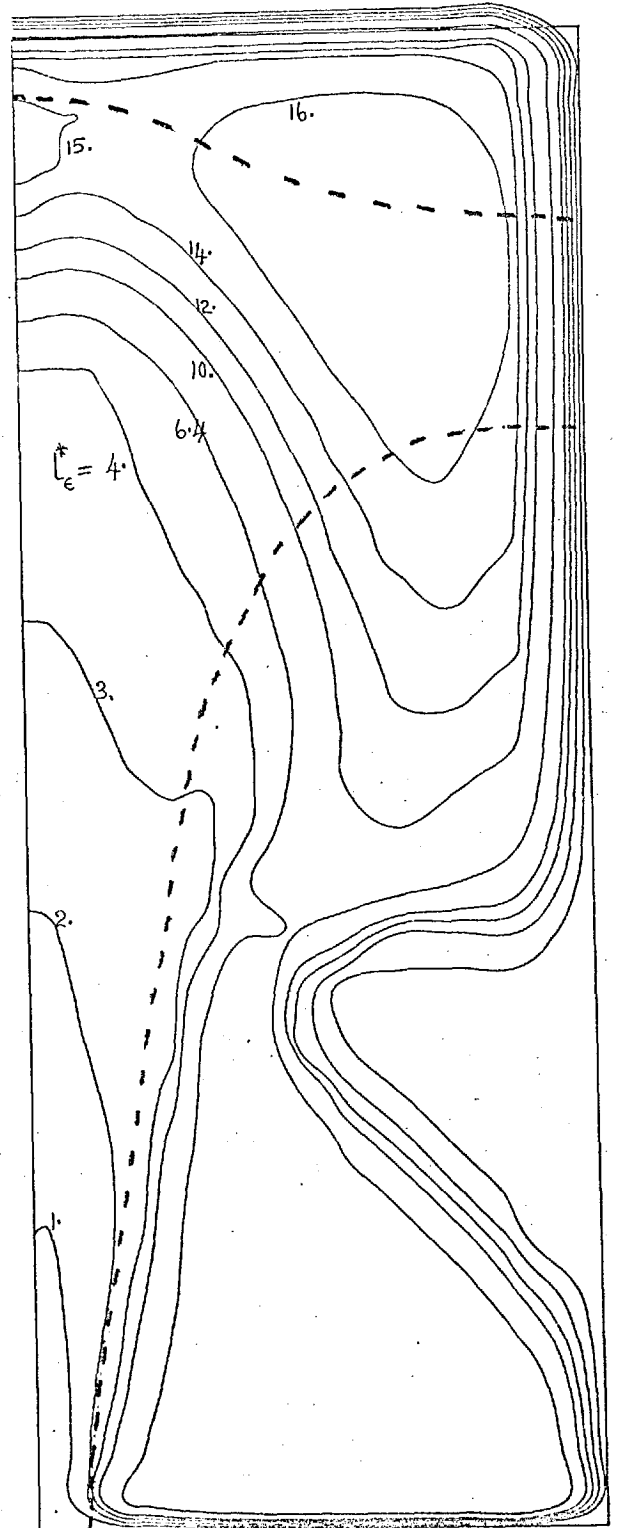
Fig. 5.8(d) Plume in stably-stratified surroundings: Contour of turbulence energy ($K^* = K/\bar{u}^2$)

Scale

1cm = 8.4m



(i) Non-radioactive plume



(ii) Radioactive plume ($\beta + \gamma$ heating)

Fig. 5.8(e) Plume in stably-stratified surroundings: Contour of dissipation length scale
($l_e^* = l_e/d$)

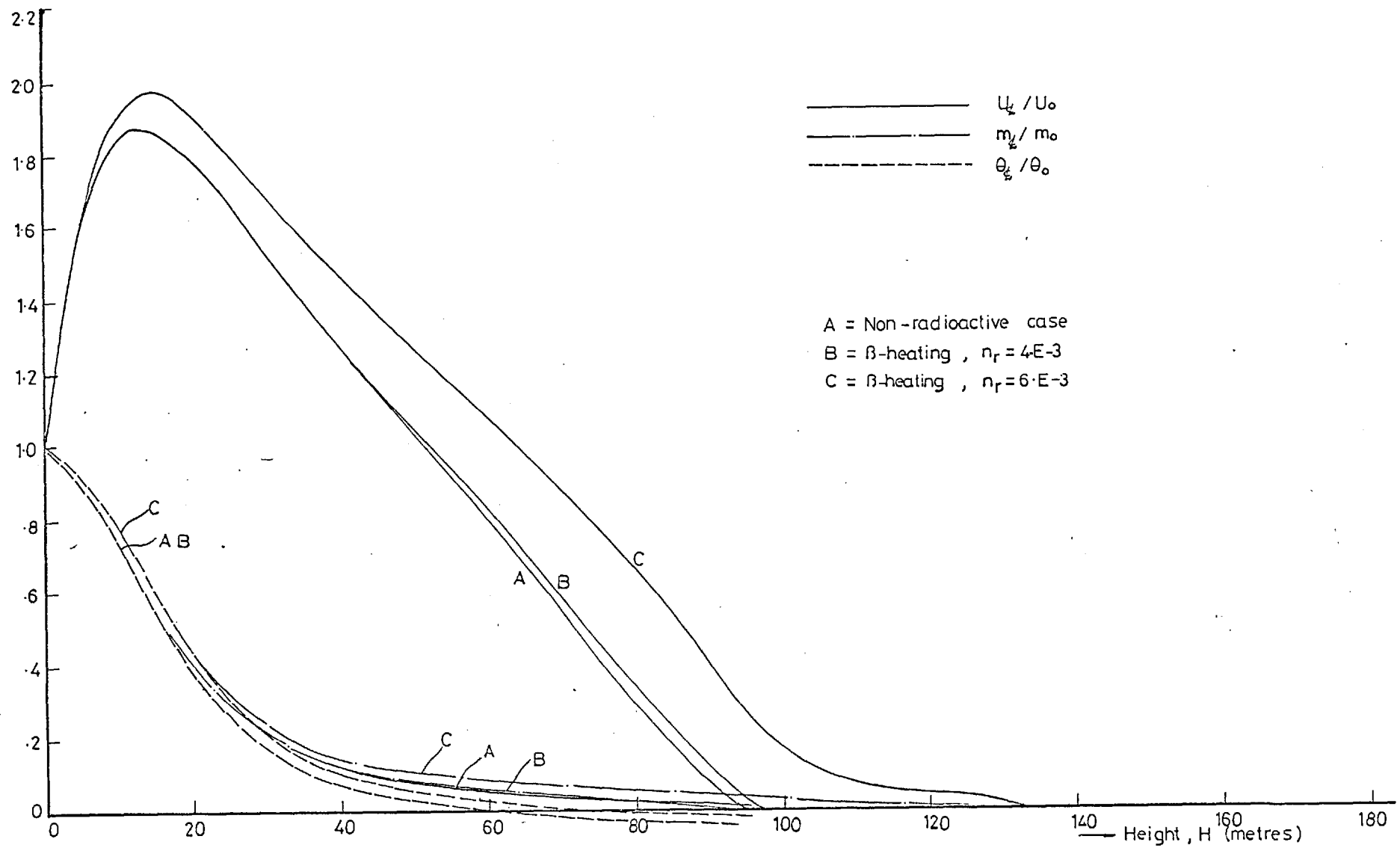


Fig. 5.9 Beta heating: Axial distribution of mean variables

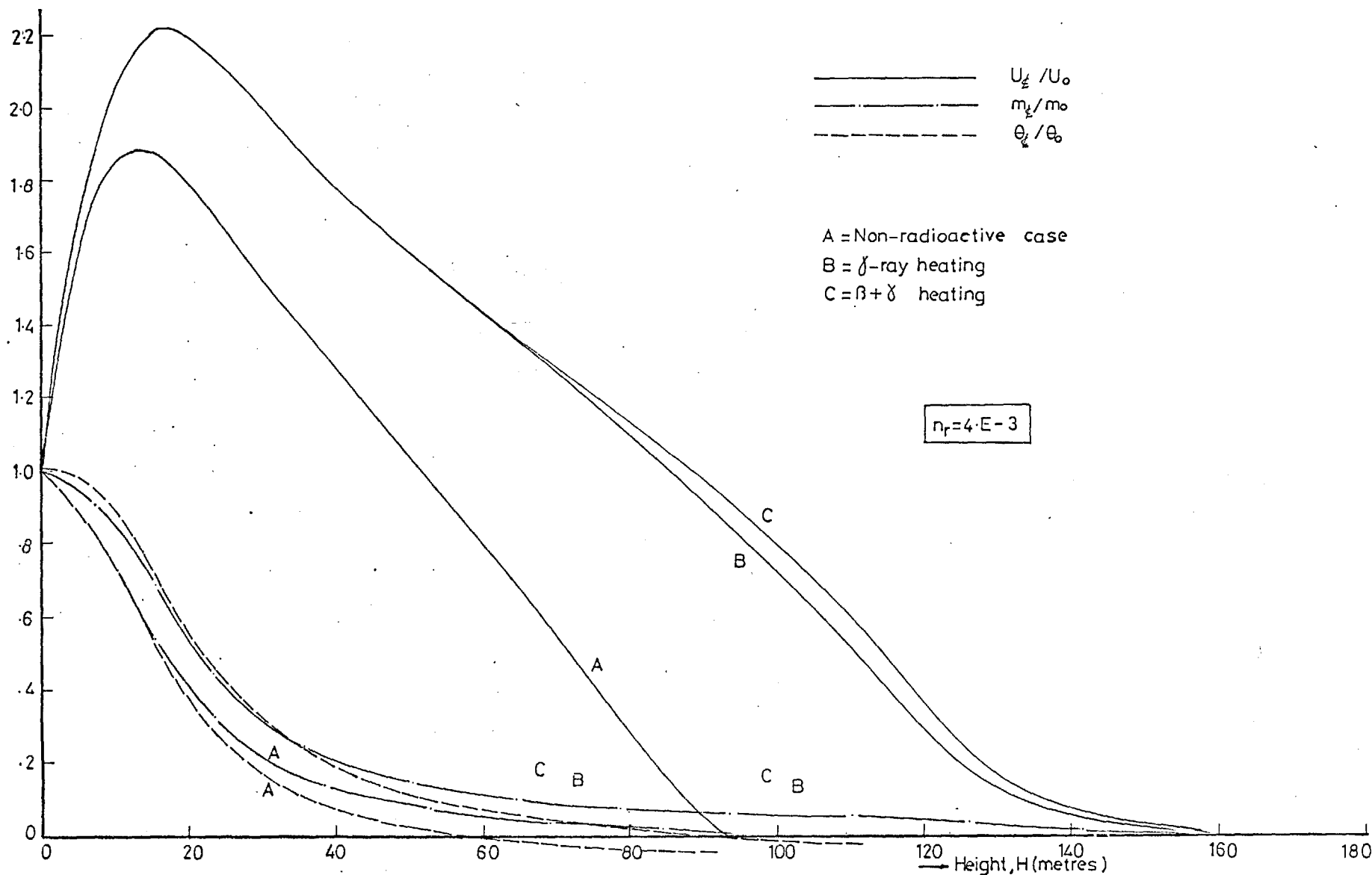


Fig. 5.10 Gamma and beta heating: Axial distribution of mean variables

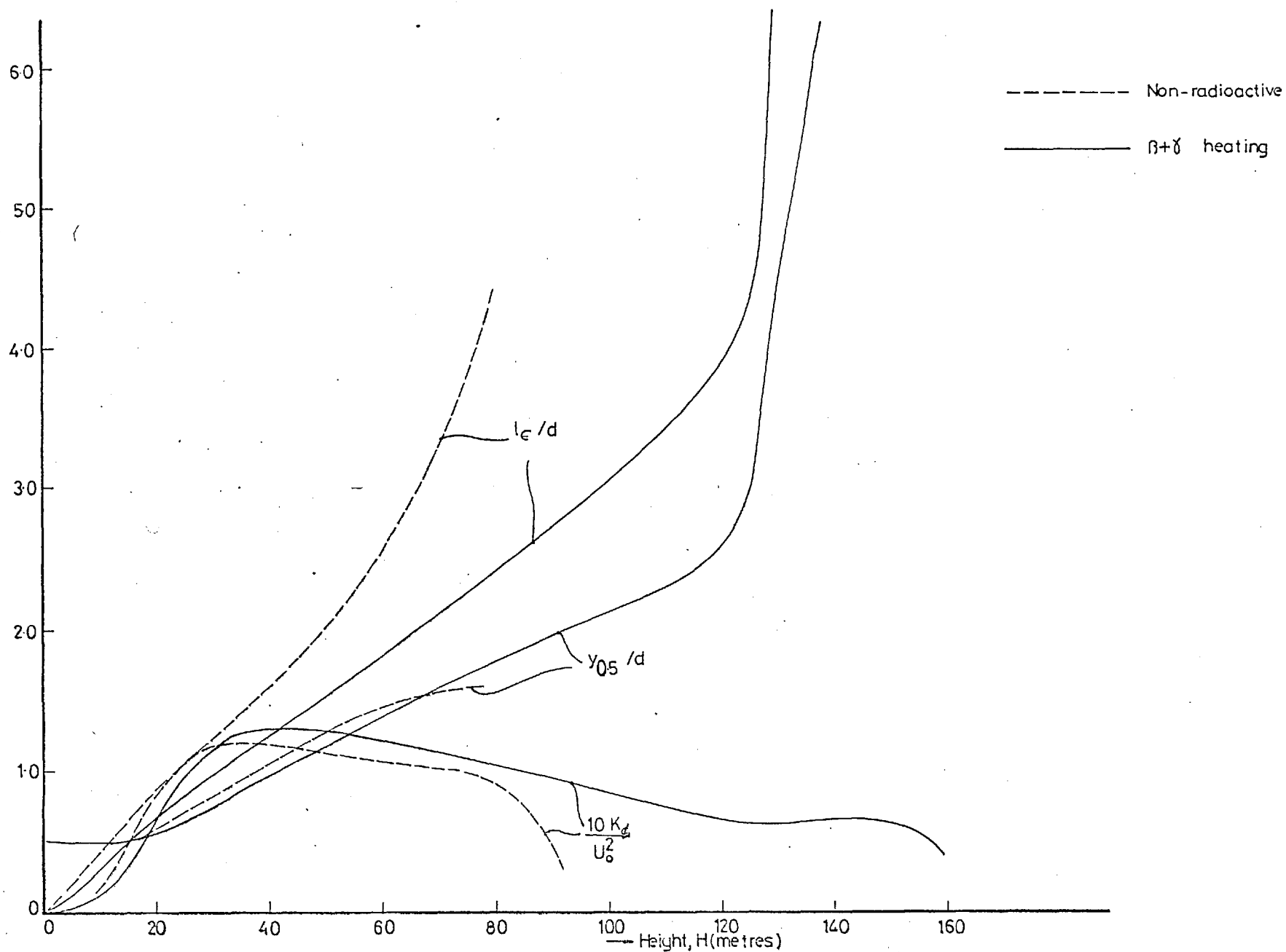


Fig. 5.11 Growth rate ($y_{0.5}/d$), and axial distribution of K and l_e

NON RADIOACTIVE
PLUME

RADIOACTIVE PLUME
($\beta + \delta$ heating)

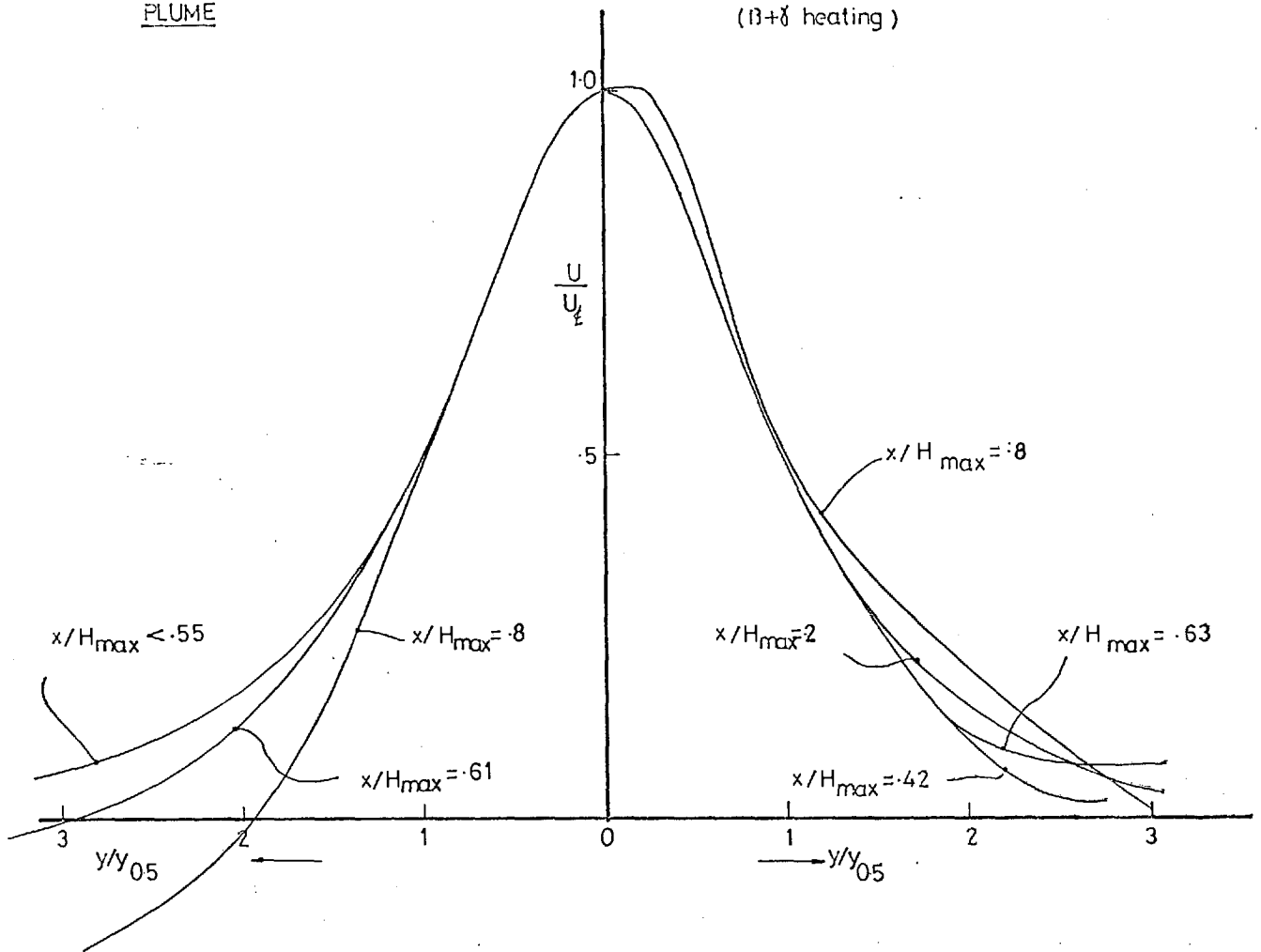


Fig. 5.12 Radial distribution of mean velocity for non-radioactive and radioactive plumes

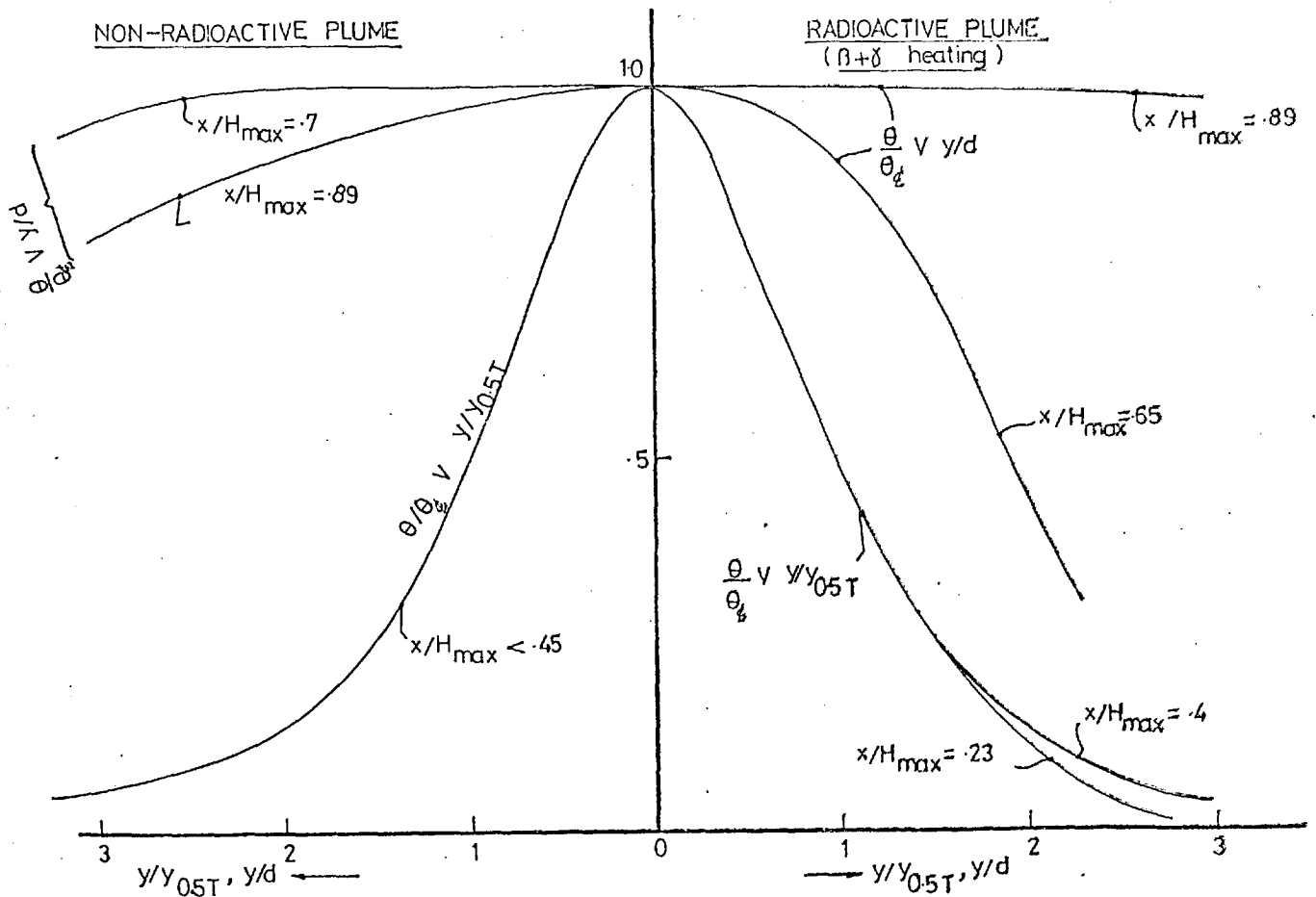


Fig. 5.13 Radial distributions of excess temperature for non-radioactive and radioactive plumes

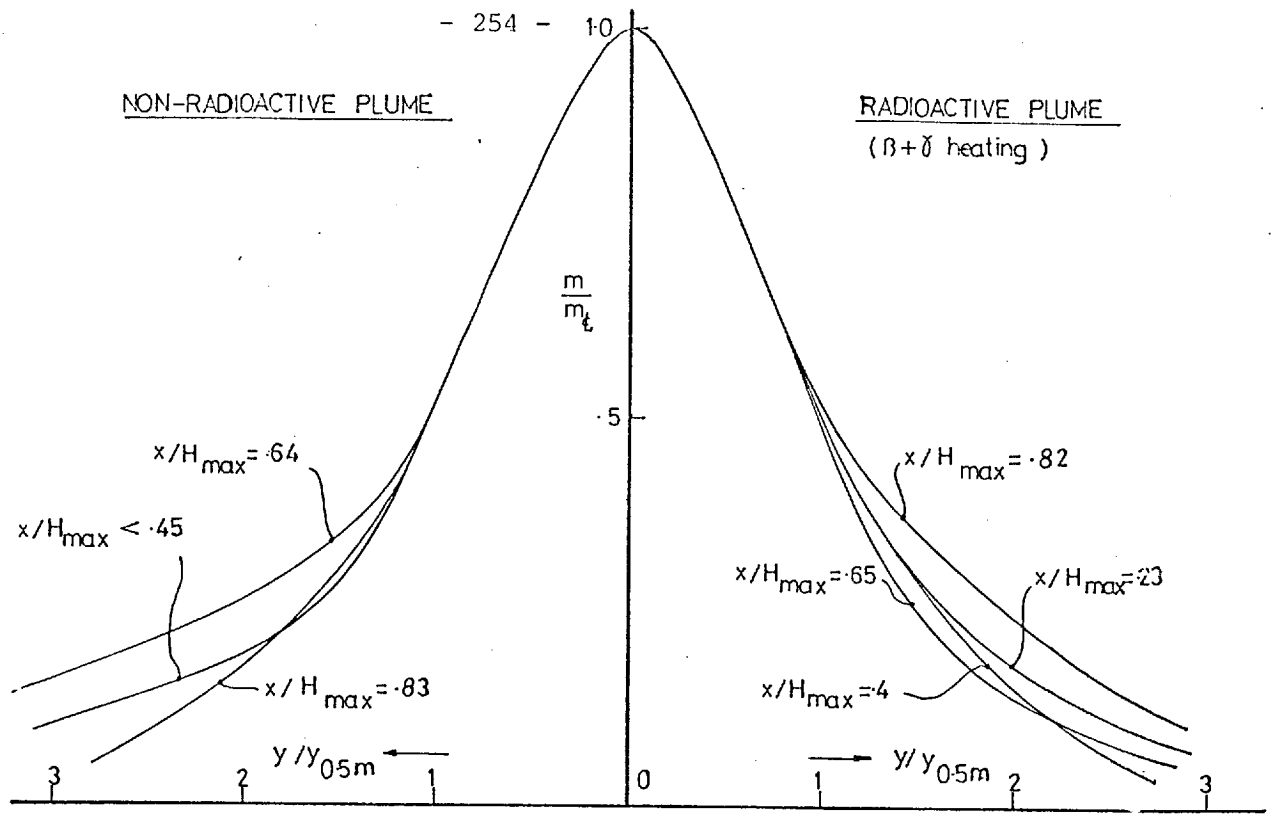


Fig. 5.14 Radial distribution of concentration of fission products for non-radioactive and radioactive plumes

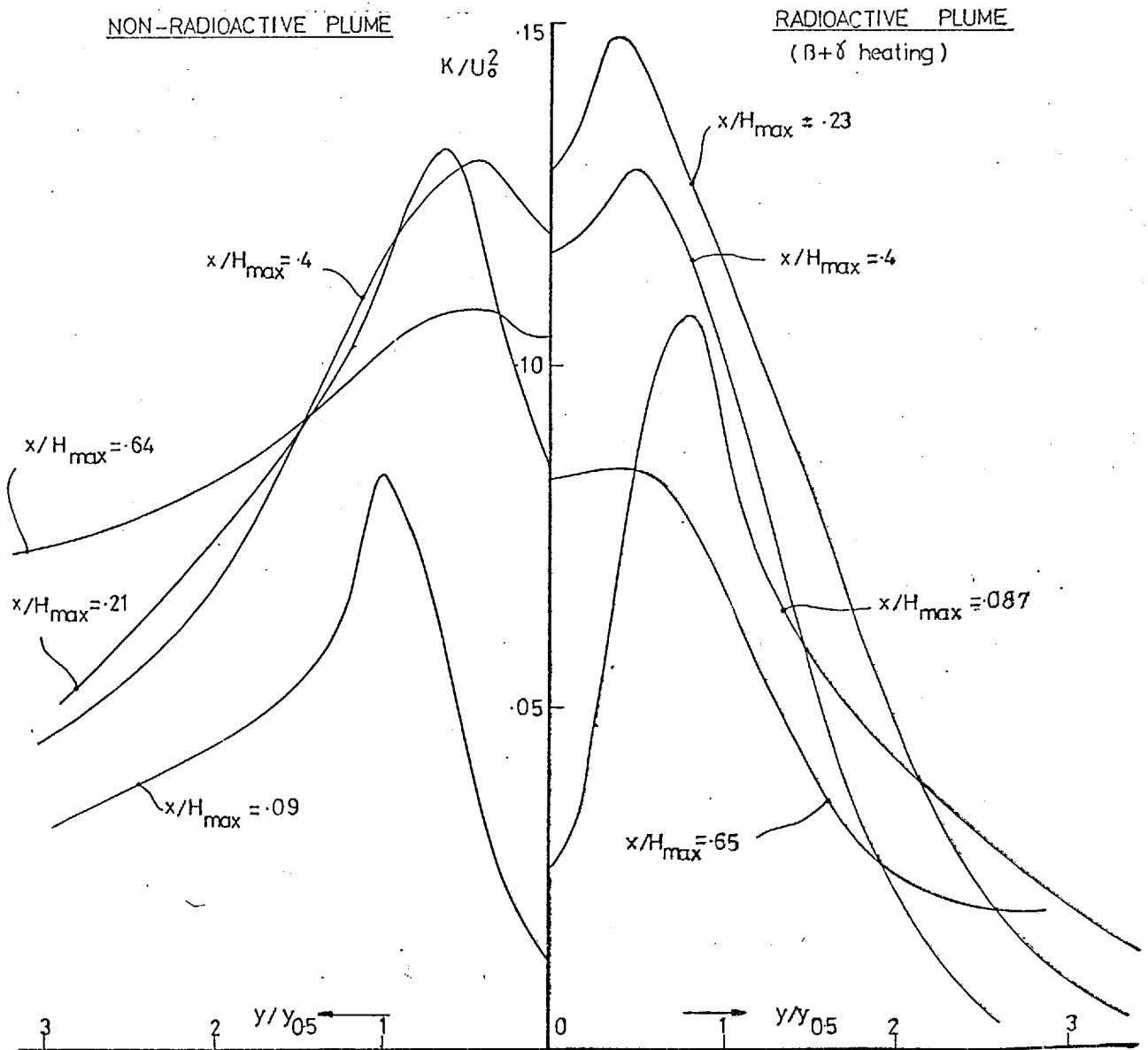
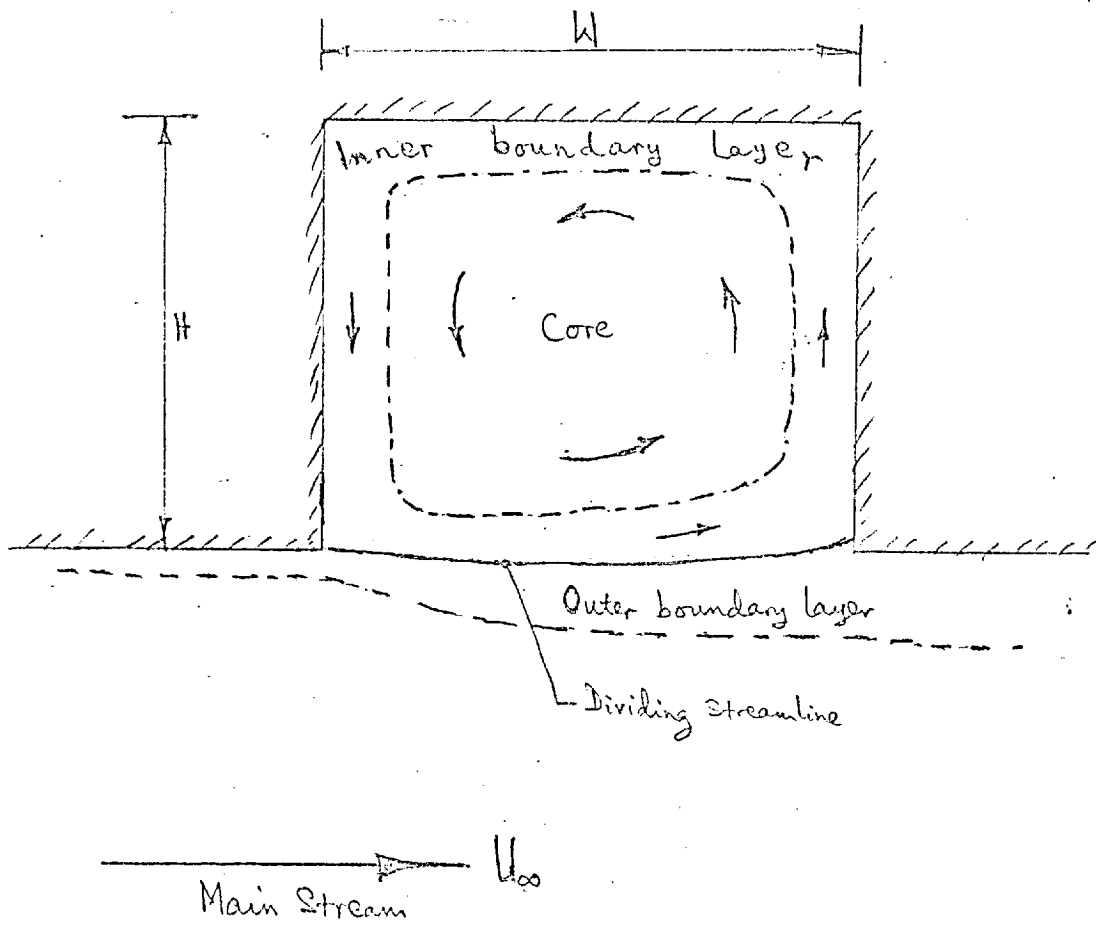
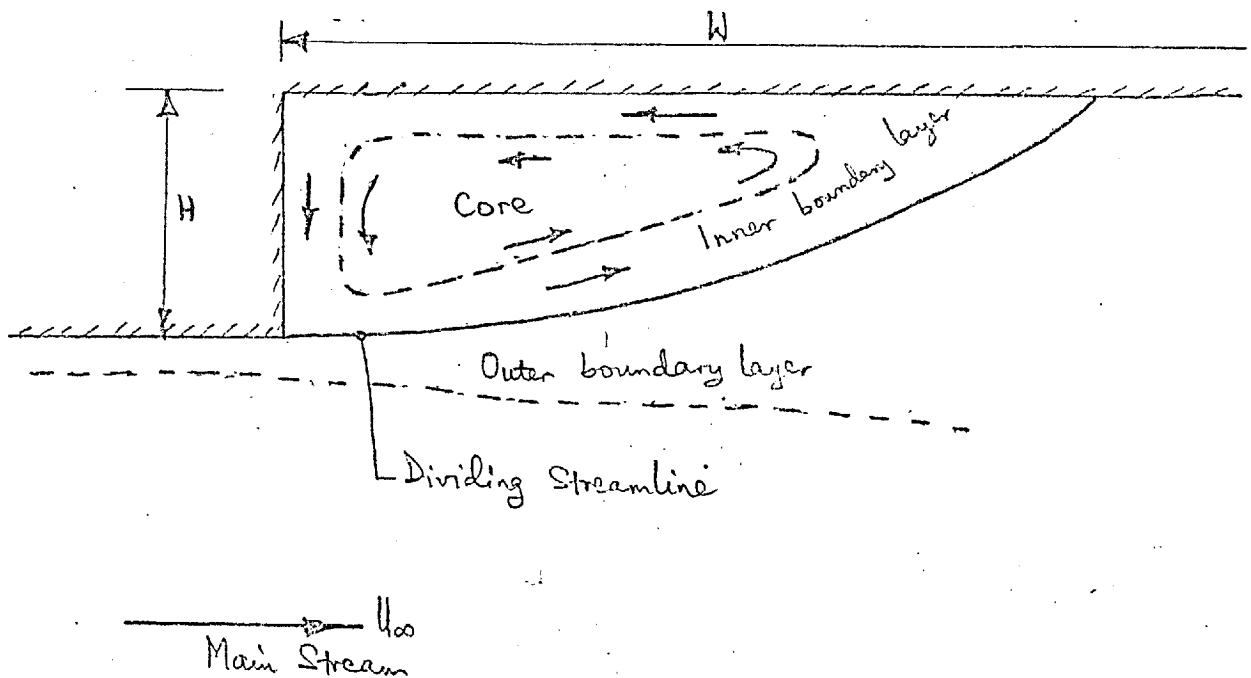


Fig. 5.15 Radial distribution of turbulence energy for non-radioactive and radioactive plumes

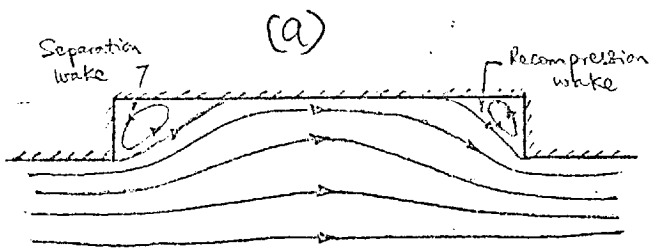


(a) Deep Cavity



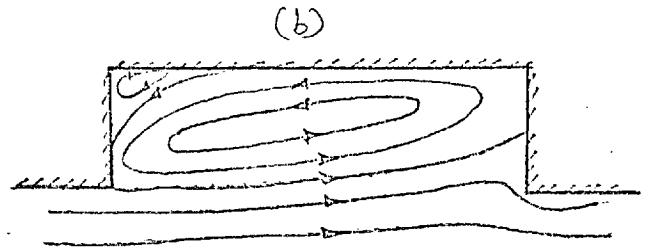
(b) Shallow Cavity

Fig. 6.1 FLOW REGIMES IN "CAVITY FLOW"



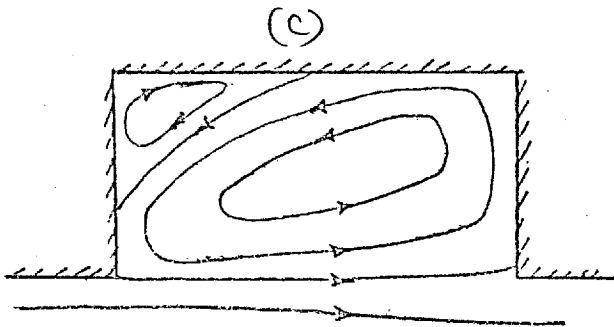
$$0 \leq H/W \leq 0.10$$

Boundary layer flow with corner eddies



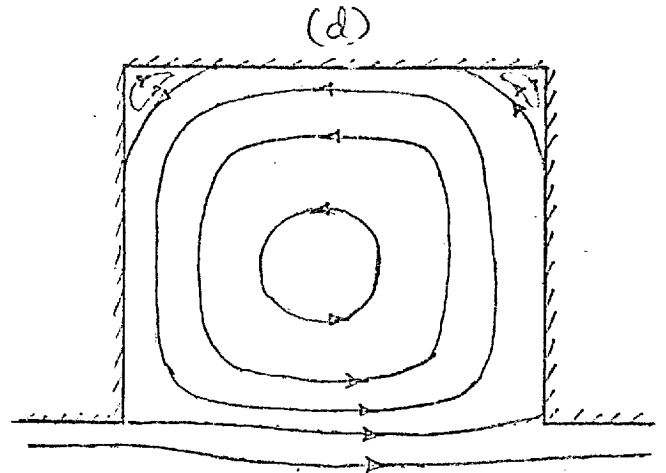
$$0.10 \leq H/W \leq 0.25$$

Flat stable eddy. Probably small eddy at roof of upstream wall



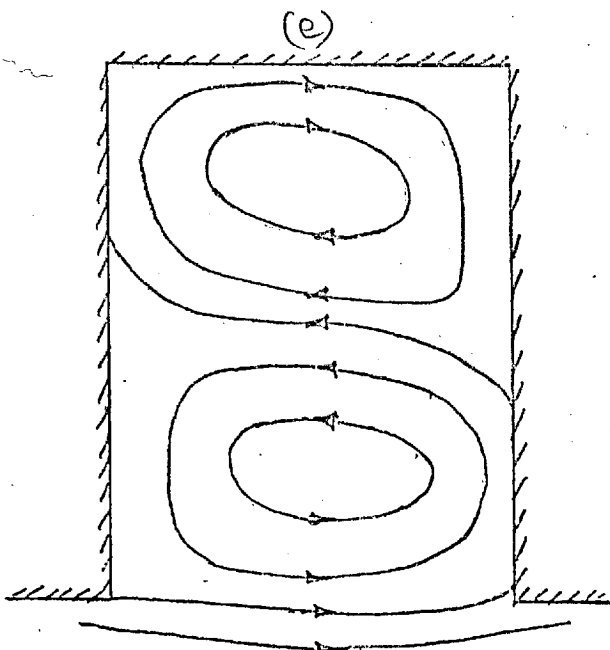
$$0.25 \leq H/W \leq 0.80$$

Unsteady main vortex, with eddy at roof of upstream wall



$$0.80 \leq H/W \leq 1.35$$

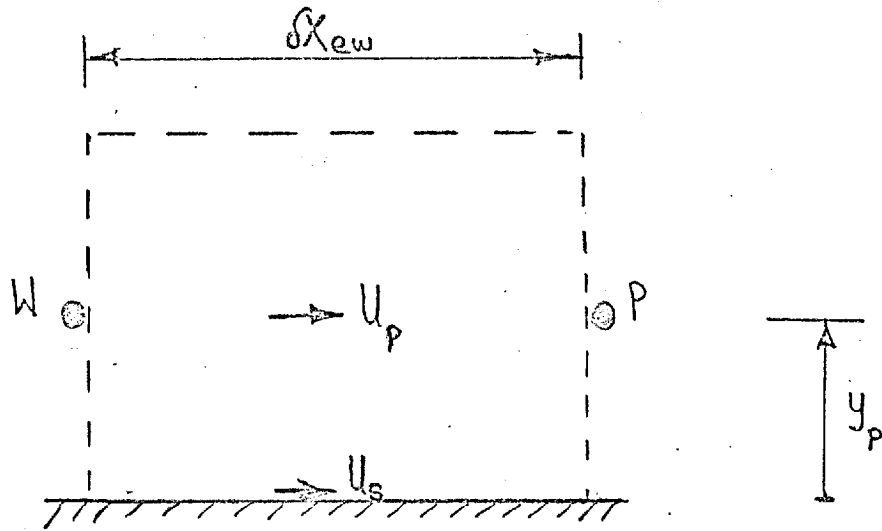
Steady, main vortex. Small corner eddies. The dividing streamline meets the downstream wall at about 1.5% of the depth below the corner, and, at the centre-section, it curves outside the cavity.



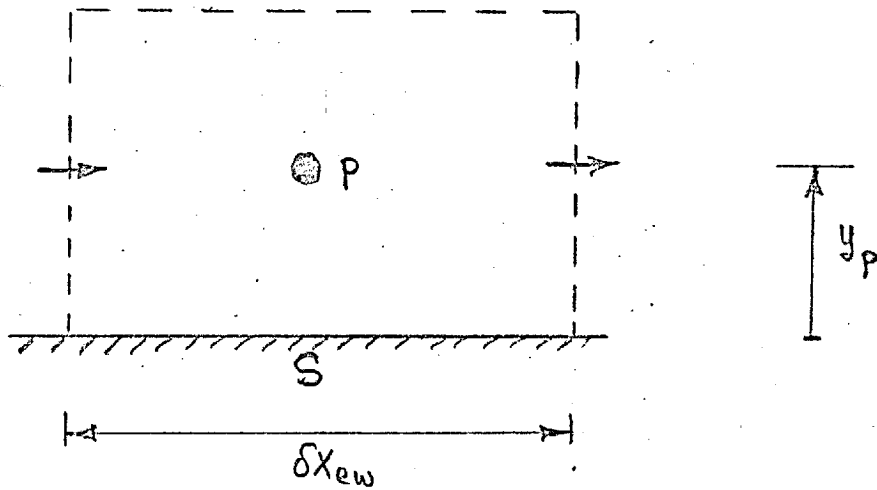
$$1.4 \leq H/W \leq 2.4$$

Two vortices of smaller strength than the vortex in a square cavity

Fig. 6.2 FLOW REGIMES IN CAVITIES OF DIFFERENT DEPTH-WIDTH RATIOS (IDENTIFIED BY MILLS, 1961).

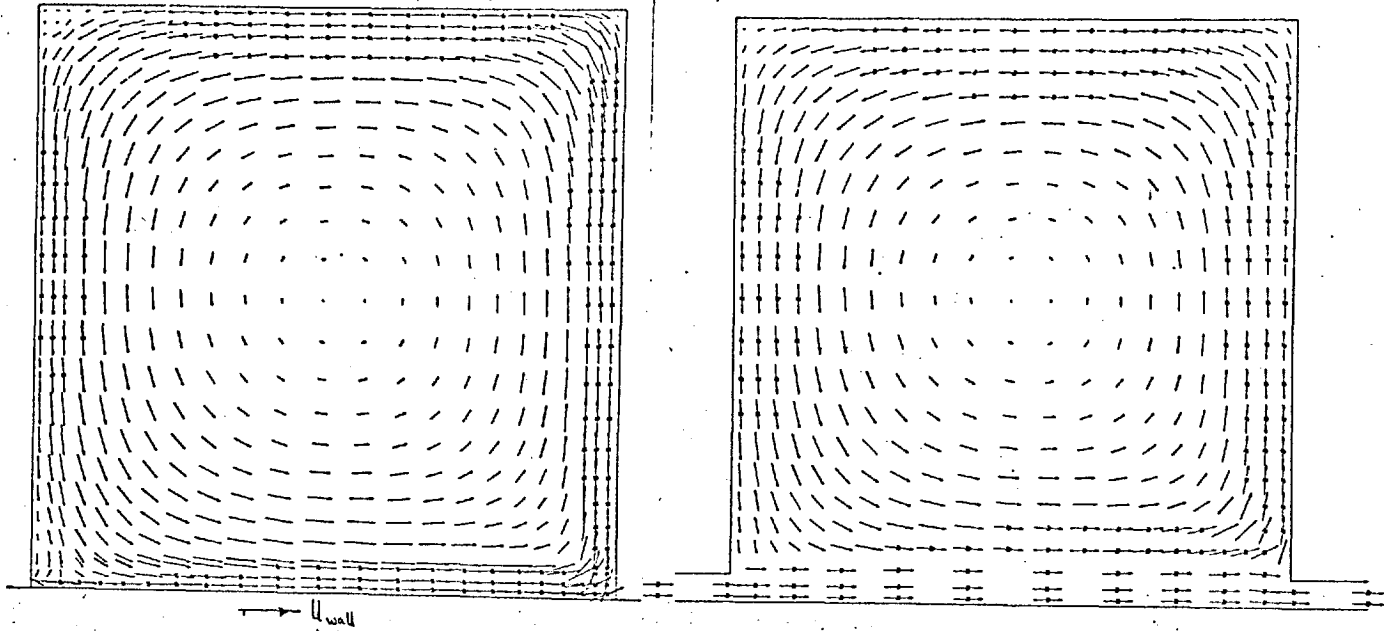


(a) Tangential-Velocity cell



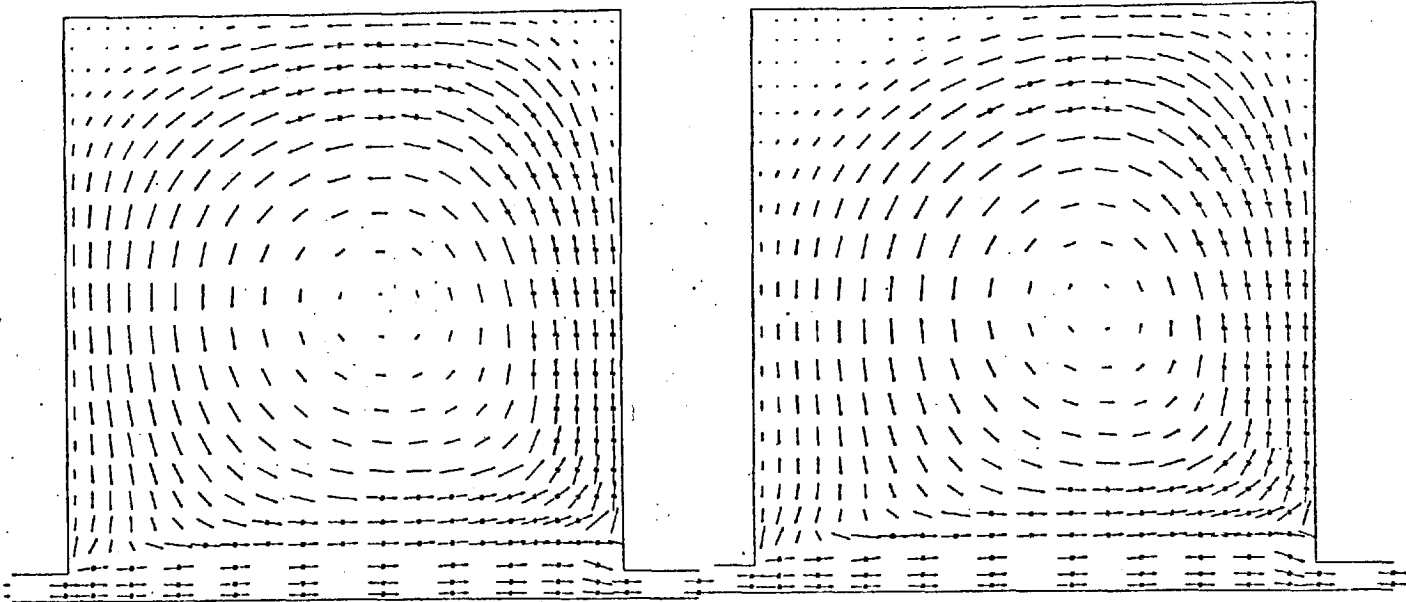
(b) Scalar cell

Fig. 6.3 Illustration of the near-wall region



(a) $Re = 2 \times 10^5$

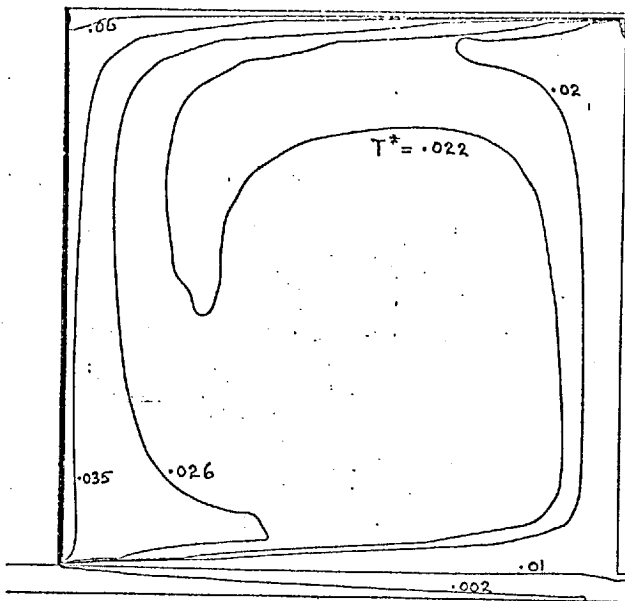
(b) $Re = 2 \times 10^5, Ar = 0.004$



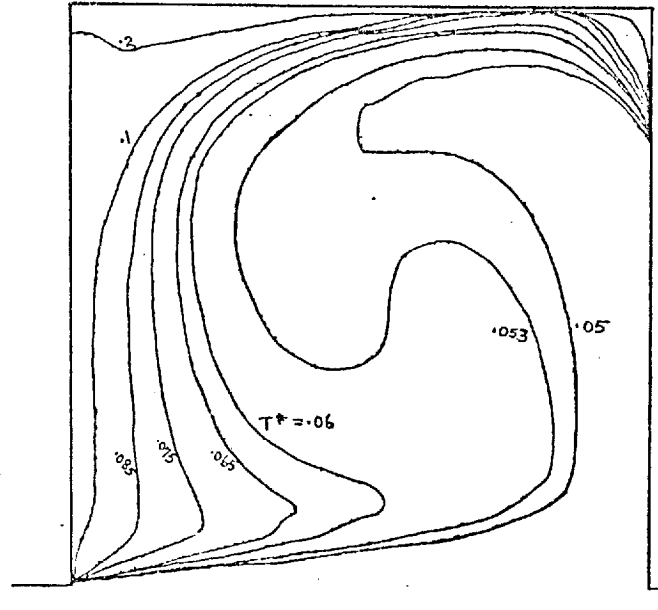
(c) $Re = 10^4$

(d) $Re = 10^4, Ar = 0.37$

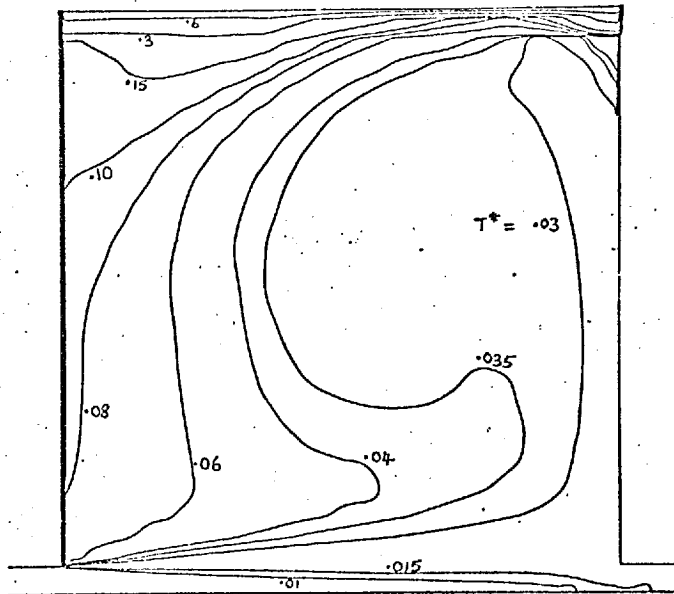
Fig. 6.4 Vector plot of flow field



(a) $Re = 2 \times 10^5$, $Ar = 0.004$



(b) $Re = 10^4$



(c) $Re = 10^4$, $Ar = 0.37$

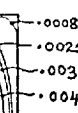
Fig. 6.5 Temperature field (Isotherms), $T^* = T - T_0 / \Delta T$



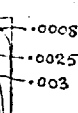
(c) $Re = 10^4$



(d) $Re = 10^4$, $Ar = 0.37$

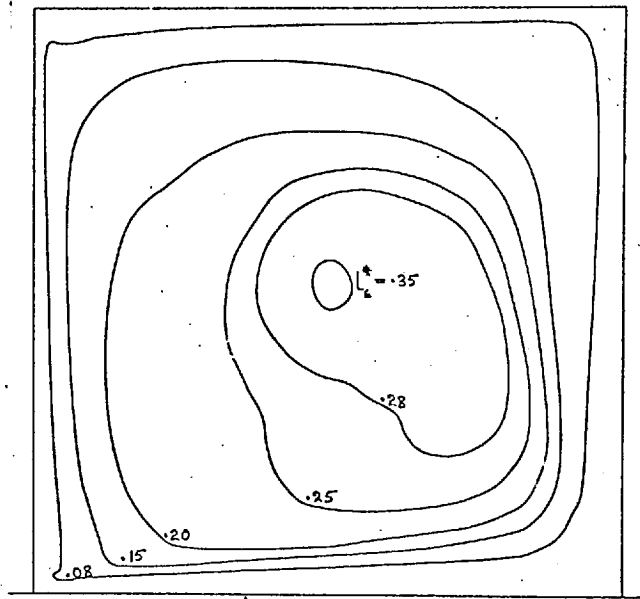


(c) $Re = 10^4$

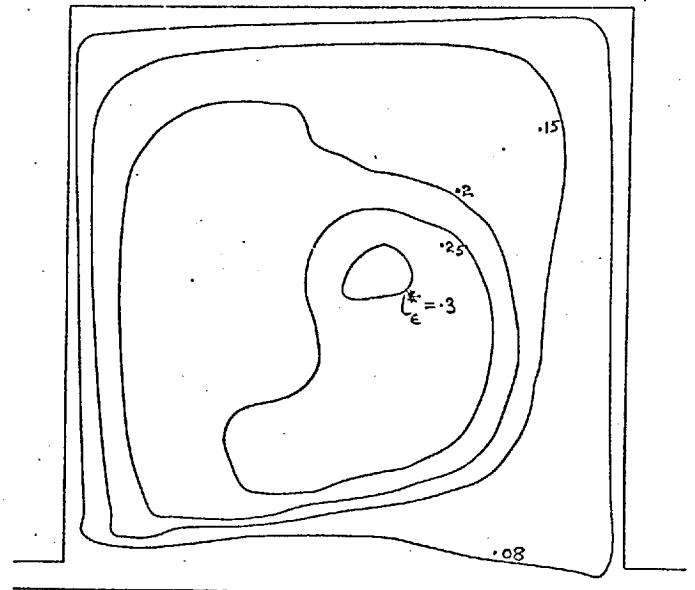


(d) $Re = 10^4$, $Ar = 0.37$

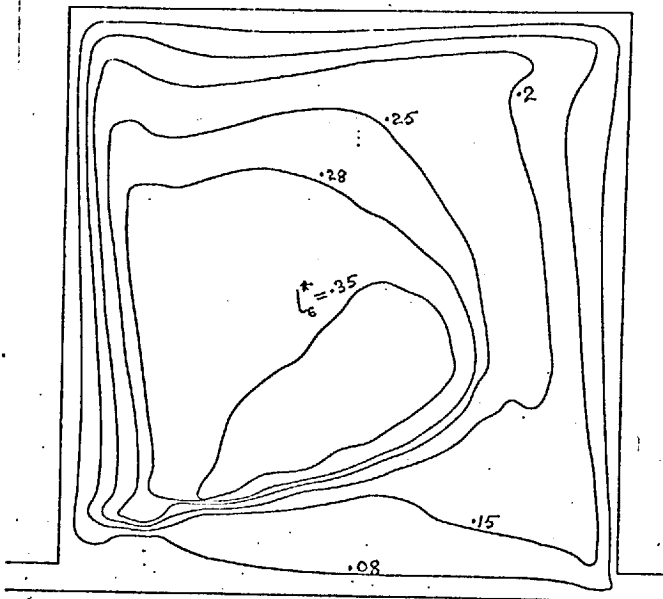
Fig. 6.6 Contour of turbulence energy, $K^* = K/U_0^2$



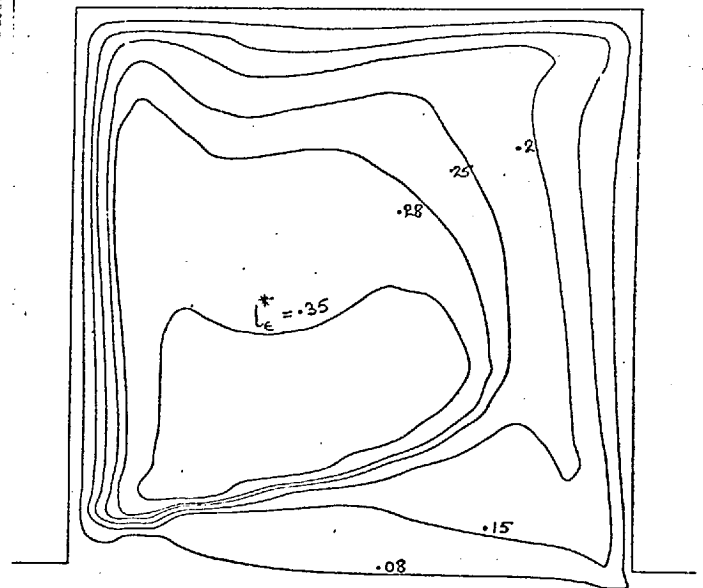
(a) $Re = 2 \times 10^5$



(b) $Re = 2 \times 10^5, Ar = .004$



(c) $Re = 10^4$



(d) $Re = 10^4, Ar = .37$

Fig. 6.7 Contour of dissipation length scale, $l_\epsilon^* = l_\epsilon/W$

Experimental data:

- (Grand, 1975) — Channel flow — $Re=2 \times 10^5$, $Ar=.004$
- △ (Mills, 1961) — Free Stream — $Re=5 \times 10^4 - 4 \times 10^5$, $Ar=0.00$

Predictions (Channel flow)

- $Re=2 \times 10^5$, $Ar=.004$
- $Re=5 \times 10^4$, $Ar=.064$
- $Re=10^4$, $Ar=.000$
- $Re=10^4$, $Ar=.37$

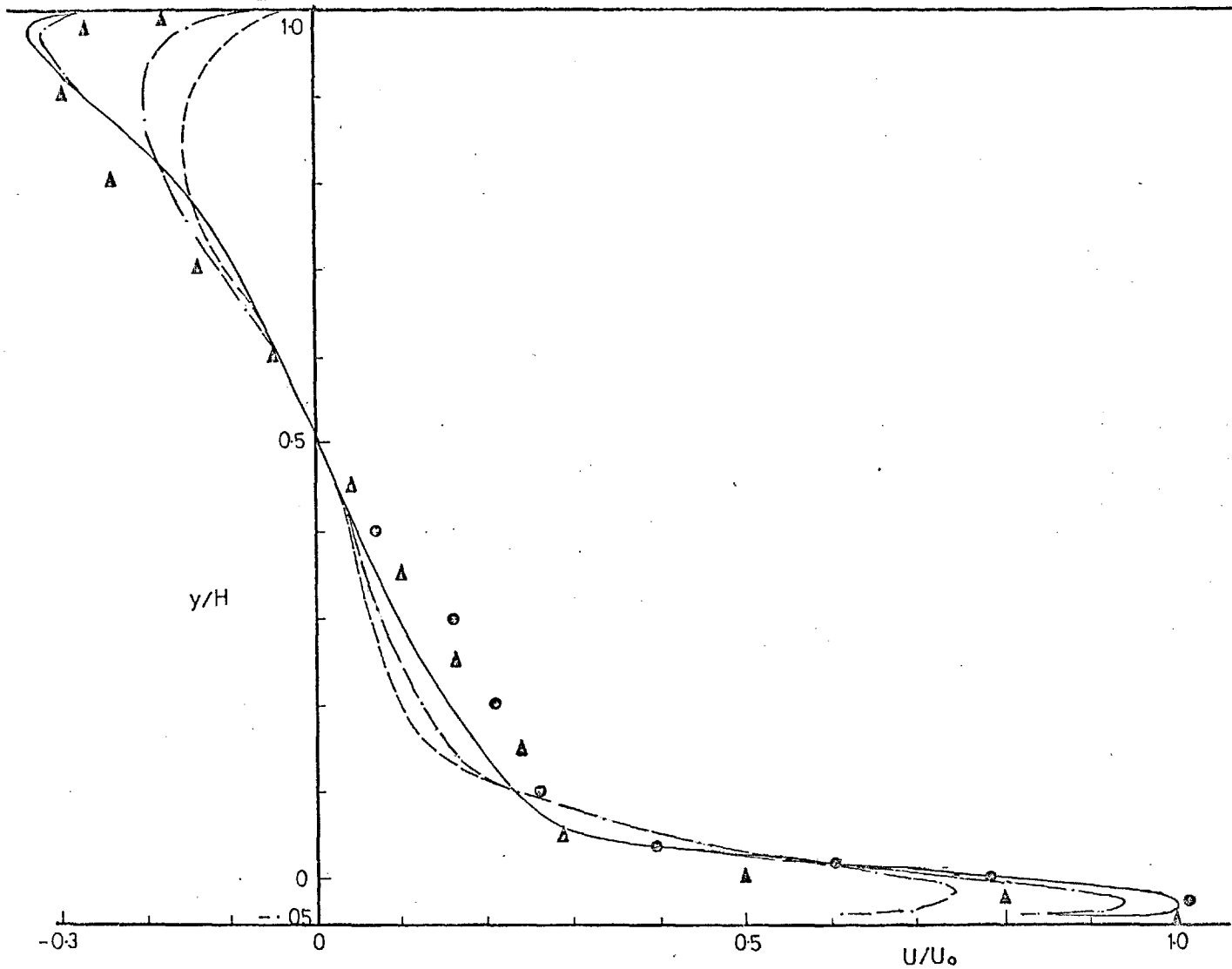


Fig. 6.8 Distribution of horizontal component of mean velocity along vertical axis ($x/w = 0.5$)

Experimental data:

- (Grand, 1975) — Channel flow — $Re = 2 \times 10^5$, $Ar = .004$
 Δ (Mills , 1961) — Free Stream — $Re = 5 \times 10^4$ to 4×10^5 , $Ar = .000$

Predictions : (Channel flow)

- $Re = 2 \times 10^5$, $Ar = .004$
 - - - - - $Re = 5 \times 10^4$, $Ar = .064$
 - - - - - $Re = 10^4$, $Ar = .000$
 - - - - - $Re = 10^4$, $Ar = .37$

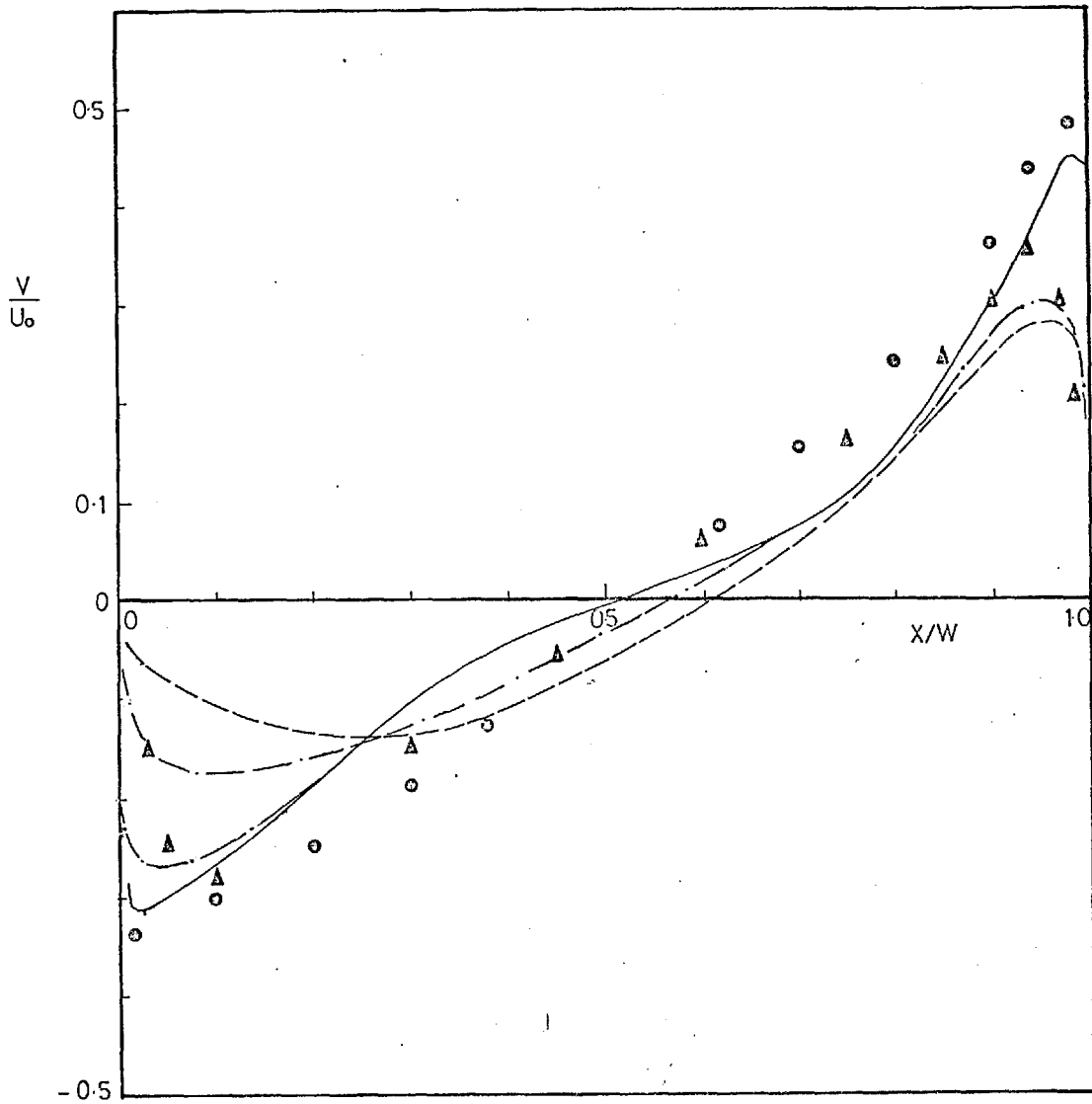
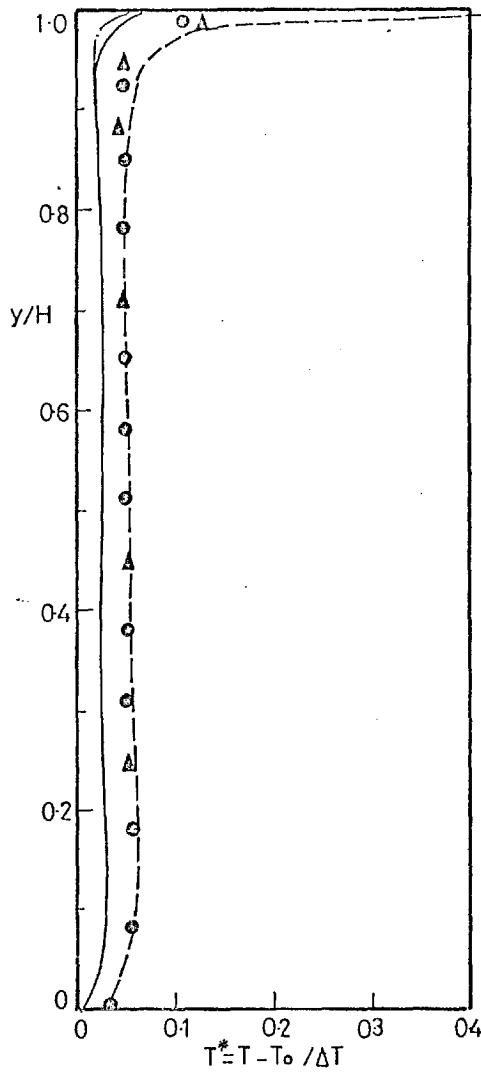


Fig. 6.9 Distribution of vertical component of mean velocity along
horizontal axis ($y/H = 0.5$)

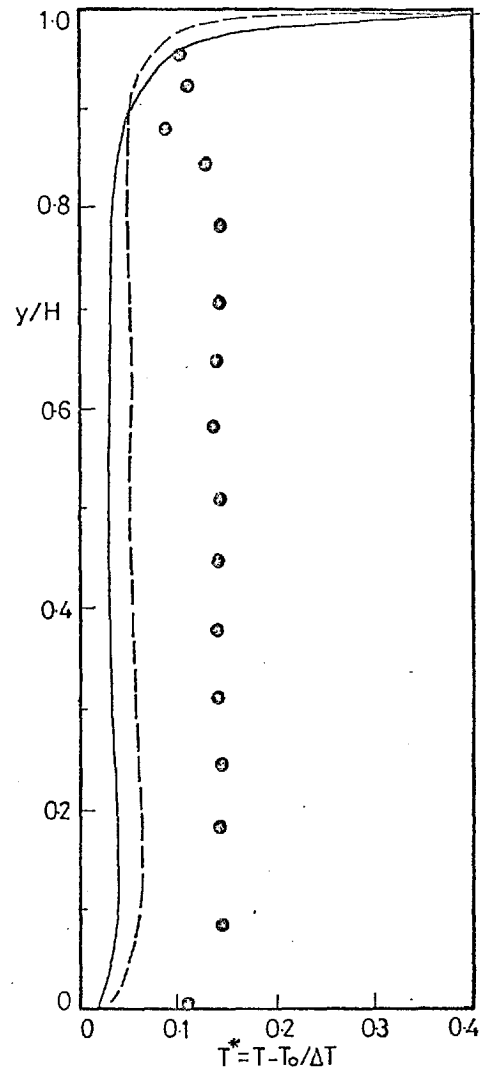


Experimental data: (Grand, 1975)

$\circ$ - $Ar = 0.064$ } $Re = 5 \times 10^4$
 Δ - $Ar = 0.085$ }

Predictions:

— $Ar = 0.064$, $Re = 5 \times 10^4$
 - - $Ar = 0.004$, $Re = 2 \times 10^5$
 . . $Ar = 0.000$, $Re = 1 \times 10^4$



Experimental data: (Grand, 1975)

$\circ$ - $Ar = 0.37$, $Re = 1 \times 10^4$

Predictions:

— $Ar = 0.37$, $Re = 1 \times 10^4$
 - - $Ar = 0.00$, $Re = 1 \times 10^4$

Fig. 6.10 Cavity-Heated Top Wall: Temperature distribution along vertical axis at $x/W = 0.5$

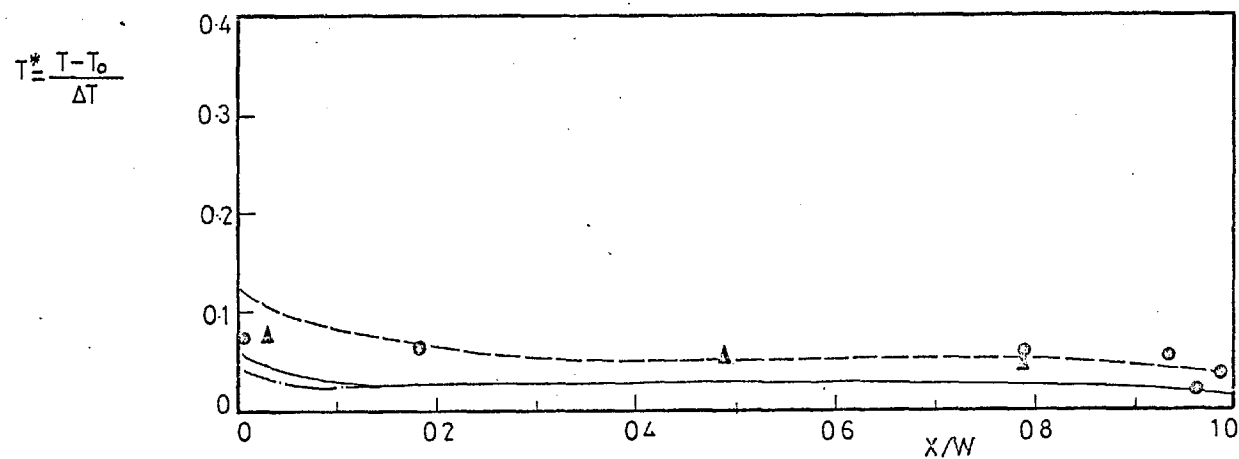
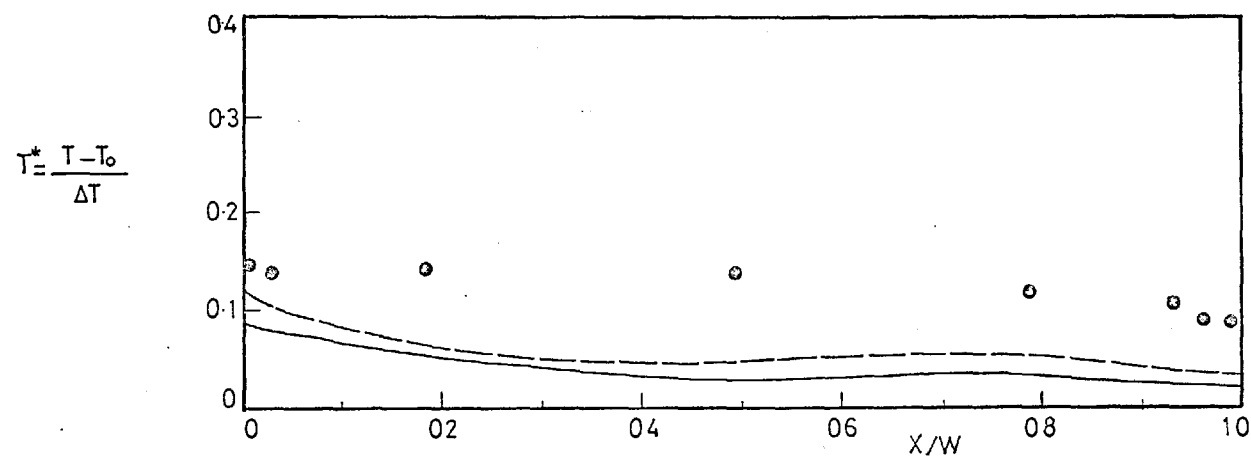


Fig. 6.11 Cavity-Heated Top Wall: Temperature distribution along horizontal axis at $y/H = 0.5$

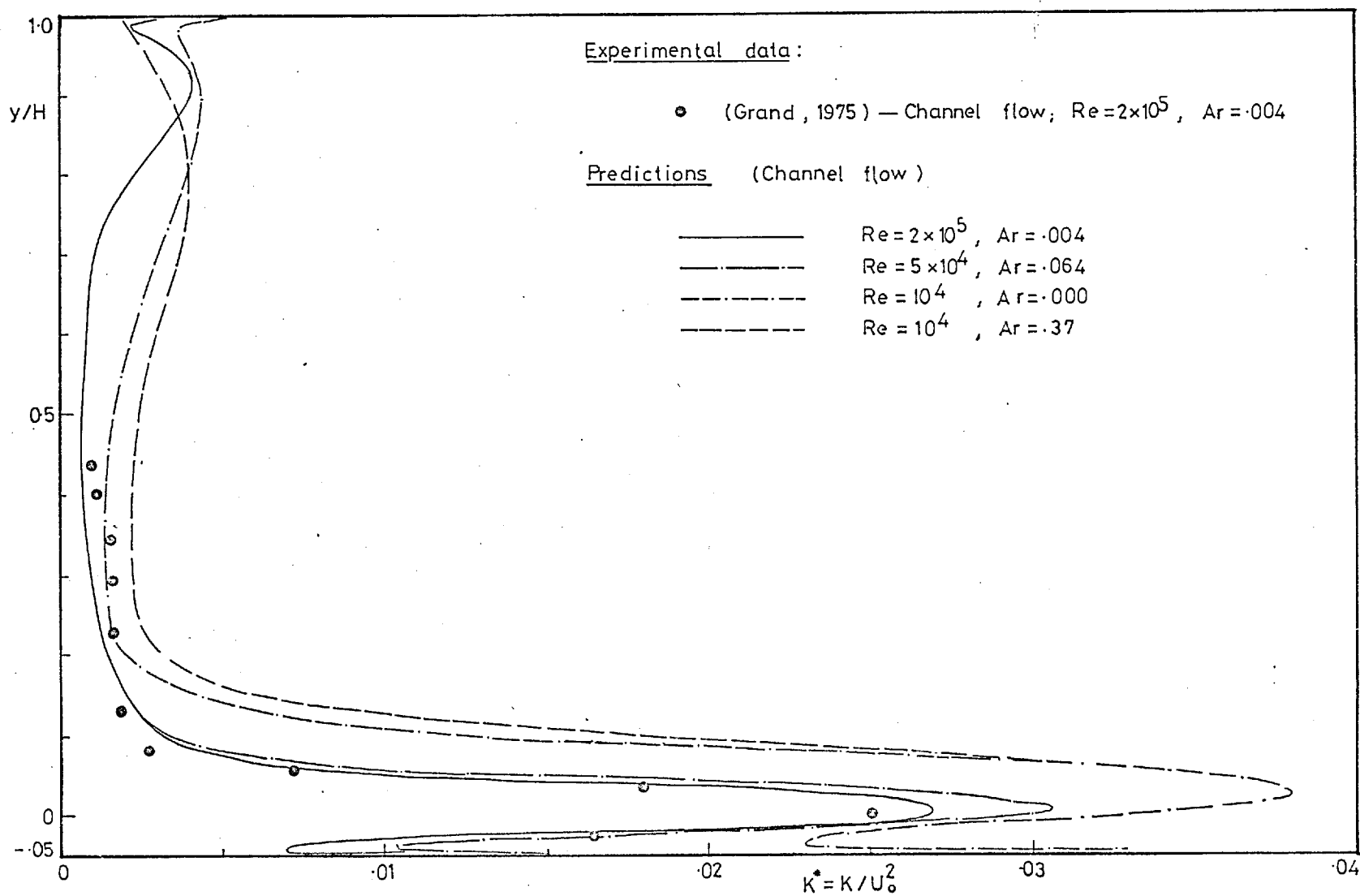


Fig. 6.12 Distribution of Turbulence Energy along vertical axis ($x/W = 0.5$)

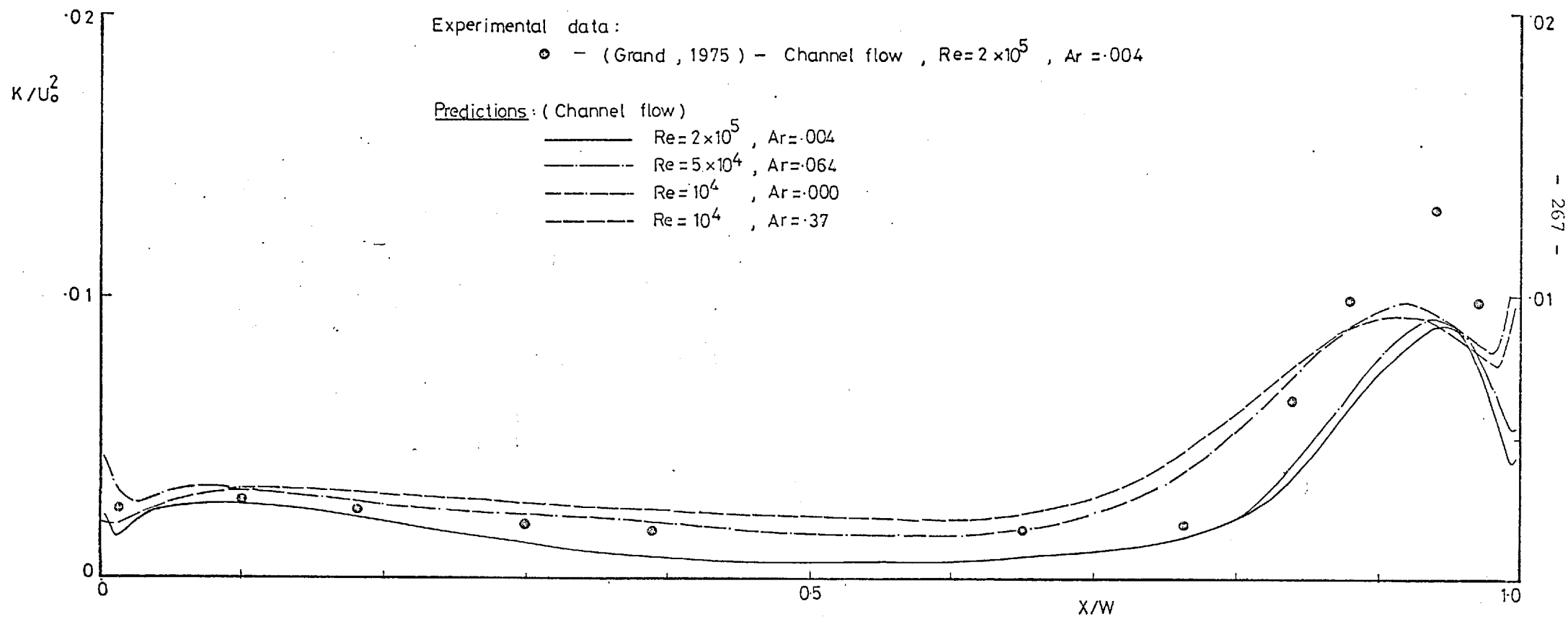


Fig. 6.13 Distribution of Turbulence Energy along horizontal axis ($y/H = 0.5$)

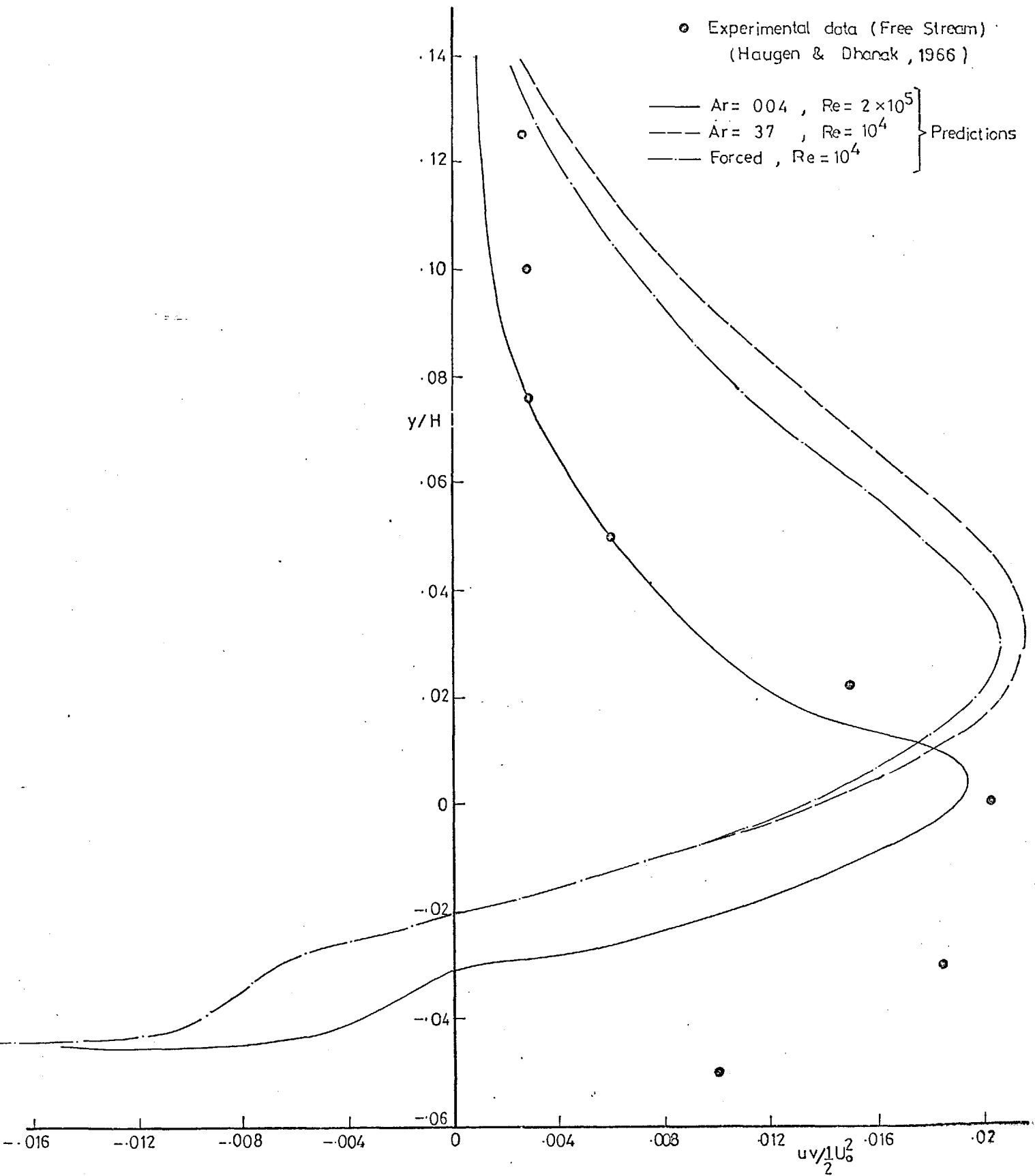


Fig. 6.14 Distribution of turbulent shear stress across shear layer ($x/W = 0.5$)

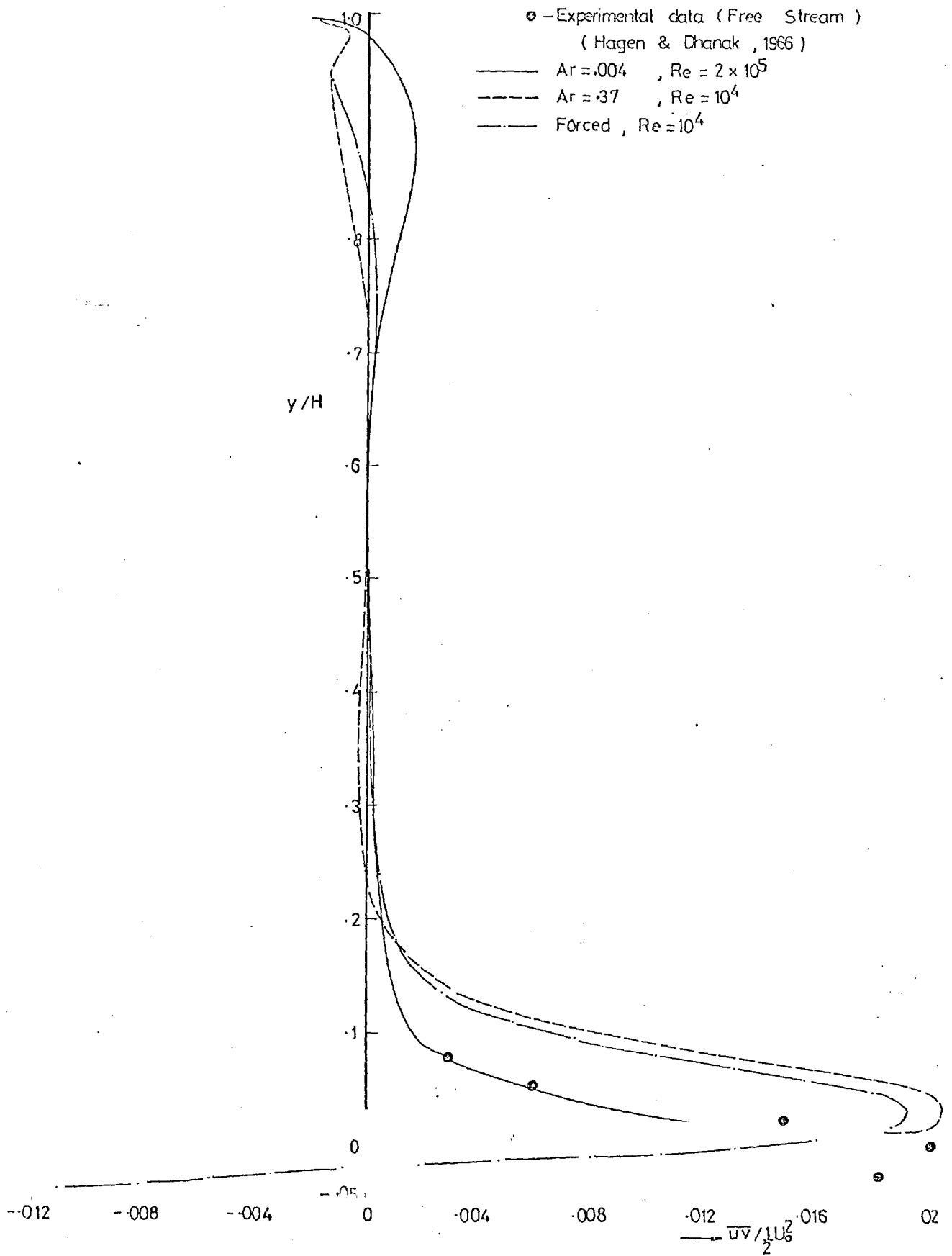


Fig. 6.15 Distribution of turbulent shear stress along vertical axis ($x/W = 0.5$)

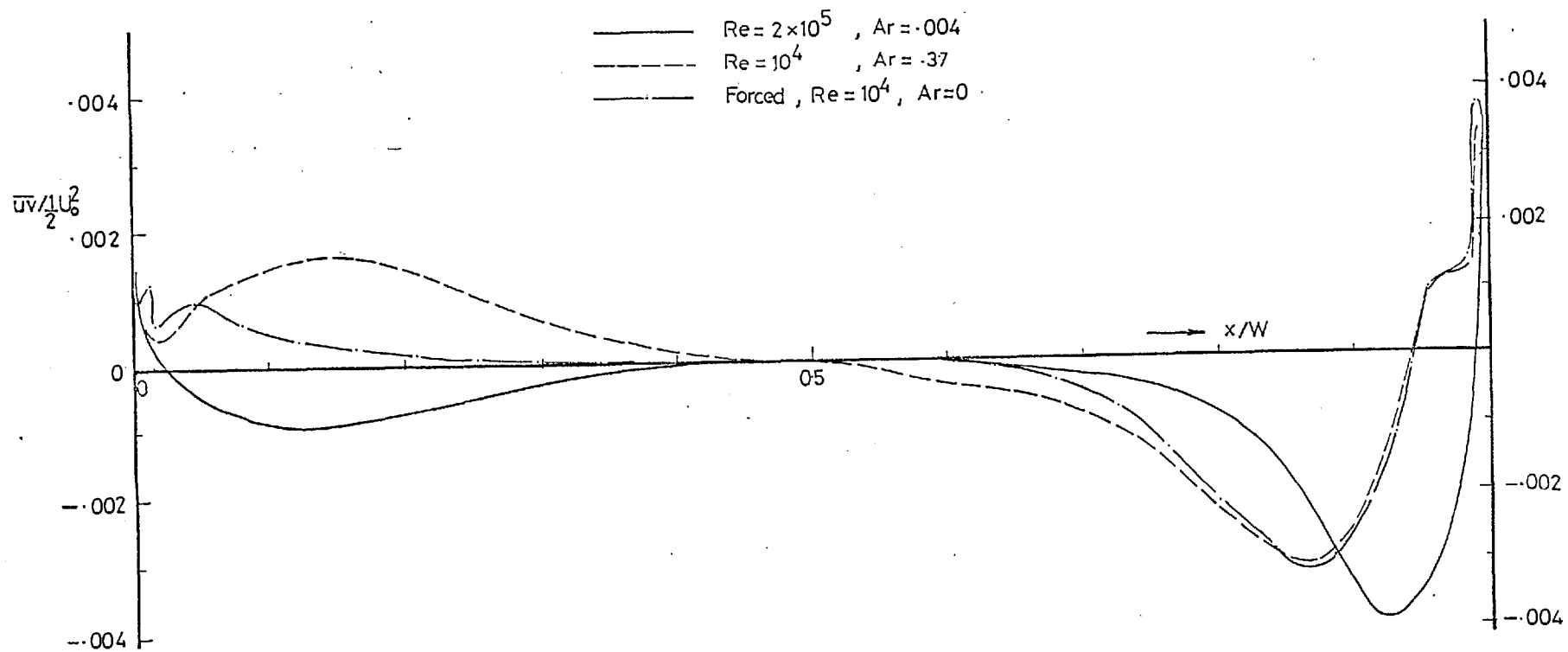


Fig. 6.16 Distribution of Turbulent shear stress along horizontal axis ($y/H = 0.5$)

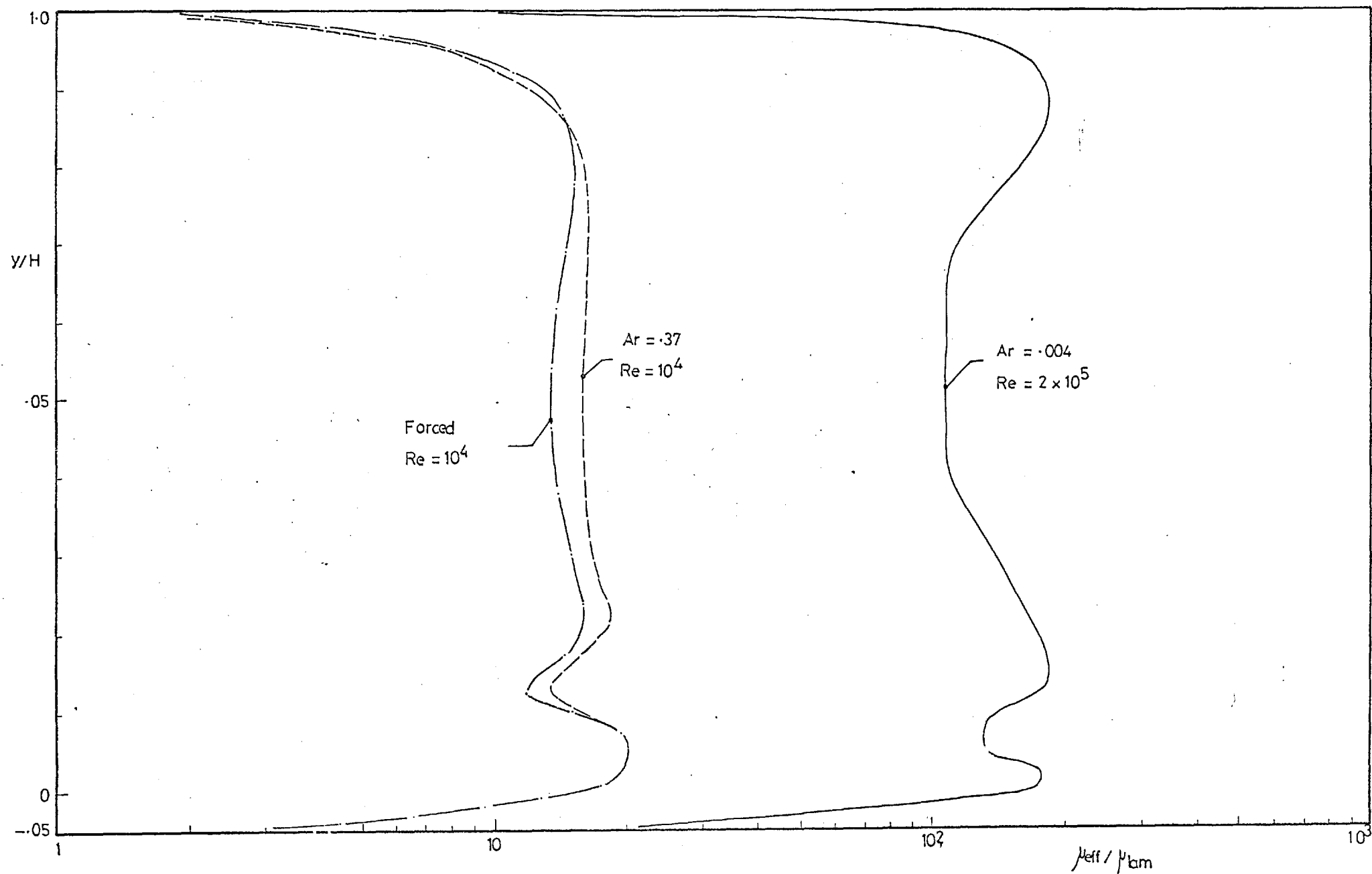


Fig. 6.17 Predicted distribution of effective viscosity along vertical axis ($x/W = 0.5$)

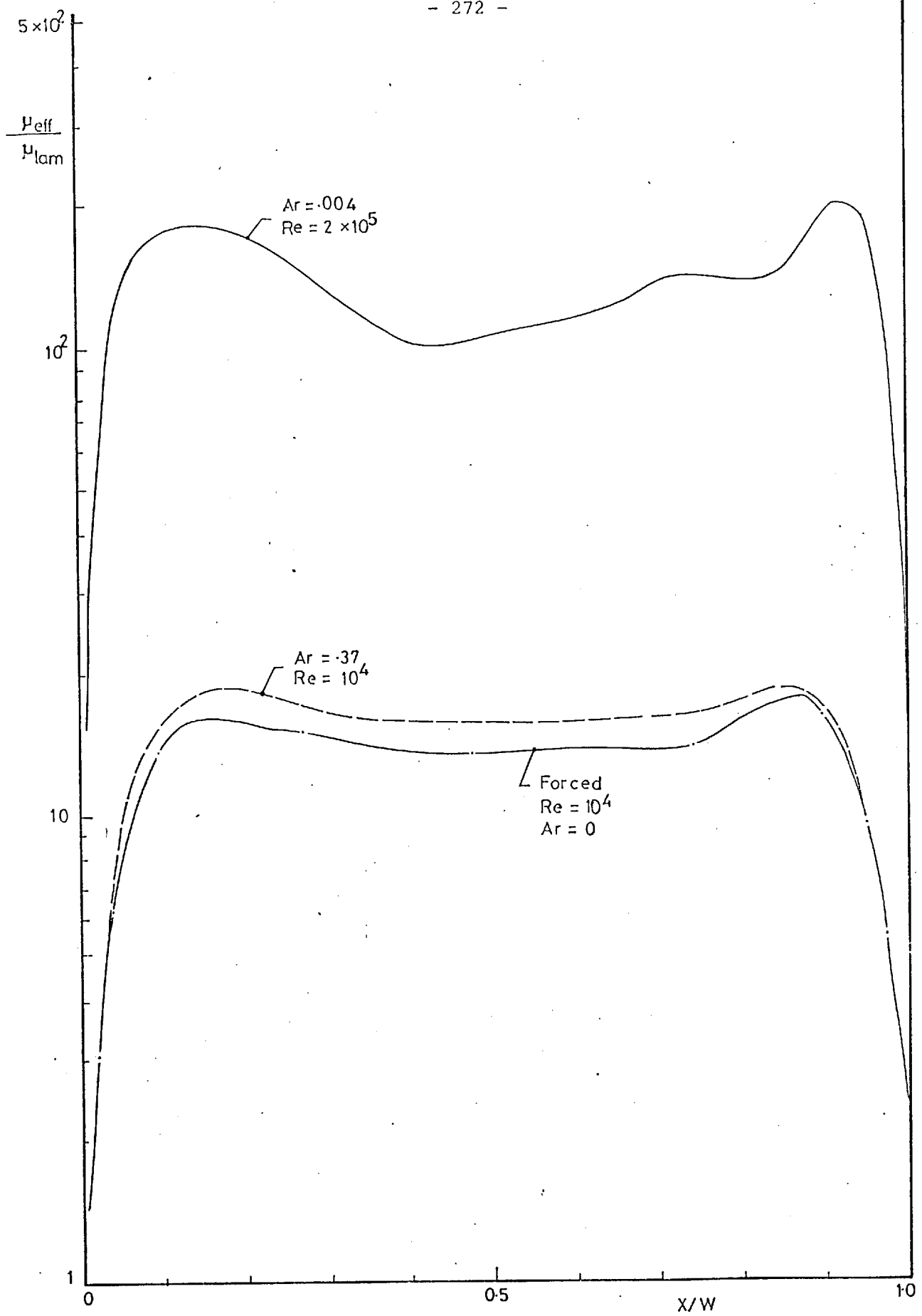
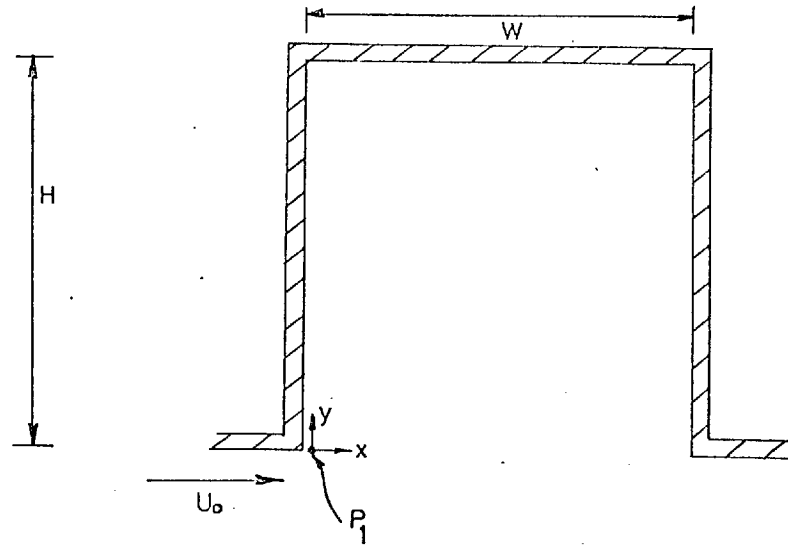


Fig. 6.18 Predicted distribution of effective viscosity along horizontal axis ($y/H = 0.5$)



Δ - Experimental data (Free Stream)
(Roshko, 1955)

— Ar = 0.04 , Re = 2×10^5
 - - - Ar = 0.37 , Re = 10^4
 - · - Forced , Re = 10^4

Predictions

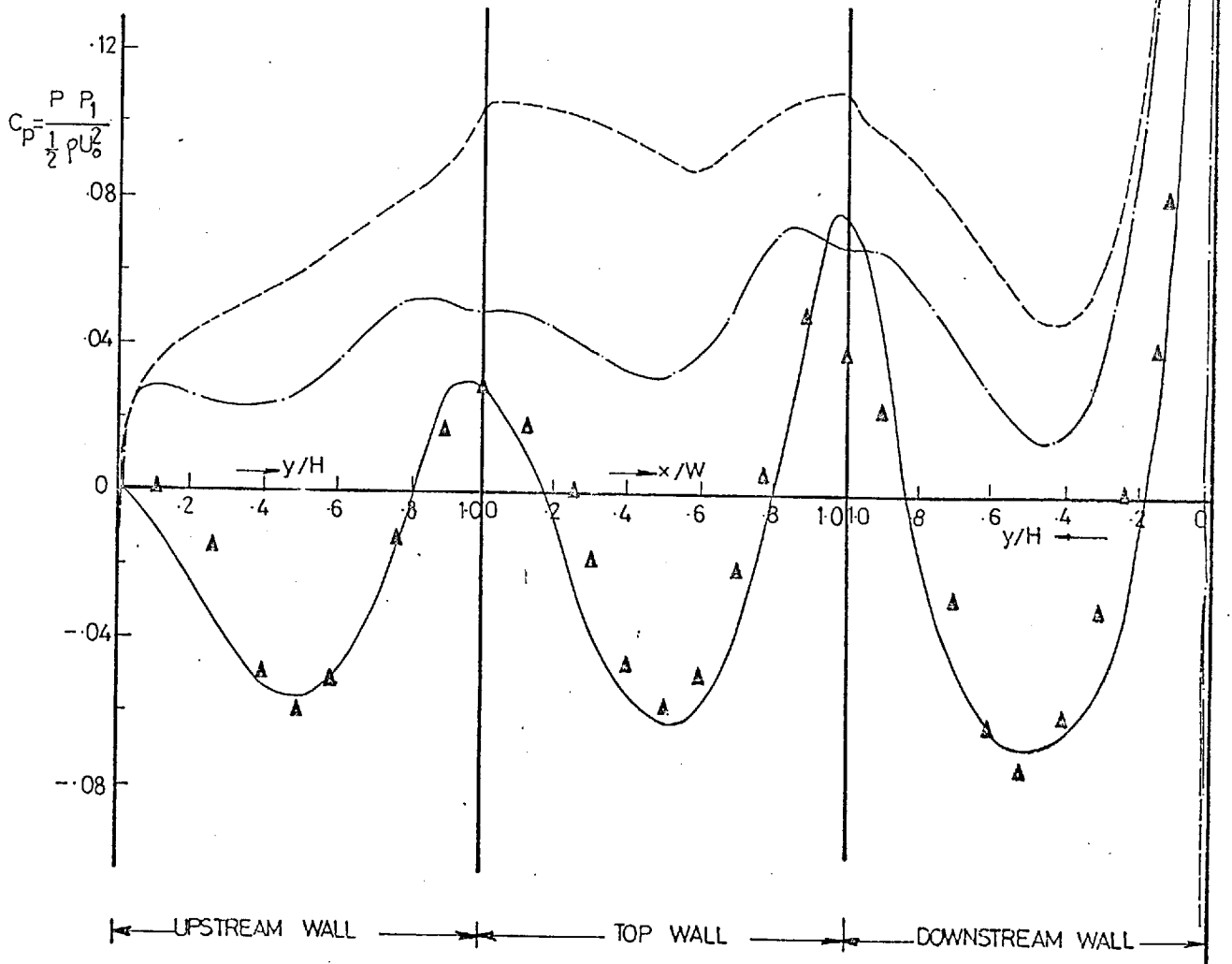


Fig. 6.19 Pressure distribution on cavity walls

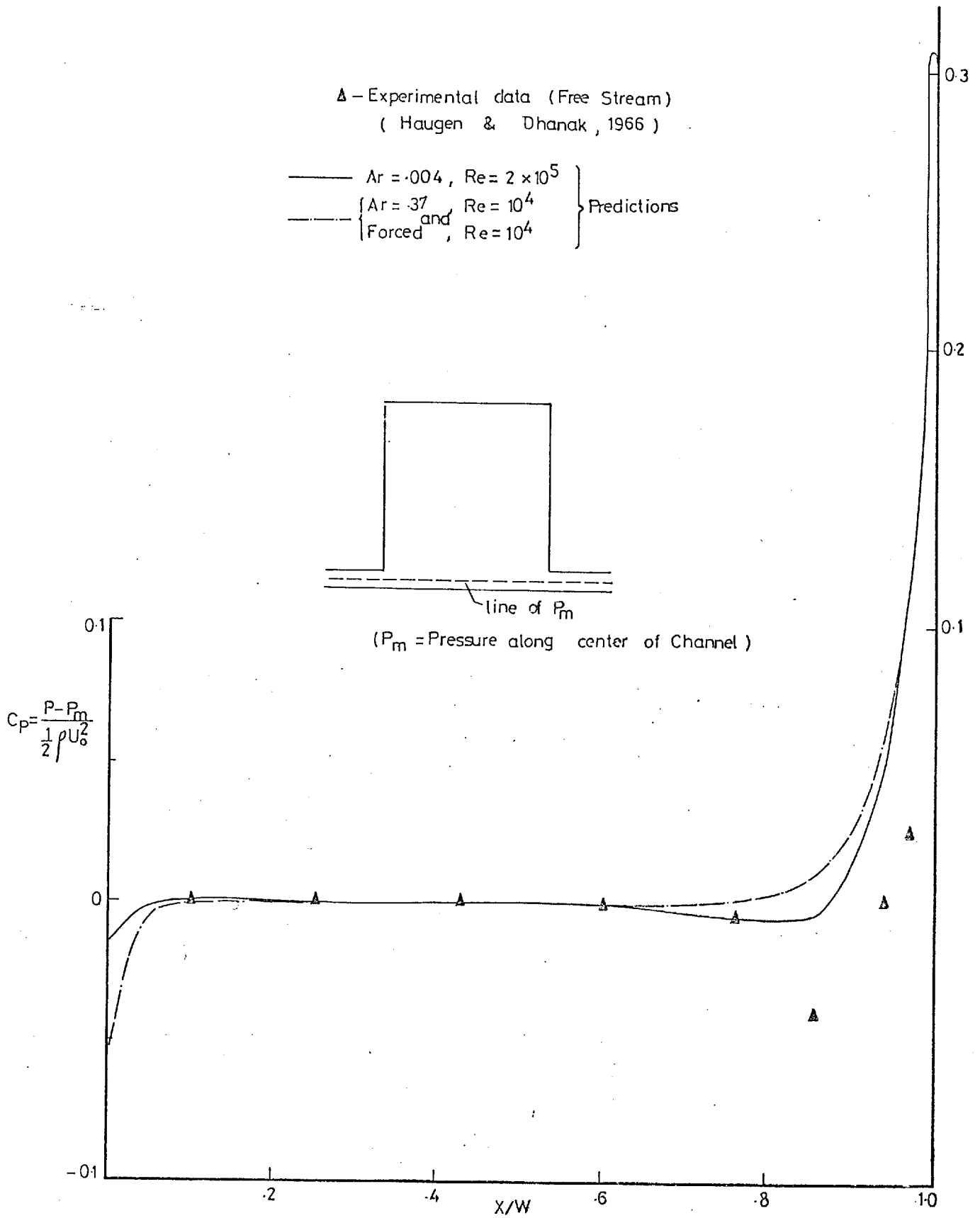


Fig. 6.20 Longitudinal Pressure Distribution in Shear layer ($y = 0$)

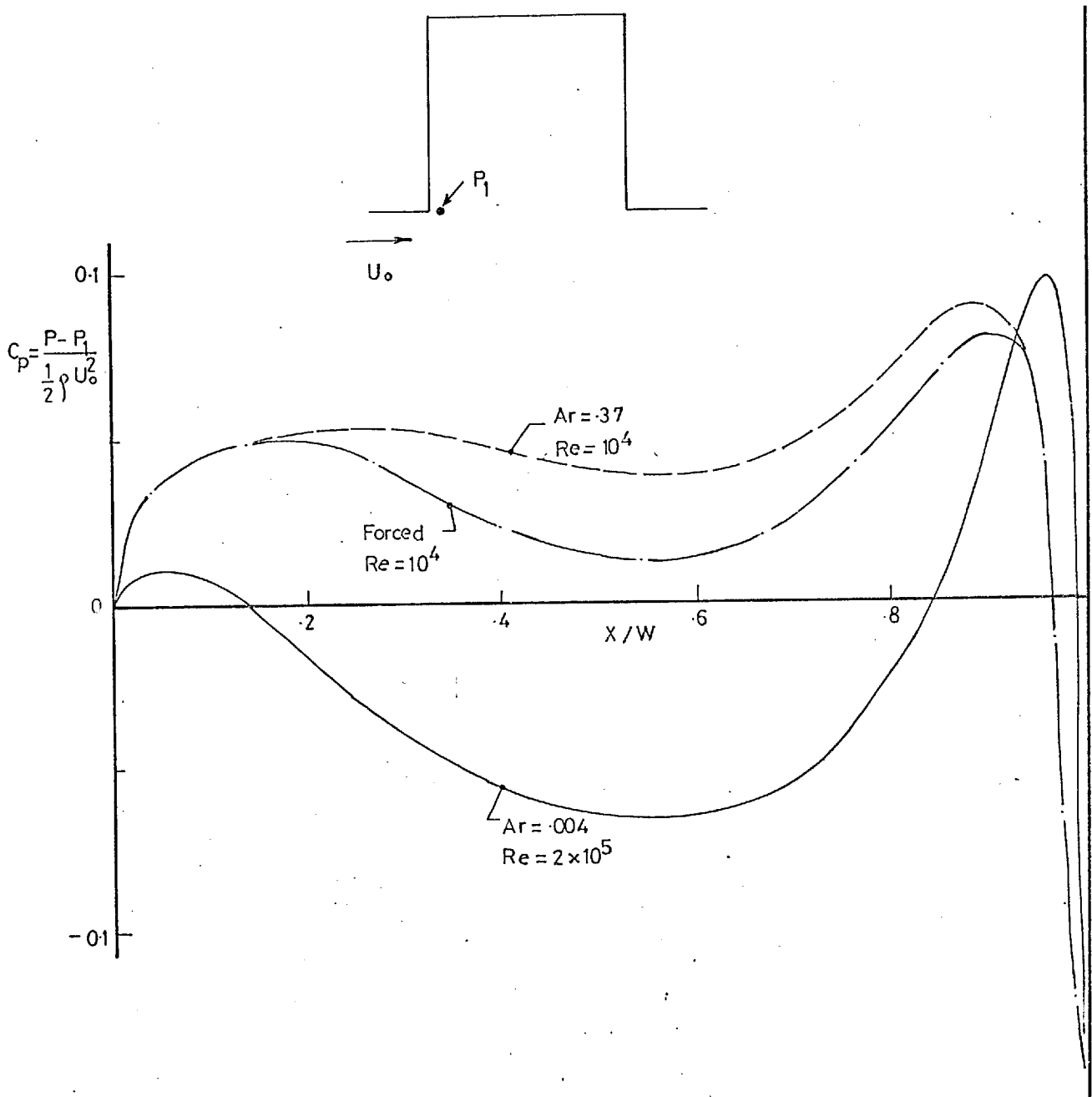


Fig. 6.21 Predicted Longitudinal Pressure Distribution in Shear Layer ($y = 0$)

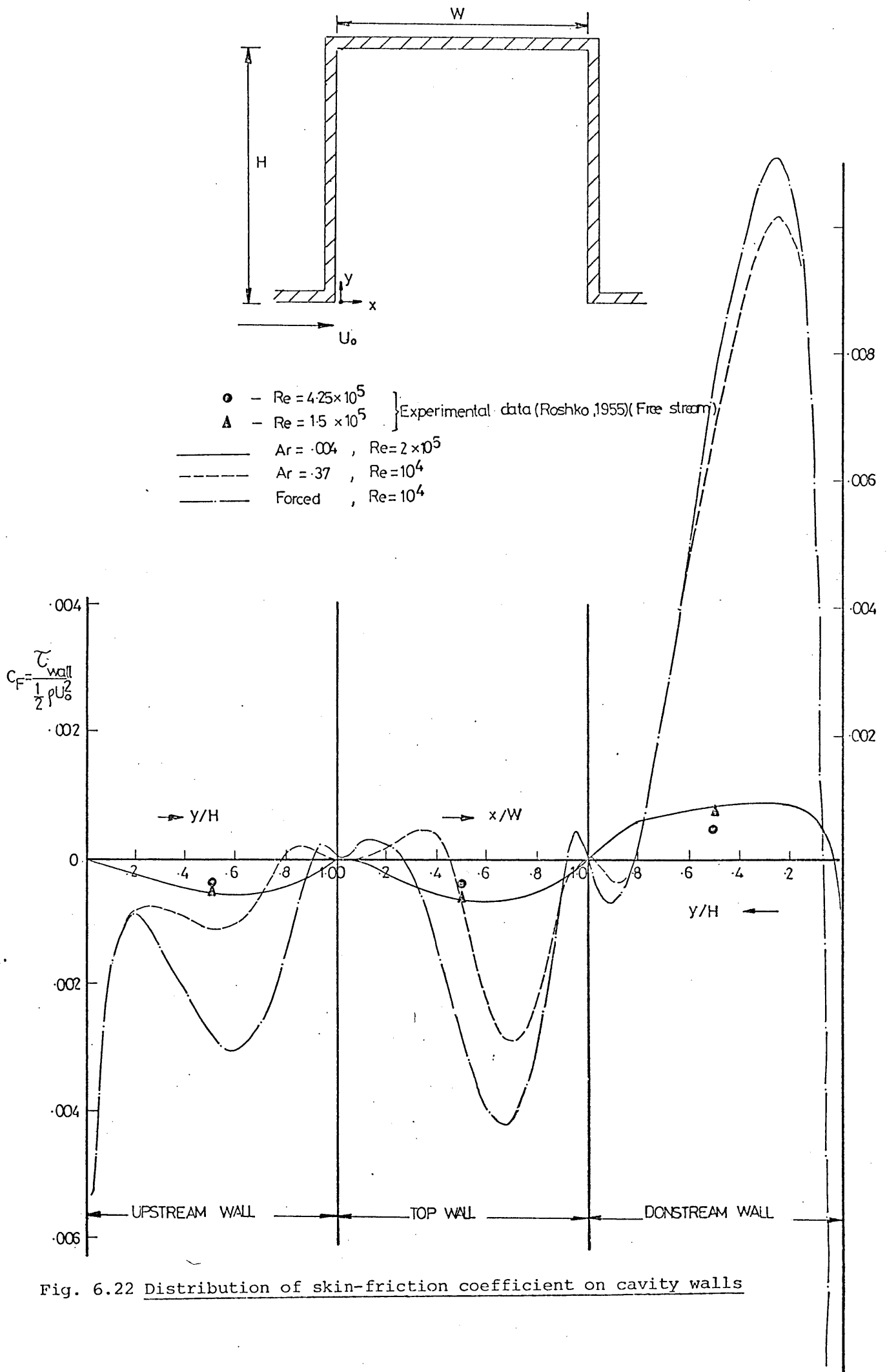


Fig. 6.22 Distribution of skin-friction coefficient on cavity walls

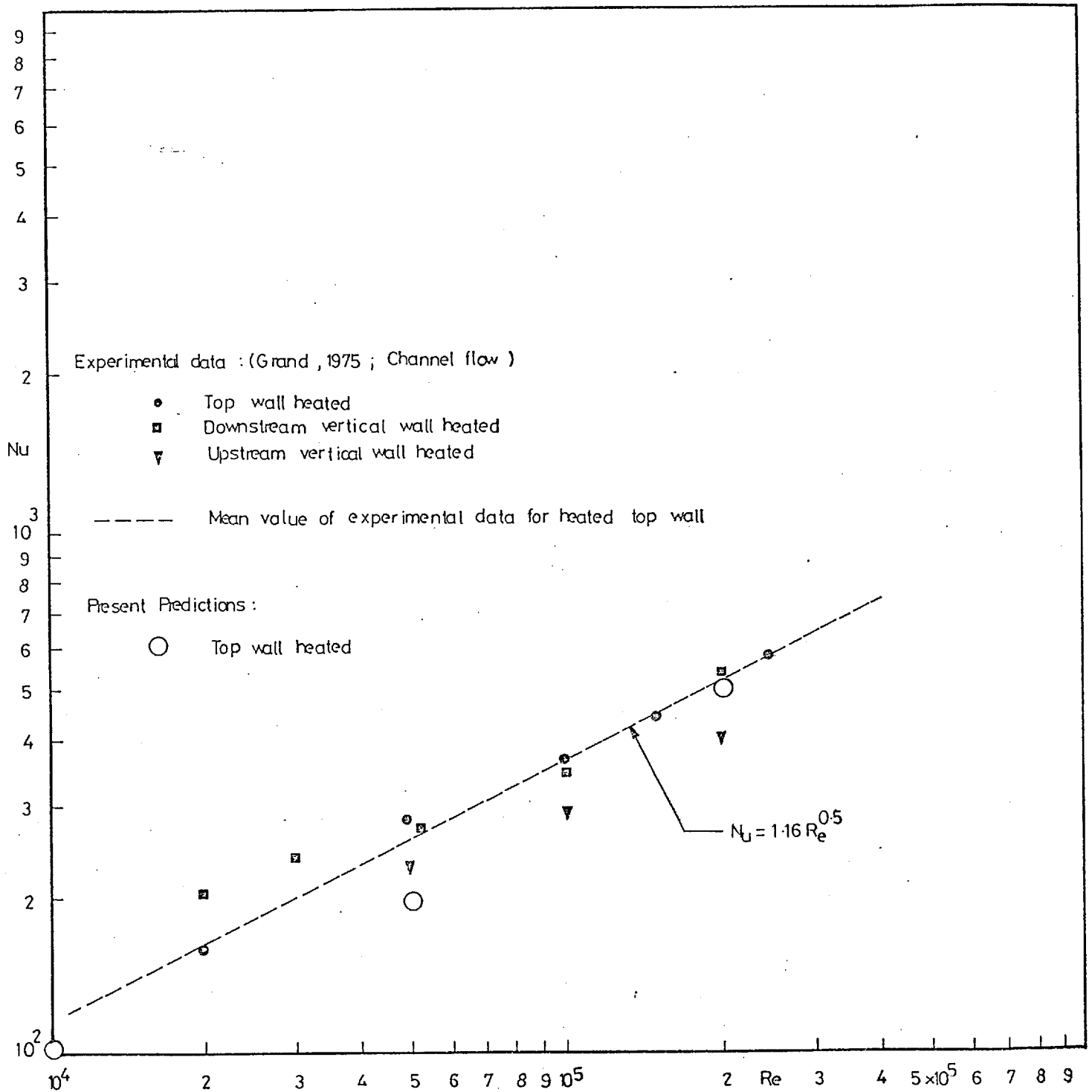


Fig. 6.23 Variation of Nusselt Number with Reynolds Number for Ar $Pr = 0.67$

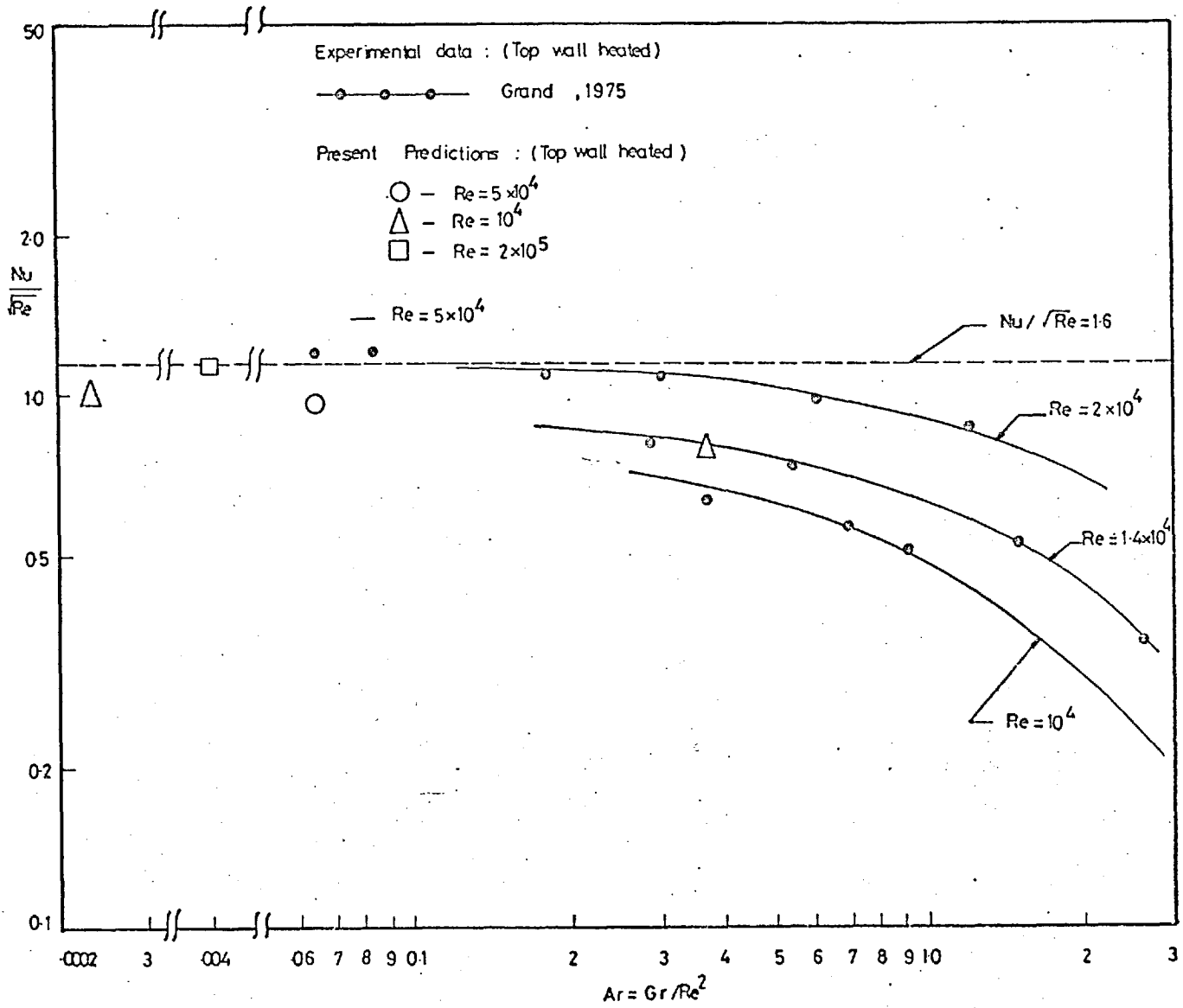


Fig. 6.24 Influence of buoyancy on heat transfer across cavity with top wall heated

- Δ - Experimental data (Grand, 1975) - Channel flow, $Re = 2 \times 10^5$
- $\circ$ - Experimental data (Mills, 1961) - Free stream
- Prediction - Channel flow, $Re = 2 \times 10^5$
- Prediction - Sliding wall, $Re = 2 \times 10^5$

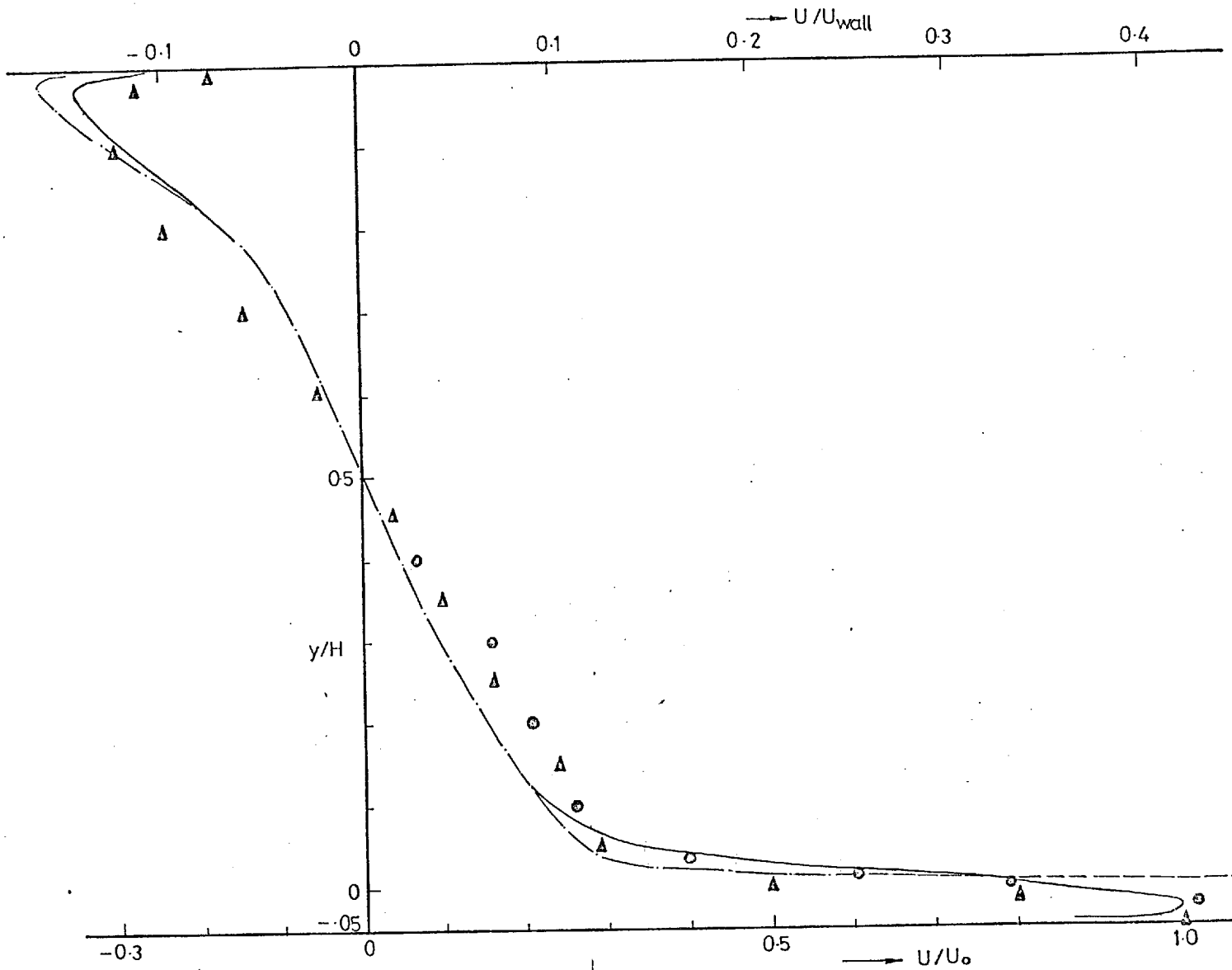


Fig. 6.25 Distribution of Streamwise component of mean velocity along vertical axis ($x/W = 0.5$)

- — Experimental data (Grand, 1975) — Channel flow, $Re = 2 \times 10^5$
- — Experimental data (Girard and Curtet, 1975) — Sliding wall, $Re = 2 \times 10^5$
- △ — Experimental data (Mills, 1961) — Free stream

- — — Prediction — Sliding wall, $Re = 2 \times 10^5$
- — — Prediction — Channel flow, $Re = 2 \times 10^5$

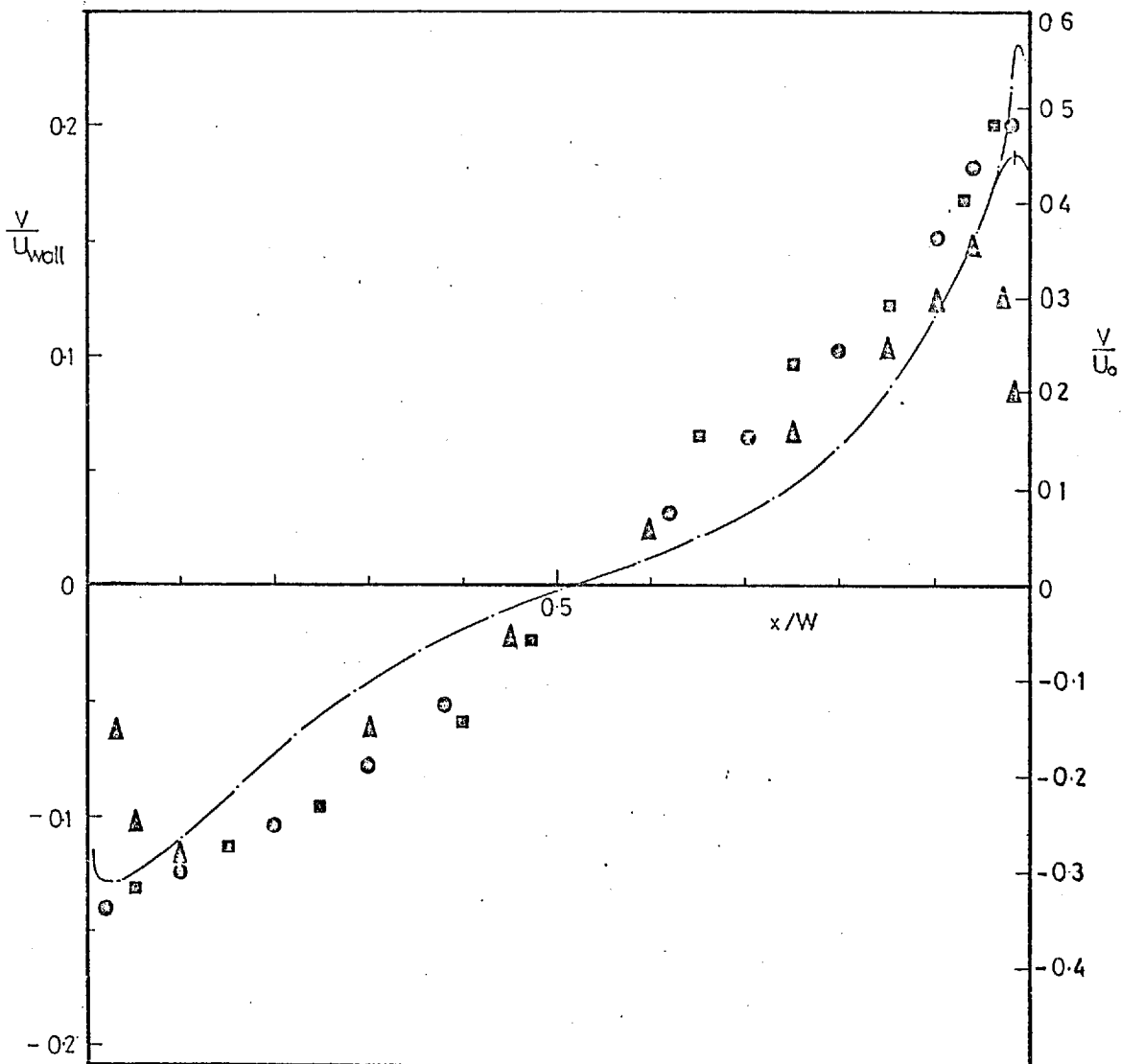
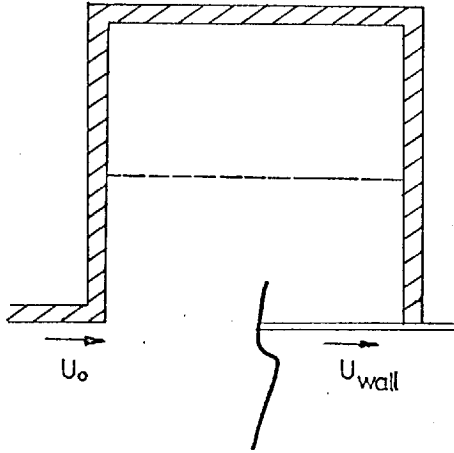


Fig. 6.26 Distribution of vertical component of mean velocity ($y/H = 0.5$)

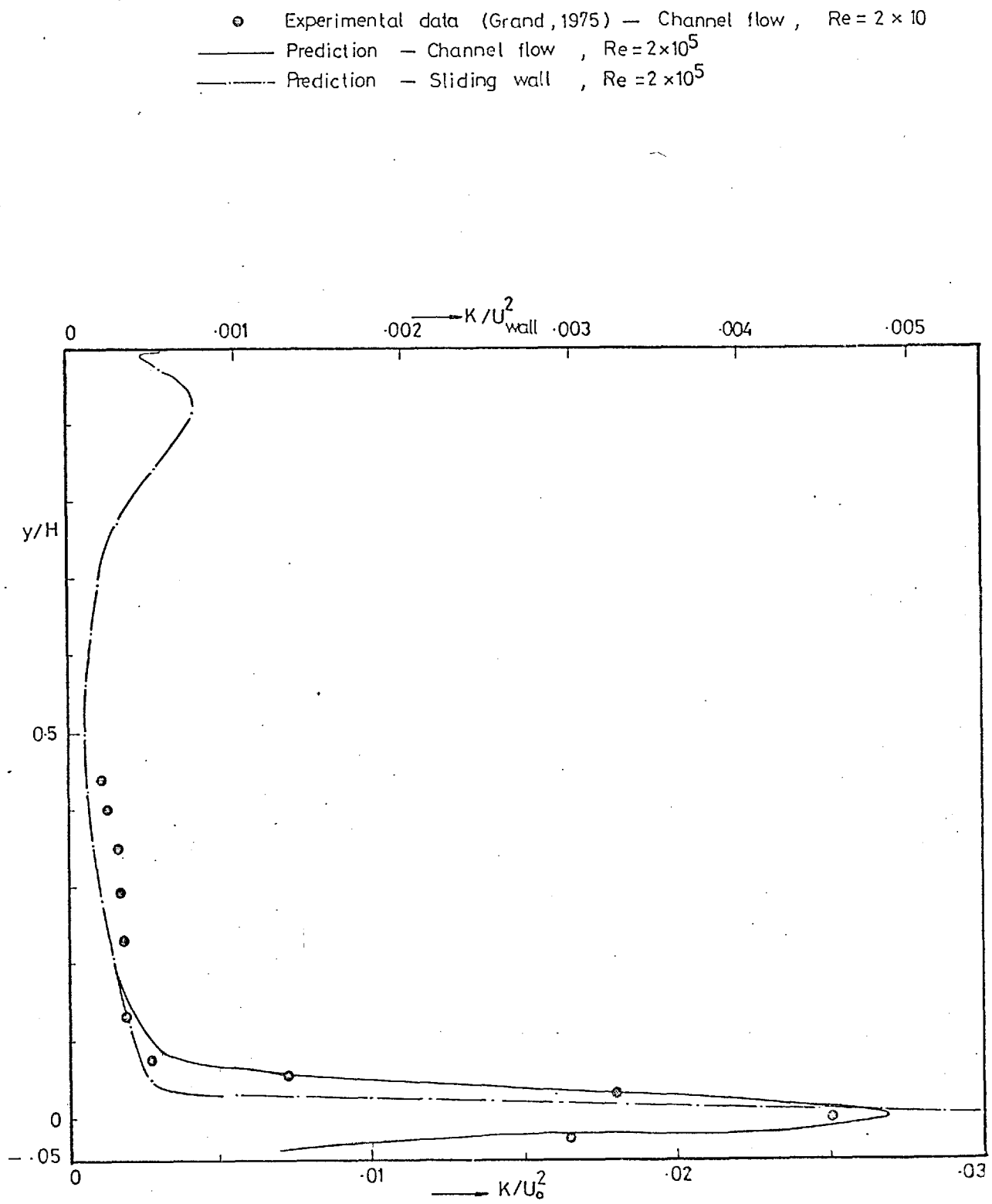


Fig. 6.27 Distribution of Turbulence Energy along vertical axis ($x/W = 0.5$)

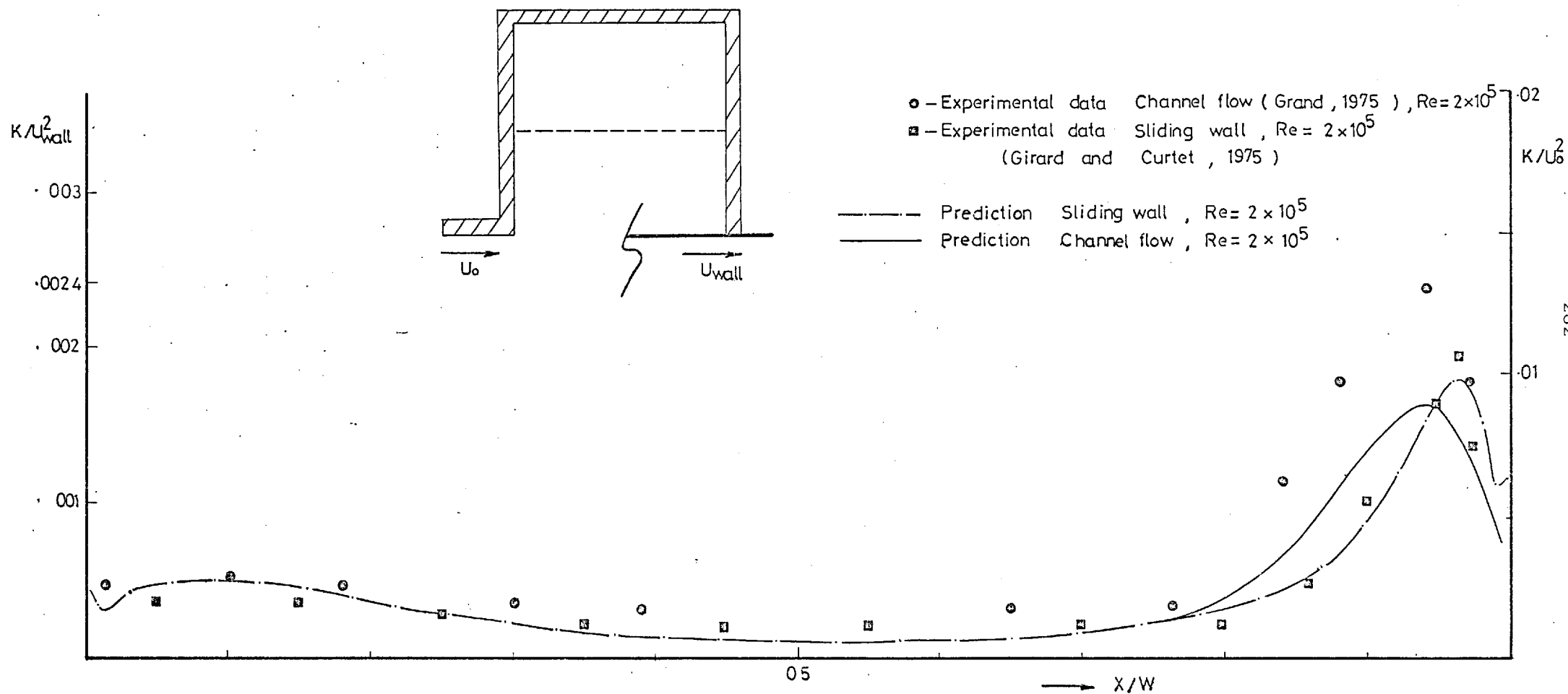


Fig. 6.28 Distribution of Turbulence Energy along horizontal axis ($y/H = 0.5$)

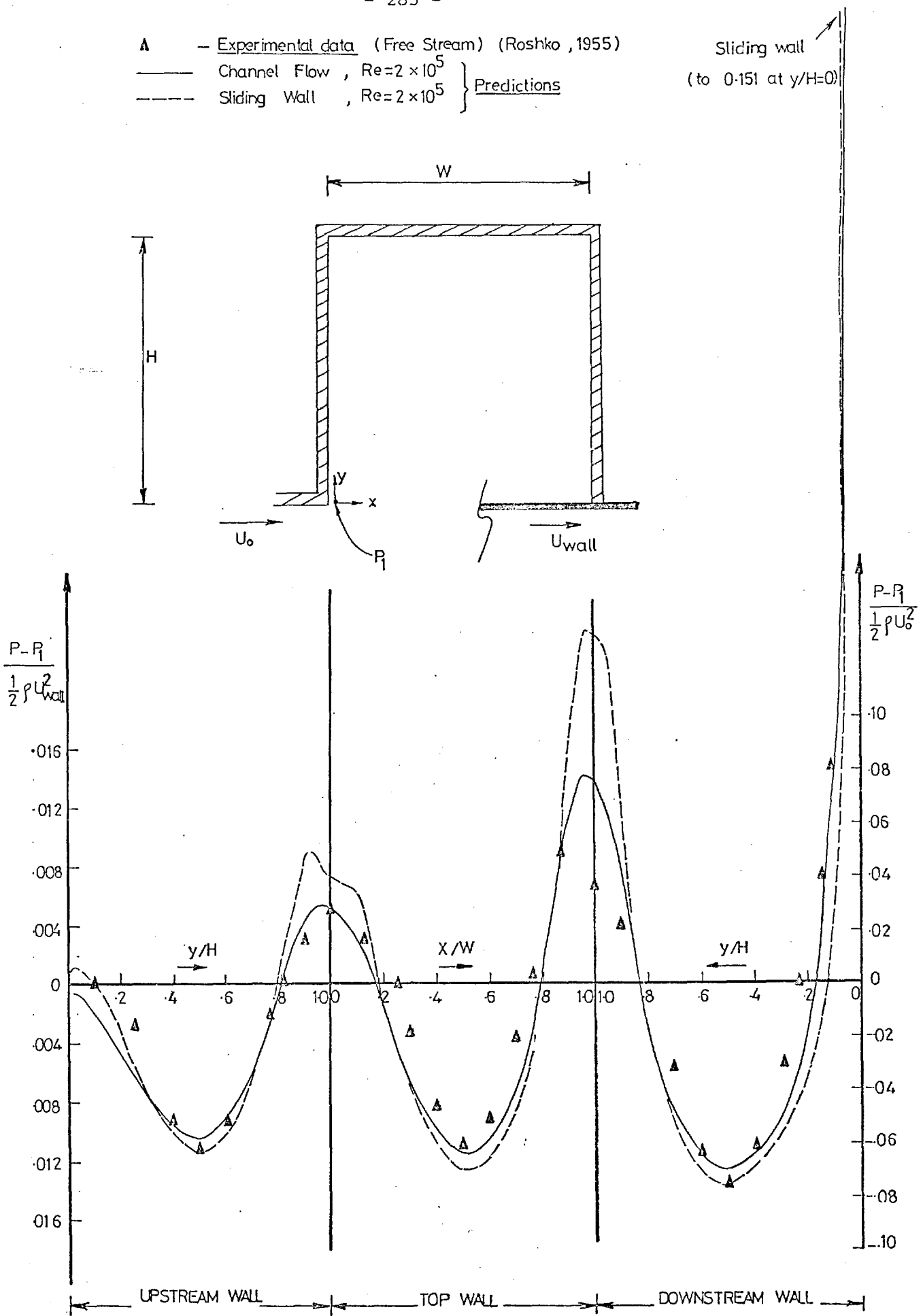


Fig. 6.29 Pressure distribution on cavity walls

$\odot$ - $Re = 4.25 \times 10^5$ } Experimental data (Free Stream)
 Δ - $Re = 1.5 \times 10^5$ } (Roshko, 1955)

— Channel flow, $Re = 2 \times 10^5$ } Predictions
 - - - Sliding wall, $Re = 2 \times 10^5$ }

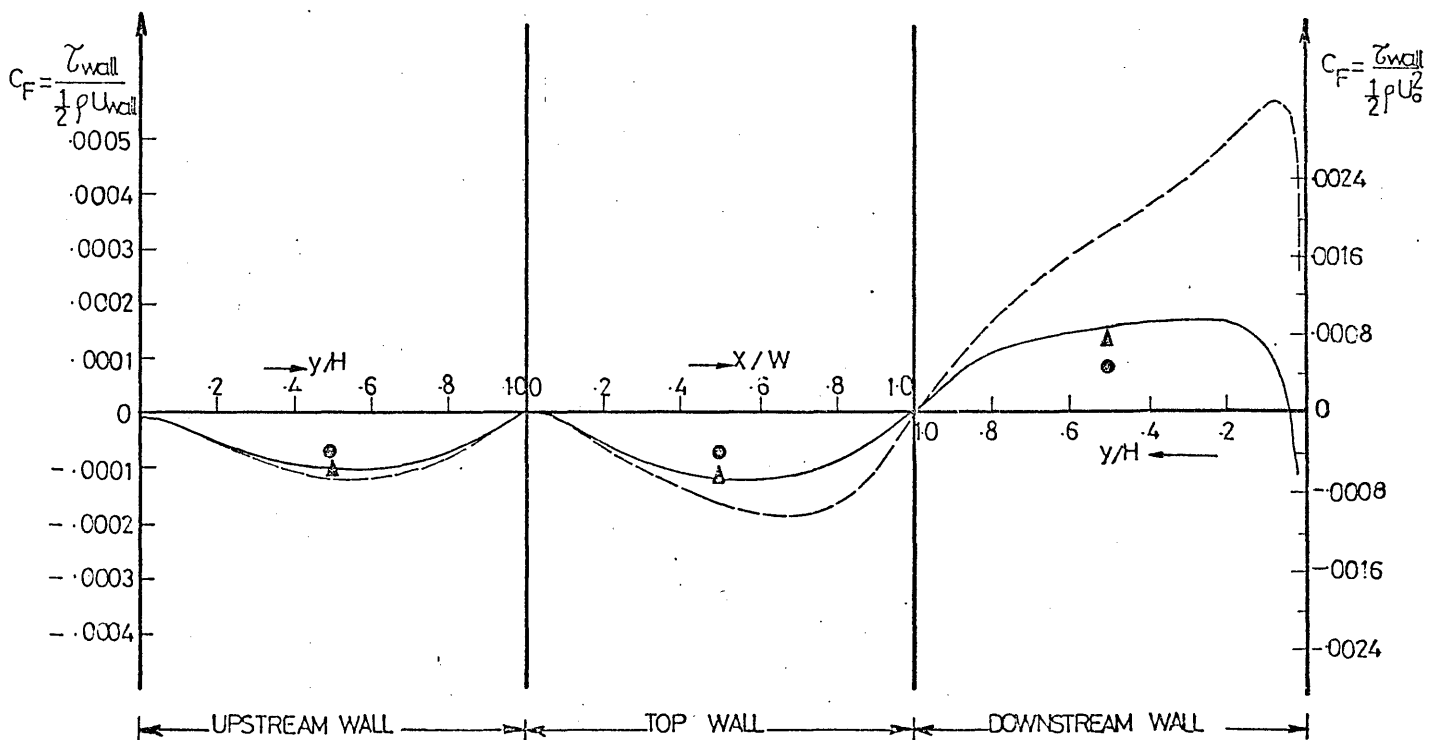
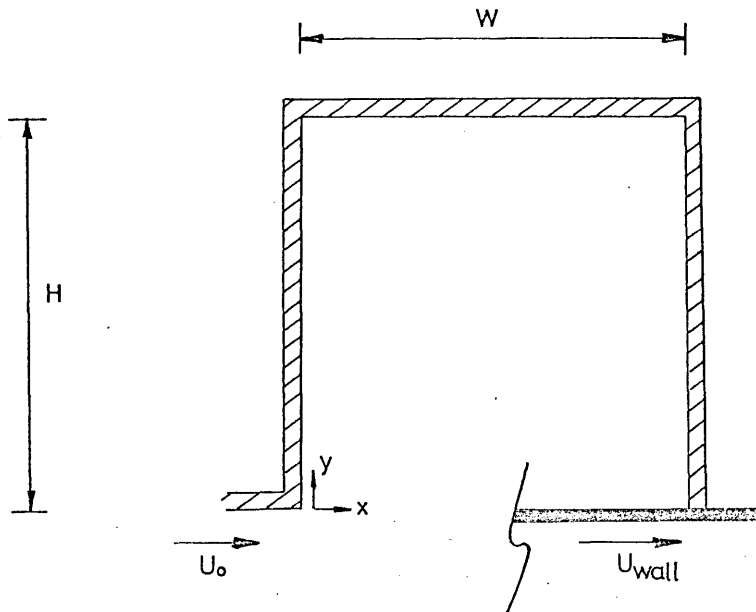


Fig. 6.30 Distribution of skin-friction coefficient on cavity walls

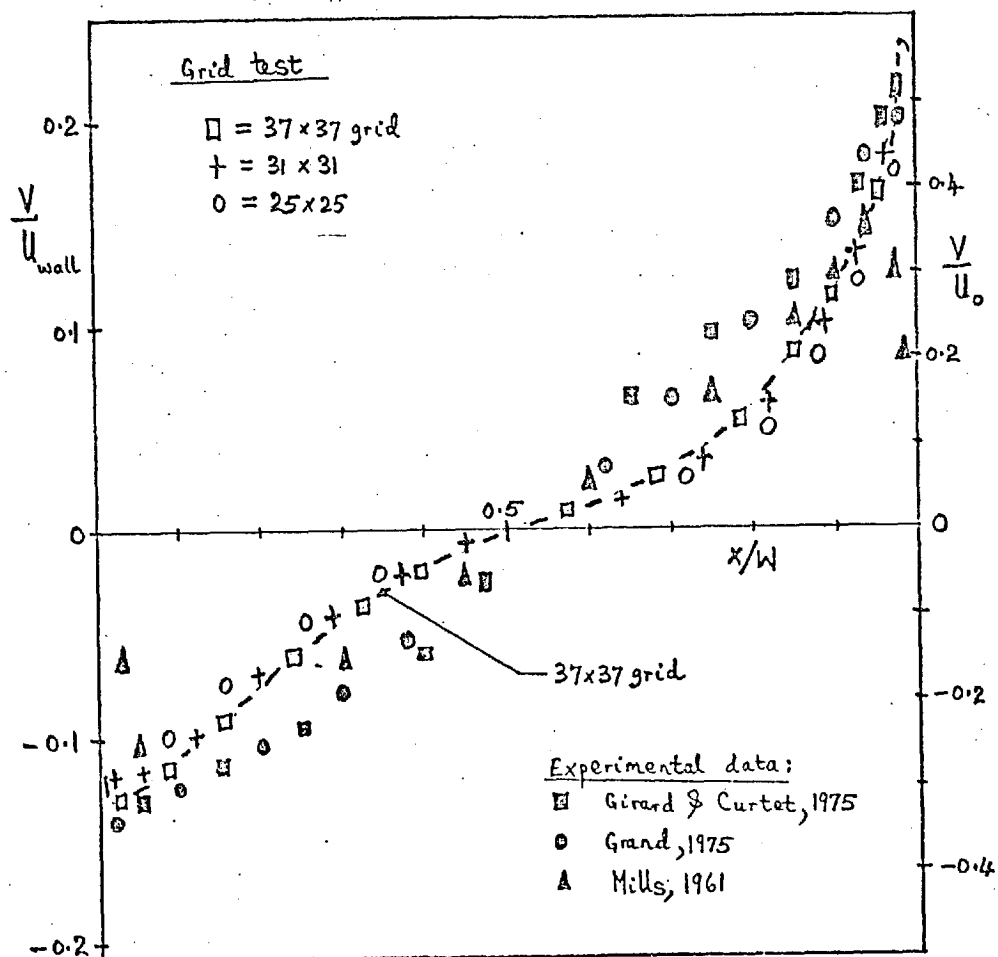
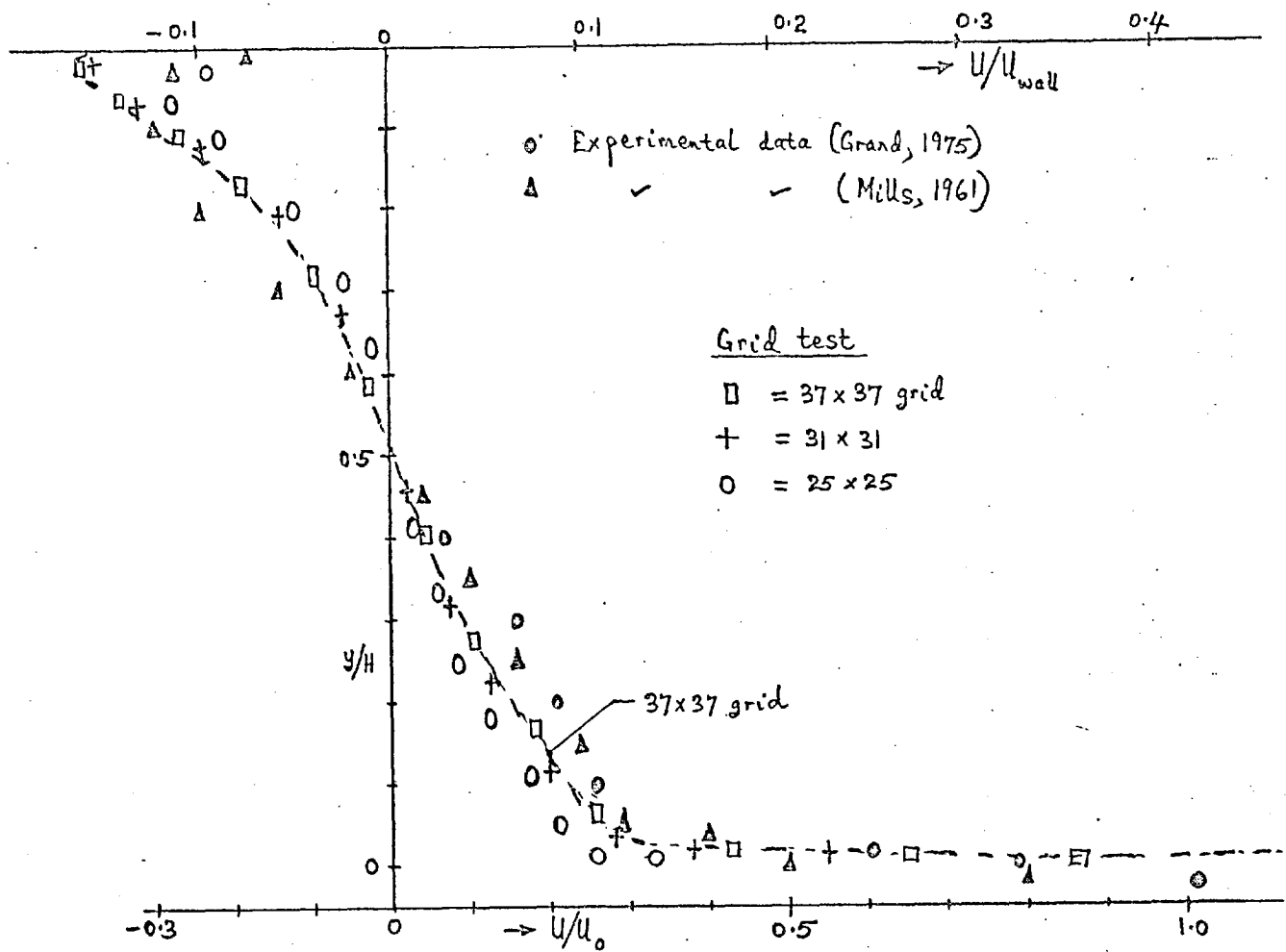


Fig. 6.31 Wall driven cavity: Grid test

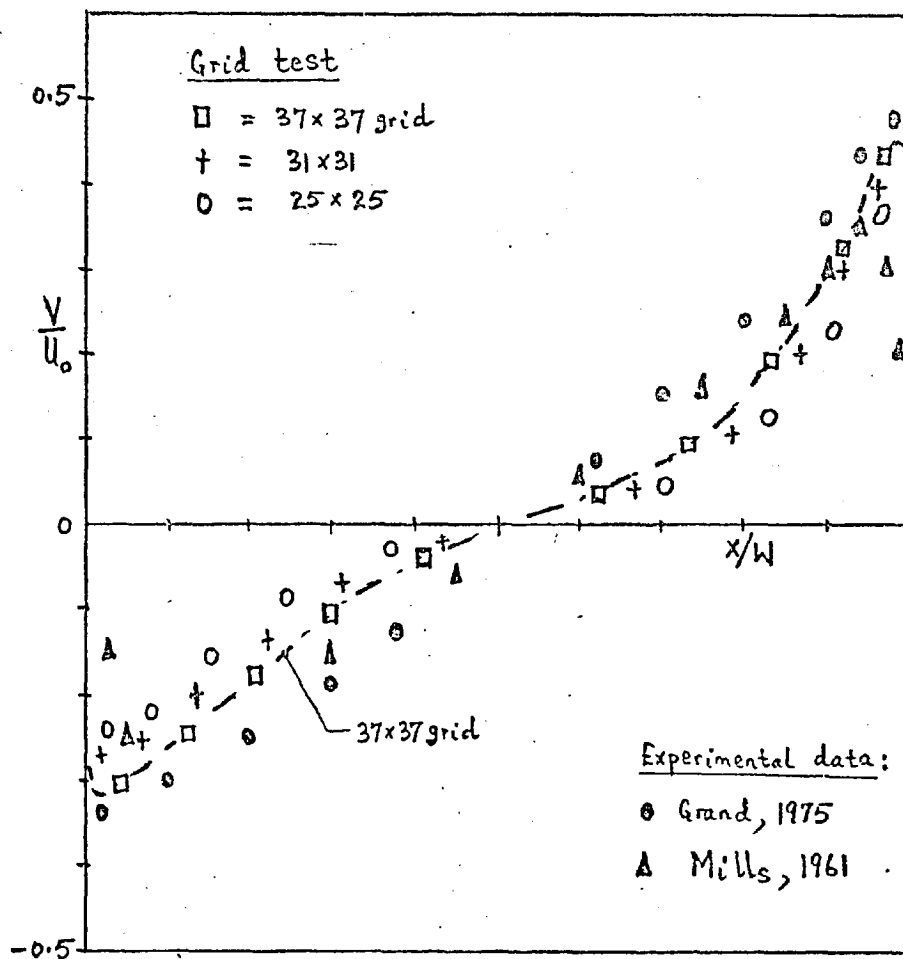
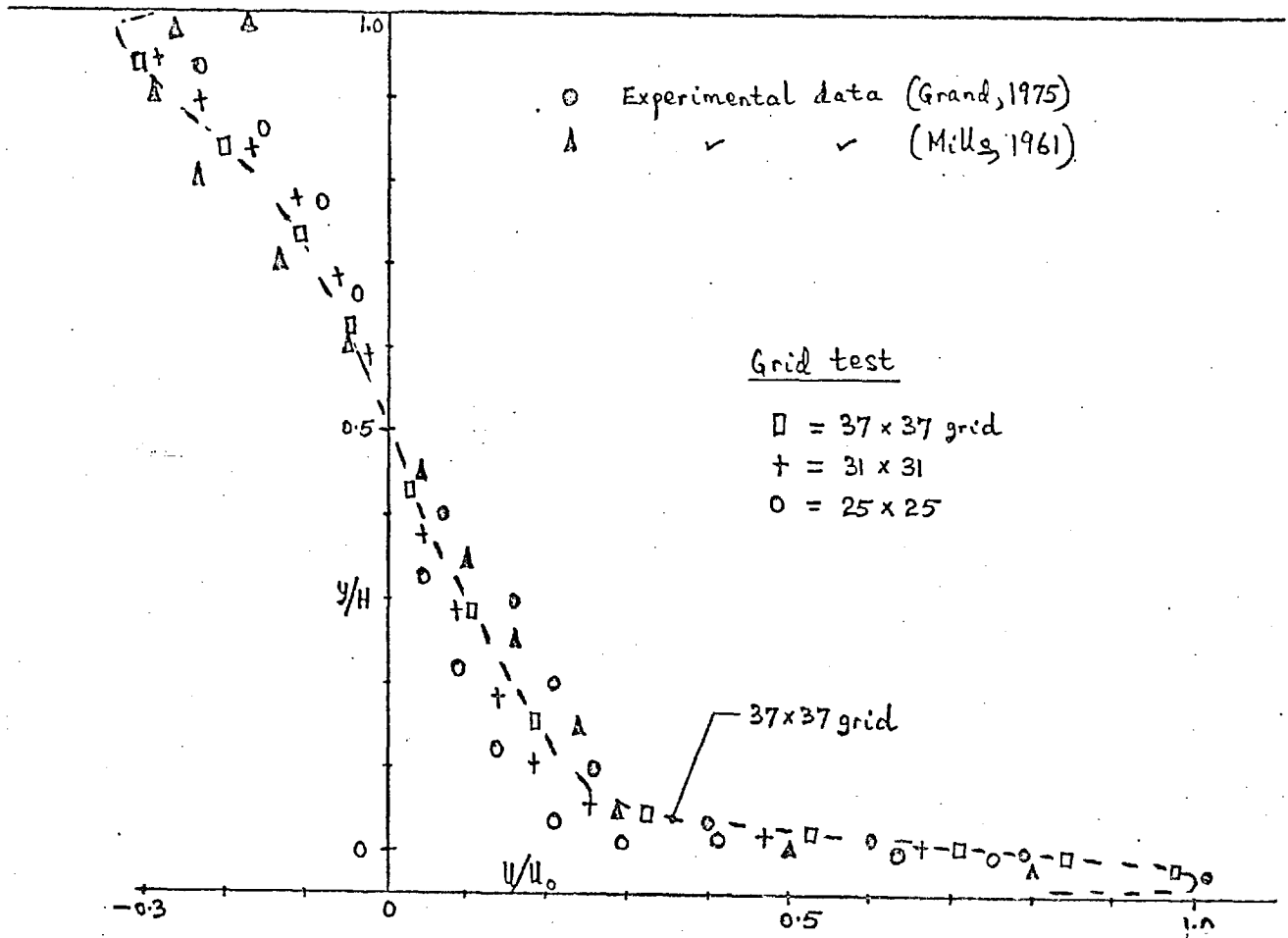


Fig. 6.32 Flow driven cavity: Grid test

APPENDIX A

EQUATIONS OF TURBULENT NATURAL CONVECTION, AND THEIR MODELLING

A.1 INTRODUCTION

The purpose of this Appendix is to indicate various ways in which, for turbulent natural convection, a closure may be effected of the time-averaged momentum and energy equations, through expressions for the Reynolds stresses $(-\overline{\rho u_i u_j})$ and energy flux $(-\overline{\rho u_j h})$.

In section A.2 are given the exact equations governing the transport of these turbulent fluxes. Then, in Section A.3 are presented the modelled forms of the equations. Finally, various versions of the modelled equations are discussed in Section A.4.

A.2 EXACT EQUATIONS GOVERNING THE TRANSPORT OF TURBULENT MOMENTUM AND ENERGY FLUXES

(a) Primary Equations

The primary unknown turbulent fluxes in the time-averaged equations of the mean motion (i.e. momentum and energy) are the Reynolds stresses $-\overline{\rho u_i u_j}$ and the energy flux $-\overline{\rho u_j h}$ (see equations 2.14 and 2.15 of Chapter 2). The turbulent energy flux $-\overline{\rho u_j h}$ can also be determined through the consideration of the fluxes $-\overline{\rho u_j T'}$ and $-\overline{\rho u_j \theta'}$, where T' and θ' are respectively the fluctuating absolute temperature and excess temperature. The fluctuating excess temperature form will be used here as buoyancy effects in a turbulence field depend, in general, not on the fluctuating absolute temperature but on the fluctuating excess temperature.

In line with the treatment given in this study, the exact equations for the turbulent fluxes will be derived for incompressible flows.

The equation for $\overline{u_i u_j}$

The starting point for the derivation of the equation for the

kinematic stress $\overline{u_i u_j}$ is the equation for the fluctuating velocity u_i , which may be derived by subtracting the equation for the time-averaged velocity U_i from the Navier-Stokes equation for the instantaneous velocity $\hat{U}_i (=U_i + u_i$; see also Chapter 2), to give:

$$\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} = -u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial}{\partial x_j} (\overline{u_i u_j} - \overline{u_i u_j}) - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + \beta \theta' g_i \quad (\text{A.1})$$

where the buoyancy term has been expressed in terms of temperature differences, and β is the coefficient of volumetric expansion.

Multiplying equation (A.1) by u_j , then time-averaging, and finally adding to the result an exactly identical equation except that i and j are inter-changed, gives the following equation for $\overline{u_i u_j}$ after some algebraic manipulation (see also Launder, 1975)

$$\begin{aligned} \underbrace{\frac{\partial \overline{u_i u_j}}{\partial t}}_{\text{rate of change}} + \underbrace{U_k \frac{\partial \overline{u_i u_j}}{\partial x_k}}_{\text{convection}} = & \underbrace{-\left\{ \overline{u_i u} \frac{\partial U_j}{\partial x_k} + \overline{u_j u} \frac{\partial U_i}{\partial x_k} \right\}}_{\text{generation by shear}} + \underbrace{\beta \left\{ \overline{u_i \theta' g_i} + \overline{u_j \theta' g_j} \right\}}_{\text{generation/destruction by buoyancy forces}} \\ & \underbrace{-2\nu \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_j}}{\partial x_k}}_{\text{viscous dissipation}} + \underbrace{\frac{p}{\rho} \left(\frac{\partial \overline{u_i}}{\partial x_k} + \frac{\partial \overline{u_j}}{\partial x_i} \right)}_{\text{pressure strain}} \\ & - \underbrace{\frac{\partial}{\partial x_k} \left\{ \overline{u_i u_j u_k} + \frac{p}{\rho} (\delta_{jk} u_i + \delta_{ik} u_j) - \nu \frac{\partial \overline{u_i u_j}}{\partial x_k} \right\}}_{\text{diffusive transport}} \end{aligned} \quad (\text{A.2})$$

where the labels indicate the physical significance of the various terms.

The equation for $\overline{u_i \theta'}$

The equation for $\overline{u_i \theta'}$ (or $\overline{u_j \theta'}$) can be obtained by starting from the equation for fluctuating excess temperature θ' , which may be derived by subtracting the time-averaged equation for the mean value $\bar{\theta}$ from the equation for instantaneous value $\hat{\theta} (= \bar{\theta} + \theta'$; see also Chapter 2), yielding:

$$\frac{\partial \theta'}{\partial t} + U_j \frac{\partial \theta'}{\partial x_j} = -u_j \frac{\partial \bar{\theta}}{\partial x_j} - \frac{\partial}{\partial x_j} (\overline{u_j \theta'} - \overline{u_j \theta'}) + \Gamma \frac{\partial^2 \theta'}{\partial x_j^2} \quad (\text{A.3})$$

Multiplying equation (A.3) by u_i , then time-averaging, and finally adding to the resulting equation another time-averaged equation obtained from θ' times equation (A.1), gives after manipulation, the following equation for $\overline{u_i \theta'}$ (see also Launder, 1975):

$$\begin{aligned}
 \underbrace{\frac{\partial}{\partial t} \overline{u_i \theta'}}_{\text{rate of change}} + \underbrace{u_j \frac{\partial \overline{u_i \theta'}}{\partial x_j}}_{\text{convection}} = & \underbrace{-\{\overline{u_i u_j} \frac{\partial \theta}{\partial x_j} + \overline{u_j \theta'} \frac{\partial u_i}{\partial x_j}\}}_{\text{generation by mean field}} + \underbrace{\beta \overline{\theta'^2} g_i}_{\text{generation/destruction by buoyancy forces}} \\
 & - \underbrace{(\Gamma + \nu) \frac{\partial \theta'}{\partial x_j} \frac{\partial u_i}{\partial x_j}}_{\text{dissipation by molecular action}} + \underbrace{\frac{p}{\rho} \frac{\partial \theta'}{\partial x_i}}_{\text{pressure scrambling}} \\
 & - \underbrace{\frac{\partial}{\partial x_j} \{ \overline{u_i u_j \theta'} + \delta_{ij} \frac{p \theta'}{\rho} + \Gamma u_i \frac{\partial \theta'}{\partial x_j} + \nu \theta' \frac{\partial u_i}{\partial x_j} \}}_{\text{diffusive transport}} \quad (A.4)
 \end{aligned}$$

The equation for $\overline{\theta'^2}$

The buoyancy term of equation (A.4) contains the unknown double correlation for fluctuating excess temperature, $\overline{\theta'^2}$. Since $\overline{\theta'^2}$ is of the same second order as both $\overline{u_i u_j}$ and $\overline{u_i \theta'}$, its equation is also necessary in a second-order closure. The equation for $\overline{\theta'^2}$ may be obtained by multiplying equation (A.3) by $2\theta'$, time-averaging and then manipulating to give (see also Tennekes and Lumley, 1972):

$$\begin{aligned}
 \underbrace{\frac{\partial \overline{\theta'^2}}{\partial t}}_{\text{rate of change}} + \underbrace{u_j \frac{\partial \overline{\theta'^2}}{\partial x_j}}_{\text{convection}} = & \underbrace{-2 \overline{u_j \theta'} \frac{\partial \theta}{\partial x_j}}_{\text{generation by mean motion}} - \underbrace{2 \Gamma \frac{\partial \theta'}{\partial x_j} \frac{\partial \theta'}{\partial x_j}}_{\text{dissipation}} \\
 & - \underbrace{\frac{\partial}{\partial x_j} \{ \overline{u_j \theta'^2} + \Gamma \frac{\partial \theta'^2}{\partial x_j} \}}_{\text{diffusive transport}} \quad (A.5)
 \end{aligned}$$

Equations (A.2), (A.4), and (A.5) contain further unknowns, namely the triple correlations $\overline{u_i u_j u_k}$, $\overline{u_i u_j \theta'}$, and $\overline{u_j \theta'^2}$. Equations for these correlations can be derived, but they again contain higher order correlations. At some stage one needs to obtain closure by making approximations. Here only equations of second-order (i.e. those of double correlation)

correlations) will be dealt with.

(b) Additional Equations

An additional equation for the length scale of turbulence is often required. And this may be obtained from the solution of an additional equation for the turbulence dissipation rate ϵ . The exact ϵ -equation therefore will also be given below. In addition, although the k -equation is contained in the general equation for $\overline{u_i u_j}$, its exact form will be presented below as it is of special interest in this study.

The equation for k

Multiplying equation (A.1) by u_i and then time-averaging gives, after manipulation, the following equation for $k (= \frac{1}{2} \overline{u_i^2})$:

$$\underbrace{\frac{\partial k}{\partial t}}_{\text{rate of change}} + \underbrace{u_j \frac{\partial k}{\partial x_j}}_{\text{convection}} = \underbrace{- \overline{u_i u_j} \frac{\partial u_i}{\partial x_j}}_{\text{generation by shear}} - \underbrace{\epsilon}_{\text{viscous dissipation}} + \underbrace{\beta \overline{u_i \theta'} g_i}_{\text{generation/destruction by buoyancy forces}}$$

$$- \underbrace{\frac{\partial}{\partial x_j} \left\{ \overline{u_j k'} + \frac{p u_j}{\rho} - \nu \left(\frac{\partial k}{\partial x_j} + \frac{\partial \overline{u_i u_j}}{\partial x_i} \right) \right\}}_{\text{diffusive transport}} \quad (A.6)$$

where $k' (= u_i^2/2)$ is the instantaneous value of the turbulence energy.

The equation for ϵ

The equation for $\epsilon (= \nu \overline{(\frac{\partial u_i}{\partial x_j})^2})$ may be obtained by differentiating equation (A.1) with respect to x_1 , multiplying by $2\nu \frac{\partial u_1}{\partial x_1}$, and then time averaging. After re-arranging the various terms, the following equation for ϵ results:

$$\underbrace{\frac{\partial \epsilon}{\partial t}}_{\text{rate of change}} + \underbrace{u_j \frac{\partial \epsilon}{\partial x_j}}_{\text{convection}} = \underbrace{-2\nu \left(\frac{\partial \overline{u_1}}{\partial x_1} \frac{\partial \overline{u_j}}{\partial x_1} + \frac{\partial \overline{u_1}}{\partial x_i} \frac{\partial \overline{u_j}}{\partial x_j} \right) \frac{\partial u_i}{\partial x_j}}_{\text{G1: generation by mean flow: production due to vortex stretching}} - \underbrace{2\nu \overline{u_j \frac{\partial u_1}{\partial x_1} \frac{\partial^2 u_i}{\partial x_j \partial x_1}}}_{\text{G2: secondary generation}}$$

$$- \underbrace{2\nu \overline{\frac{\partial u_1}{\partial x_1} \frac{\partial u_j}{\partial x_1} \frac{\partial u_i}{\partial x_j}}}_{\text{G3: generation due to "vortex stretching" by turbulence}} - \underbrace{2 \left(\nu \overline{\frac{\partial^2 u_i}{\partial x_j \partial x_1}} \right)^2}_{\text{D: viscous dissipation}} + \underbrace{2\nu \beta g_i \overline{\frac{\partial u_1}{\partial x_1} \frac{\partial \theta'}{\partial x_1}}}_{\text{generation/destruction by buoyancy forces}}$$

$$\underbrace{-\frac{\partial}{\partial x_j} \left\{ \overline{u_j \epsilon'} + \frac{\nu}{\rho} \frac{\partial p}{\partial x_1} \frac{\partial \overline{u_1}}{\partial x_1} - \nu \frac{\partial \epsilon}{\partial x_j} \right\}}_{\text{diffusive transport}} \quad (\text{A.7})$$

where $\epsilon' = \nu \left(\frac{\partial u_1}{\partial x_j} \right)^2$ is the instantaneous value of the dissipation rate. The non-buoyant form of equation (A.7) has also been derived by Tennekes and Lumley (1973) and others. The buoyant form was derived in the present study.

A.3 MODELLING THE EQUATIONS FOR THE TRANSPORT OF TURBULENT MOMENTUM AND ENERGY FLUXES

The exact equations of section A.2 cannot be used as they stand, for reasons which were explained earlier: approximations are therefore often made of various unknown terms in order to effect closure. The resultant forms are called the "modelled" equations. Modelling has been performed by various workers, e.g. Rotta (1951), Daly and Harlow (1970), Launder (1973, 1975), Launder et al (1975), Lumley (1972), Lumley and Khajeh-Nouri (1974a and b). In this section the models of Launder will be presented because they are among the few that include buoyancy.

At second-order level of closure, only the pressure-strain/scrambling, dissipation and diffusion terms require modelling. Detailed treatment has been given by Launder (1973, 1975) and Launder et al (1975); thus, here only a very brief indication will be given of their practice.

(a) Primary Equations

The equation for $\overline{u_i u_j}$

The convection, generation, and buoyancy terms in equation (A.2) are treated as exact. Of major interest is the pressure strain term which represents redistribution due to pressure fluctuations. This redistribution is controlled by three agencies:

(i) pure turbulence interactions; modelled as

$$-c_1 \left(\frac{\epsilon}{k} \right) \overline{u_i u_j} - \frac{2}{3} \delta_{ij} k \quad (\text{A.8})$$

(ii) mean strain effects; modelled as

$$-c_2 \left[-\overline{u_i u_k} \frac{\partial u_j}{\partial x_k} + \overline{u_j u_k} \frac{\partial u_i}{\partial x_k} \right] - \frac{2}{3} \delta_{ij} \left\{ -\overline{u_i u_k} \frac{\partial u_i}{\partial x_k} \right\} \quad (\text{A.9})$$

and (iii) selective influence of buoyancy; modelled as

$$-c_3 \left[\beta \{ \overline{u_i \theta' g_j} + \overline{u_j \theta' g_i} \} - \frac{2}{3} \delta_{ij} \{ \beta \overline{u_i \theta' g_i} \} \right] \quad (\text{A.10})$$

Assuming that small-scale turbulent motions are isotropic, the viscous dissipation term in equation (A.2) is approximated as

$$-\frac{2}{3} \delta_{ij} \epsilon \quad (\text{A.11})$$

For flows of high turbulence Reynolds number, the dominant component of the term expressing diffusive transport in equation (A.2) is

$\overline{u_i u_j u_k}$ and this is modelled, after Hanjalic and Launder (1972), as

$$-\overline{u_i u_j u_k} = c \frac{k}{\epsilon} \left\{ \overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l} + \overline{u_j u_l} \frac{\partial \overline{u_k u_i}}{\partial x_l} + \overline{u_i u_l} \frac{\partial \overline{u_j u_k}}{\partial x_l} \right\} \quad (\text{A.12})$$

Substituting expressions (A.8) to (A.12) into equation (A.2) and re-arranging, the final modelled form of the equation for $\overline{u_i u_j}$ becomes:

$$\underbrace{\frac{\partial \overline{u_i u_j}}{\partial t}}_{\text{rate of change}} + \underbrace{u_k \frac{\partial \overline{u_i u_j}}{\partial x_k}}_{\text{convection}} = \underbrace{\frac{\partial}{\partial x_k} \left\{ c \frac{k}{\epsilon} \left(\overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l} + \overline{u_j u_l} \frac{\partial \overline{u_k u_i}}{\partial x_l} + \overline{u_i u_l} \frac{\partial \overline{u_j u_k}}{\partial x_l} \right) \right\}}_{\text{diffusive transport}} + \underbrace{P_{ij} + P_{ijB}}_{\text{generation}} - \underbrace{\frac{2}{3} \delta_{ij} \epsilon}_{\text{dissipation}} - \underbrace{c \frac{\epsilon}{1k} \left\{ \overline{u_i u_j} - \frac{2}{3} \delta_{ij} k \right\} - \{ c_2 P_{ij} + c_3 P_{ijB} - \frac{2}{3} \delta_{ij} (c_2 P_k + c_3 P_{kB}) \}}_{\text{pressure strain}} \quad (\text{A.13})$$

where

$$P_{ij} = -\left\{ \overline{u_i u_k} \frac{\partial u_j}{\partial x_k} + \overline{u_j u_k} \frac{\partial u_i}{\partial x_k} \right\} \quad (\text{A.14})$$

= generation of $\overline{u_i u_j}$ due to shear

$$P_{ijB} = \beta \{ \overline{u_i \theta' g_j} + \overline{u_j \theta' g_i} \} \quad (\text{A.15})$$

= generation/destruction of $\overline{u_i u_j}$ by buoyancy

$$P_k = -\overline{u_i u_j} \frac{\partial u_i}{\partial x_k} \quad (\text{A.16})$$

= generation of K due to shear

$$P_{kB} = \overline{\beta u_i \theta' g_i} \quad (A.17)$$

= generation/destruction of k by buoyancy

and c_1 , c_2 , c_3 and c_s are empirical functions which are presumed to be "constants" at high turbulence Reynolds numbers.

Equation (A.13) is of the form recommended by Launder (1975).

However, Launder gave combined treatment for the generation due to shear P_{ij} and generation/destruction due to buoyancy P_{ijB} ; thus, by his treatment $c_3 = c_2$.

The equation for $\overline{u_i \theta'}$

The convection, generation, and buoyancy terms of equation (A.4) are treated as exact.

Assuming that small-scale turbulence structure is isotropic at high turbulence Reynolds number, the dissipation term in equation (A.4) tends to zero since correlation between gradients of scalar and velocity fluctuations tend to zero when local isotropy prevails.

As in the modelling of $\overline{u_i u_j}$, the pressure scrambling term in the equation for $\overline{u_i \theta'}$ is controlled by three agencies, for which the modelling is as follows:

(i) pure turbulence interactions; modelled as

$$-c_{1T} \left(\frac{\epsilon}{k} \right) \overline{u_i \theta'} \quad (A.18)$$

(ii) mean strain effects; modelled as

$$+c_{2T} \overline{u_j \theta'} \frac{\partial u_i}{\partial x_j} \quad (A.19)$$

(iii) selective influence of buoyancy; modelled as

$$-c_{3T} \overline{\beta \theta'^2} g_i$$

At high turbulence Reynolds number, the dominant part of the term describing diffusive transport in equation (A.4) is $\overline{u_i u_j \theta'}$ which, like its counterpart in the Reynolds stress equation, is approximated as

$$\overline{-u_i u_j \theta'} = c_{T\epsilon} \frac{K}{\epsilon} \left(\overline{u_j u_l} \frac{\partial \overline{u_i \theta'}}{\partial x_l} + \overline{u_i u_l} \frac{\partial \overline{u_j \theta'}}{\partial x_l} \right) \quad (A.21)$$

Substituting expressions (A.18) to (A.21) in equation (A.4), and re-arranging the terms gives the following modelled equation for $\overline{u_i \theta'}$:

$$\underbrace{\frac{\partial \overline{u_i \theta'}}{\partial t}}_{\text{rate of change}} + \underbrace{u_j \frac{\partial \overline{u_i \theta'}}{\partial x_j}}_{\text{convection}} = \underbrace{\frac{\partial}{\partial x_j} \left\{ c_{T\epsilon} \frac{K}{\epsilon} \left(\overline{u_j u_l} \frac{\partial \overline{u_i \theta'}}{\partial x_l} + \overline{u_i u_l} \frac{\partial \overline{u_j \theta'}}{\partial x_l} \right) \right\}}_{\text{diffusive transport}} + \underbrace{P_{iT} + P_{iTB}}_{\text{generation}} - \underbrace{\left\{ c_{1Tk} \frac{\epsilon}{\theta'} \overline{u_i \theta'} - c_{2T} \overline{u_j \theta'} \frac{\partial \overline{u_i}}{\partial x_j} + c_{3T} P_{iTB} \right\}}_{\text{pressure scrambling}} \quad (A.22)$$

where

$$P_{iT} = -\left\{ \overline{u_i u_j} \frac{\partial \theta}{\partial x_j} + \overline{u_j \theta'} \frac{\partial \overline{u_i}}{\partial x_j} \right\} \quad (A.23)$$

= generation of $\overline{u_i \theta'}$ by mean field

$$P_{iTB} = \beta \overline{\theta'^2} g_i \quad (A.24)$$

= generation/destruction of $\overline{u_i \theta'}$ by buoyancy

and c_{1T} , c_{2T} , c_{3T} , and c_T are empirical functions which are "constants" at high turbulence Reynolds numbers.

Again, equation (A.22) is of the same form as that recommended by Launder (1975) except that by that treatment $c_{3T} = c_{2T}$.

The equation for $\overline{\theta'^2}$.

The convection and generation terms of equation (A.5) are treated as exact. The dissipation term is assumed proportional to $\epsilon \overline{\theta'^2}/k$; that is, it takes the form:

$$-c'_{1T} \frac{\epsilon}{k} \overline{\theta'^2} \quad (A.25)$$

At high turbulence Reynolds number, diffusion by molecular conduction is negligible and the main component of diffusive transport is the part expressing diffusion due to turbulence effects, $\overline{u_j \theta'^2}$. This part is approximated by using gradient transport, and assuming the turbulence to be characterised by k and ϵ :

$$\overline{-u_j \theta'^2} = c'_{2T\epsilon} \frac{k^2}{2\epsilon} \frac{\partial \overline{\theta'^2}}{\partial x_j} \quad (A.26)$$

Making use of expressions (A.25) and (A.26), the modelled form of equation (A.5) becomes:

$$\underbrace{\frac{\partial \overline{\theta'^2}}{\partial t}}_{\text{rate of change}} + \underbrace{U_j \frac{\partial \overline{\theta'^2}}{\partial x_j}}_{\text{convection}} = \underbrace{-2\overline{u_j \theta'}}_{\text{generation by mean motion}} \frac{\partial \theta}{\partial x_j} - \underbrace{\epsilon_T}_{\text{dissipation}} + \underbrace{\frac{\partial}{\partial x_j} \left(c'_{2T} \frac{k^2}{\epsilon} \frac{\partial \overline{\theta'^2}}{\partial x_j} \right)}_{\text{diffusive transport}} \quad (\text{A.27})$$

$$\begin{aligned} \text{where } \epsilon_T &= 2\Gamma \frac{\partial \overline{\theta'}}{\partial x_j} \frac{\partial \overline{\theta'}}{\partial x_j} \quad (\text{exact}) \\ &= c'_{1Tk} \frac{\epsilon}{\overline{\theta'^2}} \quad (\text{modelled; Launder, 1975}) \end{aligned} \quad (\text{A.28})$$

and c'_{1T} and c'_{2T} are "constants" at high turbulence Reynolds numbers.

It may be noted that ϵ_T may also be obtained from its own equation as shown by e.g. Lumley and Khajeh-Nouri, (1974), instead of being simply approximated as done by Launder (1975).

(b) Additional Equations

For a closure of the foregoing modelled equations, a modelled equation for ϵ is required, and this is presented below. Also, because of the particular interest in this study in the turbulence energy k , the modelled k -equation is also given below.

The equation for k

Only the term expressing diffusive transport may be approximated in equation (A.6). At high turbulence Reynolds numbers, molecular action becomes negligible, and the dominant term is taken as $\overline{u_j k'}$ which may be approximated as (Hanjalic and Launder, 1972):

$$-\overline{u_j k'} = c_{k\epsilon} \frac{k}{\epsilon} \overline{u_j u_l} \frac{\partial k}{\partial x_l} \quad (\text{A.29})$$

Hence the modelled form of equation (A.6) may be written as:

$$\begin{aligned} \underbrace{\frac{\partial k}{\partial t}}_{\text{rate of change}} + \underbrace{U_j \frac{\partial k}{\partial x_j}}_{\text{convection}} &= \underbrace{-\overline{u_i u_j} \frac{\partial U_i}{\partial x_j}}_{\text{generation by shear}} - \underbrace{\epsilon}_{\text{dissipation}} + \underbrace{\beta \overline{u_i \theta' g_i}}_{\text{generation/destruction by buoyancy forces}} \\ &+ \underbrace{\frac{\partial}{\partial x_j} \left\{ c_{k\epsilon} \frac{k}{\epsilon} \overline{u_j u_l} \frac{\partial k}{\partial x_l} \right\}}_{\text{diffusive transport}} \end{aligned} \quad (\text{A.30})$$

where $c_{k\epsilon}$ is a "constant" at high turbulence Reynolds numbers.

The equation for ϵ

The modelling of the non-buoyant terms of equation (A.7) has been undertaken by various previous workers. Terms G1 and G2, which express the generation of ϵ from mean flow, are simulated in the manner proposed by Hanjalic and Launder (1972) as

$$-c_{1\epsilon} \frac{\epsilon}{k} \overline{u_i u_j} \frac{\partial U_i}{\partial x_j} \quad (\text{A.31})$$

The terms G3 and D in equation (A.7) may be treated as first proposed by Lumley (1970) and later supported by Rodi (1971) for high Reynolds numbers:

$$D - G3 = c_{2\epsilon} \frac{\epsilon^2}{k} \quad (\text{A.32})$$

At high turbulence Reynolds numbers, the diffusive transport in equation (A.7) is assumed to be dominated by the term $\overline{u_j \epsilon'}$ which is approximated by Hanjalic and Launder (1972) as follows:

$$\overline{u_j \epsilon'} = c_{\epsilon\epsilon} \frac{k}{\epsilon} \overline{u_j u_l} \frac{\partial \epsilon}{\partial x_l} \quad (\text{A.33})$$

Of particular interest in buoyant flows is the term expressing generation/destruction by buoyancy forces in equation (A.7). Launder (1973) has suggested that the influence of buoyancy on the dissipation rate ϵ can be assumed to be proportional to the buoyant generation of turbulence energy k . This suggestion has formed the basis of the modelling in this study of the effects of buoyancy on ϵ , which can then be expressed as:

$$c_{3\epsilon} \frac{\epsilon}{k} \beta g_i \overline{u_i \theta'} \quad (\text{A.34})$$

Now, assembling the various approximations, the modelled form of equation (A.7) becomes:

$$\underbrace{\frac{\partial \epsilon}{\partial t}}_{\text{rate of change}} + \underbrace{U_j \frac{\partial \epsilon}{\partial x_j}}_{\text{convection}} = \underbrace{-c_{1\epsilon} \frac{\epsilon}{k} \overline{u_i u_j} \frac{\partial U_i}{\partial x_j} - c_{2\epsilon} \frac{\epsilon^2}{k}}_{\text{(mean flow and turbulence generation and destruction)}} + \underbrace{c_{3\epsilon} \frac{\epsilon}{k} \beta g_i \overline{u_i \theta'}}_{\text{generation/destruction by buoyancy forces}}$$

$$+ \underbrace{\frac{\partial}{\partial x_j} \left\{ c_\epsilon \frac{k}{\epsilon} \overline{u_j u_1} \frac{\partial \epsilon}{\partial x_1} \right\}}_{\text{diffusive transport}} \quad (\text{A.35})$$

where $c_{1\epsilon}$, $c_{2\epsilon}$, $c_{3\epsilon}$ and c_ϵ are "constants" at high turbulence Reynolds numbers.

(c) The Constants of the Modelled Equations

Equations (A.13), (A.22), (A.27), (A.30) and (A.35) form a closed set of equations containing 15 empirical constants. Launder (1975) has recommended the use of some values of the constants, which have been later adopted by Chen and Rodi (1975) with slight modifications. The values given in the table below contain the modifications made by Chen and Rodi.

Equations	$\overline{u_i u_j}$ (Equation A.13)	$\overline{U_i \theta'}$ (Eq. A.22)	$\overline{\theta'^2}$ (Eq. A.27)	k (Eq. A.30)	ϵ (Equation A.35)
Constants	$c_1 \quad c_2 \quad c_3 \quad c_s$ 2.2 .55 .55 .2	$c_{1T} \quad c_{2T} \quad c_{3T} \quad c_T$ 3.2 .5 .5 .07	$c_{1T}' \quad c_{2T}'$ 1.25 .13	c_k .225	$c_{1\epsilon} \quad c_{2\epsilon} \quad c_\epsilon$ 1.45 .192 .15

Table A.1 Constants of the modelled equations

The remaining constant is c_3 of equation (A.35). Launder and Gibson (1974) and Launder (1975) assumed that there is zero influence of buoyancy on ϵ ; i.e. they effectively set $c_{3\epsilon} = 0$. On the other hand, Chen and Rodi (1975) combined the buoyancy term of equation (A.35) with the other generation term containing the constant $c_{1\epsilon}$, with therefore the implication that $c_{3\epsilon} = c_{1\epsilon}$ (=1.45). However, in this study a value of $c_{3\epsilon}$ of 1.0 is found to be more appropriate as is explained in Chapter 4.

A.4 APPLICATION OF THE MODELLED EQUATIONS

The modelled forms of the equations for Reynolds stresses, heat flux, and temperature fluctuations, may in principle be solved

directly as they stand; however they are still sufficiently complex that further simplifications are often necessary. In this section various approximations that may be applied in buoyant flows are discussed.

(a) Model I: Two-Equation Viscosity Model

In this model turbulent diffusional fluxes are assumed proportional to the gradient of mean flow properties (e.g. Hinze, 1959; Launder and Spalding, 1972)

$$-\overline{\rho u_i u_j} = \mu_t \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij} \quad (\text{A.36})$$

$$-\overline{\rho u_j \theta'} = \frac{\mu_t}{\sigma_{h,t}} \frac{\partial \theta}{\partial x_j} \quad (\text{A.37})$$

where μ_t is the turbulent viscosity, and $\sigma_{h,t}$ is the turbulent Prandtl number, which is often assumed to be a constant of order unity. The turbulent viscosity μ_t is taken to be determined by parameters characterising the turbulence field. One of the popular methods is to determine μ_t from k and ϵ ; thus from dimensional analysis:

$$\mu_t = c_\mu \rho \frac{k^2}{\epsilon} \quad (\text{A.38})$$

where c_μ is an empirical function which is a "constant" at high turbulence Reynolds numbers.

The modelled forms of the k and ϵ equations may now be written as:

k-equation

$$\underbrace{\frac{\partial k}{\partial t}}_{\text{rate of change}} + \underbrace{U_j \frac{\partial k}{\partial x_j}}_{\text{convection}} = \underbrace{\nu_t \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j}}_{\text{generation by shear}} - \underbrace{\epsilon}_{\text{dissipation}} + \underbrace{\beta \overline{u_i \theta' g_i}}_{\text{generation/destruction by buoyancy forces}} + \underbrace{\frac{\partial}{\partial x_j} \left(\frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x_j} \right)}_{\text{diffusive transport}} \quad (\text{A.39})$$

ϵ -equation:

$$\underbrace{\frac{\partial \epsilon}{\partial t}}_{\text{rate of change}} + \underbrace{U_j \frac{\partial \epsilon}{\partial x_j}}_{\text{convection}} = \underbrace{c_{1\epsilon} \frac{\nu_t \epsilon}{k} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j}}_{\text{(mean flow and turbulence generation and destruction)}} - c_{2\epsilon} \frac{\epsilon^2}{k} + \underbrace{c_{3\epsilon} \frac{\epsilon}{k} \beta g_i \overline{u_i \theta'}}_{\text{generation/destruction by buoyancy forces}} + \underbrace{\frac{\partial}{\partial x_j} \left(\frac{\nu_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_j} \right)}_{\text{diffusive transport}} \quad (\text{A.30})$$

where σ_k and σ_ϵ are additional empirical functions which are assumed to be constants at high turbulence Reynolds numbers. The turbulent heat flux $\overline{u_i \theta'}$ is derived from equation (A.37).

This is the model employed in this study.

(b) Model II: Simplified Stress Model

By neglecting the convective and diffusive transports of $\overline{u_i u_j}$ and $\overline{u_i \theta'}$, equations (A.13) and (A.22) reduce to the following forms in steady flows:

$$\begin{aligned}
 -\overline{u_i u_j} = \frac{1}{c_1} \frac{k}{\epsilon} \left[(1-c_2) \left\{ \overline{u_i u_k} \frac{\partial U_j}{\partial x_k} + \overline{u_j u_k} \frac{\partial U_i}{\partial x_k} \right\} - (1-c_3) \beta \{ \overline{u_i \theta'} g_j + \overline{u_j \theta'} g_i \} \right. \\
 \left. - \frac{2}{3} \delta_{ij} \{ c_3 \beta \overline{u_i \theta'} g_i - c_2 \overline{u_i u_k} \frac{\partial U_i}{\partial x_k} \} + \frac{2}{3} \delta_{ij} \epsilon \right] - \frac{2}{3} \delta_{ij} k \quad (\text{A.31})
 \end{aligned}$$

$$\overline{u_i \theta'} = \frac{1}{c_{1T}} \frac{k}{\epsilon} \left\{ -\overline{u_i u_j} \frac{\partial \theta}{\partial x_j} - (1-c_{2T}) \overline{u_j \theta'} \frac{\partial U_i}{\partial x_j} + (1-c_{3T}) \beta \overline{\theta'^2} g_i \right\} \quad (\text{A.32})$$

These expressions for $\overline{u_i u_j}$ and $\overline{u_i \theta'}$ may now be used to obtain closure of the time-averaged mean flow equations, with $\overline{\theta'^2}$ in equation (A.32) being derived from the solution of equation (A.27), and k and ϵ from equations (A.30) and (A.35).

This model has been employed in its boundary-layer form by Chen and Rodi (1975) for the prediction of jets and plumes (see also Chapter 4).

(c) Model III: Algebraic Stress Model

The convection and diffusion terms of the equations for $\overline{u_i u_j}$ and $\overline{u_i \theta'}$ need not be neglected as in model II, but may be assumed to be proportional to those of the k -equation in a manner similar to that first proposed by Rodi (1972):

$$\begin{aligned} \frac{D \overline{u_i u_j}}{Dt} - \mathcal{D}(\overline{u_i u_j}) &= \frac{\overline{u_i u_j}}{k} \left\{ \frac{Dk}{Dt} - \mathcal{D}(k) \right\} \\ &= \frac{\overline{u_i u_j}}{k} (p_k^B - \epsilon) \end{aligned} \quad (A.33)$$

where $p_k^B = p_k + p_{kB}$ (see equations A.16 and A.17), $\frac{D}{Dt}$ is the total derivative, and $\mathcal{D}(\phi)$ stands for the diffusive transport of ϕ . Similarly,

$$\begin{aligned} \frac{D \overline{u_i \theta'}}{Dt} - \mathcal{D}(\overline{u_i \theta'}) &= \frac{\overline{u_i \theta'^2}}{k^{\frac{1}{2}} \sqrt{\overline{\theta'^2}}} \left\{ \frac{D}{Dt} (k^{\frac{1}{2}} \sqrt{\overline{\theta'^2}}) - \mathcal{D}(k^{\frac{1}{2}} \sqrt{\overline{\theta'^2}}) \right\} \\ &= \frac{\overline{u_i \theta'}}{2 \overline{\theta'^2}} (p_T - \epsilon_T) + \frac{\overline{u_i \theta'}}{2k} (p_k - \epsilon) \end{aligned} \quad (A.34)$$

where P_T and ϵ_T are respectively the generation and dissipation terms of the $\overline{\theta'^2}$ - equation (equation A.27).

Then, assuming that the dissipation and production rates of $\overline{\theta'^2}$ are equal (with its convection and diffusion negligible), the following simple algebraic relations emerge from the modelled equations for the Reynolds stresses, heat flux and temperature fluctuations respectively:

$$(\overline{u_i u_j} - \frac{2}{3} \delta_{ij} k) / k = A (P_{ij}^B - \frac{2}{3} \delta_{ij} P_k^B) \quad (A.35)$$

$$-\overline{u_i \theta'} = A_{T\epsilon} \frac{k}{\epsilon} \overline{u_i u_j} \frac{\partial \theta}{\partial x_j} - A'_T \frac{k}{\epsilon} P_{iT}^B \quad (A.36)$$

$$\text{and, } \overline{\theta'^2} = -c'_{1T} \frac{k}{\epsilon} \overline{u_j \theta'} \frac{\partial \theta}{\partial x_j} \quad (A.37)$$

$$\text{where } P_{ij}^B = P_{ij} + P_{ijB} \quad (\text{equations A.14 and A.15})$$

$$P_{iT}^B = P_{iT} + P_{iT B} \quad (\text{equations A.23 and A.24})$$

$$A = (1 - c_2) / (c_1 - 1 + P_k^B / \epsilon) \quad (A.38)$$

$$A_T = c_{1T} + \frac{1}{2} (P_k^B / \epsilon - 1) \quad (A.39)$$

$$\text{and } A'_T = A_T (1 - c_{2T})$$

Finally, k and ϵ in equations (A.35), (A.36), and (A.37) may be derived from equations (A.30) and (A.35).

Coupling of Model III with Model I

Further simplification may be made in model III by invoking the "gradient transport hypothesis", as shown in Model I, to express only the diffusive transport of K and ϵ . That is, the diffusive transport terms in equations (A.30) and (A.35) become respectively

$$\frac{\partial}{\partial x_j} \left(\frac{v_t}{\sigma_k} \frac{\partial k}{\partial x_j} \right)$$

and,
$$\frac{\partial}{\partial x_j} \left(\frac{v_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_j} \right)$$

This modified form of Model III was employed by Gibson and Launder (1974) and Launder (1975). For horizontal buoyant flows, the scalar diffusion coefficient was shown by Gibson and Launder to be

$$v_t = \frac{0.36 + 0.165 \frac{P_k^B}{\epsilon}}{(1.2 + \frac{P_k^B}{\epsilon})} \left(\frac{k^2}{\epsilon} \right) \quad (A.41)$$

APPENDIX B

WALL REGION: ONE-DIMENSIONAL ANALYSIS OF THE INERTIAL SUBLAYER

B.1 INTRODUCTION

In this study, turbulence characteristics in the wall region is assumed to be those often associated with the inertial sublayer, and the transport of momentum, energy, and turbulence within the layer is assumed to be of one-dimensional Couette-flow type. This Appendix sets out the forms to which the equations for turbulence energy and its dissipation rate, shear stress, and heat flux reduce in the sublayer.

Much of the analysis is based on the "universal" logarithmic law of the wall; and, for a start, the essence of this law is laid down in Section B.2. Then, in Section B.3, the turbulence energy and its dissipation rate as well as the shear stress are dealt with. Finally, the equations for the transport of heat (or other scalar) are presented in Section B.4.

B.2 MEAN VELOCITY DISTRIBUTION (LOGARITHMIC LAW OF THE WALL)

In a wall region, distances from the wall are often measured in terms of the non-dimensional distance y^+ :

$$y^+ = \frac{U_\tau y}{\nu} \quad (B.1)$$

where $U_\tau = \sqrt{\tau_w/\rho}$ = friction velocity (B.2)

and τ_w is the shear stress at the wall, y is the normal distance from the wall.

The inertial sublayer is here defined to be a region where the flow is completely turbulent, and it covers the region where $y^+ > 30$, but sufficiently close to the wall that the assumption of uniform shear stress applies (see e.g. Hinze, 1959; Tennekes and Lumley, 1972).

Thus, from a one-dimensional analysis, assuming zero streamwise pressure gradient the momentum equation for the flow parallel to the wall reduces to:

$$v_t \frac{\partial U}{\partial y} = \frac{\tau}{\rho} \approx \frac{\tau_w}{\rho} \quad (B.3)$$

where v_t is the eddy kinematic viscosity.

Making a one-dimensional Couette-flow analysis, the size of the large eddies, which are responsible for the eddy viscosity in the inertial sublayer, may be assumed to vary in proportion to the distance from the wall (Hinze, 1959);

thus*

$$v_t \approx \kappa U_\tau y \quad (B.4)$$

Equation (B.3) then becomes

$$\kappa U_\tau y \frac{\partial U}{\partial y} = \frac{\tau_w}{\rho} = U_\tau^2 \quad (B.5)$$

$$\text{or,} \quad \kappa y^+ \frac{\partial U^+}{\partial y^+} = 1$$

$$\text{Hence,} \quad U^+ = \frac{1}{\kappa} \log_e y^+ + \text{const.}$$

$$\text{or,} \quad U^+ = \frac{1}{\kappa} \log_e (E y^+) \quad (B.6)$$

which is the familiar logarithmic law of the wall. Here κ is the Von Karman constant ($= 0.4187$), E ($= 9.793$ from Patel, 1965) is a constant of integration, and

$$U^+ = U/U_\tau \quad (B.7)$$

B.3 TURBULENCE ENERGY AND ITS DISSIPATION RATE, AND SHEAR STRESS

The following basic definitions may be re-called:

- ◊ Gradient transport hypothesis: (for one-dimensional flow)

$$-\overline{uv} = v_t \frac{\partial U}{\partial y} \quad (B.8)$$

- ◊ Prandtl-Kolmogorov hypothesis: (see Chapter 2, Section 2.42)

* Note that equation (B.4) is similar to the Prandtl-Kolmogorov hypothesis (equation B.9)

$$v_t = c_\mu l k^{\frac{1}{2}} \quad (B.9)$$

where l is the characteristic length scale of the turbulent eddies

• Dissipation rate: (See Chapter 2, Section 2.42)

$$\begin{aligned} \epsilon &= \nu \overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2} \\ &= k^{3/2}/l \end{aligned} \quad (B.10)$$

Turbulence energy, k

In the inertial sublayer, both convection and diffusion of k are negligible; thus

Production rate of $k \approx$ dissipation rate of k

$$\text{i.e.} \quad -\overline{uv} \frac{\partial U}{\partial y} \approx \epsilon \quad (B.11)$$

Substituting for $-\overline{uv}$ and ϵ from equations (B.8) and (B.10), then

$$v_t \left(\frac{\partial U}{\partial y}\right)^2 = \frac{k^{3/2}}{l}$$

And, using (B.9) to eliminate l , there results:

$$v_t^2 \left(\frac{\partial U}{\partial y}\right)^2 = c_\mu k^2$$

Or, making use of (B.7), then,

$$k = \frac{-\frac{1}{2} \overline{uv}}{c_\mu} \quad (B.12)$$

Further, since the shear stress is approximately uniform in the wall region ($y^+ < 200$; see for example, Gardner and Kestin, 1963)

$$-\rho \overline{uv} \approx \tau_w$$

$$\text{or,} \quad -\rho \overline{uv} = \rho U_\tau^2 \quad (\text{from equation B.2}) \quad (B.13)$$

Making use of equation (B.13) to eliminate $-\overline{uv}$ in equation (B.12), then

$$\begin{aligned} k &= c_\mu^{-\frac{1}{2}} U_\tau^2 \\ \text{or,} \quad k &= c_\mu^{-\frac{1}{2}} \left(\frac{\tau_w}{\rho}\right) \end{aligned}$$

(B.14)

Dissipation rate, ϵ

From equation (B.5),

$$\frac{\partial U}{\partial y} = \frac{U\tau}{\kappa y} \quad (B.15)$$

Substituting equations (B.13) and (B.15) into equation (B.11) yields

$$\epsilon = \frac{U^3 \tau}{\kappa y} \quad (B.16)$$

Finally, substituting for U_τ from equation (B.14) gives

$$\epsilon = c_\mu^{3/4} \frac{k^{3/2}}{\kappa y} \quad (B.17)$$

Wall shear stress, τ_w

U^+ may be re-written as

$$U^+ = \frac{U\sqrt{\tau_w \rho}}{\tau_w} \quad (B.18)$$

Since the shear stress is approximately uniform in the wall region,

U^+ may be approximated as

$$U^+ \approx \frac{U\sqrt{\tau_I \rho}}{\tau_w} \quad (B.19)$$

where,

$$\tau_I = -\rho \overline{uv}$$

or,

$$\tau_I = \rho c_\mu^{1/2} k \quad (B.20)$$

That is, τ_I is the shear stress ($-\rho \overline{uv}$) in the inertial sublayer.

Now, substituting for U^+ in equation (B.6), then

$$\frac{U\sqrt{\tau_I \rho}}{\tau_w} = \frac{1}{\kappa} \log_e (Ey^+) \quad (B.21)$$

Hence, $\tau_w = \kappa U \sqrt{\tau_I \rho} / \{\log_e (Ey^+)\}$

or, from (B.20), then

$$\tau_w = \rho \kappa c_\mu^{1/4} K^{1/2} U / \{\log_e (Ey^+)\} \quad (B.22)$$

Constraints on constants of ϵ -equation

Making use of equations (B.14) and (B.17), and neglecting convective transport of ϵ , the usual modelled equation for ϵ (see Chapter 2) reduces to the following simple form for the inertial sublayer (neglecting buoyancy effects in the layer):

$$c_1 = c_2 - \frac{\kappa^2}{\sigma_\epsilon \cdot c_\mu^{\frac{1}{2}}} \quad (\text{B.23})$$

Hence, for the inertial sublayer the turbulent Prandtl/Schmidt number

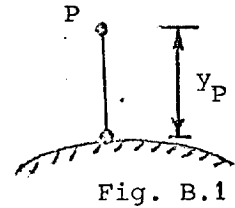
for ϵ , σ_ϵ , becomes

$$\sigma_\epsilon = \frac{\kappa^2}{(c_2 - c_1) c_\mu^{\frac{1}{2}}} \quad (\text{B.24})$$

Integration of ϵ in wall region ($\int_V dV$ used in K-equation)

Using equation (B.16), the line integral of ϵ (see Fig. B.1) may be expressed as

$$\begin{aligned} \int_0^{y_P} \epsilon dy &= \int_0^{y_P} \frac{U_\tau^3}{\kappa y} dy \\ &= \int_0^{y_P^+} \frac{U_\tau^3}{\kappa \cdot y U_\tau} \cdot d\left(\frac{y U_\tau}{v}\right) \\ &= \int_0^{y_P^+} \frac{U_\tau^3}{\kappa y^+} dy^+ \\ &= \frac{U_\tau^3}{\kappa} \log_e y^+ + \text{const} \\ &\approx \frac{U_\tau^3}{\kappa} \log_e (E y^+) \end{aligned} \quad (\text{B.25})$$



Now, making use of equation (B.6) and (B.14), equation (B.25) may be re-written as

$$\int_0^{y_P} \epsilon dy \approx c_\mu^{\frac{3}{4}} k^{\frac{3}{2}} U^+ \quad (\text{B.26})$$

where U^+ is as defined by equation (B.6).

Thus, the volume integral of ϵ in a control volume ΔV may be expressed as

$$\boxed{\int_V \epsilon dV = \frac{c_\mu^{\frac{3}{4}} k^{\frac{3}{2}} U^+}{y_P} \cdot \Delta V} \quad (\text{B.27})$$

B.4 ENERGY (OR OTHER MEAN SCALAR PROPERTY) TRANSPORT

Transport equation

For a one-dimensional Couette-flow in the inertial sublayer,

$$\dot{q}'' = \Gamma_t c_p \frac{\partial T}{\partial y} \quad (B.28)$$

where $\dot{q}''$ is the energy flux, Γ_t is the turbulent exchange coefficient, and c_p is the specific heat of the fluid. Dividing through by the wall heat flux $\dot{q}''_w$, there results

$$\frac{\dot{q}''}{\dot{q}''_w} = \frac{\Gamma_t}{\mu} \frac{\partial T^+}{\partial y^+} \quad (B.29)$$

$$\text{where} \quad T^+ = \frac{\rho U_\tau c_p (T_w - T)}{\dot{q}''_w} \quad (B.30)$$

Now, within the inertial sublayer,

$$\dot{q}'' \approx \dot{q}''_w \quad (B.31)$$

Also, making use of equation (B.4),

$$\frac{\Gamma_t}{\rho} = \frac{\nu_t}{\sigma_{h,t}} \approx \frac{\kappa y U_\tau}{\sigma_{h,t}} \quad (B.32)$$

Substituting (B.31) and (B.32) into equation (B.29), and integrating gives

$$T^+ = \frac{\sigma_{h,t}}{\kappa} \log_e y^+ + c_T \sigma_h \quad (B.33)$$

where c_T is a constant of integration and generally depends on the Prandtl number σ_h of the fluid. Equation (B.33) is often re-written as (e.g. Jayatillaka, 1967):

$$T^+ = \sigma_{h,t} \left\{ U^+ + P \left(\frac{\sigma_h}{\sigma_{h,t}} \right) \right\} \quad (B.34)$$

where the constant of integration is now expressed as a p-function which, from experimental data of smooth surfaces, is determined by Jayatillaka to be:

$$P \left(\frac{\sigma_h}{\sigma_{h,t}} \right) = 9.24 \left\{ \left(\frac{\sigma_h}{\sigma_{h,t}} \right)^{3/4} - 1 \right\} \left\{ 1 + 0.28 \exp(-0.007 \frac{\sigma_h}{\sigma_{h,t}}) \right\} \quad (B.25)$$

Wall heat flux

In the inertial sublayer, T^+ (equation B.30) may be approximated as

$$T^+ \approx \frac{\rho c_p (T_w - T) \sqrt{\tau_I / \rho}}{\dot{q}_w''} \quad (\text{B.36})$$

where τ_I is as defined by equation (B.20).

Combining equations (B.34) and (B.36) gives

$$\dot{q}_w'' = \rho c_p (T_w - T) \frac{\sqrt{\tau_I}}{\sigma_{h,t}} \left[U^+ + P \left(\frac{\sigma_h}{\sigma_{h,t}} \right) \right]$$

And, making use of equation (B.20), then

$$\dot{q}_w'' = \rho c_p (T_w - T) C_\mu^{1/4} k^{1/2} / \left\{ \sigma_{h,t} \left[\frac{1}{k} \log_e (E y^+) + P \left(\frac{\sigma_h}{\sigma_{h,t}} \right) \right] \right\} \quad (\text{B.37})$$

APPENDIX C

THE SOURCE TERMS IN THE MOMENTUM AND TURBULENCE EQUATIONS

C.1 INTRODUCTION

This Appendix presents the derivation of the complete forms of the momentum equations and the generation term of the turbulence energy equation. In Section C.2 the axial momentum equation is derived, and that for radial momentum is also derived in Section C.3. Finally, in Section C.4 is presented the derivation of the generation term in the turbulence energy equation.

C.2 AXIAL MOMENTUM

For steady state, and in terms of shear and normal stresses, the equation for axial momentum in cylindrical polar co-ordinates is (e.g. Bird et al, 1960):

$$\frac{\partial}{\partial x}(\rho U^2) + \frac{\partial}{\partial r}(\rho UV) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho UW) = - \frac{\partial P}{\partial x} - \left\{ \frac{\partial \tau_{xx}}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r}(r \tau_{rx}) + \frac{1}{r} \frac{\partial \tau_{\theta x}}{\partial \theta} \right\} \quad (C.1)$$

where x, r, θ refer to the axial, radial, and angular directions respectively; U, V, W , refer to the velocities in the axial, radial, and angular directions respectively; and the stresses τ_{xx} , τ_{rx} , and $\tau_{\theta x}$ take the following forms (e.g. Bird et al, 1960; Schlichting, 1968):

$$\tau_{xx} = -\mu \left\{ 2 \frac{\partial U}{\partial x} - \frac{2}{3} (\nabla \cdot \vec{u}) \right\} \quad (C.2)$$

$$\tau_{rx} = -\mu \left\{ \frac{\partial U}{\partial r} + \frac{\partial V}{\partial x} \right\} \quad (C.3)$$

$$\tau_{\theta x} = -\mu \left\{ \frac{\partial W}{\partial x} + \frac{1}{r} \frac{\partial U}{\partial \theta} \right\} \quad (C.4)$$

where

$$\nabla \cdot \vec{u} = \frac{\partial U}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r}(rV) + \frac{1}{r} \frac{\partial W}{\partial \theta} \quad (C.5)$$

Substituting into equation (C.1) these expressions for the stresses, and for the special case of non-swirling axi-symmetric flows ($W = 0$, and $\frac{\partial}{\partial \theta} = 0$), there results:

$$\frac{\partial}{\partial x}(\rho U^2) + \frac{\partial}{\partial r}(\rho UV) = \frac{\partial P}{\partial x} + 2\frac{\partial}{\partial x}\left(\mu\frac{\partial U}{\partial x}\right) - \frac{2}{3}\frac{\partial}{\partial x}\{\mu(\nabla \cdot \vec{U})\} + \frac{1}{r}\frac{\partial}{\partial r}(r\mu\frac{\partial U}{\partial r}) + \frac{1}{r}\frac{\partial}{\partial r}(r\mu\frac{\partial V}{\partial x}) \quad (C.6)$$

Equation (C.6) can be re-written as:

$$\frac{\partial}{\partial x}(\rho U^2) + \frac{\partial}{\partial r}(\rho UV) = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\mu\frac{\partial U}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}(r\mu\frac{\partial U}{\partial r}) + S_u$$

where

$$S_u = \frac{\partial}{\partial x}\left(\mu\frac{\partial U}{\partial x}\right) - \frac{2}{3}\frac{\partial}{\partial x}\{\mu(\nabla \cdot \vec{U})\} + \frac{1}{r}\frac{\partial}{\partial r}(r\mu\frac{\partial V}{\partial x}) \quad (C.7)$$

It may be noted that for nearly incompressible flows, from continuity

$$\nabla \cdot \vec{U} \approx 0 \quad (C.8)$$

$$\text{where } \nabla \cdot \vec{U} = \frac{\partial U}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rV) \quad (C.9)$$

C.3 RADIAL MOMENTUM

Again for steady state, and in terms of shear and normal stresses, the equation for radial momentum in cylindrical polar co-ordinates is (e.g. Bird et al, 1960):

$$\frac{\partial}{\partial x}(\rho UV) + \frac{\partial}{\partial r}(rV^2) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho WV) - \frac{\rho W^2}{r} = -\frac{\partial P}{\partial r} - \left\{ \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\partial \tau_{rx}}{\partial x} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} \right\} \quad (C.10)$$

where the stress τ_{rx} is as given in equation (C.3), while τ_{rr} , $\tau_{r\theta}$, and $\tau_{\theta\theta}$ have the following forms (Bird et al, 1960; Schlichting, 1968):

$$\tau_{rr} = -\mu\left\{2\frac{\partial V}{\partial r} - \frac{2}{3}(\nabla \cdot \vec{U})\right\} \quad (C.11)$$

$$\tau_{r\theta} = -\mu\left\{r\frac{\partial}{\partial r}\left(\frac{W}{r}\right) + \frac{1}{r}\frac{\partial V}{\partial \theta}\right\} \quad (C.12)$$

$$\tau_{\theta\theta} = -\mu\left\{2\left(\frac{1}{r}\frac{\partial W}{\partial \theta} + \frac{V}{r}\right) - \frac{2}{3}(\nabla \cdot \vec{U})\right\} \quad (C.13)$$

Substituting these stress relations into equation (C.10) for the special case of axi-symmetric flows gives:

$$\begin{aligned} \frac{\partial}{\partial x}(\rho UV) + \frac{\partial}{\partial r}(\rho V^2) &= -\frac{\partial P}{\partial r} + \frac{\partial}{\partial x}\left(\mu\frac{\partial V}{\partial x}\right) + \frac{2}{r}\frac{\partial}{\partial r}(r\mu\frac{\partial V}{\partial r}) + \frac{\partial}{\partial x}\left(\mu\frac{\partial U}{\partial r}\right) \\ &\quad - \frac{2}{3}\frac{1}{r}\frac{\partial}{\partial r}\{\mu(\nabla \cdot \vec{U})\} - 2\mu\frac{V}{r^2} \end{aligned} \quad (C.14)$$

Equation (C.14) may be re-written in the following form:

$$\frac{\partial}{\partial x} (\rho UV) + \frac{\partial}{\partial r} (\rho V^2) = - \frac{\partial P}{\partial r} + \frac{\partial}{\partial x} \left(\mu \frac{\partial V}{\partial x} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(r \mu \frac{\partial V}{\partial r} \right) + S_V \quad (C.15)$$

where

$$S_V = \frac{\partial}{\partial x} \left(\mu \frac{\partial U}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(r \mu \frac{\partial V}{\partial r} \right) - \frac{2}{3} \frac{1}{r} \frac{\partial}{\partial r} \{ \mu (\nabla \cdot \vec{U}) \} - 2 \mu \frac{V}{r^2} \quad (C.16)$$

For nearly incompressible flows, equation (C.8) applies.

C.4 GENERATION OF TURBULENCE ENERGY

In cartesian tensor notation, the generation term G of the equation for turbulence energy K is:

$$G = -\rho u_i u_j \frac{\partial U_i}{\partial x_j} \quad (C.17)$$

where, for cylindrical polar co-ordinates, i and j each refer to the directions x, r, θ .

The components of G for cylindrical polar co-ordinates, using the same stress notations as given in Sections C.2 and C.3, are (Bird et al, 1960):

$$G = \tau_{xx} \frac{\partial U}{\partial x} + \tau_{rr} \frac{\partial V}{\partial r} + \tau_{\theta\theta} \left(\frac{1}{r} \frac{\partial W}{\partial \theta} + \frac{V}{r} \right) + \tau_{r\theta} \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) + \frac{1}{r} \frac{\partial V}{\partial \theta} \right\} \\ + \tau_{\theta x} \left(\frac{1}{r} \frac{\partial U}{\partial \theta} + \frac{\partial W}{\partial x} \right) + \tau_{rx} \left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x} \right) \quad (C.18)$$

where the stresses are as follows:

$$\tau_{xx} = -\rho \overline{u^2}, \quad \tau_{rr} = -\rho \overline{v^2}, \quad \tau_{\theta\theta} = -\rho \overline{w^2} \\ \tau_{rx} = -\rho \overline{uv} = -\rho \overline{vu} \\ \tau_{\theta x} = -\rho \overline{uw} = -\rho \overline{wu} \\ \tau_{r\theta} = -\rho \overline{vw} = -\rho \overline{wv} \quad (C.19)$$

Now, employing gradient transport hypothesis to express the above turbulent stresses, then, for cylindrical polar co-ordinates, these are given by relations (C.2) to (C.4) and (C.11) to (C.13). Substituting in equation (C.18) and re-arranging the terms, yields:

$$\begin{aligned}
 G = \mu_t \left[2 \left\{ \left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial W}{\partial \theta} + \frac{V}{r} \right)^2 \right\} + \left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x} \right)^2 \right. \\
 + \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) + \frac{1}{r} \frac{\partial V}{\partial \theta} \right\}^2 + \left\{ \frac{1}{r} \frac{\partial U}{\partial \theta} + \frac{\partial W}{\partial x} \right\}^2 \\
 \left. - \frac{2}{3} \left\{ \frac{\partial U}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (rV) + \frac{1}{r} \frac{\partial W}{\partial \theta} \right\}^2 \right] \quad (C.20)
 \end{aligned}$$

where μ_t is the turbulence viscosity.

For axi-symmetric flows, expression (C.20) takes the following form:

$$G = \mu_t \left[2 \left\{ \left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial r} \right)^2 + \left(\frac{V}{r} \right)^2 \right\} + \left(\frac{\partial U}{\partial r} + \frac{\partial V}{\partial x} \right)^2 - \frac{2}{3} \left\{ \frac{\partial U}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (rV) \right\}^2 \right]$$

(C.21)

APPENDIX D

ONE-DIMENSIONAL ANALYSIS OF THE ORIGIN OF BUOYANCY INDUCED NUMERICAL INSTABILITY

The analysis of the origin of buoyancy induced instability has also been given by Caretto et al (1971). Consider one-dimensional laminar transport in the stably-stratified field shown in Fig. D.1.

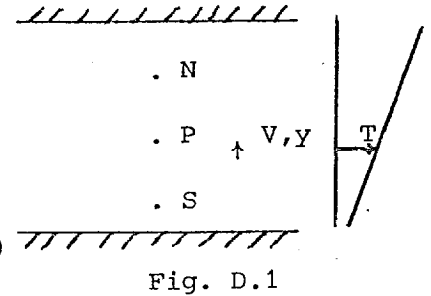
Exact equations

The exact equations are as follows:

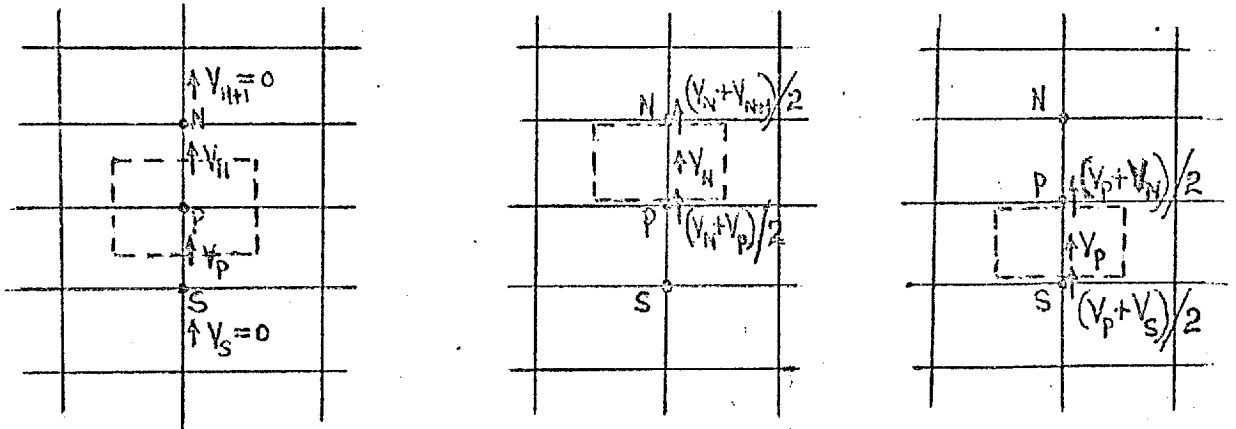
$$\text{Continuity: } \frac{\partial V}{\partial y} = 0 \quad (\text{D.1})$$

$$\text{Momentum: } \frac{\partial (VV)}{\partial y} - \nu \frac{\partial}{\partial y} \left(\frac{\partial V}{\partial y} \right) + \frac{1}{\rho} \frac{\partial P}{\partial y} + \beta g T = 0 \quad (\text{D.2})$$

$$\text{Energy: } \frac{\partial (VT)}{\partial y} - \frac{\nu}{\sigma} \frac{\partial}{\partial y} \left(\frac{\partial T}{\partial y} \right) = 0 \quad (\text{D.3})$$



Finite-difference equations



a) Main control volume b) Control volume for V_N c) Control volume for V_P

Fig. D.2

Consider the main control volume shown in Fig. D.2(a), and the possibility of a perturbation in some or all of the following four variables associated with that control volume: T_p , P_p , V_p , and V_N . Assume that all other variables in the field have their equilibrium values (i.e. undisturbed values; at equilibrium there is zero flow for the case shown in Fig. D.1.)

Then, employing "upwind" differencing for the convective terms and central differencing for the diffusive terms, the finite-difference equations for the variables associated with the main control volume are:

o Continuity about P_P : $V_P = V_N$ (D.4)

o Momentum about V_N (Fig. D.2b):

$$\left\{ \frac{v}{h} + \frac{|V_N| - V_N}{4} \right\} (-V_N) + \left\{ \frac{v}{h} + \frac{|V_N + V_P|}{4} + \frac{V_N + V_P}{4} \right\} (V_P - V_N) + \frac{1}{\rho} (P_P - P_N) + \beta gh (T_N + T_P)/2 = 0 \quad (D.5)$$

o Momentum about V_P (Fig. D.2c):

$$\left\{ \frac{v}{h} + \frac{|V_P + V_N|}{4} - \frac{(V_P + V_N)}{4} \right\} (V_N - V_P) + \left\{ \frac{v}{h} + \frac{|V_P| + V_P}{4} \right\} (-V_P) + \frac{1}{\rho} (P_S - P_P) + \beta gh (T_P + T_S)/2 = 0 \quad (D.6)$$

o Energy about T_P (Fig. D.2a):

$$\left\{ \frac{v}{\sigma h} + \frac{|V_N| - V_N}{2} \right\} (T_N - T_P) + \left\{ \frac{v}{\sigma h} + \frac{|V_P| + V_P}{2} \right\} (T_S - T_P) = 0 \quad (D.7)$$

In the foregoing equations V_S and V_{N+1} have been set to their equilibrium values of zero, and the use of the functional form $(|\phi| \pm \phi)/2$ provides an easy computational device for introducing the "upwind" differences.

Manipulation of the finite-difference equations

a) Flow equation: Setting $V = V_P = V_N$, defining the cell Reynolds number $Re = Vh/\nu$ (where h is the height of the cell), and multiplying the momentum equations through by h^2/ν^2 yields:

$$\left\{ 1 + \frac{|Re| - Re}{4} \right\} (-Re) + \frac{h^2 (P_P - P_N)}{\rho \nu^2} + \frac{\beta gh^3}{\nu^2} (T_N + T_P) = 0 \quad (D.8)$$

$$\left\{ 1 + \frac{|Re| + Re}{4} \right\} (-Re) + \frac{h^2 (P_S - P_P)}{\rho \nu^2} + \frac{\beta gh^3}{\nu^2} (T_P + T_S) = 0 \quad (D.9)$$

At equilibrium, the solution is known to be $Re = 0$, $T_P = T_{pe}$ ($=T_{ref}$), $P_P = P_{pe}$ (where subscript pe stands for equilibrium value), so that the equations above become:

$$\beta g h T_N = \frac{P_N - P_{Pe}}{\rho} - \beta g h T_{ref} \quad (D.10)$$

$$\beta g h T_S = \frac{P_{Pe} - P_S}{\rho} - \beta g h T_{ref} \quad (D.11)$$

Substituting (D.10) and (D.11) into (D.8) and (D.9) gives:

$$\left\{ 1 + \frac{|Re| - |Re|}{4} \right\} Re = \Delta P + \frac{G(T_P - T_{ref})}{2} \quad (D.12)$$

$$\left\{ 1 + \frac{|Re| + |Re|}{4} \right\} Re = -\Delta P + \frac{G(T_P - T_{ref})}{2} \quad (D.13)$$

where $G = \beta h^3 g / v^2$

$$\Delta P = h^2 (P_P - P_{Pe}) / \rho v^2$$

Adding equations (D.12) and (D.13), and dividing through by $(T_u - T_{ref})$ gives the flow (momentum/continuity) equation as

$$\boxed{\frac{T_P - T_{ref}}{T_u - T_{ref}} = \frac{4 + |Re|}{Gr} \quad Re = \frac{4 + |Re|}{Gr, \sigma} \cdot Pe} \quad (D.14)$$

where, $Gr = \beta h^3 g (T_u - T_{ref}) / v^2$

b) Energy equation: The energy equation is treated in a similar manner.

Multiplication by $\sigma h / v^2$ gives

$$T_N + T_S - 2T_P - |Pe| T_P + \left\{ \frac{|Pe| - Pe}{2} T_N + \frac{|Pe| + Pe}{2} T_S \right\} = 0 \quad (D.15)$$

The last term can be written as $|Pe| T_u$, where $T_u = T_N$ or T_S if Pe is less than, or greater than, zero respectively. Then, defining T_{ref} as the conduction average, $T_{ref} = (T_N + T_S) / 2$, yields

$$\boxed{\frac{T_P - T_{ref}}{T_u - T_{ref}} = \frac{|Pe|}{2 + |Pe|}} \quad (D.16)$$

APPENDIX E

INERTIAL RELAXATION METHOD

Exact equation

In "steady" flows where numerical instability is introduced by body forces (e.g. buoyant flows, swirling flows), the instability may be cured in part by an appropriate modification of the inertia term. Consider, for example, a one-dimensional buoyant flow in the y-direction. The exact momentum equation is:

$$\underbrace{\frac{\partial(\rho V)}{\partial t} + \frac{\partial}{\partial y}(\rho V^2)}_{\text{inertia term}} = - \frac{\partial p}{\partial y} + \underbrace{J}_{\text{diffusion term}} + \underbrace{\rho g \beta (T - T_r)}_{\text{source term}} \quad (\text{E.1})$$

where J stands for the diffusion term, and T_r is some reference temperature.

Now, for use in iterative methods for "steady" flows, it may be advantageous not to set $\partial(\rho V)/\partial t$ to zero but to use this, appropriately modified, to provide some relaxation of the source term. For this purpose, equation (E.1) may be re-written for "steady" flows as:

$$\frac{d}{dy}(\rho V^2) = - \frac{dp}{dy} + J + \underbrace{\rho g \beta (T - T_r) - \frac{d(\rho V)}{dt}}_S \quad (\text{E.2})$$

Expression S may now be treated as the new "source" term which will be expressed completely in terms of the body force.

Finite-difference form of S

In finite-difference formulation, expression S may be re-written in the form:

$$S = \rho g \beta (T - T_r) - \frac{\rho \Delta V}{\Delta t} \quad (\text{E.3})$$

where ΔV and Δt are the velocity and time changes, respectively. Then Δt may be expressed in terms of appropriately chosen velocity scale V_B and length scale L of the body-force field which allows equation (E.3) to be expressed as:

$$S = \rho g \beta (T - T_r) - \rho V_B \frac{\Delta V}{L} \quad (E.4)$$

The velocity V_B may be taken to be proportional to the ideal buoyant velocity $V_{B,i}$, where $V_{B,i}$ is the velocity of the steady motion induced by buoyancy in a non-viscous field of zero pressure gradient. From equation (E.1), then (see also Gebhart, 1971):

$$V_{B,i} \approx \sqrt{g \beta (T - T_r) L} \quad (E.5)$$

Thus,

$$V_B \approx \alpha_1 V_{B,i} = \alpha_1 \sqrt{g \beta (T - T_r) L} \quad (E.6)$$

Substituting for V_B in equation (E.4), then

$$S = \rho g \beta (T - T_r) \left\{ 1 + \frac{\alpha}{\sqrt{g \beta (T - T_r) L}} \cdot \Delta V \right\} \quad (E.7)$$

where $\alpha = -\alpha_1$ = relaxation factor,

In an iterative scheme, ΔV may be taken as the change in velocity between successive iterations.

Finite-difference momentum equations

Casting equation (E.2) into the usual finite-difference equation as indicated in the text (Chapter 3), the finite-difference equivalent of (E.2) for point P (See Fig. E.1) can now be expressed as (using expression (E.7) as the buoyancy source term):

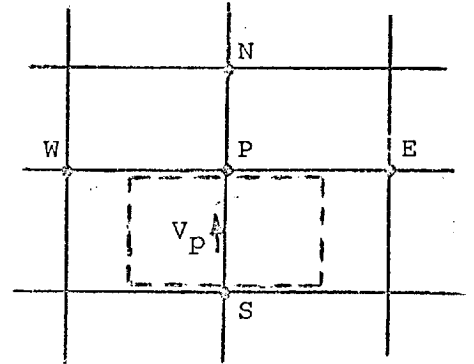


Fig. E.1

$$a_p V_p = \sum_n a_n V_n + A_{ns} (P_s - P_p) + \rho g \beta (T - T_r) \{ 1 + I (V_p^* - V_p) \} \quad (E.8)$$

$$\text{where } I = \frac{\alpha}{\sqrt{g \beta (T - T_r) L}} = \text{inertial relaxation parameter} \quad (E.9)$$

with V_p^* being the value of V_p of the previous iteration, and the coefficients a 's and area A have their usual meaning.

The stability of this method is discussed in the text (Chapter 5),

Application of the method

While the foregoing lays down the background to the inertial relaxation method, the actual form of the relaxation parameter I (equation E.9) employed in the computations is (see also Chapter 5)

$$I = \frac{\alpha_I}{\sqrt{g\beta T_r L}} \quad (E.10)$$

which makes I independent of changes in the local temperature T , and α_I is a relaxation factor. The use of expression (E.10) in place of (E.9) was found to enhance stability.

APPENDIX F

DISSIPATION LENGTH SCALE AT NOZZLE EXIT

The dissipation rate ϵ is defined as

$$\epsilon = k \frac{3/2}{l_\epsilon} \quad (F.1)$$

where l_ϵ is the "dissipation" length scale. As stated in the text (Chapter 4), the problem of predicting the correct behaviour of the initial region of a jet lies in obtaining a good estimate for ϵ . The turbulence energy can often be determined from well-documented data; the problem therefore reduces to one of determining the value of l_ϵ .

The following factors, among others, may determine the value of l_ϵ :

- (i) the geometry (i.e. shape and size) of the nozzle, and (possibly) of all upstream components e.g. smoothing screens, supply duct;
 - (ii) the Reynolds number Re of the flow
 - (iii) the temperature of the fluid
- } through influence on the boundary layer at the nozzle exit (Bradshaw 1966).

The length of the potential core of a turbulent jet varies with the turbulence intensity i at the nozzle exit, as well as being influenced by other factors. This study is restricted to low turbulence-intensity nozzle exit-flows, for which experiments (e.g. Bradshaw et al, 1964; Crow and Champagne, 1971) show that the length of the potential core is approximately constant with i . In the present study, it is found that, for fixed values of other factors, the predicted length of the core varies for different low values of i . Thus l_ϵ is also made a function of i , in order to obtain the correct low-intensity behaviour,

As stated in the text (Chapter 4), l_ϵ is assumed to obey a function of the following form:

$$l_\epsilon = \lambda d \quad (F.2)$$

$$\text{where } \lambda = \lambda \left(Re, i, \frac{\theta_0}{\theta_r}, \dots \right) \text{ and } d = \text{nozzle diameter} \quad (F.3)$$

λ is then determined in the following manner:

Intensity, i

The dependence of l_ϵ on i can be determined from the basic concept of energy transfer in the inertial subrange, which contains the medium-size eddies characterized by the fluctuating velocity u_j' and the dissipation length scale l_ϵ . This consideration directly yields equation (F.1) (e.g. Batchelor, 1960). Now, in this study k is defined at the nozzle exit to be

$$k_0 = \frac{3}{2} i^2 U_0^2 \quad (F.3)$$

where subscript 0 stands for the nozzle exit.

Then, making use of (F.2), equation (F.1) becomes

$$\epsilon_0 = \frac{(1.5)^{3/2} i^3 U_0^3}{\lambda d} \quad (F.4)$$

The effect of ϵ_0 translates itself into varying lengths of the potential core of the jet, for given value of U_0 and d . At low intensities, since experiments show that the length of the potential core is approximately constant for varying low values of i , then ϵ_0 must be approximately independent of i . This implies that

$$\lambda \propto i^3 \quad (F.5)$$

for low intensities at the jet exit.

Reynolds number, Re

It is assumed in this study that the local eddy Reynolds numbers ($Re_\epsilon = \frac{u_j' l_\epsilon}{\nu}$) of the eddies in the inertial subrange (or medium size eddies) are high. Consequently, the magnitude of l_ϵ will depend very little on laminar viscosity, μ . Hence the Reynolds number ($Re = \frac{\rho U_0 d}{\mu}$) influence will be rather small.

Generally, the nozzle-exit Re affects the boundary layer at the point of separation (Bradshaw, 1966), and the higher the Re the smaller the thickness of the initial boundary layer, and hence the greater the "effective" diameter of the nozzle. Consequently, λ may be seen to increase

slightly with increasing Re. From trial-and-error runs based only on changes to Re, it is found that approximately

$$\lambda \propto \text{Re}^{1/8} \quad (\text{F.6})$$

Temperature

For hot jets, it is found that it is only when θ is much higher than a reference value θ_r of about 197° that the influence of θ on l_e becomes appreciable. From trial-and-error runs based on changes to θ , λ was found to vary exponentially in the following manner:

$$\lambda \propto \exp\{(\theta/\theta_r)^n\} \quad (\text{F.7})$$

where $n = 3.104$, and $\theta_r = 1.97 \times 10^2$

The proposed formula for λ

Assembling expressions (F.6), (F.7), and (F.8), there results the following formula for λ :

$$\lambda = A_e \cdot \text{Re}^{1/8} i^3 \exp\{(\theta/\theta_r)^n\} \quad (\text{F.8})$$

where A_e is a constant as defined in the text (Chapter 4).

Note: In making the trial-and-error runs the data used are those indicated in the text, and for each test run the distributions of velocities, temperature, and turbulence energy are compared with the available data.