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On induced action for conformal higher spins in curved background

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Abstract

We continue the investigation of the structure of the action for a tower of conformal higher spin fields in non-trivial 4d background metric recently discussed in Grigoriev and Tseytlin (2016) [15]. The action is defined as an induced one from path integral of a conformal scalar field in curved background coupled to higher spin fields. We analyze in detail the dependence of the quadratic part of the induced action on the spin 1 and spin 3 fields, determining the presence of a curvature-dependent mixed spin 1–3 term. One consequence is that the pure spin 3 kinetic term cannot be gauge-invariant on its own beyond the leading term in small curvature expansion. We also compute the non-zero contribution of the 1–3 mixing term to the conformal anomaly c-coefficient. One is thus to determine all such mixing terms before addressing the question of possible vanishing of the total c-coefficient in the conformal higher spin theory.
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1. Introduction

Conformal higher spin (CHS) theories [1,2] in 4 dimensions generalize the Maxwell ($s = 1$) and Weyl ($s = 2$) theories to higher rank totally symmetric tensors $h_{(s)} = (h_{a_1 \dots a_s})$. They have

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a local but higher-derivative free action $S_s = \int d^4x\, h_{(s)}\, P_s\, \partial^{2s}\, h_{(s)}$ with maximal spin s gauge symmetries, $\delta h_{(s)} = \partial \varepsilon_{(s-1)} + \eta_{(2)}\, \xi_{(s-2)}$, allowing to choose a gauge where $h_{(s)}$ is transverse and traceless.

These symmetries can be systematically described and extended to non-linear interacting level by considering the coupling of CHS fields $h_{(s)} J^{(s)}$ to conserved currents of a free complex scalar theory, $J_{(s)} = \overline{\varphi}\, \mathcal{J}_{(s)}\, \varphi$, $\mathcal{J}_{(s)} = \partial^s + ...$ Integrating out the scalar field one can then obtain a local invariant interacting action for an infinite tower ($s = 0, 1, 2, ...$) of the CHS fields as an *induced* action, *i.e.* as the coefficient of the local (logarithmically UV divergent) part of the scalar effective action $S(h) = \log\det\big[-\partial^2 + \sum_s h_{(s)}\, \mathcal{J}_{(s)}\big]\big|_{UV}$ [3–6].

While it is not clear how to write down this induced action to all orders in an explicit form, few leading cubic and quartic interaction terms beyond the free $h_{(s)}\partial^{2s} h_{(s)}$ term can be found by direct diagrammatic expansion in powers of $h_{(s)}$ [5,7,8]. One can then compute some simplest 4-particle scattering amplitudes due to the exchange of the infinite tower of the CHS fields and conclude that they vanish [7,8] which may be attributed to the presence of a global conformal higher spin symmetry.

To address the question of possible anomalies in the quantum CHS theory [1,6,9] one needs to go beyond a perturbative near-flat-space expansion and determine, *e.g.*, the generalization of the CHS quadratic terms $h_{(s)}\partial^{2s} h_{(s)}$ to a curved background metric. As the free flat-space CHS theory is conformally invariant, this is relatively straightforward to do for a homogeneous conformally-flat background (S^4, (A)dS$_4$, or $\mathbb{R}\times S^3$): in this case the spin s CHS kinetic operator is known explicitly and it factorizes into a product of s second-order differential operators [9–13].[2]

The case of a general background metric appears to be much more complicated.[3] As the conformal spin 2 field h_{ab} should be the fluctuating part of the metric, $g_{ab} = \eta_{ab} + h_{ab}$, finding the background-covariant generalization of the CHS kinetic terms is equivalent to finding an infinite class of interaction terms in the above induced action containing an arbitrary power n of the spin 2 field, *i.e.* $h_{(s)} h_{(s')} (h_{(2)})^n$.

An alternative approach (which should be equivalent to a resummation of the near-flat-space expansion) was suggested in [15]. One starts with an effective particle Hamiltonian in CHS background that makes explicit the full non-linear symmetry of the theory generalizing the construction of [4]. Quantization of this Hamiltonian gives a covariant conformal scalar action in g_{ab} background coupled also to the CHS fields. This determines the background-covariant generalization of the symmetries acting on the scalar field φ and $h_{(s)}$. An underlying assumption is that the induced CHS action should admit the vacuum with the metric $g_{ab} = \eta_{ab} + h_{ab}$ satisfying the Bach-flatness condition (Weyl gravity equation of motion) with all other $h_{(s)}$ fields being zero. It then follows that the resulting CHS kinetic operator should be gauge-invariant, at least to the leading order in small curvature expansion (generalizing the $s = 3$ result of [12]).

Another consequence of the background-covariant generalization of the CHS symmetries suggested in [15] is that, in contrast to earlier expectations, the curved space analog of the CHS ∂^{2s}

[2] This factorization allowed to show that the CHS theory has vanishing 1-loop Casimir energy on $\mathbb{R}\times S^3$ [13] as well as the trivial partition function on S^4 [14]. This is consistent with the vanishing of the 1-loop conformal anomaly a-coefficient after summing up all the spin s contributions [6,9].

[3] One may attempt to construct a covariant generalization of the CHS ∂^{2s} operator by just imposing the required symmetries (covariant analog of spin s gauge invariance and Weyl symmetry). This approach suggests that a spin s CHS operator will no longer factorize if the metric is not conformally flat and allows to construct the spin 3 operator to linear order in a small curvature expansion [12].

kinetic operator should not, in general, be diagonal in spin s. In particular, the spin 1 and spin 3 fields should mix via the curvature terms like $R_{...}\nabla h_{(1)}\nabla h_{(3)} + ...$ [15]. These mixing terms vanish in a conformally flat Einstein space but are non-trivial in general.

The aim in the present paper will be to elaborate on the background-covariant approach of [15], *i.e.* to couple the CHS fields to a scalar field defined on a curved background and then directly compute the resulting induced action, explicitly determining the form of the spin 1–spin 3 mixing term anticipated in [15]. One implication of the presence of this term is that the pure spin 3 quadratic term in the induced action cannot be gauge invariant on its own, beyond the terms linear in the small curvature expansion [12] (we will confirm the result of [12] for such terms directly from the induced action approach).[4]

We will also show that this 1–3 mixing term gives a non-trivial contribution to the UV divergences and hence to the conformal anomalies of the CHS theory: while it does not contribute to the anomaly a-coefficient, it contributes to the c-coefficient. Thus similar mixing terms are to be accounted for when addressing the question of cancellation of conformal c-anomaly [9] in the full CHS theory.

We will start in section 2 with a discussion of the curved space analogs of the conserved flat space scalar field currents that can be coupled to the higher spin fields $h_{(s)}$. We will see that the candidate spin 3 current is conserved only modulo curvature terms implying that the spin 3 coupling is not gauge invariant by itself. This non-invariance can be compensated by a non-trivial transformation of the spin 1 field as, indeed, was predicted by the general analysis of [15]. We will also discuss the difference between the "on-shell" (using scalar field equation) and "off-shell" (manifest) symmetries that require introduction of extra couplings nonlinear in $h_{(s)}$ (one should be able to absorb the latter into a redefinition of the tower of CHS fields to establish an equivalence to the approach of [15]). These non-linear terms may, in principle, contribute "contact terms" to the resulting induced action.

In section 3 we will review the general structure of the induced CHS action in a curved space background starting with the well-known cases of spin 1 and 2 terms. The dependence of the quadratic part of the induced action on the spin 3 field will be studied in detail in section 4. In particular, we will compute the 1–3 mixing action by an explicit covariant background field expansion. We will also discuss the direct computation of the pure spin 3 kinetic term to the leading order in the weak curvature expansion, finding agreement with the result of [12] found from symmetry considerations.

In section 5 we will determine the contribution of the mixed 1–3 term to the Weyl-squared UV divergences, *i.e.* to the conformal anomaly c-coefficient.

Some technical details will appear in several appendices. In particular, in Appendix D we will find the traceless, conserved and (on-shell) gauge-invariant stress tensor for the free spin 3 theory verifying its conformal invariance in the flat space. In Appendix B we will show the vanishing of the linear in $h_{(3)}$ term in the CHS action in an arbitrary curved background which is consistent with the general expectation of the vanishing of the linear terms in the induced action for all spins in Bach-flat backgrounds [15].

[4] It would interesting to see how these conclusions are consistent with the supersymmetry-based approach of [16] where it was suggested to couple $\mathcal{N} = 1$ superconformal higher-spin multiplets (e.g. containing spin 3 field) with $\mathcal{N} = 1$ conformal supergravity (containing conformal spins 2, 1 and 3/2).

2. Scalar field coupled to conformal higher spins

Below we will discuss the coupling of conformal higher spin fields to bilinear currents of a complex scalar field. This is well a known story in flat space [4,5] but the case of a curved metric background is much more complicated and was addressed only recently in [15]. Here we will use a direct approach based on attempting to construct conserved traceless currents with correct flat-space limit. We will concentrate on the low spin cases $s \leq 3$.

2.1. Flat space background

Let us start with a massless complex scalar in 4d flat Minkowski space ($g_{ab} = \eta_{ab}$) with the action $S_0 = \int d^4x\, \overline{\varphi}\, \partial^2 \varphi$. As is well known, one can build in a unique way bilinear currents that are traceless totally symmetric rank s tensors $J_{a_1 \dots a_s}$ and are conserved on the scalar equations of motion $\Box\varphi = 0$, *i.e.* [17,18][5]

$$\partial^{a_1} J_{a_1 \cdots a_s} = 0, \qquad J^{a_1}{}_{a_1 \cdots a_s} = 0\,. \tag{2.1}$$

The lowest-spin examples are

$$J_a = i\, \overline{\varphi}\, \partial_a \varphi + \text{c.c.}, \qquad J_{ab} = (\overline{\varphi}\, \partial_a \partial_b \varphi - 2\, \partial_a \overline{\varphi}\, \partial_b \varphi + \text{c.c.}) + g_{ab}\, \partial^c \overline{\varphi}\, \partial_c \varphi\,, \tag{2.2}$$

$$J_{abc} = 6i \Big[\overline{\varphi}\, \partial_a \partial_b \partial_c \varphi - 9\, \partial_{(a}\, \overline{\varphi}\, \partial_b \partial_{c)}\, \varphi + 3\, g_{(ab}\, \partial^p\, \overline{\varphi}\, \partial_p \partial_{c)}\, \varphi \Big] + \text{c.c.} \tag{2.3}$$

The properties (2.1) imply that the currents may be coupled to conformal higher spin fields $h_{a_1 \cdots a_s}$ by adding to S_0 the source term

$$S_{\rm int} = \sum_s \int d^4x\, h^{a_1 \cdots a_s}(x)\, J_{a_1 \cdots a_s}\,. \tag{2.4}$$

This coupling term is then invariant under the linearized higher spin gauge and the algebraic (or "generalized Weyl") transformations

$$\delta h_{a_1 \cdots a_s} = \partial_{(a_1}\, \varepsilon_{a_2 \cdots a_s)} + g_{(a_1 a_2}\, \xi_{a_3 \cdots a_s)}\,, \tag{2.5}$$

provided one is allowed to drop terms proportional to the free scalar field equation of motion. This linearized on-shell invariance can then be extended to an off-shell invariance of $S_0 + S_{\rm int}$ if one also transforms the scalar field and adds terms linear in h_s to (2.5) (see [4,5] for a general discussion).

One may fix the algebraic invariance by imposing the traceless condition $h^{a_1}{}_{a_1 \cdots a_s} = 0$. The residual gauge transformations preserving this condition are, *e.g.*,

$$\delta h_{ab} = \partial_{(a}\, \varepsilon_{b)} - \tfrac{1}{4}\, g_{ab}\, \partial^c \varepsilon_c\,, \tag{2.6}$$

$$\delta h_{abc} = \partial_{(a}\, \varepsilon_{bc)} - \tfrac{1}{3}\, g_{(ab}\, \partial^p\, \varepsilon_{c)p}\,, \qquad \varepsilon_{ab} = \varepsilon_{ba}, \quad \varepsilon^a{}_a = 0\,. \tag{2.7}$$

[5] Below we will sometimes use shortcut notation: $h_{a_1 \cdots a_s} \equiv h_{(s)}$, $J^{a_1 \cdots a_s} \equiv J^{(s)}$, $J^{(s)}\, h_{(s)} = J^{a_1 \cdots a_s}\, h_{a_1 \cdots a_s}$ and $\partial_{a_1 \cdots a_s} = \partial_{a_1} \cdots \partial_{a_s}$. Symmetrization of indices will be the weighted one as in $A_{(a} B_{b)} = \frac{1}{2!}(A_a B_b + A_b B_a)$, *etc.*

2.2. Curved space background: spins $s \leq 3$

Switching on a curved background metric, our starting point will be the action of a conformally coupled scalar with a higher source term

$$S = S_0 + S_{\rm int} \ ,$$
$$S_0 = \int d^4x \, \sqrt{g}\, \overline{\varphi}\Big(\nabla^2 - \tfrac{1}{6}\, R\Big)\, \varphi, \qquad\qquad S_{\rm int} = \sum_s \int d^4x \, \sqrt{g}\, h^{a_1\cdots a_s}(x)\, J_{a_1\cdots a_s} \ . \quad (2.8)$$

To find suitable higher spin currents $J_{(s)}$ we shall generalize (2.1) and require the covariant conservation of the currents $\nabla^{a_1} J_{a_1\cdots a_s} = 0$ on the scalar equations of motion $(\nabla^2 - \frac{R}{6})\,\varphi = 0$ and tracelessness. We will also add the condition of local Weyl invariance of both S_0 and $S_{\rm int}$ (that implies conformal invariance in flat limit) under

$$\delta_{\rm w} g_{ab} = 2\,\omega\, g_{ab} \ , \qquad \delta_{\rm w}\varphi = -\omega\,\varphi \ , \qquad \delta_{\rm w} h_{a_1\cdots a_s} = 2\,(s-1)\,\omega\, h_{a_1\cdots a_s} \ , \quad (2.9)$$

i.e. will thus demand

$$\nabla^{a_1} J_{a_1\cdots a_s} = 0 \ , \qquad J^{a_1}{}_{a_1\cdots a_s} = 0 \ , \qquad \delta_{\rm w} \int d^4x \, \sqrt{g}\, h^{(s)}\, J_{(s)} = 0 \ . \quad (2.10)$$

In general, we will also require that the curved space currents $J^{(s)}$ have the standard flat space limit (2.2), (2.3). If the tracelessness and the covariant conservation conditions in (2.10) were possible to satisfy we would get (assuming that we may use the scalar field equations) the covariant generalization of the transformations (2.5), *i.e.*

$$\delta h_{a_1\cdots a_s} = \nabla_{(a_1}\, \varepsilon_{a_2\cdots a_s)} + g_{(a_1 a_2}\, \xi_{a_3\cdots a_s)} \ . \quad (2.11)$$

As is well known, the three conditions in (2.10) can be indeed satisfied for spins 1 and 2. The spin 1 case the current is the same as in flat space (2.2) and is again conserved on-shell, *i.e.*

$$J_a = i\,\big(\overline{\varphi}\,\nabla_a\varphi - \nabla_a\overline{\varphi}\,\varphi\big) \ , \qquad \nabla^a J_a = 0 \ . \quad (2.12)$$

The source term $\sqrt{g} g^{ab} h_a J_b$ is Weyl invariant if h_a has weight zero, in agreement with (2.9). The most general Ansatz for the spin 2 current (with correct flat limit in (2.2) for $k_1 = -2$, $k_2 = 1$) is

$$J_{ab} = (\overline{\varphi}\,\nabla_a\nabla_b\,\varphi + k_1\,\nabla_a\overline{\varphi}\,\nabla_b\varphi + \text{c.c.}) + k_2\, g_{ab}\,\nabla_c\overline{\varphi}\,\nabla^c\,\varphi + (k_3\, R_{ab} + k_4\, g_{ab}\, R)\,\overline{\varphi}\,\varphi. \quad (2.13)$$

Imposing the Weyl invariance of $\sqrt{g}\, h^{ab}\, J_{ab}$ with $\delta_{\rm w} h_{ab} = 2\,\omega\, h_{ab}$ as in (2.9) we find that $k_1 = -2$ and $k_3 = -1$. The trace condition $J^a{}_a = 0$ gives $k_2 = 1$ and $k_4 = \frac{1}{6}$. With these coefficients, the current is automatically conserved on-shell, $\nabla^a J_{ab} = 0$. One can then check that this J_{ab} is indeed the stress tensor of the conformally coupled scalar with action S_0 in (2.8)

$$J_{ab} = \frac{6}{\sqrt{g}}\,\frac{\delta S_0}{\delta g^{ab}} = (\overline{\varphi}\,\nabla_a\nabla_b\,\varphi - 2\,\nabla_a\overline{\varphi}\,\nabla_b\varphi + \text{c.c.}) + g_{ab}\,\nabla_c\overline{\varphi}\,\nabla^c\,\varphi - (R_{ab} - \tfrac{1}{6}\, g_{ab}\, R)\overline{\varphi}\,\varphi \ . \quad (2.14)$$

The higher spin cases $s \geq 3$ display new features. The most general Ansatz for the spin 3 current on a curved background is[6]

[6] Here the coefficient k_0 is introduced for generality but will be fixed to 1 later.

$$J_{abc} = 6i\Big[k_0\,\overline{\varphi}\,\nabla_{(a}\nabla_b\nabla_{c)}\,\varphi + k_1\,\nabla_{(a}\,\overline{\varphi}\,\nabla_b\nabla_{c)}\,\varphi + k_2\,g_{(ab}\,\nabla^p\,\overline{\varphi}\,\nabla_p\nabla_{c)}\,\varphi$$
$$+ k_3\,g_{(ab}\,\overline{\varphi}\,\nabla^2\,\nabla_{c)}\,\varphi + k_4\,g_{(ab}\,R\,\overline{\varphi}\,\nabla_{c)}\,\varphi + k_5\,R_{(ab}\,\overline{\varphi}\,\nabla_{c)}\,\varphi\Big] + \text{c.c.} \tag{2.15}$$

Imposing the trace condition $J^a{}_{ab} = 0$ gives

$$k_2 = -\tfrac{1}{3}\,k_1, \qquad k_4 = \tfrac{1}{12}\,k_0 + \tfrac{1}{36}\,k_1 + \tfrac{1}{3}\,k_3, \qquad k_5 = -k_0 - 3\,k_3\ . \tag{2.16}$$

Once the current is traceless the coupling

$$\int d^4x\,\sqrt{g}\,h^{abc}\,J_{abc}\ , \tag{2.17}$$

is invariant under the algebraic symmetry in (2.5), *i.e.* $\delta h_{abc} = g_{(ab}\,\xi_{c)}$, allowing to fix the traceless gauge on h_{abc},

$$h^a{}_{ab} = 0\ . \tag{2.18}$$

In this gauge the g_{ab} terms in (2.15) decouple (*i.e.* can be dropped in (2.15)) and then the Weyl invariance of (2.17) under (2.9) (*i.e.* $\delta_{\rm w} h_{abc} = 4\,\omega\,h_{abc}$) gives the constraints

$$k_1 = -9\,k_0, \qquad k_5 = -7\,k_0\ . \tag{2.19}$$

Thus, the unique traceless current that gives a Weyl invariant source term (2.17) for the traceless spin 3 field is (2.15) with

$$(k_1, k_2, k_3, k_4, k_5) = \big(-9, 3, 2, \tfrac{1}{2}, -7\big)\,k_0\ . \tag{2.20}$$

The explicit form of this J_{abc} that has the right flat space limit (2.3) is thus (we set $k_0 = 1$)

$$J_{abc} = 6i\Big[\overline{\varphi}\,\nabla_{(a}\nabla_b\nabla_{c)}\,\varphi - 9\,\nabla_{(a}\,\overline{\varphi}\,\nabla_b\nabla_{c)}\,\varphi + 3\,g_{(ab}\,\nabla^p\,\overline{\varphi}\,\nabla_p\nabla_{c)}\,\varphi$$
$$+ 2\,g_{(ab}\,\overline{\varphi}\,\nabla^2\,\nabla_{c)}\,\varphi + \tfrac{1}{2}\,g_{(ab}\,R\,\overline{\varphi}\,\nabla_{c)}\,\varphi - 7\,R_{(ab}\,\overline{\varphi}\,\nabla_{c)}\,\varphi\Big] + \text{c.c.} \tag{2.21}$$

Its covariant divergence can be simplified (using the scalar field equations of motion) to

$$\nabla_a\,J^{abc} = i\,\Big(\tfrac{8}{3}\,g^{bc}\nabla^p R\,\overline{\varphi}\nabla_p\varphi - 16\,\nabla^p\,R^{bc}\,\overline{\varphi}\nabla_p\varphi + 8\,\nabla^{(b}R^{c)}{}_p\,\overline{\varphi}\,\nabla^p\varphi$$
$$- \tfrac{4}{3}\,\nabla^{(b}R\,\overline{\varphi}\nabla^{c)}\varphi + 8\,C^{pbcq}\overline{\varphi}\nabla_p\nabla_q\varphi\Big) + \text{c.c.}\ , \tag{2.22}$$

where $C^a{}_{bcd}$ is the Weyl tensor. An equivalent form of (2.22) found in [19] is

$$\nabla_a\,J^{abc} = 8\,C^{pbcq}\,\nabla_{(p}J_{q)} + 32\,\nabla_{(p}\,C^{pbcq}\,J_{q)}\ , \tag{2.23}$$

where J_a is the spin 1 current in (2.12).

The spin-3 current is thus conserved in a conformally flat space but not in a generic curved background. However, the important observation [19,15] is that the combined spin 1 and spin 3 interaction term

$$S_{\rm int}(h_1, h_3) = \int d^4x\,\sqrt{g}\,(h^a\,J_a + h^{abc}\,J_{abc}) \tag{2.24}$$

which is invariant under spin 1 gauge transformation $\delta h_a = \partial_a\varepsilon$ in view of (2.12) can be made invariant also under the curved-space generalization of the spin 3 gauge transformation (2.7) combined with a particular Weyl tensor dependent transformation of the spin 1 field, *i.e.* under

$$\delta h_{abc} = \nabla_{(a}\,\varepsilon_{bc)} - \tfrac{1}{3}\,g_{(ab}\,\nabla^{d}\,\varepsilon_{c)d}\ , \tag{2.25}$$

$$\delta h_{a} = -8\,C_{abcd}\,\nabla^{d}\,\varepsilon^{bc} + 24\,\nabla^{d}C_{abcd}\,\varepsilon^{bc}\ . \tag{2.26}$$

Note that (2.23) and (2.26) simplify on an Einstein background ($R_{ab} = \frac{1}{4}Rg_{ab}$) as then $\nabla^{d}C_{abcd} = 0$ and thus only one Weyl tensor term survives.

Let us mention, as an aside, that one may try to determine the current (2.15) by imposing $\nabla_{a}\,J^{abc} = 0$ before other conditions. One then finds that there are no solutions unless one restricts the background to be Einstein one. In this case one finds that the coefficients in (2.15) should be $k_0 = 0$, $k_2 = -\frac{1}{2}\,k_1$, $k_4 = -\frac{1}{12}\,k_1$, $k_3 = k_5 = 0$. These values are not, however, consistent with the constraints of Weyl invariance (2.16) or tracelessness (2.19). Denoting the current (2.15) with these coefficients by $\widetilde{J}_{abc}$ we get explicitly (choosing $k_1 = -10$)

$$\widetilde{J}_{abc} = -60\,i\left[\nabla_{(a}\,\overline{\varphi}\,\nabla_{b}\nabla_{c)}\,\varphi - \tfrac{1}{2}\,g_{(ab}\,\nabla^{p}\,\overline{\varphi}\,\nabla_{p}\nabla_{c)}\,\varphi - \tfrac{1}{12}\,g_{(ab}\,R\,\overline{\varphi}\,\nabla_{c)}\,\varphi\right] + \text{c.c.} \tag{2.27}$$

This is a non-standard current as it does not reduce to (2.3) in the flat space limit. It is interesting to note that then[7]

$$h^{abc}\,\widetilde{J}_{abc} = h^{abc}\left[J_{abc} - 6\nabla_{(a}\nabla_{b}J_{c)}\right]. \tag{2.28}$$

2.3. *Formulation with manifest symmetries*

In the above discussion of the (linearized) gauge invariance of $S_{\rm int}$ in (2.4) or (2.8) we were assuming the use of the scalar field equation, *i.e.* this invariance was "on-shell" one – valid modulo terms proportional to $\frac{\delta S_0}{\delta\varphi}$. One expects that it should be possible to relax this assumption, *i.e.* to extend the invariance to a manifest (off-shell) one by (i) transforming at the same time the scalar field and (ii) adding higher order terms in the fields $h_{(s)}$.

Let us recall how that happens in the simplest vector field coupling invariant under the $U(1)$ gauge transformations: one introduces the covariant derivatives

$$\mathscr{D}_{a}\,\varphi = (\nabla_{a} + i\,h_{a})\,\varphi, \qquad\qquad \mathscr{D}_{a}\,\overline{\varphi} = (\nabla_{a} - i\,h_{a})\,\overline{\varphi}\ , \tag{2.29}$$

and then the scalar action becomes (here $J_0 \equiv \overline{\varphi}\varphi$)

$$S_0(h_1) = \int d^4x\,\sqrt{g}\,\overline{\varphi}\left(\mathscr{D}^2 - \tfrac{1}{6}\,R\right)\varphi = \int d^4x\,\sqrt{g}\left[\overline{\varphi}\left(\nabla^2 - \tfrac{1}{6}\,R\right)\varphi + h^{a}\,J_a - h^{a}h_{a}\,J_0\right]. \tag{2.30}$$

This action which is different from the sum $S_0 + S_{\rm int}$ in (2.8) by an extra "nonlinear" h_a^2 term is now manifestly invariant under $\delta h_a = \partial_a\varepsilon$ combined with $\delta\varphi = -i\,\varepsilon\varphi$.

It is easy to preserve this off-shell vector gauge invariance in the presence of also higher spin $s \geq 2$ couplings in $S_{\rm int}$ in (2.8) by just replacing $\nabla_a \to \mathscr{D}_a$ in the expression for the bilinear current $J^{(s)}$, thus getting

$$\sum_{s\geq 2} h^{(s)}\,J_{(s)}(\mathscr{D}) = \sum_{s\geq 2} h^{(s)}\Big[\,J_{(s)}(\nabla) + h^{a}\mathrm{T}_{a(s)} + O(h_a^2)\Big]\,, \tag{2.31}$$

where $\mathrm{T}_{a(s)} \equiv (\mathrm{T}_{ab_1\ldots b_s})$ is a bilinear operator that multiplies the term linear in the vector field in $J_{(s)}(\mathscr{D})$.

[7] Combined with $h^{a}\,J_a$ the coupling in (2.28) suggest some special role of the combination $h_a + 6\nabla^{b}\nabla^{c}h_{abc}$; this will be discussed further in Appendix F.

Demanding the off-shell realization of higher $s > 1$ spin symmetries will require also additional non-linear terms in the fields $h_{(s)}$. For example, it is clear how to construct the manifestly covariant coupling to h_{ab}: one is to start with S_0 in (2.8) and replace $g_{ab} \to g_{ab} + h_{ab}$; expanding in powers of h_{ab} will give at linear order the coupling to J_{ab} in (2.14) (up to normalization) plus an infinite series of higher order terms in h_{ab}. One will also be required to transform the scalar as $\delta\varphi = \varepsilon^a \partial_a \varphi$ and to modify the transformation of h_{ab} in (2.11) by order h_{ab} terms to recover the usual form of transformation of $g_{ab} + h_{ab}$ under the diffeomorphisms.

Similarly, for spin 3 one will need to supplement the transformations in (2.25), (2.26) with a transformation of the scalar field to cancel the terms proportional to $\frac{\delta S_0}{\delta\varphi}$ that were dropped in (2.23); that will then require adding also $(h_{abc})^2$ terms in the action $S_{\rm int}$ to compensate for the variation of the $h_{abc} J^{abc}$ term under this transformation of φ, *etc.*

An alternative to this procedure is to follow the approach of [4] (in flat case) and [15] (in curved background) and introduce only linear $h^{(s)}$, $J_{(s)}$ couplings but to the whole tower of the higher spin fields including the scalar h_0 coupled to $J_0 = \overline{\varphi}\varphi$ and transform both φ and $h_{(s)}$. In this case the gauge transformation of $h_{(s)}$ will contain, in addition to $\nabla\varepsilon_{(s-1)}$ term, also terms linear in $h_{(s')}$ [15]. The two approaches should be related by field redefinitions like $h_0 \to h_0 - h^a h_a$ and so on (*cf.* [8]).

Starting with an action $S(\varphi, h)$ which contains all necessary terms to be manifestly invariant under some local transformation[8] $\delta\varphi = F(\varepsilon; \varphi, h)$, $\delta h = f(\varepsilon; h)$, and then integrating out φ one should get the (full, non-local) effective action Γ in

$$Z = e^{-\Gamma(h)} = \int d\varphi\, e^{-S(\varphi,h)} \,, \tag{2.32}$$

which should be formally invariant under $\delta h = f(\varepsilon, h)$. As Γ is given just by a 1-loop determinant (φ does not have self-interactions) its logarithmically UV singular part is local and cannot contain any anomalies

$$\Gamma(h) = \log \Lambda_{\rm UV}\, \mathrm{S}(h) + \dots \,. \tag{2.33}$$

Thus $\mathrm{S}(h)$ (that we shall call the *induced* action) should be manifestly invariant under the above transformations of h.

Suppose we start instead with an action $\widetilde{S}(\varphi, h) = S_0(\varphi) + h \cdot J$ that contains only linear in h terms and is invariant under $\delta h = f(\varepsilon; h)$ only on-shell, *i.e.* up to terms proportional to the free φ equation of motion. As the terms proportional to the equations of motion contribute delta-functions to the coordinate-space correlators of J, they can be ignored as usual in the correlation functions at separated points which will thus be invariant. However, the corresponding local induced action $\widetilde{\mathrm{S}}(h)$ is no longer guaranteed to be invariant under $\delta h = f(\varepsilon; h)$.

Indeed, in the vector coupling case (*cf.* (2.30)) it is easy to see that starting just with the minimal $h^a J_a$ coupling term one gets the induced action containing non-invariant $(h^a h_a)^2$ term. Same will happen for higher spin couplings. It should be possible to eliminate such non-invariant terms by a field redefinition provided one considers $\widetilde{\mathrm{S}}(h)$ for the whole tower of the conformal higher spin fields. For example, including non-zero scalar h_0 we will get the term $(h_0 + h^a h_a)^2$ and thus non-invariant $(h^a h_a)^2$ term can be redefined away by a shift of h_0. This has, of course, an explanation in terms of the off-shell invariance of the action $S(\varphi, h)$ in (2.30) that has the term $h^a h_a J_0$ present there. Similar observations should apply to higher spin cases as well.

[8] Here ε may stand for parameters of Weyl, gauge, or algebraic symmetries as in (2.9), (2.11).

3. Structure of the induced action

Starting with the reparametrization and vector gauge invariant conformal scalar action (2.30) and integrating φ out the resulting induced action for g_{ab} and h_a (*i.e.* the coefficient of the logarithmic UV divergence in the effective action (2.33)) will take the familiar form[9]

$$\mathrm{S} = \int d^4x\,\sqrt{g}\,\Big(-\tfrac{1}{12}\,F_{ab}^2 + \tfrac{1}{120}\,C_{abcd}^2\Big)\,, \tag{3.1}$$

where $F_{ab} = \partial_a h_b - \partial_b h_a$ and C_{abcd} is the Weyl tensor. S is invariant under the reparametrizations, vector gauge symmetry and the Weyl symmetry.

One may systematically obtain S by expanding $\exp(-S)$ in (2.32) in powers of the fields $h_{(s)}$ and computing the UV singular parts of the resulting correlators of the currents on a curved background using, *e.g.*, the covariant methods of [21–30]. For example, computing the correlator of the vector current $\langle J^a J^b\rangle$ using the dimensional regularization expressions in Appendix A of [25] one finds

$$\int d^4x\,\sqrt{g}\,h^a\,\langle J_a\,J_b\rangle_{\rm UV}\,h^b = -\tfrac{1}{6}\int d^4x\,\sqrt{g}\,F_{ab}^2\,. \tag{3.2}$$

Here and below $\langle ...\rangle_{\rm UV}$ will stand for the coefficient of the logarithmically divergent (or pole $1/\varepsilon \sim \log\Lambda_{\rm UV}$) part of a correlator. The $h^a h_a$ term in the manifestly gauge-invariant action (2.30) does not contribute at h^2 order as $\langle J_0\rangle_{\rm UV} = 0$; it produces the h^4 term that cancels, however, against other h^4 contributions so the final result is in agreement with (3.1).

A similar approach can be used in the spin 2 case. If one starts with the scalar action S_0 in (2.8) and adds just a linear coupling term $h^{ab}J_{ab}$ then the coefficient of the linear in h_{ab} term in the corresponding induced action S will be given by the UV log divergent part of the 1-point function of the spin 2 current, *i.e.* $\langle J_{ab}\rangle_{\rm UV}$, which turns out (*cf.* (4.8), (4.9)) to be proportional to the Bach tensor B_{ab} (defined in Appendix A). To reproduce the gauge-covariant quadratic $h_{(2)}\mathcal{O}_4 h_{(2)}$ term one will need, in general, to add to $\langle J_{ab}J_{cd}\rangle_{\rm UV}$ also the "contact term" contribution of the quadratic $h^{ab}\mathrm{T}_{abcd}h^{cd}$ coupling term in the manifestly gauge-invariant scalar action.[10]

In the spin 2 case there is a short-cut: we may shift the metric $g_{ab} \to g_{ab} + h_{ab}$ in the conformally coupled scalar action to isolate the spin 2 coupling as in (2.8), (2.13); then the resulting dependence of S on h_{ab} should be given by the expansion of the C^2 term in (3.1)

$$\int d^4x\sqrt{g}\,C_{abcd}^2 \to \int d^4x\sqrt{g}\Big[B_{ab}(g)h^{ab} + h^{ab}\mathcal{O}_{abcd}(g)h^{cd} + ...\Big]\,, \tag{3.3}$$

where B_{ab} is the Bach tensor and the operator $\mathcal{O}_4 \sim \nabla^4 + ...$ is reparametrization and Weyl invariant and invariant under the algebraic symmetry $\delta h_{ab} = g_{ab}\xi$. The invariance of the $\mathcal{O}_4$ term under the gauge symmetry $\delta h_{ab} = \nabla_{(a}\varepsilon_{b)}$ requires the cancellation of the linear in h_{ab} term in (3.3), *i.e.* constraining g_{ab} by the condition of Bach-flatness, $B_{ab} = 0$. We also note that the quadratic coupling term $h^{ab}\mathrm{T}_{abcd}h^{cd}$ in the covariant scalar action found by shifting $g_{ab} \to g_{ab} + h_{ab}$ in S_0 in (2.8) and expanding in h_{ab} will contain (i) part with derivatives acting on h (coming from $R/6$-term) and thus giving zero contribution as $\langle J_0\rangle_{\rm UV} = 0$, and (ii) part with

[9] This expression is given by the relevant Seeley–de Witt coefficient of the conformally coupled scalar Laplacian, see, *e.g.*, [20]. Here we dropped a total derivative $\sim R^\star\,R^\star$ term.

[10] Here $\mathrm{T}_{(4)}$ is a scalar bilinear operator containing 2 derivatives. As discussed above, similar non-linear in $h_{(s)}$ terms in the scalar action $S(\varphi, h)$ can be reconstructed order by order by demanding its manifest (off-shell) invariance.

two derivatives acting on the scalar field and thus similar to (2.13), with the resulting contribution again proportional to B_{ab}. Thus its contribution can be ignored on the Bach-flat backgrounds.

Similar remarks apply to higher spin coupling terms. In general, the expansion of the induced action $\mathrm{S}(g,h)$ in powers of $h_{(s)}$ should be [15]

$$\mathrm{S}(g,h) = \mathrm{S}^{(0)}(g) + \mathrm{S}^{(1)}(g,h) + \mathrm{S}^{(2)}(g,h) + \dots , \tag{3.4}$$

$$\mathrm{S}^{(1)} = \sum_s \int d^4x \sqrt{g}\, B_{(s)}(g)\, h^{(s)} , \qquad \mathrm{S}^{(2)} = \sum_{s,s'} \int h^{(s)}\, \mathcal{O}_{s,s'}(g)\, h^{(s')} , \quad \dots \tag{3.5}$$

This action should have manifest reparametrization and Weyl symmetries. $\mathrm{S}^{(0)}(g)$ is the Weyl tensor term in (3.1) (while spin 1 term in (3.1) is included in $\mathrm{S}^{(2)}$).

Ignoring total derivatives, the coefficient of the linear term $\langle J_{(s)} \rangle_{\rm UV} \sim B_{(s)}(g)$ should be a local function of the metric g and its derivatives, which is covariantly conserved, traceless and Weyl-covariant. Explicitly, $B_a = 0$, B_{ab} is the Bach tensor and as we will show in Appendix B $B_{abc} = 0$ for any background.

As was argued in [15], for general s, the tensor $B_{(s)}(g)$ should vanish on a Bach-flat background, at least up to terms quadratic in the curvature of the background metric. The vanishing of $B_{(s)}(g)$ is required in order for the Bach-flat metric g_{ab} along with $h_{(s)} = 0$ be the vacuum of the full CHS action $\mathrm{S}(g,h)$. In that case the quadratic term $\mathrm{S}^{(2)}$ which, in general, is non-diagonal in s, s', should be invariant under the background-covariant gauge and algebraic transformations of the CHS fields generalizing (2.11) (like (2.3), *etc.*).

The operator $\mathcal{O}_{s,s'}(g)$ should, in general, receive contribution from $\langle J_{(s)} J_{(s')} \rangle_{\rm UV}$ as well as from the contact term $X_{(s)(s')} = \langle \mathrm{T}_{(s)(s')} \rangle_{\rm UV}$ coming from the quadratic term $h^{(s)} \mathrm{T}_{(s)(s')} h^{(s')}$ required for the manifest covariance of the scalar action (*cf.* (2.30), (2.31)).

We shall make the conjecture that $X_{(s)(s')} = 0$ on a Bach-flat background. As was mentioned above, this is true in spin 2 case where $X_{(2)(2')}$ is proportional to the Bach tensor. In general, since the dimension of the CHS field $h_{(s)}$ is $2 - s$ (so that the interaction action in (2.4) is dimensionless) the product $h_{(s)} h_{(s')}$ has the same dimension as $h_{(s'')}$ with $s'' = s + s' - 2$, and thus it may be possible to eliminate the $h^{(s)} \mathrm{T}_{(s)(s')} h^{(s')}$ term by a redefinition of $h_{(s'')}$ in the $h^{(s'')} J_{(s'')}$ coupling. As the linear term in $h^{(s'')}$ in S_1 in (3.5) should vanish on a Bach-flat background, the same should then apply to the contribution of $h^{(s)} \mathrm{T}_{(s)(s')} h^{(s')}$ to $\mathcal{O}_{s,s'}(g)$. Equivalently, as the scalar bilinear operator $X_{(s)(s')}$ has the same dimension as $J_{(s'')}$ it is natural to expect that the (reparametrization and Weyl covariant) expectation value $X_{(s)(s')} = \langle \mathrm{T}_{(s)(s')} \rangle_{\rm UV}$ should also vanish on a Bach-flat background, as it happened in the case of $\langle J_{(s'')} \rangle_{\rm UV}$. For example, like the h_a^2 term in (2.30) can be absorbed into a redefinition of the CHS scalar h_0, a possible $h_{(1)} h_{(3)}$ term in (2.31) may be absorbed into a redefinition of $h_{(2)}$. Indeed, we will check below that $X_{(1)(3)} = \langle \mathrm{T}_{(1)(3)} \rangle_{\rm UV}$ vanishes on a Bach-flat background. Similarly, it should be possible to absorb the $h^{(3)} \mathrm{T}_{(3)(3)} h^{(3)}$ term in the scalar action into $h^{(4)} J_{(4)}$ so that its tadpole contribution should be proportional to the variation of a linear term $B_{(4)}(g)\, h^{(4)}$ and should thus vanish along with $B_{(4)}$ if $B_{ab} = 0$.

4. Spin 3 induced action

Below we will study in detail the dependence of the quadratic part $\mathrm{S}^{(2)}$ of the induced action on the spin 3 field, and, in particular, its mixing with the spin 1 field anticipated in [15]. Our starting point will be the manifestly vector gauge covariant form of the scalar action (2.30), (2.31). We will choose h_{abc} to be traceless.

As the linear in $h_{(3)}$ term in the induced action in (3.5) vanishes (as shown in Appendix B, $\langle J_{abc}\rangle_{\rm UV}=0$) the induced action in the spin 1 plus spin 3 sector starts with a quadratic term

$$\mathrm{S}^{(2)}=\mathrm{S}_{11}+\mathrm{S}_{13}+\mathrm{S}_{33}\ , \tag{4.1}$$

where $\mathrm{S}_{ss'}$ is a term bilinear in $h_{(s)}$ and $h_{(s')}$. S_{11} is the Maxwell action in (3.1), (3.2). The 1–3 mixing term will have two contributions:

$$\mathrm{S}_{13}=\mathrm{S}^{(\rm a)}_{13}+\mathrm{S}^{(\rm b)}_{13}\ , \tag{4.2}$$

where $\mathrm{S}^{(\rm a)}_{13}$ will come from the correlator $\langle J_{(1)}\,J_{(3)}\rangle_{\rm UV}$ and $\mathrm{S}^{(\rm b)}_{13}$ from the contact term $\langle \mathrm{T}_{(1)(3)}\rangle_{\rm UV}$ (see (2.31)). Similarly, the S_{33} term

$$\mathrm{S}_{33}=\int d^4x\sqrt{g}\,h_{(3)}\mathcal{O}_6\,h_{(3)}\ , \tag{4.3}$$

will come from the correlator $\langle J_{(3)}\,J_{(3)}\rangle_{\rm UV}$ and also from the contact term $X_{(3)(3)}=\langle \mathrm{T}_{(3)(3)}\rangle_{\rm UV}$ that originates from the $h_{(3)}\mathrm{T}_{(3)(3)}h_{(3)}$ term in the manifestly covariant scalar action.

4.1. Spin 1–3 mixing term

A long straightforward calculation using covariant methods of [21–30] shows that the contribution to S_{13} coming from the UV singular part of the 2-current correlator is given by (see Appendix C for some details of this computation)

$$\mathrm{S}^{(\rm a)}_{13}=\int d^4x\,\sqrt{g}\,h^a\,\langle J_a\,J_{bcd}\rangle_{\rm UV}\,h^{bcd}=\int d^4x\,\sqrt{g}\,\mathscr{L}^{(\rm a)}_{13}\ , \tag{4.4}$$

where

$$\begin{aligned}\mathscr{L}^{(\rm a)}_{13}=&-\tfrac{74}{15}R^{bc}Rh^ah_{abc}-\tfrac{28}{15}h^a\,C_b{}^{def}C_{cdef}h_a{}^{bc}+\tfrac{116}{15}h^aC_b{}^{def}C_{cedf}\,h_a{}^{bc}\\&+\tfrac{74}{5}R_b{}^dR^{bc}h^ah_{acd}-\tfrac{38}{5}R^{bc}h^aC_{bdce}h_a{}^{de}+\tfrac{4}{5}R^{bc}h^aC_{adce}h_b{}^{de}\\&-8h^aC_a{}^e{}_b{}^fC_{cedf}\,h^{bcd}+\tfrac{1}{6}h^a\nabla^bR\nabla_ch_{ab}{}^c+\tfrac{13}{3}h^ah_{abc}\nabla^c\nabla^bR\\&+11h^a\nabla_aR^{bc}\nabla_dh_{bc}{}^d-11h^a\nabla^cR_a{}^b\nabla_dh_{bc}{}^d-\tfrac{37}{5}h^a\,h_{abc}\nabla_d\nabla^dR^{bc}\\&+10h^a\nabla_ch_{abd}\,\nabla^dR^{bc}\\&-10h^a\nabla_dh_{abc}\nabla^dR^{bc}+\tfrac{56}{5}\,h^ah_{bcd}\nabla^d\nabla_aR^{bc}-\tfrac{56}{5}h^a\,h_{bcd}\nabla^d\nabla^cR_a{}^b\\&+6h^a\nabla_bh^{bcd}\nabla_eC_{acd}{}^e+\,12h^a\nabla^dh_a{}^{bc}\nabla_eC_{bdc}{}^e\,+\tfrac{32}{5}h^ah^{bcd}\nabla_e\nabla_dC_{abc}{}^e\\&+\tfrac{24}{5}h^ah_a{}^{bc}\nabla_e\nabla_dC_b{}^d{}_c{}^e+8h^a\nabla_dC_{abce}\nabla^eh^{bcd}+8h^a\,C_{acde}\nabla^e\nabla_bh^{bcd}\\&+8h^aC_{bdce}\nabla^e\nabla^dh_a{}^{bc}\ .\end{aligned} \tag{4.5}$$

The action (4.4) is invariant under the Weyl transformations (2.9) but is not vector gauge invariant: to restore this invariance we need to add the "tadpole" contribution of the mixed 1–3 term in (2.30), *i.e.*

$$\mathscr{L}^{(\rm b)}_{13}=\langle\mathcal{T}\rangle_{\rm UV}\ , \tag{4.6}$$

$$\begin{aligned}\mathcal{T}\equiv h^d\,\mathrm{T}_{dabc}\,h^{abc}=&\big(-12\,h^a\nabla_c\nabla_bh_a{}^{bc}+84\,R^{bc}\,h^ah_{abc}\big)\overline{\varphi}\varphi\,-60\,h^a\,\nabla^ch_{abc}\,\nabla^b(\overline{\varphi}\varphi)\\&+120\,h^ah_{abc}\big[\nabla^b\overline{\varphi}\nabla^c\varphi-(\overline{\varphi}\nabla^b\nabla^c\varphi+\varphi\nabla^b\nabla^c\overline{\varphi})\big]\ .\end{aligned} \tag{4.7}$$

To compute the tadpole contribution[11] $\langle \mathcal{T} \rangle_{\rm UV}$ we note that for a conformally coupled scalar the pole part of the 2-point function vanishes, $\langle \overline{\varphi} \varphi \rangle_{\rm UV} = 0$ (see, *e.g.*, [33]). Then dropping a total derivative term we get

$$\langle \mathcal{T} \rangle_{\rm UV} = 60\, h^a\, h_{abc}\, \langle J_{ab} \rangle_{\rm UV} \ , \tag{4.8}$$

where J_{ab} is the stress tensor of the conformal scalar defined in (2.14). The UV singular part of the expectation value of the scalar stress tensor should be related to the derivative of the C^2_{abcd} logarithmic divergence in the effective action (*cf.* (3.1)) and should thus be proportional to the Bach tensor in (A.2). Indeed, as follows from eq. (6.4) in [21],[12]

$$\begin{aligned} \mathscr{L}^{(\rm b)}_{13} = \langle \mathcal{T} \rangle_{\rm UV} &= 6\, B^{ab} h_{abc}\, h^c \\ &= \big(-2\, R^{ab} R + 6\, R^a_d R^{bd} + 6 R_{ed} C^{eabd} + \nabla^a \nabla^b R - 3 \nabla^2 R^{ab} \big) h_{abc}\, h^c \ . \end{aligned} \tag{4.9}$$

Thus the tadpole contribution vanishes on a Bach-flat background in agreement with the above discussion.

Adding (4.9) to (4.5) we get a simple manifestly vector gauge invariant expression

$$\mathscr{L}_{13} = \mathscr{L}^{(\rm a)}_{13} + \mathscr{L}^{(\rm b)}_{13} = 8\, F^{ab} \Big[C_a{}^{cdp}\, \nabla_p h_{bcd} + \big(\nabla_a R^{cd} - \nabla^c R^d_a \big) h_{bcd} \Big] \ , \tag{4.10}$$

where $F_{ab} = \partial_a h_b - \partial_b h_a$ and h_{abc} is totally symmetric and traceless. Like each of the $\sqrt{g} \mathscr{L}^{(\rm a)}_{13}$ and $\sqrt{g} \mathscr{L}^{(\rm b)}_{13}$ terms, their sum $\sqrt{g} \mathscr{L}_{13}$ is also invariant under the Weyl transformations (*cf.* also the discussion at the end of Appendix B). Both $\mathscr{L}^{(\rm a)}_{13}$ and $\mathscr{L}^{(\rm b)}_{13}$ vanish on a conformally-flat Einstein space.

4.2. Spin 3 gauge invariance

As we saw in section 2, the spin 3 interaction term (2.17) is not invariant under the curved background spin 3 gauge transformation (2.11) – we need also to transform the spin 1 field in the interaction term in (2.8) according to (2.26). If we specify the metric to be, *e.g.*, the Einstein one (*i.e.* a particular Bach-flat one) then $\nabla^a C_{abcd} = 0$ and (2.25), (2.26) simplify to

$$\delta h_{abc} = \nabla_{(a}\, \varepsilon_{bc)} - \tfrac{1}{3}\, g_{(ab}\, \nabla^p\, \varepsilon_{c)p} \ , \qquad\qquad \delta h_a = -8\, C_{abcp}\, \nabla^p\, \varepsilon^{bc} \ . \tag{4.11}$$

Using (3.1) and that $R_{ab} = \frac{1}{4} R g_{ab}$ in (4.10), the quadratic part of the induced action (4.1) may then be written as

$$\mathrm{S}^{(2)} = \int d^4x\, \sqrt{g} \Big[-\tfrac{1}{12}\, F^2_{ab} + 8\, C^{abcd}\, F_{ap}\, \nabla_d h^p{}_{bc} + h_{(3)}\, \mathcal{O}_6\, h_{(3)} \Big] \ . \tag{4.12}$$

One can then check that the transformation of h_a in (4.11) in the first term in (4.12) combined with the transformation of h_{abc} in the second mixed term in (4.12) leaves the total action invariant (see Appendix E).

[11] In a massless theory, one has to be careful to isolate the UV poles from the IR ones. In curved space case, the curvature plays the role of an effective IR scale (which can be captured by a resummation of an infinite set of terms in near-flat-space expansion). Taking this into account, the tadpoles lead to non-trivial contributions to logarithmic UV divergences. Various point-splitting treatments of tadpoles in curved space are discussed in [21,31–33].

[12] See also [34] for a detailed discussion of regularization issues in the vacuum expectation value of the stress tensor.

The transformation of the second term in (4.12) under the transformation of h_a should cancel against the transformation of the last pure spin 3 term in (4.12) under the variation of h_{abc} in (4.11). As a result, the last term S_{33} in the action (4.1) (and thus the operator $\mathcal{O}_6 \sim \nabla^6 + ...$ in (4.12)) cannot be, in general, invariant under the spin 3 gauge transformations even on an Einstein background, contrary to what one might naively expect.[13]

Since the variation of h_a in (4.11) is linear in the Weyl tensor, the variation of the second term in (4.12) has the structure $C\nabla(C\nabla\varepsilon)\nabla h_{(3)}$, *i.e.* is of *second* order in the curvature. Thus the $h_{(3)}\,\mathcal{O}_6\,h_{(3)}$ term may be invariant on its own at leading linear order in the curvature in a small curvature expansion. Indeed, such an operator was constructed in [12] starting from the condition of such linearized spin 3 gauge invariance. Below we shall reproduce this result by directly computing the leading term in the induced action S_{33} in the near-flat-space expansion.[14]

4.3. Pure spin 3 term

As was discussed above, the term S_{33} in (4.1) may receive contributions from (i) the correlator $\langle J_{(3)}(x)\,J_{(3)}(x')\rangle_{\rm UV}$ and (ii) the tadpole term $X_{(3)(3)} = \langle {\rm T}_{(3)(3)}(x)\rangle_{\rm UV}$ coming from the quadratic $h_{(3)}{\rm T}_{(3)(3)}h_{(3)}$ term in the manifestly spin 3 gauge-covariant scalar action. The latter should vanish on a Bach-flat background as discussed above.

The exact computation of $\langle J_{(3)}(x)\,J_{(3)}(x')\rangle_{\rm UV}$ with spin 3 current given in (2.21) in the background-covariant approach is technically challenging and will not be attempted here. We shall discuss only the flat space case and the near-flat-space expansion to leading order in the curvature making contact with an earlier result of [12].

In the flat space limit, we have

$$S_{33}^{\rm flat} = \int d^4x\,\mathscr{L}_{33}^{\rm flat}\,, \qquad\qquad \mathscr{L}_{33}^{\rm flat} = \tfrac{1}{2}\,h^{abc}\,\langle J_{abc}\,J_{a'b'c'}\rangle_{\rm UV}^{\rm flat}\,h^{a'b'c'}. \tag{4.13}$$

The correlator $\langle J_{(3)}(x)\,J_{(3)}(x')\rangle^{\rm flat}$ of the flat-space spin 3 current (2.3) computed in the free scalar CFT is given by the transverse traceless spin 3 projector operator times $|x-x'|^{-6}$. To extract the UV pole we may use, *e.g.*, the dimensional regularization as in [35]. The resulting expression is[15]

$$\begin{aligned}\mathscr{L}_{33}^{\rm flat} = \tfrac{7}{45}\big(&h^{abc}\Box^3\,h_{abc} - \tfrac{2}{5}\,h^{abc}\,\partial_{abcdef}\,h^{def}\\ &+ \tfrac{12}{5}\,h^{abc}\Box\,\partial_{bcde}\,h_a{}^{de} - 3\,h^{abc}\,\Box^2\,\partial_{cd}\,h_{ab}{}^{d}\big)\,.\end{aligned} \tag{4.14}$$

As expected, the spin 3 CHS Lagrangian (4.14) is invariant under the gauge transformations (2.5).[16] It is also scale-invariant, and, being a flat limit of a full Weyl-invariant action, it should

[13] See Appendix F for a discussion of how this may change if one requires only the invariance under the restricted spin 3 gauge transformations with $\nabla_a\varepsilon^{ab} = 0$.

[14] The operator $\mathcal{O}_6$ found in [12] was not unique, so the matching to our result for the induced action requires fixing the remaining freedom in [12] in a particular way.

[15] The computation amounts to the evaluation of $\partial_{a_1}\partial_{a_2}\cdots G(x-x')\,\partial_{b_1}\partial_{b_2}\cdots G(x-x')$, where G is the free scalar propagator $G(x) = \frac{1}{8\pi^2\,\sigma}$ with $\sigma = \frac{1}{2}\,(x-x')^2$. Taking derivatives and using the relation $\partial^2\sigma^{-p} = 2\,p\,(p-1)\sigma^{-p-1}$, we may reduce all terms in (4.13) to the form $h^{abc}(x)\,h^{a'b'c'}(x')\,P_{abc,a'b'c'}(x-x')\,(\partial_x^2)^k\,\frac{1}{\sigma^2}$ where $P(x-x')$ is a tensor built with the displacement vector $(x-x')_a$, and $\frac{1}{\sigma^2} \sim \frac{1}{\varepsilon}\,\delta^{(4)}(x-x')$ gives the pole in dimensional regularization (see, for instance, eq. (A.1) of [25]). The final result is obtained by integrating by parts the ∂_x^2 operators and taking in the end the coincidence limit $x \to x'$.

[16] Notice that the gauge invariance fixes the coefficients in (4.14) up to an overall proportionality constant.

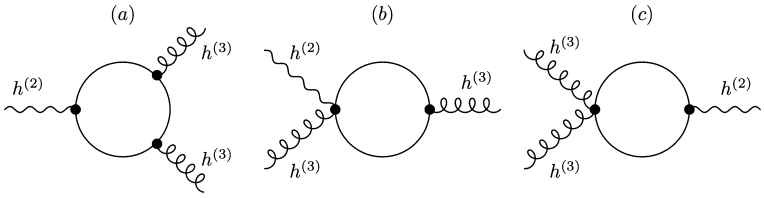


Fig. 1. Scalar field one-loop diagrams with UV divergent parts contributing to the induced action S_{233}. Dots stand for the insertions of the spin 2 or spin 3 flat-space currents in (2.2), (2.3) or the quadratic $h_{(2)}h_{(3)}$ and $h^2_{(3)}$ vertices in the scalar action required for its manifest covariance.

also have the full conformal symmetry. This is indeed the case as we demonstrate in Appendix D: the Lagrangian (4.14) admits a symmetric traceless stress tensor which is conserved and gauge invariant on the spin 3 equations of motion.

Next, we may compute the first correction to (4.14) in the near-flat-space expansion, *i.e.* at the leading order in $h_{ab} = g_{ab} - \delta_{ab}$. Schematically,

$$S_{33} = S_{33}^{\rm flat} + S_{233} + \cdots , \qquad S_{233} = \int d^4x\, h_{(2)} \left[h_{(3)}\, \partial^6 h_{(3)} + \partial h_{(3)}\, \partial_5\, h_{(3)} + \cdots \right] . \tag{4.15}$$

Since the transformation of h_a under the spin 3 gauge transformations given in (2.26) involves already one power of the curvature, the term S_{233} can not mix with the spin 1–3 term S_{13} and should thus be invariant under the linearized spin 3 gauge transformations on its own.

The first correction S_{233} is given by the sum of the three contributions shown in Fig. 1, *i.e.*

$$S_{233} = S^{(a)}_{233} + S^{(b)}_{233} + S^{(c)}_{233}. \tag{4.16}$$

To simplify the computation we shall assume that the external fields $h_{(2)}$ and $h_{(3)}$ are transverse and traceless (TT). The first diagram (a) has three flat-space current vertices that have a simple form $h^{(s)}\, \overline{\varphi}\, \partial^s\, \varphi$ (see (2.2), (2.3) and [8]). The explicit form of the vertex $h_{(2)}h_{(3)}\overline{\varphi}\varphi$ in the diagram (b) is found by expanding the curved-space source term (2.17) in powers of h_{ab}

$$\int d^4x\, \sqrt{g}\, h^{abc}\, J_{abc}(\nabla) = \int d^4x\, h^{abc}\, J_{abc}(\partial) + \int d^4x\, h^{ab} \mathcal{V}_{ab}(h_{(3)}, \varphi) + \mathcal{O}(h^2_{ab}) , \tag{4.17}$$

where the explicit form of $\mathcal{V}_{ab}$ is given in Appendix G.

The third diagram involves the $h_{(3)}h_{(3)}\overline{\varphi}\varphi$ vertex required for the manifest covariance of the scalar action under the spin 3 gauge transformations. As discussed above, its contribution is expected to vanish on a Bach-flat (*e.g.* Einstein) space. We shall impose, for simplicity, the condition that $g_{ab} + h_{ab}$ is Einstein, which (for the TT field h_{ab}) amounts to the condition $\Box h_{ab} = 0$ to be assumed below.

The explicit results for the contributions of the diagrams (a) and (b) are given by (G.2) and (G.3). The total action S_{233} turns out to be consistent with the result of [12] for the linear in curvature term in the S_{33} action where it was found by demanding the spin 3 gauge invariance to the leading order in curvature expansion.[17] As a simple illustration of the agreement, let us formally set h_{ab} to be constant (*i.e.* ignore all curvature terms). Then

[17] We have used the Mathematica notebook provided in [12] to check that they result $S^{\rm NT}_{233}$ is equivalent to ours S_{233} for a particular choice of free parameters in [12] associated with field redefinitions, total derivatives, and use of 4d identities and after restricting to TT fields and imposing the linearized Einstein space constraint $\Box h_{ab} = 0$.

$$\mathscr{L}^{\rm NT}_{233} = -\tfrac{135}{7}\big(h^{ab}h^{cde}\,\partial_{ab}\Box^2 h_{cde} + h^{ab}h_a{}^{cd}\,\Box^3 h_{bcd}\big) = \mathscr{L}_{233} = \mathscr{L}^{(a)}_{233} + \mathscr{L}^{(b)}_{233}\,, \qquad (4.18)$$
$$\mathscr{L}^{(a)}_{233} = -\tfrac{135}{7}\big(h^{ab}h^{cde}\,\partial_{ab}\Box^2 h_{cde} - h^{ab}h_a{}^{cd}\,\Box^3 h_{bcd}\big),$$
$$\mathscr{L}^{(b)}_{233} = -\tfrac{270}{7}\,h^{ab}h_a{}^{cd}\,\Box^3 h_{bcd}\,,$$

where we made a field rescaling to account for a difference in our choice of normalizations compared to [12].[18]

5. Spin 1–3 mixing term contribution to UV divergences

Starting with the induced action for the tower of CHS fields one may attempt to compute the corresponding UV divergences and thus conformal anomalies. Assuming we expand near the vacuum point (with Bach-flat metric) so that all linear terms in (3.4) vanish, the action will begin with the quadratic term $\mathrm{S}^{(2)}$ in (3.5) and thus the 1-loop correction to the CHS partition function will be expressed in terms of determinants of the operators $\mathcal{O}_{s,s'}$ in (3.5).

In general, the logarithmic divergences or conformal anomalies in curved $\mathrm{d}=4$ background are governed by the coefficients a and c in the corresponding Seeley coefficient (see, *e.g.*, [36])

$$\Gamma = -\log Z_{\rm CHS} = -\tfrac{1}{(4\pi)^2}\log\Lambda_{\rm UV}\int d^4x\,\sqrt{g}\,b_4(x) \;+\; \text{finite}\,, \qquad (5.1)$$
$$b_4 = -\,\mathrm{a}\,R^*\,R^* + \mathrm{c}\,C^2\,. \qquad (5.2)$$

To extract the coefficients a and c one may compute b_4 separately in a conformally flat Einstein background where $b_4 = -\mathrm{a}\,R^*R^*$ and in a Ricci flat background where $b_4 = (\mathrm{c}-\mathrm{a})\,C^2$.

In the conformally flat case the CHS kinetic operators are diagonal in spin and factorize into products of second-derivative operators [9,11,12] and thus the corresponding a-anomaly coefficient can be computed [9] using standard methods like in [37][19]

$$\mathrm{a}_s = \tfrac{1}{720}\,\nu_s\,(3\,\nu_s + 14\,\nu_s^2)\,, \qquad \nu_s \equiv s(s+1)\,. \qquad (5.3)$$

In the special cases of conformal spins 2 (*i.e.* Weyl graviton) and 3/2 (conformal gravitino) this factorization on (A)dS$_4$ or S^4 background was observed long ago in [38–41].

The factorization of the Weyl graviton and conformal gravitino kinetic operators turns out to hold also in a Ricci-flat background [39,1]. In [9] it was conjectured that this factorization may apply to all CHS kinetic operators in $R_{ab}=0$ background leading to the following prediction for the spin s contribution to the c-coefficient in (5.2)

$$\mathrm{c}_s = \tfrac{1}{720}\,\nu_s\,(4 - 42\,\nu_s + 29\,\nu_s^2)\,. \qquad (5.4)$$

As was argued in [12], the Ricci-flat factorization conjecture may not be true in general for $s>2$ as there should be curvature derivative dependent terms like $\nabla^n R_{\dots}$ that represent obstructions to factorization. However, such terms can not contribute to the UV divergences (5.1), (5.2) and thus to the value of c_s on dimensional grounds. Note also that the general argument in [12] was under the assumption that the CHS kinetic operator $\mathcal{O}_{2s}$ is diagonal and gauge invariant separately for each s, which is not, in general, true as we have seen above. Still, even ignoring such derivative terms the factorization conjecture for $s \geq 3$ remains to be proved.

[18] The agreement of the actions for constant h_{ab} is, in general, guaranteed if the flat space actions match (as one can generate a constant metric by a coordinate redefinition) but it still provides a formal consistency check of the two results.

[19] This was done also using the AdS$_5$ related method [6].

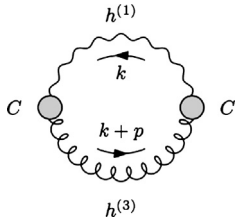


Fig. 2. One-loop diagram in the spin 1–spin 3 theory contributing to the C^2 UV divergence.

Regardless the validity of the factorization conjecture, what was not included in the previous analysis is a potential contribution to (5.1), (5.2) coming from non-diagonal mixing terms like S_{13} in (4.1), (4.10). Such mixing terms vanishing in conformally flat Einstein background can not contribute to anomaly a-coefficient but may contribute to c-coefficient. Here we will concentrate on the spin 1–3 sector discussed above. For $s=1$ (Maxwell) field we have the standard result $c_1=\frac{1}{10}$ ($\nu_1=2$ in (5.3), (5.4)) while for the diagonal $s=3$ contribution (assuming factorization of $\mathcal{O}_6$) we expect from (5.4) to get $c_3=\frac{919}{15}$ ($\nu_3=12$).

Let us consider the background metric to be generic (not necessarily Einstein). The mixed 1–3 term in the induced action (4.10) contains the $C\nabla h_{(1)}\nabla h_{(3)}$ and $(\nabla R)(\nabla h_{(1)})h_{(3)}$ vertices while the kinetic terms are $h_{(1)}(\nabla^2+...)h_{(1)}+h_{(3)}(\nabla^6+...)h_{(3)}$. It is easy to see on dimensional grounds that only the first $C\nabla h_{(1)}\nabla h_{(3)}$ mixing vertex may in principle contribute to the C^2 UV divergences in (5.2). We may thus start directly with the simple quadratic action (4.12). The corresponding additional C^2 contribution may come from the UV divergent part of the diagram in Fig. 2. As the mixing vertex contains already one factor of the Weyl tensor, to find this contribution it is sufficient to consider the flat-space spin 1 and spin 3 propagators in TT gauges. We get for the resulting contribution to the effective action in momentum representation

$$\Gamma(p)=\tfrac{1}{2}\,8^2\,n_1\,n_3\int\frac{d^{\rm d}k}{(2\pi)^{\rm d}}\,C^{aced}(-k_e-p_e)(k_a g_{bq}-k_b g_{aq})$$
$$\times\,C^{a'c'e'd'}(k_{e'}+p_{e'})(-k_{a'}g_{b'q'}+k_{b'}g_{a'q'})\,\frac{P^{q,q'}(k)}{k^2}\,\frac{P^{b}{}_{cd,}{}^{b'c'd'}(k+p)}{[(k+p)^2]^3}\,, \qquad (5.5)$$

where n_s are normalizations in the flat-space CHS quadratic actions

$$\frac{1}{2\,n_s}\int d^4x\,h_{a_1\dots a_s}\,P^{a_1\dots a_s}_{b_1\dots b_s}\,\Box^s\,h^{b_1\dots b_s}\,, \qquad (5.6)$$

while $P_{a,b}(k)$ and $P_{abc,def}(k)$ are the spin 1 and spin 3 TT projectors $P_{(s)(s')}$ in momentum representation (see, *e.g.*, eq. (3.6) and footnote 18 in [8]). Then after some standard manipulations (introducing Feynman parameters and shifting loop momentum) we get

$$\Gamma(p)=2688\,n_1\,n_3\,C_a{}^e{}_b{}^f(p)\,C_{cedf}(-p)\int_0^1 dx\int\frac{d^{\rm d}k}{(2\pi)^{\rm d}}\,\frac{k^a\,k^b\,k^c\,k^d}{\left[k^2+M^2(p,x)\right]^4}+\text{finite}. \qquad (5.7)$$

The UV divergent part is then ($\varepsilon=4-{\rm d}\to 0$)[20]

[20] We use that one can make the replacement $k_a k_b k_c k_d\to\frac{1}{24}(g_{ab}\,g_{cd}+g_{ac}\,g_{bd}+g_{ad}\,g_{bc})\,(k^2)^2$, and the standard integral:

$$\Gamma_{\rm UV} = \frac{336\, n_1\, n_3}{(4\pi)^2\, \varepsilon}\, C_{abcd} C^{abcd}\ . \tag{5.8}$$

Our normalizations of the flat-space currents in (2.2), (2.3) correspond to $n_1 = 3$ and $n_3 = \frac{7}{90}$ in (5.6) (see (4.12), (4.14)) so that the final result for the contribution of the 1–3 mixing to the coefficient c in (5.2) is

$$\mathrm{c}_{13} = \tfrac{392}{5}\ . \tag{5.9}$$

The total contribution to c from the 1–3 sector is thus $\mathrm{c}_{1+3} = \mathrm{c}_1 + \mathrm{c}_3 + \mathrm{c}_{13} = \frac{1}{10} + \frac{919}{15} + \frac{392}{5}$, *i.e.* the mixed term contribution is of the same order as the pure spin 3 one.

In general, other similar higher spin mixing terms are expected to appear and thus should also contribute to the C^2 UV divergence. Indeed, just on dimensional grounds one may have for spin s and spin s' part of the CHS action expanded near flat space (we assume the fields to be TT and ignore normalization factors, *cf.* (4.12))[21]

$$\mathrm{S}_{s+s'} = \int d^4x \Big[h_{(s)}(\partial^{2s} + ...) h_{(s)} + h_{(s')}(\partial^{2s'} + ...) h_{(s')} + \sum_n C\ \partial^n h_{(s)} \partial^{n'} h_{(s')} + ... \Big]\ , \tag{5.10}$$

where C stands for the Weyl tensor and $n + n' = s + s' - 2$ to balance dimensions. Then the UV singular C^2 contribution to the one-loop effective action is proportional to $\int d^4k \frac{k^{2n+2n'}}{k^{2s+2s'}} \sim \int \frac{d^4k}{k^4}$. *i.e.* is logarithmically divergent and thus contributes to c-coefficient in (5.2).

It then remains an open question if these mixing term contributions may change the expectation [9] that the regularized sum of all CHS contributions to the conformal anomaly c-coefficient should vanish, like that happens for the total a-coefficient [6,9].

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Appendix A. Curvature identities

In four dimensions, one has the following useful identities for the Weyl tensor

$$C^{acde} C^{b}{}_{cde} = \tfrac{1}{4} g^{ab}\ C_{cdef} C^{cdef}\,, \qquad C^{acde} C^{b}{}_{dce} = \tfrac{1}{4} g^{ab}\ C_{cedf} C^{cdef}\,. \tag{A.1}$$

$$\int \frac{d^{\rm d}k}{(2\pi)^{\rm d}} \frac{(k^2)^a}{(k^2 + M^2)^b} = \frac{\Gamma(b - a - {\rm d}/2)\Gamma(a + {\rm d}/2)}{(4\pi)^{{\rm d}/2}\Gamma(b)\Gamma({\rm d}/2)}\, (M^2)^{{\rm d}/2 + a - b}.$$

Note that in dimensional regularization in (5.1) one has $\log \Lambda_{\rm UV} \to -\frac{1}{\varepsilon}$.

21 To determine the mixing term here requires the computation of the $h_{(2)} h_{(s)} h_{(s')}$ term in the CHS action in a near flat space expansion.

The Bach tensor is defined by

$$\begin{aligned} B_{ab} &= -\tfrac{1}{4}\tfrac{1}{\sqrt{g}}\tfrac{\delta}{\delta g^{ab}}\int d^4x\,\sqrt{g}\,C^2_{abcd} \\ &= R_a{}^c R_{bc} - \tfrac{1}{3}R_{ab}R + R^{cd}\,C_{acdb} - \tfrac{1}{2}\nabla^2 R_{ab} + \tfrac{1}{6}\nabla_a\nabla_b R \\ &\quad - \tfrac{1}{4}g_{ab}(R_{cd}R^{cd} - \tfrac{1}{3}R^2 - \tfrac{1}{3}\nabla^2 R) \end{aligned} \tag{A.2}$$

Introducing the Schouten tensor $P_{ab} = \frac{1}{2}(R_{ab} - \frac{1}{6}\,R\,g_{ab})$, the Bach tensor may be written as

$$B_{ab} = \nabla^c\nabla_a\,P_{bc} - \nabla^2\,P_{ab} + P^{cd}\,C_{acdb} = \big(\nabla^c\nabla^d + \tfrac{1}{2}R^{cd}\big)\,C_{acdb}\ , \tag{A.3}$$

where the second equality follows from the Bianchi identities.[22]

Appendix B. Vanishing of spin 3 linear term in induced action

Let us consider the linear term in the induced action (3.4), (3.5)

$$\mathrm{S}_3^{(1)} = \int d^4x\,\sqrt{g}\,h^{abc}\,B_{abc}\ , \qquad\qquad B_{abc} = \langle J_{abc}\rangle_{\rm UV}\ , \tag{B.1}$$

given by the coefficient of the logarithmic divergence in the 1-point function of the spin 3 current (2.21). Since the UV singular part of the 1-point function of spin 1 current is equal to zero, $B_a = \langle J_a\rangle_{\rm UV} = 0$, the relation (2.23) implies that dimension 5 tensor B_{abc} should be covariantly conserved, $\nabla^a B_{abc} = 0$. In addition, it should parity-even (as J_{abc} in (2.21) and S_0 in (2.8) are) and Weyl-covariant with weight -2 so that $\mathrm{S}_3^{(1)}$ is Weyl-invariant (*cf.* (2.9)).

Examining the most general candidates for B_{abc} satisfying these conditions we did not find any solutions, *i.e.* we should have

$$B_{abc} = 0\ . \tag{B.2}$$

In [15] it was suggested that B_{abc} may be proportional to Eastwood–Dighton tensor

$$E_{abc} = C^{ef}{}_{cb}\nabla^d C^*_{defa} - C^{*ef}{}_{cb}\nabla^d C_{defa}\ , \qquad\qquad C^*_{abcd} = \tfrac{1}{2}\varepsilon_{abef}C^{ef}{}_{cd}\ , \tag{B.3}$$

which satisfies the Weyl-invariance and conservation conditions and vanishes on conformally Einstein spaces. However, this tensor is parity-odd and thus cannot appear in the expectation value $\langle J_{abc}\rangle_{\rm UV}$.

Let us note that in the special case of an Einstein space background $R_{ab} = \frac{1}{4}Rg_{ab}$ we do not need the condition of Weyl invariance to show that $B_{abc} = 0$. Indeed, the dimension 5 tensor B_{abc} must be constructed from the Weyl tensor and covariant derivatives, *i.e.* it should be of the form ∇CC or $\nabla^3 C$ or explicitly (ignoring g_{ab}-terms that decouple upon contraction with traceless h_{abc})

$$B_{abc} = k_1\,\nabla^e\nabla^d\nabla_c C_{adbe} + k_2\,C_a{}^{def}\nabla_c C_{bdef} + k_3\,C_a{}^{def}\,\nabla_c C_{bedf} + k_4\,C_a{}^{def}\nabla_f C_{bdce}\ . \tag{B.4}$$

The first term can be related to the last three as for an Einstein space

$$\nabla^e\nabla^d\nabla_c\,R_{adbe} = -R_b{}^{def}\nabla_d R_{aecf} + R_c{}^{def}\nabla_f R_{adbe} - R_a{}^{def}\nabla_f R_{becd}\ . \tag{B.5}$$

[22] Let us note that some papers have different signs appearing in the expression for the Bach tensor which is usually related to different conventions for the sign of the curvature.

The second and third terms do not contribute to $h^{abc}\, B_{abc}$ because of the identities like (*cf.* (A.1))

$$C_a{}^{def}\nabla_c C_{bdef} = \tfrac{1}{2}\nabla_c(C_a{}^{def} C_{bdef}) = \tfrac{1}{8}\, g_{ab}\,\nabla_c C^2\ . \tag{B.6}$$

Using the Bianchi identity we also have $C_a{}^{def}\nabla_f C_{bdce} = -C_a{}^{def}\nabla_b C_{cedf} - C_a{}^{def}\nabla_d C_{becf}$. These terms do not contribute upon contraction with a totally symmetric traceless h_{abc}.

One may also study more general linear in spin 3 terms which also involve the vector field strength F_{ab}. One finds that the only possible term with one power of F_{ab} is proportional to the combination appearing in (4.10) that we obtained by the direct computation of the induced action. The only term quadratic in F_{ab} and linear in h_{abc} must be (on dimensional and covariance grounds) proportional to $F_{ab}F^b{}_c\nabla_d h^{acd}$. Such term is not, however, Weyl invariant and thus can not appear in the induced action.

Appendix C. Background covariant computation of UV pole parts of correlators of currents

In this appendix we shall briefly explain the strategy of the computation of the UV pole (logarithmic divergence) part of the current–current correlators like $\langle J_{(s)}\, J_{(s')}\rangle_{\rm UV}$ appearing in (4.4). We shall follow the approach and notation of [25].

Starting with the explicit expression for the bilinear currents like in (2.12), (2.14), (2.21) one may express the correlator at separated points $\langle J_{(s)}(x)\, J_{(s')}(x')\rangle$ in terms of the curved space scalar propagators $G(x,x')$ getting sum of terms like

$$\nabla^n\nabla'^{n'}\, G\ \ \nabla^m\nabla'^{m'}\, G + ..., \qquad\qquad n+m=s,\ \ n'+m'=s', \tag{C.1}$$

where dots stand for other potential terms with less covariant derivatives but extra factors of curvature and its derivatives. For our purpose of extracting the UV singular part of the correlator it is sufficient to keep only the part of G which is most singular in the coincidence $x\to x'$ limit

$$G(x,x') \to \frac{\Delta^{1/2}(x,x')}{8\pi^2\,\sigma(x,x')}\ , \tag{C.2}$$

where $\sigma(x,x')$ is half the geodesic distance between x and x' and Δ (not to be confused with the Laplace operator) is defined by (see, e.g., [21] for a detailed discussion of properties of these bitensors)

$$\Delta \equiv |g|^{-1/2}\,|g'|^{-1/2}\,\det D_{ab'}\ , \qquad\qquad D_{ab'}(x,x') = -\frac{\partial^2}{\partial x^a\,\partial x'^b}\,\sigma(x,x'). \tag{C.3}$$

Using (C.2) in (C.1), we obtain terms whose denominator is a power of σ while the numerator is a tensor that involves covariant derivatives of σ and Δ.

To find the UV singular part of such terms, in the dimensional regularization approach of [25], one is to make the following replacement ($\varepsilon = 4-{\rm d}$)

$$\frac{1}{\sigma^2} \to \frac{8\pi^2}{\varepsilon}\,\delta^{(4)}(x,x')\ , \tag{C.4}$$

where $\delta^{(4)}(x,x')$ is the biscalar curved space δ-function, $\int d^4x\,\sqrt{g}\,\delta^{(4)}(x,x') = 1$. Terms with powers of $1/\sigma$ higher than 2 should be first reduced to $1/\sigma^2$ terms by iterative use of the identity

$$\frac{1}{\sigma^{p+1}} = \frac{1}{2\,p\,(p-1)}\,\Delta^{-1/2}\,(\nabla^2 - Y)\,\frac{\Delta^{1/2}}{\sigma^p}\ , \qquad\qquad Y = \Delta^{-1/2}\,\nabla^2\,\Delta^{1/2}. \tag{C.5}$$

Use of (C.4) then gives terms with powers of the differential operator in the r.h.s. of (C.5) acting on $\delta^{(4)}(x,x')$. We may then use integration by parts and take the coincidence limit $x \to x'$.

This last step is non-trivial because the coincidence limit does not commute with covariant derivatives of the biscalars σ and Δ.[23] This may be automatized to provide a table of substitution rules. Denoting by a square bracket the coincidence limit, the simplest examples are

$$[\sigma]=0\,,\qquad [\nabla_a\sigma]=0\,,\qquad [\nabla_a\nabla_b\sigma]=g_{ab}\,,\qquad [\nabla_a\nabla_b\nabla_c\sigma]=0\,,$$
$$[\nabla_a\nabla_b\nabla_c\nabla_d\,\sigma]=-\tfrac{1}{3}(R_{dbca}+R_{dacb})\,,\qquad \dots \tag{C.6}$$
$$[\Delta^{1/2}]=1\,,\qquad [\nabla_a\Delta^{1/2}]=0\,,\qquad [\nabla_a\nabla_b\Delta^{1/2}]=\tfrac{1}{6}\,R_{ab}\,,$$
$$[\nabla_a\nabla_b\nabla_c\,\Delta^{1/2}]=\tfrac{1}{12}\,(\nabla_a R_{bc}+\nabla_b R_{ac}+\nabla_c R_{bc}),\qquad \dots \tag{C.7}$$

The substitutions (C.6) and (C.7) produce a non-trivial dependence on the background curvature. When the correlator is contracted with $h_{(s)}$ fields, *i.e.* $h^{(s)}\langle J_{(s)}\,J_{(s')}\rangle_{\rm UV}\,h^{(s')}$, the integration by parts mentioned above will produce terms with derivatives acting on higher spin fields.

Appendix D. Stress tensor of the free spin 3 field in flat space

Here we shall comment on the special structure of the spin 3 flat space kinetic term in (4.14) and its stress tensor. Let us start with the general 3-parameters Lagrangian

$$\mathscr{L}=h^{abc}\Box^3 h_{abc}+k_1\,h^{abc}\,\partial_{abcdef}\,h^{def}+k_2\,h^{abc}\Box\,\partial_{bcde}\,h_a{}^{de}+k_3\,h^{abc}\,\Box^2\,\partial_{cd}\,h_{ab}{}^{d}\,, \tag{D.1}$$

and look for a symmetric traceless stress tensor $T_{ab}\sim h_{(3)}\partial^6 h_{(3)}+\cdots$ which is conserved and gauge invariant on the equations of motion following from (D.1). There are 254 possible structures in such T_{ab}, *i.e.* $(\partial^n h\,\partial^m h)_{ab}$ or $g_{ab}\,(\partial^n h\,\partial^m h)$ with $n+m=6$. The T_{ab} with required properties is found only if $\mathscr{L}$ in (D.1) is proportional to $\mathscr{L}_{33}^{\rm flat}$ in (4.14).[24] To show the converse requires an explicit calculation which gives

$$\begin{aligned}
T^{ab}=&\tfrac{1}{2}h^{bcd}\partial^a{}_{cdefp}h^{efp}-2\,h^{bcd}\partial^a{}_{def}\Box h_c{}^{ef}+\tfrac{5}{4}h^{bcd}\,\partial^a{}_e\Box^2 h_{cd}{}^e\\
&+\tfrac{1}{2}h^{acd}\partial^b{}_{cdefp}h^{efp}-2\,h^{acd}\partial^b{}_{def}\Box h_c{}^{ef}+\tfrac{5}{4}h^{acd}\,\partial^b{}_e\Box^2 h_{cd}{}^e\\
&-h^{bcd}\partial_{cdef}\Box h^{aef}-h^{acd}\,\partial_{cdef}\Box h^{bef}+h^{abc}\,\partial_{cdef}\Box h^{def}+\tfrac{5}{2}h^{bcd}\,\partial_{de}\Box^2 h^a{}_c{}^e\\
&+\tfrac{5}{2}h^{acd}\,\partial_{de}\Box^2 h^b{}_c{}^e-\,h^{abc}\partial_{de}\Box^2 h_c{}^{de}-\tfrac{5}{4}h^{bcd}\,\Box^3 h^a{}_{cd}\\
&-\tfrac{5}{4}h^{acd}\,\Box^3 h^b{}_{cd}+\eta^{ab}\Big(-\tfrac{1}{4}h^{cde}\partial_{cdefpa}h^{fpa}+\tfrac{3}{2}\,h^{cde}\partial_{defp}\Box h_c{}^{fp}\\
&-\tfrac{15}{8}h^{cde}\,\partial_{ef}\Box^2 h_{cd}{}^f+\tfrac{5}{8}h^{cde}\,\Box^3 h_{cde}\Big).
\end{aligned} \tag{D.2}$$

Here the η^{ab} term is proportional (up to a total derivative) to $\mathscr{L}_{33}^{\rm flat}$ in (4.14), as expected on general grounds.

The fact that (D.2) is not manifestly invariant under spin 3 gauge transformations may be compared with the lower spin cases. In the spin 1 case $T_{ab}=F_{ac}F^c{}_b-\frac{1}{4}g_{ab}F^2$ is traceless,

[23] Besides, one has to deal with the technical problem of separating covariant derivatives at x and x'. This may be done systematically by exploiting Synge's theorem and its multi-index generalization proved in [21].

[24] We omit trivial improvement terms not contributing to the conserved charges and obeying all the requirements automatically, *i.e.* without using the equations of motion.

symmetric and conserved on the equations of motion but also manifestly gauge-invariant. The latter feature does not automatically generalize to higher conformal spins. Let us consider the $s = 2$ case and expand the Weyl gravity action near a generic metric $g_{ab} \to g_{ab} + h_{ab}$ (with traceless h_{ab})

$$\begin{aligned} S(g;h) &= \tfrac{1}{4}\int d^4x\,\sqrt{g}\,C^2_{abcd}(g+h) \\ &= \int d^4x\,\sqrt{g}\big[\,\tfrac{1}{4}C^2_{abcd}(g) + B(g)\,h + \tfrac{1}{2}h\,\mathcal{O}_4(g)\,h + \dots\big] \\ &= S(g) + S^{(1)}(g;h) + S^{(2)}(g;h) + \dots\,, \end{aligned} \tag{D.3}$$

where B is the Bach tensor. Under the gauge variation $\delta_\varepsilon h_{ab} = \nabla_{(a}\varepsilon_{b)} - \frac{1}{4}g_{ab}\,\nabla^c\varepsilon_c$,

$$0 = S(g;h+\delta_\varepsilon h) - S(g;h) = \int d^4x\,\sqrt{g}B(g)\,\delta_\varepsilon h + S_2(g;h+\delta_\varepsilon h) - S_2(g;h) + \dots. \tag{D.4}$$

Applying $\delta/\delta g_{mn}$ and replacing the background metric by the flat one, $g_{ab} \to \eta_{ab}$, gives

$$T_{cd}(h+\delta_\varepsilon h) - T_{cd}(h) = -\Big[\frac{\delta B^{ab}}{\delta g_{cd}}\delta_\varepsilon h_{ab}\Big]_{g_{ef}=\eta_{ef}} - \Big[\frac{\delta S^{(2)}}{\delta h_{ab}}\,\frac{\delta(\delta_\varepsilon h_{ab})}{\delta g_{cd}}\Big]_{g_{ef}=\eta_{ef}} + \dots\,, \tag{D.5}$$

where T_{cd} is the stress tensor defined as the variation of the action over the metric. The first term on the r.h.s. then vanishes as B_{ab} is covariant; the second term vanishes on the equations of motion of h_{ab}. Thus T_{cd} is gauge invariant only on the equations of motion, in contrast to the spin 1 case.

Appendix E. Spin 3 gauge invariance in Einstein background

Here we provide some details of the check of the invariance of the spin 1 plus mixed spin 1–3 term in the quadratic action (4.12) in an Einstein background under the spin 3 gauge transformation in (4.11). Explicitly, we consider

$$\begin{aligned} &\delta\int d^4x\,\sqrt{g}\big(-\tfrac{1}{12}F^2_{ab} + 8\,C^{abdc}\,F_{ap}\,\nabla_c h_{bd}{}^{p}\big) \equiv \int d^4x\,\sqrt{g}\,(Q+Q'), \\ &Q = h^a\,\Delta_{a,pq}\,\varepsilon^{pq}, \qquad Q' = h^{abc}\,\Delta'_{abc,pq}\,\varepsilon^{pq}, \end{aligned} \tag{E.1}$$

where δ acts according to (4.11) and $\Delta_{a,pq}$, $\Delta'_{abc,pq}$ are differential operators containing ∇_a and $\overleftarrow{\nabla}_a$. We want to show that the part of the variation not depending on h_{abc}, *i.e.* Q, vanishes. The cancellation of Q' requires adding the variation of the last quadratic spin 3 term in (4.12) which at present is not known beyond the leading order in the curvature.

The explicit form of Q is found to be

$$\begin{aligned} Q = &-\tfrac{8}{3}R_b{}^c R_c{}^d\varepsilon_{ad}\nabla^b h^a + \tfrac{8}{3}R_a{}^c R_c{}^d\varepsilon_{bd}\nabla^b h^a + 8\,R_b{}^e{}_c{}^f C_{adef}\varepsilon^{cd}\nabla^b h^a \\ &- 8R_b{}^e{}_c{}^f C_{afde}\varepsilon^{cd}\nabla^b h^a - 8R_a{}^e{}_c{}^f C_{bdef}\,\varepsilon^{cd}\nabla^b h^a + 8R_a{}^e{}_c{}^f\,C_{bfde}\varepsilon^{cd}\nabla^b h^a \\ &+ \tfrac{8}{3}R_b{}^c R_{adce}\varepsilon^{de}\nabla^b h^a - \tfrac{8}{3}R_a{}^c R_{bdce}\varepsilon^{de}\nabla^b h^a + \tfrac{8}{3}R_b{}^c\nabla^b h^a\nabla_d\nabla_c\varepsilon_a{}^d \\ &- \tfrac{8}{3}R_a{}^c\nabla^b h^a\nabla_d\nabla_c\varepsilon_b{}^d - 8\nabla_a C_{bcde}\nabla^b h^a\nabla^e\varepsilon^{cd} + 8\nabla_b C_{acde}\nabla^b h^a\nabla^e\varepsilon^{cd} \\ &+ 8R_{bcde}\nabla^b h^a\nabla^e\nabla_a\varepsilon^{cd} - 8\,C_{bcde}\nabla^b h^a\nabla^e\nabla_a\varepsilon^{cd} - 8\,R_{acde}\nabla^b h^a\nabla^e\nabla_b\varepsilon^{cd} \end{aligned}$$

$$+8C_{acde}\nabla^b h^a\nabla^e\nabla_b\varepsilon^{cd}+\tfrac{16}{3}R_{abde}\nabla^b h^a\nabla^e\,\nabla_c\varepsilon^{cd}+\tfrac{8}{3}R_{adbe}\nabla^b h^a\nabla^e\nabla_c\varepsilon^{cd}$$
$$-\tfrac{8}{3}R_{aebd}\nabla^b h^a\nabla^e\nabla_c\varepsilon^{cd}+8R_{bcde}\nabla^b h^a\nabla^e\nabla^d\varepsilon_a{}^c+8R_{bdce}\nabla^b h^a\nabla^e\nabla^d\varepsilon_a{}^c$$
$$-8R_{acde}\nabla^b h^a\nabla^e\nabla^d\varepsilon_b{}^c-\,8R_{adce}\nabla^b h^a\nabla^e\nabla^d\varepsilon_b{}^c. \tag{E.2}$$

Integrating by parts to remove the covariant derivatives from ε_{ab} and using the Einstein-space curvature identities we arrive at

$$Q=-\tfrac{4}{3}R\varepsilon_{bc}\nabla_a\nabla^c\nabla^b h^a+\tfrac{4}{3}R\varepsilon_{ac}\nabla_b\nabla^c\nabla^b h^a+\tfrac{1}{6}R^2\varepsilon_{ab}\nabla^b h^a$$
$$+\tfrac{2}{3}R\varepsilon_{bc}\nabla^c\nabla_a\nabla^b h^a-\,\tfrac{4}{3}R\varepsilon_{ac}\nabla^c\nabla_b\nabla^b h^a-8\,\varepsilon^{de}\nabla_a C_{bdce}\nabla^c\nabla^b h^a$$
$$+8\varepsilon^{de}\nabla_b C_{adce}\nabla^c\nabla^b h^a+\tfrac{2}{3}R\varepsilon_{bc}\nabla^c\nabla^b\nabla_a h^a-8C_{bdce}\varepsilon_a{}^e\nabla^d\nabla^c\nabla^b h^a$$
$$-8C_{becd}\varepsilon_a{}^e\nabla^d\nabla^c\nabla^b h^a+8C_{adce}\varepsilon_b{}^e\nabla^d\nabla^c\nabla^b h^a+8C_{aecd}\varepsilon_b{}^e\nabla^d\nabla^c\nabla^b h^a$$
$$-\tfrac{16}{3}C_{abce}\varepsilon_d{}^e\nabla^d\nabla^c\nabla^b h^a-\tfrac{8}{3}C_{acbe}\,\varepsilon_d{}^e\nabla^d\nabla^c\nabla^b h^a+\tfrac{8}{3}C_{aebc}\varepsilon_d{}^e\nabla^d\nabla^c\nabla^b h^a$$
$$-\tfrac{16}{3}\varepsilon^{de}\nabla^c\nabla^b h^a\nabla_e C_{abcd}-\tfrac{8}{3}\varepsilon^{de}\nabla^c\nabla^b h^a\nabla_e C_{acbd}+\tfrac{8}{3}\varepsilon^{de}\nabla^c\nabla^b h^a\nabla_e C_{adbc}. \tag{E.3}$$

Symmetrizing the covariant derivatives and using the Bianchi identities gives

$$Q=\tfrac{4}{3}R_b{}^{(de)f}\,R_{cdef}\varepsilon_a{}^c\nabla^{[b}h^{a]}\,. \tag{E.4}$$

This vanishes after using $\varepsilon_{ab}=\varepsilon_{ba}$ and the identities valid in Einstein space (*cf.* (A.1))

$$R^{acde}R^b{}_{cde}=\tfrac{1}{4}g^{ab}R_{cdef}R^{cdef},\qquad R^{acde}R^b{}_{dce}=\tfrac{1}{4}g^{ab}R_{cedf}R^{cdef}. \tag{E.5}$$

Appendix F. Restricted form of spin 3 gauge invariance

If we consider the spin 3 gauge transformations with the gauge parameter constrained by $\nabla^a\varepsilon_{ab}=0$ then the transformations in (4.11) satisfy

$$\delta' h_a=-6\,\nabla^b\,\nabla^c\,\delta' h_{abc}\,,\qquad \delta'\equiv\delta\Big|_{\nabla^a\varepsilon_{ab}=0}\,. \tag{F.1}$$

This means that we can introduce a new spin-1 field $\widetilde{h}_a$ which will be neutral with respect to the restricted spin 3 gauge transformation

$$\widetilde{h}_a=h_a+6\,\nabla^b\nabla^c\,h_{abc}\,,\qquad \delta'\widetilde{h}_a=0\,. \tag{F.2}$$

Then the spin 1 and 3 interaction terms in (2.24) may be written as

$$h^a\,J_a+h^{abc}\,J_{abc}=\widetilde{h}^a\,J_a+h^{abc}\,\widetilde{J}_{abc},\qquad \widetilde{J}_{abc}=J_{abc}-6\nabla_{(a}\,\nabla_b\,J_{c)}\,, \tag{F.3}$$

where $\widetilde{J}_{abc}$ is thus the same as in (2.27), *i.e.*

$$h^{abc}\,\widetilde{J}_{abc}=60\,i\,h^{abc}\,\nabla_a\nabla_b\overline{\varphi}\,\nabla_c\varphi+\text{c.c.} \tag{F.4}$$

This term is thus invariant under the restricted gauge transformations in an Einstein background.

The quadratic part of the induced action is again of the form (4.11), but now written in terms of the new vector field $\widetilde{h}_a$ (with field strength $\widetilde{F}$)

$$\mathrm{S}^{(2)}=\int d^4x\,\sqrt{g}\Big[-\tfrac{1}{12}\,\widetilde{F}^2_{ab}+\widetilde{\mathscr{L}}_{13}+h_{(3)}\,\widetilde{\mathcal{O}}_6\,h_{(3)}\Big]\,, \tag{F.5}$$

$$\widetilde{\mathscr{L}}_{13}=\widetilde{h}^a\,\langle J_a\,\widetilde{J}_{bcd}\rangle_{\mathrm{UV}}h^{bcd}=\widetilde{F}^{ab}\big(2\,\nabla_a\,\nabla_c\,\nabla_d\,h_b{}^{cd}+8\,C_{acdp}\,\nabla^p\,h_b{}^{cd}\big)\,. \tag{F.6}$$

Note that in an Einstein background there is no nontrivial $\widetilde{h}_{(1)}h_{(3)}$ tadpole contributions so (F.6) is automatically vector gauge invariant.

Since $\delta'\widetilde{h}_a=0$, the $h_{(3)}\,\widetilde{\mathcal{O}}_6\,h_{(3)}$ term in (F.5) that should be equal to $h_{(3)}\langle\widetilde{J}\,\widetilde{J}\rangle_{\rm UV}h_{(3)}$ (up to a possible tadpole contribution) should be invariant under the restricted spin 3 gauge transformations on its own.

It is easy to see why (F.6) is spin 3 gauge invariant using (F.1):

$$\begin{aligned}\delta'\mathscr{L}_{13}&=\widetilde{F}^{ab}\big(2\,\nabla_a\,\nabla_c\,\nabla_d\,\delta' h_b{}^{cd}+8\,C_{acdp}\,\nabla^p\,\delta' h_b{}^{cd}\big)\\&=\widetilde{F}^{ab}\big(-\tfrac{1}{3}\,\nabla_a\,\delta' h_b+8\,C_{acdp}\,\nabla^p\,\delta' h_b{}^{cd}\big)\\&=\delta'\big(-\tfrac{1}{12}\,F_{ab}^2+8\,F^{ab}\,C_{acdp}\,\nabla^p\,h_b{}^{cd}\big)\Big|_{h_{(1)}\varepsilon}=0\,.\end{aligned}\tag{F.7}$$

Here we used that the two terms in the second line are as in (4.12) and thus are invariant under the spin 3 transformations modulo $O(h_{(3)}\varepsilon)$ term.

Appendix G. Some expressions used in section 4.3

Here we present the relations used in eqs. (4.16), (4.17), (4.18). The tensor $\mathcal{V}_{ab}$ in the $h_{(2)}h_{(3)}\overline{\varphi}\varphi$ vertex in (4.17) is given by

$$\begin{aligned}\mathcal{V}_{ab}=&3\,i\,(-4\,\partial_c h_{abd}\,\partial^d\overline{\varphi}\,\partial^c\varphi+4\,\partial_d h_{abc}\,\partial^d\overline{\varphi}\,\partial^c\varphi+10\,h_{bcd}\,\partial^c\overline{\varphi}\,\partial_a{}^d\varphi-10\,h_{bcd}\,\partial_a{}^d\overline{\varphi}\,\partial^c\varphi\\&+10\,h_{acd}\,\partial^c\overline{\varphi}\,\partial_b{}^d\varphi-10\,h_{acd}\,\partial_b{}^d\overline{\varphi}\,\partial^c\varphi-4\;h_{abd}\,\partial^c\overline{\varphi}\,\partial_c{}^d\varphi+4\,h_{abd}\,\partial_c{}^d\overline{\varphi}\,\partial^c\varphi\\&-7\,\partial_a h_{bcd}\,\overline{\varphi}\;\partial^{cd}\varphi-7\,\partial_b h_{acd}\,\overline{\varphi}\,\partial^{cd}\varphi+12\,\partial_d h_{abc}\,\overline{\varphi}\,\partial^{cd}\varphi+7\,\partial_a\;h_{bcd}\,\partial^{cd}\overline{\varphi}\,\varphi\\&+7\,\partial_b h_{acd}\,\partial^{cd}\overline{\varphi}\,\varphi-12\,\partial_d h_{abc}\,\partial^{cd}\overline{\varphi}\,\varphi+3\;h_{abc}\,\partial^c\overline{\varphi}\,\partial^d{}_d\varphi-3\,h_{abc}\,\partial^d{}_d\overline{\varphi}\,\partial^c\varphi\\&+7\,\partial^d{}_d h_{abc}\;\overline{\varphi}\,\partial^c\varphi-7\,\partial^d{}_d h_{abc}\,\partial^c\overline{\varphi}\,\varphi+12\,\eta_{bc}\,h_{ade}\,\partial^c\overline{\varphi}\,\partial^{de}\varphi\\&+12\eta_{ac}\;h_{bde}\,\partial^c\overline{\varphi}\,\partial^{de}\varphi-12\,\eta_{bc}\,h_{ade}\,\partial^{de}\overline{\varphi}\,\partial^c\varphi-12\,\eta_{ac}\,h_{bde}\,\partial^{de}\overline{\varphi}\,\partial^c\varphi\\&-8\,h_b{}^{cd}\,\overline{\varphi}\,\partial_{acd}\varphi+8\,h_b{}^{cd}\,\partial_{acd}\overline{\varphi}\,\varphi-8\,h_a{}^{cd}\,\overline{\varphi}\,\partial_{bcd}\varphi+8\,h_a{}^{cd}\,\partial_{bcd}\overline{\varphi}\,\varphi\\&+\;h_{ab}{}^c\,\overline{\varphi}\,\partial_c{}^d{}_d\varphi-\;h_{ab}{}^c\,\partial_c{}^d{}_d\overline{\varphi}\,\varphi)\;.\end{aligned}\tag{G.1}$$

The contribution $\mathscr{L}^{(a)}_{233}$ in (4.16) coming from the triangle diagram (a) in Fig. 1 is

$$\begin{aligned}\mathscr{L}^{(a)}_{233}=&-\tfrac{15}{7}\,h^{ab}\,(-42\Box h_a{}^{cd}\,\Box^2 h_{bcd}+35\partial_a\Box h^{cde}\;\partial_b\Box h_{cde}+42\partial_a h^{cde}\;\partial_b\Box^2 h_{cde}\\&+54\partial_a h^{cde}\;\partial_e\Box^2 h_{bcd}+18\partial_b\Box^2 h_{cde}\partial^e h_a{}^{cd}-108\;\partial_d\Box^2 h_{bce}\partial^e h_a{}^{cd}\\&-54\;\partial_e\Box^2 h_{bcd}\partial^e h_a{}^{cd}+54\partial_b\Box h_{cdf}\partial^f\Box h_a{}^{cd}-102\;\partial_d\Box h_{bcf}\partial^f\Box h_a{}^{cd}\\&-51\;\partial_f\Box h_{bcd}\partial^f\Box h_a{}^{cd}+42\Box\;h^{cde}\partial_{ab}\Box h_{cde}+24h^{cde}\partial_{ab}\Box^2 h_{cde}\\&+28\Box^2 h_{cde}\;\partial_{ab}h^{cde}-12\Box h^{cde}\partial_{be}\Box h_{acd}+\;264\partial_a{}^f h^{cde}\partial_{be}\Box h_{cdf}\\&-12\Box^2 h_{cde}\partial_b{}^e h_a{}^{cd}+88\;\partial_a{}^f h^{cde}\partial_{bf}\Box h_{cde}+72\Box h^{cde}\;\partial_{de}\Box h_{abc}\\&+72\partial_a{}^f h^{cde}\;\partial_{de}\Box h_{bcf}+54h^{cde}\partial_{de}\Box^2 h_{abc}+30\Box^2 h_{cde}\;\partial^{de}h_{ab}{}^c\\&+72\partial_a{}^f h^{cde}\;\partial_{ef}\Box h_{bcd}-18h_a{}^{cd}\;\partial^e{}_e\Box^2 h_{bcd}+48\partial_{bd}\Box h_{cef}\partial^{ef}h_a{}^{cd}\\&+48\;\partial_{bf}\Box h_{cde}\partial^{ef}h_a{}^{cd}-72\;\partial_{cd}\Box h_{bef}\partial^{ef}h_a{}^{cd}-288\partial_{df}\Box h_{bce}\partial^{ef}h_a{}^{cd}\\&-72\;\partial_{ef}\Box h_{bcd}\partial^{ef}h_a{}^{cd}+144\;\partial^f h^{cde}\partial_{abe}\Box h_{cdf}+48\partial^f h^{cde}\partial_{abf}\Box h_{cde}\\&+150\;\partial_e\Box h_{cdf}\partial_{ab}{}^f h^{cde}+50\partial_f\;\Box h_{cde}\partial_{ab}{}^f h^{cde}+90\partial_a{}^{fh}h^{cde}\partial_{bde}h_{cfh}\end{aligned}$$

$$
\begin{aligned}
&+180\,\partial_a{}^{fh}h^{cde}\partial_{beh}h_{cdf}-12\,\partial_d\Box h_{cef}\partial_b{}^{ef}h_a{}^{cd}-12\partial_f\Box h_{cde}\partial_b{}^{ef}h_a{}^{cd}\\
&+30\,\partial_a{}^{fh}h^{cde}\partial_{bfh}h_{cde}+36\,\partial^f h^{cde}\partial_{cde}\Box h_{abf}+108\partial^f h^{cde}\partial_{def}\Box h_{abc}\\
&+30\,\partial_c\Box h_{def}\partial^{def}h_{ab}{}^c+90\partial_f\,\Box h_{cde}\partial^{def}h_{ab}{}^c+12\partial_{bcd}h_{efh}\partial^{efh}h_a{}^{cd}\\
&+72\,\partial_{bdh}h_{cef}\partial^{efh}h_a{}^{cd}+36\,\partial_{bfh}h_{cde}\partial^{efh}h_a{}^{cd}-54\partial_{cdh}h_{bef}\partial^{efh}h_a{}^{cd}\\
&-108\,\partial_{dfh}h_{bce}\partial^{efh}h_a{}^{cd}-18\,\partial_{efh}h_{bcd}\partial^{efh}h_a{}^{cd}+96\partial^{fh}h^{cde}\partial_{abde}h_{cfh}\\
&+192\,\partial^{fh}h^{cde}\partial_{abeh}h_{cdf}+32\partial^{fh}h^{cde}\,\partial_{abfh}h_{cde}+48\partial^{fh}h^{cde}\,\partial_{cdeh}h_{abf}\\
&+72\partial^{fh}h^{cde}\partial_{defh}h_{abc}).
\end{aligned}
\tag{G.2}
$$

The contribution $\mathscr{L}^{(b)}_{233}$ in (4.16) coming from the bubble diagram (b) in Fig. 1 is much simpler

$$
\mathscr{L}^{(b)}_{233}=-\tfrac{270}{7}\,h^{ab}\,h_a{}^{cd}\Box^3 h_{bcd}\ . \tag{G.3}
$$

References

[1] E.S. Fradkin, A.A. Tseytlin, Conformal supergravity, Phys. Rep. 119 (1985) 233–362.
[2] E.S. Fradkin, V.Y. Linetsky, Cubic interaction in conformal theory of integer higher spin fields in four-dimensional space–time, Phys. Lett. B 231 (1989) 97.
[3] A.A. Tseytlin, On limits of superstring in $AdS_5\times S^5$, Theor. Math. Phys. 133 (2002) 1376–1389, arXiv:hep-th/0201112.
[4] A.Y. Segal, Conformal higher spin theory, Nucl. Phys. B 664 (2003) 59–130, arXiv:hep-th/0207212.
[5] X. Bekaert, E. Joung, J. Mourad, Effective action in a higher-spin background, J. High Energy Phys. 1102 (2011) 048, arXiv:1012.2103.
[6] S. Giombi, I.R. Klebanov, S.S. Pufu, B.R. Safdi, G. Tarnopolsky, AdS description of induced higher-spin gauge theory, J. High Energy Phys. 1310 (2013) 016, arXiv:1306.5242.
[7] E. Joung, S. Nakach, A.A. Tseytlin, Scalar scattering via conformal higher spin exchange, J. High Energy Phys. 02 (2016) 125, arXiv:1512.08896.
[8] M. Beccaria, S. Nakach, A.A. Tseytlin, On triviality of S-matrix in conformal higher spin theory, J. High Energy Phys. 09 (2016) 034, arXiv:1607.06379.
[9] A.A. Tseytlin, On partition function and Weyl anomaly of conformal higher spin fields, Nucl. Phys. B 877 (2013) 598–631, arXiv:1309.0785.
[10] R.R. Metsaev, Ordinary-derivative formulation of conformal totally symmetric arbitrary spin bosonic fields, J. High Energy Phys. 1206 (2012) 062, arXiv:0709.4392.
[11] R.R. Metsaev, Arbitrary spin conformal fields in (A)dS, Nucl. Phys. B 885 (2014) 734–771, arXiv:1404.3712.
[12] T. Nutma, M. Taronna, On conformal higher spin wave operators, J. High Energy Phys. 1406 (2014) 066, arXiv:1404.7452.
[13] M. Beccaria, X. Bekaert, A.A. Tseytlin, Partition function of free conformal higher spin theory, J. High Energy Phys. 1408 (2014) 113, arXiv:1406.3542.
[14] M. Beccaria, A. Tseytlin, On higher spin partition functions, J. Phys. A 48 (2015) 275401, arXiv:1503.08143.
[15] M. Grigoriev, A.A. Tseytlin, On conformal higher spins in curved background, arXiv:1609.09381.
[16] S.M. Kuzenko, R. Manvelyan, S. Theisen, Off-shell superconformal higher spin multiplets in four dimensions, arXiv:1701.00682.
[17] N.S. Craigie, V.K. Dobrev, I.T. Todorov, Conformally covariant composite operators in quantum chromodynamics, Ann. Phys. 159 (1985) 411–444.
[18] F.A. Berends, G.J.H. Burgers, H. van Dam, Explicit construction of conserved currents for massless fields of arbitrary spin, Nucl. Phys. B 271 (1986) 429–441.
[19] R. Roiban, A.A. Tseytlin, 2015, unpublished.
[20] G. 't Hooft, M. Veltman, One loop divergencies in the theory of gravitation, Ann. Inst. Henri Poincaré A, Phys. Théor. 20 (1974) 69–94.
[21] S.M. Christensen, Vacuum expectation value of the stress tensor in an arbitrary curved background: the covariant point separation method, Phys. Rev. D 14 (1976) 2490–2501.
[22] L.S. Brown, J.P. Cassidy, Stress tensor trace anomaly in a gravitational metric: general theory, Maxwell field, Phys. Rev. D 15 (1977) 2810.

[23] A.O. Barvinsky, G.A. Vilkovisky, The generalized Schwinger–Dewitt technique in gauge theories and quantum gravity, Phys. Rep. 119 (1985) 1–74.
[24] A.O. Barvinsky, Yu.V. Gusev, G.A. Vilkovisky, V.V. Zhytnikov, The one loop effective action and trace anomaly in four-dimensions, Nucl. Phys. B 439 (1995) 561–582, arXiv:hep-th/9404187.
[25] I. Jack, H. Osborn, Background field calculations in curved space–time. 1. General formalism and application to scalar fields, Nucl. Phys. B 234 (1984) 331–364.
[26] I.G. Avramidi, Covariant Methods for the Calculation of the Effective Action in Quantum Field Theory and Investigation of Higher Derivative Quantum Gravity, PhD thesis, Moscow State Univ., 1986, arXiv:hep-th/9510140.
[27] H. Osborn, General bosonic σ models and string effective actions, Ann. Phys. 200 (1990) 1.
[28] I.G. Avramidi, The covariant technique for calculation of one loop effective action, Nucl. Phys. B 355 (1991) 712–754.
[29] R. Herman, W.A. Hiscock, Renormalization of the charged scalar field in curved space, Phys. Rev. D 53 (1996) 3285–3295, arXiv:gr-qc/9509015.
[30] T. Aida, Y. Kitazawa, Two loop prediction for scaling exponents in $(2+\epsilon)$-dimensional quantum gravity, Nucl. Phys. B 491 (1997) 427–460, arXiv:hep-th/9609077.
[31] S.L. Adler, J. Lieberman, Y.J. Ng, Regularization of the stress energy tensor for vector and scalar particles propagating in a general background metric, Ann. Phys. 106 (1977) 279.
[32] S.L. Adler, J. Lieberman, Trace anomaly of the stress – energy tensor for massless vector particles propagating in a general background metric, Ann. Phys. 113 (1978) 294.
[33] R.M. Wald, Trace anomaly of a conformally invariant quantum field in curved space–time, Phys. Rev. D 17 (1978) 1477–1484.
[34] G. Gibbons, in: S. Hawking, W. Israel (Eds.), Quantum Field Theory in Curved Spacetime, Cambridge University Press, 1979.
[35] I. Jack, H. Osborn, Two loop background field calculations for arbitrary background fields, Nucl. Phys. B 207 (1982) 474–504.
[36] M. Duff, Twenty years of the Weyl anomaly, Class. Quantum Gravity 11 (1994) 1387–1404, arXiv:hep-th/9308075.
[37] S. Christensen, M. Duff, New gravitational index theorems and supertheorems, Nucl. Phys. B 154 (1979) 301.
[38] A.A. Tseytlin, Effective action in de Sitter space and conformal supergravity, Yad. Fiz. 39 (1984) 1606–1615.
[39] E.S. Fradkin, A.A. Tseytlin, Instanton zero modes and beta functions in supergravities. 2. Conformal supergravity, Phys. Lett. B 134 (1984) 307.
[40] S. Deser, R.I. Nepomechie, Gauge invariance versus masslessness in De Sitter space, Ann. Phys. 154 (1984) 396.
[41] S. Deser, R.I. Nepomechie, Anomalous propagation of gauge fields in conformally flat spaces, Phys. Lett. B 132 (1983) 321.