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Multiple scales and singular limits of perfect fluids

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Abstract. In this article, our goal is to study the singular limits for a scaled barotropic Euler system modeling a rotating, compressible and inviscid fluid, where Mach number $= \epsilon^m$, Rossby number $= \epsilon$ and Froude number $= \epsilon^n$ are proportional to a small parameter $\epsilon \to 0$. The fluid is confined to an infinite slab, the limit behavior is identified as the incompressible Euler system or a damped incompressible Euler system depending on the relation between m and n. For *well-prepared* initial data, the convergence is shown on the lifespan time interval of the strong solutions of the target system, whereas a class of generalized *dissipative solutions* is considered for the primitive system. The technique can be adapted to the compressible Navier–Stokes system in the subcritical range of the adiabatic exponent γ with $1 < \gamma \leq \frac{3}{2}$, where weak solutions are not known to exist.

1. Introduction

We study the model of a rotating fluid as described in Chemin et al. [13]. Let $T > 0$ and $\Omega = \mathbb{R}^2 \times (0, 1) \subset \mathbb{R}^3$ be an infinite slab. We consider the scaled *compressible Euler equation* in the time-space cylinder $Q_T = (0, T) \times \Omega$ describing the time evolution of the mass density $\varrho = \varrho(t, x)$ and the momentum field $\mathbf{m} = \mathbf{m}(t, x)$ of a rotating inviscid fluid with axis of rotation $\mathbf{b} = (0, 0, 1)$:

- *Conservation of mass*

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0. \tag{1.1}$$

- *Conservation of momentum*

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \frac{1}{\mathrm{Ma}^2} \nabla_x p(\varrho) + \frac{1}{\mathrm{Ro}} \mathbf{b} \times \mathbf{m} = \frac{1}{\mathrm{Fr}^2} \varrho \nabla_x G, \tag{1.2}$$

  where p stands for *pressure* and G is the gravitational potential that we consider independent of time.
- *Pressure law* The pressure p and the density ϱ of the fluid are interrelated by the standard isentropic law

$$p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma > 1. \tag{1.3}$$

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- *Boundary condition* Here we consider an impermeability condition on the horizontal boundary, i.e.,

$$\mathbf{m} \cdot \mathbf{n} = 0, \ \mathbf{n} = (0, 0, \pm 1). \tag{1.4}$$

- The scaled system contains characteristic numbers:
  Ma– Mach number,
  Ro– Rossby number,
  Fr– Froude number.
  Here we consider the following scaling:

$$\mathrm{Ma} \approx \epsilon^m, \ \mathrm{Ro} \approx \epsilon, \ \mathrm{Fr} \approx \epsilon^n \text{ for } \epsilon > 0, \ m, n > 0. \tag{1.5}$$

- *Far field condition* The condition is needed as we consider the unbounded domain Ω. Let us introduce the notation $x = (x_h, x_3)$ and $P_h(x) = x_h$. For each $\epsilon > 0$, we identify a static solution that satisfies (1.1)–(1.2) with (1.5). More precisely, a static solution is a pair $(\tilde{\varrho}_\epsilon, \mathbf{0})$, where the density profile $\tilde{\varrho}_\epsilon$ satisfies

$$\nabla_x p(\tilde{\varrho}_\epsilon) = \epsilon^{2(m-n)} \tilde{\varrho}_\epsilon \nabla_x G.$$

  In general, there are infinitely many static solutions for a given potential G. We introduce a far field condition as

$$|\varrho_\epsilon - \tilde{\varrho}_\epsilon| \to 0, \ \mathbf{m}_\epsilon \to \mathbf{0} \text{ as } |x_h| \to \infty, \tag{1.6}$$

  where $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ indicates that state variables depend on the parameter ϵ.
- *Initial data* For each $\epsilon > 0$, we supplement the initial data as

$$\varrho_\epsilon(0, \cdot) = \varrho_{\epsilon,0}, \ \mathbf{m}_\epsilon(0, \cdot) = \mathbf{m}_{\epsilon,0}.$$

- *Choice of G* As a matter of fact, the gravitational potential G can be seen as a sum of the centrifugal force proportional to the norm of the horizontal component of the spatial variable, i.e., $(x_1^2 + x_2^2)$ and the gravitational force acting in the vertical direction x_3. We omit the effect of the centrifugal force in the present paper motivated by certain meteorological models. Instead we consider

$$G(x) = -x_3 \text{ in } \Omega \tag{1.7}$$

  corresponding to the gravitational force acting in the vertical direction.

Multiple scaling We consider singular limit problems for $\epsilon \to 0$ in two multi-scale regimes:

- *Case I* Here m, n are interrelated by

$$\frac{m}{2} > n \geq 1. \tag{1.8}$$

  It implies that we are interested in low Mach and Rossby number limit in the presence of low stratification.

- *Case II* In this case, we consider $m = n = 1$, that implies the low Mach and Rossby number limit in the presence of strong stratification.

Thus, we study the effect of the *low Mach number limit* (also called *incompressible limit*), the *low Rossby number limit* and the *low Froude number limit* acting *simultaneously* on the system (1.1)–(1.2).

Formally, we observe that the low Mach number limit regime indicates that the fluid becomes *incompressible* and the low Rossby number limit indicates the fast rotation of the fluid and as a consequence of that the fluid becomes *planner* (two-dimensional).

Since solutions of the (primitive) compressible Euler systems are expected to develop singularities (shock waves) in a finite time, there are two approaches to deal with the singular limit problem.

I. The first approach is to consider *classical (strong)* solutions of the *primitive system* and expect them to converge to the classical solutions of the *target system*. Here the most important and highly non-trivial problem is to ensure that the lifespan of the strong solutions with respect to the singular parameter(ϵ) are uniformly bounded below and away from zero.
II. The second approach is based on the concept of *weak*, *measure–valued* or *dissipative* solutions of the primitive system. Given a suitable choice of initial data, one can show convergence provided that the target system admits a smooth solution.

For the first approach in the low Mach number limits, we have results of Ebin [14], Kleinermann and Majda [28], Schochet [34], and many others. For rotating fluids, there are results by Babin et al. [3,4] and Chemin et al. [13].

In the case of the second approach, most results dealing with weak solutions have been studied for the compressible Navier–Stokes system with additional consideration of the high Reynolds number limit. For rotating fluids, there are several results, see Feireisl, Gallagher and Novotný [18], Feireisl et al. [17], Feireisl and Novotný [24,25] and Li [29].

The existence of a global-in-time weak solution of the compressible Euler equation satisfying the energy inequality is still open for general initial data. Therefore, it is important to consider a *measure-valued solution* or a newly developed *dissipative solution* for this system. The concept of measure-valued solutions has been studied in various contexts, such as in the analysis of numerical schemes, etc. In the following articles by Alibert and Bouchitté [2], Gwiazda et al. [26], Březina and Feireisl [8], Březina [10], Basarić [5] et al. [21], we observe the development of the theory of measure-valued solution for various models describing compressible fluids, mainly using Young measures.

Recently Feireisl et al. [22] and Breit et al. [6] have given a new definition of generalized solutions for the compressible Euler system, which they call a *dissipative solution*. This new definition does not include Young measures.

The advantages of the second approach to singular limit problems are as follows:

- *Weak* or *measure-valued* solutions to the primitive system exist globally in time. Thus, the result depends only on the lifespan of the target problem, which may be finite.
- Since convergence holds for a large class of generalized solutions, this suggests some stability of the limit solution of the target system.

In particular, the results with generalized solutions are better in the sense that the convergence holds for a larger class of solutions and for the lifespan of the limit system.

There are a number of articles dealing with the low Mach number limit in the context of measure-valued solutions. In Feireisl et al. [20], Bruell and Feireisl [9], Březina and Mácha [11], it is shown that measure-valued solution of the primitive system, describing a compressible inviscid fluid, converges to a strong solution of the incompressible target system given suitable initial data. The 'single-scale' limit of our system, i.e., $m = 1$ and $G = 0$, was studied by Nečasová and Tong [31], again using the measure-valued solutions.

The framework of measure-valued solutions can also be applied in the context of the Navier–Stokes system. Although global in time *weak solutions* exist here, their existence is constrained by the technical condition for the adiabatic exponent $\gamma > \frac{3}{2}$. To deal with this technical constraint, Feireisl et al. [19] introduced the concept of *dissipative measure-valued solution* in terms of the Young measure. Here we use a slightly different approach, introducing a *dissipative solution* for Navier–Stokes system without explicit presence of the Young measure. In this way, we extend the convergence result to the Navier–Stokes system with high Reynolds number limit in the regime where the existence of weak solutions is unknown.

In our approach, it is very important to consider correct initial data, which are mainly called *well-prepared* and *ill-prepared* initial data. Feireisl and Novotný explain in [23] that for ill-prepared data the presence of Rossby acoustic waves plays an important role in the analysis of singular limits. For well-prepared data, on the other hand, this effect is absent. Here we are concerned with the well-prepared initial data.

Our main goal is to prove that under suitable choice of initial data a dissipative solution of a compressible rotating Euler system in low Mach and low Rossby regime converges to a strong solution of the incompressible Euler system in 2D. Hence, our plan for the article is following:

1. Derivation of limit systems.
2. Solutions of the target and primitive systems and relative energy inequality.
3. Singular limit for '*well-prepared*' data.
4. Extension to the Navier–Stokes system.

1.1. Notation and preliminaries

- Here we consider the *pressure potential* (P) as

$$P(\varrho) = \frac{a}{\gamma - 1}\varrho^{\gamma} \tag{1.9}$$

for the pressure law (1.3). Without loss of generality, we assume the 'normalized' setting for p as

$$p'(1) = 1.$$

Consideration of (1.9) leads to the following relations:

$$\varrho P'(\varrho) - P(\varrho) = p(\varrho) \text{ and } \varrho P''(\varrho) = p'(\varrho) \text{ for } \varrho > 0.$$

- Let r be a positive real-valued function and its range lie in a compact subset of $(0, \infty)$. Then for any $\varrho \geq 0$, there exists $r_1, r_2 > 0$, that depend on r such that

$$P(\varrho) - (\varrho - r)P'(r) - P(r) \geq c(r) \begin{cases} (\varrho - r)^2 \text{ for } r_1 \leq \varrho < r_2 \\ 1 + \varrho^\gamma, \text{ otherwise} \end{cases}, \tag{1.10}$$

  where $c(r)$ is a constant dependent on r. We note that if $\gamma > 2$, one can consider $1 + \varrho^2$ instead of $1 + \varrho^\gamma$ in (1.10). Taking the last observation into account, we can replace γ by $\gamma' = \min\{2, \gamma\}$ in (1.10).
- *Essential and residual part of a function:* We introduce a function $\chi = \chi(\varrho)$ such that

$$\chi(\cdot) \in C_c^\infty(0, \infty), \ 0 \leq \chi \leq 1, \ \chi(\varrho) = 1 \text{ if } \varrho_1 \leq \varrho \leq \varrho_2,$$

  where $\varrho_1, \varrho_2 > 0$. For a function $H = H(\varrho, \mathbf{u})$, we set

$$[H]_{\text{ess}} = \chi(\varrho)H(\varrho, \mathbf{u}), \ [H]_{\text{res}} = (1 - \chi(\varrho))H(\varrho, \mathbf{u}), \ \varrho \geq 0 \text{ and } \mathbf{u} \in \mathbb{R}^3.$$

2. Derivation of the target system

In this section, we provide an informal justification how we obtain the *target system*. First we note that $(\tilde{\varrho}_\epsilon, \mathbf{0})$ is a static state solution for (1.1)–(1.2) and it satisfies

$$\nabla_x p(\tilde{\varrho}_\epsilon) = \epsilon^{2(m-n)} \tilde{\varrho}_\epsilon \nabla_x G.$$

Let us consider the following asymptotic expansion:

$$\begin{aligned} \varrho_\epsilon &= \tilde{\varrho}_\epsilon + \epsilon^m \varrho_\epsilon^{(1)} + \epsilon^{2m} \varrho_\epsilon^{(2)} + \cdots, \\ \mathbf{m}_\epsilon &= \tilde{\varrho}_\epsilon \mathbf{v} + \epsilon^m \mathbf{m}_\epsilon^{(1)} + \epsilon^{2m} \mathbf{m}_\epsilon^{(2)} + \cdots. \end{aligned}$$

As a consequence of the above and Taylor's expansion, we obtain

$$p(\varrho_\epsilon) = p(\tilde{\varrho}_\epsilon) + \epsilon^m p'(\tilde{\varrho}_\epsilon)\varrho_\epsilon^{(1)} + \epsilon^{2m} \left(p'(\tilde{\varrho}_\epsilon)\varrho_\epsilon^{(2)} + \frac{1}{2} p''(\tilde{\varrho}_\epsilon)(\varrho_\epsilon^{(1)})^2 \right) + o(\epsilon^{3m}).$$

2.1. Case I: $\frac{m}{2} > n > 1$; target system: Euler system

Clearly conditions on m and n in (1.8) indicate $\lim_{\epsilon\to 0} \nabla_x P'(\tilde{\varrho}_\epsilon) = 0$. Without loss of generality, we assume

$$\tilde{\varrho}_\epsilon \approx \bar{\varrho} + \epsilon^{2(m-n)}, \text{ where } \bar{\varrho} \text{ is a positive constant.}$$

Thus, we rewrite the continuity equation and the momentum equation as

$$\bar{\varrho}\mathrm{div}_x \mathbf{v} + \epsilon^m \left(\partial_t \varrho_\epsilon^{(1)} + \mathrm{div}_x(\mathbf{m}_\epsilon^{(1)})\right) + o(\epsilon^{2m}) = 0$$

and

$$\begin{aligned} &\bar{\varrho}(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla_x)\mathbf{v}) + \nabla_x\left(p'(\bar{\varrho})\varrho_\epsilon^{(2)} + \frac{1}{2}p''(\bar{\varrho})(\varrho_\epsilon^{(1)})^2\right) + \epsilon^{m-1}\mathbf{b}\times\mathbf{m}_\epsilon^{(1)} \\ &+ \frac{1}{\epsilon^m}p'(\bar{\varrho})\nabla_x\varrho_\epsilon^{(1)} + \frac{1}{\epsilon}(\bar{\varrho}\mathbf{b}\times\mathbf{v}) - \epsilon^{m-2n}\varrho_\epsilon^{(1)}\nabla_x G + o(\epsilon) = 0, \end{aligned} \tag{2.1}$$

respectively. Let $\mathbb{H}$ be the Helmholtz projection, then we have

$$\mathbb{H}\left(\partial_t(\varrho_\epsilon \mathbf{u}_\epsilon) + \mathrm{div}_x\left(\frac{\mathbf{m}_\epsilon\otimes\mathbf{m}_\epsilon}{\varrho_\epsilon}\right) + \frac{1}{\epsilon}\mathbf{b}\times\varrho_\epsilon\mathbf{u}_\epsilon\right) = \mathbb{H}\left(\frac{1}{\epsilon^{2n}}\varrho_\epsilon\nabla_x G\right).$$

Suppose $\mathbf{m}_\epsilon \to \bar{\varrho}\mathbf{v}$ in a strong sense, multiplying the above equation by ϵ and using our standard expansion technique, we obtain

$$\mathbb{H}[\mathbf{b}\times\bar{\varrho}\mathbf{v}] = 0.$$

From the above equations, we get the following relations:

$$\begin{aligned} &\mathbf{b}\times\mathbf{v} = \nabla_x\psi, \text{ for a scalar valued function } \psi \text{ with} \\ &\psi_{x_3} = 0,\ \psi(x) = \psi(x_h),\ \nabla_{x_h}^\perp\psi = \mathbf{v}_h,\ \mathbf{v}_h = (v_1, v_2), \\ &\mathrm{div}_{x_h}\mathbf{v}_h = 0,\ v_{3_{x_3}} = 0, \text{ with } \nabla_{x_h}^\perp \equiv \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right). \end{aligned}$$

Thus, we have

$$v_{1_{x_3}} = 0,\ v_{2_{x_3}} = 0, \text{ and } \mathbf{v}(x) = \mathbf{v}(x_h).$$

The boundary condition (1.4) leads us the conclusion $v_3(x_h, x_3) = 0$ and $\mathbf{v} = (\mathbf{v}_h(x_h), 0)$. Finally, the above discussion yields the target system

$$\begin{aligned} &\mathrm{div}_{x_h}\mathbf{v}_h = 0, \\ &\partial_t\mathbf{v}_h + (\mathbf{v}_h\cdot\nabla_{x_h})\mathbf{v}_h + \nabla_{x_h}\Pi = 0, \end{aligned} \tag{2.2}$$

i.e., the incompressible Euler equation in 2D.

2.2. Case II: $m = n = 1$; target system: damped Euler system describing a stratified fluid

In this case, we first recall that, a static solution satisfies

$$\nabla_x p(\tilde{\varrho}_\epsilon) = \tilde{\varrho}_\epsilon \nabla_x G.$$

It follows that $\tilde{\varrho}_\epsilon$ is independent of ϵ and we denote it by $\hat{\varrho}$. Therefore, we rewrite the continuity equation and the momentum equation as

$$\mathrm{div}_x(\hat{\varrho}\mathbf{v}) + \epsilon\left(\partial_t \varrho_\epsilon^{(1)} + \mathrm{div}_x\left(\mathbf{v}_\epsilon^{(1)}\hat{\varrho} + \mathbf{v}\varrho_\epsilon^{(1)}\right)\right) + o(\epsilon^2) = 0$$

and

$$\begin{aligned}&\partial_t(\hat{\varrho}\mathbf{v}) + \mathrm{div}_x(\hat{\varrho}\mathbf{v}\otimes\mathbf{v}) + \nabla_x\left(p'(\hat{\varrho})\varrho_\epsilon^{(2)} + \frac{1}{2}p''(\hat{\varrho})(\varrho_\epsilon^{(1)})^2\right) + \mathbf{b}\times\mathbf{m}_\epsilon^{(1)}\\ &+\frac{1}{\epsilon}\left(\nabla_x(p'(\hat{\varrho})\varrho_\epsilon^{(1)}) + \mathbf{b}\times\hat{\varrho}\mathbf{v} - \varrho_\epsilon^{(1)}\nabla_x G\right) + o(\epsilon) = 0,\end{aligned}$$

respectively, where $\mathbf{m}_\epsilon^{(1)} = \mathbf{v}_\epsilon^{(1)}\hat{\varrho} + \mathbf{v}\varrho_\epsilon^{(1)}$.

We assume $\left(\frac{\varrho_\epsilon - \hat{\varrho}}{\epsilon}\right) \to q$ and $\mathbf{u}_\epsilon \to \mathbf{v}$ in a strong sense. Then as a consequence we have

$$\begin{aligned}&\nabla_x(p'(\hat{\varrho})q) + \mathbf{b}\times\hat{\varrho}\mathbf{v} = q\nabla_x G,\\ &\mathrm{div}_x(\hat{\varrho}\mathbf{v}) = 0,\\ &\partial_t(\hat{\varrho}\mathbf{v}) + \mathrm{div}_x(\hat{\varrho}\mathbf{v}\otimes\mathbf{v}) + \nabla_x\Pi_1 + \mathbf{b}\times\mathbf{m}_1 = 0,\\ &\partial_t q + \mathrm{div}_x\mathbf{m}_1 = 0.\end{aligned}$$

Here we get $(P''(\hat{\varrho})q)_{x_3} = 0$ and this implies $(P''(\hat{\varrho})q)(x_h, x_3) = (P''(\hat{\varrho})q)(x_h, 0)$ for $x_3 \in (0, 1)$. Consequently, from the choice of G, we obtain

$$q(x_h, x_3) = f(\hat{\varrho}(x_3))\bar{q}(x_h), \tag{2.3}$$

where f is a bounded continuous function in $(0, 1)$. Without loss of generality, we assume $\hat{\varrho} > 0$. Finally, we summarize the above discussion and derive the expected target system as

$$\begin{aligned}&\nabla_{x_h}(P''(\hat{\varrho})q) + \mathbf{b}\times\mathbf{v} = 0,\\ &\partial_t(\Delta_{x_h}(p'(\hat{\varrho})q) - q) + \nabla_{x_h}^\perp(p'(\hat{\varrho})q)\cdot\nabla_{x_h}(\Delta_{x_h}(P''(\hat{\varrho})q)) = 0,\end{aligned} \tag{2.4}$$

in $\mathbb{R}^2$, for each $x_3 \in (0, 1)$ and q in the form given in (2.3). The system (2.4) represents a stratified fluid in $\mathbb{R}^2 \times (0, 1)$ where the time evolution is given in $\mathbb{R}^2$ only and q can be viewed as a stream function of the flow.

3. Solvability of target systems, primitive system and relative energy

3.1. Target system: existence of the strong solution

In the last section, we informally obtained the expected target systems in both cases. Here we state some well-known results that prove the existence of a strong solution for these systems.

3.1.1. Case I

First, we recall the expected target system, the 2D Euler equation, i.e.,

$$\begin{aligned} &\mathrm{div}_{x_h}\mathbf{v}_h = 0, \text{ in } \mathbb{R}^2,\\ &\partial_t \mathbf{v}_h + (\mathbf{v}_h \cdot \nabla_{x_h})\mathbf{v}_h + \nabla_{x_h}\Pi = 0, \text{ in } \mathbb{R}^2. \end{aligned}$$

The following result of Kato and Lai [27] ensures existence and uniqueness for the incompressible Euler system in $\mathbb{R}^2$ for sufficiently smooth initial data.

Proposition 3.1. *Let*

$$\mathbf{v}_0 \in W^{k,2}(\mathbb{R}^2;\mathbb{R}^2),\ k \geq 3,\ div_{x_h}\mathbf{v}_0 = 0$$

be given. Then the system (2.2) *supplemented with the initial data* $\mathbf{v}_h(0) = \mathbf{v}_0$ *admits a regular solution* $(\mathbf{v}_h, \Pi)$*, unique in the class*

$$\begin{aligned} &\mathbf{v}_h \in C([0,T]; W^{k,2}(\mathbb{R}^2;\mathbb{R}^2)),\ \partial_t \mathbf{v}_h \in C([0,T]; W^{k-1,2}(\mathbb{R}^2;\mathbb{R}^2)),\\ &\Pi \in C([0,T]; W^{k,2}(\mathbb{R}^2)), \end{aligned}$$

with $div_{x_h}\mathbf{v}_h = 0$.

Alternatively, we write the system (2.2) as

$$\partial_t\ \mathrm{curl}_{x_h}\mathbf{v}_h + \mathbf{v}_h \cdot \nabla_{x_h}\mathrm{curl}_{x_h}\mathbf{v}_h = 0, \text{ in } \mathbb{R}^2,$$

i.e., vorticity formulation of the system (2.2) where the vorticity is $\mathrm{curl}_{x_h}\mathbf{v}_h$.

3.1.2. Case II

Now we consider the case $m = n = 1$. Here a static solution $\tilde{\varrho}_\epsilon$ is independent of ϵ and in particular, we choose a static solution $(\hat{\varrho} = \hat{\varrho}(x_3))$ such that

$$0 < \hat{\varrho} \in C^3(\Omega) \cap W^{1,\infty}(\Omega). \tag{3.1}$$

In this case, that expected target system is a damped variant of incompressible Euler system. It states that for each $x_3 \in [0,1]$, $(q(x_h, x_3), \mathbf{v}(x_h, x_3)(= \mathbf{v}_h(x_h), 0))$ solves

$$\nabla_x(P''(\hat{\varrho}(x_3))q) + \mathbf{b} \times \mathbf{v} = 0, \tag{3.2}$$

$$\partial_t(\Delta_{x_h}(p'(\hat{\varrho}(x_3))q) - q) + \nabla^{\perp}_{x_h}(p'(\hat{\varrho}(x_3))q) \cdot \nabla_{x_h}(\Delta_{x_h}(P''(\hat{\varrho}(x_3))q)) = 0, \quad (3.3)$$

supplemented with initial data $q(0, \cdot) = q_0$ in Ω.

If assume $q_0 \in W^{k,2}(\mathbb{R}^2)$ with $k > 4$, then for each $x_3 \in [0, 1]$, Eq. (3.3) admits strong solution. Furthermore, following Oliver [33, Theorem 3], we have for each $x_3 \in [0, 1]$, $q(\cdot, x_3) \in C([0, T]; W^{k,2}(\mathbb{R}^2)) \cap C^1([0, T]; W^{k-1,2}(\mathbb{R}^2))$.

Finally, from (2.3) we state the regularity of the target system as follows:

Proposition 3.2. *Suppose that*

$$q_0 \in W^{k,2}(\mathbb{R}^2) \text{ for } k \geq 4.$$

Then, the problem (2.4) *admits a solution* q*, unique in the class*

$$q \in C([0, T]; W^{k,2}(\mathbb{R}^2) \times C^3([0, 1])) \cap C^1([0, T]; W^{k-1,2}(\mathbb{R}^2) \times C^3([0, 1])).$$

3.2. Definition of a dissipative solution for the primitive system

Now we provide the definition of a *dissipative solution* for the primitive system of our consideration.

Definition 3.3. Let $\epsilon > 0$ and $\tilde{\varrho}_\epsilon > 0$. We say $(\varrho_\epsilon, \mathbf{u}_\epsilon)$ with

$$\varrho_\epsilon - \tilde{\varrho}_\epsilon \in C_{\text{weak}}([0, T]; L^2 + L^\gamma(\Omega)),$$
$$\varrho_\epsilon \geq 0, \ \mathbf{m}_\epsilon \in C_{\text{weak}}([0, T]; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega)),$$

is a dissipative solution to the compressible Euler equation (1.1)–(1.7) with initial data $(\varrho_{0,\epsilon}, (\varrho\mathbf{u})_{0,\epsilon})$ satisfying

$$\varrho_{0,\epsilon} \geq 0, \ E_{0,\epsilon} = \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_{0,\epsilon}|^2}{\varrho_{0,\epsilon}} + P(\varrho_{0,\epsilon}) - (\varrho_{0,\epsilon} - \tilde{\varrho}_\epsilon) P'(\tilde{\varrho}_\epsilon) - P(\tilde{\varrho}_\epsilon) \right) \mathrm{d}x < \infty, \quad (3.4)$$

if there exist the *turbulent defect measures*

$$\mathfrak{R}_{m_\epsilon} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}})), \ \mathfrak{R}_{e_\epsilon} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega})),$$

satisfying the compatibility condition

$$\lambda_1 \text{Trace}(\mathfrak{R}_{m_\epsilon}) \leq \mathfrak{R}_{e_\epsilon} \leq \lambda_2 \text{Trace}(\mathfrak{R}_{m_\epsilon}), \text{ for some } \epsilon \text{ independent } \lambda_1, \lambda_2 > 0, \quad (3.5)$$

such that the following holds:

- *Equation of continuity* For any $\tau \in (0, T)$ and any $\varphi \in C^1_c([0, T) \times \bar{\Omega})$ it holds

$$\left[\int_\Omega \varrho_\epsilon \varphi \, \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho_\epsilon \partial_t \varphi + \mathbf{m}_\epsilon \cdot \nabla_x \varphi] \, \mathrm{d}x \, \mathrm{d}t \, ;$$

- *Momentum equation* For any $\tau \in (0,T)$ and any $\boldsymbol{\varphi} \in C_c^1([0,T)\times\Omega;\mathbb{R}^d)$ with $\boldsymbol{\varphi}\cdot\mathbf{n}|_{\partial\Omega}=0$, it holds

$$\begin{aligned}\Big[\int_\Omega \mathbf{m}_\epsilon(\tau,\cdot)\cdot\boldsymbol{\varphi}(\tau,\cdot)\,\mathrm{d}x\Big]_{t=0}^{t=\tau}&=\int_0^\tau\int_\Omega\Big[\mathbf{m}_\epsilon\cdot\partial_t\boldsymbol{\varphi}+\Big(\frac{\mathbf{m}_\epsilon\otimes\mathbf{m}_\epsilon}{\varrho_\epsilon}\Big):\nabla_x\boldsymbol{\varphi}+\frac{1}{\epsilon^{2m}}p(\varrho_\epsilon)\mathrm{div}_x\boldsymbol{\varphi}+\frac{1}{\epsilon}\mathbf{b}\times\mathbf{m}_\epsilon\cdot\boldsymbol{\varphi}\Big]\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^\tau\int_\Omega\frac{1}{\epsilon^{2n}}\varrho_\epsilon\nabla_x G\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_{\overline{\Omega}}\nabla_x\boldsymbol{\varphi}:\mathrm{d}\mathfrak{R}_{m_\epsilon}\,\mathrm{d}t;\end{aligned}$$

- *Energy inequality* The total energy E_ϵ is defined in $[0,T)$ as

$$E_\epsilon(\tau)=\int_\Omega\Big(\frac{1}{2}\frac{|\mathbf{m}_\epsilon|^2}{\varrho_\epsilon}+\frac{1}{\epsilon^{2m}}(P(\varrho_\epsilon)-(\varrho_\epsilon-\tilde{\varrho}_\epsilon)P'(\tilde{\varrho}_\epsilon))-P(\tilde{\varrho}_\epsilon)\Big)(\tau,\cdot)\,\mathrm{d}x.$$

It satisfies

$$E_\epsilon(\tau)+\int_{\overline{\Omega}}\mathrm{d}\,\mathfrak{R}_{e_\epsilon}(\tau,\cdot)\le E_{0,\epsilon}\tag{3.6}$$

for a.a. $\tau>0$.

Remark 3.4. It is important to define the function

$$(\varrho_\epsilon,\mathbf{m}_\epsilon)\mapsto\frac{1}{2}\frac{|\mathbf{m}_\epsilon|^2}{\varrho_\epsilon}$$

on the vacuum set as

$$(\varrho_\epsilon,\mathbf{m}_\epsilon)\mapsto\frac{1}{2}\frac{|\mathbf{m}_\epsilon|^2}{\varrho_\epsilon}=\begin{cases}\frac{1}{2}\frac{|\mathbf{m}_\epsilon|^2}{\varrho_\epsilon} & \text{if } \varrho_\epsilon\neq 0,\ \mathbf{m}_\epsilon\neq 0,\\ 0 & \text{if } \varrho_\epsilon=0,\ \mathbf{m}_\epsilon=0,\\ \infty & \text{if } \varrho_\epsilon=0,\ \mathbf{m}_\epsilon\neq 0.\end{cases}$$

It follows from the energy inequality (3.6) that for each $\epsilon>0$,

$$\mathcal{L}^4\Big(\Big\{(t,x)\in(0,T)\times\Omega\Big|\frac{1}{2}\frac{|\mathbf{m}_\epsilon|^2}{\varrho_\epsilon}=\infty\Big\}\Big)=0,$$

where $\mathcal{L}^4$ is the Lebesgue measure in $\mathbb{R}^4$.

In order to study the existence, we first note that the impermeability boundary condition in $\mathbb{R}^2\times(0,1)$ can be transformed into periodic ones by considering the space of symmetric functions, see Ebin [15]. Here $\varrho,\mathbf{m}_h(=m_1,m_2)$ were extended as even functions in the x_3-variable defined on $\mathbb{R}^2\times\mathbb{T}^1$, while m_3 is extended as an odd function in x_3 on the same set, i.e.,

$$\begin{aligned}&\varrho(t,x_h,-x_3)=\varrho(t,x_h,x_3),\ \ \mathbf{m}_h(t,x_h,-x_3)=\mathbf{m}_h(t,x_h,x_3),\\&m_3(t,x_h,-x_3)=-m_3(t,x_h,x_3),\end{aligned}\tag{3.7}$$

for all $t\in(0,T)$, $x_h\in\mathbb{R}^2$ and $x_3\in\mathbb{T}^1$. A similar convention is adopted for the initial data. Hence, the consideration of the domain $\mathbb{R}^2\times(0,1)$ with slip boundary condition is equivalent to $\mathbb{R}^2\times\mathbb{T}^1$. We have to consider solutions in the class (3.7). The existence of a dissipative solution follows from Breit, Feireisl and Hofmanová as in [6,7].

3.3. Relative energy inequality

In our approach, relative energy functional plays an important role. We consider

$$\begin{aligned}\mathcal{E}_\epsilon(t) &= \mathcal{E}\left(\varrho_\epsilon, \mathbf{m}_\epsilon \mid \tilde{\varrho}, \tilde{\mathbf{u}}\right)(t)\\ &:= \int_\Omega \left[\frac{1}{2}\varrho_\epsilon \left|\frac{\mathbf{m}_\epsilon}{\varrho_\epsilon} - \tilde{\mathbf{u}}\right|^2 + \frac{1}{\epsilon^{2m}}(P(\varrho_\epsilon) - P(\tilde{\varrho}) - P'(\tilde{\varrho})(\varrho_\epsilon - \tilde{\varrho}))\right](t,\cdot)\,\mathrm{d}x,\end{aligned} \tag{3.8}$$

where $(\tilde{\varrho}, \tilde{\mathbf{u}})$ satisfy

$$\begin{aligned}&0 < \tilde{\varrho} \in C^\infty([0,T]) \times \overline{\Omega}) \text{ with } \tilde{\varrho} - \tilde{\varrho}_\epsilon \text{ having compact support in } [0,T] \times \overline{\Omega},\\ &\tilde{\mathbf{u}} \in C_c^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^3) \text{ with } \tilde{\mathbf{u}} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega,\end{aligned} \tag{3.9}$$

and $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ is a dissipative solution of the primitive system.

Remark 3.5. The relative energy is a coercive functional (see. Bruell et al. [9] and Březina and Feireisl [12]) satisfying the estimate

$$\begin{aligned}\mathcal{E}\left(\varrho_\epsilon, \mathbf{u}_\epsilon \mid \tilde{\varrho}, \tilde{\mathbf{u}}\right)(t) \geq &\int_\Omega \left[\left|\frac{\mathbf{m}_\epsilon}{\varrho_\epsilon} - \tilde{\mathbf{u}}\right|^2\right]_{\text{ess}} \mathrm{d}x + \int_\Omega \left[\frac{|\mathbf{m}_\epsilon|^2}{\varrho_\epsilon}\right]_{\text{res}} \mathrm{d}x\\ &+ \frac{1}{\epsilon^{2m}}\int_\Omega [(\varrho_\epsilon - \tilde{\varrho})^2]_{\text{ess}}\,\mathrm{d}x + \frac{1}{\epsilon^{2m}}\int_\Omega [1]_{\text{res}} + [\varrho_\epsilon^\gamma]_{\text{res}}\,\mathrm{d}x,\end{aligned}$$

where ess and res are considered with respect to $\varrho_1 = \frac{1}{2}\min_{(0,T)\times\Omega}\tilde{\varrho}$ and $\varrho_2 = 2\max_{(0,T)\times\Omega}\tilde{\varrho}$.

Using Definition 3.3, from (3.8) we deduce the following relative energy inequality:

$$\begin{aligned}\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\,\mathfrak{R}_{e_\epsilon}(\tau,\cdot) \leq \mathcal{E}_\epsilon(0) &- \int_0^\tau \int_\Omega (\mathbf{m}_\epsilon - \varrho_\epsilon\tilde{\mathbf{u}}) \cdot \partial_t\tilde{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\ &- \int_0^\tau \int_\Omega \left(\frac{(\mathbf{m}_\epsilon - \varrho_\epsilon\tilde{\mathbf{u}}) \otimes \mathbf{m}_\epsilon}{\varrho_\epsilon}\right) : \nabla_x\tilde{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\ &- \frac{1}{\epsilon^{2m}}\int_0^\tau \int_\Omega (p(\varrho_\epsilon) - p(\tilde{\varrho}))\mathrm{div}_x\tilde{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\ &+ \frac{1}{\epsilon^{2m}}\int_0^\tau \int_\Omega (\tilde{\varrho} - \varrho_\epsilon)\partial_t P'(\tilde{\varrho})\,\mathrm{d}x\,\mathrm{d}t\\ &+ \frac{1}{\epsilon}\int_0^\tau \int_\Omega \mathbf{b} \times \mathbf{m}_\epsilon \cdot \tilde{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\ &+ \frac{1}{\epsilon^{2m}}\int_0^\tau \int_\Omega (\tilde{\varrho}\tilde{\mathbf{u}} - \mathbf{m}_\epsilon) \cdot (\nabla_x P'(\tilde{\varrho}) - \nabla_x P'(\tilde{\varrho_\epsilon}))\,\mathrm{d}x\,\mathrm{d}t\\ &- \frac{1}{\epsilon^{2n}}\int_0^\tau \int_\Omega (\varrho_\epsilon - \tilde{\varrho})\nabla_x G \cdot \tilde{\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t\\ &- \int_0^\tau \int_\Omega \nabla_x\tilde{\mathbf{u}} : \mathrm{d}\mathfrak{R}_{m_\epsilon}(t,\cdot)\,\mathrm{d}t\,.\end{aligned} \tag{3.10}$$

3.3.1. A possible adaptation for Sobolev test functions

From Remark 3.5, it is clear that the relative energy functional is nonnegative and if $\mathcal{E}_\epsilon \to 0$ as $\epsilon \to 0$ we obtain some strong local convergence of the state variables. So we want to substitute $(\tilde{\varrho}, \tilde{u})$ suitably by the solutions of target system. We notice that the existence of strong solutions is known in the space of Sobolev functions, see Propositions 3.1 and 3.2. Therefore, following [5] we can extend the inequality (3.10) for the test functions $(\tilde{\varrho}, \tilde{\mathbf{u}})$ having Sobolev regularities, i.e., $(\tilde{\varrho} - \tilde{\varrho}_\epsilon, \tilde{\mathbf{u}}) \in C^1([0,T]; W^{k,2}(\Omega)) \times C^1([0,T]; W^{k,2}(\Omega; \mathbb{R}^3))$ with $k \geq 3$ and $\tilde{\mathbf{u}} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

4. Main results and their proofs

First we state our main results for both cases:

4.1. Main theorem: Case I

4.1.1. Properties of a static solution

First, we notice that a static solution $(\tilde{\varrho}_\epsilon, \mathbf{0})$ satisfies

$$\nabla_x p(\tilde{\varrho}_\epsilon) = \epsilon^{2(m-n)} \tilde{\varrho}_\epsilon \nabla_x G.$$

In terms of the pressure potential, we rewrite the above equation as

$$\nabla_x P'(\tilde{\varrho}_\epsilon) = \epsilon^{2(m-n)} \nabla_x G.$$

So, we obtain

$$P'(\tilde{\varrho}_\epsilon) = -\epsilon^{2(m-n)} x_3 + C,$$

where C is a constant. As a consequence of $G = (0, 0, -x_3)$, we have $\tilde{\varrho}_\epsilon(x) = \tilde{\varrho}_\epsilon(x_3)$. Without loss of generality, we consider $C = 1$. We know that $P'(s) \approx s^{\gamma-1}$ for $s \geq 0$. To reduce complication, here we assume $P'(s) = s^{\gamma-1}$, for $s \geq 0$. We also have

$$P''(\tilde{\varrho}_\epsilon) \nabla_x \tilde{\varrho}_\epsilon = \epsilon^{2(m-n)}.$$

For $0 < \epsilon < \frac{1}{2}$, we observe that a *static solution* $\tilde{\varrho}_\epsilon$ satisfies the following property:

$$\begin{aligned} &0 < \tilde{\varrho_\epsilon} \in C^\infty([0,1]), \\ &\sup_{x_3 \in [0,1]} |\tilde{\varrho}_\epsilon(x_3) - 1| \leq \epsilon^{\frac{2(m-n)}{\gamma-1}}, \quad \sup_{x_3 \in [0,1]} |\nabla_x \tilde{\varrho}_\epsilon(x_3)| \leq \epsilon^{2(m-n)}. \end{aligned} \tag{4.1}$$

Remark 4.1. Since, we are interested for the case $\epsilon \to 0$, the consideration of $0 < \epsilon < \frac{1}{2}$ is justified. Furthermore, if $\gamma > 2$ and $\epsilon < 1$ we have

$$\sup_{x_3 \in [0,1]} |\tilde{\varrho}_\epsilon(x_3) - 1| \leq \epsilon^{2(m-n)}.$$

As $m > n$, asymptotically, the static solution approaches the constant state $\tilde{\varrho} = 1$ as $\epsilon \to 0$.

4.1.2. Well-prepared data

We say that the set of initial data $\{(\varrho_{0,\epsilon}, \mathbf{m}_{0,\epsilon})\}_{\epsilon>0}$ is well-prepared if

$$\begin{aligned}
&\varrho_\epsilon(0,\cdot) = \varrho_{0,\epsilon} = \tilde{\varrho}_\epsilon + \epsilon^m \varrho_{0,\epsilon}^{(1)} \text{ with } \{\varrho_{0,\epsilon}^{(1)}\}_{\epsilon>0} \text{ is bounded in } L^2 \cap L^\infty(\Omega)\\
&\text{and } \varrho_{0,\epsilon}^{(1)} \to 0 \text{ in } L^2(\Omega),\\
&\mathbf{u}_{0,\epsilon} = \frac{\mathbf{m}_{0,\epsilon}}{\varrho_{0,\epsilon}} \to \mathbf{v}_0 = \left(\mathbf{v}_0^{(1)}, \mathbf{v}_0^{(2)}, 0\right) \text{ in } L^2(\Omega;\mathbb{R}^3) \text{ with } \mathrm{div}_{x_h}\mathbf{v}_0 = 0.
\end{aligned} \tag{4.2}$$

We provide the main result for this case.

Theorem 4.2. *Let $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ be a dissipative solution of the system (1.1)–(1.7) with $\frac{m}{2} > n \geq 1$. Moreover, we assume that the initial data are well-prepared, i.e., it satisfies (4.3) and $\mathbf{v}_0 \in W^{k,2}(\mathbb{R}^2)$ with $k \geq 3$. Then,*

$$\begin{aligned}
&ess\sup_{t\in(0,T)} \|\varrho_\epsilon - \tilde{\varrho}_\epsilon\|_{(L^2+L^{\gamma'})(\Omega)} \leq \epsilon^m c,\\
&\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{v} \begin{cases} \text{weak-(*)ly in} L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)),\\ \text{strongly in } L^1_{loc}((0,T)\times\Omega;\mathbb{R}^3), \end{cases}
\end{aligned}$$

where $c > 0$, $\gamma' = \min\{2,\gamma\}$ and $\mathbf{v} = (\mathbf{v}_h, 0)$ is the unique solution of the incompressible Euler system with initial data $\mathbf{v}_0$ in $\mathbb{R}^2$.

4.2. Main theorem: Case II

4.2.1. Well-prepared data

Let $\hat{\varrho}$ be a static solution satisfying (3.1). We say that the set of initial data $(\varrho_{0,\epsilon}, \mathbf{m}_{0,\epsilon})_{(\epsilon>0)}$ is *well-prepared* if

$$\begin{aligned}
&\varrho_{0,\epsilon} = \hat{\varrho} + \epsilon\varrho_{0,\epsilon}^{(1)},\ \{\varrho_{0,\epsilon}\}_{\{\epsilon>0\}} \text{ is bounded in } L^2\cap L^\infty(\Omega),\ \varrho_{0,\epsilon}^{(1)} \to q_0 \text{ in } L^2(\Omega),\\
&\frac{\mathbf{m}_{0,\epsilon}}{\varrho_{0,\epsilon}} \to \mathbf{v}_0 = \left(\mathbf{v}_0^{(1)}, \mathbf{v}_0^{(2)}, 0\right) \text{ in } L^2(\Omega;\mathbb{R}^3) \text{ with the relation}\\
&-\Delta_{x_h} q_0 = \hat{\varrho}\,\mathrm{curl}_{x_h} P_h(\mathbf{v}_0),\ \text{and } q_0 \in L^2(\mathbb{R}^2).
\end{aligned} \tag{4.3}$$

Theorem 4.3. *Let $(\varrho_\epsilon, \mathbf{m}_\epsilon)$ be a dissipative solution of the system (1.1)–(1.7) with $m = n = 1$. Moreover, we assume that the initial data are well-prepared, i.e., it satisfies (4.3) and $q_0 \in W^{k,2}$ with $k \geq 4$. Let $(q, \mathbf{v})$ solves (2.4) for initial data q_0. Then after taking a subsequence, the following holds*

$$\begin{aligned}
&\varrho_\epsilon^{(1)} \equiv \frac{\varrho_\epsilon - \hat{\varrho}}{\epsilon} \to q \text{ weak-(*)ly in } L^\infty(0,T;L^2 + L^{\gamma'}(\Omega)),\\
&\mathbf{m}_\epsilon \to \hat{\varrho}\mathbf{v} \text{ weak-(*)ly in } L^\infty(0,T;L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega)),
\end{aligned}$$

where $\gamma' = \min\{2, \gamma\}$. *Furthermore, we have*

$$\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{v} \text{ strongly in } L^1_{loc}((0,T)\times\Omega;\mathbb{R}^3).$$

Remark 4.4. In both cases, we note that the choice of well-prepared initial data implies (3.4), hence the consideration of the dissipative solution of the primitive system is justified.

In the following subsections, we give the proofs.

4.3. Proof of Theorem 4.2

4.3.1. Uniform bound and weak convergence

First, we note that $\tilde{\mathbf{u}} = 0$ and $\tilde{\varrho} = \tilde{\varrho}_\epsilon$ satisfy (3.9). Hence, we use them as test functions in the relative energy inequality (3.10). One the other hand, the choice of (4.2) ensures that the initial energy $E_{0,\epsilon}$ is uniformly bounded. Thus, we have the following bounds

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \right\|_{L^2(\Omega;\mathbb{R}^3)} \le C,\\
&\operatorname*{ess\,sup}_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \tilde{\varrho}_\epsilon}{\epsilon^m} \right]_{\text{ess}} \right\|_{L^2(\Omega)} \le C,\\
&\operatorname*{ess\,sup}_{t\in(0,T)} \|[\varrho_\epsilon]_{\text{res}}\|^\gamma_{L^\gamma(\Omega)} + \operatorname*{ess\,sup}_{t\in(0,T)} \|[1]_{\text{res}}\|^\gamma_{L^\gamma(\Omega)} \le \epsilon^{2m} C,
\end{aligned} \tag{4.4}$$

where C is independent of ϵ. We consider $\gamma' = \min\{2, \gamma\}$. Estimate (4.4) and the fact $\gamma' \le 2$ imply

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\varrho_\epsilon - \tilde{\varrho}_\epsilon\|_{(L^2+L^{\gamma'})(\Omega)} \le (\epsilon^m + \epsilon^{\frac{2m}{\gamma'}})C \le \epsilon^m C. \tag{4.5}$$

Equation (4.5), together with (4.1), yields

$$\varrho_\epsilon \to 1 \text{ in } L^\infty(0,T; L^q_{\text{loc}}(\Omega)) \text{ for any } 1 \le q < \gamma'. \tag{4.6}$$

Also, from the uniform bound (4.4) and (4.6) imply that

$$\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{u} \text{ weak-(*)ly in } L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)),$$

and

$$\mathbf{m}_\epsilon \to \mathbf{m} \text{ weak-(*)lyin } L^\infty(0,T; L^2 + L^{\dot{\gamma}}(\Omega;\mathbb{R}^3)),$$

passing to suitable subsequence, where $\dot{\gamma} = \min\left\{\frac{4}{3}, \frac{2\gamma}{\gamma+1}\right\}$. The strong convergence of the density (4.6) helps to obtain $\mathbf{m} = \mathbf{u}$ in the weak sense.

Finally, we may let $\epsilon \to 0$ in the continuity equation to deduce that,

$$\int_0^\tau \int_\Omega \mathbf{u} \cdot \nabla_x \varphi \, dx \, dt = 0, \ \forall \varphi \in C_c^\infty(\Omega).$$

4.3.2. Strong convergence

Here we choose proper test functions and prove that $\lim_{\epsilon\to 0}\mathcal{E}_\epsilon(t)=0$.

Taking motivation from (2.1), we consider another equation that describes a non-oscillatory part described by a variable q_ϵ, that satisfies

$$\partial_t(\Delta_{x_h}q_\epsilon-(\epsilon^{m-1})^2q_\epsilon)+\frac{1}{\epsilon^{m-1}}\nabla^\perp_{x_h}q_\epsilon\cdot\nabla_{x_h}(\Delta_{x_h}q_\epsilon-(\epsilon^{m-1})^2q_\epsilon)=0, \tag{4.7}$$

in $\mathbb{R}^2$ supplemented with initial data $q_\epsilon(0,\cdot)=q_{0,\epsilon}$ such that

$$-\Delta_{x_h}q_{0,\epsilon}+(\epsilon^{2(m-1)})q_{0,\epsilon}=\epsilon^{m-1}\mathrm{curl}_{x_h}P_h(\mathbf{v}_0)$$

Let us introduce another variable $\mathbf{v}_\epsilon$ such that $\mathbf{v}_\epsilon$ and q_ϵ are interrelated by

$$\nabla_x q_\epsilon+\epsilon^{m-1}\mathbf{b}\times\mathbf{v}_\epsilon=0. \tag{4.8}$$

Thus, initial data for $\mathbf{v}_\epsilon$ satisfy

$$-\nabla_{x_h}q_{0,\epsilon}=\epsilon^{m-1}\mathbf{b}\times\mathbf{v}_{0,\epsilon}.$$

From the hypothesis on initial data in Theorem 4.2, we have

$$\mathbf{v}_0\in W^{k,2}(\mathbb{R}^2)\text{ with }k\geq 3.$$

We observe that $\|q_{0,\epsilon}\|_{L^2(\mathbb{R}^2)}\leq C$ and $\|\nabla_x q_{0,\epsilon}\|_{L^2(\mathbb{R}^2)}\leq\epsilon^{m-1}C$. Therefore, we can consider $\{q_{0,\epsilon}\}_{\epsilon>0}$ such that $q_{0,\epsilon}\to 0$ in $L^2(\mathbb{R}^2)$ as $\epsilon\to 0$. Furthermore, we also note that $\mathbf{v}_{0,\epsilon}\to P_h(\mathbf{v}_0)$ as $\epsilon\to 0$.

In order to have a simplified notation, we consider $\omega=\epsilon^{m-1}$ and $\tilde{q}_\epsilon=\frac{q_\epsilon}{\omega}$. We rewrite (4.7) as

$$\partial_t(\Delta_{x_h}\tilde{q}_\epsilon-\omega^2\tilde{q}_\epsilon)+\nabla^\perp_{x_h}\tilde{q}_\epsilon\cdot\nabla_{x_h}(\Delta_{x_h}\tilde{q}_\epsilon-\omega^2q_\epsilon)=0, \tag{4.9}$$

We notice that Eq. (4.9) has a similar structure to that of the geophysical flows. Thus, we apply Oliver [33, Theorem 3] to ensure the existence and uniqueness of solution $\tilde{q}_\epsilon$.

In order to obtain a uniform estimate independent of ϵ, we multiply the (4.9) by q_ϵ and performing integration by parts, we get

$$\int_{\mathbb{R}^2}\left(|\nabla_{x_h}\tilde{q}_\epsilon|^2+\omega^2|\tilde{q}_\epsilon|^2\right)(t,\cdot)\,\mathrm{d}x=\int_{\mathbb{R}^2}\left(|\nabla_{x_h}\tilde{q}_{0,\epsilon}|^2+\omega^2|\tilde{q}_{0,\epsilon}|^2\right)\mathrm{d}x, \tag{4.10}$$

for a.e. $t\in(0,T)$. As the initial data for $\tilde{q}_\epsilon$ depends only on $\mathbf{v}_0$, we deduce that

$$\begin{aligned}\{-\Delta_{x_h}\tilde{q}_\epsilon\ &+\omega^2\tilde{q}_\epsilon\}_{\epsilon>0}\text{ is bounded in }C^1([0,T];W^{k-2,2}(\mathbb{R}^2))\\ &\cap C([0,T];W^{k-1,2}(\mathbb{R}^2)).\end{aligned}$$

Now, from (4.8), we also obtain

$$\{\mathbf{v}_\epsilon\}_{\epsilon>0} \text{ is bounded in } C([0,T];W^{k,2}(\mathbb{R}^2)) \cap C^1([0,T];W^{k-1,2}(\mathbb{R}^2)).$$

It is easy to verify that $\partial_t \tilde{q}_\epsilon$ satisfies the equation

$$\partial_t q_\epsilon = (\Delta_{x_h} - \omega^2)^{-1}(\mathbf{v}_\epsilon \mathrm{curl}_{x_h}\mathbf{v}_\epsilon).$$

Consequently, it yields

$$\{\partial_t \tilde{q}_\epsilon\}_{\epsilon>0} \text{ is bounded in } C([0,T];W^{k-1,2}(\mathbb{R}^2)),$$

and

$$\{\partial_t \mathbf{v}_\epsilon\}_{\epsilon>0} \text{ is bounded in } C([0,T];W^{k,2}(\mathbb{R}^2;\mathbb{R}^2)).$$

This bounds are independent of ϵ.

Therefore, we obtain the following weak convergence:

$$\mathbf{v}_\epsilon \to \mathbf{v} \text{ weakly in } C([0,T];W^{k,2}(\mathbb{R}^2)),$$

and

$$\partial_t \mathbf{v}_\epsilon \to \partial_t \mathbf{v} \text{ weakly in } C([0,T];W^{k-1,2}(\mathbb{R}^2)).$$

Since $k \geq 4$, applying Sobolev embedding theorem, we obtain

$$\mathbf{v}_\epsilon \to \mathbf{v} \text{ in } L^q(0,T;L^q_{\mathrm{loc}}(\mathbb{R}^2)). \tag{4.11}$$

We rewrite (4.9) as

$$\partial_t(\mathrm{curl}_{x_h}\mathbf{v}_\epsilon) - \omega^2\partial_t\tilde{q}_\epsilon + \mathbf{v}_\epsilon\nabla_x(\mathrm{curl}_{x_h}\mathbf{v}_\epsilon) = 0.$$

From (4.11), we infer that

$$\partial_t(\mathrm{curl}_{x_h}\mathbf{v}) + \mathbf{v}\cdot\nabla_x(\mathrm{curl}_{x_h}\mathbf{v}) = 0.$$

This is similar to (2.2). We also have

$$\Pi \in C([0,T];W^{k,2}(\mathbb{R}^2)).$$

Clearly we have the following estimates

$$\begin{aligned} &\|q_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq C,\\ &\|\partial_t q_\epsilon\|_{L^\infty(0,T;L^q(\Omega))} + \|\nabla_{x_h} q_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq \epsilon^{m-1}C, \end{aligned} \tag{4.12}$$

for $q \geq 2$. Also $\mathbf{v}_\epsilon \in C([0,T];W^{k-1,2}(\mathbb{R}^2))$ with $k \geq 4$ implies

$$\|q_\epsilon\|_{L^\infty((0,T)\times\mathbb{R}^2)} \leq C. \tag{4.13}$$

Now, we consider a suitable test function for the relative energy inequality(3.10) as

$$\tilde{\mathbf{u}} = \mathbf{V}_\epsilon = (\mathbf{v}_\epsilon, 0),\ \tilde{\varrho} = \tilde{\varrho}_\epsilon + \epsilon^m q_\epsilon,$$

where $(q_\epsilon, \mathbf{v}_3)$ satisfies (4.7) ans (4.8) and $\tilde{\varrho}_\epsilon$ is a static solution that satisfies (4.1). We use the relation between q_ϵ and $\mathbf{v}_\epsilon$ and obtain

$$\begin{aligned}
\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\, \mathfrak{R}_{e_\epsilon}(\tau, \cdot) \le \mathcal{E}_\epsilon(0)
&- \int_0^\tau \int_\Omega (\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{V}_\epsilon) \cdot (\partial_t \mathbf{V}_\epsilon + (\mathbf{V}_\epsilon \cdot \nabla_x)\mathbf{V}_\epsilon)\, \mathrm{d}x\, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left(\frac{(\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{V}_\epsilon) \otimes (\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{V}_\epsilon)}{\varrho_\epsilon} \right) : \nabla_x \mathbf{V}_\epsilon \, \mathrm{d}x\, \mathrm{d}t \\
&+ \frac{1}{\epsilon^{2m}} \int_0^\tau \int_\Omega (\tilde{\varrho} - \varrho_\epsilon) \partial_t P'(\tilde{\varrho})\, \mathrm{d}x\, \mathrm{d}t \\
&+ \frac{1}{\epsilon^m} \int_0^\tau \int_\Omega \mathbf{m}_\epsilon \cdot \nabla_x q_\epsilon (P''(\tilde{\varrho}) - P''(1))\, \mathrm{d}x\, \mathrm{d}t \\
&+ \frac{1}{\epsilon^{2m}} \int_0^\tau \int_\Omega \mathbf{m}_\epsilon \cdot (P''(\tilde{\varrho}) - P''(\tilde{\varrho_\epsilon})) \nabla_x \tilde{\varrho}_\epsilon \, \mathrm{d}x\, \mathrm{d}t \\
&- \frac{1}{\epsilon^{2n}} \int_0^\tau \int_\Omega (\varrho_\epsilon - \tilde{\varrho}) \nabla_x G \cdot \mathbf{V}_\epsilon \, \mathrm{d}x\, \mathrm{d}t \\
&- \int_0^\tau \int_{\overline{\Omega}} \nabla_x \tilde{\mathbf{u}} : \mathrm{d}\mathfrak{R}_{m_\epsilon}(t, \cdot)\, \mathrm{d}t \ = \Sigma_{i=1}^8 \mathcal{L}_i. \qquad (4.14)
\end{aligned}$$

Here we compute each term $\mathcal{L}_i$, $i = 1, \cdots, 8$ of (4.14). For term $\mathcal{L}_1$, we have

$$\mathcal{E}_\epsilon(\varrho_{0,\epsilon}, (\varrho\mathbf{u})_{0,\epsilon} \mid \tilde{\varrho}_\epsilon + \epsilon^m q_{0,\epsilon}, \mathbf{v}_0) \le \left\| \frac{(\varrho\mathbf{u})_{0,\epsilon}}{\varrho_{0,\epsilon}} - \mathbf{v}_0 \right\|^2_{L^2(\Omega)} + \left\| \varrho^{(1)}_{0,\epsilon} - q_{0,\epsilon} \right\|^2_{L^2(\Omega)}.$$

Consideration of well-prepared data yields

$$|\mathcal{L}_1| \le \xi(\epsilon). \qquad (4.15)$$

From now on, we use this generic function $\xi(\cdot)$, such that $\lim_{\epsilon \to 0} \xi(\epsilon) = 0$.

We rewrite $\mathcal{L}_2$ as

$$\begin{aligned}
\mathcal{L}_2 &= -\int_0^\tau \int_\Omega (\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{V}_\epsilon) \cdot (\partial_t \mathbf{V}_\epsilon + (\mathbf{V}_\epsilon \cdot \nabla_x)\mathbf{V}_\epsilon)\, \mathrm{d}x\, \mathrm{d}t \\
&= -\int_0^\tau \int_\Omega \mathbf{m}_\epsilon \cdot (\partial_t \mathbf{V}_\epsilon + (\mathbf{V}_\epsilon \cdot \nabla_x)\mathbf{V}_\epsilon)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad + \int_0^\tau \int_\Omega (\varrho_\epsilon - 1)\mathbf{V}_\epsilon \cdot \partial_t \mathbf{V}_\epsilon \, \mathrm{d}x\, \mathrm{d}t + \int_0^\tau \int_\Omega \partial_t \mathbf{V}_\epsilon \cdot \mathbf{V}_\epsilon \, \mathrm{d}x\, \mathrm{d}t \\
&= \mathcal{L}_{2,1} + \mathcal{L}_{2,2} + \mathcal{L}_{2,3}.
\end{aligned}$$

Using (4.12) and (4.1), we obtain

$$|\mathcal{L}_{2,2}| \le \xi(\epsilon).$$

We claim

$$\mathcal{L}_{2,1} \to -\int_0^\tau \int_\Omega \mathbf{u} \cdot \nabla_{x_h} \Pi \, \mathrm{d}x \, \mathrm{d}t \; = 0,$$

as $\epsilon \to 0$. Let, K be a compact subset of $\mathbb{R}^2$. We use (4.11) to deduce

$$\begin{aligned} &\int_0^\tau \int_{K\times(0,1)} \mathbf{m}_\epsilon \cdot (\partial_t \mathbf{V}_\epsilon + (\mathbf{V}_\epsilon \cdot \nabla_x)\mathbf{V}_\epsilon) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \to \int_0^\tau \int_{K\times(0,1)} \mathbf{u} \cdot (\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla_x)\mathbf{v}) \, \mathrm{d}x \, \mathrm{d}t \; , \end{aligned}$$

where $\mathbf{v} = (\mathbf{v}_h, 0)$. Using the fact that $\Pi \in C([0,T]; W^{k,2}(\mathbb{R}^2))$ with $k \geq 3$, we have

$$|\mathcal{L}_{2,1}| \leq \xi(\epsilon).$$

We want to estimate the term $\mathcal{L}_4$. First, we rewrite it as

$$\begin{aligned} \mathcal{L}_4 &= \frac{1}{\epsilon^m} \int_0^\tau \int_\Omega (\tilde{\varrho} - \varrho_\epsilon) P''(\tilde{\varrho}) \partial_t q_\epsilon \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^\tau \int_\Omega (P''(\tilde{\varrho}) - P''(\tilde{\varrho}_\epsilon)) \left(q_\epsilon - \frac{\varrho_\epsilon - \tilde{\varrho}_\epsilon}{\epsilon^m} \right) \partial_t q_\epsilon \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega (P''(\tilde{\varrho}_\epsilon) - P''(1)) \left(q_\epsilon - \frac{\varrho_\epsilon - \tilde{\varrho}_\epsilon}{\epsilon^m} \right) \partial_t q_\epsilon \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_0^\tau \int_\Omega q_\epsilon \partial_t q_\epsilon \, \mathrm{d}x \, \mathrm{d}t \; - \int_0^\tau \int_\Omega \frac{\varrho_\epsilon - \tilde{\varrho}_\epsilon}{\epsilon^m} \partial_t q_\epsilon \, \mathrm{d}x \, \mathrm{d}t \\ &= \mathcal{L}_{4,1} + \mathcal{L}_{4,2} + \mathcal{L}_{4,3} + \mathcal{L}_{4,4}. \end{aligned}$$

We note that, for each $x \in \Omega$ we get

$$(P''(\tilde{\varrho}) - P''(\tilde{\varrho}_\epsilon)) \leq C\epsilon^m |q_\epsilon(x)|.$$

Using the above inequality and (4.12), we deduce that

$$\begin{aligned} |\mathcal{L}_{4,1}| \leq \epsilon^m \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \tilde{\varrho}_\epsilon}{\epsilon^m} \right]_{\mathrm{ess}} - q_\epsilon \right\|_{L^2(\Omega)} \|\partial_t q_\epsilon\|_{L^\infty([0,T]\times\Omega)} \|q_\epsilon\|_{L^\infty(0,T;L^2(\Omega))} \\ + \epsilon^m \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \tilde{\varrho}_\epsilon}{\epsilon^m} \right]_{\mathrm{res}} \right\|_{L^{\gamma'}(\Omega)} \|q_\epsilon\|_{L^\infty(0,T;L^{\gamma'*}(\Omega))} \|\partial_t q_\epsilon\|_{L^\infty([0,T]\times\Omega)}, \end{aligned}$$

where $\frac{1}{\gamma'} + \frac{1}{\gamma'^*} = 1$.

Similarly, using (4.1) we have

$$|\mathcal{L}_{4,2}| \leq \epsilon^{\frac{2(m-n)}{\gamma-1}} \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \tilde{\varrho}_\epsilon}{\epsilon^m} \right]_{\text{ess}} - q_\epsilon \right\|_{L^2(\Omega)} \|\partial_t q_\epsilon\|_{L^\infty(0,T;L^2(\Omega))}$$
$$+ \epsilon^{\frac{2(m-n)}{\gamma-1}} \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \tilde{\varrho}_\epsilon}{\epsilon^m} \right]_{\text{res}} \right\|_{L^{\gamma'}(\Omega)} \|\partial q_\epsilon\|_{L^\infty(0,T;L^{\gamma'^*}(\Omega))},$$

for $1 < \gamma \leq 2$, and

$$|\mathcal{L}_{4,2}| \leq \epsilon^{2(m-n)} \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \tilde{\varrho}_\epsilon}{\epsilon^m} \right]_{\text{ess}} - q_\epsilon \right\|_{L^2(\Omega)} \|\partial_t q_\epsilon\|_{L^\infty(0,T;L^2(\Omega))}$$
$$+ \epsilon^{2(m-n)} \sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\epsilon - \tilde{\varrho}_\epsilon}{\epsilon^m} \right]_{\text{res}} \right\|_{L^{\gamma'}(\Omega)} \|\partial q_\epsilon\|_{L^\infty(0,T;L^{\gamma'^*}(\Omega))},$$

for $\gamma > 2$.

Analogously, we deduce

$$|\mathcal{L}_{4,4}| \leq \left\| \frac{\varrho_\epsilon - \tilde{\varrho}_\epsilon}{\epsilon^m} \right\|_{L^\infty(0,T;L^2+L^{\gamma'}(\Omega))} \|\partial_t q_\epsilon\|_{L^\infty(0,T;L^2\cap L^{\gamma'^*}(\Omega))},$$

where $\frac{1}{\gamma'} + \frac{1}{\gamma'^*} = 1$. We use estimate (4.12) to conclude

$$|\mathcal{L}_{4,1}| + |\mathcal{L}_{4,2}| + |\mathcal{L}_{4,4}| \leq \xi(\epsilon).$$

The equation (4.10) implies

$$\mathcal{L}_{2,3} + \mathcal{L}_{4,3} = \int_0^\tau \int_\Omega (\partial_t \mathbf{v}_\epsilon \cdot \mathbf{v}_\epsilon + q_\epsilon \partial_t q_\epsilon) \, \mathrm{d}x \, \mathrm{d}t \, = 0.$$

Therefore, combining all estimates we get

$$|\mathcal{L}_2 + \mathcal{L}_4| \leq \xi(\epsilon). \tag{4.16}$$

It is easy to verify that

$$|\mathcal{L}_3| \leq \|\nabla_{x_h} \mathbf{v}_h\|_{L^\infty(0,\tau;L^\infty(\Omega))} \int_0^\tau \mathcal{E}_\epsilon(t) \, \mathrm{d}t \, . \tag{4.17}$$

For the term $\mathcal{L}_5$, the first observation is that for $x \in \Omega$ we have

$$P''(\tilde{\varrho}(x)) - P''(1) = (\tilde{\varrho} - 1) P'''(\eta(x)),$$

where $\eta(x) \in (\min\{1, \tilde{\varrho}\}, \max\{1, \tilde{\varrho}\})$. From the choice of $\tilde{\varrho} = \tilde{\varrho}_\epsilon + \epsilon^m q_\epsilon$ and estimate (4.13), we have

$$\sup_{x\in\Omega} |P'''(\eta(x))| \leq C,$$

where C is dependent only on $\mathbf{v}_0$.

We rewrite $\mathcal{L}_5$ as

$$\begin{aligned}\mathcal{L}_5 &= \frac{1}{\epsilon^m}\int_0^\tau\int_\Omega (\tilde{\varrho}-1)P'''(\eta(x))\mathbf{m}_\epsilon\cdot\nabla_x q_\epsilon \,\mathrm{d}x\,\mathrm{d}t\\ &= \frac{1}{\epsilon^m}\int_0^\tau\int_\Omega \mathbf{m}_\epsilon\cdot\nabla_x q_\epsilon(\tilde{\varrho}_\epsilon-1)P'''(\eta(x))\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^\tau\int_\Omega q_\epsilon\mathbf{m}_\epsilon\cdot\nabla_x q_\epsilon P'''(\eta(x))\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

By using (4.1), we observe

$$\begin{aligned}|\mathcal{L}_5| \le \epsilon^{m-2n}&\|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{4/3}(\Omega;\mathbb{R}^3))}\|\nabla_x q_\epsilon\|_{L^\infty(0,T;L^2\cap L^4(\Omega;\mathbb{R}^3))}\\ &+\|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{4/3}(\Omega;\mathbb{R}^3))}\|q_\epsilon\nabla_x q_\epsilon\|_{L^\infty(0,T;L^2\cap L^4(\Omega;\mathbb{R}^3))},\end{aligned}$$

for $\gamma>2$, and

$$\begin{aligned}|\mathcal{L}_5| \le \epsilon^{\frac{2(m-n)}{\gamma-1}}&\|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{2\gamma/\gamma+1}(\Omega;\mathbb{R}^3))}\|\nabla_x q_\epsilon\|_{L^\infty(0,T;L^2\cap L^{(2\gamma/\gamma+1)'}(\Omega;\mathbb{R}^3))}\\ &+\|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{2\gamma/\gamma+1}(\Omega;\mathbb{R}^3))}\|q_\epsilon\nabla_x q_\epsilon\|_{L^\infty(0,T;L^2\cap L^{(2\gamma/\gamma+1)'}(\Omega;\mathbb{R}^3))},\end{aligned}$$

for $1<\gamma\le 2$, where $\left(\frac{2\gamma}{\gamma+1}\right)'=\frac{2\gamma}{\gamma-1}$.

In particular, $\frac{m}{2}>n\ge 1$, (4.4) and (4.12) imply

$$|\mathcal{L}_5| \le \epsilon^{2(m-n)-1}C,\ \text{for } \gamma>2,$$

and

$$|\mathcal{L}_5| \le \epsilon^{\frac{(m-2n)+(m-\gamma+1)}{\gamma-1}}C,\ \text{for } 1<\gamma<2,$$

where C is a constant depending on $\mathbf{v}_0$ in both cases. Finally we obtain

$$\mathcal{L}_5 \le \xi(\epsilon). \tag{4.18}$$

Similarly, we rewrite the term $\mathcal{L}_6$ as

$$\begin{aligned}\mathcal{L}_6 &= \frac{1}{\epsilon^{2m}}\int_0^\tau\int_\Omega \mathbf{m}_\epsilon\cdot(\tilde{\varrho}-\tilde{\varrho}_\epsilon)P'''(\zeta(x))\nabla_x\tilde{\varrho}_\epsilon\,\mathrm{d}x\,\mathrm{d}t\\ &= \frac{1}{\epsilon^m}\int_0^\tau\int_\Omega \mathbf{m}_\epsilon\cdot q_\epsilon P'''(\zeta(x))\nabla_x\tilde{\varrho}_\epsilon\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where for each x, $\zeta(x)\in(\min\{\tilde{\varrho_\epsilon},\tilde{\varrho}\},\max\{\tilde{\varrho_\epsilon},\tilde{\varrho}\})$. Using arguments similar to $\mathcal{L}_5$, we have

$$\begin{aligned}|\mathcal{L}_6| &\le \epsilon^{m-2n}\|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{2\gamma/\gamma+1}(\Omega;\mathbb{R}^3))}\|q_\epsilon\|_{L^\infty(0,T;L^2\cap L^{(2\gamma/\gamma+1)'}(\Omega))}\\ &\le \xi(\epsilon).\end{aligned} \tag{4.19}$$

Now, the choice of G implies

$$\mathcal{L}_7 = 0. \tag{4.20}$$

The compatibility of the defect measures (3.5) yields

$$|\mathcal{L}_8| \leq C \int_0^\tau \int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}_{e_\epsilon} \, \mathrm{d}t \ . \tag{4.21}$$

Therefore, combining all estimates (4.15)–(4.21), we get

$$\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\, \mathfrak{R}_{e_\epsilon}(\tau, \cdot) \leq \xi(\epsilon) + c \int_0^\tau \mathcal{E}_\epsilon(t) \, \mathrm{d}t \ + C \int_0^\tau \int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}_{e_\epsilon} \, \mathrm{d}t \ .$$

We use Grönwall's lemma to infer

$$\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\, \mathfrak{R}_{e_\epsilon}(\tau, \cdot) \leq \xi(\epsilon) C(T),$$

where $\xi(\epsilon) \to 0$ as $\epsilon \to 0$ and for a.e. τ with $0 \leq \tau \leq T$. The coercivity of the relative energy functional helps to deduce

$$\limsup_{\epsilon \to 0} \int_K \left| \frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} - \mathbf{v} \right|^2 \mathrm{d}x \leq C(T) \limsup_{\epsilon \to 0} \xi(\epsilon),$$

where $K \subset \Omega$ is a compact set. Thus, we conclude that $\mathbf{u} = \mathbf{v}_h$. Also, we obtain

$$\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \mathbf{v} \text{ strongly in } L^1_{\mathrm{loc}}((0, T) \times \Omega; \mathbb{R}^3).$$

It ends proof of Theorem 4.2.

4.4. Proof of Theorem 4.3

4.4.1. Uniform bounds and weak convergence

To obtain a uniform bound, we proceed similarly to Section 4.2.3. First, using $\tilde{\mathbf{u}} = 0$, $\tilde{\varrho} = \hat{\varrho}$ as test functions and well-prepared data(4.3), we obtain the following uniform bounds:

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t \in (0,T)} \left\| \frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \right\|_{L^2(\Omega; \mathbb{R}^3)} \leq C, \\
&\operatorname*{ess\,sup}_{t \in (0,T)} \left\| \left[\frac{\varrho_\epsilon - \hat{\varrho}}{\epsilon} \right]_{\mathrm{ess}} \right\|_{L^2(\Omega)} \leq C, \\
&\operatorname*{ess\,sup}_{t \in (0,T)} \| [\varrho_\epsilon]_{\mathrm{res}} \|^\gamma_{L^\gamma(\Omega)} + \operatorname*{ess\,sup}_{t \in (0,T)} \| [1]_{\mathrm{res}} \|^\gamma_{L^\gamma(\Omega)} \leq \epsilon^2 C.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\varrho_\epsilon^{(1)} \equiv \left(\frac{\varrho_\epsilon - \hat{\varrho}}{\epsilon} \right) \to \varrho^{(1)} \text{ weak-(*)ly in } L^\infty(0, T; L^2 + L^{\gamma'}(\Omega)), \\
&\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}} \to \hat{\mathbf{m}} \text{ weak-(*)ly in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)),
\end{aligned}$$

passing to a suitable subsequence as the case may be, here $\gamma' = \min\{2, \gamma\}$.

We also deduce that

$$\varrho_\epsilon \to \hat{\varrho} \text{ in } L^\infty(0, T; L^2 + L^{\gamma'}(\Omega)).$$

Furthermore, we have

$$\|\mathbf{m}_\epsilon\|_{L^\infty(0,T;L^2+L^{\dot{\gamma}}(\Omega;\mathbb{R}^3))} \leq C,$$

where $\dot{\gamma} = \left\{\frac{4}{3}, \frac{2\gamma}{\gamma+1}\right\}$. Eventually, for a suitable subsequence, we get

$$\mathbf{m}_\epsilon \to \mathbf{m} \text{ weak-(*)ly in } L^\infty(0, T; L^2 + L^{\dot{\gamma}}(\Omega; \mathbb{R}^3)).$$

Letting $\epsilon \to 0$ in the continuity equation, we infer the incompressibility condition in the weak sense, i.e.,

$$\int_0^\tau \int_\Omega \mathbf{m} \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \ = 0.$$

Define $\mathbf{u} = \frac{\mathbf{m}}{\hat{\varrho}}$. Multiplying momentum equation by ϵ and letting $\epsilon \to 0$, we obtain the diagnostic equation

$$\mathbf{b} \times \mathbf{u} + \nabla_x \left(P''(\hat{\varrho}) \varrho^{(1)} \right) = 0, \tag{4.22}$$

in the sense of distributions.

4.4.2. Strong convergence using relative energy inequality

Let $(q, \mathbf{v})$ be a strong solution of the above system with initial data $(q_0, \mathbf{v}_0)$ satisfying (4.3) with $k \geq 4$. Our goal is to show that $(\varrho^{(1)}, \mathbf{u}) \equiv (q, \mathbf{v})$. Here we choose appropriate test functions and will show that $\lim_{\epsilon \to 0} \mathcal{E}_\epsilon(t) = 0$.

We consider the test functions for the relative energy inequality (3.10) as

$$\tilde{\mathbf{u}} = \mathbf{v} = (\mathbf{v}_h, 0), \ \tilde{\varrho} = \hat{\varrho} + \epsilon q.$$

Thus, we rewrite the relative energy inequality in the following form:

$$\begin{aligned}
\mathcal{E}_\epsilon(\tau) + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{R}_{e_\epsilon}(\tau) &\leq \mathcal{E}_\epsilon(0) \\
&- \int_0^\tau \int_\Omega (\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{v}) \cdot (\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla_x)\mathbf{v}) \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega \left(\frac{(\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{v}) \otimes (\mathbf{m}_\epsilon - \varrho_\epsilon \mathbf{v})}{\varrho_\epsilon} \right) : \nabla_x \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t \\
&- \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (p(\varrho_\epsilon) - p(\tilde{\varrho})) \mathrm{div}_x \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t \\
&- \frac{1}{\epsilon} \int_0^\tau \int_\Omega \mathbf{b} \times \mathbf{v} \cdot (\mathbf{m}_\epsilon - \hat{\varrho} \mathbf{v}) \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\epsilon^2}\int_0^\tau\int_\Omega(\hat{\varrho}+\epsilon q-\varrho_\epsilon)P''(\hat{\varrho}+\epsilon q)\ \partial_t(\hat{\varrho}+\epsilon q)\,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{1}{\epsilon^2}\int_0^\tau\int_\Omega((\hat{\varrho}+\epsilon q)\mathbf{v}-\mathbf{m}_\epsilon)\cdot\nabla_x(P'(\hat{\varrho}+\epsilon q)-P'(\hat{\varrho}))\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^\tau\int_{\overline{\Omega}}\nabla_x\mathbf{v}:\mathrm{d}\mathfrak{R}_{m_\epsilon}(t,\cdot)\,\mathrm{d}t\,.
\end{aligned}
$$

Using the fact $\mathrm{div}_{x_h}\mathbf{v}=0$ and (3.2), we obtain

$$
\begin{aligned}
&\mathcal{E}_\epsilon(\tau)+\int_{\overline{\Omega}}\mathrm{d}\mathfrak{R}_{e_\epsilon}(\tau)\\
&\quad\le\mathcal{E}_\epsilon(0)-\int_0^\tau\int_\Omega(\mathbf{m}_\epsilon-\varrho_\epsilon\mathbf{v})\cdot(\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla_x)\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\frac{1}{\epsilon}\int_0^\tau\int_\Omega(\hat{\varrho}+\epsilon q-\varrho_\epsilon)P''(\hat{\varrho}+\epsilon q)\ \partial_t q\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_0^\tau\int_\Omega\left(\frac{(\mathbf{m}_\epsilon-\varrho_\epsilon\mathbf{v})\otimes(\mathbf{m}_\epsilon-\varrho_\epsilon\mathbf{v})}{\varrho_\epsilon}\right):\nabla_x\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\frac{1}{\epsilon^2}\int_0^\tau\int_\Omega(\hat{\varrho}\mathbf{v}-\mathbf{m}_\epsilon)\cdot\nabla_x(P'(\hat{\varrho}+\epsilon q)-P'(\hat{\varrho})-\epsilon P''(\hat{\varrho})q)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\frac{1}{\epsilon}\int_0^\tau\int_\Omega q\mathbf{v}\cdot\nabla_x(P'(\hat{\varrho}+\epsilon q)-P'(\hat{\varrho}))\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_0^\tau\int_{\overline{\Omega}}\nabla_x\mathbf{v}:\mathrm{d}\mathfrak{R}_{m_\epsilon}(t,\cdot)\,\mathrm{d}t\ =\Sigma_{i=1}^7\mathcal{L}_i.
\end{aligned}
$$

Now we want to estimate each term $\mathcal{L}_i$ for $i=1,\dots,7$. First we notice that, consideration of the well-prepared data (4.3) yields

$$
\mathcal{E}_\epsilon(\varrho_{0,\epsilon},\mathbf{m}_{0,\epsilon}\mid\bar{\varrho}+\epsilon q_0,\mathbf{v}_0)\le\left\|\frac{\mathbf{m}_{0,\epsilon}}{\varrho_{0,\epsilon}}-\mathbf{v}_0\right\|^2_{L^2(\Omega)}+\left\|\varrho^{(1)}_{0,\epsilon}-q_0\right\|^2_{L^2(\Omega)}.
$$

This implies

$$
|\mathcal{L}_1|\le c(\epsilon). \tag{4.23}
$$

Here $c(\epsilon)$ is a generic function such that $c(\epsilon)\to 0$ as $\epsilon\to 0$.

First, we rewrite two terms of $\mathcal{L}_2$ and $\mathcal{L}_3$ as

$$
\mathbf{m}_\epsilon-\varrho_\epsilon\mathbf{v}=(\mathbf{m}_\epsilon-\bar{\varrho}\mathbf{u})+((\bar{\varrho}-\varrho_\epsilon)\mathbf{u})+(\varrho_\epsilon(\mathbf{u}-\mathbf{v}))
$$

and

$$
P''(\hat{\varrho}+\epsilon q)=P''(\hat{\varrho}+\epsilon q)-P''(\hat{\varrho})+P''(\hat{\varrho}).
$$

Using the weak convergence of the variables, we obtain

$$|\mathcal{L}_2+\mathcal{L}_3| \le c(\epsilon) + \int_0^\tau \int_\Omega (\hat{\varrho}\mathbf{v}-\mathbf{m})\cdot(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla_x)\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega (q-\varrho^{(1)})\,\partial_t(P''(\hat{\varrho})q)\,\mathrm{d}x\,\mathrm{d}t\,.$$

We claim that

$$\int_0^\tau \int_\Omega (\hat{\varrho}\mathbf{v}-\mathbf{m})\cdot(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla_x)\mathbf{v})\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega (q-\varrho^{(1)})\,\partial_t(P''(\hat{\varrho})q)\,\mathrm{d}x\,\mathrm{d}t \ = 0. \tag{4.24}$$

To prove the above claim, first we observe that

$$\mathbf{v}\cdot\partial_t(\hat{\varrho}\mathbf{v}) + q\partial_t(P'(\hat{\varrho})q) = \frac{1}{2}\partial_t\left(\hat{\varrho}|\mathbf{v}|^2 + P'(\hat{\varrho})q^2\right).$$

Since $(q,\mathbf{v})$ solves (2.4), multiplying (3.3) by q we get

$$\int_0^\tau \int_\Omega \mathbf{v}\cdot\partial_t(\hat{\varrho}\mathbf{v}) + q\partial_t(P'(\hat{\varrho})q)\,\mathrm{d}x\,\mathrm{d}t \ = 0.$$

Now we use (4.22) and (3.2) to deduce

$$\int_0^\tau \int_\Omega \left(\mathbf{m}\cdot\partial_t\mathbf{v} + \varrho^{(1)}\partial_t\left(P''(\hat{\varrho})q\right)\right)\mathrm{d}x\,\mathrm{d}t \ = \int_0^\tau \int_\Omega \mathbf{b}\times\mathbf{m}\cdot\left(P''(\hat{\varrho})\mathbf{v}\right)\mathrm{d}x\,\mathrm{d}t$$

and

$$\int_0^\tau \int_\Omega \mathbf{b}\times\mathbf{m}\cdot\left(P''(\hat{\varrho})\mathbf{v}\right)\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega \mathbf{m}\cdot(\mathbf{v}\cdot\nabla_x)\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau \mathbf{m}\cdot\nabla_{x_h}\left(\frac{1}{2}|\mathbf{v}|^2\right)\mathrm{d}x\,\mathrm{d}t \ = 0.$$

Hence, we achieve (4.24) and it implies

$$|\mathcal{L}_2+\mathcal{L}_3| \le c(\epsilon). \tag{4.25}$$

From the definition of relative energy, we obtain

$$|\mathcal{L}_4| \le \int_0^\tau \mathcal{E}_\epsilon(t)\,\mathrm{d}t\,. \tag{4.26}$$

Since $0<\hat{\varrho}\in C^3([0,1])$, we verify that

$$\frac{1}{\epsilon}\nabla_x\left(P'(\hat{\varrho}+\epsilon q)-P'(\hat{\varrho})\right) \to \nabla_x\left(P''(\hat{\varrho})q\right) \text{ in } L^\infty(0,T;L^\infty\cap L^2(\Omega))$$

and

$$\frac{1}{\epsilon^2}\nabla_x(P'(\hat{\varrho}+\epsilon q)-P'(\hat{\varrho})-\epsilon P''(\hat{\varrho})q) \\ \to \nabla_x\left(\frac{1}{2}P'''(\hat{\varrho})\right) \text{ in } L^\infty(0,T;L^\infty\cap L^2(\Omega)).$$

The above relation implies

$$\lim_{\epsilon\to 0}\mathcal{L}_5=\int_0^\tau\int_\Omega(\hat{\varrho}\mathbf{v}-\mathbf{m})\cdot\nabla_x\left(\frac{1}{2}P'''(\hat{\varrho})\right)\,\mathrm{d}x\,\mathrm{d}t\ =0 \tag{4.27}$$

and

$$\lim_{\epsilon\to 0}\mathcal{L}_6=\int_0^\tau\int_\Omega q\mathbf{v}\to\nabla_x\left(P''(\hat{\varrho})q\right)\,\mathrm{d}x\,\mathrm{d}t\ =0 \tag{4.28}$$

We obtain

$$|\mathcal{L}_7|\le C\int_0^\tau\int_{\overline{\Omega}}\mathbb{I}:\mathrm{d}\mathfrak{C}_{m_\epsilon}(t,\cdot)\,\mathrm{d}t\,. \tag{4.29}$$

Combining (4.23)–(4.29), we have that

$$\mathcal{E}_\epsilon(\tau)+\int_{\overline{\Omega}}\mathrm{d}\mathfrak{C}_{e_\epsilon}\le c(\epsilon)+\int_0^\tau\mathcal{E}_\epsilon(t)\,\mathrm{d}t\ +C\int_0^\tau\int_{\overline{\Omega}}\mathrm{d}\,\mathrm{Trace}(\mathfrak{C}_{m_\epsilon})\,\mathrm{d}t\,,$$

for a.e. $\tau\in(0,T)$. Finally, using the compatibility of *turbulent defect measures* and Grönwall's lemma, we infer

$$\mathcal{E}_\epsilon(\tau)+\int_{\overline{\Omega}}\mathrm{d}\mathfrak{C}_{e_\epsilon}\le c(\epsilon)\le c(\epsilon)C(T),$$

where $c(\epsilon)\to 0$ as $\epsilon\to 0$. Therefore, we obtain our desired result

$$\lim_{\epsilon\to 0}\mathcal{E}_\epsilon(\tau)=0. \tag{4.30}$$

Now using coercivity of the relative energy functional as stated in Remark 3.5, we say

$$\varrho^{(1)}=q \text{ and } \mathbf{v}=\mathbf{u}.$$

Moreover, we use coercivity together with (4.30) to conclude

$$\frac{\varrho_\epsilon-\hat{\varrho}}{\epsilon}\to q \text{ strongly in } L^1_{\mathrm{loc}}((0,T)\times\Omega),$$
$$\frac{\mathbf{m}_\epsilon}{\sqrt{\varrho_\epsilon}}\to\sqrt{\hat{\varrho}}\mathbf{v} \text{ strongly in } L^1_{\mathrm{loc}}((0,T)\times\Omega;\mathbb{R}^3).$$

This completes the proof of Theorem 4.3.

5. Extension to the Navier–Stokes system

Here our goal is to give a proper definition of a *dissipative* solution for Navier–Stokes equation. We consider another characteristic number, i.e., Reynolds number. Additional assumption with high Reynolds number limit along with other characteristic number limits helps us to obtain the same *target system*.

5.1. Definition of dissipative solution for Navier–Stokes system

Let ϱ be the density and $\mathbf{u}$ be the velocity. In time-space cylinder $Q_T = (0, T) \times \Omega$, we consider:

- *Conservation of mass*

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0. \tag{5.1}$$

- *Conservation of momentum*

$$\begin{aligned} &\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathrm{Ma}^2}\nabla_x p(\varrho) + \frac{1}{\mathrm{Ro}}\mathbf{b} \times \varrho \mathbf{u} \\ &\quad = \frac{1}{\mathrm{Re}}\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\mathrm{Fr}^2}\varrho \nabla_x G. \end{aligned} \tag{5.2}$$

- *Constitutive relation* Here $\mathbb{S}(\nabla_x \mathbf{u})$ is the *Newtonian stress tensor* defined by

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\left(\frac{\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}}{2} - \frac{1}{d}(\mathrm{div}_x \mathbf{u})\mathbb{I}\right) + \lambda(\mathrm{div}_x \mathbf{u})\mathbb{I}, \tag{5.3}$$

  where $\mu > 0$ and $\lambda > 0$ are the *shear* and *bulk* viscosity coefficients, respectively.
- The scaled system contains all specified characteristic numbers as in (1.5) along with,
  – Re– Reynolds number.
  Here we consider,

$$\mathrm{Ma} \approx \epsilon^m,\ \mathrm{Ro} \approx \epsilon,\ \mathrm{Re} \approx \epsilon^{-\alpha},\ \mathrm{Fr} \approx \epsilon^n \text{ for } \epsilon > 0,\ m, n, \alpha > 0 \text{ and } \frac{m}{2} > n \geq 1. \tag{5.4}$$

- *Pressure law* In an isentropic setting, the pressure p and the density ϱ of the fluid are interrelated by

$$p(\varrho) = a\varrho^\gamma,\ a > 0,\ \gamma > 1. \tag{5.5}$$

- *Boundary condition* Here we consider a complete slip condition for the velocity on the horizontal boundary, i.e.,

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = [\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}]_{\tan}|_{\partial\Omega} = 0,\ \mathbf{n} = (0, 0, \pm 1). \tag{5.6}$$

- *Far field condition* Let $(\tilde{\varrho}_\epsilon, \mathbf{0})$ be a static solution, we assume the condition as

$$|\varrho - \tilde{\varrho}_\epsilon| \to 0, \ \mathbf{u} \to \mathbf{0} \text{ as } |x_h| \to \infty. \tag{5.7}$$

- *Initial data* For each $\epsilon > 0$, we supplement the initial data as

$$\varrho(0, \cdot) = \varrho_{\epsilon,0}, \ (\varrho\mathbf{u})(0, \cdot) = (\varrho\mathbf{u})_{\epsilon,0}.$$

Here we provide the definition of *dissipative solution* for the Navier–Stokes system.

Definition 5.1. Let $\epsilon > 0$ and $\tilde{\varrho}_\epsilon > 0$. We say a pair of functions $(\varrho_\epsilon, \mathbf{u}_\epsilon)$ with

$$\begin{aligned}
&\varrho_\epsilon - \tilde{\varrho_\epsilon} \in C_{\text{weak}}([0, T]; L^2 + L^\gamma(\Omega)), \ \varrho_\epsilon \geq 0,\\
&\varrho_\epsilon \mathbf{u}_\epsilon \in C_{\text{weak}}([0, T]; L^2 + L^{\frac{2\gamma}{\gamma+1}}(\Omega)),\\
&\mathbf{u}_\epsilon \in L^2(0, T; W^{1,2}(\Omega)), \ \mathbf{u}_\epsilon \cdot \mathbf{n}|_{\partial\Omega} = 0
\end{aligned}$$

is a *dissipative solution* to (5.1)–(5.7) with initial data $\left(\varrho_{0,\epsilon}, (\varrho\mathbf{u})_{0,\epsilon}\right)$ satisfying

$$\begin{aligned}
\varrho_{0,\epsilon} \geq 0, \ E_{0,\epsilon} = \int_\Omega \left(\frac{1}{2} \frac{|(\varrho\mathbf{u})_{0,\epsilon}|^2}{\varrho_{0,\epsilon}} + P(\varrho_{0,\epsilon}) - (\varrho_{0,\epsilon} - \tilde{\varrho}_\epsilon) P'(\tilde{\varrho}_\epsilon) - P(\tilde{\varrho}_\epsilon) \right) \mathrm{d}x\\
< \infty,
\end{aligned} \tag{5.8}$$

if there exist the *turbulent defect measures*

$$\mathfrak{R}_{m_\epsilon} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}})), \ \mathfrak{R}_{e_\epsilon} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega})),$$

satisfying compatibility condition

$$\lambda_1 \text{Trace}\mathfrak{R}_{m_\epsilon} \leq \mathfrak{R}_{e_\epsilon} \leq \lambda_2 \text{Trace}\mathfrak{R}_{m_\epsilon}, \ \lambda_1, \lambda_2 > 0, \tag{5.9}$$

such that the following holds:

- *Equation of continuity* For any $\tau \in (0, T)$ and any $\varphi \in C_c^1([0, T] \times \bar{\Omega})$ it holds

$$\left[\int_\Omega \varrho_\epsilon \varphi \, \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho_\epsilon \partial_t \varphi + \varrho_\epsilon \mathbf{u}_\epsilon \cdot \nabla_x \varphi] \, \mathrm{d}x \, \mathrm{d}t \, ;$$

- *Momentum equation* For any $\tau \in (0, T)$ and any $\boldsymbol{\varphi} \in C_c^1([0, T] \times \Omega; \mathbb{R}^d)$ with $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$, it holds

$$\begin{aligned}
&\left[\int_\Omega \varrho_\epsilon \mathbf{u}_\epsilon(\tau, \cdot) \cdot \boldsymbol{\varphi}(\tau, \cdot) \, \mathrm{d}x \right]_{t=0}^{t=\tau}\\
&= \int_0^\tau \int_\Omega \left[\varrho_\epsilon \mathbf{u}_\epsilon \cdot \partial_t \boldsymbol{\varphi} + \varrho_\epsilon \mathbf{u}_\epsilon \otimes \mathbf{u}_\epsilon : \nabla_x \boldsymbol{\varphi} + \frac{1}{\epsilon^{2m}} p(\varrho_\epsilon) \text{div}_x \boldsymbol{\varphi} + \frac{1}{\epsilon} \mathbf{b} \times \varrho_\epsilon \mathbf{u}_\epsilon \cdot \boldsymbol{\varphi} \right] \mathrm{d}x \, \mathrm{d}t\\
&\quad - \int_0^\tau \int_\Omega [\epsilon^\alpha \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} - \frac{1}{\epsilon^{2n}} \varrho_\epsilon \nabla_x G \cdot \boldsymbol{\varphi}] \, \mathrm{d}x \, \mathrm{d}t \ + \int_0^\tau \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_{m_\epsilon} \, \mathrm{d}t \, ;
\end{aligned}$$

- *Energy inequality* The total energy E is defined in $[0, T)$ as

$$E_\epsilon(\tau) = \int_\Omega \left(\frac{1}{2} \varrho_\epsilon |\mathbf{u}_\epsilon|^2 + \frac{1}{\epsilon^{2m}} (P(\varrho_\epsilon) - (\varrho_\epsilon - \tilde{\varrho}_\epsilon) P'(\tilde{\varrho}_\epsilon)) - P(\tilde{\varrho}_\epsilon) \right)(\tau, \cdot)\, \mathrm{d}x;$$

It satisfies

$$E_\epsilon(\tau) + \epsilon^\alpha \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\, \mathrm{d}x\, \mathrm{d}t + \int_{\overline{\Omega}} \mathrm{d}\, \mathfrak{R}_{e_\epsilon}(\tau, \cdot) \leq E_{0,\epsilon}$$

for a.a. $\tau > 0$.

Remark 5.2. The class of test functions in the momentum equation corresponds to the complete slip (or Navier slip) boundary conditions. These are necessary to avoid problems with boundary layer.

Theorem 5.3. *Suppose Ω be the domain specified above and the pressure follows (5.5). If $(\varrho_{0,\epsilon}, (\varrho\mathbf{u})_{0,\epsilon})$ satisfies (5.8), then there exists a* dissipative solution *as defined above.*

We prove the existence theorem for $\epsilon = 1$.

Proof. Here we give an extended outline of the proof. We know that the existence theory in the class of finite energy weak solutions was developed by Lions [30] and later extended by Feireisl [16] to the so far subcritical exponent $\gamma > \frac{3}{2}$. For an unbounded domain, similar result has been proposed by Novotný and Pokorný in [32].

Here our goal is to add $\delta \nabla_x \varrho^\Gamma$ in the momentum equation with $\Gamma \geq \frac{3}{2}$ so that we use the existing results to obtain a finite energy weak solution for the system and we denote it by $(\varrho_\delta, \mathbf{u}_\delta)$. Then we will show that this approximate solution converges to a dissipative solution of above described system.

This motivates the following approximate problem:

$$\begin{aligned}
&\partial_t \varrho_\delta + \mathrm{div}_x(\varrho_\delta \mathbf{u}_\delta) = 0,\\
&\partial_t(\varrho_\delta \mathbf{u}_\delta) + \mathrm{div}_x(\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) + \nabla_x p(\varrho_\delta) + \delta \nabla_x \varrho_\delta^\Gamma + \mathbf{b} \times \varrho_\delta \mathbf{u}_\delta\\
&\qquad = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_\delta) + \varrho_\delta \nabla_x G,\\
&\mathbf{u}_\delta \cdot \mathbf{n}|_{\partial\Omega} = [\mathbb{S}(\nabla_x \mathbf{u}_\delta) \cdot \mathbf{n}]_{\tan}|_{\partial\Omega} = 0,
\end{aligned}$$

with $\Gamma > \frac{3}{2}$. We assume that for each $\delta > 0$ a *static solution* of the approximate problem $\tilde{\varrho}_\delta$ has the following property

$$\sup_{x_3 \in [0,1]} |\tilde{\varrho}_\delta - \tilde{\varrho}| \leq \delta,$$

where $\tilde{\varrho} > 0$ is a static solution for the Navier–Stokes problem with $\nabla_x P'(\tilde{\varrho}) = \nabla_x G$. Further we assume that for the above-mentioned problem, initial condition

$\{\varrho_{\delta,0}, (\varrho\mathbf{u})_{\delta,0}\}$ belongs to a certain regularity class for which weak solution exists. As an additional assumption, we have

$$E^{\delta}_{0,\delta} = \int_{\Omega} \left(\frac{1}{2} \frac{|(\varrho\mathbf{u})_{0,\delta}|^2}{\varrho_{0,\delta}} + H(\varrho_{0,\delta}) - (\varrho_{0,\delta} - \tilde{\varrho}_{\delta}) H'(\tilde{\varrho}_{\delta}) - H(\tilde{\varrho}_{\delta}) \right) \mathrm{d}x$$
$$\to E_0 = \int_{\Omega} \left(\frac{1}{2} \frac{|(\varrho\mathbf{u})_0|^2}{\varrho_0} + P(\varrho_0) - (\varrho_0 - \tilde{\varrho}) P'(\tilde{\varrho}) - P(\tilde{\varrho}) \right) \mathrm{d}x,$$

where $H(s) = \frac{1}{\gamma - 1} p(s) + \delta \frac{1}{\Gamma - 1} s^{\Gamma}$.

Clearly from existence of weak solution, we have several apriori bounds, i.e.,

$$\|\varrho_{\delta} - \tilde{\varrho}_{\delta}\|_{L^{\infty}(0,T;L^2+L^{\gamma}(\Omega))} \leq C,$$
$$\|\sqrt{\varrho_{\delta}}\mathbf{u}_{\delta}\|_{L^2(\Omega;\mathbb{R}^3)} \leq C,$$
$$\|\mathbf{u}_{\delta}\|_{L^2(0,T;W^{1,2}(\Omega:\mathbb{R}^3)} \leq C,$$
$$\|H(\varrho_{\delta}) - (\varrho_{\delta} - \tilde{\varrho}_{\delta}) H'(\tilde{\varrho}_{\delta}) - H(\tilde{\varrho}_{\delta})\|_{L^{\infty}(0,T;L^1(\Omega))} \leq C.$$

As a consequence of that, we obtain

$$\|\varrho_{\delta}\mathbf{u}_{\delta}\|_{L^{\infty}(0,T;L^2+L^{2\gamma/\gamma+1}(\Omega;\mathbb{R}^3))} \leq C,$$
$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega} \delta \varrho^{\Gamma}_{\delta} \, \mathrm{d}x \leq C.$$

From the above bounds, we get

$$\varrho_{\delta} - \tilde{\varrho}_{\delta} \to \varrho - \tilde{\varrho} \text{ weak-(*)ly in } L^{\infty}(0, T; L^2 + L^{\gamma}(\Omega)),$$
$$\mathbf{u}_{\delta} \to \mathbf{u} \text{ weakly in } L^2(0, T; W^{1,2}(\Omega : \mathbb{R}^3).$$

We use Abbatiello et al. [1, Lemma 8.1 (Appendix)] to conclude

$$\varrho_{\delta}\mathbf{u}_{\delta} \to \varrho\mathbf{u} \text{ weak-(*)ly in } L^{\infty}(0, T; L^2 + L^{2\gamma/\gamma+1}(\Omega)).$$

Let us introduce the conservative variable $\mathbf{m}_{\delta} = \varrho_{\delta}\mathbf{u}_{\delta}$. In terms of momentum, we rewrite kinetic energy as

$$(\varrho_{\delta}, \mathbf{m}_{\delta}) \mapsto \frac{1}{2} \frac{|\mathbf{m}_{\delta}|^2}{\varrho_{\delta}} = \begin{cases} \frac{1}{2} \frac{|(\varrho_{\delta}\mathbf{u}_{\delta})|^2}{\varrho_{\delta}} & \text{if, } \varrho_{\delta} \neq 0, \mathbf{m}_{\delta} \neq 0, \\ 0 & \text{if } \varrho_{\delta} = 0, \ \mathbf{m}_{\delta} = 0, \\ \infty & \text{if } \varrho_{\delta} = 0, \ \mathbf{m}_{\delta} \neq 0. \end{cases}$$

As an observation, we have the above map as convex l.s.c. From energy inequality, it is worth to notice that it is ∞ only on a measure zero set in $(0, T) \times \Omega$. Using convexity of $p(\cdot)$ and $[\varrho, \mathbf{m}] \mapsto \frac{\mathbf{m}\times\mathbf{m}}{\varrho}$, and also using fact that $L^1(\Omega)$ continuously embedded in $\mathcal{M}(\bar{\Omega})$, we conclude

$$\frac{\mathbf{m}_{\delta} \times \mathbf{m}_{\delta}}{\varrho_{\delta}} \to \overline{\frac{\mathbf{m} \times \mathbf{m}}{\varrho}} \text{ weak-(*)ly in } L^{\infty}(0, T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d\times d}_{\mathrm{sym}})),$$
$$p(\varrho_{\delta}) \to \overline{p(\varrho)} \text{ weak-(*)ly in } L^{\infty}(0, T; \mathcal{M}(\overline{\Omega})),$$
$$\delta \varrho^{\Gamma}_{\delta}(t, \cdot) \to \zeta \text{ weak-(*)ly in } L^{\infty}(0, T; \mathcal{M}(\overline{\Omega})).$$

We choose

$$\mathfrak{R}_m = \left[\overline{\frac{\mathbf{m}\times\mathbf{m}}{\varrho}} - \frac{\mathbf{m}\times\mathbf{m}}{\varrho} \right] + \left[\overline{p(\varrho)} - p(\varrho) + \zeta \right] \mathbb{I},$$

and

$$\mathfrak{R}_e = \overline{\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}} - \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + \overline{P(\varrho)} - P(\varrho) + \frac{1}{\Gamma-1}\zeta.$$

The compatibility of two *turbulent defect measures* is clear from above equations. Arguing similarly as in Breit et al. [7], we obtain

$$\mathfrak{R}_m \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}})),\ \ \mathfrak{R}_e \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega})).$$

Now we are in a position to conclude that ϱ, $\mathbf{u}$, $\mathfrak{R}_m$ and $\mathfrak{R}_e$ is a *dissipative solution* for the Navier–Stokes equation. □

Finally, we state the theorem:

Theorem 5.4. *Let pressure p follows* (5.5)*. We assume that the initial data are* well-prepared*. We also consider $\mathbf{v}_0 \in W^{k,2}(\mathbb{R}^2)$ with $k \geq 3$. Let $(\varrho_\epsilon, \mathbf{u}_\epsilon)$ be a dissipative solution as in Definition* 5.1 *in $(0,T)\times\Omega$. Then,*

$$ess \sup_{t\in(0,T)} \|\varrho_\epsilon - \tilde{\varrho}_\epsilon\|_{(L^2+L^\gamma)(\Omega)} \leq \epsilon^m c,$$

$$\sqrt{\varrho_\epsilon}\mathbf{u}_\epsilon \to \mathbf{v} \begin{cases} \text{weak-(*)ly in} L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \\ \text{strongly in } L^1_{loc}((0,T)\times\Omega;\mathbb{R}^3), \end{cases}$$

where $\mathbf{v} = (\mathbf{v}_h, 0)$ and $\mathbf{v}_h$ is the unique solution of Euler system (2.2) *with initial data $\mathbf{v}_0$.*

Proof. The proof is similar as before only we have to consider few extra terms, see [24]. □

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