Integrability and RG flow in 2d sigma models

Thesis for the degree of *Doctor of Philosophy* in Physics

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Abstract

Motivated by the search for solvable string theories, we consider the problem of classifying the integrable bosonic 2d σ-models. We include non-conformal σ-models, which have historically been a good arena for discovering integrable models that were later generalized to Weyl-invariant ones. General σ-models feature a quantum RG flow, given by a 'generalized Ricci flow' of the target-space geometry.

This thesis is based on the conjecture that *integrable* σ-models are *renormalizable*, or *stable* under the RG flow. It is widely understood that classically integrable theories are stable at the leading 1-loop order with only a few parameters running.

Here we address what happens at *higher-loop orders*. We find that integrable σ-models generally remain RG-stable at higher-loops provided they receive a particular choice of finite counterterms, or quantum (α') corrections to the target-space geometry. Thought to be preserving integrability at the quantum level, these corrections are analogous to those required for higher-loop conformal invariance of gauged Wess-Zumino-Witten models.

We explicitly construct the quantum corrections restoring higher-loop renormalizability for examples of integrable η- and λ-deformed σ-models. We then consider the integrable $G \times G$ and $G \times G/H$ models and also construct a new class of integrable $G \times G/H$ models with abelian H.

We also reformulate the λ-models as σ-models on a "tripled" $G \times G \times G$ configuration space, where they become automatically renormalizable due to manifest global symmetries. In the limit when they become non-abelian dual (NAD) models, this suggests that the corresponding 'interpolating models' for NAD are also renormalizable, with 2-loop β-functions matching the group/symmetric space models.

We then present a new and different link between integrability and the RG flow in the context of σ-models with 'local couplings' depending explicitly on 2d time. Such models are naturally obtained in the light-cone gauge in string theory, pointing to the possibility of a large, new class of solvable string models.

Statement of originality

I declare that, aside from properly acknowledged or referenced contributions of others, this thesis is my own work.

This thesis is closely based on the following research papers. These papers were written as collaborations with co-authors who contributed equally to the work.

[1] B. Hoare, N. Levine and A. A. Tseytlin,
"On the massless tree-level S-matrix in 2d sigma models,"
J. Phys. A **52**, no.14, 144005 (2019) [arXiv:1812.02549].

[2] B. Hoare, N. Levine and A. A. Tseytlin,
"Integrable 2d sigma models: quantum corrections to geometry from RG flow,"
Nucl. Phys. B **949**, 114798 (2019) [arXiv:1907.04737].

[3] B. Hoare, N. Levine and A. A. Tseytlin,
"Integrable sigma models and 2-loop RG flow,"
JHEP **1912**, 146 (2019) [arXiv:1910.00397].

[4] B. Hoare, N. Levine and A. A. Tseytlin,
"Sigma models with local couplings: a new integrability – RG flow connection,"
JHEP **11**, 020 (2020) [arXiv:2008.01112].

[5] N. Levine and A. A. Tseytlin,
"Integrability vs. RG flow in $G \times G$ and $G \times G/H$ sigma models,"
JHEP **05**, 076 (2021) [arXiv:2103.10513].

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Chapter 1

Introduction

1.1 Background

The AdS/CFT correspondence is a remarkable set of dualities between string theories in Anti-de Sitter space AdS_{d+1} and conformal field theories CFT_d in one fewer dimension [6, 7, 8]. In the Poincaré coordinates for AdS,

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx^\mu dx_\mu\right) , \qquad \mu = 1, \ldots, d , \tag{1.1}$$

one may picture the CFT as existing at the "conformal boundary" $z = 0$, where the metric formally diverges but is conformally equivalent to the flat d-dimensional one. Gauge-string dualities of this type are often called "holographic" as they conjecture a mapping between the degrees of freedom on the boundary ($z = 0$) and those in the "bulk", or interior of the AdS space. A natural first check is the matching of internal symmetries: indeed the isometry group $SO(2, d-1)$ of AdS_{d+1} coincides with the conformal group of d-dimensional flat space.

Such a correspondence has several immediately interesting and surprising properties. First, the $(d+1)$th dimension (the z direction in (1.1)) seems to emerge from a d-dimensional CFT description. Second, the string theory is gravitational while the CFT is not. Thus it is possible to reformulate a complicated quantum theory of gravity in terms of a simpler and better understood CFT in flat space.

In the context of string theory, the AdS space must be embedded in a string background. A canonical example discussed in Maldacena's original proposal [6] is the type IIB background obtained in the low energy limit from a stack of N coincident D3-branes. The conjecture is that the low energy theory describing open strings ending on the D-branes — $SU(N)$ $\mathcal{N} = 4$ super Yang-Mills theory in 4d — is dual to the theory of closed strings in the near horizon region — type IIB strings on $AdS_5 \times S^5$. The theories' parameters are related by

$$\frac{R^2}{\alpha'} = \sqrt{g^2_{\mathrm{YM}} N} , \qquad g_s = \frac{g^2_{\mathrm{YM}}}{4\pi} , \tag{1.2}$$

where R is the AdS radius in (1.1), $\alpha' = l_s^2$ the squared string length, $g_{\rm YM}$ the Yang-Mills coupling and N the rank of the Yang-Mills gauge group.

Among the many examples of the AdS/CFT correspondence, the $AdS_5 \times S^5$ case is notable because it is *integrable*[1,2] in the free string ($g_s = 0$) limit, or equivalently in the planar limit ($N = \infty$ with $\lambda = g_{\rm YM}^2 N$ fixed) of the gauge theory. Here "integrability" is understood in the same sense of familiar integrable systems like the KdV or sine-Gordon model, meaning aspects of the theory can be solved exactly without recourse to perturbation theory.

On the string side (strong 't Hooft coupling λ), the Green-Schwarz action for strings in the $AdS_5 \times S^5$ background takes the Metsaev-Tseytlin form [15] equivalent to a σ-model on the semi-symmetric space $PSU(2,2|4)/[SO(1,4) \times SO(5)]$ with a particular Wess-Zumino term. It is a general fact that such models are classically integrable [16, 17, 18, 19, 20] in the sense of admitting a flat Lax connection and associated conserved charges (see Section 1.4.1 below). In spite of classic issues preventing a direct integrable quantization due to the non-ultralocality of the classical Poisson structure [21, 19], one may just *assume* the quantum theory to be integrable and bootstrap an exact integrable S-matrix for the fully gauge-fixed theory based on symmetries and consistency conditions (e.g. in an expansion around a BMN-type [22] geodesic, fixing the 'uniform light-cone' gauge; see the review [10]). This S-matrix describes scattering on a 2d plane, but the finite size (closed string) spectrum may be extracted by some Bethe ansatz equations, or more properly accounting for the "wrapping corrections" by double Wick rotating [23] and using the Thermodynamic Bethe Ansatz (TBA) [24]. The TBA (nonlinear integral) equations were reformulated as the simpler Quantum Spectral Curve [25], which is the most modern approach to computing the exact spectrum (see also the review [26]).

On the gauge side (weak coupling), the structure and origin of integrability is more mysterious. It was observed that the mixing matrix for anomalous dimensions of single-trace local operators coincides with the Hamiltonian of an integrable spin chain [27]. This conjures an image of fields inside the trace corresponding to sites of a spin chain, with their R-charges corresponding to the spins. Including more and more loop orders, the corresponding spin chain was found to remain integrable, but with longer-range interactions at each order [28]. Assuming integrability to all orders, the exact spin-chain S-matrix [29] and asymptotic Bethe ansatz equations [30] (for long operators or long spin chains) can be obtained, matching results from the string side. The relation between integrability on the gauge and string sides can be understood qualitatively in terms effective Landau-Lifshitz actions for coherent states, with the string effection action being the direct continuum limit of the spin chain one (see, e.g., [31] and refs. there).

Integrability has been extremely useful in understanding the holographic duality in the case

[1]Here we give a lightning summary of integrability in the AdS/CFT correspondence, focussing on $AdS_5 \times S^5$ (and missing many details and important references). For a comprehensive introduction, see the reviews [9, 10, 11, 12].

[2]Another well-studied, integrable, holographic string background is the type IIA background on $AdS_4 \times \mathbb{C}\mathbf{P}^3$, dual to 3d $\mathcal{N} = 6$ Chern-Simons theory [13]; see, e.g., the review [14].

of $AdS_5 \times S^5$. Not only has it allowed the solution of the spectral problem for a non-trivial 4d gauge theory (with qualitative similarities to QCD), but it has also facilitated non-trivial checks of the AdS/CFT correspondence in this example (see, e.g., [32]). Motivated by a wish to solve more string theories in curved space, and more 4d gauge theories, one could ask in which ways the $AdS_5 \times S^5$ background can be *deformed* while preserving integrability. More generally, we would like to know which string theories are integrable.

The question of classifiying integrable string backgrounds is already very difficult in the simpler case of *bosonic* strings, which will be our focus. Although focussing on bosonic strings takes a step back from the AdS/CFT correspondence that was our motivation, we note the historical fact that the integrable semi-symmetric space σ-model governing the $AdS_5 \times S^5$ background is a simple generalization of bosonic symmetric space σ-models discussed below. The motion of a bosonic string is described by a 2d σ-model coupled to 2d gravity,

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int d^2\xi \, \mathcal{L} \\ \mathcal{L} &= -\big[\sqrt{-h}h^{\mu\nu} \, \mathrm{G}_{mn}(x) + \varepsilon^{\mu\nu} \, \mathrm{B}_{mn}(x)\big] \, \partial_\mu x^m \, \partial_\nu x^n - \alpha' \sqrt{-h} \, R^{(2)} \phi(x) \ , \\ & m, n = 1, \ldots, D \ , \qquad \mu, \nu = 1, 2 \ . \end{aligned} \tag{1.3}$$

Here the scalar fields x^m may be viewed as embedding coordinates for the 2d worldsheet (traced out by a string over time) in a D-dimensional target space-time. The parameter α' plays the role of $\hbar$ (counting loops) in the σ-model. In this thesis we will set $\alpha' = 1$ for convenience, but will sometimes reinstate α' to emphasize the loop counting. The σ-model couplings $(\mathrm{G}_{mn}, \mathrm{B}_{mn})$ are respectively symmetric and antisymmetric and correspond to the background values of the target space metric and NSNS 2-form. The auxiliary 2d metric $h_{\mu\nu}$ is arbitrary and $\varepsilon_{\mu\nu}$ is the antisymmetric 2d Levi-Civita tensor.

The model (1.3) is 2d diffeomorphism-invariant and classically 2d Weyl-invariant. The role of the dilaton background $\phi(x)$, appearing coupled to the 2d curvature $R^{(2)}$ at subleading order in α', is to compensate quantum breaking of Weyl invariance from the (G, B) terms [33]. Preservation of the local Weyl invariance, which is necessary for the consistency of the quantum theory, imposes that the background fields satisfy certain 'string equations of motion' [34] (generalizing the Einstein equation),

$$\begin{aligned} & R_{mn} - \tfrac{1}{4}(\mathrm{H}^2)_{mn} + 2\nabla_m \nabla_n \phi = \mathcal{O}(\alpha') \ , \qquad (\mathrm{H}_{mnp} \equiv 3\partial_{[m}\mathrm{B}_{np]}) \\ & -\tfrac{1}{2}\nabla^p \mathrm{H}_{mnp} + \mathrm{H}_{mn}{}^p \, \nabla_p \phi = \mathcal{O}(\alpha') \ , \\ & \tfrac{1}{6}(D - 26) - \tfrac{1}{2}\nabla^2\phi + (\nabla\phi)^2 - \tfrac{1}{24}\mathrm{H}^2 = \mathcal{O}(\alpha') \ , \end{aligned} \tag{1.4}$$

which contain corrections at each order in α' (corresponding to the loop expansion of the σ-model). These equations are obtained by varying an effective gravitational action for the

massless fields,

$$S = \frac{1}{2\kappa^2}\int d^D x\, e^{-2\phi}\big(R - \tfrac{1}{12}\mathrm{H}^2 + 4(\partial\phi)^2\big) + \mathcal{O}(\alpha') \ . \tag{1.5}$$

and it has been checked to several orders in α' that this action coincides with the one obtained from the tree-level string S-matrix [35, 36, 37, 38] (see also the review [39]).

For the purposes of classical integrability, it will be sufficient to fix the 'conformal' gauge $\sqrt{-h}h^{\mu\nu} = \eta^{\mu\nu}$, obtaining a σ-model on flat 2d space,[3]

$$S = \frac{1}{4\pi\alpha'}\int d^2\xi\, \mathcal{L} = \frac{1}{4\pi\alpha'}\int d^2\xi\, \big[\mathrm{G}_{mn}(x) + \mathrm{B}_{mn}(x)\big]\, \partial_+ x^m \partial_- x^n \ , \tag{1.6}$$

where we have ignored the quantum dilaton term. At the level of classical integrability, the flat and curved 2d-space σ-models are equivalent since they are related by a redefinition $\xi^\mu \to \xi'^\mu(\xi)$ (as the σ-model is classically 2d Weyl invariant and all 2d metrics are locally equivalent up to a conformal factor). The Virasoro constraints ($T_{++} = 0$, $T_{--} = 0$) coming from the variation of the 2d metric $h_{\mu\nu}$ in (1.3) should not destroy classical integrability since these are just constraints on the initial data.

Our eventual goal is to understand which string backgrounds (G, B) (solutions of (1.4) with some choice of ϕ) correspond to integrable σ-models (1.6). At the classical level, one criterion for integrability is the 2d equations of motion taking the *zero-curvature* form [40, 41, 42, 43]

$$[\partial_+ + L_+(z), \partial_- + L_-(z)] = 0 \ , \tag{1.7}$$

where the "Lax connection" $L_\pm = L_\pm(z; x, \partial x)$ is a 1-complex-parameter family of connections constructed from the fields. For any given σ-model couplings (G, B), it is very difficult to know whether a zero-curvature representation exists, and moreover there is no general recipe for constructing the Lax connection. This tricky classification problem of integrable σ-models is the motivation for the thesis.

1.2 Stability under the RG flow

In this thesis we will relax the Weyl invariance condtion (1.4) and consider general σ-models on a flat 2d space, again asking the question of which choices of couplings (G, B) correspond to integrable theories. Although non-Weyl-invariant σ-models do not directly correspond to string backgrounds, their study is well-motivated by the historical observation that non-conformal integrable σ-models such as the PCM and the symmetric space σ-models have preceded the discovery of related conformal ones — the WZW model and the semi-symmetric space σ-model with WZ term. In other words, one may discover new examples of Weyl-invariant integrable

[3] The 2d light-cone derivatives are defined as $\partial_\pm = \partial_\tau \pm \partial_\sigma$, see Appendix A for the conventions.

theories by first discovering non-Weyl-invariant ones.

Moreover, relaxing Weyl invariance is actually helpful since it opens up a new 'diagnostic tool' for integrability: the *renormalization group (RG) flow*. The idea is that, in order to be able to identify the integrable theories, one should thoroughly investigate their properties and distinctive features. We shall focus on one conjectured property in particular: that **integrable σ-models are *stable* under the quantum RG flow** [44, 45, 46, 47, 48, 49, 50, 51, 52, 53]. Let us now explain the meaning of this statement.

General σ-models feature a quantum RG flow (breaking the scale invariance of the classical theory). The σ-model can be quantized preserving the target space covariance [54, 55] and, in the simplest case with $\mathrm{B}_{mn} = 0$, the RG flow is given by a 'generalized Ricci flow',[4]

$$\begin{aligned} &\frac{d}{dt} G_{mn} = \beta_{mn} + \nabla_{(m}\xi_{n)} \ , \qquad\qquad (B_{mn} = 0) \\ &\beta_{mn} = \alpha' \, R_{mn} + \alpha'^2 \, \tfrac{1}{2} R_{mpqr} R_n{}^{pqr} + \dots \ . \end{aligned} \tag{1.8}$$

We have included the 1-loop and 2-loop terms, and there will be further contributions at each loop order. Here $\xi^n(x)$ corresponds to a choice of renormalizations of the fields x^n, i.e. target space diffeomorphisms depending on the 2d RG scale. Now including the B-field, the β-function (1.8) generalizes, in a particular 2-loop subtraction scheme[5] to [35] (see also [56, 57, 58])

$$\begin{aligned} &\frac{d}{dt}(\mathrm{G}_{mn} + \mathrm{B}_{mn}) = \beta_{mn} + L_\xi(\mathrm{G} + \mathrm{B})_{mn} + \partial_{[m}\lambda_{n]} \ , \\ &\beta_{mn} = \alpha' \, \widehat{R}_{mn} + \alpha'^2 \, \tfrac{1}{2}\Big[\widehat{R}^{klp}{}_n \widehat{R}_{mklp} - \tfrac{1}{2} \widehat{R}^{lpk}{}_m \widehat{R}_{nklp} + \tfrac{1}{2} \widehat{R}_{kmnl} \mathrm{H}^{kpq} \mathrm{H}^l{}_{pq} \Big] + \dots \ , \end{aligned} \tag{1.9}$$

where $\mathrm{H}_{kmn} = 3\partial_{[k}\mathrm{B}_{mn]}$ and $\widehat{R}$ is the curvature of the generalized connection $\widehat{\Gamma}^k{}_{mn} = \Gamma(\mathrm{G})^k{}_{mn} - \frac{1}{2}\mathrm{H}^k{}_{mn}$. The use of $\widehat{R}$ is simply a convenient notation and one can re-express (1.9) in terms of just R, H and covariant derivatives: for example, $\widehat{R}_{mn} = R_{mn} - \frac{1}{4}\mathrm{H}_{mpq}\mathrm{H}_n{}^{pq} - \frac{1}{2} D^k \mathrm{H}_{kmn}$. Here L_ξ denotes the Lie derivative along a vector ξ, generalizing the diffeomorphism term of (1.8) above. The extra term $\partial_{[m}\lambda_{n]}$ denotes a shift of the B-field by an exact 2-form depending on the 2d RG scale; this is the usual 2-form gauge freedom, and only modifies the σ-model Lagrangian (1.6) by a total derivative.

The string equations (1.4) correspond to the Weyl-invariant case when the RG flow (1.9) trivializes, i.e. $\frac{d}{dt}(\mathrm{G}_{mn} + \mathrm{B}_{mn}) = 0$. The diffeomorphism vector and B-gauge transformation must also take a specific form [59] $\xi_n = 2\alpha'\partial_n\phi + \mathcal{O}(\alpha'^2)$, $\lambda_n = \mathcal{O}(\alpha'^2)$ in terms of the dilaton, metric and B-field. The dilaton equation in (1.4) then follows from the metric and B-field equations up to a constant term, whose correct value ensures the vanishing of the Weyl anomaly coefficient.

More precisely, we may formulate the conjecture of RG stability as follows. For a family of

[4] We denote by $t = \log\mu$ the 2d RG scale.

[5] See Section 1.4.15 for discussion of scheme dependence.

integrable models $\mathrm{G}_{mn} = \mathrm{G}_{mn}(g_\alpha)$, $\mathrm{B}_{mn} = \mathrm{B}_{mn}(g_\alpha)$ parametrized by a few couplings g_α, the claim is that the RG flow (1.9) of the metric and B-field consistently truncates to a flow of just the couplings g_α,

$$\frac{d}{dt} g_\alpha = \beta_\alpha(g) \ . \tag{1.10}$$

In other words, integrable σ-models are *renormalizable*, i.e. they are explicit solutions of the generalized Ricci flow equation.

Let us give an intuitive explanation for this conjecture. As we shall discuss below, classically integrable σ-models feature an infinite set of commuting conserved charges, as part of an intricate Poisson structure governed by a so-called Lax operator. In general one may expect this structure to be sufficiently constraining to uniquely determine the theory in question, so that the only deformations preserving it are the coupling constants g_α of the theory. Now assuming the integrable structure to exist also in the quantum theory, then it should be respected by the quantum RG flow, and hence only the couplings g_α should run under RG. Another explanation is that some integrable σ-models are understood to have exact S-matrices in the infrared, typically with a small number of parameters (see Section 1.4.12 below). One may then expect that these parameters should somehow match up with the semiclassical σ-model description, and that this matching should be preserved along the RG flow.

While there are no known counterexamples to the conjecture that integrable σ-models are renormalizable, in most examples this has only been checked to the leading 1-loop order in (1.9), and there is no general proof. In this work, we shall ask the question of what happens at *higher loop orders.* We will find that the target space geometry of the classical σ-model $\mathcal{L} = (\mathrm{G} + \mathrm{B})^{(0)}_{mn}\, \partial_+ x^m \partial_- x^n$ must generally be deformed by quantum corrections to remain a solution of the RG equation (1.9) beyond 1-loop,

$$(\mathrm{G} + \mathrm{B})^{(0)}_{mn} \to (\mathrm{G} + \mathrm{B})^{(0)}_{mn} + \alpha' (\mathrm{G} + \mathrm{B})^{(1)}_{mn} + \dots \ . \tag{1.11}$$

Equivalently, one must add to the σ-model a particular choice of finite counterterms. This is a scheme-independent statement because the required counterterms will be non-trivial in *arbitrary* covariant subtraction schemes.

Hence, while the classical Lax integrability corresponds just to the classical (uncorrected) geometry $(\mathrm{G} + \mathrm{B})^{(0)}_{mn}$, it is the quantum-corrected geometry that seems to posses the RG properties associated with integrability at the quantum level. These counterterms might then be understood as preserving 'Ward identites' for hidden integrable symmetries at the quantum level, with the expectation that these hidden symmetries are constraining the RG flow. In this sense, one may hope to interpret the corrected σ-model as being the correct choice of bare action consistent with quantum integrability.

1.3 Structure of the thesis

This thesis is structured as follows. The rest of this introductory chapter is a review of integrable 2d σ-models with a focus on classical Lax integrability, examples of integrable models and their quantum RG flows.

In Chapter 2 (based on the papers [2, 3]) we shall consider the higher-loop RG flow for some of the simplest non-trivial 'integrable deformed' models where renormalizability does not immediately follow from manifest global symmetries. For the η-deformation of S^2 ("sausage model") or H^2, we explicitly construct the counterterms required for RG stability up to 3-loop order and conjecture the all-order form of the counterterms. We also study the closely related λ-deformation based on the $\frac{SU(2)}{U(1)}$ coset and the $SU(2)$ group, finding the counterterms needed for 2-loop RG stability. We point out that these counterterms continue to serve the same function of restoring renormalizability in the non-abelian dual limit of the λ-model. This gives a prescription in these examples for the leading α'-corrections to the non-abelian duality transformation so that the non-abelian duality commutes with the RG flow, resolving prior confusion in the literature.

In Chapter 3 (based on [5]) we consider the same question for a different class of integrable σ-models on products of group spaces. We first study the integrable $G \times G$ model derived from the affine Gaudin construction (for which the 1-loop β-functions were found in [60]) and show that its condition of integrability is automatically preserved also by the 2-loop RG flow without the need for any quantum corrections. We then investigate the RG flow in the gauged $G \times G/H$ model, including the integrable $T^{1,1}$ model found in [61]. In that case we instead find that the integrability condition relating the couplings requires a certain quantum deformation to preserve renormalizability at 2-loop order. We also construct a new class of integrable $G \times G/H$ models in the case when the subgroup H is abelian. In the simplest case of $G = SU(2),\ H = U(1)$ this leads to an integrable σ-model on the $T^{1,q}$ space (with a particular B-field). This model is found to be a special case that is stable under the 2-loop RG flow without deformation — and we relate this property to its invariance under T-duality in an isometric $U(1)$ direction. This $T^{1,q}$ model may be interpreted as an integrable deformation of the GMM model (of two coupled WZW theories with generic levels) away from the conformal point.

In Chapter 4 (based on [3]) we derive an alternate formulation of the λ-model as a σ-model on a "tripled" configuration space $G \times G \times G$ by re-writing the 2d gauge field in terms of group-valued scalar fields. Our central observation is that the λ-model quantized in this formulation is automatically renormalizable to all orders without any correction, with only the deformation parameter λ running. This is in contrast to the standard σ-model found by integrating out $A_\pm$, which is 2-loop renormalizable only after adding the specific finite local counterterms discussed in Chapter 2. We compute the 2-loop β-function of the λ-model for general groups and symmetric spaces. In the non-abelian dual limit this yields an alternate formulation of

the interpolating models for non-abelian duality that is automatically renormalizable with the same 2-loop β-function as the original PCM/symmetric space σ-model.

Section 4.5 contains unpublished work on the construction of similar "tripled" or "enlarged" formulations of (i) a generalization of the λ-model to product spaces G^N and (ii) $G \times G/H$ models.

In Chapter 5 (based on [4]) we study a new and different link between integrability and the RG flow. We consider several classes of integrable σ-models (on groups and symmetric spaces, η-models, λ-models) with *local couplings* that may depend on the 2d coordinates, e.g. on time τ. We observe that (i) starting with a classically integrable 2d σ-model, (ii) formally promoting its couplings h_α to functions $h_\alpha(\tau)$ of 2d time, and (iii) demanding that the resulting time-dependent model also admits a Lax connection implies that $h_\alpha(\tau)$ must solve the 1-loop RG equations of the original theory with τ interpreted as RG time. This provides a novel example of an 'integrability–RG flow' connection. The existence of a Lax connection suggests that these time-dependent σ-models may themselves be understood as integrable. We investigate this question by studying the possibility of constructing non-local conserved charges (see also Section 5 of [4] for attempts to construct local charges). Such σ-models with D-dimensional target space and time-dependent couplings subject to the RG flow naturally appear in string theory upon fixing the light-cone gauge in a $(D+2)$-dimensional conformal σ-model with a metric admitting a covariantly constant null Killing vector and a dilaton linear in the null coordinate. This suggests the possibility of a large, new class of solvable string models.

In Chapter 6 we summarize the findings of this work and comment on the open questions, possible applications and future prospects.

Appendix A is an explanation of the conventions and notation used in this thesis.

In Appendix B (based on [2]) we discuss the effect of scheme change on the σ-model β-functions, focussing on the particular case with a 2d target space (relevant for the $\frac{SU(2)}{U(1)}$ η- and λ-models considered in Chapter 2).

In Appendix C (based on [5]) we consider deriving the classical integrability condition for the couplings of G^N models (discussed in Chapter 3) from first principles. We compare our results with the sufficient conditions for integrability obtained from affine Gaudin models. For general N we find that the dimension of the integrable space of G^N couplings coincides with that of the space of affine Gaudin models, and for the $G \times G$ case we find that the affine Gaudin models are the only integrable ones.

In Appendix D (based on [5]) we provide the explicit formulae for the 2-loop β-functions of the $G \times G$ and $G \times G/H$ models (used in Chapter 3) and explain how they were derived.

The remaining appendices, based on [4], pertain to the 'local coupling' theories discussed in Chapter 5. Appendix E contains details of the derivation of the RG flows from the condition

of existence of Lax representations for the local coupling theories.

Appendix F is about the construction of non-local charges for time-dependent σ-models other than the PCM (which was discussed in Section 5.3) — and in particular the time-dependent λ-model based on a symmetric space.

In Appendix G we consider 1d harmonic oscillators with time-dependent frequency, a special case of which was obtained in eq. (5.50) as a linearization of a 1d reduction of the time-dependent PCM.

In Appendix H we explain that the sine-Gordon model also obeys the same pattern as the σ-models discussed above (relating its integrability with local couplings to its RG flow).

1.4 Review of key concepts

In the remainder of this introductory chapter, we shall review some concepts priming the reader for the rest of the thesis. Let us highlight Section 1.4.13, which is a summary of our paper [1] attempting to use the tree-level σ-model S-matrix as another 'diagnostic tool' for integrability.

1.4.1 Classical integrability

Most models in physics cannot be solved exactly. One example is the Standard Model of particle physics, where there is no hope to make exact predictions, but rather computations are made by expanding perturbatively around a free theory. A simpler example is a many-body system in Newtonian gravity, where only the 2-body case is solvable and the 3-body and higher cases are chaotic. One could ask: *what goes right* for the 2-body problem? One explanation is that the 2-body problem is integrable[6] while the others are not. For mechanical 1d systems, integrability means there exist at least as many independent, Poisson-commuting conserved charges as the number of degrees of freedom. In that case it follows that the theory can be completely solved by quadratures.

For 2d models, and in particular the σ-models that will be our focus, there are infinitely many degrees of freedom. The analogous notion of integrability in 2d is then related to the presence of infinitely many commuting conserved charges. A theory of exactly integrable PDEs in mathematical physics was developed starting from the 1960s [40, 41, 42, 43]. The flat Lax connection is a central object, whose zero-curvature representation (1.7) expresses the relevant non-linear equation as the consistency condition for an over-determined *linear* system

$$\partial_{\pm}\psi = \psi\, L_{\pm} \ , \tag{1.12}$$

leading to the construction of explicit solutions by the "inverse-scattering" method.

If a 2d field theory admits a Lax representation (1.7), then one can immediately build non-local conserved charges using the monodromy of the Lax connection along a constant-time

[6] See also the book [62] for an introduction to classical integrability.

contour from $\sigma = a$ to $\sigma = b$,

$$\mathcal{M} = \mathcal{P} \exp \int_a^b L_\sigma \, d\sigma \ . \tag{1.13}$$

The path-ordered exponential in (1.13) is defined by the properties

$$\mathcal{M}^{-1} \partial_b \mathcal{M} = L_\sigma \big|_{\sigma=b} \ , \qquad \mathcal{M}\big|_{a=b} = 1 \ . \tag{1.14}$$

On the support of the equation of motion (1.7) of zero-curvature form, it satisfies

$$\partial_\tau \mathcal{M} = \mathcal{M} \, L_\tau(b) - L_\tau(a) \, \mathcal{M} \ . \tag{1.15}$$

Assuming a periodicity condition $L_\tau(a) = L_\tau(b)$ (e.g., following from periodic boundary conditions on the fields), then (1.15) becomes $\partial_\tau \mathcal{M} = [\mathcal{M}, L_\tau(a)]$ and hence

$$\partial_\tau \mathrm{Tr}[\mathcal{M}^n] = 0 \ , \qquad n \in \mathbb{N} \ , \tag{1.16}$$

i.e. the eigenvalues of $\mathcal{M}$ are conserved.

If additionally the boundary value is vanishing, i.e. $L_\tau(a) = L_\tau(b) = 0$ (for example with decaying boundary conditions on the plane $(a, b) = (-\infty, \infty)$), then (1.15) gives the stronger result that all the entries of $\mathcal{M}$ are conserved,

$$\partial_\tau \mathcal{M} = 0 \ . \tag{1.17}$$

In either case, the number of conserved charges is actually infinite since the entries of $\mathcal{M}$ are functions of the 'spectral parameter' z (assuming the dependence is non-trivial) so each coefficient is individually conserved in any series expansion in z. These charges are 'non-local' since the monodromy matrix is a non-local object (it is not an integral of a local density).

We note that a Lax connection valued in the algebra of G is determined up to gauge transformations $L_\pm \to k^{-1} L_\pm k + k^{-1} \partial_\pm k$, $k(\xi) \in G$, since this transformation preserves the set of equations encoded in the zero-curvature condition (1.7). The monodromy matrix (1.13) rotates covariantly under such transformations so, assuming periodicity of the gauge parameter k, the set of non-local charges is preserved.

1.4.2 Principal Chiral Model

Let us consider the simple example of the Principal Chiral Model (PCM),[7]

$$\mathcal{L} = -\tfrac{1}{2} \mathrm{h} \, \mathrm{Tr}[J_+ J_-] \ , \qquad J_\mu = g^{-1} \partial_\mu g \ , \quad g \in G \ . \tag{1.18}$$

[7] We denote the 2d indices by μ, ν and define the 2d light-cone coordinates as $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$ with derivatives $\partial_\pm = \partial_\tau \pm \partial_\sigma$. The D-dimensional target space indices are $m, n = 1, \ldots, D$. See Appendix A for conventions.

This is a special case of the general σ-model (1.6) with the target space taken to be a simple Lie group G with the bi-invariant metric (and vanishing B-field). The model has a large global $G \times G$ symmetry corresponding to left- and right-multiplications of the group,

$$g \to g_L \; g \; g_R \; , \qquad\qquad (g_L, g_R) \in G_L \times G_R \; . \tag{1.19}$$

For this model, the global symmetry (1.19) (or isometries of the target space) is sufficient to uniquely fix the form of the Lagrangian (1.18) up to the overall constant h. Hence the σ-model RG equations (1.8) are solved with just h running according to the 2-loop β-function [63, 64, 55],[8]

$$\frac{d}{dt}\mathrm{h} = c_{_G} + \tfrac{1}{2} c_{_G} \mathrm{h}^{-1} + \dots \; . \tag{1.20}$$

The constant $c_{_G}$ denotes the dual Coxeter number of the group G (see Appendix A for conventions).

The key to noticing the model's classical integrability is that the current J_μ is conserved on-shell, $\partial^\mu J_\mu = 0$, and also flat, $dJ + J \wedge J = 0$ (since $J_\mu = g^{-1}\partial_\mu g$). This information can be encoded in the flatness (1.7) of the Lax connection [43],

$$L_\pm \equiv L_\tau \pm L_\sigma = \tfrac{1}{2}(1 + z^{\pm 1})\, J_\pm \; . \tag{1.21}$$

An easy way to evaluate the monodromy matrix and extract conserved charges is by expanding in z around the value $z = -1$ where L_σ vanishes (at which point $\mathcal{M} = I$ is trivial). The expansion of the path-ordered exponential yields first the conserved Noether charge built from J_μ itself (corresponding to the G_R symmetry) and then iterated integrals of J of increasing order,

$$\begin{aligned} \mathcal{M} = I &+ \tfrac{1}{2}(z+1) \int_a^b d\sigma \, J_\tau \\ &+ \tfrac{1}{4}(z+1)^2 \Big[\int_{a<\sigma_1<\sigma_2<b} d\sigma_1 d\sigma_2 \, J_\tau(\sigma_1) J_\tau(\sigma_2) + \int_a^b d\sigma \, (J_\tau - J_\sigma) \Big] + \mathcal{O}\big((z+1)^3\big) \; . \end{aligned} \tag{1.22}$$

Each term in the expansion (or just their eigenvalues, depending on the boundary conditions) is individually conserved by the argument above, and their Poisson brackets satisfy a Yangian algebra [65].

As well as these non-local charges, integrable field theories may posses higher-spin local charges. For the example of the PCM (1.18), one can build holomorphic and anti-holomorphic

[8]In this PCM case, since the target space is isotropic, there cannot be any diffeomorphism vector ξ in (1.8) consistent with the symmetries.

local currents by contracting products of $J_\pm$ with a symmetric invariant tensor [66],

$$\partial_\pm \mathcal{J}^{(n)}_\mp = 0 \ , \qquad \mathcal{J}^{(n)}_\pm = d_{A_1\cdots A_n} J^{A_1}_\pm \cdots J^{A_n}_\pm \tag{1.23}$$

$$d_{A_1\ldots A_n} = d_{(A_1\ldots A_n)} \ , \qquad f^A{}_{B(C} d_{A_1\cdots A_{n-1})A_n} = 0 \ . \tag{1.24}$$

The nth currents $\mathcal{J}^{(n)}_\mp$ are of spin n and their (anti-)holomorphic conservation implies the conservation of the associated conserved charges,

$$\partial_\tau Q^{(n)}_\pm = 0 \ , \qquad Q^{(n)}_\pm = \int d\sigma \, \mathcal{J}^{(n)}_\pm \ , \tag{1.25}$$

which Poisson-commute with the non-local charges above [66].

A canonical choice for the invariant tensor $d_{A_1\cdots A_n}$ is given by the trace, $d_{A_1\cdots A_n} = \mathrm{Tr}[T_{(A_1}\cdots T_{A_n)}]$. The conservation of the currents (1.23) follows from the equation of motion and flatness of J_μ, written in the 2d light-cone coordinates as

$$\partial_+ J_- + \partial_- J_+ = 0 \ , \qquad \partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0 \ , \tag{1.26}$$

since these may be re-written as

$$\partial_+ J_- = -\tfrac{1}{2}[J_+, J_-] \ , \qquad \partial_- J_+ = \tfrac{1}{2}[J_+, J_-] \ . \tag{1.27}$$

1.4.3 Symmetric space sigma model

Now let us consider instead a σ-model on a homogeneous coset space G/H, obtained by gauging a subgroup H of the G_R symmetry of the PCM (here $H \subset G$ are both assumed to be simple Lie groups). The gauging procedure may be understood as (i) promoting the H global symmetry to a local one, $g \to g\, h(\xi)^{-1}$, (ii) writing down the transformation of the PCM Lagrangian (1.18), $\mathcal{L} \to -\frac{1}{2}\mathrm{h}\,\mathrm{Tr}[(J_+ - h^{-1}\partial_+ h)(J_- - h^{-1}\partial_- h)]$, and (iii) promoting the gauge parameter to a gauge connection, $h^{-1}\partial_\pm h \to A_\pm$ (no longer fixed to be pure gauge). The resulting G/H theory,

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{h}\,\mathrm{Tr}[(J_+ - A_+)(J_- - A_-)] \ , \qquad J_\pm = g^{-1}\partial_\pm g \ , \quad g \in G \ , \quad A_\pm \in \mathrm{Lie}(H) \ , \tag{1.28}$$

is invariant under the H gauge transformation

$$g \to gh \ , \quad A_\pm \to h^{-1} A_\pm h + h^{-1}\partial_\pm h \ , \qquad h(\xi) \in H \ . \tag{1.29}$$

The quadratic gauge field $A_\pm$ in (1.28) is auxilliary (appearing without derivatives) and may be integrated out to obtain

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{h}\,\mathrm{Tr}[J_+ \, P_{G/H} \, J_-] \ , \tag{1.30}$$

where $P_{G/H}$ is the projector orthogonal to Lie(H) in Lie(G) with respect to the Killing form. The H gauge transformation (1.29) then reduces to

$$g \to g\, h\ , \qquad h(\xi) \in H\ . \tag{1.31}$$

The theory also has a global G symmetry, $g \to g_L\, g$, $g_L \in G_L$ corresponding to the isometry of the coset space G/H.

If G/H is assumed to be a *symmetric space* (meaning Lie(H) is the even subalgebra under a $\mathbb{Z}_2$-grading of Lie(G)), then these global and local symmetries uniquely determine the theory (1.28) or (1.30) up to the constant h. Like the PCM above, this automatically makes the theory renormalizable to all orders with just h running. The 2-loop β-function following from (1.8) is (see App. B of [3] and particular cases in [67, 64])

$$\frac{d}{dt}\mathrm{h} = 2c_{_G} + 4c_{_G}(c_{_G} - c_{_H})\mathrm{h}^{-1}\ . \tag{1.32}$$

In contrast, for non-symmetric cosets G/H, the theory (1.30) may not be renormalizable as it should be understood as part of a multi-coupling theory $\mathcal{L} = -\frac{1}{2}\mathrm{Tr}[J_+ \left(\sum_{i=1}^n \mathrm{h}_i P_{\mathfrak{p}_i}\right) J_-]$ respecting the same symmetries, defined in terms a decomposition of the Lie algebra Lie(G) = Lie(H) + $\sum_{i=1}^n \mathfrak{p}_i$, with $\mathfrak{p}_i$ being irreducible representations under the adjoint action of H, mutually orthogonal under the trace. For general cosets we may have $n > 1$ (i.e. multi-coupling), while symmetric spaces must have $n = 1$ (single-coupling) if G,H are simple.

The symmetric space condition on G/H is also sufficient for classical integrability of the model (1.30). The corresponding equations of motion are then of the zero-curvature form (1.7) with the Lax connection [68]

$$L_\pm = B_\pm + z^{\pm 1}\, P_\pm\ , \qquad B \equiv P_H J,\quad P \equiv P_{G/H} J \tag{1.33}$$

Let us note that the symmetric space is actually a generalization of the PCM, since the PCM on a group G is equivalent to the σ-model on the symmetric space $G \times G/G$ (and is obtained upon fixing the gauge $g^{(2)} = 1$, where $(g^{(1)}, g^{(2)}) \in G \times G$, and replacing h $\to$ 2h). All of the formulas here for symmetric spaces actually apply in that case, except that $G \times G$ is not simple so the definition of $c_{_H}$ in Appendix A should be suitably modified. Indeed the symmetric space β-function (1.32) agrees with the PCM one (1.20) upon replacing $c_{_H} \to \frac{1}{2}c_{_G}$ and h $\to$ 2h. The symmetric space Lax connection (1.39) in that case is related to the PCM one (1.21) by a gauge transformation.

Both non-local [69] and local [70] charges may be constructed similarly to the PCM case

above. For the local charges, one now replaces $J \to P$ in (1.23),

$$\partial_\pm \mathcal{J}^{(n)}_\mp = 0 \ , \qquad \mathcal{J}^{(n)}_\pm = d_{A_1\cdots A_n} P^{A_1}_\pm \cdots P^{A_n}_\pm \tag{1.34}$$

$$d_{A_1\ldots A_n} = d_{(A_1\ldots A_n)} \ , \qquad f^A{}_{B(C} d_{A_1\cdots A_{n-1})A_n} = 0 \ . \tag{1.35}$$

The holomorphic conservation law in (1.34) follows again from the equation of motion and flatness of J [70] (here $D^B_\pm \equiv \partial_\pm + \mathrm{ad}_{B_\pm}$),

$$D^B_+ P_- + D^B_- P_+ = 0 \ , \qquad \partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0 \ , \tag{1.36}$$

since they now imply that

$$\partial_\pm \ P_\mp = [P_\mp, B_\pm] \ . \tag{1.37}$$

1.4.4 Constructing Lax connections for other models

The integrable models that follow will admit the *same* flat Lax connections of the form (1.21) or (1.33) as the PCM or symmetric space (for models respectively built on groups or symmetric spaces).[9] The difference is that, in each case, $J_\pm$ will be replaced with some other model-dependent quantity in terms of the fields,

$$G \ : \qquad L_\pm = \tfrac{1}{2}(1 + z^{\pm 1})\mathcal{A}_\pm \ , \qquad \mathcal{A}_\pm \in \mathrm{Lie}(G) \tag{1.38}$$

$$G/H \ : \qquad L_\pm = \mathcal{B}_\pm + z^{\pm 1}\mathcal{P}_\pm \ , \qquad \mathcal{B}_\pm \in \mathrm{Lie}(H) \ , \ \ \mathcal{P}_\pm \in \mathrm{Lie}(G)/\mathrm{Lie}(H) \tag{1.39}$$

We note that admitting Lax connections of the same form does not imply classical equivalence between these different theories (not being related by a canonical transformation); e.g. for group-based models, it simply means they possess a flat, conserved current $\mathcal{A}_\pm$.

Non-local conserved charges are then obtained from the monodromy matrix as discussed above for the PCM. Local conserved charges are also constructed similarly to the cases discussed above, with the general form of the (anti-)holomorphic currents given by

$$\partial_\pm \mathcal{J}^{(n)}_\mp = 0 \ , \qquad \mathcal{J}^{(n)}_\pm = \begin{cases} d_{a_1\cdots a_n} \mathcal{A}^{a_1}_\pm \cdots \mathcal{A}^{a_n}_\pm & (\text{group } G) \\ d_{a_1\cdots a_n} \mathcal{P}^{a_1}_\pm \cdots \mathcal{P}^{a_n}_\pm & (\text{symmetric space } G/H) \end{cases} \tag{1.40}$$

$$d_{a_1\ldots a_n} = d_{(a_1\ldots a_n)} \ , \qquad f^a{}_{b(c} d_{a_1\cdots a_{n-1})a} = 0 \ . \tag{1.41}$$

[9] Of course, not all integrable σ-models will have Lax connections of this form. For example the bi-Yang-Baxter model [71, 72] built on a group does not possess a flat, conserved current $\mathcal{A}_\pm$ so its Lax connection is not of the form (1.38).

The conservation of the currents (1.40) follows from the equations of motion

$$G: \quad \partial_+ \mathcal{A}_- + \partial_- \mathcal{A}_+ = 0 \ , \qquad F_{+-}(\mathcal{A}) = 0 \ , \tag{1.42}$$

$$G/H: \quad D_+^{\mathcal{B}} \mathcal{P}_- + D_-^{\mathcal{B}} \mathcal{P}_+ = 0 \ , \qquad F_{+-}(\mathcal{B}+\mathcal{P}) = 0 \ , \tag{1.43}$$

since these may be re-written as

$$G: \quad \partial_\pm \mathcal{A}_\mp = \tfrac{1}{2}[A_\mp, A_\pm] \ , \tag{1.44}$$

$$G/H: \quad \partial_\pm \mathcal{P}_\mp = [P_\mp, B_\pm] \ , \qquad F_{+-}(\mathcal{B}) + [\mathcal{P}_+, \mathcal{P}_-] = 0 \ . \tag{1.45}$$

1.4.5 T-duality and non-abelian duality

For σ-models with global symmetries, or isometries of the target space, one may "dualize" the symmetry to obtain a dual σ-model, with certain observables being related between the original and dual theories (see the review [73] and refs. therein). As we shall review below, the duality procedure may be carried out at the level of the path integral and applies to the quantum theory [74]; in particular duality is understood to commute with the quantum RG flow, and to map Weyl-invariant theories (string solutions) to Weyl-invariant theories. However, we stress that the procedures as discussed below are subject to quantum α' corrections, with the integrals producing quantum determinants.

First we shall consider *abelian T-duality* for a model with a $U(1)$ isometry, which we take for simplicity to be

$$\mathcal{L} = g_{ij}(x)\, \partial x^i \partial x^j + f(x)\, (\partial y)^2 \tag{1.46}$$

Here the isometry corresponds to constant shifts of y, and the x coordinates are "spectators". Gauging the isometry $\partial_\mu y \to \partial_\mu y - p_\mu$ and imposing the condition $\epsilon^{\mu\nu}\, \partial_\mu p_\nu = 0$ using a Lagrange multiplier field $\tilde{y}$ gives a gauge-invariant theory (under $y \to y + u$, $p_\mu \to p_\mu + \partial_\mu u$),

$$\mathcal{L}_{\rm int} = g_{ij}(x)\, \partial x^i \partial x^j + f(x)\, (\partial y - p)^2 + \tilde{y}\, \epsilon^{\mu\nu}\, \partial_\mu p_\nu \ . \tag{1.47}$$

Fixing the gauge $y = 0$ gives the "interpolating model",

$$\mathcal{L}_{\rm int} = g_{ij}(x)\, \partial x^i \partial x^j + f(x)\, p^2 + \tilde{y}\, \epsilon^{\mu\nu}\, \partial_\mu p_\nu \ . \tag{1.48}$$

On one hand, integrating out $\tilde{y}$ imposes that p_μ is pure gauge and locally given by $p_\mu = \partial_\mu y$, giving back the original model (1.46). On the other hand, we may instead integrate out the non-dynamical quadratic field p (after integrating by parts $\tilde{y}\, \epsilon^{\mu\nu}\, \partial_\mu p_\nu \to -\partial_\mu \tilde{y}\, \epsilon^{\mu\nu}\, p_\nu$)) to get

the dual σ-model with the $(x, \tilde{y})$ directions

$$\tilde{\mathcal{L}} = g_{ij}(x)\,\partial x^i \partial x^j + \frac{1}{f(x)}\,(\partial \tilde{y})^2 \ . \tag{1.49}$$

The dual theory still has a $U(1)$ isometry, corresponding now to shifts of $\tilde{y}$. In summary, the dual pair of models are related by $f \to \frac{1}{f}$, $y \to \tilde{y}$, and they were obtained by integrating out different fields from the interpolating theory (1.48). This procedure is easily generalized to cases with off-diagonal metric terms and B-fields, with the resulting transformation being known as the 'Buscher rules' [74].

Now let us consider *non-abelian duality* (NAD), where the above procedure is generalized to non-abelian isometries [75, 76, 77]. We shall focus on the examples of the PCM and symmetric space σ-model (considered above), and the procedure is the same for other theories with non-abelian global symmetries. Starting from one of these theories

$$\mathcal{L} = -\tfrac{1}{2}h\,\mathrm{Tr}[J_+\,P\,J_-] \ , \qquad P = \begin{cases} 1 & \text{(PCM)} \ , \\ P_{G/H} & \text{(symmetric space)} \ , \end{cases} \tag{1.50}$$

we shall dualize the G_L symmetry $g \to kg$. As above, we begin by gauging the symmetry and imposing that the gauge field is pure gauge with a Lagrange multiplier field v (here $F_{+-}(A) \equiv \partial_+ A_- - \partial_- A_+ + [A_+, A_-]$)

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{h}\,\mathrm{Tr}[(J_+ + g^{-1}A_+ g)\,P\,(J_- + g^{-1}A_- g) + v\,F_{+-}(A)] \ . \tag{1.51}$$

This resulting theory is invariant under the gauge transformation $g \to kg$, $A_\pm \to kA_\pm k^{-1} - \partial_\pm k k^{-1}$. Fixing the gauge $g = 1$, we obtain the interpolating theory

$$\mathcal{L}_{\mathrm{int}} = -\tfrac{1}{2}\mathrm{h}\,\mathrm{Tr}[A_+\,P\,A_- - v\,F_{+-}(A)] \ . \tag{1.52}$$

On one hand, integrating out v imposes that $A_\pm$ is flat and locally of the form $A_\pm = g^{-1}\partial_\pm g$, giving back the original theory (1.50). On the other hand, after integrating by parts, $A_\pm$ appears quadratically without derivatives; integrating it out yields the NAD σ-model depending on v,

$$\tilde{\mathcal{L}} = -\tfrac{1}{2}\mathrm{h}\,\mathrm{Tr}[\partial_+ v\,\frac{1}{P + \mathrm{ad}_v}\,\partial_- v] \ . \tag{1.53}$$

Unlike abelian T-duality, non-abelian duality destroys the original symmetry, meaning the original G_L symmetry is not present in the NAD theory (1.53). Thus, at least within the framework of non-abelian duality, one cannot "dualize back" from the NAD model to the original one.[10]

[10] In fact the G_L symmetry becomes a Poisson-Lie symmetry of the NAD model (1.53) (see eq. (1.72) below) and, starting from the NAD theory, one can dualize back to original theory by Poisson-Lie duality.

The path-integral derivations of abelian and non-abelian duality apply in particular at the level of the classical equations of motion (they may be formulated as canonical transformations; see, e.g., [78] and refs. therein). Hence these dualities preserve the property of classical integrability. For example the NAD of the PCM ((1.53) with $P = 1$) inherits a flat Lax connection of the same form (1.38) as the PCM with $\mathcal{A}_\pm = A_\pm(v)$ taking its value obtained in the quadratic integral in (1.52),

$$L_\pm = \tfrac{1}{2}(1 + z^{\pm 1})A_\pm(v) \ , \qquad A_\pm(v) = \frac{1}{\mathrm{ad}_v \mp 1}\partial_\pm v \ . \tag{1.54}$$

1.4.6 PCM with Wess-Zumino term

The PCM (1.18) is not the most general G-valued σ-model with $G_L \times G_R$ global symmetry (1.19). In fact, there is also a B-field coupling respecting this symmetry called the *Wess-Zumino* (WZ) term, defined by [79, 80, 81]

$$\mathrm{H} \equiv d\mathrm{B} = \tfrac{1}{12}k\,\mathrm{Tr}[J \wedge J \wedge J] \ , \qquad \mathrm{B} \equiv \tfrac{1}{2}\mathrm{B}_{mn}dx^m \wedge dx^n \ . \tag{1.55}$$

The free coefficient k is called the 'WZ level'.

One can check that $d\mathrm{H} = 0$, implying the local existence of a B-field satisfying $\mathrm{H} = d\mathrm{B}$. Although B is not globally defined (the expressions for B in different coordinate charts may be related by gauge transformations $\mathrm{B} \to \mathrm{B} + d\lambda$), the σ-model action (1.6) is well-defined up to an additive constant. Let us take the 2d space to be the Euclidean plane $\mathbb{C}$ (after analytic continuation), which we compactify to the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. Then a coordinate-invariant way to write down the WZ term's contribution to the action is[11]

$$S_{\mathrm{WZ}} = \frac{i}{2\pi}\int_{B_3} \mathrm{H} \ , \tag{1.56}$$

where B_3 is the solid ball whose boundary is S^2. One is instructed to continue the fields in an arbitrary way from the boundary into the interior of B_3 (this is always possible since Lie groups have trivial 2nd homotopy group $\pi_2(G) = 0$). Since H is closed then its integral (1.56) is independent of the choice of continuation of the fields, up to an overall topological 'winding number' constant. For compact simple groups[12] this gives an ambiguous contribution $\Delta S_{\mathrm{WZ}} = 2\pi i n k$ with n an arbitrary integer in the homotopy group $\pi_3(G) = \mathbb{Z}$, which can be shown to classify the configurations. The Euclidean path integral $Z = \int Dx\, e^{-S[x]}$ is then well-defined as long as the level k is quantized to be an integer.

We shall denote by 'PCM_k' the PCM with WZ term with level k (with $\mathcal{L}_{\mathrm{WZ}}$ being a local

[11] The factor of i in (1.56) results from the Wick rotation of the antisymmetric Levi-Civita tensor $\varepsilon^{\mu\nu}$ in the σ-model (1.3).

[12] For non-compact groups there may be no ambiguity at all and k may be any real number.

density for $S_{\rm WZ}$),

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{h}\,\mathrm{Tr}[J_+J_-] + k\,\mathcal{L}_{\rm WZ}(g)\ . \tag{1.57}$$

The PCM$_k$ is classically integrable for arbitrary values of the couplings h and k, with the Lax connection taking the same form (1.38) as the PCM in terms of a modified flat, conserved current [82]

$$L_\pm = \tfrac{1}{2}(1+z^{\pm 1})\mathcal{A}_\pm\ , \qquad\qquad \mathcal{A}_\pm = (1 \pm \tfrac{k}{h})J_\pm\ . \tag{1.58}$$

It is a general fact that WZ levels do not run under the RG flow (even for non-compact groups where they are not quantized). This is because the antisymmetric B-field terms in the σ-model β-functions (1.9) are all globally defined, so cannot be proportional to WZ terms whose B-fields are not globally defined. The WZ term does, however, modify the RG flow of the coupling h, and the PCM$_k$ has 2-loop β-function

$$\frac{d}{dt}\mathrm{h} = (1-\tfrac{k^2}{\mathrm{h}^2})\big(c_{_G} + \tfrac{1}{2}c_{_G}^2(1-3\tfrac{k^2}{\mathrm{h}^2}) + \dots\big)\ , \qquad \frac{d}{dt}k = 0\ . \tag{1.59}$$

1.4.7 Wess-Zumino-Witten model

The PCM$_k$ β-function (1.59) admits a fixed point $\mathrm{h} = k$ (with the other fixed point $\mathrm{h} = -k$ related by 2d parity or $g \to g^{-1}$), which is in fact an exactly conformal theory called the *Wess Zumino Witten* (WZW) model,

$$k\,\mathcal{L}_G(g) \equiv k\big(-\tfrac{1}{2}\,\mathrm{Tr}[J_+J_-] + \mathcal{L}_{\rm WZ}(g)\big)\ . \tag{1.60}$$

For this special value of the coupling, the $G_L \times G_R$ symmetry enhances to a "chiral gauge" symmetry [81],[13]

$$g \to g_L(\xi^-)\,g\,g_R(\xi^+)\ , \qquad g_L(\xi^-) \in G\ , \quad g_R(\xi^+) \in G\ . \tag{1.61}$$

At the WZW fixed point $h = k$, the equation of motion becomes the statement of holomorphic and anti-holomorphic conservation of the Noether currents corresponding to the chiral gauge symmetries (1.61),

$$\partial_- J_+ = 0 \iff \partial_+ K_- = 0\ , \qquad J_+ \equiv g^{-1}\partial_+ g\ , \quad K_- \equiv \partial_- g g^{-1}\ . \tag{1.62}$$

[13] In Minkowski signature $\xi^\pm$ in (1.61) are the same 2d light-cone coordinates used above. In Euclidean signature these become complex-conjugate cooordinates z, $\bar z$ on the complex plane.

This equation is explicitly solvable in terms of arbitrary G-valued (anti-)holomorpic functions,

$$g = g_1(\xi^-)\, g_2(\xi^+) \ , \tag{1.63}$$

with the currents $J_+ = J_+(\xi^+)$ and $K_- = K_-(\xi^-)$ being holomorphic and anti-holomorphic on-shell. Computing the currents' Poisson brackets and canonically quantizing, one finds that they satisfy the affine Kac-Moody algebra of G with level k, with the corresponding OPE,

$$J_+^a(z)J_+^b(z') \sim \frac{1}{k}\Big[\frac{\kappa^{ab}}{(z-z')^2} + \frac{if^{ab}{}_c J_+^c(z')}{z-z'}\Big] \qquad (+ \text{ regular terms}) \ . \tag{1.64}$$

One can show that the affine Kac-Moody algebra contains the Virasoro algebra, i.e. the current algebra OPEs (1.64) imply that a certain local operator $T(z) \equiv \frac{k^2}{2(k+c_G)}\mathrm{Tr}[:J_+J_+:(z)]$ satisfies the Virasoro OPE,

$$T(z)T(z') \sim \frac{c}{2(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial T(z')}{z-z'} \ . \tag{1.65}$$

The antiholomorphic currents K_- satisfy the same Kac-Moody algebra and generate another copy of the Virasoro algebra. Hence the WZW model is exactly conformal, being invariant under the 2d conformal algebra (two copies of the Virasoro algebra). Correspondingly the β-function (1.59) will vanish to all orders in a particular subtraction scheme (see the discussion around eq. (3.3) below). The corresponding conformal field theory, and particularly its correlation functions, were studied in [83].

1.4.8 Gauged WZW model

There is a closely related family of 2d CFTs based on coset spaces G/H [84] described by the *gauged Wess Zumino Witten* (gWZW) Lagrangian [85, 86],

$$\begin{aligned} k\,\mathcal{L}_{G/H}(g,A) &= k\big(\mathcal{L}_G(g) + \mathrm{Tr}[J_+A_- - K_-A_+ + g^{-1}A_+gA_- - A_+A_-]\big) \ , \\ J = g^{-1}dg \ , \quad K = dgg^{-1} \ , &\qquad g \in G \ , \quad A_\pm \in \mathrm{Lie}(H) \ . \end{aligned} \tag{1.66}$$

Like the symmetric space σ-model (1.28), the gWZW model depends on a quadratic 2d gauge field (that may be integrated out to obtain a σ-model) and is gauge-invariant under an action of the subgroup H,

$$g \to h^{-1}gh \ , \quad A_\pm \to h^{-1}A_\pm h + h^{-1}\partial_\pm h \ . \tag{1.67}$$

The classical gWZW σ-model (1.66) receives an infinite tower of α' corrections in order to be exactly conformal [87, 88, 89, 90, 91]. See the $SU(1,1)/U(1)$ example in eq. (2.4) and the review in Section 4.1 below.

1.4.9 Integrable η-deformation

Starting from an integrable σ-model like the PCM, one may consider squashing or deforming the target space geometry in a way that preserves the property of classical integrability. One such integrable deformation of the PCM, called the *Yang-Baxter deformation* or *η-deformation*, is given by [92]

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{h}\,\mathrm{Tr}[J_+ \frac{1}{1-\varkappa R} J_-] \ , \tag{1.68}$$

where $\varkappa$ is the deformation parameter. The deformation is built from a linear operator $R : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$, antisymmetric with respect to the Killing form (i.e. $\mathrm{Tr}[xR(y)] = -\mathrm{Tr}[R(x)y]$) and solving the modified classical Yang-Baxter equation,

$$[Rx, Ry] - R([Rx, y] + R[x, Ry]) + c^2[x, y] = 0 \ , \qquad x, y \in \mathrm{Lie}(G) \ . \tag{1.69}$$

With R satisfying these conditions, the η-deformed PCM (1.68) is classically integrable with the Lax connection again taking the the same form (1.38) as the PCM, now built from a different flat, conserved current [71]

$$L_\pm = \tfrac{1}{2}(1 + z^{\pm 1})\mathcal{A}_\pm \ , \qquad \mathcal{A}_\pm = -(1+\varkappa^2)\mathrm{Ad}_g \frac{1}{1 \pm \varkappa R} J_\pm \ . \tag{1.70}$$

One can check that the integrability of the model (i.e. the equivalence between the flatness of (1.70) and the equations of motion of (1.68)) relies on R being a solution of the Yang-Baxter equation (1.69). In general there may be various solutions of the Yang-Baxter equation and their classification is unknown. We shall focus on a particular class of canonical solutions in the "non-split inhomogeneous" case, i.e. taking $c^2 > 0$ in (1.69).[14] Given a choice of Cartan-Weyl basis $\{h_i, e_\alpha, e_{-\alpha}\}$ for $\mathrm{Lie}(G)$, where h_i are the Cartan generators, e_α the positive roots and $e_{-\alpha}$ the negative roots, we shall assume a canonical choice of R-matrix (see Section 2.2 of [71] and refs. there) solving (1.69),

$$R(h_i) = 0 \ , \qquad R(e_\alpha) = -ie_\alpha \ , \qquad R(e_{-\alpha}) = ie_{-\alpha} \ . \tag{1.71}$$

This R-matrix, and the associated η-deformed model (1.68), depend on the choice of Cartan-Weyl basis. However, if one restricts to the case of compact simple groups, then there is only choice of Cartan-Weyl basis (up to equivalence by inner automorphisms).

The associated η-deformation in (1.68) breaks the manifest global symmetry $G_L \times G_R$ of the PCM to the subgroup $G_L \times C$ where $C = U(1)^r$ is the Cartan subgroup of G commuting

[14] We shall then rescale R to set $c^2 = 1$. There are also other types of solutions with $c^2 > 0$. The "split inhomogeneous" ($c^2 < 0$) solutions can only exist for non-compact groups and include solutions of the same canonical type as (1.71); the "homogeneous" ($c = 0$) solutions include so-called "abelian" and "Jordanian" solutions.

with the R-matrix. The G_R isometry of the target space is actually deformed to a *Poisson-Lie (PL) symmetry* [92], with the associated Killing vectors $\{v_a^m\,\partial_m\}_a$ being replaced by vectors satisfying

$$L_{v_c}(G+B)_{mn} = -f^{ab}{}_c\, v_a^p\, v_b^q\, (G+B)_{mp}\, (G+B)_{qn}\ . \tag{1.72}$$

PL symmetry is thought to be related to a semi-classical limit of quantum group symmetry [93], and indeed the PL symmetries of the η-deformed models correspond to *non-local* conserved charges (that can be extracted from the monodromy matrix), the analogs of the leading Noether charges appearing in the expansion (1.22) for the undeformed models. These non-local charges are found to have a classical Poisson bracket algebra obtained from a semi-classical limit of a quantum group algebra [94, 93]. For this reason it is assumed that the full quantum η-deformed theories have quantum group symmetry, and this is the symmetry that has been assumed when bootstrapping their exact integrable S-matrices [95, 96, 97].

One can also simultaneously deform both the G_L and G_R parts of the global symmetry to obtain the integrable 'bi-Yang-Baxter' σ-model [71, 72]. The deformations can be generalized [98, 99] to include WZ terms; additionally including 'TsT transformations' [100] gives a 4-parameter integrable deformation of the PCM [101, 102].

Theories with PL symmetry can be dualized in a process called *Poisson-Lie duality* [103], generalizing non-abelian duality and abelian T-duality (see Section 1.4.5). Under PL duality, the η-models (1.68),(1.73) are mapped [104, 105, 106, 107] to another class of integrable deformed models called λ-models, which we shall discuss in the next section.

A similar integrable η-deformation exists [94] for the symmetric space σ-model,

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{h}\,\mathrm{Tr}[J_+\,P_{G/H}\,\frac{1}{1-\varkappa\,R_g P_{G/H}}\,J_-]\ , \qquad\qquad R_g \equiv \mathrm{Ad}_g^{-1} R\,\mathrm{Ad}_g\ . \tag{1.73}$$

The global G_L symmetry of the symmetric space is thus deformed to a PL symmetry, with only a Cartan subgroup remaining as a manifest isometry (and the H_R gauge invariance remaining preserved). The model is classically integrable with Lax connection of the same form (1.39) as the symmetric space in terms of different quantities $\mathcal{B}, \mathcal{P}$

$$\begin{aligned} L_\pm = \mathcal{B}_\pm + z^{\pm 1}\mathcal{P}_\pm\ , \qquad \mathcal{B}_\pm = P_H A_\pm\ , \quad \mathcal{P}_\pm = \sqrt{1+\varkappa^2}\,P_{G/H}A_\pm \\ A_\pm = \frac{1}{1\pm\varkappa R_g P_{G/H}} J_\pm \end{aligned} \tag{1.74}$$

In agreement with the conjecture that integrable σ-models are stable under the RG flow, the η-deformed models are known to be renormalizable at the leading 1-loop order with just

the couplings h and $\varkappa$ running [108, 50, 109][15]

$$G: \qquad \frac{d}{dt}\mathrm{h} = c_{_G}(1+\varkappa^2)^2 \ , \qquad \frac{d}{dt}\varkappa = c_{_G}\mathrm{h}^{-1}\varkappa(1+\varkappa^2)^2 \ , \tag{1.75}$$

$$G/H: \qquad \frac{d}{dt}\mathrm{h} = 2c_{_G}(1+\varkappa^2) \ , \qquad \frac{d}{dt}\varkappa = 2c_{_G}\mathrm{h}^{-1}\varkappa(1+\varkappa^2) \ . \tag{1.76}$$

This is a non-trivial example of the integrability–RG flow relationship because the η-deformed models are not *automatically* RG-stable due to manifest symmetries. Their stability is instead thought to be due to their hidden integrable symmetries.

1.4.10 Integrable $\boldsymbol{\lambda}$-deformation

In [110] Tseytlin considered a "universal class" of models related to gWZW models $\mathcal{L}_{G/F}$ by deformations parametrized by a constant linear operator $Q : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$,

$$\mathcal{L}_{\text{Tseytlin}} = k\big(\mathcal{L}_{G/F}(g,A) + \mathrm{Tr}[A_+(Q-1)A_-]\big) \ , \qquad g \in G \ , \quad A_\pm \in \mathrm{Lie}(F) \ . \tag{1.77}$$

The 1-loop conformal invariance conditions were computed, taking the form of polynomial constraints on Q.

Forgetting about conformal invariance, one may instead ask which choices of Q correspond to integrable theories. A particular class of integrable examples of (1.77) was later discovered [111, 112], called the λ-*model* based on a either group G or a symmetric space G/H. The model is defined by the choice $F = G$ and $Q = 1 - (\lambda^{-1} - 1)P$ in (1.77), or

$$\mathcal{L}_\lambda = k\big(\mathcal{L}_{G/G}(g,A) - (\lambda^{-1}-1)\mathrm{Tr}[A_+ \, P \, A_-]\big) \ . \tag{1.78}$$

Here k is a WZ level and the coupling λ is a deformation parameter (we will sometimes use a redefined coupling $\kappa \equiv \frac{1-\lambda}{1+\lambda}$ for convenience). The two cases of G or G/H are encoded in the choice of the projector P,

$$P = \begin{cases} 1 & (G \text{ case}) \ , \\ P_{G/H} & (G/H \text{ case}) \ , \end{cases} \tag{1.79}$$

so that the coset case has an H gauge invariance (see eq. (1.67)), while the group case has no gauge symmetry at all.

In the group case there is a global G symmetry, $g \to kgk^{-1}$, $A_\pm \to kA_\pm k^{-1}$, $k \in G$; whereas the coset case does not generally have any continuous global symmetries. In both cases, the λ-model (1.78) is invariant under a $\mathbb{Z}_2$ transformation of the fields and coupling constants [52, 104]

[15] The renormalizability and β-functions for the coset case (1.76) have only been directly studied in special cases, e.g. in [106]. However, Poisson-Lie duality has been shown to be a symmetry of the 1-loop β-functions [108] and, given that the η-models are understood to be related to the λ-models by PL duality [104, 105, 106, 107], the expression (1.76) may be derived from the corresponding λ-model one.

(see also [113, 110]),[16]

$$\begin{aligned} &k \to -k \ , \qquad\qquad \lambda \to \lambda^{-1} \ , \\ &g \to g^{-1} \ , \qquad A_+ \to A_+ - (1-\lambda^{-1})PA_+ \ , \qquad A_- \to gA_-g^{-1} - K_- \ . \end{aligned} \tag{1.80}$$

As we shall find below in Chapter 4, the preservation of this symmetry at the quantum level may require a particular choice of regularization scheme.

One may integrate out the 2d gauge field in (1.78) to obtain a σ-model depending on the scalar $g \in G$ (here $\mathcal{L}_G$ is the WZW Lagrangian),

$$\begin{aligned} &\mathcal{L}_\lambda = k\big(\mathcal{L}_G(g) + \mathrm{Tr}[J_+ M(g)^{-1} K_-]\big) \ , \qquad J \equiv g^{-1}dg \ , \quad K \equiv dgg^{-1} \ , \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad M(g) \equiv Ad_g - 1 + (1-\lambda^{-1})P \ . \end{aligned} \tag{1.81}$$

The λ-model is an integrable deformation of a WZW model on G or a gauged WZW model on G/H, obtained in the limit $\lambda = 0$ when the projected parts $PA_\pm$ decouple and are set to zero by their equations.

In another interesting limit,

$$\begin{aligned} &k \to \infty \ , \qquad \lambda = 1 - \tfrac{\mathrm{h}}{2k} \ , \qquad g = e^{-\frac{h}{2k}v} \ , \\ &v \in \mathrm{Lie}(G) \text{ and h fixed} \ , \end{aligned} \tag{1.82}$$

the λ-model becomes the non-abelian dual theory (1.53) corresponding to either the PCM or symmetric space σ-model. The theory (1.78) with unintegrated gauge field becomes the corresponding "interpolating model" (1.52).

The λ-models (1.81) (after integrating out $A_\pm$) admit flat Lax connections of the same form (1.38),(1.39) as the PCM/symmetric space, but with modified currents,

$$G: \qquad L_\pm = \tfrac{1}{2}(1+z^{\pm 1})\mathcal{A}_\pm \ , \qquad \mathcal{A}_\pm = \frac{2}{1+\lambda}A_\pm \ , \tag{1.83}$$

$$G/H: \qquad L_\pm = \mathcal{B}_\pm + z^{\pm 1}\mathcal{P}_\pm \ , \qquad \mathcal{B}_\pm = P_H A_\pm \ , \quad P_\pm = \tfrac{1}{\sqrt{\lambda}}P_{G/H}A_\pm \ , \tag{1.84}$$

where $A_\pm = A_\pm(g)$ are taken to be their on-shell values.

In line with the conjecture, the λ-model is known to be renormalizable at 1-loop with only λ running[17] [114, 53]

$$G: \qquad \frac{d}{dt}k = 0 \ , \quad \frac{d}{dt}\lambda = -\frac{2c_G}{k}\frac{\lambda^2}{(1+\lambda)^2} \ , \tag{1.85}$$

$$G/H: \qquad \frac{d}{dt}k = 0 \ , \quad \frac{d}{dt}\lambda = -\frac{c_G\lambda}{k} \ . \tag{1.86}$$

[16] Such a symmetry was also discussed for the models (1.77) in [110] (see footnotes 3 and 6 there).

[17] The 1-loop β-functions (1.85),(1.86) can also be extracted from [110]. Note that the WZ level is not running due to general arguments.

These β-functions respect the $\mathbb{Z}_2$ transformation (1.80), being invariant under the transformation $(k, \lambda) \to (-k, \lambda^{-1})$ of the couplings.

1.4.11 Integrable-deformed string backgrounds

Although we have focussed on the simplest case of bosonic theories, and relaxed the condition of Weyl invariance (i.e. that the target space is a string background), the models discussed above do admit Weyl-invariant superstring generalizations. The Green-Schwarz action for type IIB strings on $AdS_5 \times S^5$ with RR flux is a σ-model on the $\mathbb{Z}_4$ "semi-symmetric" coset $PSU(2,2|4)/[SO(1,4) \times SO(5)]$ with a particular WZ-type B-field term (here $P^{(i)}$ is the projector onto the ith graded part of the algebra) [15]

$$\begin{aligned} &S = \tfrac{1}{4\pi\alpha'} \int d^2\xi \mathcal{L} \ , \qquad \mathcal{L} = -\tfrac{1}{2} R^2 \ \mathrm{STr}\big[\sqrt{-h} h^{\mu\nu} J^{(2)}_\mu J^{(2)}_\nu + \epsilon^{\mu\nu} J^{(1)}_\mu J^{(3)}_\nu\big] \ , \\ &J^{(i)} = P^{(i)} J \ , \qquad J = g^{-1} dg \ , \qquad g \in PSU(2,2|4) \ , \end{aligned} \tag{1.87}$$

This is a generalization of the symmetric space model (1.30) above (replacing the $\mathbb{Z}_2$-automorphism by $\mathbb{Z}_4$) and indeed the bosonic part of the model (setting fermions to zero) is simply the symmetric space σ-model on $AdS_5 \times S^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$, with R being the radius of the AdS_5 and S^5 spaces.

The η-deformation [115, 116] and λ-deformation [117] can both be generalized to integrable deformations of semi-symmetric spaces. The η-deformed $AdS_5 \times S^5$ model was problematically found to be scale- but not Weyl-invariant, with the background solving only the generalized supergravity equations [96, 118, 119], and the perturbative light-cone S-matrix did not directly match the conjectured exact S-matrix [96]. It was then explained [120] that, to maintain Weyl invariance, one must in fact use the *purely fermionic* Dynkin diagram for the superalgebra $\mathfrak{psu}(2,2|4)$ to construct a *unimodular* R-matrix. In the recent paper [121], an exact S-matrix was conjectured that is different from the previous attempt, but related by a so-called 'twist'; this S-matrix indeed matches tree-level perturbative results.

Much remains to be understood about these integrable-deformed string backgrounds. For example one significant challenge is to understand the spectrum of the deformed theories (a direction in which some progress was made for the η-deformation [122] through the TBA and Quantum Spectral Curve). Examining the properties of the bosonic η-model and λ-model can provide valuable insights into their superstring counterparts. For example, the relation between the η- and λ-models by Poisson-Lie duality was first understood in the bosonic case [104, 105, 106, 107]. It is thus natural to first explore the question of quantum corrections and the higher-loop RG equations by studying the bosonic models. In Section 6.4 below, we shall comment on the possible applications of our approach to the integrable-deformed string backgrounds.

It is also worth emphasizing that the bosonic models are of interest in their own right

in the general context of investigating integrable 2d theories. The η-model has played an important role in generalizing the duality [44, 45, 46] between the deformed $O(3)$ and $O(4)$ sigma models and massive integrable QFTs to higher-rank groups and cosets [47, 48, 49] and cosets of supergroups [123]. While the dual theories are quantum-exact, the σ-model side of the duality is only understood so far to leading order in the loop expansion. Thus, after finding quantum corrections to integrable σ-models consistent with renormalizability, it may then be interesting to study their compatibility with this duality.

There are also many other known examples of integrable string backgrounds, including those based on lower-dimensional AdS spaces [124, 125] and their integrable deformations [126, 127, 128, 129, 130]; β-deformations (or TsT-deformations) [100, 131]; and more general homogeneous Yang-Baxter deformations [132, 133, 126].

1.4.12 Quantum integrable S-matrices

The presence of higher conserved charges places strict constraints on the possible scattering behaviour of a theory [134, 135] (see also the review [136]). Particle production is prohibited, i.e. the number of incoming and outgoing particles are equal; the set of initial momenta equals the set of final momenta; and the S-matrix must factorize into a product of $2 \to 2$ S-matrices. The consistency of this factorization implies that the $2 \to 2$ S-matrix satisfies the Yang-Baxter equation.

This special scattering behaviour is expected for integrable theories, and has been proven assuming the presence of at least two higher-spin conserved charges satisfying certain technical conditions [135]. S-matrices of this type may be seen as "almost free", and their existence is a special feature of 2-dimensions since in higher dimensions the presence of higher-spin charges implies triviality of the S-matrix (i.e. a free theory) by the Coleman-Mandula theorem.

For certain theories there are various arguments for the preservation of integrability at the quantum level. For example, in the case of certain principal chiral models, it has been argued [137] using consistency with global symmetry that the higher-spin conservation laws (1.23) cannot be modifed at the quantum level by anything other than a divergence, i.e. a quantum-corrected form of the current must still conserved to all orders in perturbation theory (see also [138] in the context of the $O(n)$-model, i.e. the σ-model on a sphere $S^{n-1} = \frac{O(n)}{O(n-1)}$.). The non-local charges of the $O(n)$ model were also argued to persist at the quantum level [139] and to imply factorization of the S-matrix with no particle production.

In the important paper [134], Zamolodchikov and Zamolodchikov started by assuming quantum integrability and the dynamical generation of a mass-gap for the $O(n)$-model. Using also the model's symmetries and assuming a 'minimal' solution for the phase ambiguity (with minimal number of poles on the physical sheet), they uniquely determined the exact S-matrix for non-perturbative quantum particles.

The $O(4)$-model, equivalent to the S^3 σ-model or $SU(2)$ PCM, was later solved explicitly

[140] by proving equivalence to a Thirring-type fermionic model with an infinite number of flavours, finding agreement with the Zamolodchikov-Zamolodchikov S-matrix. This exact solution was later generalized to the PCM on $SU(N)$ [141] and with arbitrary WZ term (PCM_k) [142].[18]

The Zamolodchikov-Zamolodchikov 'rational' S-matrices (with rational dependence on rapidities) are known to admit 'trigonometric' deformations [144, 145]. The 2d "sausage model" was then found [44] as a semi-classical σ-model description of the trigonometric $O(3)$ S-matrix, and this model was later understood [146] to be the $\frac{SU(2)}{U(1)}$ example of the η-model. More generally, similar trigonometric S-matrices are understood to be associated to inhomogeneous Yang-Baxter deformations and the associated q-deformed symmetries [95] (see also recent papers [97] on the light-cone S-matrix for η-deformed string backgrounds and refs. therein).

1.4.13 Factorized scattering vs. tree-level sigma model S-matrix

Given that integrable S-matrices are so constrained, one might hope to classify the integrable σ-models by (i) computing the S-matrix for a general σ-model in terms of the couplings (G, B) (ii) imposing factorization and no particle production and (iii) systematically reading off the resulting constraints on (G, B) from integrability. Like the RG flow (which is the focus of the thesis), the S-matrix could thus serve as another 'diagnostic tool' for integrability.

Here we shall summarize the paper [1] where we considered such a procedure for the leading *tree-level* S-matrix, whose factorization should correspond to *classical* integrability. The factorization of the loop-corrected S-matrix should then correspond to preservation of integrability at the quantum level. For example, for the complex sine-Gordon model, we recall that the tree-level 2-body S-matrix satisfies the Yang-Baxter equation but the 1-loop-corrected S-matrix only satisfies it if the theory is supplemented by certain quantum corrections [147]. These corrections may be traced to the leading quantum corrections to a gauged WZW model (see eq. (2.4) below) — of which the complex sine-Gordon model is a massive integrable deformation. The corrections are thus expected to correspond to a particular choice of finite counterterms consistent with quantum integrability, similar to those discussed below in the context of the RG flow.

To compute an S-matrix in perturbation theory, one must first decide on a choice of vacuum. In [1] we considered the 'trivial' vacuum $x^m = 0$ (with all fields vanishing) in an expansion of the metric and B-field around flat space, $\text{G}_{mn} + \text{B}_{mn} = \delta_{mn} + \text{h}_{mn,p}\, x^p + \text{h}_{mn,pq}\, x^p\, x^q + \dots$. This is a natural choice since it is universal to any target space geometry. However, the fields x^m expanded around this vacuum are massless 2d scalars, which are known to suffer from IR divergences at the quantum level [148]. This is related to the fact that massless particles (moving at the speed of light) on a 1d spatial line do not separate asymptotically, so an S-matrix cannot have the usual physical interpretation. Still, massless 2d S-matrices were formally discussed for

[18] See also [143], where the exact ground state energy was computed in the large-N limit.

integrable theories in the context of a finite-density TBA [149]. The S-matrix there retains the interpretation as the relative phase when one particle is moved past another.

One might still hope to define a formal *tree-level* S-matrix for such massless particles (e.g., from the classical action evaluated on a solution with special scattering boundary conditions), but we found in [1] that such an object still has IR singularities. The singularities are not true divergences but instead ambiguous "0/0" contributions due to a vertex vanishing and an internal propagator blowing up at the same time when the scattering amplitude is taken on-shell. We classified the ambiguities into two types and considered some simple approaches to regularize them (the usual $i\varepsilon$ prescription, or an alternative "massive regularization" where the particles are given a small mass). One can compute finite scattering amplitudes by regularizing the ambiguities, taking the amplitude on-shell, and finally taking the regulator to zero. However, we found that the resulting amplitudes do not have the usual connection to integrability: in various well-known integrable examples the amplitudes exhibit particle production. These findings were in agreement with earlier attempts to compute scattering amplitudes for massless 2d scalars in integrable theories [150].

Opposite conclusions have been found for massless 2d S-matrices in non-renormalizable (non scale-invariant) Nambu-like models [151]. For Nambu theory with a flat target space (but not for a curved one), each scalar appears in the action only through its derivative and thus ambiguities related to IR poles appear to be absent, not only at the tree level but also at the loop level. Here the (naively expected) relation between factorization of the massless S-matrix and integrability does appear to hold (with the Nambu action being integrable beyond tree level only in special critical dimensions). Moreover, the flat-space Nambu theory is an example of a $T\bar{T}$-deformation [152] of a free theory, for which integrability apparently constrains the RG flow even in non-renormalizable cases, or rather dictates the fixing of the counterterms to be consistent with the integrable S-matrix [153].

A similar S-matrix approach has been used [154] to rule out integrability of strings on certain (warped) AdS supergravity backgrounds by expanding around non-trivial classical solutions (BMN geodesic [22], GKP spinning string [155] or 'null cusp' [156] solutions) and fixing a light-cone gauge. These vacua give a mass to at least some fields and one may then consider particular worlsheet scattering amplitudes that do not contain any ambiguous contributions from massless fields. While this approach is not easily applied to generic backgrounds (where one does not have a good understanding of classical solutions), it would be interesting in future to consider a more general class of backgrounds than AdS ones, but still assuming enough isometries to allow a similar choice of vacuum.

1.4.14 Affine Gaudin models and local charges

There is a proposal to systematize integrable 2d theories by viewing them as realizations of affine Gaudin models [157, 158] (see also [159]). It so happens that all of the integrable σ-models

described above are examples of such theories. This is indeed suggestive they may describe a large portion of the integrable theories.

The affine Gaudin-type integrable models have flat Lax connections (1.7). They also have a systematic construction of local, higher-spin, classically conserved charges, generalizing those constructed in (1.40) for the PCM/symmetric space and models with Lax connections of those same form. The classical Poisson brackets of the spatial component of the Lax connection $\mathcal{L} \equiv L_\sigma = \frac{1}{2}(L_+ - L_-) \in \text{Lie}(G)$ take the Maillet r/s-form [21] (here z_1, z_2 are spectral parameters and σ_1, σ_2 are spatial coordinates),

$$\begin{aligned}\big\{\mathcal{L}_{\underline{1}}(z_1;\sigma_1), \mathcal{L}_{\underline{2}}(z_2;\sigma_2)\big\} = \Big(&[r_{\underline{12}}(z_1,z_2), \mathcal{L}_{\underline{1}}(z_1;\sigma_1) + \mathcal{L}_{\underline{2}}(z_2;\sigma_1)] \\ &+ [s_{\underline{12}}(z_1,z_2), \mathcal{L}_{\underline{1}}(z_1;\sigma_1) - \mathcal{L}_{\underline{2}}(z_2;\sigma_1)]\Big)\delta_{\sigma_1\sigma_2} - 2s_{\underline{12}}(z_1,z_2)\,\delta'_{\sigma_1\sigma_2} \ .\end{aligned} \tag{1.88}$$

Here the underlined $\underline{1}, \underline{2}$ indices in (1.88) denote the first and second factors in a tensor product $\text{Lie}(G)\otimes\text{Lie}(G)$, e.g. $\mathcal{L}_{\underline{1}} \equiv \mathcal{L}\otimes 1$, $\mathcal{L}_{\underline{2}} \equiv 1\otimes\mathcal{L}$. The Poisson bracket (1.88) is called 'non-ultralocal' because of the presence of the δ' term. The constant r- and s-matrices $r_{\underline{12}}(z_1,z_2) = -r_{\underline{21}}(z_2,z_1)$ and $s_{\underline{12}}(z_1,z_2) = s_{\underline{21}}(z_2,z_1)$ are respectively antisymmetric and symmetric, and they take the special "twist" form,[19]

$$r_{\underline{12}}(z_1,z_2) + s_{\underline{12}}(z_1,z_2) = \frac{1}{(z_2-z_1)\phi(z_2)}\, T_A \otimes T^A \ , \tag{1.89}$$

in terms of a model-dependent meromorphic *twist function* $\phi(z)$ of the spectral parameter. Here the 'split Casimir' $T_A \otimes T^A$, built from any set of generators T_A for G and the dual generators T^A (with respect to $\kappa_{AB} \equiv \text{Tr}[T_A T_B]$), is invariant under change of basis.

At each zero z_i of the twist function $\phi(z)$ (or more properly zeros of the 1-form $\phi(z)dz$)[20], one can automatically construct from (1.88) an infinite tower of local, higher spin charges. Assuming for simplicity the Lax connection to be valued in a B, C or D type algebra (the A type case is slightly more complicated), the following charges $\{Q^{(2n)}(z_i)\}_{n,i}$ are all in involution [157] (assuming a well-defined limit of $\phi(z)\mathcal{L}(z)$ at z_i),

$$\big\{Q^{(2n)}(z_i), Q^{(2m)}(z_j)\big\} = 0 \ , \qquad Q^{(2n)}(z_i) = \int d\sigma\, \text{Tr}\big[\big(\phi(z)\mathcal{L}(z)\big)^{2n}\big]\Big|_{z=z_i} \ . \tag{1.90}$$

For realizations of affine Gaudin models, the Hamiltonian is a linear combination of some of these charges, so all of the charges are conserved (commuting with the Hamiltonian).

For the group-based models above (PCM, PCM_k, group η-model, group λ-model) with Lax connections of the form (1.38) in terms of a flat, conserved current $\mathcal{A}_\pm$, let us re-define the spectral parameter $z = \frac{1+w}{1-w}$ for simplicity so that the Lax matrix $\mathcal{L} = L_\sigma$ has simple poles

[19] The form of (1.89) is slightly modified is so-called 'cyclotomic' examples involving twisting by a Z_T automorphism, e.g. the symmetric space σ-model with $T = 2$; here we ignore such cases.

[20] In particular the poles of $\mathcal{L}$ will be zeros of the twist function.

at finite values $w = \pm 1$. These models' twist functions all have simple zeros at these points ± 1, and take the form $\phi(w) = \frac{(1-w)(1+w)}{p(w)}$ where p is a polynomial not vanishing at ± 1. The resulting charges at $w_1 = 1$ and $w_2 = -1$ are proportional to the local charges obtained from the (anti-)holomorphic currents in (1.40) above,[21]

$$\frac{d}{dt}Q^{(2n)}(\pm 1) = 0 \ , \qquad Q^{(2n)}(\pm 1) \sim \int d\sigma \, \mathcal{J}_{\pm}^{(2n)} \ , \qquad \mathcal{J}_{\pm}^{(2n)} = \mathrm{Tr}[\mathcal{A}_{\pm}^{2n}] \ . \tag{1.91}$$

Here we have used the trace but, as in (1.40)(1.41) above, we could have instead used any symmetric, invariant tensor in (1.90).

Thus one learns from (1.90) that the higher-spin charges (1.91) are all in involution. It also follows from the Poisson bracket (1.88) that these charges are in involution with the monodromy matrix and its non-local conserved charges [157] as constructed in Section 1.4.1. A similar construction of local, conserved charges applies to models based on symmetric spaces (or other Z_T-cosets).

As well as re-interpreting known models, the affine Gaudin approach has the power to generate new integrable theories, essentially by picking new twist functions $\phi(z)$. For example, the η-deformations of symmetric space [94] and semi-symmetric space [115] σ-models were discovered in this way (based on the previously known group space case); more recent examples include the integrable G^N models [160], deformed G^N models [161] and $G \times G/H$ models [61] (see Chapter 3).

The affine Gaudin construction systematizes certain reformulations of these integrable σ-models. For a certain class of twist functions, the affine-Gaudin-type models are equivalent [162] to "E-models" [103] (a first order formulation with manifest Poisson-Lie symmetry), and can also be obtained [163, 164] from 4d Chern-Simons theory [165].

Finally we remark that the affine Gaudin formalism could lead to a more systematic understanding of the RG flow of integrable σ-models, or perhaps even a first-principles proof of their renormalizability. Along these lines, a recent paper [60] noted a pattern in the 1-loop RG flow for several integrable affine-Gaudin-type σ-models: in each case it may be written in the same universal way as a flow of the twist function. It would be interesting to try to derive this statement, or to check it beyond the leading 1-loop order. One complication is that, beyond the 1-loop order, we should expect the integrable theories to require a quantum deformation to remain RG-stable (see below); it is not yet clear how these quantum deformations would be encoded in the affine Gaudin language.

[21] We seem to have produced only half as many charges in (1.91) as in (1.40), but in fact all of the odd currents $J^{(2n+1)}$ vanish for B, C and D type groups [66]. Formally this procedure applies to the complexifed Lie group and one specifies a real form by some reality conditions. This does not affect the charges in this case since the zeros $w = \pm 1$ of the twist function are real and so the charges (1.91) are real.

1.4.15 On scheme dependence

Under covariant scheme transformations, or redefinitions of the σ-model couplings (G, B) (in terms of constants c, d, e),

$$\mathrm{G}_{mn} \to \mathrm{G}_{mn} + \alpha'(c\, R_{mn} + d\, \mathrm{H}_{mpq}\mathrm{H}_n{}^{pq}) + \dots \ , \quad \mathrm{B}_{mn} \to \mathrm{B}_{mn} + \alpha'\, e\, \nabla^p \mathrm{H}_{pmn} + \dots \ , \quad (1.92)$$

the β-function $\beta_{mn} = \frac{d}{dt}(\mathrm{G}+\mathrm{B})_{mn}$ transforms as a contravariant vector in the space of couplings. Simple power counting implies that the leading 1-loop β-function is invariant while, in general, scheme dependence will begin at the 2-loop order.[22]

In this thesis, we will generally use the 2-loop β-functions given in (1.9), which were computed in a special 2-loop scheme that effectively treats G_{mn} and B_{mn} as symmetrically as possible [166, 35] (see also [57, 58]). Indeed, the β-functions for G and B in (1.9) are given by the symmetric and antisymmetric parts of a single tensor expression. We shall refer to this G-B symmetric scheme as the "GB scheme" (see the introduction to Chapter 3 for discussion of the properties of this scheme).

It is important to emphasize, however, that our conclusions will be scheme-independent. In particular, the statement that 'a model is not renormalizable' is understood to mean that it is not renormalizable in any covariant subtraction scheme. We shall find certain finite counterterms that restore higher-loop renormalizability and, while the form of these counterterms will change in different subtraction schemes, they cannot be eliminated by covariant scheme changes such as (1.92).

[22] For the special case of vanishing $\mathrm{B} = 0$, the σ-model may be viewed as a "single-coupling" theory and its 2-loop β-function (1.8) is also invariant (see Appendix B).

Chapter 2

Quantum corrections to geometry from RG flow

As discussed in the Introduction, it was observed for some years that integrable σ-models appear to be stable under the leading 1-loop RG flow, and our purpose here is to investigate what happens at higher-loop orders. The general finding is that the integrable theories remain stable under the higher-loop RG flow provided they are supplemented with particular finite counterterms, i.e. the classical target space geometry $(\mathrm{G}+\mathrm{B})^{(0)}_{mn}$ receives certain quantum corrections,

$$(\mathrm{G}+\mathrm{B})_{mn} = (\mathrm{G}+\mathrm{B})^{(0)}_{mn} + \alpha'(\mathrm{G}+\mathrm{B})^{(1)}_{mn} + \dots \ . \tag{2.1}$$

One might first try to study the integrability–RG flow relationship for a canonical integrable example such as the PCM,

$$\mathcal{L}_{\mathrm{PCM}} = -\tfrac{1}{2}\mathrm{hTr}[J_+ J_-] \ . \tag{2.2}$$

However, the question trivializes for this simple example, because the renormalizability of the PCM follows immediately from its large $G \times G$ global symmetry (and parity invariance)

$$g \to g_L \, g \, g_R \ , \qquad (g_L, g_R) \in G_L \times G_R \ . \tag{2.3}$$

Only the curvature radius $\sqrt{h}$ can run because the theory (2.2) does not admit any deformations preserving these symmetries. This conclusion is consistent with the conjecture that integrable models are renormalizable, but it is not good evidence for it: we only used manifest global symmetry, without relying on the hidden integrable symmetries.

In order to really probe the role of integrability in constraining the RG flow, we shall consider integrable theories with less global symmetry. As we shall observe, integrability appears to be sufficient for renormalizability, even in cases without any manifest global symmetries. In this

chapter we shall focus on the integrable η- and λ-deformations introduced in Sections 1.4.9 and 1.4.10, which are classically integrable but have reduced global symmetry groups. In particular, these models are not fully constrained by their global symmetries, and renormalizability does not follow from global symmetries alone.

First in Section 2.2, we shall consider the simplest η-deformed model based on the coset $\frac{SU(2)}{U(1)}$. Then in secton 2.3, we shall consider λ-deformed models based on the coset $\frac{SU(2)}{U(1)}$ and the group $SU(2)$. As we reviewed above, these models are already known to be stable under the leading 1-loop RG flow; we shall determine the quantum corrections required to solve the 2- and higher-loop RG equation (1.9).

This chapter is based on the papers [2, 3] and closely follows the presentation there.

2.1 On quantum corrections in sigma models

Before proceeding, let us provide justification for the presence of quantum corrections (2.1) in σ-models by recalling some familiar examples where such corrections are known to occur.

One class of examples are the special integrable σ-models (1.66) corresponding to gauged WZW conformal field theories. The classical σ-models (1.66) are indeed scale-invariant, i.e. fixed-point solutions of the RG equation (1.9), at the leading 1-loop order. However, to remain conformally invariant beyond 1-loop, the target space metric and B-field require non-trivial α' corrections at each order, which may be derived exactly the underlying quantum gWZW theory (for discussion and examples see [167, 168, 169]). The simplest example is provided by the exact counterpart [87] of the familiar classical $SU(1,1)/U(1)$ gWZW metric (or 2d Euclidean black hole) [170, 171]

$$ds^2 = k(dr^2 + \tanh^2 r\, dy^2) \quad \to \quad ds^2 = (k-2)\Big[dr^2 + \big(\coth^2 r - \tfrac{2}{k}\big)^{-1} dy^2\Big] \ . \tag{2.4}$$

Here the inverse level k^{-1} (and later the inverse coupling h^{-1}) plays the role of the loop-counting parameter α' in (1.6)

As was checked directly in [172], the leading k^{-1} correction in (2.4) is required to solve the 2-loop scale invariance equation following from (1.8) (see also [173] for a 4-loop test of (2.4)).[23] The integrable cases that follow will be a direct generalization of this principle: instead of ensuring complete scale-invariance, the quantum corrections will ensure invariance under the RG flow with just a few couplings running.

It is also well-known that the classical Buscher rules for abelian T-duality are subject to quantum corrections [174, 175, 176, 177], so geometries resulting from T-duality will contain non-trivial α' corrections. For example, given the metric $ds^2 = \mathrm{h}[dx^i dx^i + \mathcal{M}(x) dy^2]$ with an

[23] In this Weyl-invariant case the diffeomorphism vector ξ_n can be written as $2\partial_n \Phi$ where the exact dilaton field Φ is given by $e^{-2\Phi} = \sinh 2r \, \big(\coth^2 r - \frac{2}{k}\big)^{1/2}$.

isometry direction y, the standard T-duality rule $\mathcal{M} \to \widetilde{\mathcal{M}} = \mathcal{M}^{-1}$ is known to be modified at the 2-loop level [174] (see also [178, 179])

$$\widetilde{\mathcal{M}} = \mathcal{M}^{-1}\big(1 + \tfrac{1}{2}\mathrm{h}^{-1}\,\partial_i \log \mathcal{M}\,\partial^i \log \mathcal{M}\big)\ . \tag{2.5}$$

In an example below in Section 2.3, this quantum correction to the T-duality transformation will be reproduced by special limits of the corrections for the λ-model. We shall also produce similar corrections to the transformation rules of non-abelian duality in some special examples.

2.2 η-model

The metric of the 2d "sausage" model[24]

$$ds^2 = \tfrac{1}{4}\mathrm{h}\Big[\frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \frac{1-r^2}{1+\varkappa^2 r^2}d\phi^2\Big]\ , \tag{2.6}$$

was originally written down in [44] as the leading semi-classical approximation to the σ-model corresponding to the massive integrable trigonometric S-matrix of [144]. It was discovered as a solution to the 1-loop RG equation (the leading Ricci flow term in (1.8)) and conjectured to be classically integrable. Its classical integrability was later shown in [46]. In [146] this model was identified as the $\frac{SU(2)}{U(1)}$ example of the η-deformation [92, 94], introduced in Section 1.4.9.[25]

The metric (2.6) is an integrable deformation of the round S^2 ($\varkappa = 0$) of radius $\sqrt{\mathrm{h}/4}$, with $\varkappa$ being the deformation parameter. The usual η-deformation correspnds to the case when $\varkappa$ is real, however the authors of [44] considered another regime of interest with $\varkappa$ imaginary, in which case the σ-model is UV stable with a free UV fixed point at $\varkappa^2 = -1$. Alternatively, we can send $\varkappa^2 \to -1$ while simultaneously expanding around $r^2 = 1$. Taking the limit in this way, the metric (2.6) reduces to the classical metric of the $\frac{SU(1,1)}{U(1)}$ gWZW model (cf (2.4)),

$$r^2 = 1 - (1+\varkappa^2)\sinh^2 x\ , \qquad \varkappa^2 \to -1\ , \qquad ds^2 \to \tfrac{1}{4}\mathrm{h}(dx^2 + \tanh^2 x\, d\phi^2)\ , \tag{2.7}$$

with h/4 becoming the level. Another useful limit is the maximal deformation limit $\varkappa \to \infty$, $\mathrm{h} \to \infty$ with $\mathrm{h}' \equiv \frac{\mathrm{h}}{\varkappa^2}$ fixed, which yields [146] the undeformed hyperbolic space H^2 with radius $\sqrt{\mathrm{h}'/4}$ (here we set $r = \frac{1}{\cosh\theta}$)

$$ds^2 = \frac{\mathrm{h}'}{4r^2}\Big[\frac{dr^2}{1-r^2} + (1-r^2)d\phi^2\Big] = \tfrac{1}{4}\mathrm{h}'\Big[d\theta^2 + \sinh^2\theta\, d\phi^2\Big]\ . \tag{2.8}$$

The metric (2.6) solves the 1-loop RG equation (1.8) with h and $\varkappa$ running as (eq. (1.76)

[24] We note that h in this chapter is equivalent to $4h$ in [2].

[25] It is interesting to note that the metric (2.6) is formally self T-dual, i.e. invariant under $\phi \to \tilde{\phi}$ and $\tilde{\phi} \to i\varkappa^{-1}\phi$, $r \to i\varkappa^{-1}r^{-1}$.

with $c_{_{SU(2)}} = 2$)

$$\frac{d}{dt}\mathrm{h} = 4(1+\varkappa^2) + \mathcal{O}(\mathrm{h}^{-1}) \ , \qquad\qquad \frac{d}{dt}\varkappa = 4\mathrm{h}^{-1}\varkappa(1+\varkappa^2) + \mathcal{O}(\mathrm{h}^{-2}) \ . \tag{2.9}$$

There is an RG-invariant combination of couplings

$$\frac{d}{dt}\nu = 0 \ , \qquad\qquad \nu \equiv \frac{\mathrm{h}}{4\varkappa} + \mathcal{O}(\mathrm{h}^0) \ . \tag{2.10}$$

In agreement with the above comments, $\varkappa^2 = -1$ is a UV fixed point (corresponding to either a free theory or a gWZW model depending how the limit is taken).

However, including also the 2-loop term in the RG equations (1.8), the σ-model on the classical sausage geometry (2.6) appears not to be renormalizable: it does not solve the 2-loop RG equations with only h and $\varkappa$ running. This is true in any covariant subtraction scheme[26] and with any diffeomorphism vector (here one only needs to consider diffeomorphism vectors of the form $\xi^m = \xi^m(r)$ since ϕ is an isometry direction).

The relation to the gWZW model suggests that, for (2.6) to remain a solution to the RG equation at higher loops with only h and $\varkappa$ running, it should similarly be modified by quantum (i.e. 1/h) corrections. Inspired by the analogy with the form of the gWZW metric in (2.4) we propose the following conjecture for the exact generalization of (2.6)

$$ds^2 = (\tfrac{1}{4}\mathrm{h} - 1 + \varkappa^2)\Big[\frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \big(\frac{1+\varkappa^2 r^2}{1-r^2} + \frac{8\varkappa^2}{\mathrm{h}}\big)^{-1} d\phi^2\Big] \ . \tag{2.11}$$

The metric (2.11) is expected to solve the RG equations to all loop orders (in a particular scheme and modulo coordinate redefinitions) with h and $\varkappa$ running according to a generalization of (2.9). The conjecture passes some obvious tests:

(i) The metric (2.11) reduces to S^2 when setting $\varkappa = 0$, now with shifted radius-squared coupling $\frac{1}{4}\mathrm{h} - 1$, which is simply related to the classical one $\frac{1}{4}\mathrm{h}$ by a finite coupling redefinition. Alternatively, one may shift $\mathrm{h} \to \mathrm{h} + 4(1-\varkappa^2)$ in (2.11) to obtain $ds^2 = \frac{1}{4}\mathrm{h}\Big[\frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \big(\frac{1+\varkappa^2 r^2}{1-r^2} + \frac{2\varkappa^2}{\frac{1}{4}\mathrm{h}+1-\varkappa^2}\big)^{-1} d\phi^2\Big]$, so that the radius-squared of S^2 remains $\frac{1}{4}\mathrm{h}$ in the $\varkappa = 0$ limit.

(ii) The metric remains flat for $\varkappa^2 = -1$.

(iii) In the non-trivial $\varkappa^2 \to -1$ limit (2.7) it reduces to the exact gWZW metric in (2.4) with level $k = \mathrm{h}/4$.

(iv) In the maximal deformation limit (2.8) it reduces to H^2, now with shifted radius-squared coupling $\mathrm{h}'/4 + 1$.

[26] As discussed in Appendix B, for 2d target spaces, the scheme dependence does not begin until 3-loops.

More importantly, one can directly check that (2.11) with the leading h^{-1} correction included,

$$ds^2 = \mathrm{h}\Big[\frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)}\Big(\frac{1}{4} - \frac{1-\varkappa^2}{\mathrm{h}}\Big) + \frac{1-r^2}{1+\varkappa^2 r^2}\Big(\frac{1}{4} - \frac{1-\varkappa^2}{\mathrm{h}} - \frac{2\varkappa^2}{\mathrm{h}}\frac{1-r^2}{1+\varkappa^2 r^2} + \dots\Big)d\phi^2\Big] \ , \tag{2.12}$$

indeed solves the 2-loop RG equation (1.8), which becomes in this case with 2d target space,[27]

$$\frac{d}{dt}\mathrm{G}_{mn} = (\tfrac{1}{2}R + \tfrac{1}{4}R^2)\mathrm{G}_{mn} + \dots + \nabla_{(m}\xi_{n)} \ , \tag{2.13}$$

for a particular diffeomorphism vector ξ_n with components,[28]

$$\xi_\phi = 0 \ , \qquad \xi_r = \frac{8\varkappa^2(1+\varkappa^2)r}{\mathrm{h}(1+\varkappa^2 r^2)^2} \ . \tag{2.14}$$

The β-functions (2.9) receive the following 2-loop corrections

$$\frac{d}{dt}\mathrm{h} = (1+\varkappa^2)\big[4 + 16\,\mathrm{h}^{-1}(1-\varkappa^2) + \mathcal{O}(\mathrm{h}^{-2})\big] \ , \tag{2.15}$$

$$\frac{d}{dt}\varkappa = \mathrm{h}^{-1}\varkappa(1+\varkappa^2)\big[4 + 16\,\mathrm{h}^{-1}(1-\varkappa^2) + \mathcal{O}(\mathrm{h}^{-2})\big] \ . \tag{2.16}$$

The RG-invariant quantity in (2.10) remains RG-invariant without 2-loop corrections,

$$\frac{d}{dt}\nu = 0 \ , \qquad \nu \equiv \frac{\mathrm{h}}{4\varkappa} + 0 + \mathcal{O}(\mathrm{h}^{-1}) \ . \tag{2.17}$$

Furthermore, the ansatz (2.11) also solves the 3-loop RG equation (1.8) with

$$\beta_{mn} = \Big[\tfrac{1}{2}R + \tfrac{1}{4}R^2 + c_1 R^3 + c_2(\nabla R)^2 + c_3 R\nabla^2 R\Big]\mathrm{G}_{mn} + c_4\nabla_m R\nabla_n R \ , \tag{2.18}$$

in a particular "natural" renormalization scheme corresponding to

$$c_1 = 0 \ , \qquad c_2 = \tfrac{1}{8} \ , \qquad c_3 = -\tfrac{1}{4} \ , \qquad c_4 = -\tfrac{1}{16} \ , \tag{2.19}$$

[27] In 2 dimensions the curvature tensor can be written as $R_{mnkl} = \frac{1}{2}R(\mathrm{G}_{mk}\mathrm{G}_{nl} - \mathrm{G}_{nk}\mathrm{G}_{ml})$. Note that any B-field on a 2d space is trivial (with zero field strength $\mathrm{H} = d\mathrm{B}$) so we use the β-function (1.8) for the case of vanishing B-field (also taking $\lambda_m = 0$).

[28] We note that the diffeomorphism vector (2.14) begins at 2-loop order; the 1-loop diffeomorphism vanishes in these coordinates.

in which the 3-loop β-functions for h and $\varkappa$ take the form

$$\frac{d}{dt}\mathrm{h} = (1+\varkappa^2)\Big[4 + 16\,\mathrm{h}^{-1}(1-\varkappa^2) + 64\,\mathrm{h}^{-2}(1-\varkappa^2)^2 + \mathcal{O}(\mathrm{h}^{-3})\Big] \ , \tag{2.20}$$

$$\frac{d}{dt}\varkappa = \mathrm{h}^{-1}\varkappa(1+\varkappa^2)\Big[4 + 16\,\mathrm{h}^{-1}(1-\varkappa^2) + 64\,\mathrm{h}^{-2}(1-\varkappa^2)^2 + \mathcal{O}(\mathrm{h}^{-3})\Big] \ . \tag{2.21}$$

The diffeomorphism vector in (2.14) receives the following 3-loop correction

$$\xi_\phi = 0 \ , \qquad \xi_r = \frac{8\varkappa^2(1+\varkappa^2)r}{\mathrm{h}(1+\varkappa^2 r^2)^2}\Big[1 + \frac{8\varkappa^2(r^2-1)}{\mathrm{h}(1+\varkappa^2 r^2)}\Big] \ . \tag{2.22}$$

This scheme (2.19) is related to the minimal subtraction scheme in [36, 37, 38],

$$c_1 = \tfrac{5}{32} \ , \qquad c_2 = \tfrac{1}{16} \ , \qquad c_3 = 0 \ , \qquad c_4 = -\tfrac{1}{16} \ , \tag{2.23}$$

by the covariant coupling redefinition,

$$\mathrm{G}^{(\mathrm{nat})}_{mn} = \big[\mathrm{G}_{mn} + \tfrac{5}{8}\,\alpha'^2(R^2)_{mn} + \tfrac{1}{2}\,\alpha'^2\nabla^2 R_{mn}\big]^{(\mathrm{min})} \ , \tag{2.24}$$

where $\mathrm{G}^{(\mathrm{nat})}_{mn} \equiv \mathrm{G}_{mn}$ is the metric (2.11) in the "natural" scheme and $\mathrm{G}^{(\mathrm{min})}_{mn}$ is the corresponding metric in the minimal scheme (see Appendix B).

We call the scheme (2.19) "natural" because the RG-invariant quantity in (2.17) remains invariant with no 3-loop corrections:

$$\frac{d}{dt}\nu = 0 \ , \qquad \nu \equiv \frac{\mathrm{h}}{4\varkappa} + 0 + 0 + \mathcal{O}(\mathrm{h}^{-2}) \ . \tag{2.25}$$

This prompts us to conjecture that, in a natural choice of subtraction scheme at each loop order, the quantity $\nu \equiv \frac{\mathrm{h}}{\varkappa}$ is an exact RG-invariant. This suggests that ν should be the parameter that appears in the exact quantum trigonometric S-matrix that generalizes the non-perturbative massive S-matrix of the $O(3)$-invariant S^2 σ-model [134].

Relatedly, we extrapolate from the obvious pattern in (2.20),(2.21) to conjecture that, in the same natural scheme, the all-loop β-functions of h and $\varkappa$ are

$$\frac{d}{dt}\mathrm{h} = \frac{1+\varkappa^2}{\frac{1}{4}\mathrm{h} - (1-\varkappa^2)}\,\mathrm{h} \ , \tag{2.26}$$

$$\frac{d}{dt}\varkappa = \frac{1+\varkappa^2}{\frac{1}{4}\mathrm{h} - (1-\varkappa^2)}\,\varkappa \ . \tag{2.27}$$

Here $\varkappa^2 = -1$ remains a fixed point, as it should to all orders since it corresponds both to flat space and the gWZW limit. For $\varkappa = 0$ we get $\frac{d}{dt}R^2_{S^2} = 1 + R^{-2}_{S^2}$ for the S^2 radius $R_{S^2} = \sqrt{\frac{1}{4}\mathrm{h} - 1}$, which agrees with the 1-loop and 2-loop coefficients in the β-function of the S^2 model. In the maximal deformation limit (2.8) we get $\frac{d}{dt}R^2_{H^2} = -1 + R^{-2}_{H^2}$ for the H^2 radius

$R_{H^2} = \sqrt{\frac{1}{4}\mathrm{h}' + 1} = \sqrt{\frac{1}{4}\mathrm{h}\varkappa^{-2} + 1}$. The fact that this is correctly related to the S^2 β-function by the analytic continuation $R^2_{H^2} = -R^2_{S^2}$ is a check of the consistency of the conjectured exact metric (2.11) and exact β-functions (2.26),(2.27). Moreover, if there is a natural scheme where (2.26),(2.27) are exact, then the S^2 and H^2 β-functions are 2-loop exact in this scheme (this may be possible in a special non-minimal scheme since the 3- and higher-loop β-function coefficients are scheme-dependent).

Let us mention that in the case of the (1,1) supersymmetric generalization of the σ-model (1.6), the first potential correction to the 1-loop β-function appears at 4 loops ($\sim \zeta(3)R^4$ in the minimal scheme [180], with this particular invariant actually vanishing in the case of 2d target space). There is also no deformation of the super-gWZW metric (to all orders) and it is thus natural to expect that the same may apply to the supersymmetric σ-model on the sausage geometry (2.6) (i.e. only h and $\varkappa$ running without adding corrections to the metric).

One may also repeat the above discussion for the η-deformation of hyperbolic space $H^2 = \frac{SU(1,1)}{U(1)}$ (or the Lorentzian signature symmetric spaces dS_2 and AdS_2). The corresponding metric is related to (2.6) by the formal analytic continuation,

$$r \to ir \ , \qquad \phi \to i\phi \ , \qquad \varkappa \to i\varkappa \ , \qquad \mathrm{h} \to -\mathrm{h} \ . \tag{2.28}$$

i.e. we find[29]

$$ds^2 = \tfrac{1}{4}\mathrm{h}\Big[\frac{dr^2}{(1+r^2)(1+\varkappa^2 r^2)} + \frac{1+r^2}{1+\varkappa^2 r^2}d\phi^2\Big] \ . \tag{2.29}$$

This inhomogeneous Yang-Baxter deformation of H^2 is based on the *split* R-matrix of Lie$(SU(1,1))$ (see footnote 14). On the other hand, if we had not continued $\varkappa \to i\varkappa$ in (2.28), we would have found Yang-Baxter deformation based on the non-split R-matrix. For S^2 only the latter case exists as a Yang-Baxter deformation in a strict sense since there is no split R-matrix of Lie$(SU(2))$ (see, e.g., [181]).

For $\varkappa = 0$ (2.29) becomes the H^2 metric, while for $\varkappa^2 = 1$ it is flat. The limit analogous to (2.7), i.e. $\varkappa \to 1$ with $r^2 \to -1 + (1-\varkappa^2)\sinh^2 t$, is now a formal limit and gives the metric $ds^2 = \frac{1}{4}\mathrm{h}(-dt^2 + \tanh^2 t\, d\phi^2)$, which is the classical metric of the $\frac{SU(1,1)}{\mathbb{R}}$ gWZW model.

In conformally-flat coordinates the metric (2.29) may be written as

$$ds^2 = \frac{\mathrm{h}}{4(p^2+\varkappa^2 q^2)}(dp^2 + dq^2) \ , \qquad r = \frac{q}{p} \ , \quad \phi = \tfrac{1}{2}\log(p^2+q^2) \ , \tag{2.30}$$

where the scaling symmetry $(p,q) \to \lambda(p,q)$ is the counterpart of ϕ-shift isometry of (2.29).

[29] Let us note that the metric (2.29) is self T-dual, i.e. invariant under $\phi \to \tilde\phi$ and $\tilde\phi \to \varkappa^{-1}\phi$, $r \to \varkappa^{-1} r^{-1}$.

The analog of the exact metric (2.11) is found by the analytic continuation (2.28),

$$ds^2 = (\tfrac{1}{4}\mathrm{h} + 1 + \varkappa^2)\Big[\frac{dr^2}{(1+r^2)(1+\varkappa^2 r^2)} + \big(\frac{1+\varkappa^2 r^2}{1+r^2} + \frac{8\varkappa^2}{\mathrm{h}}\big)^{-1} d\phi^2\Big] \ . \tag{2.31}$$

Again, formally taking $r^2 \to -1 + (1-\varkappa^2)\sinh^2 t$, the limit $\varkappa^2 \to 1$ is the exact metric of the $\frac{SU(1,1)}{\mathbb{R}}$ gWZW model.

Interestingly the exact metric (2.31) may have different properties compared to the classical metric (2.29). For example, analytically continuing further ($\varkappa \to i\varkappa$, $r \to \rho$, $\phi \to it$) to the non-split η-deformation of AdS_2, there is a singularity at $\rho^2 = \frac{\mathrm{h}-8\varkappa^2}{\varkappa^2(\mathrm{h}+8)}$ that is always present in the classical limit $\mathrm{h} \gg 1$ but absent for $\mathrm{h} < 8\varkappa^2$.

Written in the conformally-flat coordinates as in (2.30), the quantum-corrected metric (2.31) is

$$ds^2 = (\tfrac{1}{4}\mathrm{h} + 1 + \varkappa^2)\Big[\frac{dp^2+dq^2}{p^2+\varkappa^2 q^2} - \frac{\varkappa^2}{2\mathrm{h}}\,\frac{\big[d(p^2+q^2)\big]^2}{(p^2+\varkappa^2 q^2)^2(\frac{1}{4} + \frac{2\varkappa^2}{\mathrm{h}}\frac{p^2+q^2}{p^2+\varkappa^2 q^2})}\Big] \ . \tag{2.32}$$

Expanding the metric (2.32) to first subleading order in small h^{-1} one obtains

$$ds^2 = \tfrac{1}{4}\mathrm{h}\,\frac{dp^2+dq^2}{p^2+\varkappa^2 q^2} + (1+\varkappa^2)\frac{dp^2+dq^2}{p^2+\varkappa^2 q^2} - \frac{\varkappa^2}{2}\,\frac{\big[d(p^2+q^2)\big]^2}{(p^2+\varkappa^2 q^2)^2} + \mathcal{O}(\mathrm{h}^{-1}) \ . \tag{2.33}$$

2.3 $\boldsymbol{\lambda}$-model

Let us now turn to the λ-deformed σ-models. As we reviewed in Section 1.4.10, the λ-model may be constructed by starting with the G/G gauged WZW action and adding a deformation quadratic in the gauge field, breaking some or all of the gauge symmetry,

$$\begin{aligned}\mathcal{L} = k\,\Big(\mathcal{L}_G(g) + \mathrm{Tr}\big[&g^{-1}\partial_+ g A_- - A_+\partial_- g g^{-1} \\ &+ g^{-1}A_+ g A_- - A_+A_- + (\lambda^{-1}-1)A_+ P A_-\big]\Big) \ .\end{aligned} \tag{2.34}$$

One obtains a σ-model by integrating out the gauge field $A_\pm$ (here $\mathcal{L}_G$ is the WZW Lagrangian, see Section 1.4.7)

$$\mathcal{L} = k\,\Big(\mathcal{L}_G(g) + \mathrm{Tr}[J_+\, M^{-1}\, K_-]\Big) \ , \qquad M = \mathrm{Ad}_g - I + (1-\lambda^{-1})P \ . \tag{2.35}$$

Here the integration over the gauge field was done classically so the resulting σ-model (2.35) should only be understood as a classical approximation that will be subject to quantum corrections (cf. the classical gWZW geometry in eq. (2.4)). In what follows we shall construct such corrections in a manner consistent with 2-loop renormalizability in the simplest examples based on $\frac{SU(2)}{U(1)}$ and $SU(2)$.

2.3.1 $SU(2)/U(1)$ example

First let us consider the λ-model with 2d target space based on the symmetric space $\frac{G}{H} = \frac{SU(2)}{U(1)}$. We fix the $U(1)$ gauge symmetry (generated by σ_2) by choosing the following parametrization of the coset element

$$g = \exp(i\alpha\sigma_3)\exp(i\beta\sigma_2) \ , \qquad \cos\alpha = \sqrt{p^2+q^2} \ , \qquad \tan\beta = \tfrac{p}{q} \ . \tag{2.36}$$

The λ-model (2.35) then becomes

$$\mathcal{L} = \frac{k}{1-p^2-q^2}\big(\kappa\,\partial_+ p\partial_- p + \kappa^{-1}\partial_+ q\partial_- q\big) \ , \qquad \kappa \equiv \frac{1-\lambda}{1+\lambda} \ . \tag{2.37}$$

The model (2.37) again admits several useful limits. In general, in a "boost" limit that infinitely rescales the Cartan directions, the λ-model gives the T-dual of the η-model on an analytically continued coset [104] (equivalent to the Poisson-Lie duality relating the η- and λ-models discussed in Section 1.4.9). In this special 2d case, the η-deformed S^2 model is self T-dual (see footnote 25) so the T-duality may be skipped: infinitely rescaling the coordinates,[30]

$$(p,q) \to \gamma(p,\kappa q) \ , \qquad \gamma\to\infty \tag{2.38}$$

the metric becomes equivalent [104] to the η-model metric (2.30) on the analytically continued coset $\frac{SU(1,1)}{U(1)}$. Changing coordinates $(p,q) \to (r,\phi) = (q/p, \frac{1}{2}\log(p^2+q^2))$ and analytically continuing $(r,\phi)\to(ir,i\phi)$, we obtain again the η-model (2.6) on $\frac{SU(2)}{U(1)}$, with the couplings related by

$$\mathrm{h} = 4k\kappa \ , \qquad \varkappa = i\kappa \ . \tag{2.39}$$

The limit $\kappa = 1$ (or $\lambda = 0$) corresponds to the classical metric of the $\frac{SU(2)}{U(1)}$ gWZW model:

$$ds^2 = \frac{k}{1-p^2-q^2}(dp^2+dq^2) = k(d\alpha^2+\cot^2\alpha\, d\beta^2) \ , \qquad (p,q) = \cos\alpha(\sin\beta,\cos\beta) \ . \tag{2.40}$$

As discussed in Section 2.1, the gWZW metric (2.40) is known to be subject to quantum corrections at each order. Thus it is already clear that the λ-deformation (2.36) of this model should be subject to non-trivial quantum corrections.

The λ-model is known to be 1-loop renormalizable with only κ running [52, 53] (eq. (1.86) with $c_{SU(2)} = 2$ and $\kappa \equiv \frac{1-\lambda}{1+\lambda}$):

$$\frac{d}{dt}k = 0 + \mathcal{O}(k^{-1}) \ , \qquad \frac{d}{dt}\kappa = -k^{-1}(1-\kappa^2) + \mathcal{O}(k^{-2}) \ . \tag{2.41}$$

[30] This limit takes the coordinates (p,q) out of their proper coordinate range $p^2+q^2<1$ in (2.37). This may be avoided by instead considering the λ-model on the analytically continued coset $\frac{SU(1,1)}{U(1)}$.

The main observation here is that one may ensure the 2-loop renormalizability of this model by modifying the classical action (2.37) with a particular quantum correction. A natural guess is that the necessary counterterms originate from a determinant of integration over the 2d gauge field, which appears quadratically in (2.34). Such integrals are not well-defined [182], with the resulting local terms depending on a choice of regularization. One particular prescription is to write the gauge field in terms of two scalars, $A = du + d * v$ and integrate over u, v with measures M [174, 182] (see also [183]), obtaining the result

$$\int [dA]\, e^{i\int d^2\xi\,\sqrt{-h}\,(-M\,A^\mu A_\mu)} = \exp\Big[\frac{i}{4\pi}\int d^2\xi\,\sqrt{-h}\,\big(\tfrac{1}{2}(\partial\log M)^2 + \tfrac{1}{2}R^{(2)}\log M\big)\Big]\ . \qquad (2.42)$$

For generality we assumed a curved 2d background with curvature $R^{(2)}$, and we have ignored a trivial M-independent factor in (2.42). The $R^{(2)}\log M$ term in (2.42) is related to the familiar dilaton shift found in [74] (see also [184]), and vanishes on a flat 2d background.

However, the $(\partial\log M)^2$ term remains non-trivial on a flat 2d background. In this case $M = M(g)$ is the matrix that appears in the part of the λ-model action (2.34) that is quadratic in the gauge field, i.e. $\mathcal{L} = \ldots + \mathrm{Tr}A_+M(g)A_-$. It generally depends on the group element g, taking the the form $M \propto 1+(\lambda^{-1}-1)P-\mathrm{Ad}_g$ (written as an operator $\mathrm{Lie}(G)\to\mathrm{Lie}(G)$). Hence the prescription (2.42) suggests the following correction to the classical Lagrangian (2.37) of the $\frac{SU(2)}{U(1)}$ λ-model,[31]

$$\begin{aligned}&\Delta\mathcal{L} = \tfrac{1}{2}(\partial\log\det M)^2 = \tfrac{1}{2}\big[\partial\log(1-p^2-q^2)\big]^2\ ,\\ &\Delta(ds^2) = -\tfrac{1}{2}\big[d\log(1-p^2-q^2)\big]^2\ .\end{aligned} \qquad (2.43)$$

Somewhat surprisingly, in this case the 1-loop correction (2.43) happens to be independent of the deformation parameter λ.

This leads to a 1-loop corrected metric for the λ-model (2.37) given by (cf. (2.33))

$$ds^2 = k\,\frac{\kappa\,dp^2+\kappa^{-1}dq^2}{1-p^2-q^2} - \frac{1}{2}\Big[\frac{d(1-p^2-q^2)}{1-p^2-q^2}\Big]^2 + \mathcal{O}(k^{-1})\ . \qquad (2.44)$$

One can then check directly that the metric (2.44) indeed solves the 2-loop RG equations (2.13) with (recall that $\kappa \equiv \frac{1-\lambda}{1+\lambda}$)

$$\frac{d}{dt}k = 0 + k^{-1}(\kappa^{-1}-\kappa)^2 + \mathcal{O}(k^{-2})\ , \qquad \frac{d}{dt}\kappa = k^{-1}(1-\kappa^2) + 0 + \mathcal{O}(k^{-3})\ , \qquad (2.45)$$

[31] Recall that the Lagrangian $\mathcal{L} = \mathrm{G}_{mn}\partial_+x^m\partial_-x^n = -\mathrm{G}_{mn}\partial_\mu x^m\partial^\mu x^n$ has a minus sign relative to the target space metric when written in covariant 2d notation. This explains the relative minus sign between the two lines in (2.43).

and the components of the diffeomorphism vector ξ_m given by

$$\xi_p = \frac{2p}{1-p^2-q^2}\big[1+\frac{1}{k}\frac{\kappa(p^2-1)-\kappa^{-1}p^2}{1-p^2-q^2}\big] ,$$
$$\xi_q = \frac{2q}{1-p^2-q^2}\big[1+\frac{1}{k}\frac{\kappa^{-1}(q^2-1)-\kappa q^2)}{1-p^2-q^2}\big] . \qquad (2.46)$$

The β-function for the deformation parameter κ does not receive a 2-loop correction. Surprisingly, the parameter k (which has the interpretation of the level in the gWZW model) starts running at 2-loop order. Still, there is an RG invariant (cf. (2.17))

$$\frac{d}{dt}\mathrm{k} = 0 , \qquad \mathrm{k} \equiv k + (\kappa^{-1}+\kappa) + \mathcal{O}(k^{-1}) . \qquad (2.47)$$

This suggests that it is k rather than k that should be identified with level of the gWZW model in (2.34). However, in this 2d case the WZ term vanishes upon gauge fixing (any B-field is a total derivative in 2d) so the question of which is the WZ level becomes somewhat meaningless.

These results are consistent with the limits of the λ-model discussed above. Setting (cf. (2.36),(2.40))

$$(p,q) = (1+k^{-1})\cos\alpha(\sin\beta,\cos\beta) \qquad (2.48)$$

in (2.44) and taking the gWZW limit, $\kappa = 1$, we find

$$ds^2 = k(d\alpha^2+\cot^2\alpha\, d\beta^2) + 2(d\alpha^2+\frac{\cot^2\alpha}{\sin^2\alpha}\, d\beta^2) + \mathcal{O}(k^{-1}) , \qquad (2.49)$$

which precisely matches the large k expansion of the exact gWZW metric (cf. (2.4)), i.e.

$$ds^2 = (k+2)\Big(d\alpha^2+\frac{\cot^2\alpha}{1-\frac{2}{k}\cot^2\alpha}\, d\beta^2\Big) . \qquad (2.50)$$

Note that in this limit the RG invariant in (2.47) becomes $\mathrm{k} = k+2$, i.e. the shifted level of the $SU(2)/U(1)$ gWZW model.[32]

In the η-model limit, the corrected λ-model metric (2.44) reduces to the leading correction in the η-model metric (2.11) if one modifies the scaling limit (2.38) and coupling relation (2.39) by certain quantum corrections,

$$p \to \gamma p , \qquad q \to -i\gamma\varkappa\big[1-4\mathrm{h}^{-1}(1+\varkappa^2)\big]\, q , \qquad \gamma\to\infty , \qquad (2.51)$$
$$\kappa = -i\varkappa\big[1-8\mathrm{h}^{-1}(1+\varkappa^2)\big] , \qquad \tfrac{1}{4}\mathrm{h}-1+\varkappa^2 = -ik\varkappa , \qquad (2.52)$$

followed by again making the same change of variables $(r,\phi) = (q/p, \frac{1}{2}\log(p^2+q^2))$ and contin-

[32] In the $\kappa = 1$ limit the diffeomorphism vector (2.46) becomes a gradient of the exact dilaton of the gWZW model: $\xi_n = 2\partial_n\Phi$, $\Phi = -\log(1-p^2-q^2) - \frac{1}{2k}\frac{1}{1-p^2-q^2} + \mathcal{O}(k^{-2})$ (cf. footnote 23).

uing $(r, \phi) \to (ir, i\phi)$. In this limit the RG equations (2.45) reproduce the corresponding ones for the η-model couplings $\varkappa$ and h in (2.15),(2.16) with the RG invariant (2.47) being mapped to an analytic continuation of the η-model RG invariant (2.17),

$$\mathrm{k} = k + (\kappa^{-1} + \kappa) + \mathcal{O}(k^{-1}) = i\frac{\mathrm{h}}{4\varkappa} + \mathcal{O}(\mathrm{h}^{-1}) = i\nu + \mathcal{O}(\nu^{-1}) \ . \tag{2.53}$$

This may be viewed as a hint that a quantum deformation parameter that appears in the corresponding exact S-matrix should be $\mathrm{q} = e^{-\frac{i}{\mathrm{k}}} = e^{-\frac{1}{\nu}}$ (cf. [104]), which becomes $\mathrm{q} = e^{-\frac{i}{k+2}}$ at the gWZW point $\kappa = 1$.

2.3.2 $SU(2)$ example

Now let us turn to the λ-model for the group $SU(2)$ [185, 111]. Parametrizing the group element as

$$g = \exp\big[- i \arcsin\alpha \big(\cos\beta\, \sigma_2 + \sin\beta(\cos\gamma\, \sigma_3 - \sin\gamma\, \sigma_1)\big)\big] \ , \tag{2.54}$$

we obtain from (2.34) the following 3-dimensional classical σ-model metric and B-field [111][33]

$$\begin{aligned} (ds^2)^{(0)} &= k\big[\frac{d\alpha^2}{\kappa(1-\alpha^2)} + \frac{\kappa\alpha^2}{\Delta}(d\beta^2 + \sin^2\beta d\gamma^2)\big] \ , \qquad \Delta(\alpha) \equiv \kappa^2 + (1-\kappa^2)\alpha^2 \ , \\ \mathrm{B}^{(0)} &= k\big(\arcsin\alpha - \frac{\kappa^2\alpha\sqrt{1-\alpha^2}}{\Delta}\big)\sin\beta\, d\beta\wedge d\gamma \ , \\ \mathrm{H}^{(0)} &= d\mathrm{B}^{(0)} = \frac{k\alpha^2}{\sqrt{1-\alpha^2}\Delta^2}\big[2\kappa^2 + (1-\kappa^2)\Delta\big]\sin\beta\, d\alpha\wedge d\beta\wedge d\gamma \ . \end{aligned} \tag{2.55}$$

As discussed in Section 1.4.10, λ-models are known to be 1-loop renormalizable but, again, one can check that the classical background (2.55) does not solve the 2-loop RG equation (1.9) (with only k, κ running) in *any* covariant subtraction scheme and with *any* choice of diffeomorphism vector. (Here due to the $SU(2)$ global symmetry acting on β, γ, the diffeomorphism vector must be of the form $\xi^\alpha = \xi^\alpha(\alpha)$, $\xi^{\beta,\gamma} = 0$ and the B-gauge transformation must be trivial, $\partial_{[m}\lambda_{n]} = 0$.)

However, we observe that it is possible to restore the 2-loop renormalizability of the model

[33] Note that, in these coordinates, WZ term $k\mathcal{L}_{\mathrm{WZ}}(g)$ in (2.34) contributes $k(\arcsin\alpha - \alpha\sqrt{1-\alpha^2})\sin\beta\, d\beta\wedge d\gamma$ to the B-field.

by adding special quantum $(1/k)$ counterterms to this classical background,

$$
\begin{aligned}
ds^2 &= (ds^2)^{(0)} + \frac{2(1-\kappa^2)^2\alpha^4}{\kappa^2(1-\alpha^2)\Delta^2} d\alpha^2 \ , \\
\mathrm{B} &= \frac{\mathrm{k}}{k}\mathrm{B}^{(0)} - 2\big(\arctan\frac{\alpha}{\kappa\sqrt{1-\alpha^2}} - \frac{\kappa\alpha\sqrt{1-\alpha^2}}{\Delta}\big)\sin\beta\, d\beta\wedge d\gamma \ , \\
\mathrm{H} &= \frac{\mathrm{k}}{k}\mathrm{H}^{(0)} - \frac{4\kappa\alpha^2}{\sqrt{1-\alpha^2}\Delta^2}\sin\beta\, d\alpha\wedge d\beta\wedge d\gamma \ , \qquad\qquad \mathrm{k} \equiv k + \frac{4+(1+\kappa^2)^2}{4\kappa} \ .
\end{aligned}
\tag{2.56}
$$

This corrected background (2.56) indeed solves the 2-loop RG equation (1.9) with[34]

$$
\frac{d}{dt}k = 0 + \frac{(1-\kappa^2)^3(5+3\kappa^2)}{8k\kappa^2} \ , \qquad\qquad \frac{d}{dt}\kappa = \frac{(1-\kappa^2)^2}{2k}\Big[1 - \frac{(1-\kappa^2)^2}{k\kappa}\Big] \ . \tag{2.57}
$$

The only non-zero component of the diffeomorphism vector ξ^m is

$$
\xi^\alpha = \frac{\alpha(1-\alpha^2)(1-\kappa^2)}{k\Delta}\Big[\kappa + \frac{2\alpha^2\kappa^2(1-\kappa^2) - (3-2\kappa^2+\kappa^4)\Delta^2}{k\Delta^2}\Big] \ , \tag{2.58}
$$

and the B-gauge transformation in (1.9) vanishes, $\lambda_m = 0$.

Note that the coupling k defined in (2.56) does not run at 2-loop order,

$$
\frac{d}{dt}\mathrm{k} = 0 + \frac{1}{\mathrm{k}}\times 0 + \dots \ . \tag{2.59}
$$

This RG invariant k is the coefficient of $\arcsin\alpha$ (present in B_0 in (2.55)) in the 1-loop corrected background (2.56). Choosing it to be integer-valued removes the global ambiguities arising from the $\arcsin\alpha$ term. Furthermore, given that under a large transformation,[35]

$$
\delta(\arcsin\alpha) = \mathrm{sign}\,\kappa\ \delta(\arctan\frac{\alpha}{\kappa\sqrt{1-\alpha^2}}) \ , \tag{2.60}
$$

and that the coefficient of $\arctan\frac{\alpha}{\kappa\sqrt{1-\alpha^2}}$ in (2.56) is integer-valued, the arctan term does not lead to any additional ambiguities. The quantization of the flux,

$$
\frac{1}{4\pi^2}\int \mathrm{H}^{(0)} = k \ , \qquad\qquad \frac{1}{4\pi^2}\int \mathrm{H} = \mathrm{k} - 2 \ , \tag{2.61}
$$

also supports the identification of k as integer-valued.

In the case of the $\frac{SU(2)}{U(1)}$ λ-model it was possible to write the 1-loop corrections (2.43) in terms of a simple counterterm $\Delta\mathcal{L} = -\frac{1}{2}(\partial\log\det M)^2 = -\frac{1}{2}(\mathrm{Tr}[M^{-1}\partial M])^2$ in terms of the matrix M as defined in (2.35). For this $SU(2)$ case, we find however that this particular counterterm is not sufficient to restore 2-loop renormalizability (in any covariant subtraction scheme) — i.e. the required corrections (2.56) cannot be written in this form even after a covariant scheme change.

[34] The 1-loop terms in (2.57) match the general result (1.86) with $c_{_{SU(2)}} = 2$ and $\kappa \equiv \frac{1-\lambda}{1+\lambda}$.

[35] Note that $\arctan\frac{\alpha}{\kappa\sqrt{1-\alpha^2}} = \mathrm{sign}\,\kappa\ \arcsin\frac{\alpha}{\sqrt{\Delta}}$ and $\alpha = \frac{\alpha}{\sqrt{\Delta}}$ for $\alpha = 0, \pm 1$.

Instead, let us also include similar counterterms (with coefficients c_i) built out of the quantity $M^{-1}\partial M$. In addition, we may also include counterterms (with coefficients d_i) proportional to the terms present in the classical Lagrangian (2.35) (and its image under parity). As a result, we are led to the following ansatz[36]

$$\begin{aligned}\Delta\mathcal{L} = {} & c_1(\partial_+ \log\det M)(\partial_- \log\det M) + c_2\mathcal{L}_{\rm PCM}(M) + c_3\mathcal{L}_{\rm WZ}(M) \\ & + d_1\mathcal{L}_{\rm PCM}(g) + d_2\mathcal{L}_{\rm WZ}(g) + d_3{\rm Tr}[J_+M^{-1}K_-] + d_4{\rm Tr}[J_-M^{-1}K_+] \ .\end{aligned} \tag{2.62}$$

We find that this matches the required 1-loop corrected background (2.56) provided the constants c_i and d_i take the following values

$$\begin{aligned} & c_1 = -\tfrac{3+\kappa}{2(1-\kappa)} \ , \qquad c_2 = -\tfrac{2(1+\kappa)}{1-\kappa} \ , \qquad c_3 = 1 \ , \\ & d_1 = 0 \ , \quad d_2 = {\rm k} - k - 2 \ , \quad d_3 = \tfrac{1}{2}\big[{\rm k} - k + (1+\kappa^{-1})\big] \ , \quad d_4 = -\tfrac{1}{2}\big[{\rm k} - k - 3(1+\kappa^{-1})\big] \ . \end{aligned} \tag{2.63}$$

Combining the quantum counterterms (2.62) with the classical Lagrangian (2.35) allows us to represent the σ-model corresponding to the 1-loop corrected geometry (2.56) in the form

$$\begin{aligned} \mathcal{L} = {} & k\,\mathcal{L}_{\rm PCM}(g) + ({\rm k}-2)\mathcal{L}_{\rm WZ}(g) \\ & + \tfrac{1}{2}{\rm Tr}\Big[\big({\rm k} + k + (1+\kappa^{-1})\big)J_+M^{-1}K_- - \big({\rm k} - k - 3(1+\kappa^{-1})\big)J_-M^{-1}K_+\Big] \\ & - \tfrac{3+\kappa}{2(1-\kappa)}(\partial_+\log\det M)(\partial_-\log\det M) - \tfrac{2(1+\kappa)}{1-\kappa}{\rm PCM}(M) + {\rm WZ}(M) \\ M(g) = {} & \big(\tfrac{1-\kappa}{1+\kappa}\big)^2 - {\rm Ad}_g \ . \end{aligned} \tag{2.64}$$

We note that coefficients of the WZ terms in (2.64) are RG invariants, ${\rm k}-2$ (see eq. (2.59)) and 1. This is consistent with usual expectations and suggests that the representation (2.64) is a natural one.

In the WZW limit $\kappa \to 1$ when the RG invariant k in (2.56) reduces to the usual shifted WZ level

$${\rm k}\big|_{\kappa=1} = k + 2 \ , \tag{2.65}$$

the corrections to the metric and B-field in (2.56) vanish, so that the expression in (2.64) reduces to the classical WZW Lagrangian $k\big[\mathcal{L}_{\rm PCM}(g) + \mathcal{L}_{\rm WZ}(g)\big]$. Indeed, even though the coefficients $\frac{3+\kappa}{2(1-\kappa)}$ and $\frac{2(1+\kappa)}{1-\kappa}$ in (2.56) blow up in the limit $\kappa \to 1$, the corresponding M-dependent expressions vanish faster and all M-dependence disappears in the $\kappa = 1$ limit.

[36] In (2.62) and below, $\mathcal{L}_{\rm PCM}(a) \equiv -\frac{1}{2}{\rm Tr}[a^{-1}\partial_+ a\, a^{-1}\partial_- a]$ and $\mathcal{L}_{\rm WZ}$ denote the principal chiral model and Wess-Zumino Lagrangian terms (see Section 1.4.6).

η-model limit

Like the $\frac{SU(2)}{U(1)}$ example discussed above, the $SU(2)$ λ-model (2.55) admits a formal "boost" scaling limit,

$$\alpha \to \sin(\alpha + i\zeta) \ , \qquad\qquad \zeta \to \infty \ , \tag{2.66}$$

when it becomes (dropping a closed 2-form contribution to the B-field)

$$\widetilde{ds^2} = k\Big[\frac{1}{\kappa} d\alpha^2 + \frac{\kappa}{1-\kappa^2}(d\beta^2 + \sin^2\beta d\gamma^2)\Big] \ , \qquad \widetilde{\mathrm{B}} = k\cos\beta\, d\alpha \wedge d\gamma \ . \tag{2.67}$$

This geometry is related by abelian T-duality (in the α direction) to the squashed 3-sphere [186],

$$ds^2 = k\Big[\kappa(d\tilde{\alpha} - \cos\beta d\gamma)^2 + \frac{\kappa}{1-\kappa^2}(d\beta^2 + \sin^2\beta d\gamma^2)\Big] \ , \qquad \mathrm{B} = 0 \ , \tag{2.68}$$

which in turn has the interpretation as the η-deformation (1.68) of the $SU(2)$ PCM (cf. [187]). The squashed 3-sphere geometry (2.68) is an anomalous case of an η-model that is automatically renormalizable at 2-loops without needing quantum corrections (this statement was checked in App. B of [3] — and more generally for "squashed" models interpolating between a PCM and a symmetric space). As we shall now demonstrate, the quantum corrections derived above for the $SU(2)$ λ-model are consistent with this fact.

Starting from the 1-loop corrected λ-model background (2.56) and taking the same scaling limit (2.66), we get a corrected version of (2.67),

$$\begin{aligned} \widetilde{ds^2} &= \big(\frac{k}{\kappa} + \frac{2}{\kappa^2}\big) d\alpha^2 + \frac{k\kappa}{1-\kappa^2}(d\beta^2 + \sin^2\beta d\gamma^2) \ , \\ \widetilde{\mathrm{B}} &= \big[k + \frac{4 + (1+\kappa^2)^2}{4\kappa}\big] \cos\beta d\alpha \wedge d\gamma \ . \end{aligned} \tag{2.69}$$

The background (2.69) solves the 2-loop RG equations (1.9) with the same β-functions (2.57) as the λ-model and vanishing diffeomorphism vector and B-gauge transformation ($\xi_m = \lambda_m = 0$). To extend the T-duality relation between (2.67) and (2.68) to the quantum level, we use the 1-loop corrections to the T-duality transformation laws of [176] (see also [177]), generalizing the quantum correction in (2.5).[37] As a result, the T-dual of (2.69) is found to be

$$\begin{aligned} ds^2 &= \big[k\kappa + \tfrac{1}{2}(1+\kappa^2)^2\big]\big(d\tilde{\alpha} - \cos\beta d\gamma\big)^2 + \big[\frac{k\kappa}{1-\kappa^2} + \tfrac{1}{2}(1-\kappa^2)\big]\big(d\beta^2 + \sin^2\beta d\gamma^2\big) \ , \\ \mathrm{B} &= 0 \ , \end{aligned} \tag{2.70}$$

where we have rescaled $\tilde{\alpha}$. Indeed this just the squashed 3-sphere geometry (2.68) with redefined parameters,

$$(k, \kappa) \to (\hat{k}, \hat{\kappa}) \ , \qquad k = \hat{k} - \frac{3 + \hat{\kappa}^4}{2\hat{\kappa}} \ , \qquad \kappa = \hat{\kappa} + \frac{1 - \hat{\kappa}^2}{\hat{k}} \ . \tag{2.71}$$

[37] We note that, in order to directly apply the results of [176], one first has to perform a scheme change, $(\mathrm{G} + \mathrm{B})_{\mu\nu} \to (\mathrm{G} + \mathrm{B})_{\mu\nu} + \frac{1}{4}\mathrm{H}^2_{\mu\nu}$.

Thus the fact that the quantum corrected λ-model (2.56) is stable under the 2-loop RG flow is perfectly consistent, in the η-model or "boost" limit, with the fact that the squashed 3-sphere is automatically stable without quantum deformation.

We see in this example that the known quantum corrections to the T-duality transformation rules are naturally consistent with the quantum deformations of integrable models for higher-loop renormalizability. As we shall discuss in the next section, similar quantum corrections are expected for non-abelian duality.

2.3.3 Non-abelian dual limit

In a particular limit,

$$\begin{aligned} &k \to \infty \ , \qquad \lambda = 1 - \tfrac{\mathrm{h}}{2k} \ , \qquad g = e^{-\frac{h}{2k}v} \ , \\ &v \in \mathrm{Lie}(G) \text{ and h fixed} \end{aligned} \tag{2.72}$$

the λ-model (2.34) becomes the non-abelian dual (NAD) of the PCM or symmetric space (see Section 1.4.5)

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{h}\,\mathrm{Tr}[\partial_+ v\,\frac{1}{P + \mathrm{ad}_v}\partial_- v] \ . \tag{2.73}$$

It was noted earlier in the literature [188, 189, 190, 191] that the NAD theories (2.73) appear not to be renormalizable beyond the 1-loop order, prompting doubt about their inequivalence with the PCM and symmetric space σ-models. In this section, we shall resolve this confusion by observing that the classical NAD models (2.73) must be supplemented with particular finite counterterms to maintain renormalizability beyond 1-loop order. We shall produce the necessary counterterms in the $\frac{SU(2)}{U(1)}$ and $SU(2)$ examples by taking the NAD limit of the corresponding λ-model corrections constructed above.

NAD of S^2 sigma model

For the $\frac{SU(2)}{U(1)}$ λ-model in the coordinates (2.37), the NAD limit amounts to

$$p = \kappa x \ , \qquad q = 1 - \kappa^2 y \ , \qquad k \to \infty \ , \quad \kappa \to 0 \ , \qquad \mathrm{h} \equiv 4k\kappa = \text{fixed} \ . \tag{2.74}$$

The resulting theory (2.73) is the NAD of the σ-model on S^2, with metric

$$ds^2 = \frac{\mathrm{h}}{4}\,\frac{dx^2 + dy^2}{2y - x^2} \ , \tag{2.75}$$

and trivial B-field since the target space is 2d. In the NAD limit (2.74), the leading quantum correction to the λ-model in (2.44) suggests a particular correction to the NAD metric (2.75),

$$ds^2 = \frac{\mathrm{h}}{4}\,\frac{dx^2+dy^2}{2y-x^2} - \frac{1}{2}\Big[\frac{d(2y-x^2)}{2y-x^2}\Big]^2 + \mathcal{O}(h^{-1})\ , \tag{2.76}$$

which indeed solves the 2-loop RG equation (2.13) with only h running,

$$\frac{d}{dt}\mathrm{h} = 4 + \frac{16}{\mathrm{h}} + \mathcal{O}(h^{-2})\ , \tag{2.77}$$

and matching the 2-loop β-function the dual undeformed $S^2 = \frac{SU(2)}{U(1)}$ σ-model (eq. (1.32) with $c_{_{SU(2)}} = 2$, $c_{_{U(1)}} = 0$). Note that, since the coordinate redefinition in the NAD limit (2.74) depends on $\kappa = \kappa(t)$, the diffeomorphism vector gets an additional contribution compared to (2.46) and its components are given by

$$\begin{aligned}\xi_x &= \frac{4x}{2y-x^2} - \frac{1}{\mathrm{h}}\frac{8x(1+3x^2-4y)}{(2y-x^2)^2} + \mathcal{O}(\mathrm{h}^{-2})\ ,\\ \xi_y &= \frac{2(2y-1)}{2y-x^2} + \frac{1}{\mathrm{h}}\frac{8(1+2x^2-2y)}{(2y-x^2)^2} + \mathcal{O}(\mathrm{h}^{-2})\ .\end{aligned} \tag{2.78}$$

If one takes a further scaling limit,

$$x \to \gamma\cos\chi\ , \qquad y \to \tfrac{1}{2}\gamma^2 + \gamma y\ , \qquad \gamma\to\infty\ , \tag{2.79}$$

the NAD metric (2.75) degenerates to an abelian T-dual of S^2 (in the y direction),[38]

$$ds^2 = \tfrac{1}{4}\mathrm{h}\,(d\chi^2 + \frac{1}{\sin^2\chi}dy^2)\ . \tag{2.80}$$

Taking a particular 1-loop modification of the limit (2.79) (cf. (2.48)),

$$x \to \gamma(1+4\mathrm{h}^{-1})\cos\chi\ , \qquad y \to \tfrac{1}{2}\gamma^2 + \gamma(1+4\mathrm{h}^{-1})y\ , \qquad \gamma\to\infty\ , \tag{2.81}$$

and shifting the coupling $\mathrm{h}\to\mathrm{h}-8$ in the quantum-corrected NAD metric (2.76), we obtain a quantum-corrected T-dual metric (2.80),

$$ds^2 = \tfrac{1}{4}\mathrm{h}\,\big[d\chi^2 + \frac{1}{\sin^2\chi}\big(1+8\mathrm{h}^{-1}\cot^2\chi\big)dy^2\big] + \mathcal{O}(\mathrm{h}^{-1})\ , \tag{2.82}$$

which coincides with the known corrections (2.5) to the T-duality transformation rule [174] (with $\mathcal{M} = \sin^2\chi$ and $\mathrm{h}\to\frac{1}{4}\mathrm{h}$ so that (2.5) gives $\tilde{\mathcal{M}} = \frac{1}{\sin^2\chi}\big(1+2(\mathrm{h}/4)^{-1}\cot^2\chi\big)$ in this case).

[38] The T-dual metric (2.80) can also be found by directly taking the limit $k\to\infty$, $\kappa\to 0$ with $p=\cos\chi$, $q=\kappa y$ and $\mathrm{h}\equiv 4k\kappa$ fixed in the λ-model metric (2.37).

NAD of $SU(2)$ principal chiral model

In the case of the $SU(2)$ λ-model in the coordinates (2.55), the NAD limit (2.72) amounts to

$$\alpha \to \kappa\,\alpha\ , \qquad \mathrm{k}\to\infty\ , \quad \kappa\to 0\ , \qquad \mathrm{h}\equiv 4\mathrm{k}\kappa = \text{fixed} \tag{2.83}$$

where h and the new coordinate α are fixed. Taking this limit in the 1-loop-corrected λ-model background (2.56) suggests a particular quantum correction to the $SU(2)$ NAD model,

$$\begin{aligned} ds^2 &= \Big[\frac{\mathrm{h}}{4}+\frac{2\alpha^4}{(1+\alpha^2)^2}\Big]d\alpha^2+\frac{h}{4}\frac{\alpha^2}{(1+\alpha^2)}(d\beta^2+\sin^2\beta d\gamma^2)\ ,\\ \mathrm{B} &= \Big[\frac{\mathrm{h}+5}{4}\frac{\alpha^3}{(1+\alpha^2)}+\frac{2\alpha}{1+\alpha^2}-2\arctan\alpha\Big]\sin\beta d\beta\wedge d\gamma\ ,\\ \mathrm{H} &= \Big[\frac{\mathrm{h}+5}{4}\frac{\alpha^2(3+\alpha^2)}{4(1+\alpha^2)^2}-\frac{4\alpha^2}{(1+\alpha^2)^2}\Big]\sin\beta d\alpha\wedge d\beta\wedge d\gamma\ . \end{aligned} \tag{2.84}$$

Indeed it solves the 2-loop RG equations (1.9) with β-function,

$$\frac{d}{dt}\mathrm{h} = 2+2\mathrm{h}^{-1}\ , \tag{2.85}$$

matching that of the $SU(2)$ PCM (i.e. (1.20) with $c_{_{SU(2)}}=2$) and diffeomorphism vector,

$$\xi^\alpha = \frac{2\alpha}{\mathrm{h}(1+\alpha^2)}\Big[(1-\alpha^2)-\frac{4(5+5\alpha^2+3\alpha^4+\alpha^6)}{h(1+\alpha^2)^2}\Big]\ , \qquad \xi^{\beta,\gamma}=\lambda_{\alpha,\beta,\gamma}=0\ , \tag{2.86}$$

equalling the λ-model diffeomorphism vector (2.58) in the NAD limit (2.83) plus an extra contribution due to the RG-dependent rescaling of α.

We can also take the NAD limit (2.72) in the corrected Lagrangian written in the form (2.64), getting

$$\begin{aligned} \mathcal{L} &= -(\tfrac{1}{2}\mathrm{h}+\tfrac{9}{4})\mathrm{Tr}\big[\partial_+ v\,M^{-1}\partial_- v\big]-\tfrac{7}{4}\mathrm{Tr}\big[\partial_- v\,M^{-1}\partial_+ v\big]\\ &\quad -\tfrac{3}{2}(\partial_+\log\det M)(\partial_-\log\det M)-2\,\mathcal{L}_{\mathrm{PCM}}(M)+\mathcal{L}_{\mathrm{WZ}}(M)\ ,\\ M &\equiv 1+\mathrm{ad}_v\ . \end{aligned} \tag{2.87}$$

This indeed matches with (2.84) upon choosing the coordinates,

$$v = -\tfrac{i}{2}\alpha\big[\cos\beta\,\sigma_2+\sin\beta(\cos\gamma\,\sigma_3-\sin\gamma\,\sigma_1)\big]\ . \tag{2.88}$$

In a further limit of infinite shift of α,

$$\alpha\to\alpha+\ell\ , \qquad \ell\to\infty\ , \tag{2.89}$$

the NAD background (2.84) degenerates to an abelian T-dual of the PCM on $SU(2)$ (in the

α direction). The corrected NAD background then leads to a corrected T-dual background in this limit (dropping a trivial closed 2-form contribution to the B-field),

$$\widetilde{ds^2} = \tfrac{\mathrm{h}+8}{4}\, d\alpha^2 + \tfrac{\mathrm{h}}{4}\big(d\beta^2 + \sin^2\beta d\gamma^2\big)\ , \qquad \mathrm{B} = \tfrac{\mathrm{h}+5}{4}\, \cos\beta\, d\alpha \wedge d\gamma\ . \tag{2.90}$$

This background ($\mathbb{R}\times S^2$ supported by a B-field) is indeed correctly related to the $SU(2)$ PCM,

$$ds^2 = \mathrm{h}\big(d\theta^2 + \sin^2\theta\, d\psi^2 + \cos^2\theta\, d\chi^2\big)\ , \tag{2.91}$$

by (i) the 1-loop corrected T-duality transformation $\alpha \to \tilde\alpha$ of [176], (ii) shifting the coupling $\mathrm{h} \to \mathrm{h}-2$, and (iii) changing coordinates $\tilde\alpha = \psi + \chi$, $\gamma = \psi - \chi$ and $\beta = 2\theta$.

Summary

These findings extend the previous conclusions [75, 76] about the 1-loop quantum equivalence of the models related by the non-abelian duality to the 2-loop level. As for abelian T-duality, NAD (properly modified by quantum α' corrections) should also be a symmetry of the σ-model β-functions or the string effective action to all orders in α'.

Resolving earlier problems checking NAD at 2-loop level [188, 189, 190, 191], we found that the preservation of quantum equivalence requires a non-trivial 1-loop correction to the classical NAD model metric. In the $\frac{SU(2)}{U(1)}$ and $SU(2)$ cases, we explicitly constructed the corrections restoring 2-loop renormalizability, with the 2-loop β-functions matching the associated symmetric space and group models. This is a non-trivial result since these 2-loop β-functions are invariant under scheme changes (redefinitions of h) since the PCM/symmetric space and their NADs are *single-coupling* theories.

In further abelian T-dual limits, we obtained corrected versions of T-dual models matching the known quantum corrections to the T-duality rules (the simple case of (2.5) and the more general case of [176]). In these special cases, this approach serves as an alternate derivation of the leading correction to the T-duality transformation.

2.4 Discussion

We have demonstrated that, just as for gauged WZW models, the invariance of integrable σ-models under the two-loop (and higher) RG flow requires a specific quantum deformation of the classical Lagrangian (i.e. of the target space metric and B-field).

In particular, we proposed an exact metric for the η-deformation of the S^2 (and H^2) σ-model that solves the 3-loop RG flow equations and is consistent with the gWZW limit. We also found the leading-order deformation of the λ-model for $\frac{SU(2)}{U(1)}$, which solves the 2-loop RG equations and is consistent with the gWZW and η-model limits. For this example we identified the origin of the deformation as a finite counterterm resulting from the determinant of integrating over

the auxiliary 2d gauge field (the same determinant that leads to the correction of the dilaton term on a curved 2d background).

We also found the leading-order deformation of the λ-model for $SU(2)$ that solves the 2-loop RG equations, but in that case the necessary counterterm does not take the same simple and identifiable form. We checked the consistency of the corrections with the WZW model limit and the limit T-dual to the squashed 3-sphere (η-model).

In the non-abelian dual limit, the corrections for the λ-models generated predictions for leading corrections to the $\frac{SU(2)}{U(1)}$ and $SU(2)$ NAD models. These corrections were indeed found to restore the NAD models' 2-loop renormalizability, resolving the earlier problems [188, 189, 190, 191] in checking the consistency of non-abelian duality with the 2-loop RG flow.

The classical λ-model's $\mathbb{Z}_2$ transformation (1.80) is respected by the leading corrections found in the above examples. In explicit coordinates, this is seen as invariance of the counterterm in (2.44) (in the $\frac{SU(2)}{U(1)}$ case) under $k \to -k$, $\kappa \to -\kappa$, $p \to -p$; and invariance of the counterterms in (2.56) (in the $SU(2)$ case) under $k \to -k$, $\kappa \to -\kappa$, $\alpha \to -\alpha$. The resulting 2-loop β-functions (2.45) and (2.57) are thus invariant under $k \to -k$, $\kappa \to -\kappa$. As we shall see later in Chapter 4 (see eq. (4.33)), the preservation of this $\mathbb{Z}_2$ invariance requires a particular formulation of the quantum theory, and in generic subtraction schemes it would require a certain coupling redefinition to become manifest.

In the $\frac{SU(2)}{U(1)}$ case, the λ-model (2.37) is invariant under an additional $\mathbb{Z}_2$ transformation: $p \leftrightarrow q$, $\kappa \to \kappa^{-1}$ (i.e. $\lambda \to -\lambda$), associated with the subgroup $H = U(1)$ being abelian.[39] Indeed this $\mathbb{Z}_2$ is also respected by the counterterm in (2.44), and the resulting 2-loop β-functions (2.45) are then invariant under $\kappa \to \kappa^{-1}$.

In summary, while some models (e.g., on groups or symmetric spaces) have sufficient *manifest* global symmetry to ensure renormalizability to all orders, general integrable models may not have any manifest symmetries. However, the *hidden* integrable symmetries still seem to constrain the RG flow. This pattern is found to persist at higher-loop orders, and in such cases the hidden symmetries seem to require a particular choice of finite counterterms for their preservation at the quantum level.

A general open question is how to systematically construct the necessary finite counterterms. One natural approach would be to try to derive them by imposing "Ward identities" for the hidden integrable symmetries at the quantum level, i.e. requiring the choice of finite counterterms to be consistent with quantum integrability, and assuming that such counterterms would also be consistent with renormalizability. Another approach, assuming a UV fixed point and a known underlying quantum S-matrix, would be to try to systematically reconstruct the corrected σ-model from a dual massive model, as considered in [45, 49, 192].

[39] I thank B. Hoare for a discussion of this point.

An interesting open problem would be to construct an exact generalization of the 1-loop-corrected metric (2.44) for the $\frac{SU(2)}{U(1)}$ λ-model reducing in the appropriate limits to the exact gWZW metric (2.50) and the proposed exact η-model metric (2.31),(2.32). This is non-trivial given the lack of isometries in the λ-model metric and that the effective action approach used for the gWZW model [91] does not appear to apply directly to non-conformal theories.

It would be interesting to see whether the exact metric (2.11) of the $\frac{SU(2)}{U(1)}$ η-model, translated to the Hamiltonian framework, would have a natural interpretation in terms of a deformation of the classical integrable structure.

A case that was not yet considered is the higher-loop RG flow of the η-deformation of the PCM (rather than the symmetric space). For the simplest $SU(2)$ case, the η-model happens to coincide [92] with the σ-model on a 'squashed 3-sphere' [186] and is automatically stable due to manifest symmetries. This is because the η-deformation breaks the PCM's symmetry to $G_L \times U(1)^r$, i.e. $SU(2)_L \times U(1)_R$ in the $SU(2)$ case, with $U(1)_R$ happening to be a subgroup of $SU(2)$ such that $\frac{SU(2)}{U(1)}$ is a symmetric space. This implies that the $SU(2)$ η-model must coincide with the "squashed" model related to $\frac{SU(2)}{U(1)}$ — part of a general family built on symmetric spaces G/H considered in App. B of [3]. (Note that general examples from that family are renormalizable due to manifest $G_L \times H_R$ symmetry but *not* classically integrable: the $\frac{SU(2)}{U(1)}$ example is atypically integrable.) For η-models based on general groups, this coincidence does not happen, so the theory is not protected by manifest symmetries and one may expect to require additional finite local counterterms to get higher-loop renormalizability.

Finally, the η-model and the λ-model are, in general, related [193, 104, 106, 194, 107] by Poisson-Lie (PL) duality [195] (and analytic continuation). Since the scaling limits between the λ-models and (T-duals of) η-models revealed that our 2-loop results for the two classes of models are consistent, then the same observations made above for NAD are likely to apply also to PL duality. Namely, PL duality (with suitable quantum corrections) should be a symmetry not only at 1-loop order [50, 196], but also at higher loops. This expectation was recently confirmed by several papers [197] proposing 1-loop corrections to the classical PL duality rules (including NAD as a special case[40]). It would be interesting to check those results against the corrections (2.76),(2.87) found here for particular NAD models. In a related approach, it has recently been proposed [200] that there is a particular 2-loop "subtraction scheme"[41] in which PL symmetry becomes manifest. The λ- and η-models were claimed to be automatically renormalizable in that PL-scheme, suggesting the possibility to derive the corresponding quantum corrections in the standard scheme by reversing the scheme transformation.

[40] See also [198] on the leading quantum corrections to homogeneous Yang-Baxter deformations (equivalent to particular NAD transformations [133, 199]) in order for the deformation to preserve 2-loop conformal invariance.

[41] The PL-symmetric subtraction scheme of [200] is not a covariant subtraction scheme in the usual sense as it is related to the GB scheme (1.9) by a Lorentz-non-covariant transformation involving the spin connection.

Chapter 3

$G \times G$ and $G \times G/H$ sigma models

It is expected that classically integrable 2d σ-models should be stable under the renormalization group flow, the intuition being that hidden symmetries will constrain the RG evolution. Constraints on coupling constants required for integrability should thus be RG-invariant.

The aim of this chapter is to explore the connection between integrability and the RG flow on another class of examples — integrable $G \times G$ and $G \times G/H$ models that were derived from the affine Gaudin construction [160, 61]. These models may be viewed as generalizations of the PCM_k,

$$\mathcal{L}_{\mathrm{PCM}_k} = \mathrm{h}\, \mathcal{L}_{\mathrm{PCM}} + k\, \mathcal{L}_{\mathrm{WZ}}(g) \ , \qquad \mathcal{L}_{\mathrm{PCM}} = -\tfrac{1}{2}\, \mathrm{Tr}[J_+ J_-] \ , \qquad J \equiv g^{-1} dg \ , \quad g \in G \ , \tag{3.1}$$

i.e. the principal chiral model (with inverse coupling h) with the WZ term (with level k). The conformal WZW model is obtained at the special points $\mathrm{h} = \pm k$ (see Secrtions 1.4.6-1.4.7).

The PCM_k admits various integrable deformations (see, e.g., [92, 71, 72, 185, 94, 99] and Section 1.4.9 above), which have been interpreted [160] as particular cases of integrable affine Gaudin models. The affine Gaudin construction also produces natural generalizations of the PCM_k to integrable models on products of group spaces $G^N = G \times \ldots \times G$ [160].

Here we shall consider a subclass of such models defined by

$$\begin{aligned} &\mathcal{L} = -\tfrac{1}{2}\, \rho_{ij}\, \mathrm{Tr}[J_+^{(i)} J_-^{(j)}] + k_i\, \mathcal{L}_{\mathrm{WZ}}(g^{(i)}) \ , \\ &J^{(i)} \equiv {g^{(i)}}^{-1} dg^{(i)} \ , \qquad\qquad g^{(i)} \in G^N \ , \qquad i = 1, \ldots, N \ . \end{aligned} \tag{3.2}$$

We denote by $J^{(i)}$ the Maurer-Cartan 1-form corresponding to i-th copy of G and ρ_{ij} is a *constant* coupling matrix (summation over repeated i, j is assumed).

The PCM_k (3.1) corresponds to the special case $N = 1$ (with $\rho_{11} = \mathrm{h}$ and $k_1 = k$), and is integrable for any values of its couplings. However, for $N > 1$, the model (3.2) is classically integrable only for *special* couplings (ρ_{ij}, k_i) that correspond to the affine Gaudin models [160]. These are selected as the solutions of certain polynomial equations. We will focus on the first non-trivial case of $N = 2$, i.e. on $G \times G$ models.

As we shall find in Section 3.1, the classical integrability condition for $G \times G$ theories (3.2) is automatically stable under the *2-loop* RG flow in a particular subtraction scheme (extending the 1-loop results of [60]). Here the 2-loop stability is obtained without the need for any finite counterterms.

The model (3.2) is a special case of the 2d σ-model (1.6), whose 2-loop β-functions are given in the "GB scheme" in eq. (1.9) (see also Section 1.4.15 on subtraction schemes). This special 2-loop scheme effectively treats G_{mn} and B_{mn} as symmetrically as possible, with the respective β-functions being the symmetric and antisymmetric parts of a single tensor expression [166, 35] (see also [57, 58]). We note that in the context of this chapter, the diffeomorphism ($L_\xi(\mathrm{G}+\mathrm{B})_{mn}$) and B-gauge transformation ($\partial_{[m}\lambda_{n]}$) terms in (1.9) may be ignored since they automatically vanish in the examples considered below due to manifest global $G_L \times G_L$ symmetry.

Applied to the case of the PCM_k in (3.1), the 2-loop β-function in GB scheme (1.9) gives (here we set $\alpha' = 1$)[42]

$$\frac{d}{d\tau}\mathrm{h} = \mathrm{c}_{_G}\,(1 - \tfrac{k^2}{\mathrm{h}^2})\,\Big[1 + \tfrac{\mathrm{c}_G}{2\mathrm{h}}(1 - \tfrac{3\,k^2}{\mathrm{h}^2})\Big]\ , \qquad\qquad \frac{d}{d\tau}k = 0\ , \tag{3.3}$$

so that the position of the WZW fixed point $\mathrm{h} = \pm k$ remains unchanged at the 2-loop order. The 2-loop PCM_k β-function (3.3) was found in [166] using a scheme equivalent this one at the 2-loop level. To recall, part of the scheme freedom comes from the prescription of how one treats the antisymmetric 2d tensor $\varepsilon^{\mu\nu}$ appearing in the B-term in (1.6) in dimensional regularization. Ref. [166] used a 't Hooft-Veltman prescription of treating $\varepsilon^{\mu\nu}$ as effectively 2-dimensional. In [35] it was assumed that, in $d = 2 + \epsilon$ dimensions, $\varepsilon^{\mu\nu}\varepsilon_{\rho\sigma} = f(\epsilon)(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)$ where $f = 1 + f_1\epsilon + ...$, and then the GB scheme (1.9) corresponds to the choice $f_1 = -1$. As noted in [201], the scheme used in [166] is equivalent (at least at the 2-loop level) to $f(d) = \frac{1}{d-1} = 1 - \epsilon + ...$, i.e. to the choice $f_1 = -1$ [35] of the GB scheme.

The GB scheme is naturally "adapted" to the vicinity of the WZW conformal point: the derivative $\partial_{\mathrm{h}}\beta_{\mathrm{h}}\big|_{\mathrm{h}=k}$ of the β-function for h at the fixed point correctly reproduces [166] the anomalous dimension of the $\mathrm{Tr}(J_+J_-)$ operator (PCM Lagrangian) as computed [202] using the underlying infinite dimensional Kac-Moody symmetry of the WZW model. Thus this scheme is apparently consistent with the preservation of the KM symmetry in the vicinity of the conformal point.

It is then natural to expect that this scheme should also play a special role in a more general class of integrable models (3.2) containing WZW models as special limits,[43] and should facilitate preservation of the hidden integrable structure of these models at the quantum level. We will indeed see evidence for this below: the classical integrability conditions for the $G \times G$

[42] In this chapter we denote $\tau = \log\mu$ (instead of t) since we will use a coupling constant called t.

[43] Similar logic was recently used in [203] in the discussion of the 2-loop RG evolution of a "squashed" $SU(2)$ variant of PCM_k.

model (3.2) will be automatically preserved by the 2-loop RG evolution provided one uses the β-functions in the GB scheme (1.9).

We shall also study, in Section 3.2, a gauged analog of the models (3.2) defined on a coset space $G\times G/H$. This theory, which was recently derived from affine Gaudin models in [61], may be viewed as a generalization of the standard G/H symmetric space σ-model, also including WZ terms. For these $G \times G/H$ theories to be gauge invariant, the corresponding couplings must satisfy certain linear relations. In addition, for a gauge invariant model to be classically integrable, the couplings should further satisfy certain polynomial relations [61].

We will compute the RG flow for these integrable $G \times G/H$ theories, finding that they are stable under the 1-loop RG flow. However, at the 2-loop level, RG stability does not automatically arise and, in general, requires certain finite redefinitions of the couplings. These are equivalent to adding specific finite counterterms, which may be interpreted as required for preservation of integrability at the quantum level (analogous to what was observed on other examples in Chapter 2). There are still a few special cases, in particular the integrable $T^{1,1}$ model of [61], that are automatically stable under the 2-loop RG flow (see Section 3.2.2).

In Section 3.3 we shall present a new integrable σ-model with target space metric $T^{1,q} = SU(2) \times SU(2)/U(1)$ [204] and a particular B-field. The model admits as a special limit the conformal GMM model with unequal levels [205, 206]. Our central observation is that, in the case of the subgroup H in $G \times G/H$ being abelian, the gauge invariance conditions of [61] are too restrictive and there is also a second "branch" of gauge invariant theories. This allows a natural generalization of the integrable $T^{1,1}$ model of [61] to $T^{1,q}$ with a general parameter q. We demonstrate that the resulting $T^{1,q}$ model is classically integrable, admitting a Lax representation. We observe that the $T^{1,q}$ model is self-dual under T-duality in one isometry direction, and argue that this property forces it to be stable under the RG flow. We verify this fact explicitly by computing the corresponding 2-loop RG flow of the two coupling constants.

A few concluding remarks will be made in Section 3.4. Appendix C contains discussion of the integrability conditions for the G^N model (3.2) relevant to this chapter. The explicit formulae for the 2-loop β-functions of the $G \times G$ and $G \times G/H$ models used in this chapter, and their derivations, are presented in Appendix D.

This chapter is based on the paper [5] and we closely follow the presentation there.

3.1 $G \times G$ models

As mentioned above, the G^N model (3.2) is classically integrable for *special* values of its couplings (ρ_{ij}, k_i) satisfying certain polynomial relations, which originate from the affine Gaudin

construction [160]. For such values of the couplings the model admits a Lax connection of the form

$$L_+ = \alpha_i \, J_+^{(i)} \ , \qquad\qquad L_- = \beta_i \, J_-^{(i)} \ , \tag{3.4}$$

whose flatness condition, $F_{+-}(L) \equiv \partial_+ L_- - \partial_- L_+ + [L_+, L_-] = 0$, is equivalent to the equations of motion following from (3.2). Moreover, the affine Gaudin construction guarantees the existence of an infinite tower of conserved, commuting higher-spin charges [207], as reviewed above in Section 1.4.14.

Below we shall consider the simplest $N = 2$ case of the G^N model (3.2) for a simple Lie group G. We shall parametrize the 2×2 matrix ρ_{ij} in (3.2) in terms of the 4 components s, t, u, b as follows

$$\mathcal{L} = -\tfrac{1}{2} \begin{pmatrix} s & t+b \\ t-b & u \end{pmatrix}_{ij} \mathrm{Tr}[J_+^{(i)} J_-^{(j)}] + k_i \, \mathcal{L}_{\mathrm{wz}}(g^{(i)}) \ . \tag{3.5}$$

Then the affine Gaudin condition for integrability is the vanishing of a cubic polynomial [160, 60],

$$f(s,t,u,b,k_1,k_2) \equiv -t\,(s+t)\,(t+u) + b^2\,(s+t+u) + t\,k_1\,k_2 + b\,(u\,k_1 - s\,k_2) = 0 \ . \tag{3.6}$$

Let us note that the affine Gaudin conditions for integrability (e.g. (3.6) in the $N = 2$ case) are certainly sufficient for integrability. However, it is not a priori clear if they are necessary, since there could also be integrable theories of the form (3.2) that are unrelated to the affine Gaudin construction of [160]. In Appendix C we presented a check that the condition (3.6) is also *necessary* for the integrability of the $G \times G$ model (3.5), assuming the natural ansatz (3.4) for the corresponding Lax connection.

3.1.1 RG flow in $G \times G$ models

The general G^N model (3.2) has global $(G_L)^N \times G_R$ symmetry acting as

$$g^{(i)} \to u_L^{(i)} g^{(i)} u_R \ , \qquad\qquad (u_L^{(i)}, u_R) \in (G_L)^N \times G_R \ . \tag{3.7}$$

In fact, (3.2) is the most general 2-derivative local Lagrangian having this symmetry. This implies that only ρ_{ij} can run under the RG flow (with the WZ parameters k_i not renormalized as usual, since WZ terms do not have a local 2d representation that could be produced by the beta functions (1.9)).

Starting with the σ-model couplings $(\mathrm{G}_{mn}, \mathrm{B}_{mn})$ corresponding to the $G \times G$ model (3.5)

and computing the corresponding 2-loop β-functions in the GB scheme (1.9), we find

$$\begin{gathered}\frac{d}{d\tau}\rho_{ij} = \alpha'\,\beta^{(1)}_{ij} + \alpha'^2\,\beta^{(2)}_{ij} + \dots\ , \\ \beta^{(1)}_{ij} = \mathrm{c}_{_G}\,F^{(4)}_{ij}(s,t,u,b,k_1,k_2)\ , \qquad \beta^{(2)}_{ij} = \mathrm{c}^2_{_G}\,(su-t^2)^{-5}\,F^{(9)}_{ij}(s,t,u,b,k_1,k_2)\ ,\end{gathered} \tag{3.8}$$

where the matrices $F^{(4)}, F^{(9)}$ are homogeneous polynomials of degrees 4 and 9 in their arguments and $\mathrm{c}_{_G}$ is the dual Coxeter number of the group G, as in the PCM_k case in (3.3). The explicit expressions for $F^{(4)}, F^{(9)}$ are given in Appendix D.1 and also in an example below.

Remarkably, despite the complicated expressions for the β-functions, one is able to verify that the integrability condition (3.6) is, in fact, preserved by the 2-loop RG flow:

$$\frac{df}{d\tau}\Big|_{f=0} = \big(\alpha'\,\beta^{(1)}_{ij} + \alpha'^2\,\beta^{(2)}_{ij} + ...\big)\,\frac{\partial f}{\partial\rho_{ij}}\Big|_{f=0} = \alpha'\times 0 + \alpha'^2\times 0 + ...\ . \tag{3.9}$$

The vanishing of the 1-loop $\mathcal{O}(\alpha')$ term in (3.9) was already established in [60], and the vanishing of the 2-loop term is a new non-trivial result. Let us stress that this property of the integrability condition (3.6) not being deformed at the 2-loop level is specific to the GB scheme (1.9).

3.1.2 Example with $k_1 = k_2 = 0$

Let us consider the special case of the integrable $G\times G$ models (3.5),(3.6) with vanishing WZ levels, $k_1 = k_2 = 0$. In that case the integrability condition (3.6) implies that

$$b = b(s,t,u) \equiv \Big[\frac{t(t+s)(t+u)}{t+s+u}\Big]^{1/2}\ . \tag{3.10}$$

We thus obtain from (3.5) an integrable $G\times G$ model with 3 independent couplings s,t,u,

$$\mathcal{L} = -\tfrac{1}{2}\begin{pmatrix} s & t+b(s,t,u) \\ t-b(s,t,u) & u\end{pmatrix}_{ij}\mathrm{Tr}[J^{(i)}_+ J^{(j)}_-]\ . \tag{3.11}$$

Since k_i do not run, this special case of the model (3.5),(3.6) should also be stable under the RG flow, i.e. (3.11) should be renormalizable with only s,t,u running. Indeed, using for convenience the redefined couplings $(s,t,u)\to(x,y,z)$ with

$$x = s+t+u\ , \qquad y = \frac{s}{t}\ , \qquad z = \frac{u}{t}\ , \tag{3.12}$$

the 2-loop β-functions (3.8) become

$$\frac{d}{d\tau}x = 2\mathrm{c}_{_G} - \frac{\mathrm{c}_{_G}^2}{2x(yz-1)^3}\Big[16+32(y+z)+16(y^2+z^2)+88yz+68yz(y+z) \tag{3.13}$$
$$+12yz(y^2+z^2+5yz)+8y^2z^2(y+z)-y^2z^2(y+z)^2\Big]\ ,$$
$$\frac{d}{d\tau}y = F(x;y,z)\ , \qquad \frac{d}{d\tau}z = F(x;z,y)\ , \tag{3.14}$$
$$F(x;y,z) \equiv \frac{y(y+1)(y+2)}{(zy-1)^2}\Big(\frac{\mathrm{c}_{_G}}{x}\big[1-zy-3(z+1)^2\big] \tag{3.15}$$
$$-\frac{\mathrm{c}_{_G}^2}{2x^2}\frac{1}{(zy-1)^3}\Big[-z^6y^2-y^6(3zy-38z-44)+2z(y+1)(26zy+101y+58)$$
$$+20(y+1)^2-z^4(y(3y((y-14)y-100)-296)-38)$$
$$-z^3(y(y(y((y-4)y-178)-728)-708)-152)$$
$$+2z^2(y+1)(y(y(5y+89)+262)+105)\Big]\Big)\ .$$

The obvious symmetry between s and u in (3.11) is translated into the symmetry of the RG equations under $y \leftrightarrow z$.

The fact that these 2-loop β-functions are much simpler than the general (not necessarily integrable) case of (3.8) (see also Appendix D.1) suggests that a substantial simplification happens upon specifying the couplings to be at the integrable locus $f = 0$ in (3.6) (this was already observed at the 1-loop order in [60]).

3.2 $G \times G/H$ models

Let H be a subgroup of G such that G/H is a symmetric space (we assume that both G and H are simple real Lie groups). Then the Lagrangian for the gauged $G \times G/H$ model of [61] takes the form[44]

$$\mathcal{L} = -\tfrac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[P_+^{(i)}P_-^{(j)}] - \tfrac{1}{2}\,r_{ij}\,\mathrm{Tr}[B_+^{(i)}B_-^{(j)}] + k_i\,\mathcal{L}_{\mathrm{WZ}}(g^{(i)})\ , \tag{3.16}$$
$$P^{(i)} = P_{G/H}J^{(i)}\ , \qquad B^{(i)} = P_H J^{(i)}\ , \qquad J^{(i)} = {g^{(i)}}^{-1}dg^{(i)}\ . \tag{3.17}$$

Here $P_{G/H}$ and P_H are projectors to the corresponding parts of the algebra of G, and ρ_{ij}, r_{ij} are constant 2×2 matrices. The global symmetry consists of left multiplication $G_L \times G_L$, as well as the discrete $\mathbb{Z}_2$ corresponding to the symmetric space structure of G/H. The action for (3.16) is required to be gauge invariant under the local right action by an element of H (acting

[44] Our conventions in (3.16) are related to the ones of [61] by $r_{ij} \to 2\rho^{(0)}_{ij}$, $\rho_{ij} \to 2\rho^{(1)}_{ij}$ and the opposite sign for the WZ terms, i.e. $k_i \to -k_i$.

the same on both $g^{(i)}$)

$$g^{(i)} \to g^{(i)}\mathrm{w} \ , \qquad\qquad \mathrm{w}(\xi^+,\xi^-) \in H \ . \tag{3.18}$$

For general choices of G and H, gauge invariance imposes the linear constraints [61][45]

$$k_1 = -k_2 \equiv k \ , \qquad\qquad r_{ij} = \begin{pmatrix} r & -r-k \\ -r+k & r \end{pmatrix} \ . \tag{3.19}$$

The remaining free parameters of the gauge invariant model are then r, k and the 2×2 matrix ρ_{ij}.

Requiring integrability imposes further constraints which, as for the G^N models (3.2), can be obtained from the affine Gaudin construction. The following parametrization of the 6 constants r, k, ρ_{ij} in terms of 4 parameters K, x, ζ_+, ζ_- was shown in [61] to be sufficient for integrability

$$r = r_{11} = r_{22} = K\frac{\zeta_-^2-\zeta_+^2}{(1-x^2)^2} \ , \qquad\qquad r_{12} = 2K\frac{(1-\zeta_+^2)(x^2-\zeta_-^2)}{(1-x^2)^3} \ , \tag{3.20}$$

$$r_{21} = -2K\frac{(1-\zeta_-^2)(x^2-\zeta_+^2)}{(1-x^2)^3} \ , \tag{3.21}$$

$$\rho_{11} = K\frac{1-2\zeta_+^2+\zeta_-^2\zeta_+^2}{(1-x^2)^2} \ , \qquad\qquad \rho_{12} = x\, r_{12} = 2K\frac{x(1-\zeta_+^2)(x^2-\zeta_-^2)}{(1-x^2)^3} \ , \tag{3.22}$$

$$\rho_{21} = x^{-1} r_{21} = -2K\frac{(1-\zeta_-^2)(x^2-\zeta_+^2)}{x(1-x^2)^3} \ , \qquad\qquad \rho_{22} = K\frac{x^4-2\zeta_+^2x^2+\zeta_-^2\zeta_+^2}{x^2(1-x^2)^2} \ , \tag{3.23}$$

$$k = k_1 = -k_2 = -K\frac{2x^2+2\zeta_-^2\zeta_+^2-(1+x^2)(\zeta_-^2+\zeta_+^2)}{(1-x^2)^3} \ . \tag{3.24}$$

This parametrization is simply equivalent to the gauge invariance conditions (3.19) combined with the two extra polynomial integrability conditions

$$\begin{aligned} &f_1 \equiv r^2-k^2-\rho_{12}\rho_{21} = 0 \ , \\ &f_2 \equiv (r-k)^4\rho_{12} + (r-k)^2(r-k-2\rho_{11})(r-k+2\rho_{22})\rho_{21} \\ &\qquad\qquad - 2(r-k)(\rho_{11}+\rho_{22})\rho_{12}\rho_{21}^2 + (\rho_{12}+\rho_{21})\rho_{12}\rho_{21}^3 = 0 \ . \end{aligned} \tag{3.25}$$

Two simple solutions of these conditions are found by setting $r = k$ (i.e. $r_{21} = 0$ in (3.19)) and either $\rho_{21} = 0$ or $\rho_{12} = 0$.

We note that the integrable $G \times G/H$ models satisfying (3.16)(3.19),(3.16) were recently found to be obtainable [208] from 4d Chern-Simons theory [165].

[45] The special case of abelian H will be discussed below in Section 3.3.

3.2.1 RG flow in $G \times G/H$ models

The structure of the gauge invariant $G \times G/H$ action (3.16),(3.19) is protected by the right H gauge symmetry (3.18) and the global $G_L \times G_L$ and $\mathbb{Z}_2$ symmetry. This rules out all counterterms except those corresponding to renormalizations of the 6 couplings r, k, ρ_{ij} (of which k is not renormalized as usual). Let us parametrize ρ_{ij} as in (3.5),

$$\rho_{ij} = \begin{pmatrix} s & t+b \\ t-b & u \end{pmatrix} \ . \tag{3.26}$$

Computing the β-functions (1.9) corresponding to the σ-model couplings $(\mathrm{G}_{mn}, \mathrm{B}_{mn})$ for the model (3.16),(3.19),(3.26), we find for the 1-loop β-functions of the 5 running couplings

$$\frac{d}{d\tau}\mathrm{h}_p = \alpha' \beta^{(1)}_{\mathrm{h}_p} \ , \qquad\qquad \mathrm{h}_p \equiv (r, s, t, b, u) \ , \tag{3.27}$$

$$\begin{aligned} \beta^{(1)}_r = \frac{\mathrm{c}_G - \mathrm{c}_H}{(t^2-su)^2}\Big(& r^2s^2 - 2b^2t^2 - 2r^2t^2 + 2t^4 - 2b^2su - 2st^2u + r^2u^2 \\ & + 4bstk + 4btuk - s^2k^2 - 2t^2k^2 - u^2k^2\Big) + \mathrm{c}_H\big(1 - \tfrac{k^2}{r^2}\big) \ , \end{aligned} \tag{3.28}$$

$$\beta^{(1)}_s = \frac{\mathrm{c}_G}{r(t^2-su)}\big[b^2s - st^2 + r^2u + 2r(t^2-su) - 2btk + uk^2\big] \ , \tag{3.29}$$

$$\beta^{(1)}_t = \frac{\mathrm{c}_G}{r(t^2-su)}\big[- b^2t + b(s+u)k + t(r^2 - su - k^2)\big] \ , \tag{3.30}$$

$$\beta^{(1)}_b = \frac{\mathrm{c}_G}{r(t^2-su)}\big[- b(t^2+su) + t(s+u)k\big] \ , \tag{3.31}$$

$$\beta^{(1)}_u = \frac{\mathrm{c}_G}{r(t^2-su)}\big[r^2s + b^2u - t^2u + 2r(t^2-su) - 2btk + sk^2\big] \ . \tag{3.32}$$

We observe that the integrability conditions (3.25) are stable under the 1-loop RG flow (3.27)–(3.32),

$$\frac{df_a}{d\tau}\Big|_{f_1=f_2=0} = \alpha' \beta^{(1)}_{\mathrm{h}_p} \frac{\partial f_a}{\partial \mathrm{h}_p}\Big|_{f_1=f_2=0} + \mathcal{O}(\alpha'^2) = 0 + \mathcal{O}(\alpha'^2) \ , \qquad\qquad a = 1, 2 \ . \tag{3.33}$$

However, it turns out that (as for the examples discussed in Chapter 2) this property of RG stability does not automatically extend to the 2-loop order. Computing the 2-loop β-functions for the model (3.16),(3.19) in the GB scheme (1.9) (given explicitly in Appendix D.2), we find that the subleading correction to (3.33) is non-zero at general values of the couplings,

$$\beta^{(2)}_{\mathrm{h}_p} \frac{\partial f_a}{\partial \mathrm{h}_p}\Big|_{f_1=f_2=0} \neq 0 \ , \qquad\qquad a = 1, 2 \ . \tag{3.34}$$

Moreover, we checked that (3.34) is also non-vanishing in arbitrary covariant 2-loop subtraction schemes.[46]

As in other examples [2, 3], one may expect to restore the property of RG stability at the 2-loop order by adding certain finite quantum α'-corrections to the target space geometry. Because of the global and local symmetries, the only possible corrections would correspond to redefinitions $\mathrm{h}_p \to \bar{\mathrm{h}}_p$ of the couplings $\mathrm{h}_p = (r, s, t, b, u)$,

$$\mathrm{h}_p = \bar{\mathrm{h}}_p + \alpha' Q_p(\bar{\mathrm{h}}) + \dots \quad . \tag{3.35}$$

Such redefinitions may be interpreted as quantum corrections to the integrability conditions (3.25): if the original couplings h_p satisfied $f_a(\mathrm{h}) = 0$, then the corrected ones $\bar{\mathrm{h}}_p$ would satisfy a *corrected* version of the integrability conditions,

$$\bar{f}_a(\bar{\mathrm{h}}) = 0 \ , \qquad \bar{f}_a \equiv f_a + \alpha' \, Q_p \, \partial_{\mathrm{h}_p} f_a + \dots \quad . \tag{3.36}$$

3.2.2 Example: integrable deformation of GMM model on $G \times G/H$ and $T^{1,1}$ model

There are still special exceptional cases of the integrable $G\times G/H$ model (3.16),(3.19),(3.25) that are automatically stable under the 2-loop RG flow in the GB scheme. As we shall demonstrate below, one of them is the particular solution of the integrability conditions (3.25) that was studied in [61],

$$r = k \ , \qquad\qquad \rho_{12} = \rho_{21} = 0 \ , \quad \text{i.e.} \ \ t = b = 0 \ . \tag{3.37}$$

The Lagrangian of the corresponding theory (3.16),(3.19) is given by

$$\begin{aligned} \mathcal{L} = \ & -\tfrac{1}{2} \, \mathrm{Tr}\big[\mathrm{h} \, P_+ P_- + \widetilde{\mathrm{h}} \, \widetilde{P}_+ \widetilde{P}_-\big] - \tfrac{1}{2} k \, \mathrm{Tr}\big[B_+ B_- + \widetilde{B}_+ \widetilde{B}_- - 2 B_+ \widetilde{B}_-\big] \\ & + k \, \big[\mathcal{L}_{\mathrm{wz}}(g) - \mathcal{L}_{\mathrm{wz}}(\widetilde{g})\big] \ , \end{aligned} \tag{3.38}$$

where we have set[47]

$$\begin{aligned} & (g^{(1)}, g^{(2)}) \equiv (g, \widetilde{g}) \ , \quad \widetilde{P}_\pm = P_\pm(\widetilde{g}) \ , \quad \widetilde{B}_\pm = B_\pm(\widetilde{g}) \ , \\ & \mathrm{h} \equiv \rho_{11} = s \ , \quad \widetilde{\mathrm{h}} \equiv \rho_{22} = u \ . \end{aligned} \tag{3.39}$$

This is an integrable deformation of the special point $\mathrm{h} = \widetilde{\mathrm{h}} = k$ that corresponds to the conformal GMM model [205] on the homogeneous space $G \times G/H$ with equal levels.

Specializing the 1-loop and 2-loop β-functions in (3.27)–(3.32) and Appendix D.2 to this

[46] More precisely, we considered arbitrary subtraction schemes related to the GB scheme (1.9) by covariant redefinitions of G_{mn} and B_{mn} (see Section 1.4.15).

[47] Ref. [61] used the notation $(k, \mathrm{h}, \widetilde{\mathrm{h}}) \equiv (\lambda^2, \lambda_2^2, \lambda_1^2)$.

case, we find that the model (3.38) is automatically stable under 2-loop renormalization with only h and $\widetilde{\mathrm{h}}$ running,

$$\begin{aligned}
&\frac{d}{d\tau}\mathrm{h} = 2\mathrm{c}_{_G}\big(1-\tfrac{k}{\mathrm{h}}\big)\Big(1+\tfrac{1}{\mathrm{h}}\Big[2(\mathrm{c}_{_G}-\mathrm{c}_{_H})-(3\mathrm{c}_{_G}-2\mathrm{c}_{_H})\tfrac{k}{\mathrm{h}}\Big]\Big)\ ,\\
&\frac{d}{d\tau}\widetilde{\mathrm{h}} = 2\mathrm{c}_{_G}(1-\tfrac{k}{\widetilde{\mathrm{h}}})\Big(1+\tfrac{1}{\widetilde{\mathrm{h}}}\Big[2(\mathrm{c}_{_G}-\mathrm{c}_{_H})-(3\mathrm{c}_{_G}-2\mathrm{c}_{_H})\tfrac{k}{\widetilde{\mathrm{h}}}\Big]\Big)\ , \qquad \frac{d}{d\tau}k=0\ .
\end{aligned} \tag{3.40}$$

Remarkably, the RG evolution of h and $\widetilde{\mathrm{h}}$ is decoupled. Note that the structure of their β-functions is similar to the one in the PCM_k case (3.3). As expected, the GMM model $\mathrm{h}=\widetilde{\mathrm{h}}=k$ is a fixed point.

Let us consider the simplest example of this theory (3.38) with $G=SU(2)$ and $H=U(1)$ and choose the parametrization

$$g = e^{\phi_1 T_1} e^{\theta_1 T_2} e^{\psi T_3}\ , \qquad\qquad \widetilde{g} = e^{-\phi_2 T_1} e^{-\theta_2 T_2} e^{-\widetilde{\psi} T_3}\ , \tag{3.41}$$

where the $SU(2)$ generators are $T_A=\frac{i}{2}\sigma_A$ and the generator of $H=U(1)$ is T_3. We shall fix the H gauge freedom by setting $\widetilde{\psi}=0$. As a result, we get an integrable 5-dimensional σ-model (cf. (1.6))

$$\begin{aligned}
\mathcal{L} = (G_{mn}+B_{mn})\partial_+x^m\partial_-x^n &= \tfrac{1}{4}k\Big[\partial_+\psi\partial_-\psi+\cos^2\theta_1\,\partial_+\phi_1\partial_-\phi_1+\cos^2\theta_2\,\partial_+\phi_2\partial_-\phi_2\\
&\quad+2\cos\theta_1\,\partial_+\phi_1\partial_-\psi+2\cos\theta_2\,\partial_+\psi\partial_-\phi_2+2\cos\theta_1\cos\theta_2\,\partial_+\phi_1\partial_-\phi_2\Big]\\
&\quad+\tfrac{1}{4}\mathrm{h}\left[\partial_+\theta_1\partial_-\theta_1+\sin^2\theta_1\,\partial_+\phi_1\partial_-\phi_1\right]+\tfrac{1}{4}\widetilde{\mathrm{h}}\left[\partial_+\theta_2\partial_-\theta_2+\sin^2\theta_2\,\partial_+\phi_2\partial_-\phi_2\right]\ .
\end{aligned} \tag{3.42}$$

The resulting target space geometry corresponds to the $T^{1,1}$ metric and a particular B-field [61][48]

$$\begin{aligned}
ds^2_{T^{1,1}} = G_{mn}dx^m dx^n &= \tfrac{1}{4}k\,(d\psi+\cos\theta_1\,d\phi_1+\cos\theta_2\,d\phi_2)^2 \\
&\quad+\tfrac{1}{4}\mathrm{h}\,(d\theta_1^2+\sin^2\theta_1\,d\phi_1^2)+\tfrac{1}{4}\widetilde{\mathrm{h}}\,(d\theta_2^2+\sin^2\theta_2\,d\phi_2^2)\ ,
\end{aligned} \tag{3.43}$$

$$B = \tfrac{1}{2}B_{mn}dx^m\wedge dx^n = \tfrac{1}{4}k\,(d\psi+\cos\theta_1\,d\phi_1)\wedge(d\psi+\cos\theta_2\,d\phi_2)\ . \tag{3.44}$$

The 3 parameters $\mathrm{h},\widetilde{\mathrm{h}},k$ of (3.38) are thus mapped to the 3 parameters of the $T^{1,1}$ metric in [204]. To recall, the $T^{1,1}$ metric (3.43) is an Einstein space if $\mathrm{h}=\widetilde{\mathrm{h}}=\frac{3}{2}k$. It then serves as a base of a Ricci flat 6d conifold with metric $dr^2+r^2ds^2_{T^{1,1}}$ if we formally set $k=\frac{1}{9}$ so that $R_{ij}=4g_{ij}$. In general, the non-zero components of the Ricci tensor of the cone geometry $ds^2=G_{mn}(X)dX^mdX^n=dr^2+r^2g_{ij}(x)dx^idx^j$ (with $i=1,...,d$) are $R_{ij}(G)=R_{ij}(g)-(d-1)g_{ij}$. Thus it vanishes if g_{ij} is an Einstein metric with a particular value of the scalar curvature $R(g)=d(d-1)$ (this condition is satisfied, e.g., for a unit-radius sphere S^d when G_{mn} is flat).

[48] Due to differing conventions, the B-field here is opposite in sign to that in [61]. This difference is not significant, and can be removed by a parity transformation.

For the integrable $T^{1,1}$ model (3.42), the 2-loop RG equations (3.40) become (putting $\mathrm{c}_{_G} = 2,\ \mathrm{c}_{_H} = 0$)

$$\frac{d}{d\tau}\mathrm{h} = 4\big(1-\tfrac{k}{\mathrm{h}}\big)\big[1+\tfrac{4}{\mathrm{h}}\big(1-\tfrac{3k}{2\,\mathrm{h}}\big)\big]\ , \qquad\qquad \frac{d}{d\tau}\widetilde{h} = 4\big(1-\tfrac{k}{\widetilde{\mathrm{h}}}\big)\big[1+\tfrac{4}{\widetilde{\mathrm{h}}}\big(1-\tfrac{3k}{2\,\widetilde{\mathrm{h}}}\big)\big]\ . \tag{3.45}$$

As we shall discuss in Section 3.3.2, the 2-loop RG stability of this $T^{1,1}$ model may be understood as consequence of the fact that the σ-model (3.42) is self-dual under T-duality in the ψ-direction.

3.3 Integrable $T^{1,q}$ model

Let us now introduce a new integrable σ-model with target space metric $T^{1,q}$ and a particular B-field, which is a one-parameter generalization of the $T^{1,1}$ model (3.42) of [61]. Its special conformal case will be the $SU(2)\times SU(2)/U(1)$ GMM model, now with *unequal* WZ levels [205, 206] (with their ratio related to the parameter q).

Our central observation is that, starting with the $G\times G/H$ model (3.16) and considering the case when the subgroup H is *abelian*, the gauge invariance condition (3.19) of [61] is too restrictive. At the particular point $\rho_{12}=\rho_{21}=0$, there is also a second "branch" of gauge invariant models,

$$\begin{aligned}
&\mathcal{L} = -\tfrac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[P^{(i)}_{+}P^{(j)}_{-}] - \tfrac{1}{2}\,r_{ij}\,\mathrm{Tr}[B^{(i)}_{+}B^{(j)}_{-}] + k_i\,\mathcal{L}_{\mathrm{WZ}}(g^{(i)})\ ,\\
&\rho_{12}=\rho_{21}=0\ , \qquad r_{ij} = \begin{pmatrix} r & q(-r-k_1) \\ q(-r+k_1) & q^2 r \end{pmatrix}\ , \qquad q^2\equiv -k_2/k_1\ ,
\end{aligned} \tag{3.46}$$

where k_1, k_2 are assumed to be of opposite sign. The theory (3.46) is invariant under a modified gauge transformation

$$(g^{(1)},\, g^{(2)}) \to (g^{(1)}\mathrm{w}^q,\, g^{(2)}\mathrm{w})\ , \qquad\qquad \mathrm{w}=\mathrm{w}(\xi^+,\xi^-)\in H\ . \tag{3.47}$$

Here w^q is the q-th power of the abelian group element w. In the case when the abelian H is compact then, to make w^q single-valued, one should assume that $q=\sqrt{|\frac{k_2}{k_1}|}$ is an integer.[49] The reason for the restriction of H to be abelian if $q\neq 1$ is that the variation of the Lagrangian (3.16) under (3.47) with $\mathrm{w}\in H$ will be proportional to $(q-1)\mathcal{L}_{\mathrm{WZ}}(\mathrm{w})$, which vanishes for abelian H for any q. We also need to assume $\rho_{12}=\rho_{21}$ to prevent mixing between $P^{(1)}$ and $P^{(2)}$ terms, which are rotated differently by w^q and w respectively. At the value $q=1$ (i.e. $k_1=-k_2$), this model intersects with the first "branch" of gauge-invariant $G\times G/H$ theories (3.16),(3.19) considered above.

[49] More generally, one could consider a "twisted" action of the abelian subgroup, $(g^{(1)},\, g^{(2)})\to(g^{(1)}\mathrm{w}^q,\, g^{(2)}\mathrm{w}^p)$ characterized by integers p,q satisfying $q^2/p^2=-k_2/k_1$. In the $SU(2)\times SU(2)/U(1)$ example discussed below, that would lead to the $T^{p,q}$ model.

Note that, in generic cases, WZ terms present a topological obstruction to gauging [120]. There are, however, special "anomaly-free" subgroups of the WZ term's global symmetry $G_L \times G_R$ that can be gauged [209], satisfying $\mathrm{Tr}_L[T_A T_B] - \mathrm{Tr}_R[T_A T_B] = 0$. This condition is satisfied here by the gauge transformations (3.18) and (3.47) on both "branches" of theories, due to cancellation between the two copies of G in $G \times G/H$.

We claim that model (3.46) is integrable (admitting a Lax representation) if [50]

$$r = k \ . \tag{3.48}$$

In this case it becomes a generalization of (3.38) to the case of unequal levels $k, \widetilde{k}$,

$$\begin{aligned} \mathcal{L} = \ & -\tfrac{1}{2}\,\mathrm{Tr}\big[\mathrm{h}\,P_+P_- + \widetilde{\mathrm{h}}\,\widetilde{P}_+\widetilde{P}_-\big] - \tfrac{1}{2}\,\mathrm{Tr}\big[k\,B_+B_- + \widetilde{k}\,\widetilde{B}_+\widetilde{B}_- - 2\sqrt{k\widetilde{k}}\,B_+\widetilde{B}_-\big] \\ & + k\mathcal{L}_{\mathrm{wz}}(g) - \widetilde{k}\mathcal{L}_{\mathrm{wz}}(\widetilde{g}) \ , \end{aligned} \tag{3.49}$$

where we have set (cf. (3.39))

$$\begin{aligned} & (g^{(1)}, g^{(2)}) \equiv (g, \widetilde{g}) \ , \qquad \widetilde{P}_\pm = P_\pm(\widetilde{g}), \qquad \widetilde{B}_\pm = B_\pm(\widetilde{g}), \\ & \mathrm{h} \equiv \rho_{11}, \qquad \widetilde{\mathrm{h}} \equiv \rho_{22} \ , \qquad k \equiv k_1, \quad \widetilde{k} \equiv -k_2 \ , \qquad q = \sqrt{\tfrac{\widetilde{k}}{k}} \ . \end{aligned} \tag{3.50}$$

The fact that the $\widetilde{k} = k$ limit (3.38) is an integrable theory provides a first check of the integrability of (3.49): its Lax connection was obtained [61] by taking the limit $r = k$, $\rho_{12} = \rho_{21} = 0$ of the affine Gaudin Lax connection for the general integrable $G \times G/H$ model (3.16),(3.19),(3.25). It was found that certain components of the Lie(G)-valued Lax connection degenerate to zero and thus the flatness condition of the resulting connection $L_\pm$ does not imply all of the equations of motion. However, one can consider a generalized limiting procedure by infinitely rescaling the spectral parameter while taking this limit, thus obtaining a second connection $\widetilde{L}_\pm$ that "misses" a different subset of equations of motion. The flatness conditions of the two Lax connections together encode the full set of the equations of motion [61],

$$L_+(z) = B_+ + z^{-1}P_+ \ , \qquad L_-(z) = \frac{1}{kz^2 - \mathrm{h}}\Big[(k-\mathrm{h})(B_- + zP_-) + k(z^2-1)\widetilde{B}_-\Big] \ , \tag{3.51}$$

$$\widetilde{L}_-(z) = \widetilde{B}_- + z\widetilde{P}_- \ , \qquad \widetilde{L}_+(z) = \frac{1}{kz^{-2} - \widetilde{\mathrm{h}}}\Big[(k-\widetilde{\mathrm{h}})(\widetilde{B}_+ + z^{-1}\widetilde{P}_+) + k(z^{-2}-1)B_+\Big] \ . \tag{3.52}$$

The fact of having two separate Lax connections may seem unusual but should be sufficient for the integrability applications: for example, each Lax connection will lead to its own family of conserved charges. They may also be regarded as a single block-diagonal connection valued in $\mathrm{Lie}(G) \oplus \mathrm{Lie}(G)$. Indeed for $k = 0$ the two Lax connections (3.51),(3.52) become the familiar ones (1.33) of the two decoupled G/H σ-models.[51]

[50] The case $r = -k$ is also integrable since it is related to (3.48) by parity.

[51] I thank B. Hoare for this comment.

We observe that the Lax connections (3.51),(3.52) naturally generalize to the new model (3.49) with unequal levels by replacing certain factors of $k \to \widetilde{k}$,

$$L_+(z) = B_+ + z^{-1}P_+ \ , \qquad L_-(z) = \frac{1}{kz^2 - \mathrm{h}}\Big[(k-\mathrm{h})(B_- + zP_-) + \sqrt{k\widetilde{k}}(z^2-1)\widetilde{B}_-\Big] \ , \tag{3.53}$$

$$\widetilde{L}_-(z) = \widetilde{B}_- + z\widetilde{P}_- \ , \quad \widetilde{L}_+(z) = \frac{1}{\widetilde{k}z^{-2} - \widetilde{\mathrm{h}}}\Big[(\widetilde{k}-\widetilde{\mathrm{h}})(\widetilde{B}_+ + z^{-1}\widetilde{P}_+) + \sqrt{k\widetilde{k}}(z^{-2}-1)B_+\Big] \ . \tag{3.54}$$

Assuming the simplest case $G = SU(2)$, $H = U(1)$, using the same coordinate parametrization of this $SU(2) \times SU(2)/U(1)$ model as in (3.41), and fixing again the $H = U(1)$ gauge as $\widetilde{\psi} = 0$, we find the following generalization of (3.42)

$$\begin{aligned}\mathcal{L} = (G_{mn} + B_{mn})\partial_+ x^m \partial_- x^n = \tfrac{1}{4}k\Big[&\partial_+\psi\partial_-\psi + \cos^2\theta_1\,\partial_+\phi_1\partial_-\phi_1 + q^2\cos^2\theta_2\,\partial_+\phi_2\partial_-\phi_2 \\ &+ 2\cos\theta_1\,\partial_+\phi_1\partial_-\psi + 2q\cos\theta_2\,\partial_+\psi\partial_-\phi_2 + 2q\cos\theta_1\cos\theta_2\,\partial_+\phi_1\partial_-\phi_2\Big] \\ &+ \tfrac{1}{4}\mathrm{h}\left[\partial_+\theta_1\partial_-\theta_1 + \sin^2\theta_1\,\partial_+\phi_1\partial_-\phi_1\right] + \tfrac{1}{4}\widetilde{\mathrm{h}}\left[\partial_+\theta_2\partial_-\theta_2 + \sin^2\theta_2\,\partial_+\phi_2\partial_-\phi_2\right] \ , \qquad q = \sqrt{\tfrac{\widetilde{k}}{k}} \ .\end{aligned} \tag{3.55}$$

The resulting target space metric is that of the $T^{1,q}$ space [204] and the B-field is a natural generalization of the one in (3.44),

$$\begin{aligned} ds^2_{T^{1,q}} = G_{mn}dx^m dx^n &= \tfrac{1}{4}k\,(d\psi + \cos\theta_1\,d\phi_1 + q\cos\theta_2\,d\phi_2)^2 \\ &\quad + \tfrac{1}{4}\mathrm{h}\,(d\theta_1^2 + \sin^2\theta_1\,d\phi_1^2) + \tfrac{1}{4}\widetilde{\mathrm{h}}\,(d\theta_2^2 + \sin^2\theta_2\,d\phi_2^2) \ , \end{aligned} \tag{3.56}$$

$$B = \tfrac{1}{2}B_{mn}dx^m \wedge dx^n = \tfrac{1}{4}k\,(d\psi + \cos\theta_1\,d\phi_1) \wedge (d\psi + q\cos\theta_2\,d\phi_2) \ . \tag{3.57}$$

Like the $q = 1$ case [61] in (3.42), the presence of the B-field is crucial here for integrability (the $T^{1,q}$ σ-model without B-field is not integrable [210, 211]). The coordinate form of the Lax connections (3.53),(3.54) is (T_A are the $SU(2)$ generators in (3.41))

$$L'_+(z) = \cos\theta_1\,\partial_+\phi_1\,T_3 + z^{-1}\big(\partial_+\theta_1\,T_2 + \sin\theta_1\,\partial_+\phi_1\,T_1\big) \ , \tag{3.58}$$

$$\begin{aligned} L'_-(z) = \tfrac{1}{kz^2-\mathrm{h}}\Big[&(k-\mathrm{h})\big(\cos\theta_1\,\partial_-\phi_1\,T_3 + z(\partial_-\theta_1\,T_2 + \sin\theta_1\,\partial_-\phi_1\,T_1)\big) \\ &- (z^2-1)\big(\sqrt{k\widetilde{k}}\,\cos\theta_2\,\partial_-\phi_2 + k\,\partial_-\psi\big)\Big] \ , \end{aligned}$$

$$\widetilde{L}_+(z) = \tfrac{1}{\widetilde{k}z^{-2}-\widetilde{h}}\Big[(\widetilde{k}-\widetilde{\mathrm{h}})\big(-\cos\theta_2\,\partial_-\phi_2\,T_3 + z^{-1}(-\partial_+\theta_2\,T_2 + \sin\theta_3\,\partial_+\phi_3\,T_1)\big) + \sqrt{k\widetilde{k}}(z^{-2}-1)\big(\cos\theta_1\,\partial_+\phi_1 + \partial_+\psi\big)\Big] \ , \tag{3.59}$$

$$\widetilde{L}_-(z) = -\cos\theta_2\,\partial_-\phi_2\,T_3 + z\big(-\partial_-\theta_2\,T_2 + \sin\theta_2\,\partial_-\phi_2\,T_1\big) \ .$$

To simplify the expressions we followed [61] here in replacing $L_\pm$ by its gauge transformed version $L'_\pm = \mathrm{w}^{-1}L_\pm\mathrm{w} + \mathrm{w}^{-1}\partial_\pm\mathrm{w}$, with $\mathrm{w} = \exp(-\psi T_3)$.

At the special point $\mathrm{h} = k,\ \widetilde{\mathrm{h}} = \widetilde{k} = q^2 k$, the model (3.55) becomes the $SU(2)\times SU(2)/U(1)$ case of the conformal GMM model with levels $k_1 = k,\ k_2 = -\widetilde{k}$. It was pointed out in [206] that the $SU(2) \times SU(2)/U(1)$ GMM model corresponds to the $T^{1,q}$ metric and a particular B-field, and its 2-loop conformality was explicitly checked (see also [212]). The general GMM model has a current algebra symmetry [205] and is also integrable in the Lax connection sense [213]. What we have shown above is that it admits an integrable extension (3.55) away from the conformal point $\mathrm{h} = k,\ \widetilde{\mathrm{h}} = \widetilde{k}$.

3.3.1 Stability under the 2-loop RG flow

Let us now show that the integrable $T^{1,q}$ model (3.55) is stable under the 2-loop RG flow.

The general gauge invariant model (3.46) (with the $r = k$ condition (3.48) relaxed) must be stable under the RG with only $(r, \mathrm{h}, \widetilde{\mathrm{h}})$ running, due to its H gauge invariance and global $G_L\times G_L$ symmetry. Even at the point $q = 1$ or $\widetilde{k} = k$ where the two "branches" of gauge invariant theories (3.46),(3.19) intersect, the RG flow cannot 'turn on' the extra couplings ρ_{12}, ρ_{21} since this is forbidden by an extra global "center" symmetry [204, 61], $(g,\, \widetilde{g}) \to (g\, z,\, \widetilde{g}),\ z \in \mathrm{Z}(H) \subset H$ when H is abelian.[52]

We shall see that the integrable $T^{1,q}$ model, obtained by fixing $r = k$, is a "fixed line" of the RG flow. Relaxing $r = k$ has the effect of replacing $k \to r$ in the metric,[53] with k still appearing in the B-field (cf. (3.56),(3.57)),[54]

$$
\begin{aligned}
ds^2 &= \tfrac{1}{4} r\, (d\psi + \cos\theta_1\, d\phi_1 + q \cos\theta_2\, d\phi_2)^2 + \tfrac{1}{4}\mathrm{h}\, (d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2) + \tfrac{1}{4}\widetilde{\mathrm{h}}\, (d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2)\ , \\
B &= \tfrac{1}{4} k\, (d\psi + \cos\theta_1\, d\phi_1) \wedge (d\psi + q\cos\theta_2\, d\phi_2)\ , \qquad q = \sqrt{\tfrac{\widetilde{k}}{k}}\ .
\end{aligned} \tag{3.60}
$$

The corresponding 2-loop β-functions in the GB scheme following from (1.9) are ($k, \widetilde{k}$ do not

[52] Note, however, that the center symmetry alone is not sufficient to explain the stability of the $T^{1,q}$ model since it does not prevent r from running.

[53] It was recently argued by observing chaotic behaviour [214] that relaxing the $r = k$ condition indeed destroys classical integrability, as least in the equal level case $q = 1$ (i.e. $\widetilde{k} = k$). This implies that, with $q = 1$, the fixed line $r = k$ and its parity conjugate $r = -k$ are the only integrable models among the σ-models on the class of backgrounds (3.60).

[54] Rescaling $r \to r'k$ and $\psi \to \frac{1}{\sqrt{k}}\psi'$ the background (3.60) can be put into the form symmetric under $k \leftrightarrow \widetilde{k},\ \mathrm{h} \leftrightarrow \widetilde{\mathrm{h}}$:

$$
\begin{aligned}
ds^2 &= \tfrac{1}{4} r'\, (d\psi' + \sqrt{k}\cos\theta_1\, d\phi_1 + \sqrt{\widetilde{k}}\cos\theta_2\, d\phi_2)^2 + \tfrac{1}{4}\mathrm{h}\, (d\theta_1^2 + \sin^2\theta_1\, d\phi_1^2) + \tfrac{1}{4}\widetilde{\mathrm{h}}\, (d\theta_2^2 + \sin^2\theta_2\, d\phi_2^2)\ , \\
B &= \tfrac{1}{4}\, (d\psi' + \sqrt{k}\cos\theta_1\, d\phi_1) \wedge (d\psi' + \sqrt{\widetilde{k}}\cos\theta_2\, d\phi_2)\ .
\end{aligned}
$$

run)

$$\frac{d}{d\tau}r = 2(r^2-k^2)\Big[\tfrac{1}{\mathrm{h}^2} + \tfrac{\widetilde{k}}{k}\tfrac{1}{\widetilde{\mathrm{h}}^2} + \tfrac{r^2-3k^2}{r}\big(\tfrac{1}{\mathrm{h}^4} + \tfrac{\widetilde{k}^2}{k^2}\tfrac{1}{\widetilde{\mathrm{h}}^4}\big)\Big] \ , \tag{3.61}$$

$$\frac{d}{d\tau}\mathrm{h} = 4\big[1 - \tfrac{r}{2\mathrm{h}}(1+\tfrac{k^2}{r^2})\big] + \tfrac{2}{\mathrm{h}^3 r^2}\Big[8\mathrm{h}^2r^2 - 8\mathrm{h}k^2r - 12\mathrm{h}r^3 + 4r^2k^2 + 3k^4 + 5r^4 + \tfrac{\widetilde{k}}{k}\tfrac{\mathrm{h}^2}{\widetilde{\mathrm{h}}^2}(r^2-k^2)(3r^2-k^2)\Big] \ , \tag{3.62}$$

$$\frac{d}{d\tau}\widetilde{\mathrm{h}} = 4\big[1 - \tfrac{\widetilde{k}}{k}\tfrac{r}{2\widetilde{\mathrm{h}}}(1+\tfrac{k^2}{r^2})\big] + \tfrac{2}{\widetilde{\mathrm{h}}^3 r^2}\Big[8\widetilde{\mathrm{h}}^2r^2 - 8\widetilde{\mathrm{h}}\widetilde{k}kr - 12\tfrac{\widetilde{k}}{k}\widetilde{\mathrm{h}}r^3 + 4r^2\widetilde{k}^2 + 3\widetilde{k}^2k^2 + 5\tfrac{\widetilde{k}^2}{k^2}r^4 + \tfrac{\widetilde{k}}{k}\tfrac{\widetilde{\mathrm{h}}^2}{\mathrm{h}^2}(r^2-k^2)(3r^2-k^2)\Big] \ . \tag{3.63}$$

Thus $r = \pm k$ are fixed lines of (3.61), both at 1-loop and 2-loop order. The couplings $(r, \mathrm{h}, \widetilde{\mathrm{h}})$ grow linearly with $\tau \to \infty$ in the UV (reflecting asymptotic freedom), while they decrease to the GMM fixed point $(r, \mathrm{h}, \widetilde{\mathrm{h}}) = (k, k, \widetilde{k})$ in the IR. In fact, as argued in [205], the GMM model on $G \times G'/H$ is an exact CFT, assuming at least one of the cosets G/H or G'/H is a symmetric space (as is indeed the case for $SU(2) \times SU(2)/U(1)$).

Specialising to the fixed line $r = k$ of (3.61), the expressions (3.62),(3.63) simplify, giving the 2-loop β-functions of the integrable $T^{1,q}$ model (3.49),(3.55),

$$\frac{d}{d\tau}\mathrm{h} = 4\big(1-\tfrac{k}{\mathrm{h}}\big)\Big[1 + \tfrac{4}{\mathrm{h}}\big(1 - \tfrac{3k}{2\mathrm{h}}\big)\Big] \ , \qquad \frac{d}{d\tau}\widetilde{h} = 4\big(1-\tfrac{\widetilde{k}}{\widetilde{\mathrm{h}}}\big)\Big[1 + \tfrac{4}{\widetilde{\mathrm{h}}}\big(1 - \tfrac{3\widetilde{k}}{2\widetilde{\mathrm{h}}}\big)\Big] \ . \tag{3.64}$$

These are a natural generalization of the β-functions for the $T^{1,1}$ model in (3.45) to the case of $\widetilde{k} \neq k$. Like in (3.45), the RG evolution of h and $\widetilde{\mathrm{h}}$ happens to be decoupled (while this is not the case for $r \neq k$ in (3.62),(3.63)).

We note that, in addition to the $r = k$ case of the $T^{1,q}$ model, the σ-model corresponding to (3.60) admits *another* integrable limit, $\widetilde{k} = 0$. In this case it factorizes into a squashed S^3 with WZ term and a round S^2. The β-functions (3.61) and (3.62) become those of this squashed S^3 WZ-model in [203] (for 1-loop β-functions see [98, 51])[55] and the β-function (3.63) for the coefficient $\widetilde{\mathrm{h}}$ becomes that of the $S^2 = \frac{SU(2)}{U(1)}$ σ-model (eq. (1.32) with $\mathrm{c}_{_G} = 2,\ \mathrm{c}_{_H} = 0$). Taking further limits, the β-functions (3.61),(3.62),(3.63) agree with other previously known expressions:

(i) Setting $\widetilde{k} = 0$ and $r = \mathrm{h}$, we get from (3.60) the direct sum of the PCM_k (round S^3 with a WZ term) and the S^2 σ-model. In this case (3.61),(3.62) are indeed equivalent to the β-function of PCM_k, i.e. (3.3) with $\mathrm{c}_{_G} = 2$.

(ii) Setting $\widetilde{k} = 0$ (i.e. $q = 0$) and then $k = 0$, we instead get the direct sum of a squashed S^3 (with no WZ term) and a round S^2. The β-functions for r and h agree with those of the "squashed" PCM in [3] (with $G = SU(2)$ and the "squashing" parameter $\varepsilon = \frac{r}{\mathrm{h}}$):

$$\tfrac{d}{d\tau}r = 2r^2\mathrm{h}^{-2} + r^3\mathrm{h}^{-4} \ , \qquad \tfrac{d}{d\tau}\mathrm{h} = 4\big(1 - \tfrac{1}{2}r\mathrm{h}^{-1}\big) + 2\mathrm{h}^{-3}\big(8\mathrm{h}^2 - 12\mathrm{h}r + 5r^2\big) \ .$$

[55] The relation to the notation used in [203] is $\eta = k\mathrm{h}^{-1},\ \lambda^2 = 2\pi\mathrm{h}^{-1},\ \kappa = 1 - r\mathrm{h}^{-1}$.

3.3.2 Covariance under T-duality

One can argue that the RG stability of the integrable $T^{1,q}$ model (3.55), i.e. the presence of the fixed line $r = k$ of (3.61), is related to its property of being self-dual under T-duality in the isometric ψ-direction. To see this, let us write the Lagrangian (3.55) in the following form[56]

$$\mathcal{L} = \tfrac{1}{4}k\Big[(\partial_+\psi + U_+)(\partial_-\psi + V_-) - \tfrac{1}{2}U_+V_-\Big] + \widehat{\mathcal{L}}\ , \tag{3.65}$$

$$U_\pm \equiv 2\cos\theta_1\,\partial_\pm\phi_1\ , \qquad V_\pm \equiv 2q\cos\theta_2\,\partial_\pm\phi_2\ , \tag{3.66}$$

$$\begin{aligned}\widehat{\mathcal{L}} \equiv\ & \tfrac{1}{4}\mathrm{h}\Big[\partial_+\theta_1\partial_-\theta_1 + (\sin^2\theta_1 + \tfrac{k}{\mathrm{h}}\cos^2\theta_1)\partial_+\phi_1\partial_-\phi_1\Big] \\ & + \tfrac{1}{4}\widetilde{\mathrm{h}}\Big[\partial_+\theta_2\partial_-\theta_2 + (\sin^2\theta_2 + \tfrac{kq^2}{\widetilde{\mathrm{h}}}\cos^2\theta_2)\,\partial_+\phi_2\partial_-\phi_2\Big]\ .\end{aligned} \tag{3.67}$$

Starting from the interpolating Lagrangian (obtained by $\partial_\pm\psi \to A_\pm$ and adding the condition $\partial_+A_- - \partial_-A_+ = 0$ with a Lagrange multiplier $\bar\psi$),

$$\mathcal{L}_{\rm int} = \tfrac{1}{4}k\Big[(A_+ + U_+)(A_- + V_-) - \tfrac{1}{2}U_+V_- - \bar\psi(\partial_+A_- - \partial_-A_+)\Big] + \widehat{\mathcal{L}}\ , \tag{3.68}$$

and integrating out $A_\pm$, we obtain the following T-dual Lagrangian

$$\bar{\mathcal{L}} = \tfrac{1}{4}k\Big[(\partial_+\bar\psi + U_+)(\partial_-\bar\psi - V_-) + \tfrac{1}{2}U_+V_-\Big] + \widehat{\mathcal{L}}\ . \tag{3.69}$$

This is the same as the original theory (3.65), with $\psi \to \bar\psi$ and a coordinate redefinition $\phi_2 \to -\phi_2$ (under which $V_- \to -V_-$).

To appreciate the special structure of (3.65), let us relax the condition $r = k$ and go back to the general non-integrable model (3.46) corresponding to the background (3.60). Using again the notation (3.66), we find the following generalization of (3.65)

$$\begin{aligned}\mathcal{L} =& \tfrac{r+k}{8}\Big[(\partial_+\psi + U_+)(\partial_-\psi + V_-) - \tfrac{1}{2}U_+V_-\Big] \\ & + \tfrac{r-k}{8}\Big[(\partial_-\psi + U_-)(\partial_+\psi + V_+) - \tfrac{1}{2}U_-V_+\Big] + \widehat{\mathcal{L}}\ .\end{aligned} \tag{3.70}$$

Applying the T-duality $\psi \to \bar\psi$ to (3.70) we get, instead of (3.69),

$$\begin{aligned}\bar{\mathcal{L}} = \tfrac{1}{4}r\Big[&\Big(\partial_+\bar\psi + \tfrac{r+k}{2r}U_+ + \tfrac{r-k}{2r}V_+\Big)\Big(\partial_-\bar\psi - \tfrac{r-k}{2r}U_- - \tfrac{r+k}{2r}V_-\Big) \\ & + \tfrac{r+k}{4r}U_+V_- + \tfrac{r-k}{4r}U_-V_+\Big] + \widehat{\mathcal{L}}\ .\end{aligned} \tag{3.71}$$

For general values of r and k, (3.71) is different from (3.70); the only self-dual theory where (3.70) and (3.71) coincide is the $T^{1,q}$ model (3.65) corresponding to $r = k$ (or its parity-conjugate $r = -k$).

By the standard path integral argument, the T-dual models (3.70),(3.71) should be

[56] Note that $\widehat{\mathcal{L}}$ becomes simply quadratic in the fields at the GMM point $\mathrm{h} = k$, $\widetilde{\mathrm{h}} = \widetilde{k} = q^2k$.

quantum-equivalent. In general, the T-duality transformation rules are subject to quantum α' corrections [174, 178, 176] that may be attributed to extra finite counterterms resulting from integration over the auxiliary gauge field $A_\pm$ (see eq. (2.42) above). If the kinetic term of the isometric coordinate is non-trivial, i.e. the term quadratic in $A_\pm$ is $A_+M(x)A_-$, then the leading quantum correction to the effective Lagrangian is represented by the term $\Delta\mathcal{L} \sim \alpha' \partial_+ \log M \, \partial_- \log M$ (as well as a shift of the dilaton [74, 182]). In the case of (3.70) we have $M = 1$ and thus this correction is absent.

Since the model (3.70) is stable under the RG due to its symmetries, with the 3 running couplings $r, \mathrm{h}, \widetilde{\mathrm{h}}$, its T-dual (3.71) must also be stable. Given that the self-dual points $r = \pm k$ are part of both RG-stable families (3.70) and (3.71) then they must also remain in both families after the renormalization. Since they are the *only* such self-dual models, then $r = \pm k$ must be fixed lines of the RG flow. This was indeed confirmed above by the explicit computation of the β-functions leading to (3.61).

3.4 Discussion

In this chapter we discussed further instances of a close connection between the conditions of integrability and a consistent restriction of the RG flow to a subspace of couplings.

We have found the 2-loop β-functions of the 6-parameter $G \times G$ model (3.5) and have shown that its integrability condition (3.6) is automatically preserved by the RG flow. In [60], the 1-loop β-functions for this integrable model were written in a universal form in terms of the twist function, revealing a hidden simplicity. It would be interesting to see if the complicated expressions we have found for the 2-loop β-functions (see Appendix D.1) simplify on the "integrable surface" once expressed in terms of the twist function. Indeed, one may try to extend the method of [60] from 1-loop to 2-loop order, i.e. computing the Riemann tensor and thus the 2-loop β-function in terms of the twist function. It would also be interesting to investigate the connection to the "doubled" approach of [200] which studied the model (3.5),(3.6) with additional integrable η- or λ-deformation parameters turned on.

We also studied the 6-parameter gauged $G \times G/H$ model (3.16),(3.19), which is integrable under the conditions (3.25). The latter were found to be stable under the 1-loop RG flow but, in general, require a certain deformation (i.e. the addition of finite counterterms) at the 2-loop level to preserve renormalizability.

There are still some special cases in which integrable $G \times G/H$ models are automatically stable under the 2-loop RG flow. One simple example is the $T^{1,1}$ model of [61]. We also constructed a new class of integrable $G \times G/H$ models (3.49) in the case when the subgroup H is abelian (see (3.46),(3.48),(3.49)). For $G = SU(2)$ and $H = U(1)$, this led to an integrable $T^{1,q}$ model generalizing the $T^{1,1}$ model, which we also found to be stable under the 2-loop RG flow for general values of the parameter q. This model may be interpreted as an integrable deformation of the conformal GMM model with unequal levels [205]. An open question is

whether the integrable $T^{1,q}$ model admits a description in terms of affine Gaudin models (like the $T^{1,1}$ case) or if it is outside of that formalism.

Since the GMM model admits a $G \times G'/H$ generalization (with $G \neq G'$), this raises the question of whether there is a larger class of integrable $G \times G'/H$ models that flow to such conformal theories. One obvious possibility is to consider some analytic continuations, e.g., take G' to be a different real form of the complexification of G (assuming the resulting σ-model couplings G, B remain real). For example, the counterpart of the $SU(2) \times SU(2)/U(1)$ model would be $SL(2,\mathbb{R}) \times SU(2)/U(1)$.

Another interesting open question is why the integrable $G \times G$ model was automatically stable under the 2-loop RG flow, while the gauged $G \times G/H$ analog requires a non-trivial quantum deformation of the "integrable surface" in the space of couplings. This is analogous to WZW models, which are conformal without any α'-corrections, and gauged WZW models, which require a non-trivial tower of α' corrections (e.g. in eq. (2.4)) to remain conformal beyond 1-loop. One possible resolution, which we shall consider below in Section 4.5, is to reformulate the $G \times G/H$ as a theory with a gauge field and redefine the gauge field in terms of H-valued scalars; as for the "tripled" formulation of the λ-model in Chapter 4, such a reformulation could restore higher-loop renormalizability without needing any quantum corrections to the geometry.

Chapter 4

"Tripled" formulation of the λ-model

We have been studying a close link between renormalizablility, or stability under the RG flow, and integrability in 2d σ-models. Having previously been observed [44, 45, 46, 47, 48, 49] only at the 1-loop (Ricci flow) level, it is important to study this relation also at higher loop orders.

This question of higher loops was addressed above in Chapters 2 and 3, where we found that, starting from 2 loops, renormalizability typically requires a specific deformation of the classical target space geometry, which may be interpreted as the result of adding finite local counterterms. In Chapter 2, we considered the simplest examples of integrable η- and λ-deformed bosonic σ-models with 2- and 3-dimensional target spaces. In Chapter 3, we looked at another class of integrable σ-models with target spaces $G \times G$ and $G \times G/H$.

Here we shall concentrate on the λ-deformation [111, 112] and address the *general case* where the model is based on an arbitrary group G or symmetric space G/H. Its Lagrangian is (repeating eq. (1.78) for clarity)

$$\mathcal{L} = k\Big(\mathcal{L}_G(g) + \mathrm{Tr}\big[J_+A_- - A_+K_- + g^{-1}A_+gA_- - A_+A_- - (\lambda^{-1}-1)A_+PA_-\big]\Big), \tag{4.1}$$

$$J = g^{-1}dg\ , \quad K = dg\, g^{-1}\ ,$$

$$P = \begin{cases} 1\ , & \text{group space } G \\ P_{G/H}\ , & \text{symmetric space } G/H\ , \end{cases} \tag{4.2}$$

where $\mathcal{L}_G = \mathcal{L}_{\rm PCM} + \mathcal{L}_{\rm WZ}$ is the WZW Lagrangian, $g \in G$, $A_\pm \in \mathrm{Lie}(G)$ and $P_{G/H}$ is the projector onto the orthogonal complement of $\mathrm{Lie}(H)$ in $\mathrm{Lie}(G)$. Instead of λ it is often convenient to use a redefined parameter,

$$\kappa = \frac{1-\lambda}{1+\lambda}\ . \tag{4.3}$$

The limit $\lambda \to 0$ (or $\kappa \to 1$) yields the conformal WZW model on G (or gauged WZW model on G/H) with level k, which should thus be a fixed point of the RG flow. The λ-model's $\mathbb{Z}_2$ transformation (1.80) then implies that the opposite limit $\lambda \to \infty$ (or $\kappa \to -1$) should also be a fixed point. Indeed, in the group case the $\lambda \to \infty$ limit of (4.1) is conformal: it is the G/G

chiral gauged WZW model [215], which, on integrating out $A_\pm$, gives the G WZW model at level $-k$. Similarly, in the coset case we find in this limit the G/H gauged WZW model at level $-k$.

Integrating out the 2d gauge field $A_\pm$ in (4.1), i.e. reducing the model to the standard (or "physical") configuration space, one finds a σ-model with parameters k and λ (see eq. (2.35)). As we found in Chapter 2, to preserve renormalizability of the λ-model at the 2-loop level, one must make a non-trivial modification of the classical target space geometry.

In this chapter our central observation will be that, *before* integrating out $A_\pm$, the λ-model (4.1) is renormalizable without any deformation. As was done in the context of gWZW models [86, 89, 216, 217], we shall change variables from $A_\pm$ to scalar fields $h, \bar{h} \in G$ as

$$A_+ = h\,\partial_+ h^{-1},\ A_- = \bar{h}\,\partial_+ \bar{h}^{-1}\ , \tag{4.4}$$

resulting in a σ-model on the extended or "tripled" $(G \times G \times G)$ configuration space $(g, h, \bar{h})$. It may be interpreted as the sum of two decoupled parts: (i) a WZW model on G and (ii) a deformation of a WZW model on $G \times G$ by a particular left-right current interaction. This special decoupling is expected to be a manifestation of integrability and, as a result, the *manifest* global symmetries become sufficient to protect the model under the RG flow without the need for any additional finite counterterms.

The rest of the chapter is organized as follows. In Section 4.1 we shall review the previous application of the change of variables (4.4) to gauged and chiral gauged WZW models.

In Sections 4.2 and 4.3, we consider the λ-models, respectively based on groups and symmetric spaces, on the configuration space (g, A_+, A_-), i.e. with the gauge field unintegrated. By using the change of variables (4.4) to write the λ-models as certain "tripled" σ-models with extra manifest symmetries, we shall argue that they are automatically renormalizable to all orders and explicitly compute their 2-loop β-functions. In the non-abelian dual (NAD) limit (2.72), these will correctly reproduce the (scheme-invariant) 2-loop β-functions of the PCM and symmetric space σ-model, suggesting that the interpolating models for NAD are also automatically renormalizable without needing corrections.

In Section 4.4 we consider some simple examples of the models obtained in the "tripled" formulation of the λ-models based on $\frac{SU(2)}{U(1)}$ and $SU(2)$. In Section 4.5 we construct analogous "tripled" or "extended" space formulations of some other integrable σ-models (a variant of the λ-model built on G^N [161] and the $G \times G/H$ models of [61]). Some concluding remarks are made in Section 4.6.

This chapter is based on the paper [3] and we closely follow the presentation there.

4.1 Review of gWZW model

First let us review the original use of the change of variables (4.4) in the context of the gauged WZW models [86, 89, 216, 217]. The gauged WZW model is closely related to the λ-model (4.1) but with the gauge field being valued in some subalgebra $\mathrm{Lie}(F) \subset \mathrm{Lie}(G)$ (see Section 1.4.8),

$$\begin{aligned} &\mathcal{L} = k\Big(\mathcal{L}_G(g) + \mathrm{Tr}\big[J_+A_- - A_+K_- + g^{-1}A_+gA_- - A_+A_-\big]\Big) \ , \\ &g \in G \ , \quad A_\pm \in \mathrm{Lie}(F) \ . \end{aligned} \tag{4.5}$$

Changing variables $(A_+, A_-) \to (h, \bar{h})$ as in (4.4), now with with $h, \bar{h} \in F$, and using the Polyakov-Wiegmann identity[57] [140], the Lagrangian (4.5) can be written as a combination of WZW models,

$$\mathcal{L} = k\Big(\mathcal{L}_G(h^{-1}g\bar{h}) - \mathcal{L}_F(h^{-1}\bar{h})\Big) \ . \tag{4.6}$$

The change of variables in (4.4) results in a Jacobian contributing to the action as [140, 218]

$$\Delta\mathcal{L} = -2c_{_F}\Big(\mathcal{L}_H(h^{-1}\bar{h}) + \mathrm{qTr}[A_+A_-]\Big) \tag{4.7}$$

$$= -2c_{_F}\Big(\mathcal{L}_H(h^{-1}\bar{h}) + \mathrm{qTr}[\partial_+h\,h^{-1}\,\partial_-\bar{h}\,\bar{h}^{-1}]\Big) \ , \tag{4.8}$$

where $c_{_F}$ is the dual Coxeter number[58] of F and an arbitrary coefficient q parametrizes the ambiguity of adding a finite local counterterm $\mathrm{Tr}[A_+A_-]$. Combining (4.8) and the classical Lagrangian (4.6) gives

$$\mathcal{L} = k\mathcal{L}_G(h^{-1}g\bar{h}) - (k + 2c_{_F})\mathcal{L}_F(h^{-1}\bar{h}) - 2\mathrm{q}c_{_F}\big[\mathcal{L}_F(h^{-1}\bar{h}) - \mathcal{L}_F(h^{-1}) - \mathcal{L}_F(\bar{h})\big] \ . \tag{4.9}$$

Now one can choose the coefficient $q = 0$ to obtain the sum of two decoupled WZW models (e.g. after defining $\tilde{g} = h^{-1}gh$, $\tilde{h} = h^{-1}\bar{h}$)

$$L = k\mathcal{L}_G(h^{-1}g\bar{h}) - (k + 2c_{_F})\mathcal{L}_F(h^{-1}\bar{h}) \tag{4.10}$$

Expressing the gWZW model as this sum of decoupled WZW models makes manifest its properties of (i) exact conformal invariance and (ii) gauge invariance under $g \to ugu^{-1}$, $h \to uh$, $\bar{h} \to u\bar{h}$.

[57] The Polyakov-Wiegmann identity is a multiplicative identity for the WZW Lagrangian [140],
$\mathcal{L}_G(uv) = \mathcal{L}_G(u) + \mathcal{L}_G(v) - \mathrm{Tr}[u^{-1}\partial_+u\,\partial_-vv^{-1}]$.

[58] More precisely, $c_{_F}$ denotes here a rescaled version of the dual Coxeter number of F by an index factor relating to the embedding of F into G (see discussion in the final paragraph of App. A for subgroups $H \subset G$). If G and F are both classical groups with the same index for their fundamental representation (e.g. both A-type) then this index factor is trivial and $c_{_F}$ is the dual Coxeter number of F.

The exact form of the gWZW metric and B-field can be derived [89] from (4.10) (agreeing with [87, 88]) by considering the quantum effective action. The WZW model's effective action is just the classical one with a shifted level (respectively $k \to k + c_{_G}$ or $k \to k + c_{_F}$), leading to

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= (k + c_{_G})\mathcal{L}_G(h^{-1}g\bar{h}) + [-(k + 2c_{_F}) + c_{_F}]\mathcal{L}_F(h^{-1}\bar{h}) \\ &= (k + c_{_G})\mathcal{L}_G(h^{-1}g\bar{h}) - (k + c_{_F})\mathcal{L}_F(h^{-1}\bar{h}) \ . \end{aligned} \tag{4.11}$$

Transforming back to $A_\pm$, the resulting theory is not local but the "local part" gives a prescription for a corrected version of (4.5) in which $A_\pm$ can be integrated out *classically* to get an exactly conformal σ-model (of which the $SL(2,R)/U(1)$ example is given in (2.4)).

Now let us briefly consider the related G/F *chiral gauged WZW model* (cWZW) [215], for which a similar argument was carried out in [216],

$$\begin{aligned} &\mathcal{L} = k\Big(\mathcal{L}_G(g) + \mathrm{Tr}\big[J_+A_- - A_+K_- + g^{-1}A_+gA_-\big]\Big) \ , \\ &g \in G \ , \quad A_\pm \in \mathrm{Lie}(F) \ . \end{aligned} \tag{4.12}$$

This theory differs from the gWZW model (4.5) by the addition of $k\,\mathrm{Tr}[A_+A_-]$. Again changing variables as in (4.4), the extra classical term in (4.12) is proportional to the ambiguous local term with coefficient q in (4.7). Thus the result for the cWZW model is obtained by replacing $-2\mathrm{q}c_{_F} \to k - 2\mathrm{q}c_{_F}$ in (4.9),

$$\mathcal{L} = k\mathcal{L}_G(h^{-1}g\bar{h}) - (k + 2c_{_F})\mathcal{L}_F(h^{-1}\bar{h}) + (k - 2\mathrm{q}c_{_F})\big[\mathcal{L}_F(h^{-1}\bar{h}) - \mathcal{L}_F(h^{-1}) - \mathcal{L}_F(\bar{h})\big] \ . \tag{4.13}$$

Again in this case, there is a choice $\mathrm{q} = -1$ that leads to cancellations and gives a sum of decoupled WZW models [216],

$$\mathcal{L} = k\mathcal{L}_G(h^{-1}g\bar{h}) + (k + 2c_{_F})\big[\mathcal{L}_F(h^{-1}) + \mathcal{L}_F(\bar{h})\big] \ , \tag{4.14}$$

illustrating that the cWZW model is also exactly conformal.

4.2 "Tripled" λ-model based on a group

Now let us turn to the λ-model based on a group G (putting $P = 1$ in eq. (4.1)),

$$\begin{aligned} &\mathcal{L} = k\Big(\mathcal{L}_G(g) + \mathrm{Tr}\big[J_+A_- - A_+K_- + g^{-1}A_+gA_- - \lambda^{-1}A_+A_-\big]\Big), \\ &g \in G \ , \quad A_\pm \in \mathrm{Lie}(G) \ . \end{aligned} \tag{4.15}$$

This model interpolates between the gWZW model (4.5) ($\lambda = 1$) and the cWZW model (4.12) ($\lambda = \infty$) in the particular case $F = G$. Hence, upon making the same change of variables

$(A_+, A_-) \to (h, \bar{h})$ as in (4.4), the resulting theory is just the weighted average of the gWZW and cWZW results (4.9),(4.13) above,

$$\begin{aligned}\mathcal{L} =& k\,\mathcal{L}_G(h^{-1}g\bar{h}) - (k+2c_{_G})\mathcal{L}_G(h^{-1}\bar{h}) \\ & - (k(\lambda^{-1}-1)+2\mathrm{q}c_{_G})\big[\mathcal{L}_G(h^{-1}\bar{h}) - \mathcal{L}_G(h^{-1}) - \mathcal{L}_G(\bar{h})\big]\ ,\end{aligned} \tag{4.16}$$

with the new scalar fields $h, \bar{h}$ now valued in the full group G. This expression includes the Jacobian contribution (4.8), with the coefficient q parametrizing the local term ambiguity. In this case q just specifies a particular redefinition of the coupling λ, so the choice of q is not important and we shall pick $\mathrm{q} = -1$ (as in the cWZW case above).

Again using the Polyakov-Wiegmann identity (see footnote 57), we may rewrite (4.16) as

$$\mathcal{L} = k\mathcal{L}_G(\tilde{g}) + \mathcal{L}'(\hat{h}, \bar{h})\ , \qquad \tilde{g} \equiv h^{-1}g\bar{h} \in G\ , \qquad \hat{h} \equiv h^{-1} \in G\ , \tag{4.17}$$

$$\mathcal{L}' = -(k+2c_{_G})\big[\mathcal{L}_G(\hat{h}) + \mathcal{L}_G(\bar{h})\big] + k\lambda^{-1}\mathrm{Tr}[\hat{h}^{-1}\partial_+\hat{h}\,\partial_-\bar{h}\bar{h}^{-1}]\ . \tag{4.18}$$

This is a σ-model with fields valued in the extended configuration space $(\tilde{g}, \hat{h}, \bar{h}) \in G \times G \times G$. As for the gauged WZW models considered above, the first WZW term $k\mathcal{L}_G(\tilde{g})$ still decouples from the other fields in (4.17). This decoupling is a special property expected to be associated with integrability: indeed if one deformed the λ-model (4.15) in an arbitrary way preserving its global symmetry but breaking integrability, the decoupling would be lost.[59] Since the decoupled WZW term is conformal on its own, then its coefficient k will not run, and the RG flow of λ is determined by the other "truncated" part $\mathcal{L}'$.

Redefining the couplings as

$$\tilde{k} \equiv k + 2c_{_G}\ , \qquad \tilde{\lambda} \equiv \frac{k}{\tilde{k}}\lambda^{-1}\ , \tag{4.19}$$

then the truncated part,

$$\mathcal{L}'(\hat{h}, \bar{h}) = -\tilde{k}\Big(\mathcal{L}_G(\hat{h}) + \mathcal{L}_G(\bar{h}) - \tilde{\lambda}\mathrm{Tr}[\hat{h}^{-1}\partial_+\hat{h}\,\partial_-\bar{h}\bar{h}^{-1}]\Big)\ , \qquad \hat{h} \equiv h^{-1},\ \bar{h} \in G\ , \tag{4.20}$$

may be interpreted as the sum of two WZW models for the group G with the same level $-\tilde{k}$ perturbed by a current-current interaction. Classical integrability of the model (4.20) (implying integrability of the group space λ-model) was first shown in [213] (see also [219]). In fact (4.20) is a particular instance of the integrable $G \times G$ models [160] in (3.5),(3.6) above, with the couplings and fields related by

$$(k_1, k_2, s, u, t, b) = (-\tilde{k}, +\tilde{k}, -\tilde{k}, -\tilde{k}, \tilde{k}\tilde{\lambda}^{-1}, \tilde{k}\tilde{\lambda}^{-1})\ , \qquad (g^{(1)}, g^{(2)}) = (\hat{h}, \bar{h}^{-1})\ . \tag{4.21}$$

At the value $\widetilde{\lambda} = 1$ of the coupling, the model (4.20) becomes a special case ($G \times G/G$ with

[59] I thank N. Gromov for a discussion of this point.

equal levels) of the conformal GMM model [220]. A generalization of the model (4.20) was also considered in [221, 222], where it was interpreted as a special case of a "doubly λ-deformed" σ-model. Our path-integral relation between the λ-model (4.1) and the truncated model (4.20) should be equivalent (at least at the classical and 1-loop level) to the canonical equivalence between the doubly λ-deformed model and two copies of the λ-model found in [223] (upon setting one of the two λ-parameters to zero). The leading order in $1/k$ (1-loop) renormalization of similar models was also studied earlier in [224, 225].

Our central observation is that the "truncated" model (4.20) is automatically renormalizable to all orders with only $\tilde{\lambda}$ (or equivalently λ) running. This is because, thanks to the decoupling of the first WZW term in (4.17), the truncated model is fully protected under the RG flow by its manifest symmetries: the global G symmetry (present in the usual formulation (4.15) of the λ-model),

$$\hat{h} \to \hat{h}\, g_0 \ , \qquad \bar{h} \to g_0^{-1}\, \bar{h} \ , \qquad\qquad g_0 \in G \ , \tag{4.22}$$

and chiral gauge symmetries (cf. the symmetry (1.61) of the WZW model),

$$\hat{h} \to u(\sigma^-)\, \hat{h} \ , \qquad \bar{h} \to \bar{h}\, w(\sigma^+) \ , \qquad\qquad u, w \in G \ , \tag{4.23}$$

arising ubiquitously from the change of variables (4.4). These symmetries uniquely protect the structure of (4.20), with $\tilde{\lambda}$ and $\tilde{k}$ being the only free parameters. In particular, the chiral symmetries (4.23) prohibit the appearance of other current-current interaction terms under the RG flow.

We shall now directly compute the 2-loop β-function for the coupling $\tilde{\lambda}$ of the truncated σ-model (4.20) in large $\tilde{k}$ perturbation theory, demonstrating its renormalizability with only $\tilde{\lambda}$ (or λ) running. Let us emphasize that — modulo a finite renormalization, or redefinition, of λ encoded in the choice of the coefficient q in (4.7) — the RG evolution of the tripled σ-model (4.17) is exactly equivalent to that of the λ-model (4.15), since they were related at the level of the path integral including the necessary Jacobian (4.7).[60]

Let us introduce the basis $\{T_A\}$ for Lie(G) (see Appendix A for conventions). We shall use the indices A and $\bar{A}$ respectively for the two G-valued fields $\hat{h}$ and $\bar{h}$ in (4.20) (with the tangent space index for $G \times G$ denoted as $M = \{A, \bar{A}\}$). As in [226] we introduce the vielbein

$$E^M = (e^A, e^{\bar{A}}) = (\sqrt{1-\tilde{\lambda}^2}\, J^A,\ K^{\bar{A}} + \tilde{\lambda} J^A) \ , \tag{4.24}$$

$$J \equiv T_A J^A = \hat{h}^{-1} d\hat{h} \ , \qquad K^{\bar{a}} \equiv T_A K^A = d\bar{h}\bar{h}^{-1} \ , \tag{4.25}$$

[60] The β-function of the truncated $G \times G$ model (4.20) can actually be extracted from that of the more general $G \times G$ model studied in Chapter 3 (given explicitly in App. D.1). For completeness, we explicitly run through the computation again here. The results are in agreement those in App. D.1 for the special values of the parameters in (4.21).

where J and K are the currents that appear in the deformation term in (4.20). Up to permutations, the non-zero components of the metric and H-tensor of the $G\times G$ model (4.20) are (cf. (1.6))

$$\mathrm{G}_{AB}=\mathrm{G}_{\bar A\bar B}=\tfrac{\tilde k}{2}\,\mathrm{Tr}[T_AT_B]\ , \tag{4.26}$$

$$\mathrm{H}_{ABC}=-i\tfrac{\tilde k}{2}\tfrac{\sqrt{1-\tilde\lambda^2}(1+2\tilde\lambda)}{(1+\tilde\lambda)^2}f_{ABC}\ ,\qquad \mathrm{H}_{\bar A\bar B\bar C}=-i\tfrac{\tilde k}{2}f_{ABC}\ ,\qquad \mathrm{H}_{\bar A BC}=-i\tfrac{\tilde k}{2}\tfrac{\tilde\lambda}{1+\tilde\lambda}f_{ABC}\ . \tag{4.27}$$

Our aim is to compute the corresponding 2-loop β-function in (1.9). Let us formally define the torsion as $T^M=\frac12\mathrm{H}^M{}_{NP}E^N\wedge E^P$ where the tangent space index is raised with the inverse of the metric (4.26). Then, from the Cartan structure equation $dE^M+\widehat\omega^M{}_N\wedge E^N=T^M$, we obtain the torsionful spin connection $\widehat\omega^M{}_N=\widehat\omega^M{}_{NP}E^P$ with non-zero components

$$(\tilde\lambda^{-1}+1)\widehat\omega^A{}_B=\widehat\omega^{\bar A}{}_{\bar B}=if^A{}_{BC}\big(-\tfrac{\tilde\lambda}{\sqrt{1-\tilde\lambda^2}}e^C+e^{\bar C}\big)\ . \tag{4.28}$$

The non-zero components of the curvature, $\widehat R^M{}_N=d\widehat\omega^M{}_N+\widehat\omega^M{}_P\wedge\widehat\omega^P{}_N=\frac12\widehat R^M{}_{NPQ}E^P\wedge E^Q$, are then found to be

$$(\tilde\lambda^{-2}-1)\widehat R^A{}_{BDE}=-\sqrt{\tilde\lambda^{-2}-1}\,\widehat R^A{}_{BD\bar E}=\widehat R^A{}_{B\bar D\bar E}=-\frac{\tilde\lambda}{(1+\tilde\lambda)^2}f^A{}_{BC}f^C{}_{DE}\ . \tag{4.29}$$

Substituting (4.29) into the 2-loop RG equation (1.9) in the GB scheme, one obtains

$$\big[\frac{d}{dt}(G_{mn}+B_{mn})\big]dx^m\otimes dx^n=\sum_{A=\bar A=1}^{\dim G}\Big[\frac{2c_{_G}\tilde\lambda^2}{(1+\tilde\lambda)^2}+\frac{4c_{_G}^2}{\tilde k}\frac{\tilde\lambda^4(1-2\tilde\lambda)}{(1+\tilde\lambda)^5(1-\tilde\lambda)}\Big]\,\mathrm{Tr}[J\otimes K]\ . \tag{4.30}$$

Here $\otimes$ indicates that the product is not symmetrized. We conclude that only the $\tilde\lambda$-dependent term in (4.20) gets renormalized, i.e. the 2-loop RG equation (1.9) is solved with $\xi_m=\lambda_m=0$ and

$$\frac{d}{dt}\tilde k=0\ ,\qquad\qquad \frac{d}{dt}\tilde\lambda=\frac{2c_{_G}}{\tilde k}\Big[\frac{\tilde\lambda^2}{(1+\tilde\lambda)^2}+\frac{2c_{_G}}{\tilde k}\frac{\tilde\lambda^4(1-2\tilde\lambda)}{(1+\tilde\lambda)^5(1-\tilde\lambda)}\Big]\ . \tag{4.31}$$

The 1-loop term here agrees with previous results, both for the standard λ-model (eq. (1.85) with $\kappa=\frac{\tilde\lambda-1}{\tilde\lambda+1}$) [114, 53] and the "truncated" $G\times G$ model (4.20) [227, 226]).

The level $k=\tilde k-2c_{_G}$ is thus RG-invariant as it should be. The fixed points of the RG flow are as expected: $\tilde\lambda=\infty$ (i.e. $\lambda=0$), corresponding to the G WZW model with level k, and $\tilde\lambda=0$ (i.e. $\lambda=\infty$) corresponding to the G/G cWZW model, or the G WZW model with level $-k$ after integrating out the gauge field.[61]

[61] Taking $\tilde\lambda=1$ (i.e. $\lambda=1$) with finite k (in contrast to the NAD limit (2.72)) gives the trivial G/G gWZW model in which one can fix a gauge so that the remaining degrees of freedom correspond to the Cartan torus.

Expressing $\tilde{k}$ and $\tilde{\lambda}$ in (4.19) in terms of k and the coupling κ using (4.3), we find

$$\frac{d}{dt}k = 0 \ , \qquad \frac{d}{dt}\kappa = \frac{c_{_G}}{4k}(1-\kappa^2)^2\Big[1 + \frac{c_{_G}(1-\kappa)^2(1-10\kappa-3\kappa^2)}{8k\kappa}\Big] \ . \tag{4.32}$$

Note that the 2-loop term in (4.32) is scheme-dependent: it can be changed by redefining κ by a $1/k$ term (or shifting k by a finite term): for example, it will depend on the choice of the parameter q in (4.7). Even though k is not running, we effectively have a 2-coupling theory (with $1/k$ playing the role of a loop-counting parameter) so only the 1-loop term in the β-function (4.32) is scheme-independent.

In a general scheme, the symmetry $k \to -k, \ \lambda \to \lambda^{-1}$ (1.80) of the 1-loop RG equation in (4.31),(4.32) is not manifest at the 2-loop level. To preserve it requires a particular formulation of the quantum theory, i.e. a specific definition of the couplings, or choice of scheme. For example, if we redefine the parameters as

$$(\tilde{k}, \tilde{\lambda}) \to (\mathrm{k}, \bar{\lambda}) \ , \qquad \tilde{k} = \mathrm{k} + c_{_G} \ , \qquad \tilde{\lambda} = \bar{\lambda}\Big[1 + c_{_G}\mathrm{k}^{-1}\frac{1-\bar{\lambda}}{1+\bar{\lambda}}\Big] \ , \tag{4.33}$$

then the RG equation for $\bar{\lambda}$ resulting from (4.31) is

$$\frac{d}{dt}\bar{\lambda} = \frac{2c_{_G}}{\mathrm{k}}\frac{\bar{\lambda}^2}{(1+\bar{\lambda})^2}\Big[1 - \frac{2c_{_G}}{\mathrm{k}}\frac{\bar{\lambda}^2}{(1+\bar{\lambda})^2}\frac{\bar{\lambda}+\bar{\lambda}^{-1}-1}{1-\bar{\lambda}^2}\Big] \ , \tag{4.34}$$

which is invariant under the following quantum version of (1.80)

$$\mathrm{k} \to -\mathrm{k} \ , \qquad \bar{\lambda} \to \bar{\lambda}^{-1} \ . \tag{4.35}$$

Note that here $\mathrm{k} = k + c_{_G}$ is the same as the shifted level of the WZW theory. It is also worth observing that (4.33) is not the only redefinition that restores the symmetry (1.80). Indeed, even restricting to those that preserve the existence of the fixed points and the NAD limit discussed below, there are many.

In the non-abelian dual (NAD) limit (2.72) (i.e. $\kappa = \frac{1}{4}\mathrm{h}k^{-1}, \ k \to \infty$) when the λ-model (4.15) with unintegrated gauge field becomes the interpolating model (1.52) for non-abelian duality of the PCM, the β-function (4.32) becomes

$$\frac{d}{dt}\mathrm{h} = c_{_G} + \tfrac{1}{2}c_{_G}^2\mathrm{h}^{-1} \ . \tag{4.36}$$

This result matches the standard expression (1.20) for the 2-loop β-function of the PCM on a simple group G (see, e.g., [63, 64, 55]). The 2-loop term in (4.36) is scheme-independent as the NAD limit produces a single-coupling theory. Hence the scheme-dependent 2-loop term in the λ-model β-function (4.32) does still contain some invariant information, which passes a non-trivial test in this limit.

4.3 "Tripled" λ-model based on a symmetric space

Next let us consider the λ-model based on a symmetric space G/H (setting $P = P_{G/H}$ in (4.1)),

$$\begin{aligned} &\mathcal{L} = k\Big(\mathcal{L}_G(g) + \mathrm{Tr}\big[J_+A_- - A_+K_- + g^{-1}A_+gA_- - A_+\big(1-(\lambda^{-1}-1)P_{G/H}\big)A_-\big]\Big)\ , \\ &g\in G\ ,\quad A_\pm \in \mathrm{Lie}(G)\ . \end{aligned} \tag{4.37}$$

This model has a special subgroup $H \subset G$ corresponding to its gauge symmetry, however we note that the 2d gauge field $A_\pm$ is still valued in the algebra of the full group G.

We shall repeat the same steps as in the group case above, obtaining a similar tripled formulation: redefining $(A_+, A_-) \to (h, \bar h)$ according to (4.4), using the Polyakov-Wiegmann identity (see footnote 57), including the contribution from the Jacobian (4.7), and setting $\tilde g = h^{-1}g\bar h$ and $\hat h = h^{-1}$, we now get

$$\begin{aligned} \mathcal{L} = {}& k\mathcal{L}_G(\tilde g) - (k+2c_{_G})\mathcal{L}_G(\hat h\bar h) + k(\lambda^{-1}-1)\mathrm{Tr}[\hat h^{-1}\partial_+\hat h\, P_{G/H}\,\partial_-\bar h\bar h^{-1}] \\ &+ 2\mathrm{q}c_{_G}\mathrm{Tr}[\hat h^{-1}\partial_+\hat h\,\partial_-\bar h\bar h^{-1}]\ . \end{aligned} \tag{4.38}$$

Classically (i.e. for large k) this model has the expected H gauge symmetry

$$\hat h \to \hat h\, f\ ,\qquad \bar h \to f^{-1}\,\bar h\ ,\qquad\qquad f = f(\sigma^+,\sigma^-)\in H\ . \tag{4.39}$$

Since the term proportional to q in (4.38) does not respect this gauge symmetry, then we should make the choice q = 0 in the ambiguous Jacobian term in (4.7), corresponding to a natural scheme preserving the gauge symmetry (this is the same choice as in the gWZW case (4.10) above). Again applying the Polyakov-Wiegmann identity, we then obtain

$$\begin{aligned} \mathcal{L} = {}& k\mathcal{L}_G(\tilde g) - (k+2c_{_G})\big[\mathcal{L}_G(\hat h) + \mathcal{L}_G(\bar h)\big] \\ &+ \mathrm{Tr}[\hat h^{-1}\partial_+\hat h\,\big(k+2c_{_G}+k(\lambda^{-1}-1)P_{G/H}\big)\,\partial_-\bar h\bar h^{-1}]\ . \end{aligned} \tag{4.40}$$

As in the group case, the first WZW term $k\mathcal{L}_G(\tilde g)$ decouples, leaving the RG flow to be determined by the "truncated" σ-model,

$$\mathcal{L}'(\hat h,\bar h) = -\tilde k\Big(\mathcal{L}_G(\hat h) + \mathcal{L}_G(\bar h) - \mathrm{Tr}\Big[\hat h^{-1}\partial_+\hat h\,(P_H + \tilde\lambda\, P_{G/H})\,\partial_-\bar h\bar h^{-1}\Big]\Big)\ , \tag{4.41}$$

$$\tilde k \equiv k + 2c_{_G}\ ,\qquad \tilde\lambda \equiv \frac{k}{\tilde k}\lambda^{-1} + \frac{2c_{_G}}{\tilde k}\ . \tag{4.42}$$

This is a WZW model on $G\times G$, deformed a left-right current interaction term, now containing projectors P_H,$P_{G/H}$ onto the subalgebra Lie(H) its orthogonal complement. The classical integrability of the model (4.41) (implying also the integrability of the G/H λ-model) was argued in [213] (example 4 there). It is also a special case of the integrable $G\times G/H$ models [61] considered above in eqs. (3.16),(3.19),(3.25), with the couplings and fields related according

to

$$(k, r, s, u, t, b) = (-\tilde{k}, -\tilde{k}, -\tilde{k}, -\tilde{k}, \tilde{k}\tilde{\lambda}, \tilde{k}\tilde{\lambda}) \ , \qquad (g^{(1)}, g^{(2)}) = (\hat{h}, \bar{h}^{-1}) \ . \tag{4.43}$$

The truncated model (4.41) is again automatically renormalizable due to manifest symmetries. As well as the gauge symmetry (4.39) and discrete symmetries, it is also invariant under the same chiral gauge transformations (4.23) as the group case. These symmetries protect the structure of (4.41), with $\tilde{\lambda}$ and $\tilde{k}$ being the only free parameters.

We shall explicitly demonstrate the renormalizability of the symmetric space λ-model (4.37) by computing the 2-loop β-function for the truncated model (4.41).[62] The computation runs analogously to the group case above, with the only difference being that one must correctly handle the gauge invariance. We decompose the generators $\{T_A\}$ of G into the generators $\{T_\alpha\}$ of H and the remaining generators $\{T_a\}$ chosen to be orthogonal to Lie(H) under the trace, i.e. splitting the algebra index of G as $A = \{\alpha, a\}$ (see Appendix A for conventions). Since G/H is a symmetric space then the $\mathbb{Z}_2$ grading of Lie(G) implies $f_{\alpha\beta c} = f_{abc} = 0$ (i.e. the only non-zero structure constants are $f_{\alpha\beta\gamma}$ and $f_{\alpha bc}$). The tangent space index for $G \times G$ now splits as $M = \{A; \bar{A}\} = \{\alpha, a; \bar{\alpha}, \bar{a}\}$, with the unbarred and barred indices corresponding respectively to the two G-valued fields $\hat{h}$ and $\bar{h}$.

Here we use the vielbein,

$$E^M = (e^a, e^{\bar{a}}, e^\alpha, e^{\bar{\alpha}}) = (\sqrt{1 - \tilde{\lambda}^2} J^a, \ K^{\bar{a}} + \tilde{\lambda} J^a, \ J^\alpha + K^{\bar{\alpha}}, \ J^\alpha - K^{\bar{\alpha}}) \ . \tag{4.44}$$

Up to permutations, the non-zero components of the metric and H-tensor are

$$\mathrm{G}_{ab} = \mathrm{G}_{\bar{a}\bar{b}} = \tfrac{\tilde{k}}{2} \operatorname{Tr}[T_a T_b] \ , \qquad \mathrm{G}_{\alpha\beta} = \tfrac{\tilde{k}}{2} \operatorname{Tr}[T_\alpha T_\beta] \ , \tag{4.45}$$

$$\mathrm{H}_{\alpha\beta\gamma} = -i\tfrac{\tilde{k}}{2} f_{\alpha\beta\gamma} \ , \qquad \mathrm{H}_{\alpha bc} = -i\tfrac{\tilde{k}}{2} f_{\alpha bc} \ , \qquad \mathrm{H}_{\alpha\bar{b}\bar{c}} = -i\tfrac{\tilde{k}}{2} f_{\alpha bc} \ . \tag{4.46}$$

The H gauge symmetry of the model (4.41) leads to the vanishing of all $\bar{\alpha}$-components of the metric and the H-tensor.

As discussed in App. D.2, there are various possible approaches to treat the H gauge symmetry. Here we shall use the trick of lifting the degeneracy of the metric with a small parameter ε acting as a "regulator",

$$\mathrm{G}_{\bar{\alpha}\bar{\beta}} = \varepsilon \, \tfrac{\tilde{k}}{2} \operatorname{Tr}[T_\alpha T_\beta] \ , \tag{4.47}$$

computing the (torsionful) Riemann tensor, projecting out the $\bar{\alpha}$ directions and setting the regulator ε to zero. This is a short-cut for a proper gauge-fixing procedure explained in footnote

[62] Like the group case above, the RG flow for the truncated model (4.42) is a special case of the that of the $G \times G/H$ models in Chapter 3 (given explicitly in App. D.2). The results here are in agreement with those in App. D.2 for the special values of the parameters in (4.43).

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The resulting torsionful curvature has the non-zero components

$$(\tilde\lambda^{-2}-1)\widehat R^a{}_{bde}=-\sqrt{\tilde\lambda^{-2}-1}\,\widehat R^a{}_{bd\bar e}=\widehat R^a{}_{bd\bar e}=-f^a{}_{b\gamma}f^\gamma{}_{de}\ . \tag{4.48}$$

Substituting this into the RG equation (1.9), one obtains

$$\big[\frac{d}{dt}(\mathrm{G}_{mn}+\mathrm{B}_{mn})\big]\,dx^m\otimes dx^n=\sum_{i=\bar\imath=1}^{\dim G-\dim H}\Big[c_{_G}\tilde\lambda+\frac{c_{_G}}{\tilde k}\frac{\tilde\lambda(c_{_H}-(2c_{_G}-c_{_H})\tilde\lambda^2)}{1-\tilde\lambda^2}\Big]\,\mathrm{Tr}[J\otimes P_{G/H}K]\ . \tag{4.49}$$

We conclude that only the $\tilde\lambda$-dependent term in (4.41) gets renormalized, i.e. the 2-loop RG equation (1.9) is solved with $\xi_m=\lambda_m=0$ and

$$\frac{d}{dt}\tilde k=0\ ,\qquad\qquad \frac{d}{dt}\tilde\lambda=\frac{c_{_G}\tilde\lambda}{\tilde k}\Big[1+\frac{1}{\tilde k}\frac{c_{_H}-(2c_{_G}-c_{_H})\tilde\lambda^2}{1-\tilde\lambda^2}\Big]\ . \tag{4.50}$$

The 1-loop term here agrees with the standard results for the symmetric space λ-model (eq. (1.86) with $\kappa=\frac{\tilde\lambda-1}{\tilde\lambda+1}$) [114, 53]. The level k is RG-invariant, and, as expected, $\tilde\lambda=\infty$ and $\tilde\lambda=0$ are fixed points. At the fixed point $\tilde\lambda=\infty$ ($\lambda=0$) the λ-model reduces to the G/H gWZW model. The other fixed point $\tilde\lambda=0$ ($\lambda=-\frac{k}{2c_{_G}}$) is a quantum-corrected version of the 1-loop fixed point $\lambda=\infty$; this model is like a version of the G/G cWZW model with an additional H gauge symmetry, and after integrating out $A_\pm$ it corresponds to the G/H gWZW model with level $-k$.

Expressing $\tilde k$ and $\tilde\lambda$ in (4.42) in terms of k and the coupling κ using (4.3), we find

$$\frac{d}{dt}k=0\ ,\qquad\qquad \frac{d}{dt}\kappa=\frac{c_{_G}}{2k}\Big[(1-\kappa^2)+\frac{c_{_G}(1-\kappa)^2(1+4\kappa-\kappa^2)-c_{_H}(1-\kappa^4)}{2k\kappa}\Big]\ . \tag{4.51}$$

As in the group space case (4.32), the 2-loop term in the β-function (4.50) is, in general, scheme-dependent. Again, we find that the symmetry under $k\to -k$, $\lambda\to\lambda^{-1}$ in (1.80), present at the 1-loop order is absent in the 2-loop term of (4.50). However, after introducing the shifted level $\mathrm{k}=\tilde k-c_{_G}=k+c_{_G}$ as in (4.33), the 2-loop RG equation for $\tilde\lambda$ (4.50) becomes

$$\frac{d}{dt}\tilde\lambda=\frac{c_{_G}\tilde\lambda}{\mathrm{k}}\Big[1-\frac{1}{\mathrm{k}}(c_{_G}-c_{_H})\frac{1+\tilde\lambda^2}{1-\tilde\lambda^2}\Big]\ , \tag{4.52}$$

which is manifestly invariant under $\mathrm{k}\to-\mathrm{k}$, $\tilde\lambda\to\tilde\lambda^{-1}$, i.e. a quantum version of the symmetry (1.80) of the original couplings (cf. (4.35)).

In the NAD limit (2.72), when the extended-space λ-model becomes the interpolating model (1.52) for NAD of the symmetric space, we get from (4.51)

$$\frac{d}{dt}\mathrm{h}=2c_{_G}+4c_{_G}(c_{_G}-c_{_H})\mathrm{h}^{-1}\ . \tag{4.53}$$

As verified in Appendix B of [3], equation (4.53) is the correct 2-loop β-function for the G/H symmetric space σ-model (matching eq. (1.32) above and, in particular cases, the results in [67, 64]). Like the group case above, the 2-loop coefficient here is also scheme-independent in this single-coupling limit, providing a non-trivial, scheme-invariant test of the λ-model's 2-loop β-function.

4.4 Low-dimensional examples: λ-models for $SU(2)$ and $SU(2)/U(1)$

Let us study the simplest case of the truncated model (4.20) in the group case with $G = SU(2)$. Parametrizing

$$\hat{h} = \exp(i\phi\sigma_3)\exp(i\theta\sigma_1)\exp(i\psi\sigma_3) \ , \qquad \bar{h} = \exp(i\bar{\phi}\sigma_3)\exp(i\bar{\theta}\sigma_1)\exp(i\bar{\psi}\sigma_3) \ , \tag{4.54}$$

where σ_A are the standard Pauli matrices, we find the following 6-dimensional metric and B-field (with $\mathrm{B} = \frac{1}{2}\mathrm{B}_{mn}dx^m \wedge dx^n$)

$$\begin{aligned}
&ds^2 = (ds^2)_{G\times G} + (ds^2)_{\text{int}} \ , \qquad \mathrm{B} = \mathrm{B}_{G\times G} + \mathrm{B}_{\text{int}} \ , \qquad (4.55)\\
&(ds^2)_{G\times G} = -\tilde{k}(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos 2\theta d\phi d\psi + d\bar{\theta}^2 + d\bar{\phi}^2 + d\bar{\psi}^2 + 2\cos 2\bar{\theta} d\bar{\phi} d\bar{\psi}) \ ,\\
&\mathrm{B}_{G\times G} = -\tilde{k}(\cos 2\theta d\phi \wedge d\psi + \cos 2\bar{\theta} d\bar{\phi} \wedge d\bar{\psi}) \ ,\\
&(ds^2)_{\text{int}} = -2\tilde{k}\tilde{\lambda}\big[\cos 2(\psi+\bar{\phi})(d\theta d\bar{\theta} - \sin 2\theta \sin 2\bar{\theta} d\phi d\bar{\psi}) + (d\psi + \cos 2\theta d\phi)(d\bar{\phi} + \cos 2\bar{\theta} d\bar{\psi})\\
&\qquad\qquad + \sin 2(\psi+\bar{\phi})(\sin 2\theta d\phi d\bar{\theta} + \sin 2\bar{\theta} d\theta d\bar{\psi})\big] \ ,\\
&\mathrm{B}_{\text{int}} = -\tilde{k}\tilde{\lambda}\big[\cos 2(\psi+\bar{\phi})(d\theta \wedge d\bar{\theta} - \sin 2\theta \sin 2\bar{\theta} d\phi \wedge d\bar{\psi}) + (d\psi + \cos 2\theta d\phi) \wedge (d\bar{\phi} + \cos 2\bar{\theta} d\bar{\psi})\\
&\qquad\qquad + \sin 2(\psi+\bar{\phi})(\sin 2\theta d\phi \wedge d\bar{\theta} + \sin 2\bar{\theta} d\theta \wedge d\bar{\psi})\big] \ .
\end{aligned}$$

Here the "$G\times G$" terms come from the WZW models and the "int" terms come from the current-current interactions between them. One can check explicitly that this metric and B-field solve the 2-loop RG equation (1.9) with vanishing ξ^m and λ_m, and the couplings running as in (4.31) (with the dual Coxeter number $c_{_{SU(2)}} = 2$).

Next, let us consider the truncated model (4.41) obtained from the λ-model based on the symmetric space $G/H = SU(2)/U(1)$. Using the parametrization for $\hat{h}$ and $\bar{h}$ given in eq. (4.54), we find that the corresponding target space background depends on ψ and $\bar{\phi}$ only through the combination $\chi = \psi + \bar{\phi}$, which is a manifestation of the $U(1)$ gauge symmetry. The resulting

5-dimensional metric and B-field are

$$
\begin{aligned}
ds^2 &= (ds^2)_{G\times G/H} + (ds^2)_{\rm int} \ , \qquad\qquad \mathrm{B} = \mathrm{B}_{G\times G/H} + \mathrm{B}_{\rm int} \ , \qquad (4.56)\\
(ds^2)_{G\times G/H} &= -\tilde{k}(d\theta^2 + d\phi^2 + d\bar{\theta}^2 + d\bar{\psi}^2) \ , \qquad \mathrm{B}_{G\times G/H} = 0 \ ,\\
(ds^2)_{\rm int} &= -2\tilde{k}\tilde{\lambda}\big[\cos 2\chi(d\theta d\bar{\theta} - \sin 2\theta \sin 2\bar{\theta} d\phi d\bar{\psi}) + \sin 2\chi(\sin 2\theta d\phi d\bar{\theta} + \sin 2\bar{\theta} d\theta d\bar{\psi})\big]\\
&\qquad + \tilde{k} d\chi^2 - 2\tilde{k}(d\chi + \cos 2\theta d\phi)(d\chi + \cos 2\bar{\theta} d\bar{\psi}) \ ,\\
\mathrm{B}_{\rm int} &= -\tilde{k}\tilde{\lambda}\big[\cos 2\chi(d\theta \wedge d\bar{\theta} - \sin 2\theta \sin 2\bar{\theta} d\phi \wedge d\bar{\psi}) + \sin 2\chi(\sin 2\theta d\phi \wedge d\bar{\theta} + \sin 2\bar{\theta} d\theta \wedge d\bar{\psi})\big]\\
&\qquad - \tilde{k}(d\chi + \cos 2\theta d\phi) \wedge (d\chi + \cos 2\bar{\theta} d\bar{\psi}) \ .
\end{aligned}
$$

Again one can check explicitly that this metric and B-field solve the 2-loop RG equation (1.9) with vanishing ξ^m and λ_m, and the couplings running as in (4.50) (with the dual Coxeter numbers $c_{_{SU(2)}} = 2$ and $c_{_{U(1)}} = 0$).

4.5 "Tripled" formulations of other models

In principle the trick of rewriting 2d gauge fields in terms of group-valued scalars can be applied to any theory with 2d gauge fields. In this section, containing unpublished work, we shall construct analogous "tripled" or "extended" formulations of two other classes of σ-models.

4.5.1 G^N λ-models

First let us consider a generalization of the λ-model to N copies of the group G, which was recently obtained from affine Gaudin models [161],

$$
\mathcal{L} = k_i \, \mathcal{L}_{G/G}(g^{(i)}, A^{(i)}) - \tfrac{1}{2} \, \gamma_{ij} \, \mathrm{Tr}[A_+^{(i)} A_-^{(j)}] \ , \qquad\qquad i = 1, \dots n \ . \qquad (4.57)
$$

Here k_i are the WZ levels and γ_{ij} is 2×2 matrix of coupling constants. Integrating out the quadratic gauge fields $A_\pm^{(i)}$ yields a σ-model with scalar fields $g^{(i)} \in G$.

This may be viewed as an example of Tseytlin's "universal class" model (1.77) adapted to G^N, with the constant operator Q chosen to preserve the global G symmetry $(g^{(i)}, A^{(i)}) \to (ug^{(i)}u^{-1}, uA^{(i)}u^{-1})$, $u \in G$. Like the G^N models discussed in Chapter 3, the theory (4.57) is classically integrable only for special values of the coupling constants (k_i, γ_{ij}), parametrized algebraically in terms of the twist function in [161]. Although we shall not work out the details here, this parametrization may be recast as a set of polynomial equations in (k_i, γ_{ij}) (cf. the equation (3.6) for undeformed $G \times G$ models).

Like standard λ-models, the G^N λ-model (4.57) is a deformation of a WZW model (when γ_{ij} is taken to infinity while being non-degenerate) and becomes in a particular limit, $k_i \to \infty$,

$g^{(i)} = e^{-\frac{v^{(i)}}{2k_i}}$, the non-abelian dual of a particular G^N model (3.2) in its $(G_L)^N$ symmetry,

$$\mathcal{L} = -\tfrac{1}{2}\,\mathrm{Tr}[\gamma_{ij} A_+^{(i)} A_-^{(j)} - v^{(i)} F_{+-}(A^{(i)})]\ , \qquad F_{+-}(A) \equiv \partial_+ A_- - \partial_- A_+ + [A_+, A_-]\ . \quad (4.58)$$

Our observation is that the tripled formulation of the λ-model generalizes very easily to this G^N case. Let us perform the same change of variables $(A_+^{(i)}, A_-^{(i)}) \to (h^{(i)}, \bar{h}^{(i)})$ on each copy of the group, where $A_+^{(i)} = h^{(i)}\partial_+\big(h^{(i)}\big)^{-1}$, $A_-^{(i)} = \bar{h}^{(i)}\partial_-\big(\bar{h}^{(i)}\big)^{-1}$ (cf. eq. (4.4)). Including the Jacobian (4.7) for this transformation (and picking again q = −1), the resulting theory is a "tripled" σ-model with fields $(g^{(i)}, h^{(i)}, \bar{h}^{(i)}) \in G^{3N}$,

$$\begin{aligned}
&\mathcal{L} = k_i\,\mathcal{L}_G(\tilde{g}^{(i)}) + \mathcal{L}'(\hat{h}^{(i)}, \bar{h}^{(i)})\ , \qquad\qquad \tilde{g}^{(i)} \equiv \big(h^{(i)}\big)^{-1} g^{(i)} \bar{h}^{(i)}\ , \quad \hat{h}^{(i)} \equiv \big(h^{(i)}\big)\ , \\
&\mathcal{L}' = -(k_i + 2c_{_G})\big[\mathcal{L}_G(\hat{h}^{(i)}) + \mathcal{L}_G(\bar{h}^{(i)})\big] + \mathrm{Tr}\big[(k_i\delta_{ij} - \tfrac{1}{2}\gamma_{ij})\, J_+^{(i)}\, \bar{K}_-^{(j)}\big]\ , \\
&\qquad J_+^{(i)} \equiv \big(\hat{h}^{(i)}\big)^{-1}\partial_+\hat{h}^{(i)}\ , \qquad \bar{K}_-^{(i)} \equiv \partial_+\bar{h}^{(i)}\big(\bar{h}^{(i)}\big)^{-1}\ .
\end{aligned} \quad (4.59)$$

This is a particular case of the undeformed G^M model (3.2) with $M = 3N$ (after a redefinition $\bar{h}^{(i)} \to \big(\bar{h}^{(i)}\big)^{-1}$).

As in the λ-model considered above, the first WZW terms $k_i\,\mathcal{L}_G(\tilde{g}^{(i)})$ in (4.59) decouple, leaving the RG flow to be determined by the "truncated" Lagrangian $\mathcal{L}'$, which is a σ-model with fields $(h^{(i)}, \bar{h}^{(i)}) \in G^{2N}$. This truncated theory is also a particular case of the undeformed G^M model (3.2) with $M = 2N$.

Hence we have produced a quantum equivalence between (i) the λ-deformation (4.57) based on G^N models and (ii) certain G^M models with $M = 2N$ (plus N decoupled WZW models). In effect, moving from the G^N model to the related λ-deformation is equivalent to replacing $N \to 2N$, restricting to particular class of the G^{2N} model's couplings, and adding N decoupled WZW models.

Let us make some observations. As mentioned above, the G^N λ-model is only integrable for special values of the couplings solving polynomial equations. This is indeed consistent with the equivalence found here, since the same is true for the undeformed G^M models of which the tripled formulation (4.59) is a special case. Hence one should be able to understand the integrability conditions for G^N λ-models in terms of those of undeformed G^M models; in this respect the λ-deformation does not introduce any new elements. It would be interesting to understand the relation between these models at the level of affine Gaudin models and the twist functions.

Renormalizability of the G^N λ-model (4.57) does not seem obvious based alone on its manifest G symmetry, which is not sufficient to protect the form of the action. However, in the tripled formulation (4.59), there is again a decoupling of some fields $\tilde{g}^i$ and the remaining

truncated theory $\mathcal{L}'$ is protected by $G(\xi^-)^N \times G(\xi^+)^N \times G$ chiral gauge symmetry,

$$\begin{aligned} &(\hat{h}^{(i)},\ \bar{h}^{(i)}) \to (u^{(i)}(\xi^-)\,\hat{h}^{(i)}\,w,\ w^{-1}\,\bar{h}^{(i)}\,v^{(i)}(\xi^+))\ , \\ &u^{(i)}(\xi^-) \in G(\xi^-)\ , \quad v^{(i)}(\xi^+) \in G(\xi^+)\ , \quad w \in G\ , \end{aligned} \tag{4.60}$$

which uniquely fixes the theory up to the parameters (k_i, γ_{ij}). Hence it will be renormalizable to all orders with just γ_{ij} running (while the WZ levels do not run for general reasons). Since the original and tripled formulations are equivalent at the quantum level then we have proved the renormalizability of the G^N λ-model (4.57) with unintegrated gauge field to all orders.

So far we have established the renormalizability of the G^N λ-model (4.57) before imposing the integrability condition. We expect the "integrable surface" in the space of couplings (k_i, γ_{ij}) to be a stable subspace under the RG flow (cf. the $G \times G$ and $G \times G/H$ models studied in Chapter 3). It would be interesting to check this fact and to see whether the integrability conditions require quantum corrections to remain stable beyond 1-loop. The tripled formulation (4.59) is the natural arena to study these questions since it is already RG-stable with a finite set of couplings before imposing the integrability condition.

4.5.2 $G \times G/H$ models

Now let us consider the H-gauge-invariant $G \times G/H$ models considered in Section 3.2, with Lagrangian,

$$\begin{aligned} &\mathcal{L} = -\tfrac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[P_+^{(i)}P_-^{(j)}] - \tfrac{1}{2}\,r_{ij}\,\mathrm{Tr}[B_+^{(i)}B_-^{(j)}] + k_i\,\mathcal{L}_{\mathrm{wz}}(g^{(i)})\ , \\ &k_1 = -k_2 \equiv k\ , \qquad r_{ij} = \begin{pmatrix} r & -r-k \\ -r+k & r \end{pmatrix}\ , \\ &P^{(i)} = P_{G/H}J^{(i)}\ , \qquad B^{(i)} = P_H J^{(i)}\ , \qquad J^{(i)} = {g^{(i)}}^{-1}dg^{(i)}\ . \end{aligned} \tag{4.61}$$

First let us re-write the theory in terms of a 2d gauge field by starting with an ungauged $(G \times G)$-type model (here R_{ij} are arbitrary constants and $k_1 = -k_2 = -k$),

$$\mathcal{L} = -\tfrac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[P_+^{(i)}P_-^{(j)}] - \tfrac{1}{2}\,R_{ij}\,\mathrm{Tr}[B_+^{(i)}B_-^{(j)}] + k\left(\mathcal{L}_{\mathrm{wz}}(g^{(1)}) - \mathcal{L}_{\mathrm{wz}}(g^{(2)})\right)\ , \tag{4.62}$$

and gauging the H global symmetry $(g^{(1)}, g^{(2)}) \to (g^{(1)}h, g^{(2)}h)$, $h \in H$ by the same gauging procedure discussed for coset spaces in Section 1.4.3. Promoting $h \to h(\xi)$ to a local parameter,

the Lagrangian transforms as

$$\begin{aligned}
\mathcal{L} \to & -\tfrac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[P_+^{(i)}P_-^{(j)}] + k\,\big(\mathcal{L}_{\rm wz}(g^{(1)}) - \mathcal{L}_{\rm wz}(g^{(2)})\big) \\
& - \tfrac{1}{2}\,\mathrm{Tr}[R_{ij}\,B_+^{(i)}B_-^{(j)} + \partial_+ hh^{-1}(\alpha\,B_-^{(1)} + \beta\,B_-^{(2)}) \\
& \qquad\qquad + \partial_- hh^{-1}(\gamma\,B_+^{(1)} + \delta\,B_+^{(2)}) + \varepsilon\,\partial_+ hh^{-1}\,\partial_- hh^{-1}], \qquad (4.63)\\
\alpha = R_{11} + R_{21} - k\ , & \quad \beta = R_{12} + R_{22} + k\ , \\
\gamma = R_{11} + R_{12} + k\ , & \quad \delta = R_{21} + R_{22} - k\ , \quad \varepsilon = R_{11} + R_{12} + R_{21} + R_{22}\ .
\end{aligned}$$

Now promoting $\partial_+ hh^{-1} \to -A_\pm$ in the transformed Lagrangian (4.63), the resulting theory is gauge invariant (under $g \to h^{-1}gh$, $A_\pm \to h^{-1}A_\pm h + h^{-1}\partial_\pm h$, $h(\xi) \in H$),

$$\begin{aligned}
\mathcal{L}(g,A) = & -\tfrac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[P_+^{(i)}P_-^{(j)}] + k\,\big(\mathcal{L}_{\rm wz}(g^{(1)}) - \mathcal{L}_{\rm wz}(g^{(2)})\big) \qquad (4.64)\\
& - \tfrac{1}{2}\,\mathrm{Tr}[R_{ij}\,B_+^{(i)}B_-^{(j)} - A_+(\alpha\,B_-^{(1)} + \beta\,B_-^{(2)}) - A_-(\gamma\,B_+^{(1)} + \delta\,B_+^{(2)}) + \varepsilon\,A_+A_-]\ .
\end{aligned}$$

Upon integrating out the 2d gauge field $A_\pm$, one indeed obtains the desired $G \times G/H$ model (4.61) with the following identification of parameters

$$r = \frac{k^2 + k(R_{12} - R_{21}) + R_{11}R_{22} - R_{12}R_{21}}{R_{11} + R_{12} + R_{21} + R_{22}}\ , \qquad k = k\ . \qquad (4.65)$$

Hence the model (4.64) with the gauge field is equivalent to the $G \times G/H$ model (4.61) and, moreover, only one combination of the couplings R_{ij} appears after integrating out $A_\pm$. This means one is free to fix any 3 components (or combinations of components) of R_{ij} to arbitrary values, as long as $R_{11} + R_{12} + R_{21} + R_{22} \neq 0$ so that (4.65) does not diverge.

For general R_{ij} we may again change variables $(A_+, A_-) \to (h, \bar{h})$ according to (4.4) (this is like replacing $\partial_- hh^{-1} \to \partial_- \bar{h}\bar{h}^{-1}$ in (4.63)). The resulting theory is a σ-model with an enlarged configuration space $(g^{(1)}, g^{(2)}, h, \bar{h}) \in G \times G \times H \times H$,

$$\begin{aligned}
\mathcal{L}(g^{(1)}, g^{(2)}, h, \bar{h}) = & -\tfrac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[P_+^{(i)}P_-^{(j)}] + k\,\big(\mathcal{L}_{\rm wz}(g^{(1)}) - \mathcal{L}_{\rm wz}(g^{(2)})\big) \\
& - \tfrac{1}{2}\,\mathrm{Tr}[R_{ij}\,B_+^{(i)}B_-^{(j)} + \partial_+ hh^{-1}(\alpha\,B_-^{(1)} + \beta\,B_-^{(2)}) \qquad (4.66)\\
& \qquad\qquad + \partial_- \bar{h}\bar{h}^{-1}(\gamma\,B_+^{(1)} + \delta\,B_+^{(2)}) + \varepsilon\,\partial_+ hh^{-1}\partial_- \bar{h}\bar{h}^{-1}]\ .
\end{aligned}$$

One may fix the gauge symmetry as, e.g., $\bar{h} = 1$ (or $A_- = 0$) to obtain

$$\begin{aligned}
\mathcal{L}(g^{(1)}, g^{(2)}, h) = & -\tfrac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[P_+^{(i)}P_-^{(j)}] + k\,\big(\mathcal{L}_{\rm wz}(g^{(1)}) - \mathcal{L}_{\rm wz}(g^{(2)})\big) \\
& - \tfrac{1}{2}\,\mathrm{Tr}[R_{ij}\,B_+^{(i)}B_-^{(j)} + \partial_+ hh^{-1}(\alpha\,B_-^{(1)} + \beta\,B_-^{(2)})]\ , \qquad (4.67)\\
\alpha = R_{11} + R_{21} - k\ , & \qquad \beta = R_{12} + R_{22} + k\ .
\end{aligned}$$

The resulting gauge-fixed model (4.67) is a σ-model with configuration space $(g^{(1)}, g^{(2)}, h) \in G \times G \times H$. It may be described as a particular "squashed" $G \times G \times H$ model, since the $G \times G$

sector includes dependence on the projectors P_H and $P_{G/H}$.

One simple example of (4.67) obtained from the choice

$$R_{ij} = \begin{pmatrix} r & 0 \\ -r+k & 0 \end{pmatrix} \ , \tag{4.68}$$

is given by

$$\begin{aligned} \mathcal{L}(g^{(1)}, g^{(2)}, h) = & -\tfrac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[P_+^{(i)} P_-^{(j)}] + k\,\big(\mathcal{L}_{\rm wz}(g^{(1)}) - \mathcal{L}_{\rm wz}(g^{(2)})\big) \\ & -\tfrac{1}{2}\,\mathrm{Tr}[(rB_+^{(1)} + (k-r)B_+^{(2)})B_-^{(1)} + k\,\partial_+ h h^{-1} B_-^{(2)}] \ . \end{aligned} \tag{4.69}$$

Another natural choice consistent with (4.65) is

$$R_{ij} = \epsilon\,\rho_{ij} \ , \qquad r = \frac{k^2\varepsilon^{-1} + k(R_{12} - R_{21}) + \varepsilon(R_{11}R_{22} - R_{12}R_{21})}{R_{11} + R_{12} + R_{21} + R_{22}} \ . \tag{4.70}$$

Then (4.67) becomes

$$\begin{aligned} \mathcal{L}(g^{(1)}, g^{(2)}, h) = & -\tfrac{1}{2}\,\rho_{ij}\,\mathrm{Tr}[\varepsilon B_+^{(i)} B^{(j)} + P_+^{(i)} P_-^{(j)}] + k\,\big(\mathcal{L}_{\rm wz}(g^{(1)}) - \mathcal{L}_{\rm wz}(g^{(2)})\big) \\ & -\tfrac{1}{2}\,\mathrm{Tr}[\partial_+ h h^{-1}\,\big(\alpha\,B_-^{(1)} + \beta\,B_-^{(2)}\big)] \ , \\ \alpha = & \ \varepsilon(\rho_{11} + \rho_{21}) - k \ , \qquad \beta = \varepsilon(\rho_{12} + \rho_{22}) + k \ . \end{aligned} \tag{4.71}$$

These extended-space reformulations of the $G \times G/H$ models will inherit the same classical integrability and RG flow properties as the original theory (modulo the inclusion of a Jacobian of the same form as (4.7)). Hence we learn that the extended $G \times G \times H$ models (4.67) (and the particular cases (4.69) and (4.71)) should also be integrable when the couplings r, k, ρ_{ij} satisfy the integrability conditions (3.25) of the $G \times G/H$ model. Speculatively, one might hope that, after imposing the integrability conditions, there could be a decoupling of some fields (as for the λ-models studied above), which may leave the remaining theory manifestly renormalizable due to its global symmetries. Such a phenomenon would be interesting as (i) it could provide another origin for the integrability conditions (3.25) of the $G \times G/H$ model and (ii) in such a reformulation, the $G \times G/H$ model's integrability condition would become automatically stable at higher-loop orders (cf. eq. (3.34) in the standard formulation).

4.6 Discussion

Formulating the λ-model on an extended $(G \times G \times G)$ configuration space "linearizes" the RG flow, i.e. makes its renormalizability manifest without the need for extra local counterterms apart from the running of the coupling λ. This implies that the λ-model with unintegrated gauge field is renormalizable to all orders.

As well as eliminating the need for quantum corrections to remain renormalizable at higher-loop orders, the tripled formulation also explains the λ-model's renormalizability already at the 1-loop order. Whereas the standard formulation (4.1) is apparently protected under the RG flow by *hidden* symmetries, the tripled formulation is protected by *manifest* symmetries.

The decoupling of one of the WZW models $k\mathcal{L}_G(\tilde{g})$ in (4.17) and (4.40) appears to be the special property that renders the manifest global symmetries sufficient for renormalizability of the remaining "truncated" models. This property should be associated with integrability, since it would be lost under generic deformations of the λ-model (4.1) preserving its manifest symmetries but breaking integrability. It would be interesting in future to understand the precise relationship between integrability and this special decoupling in the tripled formulation.

In the non-abelian dual limit (2.72), the resulting scheme-invariant 2-loop β-functions matched the known β-functions for the PCM and symmetric space. One may regard this tripled computation as a particular scheme for the renormalization of the interpolating models (1.52) for non-abelian duality (obtained from the tripled λ-model in this limit). One might worry that the NAD limit could be too singular to commute with the RG flow (e.g., the decoupling of the $\mathcal{L}_G(\tilde{g})$ term in (4.17) breaks down in that limit), but this is not likely since one can independently argue the renormalizability of the interpolating models (1.52) through their relation to the original PCM/symmetric space models by an integral over the linear Lagrange multiplier v. This integral is not summing any loops as v only appears linearly, so the classical result $\delta(F_{+-}(A))$ is expected to be consistent with the RG flow, without needing any extra counterterms.[63] It would be interesting in future to check this fact directly, e.g. computing the RG flow for a "tripled" formulation of the interpolating models for NAD.

If one regards the interpolating models, or the corresponding tripled σ-models, as particular formulations of the corresponding NAD theories at the quantum level, then it is natural to conclude that non-abelian duality commutes with the RG flow beyond the 1-loop order on arbitrary groups G and symmetric spaces G/H, with the original and NAD models having the same 2-loop β-function. This is in line with previous 1-loop conclusions [75, 76] and resolves related confusions at the 2-loop level [188, 189, 190, 191].

Hence, for (i) the λ-models with unintegrated gauge field and (ii) the interpolating models for NAD, the non-trivial corrections that were required in the standard configuration space in Chapter 2 should be understood as finite counterterms resulting from the integral over the 2d gauge field appearing quadratically in the action. It remains to understand the general structure of those counterterms, or equivalently the correct prescription for the 2d gauge field integral consistent with integrability (generalizing (2.42), which was sufficient only in the simple $\frac{SU(2)}{U(1)}$ example).

[63] I thank A.A. Tseytlin for a discussion of this point.

The 2-loop β-functions (4.32) and (4.51) obtained through the tripled formulation are consistent with those found for particular examples of the λ-model in the standard formulation in eqs. (2.45) and (2.57), modulo a redefinition of the couplings k, λ, i.e. scheme change (see [3] for the precise redefinitions that relate them). In particular this means that the scheme-invariant 1-loop terms match exactly; moreover the 2-loop terms also match in the single-coupling NAD limit when they become scheme-invariant.

A similar "tripled" approach may also be useful for clarifying the higher-loop deformation in abelian T-duality. For example, consider the metric $ds^2 = dy^2 + a(y)dx^2$ and its classical dual $\widetilde{ds^2} = dy^2 + a(y)^{-1}d\tilde{x}^2$, with the interpolating model given by

$$-\mathcal{L} = (\partial_\mu y)^2 + a(y)(p_\mu)^2 + \tilde{x}\,\varepsilon^{\mu\nu}\partial_\mu p_\nu \ , \tag{4.72}$$

such that y is a spectator coordinate (cf. eq. (1.48)). If we integrate out $\tilde{x}$ to give $p_\mu = \partial_\mu x$, we recover the original model for x. If we integrate out p_μ, we find the T-dual model for $\tilde{x}$. If instead we set $p_1 = \partial_1 x$ and integrate out p_0 we get the "doubled" model of [228] for x and $\tilde{x}$ (equivalent to the "axial" gauge choice in the Appendix of [229]). The "tripled" model for $(x, \tilde{x}, \bar{x})$, analogous to the ones discussed above, is obtained by setting $p_\mu = \partial_\mu x + \varepsilon_{\mu\nu}\partial^\nu \bar{x}$:

$$-\mathcal{L} = (\partial_\mu y)^2 + a(y)[(\partial_\mu x)^2 - (\partial_\mu \bar{x})^2 + 2\varepsilon^{\mu\nu}\partial_\mu x \partial_\nu \bar{x}] + \partial_\mu \tilde{x}\partial^\mu \bar{x} \ . \tag{4.73}$$

This may be interpreted as a σ-model on 4-dimensional target space with pp-wave metric and B-field.

We also considered similar "tripled" or "extended" formulations for other models. We found that the tripled formulation of the λ-model generalizes easily to λ-models based on G^N. In this tripled formulation, it is clear that the G^N λ-models are renormalizable to all orders due to manifest symmetries (before imposing the integrability condition). The resulting tripled σ-models are particular class of G^{3N} models, or G^{2N} models after removing decoupled WZW models. Hence in this case we find a correspondence between the integrability and RG flow properties of G^N λ-models and certain undeformed G^{2N} models.

We reformulated the $G \times G/H$ models studied in Chapter 3 in terms of a gauge field and found a 3-parameter family of extended formulations (3 of the components of R_{ij}, with the 4th being fixed by the relation (4.65)). The resulting σ-model has configuration space $G\times G\times H\times H$, or $G \times G \times H$ after fixing the H gauge symmetry. Hence we find a correspondence between the integrability and RG properties of $G \times G/H$ models and certain "squashed" $G \times G \times H$ models. It would be interesting to study the RG flow for both G^N λ-models and $G \times G/H$ models in these new, extended formulations.

A useful extension of the tripled formulation would be to the η-model. Up to analytic

continuation, the η-model and λ-model are related by limits and T-duality [104, 107] or by Poisson-Lie duality [104, 105, 107], so it seems plausible that the tripled formulation of the λ-model could lead to a similar approach for the η-model. This could allow a generalizaton of our results for the $\frac{SU(2)}{U(1)}$ η-model in Section 2.2 to general groups and symmetric spaces.

Another obvious generalization would be to σ-models on supergroups and supercosets (see, e.g., [230, 231, 124, 53]), and to integrable-deformed superstring σ-models. See Section 6.4 for further discussion of this possibility.

Chapter 5

Sigma models with local couplings

There is a remarkable link between the integrability of 2d σ-models and their stability under RG flow, i.e. their renormalizability with finitely many running couplings. However, it remains rather mysterious how the integrability translates into simple RG behaviour.

In this chapter we shall present a new and different link between classical integrability and the RG flow (based on the paper [4] and closely following the presentation there). Consider a familiar example of an integrable 2d σ-model, the principal chiral model (PCM). Suppose we formally replace its overall coupling h by a function of 2d time τ, obtaining the Lagrangian

$$\widehat{\mathcal{L}} = -\tfrac{1}{2}h(\tau)\,\mathrm{Tr}[J_+J_-]\ . \tag{5.1}$$

The resulting time-dependent theory is not Lorentz-invariant but one may wonder if some notion of classical integrability still applies even for a non-constant $h(\tau)$. As we shall see below, this is indeed the case *provided* the function $h(\tau)$ is special: the same as the solution of the 1-loop RG equation of the standard PCM, i.e. $h(\tau) \sim \tau$.

More generally, starting with a Lorentz-invariant 2d theory that is classically integrable (i.e. admitting a Lax connection), one may formally promote its coupling constants h_α to functions $h_\alpha(\tau)$ depending on 2d time. We address the question of which functions $h_\alpha(\tau)$ allow the integrability to survive in the resulting time-dependent theory, in the sense that the resulting equations of motion still admit a Lax representation. We shall suggest a certain natural ansatz for the corresponding Lax connection generalizing that of the time-independent theory and will find that the unique functions $h_\alpha(\tau)$ preserving the integrability under this ansatz are the ones solving the 1-loop RG equations of the original theory,

$$\partial_\tau h_\alpha = \beta_\alpha(h)\ , \tag{5.2}$$

with 2d time τ playing the role of RG time $t = \log\mu$. We observe this new 'integrability–RG flow' connection in a number of non-trivial examples but will not give a general proof of it.

One may wonder how such time-dependent models (where, e.g., energy is no longer con-

served) can be integrable. In fact, integrable examples of time-dependent 1d mechanical models are known (see, e.g., [232] and refs. there). Our present interest in such 2d models is motivated by the desire to find new solvable examples of conformal σ-models representing consistent string backgrounds with Minkowski signature. One class of examples would be conformal σ-models with $(D+2)$-dimensional target space metric $ds^2 = \mathrm{G}_{mn}(x)dx^m dx^n = -2dudv + K(u,x)du^2 + dx^i dx^i$ $(x^m = (u, v, x^i),\ i = 1, ..., D)$, admitting a covariantly constant null Killing vector. In the light-cone (l.c.) gauge $u = \tau$, one then gets a model with a time-dependent potential $K(\tau, x)$, which is formally solvable in some cases if K is quadratic in x^i (see, e.g., [233] and refs. there).[64]

The most general metric with a covariantly constant null Killing vector has the Brinkmann-Walker form, $ds^2 = -2dudv + K(u,x)du^2 + A_i(u,x)dudx^i + \mathrm{G}_{ij}(u,x)dx^i dx^j$. Once one relaxes the assumption that G_{ij} is flat, the K and A_i terms can be absorbed into $\mathrm{G}_{ij}(u,x)$ by a coordinate transformation to give $ds^2 = -2dudv + \mathrm{G}_{ij}(u,x)dx^i dx^j$. We shall focus on the remarkable class of string σ-models on such backgrounds [238, 239],

$$\mathcal{L} = -2\partial_+ u \partial_- v + \mathrm{G}_{ij}(u,x)\partial_+ x^i \partial_- x^j \ . \tag{5.3}$$

Moving to a curved 2d background $h_{\mu\nu}$ and incuding also a dilaton term $\mathcal{L}_\phi = -\alpha' \sqrt{-h}\, R^{(2)}\phi(v,u,x)$, this σ-model is Weyl-invariant, i.e. $\bar\beta_{mn} = R_{mn} + 2D_m D_n \phi + ... = 0$, provided ϕ is linear in the null coordinate v and $\mathrm{G}_{ij}(u,x)$ depends on u according to the RG equation of the 'transverse' σ-model $\mathcal{L} = \mathrm{G}_{ij}(x)\partial_+ x^i \partial_- x^j$ [238, 239],

$$\partial_u \mathrm{G}_{ij} = \bar\beta_{ij}(\mathrm{G}) = \beta_{ij} + 2D_i D_j \bar\phi \ , \qquad \beta_{ij} = R_{ij} + \dots \ , \tag{5.4}$$

$$\phi = v + \bar\phi(u,x) \ . \tag{5.5}$$

The 'first-order' equation (5.4) comes from the contribution of the connection in the $D_m D_n \phi$ term in the Weyl-invariance condition due to the null Killing form of the metric and the v-term in the dilaton.[65] We have ignored an additional diffeomorphism term $D_{(i}W_{j)}$ in (5.4) that appears only at higher loop orders. The model (5.3) also admits a natural generalization to the case of non-zero 'transverse' B-field coupling $\mathrm{B}_{ij}(u,x)$: the resulting $(D+2)$-dimensional σ-model is Weyl invariant provided $\partial_u(\mathrm{G}_{ij} + \mathrm{B}_{ij}) = \bar\beta_{ij}(\mathrm{G}, \mathrm{B})$ where $\bar\beta_{ij}$ is the corresponding Weyl anomaly coefficient [238, 239].

One example of such a conformal background with a sphere S^D as the transverse space is (to 1-loop order) $ds^2 = -2dudv + u\, d\Omega_D^2$, $\phi = v + \frac{D-24}{6}u + \frac{1}{8}D\log u$.

This construction is also naturally generalized to supersymmetric σ-models [239]. On certain

[64] Another solvable example obtained as a certain Yang-Baxter deformation of flat 4d Minkowski space was found in [234]. Examples with K not depending on τ corresponding to integrable l.c.-gauge theories (sine-Gordon, Liouville, etc.) were discussed, e.g., in [235, 236, 237].

[65] The exact RG evolution equation (5.4) in the l.c. direction u may be compared to an approximate RG equation (see, e.g., [240]) appearing in the case of a 'cosmological' metric and a time-dependent dilaton which provides a 'friction' term in the string generalization of the Einstein equations.

special Kähler symmetric spaces, the supersymmetric σ-models (which then have $\mathcal{N} = (2,2)$ supersymmetry [241]) are known to have a 1-loop exact β-function [242].[66] Hence, taking one of those as the transverse theory, the background in (5.3),(5.4),(5.5) resulting from the 1-loop RG flow will be *exact* string backgrounds [239].

Starting with the classical string σ-model (5.3) in conformal gauge and fixing the residual conformal symmetry by the l.c. gauge condition $u = \tau$, we get a time-dependent theory for the transverse coordinates[67]

$$\mathcal{L}_{l.c.} = \mathrm{G}_{ij}(\tau, x)\partial_+ x^i \partial_- x^j \ , \qquad \partial_\tau \mathrm{G}_{ij} = \beta_{ij}(\mathrm{G}) \ . \tag{5.6}$$

If we now assume that the transverse σ-model $\mathcal{L} = \mathrm{G}_{ij}(x)\partial_+ x^i \partial_- x^j$ is classically integrable, then the fact that $\mathrm{G}_{ij}(\tau, x)$ is subject to the RG equation in (5.6) suggests, in view of the above 'integrability–RG flow' connection, that the l.c. gauge theory (5.6) is classically integrable (admits a flat Lax connection) and thus may potentially be solvable.

Let us note that a special class of the σ-models (5.3) with $\mathrm{G}_{ij}(u, x) = u\, g_{ij}(x)$ are obtained upon fixing the conformal gauge $h_{\mu\nu} d\xi^\mu d\xi^\nu = e^{-2v}\eta_{\mu\nu}$ (so that $\sqrt{-h}\, R^{(2)} = -2\partial_+\partial_- v$) in a theory of 2d dilaton gravity (with dilaton u) coupled to a σ-model [238] with $g_{ij}(x)$ as the target space metric,

$$\mathcal{L} = -\sqrt{-h}\, u\big[R^{(2)} + g_{ij}(x)\, h^{\mu\nu}\, \partial_\mu x^i \partial_\nu x^j\big] \ . \tag{5.7}$$

Particular 2d models of the type (5.7), with $g_{ij}(x)$ corresponding to specific symmetric spaces, appear upon dimensional reduction (in 2 Killing vector directions) of 4d Einstein gravity [245, 246, 247]. After solving $\partial_+\partial_- u = 0$ (the equation from varying v) to obtain $u(\tau, \sigma) = f^+(\xi^+) + f^-(\xi^-)$, the resulting 'local coupling' σ-model,

$$\mathcal{L} = u(\tau, \sigma)\, g_{ij}(x)\partial_+ x^i \partial_- x^j \ , \qquad u(\tau, \sigma) = f^+(\xi^+) + f^-(\xi^-) \ , \tag{5.8}$$

was shown, in some special cases [245] and for general symmetric spaces [246] (see also [247]), to be integrable in the sense of admitting a Lax pair. The spectral parameter is now replaced by a function of τ, σ expressed in terms of $f^+(\xi^+)$, $f^-(\xi^-)$ and an integration constant w (which may be viewed as a new spectral parameter — see below).

Here will go beyond this related earlier work in two important ways: (i) showing that there are other integrable σ-models with 'local couplings' (e.g., related to integrable-deformed

[66] In general, $\mathcal{N} = (2,2)$ supersymmetric σ-models on Kähler manifolds have a non-trivial 4-loop β-function coefficient [180, 243] (which may be absorbed into a non-local redefinition of the metric in the case of Ricci-flat Kähler (Calabi-Yau) manifolds [244]).

[67] Here we ignore the 'diffeomorphism vector' term that will also contribute to the time dependence of the metric; however, this term can be eliminated or modified as desired by an appropriate field redefinition in (5.6) depending on time, $x^i \to y^i(x, \tau)$ (see discussion in Section 5.1 below).

models) for which the dependence of $\mathrm{G}_{ij}(u,x)$ on $u(\tau,\sigma)$ is not simply through an overall factor as in (5.8); (ii) establishing the link between the dependence on u required for integrability and the RG evolution of couplings in the original σ-model defined by the transverse metric $\mathrm{G}_{ij}(u,x)\big|_{u=\text{const}}$.

The rest of this chapter is organized as follows. In Section 5.1 we will show that, starting from a certain class of classically integrable σ-models admitting a flat Lax connection, it is possible to find a Lax connection for the corresponding time-dependent theory obtained from the original one by replacing the couplings by functions of τ satisfying the 1-loop RG equations. We will explicitly discuss the case of the PCM, and a similar statement is also true for several classes of integrable σ-models (PCM with WZ term, symmetric spaces, η-model, λ-models) presented in Table 1.

In Section 5.2 we will demonstrate that the converse is also true: demanding the existence of a Lax pair for the time-dependent model implies that the couplings must solve the 1-loop RG equations of the original theory. Details of the derivation of the RG flow will be presented in Appendix E.

In Section 5.3 we will discuss how the monodromy matrix associated with the Lax connection can be used to construct, at least in principle, non-local conserved charges for time-dependent models.[68] We will concentrate on the PCM example with comments on other models relegated to Appendix F. To shed light on the notion of integrability in the time-dependent case we will also consider consistent 1d reductions of some simple 2d σ-models, investigating their solvability. In Appendix G we will further discuss a particular linear 1d model which is a special case of a time-dependent harmonic oscillator.

Section 5.4 contains a discussion of the results and possible extensions. In particular, we will mention that the relation between integrability of the time-dependent theory and the RG flow should extend to models with potentials, illustrating this on the example of the sine-Gordon model (see also Appendix H).

5.1 Lax connection for time-dependent sigma models

Below we shall discuss how, starting from a classically integrable model (with Lagrangian $\mathcal{L}$) admitting a Lax pair $L_\pm$, it is possible to find also a Lax pair $\widehat{L}_\pm$ for the corresponding time-dependent theory $\widehat{\mathcal{L}}$ obtained from the original one by replacing the couplings h_α by functions

[68] See also Section 5 of [4] for an attempt to construct higher-spin, local charges for the PCM with a local (σ-dependent) coupling.

$h_\alpha(\tau)$ depending on τ according to the 1-loop RG equation (5.2),[69]

$$h_\alpha \to h_\alpha(\tau) \ , \qquad \tfrac{d}{d\tau} h_\alpha(\tau) = \beta_\alpha(h(\tau)) \ , \tag{5.9}$$

$$\mathcal{L}(h_\alpha) \ \to \ \widehat{\mathcal{L}} = \mathcal{L}(h_\alpha(\tau)) \ . \tag{5.10}$$

The Lax connection must be modified to account for 'extra' terms in the equations of motion following from (5.10) that are proportional to the derivatives $\partial_\tau h_\alpha$ of the couplings. We propose a very simple ansatz for this modification: the extra terms can be effectively 'absorbed' into the spectral parameter z with the structure of the Lax connection remaining the same, i.e.

$$L_\pm(h_\alpha, z) \to \widehat{L}_\pm = L_\pm(h_\alpha(\tau), z(w;\tau,\sigma)) \ , \tag{5.11}$$

$$z \to z = z(w;\tau,\sigma) \ , \qquad h_\alpha \to h_\alpha(\tau) \ . \tag{5.12}$$

While z was regarded as constant in the original Lax connection, it is now promoted to a particular function $z(w;\tau,\sigma)$ of the 2d coordinates. Here w is an arbitrary (complex) constant playing the role of a new spectral parameter. As we shall see below, the explicit form of the function $z(w;\tau,\sigma)$ will be model-dependent.

Let us note that if the original theory is a σ-model on flat 2d Minkowski space, which is classically invariant under the 2d conformal transformations $\xi^\pm \to f^\pm(\xi^\pm)$, then the Lagrangian of the corresponding time-dependent theory (5.10) changes under such transformations as

$$\widehat{\mathcal{L}} = \mathcal{L}(h_\alpha(\tau)) \ \to \ \mathcal{L}(h_\alpha(\tau')) \ , \qquad \tau = \xi^+ + \xi^- \ \to \ \tau' = f^+(\xi^+) + f^-(\xi^-) \ . \tag{5.13}$$

In the case of the string σ-model (5.3),(5.6), this is equivalent to replacing the l.c. gauge $u = \tau$ with a more general one given by the solution $u = f^+(\xi^+) + f^-(\xi^-)$ of the equation of motion $\partial_+\partial_- u = 0$. One special choice is $u = \xi^-$ for which we get $\widehat{\mathcal{L}}_{\rm ch} = \mathcal{L}(h_\alpha(\xi^-))$. One may formally consider that theory, which only fixes one 'half' of the conformal reparametrizations, as an extreme limit $(f^+(\xi^+), f^-(\xi^-)) = (0, \xi^-)$ of (5.13).

To provide evidence for the above proposal, we have considered six examples of integrable σ-models (each introduced in Section 1.4) with the results summarized in Table 1 below. There, for each model, we give the original Lagrangian, original Lax pair, 1-loop RG equation and its solution, and the expression for the function $z(w;\tau,\sigma)$ that generalizes the original spectral parameter in such a way that (5.11) gives a Lax pair for the time-dependent theory (5.10).

The classically integrable σ-models in Table 1 are each 1-loop renormalizable assuming a particular choice of 'renormalizable' diffeomorphism vector. This means that the insertion of time dependence according to the RG flow in (5.9),(5.10) will only apply to a finite set of couplings. As discussed at the beginning of this chapter, a time-dependent σ-model arises in

[69] Here and below for simplicity we are only indicating the dependence of the Lagrangian on the couplings, suppressing dependence on fields.

the light-cone gauge from a conformal string σ-model. There, the time dependence is governed by the RG flow with a specific diffeomorphism vector (generally different from the renormalizable one) following from the $(D+2)$-dimensional dilaton solving the Weyl-invariance condition. However, these two patterns of time dependence (according to the RG flow with different diffeomorphism vectors) are simply related by a time-dependent field redefinition in the corresponding time-dependent models, and such a redefinition preserves the existence of a Lax representation.

All of our examples are built on either a group G or a symmetric space G/H. Our notation is the same as that of Section 1.4, except that here h denotes a coupling constant, or h_α to denote many of them (whereas before they were called h to distinguish from group elements). We again denote $J_\pm = g^{-1}\partial_\pm g$, where $g \in G$ is scalar field. PCM_k stands for the principal chiral model plus a Wess-Zumino (B-field) term with coefficient k (and the special case $h = \pm k$ is the conformal WZW model). The group space η-model is an integrable deformation of the PCM while the group space and symmetric space λ-models are integrable deformations of the WZW and gauged WZW models.

In this section we shall explicitly consider only the basic PCM example, postponing a more general discussion to Section 5.2, where we shall also explain that the existence of the spectral function $z(w;\tau,\sigma)$ is a highly non-trivial feature, depending on specifically choosing the coupling functions $h_\alpha(\tau)$ to solve the 1-loop RG equation.

For the PCM corresponding to a simple Lie group G

$$\mathcal{L} = -\tfrac{1}{2}h\mathrm{Tr}[J_+J_-] \ , \qquad J = g^{-1}dg \ , \quad g \in G \ , \tag{5.14}$$

the global $G\times G$ symmetry implies that only the overall coupling h runs under the 1-loop RG,[70]

$$\frac{d}{dt}h = c \ , \qquad\qquad h(t) = c\,t \ , \qquad\qquad c \equiv c_{_G}. \tag{5.15}$$

The corresponding time-dependent theory (5.10) is then given by

$$\widehat{\mathcal{L}} = -\tfrac{1}{2}c\,\tau\,\mathrm{Tr}[J_+J_-] \ . \tag{5.16}$$

As explained above, this theory arises naturally in the l.c. gauge $u = \tau$ from the conformal string σ-model (cf. (5.3),(5.6))

$$\mathcal{L} = -2\partial_+u\partial_-v - \tfrac{1}{2}c\,u\mathrm{Tr}[J_+J_-] \ , \tag{5.17}$$

where the constant c can be eliminated by a rescaling $(u,v) \to (c^{\,-1}u,\, c\,v)$.

[70]The constant $c \equiv c_{_G}$ is the dual Coxeter number of the group G (see Appendix A for the group theory conventions).

	PCM	PCM_k
Lagrangian	$\mathcal{L} = -\frac{1}{2}h\text{Tr}[J_+J_-]$	$\mathcal{L} = -\frac{1}{2}h\text{Tr}[J_+J_-] + k\,\mathcal{L}_{\text{WZ}}$
Lax connection	$L_\pm = \frac{1}{2}(1+z^{\pm 1})J_\pm$	$L_\pm = \frac{1}{2}(1+z^{\pm 1})(1\pm\frac{k}{h})J_\pm$
1-loop RG equation and its solution	$\frac{d}{dt}h = c$ $h(t) = c\,t$	$\frac{d}{dt}h = c\,(1-\frac{k^2}{h^2}),\quad \frac{d}{dt}k = 0$ $h(t) - k\tanh^{-1}\big(\frac{h(t)}{k}\big) = c\,t$
Spectral parameter function	$z = \sqrt{\frac{w+\sigma-\tau}{w+\sigma+\tau}}$	$c\,(w+\sigma-\tau) + 2zh(\tau)\,\frac{k+z(k+h(\tau))}{k(1+z)^2+h(\tau)(z^2-1)}$ $+k\log\big(\frac{(1+z)(k-h(\tau))}{(k-h(\tau))+z(k+h(\tau))}\big) = 0$

	Group space η-model	Group space λ-model
Lagrangian	$\mathcal{L} = -\frac{1}{2}h\text{Tr}[J_+\frac{1}{1-\varkappa\mathcal{R}}J_-]$	$\mathcal{L} = k\big[\mathcal{L}_{G/G}(g,A)$ $-(\lambda^{-1}-1)\text{Tr}(A_+A_-)\big]$
Lax connection	$L_\pm = \frac{1}{2}(1+z^{\pm 1})C_\pm$ $C_\pm = -(1+\varkappa^2)\text{Ad}_g\frac{1}{1\pm\varkappa\mathcal{R}}J_\pm$	$L_\pm = \frac{1}{2}(1+z^{\pm 1})\frac{2}{1+\lambda}A_\pm$ $A_\pm$ takes on-shell value
1-loop RG equation and its solution	$\frac{d}{dt}h = c\,(1+\varkappa^2)^2,\quad \frac{d}{dt}(h\varkappa^{-1}) = 0$ $\nu \equiv \frac{h(t)}{4\varkappa(t)} = \text{const}$ $\tan^{-1}\varkappa(t) + \frac{\varkappa(t)}{1+\varkappa(t)^2} = \frac{c}{2\nu}t$	$\frac{d}{dt}\lambda = -\frac{2c}{k}\frac{\lambda^2}{(1+\lambda)^2},\ \frac{d}{dt}k = 0,$ $-2\log\lambda(t) - \lambda(t) + \lambda(t)^{-1} = \frac{2c}{k}t$
Spectral parameter function	$\frac{c}{2\nu}(w+\sigma) - \frac{1-z}{1+z}\frac{\varkappa(\tau)}{1+\varkappa(\tau)^2}$ $+\tan^{-1}\big(\frac{1}{\varkappa(\tau)}\frac{1-z}{1+z}\big) = 0$	$\frac{2c}{k}(w+\sigma) + \frac{1-z}{1+z}[\lambda(\tau) - \lambda(\tau)^{-1}]$ $-2\log\big(\frac{z\lambda(\tau)-1}{z-\lambda(\tau)}\big) = 0$

	Symmetric space σ-model	Symmetric space λ-model
Lagrangian	$\mathcal{L} = -\frac{1}{2}h\text{Tr}[J_+P_{G/H}J_-]$	$\mathcal{L} = k\big[\mathcal{L}_{G/G}(g,A)$ $-\text{Tr}\big[A_+(\lambda^{-1}P_{G/H}-1)A_-\big]$
Lax connection	$L_\pm = J^H_\pm + z^{\pm 1}J^{G/H}_\pm$ $J^H_\pm = P_HJ_\pm,\ J^{G/H}_\pm = P_{G/H}J_\pm$	$L_\pm = A^H_\pm + z^{\pm 1}\frac{1}{\sqrt{\lambda}}A^{G/H}_\pm$ $A^H_\pm = P_HA_\pm,\ A^{G/H}_\pm = P_{G/H}A_\pm$ $A_\pm$ takes on-shell value
1-loop RG equation and its solution	$\frac{d}{dt}h = 2c$ $h(t) = 2c\,t$	$\frac{d}{dt}\lambda = -\frac{c}{k}\lambda$ $\lambda(t) = \exp(-\frac{c}{k}t)$
Spectral parameter function	$z = \sqrt{\frac{w+\sigma-\tau}{w+\sigma+\tau}}$	$z = \exp(-\frac{c}{2k}\tau)\sqrt{\frac{1+\exp(\frac{c}{k}[w+\sigma-\tau])}{1+\exp(\frac{c}{k}[w+\sigma+\tau])}}$

Table 1: Examples of integrable σ-models and their time-dependent generalizations. t is the RG time which is replaced by the 2d time τ in the couplings entering the Lagrangian $\widehat{\mathcal{L}}$ in (5.10).

The equations of motion for the model (5.16) written in first-order form are

$$\partial_+(\tau J_-) + \partial_-(\tau J_+) = 0 \ , \tag{5.18}$$

$$F_{+-}(J) \equiv \partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0 \ . \tag{5.19}$$

Starting from the Lax connection of the original PCM (5.14)

$$L_\pm = \tfrac{1}{2}(1 + z^{\pm 1}) J_\pm \ , \tag{5.20}$$

and making the replacement $z \to \sqrt{\frac{w+\sigma-\tau}{w+\sigma+\tau}}$ where w is a constant spectral parameter (see Table 1), we obtain the following expression for the Lax connection (5.11) of the time-dependent theory

$$\widehat{L}_\pm = \tfrac{1}{2}\big(1 + z^{\pm 1}\big) J_\pm \ , \qquad\qquad z = \sqrt{\frac{w+\sigma-\tau}{w+\sigma+\tau}} \ . \tag{5.21}$$

Indeed, its curvature is

$$\begin{aligned} F_{+-}(\widehat{L}) &\equiv \partial_+ \widehat{L}_- - \partial_- \widehat{L}_+ + [\widehat{L}_+, \widehat{L}_-] \\ &= \frac{z}{2(w+\sigma-\tau)} \Big[\partial_+(\tau J_-) + \partial_-(\tau J_+)\Big] + \tfrac{1}{4}(1+z)(1+z^{-1})\ F_{+-}(J) \ , \end{aligned} \tag{5.22}$$

so that, at any (τ, σ), its vanishing for all w implies the equations of motion (5.18),(5.19) of the time-dependent theory.

A surprising feature (shared also by the other examples in Table 1) is that, despite the Lagrangian (5.16) explicitly involving only τ, the Lax connection (5.21) also depends on the 2d spatial coordinate σ. The spatial coordinate σ and the spectral parameter w appear only through the combination $w + \sigma$. Since w is a complex constant, it cannot be eliminated by a shift of real σ. Moreover, what is important is the existence of a Lax connection depending on w, which can in principle be used to construct conserved charges, etc. Indeed, we can instead interpret the formal possibility to shift σ as the freedom to introduce a spectral parameter in the first place.

Another unusual property (shared by all the examples) is that the Lax connection has branch cuts in the spectral w-plane (e.g., for the PCM the branch cuts end at $w = -\sigma \pm \tau$). Normally, one could simply remove square roots by redefining the spectral parameter (or more formally moving to an appropriate Riemann surface on which the Lax connection is meromorphic). However, this is not possible since the positions of the branch cuts depend on (τ, σ) and one cannot redefine the spectral parameter in a way depending on (τ, σ) without changing the equations of motion encoded in the zero curvature condition.

In each case in Table 1, one can freely choose any branch of the function $z(w; \tau, \sigma)$ (in the w-plane) while still encoding the correct equations of motion in the zero-curvature equation.

For example, in the PCM case, one may choose either sign of the square root $z = \pm\sqrt{\frac{w+\sigma-\tau}{w+\sigma+\tau}}$ in the Lax connection (5.21). All similar square roots in the spectral functions below have the same branch ambiguity, corresponding to the option to reverse their sign.

Applying a general conformal transformation to the action corresponding to (5.16),(5.21) we get (cf. (5.13))

$$\widehat{\mathcal{L}} = -\tfrac{1}{2}c\,\big(f^+(\xi^+) + f^-(\xi^-)\big)\,\mathrm{Tr}[J_+J_-]\ , \tag{5.23}$$

$$\widehat{L}_\pm = \tfrac{1}{2}\big(1 + z^{\pm 1}\big)J_\pm\ , \qquad z = \sqrt{\frac{w - 2f^-(\xi^-)}{w + 2f^+(\xi^+)}}\ . \tag{5.24}$$

This theory may equivalently be obtained from the string σ-model (5.17) by picking up a generalized l.c. gauge $u = \tau' = f^+(\xi^+) + f^-(\xi^-)$. A (degenerate) limit $u = a + b\tau,\ \ a \to 1$, $b \to 0$ eliminates the explicit τ-dependence and gives back the original PCM (5.14), with (5.23) becoming the original Lax connection in (5.20) with a redefined spectral parameter $z \to \sqrt{\frac{w-2a_-}{w+2a_+}}$, where $a_\pm$ are constants satisfying $a_+ + a_- = 1$. We note that the theory (5.23) is a special case of (5.8) where G_{ij} is the group-space metric. Indeed, an equivalent expression for the Lax connection of this 'local' PCM (or symmetric space σ-model, see Table 1) was originally found in [246] where the dependence of the analog of the spectral parameter on the functions $f^+(\xi^+)$ and $f^-(\xi^-)$ was discovered. The spectral function z in (5.24) depends separately on u and its dual field $\tilde{u} = f^+(\xi^+) - f^-(\xi^-),\ du = *d\tilde{u}$, and thus separately on f^+ and f^- [246, 247].

5.2 RG flow from condition of integrability of time-dependent theory

As we have found above on several examples, if the couplings of an integrable σ-model are promoted to functions of time that solve the 1-loop RG equations, $h_\alpha \to h_\alpha(\tau),\ \partial_\tau h_\alpha = \beta_\alpha(h)$, then the Lax connection of the original model $\mathcal{L}(h_\alpha)$ admits a natural generalization to a classical Lax connection for the time-dependent model $\mathcal{L}(h_\alpha(\tau))$. Here we shall argue that the converse is also true: demanding the existence of a Lax pair for the time-dependent theory implies that $h_\alpha(\tau)$ must solve the 1-loop RG equations.

It is useful to start with a more general theory with local couplings[71] $h_\alpha(\tau, \sigma)$ depending on both τ and σ, i.e.

$$\widehat{\mathcal{L}} = \mathcal{L}(h_\alpha(\tau, \sigma))\ , \tag{5.25}$$

and demand the existence of a Lax representation for this theory. For the examples in Table 1, the original Lax connection takes a particular form (which is generic to many integrable

[71] We generally expect integrable models to have a finite number of couplings h_α. A natural way to identify these couplings is as the parameters that run under the RG flow.

σ-models built on groups G or symmetric spaces G/H):

$$G \ : \qquad L_\pm = \tfrac{1}{2}(1+z^{\pm 1})\mathcal{A}_\pm \ , \tag{5.26}$$

$$G/H \ : \qquad L_\pm = \mathcal{B}_\pm + z^{\pm 1}\mathcal{P}_\pm \ . \tag{5.27}$$

Here the connection components $\mathcal{A}_\pm \in \mathrm{Lie}(G)$, $\mathcal{B}_\pm \in \mathrm{Lie}(H)$, $\mathcal{P}_\pm \in \mathrm{Lie}(G)/\mathrm{Lie}(H)$ depend implicitly on the fields and their derivatives and the couplings h_α (e.g. for PCM, $\mathcal{A}_\pm = J_\pm$, cf. (5.20)), while the dependence on the spectral parameter z is indicated explicitly. Let us now take the following natural ansatz for the Lax connection $\widehat{L}_\pm$ corresponding to the (τ,σ)-dependent theory (5.25)

$$G \ : \qquad \widehat{L}_\pm = p_\pm(w;\tau,\sigma)\,\mathcal{A}_\pm \ , \tag{5.28}$$

$$G/H \ : \qquad \widehat{L}_\pm = q_\pm(w;\tau,\sigma)\,\mathcal{B}_\pm + r_\pm(w;\tau,\sigma)\,\mathcal{P}_\pm \ , \tag{5.29}$$

where $p_\pm, q_\pm, r_\pm$ are some functions, and $\mathcal{A}_\pm$, $\mathcal{B}_\pm$, $\mathcal{P}_\pm$ are assumed to take the same form as in (5.26),(5.27), but now depending on the coupling *functions* $h_\alpha(\tau,\sigma)$ in place of the couplings, i.e. $\mathcal{A}_\pm = \mathcal{A}_\pm(x,\partial x; h_\alpha(\tau,\sigma))$, etc.

Demanding that the flatness of the connection (5.28),(5.29) is equivalent to the equations of motion of the 'local coupling' theory (5.25), we shall first prove that it is sufficient to use just a single function $z(w;\tau,\sigma)$, so that the Lax connections (5.28),(5.29) become the same as the original ones (5.26),(5.27) with the replacement $z \to z(w;\tau,\sigma)$ as in (5.11),

$$\widehat{L}_\pm = L_\pm\big(h_\alpha(\tau,\sigma), z(w;\tau,\sigma)\big) \ . \tag{5.30}$$

Indeed, the equations of motion for the theory (5.25) contain two types of terms: the "original" terms (not depending on $\partial_\pm h_\alpha$), and "new" terms (depending on $\partial_\pm h_\alpha$). To match the "original" terms with the corresponding terms in the zero-curvature condition for $\widehat{L}_\pm$, one needs to impose the following conditions on $p_\pm$, $q_\pm$, $r_\pm$ (see Appendix E.1 for details)

$$G \ : \qquad p_+^{-1} + p_-^{-1} = 2 \ , \tag{5.31}$$

$$G/H \ : \qquad q_\pm = 1 \ , \qquad r_+ r_- = 1 \ . \tag{5.32}$$

Substituting (5.31),(5.32) into the Lax ansatz (5.28),(5.29) then gives (5.26),(5.27) but with $z \to z(w;\tau,\sigma)$, where we set $z(w;\tau,\sigma) = 2p_+ - 1$ (for models on G) and $z(w;\tau,\sigma) = 2r_+$ (for models on G/H).

All of our examples are single-coupling theories[72] except for the group η-model. Here we

[72]We do not promote the WZ level k in PCM_k or λ-models to a function of (τ,σ) since the resulting model would not be well-defined: starting with the 3d representation of the WZ term and replacing $k\mathrm{H} \to k(\tau,\sigma)\mathrm{H}$ would no longer give a local 2d action while $k\mathrm{B} \to k(\tau,\sigma)\mathrm{B}$ in the 2d action would not be globally well-defined, or locally consistent with global symmetry (invariance of the 2d action under global symmetry transformations requires dropping total derivatives).

shall specialize to the simplest case with one coupling $h(\tau,\sigma)$ but the same result will also be true for the group η-model (see Appendix E.3). To reproduce the "new" terms (proportional to $\partial_\pm h$) in the equations of motion for (5.25) from the flatness condition for the generalized Lax connection (5.30), one needs to additionally impose certain constraints on both $h(\tau,\sigma)$ and $z(w;\tau,\sigma)$. For the models in Table 1 this leads to the following two first-order differential equations for the function $z(w;\tau,\sigma)$ (here $h = h(\tau,\sigma)$)

$$\partial_\tau z = V_\tau(z,h) \ , \qquad\qquad \partial_\sigma z = V_\sigma(z,h) \ , \tag{5.33}$$

where $V_{\tau,\sigma}(z,h)$ are model-dependent functions. For all five single-coupling examples in Table 1, the consistency condition ($\partial_\tau V_\sigma - \partial_\sigma V_\tau = 0$) for the system (5.33) takes the remarkable form,

$$\partial_+\partial_- h - \frac{\beta'(h)}{\beta(h)}\,\partial_+ h\partial_- h = 0 \ , \tag{5.34}$$

where $\beta(h)$ is precisely the 1-loop β-function for the coupling h in the original model. In addition to the appearance (in this purely classical context) of the 1-loop β-functions, a remarkable feature of (5.34) that it does not depend on z (which completely factors out of the consistency condition).

The condition (5.34) may be written simply as $\partial_-(\frac{\partial_+ h}{\beta(h)}) = 0$ or $\partial_+(\frac{\partial_- h}{\beta(h)}) = 0$ and thus leads to two first-order equations,

$$\frac{\partial_+ h}{\beta(h)} = s_+(\xi^+) \ , \qquad\qquad \frac{\partial_- h}{\beta(h)} = s_-(\xi^-) \ , \tag{5.35}$$

where $s_\pm(\xi^\pm)$ are arbitrary functions of the 2d light-cone coordinates $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$. By applying a conformal transformation (i.e. redefining τ and σ), one can absorb $s_\pm$ into $\partial_\pm$, i.e. replace $s_\pm \to 1$, so that in terms of the redefined coordinates the first-order equations (5.35) take the form of the 1-loop RG equation in τ

$$\partial_\tau h = \beta(h) \ , \qquad\qquad \partial_\sigma h = 0 \ . \tag{5.36}$$

Thus, modulo a conformal transformation, the 1-loop RG solution is the only choice of local coupling $h(\tau,\sigma)$ for which the Lax connection can be uplifted to the (τ,σ)-dependent theory according to (5.30). This argument (in eqs. (5.33)-(5.36)) is demonstrated explicitly for the example of the PCM in Appendix E.2.

The same conclusion is reached of course if one starts directly with the theory where the couplings h_α depend only on time (giving $\partial_\pm \to \partial_\tau$ and $s_+ = s_- = \text{const}$ in eqs. (5.34),(5.35)): using the same ansatz for the Lax connection one finds that the only functions $h_\alpha(\tau)$ consistent with preserving integrability are the solutions of the 1-loop RG flow — modulo a rescaling of time, i.e. the part of the conformal group that does not introduce spatial dependence. By

performing a classical conformal transformation, one could instead assign a preferred role to space σ rather than time τ, and in that case the RG equation (5.36) will hold with $\tau \to \sigma$.

To finish the construction of the generalized Lax pair let us now explain how to obtain the explicit form of the spectral parameter function $z(w;\tau,\sigma)$ in (5.30). Starting with $h = h(\tau)$ that satisfies the 1-loop RG equation in (5.36) (so that the consistency conditions (5.35) of (5.33) are satisfied) one can solve the second equation in (5.33) as

$$\partial_\sigma z = V_\sigma(z, h(\tau)) \qquad \to \qquad \int \frac{dz}{V_\sigma(z, h_\alpha(\tau))} = \sigma + \ell(\tau) \ . \tag{5.37}$$

The function $\ell(\tau)$ is then fixed by substituting this solution into the first equation in (5.33). The solution for $\ell(\tau)$ leaves one free integration constant, which we call w. Finally, eq. (5.37) becomes an algebraic equation for $z(w;\tau,\sigma)$ that can be explicitly solved in some simple cases (see Table 1). Note that since the parameter w appears as an integration constant in the function $\ell(\tau)$, it always appears in the combination $w+\sigma$ with the spatial coordinate, implying that constant shifts of σ can be compensated by shifts of w.

Although we do not have a proof that the 1-loop RG flow always follows from requiring integrability of the time-dependent generalizations of integrable models, the examples in Table 1 reveal a highly non-trivial pattern. In particular, some of the 1-loop RG equations obtained in this way are quite non-trivial, not admitting simple closed form solutions.

5.3 Non-local conserved charges in time-dependent integrable models

We have shown in the previous sections that some well-known examples of classically integrable σ-models admit generalizations of their Lax connections to models with time-dependent couplings under the condition that the latter solve the 1-loop RG equation in τ. However, one may question the usefulness of the resulting Lax connections since they have several unusual properties: they depend on τ and σ explicitly; the spectral parameter only enters as a 'constant piece' of the spatial coordinate σ; and they have branch cuts in the spectral plane (whose positions depend on τ and σ).

In this section we shall argue that, nevertheless, the Lax connections can still be used to construct the 'non-local' charges typical of integrable models. However, the space-time dependence of the Lax connection renders these charges difficult to compute explicitly. We shall focus here on the case of the PCM, while the other examples from Table 1 are discussed in Appendix F.

5.3.1 Non-local charges

The derivation of non-local conserved charges from the monodromy matrix $\mathcal{M} = \mathcal{P}\exp\int_a^b d\sigma\, L_\sigma$ discussed in Section 1.4.1 still applies for Lax connections with explicit space-

time dependence. Indeed, the zero-curvature condition $\partial_\sigma L_\tau - \partial_\tau L_\sigma + [L_\sigma, L_\tau] = 0$ implies that

$$\partial_\tau \mathcal{M} = \mathcal{M}\, L_\tau(b) - L_\tau(a)\, \mathcal{M} \ . \tag{5.38}$$

Hence, if we assume a periodicity condition on the spatial domain $a < \sigma < b$,

$$L_\tau(a) = L_\tau(b) \ , \tag{5.39}$$

then we obtain

$$\partial_\tau \mathcal{M} = [\mathcal{M}, L_\tau(a)] \ , \tag{5.40}$$

and so the eigenvalues of $\mathcal{M}$ are conserved ($\partial_\tau \mathrm{Tr}[\mathcal{M}^n] = 0$). If also the value at the endpoints vanishes, $L_\tau(b) = L_\tau(a) = 0$, then all of the components of $\mathcal{M}$ are conserved ($\partial_\tau \mathcal{M} = 0$).

We shall focus on the time-dependent PCM on an infinite spatial interval $(a, b) = (-\infty, \infty)$.[73] Picking the 'negative' branch[74] of the square root, then the spectral function $z(w; \tau, \sigma)$ in (5.21) has the finite limit $z = \sqrt{\frac{w+\sigma+\tau}{w+\sigma-\tau}} \to z_\infty = -1$ at $\sigma \to \pm\infty$. Hence, assuming $J_\pm$ is bounded for large $|\sigma|$, the Lax connection $\widehat{L}_\pm = \frac{1}{2}(1 + z^{\pm 1})J_\pm$ vanishes at spatial infinity. The periodicity condition (5.39) is satisfied with $\widehat{L}_\tau$ vanishing at spatial infinity, so it follows that all the components of $\mathcal{M}$ are conserved.

The boundary condition (that $J_\pm$ is bounded), needed above for the conservation of $\mathcal{M}$, is quite weak. A further, more stringent boundary condition comes from the requirement that $\mathcal{M}$ converges as $(a, b) \to (-\infty, +\infty)$ to produce well-defined charges.

Although the monodromy is conserved, it is only defined implicitly by the first-order ordinary differential equation (1.14), i.e.

$$\mathcal{M}^{-1}\partial_\sigma \mathcal{M} = \widehat{L}_\sigma = \tfrac{1}{4}\big(1 + \big[\tfrac{w+\sigma-\tau}{w+\sigma+\tau}\big]^{1/2}\big)J_+ - \tfrac{1}{4}\big(1 + \big[\tfrac{w+\sigma-\tau}{w+\sigma+\tau}\big]^{-1/2}\big)J_- \ , \qquad \mathcal{M}\big|_{\sigma=-\infty} = I \ . \tag{5.41}$$

Such an equation generally admits a solution but it is hard to find its explicit form. One possible approach is to develop an expansion in the spectral parameter around a point where $\widehat{L}_\sigma$ vanishes (and thus the monodromy is trivial, $\mathcal{M} = I$).

In the usual time-independent PCM case (obtained by replacing $\sqrt{\frac{w+\sigma-\tau}{w+\sigma+\tau}}$ by a constant z in (5.41)), such an expansion around $z = -1$ yields the conserved 'multi-local' Yangian charges in

[73] It seems much harder to satisfy the periodicity condition (5.39) on a spatial circle due to the explicit non-periodic σ dependence in the Lax connections for the time-dependent models.

[74] We note that it is consistent to 'pick a branch' here: e.g., by assuming that w has an imaginary part, it then follows that the sign of the square root does not change from $\sigma = -\infty$ to $\sigma = \infty$. The choice of a branch is arbitrary but this negative choice is more useful for satisfying the periodicity condition and constructing conserved charges. Also, for the group η-model and PCM$_k$, which are deformations of PCM, the function z is single-valued at $\sigma = \pm\infty$, with $z_\infty \equiv z\big|_{\sigma=\infty} = -1$. It is then natural to obtain $z_\infty = -1$ for the PCM as a limit of these models.

eq. (1.22). Each term in the expansion around $z = -1$ is individually conserved because each term in the corresponding expansion of L_τ (and hence the right hand side of (5.38)) vanishes at spatial infinity.

For the time-dependent theory, the only zero around which to expand $\widehat{L}_\sigma$ in (5.41) is $w = \infty$ (again taking the negative branch of the square root). But, due to its explicit spatial dependence, the corresponding expansion of the Lax connection is a sum of terms $w^{-n}P_{n-1}$, where P_{n-1} is a polynomial of degree $n-1$ in σ. Then for any polynomial decaying boundary conditions on the fields at spatial infinity, the periodicity condition (5.39) on $\widehat{L}_\tau$ will be broken at sufficiently high orders in this expansion.

Hence, while we have shown the formal existence of conserved non-local charges defined implicitly by (5.41), it appears to be hard to compute the monodromy matrix $\mathcal{M}$ explicitly: due to the explicit σ dependence in the Lax connection, the usual expansion trick (1.22) does not easily work. It is not clear at the moment how to verify if the resulting conserved charges are infinite in number (and independent) as they should be for an integrable 2d theory.

5.3.2 1d reductions

One useful test of 2d integrability is to check the integrability of various 1d mechanical theories obtained as consistent reductions of the 2d equations of motion.

Assuming an ansatz for a classical solution of an integrable 2d model yielding a 1d system of equations (say with τ as the remaining independent variable), one may expect the resulting 1d system also to be integrable in the sense of admitting a Lax pair, $\frac{d}{d\tau}A = [B, A]$. One should further demand that the conserved charges are in involution.

In general, if the solution satisfies the periodicity condition (5.39), then the flatness of the 2d Lax connection leads to a 1d Lax pair given by $(A, B) = (\mathcal{M}, -L_\tau(\infty))$ in (5.40), evaluated on the reduction ansatz. For example, in [248], such a reduction was performed for a certain 'spinning string' ansatz in $AdS_5 \times S^5$, resulting in an integrable 1d Neumann-Rosochatius model. There (reversing the roles of τ and σ from that paper to match the above discussion) the σ-dependence of the ansatz was simple and it was possible to completely remove the σ-dependence from the Lax connection using a gauge transformation — which is essentially the same problem as computing the monodromy matrix. In our case, with explicit σ-dependence already in the Lax connection, the problem of directly removing the σ-dependence by gauge transformations would appear to be much more difficult.

Trivial reduction. In the simplest 'trivial' 1d reduction one assumes that the fields do not depend on σ. This is indeed a consistent truncation of the above time-dependent models, obtained by replacing $\partial_\pm \to \tau$ in their Lagrangians. For example, in the PCM case, setting $g = g(\tau)$ in (5.16) gives the 1d action for geodesic motion on a group space with a time-dependent radius. The solvability of geodesics in this model is obvious since the explicit time

dependence may be eliminated by a redefinition of τ

$$S_1 \sim \int d\tau\ \tau\ \mathrm{Tr}\big[(g^{-1}\partial_\tau g)^2\big] = \int d\tau'\ \mathrm{Tr}\big[(g^{-1}\partial_{\tau'} g)^2\big]\ , \qquad \tau' = \log\tau\ . \tag{5.42}$$

The resulting equation of motion $\partial_{\tau'}(g^{-1}\partial_{\tau'} g) = 0$ is solved by

$$g = g_0\, e^{\tau' u_0} = g_0\, \tau^{u_0}\ , \qquad g_0 = \text{const} \in G\ , \quad u_0 = \text{const} \in \mathrm{Lie}(G)\ . \tag{5.43}$$

The same argument equally applies for any case where the time dependence only appears as an overall factor rescaling the Lagrangian – which will follow if there are enough global symmetries that only the overall 'radius' can run under the RG flow (e.g. also in the symmetric space σ-model case).

In general, the global symmetry charges may be found from the monodromy $\mathcal{M}$. Formally, the monodromy on an infinite line does not converge upon the trivial reduction since $J_\tau = g^{-1}\partial_\tau g$ does not decay at spatial infinity (as it is now independent of σ). On a finite space interval, the periodicity condition (5.39) is not satisfied so charges will not be conserved. However, making an expansion around the zero of the Lax connection at $w = \infty$ (cf. (1.22)), the global symmetry is reflected in the σ independence of the leading $\mathcal{O}(w^{-1})$ term in $\widehat{L}_\tau$,

$$\widehat{L}_\tau = \tfrac{1}{4}(1 + \big[\tfrac{w+\sigma-\tau}{w+\sigma+\tau}\big]^{1/2})J_+ + \tfrac{1}{4}(1 + \big[\tfrac{w+\sigma-\tau}{w+\sigma+\tau}\big]^{-1/2})J_- = \tfrac{1}{2}\tau J_\sigma\, w^{-1} + \mathcal{O}(w^{-2})\ . \tag{5.44}$$

This puts the leading term on the right hand side of (5.38) in the commutator form (5.40). Since the monodromy matrix is the identity at the leading order, the right hand side of (5.38) is of $\mathcal{O}(w^{-2})$. It then follows that the $\mathcal{O}(w^{-1})$ term in the monodromy matrix, which is the global G_R Noether charge, is still conserved[75]

$$\mathcal{M} = 1 + \tfrac{1}{2} w^{-1} Q_R + \mathcal{O}(w^{-2})\ , \qquad \partial_\tau Q_R = 0\ , \qquad Q_R = \int d\sigma\ \tau J_\tau\ . \tag{5.45}$$

Non-trivial reduction. Now let us consider a non-trivial 1d reduction when the 2d fields *do* depend on σ in a particular prescribed way. Such a reduction will lead to a non-trivial 1d model, giving a more stringent test of the integrability of the time-dependent 2d model.

Starting from the time-dependent $SU(2)$ PCM (5.16) parametrized as

$$\begin{aligned} &\widehat{\mathcal{L}} = -\tfrac{c}{2}\,\tau\,\mathrm{Tr}[J_+ J_-] = c\,\tau\,\big[\partial_+\theta\partial_-\theta + \sin^2\theta(\partial_+\phi\partial_-\phi + \sin^2\phi\,\partial_+\psi\partial_-\psi)\big]\ , \\ &\quad g = n^a\sigma_a\ , \qquad \sigma_a = (I, \sigma_i)\ , \quad i = 1,2,3\ , \\ &\quad n^0 n^0 - n^i n^i = 1\ , \qquad n^a = (\cos\theta,\ i\sin\theta\cos\phi,\ i\sin\theta\sin\phi\cos\psi,\ i\sin\theta\sin\phi\sin\psi)\ , \end{aligned} \tag{5.46}$$

[75] The Noether charge for G_L symmetry is obtained similarly after a gauge transformation of the Lax connection and choosing instead the positive branch of the square root.

let us consider, e.g., the following ansatz for a classical solution

$$\theta = \theta(\tau) \ , \qquad \phi = m\sigma \ , \qquad \psi = 0 \ . \tag{5.47}$$

This leads to a consistent reduction of the 2d theory: the ϕ and ψ equations are both satisfied, while the equation for $\theta(\tau)$ follows from the 1d Lagrangian ($\dot\theta \equiv \partial_\tau \theta$)

$$\mathcal{L}_1 = c\,\tau\left(\dot\theta^2 - m^2\sin^2\theta\right) \ . \tag{5.48}$$

Thus the effective 1d model is a time-dependent analog of the 'sine-Gordon' mechanics.

Let us assume the σ-direction is an infinite line, still using the ansatz (5.47) (e.g. by taking m to be a continuous parameter and formally decompactifying ϕ). Then one can check that $J_\pm$ is oscillating but bounded at spatial infinity. Hence, as discussed above, the right hand side of (5.38) vanishes and the entries of the monodromy matrix are conserved. While again it is not easy to compute the monodromy explicitly in terms of θ and $\dot\theta$, it can be done in the small field expansion $(\theta, \dot\theta) \to \varepsilon(\theta, \dot\theta)$, $\varepsilon \ll 1$ since the Lax component $\widehat{L}_\sigma$ vanishes as $(\theta, \dot\theta) \to 0$. In this expansion the Lagrangian (5.48) becomes (after rescaling it by a factor of $c^{-1}\varepsilon^{-2}$)

$$\mathcal{L}_1' = \tau\big[\dot\theta^2 - m^2\varepsilon^{-2}\sin^2(\varepsilon\theta)\big] = \mathcal{L}^{lin} + \tfrac{1}{3}\varepsilon^2\, m^2\,\theta^4 + \mathcal{O}(\varepsilon^4) \ , \tag{5.49}$$

$$\mathcal{L}^{lin} = \tau\left[\dot\theta^2 - m^2\theta^2\right] \ . \tag{5.50}$$

The leading ('linearized') Lagrangian (5.50) describes (after the same redefinition $\tau = e^{\tau'}$ as in (5.42)) a harmonic oscillator with time-dependent frequency $m^2 e^{2\tau'}$. The corresponding $\mathcal{O}(\varepsilon)$ terms in the monodromy matrix lead to the conserved charge

$$\dot{Q}^{lin} = 0 \ , \qquad Q^{lin} = \tau\big[\alpha(\tau; w)\,\dot\theta - \dot\alpha(\tau; w)\,\theta\big] \ , \tag{5.51}$$

$$\alpha(\tau; w) = \int_{-\infty}^{+\infty} d\sigma\, e^{-im\sigma}\tfrac{i}{2s_+ s_-} \ , \qquad s_\pm \equiv \sqrt{w + \sigma \pm \tau} \ . \tag{5.52}$$

In Appendix G we use this conserved charge to construct the general solution of the linearized equation of motion (with $\theta = \alpha$ being a particular solution).

Computing the perturbative corrections to the monodromy, one then finds for the conserved

charge of the non-linear theory (5.49)

$$\dot{Q}=0\ ,\quad Q=Q^{lin}+\varepsilon^2\Big[\int d\sigma\, q^{(1)}+\int_{\sigma_1<\sigma_2} d\sigma_1 d\sigma_2\, q^{(2)}+\int_{\sigma_1<\sigma_2<\sigma_3} d\sigma_1 d\sigma_2 d\sigma_3\, q^{(3)}\Big]+\mathcal{O}(\varepsilon^4)\ ,$$
$$q^{(1)}=-\tfrac{m\,e^{-im\sigma}}{3s_+s_-}(s_+s_--w-\sigma)\,\theta^3\ , \tag{5.53}$$
$$q^{(2)}=\tfrac{im}{4s_+^1s_-^1s_+^2s_-^2}\Big[e^{-im\sigma_2}(s_+^1s_-^1-w-\sigma_1)\big[m(s_+^2s_-^2-w-\sigma_2)\theta+i\tau\dot\theta\big]-(\sigma_1\leftrightarrow\sigma_2)\Big]\,\theta^2\ ,$$
$$q^{(3)}=\tfrac{e^{-im(\sigma_1-\sigma_2+\sigma_3)}}{8s_+^1s_-^1s_+^2s_-^2s_+^3s_-^3}\big[m(s_+^1s_-^1-w-\sigma_1)+i\tau\dot\theta\big]$$
$$\times\ \big[-m(s_+^2s_-^2-w-\sigma_2)+i\tau\dot\theta\big]\big[m(s_+^3s_-^3-w-\sigma_3)+i\tau\dot\theta\big]\ ,\qquad s_\pm^n\equiv\sqrt{w+\sigma_n\pm\tau}\ .$$

This perturbative procedure of constructing the conserved charge can be extended to higher orders, computing more and more σ integrals at each step.

Let us also mention that the time-dependent $SL(2,\mathbb{R})$ PCM (cf. (5.46)),

$$\widehat{\mathcal{L}}=-c\,\tau\Big[\partial_+\rho\partial_-\rho+\tfrac{1}{2}e^{-2\rho}\,(\partial_+x^+\partial_-x^-+\partial_-x^+\partial_+x^-)\Big]\ , \tag{5.54}$$

admits a 'non-trivial' reduction that is manifestly integrable. Indeed, if we set $x^+=a\sigma,\ x^-=b\sigma,\ \rho=\rho(\tau)$ then the $x^\pm$ equations are solved and the equation for $\rho(\tau)$ follows from the 1d action (cf. (5.48))

$$S_1=-c\int d\tau\,\tau\big[(\partial_\tau\rho)^2-ab\,e^{-2\rho}\big]\ . \tag{5.55}$$

Here the explicit time dependence can be eliminated by redefining $\tau=e^{\tau'}$ and $\rho=\rho'+\tau'$ so that we end up with the standard 1d Liouville mechanics with the energy being a conserved charge.

5.4 Discussion

We have observed a surprising new connection between classical integrability and the RG flow in 2d theories: starting with an integrable theory and promoting its couplings to time-dependent functions, its Lax connection generalizes naturally to the resulting time-dependent theory only if the coupling functions solve the 1-loop RG equations of the original theory. We demonstrated this on six classes of integrable σ-models.

One interesting implication is that the 1-loop β-functions, which are normally associated with 1-loop divergences in quantum theory, can thus be obtained in these models through the classical procedure of requiring the existence of a flat Lax connection in the corresponding time-dependent theory.

As we discussed at the start of the chapter, such time-dependent models can be naturally embedded into string theory by starting with a Weyl-invariant σ-model (5.3) with two extra null directions (u,v) and fixing a l.c. gauge $u=\tau$. If the l.c. theory (5.6) is classically solvable,

it is natural to expect that the corresponding string σ-model (5.3) should be solvable, since one may move from $u = \tau$ to any other solution of the u-equation $\partial_+\partial_- u = 0$ by a conformal transformation. The v-equations $\partial_\pm v = \frac{1}{2}\mathrm{G}_{ij}(\tau, x)\,\partial_\pm x^i \partial_\pm x^j$ (following from the conformal constraints) are linear and so are also readily solvable. It is not immediately clear, however, if integrability in the sense of admitting a Lax representation should similarly lift up to the $(D+2)$-dimensional string σ-model directly.

The Lax connections for the time-dependent theories have an unusual form (depending explicitly on τ and σ, as in the special examples discussed in [245, 246]). However, given certain boundary conditions, the entries or the eigenvalues of the monodromy matrix on an infinite spatial line are conserved. Due to the explicit σ dependence in the Lax connection, it is hard to evaluate the monodromy and find the conserved charges explicitly. We considered some consistent reductions of the 2d equations of motion to time-dependent 1d systems. The 'trivial' (σ-independent) reduction of the PCM is clearly integrable due to the remaining global symmetry.

For the 'non-trivial' 1d reduction in Section 5.3.2 we evaluated the monodromy, and hence constructed a conserved charge (5.53) in perturbation theory in a small field expansion. The perturbative existence of this conserved charge (following from the existence of a Lax connection) is a non-trivial property, suggesting that the 1d theory (5.48) may be interpreted as integrable. In contrast, with generic (time-dependent) potentials added to the linearized theory (5.50), we would not expect the linearized conserved charge (5.51) to admit even a perturbative extension to the non-linear theory. The time-dependent 1d theory (5.48) is certainly not integrable in the Liouville sense as it would fail the Kovacic algorithm test used in [211, 249]. Here we are suggesting a broader notion of integrability relating to the explicit solvability of the equations of motion (allowing non-Liouvillian solutions like Bessel functions, etc.).

One way to fully establish the classical integrability of the σ-models with local couplings would be to construct an infinite tower of local conserved charges in involution, in the spirit of [137, 66, 70, 207]. In Section 5 of [4] we attempted this for the PCM with 'local couplings' in the Hamiltonian formulation. A naive application of the method used for constant couplings does not yield such charges, except in the case of the 'chiral' theories only depending on ξ^+ or on ξ^-, where it works with a slight modification (see Appendix D of [4]). Nevertheless, the form of the Lax matrix and R-matrix is suggestive of an underlying algebraic structure. Understanding it may provide additional insights into the question of integrability.

While we considered time-dependent models at the classical level only, one may wonder if they themselves are renormalizable, i.e. stable under RG flow. Renormalization of generalized σ-models with target-space metric depending on the 2d coordinates was considered in [250]. In the case we discussed (cf. (5.6)) when G_{ij} depends only on τ, i.e. $\mathcal{L} = \mathrm{G}_{ij}(\tau, x)\partial_+ x^i \partial_- x^j$, the 1-loop logarithmic counterterm is proportional to $K_1 = \mathrm{G}_{ml}\mathrm{G}^{jk}\partial_\tau x^m\, \partial_\tau \Gamma^l_{jk} + \frac{1}{4}\partial_\tau \mathrm{G}^{ij}\partial_\tau \mathrm{G}_{ij}$, where Γ^l_{jk} is the Christoffel connection of G_{ij}. Thus in general one needs to add also counterterms with one and no derivatives, i.e. $V_m(x)\partial_\tau x^m + T(x)$. However, in the 'factorized' case when

$\mathrm{G}_{ij}(\tau, x) = f(\tau)\mathrm{G}_{ij}(x)$, like in the time-dependent PCM (5.16), the above counterterm becomes an x-independent function, $K_1 \sim (\partial_\tau f)^2$. Furthermore, if f is linear in τ as in (5.16) then this K_1 is just a constant and thus such a model is renormalizable. This suggests that at least some time-dependent integrable models discussed in this chapter may have well-defined quantum generalizations.

Among other possible directions, it would be important to generalize the present investigation to other integrable σ-models, e.g. the class of models based on complex homogeneous spaces whose Ricci flow was studied in [251, 252] and the $\mathbb{Z}_T$ coset models of [18, 253]. Another interesting class of examples would be the integrable $G \times G$ and $G \times G/H$ models studied in Chapter 3 above. Furthermore, while in this chapter we have only studied 2-derivative σ-models, we expect the connection between the requirement of integrability of the time-dependent theory and the RG flow to be more general and to apply also to models with potentials. Indeed, one also finds this remarkable connection in the case of the sine-Gordon model,

$$S = \tfrac{1}{4\pi} \int d\tau d\sigma \, \tfrac{1}{g^2} \big[\tfrac{1}{2} \partial_+ x \partial_- x + m^2 \cos x \big] \ . \tag{5.56}$$

Replacing the couplings (g, m) by functions[76] of τ one can show that, for the resulting theory to be integrable (assuming a natural ansatz for a generalization of the Lax connection to the time-dependent theory — see Appendix H), the functions $(g(\tau), m(\tau))$ should be solutions of the 1-loop RG flow equation for (5.56), i.e. should be given by

$$m^2(\tau) = e^{\beta(g)\,\tau}\, m_0^2 \ , \qquad g(\tau) = g \ , \qquad \beta(g) = -2 + g^2 \ . \tag{5.57}$$

The time-dependent theory,

$$\widehat{S} = \tfrac{1}{4\pi} \int d\tau d\sigma \, \tfrac{1}{g^2} \big[\tfrac{1}{2} \partial_+ x \partial_- x + e^{\beta(g)\,\tau}\, m_0^2 \cos x \big] \ , \tag{5.58}$$

is indeed classically integrable since the explicit τ-dependence in (5.58) can be removed by a 2d conformal transformation getting back to (5.56): under $\xi^\pm \to f^\pm(\xi^\pm)$ the action (5.56) retains its form with $m^2 \to f^{+\prime}(\xi^+)\, f^{-\prime}(\xi^-)\, m^2$. Note that the non-trivial 1d reduction of the $SU(2)$ PCM in eq. (5.48) is also obtained as the 'trivial' reduction $x = x(\tau)$ of (5.58) after a redefinition $\tau \to \log \tau$, supporting the expectation that (5.48) is an integrable 1d theory. It would be interesting to explore whether this time-dependent integrability–RG flow connection applies also to other examples of integrable massive theories, such as complex sine-Gordon and Toda models.

[76] For a discussion of the sine-Gordon model with a particular 'non-integrable' local coupling $m = m(\sigma)$ see [254] and references there.

One of the features of our construction of the Lax connection for time-dependent theories is that the constant spectral parameter of the original theory is replaced by a function $z \to z(w;\tau,\sigma)$ of a new spectral parameter w and the 2d coordinates (cf. (5.20),(5.21)). This puts the spectral parameter and 2d space-time coordinates on a more equal footing, suggesting a possible interpretation from the point of view of the construction [165] of many integrable 2d theories from a 4d Chern-Simons theory, with the two extra directions related to the complex spectral parameter (see also [164] and refs. there). In that context the redefinition of z is like changing the differential structure of the 4d space, i.e. replacing $\partial_{\tau,\sigma} z = 0$ with $\partial_{\tau,\sigma} z = V_{\tau,\sigma}(z;\tau,\sigma)$ (cf. (5.33)). It would be interesting to see if the time-dependent models considered above can be reproduced from the 4d Chern-Simons construction. Indeed, the σ-dependent version of the PCM discussed above has already been obtained [255] from a closely related Chern-Simons theory on twistor space.

Chapter 6

Conclusions and future directions

6.1 Integrability vs. RG flow

This thesis is based on the interesting observation that integrable 2d σ-models appear to be stable under the leading 1-loop RG flow, i.e. they are solutions of the (generalized) Ricci flow equation (1.9) with a few couplings running. Our purpose was to investigate this property beyond the leading 1-loop order. We found in Chapter 2 for integrable η- and λ-deformed models, and in Chapter 3 for integrable $G \times G/H$ models, that generic examples will require quantum deformation of the target space geometry to remain stable under the 2-loop and higher RG flow. Thus, in each case, there is a particular finite counterterm — or particular choice of bare action — that restores renormalizability beyond 1-loop. Moreover, this is a scheme-independent statement since these counterterms cannot be eliminated by a covariant scheme transformation.

In Section 2.2, we explicitly constructed these counterterms for the simplest η-deformed S^2 (or "sausage") model. In that case, by analogy with the gauged WZW model (whose corrections are well-known, and which arises as a particular limit of the sausage model) we conjectured the exact form (2.11) of the quantum deformation to all orders in α' (or the σ-model loop expansion). We checked explicitly up to 3 loops that these corrections indeed restore renormalizability in a particular subtraction scheme.

In Section 2.3, we constructed the leading corrections restoring 2-loop renormalizability to the λ-models based on $\frac{SU(2)}{U(1)}$ and $SU(2)$. In the simpler $\frac{SU(2)}{U(1)}$ case, the corrections were understood as a familiar $(\partial \log \det M)^2$ determinant (2.43) from the quadratic integral over the λ-model's 2d gauge field in (1.78). However, such 2d integrals are known to be ill-defined [182] with a local term ambiguity depending on a choice of prescription or regularization. Indeed we find that this simple formula is not sufficient to restore renormalizability in the $SU(2)$ case, where the leading corrections take the more complicated form (2.64), or (2.56) in explicit coordinates.

In Chapter 4 we proved in general that the λ-model (1.78) in the formulation with unin-

tegrated gauge field is renormalizable to all orders, by re-writing the gauge field in terms of group valued scalars and obtaining a σ-model with a "tripled" $G \times G \times G$ configuration space. This means that the corrections required for renormalizability of the λ-model in "standard" configuration space may all be understood as finite counterterms arising from the ambiguous 2d gauge integral. Clearly the simple $(\partial \log \det M)^2$ prescription is not generally consistent with renormalizability (since it breaks down already for the $SU(2)$ case).

An interesting open problem is to find a general prescription for the finite local counterterms produced by integrals of this type, $\int dA \exp\left(i \int A_+ M A_-\right)$, consistent with the RG flow. This would give a general formula for the quantum corrections needed to restore higher-loop renormalizability for all λ-models, generalizing the $\frac{SU(2)}{U(1)}$ and $SU(2)$ results found here. One possible approach based on the work of [200] would be to try to construct the necessary counterterms from the scheme transformation between our standard 2-loop scheme (1.9) and one that is non-Lorentz-covariant but manifestly respects Poisson-Lie symmetry, in which the λ- and η-models are claimed to be automatically renormalizable.

For the $G \times G/H$ models in Section 3.2, the manifest (global and local) symmetries alone restrict the theory to a finite set of couplings (3.16),(3.19) that could run under the RG flow. The integrable theories then correspond to a "surface" in this space of couplings, specified by polynomial equations (3.25). Hence we observed that the necessary counterterms to restore 2-loop stability of the integrable surface under the RG flow can be interpreted as a quantum correction to the shape of the surface. This is like a simple, finite-dimensional analog of the general case where the space of σ-model couplings is essentially infinite (e.g., for the integrable deformed models above).

We also constructed in Chapter 3 a new "branch" of H-gauge-invariant $G \times G/H$ theories in the case when H is abelian, with independent WZ levels on the two copies of the group G. We found that a particular subfamily of this new branch is classically integrable and, in the $SU(2) \times SU(2)/U(1)$ case, these correspond to a family of integrable σ-models with target space $T^{1,q}$ supported by a particular B-field. These models are a generalization of the integrable $T^{1,1}$ models of [61], with the extra parameter q being related to the ratio of the two WZ levels.

In integrable examples with enough *manifest* symmetries, such as the undeformed PCM or symmetric space, renormalizability can be an automatic result of the symmetries. However, generic integrable σ-models may have no manifest global symmetries at all; rather, the integrable structure is related to an intricate structure of *hidden* (non-local or higher spin) symmetries. The effect of such symmetries on the RG flow is far less obvious since, e.g., it is not clear in which way they should constrain the Riemann tensor and H-field that enter the RG equations (1.9). Remarkably, it appears that hidden integrable symmetries impose drastic constraints on the RG flow, restricting the 1-loop RG flow to just a flow of the couplings of the classical theory. We have observed that this property persists also at higher loop orders provided the target space geometry is suitably deformed by quantum (α') corrections. Indeed,

such a deformation is natural given the the well-known examples of conformal σ-models where α' corrections are needed to restore higher-loop conformal invariance.

One interesting open question is to understand how and why integrability implies renormalizability, and perhaps to prove this statement, starting at the leading 1-loop order. As we discussed in the Introduction, the rationale behind the conjectured integrability–RG flow relationship is that the stability of integrable theories under the RG flow simply corresponds to preserving their integrable structure at the quantum level. In fact, it is natural that such hidden integrable symmetries should require a particular choice of finite local counterterms to be manifest in the quantum theory, just like for standard global symmetries. It would be interesting to verify this interpretation by checking explicitly that the finite counterterms required for renormalizability are actually restoring some quantum 'Ward identities' for hidden integrable symmetries. This would give an interesting semi-classical picture of quantum integrability. Such computations could be carried out, e.g., for the integrable-deformed models considered in Chapter 2, in relation to either the higher-spin charges or the non-local charges. For the η-deformed models, studying the quantum non-local charges would also be an interesting check of the conjectured quantum group symmetry of the quantum theory [95, 94, 96, 97].

The integrable $G \times G$ models studied in Section 3.1 are an anomaly to the discussion above. These theories are *automatically* stable under the 2-loop RG flow, even though (to our best knowledge) the integrability condition is not protected by any manifest symmetries. Generically we would have expected the integrable "surface" in coupling space to require some quantum deformation (like the $G \times G/H$ models) in order to remain stable beyond 1-loop. An interesting open problem is to explain this fact, which surely should not be a coincidence.

The $G \times G$ and $G \times G/H$ models demonstrate that, while integrability seems to imply renormalizability, the converse statement is not true. These models are renormalizable with a finite set of couplings but most of them are not integrable (only ones with special values of the couplings solving polynomial equations).

Classical integrability may also become anomalous at the quantum level, and we note that renormalizability seems too coarse a measure to pick up on this. For example the bosonic $\mathbb{C}\mathbf{P}^N$ model is well known to suffer such an anomaly [256] (see also [257]). However, it is still 2-loop renormalizable [258].[77]

We recall that our original motivation for studying stability under the RG flow was that it might help to *identify* integrable σ-models. In a certain sense, this is what we have been doing: starting from classically integrable theories, we have been constructing particular formulations

[77] For the $\mathbb{C}\mathbf{P}^N$ model it has been conjectured that quantum integrability can be restored by including an additional free field [259] in the classical limit (related models have also appeared in [260]). An alternative is to consider the supersymmetric $\mathbb{C}\mathbf{P}^N$ σ-model in which there is no anomaly [261].

of the quantum theories (with particular finite counterterms) that are RG-stable at higher-loop orders — with the belief that these formulations are the ones consistent with integrability at the quantum level.

It would be interesting in future to apply the integrability–RG flow connection to discover new examples of integrable σ-models. First at the level of classical integrability, this may be carried out by imposing stability under the leading 1-loop RG flow. One possible example would be for non-symmetric G/H cosets, which generally have several couplings,

$$\mathcal{L} = -\tfrac{1}{2}\,\mathrm{Tr}[J_+\,(\sum_{i=1}^{n} h_i P_{\mathfrak{p}_i})\,J_-]\ , \tag{6.1}$$

corresponding to a decomposition of the tangent space into irreducible representations $\mathfrak{p}_i$ under the adjoint action of H (see Section 1.4.3). It may be possible in certain cases to discover integrable "surfaces" in this coupling space by demanding stability under the RG flow. Indeed, renormalizability of certain non-symmetric coset spaces was previously considered in [262].

6.2 "Tripled" formulation

The "tripled" formulation of the λ-model in Chapter 4 may be viewed as an alternate formulation of the quantum theory where some fields decouple, likely as a manifestation of integrability. The remaining "truncated" theory is automatically stable under the RG flow to all orders due to manifest global symmetries, without needing any quantum deformation.

The general idea of moving to an enlarged configuration space to make manifest extra symmetries (in this case integrability) is reminiscent of T-duality-invariant formulations of string theory [228]. Such approaches are essentially 1st order Hamiltonian formulations of the σ-model, with manifest Lorentz invariance being sacrificed in favour of manifest T-duality invariance. A related 1st order formulation of Poisson-Lie symmetric σ-models known as the "E-model" makes the more general Poisson-Lie duality manifest [103].

In Section 4.5, we considered similar reformulations on enlarged configuration spaces of (i) a variant of the λ-model based on G^N and (ii) $G \times G/H$ models. In each case the resulting model was of product type, i.e. a particular class of coupled models on some space G^M; and in the $G \times G/H$ case there was some "squashing", or asymmetry between the H and G/H directions. Re-writing these models in this way is interesting as the corresponding enlarged models will have identical classical integrability and RG flow properties to the original ones. For the G^N λ-model we find an identification between its RG flow and classical integrability conditions and those of an apparently simpler undeformed G^{2N} model. We also learn that it is automatically renormalizable in the tripled formulation due to its manifest symmetries (before imposing the integrability conditions of [161]). For the $G \times G/H$ models we essentially discover a new class of classically integrable and RG-stable squashed $G \times G \times H$ models. It would be interesting to

study the higher-loop RG flows of these theories to see if the enlarged formulations render the integrability conditions automatically stable at higher-loops (as for the λ-model).

Another interesting class of σ-models are those with complex homogeneous target spaces that are equivalent [252, 263] to bosonic Gross-Neveu (GN) models [264] (see also the reviews [265]). The reformulation of these σ-models as GN models is essentially equivalent to the introduction of a 2d gauge field, so should be closely related to the "tripled" or "extended" formulations discussed here. It would be interesting to check stability under the 2-loop RG flow in that formulation (which was previously only checked at 1-loop [252]).

An interesting future direction would be to try to construct a similar tripled formulation of the η-deformed models. It seems plausible that such a formulation should exist since the η-models and λ-models are related (up to analytic continuation) by Poisson-Lie duality a [104, 105, 107], or by limits and T-duality [104, 107].

6.3 Non-abelian duality

In the non-abelian dual limit of the λ-model, the general 2-loop β-functions obtained from the tripled formulation match those of the original PCM/symmetric space σ-models. This agreement is a non-trivial check of our λ-model β-functions since the 2-loop terms are also scheme-invariant under coupling redefinitions in that single-coupling limit. It is thus natural to expect that the interpolating models for non-abelian duality are renormalizable with the same 2-loop β-functions as the original models — justified also by their relation to the original models by just an integral over a linear Lagrange multipler imposing the flatness of the gauge field.

Hence, like the λ-model, the non-trivial corrections found for the NAD models based on $SU(2)/U(1)$ and $SU(2)$ in Section 2.3.3 should be seen as resulting from the integral over the 2d gauge field. In certain limits these NAD models degenerate to abelian T-duals and in those cases our corrections matched with the known corrections to the T-duality transformation.

It would be interesting to consider more general models with non-abelian symmetries, which may have spectator coordinates (here x or u, v), for example

$$\mathcal{L} = \partial_+ x\, \partial_- x + f(x)\, \mathrm{Tr}[J_+ J_-], \qquad (J \equiv g^{-1} dg, \quad g \in G) \tag{6.2}$$

or perhaps the conformal σ-model considered in (5.17),

$$\mathcal{L} = -2\, \partial_+ u\, \partial_- v + u\, \mathrm{Tr}[J_+ J_-]\ . \tag{6.3}$$

Starting from such a theory that is integrable and renormalizable (say, for a particular choice of $f(x)$) the interpolating model for NAD is obtained by replacing $J_\pm \to A_\pm$ and adding a Lagrange multiplier constraint $\mathrm{Tr}[v\, F_{+-}(A)]$. One can then perform the same trick of going to

a tripled formulation by re-writing the gauge field in terms of group valued scalars. It would be interesting to study the RG flows for these interpolating models in the tripled formulation and to see if they still automatically match the original models (also at 2-loops) in the presence of spectator fields.

6.4 Quantum corrections and string sigma models

An important application of our results would be to the related λ- and η-deformations of superstring σ-models. For example, focussing on the simplest case of $AdS_5 \times S^5$, the λ-deformed background has been extracted from a σ-model formulation [117] and checked to be a type IIB supergravity solution [126]. Equivalently, the string σ-model is 1-loop Weyl-invariant (1-loop scale-invariance was also checked in [53]). The η-deformed $AdS_5 \times S^5$ background of [129] was also extracted and checked to be a solution of the supergravity equations.

In general, solutions of the $d = 10$ type II supergravity equations should receive α' corrections to become exact string solutions, with the first corrections appearing at order α'^3 [266]. In contrast with the undeformed $AdS_5 \times S^5$ background where corrections are prohibited by manifest symmetries,[78] it is not clear whether the deformed backgrounds will require non-trivial corrections. It may be that, due to the models' hidden integrable symmetries, the corrections can be made to vanish in some particular subtraction scheme. On the other hand, this did not happen for the bosonic analogs above, which are also integrable.

It would be interesting to study this question and to understand the structure of the α' corrections to these backgrounds. For the λ-deformation, a systematic way to do this may come from adapting the tripled formulation to the superstring case. This seems feasible since the λ-deformed $AdS_5 \times S^5$ superstring was formulated in [117] in the language of supergroups with a 2d gauge field. In such a tripled formulation, the α' corrections may be expected to vanish due to manifest symmetries after a decoupling of some fields (like the bosonic case), and the exact finiteness of the model may become manifest.

6.5 Sigma models with local couplings

In Chapter 5 we discussed a new and apparently different connection between integrability and the RG flow: upon promoting the couplings of an integrable σ-model to functions of 2d time, the resulting "local coupling" theory remains classically integrable (with a natural ansatz for its Lax connection) only when the coupling functions solve the 1-loop RG flow, with 2d time playing the role of RG time $t = \log \mu$. This curious pattern was observed in a broad

[78] Exact finiteness of the undeformed $AdS_5 \times S^5$ superstring σ-model (1.87) follows because the WZ-type $J^{(1)}J^{(3)}$ term is not renormalized due to general arguments and the $J^{(2)}J^{(2)}$ term may only be renormalized by an overall factor, but this cannot happen since the two terms are related by κ-symmetry (see footnote 4 of [267] and refs. there).

class of integrable σ-models on groups and symmetric spaces, including integrable η- and λ-deformations. It would be good to check this claim for more examples of integrable σ-models, like the $G \times G$ and $G \times G/H$ models. We also found the same result for the sine-Gordon model, and it would be interesting to check more examples of integrable theories with potentials, e.g. Toda models.

These results are surprising because it seems that the 1-loop quantum β-functions are derivable in this way from a purely classical procedure (from demanding that the local-coupling theory admits a flat Lax connection). We also noted that theories with explicit time-dependence governed by the RG flow are naturally obtained in string theory upon fixing the light-cone gauge $u = \tau$, starting from a class of Weyl-invariant σ-models [238, 239] with a covariantly constant null Killing vector and a dilaton linear in the isometry direction,

$$\mathcal{L} = -2\,\partial_+ u\,\partial_- v + \mathrm{G}_{ij}(u;x)\partial_+ x^i \partial_- x^j \ , \quad \partial_u \mathrm{G}_{ij} = \beta_{ij}(x) \ , \qquad (\phi = v + \phi(u,x)) \ . \tag{6.4}$$

Thus we find that — assuming the "transverse" (u = const) theory to be integrable — the light-cone ($u = \tau$) theory is apparently classically integrable only when the string σ-model (6.4) is 1-loop Weyl invariant. This points to a deeper meaning or explanation of our results, and to the possibility of a large, new class of solvable string backgrounds.

What remains to be clarified is whether these time-dependent (light-cone) theories may really be viewed as solvable. Although the monodromy matrix is formally conserved, it is difficult to explicitly extract the associated conserved charges because of problems making a good expansion in the spectral parameter. We also made attempts to study higher-spin local charges in the paper [4], but our results so far were inconclusive. One possible way to demonstrate solvability would be to explicitly construct some classical solutions; indeed this has already been done for the time-dependent PCM example [245] by adapting the inverse scattering approach of [42, 43].

An interesting extension would be to consider the $(2,2)$ supersymmetric σ-models on certain Kähler symmetric spaces whose β-functions are known to be 1-loop exact [242]. Supersymmetric σ-models on symmetric spaces are classically integrable in general [268]. Hence the pattern found above for bosonic theories may generalize to the supersymmetric ones. In that case the corresponding string backgrounds built from the 1-loop RG flow (which will be *exact* string solutions) may be completely solvable.

A further question is whether the local-coupling σ-models admitting Lax representations can be obtained from 4d Chern-Simons theory [165] (like all of the regular integrable σ-models considered in this thesis [163, 164]). This is a natural expectation, since the 4d perspective places the 2d space-time and the complex spectral parameter on an equal footing, while the Lax connections for the local coupling theories are related to the original ones by a space-time-dependent redefinition of the spectral parameter, $z \to z(w;\tau,\sigma)$.

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Appendix A

Conventions

Unless otherwise stated, the 2d space is assumed to be flat Minkowski space with signature $(-+)$, i.e. $\eta_{00} = -\eta_{11} = -1$ in terms of the 2d coordinates $\xi^\mu = (\xi^0, \xi^1) = (\tau, \sigma)$ with 2d incides $\mu, \nu = 1, 2$. The antisymmetric Levi-Civita symbol $\varepsilon_{\mu\nu}$ is defined by $\varepsilon^{01} = -\varepsilon^{10} = 1$ and $\varepsilon_{01} = -\varepsilon_{10} = -1$ (with indices raised/lowered by $\eta_{\mu\nu}$). We use 2d light-cone coordinates defined as $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$ and light-cone derivatives $\partial_\pm = \partial_\tau \pm \partial_\sigma$.

The σ-model action is normalized as $S = \frac{1}{4\pi\alpha'} \int d^2\xi\, \mathcal{L}$ with the usual flat-space volume form $d^2\xi = d\xi^0 \wedge d\xi^1$. The Lagrangian is

$$\mathcal{L} = -\big[\eta^{\mu\nu}\, \mathrm{G}_{mn}(x) + \varepsilon^{\mu\nu}\, \mathrm{B}_{mn}(x)\big]\, \partial_\mu x^m\, \partial_\nu x^n = \big[\mathrm{G}_{mn}(x) + \mathrm{B}_{mn}(x)\big]\, \partial_+ x^m \partial_- x^n \ . \tag{A.1}$$

The target space metric is $ds^2 = G_{mn} dx^m dx^n$. The B-field written as a 2-form is $\mathrm{B} = \frac{1}{2} dx^m \wedge dx^n$ and its field strength is $\mathrm{H} = d\mathrm{B}$.

The components of n-forms are normalized in the standard way, $H_{(n)} = \frac{1}{n!} H_{i_1 \cdots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n}$.

We take G to be a simple, real Lie group with generators T_A ($A = 1, \ldots, \dim(G)$) satisfying $[T_A, T_B] = i f^C{}_{AB} T_C$.

We use a non-degenerate bilinear form defined as

$$\mathrm{Tr} = \frac{1}{2\chi_{G,R}}\, \mathrm{tr}_R \ , \tag{A.2}$$

where tr_R is the matrix trace in some faithful representation R of G. The constant $\chi_{G,R}$ denotes the *index* of the representation R, defined by

$$\frac{1}{2\chi_{G,R}}\, \mathrm{tr}_R(T_A T_B) = -\frac{1}{2c_G}\, f^C{}_{DA} f^D{}_{CB} \ . \tag{A.3}$$

Here c_G denotes the dual Coxeter number of G: for the classical groups these are $c_{SU(N)} = N$, $c_{Sp(N)} = N + 1$ and $c_{SO(N)} = N - 2$.

The indices of the adjoint representations coincide with the dual Coxeter numbers, $\chi_{G,adj} = c_{_G}$, and those of the fundamental representations of the classical groups are $\chi_{SU(n),f} = \chi_{Sp(n),f} = \frac{1}{2}$ and $\chi_{SO(n),f} = 1$.

The structure constants then satisfy

$$f^A{}_{BC} f^B{}_{AD} = -2\, c_{_G} \mathrm{Tr}[T_C T_D] \ , \qquad f^D{}_{EA} f^E{}_{GB} f^G{}_{DC} = c_{_G}\, f_{ABC} \ , \tag{A.4}$$

where we have defined $f_{ABC} = \mathrm{Tr}[T_C T_D] f^D{}_{AB}$.

We shall sometimes consider a Lie subgroup $H \subset G$, which is also assumed to be simple. Then we split the algebra directions as $A = (\alpha, a)$ where T^α are the generators of the H and the remaining generators T^a are orthogonal to T^α with respect to the Killing form.

The structure constants $f^\alpha{}_{\beta\gamma}$ of the subgroup H then satisfy

$$f^\alpha{}_{\beta\gamma} f^\beta{}_{\alpha\delta} = -2\, c_{_H}\, \mathrm{Tr}[T_\gamma T_\delta] \qquad f^\delta{}_{\varepsilon\alpha} f^\varepsilon{}_{\kappa\beta} f^\kappa{}_{\delta\gamma} = c_{_H}\, f_{\alpha\beta\gamma} \ , \tag{A.5}$$

where $c_{_H}$ denotes a *rescaled* version of the dual Coxeter number $\frac{\chi_{H,R}}{\chi_{G,R}} c_{_H}$ of H.

Appendix B

3-loop β-function in different renormalization schemes

In general, in quantum field theories changes of a renormalization scheme are equivalent to local redefinitions of the coupling constants,[79]

$$g^i \to \hat{g}^i(g) = g^i + \alpha' \, c^i_{jk} g^j g^k + \dots \ . \tag{B.1}$$

Under (B.1) the β-functions $\beta^i \equiv \frac{d}{dt} g^i$ transform as a contravariant vector in the space of couplings:

$$\delta\beta^i \equiv \hat{\beta}^i - \beta^i = \frac{\partial \, \delta g^i}{\partial g^j} \beta^j - \frac{\partial \beta^i}{\partial g^j} \delta g^j + \dots \ , \qquad\qquad \delta g^i \equiv \hat{g}^i - g^i \ , \tag{B.2}$$

where the higher order corrections are non-linear in δg. Assuming β^i and δg^i both start at 1-loop order, i.e. are both $\mathcal{O}(\alpha')$, then $\delta\beta$ is $\mathcal{O}(\alpha'^2)$, so the 1-loop β-function is scheme-invariant. In single-coupling theories, one can further show that the 2-loop β-function is also scheme-invariant.

Now let us turn to the particular case of 2d σ-models, for which the relevant coupling redefinitions (B.1) are local redefinitions of the target space geometry. In order to preserve the manifestly covariant structure of the action (1.6) we restrict to covariant redefinitions. Restricting further to include only redefinitions with a natural interpretation in terms of changing the

[79] Here we reinstate the loop-counting parameter $\alpha' = \hbar$.

subtraction scheme,[80] we consider $\mathrm{G}_{mn} \to \hat{\mathrm{G}}_{mn}$ with

$$\begin{aligned} &\hat{\mathrm{G}}_{mn} = \mathrm{G}_{mn} + c\,\alpha' R_{mn} + d\,\alpha'^2 (R^2)_{mn} + e\,\alpha'^2 \nabla^2 R_{mn} + \mathcal{O}(\alpha'^3) \ , \\ &(R^2)_{mn} \equiv R_{mabc} R_n{}^{abc} \ , \end{aligned} \tag{B.3}$$

where c, d, e are arbitrary coefficients (for simplicity we consider the special case with vanishing B-field; for a general discussion see [35]).

At the leading $\mathcal{O}(\alpha')$ order, both $\beta_{mn} = \alpha' R_{mn} + \ldots$ and $\delta \mathrm{G}_{mn} = c\,\alpha' R_{mn} + \ldots$ only contain a single term proportional to R_{mn}. Thus at the leading $\mathcal{O}(\alpha'^2)$ order in (B.2), their contributions cancel and the 2-loop β-function is invariant.[81]

Specializing further to the case of a 2d target space (relevant for the simplest $\frac{SU(2)}{U(1)}$ η- and λ-models considered in Chapter 2), the redefinition (B.3) becomes

$$\hat{\mathrm{G}}_{mn} = \mathrm{G}_{mn} \Big[1 + \tfrac{1}{2} c\,\alpha' R + \tfrac{1}{2} d\,\alpha'^2 R^2 + \tfrac{1}{2} e\,\alpha'^2 \nabla^2 R \Big] \ . \tag{B.4}$$

Possible scheme dependence starts at 3-loop order, with (B.3) or (B.4) being the relevant redefinition at this order. The most general form of a covariant expression for a 3-loop β-function is[82]

$$\beta_{mn} = \Big[\tfrac{1}{2}\alpha' R + \tfrac{1}{4}\alpha'^2 R^2 + c_1\,\alpha'^3 R^3 + c_2\,\alpha'^3 (\nabla R)^2 + c_3\,\alpha'^3 R \nabla^2 R \Big] \mathrm{G}_{mn} + c_4\,\alpha'^3 \nabla_m R \nabla_n R \ . \tag{B.5}$$

Not all of the coefficients $c_1, \ldots, c_4$ can be scheme-dependent, since the space of legitimate schemes should be parametrized by the 3-dimensional space of coupling redefinitions (B.3). To find the values of c_i that are allowed to appear in the β-function in a particular scheme let us start from the expression for the β-function (B.5) in the minimal subtraction scheme where [36, 37, 38]

$$c_1 = \tfrac{5}{32} \ , \qquad c_2 = \tfrac{1}{16} \ , \qquad c_3 = 0 \ , \qquad c_4 = -\tfrac{1}{16} \ . \tag{B.6}$$

Implementing the metric redefinition (B.3) we find from (B.2) that the coefficients in the trans-

[80] While the term $\alpha'^2 \nabla^2 R_{mn}$ in (B.3) has no natural interpretation as a change of subtraction scheme starting from the minimal subtraction scheme, it is required in order that the transformation (B.4) be closed under inversion in the case of the σ-model with a 2d target space, i.e. in order to be able to move back to the minimal subtraction scheme starting from another scheme.

[81] Note that if one also included redefinitions of the form $\alpha' R \mathrm{G}_{mn}$ then this argument would not go through in general dimensions. However, for the special case of a 2d target space, the identity $R_{mn} = \frac{1}{2} R \mathrm{G}_{mn}$ means these terms are proportional and the 2-loop β-function remains invariant.

[82] In (B.5) we have dropped terms of the form $\nabla_{(m} V_{n)}$ since these may be absorbed into the diffeomorphism vector term in (1.8). We have excluded the term $\alpha'^2 \nabla^2 \nabla^2 R \mathrm{G}_{mn}$ in (B.5) since it does not arise naturally through changes of subtraction scheme starting from the minimal subtraction scheme.

formed β-function are given by

$$\begin{aligned} \hat{c}_1 &= \tfrac{5}{32} + \tfrac{1}{8}(c-2d) \ , & \hat{c}_2 &= \tfrac{1}{16} - \tfrac{1}{4}(c-2d) - \tfrac{1}{8}(4e+c^2) \ , \\ \hat{c}_3 &= -\tfrac{1}{8}(4e+c^2) \ , & \hat{c}_4 &= -\tfrac{1}{16} \ . \end{aligned} \tag{B.7}$$

We note that since (B.7) only depends on the redefinition parameters c, d, e through the combinations $c-2d$ and $4e+c^2$, the choice of c, d, e to give any particular values of $c_1, \ldots, c_4$ is not unique, but rather there is one free parameter. In the case of the η-model metric studied in Section 2.2, the "natural" scheme (2.19) is given by (B.7) with, e.g., $(c, d, e) = (0, \frac{5}{8}, \frac{1}{2})$; i.e. it is related to the minimal subtraction scheme by the redefinition

$$\mathrm{G}^{(\mathrm{nat})}_{mn} = \left[\mathrm{G}_{mn} + \tfrac{5}{8}\,\alpha'^2 (R^2)_{mn} + \tfrac{1}{2}\,\alpha'^2 \nabla^2 R_{mn}\right]^{(\mathrm{min})} \ . \tag{B.8}$$

Appendix C

Deriving the integrability conditions for the G^N model

It was shown in [160] that the coupled G^N model (3.2) is integrable for particular choices of the couplings (ρ_{ij}, k_i) corresponding to realisations of the affine Gaudin models. Here we shall try to demonstrate the converse statement: these affine Gaudin models are the *only* integrable cases of the coupled models (3.2).

We will assume a natural ansatz (3.4) for the Lax connection, valued in Lie(G) (here we explicitly indicate the summation over $i = 1, ..., N$)[83]

$$L_+ = \sum_i \alpha_i(z)\, J_+^{(i)} \ , \qquad\qquad L_- = \sum_i \beta_i(z)\, J_-^{(i)} \ , \tag{C.1}$$

where z is the spectral parameter. The curvature of this Lax connection takes the form

$$F_{+-}(L) = \sum_i \Big(\beta_i(1-\alpha_i)\ \partial_+ J_-^{(i)} - \alpha_i(1-\beta_i)\ \partial_- J_+^{(i)}\Big) + \sum_{i\neq j} \alpha_i\beta_j\ [J_+^{(i)}, J_-^{(j)}] \ . \tag{C.2}$$

The equations of motion of the G^N model (3.2) (for arbitrary N)

$$\begin{aligned} E_i \quad \equiv \quad \sum_j \Big(& (\rho_{ij} - \delta_{ij}k_j)\ \partial_+ J_-^{(j)} + (\rho_{ij} + \delta_{ij}k_j)\ \partial_- J_+^{(j)} \\ & + \rho_{ij}\ [J_+^{(i)}, J_-^{(j)}] + \rho_{ji}\ [J_-^{(i)}, J_+^{(j)}]\Big) \quad = \quad 0 \end{aligned} \tag{C.3}$$

We note that (C.2) and (C.3) are the *unique* ways to write these expressions without any terms of the form $[J_+^{(i)}, J_-^{(i)}]$, which have been eliminated using the identity $F_{+-}(J^{(i)}) = 0$.

[83] While (C.1) is the natural ansatz for the Lax connection arising from affine Gaudin models, it does degenerate at certain points in coupling space. For example, taking ρ_{ij} to be diagonal (i.e. decoupled PCM$_k$ models), one instead requires a Lax connection valued in Lie$(G)^N$. Thus it would also be interesting to consider other ansatze for the Lax connection.

If the model (3.2) is integrable then,[84] for some $v^i(z)$, we have

$$F_{+-}(L) = \sum_i v^i(z)\, E_i \ , \tag{C.4}$$

which implies that

$$\begin{gathered} \beta_j(1-\alpha_j) = \sum_i v^i(\rho_{ij} - \delta_{ij}k_j) \ , \qquad\qquad \alpha_j(1-\beta_j) = \sum_i v^i(-\rho_{ji} - \delta_{ij}k_j) \ , \\ \alpha_i\beta_j = (v^i - v^j)\rho_{ij} \ , \quad i \neq j \ \ \text{(no summation)} \ . \end{gathered} \tag{C.5}$$

This is a system of $N + N + (N^2 - N) = N^2 + N$ equations. Fixing the freedom to redefine the spectral parameter by setting $v^1 = z$, there are $N + N + (N-1) = 3N - 1$ "artificial" variables, $\alpha_i, \beta_i, v^{i\neq 1}$. After solving for these, there are $(N^2+N) - (3N-1) = N^2 - 2N + 1$ remaining equations to be solved for the $N^2 + N$ variables ρ_{ij}, k_i. After solving all the equations, this leaves $(N^2+N) - (N^2 - 2N + 1) = 3N - 1$ free parameters for the integrable theory (including the WZ levels, which may be continuous for non-compact groups).[85]

Thus the space of integrable models is $(3N-1)$-dimensional, which coincides with the number of free parameters following from the affine Gaudin construction [160].

Specializing to the $N = 2$ case of $G \times G$, this counting suggests a 5-dimensional space of integrable models. Then the 6 free parameters (s, t, u, b, k_1, k_2) in (3.5) should be subject to only one relation to ensure integrability. Solving the equations (C.5) in this case, one indeed obtains the condition (3.6) originally found from the affine Gaudin construction.

To summarize, for general N, the space of integrable models has the same dimension as the space of affine Gaudin models. It remains to understand if there may still be extra branches of integrable theories not corresponding to the affine Gaudin models (cf. the $G \times G/H$ models, where this seems to be the case for abelian H, see Section 3.3). For the $N = 2$ case of $G \times G$ models, we found exact matching between the space of integrable models (C.5) and the space of affine Gaudin models satisfying the condition (3.6).

[84] Here we are assuming integrability and deriving necessary conditions on the couplings. Thus we do not need to worry about whether the $v^i(z)$ in (C.4) are independent functions (which would be relevant for the converse question).

[85] One might worry that the integrability constraints on the couplings ρ_{ij}, k_i resulting from (C.4) might depend on the spectral parameter $v^1 = z$. However, this will not happen because there is a rescaling ambiguity $E_i \to c_i E_i$ in the definition of the equations of motion (C.3). One may thus rescale E_1 to effectively set $v^1 = z = 1$ in (C.4). Since the constraints on the couplings from (C.4) must be invariant under such rescalings, then they must not depend on z.

Appendix D

Explicit 2-loop β-functions for $G \times G$ and $G \times G/H$ models

Here we shall provide the explicit formulae for the 2-loop β-functions of the general $G \times G$ and $G \times G/H$ models that were used in the Chapter 3.[86] We will also briefly explain how they were derived.

D.1 $G \times G$ model

For the $G \times G$ model (3.5), let us use the notation

$$\rho_{ij} = \mathrm{h}_{(ij)} + b_{[ij]} = \begin{pmatrix} s & t \\ t & u \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \ . \tag{D.1}$$

Let the 2×2 matrix n_{ij} be the "square root" of $\mathrm{h}_{ij} = \mathrm{h}_{(ij)}$, and let m_{ij} be its inverse,

$$n_{ik}\, n_{jk} = \mathrm{h}_{ij} \ , \qquad m_{ik}\, n_{kj} = \delta_{ij} \ , \qquad\qquad n_{ij} = n_{(ij)}, \qquad m_{ij} = m_{(ij)} \ . \tag{D.2}$$

The target space metric of the σ-model (3.5) is "diagonalized" by the vielbein 1-form,[87]

$$\begin{aligned} &E^{Ai} = n_{ik}\, J^{(k)A} \ , \qquad A = 1, \dots, \dim G \ , \quad i = 1, 2 \ , \\ &J^{(k)} \equiv T_A J^{(k)A} = \big(g^{(k)}\big)^{-1} dg^{(k)} \ . \end{aligned} \tag{D.3}$$

[86] The formulae derived in this Appendix are also available in the Mathematica file attached to the arXiv submission of the paper [5].

[87] See Appendix A for the group theory conventions. The generators T_A satisfy $[T_A, T_B] = i f^C{}_{AB} T_C$ and we define $f_{ABC} = \mathrm{Tr}[T_C T_D] f^D{}_{AB}$. For simple groups G, the structure constants satisfy $f^A{}_{BC} f^B{}_{AD} = -2 c_{_G} \mathrm{Tr}[T_C T_D]$ and $f^D{}_{EA} f^E{}_{GB} f^G{}_{DC} = c_{_G} f_{ABC}$, where $c_{_G}$ is the dual Coxeter number of G.

Then the coefficients of the metric $ds^2 = \mathrm{G}_{Ai,Bj}E^{Ai}E^{Bj}$ and the 3-form $\mathrm{H} = d\mathrm{B} = \frac{1}{6}\mathrm{H}_{Ai,Bj,Ck}E^{Ai}\wedge E^{Bj}\wedge E^{Ck}$ are given by[88]

$$\mathrm{G}_{Ai,Bj} = -\tfrac{1}{2}\,\mathrm{Tr}[T_A T_B]\,\delta_{ij}\ , \tag{D.4}$$

$$\mathrm{H}_{Ai,Bj,Ck} = \tfrac{i}{2} f_{ABC}\big[k_l\, m_{il}\, m_{jl}\, m_{kl} + b_{lp}(m_{ip}\, m_{jl}\, m_{kl} + m_{il}\, m_{jp}\, m_{kl} + m_{il}\, m_{jl}\, m_{kp})\big]\ . \tag{D.5}$$

From Cartan's structure equation $dE^{Ai} + \widehat{\omega}^{Ai}{}_{Bj}\wedge E^{Bj} = T^{Ai}$ with torsion $T^{Ai} = \frac{1}{2}\mathrm{H}^{Ai}{}_{Bj,Ck}E^{Bj}\wedge E^{Ck}$, we obtain the torsionful spin connection,

$$\widehat{\omega}^{Ai}{}_{Bj} = \tfrac{i}{2} f^{A}{}_{BC} M_{ijk} E^{Ck}\ , \tag{D.6}$$

$$\begin{aligned} M_{ijk} \equiv\ & m_{il}m_{jl}n_{kl} - m_{il}n_{jl}m_{kl} - n_{il}m_{jl}m_{kl} \\ & + k_l\, m_{il}\, m_{jl}\, m_{kl} + b_{lp}(m_{ip}\, m_{jl}\, m_{kl} + m_{il}\, m_{jp}\, m_{kl} + m_{il}\, m_{jl}\, m_{kp})\ . \end{aligned} \tag{D.7}$$

The torsionful Riemann curvature $\widehat{R}^{Ai}{}_{Bj} \equiv \frac{1}{2}\widehat{R}^{Ai}{}_{Bj,Ck,Dl}E^{Ck}\wedge E^{Dl} = d\widehat{\omega}^{Ai}{}_{Bj} + \widehat{\omega}^{Ai}{}_{Ck}\wedge\widehat{\omega}^{Ck}{}_{Bj}$ is then found in terms of M_{ijk} to be

$$\begin{aligned} \widehat{R}^{Ai}{}_{Bj,Ck,Dl} = \tfrac{1}{4}\big[& 2 f^{A}{}_{BE} f^{E}{}_{CD} M_{ijp} n_{pq} m_{kq} m_{lq} \\ & + f^{A}{}_{CE} f^{E}{}_{BD} M_{ipk} M_{pjl} - f^{A}{}_{DE} f^{E}{}_{BC} M_{ipl} M_{pjk}\big]\ . \end{aligned} \tag{D.8}$$

It is then straightforward to substitute (D.5),(D.7),(D.8) into the 2-loop β-functions in the GB scheme (1.9), obtaining explicit formulae for the RG equations $\frac{d}{dt}\rho_{ij} = \beta_{ij}(n_{11}, n_{12}, n_{22}, b, k_1, k_2)$ depending on the components of n_{ij}. Using a computer symbolic algebra package (e.g. Mathematica) it is easy to rewrite these expressions in terms of the components s, t, u of the "squared" coupling $\mathrm{h}_{ij} = n_{ik}n_{jk}$ in (D.1), with all the square roots cancelling out as the Riemann tensor and the H-field must clearly be rational functions of ρ_{ij}. We thus obtain the β-functions in the form given in (3.8),

$$\begin{aligned} & \frac{d}{dt}\rho_{ij} = \alpha'\,\beta^{(1)}_{ij} + \alpha'^2\,\beta^{(2)}_{ij} + \dots\ , \\ & \beta^{(1)}_{ij} = c_{\,G}(su - t^2)^{-2}\,F^{(4)}_{ij}(s,t,u,b,k_1,k_2)\ , \qquad \beta^{(2)}_{ij} = \mathrm{c}^2_{G}(su - t^2)^{-5}\,F^{(9)}_{ij}(s,t,u,b,k_1,k_2)\ , \end{aligned} \tag{D.9}$$

where the explicit form of the homogeneous polynomials $F^{(4)}_{ij}$ and $F^{(9)}_{ij}$ is:

$$\begin{aligned} F^{(4)}_{11} = {} & -b^2 s^2 - 4 b^2 s t - 2 b^2 t^2 + s^2 t^2 + 2 s t^3 + 2 t^4 - 2 b^2 s u - 2 s t^2 u + s^2 u^2 + 2 b t^2 k_1 + 4 b t u k_1 - u^2 k_1^2 \end{aligned}$$

$$\begin{aligned} F^{(4)}_{12} = {} & 2 b^2 s t + 3 b^2 t^2 + 2 b s t^2 + 2 b t^3 - t^4 + 2 b^2 s u + b s^2 u + 2 b^2 t u + 4 b s t u + s^2 t u + 2 b t^2 u + 3 s t^2 u + b s u^2 \\ & + s t u^2 - 2 b t u k_1 - t^2 u k_1 - b u^2 k_1 - t u^2 k_1 + b s^2 k_2 + 2 b s t k_2 + s^2 t k_2 + s t^2 k_2 - t^2 k_1 k_2 \end{aligned}$$

[88] The overall factor of i in (D.5) simply reflects the fact that the vielbein (D.3) is imaginary. This makes no difference and could be eliminated by just multiplying $E^{Ai} \to iE^{Ai}$.

$$F^{(4)}{}_{21} = 2b^2st + 3b^2t^2 - 2bst^2 - 2bt^3 - t^4 + 2b^2su - bs^2u + 2b^2tu - 4bstu + s^2tu - 2bt^2u + 3st^2u - bsu^2 + stu^2 - 2btuk_1 + t^2uk_1 - bu^2k_1 + tu^2k_1 + bs^2k_2 + 2bstk_2 - s^2tk_2 - st^2k_2 - t^2k_1k_2$$

$$F^{(4)}{}_{22} = -2b^2t^2 + 2t^4 - 2b^2su - 4b^2tu - 2st^2u + 2t^3u - b^2u^2 + s^2u^2 + t^2u^2 - 4bstk_2 - 2bt^2k_2 - s^2k_2^2$$

$$F^{(9)}{}_{11} = \frac{1}{2}\left(22b^4s^3t^2 + 98b^4s^2t^3 + 115b^4st^4 - 20b^2s^3t^4 + 30b^4t^5 - 78b^2s^2t^5 - 98b^2st^6 - 2s^3t^6 - 36b^2t^7 + 7st^8 + 6t^9 + 5b^4s^4u + 64b^4s^3tu + 214b^4s^2t^2u - 6b^2s^4t^2u + 200b^4st^3u - 24b^2s^3t^3u + 30b^4t^4u - 64b^2s^2t^4u + s^4t^4u - 100b^2st^5u - 12s^3t^5u - 36b^2t^6u - 26s^2t^6u - 12st^7u + 6t^8u + 22b^4s^3u^2 - b^2s^5u^2 + 94b^4s^2tu^2 - 6b^2s^4tu^2 + 70b^4st^2u^2 + 4b^2s^3t^2u^2 + s^5t^2u^2 + 44b^2s^2t^3u^2 + 12s^4t^3u^2 + 4b^2st^4u^2 + 13s^3t^4u^2 - 6s^2t^5u^2 - 18st^6u^2 + 8b^4s^2u^3 - 4b^2s^4u^3 - 16b^2s^3tu^3 + 12b^2s^2t^2u^3 + 6s^4t^2u^3 + 12s^3t^3u^3 + 16s^2t^4u^3 - 7b^2s^3u^4 - 5s^3t^2u^4 + s^4u^5 - 30b^3st^4k_1 - 42b^3t^5k_1 + 10bst^6k_1 + 18bt^7k_1 - 40b^3s^2t^2uk_1 - 196b^3st^3uk_1 - 156b^3t^4uk_1 + 12bs^2t^4uk_1 + 44bst^5uk_1 + 68bt^6uk_1 - 2b^3s^3u^2k_1 - 68b^3s^2tu^2k_1 - 238b^3st^2u^2k_1 - 4bs^3t^2u^2k_1 - 108b^3t^3u^2k_1 - 4bs^2t^3u^2k_1 - 10bst^4u^2k_1 + 52bt^5u^2k_1 - 20b^3s^2u^3k_1 - 72b^3stu^3k_1 - 4bs^3tu^3k_1 - 4bs^2t^2u^3k_1 - 48bst^3u^3k_1 + 14bs^2tu^4k_1 + 30b^2t^4uk_1^2 - 2t^6uk_1^2 + 34b^2st^2u^2k_1^2 + 126b^2t^3u^2k_1^2 - st^4u^2k_1^2 - 10t^5u^2k_1^2 + 2b^2s^2u^3k_1^2 + 48b^2stu^3k_1^2 + 96b^2t^2u^3k_1^2 + 4st^3u^3k_1^2 - 12t^4u^3k_1^2 + 15b^2su^4k_1^2 + 13st^2u^4k_1^2 - 4s^2u^5k_1^2 - 24bt^2u^3k_1^3 - 30btu^4k_1^3 + 3u^5k_1^4 + 18b^3s^4tk_2 + 82b^3s^3t^2k_2 + 100b^3s^2t^3k_2 - 18bs^4t^3k_2 + 30b^3st^4k_2 - 58bs^3t^4k_2 - 58bs^2t^5k_2 - 18bst^6k_2 + 8b^3s^4uk_2 + 44b^3s^3tuk_2 + 40b^3s^2t^2uk_2 + 4bs^4t^2uk_2 - 4bs^2t^4uk_2 + 2b^3s^3u^2k_2 + 4bs^4tu^2k_2 + 4bs^3t^2u^2k_2 - 34b^2s^2t^3k_1k_2 - 64b^2st^4k_1k_2 - 18b^2t^5k_1k_2 + 10s^2t^5k_1k_2 + 16st^6k_1k_2 + 6t^7k_1k_2 - 8b^2s^3tuk_1k_2 - 64b^2s^2t^2uk_1k_2 - 64b^2st^3uk_1k_2 - 4s^3t^3uk_1k_2 + 4st^5uk_1k_2 - 4b^2s^3u^2k_1k_2 - 14b^2s^2tu^2k_1k_2 - 4s^3t^2u^2k_1k_2 - 4s^2t^3u^2k_1k_2 + 6bt^5k_1^2k_2 + 20bst^3uk_1^2k_2 + 24bt^4uk_1^2k_2 + 4bs^2tu^2k_1^2k_2 + 18bst^2u^2k_1^2k_2 - 6t^3u^2k_1^3k_2 + 3b^2s^5k_2^2 + 12b^2s^4tk_2^2 + 10b^2s^3t^2k_2^2 - 3s^5t^2k_2^2 - 6s^4t^3k_2^2 - 3s^3t^4k_2^2 + 2b^2s^4uk_2^2 - 6bs^3t^2k_1k_2^2 - 8bs^2t^3k_1k_2^2 - 4bs^3tuk_1k_2^2 + st^4k_1^2k_2^2 + 2s^2t^2uk_1^2k_2^2\right)$$

$$F^{(9)}{}_{12} = \frac{1}{2}\left(-32b^4s^2t^3 - 79b^4st^4 - 22b^3s^2t^4 - 38b^4t^5 - 44b^3st^5 + 18b^2s^2t^5 - 16b^3t^6 + 54b^2st^6 + 2bs^2t^6 + 32b^2t^7 + 4bst^7 - 6s^2t^7 - 15st^8 - 10t^9 - 22b^4s^3tu - 149b^4s^2t^2u - 52b^3s^3t^2u - 236b^4st^3u - 200b^3s^2t^3u - 40b^2s^3t^3u - 79b^4t^4u - 210b^3st^4u - 77b^2s^2t^4u - 44b^3t^5u + 34b^2st^5u + 6bs^2t^5u + 10s^3t^5u + 54b^2t^6u + 18bst^6u + 18s^2t^6u + 4bt^7u - 2st^7u - 15t^8u - 15b^4s^3u^2 - 7b^3s^4u^2 - 104b^4s^2tu^2 - 80b^3s^3tu^2 - 5b^2s^4tu^2 - 149b^4st^2u^2 - 240b^3s^2t^2u^2 - 86b^2s^3t^2u^2 - 3bs^4t^2u^2 - 32b^4t^3u^2 - 200b^3st^3u^2 - 204b^2s^2t^3u^2 - 12bs^3t^3u^2 - 5s^4t^3u^2 - 22b^3t^4u^2 - 77b^2st^4u^2 - 18bs^2t^4u^2 - 3s^3t^4u^2 + 18b^2t^5u^2 + 6bst^5u^2 + 16s^2t^5u^2 + 2bt^6u^2 + 18st^6u^2 - 6t^7u^2 - 15b^4s^2u^3 - 20b^3s^3u^3 + b^2s^4u^3 + bs^5u^3 - 22b^4stu^3 - 80b^3s^2tu^3 - 24b^2s^3tu^3 + 2bs^4tu^3 + s^5tu^3 - 52b^3st^2u^3 - 86b^2s^2t^2u^3 - 40b^2st^3u^3 - 12bs^2t^3u^3 - 4s^3t^3u^3 - 3s^2t^4u^3 + 10st^5u^3 - 7b^3s^2u^4 + b^2s^3u^4 - 5b^2s^2tu^4 + 2bs^3tu^4 - 3bs^2t^2u^4 - 5s^2t^3u^4 + bs^3u^5 + s^3tu^5 + 12b^3t^5k_1 + 6b^2t^6k_1 - 8bt^7k_1 - 2t^8k_1 + 56b^3st^3uk_1 + 101b^3t^4uk_1 + 49b^2st^4uk_1 + 67b^2t^5uk_1 + 10bst^5uk_1 - 27bt^6uk_1 + 3st^6uk_1 - 5t^7uk_1 + 22b^3s^2tu^2k_1 + 148b^3st^2u^2k_1 + 27b^2s^2t^2u^2k_1 + 152b^3t^3u^2k_1 + 152b^2st^3u^2k_1 + 16bs^2t^3u^2k_1 + 117b^2t^4u^2k_1 + 55bst^4u^2k_1 - s^2t^4u^2k_1 - 26bt^5u^2k_1 + 7st^5u^2k_1 - 5t^6u^2k_1 + 12b^3s^2u^3k_1 - bs^3u^3k_1 + 82b^3stu^3k_1 + 24b^2s^2tu^3k_1 + 54b^3t^2u^3k_1 + 121b^2st^2u^3k_1 + 26bs^2t^2u^3k_1 + 50b^2t^3u^3k_1 + 72bst^3u^3k_1 - 2s^2t^3u^3k_1 - 6bt^4u^3k_1 + 7st^4u^3k_1 - 2t^5u^3k_1 + 9b^3su^4k_1 + 5b^2s^2u^4k_1 + 31b^2st^2u^4k_1 + 8bs^2t^2u^4k_1 + 25bst^3u^4k_1 - 2s^2t^3u^4k_1 + 3st^4u^4k_1 - bs^2u^5k_1 - s^2tu^5k_1 - 32b^2t^3u^2k_1^2 - 19bt^4u^2k_1^2 + 3t^5u^2k_1^2 - 10b^2stu^3k_1^2 - 63b^2t^2u^3k_1^2 - 8bst^2u^3k_1^2 - 46bt^3u^3k_1^2 - 6st^3u^3k_1^2 + 3t^4u^3k_1^2 - 6b^2su^4k_1^2 - 24b^2tu^4k_1^2 - 8bstu^4k_1^2 - 24bt^2u^4k_1^2 - 9st^2u^4k_1^2 - 3bsu^5k_1^2 - 3stu^5k_1^2 + 6btu^4k_1^3 + 3t^2u^4k_1^3 + 3bu^5k_1^3 + 3tu^5k_1^3 - 54b^3s^3t^2k_2 - 152b^3s^2t^3k_2 - 50b^2s^3t^3k_2 - 101b^3st^4k_2 - 117b^2s^2t^4k_2 + 6bs^3t^4k_2 - 12b^3t^5k_2 - 67b^2st^5k_2 + 26bs^2t^5k_2 + 2s^3t^5k_2 - 6b^2t^6k_2 + 27bst^6k_2 + 5s^2t^6k_2 + 8bt^7k_2 + 5st^7k_2 + 2t^8k_2 - 9b^3s^4uk_2 - 82b^3s^3tuk_2 - 31b^2s^4tuk_2 - 148b^3s^2t^2uk_2 - 121b^2s^3t^2uk_2 - 25bs^4t^2uk_2 - 56b^3st^3uk_2 - 152b^2s^2t^3uk_2 - 72bs^3t^3uk_2 - 3s^4t^3uk_2 - 49b^2st^4uk_2 - 55bs^2t^4uk_2 - 7s^3t^4uk_2 - 10bst^5uk_2 - 7s^2t^5uk_2 - 3st^6uk_2 - 12b^3s^3u^2k_2 - 5b^2s^4u^2k_2 + bs^5u^2k_2 - 22b^3s^2tu^2k_2 - 24b^2s^3tu^2k_2 - 8bs^4tu^2k_2 + s^5tu^2k_2 - 27b^2s^2t^2u^2k_2 - 26bs^3t^2u^2k_2 + 2s^4t^2u^2k_2 - 16bs^2t^3u^2k_2 + 2s^3t^3u^2k_2 + s^2t^4u^2k_2 + b^2s^3u^3k_2 + 41b^2st^4k_1k_2 + 32b^2t^5k_1k_2 + 26bst^5k_1k_2 + 20bt^6k_1k_2 - st^6k_1k_2 + 49b^2s^2t^2uk_1k_2 + 128b^2st^3uk_1k_2 + 30bs^2t^3uk_1k_2 + 41b^2t^4uk_1k_2 + 76bst^4uk_1k_2 + 7s^2t^4uk_1k_2 + 26bt^5uk_1k_2 + 4st^5uk_1k_2 - t^6uk_1k_2 + 3b^2s^3u^2k_1k_2 + 32b^2s^2tu^2k_1k_2 - 2bs^3tu^2k_1k_2 + 49b^2st^2u^2k_1k_2 + 12bs^2t^2u^2k_1k_2 + 30bst^3u^2k_1k_2 + 8s^2t^3u^2k_1k_2 + 7st^4u^2k_1k_2 + 3b^2s^2u^3k_1k_2 - 2bs^2t^2u^3k_1k_2 - 19bt^4uk_1^2k_2 - 7t^5uk_1^2k_2 - 14bst^2uk_1^2k_2 - 24bt^3u^2k_1^2k_2 - 2st^3u^2k_1^2k_2 - 10t^4u^2k_1^2k_2 - 6bstu^3k_1^2k_2 + st^2u^3k_1^2k_2 + 3t^2u^3k_1^3k_2 - 24b^2s^4tk_2^2 - 63b^2s^3t^2k_2^2 - 24bs^4t^2k_2^2 - 32b^2s^2t^3k_2^2 - 46bs^3t^3k_2^2 - 19bs^2t^4k_2^2 + 3s^3t^4k_2^2 + 3s^2t^5k_2^2 - 6b^2s^4uk_2^2 - 3bs^5uk_2^2 - 10b^2s^3tuk_2^2 - 8bs^4tuk_2^2 - 3s^5tuk_2^2 - 8bs^3t^2uk_2^2 - 9s^4t^2uk_2^2 - 6s^3t^3uk_2^2 + 24bs^2t^3k_1k_2^2 + 19bst^4k_1k_2^2 + 10s^2t^4k_1k_2^2 + 7st^5k_1k_2^2 + 6bs^3tuk_1k_2^2 + 14bs^2t^2uk_1k_2^2 - s^3t^2uk_1k_2^2 + 2s^2t^3uk_1k_2^2 - 2t^5k_1^2k_2^2 - 4st^3uk_1^2k_2^2 - 3bs^5k_2^3 - 6bs^4tk_2^3 - 3s^5tk_2^3 - 3s^4t^2k_2^3 + 3s^3t^2k_1k_2^3\right)$$

$$
\begin{aligned}
F^{(9)}{}_{21} = \frac{1}{2}\,(&-32\,b^4\,s^2\,t^3 - 79\,b^4\,s\,t^4 + 22\,b^3\,s^2\,t^4 - 38\,b^4\,t^5 + 44\,b^3\,s\,t^5 + 18\,b^2\,s^2\,t^5 + 16\,b^3\,t^6 + 54\,b^2\,s\,t^6 - 2\,b\,s^2\,t^6 + 32\,b^2\,t^7 - 4\,b\,s\,t^7 - 6\,s^2\,t^7 - 15\,s\,t^8 - \\
&10\,t^9 - 22\,b^4\,s^3\,t\,u - 149\,b^4\,s^2\,t^2\,u + 52\,b^3\,s^3\,t^2\,u - 236\,b^4\,s\,t^3\,u + 200\,b^3\,s^2\,t^3\,u - 40\,b^2\,s^3\,t^3\,u - 79\,b^4\,t^4\,u + 210\,b^3\,s\,t^4\,u - 77\,b^2\,s^2\,t^4\,u + 44\,b^3\,t^5\,u + \\
&34\,b^2\,s\,t^5\,u - 6\,b\,s^2\,t^5\,u + 10\,s^3\,t^5\,u + 54\,b^2\,t^6\,u - 18\,b\,s\,t^6\,u + 18\,s^2\,t^6\,u - 4\,b\,t^7\,u - 2\,s\,t^7\,u - 15\,t^8\,u - 15\,b^4\,s^3\,u^2 + 7\,b^3\,s^4\,u^2 - 104\,b^4\,s^2\,t\,u^2 + \\
&80\,b^3\,s^3\,t\,u^2 - 5\,b^2\,s^4\,t\,u^2 - 149\,b^4\,s\,t^2\,u^2 + 240\,b^3\,s^2\,t^2\,u^2 - 86\,b^2\,s^3\,t^2\,u^2 + 3\,b\,s^4\,t^2\,u^2 - 32\,b^4\,t^3\,u^2 + 200\,b^3\,s\,t^3\,u^2 - 204\,b^2\,s^2\,t^3\,u^2 + 12\,b\,s^3\,t^3\,u^2 - \\
&5\,s^4\,t^3\,u^2 + 22\,b^3\,t^4\,u^2 - 77\,b^2\,s\,t^4\,u^2 + 18\,b\,s^2\,t^4\,u^2 - 3\,s^3\,t^4\,u^2 + 18\,b^2\,t^5\,u^2 - 6\,b\,s\,t^5\,u^2 + 16\,s^2\,t^5\,u^2 - 2\,b\,t^6\,u^2 + 18\,s\,t^6\,u^2 - 6\,t^7\,u^2 - 15\,b^4\,s^2\,u^3 + \\
&20\,b^3\,s^3\,u^3 + b^2\,s^4\,u^3 - b\,s^5\,u^3 - 22\,b^4\,s\,t\,u^3 + 80\,b^3\,s^2\,t\,u^3 - 24\,b^2\,s^3\,t\,u^3 - 2\,b\,s^4\,t\,u^3 + s^5\,t\,u^3 + 52\,b^3\,s\,t^2\,u^3 - 86\,b^2\,s^2\,t^2\,u^3 - 40\,b^2\,s\,t^3\,u^3 + \\
&12\,b\,s^2\,t^3\,u^3 - 4\,s^3\,t^3\,u^3 - 3\,s^2\,t^4\,u^3 + 10\,s\,t^5\,u^3 + 7\,b^3\,s^2\,u^4 + b^2\,s^3\,u^4 - 5\,b^2\,s^2\,t\,u^4 - 2\,b\,s^3\,t\,u^4 + 3\,b\,s^2\,t^2\,u^4 - 5\,s^2\,t^3\,u^4 - b\,s^3\,u^5 + s^3\,t\,u^5 + \\
&12\,b^3\,t^5\,k_1 - 6\,b^2\,t^6\,k_1 - 8\,b\,t^7\,k_1 + 2\,t^8\,k_1 + 56\,b^3\,s\,t^3\,u\,k_1 + 101\,b^3\,t^4\,u\,k_1 - 49\,b^2\,s\,t^4\,u\,k_1 - 67\,b^2\,t^5\,u\,k_1 + 10\,b\,s\,t^5\,u\,k_1 - 27\,b\,t^6\,u\,k_1 - 3\,s\,t^6\,u\,k_1 + \\
&5\,t^7\,u\,k_1 + 22\,b^3\,s^2\,t\,u^2\,k_1 + 148\,b^3\,s\,t^2\,u^2\,k_1 - 27\,b^2\,s^2\,t^2\,u^2\,k_1 + 152\,b^3\,t^3\,u^2\,k_1 - 152\,b^2\,s\,t^3\,u^2\,k_1 + 16\,b\,s^2\,t^3\,u^2\,k_1 - 117\,b^2\,t^4\,u^2\,k_1 + 55\,b\,s\,t^4\,u^2\,k_1 + \\
&s^2\,t^4\,u^2\,k_1 - 26\,b\,t^5\,u^2\,k_1 - 7\,s\,t^5\,u^2\,k_1 + 5\,t^6\,u^2\,k_1 + 12\,b^3\,s^2\,u^3\,k_1 + b^2\,s^3\,u^3\,k_1 + 82\,b^3\,s\,t\,u^3\,k_1 - 24\,b^2\,s^2\,t\,u^3\,k_1 + 54\,b^3\,t^2\,u^3\,k_1 - 121\,b^2\,s\,t^2\,u^3\,k_1 + \\
&26\,b\,s^2\,t^2\,u^3\,k_1 - 50\,b^2\,t^3\,u^3\,k_1 + 72\,b\,s\,t^3\,u^3\,k_1 + 2\,s^2\,t^3\,u^3\,k_1 - 6\,b\,t^4\,u^3\,k_1 - 7\,s\,t^4\,u^3\,k_1 + 2\,t^5\,u^3\,k_1 + 9\,b^3\,s\,u^4\,k_1 - 5\,b^2\,s^2\,u^4\,k_1 - 31\,b^2\,s\,t\,u^4\,k_1 + \\
&8\,b\,s^2\,t\,u^4\,k_1 + 25\,b\,s\,t^2\,u^4\,k_1 + 2\,s^2\,t^2\,u^4\,k_1 - 3\,s\,t^3\,u^4\,k_1 - b\,s^2\,u^5\,k_1 + s^2\,t\,u^5\,k_1 - 32\,b^2\,t^3\,u^2\,k_1^2 + 19\,b\,t^4\,u^2\,k_1^2 + 3\,t^5\,u^2\,k_1^2 - 10\,b^2\,s\,t\,u^3\,k_1^2 - \\
&63\,b^2\,t^2\,u^3\,k_1^2 + 8\,b\,s\,t^2\,u^3\,k_1^2 + 46\,b\,t^3\,u^3\,k_1^2 - 6\,s\,t^3\,u^3\,k_1^2 + 3\,t^4\,u^3\,k_1^2 - 6\,b^2\,s\,u^4\,k_1^2 - 24\,b^2\,t\,u^4\,k_1^2 + 8\,b\,s\,t\,u^4\,k_1^2 + 24\,b\,t^2\,u^4\,k_1^2 - 9\,s\,t^2\,u^4\,k_1^2 + \\
&3\,b\,s\,u^5\,k_1^2 - 3\,s\,t\,u^5\,k_1^2 + 6\,b\,t\,u^4\,k_1^3 - 3\,t^2\,u^4\,k_1^3 + 3\,b\,u^5\,k_1^3 - 3\,t\,u^5\,k_1^3 - 54\,b^3\,s^3\,t^2\,k_2 - 152\,b^3\,s^2\,t^3\,k_2 + 50\,b^2\,s^3\,t^3\,k_2 - 101\,b^3\,s\,t^4\,k_2 + 117\,b^2\,s^2\,t^4\,k_2 + \\
&6\,b\,s^3\,t^4\,k_2 - 12\,b^3\,t^5\,k_2 + 67\,b^2\,s\,t^5\,k_2 + 26\,b\,s^2\,t^5\,k_2 - 2\,s^3\,t^5\,k_2 + 6\,b^2\,t^6\,k_2 + 27\,b\,s\,t^6\,k_2 - 5\,s^2\,t^6\,k_2 + 8\,b\,t^7\,k_2 - 5\,s\,t^7\,k_2 - 2\,t^8\,k_2 - 9\,b^3\,s^4\,u\,k_2 - \\
&82\,b^3\,s^3\,t\,u\,k_2 + 31\,b^2\,s^4\,t\,u\,k_2 - 148\,b^3\,s^2\,t^2\,u\,k_2 + 121\,b^2\,s^3\,t^2\,u\,k_2 - 25\,b\,s^4\,t^2\,u\,k_2 - 56\,b^3\,s\,t^3\,u\,k_2 + 152\,b^2\,s^2\,t^3\,u\,k_2 - 72\,b\,s^3\,t^3\,u\,k_2 + 3\,s^4\,t^3\,u\,k_2 + \\
&49\,b^2\,s\,t^4\,u\,k_2 - 55\,b\,s^2\,t^4\,u\,k_2 + 7\,s^3\,t^4\,u\,k_2 - 10\,b\,s\,t^5\,u\,k_2 + 7\,s^2\,t^5\,u\,k_2 + 3\,s\,t^6\,u\,k_2 - 12\,b^3\,s^3\,u^2\,k_2 + 5\,b^2\,s^4\,u^2\,k_2 + b\,s^5\,u^2\,k_2 - 22\,b^3\,s^2\,t\,u^2\,k_2 + \\
&24\,b^2\,s^3\,t\,u^2\,k_2 - 8\,b\,s^4\,t\,u^2\,k_2 - s^5\,t\,u^2\,k_2 + 27\,b^2\,s^2\,t^2\,u^2\,k_2 - 26\,b\,s^3\,t^2\,u^2\,k_2 - 2\,s^4\,t^2\,u^2\,k_2 - 16\,b\,s^2\,t^3\,u^2\,k_2 - 2\,s^3\,t^3\,u^2\,k_2 - s^2\,t^4\,u^2\,k_2 - \\
&b^2\,s^3\,u^3\,k_2 + 41\,b^2\,s\,t^4\,k_1\,k_2 + 32\,b^2\,t^5\,k_1\,k_2 - 26\,b\,s\,t^5\,k_1\,k_2 - 20\,b\,t^6\,k_1\,k_2 - s\,t^6\,k_1\,k_2 + 49\,b^2\,s^2\,t^2\,u\,k_1\,k_2 + 128\,b^2\,s\,t^3\,u\,k_1\,k_2 - 30\,b\,s^2\,t^3\,u\,k_1\,k_2 + \\
&41\,b^2\,t^4\,u\,k_1\,k_2 - 76\,b\,s\,t^4\,u\,k_1\,k_2 + 7\,s^2\,t^4\,u\,k_1\,k_2 - 26\,b\,t^5\,u\,k_1\,k_2 + 4\,s\,t^5\,u\,k_1\,k_2 - t^6\,u\,k_1\,k_2 + 3\,b^2\,s^3\,u^2\,k_1\,k_2 + 32\,b^2\,s^2\,t\,u^2\,k_1\,k_2 + 2\,b\,s^3\,t\,u^2\,k_1\,k_2 + \\
&49\,b^2\,s\,t^2\,u^2\,k_1\,k_2 - 12\,b\,s^2\,t^2\,u^2\,k_1\,k_2 - 30\,b\,s\,t^3\,u^2\,k_1\,k_2 + 8\,s^2\,t^3\,u^2\,k_1\,k_2 + 7\,s\,t^4\,u^2\,k_1\,k_2 + 3\,b^2\,s^2\,u^3\,k_1\,k_2 + 2\,b\,s^2\,t\,u^3\,k_1\,k_2 - 19\,b\,t^4\,u\,k_1^2\,k_2 + \\
&7\,t^5\,u\,k_1^2\,k_2 - 14\,b\,s\,t^2\,u^2\,k_1^2\,k_2 - 24\,b\,t^3\,u^2\,k_1^2\,k_2 + 2\,s\,t^3\,u^2\,k_1^2\,k_2 + 10\,t^4\,u^2\,k_1^2\,k_2 - 6\,b\,s\,t\,u^3\,k_1^2\,k_2 - s\,t^2\,u^3\,k_1^2\,k_2 + 3\,t^2\,u^3\,k_1^3\,k_2 - 24\,b^2\,s^4\,t\,k_2^2 - \\
&63\,b^2\,s^3\,t^2\,k_2^2 + 24\,b\,s^4\,t^2\,k_2^2 - 32\,b^2\,s^2\,t^3\,k_2^2 + 46\,b\,s^3\,t^3\,k_2^2 + 19\,b\,s^2\,t^4\,k_2^2 + 3\,s^3\,t^4\,k_2^2 + 3\,s^2\,t^5\,k_2^2 - 6\,b^2\,s^4\,u\,k_2^2 + 3\,b\,s^5\,u\,k_2^2 - 10\,b^2\,s^3\,t\,u\,k_2^2 + \\
&8\,b\,s^4\,t\,u\,k_2^2 - 3\,s^5\,t\,u\,k_2^2 + 8\,b\,s^3\,t^2\,u\,k_2^2 - 9\,s^4\,t^2\,u\,k_2^2 - 6\,s^3\,t^3\,u\,k_2^2 + 24\,b\,s^2\,t^3\,k_1\,k_2^2 + 19\,b\,s\,t^4\,k_1\,k_2^2 - 10\,s^2\,t^4\,k_1\,k_2^2 - 7\,s\,t^5\,k_1\,k_2^2 + 6\,b\,s^3\,t\,u\,k_1\,k_2^2 + \\
&14\,b\,s^2\,t^2\,u\,k_1\,k_2^2 + s^3\,t^2\,u\,k_1\,k_2^2 - 2\,s^2\,t^3\,u\,k_1\,k_2^2 - 2\,t^5\,k_1^2\,k_2^2 - 4\,s\,t^3\,u\,k_1^2\,k_2^2 - 3\,b\,s^5\,k_2^3 - 6\,b\,s^4\,t\,k_2^3 + 3\,s^5\,t\,k_2^3 + 3\,s^4\,t^2\,k_2^3 + 3\,s^3\,t^2\,k_1\,k_2^3)
\end{aligned}
$$

$$
\begin{aligned}
F^{(9)}{}_{22} = \frac{1}{2}\,(&30\,b^4\,s\,t^4 + 30\,b^4\,t^5 - 36\,b^2\,s\,t^6 - 36\,b^2\,t^7 + 6\,s\,t^8 + 6\,t^9 + 70\,b^4\,s^2\,t^2\,u + 200\,b^4\,s\,t^3\,u + 115\,b^4\,t^4\,u + 4\,b^2\,s^2\,t^4\,u - 100\,b^2\,s\,t^5\,u - 98\,b^2\,t^6\,u - \\
&18\,s^2\,t^6\,u - 12\,s\,t^7\,u + 7\,t^8\,u + 8\,b^4\,s^3\,u^2 + 94\,b^4\,s^2\,t\,u^2 + 214\,b^4\,s\,t^2\,u^2 + 12\,b^2\,s^3\,t^2\,u^2 + 98\,b^4\,t^3\,u^2 + 44\,b^2\,s^2\,t^3\,u^2 - 64\,b^2\,s\,t^4\,u^2 + 16\,s^3\,t^4\,u^2 - \\
&78\,b^2\,t^5\,u^2 - 6\,s^2\,t^5\,u^2 - 26\,s\,t^6\,u^2 + 22\,b^4\,s^2\,u^3 - 7\,b^2\,s^4\,u^3 + 64\,b^4\,s\,t\,u^3 - 16\,b^2\,s^3\,t\,u^3 + 22\,b^4\,t^2\,u^3 + 4\,b^2\,s^2\,t^2\,u^3 - 5\,s^4\,t^2\,u^3 - 24\,b^2\,s\,t^3\,u^3 + \\
&12\,s^3\,t^3\,u^3 - 20\,b^2\,t^4\,u^3 + 13\,s^2\,t^4\,u^3 - 12\,s\,t^5\,u^3 - 2\,t^6\,u^3 + 5\,b^4\,s\,u^4 - 4\,b^2\,s^3\,u^4 + s^5\,u^4 - 6\,b^2\,s^2\,t\,u^4 - 6\,b^2\,s\,t^2\,u^4 + 6\,s^3\,t^2\,u^4 + 12\,s^2\,t^3\,u^4 + \\
&s\,t^4\,u^4 - b^2\,s^2\,u^5 + s^2\,t^2\,u^5 - 30\,b^3\,t^4\,u\,k_1 + 18\,b\,t^6\,u\,k_1 - 40\,b^3\,s\,t^2\,u^2\,k_1 - 100\,b^3\,t^3\,u^2\,k_1 + 4\,b\,s\,t^4\,u^2\,k_1 + 58\,b\,t^5\,u^2\,k_1 - 2\,b^3\,s^2\,u^3\,k_1 - 44\,b^3\,s\,t\,u^3\,k_1 - \\
&82\,b^3\,t^2\,u^3\,k_1 - 4\,b\,s^2\,t^2\,u^3\,k_1 + 58\,b\,t^4\,u^3\,k_1 - 8\,b^3\,s\,u^4\,k_1 - 18\,b^3\,t\,u^4\,k_1 - 4\,b\,s^2\,t\,u^4\,k_1 - 4\,b\,s\,t^2\,u^4\,k_1 + 18\,b\,t^3\,u^4\,k_1 + 10\,b^2\,t^2\,u^3\,k_1^2 - 3\,t^4\,u^3\,k_1^2 + \\
&2\,b^2\,s\,u^4\,k_1^2 + 12\,b^2\,t\,u^4\,k_1^2 - 6\,t^3\,u^4\,k_1^2 + 3\,b^2\,u^5\,k_1^2 - 3\,t^2\,u^5\,k_1^2 + 108\,b^3\,s^2\,t^3\,k_2 + 156\,b^3\,s\,t^4\,k_2 + 42\,b^3\,t^5\,k_2 - 52\,b\,s^2\,t^5\,k_2 - 68\,b\,s\,t^6\,k_2 - 18\,b\,t^7\,k_2 + \\
&72\,b^3\,s^3\,t\,u\,k_2 + 238\,b^3\,s^2\,t^2\,u\,k_2 + 196\,b^3\,s\,t^3\,u\,k_2 + 48\,b^3\,t^4\,u\,k_2 + 30\,b^3\,t^4\,u\,k_2 + 10\,b\,s^2\,t^4\,u\,k_2 - 44\,b\,s\,t^5\,u\,k_2 - 10\,b\,t^6\,u\,k_2 + 20\,b^3\,s^3\,u^2\,k_2 + \\
&68\,b^3\,s^2\,t\,u^2\,k_2 - 14\,b\,s^4\,t\,u^2\,k_2 + 40\,b^3\,s\,t^2\,u^2\,k_2 + 4\,b\,s^3\,t^2\,u^2\,k_2 + 4\,b\,s^2\,t^3\,u^2\,k_2 - 12\,b\,s\,t^4\,u^2\,k_2 + 2\,b^3\,s^2\,u^3\,k_2 + 4\,b\,s^3\,t\,u^3\,k_2 + 4\,b\,s^2\,t^2\,u^3\,k_2 - \\
&18\,b^2\,t^5\,k_1\,k_2 + 6\,t^7\,k_1\,k_2 - 64\,b^2\,s\,t^3\,u\,k_1\,k_2 - 64\,b^2\,t^4\,u\,k_1\,k_2 + 4\,s\,t^5\,u\,k_1\,k_2 + 16\,t^6\,u\,k_1\,k_2 - 14\,b^2\,s^2\,t\,u^2\,k_1\,k_2 - 64\,b^2\,s\,t^2\,u^2\,k_1\,k_2 - 34\,b^2\,t^3\,u^2\,k_1\,k_2 - \\
&4\,s^2\,t^3\,u^2\,k_1\,k_2 + 10\,t^5\,u^2\,k_1\,k_2 - 4\,b^2\,s^2\,u^3\,k_1\,k_2 - 8\,b^2\,s\,t\,u^3\,k_1\,k_2 - 4\,s^2\,t^2\,u^3\,k_1\,k_2 - 4\,s\,t^3\,u^3\,k_1\,k_2 + 8\,b\,t^3\,u^2\,k_1^2\,k_2 + 4\,b\,s\,t\,u^3\,k_1^2\,k_2 + 6\,b\,t^2\,u^3\,k_1^2\,k_2 + \\
&96\,b^2\,s^3\,t^2\,k_2^2 + 126\,b^2\,s^2\,t^3\,k_2^2 + 30\,b^2\,s\,t^4\,k_2^2 - 12\,s^3\,t^4\,k_2^2 - 10\,s^2\,t^5\,k_2^2 - 2\,s\,t^6\,k_2^2 + 15\,b^2\,s^4\,u\,k_2^2 + 48\,b^2\,s^3\,t\,u\,k_2^2 + 34\,b^2\,s^2\,t^2\,u\,k_2^2 + 13\,s^4\,t^2\,u\,k_2^2 + \\
&4\,s^3\,t^3\,u\,k_2^2 - s^2\,t^4\,u\,k_2^2 + 2\,b^2\,s^3\,u^2\,k_2^2 - 4\,s^5\,u^2\,k_2^2 - 24\,b\,s\,t^4\,k_1\,k_2^2 - 6\,b\,t^5\,k_1\,k_2^2 - 18\,b\,s^2\,t^2\,u\,k_1\,k_2^2 - 20\,b\,s\,t^3\,u\,k_1\,k_2^2 - 4\,b\,s^2\,t\,u^2\,k_1\,k_2^2 + t^4\,u\,k_1^2\,k_2^2 + \\
&2\,s\,t^2\,u^2\,k_1^2\,k_2^2 + 30\,b\,s^4\,t\,k_2^3 + 24\,b\,s^3\,t^2\,k_2^3 - 6\,s^2\,t^3\,k_1\,k_2^3 + 3\,s^5\,k_2^4)
\end{aligned}
$$

D.2 $G \times G/H$ model

The computation of the β-functions for the gauge invariant $G \times G/H$ model (3.16),(3.19) is similar to the $G \times G$ case above, except that one has to correctly handle the gauge invariance.

We shall again use the notation (D.1) and (D.2), with the symmetric "square root" of $\mathrm{h}_{ij} = \rho_{(ij)}$ being n_{ij}, and its inverse being m_{ij}. In the computation below, we shall denote certain combinations of n_{ij} and m_{ij} by

$$
\begin{aligned}
p_{ij} &= m_{1i}m_{1j} + m_{2i}m_{2j}\ , & \nu_{ij} &= n_{1i}m_{1j} - n_{2i}m_{2j}\ , \\
\chi_{ij} &= \tfrac{k}{2}p_{ij} + \tfrac{b}{2}(m_{1i}m_{2j} + m_{2i}m_{1j})\ , & \lambda_{ij} &= m_{1i}m_{1j} - m_{2i}m_{2j}\ .
\end{aligned}
\tag{D.10}
$$

We shall split up the generators T_A of G into $T_\alpha \in \mathrm{Lie}(H)$ and $T_a \in \mathrm{Lie}(G)/\mathrm{Lie}(H)$ (which are

orthogonal with respect to the Killing form).

Assuming from the beginning that the matrix r_{ij} satisfies the gauge invariance condition (3.19),[89] the target space metric of the σ-model (3.16) is diagonalized by the vielbein,[90]

$$\begin{aligned} &E^M = (e^\alpha, e^{\overline{\alpha}}, e^{ai}) = \Big(\sqrt{r}(B^{(1)\alpha} - B^{(2)\alpha})\,,\ B^{(1)\alpha} + B^{(2)\alpha}\,,\ n_{ik}P^{(k)a}\Big)\ , \\ &B^{(k)} \equiv T_\alpha B^{(k)\alpha} = P_H\big[\big(g^{(k)}\big)^{-1}dg^{(k)}\big]\ , \qquad P^{(k)} \equiv T_a P^{(k)a} = P_{G/H}\big[\big(g^{(k)}\big)^{-1}dg^{(k)}\big]\ . \end{aligned} \tag{D.11}$$

In this frame, the metric $ds^2 = \mathrm{G}_{MN}E^M E^N$ and the 3-form $\mathrm{H} = \frac{1}{6}\mathrm{H}_{MNP}E^M \wedge E^N \wedge E^P$ have the following non-zero components

$$\mathrm{G}_{\alpha\beta} = -\tfrac{1}{2}\,\mathrm{Tr}[T_\alpha T_\beta]\ , \qquad \mathrm{G}_{ai,bj} = -\tfrac{1}{2}\,\mathrm{Tr}[T_a T_b]\ , \tag{D.12}$$

$$\mathrm{H}_{\alpha\beta\gamma} = \tfrac{i}{2}kr^{-3/2}f_{\alpha\beta\gamma}\ , \qquad \mathrm{H}_{\alpha,bi,cj} = ir^{-1/2}\chi_{ij}f_{\alpha bc}\ . \tag{D.13}$$

The H gauge invariance is reflected in the vanishing of all $\overline{\alpha}$ components of G_{MN} and H_{MNP}, and, in particular, the fact that G_{MN} is degenerate as a result. One could explicitly fix a gauge, eliminating some target space directions and removing this degeneracy. Instead, we find it more convenient to lift the degeneracy with a small parameter ϵ acting as a regulator,[91]

$$\mathrm{G}_{\overline{\alpha}\overline{\beta}} = -\tfrac{1}{2}\epsilon\,\mathrm{Tr}[T_\alpha T_\beta]\ . \tag{D.14}$$

Computing the torsionful Riemann tensor as in Section D.1, one finds that it has a finite $\epsilon \to 0$ limit. This means that the resulting Riemann tensor for $\epsilon = 0$ is unambiguous (since there are no divergences that could create finite-term ambiguities). Finally, we project out the $\overline{\alpha}$

[89] Alternatively, one could obtain the same results by starting with r_{ij} unconstrained, i.e. without gauge invariance imposed. One could first compute the torsionful Riemann tensor for the target space geometry (3.16) with general r_{ij}, ρ_{ij}. The gauge invariance condition (3.19) would then be imposed and the resulting Riemann tensor projected onto the non-degenerate directions of the metric G_{MN}.

[90] The index M denotes all tangent space directions. In the $G \times G$ case in (D.3) we had $M = (Ai)$, while here $M = (\alpha, \bar{\alpha}, ai)$. Both G and H are assumed to be simple.

[91] The use of the "regulator" ϵ is a short-cut for the following gauge-fixing procedure. Fixing an "axial" gauge $iX^\mu(B^{(1)}_\mu + B^{(2)}_\mu) = y(\xi) \in \mathrm{Lie}\,H$ ($\mu = 1, 2$ is the 2d index), the path integral should be independent of the choice of the constant 2d vector X^μ and the algebra-valued function $y(\xi)$ of the 2d coordinates. Inserting the δ-function of the gauge fixing condition into the path integral and then integrating over X^μ and y with a Gaussian measure, i.e. $e^{-\frac{1}{2}X^\mu X_\mu - \frac{1}{2}\epsilon\int d^2\xi\,\mathrm{Tr}[yy]}$, the result should be independent of ϵ (here we assume Euclidean 2d signature but the same is true also in Minkowski signature after an analytic continuation). Integrating first over y we get $\int d^2X \exp\big[-\frac{1}{2}X^\mu X^\nu\big(\delta_{\mu\nu} - \epsilon\,\mathrm{Tr}[(B^{(1)}_\mu + B^{(2)}_\mu)(B^{(1)}_\nu + B^{(2)}_\nu)]\big)\big]$. Integrating over X_μ restores the 2d Euclidean invariance and the result to leading order in the $\varepsilon \to 0$ limit is equivalent to simply adding the regulator term $\Delta\mathcal{L} = -\frac{1}{2}\epsilon\,\mathrm{Tr}[(B^{(1)}_\mu + B^{(2)}_\mu)^2]$ corresponding to (D.14).

directions to obtain the non-zero components

$$
\begin{aligned}
&\widehat{R}^{\alpha}{}_{\beta\delta\varepsilon} = -\tfrac{1}{4r} f^{\alpha}{}_{\beta\gamma} f^{\gamma}{}_{\delta\varepsilon} + \tfrac{k^2}{4r^3}(f^{\alpha}{}_{\delta\gamma} f^{\gamma}{}_{\beta\varepsilon} - f^{\alpha}{}_{\varepsilon\gamma} f^{\gamma}{}_{\beta\delta}) \ , \\
&\widehat{R}^{\alpha}{}_{\beta,dk,el} = (\tfrac{k}{2r}\lambda_{kl} - \tfrac{1}{2}p_{kl}) f^{\alpha}{}_{\beta\gamma} f^{\gamma}{}_{de} + \tfrac{1}{r} A_{kj} A_{lj} (f^{\alpha}{}_{dc} f^{c}{}_{\beta e} - f^{\alpha}{}_{ec} f^{c}{}_{\beta d}) \ , \\
&\widehat{R}^{\alpha}{}_{bi,\delta,el} = -\tfrac{1}{2r} A_{ji} g_{jl} f^{\alpha}{}_{bc} f^{c}{}_{\delta e} - \tfrac{k}{2r^2} A_{li} f^{\alpha}{}_{\delta\gamma} f^{\gamma}{}_{be} + \tfrac{1}{r} A_{lj} C_{ji} f^{\alpha}{}_{ec} f^{c}{}_{b\delta} \ , \\
&\widehat{R}^{ai}{}_{bj,dk,el} = (C_{ij}\lambda_{kl} - \tfrac{1}{2}\delta_{ij} p_{kl}) f^{a}{}_{b\gamma} f^{\gamma}{}_{de} + \tfrac{1}{r} A_{ki} A_{lj} f^{a}{}_{d\gamma} f^{\gamma}{}_{be} - \tfrac{1}{r} A_{li} A_{kj} f^{a}{}_{e\gamma} f^{\gamma}{}_{bd} \ , \\
&\widehat{R}^{ai}{}_{bk,\delta,\varepsilon} = -\tfrac{1}{4r}\delta_{ik} f^{a}{}_{b\gamma} f^{\gamma}{}_{\delta\varepsilon} + \tfrac{1}{r} C_{ij} C_{jk} (f^{a}{}_{\delta c} f^{c}{}_{b\varepsilon} - f^{a}{}_{\varepsilon c} f^{c}{}_{b\delta}) \ , \\
&\qquad A_{ij} \equiv \tfrac{1}{4}(\nu_{ji} - \nu_{ij}) - \chi_{ij} + \tfrac{r}{2}\lambda_{ij} \ , \qquad C_{ij} \equiv -\tfrac{1}{4}(\nu_{ji} + \nu_{ij}) + \chi_{ij} + \tfrac{r}{2}\lambda_{ij} \ .
\end{aligned}
\tag{D.15}
$$

All that remains is to substitute (D.13),(D.15) into the 2-loop β-functions (1.9) in the GB scheme. The resulting expression is hard to evaluate as it contains hundreds of terms, each proportional to a contraction of the form $(ffff)_{\alpha\beta}$ or $(ffff)_{ab}$ where each f denotes a component $f_{\gamma\delta\varepsilon}$ or $f_{\gamma de}$ of the structure constants, and indices are contracted using the Killing form $\mathrm{Tr}[T_A T_B]$. One can show, however, that there are only 11 independent such contractions after accounting for the antisymmetry of the structure constants. This allows for the efficient evaluation of the resulting β-functions for r and ρ_{ij}. These are first obtained depending on n_{11}, n_{12}, n_{22} but, as discussed in subsection D.1, they may be rewritten in terms of s, t, u with all square roots cancelling. As a result, we obtain the 2-loop generalization of the 1-loop β-functions in (3.27)–(3.32),

$$
\frac{d}{dt}\mathrm{h}_p = \alpha' \beta^{(1)}_{\mathrm{h}_p} + \alpha'^2 \beta^{(2)}_{\mathrm{h}_p} \ , \qquad\qquad \mathrm{h}_p \equiv (r, s, t, b, u) \ ,
\tag{D.16}
$$

where $\beta^{(2)}_{\mathrm{h}_p}$ are given by the following expressions:

$$\beta^{(2)}{}_r =$$

$$\frac{c_H^2 \left(3 k^4 - 4 k^2 r^2 + r^4\right)}{2 r^5}$$

$$+ \frac{1}{r^3 \left(t^2 - s u\right)^4} (c_G - c_H)\, c_H$$

$$\begin{aligned}
&\left(-2 b k^3 t \left(2 (s+u) \left(t^2 - s u\right)^2 + 4 r^2 (s+u) \left(s^2 + 2 t^2 + u^2\right) + 3 r \left(s^3 u + s u^3 + s^2 \left(-t^2 + u^2\right) - t^2 \left(t^2 + u^2\right)\right)\right) - \right. \\
&\quad 2 b k r t \left(-4 r^3 \left(s^3 - s t^2 - t^2 u + u^3\right) + 2 r (s+u) \left(-5 t^4 + 6 s t^2 u - s^2 u^2 + 4 b^2 \left(t^2 + s u\right)\right) - \left(t^2 - s u\right) \left(-3 t^4 + 3 s t^2 u + b^2 \left(t^2 + 3 s u\right)\right) + \right. \\
&\qquad \left. r^2 \left(5 t^4 + s^3 u - t^2 u^2 + s u^3 - s^2 \left(t^2 + 5 u^2\right)\right)\right) + \\
&\quad k^4 \left(\left(t^2 - s u\right)^2 \left(s^2 + 2 t^2 + u^2\right) + r^2 \left(s^4 + 4 s^2 t^2 + 2 t^4 + 4 s t^2 u + 4 t^2 u^2 + u^4\right) + r \left(-s^3 t^2 + s^4 u + 3 s^2 t^2 u - t^2 u \left(3 t^2 + u^2\right) + s \left(-3 t^4 + 3 t^2 u^2 + u^4\right)\right)\right) + \\
&\quad r^2 \left(r^4 \left(s^4 - 4 s^2 t^2 + 2 t^4 + 4 s t^2 u - 4 t^2 u^2 + u^4\right) + r \left(-5 t^4 + 5 s t^2 u + b^2 \left(3 t^2 + s u\right)\right) \left(-s^2 u + t^2 u + s \left(t^2 - u^2\right)\right) + \right. \\
&\qquad 2 \left(\left(t^2 + s u\right) \left(t^3 - s t u\right)^2 + b^4 \left(t^4 + 6 s t^2 u + s^2 u^2\right) - b^2 \left(2 t^6 + 5 s t^4 u - 8 s^2 t^2 u^2 + s^3 u^3\right)\right) - \\
&\qquad 3 r^3 \left(-s^3 t^2 + s^4 u - s^2 t^2 u + t^2 u \left(t^2 - u^2\right) + s \left(t^4 - t^2 u^2 + u^4\right)\right) + \\
&\qquad \left. r^2 \left(-4 s^3 t^2 u + 3 s^4 u^2 - 4 s t^2 u \left(t^2 + u^2\right) + t^4 \left(4 t^2 + u^2\right) + s^2 \left(t^4 + 3 u^4\right) - 2 b^2 \left(3 s^2 t^2 - 2 t^4 + s^3 u + 3 t^2 u^2 + s \left(-6 t^2 u + u^3\right)\right)\right)\right) + \\
&\quad k^2 \left(-2 r^4 \left(s^4 - 2 t^4 + u^4\right) + 2 \left(t^2 - s u\right)^2 \left(t^4 - s t^2 u + b^2 \left(t^2 + s u\right)\right) + 3 r \left(-t^4 + s t^2 u + b^2 \left(3 t^2 + s u\right)\right) \left(s^2 u - t^2 u + s \left(-t^2 + u^2\right)\right) + \right. \\
&\qquad 2 r^3 \left(-s^3 t^2 + s^4 u - 3 s^2 t^2 u + 3 t^4 u - t^2 u^3 + s \left(3 t^4 - 3 t^2 u^2 + u^4\right)\right) + \\
&\qquad \left. \left. 2 r^2 \left(-5 t^6 + 6 s^3 t^2 u - 2 s^4 u^2 - 4 t^4 u^2 + 6 s t^2 u \left(t^2 + u^2\right) - s^2 \left(4 t^4 + t^2 u^2 + 2 u^4\right) + 2 b^2 \left(3 t^4 + s^3 u + 5 t^2 u^2 + s^2 \left(5 t^2 + u^2\right) + s \left(8 t^2 u + u^3\right)\right)\right)\right)\right)
\end{aligned}$$

$$+ \frac{1}{2 r \left(t^2 - s u\right)^4} (c_G - c_H)^2$$

$$\begin{aligned}
&\left(2 \left(t^2 + 4 s u\right) \left(t^3 - s t u\right)^2 - 24 b k^3 t (s+u) \left(s^2 + 2 t^2 + u^2\right) + 6 b^4 \left(t^4 + 6 s t^2 u + s^2 u^2\right) - 8 b^2 \left(t^6 + 4 s t^4 u - 5 s^2 t^2 u^2\right) + \right. \\
&\quad r^4 \left(s^4 - 4 s^2 t^2 + 2 t^4 + 4 s t^2 u - 4 t^2 u^2 + u^4\right) + 3 k^4 \left(s^4 + 4 s^2 t^2 + 2 t^4 + 4 s t^2 u + 4 t^2 u^2 + u^4\right) - 8 r t^2 \left(-2 b^2 + t^2 - s u\right) \left(-s^2 u + t^2 u + s \left(t^2 - u^2\right)\right) - \\
&\quad 8 b k t \left(-2 r^2 \left(s^3 - s t^2 - t^2 u + u^3\right) + 2 (s+u) \left(-2 t^4 + s t^2 u + s^2 u^2 + 3 b^2 \left(t^2 + s u\right)\right) + r \left(-s^3 u + s^2 \left(t^2 - 2 u^2\right) + t^2 \left(4 t^2 + u^2\right) - s \left(2 t^2 u + u^3\right)\right)\right) - \\
&\quad 4 r^2 \left(b^2 \left(3 s^2 t^2 - 2 t^4 + s^3 u + 3 t^2 u^2 + s \left(-6 t^2 u + u^3\right)\right) - t^2 \left(-s^3 u + t^2 \left(t^2 + u^2\right) + s^2 \left(t^2 + 3 u^2\right) - s \left(4 t^2 u + u^3\right)\right)\right) - \\
&\quad 4 k^2 \left(r^2 \left(s^4 - 2 t^4 + u^4\right) + 4 r t^2 \left(s^2 u - t^2 u + s \left(-t^2 + u^2\right)\right) + t^2 \left(2 t^4 - 3 s^3 u + 3 t^2 u^2 + s^2 \left(3 t^2 - 4 u^2\right) + s \left(2 t^2 u - 3 u^3\right)\right) - \right. \\
&\qquad \left. \left. 3 b^2 \left(3 t^4 + s^3 u + 5 t^2 u^2 + s^2 \left(5 t^2 + u^2\right) + s \left(8 t^2 u + u^3\right)\right)\right)\right)
\end{aligned}$$

$$\beta^{(2)}{}_s =$$

$$\frac{1}{2 r^4 \left(t^2 - s u\right)^3} c_H\, c_G$$

$$\begin{aligned}
&\left(b^4 r^2 s \left(5 t^2 + 3 s u\right) - 2 b^3 k r t \left(s \left(t^2 - s u\right) + 4 r \left(s^2 + t^2 + 2 s u\right)\right) + k^4 \left(-u \left(t^2 - s u\right)^2 + r \left(t^2 - s u\right) \left(t^2 + u^2\right) + r^2 \left(s t^2 + s^2 u + 4 t^2 u + 2 u^3\right)\right) + \right. \\
&\quad r^2 \left(5 r^2 u \left(t^2 - s u\right)^2 + s^2 t^2 u \left(-t^2 + s u\right) + r^4 \left(s t^2 + 3 s^2 u - 8 t^2 u + 4 u^3\right) + r^3 \left(-2 s^2 t^2 + t^4 + 7 t^2 u^2 + 3 s u \left(t^2 - 3 u^2\right)\right) - \right. \\
&\qquad \left. r t^2 \left(t^4 + s^3 u + s t^2 u - s^2 \left(t^2 + 2 u^2\right)\right)\right) + \\
&\quad k^2 \left(s \left(t^3 - s t u\right)^2 - 2 r^4 \left(s t^2 + 2 s^2 u + 2 t^2 u - u^3\right) - 2 r^2 \left(s^3 t^2 + 2 s t^4 - 2 t^4 u + 5 s t^2 u^2 - 2 s^2 u^3\right) + r t^2 \left(-3 t^4 + s^3 u + 5 s t^2 u - s^2 \left(t^2 + 2 u^2\right)\right) + \right. \\
&\qquad \left. 2 r^3 \left(s^2 t^2 + 3 t^4 + 4 t^2 u^2 - s u \left(t^2 + 3 u^2\right)\right)\right) - \\
&\quad 2 b k t \left(k^2 \left(-\left(t^2 - s u\right)^2 + 2 r^2 \left(s^2 + 2 t^2 + 2 s u + 3 u^2\right) + r \left(s t^2 - s^2 u + 2 t^2 u - 2 s u^2\right)\right) + \right. \\
&\qquad \left. r \left(s t^2 \left(-t^2 + s u\right) - 2 r^3 \left(s^2 + 4 s u - u^2\right) + r^2 \left(s^2 u + 10 t^2 u + 3 s \left(t^2 - 2 u^2\right)\right) + r \left(t^4 - 6 s t^2 u + s^2 \left(-4 t^2 + u^2\right)\right)\right)\right) + \\
&\quad b^2 \left(-r^2 \left(2 r^2 \left(3 s t^2 + s^2 u - 2 t^2 u + 2 s u^2\right) + s \left(5 t^4 + 2 s t^2 u + s^2 u^2\right) + r \left(-9 t^4 - s^3 u + 3 s t^2 u + s^2 \left(t^2 - 2 u^2\right)\right)\right) + \right. \\
&\qquad \left. \left. k^2 \left(-s \left(t^2 - s u\right)^2 - r \left(-3 t^4 + s^3 u + s t^2 u - s^2 \left(t^2 - 2 u^2\right)\right) + 2 r^2 \left(s^3 + s^2 u + 10 t^2 u + 3 s \left(3 t^2 + u^2\right)\right)\right)\right)\right)
\end{aligned}$$

$$- \frac{1}{2 r^2 \left(t^2 - s u\right)^3} c_G^2$$

$$\begin{aligned}
&\left(2 s^2 t^2 u \left(-t^2 + s u\right) + 4 b^4 s \left(2 t^2 + s u\right) - 12 b^3 k t \left(s^2 + t^2 + 2 s u\right) + k^4 \left(2 s t^2 + s^2 u + 6 t^2 u + 3 u^3\right) + r^2 \left(s^3 t^2 - 2 s t^4 + 4 t^4 u - 11 s t^2 u^2 + 8 s^2 u^3\right) + \right. \\
&\quad 2 r^3 \left(-2 s^2 t^2 + t^4 + 4 t^2 u^2 + 3 s u \left(t^2 - 2 u^2\right)\right) + r^4 \left(2 s t^2 + 3 s^2 u + 5 u \left(-2 t^2 + u^2\right)\right) - 2 r t^2 \left(t^4 + s^3 u - 2 s t^2 u + s^2 \left(-t^2 + u^2\right)\right) - \\
&\quad 2 b k t \left(-2 t^2 \left(3 s^2 + t^2 + 2 s u\right) + 3 k^2 \left(s^2 + 2 t^2 + 2 s u + 3 u^2\right) + r^2 \left(-3 s^2 + 4 t^2 - 4 s u + 7 u^2\right) - 2 r \left(-s t^2 + s^2 u - 6 t^2 u + 6 s u^2\right)\right) - \\
&\quad k^2 \left(3 s^3 t^2 + 2 t^4 u + s^2 \left(4 r^2 u - 2 t^2 u\right) - 4 r^2 u \left(t^2 + u^2\right) - 2 r \left(5 t^4 + 4 t^2 u^2\right) + s \left(8 t^4 + 8 r u^3 + t^2 u (10 r + u)\right)\right) - \\
&\quad b^2 \left(-k^2 \left(3 s^3 + 26 s t^2 + 4 s^2 u + 30 t^2 u + 9 s u^2\right) + 2 s \left(4 t^4 + s t^2 u + s^2 u^2\right) + 2 r \left(-5 t^4 - s^3 u + 4 s t^2 u + s^2 \left(t^2 + u^2\right)\right) + \right. \\
&\qquad \left. \left. r^2 \left(s^3 + 2 s^2 u - 12 t^2 u + s \left(4 t^2 + u^2\right)\right)\right)\right)
\end{aligned}$$

$$\beta^{(2)}{}_t =$$

$$-\frac{1}{2\,r^4\,(t^2-s\,u)^3}\,c_H\,c_G$$

$$\begin{aligned}
&(b^4\,r^2\,t\,(3\,t^2+5\,s\,u)-b^3\,k\,r\,(t^4-s^2\,u^2+4\,r\,(3\,s\,t^2+s^2\,u+3\,t^2\,u+s\,u^2))+\\
&\quad b^2\,t\,(r^2\,(-2\,t^4+4\,s\,t^2\,u+6\,s^2\,u^2+2\,r^2\,(2\,s^2+t^2-s\,u+2\,u^2)+r\,(3\,s\,t^2-7\,s^2\,u+3\,t^2\,u-7\,s\,u^2))+\\
&\qquad k^2\,(-(t^2-s\,u)^2-3\,r\,(s^2\,u-t^2\,u+s\,(-t^2+u^2))+2\,r^2\,(5\,s^2+9\,s\,u+5\,(t^2+u^2))))+\\
&\quad b\,(k^3\,((s+u)\,(t^2-s\,u)^2-2\,r^2\,(s+u)\,(s^2+6\,t^2+u^2)+r\,(s^3\,u+s^2\,(-t^2+u^2)-t^2\,(3\,t^2+u^2)+s\,(2\,t^2\,u+u^3)))+\\
&\qquad k\,r\,(t^6-4\,s\,t^4\,u+3\,s^2\,t^2\,u^2-2\,r^3\,(s+u)\,(s^2+4\,t^2-2\,s\,u+u^2)+r^2\,(-9\,t^4+3\,s^3\,u+t^2\,u^2+s^2\,(t^2-u^2)+3\,s\,(6\,t^2\,u+u^3))-\\
&\qquad\quad r\,(-3\,t^4\,u+s^3\,u^2+s\,(-3\,t^4+10\,t^2\,u^2)+s^2\,(10\,t^2\,u+u^3))))+\\
&\quad t\,(k^4\,(-(t^2-s\,u)^2+r^2\,(2\,s^2+3\,t^2+s\,u+2\,u^2)+r\,(s\,t^2-s^2\,u+t^2\,u-s\,u^2))+\\
&\qquad r^2\,(t^6-s\,t^4\,u+r^4\,(-4\,s^2+7\,t^2+s\,u-4\,u^2)+r^3\,(s+u)\,(s^2-5\,t^2+3\,s\,u+u^2)+r\,t^2\,(s\,t^2-s^2\,u+t^2\,u-s\,u^2)+\\
&\qquad\quad r^2\,(s^2\,t^2-2\,t^4-s^3\,u+2\,s\,t^2\,u+t^2\,u^2-s\,u^3))+\\
&\qquad k^2\,(-r^3\,(s+u)^3+(t^2-2\,s\,u)\,(t^2-s\,u)^2+2\,r^4\,(s^2+3\,t^2-s\,u+u^2)+r\,t^2\,(-s^2\,u+t^2\,u+s\,(t^2-u^2))+\\
&\qquad\quad r^2\,(3\,s^3\,u+s^2\,(-t^2+u^2)-t^2\,(t^2+u^2)+s\,(4\,t^2\,u+3\,u^3)))))
\end{aligned}$$

$$+\frac{1}{2\,r^2\,(t^2-s\,u)^3}\,c_G^2$$

$$\begin{aligned}
&(4\,b^4\,(t^3+2\,s\,t\,u)-6\,b^3\,k\,(s^2\,u+3\,t^2\,u+s\,(3\,t^2+u^2))+\\
&\quad b^2\,t\,(-2\,t^4+6\,s\,t^2\,u+8\,s^2\,u^2+r^2\,(3\,s^2-10\,s\,u+3\,u^2)+k^2\,(15\,s^2+16\,t^2+26\,s\,u+15\,u^2)-7\,r\,(-s\,t^2+s^2\,u-t^2\,u+s\,u^2))-\\
&\quad b\,k\,(3\,k^2\,(s+u)\,(s^2+6\,t^2+u^2)-2\,t^2\,(s\,t^2-7\,s^2\,u+t^2\,u-7\,s\,u^2)+r^2\,(s^3-5\,s^2\,u-5\,s\,u^2+u^3)-2\,r\,(-s^2\,t^2-8\,t^4+s^3\,u+8\,s\,t^2\,u-t^2\,u^2+s\,u^3))+\\
&\quad t\,(2\,t^4\,(t^2-s\,u)+r^4\,(-5\,s^2+8\,t^2+2\,s\,u-5\,u^2)+r^3\,(s+u)\,(2\,s^2-7\,t^2+3\,s\,u+2\,u^2)+k^4\,(3\,s^2+4\,t^2+2\,s\,u+3\,u^2)+\\
&\qquad r\,t^2\,(-s\,t^2+s^2\,u-t^2\,u+s\,u^2)+r^2\,(2\,s^2\,t^2-3\,s^3\,u+2\,s\,t^2\,u+2\,t^2\,u^2-3\,s\,u^3)+\\
&\qquad k^2\,(3\,s\,u\,(s^2+2\,t^2+u^2)-2\,r^2\,(s^2-2\,t^2+2\,s\,u+u^2)-3\,r\,(-s\,t^2+s^2\,u-t^2\,u+s\,u^2))))
\end{aligned}$$

$$\beta^{(2)}{}_b =$$

$$\frac{1}{2\,r^4\,(t^2-s\,u)^3}\,c_H\,c_G$$

$$\begin{aligned}
&(b^3\,r^2\,(-t^4-10\,s\,t^2\,u-5\,s^2\,u^2+2\,r\,(s^2\,u+t^2\,u+s\,(t^2+u^2)))+\\
&\quad b^2\,k\,r\,t\,(t^4+2\,s\,t^2\,u-3\,s^2\,u^2-4\,r^2\,(s^2+t^2+3\,s\,u+u^2)+8\,r\,(2\,s^2\,u+t^2\,u+s\,(t^2+2\,u^2)))+\\
&\quad b\,(k^2\,(2\,s\,u\,(t^2-s\,u)^2+r^3\,(s+u)\,(s^2+10\,t^2+u^2)+4\,r\,t^2\,(-s\,t^2+s^2\,u-t^2\,u+s\,u^2)-r^2\,(9\,s^2\,t^2+3\,t^4+3\,s^3\,u+20\,s\,t^2\,u+s^2\,u^2+9\,t^2\,u^2+3\,s\,u^3))-\\
&\qquad r^2\,((t^3-s\,t\,u)^2+r^3\,(s+u)\,(s^2-2\,t^2+u^2)+6\,r\,t^2\,(-s^2\,u+t^2\,u+s\,(t^2-u^2))-r^2\,(3\,t^4+s^3\,u-5\,t^2\,u^2+s^2\,(-5\,t^2+u^2)+s\,(4\,t^2\,u+u^3))))+\\
&\quad k\,t\,(k^2\,(-(s+u)\,(t^2-s\,u)^2-2\,r^3\,(s^2+2\,t^2+u^2)+2\,r^2\,(s+u)\,(s^2+2\,t^2+u^2)+r\,(-s^3\,u+s^2\,(t^2+u^2)+t^2\,(3\,t^2+u^2)-s\,(4\,t^2\,u+u^3)))+\\
&\qquad r\,(2\,r^3\,(s-u)^2\,(s+u)+r\,(s+u)\,(t^2-s\,u)^2-(t^3-s\,t\,u)^2+2\,r^4\,(s^2-2\,t^2+u^2)-r^2\,(-5\,t^4+3\,s^3\,u-3\,t^2\,u^2+s^2\,(-3\,t^2+u^2)+s\,(4\,t^2\,u+3\,u^3)))))
\end{aligned}$$

$$+\frac{1}{2\,r^2\,(t^2-s\,u)^3}\,c_G^2$$

$$\begin{aligned}
&(-2\,b^2\,k\,t\,(r\,t^2+8\,s\,t^2-r\,s\,u+10\,s^2\,u+8\,t^2\,u+10\,s\,u^2)+\\
&\quad k\,t\,(r^3\,(-4\,s^2+6\,t^2+2\,s\,u-4\,u^2)-2\,r\,(t^2-s\,u)\,(-k^2+s^2+3\,t^2+u^2)-r^2\,(s+u)\,(s^2+2\,t^2-4\,s\,u+u^2)-k^2\,(s+u)\,(3\,s^2+8\,t^2-2\,s\,u+3\,u^2))+\\
&\quad b^3\,(2\,(t^4+8\,s\,t^2\,u+3\,s^2\,u^2)+r\,(-s^2\,u+t^2\,u+s\,(t^2-u^2)))+\\
&\quad b\,(2\,(t^3-s\,t\,u)^2+r^3\,(s+u)\,(2\,s^2-3\,t^2-s\,u+2\,u^2)-r^2\,(-5\,s^2\,t^2+2\,t^4+3\,s^3\,u+2\,s\,t^2\,u-5\,t^2\,u^2+3\,s\,u^3)+\\
&\qquad k^2\,(15\,s^2\,t^2+10\,t^4+3\,s^3\,u+26\,s\,t^2\,u+15\,t^2\,u^2+3\,s\,u^3)+r\,(k^2-5\,t^2)\,(s^2\,u-t^2\,u+s\,(-t^2+u^2))))
\end{aligned}$$

$$
\begin{aligned}
&\beta^{(2)}{}_u = \\
&-\frac{1}{2\,r^4\,\left(t^2-s\,u\right)^3}\,c_H\,c_G \\
&\quad\Big(2\,b\,k^3\,t\,\left(-\left(t^2-s\,u\right)^2+2\,r^2\,\left(3\,s^2+2\,t^2+2\,s\,u+u^2\right)+r\,\left(2\,s\,t^2-2\,s^2\,u+t^2\,u-s\,u^2\right)\right)- \\
&\qquad k^4\,\left(-s\,\left(t^2-s\,u\right)^2-r\,\left(s^2+t^2\right)\,\left(-t^2+s\,u\right)+r^2\,\left(2\,s^3+t^2\,u+s\,\left(4\,t^2+u^2\right)\right)\right)+ \\
&\qquad 2\,b\,k\,r\,t\,\left(\left(b^2-t^2\right)\,u\,\left(t^2-s\,u\right)+2\,r^3\,\left(s^2-4\,s\,u-u^2\right)+r^2\,\left(-6\,s^2\,u+3\,t^2\,u+s\,\left(10\,t^2+u^2\right)\right)+r\,\left(t^4+s^2\,u^2-2\,t^2\,u\,\left(3\,s+2\,u\right)+4\,b^2\,\left(t^2+u\,\left(2\,s+u\right)\right)\right)\right)+ \\
&\qquad k^2\,\left(\left(b^2-t^2\right)\,u\,\left(t^2-s\,u\right)^2+r^4\,\left(-2\,s^3+2\,t^2\,u+4\,s\,\left(t^2+u^2\right)\right)+2\,r^3\,\left(-4\,s^2\,t^2+3\,s^3\,u+s\,t^2\,u-t^2\,\left(3\,t^2+u^2\right)\right)- \\
&\qquad\quad 2\,r^2\,\left(2\,s\,t^4-5\,s^2\,t^2\,u-2\,t^4\,u+2\,s^3\,u^2-t^2\,u^3+b^2\,\left(10\,s\,t^2+3\,s^2\,u+9\,t^2\,u+s\,u^2+u^3\right)\right)+ \\
&\qquad\quad r\,\left(t^2-s\,u\right)\,\left(3\,t^4+t^2\,u\,\left(-2\,s+u\right)-b^2\,\left(3\,t^2+u\,\left(2\,s+u\right)\right)\right)\Big)+ \\
&\qquad r^2\,\left(-r^4\,\left(4\,s^3-8\,s\,t^2+t^2\,u+3\,s\,u^2\right)+r^3\,\left(-7\,s^2\,t^2-t^4+9\,s^3\,u-3\,s\,t^2\,u+2\,t^2\,u^2\right)-\left(b^2-t^2\right)\,u\,\left(s\,u\,\left(t^2-s\,u\right)+b^2\,\left(5\,t^2+3\,s\,u\right)\right)+ \\
&\qquad\quad r^2\,\left(-5\,s\,\left(t^2-s\,u\right)^2+2\,b^2\,\left(-2\,s\,t^2+2\,s^2\,u+3\,t^2\,u+s\,u^2\right)\right)+r\,\left(t^6+t^4\,\left(s-u\right)\,u+s\,t^2\,u^2\,\left(-2\,s+u\right)+b^2\,\left(-9\,t^4-s\,u^2\,\left(2\,s+u\right)+t^2\,u\,\left(3\,s+u\right)\right)\right)\Big)\Big) \\
&-\frac{1}{2\,r^2\,\left(t^2-s\,u\right)^3}\,c_G^2 \\
&\quad\Big(-6\,b\,k^3\,t\,\left(3\,s^2+2\,t^2+2\,s\,u+u^2\right)+r^4\,\left(5\,s^3-10\,s\,t^2+2\,t^2\,u+3\,s\,u^2\right)+2\,r^3\,\left(4\,s^2\,t^2+t^4-6\,s^3\,u+3\,s\,t^2\,u-2\,t^2\,u^2\right)+ \\
&\qquad 2\,\left(b^2-t^2\right)\,u\,\left(s\,u\,\left(t^2-s\,u\right)+2\,b^2\,\left(2\,t^2+s\,u\right)\right)+2\,r\,\left(t^2-s\,u\right)\,\left(-t^4+t^2\,u\,\left(s+u\right)+b^2\,\left(5\,t^2+\left(s-u\right)\,u\right)\right)+k^4\,\left(3\,s^3+2\,t^2\,u+s\,\left(6\,t^2+u^2\right)\right)+ \\
&\qquad k^2\,\left(-2\,r\,\left(4\,s^2+5\,t^2\right)\,\left(-t^2+s\,u\right)+4\,r^2\,s\,\left(s^2+t^2-u^2\right)-t^2\,\left(2\,s\,t^2+s^2\,u+8\,t^2\,u-2\,s\,u^2+3\,u^3\right)+b^2\,\left(30\,s\,t^2+9\,s^2\,u+26\,t^2\,u+4\,s\,u^2+3\,u^3\right)\right)- \\
&\qquad 2\,b\,k\,t\,\left(r^2\,\left(7\,s^2+4\,t^2-4\,s\,u-3\,u^2\right)+6\,b^2\,\left(t^2+2\,s\,u+u^2\right)-2\,t^2\,\left(t^2+2\,s\,u+3\,u^2\right)-2\,r\,\left(6\,s^2\,u-t^2\,u+s\,\left(-6\,t^2+u^2\right)\right)\right)- \\
&\qquad r^2\,\left(-4\,s\,t^4+11\,s^2\,t^2\,u+2\,t^4\,u-8\,s^3\,u^2-t^2\,u^3+b^2\,\left(s^2\,u+4\,t^2\,u+u^3+2\,s\,\left(-6\,t^2+u^2\right)\right)\right)\Big)
\end{aligned}
$$

Appendix E

Derivation of RG flow from existence of Lax connection for time-dependent sigma models

Here we shall provide some further details of the derivation of the RG flow from the existence of a Lax representation for the (τ, σ)-dependent models in Section 5.

E.1 Derivation of the equations (5.31),(5.32)

To derive (5.31),(5.32), we shall ignore terms proportional to the derivatives of the couplings $\partial_\pm h_\alpha$ and match the other terms between the flatness condition of the Lax connection and the equation of motion. The algebraic equations (5.31),(5.32) follow essentially because these 'non-derivative' terms do not change upon the introduction of space-time dependence of the couplings.

Let us consider the cases of models associated to a group and to a symmetric space separately. For the group space case, the original equations of motion following from the flatness of Lax in (5.26) are the flatness and the conservation of the current $\mathcal{A}_\pm$. In the (τ, σ)-dependent model (5.25), these equations may only be modified by $\mathcal{O}(\partial h)$ terms

$$F_{+-}(\mathcal{A}) = \mathcal{O}(\partial h) \ , \qquad \partial_+ \mathcal{A}_- + \partial_- \mathcal{A}_+ = \mathcal{O}(\partial h) \ . \tag{E.1}$$

They should follow from the flatness of the Lax connection ansatz (5.28),

$$F_{+-}(\widehat{L}) = p_+ p_- F_{+-}(\mathcal{A}) + \big[p_-(1-p_+)\partial_+ \mathcal{A}_- - p_+(1-p_-)\partial_- \mathcal{A}_+\big] + \mathcal{O}(\partial p) \ . \tag{E.2}$$

Comparing first the terms with $\partial_\pm \mathcal{A}$ and $\mathcal{A}^2$, i.e. neglecting the $\mathcal{O}(\partial h)$ and $\mathcal{O}(\partial p)$ terms in

(E.1),(E.2), the matching of (E.1) and (E.2) then implies

$$p_-(1-p_+) = -p_+(1-p_-) \ . \tag{E.3}$$

This leads to the algebraic equation (5.31).

For the symmetric space case, the original equations of motion following from the vanishing of the curvature of (5.27) are the flatness condition for $(\mathcal{B}+\mathcal{P})_\pm$ and the equations $D^{\mathcal{B}}_\pm \mathcal{P}_\mp = 0$, where $D^{\mathcal{B}}_\mu$ is the covariant derivative with respect to the connection $\mathcal{B}_\mu \in \mathrm{Lie}(H)$. In the (τ,σ)-dependent model (5.25) these equations are modified, as in (E.1),

$$F_{+-}(\mathcal{B}+\mathcal{P}) = \mathcal{O}(\partial h) \ , \qquad D^{\mathcal{B}}_\pm \mathcal{P}_\mp = \mathcal{O}(\partial h) \ . \tag{E.4}$$

These should follow from the flatness of the ansatz for the Lax pair in (5.29),

$$\begin{aligned} F_{+-}(\widehat{L}) = {} & q_+q_- F_{+-}(\mathcal{B}+\mathcal{P}) + (r_- - q_+q_-) D^{\mathcal{B}}_+ \mathcal{P}_- - (r_+ - q_+q_-) D^{\mathcal{B}}_- \mathcal{P}_+ \\ & + q_-(1-q_+)\partial_+ \mathcal{B}_- - q_+(1-q_-)\partial_- \mathcal{B}_+ + (r_+r_- - q_+q_-)[\mathcal{P}_+, \mathcal{P}_-] \\ & + r_-(q_+ - 1)[\mathcal{B}_+, \mathcal{P}_-] - r_+(q_- - 1)[\mathcal{B}_-, \mathcal{P}_+] + \mathcal{O}(\partial q, \partial r) \ . \end{aligned} \tag{E.5}$$

Comparing the terms in (E.4) and (E.5) that contain $\partial\mathcal{B}$, $\partial\mathcal{P}$, $\mathcal{B}^2$, $\mathcal{P}^2$ and $\mathcal{B}\mathcal{P}$, i.e. neglecting the $\mathcal{O}(\partial h)$ terms in (E.4) and $\mathcal{O}(\partial q, \partial r)$ terms in (E.5), we conclude in particular that the coefficients of the extra $\partial\mathcal{B}$, $[\mathcal{B},\mathcal{P}]$ and $[\mathcal{P},\mathcal{P}]$ terms in $(E.5)$ must vanish. Assuming $q_\pm, r_\pm$ are not all zero (so that the Lax connection (5.29) is not identically zero) the only solution is $q_+ = q_- = 1$ and $r_+r_- = 1$, i.e. the conditions in (5.32).

E.2 RG flow in PCM case

The matching of the $\mathcal{O}(\partial h)$ terms then forces the coupling functions to solve the RG flow equations. The general structure of this argument is explained in eqs. (5.33)–(5.36), but here we shall run through it explicitly for the simplest PCM example.

Together, (5.28) and (5.31) lead to the following ansatz for the Lax connection (cf. (5.21))

$$\widehat{L}_\pm = \tfrac{1}{2}\Big(1 + [z(\tau,\sigma)]^{\pm 1}\Big) J_\pm \ , \tag{E.6}$$

whose curvature is ($F_{+-}(J) = 0$ since $J = g^{-1}dg$)

$$F_{+-}(\widehat{L}) = \tfrac{1-z^2}{4z}(\partial_+ J_- + \partial_- J_+) - \tfrac{\partial_+ z}{2z^2} J_- - \tfrac{\partial_- z}{2} J_+ \ . \tag{E.7}$$

The equation of motion for the PCM with coupling $h(\tau,\sigma)$ is (generalizing (5.18))

$$h(\partial_+ J_- + \partial_- J_+) + \partial_+ h J_- + \partial_- h J_+ = 0 \ . \tag{E.8}$$

For this to be equivalent to the vanishing of the curvature (E.7) the ratios of the different coefficients should match. This leads to the equations of the form (5.33), i.e.

$$\partial_+ z = -\tfrac{z}{2}(1-z)^2\,\tfrac{\partial_+ h}{h}\ , \qquad \partial_- z = -\tfrac{1}{2z}(1-z)^2\,\tfrac{\partial_- h}{h}\ . \tag{E.9}$$

The consistency condition $\partial_+(\partial_- z) - \partial_-(\partial_+ z) = 0$ gives

$$\tfrac{(1-z^2)^2}{2hz}\ \partial_+\partial_- h = 0\ . \tag{E.10}$$

It is remarkable that the z dependence has totally factored out (a term proportional to $\partial_+ h \partial_- h$ is absent here due to a special cancellation). Excluding the trivial cases $z(\tau,\sigma) = \pm 1$, which would not encode the correct equation of motion, we find that the Lax connection (E.6) only matches the correct equation of motion if $\partial_+\partial_- h = 0$, i.e. if $h = f^+(\xi^+) + f^-(\xi^-)$. Any such solution is related to the PCM's 1-loop RG flow, $h = c\,\tau$, by a 2d conformal transformation.

E.3 RG flow for theories with multiple couplings

In Section 5.2 we explained the derivation of the RG flow focussing on the case with only one coupling (see eqs. (5.33)–(5.36)). The same conclusion also holds for the group space η-model with two couplings h and $\varkappa$ (see Table 1), and more generally is expected to be true for multi-coupling theories.

There are multiple independent structures in the equation of motion (for the η-model these involve different powers of the R-matrix) and correspondingly in the flatness of the Lax connection. Matching the coefficients of these structures in the η-model case yields *two* pairs of equations (cf. (5.33)),

$$\partial_\tau z = U_\tau(z;h,\varkappa)\ , \qquad \partial_\sigma z = U_\sigma(z;h,\varkappa)\ , \tag{E.11}$$

$$\partial_\tau z = V_\tau(z;h,\varkappa)\ , \qquad \partial_\sigma z = V_\sigma(z;h,\varkappa)\ . \tag{E.12}$$

In general, for an N-coupling theory we would expect to find N pairs of equations.

As a system of equations for z, (E.11),(E.12) is clearly overdetermined. In two combinations of these equations the z dependence cancels to give relations between $h(\tau,\sigma)$ and $\varkappa(\tau,\sigma)$

$$\partial_\tau(\varkappa^{-1}h) = \partial_\sigma(\varkappa^{-1}h) = 0\ . \tag{E.13}$$

Eq. (E.13) implies that $\nu \equiv \frac{h}{4\varkappa}$ is a constant; this coincides precisely with the 1-loop RG invariant of the η-model (see Table 1). For an N-coupling theory, we may expect to obtain $(N-1)$ RG invariants ν_r in this way.

Then the system (E.11),(E.12) reduces to just two equations — effectively returning to the single-coupling case of equations (5.33). Again, the consistency condition for the two remaining

equations takes the remarkable form (5.34), where the β-function $\beta(h) \equiv \beta^h(h, \nu)$ is understood as a function of the coupling h and the RG invariant. As in eqs. (5.35),(5.36) it then follows (modulo a conformal transformation) that $h(\tau, \sigma)$ depends only on τ and follows the 1-loop RG flow. The same should generalize to the N-coupling case with $(N-1)$ independent RG invariants ν_r that can be chosen as constants.

Appendix F

On non-local charges in time-dependent sigma models

In Section 5.3 we discussed the construction of non-local charges in the time-dependent PCM. Here we shall comment on the other time-dependent models in Table 1, and, in particular, on the symmetric space λ-model.

The construction of the conserved monodromy matrix satisfying (5.40) works similarly, although for the models built on symmetric spaces (symmetric space σ-model and symmetric space λ-model), it is only the eigenvalues of the monodromy matrix that are conserved (because the boundary value of L_τ does not vanish). A sufficient boundary condition in all cases is that $g(\tau, \sigma) \to g_0$ at spatial infinity and that $(g - g_0)$ decays sufficiently fast so that the monodromy converges at spatial infinity. As in the PCM example it is hard to evaluate the conserved charges explicitly and thus to verify that they are infinite in number (i.e. depend non-trivially on the spectral parameter w).

Except for the symmetric space λ-model, all the other models have global symmetries. As for the PCM in (5.45), the associated charges can be obtained by expanding the monodromy around $w = \infty$.[92] In the remainder of this appendix, we shall consider the trivial reduction of the time-dependent symmetric space λ-model, which has no manifest global symmetries. In general, the eigenvalues of the monodromy matrix would be conserved on an infinite spatial line; however, in the "trivial" reduction where the fields depend only on time (see Section 5.3), the monodromy matrix does not converge at spatial infinity since L_σ does not vanish there. Below we will try to shed some light on this issue.

Let us first recall what happens for geodesics in the usual time-independent case, where the Lax connection is $L_\pm = A^H_\pm + z^{\pm 1} \frac{1}{\sqrt{\lambda}} A^{G/H}_\pm$. In the trivial reduction we have $A_\pm = A_\pm(g(\tau))$ so the periodicity condition (5.39) is satisfied and the eigenvalues of the monodromy conserved even on a finite interval of length a. The path-ordered exponential trivializes to give $\mathcal{M} = \exp(a\, L_\sigma)$

[92] This applies also for the symmetric space σ-model after applying the gauge transformation $L_\pm \to g L_\pm g^{-1} - \partial_\pm g g^{-1}$ to the Lax connection $L = J^H_\pm + z^{\pm 1} J^{G/H}_\pm$, obtaining the alternate Lax connection $L_\pm = \frac{1}{2}(1 - z^{\pm 1})(-2 g J^{G/H}_\pm g^{-1})$ of the 'group space' form (5.26), instead of (5.27) (modulo sign reversal of z).

and hence, equivalently, the eigenvalues of $L_\sigma = \frac{1}{2}(A^H_+ - A^H_- + \frac{z}{\sqrt{\lambda}}A^{G/H}_+ - \frac{z^{-1}}{\sqrt{\lambda}}A^{G/H}_-)$ are conserved. For example, let us consider the simplest $\frac{SU(2)}{U(1)}$ λ-model (with the $U(1)$ subgroup generated by σ_2), parametrized after gauge fixing as (cf. eqs. (2.36),(2.37))

$$g = e^{i\alpha\sigma_3}e^{i\beta\sigma_2} \ , \qquad \cos\alpha = \sqrt{p^2+q^2} \ , \qquad \tan\beta = \frac{p}{q} \ , \tag{F.1}$$

$$\mathcal{L} = \frac{k}{1-\lambda^2}\frac{(1-\lambda)^2(\partial p)^2 + (1+\lambda)^2(\partial q)^2}{1-p^2-q^2} \ . \tag{F.2}$$

The trivial 1d reduction, or geodesic motion, is described by the Lagrangian ($\dot{} \equiv \partial_\tau$)

$$L = \frac{k}{1-\lambda^2}\frac{(1-\lambda)^2\dot{p}^2 + (1+\lambda)^2\dot{q}^2}{1-p^2-q^2} \ . \tag{F.3}$$

There is only one independent eigenvalue l of L_σ since it is a traceless 2×2 matrix. Its expansion around, e.g., $z=1$ gives (at least) two independent conserved charges for the geodesic motion

$$\dot{Q}_1 = \dot{Q}_2 = 0 \ , \qquad l = \tfrac{1}{2(\lambda^2-1)}\sqrt{Q_1} + (z-1)^2\tfrac{\lambda}{\lambda^2-1}\frac{Q_2}{\sqrt{Q_1}} + \dots \ , \tag{F.4}$$

$$Q_1 = \frac{-(\lambda-1)^4q^2\dot{p}^2 + 2(\lambda^2-1)^2pq\dot{p}\dot{q} + (1+\lambda)^2[4\lambda(q^2-1) - (\lambda-1)^2p^2]\dot{q}^2}{(1-p^2-q^2)^2} \ , \tag{F.5}$$

$$Q_2 = \frac{(1-\lambda)^2\dot{p}^2 + (1+\lambda)^2\dot{q}^2}{1-p^2-q^2} \ , \tag{F.6}$$

with Q_2 proportional to the Hamiltonian.

Returning to the time-dependent case, the monodromy is not defined on an infinite line, and on any finite interval the periodicity condition is not satisfied due to the explicit σ dependence of the Lax connection for generic values of w. However, at certain special values of w satisfying $\exp(\frac{c}{k}w) = \pm\infty$, the σ dependence disappears to give flat connections

$$[\partial_+ + A^H_+ + \tfrac{1}{\lambda(\tau)}A^{G/H}_+ \, , \ \partial_- + A^H_- + A^{G/H}_-] = 0 \ , \tag{F.7}$$

$$[\partial_+ + A^H_+ + A^{G/H}_+ \, , \ \partial_- + A^H_- + \tfrac{1}{\lambda(\tau)}A^{G/H}_-] = 0 \ , \qquad \lambda(\tau) = \exp(-\tfrac{c}{k}\tau) \ , \tag{F.8}$$

generalizing the same expressions from the time-independent case ($\lambda(\tau)\to\lambda$). At these values the periodicity condition *is* satisfied on any finite interval. The monodromy trivializes as in the time-independent case to give $\mathcal{M} = \exp(a\,L_\sigma)$, so the eigenvalues of L_σ are again conserved.

In fact, the two flat connections (F.7),(F.8) are related by a gauge transformation, so their conserved charges are the same. Hence, the maximum number of independent charges obtained from the flat connections (F.7),(F.8) is $r = \mathrm{rank}(G)$ for a symmetric space G/H. This number is generally less than the number of fields, $\dim G/H = \dim G - \dim H$ (e.g. for $SO(n+1)/SO(n)$ we get $r = n-1 < \dim SO(n+1)/SO(n) = n$), so this is not sufficient for integrability.

For example, in the $\frac{SU(2)}{U(1)} \sim \frac{SO(3)}{SO(2)}$ case (F.2), where there are 2 fields, we only obtain $r=1$

conserved charge,

$$Q = \frac{[\lambda(\tau)-1]^4(p^2-1)\dot{p}^2 + 2[\lambda(\tau)^2-1]^2 pq\dot{p}\dot{q} + [\lambda(\tau)+1]^4(q^2-1)\dot{q}^2}{[\lambda(\tau)^2-1]^2(1-p^2-q^2)^2} \ , \tag{F.9}$$

for the trivial 1d reduction, which is a time-dependent generalization of (F.3),

$$L = \frac{k}{1-\lambda^2}\frac{(1-\lambda(\tau))^2\dot{p}^2 + (1+\lambda(\tau))^2\dot{q}^2}{1-p^2-q^2} \ , \qquad \lambda(\tau) = \exp\left(-\tfrac{c}{k}\tau\right) \ . \tag{F.10}$$

In the time-independent limit $\lambda(\tau) \to \lambda$ (obtained, e.g., by shifting $\tau \to -\frac{k}{c}\log\lambda + b\tau$ and taking $b \to 0$), this charge becomes a particular combination of the charges (F.5),(F.6) in the time-independent theory,

$$Q \to Q_1 - (\lambda-1)^2 Q_2 \ . \tag{F.11}$$

The presence of the charge (F.9) in the time-dependent 1d theory is a non-trivial property However, at least upon restricting consideration to special values of w, we have not found enough charges for the integrability of the geodesics. Since the construction of charges is subtle, depending on the value of w, boundary conditions and the choice of the spatial domain (the periodicity condition (5.39) must be satisfied and, if the interval is infinite, the monodromy must converge at infinity), it is possible that the monodromy constructed on some special domain would yield further conserved charges, but this remains to be clarified.

Appendix G

Time-dependent 1d harmonic oscillator and conserved charge

In Section 5.3 we came across a particular time-dependent, linear, 1d model (5.50). Starting with a general time-dependent linear 1d action[93]

$$S = \int d\tau \left[h(\tau)\, \dot{\theta}^2 - k(\tau)\, \theta^2 \right] , \qquad \dot{\theta} \equiv \partial_\tau \theta \ , \tag{G.1}$$

one may redefine τ as $\tau \to t(\tau)$, $\dot{t}(\tau) = h^{-1}(\tau)$ to put all time dependence in the harmonic potential term

$$S = \int dt \left[\theta'^2 - m^2(t)\, \theta^2 \right] , \qquad m^2(t) = k(\tau(t)) h(\tau(t)) \ , \quad \theta' \equiv \partial_t \theta \ . \tag{G.2}$$

The corresponding equation of motion is

$$\theta'' + m^2(t)\, \theta = 0 \ . \tag{G.3}$$

It is easy to see that for a given function $\theta_0(t)$, the quantity

$$Q = \theta_0\, \theta' - \theta_0'\, \theta \ , \tag{G.4}$$

is conserved on-shell if and only if θ_0 is a particular solution of the equation of motion (G.3). Furthermore, such a conserved charge provides a first integral for (G.3),

$$Q = \theta_0\, \theta' - \theta_0'\, \theta = C_1 = \text{const} \qquad \to \quad (\tfrac{\theta}{\theta_0})' = \tfrac{C_1}{\theta_0^2} \ . \tag{G.5}$$

[93] We do not include the term $f(\tau)\theta\dot{\theta}$ as it can be put in the form $k(\tau)\theta^2$ by adding a total derivative.

Integrating this first-order equation yields the general solution of (G.3) ($C_2 = \text{const}$),

$$\theta(t) = C_1\,\theta_0(t)\int \tfrac{dt}{\theta_0^2(t)} + \; C_2\,\theta_0(t)\ , \tag{G.6}$$

with $\theta = \theta_0$ being, of course, a special case. Thus (G.3) is solvable if a particular solution θ_0 can be constructed explicitly.

Changing back to the original parametrization (G.1) ($t \to \tau$), the conserved charge (G.5) takes the form

$$Q = h(\tau)\big[\theta_0\,\dot\theta - \dot\theta_0\,\theta\big]\ , \tag{G.7}$$

where $\theta_0 = \theta_0(\tau)$ is a particular solution of (G.1), while (G.6) becomes

$$\theta(\tau) = C_1\,\theta_0(\tau)\int \tfrac{d\tau}{h(\tau)\,\theta_0^2(\tau)} + C_2\,\theta_0(\tau)\ . \tag{G.8}$$

The linearized theory (5.50) corresponds to (G.1) with $h(\tau) = k(\tau) = \tau$. The conserved charge in (5.51),(5.52) is indeed of the form (G.7). From the monodromy matrix one finds that $Q = \tau[\gamma(\tau)\,\theta + \alpha(\tau)\,\dot\theta]$ where

$$\gamma(\tau) = \int_{-\infty}^{+\infty} d\sigma\, e^{-im\sigma}\,\tfrac{m(s_+s_- - w - \sigma)}{2\tau s_+ s_-}, \quad \alpha(\tau) = \int_{-\infty}^{+\infty} d\sigma\; e^{-im\sigma}\,\tfrac{i}{2s_+s_-}, \quad s_\pm \equiv \sqrt{w+\sigma\pm\tau}\ . \tag{G.9}$$

Let us show that $\gamma = -\dot\alpha$, as required to match (G.7). Indeed, the term proportional to $\dot\theta$ in the leading $\mathcal{O}(\varepsilon)$ expansion of the flatness equation $\partial_\tau L_\sigma - \partial_\sigma L_\tau + [L_\tau, L_\sigma] = 0$ tells us that the integrands in (G.9) satisfy $\partial_\tau[\frac{e^{-im\sigma}i}{2s_+s_-}] + \frac{e^{-im\sigma}m(s_+s_- - w - \sigma)}{2\tau s_+s_-} = \partial_\sigma\,[\frac{e^{-im\sigma}\,i(s_+s_- - w - \sigma)}{2\tau s_+s_-}]$. Integrating over σ and noting that $\frac{e^{-im\sigma}\,i(s_+s_- - w - \sigma)}{2\tau s_+s_-}$ vanishes at $\sigma = \pm\infty$,[94] we conclude that $\gamma + \dot\alpha = 0$.

Since the charge in (5.51) is conserved on-shell (from the monodromy matrix construction), it follows that $\alpha(\tau)$ is a particular solution, and thus the general solution (G.8) is in this case given by

$$\theta = C_1\,\alpha(\tau)\int \tfrac{d\tau}{\tau\,\alpha^2(\tau)} + C_2\,\alpha(\tau)\ . \tag{G.10}$$

[94] This follows from the periodicity condition (5.39) since it is the coefficient of $\dot\theta$ in a component of the leading $\mathcal{O}(\varepsilon)$ term in L_τ.

Appendix H

Lax connection in time-dependent sine-Gordon model

As discussed in Section 5.4, the sine-Gordon model,

$$\mathcal{L} = \frac{1}{g^2}\left[\frac{1}{2}\partial_+ x \partial_- x + m^2 \cos x\right] , \tag{H.1}$$

displays the same pattern as the σ-models considered above: upon promoting the couplings (g, m) to functions of 2d time τ, the Lax connection naturally generalizes to the resulting time-dependent model only if the time dependence is given (up to rescalings of time) by the 1-loop RG flow of the original model,

$$m^2(\tau) = e^{\beta(g)\,\tau} m_0^2 , \qquad g(\tau) = g , \qquad \beta(g) = -2 + g^2 . \tag{H.2}$$

Below we shall justify this claim in more detail.

As was noted in section 5.4, the time-dependent theory obtained from (H.2)

$$\widehat{\mathcal{L}} = \frac{1}{g^2}\left[\frac{1}{2}\partial_+ x \partial_- x + e^{\beta(g)\,\tau}\, m_0^2 \cos x\right] , \tag{H.3}$$

is clearly integrable since the explicit τ-dependence in (H.3) can be removed by a 2d conformal transformation getting back to (H.1). Indeed, starting with the Lax pair [269] for the original sine-Gordon model (H.1) (σ_i are Pauli matrices)

$$L_\pm = \pm\frac{i}{4}\partial_\pm x\ \sigma_3 + \frac{i}{2} z^{\pm 1}\, m \cos\tfrac{x}{2}\ \sigma_1 \pm \frac{i}{2} z^{\pm 1}\, m \sin\tfrac{x}{2}\ \sigma_2 , \tag{H.4}$$

and applying a conformal transformation $\xi^\pm \to f^\pm(\xi^\pm)$, one obtains a Lax connection for the

time-dependent theory (H.3),[95]

$$\widehat{L}_\pm = \pm\tfrac{i}{4}\partial_\pm x\ \sigma_3 + \tfrac{i}{2} z^{\pm 1}\, e^{\beta(g)\,\xi^\pm}\, m_0 \cos\tfrac{x}{2}\ \sigma_1 \pm \tfrac{i}{2} z^{\pm 1}\, e^{\beta(g)\,\xi^\pm}\, m_0 \sin\tfrac{x}{2}\ \sigma_2\ . \tag{H.5}$$

Note that this Lax connection follows the same ansatz (5.11) as in the σ-model case: it is obtained from the original Lax connection (H.4) by the replacements $z \to \widehat{z} = e^{\frac{1}{2}\beta(g)\sigma} z$ and $m_0 \to m(\tau) = e^{\frac{1}{2}\beta(g)\tau} m_0$.

Conversely, suppose we replace the couplings (g, m) in (H.1) with general functions of (τ, σ) and then demand that the resulting theory (cf. (5.25))

$$\widehat{\mathcal{L}} = \tfrac{1}{g^2(\tau,\sigma)}\big[\tfrac{1}{2}\partial_+ x \partial_- x + m^2(\tau,\sigma)\, \cos x\big] \tag{H.6}$$

admits a Lax representation. Motivated by (H.4), we shall assume the following ansatz for the Lax connection (cf. (5.28),(5.29))

$$\widehat{L}_\pm = f_\pm(\tau,\sigma)\ \tfrac{i}{4}\partial_\pm x\, \sigma_3 + \ v_\pm(\tau,\sigma)\ \tfrac{i}{2}\cos\tfrac{x}{2}\,\sigma_1 + w_\pm(\tau,\sigma)\ \tfrac{i}{2}\sin\tfrac{x}{2}\,\sigma_2\ . \tag{H.7}$$

Then matching coefficients of various terms in the zero-curvature condition for (H.7) and in the equation of motion corresponding to (H.6) leads to the following constraints on the coefficient functions in (H.7) and the coupling functions,

$$f_\pm = \pm 1\ , \qquad w_\pm = \pm v_\pm\ , \qquad \partial_\mp v_\pm = 0\ , \tag{H.8}$$

$$g(\tau,\sigma) = g = \text{const}\ , \qquad v_+ v_- = m^2(\tau,\sigma)\ . \tag{H.9}$$

It follows from (H.8),(H.9) that $v_\pm = \pm w_\pm = v_\pm(\xi^\pm)$, $m^2(\tau,\sigma) = v_+(\xi^+)\, v_-(\xi^-)$. Finally, applying a conformal transformation to (H.7), we may set, e.g., $v_\pm(\xi^\pm) \to z^{\pm 1}\, e^{\beta(g)\,\xi^\pm}\, m_0$, $m^2(\tau,\sigma) \to e^{\beta(g)\,\tau}\, m_0^2$, thus bringing the Lagrangian (H.6) and the Lax connection (H.7) to the form (H.3) and (H.5) where the time dependence is given by the 1-loop RG flow (H.2) of the standard sine-Gordon model.

[95] Note that, as for the σ-models discussed in Chapter 5 above, the dependence on the spectral parameter z in (H.5) is again correlated with a constant shift of σ.