Generation of New Exciting Regressors for Consistent On-line Estimation of Unknown Constant Parameters

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***Abstract*— The problem of estimating constant parameters from a standard vector linear regression equation in the absence of sufficient excitation in the regressor is addressed. The first step to solve the problem consists in transforming this equation into a set of scalar ones using the well-known dynamic regressor extension and mixing technique. Then a novel procedure to generate new scalar exciting regressors is proposed. The superior performance of a classical gradient estimator using this new regressor, instead of the original one, is illustrated with comprehensive simulations.**

***Index Terms*— parameter estimation, system identification, persistent excitation**

I. Introduction

It is well known that consistent estimation of constant parameters from a linear regression equation (LRE) with a gradient (or least squares) estimator is possible only if the regressor satisfies certain excitation conditions. A classical result shows that exponential convergence is possible if and only if the regressor verifies the *persistence of excitation* (PE) condition [18]—which is a uniform observability property for the associated linear time-varying (LTV) system. It has recently been shown in [3], [16] that asymptotic (but not exponential) convergence is guaranteed under the strictly weaker condition of generalized PE, the definition of which may be found in [3, Proposition 6].

Unfortunately, the PE (or the generalized PE) property is rarely satisfied in applications, hence the interest to propose new adaptation algorithms that ensure parameter convergence without PE. This research line has been intensively pursued in the last few years and some recent adaptive schemes, in which the PE assumption is obviated via the incorporation of some *off-line* data manipulation, have been reported in [4], [5], [15], [17]—see also [14] for a recent survey.

In this paper we are interested in *on-line* estimation using recursive algorithms. It is well known that, in contrast with off-line estimation schemes, on-line estimation provides, via the accumulation of past measurements and noise averaging, stronger robustness properties. Moreover, if the adaptation gain of the estimator remains bounded away from zero—a property usually referred as *alertness*[1]—it has the ability of tracking slowly time-varying parameters. We concentrate our attention to the case of a *single* uncertain parameter. Our main motivation to study the scalar case stems from the recent development of the *dynamic regressor extension and mixing* (DREM) estimator [2], which is a procedure that generates, from a q-dimensional LRE, q scalar LREs for each of the unknown parameters. It has been observed in several applications that the absence of excitation stymies the successful use of DREM. For instance, in [1] it is shown that consistent estimation of the parameters of a linear time-invariant (LTI) system with DREM is possible if and only if the original regressor is PE. Actually, since the key scalar function that defines the convergence properties of the gradient estimator in DREM is the determinant of the extended regressor, this converges in many cases *to zero*, hence we only have excitation on a finite interval.

Our main contribution is to propose a procedure to generate, from a scalar LRE, a new scalar LRE in which the *new regressor* satisfies the excitation property of non-square integrability, even in the case in which the original regressor is not sufficiently exciting—for instance an exponentially decaying signal. It is shown in [2] that non-square integrability of the (scalar) regressor is necessary and sufficient to ensure asymptotic convergence, which becomes exponential imposing the PE condition. Instrumental for the construction of the new LREs, that include some free signals, is to borrow the key idea of the *parameter estimation based observer* (PEBO) proposed in [11], later generalized in [12], which is a constructive procedure to design state observers for state-affine nonlinear systems. Then, applying the energy pumping-and-damping injection principle of [20], which was proposed as a passivity-based orbital stabilization technique for port-Hamiltonian systems, we select these signals to guarantee the desired excitation properties of the new regressor.

The remainder of the paper is organized as follows. In Section II we briefly recall the DREM procedure and review the problem of parameter estimation for a scalar LRE. Section III presents the procedure to generate the new LRE with some free signals, which are selected in Section IV to comply with the excitation injection requirement. Simulation results, which illustrate the *superior performance* of the classical gradient estimator using the new regressor, instead of the original one, are presented in Section V. The paper is wrapped-up with concluding remarks and a discussion on future research in Section VI.

Notation. I_n is the $n \times n$ identity matrix. For a vector $x \in \mathbb{R}^n$ we denote the Euclidean norm as $|x| := \sqrt{x^\top x}$. $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the absolute integrable and square integrable function spaces, respectively, and $\mathcal{L}_\infty$ represents the vector space of essentially bounded functions.

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[1]It is well known [18, Section 2.3.2] that, due to the so-called covariance wind-up problem, the alertness property is lost in standard least-squares estimators. Therefore, we concentrate on gradient-descent schemes.

II. Gradient Estimation of a Single Parameter

In this paper we deal with the problem of on-line estimation of the unknown *constant* parameters $\theta \in \mathbb{R}^q$ appearing in an LRE of the form

$$w = \psi^\top \theta, \tag{1}$$

where $w(t) \in \mathbb{R}$ and $\psi(t) \in \mathbb{R}^q$ are *measurable* signals. This problem appears in several applications including system identification [9] and adaptive control [7], [10], [18] in which, as discussed in Section I, a key requirement to achieve their objectives is that the regressor ψ is PE, a condition that is rarely satisfied in practice. Our task is then to generate new LREs that satisfy the PE requirement.

A. Generation of scalar LRE

The first step in our design is to apply the DREM procedure [2] to obtain q *scalar* LREs, one for each of the unknown parameters. Towards this end, we introduce a *linear, single-input q-output, bounded-input bounded-output (BIBO)–stable* operator $\mathcal{H}$ and define the vector $W \in \mathbb{R}^m$ and the matrix $\Psi \in \mathbb{R}^{q\times q}$ as

$$W := \mathcal{H}[w],\ \Psi := \mathcal{H}[\psi^\top].$$

Clearly, because of linearity and BIBO stability, these signals satisfy

$$W = \Psi\theta.$$

At this point the key step of regressor "mixing" of the DREM procedure is used to obtain a set of q *scalar* equations multiplying from the left the vector equation above by the *adjunct matrix*, denoted $\text{adj}\{\cdot\}$, of Ψ to get

$$y_i = \Delta\theta_i,\ i \in \bar{q} := \{1, 2, \dots, q\}, \tag{2}$$

in which we have defined

$$\begin{aligned}\Delta &:= \det\{\Psi\},\\ y &:= \text{adj}\{\Psi\}W.\end{aligned}$$

This fundamental modification has numerous advantages and DREM has been instrumental to solve many, previously open, problems—see [12]–[14] for a recent account of some of these results.

B. Properties of the gradient estimator

Motivated by the developments above, in the remaining part of the paper we consider scalar LREs of the form (2). The following property of the gradient estimator is easy to establish [2].

Proposition 1: Consider the scalar LREs (2) with[2] $y(t) \in \mathbb{R}$ and $\Delta(t) \in \mathbb{R}$ bounded, measurable, signals and $\theta \in \mathbb{R}$ an *unknown* parameter estimated on-line via the gradient descent adaptation algorithm

$$\dot{\hat{\theta}} = -\gamma\Delta(\Delta\hat{\theta} - y), \tag{3}$$

with $\gamma > 0$ the adaptation gain.

P1 The following equivalence holds true:

$$\lim_{t\to\infty} |\hat{\theta}(t) - \theta| = 0 \quad \Longleftrightarrow \quad \Delta \notin \mathcal{L}_2.$$

P2 The convergence of the estimate $\hat{\theta}$ to θ is exponential if and only if Δ is PE, that is, if there exist $T > 0$ and $\delta > 0$ such that

$$\int_t^{t+T} \Delta^2(\tau)d\tau \geq \delta,\ \forall t \geq 0.$$

[2]To simplify the notation we omit the subindex $(\cdot)_i$.

III. Generation of the New LRE

In this section, in order to generate a new LRE, we apply some filters to both sides of the original LRE (2) with some free terms to comply with the excitation requirement. Then, in Section IV we study how to design these terms to fulfill this task.

In the sequel we apply the construction used in generalized parameter estimation based observer (GPEBO) [12] to create a new LRE from the scalar LRE (2).

Proposition 2: Consider the scalar LRE (2). There exists a measurable signal $\mathcal{Y}_2(t) \in \mathbb{R}$ such that the new LRE

$$\mathcal{Y}_2 = \Phi_{21}\theta \tag{4}$$

holds, in which the *new regressor* $\Phi_{21}(t) \in \mathbb{R}$ is obtained from the solution of the ordinary differential equation

$$\begin{bmatrix}\dot{\Phi}_{11}\\ \dot{\Phi}_{21}\end{bmatrix} = \begin{bmatrix}0 & u_1\\ u_2\Delta & u_3\end{bmatrix}\begin{bmatrix}\Phi_{11}\\ \Phi_{21}\end{bmatrix}, \tag{5}$$

with initial conditions

$$\begin{bmatrix}\Phi_{11}(0)\\ \Phi_{21}(0)\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}, \tag{6}$$

and *arbitrary* bounded signals $u_i(t) \in \mathbb{R}$, $i = 1, 2, 3$.

Proof: Define the scalar dynamics

$$\dot{z} = u_2 y + u_3 z,\ z(0) = 0, \tag{7}$$

and note that, since θ is constant, we can write

$$\dot{\theta} = u_1(z - z). \tag{8}$$

Combining (7) and (8), and using (2), we can write the "virtual" LTV system

$$\dot{x} = A(t)x + b(t) \tag{9}$$

with

$$\begin{aligned}x &:= \begin{bmatrix}\theta\\ z\end{bmatrix}\\ A(t) &:= \begin{bmatrix}0 & u_1(t)\\ u_2(t)\Delta(t) & u_3(t)\end{bmatrix},\\ b(t) &:= \begin{bmatrix}-u_1(t)x_2(t)\\ 0\end{bmatrix},\end{aligned}$$

and initial conditions

$$x(0) = \begin{bmatrix}\theta\\ 0\end{bmatrix}. \tag{10}$$

Following the GPEBO approach define the dynamics

$$\dot{\xi} = A(t)\xi + b(t),\ \xi(0) = \mathbf{0}_{2\times 1} \tag{11a}$$

$$\dot{\Phi} = A(t)\Phi,\ \Phi(0) = I_2. \tag{11b}$$

Clearly, the state transition matrix $\Omega(t, s)$ of $A(t)$ is given by $\Omega(t, s) = \Phi(t)\Phi^{-1}(s)$.

The error signal

$$e := \xi - x \tag{12}$$

satisfies $\dot{e} = A(t)e$. Consequently, from (12) and the properties of the system matrix Φ defined in (11b), we get

$$\begin{aligned}x &= \xi - \Phi e(0)\\ &= \xi + \Phi\begin{bmatrix}\theta\\ 0\end{bmatrix}\\ &= \xi + \begin{bmatrix}\Phi_{11}\\ \Phi_{21}\end{bmatrix}\theta,\end{aligned}$$

where, to get the second identity, we have taken into account (10) and the initial conditions in (11a), and introduced the notation

$$\Phi = \begin{bmatrix}\Phi_{11} & \Phi_{12}\\ \Phi_{21} & \Phi_{22}\end{bmatrix}.$$

Note now that

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \Delta & 0 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} \Delta & 0 \\ 0 & 1 \end{bmatrix} \left(\xi + \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} \theta \right).$$

The proof is then completed defining

$$\mathcal{Y} = \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix} := \begin{bmatrix} y \\ z \end{bmatrix} - \begin{bmatrix} \Delta & 0 \\ 0 & 1 \end{bmatrix} \xi, \tag{13}$$

noting that

$$\mathcal{Y}_2 = z - \xi_2$$

is a measurable signal, and computing the dynamics of the first column of Φ. ∎

The procedure to generate the new LRE described in Proposition 2 is summarized in the diagram in Fig. 1.

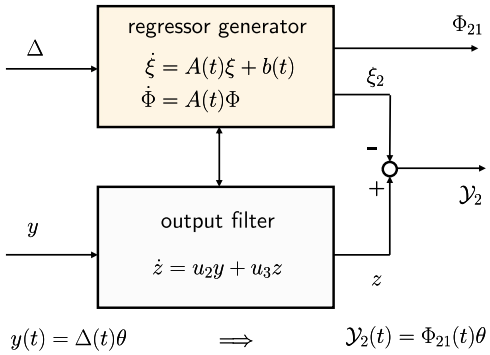


Fig. 1: Block diagram of the proposed LRE generator

IV. Improving the Excitation of the Regressor

A. Main Result

In this section we follow the basic idea of the energy pumping-and-damping injection construction used in [20] for orbital stabilization to select the signals u_i in (5) to provide excitation to the new regressor Φ_{21}.

To articulate the main result we need the following assumption.

Assumption 1: The *bounded* scalar signal Δ in (2) is interval exciting (IE), [8], [19], that is there exist a time instant $t_c > 0$ and a constant $\delta > 0$ such that

$$\int_0^{t_c} \Delta^2(\tau) d\tau \geq \delta.$$

Proposition 3: Assume Δ verifies Assumption 1. Define the signals

$$\begin{aligned} u_1 &= -\alpha\Delta \\ u_2 &= \alpha \\ u_3 &= -\tilde{V}(\Phi_{11}, \Phi_{21}), \end{aligned} \tag{14}$$

with $\alpha(t) \in \mathbb{R}$ a bounded signal such that $\alpha(t)\Delta(t) \not\equiv 0$ for some t,

$$\alpha\Delta \in \mathcal{L}_1, \tag{15}$$

and

$$\tilde{V}(\Phi_{11}, \Phi_{21}) := \frac{1}{2}(\Phi_{11}^2 + \Phi_{21}^2) - \beta,$$

where Φ_{11}, Φ_{21} are the solutions of (5) and $0 < \beta < \frac{1}{2}$. The resulting dynamics verifies the following properties.

F1 Either $\Phi_{21} \notin \mathcal{L}_2$ or

$$\Phi_{11}^2(t) + \Phi_{21}^2(t) \geq 2\beta + \varepsilon, \quad \forall t \geq 0, \tag{16}$$

for some (sufficiently small) $\varepsilon > 0$.

F2 The full state of the system—Φ, ξ and z—is bounded.

Proof: Replacing the signals u_i in (14) to (5) we get

$$\begin{aligned} \dot{\Phi} &= A(t)\Phi \\ &= \begin{bmatrix} 0 & -\alpha\Delta \\ \alpha\Delta & -\tilde{V} \end{bmatrix} \Phi. \end{aligned}$$

The dynamics of the remaining states of the regressor generator, that is, ξ and z, is given by

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & -\alpha\Delta & \alpha\Delta \\ \alpha\Delta & -\tilde{V} & 0 \\ 0 & 0 & -\tilde{V} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \alpha y \end{bmatrix}.$$

First, note that the IE Assumption 1 rules out the extreme case $y(t) = \Delta(t) \equiv 0$. Indeed, in this case all signals of the regressor generator remain at their initial conditions, which is an equilibrium. To avoid this situation we also require the technical assumption that $\alpha(t)\Delta(t) \not\equiv 0$ for some $t \geq 0$.

From the first column of the dynamic equation of Φ we immediately get

$$\dot{\tilde{V}} = -\Phi_{21}^2 \tilde{V} \leq 0, \tag{17}$$

from which we conclude boundedness of (Φ_{11}, Φ_{21}) as well as the invariance of the set

$$\Omega := \{q \in \mathbb{R}^2 | \tilde{V}(q) = 0\}.$$

Now, invoking the initial condition constraint (6) yields

$$\begin{aligned} & \frac{1}{2}(\Phi_{11}^2(0) + \Phi_{21}^2(0)) = \frac{1}{2} \\ \iff \quad & \tilde{V}(\Phi_{11}(0), \Phi_{21}(0)) + \beta = \frac{1}{2} \\ \iff \quad & \tilde{V}(\Phi_{11}(0), \Phi_{21}(0)) = \frac{1}{2} - \beta \\ \implies \quad & \tilde{V}(\Phi_{11}(0), \Phi_{21}(0)) > 0, \end{aligned}$$

where we have used the fact that $\beta < \frac{1}{2}$ to get the last implication. The latter inequality implies that the trajectory starts *outside* the disk described by the set Ω. This, together with the invariance of the set implies that the *whole trajectory* $(\Phi_{11}(t), \Phi_{21}(t))$ is outside this disk, that is,

$$\tilde{V}(t) := \tilde{V}(\Phi_{11}(t), \Phi_{21}(t)) \geq 0, \ \forall t \geq 0. \tag{18}$$

Let us, now, consider the two possible, mutually exclusive, cases:

(i) $\lim_{t\to\infty} \tilde{V}(t) = 0$

(ii) $\lim_{t\to\infty} \tilde{V}(t) \neq 0$.

Consider, first, the second case and note that, due to the invariance of the set Ω, this case also rules out the possibility that the convergence to zero happens in finite time. In this case the inequality (16) holds true.

In the first case, integrating (17) we see that the following equivalence holds true:

$$\lim_{t\to\infty} \tilde{V}(t) = 0 \quad \iff \quad \Phi_{21} \notin \mathcal{L}_2,$$

completing the proof of the claim **F1**.

Now we show boundedness of the remaining states. For the second column of the matrix Φ, note that it satisfies the dynamics

$$\begin{bmatrix} \dot{\Phi}_{12} \\ \dot{\Phi}_{22} \end{bmatrix} = \begin{bmatrix} 0 & -\alpha\Delta \\ \alpha\Delta & -\tilde{V} \end{bmatrix} \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix}, \ \begin{bmatrix} \Phi_{12}(0) \\ \Phi_{22}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Consider the function

$$W(\Phi_{12}, \Phi_{22}) := \frac{1}{2}(\Phi_{12}^2 + \Phi_{22}^2),$$

the derivative of which along the trajectories of the system (5) satisfies

$$\dot{W} = -\Phi_{22}^2 \tilde{V} \leq 0,$$

where the bound—from which we conclude the boundedness of (Φ_{12}, Φ_{22})—follows from (18).

Finally, from (12) we have

$$\begin{aligned}\xi &= x + e \\ &= \begin{bmatrix}\theta \\ z\end{bmatrix} + \Phi \begin{bmatrix}\theta \\ 0\end{bmatrix}.\end{aligned}$$

Hence, if z is bounded we conclude that ξ is also bounded completing the proof.

To prove boundedness of z we, again, consider the two scenarios **(i)** and **(ii)** described above. For the second case we consider the function

$$V_z(z) := \frac{1}{2}z^2.$$

Its time derivative along the trajectories of (5) satisfies

$$\begin{aligned}\dot{V}_z &= -\tilde{V}z^2 + \alpha y z \\ &\leq -(\tilde{V} - \frac{\epsilon}{2})z^2 + \frac{1}{2\epsilon}(\alpha y)^2 \\ &\leq -(\varepsilon - \epsilon)V_z + \frac{1}{2\epsilon}(\alpha y)^2 \\ &= -\frac{\varepsilon}{2}V_z + \frac{1}{4\varepsilon}(\alpha y)^2,\end{aligned}$$

where we have used the fact that, for all $\epsilon > 0$, we have that

$$\alpha y z \leq \frac{1}{2\epsilon}(\alpha y)^2 + \frac{\epsilon}{2}z^2,$$

to get the first inequality, the fact that $\tilde{V} \geq \frac{\varepsilon}{2}$, which follows from (16), to get the second bound and selected $\epsilon = \frac{\varepsilon}{2}$ in the final identity. The proof of boundedness of z follows, then, from the inequality above and the fact that αy is bounded.

Now, consider the scenario **(i)**. It is clear that the solution of the equation

$$\dot{z} = -\tilde{V}z + \alpha\Delta\theta, \tag{19}$$

is given by

$$z(t) = \left(\int_0^t e^{-\int_\sigma^t \tilde{V}(s)ds}\alpha(\sigma)\Delta(\sigma)d\sigma\right)\theta. \tag{20}$$

From (20) we immediately obtain

$$\begin{aligned}|z(t)| &\leq \left|\int_0^t e^{-\int_\sigma^t \tilde{V}(s)ds}\alpha(\sigma)\Delta(\sigma)d\sigma\right||\theta| \\ &\leq \int_0^t \left|e^{-\int_\sigma^t \tilde{V}(s)ds}\right||\alpha(\sigma)\Delta(\sigma)|d\sigma|\theta| \\ &\leq \int_0^t |\alpha(\sigma)\Delta(\sigma)|d\sigma|\theta|,\end{aligned}$$

where, to obtain the last bound, we have used the inequality (18), which implies that

$$\left|\exp\left(-\int_\sigma^t \tilde{V}(s)ds\right)\right| \leq 1. \tag{21}$$

Boundedness of z is concluded, from the inequality above, if $\alpha\Delta \in \mathcal{L}_1$. ∎

Clearly, invoking the equivalence in the claim **P1** of Proposition 1, we have that the condition $\Phi_{21} \notin \mathcal{L}_2$ of the claim **F1** of Proposition 2 ensures *global convergence* of a gradient estimator with the new LRE. On the other hand, the inequality (16) in the claim **F1** guarantees the following excitation properties for the new regressor Φ_{21}. If we can ensure that $\Phi_{11} \not\to \sqrt{2\beta}$ then $\Phi_{21} \not\to 0$ and, consequently, Φ_{21} is PE.[3] In this sense, the new regressor Φ_{21} is "more exciting" than the original IE regressor Δ. Although we have not been able to prove this property, it has systematically been true in all our simulations, some of which are given in Section V. Besides, in the following corollary we identify a scenario which guarantees that the new regressor $\Phi_{21} \notin \mathcal{L}_2$.

Corollary 1: Consider the filter design in Proposition 3. If $\lim_{t\to\infty}\alpha(t)\Delta(t) = 0$, then $\Phi_{21} \notin \mathcal{L}_2$.[4]

Proof: We prove this fact by contradiction assuming that $\Phi_{21} \in \mathcal{L}_2$. According to the analysis in Proposition 3, $\tilde{V}$ has a limit when $t \to +\infty$, which can be zero or a positive constant, invoking non-negativeness of $\tilde{V}$ from (18). Clearly, for the case $\Phi_{21} \in \mathcal{L}_2$, we have

$$\lim_{t\to\infty}\tilde{V}(t) = \varepsilon$$

for some $\varepsilon > 0$. From $\Phi_{21} \in \mathcal{L}_2$ and $\dot{\Phi}_{21} \in \mathcal{L}_\infty$, we have $\Phi_{21} \to 0$, and consequently

$$\lim_{t\to\infty}\Phi_{11}(t) = \sqrt{2(\beta+\varepsilon)}.$$

Rewrite now (Φ_{11}, Φ_{21}) in polar coordinates, that is

$$\Phi_{11} = r\cos\rho, \quad \Phi_{21} = r\sin\rho,$$

with

$$\begin{aligned}\rho &:= \arctan\left(\frac{\Phi_{21}}{\Phi_{11}}\right), \\ r &:= \sqrt{\Phi_{11}^2 + \Phi_{21}^2}.\end{aligned} \tag{22}$$

The transformation is well-posed and bijective almost globally except of the origin, which the above analysis proves that is not possible. In terms of the convergence of Φ_{21} and Φ_{22}, we have

$$\lim_{t\to\infty}\rho(t) = \rho_\star, \quad \lim_{t\to\infty} r(t) = \sqrt{2(\beta+\varepsilon)}, \tag{23}$$

for some constant $\rho_\star \in [0, 2\pi)$. On the other hand, the dynamics of ρ is given by

$$\begin{aligned}\dot{\rho} &= \frac{1}{1+(\frac{\Phi_{21}}{\Phi_{11}})^2}\cdot\frac{\dot{\Phi}_{21}\Phi_{11} - \dot{\Phi}_{11}\Phi_{22}}{\Phi_{11}^2} \\ &= \alpha\Delta - \frac{1}{2} + \frac{\beta}{r^2}.\end{aligned}$$

Invoking the convergence condition $\rho \to \rho_\star$ as $t \to \infty$, the boundedness of $\dot{\rho}$ and the C^1-smoothness of the solution $\rho(t)$, we have

$$\lim_{t\to\infty}\dot{\rho}(t) = \lim_{t\to\infty}\left[\alpha(t)\Delta(t) - \frac{1}{2} + \frac{\beta}{r^2(t)}\right] = 0.$$

From the convergence condition $\alpha\Delta \to 0$ and $r \to \sqrt{2(\beta+\varepsilon)}$, we have $\varepsilon = 0$, which contradicts the assumption $\varepsilon > 0$. Therefore, $\Phi_{21} \notin \mathcal{L}_2$, completing the proof. ∎

Two interesting observations are, first, that the condition $\alpha\Delta \in \mathcal{L}_1$ cannot be replaced by $\alpha\Delta \in \mathcal{L}_2$. Indeed, a counterexample to this claim is given by the selection

$$\tilde{V}(t) = e^{-t}, \ \alpha(t)\Delta(t) = \frac{1}{t+1},$$

for which $\alpha\Delta \in \mathcal{L}_2$, but $\alpha\Delta \notin \mathcal{L}_1$. In this case the system (19) has unbounded solutions. Second, the condition $\alpha\Delta \in \mathcal{L}_1$ is sufficient, but not necessary with a counterexample to this claim given by the selection

$$\tilde{V}(t) = e^{-t}, \ \alpha(t)\Delta(t) = \frac{\sin t}{t+1},$$

that does not satisfy the $\mathcal{L}_1$ condition but with associated solutions (19) bounded. Note also that $\frac{\sin t}{t+1} \in \mathcal{L}_2$!

[3] Recall that if a scalar signal is PE then it is not square integrable, but the converse is not true [2].

[4] It equivalently means that the second case in (16) does not occur.

B. Robustness

We now consider robustness of the proposed scheme.[5] For simplicity we gather all perturbations in a term $\delta(t)$, *i.e.*, the original LRE becomes

$$y_{\mathtt{N}}(t) = \Delta(t)\theta + \delta(t),$$

in which $(\cdot)_{\mathtt{N}}$ is used to represent the perturbed signal of the signal $(\cdot)$. Assuming that the selection of α is independent of the regression output y, the filter dynamics (7) becomes

$$\dot{z}_{\mathtt{N}} = u_2 y_{\mathtt{N}} + u_3 z_{\mathtt{N}},\ z_{\mathtt{N}}(0) = 0. \tag{24}$$

From linearity we have

$$z_{\mathtt{N}} = z + \delta_z,$$

with z the solution of $\dot{z} = u_2 y + u_3 z$ from $z(0) = 0$ and

$$\delta_z(t) = \int_0^t \exp\left(\int_s^t -\tilde{V}(\tau)d\tau\right)\alpha(s)\delta(s)ds. \tag{25}$$

Due to the above parameterization of the perturbation, δ does not affect $A(t)$, equivalently, the solution of $\Phi(t)$. On the other hand, the ξ-dynamics is cascaded to the filter (7), *i.e.*, for the perturbed case

$$\dot{\xi}_{\mathtt{N}} = A(t)\xi_{\mathtt{N}} + \begin{bmatrix}\alpha\Delta z_{\mathtt{N}}\\ 0\end{bmatrix} = A(t)\xi_{\mathtt{N}} + \begin{bmatrix}\alpha\Delta(z+\delta_z)\\ 0\end{bmatrix},$$

with $\xi_{\mathtt{N}}(0) = \mathbf{0}_{2\times 1}$. From superposition we have $\xi_{\mathtt{N}} = \xi + \delta_\xi$ with

$$\delta_\xi(t) := \int_0^t \Phi(t)\Phi^{-1}(s)\alpha(s)\Delta(s)\begin{bmatrix}\delta_z(s)\\ 0\end{bmatrix}ds.$$

In terms of $\mathcal{Y}_2 = z - \xi_2$, we have the perturbed new LRE as

$$\mathcal{Y}_2 = \Phi_{21}\theta + \delta_{\mathtt{N}}, \tag{26}$$

with $\delta_{\mathtt{N}} := \delta_z - \delta_{\xi,2}$. Some observations are in order.

- The perturbation δ from the original LRE (25) affects the new LRE (26) in a different way.
- Consider the perturbation δ as a zero-average high-frequency measurement noise. Noting that $\tilde{V}(t) \geq 0$ for all $t \geq 0$, roughly speaking, the filter (7) plays a role similarly to a low-pass filter (which may be analyzed via standard averaging analysis), making the new LRE robust to the high-frequency component in δ.
- For biased but bounded perturbations, *e.g.*, environmental disturbances and unmodelled dynamics, we have

$$\begin{aligned}|\delta_z| &\leq \int_0^t \left|\exp\left(\int_s^t -\tilde{V}(\tau)d\tau\right)\right||\alpha(s)||\delta(s)|ds\\ &\leq \int_0^t |\alpha(s)|\|\delta\|_\infty ds,\end{aligned}$$

in which we have used (21). Clearly, selecting $\alpha \in \mathcal{L}_1$ can guarantee $\delta_z \in \mathcal{L}_\infty$, and a vanishing signal α may also yield $\delta_\xi \in \mathcal{L}_\infty$, thus providing a BIBO stability property.

V. Simulations

In this section we provide comprehensive simulation results to verify our main claims. In all simulations the parameter for pumping-and-damping injection is selected as $\beta = 0.4$ and—following the construction of the filter proposed above—the initial conditions are fixed at

$$z(0) = 0,\ \Phi(0) = I_2,\ \xi(0) = 0.$$

[5]Robustness of the filter in Proposition 2 is studied in the BIBO sense from the perturbed term to the filtered output, which is a property of the filter regardless of collected data.

We first consider a constant unknown parameter $\theta = -5$, and test the proposed regressor generator under different excitation conditions for the original regressor Δ. Namely, they are chosen as follows:

$$\Delta_1(t) = \frac{1}{2}e^{-t}$$

$$\Delta_2(t) = \begin{cases}\frac{1}{2} & t \in [0,5]\text{s}\\ 0 & t > 5\text{s}.\end{cases}$$

Note that these signals are clearly not PE and they belong to $\mathcal{L}_2$, hence the gradient estimator (3) does not ensure parameter convergence. On the other hand, they satisfy the extremely weak condition that they are IE. For these regressor we use $\alpha(t) \equiv 1$, which verifies $\alpha\Delta \in \mathcal{L}_1$, as well as the additional condition in Corollary 1.

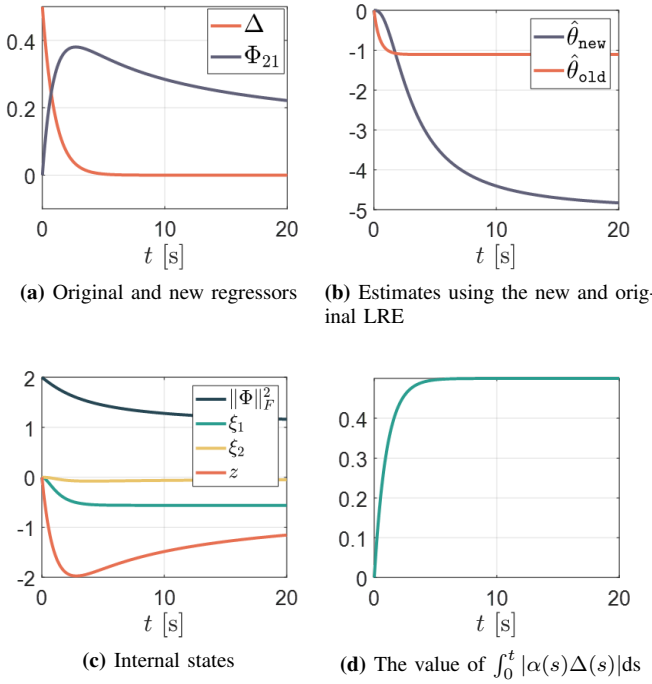
To estimate the parameter we use the standard gradient descent adaptation with the old and the new regressors, that is,

$$\dot{\hat{\theta}}_{\mathtt{old}} = \gamma\Delta(y - \Delta\hat{\theta}_{\mathtt{old}}),$$

and

$$\dot{\hat{\theta}}_{\mathtt{new}} = \gamma\Phi_{21}(\mathcal{Y}_2 - \Phi_{21}\hat{\theta}_{\mathtt{new}}),$$

selecting, in both cases, $\gamma = 2$. The initial conditions are selected as $\hat{\theta}_{\mathtt{new}}(0) = 0$ and $\hat{\theta}_{\mathtt{old}}(0) = 0$. As we can see from Figs. 2-3, the proposed regressor generator transforms the interval exciting regressors into PE regressors ensuring that the estimate $\hat{\theta}_{\mathtt{new}}$ exponentially converges to the true value. On the other hand, the estimates generated with the original regressors $\hat{\theta}_{\mathtt{old}}$ exhibit a steady state error. In these figures, we also plot the evolution of the Frobenius norm of $\Phi(t)$, that is $\|\Phi\|_F = \sqrt{\text{trace}(\Phi^\top\Phi)}$. It is also observed that all the states are bounded.



(a) Original and new regressors

(b) Estimates using the new and original LRE

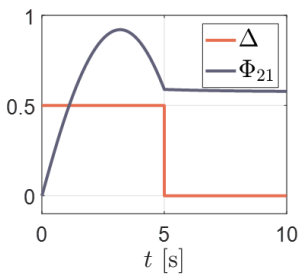
(c) Internal states

(d) The value of $\int_0^t |\alpha(s)\Delta(s)|$ds

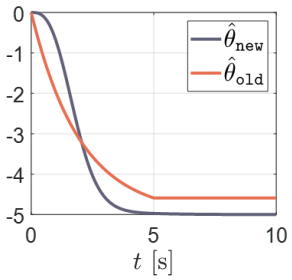
Fig. 2: Simulation results for the signal $\Delta_1(t) = \frac{1}{2}\exp(-t)$

Now, let us consider another vanishing signal

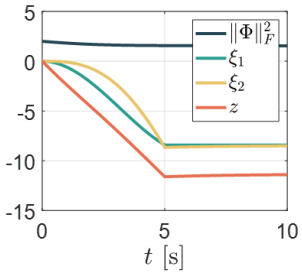
$$\Delta_3(t) = \frac{1}{2t+1}.$$



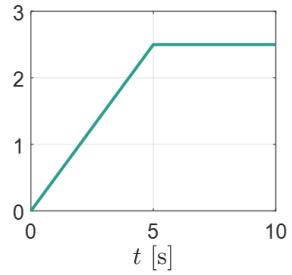
(a) Original and new regressors



(b) Estimates using the new and original LREs

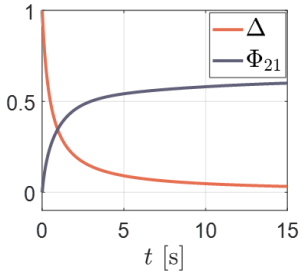


(c) Internal states

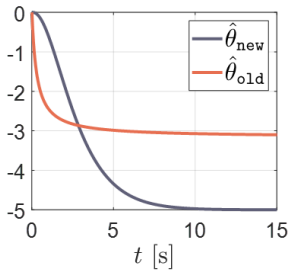


(d) The value of $\int_0^t |\alpha(s)\Delta(s)|\mathrm{ds}$

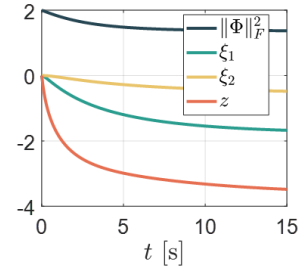
Fig. 3: Simulation results for the step signal $\Delta_2(t)$



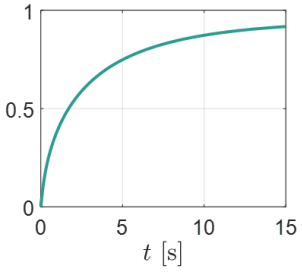
(a) Regression outputs $y(t)$ and $\mathcal{Y}_2$



(b) Estimate using the new LRE



(c) Internal states



(d) The value of $\int_0^t |\alpha(s)\Delta(s)|\mathrm{ds}$

Fig. 4: Simulation results for the signal $\Delta_3(t) = \frac{1}{2t+1}$

It is clear that a non-zero constant $\alpha(t)$ cannot guarantee the condition $\alpha\Delta \in \mathcal{L}_1$. Therefore, we parameterize α as

$$\alpha(t) = \alpha_0(t) - ky(t)z(t),$$

with $k > 0$. Then, the key equation (7) takes the form

$$\dot{z} = -(\tilde{V} + ky^2)z + \alpha_0 y,$$

with the second term in $\alpha(t)$ providing additional damping in the dynamics of z. By carefully selecting α_0, we may guarantee that $\alpha\Delta \in \mathcal{L}_1$. For Δ_3 we select $k = 0.1$ and $\alpha_0(t) = e^{-\frac{t}{10}}$ in simulation, see Fig. 4 for the corresponding results. It also illustrates the theoretical analysis.

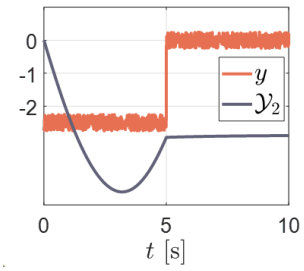
We then add high-frequency noise in the LRE output y via the "Uniform Random Number" block in Simulink/Matlab$^{\text{TM}}$. The signal-to-noise ratio (SNR) is selected around 20 dB with sampling time 0.01s. In this example we consider the signal Δ_2. The simulation results are given in Fig. 6, from which we observe that the new regressor output $\mathcal{Y}_2$ is hardly affected by the noise, as analyzed in Subsection IV-B.

As is well known, one of the main motivations of *on-line* identification is to track (slowly) time-varying parameters. To verify the alertness to the variation of the unknown parameter of the proposed procedure a simulation in which the parameter θ jumps from -5 to -4 at $t = 10$s has been carried out with the IE regressor

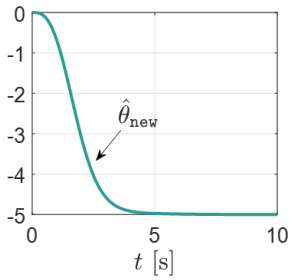
$$\Delta_4(t) = \begin{cases} \sin\left(\frac{\pi}{10}t\right), & t \in [0, 12]\text{s} \\ 0, & t > 12\text{s} \end{cases}$$

and $\gamma = 2$ and $\alpha(t) \equiv 1$.[6] The simulation results are shown in Fig. 6 from which we empirically conclude that the proposed new regressor is capable of dealing with parameter variations. The estimation $\hat{\theta}_{\texttt{old}}$

[6]We need to consider the signal being IE after the parameter variation for a while. Otherwise, the change of parameter cannot be captured by the output y.



(a) Outputs in the original and new regressors



(b) Estimates using the new LRE

Fig. 5: Simulation results in the presence of high-frequency measurement noise for the signal $\Delta_2(t)$

from the original regressor is also presented, which, as we can see, fails to get satisfactory performance due to the lack of PE.

We underscore that there is an ultimate error from $\hat{\theta}_{\texttt{new}}$, because Proposition 2 only holds when θ is constant. For a time-varying θ, a perturbation term appears in the new LRE (4), inducing the observed estimation error. As future work, it would be interesting to study the ultimate error quantitatively. To overcome this issue, re-initialization of the filter in Proposition 2 after parameter variation may be implemented, and it would be interesting to study how to re-initialize it automatically.

VI. Conclusions and Future Research

We have presented a procedure to generate a new LRE in which the new regressor has some excitation properties, even if the original regressor does not. The key idea to carry out this task is the use of the virtual dynamics $\dot{\theta} = 0$ given in (8). As seen in (5), the new LRE has some free signals u_i that we selected in a particular way in Proposition 3 to enforce the excitation-like condition of **F1**. However, further research is needed to select other signals u_i that would ensure *bona fide* excitation for a well-defined class of regressors Δ—for

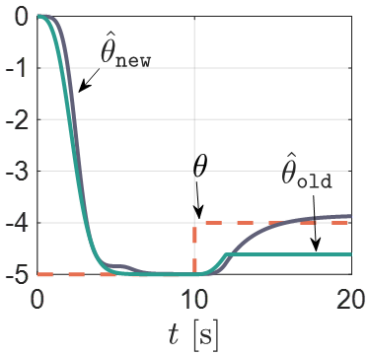


Fig. 6: Alertness to time-varying parameter θ for the IE signal $\Delta_4(t)$

instance, IE. It would be particularly interesting to relax the standing assumption (15)—that imposes constraints on the tuning function α.

A very simple extension of the result is obtained combining the elements of the signal $\mathcal{Y}$ in (13) as

$$Y := \mathcal{H}_1(\mathcal{Y}_1) + \mathcal{H}_2(\mathcal{Y}_2),$$

with $\mathcal{H}_1$ and $\mathcal{H}_2$ BIBO stable, scalar, LTV operators, to generate a new LRE

$$Y = [\mathcal{H}_1(\Delta\Phi_{11}) + \mathcal{H}_2(\Phi_{21})]\theta.$$

A topic of future research is the selection of suitable operators $\mathcal{H}_i$ to ensure excitation of the new regressor and boundedness of all signals. Note also that the extension of the results—including the application of the DREM procedure—to the case of nonlinear, *separable* parameterizations is straightforward.

Our LRE generator is restricted to the case of scalar LREs. However, mimicking the derivations of Section III it is possible to generate a new LRE also for the vector case, that is, when the original LRE is of the form (1). Indeed, using GPEBO it is possible to construct a measurable signal $e(t) \in \mathbb{R}$ and a *new regressor* $\mu(t) \in \mathbb{R}^q$, such that the new vector LRE

$$e = \mu^\top \theta$$

holds. In this case μ consists of the first q elements of the last row of the $(q+1) \times (q+1)$ matrix solution of the differential equation

$$\dot{\Phi} = \begin{bmatrix} 0_{q\times q} & u_1 \\ u_2\Delta^\top & u_3 \end{bmatrix} \Phi, \ \Phi(0) = I_{q+1},$$

with *arbitrary* functions $u_1(t) \in \mathbb{R}^q$, $u_2(t) \in \mathbb{R}$ and $u_3(t) \in \mathbb{R}$. The problem is that selecting the signals u_i to stabilize this new LTV system seems a daunting task. In any case, given the availability of DREM, the interest of such a study is highly questionable.

Another interesting line of research that we are pursuing now is to apply, to the general problem of state observation, the idea of *virtual signal injection*—that combined with GPEBO—is exploited in this paper. Indeed, the key step in the construction of our LRE generator is captured in (8), that may be interpreted as a virtual signal injection. Some encouraging preliminary results have been developed and we expect to report them in the near future.

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