# Valuation and Risk Analysis of Collateralised Debt Obligations

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## Abstract

This thesis presents CDO valuation and risk analysis viewed from three different modelling perspectives: (i) as a structural model (ii) as a non-parametric implied copula model and (ii) as a reduced form approximation model.

The structural CDO model in chapter 2 provides an alternative to the recursive loss model proposed by Andersen and Sidenius (2003). It uses the basic set up of the Vasicek homogenous pool loss distribution combined with two types of derived loss distribution adjustments. The loss distribution approximation produced by the proposed model is more accurate than the Vasicek homogenous pool model while still being faster than the recursive loss approach.

The non-parametric CDO copula model presented in chapter 3 resembles the implied copula of Hull and White (2006). The chapter shows how a binomial expansion method can be used to build marginal probability distributions and calibrate the implied copula to the iTraxx and CDX index tranches. The prices of bespoke CDO tranches and risk sensitivities calculated, using the base correlation model and the implied copula model, are compared. Even when the two models are calibrated to the same set of CDO tranche prices, they differ significantly in the way they attribute portfolio risk changes to different tranches as evidenced by differences in delta profile shapes.

The last chapter presents a dynamic portfolio model that may be used to value CDO tranches with cashflow waterfalls of arbitrary degrees of complexity using time-homogenous Markov chain processes for ratings. The approach employs a reduced-form CDO approximation technique. The accuracy of the approach is assessed under the assumption of different basis functions. The reduced form approximation can be extended to non-Gaussian copula models. To illustrate, a simplified version of the Andersen and Sidenius (2005) random factor loading model is implemented.

## Declaration statement

I, *Vladislav Peretyatkin*, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis and acknowledgements.

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**Vladislav Peretyatkin**

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# 1 Introduction to structured products

Structured products are complex instruments with payoffs that depend on the performance of portfolios of correlated underlying assets with a tranche structure, i.e. a set of rules that prescribe how cashflows from the underlying portfolio are to be split between the holders of several classes of claim of different seniorities. Collateralised Debt Obligations (CDO) are one of the standard structured products. The CDO structure allocates the principal and interest collections to a prioritised set of debt instruments which it issues to outside investors. These prioritised debt instruments are called tranches. Senior tranches are always paid allocated interest collections first. After the senior tranches have been paid in full, the mezzanine tranche is paid. The remaining interest collections may be paid to the equity tranche holders, deposited into a reserve fund or reinvested to provide extra collateral for senior note holders. Broader overview of structured products in fixed income area could be found in "Structured Credit Products" (2004) and "Credit Ratings" (2002) risk books publications.

Two major types of CDO exist — cashflow CDO and synthetic CDO. While both CDO types are par-based instruments with either a fixed or a LIBOR-linked coupon paid to compensate the investor for the credit risk assumed, the cashflow CDO have a more complex set of waterfall rules than synthetic CDO and its way of taking exposure to credit risk is quite different.

In a cashflow CDO the originator invests proceeds from issuing CDO tranches to third parties into a portfolio of diverse cash assets: bonds, loans, revolving credit facilities etc. Cashflow proceeds from these underlying assets are paid to the CDO tranche holders in the order of priority prescribed by the CDO cashflow waterfall. There is usually a fixed reinvestment period during which the proceeds from underlying assets maturing early or amounts recovered from the sale of distressed assets are reinvested into other risky assets. After the reinvestment period has passed, any notional collections or recovery from defaulted assets are directed to amortising tranches in the strict order of seniority.

The synthetic CDO structure is somewhat simpler. Synthetic CDOs take indirect credit risk exposure by referencing a portfolio of credit default swaps i.e.

no physical assets are involved. There is no reinvestment period and no early amortisation of CDO tranches. Credit events are defined by the underlying credit default swap terms and could include restructuring, bankruptcy or failure-to-pay triggers. Following an ISDA-confirmed credit event the portfolio notional is reduced by the defaulted CDS notional and the tranche subordination — by the loss amount (notional of the CDS less recovery). At maturity the investor is either paid back the full notional, part of it or, if the loss following a number of credit events has exceeded the tranche detachment point — nothing. Synthetic CDOs are sometimes called single tranche CDOs because there is no explicit waterfall dependency between tranches of different seniority, and therefore a single tranche could be issued.

High complexity of structured transaction documentation often makes it impossible for a less involved investor to fully understand the amount of credit risk embedded in a CDO tranche. To facilitate investor understanding of structured transaction risk rating agencies are often given a mandate to assign an externally derived rating which reflects the principal and coupon risk of the CDO tranche(s). The rating is typically assigned by one of the three leading rating agencies — Moody's, Standard & Poors or Fitch.

Collateralised debt obligations raise a number of modelling challenges. We can categorise these into three broad areas: (i) external credit quality assessment (investor) (ii) regulatory capital requirements (regulator) (iii) pricing and risk management (originating bank).

## 1.1 External credit quality assessment

As we explained earlier, CDO tranches of different seniority allow investors to choose the amount of credit risk they are willing to assume in return for coupon payments above the risk-free level. Conservative investors such as pension funds and insurance companies are typically interested in CDO tranches with high subordination. These assets are unlikely to suffer a loss in most market scenarios, and are supposed to deliver a small but stable yield if held to maturity. Investors with a higher appetite for risk, such as hedge funds, may take positions in mezzanine and equity tranches. While these are more likely to be affected by a sequence

of defaults in the underlying portfolio they are rewarded by significantly higher returns.

The two agents in the market — the CDO originator and the investors — have conflicting interests and often asymmetric information. Investors are interested in maximising yield without exceeding a target level of credit risk while banks are interested in higher fees, which implies that the types of assets they are likely to select for the underlying CDO portfolio are going to be more risky.

In this situation, rating agencies take the role of "due diligence managers". They independently analyse the securitised assets and use proprietary rating methodologies to assess the potential credit risk inherent in a tranche. The assessed credit risk is then mapped into discrete bins called ratings. Ratings make it much easier for investors to understand the risks associated with investing in a CDO, since the CDO tranches with publicly available ratings can be compared to other securities (e.g. rated corporate bonds), helping investors to make a risk-return decision.

The assignment of a rating to a CDO starts with collecting information about the underlying portfolio creditworthiness, i.e. the probabilities of default for each reference entity and their expected recovery rates. Rating agencies would then model dependent defaults in the CDO portfolio calculating the expected probabilities of default for each tranche and expected loss severity in the event of default. Their assumptions of default correlations are estimated off historical default rates or derived from correlations of equity indices.

A number of early papers analysed the approaches taken by rating agencies. Anderson (1997), Backman and O'Connor (1995) and Cifuentes, Murphy and Choi (1997) all looked at how the subordination and the diversity of the portfolio impact the credit rating of a CDO tranche.

Peretyatkin and Perraudin (2002) described the rating process of the three major rating agencies: Standard & Poor's, Moody's and Fitch in depth. They showed that if the market imperfectly understands the differences in rating definitions there is a significant scope for rating shopping by CDO issuers. Their findings are based on the fact that while both Standard & Poor's and Fitch rate structured transactions solely based on the probability of such transaction suffer-

ing a loss, Moody's bases its rating assessment on the concept of expected loss, which also takes into account the severity of such loss. Depending on (i) CDO tranche seniority and (ii) tranche thickness, issuers can pick off ratings from different rating agencies therefore maximising profits from the CDO structuring and issuance. Empirical evidence suggests that such rating arbitrage is taking place in synthetic CDO tranche structuring where banks have greater flexibility in choosing the tranche characteristics, see for example, Benmelech (2008).

Fender and Kiff (2004) analysed the rating process in a way similar to Peretyatkin and Perraudin (2002) and showed that imperfections in the rating methodology may lead to issuers strategically selecting certain rating agencies to get particular CDO structures rated.

Landschoot and Perraudin (2004) compared risk characteristics of ABS with the similarly rated corporate bonds. By analysing the patterns of bond and ABS price volatility over time they conclude that while corporate bonds exhibit a consistent volatility pattern, ABS tranche returns exhibit regime changes in which a particular sector deteriorates dramatically with substantial increases in risk over relatively short period in time.

The recent credit crisis of 2008-2009 and stress in the financial system cast serious doubt on the credibility of rating agencies. Whether rating agencies failure to assess the risk of structured product was due to model imperfections (e.g. correlation assumptions or incorrect parameterisation of tail events) or simple understatement of risk due to imperfectly assigned ratings for the underlying assets still remains a hot subject of discussion. Kirkpatrick (2009) reported that poor corporate governance and lack of firm-wide risk management oversight in many institutions resulted in risks not being properly disseminated to the end-recipients. He also concluded that at minimum rating agencies failed in their impartiality approach to assigning a rating to a structured product, partly because such impartiality was too often compromised by the remuneration offered by the bank in the form of a fee for assessing the credit risk of the structured product (and so "competition" for fees between rating agencies might have lead to lower rating standards). Whether the credit risk models and derived structured product ratings will stand to their historically expected performance should become more

obvious in the course of the next few years, when actual defaults rates per rating bucket as well as empirical rating transitions data become available.

## 1.2 Regulatory capital requirements

The growth of the CDO and ABS market meant that banks issuing CDO or ABS securities acquired direct or indirect exposure to structured products. For example, where a bank securitised its book of loans through a special purpose vehicle by issuing CDO notes, it would often retain the equity or the super-senior tranche in the newly created CDO, and be left with a structured exposure risk in its banking or trading books. Another driving factor behind the rapid growth of structured products was regulatory capital arbitrage. By using off-balance-sheet securitisation banks sought ways to reduce the amount of regulatory capital. Jones (2000) considered commonly used regulatory capital arbitrage techniques to lower regulatory capital requirements and the difficulties regulators face when dealing with those kind of activities under the capital requirement framework effective at the time (Basel 1). Further empirical studies analysing the scope for regulatory capital arbitrage can be found in Ambrose, LaCour-Little and Sanders (2004).

Regulatory capital arbitrage activities lead to a push to address shortcomings in the modelling of asset backed securities with particular emphasis on the on-off balance sheet consistency. The consistency condition specifies that a bank should end up with the same regulatory capital requirements whether it holds a portfolio of loans on its balance sheet or securitises them through an off balance sheet securitisation and then buys all tranches in the newly created securitisation, in other words regulatory capital should not change significantly unless the bank transfers significant amount of economic risk to the outside investors.

To understand the risk calculations behind capital charges for defaultable debt, consider rated corporate bonds or loans. For each such security Basel-2 prescribes a rating-specific capital charge called $K_{irb}$ which was derived using an asymptotic single-factor model for an infinitely granular portfolio. Being a value-at-risk estimate, it is basically a marginal portfolio loss conditional on the systemic risk factor realisation at some chosen confidence level $\alpha$ (e.g. 99%).

Early research on capital charges for asset-backed securities showed that a

single-factor model may not be appropriate for modelling capital charges for ABS with highly granular portfolios because capital charges schedule is a step function with 100% capital chargeable for tranches below $K_{irb}$ and 0% capital for tranches above $K_{irb}$. While it was acknowledged that certain senior CDO tranches may be virtually immune from actual losses even in extreme market scenarios, it was still inconceivable that such investments should be exempt from bank solvency requirements.

Gordy and Jones (2003) considered a capital allocation model assuming uncertain priority rights. They recognised securitisations as highly complex and therefore allowed for imperfect economic notion of CDO tranche priority assumed by the investors. They further assumed that a vector of tranches $\{s_1, s_2, ....s_n\}$ spanning the entire capital structure (i.e. $\sum s_i = 1$) follows the Dirichlet distribution. The Dirichlet distribution allows tranches to vary around their mean while ensuring that they always sum to 1. By varying the distribution parameters, they come up with a smooth $S$-shaped curve of capital charges while ensuring that the on/off balance sheet consistency condition is reasonably met. While their approach is easy to implement and requires minimal information about the underlying assets in the securitisation it is somewhat lacking in economic rationale, as the parameters to calibrate the model are chosen to fit a desired $S$-shaped capital charges curve, rather than such curve being derived from a structural model assuming an economically sensible set of parameters (correlation etc.).

Pykhtin and Dev (2002a) studied a single-period infinitely granular CDO model. They derived the semi-analytic distribution of losses under two-factor Gaussian model where the bank's main portfolio is driven by a single systemic risk factor and the assets in the securitised portfolio are driven by another factor. These two factors are assumed to have a pair-wise correlation of $\rho$. Further refinements feature Pykhtin and Dev (2002b) where the CDO portfolio is allowed to have a finite granularity (homogeneity assumption still applied).

Pykhtin and Dev (2002a) and (2002b) model was a big improvement on the approach proposed by Gordy and Jones but it still lacked the ability to analyse CDOs with longer maturities and could not be applied to heterogeneous CDO portfolios. Another disadvantage of their model was that it could only be used for

CDO structures which are negligible compared to the overall bank portfolio (and so tail events in the securitisation do not impact the conditional risk percentile).

Peretyatkin and Perraudin (2004) proposed a ratings-based dynamic capital charges model. As in standard ratings-based portfolio credit risk models, the ratings of defaultable claims are driven by a set of normally distributed latent variables. The correlations of these variables is taken to be that of a weighted average of equity indices, with weights being chosen based on the relative importance of sector and country risks. They assume that obligors' rating transitions between the simulation horizon and the CDO maturity are driven by a time-homogenous risk-adjusted transition matrix. Such transition matrix is calibrated to match the generic corporate spread curves and ensures consistency of pricing for underlying defaultable claims. They derive reduced form approximation functions for CDO tranches assuming a two-factor model similar to that of Pykhtin and Dev, and analyse a wide range of CDO structures (i) infinitely granular ABSs (ii) granular CDOs (iii) long-dated CDOs thereby deriving a table of consistent capital charges linked to the structured exposure rating. The output from Peretyatkin and Perraudin (2004) model formed the basis for the ratings-based ABS capital charges in Basel-2 accord (standardised approach).

## 1.3 Pricing and risk management

Collateralised Debt Obligations are an important tool for banks to manage their credit risk exposure. When a bank securitises its book of risky assets (e.g. loans, credit card receivables or mortgages) and sells tranched credit risk to outside investors it achieves two major objectives: (i) it obtains vital funding for its asset base (ii) it transfers credit risk exposure outside the banking system (however the terms of asset securitisation may require banks to retain the equity risk).

Where the bank's balance sheet is not directly involved, for example when it issues a synthetic CDO for a client, its role would involve dynamically managing the spread risk of the CDO by trading in the underlying single-name CDS. Such delta hedging technique is very similar to how equity and interest rate options are structured and hedged.

Early papers on pricing CDOs feature Duffie and Garleanu (2001) which devel-

oped a dynamic CDO valuation model. Using a jump-diffusion model of correlated hazard rates, they analysed the price sensitivity of CDO tranches to the assumption of correlation and risk-neutral diversity score and showed that equity and very senior tranches are sensitive to the correlation assumption, while for mezzanine tranches the effect is somewhat muted and not clear. They also concluded that the dispersion of hazard rates has a significant impact on CDO tranche pricing.

Longstaff and Rajan (2006) used hazard rate-based model to analyse the time series of index CDO prices. Recent dynamic CDO modelling feature Hull and White (2008) and Laurent, Cousin, and Fermanian (2008). The latter take a somewhat new angle to modelling dependency of defaults through credit contagion modelling. In a credit contagion model with jump diffusion setup, single or multiple defaults lead to changes in hazard rates for surviving obligors which in turn impact the likelihood of subsequent defaults. Such default clustering is similar in nature to modelling correlated defaults through use of copula however it has shown to have major implications on the hedging strategy where path dependency of credit derivative payoff is present.

Copula functions are well known in actuarial science and credit risk analysis and their properties could be found in Embrechts, Lindskog and McNeil (2003) and Schonbucher and Schubert (2001). Li (2000) introduced Gaussian copula methods, which formed the basis of the industry standard base correlation framework developed 4 years later. His original research showed some important definitions and basic properties of copula functions, namely that CreditMetrics approach of modelling default correlation through latent variable correlation is equivalent to using a Gaussian copula.

A natural extension to Gaussian copula is the Student t-copula, which was considered in different contexts by Andersen, Sidenius and Basu (2003), Demarta and McNeal (2005), Embrechts Lindskog and McNeil (2003), Schloegl and Kane (2005). Hull and White (2004) analysed the double-t copula, examples and discussions could also be found in Cousin and Laurent (2008). Other copulas considered include Clayton, Marshall-Olkin and Archimedian copula. The latter were mostly considered in the portfolio credit risk context by Schonbucher and Schubert (2001) and Rogge and Schonbucher (2003). Credit derivative pricing using Clayton copula

could be found in Gregory and Laurent (2003).

Peretyatkin and Perraudin (2004) proposed a dynamic multi-period CDO model where individual obligor risk-neutral default intensities are driven by a time-homogenous Markov chain process. Within their model, they proposed a CDO value approximation approach that involves running a two-stage Monte Carlo simulation similar to that of the Longstaff-Schwartz (2001) approach to valuing early exercise premium for American-style equity options.

Luciano and Schoutens (2005) developed a multivariate variance-gamma portfolio risk model. This was further applied to pricing synthetic CDOs by Moosbrucker (2006) while Baxter (2006) introduced a continuous term. In the Brownian-Variance-Gamma model of Baxter (2006) the mixture model contains (i) idiosyncratic and global jump parameters and (ii) idiosyncratic and global diffusion parameters, giving sufficient number of degrees of freedom to fit the model to the liquidly traded CDO tranche prices such as iTraxx or CDX.

Andersen and Sidenius (2005) presented the extension to the single-factor Gaussian model allowing correlation to assume a range of discrete values. The distribution of losses is built as a weighted average of single-factor Gaussian distributions, with each such distribution using its own single correlation parameter. They showed that by using a mixture of single-factor Gaussian distributions, it is possible to replicate the fat tails observed in traded CDO tranches.

# 2 Analytical approximation to constructing loss distribution in a single–factor Gaussian model

## 2.1 Gaussian single-factor models

Every credit derivative pricing model begins with the construction of the loss distribution based on the underlying portfolio characteristics such as probability of default and expected recovery rate for each reference entity combined with a chosen dependency structure. The dependency structure or copula function plays an important role in the calculation of loss distribution, as it ultimately defines how the total risk in the portfolio is split between claims of different seniority.

Gaussian copula models assume that the default probabilities for individual obligors are driven by a set of normally distributed correlated latent variables. In the single-factor Gaussian copula model, the correlation between latent variables can be viewed as exposure to a single normally distributed systemic risk factor. Single-factor Gaussian copula models have become a popular modelling tool in credit risk analysis and derivatives pricing and have lead to the development of a number of analytical techniques.

Vasicek (1991) proposed an asymptotic formula for the distribution of losses for a large portfolio of loans assuming a single-factor Gaussian copula. He considered a homogenous portfolio consisting of a large number of loans, each with the same notional, probability of default and recovery rate. Each loan's default probability of default is determined by the realisation of the normally distributed latent variable. The assumed pair-wise correlation between latent variables is shown to be equivalent to having a common systemic risk factor for all loans in the portfolio. Conditional on such systemic risk factor realisation defaults are independent and by law of large numbers the conditional default rate in the loan book should converge to the conditional average default rate. Unconditional distributions of losses under these assumptions could be computed by integrating the conditional distribution over all realisations of the systemic risk factor.

Vasicek's model is commonly referred to as the Large Homogenous Pool Model (LHPM) and his technique of conditioning played a significant role in further

research in this area (including the construction of non-Gaussian copula distributions which are outside the scope of this chapter). LHPM is still widely used in pricing ABS where the number of reference entities in the portfolio is typically large (greater than 250). Another useful application of LHPM is the calculation of approximations to global risk sensitivities (e.g. global spread widening).

Further non-Gaussian extensions to the LHPM feature a Student t-copula extension by Schloegl and O'Kane (2005), double-t copula in Hull and White (2004) and Gaussian inverse copula in Kalemanova, Schmid and Werner (2005). Vasicek's conditioning technique can be seen in numerous papers, to name the few Andersen and Sidenius (2005), Burtschell and Laurent (2007), Greenberg, Mashal and Schloegl (2004) and Gregory and Laurent (2005).

A separate but related model was the Moody's binomial expansion method first described by Cifuentes and O'Connor (1996). Moody's method approximated the loss distribution in the portfolio by analysing a portfolio of $N$ independent equally sized names for which the portfolio loss distribution is given by the binomial expansion formula. To account for the heterogeneity of the portfolio, Moody's approximated the heterogeneous portfolio characteristics with that of a homogenous portfolio with the average of the portfolio's probabilities of default. A special diversity score adjustment took into account the impact of correlation essentially approximating the distribution of $N$ correlated names by a distribution of $M$ independent and identically distributed (iid) names, where $M < N$. The diversity score adjustment was never analytically derived, but instead relied on a set of look-up tables calculated by Moody's.

Using the conditioning technique of Vasicek's model, the Moody's binomial expansion method was later refined further resulting in a so-called Homogenous Pool Model . The Homogenous Pool Model allowed for single-factor Gaussian dependency but still approximated the heterogeneous portfolio with the homogenous portfolio using the average default probabilities and average recovery rates.

Merino and Nyfeler (2002) relied on the conditioning technique of Vasicek model and derived the portfolio loss distribution using Fourier Transform method. Fourier transform technique was quickly adapted by practitioners and replaced the Monte Carlo methods previously used for heterogeneous portfolios. Application

of the Fourier transform method to pricing CDOs can be found in Gregory and Laurent (2004).

Andersen, Sidenius and Basu (2003) proposed a recursive loss technique where the loss distribution is built recursively by adding one name at a time. Having constructed the loss distribution for $n$ obligors in the portfolio it is possible to add an (n+1)th obligor using recursive formulae. Hull and White (2004) later applied the recursive loss model to price first-to-default baskets and CDO.

Greenberg, O'Kane and Schloegl (2004) considered an approximation to analytic idiosyncratic and systemic risks in CDO pricing. Their idea was to quickly construct the portfolio loss distribution using Vasicek model and analyse the impact of adding or removing an obligor to/from the portfolio using Andersen's recursive loss method.

The recursive loss model has proven to be a very useful modelling technique both for academics and practitioners for three main reasons. Firstly, the loss distribution for a portfolio of correlated defaultable names can be obtained to any chosen degree of precision. Secondly, it is relatively quick, and the calculation time can be worked out in advance given the underlying portfolio characteristics and the desired accuracy. Thirdly, it allows for quick analysis of marginal distributions by re-using the previously built loss distribution (whereas conventional methods would require calculating 2 separate distributions therefore doubling the computational time).

While the recursive loss model is deemed to be reasonably fast for most pricing and risk analysis exercises, it is still much slower than the Homogenous Pool Model. This is mainly because the recursive loss model speed is proportional to the square of the number of names in the portfolio while the homogenous pool model speed has a linear relationship to the number of names in the portfolio. The speed gain is largest when the number of names is relatively large, e.g. 125 to 250.

In this chapter, we present an extension to the homogenous pool model which we will refer to as HPM+. Our model uses the basic set up of the homogenous pool model combined with two types of derived loss distribution adjustments. It results in a very accurate loss distribution close to that obtained by the recursive loss model while retaining the speed of the homogenous pool model.

## 2.2 Models overview

### 2.2.1 Homogenous Pool Model (HPM)

We employ the single-factor Gaussian copula model default dependence with the homogenous pool framework, i.e. all names in the portfolio are assumed to have the same characteristics.

Consider a portfolio of $N$ equally weighted names, each with a default probability of $\overline{p}$ and a recovery rate of $\delta$. The time horizon for the probability of default is not relevant in this analysis because in the Gaussian copula model the time dimension of portfolio loss distribution is obtained by using a series of loss distributions each built using cumulative default probabilities for such time horizon.

Let's first assume that defaults in the portfolio are independent. The probability of $k$ out of $N$ names defaulting is given by the binomial expansion formula:

$$P(k) = C_k^N \cdot \overline{p}^k \cdot (1-\overline{p})^{N-k} \tag{2.2.1}$$

Here $C_k^N$ is a combinatorial factor defined as:

$$C_k^N = \frac{N!}{k!(N-k)!} \tag{2.2.2}$$

Assume that a name defaults if a normally distributed latent variable $X$ associated with a name falls below the cutoff point $Z$:

$$Z = \Phi^{-1}(\overline{p}) \tag{2.2.3}$$

The assumption of independence of defaults in our portfolio means that any pair of latent variables within vector $\{Z_i\}$ is independent of each other. To introduce default correlation, assume that each name's latent variable is driven by two components: a systemic component common to all names in the portfolio and an idiosyncratic (name-specific) component:

$$X_i = \sqrt{\rho} \cdot x + \sqrt{1-\rho} \cdot \varepsilon_i \tag{2.2.4}$$
$$x, \varepsilon_i \sim N(0,1)$$

The probability of default conditional on the systemic risk factor realisation $x$ is then:

$$f(\overline{p}, x) = \Phi\left(\frac{\Phi^{-1}(\overline{p}) - \sqrt{\rho} \cdot x}{\sqrt{1-\rho}}\right) \tag{2.2.5}$$

Conditional on the systemic risk factor realisation, defaults are independent, therefore:

$$P(k,\overline{p},x) = C_k^N \cdot f(\overline{p},x)^k \cdot \left(1 - f(\overline{p},x)\right)^{N-k} \tag{2.2.6}$$

The unconditional loss distribution is then simply the integral of the conditional distribution over all realisations of systemic risk factor:

$$P(k,\overline{p},x) = \int_{-\infty}^{\infty} C_k^N \cdot f(\overline{p},x)^k \cdot \left(1 - f(\overline{p},x)\right)^{N-k} \cdot \phi(x) \cdot dx \tag{2.2.7}$$

Now consider a tranche with attachment point $\alpha$ and detachment point $\beta$. Let $\tilde{L}_\alpha^\beta$ denote the expected loss on this tranche:

$$\tilde{L}_\alpha^\beta = \int_{-\infty}^{\infty} \sum_{i=\alpha}^{N} \left( C_i^N \cdot f(\overline{p},x)^i \cdot \left(1 - f(\overline{p},x)\right)^{N-i} \cdot I_\alpha^\beta(i) \right) \cdot \phi(x) \cdot dx \tag{2.2.8}$$

Here, $I_\alpha^\beta(i)$ is a *loss count* function defined as:

$$I_\beta^\alpha(i) = \begin{cases} i \cdot (1-\delta) - \alpha, & \text{if } \alpha \le i \cdot (1-\delta) \le \beta \\ (\beta - \alpha), & \text{if } i \cdot (1-\delta) \ge \beta \\ 0, & \text{if } i \cdot (1-\delta) < \alpha \end{cases} \tag{2.2.9}$$

### 2.2.2 Recursive Loss Model (RLM)

Andersen et al. (2003) proposed an analytical method for constructing the loss distribution using recursion for the case when default probabilities vary between obligors in the portfolio. To analyse their approach, consider a portfolio consisting of $N$ reference entities. Let $\{p_i\}$ be the vector of default probabilities for these names and let each name have a fixed recovery rate $\delta$. Let's initially assume that defaults are independent.

Let $P^{(i)}(j)$ denote the probability of $j$ defaults happening, where the superscript $i$ reflects the number of names used to construct the loss distribution. For the first asset, the loss distribution is simply the two possible outcomes:

$$\begin{aligned} P^{(1)}(0) &= 1 - p_1 \\ P^{(1)}(1) &= p_1 \\ &\cdots \\ P^{(1)}(j) &= 0, \quad j = 2,\ldots,N \end{aligned} \tag{2.2.10}$$

Next consider a loss distribution containing two names:

$$\begin{aligned} P^{(2)}(0) &= (1-p_1)\cdot(1-p_2) \\ P^{(2)}(1) &= p_1\cdot(1-p_2)+p_2\cdot(1-p_1) \\ P^{(2)}(2) &= p_1\cdot p_2 \\ &\cdots \\ P^{(2)}(j) &= 0, \quad j=3,...,N \end{aligned} \tag{2.2.11}$$

Equation (2.2.11) could alternatively be expressed as:

$$\begin{aligned} P^{(2)}(0) &= P^{(1)}(0)\cdot(1-p_2) \\ P^{(2)}(1) &= P^{(1)}(0)\cdot p_2 + P^{(1)}(1)\cdot(1-p_2) \\ P^{(2)}(2) &= P^{(1)}(1)\cdot p_2 + P^{(1)}(2)\cdot(1-p_2) \\ &\cdots \\ P^{(2)}(j) &= 0, \; j=3,...,N \end{aligned} \tag{2.2.12}$$

It is not difficult to generalise the formulae above using a recursive scheme. Suppose at some step $i$ we managed to build the loss distribution for $i$ names. The loss distribution for $(i+1)$ names is given by:

$$\begin{aligned} P^{(i+1)}(0) &= P^{(i)}(0)\cdot(1-p_{i+1}) \\ &\cdots \\ P^{(i+1)}(k) &= P^{(i)}(k-1)\cdot p_{i+1} + P^{(i)}(k)\cdot(1-p_{i+1}), \quad 0<k<i+1 \\ P^{(i+1)}(j) &= 0, \quad j>k \end{aligned} \tag{2.2.13}$$

Please note that the algorithm described above assumes an equally weighted loss-given default (or a constant recovery rate with equally weighted names in the portfolio). The recursive loss approach in general does not require this, and may be generalised to account for heterogeneous recovery rates or uneven name weights in the portfolio.

To analyse the case when joint default probabilities are defined by the single-factor Gaussian copula, we use the same conditioning technique as in the homogenous pool model. Conditional on systemic risk factor realisation, defaults are independent; therefore, the recursive algorithm (2.2.13) could be used to construct

the conditional loss distribution using a vector of conditional default probabilities:

$$\begin{cases} P^{(i+1)}(0,x) = \prod_{j=1}^{i+1} \big(1 - f_j(p_j,x)\big), \\ P^{(i+1)}(k,x) = P^{(i)}(k-1,x) \cdot f(p_i,x) + P^{(i)}(k,x) \cdot \big(1 - f(p_{i+1},x)\big), \\ P^{(i+1)}(j,x) = 0, \ j > k, \end{cases} \tag{2.2.14}$$

$$i = 1,2,...,N, \ \ 1 \le k \le i+1.$$

The unconditional portfolio loss distribution is then the integral of conditional loss distribution over all realisations of systemic risk factor:

$$P^{(N)}(k) = \int_{-\infty}^{\infty} P^{(N)}(k,x) \cdot \phi(x) \cdot dx, \ \ k = 0,\ldots,N \tag{2.2.15}$$

The expected loss for a CDO tranche is calculated in the same way as in (2.2.8).

## 2.3 HPM+

One important advantage of HPM over RLM is that it only requires $\alpha \times N$ multiplications to construct the loss distribution, whereas RLM requires at least $\beta \times N^2$ multiplications (where $\alpha$ and $\beta$ are some constants). In other words the speed of the HPM algorithm is linear in the number of names, while the speed of RLM algorithm is not (and as such, HPM is generally faster than RLM).

Another computational advantage of HPM is that it only needs to construct the relevant part of the loss distribution (e.g. the loss distribution between attachment and detachment points) without the need to build the distribution for irrelevant strikes. By contrast, the RLM requires a larger part of distribution to be built (because of the recursive dependence of default probabilities for higher strikes on the default probabilities for lower strikes).

The disadvantage of the HPM lies in its accuracy as it produces a somewhat approximate distribution for a significantly heterogeneous portfolio. The numerical exercise at the end of this chapter shows the extent to which the loss distribution constructed using the HPM deviates from that of RLM.

The purpose of this section is to derive analytical adjustments for the HPM-implied loss distribution so as to bring it closer to that of RLM. Because the loss

distribution could be built exactly using RLM, it will be used as a benchmark to compare the accuracy of adjustments calculated.

We derive two types of adjustments: (i) HPM portfolio heterogeneity adjustment or *type I adjustment* and (ii) HPM conditional convexity adjustment or *type II adjustment*. Both adjustments are analysed in the context of conditional loss distribution, i.e. assuming defaults are independent. Each adjustment is subsequently applied per conditional integration to arrive at an adjusted unconditional distribution.

### 2.3.1 HPM+ portfolio heterogeneity adjustment

As before, consider a heterogeneous portfolio of $N$ names. Again, let $\{p_i\}$ be the vector of default probabilities in the portfolio, $i=1..N$. Also, let the average default probability in the portfolio be $\overline{p}=\frac{1}{N}\sum_{i=1}^{N}p_i$. Let's assume that $p_i$ is *reasonably distributed* around its mean $\overline{p}$ i.e. the distribution of default probabilities around the mean is symmetric and thin-tailed and could therefore be approximated by the normal distribution (we will later show how this assumption may be relaxed). Let $E(\overline{p}-p)^2=\frac{1}{N}\sum_{i=1}^{N}(\overline{p}-p_i)^2=\sigma^2$. We can therefore write $p_i=\overline{p}+\varepsilon_i$ where $\varepsilon_i\sim N(0,\sigma^2)$.

As a starting point, consider $P(0)$ i.e. the probability that there will be no defaults in the portfolio. We can write the analytical solution for the "true" probability of this event as:

$$P(0)=(1-p_1)\cdot(1-p_2)\cdot\ldots\cdot(1-p_N) \tag{2.3.1}$$

Similarly, the HPM-implied probability is given by:

$$P(0)=C_0^N\cdot\overline{p}^{\,0}\cdot(1-\overline{p})^N \tag{2.3.2}$$

The following will therefore hold:

$$(1-p_1)\cdot(1-p_2)\cdot\ldots\cdot(1-p_N)=(1-\overline{p})^N+\gamma^{(0)} \tag{2.3.3}$$

Here $\gamma^{(k)}$ is the HPM type I adjustment. Rewrite (2.3.3) substituting for $p_i=\overline{p}+\varepsilon_i$:

$$(1-\overline{p}+\varepsilon_1)\cdot(1-\overline{p}+\varepsilon_2)\cdot\ldots\cdot(1-\overline{p}+\varepsilon_N)=(1-\overline{p})^N+\gamma^{(0)} \tag{2.3.4}$$

Expand the left-hand side of (2.3.4), ignoring terms involving $\varepsilon^{k>2}$:

$$(1-\overline{p})^N+\sum_{i=1}^{N}\sum_{j=1}^{i-1}(1-\overline{p})^{N-2}\varepsilon_i\varepsilon_j+O(\varepsilon^3)=(1-\overline{p})^N+\gamma^{(0)} \tag{2.3.5}$$

The type I adjustment is thus given by:

$$\gamma^{(0)}=\sum_{i=1}^{N}\sum_{j=1}^{i-1}(1-\overline{p})^{N-2}\cdot\varepsilon_i\varepsilon_j+O(\varepsilon^3)\approx-(1-\overline{p})^{N-2}\cdot\frac{N\cdot\sigma^2}{2} \tag{2.3.6}$$

Similarly, the type 1 adjustment for $P(1)$ is:

$$\gamma^{(1)}=-\frac{N\cdot\sigma^2}{2}\cdot\left(-2\cdot(1-\overline{p})^{N-2}+(N-2)\cdot\overline{p}\cdot(1-\overline{p})^{N-3}\right) \tag{2.3.7}$$

Or in general:

$$\gamma^{(k)}=-\frac{N\cdot\sigma^2}{2}\cdot\frac{(N-2)!}{k!(N-k)!}\cdot\left(k\cdot(k-1)\cdot\overline{p}^{k-2}\cdot(1-\overline{p})^{N-k}-\right. \tag{2.3.8}$$
$$\left.2\cdot k\cdot(N-k)\cdot\overline{p}^{k-1}\cdot(1-\overline{p})^{N-k-1}+(N-k)\cdot(N-k-1)\cdot\overline{p}^{k}\cdot(1-\overline{p})^{N-k-2}\right)$$

Proof 1 in Appendix provides derivation of formula (2.3.8). Also Proof 2 in Appendix shows that the sum of all gammas is exactly 0, i.e. the error-correction redistributes the probabilistic distribution and does not bias it.

When dealing with the independent distribution of defaults, we will write the adjustment in the following form:

$$HPM_{iid}\left(\frac{1}{N}\cdot\sum_{i=1}^{N}p_i,k\right)+\gamma^{(k)}\approx RLM_{iid}\left(\{p_i\},k\right) \tag{2.3.9}$$

### 2.3.2 HPM+ conditional convexity adjustment

Both RLM and HPM techniques deal with the default correlation by conditioning on the realisation of systemic risk factor $x$ and then integrating the conditional distribution over all realisations of systemic risk factor $x$. The two models, both perform the same integration technique but using slightly different conditional distributions.

Consider the conditional loss distribution under RLM (here $\{\cdot\}$ denotes a vector of variables):

$$P(x,k)=RLM_{iid}\left(\{f(p_i,x)\},k,x\right) \tag{2.3.10}$$

Conditional on the systemic risk factor realisation, defaults are independent; therefore we can apply equation (2.3.9) to (2.3.10):

$$RLM_{iid}\Big(\{f(p_i,x)\},k,x\Big)=HPM_{iid}\Big(\frac{1}{N}\cdot\sum_{i=1}^{N}f(p_i,x),k\Big)+\gamma^{(k)} \tag{2.3.11}$$

Consider the left-hand side of equation (2.3.11). It describes the conditional RLM model, which uses a vector of conditional default probabilities as inputs. The right-hand-side of equation (2.3.11) looks similar to the HPM model, but uses slightly different inputs. The input to the right-hand side of equation (2.3.11) is the average of conditional default probabilities while the correct HPM model input would be the conditional average (unconditional) default probability of the portfolio, i.e. $HPM\Big(f\big(\frac{1}{N}\cdot\sum_{i=1}^{N}p_i\,,\,x\big)\,,k\Big)$. We can therefore re-write equation (2.3.11) as:

$$RLM_{iid}\Big(\{f(p_i,x)\},k,x\Big)=HPM_{iid}\Big(f\Big(\frac{1}{N}\cdot\sum_{i=1}^{N}p_i,x\Big)+\omega^{(k)},k\Big)+\gamma^{(k)} \tag{2.3.12}$$

Here $f\big(\frac{1}{N}\cdot\sum_{i=1}^{N}p_i\,,\,x\big)\equiv f(\overline{p},x)$, and $\omega^{(k)}$ is the HPM type II adjustment, correcting the conditional average default probability to be equal to $\frac{1}{N}\cdot\sum_{i=1}^{N}f(p_i\,,\,x)$. So in essence, we are looking for some $\omega^{(k)}$ which solves the following equation:

$$\frac{1}{N}\cdot\sum_{n=1}^{N}f(p_i,x)=f(\overline{p},x)+\omega^{(k)} \tag{2.3.13}$$

Expand the left-hand-side of (2.3.13) around $\overline{p}$ using a second-order Taylor series expansion:

$$\frac{1}{N}\cdot\sum_{i=1}^{N}\Big(f(\overline{p},x)+\frac{\partial f}{\partial p}(\overline{p},x)\cdot(p-\overline{p})+\frac{1}{2}\cdot\frac{\partial^2 f}{\partial p^2}(\overline{p},x)\cdot(p-\overline{p})^2\Big)=f(\overline{p},x)+\omega^{(k)} \tag{2.3.14}$$

Take expectation of both sides of equation:

$$f(\overline{p},x)+\frac{1}{2}\cdot\frac{\partial^2 f}{2\partial p^2}(\overline{p},x)\cdot E(p-\overline{p})^2=f(\overline{p},x)+\omega^{(k)} \tag{2.3.15}$$

The type II HPM adjustment is therefore:

$$\omega^{(k)}=\frac{1}{2}\cdot\frac{\partial^2 f}{\partial p^2}(\overline{p},x)\cdot\sigma^2 \tag{2.3.16}$$

Expanding (2.3.16) it is easy to show that the type II HPM adjustment is:

$$\omega^{(k)}=\frac{\sigma^2}{2}\cdot\frac{\phi\Big(\frac{\Phi^{-1}(\overline{p})-\sqrt{\rho}\cdot x}{\sqrt{1-\rho}}\Big)}{\phi^2(\Phi^{-1}(\overline{p}))\cdot\sqrt{1-\rho}}\cdot\left(\Phi^{-1}(\overline{p})-\frac{\Phi^{-1}(\overline{p})-\sqrt{\rho}\cdot x}{1-\rho}\right) \tag{2.3.17}$$

Throughout this analysis we have assumed that the distribution of default probabilities in the portfolio is reasonably symmetric and thin-tailed, and therefore can be approximated by the normal distribution with the appropriate mean and variance. Such an assumption implies that the third order Taylor series expansion term in (2.3.14) is insignificant. Consider (2.3.16) when the third order Taylor series expansion is used:

$$\omega^{(k)} = \frac{1}{2}\cdot\frac{\partial^2 f}{\partial p^2}(\overline{p},x)\cdot\sigma^2 + \frac{1}{6}\cdot\frac{\partial^3 f}{\partial p^3}(\overline{p},x)\cdot E(\overline{p}-p)^3 \tag{2.3.18}$$

For a large enough sample generated by a normally distributed random variable, the (central) expectation of the third moment $E(\overline{p}-p)^3 \to 0$. In finite samples, the third moment will be different from zero but will be negligibly small.

If, however, the distribution of default probabilities in the portfolio is skewed and the third moment is significant, then the type II adjustment would include the third order term. It is not difficult to show that the adjustment including the $3^{rd}$ order term is:

$$\omega^{(k)} = \frac{E(\overline{p}-p)^2}{2}\cdot\frac{\phi(C)}{\phi^2(D)\cdot B}\cdot\left(D-\frac{C}{B}\right)+ \tag{2.3.19}$$
$$\frac{E(\overline{p}-p)^3}{6}\cdot\frac{\phi(C)}{B\cdot\phi(D)^3}\cdot\left(\frac{1}{B^4}-\frac{3}{B^2}+2\right)\cdot\left(D^2+\frac{A}{B^2}\cdot\left(\frac{2}{B^2}-3\right)\cdot D+\frac{A^2}{B^4}-\frac{1}{B^2}+1\right)$$

*where*

$$A=-\sqrt{\rho}\cdot x,\quad B=\sqrt{1-\rho},\quad C=\frac{\Phi^{-1}(\overline{p})-\sqrt{\rho}\cdot x}{\sqrt{1-\rho}},\quad D=\Phi^{-1}(\overline{p}).$$

### 2.3.3 Unconditional distribution adjustment

Sections 2.3.1 and 2.3.2 discussed how type I and type II adjustments are calculated per conditional distribution, i.e. assuming conditional independence. The unconditional distribution adjustment is as before, the integral of conditional distributions over all realisations of the systemic risk factor:

$$P(k,\overline{p},x) = \tag{2.3.20}$$
$$\int_{-\infty}^{\infty}\left(C_k^N\cdot\left(f(\overline{p},x)+\omega^{(k)}(\overline{p},x)\right)^k\cdot\left(1-f(\overline{p},x)-\omega^{(k)}(\overline{p},x)\right)^{N-k}+\gamma^{(k)}(\overline{p},x,\sigma_{cond})\right)\cdot\phi(x)\cdot dx$$

Where $\gamma^{(k)}$ and $\omega^{(k)}$ are given by (2.3.8) and (2.3.17), and where the conditional variance estimate $\sigma_{cond}$ is given by the following formula:

$$\sigma_{cond} = \phi\left(\frac{\Phi^{-1}(\overline{p}) - \sqrt{\rho}\cdot x}{\phi\left(\Phi^{-1}(\overline{p})\right)\cdot\sqrt{1-\rho}}\right)\cdot\sigma \tag{2.3.21}$$

Formula (2.3.21) can be derived using Taylor series expansion of the variance estimator:

$$\mathrm{var}\Big(f(x)\Big) = f'(x)^2\cdot x \tag{2.3.22}$$

## 2.4 Numerical techniques employed

### 2.4.1 Extracting risk-neutral probabilities of default from credit spreads

Extracting default probabilities from CDS spreads would normally require a CDS model (see for example Jarrow and Turnbull (1995) and O'Kane and Turnbull (2003)). One such simple hazard rate model will be introduced in Chapter 4, for now we stick to bond par spreads for portfolio constituents and extract risk-neutral default probabilities using an approximation formula.

We assume that the interest rate risk and credit risk are independent from each other. Under such assumption the risk-neutral probabilities implicit in bond spreads could be approximated using the following approach.

Let $C_t$ be some cashflow payable at future time $t$. The present value of this cashflow is:

$$PV(C_t) = C_t\cdot B_t\cdot\exp(-S_t\cdot t) \tag{2.4.1}$$

here $B_t$ is the price of a pure discount bond maturing at time $t$ (the expectation splits thanks to the assumption of independence between credit risk and interest rate risk).

Let $p_t^*$ be the risk-neutral (cumulative) default probability at $t$. Rewrite (2.4.1) as:

$$B_t\cdot\Big(p_t^*\cdot\delta\cdot C_t + (1-p_t^*)\cdot C_t\Big) = C_t\cdot B_t\cdot\exp(-S_t\cdot t) \tag{2.4.2}$$

Here with probability $p^*$ the name defaults and the payoff is the recovery value equal to contractual cashflow $C_t$ times recovery rate $\delta$ and with probability $(1-p^*)$ it does not default in which case the payoff is just the contractual cashflow $C_t$.

We can thus extract the risk-neutral cumulative default probabilities from bond spreads using:

$$p_t^* = \frac{1-\exp(-S_t \cdot t)}{1-\delta} \tag{2.4.3}$$

Another advantage of relying on the above approximation is it allows us to approximate the moments of $p_t^*$ with that of $S_t$. Indeed:

$$p_t^* = \frac{1-\exp(-S_t \cdot t)}{1-\delta} \approx \frac{S_t \cdot t}{1-\delta} \tag{2.4.4}$$

Therefore:

$$E\left(\left(p_t^*\right)^n\right) \approx \frac{t}{1-\delta} \cdot E\left(\left(S_t\right)^n\right) \tag{2.4.5}$$

One should note, that the above approximation does not impact the general findings of the HPM+ model and so a more accurate model to extract risk-neutral probabilities from credit spreads could be used (for example in the presence of correlation between interest rate risk and credit risk).

### 2.4.2 Integration of conditional functions

Consider the integral in (2.2.15):

$$P^N(k) = \int_{-\infty}^{\infty} f(x) \cdot d\Phi(x) \tag{2.4.6}$$

The integral in (2.4.6) cannot be evaluated analytically and therefore a numerical technique approximating the integral with a discretised version of it has to be used (e.g. Gauss-Hermite integration). Integration accuracy is not the main essence of this paper so we employ a much simpler but reasonably accurate integration technique. Consider a discretised trapezoid rule version of (2.4.6):

$$P^N(k) = \sum_{i=-LB}^{UB} f\left(\frac{(i+0.5)\cdot\Delta}{UB-LB}\right) \cdot \omega(i) \tag{2.4.7}$$

$$\omega(i) = \Phi\left(\frac{(i+1)\cdot\Delta}{UB-LB}\right) - \Phi\left(\frac{i\cdot\Delta}{UB-LB}\right)$$

*where*

$$\sum_i \omega(i) = 1$$

Suppose we want to minimise the integration error in (2.4.7) (having fixed the discretisation grid size) and suppose that the error due to approximation of $f(x)$

by evaluating $f\left(\frac{(i+0.5)\cdot\Delta}{UB-LB}\right)$ at mid-point is a slowly changing function of $x$ (or in other words is close to being constant). In that case, the weighting function which minimises the total integration error is such that the weights are equal to each other, i.e. $\omega(i)=\omega(j)$ for any $i,j$.

For a normal distribution function, such weights are obtainable by spacing the uniform distribution domain into equal intervals on [0,1] and then inverting these equally sized grid points using the cumulative normal inverse function. This ensures that the area between any two adjacent points has the same value (probability), i.e. weights are equally sized. Throughout this paper we have chosen the integration bounds to be between $[-5,5]$ with discretisation grid size of 1000.

## 2.5 Example calculations

In this section we will perform stylised calculations to test the accuracy of the HPM+ model. We therefore need to construct the loss distribution using the three models analysed: HPM, HPM+ and RLM. Our chosen test portfolio is 7 year iTraxx index constituents for June 2008. The basic portfolio statistics are shown in table 2.1. We tested the performance of our technique using a range of different portfolios and the technique performed well in most cases, however for compactness of this analysis we are going to focus on a single portfolio.

The first step is to analyse the performance of type I adjustment alone. To do this we assume that defaults in our stylised portfolio are independent, i.e. $\rho=0$. Table 2.2 shows the output for the HPM, HPM+ and RLM loss distributions under these assumptions. As can be seen from this table, the majority of the type I error is corrected by applying the type I adjustment.

Now consider a more realistic case when defaults in the portfolio are correlated with the latent variable correlation $\rho=25\%$. There are now two simultaneous sources of error in the loss distribution produced by the HPM.

Table 2.3 shows the loss distribution built using HPM, HPM+ and RLM models. The calculation time for the HPM+ loss distribution is almost 10 times shorter than that for the RLM loss distribution while the accuracy of HPM+ comes very close to RLM (the calculation time assumes that both models need to build the entire loss distribution and so the speed gain will be larger if a smaller range of

loss distribution buckets is needed).

Table 2.4 shows the percentage of the total error corrected by type I and type II adjustments. We can see that the type I adjustment is only responsible on average for 4% of total error correction, while the type II adjustment gives a large 92% error correction on average (these numbers obviously vary between different loss levels).

One can see that the heterogeneity error flips the sign at the point of six names defaulting (for this particular portfolio of credit names, of course), i.e. HPM overstates default probabilities for lower strikes and compensates that this by understating the default probabilities for higher strikes (since the total probability of default across all strikes is the same for both models). This is one of the major drawbacks of the conventional homogenous pool model in that it is unable to distinguish between pools with different dispersion of credit risk in the portfolio as soon as the average characteristics are the same.

CDO tranches, being claims of different seniority are said to be either long or short credit spread dispersion, depending on their relative seniority in the capital structure. Irrespective of the underlying portfolio characteristics, the equity tranche is always long credit spread dispersion which means that the equity tranche risk will increase with an increase in the credit spread dispersion. The reverse is true for the super-senior tranche, which is short credit dispersion, i.e. its risk reduces with the increase in the credit spread dispersion. Mezzanine tranches in that sense can be similar to equity tranche or super-senior tranche, depending on their seniority.

The impact of credit spread dispersion on portfolio loss distribution received surprisingly little focus in academic and practitioner research, basic discussions can be found in Hull and White (2006). The empirical analysis of the credit spreads dispersion impact can be found in Longstaff and Rajan (2008).

In the analysis shown in Table 2.4 the lower error correction power around the point of six names defaulting is due to smaller sensitivity of that part of capital structure to the portfolio dispersion $\sigma^2$ (and not reduction of accuracy of HPM+).

## 2.6 Credit spread dispersion sensitivity

As we briefly discussed earlier in this chapter, the credit risk embedded in tranches of different seniority can either increase or decrease with the changes in the portfolio spread dispersion. This section will consider the impact of spreads dispersion on portfolio loss distribution. As we have shown in the previous section, the point where the homogenous pool model overstates default probabilities compared to a heterogeneous pool model (RLM or HPM+) was around $\overline{k}=6$ names defaulting in the portfolio of 125 names, as seen in table 2.2. As HPM+ directly links the loss distribution to the portfolio dispersion, this feature allows us to analyse the impact of portfolio parameters on such point $\overline{k}$.

We assume the following base case portfolio characteristics: PD=11%, $\rho$=25%, $\sigma$=6%, $N$=125. Our analysis will involve varying (i) probability of default (ii) correlation (iii) dispersion and measuring the $\overline{k}$, i.e. the number of defaults in the portfolio loss distribution where the sensitivity with respect to increase in credit dispersion changes sign (and as such thin tranches spanning the capital structure below $\overline{k}$ will be long dispersion while thin tranches spanning the capital structure above $\overline{k}$ will be short dispersion).

Figure 2.1 show the impact of default correlation on $\overline{k}$. As can be seen, an increase in correlation increases $\overline{k}$. One possible interpretation for that is that high correlation makes all tranches in the capital structure look alike, and therefore with the increase in correlation, the impact of dispersion is becoming less pronounced in the lower parts of the capital structure.

Figure 2.2 shows the impact of portfolio average default probability on $\overline{k}$. The direct relationship between $\overline{k}$ and average default probability can be explained by the overall shift of the loss distribution centre (by term "centre" we mean the part of the capital structure where most probabilistic weight is concentrated) therefore resulting in a shift in $\overline{k}$.

Finally, Figure 2.3 shows the impact of dispersion on $\overline{k}$. It is an interesting fact that for a small initial change in dispersion, the point $\overline{k}$ has a non-zero intercept around 4 names defaulting with the point $\overline{k}$ increasing with the increase in portfolio dispersion.

## 2.7 Concluding remarks

Vasicek's Homogenous Pool Model has been widely used in research literature to analyse loss distribution approximations in credit risk modelling and CDO valuation. Its conditioning technique, simplicity and analytical tractability meant that a large number of research papers directly or indirectly relied on his results.

Various non-Gaussian copula versions of the Large Homogenous Pool models were proposed, see for example Kalemanova, Schmid and Werner (2005) or Schloegl and O'Kane (2005). The angle of relationship between homogeneous and heterogeneous pool modelling was however somewhat missed out.

Chapter 2 leverages off prior research in this area and extends the Vasicek's homogeneous pool model by incorporating the effect of portfolio heterogeneity using the spread dispersion in the reference portfolio. The significant increase in loss distribution accuracy comes at a small computational speed expense therefore making the approach a good alternative to the exact loss distribution methods such as the Fourier transform or the recursive loss method.

Our major finding is that the approximation error due to the assumption of portfolio homogeneity is less pronounced than the effect of conditional integration convexity, i.e. the impact of portfolio heterogeneity is significantly higher for correlated portfolios than uncorrelated ones. Furthermore, the model allows us to explore the CDO tranche sensitivity to the portfolio credit spreads dispersion changes, which is an important input to CDO tranche risk analysis.

Further research ideas include derivation of similar adjustments for non-Gaussian copulas and possibly further analysis of the impact of higher moments of portfolio spread distribution.

# 3 Implied copula model with the embedded expected loss adjustment

## 3.1 Base correlation model and its alternatives

The synthetic CDO market growth has been due in part to the development of the base correlation model, nowadays widely accepted by market participants as the industry standard. The standard base correlation model is a development of the single-factor Gaussian model first suggested by Vasicek (1991). Subsequent analytical methods such as the Fourier transform method and recursive loss model made the single-factor model quick and accurate enough to use in the semi-analytic pricing of synthetic CDOs.

Single-factor correlation cannot explain the distribution of risks in traded synthetic CDO tranches as traded tranche prices assign higher probabilities to the tails of distributions than those implied by the Gaussian copula. This limitation led to the development of the base correlation framework according to which any mezzanine CDO tranche spanning the $\{\alpha,\beta\}$ capital structure is seen as a long position in an $\{0,\beta\}$ equity tranche combined with a short position in an $\{0,\alpha\}$ equity tranche. A single-factor implied correlation is then derived for each base equity tranche. Plotting these implied correlations against equity tranche thickness results in a correlation smile similar to the volatility smile observed in equity options implied volatilities.

The base correlation model is just one of the models tailored to address the heavy tailed nature of market prices of risk. It naturally attracted significant attention from most prominent credit risk researchers and resulted in a rich set of alternative modelling approaches. They can be broadly categorised into three categories: (i) stochastic hazard rate models (ii) structural models and (iii) copula models.

Hazard-rate based models assume that defaults for each obligor are generated by a pure Poisson process with stochastic intensity. Dependency between intensities is typically modelled by correlating Brownian motion drivers and introducing common and idiosyncratic shocks to the intensity process. Papers following such

an approach include Duffie and Singleton (1999), Duffie and Garleanu (2001) and Jarrow and Yu (2001).

Structural models assume that an obligor defaults when a latent variable representing company assets falls below a certain threshold. A rich set of default dependencies could be modelled by making latent variables dependent on each other. Structural models were initially tailored to modelling portfolio credit risk but later adapted to pricing CDOs. Examples of these are Zeng and Zhang (2001) and Perraudin and Peretyatkin (2002). Frey and McNeil (2003) mapped the latent variable models to the Bernoulli mixture models facilitating simulation and statistical fitting. Baxter (2006) introduced a structural model with a Brownian-Variance-Gamma process.

The copula function approach involves defining dependency between marginal distributions for individual obligors. Copula functions are well known in actuarial science and credit risk analysis and their properties could be found in Embrechts, Lindskog and McNeil (2003) and Schonbucher and Schubert (2001). The Gaussian copula for CDO valuation was introduced by Li (2001) and the further Student-t extension by Frey and McNeil (2003). Andersen and Sidenius (2005) proposed a Gaussian mixture model, which is a generalised extension of the base correlation model. Their model allows for random Gaussian correlation, the distribution of which can be chosen to mimic the shape of the correlation smile.

The non-parametric CDO copula is mostly represented by Hull and White (2006). In their implied copula model, each obligor's hazard rate is seen as the expectation of different future hazard rate scenarios which they refer to as "hazard rate paths". The joint distribution of hazard rate paths forms dependency structure, similar to the structural model in Vasicek (1991).

In this chapter we present implied copula model resembling the model developed independently by Hull and White (2006). Our model differs significantly from Hull and White (2006) implied copula model in a number of aspects. Firstly and most importantly, our model can calibrate to the iTraxx and CDX index tranches taking into account the heterogeneity of the underlying portfolio spreads. Hull and White (2006) basic model focused on the homogenous pool case where such simplification can potentially lead to a significant distortion of the underlying

portfolio distribution, as we have shown in Chapter 2. Secondly, our model has a significant advantages over Hull and White (2006) modelling approach in that it does not need to rely on the assumption of negative dependency between probability of default and recovery rate. Hull and White (2006) had to incorporate a negative dependency between the hazard rates and recovery rates to overcome calibration difficulties. Finally, within our implied copula we directly model the dynamics of risk-neutral probabilities of default for each obligor, therefore greatly improving the calibration speed.

Similarly to the approach taken by Hull and White (2006), we do not specify the dependency between individual names' default probabilities but instead specify a series of conditionally independent joint probabilistic scenarios, which preserve each name's unconditional default probabilities at different time horizons. This allows us to explore dependency structure by fitting it to the index tranche prices.

The structure of this chapter is going to be as follows. We first explain how a synthetic CDO can be valued using the single factor Gaussian model and its base correlation extension. The bespoke CDO valuation technique is then explained outlining the weaknesses of the base correlation approach.

Using the conditioning technique presented in the single factor Gaussian model, we are then able to show how similar method could be applied to construct the loss distribution using our implied copula model. We present numerical techniques allowing to calibrate the implied copula to index tranche prices using multinomial trees.

The chapter concludes by calibrating base correlation and implied copula models and comparing them using a number of stylised portfolios. Since the suggested implied copula model explicitly incorporates the expected loss adjustment (by construction) we investigate valuation differences between two models. Our conclusions are that the two models produce substantially different tranche prices when applied to bespoke CDO pricing. We also show that the implied copula exhibits significantly less idiosyncratic risk associated with each name in the portfolio which can be seen in less convex delta profiles.

## 3.2 Base correlation framework

This section analyses the CDO pricing technique in the single-factor Gaussian model. We briefly discuss some practical aspects of applying the single-factor Gaussian model to pricing synthetic CDOs.

### 3.2.1 CDO pricing in a single–factor Gaussian model

Consider the following model set-up:

- $T$-years maturity CDO tranche with attachment point $\alpha$ and detachment point $\beta$
- portfolio consisting of $N$ credit names, each having a set of CDS spreads per tenor $C^{(i)}(\tau) = \left\{C^{(i)}(3Y), C^{(i)}(5Y), C^{(i)}(7Y), C^{(i)}(10Y)\right\}$
- assumed recovery rate $\delta$
- a series of pure discount bond prices $B(t)$
- default correlation $\rho$

To price this CDO tranche (i.e. calculate the at-the-money CDO tranche spread), we need to build the portfolio loss distribution based on the underlying portfolio default probabilities, recovery rates and the imposed dependency structure.

The first step is therefore to bootstrap the cumulative default probabilities implicit in each name's spreads. This is conventionally done by assuming some functional form of hazard rates (e.g. piece-wise linear hazard rate model) and calibrating a survival probability curve $s(t)$ such that the present value of the expected loss leg equals the present value of the payment leg for all CDS tenors:

$$\int_0^\tau -B(t)\cdot ds(t) = \int_0^\tau B(t)\cdot s(t)\cdot C^{(\tau)}\cdot dt, \quad \textit{for } \tau = 3,5..10, \tag{3.2.1}$$

The cumulative default probabilities curve $p(t)$ is then simply:

$$p(t) = 1 - s(t) \tag{3.2.2}$$

Recall that in a single-factor Gaussian model, a name with the unconditional default probability $p^{(i)}$ defaults if a normally distributed latent variable $y^{(i)}$ falls

below the threshold $Z^{(i)}$ defined by the following equation:

$$Z^{(i)} = \Phi^{-1}\left(p^{(i)}\right) \tag{3.2.3}$$

It is further assumed that each latent variable $y^{(i)}$ is driven by the name-specific idiosyncratic risk factor $\varepsilon^{(i)}$ and the systemic risk factor $x$ common to all names in the portfolio. The contribution of systemic risk factor to the total volatility of $y^{(i)}$ is proportional to $\sqrt{\rho}$, where the parameter $\rho$ is called *default correlation*:

$$y^{(i)} = \sqrt{\rho}\cdot x + \sqrt{1-\rho}\cdot\varepsilon^{(i)} \tag{3.2.4}$$
$$x \sim N(0,1),\ \varepsilon^{(i)} \sim N(0,1)$$

A normally distributed random variable $x$ can assume any value on $(-\infty,+\infty)$, but let us suppose that we actually observe some realisation $x$ of this random variable. In this case the probability of default conditional on $x$ is defined by the following function:

$$f^{(i)}(x) = \Phi\left(\frac{\Phi^{-1}(p^{(i)}) - \rho\cdot x}{\sqrt{1-\rho}}\right) \tag{3.2.5}$$

Conditional on systemic risk factor realisation defaults are independent, so we can use one of the quick analytical methods to build a conditional loss distribution (e.g. recursive loss technique described in Chapter 2).

$$D(x) = RL\left(\left\{f^{(i)}\left(p^{(i)},x,\rho\right)\right\},\delta\right) \tag{3.2.6}$$

To build the unconditional loss distribution we can simply integrate the conditional distribution over the systemic risk factor domain $(-\infty,+\infty)$:

$$D = \int_{-\infty}^{\infty} RL\left(\left\{f^{(i)}\left(p^{(i)},x,\rho\right)\right\},\delta\right)\cdot d\Phi(x) \tag{3.2.7}$$

To price a synthetic CDO, where losses can occur at different points in time, requires the construction of the cumulative loss distribution for each point in time between 0 and $T$. We will therefore need a discretisation technique for the time variable domain and for the systemic risk factor variable domain. Let's divide the time variable $t$ domain into $K$ equally sized intervals $t_1 > t_2 > \ldots > t_K \in [0;T]$, and the systemic risk factor $x$ domain into $M$ equally sized intervals $x_1 > x_2 > \ldots > x_M \in [-5;5]$ (Normal distribution is thin-tailed so it is reasonable to truncate the

variable domain to a certain range instead of considering infinite intervals). The discretised version of equation (3.2.7) thus becomes:

$$D(t_k) = \sum_m RL\left(\left\{f^{(i)}\left(p^{(i)}(t), x_m, \rho\right)\right\}, \delta\right) \cdot Q_m \tag{3.2.8}$$

where

$$Q_m = prob(x_m > x > x_{m+1}) = \Phi(x_m) - \Phi(x_{m+1}) \tag{3.2.9}$$

is the probability of conditional scenario occurrence.

Note that for any name in the portfolio, the following holds:

$$\int_{-\infty}^{\infty} f^{(i)}\left(p^{(i)}(t), x, \rho\right) \cdot d\Phi(x) = \lim_{m\to\infty} \sum_m f^{(i)}\left(p^{(i)}(t), x_m, \rho\right) \cdot Q_m = p^{(i)}(t) \tag{3.2.10}$$

In other words, integrating conditional distribution for any name in the portfolio using (3.2.7) or (3.2.8) gives us the original unconditional default probability for that name (not surprising as it is by construction — however, this is quite an important fact which we are going to use later on).

Let $L_\alpha^\beta$ be the CDO tranche loss count function, defined by:

$$L_\beta^\alpha(j) = \begin{cases} j\cdot(1-\delta) - \alpha & if \;\; \alpha \le j\cdot(1-\delta) \le \beta \\ (\beta - \alpha) & if \;\; j\cdot(1-\delta) \ge \beta \\ 0 & if \;\; j\cdot(1-\delta) < \alpha \end{cases} \tag{3.2.11}$$

Other credit derivatives may have different loss count functions, not necessarily defined by (3.2.11). The cumulative CDO tranche expected loss at time $t_k$ is then:

$$EL(t_k) = \sum_{j=0}^{N} L_\alpha^\beta(j) \cdot D(t_k)_j \tag{3.2.12}$$

where $D(t_k)$ is the probability that $j$ names default before $t_k$. Therefore the surviving CDO tranche notional curve $s(t_k)$ equivalent to the single name survival probability curve bootstrapped from CDS quotes is:

$$s(t_k) = 1 - EL(t_k) \tag{3.2.13}$$

Having built the tranche survival probability curve, the tranche at-the-money coupon $C_T$ is the coupon that sets the present value of the expected loss leg

equal to the present value of the payment leg:

$$\int_0^T ds(t)\cdot B(t) = \int_0^T s(t)\cdot C_T \cdot B(t)\cdot dt \tag{3.2.14}$$

The discretised version of (3.2.14) is:

$$\sum_k \Big(s(t_k) - s(t_{k-1})\Big)\cdot B(t_k) = \sum_k s(t_k)\cdot C_T \cdot B(t_k)\cdot (t_k - t_{k-1}) \tag{3.2.15}$$

Therefore $C_T$ could be calculated using:

$$C_T = \frac{\sum_k \Big(s(t_k) - s(t_{k-1})\Big)\cdot B(t_k)}{\sum_k s(t_k)\cdot B(t_k)\cdot (t_k - t_{k-1})} \tag{3.2.16}$$

### 3.2.2 Base correlation model

In the previous section we described the single-factor Gaussian model approach to pricing CDO tranches. In practice, the method is not straightforward to apply because different synthetic CDO tranches referencing the same portfolio of names imply different default correlations. The table below shows the implied correlations, extracted from iTraxx Series 7 tranche prices.

iTraxx Series 7 implied correlations:

| Tranche | Spread | Implied Correlation |
|---|---|---|
| 0-3% | 27.37% + 500 bp | 11.65% |
| 3-6% | 139 bp | 3.05% |
| 6-9% | 39 bp | 9.62% |
| 9-12% | 18 bp | 14.03% |
| 12-22% | 6 bp | 21.74% |

Implied correlations cannot easily be used to price other synthetic CDOs, especially when it comes to pricing CDOs with non-standard attachment/detachment points. They tend to form a disjoint curve and so conventional curve interpolation methods simply would not work. Implied correlations are also impossible to handle in the calculation of correlation sensitivities, as perturbing the correlation for one of the tranches would change the total expected loss on the entire capital structure, thereby breaking the no-arbitrage condition of the total expected loss

preservation (the sum of expected losses on all tranches in the capital structure should be equal to that of the index or sum of expected losses on all single-name CDSs in the portfolio). The latter also poses problems in the calculation of other hedge sensitivities, such as sensitivity to credit spread widening.

To overcome the above-mentioned problems McGinty et al. (2004) proposed the so-called base correlation approach. Since synthetic CDOs have a very simplistic notional-based waterfall, the expected loss on a mezzanine tranche attaching at $\alpha$ and detaching at $\beta$ is equivalent to a long position in an equity tranche with thickness $\beta$ combined with a short position in an equity tranche with thickness $\alpha$:

$$EL\left(Tr_{\alpha}^{\beta}\right) = EL\left(Tr_{0}^{\beta}\right) - EL\left(Tr_{0}^{\alpha}\right) \tag{3.2.17}$$

Therefore for a series of tranches with attachment/detachment points of $[0,\alpha_1]$ , $[\alpha_1,\alpha_2]$, $[\alpha_2,\alpha_3]$,..., $[\alpha_n,1]$, one can always calibrate implied correlations of corresponding equity tranches $[0,\alpha_1]$, $[0,\alpha_2]$,..., $[0,\alpha_n]$ such that the expected loss differences in consecutive equity tranches match the original set of mezzanine tranches. The term *base correlation* comes from the fact that equity tranches are called *base tranches*. Please note that the correlation for $[0,1]$ tranche is not defined and so the super-senior tranche price is residual, i.e. it's equal to the index value less the value of all tranches in the capital structure junior to super-senior tranche.

iTraxx Series 7, base correlations

| Tranche | Spread | Base Correlation |
|---|---|---|
| 0-3% | 27.37/500 bp | 11.65% |
| 3-6% | 139 bp | 19.54% |
| 6-9% | 39 bp | 26.05% |
| 9-12% | 18 bp | 31.52% |
| 12-22% | 6 bp | 49.59% |

Base tranche modelling is not restricted to the single factor model, see for example Hooda (2006) who used Normal-Gamma processes to the base tranche modelling and Levy base tranche modelling in Garcia et al. (2007).

Base correlations allow for ease of pricing of synthetic CDOs because they can be interpolated when pricing tranches with non-standard attachment/detachment points. Another useful feature of the base correlation approach is its no-arbitrage

preservation property when calculating correlation sensitivities. The next section will look at some of the aspects of bespoke CDO pricing using the base correlation model.

### 3.2.3 Bespoke CDOs pricing in the base correlation framework

As the market for liquid tranches developed, banks increasingly started structuring bespoke single-tranche synthetic CDOs. Bespoke CDOs typically reference portfolios which have little or no overlap with the iTraxx or CDX portfolios and their attachment and detachment points are customised. Pricing bespoke CDOs requires making assumptions about how dependency parameters are mapped from index tranche CDOs such as iTraxx or CDX into the bespoke portfolio. The key mapping assumptions required are: (i) base correlation curve interpolation technique (ii) base correlation mapping. We briefly discuss these issues below.

*Base correlation curve interpolation.* Bespoke CDO tranches structured by banks have little resemblance to the standard capital structure of index tranches. The attachment point defining the CDO subordination is usually customised to deliver an initial target rating to the customer and the tranche thickness is chosen depending on how aggressive the client is in accepting a high loss in the event of default (one default thick tranches might have 100% loss given default). Some kind of interpolation technique must therefore be chosen to select the base correlations for a bespoke CDO. One interpolation might be better than another, but the monotonicity of tranche spreads (the tranche spread should decrease with increasing subordination) is never mathematically guaranteed. There are infrequent instances where the base correlation model produces non-monotonic tranche spreads when a set of thin tranches spanning the entire capital structure of a bespoke CDO is analysed. Such occurrences may lead to arbitrage opportunities (and pose a problem even if a problematic tranche of this kind is not explicitly traded as it indirectly affects the expected loss of other tranches in the capital structure).

From the analysis of the monthly price-testing results over the last few years and conversations with a number of synthetic correlation market participants, I concluded that the major players have adopted a piece-wise cubic spline interpo-

lation technique (or slightly more elaborate version of this). Extrapolation of the base correlation curve below the lowest attachment point and beyond the highest detachment point is based on guesswork, and seems to differ from bank to bank.

*Base correlation mapping.* The average spread on a bespoke CDO can significantly differ from the average spread of the index portfolio. What kind of transformation (if any) of the base correlation smile is needed to map the base correlation from one portfolio to another? McGinty et al. (2004) was first to suggest the so-called expected loss adjustment method, which transforms the index tranche base correlation curve linearly by the ratio of the total expected loss on the bespoke CDO portfolio and the total expected loss on the index tranche portfolio. The argument in support of this kind of transformation is that portfolios with higher spreads have a higher probability of affecting more senior tranches. Thus, to make the dependency structure on the bespoke portfolio comparable to that of the index portfolio, correlations should be chosen as if the real CDO tranche strikes were higher or lower (depending on whether the average spread of the bespoke CDO is lower or higher than the spread of the index portfolio). Morini (2008) studied the dynamics of the dependency structure implicit in index tranches and suggested alternative mapping functions some of which performed better over crisis when a number of local correlation regime changes were observed over time. For example, one of the alternative mappings proposed is based on the ratio of the credit spread dispersion of index and bespoke portfolio.

As the focus of this paper is not on analysing base correlation mapping techniques, we are going to take them as given. In other words, we will use cubic spline interpolation combined with the linear expected loss adjustment in pricing bespoke CDOs in a base correlation framework when comparing the prices implied by our copula model.

## 3.3 Implied copula approach to pricing synthetic CDOs

### 3.3.1 Implied copula explained

Consider equations (3.2.8) and (3.2.15). Equation (3.2.8) joins a series of conditionally independent joint distributions subject to the simultaneous preservation

of each name's marginal distribution as shown in equation (3.2.15). In fact, any unconditional joint distribution can be represented as a probability-weighted series of conditionally independent joint distributions. Conditional independence is convenient in that it makes the loss distribution analytically tractable by using one of the quick loss distribution methods such as the recursive loss method or the Fourier transform method.

Section 3.2.1 described the single-factor Gaussian copula model where we specified the dependency structure between individual names using the correlation between latent variables driving individual names' conditional default probabilities. This model allowed us to build the (unconditional) joint distribution of default probabilities. The set of probabilistic weights $\{Q_m\}$ associated with a series of conditional distributions in (3.2.8) were therefore implied by the single-factor Gaussian dependency structure defined.

We are now going to reverse the dependency formulation in the following way. Instead of formulating the dependency structure and then inferring conditional distributions and their probabilistic weights $\{Q_m\}$ as seen in equation (3.2.8), we will form a series of joint conditionally independent distributions and their associated probabilistic weights $\{Q_m\}$, such that each reference entity preserves its unconditional default probability.

To briefly explain the concept, consider any two reference names in our portfolio. Let their cumulative unconditional default probabilities be $p^{(1)}$ and $p^{(2)}$ respectively. We are looking to find a series of $m$ conditional default probabilities $\{p_m^{(1)}\}$ and $\{p_m^{(2)}\}$ and a set of corresponding (common) probabilistic weights $\{Q_m\}$ such that:

$$\sum_m p_m^{(1)} \cdot Q_m = p^{(1)} \tag{3.3.1}$$

$$\sum_m p_m^{(2)} \cdot Q_m = p^{(2)}$$

Here each discrete realisation of variable $p^{(i)}$ represents a particular state of the economy (high/medium/low future default rate environments) and the elements in the vector $\{Q_m\}$ correspond to the probability of such future default rate environment. Since it is assumed that each probabilistic scenario is conditional, events within each scenario are independent of each other. By virtue of (3.2.8),

the (unconditional) joint distribution $D(t_k)$ can be calculated using, for example, the recursive loss method:

$$D(t_k) = \sum_m RL\left(\{p_m^{(i)}\}, \delta\right) \cdot Q_m \tag{3.3.2}$$

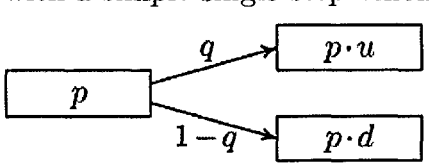
In the next section we will show how binomial expansion method can be used to construct marginal distributions where the conditional probabilistic weights $\{Q_m\}$ are common across reference entities in the portfolio.

### 3.3.2 Binomial expansion methods applied to copula modelling

Binomial tree approximation techniques has been used in various areas in finance, but they are best known in equity option semi-analytical methods. Bandreddi et al. (2007) attempted to apply the binomial tree technique to construct loss distribution for a portfolio of defaultable claims assuming single factor Gaussian copula dependency. Karoui, Jiao and Kurtz (2008) used Gaussian and Poisson approximation to derive the loss distribution for a local correlation model by Burtschell, Gregory and Laurent (2007).

Our binomial expansion method can be seen as a multi-state extension of the simple binomial tree technique. Namely, we are looking for a way of constructing the marginal distribution of default probabilities where the vector of probabilistic weights $\{Q_m\}$ is common across all reference entities in the portfolio (see condition (3.3.1)).

We start our analysis with a simple single-step binomial tree:




Here the initial default probability $p$ can increase to $p \cdot u$ with probability $q$ or decrease to $p \cdot d$ with probability $1-q$, where $p > 1$ and $0 < u < 1$ are scaling parameters. The following will therefore hold:

$$p = p \cdot u \cdot q + p \cdot d \cdot (1-q) \tag{3.3.3}$$

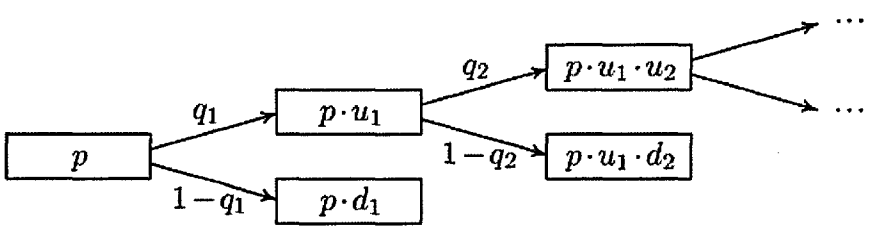
and since $p$ cancels out on both sides of the equation, we get:

$$1 = u \cdot q + d \cdot (1-q) \tag{3.3.4}$$

The fact that equation (3.3.4) is independent of $p$ implies that each state's probability is determined solely by its parameters $u$ and $d$. Therefore, if these parameters are the same for all names in our references portfolio, it can guarantee (i) that the expectation of conditional default probability for any name gives us the unconditional default probability (ii) that the probabilities of conditional scenario occurrence are common across all names in the portfolio.

*Horizontal expansion* and *vertical expansion* methods can be used to make the model more granular.

In the *horizontal expansion* method, one could expand the top leaf repeatedly as depicted below (there are other alternatives, e.g. one could expand each leaf in the tree, but that would lead to too many end-leaves):




In the scheme above, the probabilistic weight associated with the top leaf is divided into two sub-states. Each subsequent sub-state inherits decreasing probabilistic weight and (i) it is dependent on the parameters of leaves on the left hand side (ii) it will command the maximum probabilistic weight of the leaves on the right hand side.

In the *vertical expansion* method, the variable $p$ is initially viewed as a flat tree with $G$ probabilistic states $\{w_g\}$, where each state's realisation of the variable $p$ is the variable itself:

$$p = \sum_G p \cdot w_g \tag{3.3.5}$$

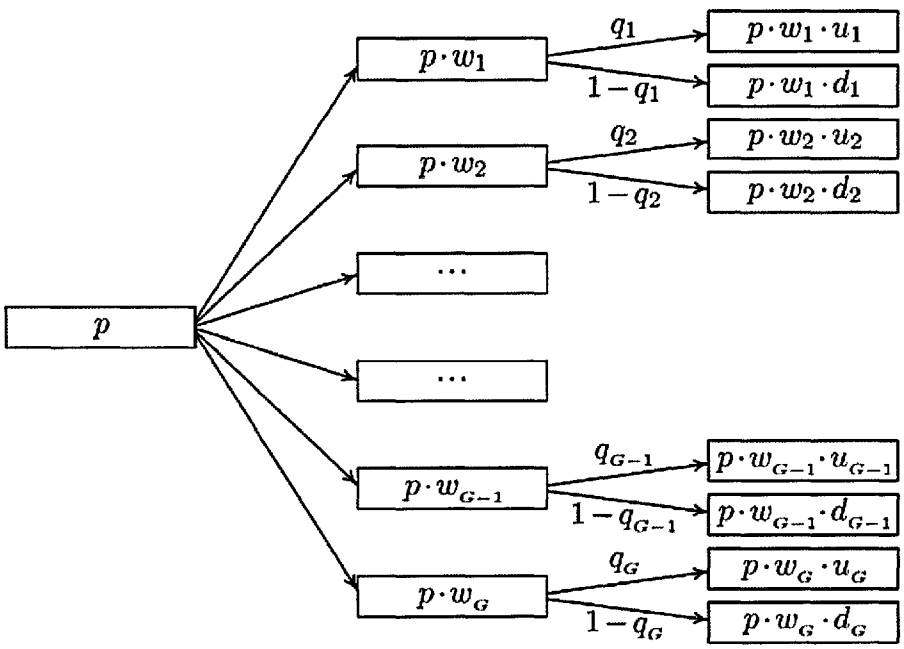
*such that*

$$\sum_G w_g = 1, \quad w_g \in [0,1]$$

Each state can then be expanded using a simple single-step binomial expansion as described in (3.3.3).

These two seemingly different ways of constructing trees are in fact equivalent in that there always exists a unique horizontal tree replicating any vertical tree

and vice versa. For an external optimisation routine however, these two ways of representing the same problem are rather different. A horizontal tree always exhibits high parameter cross-dependency in that changing any set of parameters $u_j$ and $d_j$ at some intermediate branching step $j$, $1 \le j < m$, would change the end-probabilities for all leaves having an index greater than $j$. A vertical tree with a deterministic set of probabilistic weights $\{w_G\}$ has absolutely no parameter cross-dependency. For that reason we prefer to use vertical tree representation where for simplicity weights $w_g = \{\frac{1}{G}\}$ are assumed to be equal to each other.




It is easy to show that the split of initial probabilistic weight between states in a vertical tree makes only a small difference as soon as each state's probability is small. When the number of states is sufficiently large, any set of $\{w_g\}$ satisfying $0 < w_g << 1$ is suitable. To prove this proposition, it is trivial to show that any horizontal tree with unequally distributed set of states $\{w_f\}$ can be exactly replicated by a horizontal tree based on a set of equally distributed states $\{w_G \equiv \frac{1}{G}\}$ where $\frac{1}{G}$ is the greatest common divisor of $\{w_f\}$.

Let's now quickly revert back to the CDO valuation. Let vector $\{U_m > 1\}$ and $\{0 < D_m < 1\}$ be the vectors of up-moves and down-moves (these will be chosen by the optimisation function) and let $\{w_m \equiv \frac{1}{m}\}$ be the vector of vertical tree probabilistic weights. We can now rewrite equation (3.2.8) as:

$$D(t_k) = \sum_m \left( RL\left(p^{(i)}(t) \cdot U_m, \delta\right) \cdot q_m + RL\left(p^{(i)}(t) \cdot D_m, \delta\right) \cdot (1 - q_m) \right) \cdot w_m \quad (3.3.6)$$

where $q_m$ is calculated from (3.3.4):

$$q_m = \frac{1 - D_m}{U_m - D_m} \tag{3.3.7}$$

To price a synthetic CDO tranche we will follow the same set of steps outlined in section 3.2.1 using the unconditional loss distribution built in (3.3.6).

### 3.3.3 Practical aspects of implied copula calibration

Calibrating a model with a large number of parameters (and where the number of parameters can be chosen arbitrarily) is a non-trivial exercise. To begin with, it is important to choose the number of parameters wisely: too many parameters may lead to multiple solutions, while too few parameters may result in a bad fit. Due to the discreteness of our model, even if the implied copula is calibrated to fit a set of thick tranches perfectly, the smoothness of thin tranche prices implied by the model is not guaranteed. To address these two issues, we will fit the copula model with a large number of parameters (32 states) to a large number of 1% thick tranches implied by the base correlation model (19 tranches for iTraxx and 27 tranches for CDX). This ensures as far as reasonable that these conditions are fulfilled.

We already mentioned that the model parameterisation may impact the convergence speed when a brute force fitting method is used (e.g. by minimising the sum of squared residuals from the target function values): the convergence speed is greater when the parameter cross-dependency (cross derivatives of the objective function) is small. For that reason we use a vertical tree representation of our copula function.

To further improve the calculation speed, we use a semi-analytical approximation of the copula function. Consider equation (3.3.6) where there are $2 \cdot m$ conditional probabilistic scenarios and each conditional scenario is uniquely defined by its probabilistic weight and its scaling parameter (either up-move or down-move). In chapter 2 we showed that the recursive loss model $RL(\cdot)$ is computationally expensive and requires at least $\alpha \cdot N^2$ floating point operations. Therefore, it makes sense to avoid having to rebuild the conditional loss distribution as far as possible.

We can approximate the function $RL(p^{(i)}(t) \cdot J, \delta)$ by taking a linear (or log-

linear, to take some convexity effects into account) combination of the two nearest pre-calculated functions values $RL(p^{(i)}(t) \cdot J_L, \delta)$ and $RL(p^{(i)}(t) \cdot J_R, \delta)$, where $J_L \leq J \leq J_R$. To achieve that we can pre-calculate a set of conditional scenarios for a fine grid of scaling parameters $0 > J_1 > J_2 > ... > J_{max}$ and use the approximation to quickly construct the loss distribution during the calibration exercise. Pre-calculation of the grid of conditional scenarios takes some time, especially if the grid size is large, but any further approximations of the recursive loss function are effortless.

### 3.3.4 Market data inputs

Our market data inputs are all dated 4 April 2007. The CDX and iTraxx tranche prices and index tranche reference levels are the J.P. Morgan correlation trading desk's closing prices. For underlying single-name CDS spreads, we have taken basis-adjusted Mark-it data provider CDS quotes (these are consensus average CDS quotes across a large number of market participants), containing CDS spreads for 6 month, 1 year, 2 years, 3 years, 5 years and 7 years IMM tenors. Basis adjustment is in place to ensure that the total expected loss on all tranches (index) is equal to the total expected loss on 125 constituent single-name CDSs. We have followed the market convention and linearly scaled all constituent single-name CDS spreads until the total expected loss implied by the single name CDSs matches that of quoted index reference spread.

To build the distribution of losses through time, we have chosen a 14-day discretisation, while for systemic risk factor — a 128-interval discretisation on $[-5;5]$.

Before we proceed with the calibration of the implied copula to the target tranche prices, we first price a series of 1% thick tranche spreads on [3%, 22%] for iTraxx and [3%,30%] for CDX using the base correlation model (using cubic spline interpolation to interpolate base correlations for non-standard attachment/detachment points). We then use 1% thick tranches combined with a simple Excel solver to solve for a vector of parameters $\{J\}$ such that the following objec-

tive function is minimized:

$$F = \sum_j \left( \frac{ATM_{bc}^{(j)} - ATM_{ic}^{(j)}}{ATM_{bc}^{(j)}} \right)^2 \tag{3.3.8}$$

Here $j$ is the tranche index variable, and $ATM_{ic}$ and $ATM_{bc}$ are the at-the-money spreads for implied copula model and base correlation model respectively. The results of our calibration for iTraxx and CDX index tranche prices can be found in tables 3.1 and 3.2. Thin tranche calibration results are shown in tables 3.3 and 3.4.

## 3.4 Comparative analysis

This section will look at the valuation differences of the implied copula model and the base correlation models. We aim to achieve the following two objectives (i) price a range of thin bespoke CDO tranches with the two models and see if the expected loss adjustment results in a similar CDO tranche price and (ii) compare the single name CDS hedge sensitivities implied by the two models.

### 3.4.1 Bespoke CDO pricing

We start by noting that the chosen way of modelling default dependency implicitly models discrete states of the world where every probabilistic event defines proportional changes to cumulative probabilities of default. For example in a two-state model like that in (3.3.3) with $u=2$ and $d=0.5$, we have two possible states of the world: with conditional probability $p$ cumulative probabilities of default would double and with conditional probability of $(1-p)$ they will all halve (spreads can be approximated using linear transformations of cumulative default probabilities and therefore a similar concept would apply to spread distribution). Our model therefore implicitly includes the concept of so-called expected loss adjustment.

We now turn to comparing bespoke CDO tranche pricing. To make things simple, we have selected the same set of reference names in the credit indices (CDX Series 8 and iTraxx Series 7) and doubled each name's spread. We then price a range of 1% thick tranches, in a similar way to the analysis in the previous section. Because the expected loss adjustment used in the base correlation model would require us to know base correlations for $\alpha/2$ and $\beta/2$ where $\alpha$ and $\beta$ are

the tranche attachment and detachment points, we cannot price tranches attaching below 6% as this would require us to make assumptions about the behaviour of the base correlation curve below 3%. We therefore price a series of 1% thick tranches in the range of 6%-22% for iTraxx and 6-30% for CDX. We then compare the prices implied by the base correlation model with expected loss adjustment and the implied copula model.

Tables 3.5 and 3.6 show the results of this analysis. It's quite clear that the pricing given by both models is similar, but not to a level that the differences can be considered insignificant. The implied copula model attributes more risk to the super-senior tranches than the base correlation model with the expected loss adjustment does (and therefore less risk is attributed to junior tranches).

### 3.4.2 Analysis of hedge sensitivities

Accurate calculation of single-name CDS sensitivities is important in managing the risk on a book of synthetic CDOs. As single-name CDS sensitivities are implied by the model, a CDO trader hedging CDO position must put some faith into the correctness of his hedges, and is therefore vulnerable to the potential model error.

Since the sum of CDO hedges across the entire capital structure is equal to that of the index, such a measure (sum of hedges) must be the same across all models (provided that they are arbitrage-free). Therefore what we are going to analyse is (i) how each model splits the total index hedge notional between its tranche components (senior tranches versus subordinated tranches) and (ii) distribution of index tranche hedge national between individual names in the portfolio (wide spread names versus tight spread names). In the previous section we already saw that the two models produce different prices for bespoke CDO portfolios. As hedge notional is somewhat dependent on the amount of expected loss associated with the tranche, comparing the base correlation model and the implied copula model as applied to a bespoke CDO would not necessarily yield meaningful conclusions. We will therefore concentrate on comparing hedges for the base correlation model and the implied copula model for standard index tranches only (where both models produce similar tranche spreads).

Single-name delta hedge sensitivities can be defined either as the tranche price

change with respect to 1bp CDS spread widening, or as the single-name CDS notional which hedges such widening. In our analysis, we concentrate on the former risk measure to simplify the analysis (in fact, the latter would produce roughly the same result but the output would be on a different scale).

Table 3.7 shows the average absolute delta per name for the base correlation model and the implied copula model. The implied copula model attributes slightly more spread risk to equity tranches (and therefore less risk to senior tranches). Because the largest part of the total index spread risk is concentrated in the equity tranches, in relative terms it means that the hedge notionals for them do not vary much between the implied copula and base correlation models. However, while senior tranches might have small absolute difference in hedge notionals, in relative terms that difference is quite pronounced, especially when expressed as the tranche leverage (the total single-name CDS hedge notional divided by the tranche notional).

Figures 3.1-3.5 show the delta profiles graphically (single-name delta as a function of single-name CDS spread). Practitioner research into the properties of the tranche delta profiles under the base correlation model showed that the base correlation model's tranche delta profile assumes one of two shapes: the equity tranche delta profile is usually an increasing function of the spread, while the mezzanine and the senior tranche delta profiles are hump-shaped. The implied copula model's delta profile is slightly different: the equity tranche delta profile is similar to the base correlation model's delta profile, while the mezzanine and the senior tranche delta profiles decrease monotonically with the spread.

The convexity of the delta profile shows a degree of differentiation between names in the portfolio. Implied copula model's delta profiles are significantly less convex, implying that the implied copula model exhibits significantly less idiosyncratic risk behaviour, i.e. the tranche price sensitivity with respect to wide spread changes is similar to the price sensitivity with respect to a tight spread name changes.

## 3.5 Concluding remarks

The copula modelling approach was studied in the analysis of portfolio credit risk by numerous researchers, to name the few are Embrechts, Lindskog and McNeil (2001) and Schonbucher and Schubert (2001).

Li (2000) introduced the Gaussian copula for CDO valuation, but what is more important, he showed how copula function can be applied to any default modelling effortlessly. His pioneering approach to copula CDO modelling produced numerous follow-up models, such as Frey and McNeil (2003) and Andersen and Sidenius (2005).

Hull and White (2006) empirical copula modelling is a somewhat different approach. Unlike the approach taken by Li (2000), Hull and White (2006) directly specify the series of marginal distributions in such a way that they form a joint distribution and ultimately define a copula.

Chapter 3 proposes an empirical CDO pricing copula model with a setup similar to Hull and White (2006) implied copula model. It differs from Hull and White (2006) in that it is constructed directly using cumulative default probabilities at different time horizons and therefore is much simpler and much easier to calibrate to the observed tranche prices. This is in fact reaffirmed by the later paper by Hull and White (2008), where the dynamic CDO modelling is done directly off the cumulative default probabilities and where a different type of multi-period recombining binomial tree is used to calibrate the model.

We present a number of numerical techniques facilitating the ease of implied copula construction and propose techniques to ensure that the resulting loss distribution is smooth.

We show that even when both the base correlation and the implied copula models are calibrated to the same market data, the implied copula model exhibits significantly less idiosyncratic risk as evidenced by flatter delta profiles. That leads us to conclude that the model choice can have a significant impact on the CDO hedging strategy.

# 4 Reduced form approximation technique in a dynamic CDO model

## 4.1 Introduction

As we have seen in earlier chapters, significant amount of research in CDO modelling has been dedicated to modelling correlated credit portfolios. Early research papers analysed the portfolio credit risk mostly in the context of Value-at-Risk calculations, see for example Embrechts, McNeil and Straumann (2001), Kiesel, Perraudin and Taylor (2003) and Bingham, N., Kiesel, R., Schmidt. R (2003). Further discussions and applications to CDO valuation include Embrechts, Lindskog and McNeil (2003), Frey and McNeil (2003). Applications to CDO include Li (2001), Duffie and Garleanu (2001), Hull and White (2006) and Baxter (2006).

Most of the papers discussing CDO valuation techniques either consider static models or where a dynamic model is developed, the approach taken does not yield tractable results at some intermediate periods in time. In this respect Burtschell, Gregory and Laurent (2005) noted that "*copula models fail to provide satisfactory dynamics of credit spreads and exhibit various kinds of unsatisfactory time instability*".

There are however many areas in finance where the time dynamics of CDO tranche prices is of primary interest. These include, but are not limited to, value-at-risk and counterparty credit risk exposure calculations.

The proposed dynamic portfolio modelling approach to the valuation and modelling of CDO tranches with both trivial and non-trivial cashflow waterfalls follows the spirit of the Duffie and Garleanu (2001) CDO model. While their model simulates stochastic hazard rates (or in other words the dynamics of future pure Poisson process), our model could be seen as a multi-obligor version of the ratings-based credit derivatives pricing models of the type first suggested by Jarrow, Lando and Turnbull (1997) and Kijima and Komoribayashi (1998). Assuming that risk-neutral defaults are driven by a time-homogenous transition matrix we incorporate correlation between defaults by following an ordered probit modelling approach, in the spirit of the credit risk model by Gupton, Finger and Bhatia (1997).

Within our model setup, we are able to derive the reduced-form CDO value approximations, which allow for quick and accurate analysis of CDO price dynamics at different time horizons. Our method of approximating CDO tranche prices resembles Longstaff and Schwartz (2001) least squares approach to valuing early exercise premium in American style equity options.

The proposed model can be applied to CDO structures with strong path dependency as well as in calculation of bank solvency and counterparty credit risk, where tail statistics are often obtained based on a portfolio distribution built using Monte Carlo methods. We show that the reduced form approximation technique presented can be generalised to incorporate non-Gaussian copula models and as an example implement a simplified version of Andersen and Sidenius (2005) random correlation model.

## 4.2 Static versus dynamic CDO models

A typical approach to valuing a synthetic CDO assumes a static model whereby at any moment in time the split of the portfolio total expected loss between different parts of the capital structure is determined by a static dependency structure (referred to as copula).

Static models are often analytically tractable, allowing for relatively quick calculation of risk sensitivities, which are an important tool allowing CDO traders to make hedging decisions. Their implementation is commonly based on the analytical methods to constructing loss distributions that we analysed in Chapter 2.

However, in terms of applicability, static models cannot be used where the path dependence of CDO tranche payoff is present. This generally limits them to the case of a simple synthetic CDO where principal write-downs follow a simplistic notional-based waterfall with no early amortisation features. Path dependency of tranche payoffs is observed in tranches with cashflow waterfalls, and may include realised loss-based or expected loss-based sets of triggers leading to early amortisation of senior tranches thus increasing the risk of subordinated tranches. Gallagher, Gleeson, Kenyon and Litchers. (2009) noted lack of models allowing for path-dependent cashflow waterfalls. They propose an approach to valuing cash-

flow CDOs which is however limited determining the at-the-money coupons on amortising cashflow CDOs, which is coming short of incorporating full path dependency (e.g. such features as non-sequential redistribution of principal following over-collateralisation test failure).

Even if a static CDO model is used to value a cashflow CDO with weak path dependency it is usually still not fast enough to serve a given risk analysis purpose. For example calculation of the counterparty credit risk exposure calculation may require running a large number of Monte Carlo simulations with multiple risk factors. Each such simulation requires re-valuing CDO tranche using conditional risk factor realisations (e.g. spreads, correlations or interest rates), which might be a time-consuming operation.

## 4.3 Ratings–based portfolio model

Before we proceed with the explanation of our reduced-form approximation technique, we first need to describe the basic underlying model, allowing us to simulate dependent defaults using time homogenous Markov chain processes and how to value a CDO tranche within such model setup.

Ratings based models were introduced by Jarrow, Lando and Turnbull (1997) and Lando (1998) and later extended by Kijima and Komoribayashi (1998). All of those authors derived adjustments to the historic transition matrix such that the pattern of default probabilities implicit in the Markov chain process mimics that of the bond spreads, resulting in time-varying risk-adjusted transition matrices. Our approach differs in that it assumes a time-homogenous risk-adjusted transition matrix, which simultaneously fits the term structure of spreads for a set of tenors and rating buckets. The "Risk-neutral transition matrix calibration" section in the chapter will explain how such transition matrix is calibrated to CDS spreads.

We start by assuming that the single-period risk-neutral default probabilities for each obligor in the CDO portfolio are driven by a homogenous Markov chain process with the transition matrix $A=\{a_{i,j}\}$. The transition matrix consists of $s=1..D$ states, where state $s=1$ corresponds to the state whith the lowest default probability and $s=D$ corresponds to the "in default" state, which is also an absorbing state.

Consider a portfolio of $N$ obligors. Let each obligor in the portfolio be assigned its initial state $s_i(0)$ and let the $t$-period state be denoted $s_i(t)$ where $t$ can assume values in $[0..T]$. To model state transitions for obligor $i$, let us assume that the obligor's state $s_i(t+1)$ is determined by a realisation of a normally distributed latent variable $z_i \sim N(0,1)$.

Define a matrix of cutoff points $C = \{c_{i,j}\}$ such that:

$$c_{i,j} = \Phi^{-1}\left(\sum_{k=1}^{j} A_{i,k}\right) \tag{4.3.1}$$

$$c_{i,0} = -\infty$$

For a random realisation of a latent variable $z_i$ the state $s_i(t+1)$ is determined by such index $k$ that:

$$c_{s(t),k-1} < z_i < c_{s(t),k} \tag{4.3.2}$$

To introduce dependency between obligors' state transitions which will ultimately determine the dependency between defaults, let us assume that a vector of individual obligors' latent variables is normally distributed $Z \sim N(0,1,\rho)$ where $\rho$ is a constant pair-wise correlation between each pair of latent variables.

The state transition modelling will therefore involve the following steps:

- for each time grid point, draw a vector of normally distributed correlated random variables $Z = \{z_i\}$.
- based on the previous time grid period's state $s_i(t-1)$, determine the new obligor state $s_i(t)$, based on (4.3.2)
- update obligor state indicators and repeat the same procedure for next time period until maturity $T$ is reached.

More complex correlation structures could be introduced by assuming that the vector of latent variables follows multivariate normal distribution with correlation matrix $\Omega$, for more details see Gupton, Finger and Bhatia (1997) and Peretyatkin and Perraudin (2004).

## 4.4 CDO valuation

Consider a simple synthetic CDO tranche with attachment point $\alpha$ and detachment point $\beta$. Suppose $M$ Monte Carlo simulations to final maturity $T$ were run according to (4.3.2) and consider some Monte Carlo simulation $m$, where $m \in [1..M]$.

The cumulative portfolio loss for each period can be calculated using:

$$\Theta(m,t) = \sum_{i=1}^{N} \omega_i \cdot (1-\delta) \cdot I\Big(s_i(t) = D\Big) \tag{4.4.1}$$

Here $\omega_i$ is the weight of obligor $i$ in the portfolio and $I(\cdot)$ is the indicator function assuming value 1 when the argument evaluates to *true* and 0 otherwise.

For a Monte Carlo simulation index $m$ and a simulation horizon $t$, the tranche cumulative loss is:

$$\Delta(m,t) = \frac{\min\Big(\max\big(\Theta(m,t)-\beta, 0\big), \beta-\alpha\Big)}{\beta-\alpha} \tag{4.4.2}$$

Let's assume that the discount factor for period $T$ is $B(t)$. The present value of the total tranche expected loss is:

$$P(m) = \sum_{t=1}^{T} \big(\Delta(m,t) - \Delta(m,t-1)\big) \cdot B(t) \tag{4.4.3}$$

And the CDO tranche price is then the average expected loss across $M$ Monte Carlo simulations:

$$P(\alpha,\beta) = \frac{1}{M} \cdot \sum_{m=1}^{M} P(m) \tag{4.4.4}$$

The formulae above are not restricted to a synthetic CDO waterfall but can easily be extended to cashflow waterfalls of any complexity, such as CLO or ABS. For simplicity of analysis, in this paper we will stick to the synthetic CDO waterfalls only.

## 4.5 Reduced–form approximation technique

In a Monte Carlo model where risk-neutral (or historic) rating transitions are simulated to some intermediate time horizon $\tau < T$, there is often a need to calculate the price of a CDO based on the time $\tau$ simulated CDO portfolio characteristics.

Such calculations arise for example in the calculation of value-at-risk statistics or counterparty future exposure risk profiles.

In the absence of analytical approximation to the CDO tranche value there is need to run a nested Monte Carlo simulation working out the tranche average discounted payoff from some intermediate time horizon $\tau$ to CDO maturity $T$. Such nested Monte Carlo simulation is obviously computationally intensive and alternative CDO value approximation techniques are sought.

Our approach to approximating CDO tranche value is similar in nature to the Longstaff and Schwartz (2001) approach to valuing early exercise premia for American-style equity options.

To explain our technique, consider a Monte Carlo simulation analysis where state transitions are simulated using a risk-neutral transition matrix $A$ to CDO maturity $T$ and where for each Monte Carlo simulation we calculate the time $\tau$ present value of the realised tranche loss $L$. In this simulation, we can observe two types of payoffs: (i) if the portfolio loss is less than the tranche subordination $\alpha$ then the tranche loss will be 0 and (ii) if the portfolio loss is greater than the tranche subordination $\alpha$ the tranche loss will be some quantity $0<\delta\leq 1$.

A simple way of relating expected tranche payoff $L$ to the underlying portfolio characteristics vector $\{X\}$ is by regressing $L$ on $X$ in the following way:

$$L=\gamma\cdot X \tag{4.5.1}$$

While very simple and quick to calibrate this method would work reasonably well if the payoff function is linear in $X$. Non-linearity itself is not a problem for simple regression, as non-linear transformations of vector $\{X\}$ could be added to the vector of explanatory variables to account for non-linearity. That, however is likely to increase the risk of over-fitting and may result in poor out-of-sample fit.

As an alternative to simple regression we can consider modelling CDO tranche payoff using logistic regression. Let variable $q^*(X)$ denote the probability of losses in the portfolio exceeding $\alpha$, and let the tranche loss conditional on portfolio losses being greater than $\alpha$ be some quantity $\delta(X)$. In this case the CDO expected loss can be expressed as:

$$P=\big(1-q^*(X)\big)\cdot 0+q^*(X)\cdot\delta(X) \tag{4.5.2}$$

Given restriction $0 \leq q^* \leq 1$, we assume that the probability of losses exceeding the attachment point $\alpha$ is a logistic regression on a vector of explanatory variables $\{X\}$:

$$q^*(X) = \frac{\exp(\beta X)}{1+\exp(\beta X)} \tag{4.5.3}$$

Moreover, we assume that, conditional on total portfolio loss being greater than tranche subordination $\alpha$, the tranche recovery $\delta(X)$ is a linear regression on a vector of explanatory variables $\{X\}$:

$$\delta = \varphi \cdot X \tag{4.5.4}$$

We can calibrate the logistic regressions in (4.5.3) using the maximum likelihood estimation. To achieve this, we maximise the following log-likelihood function numerically:

$$LL = \sum_{m=1}^{M} \ln\Big(1+\exp(\beta X)\Big) \cdot I\Big(P(m)>0\Big) + \Big(\beta X + \ln(1+\exp(\beta X))\Big) \cdot I\Big(P(m)=0\Big) \tag{4.5.5}$$

Calibration of (4.5.4) can be carried out using a conditional regression. One way of achieving this is to select simulations where the tranche loss $P(m)$ is strictly greater than zero and estimate $\delta$. by regressing non-zero tranche losses on the vector of independent variables $\{X\}$. This approach, however, may not be accurate for senior tranches, where the percentage of instances in which a CDO tranche suffers principal losses is small.

Consider the pricing function in (4.5.2) again. Let's assume that for each Monte Carlo simulation in our calibration sample, we know $q^*(X)$. The optimal calibration would therefore be to minimise the sum of squared residuals between the projected and realised payoff solving for the following least squares problem:

$$\min \sum_{m=1}^{M} \Big(\overline{P(m)} - q^*(m) \cdot (\gamma \cdot X(m))\Big)^2 \tag{4.5.6}$$

As we don't have the $q^*(X)$, we suggest approximating the unknown vector $\{q^*\}$ with the estimated probabilities given by equation (4.5.3) therefore looking for a solution to minimize the following objective function:

$$\min \sum_{m=1}^{M} \left(\overline{P}(m) - \frac{\exp(\overline{\beta} \cdot X(m))}{1+\exp(\overline{\beta} \cdot X(m))} \cdot (\gamma \cdot X(m))\right)^2 \tag{4.5.7}$$

Other basis functions relating the tranche payoff $L$ to the vector of explanatory variables $X$ can be assumed. We consider one other such basis function to be probit regression. With a setup similar to logistic regression, we model the equation (4.5.2) instead as:

$$q^*(X) = \Phi(\theta \cdot X) \tag{4.5.8}$$

Similarly to logistic regression we calibrate (4.5.8) by numerically maximising the following log-likelihood function:

$$LL = \sum_{m=1}^{M} \ln\Big(\Phi(\beta X)\Big) \cdot I\Big(P(m>0)\Big) + \Big(1-\Phi\big(\beta X\big)\Big) \cdot I\Big(P(m)=0\Big) \tag{4.5.9}$$

## 4.6 Choice of explanatory variables

A range of different variables could be used to relate the payoff functions in (4.5.3), (4.5.4) and (4.5.8) to the underlying portfolio characteristics. The choice of variables would depend somewhat on the degree of complexity of CDO modelling. If a single risk-neutral transition matrix is used for all obligors, one could easily use the percentages of the portfolio notional in different transition states as state variables.

As our analysis is more general and assumes multiple sector-specific transition matrices, a given rating category no longer implies a single definitive measure of risk. To overcome this problem, we adapt the expected loss bucketing technique where the default leg of a CDS in (4.7.4) is mapped to a deterministic expected loss grid. To understand the bucketing approach let $L_1 < L_2 < ... < L_n$ be a strictly increasing pre-defined expected loss grid, and let $V_1, V_2, ..., V_n$ be our portfolio expected loss buckets. Consider a reference entity $i$ with a simulated rating $r$ and risk-adjusted transition matrix $A$. The expected loss $EL_{i,r}$ for this name is given by:

$$EL_{i,r} = \sum_{t=1}^{T} \Big(A^t(r,D) - A^{t-1}(r,D)\Big) \cdot \frac{1}{2}(B_{t-1} + B_t) \cdot (1-\Delta) \tag{4.6.1}$$

The expected loss here is simply a discounted expectation of future losses implied by the risk-adjusted transition matrix and incorporates the following characteristics of the reference entity risk profile (i) the expected timing of losses (ii) loss frequency and (iii) loss severity.

Assuming some pre-defined expected loss grid $\{L_i\}$ and the expected loss vector $\{V_i\}$, we will iterate through each obligor in the portfolio, finding two bracketing expected loss buckets in the expected loss grid such that:

$$L_k < EL_{i,r} < L_{k+1} \tag{4.6.2}$$

The two corresponding expected loss buckets are subsequently incremented using linear fractions of the name's expected loss $L_k$:

$$\begin{aligned} V_k &= V_k + \frac{L_{k+1} - EL_{i,r}}{L_{k+1} - L_k} \cdot EL_{i,r} \\ V_{k+1} &= V_{k+1} + \frac{EL_{i,r} - L_k}{L_{k+1} - L_k} \cdot EL_{i,r} \end{aligned} \tag{4.6.3}$$

## 4.7 Performance analysis

### 4.7.1 Market data inputs and portfolio

Our portfolio consists of 125 reference entities (similarly to traded index portfolios such as iTraxx or CDX), equally split between five sectors: consumer goods, consumer services, financials, industrials and government. In terms of credit quality, the portfolio is going to be mainly investment grade with a small percentage of high-yield names (below BBB-). The recovery rate is assumed to be 40%, which is an industry standard assumption for investment-grade senior unsecured debt.

Portfolio composition

| | Consumer Goods | Consumer Services | Financials | Industrials | Government | Sub-total |
|---|---|---|---|---|---|---|
| BBB | 10.40% | 10.40% | 10.40% | 10.40% | 10.40% | 52.00% |
| BBB– | 8.00% | 8.00% | 8.00% | 8.00% | 8.00% | 40.00% |
| BB | 1.60% | 1.60% | 1.60% | 1.60% | 1.60% | 8.00% |
| Sub-total | 20.00% | 20.00% | 20.00% | 20.00% | 20.00% | |

Our analysis will be similar to the analysis on capital charges for asset backed securities described in Peretyatkin and Perraudin (2004), namely we are going to simulate historic rating transitions to the one-year simulation horizon following that the rating transitions are going to be simulated on a risk-adjusted basis to the remaining four years CDO maturity. The idea of this kind of analysis is to consider a combination of actual losses (due to defaults eating into the tranche notional) and mark-to-market losses on a CDO tranche at a pre-defined investment horizon.

The historical transition matrix used is the all-obligor Moody's transition matrix for 1983-2007.

We further assume 25% default correlation for historic rating transition dynamics, which is a conservative estimate of default correlation in the times of market disruption. The synthetic correlation for modelling rating transitions after the simulation horizon will be 50%, which is the approximate equity tranche correlation implied in traded index tranches at the time.

The CDO tranche structure will mimic that of the CDX index, having the following six tranches covering the entire capital structure of the CDO portfolio: 0-3%, 3-7%, 7-10%, 10-15%, 15-30%, 30-100%. The choice of tranches in this analysis is purely arbitrary and any set of tranches (including incomplete or overlapping CDO capital structures) can be chosen.

### 4.7.2 Risk–neutral transition matrix calibration

**4.7.2.1.** ***Risk-neutral transition matrix calibration technique used*** The time homogeneity property of the risk-neutral transition matrix implies that the multi-period transition probabilities are defined by the last column of the $t$-power of the single period transition matrix $A$. Effectively we have a problem of inferring an unobserved transition matrix given knowledge about its properties and selective information about some elements of it. The three key properties of the transition matrix we require are:

(i) to be a transition matrix, the sum of probabilities in each row should be equal to 1

(ii) to be a probabilistic matrix, each element should be between 0 and 1

(iii) default state is absorbing, i.e. all off-diagonal elements in the last row are 0 (this a typical assumption in modelling defaultable claims, however for certain models multiple emergence from default could be allowed, in which case the last row in the transition matrix will be different)

The elements we "observe" are target default probabilities being the elements in the last column of the $t$-power of the transition matrix, $\{t = 1..N\}$. These probabilities are cumulative and can be bootstrapped from liquid defaultable instruments such as bonds or CDS.

A direct approach to fitting such a transition matrix involves setting up a minimization routine of the following kind:

$$\min F\left(\sum_{t=1}^{T}\sum_{r=1}^{m-1} a_{r,D}^{t} - \delta_{r,t}\right) \tag{4.7.1}$$

*subject to*

$$0 \leq a_{i,j} \leq 1,$$

$$\sum_{j=1}^{m} a_{i,j} = 1$$

While it is seemingly easy to formulate the required set of restrictions, it's not that straightforward to implement them in practice. Restrictions in (4.7.1) state that each variable is range-bound and in addition has restrictions imposed by the total sum of variables in each row, and so the relationship is somewhat circular.

Range bound restrictions on each variable are straightforward to implement, while the second restriction can be incorporated via rescaling of the elements in the transition matrix such that the sum of elements in each row is always equal to 1. That however leads to numerical instabilities as variables must be "free" for optimisation routine to operate on, i.e. they are only meant to change when the optimisation routine decides and so this solution is a no-goer unfortunately.

Another possible solution is to define one of the variables (say diagonal element) to be residual, i.e. equal to 1 minus sum of all other elements in the row and restrict changes for each variable locally to be such that the residual element does not tip it over 1 or below 0, such approach however requires having dynamic restrictions on each variable which again, may confuse the optimisation routine and lead to numerical instabilities of different kind (cross-derivatives in some cases might be impossible to calculate and therefore continuity of cross-derivatives will not be guaranteed). The only workable solution that we found to work is to indirectly enforce the rows to sum to 1 by imposing an extra high penalty to the objective function each time the sum of elements in any row deviates from 1 — this forces the optimisation routine to adjust for changes in one of elements by subtracting from or adding to other elements in the same row, however such penalty slows the algorithm significantly and still does not provide a 100% guarantee that the resulting matrix will be truly transitional in the probabilistic sense.

We however found it much easier to set up a nested search routine in the following way:

- global search is finding a solution to a series of local searches and stops when either the maximum number of iterations or the desired accuracy is achieved
- local search finds a solution to a locally linearlised version of (4.7.1) subject to *locally constant* restrictions. The local search operates in small increments so as to minimize any higher-order effects (the objective function is highly non-linear in its arguments therefore it is important that any incremental changes in the local search are small).

To understand how the local search is set up, let $\overline{A}$ be the objective risk-adjusted transition matrix and let $A$ be the current estimate of $\overline{A}$. The difference matrix $E$ is defined as:

$$E \equiv \overline{A} - A \tag{4.7.2}$$

We can write the equation (4.7.1) in the following form:

$$\begin{cases} (A+E)_{col(D)} = \delta_{col(1)} \\ (A+E)^2_{col(D)} = \delta_{col(2)} \\ \ldots\ldots\ldots\ldots\ldots \\ (A+E)^T_{col(D)} = \delta_{col(T)} \end{cases} \tag{4.7.3}$$

Here col(.) is an operator selecting a column from matrix. Let's expand the power function in (4.7.3) which should give us the following set of simultaneous equations:

$$\begin{cases} (A+E)_{col(D)} = \delta_{col(1)} \\ (A^2+EA+AE+E^2)_{col(D)} = \delta_{col(2)} \\ \ldots\ldots\ldots\ldots\ldots \\ (A^T+EA^{T-1}+AEA^{T-2}+\ldots+EAE^{T-2}+AE^{T-1}+E^T)_{col(D)} = \delta_{col(T)} \end{cases} \tag{4.7.4}$$

Now let's assume that matrix $E$ is small and therefore $E^m A^T E^n \approx 0$ for any $m+n>1$. We then have the following restricted linear problem:

$$\begin{cases} (A^0 E A^0)_{col(D)} = P_{col(1)} - A_{col(D)} \\ (A^0 E A^1 + A^1 E A^0)_{col(D)} = P_{col(2)} - A^2_{col(D)} \\ (A^0 E A^2 + A^1 E A^1 + A^2 E A^0)_{col(D)} = P_{col(2)} - A^3_{col(D)} \\ \ldots\ldots\ldots\ldots \\ (A^0 E A^{T-1} + A^1 E A^{T-2} + \ldots + A^{T-1} E A^0)_{col(D)} = P_{col(T)} - A^T_{col(D)} \end{cases} \tag{4.7.5}$$

or alternatively:

$$\begin{cases} A^0 E A^0_{col(D)} = P_{col(1)} - A_{col(D)} \\ (A^0 + A^1) E (A^1 + A^0)_{col(D)} = P_{col(2)} - A^2_{col(D)} \\ (A^0 + A^1 + A^2) E (A^2 + A^1 + A^0)_{col(D)} = P_{col(2)} - A^3_{col(D)} \\ \ldots\ldots\ldots\ldots \\ (A^0 + A^1 + \ldots + A^{T-1}) E (A^{T-1} + \ldots + A^2 + A^1)_{col(D)} = P_{col(T)} - A^T_{col(D)} \end{cases} \tag{4.7.6}$$

It's now easy to see how the system of simultaneous equations is constructed and that to calculate all these equations it's only necessary to calculate $T$ subsequent powers of matrix $A$ per each local search iteration, which is important bearing in mind that, generally speaking, the objective function is expensive to calculate.

It is tempting to try and simply solve for $E$ using a single local iteration and call the job done, however one should not forget that while the distance between the current estimate of $A$ and target matrix $\overline{A}$ may be substantially large, the equation above only holds if such difference is small. In other words — in the current local search iteration we are only allowed to approach the solution in a small incremental step and the global iteration is in place to get as close to the true solution as possible in $M$ such small steps.

Let the local step size be some quantity $\lambda$, where $\lambda$ is some small number such as 0.01. Our local optimisation is therefore to find the solution to the following

restricted linear problem:

$$\begin{cases} \left(\sum_{t=0}^{0} A^t\right) E \left(\sum_{t=0}^{0} A^t\right)_{col(D)} = P_{col(1)} - A_{col(D)} \\ \left(\sum_{t=0}^{1} A^t\right) E \left(\sum_{t=0}^{1} A^t\right)_{col(D)} = P_{col(2)} - A^2_{col(D)} \\ \left(\sum_{t=0}^{2} A^t\right) E \left(\sum_{t=0}^{2} A^t\right)_{col(D)} = P_{col(2)} - A^3_{col(D)} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \left(\sum_{t=0}^{T-1} A^t\right) E \left(\sum_{t=0}^{T-1} A^t\right)_{col(D)} = P_{col(T)} - A^T_{col(D)} \end{cases} \tag{4.7.7}$$

subject to the following constant restriction imposed on the elements of matrix $E$:

$$-\min\left(\lambda_i, a_{i,j}\right) \leq \varepsilon_{i,j} \leq \min\left(\lambda_i, 1 - a_{i,j}\right) \tag{4.7.8}$$

We set the diagonal element in the matrix $E$ be equal to 1 minus sum of all off-diagonal elements, therefore ensuring that rows in matrix $E$ always sum to 0 and therefore the next step solution matrix $A$ retains its transition matrix properties. One last restriction we impose is the maximum step size for each row so that for row $i$:

$$\lambda_i = \min\left(\lambda, \frac{a_{i,i}}{D-1}\right) \tag{4.7.9}$$

The above restriction ensures that the diagonal element $a_{i,i}$ in matrix $A$ is always large enough to absorb the scenario where all elements in row $i$ of difference matrix $E$ going up by the maximum amount $\lambda_i$ (and therefore the diagonal element of $E$ becoming $-\lambda_i(D-1)$ which we want to be floored at the size of the corresponding element $a_{i,i}$ of the current estimate of $A$).

To estimate the transition matrix using the above described approach, we therefore need to start with some initial guess matrix $A^{(ig)}$ (the only condition on such matrix being that it should fulfil the transition matrix criteria) and perform a number of operations on this matrix through global search algorithm: $A^{(g)} = A^{(ig)} + E^{(1)} + E^{(2)} + \ldots + E^{(g)}$, namely let's assume that we reached some global iteration step $g$:

— calculate linear equation coefficients using $A^{(g)}$ and equation (4.7.7)

— define local search restrictions as per (4.7.8)

— run restricted optimisation to find the best fit solution to the local search solution matrix $E^{(g)}$ : such matrix imperfectly solves the system of simultaneous linear equations defined in (4.7.7) and adheres to the set of restrictions defined in (4.7.8)

— calculate next global step estimate as $A^{(g+1)} = A^{(g)} + E^{(g)}$

— if the global search criteria is not fulfilled after step $g$, repeat the steps above, otherwise stop and return $A^{(g+1)}$.

Calibration technique using the above described approach is fairly robust — starting with the identity matrix as the initial guess and using $\lambda$ of 5%, we typically find the solution in less than 20 global iterations. Each local iteration requires a relatively large number of steps, typically between 50 and 100, however each such local search calculation is very quick, so the entire fitting procedure takes less than 1 minute on an average specification PC.

**4.7.2.2. *Risk-neutral transition matrix calibration: market data inputs***

As a source of risk-neutral prices of credit risk, we have chosen CDS spreads for the most liquidly traded reference entities. Mark-it data provider produces so-called composites by convention CDS spreads by analysing a number of major dealers' end-of-day CDS prices. The spreads are at-the-money CDS spreads for a set of tenors covering a number of sectors, among which we are primarily interested in consumer goods, consumer services, financials, industrials and government.

For a chosen reference date of 24 March 2010 there were a total of 1751 relevant reference entity rows of data, each containing CDS spreads for IMM roll maturities of IMM 1y, IMM 2y.. IMM 10y. The IMM roll date specifies maturity for a CDS contract given a CDS trade date. For a known trade date the maturity is determined by finding the nearest *next* 20-March, 20-June, 20-September, 20-December date. As the reference date (which is an assumed trade date for the Mark-it CDS spreads consensus dataset) is 24 of March, this is rolled to the 20-June (as 24 March is after 20 March), so a 2y CDS contract maturity would be defined as 20-June-2012, i.e. have an end-maturity of approximately two years and three months (and not two calendar years). CDS spreads are quoted in annualised terms but are paid quarterly on each IMM date.

The additional information pertinent to our analysis is rating, sector and expected recovery rate, which is contained in the Mark-it data file. While both the sector name and the expected recovery rate are available for every obligor, rating information is only available for about 80% of data points. Where no rating is available we use the average rating category instead, which is the average consensus rating (rather than that assigned by S&P or Moody's).

To calibrate sector-specific risk-adjusted transition matrices we need the cumulative risk- neutral default probabilities for each sector, rating category and maturity bucket which are to be bootstrapped from CDS spreads. Our bootstrap technique is based on a simple but accurate piece-wise constant hazard rate model. Let $\tau_1, \tau_2, ..., \tau_T$ be CDS tenors corresponding to the observed CDS spreads $\{C_\tau\}$.

Let $h_i$ be the instantaneous piece-wise constant default probability for some period $[\tau_i, \tau_{i+1}]$. Conditional on surviving by $\tau_i$, the constant hazard rate model implies the following equation for the conditional default probability at $\tau_{i+1}$:

$$\overline{F}(\tau_i, \tau_{i+1}) = \int_{\tau_i}^{\tau_{i+1}} h_i \cdot e^{-h_i \cdot x} \cdot dx \tag{4.7.10}$$

Therefore for the first tenor $\tau_1$ the unconditional cumulative default probability is the same as the conditional default probability in (4.7.10) and for the subsequent tenors, the unconditional default probabilities could be calculated iteratively as:

$$F(\tau_{i+1}) = \big(1 - F(\tau_i)\big) \cdot \int_{\tau_1}^{\tau_2} h_2 \cdot e^{-h_2 \cdot x} \cdot dx \tag{4.7.11}$$

The bootstrapping technique involves the iterative calculation of the vector of piece-wise constant hazard rates $\{h_\tau\}$ such that the payment leg (risky discounted CDS spread) is equal to the loss leg (discounted losses defined by the vector of hazard rates):

$$\sum_{i=0}^{T} \int_{\tau_i}^{\tau_{i+1}} dF(t) \cdot B(t) \cdot (1 - \Delta) = \sum_{i=0}^{T} \int_{\tau_i}^{\tau_{i+1}} \big(1 - F(t)\big) \cdot C_\tau \cdot B(t) \cdot dt \tag{4.7.12}$$

Here $B(t)$ is a pure discount bond price for some time $t$, and $\Delta$ is the expected recovery rate.

Once the piece-wise constant hazard rates for each obligor have been bootstrapped, we can calculate the risk-neutral cumulative default probabilities for a set

of fixed annual maturities, $t=1,2..T$ by evaluating equation (4.7.11) at $t=1,2..T$.

Finally, we average the default probabilities for each rating category and tenor (sector- specific) to arrive at a set of target cumulative default rates $P\{r,t\}$ per rating and maturity bucket. They are then used in equation (4.7.1) to calibrate sector-specific risk-adjusted transition matrices.

### 4.7.3 Calibration sample and expected loss grid

Calibration of reduced-form approximation function parameters for simple regression, logistic regression and probit regression is done on a random sample of 500,000 Monte Carlo simulations (this sample will be referred to as the *calibration sample*). For every Monte Carlo simulation in our calibration sample, we store the realised expected loss vector and realised loss for each CDO tranche to be used in least squares, logistic and probit reduced form approximation functions calibrations. We choose a simple exponential expected loss grid such that for some scaling parameter $k \in (0,1]$ and decay parameter $\alpha \in (0,1)$ the expected loss grid forms the following vector:

$$L_1 = k \cdot \alpha^n, \;\; L_2 = k \cdot \alpha^{n-1}, \;\; ..., \;\; L_n = k \cdot \alpha^0 \tag{4.7.5}$$

It is important that the expected loss for each reference entity in the portfolio is always bracketed by the expected loss grid. To ensure that this condition is met, we need to have an idea of the minimum and maximum expected loss in the portfolio. We can find the minimum expected loss $EL_{min}$ in the portfolio by iterating through each obligor and evaluating the expected loss to four year maturity. The maximum expected loss is known — the expected loss for an in-default reference entity at the simulation horizon would simply be the loss given default, i.e. $1-\Delta$ (or $1-\Delta_{min}$ when using different recoveries for different names in the portfolio). We can therefore solve for $\alpha$ and $k$ using the following two equations:

$$\begin{aligned} k &= 1-\Delta \\ \alpha &= \left(\frac{EL_{min}}{1-\Delta}\right)^{\frac{1}{n}} \end{aligned} \tag{4.7.6}$$

We choose a 14-variable expected loss grid with the constant term added as an extra variable. Using the stored set of expected loss vectors and the realised tranche losses, we calibrate simple regression, logistic and probit model parameters as in (4.5.2), (4.5.1) and (4.5.8).

### 4.7.4 Out–of–sample testing

To assess the accuracy of the three approximations being analysed, we need a range of random portfolios for which the true CDO values are known. Ideally we need a large set of such test portfolios in order to be able to analyse the statistics of the pricing function approximations, such as for example the mean squared error of the approximation function.

We construct 10,000 such random portfolios by running yet another Monte Carlo simulation, using a different pseudo-random sequence to a one-year horizon (similar to constructing the calibration sample, except with shorter maturity). In this Monte Carlo simulation we store (i) realised portfolio characteristics (ratings for each reference entity in the portfolio) and (ii) realised expected loss grid as per (4.7.5). We then run 100,000 nested Monte Carlo simulations to the remaining 4Y CDO maturity for each of the 10,000 random portfolios using the previously stored portfolio rating distributions as starting points for the nested Monte Carlo simulations. The average payoff for each tranche gives the estimate of the true CDO value. The stored realised expected loss vector is used to calculate the reduced-form approximations functions in (4.5.2) and (4.5.1) which are then compared to the obtained true values.

The true value statistics are show in table 4.11. Please note that throughout this analysis we think of CDO tranche value as representing the loss leg only. For example the equity tranche value of 0.7061 implies the expected loss of 0.7061 units of CDO tranche notional for each unit of notional held (and so in the absence of coupon payments, the cash value of a CDO tranche as a bond would be $1-0.7061=0.2939$).

### 4.7.5 Degree of non–linearity of CDO pricing functions

To get an idea of how many (if any) additional variables are needed to account for the non-linearity of CDO tranche payoff, we first calibrate a simple regression model in (4.5.4) using only linear regressors (and a constant). We then add extra variables (non-linear transformation of the initial set of variables) to our expected loss grid and recalibrate the model measuring the out-of-sample root mean squared error.

We consider adding the following additional variables: (i) squared regressors (ii) expected loss dispersion, calculated as $\frac{1}{N} \cdot \sum (EL_i - \overline{EL})^2$. The results of this analysis are presented in table 4.12. Interestingly, pretty much all of the non-linearity is captured by the dispersion parameter, which is a model with $N+1$ variables versus the model where the number of variables is $2 \cdot N$.

The intuition behind the importance of the dispersion parameter lies in the following example. Consider an investor choosing to invest in one of the two equity tranches, both having the same thickness but referencing different portfolios. Both portfolio $A$ and portfolio $B$ have the same average expected loss, but whereas in portfolio $A$ all obligors are equal to each, portfolio $B$ contains a few heavily distressed names offset by the remaining better-than-average names. The dispersion for portfolio $A$ is therefore 0 while the dispersion for portfolio $B$ is some positive number.

Independent of the assumed correlation between reference entities in the portfolio it is not difficult to see that the portfolio $B$ equity tranche will attract more risk than the portfolio $A$ equity tranche. This is because the equity tranche is associated with the risk of the few lowest credit quality names and not the average risk of the portfolio.

In addition to adding the expected loss dispersion variable we have also analysed the performance of the reduced form approximation using higher moments of expected loss distribution such as skewness. Those variables however do not seem to add significant explanatory power beyond that of the expected loss dispersion.

### 4.7.6 Logistic/simple regression/probit model comparison

Having established the choice of independent variables we are now going to compare different basis functions in (4.5.2), (4.5.1) and (4.5.8) by measuring the mean squared error of the difference between the true value and the approximation.

We analyse the mean squared error for the entire sample as well as for the left and right tails of the tranche value distribution by selecting a sub-sample of the top 10% and bottom 10% of CDO tranche values. The results of this analysis are presented in tables 4.13–4.15. As can be seen from the root mean square deviation estimates, the logistic regression generally performs better than simple regression. Probit regression tends to outperform the simple regression but is somewhat less accurate than logistic regression.

### 4.7.7 Generalisation to other dynamic models

We mentioned earlier in this chapter that our approach to approximating the CDO tranche value could be extended to accommodate other non-Gaussian modelling techniques. For example, the Duffie and Garleanu (2001) approach to modelling stochastic hazard rates could be easily implemented within our framework without any major changes to the basic set-up.

As an additional exercise we implement a model similar to that proposed by the Andersen and Sidenius (2005) by allowing the latent variable correlation parameter in the risk-neutral measure to be random. Let's take a simple case where there are four discrete correlation states, with the first three states representing low, medium and high-correlation environments and the last state representing a global jump event where a large number of obligors in the portfolio default simultaneously:

Assumed correlation scenarios

| Correlation value | Probabilistic weight |
|---|---|
| 10% | 0.2 |
| 40% | 0.3 |
| 60% | 0.4 |
| 100% | 0.1 |

Using the same portfolio as in the previous analysis, we compare the true val-

ues obtained using a nested Monte Carlo to those obtained using the reduced form approximation function. Table 4.17 shows the results of reduced form regression approximation functions, which mimic the accuracy of the reduced form approximation approach in the single factor Gaussian model.

### 4.7.8 Calculation of risk sensitivities

A number of risk sensitivities with respect to parameter changes can be analytically derived from the derived reduced form approximation functions (by taking partial derivatives with respect to such parameters). This is however restricted to those factors which are assumed to be dynamic within the underlying model, (e.g. credit spread changes) and excludes factors assumed to be constant (e.g. correlation in a single-factor Gaussian model).

Where risk sensitivities cannot be derived because the reduced form approximation technique does not explicitly depend on them, this can be solved by allowing those parameters to change dynamically (even if such changes are small). Parameters being dynamic is not however equivalent to parameters being stochastic, for example the correlation in the Andersen and Sidenius (2005) models is stochastic but not dynamic, therefore the conditional function in equation (4.5.2) would not include conditional realisation of correlation distribution since such distribution remains constant through time (in the distribution sense).

## 4.8 Concluding remarks

While numerous research papers offered different copula-based CDO models, only the Longstaff and Schwartz (2001) model and a vintage of recent papers by Laurent et al. (2008), Longstaff and Rajan (2008) and Hull and White (2008) consider dynamic CDO modelling therefore branching out from the conventional copula modelling on which the majority of research to date is based. Dynamic CDO models are inevitably more complex however they provide intuition into CDO value dynamics over time, which is important in calculation of risk measures at intermediate time horizons such as Value-at-Risk or counterparty credit risk statistics. The analytical tractability of dynamic CDO models however is often limited to the case where all obligors in the portfolio are assumed to be the same, i.e. a

homogeneous portfolio case is considered. Realistic portfolios typically feature significant heterogeneity of credit spreads therefore making such dynamic modelling an approximation at best.

Chapter 4 introduces a dynamic CDO model based on homogenous Markov chain process. Its key contribution is in the reduced form pricing technique similar to Longstaff and Schwartz (2001) least squares estimator approach. The proposed technique (i) allows to analyse CDO values at intermediate time horizons and (ii) can handle non-trivial cashflow waterfalls beyond synthetic CDOs.

We analyse the properties of different basis functions and propose the expected loss bucketing technique. The generic nature of our approach implies that it can be applied in different CDO modelling frameworks and is not limited to the dynamic CDO model described. To demonstrate how it can be extended to non-Gaussian copula models, we implement a simplified version of Andersen and Sidenius (2005) random factor loading model.

# 5 Conclusion

This thesis presents CDO valuation and risk analysis viewed from three different modelling perspectives: (i) as a structural model (ii) as a non-parametric implied copula model and (ii) as a reduced form approximation model.

The structural CDO model in chapter 2 provides an alternative to the recursive loss model proposed by Andersen and Sidenius (2003). It uses the basic set up of the Vasicek homogenous pool loss distribution combined with two types of derived loss distribution adjustments. The loss distribution approximation produced by the proposed model is more accurate than the Vasicek homogenous pool model while still being faster than the recursive loss approach. Using the derived semi-analytical adjustment to the homogenous portfolio distribution we have shown how portfolio credit risk dispersion affects tranches of different seniorities.

The non-parametric CDO copula model presented in chapter 3 resembles the implied copula of Hull and White (2006). Our implied copula is much simpler than that of Hull and White (2006) model and can be easily calibrated to index tranche prices such as CDX or Itraxx. The implied copula model proposed is based on the binomial expansion technique as a method of building marginal probability distributions. As an exercise we calibrated the implied copula to the iTraxx and CDX index tranches and compared bespoke CDO tranche prices and risk sensitivities between the base correlation model and the implied copula model. We have concluded that even when the two models are calibrated to the same set of CDO tranches, they differ significantly in the way they attribute portfolio risk changes to different tranches as evidenced by differences in delta profile shapes.

The last chapter presented a dynamic portfolio model that may be used to value CDO tranches with cashflow waterfalls of arbitrary degrees of complexity using time-homogenous Markov chain processes. Using the reduced-form CDO approximation technique we analysed three basis functions: linear regression, logistic regression and probit model. We have shown that the CDO tranche payoff is reasonably explained by the portfolio expected loss profile when an extra portfolio risk dispersion variable is added to the list of explanatory variables. The reduced form approximation was then extended to non-Gaussian copula models by implementing a simplified version of the Andersen and Sidenius (2005) random factor

loading model.

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# Appendix

Math 2.1 Derivation of Equation (2.3.8)

Our model assumes that $p=\overline{p}+\varepsilon_i$ where $\varepsilon_i \sim N(0,\sigma)$. Consider the following expectation:

$$E(\varepsilon_1+\varepsilon_2+...+\varepsilon_N)^2=0 \qquad (M.2.1.1)$$

Expand the quadratic polynomial in (M.2.1.1) and group the cross-product terms as follows:

$$E\sum_{i=1}^{N}\varepsilon_i+2\cdot E\left(\sum_{i=1}^{N}\sum_{j=1}^{i-1}\varepsilon_i\varepsilon_j\right)=0 \qquad (M.2.1.2)$$

Thus the expected value of the sum of all cross-products is:

$$E\left(\sum_{i=1}^{N}\sum_{j=1}^{i-1}\varepsilon_i\varepsilon_j\right)=-\frac{N\cdot\sigma^2}{2} \qquad (M.2.1.3)$$

For a vector of default probabilities $\{p_i\}$ the probability of exactly $k$ events is:

$$P(k)=\frac{1}{k!\cdot(N-k)!}\sum_{s\in S(N)}\left(\prod_{i=1}^{k}p_{s(i)}\cdot\prod_{i=k+1}^{N}\left(1-p_{s(i)}\right)\right) \qquad (M.2.1.4)$$

where $S(N)$ is the set of all transpositions on the set $\{1,2,...,N\}$. Notice that (M.2.1.4) represents a symmetrical polynomial of the variables $\varepsilon_1,\varepsilon_2,...,\varepsilon_N$, which is an important property which we are going to rely upon later on in this derivation. By construction $p_i=\overline{p}+\varepsilon_i$ therefore:

$$P(k)=\frac{1}{k!\cdot(N-k)!}\sum_{s\in S(N)}\left(\prod_{i=1}^{k}(\overline{p}+\varepsilon_{s(i)})\cdot\prod_{i=k+1}^{N}\left(1-\overline{p}-\varepsilon_{s(i)}\right)\right) \qquad (M.2.1.5)$$

Let's expand the above polynomial ignoring $\varepsilon_i^r$ for $r\neq 0,2$:

$$P(k)=C_k^N\cdot\overline{p}^k\cdot(1-\overline{p})^{N-k}+C_k^N\cdot\Big(C_2^k\cdot\overline{p}^{k-2}\cdot(1-\overline{p})^{N-k}\cdot\varepsilon_i\varepsilon_j\ - \qquad (M.2.1.6)$$
$$C_1^k\cdot\overline{p}^{k-1}\cdot C_1^{N-k}\cdot(1-\overline{p})^{N-k-1}\cdot\varepsilon_i\varepsilon_j\ +\ C_2^{N-k}\cdot(1-\overline{p})^{N-k-2}\cdot\overline{p}^k\cdot\varepsilon_i\varepsilon_j\Big)$$

The latter term in the equation (M.2.1.6) is our $\gamma^{(k)}$; namely:

$$\gamma^{(k)}=C_k^N\cdot\Big(C_2^k\cdot\overline{p}^{k-2}\cdot(1-\overline{p})^{N-k}\cdot\varepsilon_i\varepsilon_j\ - \qquad (M.2.1.7)$$
$$C_1^k\cdot\overline{p}^{k-1}\cdot C_1^{N-k}\cdot(1-\overline{p})^{N-k-1}\cdot\varepsilon_i\varepsilon_j\ +\ C_2^{N-k}\cdot(1-\overline{p})^{N-k-2}\cdot\overline{p}^k\cdot\varepsilon_i\varepsilon_j\Big)$$

In equations (M.2.1.6) and (M.2.1.7) the notion of $\varepsilon_i\varepsilon_j$ product is *notional* in that the equation is only showing the sum of coefficients in front of cross-product terms $\varepsilon_i\varepsilon_j$ for $i \neq j$. Because the calculation of $P(k)$ is based on counting all possible $k$-subset combinations out of $N$ in the event space $S(N)$, the sum of cross-product terms $\varepsilon_i\varepsilon_j$ will always be *complete*, in that the cross-product summation will always be over all possible combinations of $i$ and $j$. Therefore while being fully aware that $E(\varepsilon_i,\varepsilon_j)=0$ for any $i \neq j$, we can derive a *notional quantity* of $\varepsilon_i\varepsilon_j$ from (M.2.1.1) as:

$$\varepsilon_i \cdot \varepsilon_j = -\frac{2 \cdot N \cdot \sigma^2}{2 \cdot (N-1) \cdot N} = -\frac{\sigma^2}{N-1} \qquad (M.2.1.8)$$

and substitute it into (M.2.1.6) or (M.2.1.7). As we already mentioned, such substitution is only possible when the sum of cross-products is known to be *complete.* After the substitution, we obtain the following expression for $\gamma^{(k)}$:

$$\gamma^{(k)} = \frac{N!}{(N-k)!\cdot k!} \cdot \frac{-\sigma^2}{N-1} \cdot \left( \frac{k\cdot(k-1)}{2} \cdot \overline{p}^{k-2} \cdot (1-\overline{p})^{N-k} - \right. \qquad (M.2.1.9)$$

$$\left. k \cdot \overline{p}^{k-1} \cdot (N-k) \cdot (1-\overline{p})^{N-k-1} + \overline{p}^{k} \cdot \frac{(N-k)\cdot(N-k-1)}{2} \cdot (1-\overline{p})^{N-k-2} \right)$$

or alternatively

$$\gamma^{(k)} = -\frac{\sigma^2 \cdot N}{2} \cdot \frac{(N-2)!}{(N-k)!\cdot k!} \cdot \left( k \cdot (k-1) \cdot \overline{p}^{k-2} \cdot (1-\overline{p})^{N-k} - \right. \qquad (M.2.1.10)$$

$$\left. 2 \cdot k \cdot \overline{p}^{k-1} \cdot (N-k) \cdot (1-\overline{p})^{N-k-1} + \overline{p}^{k} \cdot (N-k) \cdot (N-k-1) \cdot (1-\overline{p})^{N-k-2} \right)$$

that is exactly the expression (2.3.8).

Math 2.2 Sum of gamma terms in Equation (2.3.8) is exactly zero

To prove that all gammas sum to zero, let's sum the gamma terms over $k=0,1,...,N$:

$$\sum_{k=0}^{N} \gamma^{(k)} =$$

$$-\frac{\sigma^2 \cdot N \cdot (N-2)!}{2} \cdot \sum_{k=0}^{N} \frac{1}{(N-k)! \cdot k!} \cdot \Big($$

$$k \cdot (k-1) \cdot \overline{p}^{k-2} \cdot (1-\overline{p})^{N-k} \; - \; 2 \cdot k \cdot (N-k) \cdot \overline{p}^{k-1} \cdot (1-\overline{p})^{N-k-1} \; +$$

$$(N-k) \cdot (N-k-1) \cdot (1-\overline{p})^{N-k-2} \cdot \overline{p}^{k} \Big) \; =$$

$$G \cdot \sum_{k=2}^{N} \frac{\overline{p}^{k-2} \cdot (1-\overline{p})^{N-k}}{(N-k)! \cdot (k-2)!} \; - \; 2 \cdot G \cdot \sum_{k=1}^{N-1} \frac{\overline{p}^{k-1} \cdot (1-\overline{p})^{N-k-1}}{(N-k)! \cdot (k-1)!} \; + \; G \cdot \sum_{k=0}^{N-2} \frac{\overline{p}^{k} \cdot (1-\overline{p})^{N-k-2}}{(N-k)! \cdot k!} = ...$$

Here $G = -\frac{\sigma^2 \cdot N \cdot (N-2)!}{2}$ and the summation bounds change so as to exclude cases when the corresponding term evaluates to 0. For example the first term becomes the summation from $k=2$ to $N$, because it's 0 for $k=0$ and 1.

It's then easy to prove that the summation above evaluates to 0 if we simply redefine the variable $k$ for each summation as follows:

$$... = \underset{\substack{\text{substitute} \\ k-2=t}}{G \cdot \sum_{t=0}^{N-2} \frac{\overline{p}^{t} \cdot (1-\overline{p})^{N-t-2}}{(N-t-2)! \cdot t!}} \; - \; \underset{\substack{\text{substitute} \\ k-1=t}}{2 \cdot G \cdot \sum_{t=0}^{N-2} \frac{\overline{p}^{t} \cdot (1-\overline{p})^{N-t-2}}{(N-t-2)! \cdot t!}} \; + \; \underset{\substack{\text{substitute} \\ k=t}}{G \cdot \sum_{t=0}^{N-2} \frac{\overline{p}^{t} \cdot (1-\overline{p})^{N-t-2}}{(N-t-2)! \cdot t!}} =$$

$$= G \cdot \left( \sum_{t=0}^{N-2} \frac{\overline{p}^{t} \cdot (1-\overline{p})^{N-t-2}}{(N-t-2)! \cdot t!} \right) \cdot (1-2+1) \; = \; 0$$

that is exactly what is required.

Table 2.1 Stylised portfolio statistics (iTraxx 7Y)

| | |
|---|---|
| Number of names | 125 |
| Mean spread | 99 bp |
| Minimum spread | 4.7 bp |
| Maximum spread | 185 bp |
| Spread standard deviation | 34 bp |
| Maturity (years) | 7 |
| Assumed recovery rate | 40% |

Table 2.2 Comparison of HPM, HPM+ and RLM models performance (type I adjustment only, iid loss distribution)

| **Default Count** | **HPM** | **HPM+** | **RLM** |
|---|---|---|---|
| 0 | 0.00000 | 0.00000 | 0.00000 |
| 1 | 0.00001 | 0.00001 | 0.00001 |
| 2 | 0.00005 | 0.00004 | 0.00004 |
| 3 | 0.00025 | 0.00023 | 0.00023 |
| 4 | 0.00094 | 0.00089 | 0.00090 |
| 5 | 0.00286 | 0.00274 | 0.00274 |
| 6 | 0.00716 | 0.00693 | 0.00693 |
| 7 | 0.01523 | 0.01488 | 0.01488 |
| 8 | 0.02811 | 0.02768 | 0.02768 |
| 9 | 0.04572 | 0.04532 | 0.04532 |
| 10 | 0.06637 | 0.06615 | 0.06615 |
| 11 | 0.08684 | 0.08692 | 0.08691 |
| 12 | 0.10323 | 0.10366 | 0.10365 |
| 13 | 0.11230 | 0.11298 | 0.11298 |
| 14 | 0.11242 | 0.11321 | 0.11321 |
| 15 | 0.10411 | 0.10482 | 0.10482 |
| 16 | 0.08957 | 0.09006 | 0.09006 |
| 17 | 0.07187 | 0.07208 | 0.07208 |
| 18 | 0.05396 | 0.05392 | 0.05392 |
| 19 | 0.03803 | 0.03782 | 0.03782 |
| 20 | 0.02522 | 0.02494 | 0.02494 |
| ... | ... | ... | ... |
| 125 | 0.00000 | 0.00000 | 0.00000 |

Table 2.3 Comparison of HPM, HPM+ and RLM models performance (both type I and type II adjustments are applied)

| **Default Count** | **HPM** | **HPM+** | **RLM** |
|---|---|---|---|
| 0 | 0.0458 | 0.0417 | 0.0417 |
| 1 | 0.0555 | 0.0526 | 0.0525 |
| 2 | 0.0571 | 0.0554 | 0.0553 |
| 3 | 0.0557 | 0.0549 | 0.0548 |
| 4 | 0.0532 | 0.0530 | 0.0529 |
| 5 | 0.0502 | 0.0504 | 0.0504 |
| 6 | 0.0471 | 0.0476 | 0.0475 |
| 7 | 0.0440 | 0.0447 | 0.0446 |
| 8 | 0.0410 | 0.0418 | 0.0418 |
| 9 | 0.0382 | 0.0390 | 0.0390 |
| 10 | 0.0356 | 0.0364 | 0.0364 |
| 11 | 0.0331 | 0.0339 | 0.0339 |
| 12 | 0.0308 | 0.0316 | 0.0316 |
| 13 | 0.0286 | 0.0294 | 0.0295 |
| 14 | 0.0267 | 0.0274 | 0.0274 |
| 15 | 0.0248 | 0.0255 | 0.0255 |
| 16 | 0.0231 | 0.0237 | 0.0238 |
| 17 | 0.0215 | 0.0221 | 0.0221 |
| 18 | 0.0200 | 0.0205 | 0.0206 |
| 19 | 0.0186 | 0.0191 | 0.0192 |
| 20 | 0.0174 | 0.0178 | 0.0178 |
| ... | ... | ... | ... |
| 125 | 0.0000 | 0.0000 | 0.0000 |
| **Calculation time** | 1.9 | 2 | 17.5 |

Table 2.4 Percentage of total HPM error corrected by type I & type II adjustments

| Default Count | HPM+ type I adj. | HPM+ type II adj. | HPM+ type I & II adj. |
|---|---|---|---|
| 0 | 3.1% | 96.6% | 99.7% |
| 1 | 2.8% | 94.7% | 97.5% |
| 2 | 2.6% | 91.7% | 94.4% |
| 3 | 2.4% | 86.1% | 88.5% |
| 4 | 1.6% | 62.2% | 63.7% |
| 5 | 3.8% | 54.5% | 58.3% |
| 6 | 3.0% | 84.7% | 87.6% |
| 7 | 2.8% | 91.0% | 93.7% |
| 8 | 2.7% | 94.0% | 96.7% |
| 9 | 2.7% | 95.9% | 98.6% |
| 10 | 2.7% | 97.2% | 100.0% |
| 11 | 2.7% | 96.1% | 98.8% |
| 12 | 2.7% | 95.1% | 97.8% |
| 13 | 2.7% | 94.2% | 96.9% |
| 14 | 2.7% | 93.3% | 96.0% |
| 15 | 2.7% | 92.5% | 95.1% |
| 16 | 2.7% | 91.6% | 94.3% |
| 17 | 2.7% | 90.8% | 93.5% |
| 18 | 2.7% | 90.0% | 92.7% |
| 19 | 2.7% | 89.1% | 91.8% |
| 20 | 2.7% | 88.3% | 91.0% |
| 21 | 2.7% | 87.3% | 90.0% |
| 22 | 2.7% | 86.3% | 89.1% |
| 23 | 2.8% | 85.2% | 88.0% |
| 24 | 2.8% | 84.0% | 86.8% |
| ... | ... | ... | ... |

Table 3.1 iTraxx Series 7 7Y index tranche calibration results

| Tranche | Target Tranche Prices | | Fitted Tranche Prices | |
|---|---|---|---|---|
| | Upfront | Spread | Upfront | Spread |
| 0-3% | 27.4 | 500 | 27.5 | 500 |
| 3-6% | | 139 | | 141 |
| 6-9% | | 38 | | 37 |
| 9-12% | | 18 | | 18 |
| 12-22% | | 6 | | 6 |
| 22-100% | | 1.2 | | 1.4 |

Table 3.2 CDX Series 8 7Y index tranche calibration results

| Tranche | Target Tranche Prices | | Fitted Tranche Prices | |
|---|---|---|---|---|
| | Upfront | Spread | Upfront | Spread |
| 0-3% | 43.3 | 500 | 43.9 | 500 |
| 3-7% | | 246 | | 245 |
| 7-10% | | 52 | | 52 |
| 10-15% | | 26 | | 26 |
| 15-30% | | 9 | | 9 |
| 30-100% | | 2.4 | | 2.5 |

Table 3.3 Thin CDX Series 8 7Y tranche prices implied by the base correlation model and the implied copula model[1]

| Tranche | Base Correlation Price | Implied Copula Price |
|---|---|---|
| 0-3% | 43.85%+500bp | 43.26%+500bp |
| 3-4% | 456 | 472 |
| 4-5% | 265 | 265 |
| 5-6% | 163 | 159 |
| 6-7% | 105 | 100 |
| 7-8% | 69 | 67 |
| 8-9% | 48 | 50 |
| 9-10% | 39 | 40 |
| 10-11% | 34 | 33 |
| 11-12% | 29 | 29 |
| 12-13% | 25 | 25 |
| 13-14% | 21 | 22 |
| 14-15% | 19 | 19 |
| 15-16% | 16 | 17 |
| 16-17% | 14 | 14 |
| 17-18% | 12 | 12 |
| 18-19% | 11 | 10 |
| 19-20% | 10 | 9 |
| 20-21% | 9 | 8 |
| 21-22% | 8 | 8 |

[1] Cubic spline method was used to interpolate base correlation for non-standard strikes.

Table 3.4 Thin iTraxx Series 7 7Y index tranche prices implied by the base correlation and the implied copula model[2]

| Tranche | Base Correlation Price | Implied Copula Price |
|---|---|---|
| 0-3 | 27.4%+500bp | 27.27%+500bp |
| 3-4% | 218 | 235 |
| 4-5% | 123 | 122 |
| 5-6% | 76 | 74 |
| 6-7% | 51 | 52 |
| 7-8% | 37 | 38 |
| 8-9% | 28 | 28 |
| 9-10% | 22 | 22 |
| 10-11% | 18 | 18 |
| 11-12% | 15 | 15 |
| 12-13% | 12 | 12 |
| 13-14% | 11 | 10 |
| 14-15% | 9 | 9 |
| 15-16% | 8 | 7 |
| 16-17% | 7 | 6 |
| 17-18% | 5 | 5 |
| 18-19% | 4 | 4 |
| 19-20% | 3 | 4 |
| 20-21% | 2 | 4 |
| 21-22% | 1 | 4 |

[2]Cubic spline method was used to interpolate base correlation for non-standard strikes.

Table 3.5 Thin bespoke CDO tranche prices implied by the base correlation and the implied copula model[3]

| Tranche | Base Correlation Price | Implied Copula Price |
|---|---|---|
| 0-6% | 49.48+500bp | 49.40+500bp |
| 6-7% | 593 | 568 |
| 7-8% | 432 | 390 |
| 8-9% | 317 | 281 |
| 9-10% | 235 | 213 |
| 10-11% | 177 | 163 |
| 11-12% | 136 | 124 |
| 12-13% | 106 | 93 |
| 13-14% | 84 | 73 |
| 14-15% | 69 | 61 |
| 15-16% | 57 | 53 |
| 16-17% | 48 | 47 |
| 17-18% | 41 | 42 |
| 18-19% | 35 | 38 |
| 19-20% | 31 | 34 |
| 20-21% | 28 | 32 |
| 21-22% | 25 | 29 |
| 22-23% | 23 | 27 |
| 23-24% | 21 | 25 |
| 24-25% | 19 | 22 |
| 25-26% | 17 | 21 |
| 26-27% | 16 | 19 |
| 27-28% | 15 | 18 |
| 28-29% | 14 | 16 |
| 29-30% | 12 | 14 |

[3]Bespoke CDO portfolio was formed by doubling the spread for each name in the CDX index portfolio

Table 3.6 Thin bespoke CDO tranche prices implied by the base correlation and the implied copula model[4]

| Tranche | Base Correlation Price | Implied Copula Price |
|---|---|---|
| 0-6% | 31.91%+500bp | 30.90%+500bp |
| 6-7% | 252 | 231 |
| 7-8% | 176 | 163 |
| 8-9% | 127 | 121 |
| 9-10% | 95 | 93 |
| 10-11% | 72 | 74 |
| 11-12% | 57 | 60 |
| 12-13% | 45 | 50 |
| 13-14% | 37 | 41 |
| 14-15% | 31 | 33 |
| 15-16% | 26 | 27 |
| 16-17% | 22 | 23 |
| 17-18% | 19 | 21 |
| 18-19% | 16 | 19 |
| 19-20% | 14 | 19 |
| 20-21% | 13 | 18 |
| 21-22% | 11 | 18 |

Table 3.7 Average CR01 for CDX index tranches

| | 0-3% | 3-7% | 7-10% | 10-15% | 15-30% |
|---|---|---|---|---|---|
| Implied Copula | 0.1782% | 0.0777% | 0.0164% | 0.0068% | 0.0021% |
| Base Correlation | 0.1736% | 0.0658% | 0.0190% | 0.0094% | 0.0033% |

[4]Bespoke CDO portfolio was formed by doubling the spread for each name in the iTraxx index portfolio

Table 4.1 Consumer goods generic CDS spreads[5]

| | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y |
|---|---|---|---|---|---|---|
| AAA | 15 | 20 | 25 | 31 | 37 | 42 |
| AA | 20 | 26 | 34 | 42 | 49 | 55 |
| A | 27 | 33 | 40 | 48 | 56 | 62 |
| BBB+ | 39 | 47 | 56 | 67 | 76 | 85 |
| BBB | 51 | 62 | 73 | 85 | 96 | 107 |
| BBB– | 106 | 118 | 129 | 141 | 151 | 158 |
| BB+ | 161 | 175 | 185 | 197 | 206 | 209 |
| BB | 217 | 231 | 241 | 253 | 262 | 260 |
| BB– | 261 | 282 | 301 | 323 | 338 | 335 |
| B+ | 306 | 332 | 361 | 393 | 415 | 410 |
| B | 350 | 382 | 421 | 463 | 492 | 485 |
| B– | 815 | 712 | 645 | 622 | 612 | 559 |
| CCC | 1280 | 1043 | 868 | 782 | 733 | 633 |

[5]Markit Partners composites by convention download for March 24, 2010

Table 4.2 Consumer services generic CDS spreads

| | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y |
|---|---|---|---|---|---|---|
| AAA | 13 | 17 | 20 | 25 | 29 | 33 |
| AA | 17 | 22 | 27 | 33 | 39 | 45 |
| A | 32 | 38 | 44 | 51 | 58 | 64 |
| BBB+ | 42 | 50 | 59 | 68 | 76 | 83 |
| BBB | 51 | 62 | 74 | 85 | 95 | 102 |
| BBB– | 96 | 113 | 131 | 147 | 159 | 164 |
| BB+ | 141 | 164 | 188 | 208 | 224 | 226 |
| BB | 187 | 214 | 245 | 270 | 288 | 288 |
| BB– | 252 | 280 | 318 | 344 | 361 | 349 |
| B+ | 316 | 347 | 391 | 418 | 434 | 411 |
| B | 381 | 413 | 465 | 492 | 507 | 473 |
| B– | 595 | 584 | 611 | 625 | 627 | 563 |
| CCC | 808 | 756 | 757 | 758 | 747 | 653 |

Table 4.3 Financials generic CDS spreads.

| | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y |
|---|---|---|---|---|---|---|
| AAA | 45 | 51 | 58 | 64 | 71 | 77 |
| AA | 60 | 68 | 78 | 86 | 94 | 102 |
| A | 90 | 98 | 108 | 117 | 123 | 130 |
| BBB+ | 110 | 120 | 132 | 143 | 150 | 155 |
| BBB | 130 | 142 | 156 | 168 | 177 | 180 |
| BBB– | 178 | 192 | 208 | 221 | 230 | 230 |
| BB+ | 226 | 241 | 261 | 274 | 284 | 279 |
| BB | 275 | 291 | 313 | 327 | 337 | 328 |
| BB– | 493 | 459 | 451 | 451 | 449 | 421 |
| B+ | 712 | 628 | 588 | 574 | 561 | 514 |
| B | 931 | 796 | 726 | 698 | 674 | 606 |
| B– | 1133 | 994 | 906 | 840 | 808 | 695 |
| CCC | 1335 | 1192 | 1085 | 983 | 942 | 783 |

Table 4.4 Industrials generic CDS spreads

| | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y |
|---|---|---|---|---|---|---|
| AAA | 28 | 32 | 37 | 46 | 53 | 57 |
| AA | 33 | 37 | 42 | 48 | 54 | 58 |
| A | 36 | 43 | 50 | 57 | 65 | 74 |
| BBB+ | 51 | 60 | 70 | 79 | 90 | 98 |
| BBB | 67 | 77 | 90 | 102 | 114 | 121 |
| BBB– | 100 | 110 | 126 | 142 | 156 | 163 |
| BB+ | 132 | 143 | 163 | 182 | 199 | 206 |
| BB | 164 | 177 | 200 | 223 | 241 | 248 |
| BB– | 256 | 262 | 281 | 293 | 308 | 310 |
| B+ | 347 | 347 | 362 | 363 | 375 | 372 |
| B | 438 | 432 | 443 | 434 | 442 | 434 |
| B– | 661 | 649 | 646 | 631 | 614 | 564 |
| CCC | 885 | 865 | 849 | 828 | 785 | 693 |

Table 4.5 Government generic CDS spreads

| | 1Y | 2Y | 3Y | 4Y | 5Y | 7Y |
|---|---|---|---|---|---|---|
| AAA | 45 | 50 | 55 | 60 | 68 | 66 |
| AA | 47 | 51 | 55 | 62 | 70 | 75 |
| A | 107 | 113 | 120 | 127 | 134 | 140 |
| BBB+ | 121 | 127 | 134 | 141 | 147 | 154 |
| BBB | 135 | 141 | 149 | 155 | 160 | 168 |
| BBB– | 177 | 185 | 188 | 196 | 200 | 204 |
| BB+ | 219 | 228 | 228 | 237 | 240 | 241 |
| BB | 261 | 271 | 268 | 278 | 279 | 277 |
| BB– | 334 | 341 | 335 | 339 | 336 | 325 |
| B+ | 406 | 410 | 402 | 401 | 393 | 372 |
| B | 479 | 480 | 469 | 462 | 450 | 419 |
| B– | 588 | 575 | 618 | 544 | 571 | 532 |
| CCC | 698 | 669 | 766 | 626 | 691 | 645 |

Table 4.6 Consumer goods risk-adjusted transition matrix

| | AAA | AA | A | BBB+ | BBB | BBB– | BB+ | BB | BB– | B+ | B | B– | CCC | D |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| AAA | 0.784 | 0.079 | 0.064 | 0.000 | 0.022 | 0.040 | 0.000 | 0.000 | 0.000 | 0.001 | 0.004 | 0.000 | 0.003 | 0.002 |
| AA | 0.023 | 0.886 | 0.018 | 0.015 | 0.014 | 0.013 | 0.002 | 0.002 | 0.002 | 0.003 | 0.007 | 0.004 | 0.007 | 0.003 |
| A | 0.027 | 0.022 | 0.867 | 0.018 | 0.016 | 0.014 | 0.004 | 0.004 | 0.002 | 0.004 | 0.004 | 0.009 | 0.006 | 0.004 |
| BBB+ | 0.030 | 0.026 | 0.025 | 0.824 | 0.018 | 0.016 | 0.008 | 0.008 | 0.005 | 0.006 | 0.008 | 0.014 | 0.006 | 0.006 |
| BBB | 0.033 | 0.026 | 0.026 | 0.026 | 0.793 | 0.016 | 0.009 | 0.010 | 0.006 | 0.009 | 0.011 | 0.017 | 0.010 | 0.008 |
| BBB– | 0.036 | 0.025 | 0.024 | 0.023 | 0.021 | 0.750 | 0.015 | 0.014 | 0.006 | 0.013 | 0.012 | 0.028 | 0.016 | 0.017 |
| BB+ | 0.026 | 0.021 | 0.019 | 0.016 | 0.015 | 0.014 | 0.742 | 0.021 | 0.013 | 0.019 | 0.014 | 0.034 | 0.021 | 0.026 |
| BB | 0.018 | 0.016 | 0.013 | 0.010 | 0.008 | 0.007 | 0.017 | 0.749 | 0.018 | 0.025 | 0.021 | 0.035 | 0.028 | 0.035 |
| BB– | 0.019 | 0.017 | 0.014 | 0.014 | 0.015 | 0.021 | 0.022 | 0.025 | 0.500 | 0.081 | 0.103 | 0.087 | 0.042 | 0.041 |
| B+ | 0.000 | 0.000 | 0.000 | 0.004 | 0.008 | 0.014 | 0.025 | 0.030 | 0.014 | 0.547 | 0.136 | 0.102 | 0.074 | 0.046 |
| B | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.003 | 0.000 | 0.021 | 0.000 | 0.691 | 0.123 | 0.110 | 0.052 |
| B– | 0.020 | 0.020 | 0.020 | 0.026 | 0.030 | 0.034 | 0.044 | 0.049 | 0.034 | 0.046 | 0.062 | 0.124 | 0.366 | 0.126 |
| CCC | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.813 | 0.187 |
| D | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 |

Table 4.7 Consumer services risk-adjusted transition matrix

| | AAA | AA | A | BBB+ | BBB | BBB– | BB+ | BB | BB– | B+ | B | B– | CCC | D |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| AAA | 0.791 | 0.079 | 0.064 | 0.000 | 0.022 | 0.040 | 0.000 | 0.000 | 0.000 | 0.000 | 0.002 | 0.000 | 0.000 | 0.002 |
| AA | 0.023 | 0.896 | 0.018 | 0.015 | 0.014 | 0.012 | 0.002 | 0.001 | 0.001 | 0.002 | 0.006 | 0.003 | 0.004 | 0.003 |
| A | 0.027 | 0.022 | 0.869 | 0.018 | 0.016 | 0.014 | 0.004 | 0.004 | 0.002 | 0.003 | 0.004 | 0.008 | 0.005 | 0.005 |
| BBB+ | 0.030 | 0.026 | 0.025 | 0.828 | 0.018 | 0.016 | 0.007 | 0.007 | 0.004 | 0.006 | 0.007 | 0.013 | 0.005 | 0.007 |
| BBB | 0.033 | 0.026 | 0.026 | 0.026 | 0.796 | 0.016 | 0.009 | 0.009 | 0.006 | 0.009 | 0.011 | 0.016 | 0.008 | 0.008 |
| BBB– | 0.035 | 0.025 | 0.024 | 0.022 | 0.021 | 0.739 | 0.015 | 0.015 | 0.008 | 0.015 | 0.015 | 0.031 | 0.021 | 0.015 |
| BB+ | 0.021 | 0.017 | 0.015 | 0.013 | 0.012 | 0.012 | 0.731 | 0.022 | 0.016 | 0.024 | 0.020 | 0.043 | 0.032 | 0.023 |
| BB | 0.009 | 0.007 | 0.004 | 0.002 | 0.001 | 0.003 | 0.014 | 0.763 | 0.021 | 0.031 | 0.029 | 0.046 | 0.041 | 0.031 |
| BB– | 0.014 | 0.012 | 0.010 | 0.009 | 0.010 | 0.017 | 0.020 | 0.024 | 0.514 | 0.082 | 0.105 | 0.092 | 0.050 | 0.041 |
| B+ | 0.000 | 0.000 | 0.000 | 0.003 | 0.007 | 0.014 | 0.025 | 0.030 | 0.014 | 0.537 | 0.136 | 0.106 | 0.078 | 0.051 |
| B | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.006 | 0.000 | 0.022 | 0.000 | 0.670 | 0.125 | 0.108 | 0.061 |
| B– | 0.008 | 0.008 | 0.009 | 0.016 | 0.021 | 0.028 | 0.041 | 0.049 | 0.037 | 0.052 | 0.070 | 0.192 | 0.366 | 0.103 |
| CCC | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.814 | 0.186 |
| D | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 |

Table 4.8 Financials risk-adjusted transition matrix

| | AAA | AA | A | BBB+ | BBB | BBB– | BB+ | BB | BB– | B+ | B | B– | CCC | D |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| AAA | 0.785 | 0.079 | 0.064 | 0.000 | 0.022 | 0.040 | 0.000 | 0.000 | 0.000 | 0.000 | 0.002 | 0.000 | 0.001 | 0.007 |
| AA | 0.023 | 0.882 | 0.018 | 0.015 | 0.014 | 0.012 | 0.002 | 0.002 | 0.001 | 0.003 | 0.007 | 0.004 | 0.007 | 0.009 |
| A | 0.027 | 0.022 | 0.854 | 0.018 | 0.016 | 0.014 | 0.004 | 0.004 | 0.002 | 0.004 | 0.004 | 0.009 | 0.007 | 0.014 |
| BBB+ | 0.030 | 0.026 | 0.025 | 0.811 | 0.018 | 0.016 | 0.008 | 0.008 | 0.005 | 0.006 | 0.007 | 0.014 | 0.008 | 0.017 |
| BBB | 0.033 | 0.026 | 0.026 | 0.026 | 0.775 | 0.016 | 0.010 | 0.010 | 0.006 | 0.010 | 0.011 | 0.018 | 0.012 | 0.020 |
| BBB– | 0.036 | 0.025 | 0.024 | 0.022 | 0.021 | 0.732 | 0.015 | 0.015 | 0.007 | 0.014 | 0.013 | 0.030 | 0.020 | 0.027 |
| BB+ | 0.024 | 0.019 | 0.017 | 0.014 | 0.014 | 0.013 | 0.731 | 0.021 | 0.014 | 0.021 | 0.016 | 0.037 | 0.024 | 0.035 |
| BB | 0.014 | 0.012 | 0.008 | 0.005 | 0.004 | 0.005 | 0.015 | 0.758 | 0.019 | 0.027 | 0.024 | 0.037 | 0.029 | 0.042 |
| BB– | 0.023 | 0.021 | 0.018 | 0.017 | 0.018 | 0.024 | 0.026 | 0.028 | 0.449 | 0.079 | 0.100 | 0.085 | 0.045 | 0.066 |
| B+ | 0.007 | 0.007 | 0.007 | 0.006 | 0.006 | 0.012 | 0.023 | 0.029 | 0.014 | 0.469 | 0.137 | 0.113 | 0.091 | 0.079 |
| B | 0.001 | 0.001 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.009 | 0.000 | 0.621 | 0.145 | 0.134 | 0.089 |
| B– | 0.001 | 0.001 | 0.001 | 0.004 | 0.008 | 0.014 | 0.025 | 0.031 | 0.018 | 0.033 | 0.052 | 0.284 | 0.382 | 0.146 |
| CCC | 0.006 | 0.006 | 0.006 | 0.006 | 0.005 | 0.005 | 0.005 | 0.005 | 0.004 | 0.003 | 0.003 | 0.001 | 0.736 | 0.210 |
| D | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 |

Table 4.9 Industrials risk-adjusted transition matrix

| | AAA | AA | A | BBB+ | BBB | BBB– | BB+ | BB | BB– | B+ | B | B– | CCC | D |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| AAA | 0.771 | 0.079 | 0.064 | 0.000 | 0.022 | 0.041 | 0.001 | 0.001 | 0.002 | 0.002 | 0.004 | 0.004 | 0.005 | 0.004 |
| AA | 0.023 | 0.891 | 0.018 | 0.015 | 0.014 | 0.012 | 0.002 | 0.001 | 0.001 | 0.002 | 0.006 | 0.003 | 0.006 | 0.005 |
| A | 0.027 | 0.022 | 0.865 | 0.017 | 0.016 | 0.014 | 0.004 | 0.004 | 0.002 | 0.003 | 0.004 | 0.009 | 0.006 | 0.005 |
| BBB+ | 0.031 | 0.026 | 0.025 | 0.818 | 0.018 | 0.016 | 0.008 | 0.008 | 0.005 | 0.007 | 0.008 | 0.015 | 0.008 | 0.008 |
| BBB | 0.033 | 0.026 | 0.026 | 0.026 | 0.781 | 0.016 | 0.010 | 0.010 | 0.007 | 0.010 | 0.012 | 0.019 | 0.013 | 0.010 |
| BBB– | 0.035 | 0.025 | 0.024 | 0.022 | 0.021 | 0.741 | 0.015 | 0.015 | 0.007 | 0.015 | 0.014 | 0.031 | 0.021 | 0.015 |
| BB+ | 0.023 | 0.018 | 0.016 | 0.013 | 0.013 | 0.013 | 0.743 | 0.022 | 0.015 | 0.022 | 0.018 | 0.039 | 0.026 | 0.020 |
| BB | 0.012 | 0.010 | 0.007 | 0.004 | 0.003 | 0.004 | 0.015 | 0.777 | 0.019 | 0.028 | 0.026 | 0.040 | 0.030 | 0.025 |
| BB– | 0.027 | 0.025 | 0.023 | 0.021 | 0.022 | 0.028 | 0.029 | 0.030 | 0.466 | 0.078 | 0.098 | 0.079 | 0.034 | 0.040 |
| B+ | 0.021 | 0.020 | 0.020 | 0.019 | 0.018 | 0.022 | 0.032 | 0.036 | 0.017 | 0.435 | 0.134 | 0.100 | 0.073 | 0.053 |
| B | 0.020 | 0.020 | 0.019 | 0.019 | 0.019 | 0.017 | 0.015 | 0.013 | 0.016 | 0.003 | 0.531 | 0.131 | 0.109 | 0.066 |
| B– | 0.010 | 0.010 | 0.010 | 0.016 | 0.020 | 0.025 | 0.036 | 0.042 | 0.027 | 0.041 | 0.059 | 0.215 | 0.373 | 0.116 |
| CCC | 0.010 | 0.010 | 0.010 | 0.009 | 0.009 | 0.008 | 0.008 | 0.007 | 0.005 | 0.004 | 0.003 | 0.001 | 0.727 | 0.189 |
| D | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 |

Table 4.10 Government risk-adjusted transition matrix

| | AAA | AA | A | BBB+ | BBB | BBB- | BB+ | BB | BB- | B+ | B | B- | CCC | D |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| AAA | 0.788 | 0.079 | 0.063 | 0.000 | 0.021 | 0.039 | 0.000 | 0.000 | 0.000 | 0.000 | 0.002 | 0.001 | 0.001 | 0.007 |
| AA | 0.024 | 0.886 | 0.019 | 0.016 | 0.015 | 0.013 | 0.003 | 0.002 | 0.000 | 0.002 | 0.005 | 0.003 | 0.004 | 0.007 |
| A | 0.026 | 0.021 | 0.834 | 0.019 | 0.017 | 0.015 | 0.006 | 0.006 | 0.004 | 0.007 | 0.008 | 0.012 | 0.007 | 0.017 |
| BBB+ | 0.030 | 0.027 | 0.024 | 0.803 | 0.019 | 0.017 | 0.009 | 0.009 | 0.005 | 0.007 | 0.009 | 0.015 | 0.006 | 0.020 |
| BBB | 0.035 | 0.029 | 0.027 | 0.027 | 0.783 | 0.015 | 0.008 | 0.008 | 0.004 | 0.007 | 0.009 | 0.016 | 0.010 | 0.022 |
| BBB- | 0.039 | 0.028 | 0.026 | 0.024 | 0.022 | 0.736 | 0.013 | 0.013 | 0.005 | 0.012 | 0.011 | 0.026 | 0.016 | 0.028 |
| BB+ | 0.029 | 0.025 | 0.020 | 0.017 | 0.016 | 0.014 | 0.729 | 0.021 | 0.013 | 0.018 | 0.012 | 0.032 | 0.019 | 0.035 |
| BB | 0.022 | 0.020 | 0.013 | 0.009 | 0.008 | 0.006 | 0.015 | 0.749 | 0.018 | 0.024 | 0.020 | 0.032 | 0.022 | 0.042 |
| BB- | 0.031 | 0.030 | 0.025 | 0.024 | 0.024 | 0.029 | 0.029 | 0.030 | 0.445 | 0.078 | 0.097 | 0.076 | 0.028 | 0.054 |
| B+ | 0.026 | 0.025 | 0.024 | 0.023 | 0.022 | 0.026 | 0.035 | 0.039 | 0.019 | 0.400 | 0.132 | 0.097 | 0.068 | 0.066 |
| B | 0.024 | 0.024 | 0.023 | 0.023 | 0.022 | 0.020 | 0.018 | 0.016 | 0.018 | 0.005 | 0.497 | 0.129 | 0.104 | 0.077 |
| B- | 0.010 | 0.010 | 0.012 | 0.017 | 0.021 | 0.026 | 0.038 | 0.044 | 0.029 | 0.043 | 0.061 | 0.224 | 0.369 | 0.095 |
| CCC | 0.006 | 0.006 | 0.007 | 0.007 | 0.006 | 0.006 | 0.006 | 0.005 | 0.004 | 0.003 | 0.003 | 0.001 | 0.767 | 0.174 |
| D | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 |

Table 4.11 Single factor Gaussian Copula Model simulated tranche true value statistics

| | 0-3% | 3-7% | 7-10% | 10-15% | 15-30% | 30-100% |
|---|---|---|---|---|---|---|
| Mean | 0.7061 | 0.4034 | 0.2545 | 0.1613 | 0.0593 | 0.0023 |
| Minimum | 0.540 | 0.244 | 0.134 | 0.076 | 0.023 | 0.001 |
| Maximum | 1.000 | 1.000 | 0.972 | 0.686 | 0.259 | 0.012 |
| Mean estimator standard error, % of mean | 0.22% | 0.47% | 0.47% | 0.66% | 0.36% | 0.50% |
| Tranche volatility, % of mean | 11.0% | 18.3% | 21.4% | 23.2% | 25.9% | 30.0% |
| 10% percentile | 0.640 | 0.345 | 0.211 | 0.131 | 0.046 | 0.002 |
| 90% percentile | 0.814 | 0.483 | 0.310 | 0.199 | 0.075 | 0.003 |

Table 4.12 Root mean square deviation (% of mean) for simple regression

| | 0-3% | 3-7% | 7-10% | 10-15% | 15-30% | 30-100% |
|---|---|---|---|---|---|---|
| Linear terms only | 2.13% | 1.36% | 2.41% | 3.18% | 4.24% | 7.03% |
| With quadratic regressors | 1.52% | 1.39% | 2.23% | 2.66% | 3.35% | 8.09% |
| With expected loss dispersion | 1.62% | 1.33% | 2.18% | 2.61% | 3.10% | 5.53% |

Table 4.13 Root mean square deviation (% of selected sample mean) for logistic regression

| | 0-3% | 3-7% | 7-10% | 10-15% | 15-30% | 30-100% |
|---|---|---|---|---|---|---|
| All observations | 1.11% | 1.80% | 1.89% | 2.04% | 2.20% | 4.50% |
| 10% percentile | 1.82% | 2.13% | 2.04% | 1.99% | 2.22% | 9.24% |
| 90% percentile | 0.71% | 2.58% | 2.77% | 3.31% | 3.26% | 4.94% |

Table 4.14 Root mean square deviation (% of selected sample mean) for simple regression

| | 0-3% | 3-7% | 7-10% | 10-15% | 15-30% | 30-100% |
|---|---|---|---|---|---|---|
| All observations | 1.62% | 1.33% | 2.18% | 2.61% | 3.10% | 5.53% |
| 10% percentile | 1.65% | 1.47% | 2.15% | 2.57% | 2.95% | 8.00% |
| 90% percentile | 1.83% | 1.82% | 3.51% | 4.39% | 4.87% | 6.71% |

Table 4.15 Root mean square deviation (% of selected sample mean) for probit regression

| | 0-3% | 3-7% | 7-10% | 10-15% | 15-30% | 30-100% |
|---|---|---|---|---|---|---|
| All observations | 0.70% | 2.69% | 2.20% | 1.85% | 2.24% | 4.57% |
| 10% percentile | 1.17% | 3.10% | 1.95% | 1.96% | 2.16% | 8.76% |
| 90% percentile | 0.48% | 5.09% | 2.27% | 2.98% | 3.46% | 5.16% |

Table 4.16 Random factor loading model simulated true value statistics

| | 0-3% | 3-7% | 7-10% | 10-15% | 15-30% | 30-100% |
|---|---|---|---|---|---|---|
| Mean | 0.6831 | 0.4060 | 0.2338 | 0.1376 | 0.0577 | 0.0063 |
| Minimum | 0.5552 | 0.2579 | 0.1282 | 0.0742 | 0.0300 | 0.0031 |
| Maximum | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.8211 | 0.0552 |
| Tranche volatility, % of mean | 11.5% | 18.9% | 26.7% | 29.1% | 26.4% | 18.2% |
| 10% percentile | 0.6172 | 0.3417 | 0.1851 | 0.1081 | 0.0466 | 0.0052 |
| 90% percentile | 0.793 | 0.492 | 0.297 | 0.172 | 0.070 | 0.008 |

Table 4.17 Root mean square deviation (% of selected sample mean) for random factor loading model

| | 0-3% | 3-7% | 7-10% | 10-15% | 15-30% | 30-100% |
|---|---|---|---|---|---|---|
| Logistic | 1.29% | 1.81% | 2.37% | 3.37% | 5.70% | 9.23% |
| Probit | 1.73% | 2.95% | 3.12% | 3.39% | 4.78% | 9.64% |
| Regression | 1.81% | 1.65% | 2.69% | 4.18% | 8.64% | 9.72% |

Figure 2.1 Minimum sensitivity point [$k$] as a function of portfolio default correlation [$\rho$]

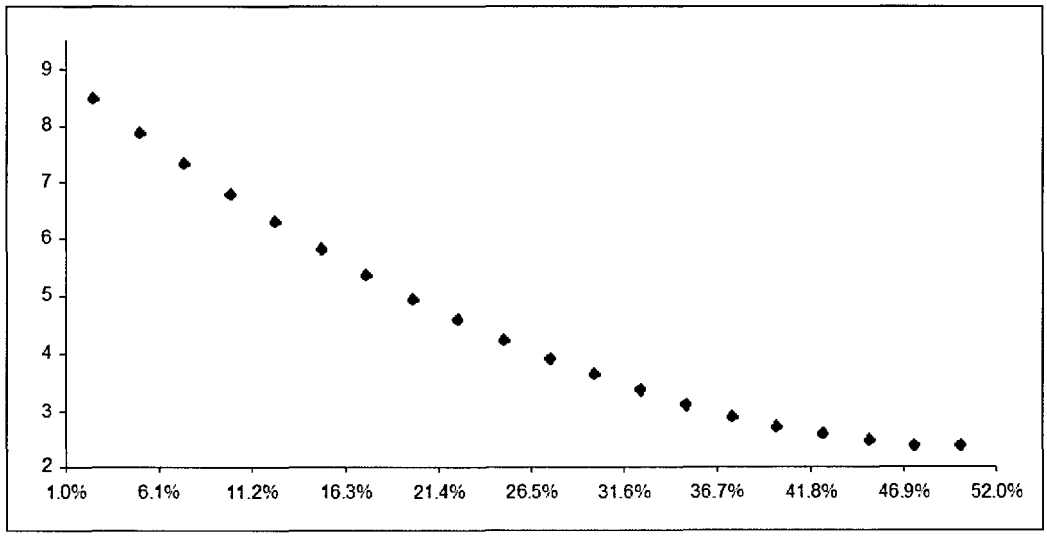



Figure 2.2 Minimum sensitivity point [$k$] as a function of the portfolio average default probability

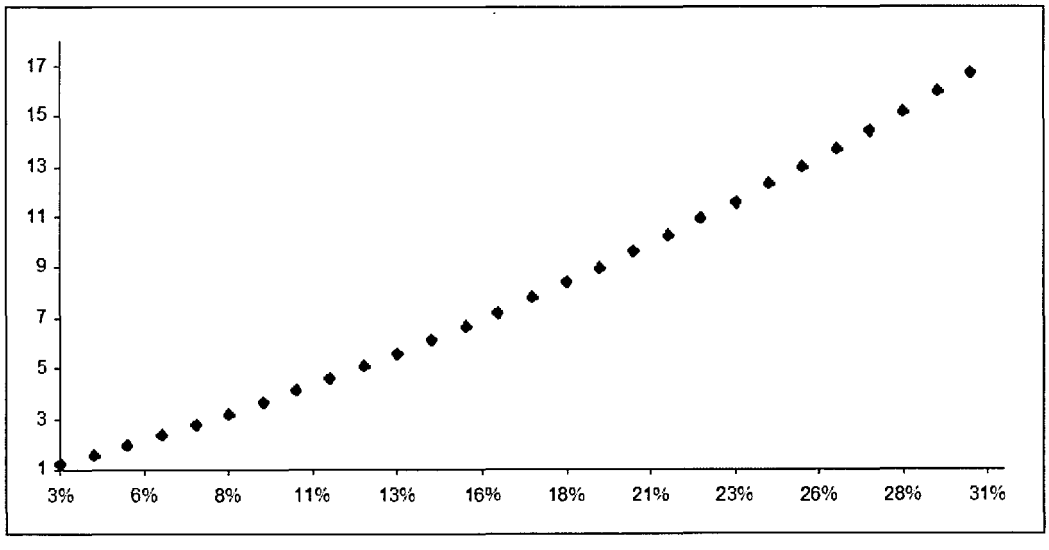


Figure 2.3 Minimum sensitivity point $[k]$ as a function of the portfolio dispersion $[\sigma]$

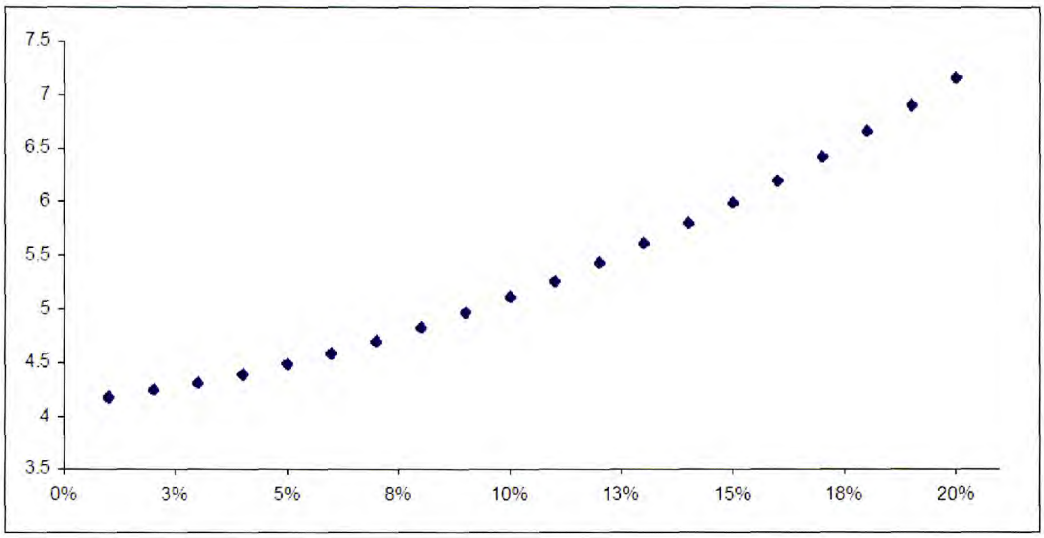



Figure 3.1 CDX 0–3% tranche delta profile

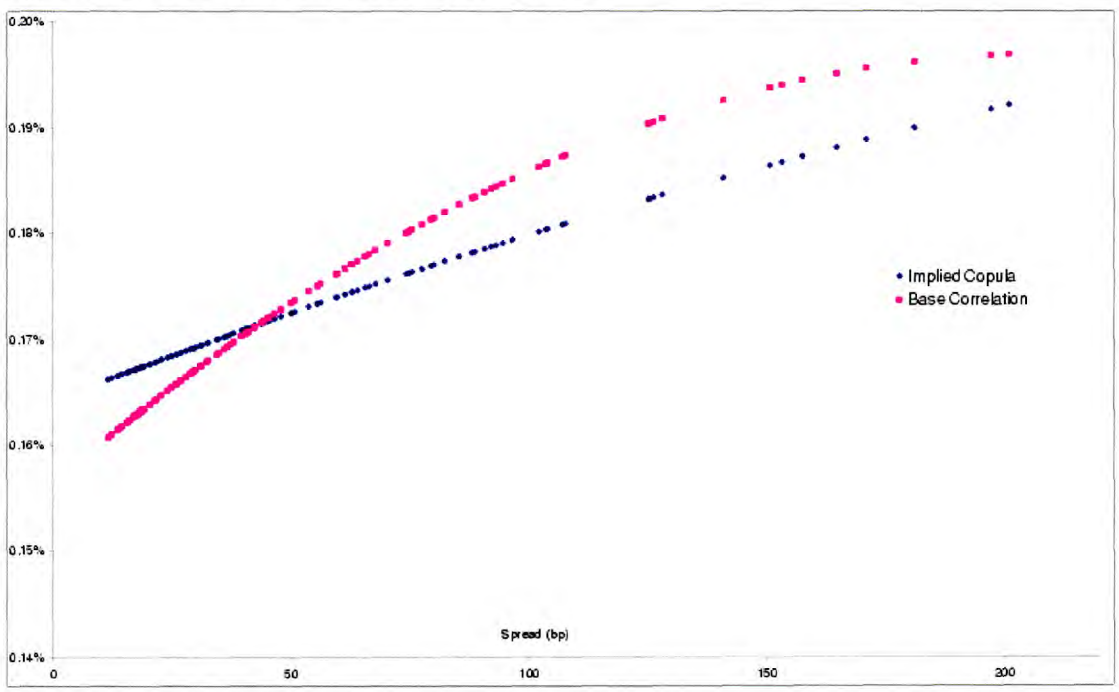


Figure 3.2 CDX 3–7% tranche delta profile

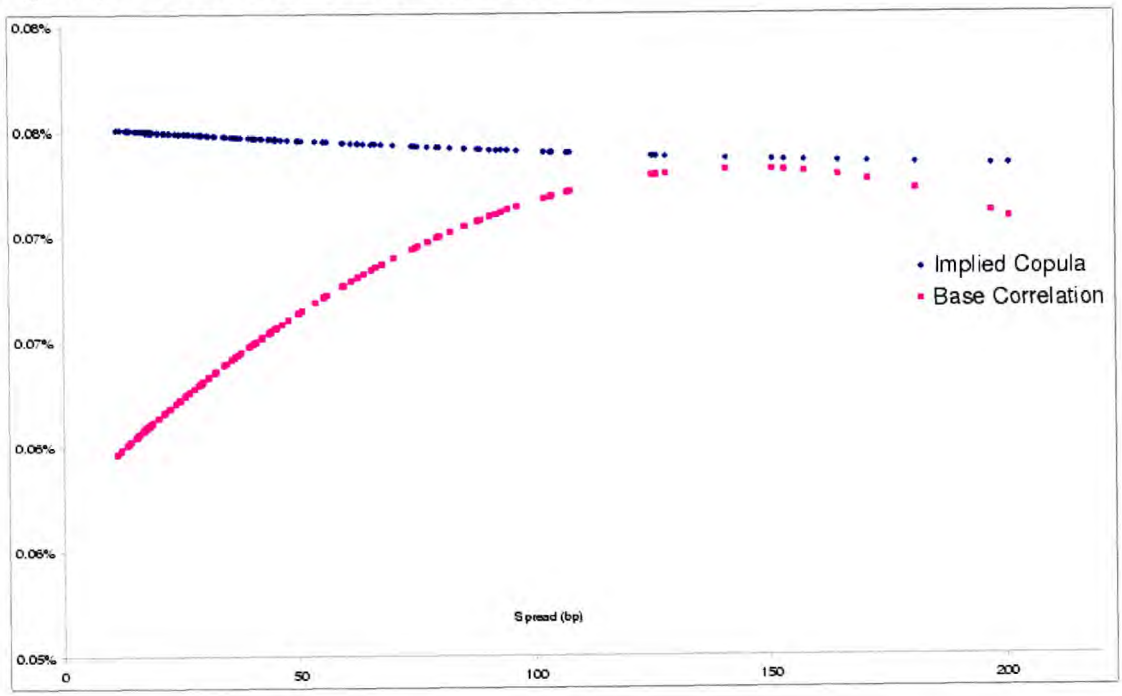



Figure 3.3 CDX 7–10% tranche delta profile

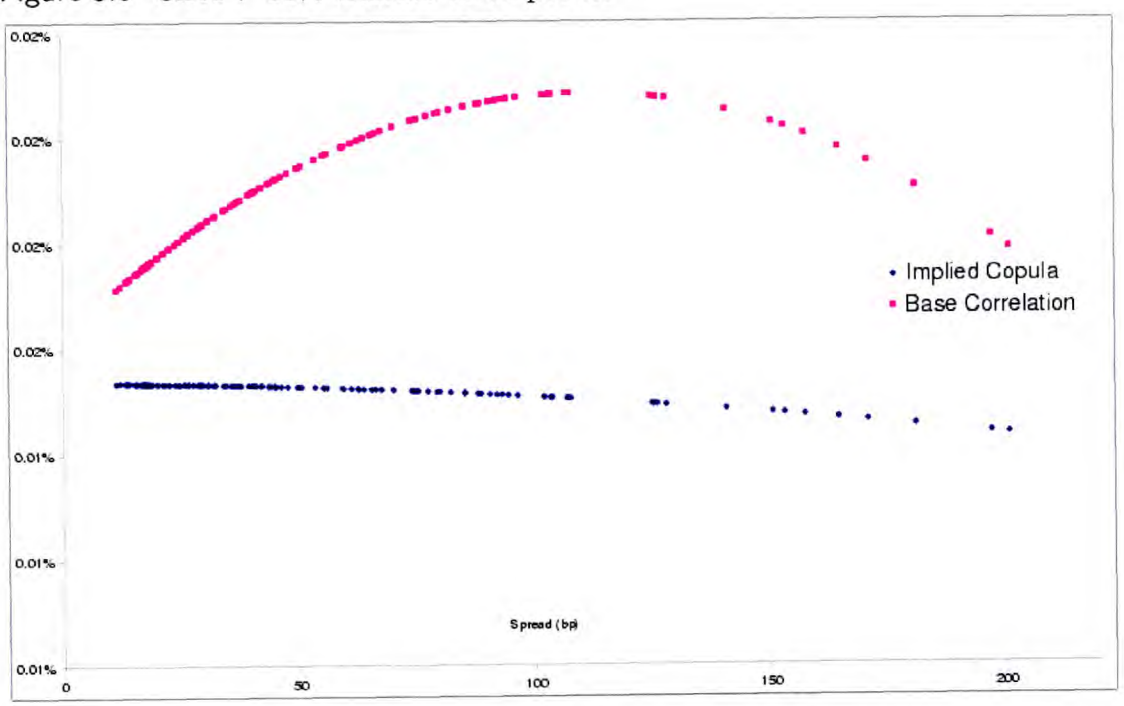


Figure 3.4 CDX 10–15% tranche delta profile

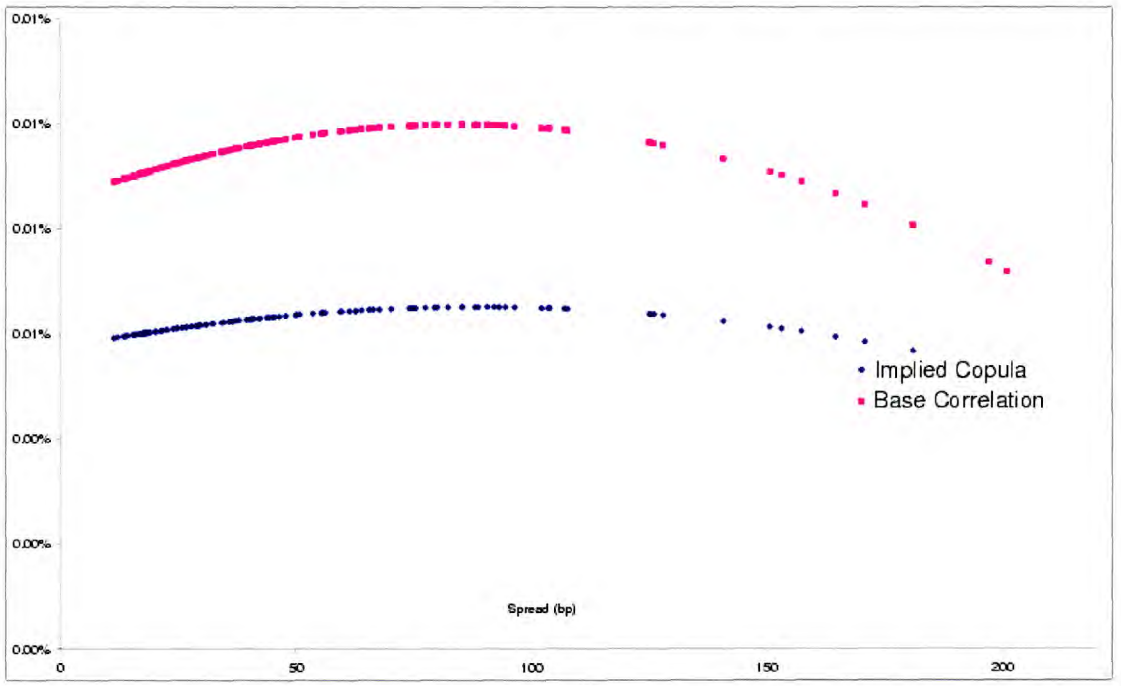



Figure 3.5 CDX 15–30% tranche delta profile

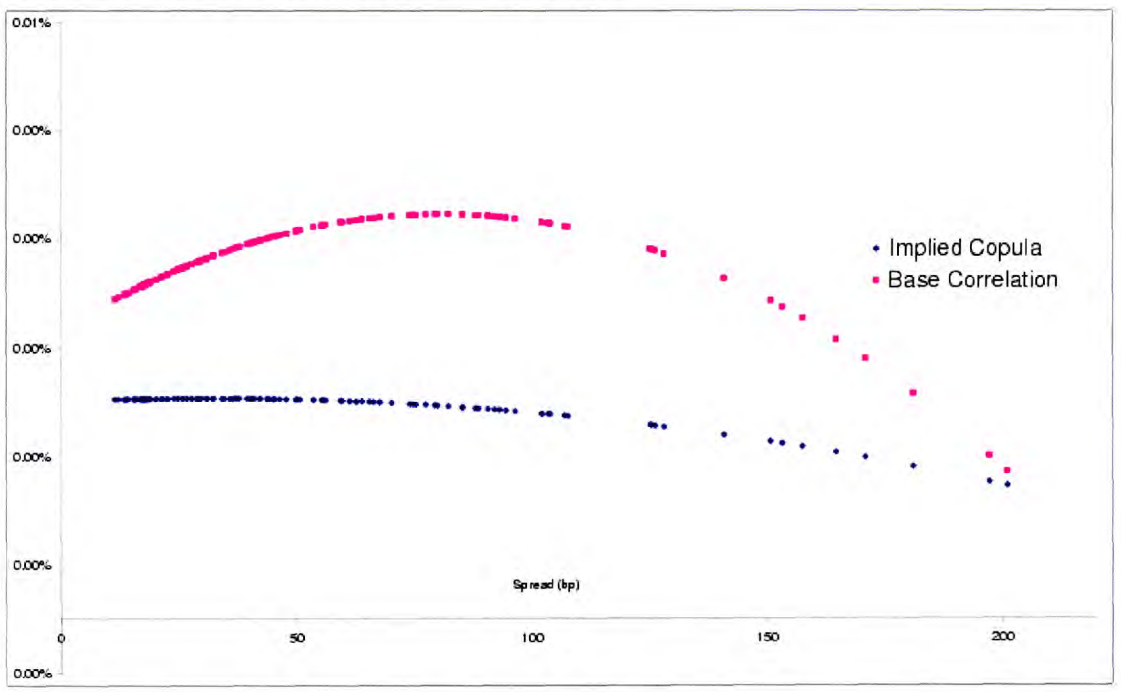


Figure 4.1 Logistic approximation for the 0–3% tranche

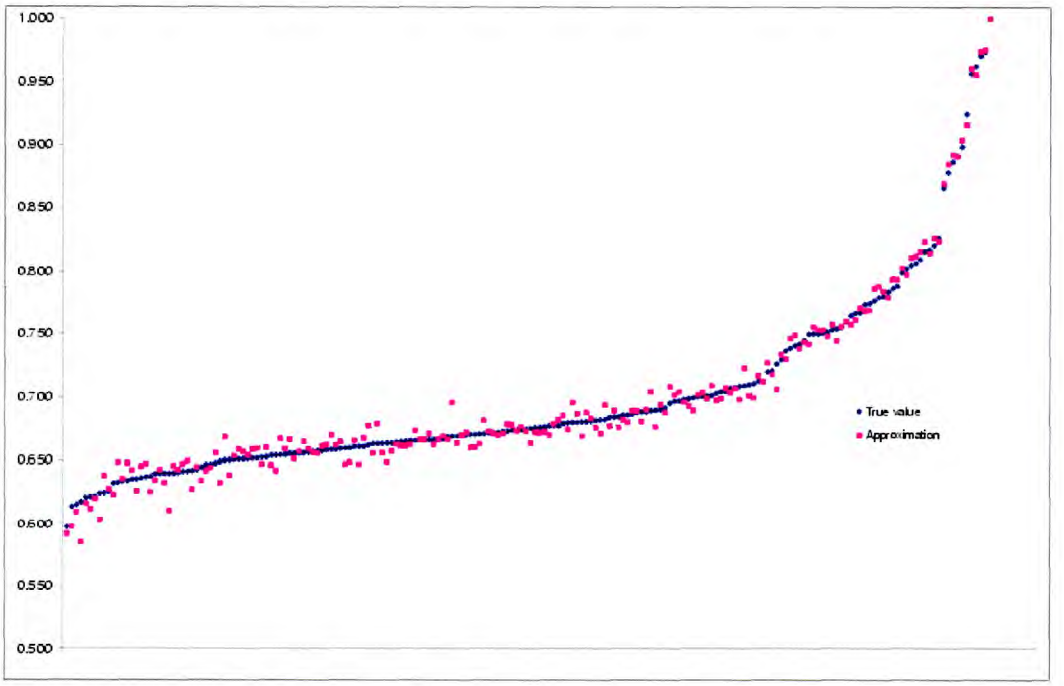



Figure 4.2 Logistic approximation for the 3–7% tranche

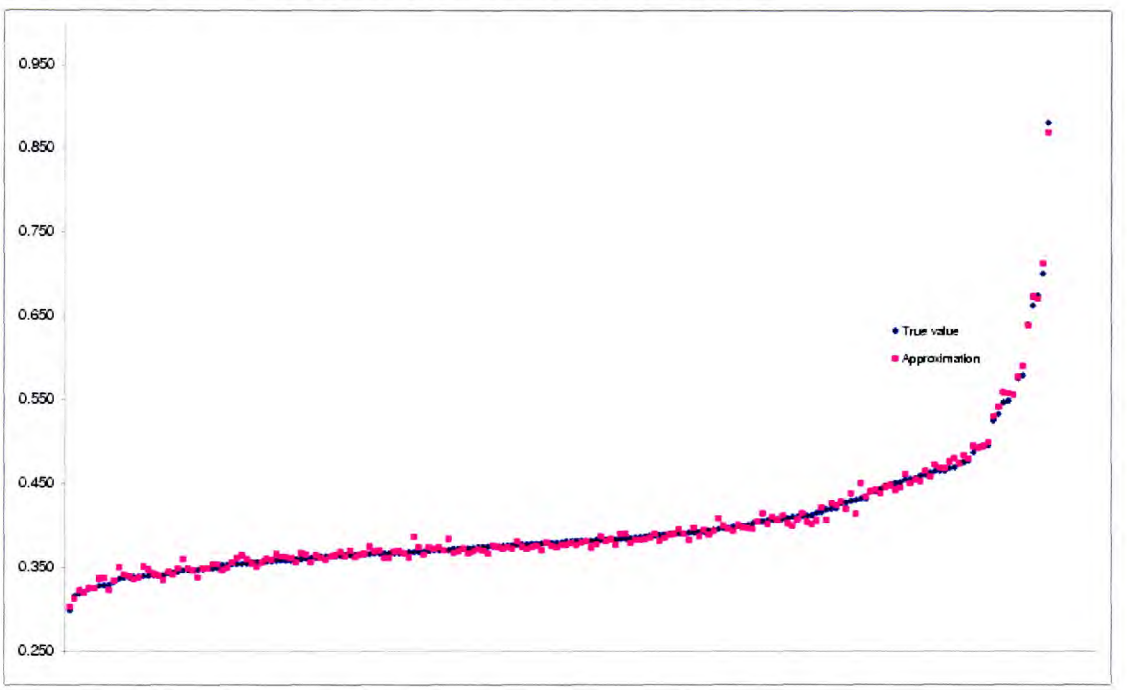


Figure 4.3 Logistic approximation for the 7–10% tranche

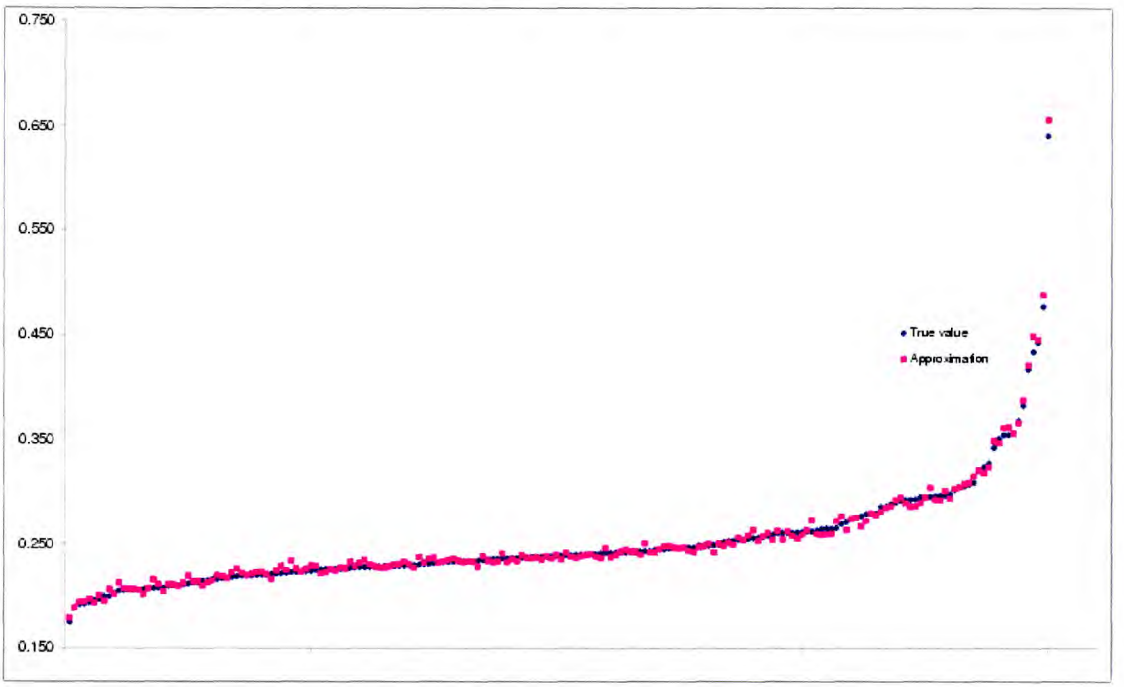



Figure 4.4 Logistic approximation for the 10–15% tranche

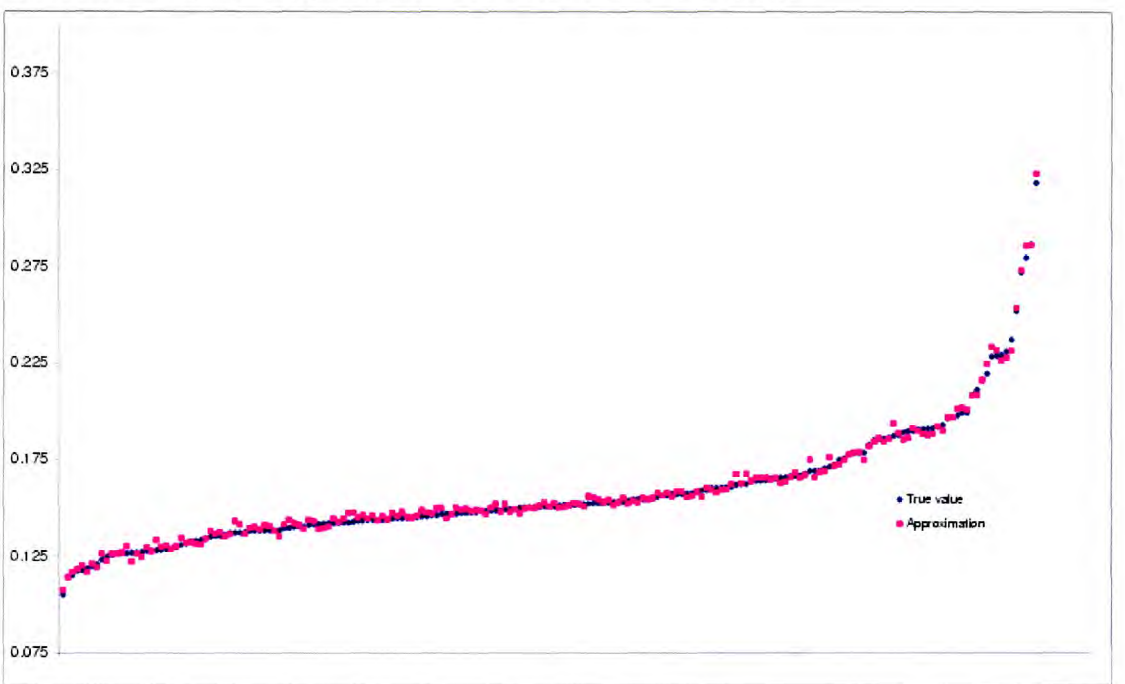


Figure 4.5 Logistic approximation for the 15–30% tranche

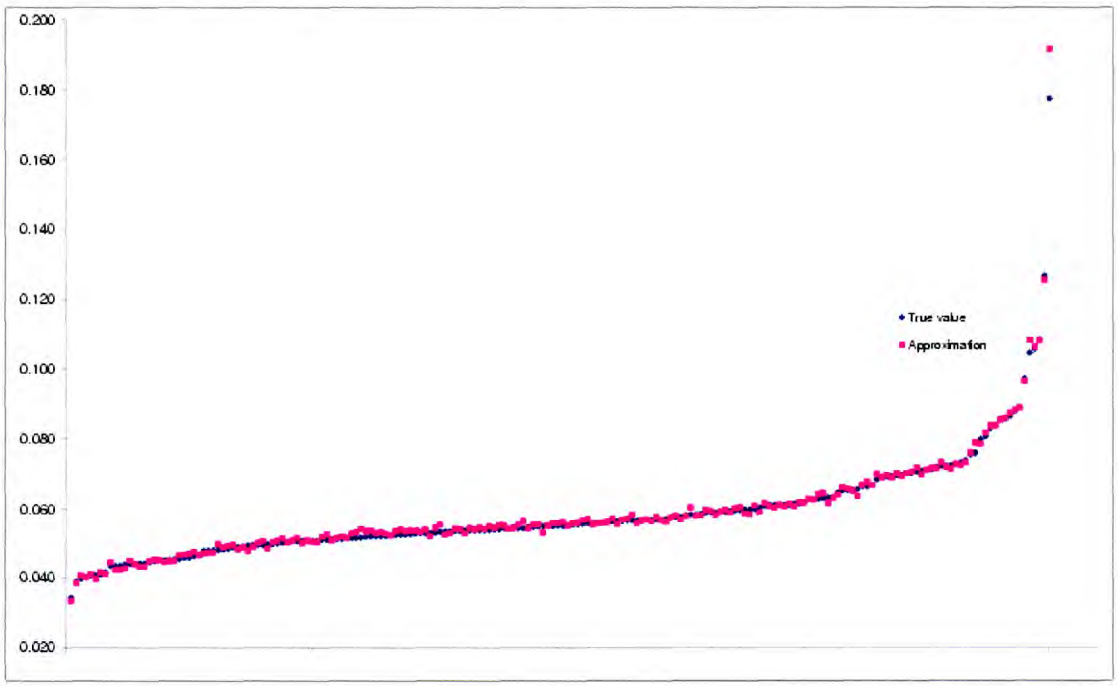



Figure 4.6 Logistic approximation for the 30–100% tranche

