

NEW METHODS OF DEFINING THE CHARACTERISTICS
OF A REFLECTED SIGNAL BY SEISMIC TRACE ANALYSIS

THESIS

submitted by

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ABSTRACT

The aim of this thesis is to investigate the possibility of analysing a given seismic trace in order to obtain its components. It seems that various characteristics of the components of a seismic trace can be obtained by several different methods. In all these methods the basic model of the seismic trace is taken as mathematically represented by a convolution.

The first method deals with analysis with regard to differential moveout of the signal and correlated noise. The assumption that is made in this analysis is that the moveout function of the signal and the correlated noise from trace to trace can be represented by a linear function. If this is so it is possible to separate the signal and correlated noise on the basis of a Fourier Transformation. If the linear relationship is not fulfilled this method will fail though it is possible to give an approximate solution if it is quadratic.

The second method deals with the analyses of a trace when it can be represented as a convolution of a minimum delay wavelet with a random spike series.

The third method deals with the analyses of a trace when it can be represented as a sum of two convolutions represented by the convolution of the signal wave and its arrival time series and the noise wave and its arrival time series. If the two arrival time series can be regarded as completely random a complete analysis can be made which provides the wavelets representing the signal and noise in their

autocorrelation form. If this is not satisfied the general method breaks down. However under certain considerations it is still possible to obtain the signal and noise wavelets.

The fourth method deals with the analyses of a trace when it can be represented by a Legendrian Polynomial.

The standard procedure in data processing is to use square error as a minimising criterion for filter design. The aim of the fifth method is to provide a different criterion which would give comparable results.

The final section explains the mathematical considerations that are necessary to implement some of the methods explained. Particular emphasis is placed on inverse matrix solution as this is the basis of all filter evaluations. The analysis is made with regard to matrix inversion which indirectly influence the solution of equations involved in the analysis of the seismic trace representation.

CONTENTS

	page
<u>ABSTRACT</u>	1
<u>CONTENTS</u>	2
<u>CHAPTER 1</u> F-K ANALYSIS	4
1.0 Introduction	4
1.1 Derivation of Fourier Transform for linear moveouts.	4
1.1.1 Analysis of the transform	9
1.1.2 Analysis of the sampled function	13
1.1.3 Illustrative example	29
1.1.4 Discussion of the results	33
1.2 Derivation of Fourier Transform for quadratic moveout	38
<u>CHAPTER 2</u> WAVELET CONTRACTION	40
2.0 Introduction	40
2.1 Time series decomposition	40
2.2 Discussion on the assumptions and modifications	43
2.3 Alternate solutions	47
2.4 General solution	56
<u>CHAPTER 3</u> CORRELATION ANALYSIS	66
3.0 Introduction	66
3.1 Derivation of the correlations	66
3.2 Breakdown of the theoretical case	69
3.3 Solution based on modified conditions	74
3.4 Illustrative examples	78

<u>CHAPTER 4</u>	POLYNOMIAL ANALYSIS	86
4.0	Introduction	86
4.1	Polynomial representation	86
4.2	Illustrative examples	96
4.3	Conclusions	101
<u>CHAPTER 5</u>	ABSOLUTE ERROR ANALYSIS	102
5.0	Introduction	102
5.1	Square Error Analysis	102
5.2	Illustrative examples	105
5.3	Absolute error analysis	106
5.4	Illustrative examples	107
5.5	Discussion of results	108
5.6	Conclusions	111
<u>CHAPTER 6</u>	STABILITY	113
6.0	Introduction	113
6.1	Multichannel theory	113
6.2	Conditions of compatibility	113
6.3	Stability criterion of operator	122
6.4	Modified inverse	125
6.5	Illustrative examples	128
6.6	Discussion of results	134
6.7	Stability of autocorrelation matrix	135
<u>CHAPTER 7</u>	CONCLUSIONS	140
<u>ACKNOWLEDGEMENTS</u>		141
<u>REFERENCES</u>		142
<u>APPENDIX</u>		143

CHAPTER 1

F-K ANALYSIS

1.0 Introduction

It is known that signal and correlated noise have different move out functions. If a double Fourier transformation is made on a set of given seismic traces to transform from the time-distance domain to frequency-wave number domain it is seen that the signal and the correlated noise are separated on the different move out basis. Assuming a linear move out, B. A. Haines in his Ph. D. thesis, 'Studies of Auto-Matrix & F. T. Analysis', has shown examples of this transformation. In this chapter a mathematical representation of the transformation is given. It is also demonstrated how the separation occurs and the critical points at which this takes place. The approach is different to that of Haines in which no mathematical justification is given for his contouring pattern. It is shown that if differentiated values are plotted and each trace is taken independently a much clearer pattern (with mathematical justification) is obtained for the differential move out.

1.1 Derivation of Fourier Transform for Linear Move Out

The seismic model chosen for the computation of synthetic data is defined by the following relation:

$$x_i(t) = s(t - \alpha_i) + \frac{1}{\sqrt{p}} r(t - \beta_i) + \frac{1}{\sqrt{v}} n_i(t) \quad (i=1,2,\dots,N) \quad (1.1)$$

where i = trace number (or channel number)

N = total number of traces

t = discrete time index, at unit sampling interval

$x_i(t)$ = output trace on channel i

$s(t)$ = signal trace component

$r(t)$ = correlated noise trace component

$n_i(t)$ = uncorrelated noise trace component

α_i = signal move out on channel i

β_i = correlated noise move out on channel i

p = signal to correlated noise power ratio

v = signal to uncorrelated noise power ratio

The three constituent traces $s(t)$, $r(t)$ and $n(t)$ were constructed from wavelets and random spike series according to

$$s(t) = b_1(t) * \psi(t)$$

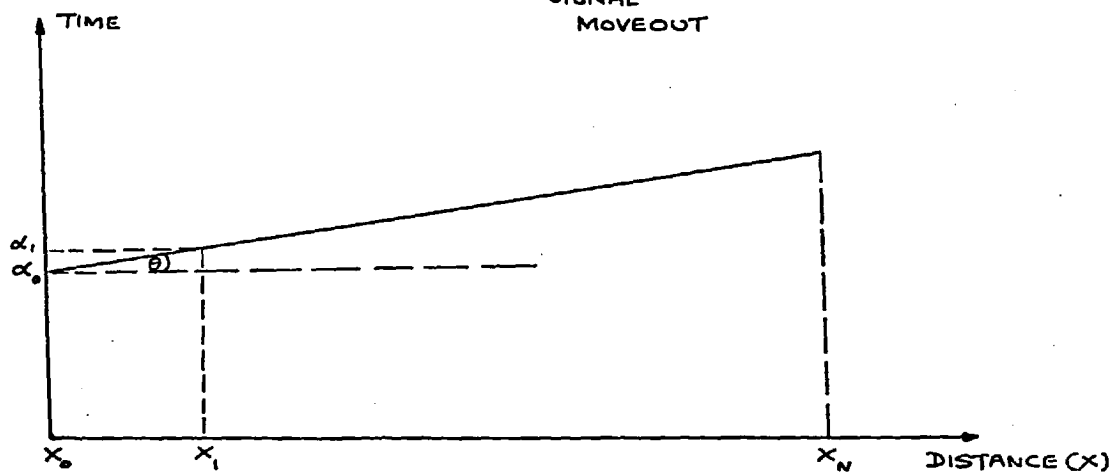
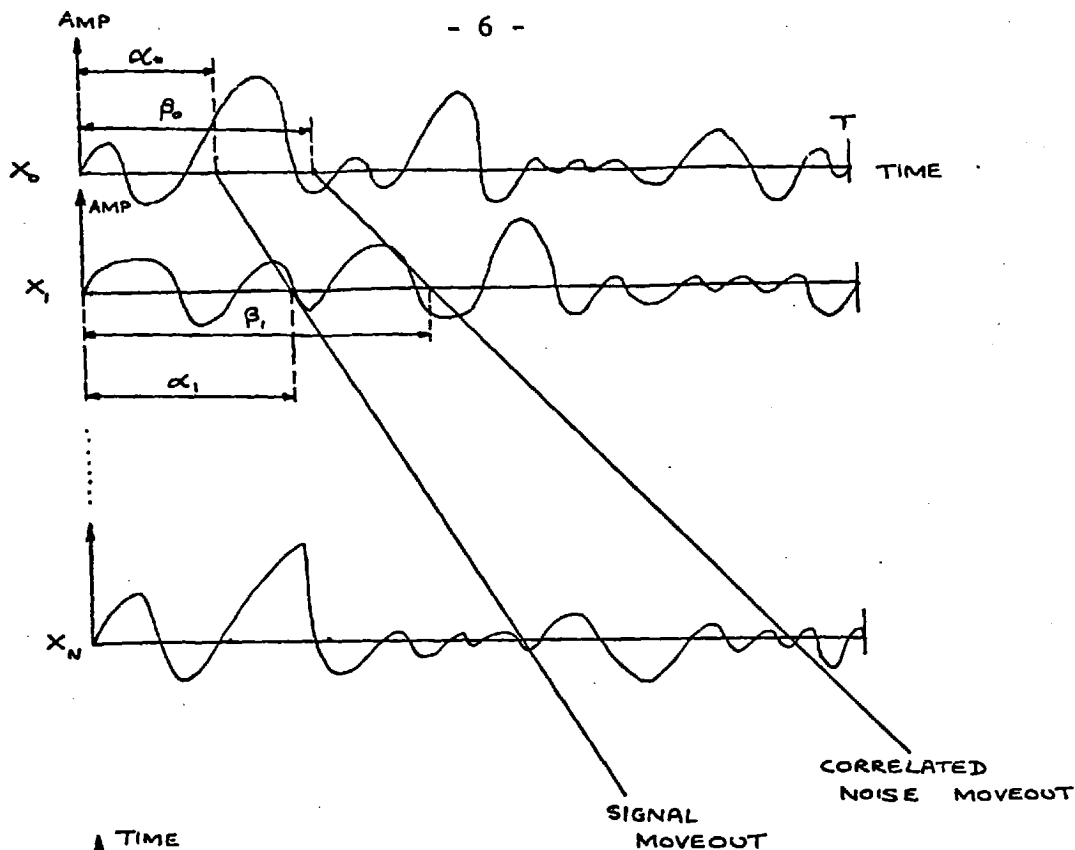
$$r(t) = b_2(t) * \xi(t)$$

$$n_i(t) = b_3(t) * \eta_i(t)$$

The following analysis can be performed under the assumption of linear move out for the signal and the correlated noise. Equation 1.1 and the above assumption give rise to the diagrammatic representation as given in fig. 1.01.

NOTE: $b_1(t)$, $b_2(t)$, $b_3(t)$; $\psi(t)$, $\xi(t)$, $\eta_i(t)$

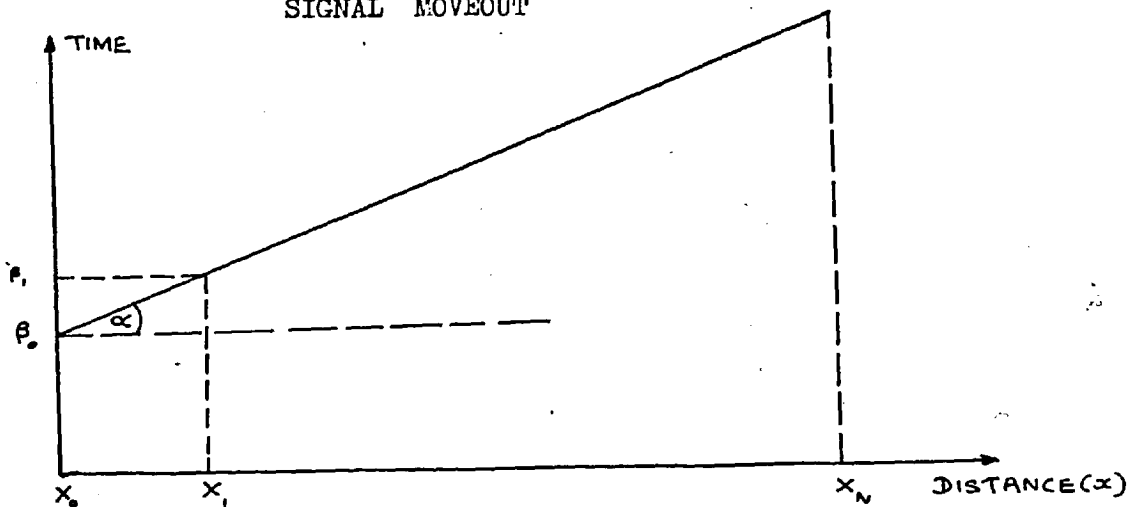
ARE THE WAVELETS AND SPIKE SERIES FOR SIGNAL
CORRELATED NOISE AND UNCORRELATED NOISE.



$$\tan \theta = m_s$$

$$\alpha_r = m_s x_r + \alpha_0$$

SIGNAL MOVEOUT



$$\tan \alpha = m_c$$

$$\beta_r = m_c x_r + \beta_0$$

CORRELATED NOISE MOVEOUT

FIG. 1.01

The exact double Fourier Transformation of 1.1 is

$$A(f, k) = \int_0^{X_N} \int_0^T \left\{ s(t - \alpha_i) + \frac{1}{\sqrt{p}} r(t - \beta_i) + \frac{1}{\sqrt{v}} n_i(t) \right\} dt dx e^{-2\pi i(f t + k x)}$$

Consider a typical term

$$\begin{aligned} & \int_0^{X_N} \int_0^T s(t - \alpha_i) e^{-2\pi i(f t + k x)} dt dx \\ &= \int_0^{X_N} \int_0^T b_1(t - \alpha_i) * \psi(t - \alpha_i) e^{-2\pi i(f t + k x)} dt dx \end{aligned}$$

Let $t - \alpha_i = t_1$

$$\begin{aligned} &= \int_0^{X_N} \int_{-\alpha_i}^{T - \alpha_i} \{ (b_1(t_1) * \psi(t_1)) e^{-2\pi i f(t_1 + \alpha_i)} \} dt_1 e^{-2\pi i k x} dx \\ &= \int_0^{X_N} e^{-2\pi i k x} e^{-2\pi i f \alpha_i} \left\{ \int_{-\alpha_i}^{T - \alpha_i} (b_1(t_1) * \psi(t_1)) e^{-2\pi i f t_1} dt_1 \right\} dx \quad (1.2) \end{aligned}$$

Comparing 1.2 with results of Appendix A

$$= \int_0^{X_N} e^{-2\pi i k x} e^{-2\pi i f \alpha_i} (f_{b_1}(f) - E_s) dx (f_{b_1}(f))$$

where E_s is defined in Appendix A.

Substituting for α_i from fig. 1.01

$$\begin{aligned} &= \int_0^{X_N} e^{-2\pi i k x} e^{-2\pi i f(m_s x_i + \alpha_0)} \{ f_{b_1}(f) \} \{ f_{\psi}(f) - E_s \} dx \\ &= f_{b_1}(f) \{ f_{\psi}(f) - E_s \} e^{-2\pi i f \alpha_0} \left\{ \frac{e^{-2\pi i x(k + m_s f)}}{-2\pi i(k + m_s f)} \right\} \int_0^{X_N} \end{aligned}$$

$$= f_{b1}(f) \{f_{\psi}(f) - E_s\} e^{-2\pi i f \alpha_0} \left\{ \frac{e^{-2\pi i X_N(k+m_s f)} - 1}{-2\pi i (k+m_s f)} \right\}$$

Therefore

$$\begin{aligned} A(f, k) = & \frac{f_{b1}(f) \{f_{\psi}(f) - E_s\} e^{-2\pi i f \alpha_0}}{2\pi i (k+m_s f)} \left\{ \frac{e^{-2\pi i X_N(k+m_s f)}}{1-e} \right\} \\ & + \frac{f_{b2}(f) \{f_{\psi}(f) - E_c\} e^{-2\pi i f \beta_0}}{\sqrt{p} 2\pi i (k+m_c f)} \left\{ \frac{e^{-2\pi i X_N(k+m_c f)}}{1-e} \right\} + \dots \\ & + \frac{1}{\sqrt{v}} f_{b3}(f) \int_0^{X_N} \{ \eta_x(f) - E_v \} e^{-2\pi i k x} dx \end{aligned} \quad (1.3)$$

for $k \neq -m_s f$, $k \neq -m_c f$ and $k \neq 0$.

If $k = f = 0$

$$A(0,0) = f_{b1}(0) \{f_{\psi}(0) - E_s\} + \frac{f_{b2}(0) \{f_{\psi}(0) - E_c\}}{\sqrt{p}} + \frac{1}{\sqrt{v}} f_{b3}(0) \int_0^{X_N} \{ \eta_x(0) - E_v \} dx$$

And if $k = -m_s f$

$$\begin{aligned} A(f, k) = & f_{b1}(f) \{f_{\psi}(f) - E_s\} e^{-2\pi i f \alpha_0} \\ & + \frac{f_{b2}(f) \{f_{\psi}(f) - E_c\} e^{-2\pi i f \beta_0}}{\sqrt{p} 2\pi i (-m_s f + m_c f)} \left(\frac{e^{-2\pi i X_N(-m_s f + m_c f)}}{1-e} \right) \\ & + \frac{1}{\sqrt{v}} f_{b3}(f) \int_0^{X_N} \{ \eta_x(f) - E_v \} e^{-2\pi i k x} dx \end{aligned}$$

Similar expression for $k = -m_c f$.

1.1.1 Analysis of the Transform

Expanding 1.3

$$A(f,k) = \frac{f_{b1}(f) \{f_{\psi}(f) - E_s\}}{2\pi i(k+m_s f)} (\cos 2\pi f \alpha_0 - i \sin 2\pi f \alpha_0) \\ \{1 - \cos 2\pi X_N(k+m_s f) + i \sin 2\pi X_N(k+m_s f)\} \\ + \text{similar expression for correlated noise} \\ + \frac{1}{\sqrt{V}} f_{b3}(f) \int_0^{X_N} (\eta_x(f) - E_v) (\cos 2\pi kx - i \sin 2\pi kx) dx$$

$$\text{Real } A(f,k) = \frac{f_{b1}(f) \{f_{\psi}(f) - E_s\}}{-2\pi(k+m_s f)} \{-\cos 2\pi f \alpha_0 \sin 2\pi X_N(k+m_s f) \\ + \sin 2\pi f \alpha_0 - \sin 2\pi f \alpha_0 \cos 2\pi X_N(k+m_s f)\} \\ + \text{similar expression for correlated noise} \\ + \frac{1}{\sqrt{V}} f_{b3}(f) \int_0^{X_N} \{\eta_x(f) - E_v\} \cos 2\pi kx dx$$

$$\frac{\partial}{\partial k} (\text{Real } A(f,k)) = \frac{f_{b1}(f) \{f_{\psi}(f) - E_s\}}{(-2\pi(k+m_s f))^2} [-\cos 2\pi f \alpha_0 \sin 2\pi X_N(k+m_s f) \\ + \sin 2\pi f \alpha_0 - \sin 2\pi f \alpha_0 \cos 2\pi X_N(k+m_s f)] (-1) + (k+m_s f) \\ (-2\pi X_N \cos 2\pi f \alpha_0 \cos 2\pi X_N(k+m_s f) + 2\pi X_N \sin 2\pi f \alpha_0 \sin 2\pi X_N(k+m_s f)) \\ + \text{similar term for correlated noise} \\ + \frac{1}{\sqrt{V}} f_{b3}(f) \int_0^{X_N} \{\eta_x(f) - E_v\} \sin 2\pi kx (-2\pi x) dx$$

$$= \frac{f_{b1}(f) \{f_{\psi}(f) - E_s\}}{-2\pi(k+m_s f)^2} \left[\cos 2\pi f \alpha_o \sin 2\pi X_N(k+m_s f) - \sin 2\pi f \alpha_o \{1 - \cos 2\pi X_N(k+m_s f)\} \right. \\ \left. - 2\pi X_N(k+m_s f) \{ \cos(2\pi f \alpha_o + 2\pi X_N(k+m_s f)) \} \right] + \text{similar term for correlated noise + et} \\ \text{as } (k+m_s f) = I \text{ where } I \text{ is an integer}$$

$$\sin 2\pi X_N(k+m_s f) \rightarrow 0$$

$$\cos 2\pi X_N(k+m_s f) \rightarrow 1 \quad \text{and} \quad \cos \{2\pi f \alpha_o + 2\pi X_N(k+m_s f)\} \rightarrow \cos(2\pi f \alpha_o)$$

Therefore

$$\frac{\partial}{\partial k} (\text{real } A(f, k)) = (..) \frac{-2\pi X_N(k+m_s f) \cos 2\pi f \alpha_o}{(k+m_s f)^2}$$

$$\text{as } k \rightarrow -m_s f$$

$$+ \text{etc.} \rightarrow -\infty \text{ for } k = -m_s f + \delta$$

$$\rightarrow \infty \text{ for } k = -m_s f - \delta$$

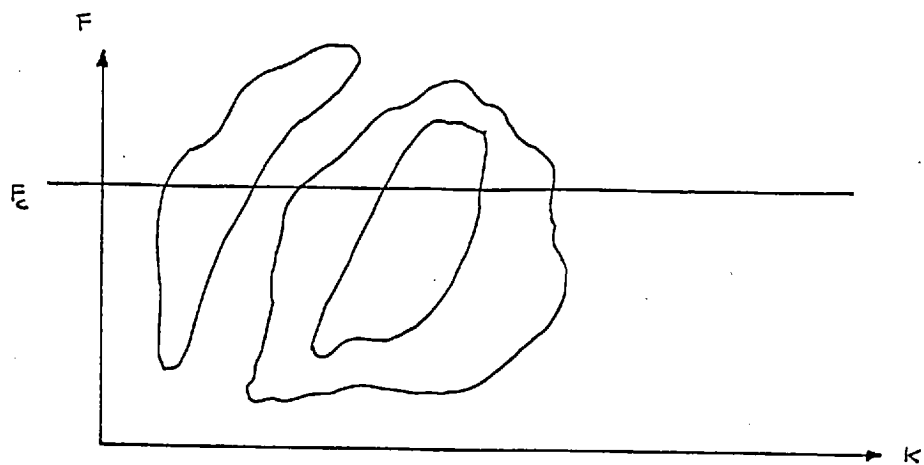
On a similar argument

$$\frac{\partial}{\partial k} (\text{real } A(f, k)) \rightarrow -\infty \quad \text{for } k = -m_c f + \delta$$

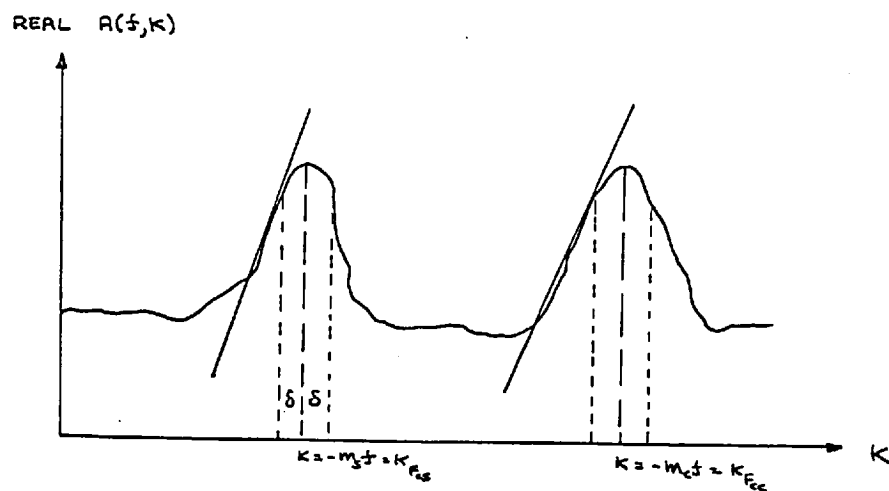
$$\text{as } k \rightarrow -m_c f$$

$$\rightarrow \infty \quad \text{for } k = -m_c f - \delta$$

For a constant value of f , the gradient of the graph of real $A(f, k)$ against k should have a steep gradient at values of k just less or greater than $k = -m_s f$ and $k = -m_c f$



CONTOURING OF REAL $A(f, k)$



AT $F = F_c$

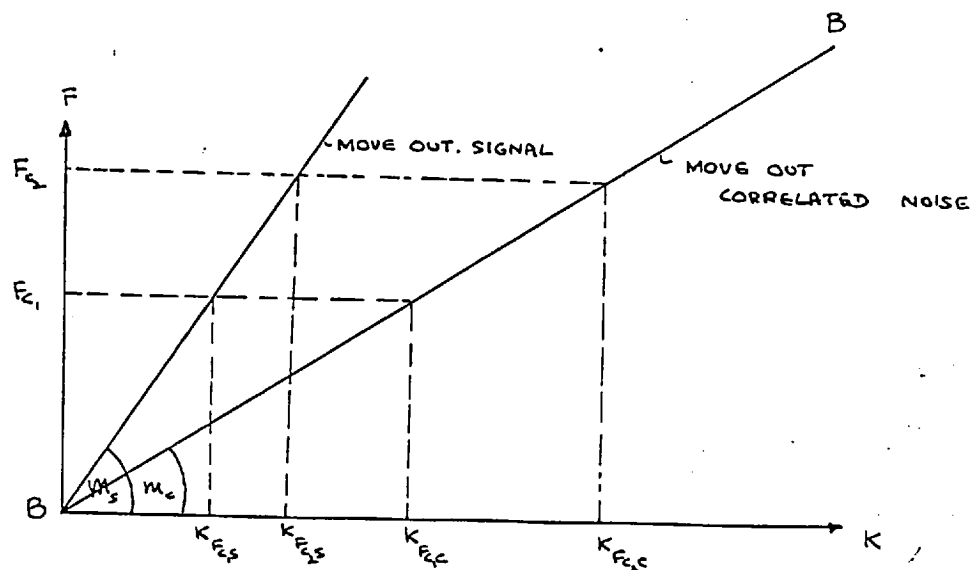


FIG. 1.02

$$\text{Similarly, } \text{Imag } A(f, k) = \frac{f_{b1}(f) \{f_{\psi}(f) - E_s\}}{-2\pi(k+m_s f)}$$

$$\begin{aligned} & \left\{ \cos 2\pi f \alpha_0 - \cos 2\pi f \alpha_0 \cos 2\pi X_N(k+m_s f) + \sin 2\pi f \alpha_0 \sin 2\pi X_N(k+m_s f) \right\} + \dots \\ & + \frac{1}{\sqrt{v}} f_{b3}(f) \int_0^{X_N} \{ \eta_x(f) - E_v \} \sin(2\pi k x) dx \end{aligned}$$

$$\frac{\partial}{\partial k} \text{Imag } A(f, k) = \frac{f_{b1}(f) \{f_{\psi}(f) - E_s\}}{-2\pi(k+m_s f)^2}$$

$$\begin{aligned} & \left[(k+m_s f) \{ 2\pi X_N \cos 2\pi f \alpha_0 \sin 2\pi X_N(k+m_s f) \right. \\ & \quad \left. + 2\pi X_N \sin 2\pi f \alpha_0 \cos 2\pi X_N(k+m_s f) \} \right. \\ & \quad \left. - \{ \cos 2\pi f \alpha_0 - \cos 2\pi f \alpha_0 \cos 2\pi X_N(k+m_s f) + \sin 2\pi f \alpha_0 \sin 2\pi X_N(k+m_s f) \} \right] \\ & = \frac{1}{\sqrt{v}} f_{b3}(f) \int_0^{X_N} \{ \eta_x(f) - E_v \} \cos 2\pi k x (2\pi x) dx \\ & = \frac{f_{b1}(f) \{f_{\psi}(f) - E_s\}}{-2\pi(k+m_s f)^2} \left[2\pi X_N(k+m_s f) \sin(2\pi f \alpha_0 + 2\pi X_N(k+m_s f)) \right. \\ & \quad \left. - \{ \cos 2\pi f \alpha_0 (1 - \cos 2\pi X_N(k+m_s f)) + \sin 2\pi f \alpha_0 \sin 2\pi X_N(k+m_s f) \} \right] + \text{etc.} \end{aligned}$$

$\rightarrow \infty$ as $k \rightarrow -m_s f$ as before.

1.1.2 Analysis of the Sampled Function

$$\begin{aligned}
 A(f, k) &= \sum_{x=0}^{X_N} \sum_{t=0}^T s(t - \frac{\alpha}{x}) + \frac{1}{\sqrt{p}} r(t - \frac{\beta}{1}) + \frac{1}{\sqrt{v}} n_i(t) e^{-2\pi i(f t + k x)} \\
 &= f_{b1}(f) \{f_{\psi}(f) - E_s\} e^{-2\pi i f \alpha_0} \sum_{x=0}^{X_N} e^{-2\pi i x(k + f m_s)} \\
 &\quad + f_{b2}(f) \frac{\{f_{\psi}(f) - E_c\}}{\sqrt{p}} e^{-2\pi i f \beta_0} \sum_{x=0}^{X_N} e^{-2\pi i x(k + f m_c)} \\
 &\quad + \frac{f_{b3}(f)}{\sqrt{v}} \sum_{x=0}^{X_N} \eta_x(f) e^{-2\pi i k x} \tag{1.4}
 \end{aligned}$$

for all k, f .

From 1.4, expanding the equation:

$$\begin{aligned}
 A(f, k) &= f_{b1}(f) \{f_{\psi}(f) - E_s\} \{ \cos 2\pi f \alpha_0 - i \sin 2\pi f \alpha_0 \} \\
 &\quad \sum_{x=0}^{X_N} \{ \cos 2\pi x(k + f m_s) - i \sin 2\pi x(k + f m_s) \} + \text{similar term for correlated noise} \\
 &\quad \dots + \frac{f_{b3}(f)}{\sqrt{v}} \sum_{x=0}^{X_N} (\cos 2\pi k x - i \sin 2\pi k x) \eta_x(f) \tag{1.5}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \text{Real}\{A(f, k)\} &= f_{b1}(f) \{f_{\psi}(f) - E_s\} \\
 &\quad \left[\sum_{x=0}^{X_N} \{ (\cos 2\pi f \alpha_0 \cos 2\pi x(k + f m_s) - \sin 2\pi f \alpha_0 \sin 2\pi x(k + f m_s)) \} \right] + \dots \\
 &\quad + \frac{f_{b3}(f)}{\sqrt{v}} \sum_{x=0}^{X_N} \eta_x(f) \cos 2\pi k x
 \end{aligned}$$

$$\text{Imag } A(f, k) = f_{b1}(f) \{f_{\psi}(f) - E_s\}$$

$$\left[\sum_{x=0}^{X_N} \{-\cos 2\pi f \alpha_0 \sin 2\pi x(k + fm_s) - \sin 2\pi f \alpha_0 \cos 2\pi x(k + fm_s)\} \right] + \dots$$

$$- \frac{f_{b3}(f)}{\sqrt{v}} \sum_{x=0}^{X_N} \eta_x(f) \sin 2\pi kx$$

Therefore,

$$\text{Real}\{A(f, k)\} = f_{b1}(f) \{f_{\psi}(f) - E_s\} \sum_{x=0}^{X_N} \cos 2\pi (f \alpha_0 + x(k + fm_s)) + \dots$$

$$+ \frac{f_{b3}(f)}{\sqrt{v}} \sum_{x=0}^{X_N} \eta_x(f) \cos 2\pi kx \quad (1.6)$$

$$\text{Imag}\{A(f, k)\} = f_{b1}(f) \{f_{\psi}(f) - E_s\} \sum_{x=0}^{X_N} -(\sin 2\pi (f \alpha_0 + x(k + fm_s))) + \dots$$

$$- \frac{f_{b3}(f)}{\sqrt{v}} \sum_{x=0}^{X_N} \eta_x(f) \sin 2\pi kx \quad (1.7)$$

The absolute expansion of 1.5 is:

$$A(f, k) = \{f_{b1\psi}(R) + if_{b1\psi}(I)\}$$

$$\sum_{x=0}^{X_N} (\cos 2\pi (f \alpha_0 + x(k + fm_s)) - i \sin 2\pi (f \alpha_0 + x(k + fm_s))) + \dots + (\text{CORR. NOISE})$$

$$+ \sum_{x=0}^{X_N} \{f_{b3\eta}(R) + if_{b3\eta}(I)\} \{\cos 2\pi kx - i \sin 2\pi kx\} \quad (1.8)$$

where $f_{b1\psi}(R) + if_{b1\psi}(I) = f_{b1}(f) \{f_{\psi}(f) - E_s\}$

and $f_{b3\eta}(R) + if_{b3\eta}(I) = \frac{f_{b3}(f)}{\sqrt{v}} \eta_x(f)$

Since the Fourier Transform has a Real part as well as an Imaginary part.

Therefore,

$$\begin{aligned}
 \text{Real}\{A(f,k)\} &= \sum_{x=0}^{X_N} \{f_{b1\psi}(R) \cos 2\pi(G) + f_{b1\psi}(I) \sin 2\pi(G)\} + \dots \\
 &\quad \text{WHERE } G = f\alpha_0 + x(k+f m_s) \\
 &+ \sum_{x=0}^{X_N} \{f_{b3\eta}(R) \cos 2\pi kx + f_{b3\eta}(I) \sin 2\pi kx\} \\
 &= A \sum_{x=0}^{X_N} \cos \{2\pi(f\alpha_0 + x(k+f m_s)) - \theta_A\} + \dots \\
 &+ \sum_{x=0}^{X_N} B_{x,f} \cos(2\pi kx - \theta_B) \quad (1.9)
 \end{aligned}$$

$$\begin{aligned}
 \text{where } A \cos \theta_A &= f_{b1\psi}(R); & A \sin \theta_A &= f_{b1\psi}(I); \\
 B_{x,f} \cos \theta_B &= f_{b3\eta}(R); & B_{x,f} \sin \theta_B &= f_{b3\eta}(I).
 \end{aligned}$$

On the same basis,

$$\begin{aligned}
 \text{Imag}\{A(f,k)\} &= \sum_{x=0}^{X_N} \{-f_{b1\psi}(R) \sin 2\pi(G) + f_{b1\psi}(I) \cos 2\pi(G)\} + \dots \\
 &+ \sum_{x=0}^{X_N} \{-f_{b3\eta}(R) \sin 2\pi kx + f_{b3\eta}(I) \cos 2\pi kx\} \\
 &= C \sum_{x=0}^{X_N} -\sin \{2\pi(f\alpha_0 + x(k+f m_s)) - \theta_C\} + \dots - \sum_{x=0}^{X_N} D_{x,f} \sin(2\pi kx - \theta_D) \\
 &\quad (1.10)
 \end{aligned}$$

$$\begin{aligned}
 \text{where } f_{b1\psi}(R) &= C \cos \theta_C; & f_{b1\psi}(I) &= C \sin \theta_C; \\
 D_{x,f} \cos \theta_D &= f_{b3\eta}(R); & D_{x,f} \sin \theta_D &= f_{b3\eta}(I).
 \end{aligned}$$

Equations 1.6 and 1.7 are similar to equations 1.9 and 1.10, the only difference being the phase shift of $\Theta_A, \Theta_B, \Theta_C, \Theta_D$ which will shift the curves represented in 1.6 and 1.7 by angles Θ_A, Θ_B , etc. The form of the curves 1.6 and 1.7 will remain the same. Hence all the properties of 1.9 and 1.10 will remain intact in 1.6 and 1.7 but with a relevant shift. The analysis will be restricted to equations 1.6 and 1.7 for easier mathematical representation and computing. Referring to fig. 1.01 we can make $\alpha_0=0$ without any loss of generality, i.e. the time is measured from the arrival of the signal in the first trace. Since the origin of time is arbitrarily chosen, the mathematics can be simplified by choosing the origin at the arrival of the signal. On this basis equations 1.6 and 1.7 can be rewritten as:

$$\begin{aligned} \text{Real}\{A(f,k)\} = & A(f) \sum_{x=0}^{X_N} \cos 2\pi x(k+fm_s) + B(f) \sum_{x=0}^{X_N} \cos 2\pi(f\beta_0 + x(k+fm_c)) \\ & + C(f) \sum_{x=0}^{X_N} D_x(f) \cos 2\pi kx \end{aligned} \quad (1.11)$$

$$\begin{aligned} \text{Imag}\{A(f,k)\} = & A(f) \sum_{x=0}^{X_N} -\sin 2\pi x(k+fm_s) - B(f) \sum_{x=0}^{X_N} \sin 2\pi(f\beta_0 + x(k+fm_c)) \\ & - C(f) \sum_{x=0}^{X_N} D_x(f) \sin 2\pi kx \end{aligned} \quad (1.12)$$

where β is now measured from the new origin. Consider equation 1.12 when $f=0$ i.e. on the k axis,

$$\text{Imag}\{A(0,k)\} = A(0) \sum_{x=0}^N -\sin 2\pi x(k) - B(0) \sum_{x=0}^N \sin 2\pi xk - C(0) \sum_{x=0}^N D_x(0) \sin 2\pi kx$$

Consider a numerical value for $x=9$ (say). The analysis can be varied depending on the number of traces, i.e. the value of N used.

Therefore a typical term for signal and correlated noise in its expanded form would be:

$$= \text{Constant} \sum_{x=0}^9 \sin(2\pi kx) = C \left\{ \sin 0 + \sin 2\pi k + \sin 4\pi k + \dots \right. \\ \left. + (\sin 8\pi k + \sin 10\pi k) + \dots + \sin 18\pi k \right\}$$

Grouping the Sin terms

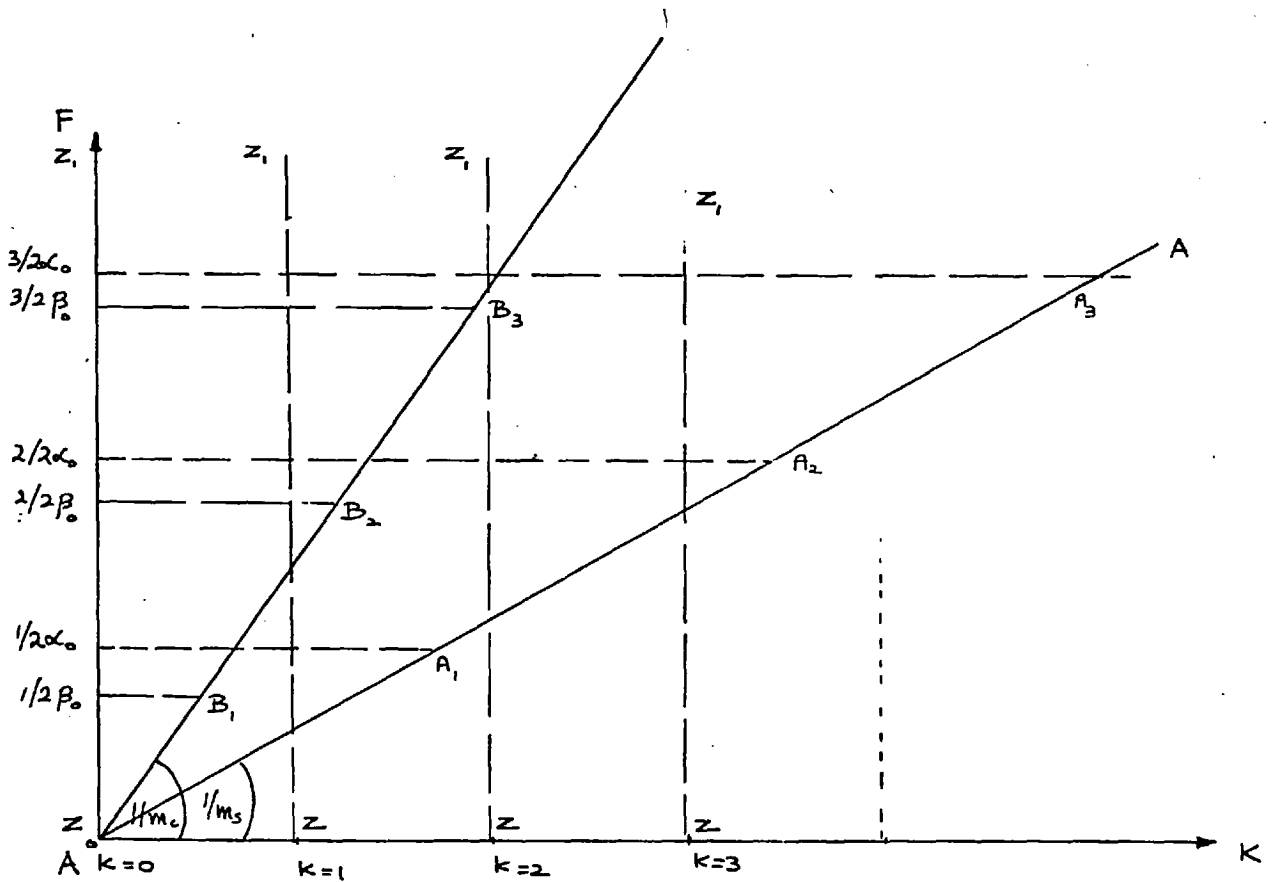
$$= C \left\{ \frac{1}{2} (\sin 9\pi k \cos 9\pi k + \sin 9\pi k \cos 7\pi k + \dots + \sin 9\pi k \cos \pi k) \right\} \\ = C \frac{\sin 9\pi k}{2} (\cos 9\pi k + \cos 7\pi k + \dots + \cos \pi k)$$

If we were to choose the value of k as multiples of 1/9 we would have

$$= \frac{C}{2} \sin m (\dots) \text{ where } m \text{ is now an integer.}$$

But $\sin \pi m = 0$ for all integer values of m. Therefore on the $f=0$ axis at all values of k which are multiples of 1/9 we would have the correlated noise and the signal terms zero. The uncorrelated noise terms would not vanish on these points because each Sin term is multiplied by $D_x(0)$ before summation and hence the total will not vanish. Therefore on these points $\text{Imag } A(f,k)$ would give the Sin representation of the uncorrelated noise wave form. Also uncorrelated noise is zero for all f where k is an integer since $\sin 2\pi kx = 0$ for all integer k as x is an integer. Therefore for every f on lines $k=0,1,2$, etc. only the signal and correlated noise terms will be obtained. On examining the summation for uncorrelated noise it can be seen that for $f \neq 0$ for various values of k the uncorrelated noise wave form will retain its Sin wave form with a multiplying factor depending on $C_x(f)$ and $D_x(f)$. Therefore if we could find these multiplying factors for various values of f, since we know the basic shape of the curve at $f=0$ we can eliminate the uncorrelated noise terms in the (f,k) plot. As shown earlier at $k=0$ only the correlated and the signal values will be obtained. But at $k=(0+\delta)$ the uncorrelated terms will also be added on. But at $k=(0+\delta)$ the values of signal and correlated noise will not change

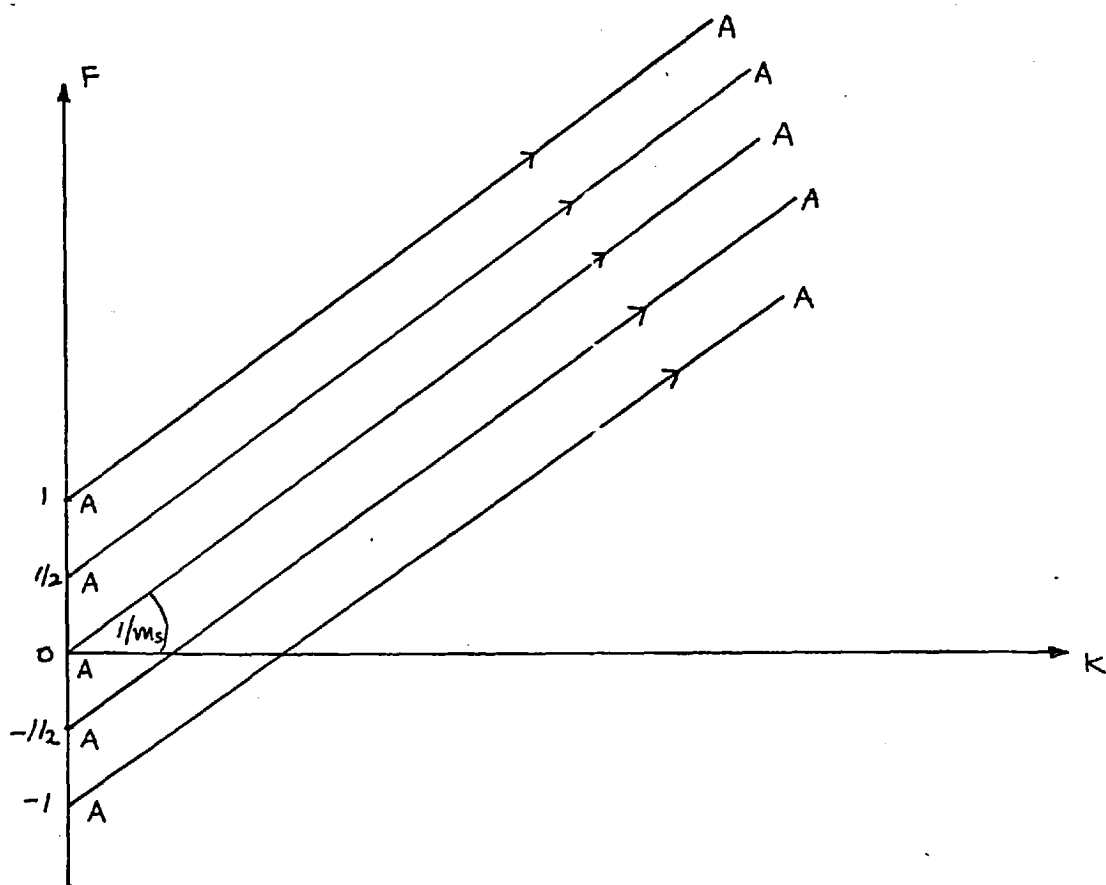
much and hence can be taken as equal to values at $k=0$. Therefore by subtracting the values at $k=0$ from values at $k=(0+\delta)$ we can obtain the multiplying factors for the uncorrelated noise and by multiplying the corresponding co-ordinates of the basic shape of the uncorrelated wave form we can obtain the contour map of the uncorrelated noise only. The points explained so far will be clarified by working through a numerical example.



PLOT OF IMAG (F,K)

FIG. 1.03

The crucial points on the 'F-K' plot are as shown above. Every point on lines marked ZZ_1 have uncorrelated noise zero. On points marked $A_1, A_2, A_3 \dots$ signal component is zero. On points marked $B_1, B_2, B_3 \dots$ the correlated component is zero. If α_0 is zero as is the case considered here every point on line AA will have a signal component zero. If α_0 is zero from equation 1.12 it can be seen that the signal component is zero in all lines represented by the set of equations $K + fm_s = \frac{I}{2}$ where I is an integer. On this condition the signal component will be zero in all points on lines such as AA as shown below.



PLOT OF IMAG (F,K)

$$\alpha_0 = 0$$

FIG. 1.04

A typical term for Real (A(f,k)) is

$$\text{Real (A(f,k))} = C \sum_{x=0}^{X_N} \cos 2\pi (f\alpha_o + x(k+fm_s)) + \dots$$

$$\frac{\partial \text{Real (A(f,k))}}{\partial k} = 2\pi C \sum_{x=0}^{X_N} x \sin 2\pi (f\alpha_o + x(k+fm_s)) + \dots$$

$$\therefore \frac{\partial \text{Real (A(f,k))}}{\partial k} = 0 \text{ when } f\alpha_o + x(k+fm_s) = I$$

for all x from $0 \rightarrow X_n$ when I is an integer. If $f\alpha_o$ is an integer, this condition is satisfied for all x when $k + fm_s = \text{integer}$ which would give the turning point. As could be seen by comparison of eqⁿ 1.3 and the conditions derived from it the conclusions are slightly different from that which is derived from equation 1.6.

The infinite gradient conditions do not occur due to the summation and instead a turning point is obtained with the additional requirement that $f\alpha_o$ etc... should be integer as well. If $f\alpha_o$ was not an integer $k + fm_s = I$ line would not pass through the turning points of various curves. But the deviation is not very much as could be seen by inspection of figure 1.6. In any case only a probable region of occurrence can be designated anyway.

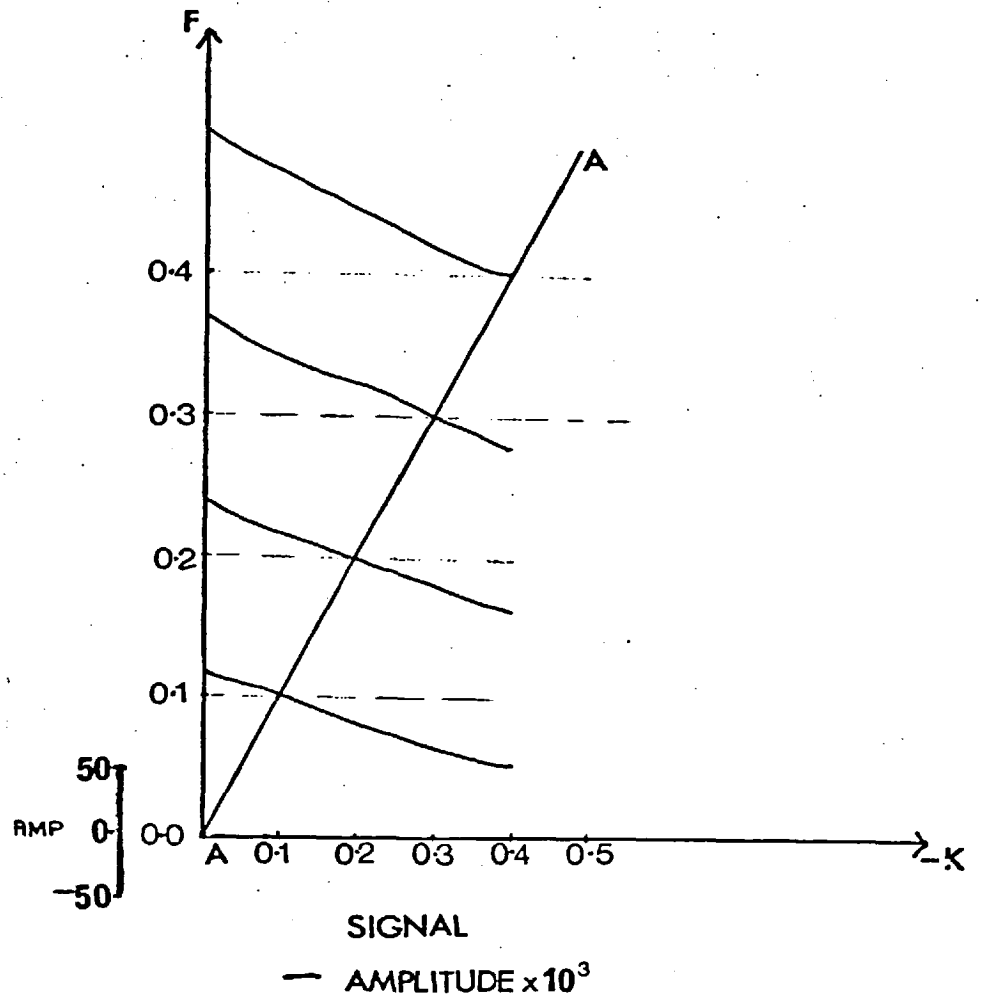
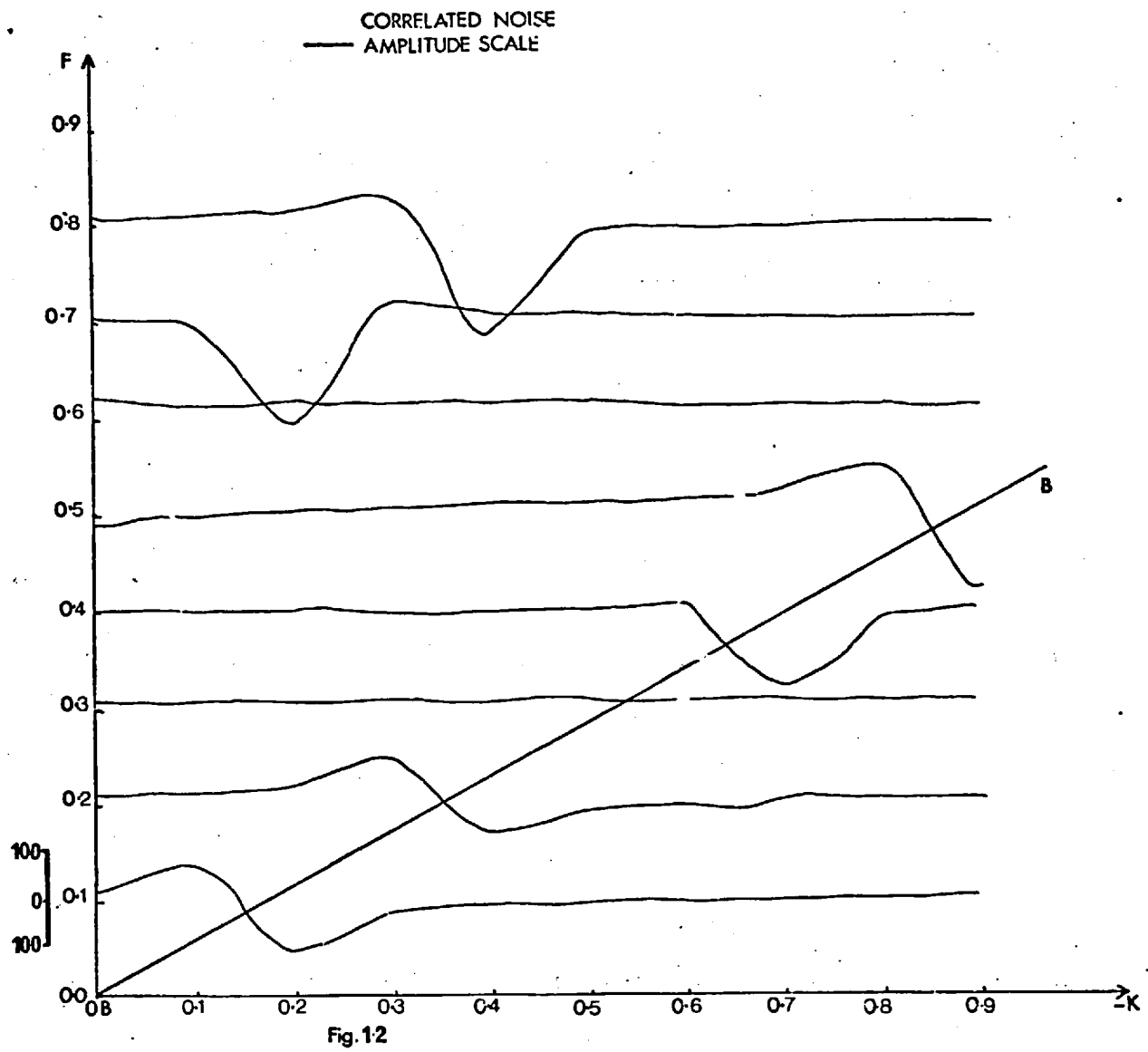


Fig.11



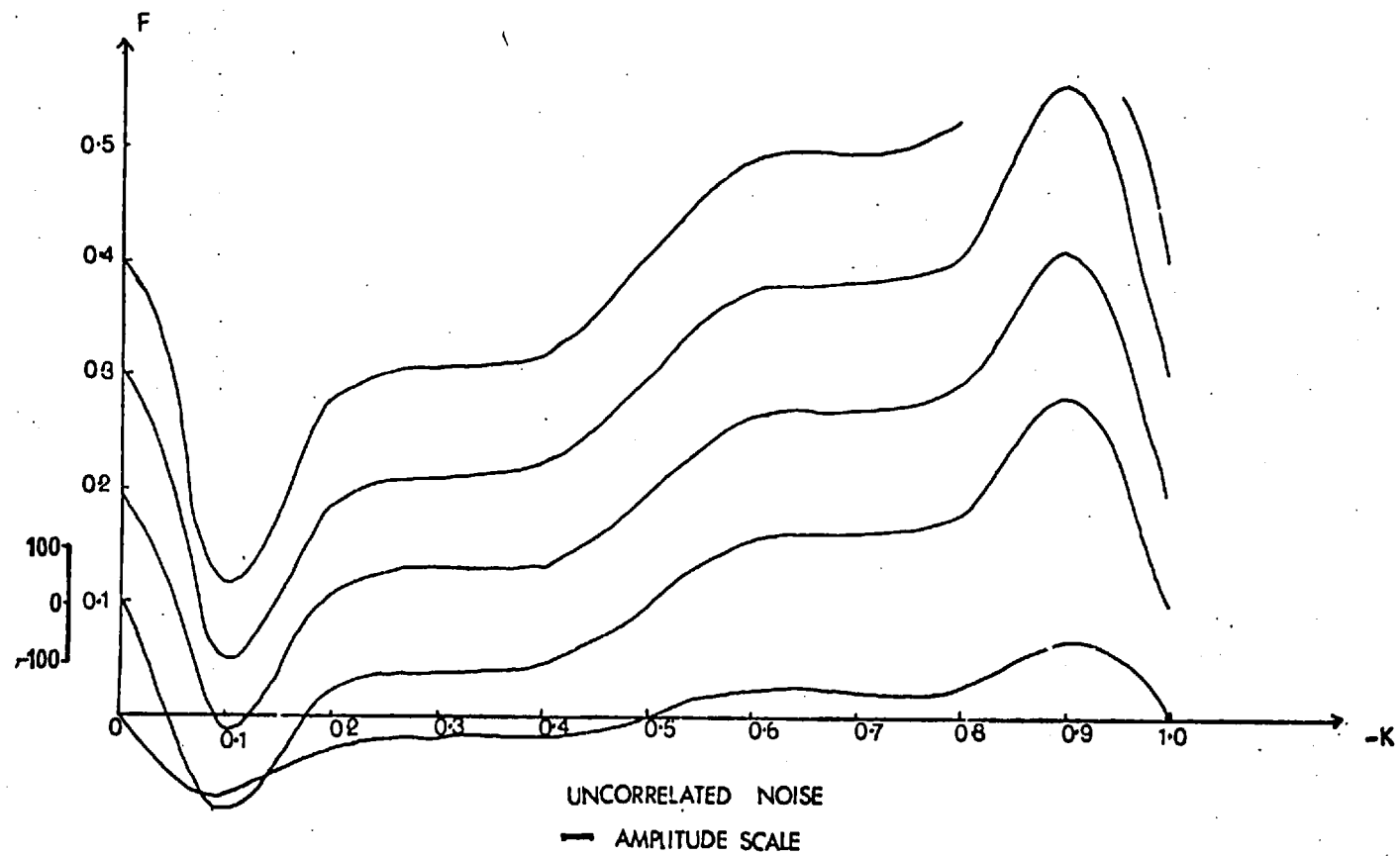
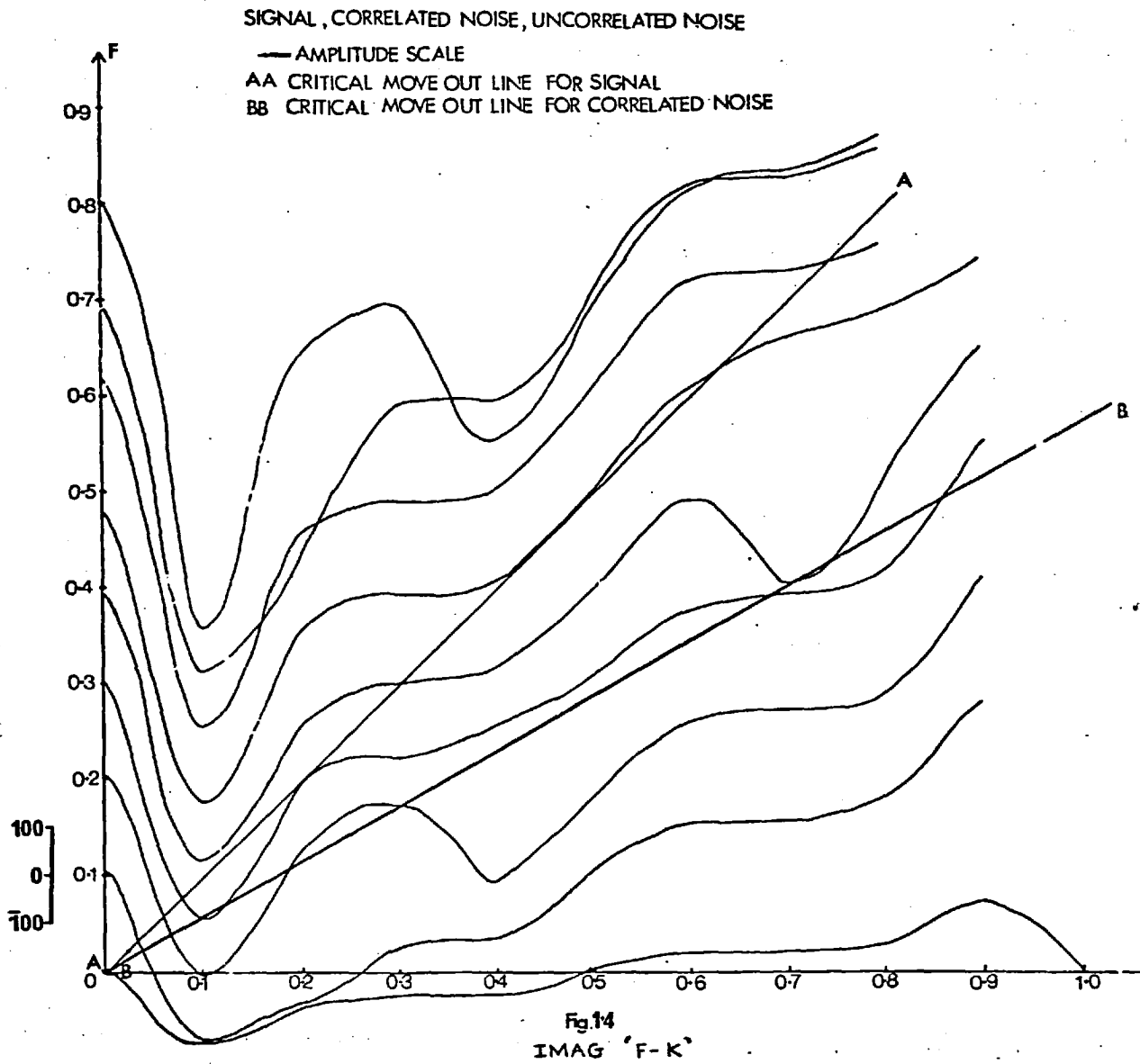


Fig.13



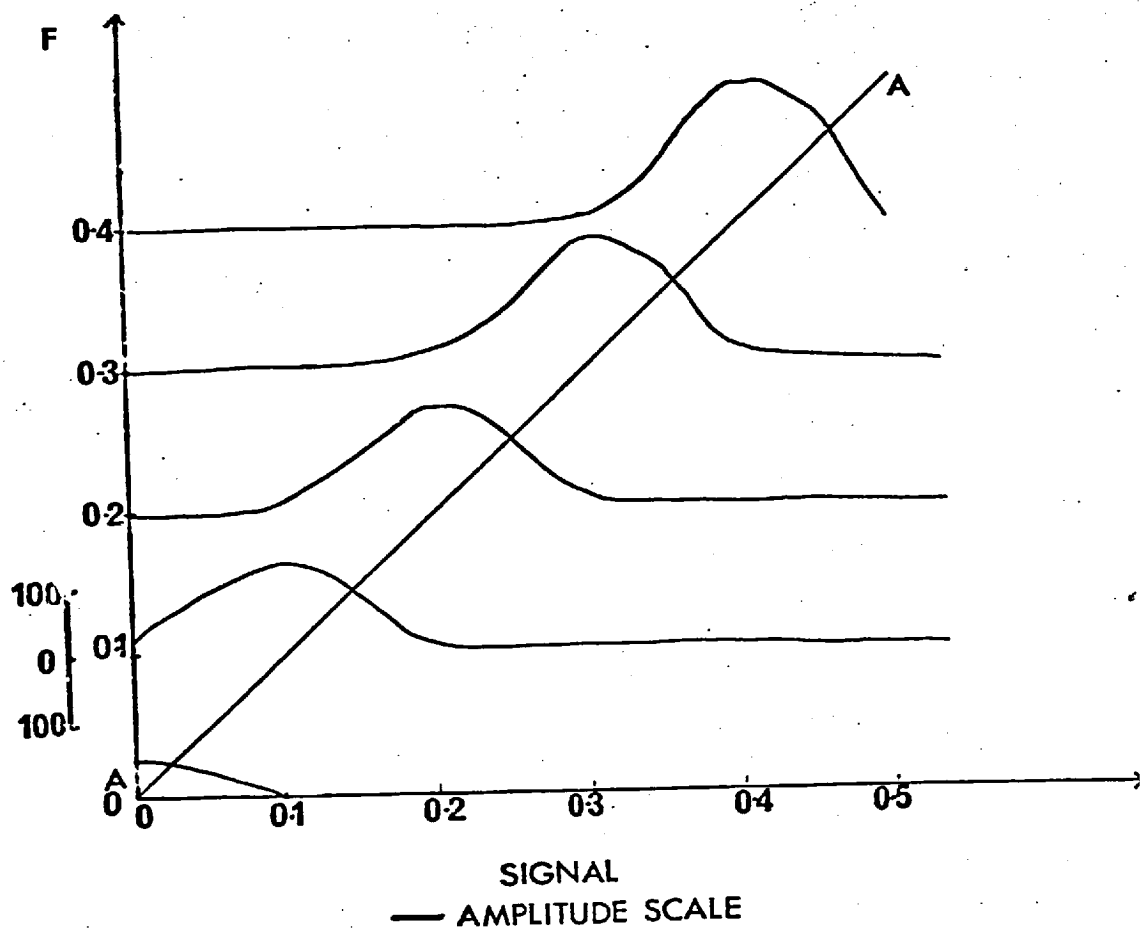


Fig.15

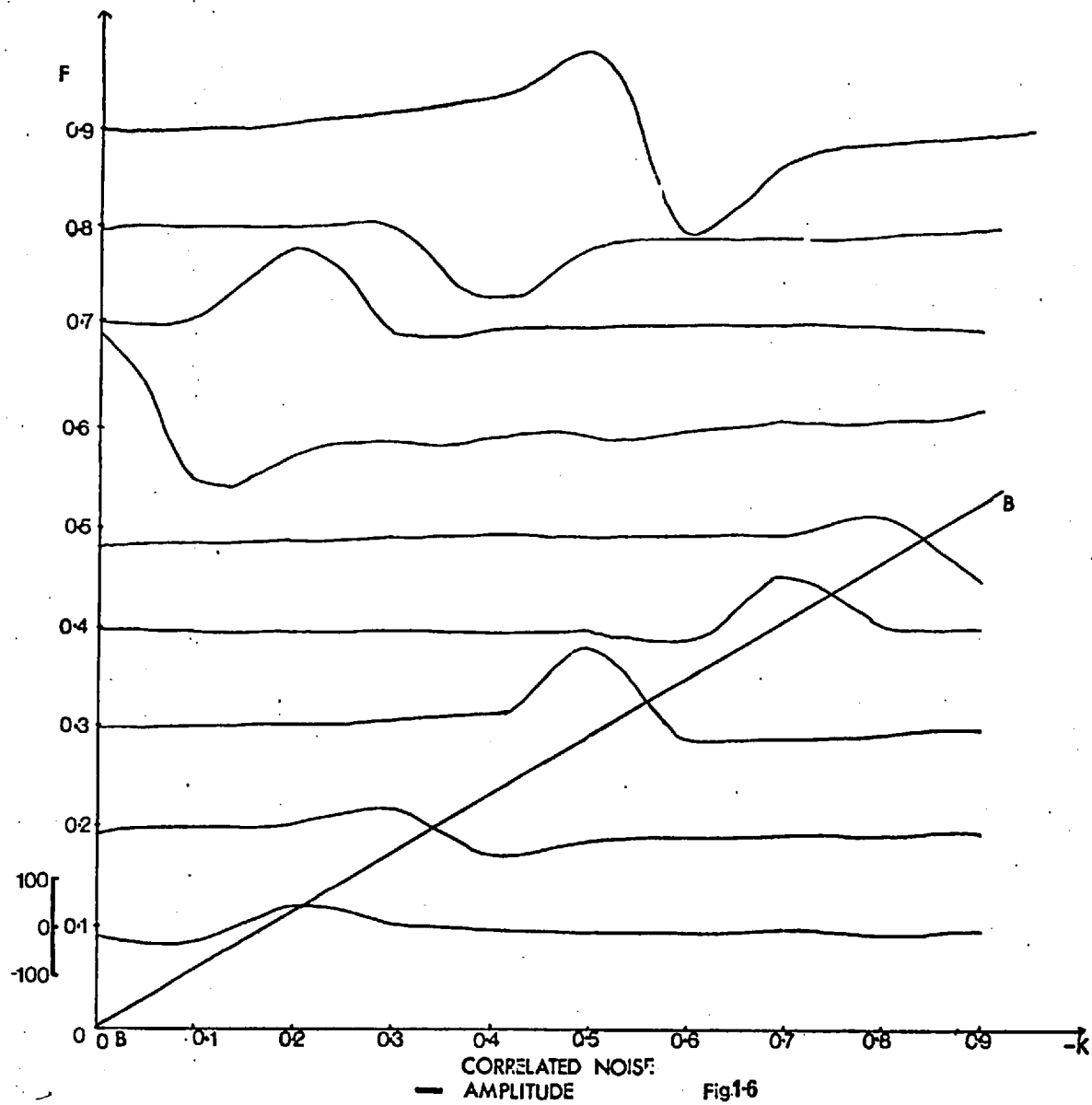


Fig1-6

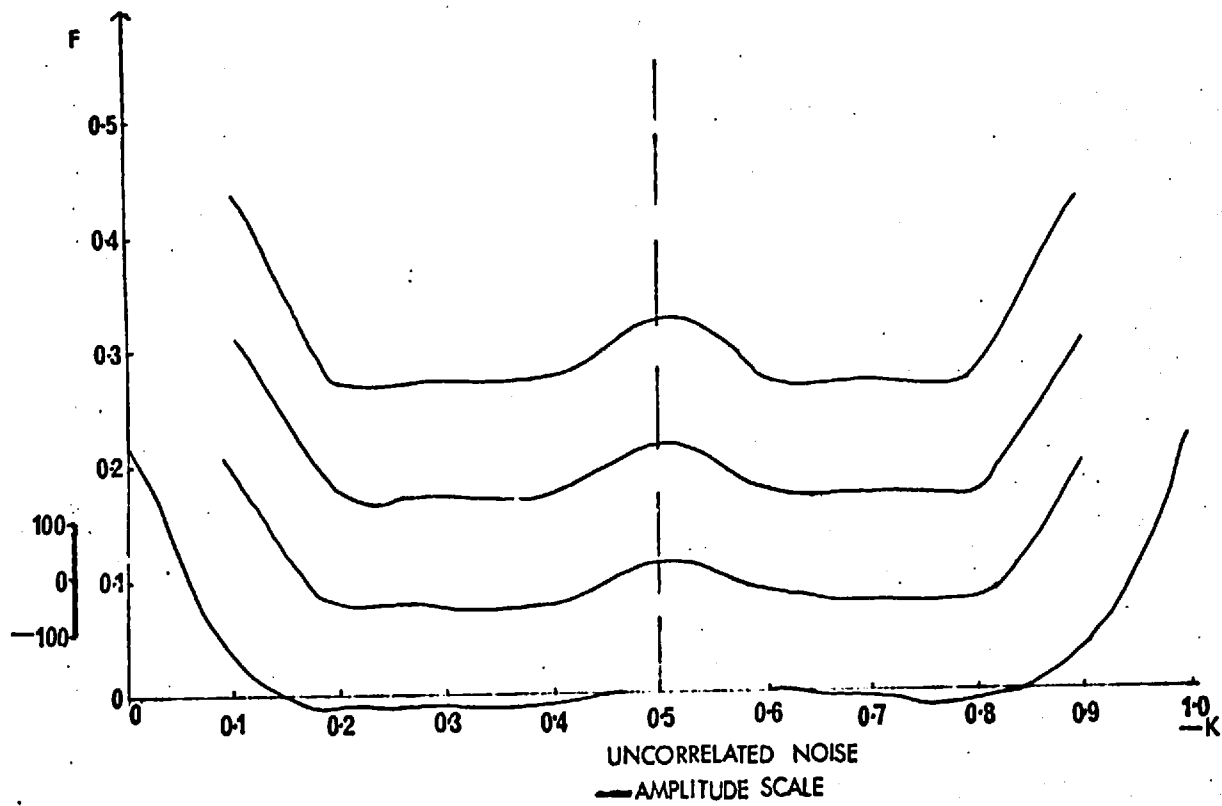


Fig.17

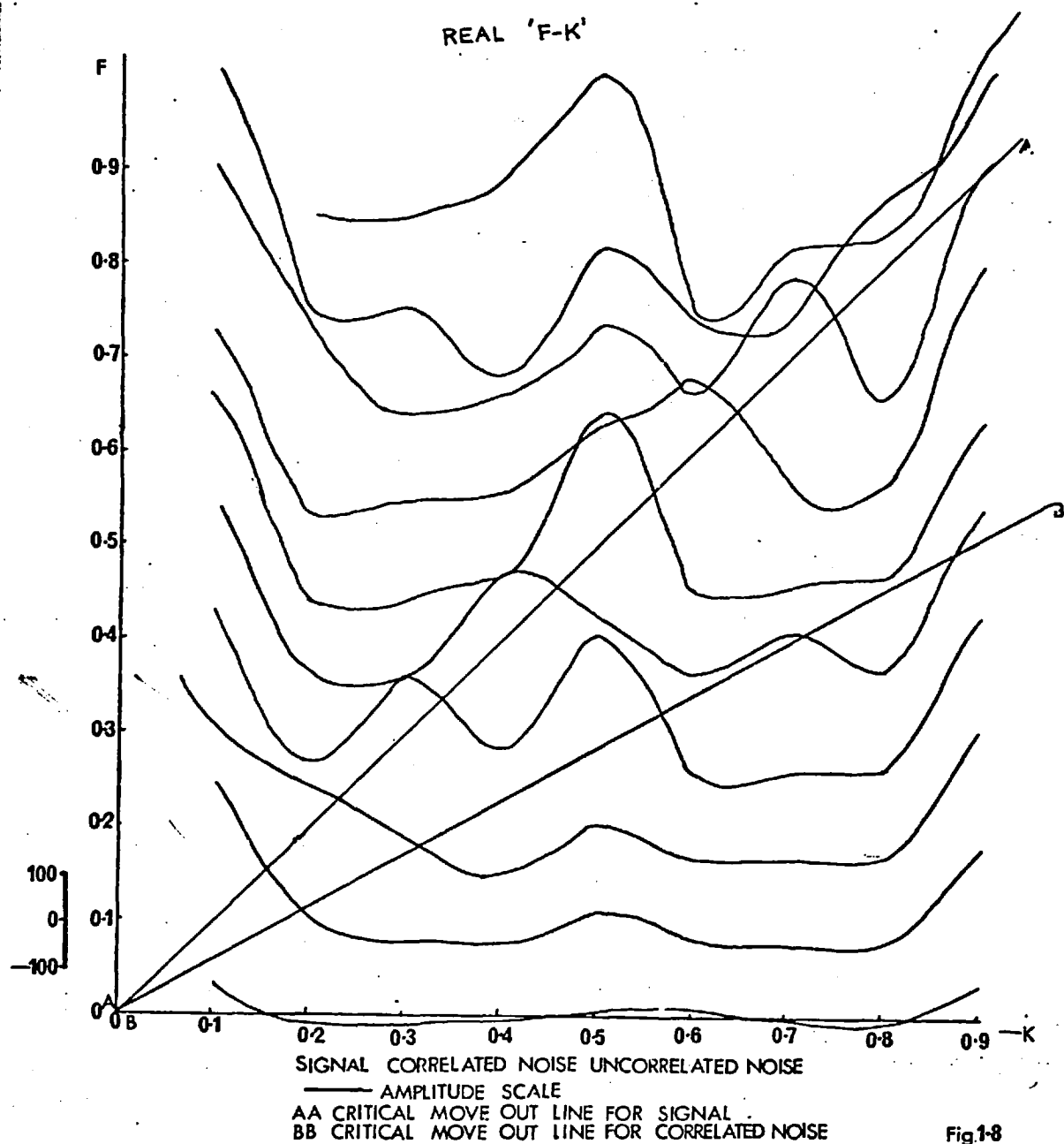


Fig.18

1.13 Illustrative Example

For the numerical example considered now the values were chosen as follows:

$$m_s = \tan 45^\circ ; m_c = \tan 60^\circ ; \alpha_0 = 0 ; \beta_0 = 3.0 \text{ secs.}$$

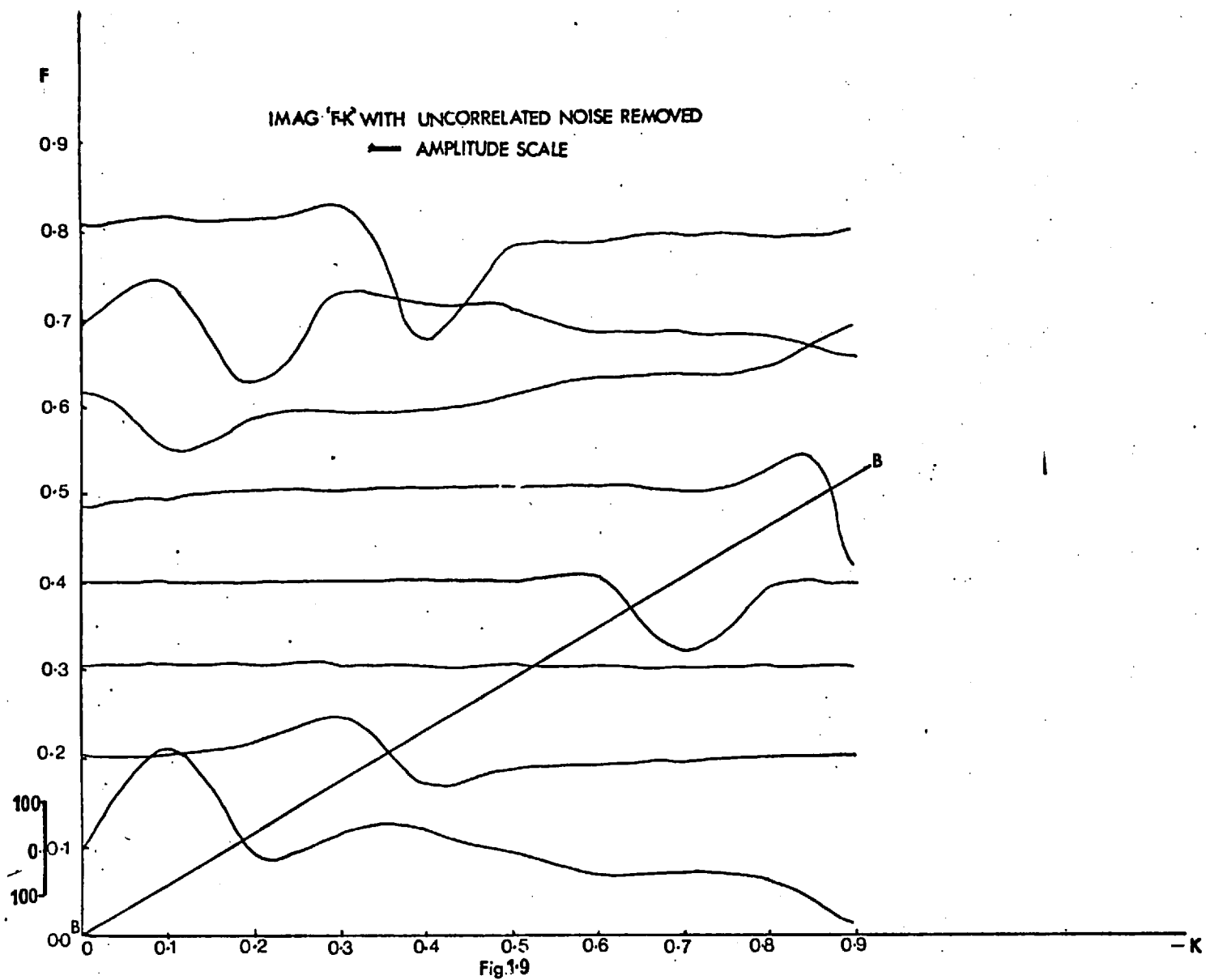
Ten traces were chosen. Initial summation phase angle was taken as zero for signal and correlated noise. Fig. 1.1 indicates the theoretical picture of the Imag (F-K) plot that would be obtained if there were no noise. As can be seen the zero magnitude contour is a straight line through the origin with an angle gradient of 1 corresponding to $m_s = \tan 45^\circ$. The curves are the graphs obtained for constant values of F for F=0, 0.1,... for various values of K=0,0.1,0.2 etc. Fig. 1.2 shows the theoretical picture of the Imag (F-K) plot that would be obtained if there were no signal and no uncorrelated noise. The change of gradient as explained in fig. 1.02 is seen along the line BB which is at an angle $(1/m_c)$ where m_c is the move out gradient for correlated noise. Fig. 1.3 shows the theoretical picture of the Imag (F-K) plot that would be obtained if there were no signal and no correlated noise. It can be seen that the curve is symmetrical (mirror image) about the point $k=-0.5$ and zero at $k=0,1$. For different values of F the curve retains its basic form as in $f=0$ but the ordinates are multiplied by a constant factor denoted by the multiplying factor in the discussion before. Fig. 1.4 shows the actual picture that would be obtained in practice. It can be seen that the move out gradient variations have been considerably distorted due to the uncorrelated noise component and it is not easy to draw the lines AA and BB though a probable region of occurrence can be designated. Figs. 1.5, 1.6, 1.7, 1.8 show the same considerations as above but plotted for the real (F,K). Comparing fig. 1.5 with fig. 1.1 the move out gradient change is shown much better in 1.5 than in 1.1. This is due to the fact that the real component is a Cos form which gives a unit at the critical value and the Imag component

is a Sin form which gives a zero at the critical value. This is reflected in the practical (F-K) plot fig. 1.8 where the signal move out line AA can be drawn much easier than in the corresponding fig. 1.4. Therefore for signal move out line it is much better to consider Real 'F-K' than the imaginary F-K. The uncorrelated noise also exhibits the same characteristics as before the only difference between fig. 1.7 and fig. 1.3 is that fig. 1.7 is symmetrical about line $k=0.5$ and fig. 1.3 gives mirror image about line $k=0.5$. As can be seen the uncorrelated noise component distorts the F-K plot quite considerably and hence it is not easy to pick out the move out lines AA and BB for filtering purposes. The following is an attempt to get rid of the uncorrelated noise components.

Imag (F,K) was calculated for $F=0, 0.1...0.9$ with $k=0, (0+\delta), (0+2\delta)$ etc. where δ is a small number. In the example δ was chosen as 10^{-4} . It was seen that the values of signal and correlated components hardly changed for constant F with increasing K. Whereas the uncorrelated component still assumed its Sin form with the appropriate multiplying factor. Since uncorrelated component for $k=0$ is zero the values of Imag (F,K) at $k=0+\delta$ was subtracted from Imag (F,K) at $k=0$. This should give the values of the uncorrelated component for $k=(0+\delta)$ for different values of F. Taking the basic form of the Sin function as at $f=0$ the corresponding multiplying factors for various F were obtained by the corresponding division of the subtracted term.

The following is a comparison of the derived and actual multiplying factors for various f.

F =	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Actual	1	3.92	1.16	1.21	1.11	1.17	0.91	1.22	1.07	1.04
Derived	1	2.6	1.15	1.19	1.11	1.15	1.08	1.07	1.05	1.03



As can be seen a reasonable correspondence is obtained between the two sets of values. Using these derived multiplying factors the uncorrelated components were calculated using the basic form as at $f=0$ and values thus obtained were subtracted from the Imag 'F-K' plot. Thus the residual should represent the signal and correlated noise only with the uncorrelated noise component removed. The results are plotted in fig. 1.9. A comparison of fig. 1.4 and fig. 1.9 shows the obvious improvements obtained by this process. In fact a comparison of fig. 1.9 and fig. 1.2 shows how well the filtering has succeeded. It is also observed that in the Imag 'F-K' plot the contribution from the signal is very small due to the fact that $\alpha_0=0$ and hence in practice the filtered version of Imag F-K would almost give the correlated noise component only as seen by a comparison of figs. 1.9 and 1.2. By employing fig. 1.9 we can find the signal component move out line as in the 'Real Case' the signal components are enhanced due to the Cos term.

The separation of move out components of signal and correlated noise immersed in uncorrelated noise can be carried out in the following manner. Obtain the real and imaginary component of the double Fourier Transform as given in equation 1.1. Plot these values for $f=0, 0.1, \dots, 0.9$ and $k=0, 0.1 \dots 0.9$ as in figs. 1.4 and 1.8. A line such as AA passing through the inflexion part of the various curves should give the move out line for the signal from fig. 1.8. Obtain Imag. (F,K) for $F=0, 0.1 \dots 0.9$ for $K=0, (0+\delta), (0+2\delta) \dots (0+9\delta)$ where δ is a small number. Subtract Imag (F,0) from Imag $\{F, (0+\delta)\}$ for $F=0, 0.1 \dots 0.9$. Obtain the multiplying factors by division of the various values thus obtained with the basic curve at Imag (0,0). Taking Imag (0,K) as the basic uncorrelated component curve construct the diagram as in fig. 1.3 by multiplying Imag (0,K) by the relevant multiplying factors for $f=0, 0.1 \dots$. Subtract curves in fig. 1.3 from curves in fig. 1.4 to obtain fig. 1.9.

A line such as BB passing through the inflexion part of the various curves should give the move out line for correlated noise.

1.1.4 Discussion of the Results

The numerical example quoted assumes that all phase shifts are zero, i.e. $\theta_A = \theta_B = \theta_D = 0$. But in practice this will not be so. Hence in practice it is not possible to remove the uncorrelated terms from the F-K diagram as illustrated before. However if the angles of shifts are quite small the above procedure would not give too much of an error. An appropriate method of calculating the phase angles and thus to decide if filtering procedure should be adopted is given below.

Equations 1.10 and 1.9 are rewritten as:

$$\text{Real}\{A(f,k)\} = \sum_{x=0}^{X_N} \{f_{b1\psi}(R) \cos(P) + f_{b1\psi}(I) \sin(P)\} + \dots$$

WHERE $P = 2\pi G$

$$+ \sum_{x=0}^{X_N} \{f_{b3\eta}(R) \cos 2\pi kx + f_{b3\eta}(I) \sin 2\pi kx\} \quad (1.13)$$

$$\text{Imag}\{A(f,k)\} = \sum_{x=0}^{X_N} \{-f_{b1\psi}(R) \sin(P) + f_{b1\psi}(I) \cos(P)\} + \dots$$

$$+ \sum_{x=0}^{X_N} \{-f_{b3\eta}(R) \sin 2\pi kx + f_{b3\eta}(I) \cos 2\pi kx\} \quad (1.14)$$

In equation 1.13 let $f_{b1\psi}(R) = A \cos \theta_A$; $f_{b1\psi}(I) = A \sin \theta_A$;

and $f_{b3\eta}(R) = H \sin \theta_H$; $f_{b3\eta}(I) = H \cos \theta_H$.

Comparing the conditions in equations 1.9 and 1.10 it can be seen that the substitutions are the same except for θ_B (phase shift in uncorrelated noise) is now $90 - \theta_B$

$$\text{i.e. } \theta_H = \frac{\pi}{2} - \theta_B \quad (1.15)$$

From equation 1.9

$$\text{Real}\{A(0,0)\} = A \cos \theta_A + A_1 \cos \theta_{A_1} + \sin \theta_H \sum_{x=0}^{X_N} B_{x,0} \quad (1.16)$$

From equation 1.10

$$\text{Imag}\{A(0,0)\} = A \sin \theta_A + A_1 \sin \theta_{A_1} + \cos \theta_H \sum_{x=0}^{X_N} B_{x,0} \quad (1.17)$$

$$\frac{\partial}{\partial k} \{ \text{Real}\{A(f,k)\} \}_{f=0, k=0} = +A \sum_{x=0}^{X_N} 2\pi x \sin \theta_A + A_1 \sum_{x=0}^{X_N} 2\pi x \sin \theta_{A_1} + \sum_{x=0}^{X_N} B_{x,0} \cos \theta_H 2\pi x \quad (1.18)$$

$$\frac{\partial}{\partial k} \text{Imag}\{A(f,k)\}_{f=0, k=0} = -A \sum_{x=0}^{X_N} \cos \theta_A 2\pi x + A_1 \sum_{x=0}^{X_N} 2\pi x \cos \theta_{A_1} - \sum_{x=0}^{X_N} B_{0,x} \sin \theta_H 2\pi x \quad (1.19)$$

$$(1.17) \times \sum_{x=0}^{X_N} 2\pi x - (1.18) \text{ gives}$$

$$\begin{aligned} \text{Imag}\{A(0,0)\} \sum_{x=0}^{X_N} 2\pi x + \frac{\partial}{\partial k} \text{Real}\{A(f,k)\}_{f=0, k=0} \\ = \cos \theta_H \sum_{x=0}^{X_N} 2\pi x \sum_{x=0}^{X_N} B_{0,x} - \left\{ \sum_{x=0}^{X_N} B_{0,x} 2\pi x \right\} \cos \theta_H \end{aligned} \quad (1.20)$$

$$(1.16) \times \sum_{x=0}^{X_N} 2\pi x + (1.19) \text{ gives}$$

$$\begin{aligned} \text{Real}\{A(0,0)\} \sum_{x=0}^N 2\pi x + \frac{\partial}{\partial k} \text{Imag}\{A(f,k)\}_{f=0, k=0} \\ = \sin \theta_H \sum_{x=0}^N 2\pi x \sum_{x=0}^{X_N} B_{0,x} + \left\{ \sum_{x=0}^{X_N} B_{0,x} 2\pi x \right\} \sin \theta_H \end{aligned} \quad (1.21)$$

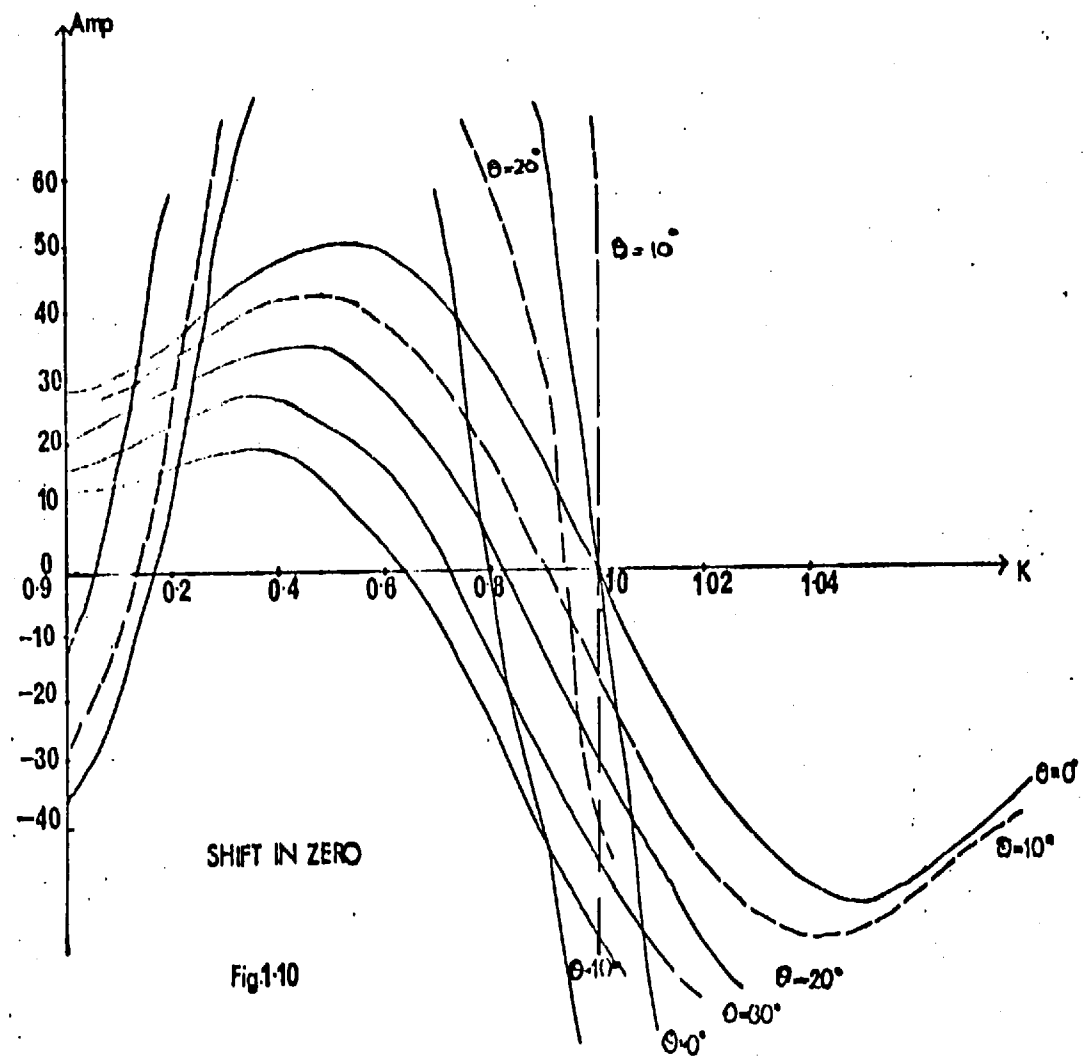
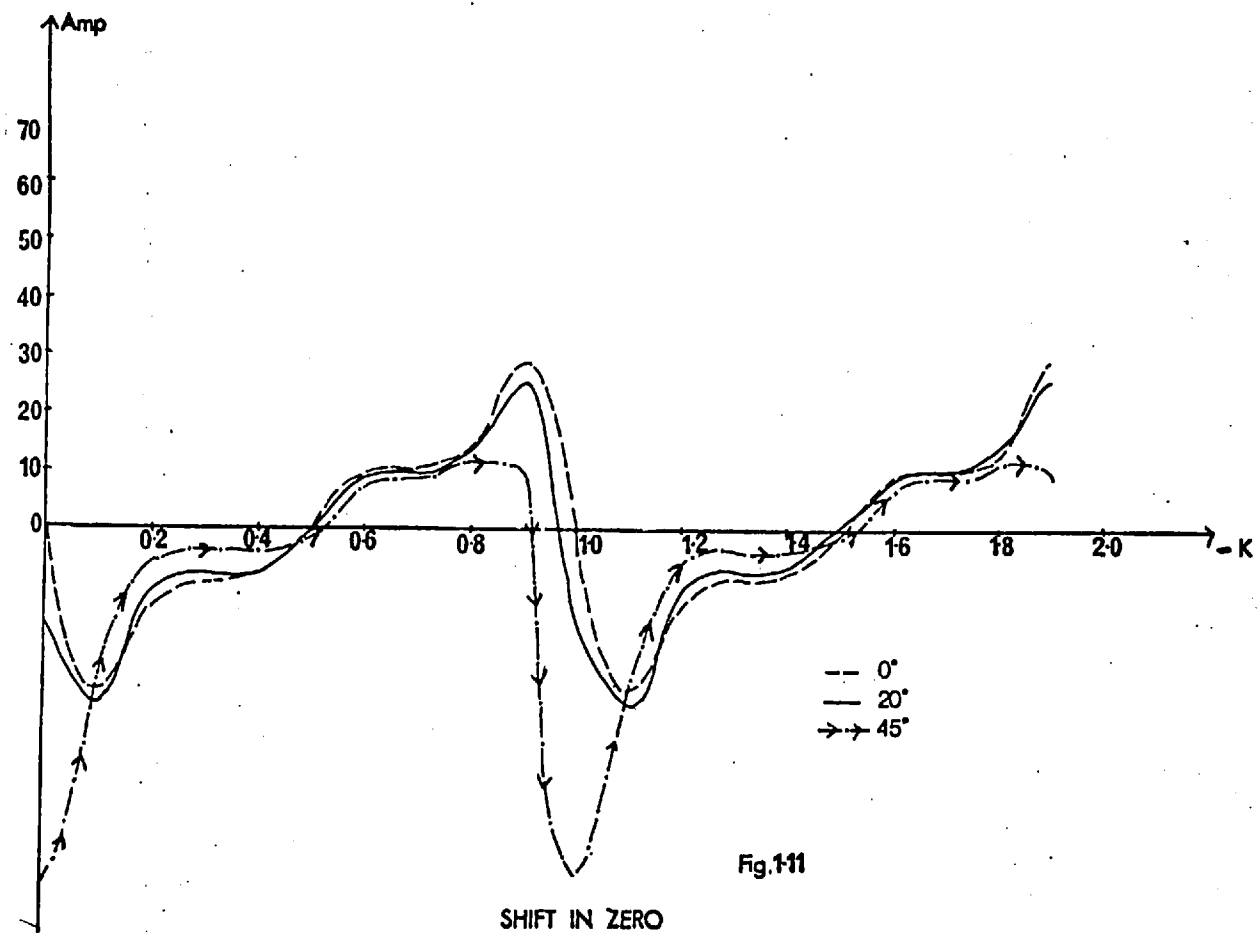


Fig.1-10



$$\begin{aligned}
 (1.21) \quad (1.20) \text{ gives } & \frac{2\pi \text{Real}\{A(0,0)\} \sum_{x=0}^N x + \frac{\partial}{\partial k} \text{Imag}\{A(f,k)\}_{f=0, k=0}}{2\pi \text{Imag}\{A(0,0)\} \sum_{x=0}^N x - \frac{\partial}{\partial k} \text{Real}\{A(f,k)\}_{f=0, k=0}} = \tan \Theta_H
 \end{aligned}
 \tag{1.22}$$

In equation 1.22 the only unknown is Θ_H and hence can be evaluated. From equation 1.15 Θ_B can be estimated and this is the required phase shift of the uncorrelated noise wave forms. Unfortunately there is no direct relationship between the phase shift Θ_B and the point co-ordinate of k at which the uncorrelated wave forms become zero at $f=0$. For this would depend on the form of the summation $\sum_{x=0}^N B_{x,0}$. Computer programmes were run to determine the shift in zero for various values of Θ_B for two different curves. Fig. 1.11 shows the form of the curve for phase shift 0° , 20° and 45° . It can be seen that the zero shift is about 0.03 for 20° and 0.08 for 45° . The part of the curve between $k=-0.99$ and $k=-1.01$ is depicted in fig. 1.10 for two curves for phase shifts 0° , 10° , 20° , 30° , 40° . It can be seen that for about phase shifts of 10° the zero point is almost independent of the curve and quite a difference is seen at phase shift 40° . These results are subjected to round off errors as well, as angular differences in this region are quite small. Therefore if it turns out that the estimate of Θ_B is less than about 10° we can assume that the zero point is independent of the curve and have a fixed value of k for the relevant value of Θ_B . This can be worked out for any curve and used in any subsequent analysis and it can be assumed that the zero point is independent of the curve. If $\Theta_B > 10^\circ$ the shift in zero is too much dependent on the form of the curve and hence the above procedure cannot be used. In this case an estimate has to be made of the probable zero point on the uncorrelated curve and use of the values of $A(f,k)$ to evaluate the multiplying constants. Unfortunately there is no mathematical way of finding the precise point as it would depend on the curve $B_{f,x}$ which is not known.

1.2 Quadratic Moveout

The general formula for moveout for signal is $T(x, t_0) = \frac{x^2}{2t_0 v^2(t_0)}$

The general formula for moveout for correlated noise is

$$T(x, t_0) = \frac{x^2}{2t_0 v^2(t_0)} + A_c$$

These two formulae are derived in 'Geophysical Prospecting' vol:18, 1970 by Robinson. Therefore the general formula for moveout is

$$T = \alpha_0 x^2 + \beta$$

From equation 1.2

$$A(f, k) = \int_0^\infty e^{-2\pi i k x} e^{-2\pi i f (\alpha_0 x^2 + \beta)} \{f_{b1}(f) f_{b2}(f)\} dx + \dots$$

$$+ \frac{1}{\sqrt{v}} f_{b3}(f) \int_0^\infty x(f) e^{-2\pi i k x} dx \quad (1.23)$$

Considering a typical term inside the integration sign,

$$e^{-2\pi i (kx + f\alpha_0 x^2 + f\beta)} \quad \text{can be written as}$$

$$e^{-2\pi i f \alpha_0 \left(x + \frac{k}{2f\alpha_0}\right)^2} e^{-D} \quad (1.24)$$

$$\text{where } D = 2\pi i f \beta - \frac{2\pi i f \alpha_0 k^2}{4f^2 \alpha_0^2}$$

$$\text{From Appendix B} \quad \int_0^\infty e^{-k_1(a+x)^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{k_1}} - A \quad (1.25)$$

From comparison of 1.23, 1.24, 1.25,

$$k_1 = 2\pi i f \alpha_0 \quad \text{and} \quad a = k/2f\alpha_0 \quad (1.26)$$

$$\text{Therefore } \int_0^{\infty} e^{-2\pi i k x} e^{-2\pi i f (\alpha_0 x^2 + \beta)} dx$$

$$= e^{-D} \left\{ \frac{1}{2} \sqrt{\frac{\pi}{2\pi i f \alpha_0}} - \frac{1}{2} \left\{ \frac{k}{2f\alpha_0} \left\{ 1 - \frac{(2\pi i f \alpha_0)}{3} \left\{ \frac{k}{2f\alpha_0} \right\}^2 + \frac{(2\pi i f \alpha_0)^2}{5 \cdot 2!} \left(\frac{k}{2f\alpha_0} \right)^4 - \dots \right\} \right\} \right\}$$

Therefore for quadratic moveout the general form of the (f,k) diagram is given by:

$$A(f,k) = \{f_{b1}(f)\} \{f_{\varphi}(f)\} e^{-D_{\text{signal}}} \left\{ \frac{1}{2} \sqrt{\frac{\pi}{k_{1,\text{signal}}}} - A_{\text{signal}} \right\}$$

$$+ \{f_{b2}(f)\} \{f_{\varphi}(f)\} e^{-D_{\text{C.N}}} \left\{ \frac{1}{2} \sqrt{\frac{\pi}{k_{1,\text{C.N}}}} - A_{\text{C.N}} \right\}$$

$$+ \frac{1}{\sqrt{v}} f_{b3}(f) \int_0^{\infty} f_x(f) e^{-2\pi i k x} dx \quad (1.27)$$

where A, k_1 , D are defined in 1.24, 1.26 and appendix B. Equation 1.27 is the final form of the Double Fourier Transform plot. A point of interest is that $A(f,k) \rightarrow \infty$ as $f \rightarrow 0$. This can be seen by examining the equation 1.27. It is recalled that in the case of linear moveout this did not happen. Hence if $A(f,k) \rightarrow \infty$ as $f \rightarrow 0$ we can conclude that the moveout is non linear.

CHAPTER 2

WAVELET CONTRACTION

2.0 Introduction

The basis of this chapter is the predictive decomposition theorem as illustrated by Robinson ('Geophysics', 1967). The problem that is solved in here is the decomposition of a given seismic trace assuming it as a convolution of a minimum delay wavelet and an impulse series. Robinson's method is explained briefly. The basis of this method is the Wold Decomposition Theorem in which only minimum delay conditions are assumed for the derivation of the equivalent seismic case. But unfortunately in Robinson's derivation a more strict definition of strictly minimum delay condition has to be assumed. The reason for this is the inverse deconvolution involved in the solution. Two methods by which this could be overcome are explained. These two methods assume only minimum delay conditions and roots of the Z transform equal to unity can be thus dealt with. Another important assumption necessary in Robinson's derivation is randomness of the impulse series. It is necessary that the impulse series should have zero auto correlation for every lag other than zero lag and have unity for zero lag. It is not certain how far this condition is satisfied in practice. If this condition is not satisfied his method is not applicable. Though a general decomposition is not possible in this case, under certain considerations decomposition is still possible.

2.1 Time Seismic Decomposition

According to the Predictive Decomposition Theorem (Robinson, 'Geophysics', 1967) a given non deterministic stationary process (in the example, the given time series) x_t ($-\infty < t < \infty$) with discrete time parameter t having an absolutely continuous spectral distribution can be represented by:

$$x_t = b_0 z_t + b_1 z_{t-1} + \dots \quad (2.1)$$

where the components have the following properties:

- a) the variables z_t have zero mean $E[z_t] = 0$; unit variance $E[z_t^2] = 1$ and are mutually uncorrelated $E[z_t z_s] = 0$ for $s \neq t$
- b) $b_0 > 0$ and $b_0^2 + b_1^2 + \dots < \infty$
- c) the operator $b_0, b_1 \dots$ is minimum delay. (Note: minimum delay is defined as a sequence of values whose Z transform has all its roots greater than or equal to 1 and strictly minimum delay is a sequence of values whose Z transform has all its roots greater than 1)

In the seismic equivalent of the above time series, $b_0, b_1 \dots$ is the common minimum delay wavelet and z_t is the strength or the amplitude of the wavelet which arrives at time t . Decomposition is an attempt to derive the arrival times of the wavelet i.e. z_t , given the time series values x_t . Given the absolutely continuous spectral distribution conditions the decomposition can be performed in the following way. (Robinson, Geophysics, 1967)

- 1) Compute the autocorrelation of the given time series.

$$\phi(\tau) = E[x_t x_{t+\tau}] = \sum_{t=-\infty}^{\infty} \sum_{k=0}^{\infty} b_k E[z_{t-k}] \sum_{N=0}^{\infty} b_N E[z_{(t+\tau)-N}]$$

Since it is assumed that $E[z_t z_s] = 0$ for $t \neq s$ (see condition (a)) the only non zero value of $\phi(\tau)$ occurs when $t-k=t+\tau-N$ or when $N=\tau+k$. Therefore,

$$\phi(\tau) = \sum_{t=-\infty}^{\infty} \sum_{k=0}^{\infty} b_k E[z_{t-k}] E[z_{t-k}] b_{\tau+k} = \sum_{t=-\infty}^{\infty} b_t b_{t+\tau}$$

from the unit variance condition in (a). This means that the autocorrelation of the minimum delay wavelet b_t is the same as the autocorrelation of the time series x_t .

- 2) Fourier Transform the autocorrelation to obtain the power spectrum.

$$\text{Power spectrum } \Phi(w) = \sum_{\tau=-\infty}^{\infty} \phi(\tau) e^{-i w \tau} = |B(w)|^2$$

Although there are many wavelets b_t which yield the energy spectrum $\Phi(w)$,

only one of these is minimum delay. Therefore it is assumed that this is the unique minimum delay wavelet which generates the physical phenomenon. The required minimum delay wavelet is obtained as follows. The power transfer function of the minimum delay wavelet is given by:

$$\psi(w) = \sum_{\tau=-m}^m r_{\tau} e^{-i w \tau} \geq 0, \quad r_0 > 0$$

where

$$r_{\tau} = \sum_{t=0}^m a_t a_{t+\tau}, \quad r_{\tau} = r_{-\tau}$$

Form the polynomial

$$z^m \psi(z) = \sum_{\tau=-m}^m r_{\tau} z^{\tau+m}$$

and solve for its roots $z_1, z_2 \dots z_{2m}$. Let $z'_1, z'_2 \dots z'_m$ be those roots $z_k (k=1, 2, \dots, 2m)$ of modulus greater than 1 and also those z_k of modulus 1 counted half as many times. If there are s roots greater than 1, there will be s roots less than 1 and also there will be an even number of roots equal to 1. (For proof see Robinson 1967)

Form the polynomial

$$B(z) = (z - z'_1)(z - z'_2) \dots (z - z'_m) = \sum_{s=0}^m b_s z^s$$

and the required wavelet values are then given by $b_1, b_2 \dots b_m$. This will give the minimum delay wavelet b_s .

3) Evaluate the inverse wavelet a_s such that $a_t = 0$ for $t < 0$; $a_0 b_0 = 1$

for normalising $b_0 = 1$ and $\sum_{s=0}^t a_s b_{t-s} = 0$ for $t = 1, 2, 3 \dots$

4) The inverse wavelet a_s convolved with x_t will remove the wavelet b_s and the amplitude factors $\tilde{z}_t$ will remain as:

$$\tilde{z}_t = \sum_{s=0}^{\infty} a_s x_{t-s}$$

for

$$\sum_{s=0}^{\infty} a_s x_{t-s} = \sum_{s=0}^{\infty} a_s \sum_{\tau=0}^{\infty} b_{\tau} \tilde{z}_{t-s-\tau}$$

$$= \sum_{s=0}^{\infty} a_s \sum_{n=s}^{\infty} b_{n-s} z_{t-n}$$

Recalling $b_r=0$ for $r < 0$

Since $\sum_{s=0}^t a_s b_{t-s} = 0$ for $t \neq 0$

$$\sum_{s=0}^{\infty} a_s x_{t-s} = \sum_{n=0}^{\infty} \left[\sum_{s=0}^{\infty} a_s b_{n-s} \right] z_{t-n} = z_t$$

2.2 Discussion on the Assumptions and Modifications

In this derivation several points become noticeable. In the original decomposition theorem b_s need be only minimum delay but in the inverse evaluation (step 3) b_s has to be strictly minimum delay. If b_s is not strictly minimum delay it has no finite inverse, as a root of $z=1$ will give a divergent inverse series. Hence the above derivation is valid only if b_s is assumed to be strictly minimum delay. In this case it will have a finite inverse a_s and the convolution $a_s * x_t$ is a valid process. This seems an unfortunate restriction as the decomposition theorem is valid for minimum delay conditions as well. This difficulty could be overcome as follows.

The minimum delay condition would mean that the Z transform of the wavelet will have one or several roots equal to 1. This would be represented by a convolution of a strictly minimum delay wavelet with a n equi delay wavelets where n is the multiplicity order of roots $z=1$ in the Z transform. The Z transform of

$$\Phi(w) = \sum_{\tau=-\infty}^{\infty} \phi(\tau) e^{-iw\tau}$$

would give $\Phi(z) = \sum_{\tau=-m}^m \phi(\tau) z^{\tau}$

Therefore $z^m \Phi(z) = \sum_{\tau=-m}^m \phi(\tau) z^{\tau+m}$

From this polynomial solve for its roots $z_1, z_2 \dots z_m$.

Let $(z'_1, z'_2 \dots z'_{m-n})$ be roots greater than 1 and $(z'_{m-n+1} \dots z'_m)$ be roots equal to 1. (note: the total number of roots $2m$ can be divided into three batches; a set of roots k (say) greater than 1 and k roots less than 1 (if z is a root, $\frac{1}{z}$ is also a root) and $(2m-k)$ roots equal to 1. $(2m-k)$ will always be an even number.) This would mean that the wavelet b_g consists of a convolution of a strictly minimum delay wavelet with Z transform roots $(z'_1, z'_2 \dots z'_{m-n})$ and n equi delay wavelets.

On this basis the decomposition theorem can be written from

$$x_t = \sum_{k=0}^{\infty} b_k z_{t-k}$$

to

$$x_t = \sum_{k=0}^{\infty} [c_k * n \text{ (equi delay wavelets)}] z_{t-k}$$

where c_k is strictly minimum delay with Z transform roots $(z'_1, z'_2 \dots z'_{m-n})$

To Evaluate the Impulse Series

Method 1 : If x_t is composed of a number of equi delay wavelets as shown by the factorisation of the auto-correlation, its Z transform should have roots = 1 of multiplicity equal to or greater than n . If this is so eliminate the wave obtained by convolving n equi delay wavelets from the time series. This can be done by the subtraction of the relevant Z transform. This would leave

$$x'_t = \sum_{k=0}^{\infty} c_k z_{t-k}$$

where x'_t is the new time series obtained after elimination of n equi delay wavelets. This is the same form as Robinson's derivation and his method is now applicable as c_k is strictly minimum delay. If the Z transform of the given time series does not contain at least n roots equal to 1, one of two conclusions can be arrived at:

a) the given time series does not contain a convolution of n equi delay wavelets.

b) the values of x_t are such that it has n equi delay wavelets but it is an addition of another component series, x'_t (say) where $(x_t - x'_t)$ has a Z transform with n roots equal to 1. But any time series which has a continuous spectral distribution can be represented by a convolution. (Robinson, 1967). Hence the assumption (a) is not valid. Hence only assumption (b) is valid. x'_t can be found as a polynomial in Z (Z transform) with coefficients $A_0, A_1 \dots A_n$ by the condition that

$$f(z) = x(z) - x'(z)$$

together with its n derivatives are zero for $z=1$. Hence the modified x_t series will have exactly n roots equal to 1 and can be treated as before.

Method 2 : Compute the inverse of the strictly minimum delay wavelet values given by c_k ($k=1, 2 \dots (m-n)$) as d_k where

$$c_0 d_0 = 1 \quad \text{and} \quad d_t = 0 \text{ for } t < 0$$

and

$$\sum_{s=0}^{m-n} d_s c_{t-s} = 0 \quad \text{for } t=1, 2, 3 \dots$$

Compute the convolution $(d_k * x_t)$ which would give the series obtained by the convolution of the impulse series $\tilde{x}_t$ (unknown) with n (known) equi delay wavelets. Knowing the value of n , $\tilde{x}_t$ can be obtained as shown in the example below.

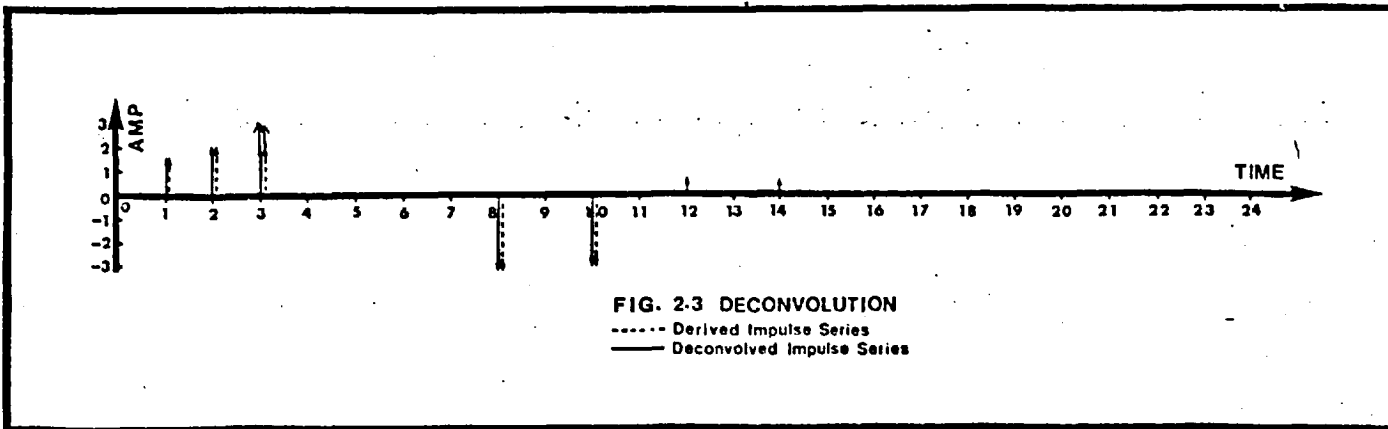
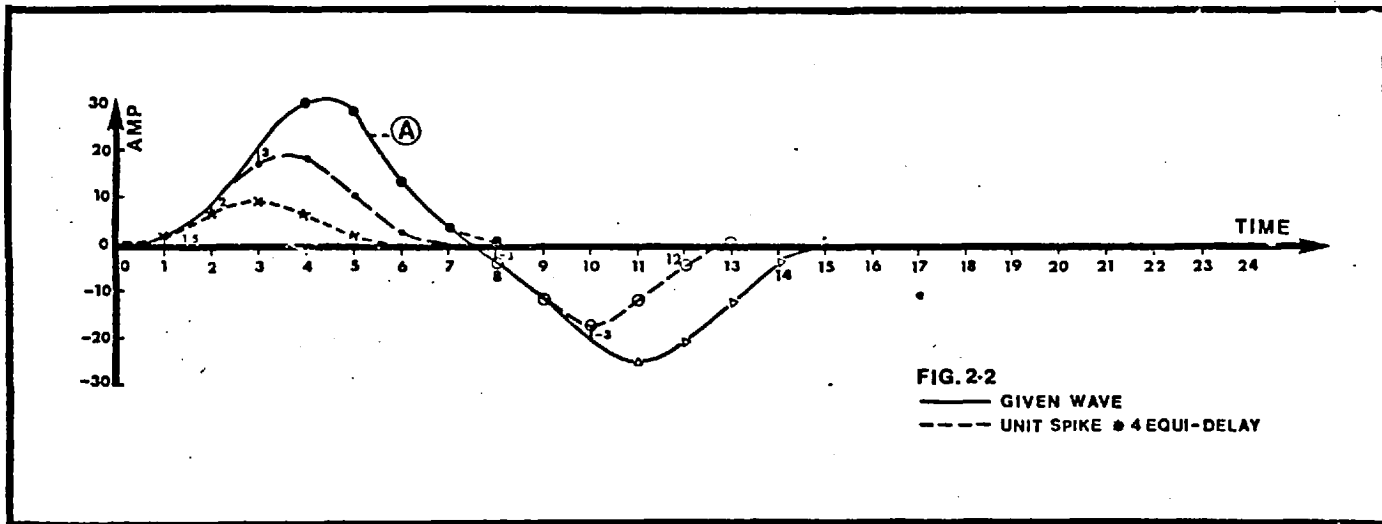
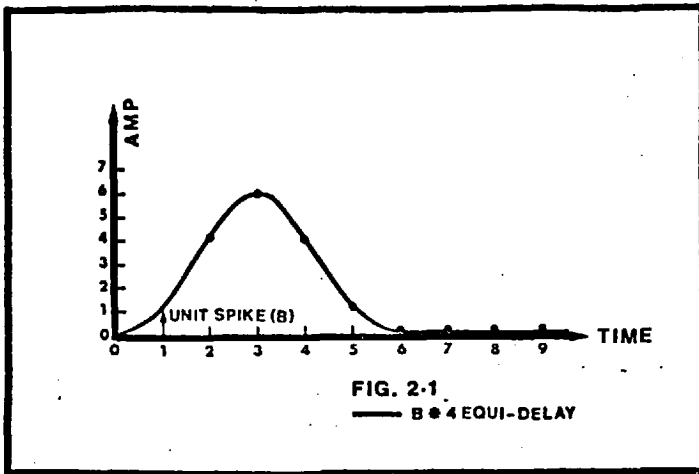
Let us assume a value for n (say 4) i.e. the auto-correlation factorisation gives 8 roots equal to 1. This would mean that the minimum delay wavelet has 4 equi delay convolutions. The Z transform of 4 equi delay convolutions would be

$$f_1(z) = (z + 1)^4 = 1 + 4z + 6z^2 + 4z^3 + z^4$$

i.e. the time series (1, 4, 6, 4, 1)

An impulse of amplitude equal to unity arriving at time $t=1$ when convolved with $f_1(z)$ will give the series

$$f_2(z) = (z+1)^4 \times (1z) = z + 4z^2 + 6z^3 + 4z^4 + z^5$$



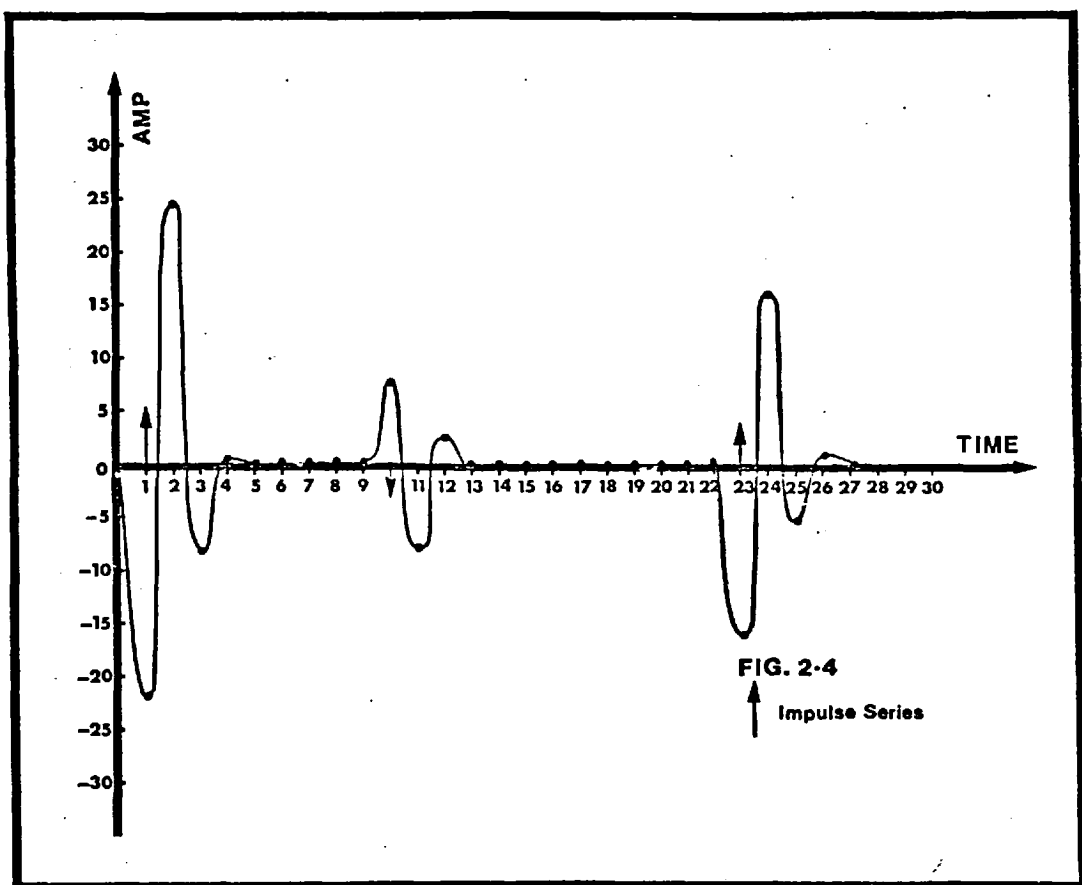
This series is represented in fig 2.1. This curve is the basic representation of the 4 equi delay convolutions. Consider a series y_t obtained by convolving the given series x_t with the inverse of the strictly minimum delay wavelet c_k obtained by the factorisation of the power spectrum as outlined earlier on. Let the digitised values of $y_t = (0, 1.5, 8.0, 20.0, 30, 27.5, 14, 3, -3, -12, -21, -24, -21, -12, -3, 0, 0...)$

This curve is represented as curve A in fig. 2.2. It is obvious that the first arrival is at time $T=1$ and is of magnitude 1.5. In fig. 2.2 plot the curve in fig. 2.1 but magnified 1.5 times as the amplitude of arrival is 1.5. This is shown as the first dashed lined curve. If there were no other arrivals this curve should match the curve in fig. 2.2 exactly. The discrepancy is due to the later arrivals. Examining the difference at $T=2$ it can be seen that an amplitude difference of 2 is obtained between the curves. Hence there is an arrival at $T=2$ of magnitude 2. Insert the curve in fig. 2.1 in fig. 2.2 but with origin at $T=1$ and magnified 2 times as the amplitude is now 2. If there are no further arrivals the sum of the first dashed curve and the second dashed curve should give curve A. Any discrepancy is due to a third arrival at the first point of discrepancy whose magnitude is the difference in values of the two curves. This procedure can be continued until the end of the curve A is obtained by matching the co-ordinates. The results are shown in fig. 2.3 where the impulse series is obtained as $(0, 1.5, 2, 3, 0, 0, 0, 0, -3, 0, -3)$. This is the required z_t series.

2.3 Alternative Solution

It will be recalled that an essential condition in the derivation is the restrictions imposed on the impulse series. An impulse series can be evaluated in this way only if its values are such that

- 1) they are uncorrelated
- 2) have unit variance
- 3) zero mean.



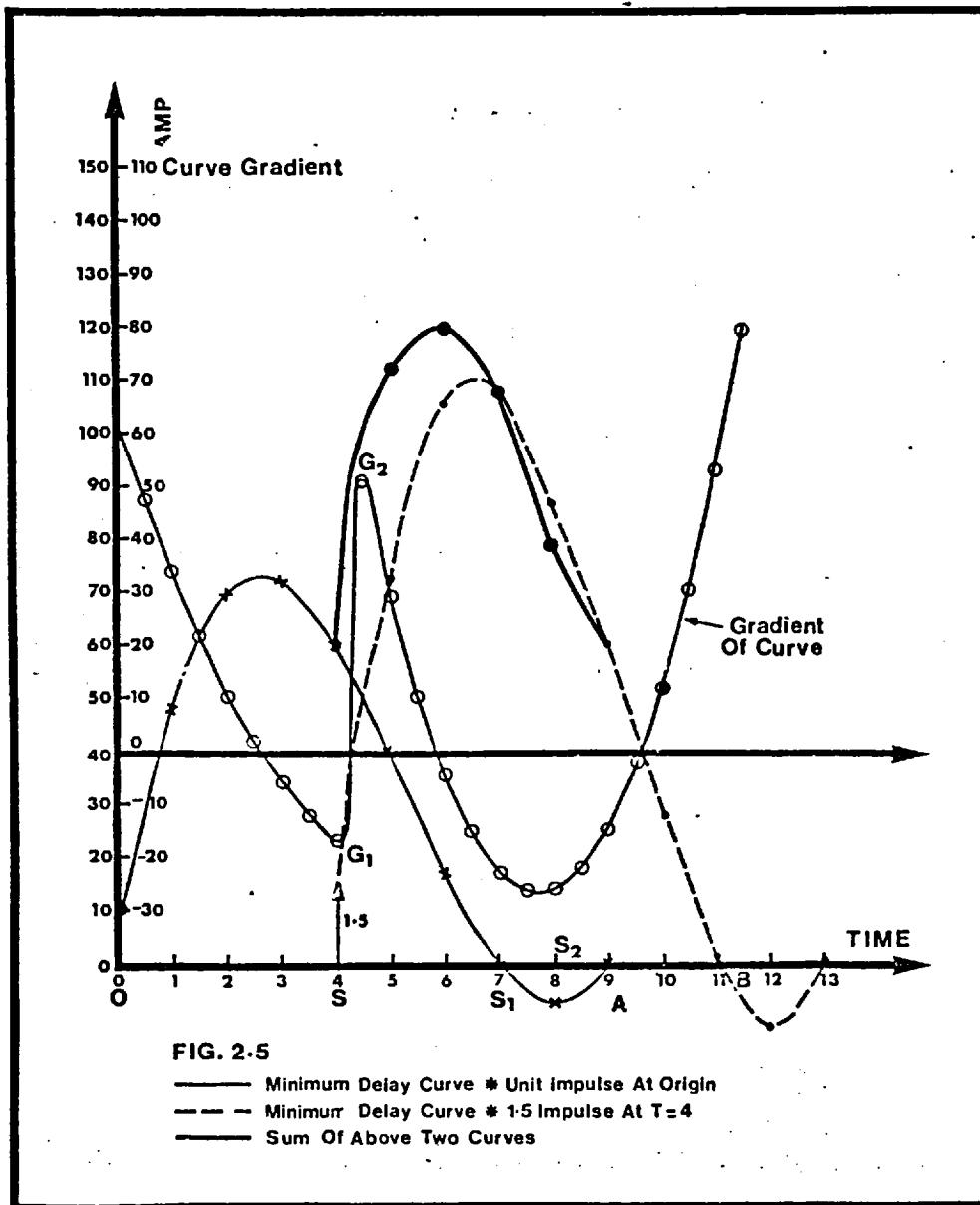
It would be quite helpful if an impulse series which does not have the above properties can be evaluated as well. Though it is not possible to do this all the time it is sometimes possible to evaluate an impulse series under certain conditions even though it does not satisfy the above conditions. These special cases will now be discussed.

It is quite important to make certain that the minimum delay wavelet evaluated from any method is indeed the unique wavelet involved in the convolution. In the case of a minimum delay restriction this is always so as a given auto-correlation will yield only one minimum delay wavelet though it may yield infinite number of wavelets which are not minimum delay. However, under non minimum delay conditions the solution can be tested as follows. Form the Z transform of the wavelet obtained, and compute its roots. Compute the roots of the Z transform of the given series. If all the roots of the wavelet are included as roots of the given time series it is not unreasonable to assume that the solution is correct. In this way the obtained solution can be tested for correctness. In the following discussion it is assumed that the given time series is a convolution of a finite wavelet and a finite impulse series.

Case 1 : Let the wavelet be represented by $b_0, b_1 \dots b_m$. Let the first arrival be at time $t=0$ (there is no loss of generality in this assumption as the origin of the given time series can be shifted to the first non zero value) and let the second arrival be at time $t=T$. If $T \geq m$ the wavelet shape can be evaluated quite simply. Consider a time series $x_t = (0, -24, 24.8, -8, 0.8, 0, 0, 0, 0, 0, 7.8, -8.06, 2.6, -0.260, 0, 0, 0, 0, 0, 0, 0, 0, 0, -16.2, 16.74, -5.4, 0.540, 0 \dots)$

This series is shown in fig. 2.4. By inspection it can be seen that the required wavelet shape whose Z transform is

$$f(z) = -24 + 24.8z - 8z^2 + 0.8z^3$$



This can be verified by checking the roots of the time series with the derived wavelet roots. Dividing the Z transform of the given series, i.e.

$$f_T(z) = -24z + 24.8z^2 - 8z^3 + 0.8z^4 + 7.8z^{10} - 8.06z^{11} + 2.6z^{12} - 0.26z^{13} \\ - 16.2z^{23} + 16.74z^{24} - 5.4z^{25} + 0.540z^{26}$$

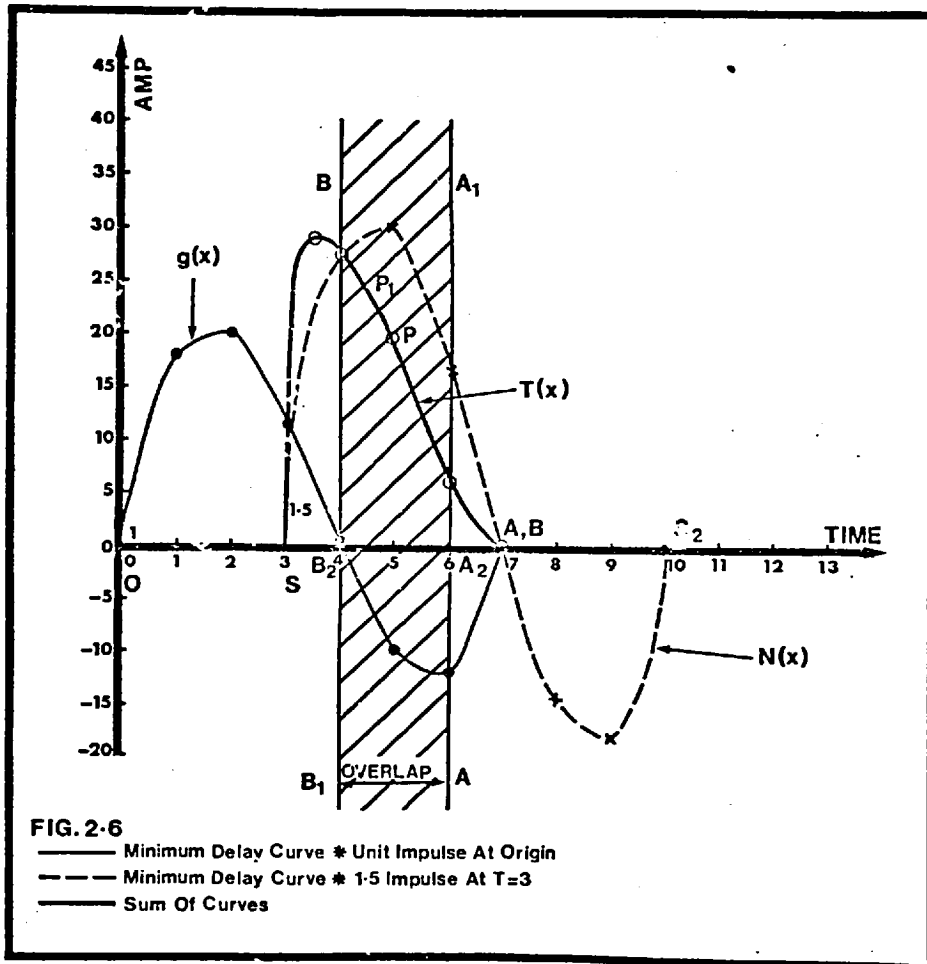
by $f(z)$, the required impulse series is obtained as its Z transform as

$$f_I(z) = z - 0.325z^{10} + 0.675z^{23}$$

The arrival times are represented in fig. 2.4 with a unit amplitude arrival at time $t=1$.

Case 2 : Let the wavelet be represented by $(b_0, b_1 \dots b_m)$ i.e. assume a length of m time units for the wavelet point A in fig. 2.5. Let the given time series meet the time axis for the first time at a point B (see fig. 2.5) where B is at time units G . Assume that there are only two arrivals before time units G and that $m < G$. Under these conditions the wavelet values can be evaluated as follows. Plot the values of the gradient of the given time series. The values when joined together should produce a smooth curve except at the point of arrival of the second impulse when it should give a sudden jump in value. This is clearly illustrated in the plot of gradient shown in fig. 2.5. Hence the arrival time of the second impulse can be marked at point s . Find the time interval from S to B and mark off a similar time interval from origin O to point S_1 in fig. 2.5. Under the assumptions made above this point S_1 is the point at which the wavelet would meet the time axis for the first time. Let the amplitude of the second impulse be $B \times$ (amplitude of first impulse). The amplitude of the first impulse can be taken as unity without loss of generality.

If the digitised values of the required wavelet is represented by X_N and the time series values by S_L (i.e. S_L is the amplitude co-ordinate at time L) then the co-ordinate relationship for the period O to B can be given by



$$S_L = X_L + BX_{L-OS} \quad (2.1) \quad \text{for time } T \geq OS$$

and $S_L = X_L \quad \text{for time } T \leq OS$

at time $T=OS_1$, $X_L=0$ from conditions assumed.

Therefore at point S_1

$$S_L = BX_{L-OS}$$

If $OS_1 - OS < 0$, the value of B can be obtained. In the example in fig. 2.5

$$OS = 4 \text{ time units}$$

$$SF = 7 \text{ time units}$$

Therefore $SB = OS_1 = 7$

and $OS_1 - OS = 7 - 4 = 3$ is less than OS and hence the conditions are satisfied.

$$S_L = 108 \text{ at } L = OS_1$$

and $X_3 = 72 \quad \text{for } L-OS = 3$

Therefore

$$B \cdot \frac{S_L}{X_3} = \frac{108}{72} = 1.5$$

Therefore the amplitude of the arrival at time $T=4$ is 1.5. Since the value of B is known, using the equation 2.1 the values of X_L up to S_2 where $SS_2 = OS$ can be obtained and these are the required values on the wavelet and hence wavelet shape can be determined. Once the wavelet shape is known the impulse series can be evaluated by Z transform division as shown in case 1. The time series chosen for this example has values (0, 48, 70, 72, 60, 112, 120, 108, 78, 60, 27, 0, -12, 0,...etc.)

Case 3 : In the previous example it was necessary for point B to be further away than the point A from the origin. Even if this was not so (i.e. if $OA > OB$) the wavelet values can be obtained if $S_2A > AA_2$. Consider the example in fig. 2.6. As before by plotting the gradient values the second arrival can be established. Let us assume that the wavelet values are represented by $g(x)$ for $n = 0, 1, 2, \dots$ etc. and the time series values

by $T(x)$ for $x=0,1,2\dots$ etc. Let the amplitude of second arrival at point S (found earlier) be B. Draw line AA_1 at a distance $SA_2 = OS$. Find the point A where $OA=SS_2$. Draw the line BB_1 at a distance $AB_2 = S_2A$. The region AA_1BB_1 is the overlap region. If the curve values from A to S_2 are represented by $N(x)$ for $x = OA_2 \rightarrow OS_2$, the co-ordinates of any point P in the overlap region AA_1BB_1 is satisfied by the relationship:

$$B g(x)_{\text{at } p} + \frac{1}{B} N(x)_p = T(x)_p \quad (2.2)$$

For example consider the point p on the time-curve at a time interval 5 within the overlap region. At this point $T(x) = 20$.

The point p is at a time distance 2 from S and a time distance of 2 from A.

$$g(x)_{\text{at } x=2} = 20$$

and

$$N(x)_{\text{at } x=2 \text{ from } S_2 \text{ origin}} = -15$$

Therefore from equation 2.2

$$B \times 20 - \frac{1}{B} \times 15 = 20 \quad (2.3)$$

which gives $B = 1.5$ or $-\frac{1}{2}$

The ambiguity of values for B can be resolved by considering another point p_1 (say) within the overlap region. At point p_1 , $T(x) = 23.5$ and point p_1 is 1.5 time units from S and 2.5 time units from A.

$$\text{Therefore } g(x)_{\text{at } x=1.5} = 18$$

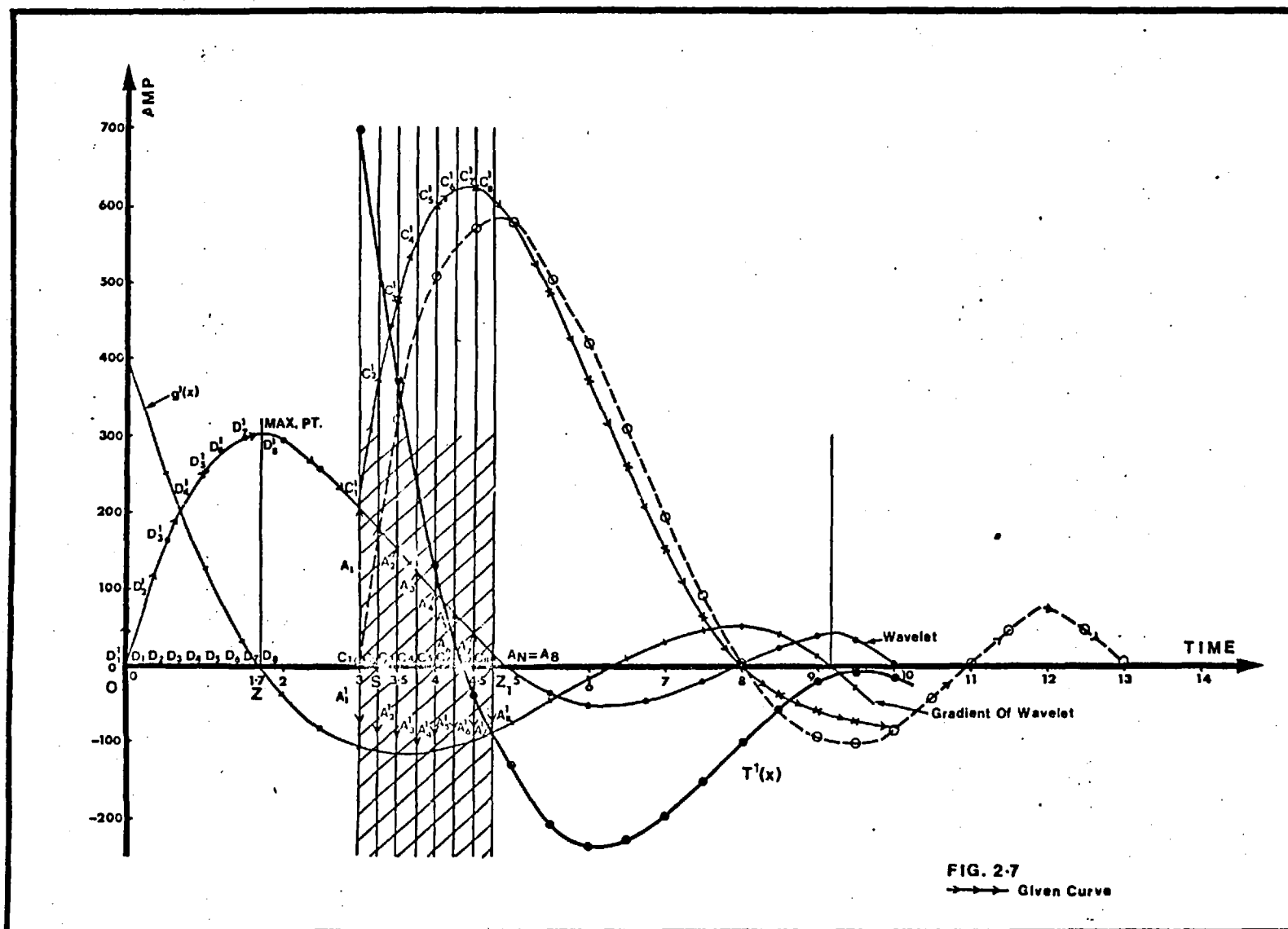
and

$$N(x)_{\text{at } x=2.5 \text{ from } S_2 \text{ origin}} = -10$$

$$\text{Therefore } 17B - \frac{10}{B} = 23.5$$

$$\text{which gives } B = 1.5 \text{ or } -0.05 \quad (2.4)$$

From results 2.3 and 2.4 B can be evaluated as 1.5. Knowing the value of B the wavelet shape can be computed and hence the impulse series as before. The time series considered for this example is 0, 16, 20, 15, 27, 20, 6, 0, -15, -17, 0, etc.



It should be noticed that the position of B (i.e. the point at which the given curve intersects the time axis) is quite irrelevant to the calculations. In the illustrated example B and A are identical. This is a coincidence and is not used in the calculations. The derivation is valid even if $OA > OB$. The only condition necessary is that $S_2A > AA_2$ or otherwise there would be no overlap region.

2.4 General Solution

Case 4 This is the most general case when no assumptions are made. An exact solution is not possible. However an approximate solution can be obtained as follows.

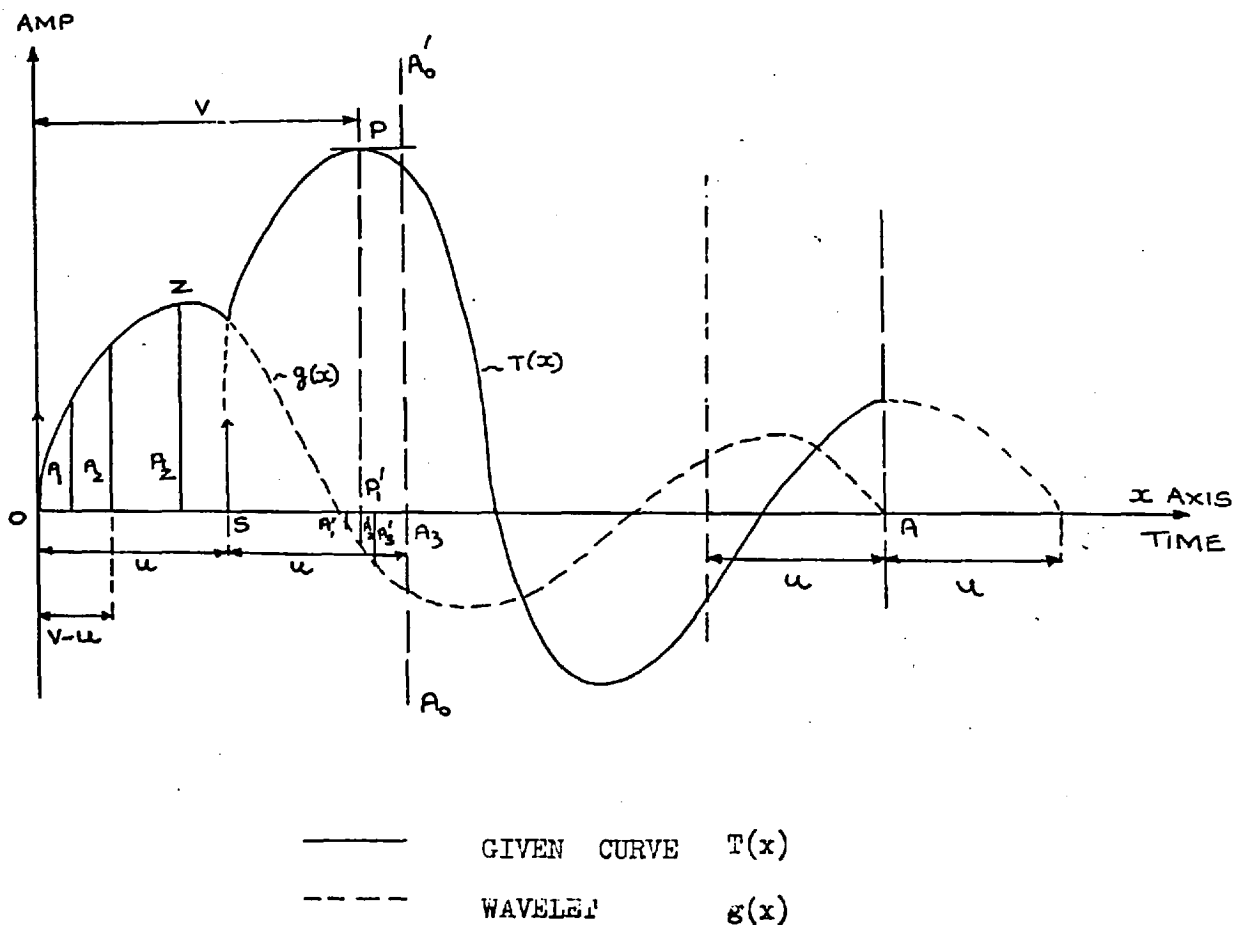


FIG. 2.8

Consider fig. 2.8 where $T(x)$ is the given time series. As before point S can be found. Draw $A_0 A_0'$ at a distance $SA_3 = OS = u$. Let p be a turning point of the given time series and let $OP = v$ where $v \leq 2u$.

If $v \leq 2u$ the following procedure can be adopted. The time series curve $T(x)$ can be represented by

$$T(x) = g(x) \text{ for time } 0 \rightarrow S \text{ (S is a known point)}$$

$$\text{and} \quad T(x) = Bg(x-u) + g(x) \text{ for time } > u. \quad (2.5)$$

where B is the amplitude of the impulse at time u.

Therefore for time $T > u$

$$\frac{dT(x)}{dx} = Bg'(x-u) + g'(x) \quad (2.51)$$

At the point p of the curve

$$\frac{dT(x)}{dx} = \text{gradient of time series at p.}$$

Therefore at point p

$$B g'(x-u)_{x=v} + g'(x)_{x=v} = 0 \quad (2.6)$$

Let A_1', A_2', A_3' be the co-ordinates of the wavelet at points $(p-\delta t), p, (p+\delta t)$ where δt is a small increment in the time axis.

Let T_1, T_2, T_3 be the corresponding co-ordinates of the time series curve $T(x)$, and let A_1, A_2, A_3 be the co-ordinates as shown.

Then form equation 2.5 applying to point $(p-\delta t), p, (p+\delta t)$.

$$BA_1 + A_1' = T_1 \quad (2.7)$$

$$BA_2 + A_2' = T_2 \quad (2.8)$$

$$BA_3 + A_3' = T_3 \quad (2.9)$$

AB = TANGENT AT POINT P'_1

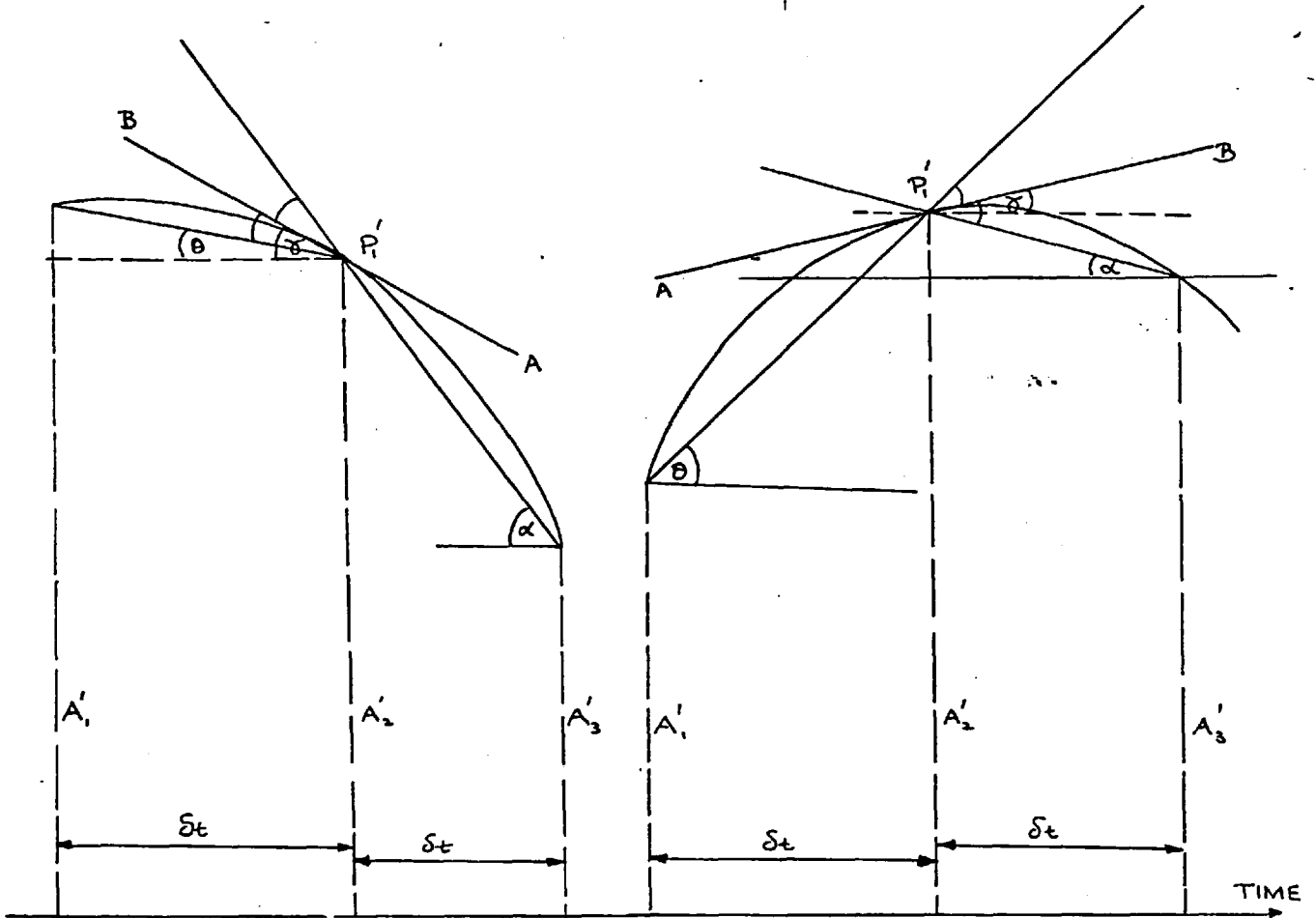


FIG. 2.9

FIG. 2.10

Consider the figs. 2.9, 2.10 where the region containing the point p'_1 is shown enlarged. Depending on the position of p'_1 three configurations can be drawn. Fig. 2.9 represents the conditions when $A'_1 > A'_2 > A'_3$

and fig. 2.10 represents the conditions when $A'_2 > A'_1$, $A'_2 > A'_3$

Another configuration representing the condition $A'_1 < A'_2 < A'_3$ can also be drawn. Conditions for this case can easily be modified.

Considering fig. 2.9 for small δt

$$\tau = \theta + (\alpha - \theta)/2 \quad (2.10)$$

$$\text{But} \quad \tan \theta = \frac{A'_1 - A'_2}{\delta t} = A'_{12} \quad (2.11)$$

t can be chosen as unity without any loss of generality

$$\text{and} \quad \tan \alpha = \frac{A_2' - A_3'}{\delta t} = \frac{A_2' - A_3'}{A_{23}} \quad (2.12)$$

$$\text{Therefore} \quad \tan \frac{\theta}{2} = (-1 + \sqrt{1 + A_{12}^2}) / A_{12}$$

$$\text{and} \quad \tan \frac{\alpha}{2} = (-1 + \sqrt{1 + A_{23}^2}) / A_{23}$$

From 2.10

$$\begin{aligned} \tan \theta + \frac{\tan \frac{\alpha}{2} - \tan \frac{\theta}{2}}{1 + \tan \frac{\alpha}{2} \tan \frac{\theta}{2}} \\ \text{Tan } \gamma = \frac{1 - \tan \theta \frac{\tan \frac{\alpha}{2} - \tan \frac{\theta}{2}}{1 + \tan \frac{\alpha}{2} \tan \frac{\theta}{2}}}{1 + \tan \frac{\alpha}{2} \tan \frac{\theta}{2}} \end{aligned} \quad (2.13)$$

But $g'(x)_{x=p} = \tan \gamma$. Therefore $g'(x)_{x=p}$ can be formed in terms of

A_1', A_2', A_3' . Also $g'(x-u)_{x=v}$ is known since the curve from $0 \rightarrow S$

is known.

Therefore in equation 2.6 B is unknown and $g'(x-u)$ is known

and $g'(x)$ can be written in terms of A_1', A_2', A_3' .

Therefore in equations 2.7, 2.8, 2.9, 2.6, the unknowns are A_1', A_2', A_3', B

and hence can be solved to obtain the wavelet co-ordinates A_1', A_2', A_3' , etc. and B the amplitude of the impulse.

Equation 2.10 has to be slightly modified for fig. 2.10 to

$$\gamma = \theta - \frac{\theta + \alpha}{2}$$

and similar modification can be applied for the third configuration

when $A_1' < A_2' < A_3'$.

If it so happens that a maximum point Z occurs in the region OS the above calculations can be modified to give a less tedious calculation.

Mark a point Z_1 on the curve such that $SZ_1 = OZ$ on the time axis.

At point Z_1 equation 2.51 would give

$$\frac{dT(x)}{dx} = g'(x) = \tan \gamma$$

because $g'(x-u) = 0$ as Z_1 is a maximum point.

Hence equation 2.13 would give an equation involving A'_1, A'_2, A'_3 only and in conjunction with equations 2.7, 2.8, 2.9 these values can be solved. An alternative to the above is as follows. Integrating equation 2.51

$$\int T'(x)dx = B \int g'(x-u)dx + \int g'(x)dx \quad (2.14)$$

Plot the differentiated curve values as shown below:

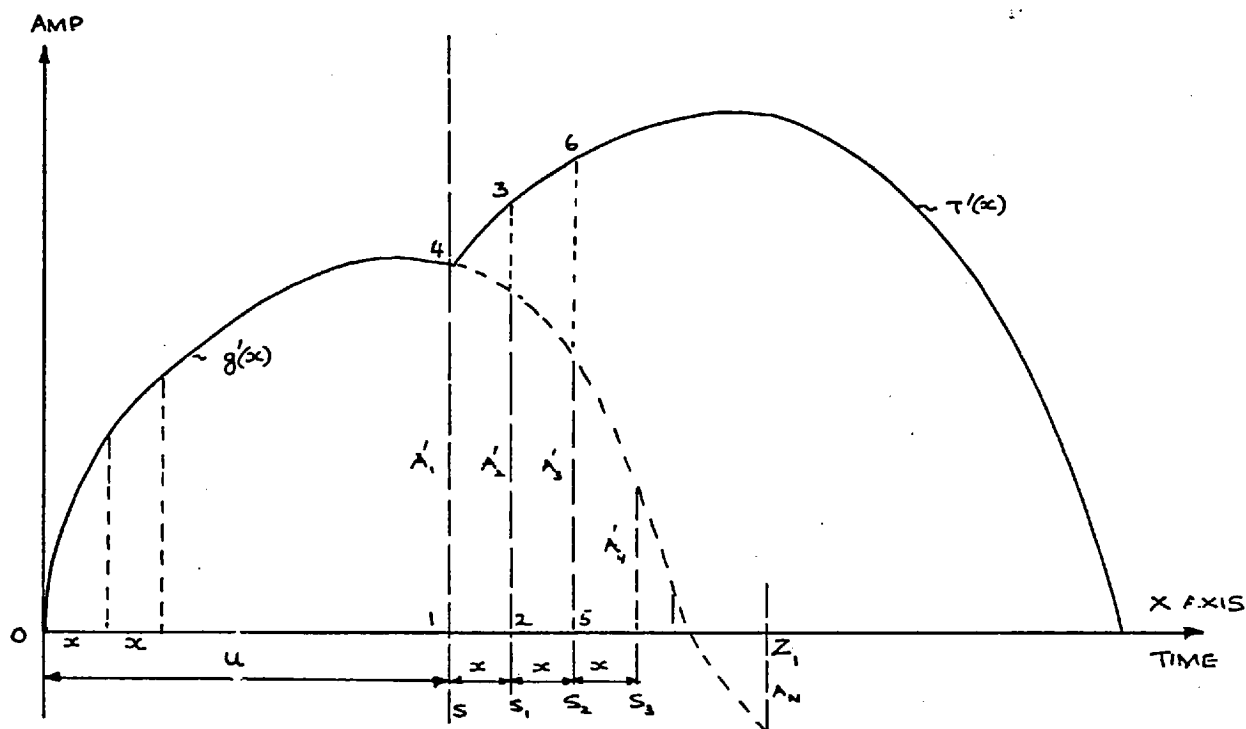


FIG. 2.11

Let $A'_1, A'_2 \dots A'_N$ be the co-ordinate values of the differentiated wavelet form A'_1 at $x=u$ to A'_N at Z_1 where $SZ_1 = OZ$ on the time axis.

The values of $A'_1 \times A'_N$ are known as shown earlier. If x is chosen small enough the following approximation can be made valid to equation 2.14

$$\int_{S_1}^{S_2} T'(x)dx = B \int_0^x g'(x-u) + \frac{A'_1 + A'_2}{2} x \quad \text{for area 1234}$$

$$\int_{S_1}^{S_2} T'(x)dx = B \int_x^{2x} g'(x-u) + \frac{A'_2 + A'_3}{2} x \quad \text{for area 2365}$$

.

$$\int_{S_{N-1}}^{S_N} T'(x)dx = B \int_{(N-1)x}^{Nx} g'(x-u) + \frac{A'_{N-1} + A'_N}{2} x \quad \text{for last area.}$$

In this set of equations there will be (N-1) unknowns consisting of the unknown co-ordinates of differentiated wavelet from $(A'_2, A'_3 \dots A'_{N-1})$ with the amplitude of the second impulse B and there are (N-1) equations. Hence the unknowns can be solved to give the required wavelet values of the differentiated curve. The value of B the amplitude of the impulse will also be obtained and hence the wavelet values can be derived.

An example is evaluated now to illustrate the required steps for the general case. Let the given time series be as shown in fig. 2.7. Only the relevant part of the curve is shown as the rest of the curve is irrelevant to the argument. The conditions that have to be satisfied by this curve are as follows:

- (1) It is assumed that the curve is a representation of a convolution of a wavelet $b_t (t=0, 1 \dots n)$ with an impulse series.
- (2) The wavelet is assumed to be finite in the sense that n is a finite number. The first impulse is at $t=0$.
- (3) There are at the most only two arrivals between $t=0$ and $t=n$, where n is defined as above.

For this particular calculation it is assumed that a maximum of the wavelet occurs before the second arrival. This is not a necessary condition but otherwise the calculations are quite tedious. Under the

above conditions the impulse series can be evaluated as follows:

Step 1 : Plot the gradient of the curve. This is shown in fig. 2.7.

Examining this diagram a sudden break in the curve at $T=3$ is shown which implies that the second arrival is at $T=3$.

Step 2 : From fig. 2.7 find the time at which the first maximum of the curve occurs. By inspection this is seen to be at $T=1.7$. This is verified by noting the value at which the differentiated curve cuts the time axis, for at a maximum point the differentiated value should be zero. Fig. 2.7 shows that at $T=1.7$ the value of this curve is zero; hence the original curve has a maximum at $T=1.7$. Mark this point Z on fig. 2.7 where $OZ=1.7$ and $OS=3$. Mark Z_1 such that $SZ_1 = OZ$.

Step 3 : Divide the region SZ_1 and OZ into a convenient equidistant values. In this example an interval of $T=0.25$ was chosen as shown. It should be obvious that the smaller the interval the better the results would be. Evaluate the areas $D_1 D'_1 D_2 D'_2$; $D_2 D'_2 D_3 D'_3$; ... $D_7 D'_7 D_8 D'_8$: and $C_1 C'_1 C_2 C'_2$; $C_2 C'_2 C_3 C'_3$; ... $C_7 C'_7 C_8 C'_8$. Let the above areas be denoted as $AA(1)$, $AA(2)$, ... $AA(7)$, and $BB(1)$, $BB(2)$, ... $BB(7)$.

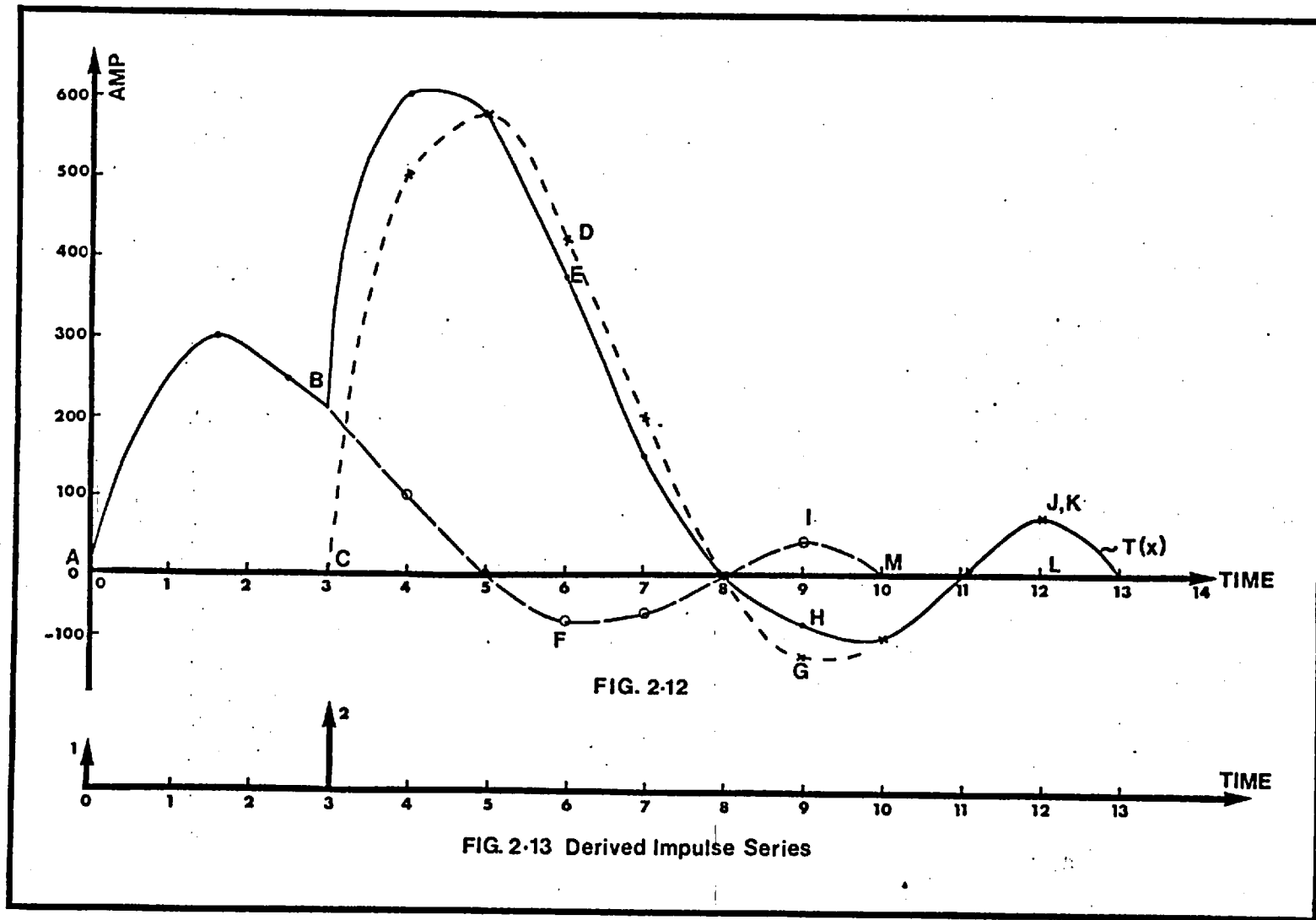
If the differentiated wavelet curve has co-ordinates A_1' A_2' A_3' A_4' A_5' A_6' A_7' A_8' at $T=OC_1, OC_2, OC_3, OC_4, OC_5, OC_6, OC_7, OC_8$, the following approximations can be made as shown earlier. If the impulse magnitude at $T=3$ is B,

$$\frac{A_1' + A_2'}{2} (0.25) + B \times AA(1) = BB(1)$$

$$\frac{A_2' + A_3'}{2} (0.25) + B \times AA(2) = BB(2)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\frac{A_7' + A_8'}{2} (0.25) + B \times AA(7) = BB(7)$$



As such there are seven equations and seven unknowns in $B, A'_2, A'_3 \dots A'_7$ and can be easily solved.

In the example the following values were obtained for the relevant quantities:

AA(1) = 89.906	BB(1) = 152.250	$A'_1 = -107$
AA(2) = 70.750	BB(2) = 112.781	$A'_8 = -97$
AA(3) = 53.609	BB(3) = 78.234	
AA(4) = 38.391	BB(4) = 48.328	
AA(5) = 25.000	BB(5) = 22.781	
AA(6) = 13.344	BB(6) = 1.313	
AA(7) = 3.328	BB(7) = -16.359	

The seven equations were solved by the matrix inversion method and a value of $B = 2.026$ was obtained.

Step 4 : Time series curve is plotted again in fig. 2.12 as $T(x)$.

Starting with origin as $T=3$ plot the part of the curve AB but multiplied by 2.026 as curve CD. By the subtraction of curve CD from curve BE obtain part of curve EF. Multiply the co-ordinates of curve EF by 2.026 and plot them with origin at $T=6$ as curve DG. Subtract curve DG from EH and plot these values with origin at $T=6$ as FI. Multiply curve FI by 2.026 and plot the values with origin at $T=9$ as curve GJ. Subtract curve GJ from HK and plot it as IL with origin as $T=9$. Since value of curve ABFIL is zero beyond $T=10$, the required form of the wavelet is obtained as ABFIM.

Step 5 : Obtain the Z transform of the given time series as $f_T(z)$ and the Z transform of the wavelet obtained as shown above as $f_W(z)$.

Divide $f_T(z)$ by $f_W(z)$ and the result will be the Z transform of the impulse series. In the above example considering only the part of the curve containing the two impulses

$$f_T(z) = 0 + 252z + 288z^2 + 210z^3 + 600z^4 + 576z^5 + 372z^6 + 150z^7 - 60z^9 - 84z^{10} + 72z^{12} \dots$$

$$f_W(z) = 0 + 252z + 288z^2 + 210z^3 + 96z^4 - 48z^6 - 42z^7 + 36z^9$$

Division of $f_T(z)$ by $f_W(z)$ gives $f_I(z) = 1+2z^3$ and hence the impulses can be obtained as shown in fig 2.13 where an arrival of magnitude 1 is obtained at $T=0$ and an arrival of magnitude 2 is obtained at $T=3$.

The example considered was built up theoretically so as to satisfy the conditions 1, 2, 3 and this build-up and the subsequent decomposition process is illustrated in fig. 2.7.

CHAPTER 3

CORRELATION ANALYSIS

3.0 Introduction

This chapter is based on an article by B. P. Dash and OBAIDULLAH K on 'Determination of Signal and Noise Characteristics using Correlation Theory'. The method used by these authors is briefly indicated. The article is an attempt to evaluate the signal characteristics in the form of its auto correlation when it is immersed in noise. The authors have come to the conclusion that within the length of the signal wavelet the auto correlation of the signal matches with the cross correlation of the traces. It was noticed that the practical results do not tally with the theoretical solution. The purpose of this chapter is to explain the discrepancy and to illustrate the conclusions with examples.

3.1 Derivation of the correlations

In an article entitled 'Determination of Signal and Noise statistics using Correlation Theory' by B. P. Dash and OBAIDULLAH K in 'Geophysics' Vol:35, 1970, it has been claimed that the noise and signal characteristics can be obtained by correlation analysis. The method is as follows.

Consider two seismic traces $x(t)$ and $y(t)$ having a common signal component $s(t)$ and different noise components $n(t)$ and $m(t)$.

These may be represented by the following two equations:

$$x(t) = s(t) + n(t) \quad (3.1)$$

$$y(t) = s(t) + m(t) \quad (3.2)$$

The auto correlations $c_{xx}(t)$ and $c_{yy}(t)$ of the two equations are:

$$c_{xx}(\tau) = \int_{-\infty}^{\infty} x(t) x(t+\tau) dt \quad (3.3)$$

$$c_{yy}(\tau) = \int_{-\infty}^{\infty} y(t) y(t+\tau) dt \quad (3.4)$$

or writing it fully,

$$C_{xx}(\tau) = \int_{-\infty}^{\infty} [s(t) + n(t)] [s(t+\tau) + n(t+\tau)] dt \quad (3.5)$$

and

$$C_{yy}(\tau) = \int_{-\infty}^{\infty} [s(t) + m(t)] [s(t+\tau) + m(t+\tau)] dt \quad (3.6)$$

By expanding the terms in the brackets in 3.5 and 3.6 and putting

$$C_{ss}(\tau) = \int_{-\infty}^{\infty} s(t) s(t+\tau) dt$$

$$C_{sn}(\tau) = \int_{-\infty}^{\infty} s(t) n(t+\tau) dt$$

$$C_{ns}(\tau) = \int_{-\infty}^{\infty} n(t) s(t+\tau) dt$$

$$C_{nn}(\tau) = \int_{-\infty}^{\infty} n(t) n(t+\tau) dt$$

we have

$$C_{xx}(\tau) = C_{ss}(\tau) + C_{sn}(\tau) + C_{ns}(\tau) + C_{nn}(\tau) \quad (3.7)$$

and

$$C_{yy}(\tau) = C_{ss}(\tau) + C_{sm}(\tau) + C_{ms}(\tau) + C_{mm}(\tau) \quad (3.8)$$

Where $C_{ss}(\tau)$ is the same as in equation 3.7 and

$$C_{sm}(\tau) = \int_{-\infty}^{\infty} s(t) m(t+\tau) dt$$

$$C_{ms}(\tau) = \int_{-\infty}^{\infty} m(t) s(t+\tau) dt$$

$$C_{mm}(\tau) = \int_{-\infty}^{\infty} m(t) m(t+\tau) dt$$

Next, taking the cross correlation of the traces $x(t)$ and $y(t)$ we have

$$C_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y(t+\tau) dt$$

and

$$C_{yx}(\tau) = \int_{-\infty}^{\infty} y(t) x(t+\tau) dt$$

On substituting for $x(t)$ and $y(t)$ in terms of $s(t)$, $n(t)$, $m(t)$ we may write

$$C_{xy}(\tau) = C_{ss}(\tau) + C_{ns}(\tau) + C_{sm}(\tau) + C_{nm}(\tau) \quad (3.9)$$

and

$$C_{yx}(\tau) = C_{ss}(\tau) + C_{sn}(\tau) + C_{ms}(\tau) + C_{mn}(\tau) \quad (3.10)$$

where $C_{ss}(\tau)$, $C_{ns}(\tau)$, $C_{sn}(\tau)$, $C_{sm}(\tau)$ and $C_{ms}(\tau)$ are the same as in equations 3.7 and 3.8,

while
$$C_{nm}(\tau) = \int_{-\infty}^{\infty} n(t) m(t+\tau) dt$$

and
$$C_{mn}(\tau) = \int_{-\infty}^{\infty} m(t) n(t+\tau) dt$$

Since we have assumed earlier that the signal and noise are uncorrelated and that the noises $n(t)$ and $m(t)$ in the two traces are also uncorrelated it follows that in equations 3.7, 3.8, 3.9 and 3.10,

$$C_{sn}(\tau) = C_{ns}(\tau) = C_{sm}(\tau) = C_{ms}(\tau) = C_{nm}(\tau) = C_{mn}(\tau) = 0$$

Thus we have

$$C_{xx}(\tau) = C_{ss}(\tau) + C_{nn}(\tau) \quad (3.11)$$

$$C_{yy}(\tau) = C_{ss}(\tau) + C_{mm}(\tau) \quad (3.12)$$

and

$$C_{xy}(\tau) = C_{ss}(\tau) \quad (3.13)$$

$$C_{yx}(\tau) = C_{ss}(\tau) \quad (3.14)$$

From equations 3.13 and 3.14

$$C_{ss}(\tau) = \frac{1}{2}[C_{xy}(\tau) + C_{yx}(\tau)] \quad (3.15)$$

And from equations 3.11 and 3.12 we have by substituting the values of $C_{ss}(\tau)$ for equation 3.15

$$C_{nn}(\tau) = C_{xx}(\tau) - \frac{1}{2}[C_{xy}(\tau) + C_{yx}(\tau)] \quad (3.16)$$

and

$$C_{mm}(\tau) = C_{yy}(\tau) - \frac{1}{2}[C_{xy}(\tau) + C_{yx}(\tau)] \quad (3.17)$$

Hence it is claimed that it is possible to determine the auto correlations of the signal and noise from two similar traces under the assumption made earlier. In the article quoted above a theoretical example has been given to illustrate the application of the above method. In this example the two traces were represented by

$$x(t) = \alpha(t) * A(t) + K_1 \beta(t) * B(t) \quad (3.18)$$

$$y(t) = \alpha(t) * A(t) + K_2 \beta(t) * C(t) \quad (3.19)$$

where the series $s(t)$ is considered as the convolution of a wavelet $\alpha(t)$ with a random series $A(t)$. Similarly, series $n(t)$ is considered as the result of the convolution of a wavelet $\beta(t)$ with a random series $B(t)$ while series $m(t)$ is the convolution of $\beta(t)$ with a random series $C(t)$. It was also assumed that random series $A(t)$, $B(t)$ and $C(t)$ are mutually uncorrelated. As would be seen from the above analysis the cross correlation of x and y traces should coincide with the auto correlation of the signal as shown by equation 3.13. But when the above method was applied it was found (as shown in the article quoted) that the cross correlation function was non converging whereas the auto correlation function of the signal is a definite converging finite function. The authors have observed that in the interval of the signal duration this cross correlation diagram is similar to the auto correlation of the signal and hence they have come to the conclusion that the above method would give the auto correlation of the signal as far as the length of the signal.

3.2 Breakdown of the Theoretical Case

It will be granted that if all the assumptions regarding various series were fulfilled the cross correlation function will coincide exactly with the auto correlation function as given in equation 3.13. The fact that this does not occur means that in reality one or many of the assumptions made are not fulfilled.

Consider equation 3.9:

$$C_{xy}(\tau) = C_{ss}(\tau) + C_{ns}(\tau) + C_{sm}(\tau) + C_{nm}(\tau)$$

Rewriting this equation in the expanded form,

$$C_{xy}(\tau) = C_{ss}(\tau) + \int n(t) s(t+\tau) dt + \int s(t) m(t+\tau) dt + \int n(t) m(t+\tau) dt$$

From obvious considerations none of the integrations can be carried out from $-\infty$ to $+\infty$ as all time series are finite and all integrations will be performed as summations as the time series is given in digitised values.

Let us denote the length of wavelet β as consisting of $(N+1)$ values

from $t=0$ to $t=N$ and zero elsewhere, i.e. let $\beta(t)$ be represented by

$\beta(t)$ for $t=0$ to N . Similarly let the length of α be $(L+1)$

" " " " B " $(M+1)$

" " " " A " $(S+1)$

" " " " C " $(P+1)$

Under these restrictions and with the understanding that

$$n(t) = K_1 \beta(t) * B(t)$$

and $m(t) = K_2 \beta(t) * C(t)$

and $s(t) = \alpha(t) * A(t)$

we can write say $C_{ns}(\tau)$ as

$$C_{ns}(\tau) = \sum_{t=0}^{M+N} \left[K_1 \beta(t) * B(t) \right] \left[\alpha(t+\tau) * A(t+\tau) \right]$$

where the integration sign has been replaced by summations. Therefore,

$$C_{ns}(\tau) = \sum_{t=0}^{M+N} \left\{ K_1 \left[\sum_{s=0}^N \beta(s) B(t-s) \right] \left[\sum_{G=0}^L \left\{ \alpha(G+\tau) A(t+\tau-G) \right\} \right] \right\}$$

Expanding the above,

$$C_{ns}(\tau) = K_1 \sum_{t=0}^{M+N} \left[\beta_0 B_t + \beta_1 B_{t-1} + \dots + \beta_N B(t-N) \right] \cdot$$

$$\left[\alpha(\tau) A(t+\tau) + \alpha(\tau+1) A(t+\tau-1) + \dots + \alpha(L+\tau) A(t+\tau-L) \right]$$

$$\begin{aligned} &= \left\{ \left[\beta_0 B_0 \right] \left[\alpha(\tau) A(\tau) + \alpha(\tau+1) A(\tau-1) + \dots + \alpha(L+\tau) A(\tau-L) \right] \right\}_{K_1} \\ &+ \left\{ \left[\beta_0 B_1 + \beta_1 B_0 \right] \left[\alpha(\tau) A(\tau+1) + \alpha(\tau+1) A(\tau) + \dots + \alpha(L+\tau) A(\tau-L+1) \right] \right\}_{K_1} \\ &+ \dots \\ &+ \left\{ \left[\beta_0 B_N + \beta_1 B_{N-1} + \dots + \beta_N B_0 \right] \left[\alpha(\tau) A(\tau+N) + \dots + \alpha(L+\tau) A(\tau-L+N) \right] \right\}_{K_1} \\ &+ \dots \\ &+ \left\{ \left[\beta_0 B_M + \beta_1 B_{M-1} + \dots + \beta_N B(M-N) \right] \left[\dots \right] \right\}_{K_1} \\ &+ \dots \\ &+ \left\{ \left[0 \quad 0 \quad + \beta_N B(M) \right] \left[\alpha(\tau) A(\tau+M+N) + \dots + \alpha(L+\tau) A(\tau-L+M+N) \right] \right\}_{K_1} \end{aligned}$$

where the outer brackets enclose the term for the summation $t=0$, $t=1$, etc.

up to $t=(M+N)$. The above multiplication can be written in a compact matrix notation as:

$$\begin{bmatrix} B_0 & 0 & 0 & 0 \\ B_1 & B_0 & & \\ \cdot & \cdot & \cdot & \cdot \\ B_N & B_{N-1} & \cdot & B_0 \\ \cdot & \cdot & \cdot & \cdot \\ B_M & \cdot & \cdot & B_{M-N} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & B_M \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \beta_N \end{bmatrix} \times \begin{bmatrix} A(\tau) & A(\tau-1) & \cdot & A(\tau-L) \\ A(\tau+1) & A(\tau) & \cdot & A(\tau-L+1) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A(\tau+M+N) & A(\tau+M-1+N) & \cdot & A(\tau+M-L+N) \end{bmatrix} \begin{bmatrix} \alpha(\tau) \\ \alpha(\tau+1) \\ \alpha(\tau+2) \\ \cdot \\ \cdot \\ \alpha(\tau+L) \end{bmatrix} \quad (K_1)$$

where each row obtained by the matrix multiplication should be added on to the subsequent row. The terms in the above multiplication can be regrouped to obtain:

$$\begin{aligned}
 \frac{1}{K_1} c_{ns}(\tau) &= \left\{ \beta_0 \alpha(\tau) [B_0 A_\tau + B_1 A_{\tau+1} + \dots + B_M A_{\tau+M}] + \beta_0 \alpha_{\tau+1} \dots + \beta_0 \alpha_{\tau+L} [B_0 A_{\tau-L} + \dots + B_M A_{\tau-L+M}] \right. \\
 &\quad + \beta_1 \alpha(\tau) [B_0 A_{\tau+1} + B_1 A_{\tau+2} + \dots] + \beta_1 \alpha_{\tau+1} \dots + \dots \\
 &\quad + \dots \\
 &\quad \left. + \beta_N \alpha(\tau) [B_0 A_{\tau+N} + \dots] + \beta_N \alpha_{\tau+1} \dots + \dots \right\} \\
 &= \left\{ \beta_0 \alpha \left[\phi_{BA}(\tau) \right] + \beta_0 \alpha_{\tau+1} \left[\phi_{BA}(\tau-1) \right] + \dots + \beta_0 \alpha_{\tau+L} \left[\phi_{BA}(\tau-L) \right] \right. \\
 &\quad + \beta_1 \alpha_{\tau+1} \left[\phi_{BA}(\tau+1) \right] + \beta_1 \alpha_{\tau+1} \left[\phi_{BA}(\tau) \right] + \dots + \dots \\
 &\quad + \dots \\
 &\quad \left. + \beta_N \alpha_{\tau+1} \left[\phi_{BA}(\tau+N) \right] + \beta_N \alpha_{\tau+1} \left[\phi_{BA}(\tau+N-1) \right] + \dots + \beta_N \alpha_{\tau+L} \left[\phi_{BA}(\tau+N-L) \right] \right\}
 \end{aligned}$$

where $\phi_{BA}(\tau)$ is the cross correlation of the wavelets B and A at a lag τ .

By observing the cross correlation function values involved in the expression it can be seen that the lowest lag is $(\tau-L)$ obtained at the top right corner of the correlation matrix, and the highest lag is $(\tau+N)$ obtained at the lower left corner of the correlation matrix. The correlation matrix will thus contain all lags from $\tau=(\tau-L)$ to $\tau=(\tau+N)$ multiplied by the appropriate coefficients.

A similar analysis can be performed for the correlation values

of $C_{sm}(\tau)$ and $C_{nm}(\tau)$. By following exactly the same procedure it can be shown that for C_{sm} the correlation involved is $\phi_{AC}(\tau)$ and the relevant lags are from

$$T = \tau - N \text{ to } T = \tau + L$$

and for C_{nm} the correlation involved is $\phi_{BC}(\tau)$ with the relevant lags from

$$T = \tau - N \text{ to } T = \tau + N$$

From equation 3.9 if we were to have perfect matching between $C_{xy}(\tau)$ and $C_{ss}(\tau)$ we should have $C_{ns}(\tau)$, $C_{sm}(\tau)$ and $C_{nm}(\tau)$ all zero for every τ from time $t=0$ to τ . But as shown above this would only happen if each cross correlation term in the expressions for $C_{ns}(\tau)$ etc. are identically zero for all lags.

Hence the following conclusion can be arrived at: If the length of the signal is such that it has an auto correlation function for lags $t=0$ to $t=\tau$ and is zero for lags $t>\tau$ (i.e. as we have taken the length of the signal as $(L+S)$ the auto correlation function will have values for $t=0$ to $t=\tau=L+S$ the cross correlation of trace x and trace y will coincide exactly with the auto correlation of the signal up to lag from $t=0$ to $t=\tau=L+S$ (in this instance), provided:

- (1) $\phi_{BA}(\tau) = 0$ for $\tau = \tau - L$ to $\tau + N$
 $= -L$ to $(L+N+S)$ in our example.
- (2) $\phi_{AC}(\tau) = 0$ for $\tau = \tau - N$ to $\tau + L$
 $= -N$ to $2L+S$ in our example.
- (3) $\phi_{BC}(\tau) = 0$ for $\tau = \tau - N$ to $\tau + N$
 $= -N$ to $L+N+S$ in our example.

If all conditions (1), (2) and (3) are satisfied, the entire auto correlation function can be recovered from the cross correlation diagram. If the relevant cross correlations are not zero, the coincidence will be only up to that lag which would give the least common range for the zero ranges of the three cross correlation functions $\phi_{BA}(\tau)$, $\phi_{AC}(\tau)$ and $\phi_{BC}(\tau)$.

As can be seen if our original assumption of non correlation is true for the entire range $\tau = -\infty$ to $\tau = +\infty$, exact coincidence will be obtained from $\tau = -\infty$ to $\tau = +\infty$. But because of the truncation this is not satisfied and errors will occur at a point relative to the first non zero cross correlation coefficient. Therefore the success of the above method lies in how far the assumptions are valid and if it so happens that the three random series give cross correlation values zero up to sufficient lags to fulfil the three conditions (1), (2) and (3), the method is 100% successful and the method will be successful in varying degrees depending on the length of the lag. It is satisfying to notice that non correlation of the random series is only required within a specific range and not from $t = -\infty$ to $t = +\infty$, for if this is the case the method is completely invalid and it is not unreasonable to expect non correlation within a certain range but it is highly improbable that there is complete non correlation in the entire time lag.

Hence, as shown, the auto correlation of the signal can be recovered only if the three conditions (1), (2) and (3) are satisfied and since there is no way of verifying whether this was in fact so, the method cannot be applied in all cases. Even if the conditions (1), (2) and (3) were satisfied and the cross correlation diagram would coincide with the auto correlation diagram at least up to the required length, it still remains to find the point at which the first error occurs for there is no way given of finding the point at which coincidence ceases. But in practice the second problem can be overcome by visual inspection of the correlation diagram. The signal auto correlation would have zero values beyond a certain maximum lag say τ . Therefore if the cross correlation diagram satisfies coincidence up to and including this lag τ , beyond this it should have several zero amplitudes. In fact the cross correlation will continue to be zero beyond lag τ until the first error occurs and beyond this the correlation diagram is at error. Hence if a series of zero amplitudes are obtained in the cross correlation diagram,

it is reasonable to assume that the diagram represents the auto correlation of the signal up to the lag at which the first two consecutive zero amplitudes occur.

3.3 Solution based on modified conditions

The following is an attempt to find an alternative way of obtaining the noise and the signal wavelets assuming the only condition that the trace can be represented by convolution of a wavelet and an impulse series. It is assumed that several traces are available which have the same statistics in the sense that it has the same signal wavelet and the same noise wavelet, and the signal arrivals are the same in all traces but noise wavelets arrive at different times in different traces. Recalling equations 3.18 and 3.19, let the two traces $x(t)$ and $y(t)$ be:

$$x(t) = \alpha(t) * A(t) + K_1 \beta(t) * B(t) \quad (3.18)$$

$$y(t) = \alpha(t) * A(t) + K_2 \beta(t) * C(t) \quad (3.19)$$

with the same meaning for the symbols.

3.18 - 3.19 would give:

$$\begin{aligned} f(t) = x(t) - y(t) &= \beta(t) * [K_1 B(t) - K_2 C(t)] = \beta(t) * D(t) \\ &= \int_0^N \beta(s) D(t-s) dt \\ &= [\beta_0^D t + \beta_1^D t_{-1} + \dots + \beta_N^D t_{-N}] \end{aligned}$$

$$\text{and } f(t+\tau) = [\beta_0^D t+\tau + \beta_1^D t+\tau-1 + \dots + \beta_N^D t+\tau-N]$$

$$\text{Therefore, } \Phi_{ff}(\tau) = \int_{t=-\infty}^{t=\infty} f(t) f(t+\tau) dt$$

In the digitised form, we convert integrations into summations. Therefore,

$$\Phi_{ff}(\tau) = \sum_{t=0}^{M+N} (\beta_0^D t + \beta_1^D t_{-1} + \dots + \beta_N^D t_{-N}) (\beta_0^D t+\tau + \beta_1^D t+\tau-1 + \dots + \beta_N^D t+\tau-N)$$

It should be noticed that the length of the impulse series $D(t)$ is $(M+1)$ and hence the summation is from $t=0$ to $t=M+N$, for $M \geq P$.

The above expression can be written in the expanded form as:

provided (i) $\phi_{DD}(r) = 0$ for $r = -(N-\tau)$ to $r = (N+\tau)$
 and (ii) $\phi_{DD}(0) = 1$ (3.27)

In reality as the maximum negative lag $(N-\tau)$ is less than the maximum positive lag $(N+\tau)$ we can assume that the negative lag zero is automatically satisfied if the positive lag condition is satisfied. This is assuming the symmetry of the auto correlation function on either side of the zero lag axis.

On this basis the conclusion quoted in equation 3.27 can be rewritten as follows:

$\phi_{ff}(\tau) = \phi_{\beta\beta}(\tau)$ from $\tau=0$ to τ
 provided (i) $\phi_{DD}(r) = 0$ for $r=1$ to $r=(N+\tau)$
 and (ii) $\phi_{DD}(0) = 1$ (3.28)

The second condition is not a serious handicap as a constant value of $\phi_{DD}(0) = K$ say, would merely amplify the relevant auto correlation values by a constant amount and hence the basic shape of the auto correlation diagram would be kept intact though amplified by a constant amount.

Let us say that the impulse series D is such that it has $\phi_{DD}(\tau) = 0$ from $\tau=1$ to $\tau=G$ and $\phi_{DD}(\tau) \neq 0$ at $\tau=(G+1)$ and for the sake of simplicity assume $\phi_{DD}(0) = 1$. From consideration of equation 3.28 we get: $N+\tau = G$ and therefore, $\tau=(G-N)$.

Therefore if we were to plot the auto correlation of trace f it would coincide with the unknown correlation of β from lag $\tau=0$ to lag $\tau=(G-N)$ and thereafter, the coincidence will cease. In fact, by examining the conditions given in equations 3.22, 3.23 and 3.24, we can deduce errors for all subsequent lags as follows. Let $(G) = r$, i.e. $\phi_{DD}(\tau) = 0$ for $\tau=1$ to $\tau=r$ and $\phi_{DD}(\tau) \neq 0$ at $\tau=(r+1)$. Considering the form of the set of equations in equation 3.26 it can be seen that the only error term involved is at the highest lag, i.e. $\phi_{DD}(r+1)$ on the lower left coefficient, for all $\phi_{DD}(\tau)$ from $\tau=1$ to $\tau=r$ will be zero.

Therefore the error in the lag $\tau = (r+1)$ is:

$$\beta_0 \beta_N [\phi_{DD}(r+1)]$$

The error in the lag $\tau=(r+2)$ -is:

$$\phi_{DD}(r+2)(\beta_0 \beta_N) + \phi_{DD}(r+1)[\beta_{N-1} \beta_0 + \beta_N \beta_1]$$

The error in the lag $\tau=(r+3)$ -is:

$$\phi_{DD}(r+3)(\beta_0 \beta_N) + \phi_{DD}(r+2)(\beta_{N-1} \beta_0 + \beta_N \beta_1) + \phi_{DD}(r+1)[\beta_{N-2} \beta_0 + \beta_{N-1} \beta_1 + \beta_N \beta_2]$$

and the error in the lag $\tau=(r+G)$ -say, is:

$$\phi_{DD}(r+G)\beta_0 \beta_N + \phi_{DD}(r+G-1)(\beta_{N-1} \beta_0 + \beta_N \beta_1) + \dots + \phi_{DD}(r+1)[\beta_{N-(G-1)} \beta_0 + \beta_{N-G+2} \beta_1 + \dots + \beta_N \beta_{G-1}]$$

The above expressions can be written in a compact matrix as follows. The errors involved in the evaluation of the auto correlation values from lag

$\tau=(r+1)$ -to $\tau=(r+G)$ -are given by the matrix multiplication:

$$\begin{bmatrix} \sum_{s=0}^0 \beta_s \beta_{N-(0-s)} & 0 & 0 & 0 & \dots & 0 \\ \sum_{s=0}^1 \beta_s \beta_{N-(1-s)} & \sum_{s=0}^0 \beta_s \beta_{N-(0-s)} & 0 & 0 & \dots & 0 \\ \sum_{s=0}^2 \beta_s \beta_{N-(2-s)} & \sum_{s=0}^1 \beta_s \beta_{N-(1-s)} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{s=0}^{G-1} \beta_s \beta_{N-[(G-1)-s]} & \dots & \dots & \sum_{s=0}^0 \beta_s \beta_{N-(0-s)} & \dots & \dots \end{bmatrix} \begin{bmatrix} \phi_{DD}(r+1) \\ \phi_{DD}(r+2) \\ \phi_{DD}(r+3) \\ \vdots \\ \phi_{DD}(r+G) \end{bmatrix} = \begin{bmatrix} E(r+1) \\ E(r+2) \\ E(r+3) \\ \vdots \\ E(r+G) \end{bmatrix}$$

where $E(r+1)$ is the error at lag $\tau=(r+1) - n$

and $E(r+2)$ " " " " " $\tau=(r+2)-n$ etc.

3.4 Illustrative examples

The application of the above method will now be illustrated with a theoretically constructed example. The problem to be solved is as follows. Given a set of traces (at least four) which contain a signal and noise wavelets, is it possible to obtain the characteristics of the signal

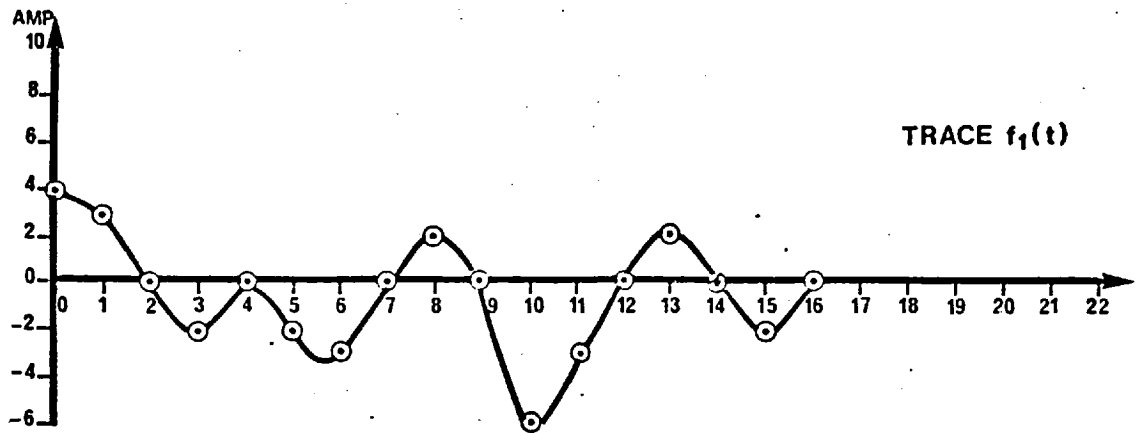


FIG. 3.1

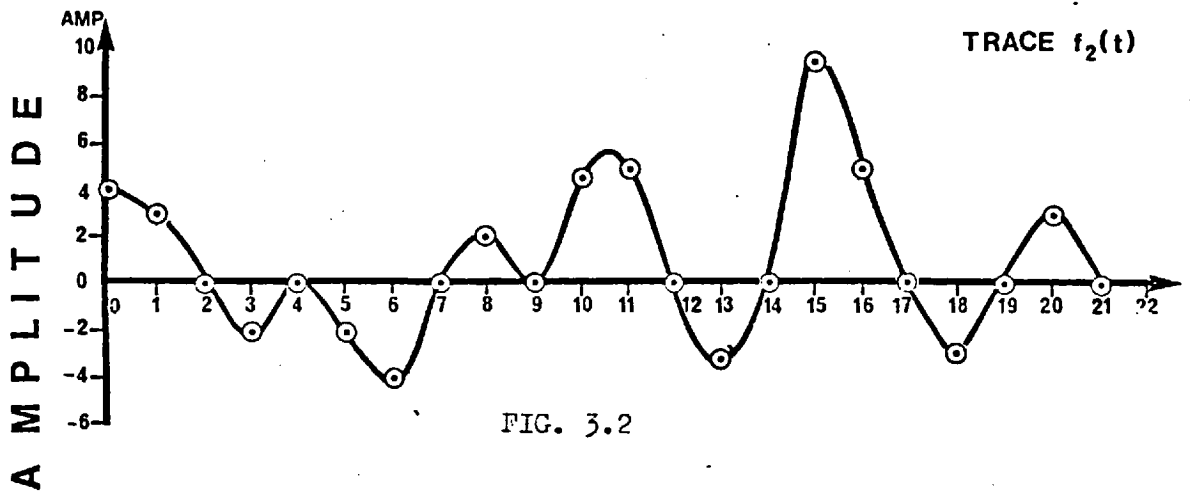
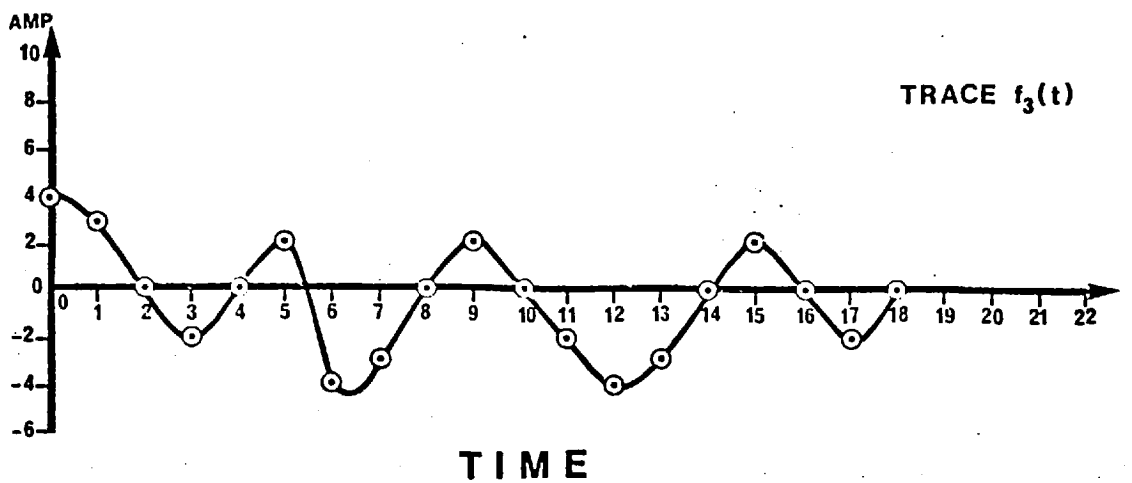


FIG. 3.2



TIME

FIG. 3.3

Fig.1 Traces Denoted As $f_1(t)$, $f_2(t)$, $f_3(t)$.

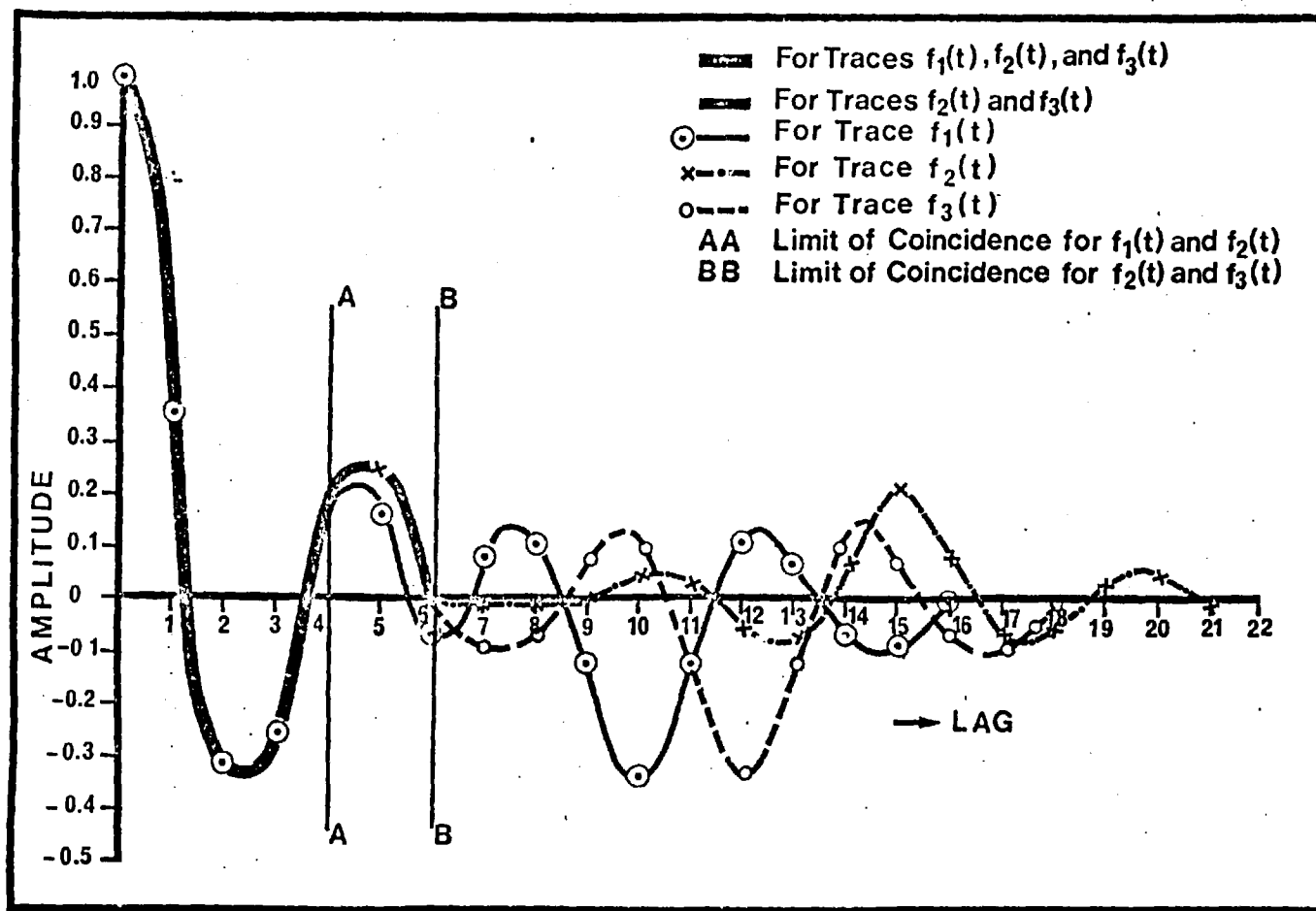


Fig. 2 Auto Correlation of Traces $f_1(t)$, $f_2(t)$, and $f_3(t)$.

and the noise wavelets in the form of their auto correlation function?

The following assumptions are made:

- (i) the form of the signal wavelet is the same in all four traces.
- (ii) the arrival times of the signal wavelet are the same in all four traces.
- (iii) the noise wavelet is the same in all four traces.
- (iv) the trace can be mathematically represented by the sum of two convolutions of the signal with a random series (corresponding to the arrival times) and the noise with a random series (corresponding to the arrival times).

No restrictions are placed on the properties of the wavelets for signal and noise.

Following the notation used before we have four traces denoted by $x(t)$, $y_1(t)$, $y_2(t)$, $y_3(t)$, and hence we can form

$$f_1(t) = x(t) - y_1(t)$$

$$f_2(t) = x(t) - y_2(t)$$

and

$$f_3(t) = x(t) - y_3(t)$$

These three traces are presented as time series in figs. 3.1, 3.2 and 3.3. The auto correlation diagrams for these three traces are presented in fig. 3.4. By inspection of fig. 3.4 it can be seen that auto correlations of trace 1 and trace 2 coincide up to lag 4 (denoted by line AA) and auto correlations of trace 2 and trace 3 correspond up to lag 6 (denoted by line BB). Hence it is obvious that the auto correlation of the noise wavelet $\beta(t)$ has been correctly evaluated at least up to lag 4 for coincidence is obtained in all three traces up to this lag. Therefore $\tau=4$ as used in equation 3.28.

The analysis will now be made with regard to trace 1. Considering condition 3.28 the impulse series contained in trace 1 (denoted by $D(t)$) must be such that its auto correlation function should give zero for all lags from unity to $(N+4)$ where N is the length of the noise wavelet. In other words $\phi_{DD}(\tau)=0$ for $\tau=1$ to $(N+4)$. At present there is no direct way of evaluating the coefficients of the impulse series to

satisfy the above condition, but the required values can be obtained by a trial and error method as illustrated below. Obtain the Z transform of trace 1. This would be a polynomial in Z with its highest power as 15. Factorise this polynomial and obtain its 15 roots. This was done using the CDC 6400 and the results are tabulated in the computer program. By a similar method obtain the 20 roots of trace 2. Trace 1 was obtained by the convolution of $\beta(t)$ with an impulse series $D_1(t)$ and hence can be represented by its Z transform as $\beta(Z).D_1(Z)$. Similarly the trace 2 can be represented by $\beta(Z).D_2(Z)$. Therefore any root obtained in trace 1 and not on trace 2 is part of the impulse series $D_1(t)$ as a root which is part of the wavelet should also be present as a root in trace 2 since the same wavelet was assumed. Therefore collect all the roots in (1) which are not present in (2) and obtain a polynomial $f(Z)$ by factor multiplication. In the example the required roots are 4,7,8,9,10,11. Since we do not know the length of the noise wavelet we will have to assume certain values and check whether this value would give coefficients to satisfy condition 3.28. A table can be prepared to illustrate the conditions necessary for various values of N. If $\beta(t)$ is such that $\beta(N)$ is the first zero value beyond which it is all zero, i.e. $\beta(r)=0$ for $r=N, (N+1), (N+2) \dots$ and so on, and the impulse series is such that its last impulse arrives at time $t=M$ i.e. its Z transform has a highest power of $(M-1)$ we can form the following table for trace 1 for different values of N:

Length of Impulse Series T=0, T=M	No. of Roots of Time Series	Length of Time Series T=0, T=s	Length Of Wavelet $\beta(t)$ T=0, T=N	$\phi_{DD}^* = 0$ Value of τ
6	15	17	12	15
7	15	17	11	14
8	15	17	10	13
9	15	17	9	12
10	15	17	8	11
11	15	17	7	10
12	15	17	6	9
13	15	17	5	8
14	15	17	4	7
15	15	17	3	6
16	15	17	2	5
17	15	17	1	4

Column 5 is obtained by adding 3 (since the coincidence ceases beyond $\tau=4$) to column 4. Depending on the value of N in column 4 the condition in column 5 would change as shown. Theoretically all the values of N are possible and we will have to check for each value of N. But practically we can exclude certain possibilities straight away. For instance the row 1 requires an impulse series of length only 6 to have a non zero auto correlation value at a lag 16, which is quite impossible and hence N=12 can be excluded immediately. On this basis we can also exclude N=11, 10, 9, 8, and hence we need only consider N=7 downwards. For instance let us say the correct value of N was in fact 7. This would mean that the length of the impulse series is 11 and thus it would have 10 roots for its factors in its Z transform. But we have already found 6 roots and hence will have to select 4 roots from the rest in order that the time series obtained by the Z transform multiplication of these roots would give an auto correlation function which would give zero for lag $\tau=1$ to lag $\tau=10$. If it is possible to do so (we will have to try all the possible combinations) this series is the impulse series involved in trace 1 and hence the multiplication of the rest of the roots in its Z transform would give the required wavelet shape of the noise wavelet. If none of the combinations satisfy

this condition we will have to discard $N=7$ and check for $N=6$. By this way we will find the right combination of roots which would give the necessary condition given in column 5 and thus we can evaluate the wavelet shape. Hence we can evaluate the impulse series as well as the arrival times of the noise wavelet. Therefore the complete shape of the noise wavelet as well as its impulse series in trace 1 and trace 2 can be evaluated. Rewriting the equations 3.18 and 3.19 in their Z transform notation:

$$x(Z) = \alpha(Z) A(Z) + K_1 \beta(Z) B(Z) \quad (3.29)$$

$$y(Z) = \alpha(Z) A(Z) + K_2 \beta(Z) C(Z) \quad (3.30)$$

In equations 3.29 and 3.30, $x(Z)$, $y(Z)$, $\beta(Z)$, $B(Z)$ and $C(Z)$ are all known functions.

Let $\alpha(Z)A(Z)=F_A(Z)$ and $\beta(Z)B(Z)=F_B(Z)$ and $\beta(Z)C(Z)=F_C(Z)$. Therefore:

$$F_A(Z) = x(Z) - K_1 F_B(Z) \quad (3.31)$$

and
$$F_A(Z) = y(Z) - K_2 F_C(Z) \quad (3.32)$$

Comparison of corresponding coefficients of equal powers of Z in the right hand side of equations 3.31 and 3.32 would give the full Z transform expression of $F_A(Z)$. Therefore:

$$F_A(t) = \alpha(t) * A(t) \quad (3.33)$$

Thus the series $F_A(t)$ can be found. To evaluate $\alpha(t)$ and $A(t)$ separately the procedures illustrated can be repeated. But unfortunately in this case we have only one series and therefore we have to assume that the impulse series $A(t)$ satisfies the required condition 3.28 i.e. if the length of $\alpha(t)$ is N_α then we will have to assume that $A(t)$ is a series such that $\phi_{AA}(\tau)=0$ for $\tau=1$ to $(N_\alpha + N_\alpha)$, If this is so, condition 3.28 is satisfied from $\tau=0$ to $\tau=N_\alpha$ (i.e. the full auto correlation function of the signal wavelet $\alpha(t)$). We could verify whether this condition has been in fact satisfied by observing the sequence of zero values that should be obtained for the auto correlation of $F_A(t)$ beyond a certain $\tau=s$ before the first

error values are obtained. If this condition is not satisfied we cannot obtain the full auto correlation of $\mathcal{L}(t)$ but we can still obtain it partially up to a certain lag but we cannot be sure at what lag the coincidence ceases.

Thus we have obtained the form of the noise wavelet and its arrival times and the signal wavelet (under certain conditions) without assuming any prior conditions for its arrival form presented as an impulse series.

Therefore this method would give results even under conditions where the complete non correlation of random series is invalid provided non correlation is obtained within certain specified periods.

CHAPTER 4

POLYNOMIAL ANALYSIS

4.0 Introduction

There have been many attempts to separate the signal from correlated noise on the basis of differential moveouts. This chapter deals with a new attempt to achieve the separation. The method used is the polynomial approximation. Under certain conditions (this is almost always satisfied in the seismic case), it is possible to analyse a seismic trace by representing it as a Legendrian Polynomial. If the usual assumptions regarding signal, correlated noise and uncorrelated noise^{1.2 REPRESENTATION AS CONVOLUTION} are made it is possible to work out the different moveout functions of the signal and the correlated noise from trace to trace. This method is different to all other methods in that no prior formula is assumed for the moveouts, and moveout from trace to trace is evaluated and hence the complete moveout function can be formed. Under certain conditions it is possible to eliminate uncorrelated noise. This is effected by repeated differentiation.

4.1 Polynomial Representation

The following is an attempt to separate the signal and the correlated noise represented in a seismic trace by means of their differential moveouts. It is known that signal and correlated noise have different moveouts and by investigating the moveout patterns it is possible to separate the signal and the correlated noise. This is the basis of velocity filtering as was analysed in Chapter 1. An unfortunate necessity that was apparent in the previous analysis was that the move out function of the signal and the correlated noise had to be known beforehand. It was demonstrated how by assuming linear variation the signal and the noise components were separated. But when even a simple extension of a quadratic moveout was incorporated the analysis became quite involved and no concrete conclusions could be obtained. For any filtering the moveout functions have to be known beforehand. The following is an attempt to

follow the moveout of the signal and the correlated noise from trace to trace and hence not assume any prior formula for the moveout functions. The model for the seismic trace is as before:

$$x_i(t) = s(t-\alpha_i) + \frac{1}{\sqrt{p}} r(t-\beta_i) + \frac{1}{\sqrt{1}} n_i(t) \quad (i=1,2,\dots,N)$$

where signal $s(t)=b_1(t)*\psi(t)$

correlated noise $r(t)=b_2(t)*\gamma(t)$

uncorrelated noise $n_i(t)=b_3(t)*\eta_i(t)$

with the usual meanings for the various symbols. Let α_0 be the moveout of the signal in the very first trace, and β_0 the moveout for the correlated noise in the first trace. Let α_s be the moveout of the signal from the first trace to another trace and β_s be the corresponding moveout of the uncorrelated noise. The problem to be solved is: is there any way of estimating the value of α_s and β_s ? If it is possible to estimate the shift of the signal and the correlated noise from trace to trace the complete moveout function for signal and correlated noise can be worked out and velocity filtering can be carried out very successfully, as a major problem in velocity filtering is the lack of knowledge of the precise nature and form of the moveout functions. An important consideration is the assumption inherent in all seismic work, i.e. the wave form of signal and correlated noise, i.e. $s(t)$ and $r(t)$ remain the same in all traces but with different shifts denoted by their movecuts. On the other hand uncorrelated noise is completely random and its wave form is completely different from trace to trace. The method to be adopted in the analysis is the method of polynomial representation. A given set of points can be represented by an appropriate polynomial provided it satisfies certain considerations. Various polynomial representations have been made and the one adopted here is the LAGRANGIAN Polynomial. A given curve $f(x)$ can be approximated by a LAGRANGIAN Polynomial which takes the same value as $f(x)$ at a number of selected points $x_1, x_2, \dots, x_n$ in a given range (a, b) . The polynomial is written as:

$$L(f) = \sum_{j=1}^n l_j(x) f(x_j)$$

where the polynomials $l_j(x)$ are defined as:

$$l_j(x) = \pi_j(x)/\pi_j(x_j)$$

where $\pi(x) = (x-x_1)(x-x_2)\dots(x-x_n)$

and $\pi_j(x) = \pi(x)/(x-x_j)$

The error of the Lagrangian Polynomial is written in the form:

$$f(x) - L(f) = \frac{\pi(x)}{n!} f^{(n)}(\xi)$$

where ξ is some number between the least and the greatest of $x_1, x_2, \dots, x_n, x$ and the continuity of $d^n f/dx^n$ is assumed. The only consideration relevant here is the convergence condition which is proved in 'Methods of Numerical Approximations' by D. C. Handscomb, on page 23 as follows: If x and $x_1, x_2, \dots, x_n$ lie in the real interval (a, b) then $L_n(f)$ converges uniformly to $f(x)$ if $f(z)$ has no singularities in the complex plane within a distance of $(b-a)$ from any point of the interval. It could be seen that the three component curves in a seismic trace are all convolutions of wavelets with impulse series and this conforms to the consideration stated before. Therefore we can represent a seismic trace as a Lagrangian Polynomial.

Let us represent the signal component as a polynomial of degree N and the correlated component as a polynomial of degree M and the uncorrelated component as a polynomial of degree L .

On this basis the first trace can be written as:

$$Y_{tX_1} = [A_N(t-\alpha_0)^N + A_{N-1}(t-\alpha_0)^{N-1} + \dots + A_0 + \frac{1}{\sqrt{p}} [B_M(t-\beta_0)^M + B_{M-1}(t-\beta_0)^{M-1} + \dots + B_0]] + \frac{1}{\sqrt{i}} (L_L t^L + L_{L-1} t^{L-1} + \dots + L_0)$$

where X_1 is the first trace (the traces are denoted as $X_1, X_2, \dots, X_N$ etc.) and Y_{tX_1} is the amplitude of the X_1 th trace at time $t=t$.

The signal to noise ratio values $\sqrt{p} \times \sqrt{i}$ can be incorporated into the polynomial coefficients and hence the seismic trace can be represented as:

$$Y_{tX_1} = [A_N(t-\alpha_0)^N + A_{N-1}(t-\alpha_0)^{N-1} + \dots + A_0] + [B_M(t-\beta_0)^M + B_{M-1}(t-\beta_0)^{M-1} + \dots + B_0] \\ + [L_{X_1} t^{X_1} + L_{X_1-1} t^{X_1-1} + \dots + L_0]$$

where α_0 is the delay in signal arrival

and β_0 is the delay in correlated noise arrival.

The second trace can be represented as X_2 in which the amplitudes are given by signal delay α_s and correlated noise delay β_s as:

$$Y_{tX_2} = \{A_N [t-(\alpha_0+\alpha_s)]^N + A_{N-1} [t-(\alpha_0+\alpha_s)]^{N-1} + \dots + A_0\} \\ + \{B_M [t-(\beta_0+\beta_s)]^M + B_{M-1} [t-(\beta_0+\beta_s)]^{M-1} + \dots + B_0\} \\ + \{L_{X_2} t^{X_2} + L_{X_2-1} t^{X_2-1} + \dots + L_0\}$$

The form of the subsequent traces Y_{tX_3} , Y_{tX_4} ..etc. will be in the same format with different delay α_s and β_s and different polynomials for the uncorrelated noise. For the following analysis it is necessary that the highest power of the Legrangian Polynomial for the uncorrelated noise, ie X_1 in trace X_1 and X_2 in trace X_2 , should be less than both M and N . This could be easily verified as shown later in the analysis. It is also assumed for the sake of analysis that $M > N$. Rewriting the equation of the first trace:

$$Y_{tX_1} = y_1 = \left[A_N(t-\alpha_0)^N + \dots + A_0 \right] + \left[B_M(t-\beta_0)^M + \dots + B_0 \right] + U.N$$

$$\frac{\partial y_1}{\partial t} = \left[N A_N(t-\alpha_0)^{N-1} + \dots + A_1 \right] + \left[M B_M(t-\beta_0)^{M-1} + \dots + B_1 \right] + U.N$$

$$\frac{\partial^2 y_1}{\partial t^2} = \left[N(N-1) A_N(t-\alpha_0)^{N-2} + \dots + 2A_2 \right] + \left[M(M-1) B_M(t-\beta_0)^{M-2} + \dots + 2B_2 \right] + U.N$$

$$\frac{\partial^N y_1}{\partial t^N} = \left[{}^N P_N A_N \right] + \left[{}^M P_N B_M(t-\beta_0)^{M-N} + \dots + {}^N P_N B_N \right]$$

$$\frac{\partial^M y_1}{\partial t^M} = \left[0 \right] + \left[{}^M P_M B_M \right]$$

where ${}^A P_B = (A)(A-1)(A-2) \dots (A-(B-1))$

It should be noted that the uncorrelated noise contributions drop off after the (L+1)th derivative and the signal component distributions drop off after the (N+1)th derivative and the (M+1)th derivative, and thereafter are all zero.

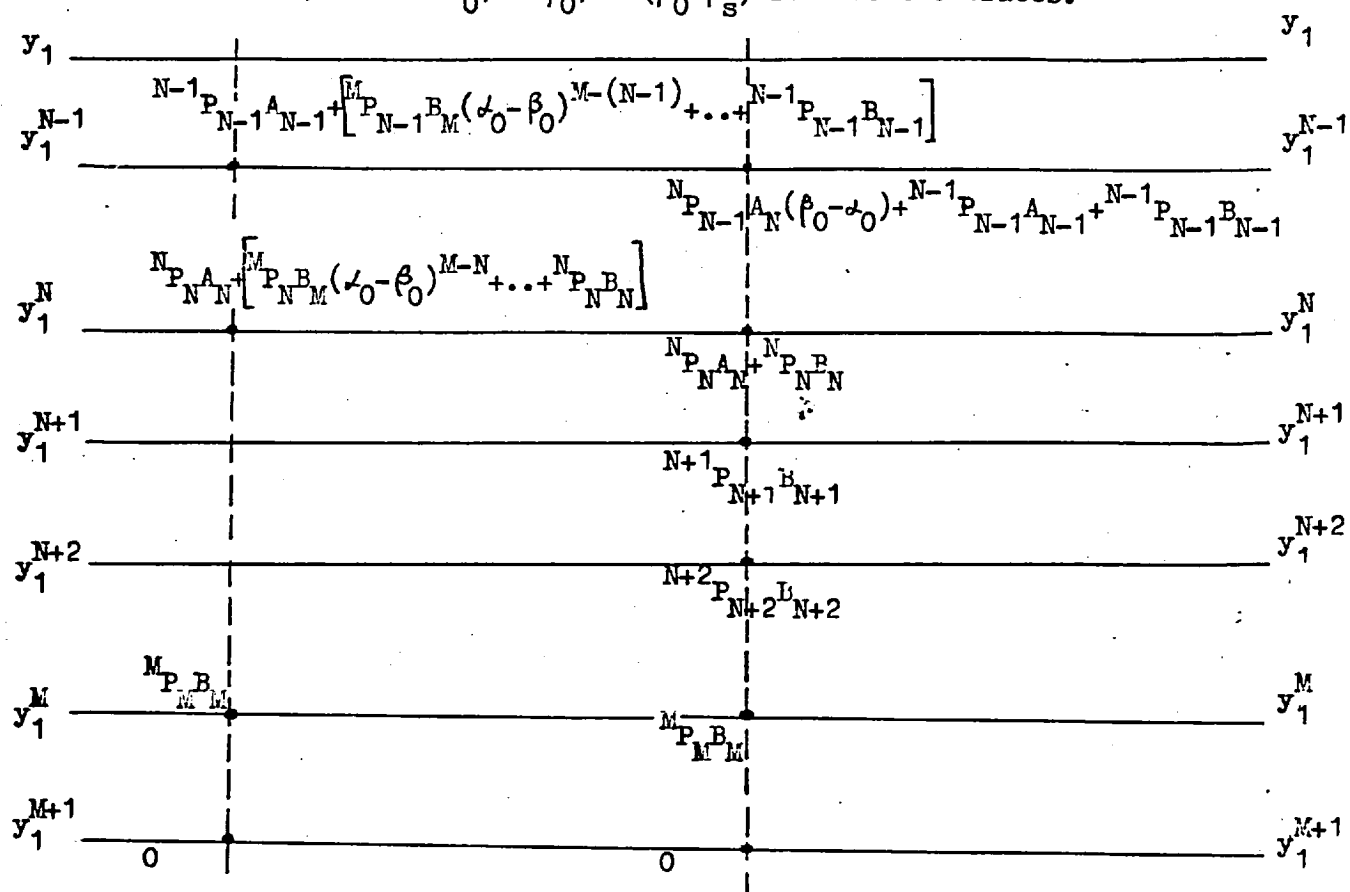
The above equations can be written in a matrix form as follows:

$$\begin{array}{c}
 \begin{array}{c} (M+1)(1) \\ y_1^0 \\ y_1^1 \\ y_1^2 \\ \vdots \\ y_1^N \\ \vdots \\ y_1^M \end{array}
 \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}
 \begin{array}{c} (M+1)(N+1) \\ (N+1)(1) \end{array}
 \begin{array}{c} (t-\alpha_0)^N \quad (t-\alpha_0)^{N-1} \quad \dots \quad {}^0P_0 \\ {}^N P_1 (t-\alpha_0)^{N-1} \quad {}^{N-1} P_1 (t-\alpha_0)^{N-2} \quad \dots \quad {}^1 P_1 \\ {}^N P_2 (t-\alpha_0)^{N-2} \quad {}^{N-1} P_2 (t-\alpha_0)^{N-3} \quad \dots \quad {}^2 P_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ {}^N P_N \quad 0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ 0 \end{array}
 \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}
 \begin{array}{c} (M+1)(N+1) \\ (N+1)(1) \end{array}
 \begin{array}{c} A_N \\ A_{N-1} \\ \vdots \\ A_0 \\ \vdots \\ \vdots \end{array}
 \end{array}
 +
 \begin{array}{c}
 \begin{array}{c} (M+1)(M+1) \\ (M+1)(1) \end{array}
 \begin{array}{c} (t-\beta_0)^M \quad (t-\beta_0)^{M-1} \quad \dots \quad {}^0P_0 \\ {}^M P_1 (t-\beta_0)^{M-1} \quad {}^{M-1} P_1 (t-\beta_0)^{M-2} \quad \dots \quad {}^1 P_1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ {}^M P_N (t-\beta_0)^{M-N} \quad \dots \quad {}^N P_N \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ {}^M P_M \end{array}
 \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}
 \begin{array}{c} (M+1)(M+1) \\ (M+1)(1) \end{array}
 \begin{array}{c} B_M \\ B_{M-1} \\ \vdots \\ B_N \\ \vdots \\ B_0 \end{array}
 \end{array}
 + U.N$$

where y^N is the Nth differential. It is observed that y^M is the last non zero differential and y^{M+C} is zero for all $C \geq 1$.

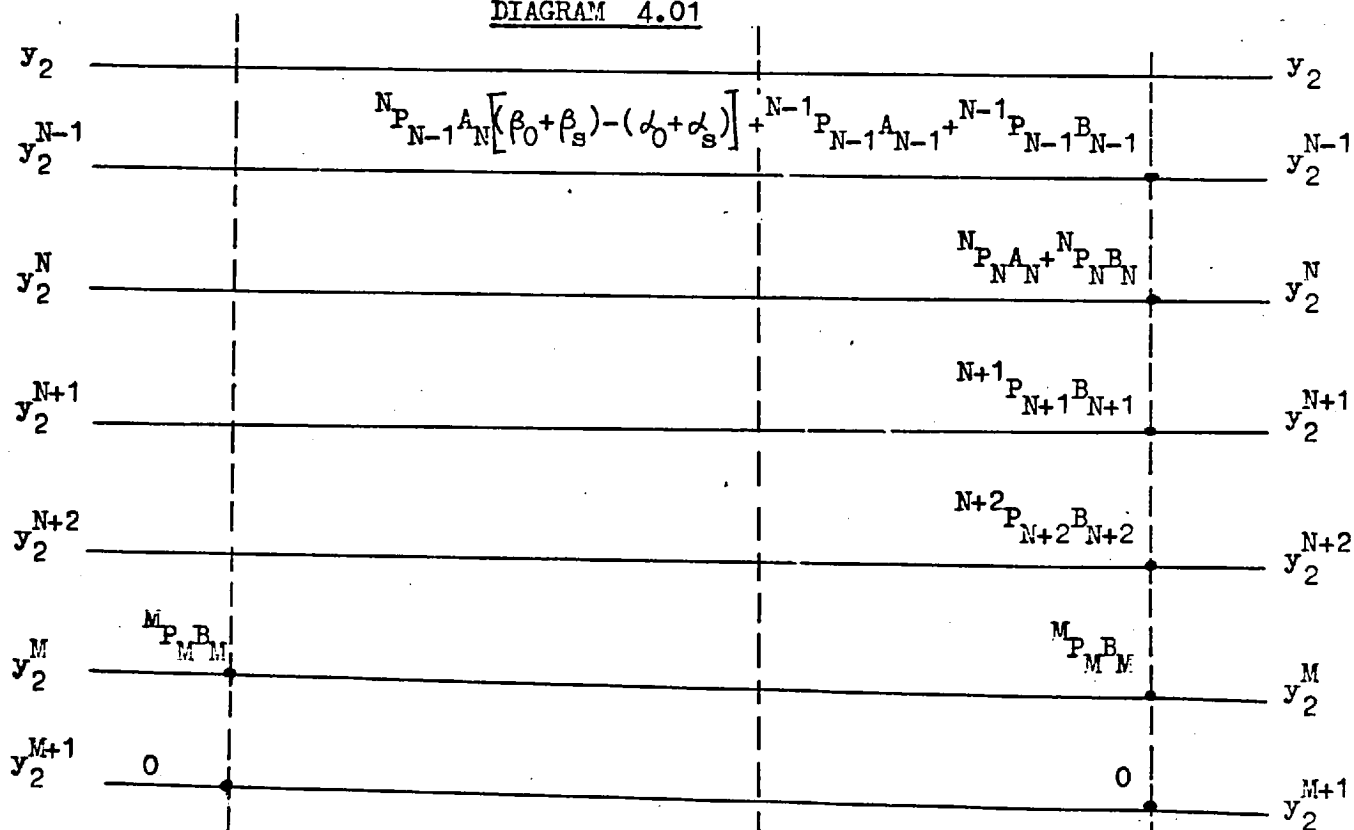
A similar expansion can be made for trace 2, the only difference being α_0 above is replaced by $(\alpha_0 + \alpha_s)$ and β_0 is replaced by $(\beta_0 + \beta_s)$.

The diagram below shows the differential values at three important co-ordinates, i.e. $t=\alpha_0$, $t=\beta_0$, $t=(\beta_0+\beta_s)$ for the two traces.



FOR TRACE 1

DIAGRAM 4.01



FOR TRACE 2

DIAGRAM 4.02

$t=\alpha_0$

$t=\beta_0$

$t=\beta_0+\beta_s$

β_s

The following values can be obtained from inspecting the diagrams 4.01 and 4.02.

- 1) In both diagrams the first zero derivative for all time $t=t$ occurs at y_{M+1} . Therefore by inspection the value of $(M+1)$ can be obtained. But M is the highest power of the polynomial forming the correlated noise component. Therefore the value of M can be found.
- 2) The values at $t=\beta_0$ from y^N to y^{M+1} in trace 1 is exactly repeated at $t=(\beta_0+\beta_s)$ in trace 2. Therefore by locating the value of t at which the amplitude values at $t=\beta_0$ in trace 1 are repeated in trace 2 we can locate the point $t=(\beta_0+\beta_s)$ in trace 2. But β_0 is known, therefore β_s can be estimated. N.B. there is only one time $t=(\beta_0+\beta_s)$ at which all the values are repeated as this is the crucial shift of time of the correlated noise curve.
- 3) Once the coincident times $t=\beta_0$ in trace 1 and $t=\beta_0+\beta_s$ are located the first non equal value on these two lines would occur at the derivative at which the signal component makes a contribution. In the example this would occur at the $(N-1)$ th derivative point. Therefore the curve values which give the first non coincident values on lines $t=\beta_0$ in trace 1 and $t=\beta_0+\beta_s$ in trace 2 would be the curve derivative $(N-1)$. Therefore N can be evaluated and hence the highest power of the signal polynomial.

- 4) The value of the y_1^{N-1} derivative curve in trace 1 at $t=\beta_0$ is say Z_1 .

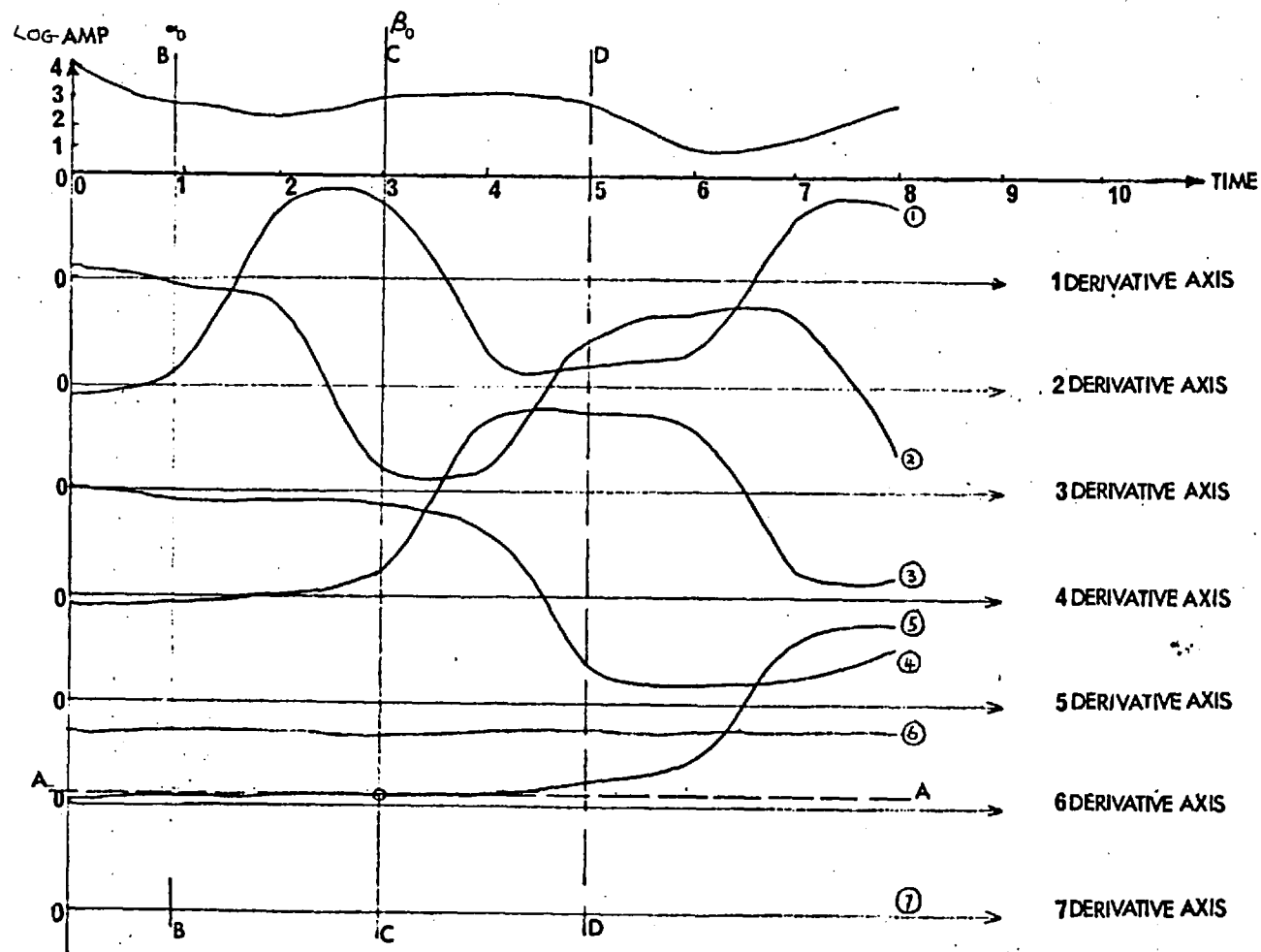
$$Z_1 = {}^N P_{N-1} A_N (\beta_0 - \alpha_0) + {}^{N-1} P_{N-1} A_{N-1} + {}^{N-1} P_{N-1} B_{N-1} \quad (4.3)$$

The value of the y_2^{N-1} derivative curve in trace 2 at $t=\beta_0+\beta_s$ is say Z_2 .

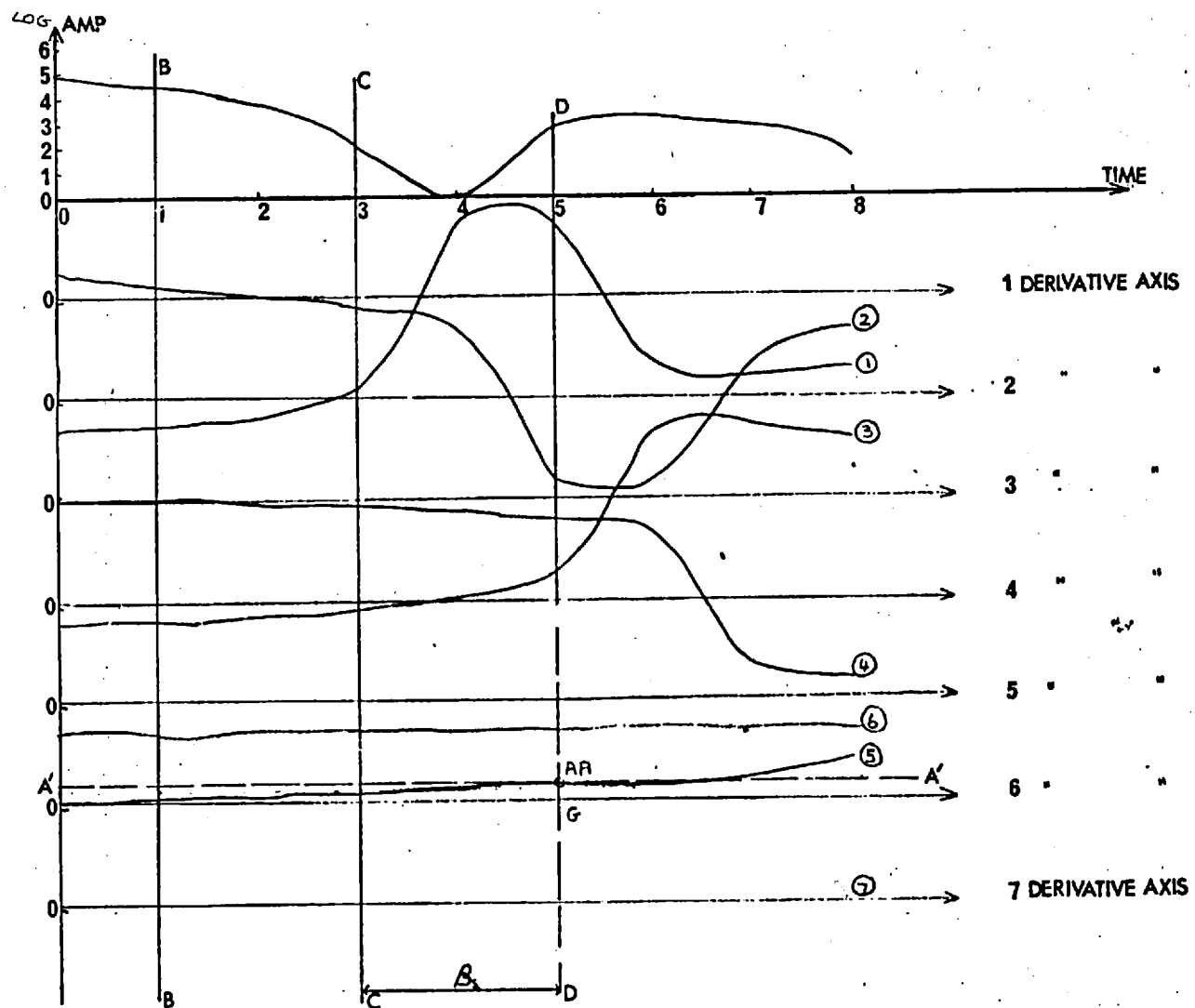
$$Z_2 = {}^N P_{N-1} A_N [(\beta_0+\beta_s) - (\alpha_0+\alpha_s)] + {}^{N-1} P_{N-1} A_{N-1} + {}^{N-1} P_{N-1} B_{N-1} \quad (4.4)$$

(4.3) - (4.4) gives:

$$Z_1 - Z_2 = {}^N P_{N-1} A_N (\alpha_s - \beta_s) \quad (4.5)$$



TRACE 1
Fig. 41



TRACE 2
Fig. 42

Now A_N is the coefficient of the highest power of the polynomial representing the signal component and hence can be made unity without any loss of generality. Therefore equation 4.5 reduces to

$$Z_1 - Z_2 = \sum_{N-1}^N (\alpha_s - \beta_s) \quad (4.6)$$

In equation 4.6 the only unknown is α_s and hence can be estimated. This is the required shift in signal from trace 1 to trace 2. Therefore we have evaluated α_s and β_s the required shift in signal and correlated noise in the two traces. By repeating this procedure for all subsequent traces the corresponding shifts α_s , β_s can be obtained and hence a moveout functions for α_s and β_s can be formed for all the required traces.

4.2 Illustrative Examples

A numerical example will be given to illustrate the various points mentioned above. The problem is as follows. Two traces as represented in the topmost curves of figs. 4.1 and 4.2 are given. (The amplitude values are plotted on a logarithmic scale.) It can be assumed that the traces have components signal, correlated noise and uncorrelated noise with the usual assumptions. Determine the moveout from trace 1 to trace 2 for the signal and the correlated noise.

The amplitude values for the two traces from time $t=0$ to $t=8$ are given. Using these values a Legendrian Polynomial was fitted to the two traces. The RESULTS ARE illustrated in TABLES 1 and 2. A seventh order polynomial was thought to be sufficient and as can be seen from the results a good fit is obtained with only a six order polynomial. In a given example a high order polynomial should be used to obtain the maximum accuracy and as in the present example if the order is far too high the coefficients of the high powers will come out as small values and hence the order can be reduced further on these results to save computer time. Using these expressions for the traces successive differentiations were performed and the results are displayed

PROGRAM 1

Trace 1 in its Z transform form

$$f_1(Z) = 11523 + 630Z + 191Z^2 + 1512Z^3 + 1683Z^4 + 818Z^5 + 15Z^6 + 36Z^7 + 707Z^8$$

which gave a 7th order polynomial $f_1(x)$ as

$$\text{Coefficient of } x^0 = 11523$$

$$\text{" } x^1 = -21136$$

$$\text{" } x^2 = 13649$$

$$\text{" } x^3 = -3925$$

$$\text{" } x^4 = 556$$

$$\text{" } x^5 = -38$$

$$\text{" } x^6 = 1.0$$

$$\text{" } x^7 = .0$$

PROGRAM 2

Trace 2 in its Z transform form

$$f_2(Z) = 147932 + 49491Z + 10710Z^2 + 191Z^3 + 0Z^4 + 1467Z^5 + 1706Z^6 + 855Z^7 + 36Z^8$$

which gave a 7th order polynomial $f_2(x)$ as

$$\text{Coefficient of } x^0 = 147932$$

$$\text{" } x^1 = -143063$$

$$\text{" } x^2 = 53724$$

$$\text{" } x^3 = -10049$$

$$\text{" } x^4 = 996$$

$$\text{" } x^5 = -50$$

$$\text{" } x^6 = 1.0$$

$$\text{" } x^7 = -.0$$

PROGRAM 3 TRACE 1

LOGARITHMIC AMPLITUDE VALUES

[illegible]

PROGRAM 4 TRACE 2

LOGARITHMIC AMPLITUDE VALUES

Time	t=0	t=1	t=2	t=3	t=4	t=5	t=6	t=7	t=8		
5.1	4.6	4.0	2.2	0	3.1	3.2	2.9	1.5		Given trace	
-5.1	-4.7	-4.3	-3.5	3.1	3.0	-2.6	-3.0	-2.6		1st derivative	
5.0	4.7	4.4	3.9	3.0	-3.1	-3.0	1.8	2.9		2nd	"
-4.7	-4.5	-4.3	-4.0	-3.6	-2.8	3.0	3.0	2.5		3rd	"
4.3	4.2	4.1	3.9	3.7	3.4	2.9	-2.6	-3.0		4th	"
-3.7	-3.7	-3.6	-3.5	-3.4	-3.3	-3.2	-2.9	-2.3		5th	"
2.8	2.8	2.8	2.8	2.8	2.8	2.8	2.8	2.8		6th	"
0	0	0	0	0	0	0	0	0		7th	"

in figs. 4.1 and 4.2. As before the amplitudes were plotted on a logarithmic scale. The programs 3 and 4 illustrate the method used. The values of α_0 and β_0 are given as $t=1$ and $t=3$. The lines BB and CC were drawn at these timing values.

From the results displayed in figs. 4.1 and 4.2 it is seen that the first zero derivative is at the seventh curve and therefore from 4.1 $M=6$. Obtain the amplitude of a non zero derivative curve near to the last non zero curve at $t=\beta_0$ in trace 1. The fifth derivative curve was chosen and the required point is the intersection of AA and CC. Draw a line A'A' on the corresponding derivative curve in trace 2 at the same amplitude. This is at the fifth derivative curve in trace 2. The intersection of the fifth derivative curve and A'A' in trace 2 is marked AA. The line at ordinate AA is equivalent to $(\beta_0 + \beta_s)$ from trace 2. This could be seen to be $t=5$ at point G in fig. 4.2 since $\beta_0=3$ and $\beta_s=2$. Therefore the shift in the correlated noise from trace 1 to trace 2 is two time units. Referring to program 3 the relevant values of different derivatives for trace 1 at $t=\beta_0$ is marked enclosed. Referring to program 4 the relevant values of different derivatives for trace 2 at $t=\beta_0 + \beta_s$ (now known) is marked enclosed. By inspecting these figures it can be seen that the first discrepancy occurs at the third derivative term. In trace 1 it is -2.9020 and in trace 2 it is -2.8887.

Therefore by consideration 3, $N=4$.

The first non coincident value in trace 1 at $t=\beta_0$ is -798 and the corresponding value in trace 2 at $t=\beta_0 + \beta_s$ is -774.

Therefore from equation 4.6:

$$(-798) - (-774) = {}^4P_3 [\alpha_s - \beta_s] = 4 \times 3 \times 2 [\alpha_s - 2]$$

$$\text{Therefore } -24 = +24 [\alpha_s - 2]$$

$$\text{Therefore } \alpha_s = 1$$

Therefore there is a shift of 1 time unit from trace 1 to trace 2 in the signal.

Thus we have solved the problem, namely we have estimated the move out shift from trace to trace for the signal and the correlated noise.

It is recalled that the highest power of the Legrangian Polynomial representing the uncorrelated noise component should have a value less than M or N. Otherwise differentiation would not eliminate this component. If $L > M$ the above analysis is not valid and we cannot obtain the move outs as shown. But we could in fact verify whether this is so. If $L > M$ or $L > N$ it will be impossible to find the same set of values for differentiated curves in trace 1 and for trace 2, as the two uncorrelated noise curves in trace 1 and trace 2 are quite independent. If this happens to be so we can conclude that L is in fact greater than M or N and the method is now not valid. But in practice we would have many traces and we may be able to work out the move out for those traces in which $L < M$ and $L < N$ and by interpolation it is possible to obtain the move out values for those traces for which $L > M$ or $L > N$.

Therefore the move out analysis procedure can be summarised as follows. Consider the reference curve and any other curve. Fit a Legrangian Polynomial for the two traces. The degree can be tested for best and most economical value. By using considerations 1, 2, 3, 4, obtain α_s, β_s . Repeat for another trace and the reference trace and obtain α_s and β_s and so on for all traces. Hence a move out curve representing the values so obtained for α_s will give the move out curves for signal and the values for β_s would give the move out curve for correlated noise. Knowing these two curves filters can be used to separate the signal and the noise on the basis of their different move out curves.

4.3 Conclusion

In the above analysis the order of the polynomial representing the correlated noise component (M) was assumed to be higher than the order of the polynomial representing the signal (N). If this was not so and if $N > M$ the analysis is still valid. The differentiation will now eliminate the correlated noise component and hence will give the signal move out equal to β_s . The two move out curves for signal and correlated noise components can be worked out exactly as before but the mathematics does not differentiate between the two move out curves as belonging to the signal and correlated noise. Practical considerations will have to be employed to identify signal move out and noise move out separately. In the illustrative example α_0 and β_0 was assumed known. If this was not so the relative shifts of move out can be obtained from trace to trace but the absolute value cannot be obtained. However as regards filtering, only the relative curves are necessary as move outs are always with reference to a trace. This is illustrated in the values obtained in programs 3 and 4 where for any value of time (t) the values obtained for the differentiated curves for program 3 are repeated for time (t+2) in program 4 implying a constant shift $t=2$ from trace 1 to trace 2 as before.

CHAPTER 5

ABSOLUTE ERROR ANALYSIS

5.0 Introduction

The conventional way of evaluating prediction error operators is to use least square error minimisation criterion. A quiet zone of the trace is selected and a set of operators is calculated to predict the future values of the time series. By comparing the predicted values and the actual values an error series can be formed and the reflection zones can be picked off from this error series. The method is explained by Wadsworth et al. ('Geophysics' (1953)). An important feature in this derivation is the use of least square error criterion to obtain the values of the operators. No justification is given for the use of this particular criterion and it is assumed that it is the most common way of evaluating the operators. Results obtained from this method are good enough to justify the method. However it would be convenient if an alternative method were available to test the results obtained using this method. In this chapter absolute error is used as a minimising criterion. The equations are developed using this as a criterion and the operators are calculated from the ensuing matrix equations. It is shown how this method gives comparable results with the conventional method and it may be used to resolve ambiguities in the conventional method.

5.1 Square error analysis

An alternative to the usual least square error criterion of prediction is investigated here. It is useful to illustrate briefly the method normally used in this context. The method followed is well illustrated in the article on 'Detection of Reflection on Seismic Records' by Wadsworth et. al. in 'Geophysics' (Vol. 18, 1953). The problem to be solved is; 'given information in the form of a seismogram, separate the desired information, i.e. reflection times, from the total information

present in the seismogram'. A statistical approach is used here as opposed to a deterministic approach. Normal statistical analysis is valid in the case of stationary processes but in our example the time series obtained in the form of a seismogram is non stationary in the sense that the origin is tied down to the time of the explosion. The difficulty is got over by dividing the record into several sections and considering each time interval of these subdivided seismograms as approximately stationary. The traces contained in these intervals are used to determine an optimum linear operator according to some criterion. The operator will thus contain the dynamic characteristics of the traces in the given interval called the operator interval. The type of operator used in this example is the pure prediction operator which predicts the future values of one trace from its past values. As the operator will contain the dynamics of the traces in the given time interval, any disturbance in the dynamics will be detected by the operator depending on how closely it can reconstruct the trace in the given operator interval. Having thus evaluated the operator over a given time interval, it is now used to predict the values of the rest of the trace. In this context prediction error is defined as the difference between the predicted value and the actual value of the trace. This error gives a measure of the dynamic uniformity of the rest of the trace as compared to the trace in the operator interval in the sense that a low error would indicate a small dynamic change, and a high error would indicate a large dynamic change. Hence the operator can be used to predict the values of the entire trace and consequently the prediction errors and the series of errors called prediction error series would contain information regarding the dynamic uniformity, and this is the required information series. By experience and experiments it was found that dynamic characteristics remained fairly constant in the sense that one operator would apply to several sections except during intervals corresponding to a reflection. This

would mean that an operator evaluated for a non reflection interval when applied to a region where reflections occur, would give high errors of prediction. Therefore an error series as defined above should give high values at regions of reflections and should give uniform values or near zero values at non reflection zones, provided the operator was evaluated at a non reflection zone. Hence by examining an error time series one is able to pick off the arrival of reflection at places of high error.

The following analysis with slight modifications is reproduced from Wadsworth's article. The linear operator which predicts the future values of trace x from its past values is expressed mathematically as

$$\hat{x}_{i+k} = \sum_{s=0}^M a_s x_{i-s} \quad (5.1)$$

where $\hat{x}_{i+k}$ is the approximated value of the future value x_{i+k}

where k is the number of time units ahead which the operator predicts.

The $(M+1)$ past values of the trace x are shown by x_{i-s} where $s=0,1,...M$.

The $(M+1)$ values of the operator given by a_s ($s=0,...M$) are constants determined in an optimum sense. This criterion is normally chosen as

the Gauss Method of least squares. In this context we wish to determine the constant a_s in such a way that the sum of the square errors between

the actual value x_{i+k} and the predicted value $\hat{x}_{i+k}$ is minimised. The

summation is taken over the time interval defined by the operator time

interval. Let us assume that this time interval defined as the operator interval has a time duration nh seconds, where h is the time interval

between successive observations in the digitised form of the continuous

curve represented in the seismogram. Hence the values of x_i will consist of $i=N$ to $i=N+n-1$ where $i=N$ is an arbitrary origin.

Accordingly, the minimising error criterion will be represented as

$$I = \sum_{i=N-k}^{N+n-1-k} (x_{i+k} - \hat{x}_{i+k})^2 \quad (5.2)$$

Substituting equation 5.1 into equation 5.2

$$I = \sum_i (x_{i+k} - \sum_s a_s x_{i-s})^2 \quad (5.3)$$

where the summation i is from $i=N-k$ to $i=N+n-1-k$
and " " " " " $s=0$ to $s=M$.

To carry out the minimisation we set the partial derivative of I with respect to a_s to zero for $s=0, \dots, M$ which would give from equation 5.3

$$\sum_s (a_s \sum_i x_{i-s} x_{i-r}) = \sum_i x_{i-r} x_{i+k} \quad (5.4)$$

for $r=0, 1, \dots, M$.

This combined equation would give $(M+1)$ equations for the $(M+1)$ unknowns a_s ($s=0, \dots, M$) and hence can be solved. Two criterions of errors will be illustrated:

- 1) The square error $I = (x_{i+k} - \hat{x}_{i+k})^2$ (5.5)
- 2) The running average of squared errors E_i defined as

$$E_i = \frac{1}{2P} \sum_{j=i-P}^{i+P-1} (x_j - \hat{x}_j)^2 \quad (5.6)$$

where $2P$ is the number of elements in each average.

5.2 Illustrative example

The trace designated as N750 in fig. 5 of Wadsworth's article is chosen as an example. The above trace is reproduced on an enlarged scale in fig. 5.00. The digitised values were read off from this diagram at an interval of $h = .005$ seconds. Prediction distance = .005 seconds = 1 unit of time. The past lags = $M = 4$. The origin is designated as $X_N = X_1$ in fig. 5.00. Therefore $N=1$ and $N+n-1$ was chosen as equal to 26 which gives a value $n=26$ with an operator interval of

$nh = 26 \times 5 = 0.130$ seconds. The equations 5.4 were solved in the CDC 6400 computer using a matrix inversion method. The five operator coefficients for a_s ($s=0, \dots, M$) were obtained as (1.26, -1.00, 0.49, 0.10, 0.15). Using these operator values and equation 5.1 the predicted curve was obtained as shown in fig. 5.00. The error criteria 5.5 and 5.6 were also calculated and are plotted in fig. 5.01 and fig. 5.02. The values of x_i from $i=0.85$ to $i=1.075$, only are calculated as this example is made only to illustrate the method. In fig. 5.00 reflection arrival zone observed visually from the records is marked as a hatched area. As can be seen from examining the error diagrams this area is clearly brought out by its high values. Hence it can be concluded that a reflection has occurred in the region from time = 1.0 to time = 1.04 which confirms the visual observation. By optimising the parameters involved, i.e. the number of operators, operator length, etc. and by using the adjacent traces as well, a better representation of the two error series can be obtained as shown in Wadsworth's article. However, even with just one trace and only five operator coefficients a reasonable picture can be obtained as seen in fig. 5.01 and 5.02.

5.3 Absolute error criterion

IT would be useful if we can confirm these results with an alternative method. This will be attempted in the following section of this chapter.

As will be recalled the prediction error was defined as square of the error

$$I_{i+k} = (x_{i+k} - \hat{x}_{i+k})^2$$

and the minimising was done in order to find the operator values which would give the least value for the sum of the squares of errors. Instead of using the square error the sum of the absolute errors will be minimised to obtain the operator coefficients. In fact the absolute error can be

The values containing $x_1 \rightarrow x_{10}$ consist of the operator interval, and
 " " " $x_{11} \rightarrow x_{20}$ " " " 'predicted' present values.

The set of equations 5.9 would reduce to

$$\begin{pmatrix} x_{10} & x_9 & \dots & x_1 \\ x_{11} & x_{10} & \dots & x_2 \\ \dots & \dots & \dots & \dots \\ x_{19} & x_{18} & \dots & x_{10} \end{pmatrix}^{10 \times 10} \begin{pmatrix} a_0 \\ a_1 \\ \dots \\ a_9 \end{pmatrix}^{10 \times 1} = \begin{pmatrix} x_{11} \\ x_{12} \\ \dots \\ x_{20} \end{pmatrix}^{10 \times 1} \quad (5.10)$$

The matrix equation 5.10 can be solved by the matrix inversion method. This was done on the CDC 6400 computer using the matrix inversion method and the following values were obtained for the ten operator coefficients.
 $a_g = (-0.27, -0.46, -0.25, -0.42, 0.27, 0.68, 0.00, 0.62, 0.73, 0.07).$

5.5 Discussion of results

THE above values were used in equation 5.1 to predict the trace curve values and the result is plotted in fig. 5.10. As can be seen from this diagram the predicted curve approximates the actual curve reasonably well in the region outside the interval $t = 1.00$ to $t = 1.04$. This is shown more clearly in the error diagram where there are high peak values in this region. A disturbing feature in this error diagram is that even in the region after a reflection zone errors seem^s to be still very high which gives a confused picture. This is also seen in fig. 5.11 where there are two small humps after the reflected arrival region. Ideally there should be no peaks apart from the reflection zone. An improvement can be made in the following way. It must be noticed that the future values are predicted using the present and past values. This would work quite well up to and including the reflection zone as seen in figs. 5.11 and 5.12. But when the values beyond this region are calculated, the values used will be the values in the reflection zone region. Obviously

these values do not have the same dynamics of the quiet zone and hence, if we were to use these values to predict, we would get values which are quite different to that which would be obtained in the quiet zone and hence, in the error diagram this region would give high errors as seen in fig. 5.11 and 5.12. But if we were to use the predicted values in the reflection zone to predict the values beyond the reflection zone, this error should not arise. This was done as modification 1 in figs. 5.10, 5.11, 5.12, and the results can be seen to be much better than the original representation. The fluctuations beyond the reflection zone are reduced and the running error fig. 5.12 gives a better picture. This was extended to the whole curve, i.e. the predicted values were used to predict the future values and the results are quite good as seen in fig. 5.11 and 5.12. The fluctuations in fig. 5.11 are much reduced beyond the reflection zone and the peak in fig. 5.12 is quite definite, and it converges to low levels fairly soon beyond the reflection zone. As can be seen the high error corresponds to the region $\text{time} = 1.0 \rightarrow \text{time} = 1.04$ and the results obtained previously are now confirmed. In general the peaks in the error diagram tend to be more definite than in the square error criterion, and a more definite picture is obtained because of its higher values.

To test the method further the analysis was done to the entire trace designated as N750 in Wadsworth's article and represented as fig. 6 in his article. The enlarged form of this trace is represented in fig. 5.20. First the squared error criterion was applied and the original operator values were used to predict the entire curve from $T = 0.85$ to $T = 1.235$. The resulting curve is plotted in fig. 5.20, and it can be seen that the curve matches the original curve except in the regions $T = 1.00$ to $T = 1.04$ and $T = 1.15$ to $T = 1.23$ which in fact corresponds to the two reflections observed in the seismogram. The error figs. 5.21 and 5.22 illustrates this further. Fairly high fluctuations can be seen just after $t = 1.05$ in fig. 5.22 and this is due to the fact that wrong values

in the reflection zone were used to predict these values. This should not correspond to a reflection but due to the error in the evaluation as explained by Wadsworth. This is an unfortunate feature in the conventional method. The fig. 5.22 gives peaking values in the region $1.00 \rightarrow 1.04$ but is fairly well spread out due to the error mentioned above. Increasing trend is observed after $t = 1.15$ in fig. 5.22 which suggests an arrival around this region confirming the visual observation.

The exercise was repeated with the second method as explained before. A further modification was incorporated into the calculation. From the results obtained in the previous example it was obvious that from time $t = 1.00$ to about $t = 1.04$ a reflection zone was situated. This would mean the formula 5.1 should not be applied to the observed values beyond this region, otherwise fluctuations would occur in the error diagram. Using the predicted values in the region, and the present values in the region immediately after this, a further set of operators were calculated to be applied to the rest of the series from $t = 1.04$ onwards. This would get over the fact of using wrong values, and these operator values should reproduce the rest of the curve until the next reflection comes when the process can be repeated. The results are shown in figs. 5.20, 5.21 and 5.22. From fig. 5.20 it can be seen that the predicted curve agrees with the original curve within times $t = 0.85 \rightarrow t = 1.00$, and $t = 1.04 \rightarrow t = 1.11$, (This is the operator interval for the second set of operator values) and from $t = 1.14$ to $t = 1.17$.

The error figs. 5.21 and 5.22 shows the results quite clearly. The fluctuation in zones adjacent to the reflection zones are reduced and the error values are quite high and give definite peaks. Fig. 5.22 gives a quite clear picture of this. The high peak near $t = 1.02$ is quite definite, and so is the gradual climbing up of the values beyond $t = 1.16$. The record length was not long enough to compute error values beyond $t = 1.21$ but the peaking tendency near $t = 1.2$ is clearly shown. The error diagram gives low values everywhere except at the region between

$t = 1.1$ and $t = 1.14$. A high peak is obtained in fig. 5.21 and the build-up of values in fig. 5.22 near the region is quite marked. A reflection was not observed visually in this region and it is not possible to verify whether in fact there is a reflection in this region as indicated by this diagram. The mean square error criterion method does not show this as a reflection zone and it would be interesting to verify whether in fact there is a reflection in this region. Wadsworth has in his example done a more comprehensive test in his paper and the results reproduced in his article give an interesting picture. In the diagram designated as fig. 5 it can be seen that in fact in the region 1.1 to 1.15 the two curves, predicted and actual, are quite far apart relatively. Referring to fig. 7 in his article the lower left-hand diagram is interesting as it gives a peak at the region 1.1 to 1.14. Unfortunately in the least square error criterion method the peak does not come to the required high values to be regarded as a reflection. But the very fact that a peak of a fairly high value occurs at this point suggests that a reflection could occur at this point. The absolute error method on the other hand suggests quite clearly that a reflection could occur in this region, and combining the two methods, a reflection is quite likely at this region which was not observed visually. This exemplifies another quality in the absolute error method. It tends to have higher error values at reflection zones than the square error method and hence, even faint reflection zones are brought forward more clearly. This is clearly shown by observing the region from 1.2 to 1.24 in figs. 5.22 and 5.21. The absolute error curves give quite high peaking values and the square error curve does not give as clear a picture.

5.6 Conclusion

It does not follow that the absolute error criterion should always be preferred to the usual square error criterion. The two methods should give complementary results and in doubtful regions the combination of the two methods should resolve the ambiguity. As in the region from

1.1 to 1.14, square error showed only a very slight indication of a peaking error but the absolute error diagram was quite definite in this region.

Hence by using both methods to analyse a given trace, more definite information can be obtained as opposed to the application of one of the two methods.

CHAPTER 6

STABILITY

6.0 Introduction

The purpose of this chapter is to illustrate the conditions under which the multichannel analysis of seismic data is feasible. The condition of feasibility can be mathematically analysed as equivalent to the inversion of a relevant matrix. The matrix thus formed is important as regards the analysis of filters as well. In fact the stability of a filter can be related to the stability of the auto correlation matrix. AN OPERATOR IS CALLED STABLE IF IT HAS A FINITE INVERSE.

6.1 Multichannel Theory



A box with N inputs and M outputs is shown in fig. 6.0. Exciting input channel 1 with a spike $(1, 0, \dots)$ a multichannel response

FIG. 6.0

of all M outputs is expected whose Z transforms are $A_{11}(Z)$, $A_{21}(Z)$, ..., $A_{M1}(Z)$. Introducing a spike on input channel K will result in M different outputs $A_{1K}(Z)$, $A_{2K}(Z)$, ..., $A_{MK}(Z)$. The behaviour of the box is fully described by the $M \times N$ responses $A_{ij}(Z)$, ($i = 1, 2, \dots, M$; $j = 1, \dots, N$). For a multichannel filter the principle demands that the multichannel output is connected with the multichannel input by the following matrix multiplication:

$$\begin{array}{c}
 \begin{array}{c} M \times N \\ \left| \begin{array}{cccc} A_{11}(Z) & A_{12}(Z) & \dots & A_{1N}(Z) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{M1}(Z) & \dots & \dots & A_{MN}(Z) \end{array} \right| \end{array}
 \begin{array}{c} N \times 1 \\ \left| \begin{array}{c} x_1(Z) \\ x_2(Z) \\ \cdot \\ \cdot \\ x_N(Z) \end{array} \right| \end{array}
 =
 \begin{array}{c}
 \begin{array}{c} M \times 1 \\ \left| \begin{array}{c} Y_1(Z) \\ Y_2(Z) \\ \cdot \\ \cdot \\ Y_M(Z) \end{array} \right| \end{array}
 \end{array}
 \quad (6.1)$$

6.2 Conditions of Compatibility

The Z transform of a multichannel filter therefore has the mathematical representation of a polynomial matrix. It may be written in the form of a matrix polynomial:

$$A(Z) = a_0 + a_1 Z + \dots + a_s Z^s$$

where each coefficient is a constant ($M \times N$) matrix. If the input series

$X_N(Z)$ has amplitude values up to time $t=s_1$ i.e. $X_N(t)$ is for $t=0, 1, 2, \dots, S_1$, the output values $Y_N(Z)$ should be specified up to $t=(s+s_1)$ i.e. $Y_N(t)$ is for $t=0, 1, 2, \dots, S_2$, where $s_2=(s+s_1)$. This is the compatibility condition for the matrix multiplication given above. This is clearly seen in the expanded version of equation 6.1 shown below:

$$\left[\begin{array}{c|c|c} \begin{matrix} A_{11t_0} & A_{12t_0} & \dots & A_{1Nt_0} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{M1t_0} & A_{M2t_0} & \dots & A_{MNT_0} \end{matrix} & \begin{matrix} A_{11t_1} & \dots & A_{1Nt_1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A_{M1t_1} & \dots & A_{MNT_1} \end{matrix} & \begin{matrix} A_{11t_s} & \dots & A_{1Nt_s} \\ \vdots & & \cdot \\ \cdot & & \cdot \\ A_{M1t_s} & \dots & A_{MNT_s} \end{matrix} \\ + & Z + \dots + & Z^s \end{array} \right] X$$

$$\left[\begin{array}{c|c|c} \begin{matrix} B_{1t_0} \\ \cdot \\ \cdot \\ B_{Nt_0} \end{matrix} & \begin{matrix} B_{1t_1} \\ \cdot \\ \cdot \\ B_{Nt_1} \end{matrix} & \begin{matrix} B_{1ts_1} \\ \cdot \\ \cdot \\ B_{Nts_1} \end{matrix} \\ + & Z + \dots + & Z^{s_1} \end{array} \right] = \left[\begin{array}{c|c|c} \begin{matrix} O_{1t_0} \\ \cdot \\ \cdot \\ O_{Mt_0} \end{matrix} & \begin{matrix} O_{1t_1} \\ \cdot \\ \cdot \\ O_{Mt_1} \end{matrix} & \begin{matrix} O_{1ts_2} \\ \cdot \\ \cdot \\ O_{Mts_2} \end{matrix} \\ + & Z + \dots + & Z^{s_2} \end{array} \right]$$

By equating the corresponding coefficients of the powers of Z , $M(s_2+1)$ equations can be formed for the filter coefficient unknowns $M \times N \times (s+1)$. This would mean that for a full solution of the unknown filter coefficients the following conditions should be satisfied:

$$M(s_2+1) = (M \times N)(s+1) \quad (6.2)$$

also for matrix multiplication

$$s_2 = s+s_1 \quad (6.3)$$

Consider a case where N = number of input channels and $(1+s_1)$ the number of digitised values of each input series and $(1+s)$ the number of filter coefficients. The convolution formula can then be written as:

$$Y_{jt} = \sum_{i=1}^N \sum_{s=0}^s f_{i,j,s} X_{i,(t-s)}$$

$$= \sum_{i=1}^N f_{i,j,0} X_{i,t} + f_{i,j,1} X_{i,(t-1)} + \dots + f_{i,j,s} X_{i,(t-s)}$$

where Y_{jt} = output at j th channel at time t and $f_{i,j,t}$ is the filter

coefficient for i^{th} input, j^{th} output at time t .

Consider one output channel only, i.e. $j=1$. The output values on this single channel are given by:

$$\begin{aligned}
 Y_0 &= \sum_{i=1}^N f_{i,0} X_{i,0} && \text{for time } t=0 \\
 Y_1 &= \sum_{i=1}^N (f_{i,0} X_{i,1} + f_{i,1} X_{i,0}) && \text{for time } t=1 \\
 Y_2 &= \sum_{i=1}^N (f_{i,0} X_{i,2} + f_{i,1} X_{i,1} + f_{i,2} X_{i,0}) && \text{for time } t=2 \\
 &\cdot && \cdot \\
 &\cdot && \cdot \\
 Y_{s_2} = Y_{s+s_1} &= \sum_{i=1}^N f_{i,0} X_{i,s_2} + f_{i,1} X_{i,(s_2-1)} + \dots + f_{i,s} X_{i,(s_2-s)} && \text{for time } t=s_2
 \end{aligned}$$

Expanding the above equations:

WITH $s_2 = S_2$, $S = S$ AND $s_1 = S_1$

$$\begin{aligned}
 Y_0 &= (f_{10} X_{10} + f_{20} X_{20} + \dots + f_{N0} X_{N0}) \\
 Y_1 &= (f_{10} X_{11} + f_{20} X_{21} + \dots + f_{N0} X_{N1}) + (f_{11} X_{10} + f_{21} X_{20} + \dots + f_{N1} X_{N0}) \\
 &\cdot && \cdot \\
 &\cdot && \cdot \\
 Y_S &= (f_{10} X_{1S} + f_{20} X_{2S} + \dots + f_{N0} X_{NS}) + \cdot && \cdot && + (f_{1S} X_{10} + f_{2S} X_{20} + \dots + f_{NS} X_{N0}) \\
 &\cdot && \cdot && \cdot && \cdot && \cdot && \cdot && \cdot \\
 Y_{S_2} &= (f_{10} X_{1S_2} + f_{20} X_{2S_2} + \dots + f_{N0} X_{NS_2}) + \cdot && \cdot && + (f_{1S} X_{1, (S_2-S)} + \dots + f_{NS} X_{N, (S_2-S)})
 \end{aligned}$$

which can be written in the expanded matrix form as:

$$\begin{array}{c}
 \begin{array}{c} (S_2+1), N \\ \left[\begin{array}{c|c} X_{10} \dots X_{N0} \\ X_{11} \dots X_{N1} \\ \vdots \\ X_{1S_2} \dots X_{NS_2} \end{array} \right] \end{array} \quad \begin{array}{c} N, 1 \\ \left[\begin{array}{c|c} f_{10} \\ f_{20} \\ \vdots \\ f_{N0} \end{array} \right] \end{array} \quad \begin{array}{c} 0 \\ X_{10} \dots X_{N0} \\ \vdots \\ X_{1(S_2-1)} \dots X_{N(S_2-1)} \end{array} \quad \begin{array}{c} 0 \\ \vdots \\ X_{1(S_2-S)} \dots X_{N(S_2-S)} \end{array} \quad \begin{array}{c} f_{11} \\ f_{21} \\ \vdots \\ f_{N1} \end{array} \quad \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \quad \begin{array}{c} f_{1S} \\ f_{2S} \\ \vdots \\ f_{NS} \end{array} \quad \begin{array}{c} S_2+1, 1 \\ \left[\begin{array}{c} Y_0 \\ Y_1 \\ \vdots \\ Y_{S_2} \end{array} \right] \end{array}
 \end{array}$$

which can be combined as:

$$\begin{array}{c}
 \begin{array}{c} (1+S_2), 1 \\ N(1+S), 1 \\ (1+S_2), N(1+S) \end{array} \\
 \left[\begin{array}{c|c} X_{10} \ X_{20} \dots X_{N0} \ 0 \ 0 \dots 0 \dots \\ X_{11} \ X_{21} \dots X_{N1} \ X_{10} \ X_{20} \dots X_{N0} \\ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ X_{10} \ X_{20} \dots X_{N0} \\ X_{1S_2} \ X_{2S_2} \dots X_{NS_2} \ X_{1, S_2-1} \ X_{2, S_2-1} \dots X_{N, S_2-1} \dots X_{1, S_2-S} \ X_{2, S_2-S} \dots X_{N, S_2-S} \end{array} \right] \quad \begin{array}{c} f_{10} \\ f_{N0} \\ f_{11} \\ f_{N1} \\ f_{1S} \\ f_{NS} \end{array} \quad \begin{array}{c} Y_0 \\ Y_{S_2} \end{array}
 \end{array}$$

From the above analysis it can be seen that the multichannel filter coefficients can be found by solving the matrix equation:

$$[X][F] = [Y]$$

where X = input matrix of order $(1+S_2), N(S+1)$

and F = filter coefficient matrix of order $N(S+1), 1$

and Y = output matrix of order $(1+S_2), 1$.

On this basis two possible cases can be analysed for solution.

Case 1

$$M(S_2+1) = (M \times N)(S+1)$$

In the problem analysed M was taken as unity and as such the condition above reduces to:

$$S_2+1 = N(S+1)$$

Under this condition the matrix multiplication is compatible and the solution is given by:

$$[F] = [Y][X]^{-1}$$

and the values for F can be uniquely determined depending on the form of the inverse matrix X . A real unique solution can be obtained only if X^{-1} exists. This will be so only if $\det X \neq 0$. This stability criterion is entirely dependent on the relative values of S , S_1 and S_2 .

The output value at maximum time i.e. at $t=S_2$ is dependent on the knowledge of the input values up to $t=S_2$. If only the values up to $t=S_1$ are specified as inputs and it cannot be assumed that input values are zero for $t>S_1$ the output values can only be specified up to $S_2=S_1$. But if it can be assumed that the input time series is zero for $t>S_1$ the output values can be specified up to a maximum $S_2=S+S_1$. The matrix formation X will be dependent on this criterion.

(a) if it can be assumed that the input series values are zero for $t>S_1$.

The output formulae would take the form:

$$Y_0 = (f_{10}X_{10} + f_{20}X_{20} + \dots + f_{N0}X_{N0})$$

$$Y_1 = (f_{10}X_{11} + f_{20}X_{21} + \dots + f_{N0}X_{N1}) + (f_{11}X_{10} + f_{21}X_{20} + \dots + f_{N1}X_{N0})$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Y_S = (f_{10}X_{1S} + f_{20}X_{2S} + \dots + f_{N0}X_{NS}) + \cdot \quad \cdot \quad \cdot \quad \cdot \quad + (f_{1S}X_{10} + f_{2S}X_{20} + \dots + f_{NS}X_{N0})$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Y_{S_2} = (f_{10}X_{1S_2} + f_{20}X_{2S_2} + \dots + f_{N0}X_{NS_2}) + \cdot \quad \cdot \quad + (f_{1S_2}X_{1, (S_2-S)} + \dots + f_{NS_2}X_{N, (S_2-S)})$$

This would give X as:

$$\begin{vmatrix} X_{10} & X_{20} & \dots & X_{N0} & 0 & 0 & & & & & \\ X_{11} & X_{21} & \dots & X_{N1} & X_{10} & X_{20} & \dots & X_{N0} & & & \\ \cdot & \cdot & \cdot & \cdot & X_{11} & X_{21} & \dots & X_{N1} & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ X_{1S_1} & X_{2S_1} & \dots & X_{NS_1} & \cdot & \cdot & \cdot & \cdot & X_{10} & \dots & X_{N0} \\ 0 & 0 & 0 & 0 & X_{1S_1} & X_{2S_1} & \dots & X_{NS_1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & & & & & X_{1S_1} & \dots & X_{NS_1} \end{vmatrix} \quad (1+S_2), N(I+S)$$

This would give $N(1+S)$ number of filter coefficients providing the matrix above has an inverse, i.e. provided $\det(X) \neq 0$.

(b) If it cannot be assumed that the input series values are zero for $t > s_1$, This would mean that output values can only be specified up to $t = s_2 = s_1$.

Two cases can be distinguished:

(i) when $s_1 > s$ the input matrix will take the form:

$$\left[\begin{array}{cccccc} X_{10} & X_{20} & \dots & X_{N0} & 0 & 0 & 0 & (1+S_1)^N(1+S) \\ X_{11} & X_{21} & \dots & X_{N1} & X_{10} & X_{20} & \dots & X_{N0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & X_{10} \dots X_{N0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ X_{1S_1} & X_{2S_1} & \dots & X_{NS_1} & X_1(S_1-1) & X_2(S_1-1) & \dots & X_N(S_1-1) \dots X_1(S_1-S) \dots X_N(S_1-S) \end{array} \right]$$

In this case if $(1+S_1)^N(1+S)$ the coefficient matrix is square and a unique solution for the $N(1+S)$ filter coefficients is possible. If this is not so a redundant number of filter coefficients will be obtained as the number of unknown filter coefficients are greater than or equal to the number of equations available. This condition is discussed later on in this chapter.

(ii) when $s > s_1$, the input matrix will take the form:

$$\left[\begin{array}{cccccc} X_{10} & X_{20} & \dots & X_{N0} & 0 & 0 & 0 & (1+S_1)^N(1+S_1) \\ \cdot & \cdot & \cdot & \cdot & X_{10} & X_{20} & \dots & X_{N0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ X_{1S_1} & X_{2S_1} & \dots & X_{NS_1} & X_1(s_1-1) & \dots & X_N(s_1-1) & (X_{10} \ X_{20} \ \dots \ X_{N0}) \end{array} \right]$$

In this case the number of filter coefficients, i.e. $N(1+S)$ are more than the number of equation $(1+S_1)$ available. Hence there is no unique solution and certain number of filter coefficients will have redundant values. This condition is discussed later on in this chapter.

Case 2

If condition 2 is not satisfied the problem cannot be solved uniquely. However a solution is possible as shown below. Consider a 6-channel input with each input having 10 digitised amplitudes, i.e. $s_1=9$. Let the desired single channel output be a spike at zero time and specified as zero for time up to $t=10$. For convenience select s as equal to s_2 , i.e. 10.

Hence the relevant equations are:

$$Y_0 = (f_{10}X_{10} + f_{20}X_{20} \dots + f_{60}X_{60})$$

$$Y_1 = (f_{10}X_{11} + f_{20}X_{21} \dots + f_{60}X_{61}) + (f_{11}X_{10} + f_{21}X_{20} \dots + f_{61}X_{60})$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Y_{S_2-S} = (f_{10}X_{1S} + f_{20}X_{2S} \dots + f_{60}X_{6S}) + \dots + (f_{1S}X_{10} + f_{2S}X_{20} \dots + f_{6S}X_{60})$$

which gives the matrix multiplication as:

$$\begin{bmatrix} X_{10} & X_{20} & \dots & X_{60} \\ \cdot & \cdot & \cdot & X_{10} & X_{20} & \dots & X_{60} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ X_{1S_1} & X_{2S_1} & \dots & X_{6S_1} & X_{1,S_1-1} & X_{2,S_1-1} & \dots & X_{6,S_1-1} & (X_{10} & X_{20} & \dots & X_{60}) \end{bmatrix} \begin{bmatrix} f_{10} \\ f_{60} \\ \cdot \\ \cdot \\ \cdot \\ f_{1S} \\ f_{6S} \end{bmatrix} = \begin{bmatrix} Y_0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ Y_S \end{bmatrix}$$

For the parameter specified above the matrix reduces to:

$$\begin{array}{c} \begin{bmatrix} X_{10} & X_{20} & \dots & X_{60} & 0 & 0 & 0 \\ X_{11} & X_{21} & \dots & X_{61} & X_{10} & X_{20} & \dots & X_{60} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ X_{19} & X_{29} & \dots & X_{69} \\ X_{110} & X_{210} & \dots & X_{610} & X_{19} & X_{29} & \dots & X_{69} & X_{10} & X_{20} & \dots & X_{60} \end{bmatrix} \begin{array}{c} 11 \times 66 \\ 66 \times 1 \\ 11 \times 1 \end{array} \end{array} = \begin{array}{c} \begin{bmatrix} f_{10} \\ \cdot \\ f_{60} \\ f_{11} \\ \cdot \\ f_{61} \\ \cdot \\ f_{110} \\ \cdot \\ f_{610} \end{bmatrix} \begin{array}{c} 66 \times 1 \\ 11 \times 1 \end{array} \end{array}$$

Since only 11 output values are specified 11 equations will be available with a solution for 11 filter coefficients from a total of 66 required values. This would mean that 55 values of the filter coefficients are redundant in the sense that they are indeterminate uniquely and as such any arbitrary values could be incorporated. For convenience these values were taken as zero. The zero coefficients can be taken arbitrarily and the

MULTICHANNEL FILTERING

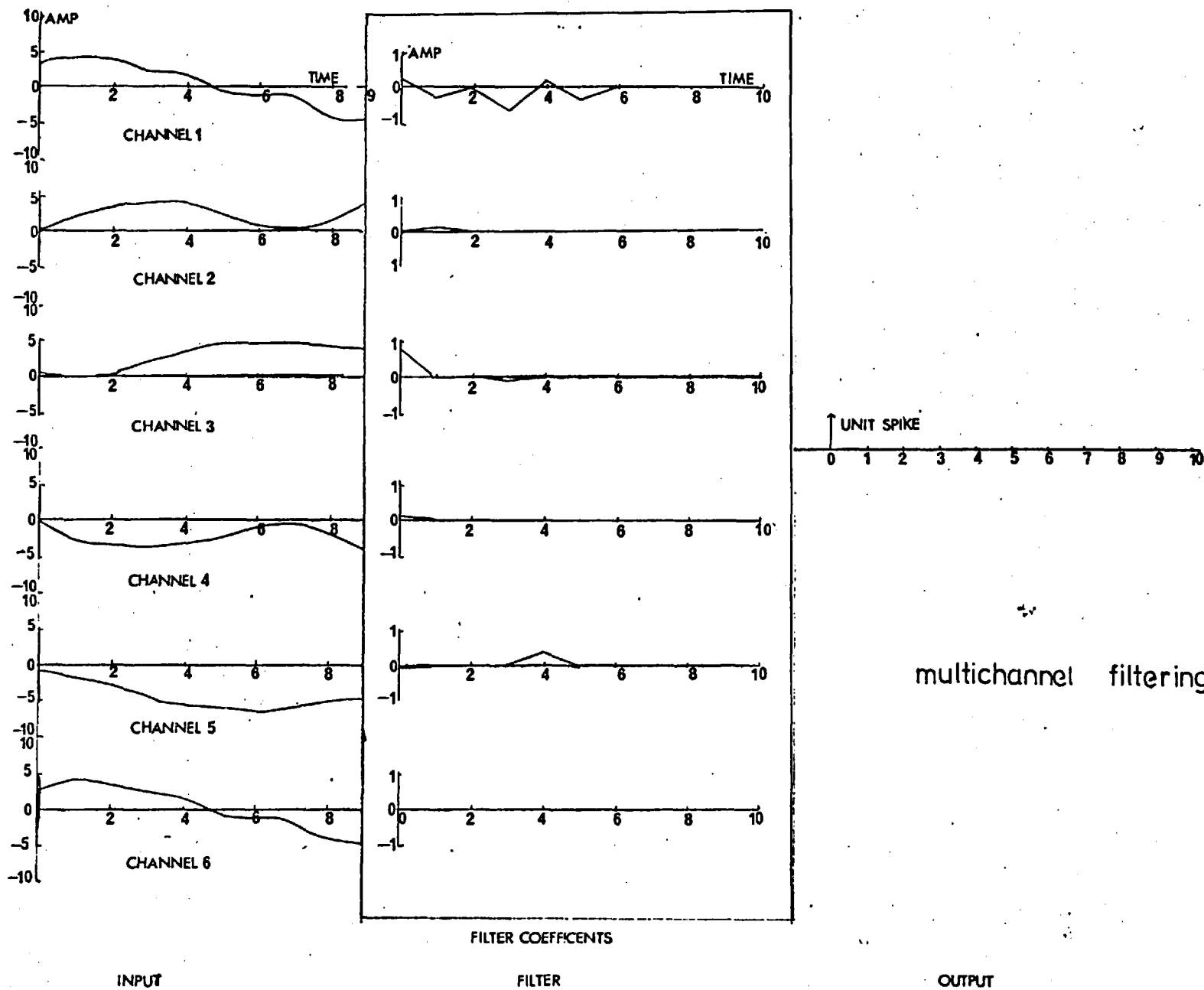


Diagram 64

results are as shown where the zero values were taken on random coefficient values. The results are displayed in fig. 61 and it can be seen that a perfect representation of the desired output, i.e. unit spike at zero time was obtained. It should be noted that the resulting output is valid only upto $t=10$ and beyond this the output values could be anything as these values were not taken into consideration. As noticed in section (i) of Case 1 the values of $X_{110}, X_{210}, \dots, X_{610}$ were taken as zero to satisfy the compatibility condition 6.3 in order to obtain outputs up to time $t=10$. Inputs were specified only up to $t=9$.

6.3 Stability Criterion of Operator

This redundant condition will now be discussed with special reference to the stability criterion of the operator values.

Let a function be represented by its Z transform as:

$$f_a(Z) = a_0 + a_1 Z + a_2 Z^2 + \dots + a_m Z^m + \dots = \sum_{m=0}^{\infty} a_m Z^m$$

The function is called stable provided its energy represented by $a_0^2 + a_1^2 + \dots$ is finite. From this it follows that the magnitudes of amplitudes of a stable function asymptotically approaches zero as the time index increases, i.e. $|a_t| \rightarrow 0$ as $t \rightarrow \infty$. In this discussion the word stability as applied to multichannel operators will be given a broader meaning. A multichannel operator is called stable if it has a finite inverse. The conditions necessary for the above condition will now be discussed. Let the finite operator be written in the Z transform form as:

$$A(Z) = a_0 + a_1 Z + \dots + a_s Z^s$$

where a_r is a matrix of dimension $(M \times N)$

where M = number of outputs

and N = number of inputs.

The following analysis is valid for $M=N$ only and let us say $M=N=p$. In this case a_r becomes a square matrix $(p \times p)$. With this restriction we can write $A(Z)$ as:

$$A(Z) = \begin{vmatrix} a_{11t_0} + a_{11t_1}Z + \dots + a_{11t_s}Z^s & a_{12t_0} + a_{12t_1}Z + \dots + a_{12t_s}Z^s & \dots & a_{1pt_0} + a_{1pt_1}Z + \dots + a_{1pt_s}Z^s \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1t_0} + a_{p1t_1}Z + \dots + a_{p1t_s}Z^s & a_{p2t_0} + a_{p2t_1}Z + \dots + a_{p2t_s}Z^s & \dots & a_{ppt_0} + a_{ppt_1}Z + \dots + a_{ppt_s}Z^s \end{vmatrix}$$

which is a $(p \times p)$ matrix with polynomial entries.

Hence

$$A^{-1}[Z] = \frac{\text{adj } A[Z]}{\det A[Z]}$$

Since $A[Z]$ has a square matrix form, its determinant can be evaluated. If M/N it would not have a determinant which can be evaluated and hence the following analysis will not be valid. Since $\text{adj } A[Z]$ is a matrix with polynomial entries the stability of $A^{-1}[Z]$ is entirely dependent on the denominator, i.e. on $\det A[Z]$. $\det A[Z]$ is a polynomial of degree at most n where $n = Sp$. Hence the determinantal equation $\det A[Z] = 0$, can be assumed to have n roots denoted by:

$$Z_1, Z_2 \dots Z_n \text{ where } n = Sp$$

It is not necessary that Z_i , $i=1, n$ should all be distinct but the argument which follows is not dependent on this criterion. Hence $\det A[Z]$ can be written in the form:

$$\det A[Z] = C(Z-Z_1)(Z-Z_2)\dots(Z-Z_n)$$

where C is a constant.

Therefore

$$A^{-1}[Z] = \frac{\text{adj } A(Z)}{C(Z-Z_1)(Z-Z_2)\dots(Z-Z_n)}$$

which can be factorised in the partial fraction form as:

$$A^{-1}[Z] = \frac{u_1}{Z-Z_1} + \frac{u_2}{Z-Z_2} + \dots + \frac{u_n}{Z-Z_n}$$

If $|Z_i| > 1$ the fraction $1/(Z-Z_i)$ can be expanded as:

$$\frac{1}{Z-Z_i} = -Z_i^{-1} - Z_i^{-2}Z - Z_i^{-3}Z^2 + \dots$$

This is a series with finite coefficients as Z_i^{-r} is a finite quantity for positive r provided $|Z_i| > 1$. Also it can be seen that $Z_i^{-r} \rightarrow 0$ as $r \rightarrow \infty$ provided $|Z_i| > 1$. But this is the condition of stability required. It is also seen that the expansion does not contain negative powers of Z which implies that the inverse does not have an anticipation component and hence is realisable. Thus an operator is realisable and stable if its Z transform determinantal equation has all its roots greater than one. This in fact is the minimum delay condition. If the determinantal equation has all its roots less than one the operator is non realisable in the sense that it will require an anticipation component for stability. If $|Z_i| < 1$ the expansion $1/(Z-Z_i)$ can be written as:

$$\frac{1}{Z-Z_i} = Z^{-1} + Z_i Z^{-2} + Z_i^2 Z^{-3} + \dots$$

This is a series with finite coefficients as Z_i^r is a finite quantity for positive r provided $|Z_i| < 1$. Also it can be seen that $Z_i^r \rightarrow 0$ for $r \rightarrow \infty$ provided $|Z_i| < 1$. But this is the condition of stability required. Hence if the determinantal equation has roots less than one, the operator inverse can be represented by a stable but nonrealisable inverse. This is the condition of maximum delay.

If some of the roots of the determinantal equation are greater than unity in magnitude while the others are less than unity in magnitude then one or other of the two expansions of $1/(Z-Z_i)$ for each Z_i must be made depending on the magnitude of Z_i . This is the case of $A[Z]$ being mixed delay; when its inverse can be represented as a stable function involving both a realisable component and a purely nonrealisable component. If the determinantal equation has any roots equal to one the partial fraction expansion can be written in either form as:

$$-1 + Z + Z^2 + \dots$$

or
$$Z^{-1} + Z^{-2} + Z^{-3} + \dots$$

In either case the coefficients of the Z transform $a_i \rightarrow 0$ as $i \rightarrow \infty$

as the coefficient is always unity. Hence this contradicts the definition of condition of stability. Therefore in this case the filter is unstable as defined previously.

As mentioned earlier the above analysis is valid only in the case where the operator coefficients are constant square matrices. An attempt is made now to analyse the case when this is not so. Consider a series whose Z transform is given as $A[Z] = a_0 + a_1 Z + a_2 Z^2 + \dots + a_m Z^m$ where a_r is a scalar quantity. The inverse of the above is defined in such a way as to satisfy the relationship $A(Z).B(Z) = 1$ where $B(Z)$ is called the inverse of $A(Z)$. This can be interpreted as single channel inverse deconvolution. As seen earlier, in the case of multichannel inputs, a_r is a matrix of order $(M \times N)$ instead of a simple scalar value. If a_r is a square matrix, i.e. number of inputs = number of outputs, the inverse of $A(Z)$ could be defined as to satisfy the relationship $A(Z) B(Z) = I$ where $A(Z)$ is a square matrix of order p (say) and I is an identity matrix (i.e. a matrix whose components are all zero except for the values in the main diagonal which are equal to 1) of order p . In this case $B(Z)$ is the inverse of $A(Z)$. The stability of this was discussed previously.

6.4 Modified Inverse

In the case where $M \neq N$ the above definition does not hold as $A(Z)$ matrix would not have an inverse. However a slight modification can be made to the definition to accommodate this if the inverse can be defined in the following way. A matrix $A(Z)^{M \times N}$ is said to have an inverse if a matrix of order $B(Z)^{N \times M}$ can be found such that

$$A(Z)^{M \times N} B(Z)^{N \times M} = I^{M \times M}$$

is satisfied where I is an identity matrix of order $M \times M$ as before. The stability of $A(Z)$ will now be discussed with the above definition as reference. The matrix multiplication that has to be satisfied is:

$$\begin{array}{ccc}
 \begin{array}{c} \text{M} \times \text{N} \\ \left| \begin{array}{ccc} A_{11}(Z) & A_{12}(Z) & \dots A_{1N}(Z) \\ A_{21}(Z) & A_{22}(Z) & \dots A_{2N}(Z) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ A_{M1}(Z) & A_{M2}(Z) & \dots A_{MN}(Z) \end{array} \right| \end{array} & \begin{array}{c} \text{N} \times \text{M} \\ \left| \begin{array}{ccc} B_{11}(Z) & B_{12}(Z) & \dots B_{1M}(Z) \\ B_{21}(Z) & B_{22}(Z) & \dots B_{2M}(Z) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ B_{N1}(Z) & B_{N2}(Z) & \dots B_{NM}(Z) \end{array} \right| \end{array} & \begin{array}{c} \text{M} \times \text{M} \\ \left| \begin{array}{ccc} 1 & 0 & 0 \dots 0 \\ 0 & 1 & 0 \dots 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{array} \right| \end{array} \\
 A(Z) & B(Z) & I
 \end{array}$$

The suffix Z will be dropped from now onwards but it should be borne in mind that a coefficient A_{11} in fact refers to the Z transform of A_{11} .

The above matrix multiplication would give rise to the set of equations:

$$\begin{array}{lcl}
 A_{11} B_{11} + A_{12} B_{21} + \dots + A_{1N} B_{N1} & = & 1 \\
 A_{11} B_{12} + A_{12} B_{22} + \dots + A_{1N} B_{N2} & = & 0 \\
 \cdot & & \cdot \\
 \cdot & & \cdot \\
 A_{11} B_{1M} + A_{12} B_{2M} + \dots + A_{1N} B_{NM} & = & 0 \\
 A_{21} B_{11} + A_{22} B_{21} + \dots + A_{2N} B_{N1} & = & 0 \\
 A_{21} B_{12} + A_{22} B_{22} + \dots + A_{2N} B_{N2} & = & 1 \\
 \cdot & & \cdot \\
 \cdot & & \cdot \\
 A_{M1} B_{11} + A_{M2} B_{21} + \dots + A_{MN} B_{N1} & = & 0 \\
 \cdot & & \cdot \\
 A_{M1} B_{1M} + A_{M2} B_{2M} + \dots + A_{MN} B_{NM} & = & 1
 \end{array}$$

This set of equations can be manipulated to give a combined matrix multiplication as:

$$\begin{array}{c|c|c|c|c}
 & (M \times M), (N \times M) & (N \times M), 1 & (M \times M), 1 & \\
 \hline
 \begin{array}{c}
 A_{11} \ A_{12} \dots A_{1N} \quad 0 \quad 0 \quad 0 \\
 0 \quad 0 \quad 0 \quad A_{11} \ A_{12} \dots A_{1N} \\
 \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 \quad \quad 0 \quad \quad \quad 0 \quad \quad \quad 0 \quad A_{11} \ A_{12} \dots A_{1N} \\
 A_{21} \ A_{22} \dots A_{2N} \\
 0 \quad 0 \quad 0 \quad A_{21} \ A_{22} \dots A_{2N} \\
 \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 A_{M1} \ A_{M2} \dots A_{MN} \quad \quad 0 \\
 0 \quad 0 \quad 0 \\
 \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 \quad \quad \quad \quad 0 \quad A_{M1} \ A_{M2} \dots A_{MN}
 \end{array}
 &
 \begin{array}{c}
 B_{11} \\
 B_{21} \\
 \cdot \\
 \cdot \\
 B_{N1} \\
 B_{12} \\
 B_{22} \\
 \cdot \\
 \cdot \\
 B_{N2} \\
 B_{1M} \\
 B_{2M} \\
 \cdot \\
 \cdot \\
 B_{NM}
 \end{array}
 &
 \begin{array}{c}
 1 \\
 0 \\
 \cdot \\
 \cdot \\
 0 \\
 0 \\
 1 \\
 \cdot \\
 \cdot \\
 0 \\
 0 \\
 0 \\
 \cdot \\
 \cdot \\
 1
 \end{array}
 &
 \begin{array}{c}
 \\
 \\
 \\
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 \\
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 \end{array}
 &
 \begin{array}{c}
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 \end{array}
 \end{array}$$

Therefore the stability condition will depend on the condition whether the above set of equations have a definite solution or not. This problem is analysed in 'Theory of Equations' by Uspensky and only the final conclusions will be stated here. Let the matrix

$$\begin{vmatrix}
 A_{11} & A_{12} \dots A_{1N} & \dots & \dots \\
 \cdot & \cdot & \cdot & \dots \\
 \cdot & \cdot & \cdot & \dots \\
 \cdot & \cdot & \cdot & \dots & A_{M1} & A_{M2} \dots A_{MN}
 \end{vmatrix}$$

be called the coefficient matrix and the matrix

$$\begin{vmatrix}
 A_{11} & A_{12} \dots A_{1N} & \dots & \dots & 1 \\
 \cdot & \cdot & \cdot & \dots & 0 \\
 \cdot & \cdot & \cdot & \dots & 0 \\
 \cdot & \cdot & \cdot & \dots & 0 \\
 \cdot & \cdot & \cdot & \dots & 0 \\
 \cdot & \cdot & \cdot & \dots & 1
 \end{vmatrix}$$

be called the augmented matrix.

It can be seen that the augmented matrix is formed by the addition of the column matrix on the right hand side of matrix equation to the coefficient matrix. The matrix equations will give definite results if and only if the rank of the coefficient matrix is the same as the rank of the augmented matrix. If this is not so the equations are not compatible and hence there is no solution for the inverse values B_{11} etc. and hence by inverse definition there is no inverse for A. If the rank is the same as the number of unknowns in B there will be a unique solution and if rank is less than the number of unknowns there will be redundant number of values but a solution is possible.

6.5 Illustrative examples

The following examples will illustrate the conditions that could arise on these conditions. Consider a one input two output case where filter coefficients are $A_{11}=1$ and $A_{21}=K$. The question to be answered is, whether this filter has a stable inverse. If the filter has a stable inverse, its coefficients should satisfy the matrix equation,

$$\begin{array}{c} 2 \times 1 \\ \left| \begin{array}{c} A_{11} \\ A_{21} \end{array} \right| \end{array} \begin{array}{c} 1 \times 2 \\ (B_{11} \ B_{12}) \end{array} = \begin{array}{c} 2 \times 2 \\ \left| \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right| \end{array}$$

which in the expanded form would give:

$$\begin{array}{c} 4 \times 2 \\ \left| \begin{array}{cc} 1 & 0 \\ 0 & 1 \\ K & 0 \\ 0 & K \end{array} \right| \end{array} \begin{array}{c} 2 \times 1 \\ \left| \begin{array}{c} B_{11} \\ B_{12} \end{array} \right| \end{array} = \begin{array}{c} 4 \times 1 \\ \left| \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right| \end{array}$$

where the coefficient matrix is

$$\left| \begin{array}{cc} 1 & 0 \\ 0 & 1 \\ K & 0 \\ 0 & K \end{array} \right|$$

Therefore the coefficient matrix is:

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ 1.5 & -.3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1.5 & -.3 \\ 1.8 & 4 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1.8 & 4 \end{vmatrix}$$

and the augmented matrix is:

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 0 \\ 1.5 & -.3 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1.5 & -.3 & 0 \\ 1.8 & 4 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1.8 & 4 & 1 \end{vmatrix}$$

From computer analysis the rank of the coefficient matrix was found to be 6 with rows 1, 2, 3 redundant. The rank of the augmented matrix was found to be 7 with rows 2, 3 redundant. As the rank of the coefficient matrix is not the same as the rank of the augmented matrix the filter has no stable inverse. This example illustrates an important consequence of the rank requirement for solution of the matrix equation. The matrix equation will not have a solution if M is greater than N, i.e. if the number of outputs is greater than the number of inputs. This can be proved in the following way. Consider the matrix equations:

(MxM), (NxM)								NxM, 1		MxM, 1	
A ₁₁	A ₁₂	...	A _{1N}					B ₁₁		1	
				A ₁₁	A ₁₂	...	A _{1N}	B ₂₁		0	
:	:	:	:	:	:	:	:	:			
:	:	:	:	:	:	:	:	:			
				A ₁₁	A ₁₂	...	A _{1N}	B _{N1}		0	
A ₂₁	A ₂₂	...	A _{2N}					.			
				A ₂₁	A ₂₂	...	A _{2N}	.			
:	:	:	:	:	:	:	:	.			
:	:	:	:	:	:	:	:	.			
:	:	:	:	:	:	:	:	B _{1M}		0	
A _{M1}	A _{M2}	...	A _{MN}					B _{2M}		0	
:	:	:	:	:	:	:	:				
:	:	:	:	:	:	:	:	B _{NM}		1	
				A _{M1}	A _{M2}	...	A _{MN}				

It can be seen that the number of unknowns B₁₁ etc. are (NxM) in magnitude. Hence the rank cannot be greater than (NxM) as for x unknowns there cannot be more than x independent equations for a stable solution. But the coefficient matrix is of order (MxM), (NxM) which means that there are MxM rows corresponding to (MxM) equations. From the above consideration (MxM = (NxM)) as regards number of independent equations. Hence out of (MxM) rows at least (MxM) - (NxM) rows are dependent. The only way in which the coefficient matrix will have a rank less than (MxM) is when a set of values X₁, X₂, ..., X_M can be found such that:

$$X_1 A_{11} + X_2 A_{21} + X_3 A_{31} + \dots + X_M A_{M1} = 0$$

$$X_1 A_{12} + X_2 A_{22} + X_3 A_{32} + \dots + X_M A_{M2} = 0$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$X_1 A_{1N} + X_2 A_{2N} + X_3 A_{3N} + \dots + X_M A_{MN} = 0$$

where X₁, X₂, ..., X_M can take zero values as well. The trivial solution for the above set of equations is of course valueless. If this is so

the coefficient matrix will have M redundant equations provided

$(A_{r1} A_{r2} \dots A_{rM}) \neq C(A_{s1} A_{s2} \dots A_{sM})$ for any $r, s = 1, 2, 3 \dots M$ and $r \neq s$

where C is a constant. If $(A_{r1} A_{r2} \dots A_{rM}) = C(A_{s1} A_{s2} \dots A_{sM})$ G times

for $r, s = 1, 2, \dots M$ and $r \neq s$ then the coefficient matrix will have $(M+GM)$

redundant equations. Therefore if $(M+GM) \geq (M \times M - NM)$ the above

condition is satisfied, i.e. rank is less than or equal to the number

of unknowns. But not only must the rank be less than or equal to the

number of unknowns, it must also be equal to the rank of the augmented

matrix. For this to happen the following set of equations must be satisfied:

$$\begin{array}{lcl} X_1 A_{11} + X_2 A_{21} + \dots + X_M A_{M1} = 0 & X_1(0) + X_2(1) + \dots + X_M(0) = 0 \\ X_1 A_{12} + X_2 A_{22} + \dots + X_M A_{M2} = 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ X_1 A_{1N} + X_2 A_{2N} + \dots + X_M A_{MN} = 0 & X_1(0) + X_2(0) + \dots + X_M(1) = 0 \\ X_1(1) + X_2(0) + \dots + X_M(0) = 0 \end{array}$$

This is clearly impossible. As before the trivial solution is valueless.

Therefore the rank of the augmented matrix will always be at least greater

than or equal to $(1 + \text{rank of coefficient matrix})$ which means that the

set of equations are incompatible, and hence no solution can be obtained.

Example 2

Now consider a case of 3 inputs and 2 outputs with the coefficient of the filter given as:

$$A_{11}=1 \quad A_{21}=1.5 \quad A_{12}=1 \quad A_{22}=-0.3 \quad A_{13}=1.8 \quad A_{23}=4.0$$

which gives the matrix equation as:

$$\begin{array}{c|ccc|ccc|c} & 4 \times 6 & 6 \times 1 & 4 \times 1 \\ \hline \begin{array}{cccccc} 1 & 1 & 1.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1.8 \\ 1.5 & -0.3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & -0.3 & 4 \end{array} & \begin{array}{c} B_{11} \\ B_{21} \\ B_{31} \\ B_{12} \\ B_{22} \\ B_{32} \end{array} & = & \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \end{array}$$

Therefore the coefficient matrix is:

$$\begin{vmatrix} 1 & 1 & 1.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1.8 \\ 1.5 & -.3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & -.3 & 4 \end{vmatrix}$$

and the augmented matrix is:

$$\begin{vmatrix} 1 & 1 & 1.8 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1.8 & 0 \\ 1.5 & -.3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & -.3 & 4 & 1 \end{vmatrix}$$

This was solved by using the CDC 6400 computer. The rank of the coefficient matrix turned out to be 4 with columns 4,1 made redundant. The rank of the augmented matrix was 4 with columns 4,1,7 made redundant. On the above basis the matrix would reduce to the modified form as:

$$\begin{vmatrix} 1 & 1.8 & 0 & 0 \\ 0 & 0 & 1 & 1.8 \\ -.3 & 4 & 0 & 0 \\ 0 & 0 & -.3 & 4 \end{vmatrix}^{4 \times 4} \begin{vmatrix} B_{21} \\ B_{31} \\ B_{22} \\ B_{32} \end{vmatrix}^{4 \times 1} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix}^{4 \times 1}$$

with coefficients B_{11} and B_{12} made zero.

The inverse of this matrix is:

$$\begin{vmatrix} .88 & -0 & -.4 & 0 \\ -.7 & -0 & .22 & 0 \\ 0 & .88 & 0 & -.4 \\ 0 & .07 & 0 & .22 \end{vmatrix}$$

which gives $B_{21}=.88$ $B_{31}=.07$ $B_{22}=-.4$ $B_{32}=.22$.

There fore the multichannel filter represented by its Z transform as

$$\begin{vmatrix} 1 & 1 & 1.8 \\ 1.5 & -.3 & 4 \end{vmatrix}^{2 \times 3}$$

has a stable inverse represented by its Z transform as

$$\begin{array}{c|c} & \begin{array}{c} 3 \times 2 \end{array} \\ \hline \begin{array}{c} 0 \quad 0 \\ .88 \quad -.4 \\ .07 \quad .22 \end{array} & \end{array}$$

In this particular example the coefficients of the filter was taken to be independent of Z for simplicity. In the real case the coefficient matrix and therefore the inverse would contain polynomial Z as coefficient and their stability can be discussed in exactly the same way as before when the number of inputs and the outputs were the same.

6.6 Discussion of Results

From the foregoing analysis it can be seen that the stability criterion of filters rests mainly on the question of inversion of a matrix. In the case of equal magnitudes of input and output channels the matrix in question is square and in the case of an N input M output case the matrix in question is of order (MxN). The square matrix will have a finite unique inverse unless it is singular. The singularity condition and the rectangular matrix formulation can be discussed with reference to the so called 'g' inverse notation as explained in 'Matrix Theory' by Gantmacher. The method of obtaining this modified inversion of a matrix is explained which is valid for the singular as well as the rectangular case.

If we redefine the inverse of matrix A as matrix G which satisfies the relationship $AGA = A$ the analysis can be made general since A can now be singular or rectangular. On this basis for the multichannel normal equation formula represented by

$$AX = Y$$

where A = filter matrix, X = input matrix, Y = output matrix, the following conditions can be derived for consistency. On the Z transform basis the above equation reduces to:

$$[A][X] = [Y] \quad (6.4)$$

where A = MxN matrix and

X = a column matrix of order N (= input channels) and

Y = a column matrix of order M (= output channels).

The consistency requirement for 6.4 to have a solution is:

$$[Y]^{M \times 1} [G_X]^{1 \times N} [X]^{N \times 1} = [Y]^{M \times 1}$$

where G_X is defined as $XG_X X = X$ and the solution is:

$$A = YG_X + Z[I - XG_X]$$

where Z is arbitrary. This is assuming that the compatibility condition is satisfied. The above derivation can be made more meaningful if we expand 6.4 and represent the expansion as shown earlier where the input values are in a matrix form and the required filter coefficients are in a column matrix. On this basis the multichannel equation can be rewritten:

$$X^{N \times M} A^{M \times 1} = Y^{N \times 1}$$

where X = input matrix of $(N \times M)$

where N = number of digitised inputs in each channel

and M = number of channels $\times N$

and A = column matrix of filter coefficients of order M

and Y = column matrix of output values of order N (only one output channel is considered)

Therefore $A = X_g Y$

where X_g is defined as $XX_g X = X$.

Assuming the compatibility condition is satisfied and the rank of X is r with

$$r \leq \text{minimum}(N, M)$$

the required consistency condition is

$$XX_g Y = Y$$

and the general solution is

$$A = X_g Y + (I - X_g X)Z$$

where Z is arbitrary.

6.7 Stability of the Auto correlation Matrix

The discrete multichannel normal equation can be written as:

$$\begin{aligned}
 a_0 \phi_{xx}(0) + a_1 \phi_{xx}(-1) + \dots + a_m \phi_{xx}(-m) &= \phi_{zx}(0) \\
 a_0 \phi_{xx}(1) + a_1 \phi_{xx}(0) + \dots + a_m \phi_{xx}(1-m) &= \phi_{zx}(1) \\
 \vdots &\vdots \\
 a_0 \phi_{xx}(m) + a_1 \phi_{xx}(m-1) + \dots + a_m \phi_{xx}(0) &= \phi_{zx}(m)
 \end{aligned}$$

with the usual meanings for the symbols and with the filter coefficient $a_s (s=0, m)$ as $N \times M$ matrices and $\phi_{xx}(s)$, $(s=-m, m)$ as $N \times N$ matrices and all $\phi_{zx}(s)$, $s=0, m$ as $N \times M$ matrices.

The filter coefficients are given by the solution of the matrix multiplication

$$\begin{array}{ccc}
 \left| \begin{array}{ccc} \phi_{xx}(0) & \phi_{xx}(-1) & \dots \phi_{xx}(-m) \\ \phi_{xx}(1) & \phi_{xx}(0) & \dots \phi_{xx}(1-m) \\ \vdots & \vdots & \vdots \\ \phi_{xx}(m) & \phi_{xx}(m-1) & \dots \phi_{xx}(0) \end{array} \right|^{(m+1), (m+1)} & \begin{array}{c} \left| \begin{array}{c} a_0 \\ a_1 \\ \vdots \\ a_m \end{array} \right|^{(m+1), 1} \\ = \\ \left| \begin{array}{c} \phi_{zx}(0) \\ \phi_{zx}(1) \\ \vdots \\ \phi_{zx}(m) \end{array} \right|^{(m+1), 1} \end{array} \\
 [X] & [A] & [Z]
 \end{array}$$

It can be seen that the form of the solution is dependent of the inversion of matrix X which is the auto correlation matrix of the inputs. Assuming the compatibility conditions mentioned earlier are satisfied the inverse of the auto correlation matrix and hence the solution of the matrix equation can be discussed in the following way. It is important to note that the auto correlation matrix is symmetrical and hence its inverse evaluation can be discussed with reference to symmetrical matrix inverse evaluation. The singularity condition obtained when $\det A = 0$ can be incorporated into one single analysis if the generalised inverse is evaluated instead of the normal inverse.

The evaluation of the generalised inverse with reference to a symmetric matrix will now be discussed and the results applied to the auto correlation matrix. If it so happens that a matrix A is symmetric idempotent then A^G (generalised inverse of A) is equal to A , i.e. A is its own unique g inverse. In this special case no problem of stability

would arise as the inverse is easily obtained as the matrix itself. If this is not so A^G can be found as follows. If A is symmetric it is reducible to leading diagonal canonical form

$$HAH' = \begin{vmatrix} \Lambda & 0 \\ 0 & 0 \end{vmatrix} \quad \text{i.e.} \quad A = H' \begin{vmatrix} \Lambda & 0 \\ 0 & 0 \end{vmatrix} H = H_1' \Lambda H_1$$

where Λ is a diagonal matrix of the non zero latent roots of A and H is orthogonal and partitioned conformably as $H' = [H_1' \ H_2']$. In this case

$$A^G = H' \begin{vmatrix} \Lambda^{-1} & 0 \\ 0 & 0 \end{vmatrix} H = H_1' \Lambda^{-1} H_1$$

The above results will be verified through a simple example to illustrate the procedure involved. Let $A = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}^{2 \times 2}$ be a symmetric matrix.

The characteristic equation is

$$[A - \lambda I]K = 0 \quad (6.5)$$

where λ is a scalar. This represents a homogenous system of 2 equations with 2 unknowns which has a non trivial solution for K if and only if the coefficient matrix has rank $r \leq 2$, that is if and only if

$$\det (A - \lambda I) = 0$$

Therefore

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

give

$$(2-\lambda)(2-\lambda) - 1 = 0$$

$$4 + \lambda^2 - 4\lambda - 1 = 0$$

$$\text{Therefore} \quad \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1) = 0$$

$$\text{Therefore} \quad \lambda = 3 \text{ Or } \lambda = 1$$

Substituting these values in 6.5

$$\text{for } \lambda = 3 \quad \left[\begin{array}{cc|cc} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 3 \end{array} \right] K = \left[\begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array} \right] K = 0 \quad (6.51)$$

which gives the characteristic vector associated with $\lambda = 3$ as

$$3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

obtained by solving the system 6.51.

Similarly for $\lambda = 1$ the characteristic vector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\text{Therefore} \quad \wedge = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Normalising the two characteristic vectors we obtain

$$H = \begin{bmatrix} -1/\sqrt{2} & 3/\sqrt{18} \\ 1/\sqrt{2} & 3/\sqrt{18} \end{bmatrix} \quad \text{and} \quad H' = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 3/\sqrt{18} & 3/\sqrt{18} \end{bmatrix}$$

In this particular case $H_1' = H'$

$$\begin{aligned} A^G &= H_1' \wedge^{-1} H_1 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 3/\sqrt{18} & 3/\sqrt{18} \end{bmatrix}^{2 \times 2} \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}^{2 \times 2} \begin{bmatrix} -1/\sqrt{2} & 3/\sqrt{18} \\ 1/\sqrt{2} & 3/\sqrt{18} \end{bmatrix}^{2 \times 2} \\ &= \begin{bmatrix} -1/\sqrt{2} & 1/3\sqrt{2} \\ 3/\sqrt{18} & 1/\sqrt{18} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 3/\sqrt{18} \\ 1/\sqrt{2} & 3/\sqrt{18} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} 1/6 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} 1/3 \end{aligned}$$

Therefore the required generalised inverse of A is $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} 1/3$

$$\text{As a check} \quad A^G A = 1/3 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Expanding the multichannel normal matrix multiplication equation we will have an input autocorrelation matrix of order $N(M+1)$, $N(M+1)$ and a filter coefficient matrix of magnitude $N(M+1)$, M and an output cross correlation

matrix of order $N(M+1)$, M . The stability of the solution can be tested as follows. Obtain the 'g inverse' of the autocorrelation matrix X as illustrated above. It should be noted that a 'g inverse' would always exist for any matrix singular or not. Denoting this by X^E and the cross correlation matrix by Z obtain the matrix multiplication XX^EZ . If this matrix is equal to the matrix Z the normal equations can be solved uniquely and the general solution is then given by

$$A = X^EZ + (I - X^EX) Z_1$$

where Z_1 is to be evaluated.

If the condition stated above is not satisfied the matrix multiplication is not compatible and hence no solution can be obtained for the filter coefficients.

It is worth noting in passing that an unstable solution, i.e. a condition which does not satisfy the requirement stated above in the case of the multichannel equation can be made stable if the structure of the auto correlation matrix is taken into consideration. In the formation of the characteristic equation the coefficient of the second highest power of the polynomial in the latent roots is the trace of the auto correlation matrix. But the trace of the auto correlation matrix is the sum of the auto correlations at zero lag for all channels. If the seismic model was chosen as a sum of the signal correlated noise and uncorrelated noise the auto correlation terms will have uncorrelated noise characteristics only at zero lag, as at all other lags the auto correlation of the uncorrelated noise is zero. Hence addition of uncorrelated noise will only have an effect on the trace of the auto correlation matrix and by this variation it is possible to alter the inverse of value of the matrix instead of changing the entire structure of the matrix. Therefore in the case of an unstable solution a simple change of the uncorrelated noise component could make the inverse matrix solution compatible with the condition stated before.

CHAPTER 7

CONCLUSION

From the foregoing analysis it can be seen that a given seismic trace can be analysed using different mathematical techniques. The basic model for the given trace was always assumed to be that due to the summation of three components: signal, correlated noise and uncorrelated noise. It was also assumed that the signal and correlated noise were convolutions of wavelets and random series. As could be seen there is no absolute way of determining the components of the trace without making any assumptions. But different assumptions regarding the characteristics of the components enable one to decompose the trace mathematically in different ways. As again there is no way of telling which assumptions are justified in a given trace. It may be possible to arrive at the appropriate assumptions (and hence the correct mathematical decomposition) by considering the geological implications of each assumption. The purpose of this thesis was to demonstrate that under different assumptions which are valid normally, it is possible to decompose a given trace to its components. At least in theoretically constructed cases the results of the decomposition were quite good. It remains to be seen how these methods would apply in the case of real data.

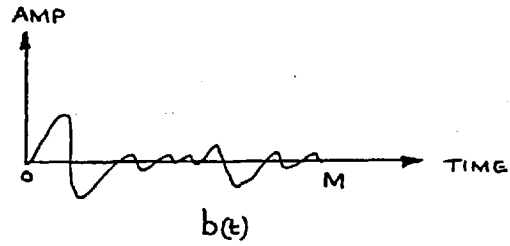
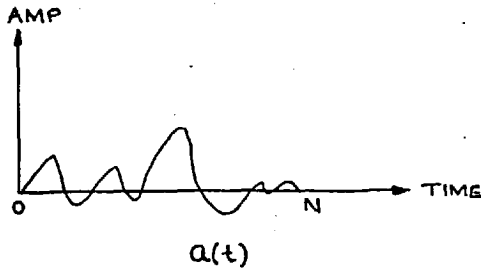
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APPENDIX A



For series $a(t)$ $a(t)=0$ for $t < 0$ or $t > N$ (1A)

For series $b(t)$ $b(t)=0$ for $t < 0$ or $t > M$ (2A)

Convolution of $a(t)$ and $b(t)$ would give

$$y(t) = a(t) * b(t) = \int_0^N a(\tau) b(t-\tau) d\tau \quad (3A)$$

Fourier Transform of $a(t)$ is $f_a(f) = \int_0^N a(t) e^{-2\pi i f t} dt$ (4A)

" " " $b(t)$ is $f_b(f) = \int_0^M b(t) e^{-2\pi i f t} dt$ (5A)

" " " $y(t)$ is $f_y(f) = \int_0^{M+N} y(t) e^{-2\pi i f t} dt$

$$= \int_0^{M+N} \left\{ \int_0^N a(\tau) b(t-\tau) d\tau \right\} e^{-2\pi i f t} dt$$

$$= \int_0^{M+N} a(\tau) \left\{ \int_0^N b(t-\tau) e^{-2\pi i f t} dt \right\} d\tau$$

Substituting $t-\tau = t$

$$f_y(f) = \int_0^{M+N} a(\tau) \left\{ \int_{-\tau}^{N-\tau} b(t) e^{-2\pi i f (t+\tau)} dt \right\} d\tau$$

$$= \left\{ \int_0^{M+N} a(\tau) e^{-2\pi i f \tau} d\tau \right\} \left\{ \int_{-\tau}^{N-\tau} b(t) e^{-2\pi i f t} dt \right\}$$

$$= \{f_a(f)\} \left\{ \int_{-\tau}^M b(t) e^{-2\pi i f t} dt - \int_{N-\tau}^M b(t) e^{-2\pi i f t} dt \right\}$$

$$= f_a(f) f_b(f) - \int_0^{M+N} a(\tau) e^{-2\pi i f \tau} [E] d\tau$$

$$= f_a(f) \{ f_b(f) - E \} \quad (6A)$$

where E is given by

$$E \sum_{\tau=0}^{M+N} A(\tau) d\tau = \sum_{\tau=0}^{M+N} A(\tau) E(\tau)$$

$$\text{with } A(\tau) = a(\tau) e^{-2\pi i f \tau}$$

$$\text{and } E(\tau) = \sum_{t=N-\tau}^{t=M} b(t) e^{-2\pi i f t}$$

APPENDIX B

Consider $I = \int_0^{\infty} e^{-k_1(a+x)^2} dx$

Substituting $a+x = u$

$$I = \int_a^{\infty} e^{-k_1 u^2} du = \int_0^{\infty} e^{-k_1 u^2} du - \int_0^a e^{-k_1 u^2} du$$

Now $\int_0^{\infty} e^{-k_1 x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{k_1}}$

Therefore, $I = \frac{1}{2} \sqrt{\frac{\pi}{k_1}} - A$ where $A = \int_0^a e^{-k_1 u^2} du$

Consider $B = \int e^{-k_1 u^2} du$

Let $k_1 u^2 = \alpha$

$$\begin{aligned} \text{Therefore, } B &= \int \frac{e^{-\alpha} d\alpha}{2 k_1^{\frac{1}{2}} \alpha^{\frac{1}{2}}} = \frac{1}{2 k_1^{\frac{1}{2}}} \int \frac{1}{\alpha^{\frac{1}{2}}} \left\{ 1 - \alpha + \frac{\alpha^2}{2!} - \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} - \dots \right\} d\alpha \\ &= \frac{1}{2 k_1^{\frac{1}{2}}} \int \left\{ \alpha^{-\frac{1}{2}} - \alpha^{\frac{1}{2}} + \frac{\alpha^{3/2}}{2!} - \frac{\alpha^{5/2}}{3!} + \dots \right\} d\alpha \\ &= \frac{\alpha^{\frac{1}{2}}}{2 k_1^{\frac{1}{2}}} \left\{ \frac{1}{\frac{1}{2}} - \frac{\alpha}{3/2} + \frac{\alpha^2}{5/2 \cdot 2!} - \dots \right\} \\ &= \frac{k_1^{\frac{1}{2}} u}{2 k_1^{\frac{1}{2}}} \left\{ \frac{1}{\frac{1}{2}} - \frac{(k_1 u^2)}{3/2} + \frac{(k_1 u^2)^2}{5/2 \cdot 2!} - \dots \right\} \end{aligned}$$

$$\begin{aligned} \text{Therefore } A &= \left[\frac{u}{2} \left\{ \frac{1}{\frac{1}{2}} - \frac{(k_1 u^2)}{3/2} + \frac{(k_1 u^2)^2}{5/2 \cdot 2!} - \dots \right\} \right]_0^a \\ &= \frac{a}{4} \left\{ \frac{1}{\frac{1}{2}} - \frac{k_1 a^2}{3/2} + \frac{k_1^2 a^4}{5/2 \cdot 2!} - \dots \right\} \end{aligned}$$

If the N^{th} term of the expansion in A is A_N and the $(N-1)^{\text{th}}$ term of the expansion is A_{N-1} , then

$$A_N = \frac{a}{4} \left\{ \frac{(-1)^{N+1} (k_1 a^2)^{N-1}}{(2N-1)/2 \cdot (N-1)!} \right\}$$

$$A_{N-1} = \frac{a}{4} \left\{ \frac{(-1)^N (k_1 a^2)^{N-2}}{(2N-3)/2 \cdot (N-2)!} \right\}$$

For convergence of series A

$$\left| \frac{A_N}{A_{N-1}} \right| < 1 \quad \text{as } N \rightarrow \infty$$

$$\text{But } \left| \frac{A_N}{A_{N-1}} \right| = \frac{k_1^2 (2N-3)}{(2N-1)(N-1)} = \frac{ka^2(2-3/N)}{(2N-1)(1-1/N)} \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

Therefore A is convergent.

Therefore $I = \frac{1}{2} \sqrt{\frac{\pi}{k_1}} - A$ is a valid representation.

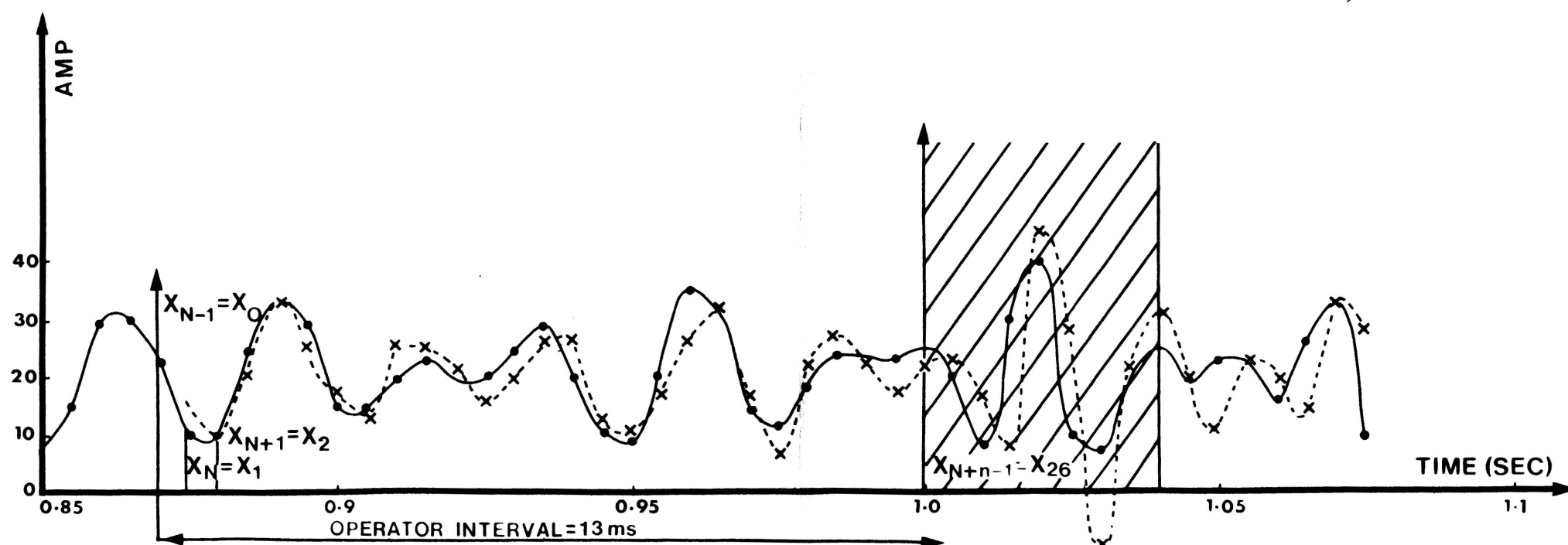


FIG. 5.00 CURVE REPRODUCTION
SQUARED ERROR

— Seismogram Trace
- - - Predicted Trace
▨ Reflection Arrival Zone

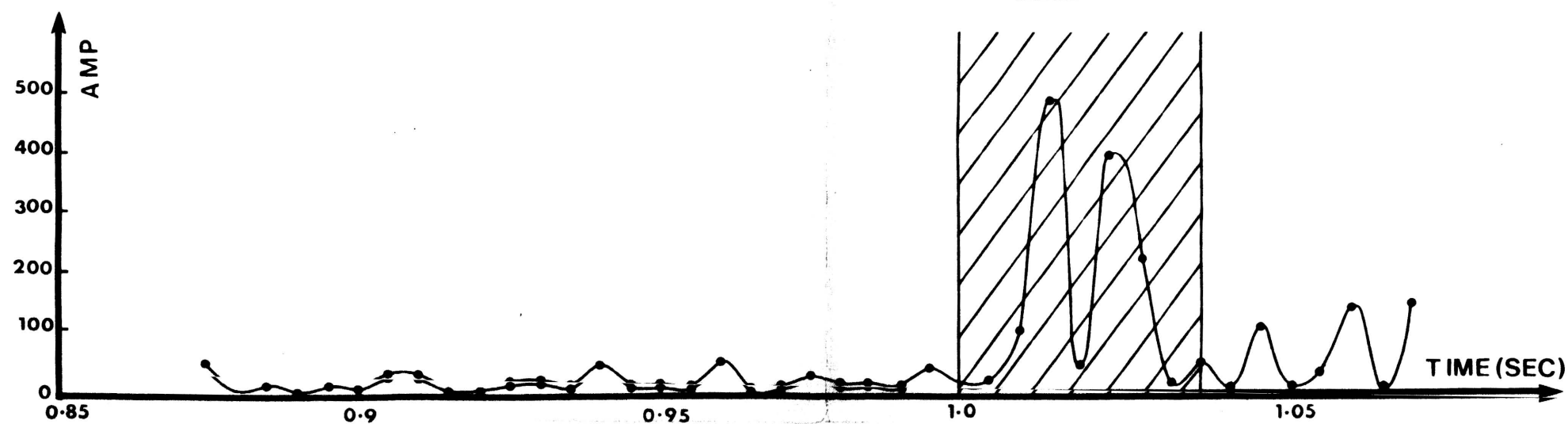


FIG. 5.01 SQUARE ERROR

▨ Reflection Zone

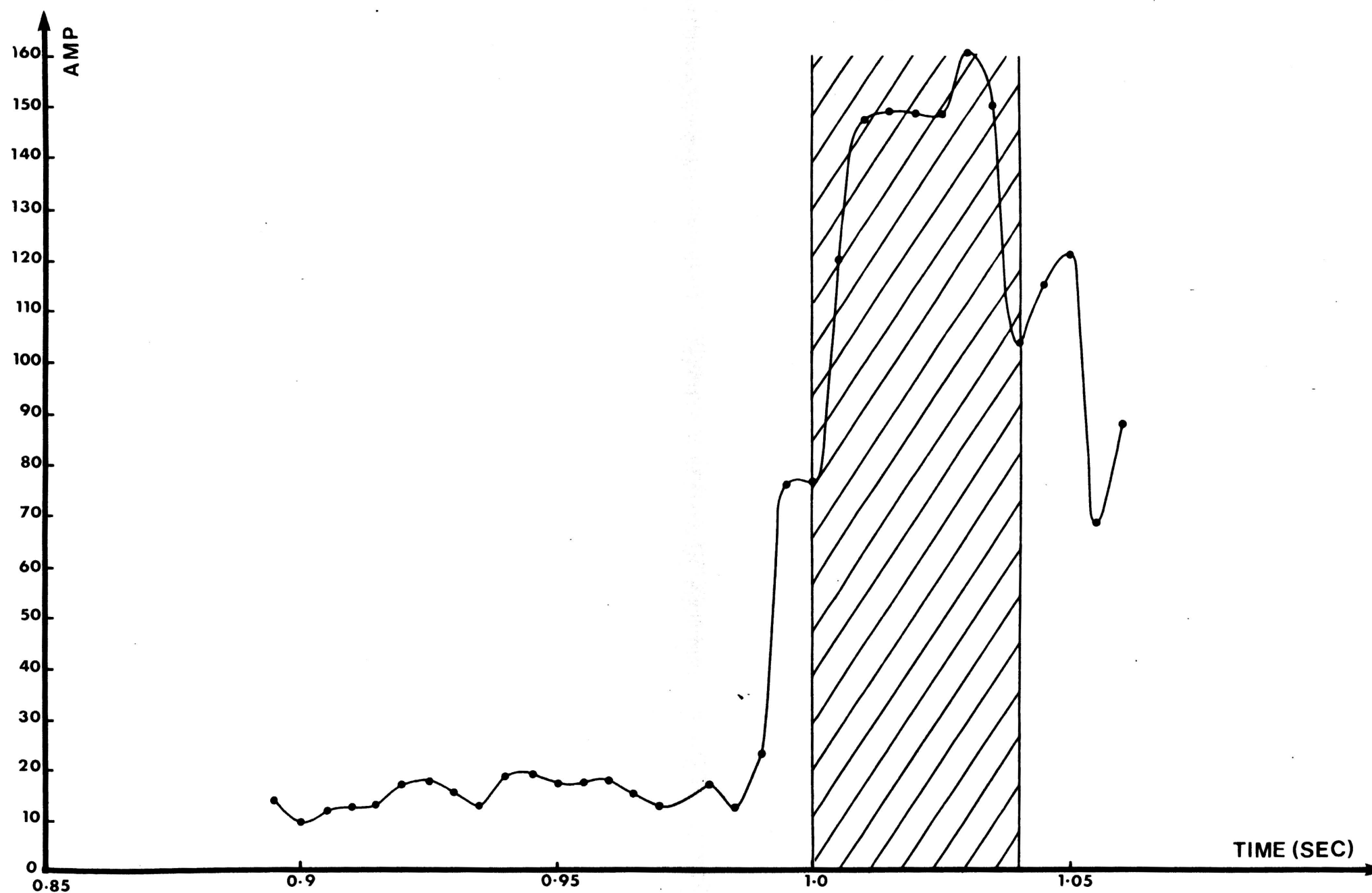


FIG. 5.02 RUNNING SQUARE ERROR

▨ Reflection Zone

