DISTANCE BOUNDING AND HEEGAARD FLOER HOMOLOGY METHODS IN REDUCIBLE DEHN SURGERY

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by

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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To my mother, Maria Teresa.

ABSTRACT

Motivated by the formation of certain link types during Hin-mediated DNA recombination experiments, we consider tangle equations where one of the products is a connect-sum of 2-bridge links. We are thus led to study Dehn surgeries on knots in lens spaces that yield connect-sums of lens spaces. Using 3-manifold methods, we prove, for certain classes of knot exteriors, a distance bound on Dehn surgery slopes. (This proof complements an algebraic-geometric proof of a more general statement due to Boyer and Zhang [6]). Analysing the known examples of connect-sums of lens spaces surgeries, together with some sample calculations, we conjecture that if surgery on a knot $K \subset L(p,q)$, with $p, q \neq 1, 2$, yields $L(p,q)\#L(t,1)$, then the knot has reducible exterior. Using Heegaard Floer homology, we prove a special case of the conjecture, as well as some surgery obstructions. We then apply our results, in the spirit of the tangle model of Ernst and Sumners [67], to the problem of Hin-mediated DNA recombination, where we characterise its distributive recombination step.

TABLE OF CONTENTS

Abstract **6**

1 Introduction and statement of results **8**

2 Basic concepts and definitions **11**
2.1 Tangles 11
2.2 Tangle equations 16

3 Dehn surgery results **22**
3.1 Preliminaries 22
3.2 Distance bounds for reducible surgeries 26
3.3 Examples and conjectures 33

4 Heegaard Floer homology results **39**
4.1 Preliminaries on Heegaard Floer homology 39
4.1.1 Maps from cobordisms and surgery exact sequences 44
4.1.2 The surgery exact sequence 46
4.1.3 The correction term 48
4.2 Calculation of $HF^+(Y_0)$ 49
4.2.1 Floer homology with twisted coefficients 58
4.3 Knot Floer homology and the knot genus 64
4.4 Using $l_{k,\mathfrak{t}}$ as an obstruction 69
4.5 A uniqueness result 74

5 Consequences for tangle equations and applications to biology **80**
5.1 Consequences 80
5.2 Applications to biology 81
5.2.1 Background in biology 81
5.2.2 The Hin recombinase 86

CHAPTER 1 INTRODUCTION AND STATEMENT OF RESULTS

The problem of protein-mediated DNA recombination can be modelled with tangles, as introduced in a seminal paper of Ernst and Sumners [67]. There, the authors used information about the knot/link types L_i of a series of circular DNA molecules resulting from experiments, and assembled this information into a set of tangle equations

$$N(O+P) = L_0$$

$$N(O+R) = L_1$$

$$N(O+2R) = L_2$$

$$\vdots$$

$$N(O+mR) = L_m$$

They were able "solve" these equations for the tangles O, P, and R, thereby obtaining schematics of the mechanism of protein action. For some of the proteins they treated, the model predicted a unique mechanism, and this was in accordance with the proposed biological theory. After the initial success of the model, the study of tangle equations became an area of active mathematical research, with contributions found, for example, in [8, 9, 14, 15, 18, 67, 71].

All of the previous analyses are limited to the case where the reaction products L_i have the type of a 2-bridge link. However, certain proteins, such as the Hin invertase [37], are known to be able to produce connect-sums of 2-bridge links as by-products of their action. One is motivated, then, to consider solving tangle equations of the form

$$N(O+P) = L_1$$

$$N(O+R) = L_2 \# L_3$$

where the L_i are 2-bridge links, to model the occurrence of these DNA recombination products.

To solve tangle equations, one passes to double branched covers, and considers the corresponding "upstairs" problem of gluing 3-manifolds along their boundary. If the tangles P and R are rational, this becomes a Dehn surgery problem. In our case, the analysis of the above equations then turns into the problem of studying knots in lens spaces with a connect-sum of lens spaces surgery. This is the central topic of study in this thesis.

Our work is organised as follows: in Chapter 2, we review the basic definitions and concepts involved regarding tangles, Dehn surgery, and 3-manifolds. In Chapter 3, after introducing more terminology, we use 3-manifold methods to prove, for certain classes of knot exteriors, that if surgery on a knot in a lens space yields a connect-sum of lens spaces, then the surgery slope is integral. (This proof complements an algebraic-geometric proof of a more general statement due to Boyer and Zhang [6]). Then, we present and discuss known examples of the phenomenon we're studying. We are able to show the following:

Theorem 3.3.2. *Let $L(p,q)$ be any lens space, with $p \neq \pm 1, \pm 2$. Then, no knot $K \in L(p,q)$ with Seifert fibered exterior has a surgery giving $L(p,q)\#L(t,1)$.*

The result above, in tandem with the other available examples, leads us to make the following conjecture:

Conjecture 3.3.3. *Let $K \subset L(p,q)$ be a knot such that $E(K)$ is irreducible. Suppose that $p \neq 1, 2$ (that is, $L(p,q)$ is not $\mathbb{R}P^3$ or S^3). Then, surgery on K never produces $L(p,q)\#L(t,1)$.*

In Chapter 4, we use Heegaard Floer homology, an invariant introduced by Ozsváth and Szabó in [56, 57], to study our problem. We start by giving a quick review of the construction, together with an overview of its properties. Then, we use it to we prove the following result:

Theorem 4.3.7. *Let $K \subset Y$ be a nullhomologous knot in an L-space Y, and suppose that integral $p > 0$ Dehn surgery Y_p on K gives $Y\#L(p,1)$. Then, K is the unknot.*

We remark that the class of L-spaces includes the lens spaces, and connect-sums thereof. This theorem is a generalisation of a result of Kronheimer, Mrowka, Ozsváth and Szabó for knots in S^3 [44]. After this, we obtain some results on how Heegaard Floer homology can be used as an obstruction to having connect-sums of lens space surgeries. Putting these together with the previous theorem and a homological argument, we are able to show:

Theorem 4.4.10. *Let Y be an L-space such that $|H_1(Y;\mathbb{Z})|$ is an odd prime, and let t be a positive integer congruent to 3 modulo 4. Then, if integral surgery on a knot $K \subset Y$ produces $Y\#L(t,1)$, K is the unknot.*

This confirms our conjecture in this case. In addition to this, using a theorem of Ni [53] on detecting fibered knots, together with work of Baker [2] and Morimoto [51], we show that some genus one fibered knots in certain lens spaces are characterised by having a certain reducible surgery.

Finally, in Chapter 5, we employ the machinery developed in the previous sections to the problem of solving tangle equations. Then, we move on to consider an application of this to Hin-mediated action: we start by giving the reader a quick tour through the essential biology behind the problem. Then, we solve the "traditional" tangle equations arising from processive recombination, as an illustration of the tangle model. We conclude by using our results to solve the equations arising from distributive recombination, this being the case that produces connect-sums of two-bridge links. Our calculation reveals a unique distributive mechanism, consistent with the simplest biological explanation.

CHAPTER 2 BASIC CONCEPTS AND DEFINITIONS

2.1 Tangles

The study of tangles began with the work of Conway [12] in the 1960s, who introduced them as the building blocks for an alternative approach to tabulating knots. Later, Montesinos [50] used them to study the properties of a wide class of knots (which came to bear his name) and their double branched covers. We will say more on this in the next chapter. In the 1980s, further work was done by Lickorish [46] and Bleiler [3]. Another important reference, although unpublished, is the monograph by Bonahon and Siebenmann [5].

We begin by giving the definition of a tangle, as in [30]:

Definition 2.1.1. *A tangle is a pair (B, A), where B is S^3 minus the interiors of a finite number of disjoint 3-balls, and A is a properly embedded 1-manifold in B which meets each component of ∂B in four points.*

Recall that a submanifold $A \subset B$ is properly embedded if $A \cap \partial B = \partial A$. Also, note that we use a definition which is slightly more general than the one given in [67], where one chooses to remove only one ball from S^3, and only considers the case where 1-submanifold is a pair of arcs. Now, it turns out to be convenient, in many situations, to think of ∂B as a rigid object. For that, we have the following:

Definition 2.1.2. *A tangle (B, A) is said to be **marked** if every component $S \subset \partial B$ is equipped with an identification $\psi_S : (S, (S \cap A)) \longrightarrow (S^2, P)$, where (S^2, P) is the unit 2-sphere in Euclidean space, centred at the origin, together with a choice of 4 distinguished points $P = \{NW, SW, NE, SE\}$.*

As is usual in knot theory, we think of marked tangles as being represented by a projection onto the plane, where the distinguished points are arranged so as to always lie in the configuration of Figure 2.1.

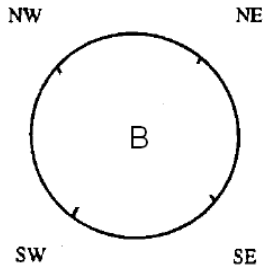


Figure 2.1:

We will consider two notions of equivalence between tangles, which are described in the next two definitions:

Definition 2.1.3. *Two tangles (B_1, A_1), (B_2, A_2) are said to be **homeomorphic** if there exists an orientation-preserving homeomorphism $h : B_1 \longrightarrow B_2$ sending A_1 to A_2.*

Definition 2.1.4. *Two tangles (B_1, A_1), (B_2, A_2), are said to be **isomorphic** if they are homeomorphic via h (in the sense of the previous definition), and h fits into the following commutative diagram, for every boundary component $S_i \subset \partial B_i$*

$$\begin{array}{ccc} B_1 & \xrightarrow{h} & B_2 \\ \downarrow{\scriptstyle \psi_{S_1}} & & \downarrow{\scriptstyle \psi_{S_2}} \\ S^2 & \xrightarrow{Id} & S^2 \end{array}$$

Remark 2.1.5. Even though we have allowed for tangles to have multiple boundary components, we will almost exclusively think of those which have a single boundary component. This is assumed to hold, unless otherwise stated.

Given marked tangles $T_1 = (B, A)$ and $T_2 = (B', A')$, one can define their **tangle sum** by connecting the point NE (resp. SE) of T_1 with the point NW (resp. SW) of T_2 as indicated in Figure 2.2. We denote the resulting tangle by $T_1 + T_2$. Similarly, given a tangle $T = (B, A)$, we can form its **numerator closure** by connecting the "northern" marked points by an arc and the "southern" marked points by another arc, as is also shown in Figure 2.2. One obtains a tangle (S^3, L), where L is some link. We tend to think of this as the link L, rather than the pair (S^3, L).

Following Lickorish [46], it is convenient to partition unmarked tangles into three pairwise disjoint classes described in the definition below:

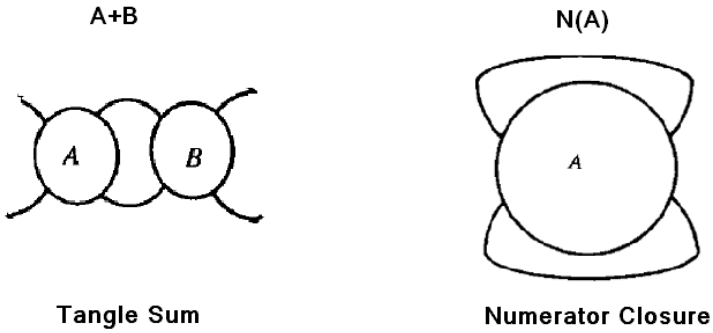


Figure 2.2: Operations on tangles.

Definition 2.1.6. *Let (B,t) be an unmarked tangle. Then:*

1. *(B,t) is said to be **rational** if it is homeomorphic to $(D^2 \times I, \{x,y\} \times I)$, where x,y are interior to D^2. Notice this implies that the submanifold t contains no loops.*

2. *(B,t) is said to be **locally knotted** if there exists an embedded sphere S in B bounding a ball E, such that S intersects t transversally in two points, and such that $(E, E \cap t)$ is not homeomorphic to $(D^2, \{x\} \times I)$.*

3. *(B,t) is said to be **prime** if it is neither rational nor locally knotted.*

Examples are given in Figure 2.3:

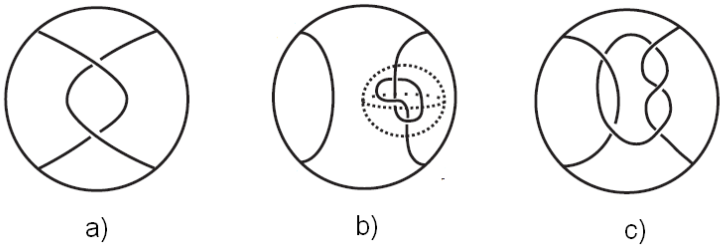


Figure 2.3: a) A rational tangle. b) A locally knotted tangle. c) A prime tangle.

The class of rational tangles is the simplest one. It turns out that these objects can be thought of as being obtained by an alternating series of horizontal and vertical twists on two basic tangles. The construction is as follows: consider the marked tangles below, which we define to be $T(0/1)$, $T(\infty)$, respectively:

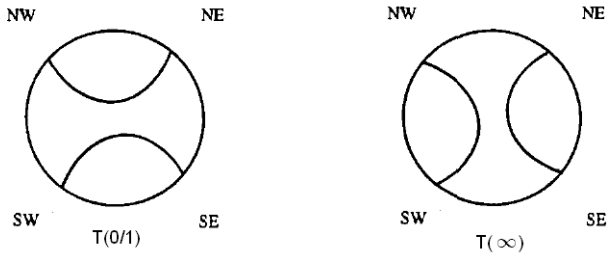
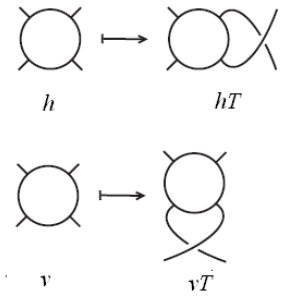


Figure 2.4:

Then, one constructs "more complex" rational tangles by applying an alternating series of horizontal and vertical half twists, encoded in a list of integers $a_1, a_2, ..., a_n$, to either $T(\infty)$ or $T(0/1)$, depending on the parity of n. Our convention for the half twists is depicted below:



$h-$ horizontal half twist

$v-$ vertical half twist

Figure 2.5: Schematic of the twisting operations on tangles.

Negative powers of these operations correspond, of course, to twisting in the opposite sense. The name rational comes from the fact that one records the information about the sequence of twists in a continued fraction expansion:

Definition 2.1.7. *Given a list of integers, $a_1, a_2, ..., a_n$, such that $a_i \neq 0$, for $2 \leq i \leq n$, define the rational tangle $T(\alpha/\beta)$ to be*

$$T(\alpha/\beta) = \begin{cases} h^{a_1}v^{a_2}...v^{a_n}T(\infty), & \text{if } n \text{ is even} \\ h^{a_1}v^{a_2}...h^{a_n}T(0/1), & \text{if } n \text{ is odd} \end{cases}$$

where $\alpha/\beta = a_1 + \frac{1}{a_2 + \frac{1}{...+\frac{1}{a_n}}}$

The fraction α/β is taken in lowest terms, of course. With this in place, Conway shows in [12] that this continued fraction expansion completely characterises the rational tangle, up to isomorphism:



Figure 2.6: The rational tangles $T(5/14)$ and $T(2/3)$.

Theorem 2.1.8. *The construction above sets up a one-to-one correspondence between isomorphism classes of rational tangles and the extended rational numbers* $\mathbb{Q} \cup \{\infty\}$.

We remark that the doctoral dissertation of Ernst also contains a full and detailed proof of this fact [17]. Other proofs exist, namely that of Goldman and Kauffman, and can be found in [25].

Remark 2.1.9. Our sign convention for twists is that of Gordon [30]. We warn the reader that Ernst and Sumners [67] use the opposite sign convention for defining twists. In effect, our $T(\alpha/\beta)$ tangle will be a $T(-\alpha/\beta)$ tangle in [67].

There is a notion of distance between rational tangles that will be useful to us:

Definition 2.1.10. *Given rational tangles* $T_1 = T(\alpha_1/\beta_1)$, $T_2 = T(\alpha_2/\beta_2)$, *their distance is defined to be* $d(T_1, T_2) = |\alpha_1\beta_2 - \alpha_2\beta_1|$.

Given a tangle $T = (B, A)$, not necessarily with only one boundary component, we may consider the double cover of B, branched over A. This is a compact 3-manifold, whose boundary is a collection of tori (thanks to the fact that A intersects each component of ∂B in four points). We denote it by $\widetilde{T}$, and refer to it simply as the **double branched cover** of T. Note that this includes the notion of double cover of S^3, branched along a knot or link L. In this particular case, we abuse notation and write $\widetilde{L}$, and often merely refer to this as the double branched cover of L. An archetypal example, where this construction can be explicitly visualised, is shown in the figure below:

The operations of tangle sum and numerator closure admit interpretations in the double branched cover world. It is easy to see that, given marked tangles A and B, $A + B$ lifts to $\widetilde{A} \cup_F \widetilde{B}$, a manifold obtained by attaching $\widetilde{A}$ and $\widetilde{B}$ along an annulus F in their boundary. Moreover, the numerator sum $N(A + B)$ has the interpretation upstairs of gluing the two manifolds along their torus boundary T. That is, $N(A+B)$

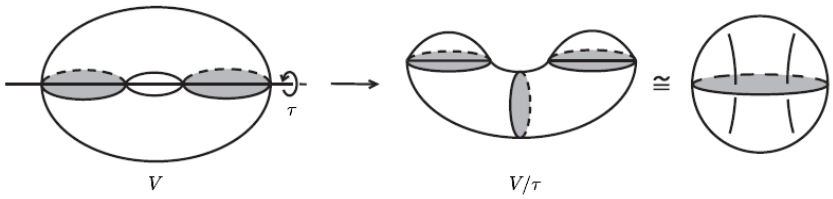


Figure 2.7: A rational tangle has a solid torus as its double branched cover, induced by the involution τ.

lifts to $\widetilde{A} \cup_T \widetilde{B}$, which is the double cover of S^3, branched along the link obtained by constructing $N(A+B)$.

2.2 Tangle equations

With the basic definitions laid out, we can begin discussing the problem of solving **tangle equations**. These are expressions of the form

$$N(O+P) = L$$

where O, P are some marked tangles with one boundary component, and L is some link in S^3. We think of the link L as being known, and the tangles being the variables one wishes to solve for. In this level of generality, it is not reasonable to expect a complete solution to this problem. However, either by restricting ourselves to particular families of links, or by restricting to certain classes of tangles, one can sometimes derive interesting conclusions. One may also wish to introduce more information and consider systems of tangle equations

$$\begin{cases} N(O+T_1) = L_1 \\ \qquad\qquad ... \\ N(O+T_n) = L_n \end{cases}$$

One class of links that is particularly wieldy to analysis is that of **two-bridge links**. Recall that a link L is said to two-bridge if it admits a projection onto the x, y plane with two maxima for its "height" coordinate y. It can be shown that every such link is the numerator closure of some rational tangle $T(p/q)$, and we denote this by $b(p,q)$. In particular, $b(1,q)$ is the unknot, and $b(0,1)$ is the unlink of two components. Also, $b(p,q) = b(-p,-q)$. We remark that there are other ways

of arriving at the number p/q, by presenting the link as a 4-plat, and calculating an associated continued fraction expansion, but we will not need to consider this. See [10] and references therein for a more complete treatment. 2-bridge links are completely classified in terms of the data p/q, as shown by Schubert:

Theorem 2.2.1. *([28]) The links $b(p,q)$, $b(p',q')$, are isotopic if and only if $p = p'$ and $q^{\pm 1} \equiv q' \bmod p$, or $p = -p'$ and $q^{\pm 1} \equiv -q' \bmod p$.*

As remarked before, solving tangle equations involving 2-bridge links is very much a tractable problem, with notable contributions to be found in [8, 9, 14, 15, 18, 67, 71]. Previously untreated, though, is the case where one of the links involved is the connect-sum of two 2-bridge links. This is what we consider in this thesis: our task is to solve a system of tangle equations of the form

$$\begin{cases} N(O+P) = b(p,q) \\ N(O+R) = b(r,s)\#b(t,u) \end{cases}$$

where $L_1 \# L_2$ denotes the connect-sum of the links L_1 and L_2.

Remark 2.2.2. In general, the connect-sum of two links is not well-defined, as it depends on a choice of components $K_i \subset L_i$, but since any 2-bridge link admits an isotopy interchanging its components, this will not be a problem for us.

Following the tradition of studying a problem in knot theory in terms of some associated 3-manifold data, one studies tangle equations by looking at their double branched covers *i.e.* one considers the problem $\widetilde{O} \cup_T \widetilde{P} = \widetilde{L}$, and then translates information about this back into information about the possible tangle solutions. Now, it is a well-known elementary fact that the double branched cover of $b(p,q)$ is the lens space $L(p,q)$ (see, for example, [65]). In fact, we will take this to be our the definition of a lens space. There are many other equivalent definitions, see [65] for a more thorough discussion. This, together with the elementary fact that $\widetilde{b(r,s)\#b(t,u)} = \widetilde{b(r,s)}\#\widetilde{b(t,u)}$, implies that we are considering the problem

$$\begin{cases} \widetilde{O} \cup_T \widetilde{P} = L(p,q) \\ \widetilde{O} \cup_T \widetilde{R} = L(r,s)\#L(t,u) \end{cases} \tag{2.1}$$

In this spirit, it is useful to recast Definition 2.1.6 in terms of some topological properties of double branched covers. First, recall the following facts:

Definition 2.2.3. *Let M be a compact, orientable 3-manifold:*

1. *M is said to be **reducible** if it admits an embedded sphere $S \subset M$ that does not bound a ball. Otherwise, M is said to be **irreducible**.*

2. *M is said to be **∂-reducible** if there is a loop γ in some boundary component $F \subset \partial M$ representing a nontrivial element of $\pi_1(F)$ with the property that it bounds a disk in M. Otherwise, M is said to be **∂-irreducible**.*

Then, we have the following result of Lickorish:

Proposition 2.2.4. *([46]) The following facts hold for a tangle T:*

1. *T is rational $\Leftrightarrow \widetilde{T}$ is a solid torus.*

2. *T is locally knotted $\Leftrightarrow \widetilde{T}$ is a reducible manifold.*

3. *T is prime $\Leftrightarrow \widetilde{T}$ is an irreducible, ∂-irreducible manifold.*

With this, if we assume that O is prime, the following restriction can be placed on P and R:

Proposition 2.2.5. *If O is prime, then neither P nor R can be prime:*

Proof. The case where P is prime is proved in Lemma 3.1 of [67], so we do not give the details. If R is prime, then gluing $\widetilde{O}$ and $\widetilde{R}$ along their torus boundary creates a reducing sphere. Since both $\widetilde{O}$ and $\widetilde{R}$ are void of reducing spheres by definition, there are properly embedded planar surfaces with boundary $F_O \subset \widetilde{O}$ and $F_R \subset \widetilde{R}$ such that $F_O \cup F_R$ is a sphere. We may assume that all of the boundary components of F_O (resp. F_R) represent nontrivial elements of $\pi_1(\partial\widetilde{O})$ (resp. $\pi_1(\partial\widetilde{R})$). Intuition suggests that if we glue two planar surfaces to produce a sphere, there have to some disk components, and this can proved as follows: we have that the Euler characteristic $\chi(F_O \cup F_R) = \chi(F_O) + \chi(F_R) = 2$. Now, a connected planar surface with boundary has positive Euler characteristic if and only if it is a disk , so that if $\chi(F_O)+\chi(F_R) = 2$ is to hold, then one of the surfaces, say F_O, has to have disk components. But this implies that $\widetilde{O}$ is ∂-reducible, contrary to our assumption. □

In spite of this result, our problem is still too general to be attacked. Nevertheless, the case where P and R are both rational turns out to be tractable. Moreover, this is of relevance when modelling DNA recombination events, as we will see in Chapter 5. So, to obtain a tractable problem, and motivated by our future biological applications, from now on we assume that P and R are rational tangles. We will put some constraints in the parameters allowed for the 2-bridge links. Firstly, we assume that $r, t \neq \pm 1$, since this was already solved in [14]. That is, we really demand to be working with "true" connect-sums. Secondly, we want to avoid working with split links, so we impose that $p, r, t \neq 0$.

Problem. *Study the existence and character of solutions to the tangle equations*

$$\begin{cases} N(O+P) = b(p,q) \\ N(O+R) = b(r,s)\#b(t,u) \end{cases} \tag{2.2}$$

assuming that $r, t \neq \pm 1$, $p, r, t \neq 0$, and that P and R are rational.

Discussing this problem, and its applications to DNA recombination, will be the main motivation of this thesis.

Since the double branched cover of a rational tangle is a solid torus, our problem has a natural interpretation in terms of **Dehn fillings**, the definition of which we recall:

Definition 2.2.6. *Let M be a manifold such that ∂M is a torus, and let $\alpha \subset \partial M$ be a non-trivial isotopy class of unoriented simple closed curves. Consider the manifold $M(\alpha) = M \cup_\partial (S^1 \times D^2)$, where the boundary identification is determined by gluing the meridian of $S^1 \times D^2$ to α. Then, $M(\alpha)$ is said to have been obtained by **Dehn filling** on M along the **slope** α.*

Given slopes α, $\beta \subset \partial M$, one defines their **distance** $d(\alpha, \beta)$ as their minimal geometric intersection number. Writing $\alpha = p_\alpha e_1 + q_\alpha e_2$ and $\beta = p_\beta e_1 + q_\beta e_2$ in terms of a basis e_1, e_2 of $H_1(\partial M; \mathbb{Z})$, it is customary to parameterise the slopes α by $p_\alpha / q_\alpha \in \mathbb{Q} \cup \{\infty\}$. Notice that a canonical choice of basis is not always available. Nevertheless, it is a well-known and straightforward fact that, for any choice of basis, $d(\alpha, \beta) = |p_\alpha q_\beta - p_\beta q_\alpha|$ (see, for example, [7]). Now, considering the expression $N(O+P)$ for all different rational tangles P corresponds to accounting for all the

possible Dehn fillings on $\widetilde{O}$. Given a rational tangle $T(p/q)$, it can be seen (see, for example, [30]) that the lifts of the $SW - NW$ and $NW - NE$ arcs, oriented so that their signed intersection number is $+1$, give a choice of basis μ, λ, for $H_1(\partial\widetilde{T(p/q)};\mathbb{Z})$. Observing that, for the $T(\infty)$ tangle, the meridian disk for $\widetilde{T(\infty)}$ is bound by μ, and that the horizontal and vertical half twists downstairs lift to Dehn twists along μ and λ, respectively, it is not hard to show that the slope $p\mu + q\lambda$ on $\partial\widetilde{T(p/q)}$ defines the meridian disk for the solid torus $\widetilde{T(p/q)}$. Therefore, given the tangle equations

$$\begin{cases} N(O+P) = L_1 \\ N(O+R) = L_2 \end{cases}$$

lifting to some Dehn fillings

$$\begin{cases} \widetilde{O}(\alpha) = \widetilde{L_1} \\ \widetilde{O}(\beta) = \widetilde{L_2} \end{cases}$$

we have the following immediate consequence:

Lemma 2.2.7. *$d(P,R) = d(\alpha,\beta)$.*

One point of view which is sometimes more convenient is the following: notice that the core of the solid torus defined by the lift of P is a connected simple closed curve *i.e,* a knot K in $\widetilde{L_1}$. Then, the pair of Dehn fillings $\{\widetilde{O}(\alpha), \widetilde{O}(\beta)\}$ can be interpreted as obtained by removing a regular neighborhood νK of K from $\widetilde{L_1}$ and attaching a solid torus to the boundary to create $\widetilde{L_2}$. Under this point of view, the slope α can be thought of as defining the meridian μ_K of the knot $K \subset \widetilde{L_1}$. We then say that $\widetilde{L_2}$ is obtained by **β-Dehn surgery** on $K \subset \widetilde{L_1}$. Of course, a similar discussion holds if we interchange $\widetilde{L_1}$ and $\widetilde{L_2}$. It will be useful to have the following definition:

Definition 2.2.8. *If $d(\beta, \mu_K) = 1$, the β-Dehn surgery on K is said to be **integral**.*

Even though in a canonical basis for slopes is generally not available, there are situations where one does have such a choice. For example, if the knot $K \subset \widetilde{L_1}$ is nullhomologous, we know that K bounds a surface F, known as a **Seifert surface**. Denote $\widetilde{L_1}\backslash\nu K$ by $E(K)$. The slope on $\partial E(K)$ determined by ∂F gives a natural choice of longitude, λ_K. Moreover, λ_K intersects μ_K once. Orienting these such that the signed intersection $\langle \mu_K, \lambda_K \rangle = +1$, in terms of the orientation on $\partial E(K)$ induced

by an orientation on $\widetilde{L_1}$, we have that $\{\mu_K, \lambda_K\}$ is a basis for $H_1(\partial E(K); \mathbb{Z})$. Then, Dehn surgery on $\alpha = p\mu_K + q\lambda_K$ is said to be the result of p/q **Dehn surgery** on $K \subset \widetilde{L_1}$. The surgery is integral when $q = \pm 1$, of course. Also, note that, if we are thinking of this in terms of lifts of rational tangle attachments, this basis may not be related at all to the basis given to us by the tangle.

Remark 2.2.9. Our convention for parameterising Dehn surgery slopes follows the one of Ozsváth and Szabó [57], and it is opposite to that of Gordon, who demands that $\langle \mu_K, \lambda_K \rangle = -1$ [30]. As an example, $L(p, q)$ (with repect to our orientation convention for lens spaces) will be the result of p/q Dehn surgery on the unknot in S^3. In the convention of [30], $L(p, q)$ is the result of $-p/q$ surgery on the unknot in S^3.

With this in place, we will largely exchange our problem in tangles with the problem of studying when a manifold has a pair of fillings α, β, yielding a lens space and a connect-sum of lens spaces, or equivalently, when a knot in a lens space has a connect-sum of lens spaces surgery. This conversion is not completely one sided: as we go along, we will see that tangles provide a very useful way to visualise topological properties of 3-manifolds, especially when trying to construct examples.

CHAPTER 3 DEHN SURGERY RESULTS

In this section, we work in the following setting: we are given an irreducible, ∂-irreducible manifold M with boundary a torus, together with a pair of slopes $\alpha, \beta \in \partial M$ such that

$$M(\alpha) = L(p, q)$$

$$M(\beta) = L(r, s) \# L(t, u)$$

We remark once again that we exclude $S^2 \times S^1$ from the definition of lens space (so that $p, r, t \neq 0$), and that we want true connect-sums, so that $r, t \neq 1$. We will consider our manifolds to be oriented.

3.1 Preliminaries

Our aim is to discuss the result that, if such a pair (α, β) exists under the conditions above, then $d(\alpha, \beta) = 1$. Before we do this, though, we will remind the reader of some relevant terminology concerning essential surfaces, Seifert fibered spaces and cable spaces. We start by reminding the reader of the definition of an essential surface:

Definition 3.1.1. *Let $F \subset M$ be a properly embedded surface. We say that F is **compressible** if there exists an embedded disk $D \subset M$ such that $D \cap F = \partial D$ and ∂D does not bound a disk in F. If no such disk D exists, F is said to be **incompressible**.*

Definition 3.1.2. *A properly embedded surface $F \subset M$ is **essential** if*

- *$F \cong S^2$ and it does not bound a ball B^3 in M.*
- *$F \cong D^2$ and ∂D^2 does not bound a disk in ∂M.*
- *F is not a sphere or a disk, F is incompressible and not boundary parallel. By boundary parallel we mean that $\exists$ an isotopy of M taking F to ∂M.*

Next, we recall the definition of a **Seifert fibered space**, and some of their basic properties. This is by no means intended to be a comprehensive account, and the reader is directed to [7] for many more details on these manifolds, including a complete classification.

Definition 3.1.3. *A 3-manifold M is said to be Seifert fibered if it admits a foliation by circles with the property that every leaf (henceforth denoted fiber) is modeled on a (p,q)-fibered solid torus neighborhood. That is, every fiber l has a neighborhood $\mathcal{N}(l)$ consisting of fibers, together with a homeomorphism $\phi : \mathcal{N}(l) \to (D^2 \times I, \rho)$, where ρ is a rotation of D^2 by an angle $\frac{2\pi p}{q}$. Here, p, q are relatively prime integers such that $0 \leq p < q$, and $(D^2 \times I, \rho)$ is the quotient space formed by identifying $D^2 \times \{0\}$ with $D^2 \times \{1\}$ via ρ. We require that the homeomorphism be fiber preserving, that is, ϕ must take l to the core $\{0\} \times I$, and it must take other leaves to the union of arcs $x \times I \cup ... \cup \rho^{q-1}(x) \times I$.*

If a fiber is modeled on a $(0,1)$ solid torus, we say that it is a **regular fiber**. If it is modeled on a (p,q) torus with $q \neq 1$, we say that it is an **exceptional fiber** of type (p,q). In particular, q is said to be **multiplicity** of the exceptional fiber.

It is easy to see that a compact Seifert fibered space has finitely many exceptional fibers, and these are isolated, in the sense that we can always find a fibered neighborhood containing no other exceptional fibers. Moreover, collapsing all the fibers to a point, one can define a map $\Pi : M \to F$, where F is some compact surface, and it is defined to be the **orbit surface** of M. It turns out that Π is a branched covering map, with branch points upstairs the exceptional fibers of M. This map induces a homomorphism

$$\Pi : \pi_1(M) \to \pi_1(F) \tag{3.1}$$

and it can be shown that this is surjective.

We will adopt the following standard notational shorthand: a Seifert fibered space M with orbit surface F and exceptional fibers of type $(p_1, q_1), ..., (p_n, q_n)$ is said to be of **type** $F((p_1, q_1), ..., (p_n, q_n))$. If it has no exceptional fibers, we will write $M = F(\emptyset)$. Sometimes, we will have no need for the information contained in the p_i, and then we will simply write $M = F(q_1, ..., q_n)$. Whenever we specify only integer parameters

in the bracket, it is to be understood that these stand for the multiplicities of the exceptional fibers.

Remark 3.1.4. The symbol $M = F((p_1, q_1), ..., (p_n, q_n))$ doesn't always determine the Seifert fibered space up to homeomorphism. Moreover, a given homeomorphism type of M may carry many inequivalent Seifert fibrations. Examples of this are, of course, the solid torus, and the lens spaces. See [7] for more details.

It will be useful to single out a collection of Seifert fibered spaces whose topology is very simple:

Definition 3.1.5. *A Seifert fibered space is said to be* ***small*** *if it is of the following types:*

1. $D^2(q_1, .., q_m)$, $m \leq 2$ *(orbit surface a disk).*
2. $A^2(q)$ *or* $A^2(\emptyset)$ *(orbit surface an annulus).*
3. $\mathbb{R}P^2(q)$, $\mathbb{R}P^2(\emptyset)$ *(orbit surface* $\mathbb{R}P^2$*).*
4. $B(\emptyset)$*, (orbit surface the Möbius band).*
5. $P(\emptyset)$ *(orbit surface the "pair of pants").*

If M has boundary, it can be shown that this consists of a disjoint union of tori and Klein bottles (we will be exclusively concerned with Seifert fibered spaces with torus boundary). It can also be shown that all fibers on any boundary component are regular. Now, Dehn fillings along a torus boundary component of M are well understood, and their "qualitative" properties depend only on the minimal intersection *i.e.* the distance $d(\alpha, h)$, between the filling slope α and that of a regular fiber $h \subset \partial M$. In particular, we have the following well known result:

Theorem 3.1.6. *Let M be a Seifert fibered space of type $F(q_1, ..., q_n)$, and let α be a slope on a torus boundary component $T \subset \partial M$. Then, $M(\alpha)$ has the following properties:*

1. $M(\alpha) = \hat{F}(q_1, ..., q_n, m)$*, if* $d(\alpha, h) = m > 1$*. $\hat{F}$ denotes F with one of its boundary components capped with a disk.*

2. $M(\alpha) = \hat{F}(q_1, ..., q_n)$*, if* $d(\alpha, h) = 1$*.* $\hat{F}$ *is as before.*

3. $M(\alpha)$ *is not Seifert fibered, if* $\alpha = h$*.*

In fact, the "forbidden" filling $\alpha = h$ typically dissolves $M(\alpha)$ into a reducible manifold, as is shown by the following result of Heil [38]:

Theorem 3.1.7. *Let* M *be an orientable Seifert fibered space of type* $F_g(q_1, ..., q_n)$*, where* F_g *is an orientable surface of genus* g *with* r *torus boundary components. Let* h *be a regular fiber on one of the components* $T \subset \partial M$*, and suppose we do the Dehn filling* $M(h)$*. Then,* $M(h) \simeq L_1 \# ... \# L_r \# (S^2 \times S^1)^{2g} \# (S^1 \times D^2)^{r-1}$*, where the* L_i *are lens spaces not homeomorphic to* S^3 *or* $S^2 \times S^1$*, and* Y^k *means* $\underbrace{Y \# ... \# Y}_{\text{k times}}$*.*

In fact, using tangles, we can say a little bit more about the lens spaces appearing in the statement of the theorem, at least when the orbit surface is a disk or an annulus. It is well-known (see [30] or [50]) that $D^2((p_1, q_1), ..., (p_n, q_n))$ is homeomorphic to the double branched cover of the **Montesinos tangle** $\mathcal{M}((p_1, q_1), ..., (p_n, q_n)) = T(p_1/q_1) + ... + T(p_n/q_n)$, depicted in the figure below:

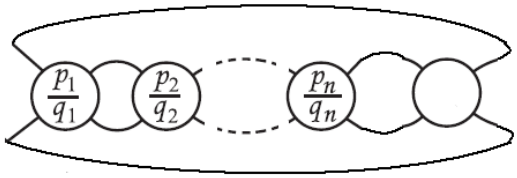


Figure 3.1: The Montesinos tangle $\mathcal{M}((p_1, q_1), ..., (p_n, q_n)) = T(p_1/q_1) + ... + T(p_n/q_n)$. Adapted from [30].

Now, rational tangle attachments to our Montesinos tangles correspond upstairs to Dehn fillings. This provides a convenient method for visualising Dehn fillings via double branched covers. With this technique, it is easy to see that the non-Seifert Dehn filling corresponds to plugging in the $T(\infty)$ tangle, and that this results upstairs in $-L(q_1, p_1) \# ... \# - L(q_n, p_n)$. Notice the permutation of parameters p_i, q_i, and the orientation reversal.

We will also need to introduce the notion of a **cabled manifold**. We start with the following:

Definition 3.1.8. *A Seifert fibered space over the annulus, with one exceptional fiber of type (p, q), is said to be a **cable space**, and it is denoted by $C_{p,q}$. This has boundary consisting of two tori. Note that having one exceptional fiber implies that $q \geq 2$.*

It is convenient, for visualisation purposes, to think about cable spaces as the manifolds obtained from a solid torus $S^1 \times D^2$ by removing a regular neighborhood of a (p, q) curve in one of the level tori $S^1 \times r$, $0 < r < 1$. Here, we think of D^2 as the unit disk in the plane, parameterised with polar coordinates (r, θ). With this in place, let M be a manifold with boundary a single torus T, and let $C_{p,q}$ be a cable space. Then, $N = C_{p,q} \cup M$ is said to be a (p, q) **cable on** M. Note that cable spaces admit a self-homeomorphism interchanging their boundary components, so that the operation of cabling doesn't depend on the choice of boundary component on $C_{p,q}$.

3.2 Distance bounds for reducible surgeries

With these basic definitions in place, we are ready to start. Let us take M, α, β, as described in the beginning of the section. We will suppose, throughout the argument, that $d(\alpha, \beta) > 1$. The idea is to use some decomposition and classification results on 3-manifolds to rule out as many families to which M might belong as possible. We start by showing that $d(\alpha, \beta) > 1$ is untenable if M is a Seifert fibered space:

Proposition 3.2.1. *Let M be a Seifert fiberd space such that ∂M is a torus, and suppose that, for some slopes α, β, $M(\alpha) = L(p, q)$ and $M(\beta) = L(r, s) \# L(t, u)$, with $r, t \neq 1$. Then $d(\alpha, \beta) = 1$.*

Proof. It is a well-known fact (see [7]) that the only non prime Seifert fibered space is $\mathbb{R}P^3 \# \mathbb{R}P^3 = L(2, 1) \# L(2, 1)$. So, let us begin by assuming that $r, t \neq 2$. Then, since in this case $M(\beta)$ does not admit a Seifert fibration, we have that β is isotopic to a regular fiber in ∂M. Therefore, $M(\alpha)$ inherits a Seifert fibration from M. Since this is a lens space, the fundamental group in finite, so that the surjectivity of the map $\Pi : M \to F$ in (3.1) implies that the orbit surface must be a sphere or $\mathbb{R}P^2$. A more careful analysis of the fundamental group shows that if M is of the type $S^2(q_1, q_2, q_3)$, or $\mathbb{R}P^2(q)$, then $\pi_1(M)$ is non-abelian. See [7] for the details. This implies that M either fibers over the disk with at most 2 exceptional fibers, or over the Möbius band with no exceptional fibers. Let us focus on the disk orbit surface case. Now, if M fibered over the disk with at most one exceptional fiber, it would be a solid torus.

Since no Dehn filling on a solid torus results in a non-prime manifold, this case cannot happen. Therefore, $M = D^2(q_1, q_2)$ has exactly two exceptional fibers, and α must become a regular fiber in $M(\alpha)$. From Theorem 3.1.6, this is equivalent to saying that α intersects a regular fiber on ∂M once, that is to say, $d(\alpha, \beta) = 1$, which is what we wanted to prove.

The case $B(\emptyset)$ is quite interesting: this is the orientable circle bundle over the Möbius band, or the orientable I-bundle over the Klein bottle. It can be given an alternative fibering of type $D^2(2, 2)$, so that this really is covered by the previous discussion. Incidentally, the Seifert structure on $L(2, 1)\#L(2, 1) = \mathbb{R}P^3\#\mathbb{R}P^3$ comes from this: $\mathbb{R}P^3\#\mathbb{R}P^3 = \mathbb{R}P^2(\emptyset)$, and this is obtained by adding a regular fiber to $B(\emptyset)$ in a way that corresponds precisely to attaching a solid torus along a regular fiber of its $D^2(2, 2)$ fibration. Thus, the $r = 2,\ t = 2$ case is not different from the others, and the result follows. □

Next, we consider the case where M has essential tori. To study this, we will decompose M along these surfaces, in order to facilitate the analysis. It should be intuitively clear that one cannot go on decomposing a compact manifold along non-isotopic essential tori forever. Since $\pi_1(M)$ is finitely generated, and $\pi_1(S)$ injects into $\pi_1(M)$ whenever S is an essential surface, the rank of $\pi_1(M)$ should give some bound on the maximal number on non-isotopic essential surfaces S of a fixed genus in M. Indeed, Haken's theory of normal surfaces [34] has been used to obtain strong finiteness results on essential surfaces in M. We will use the following result for torus decompositions, the proof of which can be found in [35]:

Theorem 3.2.2. *Let M be irreducible, compact and connected. Then, there exists a finite, pairwise disjoint collection of tori T such that the closure of each component of $M - T$ is atoroidal.*

Remark 3.2.3. There exists, indeed, a stronger uniqueness result, which requires one to possibly include annuli in the collection. This is the famous JSJ decomposition theorem [40, 41]. Since we want to avoid using annuli to chop our manifold open, we have opted for the more elementary version given above, which will suffice for our purposes.

Using the previous theorem, we decompose M along its essential tori, and obtain pieces $v_0, ..., v_n$. In fact, we may organise these into a graph $\mathcal{G}(M)$, where the vertices

are precisely the v_i, and two vertices are connected by an edge if they were originally glued along one of the essential tori that we deleted. Observe the following:

Lemma 3.2.4. *All closed surfaces in M are separating.*

Proof. If some closed surface T were non-separating, it would also be non-separating in $M(\alpha) = L(p,q)$. Now, being non-separating means that T is not a boundary, so it would generate a non-trivial class in $H_2(L(p,q);\mathbb{Z})$. But this is zero, since lens spaces are rational homology spheres. □

This implies that removing an edge from $\mathcal{G}(M)$ disconnects the graph, and this is the characterising property of a tree. We may define the **root** of $\mathcal{G}(M)$, v_0, to be the vertex associated to the component that contains ∂M. Similarly, we may take the **leaves** to be the vertices containing only one boundary component. These may be thought of, loosely, as the places where the essential torus decomposition cannot go any "deeper".

The fact that $\mathcal{G}(M)$ has a tree structure suggests that we may somehow argue inductively starting at the root and finishing at the leaves. Let us start on the manifold corresponding to v_0. This is irreducible and has no essential tori by construction, but it may have essential annuli. If that is the case, it turns out that v_0 is a small Seifert fibered space, as is shown in the following [30]:

Proposition 3.2.5. *Let M be an irreducible 3-manifold whose boundary consists of tori. If M contains an essential annulus then M either contains an essential torus or is a small Seifert fibered space.*

Now, if our Dehn fillings satisfy the condition $d(\alpha,\beta) > 1$, we can conclude that v_0 must be a small Seifert fibered space.

Proposition 3.2.6. *If $d(\alpha,\beta) > 1$, then v_0 is small Seifert fibered.*

Proof. Let $\{e_0^i\}_{i=0}^n$ denote the boundary components of $v_0(\alpha)$. We know that, upon Dehn filling, every one of these components will compress in $L(p,q)$. We want to show that at least one compresses in $v_0(\alpha)$. Take e_0^1, and consider a compressing disk for it, D, in $L(p,q)$. Since e_0^1 separates $L(p,q)$, $int(D)$ must be contained in the component that contains ∂M or in the other component. If the latter happens, e_0^1 would compress in M, which is a contradiction. Thus we may assume D is in the component which

contains ∂M, but not necessarily in $v_0(\alpha)$. If D is not in $v_0(\alpha)$, then it must intersect some other boundary components of $v_0(\alpha)$. This intersection is a disjoint collection of loops, and we may consider an innermost such loop, which we denote by l. Such a loop l bounds a disk $E \subset D$, and $l \subset e_0^m$, for some m. Moreover, we may assume that l is essential in e_0^m. Applying the separation argument above, we conclude that E is a compressing disk for e_0^m in $v_0(\alpha)$, since it doesn't intersect any of the other boundary components, by construction. Now, $v_0(\alpha)$ is an irreducible manifold whose boundary consists of tori, and we know that one of these tori compresses. By a standard result (see [30]), $v_0(\alpha)$ must be a solid torus. This shows that v_0 has exactly two boundary components. Exactly the same argument shows that e_0^m compresses in $v_0(\beta)$.

We can now appeal to a result of Wu (Theorem 1 of [74]), obtained using combinatorial techniques of intersection graphs. The result states the following: let X be a manifold with at least two incompressible boundary components, one of which must be a torus T. Denote any other component by S, and suppose that X does not contain an essential annulus with one boundary component on T and the other on S. Then, if X has a pair of Dehn fillings α and β on T such that S compresses for both of these fillings, the conclusion is $d(\alpha, \beta) \leq 1$. From this, we see that v_0 must contain an essential annulus whenever $d(\alpha, \beta) > 1$, and since all its boundary components are tori, it must be a small Seifert fibered space, by Proposition 3.2.5. □

In fact, we can say exactly what Seifert fibered space v_0 is.

Proposition 3.2.7. *If v_0 is a Seifert fibered space, then it is a cable space.*

Proof. We've shown in the previous proposition that v_0 can only have two boundary components. The only small Seifert fibered manifold with this property is the one with orbit surface an annulus, and at most one exceptional fiber. If it has no exceptional fibers, this is just a torus times an interval $T^2 \times I$. Notice that one of the boundary components is precisely ∂M. But this would mean that $\partial v_0 - \partial M$ is boundary parallel, contradicting the fact that it is essential. Hence, v_0 must have precisely one exceptional fiber, that is, it must be a cable space. □

Now, it is straightforward to see what types of manifolds we obtain from $C_{p,q}$ by Dehn filling. If we denote the isotopy class of a regular fiber on $\partial C_{p,q}$ by h, then it follows immediately from Theorem 3.1.6, together with the discussion following it, that $C_{p,q}(\alpha)$ is one of the following:

1. A Seifert fibered space of type $D^2(q,m)$, if $d(\alpha,h)=m>1$.

2. A Seifert fibered space of type $D^2(q)$, *i.e.* a solid torus if $d(\alpha,h)=1$.

3. $S^1\times D^2\#-L(q,p)$ if $\alpha=h$ (this is the filling that destroys the fibration).

Now, in our case we want both $C_{p,q}(\alpha)$ and $C_{p,q}(\beta)$ to compress, so we are forbidden from having case (1). Also, notice that we may not pick α or β to be equal to h, since then we would have to be in case (2), which violates the constraint $d(\alpha,\beta)>1$ that we are imposing (of course, when we discuss the case $d(\alpha,\beta)=1$, this setup will be a rich source of examples of reducible fillings). However, we also see that we can pick α and β at arbitrarily high distance and such that $v_0(\alpha)$ and $v_0(\beta)$ are solid tori. If we do that, then we have induced Dehn fillings γ, δ, on the vertex of $\mathcal{G}(M)$ adjacent to v_0, say, v_1. Now, we would like to repeat the previous arguments inductively at each node of $\mathcal{G}(M)$ until we reach a leaf v_l. The only thing barring us from doing this is the possibility that the distance between the induced fillings might accidentally drop to one at some stage of the process, leading to a breakdown of the inductive reasoning. However, we can show that this never happens. The basic ingredient is the following result of Gordon ([31], Lemma 3.3), which is an elementary homological calculation:

Proposition 3.2.8. *let J be a simple closed curve in a solid torus V which does not lie inside a ball, and let $Y=V-N(J)$, where $N(J)$ is a regular open neighborhood of J. We denote by $(J;r)$ the result of performing $r=m/n$ Dehn surgery on Y on the boundary component created by removing $N(J)$. Furthermore, let μ, λ be a standard meridian/longitude basis for ∂V. Then, the kernel of the inclusion $i_*:H_1(\partial(J;r))\to H_1((J;r))$ is generated by:*

$$\begin{cases} \frac{n\omega^2}{gcd(\omega,m)}\lambda+\frac{m}{gcd(\omega,m)}\mu, & \omega\neq 0 \\ \mu, & \omega=0 \end{cases}$$

where ω is the winding number of J.

In our situation, we have that $(J;r)$ is a solid torus, and $\omega=q\geq 2$ is the multiplicity of the exceptional fiber. Moreover, the generators of the kernel of the inclusion $i_*:H_1(\partial(J;r);\mathbb{Z})\to H_1((J;r);\mathbb{Z})$ are nothing but the slopes that bound disks, since we are taking $(J;r)$ to be a solid torus. That is, they are the slopes that determine the Dehn fillings γ and δ, given in terms on μ, λ. Now, write $\alpha=m_\alpha/n_\alpha$

and $\beta = m_\beta/n_\beta$ in terms of the canonical basis on $\partial C_{p,q}$ (it is not hard to compute these explicitly, but we have no need for it, see [30] for details). Since γ and δ are the slopes that bound disks, we have that

$$\gamma = \frac{n_\alpha q^2}{gcd(\omega, m_\alpha)}\lambda + \frac{m_\alpha}{gcd(\omega, m_\alpha)}\mu, \qquad \delta = \frac{n_\beta q^2}{gcd(\omega, m_\beta)}\lambda + \frac{m_\beta}{gcd(\omega, m_\beta)}\mu$$

Then,

$$d(\gamma, \delta) = \begin{vmatrix} \frac{n_\alpha q^2}{gcd(\omega,m_\alpha)} & \frac{m_\alpha}{gcd(\omega,m_\alpha)} \\ \frac{n_\beta q^2}{gcd(\omega,m_\beta)} & \frac{m_\beta}{gcd(\omega,m_\beta)} \end{vmatrix} = \frac{q^2}{gcd(q, m_\alpha)gcd(q, m_\beta)}|m_\alpha n_\beta - m_\beta n_\alpha| =$$

$$= \frac{q^2}{gcd(q, m_\alpha)gcd(q, m_\beta)} d(\alpha, \beta) \geq d(\alpha, \beta)$$

.

It follows that we may argue inductively, since the hypothesis $d(\alpha, \beta) > 1$ is never violated upon cabling.

Notice our argument implies that $\mathcal{G}(M)$ is a linear graph, since each vertex which is not the root or a leaf has valence 2. Moreover, each such vertex corresponds to a cable space. We call a manifold built by gluing cable spaces along boundary components in a "linear fashion" an **iterated cable**. We conclude by making some remarks about the leaf v_F of $\mathcal{G}(M)$ (notice that our argument shows that there can be only one). This is an irreducible, ∂-irreducible manifold with torus boundary. Now, if it has an essential annulus, it must be a small Seifert fibered space, by Proposition 3.2.5. But this is at odds with Proposition 3.2.1 (keeping in mind that we have two induced fillings at distance greater than one). In fact, it follows from Thurston's geometrisation theorem for manifolds with boundary [69] that v_F must be hyperbolic, since it has no essential surfaces of non-negative Euler characteristic (we've constructed it to be void of essential tori, spheres and disks, and we've shown that it follows from $d(\alpha, \beta) > 1$ that it must be void of essential annuli). Putting everything together, we've shown the following result:

Theorem 3.2.9. *Let M be an irreducible, ∂ irreducible manifold with boundary a*

single torus. Suppose that, for a pair of slopes $\alpha, \beta \in \partial M$ such that $d(\alpha, \beta) > 1$, we obtain

$$M(\alpha) = L(p, q)$$

$$M(\beta) = L(r, s) \# L(t, u)$$

Then, M is either hyperbolic or an iterated cable on a hyperbolic manifold.

Observe that, at each stage of the iterated cabling, we must end up with a pair of solid torus filling (at ever increasing distances), so that, in some sense, the problem reduces to the case where M is hyperbolic.

Remark 3.2.10. We point out that the hypothesis that M be irreducible is necessary: one can easily take $M = D^2(x, y) \# L(s, t)$, and as we've seen, there are infinitely many fillings on $D^2(x, y)$, at arbitrarily high distance, yielding lens spaces (any pair of fillings at distance one from the regular fiber will do). We should also remark that, by the cyclic surgery theorem [13], these are the only counterexamples.

As demonstrated above, the proof of Theorem 3.2.9 relies on purely geometric topological techniques. This proof complements an algebro-geometric proof of a more general result of Boyer and Zhang [6]. Their techniques involve using some technical algebraic geometry and group representation theory to extract information from the fundamental group, and the philosophy of the method is roughly the following: from $\pi_1(M)$, one constructs the $PSL_2(\mathbb{C})$**-character variety**, an affine algebraic variety whose points correspond to characters of representations of $\pi_1(M)$ into $PSL_2(\mathbb{C})$ (the theory is more general than this, but $PSL_2(\mathbb{C})$ turns out to be the relevant Lie group for studying 3-manifolds). From this, it is possible to construct a seminorm (known as a Culler-Shalen seminorm) on $H_1(\partial M, \mathbb{R})$, and this object turns out to naturally encode information about Dehn fillings on M. In particular, distance bounds between particular types of Dehn fillings can be obtained from this. For further details, the reader is referred to Boyer and Zhang's paper [6]. One of their results is:

Theorem 3.2.11. (*[6], Theorem 1.2*) *Let M be a compact, connected, orientable, irreducible 3-manifold with ∂M a torus. Assume that M is neither a simple Seifert fibered manifold nor a cable on $I(K)$, the twisted I-bundle over the Klein bottle. Fix slopes r_1 and r_2 on ∂M and suppose that $M(r_1)$ is a reducible manifold. Then,*

1. *If $M(r_2)$ has a cyclic fundamental group, then $d(r_1, r_2) \leq 1$.*

2. *If r_2 is a finite filling slope, then $d(r_1, r_2) \leq 5$ unless $M(r_1) = \mathbb{R}P^3 \# \mathbb{R}P^3$. and $\pi_1(M(r2))$ is a D-type group or a Q-type group.*

3. *If M contains an essential torus and r_2 is a finite filling slope, then $d(r_1, r_2) \leq 1$ unless M is a cable on a simple, non-Seifert manifold M_1 for which there are slopes r_1', r_2' on ∂M_1 such that $M_1(r_1') = \mathbb{R}P^3 \# \mathbb{R}P^3$, and $M_1(r_2')$ has a D-type or Q-type fundamental group, and $d(r_1', r_2') \geq 8$.*

Indeed, it follows from our previous discussion that the exclusion of both Seifert manifolds and cables on $I(K)$ is only necessary to obtain conclusions (2) and (3) of the theorem.

Remark 3.2.12. For completeness, we remark that groups of Q-type and of D-type have presentations of the form $(x, y : x^2 = (xy)^3 = y^n, x^4 = 1)$ and $(x, y : x^{2k} = 1, y^{2n+1} = 1, xyx^{-1} = y^{-1})$ for some n, k (compare with the finite quaternion group and the dihedral group, respectively), and they form two of the families of finite groups that act freely on S^3. See Milnor's wonderful article [49] for more details.

Unfortunately, our methods cannot say anything about the hyperbolic case, so that it is really necessary to appeal to $PSL_2(\mathbb{C})$-character varieties to show the bound $d(\alpha, \beta) \leq 1$ holds in general.

3.3 Examples and conjectures

The bound $d(\alpha, \beta) \leq 1$ is sharp, in the sense that there are examples realising it. In this section, we collect such examples of knots in lens spaces having connect-sum lens space surgeries, and will discuss any conjectures that might be made from this data.

We begin with the obvious examples *i.e.* when the knot K has a complement that is Seifert fibered. In Proposition 3.2.1 we showed that, if this is the case, then $E(K)$ must fiber over the disk with two exceptional fibers. In fact, we can use the tangle calculus to classify all the examples:

Proposition 3.3.1. *Fix relatively prime integers p, q. Let c^{ϵ_a} denote either c mod a or c^{-1} mod a. Then, a knot $K \subset L(p, q)$ whose exterior is Seifert fibered has a surgery giving $L(r, s) \# L(t, u)$ if and only if we can write $(p, q) = \pm(trm - ts^{\epsilon_r} -$*

$ru^{\epsilon_t}, -(ty - u^{\epsilon_t}x)$), where m is some integer, and x, y are integers subject to the constraint $(rm - s^{\epsilon_r})x - ry = 1$.

Proof. Since we know that $E(K)$ fibers over the disk with two exceptional fibers, $E(K)$ is homeomorphic to the double branched cover of a Montesinos tangle $\mathcal{M}(r_1, r_2)$, for some rational numbers $r_i = f_i/g_i$. Since we want this to have a $L(r, s)\#L(t, u)$ surgery upstairs, we must take the parameters in our Montesinos tangle to be $r_1 = (-s^{\epsilon_r} + ir)/r$ and $r_2 = (-u^{\epsilon_t} + jt)/t$. In fact, the choices of i, j can be thought of as horizontal twists on our Montesinos tangle, to we can push them all into r_1 to write, without loss of generality, $r_1 = (-s^{\epsilon_r} + mr)/r$ and $r_2 = -u^{\epsilon_t}/t$. Speaking in fancier terms, it is known that the exceptional fiber types, together with the orbit surface, aren't enough information to determine a lens space. One needs to specify a "section obstruction", which comes as an integer m (see [7]). Pictorially, this integer can be interpreted downstairs as a choice of horizontal half-twists to be given to the Montesinos tangle, and of course, different choices of m will affect the lens space one obtains upon numerator closure. Now, it is well known that

$$N(T(f_1/g_1) + T(f_2/g_2)) = b(f_1g_2 + g_1f_2, -(xf_2 + yg_2)) \tag{3.2}$$

with $f_1x - g_1y = 1$ (see [30] or [67], but be reminded that the sign convention in the latter is opposite to the one we've picked). The proposition follows by direct substitution. □

From this result, it looks rather hard to find any patterns. Surprisingly, though, we are able to to prove the following:

Theorem 3.3.2. *Let $L(p, q)$ be any lens space, with $p \neq \pm 1, \pm 2$. Then, no knot $K \in L(p, q)$ with Seifert fibered exterior has a surgery giving $L(p, q)\#L(t, 1)$.*

Proof. This is essentially solving the equations

$$p = -p - q^{\epsilon_p}t + tpm \quad \textit{and} \quad x - ty = q$$

or

$$p = p + q^{\epsilon_p}t - tpm \quad \textit{and} \quad ty - x = q$$

subject to the appropriate constraints. From the first equation in the second set, we get that $t(-q^{\epsilon_p}+pm)=0$, and this implies that p and q^{ϵ_p} have factors in common, which is a contradiction. So, it suffices to look at the first set of equations. Let us consider the case $q^{\epsilon_p}=q$ first. One readily gets that

$$p=\frac{t}{2}(pm-q) \tag{3.3}$$

Substituting this into the constraint $(-q+pm)x-py=1$, we get

$$\frac{2p}{t}x-py=1 \tag{3.4}$$

This is only possible if $t=p$ or $t=2p$. Thus, the proof breaks into two cases:

Case 1: $t=p$. From $x-ty=q$, we have $x-py=q$. We also have from (3.4) that $2x-py=1$. From this we obtain that $x=1-q$, and also that $2q=1-py$. From (3.3), we know that $2q=2pm-4$. Hence, $p(2m-y)=5$, so $p=\pm 5$. We do the $p=5$ case (the $p=-5$ case follows exactly the same argument). Our lens space is of the form $L(5,q)$, and using $x=1-q$ it must also be the case that our lens space is of the form $L(5,1-q)$, since $x\equiv q \ mod\ p$. It remains to check that we cannot solve $q(1-q)\equiv 1 \ mod\ 5$, and that is indeed true.

Case 2: $t=2p$. This time, running through the same methodology, we end up with the equations $pm=1+q$, $py=1-q$, whence p divides 2. Thus $p=\pm 2$ is the only possibility.

It remains to check everything again for $q^{\epsilon_p}=q^{-1} \ mod\ p$. Observe that the argument is the same up to and including (3.3). Now we check two cases, as before (we drop $mod\ p$ from $q^{-1} \ mod\ p$ for clarity):

Case 3: $t=p$. This time, we end up with $2q=1-py$ and $q^{-1}=pm-2$. Multiplying both equations gives $2qq^{-1}=(1-py)(pm-2)$, whence $2\equiv -2 \ mod\ p$. this is only possible if $p=\pm 2$ or $p=\pm 4$. Let us deal with $p=\pm 4$. Then, it is also possible to show that $x=1-q^{-1}$, and then argument goes exactly like in the case $p=\pm 5$ discussed above.

Case 4: $t=2p$. Along similar lines as before, we end up with $q=1+py$, $q^{-1}=pm-1$. This time, multiplying both equations yields $qq^{-1}=(1+py)(pm-1)\equiv$

-1 *mod* p, which is a contradiction (unless $p = \pm 2$). □

The case $p = \pm 2$ cannot be excluded in the Theorem, as it admits a pair of solutions. Solving $-2 = 2 - t + 2tm$ over the integers, one ends up with two solution pairs $(t, m) \in \{(4, 1), (-4, 0)\}$. Hence, we get two knots K, K^* in $L(2, 1)$ with a surgery giving $L(2, 1)\#L(4, 1)$ and $L(2, 1)\#L(-4, 1)$, respectively. In fact, these knots are mirror images of each other, as is demonstrated in the figure below.

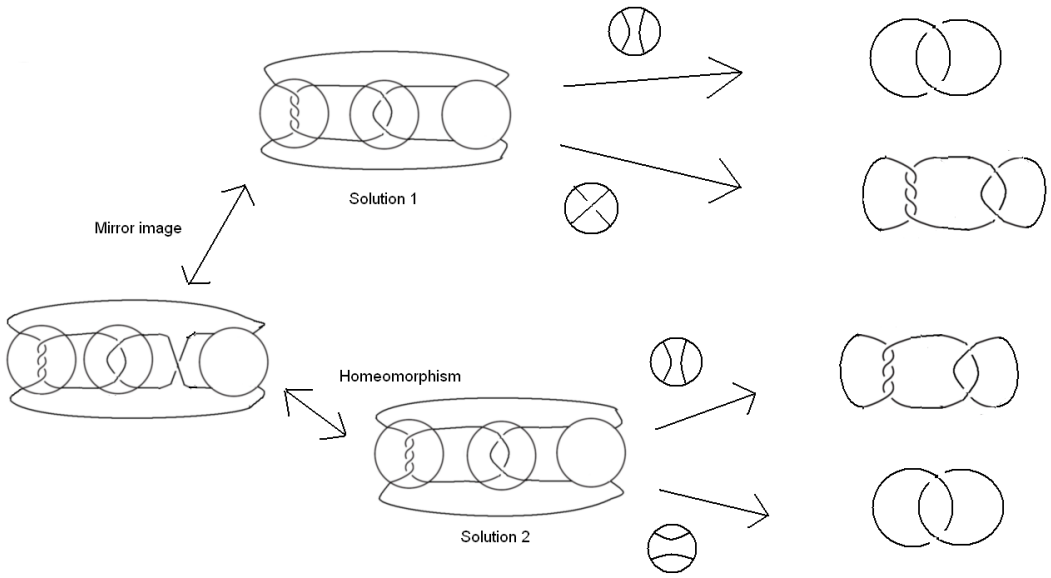


Figure 3.2: Lifting to double branched covers, we see a pair of knots K, K^* in $L(2, 1)$ with surgeries giving $L(2, 1)\#L(4, 1)$ and $L(2, 1)\#L(-4, 1)$, respectively. By applying an elementary reparametrisation of the boundary of the knot complement (induced by the homeomorphism in the figure), we see that the two knots are mirror images of each other.

Finally, we discuss the situation for hyperbolic knots. For knots in the three sphere, there is the famous **cabling conjecture**, formulated by González-Acuña and Short [29], which asserts that a knot $K \subset S^3$ has a reducible surgery if and only if it is a cable knot. So, hyperbolic knots are conjectured to not have reducible surgeries. If one demands that the surgered manifold be a connect-sum of lens spaces, then the conjecture has been established by Greene [32], by building on work of Matignon and Sayari [47]. Unfortunately, there is no hope for such a nice result to hold in an arbitrary lens space: there are examples of hyperbolic knots in lens spaces with connect-sum of lens space surgeries. The first infinite family was constructed by

Eudave-Muñoz and Wu in [20], and a tangle surgery picture of it is shown in the figure below. It is interesting to observe that the reducible manifold obtained is always $-L(3,1)\#L(2,1)$.

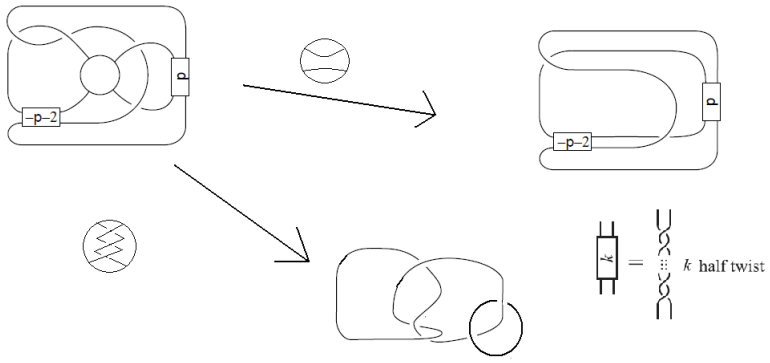


Figure 3.3: Examples of an infinite family of tangles whose double branched covers are complements of knots $K \subset L((p-1)(p-3)+1, p+3)$ having a $-L(3,1)\#L(2,1)$ surgery. Adapted from [20].

Recently, Kang [43] has constructed slightly more complicated examples of knots in lens spaces with connect-sum lens space surgeries. Their tangle surgery descriptions are shown in Figure 3.4.

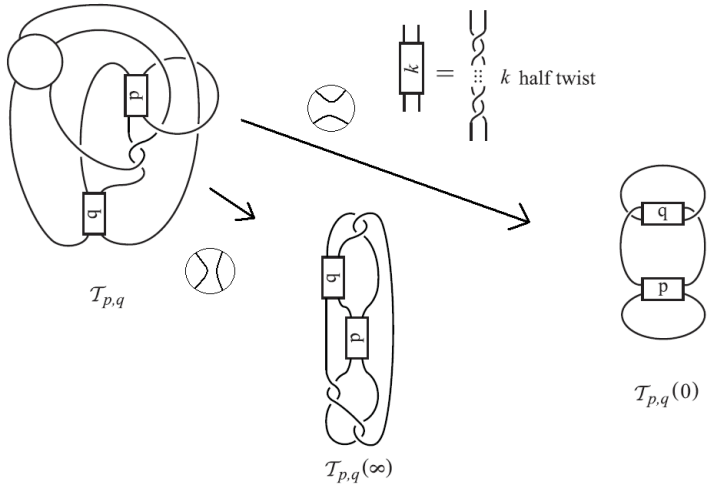


Figure 3.4: Examples of an infinite family of tangles $\mathcal{T}_{p,q}$, whose double branched covers are complements of knots $K_{p,q} \subset L((3p+2)(-2q+1)+6, (3p+2)q-3)$ with a reducible surgery giving $L(p,1)\#L(q-2,1)$. Adapted from [43].

What can one possibly hope to conjecture from this? Our understanding of the phenomenon of integral surgery on knots yielding reducible manifolds is quite limited,

as we do not have many sources of examples besides the "obvious" ones. Moreover, given what we do have, it is hard to spot a pattern. Perhaps the most salient feature from the hyperbolic examples is that they all produce reducible manifolds of the type $L(p,1)\#L(t,1)$. On the other hand, this is counterbalanced by the fact that in the Seifert examples one can easily produce $L(r,s)\#L(t,u)$ with arbitrary integer parameters. Notice, though, that none of these examples produce $L(r,s)\#L(t,1)$ from surgery on $L(r,s)$, except when $r=2$ and the knot has Seifert fibered exterior. Moreover, we saw that there are only two examples then, and they are mirror images of each other. Given this evidence, we conjecture the following:

Conjecture 3.3.3. *Let $K\subset L(r,s)$ be a knot such that $E(K)$ is irreducible. Suppose that $r\neq\pm1,\pm2$ (that is $L(r,s)$ is not $\mathbb{R}P^3$ or S^3). Then, surgery on K never produces $L(r,s)\#L(t,1)$.*

In the next section, we will prove this conjecture, provided we assume that $[K]\sim 0$ in $H_1(L(r,s);\mathbb{Z})$ and that the surgery coefficient is positive. Moreover, by imposing some additional number-theoretic constraints on r and t, we will be able to drop the assumptions on the homology class and on the sign of the surgery coefficient, whenever these constraints apply. Our results will hold for a class of rational homology spheres which includes the lens spaces, but is not limited to that. The technique we will use is Heegaard Floer homology, a quite radical departure from the geometric methods used so far, and it is to this that we turn next.

CHAPTER 4 HEEGAARD FLOER HOMOLOGY RESULTS

In this section, we study knots in lens spaces with connect-sum of lens space surgeries via Heegaard Floer homology. We begin by recalling the construction of this invariant of 3-manifolds, together with some of its most relevant and useful properties. In particular, we will focus heavily on their surgery exact sequence. Using this, we compute the Heegaard Floer homology for the zero-surgery of the knots we're interested in. Then, we will see how this relates to knot Floer homology [60], an invariant of knots that carries geometric information such as the genus, and whether a knot is fibered or not. This will lead to a proof of Conjecture 3.3.3 in the special cases mentioned at the end of the previous section. Then, we briefly discuss how we can use Heegaard Floer homology as an obstruction to obtaining certain connect-sums on lens spaces from surgery on certain knots in a lens space. We conclude by highlighting how the fact that knot Floer homology detects fibered knots can be used to show that certain knots in lens spaces constructed by Morimoto [51] and Baker [2] are characterised by their reducible surgeries.

We emphasise once again that all manifolds in this section are oriented.

4.1 Preliminaries on Heegaard Floer homology

In a now classic series of papers [56–59], Ozsváth and Szabó started what was perhaps the biggest thrust in 3-manifold topology over the last decade by introducing Heegaard Floer homology. This is an invariant of 3-manifolds, inspired by the revolutionary work on symplectic geometry of Floer [21–23] and Gromov [27]. Roughly speaking, the construction is based on an analysis of the critical points of a certain functional defined on a symplectic manifold that is associated to a 3-manifold Y. These critical points satisfy a certain "Cauchy-Riemann type" equation, and "counting" solutions to this equation, with appropriate boundary conditions, turns out to give a homology theory that is an invariant of Y. These invariants have enjoyed great success in answering several outstanding question regarding the topology of 3-manifolds, mainly due to

their computability. In the following, we briefly sketch the construction of Heegaard Floer homology, and we collect some of its elementary properties, which we will refer to extensively. All of these can be found scattered throughout Ozsváth and Szabó's foundational papers on the subject [56–58]. Also, several other detailed introductory articles are available, and for this the author suggests having a look at [48], [55], and [66].

The starting point of the construction is a Heegaard diagram for a closed, oriented 3-manifold Y. This is a triple $(\Sigma_g, \overrightarrow{\alpha}, \overrightarrow{\beta}, z)$, where Σ_g is a genus g Heegaard surface for Y, and $\overrightarrow{\alpha}$ (resp. $\overrightarrow{\beta}$) is a collection of pairwise disjoint, simple closed curves $\alpha_1, ..., \alpha_g$ (resp. $\beta_1, ..., \beta_g$) on Σ_g, linearly independent over $H_1(\Sigma_g; \mathbb{Z})$, which encode the instructions for attaching 3-dimensional 2-handles to $\Sigma_g \times [0, 1]$ so as to produce Y minus two balls, which we cap to produce Y (the handles corresponding to the α curves are attached to, say, $\Sigma_g \times \{0\}$, and the ones corresponding to the β curves are attached to $\Sigma_g \times \{1\}$). Notice that there is an additional piece of information, a point $z \in \Sigma_g$, the relevance of which will be explained soon. Associated to Σ_g, one produces the symmetric product $Sym^g(\Sigma_g) \equiv \underbrace{\Sigma_g \times ... \times \Sigma_g}_{\text{g times}}/S_g$, where S_g is the permutation group of g symbols. It can be shown that this is a complex manifold of real dimension $2g$. Moreover, it can be easily seen that the $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ curves miss the diagonal (thanks to the fact that they are pairwise disjoint), so that they lift to g-dimensional tori $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$ in $Sym^g(\Sigma_g)$. Since they are of complementary dimension, it can be arranged that they intersect transversally in a finite number of points, the set of which is denoted by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

One then defines $CF^\infty(Y)$ to be the free $\mathbb{Z}$-module generated by pairs $[x, i]$, where $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and $i \in \mathbb{Z}$. This is endowed with the action by a formal variable U, given by $U[x, i] = [x, i - 1]$, thereby turning $CF^\infty(Y)$ into a $\mathbb{Z}[U, U^{-1}]$-module.

After this, one must equip $CF^\infty(Y)$ with a boundary operator ∂^∞, in order to get a homology theory. We discuss, in the briefest of manners, how this is done: consider a map $u : \mathbb{D} \longrightarrow Sym^g(\Sigma_g)$, where $\mathbb{D}$ is the closed unit disk in the complex plane. Let $e_+ \subset \partial\mathbb{D}$ be the arc with $Re(z) \geq 0$, and let $e_- \subset \partial\mathbb{D}$ be the arc with $Re(z) \leq 0$. Given $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, a map u as above, satisfying the boundary conditions

$$u(-i) = x, \ u(i) = y$$

$$u(e_+) \subset \mathbb{T}_\alpha,\ u(e_-) \subset \mathbb{T}_\beta$$

is said to be a **Whitney disk** connecting x to y. One requires that these Whitney disks satisfy an additional analytical condition, which we describe for the sake of completeness: choose a path $(\mathcal{J}_s)_{s\in[0,1]}$ of almost-complex structures on $Sym^g(\Sigma_g)$. Identifying $\mathbb{D}\backslash\{\pm i\}$ with $[0,1]\times\mathbb{R}$, a Whitney disk u is said to be $\mathcal{J}_s$-holomorphic if it satisfies, for all $(s,t)\in[0,1]\times\mathbb{R}$, the following "Cauchy-Riemann type" equation

$$\frac{\partial u}{\partial s} + \mathcal{J}_s \frac{\partial u}{\partial t} = 0$$

Now, denote the set of homotopy classes of Whitney disks connecting x and y by $\pi_2(x,y)$. Then, given $\phi\in\pi_2(x,y)$, we consider the moduli space of $\mathcal{J}_s$-holomorphic Whitney disks in the homotopy class ϕ, and denote it by $\mathcal{M}_{\mathcal{J}_s}(\phi)$. It is readily seen that a constant shift $t\mapsto t+c$ induces a free $\mathbb{R}$ action on $\mathcal{M}_{\mathcal{J}_s}(\phi)$, so that one preferably works with a "smaller" version $\widehat{\mathcal{M}}_{\mathcal{J}_s}(\phi) = \mathcal{M}_{\mathcal{J}_s}(\phi)/\mathbb{R}$.

The moduli spaces of Whitney disks may seem like hopelessly complicated objects. However, if we restrict ourselves to disks with Maslov index $\mu(\phi)=1$, it is possible to show that $\widehat{\mathcal{M}}_{\mathcal{J}_s}(\phi)$ is a compact, zero-dimensional manifold (see [56], Theorem 3.4).

The base point z comes into the construction by considering the complex submanifold

$$V_z = \{z\}\times Sym^{g-1}(\Sigma_g) \hookrightarrow Sym^g(\Sigma_g)$$

Then, given $\phi\in\pi_2(x,y)$, we consider the intersection number $n_z(\phi)$ of the image of any representative u of ϕ with V_z. Since V_z is a complex manifold, this intersection can be shown to be always non-negative.

The boundary operator is then defined by counting the number of components in $\widehat{\mathcal{M}}_{\mathcal{J}_s}(\phi)$, in the following sense:

$$\partial^\infty[x,i] = \sum_{\{y\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1\}} \#\widehat{\mathcal{M}}_{\mathcal{J}_s}(\phi)[y, i-n_z(\phi)] \tag{4.1}$$

The count $\#\widehat{\mathcal{M}}_{\mathcal{J}_s}(\phi)$ is signed, in the sense that each component is endowed with a ±1. We won't describe how this sign is defined, and refer the reader to [56] for details. Now, it is far from obvious from the definition that this should square to zero, but it is shown in [56] that this is the case. The resulting homology theory is

denoted by $HF^\infty(Y)$. Again, it is also shown in [56] that this doesn't depend on the various choices made in the construction, so that this is a *bona fide* invariant of 3-manifolds. Moreover, note that the U action on $CF^\infty(Y)$ is a chain map, so that $HF^\infty(Y)$ is also a $\mathbb{Z}[U, U^{-1}]$-module.

Heegaard Floer homology has the additional extremely useful property of splitting along $Spin^c$ structures. Rather than give the usual definition of these involving principal $Spin^c$-bundles, which is valid in any dimension, (see [4], for example), we give an equivalent formulation for 3-manifolds, introduced by Turaev in [70]:

Definition 4.1.1. *Given an oriented 3-manifold Y, a $Spin^c$ structure on Y is a homotopy class of vector field which is nowhere-vanishing in the complement of finitely many balls.*

We denote the set of all $Spin^c$ structures on a manifold Y by $Spin^c(Y)$. It is well-known that this is a principal homogeneous space over $H^2(Y;\mathbb{Z})$. That is, if we fix a "reference" $Spin^c$ structure $\mathfrak{t}_0$, then we can write any other $Spin^c$ structure $\mathfrak{t}$ as $\mathfrak{t} = \mathfrak{t}_0 + g$, where $g \in H^2(Y, \mathbb{Z})$. In particular, we can think as the "difference" between two $Spin^c$ structures as a well-defined element in second cohomology. Moreover, $Spin^c$ structures on Y are in one-to-one correspondence with elements in $H^2(Y;\mathbb{Z})$.

To any $Spin^c$ structure $\mathfrak{t}$ on a manifold Y, we may consider its first Chern class $c_1(\mathfrak{t}) \in H^2(Y;\mathbb{Z})$. If Y is a 3-manifold, we can think of this arising as follows: taking a vector field v representing $\mathfrak{t} \in Spin^c(Y)$, define the conjugate $Spin^c$ structure $\bar{\mathfrak{t}}$ as being represented by the vector field $-v$. Then, $c_1(\mathfrak{t}) = \mathfrak{t} - \bar{\mathfrak{t}}$ (see [56], Section 3).

We now briefly sketch how the splitting of Heegaard Floer homology along $Spin^c$ structures comes to be: it is not hard to show that, in $\mathbb{Z}$ coefficients,

$$\frac{H_1(Sym^g(\Sigma_g))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \simeq \frac{H_1(\Sigma)}{[\alpha_1]..[\alpha_g][\beta_1]..[\beta_g]} \simeq H_1(Y)$$

Now, pick paths $a \subset \mathbb{T}_\alpha$ and $b \subset \mathbb{T}_\beta$ joining $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. These determine a class in $H_1(Sym^g(\Sigma_g))$, which depends on the choice of paths. However, quotienting out by $H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)$ gets rid of this dependence, so that the net result is a well-defined map

$$\epsilon : (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \longrightarrow H_1(Y)$$

Thus, there will be Whitney disks connecting x and y only if $\epsilon(x,y) = 0$. Now, by thinking of $Spin^c$ structures on 3-manifolds as homotopy classes of vector fields, Ozsváth and Szabó construct, in section 3 of [56], a map $\mathfrak{s}_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow Spin^c(Y)$ with the property that $\mathfrak{s}_z(y) - \mathfrak{s}_z(x) = PD(\epsilon(x,y))$. It follows from this that intersection points connected by Whitney disks must land on the same $Spin^c$ structure, so that $CF^\infty(Y)$ splits a a sum of subcomplexes $CF^\infty(Y,\mathfrak{t})$ indexed by $Spin^c$ structures, and consequently, we have a splitting

$$HF^\infty(Y) = \bigoplus_{\mathfrak{t} \in Spin^c(Y)} HF^\infty(Y,\mathfrak{t})$$

It is possible to derive other Heegaard Floer homology theories from HF^∞. The non-negativity of $n_z(\phi)$ ensures that the submodule $CF^-(Y,\mathfrak{t}) \subset CF^\infty(Y,\mathfrak{t})$, generated by elements $[x,i]$ with i strictly negative, inherits the structure of a subcomplex, giving rise to a "minus" Floer homology theory $HF^-(Y,\mathfrak{t})$. One can then also consider the quotient complex $CF^+(Y,\mathfrak{t}) = CF^\infty(Y,\mathfrak{t})/CF^-(Y,\mathfrak{t})$, and this also gives rise to the "plus" Floer homology, $HF^+(Y,\mathfrak{t})$. Finally, one can also construct a "simpler" version, $\widehat{HF}(Y,\mathfrak{t})$, by considering a chain complex formed by elements of the form $[x,0]$, by declaring the U action to be trivial, and by restricting the count of holomorphic disks in equation (4.1) to homotopy classes $\phi \in \pi_2(x,y)$ with $n_z(\phi) = 0$. It is easy to see from this that there are short exact sequences

$$0 \longrightarrow \widehat{CF}(Y,\mathfrak{t}) \xrightarrow{\iota} CF^+(Y,\mathfrak{t}) \xrightarrow{U^+} CF^+(Y,\mathfrak{t}) \longrightarrow 0$$

and

$$0 \longrightarrow CF^-(Y,\mathfrak{t}) \xrightarrow{\iota} CF^\infty(Y,\mathfrak{t}) \xrightarrow{\pi} CF^+(Y,\mathfrak{t}) \longrightarrow 0$$

giving rise to exact triangles

$$\begin{array}{ccc} \widehat{HF}(Y,\mathfrak{t}) & \longrightarrow & HF^+(Y,\mathfrak{t}) \\ & \nwarrow \quad \swarrow & \\ & HF^+(Y,\mathfrak{t}) & \end{array} \tag{4.2}$$

and

$$\begin{array}{ccc} HF^-(Y,\mathfrak{t}) & \longrightarrow & HF^\infty(Y,\mathfrak{t}) \\ & \nwarrow \quad \swarrow & \\ & HF^+(Y,\mathfrak{t}) & \end{array} \tag{4.3}$$

4.1.1 Maps from cobordisms and surgery exact sequences

One aspect of the theory that is extremely useful is its functoriality under cobordisms. To be precise, given a cobordism $W : Y_1 \longrightarrow Y_2$ between 3-manifolds Y_1 and Y_2, and equipping W with a $Spin^c$ structure $\mathfrak{s}$, we may consider the restriction of $\mathfrak{s}$ to the boundary components, $\mathfrak{t}_1 = \mathfrak{s}|_{Y_1} \in Spin^c(Y_1)$ and $\mathfrak{t}_2 = \mathfrak{s}|_{Y_2} \in Spin^c(Y_2)$. With this in place, one can construct certain maps induced by W:

$$F^\circ_{W,\mathfrak{s}} : HF^\circ(Y_1, \mathfrak{s}|_{Y_1}) \longrightarrow HF^\circ(Y_2, \mathfrak{s}|_{Y_2}) \tag{4.4}$$

The superscript $\circ$ stands for any of the Heegaard Floer theories. These maps are obtained by "counting holomorphic triangles", which can be thought of as a generalisation of counting Whitney disks in the definition of ∂° for HF°. We will largely avoid discussing these, and refer the reader to [59] for details. With this said, the maps induced from cobordisms enjoy some very strong naturality properties with respect to the triangles (4.2) and (4.3), as is reflected in the following theorem:

Theorem 4.1.2. *([58]) Suppose that $W : (Y_1, \mathfrak{t}_1) \longrightarrow (Y_2, \mathfrak{t}_2)$ is a cobordism, and let $\mathfrak{s} \in Spin^c(W)$ extend $\mathfrak{t}_1$ and $\mathfrak{t}_2$. Then, we have the following commutative diagram*

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & HF^-(Y_1,\mathfrak{t}_1) & \xrightarrow{\iota} & HF^\infty(Y_1,\mathfrak{t}_1) & \xrightarrow{\pi} & HF^+(Y_1,\mathfrak{t}_1) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle F^-_W} & & \downarrow{\scriptstyle F^\infty_W} & & \downarrow{\scriptstyle F^+_W} & & \\ \ldots & \longrightarrow & HF^-(Y_2,\mathfrak{t}_2) & \xrightarrow{\iota} & HF^\infty(Y_2,\mathfrak{t}_2) & \xrightarrow{\pi} & HF^+(Y_2,\mathfrak{t}_2) & \longrightarrow & \ldots \end{array} \tag{4.5}$$

We also have a commutative diagram

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \widehat{HF}(Y_1,\mathfrak{t}_1) & \xrightarrow{\iota} & HF^+(Y_1,\mathfrak{t}_1) & \xrightarrow{U^+} & HF^+(Y_1,\mathfrak{t}_1) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle \widehat{F}_{W,\mathfrak{s}}} & & \downarrow{\scriptstyle F^+_{W,\mathfrak{s}}} & & \downarrow{\scriptstyle F^+_{W,\mathfrak{s}}} & & \\ \ldots & \longrightarrow & \widehat{HF}(Y_2,\mathfrak{t}_2) & \xrightarrow{\iota} & HF^+(Y_2,\mathfrak{t}_2) & \xrightarrow{U^+} & HF^+(Y_2,\mathfrak{t}_2) & \longrightarrow & \ldots \end{array} \tag{4.6}$$

Besides providing a deep connection with 4-manifold topology, the cobordism maps allow one to turn HF° into graded invariants in certain situations. We begin by recalling that, when Y is equipped with a torsion $Spin^c$ structure $\mathfrak{s}$ (that is, one for which $c_1(\mathfrak{s}) \in H^2(Y;\mathbb{Z})$ is a torsion element), one can define a relative $\mathbb{Z}$ grading on the homogeneous elements of $HF^\circ(Y,\mathfrak{s})$, and this is denoted by $gr(x,y)$ (see [56]). In fact, it is always possible, without requiring that $\mathfrak{s}$ be torsion, to define an absolute $\mathbb{Z}_2$ grading, but we will have no need for it here (see [57] for details). The cobordism map allows us to lift this relative grading to an absolute $\mathbb{Q}$ grading, whilst also providing us with a recipe to compute it, as is described by the following fundamental result proved in [58]. For convenience, we state it only for rational homology spheres:

Theorem 4.1.3. *Let Y be a rational homology sphere. Then, its Floer homology invariants $HF^\circ(Y,\mathfrak{t})$ can be endowed with a $\mathbb{Q}$ grading, $\widetilde{gr}$, uniquely characterised by the following properties:*

- *Homogeneous elements in $HF^+(S^3)$ in the kernel of $U^+ : HF^+(S^3) \longrightarrow HF^+(S^3)$ have grading zero (this can be seen as a normalisation procedure).*
- *$gr(x,y) = \widetilde{gr}(x) - \widetilde{gr}(y)$.*
- *$\widetilde{gr}$ is preserved by the maps ι, π, while U decreases $\widetilde{gr}$ by two, and the coboundary maps decrease $\widetilde{gr}$ by one.*
- *Suppose that $W : (Y_1,\mathfrak{t}_1) \longrightarrow (Y_2,\mathfrak{t}_2)$ is a cobordism between rational homology spheres, and let $\mathfrak{s} \in Spin^c(W)$ extend $\mathfrak{t}_1$ and $\mathfrak{t}_2$. Then, for $\xi \in HF^\circ(Y_1,\mathfrak{t}_1)$, we have*

$$\widetilde{gr}(F^\circ_{W,\mathfrak{s}}(\xi)) - \widetilde{gr}(\xi) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4} \tag{4.7}$$

where χ is the Euler characteristic and σ is the signature.

4.1.2 The surgery exact sequence

We will now be mainly interested in HF^+, though we will refer back to the other theories occasionally. The invariants HF° are usually very difficult to compute from the definition, as they essentially involve solving a partial differential equation. Nevertheless, there is an extremely useful tool that facilitates calculations, which comes in the form of a **surgery exact sequence**. Given a nullhomologous knot $K \subset Y$, this sequence relates the HF^+ invariants of Y and those of the manifolds obtained by integral Dehn surgery on K. Let Y_i denote the result of performing i-framed Dehn surgery on $K \subset Y$. Recall that this gives rise to a cobordism $W^i : Y \longrightarrow Y_i$, interpreted as adding an i- framed four-dimensional 2-handle to $Y \times [0,1]$ along $K \subset Y \times \{1\}$. One may reverse the orientation and consider $W_i : Y_i \longrightarrow Y$ instead. Now, given a $Spin^c$ structure $\mathfrak{t}_{Y_i}$ on Y_i, there exists a unique $Spin^c$ structure $\mathfrak{t}_Y$ such that the pair $(\mathfrak{t}_{Y_i}, \mathfrak{t}_Y)$ extends to a $Spin^c$ structure on W_i. Letting $S(\mathfrak{t}_{Y_i}, \mathfrak{t}_Y)$ denote the set of $Spin^c$ structures on W_i that extend a given pair $(\mathfrak{t}_{Y_i}, \mathfrak{t}_Y)$, one may consider the map

$$F_{W_i,\mathfrak{t}_{Y_i}} = \sum_{\mathfrak{s} \in S(\mathfrak{t}_{Y_i}, \mathfrak{t}_Y)} F^+_{W_i,\mathfrak{s}}$$

where individual summands are the suitable maps induced by the surgery cobordism W_i, in the sense of Equation (4.4). Even though this sum is infinite, it can be shown to be well-defined (see [57, 58]).

Let $p > 0$ be an integer. One may also consider a map between $HF^+(Y_0)$ and $HF^+(Y_p)$, in a similar spirit to the maps described before. Coarsely speaking, one constructs a cobordism X between $-(Y_0 \sqcup L(p,1))$ and Y_p, and then tries to count holomorphic triangles with appropriate boundary conditions. To get a well-defined map between $HF^+(Y_0)$ and $HF^+(Y_p)$, a $Spin^c$ structure $\mathfrak{u}$ on $L(p,1)$ must be fixed, and we refer the reader to [57] for a canonical way of doing this. With this in place, it can be shown that, given $\mathfrak{t}_0 \in Spin^c(Y_0)$, there exists a unique $Spin^c$ structure $\mathfrak{t}_{Y_p}$ on Y_p such that the triple $(\mathfrak{u}, \mathfrak{t}_0, \mathfrak{t}_{Y_p})$ extends to a $Spin^c$ structure across X, and that every $\mathfrak{t}_{Y_p}$ is realised in this way. With the "canonical" choice of $\mathfrak{u}$ understood, this procedure defines a surjective assignment

$$Q : Spin^c(Y_0) \longrightarrow Spin^c(Y_p) \tag{4.8}$$

Now, given a fixed $\mathfrak{t}_{Y_p} \in Spin^c(Y_p)$, let

$$HF^+(Y_0, [\mathfrak{t}_{Y_p}]) = \sum_{\mathfrak{t} \in Q^{-1}(\mathfrak{t}_{Y_p})} HF^+(Y_0, \mathfrak{t})$$

With this notation, one can finally define a map

$$M_{\mathfrak{t}_{Y_p}} : HF^+(Y_0, [\mathfrak{t}_{Y_p}]) \longrightarrow HF^+(Y_p, \mathfrak{t}_{Y_p})$$

by, roughly speaking, summing all the holomorphic triangle counts obtained by varying the triple $(\mathfrak{u}, \mathfrak{t}, \mathfrak{t}_{Y_p})$ over all $\mathfrak{t} \in Q^{-1}(\mathfrak{t}_{Y_p})$. The statement of the surgery exact sequence is then:

Theorem 4.1.4. *([57], Theorem 9.19) Let $K \subset Y$ be a nullhomologous knot, and let p be a positive integer. Consider $\mathfrak{t}_{Y_p} \in Spin^c(Y_p)$, and let $\mathfrak{t}_Y \in Spin^c(Y)$ be the $Spin^c$ structure such that the pair $(\mathfrak{t}_{Y_p}, \mathfrak{t}_Y)$ extends to a $Spin^c$ structure on W_p. Then, there is the following U^+ equivariant exact sequence*

$$... \longrightarrow HF^+(Y_0, [\mathfrak{t}_{Y_p}]) \overset{M_{\mathfrak{t}_{Y_p}}}{\longrightarrow} HF^+(Y_p, \mathfrak{t}_{Y_p}) \overset{F_{W_p, \mathfrak{t}_{Y_p}}}{\longrightarrow} HF^+(Y, \mathfrak{t}_Y) \longrightarrow ... \tag{4.9}$$

In fact, Ozsváth and Szabó only state the theorem for knots in an integral homology sphere, but remark that the more general form which we use also holds. See [36], Section 4.3, for a related sequence involving $\widehat{HF}$.

Remark 4.1.5. It turns out that the theorem is proved by constructing the maps at the chain level, and then by showing that there is a certain short exact sequence of chain complexes at the level of CF^+, from which the result follows. With this understood, the unlabelled map corresponds to the connecting homomorphism of this short exact sequence.

There is also a sequence for negative surgeries, which takes the following similar form:

Theorem 4.1.6. *([57], remark after Theorem 9.19) For $p < 0$ integral surgeries, we have an exact sequence*

$$... \longrightarrow HF^+(Y_0, [\mathfrak{t}_{Y_p}]) \longrightarrow HF^+(Y, \mathfrak{t}_Y) \overset{F_{W_p, \mathfrak{t}_{Y_p}}}{\longrightarrow} HF^+(Y_p, \mathfrak{t}_{Y_p}) \longrightarrow ... \tag{4.10}$$

4.1.3 The correction term

The final ingredient in Heegaard Floer homology that we will need to discuss is the invariant known as the **correction term**, which we introduce in the following: let Y be a rational homology sphere. Then, according to Theorem 4.1.3, its HF° invariants admit an absolute $\mathbb{Q}$ grading. Recall the grading-preserving map

$$HF^\infty(Y,\mathfrak{t}) \xrightarrow{\pi} HF^+(Y,\mathfrak{t})$$

Definition 4.1.7. *The* ***correction term*** $d(Y,\mathfrak{t})$ *for* $HF^+(Y,\mathfrak{t})$ *is defined to be the minimal absolute grading of any non-torsion element* $\xi \in HF^+(Y,\mathfrak{t})$ *belonging to* $Im(\pi)$.

Correction terms behave nicely with respect to connect-sums. Consider the manifold $Y_1\#Y_2$. Then, interpreting $Spin^c$ structures as homotopy classes of vector fields, we see that a $Spin^c$ structure $\mathfrak{t}$ in $Y_1\#Y_2$ naturally gives rise to $Spin^c$ structures $\mathfrak{t}_i$ in Y_i. The converse can also be seen to hold. Therefore, we can write $\mathfrak{t} \in Spin^c(Y_1\#Y_2)$ as $\mathfrak{t}_1\#\mathfrak{t}_2$. With this notation, we have the following result.

Theorem 4.1.8. *([59], Theorem 4.3)* $d(Y_1\#Y_2,\mathfrak{t}_1\#\mathfrak{t}_2) = d(Y_1,\mathfrak{t}_1) + d(Y_2,\mathfrak{t}_2)$.

Two other properties of correction terms that are quite useful are the following:

Theorem 4.1.9. *([59], Proposition 4.2)* $d(Y,\mathfrak{t}) = d(Y,\bar{\mathfrak{t}})$ *and* $d(-Y,\mathfrak{t}) = -d(Y,\mathfrak{t})$.

It is sometimes useful to consider a finitely generated version of HF^+, known as the reduced Floer homology. It is defined as $HF^+_{red}(Y,\mathfrak{t}) \simeq HF^+(Y,\mathfrak{t})/Im(\pi)$. Now, there is a class of rational homology spheres for which the correction term completely determines the HF^+ invariants, and this will be pivotal in the rest of this section:

Definition 4.1.10. *A rational homology sphere* Y *for which* $HF^+_{red}(Y,\mathfrak{t}) = 0$, *for all* $\mathfrak{t} \in Spin^c(Y)$, *is said to be an* ***L-space***.

It is shown in [56] that, for any rational homology sphere Y, $HF^\infty(Y,\mathfrak{t}) \simeq \mathbb{Z}[U,U^{-1}]$. Then, it is easy to see that, whenever $HF^+_{red}(Y,\mathfrak{t}) = 0$, $HF^+(Y,\mathfrak{t}) \simeq \mathbb{Z}[U,U^{-1}]/U\cdot\mathbb{Z}[U]$. It is very useful to think of this as Laurent polynomials with strictly non-positive powers, with the homogeneous element a_nU^{-n} supported in degree $d(Y,\mathfrak{t}) + 2n$. We will adopt this point of view in the subsequent discussion.

The class of L-spaces turns out to be quite broad. In particular, we have the following key facts (see [56–58] for the proofs).

Theorem 4.1.11. *The family of L-spaces contains the lens spaces, and it is closed under connect-sums.*

Having laid out all the tools from Heegaard Floer homology that we need, we can begin with the program described at the beginning of the chapter. Recall that this involves two steps:

1. Calculation of $HF^+(Y_0)$.
2. Relating $HF^+(Y_0)$ to the knot genus $g(K)$.

The strategy is that $HF^+(Y_0)$ will give enough information to conclude, under certain hypotheses, that $g(K) = 0$, which of course shows that K must be the unknot. Item (1) will involve using the surgery exact sequence, whereas item (2) will involve a relationship of $HF^+(Y_0)$ with knot Floer homology, an invariant of knots which we discuss in a later section.

4.2 Calculation of $HF^+(Y_0)$

In [55], Ozsváth and Szabó sketch an argument for this calculation in the case of knots in S^3. We will see that, after taking care of a few minor issues arising from the presence of torsion, that their argument can be quite readily adapted to the case where Y is an arbitrary L-space, and K is a nullhomologous knot. From now on, Y will taken to be a rational homology sphere.

Since the properties of the map connecting $HF^+(Y_p)$ and $HF^+(Y)$ depend heavily on the algebraic topology of $W_p : Y_p \longrightarrow Y$, we describe it in what follows. Recall from the previous subsection that W_p is actually the orientation-reversed version of the usual 2-handle cobordism W^p induced by integral surgery.

Lemma 4.2.1. *Let W^p be the surgery cobordism induced by doing integral $p > 0$ surgery on nullhomologous knot $K \subset Y$, where Y is a rational homology sphere. Then, $\sigma(W^p) = 1$ and $\chi(W^p) = 1$.*

Proof. Since we're calculating homotopy invariants, we may as well work at the level of cell complexes. Up to homotopy, $W^p = Y \cup D^2$, and $Y \cap D^2$ is a circle. Recall

that $\chi(Y) = 0$ for any closed orientable 3-manifold. Then, $\chi(W^p) = \chi(Y) + \chi(D^2) - \chi(S^1)$=1. Now, consider the following portion of the Mayer-Vietoris sequence for $W^p = Y \cup D^2$:

$$0 \longrightarrow H_2(Y) \longrightarrow H_2(W^p) \longrightarrow H_1(S^1) \longrightarrow H_1(Y) \longrightarrow H_1(W^p) \longrightarrow H_0(S^1)$$

Now, $H_2(Y) = 0$, since Y is a rational homology sphere. Also, $H_1(W^p)$ is the quotient of $H_1(Y)$ by adjoining the relation $p[K] = 0$, where $[K] \in H_1(Y)$. Since K is nullhomologous, we have that $H_1(W^p) \simeq H_1(Y)$, which is torsion. So we have

$$0 \longrightarrow H_2(W) \longrightarrow \mathbb{Z} \longrightarrow H_1(Y) \longrightarrow H_1(Y) \longrightarrow \mathbb{Z}$$

Since $H_1(Y)$ is torsion, the map $H_1(Y) \longrightarrow \mathbb{Z}$ is trivial. Now, the map $H_1(Y) \longrightarrow H_1(Y)$ is a surjective map between two finite sets of the same size, so it must be an injection. Hence the sequence breaks in the middle, yielding $0 \longrightarrow H_2(W^p) \longrightarrow \mathbb{Z} \longrightarrow 0$, so $H_2(W^p) = \mathbb{Z}$. Hence $\sigma(W^p) = \pm 1$. Since we're doing positive surgery, the intersection form for $H_2(W^p)$ is positive definite, which in this case is the same as saying that $\sigma(W^p) = 1$. □

Notice that, since W_p is just W^p with orientation reversed, we have that $\sigma(W_p) = -1$.

Now, we would like a more friendly way to label the $Spin^c$ structures on W_p, and to understand how they restrict to the boundary components. Recall that $Spin^c$ is an affine space over the second cohomology with integer coefficients. That is, given a reference $\mathfrak{s}_o \in Spin^c(W_p)$, any other $\mathfrak{s}$ can be written as $\mathfrak{s} = \mathfrak{s}_o + \omega$, for $\omega \in H^2(W_p; \mathbb{Z})$. Moreover, we have the following relation when restricting $Spin^c$ structures to boundary components: $(\mathfrak{s}_o + \omega)|_\partial = \mathfrak{s}_o|_\partial + i^*(\omega)$, where i^* is the map on cohomology induced by inclusion. In addition to this, there is the well-known first Chern class formula (see [4])

$$c_1(\mathfrak{s}_o + \omega) = c_1(\mathfrak{s}_o) + 2\omega \tag{4.11}$$

We know from the universal coefficient theorem that $H^2(W_p; \mathbb{Z}) \simeq \mathbb{Z} \oplus H^2(Y)$, but it will help to have a specific choice of generators, especially one for the $\mathbb{Z}$ factor.

This is best understood in terms of second homology, translated through Poincaré duality. Consider the following Poincaré duality commutative diagram ($\mathbb{Z}$ coefficients throughout)

$$\begin{array}{ccc} H^2(W_p) & \xrightarrow{i^*} & H^2(\partial W_p) \\ \downarrow \cap[M] & & \downarrow \cap[\partial M] \\ H_2(W_p, \partial W_p) & \xrightarrow{\partial_*} & H_1(\partial W_p) \end{array}$$

The group $H_2(W_p, \partial W_p) \simeq \mathbb{Z} \oplus H_1(Y)$ can be thought of being generated by the relative 2-chain $[\mu \times I] \subset Y \times I \subset W_p$, where μ is a meridian of K, together with relative 2-chains $[g \times I]$, where $g \times \{0\}$ is a curve representing the homology class $[g] \in H_1(Y)$, and $g \times \{1\}$ represents $[g] \in H_1(Y_p)$. The former generates the $\mathbb{Z}$ factor, whereas the latter generate the $H_1(Y)$ summand. Also, notice that $\partial_*([\mu \times I])$ restricted to the $H_1(Y_p)$ factor gives us a choice of generator for the $\mathbb{Z}_p$ summand of $H_1(Y_p)$ (represented by $\mu \subset Y_p$, of course). Put together, this gives us a concrete way of thinking of the map $H_2(W_p, \partial W_p) \xrightarrow{\partial_*} H_1(Y) \oplus H_1(Y_p)$ as

$$(n, g) \xrightarrow{\partial_*} (g, g, n \ mod \ p)$$

Now, applying Poincaré duality, we get a choice of generator $PD([\mu \times I])$ for the $\mathbb{Z}$ factor of $H^2(W_p)$, and $i^*(PD([\mu \times I])) = PD(\mu)$ becomes a generator for the $\mathbb{Z}_p$ factor in $H^2(Y_p)$. A similar remark holds for the $H^2(Y)$ summand of $H^2(W_p)$, but we won't worry much about it, since torsion elements will play a minor role in what is to follow. Hence, we can also think about the map i^* as

$$(n, g) \xrightarrow{i^*} (g, g, n \ mod \ p)$$

This lets us see that, for a torsion element $g \in H^2(W_p)$ and for $\mathfrak{s} \in Spin^c(W_p)$, $(\mathfrak{s}+g)|_Y = \mathfrak{s}|_Y + g$, whereas $(\mathfrak{s}+kPD([\mu \times I]))|_Y = \mathfrak{s}|_Y$ so that adding torsion elements to $\mathfrak{s}$ changes the underlying $Spin^c$ structure $\mathfrak{s}|_Y$ in the surgery exact sequence (4.9), but adding multiples of $PD([\mu \times I])$ does not.

Now that we have generators for $H^2(W_p)$ and $H^2(Y_p)$, it will be useful to have a reference $Spin^c$ structure on W_p. Recall that $H_2(W_p) \simeq \mathbb{Z}$, and this is generated by a

surface Σ, obtained by capping a Seifert surface for the knot K with the core of the 2-handle used to define W_p. Now, it is shown in Lemma 7.11 of [59] that there exists a $Spin^c$ structure $\mathfrak{s}_o$ on W_p with the property that $\langle c_1(\mathfrak{s}_o), [\Sigma]\rangle = p$. In fact, since adding torsion doesn't affect this evaluation, there is one such $Spin^c$ structure per element $g \in H^2(Y)$, and in view of the remarks above, among all $Spin^c$ structures on W_p that lift a given $\mathfrak{t} \in Spin^c(Y)$, there is exactly one with this property.

With all of this said, we are going to relabel things in the surgery exact sequence (4.9). The following is essentially an adaptation of the discussion in [55]. Fix a $Spin^c$ structure $\mathfrak{t}$ on Y, and consider its unique lift to W_p, $\mathfrak{s}_{o,\mathfrak{t}}$, with the property that $\langle c_1(\mathfrak{s}_{o,\mathfrak{t}}), [\Sigma]\rangle = p$. This induces a $Spin^c$ structure on Y_p, which we label as $(0, \mathfrak{t})$. Now, adding multiples of $PD([\mu \times I])$ to $\mathfrak{s}_{o,\mathfrak{t}}$ will give us different $Spin^c$ structures on Y_p compatible with $\mathfrak{t}$. Writing $\mathfrak{s}_{N,\mathfrak{t}} = \mathfrak{s}_{o,\mathfrak{t}} + N \cdot PD([\mu \times I])$, and noticing that $p \cdot PD([\mu \times I]) \in ker\ i^*$, we see that the restriction of $\mathfrak{s}_{N,\mathfrak{t}}$ to Y_p depends only on the value of N *mod* p. So, given $\mathfrak{t}_{Y_p} \in Spin^c(Y_p)$, we label it by $(i, \mathfrak{t})$, where $i \in \mathbb{Z}_p$ is the reduction on N modulo p. Along these lines, we write the cobordism map induced by $\mathfrak{s}_{N,\mathfrak{t}}$ as $F^+_{N,\mathfrak{t}}$, so that the map $F_{W_p,\mathfrak{t}_{Y_p}}$ in (4.9) is written as

$$F_{W_p,\mathfrak{t}_{Y_p}} = F^+_{(i,\mathfrak{t})} = \sum_{k\in\mathbb{Z}} F^+_{i+kp,\mathfrak{t}}$$

Next, we calculate the degree shift from Equation (4.7) induced by each $F^+_{i+kp,\mathfrak{t}}$. From Lemma 4.2.1 and Theorem 4.1.3, this is just

$$\frac{c_1(\mathfrak{s})^2 + 1}{4} \tag{4.12}$$

We need to say a word about squaring classes ω in $H^2(W_p; \mathbb{Z})$. Usually, when the 4-manifold W is closed, one simply defines ω^2 as $\langle \omega \smile \omega, [W]\rangle$. This is integer-valued, and vanishes on torsion elements. When W has boundary, one must work with relative cohomology classes for the above evaluation to make sense. Interestingly enough, one ends up with a $\mathbb{Q}$-valued form. To define squaring on $H^2(W_p; \mathbb{Z})$, we begin by defining it on the kernel of i^*, where there is an identification with $H^2(W, \partial W)$, and then extend to obtain a quadratic form. We proceed as follows: consider the following portion of the long exact sequence of cohomology for W_p:

$$0 \longrightarrow H^2(W, \partial W) \xrightarrow{j^*} H^2(W) \xrightarrow{i^*} H^2(\partial W)$$

To simplify notation, let $PD([\mu \times I]) = G$. We know that $i^*(pG) = 0$, so $pG = j^*(H)$, where H is a generator of $H^2(W, \partial W)$, chosen to be Poincaré dual to $[\Sigma]$, and uniquely determined since j^* injects. Define the square $(pG)^2 = \langle H \smile H, [W, \partial W]\rangle = [\Sigma]\bullet[\Sigma] = -p$, where the second equality follows from Poincaré duality (here, $\bullet$ denotes the intersection form). Demanding that this be quadratic, we see that $G^2 = -\frac{1}{p}$, and more generally, that $(nG)^2 = -\frac{n^2}{p}$. Now, it can be seen that this operation doesn't depend on adding torsion elements $g \in H^2(W_p, \mathbb{Z})$. Consider $G + g$, and let $o(g)$ denote the order of g. Then, $(o(g)(G + g))^2 = (o(g)(G))^2 = -\frac{o(g)^2}{p}$. Hence, if we demand that this squaring operation be quadratic on the whole of $H^2(W_p; \mathbb{Z})$, we must have $(G + g)^2 = G^2$.

Now, we need to determine $c_1(\mathfrak{s}_{N,\mathfrak{t}})$. Since $\langle c_1(\mathfrak{s}_{o,\mathfrak{t}}), [\Sigma]\rangle = p$, and $\langle G, [\Sigma]\rangle = -1$, we must have that $c_1(\mathfrak{s}_{N,\mathfrak{t}}) = -(p + 2(i + kp))G + g$, where g is some torsion element (and of course, varying through all g one jumps in between all $\mathfrak{t}$). From this, we conclude that

$$c_1(\mathfrak{s}_{N,\mathfrak{t}})^2 = \frac{-(2(i + kp) + p)^2}{p}$$

whence the map $F^+_{i+kp,\mathfrak{t}}$ shifts degree, just like in the S^3 case, by

$$\frac{p - (2(i + kp) + p)^2}{4p} \tag{4.13}$$

Now we are left with the task of writing $HF^+(Y_0)$ in terms of this labelling. Here, $Spin^c(Y_0) \simeq \mathbb{Z} \oplus H^2(Y)$. We (partially) identify $\mathfrak{m} \in Spin^c(Y_0)$ with $\mathbb{Z}$ as follows: a closed surface $[\hat{F}]$ representing a generator for $H_2(Y_0) \simeq \mathbb{Z}$ may be obtained by capping a Seifert surface F for K with the core of the solid torus used to perform Dehn surgery (since the framing is zero, it does give an embedded surface). Now, $\langle c_1(\mathfrak{m}), [\hat{F}]\rangle = 2j$, for some integer j. Moreover, we can realise the coboundary map in (4.9) as a map induced by the cobordism associated to the 0-surgery, $W_0 : Y \longrightarrow Y_0$ (this is not used in the proof of the surgery exact sequence, but it is remarked in [57], Section 7, that it can be done). So, $\mathfrak{m}$ must be compatible with $\mathfrak{t} \in Spin^c(Y)$. With this said, the pair $(j, \mathfrak{t})$ determines $\mathfrak{m}$, and so we label it this way. Now, we would

like a nice description of the map $Q : Spin^c(Y_0) \longrightarrow Spin^c(Y_p)$ in (4.8) in terms of our labeling conventions. Indeed, there is such a nice description (almost as nice as we could hope for), as is given in the following result that we quote:

Proposition 4.2.2. *([57], Lemma 7.10) With the choices above, $Q(j, \mathfrak{t})$ extends across W as a $Spin^c$ structure $\mathfrak{s}$ with $\pm\langle c_1(\mathfrak{s}), [\Sigma]\rangle \equiv 2j + p \ (mod\ 2p)$.*

Of course $Q(j, \mathfrak{t})$ only depends on j mod p. With all the labeling conventions in place, we can finally rewrite the surgery exact sequence (4.9) in the following form

$$... \longrightarrow \bigoplus_{k\in\mathbb{Z}} HF^+(Y_0, i + kp, \mathfrak{t}) \longrightarrow HF^+(Y_p, Q(i, \mathfrak{t})) \stackrel{F^+_{Q(i,\mathfrak{t})}}{\longrightarrow} HF^+(Y, \mathfrak{t}) \longrightarrow ... \quad (4.14)$$

Now, we would like to say that $Q(i, \mathfrak{t}) = (i, \mathfrak{t})$, but this is not necessarily true, due to the sign indeterminacy in the above proposition. The best we can do is $Q(i, \mathfrak{t}) = (\pm i, \mathfrak{t})$. Notice, however, that the sign indeterminacy gets squared away in Equation (4.14), so that the properties of $F^+_{Q(i,\mathfrak{t})}$ are the same as those of $F^+_{(i,\mathfrak{t})}$ *i.e.* we may effectively interchange them. Also, notice that $Q(0, \mathfrak{t}) = (0, \mathfrak{t})$.

Now we can start doing computations. Since we've seen that the various properties of the surgery exact sequence don't really depend on $\mathfrak{t} \in Spin^c(Y)$, things will be really similar to the case of knots in S^3. Indeed, we follow the argument sketched in [55], clarifying all the points and introducing the necessary modifications when these are called for. In order to keep things simple, we will work with Floer homology in $\mathbb{Z}_2$ coefficients. This does not really impose a limitation, since the information about the knot K contained in $HF^+(Y_0)$ is kept in the $\mathbb{Z}_2$ version, as we shall see. Our first order of business is to verify that the maps $F^+_{i+kp,\mathfrak{t}}$ are in fact nonvanishing. This is indeed always true, thanks to the following result from [57]:

Theorem 4.2.3. *([57], Proposition 9.3) Let Y be a closed oriented 3-manifold, and let $K \subset Y$ be a knot, framed so that the surgery cobordism $W(K)$ has $b_2^- = 1$. If $HF^\infty(Y)$ is standard, then for every $Spin^c$ structure $\mathfrak{s}$ whose restriction to the boundary components Y and $Y(K)$ is torsion, the map $F^\infty_{W(K),\mathfrak{s}}$ is an isomorphism.*

Here, HF^∞ is **standard** if, for each torsion $Spin^c$ structure $\mathfrak{t}$, we have that

$$HF^\infty(Y, \mathfrak{t}) \simeq (\Lambda^{b_1(Y)} H^1(Y, \mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}]$$

Clearly, the hypotheses of the above result are satisfied in our case. We also remark that the theorem is proved for an arbitrary finite field, from which the $\mathbb{Z}$ case is deduced, so that it remains true in $\mathbb{Z}_2$ coefficients. Moreover, it is seen from the commutative square below, which is a portion of (4.5), that this implies nonvanishing of the corresponding maps on HF^+

$$\begin{array}{ccccc} \ldots \longrightarrow & HF^\infty(Y_1,\mathfrak{t}_1) & \xrightarrow{\pi} & HF^+(Y_1,\mathfrak{t}_1) & \longrightarrow \ldots \\ & \downarrow{\scriptstyle F^\infty_{W,\mathfrak{s}}} & & \downarrow{\scriptstyle F^+_{W,\mathfrak{s}}} & \\ \ldots \longrightarrow & HF^\infty(Y_2,\mathfrak{t}_2) & \xrightarrow{\pi} & HF^+(Y_2,\mathfrak{t}_2) & \longrightarrow \ldots \end{array}$$

Now, we start using the hypotheses that Y_p is an L-space. In $\mathbb{Z}_2$ coefficients, this means that $HF^+(Y_p,\mathfrak{t}) = \mathbb{Z}_2[U,U^{-1}]/U\cdot\mathbb{Z}_2[U]$ for all $Spin^c$ structures. It will be useful to have the following result considering endomorphisms of $\mathbb{Z}_2[U,U^{-1}]/U\cdot\mathbb{Z}_2[U]$:

Lemma 4.2.4. *([55]) Let $\mathbb{F}$ be a field. Then, every endomorphism ψ of $\mathbb{F}[U,U^{-1}]/U\cdot\mathbb{F}[U]$ can be described as multiplication by a power series $\sum_{i=0}^{\infty} a_i\cdot U^i$, where $a_i\in\mathbb{F}$. In particular, every non trivial endomorphism is surjective, and $\ker\psi\simeq\mathbb{F}[U]/U^l$, where $l=min\{i|a_i\neq 0\}$.*

We will think of $F^+_{(i,\mathfrak{t})}$ as an endomorphism of $\mathbb{Z}_2[U,U^{-1}]/U\cdot\mathbb{Z}_2[U]$, where the individual summands $F^+_{i+kp,\mathfrak{t}}$ take the role of the a_iU^i in the previous lemma, and where the precise power of U can be determined by the grading shift. Now, to have all maps $F^+_{i+kp,\mathfrak{t}}$ nonvanishing is not in itself a sufficient condition for $F_{W_p,(i,\mathfrak{t})}$ to be nontrivial. If we have repeated grading shifts, the maps may cancel in pairs, so we have to be careful. Indeed, this is seen to happen. The following result is elementary:

Lemma 4.2.5. *Let m and k be different integers, and assume that both are congruent to i mod p. If $(p+2m)^2=(p+2k)^2$, then $i=0$, or $i=\frac{p}{2}$*

Proof. The expression $(p+2m)^2=(p+2k)^2$ is equivalent to $(p+m+k)(m-k)=0$. Since are assuming that $m\neq k$, we have that $p+m+k=0$. Now, write $m=ap+i$, $k=bp+i$, where $i<p$. Then, substituting in, we have

$$(a+b)p+2i=-p$$

Thus, $p\mid 2i$, which implies the result (keeping in mind that $i<p$). □

Now, fix $0 \leq i < p$, and consider the assignment

$$i + kp \mapsto -\frac{(2(i+kp)+p)^2}{p}$$

When $i = 0$, this sets up a 2:1 correspondence over the integers (if we see k as the argument of the assignment). If $i = \frac{p}{2}$, the correspondence is 2:1 except at $k = -1$. Moreover, notice that this is where the maximum grading shift is attained (this will be important soon). If $i \neq 0, \frac{p}{2}$, then the assignment is 1:1. The effect of this is that when $i = 0$, the maps $F_{(0,\mathfrak{t})}$ are trivial, since for each grading shift there exist exactly two maps carrying such grading, and they cancel upon adding (recall that we are working in $\mathbb{Z}_2$ coefficients). Moreover, recall that $Q(0,\mathfrak{t}) = (0,\mathfrak{t})$. For $i \neq 0$, there are always "unpaired" gradings, so the map $F_{(i,\mathfrak{t})}$ is nontrivial *i.e.* surjective, thanks to Lemma 4.2.4. Therefore, the surgery exact sequence breaks up into a short exact sequence, depending on i:

- **Case 1**: $(i,\mathfrak{t})$, $i \neq 0$. We have the short exact sequence

$$0 \longrightarrow \bigoplus_{k\in\mathbb{Z}} HF^+(Y_0, i+kp, \mathfrak{t}) \longrightarrow HF^+(Y_p, Q(i,\mathfrak{t})) \xrightarrow{F_{(i,\mathfrak{t})}} HF^+(Y,\mathfrak{t}) \longrightarrow 0$$

- **Case 2**: $(0,\mathfrak{t})$. We have the short exact sequence

$$0 \longrightarrow HF^+(Y,\mathfrak{t}) \longrightarrow \bigoplus_{k\in\mathbb{Z}} HF^+(Y_0, kp, \mathfrak{t}) \longrightarrow HF^+(Y_p, (0,\mathfrak{t})) \longrightarrow 0$$

We focus on Case 1 first. In this regime, $\bigoplus_{k\in\mathbb{Z}} HF^+(Y_0, i+kp, \mathfrak{t})$ is isomorphic, as a $\mathbb{Z}_2[U]$ module, to $ker\ F^+_{(i,\mathfrak{t})} \simeq \mathbb{Z}_2[U]/U^{l_{i,\mathfrak{t}}}$, for some integer $l_{i,\mathfrak{t}} \geq 0$, and our task is to determine $l_{i,\mathfrak{t}}$. If we use Lemma 4.2.4 to think of $F_{(i,\mathfrak{t})}$ as multiplication by a power series, then the individual terms are given by the homogeneous summands $F^+_{i+kp,\mathfrak{t}}$, and the one that determines $l_{i,\mathfrak{t}}$ is precisely the summand that carries the least grading. That is, we take the value $k = -1$ in Equation (4.13), to get $\frac{p-(2i-p)^2}{4p}$. Notice that this is exactly the value of the correction term $d(L(p,1),i)$, under the identification $Spin^c(L(p,1)) \longleftrightarrow \mathbb{Z}_p$ given in Proposition 4.8 of [57] (we will say more about this later).

We now see what the actual value of $l_{i,\mathfrak{t}}$ is: since $F_{(i,\mathfrak{t})}$ is surjective, we must have the following situation

$$HF^+_{d(L(p,1),i)+d(Y,\mathfrak{t})}(Y_p, Q(i,\mathfrak{t})) \longrightarrow HF^+_{d(Y,\mathfrak{t})}(Y,\mathfrak{t})$$

where the subscripts denote a particular grading (there are many terms in other gradings hitting $HF^+_{d(Y,\mathfrak{t})}(Y,\mathfrak{t})$, but we won't need to consider them). Since $d(Y,\mathfrak{t})$ is the least grading of any element in $HF^+(Y,\mathfrak{t})$, we must have that the terms from $HF^+(Y_p, Q(i,\mathfrak{t}))$ supported in gradings from $d(Y_p, Q(i,\mathfrak{t}))$ up to and excluding $d(L(p,1),i)+d(Y,\mathfrak{t})$ constitute the kernel of $F_{(i,\mathfrak{t})}$. Since U carries grading -2 *i.e.* gradings increase in steps of two, we see that

$$l_{i,\mathfrak{t}} = \frac{d(L(p,1),i) + d(Y,\mathfrak{t}) - d(Y_p, Q(i,\mathfrak{t}))}{2} \qquad i \neq 0 \tag{4.15}$$

i.e. the situation is the same as for knots in S^3 (Theorem 7.2 of [57]), except for the fact that there are more $Spin^c$ structures to account for.

We've pinned down the structure of $HF^+(Y_0)$. Now, we'd like to to say something about the $Spin^c$ structures in which these groups are supported. In order to do this, we must pause the discussion of Case 1, and begin discussing Case 2. Here, we see that $HF^+(Y,\mathfrak{t})$ injects into $\bigoplus_{k\in\mathbb{Z}} HF^+(Y_0, kp, \mathfrak{t})$, so that the latter is infinitely generated as a $\mathbb{Z}_2$ vector space. Although this is not very illuminating, we can at least say that $\bigoplus_{k\in\mathbb{Z}} HF^+(Y_0, kp, \mathfrak{t})$ is supported in a single $Spin^c$ structure, by following the argument in [55]. It can be readily seen from diagram (4.6) that vanishing of the cobordism map on HF^+ implies vanishing of the cobordism map on $\widehat{HF}$. Using the exact sequence for $\widehat{HF}$, which splits in exactly the same way along $Spin^c$ structures as the one for HF^+, we get

$$0 \longrightarrow \widehat{HF}(Y,\mathfrak{t}) \longrightarrow \bigoplus_{k\in\mathbb{Z}} \widehat{HF}(Y_0, kp, \mathfrak{t}) \longrightarrow \widehat{HF}(Y_p, (0,\mathfrak{t})) \longrightarrow 0$$

Now, $\widehat{HF}(Y_p,(0,\mathfrak{t})) \simeq \widehat{HF}(Y,\mathfrak{t}) \simeq \mathbb{Z}_2$, since both are L-spaces. Hence, we have that $\bigoplus_{k\in\mathbb{Z}} \widehat{HF}(Y_0, kp, \mathfrak{t}) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Now, it is shown in Proposition 5.1 of [57] that the Euler characteristic of $\widehat{HF}$, $\chi(\widehat{HF}(Z,\mathfrak{t}))$, is equal to zero whenever $b_1(Z) > 0$. This immediately implies that $\bigoplus_{k\in\mathbb{Z}} \widehat{HF}(Y_0, kp, \mathfrak{t})$ is supported in a single $Spin^c$ structure. Now, it is an easy consequence of diagram shown in equation (4.2) that $\widehat{HF}$ vanishes

if and only if HF^+ vanishes, so that $\bigoplus_{k\in\mathbb{Z}} HF^+(Y_0, kp, \mathfrak{t})$, too, is supported in a single *Spin*^{c} structure. In fact, we can say a little bit more about the support *Spin*^{c} structure: in [55], the conjugation symmetry of the invariants is used to show that this is in fact supported a torsion *Spin*^{c} structure, corresponding to $k = 0$. In our case, the argument breaks down, since the *Spin*^{c} structure conjugate to $(kp, \mathfrak{t})$ is $(-kp, \bar{\mathfrak{t}})$, not $(-kp, \mathfrak{t})$. However, we can still appeal to Theorem 5.2 of [57], which states that $HF^+(Y, \mathfrak{t})$ is finitely generated as an abelian group whenever $b_1(Y) > 0$ and $\mathfrak{t}$ is non-torsion, to conclude that $HF^+(Y_0, kp, \mathfrak{t})$ is nonzero only when $k = 0$.

We now treat the support $i + kp$ of $HF^+(Y_0, i + kp, \mathfrak{t})$, for $0 < i < p$, as promised:

Proposition 4.2.6. *Let* $0 < i < p$. *If* $k \notin \{0, -1\}$, *then* $HF^+(Y_0, i + kp, \mathfrak{t}) = 0$.

Proof. Let $j = i + kp$, and suppose that $HF^+(Y_0, j, \mathfrak{t})$ is non-vanishing for some $k \notin \{0, -1\}$. Observe that $k \notin \{0, -1\}$, is equivalent to $|j| > p$. So, let us consider $|j|$-surgery on K. The inequality implies that this manifold is an L-space, by Lemma 7.12 of [57] (the proof given there is for integral homology spheres, but it adapts to rational homology spheres with no changes). Consider the surgery exact sequence for $|j|$-surgery on K, and observe that this doesn't affect the isomorphism type of $HF^+(Y_0)$, but it does affect the labelling. Then, under the labelling of the *Spin*^{c} structures on Y_0 induced by $|j|$ surgery, we would have $HF^+(Y_0, |j|, \mathfrak{t})$ or $HF^+(Y_0, -|j|, \mathfrak{t})$ nonvanishing. But this contradicts the fact that $\bigoplus_{n\in\mathbb{Z}} HF^+(Y_0, n|j|, \mathfrak{t})$ is supported in a torsion *Spin*^{c} structure. □

Remark 4.2.7. We make the observation that, since $\mathbb{Z}_2[U]/U^{l_{i,\mathfrak{t}}}$ is an irreducible $\mathbb{Z}_2[U]$ module, $\bigoplus_{k\in\mathbb{Z}} HF^+(Y_0, i + kp, \mathfrak{t})$ can only be supported in a single *Spin*^{c} structure.

This concludes the analysis of Case 1. In the next subsection, we tackle Case 2 by using an appropriate coefficient system to obtain an answer that is "finitely generated".

4.2.1 Floer homology with twisted coefficients

The infinite-dimensionality of $HF^+(Y_0)$ in torsion *Spin*^{c} structures is rather undesirable. Ideally, one would like to work with a version that is finite dimensional. There is a way to achieve this, at the expense of introducing a minor technical detail: one

must work with a twisted coefficient system. In the following, we sketch this construction, highlighting its useful features. We follow the treatment in Yi Ni's paper [54].

In [57], Section 8, Ozsváth and Szabó introduce a generalisation of the Heegaard Floer homology invariants, by using the group ring $\mathbb{Z}[H^1(Y;\mathbb{Z})]$ to turn $CF^+(Y,\mathfrak{t})$ into a $\mathbb{Z}[H^1(Y;\mathbb{Z})]$ module, denoted by $\underline{CF}^+(Y,\mathfrak{t})$. Moreover, they show how there is an interpretation of the boundary map ∂^+ in this regime, giving rise to "twisted" Floer homology invariants $\underline{HF}^+(Y,\mathfrak{t})$. In fact, there are twisted theories $\underline{HF}^\circ$ for all other flavors of Heegaard Floer homology, but we won't need to discuss them here. Now, it is a standard fact in algebraic topology (see [16]) that, given a chain complex C over a commutative ring R, and a R-module M_R, one can form the homology of C, "twisted" by M_R, as

$$H_*(C \otimes_R M_R)$$

Recall that we work in $\mathbb{Z}_2$ coefficients. So, given any $\mathbb{Z}_2[H^1(Y;\mathbb{Z})]$-module M, one can form the Floer homology with coefficients twisted by M

$$\underline{HF}^+(Y,\mathfrak{t},M) = H_*(\underline{CF}^+(Y,\mathfrak{t}) \otimes_{\mathbb{Z}_2[H^1(Y;\mathbb{Z})]} M)$$

As is described in [54] (see also [60]), there is a natural choice of module M that is useful. Consider the "universal Novikov ring"

$$\Lambda = \{\sum_{a_r \in \mathbb{R}} a_r t^r, \#\{a_r | a_r \neq 0, r \leq c\} < \infty, \textit{for any } c \in \mathbb{R}\}$$

Notice that Λ is a field. Given a cohomology class $\omega \in H^2(Y;\mathbb{R})$, one may turn Λ into a $\mathbb{Z}_2[H^1(Y;\mathbb{Z})]$-module as follows: there is a homomorphism from $H^1(Y;\mathbb{Z})$ into $\mathbb{R}$ obtained by viewing $h \in H^1(Y;\mathbb{Z})$ as an element of $H^1(Y;\mathbb{R})$ and performing the evaluation $\langle h \cup \omega, [Y]\rangle$. It is a standard fact of algebra that this induces a homomorphism $\mathbb{Z}_2[H^1(Y;\mathbb{Z})] \to \Lambda$ obtained by the assignment

$$\sum a_h h \mapsto \sum a_h t^{\langle h \cup \omega, [Y]\rangle} \tag{4.16}$$

One denotes this module by Λ_ω, to emphasise the point that the module structure

depends on ω, and the corresponding Heegaard Floer invariants by $\underline{HF}^+(Y, \mathfrak{t}, \Lambda_\omega)$. Now, this construction enjoys many of the properties of the "untwisted case". Namely, there are twisted maps induced by cobordisms, and there are surgery exact sequences analogous to the ones described before. We refer the reader to [57] for a description of these. In particular, we have the following enhanced version of the exact sequence for integral surgeries:

Theorem 4.2.8. *([54], Theorem 2.2) Let $K \subset Y$ be a nullhomologous knot, and let $\gamma \in Y - K$ be a simple closed curve. Observe that γ also lives inside any manifold obtained by $r \in \mathbb{Q}$ surgery on K. Let $\omega_r \in PD(\gamma) \in H^2(Y_r, \mathbb{R})$ and $\omega \in PD(\gamma) \in H^2(Y, \mathbb{R})$. Let $p > 0$ be an integer. Then, there is the following exact sequence*

$$... \longrightarrow \underline{HF}^+(Y_0, [\mathfrak{t}_{Y_p}], \Lambda_{\omega_0}) \longrightarrow \underline{HF}^+(Y_p, \mathfrak{t}_{Y_p}, \Lambda_{\omega_p}) \xrightarrow{\underline{F}} \underline{HF}^+(Y, \mathfrak{t}_Y, \Lambda_\omega) \longrightarrow ...$$

Recall that, for a rational homology sphere Z, $H^1(Z; \mathbb{Z}) = H^2(Z; \mathbb{R}) = 0$, so the assignment (4.16) is necessarily trivial, meaning that Λ is a trivial $\mathbb{Z}_2[H^1(Z; \mathbb{Z})]$-module (also, we've dropped the ω subscript since the construction no longer depends on it). Moreover, $\underline{CF}^+$ is also a trivial $\mathbb{Z}_2[H^1(Z; \mathbb{Z})]$-module, so that the construction is really "not twisted". Indeed, we have

$$\underline{HF}^+(Z, \mathfrak{t}, \Lambda) = H_*(CF^+(Z, \mathfrak{t})) \otimes \Lambda = HF^+(Z, \mathfrak{t}) \otimes \Lambda$$

Then, the surgery exact sequence with twisted coefficients for positive integral surgery on a nullhomologous knot K in a rational homology Y becomes

$$... \longrightarrow \bigoplus_{k \in \mathbb{Z}} \underline{HF}^+(Y_0, i + kp, \mathfrak{t}, \Lambda_0) \longrightarrow HF^+(Y_p, Q(i, \mathfrak{t})) \otimes \Lambda \xrightarrow{\underline{F}} HF^+(Y, \mathfrak{t}) \otimes \Lambda \longrightarrow ...$$

Notice that, under the assumption that Z is an L-space, $HF^+(Z, \mathfrak{t}) \otimes \Lambda \simeq \Lambda[U, U^{-1}]/U \cdot \Lambda[U]$. Moreover, the map $\underline{F}$ also admits a nice description in terms of the untwisted maps (this is independent of the L-space condition, we only need it going between rational homology spheres). As is shown in Section 3 of [60],

$$\underline{F} = t^c \sum_{k\in\mathbb{Z}} F^+_{i+kp,\mathfrak{t}} \cdot t^{n_k}$$

where $n_k = \frac{\langle c_1(\mathfrak{s}_{i+kp},\mathfrak{t})\cup PD[\mu\times I],[W]\rangle}{2}$, and $c \in \mathbb{R}$. Now,

$$\frac{\langle c_1(\mathfrak{s}_{i+kp},\mathfrak{t})\cup PD[\mu\times I],[W]\rangle}{2} = \frac{p+(2+ikp)}{p^2}\cdot\frac{\langle pPD[\mu\times I]\smile pPD[\mu\times I],[W]\rangle}{2} =$$

$$= -\frac{p+(2+ikp)}{2p}$$

and when $i = 0$ this is -$\frac{(2k+1)}{2}$. The grading shift given in equation (4.13), for the case when $i = 0$, is minimised when $k = -1$ and $k = 0$ simultaneously, so that $n_{-1} = \frac{1}{2}$ and $n_0 = -\frac{1}{2}$. What is important here is that n_{-1} and n_0 are different, so that the lowest degree terms in $\underline{F}$ no longer cancel (in fact, since n_k is a linear function of k, none of the terms cancel, but we won't need this). Also, since $F^+_{0,\mathfrak{t}}$ and $F^+_{-p,\mathfrak{t}}$ both carry the same grading, we can think of them as being the same $\Lambda[U,U^{-1}]/U\cdot\Lambda[U]$ map. With this, we can write

$$\underline{F} = (t^{c+\frac{1}{2}} + t^{c-\frac{1}{2}})\cdot F^+_{0,\mathfrak{t}} + \phi \tag{4.17}$$

where ϕ is a map whose individual summands carry degrees strictly higher than that of $F^+_{0,\mathfrak{t}}$. Now we're in exactly the same situation as in Case 1: we've certified that the cobordism map is nonzero, and since Λ is a field, we can apply Lemma 4.2.4 and see that $\underline{F}$ is indeed surjective, so that

$$\bigoplus_{k\in\mathbb{Z}} \underline{HF}^+(Y_0,kp,\mathfrak{t},\Lambda_0) \simeq Ker\ \underline{F} \simeq \Lambda[U]/U^{l_{0,\mathfrak{t}}}$$

where $l_{0,\mathfrak{t}} \geq 0$ is an integer. One sees immediately that $l_{0,\mathfrak{t}}$ is calculated by the formula

$$l_{0,\mathfrak{t}} = \frac{d(L(p,1),0) + d(Y,\mathfrak{t}) - d(Y_p,(0,\mathfrak{t}))}{2} \tag{4.18}$$

So, a formula similar to 4.15 holds in the twisted case: this is what we would have obtained if we had naïvely assumed that we didn't have to worry about cancellation of maps when $i = 0$.

Now, we would also like to conclude that $\bigoplus_{k\in\mathbb{Z}} \underline{HF}^+(Y_0, kp, \mathfrak{t}, \Lambda_0)$ is only supported in the torsion $Spin^c$ structure $k = 0$, like in the untwisted case. This, indeed, is seen to be the case:

Proposition 4.2.9. *For any manifold Y and any $Spin^c$ structure $\mathfrak{t}$ on Y, $HF^+(Y, \mathfrak{t}) = 0$ implies that $\underline{HF}^+(Y, \mathfrak{t}, \Lambda_\omega) = 0$.*

Proof. First notice that $\underline{\widehat{HF}}$ vanishes if and only if $\underline{HF}^+$ vanishes. This follows immediately from the triangle analogous to (4.2) in twisted coefficients. So it suffices to show the result for $\underline{\widehat{HF}}$ (in fact, the only advantage in doing this step is a considerable ease of notation).

We can prove the result directly at the level of chain complexes. Namely, we will show that that passing to twisted coefficients doesn't create new cycles, and also doesn't destroy any old boundaries. Recall that the twisted chain complex $\underline{\widehat{CF}}$ is defined as the free $\mathbb{Z}_2[H^1(Y, \mathbb{Z})]$-module on the intersection points $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The twisted boundary map is written as (see, for example, Section 8 of [57])

$$\underline{\widehat{\partial}}(x) \equiv \sum_{\{y\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1, n_z(\phi)=0\}} \#\widehat{\mathcal{M}}(\phi) y \cdot t^{\int_\nu \phi}$$

where ν is a 2-chain representing ω. Compare with the untwisted differential on $\widehat{CF}$:

$$\widehat{\partial}(x) \equiv \sum_{\{y\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta\}} \sum_{\{\phi\in\pi_2(x,y)|\mu(\phi)=1, n_z(\phi)=0\}} \#\widehat{\mathcal{M}}(\phi) y$$

From this, it is immediately seen that if x is a cycle in $\underline{\widehat{HF}}$, then it is a cycle in $\widehat{HF}$, and that if x is a boundary in $\widehat{HF}$, then it is also a boundary in $\underline{\widehat{HF}}$. Thus all cycles in $\underline{\widehat{HF}}$ are boundaries, and the same holds for $\underline{HF}^+$. □

We can summarise our calculation of $HF^+(Y_0)$ in the following result (compare with [57], Theorem 7.2):

Theorem 4.2.10. *Let $K \subset Y$ be a nullhomologous knot in an L-space Y, and suppose that integral $p > 0$ Dehn surgery on K gives an L-space Y_p. Then, for each $0 < i < p$, $HF^+(Y_0, i + kp, \mathfrak{t})$ is only possibly nonvanishing when k is either 0 or -1 (but not both), in which case we have that $HF^+(Y_0, i + kp, \mathfrak{t}) \simeq \mathbb{Z}_2[U]/U^{l_{i,\mathfrak{t}}}$, where*

$$l_{i,\mathfrak{t}} = \frac{d(L(p,1), i) + d(Y, \mathfrak{t}) - d(Y_p, Q(i, \mathfrak{t}))}{2} \tag{4.19}$$

Moreover, $\underline{HF}^+(Y_0, kp, \mathfrak{t}, \Lambda_0)$ *is possibly nontrivial only when* $k = 0$*, in which case we have* $\underline{HF}^+(Y_0, 0, \mathfrak{t}, \Lambda_0) \simeq \Lambda[U]/U^{l_{0,\mathfrak{t}}}$*, where*

$$l_{0,\mathfrak{t}} = \frac{d(L(p,1),0) + d(Y,\mathfrak{t}) - d(Y_p,(0,\mathfrak{t}))}{2} \tag{4.20}$$

Remark 4.2.11. In Theorem 7.2 of [57], Ozsváth and Szabó show that $HF^+(S^3_0, m) \neq 0$ implies the fact that $|m| \leq p/2$ (in fact, their argument works for any L-space integral homology sphere). However, this does not readily generalise to our case, since a crucial step in their argument is the fact that $HF^+(S^3_0, m) \simeq HF^+(S^3_0, -m)$, due to conjugation symmetry. The best we can do is say that $HF^+(Y_0, i + kp, \mathfrak{t}) \simeq HF^+(Y_0, -(i + kp), \bar{\mathfrak{t}})$. Nevertheless, there are some restricted situations where we always have $\mathfrak{t} = \bar{\mathfrak{t}}$. Namely, if $H^2(Y;\mathbb{Z}) \simeq \mathbb{Z}_2^n$, then we see that $c_1(\mathfrak{t}) = 0$ for all *Spin*^{c} structures, so that $\mathfrak{t} - \bar{\mathfrak{t}} = c_1(\mathfrak{t})$ implies that all *Spin*^{c} structures are self-conjugate. Another way of saying this is that every *Spin*^{c} is a *Spin* structure (we do not give the definition of a *Spin* structure here, see [4] for details). This holds, for example, for $\mathbb{R}P^3$ and connect-sums thereof, so that in this case we may arrive at the conclusion that $HF^+(Y_0)$ is supported in "levels" m such that $|m| \leq p/2$.

Question. *Is it possible to show that* $|m| \leq p/2$ *in general, or can one construct any counterexamples to this claim?*

We conclude by remarking that a result exactly analogous to Theorem 4.2.10 holds in the context of negative surgeries, and we state it below:

Theorem 4.2.12. *Let* $K \subset Y$ *be a nullhomologous knot in an* L*-space* Y*, and suppose that integral* $-p < 0$ *Dehn surgery on* K *gives an* L*-space* Y_{-p}*. Then, for each* $0 < i < p$*,* $HF^+(Y_0, i + kp, \mathfrak{t})$ *is only possibly nonvanishing when* k *is either* 0 *or* -1 *(but not both), in which case we have that* $HF^+(Y_0, i + kp, \mathfrak{t}) \simeq \mathbb{Z}_2[U]/U^{l_{i,\mathfrak{t}}}$*, where*

$$l_{i,\mathfrak{t}} = \frac{d(L(p,1),i) + d(Y_{-p},Q(i,\mathfrak{t})) - d(Y,\mathfrak{t})}{2} \tag{4.21}$$

Moreover, $\underline{HF}^+(Y_0, kp, \mathfrak{t}, \Lambda_0)$ *is possibly nontrivial only when* $k = 0$*, in which case we have* $\underline{HF}^+(Y_0, 0, \mathfrak{t}, \Lambda_0) \simeq \Lambda[U]/U^{l_{0,\mathfrak{t}}}$*, where*

$$l_{0,\mathfrak{t}} = \frac{d(L(p,1),0) + d(Y_{-p},(0,\mathfrak{t})) - d(Y,\mathfrak{t})}{2} \tag{4.22}$$

Proof. This time, the cobordism map is $F^+_W : HF^+(Y, \mathfrak{s}|_Y) \longrightarrow HF^+(Y_{-p}, \mathfrak{s}|_{Y_{-p}})$. Since the surgery is negative, W is automatically negative-definite, so that all labeling conventions are as before. In particular, Theorem 4.2.3 still holds. The rest follows exactly as before. □

Having calculated $HF^+(Y_0)$ for knots in L-spaces with L-space surgeries, we turn our attention in the next section to the question of extracting information about the knot K from the calculation.

4.3 Knot Floer homology and the knot genus

In [60], Ozsváth and Szabó introduce an invariant of knots derived from Heegaard Floer homology, known as **knot Floer homology**. Here, we recall the construction with $\mathbb{Z}_2$ coefficients. The starting observation is that placing an additional basepoint w in a Heegaard diagram for Y defines a knot $K \subset Y$ by "appropriately" connecting z and w (see [60] for the details). Suppose that K is nullhomologous, and denote a Seifert surface for K by F. With the help of w, one can define a map from intersection points to $Spin^c$ structures on the zero-surgery on K (see, once again, [60])

$$s_{w,z} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow Spin^c(Y_0)$$

analogous to the map s_z described before. For a fixed $Spin^c$ structure $\mathfrak{t}$ on Y, one defines a chain complex $C(\mathfrak{t}) = CFK^\infty(Y, K, \mathfrak{t})$, which is a free $\mathbb{Z}_2$ module generated by $[x, i, j]$, where $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is such that $s_{w,z}(x)$ extends $\mathfrak{t}$, and $i, j \in \mathbb{Z}$, together with $s_{w,z}(x)$, satisfy the following constraint:

$$\frac{\langle c_1(s_{w,z}(x)), [\widehat{F}]\rangle}{2} + (i - j) = 0$$

The complex $C(\mathfrak{t})$ can be endowed with the differential

$$\partial[x, i, j] \equiv \sum_{\{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta\}} \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi) = 1\}} \#(\widehat{\mathcal{M}}(\phi))[y, i - n_z(\phi), j - n_w(\phi)]$$

that is in every way like ∂^∞, only that one chooses to record the information about the intersection number $n_w(\phi)$ in the second variable. The non-negativity of the $n_z(\phi)$ and $n_w(\phi)$ endows $CFK^\infty(Y, K, \mathfrak{t})$ with a $\mathbb{Z} \oplus \mathbb{Z}$ filtration $\mathcal{F}$, where $\mathcal{F}([x, i, j]) = (i, j)$. Just like we can derive the chain complexes CF^-, CF^+, $\widehat{CF}$ from CF^∞, we can

perform analogous constructions in the context of these "knot-enhanced complexes": namely, there is a subcomplex $CFK^-(Y,K,\mathfrak{t}) \subset CFK^\infty(Y,K,\mathfrak{t})$ obtained by demanding that $i<0$, and there is a corresponding quotient complex $CFK^+(Y,K,\mathfrak{t})$.

At this point, it is convenient to introduce the following notation: $C(\mathfrak{t})\{S\}$ will denote the subset of $CFK^\infty(Y,K,\mathfrak{t})$ formed by generators whose filtration levels satisfy a condition S on the (i,j) plane. For example, $CFK^+(Y,K,\mathfrak{t})$ is identified with $C(\mathfrak{t})\{i \geq 0\}$. An important specialisation of this is taken by further requiring that $i=0$, and considering an induced boundary map ∂_K that counts holomorphic disks subject to the constraint $n_w(\phi) = n_z(\phi) = 0$. Then, one defines the following homology theory

$$\widehat{HFK}(Y,K,\mathfrak{t},d) = H_*\left(C(\mathfrak{t})\{i=0, j\leq d\}/C(\mathfrak{t})\{i=0, j\leq d-1\}, \partial_K\right)$$

known as the **knot Floer homology** of K. Indeed, it is shown in [60] that this is a knot invariant. It has the remarkable property of detecting the Seifert genus of a knot. This was first established for knots in S^3 by Ozsváth and Szabó in [62]:

Theorem 4.3.1. *Let $K \subset S^3$ be a knot in S^3, and let $g(K)$ denote its Seifert genus. Then,*

$$g(K) = max\{d \mid \widehat{HFK}(S^3,K,d) \neq 0\}$$

Notice that we suppressed the $Spin^c$ structure on S^3 from the notation, since there is only one in this case. Indeed, this statement admits a generalisation to an arbitrary rational homology sphere Y, as was shown later by Ni in [52]: in particular, if we write

$$\widehat{HFK}(Y,K,d) = \bigoplus_{\mathfrak{t}\in Spin^c(Y)} \widehat{HFK}(Y,K,\mathfrak{t},d)$$

then we have the following:

Theorem 4.3.2. (*[52], Theorem 3.1*) *Suppose K is a null-homologous knot in a closed rational homology sphere Y. Then*

$$g(K) = max\{i \mid \widehat{HFK}(Y,K,i) \neq 0\}$$

Remark 4.3.3. Ni proves his theorem for knot Floer homology with coefficients in $\mathbb{Z}_2$ (see Section 4 of his paper). This justifies our use of $\mathbb{Z}_2$ coefficients throughout this thesis.

The strategy now is to relate knot Floer homology to $HF^+(Y_0)$. The relationship hinges on the fact that, for large integral surgeries on a knot K, the surgery exact sequence (4.14) can be identified with a suitable exact sequence at the level of knot Floer homology. More precisely, we have the following: define $B^+_{\mathfrak{t}} = C(\mathfrak{t})\{i \geq 0\}$ and $A^+_{\mathfrak{t},k} = C(\mathfrak{t})\{max(i, j-k) \geq 0\}$. Notice that $B^+_{\mathfrak{t}}$ can be identified with $CF^+(Y, \mathfrak{t})$. Then, one can construct chain maps $v^+_{\mathfrak{t},k} : A^+_{\mathfrak{t},k} \longrightarrow B^+_{\mathfrak{t}}$ and $h^+_{\mathfrak{t},k} : A^+_{\mathfrak{t},k} \longrightarrow B^+_{\mathfrak{t}}$ with the following remarkable property:

Theorem 4.3.4. *([54], Theorem 2.4) Let $K \subset Y$ be a nullhomologous knot in a rational homology sphere. Then, there exists a positive integer $N(K)$ such that, for all $m \geq N(K)$, we have that $CF^+(Y_m, (i, \mathfrak{t}))$ is represented by the chain complex $A^+_{\mathfrak{t},k}$, with $k \equiv i \ mod \ m$ and $|k| \leq \frac{m}{2}$, in the sense that there are isomorphisms*

$$\Phi_{m,k} : CF^+(Y_m, (i, \mathfrak{t})) \longrightarrow A^+_{\mathfrak{t},k}$$

Moreover, if $\mathfrak{s}_k$ and $\mathfrak{g}_k$ are Spin^c structures over the surgery cobordism W_m compatible with $\mathfrak{t}$ and $(i, \mathfrak{t})$, and such that

$$\langle c_1(\mathfrak{s}_k), [\Sigma] \rangle = 2k - m \quad and \quad \langle c_1(\mathfrak{g}_k), [\Sigma] \rangle = 2k + m$$

then the chain maps $v^+_{\mathfrak{t},k}$, $h^+_{\mathfrak{t},k}$ described above represent the cobordism maps $F^+_{\mathfrak{s}_k}$ and $F^+_{\mathfrak{g}_k}$, respectively.

What this result is saying is that, for m sufficiently large, we can effectively replace the surgery exact sequence

$$... \longrightarrow \bigoplus_{l \in \mathbb{Z}} HF^+(Y_0, i + lm, \mathfrak{t}) \longrightarrow HF^+(Y_m, Q(i, \mathfrak{t})) \stackrel{F^+_{Q(i,\mathfrak{t})}}{\longrightarrow} HF^+(Y, \mathfrak{t}) \longrightarrow ... \quad (4.23)$$

with the following sequence

$$... \longrightarrow \bigoplus_{l\in\mathbb{Z}} HF^+(Y_0, i+lm, \mathfrak{t}) \longrightarrow H_*(A^+_{\mathfrak{t},k}) \xrightarrow{F} H_*(B^+_{\mathfrak{t}}) \longrightarrow ... \tag{4.24}$$

where $F = v^+_{\mathfrak{t},k} + h^+_{\mathfrak{t},k} + H.O.T.$ With the help of this, the following result can be obtained, as shown in [53]:

Proposition 4.3.5. *Suppose that Y is an L-space. Let $g > 1$ be the largest integer for which $\widehat{HFK}(Y, K, \mathfrak{t}, g)$ is nontrivial (notice that we don't need to know that it equals the knot genus here). Then, we have the following isomorphism of $\mathbb{Z}_2$ vector spaces: $HF^+(Y_0, g-1, \mathfrak{t}) \simeq \widehat{HFK}(Y, K, \mathfrak{t}, g)$.*

Not surprisingly, the argument breaks down when $g = 1$. To derive a similar statement in this case, one adapts the proof of the above result to twisted coefficients, and the following can be shown:

Proposition 4.3.6. *([53]) Let $K \subset Y$ be a nullhomologous knot in an L-space such that $g(K) = 1$. Then, we have that*

$$rank_\Lambda(\underline{HF}^+(Y_0, 0, \mathfrak{t}, \Lambda_0)) = rank_{\mathbb{Z}_2}(\widehat{HFK}(Y, K, \mathfrak{t}, 1))$$

.

Putting all of these results together, we are finally able to prove:

Theorem 4.3.7. *Let $K \subset Y$ be a nullhomologous knot in an L-space Y, and suppose that integral $p > 0$ Dehn surgery Y_p on K gives $Y \# L(p, 1)$. Then, K is the unknot.*

Proof. By Theorem 4.1.11, $Y_p = Y \# L(p, 1)$ is also an L-space. In this case, we calculated in Theorem 4.2.10 that, for non-torsion $Spin^c$ structures, $HF^+(Y_0)$ has the form

$$HF^+(Y_0, i + kp, \mathfrak{t}) = \mathbb{Z}[U]/U^{l_{i,\mathfrak{t}}}$$

where $l_{i,\mathfrak{t}}$ is a non-negative integer determined by Equation 4.19. There is also the corresponding result for torsion $Spin^c$ structures,

$$\underline{HF}^+(Y_0, 0, \mathfrak{t}, \Lambda_0) = \mathbb{Z}[U]/U^{l_{0,\mathfrak{t}}}$$

where $l_{0,\mathfrak{t}}$ is computed via Equation 4.20. Substituting Y_p for $Y\#L(p,1)$ and using Theorem 4.1.8, we see that $d(Y_p, Q(k,\mathfrak{t})) = d(Y,\mathfrak{m}) + d(L(p,1),\mathfrak{n})$ for some $\mathfrak{m} \in Spin^c(Y)$ and $\mathfrak{n} \in Spin^c(L(p,1))$. Now, we don't necessarily know what these $Spin^c$ structures are, but this will not be important. What is important to notice is that as we run through all the $Spin^c$ structures in Y_0, we run once through all the $Spin^c$ structures in $Y\#L(p,1)$, we run $N = |H^2(Y,\mathbb{Z})|$ times through all the $Spin^c$ structures in $L(p,1)$, and we run p times through all the $Spin^c$ structures on Y. Thus, summing over all $Spin^c$ structures in Y_0, we get

$$\sum_{\{k\in\mathbb{Z},\mathfrak{t}\in Spin^c Y\}} 2l_{k,\mathfrak{t}} = N\sum_{k\in\mathbb{Z}/p} d(L(p,1),k) + p\sum_{\mathfrak{t}} d(Y,\mathfrak{t}) -$$

$$\left(p\sum_{\mathfrak{m}} d(Y,\mathfrak{m}) + N\sum_{\mathfrak{n}} d(L(p,1),\mathfrak{n})\right) = 0$$

Since the $l_{k,\mathfrak{t}}$ are non-negative by definition, they must all be zero. Thus, $HF^+(Y_0)$ is trivial for non torsion $Spin^c$ structures, and $\underline{HF}^+(Y_0)$ is trivial for torsion $Spin^c$ structures. An application of Propositions 4.3.5 and 4.3.6 now implies that $g(K) = 0$, that is, K is the unknot. □

Remark 4.3.8. It is interesting to compare how the proof of this result differs from that of the analogous statement for knots in S^3 (see [60]) : there, the authors avoid an appeal to twisted coefficients by using the fact that if a knot $K \subset S^3$ has an L-space surgery, then its Knot Floer homology is determined by its Alexander polynomial, which has to be trivial for positive $L(p,1)$ surgeries. Perhaps an analogous fact also holds for knots in L-spaces, but lacking such a general result, we chose to argue in the way just presented.

We can also prove a result similar to Theorem 4.3.7 for negative surgeries, this time with $L(p,1)$ replaced by $-L(p,1)$.

Corollary 4.3.9. *Let $K \subset Y$ be a nullhomologous knot in an L-space Y, and suppose that $-p < 0$ Dehn surgery Y_{-p} on K gives $Y\# - L(p,1)$. Then, K is the unknot.*

Proof. This follows exactly the same methodology of Theorem 4.3.7, using the surgery exact sequence in (4.10), so we omit the details. □

4.4 Using $l_{k,\mathfrak{t}}$ as an obstruction

Recall equations (4.19) and (4.21), and their twisted coefficient analogues. These compute the dimension of $HF^+(Y_0)$ as a vector space over $\mathbb{Z}_2$ or Λ in the different $Spin^c$ structures. It is clear that we can use the fact that these are non-negative integers as an obstruction to obtaining certain L-spaces by surgery on a nullhomologous knot in Y. However, this is hard to do in practice, for two reasons: firstly, it can be very challenging to determine the correction terms for an L-space, and even when this is possible, one often ends up with complicated recursive formulae which are unsuitable for computations. Secondly, and perhaps more importantly, it is nearly impossible to pin down the correspondence $d(Y_p, \mathfrak{s}|_{Y_p}) \leftrightarrow d(Y, \mathfrak{s}|_Y)$ in practice, so that one often has to check all possible combinations. Needless to say, the complexity of these computations quickly becomes overwhelming. When dealing with lens spaces, though, the situation becomes slightly more tractable. To begin, there is the following formula for correction terms for $L(p,q)$ (which we've already implicitly used to identify the grading shift):

Theorem 4.4.1. *([57], Proposition 4.8) There is a correspondence*

$$Spin^c(L(p,q)) \xleftrightarrow{\phi} \mathbb{Z}_p = \{0, 1, .., p-1\}$$

such that the following formula holds:

$$d(L(p,q), i) + d(L(q,r), j) = \frac{(2i + 1 - p - q)^2 - pq}{4pq} \tag{4.25}$$

Here, r is the reduction mod q of p, and j is the reduction mod q of i.

This provides a method, albeit still a messy one, to compute the correction terms for lens spaces recursively. However, if one wishes to uses this to obstruct obtaining $L(a,b)\#L(c,d)$ from integral surgery on a knot $K \subset L(p,q)$, it is still an outstanding challenge to relate $d(Y_p, \mathfrak{s}|_{Y_p}) \leftrightarrow d(Y, \mathfrak{s}|_Y)$ to this numbering.

Remark 4.4.2. In this section, **whenever a letter stands for an integer**, it is assumed that the integer is positive, unless accompanied by a minus sign. We will sometimes reinforce this point, especially when stating theorems.

The situation improves if one picks $q = b = d = 1$. To begin, one gets a closed formula for the correction terms in equation (4.25):

Lemma 4.4.3. $\sum_{i\in\mathbb{Z}_p} d(L(p,1),i) = \frac{p^2}{4} - \frac{p}{4} + \frac{1-p^2}{6}$

Proof. This is an elementary sum using well-known formulae:

$$\sum_{i\in\mathbb{Z}_p} d(L(p,1),i) = \sum_{i=0}^{p-1} \frac{(2i-p)^2-p}{4p} = \frac{1}{p}\sum_{i=0}^{p-1} i^2 - \sum_{i=0}^{p-1} i + \frac{p^2}{4} - \frac{p}{4}$$

$$= \frac{(p-1)(2p-1)-3p(p-1)}{6} + \frac{p^2}{4} - \frac{p}{4} = \frac{1-p^2}{6} + \frac{p^2}{4} - \frac{p}{4}$$

□

Using this, we can get a closed form obstruction to obtaining $M = L(a,1)\#L(b,1)$ from $+p$ surgery on a knot $K \subset L(m,1)$ by adding over all $Spin^c$ structures. Namely, if we add all the $l_{k,\mathfrak{t}}$ associated to the zero-surgery on K, we get

$$\sum_{\{k\in\mathbb{Z},\mathfrak{t}\in Spin^c(M)\}} 2l_{k,\mathfrak{t}} = m(\frac{p^2}{4} - \frac{p}{4} + \frac{1-p^2}{6}) + p(\frac{m^2}{4} - \frac{m}{4} + \frac{1-m^2}{6}) -$$

$$-a(\frac{b^2}{4} - \frac{b}{4} + \frac{1-b^2}{6}) - b(\frac{a^2}{4} - \frac{a}{4} + \frac{1-a^2}{6}) =$$

$$\frac{pm^2+mp^2-ba^2-ab^2}{4} + \frac{p+m-a-b}{6} - \frac{pm^2+mp^2-ba^2-ab^2}{6}$$

$$= (p+m-a-b)\frac{(pm+2)}{12}$$

keeping in mind that $ab = pm$, for homological reasons. Thus, we have

Proposition 4.4.4. *If $L(a,1)\#L(b,1)$ is obtained from $+p$ surgery on a nullhomologous knot $K \subset L(m,1)$, then we must have that $(p+m-a-b)(pm+2)$ is a non-negative integer congruent to 0 mod 24. In particular, $p+m \geq a+b$, with equality if and only if K is the unknot.*

As an example, we can see that $L(3,1)\#L(85,1)$ cannot be obtained by positive surgery on a nullhomologous knot in $L(15,1)$. Here, the surgery coefficient would have to be $+17$, and $15+17 < 85+3$. Notice that, turning this around, we can also see that $L(17,1)\#L(15,1)$ cannot be obtained by positive surgery on a nullhomologous knot in $L(3,1)$. This time, we would get on the left hand side of the congruence $14392 \equiv 0$ *mod* 24, which is clearly not true.

In fact, when we take m to be prime, the following is worth pointing out:

Corollary 4.4.5. *Let m be prime. Then, if $L(a,1)\#L(b,1)$ is obtained from $+p$ surgery on a nullhomologous knot $K \subset L(m,1)$, we must have that $m \leq min(a,b)$.*

Proof. Clearly, for some factorisation $p = p_1p_2$, we must have, without loss of generality, $a = p_1m$, $b = p_2$, since $ab = pm$. Then, $m + p - (a + b) = (p_1 - 1)(p_2 - m)$, whence the result. □

Of course, there is an obstruction similar in spirit for negative surgeries, whose proof we do not include, since it is very similar to that of the previous result. It takes the form:

Proposition 4.4.6. *If $L(a,1)\#L(b,1)$ is obtained from $-p$ surgery on a nullhomologous knot $K \subset L(m,1)$, then we must have that $(p+a+b-m)(pm+2)+4(m-p)-6pm$ is a non-negative integer congruent to 0 mod 24.*

If we try to use only the positivity of the quantity above as an obstruction, we end up with the inequality

$$(p+a+b-m) \geq \frac{6pm + 4(p-m)}{pm+2} \Leftrightarrow (ab+2)(a+b-m) \geq ab(6+\frac{2-ab}{m})-4m \quad (4.26)$$

which, unfortunately, is not very useful in general.

So far, we've been summing over correction terms, thereby losing a lot of information. As we've remarked, the correspondence $d(Y_p, \mathfrak{s}|_{Y_p}) \leftrightarrow d(Y, \mathfrak{s}|_Y)$ is very hard to pin down in practice, so that the method of summing correction terms seems to be the only evident one. In the following discussion, we suppose that $|H^2(Y;\mathbb{Z})|$ is odd, and that p is also odd. Recall that the space of $Spin^c$ structures on a manifold Y is a principal homogeneous space over its second cohomology with integer coefficients. Now, when $|H^2(Y;\mathbb{Z})|$ is odd, the map

$$H^2(Y;\mathbb{Z}) \xrightarrow{\times 2} H^2(Y;\mathbb{Z})$$

is an isomorphism. That means we can talk about "dividing by 2". Moreover, under this situation, $c_1(\mathfrak{t} + g) = c_1(\mathfrak{t}) \Leftrightarrow 2g = 0 \Leftrightarrow g = 0$ (the last implication following from the absence of 2-torsion), meaning that $Spin^c$ structures are determined by their first Chern classes. Hence, the assignment $\mathfrak{t} \longrightarrow c_1(\mathfrak{t})/2$ can be thought of as giving

$Spin^c(Y)$ a natural choice of group structure. Under this correspondence, the unique $Spin^c$ structure with trivial Chern class plays the role of the identity element. This $Spin^c$ structure can be thought of as coming from the unique $Spin$ structure on Y (we haven't defined $Spin$ structures here, but for the purposes of this discussion we can think of them as $Spin^c$ structures with trivial first Chern class. See [4]). Denote this special $Spin^c$ structure on Y by $\mathfrak{m}_Y$. Then, it follows from the discussion at the beginning of Section 4.2 that any lift of $\mathfrak{m}_{Y_p}$ on Y_p to the surgery cobordism W_p must restrict back to $\mathfrak{m}_Y$ on Y. That is, we can be sure that $\mathfrak{m}_Y$ and $\mathfrak{m}_{Y_p}$ always get paired together in Equations (4.20) and (4.22). Moreover, the third term in the equations is readily seen to be $d(L(p,1),0)$, in this case. It follows that we have

$$l_{0,\mathfrak{m}_Y} = \frac{d(L(p,1),0) + d(Y,\mathfrak{m}_Y) - d(Y_p,\mathfrak{m}_{Y_p})}{2} \tag{4.27}$$

for positive surgeries, and that

$$l_{0,\mathfrak{m}_Y} = \frac{d(L(p,1),0) + d(Y_{-p},\mathfrak{m}_{Y_{-p}}) - d(Y,\mathfrak{m}_Y)}{2} \tag{4.28}$$

for negative surgeries. With this, we can show the following:

Proposition 4.4.7. *Let m, p be odd. Then, if $L(a,1)\#L(b,1)$ is obtained by $-p$ surgery on a nullhomologous knot in $L(m,1)$, $p+a+b-m-2$ is a non-negative integer congruent to 0 modulo 8.*

Proof. The natural inclusions $Y_1, Y_2 \hookrightarrow Y_1\#Y_2$ provide us with a natural identification $H^2(Y_1;\mathbb{Z}) \oplus H^2(Y_2;\mathbb{Z}) = H^2(Y_1\#Y_2;\mathbb{Z})$, under which we can think of $c_1(\mathfrak{t}_1\#\mathfrak{t}_2)$ as $c_1(\mathfrak{t}_1) \oplus c_1(\mathfrak{t}_2)$. It follows from this that $\mathfrak{m}_{Y_1\#Y_2} = \mathfrak{m}_{Y_1}\#\mathfrak{m}_{Y_2}$, whence

$$d(L(a,1)\#L(b,1),\mathfrak{m}_{L(a,1)\#L(b,1)}) = d(L(a,1),\mathfrak{m}_{L(a,1)}) + d(L(b,1),\mathfrak{m}_{L(b,1)})$$

$$= \frac{a-1}{4} + \frac{b-1}{4}$$

Now, to see that $d(L(p,1),\mathfrak{m}_{L(p,1)}) = d(L(p,1),0)$, under the correspondence $\mathbb{Z}_p \overset{\phi}{\longleftrightarrow} Spin^c(L(p,1))$ of Proposition 4.4.1, consider $+p$-framed surgery on the unknot in S^3. We know that all the l_k must be equal to zero in this case, so that, in particular, $d(L(p,1),\mathfrak{m}_{L(p,1)}) = d(L(p,1),0)$. Recalling that $d(L(p,1),0) = \frac{p-1}{4}$, the result now follows from direct substitution into (4.28). □

This gives us a piece of information that is not revealed by Proposition 4.4.6. We could also deduce from this that $p + m - a - b$ is a non-negative integer congruent to 0 modulo 8 for positive surgeries, but in this case we already know that this must hold, from Proposition 4.4.4 (keeping in mind that $pm + 2$ is odd).

More generally, we can show the following:

Proposition 4.4.8. *Let Y be an L-space with $|H^2(Y;\mathbb{Z})|$ odd, and let p be a positive odd integer. If $Y\#L(p,1)$ can be obtained by $-p$ surgery on a nullhomologous knot $K \subset Y$, then $p \equiv 1$ mod 4.*

Proof. We have $\mathfrak{m}_{Y\#L(p,1)} = \mathfrak{m}_Y \# \mathfrak{m}_{L(p,1)}$. Plugging in the appropriate values of the correction terms into (4.28), the result follows immediately. □

We now wish to apply these results to prove a special case of Conjecture 3.3.3. One of the limitations of our approach is that we are assuming that the knot K is nullhomologous. In the following, we see that if $|H_1(Y;\mathbb{Z})|$ has prime order, then we can drop this assumption:

Proposition 4.4.9. *Let Y be a rational homology sphere such that $|H_1(Y;\mathbb{Z})|$ is a prime p. Then, if integral surgery on a knot $K \subset Y$ yields a manifold Z such that p divides $|H_1(Z;\mathbb{Z})|$, K is nullhomologous.*

Proof. Let $[K]$ be the homology class of the knot. Then, $[K]$ is either zero (*i.e* K is nullhomologous), or $[K]$ generates $H_1(Y;\mathbb{Z})$. Now, it is a well-known fact that, if the latter holds, $H_1(E(K);\mathbb{Z}) \simeq \mathbb{Z}$ (see [63] for a proof). Consider the kernel of the inclusion-induced homomorphism

$$i_* : H_1(\partial E(K)) \longrightarrow H_1(E(K))$$

It can be shown that this is an infinite cyclic group (this result is sometimes known as the half lives, half dies lemma, and it is proved in [35]). Let l denote a generator. Since $H_1(E(K)) \simeq \mathbb{Z}$, it is readily seen that l is a primitive element of $H_1(\partial E(K))$. That is, it is represented by a simple closed curve, which we denote by l in an abuse of notation. Let α be the slope on the boundary defining the surgery. Now, it is also well-known (see [73]) that, for α Dehn filling on $E(K)$,

$$|H_1(E(K)(\alpha);\mathbb{Z})| = d(\alpha, l) \cdot Tor(H_1(E(K);\mathbb{Z}))$$

Under our hypotheses, it follows that $d(\alpha, l) = pj$, for some positive integer j. Moreover, the same equation tells us that, for a meridian μ for K, $d(\mu, l) = p$. Since the surgery is assumed to be integral, $d(\alpha, \mu)$=1, so that we may take $\{\alpha, \mu\}$ (oriented any way we like) as a basis for $H_1(E(K); \mathbb{Z})$. Now, if we write l in terms of this basis, we have that

$$l = \langle l, \mu\rangle\alpha + \langle l, \alpha\rangle\mu = \pm d(\mu, l)\alpha \pm d(\alpha, l)\mu = p(\pm\alpha \pm j\mu)$$

This contradicts the fact that l is a simple closed curve, and the result is established. □

With this in place, we have:

Theorem 4.4.10. *Let Y be an L-space such that $|H_1(Y; \mathbb{Z})|$ is an odd prime or one, and let t be a positive integer congruent to 3 modulo 4. Then, if integral surgery on a knot $K \subset Y$ produces $Y\#L(t, 1)$, K is the unknot.*

Proof. Put together Proposition 4.4.9, Theorem 4.3.7, and Proposition 4.4.8. □

Remark 4.4.11. There is no hope to exclude the case where $t \equiv 1$ *mod* 4 in general. As an example, it is well known that -5 surgery on the left-handed trefoil knot $T \subset S^3$ gives $L(5, 1)$, so that -5 surgery on $Y\#E(T)$ gives $Y\#L(5, 1)$, for any 3-manifold Y.

This prompts us to discuss the state of Conjecture 3.3.3. On one hand, we've proved a special case of it for a class of rational homology spheres far wider than that of the lens spaces. On the other hand, the example above shows that the methods we use are in themselves not sufficient to obtain a full proof. At the heart of the matter lies the fact that, even though Floer homology is good at detecting homologically essential spheres (such as in $S^2 \times S^1$), it is not designed to be a very good detector of reducing spheres in general. For instance, it cannot see the difference between S^3 and $\Sigma\# - \Sigma$, (here, Σ stands for the Poincaré homology sphere). These are L-spaces, both with correction term equal to zero in their unique $Spin^c$ structure. Should the conjecture be true, then, it is likely that we may need to use other methods not considered here to prove it.

4.5 A uniqueness result

Heegaard Floer homology is an obstruction at heart, at least as far as the study of surgery on 3-manifolds is concerned. Therefore, its power as a tool for proving

affirmative results is rather limited. Nevertheless, we did manage to produce an instance of this in our Dehn surgery characterisation of the unknot. In spirit, our argument relied on Floer homology's ability to detect a geometric property that is carried by a single topological object, namely, the unknot is the only knot with Seifert genus equal to zero. There is another instance where Floer homology captures a geometric property of knots, and that is fiberedness. In particular, Ni shows the following in [53]:

Theorem 4.5.1. *Let $K \subset Y$ be a nullhomologous knot with genus g in an arbitrary 3-manifold such that $E(K)$ is irreducible, and suppose that $rk(\widehat{HFK}(Y, K, g)) = 1$. Then, K is a fibered knot.*

It turns out that genus 1 fibered knots in lens spaces are very rare. Building on work of Morimoto [51], Baker shows in [2] that any lens space (taken as an oriented manifold) contains at most 3 genus one fibered knots. In particular, for $m \neq 4$, $L(m, 1)$ contains exactly two genus one fibered knots, which are described below:

Definition 4.5.2. *For each $m \in \mathbb{N}$, define two knots K_m, $K'_m \subset L(m, 1)$ by the surgery descriptions in Figure 4.1.*

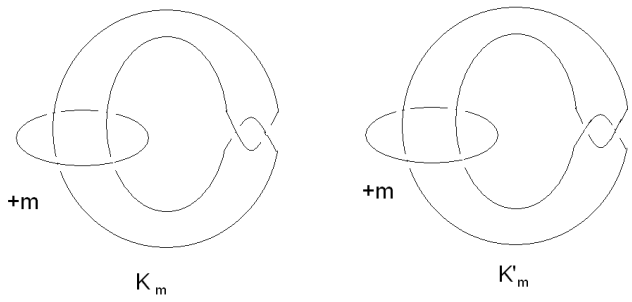


Figure 4.1: The genus one fibered knots in $L(m, 1)$.

The reader should interpret the picture as instructions for performing $+m$ surgery on the component of the link to which the surgery coefficient is attached. The remaining component is then a knot in $L(m, 1)$.

Remark 4.5.3. For completion, we remark that $L(4, 1)$ contains a third genus one fibered knot, and we refer the reader to Morimoto's paper [51] for its surgery description and further details.

Now, we can verify that some of these knots admit reducible surgeries: in order to do so, it is convenient to notice that the link presenting their surgery description (the Whitehead link) is strongly invertible, so that we may quotient out by a strong involution and pass to a tangle picture. Using the obvious strong involution on K_m, we get the following picture:

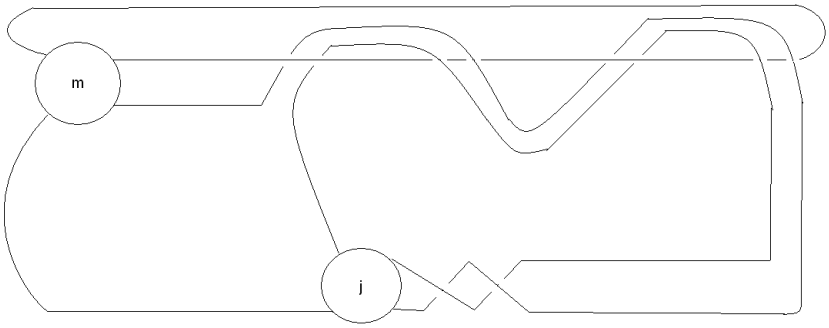


Figure 4.2: The result (after a few isotopies) of quotienting out the surgery description of K_m by the obvious strong involution. Notice that the blackboard framing in Figure 4.1 is $+2$, so that inserting $m, j \in \mathbb{Z}$ twists in the appropriate places gives a knot whose double branched cover is the result of $j+2$ surgery on $K_m \subset L(m,1)$.

From this, it can be easily seen that $+3$ surgery on K_3 gives $L(3,2)\#L(3,2)$, by setting $m = 3$, $j = 1$. Similarly, it can be seen, by setting $m = 2$, $j = 2$, that $+4$ surgery on K_2 yields $-L(4,1)\#L(2,1)$, and that $+1$ surgery on K_6 gives $-L(3,1)\#L(2,1)$, by putting $m = 6$, $j = -1$. This is shown in Figure 4.3 below:

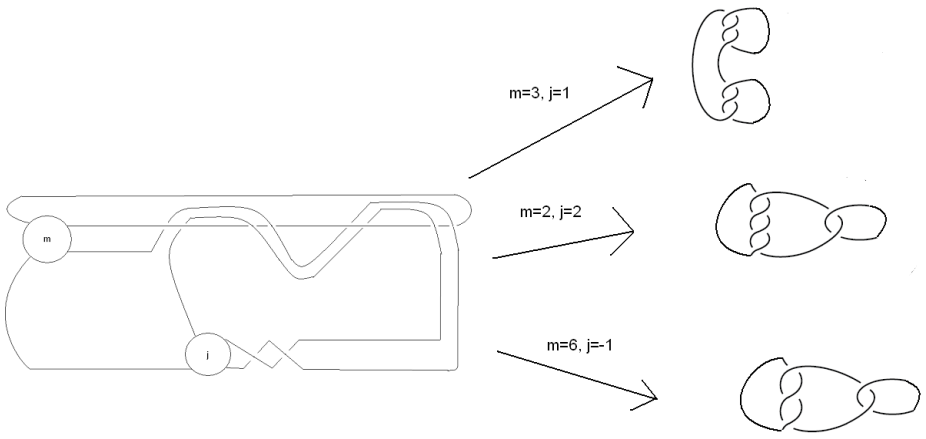


Figure 4.3: Reducible surgeries on some of the K_m.

Remark 4.5.4. The fact that K_3 and K_2 have reducible surgeries is not surprising, since the exteriors of these knots are Seifert fibered. This can be seen by inserting $m = 3$, $m = 2$ into the corresponding tangle diagrams (and leaving the j slot unfilled). After isotopy, we end up with the following:



Figure 4.4: The tangle description of $E(K_3)$ and $E(K_2)$, respectively. We see that $E(K_3)$ is a Seifert fibered space of type $D^2(3,3)$, and that $E(K_2)$ is a Seifert fibered space of type $D^2(4,2)$. In fact, the latter is one of the exceptional cases in Theorem 3.3.2.

The paucity of genus one fibered knots in lens spaces, combined with these examples, can be put together with calculations of the $l_{j,\mathfrak{t}}$ to obtain the following uniqueness results:

Proposition 4.5.5. *$K_3 \subset L(3,1)$ (resp. $K_6 \subset L(6,1)$) is the unique knot with the property that Dehn surgery on it gives $-(L(3,1)\#L(3,1))$ (resp. $-L(3,1)\#L(2,1)$). Moreover, $K_2 \subset L(2,1)$ and the unknot $U_2 \subset L(2,1)$ are the only knots in $L(2,1)$ with the property that surgery on them gives $L(-4,1)\#L(2,1)$.*

Proof. We start with surgery on $L(3,1)$ giving $-(L(3,1)\#L(3,1))$. For homological reasons, any knot with such a surgery must be nullhomologous, and the surgery coefficient is ± 3. Now, the correction terms for $L(3,1)$ are $\{\frac{1}{2}, -\frac{1}{6}, -\frac{1}{6}\}$. Using the negative surgeries exact sequence (in particular, Equation (4.21)), we see that $\sum l_{j,\mathfrak{t}}$ is not an integer, so this case cannot happen. Using the positive surgeries exact sequence (in particular, Equation (4.19)), we see that $\sum l_{j,\mathfrak{t}} = 2$. Now, a straightforward (and tedious) calculation analysing all combinations shows that the only viable possibility is for us to have

$$l_{0,0} = \frac{d(L(3,1),0) + d(L(3,1),0) - d(-(L(3,1)\#L(3,1)),0\#0)}{2} = 1$$

and $l_{j,\mathfrak{t}} = 0$ for all other *Spin*^{c} structures. In particular, since the $l_{j,\mathfrak{t}}$ are supported only in a torsion *Spin*^{c} structure, we conclude that $g(K) = 1$, by combining Theorem

4.3.2 and Proposition 4.3.6. Moreover, $\widehat{HFK}(Y,K,1)$ has rank one, and a knot in $L(3,1)$ with a surgery giving $-(L(3,1)\#L(3,1))$ obviously has an irreducible complement. We then apply Theorem 4.5.1 to conclude that the knot must be fibred. Thus, we have at most two knots with such a surgery, K_3 and K_3'. We've already seen that $+3$ surgery on K_3 gives $-(L(3,1)\#L(3,1))$. To check that $+3$ surgery on K_3' doesn't yield $-(L(3,1)\#L(3,1))$, we reflect its surgery description, to take advantage of our description of surgeries on K_m. Plugging in $m=-3$ and $j=-5$ should then give $b(3,1)\#b(3,1)$. After some simplification, what one obtains is the following:

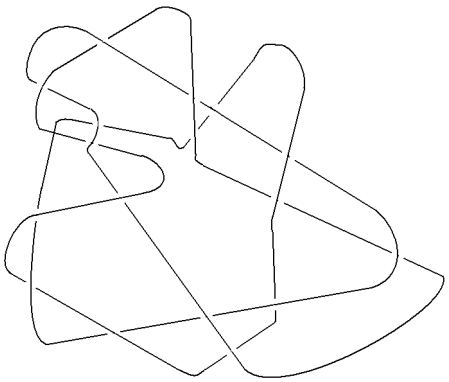


Figure 4.5: Result of inserting $m=-3$, $j=-5$ in Figure 4.2.

Using the computer software Knotplot, we calculate the HOMFLY polynomial of this knot to be $P(L,M)=10L^{-20}+...+M^{16}L^{-16}$. This is clearly not the HOMFLY polynomial of $b(3,1)\#b(3,1)$. We conclude that K_3 is the unique knot in $L(3,1)$ with a surgery giving $-(L(3,1)\#L(3,1))$.

The proof for $K_6\subset L(6,1)$ follows the same line of argument, so we omit the details.

For the final case, using Equation (4.19) we obtain that $\sum l_{j,\mathfrak{t}}=2$, and applying the same line of argument as before we conclude that our knot must be genus one fibered. This time, to check that $+4$ surgery on K_2' doesn't give $-L(4,1)\#L(2,1)$, simply notice that K_2' is the mirror image of K_2 (keep in mind that $L(2,1)$ admits an orientation reversing homeomorphism, so that we can talk about mirror images). If $+4$ surgery on K_2' yielded $-L(4,1)\#L(2,1)$ then K_2 would have -4 surgery giving $L(2,1)\#L(4,1)$. But this would mean that K_2 has two reducible surgeries at distance 8, which contradicts Theorem 3.2.11. Now, using Equation 4.21 for negative surgeries, we get that $\sum l_{j,\mathfrak{t}}=0$, and Corollary 4.3.9 implies that our knot is the unknot, thereby

completing the proof of the proposition. □

Remark 4.5.6. It is interesting to compare the proposition above with an analogous result for knots in S^3, which states that the trefoil knot is the unique knot in S^3 with a surgery yielding the Poincaré homology sphere.

It is natural to ask if other K_m', K_m have reducible surgeries. This is not the case: it is shown in Theorem 1.2 of [24] that the only pairs of Dehn surgery slopes on both components of the Whitehead link giving a reducible manifold are $\{+3, +3\}$, $\{+1, +6\}$ and $\{+2, +4\}$ (the order doesn't matter, since there is an isotopy interchanging both components), and these are precisely all the examples we have described.

CHAPTER 5 CONSEQUENCES FOR TANGLE EQUATIONS AND APPLICATIONS TO BIOLOGY

5.1 Consequences

We now discuss the consequences of the work in the previous two sections to the existence and character of solutions to our tangle equations, which we recall:

$$\begin{cases} N(O+P) = b(p,q) \\ N(O+R) = b(r,s)\#b(t,u) \end{cases} \tag{5.1}$$

Also, the reader is reminded that P and R are assumed to be rational, and that $r, t \neq \pm 1$, $p, r, t \neq 0$. We have the following results:

Theorem 5.1.1. *If there are prime tangle solutions O to Equation (5.1), then we have that $d(P,R) = 1$.*

Proof. This follows immediately by putting together Theorem 3.2.11 and Lemma 2.2.7. □

Recalling the definition of a Montesinos tangle, we can classify all such solutions. This is essentially a restatement of Proposition 3.3.1. In particular, recall the notation used in the proof of that result:

Theorem 5.1.2. *For a fixed choice knots/links $b(p,q)$, $b(r,s)\#b(t,u)$, in Equation (5.1), there exists a solution set $\{P,R,O\}$ such that O is a Montesinos tangle if and only if $R = T(\infty)$, $O = \mathcal{M}(-s^{\epsilon_r}/r, -u^{\epsilon_t}/t)$, and $(p,q) = (trm - ts^{\epsilon_r} - ru^{\epsilon_t}, -(ty - u^{\epsilon_t}x))$, where m, x, y are some integers subject to the constraint $(rm - s^{\epsilon_r})x - ry = 1$. If a solution exists, then $P = T(m/1)$.*

Proof. This is the proof of the aforementioned proposition, with the understanding that the m horizontal twists may be "pushed out" of the Montesinos tangle to define R. □

Next, we consider the special case $p = r$, $q = s$, $u = 1$:

$$\begin{cases} N(O + P) = b(p, q) \\ N(O + R) = b(p, q)\#b(t, 1) \end{cases} \tag{5.2}$$

There are clearly locally knotted solutions here, as we may consider solving the equations

$$\begin{cases} N(O' + P) = b(1, 0) \\ N(O' + R) = b(t, 1) \end{cases}$$

and then introducing an appropriately knotted arc into O' to obtained the desired solutions. It would be nice to show that only such locally knotted solutions may occur, but unfortunately, it follows from the discussion after Theorem 3.3.2 that this is not the case, in general. However, we can prove:

Theorem 5.1.3. *Let $K \subset S^3$ be an alternating knot with determinant $\Delta(K)$ an odd prime or one, and let t be a positive integer such that $t \equiv 3$ mod 4. Then, there are no prime tangle solutions O to the tangle equations*

$$\begin{cases} N(O + P) = K \\ N(O + R) = K\#b(t, 1) \end{cases}$$

Proof. It is a well known fact (see [65]) that $|H_1(\widetilde{K}); \mathbb{Z}| = \Delta(K)$. Moreover, it is shown in [61] that the double branched cover of an alternating knot/link is an L-space. The result now follows immediately from Theorem 4.4.10. □

5.2 Applications to biology

We now consider the final part of this thesis, which concerns itself with the application of some of the ideas we've treated to the problem of site-specific recombination in molecular biology.

5.2.1 Background in biology

The reader is probably familiar with DNA as a double helical molecule that contains information on how to synthesise proteins, which in turn determine the shape and

function of all known living organisms. Roughly speaking, it consists of complementary base pairs $A \sim T$, $G \sim C$ arranged into units called nucleotides. These turn out to be assembled in a "spiraling staircase" pattern depicted below:

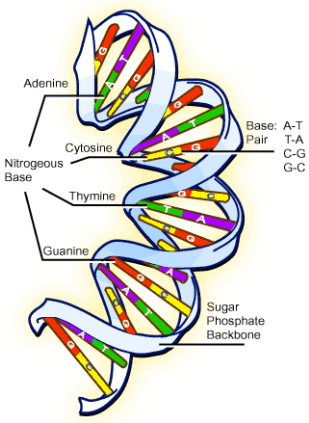


Figure 5.1: The famous DNA helix.

One feature of DNA that may not be known to the reader is that, in many organisms, the backbone of the double helix can be circular, rather than linear [11]. That is, DNA can (and certainly does) form closed loops, as schematised in the figure below. This occurs, for example, in bacterial genomic DNA, and in human mitochondrial DNA.



Figure 5.2: Schematic of a closed circular DNA configuration.

A moment's thought will convince the reader that the occurrence of closed circular DNA can give rise to topologically interesting behaviour, for example, as far as knotting of these loops is concerned. One process during which this occurs is known as **site-specific recombination (SSR)**. In a nutshell, SSR is a localised enzyme-driven reaction in which two disjoint short double stranded DNA segments are cleaved, recombined in some way, and then resealed. It is mediated by proteins known as **recombinases**.

Recombinases are the enzymes involved in this "precision surgery" of DNA. They act on short sequences, and perform operations which generally fall in to the categories of **excision** (removal of a segment), **integration** (insertion of a segment), or **inversion** (reversal of a segment). All these processes are of vital importance for cells. For instance, some viruses (e.g. bacteriophage λ infecting *E. coli* [67]) use integration to insert their DNA into the hosts, thereby infecting them. On the other hand, inversion is very useful for regulating gene expression: recombinases can be used to invert a DNA segment, and hence activate/halt the expression of a certain gene. An example of this is the variation on the *flagella* of *Salmonella* [45], which we will discuss later. Alternatively, inversion is an important feature in the transition of certain viruses from dormant to active state [1].

A site-specific recombination reaction involves several ingredients: a recombinase, two short double stranded DNA sequences, which may or may not be part of the same molecule, and possibly some accessory proteins that bind to the enzyme-DNA complex to facilitate the reaction (see, for example, [37]). Recombinases are specialised enzymes, in that they identify the identical DNA segments and bind exclusively to these sites, bringing them together and aligning them. They perform double stranded breaks, and then recombine the strands in an enzyme-specific manner. If circular DNA is used as a substrate for the reaction, one often obtains products that are knotted or linked. We will see plenty of examples of this phenomenon.

The process of site-specific recombination can be described as follows: in a first step, recombinases detect each individual specific DNA sequence and bind to it. Typically, four enzyme molecules bind to the sites (two for each separated segment, one in each strand of each segment). The two segments are then brought together, either by enzyme action or random thermal motion [67], and juxtaposed. The juxtaposed sites are known as the **crossover sites**. The enzyme-DNA complex that is formed is known as the **synaptic complex**. The particular region consisting of trapped DNA together with the enzyme is known as the **synaptosome**. In a second stage of the reaction, DNA strands are cleaved, exchanged and resealed within the synaptosome complex. A final stage of the reaction involves the breaking down of the synaptosome, and the disassociation of the protein from the DNA. Before this occurs, the protein can immediately cleave again and perform one more round of recombination. This phenomenon is known as **processive recombination**. Naturally, proteins may act

on DNA which has already undergone one or more full recombination events, and this is known as **distributive recombination**.

Given what was said above, it is natural to ask the following questions:

Question. *What sort of strand exchange mechanism does the protein employ?*

Question. *What is the configuration of the synaptosome during the reaction?*

In order to answer the questions above, Ernst and Sumners [67] introduced the **tangle model**, which combines information about DNA knotting/linking experiments with topological reasoning to reveal which mechanisms are possible *a priori*. In many cases, the model turns out to predict a unique solution, consistent with biological reasoning.

The starting point for the tangle model is information from different knot/link types arising from performing experiments where one adds a particular recombinase to a collection of unknotted circular DNA molecules in solution. After some time, the formation of a variety of products representing different knot/link types is usually observed. Using the technique of electron microscopy, it is sometimes possible to "take a photograph" of the products, as exemplified in the figure below:

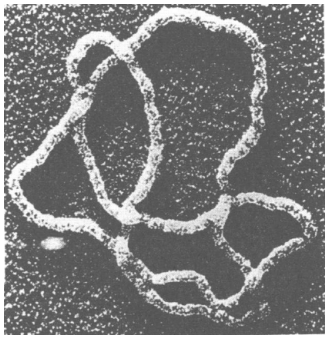


Figure 5.3: Electron micrograph of a RecA coated DNA knot, from [67]. The figure displays only the knotted central axis of the molecule, coated with Rec-A protein.

Remark 5.2.1. There are other experimental techniques to ascertain the knot type, but we will not discuss them. See [33] for a treatment of this.

Figure 5.4 motivates the tangle model, as one can see in it a "ball" (the heavy black dot) with some arcs emanating from it (represented by the heavy black lines). This suggests that one may think of site-specific DNA recombination using tangles.

The model is then formulated as follows: start with an initial unknotted substrate. One thinks of the synaptosome as a tangle P, which, together with the unbound DNA, specify the tangle equation

$$N(O + P) = b(1, 0)$$

The strand exchange mechanism in a recombination event is then modelled by replacing the P tangle by a tangle R, so that we have

$$N(O + R) = K_1$$

for some knot K_1.



Figure 5.4: Electron micrograph of the synaptic complex, from [67].

In a similar fashion, successive recombination events are written as

$$N(O + 2R) = K_2$$

$$\vdots$$

$$N(O + mR) = K_m$$

where mR is understood to be $\underbrace{R + ... + R}_{\text{m times}}$. The aim of the tangle model is to find solutions for O, P, and R, given information on the knot types $K_1, ..., K_m$. As we already remarked, this approach was carried out in several contexts, most notably, in [8, 9, 26, 67, 71, 72]. We will consider recombination experiments mediated by the protein Hin, carried out in [37], which we describe below.

5.2.2 The Hin recombinase

Hin is a site-specific recombinase from the serine family. It is found *in vivo* in members of the *Salmonella* family of bacteria, for example, in *Salmonella enterica* [45]. Its primary function is to regulate the expression of the so-called *fliC* and *fliB* genes. These genes are involved in the synthesis of the *flagellum* of the bacteria. The *flagellum* is a tail-like structure present in some bacteria, and it is their primary source of locomotion through media. A primary constituent of the *flagellum* is the protein known as flagellin. This helps the former achieve a helicoidal structure, which favours locomotion through media. It is precisely the synthesis of this protein that *fliC* and *fliB* regulate. Depending on the state of activity of these genes, slightly different versions of flagellin will be assembled. This phenomenon is known as **flagellar phase variation**. It is an important process, since *Salmonella* uses different versions of flagellin to elude the immune system of infected organisms.

Recombination by Hin inverts a particular DNA sequence, thereby acting as an "on-off" switch for gene expression. It acts on a so-called H segment, containing the regions that code for *fliB* and *fliC* [33]. It is known that, when H is in standard position, expression of *fliC* is inhibited, resulting in one type of flagellar phase, whereas when H is in inverted position, *fliB* is inhibited and *fliC* is expressed, resulting in the alternative flagellar variety. This is described in more detail by Kutsukake *et al* in [45]. H is a 996 base pair sequence, with two specific sites, each of length 26 base pairs, at its extremes. These sites are designated *hixL* and *hixR*, and it is known that these are the specific sites that Hin binds to. Figure 5.5 illustrates the known base pair sequence for *hixL*. The two letter subsequence AA (or TT, depending on the strand) adjacent to the vertical red line is the region where DNA cleavage takes place. Experiemental evidence suggests that cleavage occurs along the region specified by the red line. The *hixR* site has a slightly different sequence, but it also contains the cleavage sites highlighted.

The Hin mediated inversion reaction is catalysed by the so-called enhancer proteins HU and Fis. In the H segment, there is also a 65 base pair long **enhancer sequence**. It has been suggested [42] that, during the reaction, two pairs of molecules of Fis bind to the enhancer sequence to help form the synaptic complex: see Figure 5.6. The synaptosome consists then of four subunits of Hin, bound in pairs to each of the recombination sites. Following the terminology of [42], we will henceforth name it

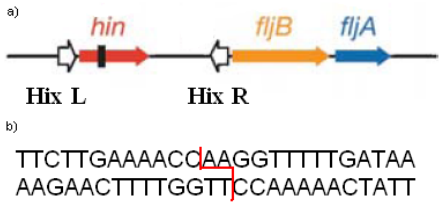


Figure 5.5: a) Schematic of the H sequence, together with the *hixL* and *hixR* sequence. The *hixL* sequence. b) The *hixL* sequence in detail, together with the cleavage sites indicated by red lines. Adapted from [33].

the **invertasome**.



Figure 5.6: Proposed model for the configuration of the invertasome. Black lines represent double stranded DNA segments. The white balls are the four subunits of Hin, whereas the solid black balls are units of Fis bound to the enhancer sequence (depicted by the white rectangle).

In [37], Heichman *et al* performed a series of experiments with Hin *in vitro*, on both "standard" and mutated circular DNA molecules. To create the mutants, they introduced a small change at the cleavage site of *hixL*, replacing AA by AT. These molecules were then recombined by Hin, and some of the resulting knot types were analysed both by agarose gel techniques and by electron microscopy. The electron microscopy results for recombination on mutated sites are displayed in Figure 5.7:

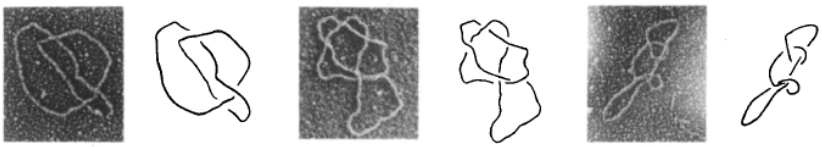


Figure 5.7: Rec-A enhanced electron microscopy images of the products of the pRJ862 reaction. From left to right, one sees a trefoil knot, the twist knot $b(7,3)$, and a composite knot $b(3,1)\#b(3,1)$. Taken from [37].

There is also the occurrence of a 7 crossing knot/link, the type of which was not imaged via electron microscopy. We will denote this by $K_?$. Following the discussion in Heichman's article, we have a sequence of reaction products as $b(1,0) \rightarrow$

$b(3,1) \to b(7,3) \to K_?$. This ordering in the products is justified by the fact that they correspond to increasingly fading lines in the gel electrophoresis experiments. One postulates that fainter lines should correspond to products of later stages of recombination, since these should be more rare.

Given the experimental information, one may model the $b(1,0) \to b(3,1) \to b(7,3) \to K_?$ steps of the reaction with the Ernst and Sumners tangle model, as follows:

$$\begin{cases} N(O+P) = b(1,0) \\ N(O+R) = b(3,1) \\ N(O+R+R) = b(7,3) \\ N(O+R+R+R) = K_? \end{cases} \tag{5.3}$$

Less straightforward, perhaps, but more exciting, is to account for the formation of $b(3,1)\#b(3,1)$ in terms of the tangle model. We begin by remarking the following:

Proposition 5.2.2. *There are no tangle solutions to the equations*

$$\begin{cases} N(Q+P) = b(1,0) \\ N(Q+R) = b(3,1)\#b(3,1) \end{cases}$$

with P and R rational.

Proof. This follows at once from the fact that surgery on a knot in S^3 must produce a manifold with cyclic first homology. □

Thus, this product cannot come from processive recombination: it must have been formed in a distributive recombination event. One reasonable biological explanation is that Hin acts on a plasmid with the knot type $b(3,1)$ and keeps the knot "localised", away from the recombination sites. However, other possibilities could occur *a priori*. The serine recombinases (including Hin) utilise a *topological filter model* [64]: the topology of the substrate can promote or inhibit the formation of the synaptosome.

Question. *When Hin acts on a knotted substrate such as b(3,1), does the topological filter necessitate a different amount of supercoiling within the synaptic complex to*

facilitate the assembly of the synaptosome? Rephrasing in terms of the tangle model, then, we ask if the "outside tangle" Q must compensate for the knot crossings of a knotted substrate.

We model this recombination step with the following tangle equations:

$$\begin{cases} N(Q+P) = b(3,1) \\ N(Q+R) = b(3,1)\#b(3,1) \end{cases} \tag{5.4}$$

The reader should note that we are not investigating the possibility of an alternative strand exchange mechanism, but rather, the formation of an alternative configuration for the outside bound DNA. Therefore, Q is not the same as O above, but P and R will be kept the same *a priori*. Our aim is, in the spirit of similar tangle analyses, to try to solve the tangle equations above, so as to list all the topologically possible reaction mechanisms for Hin recombination on mutated sites.

We start by solving (5.3), the equations for processive recombination. We begin by showing that O is rational:

Lemma 5.2.3. *O is rational.*

Proof. Since $N(O+P) = b(1,0)$, neither O nor P can be locally knotted. Suppose that O is prime. Then, by the same argument as in Lemma 3.1 of [67], P must be rational. That implies that $\widetilde{O}$ is homeomorphic to the complement of a nontrivial knot in S^3. Now, R cannot be locally knotted, otherwise we would see a $b(3,1)$ summand in $N(O+R+R)$. By the same argument as in Lemma 3.1 of [67] again, R must be rational. Taking double branched covers, we conclude that there exists a nontrivial knot in S^3 with a surgery giving $L(3,1)$. But this is at odds with Theorem 4.4.10. We conclude that O must be rational. □

Next, we show that R must be rational.

Lemma 5.2.4. *R is rational.*

Proof. This is exactly the same argument as in [67], Theorem 6.1. □

The last lemma we'll need is the following:

Lemma 5.2.5. *R is of the form $T(m/1)$.*

Proof. Consider $N(O+R+R+R) = K_?$. Since $N(O+R)$ and $N(O+R+R)$ are both knots, it follows that $N(O+R+R+R)$ is also a knot. Now, since $K_?$ has seven crossings, an appeal to knot tables shows that it is either a 2-bridge knot or $b(3,1)\#b(5,2)$ (up to mirror image). Now, since O and R are both rational, then $O+R+R$ and $R+R+R$ are both Montesinos tangles. We know that there exists a unique rational tangle attachment to a Montesinos tangle yielding a composite knot upon numerator closure, and this is precisely $T(\infty)$. If $N(O+R+R+R) = b(3,1)\#b(5,2)$, then either $R = T(\infty)$ or $O = T(\infty)$. In both cases it is easily seen that we can never get $b(3,1)\#b(5,2)$ from $N(O+R+R+R)$. Therefore, $K_?$ is a 2-bridge knot, and its double branched cover is a lens space. If R weren't of the form that is claimed, $\widetilde{K_?}$ would be a Seifert fibered space over the sphere with at least 3 exceptional fibers, and we know that cannot be a lens space. The result follows. □

Now, we can combine all of this into a calculation which produces all possible solution pairs $\{O, R\}$. In fact, we find a unique one.

Proposition 5.2.6. *$O = T(1/2)$, and $R = T(-2)$.*

Proof. Let $O = T(u/v)$, $R = T(r/1)$. Solving $N(T(u/v) + T(r/1)) = b(3,1)$ and $N(T(u/v)+T(r/1)+T(r/1)) = b(7,3)$ as in Proposition 3.3.1, we have, respectively,

$$|u + rv| = 3 \qquad \textit{and} \qquad |u + 2rv| = 7$$

Solving these equations over the integers, one finds that $\{O, R\}$ is one of the following pairs: $\pm\{13/5, -2\}$, $\pm\{13/10, -1\}$, $\pm\{13, -10\}$, $\pm\{13/2, -5\}$, $\pm\{1/2, -2\}$, $\pm\{-1/2, 2\}$. Now, $N(O+R+R+R)$ must yield a 7-crossing, 2-bridge knot. Consulting the table at the end of [10], we find that $\{1/2, -2\}$ is the only consistent solution. □

Incidentally, we have characterised the knot $K_?$:

Corollary 5.2.7. *Under the hypotheses of the tangle model, $N(O+R+R+R) = b(11,5)$.*

Proof. This is a straightforward computation. □

Unfortunately, little can be said about the tangle P. In fact, it admits infinitely many rational and prime solutions (see [67]). This is essentially due to the fact that

it only appears once in the equations, and this does not give enough information to determine its type. We adopt the common (and biologically most reasonable) approach and set $P = T(0)$.

The result obtained for R is consistent with the reaction mechanism proposed in Heichman's paper [37]: there, it is suggested that the action of Hin is obtained by cutting the DNA strands, and effecting a negative half twist on the DNA strands. In the mutated sites case that we are treating, the mismatch in the DNA sequence after the half twist makes it impossible to reseal the DNA, causing Hin to perform an additional round of recombination before resealing. See Figure 5.8.



Figure 5.8: Mismatch in the DNA sequence after one attempted round of recombination. Taken from [37]

The solution for O can also be interpreted in the context of the model for the invertasome: the two crossings in $T(1/2)$ are consistent with the fact that the enhancer sequence does indeed acquire the predicted relative position with respect to the crossover sites.

We now turn to the tangle equations (5.4) for the distributive recombination case.

Proposition 5.2.8. *Assuming that $P = T(0)$, $R = T(-2)$, then all the tangle solutions to (5.4) are the ones in the figure below:*

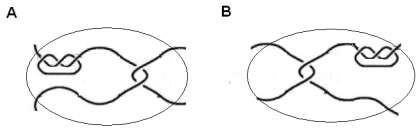


Figure 5.9: The only topological possibilities for Q.

Proof. Clearly, there are no rational solutions for Q. Now, since $d(T(0), T(-2)) = 2$, Theorem 5.1.1 implies that there are no prime tangle solutions. To compute locally knotted solutions, we excise the common $b(3, 1)$ factor to obtain

$$\begin{cases} N(Q' + P) = b(1, 0) \\ N(Q' + R) = b(3, 1) \end{cases}$$

The same proof as in Lemma 5.2.3 shows that Q' is rational. Letting $Q' = T(u/v)$, and performing a computation similar to that of Proposition 5.2.6, we get that $u/v \in \{-1, \frac{1}{2}\}$. Only $u/v = \frac{1}{2}$ yields a solution with the right chirality. □

The results suggest that the action of Hin during distributive recombination is essentially identical to that of processive recombination. In particular, this seems to indicate the formation of composite knots occurs as Hin traps a DNA molecule which has already undergone one round of recombination, with the existing DNA knot being localised outside of the invertasome. The invertasome maintains a configuration that is identical to the processive recombination steps. This is consistent with the fact that Hin is a highly restrictive protein, and the invertasome tends to be productive only under very special conditions [42]. These results are what one would naturally expect of distributive recombination yielding composite knots.

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