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**Direct simulation of entrainment
in forced fountains and cumulus clouds**

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Abstract

To investigate the entrainment between a shallow cumulus cloud and its environment, and the entrainment between the updraft and subsiding shell of the shallow cumulus cloud, the thesis is divided into two parts. In the first part, the entrainment of a simple analogue, a single-phase forced fountain, is examined; this has a similar structure to the cumulus cloud but is dynamically simpler. Furthermore, this flow is examined in a quiescent environment of uniform density. In the second part, the focus is moved to the shallow cumulus cloud, for which the dynamics are affected by the stratified atmosphere. The shallow cumulus cloud not only transports the temperature but also the humidity, involving the water phase change where the latent heat will influence the temperature, which can be regarded as a ‘moist fountain’. In order to examine full-scale details, both stages use direct simulation to study the fountain and cumulus cloud.

Declaration

I herewith certify that all material in this thesis which is not my own work has been properly acknowledged.

Jingzi Huang

October 2023

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Chapter 1

Introduction

1.1 Motivation

Clouds are attractive due to their immense beauty and variety. Moreover, they play an important role in Earth's climate and daily weather. Clouds are saturated portions of the atmosphere where small water droplets or ice crystals are visibly suspended (Stull, 2017) and typically cover about two-thirds of our planet. Therefore, they are essential parts of Earth's water cycle (Siebesma, 1998) in which water evaporated from the earth condenses and is temporarily stored within clouds, and is then brought back to the ground level via precipitation. Moreover, clouds are important in determining the temperature of the Earth, they block sunlight and heat during the daytime thereby having a cooling effect on the Earth, and they trap the heat at night keeping in warmth (Ramanathan *et al.*, 1989). Life, as we know it on Earth, would not exist without clouds.

Clouds vary in shape, leading to an intricate classification based on their formation, height, thickness and organisation. The World Meteorological Organisation (WMO) published an International Cloud Atlas listing detailed cloud categories broken into genera, species and varieties. Figure 1.1 presents some typical clouds that we can often see. Among them, the cumulus cloud is a common genus which is formed by an updraft. The updraft lifts the warm and moist air which condenses to form the cumulus cloud (Siebesma, 1998). Typically within the atmosphere, the higher the air is lifted, the

(a)



(b)



(c)



Figure 1.1: Three typical genera of clouds, (a) cumulus which are formed by vertical convection, (b) stratus which are grey cloud layers formed by horizontal advection, (c) cirrus which appear like curved white filaments, are high and contain ice crystals. Figures reproduced from International Cloud Atlas, WMO.

more easily the condensation occurs. However, the cumulus cloud is not simply an updraft, at the edge, more evaporation (as the cloudy air is more easily mixed with the ambient clear air there) occurs which cools the air and eventually makes it colder than its environment and falls, forming a downflow called ‘subsiding shell’ around the updraft (Jonas, 1990; Heus & Jonker, 2008). Figure 1.2(a) sketches this up-down flow structure of the cumulus cloud with the upward arrow showing the updraft and the downward arrow showing the subsiding shell around. Further details on the formation of these clouds and their subsiding shell are presented in §2.2.

Precipitation of the cumulus cloud will typically affect the cloud’s behaviour, the clouds focused on in this thesis are shallow cumulus clouds, where the ‘shallow’ de-

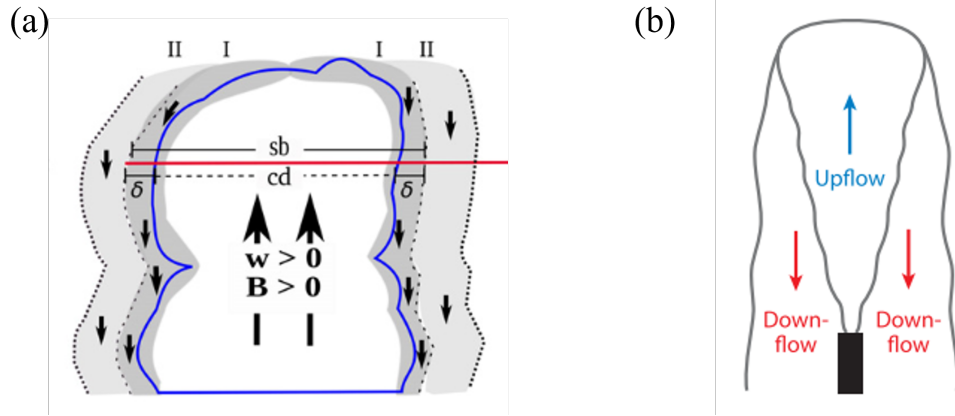


Figure 1.2: The sketches of the up-down flow structure of the (a) cumulus cloud and (b) fountain, respectively. (a) Katzwinkel *et al.* (2014) sketched the cloud updraft (in white and labelled as upward arrows) shrouded by the subsiding shell (in grey and labelled as downward arrows) for an active cloud. (b) Hunt & Burridge (2015) sketched the fountain upflow developed from the source is shrouded by the fountain downflow that is reversed by the negative buoyancy, in the quasi-steady state.

scribes the vertical extent of the cumulus cloud is insufficient for precipitation, e.g., we do not study rainy clouds. Shallow cumulus clouds impact the atmosphere by vertically transporting heat and humidity, a key process that affects this vertical transport is the inflow of environmental air into the clouds and the outflow of cloudy air into the environment, termed entrainment and detrainment, respectively, which controls the growth rate of the clouds (Siebesma *et al.*, 2003; De Rooy *et al.*, 2013). However, uncertainties associated with existing entrainment parametrisations of shallow cumulus clouds are significant (De Rooy *et al.*, 2013), and for example, propagate into climate prediction (Murphy *et al.*, 2004; Klocke *et al.*, 2011). These uncertainties arise, in part, due to the insufficient understanding of the physics under the entrainment of the clouds, which is something that this thesis sets out to address.

The entrainment process is not a unique problem to atmospheric science, it is also a generic challenge in fluid mechanics, even for canonical flows (Fernando, 1991) as the entrainment depends on specific cases. In the fluid community, entrainment is usually described as ‘turbulent entrainment’, with the term ‘turbulent’ emphasising the physical mechanism by which the entrainment occurs — the turbulent entrainment usually

refers to the transport of fluid from regions of relatively low to relatively high levels of turbulence via either the small-scale eddy nibbling or large-scale engulfment (Da Silva *et al.*, 2014). Pioneered by an early entrainment model (Morton *et al.*, 1956) which characterised the bulk entrainment of mass by a plume by relating the fluid entrainment velocity across the plume edge to local characteristic scales via a constant entrainment coefficient, numerous studies on the entrainment of various canonical flows have been performed, e.g., jets (Abramovich *et al.*, 1984; Van Reeuwijk & Holzner, 2014), plumes (List, 1982; Woods, 2010), fountains (Bloomfield & Kerr, 2000; Williamson *et al.*, 2011; Shrinivas & Hunt, 2014), boundary layer flow (Sillero *et al.*, 2013) and inclined gravity currents (Krug *et al.*, 2015). For example, figure 1.3 presents the entrainment of an axisymmetric jet in the steady state (Craske & Van Reeuwijk, 2015). On the right half which presents the jet in a Reynolds-averaged view, the entrainment is indicated by the streamlines showing the environmental fluid flowing into the jet core, across the jet edge (the inclined grey line). The radial expansion of the jet in the vertical direction is accompanied by the entrainment. The left half presents the instantaneous jet by the iso-surface of the velocity, indicating the jet has a relatively high level of turbulence compared with the environment, and the entrainment in the jet occurs via the turbulence.

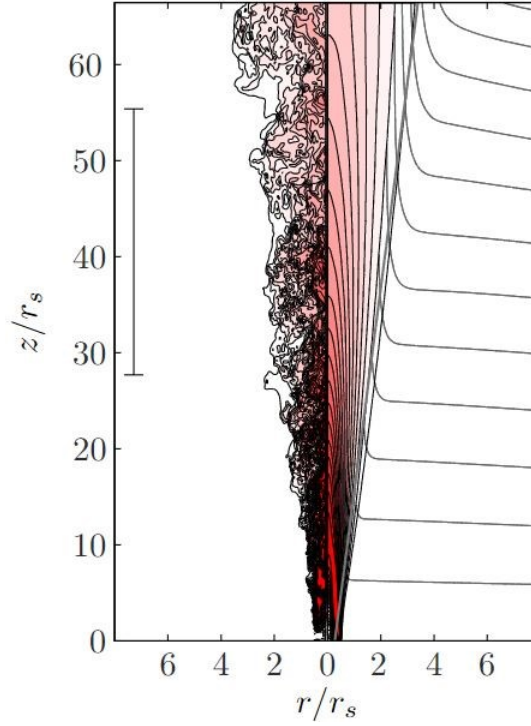


Figure 1.3: Isoregions of instantaneous longitudinal velocity (left-hand side) and Reynolds-averaged longitudinal velocity (right-hand side) of a jet, with darker shades corresponding to larger values. A solid grey line on the right-hand side indicates a theoretical jet width. Black lines on the right-hand side are the streamlines of the induced environmental flow. Figure reproduced from Craske (2016).

One of the main challenges of turbulent entrainment is at which interface it occurs and is quantified. This interface separates high-level and low-level turbulent regions but is not a physical boundary and is therefore always somewhat arbitrary (Da Silva *et al.*, 2014). The turbulent entrainment is generally investigated from a local or a global perspective (Van Reeuwijk *et al.*, 2021). Here, local entrainment refers to the transport across an interface obtained via an instantaneous indicator field χ and global entrainment describes the transport across an interface obtained via a Reynolds-averaged indicator field $\bar{\chi}$. An example of global entrainment is shown in the right half of a jet figure 1.3, where the interface is defined by a small percentage of the characteristic local jet velocity w_m , i.e. $\bar{\chi} = \bar{w}$ and $\bar{\chi}_0 = \varepsilon w_m$, where ε is a small value. Local entrainment is typically described using the enstrophy as the indicator field, i.e. $\chi = \omega^2$. For turbulent/non-turbulent interface (TNTI, e.g., the edge of a turbulent flow in a non-

turbulent environment), there is a range of threshold values for which the (conditional) statistics are independent of the threshold values (Da Silva *et al.*, 2014). While, for a turbulent/turbulent interface (TTI, e.g., the edge of a turbulent flow in a turbulent environment), the interface may be defined on another kind of indicator field (such as the velocity field). Examples are the cylinder wake (Baj & Buxton, 2019; Kankanwadi & Buxton, 2020), and jets (Gaskin *et al.*, 2004; Khorsandi *et al.*, 2013) in a turbulent background. The main contribution of Van Reeuwijk *et al.* (2021) is that it provides a universal connection between global and local entrainment. In this thesis, we will investigate the flows from both perspectives, seeking the consistency and difference between them.

The physics of entrainment across TNTIs and TTIs differs. From the global perspective — the Reynolds-averaged view (specifically, time average and azimuthally average, see (3.6), throughout this thesis on the axisymmetric flows) — the exchanges (of mass, momentum, and buoyancy) across TNTI are associated with a mean flow (Morton *et al.*, 1956; Bloomfield & Kerr, 2000). However, the physics of exchanges across TTI is much more complex, since there is the possibility of turbulent exchanges as well as exchanges due to pressure (Van Reeuwijk *et al.*, 2021). Additionally, as both sides are turbulent, the entrainment across the TTI is more likely to be bi-directional than that of the TNTI (Kankanwadi & Buxton, 2020; Milton-McGurk *et al.*, 2021), i.e., entrainment and detrainment may co-exist at a TTI. The TNTI has received more focus as it is usually the kind of natural boundary between a flow and its environment, (see Da Silva *et al.*, 2014, for a review), while the study on the entrainment across TTI is still in its infancy (Kankanwadi & Buxton, 2020; Van Reeuwijk *et al.*, 2021). A precise introduction to turbulent entrainment will be given in §3.1.

Shallow cumulus clouds involve both the entrainment across a TNTI and a TTI. The envelope of the cloud, separating the cloud and the free troposphere is a TNTI (Siebesma, 1996; Romps, 2010; Nair *et al.*, 2021). Conversely, the interface between the cloud updraft and subsiding shell (Figure 1.2 a) is a TTI as both sides of it are turbulent. The entrainment across both the TTI and the TNTI affects the life cycle of the cloud. Historically, meteorologists used steady-state plume models to describe

shallow cumulus clouds (Morton *et al.*, 1956; Squires & Turner, 1962). However, as the cloud is transient, behaving apparently differently over its life cycle from developing to dissipating, the steady-state model is not appropriate. It is also more useful to model an ensemble cloud field which can be incorporated into climate and weather prediction models (Arakawa & Schubert, 1974). Therefore, the atmospheric community has switched their focus to the cloud ensemble and applied a conditional sampling method to identify the cloud from the environment (e.g., examining whether an air parcel contains liquid water) (Siebesma & Cuijpers, 1996; Roode *et al.*, 2012). Experiment campaigns on cloud studies provide valuable data on the cloud and atmosphere (Holland & Rasmusson, 1973; Rauber *et al.*, 2007) but are extremely hard and expensive due to the measurement techniques. Therefore, the large eddy simulation (LES) has been a mature approach to studying the entrainment of the cloud ensemble since the 1990s (Cuijpers & Duynkerke, 1993; Siebesma & Cuijpers, 1996; Siebesma *et al.*, 2003), whose resolution is high enough to resolve the largest eddies over a wide range of cloud fields.

The ultimate aim of this thesis is to apply a fluid mechanics approach to shallow cumulus clouds, focusing on an individual shallow cumulus cloud in a digital cloud chamber so that we can accurately quantify the entrainment and mixing and avoid the variety of cloud heights. Direct numerical simulation (DNS), which is able to capture the full range of active scales, is used to provide the data that enables in-depth understanding of the cloud and its interaction with the environment. However, the simulation of a single shallow cumulus cloud is far from trivial as it requires i) the transport of two scalars (temperature and humidity); ii) phase changes (condensation/evaporation of water) and iii) a stratified environment. Therefore, the focus of the first part of this thesis is on a simpler flow: a turbulent forced fountain in a quiescent unstratified environment.

Turbulent fountains are beautiful localised canonical flows with a similar up-down flow structure as cumulus clouds (figure 1.2b). In single-phase flows, a fountain is created when ejecting fluid which is heavier than the ambient upwards against gravity. Initially, this creates a negatively buoyant rising jet, but eventually, the fluid has no kinetic energy left and starts falling, causing the formation of a downflow shroud-

ing the central upflow. The interface between the turbulent fountain and the quiescent environment is a TNTI and the interface between upflow and downflow is a TTI. Fountains have wide applications in industry and atmosphere, for example, determining the environment within the disposal of effluent into the ocean (Koh & Brooks, 1975), the replenishment of magma chambers in the Earth’s crust (Turner, 1986), within buildings (Burridge & Hunt, 2016), water recirculation and purification (Williamson *et al.*, 2011) and the forced heating or cooling of enclosed spaces (Baines *et al.*, 1990). However, the entertainment of forced fountains (Bloomfield & Kerr, 2000; Debugne & Hunt, 2016; Burridge & Hunt, 2016; Williamson *et al.*, 2011) has not received as much focus as other canonical flows like the jets and plumes. The measurement of the local entrainment of the forced fountains is lacking, especially across the TTI between the upflow and downflow. Moreover, there is rare study on the exchange of momentum and buoyancy of the fountains that would be coupled by the entrainment. A precise introduction to forced fountains and relevant studies will be given in Chapter 3.

1.2 Project scope

The aim of this thesis is to investigate entrainment across TNTIs and TTIs within forced fountains and shallow cumulus clouds, and hence contribute to the understanding of localised convective flows of relevance to atmospheric processes. To meet this aim, the following objectives are set:

1. Study the global entrainment across the turbulent/non-turbulent interface (TNTI), and turbulent/turbulent interface (TTI) within a turbulent forced fountain using Reynolds-averaged statistics;
2. Develop a methodology based on conditional averaging to quantify instantaneous and local exchanges across arbitrary interfaces;
3. Study the local entertainment of the fountain across the TNTI and TTI with the conditional-averaged statistics and compare the results with Reynolds-averaged statistics;

4. Develop a digital cloud chamber to study an idealised shallow cumulus cloud within a quiescent stratified environment, including phase change;
5. Explore the dynamics of the idealised shallow cumulus cloud, including examining relevant interfaces within and investigating the global and local entrainment using Reynolds-average and conditional-average statistics, respectively.

Direct Numerical Simulation (DNS) will be employed to study the both forced fountain and shallow cumulus cloud to provide high-resolution access to the full three-dimensional data.

1.3 Thesis outline

The thesis is structured as follows. In Chapter 2, we provide the relevant background, including derivation of the key thermodynamics associated with shallow cumulus clouds, and an introduction to the DNS code SPARKLE. We then derive and validate an algorithm that calculates the integral over an arbitrary iso-surface, which will be employed in measuring local entrainment in conditional statistics. Chapter 3 explores the forced fountain with Reynolds-averaged statistics, identifying the internal structures of the fountain and examining the net entrainment of volume, momentum and buoyancy between the structures via interfaces. Chapter 4 explores the same fountain with conditional-averaged statistics. The algorithm presented and validated in Chapter 2 is applied here to measure the local entrainment of volume, momentum and buoyancy via the equal-definition interfaces in Reynolds-averaged statistics. Chapter 5 examines a shallow cumulus cloud by adding moisture and a stratified atmosphere in the simulation domain. The Reynolds-averaged statistics and conditional-averaged statistics are exploited to examine the cloud, with a comparison made of the two statistics. Finally, Chapter 6 draws conclusions from the thesis.

1.4 Publications of work

The work in Chapter 3 and 4 have been published in the Journal of Fluid Mechanics, and listed below, respectively.

Huang, J., Burridge, H.C., & van Reeuwijk, M. (2023a). The internal structure of forced fountains. *J. Fluid Mech.*, **961**, A31.

Huang, J., Burridge, H.C., & van Reeuwijk, M. (2023b). Local entrainment across a TNTI and a TTI in a turbulent forced fountain. *J. Fluid Mech.*, **977**, A13.

Chapter 2

Background

This chapter provides the fundamental background for the work in subsequent chapters. §2.1 presents the governing equations for turbulent fountains and atmospheric flows that are implemented in the DNS code. §2.2 provides essential knowledge on thermodynamics including the variables, condensation scheme in phase change and adiabatic process of moisture air. §2.3 introduces the DNS code that will be used.

2.1 The governing equations

This section introduces the governing equation of the turbulent fountains and the atmospheric flows, respectively. However, the following assumptions needed to be noted in the modelisation strategies in advance. The Boussinesq approximation is made for all kinds of flows, where the density difference only matters in the buoyancy force. In the atmospheric flows, specifically considering the shallow cumulus clouds, no aerosol nor supersaturation, no precipitation nor ice is considered in the clouds. Moreover, a bulk condensation scheme is applied to model the water phase change in the clouds. These assumptions will be noted again near the relevant equations in the following sections.

2.1.1 Turbulent fountains

The governing equations for turbulent fountains used in this thesis are the incompressible Navier-Stokes in the Boussinesq approximation (Bird *et al.*, 2002; Turner, 1973):

$$\nabla \cdot \mathbf{u} = 0, \quad (2.1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{u}\mathbf{u} + \nabla p = \nu \nabla^2 \mathbf{u} + b\mathbf{e}_z, \quad (2.2)$$

$$\frac{\partial b}{\partial t} + \nabla \cdot \mathbf{u}b = \kappa \nabla^2 b, \quad (2.3)$$

representing the conservation of mass, momentum and buoyancy, respectively. The term $\mathbf{u}(\mathbf{x}, t)$ is the fluid velocity of the spatial coordinates $\mathbf{x}$ and time t , ν is kinematic viscosity, κ is the diffusivity of the air (we assume the fountains are air flows), and $\mathbf{e}_z$ is the unit vector in the vertical direction. The Boussinesq approximation assumes that the density differences in the flow are small relative to a reference density ρ_0 , and solely manifest themselves in the buoyancy force b . Note that, $p(\mathbf{x}, t) = (p_t - p_e)/\rho_0$ is the deviatoric kinematic pressure, where p_t is the local total pressure and p_e is the local hydrostatic pressure given as

$$\frac{dp_e}{dz} = -\rho_e g, \quad (2.4)$$

where g is the gravity. As the vertical length scale of the fountain is small relative to the height over which hydrostatic variation in environmental density ρ_e occurs, herein, we assume constant ρ_e , i.e., the unstratified environment. The buoyancy $b(\mathbf{x}, t)$ is defined by the densities and further associated with the temperature (Turner, 1973; Tritton, 1988)

$$b(\mathbf{x}, t) \equiv \frac{\rho_e - \rho}{\rho_0} g \equiv \beta(T - T_e), \quad (2.5)$$

where $\rho(\mathbf{x}, t)$, $T(\mathbf{x}, t)$ are the fluid density and temperature, respectively. The environmental temperature T_e is constant according to the assumption of constant p_e , and β is the expansion coefficient of an ideal gas which is approximately $1/300$.

2.1.2 Atmospheric flows

The governing equations for atmospheric flows are greater complexity than those of turbulent fountains, due to the large density variations over the height (i.e., the stratified environment) and the phase changes of water. The main challenge is that the reference density ρ_0 has to be a function of height, $\rho_0(z)$, to fit the Boussinesq assumption in the stratified environment, which produces additional pressure gradient terms in the momentum conservation equation. Nevertheless, by choosing appropriate temperature definitions, e.g., the potential temperature which is the temperature considering the pressure effect, it is possible to obtain a system of equations that is very similar to those of turbulent fountains (Wyngaard, 2010):

$$\nabla \cdot \mathbf{u} = 0, \quad (2.6)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \mathbf{u}\mathbf{u} + \nabla p = \nu \nabla^2 \mathbf{u} + b\mathbf{e}_z, \quad (2.7)$$

$$\frac{\partial \theta_l}{\partial t} + \nabla \cdot \mathbf{u}\theta_l = \kappa \nabla^2 \theta_l, \quad (2.8)$$

$$\frac{\partial q_t}{\partial t} + \nabla \cdot \mathbf{u}q_t = D \nabla^2 q_t, \quad (2.9)$$

where κ is the molecular diffusivity of the air and D is the molecular diffusivity of water vapour. Note that, in the simulation, D is also set to be the molecular diffusivity of air as the total mass of water (including the vapour) only takes a very small portion of the total cloud mass (as the total water specific humidity is often in an order of 10^{-3}), so that, the cloud diffuses closer to a flow of air rather than a flow of water (vapour) (Nair *et al.*, 2020). The last two equations present the transport of two conserved variables of the moist air during the motion — the liquid water potential temperature $\theta_l(\mathbf{x}, t)$ and the total water specific humidity $q_t(\mathbf{x}, t)$, respectively. For the moist air, the temperature is not directly used to determine the buoyancy b . The idea is that the buoyancy of the moist air equals the buoyancy of an equivalent dry air which has the same energy as the moist air, resulting in

$$b \equiv \beta(\theta_v(\mathbf{x}, t) - \theta_{v,e}(z)), \quad (2.10)$$

where $\theta_v(\mathbf{x}, t)$ and $\theta_{v,e}(z)$ are the virtual potential temperature of the moist air (i.e., the potential temperature of the equivalent dry air) of the flow and environment, respectively. A detailed introduction of the above temperature and humidity variables can be found in the next section.

2.2 Atmospheric Thermodynamics

2.2.1 Thermodynamics variables

Moist air, i.e. air in the presence of water, plays an important role in the atmosphere. Moist air is a mixture of dry air, water vapour and liquid water (ice will not be considered here). The dry air and water vapour comprise the gas portion of the mixture while liquid water exists as suspended droplets (Wyngaard, 2010). Hereafter we denote the dry air, the water vapour and liquid water by subscript ‘ d ’, ‘ v ’ and ‘ l ’, respectively. Now consider a moist air mixture with mass m , volume V and temperature T , the mass relation is

$$m = m_d + m_v + m_l. \quad (2.11)$$

Generally, the word ‘humidity’ refers to how much water (in any phase) is contained within the air mixture. Therefore, the water vapour specific humidity q_v is defined as the ratio of the water vapour mass to the total mixture mass; and the liquid water specific humidity q_l is defined as the ratio of the liquid water mass to the total mixture mass

$$q_v = \frac{m_v}{m}, \quad q_l = \frac{m_l}{m}, \quad (2.12)$$

respectively. The total water specific humidity q_t is their summation (Betts, 1973)

$$q_t = \frac{m_v + m_l}{m} = q_l + q_v. \quad (2.13)$$

Note that, as the water transfers between the gas and liquid, q_t is conserved if there is no external water exchange, e.g., via precipitation.

Dalton's law states that in a mixture of non-reacting gases, the total pressure p_m is equal to the sum of the partial pressures of the individual gases (Siebesma, 1998). The partial pressure for the dry air p_d and the water vapour p_v are given by the ideal gas law

$$p_d = \frac{m_d}{V} R_d T, \quad p_v = \frac{m_v}{V} R_v T, \quad (2.14)$$

respectively. Here, the volume V and temperature T are the values of the air mixture, which are the same for the two portions. Furthermore, $R_d \approx 287.05 \text{ J kg}^{-1} \text{ K}^{-1}$ is the specific gas constant for dry air, and $R_v \approx 461.50 \text{ J kg}^{-1} \text{ K}^{-1}$ is the specific gas constant of the water vapour. The total pressure of the air mixture p_m then gives

$$p_m = p_v + p_d = \frac{m_v}{V} R_v T + \frac{m_d}{V} R_d T = \frac{m}{V} R_m T. \quad (2.15)$$

where R_m is the specific gas constant of the mixture given by

$$R_m = \frac{m_v}{m} R_v + \frac{m_d}{m} R_d = (1 - q_t) R_d + q_v R_v. \quad (2.16)$$

Alternatively, (2.15) can be rewritten as $p_m = \frac{m}{V} R_d T_v$ where T_v is the virtual temperature defined as

$$T_v = \frac{R_m}{R_d} T = \left(1 - q_t + q_v \frac{R_v}{R_d} \right) T \approx (1 + 0.608 q_t - 1.608 q_l) T. \quad (2.17)$$

The virtual temperature provides the bridge connecting the moist air and dry air — it is an equivalent temperature of the dry air with the same density as the moist air.

In an adiabatic process, the air does not exchange heat with the ambient. However, this does not mean the temperature of the air is always constant. In the 'dry' adiabatic process where no liquid water is involved in the air, the adiabatic expansion or compression due to the local pressure change will change the temperature. In the 'moist' adiabatic process, besides the pressure, the latent heat from the phase change will also change the temperature. The potential temperature is defined to be conservative during the adiabatic processes, by considering the pressure.

During the dry adiabatic process, a potential temperature θ (unit ‘K’) is utilised on an unsaturated air parcel with pressure p_m and (absolute) temperature T . The potential temperature θ is defined as the temperature that the air parcel would have if it were brought adiabatically to the standard reference pressure p_r , generally, $p_r = 1000\text{mb}$. The potential temperature is written as below by considering the conservation of the energy (Siebesma, 1998),

$$\theta = T \left(\frac{p_r}{p_m} \right)^{\frac{R_d}{c_p}}, \quad (2.18)$$

where $c_p = 1003\text{Jkg}^{-1}\text{K}^{-1}$ is the specific heat capacity of dry air at constant pressure. Here $(p_m/p_r)^{R_d/c_p}$, is typically referred to as the Exner function Π and is an important parameter in atmospheric modelling.

During the moist adiabatic process, a liquid water potential temperature θ_l (unit ‘K’) is utilised on a saturated air parcel, apart from the pressure, it takes the liquid water and potential phase change into account. Betts (1973) derived

$$\theta_l = \theta \exp \left(-\frac{L}{c_p} \frac{q_l}{T} \right), \quad (2.19)$$

where L is the latent heat of the vaporisation of water. Sommeria & Deardorff (1977) gave a more common approximate form of θ_l by using Taylor expansion around $q_l = 0$, which is

$$\theta_l \approx \theta - \frac{L}{c_p} \frac{\theta}{T} q_l. \quad (2.20)$$

Note that θ_l reduces to θ if there is no liquid water in the air ($q_l = 0$), which indicates that the liquid water potential temperature θ_l of an air parcel conserves in both the dry and moist adiabatic processes.

Combining (2.18) and (2.20) to eliminate T , we directly link the potential temperature to the pressure (via Π)

$$\theta_l = \theta - \Pi \frac{L}{c_p} q_l. \quad (2.21)$$

The virtual potential temperature can be obtained by dividing (2.17) by the Exner

function Π with the result

$$\theta_v = \theta (1 + 0.608q_t - 1.608q_l) , \quad (2.22)$$

which represents an equivalent potential temperature of a dry air parcel with the same density.

2.2.2 Bulk condensation scheme

As stated in §2.1.2, q_t and θ_l are chosen to be the prognostic variables. In order to partition q_t into the water vapour specific humidity q_v and the liquid water specific humidity q_l , a condensation scheme needs to be specified. In this thesis, one of the simplest schemes is chosen, namely a bulk condensation scheme due to Sommeria & Deardorff (1977), which avoids having to deal with aerosols and supersaturation. Its key assumption is that the partitioning between liquid water and water vapour is controlled by the saturated specific humidity q_s which is a function of the pressure and temperature, i.e., $q_s \equiv q_s(T, p)$. Then, the liquid water specific humidity q_l is determined as

$$q_l = \begin{cases} q_t - q_s, & \text{if } q_t > q_s, \\ 0, & \text{if } q_t \leq q_s. \end{cases} \quad (2.23)$$

The saturated specific humidity $q_s(T, p)$ is calculated using a Taylor expansion around $T = T_l$, where $T_l = \Pi\theta_l$ (Siebesma, 1998)

$$q_s(T, p) = q_{sl} + \left(\frac{\partial q_s}{\partial T} \right)_{T=T_l} (T - T_l) + O(\Delta T^2), \quad (2.24)$$

where q_{sl} can be determined from Tetens equation (Tetens, 1930)

$$q_{sl} = \frac{e_s(T_l)}{p - \left(1 - \frac{R_d}{R_v}\right) e_s(T_l)} \frac{R_d}{R_v}. \quad (2.25)$$

For typical temperatures in the atmosphere boundary layer, Murray (1967) approximated e_s as follow

$$e_s(T) = e_{s,0} \exp \left(a \frac{T - T_{trip}}{T - b} \right), \quad (2.26)$$

where $e_{s,0}$, a , b , T_{trip} are all empirical constants, with

$$e_{s,0} = 610.78 \text{Pa}, \quad T_{trip} = 273.16 \text{K}, \quad a = 17.27, \quad b = 35.86.$$

From Clausius-Clapeyron equation (Sommeria & Deardorff, 1977), the gradient $\frac{\partial q_s}{\partial T}$ gives

$$\left(\frac{\partial q_s}{\partial T} \right)_{T=T_l} = \frac{L q_{sl}}{R_v T_l^2}. \quad (2.27)$$

Finally, substituting (2.25) and (2.27) into (2.24), results in

$$q_s = q_{sl} \frac{1 + \beta_1 q_t}{1 + \beta_1 q_{sl}}, \quad (2.28)$$

where

$$\beta_1 = \left(\frac{L}{R_v T_l} \right) \left(\frac{L}{c_p T_l} \right). \quad (2.29)$$

2.2.3 Hydrostatic pressure

The pressure of the air parcel p_m is assumed to be equal to the local hydrostatic pressure p_e as we see in the data that the pressure fluctuation in the air is relatively small. The hydrostatic pressure p_e can be integrated from the hydrostatic balance (2.4)

$$\frac{dp_e(z)}{dz} = -\rho(z)g = -\frac{p_e(z)g}{R_d T_{v,e}(z)} = -\frac{p_e(z)g}{R_d \theta_{v,e}(z) \left(\frac{p_e(z)}{p_r} \right)^{\frac{R_d}{c_p}}}. \quad (2.30)$$

Here, the subscript ‘ e ’ represents the environmental values, and the gas constant of the dry air R_d is used as the atmosphere is assumed to be unsaturated. Note that

$$\frac{dp_e(z)^{\frac{R_d}{c_p}}}{dz} = \frac{R_d}{c_p} p_e(z)^{\frac{R_d}{c_p}-1} \frac{dp_e}{dz}, \quad (2.31)$$

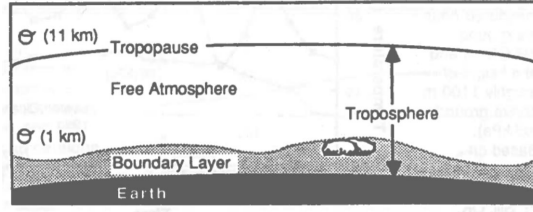


Figure 2.1: The troposphere structure. Figure reproduced from Stull (1988).

which implies that (2.30) can be rewritten as

$$\frac{dp_e(z) \frac{Rd}{c_p}}{dz} = \frac{gp_r \frac{Rd}{c_p}}{c_p \theta_{v,e}(z)}. \quad (2.32)$$

This expression allows us to calculate the pressure by integration of the ODE above if the atmosphere's virtual potential temperature $\theta_{v,e}(z)$ is given.

2.2.4 Atmospheric structure

The atmosphere generally consists of four layers: from low to high, namely, the troposphere, stratosphere, mesosphere, and thermosphere (Stull, 2017). The troposphere is the lowest layer of the atmosphere where humans live and weather happens (see figure 2.1). The height of the troposphere varies between 8km-11km above sea level, depending on the weather. It contains 75% of the atmosphere's mass and 99% of the water vapour. In the troposphere, the temperature and pressure decrease with height. The troposphere can be divided into two parts, an atmosphere boundary layer (ABL) near the Earth's surface and a free atmosphere above it. The ABL is around 500 – 1000 meters height above sea level, which is directly influenced by the presence of the Earth's surface. In the ABL, the dry and moist air mix, preparing for cloud formation. Above the troposphere lies the stratosphere with a transition boundary between them called the tropopause. The stratosphere houses the ozone layer that absorbs the sun's ultraviolet radiation, therefore, the temperature increases with height within this region. An aircraft can stay in the troposphere and sometimes reach the stratosphere. Most clouds are located in the troposphere and taller clouds could reach the bottom of the stratosphere.

The vertical atmospheric temperature structure is typically complex due to the work of the air expansion and the latent heat during the phase change (Siebesma, 1998). However, assuming hydrostatic conditions, we can estimate two lapse rates, one for a dry adiabatic process and one for a moist adiabatic process.

Consider an unsaturated air parcel of the atmosphere which rises adiabatically. Due to the decrease in pressure, the air parcel does work to enlarge its volume, consuming internal energy, hence, the air parcel temperature decreases, due to the first law of thermodynamics:

$$c_p dT = \frac{V}{m} dp. \quad (2.33)$$

Combined with hydrostatic equilibrium (2.4), this equation can be written as

$$\frac{dT}{dz} = -\frac{g}{c_p} \approx -9.8^\circ\text{C/km}, \quad (2.34)$$

which means the temperature of an unsaturated air parcel reduces about 9.8°C/km when it rises adiabatically. This rate is called the dry adiabatic lapse rate. When the air parcel becomes saturated, condensation occurs, and then the parcel experiences a moist adiabatic process. During the condensation, latent heat is released which compensates for the temperature reduction. Therefore, the overall air temperature reduction rate which is now called moist adiabatic lapse rate is approximately -5.0°C/km . For the detailed derivation of the value, one can refer to (Siebesma, 1998; Wyngaard, 2010).

It should be noted that the above lapse rates refer to the absolute temperature T , showing how T reduces with the height in the atmosphere. Alternatively, we can also look at the lapse rate from the virtual potential temperature θ_v , sight to consider the pressure and humidity of the fluid. From equation (2.22), in the dry adiabatic process, note that θ and q_t converse during the process and $q_l = 0$, so that the dry adiabatic lapse rate for θ_v is zero. Whereas in the moist adiabatic process, with the latent heat released, θ_v will increase, so that the moist adiabatic lapse rate for θ_v is positive. Figure 2.2 shows the temperature variation based on the virtual potential temperature.

However, not all environmental lapse rates can form clouds, they must satisfy appropriate atmospheric stability. To access atmosphere stability, we consider a rising air

parcel whose temperature equals the local environmental temperature initially. The air parcel has either a dry lapse rate (if unsaturated) or a moist lapse rate (if saturated) during the adiabatic process, while the environmental temperature reduces with the height at its own environmental lapse rate. Here are three cases:

- I. When the environmental lapse rate is smaller than both the dry and moist lapse rate, the air parcel, no matter unsaturated or saturated, will always be cooler than the environment so that it sinks (back to the ground). This condition of the environment is referred as to absolute stability. This is usually the case at the top of the cloud, where absolute stability forms an inversion layer that prevents the cloud from rising.
- II. Conversely, when the environmental lapse rate is greater than both the dry and moist lapse rate, the air parcel will always be warmer than the environment so that it continuously rises. This condition is referred as to absolute instability. This is usually the condition in which the cumulus congestus and cumulonimbus can be observed.
- III. When the environmental lapse rate is in the middle, smaller than the dry lapse rate but greater than the moist lapse rate, the air parcel will be cooler than the environment and sink when it is unsaturated while, it will be warmer than the environment and rise when it is saturated. This condition is referred as to conditional instability. This is usually the case where the active shallow cumulus clouds form (figure 2.2).

2.2.5 Cumulus cloud formation

As we know, the cloud is the saturated portion of the atmosphere with tiny droplets or ice crystals. In this section, we summarise how the cumulus cloud forms in the atmosphere of conditional instability by considering a moist air parcel ascending adiabatically, Siebesma (1998) is referred for further detailed introduction. However, again, the precipitation and ice are not considered here.

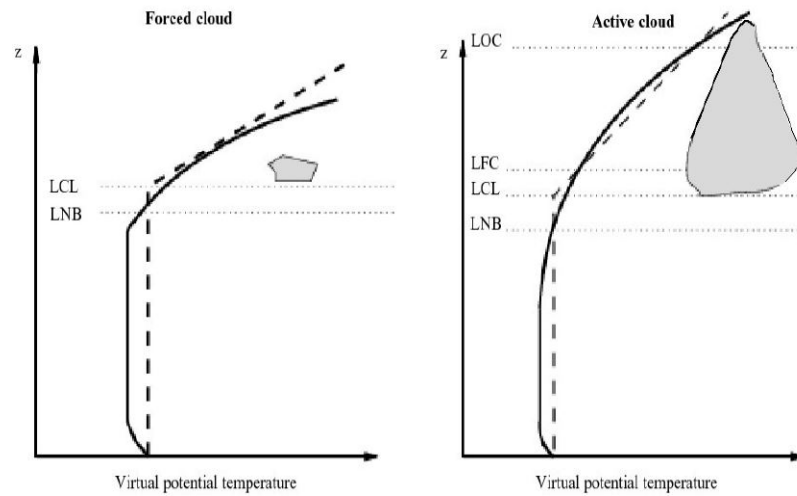


Figure 2.2: The virtual potential temperature with the height, the solid line is for the environment, and the dashed line is for the air parcel. The approximated level height is from BOMEX data. The level of neutral buoyancy (LNB), the lifting condensation level (LCL), the level of free convection (LFC) and the limit of convection (LOC) are labelled. Figure reproduced from Siebesma (1998).

When an unsaturated air parcel ascends from the Earth's surface, its buoyancy, which relates to the temperature difference from the atmosphere, determines how far it will rise. Figure 2.2 sketch this process in view of the virtual potential temperature θ_v . The vertical dashed line represents the dry lapse rate which is constant with height, and the inclined dashed line represents the moist lapse rate which increases with height. The joint of them represents the air parcel becoming saturated. The solid line represents the environmental lapse rate of absolute stability (left) and conditional instability (right), respectively.

At the beginning of the rise, the temperature of the air parcel follows the dry adiabatic lapse rate. When it reaches a level of neutral buoyancy (LNB), where the air parcel has the same density as its local environment, and if it has enough momentum to continue rising to overshoot the LNB, it will reach a higher level where the parcel becomes saturated and condensation begins, therefore, cloud forms. After crossing this higher level, the latent heat released heats the air parcel then it follows the moist adiabatic lapse rate. This higher level is called the lifting condensation level (LCL). Note that, between the LNB and LCL, the air parcel follows the dry adiabatic lapse rate and

the air parcel is still negatively buoyant (colder than the environment).

Above the LCL, if the environmental lapse rate is larger than the moist adiabatic lapse rate of the air parcel (figure 2.2 left), i.e., the environment is absolutely stable, the parcel remains negatively buoyant and will not rise too far. Therefore, the cloud formed in this case is called a forced cloud. However, as shown in the figure 2.2 (right) which we are more interested in, if the environmental lapse rate is smaller than the moist adiabatic lapse rate of the air parcel (i.e., the environment is of conditional instability). The air parcel will become positively buoyant when it reaches a certain level which is called the level of free convection (LFC). The energy it requires to overcome the negative buoyancy from LCL to LFC is called Convective Inhibition Energy (CIN). After crossing the LFC, the positive buoyancy lifts the saturated air parcel rises and condenses, the released latent heat in return feeds the positive buoyancy, therefore, an active cumulus cloud forms. This loop continues until the air parcel reaches an inversion layer which is absolutely stable with a very large environmental lapse rate. Therefore, the air parcel becomes negatively buoyant and eventually is forced to stop so that the cloud top is marked. The bottom of the inversion layer is located where the air parcel is neutral buoyant again, which is called the limit of convection (LOC). The total energy that drives the air parcel from the LFC to LOC is called the Convective Available Potential Energy (CAPE).

Note that, in fact, the rising of the air parcel is a competition between condensation and evaporation as the process is not adiabatic. Meanwhile, on its way up, the parcel mixes with the environmental air and evaporates which consumes part of the latent heat and liquid water. This prevents the cloud from rising and is the reason for the cloud dissipation (i.e., all the liquid water is evaporated in the air). Generally, the evaporation occurs near the cloud edge and the top where the cloud interacts with the environment the most. When evaporation dominates the phase change, it mainly reduces the temperature, so that the saturated parcel becomes negatively buoyant which further reverses the parcel motion, therefore, the subsiding shell forms at the cloud edge. However, note that, the main reason for the subsiding shell has been controversial for many years, the evaporation-driven one introduced above is a mainstream view and also the view held

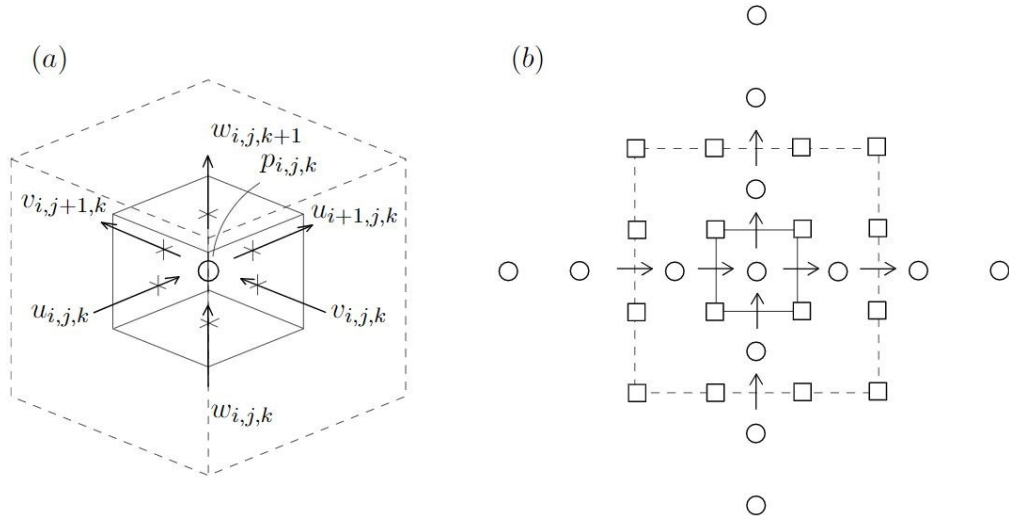


Figure 2.3: Staggered arrangement of variables relative to a single computational cell, where $\rightarrow$ denotes the location of a flux, $\circ$ the location of the transported quantity and, in the case for which represents a velocity, $\square$ represents the location of lateral transporting velocities. (a) Control volume; (b) Typical two-dimensional section figure reproduced from Craske & Van Reeuwijk (2015).

by this thesis (see Chapter 5), although, some argued the subsiding shell is mainly due to the dynamic force from the top.

2.3 DNS code SPARKLE

The DNS code used to solve the governing equations, (2.1) – (2.3) for turbulent fountains and (2.6) – (2.9) for cumulus clouds, is called SPARKLE. The code is based on a symmetry-preserving finite volume discretisation (Verstappen & Veldman, 2003), and is accurate to fourth-order in space. To handle a large simulation with about a billion nodes, a matrix of processes is employed, and the parallel programming with Message Passing Interface (MPI) is used to realise the communication between the processes. Each process computes an equal-sized sub-region of the whole simulation domain. A two-dimensional decomposition (pencil decomposition) is employed to parallelise the computations. Specifically, in order to take Fourier transforms to solve the Poisson equation, the decomposition first aligns on the x -direction and each process handles a subset of the y - z plane and the entire x domain to perform the Fourier transforms. Then

the Fourier transforms are performed followed by aligning the decomposition on the y -direction and on the z -direction, in sequence (see Li, 2010, for details). To adapt the fourth-order scheme in space, each process handles a set of interior cells surrounded by 3 ghost nodes.

Variables are arranged in a staggered manner over the grid with the transported variables defined on the centre of a control volume (shown as the circles in the figure 2.3) while the velocity components (u, v, w which are perpendicular to each other) and fluxes are defined on face centres (shown as the arrows in the figure 2.3). Details of the numerical scheme, parallelisation and boundary conditions are provided in (Craske & Van Reeuwijk, 2015). The High-performance computing (HPC) service used for all the simulations in this thesis is the UK National Supercomputing Service, Archer2.

To capture detailed insight into the behaviour of the turbulence, the DNS solves the governing equations with a spatial resolution up to the Kolmogorov scale $\eta_K = (\nu^3/\varepsilon)^{1/4}$, which represents the smallest length scale of the turbulent energy dissipation, where ν is the viscosity and ε is the rate of dissipation of turbulent kinetic energy (Pope, 2000). This requires the grid size should be characterised by the Kolmogorov scales, therefore, the DNS is usually computationally expensive (Ferziger *et al.*, 2020).

SPARKLE has been used on various supercomputing platforms and systems throughout Europe. Recent utilisation includes axisymmetric plumes (Craske & Van Reeuwijk, 2016), plumes in crossflow (Jordan *et al.*, 2022) and subsiding shells of the cloud (Nair *et al.*, 2021).

The thermodynamics required to simulate cumulus clouds were lifted from the open-source Dutch Atmospheric LES code (DALES; Heus *et al.*, 2010) and were used in Nair *et al.* (2020). The code contribution of this thesis, apart from the conditional averaging framework (§4.2), is to include a z -dependent hydrostatic pressure and therefore also z -dependent saturation pressure (§2.2.3). The full procedure is summarised below with the θ_t and q_t as the prognostic variables

I. Calculate the properties of the environment.

1. calculate environmental virtual potential temperature $\theta_{v,e}$ using (2.21) and

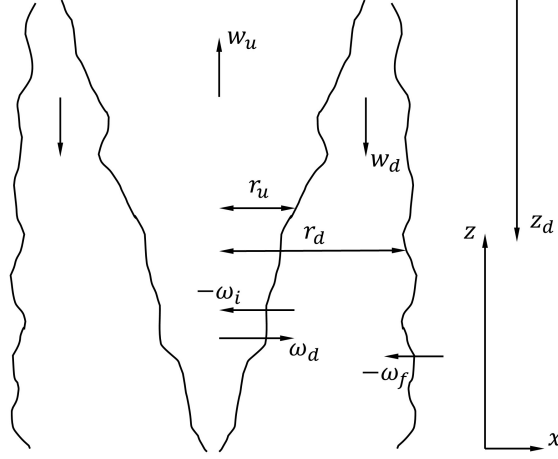


Figure 2.4: Fountain model BK00, with the arrows indicating the direction of variables.

(2.22), noting that the environmental total water specific humidity $q_{l,e} = 0$ as it is the dry environment.

2. calculate local hydrostatic pressure $p_h(z)$ using (2.32) by integration, with the boundary condition $p_h(0) \approx 95712$ Pa, estimated at a reasonable cloud base level where the origin locates.

II. Calculate the properties of the moist air.

1. calculate saturation point q_s of the air using (2.28), combined with (2.29), (2.25), (2.26).
2. calculate liquid water specific humidity q_l of the air using (2.23).
3. calculate potential temperature θ of the air using (2.21).
4. calculate virtual potential temperature θ_v of the air using (2.22)

III. Calculate the buoyancy b of the air using (2.10).

2.4 Fountain models

The fountain model established in Bloomfield & Kerr (2000) (hereafter referred as to BK00) will be the main comparison to our fountain throughout the Chapter 3 and 4, for

example, a direct comparison of the entrainment coefficients. Therefore, this section introduces this fountain model as preliminary.

The BK00 model structured the fountain as an upflow and a surrounding downflow in a Reynolds-averaged aspect, as shown in figure 2.4. Here, we present the model in our notation (consistent with the Chapter 3 and 4): r_u and r_d are the radius of upflow and downflow, respectively; w_u and w_d are the bulk streamwise velocity of upflow and downflow as functions of z , respectively, while positive w_d points downwards as indicated by the arrow. The BK00 model assumed the upflow and downflow both entrain from the ambient — ω_i characterise the entrainment velocity of the upflow, and ω_d and ω_f characterise the entrainment velocity of the downflow from its two sides. Note that the positive values of the entrainment velocities are all in line with the direction of x -axis. As these velocities are defined under our outward horizontal direction (to be consistent with the governing equations in §2.1.1), ω_i and ω_f here should associate with a negative sign to represent the entrainment velocity into the respective flow.

In the model of BK00, the integral fluxes are defined equivalently to those defined in (3.13)–(3.15) and note that the BK00 model assumed (top-hat) uniform fluid properties and velocities in each flow region. BK00 directly applied integral conservation laws for volume, momentum and buoyancy fluxes which resulted in

$$\frac{dQ_u}{dz} = 2r_u(-\omega_i) - 2r_u\omega_d, \quad (2.35a)$$

$$\frac{dQ_d}{dz_d} = 2r_d(-\omega_f) - 2r_u(-\omega_i) + 2r_u\omega_d, \quad (2.35b)$$

$$\frac{dM_u}{dz} = B_u + 2r_u(-\omega_i)(-w_d) - 2r_u\omega_d w_u, \quad (2.35c)$$

$$\frac{dM_d}{dz_d} = B_d + 2r_u\omega_d(-w_u) - 2r_u(-\omega_i)w_d, \quad (2.35d)$$

$$\frac{dF_u}{dz} = 2r_u(-\omega_i)(-b_d) - 2r_u\omega_d b_u, \quad (2.35e)$$

$$\frac{dF_d}{dz_d} = 2r_u\omega_d(-b_u) - 2r_u(-\omega_i)b_d. \quad (2.35f)$$

The equations of the upflow were first solved numerically until a fountain top which was assumed of infinite width and zero-momentum flux. Then the results of the top

were used as the initial value to solve the equations of the downflow until the bottom. The equations were solved iteratively until the results converged.

As is typical, BK00 considered mean integral fluxes only, and thus the turbulent and pressure terms $M'_u, P_u, F'_u, M'_d, P_d$ and F'_d in (3.12) and (3.14) are identically zero. Momentum and buoyancy entrainment from the environment is assumed to be negligible. Moreover, as the entrainment velocities were assumed horizontally, therefore, the Leibniz contributions to the entrainment were not considered in the model.

There are some comments on the BK00 model after we finish the present study but worth pointing out here. We note that the BK00 model assumes top-hat conditions which imply that the upflow region has a width $r_u = Q_u/M_u^{1/2}$; in reality, the profiles are not top-hat and thus the actual internal boundary is located at a greater radial position and equivalent to $r_i \approx 1.25r_u$. Another consequence of the top-hat assumption is that the momentum and buoyancy on the boundary are equivalent to their values in the upflow or downflow.

Another fountain model we will mention is Shrinivas & Hunt (2014), hereafter referred as to SH14. Its governing equations are similar to those of BK00 but with several differences: first, it assumed the ω_d is negligible compared with the ω_i (although in the following studies, we find this is not true); second, SH14 assumed a hemispheric fountain cap above a fountain cap base level — the governing equations were solved until the base level and an additional equation of fountain cap calculated the downflow flux to continue solving the model.

Chapter 3

The internal structure of forced fountains

3.1 Introduction

Understanding the process of turbulent mixing within fluid flows is a ubiquitous challenge in the modelling of environmental processes, and in our exploitation of similar physical processes to meet societal needs and demands (Fernando, 1991). Frequently, the understanding of the process of turbulent mixing between distinct bodies of fluid is approached by seeking to investigate the exchanges of the physical properties between these fluid bodies. In some circumstances, the region where these exchanges occur is robustly defined by the constraints imposed by the presence of solid boundaries; with the fluid behaviour in the proximity of that region being significantly altered by these boundaries. Examples range in scale, at least, from the flow through the windows of our buildings (Gladstone & Woods, 2001) to the regions that occur when continental landmasses and the ocean floor conspire to create so-called ‘straits’, which affects the dynamics of our global oceans (Finnigan & Ivey, 1999, 2000). However, an even wider range of exchanges between distinct bodies of fluid can be driven by turbulence in the absence of any solid boundary. The study of such exchanges is long-standing and on-going (see Dimotakis, 2005, for a broad review). It is this type of exchange, specifically

that within a canonical turbulent forced fountain, which forms the focus of this study.

The term ‘turbulent fountain’ describes the flow in which the buoyancy force opposes the initial momentum flux of an ejection from a localised source; for example, a negatively buoyant high Reynolds number jet continuously injected upwards into a uniform quiescent environment will produce a turbulent fountain (Hunt & Burridge, 2015). In this frame of reference, quasi-steady fountains consist of an inner core, herein the ‘upflow’, shrouded by a returning counterflow, herein the ‘downflow’ (see, for example Turner, 1966; Mcdougall, 1981). Amongst other applications (see Hunt & Burridge, 2015), turbulent fountains play important roles in determining the environment within our buildings (e.g. Lin & Linden, 2005), water recirculation and purification (e.g. Bloomfield & Kerr, 2000), and the forced heating or cooling of enclosed spaces (Baines *et al.*, 1990). The dynamics of high-Reynolds number high-Péclet number miscible fountains can be described by consideration of a single dimensionless parameter (Burridge & Hunt, 2014): the source Froude number

$$Fr_0 = \frac{M_0^{5/4}}{Q_0|F_0|^{1/2}} = \frac{w_0}{\sqrt{|b_0|r_0}}, \quad (3.1)$$

where Q_0, M_0, F_0 are the integral source fluxes, per unit π , of volume, momentum, and buoyancy, respectively; $b_0 = F_0/Q_0$ is the source buoyancy scale, $r_0 = Q_0/M_0^{1/2}$ is the source radial scale, and $w_0 = M_0/Q_0$ is the characteristic vertical velocity at the source. Note that r_0 is the physical radius of the source only if one takes the velocity profile at the source to be uniform (i.e. top-hat). These fountains: a) offer a rich variety of turbulent exchanges across non-solid boundaries, b) are canonical flows and as such can inform our fundamental understanding of the physics of these processes, and c) have been relatively unexplored, compared with other canonical flows, like plumes or jets.

Exchanges driven by turbulence, in the absence of a solid boundary, have traditionally been studied from perspectives driven by two distinct motivations. Firstly, that of understanding, and often modelling, the fundamental physics for these processes, e.g., the modelling of gravitational convection of turbulent plumes and jets (e.g. Mor-

ton *et al.*, 1956); or secondly, from a need to account for the effects of these physical processes within societal applications, e.g., the modelling of atmospheric dynamics for weather prediction (for example Neggers *et al.*, 2002). These different motivations have led to subtly different meanings of the same key terms within different fields of scientific literature. As such, we devote some effort to clarifying, first the history, and then our usage of these terms, in particular ‘entrainment’.

When driven by the desire to develop widely applicable models for the effects of turbulence, the term ‘entrainment’ has classically referred exclusively to the drawing in of mass (volume) by a region of turbulent fluid from an ambient background (Morton *et al.*, 1956); implicitly this occurs across what is now termed a ‘turbulent/non-turbulent interface (TNTI)’. This physical process is relevant to a broad class of canonical flows, e.g., jets (Abramovich *et al.*, 1984), plumes (Priestley & Ball, 1955; Morton *et al.*, 1956), temporal jets (Van Reeuwijk & Holzner, 2014), clouds (De Rooy *et al.*, 2013), growing regions of turbulence driven by an oscillating grid (Thompson & Turner, 1975), and between fountain flows and the environment at their envelope (Burridge & Hunt, 2016); for a thorough review see Da Silva *et al.* (2014). By definition, a quantity that is exchanged from the background into the region of interest is referred to as being entrained, or ‘entrainment’; if it leaves the region of interest it is often referred to as being detrained or ‘detrainment’ — we adhere to this usage of these terms.

In the context of these scientific endeavours, the term ‘turbulent entrainment’ has historically been used to emphasise the physical mechanism by which this process occurs. However, this ‘entrainment’ is classically associated with a mean inflow velocity across the TNTI which is related to a characteristic velocity of the flow within the region of interest, in this case, the only region of turbulent fluid, via an entrainment coefficient (Turner, 1986). This inflow might not only be associated with fluxes of mass (volume), but also with other quantities such as momentum and buoyancy (Talluru *et al.*, 2022). This can be insignificant, as in the canonical cases of plumes or jets when the background is a quiescent uniform environment, but becomes significant, for example, whenever the background is stratified or in motion. These considerations are pertinent to the modelling of the momentum and buoyancy exchanges between the upflow and

downflow within fountains (Bloomfield & Kerr, 2000; Shrinivas & Hunt, 2014).

More recently, some studies have focused on cases where the background fluid is also turbulent (rotational) so that the exchanges occur across a turbulent/turbulent interface (TTI) (Kankanwadi & Buxton, 2020; Van Reeuwijk *et al.*, 2021). In cases that feature a TTI, the use of the term ‘turbulent entrainment’ referring to the transport by the mean properties of a turbulent flow region is no longer helpful, since the flow is turbulent on both sides of the interface (TTI).

An unambiguous terminology is thus needed to describe the different exchanges when dealing with TTIs. We will refer to the exchanges into our region of interest (which must be defined explicitly in each case) due to the mean flow properties as ‘mean entrainment’ and the exchanges due to turbulence as ‘turbulent entrainment’. However, to avoid a conflict with the classical usage of the terms, we shall not use the term ‘turbulent entrainment’ in isolation, instead we always explicitly specify the quantity that is being exchanged, e.g. ‘mean volume entrainment’ and ‘turbulent momentum entrainment’, etc.

The aforementioned entrainment terms can be defined rigorously using the example of a slender axisymmetric steady-state jet/plume/fountain developing in the direction of the coordinate z , for which the integral continuity equation and transport equation of a generic scalar, or vector component, X in absence of sources are given by (see Van Reeuwijk *et al.*, 2021)

$$\frac{d}{dz} \int_{\bar{\Omega}} \bar{w} \, dA = - \oint_{\partial \bar{\Omega}} \bar{V}_g \, d\ell, \quad (3.2)$$

$$\frac{d}{dz} \int_{\bar{\Omega}} (\bar{w}\bar{X} + \overline{w'X'}) \, dA = - \oint_{\partial \bar{\Omega}} (\bar{V}_g \bar{X} + \overline{V'_g X'}) \, d\ell. \quad (3.3)$$

Here, the velocity component in the z -direction is denoted w , and integration is performed over a domain $\bar{\Omega}$, orthogonal to the z -direction, that encompasses the flow within the region of interest; $\partial \bar{\Omega}$ denotes the boundary of $\bar{\Omega}$. The overbar indicates the domain is chosen based on a Reynolds-averaged quantity: in the case of jets/plumes, this Reynolds-averaged quantity is typically a small fraction ε of the centreline velocity $\bar{w}_c$ (Van Reeuwijk *et al.*, 2016), i.e. $\bar{\Omega}$ is the domain for which $\bar{w} > \varepsilon \bar{w}_c$. Since the

flow is statistically axisymmetric and time-independent, the interface can equally be described using a single-valued function $r = h_t(z)$.

The instantaneous entrainment velocity across the interface is denoted V_g , which is the difference between the local fluid velocity and the boundary velocity. Following Van Reeuwijk *et al.* (2021), we write this as

$$V_g = u - \frac{Dh_t}{Dt} = u - w \frac{dh_t}{dz}, \quad (3.4)$$

where Dh_t/Dt is the material derivative representing the boundary velocity, u and w are the *instantaneous* radial and streamwise velocity components of fluid on the interface $\partial\bar{\Omega}$, respectively. For the case under consideration, h_t is time-independent due to consideration of a boundary defined by a Reynolds-averaged quantity, in which case this term then simplifies as per the right-hand side (RHS) of (3.4). Reynolds-averaging the expression above results in $\bar{V}_g = \bar{u} - \bar{w} \frac{dh_t}{dz}$. Note that since we choose the interface based on a threshold of velocity, the last term is negligibly small compared to the first, and we recover the classical entrainment velocity $\bar{V}_g = \bar{u}$ for jets/plumes.

The RHS of (3.2) represents the *mean volume entrainment*, as the transport is due to the mean quantity $\bar{V}_g$, in this case $\bar{V}_g = \bar{u}$. The first term on the RHS of (3.3) represents the *mean scalar entrainment* term $\bar{V}_g \bar{X}$ and the second term of the RHS is the *turbulent scalar entrainment* term $\overline{V'_g X'}$ (by definition X can also be a vector component, e.g. w ; we omit the ‘vector’ in the name here for simplicity). Note that volume entrainment can only be caused by the mean flow, but *all* other transported properties (momentum, buoyancy, etc.) can have both a mean and a turbulent component.

The terms for entrainment are illustrated schematically in figure 3.1. The figure illustrates the entrainment from an ambient of two cases: a) the ambient is non-turbulent so that the interface is a TNTI, for which only mean volume entrainment and mean scalar entrainment can occur at the interface (e.g. the interface between a turbulent fountain and the quiescent environment); b) the ambient is turbulent so that the interface is a TTI (e.g. the interface between the upflow and downflow inside a turbulent fountain), for which, besides mean volume entrainment and mean scalar entrainment,

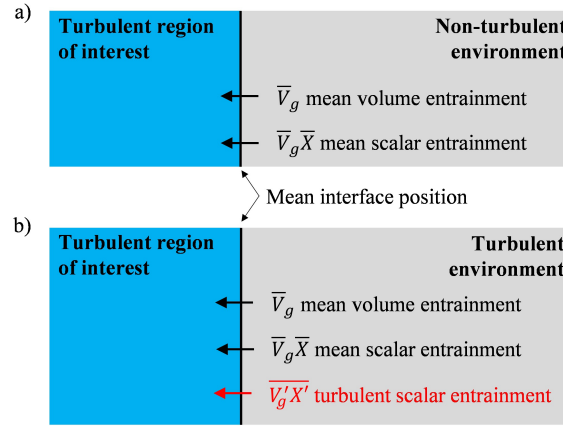


Figure 3.1: Schematic illustrating the entrainment of volume and scalars across interfaces separating the flow within the chosen region of interest, which is by definition turbulent, and the environment. Two different cases emphasise our chosen terminology: a) a non-turbulent environment, and b) a turbulent environment. Transport or exchanges across the interface, here all termed ‘entrainment’, are illustrated by arrows; for each entrainment, the associated transport term and its naming convention are highlighted. Black arrows represent the entrainment that is associated with mean flow, while the red arrow represents that associated with turbulence.

turbulent scalar entrainment can occur at the interface. Instantaneous exchanges at the TTI can and do occur in both directions (as allowed in the BK00 model of a fountain or measured at cloud edge De Rooy *et al.*, 2013), but the Reynolds averaged quantities discussed here are the net value of these bi-directional exchanges.

In some circumstances, the fluid on one particular side of the TTI has a larger turbulence intensity and so the choice of the ‘region of interest’ (or ‘flow region’) and ‘background’ is obvious. However, for more complex fluid flows this is not so straightforward. For example, in a fountain one can distinguish between an upflow and a downflow, which are modelled separately (BK00). Here, in the upflow and in the downflow, turbulence levels can be of comparable turbulence intensity, with standard metrics switching from being typically larger in the upflow near the source, to the downflow further from the source. Another example is a cumulus cloud, where it is also difficult to distinguish on which side of the interface between the cloud core and descending shell the turbulence intensity is largest (De Rooy *et al.*, 2013; Heus & Jonker, 2008; Nair *et al.*, 2020). Thus in order to distinguish ‘entrainment’ and ‘detrainment’, one

must always clearly define the ‘region of interest’, as this will determine whether one should use the term entrainment (exchange into the region) or detrainment (exchange out of the region). For completeness, we note that should the exchanges illustrated in figure 3.1(b) be evaluated to be negative then, following our convention, this would result in the reversal of the direction of the arrows, and the exchanges would thus be termed ‘detrainment’.

The aim of this work is to study the internal structure of a turbulent forced fountain; to do so, we use direct numerical simulation (DNS) to investigate a highly forced fountain. The structure of the chapter is as follows: in §3.2 we describe the numerical simulation details. In §3.3 we describe the instantaneous fountain observation and discuss the internal structure of a Reynolds-averaged fountain, we divide the fountain into three regions of interest with two turbulent/turbulent interfaces. In §3.4 we introduce the integral quantities of a fountain and appropriate conservation equations in order to find the flux and entrainment budgets of mass, momentum and buoyancy. In §3.5 we examine the vertical variation in fluxes within the three flow regions. In §3.6 we discuss entrainment across the interfaces respectively, including that near the fountain top; ending with a comparison of the entrainment coefficients from our DNS to those of other studies. Concluding remarks are made in §3.7.

3.2 Simulation details

We simulate an axisymmetric fountain driven by an isolated source of uniform (top-hat) vertical upwards velocity w_0 and opposing (negative) buoyancy b_0 where buoyancy is defined as $b = g(\rho_e - \rho)/\rho_a$, in which ρ and ρ_e denote the density of fountain flow and environment, respectively, and ρ_a is a reference density. The domain has size $L_x \times L_y \times L_z = 160r_0 \times 160r_0 \times 100r_0$ where r_0 is the circular source radius. The coordinate system has its origin in the bottom corner with the z -direction upwards (figure 3.2), and the source is located in the centre at $(L_x/2, L_y/2, 0)$. The source fluxes, per unit π , of volume Q_0 , (specific) momentum M_0 , (specific) buoyancy F_0 , and (again per unit π) the

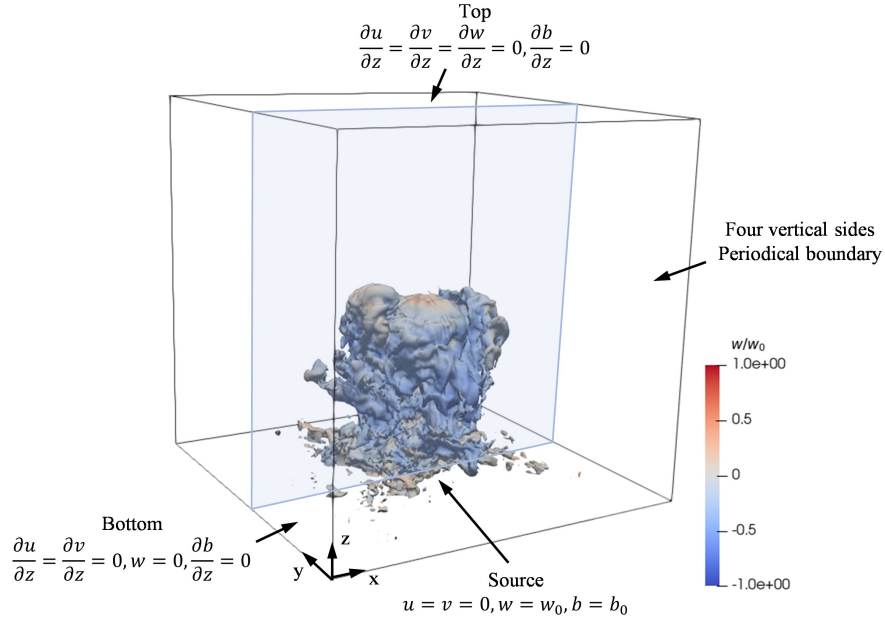


Figure 3.2: The whole simulation domain with an illustration of an instantaneous fountain depicted by plotting an isosurface of buoyancy, coloured according to the local vertical velocity.

integral buoyancy B_0 , can be defined as

$$Q_0 = w_0 r_0^2, \quad M_0 = w_0^2 r_0^2, \quad F_0 = w_0 b_0 r_0^2, \quad B_0 = b_0 r_0^2. \quad (3.5)$$

The fountain source is such that the source Froude number is $Fr_0 = 21$ which sits well above the threshold ($Fr_0 \gtrsim 4.0$) for the region described as forced fountains (Burridge & Hunt, 2012). The Reynolds number is $Re_0 = w_0 r_0 / \nu = 1\,667$, sufficient such that our simulations sit within a Reynolds number regime ($Re_0 \gtrsim 1\,000$) for which the experiments of Burridge *et al.* (2015) demonstrate that the bulk behaviour of the fountain, in particular the fountain rise height, is broadly independent of Re_0 , and the flow can be regarded as high Reynolds number, or ‘turbulent’.

For these turbulent highly forced fountains universal scalings exist based on consideration of a point source fountain of momentum flux, M_0 , and opposing buoyancy flux, F_0 . By consideration of the length scale $L_F = M_0^{3/4} / |F_0|^{1/2} = r_0 Fr_0$ (*cf.* Turner, 1966, who originally defined the length scale based on physical fluxes, rather than the fluxes

per unit π , i.e., there is a ratio of $\pi^{1/4}$ between the two length scales) and time scale $T_F = M_0/|F_0| = \sqrt{r_0/|b_0|} Fr_0$ (Burridge & Hunt, 2013), results for forced fountains are therefore universal when scaled on Fr_0 . Herein, we use these scales to normalise all quantities, making our findings directly applicable to all turbulent fountain flows with source Froude number $Fr_0 \gtrsim 4$ (Hunt & Burridge, 2015). For example, we normalise all volume fluxes using $Q_F = L_F^3/T_F = M_0^{5/4}/|F_0|^{1/2}$ ($Q_F \propto Q_0 Fr_0$ Baines *et al.*, 1993) and for all quantities concerning the entrainment of volume, momentum and buoyancy, we use $q_F = Q_F/L_F = M_0^{1/2}$, $m_F = M_0/L_F$, and $f_F = F_0/L_F$, respectively. The size of our cuboidal domain in these units, is $L_x \times L_y \times L_z = (7.55 L_F)^2 \times 4.72 L_F$.

The incompressible Navier-Stokes equations under the Boussinesq approximation are solved numerically using DNS on a uniform Cartesian grid of $N_x \times N_y \times N_z = 1280^3$ cells. The grid size is resolved smaller than two Kolmogorov scales, such as $\Delta x = L_x/N_x < 2\eta_K$. Here the Kolmogorov scale $\eta_K = (\nu^3/\varepsilon_z)^{1/4}$, where ε_z is the averaged integral dissipation rate of turbulent kinetic energy over the entire fountain height. The code employed is SPARKLE, which uses a spatial discretization of fourth-order accuracy and a third-order Adams-Bashforth time integration scheme. Details of the numerical method used in SPARKLE can be found in Craske & Van Reeuwijk (2015).

A Dirichlet boundary condition $w = w_0, b = b_0$ is applied at the source to ensure the uniform source value. We initiate the turbulence by applying an uncorrelated perturbation of 1% to the velocities in the first cell above the source. The boundary condition at the domain top has a Neumann condition on velocity and buoyancy, and at the domain bottom, the boundary condition is ‘free slip’ with a Neumann condition on buoyancy (except at the source). The four sidewalls of the domain are periodic, raising potential concerns that fluid leaving from one side and re-entering at the opposite can potentially influence the fountain flow — we mitigate this concern in two ways. First, we ensure that the domain is sufficiently large that the dynamics of the fountain are unaffected by any transport at the domain edges. To do this, we checked a variety of fountain flow statistics for a domain that is about 50% smaller. We found no substantial differences between the domain in terms of the fountain (outer boundary) and the exchanges across the internal boundaries (defined later on). These results are presented in appendix §D.

Second, to avoid any accumulation of buoyancy within the domain affecting our results, a buoyancy ‘sink’ region is set within a thin region (of height r_0) at the bottom of the domain and, to avoid affecting the source, only within the region $r \geq 9r_0$ — within this sink region the fluid buoyancy is gradually adjusted by a ‘nudging’ process towards the environmental value, i.e. the buoyancy is nudged to the zero in this region (Stevens *et al.*, 2014).

Observations show that the longest timescales in a forced fountain flow are the timescale required to establish a statistically steady state (found here to be approximately $5T_F$), followed by the timescales of the large-scale low-frequency fluctuations near the fountain top. The latter timescale is approximately $2T_F$ (Burridge & Hunt, 2013). We ran our simulations for a duration of $16T_F$ in order to allow sufficient time for the flow to reach a statistically steady state and then experience a number of the largest scale fluctuations in the region near the top of the fountain, a region which we describe as the ‘fountain cap’. Instantaneous statistics were obtained at a vertical plane at the centre of the domain with time increments of $0.09T_F$. Azimuthally Reynolds averaged data was computed and stored every $0.14T_F$ once the fountain is in a statistically steady state by partitioning the domain into concentric cylindrical shells and averaging over all cells lying within a given shell (Craske & Van Reeuwijk, 2015; Van Reeuwijk *et al.*, 2016). Therefore, the Reynolds average is defined as

$$\bar{X}(r, z) \equiv \frac{1}{2\pi T} \int_0^T \int_0^{2\pi} X(r, \theta, z, t) d\theta dt, \quad (3.6)$$

where X is an arbitrary field, and θ is the azimuthal direction.

3.3 Flow observations and internal structure

3.3.1 Observations of instantaneous fountain

A screen shot of the fountain from the initial transient state to the statistically steady state is shown in figure 3.3, where the initial negatively buoyant jet reaches the highest around figure 3.3(c), then fluid falls to form a quasi-steady downflow. Figure 3.4 shows

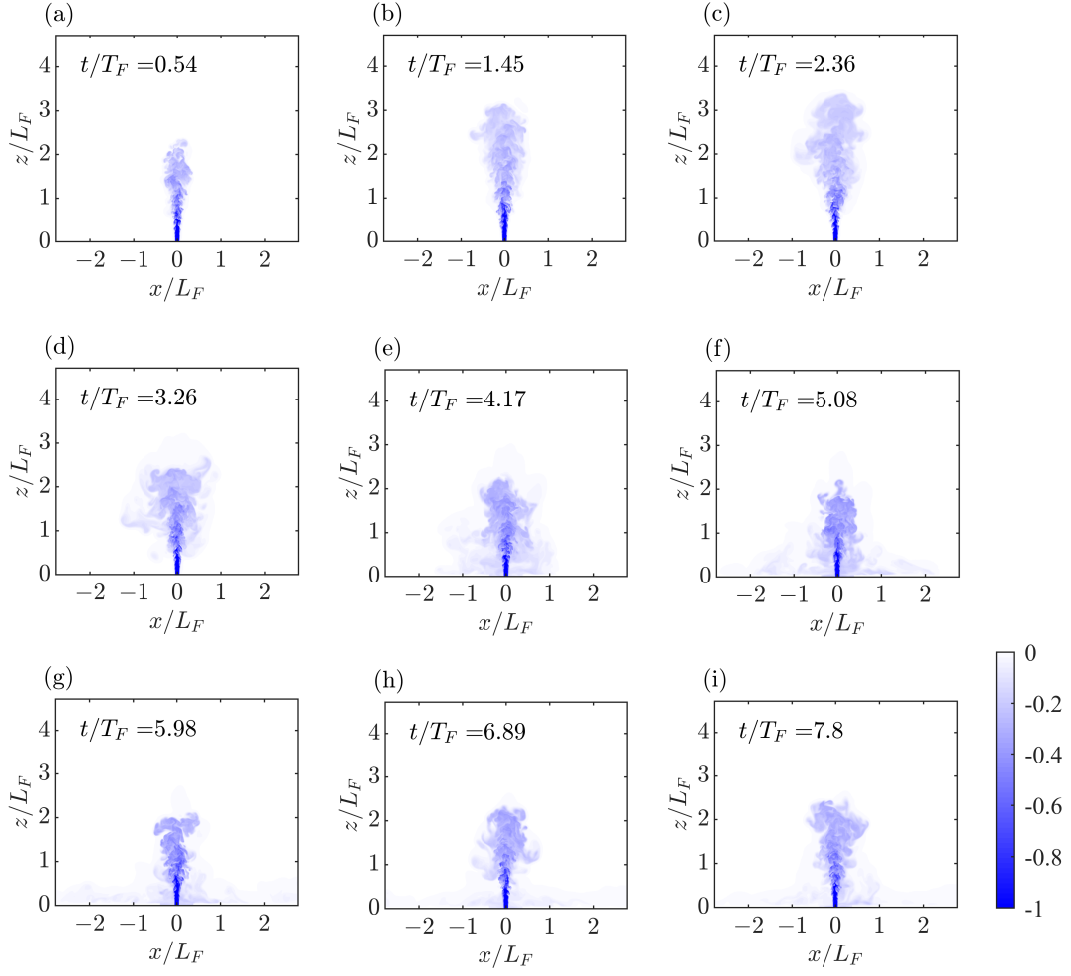


Figure 3.3: A screen shot of the fountain from its initial to the statistically steady state. The colour indicates the normalised buoyancy field of the fountain.

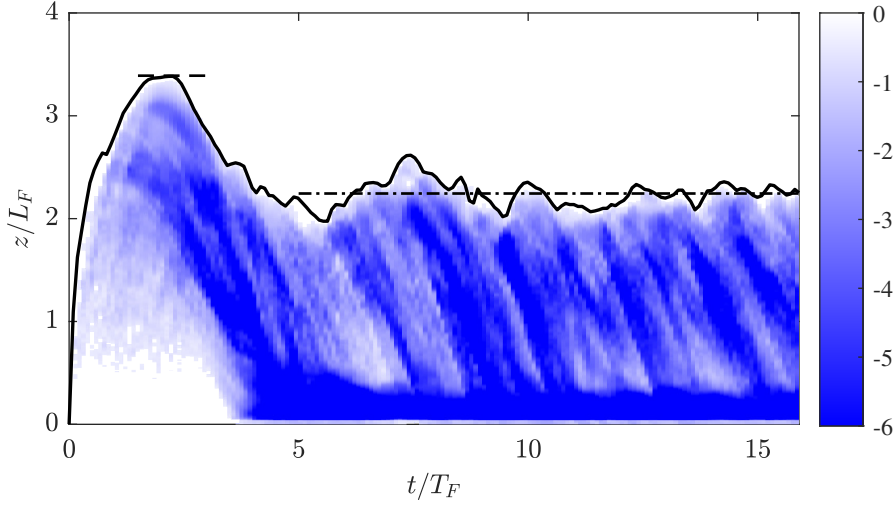


Figure 3.4: Time series of normalised local integral buoyancy, $\mathcal{B}(z,t)/\mathcal{B}_0$, within the fountain. The instantaneous fountain height is outlined by the solid line. The horizontal dashed line marks the initial height and the dash-dotted line marks mean steady height. Within the figure, the darker shades of blue represent the greater negative buoyancy.

a time series of the evolution of the vertical distribution of the horizontally integrated buoyancy within the fountain $\mathcal{B}$ (per unit π), which is defined as

$$\mathcal{B}(z,t) = \frac{1}{\pi} \int_0^{L_x} \int_0^{L_y} b(x,y,z,t) dx dy \approx \int_0^{L_x} b(x,z,t) \left| x - \frac{L_x}{2} \right| dx, \quad (3.7)$$

for practical reasons (namely, instantaneous data is taken at a central vertical plane), our evaluation comes via the approximation in (3.7)).

Fountain flows typically feature a region at the fountain cap which is characterised by the formation and collapse of large-scale structures within the fountain flow as is observed in figure 3.4. The fountain height is determined by a threshold $0.01 \mathcal{B}(z_b)$, where $\mathcal{B}(z_b)$ is the time-averaged integral buoyancy at a height $z_b/L_F = 1.63$ which we define as the fountain cap base (see §3.3.2). During $t/T_F \lesssim 5.0$ which we regard as the initial transient period, the upflow reaches an initial height at $t/T_F \approx 2.2$, and the downflow is yet to be fully developed until $t/T_F \approx 4.0$. After $t/T_F \approx 5.0$, the fountain can be said to have reached a statistically steady state, and the fountain height fluctuates around a mean steady height. The ratio of initial to steady heights is approximately 3:2, in line with experimental measurements (Turner, 1966; Burrige & Hunt, 2012). Note

that the heights referred to here are taken from measurements only in the central vertical plane. Figure 3.4 shows the characteristic (dark blue) inclined stripes that indicate the falling of distinct negatively buoyant fluid structures within the downflow, as previously discussed for fountains in statistically steady state by Burrridge & Hunt (2013); Mingotti & Woods (2016). The thin region, $z/L_F \lesssim 0.05$ in which the local buoyancy is nudged to the environmental value in order to avoid the accumulation of buoyancy in the domain can also be observed.

3.3.2 Internal structure of Reynolds-averaged fountain

Once the fountain has reached a statistically steady state, its averaged form is governed by the axisymmetric Reynolds-averaged Navier-Stokes equations (2.1)-(2.3) in the high-Reynolds number limit, namely

$$\frac{1}{r} \frac{\partial(r\bar{u})}{\partial r} + \frac{\partial\bar{w}}{\partial z} = 0, \quad (3.8a)$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r\bar{u}\bar{w} + r\overline{u'w'}) + \frac{\partial}{\partial z}(\bar{w}\bar{w} + \overline{w'w'}) = -\frac{\partial\bar{p}}{\partial z} + \bar{b}, \quad (3.8b)$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r\bar{u}\bar{b} + r\overline{u'b'}) + \frac{\partial}{\partial z}(\bar{w}\bar{b} + \overline{w'b'}) = 0, \quad (3.8c)$$

where $\bar{\cdot}$ represents the time average (mean): $\bar{b}$ is the mean buoyancy. The mean velocity components $(\bar{u}, \bar{w})$ correspond to the radial direction r and vertical direction z respectively. The mean pressure $\bar{p}$ is the kinematic pressure from which the hydrostatic pressure field resulting from the environmental density has been subtracted.

Figure 3.5(a) shows the internal structure of the Reynolds-averaged fountain in its statistically steady state with the shaded colour indicating the mean vertical velocity, $\bar{w} = \bar{w}(r, z)$, at that location. We define the point at which the mean vertical velocity on the centreline first falls to zero to be the ‘height of the upflow’, denoted z_i and we find $z_i/L_F \approx 2.22$. Some works (e.g. BK00) attempted to model fountains only via the application of ‘plume-like’ flow theory, whilst others acknowledged the need to model a ‘cap’ region near the fountain top via distinct considerations (e.g. McDougall, 1981, and SH14). The fountain cap region can be considered to be the region of flow enclosed

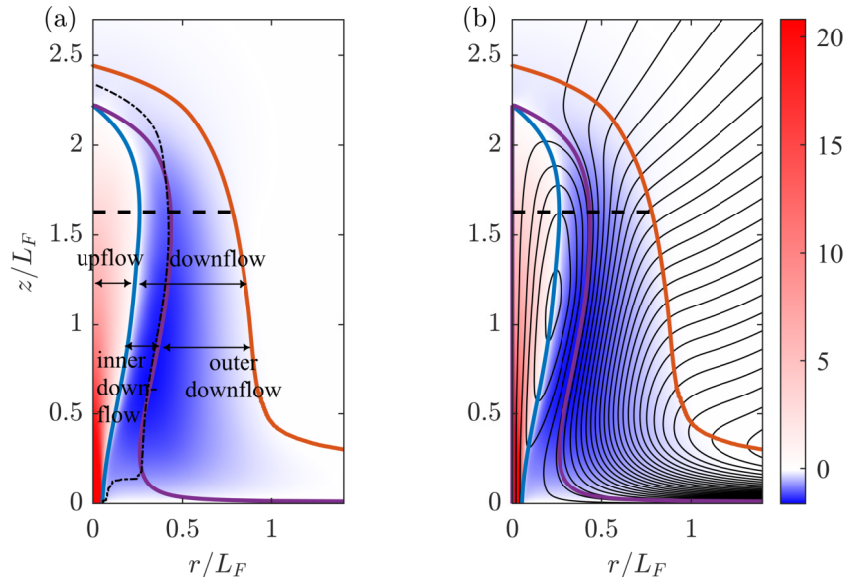


Figure 3.5: The normalised mean vertical velocity $\bar{w}/w_F$, where $w_F = M_0^{1/4}|F_0|^{1/2}$, within the time-averaged fountain marked by colour with regions of: upwards velocities shaded red, downward shaded blue, and zero vertical velocity being white. (a) Highlights the internal structure of the fountain by overlaying the vertical velocities with relevant boundaries: the inner boundary (blue line), outer boundary (red line), and the separatrix (purple line). The thin dash-dotted line marks the loci of points where the maximum downflow vertical velocity occurs. (b) the vertical velocities are overlaid with velocity streamlines. The fountain cap base is marked by the horizontal dashed line.

by the outer boundary above some height z_b , which we term the fountain cap base. The definition used to determine the height of the fountain cap base varies between studies; for example, Mcdougall (1981) theoretically modelled the cap region as a hemisphere whose radius is half of its base vertical height, SH14 and Hunt & Debugne (2016) defined the cap base locating at where the local Froude number of upflow is equal to $\sqrt{2}$. In their experimental study of fountains, Talluru *et al.* (2022) investigated the sensitivity of their findings to the choice of the cap base definition, dependent on a local Froude number condition (in the upflow); they found that the ratio of the volume flux entrained within the cap, to that in the upflow at the cap base, was insensitive to the location of the cap base when the local Froude number condition was chosen to be in the range 1.2–1.8. The numerical study of Awin *et al.* (2018) defined the cap base at the location where the radius of upflow is the widest. Specifically for our fountain, the locations of the cap base following each of the three above definitions from the existing literature are all within the region $1.60 \leq z/L_F \leq 1.65$. Therefore, our precise choice of definition is of little significance and, somewhat arbitrarily, we choose to define the fountain cap base as the point of maximum upflow radius, giving, $z_b/L_F = 1.63$.

The downflow of the steady fountain and the environment are separated by the outer boundary of radius $r_f = r_f(z)$, this also constitutes the boundary of fountain flow. The outer boundary is defined by the loci of points at which the mean buoyancy $\bar{b} = 0.01\bar{b}_{cc}$, where $\bar{b}_{cc} = \bar{b}(0, z_b)$ is the centreline mean buoyancy at the fountain cap base level which is about $0.36b_0$. The sensitivity of our results to the constant within the threshold $\bar{b} = 0.01\bar{b}_{cc}$, that was used to determine the outer boundary is presented in appendix §D; this sensitivity is not deemed significant. The top of the outer boundary (also the final steady height of the entire time-averaged fountain) is located at $z_f/L_F \approx 2.45$, i.e. $z_f/r_0 \approx 2.45Fr_0$, which is close to the value 2.46 reported by Burrige & Hunt (2012), and also agrees well with other experimental literature (see Turner, 1966; Mizushima *et al.*, 1982; Baines *et al.*, 1990). The outer boundary generally increases from the top to bottom as, too, does the area of downflow. We find that the height of the cap region (from cap base to the top) is $z_f - z_b = 0.82L_F$, which comprises about one-third of the total fountain height. We note that the fountain cap region is not our primary focus but

we do include some analysis in §3.6.2, and we mark the location of the fountain cap base in all relevant figures for completeness.

Below the cap, the flow can be divided into two regions separated by an internal boundary, or interface. Herein, we investigate two different definitions of this internal boundary. The first is to decompose the fountain into an upflow region where $\bar{w}$ is positive (shaded red in figure 3.5) and a downflow region where $\bar{w}$ is negative (shaded blue), that are separated by the inner velocity boundary r_i defined by the loci of points at which $\bar{w} = 0$. The radial position of this inner velocity boundary (hereinafter ‘inner boundary’) r_i increases to a maximum width at $z/L_F \approx 1.63$ (in fact the definition of cap base) and decreases to zero above, at the point at which the mean centreline velocity reduces to zero. Shown in figure 3.5(b) are the streamlines of the mean velocity, $\{\bar{u}, \bar{w}\}$. By construction, the streamlines are horizontal on the inner boundary since the vertical velocity is zero.

The streamlines in figure 3.5(b) then provide our second definition of the internal boundary. The streamlines illustrate that the flow being injected from the source is, in the mean, separated from the entrained ambient flow by a streamline that originates from the stagnation point at the top of the inner boundary. This separatrix, which we denote $r_s(z)$, is highlighted by the purple line in figure 3.5(a) and defines our second internal boundary. The streamlines outside the separatrix show the environmental fluid is entrained into the downflow and eventually leaves the domain from the side. The streamlines cross the outer boundary at some angle to the horizontal indicating that the entrainment velocity has some vertical component. The streamlines inside the separatrix either originate from the source or are closed loops within the upflow, crossing the inner boundary twice — this highlights that, in the mean, the upflowing core entrains near the source and then detrains fluid into downflow further from the source (above $z/L_F \approx 1.20$).

By construction, there is no mean exchange across the separatrix, and so the volume flux inside r_s is conserved. Exchange across the separatrix r_s can thus only occur by turbulence. By definition, the separatrix intersects the centreline at the height $z = z_i$,

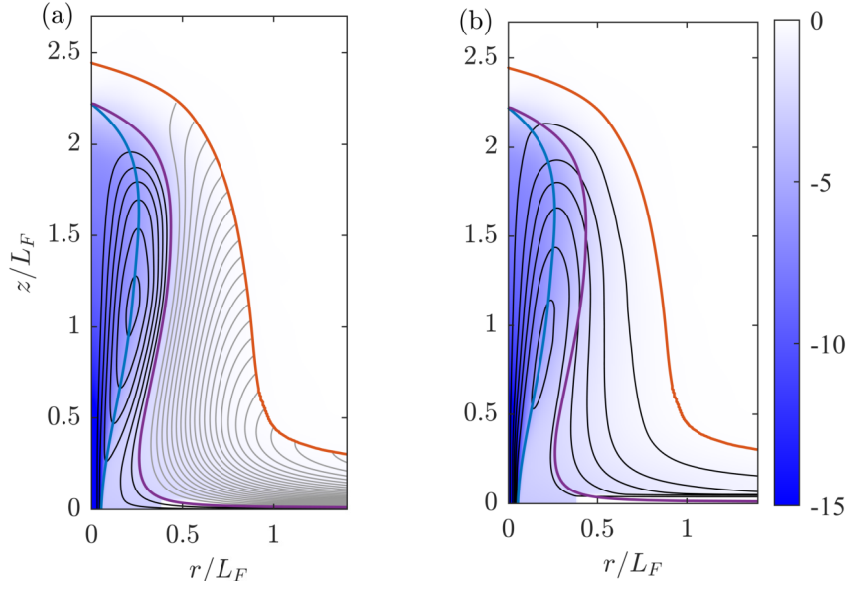


Figure 3.6: The normalised mean buoyancy field $\bar{b}/b_F$, $b_F = M_0^{-5/4}|F_0|^{3/2}$, overlaid with three fountain boundaries presented in figure 3.5 and: (a) field lines of mean buoyancy flux $\{\bar{u}\bar{b}, \bar{w}\bar{b}\}$, where the grey lines mark the field lines outside the separatrix; (b) field lines of total buoyancy flux $\{\bar{u}\bar{b} + \overline{u'b'}, \bar{w}\bar{b} + \overline{w'b'}\}$.

i.e. $r_s(z_i) = r_i(z_i) = 0$. The separatrix divides the downflow into an ‘inner downflow region’ (the region between the inner boundary and separatrix), and an ‘outer downflow region’ (the region between the separatrix and outer boundary). The outer downflow region contains all mass (volume) that has been entrained from the environment. These regions and boundaries are all shown in figure 3.5(a). Except close to the fountain top, the location of the separatrix (purple line) is very similar to that of the vertical velocity maximum (thin black dashed line). This suggests that the fluid beyond the vertical velocity maximum is entrained from the environment.

The relevance of the separatrix r_s to fountain flows is highlighted by consideration of the field lines of mean buoyancy flux $\{\bar{u}\bar{b}, \bar{w}\bar{b}\}$, and total buoyancy flux $\{\bar{u}\bar{b} + \overline{u'b'}, \bar{w}\bar{b} + \overline{w'b'}\}$, plotted in figures 3.6(a) and 3.6(b), respectively, both with the buoyancy field overlaid. Here we note that, as (3.8c) demonstrates, only the total buoyancy flux is a conserved quantity. This implies that for the field lines of mean buoyancy, there is a sink-source term and the total buoyancy between the two field lines need not

be conserved. As the mean buoyancy flux is the product of the velocity field and the scalar buoyancy field, the field lines of mean buoyancy flux are similar to the streamlines inside the fountain presented in figure 3.5(b) within the separatrix. As the environment has neutral (zero) buoyancy, the fountain source is the only source of (negative) buoyancy. Therefore it is worth noting from figure 3.6(a) that: firstly, no field lines exist outside the outer boundary r_f ; and secondly, the buoyancy injected from the source cannot be transported outside of the separatrix r_s by the mean buoyancy fluxes. However, we can observe there is mean transport of (negative) buoyancy flux within the outer downflow beyond, as indicated by the grey field lines between the purple boundary, r_s , and red boundary, r_f . Combining observations from figure 3.6(a) and figure 3.6(b) highlights that it is only turbulence that transfers the negative buoyancy across r_s into the outer downflow region, i.e. *all* the negative buoyancy within the outer downflow is the result of turbulent exchange across r_s . The field lines of total buoyancy flux in figure 3.6(b) evidence the transfer of buoyancy across the separatrix due to turbulence. Since the total buoyancy flux within the fountain is conserved, the buoyancy flux in the upflow equals the buoyancy flux in the downflow.

Note that the streamlines and field lines indicate that, as expected, the dynamics of the outer downflow below $z/L_F \approx 0.5$ are visibly affected by the (free-slip) bottom boundary, e.g. the flow is forced radially outwards in that region. To ensure the absence of boundary effects in the data presented herein, we therefore only display data from the outer downflow region for $z/L_F > 0.66$ in all further analyses. However, the inner downflow region is not likely to be significantly affected since this fluid is entrained by the upflow. We, therefore, consider our results for the inner downflow region to be valid over the full height.

3.4 Integral equations

The theoretical work in this section presents exact transport budgets for each flow region. We integrate (3.8a)-(3.8c) over the radial coordinate r between two radial locations $r_1(z)$ and $r_2(z)$ and employ the Leibniz integration rule to give conservation

equations (per unit π) for the fluxes of volume, momentum, and buoyancy as

$$\frac{dQ}{dz} = q_1 - q_2, \quad (3.9a)$$

$$\frac{d(M + M' + P)}{dz} = B + m_1 - m_2, \quad (3.9b)$$

$$\frac{d(F + F')}{dz} = f_1 - f_2. \quad (3.9c)$$

The integral quantities, per unit π , are defined as

$$\begin{aligned} Q &\equiv 2 \int_{r_1}^{r_2} \bar{w} r dr, & B &\equiv 2 \int_{r_1}^{r_2} \bar{b} r dr, \\ M &\equiv 2 \int_{r_1}^{r_2} \bar{w}^2 r dr, & M' &\equiv 2 \int_{r_1}^{r_2} \overline{w'^2} r dr, & P &\equiv 2 \int_{r_1}^{r_2} \bar{p} r dr, \\ F &\equiv 2 \int_{r_1}^{r_2} \bar{w} \bar{b} r dr, & F' &\equiv 2 \int_{r_1}^{r_2} \overline{w' b'} r dr, \end{aligned} \quad (3.10)$$

where B is the integral mean buoyancy. The quantities M , M' and P correspond to the integral mean momentum flux, turbulent momentum flux and pressure. Moreover, F and F' denote the integral mean buoyancy flux and turbulent buoyancy flux. Depending on the region of interest, the inner boundary can take the locations $r_1(z) = \{0, r_i(z), r_s(z)\}$, and the outer boundaries $r_2(z) = \{r_i(z), r_s(z), r_f(z)\}$, with the only condition being that $r_2(z) > r_1(z)$. The perpendicular exchanges of volume, momentum and buoyancy across the boundary r_j (with $j = i, f$, or s) are defined as

$$q_j = 2r \left(\bar{u} - \bar{w} \frac{dr}{dz} \right) \Big|_{r_j}, \quad (3.11a)$$

$$m_j = 2r \left(\bar{u} \bar{w} - \bar{w}^2 \frac{dr}{dz} \right) \Big|_{r_j} + 2r \left(\overline{u' w'} - \overline{w'^2} \frac{dr}{dz} \right) \Big|_{r_j} + 2r \left(-\bar{p} \frac{dr}{dz} \right) \Big|_{r_j} \text{ and,} \quad (3.11b)$$

$$f_j = 2r \left(\bar{u} \bar{b} - \bar{w} \bar{b} \frac{dr}{dz} \right) \Big|_{r_j} + 2r \left(\overline{u' b'} - \overline{w' b'} \frac{dr}{dz} \right) \Big|_{r_j}, \quad (3.11c)$$

respectively. The terms associated with the radial change, i.e. dr/dz , are ‘Leibniz terms’. As discussed in §3.1, the perpendicular exchange of volume flux q_j (3.11a) only comprises a mean contribution, and the mean entrainment velocity $\bar{V}_g$, see (3.4), is

clearly recognisable. The term m_j , (3.11b), is the perpendicular exchange of momentum flux which is comprised of a mean momentum flux (the first term on the RHS; note that this term is equal to $2r\bar{V}_g\bar{w}$ evaluated at r_j) and turbulent momentum flux terms (the second and third terms on the RHS; with the second term equal to $2r\overline{V'_g w'}$ evaluated at r_j and the third term accounting for the effects of pressure which are typically small compared to the second term). The term f_j , (3.11c), is referred to as the perpendicular exchange of buoyancy flux which is comprised of one term for mean buoyancy entrainment (the first term on the RHS; equal to $2r\bar{V}_g\bar{b}$ evaluated at r_j) and one for turbulent buoyancy entrainment (the second term on the RHS; equal to $2r\overline{V'_g b'}$ evaluated at r_j). Whether these exchanges are termed ‘entrainment’ or ‘detrainment’ depends solely on the chosen ‘region of interest’ and the sign of the flux.

In the remainder of this section, the budgets for the upflow, downflow and inner downflow are presented by substituting appropriate boundaries into the equations above. For completeness, we note that since we use a buoyancy that is consistently defined relative to the uniform ambient fluid our buoyancy formulation corresponds to both that termed ‘BFI’ by BK00, which is also that used by SH14.

3.4.1 Upflow region

The upflow budgets can be obtained by integrating from the centreline to the inner boundary, i.e. $r_1(z) = 0$ and $r_2(z) = r_i(z)$. Denoting upflow quantities with subscript ‘ u ’, it immediately follows that (3.9a–3.9c) for the upflow region become

$$\frac{dQ_u}{dz} = -q_i, \quad (3.12a)$$

$$\frac{d(M_u + M'_u + P_u)}{dz} = B_u - m_i, \quad (3.12b)$$

$$\frac{d(F_u + F'_u)}{dz} = -f_i. \quad (3.12c)$$

By stipulating that $Q_u = r_u^2 w_u$, $M_u = r_u^2 w_u^2$ and $B_u = r_u^2 b_u$, the characteristic (top-hat)

upflow vertical velocity w_u , buoyancy b_u , and radius r_u are defined

$$w_u \equiv \frac{M_u}{Q_u}, \quad b_u \equiv \frac{B_u M_u}{Q_u^2}, \quad \text{and} \quad r_u \equiv \frac{Q_u}{M_u^{1/2}}. \quad (3.13)$$

These natural characteristic scales will be used to investigate the entrainment coefficients in §3.6.3.

3.4.2 Downflow region

The downflow budget (i.e. including both the inner and outer downflow), can be obtained by integrating from the inner boundary to the fountain edge, i.e. $r_1(z) = r_i(z)$ and $r_2(z) = r_f(z)$. As the flow direction is opposite to the upflow, we define a downward coordinate z_d . A transformation $z \rightarrow -z$ implies that $w \rightarrow -w$, $b \rightarrow -b$ in order to achieve invariance of the equations (3.8a–3.8c). From the definitions, (3.10), it now follows that the integral downflow quantities are given by $Q_d = -Q$, $M_d = M$, $F_d = F$, $B_d = -B$ and $d/dz_d = -d/dz$, with the transformation of the turbulent quantities following an equivalent form. Substitution into (3.9a)–(3.9c) results in

$$\frac{dQ_d}{dz_d} = q_i - q_f, \quad (3.14a)$$

$$\frac{d(M_d + M'_d + P_d)}{dz_d} = B_d + m_f - m_i, \quad (3.14b)$$

$$\frac{d(F_d + F'_d)}{dz_d} = f_f - f_i. \quad (3.14c)$$

The radial coordinate remains unchanged in this transformation. Therefore, a positive volume flux q_f contributes negatively to the downflow budget. Note that at the outer boundary, the term ' m_f ' can be interpreted as $-(-m_f)$, where $-m_f$ is an outwards flux of downward momentum (consider (3.11b) at $r_j = r_f$ under coordinate z_d). Similar understanding holds true for the buoyancy flux term f_f in (3.14c). At the inner boundary, the exchange of volume flux, q_i , contributes oppositely to the upflow and downflow (3.12a, 3.14a) as the volume flux exchanged out of the upflow will increase the volume flux in the downflow and *vice versa*. However, in terms of the upflow, m_i

represents an outward flux of upward momentum which reduces the upflow momentum flux, so that a negative sign is correctly associated with m_i in (3.12b); with respect to the downflow and the coordinate z_d , $-m_i$ denotes a flux of downward momentum into the downflow (radially outward) at the inner boundary, this contributes positively to the downflow momentum. Hence the sign convention in (3.12b) and (3.14b), with equivalence, for the term f_i in (3.12c) and (3.14c).

The downflow region has a characteristic annular cross-section area, which can be denoted (per unit π) as $r_d^2 - r_u^2$ (BK00); thus, the fluxes can be written as $Q_d = (r_d^2 - r_u^2)w_d$, $M_d = (r_d^2 - r_u^2)w_d^2$, $B_d = (r_d^2 - r_u^2)b_d$ and it follows that the natural characteristic scales are

$$w_d \equiv \frac{M_d}{Q_d}, \quad b_d \equiv \frac{B_d M_d}{Q_d^2}, \quad \text{and} \quad r_d \equiv \sqrt{\frac{Q_u^2}{M_u} + \frac{Q_d^2}{M_d}}. \quad (3.15)$$

3.4.3 Inner downflow region

The transport budgets for the inner downflow region can be obtained by integrating from the inner boundary to the separatrix, i.e. $r_1(z) = r_i(z)$ and $r_2(z) = r_s(z)$. As the flow in this region is by definition downward, the transformation applied to §3.4.2 is required. Denoting inner downflow quantities with subscript ‘*id*’, we obtain

$$\frac{dQ_{id}}{dz_d} = q_i - q_s, \quad (3.16a)$$

$$\frac{d(M_{id} + M'_{id} + P_{id})}{dz_d} = B_{id} + m_s - m_i, \quad (3.16b)$$

$$\frac{d(F_{id} + F'_{id})}{dz_d} = f_s - f_i. \quad (3.16c)$$

3.5 Integral quantities

The vertical variation in the normalised integral quantities in the upflow, downflow, and inner downflow (the portion of the downflow that falls within the separatrix) are shown in figure 3.7. Buoyancy fluxes and integral buoyancy are normalised by absolute values of B_F and F_0 , respectively, in order to accurately reflect the opposing or assisting action

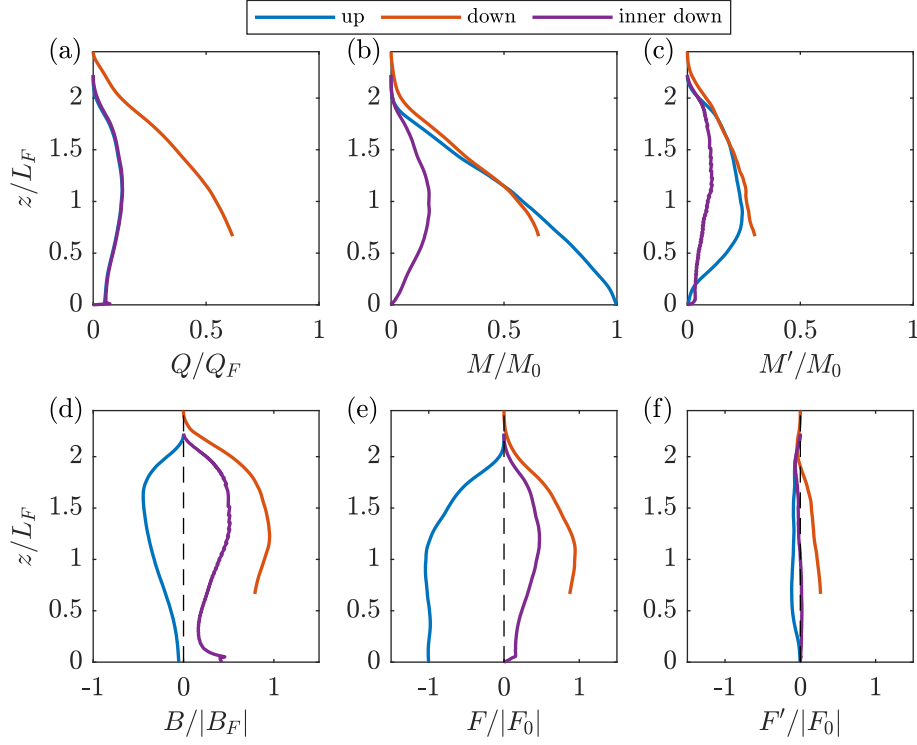


Figure 3.7: Integral quantities for the upflow, downflow and inner downflow region, each normalised by the relevant source scales. The vertical dashed line marks fluxes of zero where appropriate.

of the buoyancy force on each of the flows. We choose to discuss any flow regions in the direction of their flow, i.e. we discuss any downward flowing regions from top to bottom (that is in the direction of the downwards coordinate z_d).

Figure 3.7(a) shows that below $z/L_F \approx 1.11$, the upflow volume flux increases, suggesting a net volume entrainment into the upflow in this region. Above $z/L_F \approx 1.11$ the volume flux in the upflow decreases due to the volume detrainment into the downflow. The figure also shows that the volume flux in the upflow is identical to that within the inner downflow; this confirms that, by construction, the volume flux is conserved within the separatrix. The volume flux in the downflow increases from the top of the fountain showing net volume entrainment by the downflow. Noting that the downflow consists of both inner- and outer-downflow regions, the difference between the downflow and the inner downflow, reassuringly, confirms that fluid from the environment is entrained into the fountain outer downflow over the full height of the outer boundary. Careful

observation of the streamlines in figure 3.5(a) confirms these above observations.

Figure 3.7(b) plots the normalised vertical evolution of mean vertical momentum fluxes. The mean momentum flux in the upflow, M_u , decreases, to zero at the top of the upflow z_i , as the buoyancy acts to decelerate the flow; whilst this buoyancy then acts to enhance the mean momentum flux in the downflow M_d . The momentum flux within the inner downflow M_{id} is also shown to increase from the top of the fountain until the height $z/L_F \approx 0.88$, below which M_{id} then decreases back towards zero. The magnitudes of the mean momentum flux M_u and M_d are, perhaps surprisingly, approximately equal (despite the significant differences between the characteristic velocity and radial scales within the two flows); this suggests that, broadly speaking, fountain fluid roughly attains the same momentum flux on the way back down that it originally had on the way up.

Figure 3.7(c) plots the normalised turbulent vertical momentum flux. The maximum turbulent momentum flux in upflow M'_u occurs relatively low down within the fountain (about $z/L_F \approx 0.88$, i.e. about a third of the height of the fountain), it might be affected by the buoyancy accumulation in the intrusion region. The turbulent momentum flux in the downflow M'_d continuously increases from the fountain top, in line with the suggestion that, since the buoyancy increases the mean momentum flux, the turbulent momentum flux broadly scales with the mean momentum flux. Again, for heights $z/L_F \lesssim 1.92$, there is a surprisingly close agreement between the magnitude of the turbulent momentum fluxes in the upflow and downflow, perhaps suggesting that both these turbulent fluxes might be predominantly driven by the mean momentum fluxes over much of the fountain height. In-depth analysis of the data in figures 3.7(b) and 3.7(c) show that the ratio (not shown explicitly in the figure) of turbulent and mean momentum flux M'/M is greatest near the fountain top, suggesting analysis of the dynamics via mean flow statistics might be insufficient in this region.

Figure 3.7(d) shows that the integral buoyancy in the upflow B_u is negative, while in the downflow B_d is positive. Combining (3.12b) and (3.14b), confirms that the integral buoyancy is, in fact, a sink for M_u but a source for M_d . The magnitudes of all the three flow regions peak near $z/L_F \approx 1.50$. The inner downflow region contains nearly half

of the integral buoyancy in the downflow, evidence that the buoyancy within the inner downflow is more concentrated than the outer downflow (i.e., the inner downflow is denser than the outer downflow).

Figure 3.7(e) and (f) show the mean and turbulent vertical buoyancy fluxes, respectively. Unlike the momentum fluxes, the turbulent buoyancy fluxes are much smaller than the mean in all flow regions, suggesting that the turbulence affects the buoyancy fluxes less than it does the momentum fluxes. The upflow mean buoyancy flux F_u is negative due to its opposing buoyancy; it is noteworthy that for $z/L_F \lesssim 1.20$, the mean buoyancy flux in the upflow is approximately constant suggesting that in this region, whilst the integral buoyancy within the upflow is increasing (figure 3.7(d)), a particular interplay between radial growth, upflow velocity, and entrainment from the downflow, must exist to maintain this approximately constant flux which then breaks down at greater heights. When looking at the mean quantities, the inner downflow buoyancy flux is approximately half that of the total downflow. However, the turbulent buoyancy flux in the inner downflow region is a much smaller fraction of the total, implying that most of the downflowing turbulent buoyancy flux is transported in the outer downflow region. Finally, figure 3.7(e) and (f) show that, although relatively weak, the turbulent buoyancy fluxes act in the same direction as the mean buoyancy fluxes within both the upflow and downflow.

3.6 Entrainment

The rich variety of exchanges, herein ‘entrainment’, between the fountain upflow, the returning downflow, and the fountain cap is crucial in determining the bulk dynamics of fountain flows, including the interactions with the environment. We now examine these in detail.

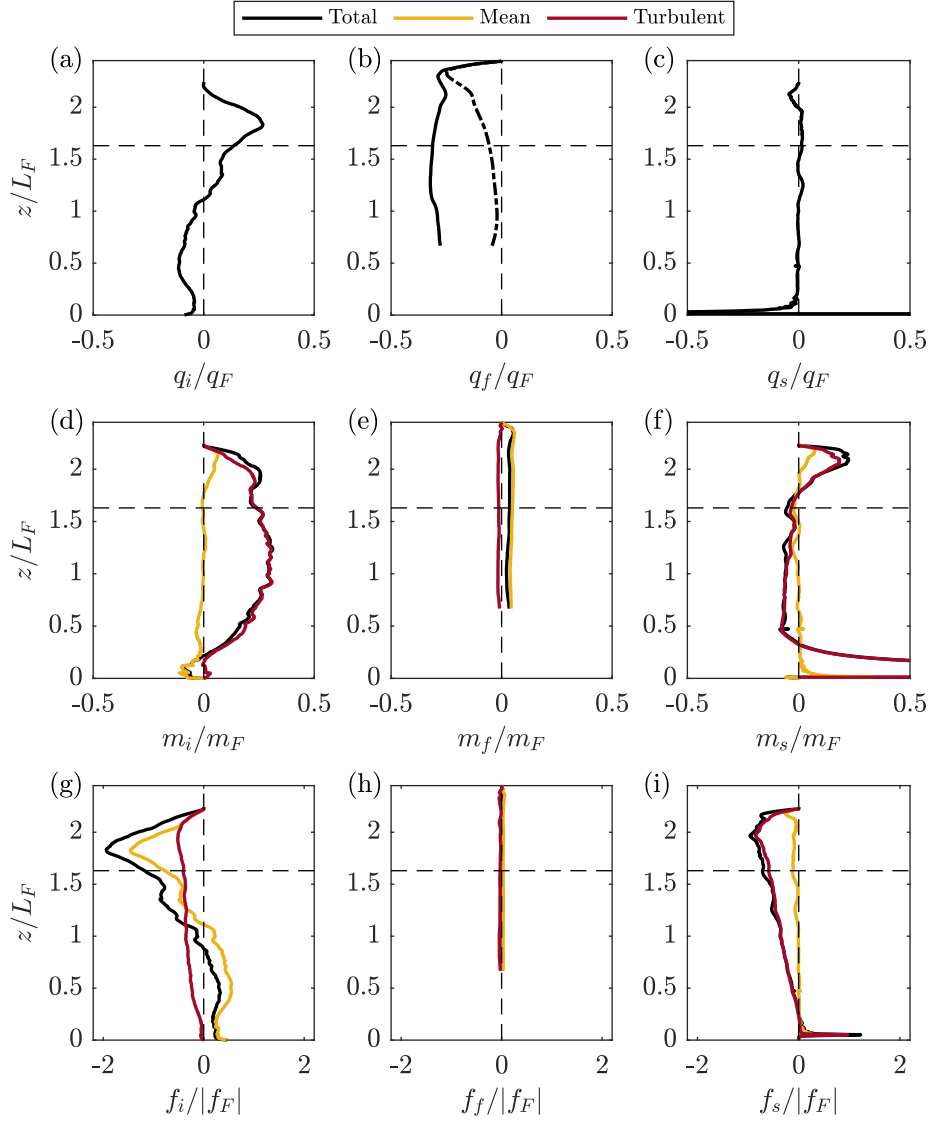


Figure 3.8: The vertical variation of the normalised (a–c) volume entrainment, (d–f) total momentum entrainment and (g–i) total buoyancy entrainment at the (left column) inner boundary, (middle column) outer boundary and (right column) the separatrix respectively. The mean and turbulent components of momentum and buoyancy entrainment are included. The total volume entrainment is also the mean. The dash-dotted line in (b) shows the vertical component of mean volume entrainment. The horizontal dashed line marks the fountain cap base. The vertical dashed line marks the line of zero exchange.

3.6.1 Vertical evolution of entrainment

Figure 3.8 shows the entrainment terms, see (3.11), at each of the defined internal boundaries r_i , r_s and the outer boundary r_f . As expected all entrainment terms approach zero at the top of their respective boundary (i.e. where these boundaries also approach $r = 0$).

Results for the volume exchange at the inner boundary (the term q_i , figure 3.8(a)) are reassuringly supportive of the observation of the fluxes in figure 3.7(a). We note that for the upflow region, there is a change from net entrainment to net detrainment at $z/L_F \approx 1.11$, where the local Froude number of upflow is 2.87, about twice of the value at the fountain cap base. By construction, the mean momentum entrainment is close to zero at the inner boundary (figure 3.8(d)), but the momentum exchange associated with turbulence by the upflow is significant over much of the fountain height — this acts to reduce both the momentum fluxes of the upflow (3.12b) and downflow (3.14b), either by entrainment of downward momentum fluctuations into, or detrainment of upward momentum fluctuations from the upflow; which of these is dominant could be rigorously determined by investigations of the instantaneous transports, for example, via a quadrant plot but this is beyond the scope of the current study. Alternatively, in terms of the turbulent stress $-\overline{u'w'}$ on the boundary, the interpretation of the reduction of upflow momentum flux is that it experiences substantial shear by the downflow, thereby decelerating the upflow. As the upflow and downflow are opposite to one another, the shear will also decelerate the downflow.

The buoyancy exchange at the inner boundary (figure 3.8(g)) exhibits significant contributions from both the mean flow and turbulence. The mean component of f_i exhibits a similar variation to that of q_i (figure 3.8(a)), consistent with the view that buoyancy can be both entrained and detrained by the mean flow at different heights according to $\overline{V}_g \overline{b}$. By contrast, the turbulent component of f_i is negative at all heights; suggesting that with respect to the upflow, there is either detrainment of negative buoyancy fluctuations, or entrainment of positive buoyancy fluctuations.

A schematic illustration of the net flux q_j , m_j and f_j (with $j = i, f$, or s) shown in

figure 3.9 is intended to clarify the sign of the exchanges. During the following discussion, this sketch is intended to act as a useful visual reference of the exchanges. The arrows indicate the sign of the exchanges of volume, momentum and buoyancy between the vertical fluxes of relevant flow regions, at various heights within the fountain. Taking the above inner boundary as an example, in figure 3.9(a), the volume exchange across the inner boundary (blue line) q_i below $z/L_f = 1.11$ has a leftward arrow which denotes a negative flux (in this case radially inward). Combined with (3.12a), this illustrates that q_i increases the volume flux in the upflow in that region.

At the outer boundary (figure 3.8(b)), there is mean volume entrainment from the environment into the downflow. Inspired by the streamline patterns shown in figure 3.5(b), we include results for the vertical component of volume entrainment at the outer boundary within figure 3.8(b); for which the contribution is significant within the region $z/L_F \gtrsim 1.63$, i.e. within the fountain cap region. Our results highlight that the volume entrainment into the fountain from the environment is generally greater than the volume exchange within the fountain (see figure 3.8(a)). Figure 3.8(e) shows there is very low momentum entrainment into the downflow at the outer boundary due to low velocities and levels of turbulence beyond; reassuringly, the buoyancy entrainment at the outer boundary, f_f , is shown to be negligible (figure 3.8(h)).

Along the separatrix there is, by construction, no volume exchange associated with the mean flow (see figures 3.8(c), (f), and (i)). For $z/L_F \gtrsim 1.77$, m_s is positive, becoming negative at lower heights and lessening the inner downflow momentum flux (3.16b). Despite the separatrix being at larger radial locations, the magnitude of turbulent momentum exchange there (figure 3.8(f)), is still markedly smaller than that at the inner boundary (figure 3.8(d)), suggesting weaker turbulence at the separatrix than at the inner boundary. Figure 3.8(i) shows that f_s acts to reduce the buoyancy flux in the inner downflow. Note that the figure shows f_s exhibits a linear decrease with the height up to $z/L_F \approx 1.95$, which by adding the integral buoyancy conservation equation (3.12c) and (3.16c), implying that the total vertical buoyancy flux $F + F'$ of the fountain inside the separatrix grows quadratically with the height (for which, in figure 3.7(e) one would need to combine the mean buoyancy flux in the upflow, blue line, with that of the inner

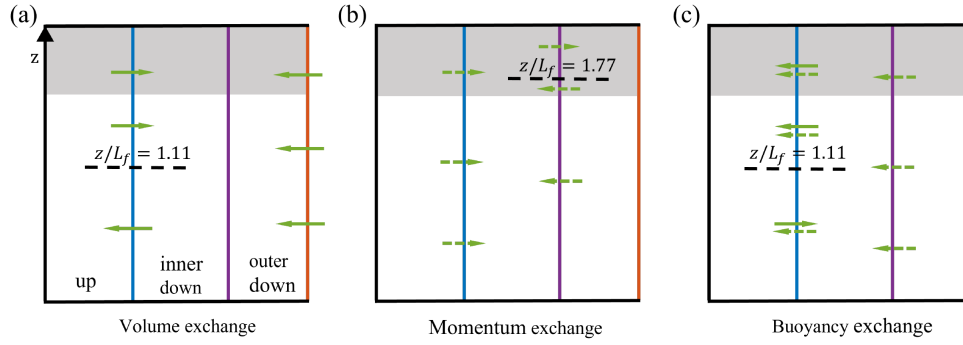


Figure 3.9: Schematic illustration depicting the sign of the exchange fluxes across the identified boundaries: right being positive and left being negative of the exchanges across the relevant boundaries within the fountain: (a) volume exchange, (b) momentum exchange and (c) buoyancy exchange. From left to right, each pane contains vertical lines: black marks the fountain’s centre line, blue illustrates the inner boundary, purple the separatrix, and orange the outer boundary. Green horizontal solid arrows represent the fluxes associated with the mean flow, while green dashed arrows represent those associated with turbulence. The dashed black lines mark any height at which the exchange changes sign. The grey areas represent the fountain cap region. The vertical coordinate is drawn approximately to scale, but the scale of the arrows itself does not represent the magnitude of the entrainment exchanges.

downflow, purple line).

3.6.2 Integral entrainment in the fountain cap region

As one might expect, the vertical evolution of entrainment of the various quantities (figure 3.8) shows significant changes within the fountain cap region, e.g. sharp peaks in volume and buoyancy exchanges, and the momentum exchange m_s changes the sign around the cap base (figure 3.8(f)). Note that in this region, Williamson *et al.* (2011) suggested that the momentum transport is significantly affected by the pressure gradient, and is also the region dominated by low-frequency fluctuations that complicate the flow dynamics (e.g. Williamson *et al.*, 2011; Burridge & Hunt, 2013). In this subsection, we focus on fluxes apparent upon integrating the entrainment over the height of this fountain cap region.

	$\frac{Q_{cap,j}}{Q_F}$	$\frac{Q_{cap,j}}{Q_{ub}}$	$\frac{M_{cap,j}}{M_0}$	$\frac{M_{cap,j}}{M_{ub}}$	$\frac{F_{cap,j}}{ F_0 }$	$\frac{F_{cap,j}}{ F_{ub} }$
Fluxes across inner boundary, r_i	0.09	1.00	0.12	0.34	-0.75	-1.00
Fluxes across separatrix, r_s	0.00	0.00	0.05	0.14	-0.45	-0.60
Fluxes across outer boundary, r_f	-0.22	-2.40	0.02	0.05	0.00	0.00

Table 3.1: Integral volume, momentum and buoyancy entrainment across the boundaries within the fountain cap region, normalisation both by the relevant forced fountain scales and by the upflowing fluxes across at the cap base are presented for convenience.

The integral entrainment across the boundaries over the fountain cap is

$$Q_{cap,j} = \int_{z_b}^{z_t} q_j dz, \quad M_{cap,j} = \int_{z_b}^{z_t} m_j dz, \quad F_{cap,j} = \int_{z_b}^{z_t} f_j dz, \quad (3.17)$$

where $z_t = z_i$ for inner boundary and separatrix, while $z_t = z_f$ for outer boundary. These quantities show the volume, momentum and buoyancy exchanges, respectively, across the corresponding boundary integrated over the cap region. The values are presented in table 3.1 with two normalisations: firstly, by the relevant forced fountain scales, e.g., Q_F , M_0 , and $|F_0|$, and secondly, by the upflowing fluxes at the cap base, namely,

$$Q_{ub} = Q_u(z_b) \approx 0.09Q_F, \quad (3.18)$$

$$M_{ub} = M_u(z_b) + M'_u(z_b) \approx 0.36M_0 \text{ and,} \quad (3.19)$$

$$F_{ub} = F_u(z_b) + F'_u(z_b) \approx -0.71|F_0|. \quad (3.20)$$

These upflowing fluxes at the cap base highlight that around one-third of the source momentum flux reaches the cap region; and nearly three-quarters of the fountain's total buoyancy flux passes through the cap region. These facts, combined with the results in table 3.1, highlight the importance of the cap region for the dynamics of fountains. Figure 3.9 is intended to aid interpretation of table 3.1 by indicating the directions of $Q_{cap,j}$, $M_{cap,j}$ and $F_{cap,j}$.

Table 3.1 shows that the integral volume entrainment by the cap is $Q_{cap,f} = -0.22Q_F$, combining this with the results for the bulk fountain entrainment (i.e. that the total entrainment flux for these highly forced fountains is approximately $0.77Q_F$ Burridge &

Hunt, 2016), would suggest that more than a quarter of all fountain’s volume entrainment occurs within the cap (we note that our bottom boundary condition prohibits us making our own estimates of the bulk volume entrainment by the fountain). Meanwhile, $Q_{cap,f} = -2.40Q_{ub}$ shows that the integral volume entrainment from the environment is about two and a half times larger than that injected at the cap base, agreeing with the data taken from Awin *et al.* (2018) and from Talluru *et al.* (2022) for a turbulent forced fountain.

Considering the exchanges across the inner boundary within the cap region, we note that $Q_{cap,i}/Q_{ub} = 1.00$ confirms that, as expected, all upcoming volume flux is detrained into the downflow region. The momentum flux detrained out of upflow, $M_{cap,i}$, is approximately 35% of the upflowing momentum flux, suggesting around two-thirds of vertical momentum flux that enters the cap is acted on by the pressure gradient and opposing buoyancy. The integral buoyancy detrainment $F_{cap,i}/|F_{ub}| = -1.00$ shows the buoyancy flux injected into the cap is all detrained into the downflow; furthermore, more than half of this is then detrained from the inner downflow to the outer downflow ($F_{cap,s}/|F_{ub}| = -0.60$).

3.6.3 Entrainment coefficients

In this section, we calculate the entrainment coefficients from our DNS data, and compare them to those employed in other studies, in particular the fountain model BK00. Classically, entrainment coefficients have been used to represent the rate of dilution of a flow; this approach has proved valuable to the modelling of a broad set of classes of turbulent free-shear flows. This concept was popularised by Morton *et al.* (1956) (referred to as the MTT entrainment model), who successfully characterised the bulk entrainment of mass by a plume by relating the fluid entrainment velocity across the plume edge to local characteristic scales via a constant entrainment coefficient. The application of this concept to bulk-averaged models of fountains came somewhat later (Mcdougall, 1981) and continues to be developed, e.g. BK00 and SH14. In the most general case presented to date, these apply the entrainment coefficients to estimate the mass, momentum and

buoyancy exchanges of the upflow and the downflow. Typically, the entrainment coefficients in models were either assumed constant with values determined by analogy to a more simple canonical flow (e.g. a jet or a line plume, as in the case of McDougall, 1981, BK00), or approximated empirically based on either experimental or numerical studies of fountains (e.g. SH14).

Notably, BK00 represented the fountain upflow and downflow as two separate flows: an upward negatively buoyant jet, and a downward (annular) line plume, respectively; which both entrain via their boundaries. As a result, unlike our Reynolds-averaged statistics, they parameterised two-way entrainment at the inner boundary: the upflow entrains from the downflow with a velocity ω_i while the downflow entrains from the upflow with a velocity ω_d . Meanwhile, the downflow also entrains from the environment via the outer boundary with a velocity ω_f , see figure 2.4. DNS, or high fidelity experiments, could enable conditionally sampled data to directly inform such a parameterisation (e.g. to parameterise simultaneous entrainment and detrainment of the upflow) should use of such a parameterisation be deemed necessary. In order that readers can check consistency with relative ease, the BK00 model and governing equations are reproduced in appendix §2.35 using the notation and coordinate system presented herein.

BK00 considered two formulations of body force acting on the upflow, and two formulations of velocity scale characteristic of entrainment; in combination, this leads to four different cases to consider. In terms of the body force, the first formulation (BFI) assumed the buoyancy force acting on the upflow depends directly on the density of the environment, that is in our notation B_u as evident in (3.12b) and (2.35c). The second, BFII, assumed the buoyancy force is relative to the local density difference between the upflow and downflow, that is in our notation $B_u + B_d$. In agreement with SH14, the two buoyancy forces generally lie in close quantitative and qualitative agreement. We, therefore, consider only BFI. In terms of the characteristic entrainment velocity, the first formulation considered by BK00, which we denote ‘EI’, assumed the mean entrainment velocity should be proportional to the difference in velocity between the upflow and downflow, so that the three entrainment velocities in BK00 can be written

as

$$-\omega_i^I = \alpha_{i00}(w_u + w_d), \quad \omega_d^I = \alpha_{f00}w_d, \quad -\omega_f^I = \alpha_{f00}w_d, \quad (3.21)$$

where, α_i and α_f are the mean volume entrainment coefficient of the upflow and downflow, respectively, the subscript ‘00’ denotes the variables as used in BK00, the superscript ‘I’ denotes the entrainment formulation EI. The negative sign in ω_i and ω_f indicates entrainment velocity into upflow and downflow (see appendix §2.35 for details). Note that BK00 parameterised the downflow as having the same entrainment coefficient α_{f00} at both boundaries — the physical reasoning for this remains unclear.

Neglecting all the turbulent and Leibniz terms, and taking entrainment parameterisation EI, (3.12a) becomes

$$q_i = 2\alpha_i^I r_u(w_u + w_d), \quad q_f = 2\alpha_f^I r_d w_d, \quad (3.22)$$

where α_i^I and α_f^I are the entrainment coefficients for the upflow and downflow, respectively, that are evident from our DNS data under entrainment formulation EI. When applying entrainment formulations presented here, it should be clarified that BK00 prescribed the same constant entrainment coefficients, α_{i00} and α_{f00} , and then applied two formulations to calculate entrainment velocities apparent within their fountain model; for our DNS data we measured the actual fluxes entrained and then directly determined the relevant entrainment coefficients. Substituting (3.21) into (2.35a) and combining with (3.12a), provides the relationships between our entrainment coefficients and those of BK00, under formation EI, as

$$\alpha_i^I = \alpha_{i00} - \alpha_{f00} \frac{w_d}{w_u + w_d}, \quad \text{and} \quad \alpha_f^I = \alpha_f = \alpha_{f00}. \quad (3.23)$$

In their second entrainment formulation ‘EII’, BK00 assumed the mean entrainment velocity is related to a characteristic velocity depending only on the relevant fraction of the total shear, i.e. the sum of the magnitude of velocity on either side of the interface.

Thus the entrainment equations of BK00 under EII are

$$-\omega_i^{\text{II}} = \alpha_{i00} w_u, \quad \omega_d^{\text{II}} = \alpha_{f00} w_d, \quad -\omega_f^{\text{II}} = \alpha_{f00} w_d, \quad (3.24)$$

with the relevant equations for our data being

$$q_i = 2\alpha_i^{\text{II}} r_u w_u, \quad q_f = 2\alpha_f^{\text{II}} r_d w_d. \quad (3.25)$$

Therefore, the relationships between our entrainment coefficients and those of BK00, under formation EII, become

$$\alpha_i^{\text{II}} = \alpha_{i00} - \alpha_{f00} \frac{w_d}{w_u}, \quad \text{and} \quad \alpha_f^{\text{II}} = \alpha_f = \alpha_{f00}. \quad (3.26)$$

We note that, BK00 assumed constant entrainment coefficients throughout the height of the fountain and calibrated their model (against a single statistic of fountain behaviour; namely, the fountain rise height), for the four combinations of body forces and entrainment formulations, always taking the same values for each of their two entrainment coefficients, specifically $\alpha_{i00} = 0.085$ and $\alpha_{f00} = 0.147$ — for this reason our notation does not distinguish between entrainment formulations when denoting the coefficients of the BK00 model.

The other fountain model worthy of consideration is that of SH14; they assumed that only entrainment by the upflow occurred at the inner boundary (i.e. only inward exchanges over the full height of their upflow $z/L_F \leq 1.62$). With this assumption, (3.23) and (3.26) reduce to $\alpha_i^{\text{I}} = \alpha_i^{\text{II}} = \alpha_{i00}$. SH14 took the inner boundary entrainment coefficient to be $\alpha_i = 0.06$, and $\alpha_f = 0.15$ for the entrainment coefficient at downflow at outer boundary (these values being loosely based on those reported in the studies of Williamson *et al.*, 2011; Burrige & Hunt, 2013) — the precedents set by SH14's model were followed by Debugne & Hunt (2016).

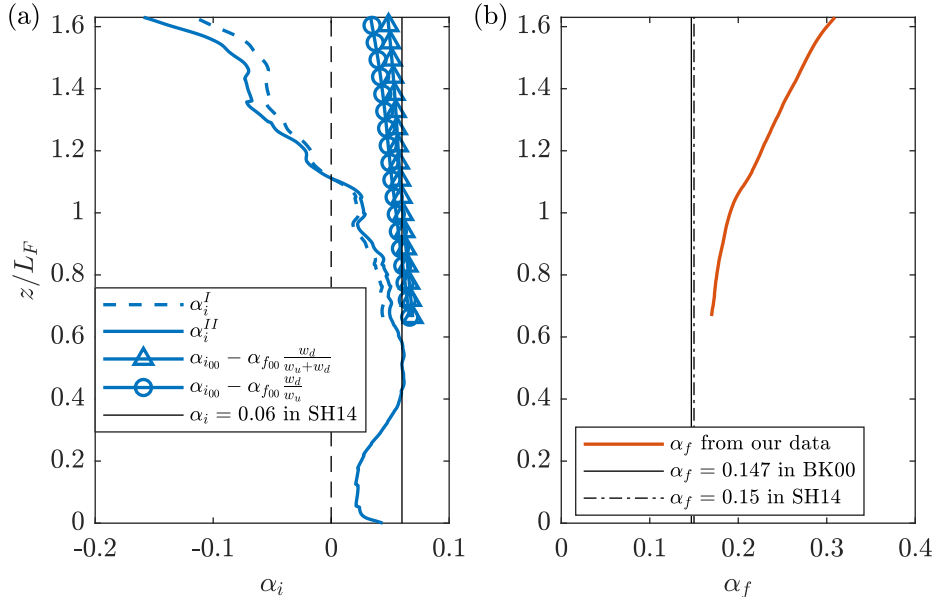


Figure 3.10: The vertical evolution of entrainment coefficients both calculated based on Reynolds averaging our DNS data (α_i^I , α_i^{II} , and α_f , see (3.22) and (3.25)) and, for direct comparison, the appropriately adjusted entrainment coefficients used in BK00, SH14, and Debugne & Hunt (2016) (who used same entrainment coefficients of SH14) are also plotted. a) The entrainment coefficients of the upflow at the inner boundary (α_i^I and α_i^{II} , shown as blue dashed line and solid line, respectively), the entrainment coefficients of BK00 presented in adjusted form (lines with mark, using velocity scales from our DNS and taking the entrainment constants from BK00, namely $\alpha_{i00} = 0.085$ and $\alpha_{f00} = 0.147$, see (3.23) and (3.26)), and the value used in both SH14 and Debugne & Hunt (2016), $\alpha_i = 0.06$ (vertical black line). The vertical dashed line marks zero coefficient. b) The entrainment coefficient of the downflow at the outer (orange line) calculated from (3.22) and (3.25) — which give the identical result, and the entrainment coefficients used in BK00 and SH14 (thin black solid line and dashed line, respectively).

Examination of figure 3.10, which plots our measurements of the entrainment coefficients and those used in fountain models thus far, challenges the validity of the entrainment assumptions that underpin the important contributions to fountain modelling of BK00, SH14, and Debugne & Hunt (2016). The figure plots the values of α_i and α_f , according to both entrainment formulations, i.e. calculated from our DNS data according to (3.22) and (3.25). Firstly, comparison of our data to that of Milton-McGurk *et al.* (2022) yields similar values — Milton-McGurk *et al.* (2022), to our knowledge, being the only other study to report the vertical variation of entrainment coefficient at the inner boundary of highly forced fountains ($Fr \geq 15$) based on measurements (therein

under entrainment formulation EI and reported as a variation with the local Richardson number of the upflow). Both studies determine that the entrainment coefficient changes sign at a particular height within the upflow. Secondly, irrespective of the entrainment formulation chosen, the notable variation in our measurements of α_i and α_f with height renders the assumption of constant α_f in BK00, and constant α_i and α_f in SH14, as questionable. Thirdly, (3.23) and (3.26) provide the relationships that must hold true for the entrainment modelling at the inner boundary of BK00 to be valid.

Figure 3.10(a) plots the LHS of (3.23) and (3.26), i.e. the equivalent entrainment coefficients calculated from our data, namely, α_i^I and α_i^{II} . In addition, it shows the profiles that follow from the existing theory, by substituting the constants used in BK00, i.e. $\alpha_{i00} = 0.085$ and $\alpha_{f00} = 0.147$ into the RHS of (3.23) and (3.26), and using our measured values of the velocity scales. It is clear from the figure that these relationships do not hold true for the values taken; furthermore, we have tested various other choices for the entrainment constants within the BK00 formulation and there is an equally poor agreement for all physically reasonable entrainment constants. These facts highlight that an integral model of fountains, that captures the exchanges within the fountain, is yet to be developed, or at least parameterised appropriately. Integral models, to date, have been parameterised based on matching a single measurable statistic of fountain behaviour, the fountain height (albeit sometimes both the initial and steady rise heights, e.g. Debugne & Hunt, 2016), but our results suggest that no integral model has even been able to qualitatively captured the appropriate exchanges within fountains. As such, the agreement with the calibrated metric might be unique and all other fountain metrics from such models may be questionable. We have shown, §3.6.1, that the total buoyancy exchanges, and the mean volume entrainment at the inner boundary, broadly follow a similar dependence of the vertical coordinate (see figures 3.8(a) and (g)); combined with figure 3.10, it shows that fountain models are yet to provide the capability to predict these fundamental exchanges which must affect the total weight of fountain fluid supported by a given source momentum flux. This highlights a major research challenge in fountain modelling, future models of fountains should seek validation via multiple statistics, not a single criterion (e.g. fountain height) as has been carried out

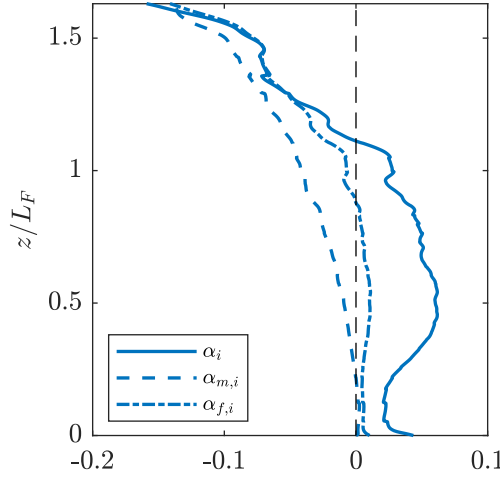


Figure 3.11: The entrainment coefficients for the upflow from our data: volume entrainment α_i from (3.26), momentum $\alpha_{m,i}$ obtained from the budget (3.27a) and buoyancy $\alpha_{f,i}$ from the budget (3.27b); i.e. all consistent with BK00’s EII formulation.

previously.

In addition, by their very construction, bulk averaged fountain models (e.g. BK00 and SH14) did not account for the turbulent exchanges within fountains, e.g. turbulent momentum entrainment and turbulent buoyancy entrainment. We have shown in §3.5 and §3.6.1 that there is significant integral vertical turbulent momentum flux (M' , see figure 3.7(c)), and the turbulent momentum exchanges and turbulent buoyancy exchanges at the inner boundary (turbulent component of m_i and f_i , respectively, see figure 3.8(d) and (g)). Hence, we challenge future models of fountains to account for these exchanges driven by turbulence, at the very least implicitly, within their parameterisation.

Following the BK00 model, but considering entire budgets (we choose to consider only EII, a choice of little consequence), the total momentum entrainment (3.12b) and total buoyancy entrainment (3.12c) are

$$m_i = 2\alpha_{i,m}r_u w_u^2, \quad (3.27a)$$

$$f_i = 2\alpha_{i,f}r_u w_u b_u. \quad (3.27b)$$

Figure 3.11 shows entrainment coefficients for momentum $\alpha_{i,m}$ and buoyancy $\alpha_{i,f}$ given by (3.27), respectively. The figure, for contrast, also re-plots α_i^H , from (3.26), the plot clearly highlights that taking the same entrainment coefficient to parameterise exchanges of volume, momentum and buoyancy is inappropriate. More research is clearly needed to determine a suitable route to developing appropriate models of turbulent fountains; our leading suggestion is that the latest understanding of transports across both TNTIs and TTIs be considered in depth. Note that we do not present the entrainment coefficients decomposed into mean and turbulent components due to the lack of self-similarity of the radial profiles within the fountain (*cf.* Milton-McGurk *et al.*, 2022).

3.7 Conclusion

The structure of a forced fountain was investigated by decomposing the fountain into regions based on two distinct internal boundaries. The first comprised the classical decomposition into an upflow and a downflow, via the internal boundary $r_i(z)$ characterised by the loci of zero vertical velocity. The second exploited a separatrix, a streamline $r_s(z)$ that forms the boundary between the flow emitted from the source and entrained from the environment. Both of these boundaries are turbulent/turbulent interfaces (TTIs), which can have both mean and turbulent contributions to entrainment across their boundary. For the internal boundaries investigated here, we showed that for the boundary r_i , entrainment fluxes are large and consist both of mean and turbulent components. However, the boundary r_s by construction exhibits zero mean entrainment, since it is a streamline; hence, only turbulent exchanges can occur across.

The findings reported herein indicate that turbulent entrainment fluxes across the internal boundary are of similar magnitude to, and sometimes larger than, the mean entrainment fluxes. This has important implications for the modelling of turbulent entrainment. Indeed, the classical view of entrainment considers mean entrainment only, which then implies that an associated entrainment velocity can be defined across the

interface, and entrainment of all other quantities (e.g. momentum, buoyancy) can then simply be obtained by multiplying that entrainment velocity by some local characteristic value of that quantity. The DNS data broadly supported this observation, in that the vertical variation of the fluxes of mean volume entrainment and mean buoyancy entrainment across r_i exhibited a similar form (note the mean momentum entrainment was zero by construction on r_i). However, the vertical variation of turbulent entrainment fluxes across r_i exhibits an entirely different form to the mean entrainment fluxes and therefore requires different modelling. Even for the ‘classical’ (mean volume) entrainment flux, the agreement between our DNS data and existing models was poor when the entrainment flux was normalised by suitable local characteristic scales, suggesting that parameterisation following MTT and deploying a constant entrainment coefficient is inappropriate for fountains and needs to be reconsidered. We further showed that different magnitudes of values are required for the respective entrainment coefficients to appropriately parameterise either mean entrainment, turbulent momentum entrainment, or turbulent buoyancy entrainment. This reinforces our suggestion that alternative modelling strategies, perhaps guided by more detailed studies of the transport across TNTIs and TTIs, might be fruitful if predictions of fountain behaviour, beyond their rise heights, are required. This is of relevance to many applications of fountains, for example, those produced within Under Floor Air Distribution Systems (Lin & Linden, 2005) where predictions of fountain rise height only determine the vertical extent within the room that the cool air might rise; the cooling effect that occupants feel is further affected by the temperature of the rising and falling cooler air — this depends both on the fountain’s mixing with the environment and the mixing within the fountain’s internal structure.

Finally, we assert that without greater regard to the modelling of the fountain cap region, insufficient progress might be made. In this region the flow is entirely dominated by the formation and collapse of large-scale turbulent structures — characterised by periodic fluctuations in fountain rise height and responsible for the dark bands typically seen in time series images of fountain (e.g. see figure 3.4 and Burridge & Hunt, 2012). Fountain modelling to date has focused on models that reflect only quantities of the

mean flow — our results suggest that both turbulent exchanges, and the transient nature of the fountain cap region, may be of importance in future modelling efforts.

Chapter 4

Local entrainment across a TNTI and a TTI in a turbulent forced fountain

4.1 Introduction

The majority of research on turbulent entrainment has focused on turbulent/non-turbulent interfaces (TNTIs), where a turbulent fluid interacts with a non-turbulent ambient. Many canonical flows fall into this category, including jets, plumes, mixing layers, boundary layers and gravity currents (Turner, 1986; Fernando, 1991; Da Silva *et al.*, 2014). However, in many practical situations, the ambient is in fact turbulent, which implies the presence of a turbulent/turbulent interface (TTI) demarcating regions of different turbulence intensity. The physics of TTIs is substantially more complex than that of TNTIs. The key difference between TNTIs and TTIs is that TTIs feature turbulent transport of momentum, buoyancy and other scalars (Van Reeuwijk *et al.*, 2021), while TNTIs feature mean transport only. These exchanges have been found to be affected by the turbulence intensity on both sides (Gaskin *et al.*, 2004; Kankanwadi & Buxton, 2020) and can lead to transport in the opposite direction to the mean entrainment components (see Chapter 3).

Turbulent forced fountains (Hunt & Burridge, 2015) are canonical flows that feature both TNTIs and TTIs. In Chapter 3, we present an analysis of a Reynolds-averaged tur-

bulent forced fountain demonstrated that the TTI between the fountain upflow (the up-flowing core rising from the source) and downflow (which shrouds the upflowing core), hereafter referred to as the ‘inner boundary’ of the fountain, exhibits strong turbulence on both sides. The turbulence was shown to drive exchanges of volume, momentum and buoyancy in either direction. For example, we found that below a particular height across the inner boundary, there is volume entrainment into the upflow, whilst above it, there is volume detrainment out of the upflow, which raises questions as to the extent, and importance, of the local instantaneous bi-directional exchanges that underlie statistics of the Reynolds-averaged exchanges within fountains. The entrainment parametrisations of the Bloomfield & Kerr (2000) model were investigated, and it was shown that particularly the momentum exchanges across the inner boundary were poorly represented. The parametrisations however, feature different velocities for entrainment and detrainment, which it was not possible to study using a Reynolds-averaged approach.

Here, we use the data from Chapter 3 to conditionally sample the flow field to analyse the instantaneous behaviour of the fountain. We present a methodology, based on Yurtoglu *et al.* (2018); Van Reeuwijk *et al.* (2021); Blakeley *et al.* (2022), that allows us to measure the magnitude and direction of the local instantaneous exchanges of volume, momentum and buoyancy. We are particularly interested in the exchanges across two of the natural interfaces within fountain flows: the inner boundary (as introduced above) and the ‘outer boundary’ of the fountain, which is the fountain envelope that contains (almost) all of the fountain fluid. The latter is a well-defined TNTI. The advantage of working with the instantaneous exchanges is that we can conditionally sample based on entrainment or detrainment events.

This chapter is laid out as follows. We introduce the framework of equations underlying our analysis and algorithms in §4.2, and the simulation details in §4.3. In §4.4, we show the results including the boundary statistics and conditional entrainment statistics. We compare the entrainment coefficients with those of the Reynolds-averaged fountain and fountain model, and finally, investigate the normalised entrainment density. Concluding remarks are made in §4.5.

4.2 Theoretical framework

4.2.1 Interface definition

Quantifying entrainment and detrainment simultaneously requires a precise definition of the domain of interest Ω across whose boundary fluid mass, and other properties, can be exchanged. Herein, this domain is defined as all points for which a field $\chi(\mathbf{x}, t)$ is larger than a threshold value χ_0 , where $\mathbf{x} = (x, y, z)$ denotes the Cartesian location, and t denotes time. Note that at any instant, the flow domain Ω may consist of many disconnected regions. The boundary surface will be denoted $\partial\Omega$, on which $\mathbf{N} = \nabla\chi/|\nabla\chi|$ is a three-dimensional unit normal pointing into the domain of interest (see figure 4.1). We define a masking function

$$I = H(\chi - \chi_0), \quad (4.1)$$

where H is the Heaviside function. Inclusion of this masking allows integration over the domain of interest (which is inherently irregular and three-dimensional, and potentially can consist of numerous disconnected regions) to be replaced by integration over an unbounded domain, via $\int_{\Omega} X dV = \int IX dV$, where X is an arbitrary scalar or vector component field.

4.2.2 Conditionally averaged plume equations

Integration of the governing equations (2.1)-(2.3) over the domain Ω in the x - y plane, in the high Reynolds number and Péclet number limit, results in (Van Reeuwijk *et al.*, 2021):

$$\frac{\partial \hat{A}}{\partial t} + \frac{\partial \hat{Q}}{\partial z} = -\hat{q}, \quad (4.2)$$

$$\frac{\partial \hat{Q}}{\partial t} + \frac{\partial \hat{M}}{\partial z} + \frac{\partial \hat{P}}{\partial z} = \hat{B} - \hat{m}, \quad (4.3)$$

$$\frac{\partial \hat{B}}{\partial t} + \frac{\partial \hat{F}}{\partial z} = -\hat{f}. \quad (4.4)$$

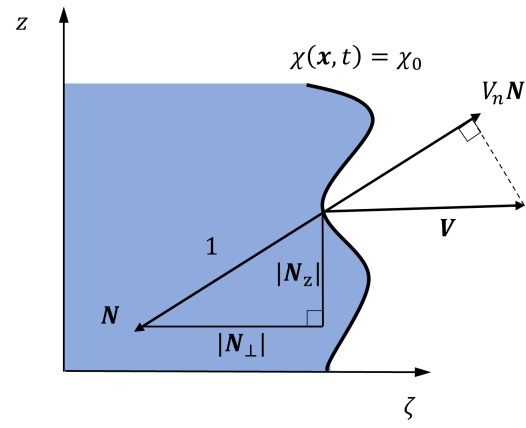


Figure 4.1: Definition sketch of interface properties. The bold black line highlights the interface determined by the threshold in the field $\chi(\mathbf{x}, t)$. The coloured region represents the inside of the domain ($\chi > \chi_0$) and ζ is the direction perpendicular to the flow direction z . The local normal vector $\mathbf{N}$ and relative velocity $\mathbf{V}$ are shown, including their decomposition into components. The length of the unit normal vector $\mathbf{N}$ is 1 by definition

Where

$$\hat{A}(z, t) = \frac{1}{\pi} \int_{\Omega} dA, \quad \hat{Q}(z, t) = \frac{1}{\pi} \int_{\Omega} w dA, \quad \hat{M}(z, t) = \frac{1}{\pi} \int_{\Omega} w^2 dA, \quad (4.5)$$

$$\hat{P}(z, t) = \frac{1}{\pi} \int_{\Omega} p dA, \quad \hat{B}(z, t) = \frac{1}{\pi} \int_{\Omega} b dA, \quad \hat{F}(z, t) = \frac{1}{\pi} \int_{\Omega} wb dA, \quad (4.6)$$

are the instantaneous surface area, vertical volume flux, vertical momentum flux, integral pressure, integral buoyancy and vertical buoyancy flux, respectively. The $\hat{\cdot}$ symbols emphasise that these are instantaneous counterparts of the classical Reynolds-averaged integral quantities in Chapter 3, and the integrals have been divided by π to remain consistent with classical plume theory (Hunt & Kaye, 2005; Van Reeuwijk *et al.*, 2016; Burridge *et al.*, 2017). The hatted lowercase symbols signify instantaneous exchanges

across the boundary $\partial\Omega$ and are defined as

$$\hat{q}(z, t) = \frac{1}{\pi} \oint_{\partial\Omega} \frac{V_n}{|\mathbf{N}_\perp|} d\ell, \quad (4.7)$$

$$\hat{m}(z, t) = \frac{1}{\pi} \oint_{\partial\Omega} \frac{V_n w - p N_z}{|\mathbf{N}_\perp|} d\ell, \quad (4.8)$$

$$\hat{f}(z, t) = \frac{1}{\pi} \oint_{\partial\Omega} \frac{V_n b}{|\mathbf{N}_\perp|} d\ell, \quad (4.9)$$

which represent the instantaneous volume, vertical momentum and buoyancy fluxes perpendicular to the interface, respectively.

Within the line integrals above, V_n is the local entrainment velocity across the boundary, which is defined by projecting the relative velocity $\mathbf{V} = \mathbf{v} - \mathbf{u}$, where $\mathbf{v}$ is the interface velocity, $\mathbf{u}$ is the fluid velocity, and V_n is the component of $\mathbf{V}$ projected onto $\mathbf{N}$, i.e., $V_n = \mathbf{V} \cdot \mathbf{N}$. This velocity can be linked to the scalar field χ using (Dopazo *et al.*, 2007; Holzner & Lüthi, 2011)

$$V_n = -\frac{1}{|\nabla\chi|} \frac{D\chi}{Dt}, \quad (4.10)$$

where $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the material derivative. The vector $|\mathbf{N}_\perp|$ is the magnitude of the three-dimensional normal $\mathbf{N}$ in the $x-y$ plane (4.1). Its function in the integrands of (4.7)-(4.9) is to account for the local surface area of the interface since $dS = |\mathbf{N}_\perp|^{-1} d\ell dz$ (Van Reeuwijk *et al.*, 2021). Note that vectors with a ‘perpendicular’ symbol ($\perp$) as a subscript denote the horizontal (two-dimensional) components perpendicular to the z -direction, e.g. $\mathbf{N} = [\mathbf{N}_\perp, N_z]^T$ or $\nabla_\perp = [\partial/\partial x, \partial/\partial y]^T$. When the relative velocity satisfies $V_n < 0$, due to the inward-pointing normal, entrainment of the properties (specifically, in the integrals above, are volume, momentum and buoyancy, respectively) into the domain of interest is occurring; conversely, $V_n > 0$ indicates the occurrence of detrainment. We note that $\mathbf{N}_\perp = 0$ where the surface is perfectly horizontal, which renders the integrands of (4.7)–(4.9) infinite. However, subsequent integration over z remains finite (see Van Reeuwijk *et al.*, 2021, for further discussion). Moreover, it is very unlikely that surface elements are perfectly horizontal, so occur-

rences of the singular integrand are rare.

Note that, unlike the other two integrals, the integral momentum exchange (4.8) comprises two terms that affect the momentum from two different mechanisms — we separate this exchange as

$$\hat{m} = \hat{m}_V + \hat{m}_p, \quad \text{where} \quad \hat{m}_V = \frac{1}{\pi} \oint_{\partial\Omega} \frac{V_n w}{|\mathbf{N}_\perp|} d\ell, \quad \hat{m}_p = \frac{1}{\pi} \oint_{\partial\Omega} \frac{-p N_z}{|\mathbf{N}_\perp|} d\ell. \quad (4.11)$$

The first term, $\hat{m}_V$, is induced by the relative velocity transferring the momentum across the boundary. Consistent with the volume (4.7) and buoyancy, (4.9), this represents the momentum entrainment when $V_n < 0$, as opposed to detrainment when $V_n > 0$. The additional term $\hat{m}_p$ is the momentum exchange induced by the pressure at the boundary. Noting the signs, (4.3) indicates that the pressure effect $\hat{m}_p$ tends to reduce the vertical momentum flux $\hat{M}$ within the flow.

Time-averaging the integral equations (4.2)–(4.4) yields the conditionally averaged plume equations

$$\frac{d\langle\hat{Q}\rangle}{dz} = -\langle\hat{q}\rangle, \quad (4.12)$$

$$\frac{d\langle\hat{M}\rangle}{dz} + \frac{d\langle\hat{P}\rangle}{dz} = \langle\hat{B}\rangle - \langle\hat{m}_V\rangle - \langle\hat{m}_p\rangle, \quad (4.13)$$

$$\frac{d\langle\hat{F}\rangle}{dz} = -\langle\hat{f}\rangle. \quad (4.14)$$

Where $\langle\cdot\rangle = \frac{1}{T} \int_0^T \cdot dt$ is the time-averaging operator, with T the averaging duration. These equations are identical in form to the classical plume equations of the boundary fluxes (e.g. Van Reeuwijk & Craske, 2015, before parameterisation), but note that no Reynolds averaging has been used, therefore $\langle\hat{M}\rangle$ and $\langle\hat{F}\rangle$ represent the total fluxes rather than the Reynolds-averaged mean fluxes.

4.2.3 Pointwise exchange rates

Following Yurtoglu *et al.* (2018), we convert the line integral to a surface integral, since surface integrals are simpler to evaluate computationally than line integrals. To do so, we first define an $x-y$ plane normal $\mathbf{n} = \nabla_{\perp}\chi/|\nabla_{\perp}\chi|$ that again points into the domain of interest. For the line integral of an arbitrary scalar or vector component X , after using $\mathbf{n} \cdot \mathbf{n} = 1$, the two-dimensional divergence theorem (taking into account the inward-pointing normal) and partial integration, we obtain

$$\oint_{\partial\Omega} X \, d\ell = \oint_{\partial\Omega} X \mathbf{n} \cdot \mathbf{n} \, d\ell = - \int I \nabla_{\perp} \cdot (X \mathbf{n}) \, dA = \int X \nabla_{\perp} I \cdot \mathbf{n} \, dA. \quad (4.15)$$

Here, the term $\nabla_{\perp} I \cdot \mathbf{n}$ represents a delta distribution that samples the boundary $\partial\Omega$.

When substituting $X = 1$ into (4.15), the perimeter of the boundary $\partial\Omega$ can be written as a surface integral

$$\pi \hat{l} = \int e_l \, dA, \quad e_l = \nabla_{\perp} I \cdot \mathbf{n} \quad (4.16)$$

where e_l is the pointwise boundary length. The quantity $e_l(\mathbf{x}, t)$ is local and instantaneous, and is zero everywhere except at the position of the interface, $\partial\Omega$. Due to the presence of the delta distribution, the expression is singular in the theoretical limit, while its numerical approximation will be dependent on the spatial resolution — hence only spatial integrals can be examined meaningfully.

Similarly, $\hat{q}$, $\hat{m}$ and $\hat{f}$ can be expressed in terms of surface integrals as

$$\pi \hat{q} = \int e_q \, dA, \quad \pi \hat{m} = \int e_m \, dA, \quad \pi \hat{f} = \int e_f \, dA, \quad (4.17)$$

$$e_q = \frac{V_n}{|\mathbf{N}_{\perp}|} e_l, \quad e_m = \underbrace{\frac{V_n w}{|\mathbf{N}_{\perp}|} e_l}_{e_{m,v}} + \underbrace{\frac{-p N_z}{|\mathbf{N}_{\perp}|} e_l}_{e_{m,p}}, \quad e_f = \frac{V_n b}{|\mathbf{N}_{\perp}|} e_l. \quad (4.18)$$

Here, e_q is the pointwise volume exchange rate (of dimension T^{-1}), e_m is the pointwise vertical-momentum exchange rate (of dimension LT^{-2}), and e_f is the pointwise buoyancy exchange rate (of dimension LT^{-3}). Note that, consistent with $V_n(\mathbf{x}, t)$, a negative

value of $e_q(\mathbf{x}, t)$ represents entrainment of volume flux into the domain of interest, and a positive value represents detrainment.

As the fountain is statistically axisymmetric (e.g. Craske & Van Reeuwijk, 2015; Van Reeuwijk *et al.*, 2016), we introduce a Reynolds-averaging operator $\bar{\cdot}$ defined in (3.6).

Time averaging of (4.16) and (4.17), and using the definition of the Reynolds average above yields

$$\langle \hat{l} \rangle = 2 \int_0^\infty \bar{e}_l r dr, \quad \langle \hat{q} \rangle = 2 \int_0^\infty \bar{e}_q r dr, \quad (4.19)$$

$$\langle \hat{m} \rangle = 2 \int_0^\infty \bar{e}_m r dr, \quad \langle \hat{f} \rangle = 2 \int_0^\infty \bar{e}_f r dr. \quad (4.20)$$

These provide an explicit link between the time-averaged boundary integral (e.g. $\langle \hat{q} \rangle$) and the integral of the Reynolds-averaged pointwise exchanges (e.g., $\bar{e}_q$) over the horizontal area. Since Reynolds averaging involves integration over the azimuthal direction, the quantities $\bar{e}$ are not resolution-dependent and have a clearly defined meaning: they represent the interface length or the net exchange across the interface at radius r , respectively. Radially integrating the Reynolds-averaged pointwise length $\bar{e}_l$ and net exchange $\bar{e}_q, \bar{e}_m, \bar{e}_f$ then provides the averaged integral length $\langle \hat{l} \rangle$ and net exchange $\langle \hat{q} \rangle, \langle \hat{m} \rangle, \langle \hat{f} \rangle$, respectively, on a particular horizontal plane.

4.2.4 Splitting entrainment and detrainment processes

Some parametrisations distinguish between entrainment and detrainment definitions, e.g. for turbulent fountains (Bloomfield & Kerr, 2000) or turbulent clouds (De Rooy & Siebesma, 2008). The current framework allows explicit calculation of these quantities. Using that $e_q < 0$, $e_q > 0$ (whose sign is consistent with V_n) represent entrainment and detrainment, respectively, the pointwise exchange rates can be written as

$$e_X = e_X^+ + e_X^-, \quad e_X^- = e_X H(-e_q), \quad e_X^+ = e_X H(e_q), \quad (4.21)$$

where X is l, q, m , or f . With (4.16) and (4.17), this implies that the boundary integrals can be decomposed as

$$\hat{l} = \hat{l}^+ + \hat{l}^-, \quad \hat{q} = \hat{q}^+ + \hat{q}^-, \quad \hat{m} = \hat{m}^+ + \hat{m}^-, \quad \hat{f} = \hat{f}^+ + \hat{f}^-. \quad (4.22)$$

The momentum exchange term $\hat{m}$ can be split into an entrainment term $\hat{m}_V = \int e_{m,V} dA$ and a pressure term $\hat{m}_p = \int e_{m,p} dA$, which each can be decomposed as

$$\hat{m}_V = \hat{m}_V^+ + \hat{m}_V^-, \quad \hat{m}_p = \hat{m}_p^+ + \hat{m}_p^-. \quad (4.23)$$

Note that the superscript above represents only the integral over the area where fluid entrainment (indicated by ‘ $-$ ’) and detrainment (indicated by ‘ $+$ ’) occur, but not the sign of the term itself. Due to the linearity of the time-averaging operator, (4.22) also hold for the averaged quantities.

A validation of this theoretical framework is presented in Appendix §A by testing the entrainment of an axisymmetric jet.

4.3 Case description and boundary specification

For the data analysis, use will be made of the highly resolved simulation of a turbulent forced fountain described in §3.2, including the simulation settings, boundary and initial conditions and the normalisation, therefore, they are not repeated here.

As introduced, we focus on entrainment statistics across two instantaneous fountain boundaries — the outer boundary and the inner boundary. The instantaneous outer boundary (hereafter denoted by the subscript f) is identified using the buoyancy, i.e. $\chi = |b|$ with $\chi_0 = |b_c|$, where $|b|$ represents the absolute value of the buoyancy field. The threshold is set to $|b_c| = 3.6 \times 10^{-3} |b_0|$, consistent with the threshold in Chapter 3, which is deduced from the buoyancy at the height of the cap base. Chapter 3 reported that the outer boundary encloses almost all (negatively) buoyant fluid, and showed that reasonable variation of the threshold does not influence statistics significantly, including

those concerning entrainment by the fountain.

The inner boundary (hereafter denoted by the subscript i) is harder to define. It is intuitive to argue that $w > 0$ is the natural criterion. Indeed, this is the default criterion to identify the upflow when analysing Reynolds-averaged quantities (Williamson *et al.*, 2011, and in Chapter 3). However, since the flow is turbulent, a threshold of zero would result in unwanted fluid being contained in the upflow; for example, slowly falling rotating turbulent fluid may contain regions of fluid that are, instantaneously, travelling upwards, i.e. patches of fluid with vertical velocity $w > 0$ are contained within the turbulent downflow — we see evidence of this within our data. In addition, there are irrotational patches of fluid within the ambient for which $w > 0$, which renders this criterion unhelpful. One could consider more complex composite criteria to solve this issue (for example, conditioning on both the velocity and buoyancy field simultaneously); however, this would require sophisticated and more computationally intensive algorithms that are beyond the scope of the current study.

Herein, we opt for a simpler, yet effective, solution by using $\chi = w$ with a small positive threshold $\chi_0 = 0.07w_0$. Selection of this threshold was based on a sensitivity study — there being no uniquely appropriate choice, mathematically. We tested thresholds in the range $0 < \chi_0 \leq 0.10w_0$. As expected, lower thresholds within this range gave rise to the inclusion of unwanted fluid within the domain of interest; for example, fluid clearly within the downflow being included in the fountain upflow, and *vice versa*. Thresholds in the range $0.07w_0 \leq \chi_0 \leq 0.09w_0$ were deemed broadly suitable, with a threshold 0.07 being selected as the smallest threshold that captures the most instantaneous upflow region but is broadly unaffected by the presence of unwanted fluid. For our data, this threshold is relatively small compared to the characteristic velocity over most heights of the fountain. In addition, as we demonstrate in §4.4, our choice is not expected to have substantially affected the vertical profile of exchanges.

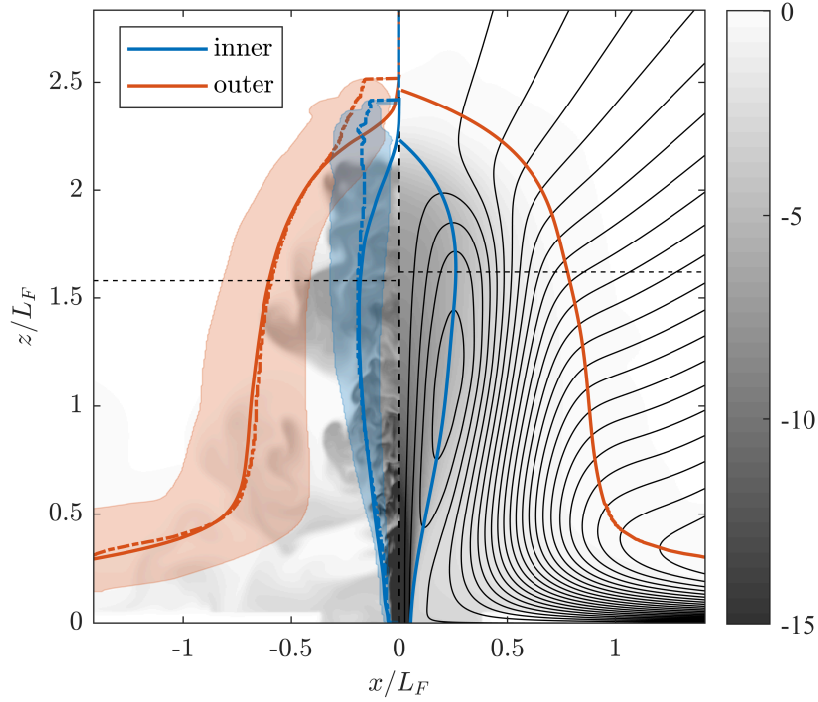


Figure 4.2: The forced fountain. Left half: a snapshot of the instantaneous buoyancy field together with the time-averaged characteristic inner and outer radii $\langle \hat{r}_i \rangle$, $\langle \hat{r}_f \rangle$ (solid lines), and the location of r_{50} of the fountain outer boundary and the upflow (dash-dotted lines), with the coloured band marking the interval between r_{95} (left bound) and r_5 (right bound). Right half: Reynolds-averaged buoyancy field, together with interface positions of the inner and outer boundary r_i , r_f inferred from Reynolds-averaged statistics, and the streamlines (black lines). The horizontal dashed lines on both sides present the location of the fountain cap base of the time-averaged conditional fountain and Reynolds-averaged fountain, respectively.

4.4 Results

4.4.1 Boundary statistics

Figure 4.2 shows a greyscale image of the forced fountain constructed in two halves; the left and right halves show snapshots of the instantaneous and Reynolds-averaged buoyancy fields, respectively. The greyscale map of the buoyancy field is normalised by $|b_0|/Fr_0$. A thin white layer at the bottom over the horizontal region relatively far from the source, namely $|x/L_F| \geq 0.4$, can be observed, due to the manually set nudging area that avoids accumulation of buoyancy at the bottom boundary (Chapter

3). Overlaid on the right-hand side are the Reynolds-averaged inner boundary r_i , the outer boundary r_f , and the streamlines: the inner boundary r_i is composed of the points of zero Reynolds-averaged vertical velocity, and the outer boundary r_f is composed of the points where the Reynolds-averaged (absolute) buoyancy reduces to $|b_c|$, the same threshold as we define for the instantaneous boundary.

In order to develop an understanding of the location of the instantaneous interface, the radial locations associated with the areas within the inner and outer boundaries are determined according to

$$\langle \hat{r}_i \rangle^2 = \langle \hat{A}_i \rangle, \quad \langle \hat{r}_f \rangle^2 = \langle \hat{A}_f \rangle, \quad (4.24)$$

where $\langle \hat{A}_i \rangle$ and $\langle \hat{A}_f \rangle$ are the time-averaged areas of the upflow and the whole fountain, respectively. These boundaries are marked by solid blue and orange lines on the left side of figure 4.2, respectively. At the particular instant shown, the outer boundary of the fountain (as inferred from the instantaneous buoyancy field) can, for example, be seen to lie well within, and well outside the solid orange line marking $\langle \hat{r}_f \rangle$.

In order to quantify the variability of the instantaneous interface location, we selected three radial locations r_5, r_{50} and r_{95} , for both the inner (blue) and outer (orange) boundaries, that correspond to 5%, 50% and 95% of the contribution to $\langle \hat{l} \rangle$, specifically via $2 \int_0^{r_\alpha} \bar{e}_l r dr = \frac{\alpha}{100} \langle \hat{l} \rangle$ where α is a percentage. Note that near the fountain top, i.e. $z/L_F \gtrsim 1.9$ for the inner boundary and $z/L_F \gtrsim 2.3$ for the outer boundary, the vertical variation of r_α is not reliable as there are insufficient samples for accurate statistics. The band between r_{95} and r_5 shows that the instantaneous boundaries of both the outer and inner boundaries exhibit broad fluctuations, with r_{95} about twice as large as r_5 for both boundaries. Note that for $z/L_F \gtrsim 1$, r_{95} is close to the Reynolds-averaged boundary (i.e. r_i or r_f in the right half of figure 4.2), and r_{50} is very close to the time-averaged characteristic radius (i.e. $\langle \hat{r}_i \rangle$ or $\langle \hat{r}_f \rangle$).

The total height of the fountain, indicated by the time-averaged outer boundary $\langle \hat{r}_f \rangle$, is $z/L_F \approx 2.40$, which is consistent with the Reynolds-averaged statistics on the right and the heights reported by the literature (see Hunt & Burridge, 2015, for a review). In

the statistically steady state, the fluid in the region near the top experiences a number of large-scale fluctuations characterised by the formation and collapse of large-scale fluid structures, i.e., the rising fluid stops and begins to fall near the top. We identify this near-top region as a ‘fountain cap’ to consider the fountain fluid that is influenced by the dynamics near the top. The fountain cap is defined above a height that is termed as the fountain cap base. Consistent with Chapter 3, we define the fountain cap base at a height where the averaged radius of upflow is the largest. Using this definition, the present conditional statistics yield a time-averaged cap base location $z/L_F = 1.58$, very close to the height $z/L_F = 1.62$ from the Reynolds-averaged statistics (Chapter 3). The two cap bases are labelled with horizontal dashed lines on both sides of figure 4.2. Indeed, different definitions of the fountain cap base have been proposed in the literature (Awin *et al.*, 2018; Shrinivas & Hunt, 2014; Mcdougall, 1981). However, Chapter 3 have clarified that all these different definitions resulted in some very similar fountain cap bases to our fountain statistics. Hereafter, the cap base will be marked in the figures when necessary, as a reminder of the influence of the top.

4.4.2 Entrainment statistics

Figure 4.3 shows an instantaneous snapshot of the pointwise exchange rate e_q across the fountain outer boundary and inner boundary, at the three selected planes: one central vertical plane, and two different horizontal planes at $z/L_F = 1.62$ and 1.18 , which are approximately the fountain cap base and the fountain half-height, respectively.

Here, e_q is displayed using a symmetric colour map that ranges from blue ($e_q < 0$, entrainment) to red ($e_q > 0$, detrainment); a colour bar is not shown since e_q contains a delta distribution implying that its value depends on the grid size. As expected, figure 4.3 shows that there is only instantaneous entrainment at the outer boundary. Note that no results are shown below $z/L_F = 0.66$; in this region, the downflow is affected by the bottom boundary (see the outer boundary and streamlines in figure 4.2), so that this region is excluded from all further analyses. At the inner boundary, entrainment (blue) and detrainment (red) co-exist instantaneously. Although the value of e_q is not

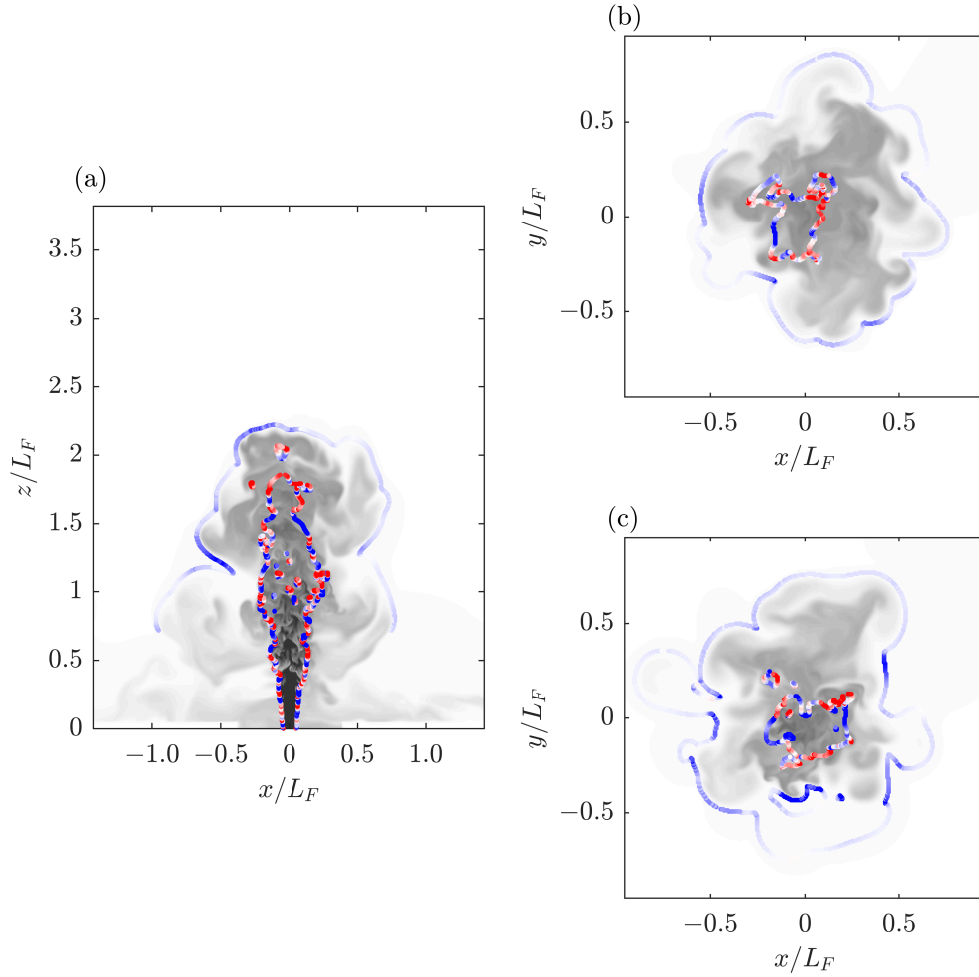


Figure 4.3: Instantaneous pointwise volume exchange e_q across the inner and outer boundaries using a symmetric colour scale ranging from blue (entrainment) to red (detrainment). The darker colour represents the larger magnitude of the exchange. Plotted in the background is the instantaneous buoyancy field: (a) central vertical plane, (b) horizontal plane $z/L_F = 1.62$, (c) horizontal plane $z/L_F = 1.18$.

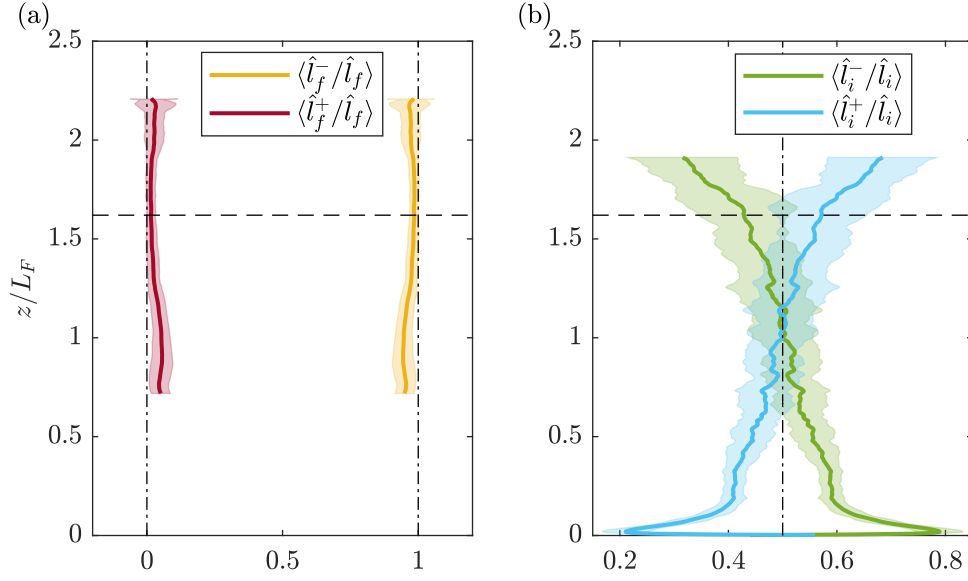


Figure 4.4: The time-averaged length fraction of entrainment and detrainment to the total perimeter across (a) the outer boundary and (b) the inner boundary. The coloured band marks the first standard deviation interval of the time average. The ratio very near the top of the upflow, i.e. $z/L_F \gtrsim 1.9$, and the fountain, i.e. $z/L_F \gtrsim 2.3$, is not shown due to the lack of sampled data. The horizontal dashed lines represent the heights of the fountain cap base.

meaningful in itself, the relative magnitude is worthy of comparison. The magnitude of e_q at the inner boundary is generally larger than that at the outer boundary, which is in accord with the stronger local turbulence at the inner boundary evident within our data (not shown), presumably due to the shearing interactions between the rising upflow and falling downflow. The colour also indicates the momentum and buoyancy exchanges. At the inner boundary the buoyancy is always negative and the vertical velocity is $0.07w_0$ due to the threshold, therefore in the blue area, these properties are entrained into the upflow, while in the red area they are detrained. However, the pressure across the inner boundary cannot be predicted, so that, the momentum exchange driven by the pressure requires more specific statistics.

Figure 4.4 shows the time-averaged length fractions of the entrainment and detrainment to the total boundary perimeter, respectively. Here, $\langle \hat{l}_i^- / \hat{l}_i \rangle$ and $\langle \hat{l}_i^+ / \hat{l}_i \rangle$ are the ratios of inner boundary, and $\langle \hat{l}_f^- / \hat{l}_f \rangle$ and $\langle \hat{l}_f^+ / \hat{l}_f \rangle$ are the ratios of the outer boundary. At the outer boundary, consistent with the observation in figure 4.3, most length of the

boundary entrains, while only an insignificant ratio of the length detrains. At the inner boundary, reassuringly, both entrainment and detrainment occur. Near the source, the entrainment length is larger than the detrainment length. However, generally, the entrainment length decreases and the detrainment length increases with the height — at $z/L_F \approx 1.00$, half of the total boundary entrains and half detrains. Above that, the detrainment length is longer than the entrainment length. This indicates that near the source, the entrainment dominates the exchange, while near the top, the detrainment dominates.

Figure 4.5(a) shows the vertical variation of the time-averaged volume entrainment and detrainment at the outer boundary, $\langle \hat{q}_f^- \rangle$, $\langle \hat{q}_f^+ \rangle$, respectively. The width of the correspondingly coloured shading highlights one standard deviation about the time-averaged mean. Also shown in the figure is the net volume exchange q_f across the Reynolds-averaged boundaries r_f (Chapter 3). Reassuringly, only entrainment is significant, with $\langle \hat{q}_f^- \rangle = \langle \hat{q}_f \rangle$. It can be seen that the Reynolds-averaged net entrainment q_f is generally greater than the conditional time-averaged entrainment $\langle \hat{q}_f^- \rangle$, especially near the top (e.g., $z/L_F \gtrsim 2.0$). Clearly, this will provide a larger total entrainment flux when integrating from a reference height up to the fountain height. The reason is that the radial position of the time-averaged instantaneous interface $\langle \hat{r}_f \rangle$ is smaller than the Reynolds-averaged interface r_f , particularly near the top (see figure 4.2), which in turn implies a lower amount of entrained fluid. Reassuringly, if we use the position of $\langle \hat{r}_f \rangle$ as the interface to calculate the Reynolds-averaged entrainment flux q_f , then we find that it is close to $\langle \hat{q}_f^- \rangle$.

Figures 4.5(b,c) show the momentum and buoyancy exchanges across the outer boundary, respectively. As expected, and consistent with the Reynolds average, there are insignificant time-averaged momentum and buoyancy exchanges across the outer boundary due to the very low magnitude of the momentum, pressure and buoyancy relative to the fountain there. Figure 4.5(d) shows the volume exchanges at the inner boundary, and highlights that the net exchange $\langle \hat{q}_i \rangle$ agrees closely with the Reynolds-average net exchange q_i as reported in Chapter 3. This provides confirmation that the methodology concerning the analysis of the instantaneous measurements, including the

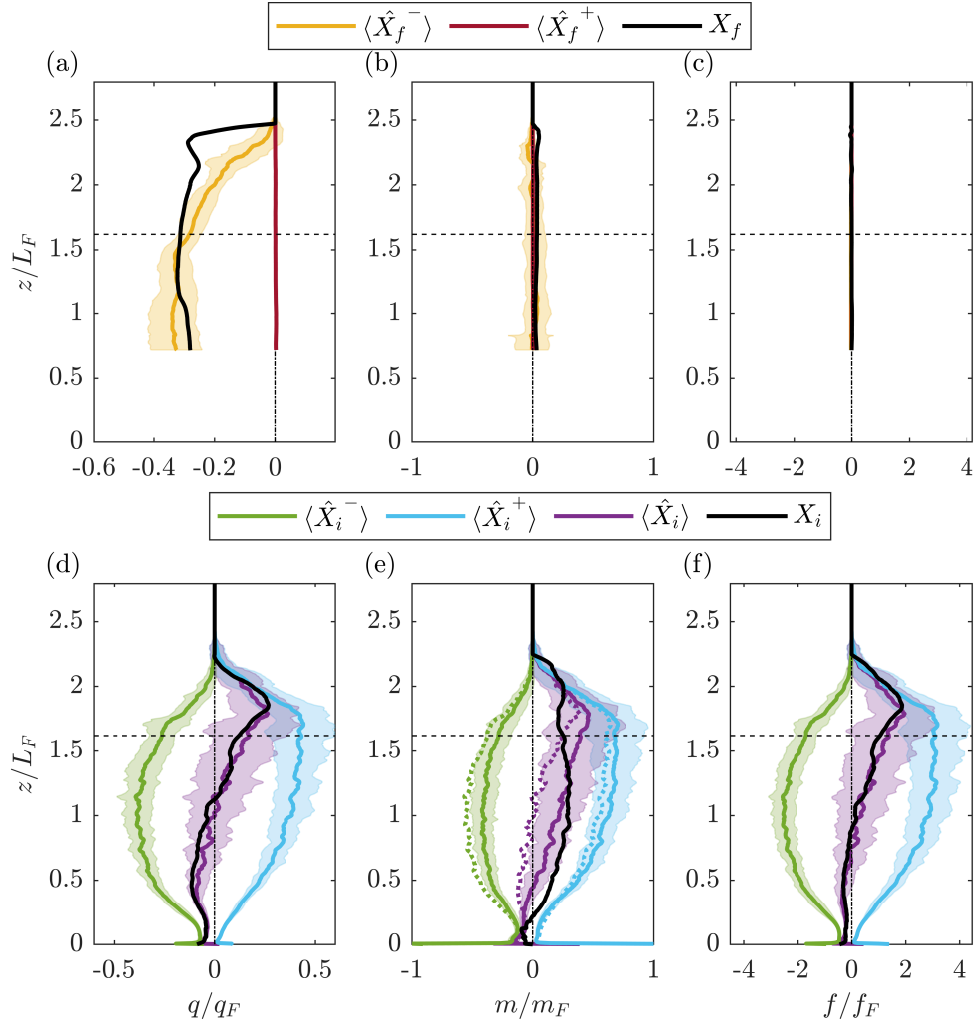


Figure 4.5: The time-averaged entrainment (with superscript $-$, yellow line in top panes a-c and green line in bottom panes d-f), detrainment (with superscript $+$, brown line in top panes and blue line in bottom panes), and the net exchange (the sum of entrainment and detrainment, purple line in bottom panes) of volume (a,d), momentum (b,e) and buoyancy (c,f) variation with the height. The top panes present those at the outer boundary and the bottom panes present the inner boundary. The exchanges are compared with the net exchanges of the Reynolds-averaged fountain (black solid lines). The coloured band marks the first standard deviation interval of the time average. The dotted lines in (e) represent the time-averaged momentum exchange associated with the relative velocity $\langle \hat{m}_V \rangle$ and its segregation. The difference between the dotted line and the solid line indicates the momentum exchange associated with the pressure, i.e. $\langle \hat{m}_{p,i} \rangle$ and its segregation.

choice of velocity threshold, is appropriate. When $\langle \hat{q}_i \rangle$ is segregated into entrainment and detrainment events, it becomes apparent that at almost all heights, $\langle \hat{q}_i \rangle$ is small compared with either the entrainment fluxes $\langle \hat{q}_i^- \rangle$ or the detrainment fluxes $\langle \hat{q}_i^+ \rangle$ — evidence that entrainment and detrainment co-exist at the inner boundary, and that, typically, the two fluxes are comparable. The net volume exchange $\langle \hat{q}_i \rangle$ intersects zero at $z/L_F \approx 1.0$, close to its Reynolds-averaged counterpart. Note that this is also very close to the location where entrainment and detrainment have an equal length in figure 4.4(b). The time-averaged instantaneous entrainment peaks at $\langle \hat{q}_i^- \rangle \approx -0.39q_F$ (at $z/L_F \approx 0.82$), and the detrainment peaks at $\langle \hat{q}_i^+ \rangle \approx 0.43q_F$ (at $z/L_F \approx 1.31$).

Figure 4.5(e) shows the time-averaged momentum exchange $\langle \hat{m}_i \rangle$ at the inner boundary, including the decomposed contributions of entrainment, $\langle \hat{m}_i^- \rangle$, and detrainment, $\langle \hat{m}_i^+ \rangle$. For $z/L_F \gtrsim 0.45$, $\langle \hat{m}_i \rangle$ is positive, indicates that the momentum exchange at the inner boundary decreases the upward vertical momentum flux within the upflow over most of the fountain height. This is broadly consistent with the Reynolds-averaged quantity m_i at the inner boundary (black line, obtained in Chapter 3). Figure 4.5(f) shows the negative buoyancy exchanges at the inner boundary, with the horizontal axis normalised by f_F (which is negative); therefore, the direction of exchange is aligned with that of the volume exchange. The quantity $\langle \hat{f}_i^- \rangle$ represents the negative buoyancy entrained into the upflow, while $\langle \hat{f}_i^+ \rangle$ represents the negative buoyancy detrained out of the upflow. The net $\langle \hat{f}_i \rangle$ generally agrees well with the Reynolds average f_i with the zero point at $z/L_F \approx 0.90$, slightly lower than that of $\langle q_i \rangle$. The buoyancy entrainment peaks at $\langle \hat{f}_i^- \rangle \approx -0.39f_F$ (at $z/L_F \approx 0.82$), and the detrainment peaks at $\langle \hat{f}_i^+ \rangle \approx 0.43f_F$ (at $z/L_F \approx 1.31$).

The net momentum exchange $\langle \hat{m}_i \rangle$ is comprised of two contributions: one due to entrainment $\langle \hat{m}_{V,i} \rangle$, and one due to pressure $\langle \hat{m}_{p,i} \rangle$ (§4.2). These quantities are examined in figure 4.6(a), and shown, for comparison, alongside the Reynolds-averaged net momentum exchange m_i (Chapter 3), and the integral buoyancy of the upflow $\langle \hat{B}_u \rangle$, the latter being a term on the right-hand-side of momentum equation (4.13). The figure shows that the exchange types are equal in magnitude and comparable to the integral buoyancy, so no terms are negligible *a priori*. The quantity $\langle \hat{m}_{V,i} \rangle$ is trivially related

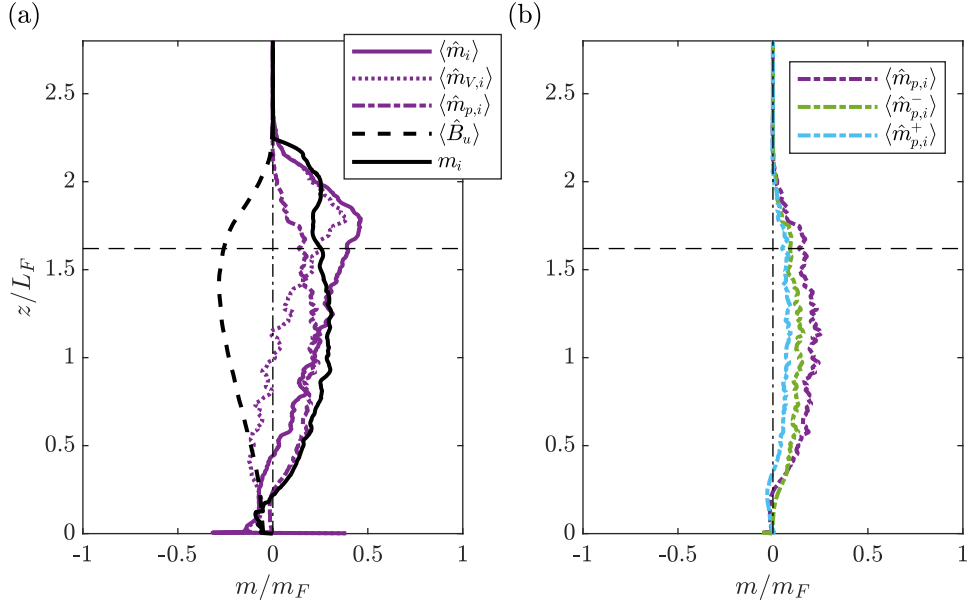


Figure 4.6: (a) The time-averaged net momentum exchange $\langle \hat{m}_i \rangle$ and its components net $\langle \hat{m}_{V,i} \rangle$ and net $\langle \hat{m}_{p,i} \rangle$, overlaid with the integral negative buoyancy of the upflow $\langle \hat{B}_u \rangle$, and the Reynolds-averaged momentum exchange m_i . (b) The pressure effect of net momentum exchange $\langle \hat{m}_{p,i} \rangle$ conditioned to the entrainment $\langle \hat{m}_{p,i}^- \rangle$ and detrainment $\langle \hat{m}_{p,i}^+ \rangle$ components.

to $\langle \hat{q}_i \rangle$ since substitution of the constant threshold value $0.07w_0$ (which identifies the inner boundary) into (4.11), and exploiting (4.7), yields directly $\langle \hat{m}_{V,i} \rangle = 0.07w_0 \langle \hat{q}_i \rangle$. This fact is borne out by our data (*cf.* the shape of $\langle \hat{m}_{V,i} \rangle$ marked by the dotted lines in figure 4.5(e), and that of $\langle \hat{q}_i \rangle$ in figure 4.5(d)).

Note that the difference between the dotted lines and solid lines in figure 4.5(e) indicates the pressure-induced momentum exchange (e.g., $\langle \hat{m}_{p,i}^- \rangle$ and $\langle \hat{m}_{p,i}^+ \rangle$). These data demonstrate that conditional momentum entrainment or detrainment is caused primarily by the decomposed components associated with entrainment or detrainment velocities. However, there is no intrinsic physical relevance to the term $\langle \hat{m}_{V,i} \rangle$, because of the need to impose a non-zero vertical velocity threshold (in our case, $0.07w_0$) to avoid unexpected fluid in the upflow (see §4.3). In theory, a zero vertical velocity threshold value is a natural choice for the inner boundary, which would result in the term $\langle \hat{m}_{V,i} \rangle$ being zero, leaving only the pressure exchange term $\langle \hat{m}_{p,i} \rangle$ of the dynamically active terms within the momentum exchange. Figure 4.6(a) compares the net momentum ex-

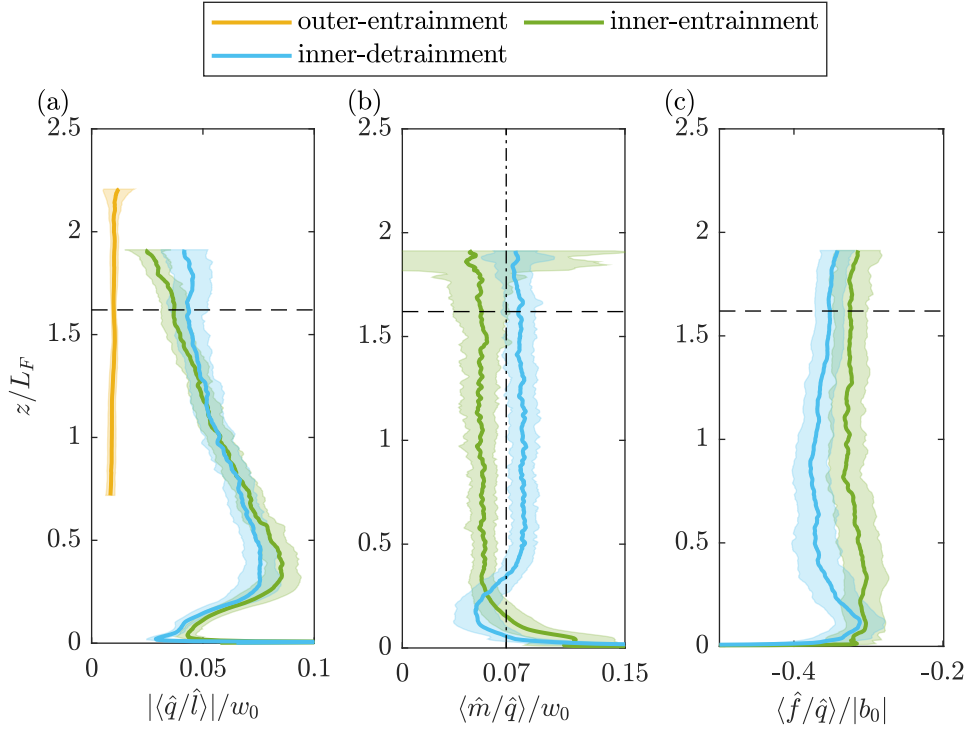


Figure 4.7: (a) Time-averaged absolute entrainment and detrainment velocity normalised with the fountain source vertical velocity w_0 . (b) Exchanged vertical velocity w_i normalised by w_0 . (c) Exchanged buoyancy b_i at the inner boundary normalised by the absolute source buoyancy $|b_0|$.

change across the inner boundaries when conditionally averaged, $\langle \hat{m}_{p,i} \rangle$, with that for the Reynolds-averaged net momentum exchange m_i , which in the case of Reynolds averaging is indeed to set the loci of zero vertical velocity (see Chapter 3). The conditionally averaged $\langle \hat{m}_{p,i} \rangle$ term agrees well with the Reynolds-averaged momentum exchange m_i below $z/L_F \approx 0.8$, and marginally lower than m_i above that.

Figure 4.6(b) shows the pressure exchange term decomposed in its entrainment and detrainment terms. As can be seen, both $\langle \hat{m}_{p,i}^- \rangle$ and $\langle \hat{m}_{p,i}^+ \rangle$ are positive. Recall that the conditioning is based on the local value of V_n (§4.2) and the pressure has a negative sign in momentum equation (4.13), the fact that both are positive implies that pressure acts to remove momentum from the upflow both during entrainment and detrainment events. From the observation that $\langle \hat{m}_{p,i}^- \rangle$ is slightly larger than $\langle \hat{m}_{p,i}^+ \rangle$, we can conclude that the pressure effect is slightly larger during entrainment events.

Figure 4.7 shows the time-averaged ratio of the integrals, which represents the local entrainment/detrainment velocity $\hat{q}/\hat{l}$, the locally exchanged vertical velocity $\hat{m}/\hat{q}$, and the locally exchanged buoyancy $\hat{f}/\hat{q}$, respectively. Note that the exchanged vertical velocity (momentum) and buoyancy across the outer boundary are not presented since these are insignificant. Figure 4.7(a) shows the local entrainment/detrainment velocities. As in earlier figures, the thick lines represent the mean value, and the shaded bands represent the range from the first standard deviation interval of the time average. The entrainment velocity at the outer boundary $\langle \hat{q}_f^- / \hat{l}_f^- \rangle$ is approximately constant over the entire fountain height and has very little fluctuation with time. Note that here, the absolute value is presented to show the magnitude of the velocity. At the inner boundary, beyond the near-source region, the entrainment velocity $\langle \hat{q}_i^- / \hat{l}_i^- \rangle$ and detrainment velocity $\langle \hat{q}_i^+ / \hat{l}_i^+ \rangle$ both decrease approximately linearly with height. Both are large compared with the entrainment velocity at the outer boundary. Combined with figure 4.4(b), we find that below $z/L_F \approx 1.0$, both the entrainment length and entrainment velocity are greater than those of detrainment, while above that, they are the opposite.

Data for the averaged exchanged velocity $\langle \hat{m}_i / \hat{q}_i \rangle / w_0$ are shown in figure 4.7(b), and indicate that over most of the fountain height, i.e., $z/L_F \gtrsim 0.45$, the exchanged velocity is approximately 0.05 for entrainment and 0.08 for detrainment. Note that these averaged quantities would both be the constant 0.07 were it not for the contribution due to pressure within the momentum exchange. Figure 4.7(c) examines the time-averaged entrained and detrained buoyancy over the inner boundary $\langle \hat{f}_f^- / \hat{q}_f^- \rangle$ and $\langle \hat{f}_f^+ / \hat{q}_f^+ \rangle$, respectively. The values are normalised by the absolute source value $|b_0|$ to keep the negative sign of the buoyancy. We note that over the main interval of the upflow, $0.5 \leq z/L_F \leq 2.0$, the time-averaged entrained buoyancy is approximately $-0.33|b_0|$ while the detrained buoyancy is $-0.37|b_0|$, i.e., the averaged detrained buoyancy is always more negative than the averaged entrained buoyancy. This is because the detrainment transfers the upflow buoyancy outwards, while the entrainment transfers the downflow buoyancy into the upflow. The downflow buoyancy is inherently less negative than the upflow.

4.4.3 Entrainment coefficients

In this subsection, we recalculate the entrainment coefficients of the model of BK00 using conditional statistics, and compare them with measurements.

The BK00 parametrisation for the downflow entrainment flux across the outer boundary is given by

$$\langle \hat{q}_f^- \rangle = 2r_d \omega_f, \quad (4.25)$$

with r_d the characteristic radius of the downflow, and ω_f the entrainment velocity given by

$$\omega_f = -\gamma |w_d|, \quad (4.26)$$

where $|w_d|$ is the absolute value of the characteristic downflow velocity, γ is the entrainment coefficient at the outer boundary, and the negative sign represents the entrainment velocity pointing into the flow in our framework.

At the inner boundary, BK00 assume that the upflow entrains from, and detrains into, the downflow with velocities ω_i and ω_d , respectively, producing an entrainment and detrainment flux given by

$$\langle \hat{q}_i^- \rangle = 2r_u \omega_i, \quad \langle \hat{q}_i^+ \rangle = 2r_u \omega_d, \quad (4.27)$$

where r_u is the characteristic radius of the upflow. BK00 parametrised the detrainment velocity as

$$\omega_d = \beta |w_d|, \quad (4.28)$$

where β is the detrainment coefficient. For the entrainment velocity ω_i , BK00 proposed two possible parametrisations:

$$\omega_i^I = -\alpha^I (w_u + |w_d|), \quad \omega_i^{II} = -\alpha^{II} w_u, \quad (4.29)$$

with associated entrainment coefficients α^I and α^{II} , respectively. The first parametrisation assumed that the entrainment velocity should be proportional to the difference in velocity between the upflow and downflow, while the second assumed that the entrain-

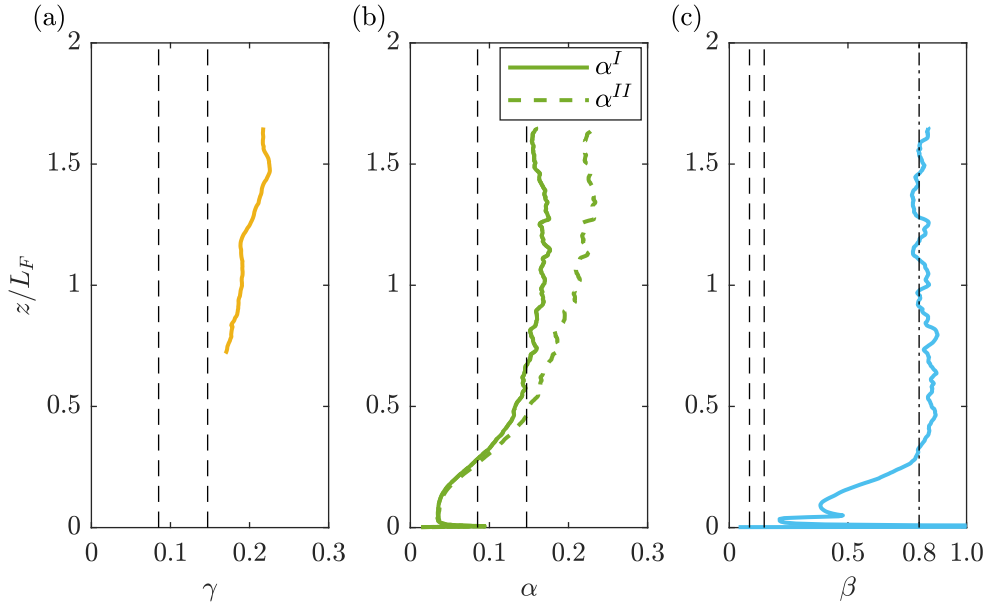


Figure 4.8: The entrainment and detrainment coefficients at (a,b) the inner boundary and (c) the outer boundary, overlaid with vertical dashed lines showing the entrainment coefficients used in BK00, 0.085 and 0.147. All the coefficients are shown up to the fountain cap base $z/L_F = 1.62$.

ment velocity depend solely on the velocity within the upflow. Here, the characteristic values are associated with the vertical fluxes of the upflow (subscript with u) and downflow (subscript with d) (Bloomfield & Kerr, 2000), given by

$$w_u \equiv \frac{\langle \hat{M}_u \rangle}{\langle \hat{Q}_u \rangle}, \quad r_u \equiv \frac{\langle \hat{Q}_u \rangle}{\langle \hat{M}_u \rangle^{1/2}}, \quad w_d \equiv \frac{\langle \hat{M}_d \rangle}{\langle \hat{Q}_d \rangle}, \quad r_d \equiv \sqrt{\frac{\langle \hat{Q}_u \rangle^2}{\langle \hat{M}_u \rangle} + \frac{\langle \hat{Q}_d \rangle^2}{\langle \hat{M}_d \rangle}}. \quad (4.30)$$

BK00 assumed that the downflow entrains similarly to a line plume, therefore taking $\gamma_{00} = \beta_{00} = 0.147$ according to the entrainment coefficient of the line plume, where the subscript 00 represents BK00. They assumed the upflow entrainment coefficient $\alpha_{00}^I = \alpha_{00}^{II} = 0.085$, which is the average of the jet value and plume value. With these coefficients, combining the exchange equations with the governing equations, the fountain model, which is a closed set of equations, can be solved. The result presented in BK00 was tuned to agree with the experimental data for the statistics of the steady mean fountain height. A sensitivity examination revealed that the predicted fountain height depends strongly on the choice of γ_{00} , but is almost not affected by β_{00} .

Figure 4.8 presents α , β and γ calculated using (4.25)–(4.30). Figure 4.8(a) shows that the coefficient γ is marginally greater than γ_{00} but generally smaller than 0.20, which supports the BK00 assumption that the downflow at the outer boundary is likely to entrain as a line plume. Figure 4.8(b) shows that near the source, below $z/L_F \approx 0.6$, the upflow entrainment coefficients α^I and α^{II} increase from below 0.085 to over 0.147. Above that, the two coefficients become approximately constant with the height: $\alpha^I \approx 0.16$ and $\alpha^{II} \approx 0.21$, respectively. The coefficient α^I is more constant with height than α^{II} , which suggests that the parametrisation I, which accounts for the total shear between the upflow and downflow, is the better parametrisation.

Figure 4.8(c) shows that the detrainment coefficient of the upflow is $\beta \approx 0.80$ beyond the near source field, which is much larger than β_{00} . This is because in our data, the detrainment flux is comparable with the entrainment flux (as shown in figure 4.5d), but the characteristic downflow velocities w_d are small compared with the characteristic upflow velocities w_u , resulting in the large detrainment coefficients determined herein. However, as mentioned before, BK00 showed that β does not have a significant effect on predicting the fountain height.

As the BK00 model is a bulk model, the momentum and buoyancy exchange across the inner boundary is modelled as the product of the volume exchange and the characteristic velocity and buoyancy, where the characteristic values are, in effect, assumed to be those transported, i.e.

$$\langle \hat{m}_i^- \rangle = \langle \hat{q}_i^- \rangle w_d, \quad \langle \hat{f}_i^- \rangle = \langle \hat{q}_i^- \rangle b_d, \quad (4.31)$$

$$\langle \hat{m}_i^+ \rangle = \langle \hat{q}_i^+ \rangle w_u, \quad \langle \hat{f}_i^+ \rangle = \langle \hat{q}_i^+ \rangle b_u. \quad (4.32)$$

Equations in the first row model the momentum and buoyancy entrained into the upflow, respectively, and the last two equations model those entrained into the downflow. Note that BK00 did not attempt to model the pressure effect on the momentum exchange. Consistent with (4.30), the characteristic buoyancy is defined as

$$b_u \equiv \frac{\langle \hat{B}_u \rangle \langle \hat{M}_u \rangle}{\langle \hat{Q}_u \rangle^2}, \quad b_d \equiv \frac{\langle \hat{B}_d \rangle \langle \hat{M}_d \rangle}{\langle \hat{Q}_d \rangle^2}. \quad (4.33)$$

Figure 4.9 compares the observed volume exchange and buoyancy exchange with the predictions of the entrainment parametrisations using the parameter values $\alpha^I = 0.16$, $\beta = 0.80$ and $\gamma = 0.19$. (We choose α^I and its corresponding parametrisation simply due to the coefficient α^I being more constant with height than α^{II} over most fountain heights. However, we believe that α^{II} and parametrisation II will not make substantial difference.) Here, the subscript M denotes the predicted fluxes. Note that the predictions for the quantities of the upflow start above $z/L_F = 0.66$ since the model requires characteristic values of the downflow in order to calculate predictions of the upflow — we attain reliable data for the downflow only above $z/L_F = 0.66$. Figures 4.9(a,b) show that the predicted volume exchanges are in agreement with the simulation for both boundaries, which shows that the revised coefficients capture the volume entrainment and detrainment of the fountain well. Figure 4.9(c) shows that the model tends to underpredict the negative buoyancy entrained into the upflow, and overpredict the negative buoyancy detrained out of the upflow. This is because the characteristic buoyancy cannot represent the local transport buoyancy at the boundary appropriately — considering the distance from the centreline, the characteristic buoyancy of the upflow, b_u , is more negative than the local boundary value, which is, further, more negative than the characteristic buoyancy of the downflow, b_d . The net exchange of negative buoyancy tends to be overpredicted by the model.

The prediction of the momentum exchanges is more challenging, as mentioned before and argued in the following discussion. First, a substantial part of the momentum exchange at the inner boundary is determined by $m_{V,i}$. Since the inner boundary was defined using a threshold $0.07w_0$, $m_{V,i}$ is given simply by $0.07w_0\langle\hat{q}_i\rangle$. This relation is shown in figure 4.10(a) and reassuringly shows excellent agreement between the parametrisation and the simulation data. However, this threshold was chosen by trial and error, and did not emerge from any fundamental insight or theoretical considerations; naturally, as mentioned, one would — for the momentum exchange only — desire to select a zero velocity threshold, in which case only the pressure term $\langle\hat{m}_{p,i}\rangle$ would contribute to the momentum exchange. Therefore, the net momentum exchange predicted by BK00, (4.31), is compared with the pressure term $\langle\hat{m}_{p,i}\rangle$ in figure 4.10(b).

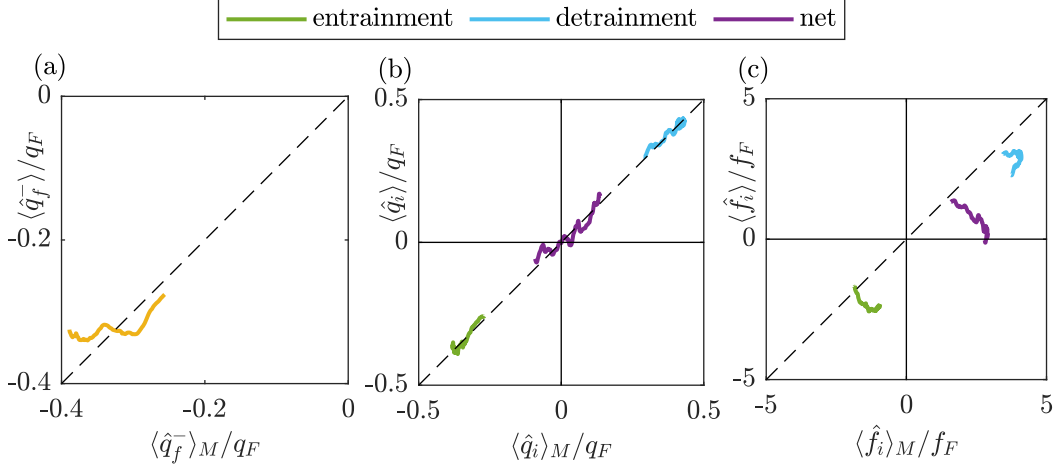


Figure 4.9: The modelled exchanges compared with the simulation exchanges for: (a) volume flux across the outer boundary; (b) volume flux across the inner boundary; and (c) buoyancy exchange across the inner boundary. The data presented here are for $0.66 \leq z/L_F \leq 1.62$, and the parameter values used are $\alpha^I = 0.16$, $\beta = 0.80$ and $\gamma = 0.19$. The diagonal dashed lines represent perfect agreement.

Although we are aware that these two plotted quantities are induced by different mechanisms (note that the modelled term $\langle \hat{m}_i \rangle_M$ is associated with the entrainment velocity), their comparison is still worthy since they both represent the momentum exchange across the inner boundary. The figure shows that the parametrisation grossly overpredicts the simulation data. However, we note that this does not mean that $\langle \hat{m}_{p,i} \rangle$ is negligible — figure 4.6 shows that this quantity is of magnitude similar to the buoyancy term $\langle \hat{B} \rangle$. Our results therefore suggest that the BK00 parametrisation is not appropriate for predicting the momentum exchange.

4.4.4 Normalised entrainment density

By rearranging (4.19) to be $\int_0^\infty 2\bar{e}_l r / \langle \hat{l} \rangle dr = 1$, the quantities $2\bar{e}_l r / \langle \hat{l} \rangle$ evidence how $\langle \hat{l} \rangle$ is distributed over the radial direction r , i.e. $2\bar{e}_l r / \langle \hat{l} \rangle$ represents a normalised exchange length density. Figure 4.11 shows the radial profiles of the normalised exchange length density at the two boundaries conditioned by entrainment and detrainment events, for both the inner and outer boundaries. Data are shown at the two vertical planes depicted in figure 4.3, i.e. $z/L_F = 1.18$ and 1.62 ; vertical dash-dotted lines mark the local

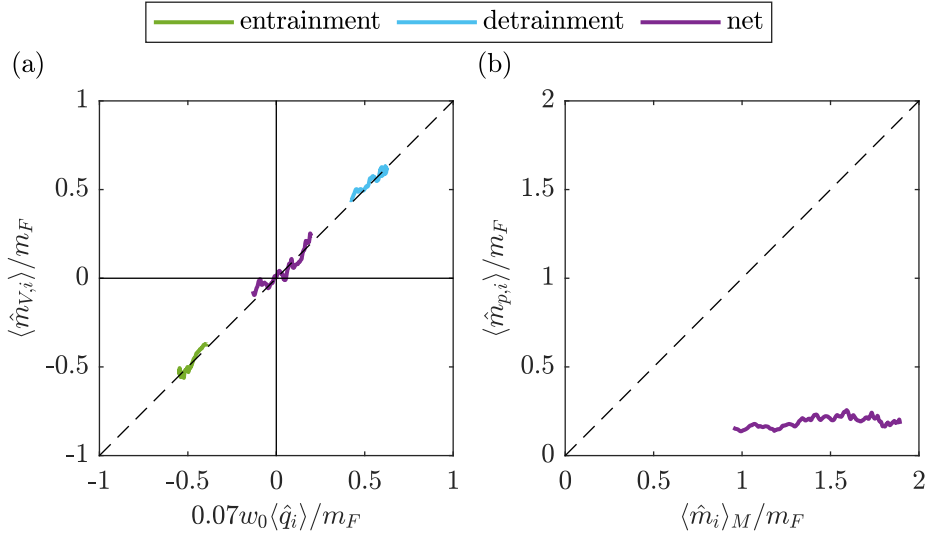


Figure 4.10: (a) The product of the volume exchange $\langle \hat{q}_i \rangle$ (including the conditional components) and the constant threshold $0.07w_0$ compared with the measured momentum exchange $\langle \hat{m}_{V,i} \rangle$. (b) The modelled net momentum exchange compared with the measured net momentum exchange $\langle \hat{m}_{p,i} \rangle$. The diagonal dashed lines represent perfect agreement.

time-averaged characteristic radii $\langle \hat{r}_f \rangle$ and $\langle \hat{r}_i \rangle$. Figures 4.11(a,b) show the normalised exchange length associated with the outer boundary $2\bar{e}_{l,f}r/\langle \hat{l}_f \rangle$ at both heights. Here, the first subscript l represents the exchange length, and the second subscript f indicates that it is associated with the outer boundary; similarly hereinafter. Consistent with figure 4.4(a), there is mainly entrainment occurring at the fountain outer boundary, thus $\bar{e}_{l,f} \approx \bar{e}_{l,f}^-$ and $\langle \hat{l}_f \rangle \approx \langle \hat{l}_f^- \rangle$. The location of the time-averaged characteristic radius $\langle \hat{r}_f \rangle$ is close to the location of the maximum value of $2\bar{e}_{l,f}r/|\langle \hat{l}_f \rangle|$ at both heights. The profiles can be approximated well by a Gaussian profile (black dashed line)

$$\frac{2\bar{e}r}{|\langle \hat{l} \rangle|} \approx \text{sgn}(\langle \hat{l} \rangle) \frac{1}{\sqrt{2\pi}\sigma_g} \exp\left(-\frac{(r-r_g)^2}{2\sigma_g^2}\right), \quad (4.34)$$

of which the distribution parameters $r_g/\langle \hat{r}_f \rangle$ and $\sigma_g/\langle \hat{r}_f \rangle$ are listed within the figure. Note that $\langle \hat{r}_f \rangle/L_F = 0.60, 0.69$ at $z/L_F = 1.62, 1.18$, respectively.

Figures 4.11(c,d) show the normalised exchange length associated with the inner boundary at both heights. Like the outer boundary, the normalised exchange length

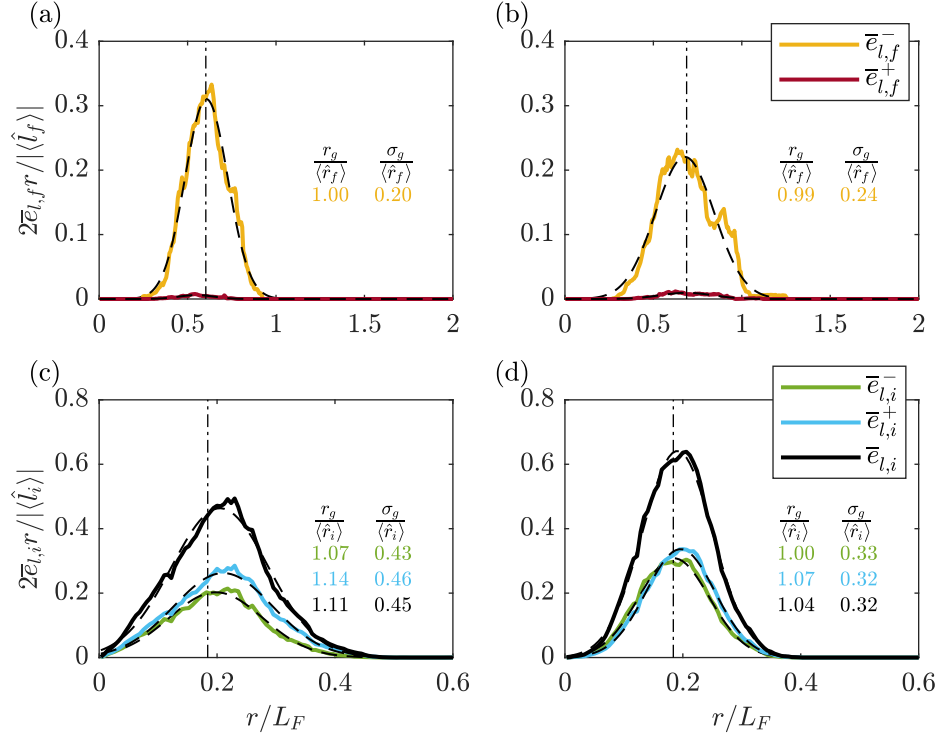


Figure 4.11: Normalised exchange length density $2\bar{e}_{l,f}r/|\langle\hat{l}_f\rangle|$ as a function of r associated with the outer boundary, plotted at (a) $z/L_F = 1.62$ and (b) $z/L_F = 1.18$. Normalised exchange length density $2\bar{e}_{l,i}r/|\langle\hat{l}_i\rangle|$ associated with the inner boundary at (c) $z/L_F = 1.62$ and (d) $z/L_F = 1.18$. Vertical dash-dotted lines mark the local time-averaged characteristic radius of the outer boundary $\langle\hat{r}_f\rangle$ and inner boundary $\langle\hat{r}_i\rangle$, correspondingly. Gaussian profiles fitted to the data are marked as dashed curves, whose distribution parameters are listed in the figure in the corresponding colour.

density $2\bar{e}_{l,i}r/|\langle\hat{l}_i\rangle|$ at the inner boundary is also well fitted by a Gaussian profile. The exchange length density peaks near the location of the time-averaged characteristic radius $\langle\hat{r}_i\rangle$ at both heights. When segregating to the conditional exchange length densities, the entrainment length density $2\bar{e}_{l,i}^-r/|\langle\hat{l}_i\rangle|$ and the detrainment length density $2\bar{e}_{l,i}^+r/|\langle\hat{l}_i\rangle|$ are also both well approximated by Gaussian profiles, i.e., the entire exchange perimeter, which is Gaussian-like, results from the superposition of two Gaussian-like profiles each associated with entrainment or detrainment. Note that the normalisation uses the total length $|\langle\hat{l}_i\rangle|$ to compare directly the entrainment and detrainment densities, implying that (4.34) needs to be multiplied by an enhancement $|\langle\hat{l}_i^-\rangle|/|\langle\hat{l}_i\rangle|$ or $|\langle\hat{l}_i^+\rangle|/|\langle\hat{l}_i\rangle|$ for the entrainment and detrainment expressions, respectively. Again, the distribution parameters for the Gaussian profile, here normalised by the averaged radius of the inner boundary, $r_g/\langle\hat{r}_i\rangle$ and $\sigma_g/\langle\hat{r}_i\rangle$, are listed in the figure. We find the radius $\langle\hat{r}_i\rangle/L_F = 0.19, 0.17$ at $z/L_F = 1.62, 1.18$, respectively. The parameters confirm that the total and conditional profiles have practically identical r_g and σ_g , so they can remain in the Gaussian profile while in the superposition.

Analogous to the length, rearranging (4.19) and (4.20) in terms of q, m and f , respectively, we obtain the normalised exchange density of volume $2\bar{e}_qr/|\langle\hat{q}\rangle|$, momentum $2\bar{e}_mr/|\langle\hat{m}\rangle|$, and buoyancy $2\bar{e}_fr/|\langle\hat{f}\rangle|$, respectively, presented in figure 4.12. Figures 4.12(a,b) show the volume exchange density associated with the outer boundary $2\bar{e}_{q,f}r/|\langle\hat{q}_f\rangle|$ at both heights. The absolute value of $\langle\hat{q}_f\rangle$ is used in order to keep the signs of the exchange density consistent with the rest of the chapter: negative represents entrainment, while positive represents detrainment, as labelled in the figure. Consistent with figure 4.5(d), there is only volume entrainment at the fountain boundary, thus $\bar{e}_{q,f} \approx \bar{e}_{q,f}^-$. Again, the profiles can be approximated well by a Gaussian profile with the distribution parameters $r_g/\langle\hat{r}_f\rangle$ and $\sigma_g/\langle\hat{r}_f\rangle$. The location of the time-averaged characteristic radius $\langle\hat{r}_f\rangle$ is close to the location of the maximum value of $2\bar{e}_{q,f}r/|\langle\hat{q}_{q,f}\rangle|$ at both heights, as reassured by the ratio $r_g/\langle\hat{r}_f\rangle$ being unity. As figure 4.5(e),(f) have shown, the momentum and buoyancy exchanges at the outer boundary are insignificant, therefore the momentum exchange density and buoyancy exchange density at the outer boundary are not presented here.

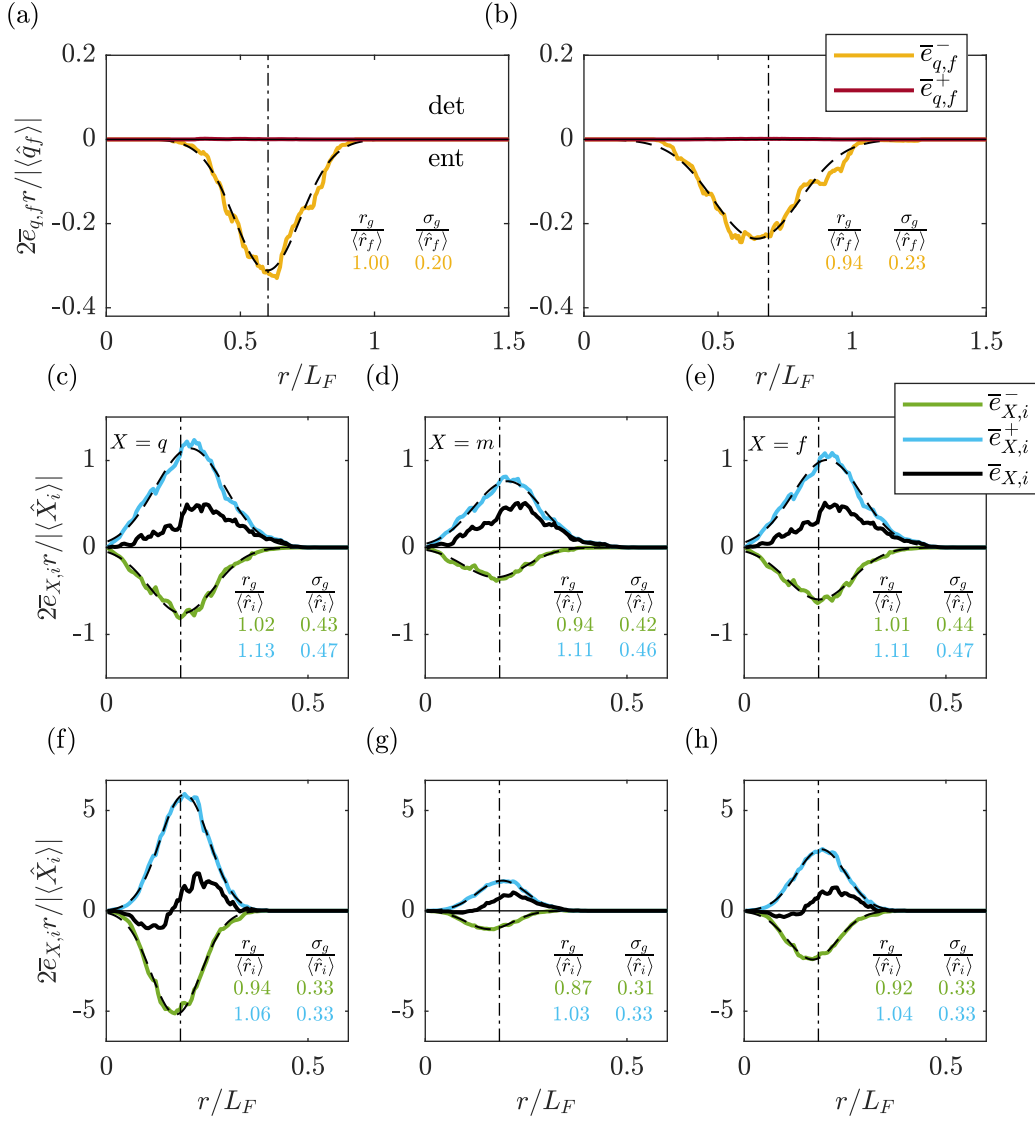


Figure 4.12: Normalised volume exchange density $2\bar{e}_{q,f}r/|\langle\hat{q}_f\rangle|$ as a function of r associated with the outer boundary, plotted at (a) $z/L_F = 1.62$ and (b) $z/L_F = 1.18$. Normalised (c, f) volume exchange density $2\bar{e}_{q,i}r/|\langle\hat{q}_i\rangle|$, (d, g) momentum exchange density $2\bar{e}_{m,i}r/|\langle\hat{m}_i\rangle|$ and (e, h) buoyancy exchange density $2\bar{e}_{f,i}r/|\langle\hat{f}_i\rangle|$ associated with the inner boundary, at (c–e) $z/L_F = 1.62$ and (f–h) $z/L_F = 1.18$. Vertical dash-dotted lines mark the local time-averaged characteristic radius of the outer boundary $\langle\hat{r}_f\rangle$ and inner boundary $\langle\hat{r}_i\rangle$, correspondingly. Gaussian profiles fitted to the data are marked as dashed curves. As labelled in (a), the area above the horizontal axis belongs to detrainment, while that below belongs to entrainment for all the exchanges.

Figures 4.12(c–h) show the radial variation of volume exchange density, momentum exchange density and buoyancy exchange density associated with the inner boundary, respectively, at both heights. Here again, due to the normalisation, each profile has been multiplied by a corresponding enhancement, e.g., $|\langle \hat{q}_i^- \rangle|/|\langle \hat{q}_i \rangle|$. It is striking that all the conditional entrainment and detrainment densities of volume, momentum and buoyancy (coloured lines), at both heights, are well approximated by the Gaussian profiles (distribution parameters $r_g/\langle \hat{r}_f \rangle$ and $\sigma_g/\langle \hat{r}_f \rangle$ are marked within the figure). However, all the net normalised exchange densities, which are the superposition of the entrainment and detrainment events, are clearly non-Gaussian-like (solid black lines). The net profile shape depends strongly on z — for $z/L_f = 1.18$ (figures 4.12f,h), there is remarkable entrainment of volume and buoyancy for $r \lesssim \langle \hat{r}_i \rangle$ and detrainment further away from the fountain centreline. However, at the higher level, $z/L_f = 1.62$ (figures 4.12c,e), insignificant net entrainment of volume and buoyancy is observed. Since for each case, the distribution parameters $\sigma_g/\langle \hat{r} \rangle$ in the conditional entrainment and detrainment profiles are approximately identical, the non-Gaussian-like profile of net exchange is due to the distribution parameter $r_g/\langle \hat{r}_i \rangle$ in the two conditional profiles being different (i.e., the difference is generally greater than 0.10). Here, the fact that for all the exchanges, the entrainment and detrainment exchange densities have a larger amplitude than the net exchange density confirms once more that the conditional entrainment and detrainment terms are large compared with the net (consistent with figures 4.5(d–f)).

The second feature noted for all kinds of exchanges, i.e., of volume, momentum and buoyancy, is that $r_g/\langle \hat{r}_i \rangle$ of conditional profiles at both heights are all close to 1, with $r_g/\langle \hat{r}_i \rangle$ of the entrainment profile always smaller than that of the detrainment. This shows that the maximum value of the entrainment (and detrainment) density is close to the location of the time-averaged characteristic radius $\langle \hat{r}_i \rangle$. However, the conditional entrainment density peaks closer to the fountain centreline than the detrainment, which shows the propensity for net entrainment closer to the fountain centreline, and detrainment further away. This is possible due to the anisotropy and inhomogeneity of the turbulence within the fountain, which are worthy of further study.

Moreover, we note that figures 4.12(d,g) show the normalised momentum exchange

densities at the inner boundary $2\bar{e}_{m,i}r/|\langle\hat{m}_i\rangle|$, including the entrainment and detrainment events. As illustrated before, the momentum exchange should be consistent with the volume exchange if only $\langle\hat{m}_V\rangle$ was considered. Hence the difference between the profiles of volume (figures 4.12c,f) and momentum (figures 4.12d,g) is driven by the pressure effect. The pressure does not affect entrainment and detrainment significantly on the Gaussian fit. However, the parameter $r_g/\langle\hat{r}_i\rangle$ indicates that the pressure makes the peaks of conditional momentum densities, especially the entrainment density, occur closer to the centreline compared with the volume. Additionally, the pressure also affects the amplitude enhancement significantly.

Recall that the net normalised exchange length densities (figure 4.11) are approximately Gaussian-like, while net normalised exchange densities of volume, momentum and buoyancy are not (figure 4.12), due to the relatively large difference between the distribution parameters $r_g/\langle\hat{r}\rangle$ of entrainment and detrainment events. This is suspected to be influenced by the averaged exchange velocity, momentum and buoyancy across the boundary that separates $r_g/\langle\hat{r}\rangle$ of entrainment and detrainment further. Therefore, figure 4.13 shows the radial distribution of the averaged entrainment velocity $\bar{e}_q/\bar{e}_l$, averaged exchanged vertical velocity $\bar{e}_m/\bar{e}_q$ and averaged exchanged buoyancy $\bar{e}_f/\bar{e}_q$ across the interface, segregated into entrainment and detrainment where relevant (i.e., over the intrusion level to the cap base $0.66 \leq z/L_F \leq 1.62$). Note that the radial axis represents the normalised distance to the time-averaged boundary because we have shown that r_g is always close to the time-averaged boundary $\langle\hat{r}_i\rangle$ or $\langle\hat{r}_f\rangle$ in figure 4.11 and 4.12. Over the range of two standard deviations, $-2.0 \leq (r - r_g)/\sigma_g \leq 2.0$, which contains approximate 95% total entrainment or detrainment, the radial profiles of different heights (which are indicated by transparency) collapse well to a single profile, indicating that the averaged transport properties depend solely on the radial location rather than the height — such as figure 4.13(a), which shows that the entrainment velocity associated with the outer boundary gets lower as it gets further away from the fountain centre, regardless of the height. Figure 4.13(b) shows that the entrainment and detrainment velocities intersect at $(r - r_g)/\sigma_g \approx -1.0$, and it is clear that further away from the centreline, over $1.0 \lesssim (r - r_g)/\sigma_g \leq 2.0$, the averaged detrainment velocity is greater than

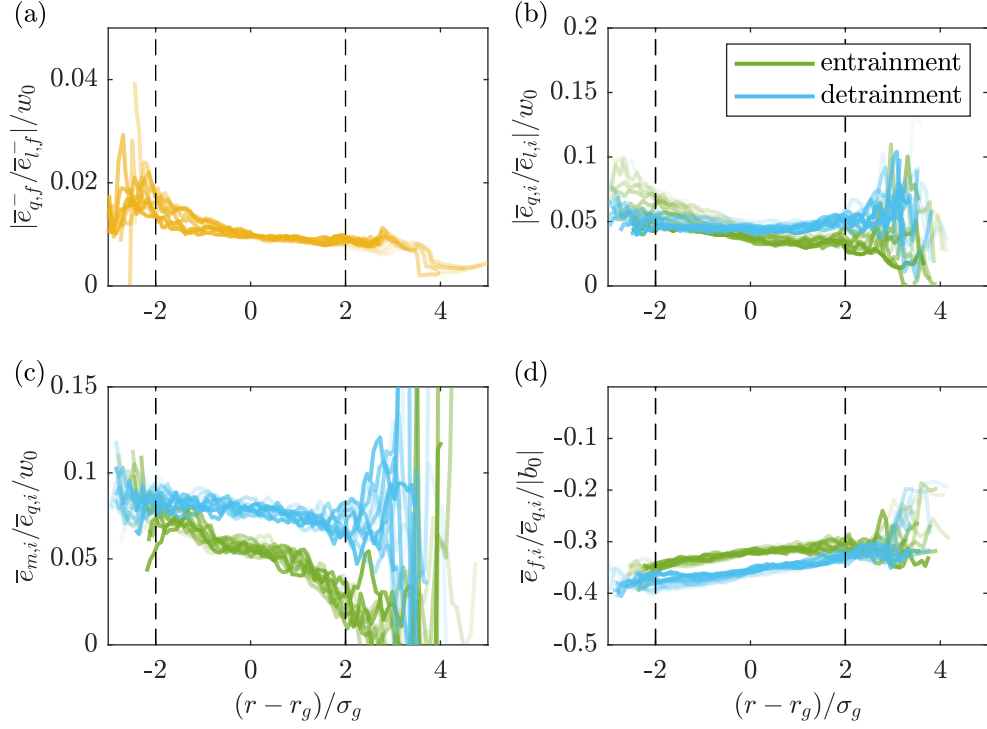


Figure 4.13: The radial profiles of (a) the averaged entrainment velocity at the outer boundary, (b) the inner boundary, the averaged entrainment and detrainment velocity, (c) the averaged entrained and detrained vertical momentum, and (d) the averaged entrained and detrained buoyancy. The velocities in (a,b) are shown in absolute values. The vertical axis is normalised with the source values w_0 or $|b_0|$, and the radial axis is rescaled by the corresponding distribution parameters in the Gaussian profile, associated with the distance from the location of maximum density r_g . The vertical dashed lines mark the locations of two standard deviations of the density distribution. The lines represent the data from different heights between the intrusion level near the bottom and the cap base, the darker colour representing the higher level.

the averaged entrainment velocity. Note that the velocities across both inner boundary and outer boundary seem not to vary substantially with the radius, indicating that the movement of the interface causes the Gaussian profile for the volume entrainment and detrainment rather than the distribution of the velocities. Figures 4.13(c,d) show the magnitude of transport properties, including vertical velocity and buoyancy. The detrained buoyancy is more negative than the entrained buoyancy, which is consistent with the results from figure 4.7. Beyond this region, $-2.0 \leq (r - r_g)/\sigma_g \leq 2.0$, the data scatter due to the insufficient samples there.

4.5 Conclusion

Data from a direct numerical simulation of a forced fountain were used to examine the instantaneous local exchanges of volume, momentum and buoyancy at the inner boundary (TTI) and the outer boundary (TNTI), conditioning the data on entrainment and detrainment events. Consistent with expectation and previous fountain studies (Bloomfield & Kerr, 2000; Shrinivas & Hunt, 2014) and our Chapter 3, the evidence shows that at the outer boundary, there is only entrainment into the fountain. At the inner boundary, the volume entrainment and the volume detrainment co-exist. The statistics show further that the magnitude of detrainment is similar to entrainment, being smaller than entrainment at lower heights, and larger than entrainment at higher heights — crucially, though, the net volume exchange is typically much smaller than both entrainment and detrainment. This is also observed in the exchanges of momentum and buoyancy. At TTIs such as the inner boundary of the fountain, the entrainment and detrainment fluxes transfer different fluid properties across the interface in different directions; examining only the net exchanges risks missing these potentially important transfers. The methodology presented herein provides insight into these processes, and we hope that it will be used to improve the understanding and modelling of the exchanges across TTIs and TNTIs in fountains and other canonical turbulent flows.

The BK00 entrainment parametrisations for forced fountains were tested based on the conditional exchanges quantified herein; thereby allowing, for the first time, entrain-

ment and detrainment coefficients to be determined explicitly from data. The parameter values obtained were not too dissimilar to those used previously, despite being based on conditional statistics rather than Reynolds-averaged statistics. The BK00 parametrisation was unable to capture the momentum exchanges at the inner boundary due to them not modelling the contribution from the pressure.

It is interesting to compare the results of this study — which uses conditional statistics (CS) — with Chapter 3, which examined the same forced fountain data but using Reynolds-averaged statistics (RAS). Both approaches identified the inner and outer boundaries of the fountain using consistent velocity and buoyancy thresholds, respectively. The RAS boundaries are marginally wider than the CS boundaries due to intermittency effects. The net volume and buoyancy exchanges agree well between CS and RAS at both boundaries. In both approaches, the momentum exchange is poorly represented by the BK00 parametrisation. For RAS, the BK00 model is unable to capture the turbulent exchange of momentum across the inner layer. For CS, the BK00 model is unable to capture the momentum exchange due to pressure. In terms of the entrainment coefficients, both RAS and CS had similar values for the entrainment coefficient of the outer boundary. However, the coefficients at the inner boundary are much more complex due to the co-existence of entrainment and detrainment. The net entrainment coefficient obtained from the net volume entrainment with RAS varies in sign with height, and cannot therefore, be used to present the momentum and buoyancy exchange. An advantage of CS is that it becomes possible to segregate the net exchanges into entrainment and detrainment events, and it further allows for the calculation of the entrainment and detrainment coefficients of the volume exchange individually. Although the values differed somewhat from the BK00 values, the coefficients from the conditional study were almost independent of the height, at least within a portion of the fountain height. Moreover, the coefficients well predicted the volume exchanges and broadly predicted the buoyancy exchanges. Therefore, the BK00 model, although derived using a Reynolds-averaged description, seems to better model forced fountains when applied to conditionally averaged statistics.

The radial profiles of the net exchange densities of volume, momentum and buoy-

ancy associated with the inner boundary are not Gaussian-like and show a propensity for net entrainment closer to the fountain centreline and net detrainment further away. However, examination of the conditional entrainment and detrainment exchange rates shows that these conditional profiles are in fact Gaussian-like, with entrainment having its maximum closer to the fountain centreline than detrainment. As the standard deviations are nearly indistinguishable, this offset fully explains the non-Gaussian shape of the radial profiles of the net exchanges. Across both interfaces, entrainment and detrainment events are most likely to occur at locations close to the local characteristic radius of the interface. This finding might challenge the notion that large-scale engulfment events at large radii are a significant source of entrainment (Burridge *et al.*, 2017), but a further examination of the peculiarities of particular canonical flows is warranted before any universal conclusions can be drawn.

Chapter 5

Entrainment and detrainment in shallow cumulus clouds

5.1 Introduction

The entrainment and detrainment in shallow cumulus clouds are investigated in this chapter. As discussed in Chapters 1 and 2, shallow cumulus clouds have a similar structure as forced fountains — an upward-flowing core (referred to as updraft) shrouded by a counterflowing downflow (referred to as subsiding shell), but are much more complex due to the additional humidity transport, phase change of water and the stratified environment.

Entrainment is very important to cloud formation since it influences the weather and climate as it dilutes cloud properties and controls the cloud's life. Unlike at the fountain outer boundary where the environmental air is entrained into the fountain, at the cloud edge entrainment and detrainment co-exist locally (De Rooy *et al.*, 2013) due to evaporation which produces evaporative cooling resulting in negative buoyancy and even negative velocity. However, a fundamental understanding of the entrainment is insufficient and parametrization of entrainment in cumulus clouds is still an active topic of research. Most cloud studies focus on a cloud ensemble with large eddy simulation (Siebesma *et al.*, 2003; Roode *et al.*, 2012), which is not accurate for studying the

details of an individual cloud (e.g. subsiding shell). Moreover, current studies on an individual cloud focus on a certain height at the cloud edge (Abma *et al.*, 2013; Nair *et al.*, 2020) which is not able to incorporate the full vertical evolution of entrainment and cloud dynamics. Lastly, cloud studies usually take the sub-cloud layer into account, which determines the initial conditions of the clouds.

In order to investigate the entrainment between an individual cloud and its environment, and also the entrainment between the updraft and subsiding shell within the cloud, we will create a digital cloud chamber using DNS. These controlled conditions will allow use of the diagnostics used in the previous chapters for the studies on local and global entrainment. Similar to the fountain cubic domain, we will have a maintained source at the centre of the chamber bottom which is regarded as the cloud base. The source injects not only warm but moist fluid (i.e., to form a ‘moist fountain’) which are determined by prognostic liquid water potential temperature θ_l and total water specific humidity q_t , respectively. Additionally, setting the source (cloud base) at the domain bottom avoids the dynamics in the sub-cloud layer and permits focus solely on the cloud layer, above the level of free convection (LFC). However, this requires real atmospheric data of the source conditions that already represent the consequence from the sub-cloud layer. The atmosphere will need to be stratified to avoid the cloud rising up to the tropopause like a cumulonimbus cloud. Detailed simulation conditions are introduced in the §5.2. Eventually, the governing equations (2.6) — (2.9) are solved in the DNS with the buoyancy calculation involving a bulk condensation scheme. Results are discussed in §5.3 and 5.4 and conclusions are drawn in §5.5.

5.2 Simulation details

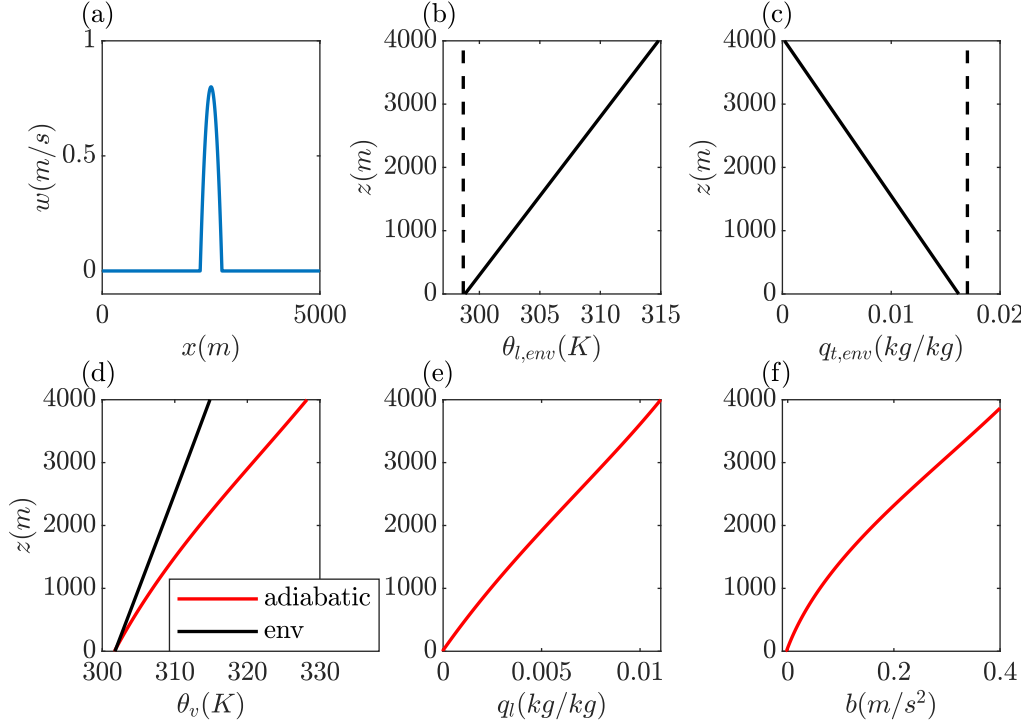


Figure 5.1: (a) The initial parabolic profile of vertical velocity w at the source. (b, c) The solid lines represent the linear initial profile of environmental liquid water potential temperature $\theta_{l,env}$ and environmental total specific humidity $q_{t,env}$, respectively. The vertical dashed lines represent the states of the cloud parcel injected from the source under the adiabatic process. (d, e, f) The red lines show the virtual potential temperature θ_v , liquid water specific humidity q_l and buoyancy b of the cloud parcel from the source under the adiabatic process, respectively. The black line in (d) shows the initial profile of the environmental virtual potential temperature $\theta_{v,env}$.

The simulation is carried out at full scale. The atmospheric environment and parameters are inspired by the Barbados Oceanographic and Meteorological Experiment (BOMEX; Holland & Rasmusson, 1973), the Rain in Shallow Cumulus Clouds (RICO; Rauber *et al.*, 2007) field campaign and LES simulations (Siebesma *et al.*, 2003). As the atmospheric data vary with time, region and measurement strategy, the data from each reference have some reasonable differences giving us some freedom in specifying initial profiles. Note that in the BOMEX case, there is a thermal inversion layer at the top of the domain where the atmosphere is absolutely stable to ensure all the clouds

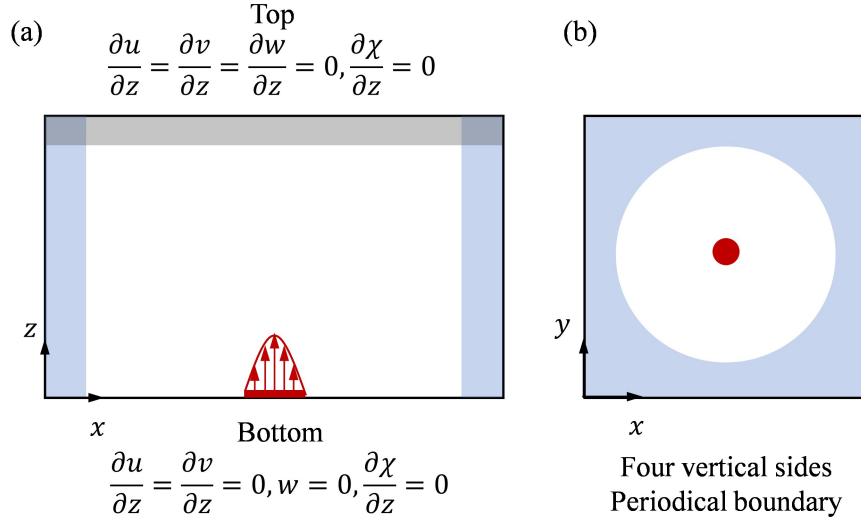


Figure 5.2: A central vertical view (a) and a plane view (b) of the simulation domain with the boundary conditions labelled. The light blue shade is the vertical nudging that forces the liquid water potential temperature θ_l and total specific humidity q_t there to the initial environmental values. The horizontal grey shade is the nudging area that forces the vertical velocity w there to zero. The red represents the location of the source.

are colder than the environment and stop rising. This leads to a piecewise function of environmental value comprised of two linear profiles. However, this is not always the case — the RICO campaign measured that a portion of the active clouds have already stopped rising due to evaporative cooling before reaching the inversion layer. For the case considered here, we want to minimise the number of lengthscales in the problem, and will therefore aim to use linear profiles of liquid water potential temperature (without an inversion layer) and the environmental total water specific humidity. This will require some tuning of the source strength to ensure the cloud stays within the simulation domain. Inspired by the RICO case, the linear profiles of the environmental liquid water potential temperature $\theta_{l,e}$ (K) and the environmental total water specific humidity q_t (kg/kg) are set as

$$\theta_{l,e} = 298.81 + 0.0040z, \quad q_{t,e} = 0.0162 - 4.0 \times 10^{-6}z, \quad (5.1)$$

where z (m) is the height from the cloud base. These profiles are shown in figure 5.1(b,c).

The source is assumed circular and the inflow conditions are estimated from RICO at the cloud base, given as (marked as the dashed lines in figure 5.1(b,c))

$$\theta_{l,0} = 298.667\text{K}, \quad q_{t,0} = 0.0170\text{kg/kg}. \quad (5.2)$$

The fluid is assumed to be injected vertically from the source with a parabolic vertical velocity as a consequence of the sub-cloud layer dynamics (shown as figure 5.1(a))

$$w_0(r) = w_m \left(1 - \left(\frac{r}{r_0} \right)^2 \right), \quad (5.3)$$

where $w_m = 0.8\text{m/s}$ is the centreline velocity, r is the distance from the centreline in the source area, and r_0 is the radius of the source. The width of a shallow cumulus cloud is normally less than 1000m (Rodts *et al.*, 2003) but varies in a wide range. We set the height of the chamber to be 4000m (above where $q_{t,e}$ reduces to negative which is impossible), and together with the other parameters, we set $r_0 = 250\text{m}$ which creates a relatively active and high cumulus cloud that covers about half of the domain height in statistically steady state (and will thus not be contaminated by interactions with the top of the domain). Therefore, we will use a domain size $L_x \times L_y \times L_z = 5000\text{m} \times 5000\text{m} \times 4000\text{m}$ with the source located in the centre at $(L_x/2, L_y/2, 0)$. The horizontal size is determined according to the periodic boundary conditions of side walls which will be illustrated later.

Figure 5.1(d) shows the virtual potential temperature θ_v of the fluid parcel rising from the source adiabatically. The adiabatical process of a moist air parcel has been introduced in §2.2. We set the cloud buoyancy to zero at the source in order to represent the active cloud from the level of free convection (LFC) under the adiabatic condition, and the figure shows the initial parameters control this well. However, because of the mixing of the cloud with its environment and the subsequent cooling due to the evaporation of cloud water, in the simulation, the buoyancy is slightly negative close to the cloud base which we cannot explicitly predict beforehand. Therefore, the cloud base (where q_l of the air begins to be larger than zero) does not exactly lie on the domain

bottom but is slightly higher across a negative buoyant zone. However, based on test simulations with slightly different source conditions around, this negative buoyant zone does not influence the dynamics of the cloud (e.g., the cloud height) in a significant way. Figures 5.1(e,f) show the q_l and buoyancy b of the fluid parcel under the adiabatic process, which both monotonically increase under the condensation.

Figure 5.2 shows the domain, the boundary conditions and the nudging regions. Following (5.2) and (5.3), a Dirichlet boundary condition $w(r) = w_0(r)$, $\theta_l = \theta_{l,0}$, $q_t = q_{t,0}$ is applied at the source to ensure the uniform source value. Considering that the fluid from the sub-cloud layer is highly turbulent, we initiate the turbulence by applying an uncorrelated perturbation of 20% to the velocities in the first cell above the source. The boundary condition at the domain top has a Neumann condition on velocity, potential liquid water temperature and total water specific humidity, and at the domain bottom, the boundary condition is ‘free slip’ with a Neumann condition on the temperature and humidity (except at the source). The four sidewalls of the domain are periodic to allow the fluid to leave from one side and re-enter at the opposite side. We ensure that the domain $L_x \times L_y \times L_z = 5000\text{m} \times 5000\text{m} \times 4000\text{m}$ is sufficiently large that the dynamics of the cloud are unaffected by any transport at the domain edges. In order to avoid spurious reflection at the top of the domain that will affect the cloud, a nudging area of vertical velocity is set within the 200m from the top (shown as the grey area in figure 5.2) to force the vertical velocity to zero, i.e., over the range $z = 3800 - 4000\text{m}$. Moreover, to avoid any accumulation of temperature and humidity within the domain affecting our results, a full-height vertical nudging area of θ_l and q_t is set within the region $r \geq 8r_0$, where the fluid temperature and humidity are gradually adjusted by a ‘nudging’ process towards the initial local environmental value with a time scale of 360s. This is shown as the light blue area in figure 5.2.

The incompressible Navier-Stokes equations under the Boussinesq approximation (2.6) – (2.9) are solved numerically using DNS on a uniform Cartesian grid of $N_x \times N_y \times N_z = 2048^3$ cells. The code employed is SPARKLE, which has been introduced in §2.3. A sketch of an instantaneous 3-D cloud in the domain is shown in the figure 5.3.

The simulation was run for a duration of 7200s in order to allow sufficient time

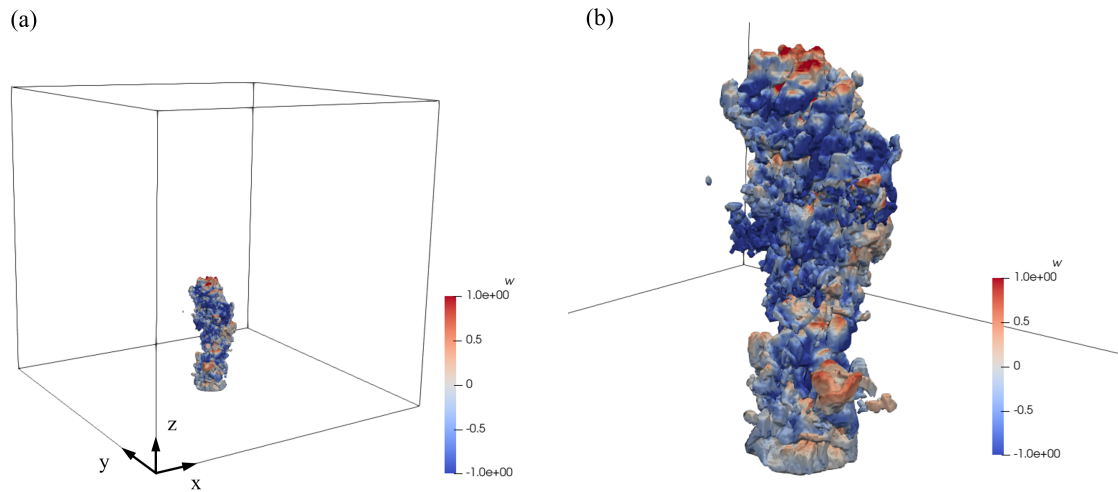


Figure 5.3: The whole simulation domain with an illustration of an instantaneous cloud depicted by plotting an isosurface of liquid water specific humidity at $q_l = 10^{-12} \text{ kg/kg}$, coloured according to the local vertical velocity, where (b) is a magnified view of the cloud.

for the cloud to reach a statistically steady state. Instantaneous statistics were obtained at a vertical plane at the centre of the domain with every 60s. Azimuthally Reynolds averaged data was computed and stored once the cloud is in a statistically steady state by partitioning the domain into concentric cylindrical shells and averaging over all cells lying within a given shell (Craske & Van Reeuwijk, 2015; Van Reeuwijk *et al.*, 2016).

5.3 Cloud observations and internal structure

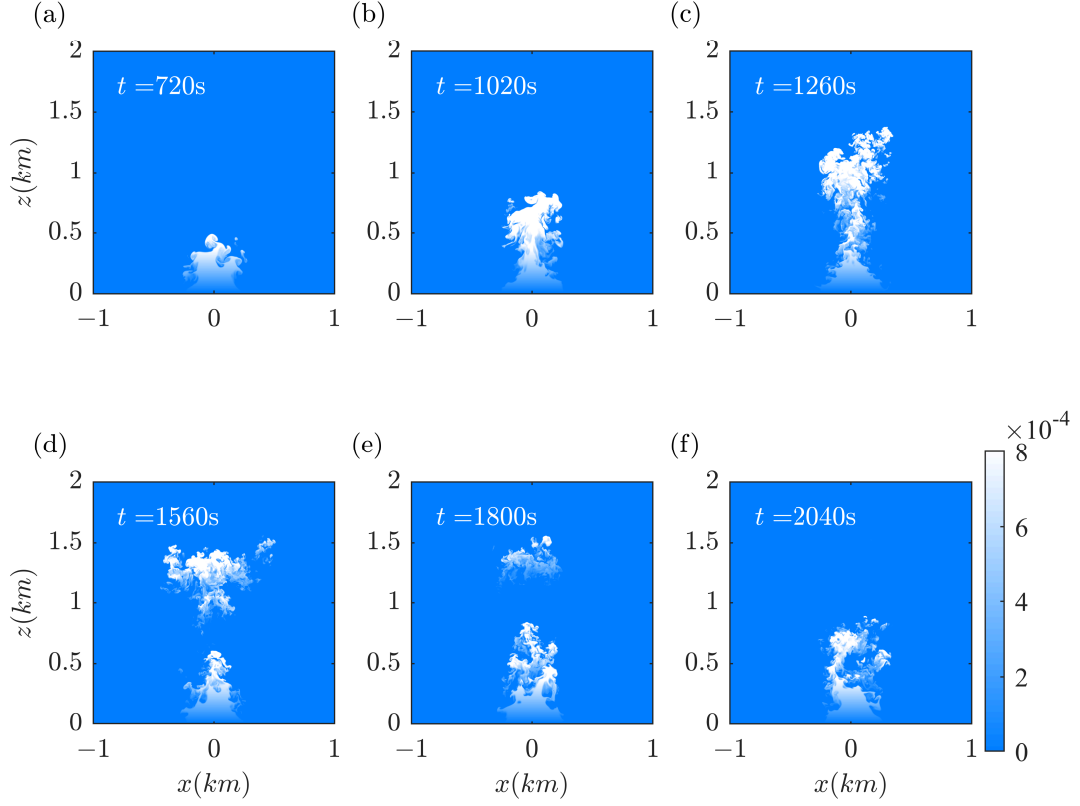


Figure 5.4: A series of screenshots of the instantaneous cumulus cloud, shown as the liquid water specific humidity q_l field. The figures are taken over the time period $t = 720s - 2040s$.

Figure 5.4 shows several snapshots of the instantaneous liquid water specific humidity q_l field, which can represent the instantaneous visible cloud. This figure shows that at around $t = 720s$, a cloud pulse is about to form from the base under the initial momentum and the latent heat from condensation. After that until around $t = 1260s$, this cloud pulse rises under the positive buoyancy with the ‘bright’ front of the cloud showing a huge amount of liquid water at the cloud head. However, in the middle of the cloud, e.g., over $z = 0.25 - 0.75km$ in figure 5.4(c), there seems to be less liquid water. This might be because more liquid water evaporates at the edge than condenses over this range, resulting in a less cloudy air supply underneath. Therefore, at $t = 1560s$, the cloud top detaches from the base when the cloudy air in the middle totally evaporates,

around this time, the cloud is about to reach its maximum height as the evaporative cooling and the subsequent strong negative buoyancy involved begin to dominate the motion of the cloudy air. At $t = 1800$ s, the cloud top is dissipating, and until $t = 2040$ s, it shows that the current cloud pulse is totally dissipated while the next pulse is about to rise from the base, again. This quasi-periodic process occurs continuously in the simulation even though the fluid is injected from the source at a constant rate.

It is worth noting the ‘invisible’ effects between the previous cloud and the coming cloud: Not shown here but seen in the velocity field, there is a downflow induced by the dissipated cloud under the negative buoyancy, which will prevent the coming cloud from rising. However, the previous cloud moistened the environment which makes the coming cloud less likely to evaporate, this, on the other side, promotes the coming cloud to rise further. We note that in a statistically steady state, the detachment of the cloud top is no more apparent than it was at the beginning, showing that the moistening effect of the previous clouds indeed tends to retain the cloud in the environment. However, the maximum cloud height does not change significantly, suggesting that the evaporation at the top is strong enough to dominate the cloud height.

Figures 5.5(a,b) show an instantaneous cross-section of the buoyancy field and vertical velocity field, respectively, laid with the outline marking the cloud edge. In the environment, internal waves are observed within these cross-sections as inclined stripes, which are induced by the turbulence generated within the cloud. The cloud boundary is defined using $q_l = 1 \times 10^{-12}$ kg/kg. Figure 5.5(a) shows that for $z < 1000$ m, as they have not mixed with enough environmental air on their way up, most cloud fluids are positive buoyant because of the latent heat released. Nevertheless, we notice that there is a low portion of negative buoyancy within the cloud in this range, for which the evaporative cooling from the mixing process is already sufficient to reverse their buoyancy. However, above this height, i.e., $z > 1000$ m, the positive buoyancy and negative buoyancy are almost half-half within the cloud, which indicates that as the fluids rise, they mix with more environmental air, so that the evaporation gets strong, producing more evaporative cooling which dominates the buoyancy variation over the latent heat. Therefore, a larger negative buoyant area is observed. As expected, the negative-

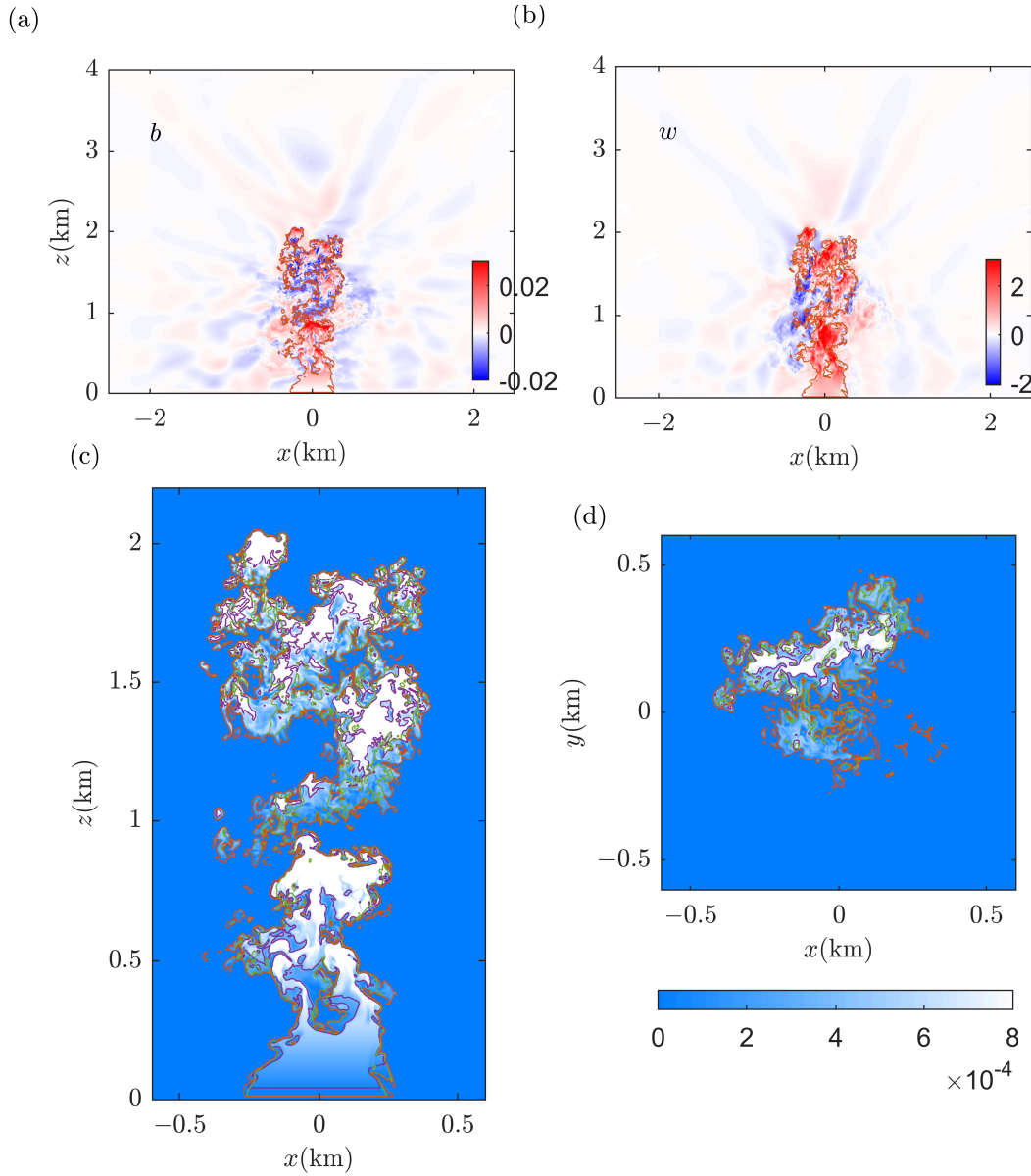


Figure 5.5: The instantaneous buoyancy b field (a) and vertical velocity field w (b), overlaid with the cloud boundary which is defined by an isosurface of liquid water specific humidity q_l . The central vertical plane (c) and a horizontal plane (d) of q_l field, overlaid with the cloud boundary (red), cloud core boundary (purple), and cloud updraft boundary (green).

buoyant area above $z = 1000\text{m}$ is not only larger but also more negative than that below $z = 1000\text{m}$. Generally, the negative-buoyant area (blue) is below the positive-buoyant area (red) and they are not well mixed, this may indicate the order of the evaporation, as shown in the figure 5.4, for a cloud pattern, the front is moister and more positively buoyant so that is the hardest or the latest to become negatively buoyant. Note that turbulence is observed closely outside the cloud, which shows that the cloud dynamics also influence no-liquid air nearby. Some of this no-liquid air may previously belong to the cloud, but at this moment, it has totally evaporated.

Figure 5.5(b) shows the vertical velocity distribution at the same time. Here, within the cloud, there is a great portion of the positive-velocity area, especially below $z < 1000\text{m}$ where the negative buoyancy is not able to reverse the upward velocity. Above $z > 1000\text{m}$, there is a small area of negative velocity (which is regarded as the subsiding shell) near the edge of the cloud where the negative buoyancy has reversed the velocity. We note that there is a strong falling flow on the left outside the cloud, over $z = 500\text{m} - 1500\text{m}$. This evidences the ‘invisible’ previous cloud fluids — these fluids have totally evaporated, and the evaporative cooling has already reversed their motion resulting in this downward flow, however, as they do have liquid water, this motion is ‘invisible’.

Figures 5.5(c, d) show the liquid water specific humidity q_l , the visible cloud, at a central vertical plane and a horizontal plane at height $z = 1000\text{m}$, respectively. The coloured lines outline the different cloud regions: the entire cloud (red lines), consistent with the cloud boundary, where $q_l > 1 \times 10^{-12}\text{kg/kg}$. It almost encloses all the air that contains liquid water; the cloud core, where the positive buoyant area within the cloud, i.e., $q_l > 1 \times 10^{-12}\text{kg/kg}, b > 1 \times 10^{-12}\text{m/s}^2$; the cloud updraft, where the positive-vertical-velocity area within the cloud, i.e., $q_l > 1 \times 10^{-12}\text{kg/kg}, w > 1 \times 10^{-12}\text{m/s}$. The boundaries show that the cloud core is generally enclosed within the cloud updraft and the latter is enclosed within the entire cloud. This is consistent with the observation from the buoyancy and velocity field in figures 5.5(a, b), that the rising fluid becomes negatively buoyant first and gradually decelerates to fall. The area between the cloud core and cloud updraft indicates the fluid that already becomes negatively buoyant under the evaporation but has not decelerated sufficiently to fall. This negatively buoyant

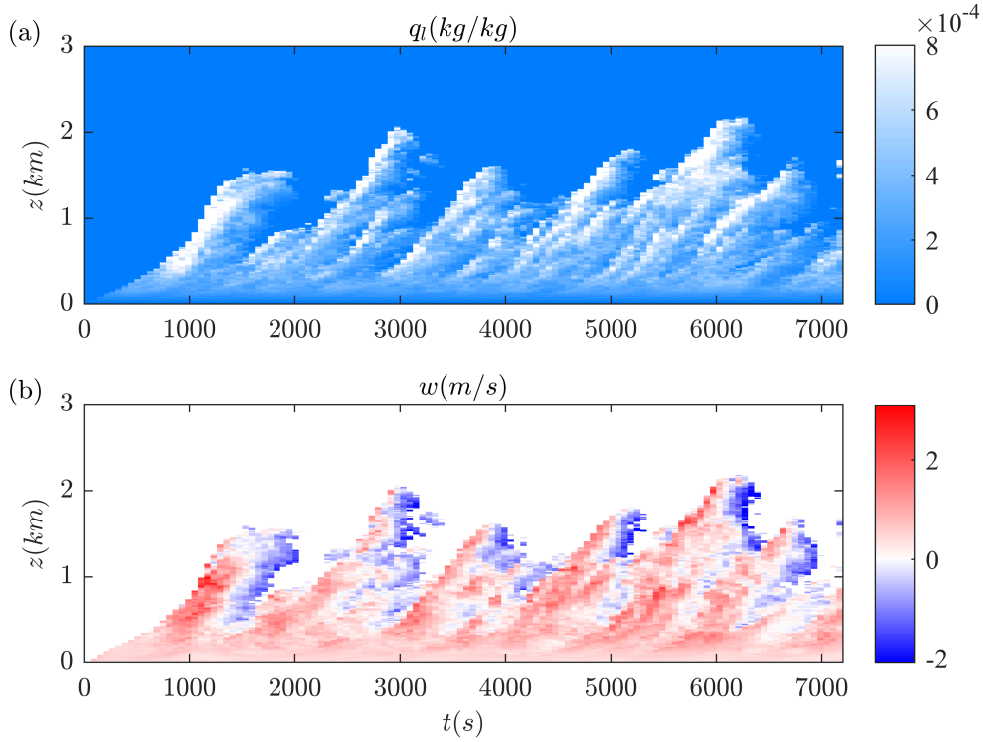


Figure 5.6: The time series of the cloud-averaged q_l and w over the horizontal plane.

region is usually called the shell and the negatively vertical-velocity region is called the subsiding shell. Note that, figure 5.5(c) near the cloud base, the base of the cloud core boundary is slightly higher than the updraft and the entire cloud, which is due to the initial condition of the source and the environment — the buoyancy of the fluid from the source is slightly negatively buoyant as explained before.

Figure 5.6(a) shows the time series of the cloud averaged q_l . A series of cloud pulses can be observed (shown as inclined lines), which evidences the cloud cycle from developing to dissipation. The gradient of the inclined lines indicates the rising velocity of the cloudy air. At a time instance, for example, $t = 1800s$, the blue area between the pulses (at around $z = 800m - 1000m$) indicates the dissipation of the cloudy air, this is consistent with the observation from the screenshot (see the detachment of the cloud top from the base, figure 5.4e). In the early stage, from $t = 0 - 3000s$, there are apparently blue gaps between the pulses which indicates the great dissipation (evaporation) in the middle of the cloud, however, over $3000s - 7000s$, there are fewer blue gaps between

the pulses, indicating that the cloud is easier to form and retain. This is due to the previous process of the cloud, recall the discussion on the two anti-effects between the two neighbour pulses, in this period the two effects seem to reach a balance so that the gaps between the pulses become stable. Therefore, the period over 3000s – 7000s is regarded as the statistically steady state for further statistics.

Figure 5.6(b) shows the cloud averaged vertical velocity w where the pulses also can be observed. The negative velocity area evidences the existence of the subsiding shell. In terms of a single pulse, the negative area generally appears after the positive area in time, showing the process that the rising cloud eventually becomes downwards under its evaporation. Note that, during the statistically steady state, i.e., $t = 3000 - 7000s$, the large negative velocity is more concentrated near the cloud top, because the huge evaporative cooling dominates the dynamics over the heat from condensation. Moreover, at a time instance, we see the negative velocity (blue area) and positive velocity (red area) attach at some heights, indicating the falling fluid meets the rising fluid there, which reassures the interaction between the two neighbour pulses.

The Reynolds-averaged data is obtained over the statistically steady state. Figure 5.7(a) shows the Reynolds-averaged ‘visible’ cloud by plotting the $\overline{q_l}(r, z)$ field as the cloud is the saturated portion of the atmosphere with liquid water. Consistent with the identification of the instantaneous cloud, the Reynolds-averaged cloud edge r_{cld} is also identified at the loci where $\overline{q_l} = 1 \times 10^{-12} \text{kg/kg}$, i.e., the Reynolds-averaged cloud is the field where $\overline{q_l} \geq 1 \times 10^{-12} \text{kg/kg}$, which is enclosed by the orange line in the figure, indicating the Reynolds-averaged cloud height is approximately 2750m. However, note the magnitude of Reynolds-averaged liquid water specific humidity is generally lower than that of the instantaneous cloud (for example, shown in (figure 5.5c, note that the Reynolds-averaged cloud has a smaller colour bar), and the large humidity mainly concentrates near the centerline, ranging over $r \approx 0 - 200\text{m}$, $z \approx 200\text{m} - 1500\text{m}$.

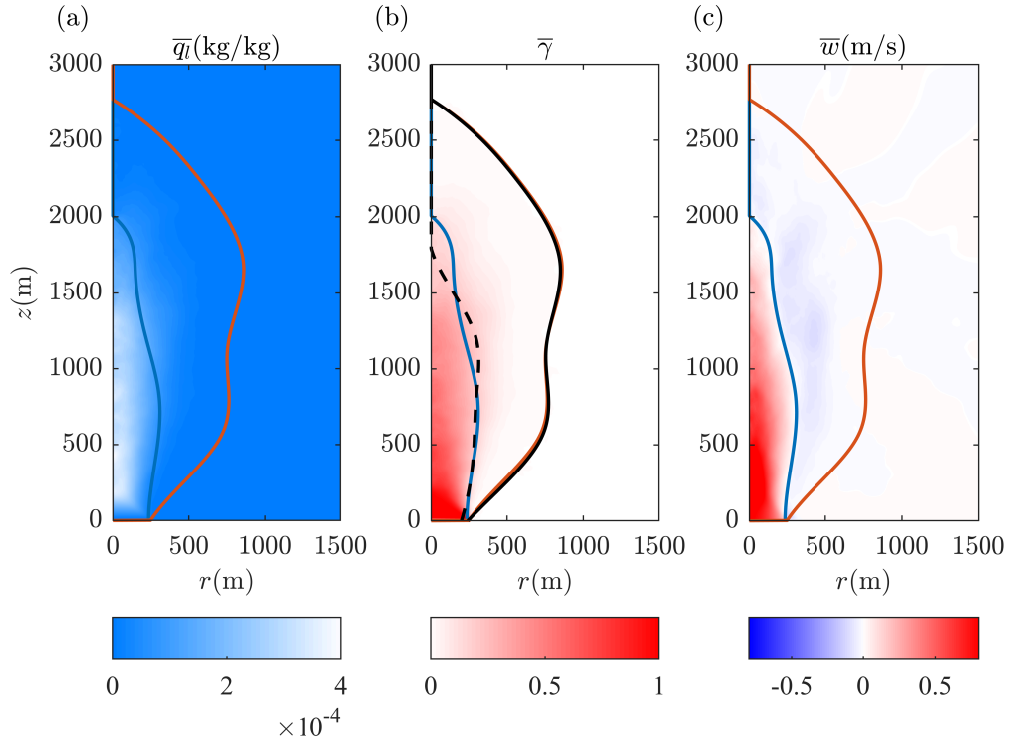


Figure 5.7: The Reynolds-averaged field of liquid water specific humidity $\bar{q}_l$ (a), intermittency $\bar{\gamma}$ (b) and vertical velocity $\bar{w}$ (c), overlaid with the cloud boundary (red line) and the cloud updraft boundary (blue line). The solid line and dashed line in (b) show the loci where $\bar{\gamma} = 1 \times 10^{-4}$ and $\bar{\gamma} = 0.2$, respectively.

These are because of the intermittency of the Reynolds-averaged statistic. As already shown in figure 5.6, the cloud comprises a series of intermittent pulses, so that, for a specific location in the field, it is sometimes cloudy but sometimes not. For example, the screenshot (figure 5.4) shows that at the Reynolds-averaged cloud top height, it is not always cloudy compared with the near-source field. Therefore, when taking the Reynolds average (time average), the magnitude of the local property is ‘diluted’, this effect is even significant for where is rarely cloudy. However, oppositely, near the low centerline, as it remains cloudy for relatively a long time, there is greater Reynolds-averaged $\bar{q}_l$. Meanwhile, the intermittency results in a larger diameter of the Reynolds-averaged cloud compared with the width of the instantaneous cloud, as the definition of Reynolds-averaged cloud ($\bar{q}_l \geq 1 \times 10^{-12} \text{kg/kg}$) contains nearly all the loci that have ever been cloudy, while the instantaneous cloud ($q_l \geq 1 \times 10^{-12} \text{kg/kg}$) only contains

the cloudy loci at a particular instance, although the threshold value is the same.

Therefore, we count a cloudy index $\gamma(\mathbf{x}, t) = H(q_l \geq 1 \times 10^{-12})$, where H is the Heaviside function, then show the intermittency by plotting the Reynolds average of the index $\bar{\gamma}(r, z)$ (see figure 5.7b), which represents the ratio of the cloudy time to the total time. It shows generally, the index γ is larger when it is close to the cloud centerline — near the source, there is almost always cloudy (i.e., the air contains liquid water), while near the cloud edge and top, it is rarely cloudy. The isosurfaces of $\bar{\gamma} = 1 \times 10^{-4}$ and 0.2 are labelled with the black solid line and dashed line, respectively, which enclose the region that is cloudy over 99.9999% and 80% of the total time in the statistically steady state, respectively. Surprisingly, the surface of $\bar{\gamma} = 1 \times 10^{-4}$ perfectly matches the cloud boundary, which confirms that the cloud edge contains almost all the places that have ever been cloudy. The area between $\bar{\gamma} = 1 \times 10^{-4}$ and $\bar{\gamma} = 0.2$ represents the region that is cloudy no more than 20% of the total period. This significant area (note that the area is the square of the radius) evidences that the intermittency indeed results in an ‘inflated’ Reynolds-averaged cloud by including the places that are rarely cloudy. In other words, the Reynolds-averaged cloud that is enclosed by the boundary we defined, in fact, counts the air parcel that is not cloudy, which is the key difference from the conditionally sampled cloud.

Figure 5.7(c) plots the Reynolds-averaged vertical velocity field $\bar{w}$, which evidences that there is a subsiding shell surrounding the cloud updraft. The updraft boundary r_{up} , labelled as the blue line, is the interface separating the updraft and subsiding shell. Therefore, by definition, it marks all the loci that $\bar{w} = 0$ within the cloud, i.e., the updraft encloses the places that $\bar{w} \geq 0$. Note that, the updraft boundary is very close to the interface $\bar{\gamma} = 0.2$ (see figure 5.7b) — in fact, this is the reason for choosing $\bar{\gamma} = 0.2$. It shows that the updraft contains liquid water more often than the subsiding shell, for the latter, is non-cloudy over almost 80% of the total time. The figure also shows that the area of the subsiding shell is much larger than the updraft but the magnitude of the velocity is much smaller.

Unlike the instantaneous cloud, in the Reynolds-averaged cloud, we do not define the cloud core region for the following reasons. First, we find that the boundary of

$\bar{w} = 0$, and $\bar{b} = 0$ (which is usually used to define the boundary of the core) almost coincide at most heights, which leads to an insignificant difference between core and updraft. Second, as the buoyancy in the simulation, and therefore, in all the results throughout, is calculated referring to the initial environmental values, at some cloud heights, the environment is slightly buoyant and the subsiding shell can be positively buoyant, implying the threshold $\bar{b} = 0$ does not work. Therefore, the cloud updraft is appropriate and sufficient to represent the internal structure of the Reynolds-averaged cloud.

As explained above, how the buoyancy is calculated, i.e., the choice of the environmental value $\theta_{v,e}$ in (2.10), would affect the results — in the present DNS, the $\theta_{v,e}(z, 0)$ is the initial value, while in the LES, the $\theta_{v,e}(z, t)$ is time-dependent, this is different is due to the different simulation settings. However, the overall effect of buoyancy and pressure is independent of the different choices, we explain and derive this in Appendix §B. The term $\bar{b} - d\bar{p}/dz$ objectively shows the total external force on the cloud regardless of the environmental value. Figure 5.8 plots the field of the total external force $\bar{F}_{ex} = \bar{b} - d\bar{p}/dz$. It shows that there is no external force out of the cloud, which confirms that, the reference effect on buoyancy and that on the pressure cancels, i.e., the cloud dynamics are not affected by this reference. The field shows that, within the cloud, below $z \approx 1100\text{m}$, there is mostly a positive region, where the force raises the cloud parcel, while above that height, there is mostly a negative region, where the force prevents the rising. It is interesting to note that, in the subsiding shell, there is a portion that the overall force of buoyancy and pressure is positive (e.g., below $z \approx 1100\text{m}$), which will prevent the cloud from falling. This is because, as illustrated in the intermittency, the Reynolds-averaged cloud accounts for the dynamics of the air parcels after they are no longer saturated.

Next, we examine the dynamics of the fluid in the cloud with the streamlines of the mean velocity and the force field. As streamlines show, the fluid from the source rises in the updraft lifted by the latent heat from the condensation, except very close to the source edge (note that, the two streamlines near $r = 250\text{m}$), the fluid immediately mixes with the environment, cools and reverses into the downflow then intrudes horizontally

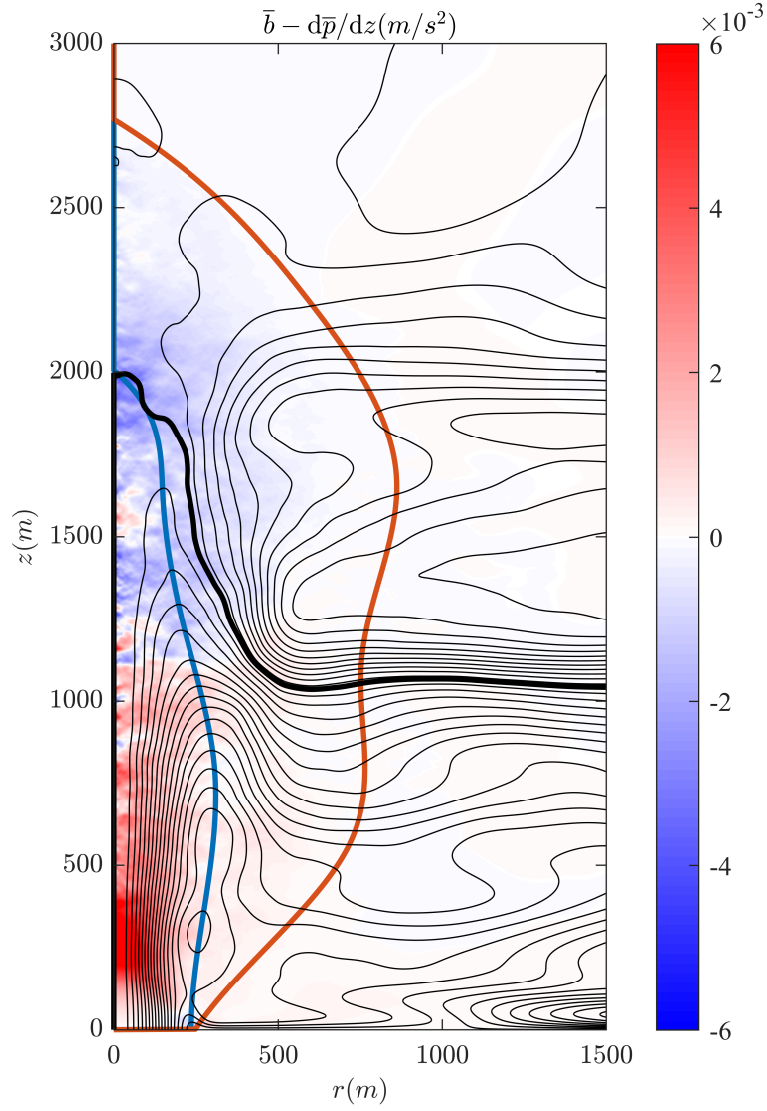


Figure 5.8: The Reynolds-averaged field (unit m/s^2) of the total external force $\bar{b} - d\bar{p}/dz$ which represents an overall effect by buoyancy and pressure, overlaid with cloud boundary (red), updraft boundary (blue) and the streamlines. The thick streamline is the one that is from the source centre.

along the bottom. For the fluid going upwards, it not only condenses but also evaporates. When it reaches the maximum height, where the evaporative cooling has reversed the buoyancy, the fluid begins to fall. Therefore, we see that the streamlines from the source bend after reaching maximum heights.

During the fall, as continuously mixes and evaporates, the fluid becomes less saturated and finally unsaturated (the re-condensation is not considered here, in fact, it can be observed in the simulation but is not significant). After being unsaturated, the evaporation stops and the fluid parcel will fall to a level where the environmental virtual potential temperature equals its virtual potential temperature, i.e., the neutral-buoyant level (NBL) of the parcel. The parcel overshoots the NBL due to the non-zero momentum but receives a restoring force below the NBL as it becomes positively buoyant again. This restoring force, therefore, raises the parcel back to the NBL, but again, due to the momentum, the parcel over-shoots the NBL and then receives a downward restoring force towards the NBL. Eventually, the parcel oscillates around the NBL and gradually stops at the NBL. Note that, the height of the NBL varies during the oscillation but not significantly. In the figure, this is shown as the horizontal intrusion (e.g., in an area $r > 500\text{m}$, $z \approx 800 - 1200\text{m}$) after the streamlines bend solely under the horizontal pressure gradient. The background external force field, with an upper negative region and a bottom positive region, generally represents the time-averaged restoring force around the NBL. However, a small portion of the falling fluid is re-entrained into the updraft instead of into the horizontal intrusion, shown as the circular streamlines across the updraft boundary. We plot the streamlines showing the equidistant stream function so that the density of the streamlines is proportional to the magnitude of the entrainment and detrainment flux — at the updraft boundary, great detrainment is concentrated near the top of the updraft, while at the cloud boundary, the large entrainment occurs near $z \approx 2000\text{m}$ and the large detrainment occurs around $z \approx 1100\text{m}$. Note that, the existence of the subsiding shell below the horizontal intrusion level seems to support the view that the subsiding shell is formed due to the local evaporative cooling rather than the dynamic force from the cloud top.

The streamline from the source centre ($r = 0$) is a separatrix (labelled with a thick

line) where the fluid from the source can reach the highest by the mean flow. Above this separatrix, the streamlines come from the environment via the top of the cloud and leave the cloud via the middle of the cloud. This indicates that the environmental fluid is entrained into the subsiding shell near the top, and after falling some distance, it is then detrained into the environment. From the background force field, this fluid experiences a downward (negative) force when it is barely entrained into the cloud as it mixes with the cooler fluid inside the cloud, but eventually stops to fall and horizontally intrude to leave the cloud under an upward (positive) force. Note that, the streamlines from the source (below the thick separatrix) do not intersect with the streamlines from the environment (above the thick separatrix), suggesting that the mean flow cannot exchange the properties between these two parts of fluid, i.e., properties are only possible to exchange by the turbulence across the separatrix.

5.4 Entrainment and detrainment

We first present the volume (mass) conservation equations of the cloud regions in Reynolds-averaged form and conditional averaged form. The Reynolds-averaged conservation equations (per unit π) of the volume flux of the updraft and the subsiding shell can be obtained by integrating the mass conservation (2.6) over the updraft region (over $r = 0$ to $r = r_{\text{up}}$) and the subsiding shell region (over $r = r_{\text{up}}$ to $r = r_{\text{cld}}$), respectively, which are

$$\frac{dQ_{\text{up}}}{dz} = -q_{\text{up}}, \quad \frac{dQ_{\text{shell}}}{dz} = q_{\text{up}} - q_{\text{cld}}, \quad (5.4)$$

where Q_{up} and Q_{shell} are the Reynolds-averaged integral vertical volume flux (per unit π) of the updraft and the subsiding shell, respectively, given as,

$$Q_{\text{up}} \equiv 2 \int_0^{r_{\text{up}}} \bar{w} r dr, \quad Q_{\text{shell}} \equiv 2 \int_{r_{\text{up}}}^{r_{\text{cld}}} \bar{w} r dr, \quad (5.5)$$

and q_{up} and q_{cld} are the perpendicular exchange of volume across the updraft boundary and the cloud boundary, respectively, which are termed ‘entrainment’ or ‘detrainment’,

namely

$$q_{\text{up}} = 2r \left(\bar{u} - \bar{w} \frac{dr}{dz} \right) \Big|_{r_{\text{up}}}, \quad q_{\text{cld}} = 2r \left(\bar{u} - \bar{w} \frac{dr}{dz} \right) \Big|_{r_{\text{cld}}}. \quad (5.6)$$

Note that, the positive exchange q_{up} or q_{cld} tends to decrease the volume flux Q_{up} or Q_{shell} which is specifically referred as to the ‘detrainment’, while the negative q_{up} or q_{cld} work oppositely, referred as to the ‘entrainment’. The subsiding shell entrains or detrains via both the updraft boundary and the cloud boundary, and apparently, Q_{shell} is negative as the subsiding shell develops opposite the z direction. The term associated with the radial change, i.e. dr/dz , is the ‘Leibniz term’. As the definition, the volume flux of the entire cloud, $Q_{\text{cld}} = Q_{\text{up}} + Q_{\text{shell}}$, is positive when volume flux goes upwards.

The integral area of the updraft A_{up} and subsiding shell A_{shell} are defined as

$$A_{\text{up}} \equiv 2 \int_0^{r_{\text{up}}} r dr, \quad A_{\text{shell}} \equiv 2 \int_{r_{\text{up}}}^{r_{\text{cld}}} r dr, \quad (5.7)$$

respectively, therefore, the area of the entire cloud is $A_{\text{cld}} = A_{\text{up}} + A_{\text{shell}}$.

Analogously, the conditional conservation equations (per unit π) of the volume flux of the updraft and subsiding shell can be obtained by integrating (2.6) over the instantaneous updraft region and subsiding shell region, respectively, which are given by

$$\frac{d\hat{A}_{\text{up}}}{dt} + \frac{d\hat{Q}_{\text{up}}}{dz} = -\hat{q}_{\text{up}}, \quad \frac{d\hat{A}_{\text{shell}}}{dt} + \frac{d\hat{Q}_{\text{shell}}}{dz} = \hat{q}_{\text{up}} - \hat{q}_{\text{cld}}, \quad (5.8)$$

recall that, to be consistent with Chapter 4, we use a top hat to note the instantaneous integral value — the instantaneous volume flux of the updraft $\hat{Q}_{\text{up}}$ and the subsiding shell $\hat{Q}_{\text{shell}}$, per unit π , give

$$\hat{Q}_{\text{up}} = \frac{1}{\pi} \int_{\Omega_{\text{up}}} w dA, \quad \hat{Q}_{\text{shell}} = \frac{1}{\pi} \int_{\Omega_{\text{shell}}} w dA, \quad (5.9)$$

respectively, where Ω_{up} and Ω_{shell} are the instantaneous sampled updraft (where $w \geq 0, q_l > 0$) and subsiding shell (where $w < 0, q_l > 0$), respectively. The term $\hat{q}_{\text{up}}$ and $\hat{q}_{\text{cld}}$ are the entrainment (like the Reynolds average, for negative values) or detrainment (for positive values) across the updraft boundary and the cloud boundary, respectively.

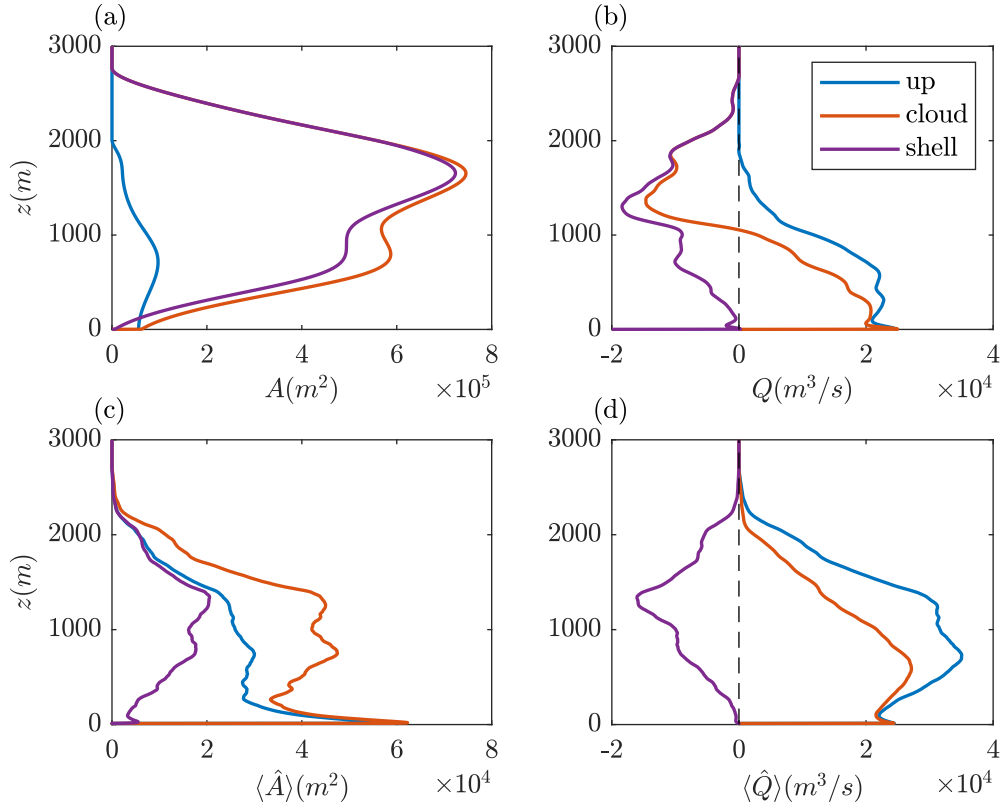


Figure 5.9: The area (a,c) and the vertical volume flux (b,d) of the updraft, subsiding shell and entire cloud, under the Reynolds-averaged integrals (a,b), and the conditional averaged integrals (c,d).

The instantaneous sampled area of the updraft and subsiding shell is

$$\hat{A}_{\text{up}} = \frac{1}{\pi} \int_{\Omega_{\text{up}}} dA, \quad \hat{A}_{\text{shell}} = \frac{1}{\pi} \int_{\Omega_{\text{shell}}} dA, \quad (5.10)$$

respectively, therefore, the total area of the entire cloud is $\hat{A}_{\text{cld}} = \hat{A}_{\text{up}} + \hat{A}_{\text{shell}}$.

Accounting for the whole statistically steady state, we take the time average of the instantaneous properties over the period, which is noted by a time-averaging operator, $\langle \cdot \rangle = \frac{1}{T} \int_0^T \cdot dt$, which is also consistent with the manner in Chapter 4. As the operator is linear, the above equations also hold for the time-averaged conditional properties.

Figure 5.9 shows the areas and volume flux of the updraft, subsiding shell and the entire cloud under both statistics, respectively. Figure 5.9(a) shows the Reynolds-averaged areas. The subsiding shell is dramatically wider than the updraft taking the

main part of the entire cloud, which is because of the intermittency as we observed from figure 5.7. The area of updraft grows below $z \approx 800\text{m}$ but decreases above that as sufficient evaporation inverses the direction of the flow. Examining from top to bottom, in line with the direction of the subsiding shell, the area of the subsiding shell area grows and reaches its maximum near $z \approx 1700\text{m}$, indicating that the liquid water is rich there so that the strong evaporation leads to a wide subsiding shell. Below this height, the subsiding shell becomes narrower as the liquid water is insufficient in the cloudy air — the shell evaporates to dissipation.

Figure 5.9(b) shows the Reynolds-averaged volume flux within the cloud regions. Although with a larger area, over the most updraft heights, i.e., below $z \approx 800\text{m}$, the updraft has greater volume flux (in magnitude) than the subsiding shell due to the larger rising velocity compared with the falling velocity within the shell (figure 5.7c). Note that, very near the source region below $z \approx 10\text{m}$, there is a slight decrease in Q_{up} which is due to the flow immediately reversing at the updraft edge, as we observed at the source edge in the streamline (figure 5.8). In general, from that height to $z \approx 600\text{m}$, Q_{up} increase, while above $z \approx 600\text{m}$ Q_{up} decreases, indicating that there is entrainment in the updraft below that height but detrainment above, consistent with the observation from the streamlines. However, the cloud top is dominated by the downward volume flux Q_{shell} as there is only the subsiding shell near the top (e.g., $z \gtrsim 1800\text{m}$). In line with the evolution of the area, there is an increasing falling volume flux Q_{shell} in the subsiding shell above $z \gtrsim 1300\text{m}$ but decreasing Q_{shell} below. As a result, the entire volume flux decreases from the cloud base to a negative value (representing a downward flux) until $z \gtrsim 1300\text{m}$, indicating a cloud detrainment, and increases to zero in $z \gtrsim 1300\text{m}$ under the cloud entrainment, again consistent with the streamlines.

Figure 5.9(c) shows the conditionally averaged area of the cloud regions. Note that, as expected, the areas sampled here are greatly smaller than the corresponding Reynolds-averaged ones (note that the x -axis is even shorter than figure 5.9a). Because the cloud is intermittent, the conditional statistics only sample the instantaneous cloudy area, while the Reynolds-averaged statistics enclose the area as long as it was cloudy in history. However, note that at the source where the cloud is almost not intermittent

($\gamma \approx 1$), the close area A_{up} and $\langle \hat{A}_{\text{up}} \rangle$ are very close. The figure shows that, below $z \approx 1300\text{m}$, the updraft is wider than the subsiding shell which is different from the result in the Reynolds average. However, above $z \approx 1300\text{m}$, the updraft and the subsiding shell nearly have the same area and vary with a similar gradient. The top of the updraft and the subsiding shell is almost at the same height — this is generally consistent with the observation from the instantaneous cloud (figure 5.5).

Figure 5.9(d) shows the conditionally averaged volume fluxes. Although the areas resulted from both statistics vary greatly, the volume fluxes from them are generally in the same order, especially the subsiding shell, indicating that the mean Reynolds-averaged vertical velocity is much smaller.

The volume fluxes within the updraft $\langle \hat{Q}_{\text{up}} \rangle$ grows with the height until $z \approx 700\text{m}$, suggesting the updraft entrains in that range while generally detrains above that. The volume flux of the entire cloud $\langle \hat{Q}_{\text{cld}} \rangle$ grows with the height until $z \approx 600\text{m}$, suggesting at the cloud boundary, the cloud entrains in that range while detrains above that.

In terms of the entrainment (and detrainment), similar to Chapter 3 and 4, we calculate the Reynolds-averaged entrainment at the updraft boundary and cloud boundary via the local values using (5.6), however, as already illustrated in Chapter 3, this Reynolds-averaged entrainment is the ‘net’ entrainment, but is not able to tell the co-exist entrainment and detrainment (Romps, 2010; De Rooy *et al.*, 2013) separately as it only takes one averaged volume exchange at the boundaries over the entire time period. Therefore, employing the algorithm introduced and applied in Chapter 4, the conditional statistics is able to achieve this, we conditionally sample the local entrainment and detrainment instantaneously and then take the time average, for example, at the cloud boundary, we have (Romps, 2010)

$$\hat{q}_{\text{cld}}^- = \frac{1}{\pi} \int_{\partial\Omega_{\text{cld}}^-} V_n dl, \quad \hat{q}_{\text{cld}}^+ = \frac{1}{\pi} \int_{\partial\Omega_{\text{cld}}^+} V_n dl, \quad (5.11)$$

where $\hat{q}_{\text{cld}}^-$ and $\hat{q}_{\text{cld}}^+$ are the instantaneous entrainment and detrainment at the cloud boundary, respectively. The boundary $\partial\Omega_{\text{cld}}^-$ and $\partial\Omega_{\text{cld}}^+$ are the entrainment area and detrainment area along the cloud boundary, respectively, which are defined by the direc-

tion of the entrainment velocity whose value is V_n (e.g., the entrainment area is where the entrainment velocity points into the cloud field). The value of V_n is defined via (4.10) using q_l as the scalar field. The cloud field is sampled at where $q_l > 1 \times 10^{-12} \text{kg/kg}$. Therefore, the net entrainment gives $\hat{q}_{\text{cld}} = \hat{q}_{\text{cld}}^- + \hat{q}_{\text{cld}}^+$, note that the resulting entrainment $\hat{q}_{\text{cld}}$ should also satisfy

$$\frac{d\hat{A}_{\text{cld}}}{dt} + \frac{d\hat{Q}_{\text{cld}}}{dz} = -\hat{q}_{\text{cld}}, \quad (5.12)$$

(according to (5.9)) due to the continuity. And applying the time average, the equations (5.9) become

$$\frac{d\langle\hat{Q}_{\text{up}}\rangle}{dz} = -\langle\hat{q}_{\text{up}}\rangle, \quad \frac{d\langle\hat{Q}_{\text{shell}}\rangle}{dz} = \langle\hat{q}_{\text{up}}\rangle - \langle\hat{q}_{\text{cld}}\rangle. \quad (5.13)$$

By definition, the conditional entrainment and detrainment across the updraft boundary can be obtained in a similar way, i.e., $\hat{q}_{\text{up}}$ can be split as something like $\hat{q}_{\text{up}}^-$ and $\hat{q}_{\text{up}}^+$. However, instantaneously sampling the updraft region is hard as the fluid belonging to the subsiding shell will be sampled if the nature criterion $w > 0$ is applied, due to the flow is turbulent. Therefore, we solely calculate the net conditional entrainment at the updraft boundary $\hat{q}_{\text{cld}}$ using (5.9).

Figure 5.10(a) shows Reynolds-averaged entrainment across the updraft boundary q_{up} and cloud boundary q_{cld} , respectively. These have been inferred by the streamlines (figure 5.8) but are now shown quantitatively. At the updraft boundary, except a few entrainment below $z \approx 600\text{m}$, there is almost always the detrainment from the updraft. At the cloud boundary, generally, below $z \approx 1300\text{m}$ q_{cld} is positive, the cloud detrainment occurs, while above $z \approx 1300\text{m}$, q_{cld} is negative, the cloud entrainment occurs. The maximum detrainment and entrainment occur near $z \approx 1100\text{m}$ and $z \approx 2000\text{m}$, respectively, consistent with the region where the density of streamlines is large (figure 5.8).

Figure 5.10(b) shows conditional averaged volume entrainment across the boundaries, $\langle\hat{q}_{\text{up}}\rangle$ and $\langle\hat{q}_{\text{cld}}\rangle$. At the updraft boundary, it shows that there is entrainment into the updraft below $z \approx 700\text{m}$ and detrainment out of the updraft above. This is generally similar to the Reynolds-averaged entrainment q_{up} , as the updraft is less intermittent.

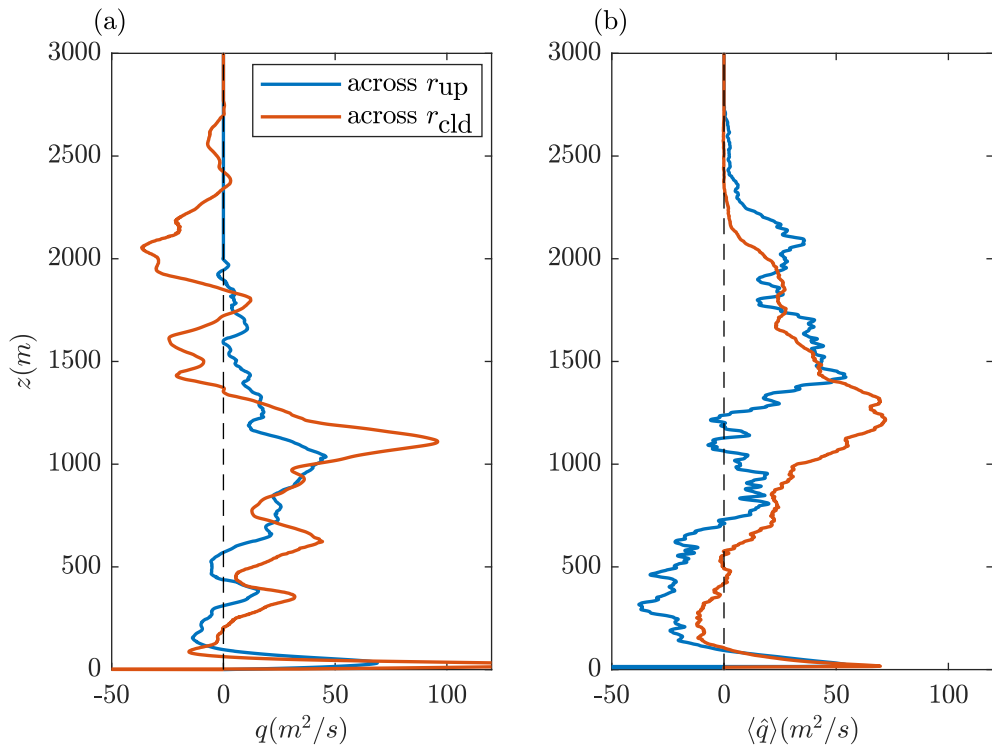


Figure 5.10: The entrainment across the updraft boundary and the cloud boundary, (a) the Reynolds-averaged entrainment, (b) the conditionally averaged entrainment.

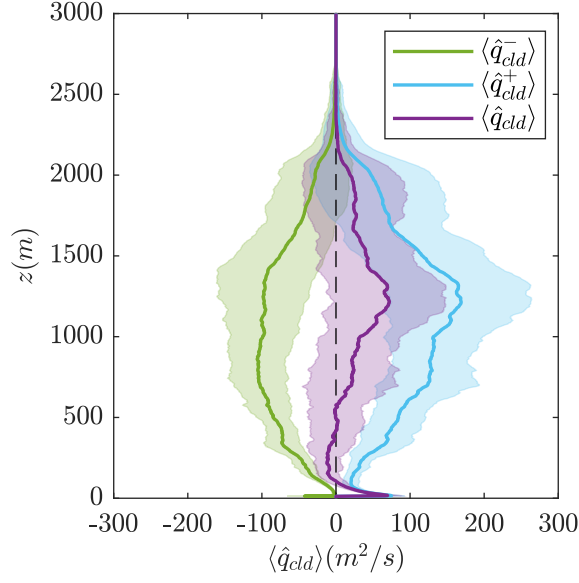


Figure 5.11: The conditionally time-averaged net entrainment $\langle \hat{q}_{cld} \rangle$, segregated into the entrainment $\langle \hat{q}_{cld}^- \rangle$ and detrainment $\langle \hat{q}_{cld}^+ \rangle$ events. The coloured band marks the first standard deviation interval of the time average.

However, the results at the cloud boundary are apparently different, even opposite, due to the great intermittency in the entire cloud. It shows that the cloud entrains at a lower level while detrains higher, with the turning point at around $z \approx 500\text{m}$. Above $z \approx 1500\text{m}$, the detrainment at the updraft boundary is generally greater than the detrainment of the cloud boundary, suggesting that the flow detrains from the updraft and then detrains into the environment, only a small portion of it falls into the subsiding shell near the top. Note that, although the boundary radii are dramatically different, again, the magnitude of the flux — Reynolds-averaged entrainment and the conditional entrainment are similar.

Figure 5.11 shows the entrainment and detrainment events underly the net entrainment $\langle \hat{q}_{cld} \rangle$, which evidences that the entrainment and detrainment co-exist at the cloud boundary and their magnitudes cannot be ignored in comparison with the net. The entrainment peaks at around $z \approx 900\text{m}$ while the detrainment peaks at around $z \approx 1300\text{m}$. The coloured band showing the first standard deviation of the time average suggests a wide time variation of the entrainment and detrainment with the time due to the inter-

mittency.

The parameterisation of the entrainment is different between the atmosphere community and the fluid community. In the fluid community, like Chapter 3 and 4, we usually use an entrainment coefficient to parameterise the entrainment which is dimensionless (Morton *et al.*, 1956). In the atmosphere community, a (fractional) entrainment rate ε and a (fractional) detrainment rate δ are used which represents the ratio of the volume entrainment and detrainment flux per unit height to the local vertical volume flux Q_{cld} , i.e.,

$$\varepsilon = \frac{E}{\langle \hat{Q}_{\text{cld}} \rangle}, \quad \delta = \frac{D}{\langle \hat{Q}_{\text{cld}} \rangle}, \quad (5.14)$$

where E and D are local entrainment and detrainment of the cloud, via the outer boundary, respectively, so that the net entrainment is $E - D$, here we use the conditional values, e.g., $\langle \hat{Q}_{\text{cld}} \rangle$ as its customary in atmosphere community (Siebesma, 1998; Siebesma *et al.*, 2003). Note that, to be consistent with atmosphere community (as we will use their assumption), hereafter the entrainment terms E and ε , and detrainment terms D and δ are both positive, remarking the magnitude of the inflow and outflow of the cloud, respectively. This is different from the signs of entrainment and detrainment of fountains in the previous characters where the direction is involved in the values. The dimension of entrainment ε and detrainment rate δ is m^{-1} .

In the atmosphere community, the local entrainment E and detrainment D of the cloud are usually hard to measure directly, instead, a bulk assumption (Betts, 1973) is used to determine the entrainment rate which assumes the entrainment is associated with the gradient of the bulk mean of property $\langle \chi_c \rangle$, χ_e , within the cloud updraft. The assumption writes

$$\frac{d\langle \chi_c \rangle}{dz} = -\varepsilon(\langle \chi_c \rangle - \chi_e), \quad (5.15)$$

the detrainment rate δ is then determined coupled with the volume (mass) conservation equation as

$$\frac{d\langle \hat{Q}_{\text{cld}} \rangle}{dz} = -(\varepsilon - \delta)\langle \hat{Q}_{\text{cld}} \rangle, \quad (5.16)$$

where χ can be either the θ_l or q_t , χ_e is the initial value of environment, $\langle \chi_c \rangle$ is the time average of χ_c which is the bulk mean of the cloud

$$\chi_c(z, t) = \frac{\int_{\Omega(z)_{\text{cld}}} \chi(\mathbf{x}, t) dA}{\hat{A}_{\text{cld}}(z)}. \quad (5.17)$$

Some literature referred χ_c as to the bulk mean of the cloud updraft (Roode *et al.*, 2012), or of the cloud core which is the positively buoyant region in the cloud (Siebesma *et al.*, 2003). However, this would not make a significant difference, as the values in the subsiding shell affect the bulk mean very less.

However, with the DNS data and the algorithm, we have already explicitly obtained the entrainment and detrainment which allows us to directly calculate the rates via (5.14), where $E = |\langle \hat{q}_{\text{cld}}^- \rangle|$ and $D = |\langle \hat{q}_{\text{cld}}^+ \rangle|$.

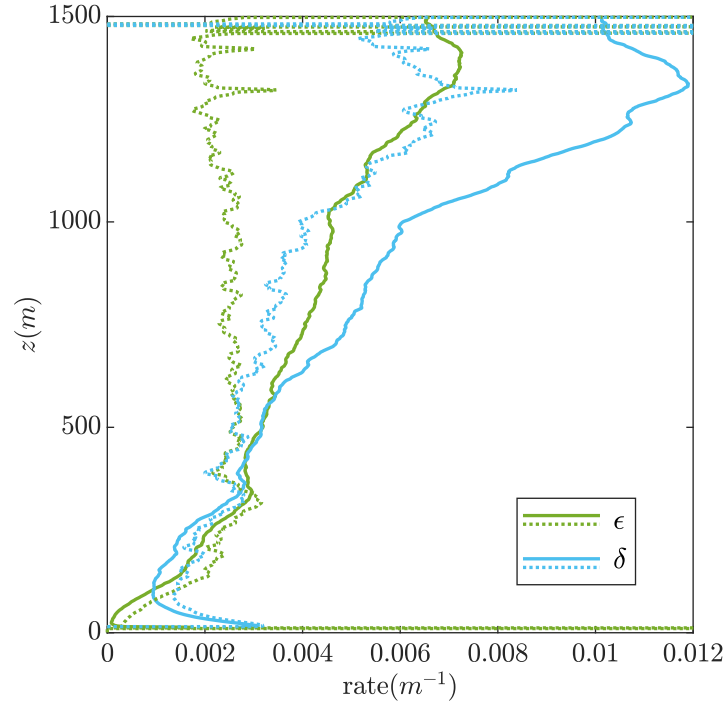


Figure 5.12: The vertical evolution of the entrainment rate ϵ and detrainment calculated δ (unit m^{-1}) using the conditional data, via the direct measurement (solid lines) and the bulk assumption (dashed lines).

With the conditionally averaged data, figure 5.12 presents the entrainment and detrainment rates following bulk assumption (5.15) and (5.16), and the rates calculated

from the definition and measurement (5.14), respectively. We suppose $\chi = q_t$, not shown here but we see in our result, the rates coincide when $\chi = \theta_l$, i.e., the rates seem not to depend on the choice of χ . Note that, these results are only shown up to $z = 1500\text{m}$, as above the level, the time-averaged $\langle \chi \rangle$ is greatly influenced by the intermittency.

The results from the bulk assumption show that the detrainment rate generally increases but the entrainment rate generally decreases with height, and the magnitude of the rates is mainly in the range $0.002 - 0.006$. These results are generally consistent with Siebesma *et al.* (2003), which shows that, under the bulk assumption, our conditional data on an individual cloud gives similar entrainment and detrainment rates to those from the data on a cloud ensemble, indicating that the rates obtained from the cloud ensemble (LES) are possible to be applied on an individual cloud.

However, the calculation directly from the measurement shows that, both the entrainment rate and detrainment rate grow with height, and the magnitudes are larger than the corresponding rates from the bulk assumption. Note that, both the detrainment rate exceeds the entrainment rate at around $z \approx 500\text{m}$ which is consistent with the evolution of the net entrainment. The bulk assumption underestimates the rates because it overestimates the scalar that detrained out of the cloud (see appendix C for a detailed derivation).

5.5 Conclusion

To investigate the entrainment across the cloud edge (a TNTI) and the interface between the updraft and subsiding shell (a TTI) of an individual shallow cumulus cloud, we established a direct simulation with a maintained source located at the domain bottom continuously injecting the warm and moist air into the dry, cold and stratified background, and examined it from both Reynolds-averaged statistics (a global perspective) and conditional averaged statistics (a local perspective).

We observed that the maintained digital cloud is intermittent as a series of stable cloud pulses appeared. This is in line with the observation from Heus & Jonker (2008)

during a real cloud life cycle. The series of cloud pulses indicate a two-side effect of the previous pulse on the current pulse: on the one hand, the previous pulse forms downflow preventing the current pulse from rising, on the other hand, the previous pulse moistens the ambient making the current pulse more easily to form. In the statistically steady state, the two effects reach a balance so that the pulses appear stable. In reality, this effect may occur between the previous cloud and the current cloud.

The shallow cumulus cloud behaviours were then examined via Reynolds-averaged statistics and conditional averaged statistics, respectively, which produce apparently different views due to the intermittency of the cloud, although the two statistics used a similar manner to identify the cloud regions, e.g., the cloud region is identified as where the liquid water specific humidity is positive, while the cloud updraft and subsiding shell are identified as where the vertical velocity is positive and negative in the cloud, respectively.

In the Reynolds average, the cloud regions, including the updraft and subsiding shell were identified according to the Reynolds-averaged q_l field. By the definition of the cloud region, $q_l > 0$, and due to the intermittency — it is not always cloudy ($q_l > 0$) at a certain location, the Reynolds-averaged cloud regions almost contained all the areas that had been ever cloudy, which led to a thick and tall subsiding shell shrouding a relatively short updraft at the centre. However, the intermittency shows that the subsiding shell was very sparse as the loci inside it were not cloudy in approximately 30% of the total time. Therefore, the Reynolds-averaged statistics provided a view of an air parcel throughout its cloudy and non-cloudy periods.

The streamlines generated from the Reynolds-averaged velocity illustrated the motion of the air parcels. All the fluid can be separated into two independent portions via the mean flow: those from the cloud source and those entrained from the environment. The rising fluid from the source falls due to the evaporation but eventually horizontally intrudes at a neutral-buoyant level, leading to net detrainment near the top of the updraft and further net detrainment in the lower half of the cloud. However, the fluid entrained from the environment led to net entrainment at the top half of the cloud.

However, as is customary in the atmospheric community, the conditional averaged

statistics sampled instantaneous cloud regions and took the time average, while the not-cloudy air is not considered. This gave a much narrower cloud and also the updraft and subsiding shell compared with the Reynolds average, and a totally different insight into the entrainment — over the approximately lower half of the cloud, the cloud net entrains, while over the top half, it net detrains.

The net entrainment was split into entrainment and detrainment via a bulk assumption (Betts, 1973), and compared with the direct measurement using the conditional averaged data. Generally, all the results are in the same order as the LES (Siebesma *et al.*, 2003; Roode *et al.*, 2012; De Rooy *et al.*, 2013). It suggests that with the bulk assumption, the entrainment and detrainment rates resulted from the data on a cloud ensemble also suit for an individual cloud. However, the direct measurements show that, the bulk assumption tends to underestimate the rates.

Chapter 6

Conclusions

The aim of this thesis was to investigate entrainment across TNTIs and TTIs within forced fountains and shallow cumulus clouds — two physically different flows but each with a similar upward-flowing core shrouded by a counterflowing downflow. Five objectives were set (see §1.2) that, upon completion, combine to ensure the aim was satisfied. These objectives can be regarded as having been completed as follows:

1. The study of global entrainment across the outer boundary (a natural boundary between the fountain and its environment, a TNTI) and the inner boundary (a boundary between the upflow and downflow, a TTI) of a turbulent forced fountain using Reynolds-averaged statistics provided numerous insights. These insights include that the entrainment coefficient at the outer boundary was broadly constant and consistent with the previous measurements (e.g. Williamson *et al.*, 2011), and the values used in previous models (Bloomfield & Kerr, 2000; Shrinivas & Hunt, 2014). However, at the inner boundary, two features were highlighted: one is that, besides the volume (mass) exchange, significant exchanges of momentum and buoyancy also occur at the inner boundary; the sign of these exchanges changes depending on the height examined within the fountain, i.e., the exchanges are bi-directional. Unlike the mass, the Reynolds-averaged momentum and buoyancy exchanges can be composed into a mean-flow contribution and a turbulent contribution — the turbulent contribution was not incorporated within

any fountain models, to date (e.g. Bloomfield & Kerr, 2000; Shrinivas & Hunt, 2014). Moreover, the momentum and buoyancy exchanges are not well modelled by taking the entrainment coefficient appropriate for the mass exchange, and the turbulent contribution of the momentum and buoyancy exchanges cannot be parameterised in the same manner as the mean-flow contribution, indicating instead of MTT model (Morton *et al.*, 1956), alternative modelling strategies on the entrainment across a TTI. In addition, apart from the inner and outer boundaries, a third physically meaningful boundary was identified — this was a TTI, defined by the critical streamline within the fountain over which no mean-flow transport occurs.

2. A method to conditionally average instant exchanges was successfully developed (§4.2) and the research within this thesis highlights the challenges when using boundaries based on velocity magnitudes due to the turbulent nature of the flows. In particular, in these buoyancy-driven counterflows, and the shear flows created with the upflowing core, just away from the instantaneous boundary defined by the location of zero vertical velocity, fluid velocities can be dominated by turbulence — often in the form of coherent eddies — which can result in different patches of fluid, on both sides of the boundary, having both upward and downward velocities, simultaneously. A sensitivity study successfully identified appropriate threshold values for which the results were relatively unaffected by the precise choice of threshold.
3. The conditionally-averaged statistics obtained from the method described above (point 2), provided an alternate method to analyse the fountain. The instantaneous inner boundary and outer boundary of the fountain were sampled (in a manner as consistent as possible with the boundaries of the Reynolds averaged analysis). The location of the boundaries and the net entrainment from the conditional statistics were, for the most part, close to those from the Reynolds averaged data, showing the consistency of the global and local net entrainment of the fountain. However, conditional-averaged statistics were able to distinguish

local entrainment and detrainment events so that they could be averaged separately and compared to the net entrainment; such capabilities enabled a direct comparison with the fountain model BK00. This comparison showed that the entrainment coefficients computed with the conditional data were broadly independent of the height and were able to broadly predict the mass and buoyancy exchanges. This might be taken to indicate that the BK00 model, which to date has only been parameterised based on Reynolds-averaged data, is more appropriately parameterised using conditional-averaged data. Moreover, examination based on the radial location of the inner boundary highlighted that the probability density functions of the conditional entrainment and detrainment exchanges were found to be Gaussian-like and also independent of the height (when appropriately scaled), providing the possibility of future models of incorporating parameterisations of the radial location of these exchanges.

4. A digital cloud chamber was developed to study an idealised shallow cumulus cloud which can be regarded as a ‘moist fountain’, including the thermodynamics of liquid-vapour phase changes. More complex than the fountain in the quiescent environment, the source of the cloud injects not only a buoyancy flux (via a temperature difference) but also moist fluid into the domain (and hence the possibility that the dynamics are affected by the latent heat of phase change). In addition, the environment within the digital cloud chamber has a stratification broadly representative of the troposphere level of the atmosphere. As such, the phase change of the water would occur, from which the latent heat would affect the fluid motion by influencing the buoyancy. Despite the clouds experiencing different stages during the life cycle (Katzwinkel *et al.*, 2014), we focused on a shallow cumulus cloud in its actively growing stage by letting the maintained cloud source continuously inject fluid, and learning the behaviours when the cloud reached the quasi-steady state.
5. Replicating the analysis of the single-phase fountain on the shallow cumulus cloud allowed investigation via both Reynolds-averaged statistics and conditional

statistics. A general conclusion is that, although the shallow cumulus cloud and fountain share a similar structure and study routine and the meteorologists learnt the cumulus cloud as a thermal plume at early times, their behaviours are greatly different, so that, the results from the fountain chapters are hard to adapt in the cloud. The main reason is that the cloud is set in a stratified environment, and because of this, the cloud is apparently intermittent, shown as a series of pulses because of the latent heat (buoyancy) accumulation at the lower level and dominated evaporation at the higher level. As a consequence, unlike the fountain, due to the intermittency, the results from two statistics of the cloud were not consistent, including that the Reynolds-averaged cloud region was significantly wider than that identified by conditional averaging (in which the instantaneous area of cloudy air was time-averaged), and the entrainment behaviours of the shallow cumulus cloud examined from the perspectives of Reynolds averaging and conditional averaging are apparently different, i.e., the global and local entrainment of the shallow cumulus cloud across the corresponding interface (including the TNTI and TTI) lack of agreement. At last, the entrainment and detrainment rates are calculated via a bulk assumption (Betts, 1973) and a direct measurement (using the boundary integral framework in Chapter 4), respectively, and compared.

The thesis highlights the benefits of investigating simple analogues of complex atmospheric flows when wishing to deepen understanding of the underlying processes which might be required to be better understood for more appropriate representation in weather and climate models. In this case, shallow cumulus clouds were abstracted to an analogue digital cloud chamber, and further simplified to a single-phase flow, i.e. a turbulent forced fountain. The understanding of the turbulent fountains via the Reynolds-averaged and conditionally-averaged statistics was directly compared with the fountain models which offered alternative entrainment coefficients and potential parameterisation of entrainment being worth future studies. Moreover, the boundary integral framework provided an avenue to distil the entrainment at TNTI and TTI of any other kinds of flows. However, translating the parameterisation of the fountains to

the shallow cumulus clouds is non-trivial, mainly due to differences in their physics and set-up; e.g. the increased intermittency that arises in the clouds. That said, the thesis has provided a better understanding of these idealised shallow cumulus clouds via the Reynolds and conditional statistics. However, the complexity of these cloud dynamics still provides challenges for future cloud study. Concluded from the work on both turbulent fountain and shallow cumulus cloud, multiple future works which can be opened up with this thesis are summarised below

1. Potential parameterisation of global entrainment across the TTIs. Entrainment across the TTIs has received far less interest than TNTIs but it is usually the case inside the flow. As we remarked in Chapter 3, the MTT model which was developed across a TNTI, associating the entrainment via the characteristic properties and an entrainment coefficient, did not work ideally in predicting the entrainment of momentum and buoyancy across the TTI in the fountain (i.e., the inner boundary). Therefore, an alternative method to model entrainment across TTI, specifically, the turbulent component of exchanged flux, is worth being explored.
2. Entrainment at the fountain top. As we mentioned in Chapter 3, the fluid reverses at the fountain top, so that the flow at the top behaves differently from the jet-like upflow and downflow below the top, and also note that in Chapter 3, the discussion on the entrainment usually up to the fountain cap base level, aiming to avoid the complexity near the top. However, the fountain cap region plays an important role, not only because it is responsible for more than a quarter of the total fountain entrainment as we found, but also because it is a key part of converging the fountain modelling (Mcdougall, 1981; Debugne & Hunt, 2016). DNS in this thesis has provided the all data at the fountain top which makes this study promising.
3. A classical fountain in the stratified environment. This is more likely a middle step between the fountain and the cloud, which has not been well-investigated in my project due to the time limit. However, the study routine simply follows that of the fountain but applies the stratified environment that we have already

done similarly in the cloud. We note that, the horizontal intrusion observed in the cloud would also appear in this case in some appropriate conditions (Bloomfield & Kerr, 2000). To the author's knowledge, the entrainment of the fountain in the stratified environment is still an open question but can be often seen in nature, examples including volcanic eruption columns (Woods, 2010) and turbidity currents which run down the continental shelf (Allen, 1971).

4. The effects of turbulent coherent structure and turbulent intensity on the entrainment. As the entrainment is induced by the turbulence, the role of the properties of the turbulence that plays in the entrainment, like shear, vorticity, turbulent structure, and turbulent intensity, are worth exploring. A deeper understanding of the turbulent entrainment would be beneficial by doing so. As a start, in Chapter 3, we mentioned that the turbulent component of the Reynolds-averaged momentum exchange can be explained from the perspective of turbulent shear, which further would be possible to link to the turbulent intensity. The DNS is able to provide all the 3-D data that are needed.
5. Entrainment at the cloud top and entrainment of the stratocumulus clouds. Unlike the lateral entrainment at the cloud edge, the cloud-top entrainment is usually vertical and DNS has been suggested to be a complementary tool to LES for stratocumulus cloud research (Mellado, 2017). The digital cloud chamber in the thesis can be used to study such as the turbulence and phase change of water in the cloud-top entrainment zone.
6. Simple shallow cumulus cloud models. Valid simple cloud models would be greatly useful in weather and climate modelling and predicting. Starting from the fountain model perhaps could be an initial step as the two flows have a similar structure. However, apparently, the cloud model would be more complex due to its stratified environment, phase change of water, and normalisation — note that the cloud study in this thesis was dimensional. The current digital cloud chamber is able to provide all the detailed data it needs and is possible to apply further conditions, like cross-wind.

7. A Lagrangian study of interfaces at the edges of cumulus clouds. The SPARKLE was already complemented with a Lagrangian particle tracking routine, and Nair *et al.* (2021) used the Lagrangian tracer to investigate the entrainment and detraining across the edges of a cumulus cloud set in a uniform environment. After establishing the cloud chamber in this thesis, the Lagrangian study could be presented again on the vertical shallow cumulus cloud to trace how the particles are transported within the cloud.

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Appendices

A Validation of the boundary integral algorithm

The algorithm introduced in §4.2 provides the approach to directly calculate the entrainment (and detrainment) by integrating the entrainment velocity over the interface, which allows us to split the entrainment and detrainment across the interface (e.g., the inner boundary of the fountain) where both of them co-exist. This section is to validate the algorithm by checking the instantaneous volume (mass) conservation (4.2) whose right-hand side is calculated via (2.10)(2.13)(2.20)(2.21) which is the core of the algorithm — involving the calculation of entrainment velocity (Van Reeuwijk *et al.*, 2021) and treatment of the line integral (Yurtoglu *et al.*, 2018).

An axisymmetric jet will be utilised to check the volume conservation, whose (characteristic) radius r_m is parametrised as (Morton *et al.*, 1956; Van Reeuwijk *et al.*, 2016)

$$r_m(z) = 2\alpha z, \quad (\text{A.1})$$

where α is the entrainment coefficient and z is the height. The velocity $w(r, z)$ is

$$w(r, z) = 2w_m \exp\left(-2\frac{r^2}{r_m^2}\right), \quad \text{where} \quad w_m = \frac{\sqrt{M_0}}{2\alpha z}, \quad (\text{A.2})$$

where w_m is characteristic velocity, M_0 is the source momentum flux. The radial velocity $u_r(r, z)$ satisfies the continuity equation

$$\frac{1}{r} \frac{\partial(ru_r)}{\partial r} + \frac{\partial w}{\partial z} = 0, \quad (\text{A.3})$$

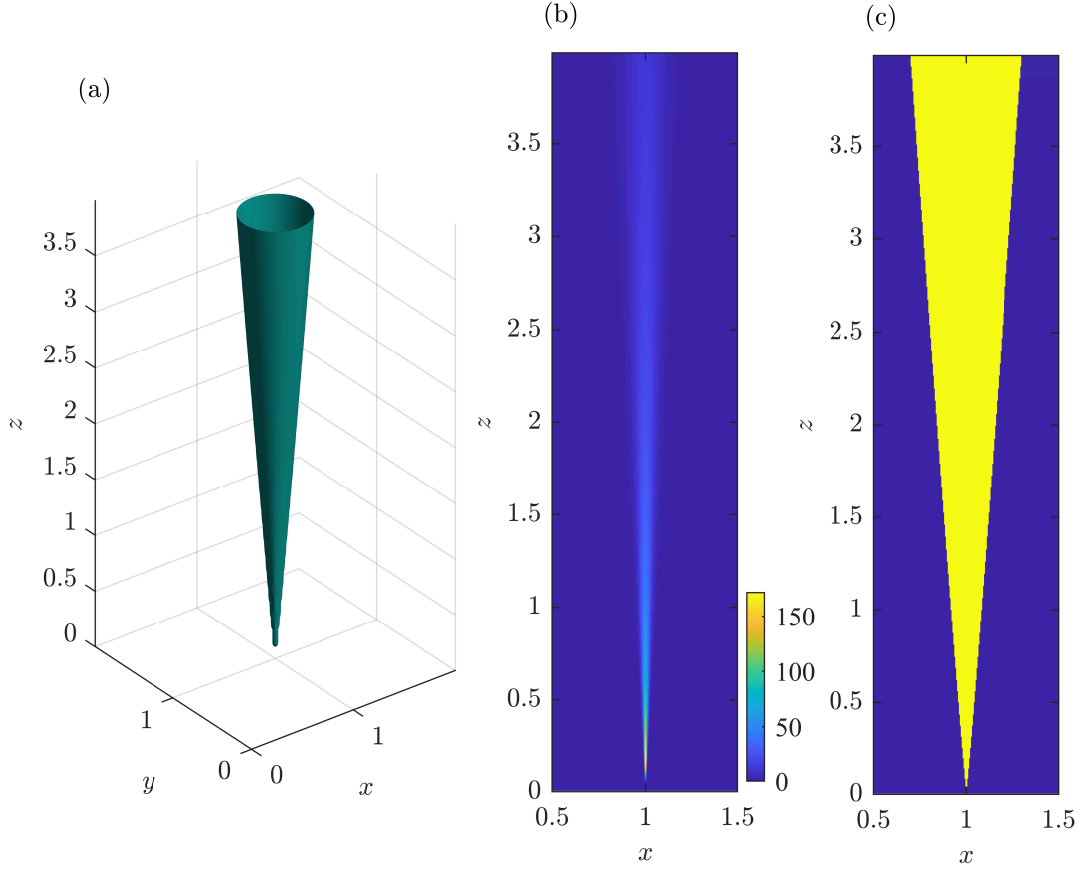


Figure A.1: (a) A 3-D isosurface of the jet at $w = 0.01$. (b) A central plane view of the vertical velocity w at $y = 1$. (c) The masking function that identifies the jet region with the threshold $w_0 = 0.01$. The jet region is coloured yellow while the rest is blue.

and is set to be zero at the jet centreline due to the axisymmetry, i.e., $u_r(0, z) = 0$. The jet is converted into the Cartesian coordinate $L_x \times L_y \times L_z$ where the algorithm is developed — the point source of the jet is located at the centre of the bottom $(x_0, y_0, 0)$, where $x_0 = L_x/2, y_0 = L_y/2$ and the radius is $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$. A 3-D plot of the isosurface of $w = 0.01$ is shown in figure A.1(a), accompanied by a cross-section of the vertical velocity w field at a centre plane $y = y_0$. Velocity fields of u, v, w at a horizontal plane $z = 2.5$ is shown in figure A.2(a,b,c), respectively.

The test is divided into two parts and in each part, the two sides of (4.2) are calculated and compared. In the first part, the jet is steady as shown here, and the entrainment $q(z)$ will be calculated via the algorithm, and compared with the parameterisation (theoretical) value $q_1(z) = 2\alpha w_m r_m$ and also the gradient of volume flux $q_2(z) = dQ(z)/dz$

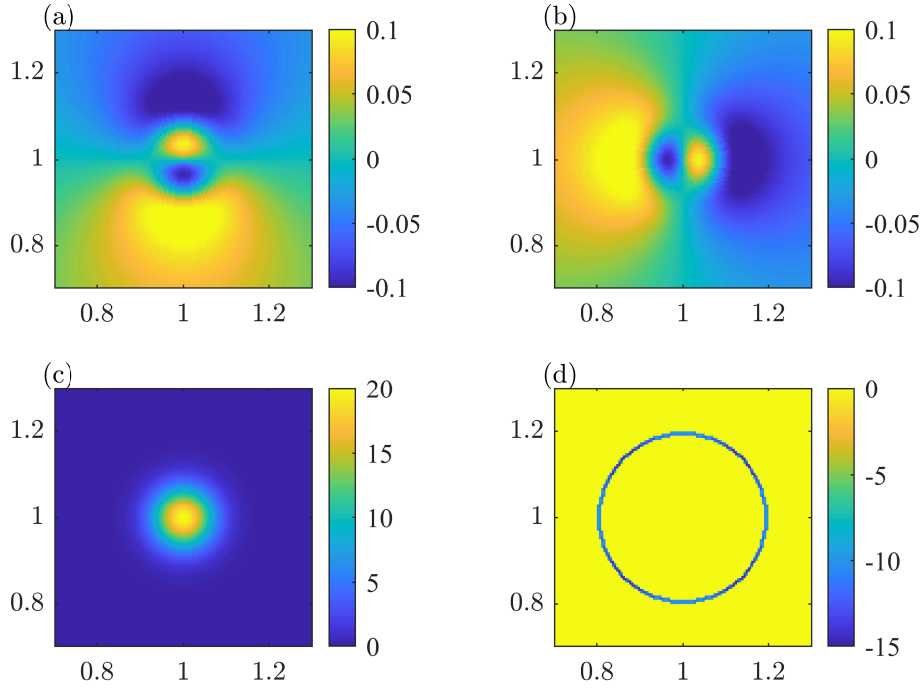


Figure A.2: The horizontal plane view at the height $z = 2.5$ of the velocity (a) v , (b) u , (c) w , and (d) the view of the pointwise entrainment e at the same height.

according to the continuity equation. Note that, although with a hat symbol, in this case, the variables do not change with time, e.g. $d\hat{A}(z)/dt = 0$. In the second part, the jet is unsteady whose entrainment coefficient α begins to vary with time, therefore, the entrainment is integrated over a moving boundary. In this case, the calculated instantaneous entrainment $\hat{q}(z, t)$ via our algorithm will be only compared with $\hat{q}_2 = d\hat{A}(z, t)/dt + d\hat{Q}(z, t)/dz$ according to the continuity equation, as the parameterisation result is only available for the steady jet. In both cases, the performance of different spatial interpolations will be tested, and additionally, in the second case, the different time-differentiation schemes will also be tested.

In the first part, to implement the algorithm, the jet is identified by the field $\chi = w$ with the threshold value $\chi_0 = 0.01$ which is a very small velocity compared with the centreline velocity. The momentum flux $M_0 \equiv 1$ and the entrainment coefficient is set as $\alpha \equiv 0.02$. Then the masking function of the jet is obtained and a centre plane ($y = L_y/2$) of it is shown in figure A.1(c). Following (4.2), the pointwise entrainment

rate e_q is obtained at each height, and that at $z = 2.5$ is shown in figure A.2(d). This shows that with the masking function, only the flux across the jet boundary is detected and is zero in the rest, enabling to transfer the line integral to the plane integral. The calculated q via the different spatial interpolation, forward scheme, backward scheme and central scheme is compared in figure A.3 overlaid with the theoretical value q_1 . Note that, the gradient of the volume flux q_2 also depends on the schemes, only the central scheme of it is shown in the figure for clearance which is sufficient to show the volume conservation. The figure shows that all the results agree well in the effective height (e.g., $z \geq 2$ away from the near-source effect), the central scheme is closer to theoretical $-2\alpha r_m w_m$ than the forward scheme and backward scheme — the latter two are almost the same. Note that, the volume flux gradient $-dQ/dz$ shows more scatters.

Therefore, in the second part, we choose the central scheme as the spatial interpolation and keep all the settings and parameters the same as those in the first part, except the entrainment coefficient α , which now varies with the time t as $\alpha = 0.02 + 0.01t$. This produces an expanding jet. Not that, we are not using the time-integration scheme as the states of the jet at each time step are known. Instead, we run three time steps with each step $\Delta t = 0.1$ from $t = 0$, and for time differentiation d/dt , we choose a backward scheme, e.g., $(\chi(\Delta t) - \chi(0))/\Delta t$, and a central scheme, e.g., $(\chi(2\Delta t) - \chi(0))/(2\Delta t)$ as we have shown that the forward scheme is insignificantly different from the backward scheme. Figure A.4 shows the calculated $\hat{q}$ and $d\hat{A}/dt + d\hat{Q}/dz$ according to the note that the latter also depends on the time scheme. However, the figure shows that the time scheme does not affect much on $d\hat{A}/dt + d\hat{Q}/dz$, while entrainment $\hat{q}$ from the central scheme seems to agree better with the term $d\hat{A}/dt + d\hat{Q}/dz$.

Therefore, the validation shows that the central scheme on both spatial and time differentiation is sufficiently accurate for the algorithm to calculate the line integral.

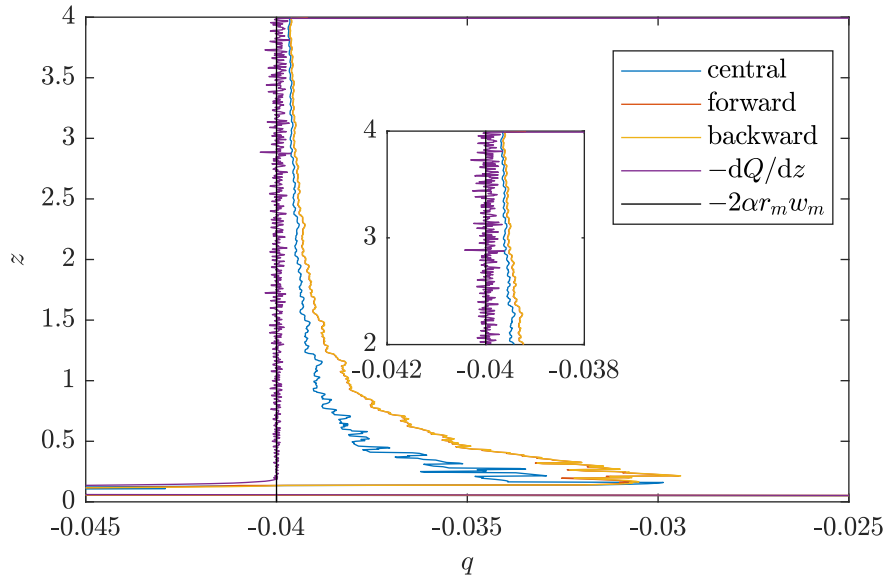


Figure A.3: The entrainment of the steady jet obtained via central, forward, backward spatial schemes, overlaid with the parameterisation value $-2\alpha r_m w_m$ and the volume flux gradient $-dQ/dz$.

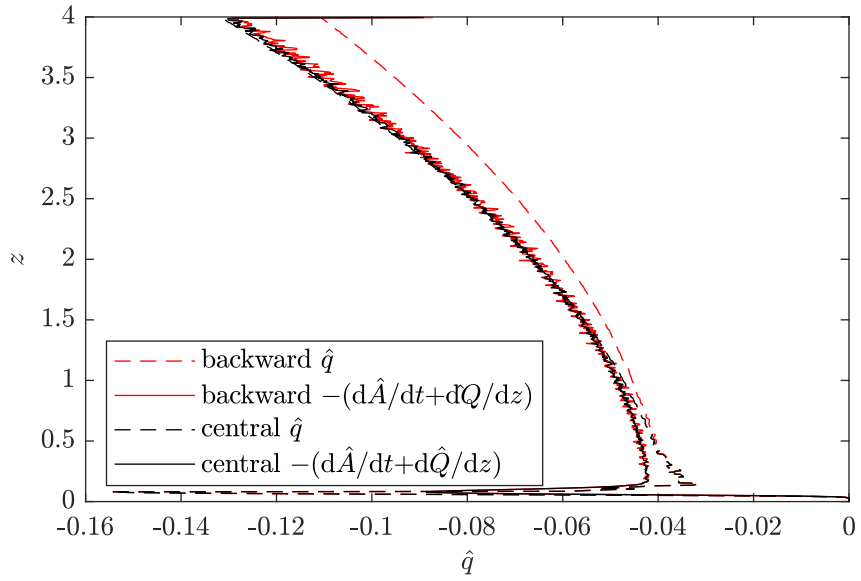


Figure A.4: The entrainment of the unsteady jet obtained via central (black) and backward (red) time schemes, overlaid with the volume variation $-(d\hat{A}/dt + d\hat{Q}/dz)$.

B Buoyancy calculation

Attention should be paid to the environmental virtual temperature $\theta_{v,e}(z)$ when calculating the buoyancy, practically, how to determine this environmental value? In the LES, as it considers a large area of the cloud field, $\theta_{v,e}$ is determined as a slab-mean value of the entire field including the cloudy air and dry air. However, in the DNS which only considers one single cloud, the slab-mean value is not meaningful as it depends on the domain size — the cloud motion affects less on the environment in a larger domain. Therefore, we choose the average virtual temperature within the vertical nudging area (light blue area) as the local environmental virtual temperature $\theta_{v,e}(z)$, where $\theta_t(z)$ and $q_t(z)$ are forced to their initial values.

Apparently, the different choices for the environmental virtual temperature will lead to different buoyancy and (modified) pressure in the (2.2), which requires some carefulness when examining or comparing such values with LES. However, here, we will show that this effect is cancelled when considering pressure and buoyancy together in the term $b\mathbf{e}_z - \nabla p$, so that will not affect the velocity.

$$b = -g \frac{\rho - \rho_e}{\rho_0} = g\beta(\theta_v - \theta_{v,e}), \quad (\text{B.4})$$

where θ_v is the virtual potential temperature, ρ_e is the environmental pressure, ρ_0 is a reference pressure. On the z -direction, the momentum equation writes

$$\frac{Dw}{Dt} = -\frac{dp}{dz} + b, \quad (\text{B.5})$$

where p is the modified pressure which is associated with the total local pressure p_t subtracted from the local hydrostatic pressure, $p = (p_t - p_e)/\rho_0$. Therefore, the right-

hand-side of the above equation can be written as

$$-\frac{dp}{dz} + b = -\frac{1}{\rho_0} \frac{d(p_t - p_e)}{dz} - g \frac{\rho - \rho_e}{\rho_0} \quad (\text{B.6})$$

$$= -\frac{1}{\rho_0} \frac{dp_t}{dz} + \frac{1}{\rho_0} \frac{dp_e}{dz} - g \frac{\rho - \rho_e}{\rho_0} \quad (\text{B.7})$$

$$= -\frac{1}{\rho_0} \frac{dp_t}{dz} - g \frac{\rho_e}{\rho_0} - g \frac{\rho - \rho_e}{\rho_0} \quad (\text{B.8})$$

$$= -\frac{1}{\rho_0} \left(\frac{dp_t}{dz} + \rho g \right). \quad (\text{B.9})$$

Here, the hydrostatic pressure

$$\frac{dp_e}{dz} = -\rho_e g, \quad (\text{B.10})$$

is substituted. The equation (B.9) shows that the external force, including the pressure and buoyancy (gravity) of the fluid, is independent of the choice of the environmental reference, i.e., it is independent of $\rho_e(z, t)$ or $\rho_e(z, 0)$. Therefore, although the different environmental references indeed affect the individual pressure gradient and buoyancy, the effect cancels within the total external force and does not influence the velocity.

C The bulk assumption for entrainment rates

The bulk assumption assumes a bulk cloud area and a bulk environmental area in a steady state. At the interface between the cloud and environment, entrainment E (de-entrainment D) occurs, carrying bulk environmental property χ_e into (cloud property $\langle \chi_c \rangle$ out of) the cloud. Considering volume conservation and scalar conservation, we have

$$\frac{d\langle \hat{Q}_{\text{cld}} \rangle}{dz} = E - D \equiv \varepsilon \langle \hat{Q}_{\text{cld}} \rangle - \delta \langle \hat{Q}_{\text{cld}} \rangle, \quad (\text{C.11})$$

$$\frac{d\langle \hat{Q}_{\text{cld}} \rangle \langle \chi_c \rangle}{dz} = E \chi_e - D \langle \chi_c \rangle, \quad (\text{C.12})$$

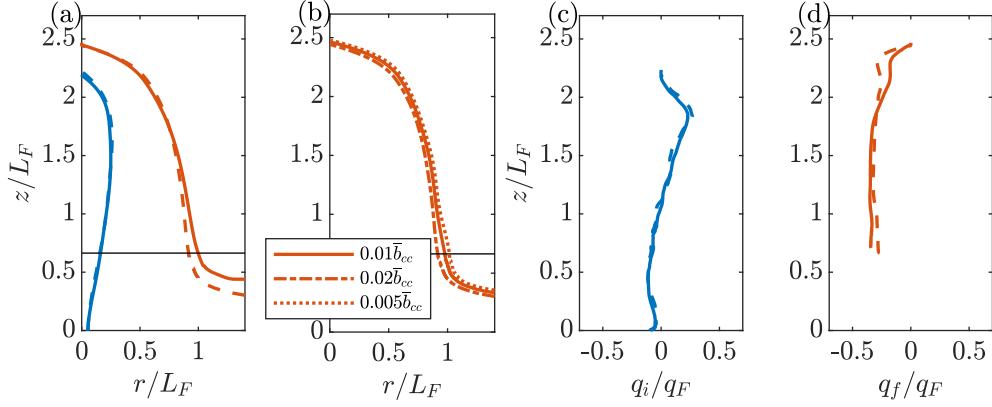


Figure D.1: (a) Fountain boundaries, inner boundary r_i (blue) and outer r_f (orange) both from the standard domain (solid lines) and a small domain (dashed lines). (b) From the standard domain, the fountain outer boundary using a buoyancy threshold of $0.01 \bar{b}_{cc}$ (solid line), $0.02 \bar{b}_{cc}$ (dot-dashed line), and $0.005 \bar{b}_{cc}$ (dotted line). The horizontal black line marks $z/L_F = 0.66$ below where the data of downflow is discarded. Mean volume entrainment at inner boundary q_i (c) and at outer boundary q_f (d) of the standard domain (solid lines) and small domain (dashed lines) respectively.

respectively. It also assumes that the flux of the scalar $\langle \hat{Q}_{\text{cld}} \chi_c \rangle = \langle \hat{Q}_{\text{cld}} \rangle \langle \chi_c \rangle$. Expanding the left-hand side of (C.12), and cancelling D , it can be simplified as

$$\frac{d\langle \chi_c \rangle}{dz} = E(\chi_e - \langle \chi_c \rangle). \quad (\text{C.13})$$

However, note that in fact, instead of boundary value, (C.12) uses the bulk mean value of the scalar $\langle \chi_c \rangle$ to present that detrained out of the cloud, which is definitely greater than the local boundary value. According to the above derivation, this overestimation results in lower entrainment ε and detrainment rate δ .

D Comparison with smaller flow domain

To validate that the effect of the domain size, in particular the side-wall periodic boundary condition, was insignificant in our results we simulated identical source conditions in a ‘small’ domain (of volume $(100r_0)^3$ and grid size 1024^3); we refer to the domain used to simulate all other results presented herein as ‘standard’.

Figure D.1(a) plots the fountain inner boundary, r_i , and outer boundary, r_f , for sim-

ulations in both the standard and small domains. The effects of domain size on the fountain boundaries are negligible, particularly when one notes that we discard all results for the downflow (including r_f) below the height $z/L_F = 0.66$, with the typical change in the radial location of the boundaries with domain size being less than 7%. Moreover, by integrating the volume of fountain $V = \int_0^{z_f} 2\pi r_f(z) dz = 82730 r_0^3 \approx 8.70 L_F^3$, the fountain occupies around 3% of the standard domain, suggesting the standard domain is of an appropriate size.

To check the location of the outer boundary is relatively insensitive to the choice of buoyancy threshold ($\bar{b} = 0.01\bar{b}_{cc}$), figure D.1(b) plots the outer boundary using both a larger threshold, $\bar{b} = 0.02\bar{b}_{cc}$, and a smaller threshold, $\bar{b} = 0.005\bar{b}_{cc}$. The figure evidences that the outer boundary is indeed insensitive on the choice threshold, within our range of interest and especially above $z/L_F = 0.66$. We therefore conclude that our outer boundary, r_f , contains the vast majority of fountain fluid and is appropriate.

To ensure the entrainment at the boundaries is also less influenced by the domain, figure D.1(c) and D.1(d) plot the mean volume entrainment q_i and q_f of two domains respectively. The lines agree well though there is a relatively greater difference in q_f near the fountain top. By calculating the integral volume entrainment of the fountain cap Q_{cap} , the value of the small domain is about $-0.19Q_F$, not far from the value of the standard domain which is $-0.22Q_F$.