

A MIXED INTEGRAL DIFFERENTIAL METHOD
FOR LOW FREQUENCY ELECTROMAGNETIC FIELDS

By

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ABSTRACT

This thesis is concerned with the calculation of low frequency electromagnetic fields, with particular reference to electrical machines where the solution domain is infinite. The analysis is limited to the two dimensional case only. The function chosen for solution is $\underline{H}$, the magnetic field intensity vector, and the solution philosophy attempts to bring together the strengths of both the differential and integral finite element approaches into a cohesive solution technique.

Two topics dominate the initial discussion, the treatment of normal derivatives of $\underline{H}$ components, and the posing of the problem. The development of the method was undertaken using the single infinitely long isolated bus bar problem in order that the calculated results might be substantiated by analytic results.

The method and its limitations having been established the remainder of the thesis is devoted to the enhancement of the method so that a wide range of low frequency electromagnetic field problems might be amenable to the technique. The theory required for the introduction of relative permeabilities greater than zero, although not non-linear, is presented along with one analytic and two practical problems for which comparisons were made: an infinitely permeable isolated bus bar, an E-core

levitator and a Xi-core levitator on open circuit. A generalised two dimensional interface condition for $\underline{H}$ is presented along with a discussion of the possibility of extending the ideas from two to three dimensional interfaces and corners. The equations relating the components of $\underline{H}$ at a three dimensional square corner are then given. Finally the method is adapted in order that problems involving eddy currents may be treated. Theoretical and experimental results are compared for three practical engineering problems which are, laminations under axial excitation, a single phase E-core levitator and a single phase Xi-core levitator.

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The heavens declare the glory of God,
and the firmament sheweth His handiwork.

The law of the Lord is perfect,
converting the soul:
The testimony of the Lord is sure,
making wise the simple.

Psalm 19 vs 1 and 7

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CHAPTER 1

INTRODUCTION

1.1 A MIXED INTEGRAL DIFFERENTIAL METHOD?

For many years now there has been, for varying reasons, interest in the calculation of magnetic and electromagnetic fields. If one were to say there have been a plethora of techniques, and each with its own variations, the case would be overstated but the bewildering array of useful approaches is certainly daunting. Moreover this array of techniques taciturnly declares the complexity of the elegant mathematical description of electromagnetic phenomena, as well as the fact that no one technique is held to be the panacea of electromagnetics. It is precisely these reasons that foster the present research into new techniques, and the enhancement or adaptation of those that have already proved useful. This in itself further complicates the situation and only adds to the bewilderment of the newcomer to this particular branch of science. It is with the intention of clarifying the situation and putting the work of this thesis into its correct historic context that a broad classification of available techniques will be given.

Three categories suffice initially and these are,

'analytic methods', 1.1, 1.2, 1.3, 'numerical methods', 1.4, 1.5, 1.6 and those methods which combine elements of the two former categories, 'combination methods', 1.7, 1.8, 1.9.

'Analytic methods' are essentially very limited in the range of problems that they can be applied to and have been largely superseded by approaches encompassed in the other two categories.

'Combination methods' require that one field at least be calculated analytically and the final field solution be representable as the super-position of at least two fields.

Consideration of the above two categories will be waived in order to give a fuller expose of the more popular 'numerical methods' which is more relevant to this thesis.

The great wealth of 'numerical methods' that one sees around today owe their existence and popularity to the advent of the digital computer and it is with such methods that the work of this thesis is concerned. To aid discussion and to highlight the setting of this thesis a subdivision of this category will be made.

As before three sections will suffice and they are,

'differential methods', 'integral methods', and 'hybrid methods', whose definition will be left until later. Essentially this subdivision is made on a chronological basis and it is hoped that the progression of thought and motivation behind such a development can be shown by considering each section in turn.

The theory or calculus of finite differences stems from the work of Newton. The theory is extensive but its application to second order differential equations has flourished only since the advent of the digital computer. Prior to this the work by Southwell^{1.10} on relaxation, initially a hand technique for the approximate solution of a set of linear equations, had set the scene for the frenzied activity that was to come. At the centre of such concentrated activity was the computer implementation of finite difference techniques and the solution of the resulting equations. Finite difference techniques were the first widely implemented procedures for the solution of many types of second order partial differential equations, and are classified as differential methods. Although finite difference techniques were widely used they had certain limitations. The two most relevant being, the difficulty of coping with anything but regular boundaries and interfaces, and the difficulty of dealing with infinite regions. With reference to the former limitation, theoretically this

should only have been a problem up to 1953 when Macneal^{1.11} published his paper.

Chronologically the finite element differential methodology was the next group of methods to creep over the horizon. Although, as Zienkiewicz^{1.12} has pointed out, these techniques overcome some of the problems encountered by finite difference analysts in no sense can it be said that the differential finite element methodology was developed to surmount any of these problems; essentially the two approaches were being developed at the same time; the differential finite element techniques were developed as an alternative methodology, albeit initially at a slower rate. It is interesting to follow the historical development of finite element techniques as given by Key and Krieg^{1.13}. They show that the seeds of the approaches can be traced back perhaps as far as Euler and that their development was undertaken independently by three separate groups, structural engineers, applied mathematicians and physicists. Although finite element techniques were not developed as superior to finite difference techniques in many areas of present day research they are seen in just that light. A better assessment would be to see finite difference techniques as but special cases of the more general finite element differential methodology since in recent years it has

been found that certain applications of non conforming trial functions give nothing more than the standard finite difference approximation^{1.14}. Differential methods are here to stay and the differential finite element methodology is by far the most general that exists and can encompass many classical techniques by utilising different constructions.

Although the finite element differential methodology is superior in many respects to finite difference techniques, both differential methodologies until recently were unable to cope satisfactorily with far boundaries, the second limitation as outlined above. It was^{1.15}, and still is today^{1.16}, in many applications customary to cope in differential terms by defining a boundary a finite distance from the problem on which the function being calculated, its normal derivative, or some combination of the two is assumed known. Some workers facing this type of problem turned their attention to classical integral equations and applied the essence of the differential finite element methodology to achieve an approximate solution; for they realised that the solution to a certain class of problems could be invariably represented by a boundary integral equation. Two of the first papers to appear in the literature were by Jaswon^{1.17,1.18} and since then research into integral techniques has mushroomed as can be gauged by a cursory look at the extensive

list of references given by Jaswon and Symm^{1.19} in just one area of applicability. Finite element techniques are nowadays applied to all types of problems and not only those which give rise to boundary integral equations. In fact finite element integral techniques embody both volume and boundary integral approaches.

Care has been taken to define the finite element methodology as 'differential' or latterly as 'integral' for the term finite element is understood by many today to encompass both.

It seems that when any new technique is fully investigated that certain limitations come to light and finite element integral techniques have by no means been exempt from such discoveries. A relevant comment to this effect is made by Biddlecombe^{1.20} who with reference to the application of integral equations to magnetic and electromagnetic phenomena highlights three classes of problem which may well prove more amenable to differential techniques. The three classes of problem stated are

- (1) those with small air gaps
- (2) those with high permeabilities
- (3) those with eddy currents in iron.

It may now be noted that both differential and integral approaches have their limitations but of interest is that on many occasions it appears that where one

technique is comparatively weak the other comparatively strong.

Within the last eight years the increasing awareness that differential and integral techniques are not mutually exclusive, for any given problem, has led to research into 'hybrid techniques' where the philosophy has been to try and combine strengths of both approaches into a cohesive solution technique. Research into the possibilities and practicalities of such a marriage is still in its early stages and thus the literature is somewhat limited. The best and two most comprehensive papers have been produced by Zienkiewicz et al^{1.21,1.22} within the last four years. Applications of such techniques can be found in microwave analysis^{1.23}, fluid dynamics^{1.24}, and magnetics^{1.25}.

The work of this thesis is concerned with the application of a 'hybrid technique' to the calculation of electromagnetic fields. The basic philosophy has been to apply differential techniques in regions where current is flowing or where material properties may well be field dependent, and to apply integral techniques only in regions where volume integrals may be transformed into surface integrals.

Immediately certain expected gains can be stated.

- (1) There is no need to mesh the integral

domain.

- (2) If needed the field in the integral domain should be easily constructable from the boundary integral equation.
- (3) Fewer nodes modelling integral regions will allow a finer mesh in regions of interest.

Of course there must be some price paid for the expected gains and several points need to be noted.

- (1) The implementation of the method may well be more complicated than the standard finite element constructions.
- (2) Integral equations always lead to fully populated matrices.
- (3) Asymmetrical matrices must be expected.

The implications of each of these points may be far reaching but individual tastes and biases will either accentuate or diminish the relative importance of each point and its ramifications.

Having stated the kind of solution methodology to be investigated two important questions arise which will be dealt with in the following sections. Firstly, the division of the problem space so as to cater for each technique, and secondly, the field describing functions to be used in the solution domain of each method.

1.2 DIVISION OF THE PROBLEM SPACE

The philosophy behind the division of the problem space that has been followed in this thesis is but one of many

that could have been chosen. This by no means implies though that an unreasoned or arbitrary choice has been made as it is hoped will be shown. It is intended to designate the region where the integral approach will be used as 'exterior' and the compliment of this region as 'interior'.

The primary advantage of using an integral method for the field in the exterior region is that regions of air, where explicit knowledge of the field is seldom required, can be handled without recourse to volume discretisation. The governing equation for $\underline{H}$ (the magnetic field intensity) in air is the vector Laplasian. This arises as follows,

$$\underline{\nabla} \times \underline{H} = \underline{0} \quad (1.1)$$

$$\underline{\nabla} \cdot \underline{H} = 0 \quad (1.2)$$

Operating on (1.1) by $\underline{\nabla} \times$ and using a vector identity it can be seen that,

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{H}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{H}) - \nabla^2 \underline{H} = \underline{0} \quad (1.3)$$

Combining (1.3) and (1.2) brings the desired result,

$$\nabla^2 \underline{H} = \underline{0} \quad (1.4)$$

It would therefore seem appropriate that for any region to be designated as part of the exterior the governing equation for $\underline{H}$ in that region should be (1.4). Therefore (1.4) is a necessary condition, but it is certainly not sufficient. For it is possible to conceive theoretical situations magnetically and electrically different from air where (1.4) is the governing equation for $\underline{H}$.

It therefore remains to decide on sufficient conditions to determine the extent of either region in any given problem.

The decision was taken that initially the integral domain should be as simple as possible to implement. In order that this might be so integral to integral interfaces were not to be permitted. The Greens integral equation for an arbitrary $\underline{H}$ component is of the form,

$$\alpha H_c(p,q) = \int_{\Gamma} H_c \frac{\partial G}{\partial n} d\Gamma - \int_{\Gamma} \frac{\partial H_c}{\partial n} G d\Gamma \quad c \in \{x,y,z\} \quad (1.5)$$

where α is the interior angle at (p,q) , Γ is the surface bounding the exterior domain, G is the relevant two or three dimensional Greens function and $\frac{\partial}{\partial n}$ is the outward directed normal derivative with respect to the integral domain. In order to eliminate integral to integral interfaces it becomes clear from equation (1.5) that

(1) $\underline{H}$ must be continuous, and

(2) $\frac{\partial \underline{H}}{\partial n}$ must be continuous.

The above two conditions are satisfied quite generally for regions where the following equations hold

$$\mu_r \text{ (relative permeability) } = 1 \quad (1.6)$$

$$\underline{\nabla} \times \underline{H} = \underline{0} \quad (1.7)$$

and together with

$$\underline{\nabla} \cdot \underline{B} = 0 \quad (1.8)$$

give the necessary and sufficient conditions for any region to be nominated as part of the exterior.

Practically this means that the exterior region comprises empty space and to maximise its extent the integral differential interfaces will be pushed onto the material interfaces.

The term exterior may be found confusing by some but the term indicates any region not adjudged part of the interior, both finite and infinite regions.

1.3 A BRIEF SURVEY OF FIELD DESCRIBING FUNCTIONS

At power frequencies the linkages between the currents and fluxes may be expressed as

$$\underline{\nabla} \times \underline{H} = \underline{J} \quad (1.9)$$

$$\underline{\nabla} \times \underline{E} = -\dot{\underline{B}} \quad (1.10)$$

where the dot signifies differentiation with respect to time. Added to the two above equations are the two constitutive relationships

$$\underline{B} = \mu_0 \mu_r \underline{H} \quad (1.11)$$

$$\underline{J} = \sigma \underline{E} \quad (1.12)$$

where $\mu_r(x,y,z,t)$, $\sigma(x,y,z,t)$ are related to the properties of each material and μ_0 is the primary constant. The above four equations are not sufficient to define any of the four vector quantities, but the addition of the continuity conditions

$$\underline{\nabla} \cdot \underline{B} = 0 \quad (1.13)$$

$$\underline{\nabla} \cdot \underline{J} = 0 \quad (1.14)$$

uniquely defines the problem to additive constants,

which are determined by the conditions at infinity or other suitable surfaces. The determination of any electromagnetic problem can be reduced to finding $\underline{H}(\underline{B})$ or $\underline{E}(\underline{J})$ assuming the material properties are known. Although it is sufficient to calculate $\underline{H}(\underline{B})$ or $\underline{E}(\underline{J})$ perhaps the majority of work published is concerned with the calculation of the latter^{1.26,1.27,1.28}, and then where necessary the deriving of the former. It has also been seen to be an advantage not to consider $\underline{E}$ directly, but a generalised $\underline{E}$ vector $\dot{\underline{A}}$ plus an associated scalar ϕ . Effectively $\underline{E}$ is partitioned in an additive manner. This allows a wide choice of doublets $(\dot{\underline{A}}, \phi)$ so as to maximise the possibility of finding the easiest route to a solution. The doublet may be defined as follows

$$\underline{\nabla} \times \underline{A} = \underline{B} \quad (1.15)$$

Now use (1.14)

$$\underline{\nabla} \times \underline{E} = -\dot{\underline{B}} = -\underline{\nabla} \times \dot{\underline{A}} \quad (1.16)$$

$$\underline{\nabla} \times (\underline{E} + \dot{\underline{A}}) = \underline{0} \quad (1.17)$$

Now integrate (1.17)

$$\underline{E} = -\dot{\underline{A}} + \text{grad } \phi \quad (1.18)$$

Many doublets $(\dot{\underline{A}}, \phi)$ may be found to satisfy (1.18) depending on the value of $\underline{\nabla} \cdot \underline{A}$. Carpenter^{1.29} discusses four separate choices for $(\dot{\underline{A}}, \phi)$.

- (1) That pair determined by what has become known as the Coulomb gauge $\underline{\nabla} \cdot \underline{A} = 0$
- (2) That pair which reduces the number of components of $\underline{A}$ by one and one only, $\underline{\nabla} \cdot \underline{A} \neq 0$ everywhere.
- (3) That pair given by the conductivity gauge $\underline{\nabla} \cdot (\sigma \underline{A}) = 0$

(4) That pair given by the permeability gauge $\underline{\nabla} \cdot (\underline{A}/\mu) = 0$

As was intimated earlier much work has been done using $(\underline{\dot{A}}, \emptyset)$ doublets but the majority of this work is in two dimensions. Some workers^{1.30,1.31,1.32} have considered the extension of these ideas to three dimensions, but not a great deal has been published because of the problems brought about by the complicated magnetic interface conditions and the importance that the surface charges now bring to the problem. Such difficulties associated with the generalised $\underline{E}$ formulation, complicated by the desire to produce three dimensional computer analysis packages, have led some workers back to consider the possibility of generalised $\underline{H}$ formulations^{1.33,1.34,1.35}.

The ideas appear essentially analogous to the generalised $\underline{E}$ formulation but the differences between μ_r , the relative permeability, and σ , the conductivity, make the formulations far from similar in operation. The generalised $\underline{H}$ vector, $\underline{T}$, and its associated scalar, Ω , are constructed analogously to $(\underline{\dot{A}}, \emptyset)$ thus,

$$\underline{\nabla} \times \underline{T} = \underline{J} \quad (1.19)$$

we now use (1.9)

$$\underline{\nabla} \times \underline{H} = \underline{J} = \underline{\nabla} \times \underline{T} \quad (1.20)$$

$$\underline{\nabla} \times (\underline{H} - \underline{T}) = \underline{0} \quad (1.21)$$

Now integrate

$$\underline{H} = \underline{T} - \text{grad}\Omega \quad (1.22)$$

Once again, $\underline{H}$ has been partitioned in an additive manner, and depending on the value of $\underline{\nabla} \cdot \underline{T}$ many doublets $(\underline{T}, \Omega)$ may be found. The work by Carpenter^{1.36} in this field has highlighted two distinct advantages of this approach,

- (1) The separation of magnetic and electric conditions to Ω and $\underline{T}$ respectively.
- (2) The reduction of $\underline{T}$ to one or two components for two or three dimensional work respectively.

Although the concepts are certainly more intellectually demanding than the generalised $\underline{E}$ formulation, the initial work shows promise, and already three dimensional applications are appearing in the literature^{1.37,1.38}.

It is worth pausing here to say that although the eddy current problem may be described in terms of $\underline{B}$ or $\underline{J}$ the resulting descriptions are more clumsy and less advantageous than their associated vector alternatives.

Summarising it has been shown that any electromagnetic problem can be solved by the evaluation of either $\underline{H}$ or $\underline{E}$. It can also be seen from the literature that it has been generally acceptable to partition in an additive manner the base vector. The options where there seems to have been little concerted effort are those of

- (1) a multiplicative partition of the base vector
- (2) no partition of the base vector at all.

This sets the background picture from which the choice of field describing function was made for this thesis.

The next section discusses the reasons for the choice made.

1.4 THE CHOICE OF A FUNCTION

Essentially as has just been shown, there are two base vectors to choose between, namely $\underline{H}$ and $\underline{E}$; that choice having been made there then remains three possible ways of proceeding (combinations being ignored) which give a possible six alternatives to choose between. In order to be systematic each alternative will be dealt with. It must be stated though that the almost clinical divisions which have been implied are nothing more than the building bricks for any solution. Therefore, for example, if $\underline{H}$ was the vector chosen an additive and multiplicative principle might well be encompassed to give the final solution.

The vector $\underline{E}$ has essentially been used via the doublet $(\underline{\dot{A}}, \emptyset)$ in perhaps the majority of electromagnetic procedures which cope with two dimensional problems of the usual transverse magnetic type. The great advantage of not using $\underline{E}$ itself but dividing $\underline{E}$ is that the surface charges can be dealt with by $\emptyset$ the scalar potential leaving $\underline{A}$ to be associated with $\underline{B}$ and the magnetic interface conditions. This is more significant than many realise since a two dimensional transverse magnetic problem will more than likely be a three dimensional electric problem. Therefore the associating

of A with B allows A the freedom to act like a scalar. A three dimensional electric problem explicitly introduces the surface charge term into the formulation; the author believes that because of the surface charges the vector E would not be a wise choice of vector function.

It would therefore seem necessary if choosing E to make some suitable partition of the vector in either an additive or multiplicative way. There would seem to be no precedent nor advantage in a multiplicative division of E and therefore consideration will only be given to the more viable additive partitioning.

One disadvantage of the $(\dot{\underline{A}}, \emptyset)$ breakdown of E is the apparent immediate move from three to four functions, but as Carpenter^{1.29} has pointed out it is possible to reduce the components of A by one in certain circumstances; however the ensuing system is somewhat restricted in application. Another problem in three dimensional work is the coupling of components that may take place at magnetic interfaces. Also the electrostatic field cannot be ignored and must be dealt with either by $\emptyset$ or by inclusion in A by some suitable gauge transformation. In the nature of the approach though is perhaps its own built in salvation, since there are yet very many doublets $(\dot{\underline{A}}, \emptyset)$ depending on V.A which remain untried; amidst such a store of possibilities some will always want to prospect. The

present author though has preferred another way.

Before moving on to consider the vector $\underline{H}$ it must be said that there has been no intention to cast aspersions on those who are or who have laboured to enhance any of the approaches mentioned above or those on which the limelight is about to fall.

As with $\underline{E}$ there are two possible partitions of $\underline{H}$: the multiplicative partition, although having some precedent^{1.39} in using the magnetisation vector and susceptibility, would seem to have little to offer in a general three dimensional eddy current problem. As far as an additive partition of $\underline{H}$ is concerned the possibilities revolve around the $(\underline{T}, \Omega)$ doublet that has been mentioned earlier. From the work that has already been done it can be seen that this method is viable for certain two and three dimensional applications, although both areas still need a great deal of further investigation. The disadvantages of this method revolve around the fact that for a genuine three dimensional problem four functions are required. It may well be possible in theory to reduce this to three, but the complications involved may well make the method unmanageable. It must also be said that in order to choose the doublet $(\underline{T}, \Omega)$ to be the most cost effective (i.e. restricting $\underline{T}$ to conductors, reducing the number of components of $\underline{T}$) the understanding of the concepts

involved must be more than superficial or cursory. It is not obvious at all that this is a method for the engineer.

Three major considerations decided the author to investigate the possibilities of calculating the vector $\underline{H}$.

- (1) $\underline{H}$ has a maximum of three components.
- (2) The apparent simplicity of applying the magnetic interface conditions.
- (3) The disadvantages of using any other approach.

At this point in the project it was important to emphasize the advantages and to minimise the disadvantages. Therefore the drawbacks of the method will be left until their chronological position in the text, where their setting and their consequences can be more fully explained and hopefully the reader be more appreciative of the points being made. It would now seem profitable to move on to discuss the details of the project.

CHAPTER 2

A CHOICE OF TECHNIQUES

2.1 'INTERIOR' DOMAIN

The decision had been taken to use a differential technique in the 'interior' domain (see 1.3) since experience had shown that such techniques are best suited for regions where the material properties may be field dependent. However this still leaves the choice of methodology to be made and effectively this choice rests between the Finite Difference or Finite Element approaches.

2.1.1 Finite Differences

Finite Difference techniques have proved themselves over the years to be very effective in a variety of different applications. Initial-value problems, hyperbolic and parabolic systems are treated in some detail by Richtmyer and Morton^{2.1}, along with theoretical topics such as accuracy and convergence. Methods for elliptic systems as well as others may be found in Forsythe and Wasow^{2.2}.

In some ways finite difference techniques are in essence the same as finite element techniques; both

replace the infinite degrees of freedom of the continuum problem by a finite number of unknown parameters, both assemble relationships between these parameters to model the system and the resulting set of equations is then solved directly, iteratively or by a combination of the two. The similarities between the techniques require a closer look in order that some distinctions might be made and conclusions may well be problem dependent.

One merit of the finite difference approach is that the philosophy is comparatively easy to understand and to implement. Of course the more complex the problem the more this merit wanes and two distinct drawbacks of finite difference techniques become apparent.

- 1) Irregular meshes require careful attention.
- 2) Irregular boundaries require special treatment.

With regard to (1) above, as long as the boundaries of the problem coincide with coordinate surfaces solution is in principle straightforward, although the coordinate system may not be standard. Examples of non standard coordinate systems are polar and triangular. Salvadori and Baron^{2,3} give examples of these difference operators along with others and their associated orders of error. Certainly the triangular coordinate system may well commend itself as an alternative, in certain circumstances, to finite element techniques.

With reference to (2) above, when the boundaries do not coincide with coordinate surfaces special difference operators must be developed to handle the specific problem, and if boundary modifications occur a new operator is in general required. A series of problems therefore may have the same governing equation but different irregular boundaries and hence a series of specific operators would need to be developed. This inflexibility of finite difference techniques is a distinct disadvantage.

2.1.2 Finite Elements

The finite element methodology has grown in general acceptance over the past twenty years since the term was first coined^{2.4,2.5}. It is by far the most general methodology around today and is applied to both integral^{2.6} and differential equations^{2.7}, encompassing within its many forms the method of moments, the Galerkin weighting process and even finite difference techniques. As far as the 'interior' domain is concerned the interest centres around that sub-class of finite element techniques that are applicable to differential equations and for this section that is the context in which the term will be used.

Two classes of finite element approach warrant close attention, these are Variational techniques and the

Weighted Residual technique.

2.1.2.1 Variational Techniques

Variational techniques require the finding of a functional, Π , which is of integral form, and following Zienkiewicz^{2.8},

$$\Pi = \int_{\Omega} F(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \dots) d\Omega + \int_{\Gamma} F_1(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \dots) d\Gamma \quad (2.1)$$

(where F and F_1 are known operators) for which a stationary point is sought such that when

$$\delta \Pi = 0 \quad (2.2)$$

$$u = v(x, y, z, t) \quad (2.3)$$

where v is the required solution to the continuum problem for which the functional was formed. This now allows for the approximate solution of the continuum problem by the technique associated with Rayleigh^{2.9} and Ritz^{2.10}.

Assuming a trial function expansion for u ,

$$u = \sum_i N_i a_i \quad (2.4)$$

where a_i is the function value at node i and N_i is the basis function also at node i . When equation (2.4) is substituted into (2.1) and the variation is performed a set of equations for the unknown a_i 's is produced whose nature is dependent on the functional. For example if the functional is quadratic the equations are of linear form,

$$K\mathbf{a} = \mathbf{f} \quad (2.5)$$

where $\mathbf{a}$ is the vector of unknowns a_i and $\mathbf{f}$ the vector of right hand sides.

The method has been used greatly over the past fifteen years and has proved a useful tool. The one cloud that has overshadowed the method is 'the finding of a functional'. For some problems the physical understanding has led to a functional based on 'a total physical quantity' which is minimised. Perhaps the best known classically is the minimisation of the total potential energy in mechanical systems^{2.11}. Although there are 'natural variational principles' there are well defined problems for which there are not and in these cases 'contrived variational principles' have been constructed^{2.8}. These involve the introduction of Lagrange Multipliers^{2.12}, Penalty Functions^{2.13} and Least Squares^{2.14}; a clear expose of the ideas behind the progressive application of these techniques and some of the drawbacks is given in reference 2.8.

The main advantage of the Variational technique is that whenever a variational principle exists the stiffness matrix arising is symmetric^{2.8}. This is of great practical importance as the numerical solution of any problem thus formed can be obtained with a reduction in computer storage of about a half, or perhaps, looking at it another way much finer meshes can be

employed with all their attendant advantages.

2.1.2.2 Weighted Residual Technique

The Weighted Residual technique or 'Weak' formulation is based on an extremely simple premise, all that is required is a description of the problem

$$A(\underline{u}) = 0 \quad (2.6)$$

(where A is a differential or integral operator and $\underline{u}$ the function on which it acts) and an arbitrary set of functions $\underline{V}$, and then the assertion is made that,

$$\int_{\Omega} \underline{V}^T A(\underline{u}) d\Omega = 0 \quad (2.7)$$

where Ω is the domain of the problem; the naming of the technique can now be seen to be quite appropriate. It was first described by Crandall^{2.15} but more recently Finlayson^{2.16,2.17} has extensively covered the method.

Equation (2.7) forms the starting point for any Weighted Residual approach; for many physical problems though equation (2.7) requires such a high continuity of basis function^{2.8} that whenever practical an integration by parts is applied to give

$$\int_{\Omega} C(\underline{V}^T) F(\underline{u}) d\Omega + \int_{\Gamma} D(\underline{V}^T) E(\underline{u}) d\Gamma = 0 \quad (2.8)$$

where Γ is the boundaries of the problem space.

Equation (2.8) is termed the 'Weak' formulation since the continuity required by the function being sought is weakened. The boundary conditions may be expressed as,

$$B(\underline{u}) = 0 \quad \text{on } \Gamma \quad (2.9)$$

(where B is a differential operator and $\underline{u}$ the function on which it acts) and then choosing another arbitrary function set $\underline{W}$,

$$\int_{\Gamma} \underline{W}^T B(\underline{u}) d\Gamma = 0 \quad (2.10)$$

The satisfaction of equations (2.7) and (2.10) is sufficient to solve the problem and these may be brought together to express the solution of the problem as the solution of,

$$\int_{\Omega} \underline{V}^T A(\underline{u}) d\Omega + \int_{\Gamma} \underline{W}^T B(\underline{u}) d\Gamma = 0 \quad (2.11)$$

The function sets $\underline{V}$ and $\underline{W}$ are at the discretion of the analyst but some common choices give recognisable methods,

- 1) $W_j = \delta_j^i$ the Kronecker Delta function -
point collocation
 $i = j \quad \delta_j^i = 1 \quad i \neq j \quad \delta_j^i = 0$
- 2) $W_j = N_j$ the basis function for node j -
Galerkin method
- 3) $W_j = x^j$ - method of moments

The Weighted Residual technique as for the Variational techniques gives rise to a set of equations for the unknown parameters a_i , of the form (2.5), which can be solved in a number of different ways.

The two major advantages of the Weighted Residual technique are firstly its ease of application and secondly its wide range of applicability. The main disadvantage

is the form of the stiffness matrix which in general will not be symmetric.

2.1.3 Conclusions

Comparisons between the finite difference and finite element techniques led the author to prefer the finite element technique for two reasons. Firstly, and from a general standpoint, the finite element technique is more flexible than the finite difference technique. Secondly, and more specifically, the hybrid approach would require the integral technique to be linked with the differential technique and the finite element approach promised a more natural combining of the two generally different approaches.

Making a choice between the two possible finite element methodologies proved more difficult since the one major advantage of the Variational technique, that of deriving a symmetric matrix, would probably be lost when the integral and differential regions were brought together. However because of the greater flexibility and ease of application of the Weighted Residual technique this was the approach chosen by the author.

2.2 'EXTERIOR' DOMAIN

It had been decided to employ an integral technique in the 'exterior' domain (see 1.3) in order to dispose of the necessity of discretising air regions.

Three functions $\underline{H}$, $\underline{A}$, and Ω were considered as possibilities to represent the 'exterior' field and irrespective of the function chosen the governing equation for the 'exterior' field would be Laplacian; therefore the integral techniques examined were limited to those which are applicable to Laplacian fields.

Consideration of the consistency of the 'interior' and 'exterior' approximations will be left until later as the characteristics of the four techniques are our present main concern.

2.2.1 Recursive Technique (ballooning)

Although the decision had been taken to use an integral technique for the 'exterior' region it was pointed out to the author that the main reason for this was to remove the necessity of discretising what might well be a large region, and consequently any technique which would allow this same freedom should be considered. Therefore although the approach of this section is not an integral technique it is examined because it

essentially exhibits this same ability. The approach examined in this section has become known, rather affectionately, by its exponents as 'ballooning'; this term is descriptive of its mode of operation rather than any comment on its theoretical derivation. A less enigmatic and more descriptive title might be 'a finite element recursive technique'.

The recursive technique is certainly attractive for a limited class of problem but certain questions still remain unanswered which hinder its general acceptance. Before moving on to discuss the relative merits and demerits of the approach there is a need that the theoretical formulation be stated more carefully since the first two published papers^{2.18,2.19} are at best rather misleading and at worst incorrect.

Fig.2.1 represents a meshed region, A, and an infinite domain, B, in which the technique will be applied. For the sake of argument the field equation for region B is Laplaces equation for ϕ , a scalar potential.

Fig.2.2 is similar to that used in reference 2.18 and shows a point P from which radial lines are drawn that pass through nodes lying on $\partial\Omega^1$, the perimeter of region A, defining intermediary nodes on all other perimeters $\partial\Omega_n^i$ $i = 1,2$. Such a point is termed a star point and it must be chosen such that any radial

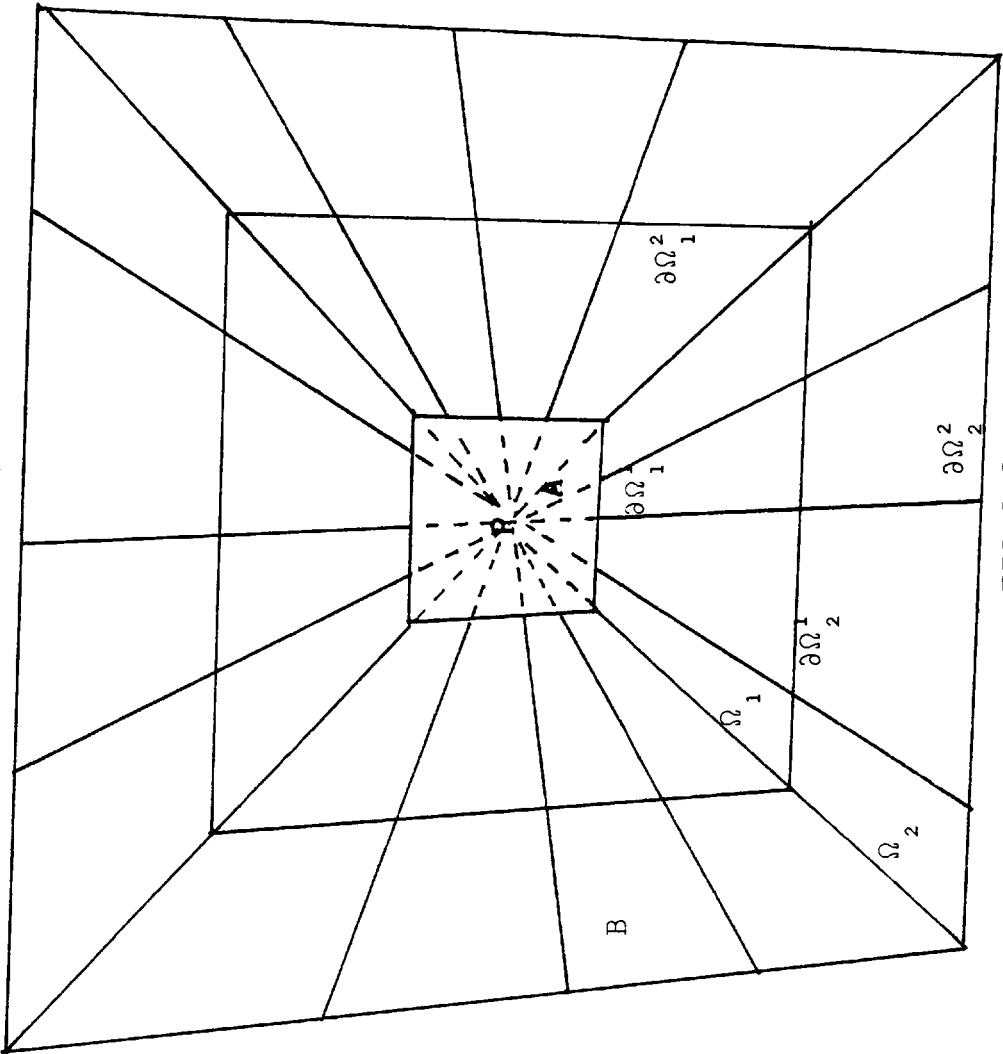


FIG.2.2

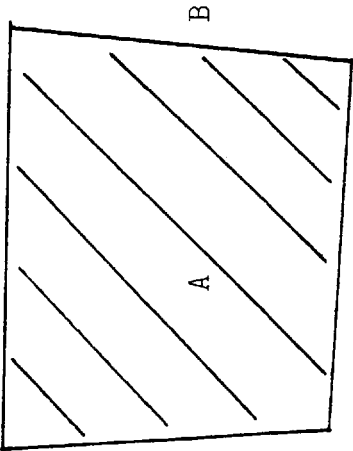


FIG.2.1

line emanating from it cuts $\partial\Omega_1^1$ once only. Fig.2.2 also shows two annuli Ω_1 and Ω_2 which are constructed such that their defining lines are parallel to those defining $\partial\Omega_1^1$. If the nodes on $\partial\Omega_1^1$ are mapped onto those of $\partial\Omega_1^2$ by K then those on $\partial\Omega_2^1$ are likewise mapped onto those of $\partial\Omega_2^2$ by K . This means that the nodes on $\partial\Omega_2^2$ are related to those on $\partial\Omega_1^1$ by K^2 .

The Weighted Residual approximation for Laplaces equation in Ω_1 and Ω_2 , after an integration by parts gives,

$$\int_{\Omega_1} \nabla W_1 \cdot \nabla \phi \, d\Omega_1 - \int_{\Gamma_1 = \partial\Omega_1^1} W_1 \frac{\partial \phi}{\partial n} \, d\Gamma_1 - \int_{\Gamma_2 = \partial\Omega_1^2} W_1 \frac{\partial \phi}{\partial n} \, d\Gamma_2 = 0 \quad (2.12)$$

$$\int_{\Omega_2} \nabla W_2 \cdot \nabla \phi \, d\Omega_2 - \int_{\Gamma_1 = \partial\Omega_2^1} W_2 \frac{\partial \phi}{\partial n} \, d\Gamma_1 - \int_{\Gamma_2 = \partial\Omega_2^2} W_2 \frac{\partial \phi}{\partial n} \, d\Gamma_2 = 0 \quad (2.13)$$

where W_1 and W_2 are weighting functions.

Using similar notation to that in reference 2.18 equations (2.12) and (2.13) may be written in matrix form as follows, (assuming a Galerkin type evaluation of the integrals).

$$\begin{bmatrix} S_{11}^1 & S_{12}^1 & S_{13}^1 \\ S_{21}^1 & S_{22}^1 & S_{23}^1 \\ S_{31}^1 & S_{32}^1 & S_{33}^1 \end{bmatrix} \begin{bmatrix} \phi_1^1 \\ \phi_2^1 \\ \phi_i^1 \end{bmatrix} = \begin{bmatrix} \underline{I}_1^1 \\ \underline{I}_2^1 \\ \underline{0} \end{bmatrix} \quad (2.14)$$

$$\begin{bmatrix} S_{11}^2 & S_{12}^2 & S_{13}^2 \\ S_{21}^2 & S_{22}^2 & S_{23}^2 \\ S_{31}^2 & S_{32}^2 & S_{33}^2 \end{bmatrix} \begin{bmatrix} \phi_1^2 \\ \phi_2^2 \\ \phi_i^2 \end{bmatrix} = \begin{bmatrix} \underline{I}_1^2 \\ \underline{I}_2^2 \\ 0 \end{bmatrix} \quad (2.15)$$

where ϕ_1^1 are the potentials on $\partial\Omega_1^1$, ϕ_2^1 the potentials on $\partial\Omega_2^1$, ϕ_i^1 the nodes in the interior of region 1, $\underline{I}_1^1$ the contributions from the boundary integral over $\partial\Omega_1^1$, $\underline{I}_2^1$ the contributions from the boundary integral over $\partial\Omega_2^1$ and similarly for the notation in (2.15). From (2.14) and (2.15) the following points should be noted,

$$\phi_2^1 \equiv \phi_1^2 \quad (2.16)$$

and if

$$\underline{W}_1 = \underline{W}_2 \quad (2.17)$$

then

$$S_{kj}^1 = S_{kj}^2 \quad k, j = 1, 2, 3, \quad (2.18)$$

and

$$\underline{I}_2^1 = -\underline{I}_1^2 \quad (2.19)$$

since the outward normals for the two regions are in opposite directions.

In reference 2.18 the counterparts of equations (2.14) and (2.15) have zero right hand sides, this is because the boundary integrals in (2.12) and (2.13) have been omitted from the formulation. It may be thought by some that it is pedantic to mention such an omission since their inclusion makes no material difference to the applications outlined in both references. However the author believes that their inclusion is mathematically

correct, illuminates the method and protects the unwary from danger.

Equations (2.14) and (2.15) may be combined as described in reference 2.18 to produce the building block equation of the approach,

$$\begin{bmatrix} S_{11}^j & S_{12}^j \\ S_{21}^j & S_{22}^j \end{bmatrix} \begin{bmatrix} \underline{\phi}_1^1 \\ \underline{\phi}_2^j \end{bmatrix} = \begin{bmatrix} \underline{I}_1^1 \\ \underline{I}_2^j \end{bmatrix} \quad (2.20)$$

where j denotes the number of such combinations. The next step is to combine equation (2.20) for the two adjacent, geometrically similar annuli j and $j+1$. As outlined in reference 2.18 the combination is achieved by addition thus,

$$\begin{bmatrix} S_{11}^j & S_{12}^j & 0 \\ S_{21}^j & S_{22}^j + S_{11}^{j+1} & S_{12}^{j+1} \\ 0 & S_{21}^{j+1} & S_{22}^{j+1} \end{bmatrix} \begin{bmatrix} \underline{\phi}_1^1 \\ \underline{\phi}_2^j \\ \underline{\phi}_2^{j+1} \end{bmatrix} = \begin{bmatrix} \underline{I}_1^1 \\ \underline{I}_2^j + \underline{I}_1^{j+1} \\ \underline{I}_2^{j+1} \end{bmatrix} \quad (2.21)$$

where,

$$\underline{\phi}_2^j \equiv \underline{\phi}_1^{j+1} \quad (2.22)$$

If the further assumption is made that the weighting functions are equivalent for both annuli

$$\underline{W}^j = \underline{W}^{j+1} \quad (2.23)$$

then,

$$S_{k\ell}^j = S_{k\ell}^{j+1} \quad k, \ell = 1, 2, 3, \quad (2.24)$$

and,

$$\underline{I}_2^j = -\underline{I}_1^{j+1} \quad (2.25)$$

since the outward normals for the two regions are in opposite directions. It is precisely at this point that the inclusion of the right hand side terms illuminates the method. In both earlier references the reader is led through the technique with no mention as to why the counterpart equations to (2.20) are added together. Since in both references the counterpart of equation (2.20) appears with zero right hand side the possibility of subtraction would seem to exist. However from equation (2.25) it can be seen that by adding the equations to form (2.21) the following relationship arises,

$$S_{21}^j \underline{\phi}_1^1 + (S_{22}^j + S_{11}^{j+1}) \underline{\phi}_2^j + S_{12}^j \underline{\phi}_2^{j+1} = \underline{0} \quad (2.26)$$

which permits in turn equation (2.21) to be reduced to the form (2.20).

There are two pitfalls that the unwary could easily stumble into if the right hand side terms are ignored and it is the intention to highlight these as part justification for this expose.

When the recursion is adjudged to be complete, after $j=k$ combinations, the assumption is made that,

$$S_{12}^k \underline{\phi}_2^k = \underline{0} \quad (2.27)$$

and this leaves the equation, (correctly expressed)

$$S_{11}^k \underline{\phi}_1^1 = \underline{I}_1^1 \quad (2.28)$$

since all intermediate $\underline{\phi}$'s and $\underline{I}$'s have been eliminated.

The Weighted Residual form of the governing equation for region A, with reference to fig.2.2, may be written,

$$\begin{bmatrix} S_{11}^A & S_{12}^A \\ S_{21}^A & S_{22}^A \end{bmatrix} \begin{bmatrix} \underline{Q}^A \\ \underline{\phi}_A \end{bmatrix} = \begin{bmatrix} \underline{R} \\ \underline{I}_A \end{bmatrix} \quad (2.29)$$

where,

$\underline{Q}^A$ represents the unknown parameters for region A

$\underline{\phi}_A$ represents the scalar ϕ interface terms, as seen from region A

$\underline{R}$ the vector of sources arising from the interior of region A

$\underline{I}_A$ the vector of sources arising from the interfaces, as seen from region A.

Equations (2.28) and (2.29) represent the total system, but it can be seen that their combination is in general very much problem dependent. To be more specific, interface conditions will be required for ϕ and, more surprisingly for the unwary, also for $\frac{\partial \phi}{\partial n}$; it is only when the interior and exterior normal derivatives of ϕ have

the following relationship,

$$\frac{\partial \phi^I}{\partial n} = - \frac{\partial \phi^E}{\partial n} \quad (2.30)$$

that $\underline{I}_1^1$ can be ignored as a natural boundary condition.

The other pitfall is in fact one into which the writers of the original paper may well have fallen. The case to consider is that of parallel bus bars carrying opposite currents, of which only one quadrant needs to be treated (see fig.2.3). Using similar notation to that in reference 2.18 the interface for the 'super' element is $I_1 I_2 I_3 I_4 = I$ and for the first annulus, Ω_1 , the process would give in terms of $\underline{A}$, the magnetic vector potential,

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} \underline{A}_Z^1 \\ \underline{A}_Z^2 \\ \underline{A}_Z^i \end{bmatrix} = \begin{bmatrix} \underline{I}^1 \\ \underline{I}^2 \\ \underline{I}^i \end{bmatrix} \quad (2.31)$$

Equation (2.31) differs from equations (2.14) and (2.15) by a term, $\underline{I}^i$, in the right hand side vector and this is produced because there is an integral, D , over $\Gamma_1 = I_1 I_1^1$ and $\Gamma_2 = I_4 I_4^1$.

$$D = \int_{\Gamma_1} W_1 \frac{\partial A}{\partial n} dz d\Gamma_1 + \int_{\Gamma_2} W_1 \frac{\partial A}{\partial n} dz d\Gamma_2 \quad (2.32)$$

From the conditions of the problem equation (2.32) may

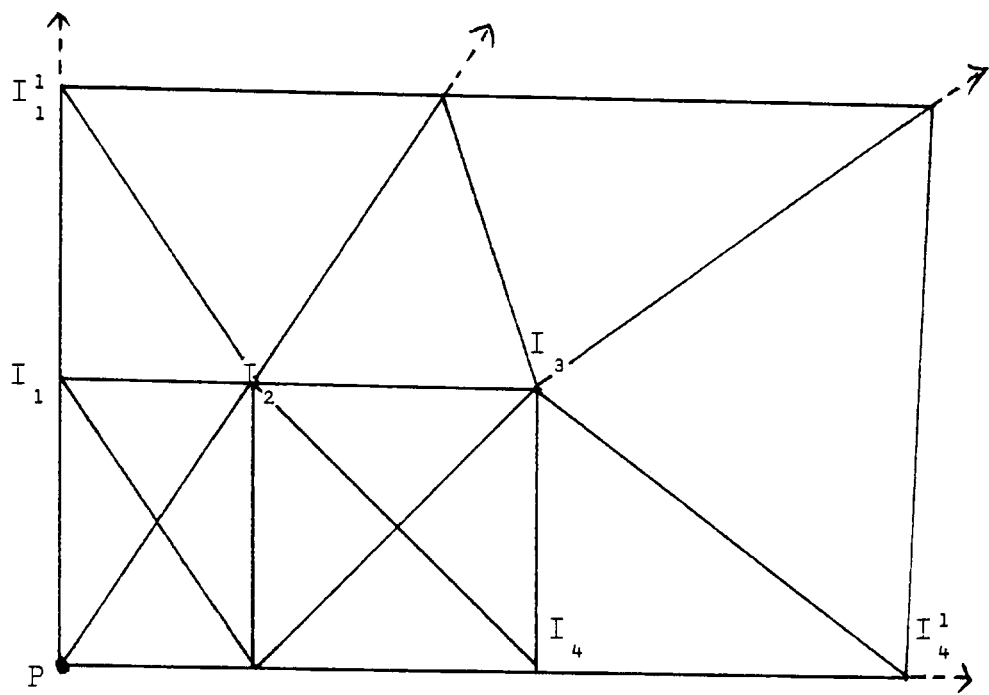


FIG.2.3

be reduced, since along $I_4 I_4^1$,

$$\frac{\partial A}{\partial n} z = 0 \quad (2.33)$$

the other integral comprising (2.32) is in general certainly not zero, since the condition along $I_1 I_1^1$ is

$$A_z = 0 \quad (2.34)$$

Elements of D will occur in I^1 and I^2 and so irrespective of whether linear or higher order elements are being considered the method as described breaks down, unless explicitly the condition (2.34) is incorporated in each annulus. The force of this point is that in neglecting the right hand side terms this feature of the formulation would not occur to the user of the method. In the double bus bar problem to ignore the integrals comprising D is in practice no different to assuming that,

$$\frac{\partial A}{\partial n} z = 0 \quad (2.35)$$

from the point I_1 outwards.

The merits of the method are firstly its simplicity, secondly it is consistent with so termed differential finite element interior approaches, thirdly the geometric nature of the annuli means that successive S matrices are equivalent and therefore although the coordinates bounding the annuli will be large there is no consequent increase in the rounding error of the representation.

The demerits of the approach may be divided into two classes, firstly unanswered questions and secondly justifiable reservations. The unanswered questions are,

- 1) Is the method viable in three dimensions?
- 2) How is the optimum, or even a reasonable, star point to be chosen?

The justifiable reservations are,

- 1) Is it worth implementing a technique that will not cope with all air regions? e.g. closed air regions as in the end region of an electrical machine.
- 2) Having modelled the field in the air regions in this way how is the field to be reconstructed from the results if required?

The implementation of the 'ballooning' technique would of necessity, due to the interface conditions for $\frac{\partial \underline{H}}{\partial n}$, require an approach that in general (including fairly simple problems) solved completely for $\underline{H}$ as distinct from an approach that solved the components of $\underline{H}$ independently. The advantages of a technique able to solve for individual components of $\underline{H}$ are obvious and since such an approach had not been discounted the recursive technique seemed to have little to offer.

2.2.2 The Simple Layer Potential

It will be helpful to define what is meant by a simple layer potential before moving on to discuss its possible use.

A unit simple source at point $\underline{r}_1$ generates the Newtonian potential,

$$G(\underline{r}, \underline{r}_1) = |\underline{r} - \underline{r}_1|^{-1} \quad (2.36)$$

in three dimensions, or,

$$G(\underline{r}, \underline{r}_1) = \ln(|\underline{r} - \underline{r}_1|) \quad (2.37)$$

in two dimensions, at the field point $\underline{r}$. Equations (2.36) and (2.37) satisfy Laplace's equation everywhere except at the source point $\underline{r}_1$. More generally G satisfies Poisson's equation

$$\nabla^2 G(\underline{r}, \underline{r}_1) = \alpha \delta(\underline{r} - \underline{r}_1) \quad (2.38)$$

where

$$\alpha = \begin{cases} 2\pi & \text{in two dimensions} \\ -4\pi & \text{in three dimensions} \end{cases} \quad (2.39)$$

and δ is Dirac's^{2.20} delta function. The simple layer potential is now produced by assuming a continuous surface distribution of simple sources of density $\sigma(\underline{r}_1)$ such that in a region, R ,

$$\phi(\underline{r}) = \int_{\Gamma = \partial R} G(\underline{r}, \underline{r}_1) \sigma(\underline{r}_1) d\Gamma \quad \underline{r} \in R + \partial R \quad (2.40)$$

where Γ is the surface of sources. Equation (2.40) satisfies Laplace's equation everywhere except on Γ and this allows the modelling of harmonic functions^{2.21}.

For the application in hand there were three possible functions that could be represented by a similar relationship to (2.40), $\underline{A}$ the magnetic vector potential, Ω a scalar potential defined such that

$$\underline{H} = - \nabla \Omega \quad (2.41)$$

and $\underline{H}$ the magnetic field strength. The relationship between $\underline{H}$ and $\underline{A}$ and between $\underline{H}$ and Ω is differential in nature and therefore if either $\underline{A}$ or Ω were represented in similar fashion to (2.40) difficulties would arise in deriving $\underline{H}$ at corner points because of the inherent singularity in G . It is worth noting at this point that as long as Γ is a closed surface the singularity in G is integrable, and it is only when G is differentiated with respect to the field point and then only at corner points that the singularity produces problems. Since both $\underline{A}$ and Ω give problems if represented in similar fashion to (2.40) $\underline{H}$ was the chosen function for consideration.

The differential interior approximation for any component of $\underline{H}$ will in general produce direct reference to the other components not explicitly under consideration. For each component of $\underline{H}$ a different source density is required and in order to complete the formulation

relationships need to be found between these source densities.

In transverse two dimensional magnetic problems the discrete interior equations for H_x make explicit reference to boundary H_y , and for representations of these functions similar to (2.40) two source densities are required, σ_x and σ_y . The link between these source densities can be established by use of the equations in the exterior

$$\underline{\nabla} \cdot \underline{H} = 0 \quad (2.42)$$

and

$$\underline{\nabla} \times \underline{H} = 0 \quad (2.43)$$

However in two dimensions these latter two equations reduce to

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0 \quad (2.44)$$

and

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \quad (2.45)$$

respectively. Therefore if H_x is expressed as,

$$H_x = \int_{\Gamma} G(\underline{r}, \underline{r}_1) \sigma_x \, d\Gamma \quad (2.46)$$

where Γ is the bounding surface of the exterior region, equations (2.44) and (2.45) can be utilised to give,

$$H_y = \int_{\Gamma} F(\underline{r}, \underline{r}_1) \sigma_x \, d\Gamma \quad (2.47)$$

In two dimensions

$$G(\underline{r}, \underline{r}_1) = \ln(|\underline{r} - \underline{r}_1|) \quad (2.48)$$

and,

$$F(\underline{r}, \underline{r}_1) = \tan^{-1} \left(\frac{y-y_1}{x-x_1} \right) \pm \frac{\pi}{4} \quad (2.49)$$

where the derivation of (2.49) is given in appendix A.

The function (2.49) is particularly awkward to deal with because of the plus and minus sign which arises because of two regimes

$$\tan^{-1} \left(\frac{y-y_1}{x-x_1} \right) + \tan^{-1} \left(\frac{x-x_1}{y-y_1} \right) = \pm \frac{\pi}{2} \quad (2.50)$$

As well as the strange functional relationship (2.49), reference 2.21 gives two reasons for avoiding a formulation based on equation (2.40),

- 1) σ becomes infinite in magnitude whenever $\frac{\partial \phi}{\partial n}$ becomes infinite e.g. at a corner.
- 2) σ may become infinite whilst ϕ and $\frac{\partial \phi}{\partial n}$ remain finite.

For these reasons this formulation was abandoned at this stage.

2.2.3 The Double Layer Potential

Once again it will be helpful to define what is meant by a double layer potential before moving on to discuss its possible use.

The double layer potential is constructed in much the

same way as the simple layer potential except that $G(\underline{r}, \underline{r}_1)$ is replaced by $G'(\underline{r}, \underline{r}_1)$ the normal derivative of the former where the outward normal from the region of interest is that considered; the potential in a region R is written as,

$$\psi(\underline{r}) = \int_{\Gamma = \partial R} G'(\underline{r}, \underline{r}_1) \mu(\underline{r}_1) d\Gamma \quad \underline{r} \in R \quad (2.51)$$

where $\mu(\underline{r}_1)$ is the density of the sources. Comparison of equations (2.40) and (2.51) shows that the former is valid for $\underline{r} \in R + \partial R$ whereas the latter is only true for $\underline{r} \in R$. When $\underline{r}$ is brought onto the boundary of $R, \partial R$ for equation (2.51) the integral jumps by $\alpha\mu(\underline{r})$ so that the boundary relationship becomes,

$$\psi(\underline{r}) = \int_{\Gamma = \partial R} G'(\underline{r}, \underline{r}_1) \mu(\underline{r}_1) d\Gamma + \alpha\mu(\underline{r}) \quad \underline{r} \in \partial R \quad (2.52)$$

where

$$\alpha = \begin{cases} 2\pi & \text{in three dimensions} \\ \pi & \text{in two dimensions} \end{cases} \quad (2.53)$$

Therefore equations (2.51) and (2.52) represent the double layer potential and this too can be used to represent harmonic functions^{2.21}.

The comments made in section 2.2.2 concerning the choice of function are just as relevant here since differentiation of equation (2.52) with respect to the field

point will produce difficulties with the inherent singularity in G' . Moreover differentiating (2.52) introduces derivatives of the source density. Therefore for these reasons the function considered was $\underline{H}$.

Consideration of this representation was abandoned at about this stage since reference 2.21 shows that equation (2.51) has an asymptotic behaviour, as $|\underline{r} - \underline{r}_1|$ becomes infinite, of $O(|\underline{r} - \underline{r}_1|^{-2})$. This is reduced to $O(|\underline{r} - \underline{r}_1|^{-1})$ by writing,

$$\psi(\underline{r}) = \int_{\Gamma = \partial R} G'(\underline{r}, \underline{r}_1) \mu(\underline{r}_1) d\Gamma + \frac{c}{|\underline{r}|} \quad \underline{r} \in R \quad (2.54)$$

where c is a constant chosen so that,

$$\int_{\Gamma = \partial R} \lambda(\underline{r}) \psi(\underline{r}) d\Gamma - c \int_{\Gamma = \partial R} \frac{\lambda(\underline{r})}{|\underline{r}|} d\Gamma = 0 \quad \underline{r} \in R \quad (2.55)$$

where $\lambda(\underline{r})$ satisfies

$$\int_{\Gamma = \partial R} G(\underline{r}, \underline{r}_1) \lambda(\underline{r}) d\Gamma = 1 \quad \underline{r} \in \partial R \quad (2.56)$$

However the amount of extra computation that the above equations would introduce was adjudged to be prohibitive and therefore this formulation was investigated no further.

2.2.4 The Greens Formula

Once again a short introduction to what is meant by Greens formula will be helpful before its usefulness is discussed.

The Greens formula may be constructed from the identity

$$-\psi \nabla^2 \phi = \underline{\nabla} \psi \cdot \underline{\nabla} \phi - \underline{\nabla} \cdot (\psi \underline{\nabla} \phi) \quad (2.57)$$

Since (2.57) is an identity ψ and ϕ can be interchanged and this leads to

$$-\phi \nabla^2 \psi = \underline{\nabla} \phi \cdot \underline{\nabla} \psi - \underline{\nabla} \cdot (\phi \underline{\nabla} \psi) \quad (2.58)$$

and now if (2.57) is subtracted from (2.58) this gives rise to

$$(\psi \nabla^2 \phi - \phi \nabla^2 \psi) = (\underline{\nabla} \cdot (\psi \underline{\nabla} \phi) - \underline{\nabla} \cdot (\phi \underline{\nabla} \psi)) \quad (2.59)$$

The identity (2.59) is integrated over the region of interest, R , and by an application of the Divergence Theorem^{2.22} to the right hand side one form of the Greens integral is arrived at,

$$\int_R (\psi \nabla^2 \phi - \phi \nabla^2 \psi) dR = \int_{\Gamma} (\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}) d\Gamma \quad (2.60)$$

where Γ denotes the boundary of the region R . It is precisely from this point that reference 2.22 derives the formula that has primarily become known as Greens formula,

$$\begin{aligned} \alpha \phi(\underline{r}) = & \int_{\Gamma} \phi(\underline{r}_1) G'(\underline{r}, \underline{r}_1) d\Gamma \\ & - \int_{\Gamma} \frac{\partial \phi}{\partial n}(\underline{r}_1) G(\underline{r}, \underline{r}_1) d\Gamma \end{aligned} \quad (2.61)$$

where

$$\alpha = \begin{cases} -4\pi & \text{for three dimensions} \\ 2\pi & \text{for two dimensions} \end{cases} \quad (2.62)$$

and

$$G'(\underline{r}, \underline{r}_1) = \frac{\partial G}{\partial n}(\underline{r}, \underline{r}_1) \quad (2.63)$$

where $\frac{\partial}{\partial n}$ is the outward normal and where in (2.60) ϕ has been regarded as a harmonic function and ψ as $G(\underline{r}, \underline{r}_1)$.

Of the three functions under consideration once again only H appeared to be viable since differentiation of (2.61) produces much the same problems with the singularity as have been mentioned earlier and thus H was chosen as the function to be considered.

Equation (2.61) seemed to offer a straight forward approach to the exterior formulation and because of the $\frac{\partial H}{\partial n}$ term that would appear seemed naturally designed to link with the interior approximation.

2.2.5 Conclusion

Of the four techniques briefly considered above only two appear to offer a reasonable approach that will link with the interior formulation whilst the other two exclude themselves because of the problems they raise.

The simple layer potential has three attendant difficulties as far as the formulation under consideration is concerned,

- 1) the source density, σ , may well become infinite at corners
- 2) the source density, σ , may well become infinite when both ϕ and $\frac{\partial \phi}{\partial n}$ are finite
- 3) the strange nature of (2.49) requires careful treatment.

The double layer potential has one major difficulty, that of the functions behaviour at infinity, and although it is possible to circumvent this problem the extra calculation involved seemed prohibitive.

Having disposed of the simple and double layer potential formulations this left only the recursive technique and the Greens formula to choose between, and the author chose the Greens formula for two reasons,

- 1) the Greens formula permits the possibility, quite generally, of solving for individual components of $\underline{H}$.
- 2) The field in the exterior region is easily constructed from the results.

CHAPTER 3INITIAL SOLUTIONS3.1 MAXWELLS EQUATIONS

The mathematical formulation of electromagnetic fields is expressed by the four differential equations,

$$\underline{\nabla} \times \underline{H} = \underline{J} + \underline{\dot{D}} \quad (3.1)$$

$$\underline{\nabla} \times \underline{E} = -\underline{\dot{B}} \quad (3.2)$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad (3.3)$$

$$\underline{\nabla} \cdot \underline{D} = \rho \quad (3.4)$$

where the dot indicates differentiation with respect to time, which are attributed to Maxwell because they represent his synthesis of all previous developments which he achieved by the introduction of the term $\underline{\dot{D}}$ ^{3.1}. In the above equations the symbols represent the following quantities,

$\underline{H}$: the magnetic field intensity

$\underline{J}$: the current density

$\underline{D}$: the electric displacement vector

$\underline{E}$: the electric field intensity

$\underline{B}$: the magnetic flux density.

Equations (3.1) to (3.4) are far too complex to solve without first making some justifiable assumptions. For the purpose of this thesis displacement current is ignored because only low frequency situations are

considered, and the relationships between the vectors $\underline{B}$ and $\underline{H}$, $\underline{J}$ and $\underline{E}$ are taken to be

$$\underline{B} = \mu_0 \mu_r \underline{H} \quad (3.5)$$

$$\underline{J} = \sigma \underline{E} \quad (3.6)$$

where the quantities as yet undefined are,

μ_0 : the permeability of free space

μ_r : the relative permeability

σ : the electrical conductivity.

3.2 THE GOVERNING EQUATION FOR $\underline{H}$ IN AN ISOTROPIC MAGNETOSTATIC REGION WITH CONSTANT PERMEABILITY

The problem examined in this chapter is an isotropic magnetostatic situation with constant permeability throughout the solution domain and so Maxwells equations reduce to,

$$\underline{\nabla} \times \underline{H} = \underline{J} \quad (3.7)$$

$$\underline{\nabla} \times \underline{E} = \underline{0} \quad (3.8)$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad (3.9)$$

Since it has been assumed that the permeability is constant, equation (3.9) reduces to

$$\underline{\nabla} \cdot \underline{H} = 0 \quad (3.10)$$

(In general equation (3.10) holds within regions of constant permeability but not at the material interface. In fact (3.10) only holds at material interfaces where μ_r is continuous across the interface.) Combining equations (3.6), (3.7) and (3.8) produces,

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{H}) = \sigma (\underline{\nabla} \times \underline{E}) = \underline{0} \quad (3.11)$$

which must be seen as holding away from material interfaces. Equations (3.10) and (3.11) are the equations that $\underline{H}$ must satisfy in such a region as set out above, but it is more useful to transform equation (3.11) so that (3.10) can be combined directly to produce a single vector governing equation for $\underline{H}$.

The following vector identity is used,

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{H}) \equiv \underline{\nabla}(\underline{\nabla} \cdot \underline{H}) - \nabla^2 \underline{H} \quad (3.12)$$

It now becomes clear that a combination of (3.10), (3.11) and (3.12) produces the vector Laplacian as the governing equation for $\underline{H}$,

$$\nabla^2 \underline{H} = \underline{0} \quad (3.13)$$

3.2.1 Observations About the Governing Equation for $\underline{H}$

Equation (3.13) immediately raises two important questions that need to be dealt with,

- 1) where do the source terms arise from?
- 2) does a solution of equation (3.13) of necessity imply (3.10)?

In order to answer the above questions and perhaps even to make sense of them consider an isolated infinitely long bus bar as given in fig.3.1 where the rectangle

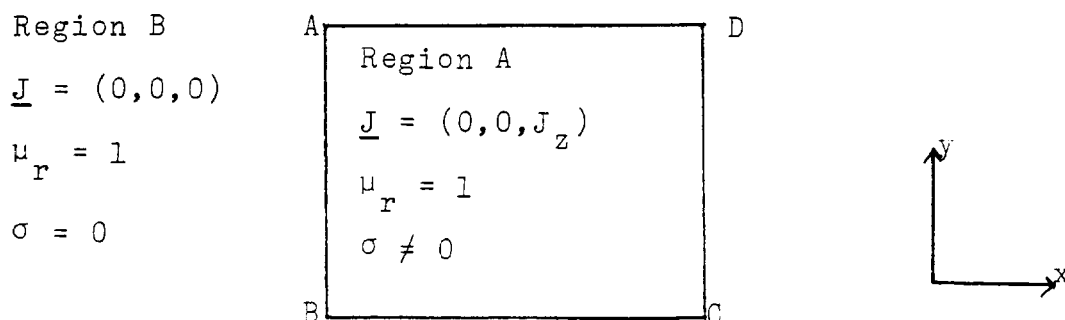


FIG.3.1

ABCD represents a cross-section of the conductor in the x, y plane and where J_z the current density is given, and region B is infinite in extent.

The force of the first question can now be seen since equation (3.13) is the governing equation for $\underline{H}$ in regions A and B and to the unwary it may well seem that $\underline{H}$ is identically zero since as yet no source terms have been produced. This is in fact caused by hitherto neglecting the conditions at the material interface. As was mentioned earlier, in general, equations (3.10) and (3.11) and hence (3.13) are valid only away from material interfaces since at these surfaces discontinuities in σ and μ_r may well occur. In the situation depicted in fig.3.1 at the material interface ABCD there is a discontinuity in σ although not in μ_r . Therefore the equation (3.10) is valid for the interface ABCD but not equation (3.11). In constructing an alternative equation to (3.11), for the interface ABCD, equations (3.6), (3.7) and (3.8) are once again combined, but this time give,

$$\begin{aligned}\underline{\nabla} \times (\underline{\nabla} \times \underline{H}) &= (\underline{\nabla} \times \sigma \underline{E}) = \underline{\nabla} \sigma \times \underline{E} + \sigma \underline{\nabla} \times \underline{E} \\ &= \underline{\nabla} \sigma \times \underline{E}\end{aligned}\quad (3.14)$$

Therefore the two equations for the interface are (3.10) and (3.14) and once again these may be combined with equation (3.12) to give a single vector governing equation for $\underline{H}$,

$$\nabla^2 \underline{H} = - \underline{\nabla} \sigma \times \underline{E} \quad (3.15)$$

Moreover it may be considered that (3.15) is the governing equation for $\underline{H}$ for any region in fig.3.1 since away from the interface,

$$\underline{\nabla} \sigma = \underline{0} \quad (3.16)$$

giving (3.13). Equation (3.15) thus provides the source terms that were missing but also points to how carefully the material interfaces must be treated with regard to $\underline{H}$. In fact it is precisely the conditions at material interfaces that generate $\underline{H}$. Equation (3.15) highlights the dependence of $\underline{H}$ on $\underline{J}$ and implicitly the solution of (3.7), but what about (3.10)?

The second question is best approached by assuming a solution, $\underline{H}'$, of (3.15), the governing equation for $\underline{H}$, for the situation set out in fig.3.1, such that

$$\underline{\nabla} \cdot \underline{H}' = f \neq 0 \quad \text{in regions A and B} \quad (3.17)$$

where f is an unknown function. However since (3.7) and (3.8) are satisfied by $\underline{H}'$, within both regions A and B

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{H}') = 0 \quad (3.18)$$

and since (3.15) is also satisfied,

$$\underline{\nabla} (\underline{\nabla} \cdot \underline{H}') = \underline{0} \quad \text{in regions A and B} \quad (3.19)$$

Equation (3.19) gives rise to

$$f = k \quad \text{a constant} \quad (3.20)$$

which may well be different in each region. The constant, k , is now trivially zero as long as $\underline{H}'$ satisfies the Gauss divergence theorem,^{3.2}

$$\oint_{\Gamma = ABCD} \underline{H}' \cdot \underline{n} \, d\Gamma = 0 \quad (3.21)$$

3.2.2 Material Surfaces

As was shown in section 3.2.1 the conditions on the material surfaces define $\underline{H}$, but the equation derived for $\underline{H}$ to satisfy on such surfaces (3.15) is of limited use in practice since $\nabla\sigma$ becomes infinite. However another line of investigation shows how to implement the surface conditions. Consider the surface CD in fig.3.1, to the left of this surface in region A

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \quad (3.22)$$

but to the right of this surface in region B

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \quad (3.23)$$

For simplicity the surface chosen is in the z-y plane and so the standard condition for H_x on CD is

$$\mu_A H_x^A = \mu_B H_x^B \quad (3.24)$$

where,

$$\mu_A = \mu_B = \mu_r = 1 \quad (3.25)$$

such that the A and B indicate the region involved. Since equation (3.24) holds along CD and the relative permeabilities of both regions are unity,

$$\frac{\partial H_x^A}{\partial y} = \frac{\partial H_x^B}{\partial y} \quad (3.26)$$

Allowing equations (3.22) and (3.23) to move onto CD and subtracting gives,

$$\left(\frac{\partial H_y^A}{\partial x} - \frac{\partial H_x^A}{\partial y} \right) - \left(\frac{\partial H_y^B}{\partial x} - \frac{\partial H_x^B}{\partial y} \right) = J_z \quad (3.27)$$

where once again A and B indicate the region involved.

Combining equations (3.26) and (3.27) gives,

$$\frac{\partial \underline{H}_y^A}{\partial x} - \frac{\partial \underline{H}_y^B}{\partial x} = J_z \quad (3.28)$$

and thus the discontinuity in σ has produced a discontinuity in $\frac{\partial \underline{H}_y}{\partial x}$ on the surface CD. Similarly all of the material surface conditions that generate $\underline{H}$ can be seen as discontinuities in $\underline{H}$ or its derivatives, and it will now be shown how the mathematics takes care of these points.

3.3 THE WEIGHTED RESIDUAL APPROXIMATION TO THE VECTOR LAPLACIAN

The starting point for the Weighted Residual approximation is,

$$\int_{\Omega} W \nabla^2 \underline{H} \, d\Omega = \underline{0} \quad (3.29)$$

where W is the weighting function chosen, a scalar only for convenience, and Ω is the region in which the vector Laplacian holds. Using the following identity,

$$- W \nabla^2 \underline{H} \equiv \underline{\nabla W} \cdot \underline{\bar{\nabla} H} - \underline{\bar{\nabla}} \cdot W \underline{\bar{\nabla} H} \quad (3.30)$$

where $\underline{\bar{\nabla}}$ is a dyadic operator^{3.3} and the Gauss divergence theorem, equation (3.29) reduces to

$$\int_{\Omega} \underline{\nabla W} \cdot \underline{\bar{\nabla} H} \, d\Omega - \int_{\Gamma} W (\underline{n} \cdot \underline{\bar{\nabla}}) \underline{H} \, d\Gamma = \underline{0} \quad (3.31)$$

where Γ is the bounding surface of Ω .

Essentially (3.31) is the form being sought but the equation may be more readily appreciated in its component form. The form of the equation for each cartesian component is the same and therefore only that for H_x (where $\underline{H} = (H_x, H_y, H_z)$) is given,

$$\int_{\Omega} \underline{\nabla W} \cdot \underline{\nabla H_x} \, d\Omega - \int_{\Gamma} W \frac{\partial H_x}{\partial n} \, d\Gamma = 0 \quad (3.32)$$

where $\frac{\partial H_x}{\partial n}$ is the outward normal derivative of H_x on Γ with respect to Ω .

It will be remembered by the reader that with respect to fig.3.1 the vector Laplacian for $\underline{H}$ holds in regions A and B but not on the material surface ABCD and yet if equation (3.32) were applied to region A the material surface ABCD ($=\Gamma$) is introduced quite naturally. It now becomes clear that relating the values of $\underline{H}$ and of $\frac{\partial \underline{H}}{\partial n}$ across a material surface is sufficient to acknowledge the electromagnetic conditions at such surfaces.

The material surface conditions pertaining for each $\underline{H}$ component, apart from at magnetic corners, can be simply derived from the standard conditions: continuity of normal $\underline{B}$, continuity of tangential $\underline{H}$. It is worth noting that in general this means that the components of $\underline{H}$ become linked on material surfaces. For the situations met in this chapter each component of $\underline{H}$ is continuous, since the relative permeability has been

taken to be unity, however, more general conditions relating $\underline{H}$ components across material surfaces including magnetic corners are examined in chapter five.

Although the material surface conditions pertaining for each $\underline{H}$ component are fairly standard ideas the similar relationships for the normal derivatives of $\underline{H}$ components are not so standard and the method of construction needs to be detailed.

3.4 MATERIAL SURFACE CONDITIONS FOR $\frac{\partial \underline{H}}{\partial n}$ (THE NORMAL DERIVATIVE OF $\underline{H}$ COMPONENTS)

It should be noted that $\frac{\partial \underline{H}}{\partial n}$ occurs in equation (3.32) in an integral over the material surface and therefore at corners the normal derivative will always be defined since it is related to a surface facet. However the actual derivative at the corner may become infinite and this is discussed in section 3.4.3.

In this section the discussion will be limited to a two dimensional cartesian coordinate system. Fig.3.2 represents an interface such that region 1 contains material 1 ($\mu_r = \mu_1$, $J_z = J_z^1$) and similarly for region 2 ($\mu_r = \mu_2$, $J_z = J_z^2$) where μ_1 and μ_2 are constant and J_z^1 and J_z^2 are known. The interface Γ between the two materials is in the y-z plane, where the z direction is into the plane of the paper. The vector $\underline{H}$ is of the form $(H_x, H_y, 0)$

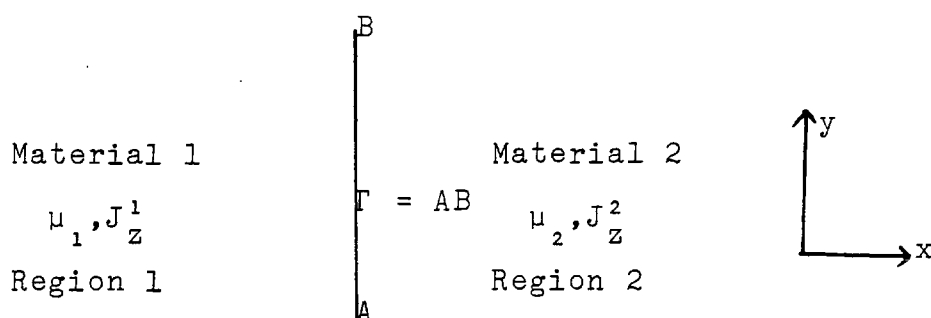


FIG. 3.2

and therefore with reference to fig.3.2 there are two relationships to be found for $\frac{\partial \underline{H}_x}{\partial x}$ and the other for $\frac{\partial \underline{H}_y}{\partial x}$ the normal derivatives of the two components of $\underline{H}$ with respect to the interface Γ . Firstly the relationship for $\frac{\partial \underline{H}_x}{\partial x}$ on Γ , the continuity of tangential $\underline{H}$, produces

$$H_y^1 = H_y^2 \quad (3.33)$$

but this condition holds between the two extreme points of Γ , and therefore

$$\frac{\partial H_y^1}{\partial y} = \frac{\partial H_y^2}{\partial y} \quad (3.34)$$

Equation (3.34) is somewhat reminiscent of $\underline{\nabla} \cdot \underline{H}$ but the desire to introduce this equation must be restrained long enough to appreciate that,

$$\underline{\nabla} \cdot \underline{B} = 0 \quad (3.35)$$

is the actual equation required since (3.35) is true for all space including the material surfaces whereas

$$\underline{\nabla} \cdot \underline{H} = 0 \quad (3.36)$$

does not necessarily hold on material surfaces.

Equation (3.35) gives the following relationship on Γ

$$\mu_1 \frac{\partial H_1^1}{\partial x} + \mu_1 \frac{\partial H_1^2}{\partial y} = 0 \quad (3.37)$$

which when combined with (3.34) gives,

$$\mu_1 \frac{\partial H_1^1}{\partial x} = -\mu_1 \frac{\partial H_1^2}{\partial y} \quad (3.38)$$

However since,

$$\mu_2 \left(\frac{\partial H_2^1}{\partial x} + \frac{\partial H_2^2}{\partial y} \right) = 0 \quad (3.39)$$

equation (3.38) reduces to

$$\frac{\partial H_1^1}{\partial x} = \frac{\partial H_2^1}{\partial x} \quad (3.40)$$

This is precisely the result that is obtained if equation (3.36) is implemented instead of (3.35), however if the permeability is not constant equation (3.36) cannot be used and so the general technique has been followed rather than a problem dependent approach.

Secondly the relationship for $\frac{\partial H_1^2}{\partial x}$ on Γ , the continuity of normal $\underline{B}$ produces,

$$\mu_1 H_1^1 = \mu_2 H_2^1 \quad (3.41)$$

but once again the condition holds between the two extreme points of Γ , and therefore

$$\mu_1 \frac{\partial H_1^1}{\partial y} = \mu_2 \frac{\partial H_2^1}{\partial y} \quad (3.42)$$

Equation (3.42) is reminiscent of $\underline{\nabla} \times \underline{H}$ and therefore for region 1 equation (3.7) gives

$$\frac{\partial H_1^2}{\partial x} - \frac{\partial H_1^1}{\partial y} = J_z^1 \quad (3.43)$$

which when combined with (3.42) produces,

$$\frac{\partial H_y}{\partial x} = \frac{\mu_2}{\mu_1} \frac{\partial H_x}{\partial y} + J_z^1 \quad (3.44)$$

however in region 2 equation (3.7) gives,

$$\frac{\partial H_y^2}{\partial x} - \frac{\partial H_x^2}{\partial y} = J_z^2 \quad (3.45)$$

so that combining equations (3.44) and (3.45) gives,

$$\frac{\partial H_y^1}{\partial x} = \frac{\mu_2}{\mu_1} \left\{ \frac{\partial H_y^2}{\partial x} - J_z^2 \right\} + J_z^1 \quad (3.46)$$

Equations (3.40) and (3.46) represent the conditions being sought and similar results hold for an interface in the x-z plane.

Having seen the principles behind these two special cases the more general two dimensional situation fig.3.3 can now be analysed, where the conditions are the same as fig.3.2 and only the geometry is changed.

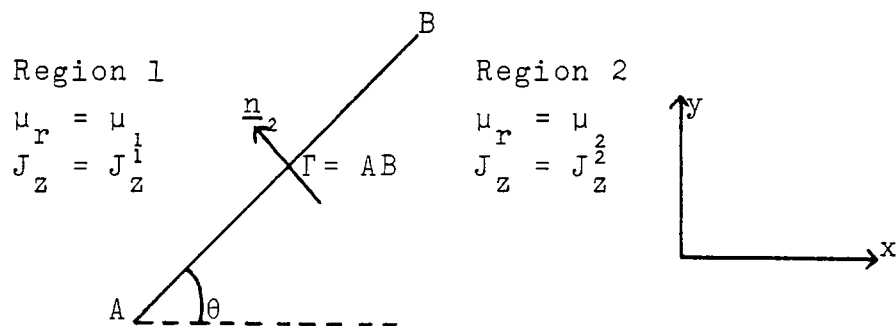


FIG.3.3

The standard two dimensional interface conditions, continuity of normal $\underline{B}$ and tangential $\underline{H}$, with respect to fig.3.3 become respectively,

$$B_y^1 \cos \theta - B_x^1 \sin \theta = B_y^2 \cos \theta - B_x^2 \sin \theta \quad (3.47)$$

and,

$$H_y^1 \sin \theta + H_x^1 \cos \theta = H_y^2 \sin \theta + H_x^2 \cos \theta \quad (3.48)$$

The relationships being sought are between $\frac{\partial H_x^1}{\partial n_1}$ and $\frac{\partial H_x^2}{\partial n_2}$, and $\frac{\partial H_y^1}{\partial n_1}$ and $\frac{\partial H_y^2}{\partial n_2}$ where $\frac{\partial}{\partial n_i}$ is the outward normal as viewed from region $i(=1,2)$ and such that,

$$\frac{\partial}{\partial n_1} = - \frac{\partial}{\partial n_2} \quad (3.49)$$

The starting point is the general two dimensional result,

$$\frac{\partial H_q^i}{\partial n_2} = \frac{1}{r} \frac{\partial H_q^i}{\partial \theta} = \frac{1}{r} \left[\frac{\partial H_q^i}{\partial x} \frac{dx}{d\theta} + \frac{\partial H_q^i}{\partial y} \frac{dy}{d\theta} \right] \quad (3.50)$$

where $q \in (x,y)$ and $i = 1,2$. The result is combined with,

$$x = r \cos \theta \quad (3.51)$$

and,

$$y = r \sin \theta \quad (3.52)$$

to produce,

$$\frac{\partial H_q^i}{\partial n_2} = \frac{\partial H_q^i}{\partial y} \cos \theta - \frac{\partial H_q^i}{\partial x} \sin \theta \quad (3.53)$$

Equation (3.53) is combined with the $\underline{\nabla} \times \underline{H}$ and $\underline{\nabla} \cdot \underline{B}$ equations to produce the following two results on substitution for q ,

$$\frac{\partial H_x^i}{\partial n_2} = \frac{\partial H_y^i}{\partial x} \cos \theta + \frac{\partial H_y^i}{\partial y} \sin \theta - J_z^i \cos \theta \quad (3.54)$$

and,

$$- \frac{\partial H_y^i}{\partial n_2} = \frac{\partial H_x^i}{\partial x} \cos \theta + \frac{\partial H_x^i}{\partial y} \sin \theta + J_z^i \sin \theta \quad (3.55)$$

acknowledging the fact that $\underline{\nabla} \cdot \underline{H}$ is zero.

It may appear, perhaps, that equations (3.54) and (3.55) have little advantage, if any, over their close relatives produced by expanding (3.53) directly, however the general two dimensional result,

$$\frac{\partial H_q^i}{\partial t_2} = \frac{\partial H_q^i}{\partial r} = \frac{\partial H_q^i}{\partial x} \frac{dx}{dr} + \frac{\partial H_q^i}{\partial y} \frac{dy}{dr} \quad (3.56)$$

(where $\frac{\partial}{\partial t_2}$ is the tangential derivative that is right handedly consistent with $\frac{\partial}{\partial n_2}$), provides the link to a useful result. Equation (3.56) on substitution of (3.51) and (3.52) reduces to,

$$\frac{\partial H_q^i}{\partial r} = \frac{\partial H_q^i}{\partial x} \cos \theta + \frac{\partial H_q^i}{\partial y} \sin \theta \quad (3.57)$$

Combining equations (3.54) and (3.55) independently with (3.57) produces the two equations,

$$\frac{\partial H_x^i}{\partial n_2} = \frac{\partial H_y^i}{\partial t_2} - J_z^i \cos \theta \quad (3.58)$$

and,

$$\frac{\partial H_y^i}{\partial n_2} = \frac{\partial H_x^i}{\partial t_2} - J_z^i \sin \theta \quad (3.59)$$

Having found the relationships (3.58) and (3.59) it now only remains to link the normal derivatives across the interface. The next step therefore is to turn the spotlight back to equations (3.47) and (3.48) and to recall that it is legitimate to differentiate those equations in the tangential direction, and this leads to,

$$\mu_1 \frac{\partial H_y^1}{\partial t_2} \cos \theta - \mu_1 \frac{\partial H_x^1}{\partial t_2} \sin \theta = \mu_2 \frac{\partial H_y^2}{\partial t_2} \cos \theta - \mu_2 \frac{\partial H_x^2}{\partial t_2} \sin \theta \quad (3.60)$$

and,

$$\frac{\partial H_1^1}{\partial t_2} \sin\theta + \frac{\partial H_1^1}{\partial t_2} \cos\theta - \frac{\partial H_2^2}{\partial t_2} \sin\theta + \frac{\partial H_2^2}{\partial t_2} \cos\theta \quad (3.61)$$

Equations (3.60) and (3.61) combined with (3.58) and (3.59) provide the relationships being sought. It may appear that only one of equations (3.60) and (3.61) is required, and certainly as long as $\sin\theta$ or $\cos\theta$ does not equal zero either will suffice. However when one or other of $\cos\theta$ and $\sin\theta$ is zero then both equations are required. One possible combination of these equations using both (3.60) and (3.61) gives rise to the following two equations,

$$\begin{aligned} \mu_1 \frac{\partial H_1^1}{\partial n_2} = & (\mu_2 \cos^2\theta + \mu_1 \sin^2\theta) \left[\frac{\partial H_2^2}{\partial n_2} + J_z^2 \cos\theta \right] \\ & - (\mu_2 - \mu_1) \sin\theta \cos\theta \frac{\partial H_2^2}{\partial t_2} \\ & - J_z^1 \mu_1 \cos\theta \quad (3.62) \end{aligned}$$

and,

$$\begin{aligned} \mu_1 \frac{\partial H_1^1}{\partial n_2} = & + (\mu_2 - \mu_1) \sin\theta \cos\theta \frac{\partial H_2^2}{\partial t_2} \\ & + (\mu_1 \cos^2\theta + \mu_2 \sin^2\theta) \\ & \left[\frac{\partial H_2^2}{\partial n_2} + J_z^2 \sin\theta \right] - J_z^1 \mu_1 \sin\theta \quad (3.63) \end{aligned}$$

3.4.1 Observations About the Material Surface Conditions

Equations (3.62) and (3.63) need to be tested in some fairly simple examples in order to generate faith in their correctness. Two situations will be considered, firstly if it is assumed that the material properties as well as the current density are continuous across the interface in fig.3.3 the interface is effectively

removed and AB simply becomes a line segment in a permeable conductor. Across such a line segment both $\underline{H}$ and its derivatives are continuous. Substituting these conditions into equations (3.62) and (3.63) gives,

$$\mu_1 \frac{\partial H_x^1}{\partial n_2} = \mu_1 \frac{\partial H_x^2}{\partial n_2} \quad (3.64)$$

and,

$$\mu_1 \frac{\partial H_y^1}{\partial n_2} = \mu_1 \frac{\partial H_y^2}{\partial n_2} \quad (3.65)$$

as was anticipated. Secondly if in fig.3.3 θ is set to 90° the situation depicted by fig.3.2 ensues and therefore the results given by equations (3.40) and (3.46) should be expected. On putting

$$\theta = 90^\circ \quad (3.66)$$

in equations (3.62) and (3.63) these equations reduce to,

$$\frac{\partial H_x^1}{\partial n_2} = \frac{\partial H_x^2}{\partial n_2} \quad (3.67)$$

and,

$$\frac{\partial H_y}{\partial n_2} = \frac{\mu_2}{\mu_1} \left(\frac{\partial H_y^2}{\partial n_2} + J_z^2 \right) - J_z^1 \sin \theta \quad (3.68)$$

which are the expected equations.

3.4.2 Modelling $\frac{\partial H}{\partial n}$ and its Implications

Equations (3.62) and (3.63) represent one way of modelling $\frac{\partial H}{\partial n}$ in a two dimensional problem, but inadvertently in deriving equations (3.62) and (3.63) a possible alternative arose which is expressed by equations (3.58) and (3.59).

The only apparent objection to using equations (3.58), (3.59) in preference to (3.62), (3.63) is that the former pair introduce explicitly the other vector component of $\underline{H}$ not under consideration, whereas, apparently the latter pair of equations maintains the purity of the formulation for that component of $\underline{H}$ being considered. This latter observation, although true, needs to be placed in the context of the total formulation where, in general, explicit reference will be made to components of $\underline{H}$ not under consideration. Having taken note of the merits of both pairs of equations the author considered that the former pair, (3.58), (3.59), offered a slightly better approach for three reasons,

- 1) they are less complex in nature
- 2) the reduction in complexity reduces the numerical computation involved
- 3) they permit the handling of derivatives of $\underline{H}$ that become infinite (section 3.4.3).

There are two important implications of this choice, the first of which has already been alluded to, that is the explicit appearance of the component of $\underline{H}$ not under consideration, the second is the explicit appearance of $\underline{J}$ the current density. In fact the appearance of $\underline{J}$ would have occurred whichever choice had been made for $\frac{\partial \underline{H}}{\partial n}$. The appearance of both, the component of $\underline{H}$ not under consideration and $\underline{J}$, should have been expected since they express nothing more than the fact that $\underline{H}$ is a vector dependent on the currents in the problem.

Moreover the occurrence of the $\underline{H}$ component poses no great problems but the appearance of $\underline{J}$ although necessary in magnetostatic problems, as was discussed earlier in this chapter, might lead to difficulties in eddy current problems where $\underline{J}$ is not completely known. This situation is discussed further in chapter six.

3.4.3 Points at which Derivatives of $\underline{H}$ Become Infinite

In the derivation of the Weighted Residual form of the vector Laplacian the assumption was made that the derivatives of $\underline{H}$ were finite and this is not always true, in particular it can be shown (appendix B) that at material corners the derivatives can become infinite. This requires that the Weighted Residual form of the vector Laplacian (3.31) be justified for regions containing derivatives of $\underline{H}$ that become infinite. There are two cases that need consideration, the first is where the derivative of $\underline{H}$ that becomes infinite lies within the region and the second is where it lies on the boundary. The analysis that follows is for a two dimensional situation only.

Consider fig.3.4 where A is an arbitrary region such that without loss of generality at point P $\frac{\partial H_x}{\partial x}$ becomes infinite. Assume that,

$$\nabla^2 H_x = 0 \quad (3.69)$$

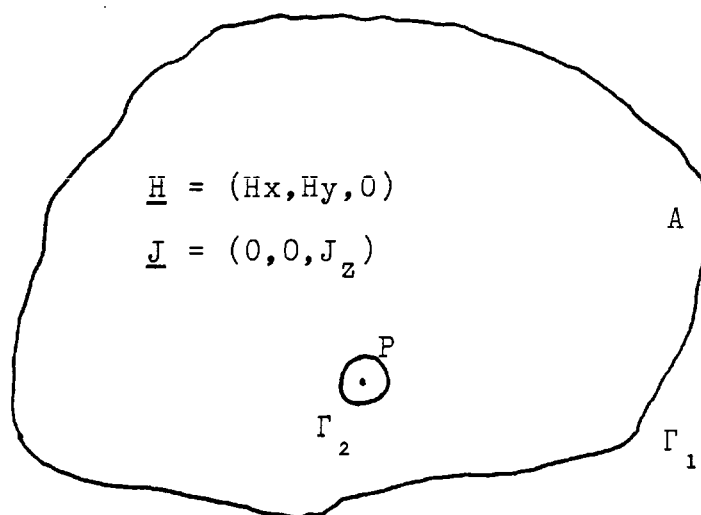


FIG.3.4

is the governing equation for H_x in A . The Weighted Residual form of (3.69) may be taken from equation (3.31) to be,

$$\int_{A - P(\epsilon)} \underline{\nabla W} \cdot \underline{\nabla H_x} \, dA - \int_{\Gamma} W \frac{\partial H_x}{\partial n} \, d\Gamma = 0 \quad (3.70)$$

where W is the weighting function, $P(\epsilon)$ the area associated with the circle centred on P and Γ the perimeter of the area A . Since $\frac{\partial H_x}{\partial x}$ becomes infinite at P , Γ consists of Γ_1 and Γ_2 (the perimeter of the circle at P with radius ϵ). If therefore the second integral in (3.70) is expanded it may be written as,

$$I = \left(I_1 = \int_{\Gamma_1} W \frac{\partial H_x}{\partial n} \, d\Gamma_1 \right) + \left(I_2 = \int_{\Gamma_2} W \frac{\partial H_x}{\partial n} \, d\Gamma_2 \right) \quad (3.71)$$

The second integral in (3.71), I_2 , encapsulates the problem point P and it is the assertion of the author that this point may be included in the formulation since as $\epsilon \rightarrow 0$, so does I_2 . The first step in the proof is to expand W in a Taylor^{3.4} series centred on P,

$$W(\epsilon, \theta) = W(0, \theta) + \epsilon \frac{\partial W}{\partial r}(0, \theta) + O(\epsilon^2) \quad (3.72)$$

The second step is to replace $\frac{\partial Hx}{\partial n}$ in I_2 by,

$$\begin{aligned} \frac{\partial Hx}{\partial n} &= \frac{\partial Hy}{\partial t} - J_z \cos(\theta - 90^\circ) = \\ &= -\frac{1}{\epsilon} \frac{\partial Hy}{\partial \theta} + J_z \sin \theta \end{aligned} \quad (3.73)$$

in accordance with (3.58). Bringing together (3.72) and (3.73), I_2 becomes,

$$I_2 = \int_0^{2\pi} \left(W(0, \theta) + \epsilon \frac{\partial W}{\partial r}(0, \theta) + O(\epsilon^2) \right) \left(-\frac{1}{\epsilon} \frac{\partial Hy}{\partial \theta} + J_z \sin \theta \right) \epsilon d\theta \quad (3.74)$$

which reduces to ,

$$I_2 = -W(0, \theta) Hy(\epsilon, \theta) \Big|_0^{2\pi} + O(\epsilon) \quad (3.75)$$

Since Hy is single valued (3.75) reduces to,

$$I_2 = O(\epsilon) \quad (3.76)$$

and it is straightforward to observe that as ϵ tends to zero so does I_2 . As $\epsilon \rightarrow 0$, $P(\epsilon) \rightarrow 0$ and therefore equation (3.70) becomes that which is expected for A.

The second case to consider is where the point, P, of fig.3.4 lies on the boundary of the region (e.g. a

corner) as shown in fig.3.5. The only change necessary to cope with this situation is the integration limits as given in (3.74) which for Γ_2 as shown in fig.3.5

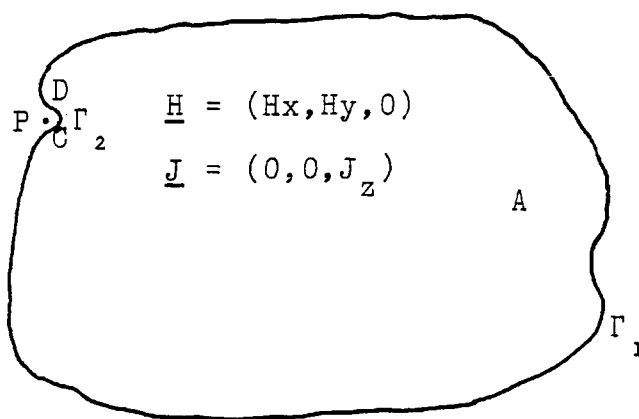


FIG.3.5

become,

$$I_2 = \int_{\beta}^{\alpha} \left[W(0, \theta) + \epsilon \frac{\partial W(0, \theta)}{\partial r} + O(\epsilon^2) \right] \left[-\frac{1}{\epsilon} \frac{\partial H_y}{\partial \theta} + J_z \sin \theta \right] \epsilon d\theta \quad (3.77)$$

where α is the angle of the line PC and β of the line PD. Equation (3.77) reduces to,

$$I_2 = -W(0, \theta) H_y(\epsilon, \theta) \Big|_{\beta}^{\alpha} + O(\epsilon) \quad (3.78)$$

which gives,

$$I_2 = -W(0, \theta) (H_y(\epsilon, \alpha) - H_y(\epsilon, \beta)) + O(\epsilon) \quad (3.79)$$

It now becomes obvious that as ϵ tends to zero so does I_2 . Of course in conjunction with I_2 as ϵ tends to zero so does $P(\epsilon)$ the area of the circular segment surrounding

P; and therefore irrespective of occurrences of infinite $\frac{\partial H_x}{\partial x}$ the following equation holds in any region A, where there are no discontinuities in material properties and (3.69) is the governing equation for H_x ,

$$\int_A \underline{\nabla W} \cdot \underline{\nabla H_x} \, dA - \int_{\Gamma} W \frac{\partial H_x}{\partial n} \, d\Gamma = 0 \quad (3.80)$$

similarly for H_y in a two dimensional situation.

Not only does (3.80) serve as the fundamental equation for the differential solution but also for the Greens integral formulation. For the two dimensional Greens function,

$$G(\underline{r}, \underline{r}_1) = \ln |\underline{r} - \underline{r}_1| \quad (3.81)$$

the following equations hold^{3.5},

$$\nabla^2 G = 0 \quad \text{for } \underline{r} \neq \underline{r}_1 \quad (3.82)$$

$$\nabla^2 G = 2\pi \quad \text{for } \underline{r} = \underline{r}_1 \quad (3.83)$$

Combining the above two equations with the identity

$$- H_x \nabla^2 G = \underline{\nabla H_x} \cdot \underline{\nabla G} - \nabla \cdot (H_x \underline{\nabla G}) \quad (3.84)$$

and integrating over region A in fig.3.4, and implementing the Gauss divergence theorem gives,

$$\begin{aligned} \int_A \underline{\nabla H_x}(\underline{r}_1) \cdot \underline{\nabla G}(\underline{r}, \underline{r}_1) \, dA - \int_{\Gamma} H_x(\underline{r}_2) \frac{\partial G(\underline{r}, \underline{r}_2)}{\partial n} \, d\Gamma \\ = - 2\pi H_x(\underline{r}) \end{aligned} \quad (3.85)$$

where $\underline{r}$ lies within A. If $\underline{r}$ is pushed onto Γ (3.85) becomes,

$$\int_A \underline{\nabla} Hx(\underline{r}_1) \cdot \underline{\nabla} G(\underline{r}, \underline{r}_1) dA - \int_{\Gamma} Hx(\underline{r}_2) \frac{\partial G(\underline{r}, \underline{r}_2)}{\partial n} d\Gamma$$

$$= -\alpha Hx(\underline{r}) \quad (3.86)$$

where α is the angle interior to A at the boundary point $\underline{r}$.

If in equation (3.80) W is chosen to be G the subtraction of (3.80) and (3.85) or (3.86) gives,

$$\beta Hx(\underline{r}) = \int_{\Gamma} Hx(\underline{r}_2) \frac{\partial G(\underline{r}, \underline{r}_2)}{\partial n} d\Gamma - \int_{\Gamma} G(\underline{r}, \underline{r}_2) \frac{\partial Hx(\underline{r}_2)}{\partial n} d\Gamma \quad (3.87)$$

where

$$\beta = \begin{cases} 2\pi & \underline{r} \in A \\ \alpha & \underline{r} \in \Gamma \end{cases} \quad (3.88)$$

Equation (3.87) is the two dimensional Greens integral formula for Hx . The beauty of the above derivation though is that (3.87) is constructed using (3.80) and therefore (3.87) holds at places where derivatives of H become infinite.

3.5 THE INFINITELY LONG ISOLATED CONDUCTOR PROBLEM (ANALYTIC SOLUTION)

In any project there has to come a time when ideas are put to the test and for this project the first problem chosen for consideration was the infinitely long isolated conductor problem. Two reasons motivated this

choice firstly, and most importantly, an analytic solution exists for this problem, and secondly the whole domain contained material whose relative permeability was unity. This latter point is important due to the standard continuity conditions breaking down at material corners where there is a discontinuity in the relative permeability. This point is discussed in detail in chapter five.

Fig.3.6 shows the situation under consideration; a single copper conductor positioned parallel to the z -axis, that is assumed to be infinitely long and surrounded by air.

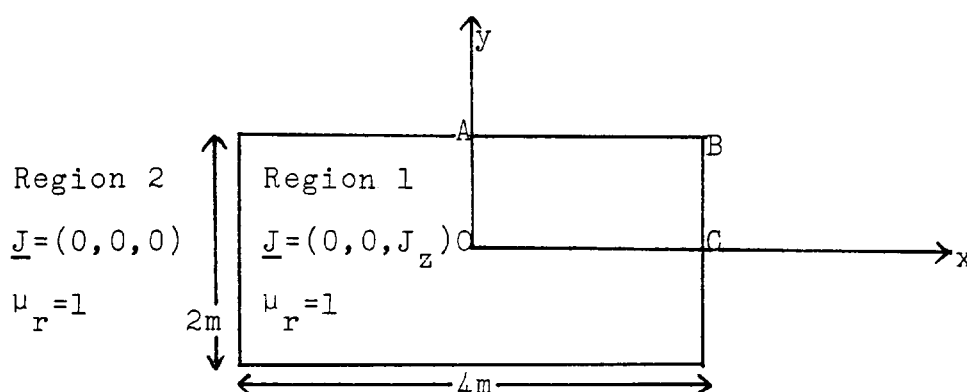


FIG.3.6

3.5.1 The Analytic Solution

The discussion given here follows that given by Binns and Lawrenson^{3.6}. Consider a rectangular bus bar of sides $2a$ and $2b$ carrying a current I . The current

density of the bus bar is then $I/4ab$, and therefore the current carried by any filament in the bar of cross section $dx'dy'$ is $(I/4ab)dx'dy'$. The vector potential function, $\underline{A} = (0,0,A_z)$ at a distance r from a line current i is,

$$A_z = \frac{\mu_0}{2\pi} i \ln r \quad (3.89)$$

where μ_0 is the relative permeability of free space. Therefore the vector potential function at a point (x,y) with respect to the rectangular bus bar is,

$$A_z = \frac{I\mu_0}{8\pi ab} \int_a^a \int_b^b \ln r \, dx'dy' \quad (3.90)$$

which becomes on expanding r ,

$$A_z = \frac{I\mu_0}{16\pi ab} \int_a^a \int_b^b \ln \left((x-x')^2 + (y-y')^2 \right) dx'dy' \quad (3.91)$$

and this can be integrated in terms of simple functions as given by Strutt^{3.7}. Having obtained A_z , $\underline{H} = (H_x, H_y, 0)$ is given by,

$$H_x = \frac{1}{\mu_0} \frac{\partial A}{\partial y} \quad (3.92)$$

and,

$$H_y = - \frac{1}{\mu_0} \frac{\partial A}{\partial x} \quad (3.93)$$

and the resulting expressions given by Binns and Lawrenson are,

$$H_x = \frac{I}{8\pi ab} \left((y+b)(\theta_1 - \theta_2) - (y-b)(\theta_4 - \theta_3) + (x+a)\ln \frac{r_2}{r_3} - (x-a)\ln \frac{r_1}{r_4} \right) \quad (3.94)$$

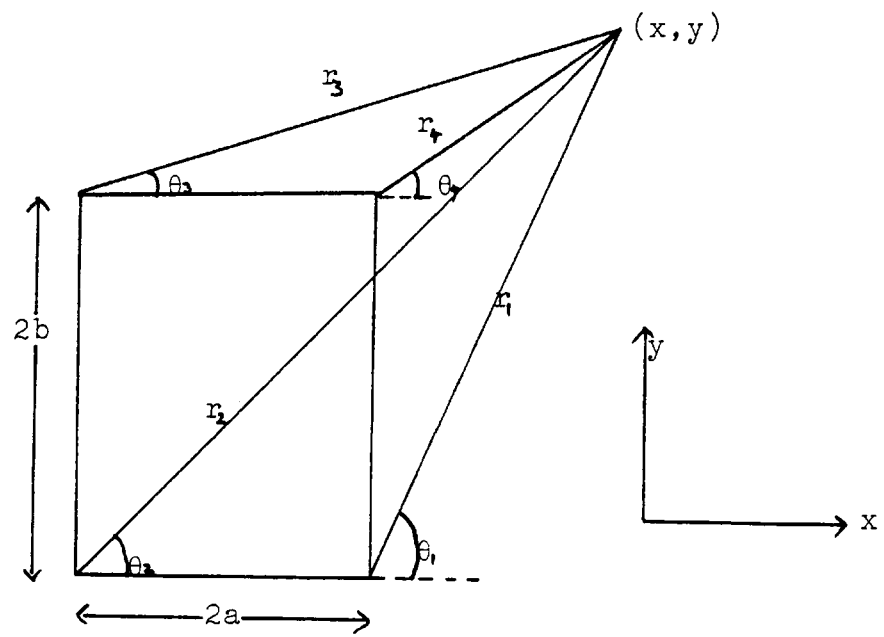


FIG. 3.7

$$H_y = - \frac{I}{8\pi ab} \left((x+a)(\theta_2 - \theta_3) - (x-a)(\theta_1 - \theta_4) + (y+b)\ln \frac{r_2}{r_1} - (y-b)\ln \frac{r_3}{r_4} \right) \quad (3.95)$$

where the undefined symbols are given in fig.3.7.

3.5.2 The Integral Equation Solution for $\underline{H}$

Initially the Binns and Lawrenson solution was used by the author but as his competence grew in dealing with $\underline{H}$ it soon became apparent that a direct $\underline{H}$ solution for general conductor shapes was fairly easily constructable. Consider fig.3.8 where $\Gamma = abcde$ represents a general two dimensional conductor (five corners an arbitrary choice) infinitely long in the z-direction.

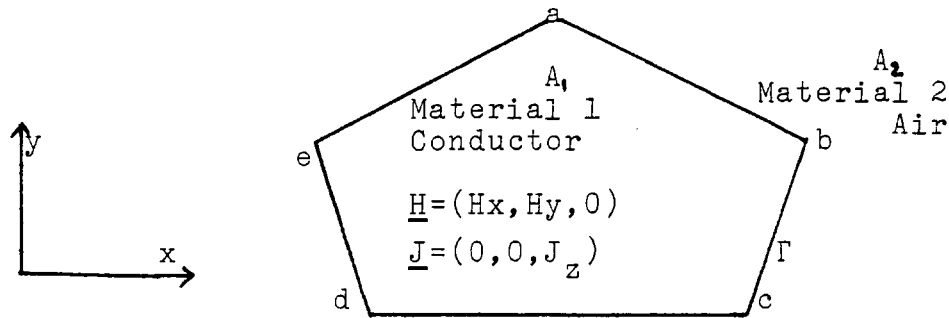


FIG.3.8

The governing equation for regions A_1 and A_2 is the vector Laplacian for $\underline{H}$. The details will only be set out for H_x . The governing equation for H_x in regions A_1 and A_2 is,

$$\nabla^2 H_x = 0 \quad (3.96)$$

The Greens integral formula for (3.96) applied to regions A_1 and A_2 respectively is,

$$\alpha_1 H_1^1(\underline{p}_1) = \int_{\Gamma} H_1^1 \frac{\partial G}{\partial n_1} d\Gamma - \int_{\Gamma} \frac{\partial H_1^1}{\partial n_1} G d\Gamma \quad \underline{p}_1 \in A_1 \quad (3.97)$$

$$\alpha_2 H_2^2(\underline{p}_2) = \int_{\Gamma} H_2^2 \frac{\partial G}{\partial n_2} d\Gamma - \int_{\Gamma} \frac{\partial H_2^2}{\partial n_2} G d\Gamma \quad \underline{p}_2 \in A_2 \quad (3.98)$$

where,

$$\alpha_i = \begin{cases} 2\pi & \underline{p}_i \in A_i \\ 0 & \underline{p}_i \notin A_i \\ \text{interior angle at } p_i & \underline{p}_i \in \Gamma \end{cases} \quad i=1,2 \quad (3.99)$$

$$\underline{p}_i = (x_{p_i}, y_{p_i}) \quad i=1,2 \quad (3.100)$$

$$G = \frac{1}{2} \ln \left[(x-x_{p_i})^2 + (y-y_{p_i})^2 \right], \quad (x,y) \in \Gamma \quad (3.101)$$

$$\frac{\partial}{\partial n_1} = - \frac{\partial}{\partial n_2} \quad (3.102)$$

The interface conditions for the conductor may be written as,

$$H_1^1 = H_2^2 \quad (3.103)$$

and,

$$H_1^1 = H_2^2 \quad (3.104)$$

since $u_r = 1$ throughout the solution domain. Therefore when equations (3.97) and (3.98) are added together the result is,

$$2\pi H_x(\underline{p}) = - \int_{\Gamma} \left(\frac{\partial H_1^1}{\partial n_1} - \frac{\partial H_2^2}{\partial n_1} \right) G d\Gamma \quad \underline{p} \in A_1 \cup A_2 \cup \Gamma \quad (3.105)$$

since

$$\alpha_1 + \alpha_2 = 2\pi \quad (3.106)$$

If the normal derivative for region A_1 over a line segment is defined such that

$$\frac{\partial}{\partial n_1} = \frac{1}{r} \frac{\partial}{\partial \theta} \quad (3.107)$$

where θ varies anti-clockwise (see fig.3.9),

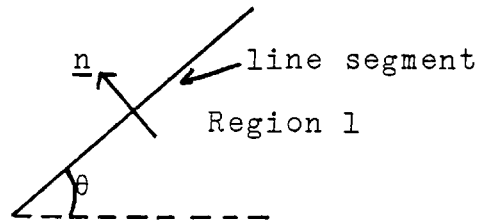


FIG.3.9

the following results hold,

$$\frac{\partial H_1^x}{\partial n_1} = \frac{\partial H_1^y}{\partial t_1} - J_z \cos \theta \quad (3.108)$$

$$\frac{\partial H_1^x}{\partial n_1} = \frac{\partial H_1^y}{\partial t_1} \quad (3.109)$$

where $\frac{\partial}{\partial n_1}$ and $\frac{\partial}{\partial t_1}$ form a right hand system. Remembering (3.104), equations (3.108) and (3.109) give,

$$\frac{\partial H_1^x}{\partial n_1} - \frac{\partial H_1^x}{\partial n_1} = -J_z \cos \theta \quad (3.110)$$

Bringing together (3.105) and (3.110) gives the required result,

$$2\pi H_x(\underline{p}) = \int_{\Gamma} J_z G \cos \theta d\Gamma \quad \underline{p} \in A_1 \cup A_2 \quad (3.111)$$

and the similar analysis for H_y gives,

$$2\pi Hy(\underline{p}) = \int_{\Gamma} J_z G \sin\theta \, d\Gamma \quad \underline{p} \in A_1 \cup A_2 \cup \Gamma \quad (3.112)$$

The author has performed the integrations for a rectangular conductor and confirmed that the solutions are identical to those given by Binns and Lawrenson.

3.6 THE INFINITELY LONG ISOLATED CONDUCTOR PROBLEM (APPROXIMATE SOLUTION)

Fig.3.6 sets out the situation under consideration. The problem exhibits symmetry about the x and y axis and this symmetry was exploited in seeking a solution; only the top right-hand quadrant was considered. The current density, J_z , was chosen to be 1.0 A/m^2 and the boundary conditions imposed on the problem were as follows,

$$1) \quad \text{on the line } x = 0; \quad Hy = 0 \quad (3.113)$$

$$2) \quad \text{on the line } y = 0; \quad Hx = 0 \quad (3.114)$$

$$3) \quad \text{on a bounding are at infinity; } Hx, Hy=0 \quad (3.115)$$

It is worth noting at this point that condition (3.115) implies

$$4) \quad \text{at infinity } \frac{\partial Hx}{\partial n}, \frac{\partial Hy}{\partial n} = 0 \quad (3.116)$$

when it is combined with equations (3.58) and (3.59).

As has been expressed earlier the principle of solution is to seek a differential approximation in region 1 (OABC), and an integral approximation in region 2, such

that the two approximations suitably combined produce a solution to the problem. To this end the following discussion will look at each of the three points above i.e. interior approximation, exterior approximation and combining the two.

3.6.1 Interior Approximation

Without loss of generality it is in order to set out the approach for only one component of $\underline{H}$, and for the ensuing discussion H_x is that chosen component. The governing equation for H_x in region 1 is,

$$\nabla^2 H_x = 0 \quad (3.117)$$

and the Weighted Residual form of this equation is,

$$\int_{\Omega} \underline{\nabla W} \cdot \underline{\nabla H_x} \, d\Omega - \int_{\Gamma} W \frac{\partial H_x}{\partial n} \, d\Gamma = 0 \quad (3.118)$$

where W is the weighting function, Ω the domain in which (3.117) holds and Γ the bounding perimeter of Ω .

Before (3.118) can be approximated three choices must be made,

- 1) the polygonal elements to be used,
- 2) the order of polynomial approximation to be used in each element,
- 3) the weighting functions to be used.

Triangular elements were chosen for two standard reasons, the ease of representing irregular boundary shapes, and

their simplicity would form a good base on which to build.

The order of polynomial approximation was taken to be linear as it was thought that this would be adequate initially to determine the behaviour of the solution.

The third choice made was that of setting

$$\underline{W} = \underline{N} \quad (3.119)$$

where $\underline{N}$ is the vector of linear shape functions (see appendix C). This restriction of $\underline{W}$ to a local definition gives a banded sparse matrix and this is a recognised advantage^{3.8} of this particular approach. However for the authors application the advantage is minimised by the inclusion of a fully populated matrix representing the exterior. This choice for $\underline{W}$ makes the method being implemented that which has become known as the Galerkin-Weighted Residual process.

It now remains to implement the choices made and to produce from equation (3.118) a series of algebraic equations for the unknown parameters. To this end over the domain Ω which has been divided into m triangular elements the approximation for Hx may be written as,

$$H_i = \sum_{e=1}^m \sum_{j=1}^3 N_j^e H_{x_j}^e \quad (3.120)$$

where N_j^e is the shape function at node j in element e and $H_{x_j}^e$ is the Hx value at the same node. Therefore when specifically,

$$W = N_j^e \quad 1 \leq e \leq m, \quad 1 \leq j \leq 3 \quad (3.121)$$

equation (3.104) reduces to,

$$\int_{\Omega^e} \nabla N_j^e \cdot \nabla \left(\sum_{k=1}^3 N_k^e Hx_k^e \right) d\Omega^e - \int_{\Gamma^e} N_j^e \frac{\partial Hx}{\partial n} d\Gamma^e = 0 \quad (3.122)$$

since

$$N_j^e = 0 \quad (3.123)$$

outside of element e . In fact the second integral in (3.122) is zero unless element e contains at least one side on Γ , the bounding surface of Ω . Substitution is made for $\frac{\partial Hx}{\partial n}$ in the second integral of (3.122) from equation (3.58) to produce

$$\sum_{k=1}^3 \int_{\Omega^e} \nabla N_j^e \cdot \nabla (N_k^e Hx_k^e) d\Omega^e - \int_{\Gamma^e} l_n N_j^e \left(\frac{\partial Hy}{\partial t} - J_z \cos \theta \right) d\Gamma^e = 0 \quad (3.124)$$

It is worth while pausing at this juncture to highlight the step that has just been taken. Three points need to be made, firstly it has been assumed that the normal derivative is consistent with the definition given in fig.3.9 and

$$l_n = \begin{cases} +1 & \text{anti-clockwise direction} \\ -1 & \text{clockwise direction} \end{cases} \quad (3.125)$$

Secondly it should be noted that the conditions across

a material surface for $\frac{\partial H_x}{\partial n}$ have been transformed to the more standard conditions for the component H_y since

$$\int_{\Gamma^e} N_j^e \frac{\partial H_y}{\partial t} d\Gamma^e = N_j^e H_y \bigg|_{\Gamma^e} - \int_{\Gamma^e} \frac{\partial N_j^e}{\partial t} H_y d\Gamma^e \quad (3.126)$$

Thirdly having expressed the integral of $\frac{\partial H_y}{\partial t}$ as above it now looks fairly natural to approximate H_y in a linear manner, over a line segment, and this is precisely what the author did. In fact a linear approximation for H_y is the same as assuming a constant value for $\frac{\partial H_y}{\partial t}$ over the line segment.

The linear nature of H_y over Γ may be expressed as,

$$H_y(x,y) = \sum_{e=1}^m \sum_{j=1}^3 N_j^e(x,y) H_{y_j}^e \quad (x,y) \text{ on } \Gamma \quad (3.127)$$

since for nodes in Ω not on Γ the shape function is zero for (x,y) on Γ . Therefore equation (3.124) becomes,

$$\begin{aligned} \sum_{k=1}^3 \int_{\Omega^e} \nabla N_j^e \cdot \nabla (N_k^e H_{x_k}^e) d\Omega^e \\ - \int_{\Gamma^e} \ell_n N_j^e \left[\frac{\partial}{\partial t} \left(\sum_{n=1}^3 N_n^e H_{y_n}^e \right) - J_z \cos \theta \right] d\Gamma^e = 0 \end{aligned} \quad (3.128)$$

When each of the equations has been assembled the matrix form of the equations may be written

$$\underline{A} \underline{H_x^I} + \underline{B} \underline{H_x^b} = \underline{C} \underline{H_y^b} + \underline{J_z^x} \quad (3.129)$$

where $\underline{H_x^I}$ is the vector of H_x nodal values completely internal to the model, $\underline{H_x^b}$ is the vector of H_x nodal

values on Γ , $\underline{H}_y^b$ is the vector of H_y nodal values on Γ , and $\underline{J}_z^x$ is the vector of right hand sides arising from J_z (the current density in the z direction) when explicitly seeking H_x . For the interior approximation a mesh of nodes and triangles is required and fig.3.10 sets out the first mesh that was used which covers the top right quadrant of the rectangular conductor given in fig.3.6.

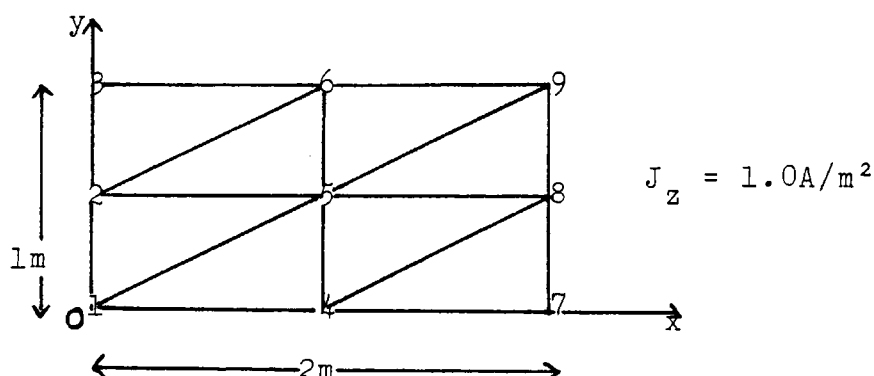


FIG.3.10

The numbers given are the node numbers for the nodes in the mesh. The above definitions for $\underline{H}_x^I$, $\underline{H}_x^b$, $\underline{H}_y^b$ can now be made a little clearer with respect to fig.3.10,

$$\left. \begin{aligned} \underline{H}_x^I &= (H_x^1) \\ \underline{H}_x^b &= (H_x^2, H_x^3, H_x^4, H_x^5, H_x^6) \\ \underline{H}_y^b &= (H_y^7, H_y^8, H_y^9, H_y^{10}, H_y^{11}) \end{aligned} \right\} \quad (3.130)$$

where H_x^i is the nodal value of H_x at node i and similarly for H_y .

3.6.2 Exterior Approximation

Since in the exterior region the governing equation for $\underline{H}$ is Laplaces (see chapter one) the exterior approximation for Hx is based on the two dimensional Greens integral formula,

$$\alpha Hx(\underline{p}_1) = \int_{\Gamma} Hx^b \frac{\partial G}{\partial n} d\Gamma - \int_{\Gamma} \frac{\partial Hx^b}{\partial n} G d\Gamma \quad (3.131)$$

where,

$$G = \frac{1}{2} \ln \left((x-x_1)^2 + (y-y_1)^2 \right) \quad (3.132)$$

$$\alpha = \begin{cases} 2\pi & \underline{p}_1 \text{ in the exterior region} \\ 0 & \underline{p}_1 \text{ not in the exterior region} \\ \text{angle at } \underline{p}_1 & \underline{p}_1 \text{ on boundary of} \\ & \text{exterior region} \end{cases} \quad (3.133)$$

$$\underline{p}_1 = (x_1, y_1) \quad (3.134)$$

and the superscript b indicates values on the boundary Γ . In practice Γ not only incorporates the near boundaries but for infinite regions the boundary at infinity as well. However for the problem under consideration, described in section 3.6, the contribution to the integral is zero and so this part of Γ will be ignored for the rest of the discussion.

The first step is to replace $\frac{\partial Hx^b}{\partial n}$ in equation (3.131) in accordance with (3.58), the normal defined as given in fig.3.9, noting that in the exterior there is no current

density and so (3.131) becomes,

$$\alpha H_x(p_1) = \int_{\Gamma} H_x \frac{\partial G}{\partial n} d\Gamma - \int_{\Gamma} \frac{\partial H_y}{\partial t} G d\Gamma \quad (3.135)$$

It now becomes apparent that two decisions have to be made,

- 1) how to approximate H_x^b ?
- 2) how to approximate $\frac{\partial H_y^b}{\partial t}$?

Equation (3.135) is required in some sense to join with the interior approximation to produce a solution and therefore it seemed reasonable to operate (3.123) on the boundary of the already defined interior mesh. Therefore for example if fig.3.10 is considered, Γ is seen to be (3,6,9,8,7), or more precisely, the line segment thus described. Not only did it seem sensible to use the same geometric boundary but also the same node placements. Having decided the boundary and node placements it still remained to choose the order of approximation. However the former decisions really only admit a linear approximation since any higher order approximation would require restrictions on the number of boundary nodes, hence a linear approximation was chosen to allow the greatest flexibility.

The second point at issue is very similar to the first and only differs in that there would seem to be two possible approaches,

1) deal directly with $\frac{\partial H_y^b}{\partial t}$

$$2) \text{ or note that } \int_{\Gamma} G \frac{\partial H_y^b}{\partial t} d\Gamma = \int_{\Gamma} H_y^b G d\Gamma - \int_{\Gamma} H_y^b \frac{\partial G}{\partial t} d\Gamma \quad (3.136)$$

In fact it turns out that a linear approximation for H_y will give the same result in either case although the second is a little more tricky to handle since $\int_{\Gamma} H_y^b G d\Gamma$ is a principal valued integral. The choice then that was made was to deal directly with $\frac{\partial H_y^b}{\partial t}$ and use a linear approximation for H_y over each line segment.

On implementing the decisions taken the matrix equation for (3.135) becomes,

$$A^E \underline{H_x^b} = B^E \underline{H_y^b} \quad (3.137)$$

3.6.3 Combining the Two Approximations

The discussion above has been concerned with detailing the steps of the interior and exterior approximations for H_x only and for the interior the matrix form of the equation is,

$$A \underline{H_x^I} + B \underline{H_x^b} = C \underline{H_y^b} + \underline{J_z^x} \quad (3.138)$$

and for the exterior,

$$A^E \underline{H_x^b} = B^E \underline{H_y^b} \quad (3.139)$$

Fig.3.10 shows the first mesh considered and the conditions enforced were as follows,

$$1) \text{ along } O_x \quad H_x = 0$$

$$2) \text{ along } O_y \quad H_y = 0$$

because of the symmetry of the problem, and.

$$3) \text{ at infinity} \quad H_x = H_y = 0$$

which give

$$4) \text{ at infinity} \quad \frac{\partial H_x}{\partial n} = \frac{\partial H_y}{\partial n} = 0$$

All of these conditions are reflected in the form of the matrices in equations (3.138) and (3.139). These two equations were solved simultaneously and the results set out consistently with fig.3.10 are,

<u>Hx (calculated)</u>			<u>Hx (Binns and Lawrenson)</u>		
-.7151	-.6720	-.4513	-.7206	-.6763	-.4234
-.3512	-.3256	-.2169	-.3545	-.3275	-.2112
0.0	0.0	0.0	0.0	0.0	0.0

TABLE 3.1

where the results from the Binns and Lawrenson result were calculated to full machine accuracy which is about twelve significant figures. The absolute percentage error at each node compared with the analytic result, calculated as,

$$\left| \frac{\text{analytic value} - \text{calculated value}}{\text{analytic value}} \right| \times 100$$

gives,

.76	.64	6.6
.93	.58	2.7
0.0	0.0	0.0

TABLE 3.2

which shows that the greatest error occurs at the corner. From the values of $\underline{H}_y^b$ ensuing from the solution of equations (3.138) and (3.139) the following solution for H_y was constructed.

<u>H_y (hybrid technique)</u>			<u>H_y (Binns and Lawrenson)</u>		
0.0	.2666	.6776	0.0	.2566	.5513
0.0	.3145	.7579	0.0	.2979	.7235
0.0	.3301	.7856	0.0	.3134	.7628

TABLE 3.3

The absolute percentage error at each node compared with the analytic result is,

0.0	3.9	22.9
0.0	5.57	4.75
0.0	5.33	2.99

TABLE 3.4

The absolute percentage errors shown above are very much worse than those for Hx and this is not suprising seeing equations (3.138) and (3.139) are primarily equations for Hx. If the similar equations to (3.138) and (3.139) are set up for Hy and solved the absolute percentage comparisons between the calculated and analytic results are,

0.0	1.09	.29
0.0	.03	1.88
0.0	.03	1.52

TABLE 3.5

which are considerably better and point to the fact that it is probably better to solve for each component individually.

Equation (3.139) has been regarded as an equation for Hx, however it is possible to regard it as an equation for Hy and therefore the author constructed a solution for Hy using the Hy counterpart equation to (3.138) and equation (3.139), the results are,

<u>Hy (calculated)</u>			<u>Hy (Binns and Lawrenson)</u>		
0.0	.2357	.6428	0.0	.2566	.5513
0.0	.2940	.7461	0.0	.2979	.7235
0.0	.3129	.7765	0.0	.3134	.7628

TABLE 3.6

which gives absolute percentage errors at each node of,

0.0	8.14	16.6
0.0	1.31	3.12
0.0	0.16	1.8

TABLE 3.7

which are obviously worse than those in table 3.5. The same finding is arrived at if equation (3.138) is solved together with the H_y counterpart to (3.139) for H_x . In all cases when a particular component has been solved for and the other component is constructed from the boundary values (e.g. $\underline{H_y^b}$ in (3.138), (3.139)) the resulting solution has been found to be very indifferent (e.g. tables 3.3, 3.4).

The above series of comparisons were also made for the finer mesh shown in fig. 3.11,

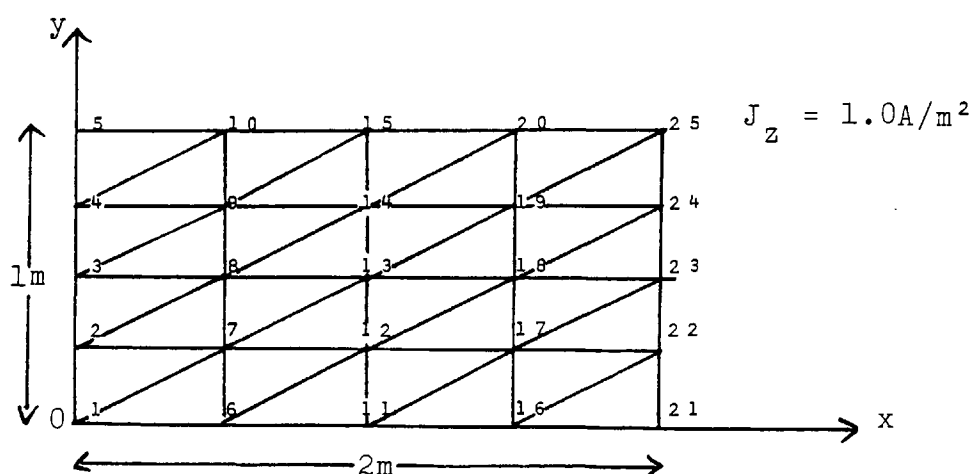


FIG. 3.11

<u>Hx (calculated)</u>					<u>Hx (Binns and Lawrenson)</u>				
-.7191	-.7090	-.6752	-.6038	-.4367	-.7206	-.7104	-.6763	-.6042	-.4234
-.5342	-.5257	-.4972	-.4373	-.3199	-.5355	-.5269	-.4981	-.4368	-.3171
-.3536	-.3475	-.3270	-.2851	-.2118	-.3545	-.3483	-.3275	-.2843	-.2112
-.1760	-.1728	-.1622	-.1408	-.1056	-.1765	-.1732	-.1624	-.1403	-.1055
0	0	0	0	0	0	0	0	0	0

TABLE 3.8

and the same conclusions were arrived at. The best solution for H_x was produced when the equations solved were for,

$$\nabla^2 H_x = 0 \quad (3.140)$$

in both the exterior and interior regions and similarly for H_y . The results produced for H_x are shown in table 3.8. The absolute percentage errors for the results in table 3.8 are given in table 3.9.

.2	.2	.16	.07	3.14
.24	.23	.18	.11	.88
.25	.23	.15	.28	.28
.28	.23	.12	.36	.09
0	0	0	0	0

TABLE 3.9

The similar results for H_y are shown in table 3.10. The absolute percentage errors for the results in table 3.10 are given in table 3.11.

0	.32	.31	.33	.33
0	.07	.07	.02	.78
0	.07	.03	.08	.54
0	.07	.03	.18	.41
0	.13	.03	.22	.38

TABLE 3.11

	<u>Hy (calculated)</u>			
0	.1254	.2558	.3958	.5531
0	.1354	.2795	.4446	.6621
0	.1433	.2980	.4802	.7199
0	.1482	.3095	.5015	.7503
0	.1499	.3135	.5085	.7599

	<u>Hy (Binns and Lawrenson)</u>			
0	.1258	.2566	.3971	.5513
0	.1355	.2797	.4445	.6673
0	.1432	.2979	.4806	.7238
0	.1481	.3094	.5024	.7534
0	.1497	.3134	.5096	.7628

TABLE 3.10

It is encouraging to compare tables 3.9 and 3.2 and tables 3.11 and 3.5 and observe that for a finer mesh the percentage errors have dropped as should be expected.

3.6.4 Conclusions

There are three conclusions to be drawn from the above solutions,

- 1) In principle the method looks favourable and is worth exploring.
- 2) The solutions solving $\nabla^2 H_x = 0$ in one region and $\nabla^2 H_y = 0$ in the other give reasonable results but are somewhat lacking in theoretical justification.
- 3) The solutions solving $\nabla^2 H_x$ or $\nabla^2 H_y = 0$ throughout the whole domain give good solutions for their respective components; but the alternate component results are variable.

CHAPTER 4THE FORMULATION FOR TWO DIMENSIONAL
MAGNETOSTATIC PROBLEMS ($\mu_r = 1$)4.1 NOT ENOUGH EQUATIONS

The initial formulations as set out in chapter three gave promising results, but the simplicity of the model had disguised one important feature of the formulation. The model of the single bus bar as considered in chapter three took into account the symmetry of the problem and imposed boundary conditions on the quadrant in order to find a solution. If those boundary conditions are removed and the whole problem is treated the interior discrete representation exhibits the strange phenomenon of being devoid of one equation. A simple example will suffice to show what is meant.

Consider the mesh as given in fig.4.1 to be that describing the single bus bar situation. The governing

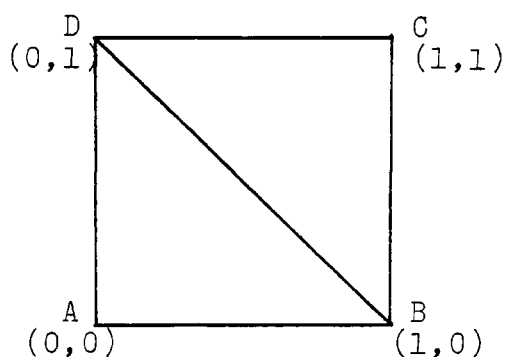


FIG.4.1

equation for $\underline{H}$ in the bus bar being

$$\nabla^2 \underline{H} = \underline{0} \quad (4.1)$$

The governing equation for H_x as taken from (4.1) is

$$\nabla^2 H_x = 0 \quad (4.2)$$

Without loss of generality the component H_x will be used in the example. The Weighted Residual form of (4.2) is

$$-\int_{\Omega} W \nabla^2 H_x \, d\Omega = \int_{\Omega} \underline{\nabla W} \cdot \underline{\nabla H_x} \, d\Omega - \int_{\Gamma} W \frac{\partial H_x}{\partial n} \, d\Gamma = 0 \quad (4.3)$$

where Ω is the area of the bus bar, Γ its perimeter and W the weighting function. The linear approximation for H_x over triangle ABD is

$$H_x = H_x^A(1-x-y) + xH_x^B + yH_x^D \quad (4.4)$$

therefore if W becomes $1-x-y$, x, y in turn the integrals over Ω in (4.3) $\underline{I}^{ABD}$ may be represented by the matrix,

$$\underline{I}^{ABD} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} H_x^A \\ H_x^B \\ H_x^D \end{bmatrix} \quad (4.5)$$

since the integral is zero over BCD. The similar integrals over BCD, $\underline{I}^{BCD}$, may be represented as

$$\underline{I}^{BCD} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} H_x^B \\ H_x^C \\ H_x^D \end{bmatrix} \quad (4.6)$$

Combining these equations the total integrals over Ω , $\underline{I}$, using the Galerkin approach may be written

$$\underline{I} = \begin{bmatrix} 1 & -0.5 & 0 & -0.5 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ -0.5 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} H_x^A \\ H_x^B \\ H_x^C \\ H_x^D \end{bmatrix} \quad (4.7)$$

With respect to the integral equation (4.3) there only remains the evaluation of the integral over the boundary. When point A is being considered

$$W = 1-x-y \quad (4.8)$$

and there exists two sides AB and AD over which W is non zero. On AB $\frac{\partial H_x}{\partial n}$ may be written

$$\frac{\partial H_x}{\partial n} = - \frac{\partial H_x}{\partial y} = - \frac{\partial H_y}{\partial x} + J \quad (4.9)$$

and then assuming H_y to have a linear approximation

$$\begin{aligned} \int_{AB} W \frac{\partial H_x}{\partial n} d\Gamma &= \int_0^1 (1-x)(H_y^A - H_y^B + J) dn \\ &= \frac{H_y^A - H_y^B}{2} + \frac{J}{2} \end{aligned} \quad (4.10)$$

On AD $\frac{\partial H_x}{\partial n}$ may be written

$$\frac{\partial H_x}{\partial n} = - \frac{\partial H_x}{\partial x} = \frac{\partial H_y}{\partial y} \quad (4.11)$$

and so as above,

$$\int_{AD} W \frac{\partial H_x}{\partial n} d\Gamma = \int_0^1 (1-y)(H_y^B - H_y^A) dy = \frac{H_y^B - H_y^A}{2} \quad (4.12)$$

If the process is continued for all faces the equation (4.3) is approximated by the matrix equation,

$$\begin{bmatrix} 1 & -0.5 & 0 & -0.5 \\ -0.5 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & -0.5 \\ -0.5 & 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} H_x^A \\ H_x^B \\ H_x^C \\ H_x^D \end{bmatrix} - \begin{bmatrix} 0 & -0.5 & 0 & 0.5 \\ 0.5 & 0 & -0.5 & 0 \\ 0 & 0.5 & 0 & -0.5 \\ -0.5 & 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} H_y^A \\ H_y^B \\ H_y^C \\ H_y^D \end{bmatrix} = \begin{bmatrix} J/2 \\ J/2 \\ -J/2 \\ -J/2 \end{bmatrix} \quad (4.13)$$

The above set of equations demonstrates that one member of the set is a linear combination of the others.

It must be stated at this juncture that only areas where $\underline{H}$ satisfies Laplaces equation exhibit this phenomenon. In general the above result can be proved from the mathematics as follows. Consider a Laplacian region, Ω , such that (4.3) holds. One equation for element e for region Ω may be written as,

$$\int_{\Omega^e} \underline{\nabla W_i^e} \cdot \underline{\nabla H_x^e} d\Omega^e - \int_{\Gamma^e} W_i^e \frac{\partial H_x^e}{\partial n} d\Gamma^e = 0 \quad (4.14)$$

where $1 \leq e \leq N$ and Ω has been divided into N linear triangular elements, W_i^e is the shape function at node i in element e for $1 \leq i \leq 3$ and H_x^e is the approximation to H_x over element e .

Element shape functions exhibit the behaviour^{4.1},

$$\sum_{i=1}^3 W_i^e = 1 \quad (4.15)$$

therefore if the equations at each node over the region, Ω , are added together as follows,

$$\begin{aligned} \sum_{e=1}^N \sum_{i=1}^3 \int_{\Omega^e} \underline{\nabla W_i^e} \cdot \underline{\nabla H_x^e} d\Omega^e \\ - \sum_{e=1}^N \sum_{i=1}^3 \int_{\Gamma^e} W_i^e \frac{\partial H_x^e}{\partial n} d\Gamma^e = 0 \end{aligned} \quad (4.16)$$

(4.16) may be reduced to

$$\begin{aligned} \sum_{e=1}^N \int_{\Omega^e} \underline{\nabla} \left(\sum_{i=1}^3 W_i^e \right) \cdot \underline{\nabla H_x^e} d\Omega^e \\ - \sum_{e=1}^N \int_{\Gamma^e} \sum_{i=1}^3 W_i^e \frac{\partial H_x^e}{\partial n} d\Gamma^e = 0 \end{aligned} \quad (4.17)$$

so that when (4.15) is included (4.17) reduces to

$$\int_{\Gamma} \frac{\partial H_x}{\partial n} d\Gamma = 0 \quad (4.18)$$

As was discussed in chapter three $\frac{\partial H_x}{\partial n}$ is replaced by,

$$\frac{\partial H_x}{\partial n} = \frac{\partial H_y}{\partial t} - J \cos \theta \quad (4.19)$$

and therefore (4.19) may be written (for constant J)

$$\int_{\Gamma} \frac{\partial H_y}{\partial t} dt - J \int_{\Gamma} \cos \theta dt = 0 \quad (4.20)$$

which is identically zero for single valued H . Hence it has been shown that one equation in the set is the sum of the other equations in the set. Therefore the equations representing the interior for H_x or H_y plus the equations representing H_x or H_y in the exterior are apparently insufficient for solution. This can be characterised more clearly in the following way, assume that the interior region has N nodes and that

$$N = n_I + n_B \quad (4.21)$$

where n_I are the nodes totally interior to the region and n_B are those nodes comprising the boundary. Then the interior equations provide $n_I + n_B - 1$ distinct equations and the exterior integral provides n_B distinct equations. The number of unknowns for the problem is $n_I + 2n_B$, $n_I + n_B$ unknowns for the actual component being solved, and n_B unknowns for the other component arising on the boundary due to the normal derivative term being expressed as a tangential derivative of the other component. The insufficiency now becomes clear since there are $n_I + 2n_B$ unknowns and a maximum of $n_I + 2n_B - 1$ distinct equations. At this stage in the

investigation the insufficiency seemed to arise from the interior set of equations and therefore efforts were made to find the missing relationship.

Two attempts were made to introduce the extra information required and it is instructive to work each of these through for the simple example given earlier (fig.4.1).

4.1.1 Applying Stokes Theorem to $\nabla \times \underline{H} = \underline{J}$

It was decided that the application of Stokes Theorem to

$$\nabla \times \underline{H} = \underline{J} \quad (4.22)$$

may well provide the extra information required. The transformation of (4.22) by Stokes Theorem produces

$$\oint \underline{H} \cdot d\underline{t} = + I = + \underline{J} \times (\text{area of region}) \quad (4.23)$$

When equation (4.23) is applied to fig.4.1 the following equation results,

$$H_x^A + H_x^B - H_x^C - H_x^D - H_y^A + H_y^B + H_y^C - H_y^D = +2J \quad (4.24)$$

However equation (4.24) is already part of the equation set (4.13) as can clearly be seen by taking the sum of the first two numbers of that set.

Although the model is simple it can easily be seen that any larger model consisting entirely of cells as given

by fig.4.1 would also exhibit the same numerical behaviour. Hence if it can be shown that for an arbitrary linear triangle the application of equation (4.23) brings no new information to the system then it follows that for an arbitrary system approximated by linear triangles equation (4.23) is already incorporated in the system.

Fig.4.2 represents an arbitrary triangle for which the

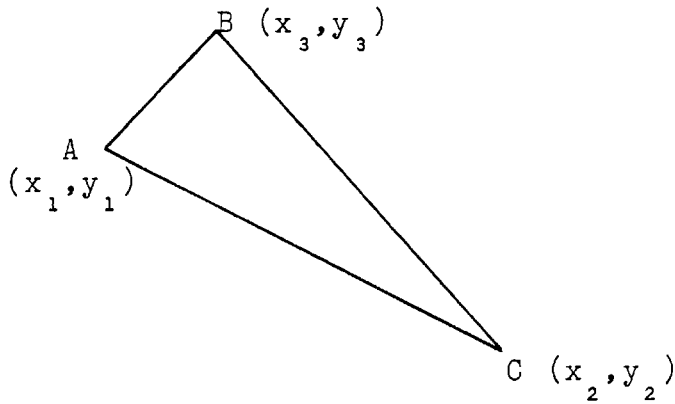


FIG.4.2

governing equation for H_x is,

$$\nabla^2 H_x = 0 \quad (4.25)$$

The Weighted Residual form of (4.25) becomes

$$\int_{\Omega} \underline{\nabla W} \cdot \underline{\nabla H_x} \, d\Omega - \int_{\Gamma} W \frac{\partial H_x}{\partial n} \, d\Gamma = 0 \quad (4.26)$$

where W is the chosen weighting function, Ω is the area of the triangle and Γ its perimeter. The shape function for a general linear triangle may be expressed as follows^{4.1},

$$L_1 = (a_1 + b_1 x + c_1 y)/2\Delta \quad (4.27)$$

$$L_2 = (a_2 + b_2 x + c_2 y)/2\Delta \quad (4.28)$$

$$L_3 = (a_3 + b_3 x + c_3 y)/2\Delta \quad (4.29)$$

where

$$\Delta = \frac{1}{2} \det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \text{area of triangle ABC} \quad (4.30)$$

and

$$a_1 = x_2 y_3 - x_3 y_2 \quad (4.31)$$

$$b_1 = y_2 - y_3 \quad (4.32)$$

$$c_1 = x_3 - x_2 \quad (4.33)$$

such that a_i, b_i, c_i for $i=2,3$ are gained by cyclic permutation of (4.31) to (4.33). Hence the Galerkin approximation to the area integral in (4.26) may be written as,

$$\frac{1}{4\Delta} \times \begin{pmatrix} b_1 b_1 + c_1 c_1 & b_1 b_2 + c_1 c_2 & b_1 b_3 + c_1 c_3 \\ b_2 b_1 + c_2 c_1 & b_2 b_2 + c_2 c_2 & b_2 b_3 + c_2 c_3 \\ b_3 b_1 + c_3 c_1 & b_3 b_2 + c_3 c_2 & b_3 b_3 + c_3 c_3 \end{pmatrix} \begin{bmatrix} Hx^1 \\ Hx^2 \\ Hx^3 \end{bmatrix}$$

where W has been chosen to be L_1, L_2 and L_3 respectively and

$$Hx = \sum_{i=1}^3 L_i Hx^i \quad (4.34)$$

where Hx^i is the value of Hx at node i .

For the second integral in (4.26)

$$\frac{\partial Hx}{\partial n} = \frac{\partial Hy}{\partial t} - J \cos \theta \quad (4.35)$$

In equation (4.35) $\cos\theta$ may be represented as

$$\cos\theta_{ij} = \frac{x_i - x_j}{d_{ij}} = - \frac{c_{j+1}}{d_{ij}} \quad (4.36)$$

where d_{ij} is the distance from node i to node j , $j=i+1$ in a cyclic sense and $j+1$ is to be understood in a cyclic sense. Bringing all these definitions together the approximation to the second integral may be written

$$- \begin{bmatrix} 0 & 0.5 & -0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & -0.5 & 0 \end{bmatrix} \begin{bmatrix} H_y^1 \\ H_y^2 \\ H_y^3 \end{bmatrix} + \begin{bmatrix} \frac{Jc}{2}_3 + \frac{Jc}{2}_2 \\ \frac{Jc}{2}_1 + \frac{Jc}{2}_3 \\ \frac{Jc}{2}_2 + \frac{Jc}{2}_1 \end{bmatrix}$$

Therefore the total approximation to (4.26) becomes

$$\begin{aligned} & \frac{1}{4\Delta} \times \begin{pmatrix} b_1 b_1 + c_1 c_1 & b_1 b_2 + c_1 c_2 & b_1 b_3 + c_1 c_3 \\ b_2 b_1 + c_2 c_1 & b_2 b_2 + c_2 c_2 & b_2 b_3 + c_2 c_3 \\ b_3 b_1 + c_3 c_1 & b_3 b_2 + c_3 c_2 & b_3 b_3 + c_3 c_3 \end{pmatrix} \begin{bmatrix} H_x^1 \\ H_x^2 \\ H_x^3 \end{bmatrix} \\ & + \begin{bmatrix} 0 & 0.5 & -0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & -0.5 & 0 \end{bmatrix} \begin{bmatrix} H_y^1 \\ H_y^2 \\ H_y^3 \end{bmatrix} = - \begin{bmatrix} \frac{Jc}{2}_1 \\ \frac{Jc}{2}_2 \\ \frac{Jc}{2}_3 \end{bmatrix} \quad (4.37) \end{aligned}$$

since

$$c_1 + c_2 + c_3 = 0 \quad (4.38)$$

Attention is now turned to the form of

$$\underline{\nabla} \times \underline{H} = \underline{J} \quad (4.39)$$

over a general triangle as given by fig4.2. Applying Stokes Theorem to (4.39) produces

$$\int_{\Gamma} \underline{H} \cdot d\underline{t} = + J \times \Delta = I \quad (4.40)$$

where $\underline{H} \cdot d\underline{t}$ produces the tangential $\underline{H}$. Given a general side as in fig.4.3 the tangential $\underline{H}$ may be expressed as

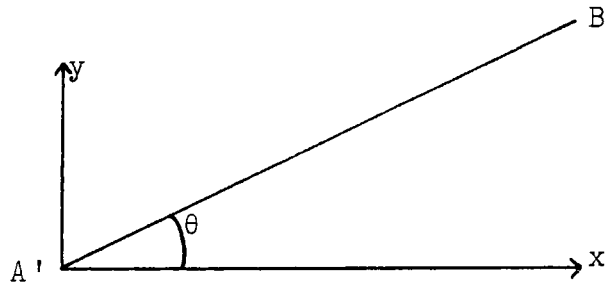


FIG.4.3

$$H_t = H_x \cos \theta + H_y \sin \theta \quad (4.41)$$

where

$$\cos \theta = \frac{x_{B'} - x_{A'}}{d_{A'B'}} \quad (4.42)$$

and

$$\sin \theta = \frac{y_{B'} - y_{A'}}{d_{A'B'}} \quad (4.43)$$

where $d_{A'B'}$ is the distance from A' to B' . Combining equations (4.41), (4.42), (4.43), equation (4.40) may be approximated by,

$$\begin{aligned} & \frac{c_1}{2} (H_x^1 + H_x^2) - \frac{b_1}{2} (H_y^1 + H_y^2) + \frac{c_2}{2} (H_x^2 + H_x^3) - \frac{b_2}{2} (H_y^2 + H_y^3) \\ & + \frac{c_3}{2} (H_x^3 + H_x^1) - \frac{b_3}{2} (H_y^3 + H_y^1) = + J \times \Delta \quad (4.44) \end{aligned}$$

Equation (4.44) can be simplified to,

$$c_1 Hx - b_1 Hy + c_2 Hx - b_2 Hy + c_3 Hx - b_3 Hy = -2J\Delta \quad (4.45)$$

by using

$$b_1 + b_2 + b_3 = 0 \quad (4.46)$$

and (4.38).

It now remains to show that equation (4.45) is already contained in the set of equations (4.37). In order to show this multiply the first equation in (4.37) by b_2 , the second equation by b_1 , and subtract. This gives

$$\begin{aligned} \frac{1}{4\Delta} \left(c_1 (c_1 b_2 - b_1 c_2) Hx + c_2 (c_1 b_2 - b_1 c_2) Hx \right. \\ \left. + c_3 (c_1 b_2 - b_1 c_2) Hx \right) + \frac{b_2}{2} Hy + \frac{b_1}{2} Hy - \frac{(b_1 + b_2)}{2} Hy \\ = -\frac{J}{2} (c_1 b_2 - b_1 c_2) \end{aligned} \quad (4.47)$$

which reduces to (4.45) since

$$c_1 b_2 - b_1 c_2 = -2\Delta \quad (4.48)$$

Hence it has been shown that for an arbitrary triangle equation (4.40) adds no new information and therefore for any region for which Laplaces equation holds for H no extra information can be gained by applying (4.40) if the region is approximated using linear triangles.

4.1.2 Applying the Gauss Divergence Theorem to $\nabla \cdot H = 0$

The second attempt to find the extra information required involved applying the Gauss divergence theorem to

$$\nabla \cdot H = 0 \quad (4.49)$$

which produces,

$$\int_{\Gamma} \underline{H} \cdot \underline{n} \, d\Gamma = 0 \quad (4.50)$$

Considering the example based on fig.4.1 application of (4.50) produces,

$$-H_y^A - H_y^B + H_x^B + H_x^C + H_y^C + H_y^D - H_x^D - H_x^A = 0 \quad (4.51)$$

However equation (4.51) is already part of the equation set (4.13) as can clearly be seen by taking the sum of the second and third members of that set.

The discussion as set out for equation (4.23) once again applies and therefore the form of (4.50) as applied to a general triangle such as is given in fig.4.2 will be found. Given a general line segment as in fig.4.3 the normal $\underline{H}$ component may be expressed as,

$$H_n = H_y \cos\theta - H_x \sin\theta \quad (4.52)$$

where $\cos\theta$ and $\sin\theta$ are consistent with equations (4.42) and (4.43). Substituting (4.52) into (4.50) and integrating around the general triangle as given in fig.4.2 the approximation may be written,

$$\begin{aligned} \frac{c_1}{2} (H_y^1 + H_y^2) + \frac{b_2}{2} (H_x^1 + H_x^2) + \frac{c_2}{2} (H_y^2 + H_y^3) \\ + \frac{b_1}{2} (H_x^2 + H_x^3) + \frac{c_3}{2} (H_y^1 + H_y^3) \\ + \frac{b_3}{2} (H_x^1 + H_x^3) = 0 \end{aligned} \quad (4.53)$$

Equation (4.53) can be simplified to,

$$b_1 H_x^1 + b_2 H_x^2 + b_3 H_x^3 + c_1 H_y^1 + c_2 H_y^2 + c_3 H_y^3 = 0 \quad (4.54)$$

using equations (4.38) and (4.46). Equation (4.54) can now easily be seen to be part of the equation set (4.37) by multiplying the first equation in (4.37) by c_2 , the second by c_1 and subtracting,

$$\begin{aligned} \frac{1}{4\Delta} \left(b_1(b_1c_2 - c_1b_2)Hx^1 + b_2(b_1c_2 - c_1b_2)Hx^2 \right. \\ \left. + b_3(b_1c_2 - c_1b_2)Hx^3 \right) + \frac{c_1Hy^1}{2} + \frac{c_2Hy^2}{2} \\ - (c_1 + c_2) \frac{Hy^3}{2} = 0 \end{aligned} \quad (4.55)$$

Equations (4.38) and (4.48) and equation (4.55) can be seen to be the same as equation (4.54). Hence equation (4.50) brings no new information for any shape constructed by linear triangles where the governing equation for $\underline{H}$ is the vector Laplacian.

4.2 SUFFICIENT CONDITIONS TO DETERMINE $\underline{H}$

The problem of insufficient equations to determine H_x prompted a theoretical look at exactly what conditions are required to determine H_x for the situation depicted in fig.4.1. The governing equation for H_x is

$$\nabla^2 H_x = 0 \quad (4.56)$$

both within the conductor and exterior to it. It must be pointed out that the following discussion is only concerned with single valued $\underline{H}$.

The way to proceed is to assume that two different solutions for H_x , H_x^1 and H_x^2 , exist both satisfying the

conditions for the single isolated bus bar. This means that a further solution exists, namely

$$Hx^3 = Hx^1 - Hx^2 \quad (4.57)$$

and the task in hand is to find sufficient conditions to force

$$Hx^3 \equiv 0 \quad (4.58)$$

for then Hx^1 and Hx^2 are equal and the conditions determine a unique Hx .

The governing equation for Hx^3 is Laplaces equation and this holds throughout the solution domain with the possible exception of the conductor surface. However if Hx^3 and $\frac{\partial Hx^3}{\partial n}$ are continuous across the conductor surface and Hx^3 is zero at infinity then Hx^3 is identically zero. This is easily proved as follows, consider the identity

$$-\psi \nabla^2 \phi \equiv \underline{\nabla \psi} \cdot \underline{\nabla \phi} - \nabla(\psi \nabla \phi) \quad (4.59)$$

and integrate over an arbitrary surface, A , bounded by Γ , such that,

$$- \int_A \psi \nabla^2 \phi \, dA = \int_A \underline{\nabla \psi} \cdot \underline{\nabla \phi} \, dA - \int_{\Gamma} \psi \frac{\partial \phi}{\partial n} \, d\Gamma \quad (4.60)$$

after an application of the Gauss divergence theorem. If this is now applied specifically to fig.4.1 where ψ and ϕ are both chosen to be Hx^3 , and A is chosen in turn to be the conductor, Ω_c , and the infinite region, Ω_I , the two resulting equations are,

$$- \int_{\Omega_c} Hx^3 \nabla^2 Hx^3 \, d\Omega_c = \int_{\Omega_c} \underline{\nabla Hx^3} \cdot \underline{\nabla Hx^3} \, d\Omega_c - \int_{\Gamma_c} Hx^3 \frac{\partial Hx^3}{\partial n} \, d\Gamma_c \quad (4.61)$$

$$\int_{\Omega_I} Hx^3 \nabla^2 Hx^3 d\Omega_I = \int_{\Omega_I} \underline{\nabla Hx^3} \cdot \underline{\nabla Hx^3} d\Omega_I - \int_{\Gamma_I} Hx^3 \frac{\partial Hx^3}{\partial n} d\Gamma_I \quad (4.62)$$

where Γ_c and Γ_I are the boundaries of Ω_c and Ω_I respectively. Adding these two equations gives,

$$\begin{aligned} \int_{\Omega_c} \underline{\nabla Hx^3} \cdot \underline{\nabla Hx^3} d\Omega_c + \int_{\Omega_I} \underline{\nabla Hx^3} \cdot \underline{\nabla Hx^3} d\Omega_I - \int_{\Gamma_c} Hx^3 \frac{\partial Hx^3}{\partial n} d\Gamma_c \\ - \int_{\Gamma_I} Hx^3 \frac{\partial Hx^3}{\partial n} d\Gamma_I = 0 \end{aligned} \quad (4.63)$$

The contour Γ_I may be divided such that,

$$\Gamma_I = \Gamma_c + \Gamma_\infty \quad (4.64)$$

where Γ_∞ is the boundary at infinity and so equation (4.63) gives,

$$\begin{aligned} \int_{\Omega_c} \underline{\nabla Hx^3} \cdot \underline{\nabla Hx^3} d\Omega_c + \int_{\Omega_I} \underline{\nabla Hx^3} \cdot \underline{\nabla Hx^3} d\Omega_I \\ - \int_{\Gamma_c} \left[\left(Hx^3 \frac{\partial Hx^3}{\partial n} \right)_c + \left(Hx^3 \frac{\partial Hx^3}{\partial n} \right)_I \right] d\Gamma_c \\ - \int_{\Gamma_\infty} Hx^3 \frac{\partial Hx^3}{\partial n} d\Gamma_\infty = 0 \end{aligned} \quad (4.65)$$

If the following conditions hold,

- 1) Hx^3 continuous across Γ_c
- 2) $\frac{\partial Hx^3}{\partial n}$ continuous across Γ_c
- 3) $Hx = 0$ at infinity

equation (4.65) reduces to,

$$\int_{\Omega_c + \Omega_I} (\nabla Hx^3)^2 d\Omega = 0 \quad (4.66)$$

which gives,

$$H\dot{x} = k \quad \text{a constant,} \quad \text{over } \Omega_c + \Omega_I \quad (4.67)$$

but since condition three holds,

$$H\dot{x} \equiv 0 \quad \text{over } \Omega_c + \Omega_I \quad (4.68)$$

It now remains to interpret the three conditions in terms of $H\dot{x}$ or $H\dot{y}$.

Condition one gives,

$$(H\dot{x} - H\dot{y})_c = (H\dot{x} - H\dot{y})_I \quad (4.69)$$

and this is trivially satisfied as long as Hx is continuous across the conductor surface.

Condition two gives,

$$\left(\frac{\partial H\dot{x}}{\partial n} - \frac{\partial H\dot{y}}{\partial n} \right)_c = \left(\frac{\partial H\dot{x}}{\partial n} - \frac{\partial H\dot{y}}{\partial n} \right)_I \quad (4.70)$$

which can be reduced to,

$$\left[\left(\frac{\partial H\dot{y}}{\partial t} - J \cos\theta \right) - \left(\frac{\partial H\dot{y}}{\partial t} - J \cos\theta \right) \right]_c = \left[\frac{\partial H\dot{y}}{\partial t} - \frac{\partial H\dot{y}}{\partial t} \right]_I \quad (4.71)$$

Since the two solutions are for the same current and geometry equation (4.71) is satisfied as long as $H\dot{y}$ is continuous across the conductor surface.

Condition three is trivially satisfied by assuming that $\underline{H}$ is zero at infinity.

Therefore sufficient conditions to determine Hx uniquely are,

- 1) H_x continuous across the conductor surface
- 2) H_y continuous across the conductor surface
- 3) H_x zero at infinity.

These conditions are all included in the interior and exterior approximations and therefore H_x must be determined uniquely by the equations and therefore something has been overlooked.

4.3 ANOTHER LOOK AT THE EQUATIONS

In the last section three sufficient conditions were found to determine H_x uniquely in the limited class of problems under consideration in this chapter. However in bringing together the integral and differential approximations all three conditions become explicitly part of the formulation.

First of all the matrix equations (for H_x) for the two domains (see fig.4.1) may be written as follows,

$$A^I \underline{H_x}^I = B^I \underline{H_y}^{I^b} + \underline{J_z}^x \quad (4.72)$$

from the differential region where $\underline{H_x}^I$ and $\underline{H_y}^{I^b}$ are the vectors of interior and boundary nodal values for their respective components and $\underline{J_z}^x$ is the vector of right hand sides arising from J_z the given current density, and

$$A^E \underline{H_x}^{E^b} = B^E \underline{H_y}^{E^b} \quad (4.73)$$

from the integral region where $\underline{H_x}^{E^b}$ and $\underline{H_y}^{E^b}$ are the vectors of exterior boundary nodal values where the interface

between the differential and integral domain is the conductor surface. This latter equation (4.73) has already taken into account the fact that H_x is zero at infinity since the contributions from the boundary at infinity have been put equal to zero. Let the contributions from this far boundary, as seen from the differential domain, be I so that,

$$I = \int_{\Gamma_{\infty}} H_x \frac{\partial G}{\partial n} d\Gamma_{\infty} - \int_{\Gamma_{\infty}} \frac{\partial H_x}{\partial n} G d\Gamma_{\infty} \quad (4.74)$$

The outward normal derivative, $\frac{\partial}{\partial n}$, and G , the two dimensional Greens function, may be regarded as,

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial R} \quad (4.75)$$

$$G = \ln R \quad (4.76)$$

where R is very large and is approximately the distance from the far boundary to the differential domain. Hence (4.74) becomes,

$$I \approx \int_0^{2\pi} H_x \frac{2}{R} R d\theta - \int_0^{2\pi} 2 \cdot \frac{\partial H_x}{\partial n} \ln R R d\theta \quad (4.77)$$

where $d\Gamma_{\infty}$ has been replaced by $R d\theta$. Since

$$\frac{\partial H_x}{\partial n} = \frac{\partial H_x}{\partial R} \quad (4.78)$$

it follows that,

$$\frac{\partial H_x}{\partial n} = -\frac{\partial H_y}{\partial t} = -\frac{1}{R} \frac{\partial H_y}{\partial \theta} \quad (4.79)$$

so that for single valued H the second integral is zero (for H_y the order of $1/R^n$ $n>0$ for large R) leaving,

$$I \approx 2 \int_0^{2\pi} H_x d\theta \quad (4.80)$$

It now follows that in general only if H_x is of order $1/R^n$ $n > 0$ for large R may I be put equal to zero. It should be noted that I zero still permits the integral of $\underline{H} \cdot d\underline{\ell}$ to be non zero. This leaves two conditions yet to be incorporated.

1) the continuity of H_x across the conductor surface

2) the continuity of H_y across the conductor surface.

Both of these conditions are simply incorporated since they give respectively,

$$\underline{H}_x^E = \underline{H}_x^I \quad (4.81)$$

and,

$$\underline{H}_y^E = \underline{H}_y^I \quad (4.82)$$

thus bringing the equations for H_x together they are,

$$A^I \underline{H}_x^I = B^I \underline{H}_y^I + \underline{J}_z^x \quad (4.83)$$

from the differential region, and

$$A^E \underline{H}_x^E = B^E \underline{H}_y^E \quad (4.84)$$

from the integral region, where all three sufficient conditions have been applied and yet the system of equations formed by (4.83) and (4.84) cannot be solved since the resulting matrix is singular.

The key to the issue is the fact that only the conditions to determine H_x uniquely have been applied, nothing has been said about H_y apart from the fact that it is continuous across the conductor surface. In fact an examination of the equations shows that there are enough

equations to determine $\underline{Hx}^I$ but not $\underline{Hy}^I$.

4.3.1 Too Few Equations for $\underline{Hy}^I$

The $\underline{Hy}^I$ in the heading refers to the terms in equations (4.83) and (4.84) and it is the purpose of this section to show why $\underline{Hy}^I$ is not determined.

Equation (4.83) arises from the differential region which in this case is the conductor as given in fig.4.1.

The $B^I \underline{Hy}^I$ terms arise from the treatment of $\int_{\Gamma_c} W \frac{\partial Hx}{\partial n} d\Gamma_c$ where W is the weighting function and Γ_c the conductor surface. Ignoring the source terms arising from the current density this integral transforms to $\int_{\Gamma_c} W \frac{\partial Hy}{\partial t} d\Gamma_c$.

If now, however, this latter integral is evaluated assuming $\frac{\partial Hy}{\partial t}$ is constant over a surface facet its matrix approximation may be written $B_1^I \frac{\partial \underline{Hy}^I}{\partial t}$. The connection between $B^I \underline{Hy}^I$ and $B_1^I \frac{\partial \underline{Hy}^I}{\partial t}$ is established by recalling that in the construction of the former,

$$\underline{Hy}^{ij} = \underline{Hy}^i \left(1 - \frac{t}{d}\right) + \underline{Hy}^j \frac{t}{d} \quad 0 < t < d \quad (4.85)$$

where $\underline{Hy}^{ij}$ is the local variation of Hy between nodes i and j , and d the distance between them so that,

$$\frac{\partial \underline{Hy}^{ij}}{\partial t} = \frac{\underline{Hy}^j - \underline{Hy}^i}{d} \quad (\text{a constant}) \quad (4.86)$$

Therefore the following may be written,

$$B^I \underline{Hy}^I = B_1^I C \underline{Hy}^I \quad (4.87)$$

where,

$$C = \begin{bmatrix} \frac{1}{d_1} & -\frac{1}{d_1} & \cdot & \cdot & \cdot & \cdot \\ 0 & \frac{1}{d_2} & -\frac{1}{d_2} & \cdot & \cdot & \cdot \\ 0 & 0 & \frac{1}{d_3} & -\frac{1}{d_3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \frac{1}{d_{n-1}} & -\frac{1}{d_{n-1}} \\ -\frac{1}{d_n} & \cdot & \cdot & \cdot & 0 & \frac{1}{d_n} \end{bmatrix} \quad (4.88)$$

where n is the number of boundary segments and d_i the length of the i th segment. The matrix C is such that each row sums to zero and this feature is passed onto B^I so that it too exhibits the same phenomenon. Moreover exactly the same is true of the exterior approximation too,

$$\int_{\Gamma_c} \frac{\partial H_x}{\partial n} G \, d\Gamma_c = \int_{\Gamma_c} \frac{\partial H_y}{\partial t} G \, d\Gamma_c \quad (4.89)$$

so that,

$$B_1^E \frac{\partial H_y^E}{\partial t} = B_1^E C \underline{H_y^E}^b = B^E \underline{H_y^E}^b \quad (4.90)$$

Therefore with both B^E and B^I possessing the feature that the entries in each row sum to zero $\underline{H_y^I}^b$ must be left undetermined.

4.3.2 Understanding the $\underline{H_y^b}$ Terms

In order to appreciate the $\underline{H_y^b}$ terms that arise in equations (4.83) and (4.84) it is important that the steps that introduce them are understood.

Both the differential and integral approximations introduce the term $\frac{\partial H_x}{\partial n}$ and it is this term that must be modelled. If this term was treated as it stands the following conditions would be applied,

1) the interface conditions given in chapter three

$$2) \int_{\Gamma_c} \frac{\partial H_x}{\partial n} d\Gamma_c = 0, \quad \Gamma_c \text{ the conductor surface.}$$

In transforming $\frac{\partial H_x}{\partial n}$ to $\frac{\partial H_y}{\partial t}$ the interface conditions are simplified and the integral is trivially satisfied for single valued H_y and it appears that all the conditions have been met. However it is easy to miss the point that the equations are only interested in $\frac{\partial H_y}{\partial t}$ so that although it is perfectly valid to write

$$\frac{\partial H_y^{ij}}{\partial t} = \frac{H_y^i - H_y^j}{d} \quad (4.86)$$

it is equally correct to write,

$$\frac{\partial H_y^{ij}}{\partial t} = \frac{a^i - b^j}{d} \quad (4.91)$$

where a^i, b^j are related to H_y^i, H_y^j by

$$a^i = H_y^i + c \quad (4.92)$$

$$b^j = H_y^j + c \quad (4.93)$$

where c is an arbitrary constant.

Therefore since $\frac{\partial H_y}{\partial t}$ is the important function any one of the H_y 's may be arbitrarily chosen to be a given constant. As long as the same constant is chosen for both the integral and differential regions the new functions are still continuous across the interface.

4.4 THE FORMULATION IMPLEMENTED

Having discovered the extra condition required some examples were tried to justify the theory by results. However two distinct situations arose,

- 1) for odd numbered boundary noded shapes the formulation worked perfectly
- 2) for even numbered boundary noded shapes another difficulty arose.

4.4.1 Odd Numbered Boundary Noded Shapes

An odd numbered boundary noded shape is simply a meshed shape with an odd number of nodes on the boundary. Fig.4.4 shows the mesh considered. The example is consistent with that given in chapter three, the infinitely long rectangular conductor carrying a current density of 1 A/m² and of dimensions 2metres by 4metres. In this case though the whole conductor is analysed and only the results for H_x are presented. The method is as described above where the equations in the differential and integral regions are constructed and solved simultaneously with one of the H_y 's chosen.

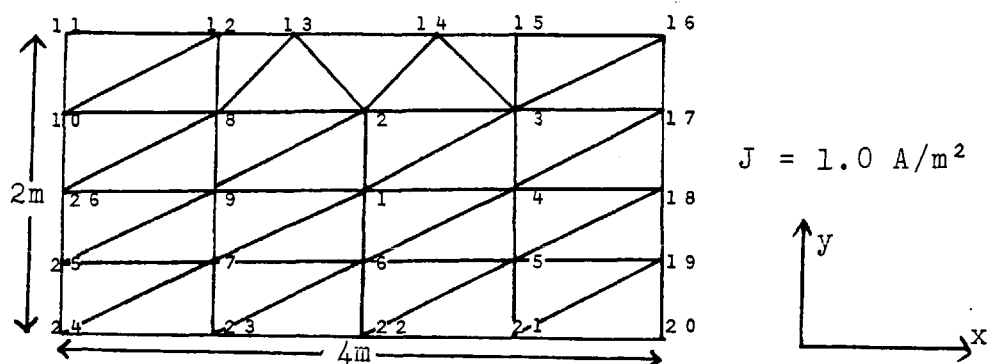


FIG.4.4

Table 4.1 sets out the results for two solutions obtained with a different $\underline{H_y^b}$ chosen for each solution. The values of calculated H_x at each node, for the two runs, were the same up to the rounding error of the machine which is about twelve significant figures. However the values of the $\underline{H_y^b}$'s for each solution were vastly different and these are presented in tables 4.2 and 4.3 where the $\underline{H_y^b}$ chosen is indicated. Commenting earlier the point was made that $\frac{\partial H_y}{\partial t}$ is the important function as opposed to H_y and it can be seen from tables 4.2 and 4.3 that although the nodal values themselves are different the difference between adjacent nodal values is the same within the limits of the four decimal places given.

As far as accuracy is concerned, as compared with the analytic result, apart from a minor effect generated

Hx (calculated)

-.4512	-.6707	-.7093	-.7093	-.6707	-.4512
-.2170	-.3269		-.3514	-.3269	-.2170
-9 E-5	-7 E-4		-3 E-4	-7 E-4	-9 E-5
.2168	.3253		.3510	.3253	.2168
.4513	.6720		.7152	.6720	.4513

TABLE 4.1

$\overset{b}{H\bar{y}}_1$'s

.1444	.4185	.5841	.7521	.9176	1.1917
-.2229					1.5591
0 ← chosen					1.3361
-.2234					1.5595
.1435	.4180	.6681	.9181		1.1926

TABLE 4.2

$\overset{b}{H\bar{y}}_1$'s

-1.4146	-1.1406	-.9750	-.8070	-.6414	-.3673
-1.7820					0
-1.5599				chosen	-.2229
-1.7825					5 E-4
-1.4156	-1.1411	-.8910		-.6410	-.3664

TABLE 4.3

by the symmetry of the mesh (fig.4.4), the results are within a fraction of one percent of those given in table 3.1 in chapter three which are within three percent everywhere except at the corner, where the error is about seven percent.

The approach adopted for the solution of the problem represented by fig.4.4 will work for all odd numbered boundary noded shapes whose relative permeability is unity.

4.4.2 Even Numbered Boundary Noded Shapes

An even numbered boundary noded shape is simply a meshed shape that has an even number of boundary nodes. The author has found that these particular shapes may be divided into two classes,

- 1) shapes whose boundary meshing is symmetrical
- 2) shapes whose boundary is meshed in a non symmetrical manner.

For the latter class the technique used in the last section is perfectly adequate for solution but for the former class the interior(differential) together with the exterior(integral) equations are deficient by two equations instead of one. An examination of the matrices indicates that the deficiency is once again in the $\underline{H}_y^b$ terms. Recapping, the differential equation for H_x may be written in matrix form as,

$$A^I \underline{H}_x^I = B^I \underline{H}_y^I + \underline{J}_z^x \quad (4.94)$$

and the integral equation similarly as,

$$A \underline{E}_{Hx}^b = B \underline{E}_{Hy}^b \quad (4.95)$$

where the interface conditions have been included in the latter equation and the nomenclature is as described before.

For the first class outlined above the author has been unable to extend the method discussed in the last section to produce a satisfactory solution. The reason for this is that the two possible sources of extra relationships to determine the $\underline{E}_{Hy}^b$ terms, namely the differential and integral equations for H_y , provide many possible equations and there appears to be no theoretical reasons for preferring one above another. In conjunction with this the results obtained are very much dependent on the equations chosen to supplement (4.94) and (4.95). To illustrate this consider the mesh in fig.4.5 which is

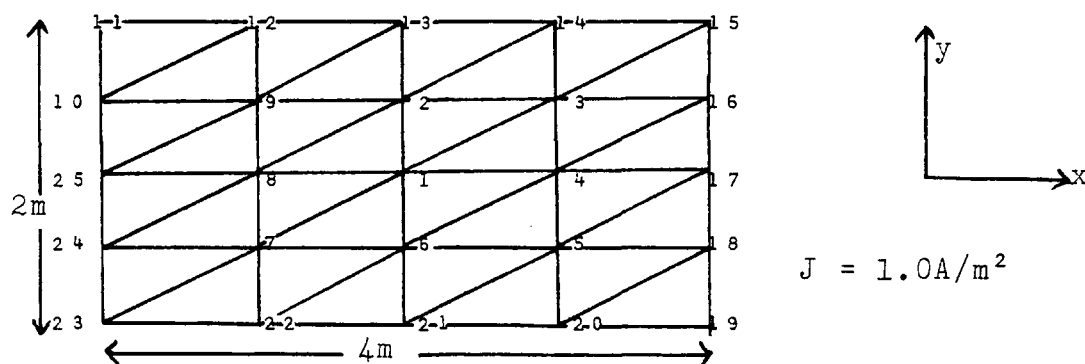


FIG.4.5

once again of the infinitely long isolated rectangular conductor, where on this occasion the boundary has

symmetrically placed nodes. From the Weighted Residual form of,

$$\nabla^2 H_y = 0 \quad (4.96)$$

the governing equation for H_y over the conductor region, the following nodal equation (for node 19),

$$1.25H_y^9 - 0.25H_y^0 - H_y^8 + 0.5H_x^0 - 0.5H_x^8 = 0.25 \quad (4.97)$$

where the numbers indicate the nodal value (fig.4.5) concerned, was incorporated with (4.94) and (4.95) in order to obtain a solution. Three solutions were obtained where for each solution a different H_y^b was chosen. The results for H_x^I are presented in table 4.4, the results were the same for each solution to within the rounding error of the computer. However the results for H_y^b were different for each solution and tables 4.5, 4.6 and 4.7 give the results consistent with fig.4.5, the node chosen in each case is indicated. Once again, as is dictated by the equations, although the H_y^b values are different the difference of adjacent nodal values is within the accuracy presented, the same for each solution. The important feature though of these results is that they demonstrate the potential for different solutions, depending on the equations chosen. The results given in tables 4.4 to 4.7 were found utilising (4.97) which is of course but one possibility, another might be perhaps the equation at node 23,

$$1.25H_y^3 - 0.25H_y^2 - H_y^4 + 0.5H_x^4 - 0.5H_x^2 = -0.25 \quad (4.98)$$

However if the above equation (4.98) is evaluated using the results given in tables 4.4 to 4.7 the following

<u>Hx (calculated)</u>				
-.4839	-.6773	.7151	-.6667	-.4187
-.2161	-.3276	-.3512	-.3236	-.2177
0	0	0	0	0
.2161	.3276	.3512	.3236	.2177
.4839	.6773	.7151	.6668	.4187

TABLE 4.4

<u>$\underline{H_y^b}$'s</u>				
-.5236	-.9309	0	-.4298	.5274
-1.4921				.2922
-.7385		chosen		.5999
-1.4921		↓		.2922
-.5236	-.9309	0	-.4298	.5274

TABLE 4.5

<u>$\underline{H_y^b}$'s</u>				
.4073	0	.9309	.5011	1.4583
-.5612				1.2231
.1924		chosen		1.5308
-.5612				1.2231
.4073	0 ←	.9309	.5011	1.4583

TABLE 4.6

<u>$\underline{H_y^b}$'s</u>				
-.8158	-1.2231	-.2922	-.7220	.2352
-1.7843				0
-1.0307			chosen →	.3077
-1.7843				0
-.8158	-1.2231	-.2922	-.7220	.2352

TABLE 4.7

result is obtained,

$$0.839725 \approx -0.25 \quad (4.99)$$

showing the inconsistency of the equations and also that if (4.98) was implemented instead of (4.97) generally a different result would ensue, not only for $\underline{H_y^b}$ but also for $\underline{H_x^I}$.

4.5 UNIQUENESS OF THE EXTERIOR H_y FIELD

Thus far the $\underline{H_y^b}$ terms have been regarded as being related to the actual H_y values by a constant, c , chosen such that one of the $\underline{H_y^b}$ terms has a known value. This understanding of the situation regarding the $\underline{H_y^b}$ terms is adequate as long as the boundaries on which they arise permit continuity of H_y . If however the boundary on which these terms arise is permeable the assumed constant field, c , will in general affect the H_x component. This therefore prompted investigation into an adaptation of the approach used thus far able to cope with permeable materials.

All that is required in order that permeable materials may be treated are suitable conditions to determine the exterior (integral) H_y uniquely.

As was discussed in chapter one the governing equation for the exterior field is Laplace's equation for $\underline{H}$ which gives, without loss of generality, for H_y ,

$$\nabla^2 H_y = 0 \quad (4.100)$$

Assume that there exists two solutions for H_y in the exterior H_y^1 and H_y^2 , then there also exists a third solution,

$$H_y^3 = H_y^1 - H_y^2 \quad (4.101)$$

The identity

$$-\psi \nabla^2 \phi = \underline{\nabla \psi} \cdot \underline{\nabla \phi} - \nabla \cdot (\psi \underline{\nabla \phi}) \quad (4.102)$$

gives when integrated over the exterior domain, Ω , and on application of the Gauss' divergence theorem,

$$- \int_{\Omega} \psi \nabla^2 \phi \, d\Omega = \int_{\Omega} \underline{\nabla \psi} \cdot \underline{\nabla \phi} \, d\Omega - \int_{\Gamma} \psi \frac{\partial \phi}{\partial n} \, d\Gamma \quad (4.103)$$

where Γ is the bounding surface of Ω and $\frac{\partial}{\partial n}$ is the outward normal derivative. If both ψ and ϕ are put equal to H_y^3 equation (4.103) becomes,

$$\int_{\Omega} (\underline{\nabla H_y^3})^2 \, d\Omega - \int_{\Gamma} H_y^3 \frac{\partial H_y^3}{\partial n} \, d\Gamma = 0 \quad (4.104)$$

Since $\frac{\partial H_y^3}{\partial n}$ may be replaced by $\frac{\partial H_x^3}{\partial t}$, where

$$H_x^3 = H_x^1 - H_x^2 \quad (4.105)$$

then (4.104) may be written,

$$\int_{\Omega} (\underline{\nabla H_y^3})^2 \, d\Omega - \int_{\Gamma} H_y^3 \left(\frac{\partial H_x^1}{\partial t} - \frac{\partial H_x^2}{\partial t} \right) \, d\Gamma = 0 \quad (4.106)$$

However if H_x is uniquely defined then so is $\frac{\partial H_x}{\partial t}$ so that (4.106) becomes,

$$\int_{\Omega} (\underline{\nabla H_y^3})^2 \, d\Omega = 0 \quad (4.107)$$

so that over Ω ,

$$H\dot{y}^3 = k \quad \text{a constant} \quad (4.108)$$

If therefore the exterior region extends to infinity where $H\dot{y}$ is zero trivially

$$H\dot{y}^1 = H\dot{y}^2 \quad (4.109)$$

and $H\dot{y}$ is unique, however for enclosed spaces it appears that there is a possibility of a non unique $H\dot{y}$. In this thesis though the exterior field always extends to infinity. The two conditions therefore sufficient to give a unique $H\dot{y}$ are,

- 1) a uniquely defined $\frac{\partial Hx}{\partial t}$
- 2) $H\dot{y}$ zero at infinity.

4.6 A SUPPLEMENTARY FORMULATION

Recapping, the matrix equations arising for Hx are, from the differential region,

$$A^I_{H\dot{X}} = B^I_{H\dot{Y}} + \underline{J}_Z^X \quad (4.110)$$

and from the integral region,

$$A^E_{H\dot{X}} = B^E_{H\dot{Y}} \quad (4.111)$$

where the interface conditions have been included in this latter equation and the nomenclature is as defined earlier in the chapter. For the two equations above (4.110) and (4.111) the term in $\frac{\partial Hx}{\partial n}$ has been transformed to $\frac{\partial H\dot{y}}{\partial t}$ which in turn gives nodal $H\dot{y}$ values. However in the last section it was shown that if $H\dot{y}$ is zero at infinity and $\frac{\partial Hx}{\partial t}$ is uniquely defined then so is the exterior $H\dot{y}$. This therefore suggests that the exterior

Greens integral for H_y ,

$$\begin{aligned} \alpha H_y(p) = & \int_{\Gamma} H_y(\underline{p}_1) \frac{\partial G}{\partial n}(\underline{p}, \underline{p}_1) d\Gamma \\ & - \int_{\Gamma} \frac{\partial H_y}{\partial n}(\underline{p}_1) G(\underline{p}, \underline{p}_1) d\Gamma \quad \underline{p}_1 \in \Gamma \end{aligned} \quad (4.112)$$

where

$$\alpha = \begin{cases} 2\pi & \underline{p} \in \Omega \\ 0 & \underline{p} \notin \Omega + \Gamma \\ \text{interior angle at } \underline{p} & \underline{p} \in \Gamma \end{cases} \quad (4.113)$$

and Ω is the exterior region, may be used in situations where $\frac{\partial H_x}{\partial t}$ is uniquely defined to give a uniquely defined H_y .

The matrix form of equation (4.112) is,

$$C^E_{HY} \underline{E}^b = D^E_{\frac{\partial H_x}{\partial t}} \underline{E}^b \quad (4.114)$$

and since H_x is a single valued function associated with (4.114) is the equation,

$$\int_{\Gamma} \frac{\partial H_x}{\partial t} d\Gamma = 0 \quad (4.115)$$

Once again if $\frac{\partial H_x}{\partial t}$ is replaced by

$$\frac{\partial H_x^{ij}}{\partial t} = \frac{a^i - b^j}{d_{ij}} \quad (4.116)$$

where i and j are adjacent boundary nodes, a^i and b^j are constants associated with the relevant node and d_{ij} is the distance between the nodes, then equation

(4.114) may be rewritten,

$$C_{HY}^{EI^b} = D^E C \underline{K}^E \quad (4.117)$$

where C is the matrix given in section 4.3.1 (equation (4.88)) and $\underline{K}^E$ is a vector of constants associated with the boundary nodes of Ω . The equation (4.117) incorporates the condition (4.115) and replaces it by the condition that one of the constants $\underline{K}^E$ is arbitrary for each complete boundary in Ω .

Collecting together the equations for the formulation, they are without loss of generality for H_x ,

$$A_{HX}^{II} = B_{HY}^{II^b} + \underline{J}_Z^x \quad (4.118)$$

from the differential domain, and

$$A_{HX}^{EI^b} = B_{HY}^{EI^b} \quad (4.119)$$

from the integral domain, and

$$C_{HY}^{EI^b} = D^E C \underline{K}^E \quad (4.120)$$

from the integral domain for H_y , where the interface conditions for H_y have been included and associated with this last equation is at least one arbitrary constant. It now remains to try the formulation and see what results are produced.

4.7 THE SUPPLEMENTARY FORMULATION IN ACTION

A similar situation arises as for the formulation discussed in section 4.4 where for odd numbered boundary noded shapes the formulation works admirably, but for even numbered boundary noded shapes the difficulties are more acute than experienced before.

4.7.1 Odd Numbered Boundary Noded Shapes

The problem considered is the same as in section 4.4.1, and fig.4.4 sets out the mesh. The problem is the isolated infinitely long rectangular conductor carrying a current of 1.0A/m^2 and the results for H_x only will be presented. The formulation is as set out in equations (4.118), (4.119) and (4.120) where the surface of the conductor is used as the boundary between the integral and differential domains. Two solutions were obtained and the H_x values for both were the same and are presented in table 4.8. Comparing table 4.8 with table 4.1 shows that the H_x results are exactly the same for this formulation and the earlier one. In fact the solutions are the same to within the rounding error of the computer. The results for $\overset{b}{H_y}$ are also the same for both solutions and the results are presented in table 4.9. Comparing the results in table 4.9 with the analytic results in table 3.6 gives an unfavourable comparison, however it is instructive to notice that the adjacent nodal differences for table 4.9 are the same as for tables 4.2 and 4.3 and therefore $\frac{\partial \overset{b}{H_y}}{\partial t}$ is still the important function as far as the results are concerned. Moreover this suggests that the finer the mesh the closer $\overset{b}{H_y}$ will approach the actual values of H_y . The values of $\underline{K}^E$ for each solution are presented in tables 4.10 and 4.11 and the chosen value is indicated. The two tables show that although the nodal

<u>Hx (calculated)</u>					
-.4512	-.6707	-.7093	-.7093	-.6707	-.4512
-.2170	-.3269		-.3514	-.3269	-.2170
-9 E-5	-7 E-4		-3 E-4	-7 E-4	-9 E-5
.2168	.3253		.3510	.3253	.2168
.4513	.6720		.7152	.6720	.4513

TABLE 4.8

<u>$\frac{b}{Hy's}$</u>					
-.5237	-.2496	-.0840	-.0840	.2496	.5237
-.8910					.8910
-.6681					.6681
-.8915					.8915
-.5246	-.2501	0		.2501	.5246

TABLE 4.9

<u>K^E</u>					
-.9211	-1.0682	-1.1082	-1.1082	-1.0682	-.9211
-.7260					-.7260
0	← chosen				0
-.6956					-.6956
1.2553	-.9644	1.4814		-.9644	1.2553

TABLE 4.10

<u>K^E</u>					
-.1951	-.3422	-.3822	-.3822	-.3422	-.1951
0					0
.7260				chosen	.7260
.0304					.0304
1.9813	-.2384	2.2074		-.2384	1.9813

TABLE 4.11

values are different for the two solutions the differences in adjacent nodal values for the two results are the same. This is precisely as expected, a unique approximation to $\frac{\partial H_x}{\partial t}$ has given rise to a unique approximation to H_y for the particular mesh in question.

4.7.2 Even Numbered Boundary Noded Shapes

For even numbered boundary noded shapes the formulation encapsulated in equations (4.118), (4.119) and (4.120) fails since there is not enough information to define the problem uniquely. This demonstrates itself by a deficiency of two equations and since only one $\underline{K}^E$ value is arbitrary another condition is required. The author has been unable to find supplementary satisfactory conditions. Of course this is assuming that the whole problem is treated, and apart from $\underline{H}$ zero at infinity no other boundary condition is given.

Although the above statement is generally true there is one very special case for which the formulation can be made to work. The situation is that of a single shape with an even number of boundary nodes distributed in a symmetrical manner. For example consider the mesh given in fig.4.5 of the isolated infinitely long rectangular conductor. The equations for H_x may be written as follows, over the differential region (the conductor),

$$A^I \underline{H_x}^I = B^I \underline{H_y}^{I^b} + \underline{J_z}^x \quad (4.121)$$

and over the infinite exterior integral region using the conductor surface,

$$A^E \underline{Hx}^b = B^E \underline{Hy}^b \quad (4.122)$$

where the interface conditions have been included in this latter equation. The exterior integral for Hy becomes over the same domain as that for Hx ,

$$C^E \underline{Hy}^b = R^E \underline{K}^E \quad (4.123)$$

where one of the constants $\underline{K}^E$ is arbitrary. As has been discussed earlier in this chapter equations (4.121) and (4.122) are deficient by two equations. However the matrix R^E has rank $n_b - 2$, where n_b is the number of nodes on the boundary, while C^E has rank n_b and therefore there exists a permutation matrix S such that SR^E has the form,

$$SR^E = \begin{bmatrix} U \\ V \end{bmatrix} \quad (4.124)$$

where U is a fully populated matrix of order $(n_b - 2 \times n_b)$ and V is a zero matrix of order $(2 \times n_b)$. Thus the equation,

$$SC^E \underline{Hy}^b = SR^E \underline{K}^E \quad (4.125)$$

produces the necessary two equations relating $\underline{Hy}^b$ to complete the solution i.e. the bottom two equations.

Of course the permutation matrix S is very much problem dependent and therefore the question arises as to whether or not there is an easier way to construct the solution. It turns out to be instructive to follow the natural development from (4.125), that is the two equations

relating $\underline{H}_y^{\text{I}^b}$ are combined with equations (4.121) and (4.122) and a solution is produced for $\underline{H}_x^{\text{I}}$ and $\underline{H}_y^{\text{I}^b}$. From this solution (4.125) can be reduced to give,

$$\underline{U}\underline{K}^{\text{E}} = \underline{W} \quad (4.126)$$

where $\underline{W}$ is the evaluation of $\text{SC}^{\text{E}}_{\underline{H}_y^{\text{I}^b}}$. The matrix $\underline{U}$ is of order $(n_b-2 \times n_b)$ and of rank n_b-2 therefore an arbitrary choice for two values of $\underline{K}^{\text{E}}$ will allow the remaining $\underline{K}^{\text{E}}$ to be found, as long as the reduced $\underline{U}$ matrix maintains a rank of n_b-2 . This of course gives the key to a much simpler solution, the choice of two $\underline{K}^{\text{E}}$ values initially, allows from equation (4.123) $\underline{H}_y^{\text{I}^b}$ to be replaced by n_b-2 unknown parameters (the reduced $\underline{K}^{\text{E}}$) and thus the modified equations (4.121) and (4.122) can be solved for $\underline{H}_x^{\text{I}}$ and the reduced $\underline{K}^{\text{E}}$ vector. The solution for $\underline{H}_y^{\text{I}^b}$ can thus be found from (4.123). Tables 4.12 to 4.15 present the results for two solutions that were obtained. The $\underline{H}_x^{\text{I}}$ and $\underline{H}_y^{\text{I}^b}$ values were exactly the same for both solutions and are given in tables 4.12 and 4.13. The results given in table 4.12 are exactly those given in table 3.1 and therefore the comparison with the analytic results is as given in table 3.2. The $\underline{K}^{\text{E}}$ values given in tables 4.14 and 4.15 are a little stranger than those that have been obtained before but a pattern still exists. The values are reflectional about a vertical line through the centre of the model and the difference of nodal values reflectional about a horizontal line through the centre of the model is the same for both tables.

<u>Hx (calculated)</u>				
-.4513	-.6720	-.7151	-.6720	-.4513
-.2169	-.3256	-.3512	-.3256	-.2169
0	0	0	0	0
.2169	.3256	.3512	.3256	.2169
.4513	.6720	.7151	.6720	.4513

TABLE 4.12

<u>$\frac{H_y^b}{H_y^s}$</u>				
-.5255	-.2505	0	.2505	.5255
-.8921				.8921
-.6692				.6692
-.8921				.8921
-.5255	-.2505	0	.2505	.5255

TABLE 4.13

<u>K^E</u>				
-2.3057	7.3700	-2.6038	7.3700	-2.3057
5.1720				5.1720
0	← chosen			0
5.2027				5.2027
-1.3179	7.4810	→ 0	7.4810	-1.3179

TABLE 4.14

<u>K^E</u>				
-1.0833	-.0775	-1.2980	-.0775	-1.0833
-.0307				-.0307
0	← chosen			0
0	← chosen			0
1.0906	.0335	1.3057	.0335	1.0906

TABLE 4.15

4.8 CONCLUSIONS

It has been shown that two closely related formulations can be used to construct a solution for two dimensional magnetostatic problems where the relative permeability of the solution domain is unity. No comparison of the techniques has been given because essentially they serve different situations. The former formulation discussed in section 4.4 is only viable in situations where the field components have only a loose link or specific boundary values are known. The latter formulation is useful where the field components have a closer relationship such as at permeable material interfaces or on the surfaces of eddy current regions. However in both cases it is better if the shapes undergoing analysis have an odd number of boundary nodes. For shapes with an even number of boundary nodes the formulation is far from adequate and obviously the nature of this particular difficulty requires more investigation.

CHAPTER 5

PERMEABLE MATERIALS

5.1 INTRODUCTION

Permeable materials may be regarded as linear or non-linear which is an expression of the relationship between $\underline{H}$ and $\underline{B}$. For many practical problems it is necessary to analyse non-linear materials and there are different ways in which this has been achieved^{5.1,5.2,5.3}. However in this thesis the analysis has been restricted to situations where μ is constant for each material. Although this is a very restrictive choice it is enough to introduce a major difficulty in considering a solution for $\underline{H}$ directly; the author also believes that there is no great difficulty in lifting this restriction.

Hitherto all of the situations considered have avoided discontinuities in the relative permeability. This has been by design since it was recognised at a very early stage that as $\underline{H}$ was the vector being sought the standard interface conditions fail at edges or corners in three dimensional problems or at corners in two dimensional problems. The discussion will begin with an examination of a two dimensional corner and then a few comments will be made concerning an extension to three dimensional situations.

5.1.1 The Failure of the Standard Interface Conditions at Two Dimensional Corners

The standard interface conditions are continuity of tangential $\underline{H}$ and continuity of normal $\underline{B}$. For ease of explanation a right angled corner is considered as shown in fig.5.1.

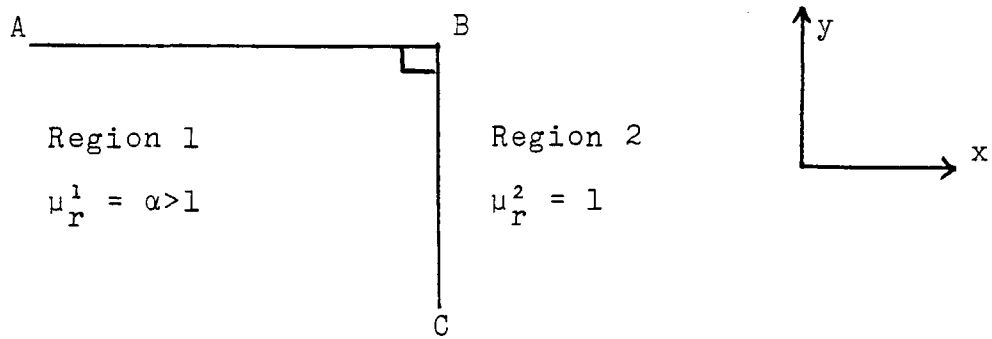


FIG.5.1

The standard interface conditions give, along AB,

$$H_x^1 = H_x^2 \quad (5.1)$$

$$\mu_r^1 H_y^1 = \mu_r^2 H_y^2 \quad (5.2)$$

and along BC,

$$\mu_r^1 H_x^1 = \mu_r^2 H_x^2 \quad (5.3)$$

$$H_y^1 = H_y^2 \quad (5.4)$$

where the numbers indicate the region and $\underline{H} = (H_x, H_y, 0)$.

The inconsistency of these equations at the point B now becomes clear and this inconsistency only disappears when $\alpha=1$.

The difficulty pointed out above at the point B in fig. 5.1 is not confined to a formulation seeking $\underline{H}$, and

therefore it was thought that an examination of how this problem is dealt with by the more popular potential formulations might yield valuable information.

5.1.2 Potential Formulation at a Two Dimensional Corner

The vector $\underline{A}$ is defined such that,

$$\underline{\nabla} \times \underline{A} = \underline{B} \quad (5.5)$$

and for two dimensional transverse magnetic problems $\underline{A}$ may be considered as,

$$\underline{A} = (0, 0, A_z) \quad (5.6)$$

The conditions along AB as in fig.5.1, equations (5.1) and (5.2) give respectively in terms of A_z ,

$$\frac{1}{\mu_r} \frac{\partial A_z^1}{\partial y} = \frac{\partial A_z^2}{\partial y} \quad (5.7)$$

and,

$$\frac{\partial A_z^1}{\partial x} = \frac{\partial A_z^2}{\partial x} \quad (5.8)$$

since μ_r^2 is unity, and along BC equations (5.3) and (5.4) give respectively,

$$\frac{\partial A_z^1}{\partial y} = \frac{\partial A_z^2}{\partial y} \quad (5.9)$$

and

$$\frac{1}{\mu_r} \frac{\partial A_z^1}{\partial x} = \frac{\partial A_z^2}{\partial x} \quad (5.10)$$

These above four equations rationalise to give,

$$A_z^1 = A_z^2 \quad (5.11)$$

and,

$$\frac{1}{\mu_r} \frac{\partial A_z^1}{\partial n} = \frac{\partial A_z^2}{\partial n} \quad (5.12)$$

where $\frac{\partial}{\partial n}$ is the outward normal from region 1. From equation (5.11) it can be seen that continuity of normal $\underline{B}$ is assured since A_z itself is continuous, however this passes no comment on the relationship of $\underline{H}$ across the interface at B (see fig.5.1). As far as equation (5.12) is concerned this is precisely the natural boundary condition for A_z and is thus only satisfied in an average sense and therefore likewise the continuity of tangential $\underline{H}$.

The two dimensional $\underline{A}$ formulation considers the interface conditions for $\underline{H}$ along surface facets and is never concerned with what happens to $\underline{H}$ at the corner. The elegance with which the difficulty of interface conditions for $\underline{H}$ at a two dimensional corner is side stepped can only be admired but unfortunately sheds no light on how to cope in terms of $\underline{H}$.

The two dimensional formulation for Ω defined such that,

$$\underline{H} = -\nabla\Omega \quad (5.13)$$

turns out to give the reverse results to $\underline{A}$, continuity of tangential $\underline{H}$ is assured since Ω is continuous while the continuity of normal $\underline{B}$ is satisfied in an average sense since it gives rise to the natural boundary condition for Ω . However once again no real light is shed on how to cope in terms of $\underline{H}$.

5.1.3 Interface Conditions for $\underline{H}$ at a Two Dimensional Corner

Fig.5.2 represents a general two dimensional corner point c . The equations that are to be satisfied at the

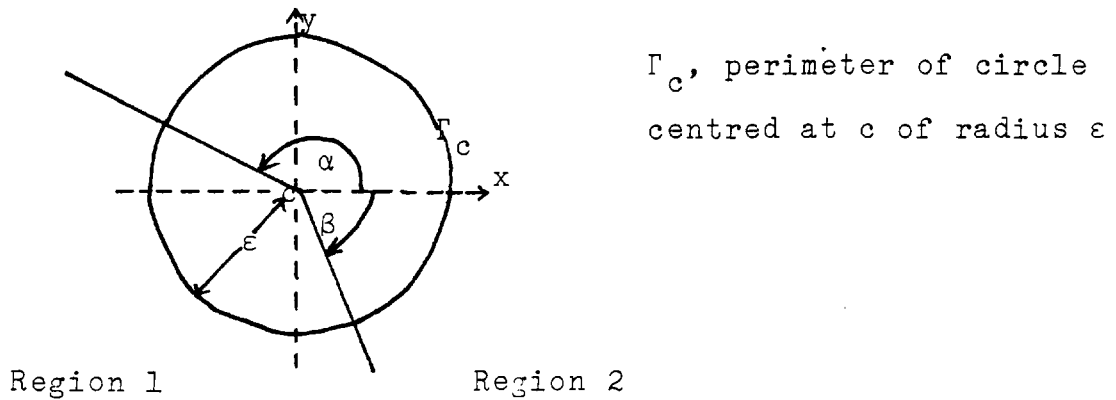


FIG.5.2

point c are,

$$\underline{\nabla} \cdot \underline{B} = 0 \quad (5.14)$$

and,

$$\underline{\nabla} \times \underline{H} = \underline{J} \quad (5.15)$$

Equation (5.14) is applied over the circle in fig. 5.2 by means of Gauss' divergence theorem^{5.4} such that,

$$\int_{\Gamma_c} B_n d\Gamma_c = 0 \quad (5.16)$$

where B_n is the outward normal component with respect to Γ_c . For the two dimensional case considered B_n may be expressed as follows,

$$B_n = B_x \cos \theta + B_y \sin \theta \quad (5.17)$$

where θ is measured from the x axis in an anti-clockwise direction. If the assumption is made that ϵ , the radius of the circle centred at c, is small B_x and B_y may be expressed as follows on Γ_c ,

$$B_q(\epsilon, \theta) = B_q(0, \theta) + \epsilon \frac{\partial B_q(\epsilon, \theta)}{\partial r} - \frac{\epsilon^2}{2} \frac{\partial^2 B_q(\epsilon, \theta)}{\partial r^2} + O(\epsilon) \quad (5.18)$$

where $q \in (x, y)$. When equations (5.18) and (5.17) are combined (5.16) can be re-expressed as,

$$\int_{-\beta}^{\alpha} \left(\mu_2 (H_x^2 \cos \theta + H_y^2 \sin \theta) + O(\epsilon) \right) \epsilon d\theta + \int_{\alpha}^{\beta} \left(\mu_1 (H_x^2 \cos \theta + H_y^2 \sin \theta) + O(\epsilon) \right) \epsilon d\theta = 0 \quad (5.19)$$

where the super fix c indicates the value at point c, and the numbers the region from which each quantity has been drawn. Since ϵ is non zero equation (5.19) may be divided through by it, and since ϵ is small the terms of order ϵ can be grouped outside the integrals to give,

$$\int_{-\beta}^{\alpha} \left(\mu_2 (H_x^2 \cos \theta + H_y^2 \sin \theta) \right) d\theta + \int_{\alpha}^{\beta} \left(\mu_1 (H_x^2 \cos \theta + H_y^2 \sin \theta) \right) d\theta + O(\epsilon) = 0 \quad (5.20)$$

Since the two left hand integrals are independent of ϵ the term $O(\epsilon)$ must be identically zero and hence,

$$\begin{aligned} \mu_2 \int_{-\beta}^{\alpha} (Hx^c \cos \theta + Hy^c \sin \theta) d\theta \\ + \mu_1 \int_{\alpha}^{\beta} (Hx^c \cos \theta + Hy^c \sin \theta) d\theta = 0 \quad (5.21) \end{aligned}$$

which gives after integration,

$$\begin{aligned} \mu_2 \left\{ Hx^c (\sin \alpha + \sin \beta) - Hy^c (\cos \alpha - \cos \beta) \right\} \\ = \mu_1 \left\{ Hx^c (\sin \alpha + \sin \beta) - Hy^c (\cos \alpha - \cos \beta) \right\} \quad (5.22) \end{aligned}$$

where the super fix c has been dropped but the Hx and Hy values are at the point c. Equation (5.15) is applied over the circle in fig.5.2 by means of Stokes' Theorem^{5.5} such that,

$$\int_{\Gamma_c} \underline{H} \cdot d\underline{l} = I \quad (5.23)$$

where I is the enclosed current within the circle and $d\underline{l}$ is a vector tangential to Γ_c moving in an anti-clockwise direction. The term $\underline{H} \cdot d\underline{l}$ gives rise to tangential $\underline{H}$, H_t , which may be expressed as,

$$H_t = H_y \cos \theta - H_x \sin \theta \quad (5.24)$$

where θ is in the anti-clockwise sense and measured from the x axis. Utilising equations (5.24) and (5.18) equation (5.23) becomes,

$$\begin{aligned} \int_{-\beta}^{\alpha} (Hy^c \cos \theta - Hx^c \sin \theta + O(\epsilon)) \epsilon d\theta \\ + \int_{\alpha}^{\beta} (Hy^c \cos \theta - Hx^c \sin \theta + O(\epsilon)) \epsilon d\theta = I^1 + I^2 \quad (5.25) \end{aligned}$$

where the super fix c indicates the value at point c and the numbers the region from which each quantity has been drawn. Both I^1 and I^2 are dependent on ϵ^2 and since ϵ is not zero equation(5.25) may be divided through by it to give,

$$\int_{-\beta}^{\alpha} (H\tilde{y}^c \cos\theta - H\tilde{x}^c \sin\theta) d\theta + \int_{\alpha}^{\beta} (H\tilde{y}^c \cos\theta - H\tilde{x}^c \sin\theta) d\theta + O(\epsilon) = 0 \quad (5.26)$$

where since ϵ is small the terms of order ϵ have been grouped outside the integrals. Since the two integrals in (5.26) are independent of ϵ the term $O(\epsilon)$ must be identically zero and therefore after integration (5.26) gives,

$$\begin{aligned} & H\tilde{y}(\sin\alpha + \sin\beta) + H\tilde{x}(\cos\alpha - \cos\beta) \\ & = H\tilde{y}(\sin\alpha + \sin\beta) + H\tilde{x}(\cos\alpha - \cos\beta) \end{aligned} \quad (5.27)$$

where the super fix c has been dropped but the Hx and Hy values are at the point c.

Using the following trigonometric identities,

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \quad (5.28)$$

$$\cos\alpha - \cos\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right) \quad (5.29)$$

equations (5.22) and (5.27) can be re-expressed

respectively as,

$$\begin{aligned} \mu_2 (H\hat{x}\cos(\frac{\alpha-\beta}{2}) - H\hat{y}\sin(\frac{\beta-\alpha}{2})) \\ = \mu_1 (H\hat{x}\cos(\frac{\alpha-\beta}{2}) - H\hat{x}\sin(\frac{\beta-\alpha}{2})) \end{aligned} \quad (5.30)$$

$$\begin{aligned} H\hat{y}\cos(\frac{\alpha-\beta}{2}) + H\hat{x}\sin(\frac{\beta-\alpha}{2}) \\ = H\hat{y}\cos(\frac{\alpha-\beta}{2}) + H\hat{x}\sin(\frac{\beta-\alpha}{2}) \end{aligned} \quad (5.31)$$

where both equations (5.30) and (5.31) have been divided through by $\sin(\frac{\alpha+\beta}{2})$. This division is only acceptable as long as $\sin(\frac{\alpha+\beta}{2})$ is not zero. If $\sin(\frac{\alpha+\beta}{2})$ is zero then,

$$\alpha + \beta = 2n\pi \quad (5.32)$$

for any integer n , and equation (5.32) is satisfied only when the interface defined by α and β is impractical and thus the division is justified. Equations (5.30) and (5.31) combine to give

$$\begin{aligned} 2H\hat{y} = \left(1 - \frac{\mu_1}{\mu_2}\right) (H\hat{y}\cos(\alpha-\beta) - H\hat{x}\sin(\alpha-\beta)) \\ + \left(\frac{\mu_1}{\mu_2} + 1\right) H\hat{y} \end{aligned} \quad (5.33)$$

and,

$$\begin{aligned} 2H\hat{x} = \left(\frac{\mu_1}{\mu_2} - 1\right) (H\hat{x}\cos(\alpha-\beta) + H\hat{y}\sin(\alpha-\beta)) \\ + \left(\frac{\mu_1}{\mu_2} + 1\right) H\hat{x} \end{aligned} \quad (5.34)$$

as the relationship for $H\hat{y}$ and $H\hat{x}$ in terms of $H\hat{y}$ and $H\hat{x}$. Equations (5.33) and (5.34) represent the interface conditions for Hx and Hy at a general two dimensional

corner. It is perhaps worth pointing out that α is described in an anti-clockwise direction from the x axis whereas β is described in a clockwise direction.

5.1.4 An Examination of the General Two Dimensional Interface Conditions

Having derived equations (5.33) and (5.34) the first objective is to gain some confidence in them. The angles α and β in fig.5.2 may be chosen to be any value such that

$$0 < \alpha + \beta < 2\pi \quad (5.35)$$

since within this constraint practical interfaces ensue. Therefore if α is chosen to be 180° and β to be 0° the interface conditions give,

$$H_y^2 = \frac{\mu_1}{\mu_2} H_y^1 \quad (5.36)$$

and,

$$H_x^2 = H_x^1 \quad (5.37)$$

as should be expected.

For a straight line interface at any angle,

$$\alpha + \beta = \pi \quad (5.38)$$

and equations (5.33) and (5.34) reduce to,

$$2H_y^2 = \left(1 - \frac{\mu_1}{\mu_2}\right) (H_y^1 \cos(\alpha - \pi) - H_x^1 \sin(\alpha - \pi)) + \left(\frac{\mu_1}{\mu_2} + 1\right) H_y^1 \quad (5.39)$$

and,

$$2H_x^2 = \left(\frac{\mu_1}{\mu_2} - 1\right) (H_x^1 \cos(\alpha - \pi) + H_y^1 \sin(\alpha - \pi)) + \left(\frac{\mu_1}{\mu_2} + 1\right) H_x^1 \quad (5.40)$$

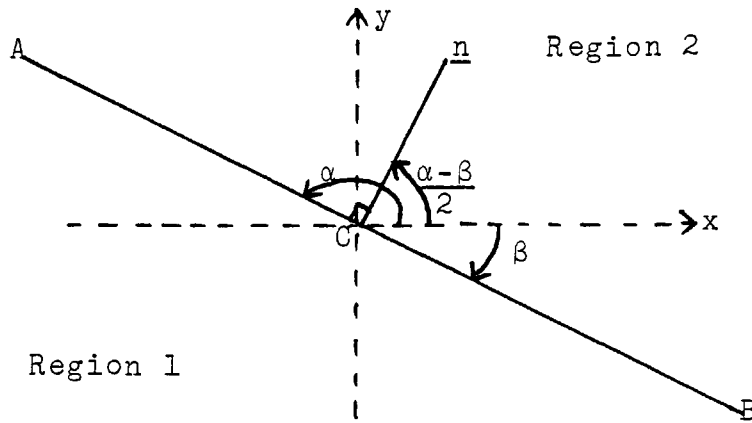


FIG.5.3

The equations (5.39) and (5.40) hold for the straight line interface depicted in fig.5.3. Applying continuity of normal $\underline{B}$ at point C in fig.5.3 across the straight line interface gives,

$$\begin{aligned} B_x^1 \cos\left(\frac{\alpha-\beta}{2}\right) + B_y^1 \sin\left(\frac{\alpha-\beta}{2}\right) \\ = B_x^2 \cos\left(\frac{\alpha-\beta}{2}\right) + B_y^2 \sin\left(\frac{\alpha-\beta}{2}\right) \end{aligned} \quad (5.41)$$

while continuity of tangential $\underline{H}$ gives,

$$\begin{aligned} H_y^1 \cos\left(\frac{\alpha-\beta}{2}\right) - H_x^1 \sin\left(\frac{\alpha-\beta}{2}\right) \\ = H_y^2 \cos\left(\frac{\alpha-\beta}{2}\right) - H_x^2 \sin\left(\frac{\alpha-\beta}{2}\right) \end{aligned} \quad (5.42)$$

From equations (5.41), (5.42) and (5.38) it is hoped to derive equations (5.39) and (5.40) in order that the generalised interface equations might be justified.

However before taking those steps it is instructive to notice that equations (5.41) and (5.42) are precisely

the same as (5.30) and (5.31) the equations used to derive the general interface conditions, and this of necessity implies the hoped for consistency i.e. equations (5.41), (5.42) and (5.38) will give equations (5.39) and (5.40). However there is a more important point here, both equations (5.41) and (5.42) are dependent on $\frac{\alpha-\beta}{2}$ and therefore the general interface conditions at any given point are not dependent on the faces that meet there solely but on their angular difference. For example in fig.5.3 the conditions at C are referred to the straight line segment ACB, such that AC has angular displacement α from the x axis and CB β , however if AC is adjusted so that its angular displacement is $\alpha+\gamma$ the conditions remain the same at C if CB is also adjusted so that its angular displacement is $\beta+\gamma$. This is best appreciated in diagrammatic form and fig.5.4 shows a number of different faces meeting at C for which the conditions (5.41) and (5.42) hold.

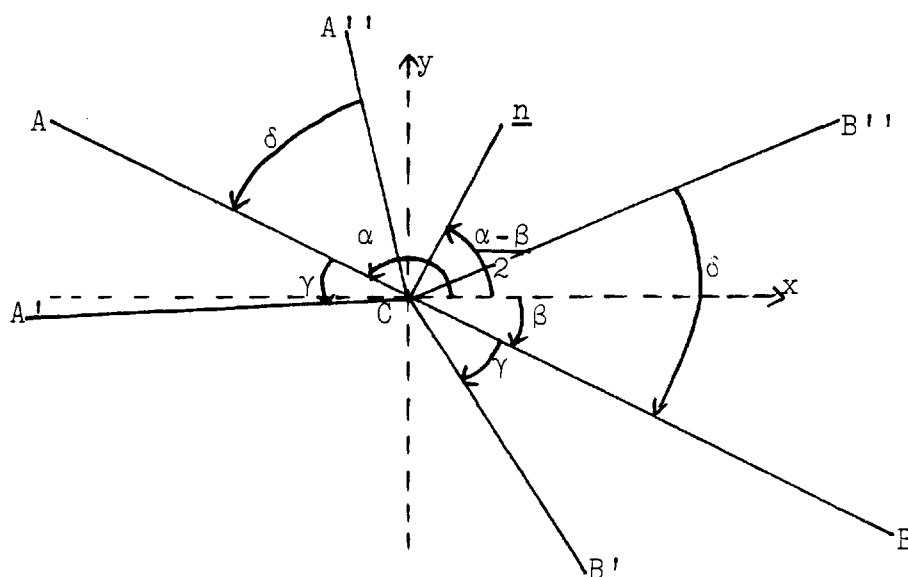


FIG.5.4

If conditions (5.41) and (5.42) are required to hold at C then the radial line having an angle of $\frac{\alpha-\beta}{2}$ from the x axis must bisect the two faces that meet at C. This suggests an alternative construction for the interface conditions at a general corner C. Consider fig.5.5 which shows a general two dimensional corner, C, constructed by the two faces AC and CB, at which the

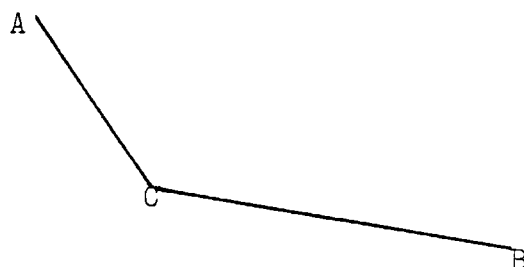


FIG.5.5

interface conditions are required. The first step is to construct a line passing through C so that as shown in fig.5.6 the angles between the constructed line and the two faces AC, CB are equal. The interface conditions at C with respect to A'CB' the straight line

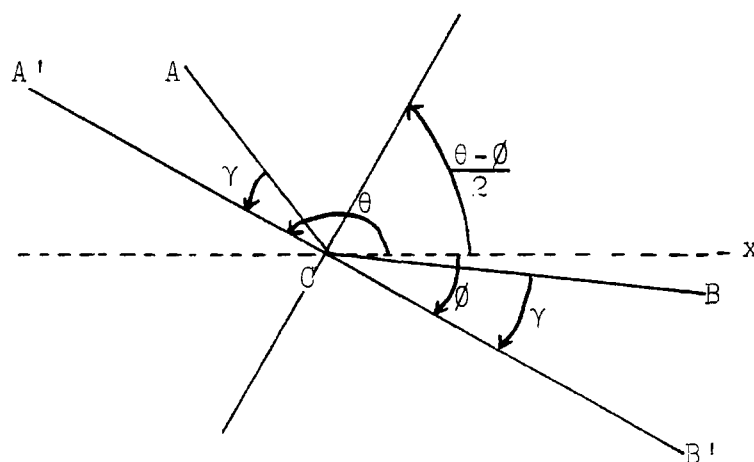


FIG.5.6

segment are equations (5.41) and (5.42) where

$$\alpha = \theta \quad (5.43)$$

and,

$$\beta = \emptyset \quad (5.44)$$

However these equations depend on $\frac{\theta - \emptyset}{2}$ which has the same value as $\frac{(\theta - \gamma) - (\emptyset - \gamma)}{2}$ where $(\theta - \gamma)$ and $(\emptyset - \gamma)$ are the angular displacements of AC and CB respectively. Therefore equations (5.41) and (5.42) with

$$\alpha = \theta - \gamma \quad (5.45)$$

and

$$\beta = \emptyset - \gamma \quad (5.46)$$

give the interface conditions at C for fig.5.5. Essentially it appears that the corner averages out the contributions from the two faces.

5.1.5 The General Two Dimensional Interface

Conditions when $\mu_2 \gg \mu_1$

As was stated at the beginning of this chapter, this thesis is only concerned with materials whose relative permeability may be considered constant. However it should be pointed out that the derivation of the interface conditions contained in section 5.1.3 assumed that

$$\underline{B} = \mu \underline{H} \quad (5.47)$$

which is a broader condition than just μ constant and therefore the conditions are more widely applicable than their use in this thesis.

Having made the decision to use materials with constant permeabilities this means that in many practical situations, without loss of generality, $\mu_2 \gg \mu_1$. It is therefore important to see how the interface conditions react and if they give the expected results in this limiting situation.

Two situations will be considered the first of these shown in fig.5.7, that of an interface in the y-z plane, z the plane into the paper.

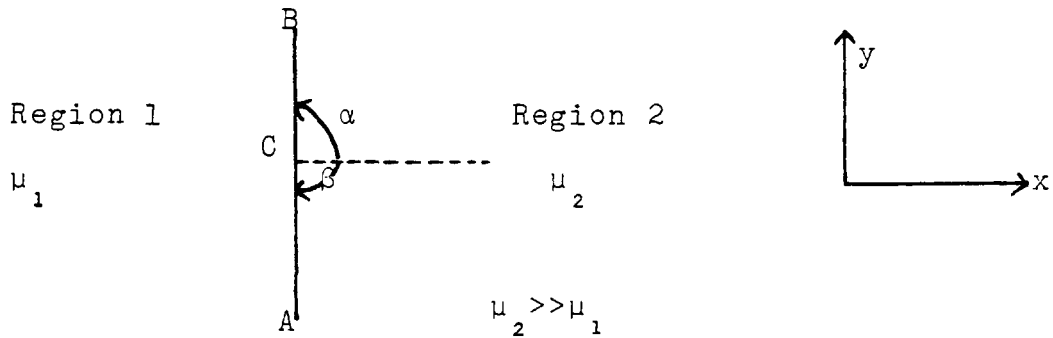


FIG.5.7

Recapping, the general two dimensional interface conditions are,

$$2H\hat{y} = \left(1 - \frac{\mu_1}{\mu_2}\right) (H\hat{y}\cos(\alpha-\beta) - H\hat{x}\sin(\alpha-\beta)) + \left(\frac{\mu_1}{\mu_2} + 1\right) H\hat{y} \quad (5.48)$$

and,

$$2H\hat{x} = \left(\frac{\mu_1}{\mu_2} - 1\right) (H\hat{x}\cos(\alpha-\beta) + H\hat{y}\sin(\alpha-\beta)) + \left(\frac{\mu_1}{\mu_2} + 1\right) H\hat{x} \quad (5.49)$$

With reference to fig.5.7,

$$\alpha = \beta = \frac{\pi}{2} \quad (5.50)$$

and therefore (5.48) reduces to,

$$2H\hat{y} = \left(1 - \frac{\mu_1}{\mu_2}\right)H\hat{y} + \left(\frac{\mu_1}{\mu_2} + 1\right)H\hat{y} = 2H\hat{y} \quad (5.51)$$

irrespective of the value of $\frac{\mu_1}{\mu_2}$. This is to be expected

since tangential $\underline{H}$ must be continuous irrespective of the values of the relative permeabilities. However there is a small trap to beware of, if equation (5.48) is rewritten for $H\hat{y}$ it becomes,

$$2H\hat{y} = \left(1 - \frac{\mu_2}{\mu_1}\right) \left(H\hat{y}\cos(\alpha-\beta) - H\hat{x}\sin(\alpha-\beta)\right) + \left(\frac{\mu_2}{\mu_1} + 1\right) H\hat{y} \quad (5.52)$$

This equation gives rise to the temptation to say, that since $\mu_2 \gg \mu_1$ and therefore $\frac{\mu_2}{\mu_1} \gg 1$, equation (5.52) may be reduced to,

$$2H\hat{y} = - \frac{\mu_2}{\mu_1} \left(H\hat{y}\cos(\alpha-\beta) - H\hat{x}\sin(\alpha-\beta)\right) + \frac{\mu_2}{\mu_1} H\hat{y} \quad (5.53)$$

which in regard to fig.5.7 gives,

$$2H\hat{y} = - \frac{\mu_2}{\mu_1} H\hat{y} + \frac{\mu_2}{\mu_1} H\hat{y} = 0 \quad (5.54)$$

Of course (5.54) is an attractive result but unfortunately this is not the correct way to justify it since if the ignored terms are reinstated (5.54) becomes,

$$2H\hat{y} = \left(1 - \frac{\mu_2}{\mu_1}\right) H\hat{y} + \left(\frac{\mu_2}{\mu_1} + 1\right) H\hat{y} = 2H\hat{y} \quad (5.55)$$

Until the exact values of the cos and sin functions are

known no limiting equations can safely be constructed.

Turning attention to (5.49) this reduces to (with respect to fig.5.7),

$$2H\dot{x} = \left(\frac{\mu_1}{\mu_2} - 1 \right) H\dot{x} + \left(\frac{\mu_1}{\mu_2} + 1 \right) H\dot{x} = 2 \frac{\mu_1}{\mu_2} H\dot{x} \quad (5.56)$$

In the limit $\frac{\mu_1}{\mu_2} \rightarrow 0$ equation (5.56) gives

$$H\dot{x} = 0 \quad (5.57)$$

for finite $H\dot{x}$.

If equation (5.49) is rewritten for $H\dot{x}$ it becomes,

$$2H\dot{x} = \left(\frac{\mu_2}{\mu_1} - 1 \right) \left(H\dot{x} \cos(\alpha - \beta) + H\dot{y} \sin(\alpha - \beta) \right) + \left(\frac{\mu_2}{\mu_1} + 1 \right) H\dot{x} \quad (5.58)$$

which with respect to fig.5.7 gives,

$$H\dot{x} = \frac{\mu_2}{\mu_1} H\dot{x} \quad (5.59)$$

In the limit $\left(\frac{\mu_2}{\mu_1} \rightarrow \infty \right)$ $H\dot{x}$ can only be finite if $H\dot{x}$ is zero, as given in (5.57); this shows the consistency of the equations and also that in the limit, $H\dot{x}$ depends on the solution of the whole problem, since the limiting value of $\frac{\mu_2}{\mu_1} H\dot{x}$ is unknown.

The important point to notice though is that as $\frac{\mu_1}{\mu_2} \rightarrow 0$ the interface conditions begin to produce a boundary condition and this ought to show up in the results and will be a pointer to the fact that they are working correctly.

The second situation is as given in fig.5.8, which shows a two dimensional right angle corner.

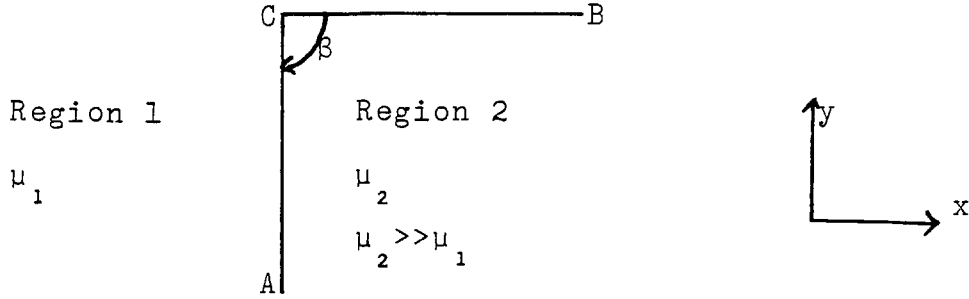


FIG.5.8

With respect to fig.5.8

$$\alpha = 0 \quad (5.60)$$

and

$$\beta = \frac{\pi}{2} \quad (5.61)$$

so that the interface conditions (5.48) and (5.49) give,

$$2H_y^2 = 1 - \frac{\mu_1}{\mu_2} H_x^1 + \frac{\mu_1}{\mu_2} + 1 H_y^1 \quad (5.62)$$

and,

$$2H_x^2 = 1 - \frac{\mu_1}{\mu_2} H_y^1 + \frac{\mu_1}{\mu_2} + 1 H_x^1 \quad (5.63)$$

As $\frac{\mu_1}{\mu_2} \rightarrow 0$ the above two equations become respectively,

$$2H_y^2 = H_x^1 + H_y^1 \quad (5.64)$$

and,

$$2H_x^2 = H_y^1 + H_x^1 \quad (5.65)$$

which gives the condition,

$$H_y^2 = H_x^2 \quad (5.66)$$

This condition could have been anticipated since as

$\frac{\mu_1}{\mu_2} \rightarrow 0$ the $\underline{H}$ field of region 2 tends to zero. However if we rewrite (5.62) and (5.63) in terms of H_y^1 and H_x^1 respectively,

$$2H_y^1 = \left(1 - \frac{\mu_2}{\mu_1}\right) H_x^2 + \left(\frac{\mu_2}{\mu_1} + 1\right) H_y^2 \quad (5.67)$$

and,

$$2H_x^1 = \left(1 - \frac{\mu_2}{\mu_1}\right) H_y^2 + \left(\frac{\mu_2}{\mu_1} + 1\right) H_x^2 \quad (5.68)$$

Equations (5.67) and (5.68) imply that in the limit,

$$H_y^1 = H_y^2 \quad (5.69)$$

and

$$H_x^1 = H_x^2 \quad (5.70)$$

However this is somewhat misleading since if

$$H_x^2 = 0 - \epsilon \quad (5.71)$$

and

$$H_y^2 = 0 + \epsilon \quad (5.72)$$

where ϵ is a small amount, equation (5.67) gives

$$H_y^1 = \frac{\mu_2}{\mu_1} \epsilon = \frac{\mu_2}{\mu_1} H_y^2 \quad (5.73)$$

Therefore depending on the exact nature of the field it appears that at a square corner one of the standard continuity conditions will dominate the other and be the effective condition at that corner for the condition $\mu_2 \gg \mu_1$. This is another feature that should be seen in the results if the conditions are working correctly.

5.2 CONSIDERATION OF INTERFACE CONDITIONS IN THREE DIMENSIONAL SITUATIONS

Essentially in this section two different situations will be considered, firstly three dimensional edges, and secondly a square box type corner. The problem of constructing interface conditions for $\underline{H}$ at three dimensional corners is for a general corner extremely complex, therefore only a simple situation is presented.

The author believes that the complexity of the correct three dimensional interface conditions for $\underline{H}$ at edges and corners is a severe drawback to implementing a three dimensional formulation directly for $\underline{H}$, and it appears that to make such a formulation viable an easier treatment of these interfaces is required.

5.2.1 Three Dimensional Edges

In principle a three dimensional edge can be dealt with in a similar manner to a two dimensional corner. Consider fig.5.9 which shows a semi-infinite slab with four corners (ABCD) where (x',y',z') are a chosen local coordinate system such that AB is parallel to the y' axis. The point E is a general point (except A or B) on the edge AB and is the focal point for the construction of the interface conditions.

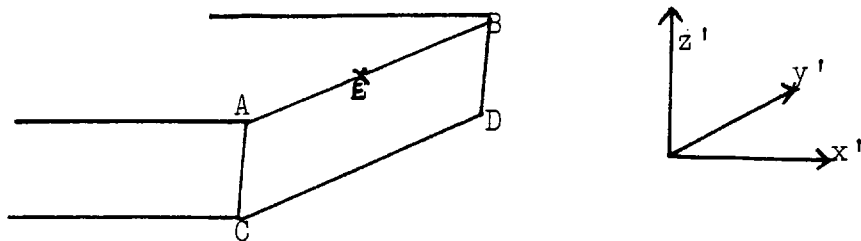


FIG. 5.9

Consider a cylinder of radius ϵ and length 2δ centred on E as shown in fig. 5.10, where that part of the cylinder that passes through the slab has been shaded,

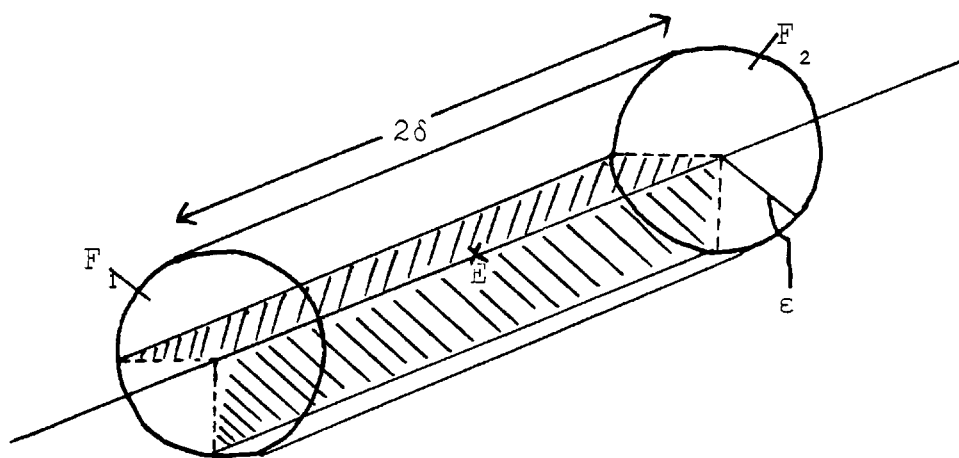


FIG. 5.10

and F_1, F_2 are the two circular faces of the cylinder. Applying Gauss' divergence theorem to the cylinder in fig.5.10 the following equation ensues,

$$\int_{F_1} (-B_{y'}) dF_1 + \int_{F_2} B_{y'} dF_2 + \int_S B_{cyl} dS = 0 \quad (5.74)$$

where $\underline{B}'$ is locally defined to be consistent with the axis i.e. $(B_{x'}, B_{y'}, B_{z'})$, and where B_{cyl} is the normal component of $\underline{B}'$ with respect to S the rounded surface of the cylinder. If for simplicity,

$$\delta = \epsilon \quad (5.75)$$

equation (5.74) may be expanded to give,

$$\begin{aligned} \int_{F_2} (B_{y'}^E + O(\epsilon)) dF_2 - \int_{F_1} (B_{y'}^E + O(\epsilon)) dF_1 \\ + \int_{-\epsilon}^{+\epsilon} \left(\int_0^{2\pi} B_{cyl} \epsilon d\theta \right) dy' = 0 \end{aligned} \quad (5.76)$$

where the E super fix implies the value at the point E, and θ is an angle in the $x'-z'$ plane centred at E. If B'_{cyl} is the value of B_{cyl} on the $x'-z'$ plane passing through E, and since the areas of faces F_1 and F_2 are of order ϵ^2 , equation (5.76) becomes,

$$\begin{aligned} (B_{y'}^E \pi \epsilon^2 + O(\epsilon^3)) - (B_{y'}^E \pi \epsilon^2 + O(\epsilon^3)) \\ + \epsilon^2 \int_0^{2\pi} B'_{cyl} d\theta + O(\epsilon^3) = 0 \end{aligned} \quad (5.77)$$

which on dividing through by ϵ produces,

$$\epsilon \int_0^{2\pi} B'_{\text{cyl}} d\theta + O(\epsilon^2) = 0 \quad (5.78)$$

where the integral is nothing more than the divergence theorem applied to a circle centred at E in the x' - z' plane with radius ϵ . A similar derivation can be made for

$$\underline{\nabla} \times \underline{H} = \underline{J} \quad (5.79)$$

and therefore the interface conditions at point E in fig.5.9 for $H_{x'}$ and $H_{z'}$ can be constructed from an analysis of the two dimensional corner shown in fig.5.11 which shows a section through E in the x' - z' plane.

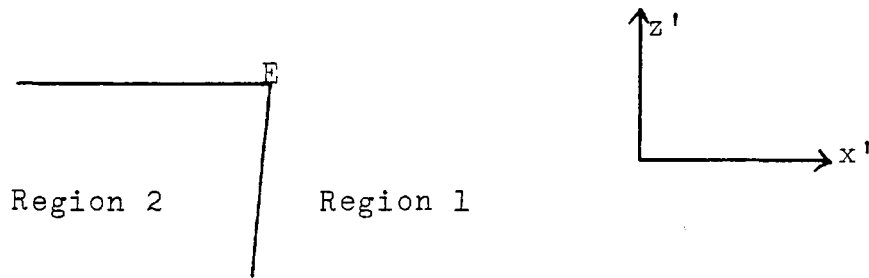


FIG.5.11

The component $H_{y'}$ is continuous across the interface at E.

Since both $\underline{B}'$ and the coordinate system have been locally defined two transformations are now required to find the actual interface conditions for the global $\underline{B}$. One transformation takes $(x', y', z') \rightarrow (x, y, z)$, the global

cartesian coordinate system, and the other takes $(B_{x'}, B_{y'}, B_{z'}) \rightarrow (B_x, B_y, B_z)$ the globally defined $\underline{B}$.

5.2.2 Interface Conditions for $\underline{H}$ at a Practical Three Dimensional Corner

In principle the approach used for two dimensional corners is easily extended to a three dimensional corner. Recapping, the two equations to apply at the corner are,

$$\underline{\nabla} \cdot \underline{B} = 0 \quad (5.80)$$

and,

$$\underline{\nabla} \times \underline{H} = \underline{J} \quad (5.81)$$

which in integral form give rise to, respectively,

$$\int_V \underline{\nabla} \cdot \underline{B} \, dV = \int_S \underline{n} \cdot \underline{B} \, dS \quad (5.82)$$

Gauss' divergence theorem, and,

$$\int_V \underline{\nabla} \times \underline{H} \, dV = \int_S \underline{n} \times \underline{H} \, dS \quad (5.83)$$

where V is a volume in which (5.80) and (5.81) hold, and S its enclosing surface. Although in principle equations (5.82) and (5.83) provide enough information to determine the corner condition in practice the application of the equations may well be far from simple.

Consider fig.5.12 which shows part of a cube and in particular one of its corners A. The edges of the cube

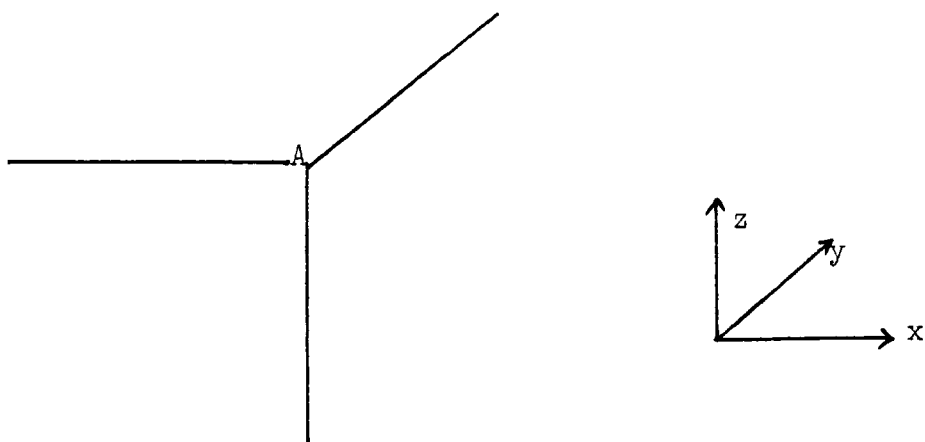


FIG.5.12

are parallel to the coordinate system displayed, and A is an exterior corner. The interface conditions for $\underline{H}$ at point A will be constructed by an application of equations (5.82) and (5.83) to a small cube centred on A of length, breadth and depth 2ε as shown in fig.5.13.

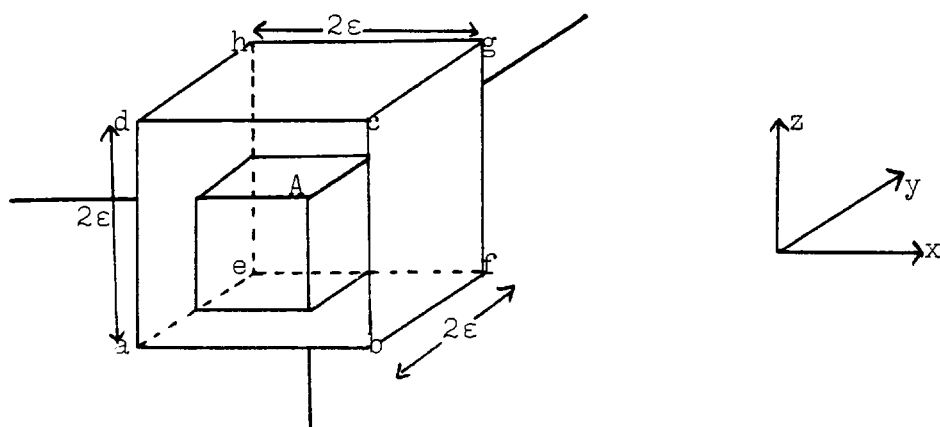


FIG.5.13

The first equation dealt with will be (5.82). On the face abcd the normal component of $\underline{B}$ is $-B_y$ and for ϵ small, B_y on abcd may be written,

$$B_y = B_y(x_A, y_A, z_A) + O(\epsilon) \quad (5.84)$$

where the coordinates of the point A have been taken to be (x_A, y_A, z_A) . Therefore the contribution to equation (5.82) becomes,

$$- \int_{abcd} B_y \, dx dz = - B_y(x_A, y_A, z_A) \epsilon^2 + O(\epsilon^3) \quad (5.85)$$

For reasons of convenience from now on $B_y(x_A, y_A, z_A)$ will be abbreviated to B_y^A . In fig. 5.13 there are two different regions, that exterior to the large cube, region 1, and the large cube itself, region 2. In equation (5.85) B_y is essentially a quantity arising from region 1 and therefore B_y^A will also be given a super fix indicating from which region it arises, hence equation (5.85) may now be written,

$$- \int_{abcd} B_y \, dx dz = - B_y^{A,1} \epsilon^2 + O(\epsilon^3) \quad (5.86)$$

Of the other five faces of the small cube two of them are totally within region 1, bfgc and dhgc, and a similar analysis as that above for these two faces produces a contribution to (5.82) from bfgc of

$$\int_{bfgc} B_x \, dy dz = B_x^{A,1} \epsilon^2 + O(\epsilon^3) \quad (5.87)$$

and a contribution to (5.82) from dhgc of,

$$\int_{dhgc} B_z \, dx dy = B_z^{A,1} \epsilon^2 + O(\epsilon^3) \quad (5.88)$$

For the remaining three faces of the small cube, abfe, efgh, aehd part of the face lies in region 1 and part in region 2. The above technique used for the three faces totally in region 1 may be adapted to handle, for example, the contribution from abfe, one of the faces in both regions, to (5.82) which may be written,

$$\int_{abfe} -B_z \, dx dy = \int_{\Omega_1} -B_z^1 \, dx dy + \int_{\Omega_2} -B_z^2 \, dx dy \quad (5.89)$$

where Ω_i is the area of abfe in region i and B_z^i is B_z pertaining to the i th region $i=1,2$. Equation (5.89) can be made clearer by reference to fig.5.14 which is a plan view of the face abfe showing which part of the

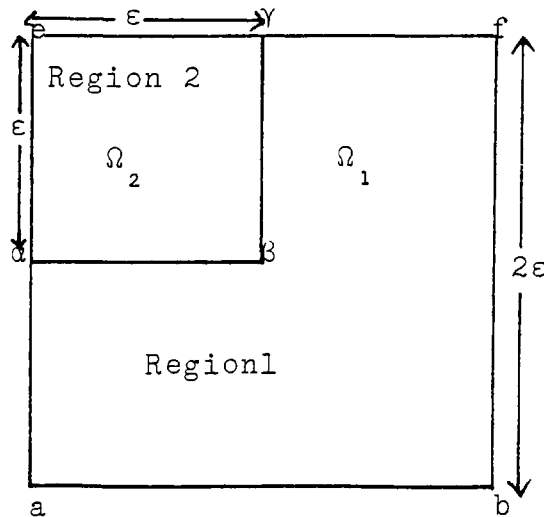


FIG.5.14

face is in which region so that equation (5.89) becomes

$$\int_{abfe} -B_z \, dx dy = \int_{\alpha\beta\gamma fba} -B_z^1 \, dx dy + \int_{\alpha\beta\gamma e} -B_z^2 \, dx dy \quad (5.90)$$

Evaluating each of the right hand integrals as before gives,

$$\int_{abfe} -B_z \, dx dy = -B_z^{A,1} \frac{3\varepsilon^2}{4} - B_z^{A,2} \frac{\varepsilon^2}{4} + O(\varepsilon^3) \quad (5.91)$$

Following the procedure for face abfe for faces efgh and aehd produces respectively the following contribution to (5.82),

$$\int_{efgh} B_y \, dx dz = B_y^{A,1} \frac{3\varepsilon^2}{4} + B_y^{A,2} \frac{\varepsilon^2}{4} + O(\varepsilon^3) \quad (5.92)$$

and,

$$\int_{aehd} -B_x \, dy dz = -B_x^{A,1} \frac{3\varepsilon^2}{4} - B_x^{A,2} \frac{\varepsilon^2}{4} + O(\varepsilon^3) \quad (5.93)$$

The six integrals over the six faces ((5.86), (5.87), (5.88), (5.91), (5.92), (5.93)) represent an evaluation of (5.82) over the small cube. Therefore adding the six contributions and dividing by ε^2 gives,

$$\begin{aligned} -B_y^{A,1} + B_x^{A,1} + B_z^{A,1} - \frac{3B_z^{A,1}}{4} - \frac{B_z^{A,2}}{4} + \frac{3B_y^{A,1}}{4} + \frac{B_y^{A,2}}{4} \\ - \frac{3B_x^{A,1}}{4} - \frac{B_x^{A,2}}{4} + O(\varepsilon) = 0 \end{aligned} \quad (5.94)$$

which contracts with ε tending to zero to give,

$$B_x^{A,1} - B_y^{A,1} + B_z^{A,1} = B_x^{A,2} - B_y^{A,2} + B_z^{A,2} \quad (5.95)$$

Turning to the second equation (5.83) to be applied over the small cube, the same approach as has been used above for $\text{div} \underline{B}$ can be used for each component of the vector equation (5.83) and the resulting equations are,

$$H_Y^A,1 + H_Z^A,1 = H_Y^A,2 + H_Z^A,2 \quad (5.96)$$

$$H_Z^A,1 - H_X^A,1 = H_Z^A,2 - H_X^A,2 \quad (5.97)$$

$$H_X^A,1 + H_Y^A,1 = H_X^A,2 + H_Y^A,2 \quad (5.98)$$

The first observation to be made about the above three equations is that only two of them are independent. This should be expected since a curl equation is generally not sufficient to define a vector field. Therefore the corner interface conditions for $\underline{H}$ at A (fig.5.12) can be found by making suitable eliminations between equations (5.95), (5.96) and (5.97).

For a more general three dimensional corner than that shown in fig.5.12 equations (5.82) and (5.83) become much more difficult to apply and therefore no generalisation of these corner concepts has been attempted.

5.3 CONDITIONS FOR UNIQUE $\underline{H}$

In chapter four the conditions for defining a unique $\underline{H}$ were discussed for the limited type of problem that that chapter was concerned with. In this chapter it is intended to do the same.

Consider a two dimensional solution domain, Ω , for a magnetostatic problem which contains conductors, air

and materials whose relative permeability is constant, but greater than unity. The equations satisfied by $\underline{H}$ within each region are,

$$\underline{\nabla} \cdot \underline{H} = 0 \quad (5.99)$$

and,

$$\underline{\nabla} \times \underline{H} = \underline{J} \quad (5.100)$$

Since the problem is magnetostatic in nature,

$$\underline{\nabla} \times \underline{E} = \underline{0} \quad (5.101)$$

and since,

$$\underline{J} = \sigma \underline{E} \quad (5.102)$$

the governing equations for $\underline{H}$ within each region are,

(5.99) and,

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{H}) = \underline{\nabla} \times \underline{J} = \sigma (\underline{\nabla} \times \underline{E}) = \underline{0} \quad (5.103)$$

assuming the materials are isotropic.

Assume that two distinct non trivial solutions $\underline{H}_1$ and $\underline{H}_2$ exist, both zero at infinity and satisfying equations (5.99) and (5.100) within each region, then a third non trivial solution also exists given by,

$$\underline{H}_3 = \underline{H}_1 - \underline{H}_2 \quad (5.104)$$

The task in hand is to force $\underline{H}_3$ to be identically zero.

The governing equations for $\underline{H}_3$ are (5.99) and (5.103)

and initially attention will be given to (5.103).

Consider the vector identity,

$$\begin{aligned} \underline{\nabla} \cdot (\underline{P} \times (\underline{\nabla} \times \underline{Q})) &\equiv (\underline{\nabla} \times \underline{P}) \cdot (\underline{\nabla} \times \underline{Q}) \\ &\quad - \underline{P} \cdot \underline{\nabla} \times (\underline{\nabla} \times \underline{Q}) \end{aligned} \quad (5.105)$$

and make the assignments

$$\underline{P} = \underline{Q} = \underline{H}_3 \quad (5.106)$$

and then integrate (5.105) over any region Ω_i (Ω_i part of Ω) where (5.99) and (5.103) are satisfied, to give,

$$\begin{aligned} \int_{\Omega_i} \nabla \cdot (\underline{H}_3 \times (\nabla \times \underline{H}_3)) d\Omega_i &= \int_{\Omega_i} (\nabla \times \underline{H}_3)^2 d\Omega_i \\ &- \int_{\Omega_i} \underline{H}_3 (\nabla \times \nabla \times \underline{H}_3) d\Omega_i \end{aligned} \quad (5.107)$$

Using Gauss' divergence theorem, and since $\underline{H}_3$ satisfies (5.103), equation (5.107) reduces to,

$$\int_{S_i} \Delta \cdot \underline{H}_3 \times (\nabla \times \underline{H}_3) dS_i = \int_{\Omega_i} (\nabla \times \underline{H}_3)^2 d\Omega_i \quad (5.108)$$

where S_i is the bounding surface of Ω_i . The integrand of the surface integral is a scalar triple product and therefore,

$$\underline{n} \cdot \underline{H}_3 \times (\nabla \times \underline{H}_3) = - \underline{H}_3 \cdot \underline{n} \times (\nabla \times \underline{H}_3) \quad (5.109)$$

However $\underline{n} \times (\nabla \times \underline{H})$ gives a vector equal in magnitude to tangential $\underline{J}$ and since the region is magnetostatic either $\underline{J}$ is given or is zero and therefore the surface integral is zero. Hence equation (5.108) reduces to

$$\int_{\Omega_i} (\nabla \times \underline{H}_3)^2 d\Omega_i = 0 \quad (5.110)$$

However this is true for all Ω_i in Ω and therefore,

$$\int_{\Omega} (\nabla \times \underline{H}_3)^2 d\Omega_i = 0 \quad (5.111)$$

and so

$$\underline{\nabla} \times \underline{H}_3 = \underline{0} \quad (5.112)$$

over Ω the solution domain, but this leaves the possibility that over Ω_i

$$\underline{H}_3 = -\underline{\nabla}\phi_i \quad (5.113)$$

Since μ_r , the relative permeability, is constant over each material,

$$\mu_r \underline{\nabla} \cdot \underline{H}_3 = 0 \quad (5.114)$$

and therefore combining equations (5.113) and (5.114) gives,

$$\mu_r \nabla^2 \phi_i = 0 \quad (5.115)$$

Using the vector identity,

$$-\psi \nabla^2 \theta \equiv \underline{\nabla} \theta \cdot \underline{\nabla} \psi - \underline{\nabla} \cdot \psi \underline{\nabla} \theta \quad (5.116)$$

where,

$$\frac{\psi}{\mu_r} = \theta = \phi_i \quad (5.117)$$

and integrating over Ω_i gives,

$$\begin{aligned} - \int_{\Omega_i} \mu_r \nabla^2 \phi_i \, d\Omega_i &= \int_{\Omega_i} \mu_r (\underline{\nabla} \phi_i)^2 d\Omega_i \\ &- \int_{S_i} \mu_r \phi_i \frac{\partial \phi_i}{\partial n_i} dS_i = 0 \end{aligned} \quad (5.118)$$

where the integral over S_i has been transformed using Gauss' divergence theorem. The integral over S_i contains ϕ_i and $\mu_r \frac{\partial \phi_i}{\partial n_i}$ and these are related to tangential $\underline{H}$ and normal $\underline{B}$ and therefore adding together similar equations to (5.118) for all regions gives,

$$\sum_i \int_{\Omega_i} \mu_r (\underline{\nabla} \phi_i)^2 d\Omega = 0 \quad (5.119)$$

since $\underline{H}_3$ is zero at infinity and assuming the standard continuity conditions are applied. This last equation gives,

$$\nabla \phi_i = 0 \quad \text{over } \Omega_i \quad (5.120)$$

and therefore

$$\underline{H}_3 \equiv 0 \quad (5.121)$$

which gives a unique $\underline{H}$. The conditions applied were,

- 1) known current density
- 2) continuity of tangential $\underline{H}$
- 3) continuity of normal $\underline{B}$
- 4) $\underline{H}$ zero at infinity

and therefore these four conditions give a unique $\underline{H}$ for the limited class of problem under consideration in this chapter.

5.4 AN ISOLATED INFINITELY LONG PERMEABLE BUS BAR

In order to gain some experience of the interface conditions for corners, given earlier in this chapter, it was decided to examine a problem with an analytic solution. The problem chosen was that of the isolated infinitely long permeable bus bar. Fig.5.15 shows the top right hand quadrant of a cross sectional view of the bus bar.

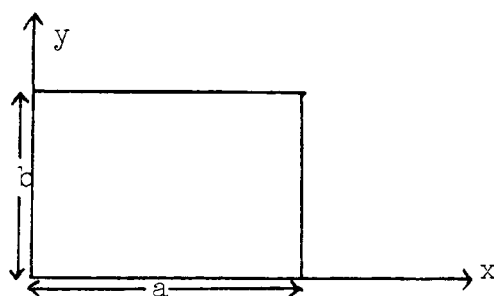


FIG.5.15

The analytic solution given by Binns and Lawrenson^{5.6} is for an infinitely permeable bus bar and the boundary conditions imposed on the problem are,

$$B_x = 0 \quad \text{when } x=a \quad \text{and } y=0 \quad (5.122)$$

and,

$$B_y = 0 \quad \text{when } y=b \quad \text{and } x=0 \quad (5.123)$$

The analytic solution is in terms of $\underline{A} = (0,0,A_z)$, the magnetic vector potential, and therefore the solutions for H_x and H_y derived by differentiation are,

$$H_x = -\frac{\mu_0 J}{2} \sum_{k=0}^{\infty} \frac{(-1)^k 16a}{(2k+1)^2 \pi^2 \cosh \frac{(2k+1)\pi b}{2a}} F_1(k,x,y) \quad (5.124)$$

and,

$$H_y = -\mu_0 J \quad x + \sum_{k=0}^{\infty} \frac{(-1)^k 8a}{(2k+1)^2 \pi^2 \cosh \frac{(2k+1)\pi b}{2a}} F_2(k,x,y) \quad (5.125)$$

where,

$$F_1(k,x,y) = \sinh \frac{(2k+1)\pi y}{2a} \cos \frac{(2k+1)\pi x}{2a} \quad (5.126)$$

and,

$$F_2(k,x,y) = \cosh \frac{(2k+1)\pi y}{2a} \sin \frac{(2k+1)\pi x}{2a} \quad (5.127)$$

Since the denominator contains the term $(2k+1)^2$ the series are quickly convergent.

5.4.1 The Actual Problem and the Analytic Results

The actual problem solved was a square isolated infinitely long permeable bus bar of side 38.5mm as shown in fig.5.16, with a current density of 1.0A/mm^2 .

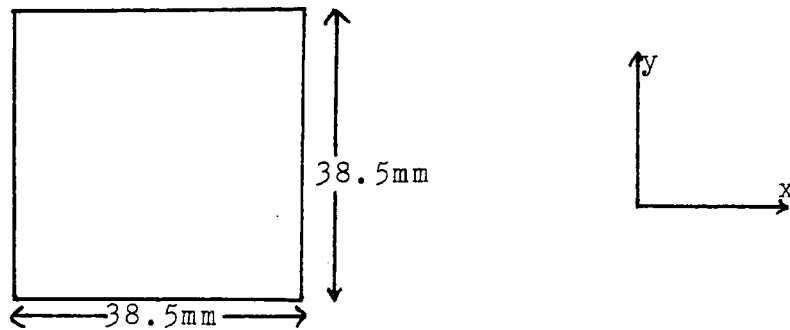


FIG.5.16

Before calculating the analytic results the finite element grid for the approximate solution must be known and figs. 5.17 and 5.18 show the two grids that were used for solution, where the mesh in each case is over the whole bus bar. Therefore for each of these grids the analytic result for H_y was constructed for the top right hand quadrant. Only H_y was constructed because the bus bar is square. Tables 5.1 and 5.2 give the analytic results for the top right hand quadrant with respect to the grids in figs.5.17 and 5.18, where the series in (5.125) was summed until there was no change in the fifth decimal place. The maximum number of terms required to achieve this accuracy was sixteen and

Meshes used for approximate solution of permeable bus
bar problem.

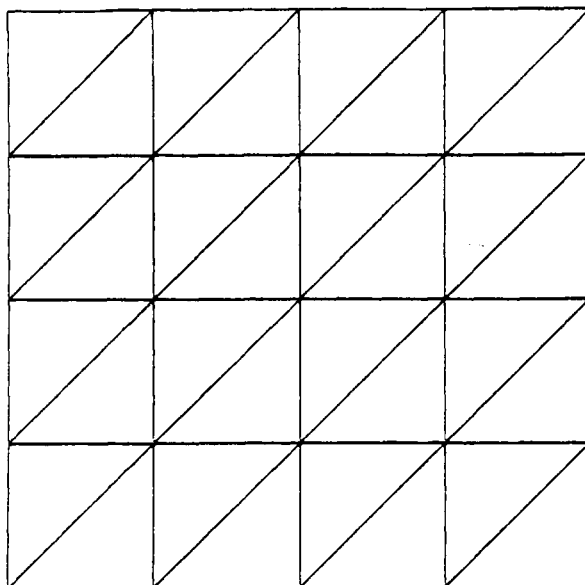


FIG. 5.17

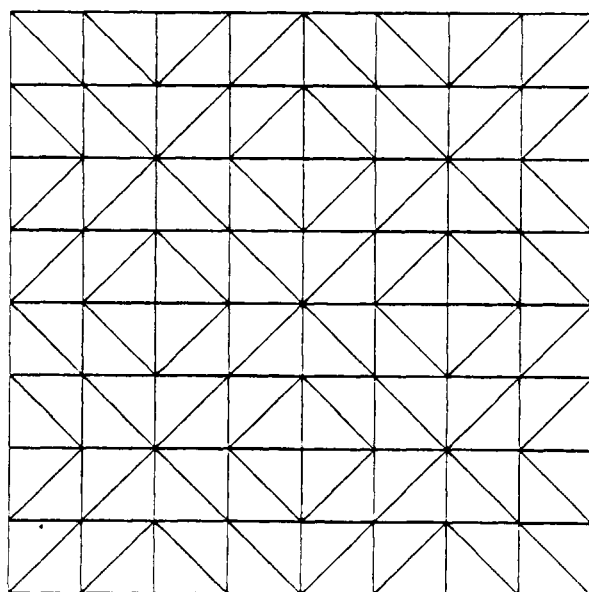


FIG. 5.18

Analytic and calculated results for H_y for the permeable
bus bar meshes.

0.0	0.0	0.0
0.0	-3.9203	-10.8192
0.0	-5.2434	-12.9829

TABLE 5.1

0.0	0.0	0.0	0.0	0.0
0.0	-1.0013	-2.2265	-4.0704	-7.5331
0.0	-1.8003	-3.9203	-6.7612	-10.8192
0.0	-2.2934	-4.9183	-8.2125	-12.4723
0.0	-2.4579	-5.2434	-8.6693	-12.9829

TABLE 5.2

0.0	0.0	0.0
0.0	-3.6047	-9.6125
0.0	-4.8063	-12.0156

TABLE 5.3

0.0	-3.1×10^{-3}	-4.5×10^{-3}	-6.1×10^{-3}	-9.2×10^{-3}
0.0	-.9914	-2.1944	-3.9446	-7.0004
0.0	-1.7679	-3.8357	-6.5743	-10.4953
0.0	-2.2444	-4.8064	-8.0214	-12.2247
0.0	-2.4032	-5.1240	-8.4802	-12.7541

TABLE 5.4

and this underlines the earlier comment concerning the convergence of the series.

5.4.2 The Calculation Method

The governing equation for $\underline{H}$ for the problem given in fig.5.16 is Laplaces equation and in chapter four two solution techniques were discussed. Recapping, for the first approach the two matrix equations for the differential and integral domains for H_y may be written,

$$A_{y\underline{H}\underline{y}}^I = B_{x\underline{H}\underline{x}}^I + \underline{J}_y \quad (5.128)$$

for the differential domain, and

$$A_{y\underline{H}\underline{y}}^E = B_{x\underline{H}\underline{x}}^E \quad (5.129)$$

for the integral domain, where the super fixes I and E indicate interior(differential) and exterior(integral) values, and b contributions on the boundary between the two. $\underline{J}_y$ is the vector of right hand sides arising from $\underline{J}$ the known current density when constructing the Weighted Residual formulation for,

$$\nabla^2 H_y = 0 \quad (5.130)$$

the governing equation for H_y in both the differential and integral domain. In equations (5.128) and (5.129) it has been assumed that the boundary between the differential and integral domain is on the surface of the bus bar. This is because the relationship between corner H_y values across the material surface interface introduce reference to H_x corner values; and therefore

if the boundary between the differential and integral domain is on the surface, those H_x values arising at the corners can be included with those H_x values arising from the transformation of $\frac{\partial H_y}{\partial n}$ to $\frac{\partial H_x}{\partial t}$ (the tangential derivative). The material surface interface conditions may be expressed as,

$$\underline{H_y^I}^b = C \underline{H_y^E}^b + D \underline{H_x^E}^b \quad (5.131)$$

and,

$$\underline{H_x^I}^b = U \underline{H_x^E}^b + V \underline{H_y^E}^b \quad (5.132)$$

The next step for the first approach having included the interface conditions, would be to arbitrarily nominate one value of $\underline{H_x^E}^b$ since these represent $\frac{\partial H_x}{\partial t}$. However equations (5.131) and (5.132) demonstrate that the actual values of $\underline{H_x^E}^b$ have now become important themselves and therefore it is no longer possible to arbitrarily choose one value of $\underline{H_x^E}^b$. Therefore the second approach as outlined in chapter four was used for the solution of the problem. This involves adding to equations (5.128), (5.129), (5.131) and (5.132) the integral domain approximation to the governing equation for H_x ,

$$\nabla^2 H_x = 0 \quad (5.133)$$

which in matrix form may be written,

$$A_{x \underline{H_x^E}^b}^E = B_{y \underline{H_y^E}^b}^E \quad (5.134)$$

The terms $\underline{H_y^E}^b$ are regarded as constructing $\frac{\partial H_y}{\partial t}$ and therefore one of them is arbitrarily chosen. This second technique works for odd numbered boundary noded shapes and with a slight modification for even numbered

boundary noded shapes possessing at least one line of symmetry in the mesh, as was discussed in chapter four. The two meshes used are given in figs.5.17 and 5.18 and both fall into the latter category; therefore the technique used is that for even numbered boundary noded shapes possessing symmetry in the mesh. The technique explicitly possesses each of the four points discussed in section 5.3 which assure a unique $\underline{H}$.

5.4.3 The Calculated Results

The meshes used for solution are given in figs.5.17 and 5.18 and both meshes cover the whole bus bar. Tables 5.3 and 5.4 present the calculated results for H_y in the bus bar with respect to the top right hand quadrant of those meshes where 0.0 has been substituted for any absolute value less than 1.0×10^{-6} . The results presented in table 5.3 are for a relative permeability of 10^6 while for the results in table 5.4 it was reduced to 10^3 . This was because the increased value for the relative permeability (10^6) was to within .001% producing the same results as for the reduced value (10^3), and therefore it was adjudged that the lower value was adequate. The absolute percentage errors at each node, in the top right hand quadrant of each mesh, defined as

$$H_y^{ape} = \frac{(H_{y_{ana}} - H_{y_{cal}})}{H_{y_{ana}}} \times 100 \quad (5.135)$$

are presented in tables 5.5 and 5.6. These results demonstrate that considerable improvement is brought about by refining the mesh and they give substantial support for the validity of the interface analysis.

Percentage errors for H_y

0.0	0.0	0.0
0.0	8.1	11.2
0.0	8.3	7.5

TABLE 5.5

Percentage errors for H_y

0.0	0.0	0.0	0.0	0.0
0.0	1.0	1.4	3.1	7.1
0.0	1.8	2.2	2.8	3.0
0.0	2.1	2.3	2.3	2.0
0.0	2.2	2.3	2.2	1.8

TABLE 5.6

There are two other points to mention with reference to the comments made in section 5.1.4. As was anticipated in that section at the top surface of the bus bar H_y is approaching zero and the computer program is almost seeing a boundary condition along that edge. Secondly at the top right hand corner the standard continuity condition, continuity of normal $\underline{B}$, is the dominant

condition. It was shown in section 5.1.4 that one of the standard conditions would dominate depending on the relative signs of H_x and H_y .

5.5 AN ANALYSIS OF AN E-CORE MACHINE ON OPEN CIRCUIT

An E-core machine is a transverse flux machine and the geometry of the right half of the machine considered is given in fig.5.19. The analysis is two dimensional and the problem is treated as magnetostatic with a given current density in the z-direction and for solution purposes non conducting iron forms the stator. The relative permeability of the stator iron is taken to be 1000. The treatment of the iron as part of the finite element mesh enables the corner interface conditions to be tested in a practical problem and also allows the testing of the technique employed in more stringent circumstances than if the iron presented nothing more than a boundary condition. Since the machine is symmetric only the right half of the machine is analysed. For all the solutions presented the boundary between the differential and integral domain was taken to be the material surfaces for the same reason as was discussed when analysing the isolated permeable bus bar. The measured results used for comparison were taken by Wyatt^{5.7}. The measurements were made with a current of 800 Amps R.M.S. in the slot at a height of 6mm above the stator.

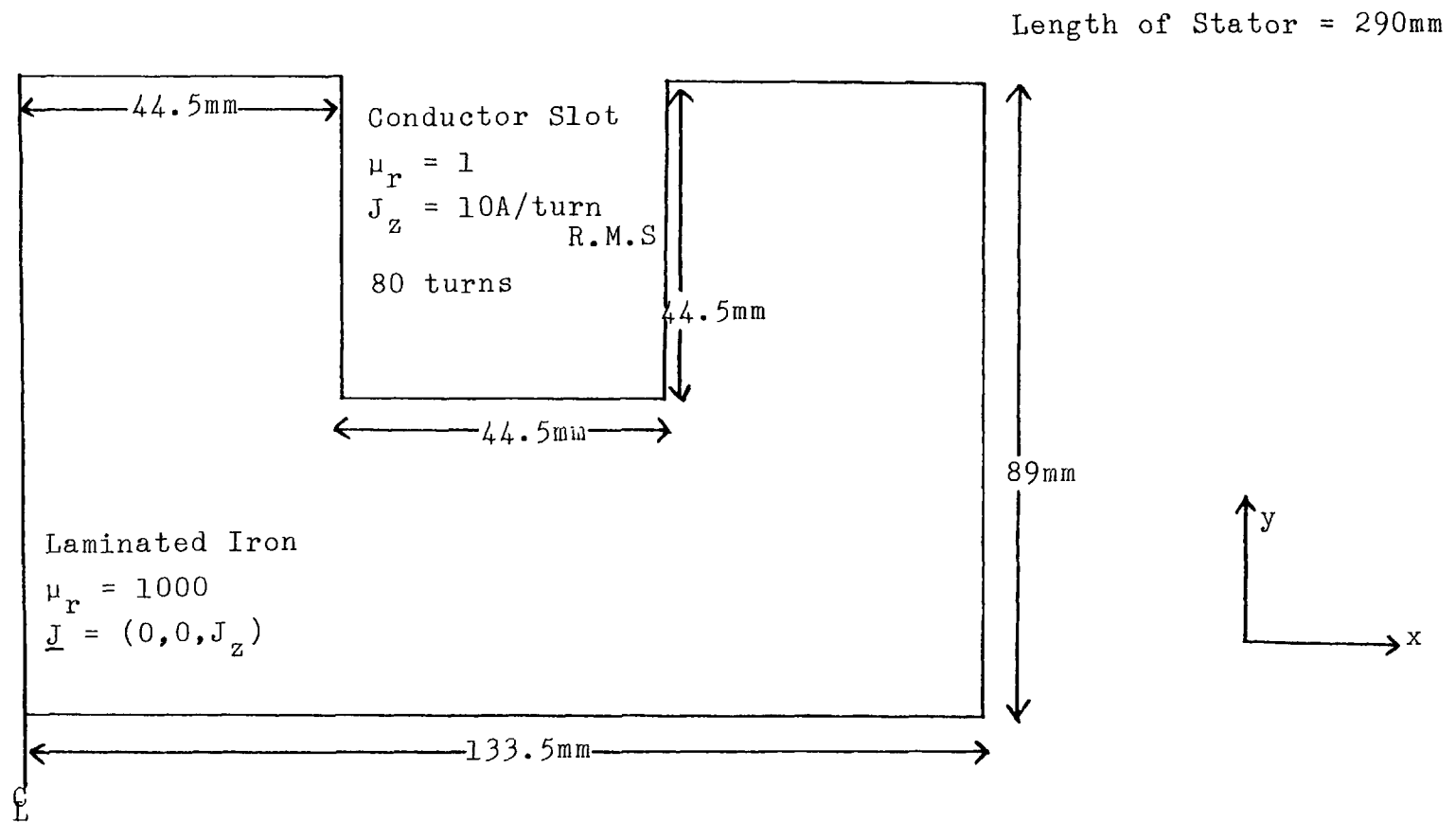


FIG.5.19

An E-Core on Open Circuit

5.5.1 Solutions for Hx

The conditions imposed on the problem were that on the centre line of the machine

$$H_x = \frac{\partial H_y}{\partial x} = 0 \quad (5.136)$$

and at infinity,

$$H_x = H_y = 0 \quad (5.137)$$

The method used for solution is in principle the same as that employed for the isolated permeable bus bar problem, adapted for Hx. The one small difference is that since the meshing for the conductor (see fig.5.20) has an odd number of boundary nodes only one value is required to set the level for $\frac{\partial H_y}{\partial t}$. The E-core mesh highlights the points that were made in chapter four concerning odd and even numbered boundary noded shapes, since if the conductor in this problem had an even number of boundary nodes no way is known to the author of producing a unique result. This is because the special case for even numbered boundary noded shapes no longer pertains, since there are two shapes in the solution domain.

The first solution produced was for the mesh of the right half of the machine given in fig.5.20. Initially a very simple mesh was chosen to see how close the solution was to the measured results. Fig.5.21 gives the comparison between the measured and calculated results.

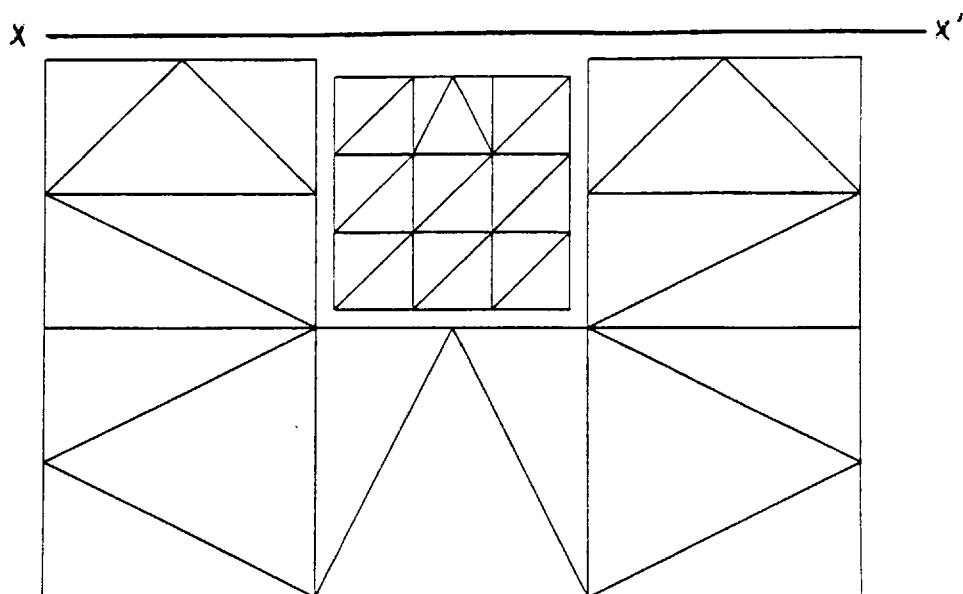
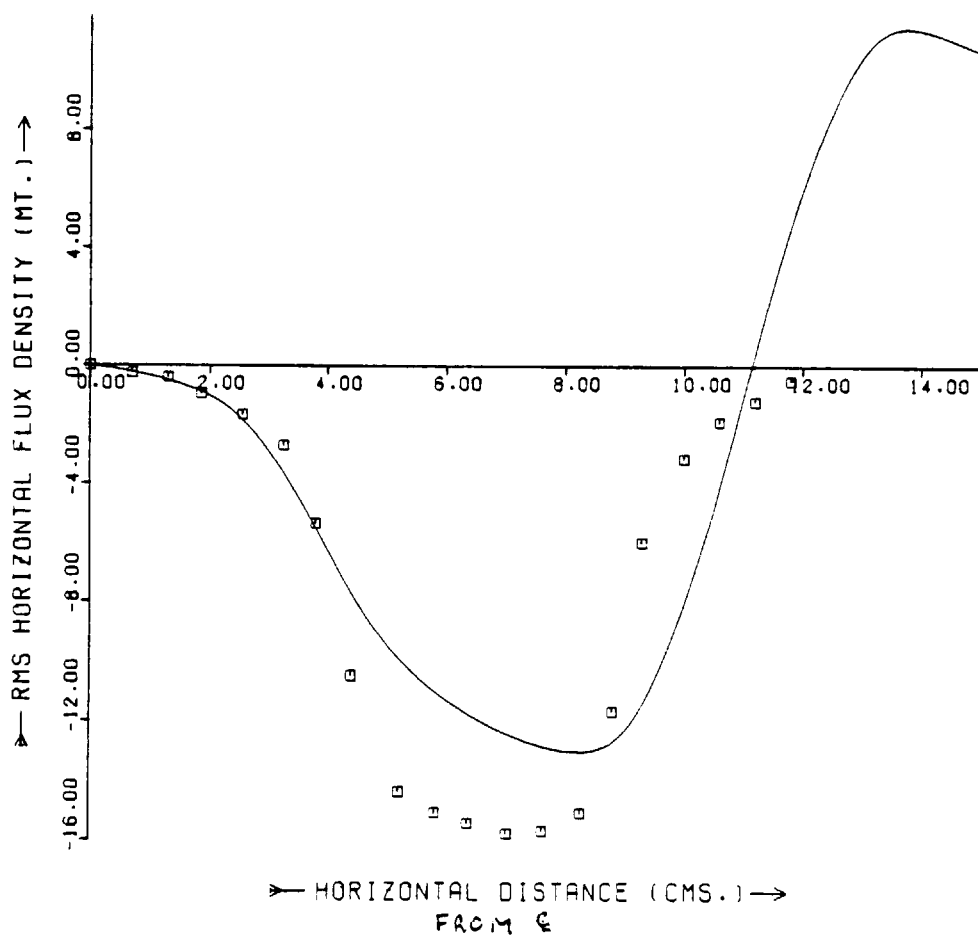


FIG.5.20 finite element mesh of right half of E-core



□ measured
 — calculated

FIG.5.21 R.M.S. horizontal flux density for FIG.5.20 along xx'

The result certainly gives room for improvement but is encouraging from the point of view that at least it is in the right area. It is worth pointing out that the calculated result for H_x is a continuous function which is in marked contrast to many potential formulations.

Figs. 5.22 to 5.25 present two more solutions to the same problem giving better correlation between measured and calculated results. The calculated result given in fig. 5.23 shows an oscillatory nature which it was thought was probably due to the surface meshing interval. Therefore for the mesh given in fig. 5.24 the surface meshing interval was altered and fig. 5.25 shows the consequent change in result.

5.5.2 Solutions for H_y

The conditions imposed on the problem were those given in (5.136) and (5.137). The method of solution is that as first outlined in chapter four. This first method can be used on this occasion since H_x is zero on the centre line of the machine gives enough conditions to determine the equations for the iron. Three solutions are presented consistent with the meshes used for the solution of H_x , figs. 5.20, 5.22 and 5.24. The results are given in figs. 5.26, 5.27 and 5.28, and show a reasonable accuracy has been obtained. It is curious to note that on the whole the solutions for H_y do not display the same oscillatory nature as those for H_x although

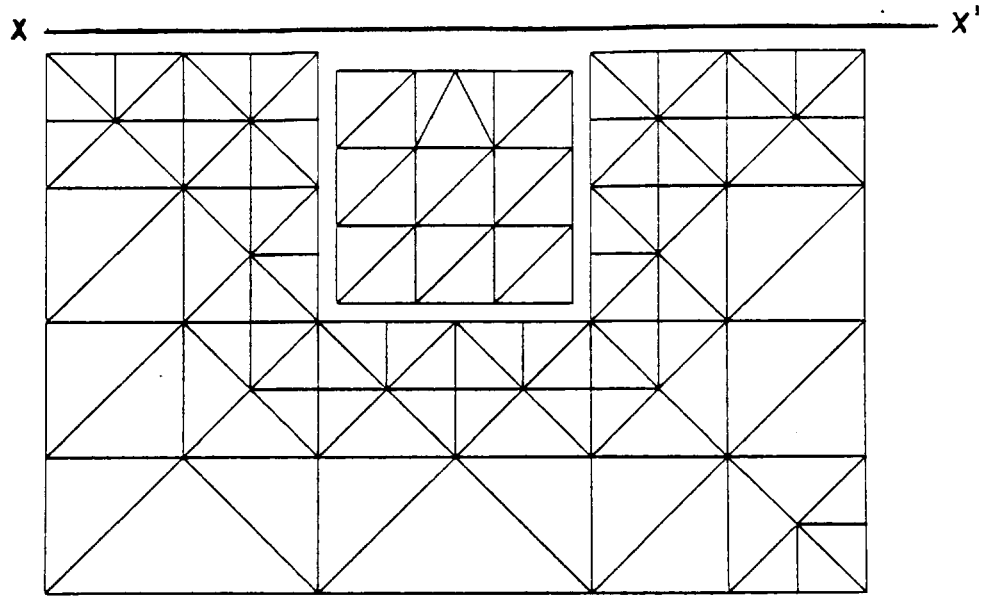
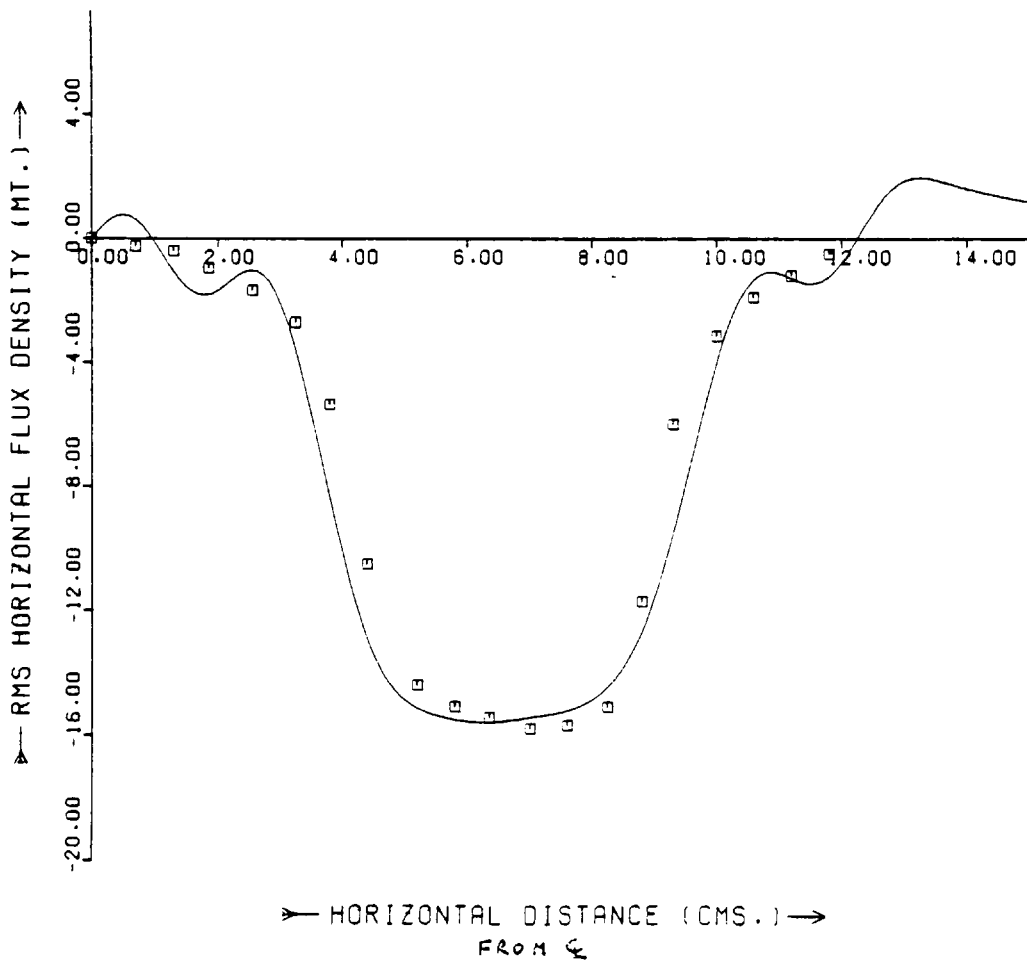


FIG.5.22 finite element mesh of right half of E-core



☐ measured
☒ calculated

FIG.5.23 R.M.S. horizontal flux density for FIG.5.22 along XX'

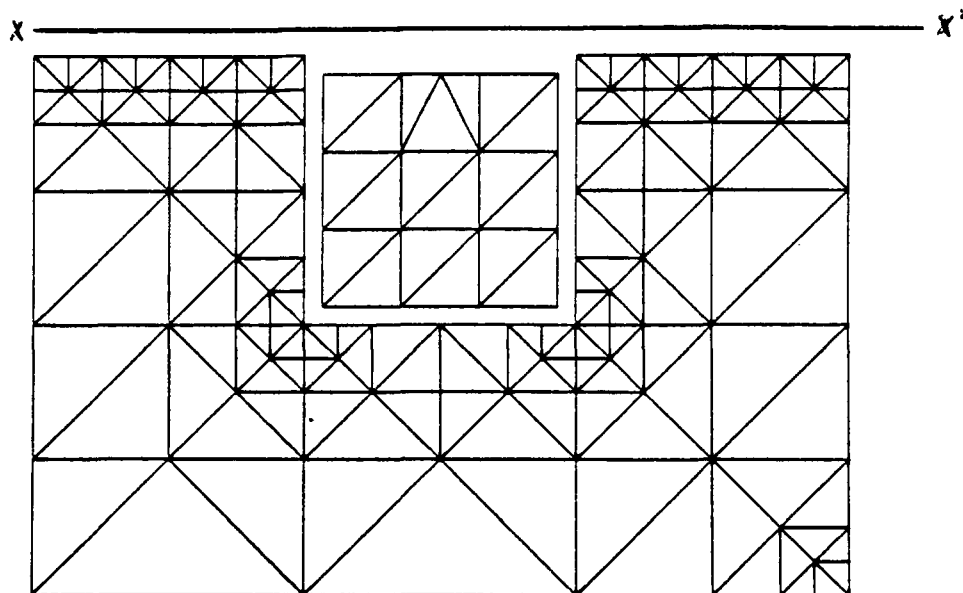
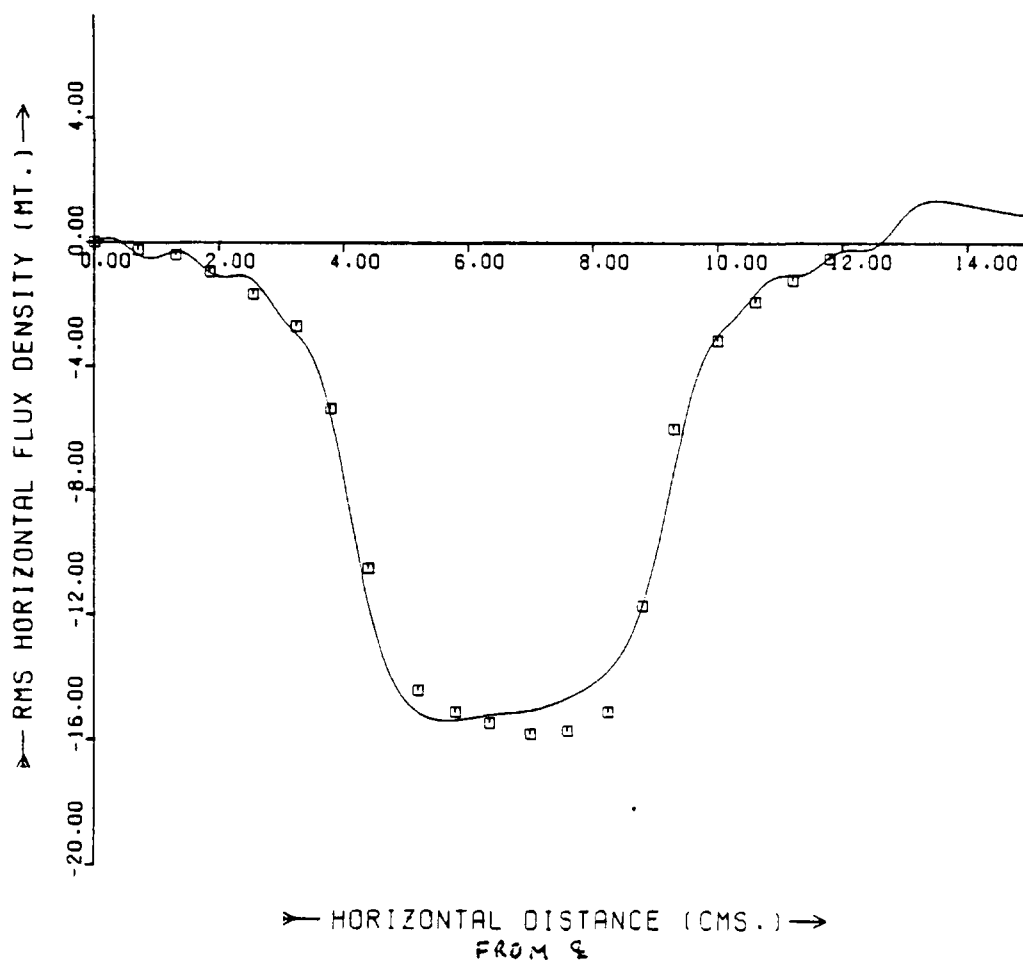


FIG.5.24 finite element mesh of right half of E-core



□ measured
— calculated

Fig.5.25 R.M.S. horizontal flux density
for FIG.5.24 along xx'

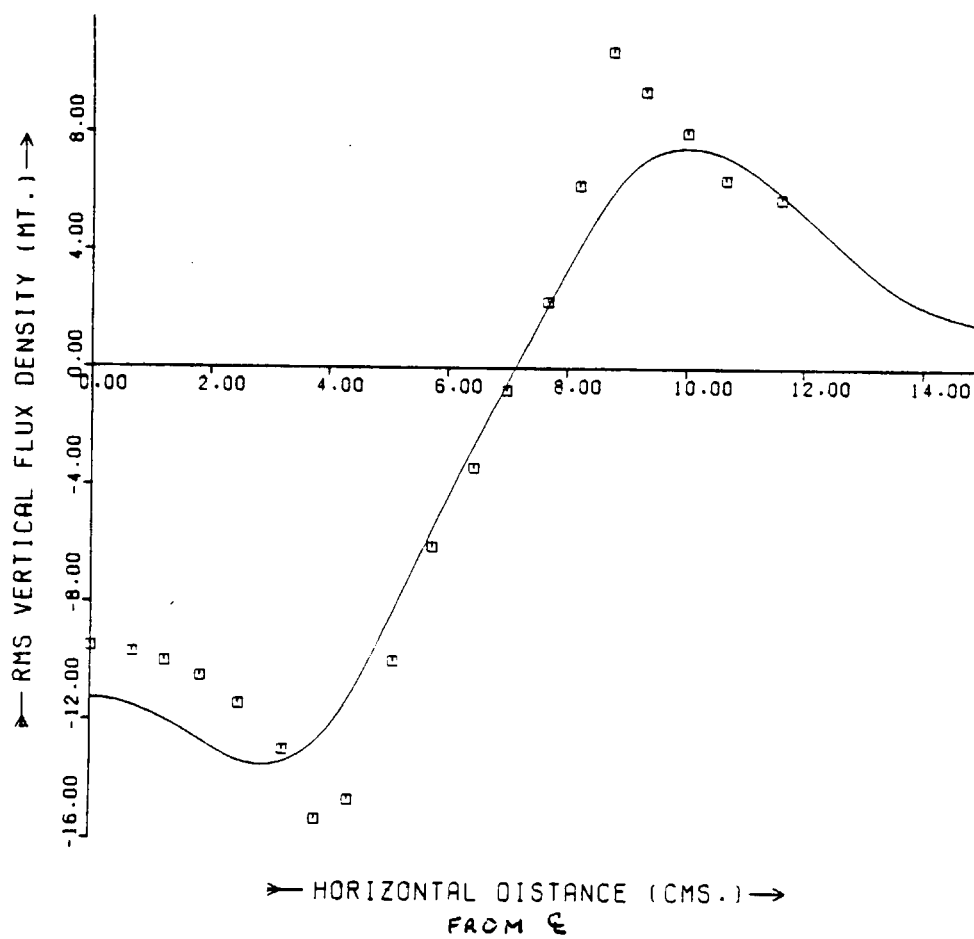


FIG. 5.26 R.M.S. vertical flux density for FIG. 5.20
along xx'

□ measured

— calculated

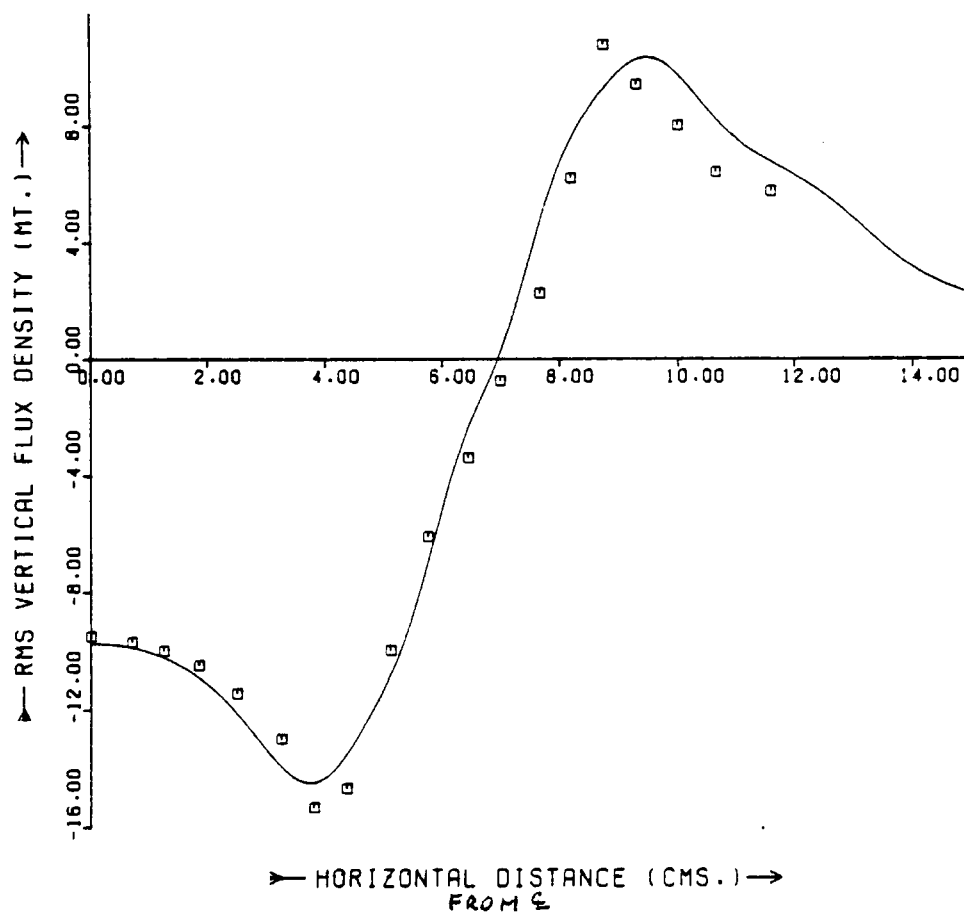


FIG.5.27 R.M.S. vertical flux density for FIG.5.22
along xx'

□ measured

— calculated

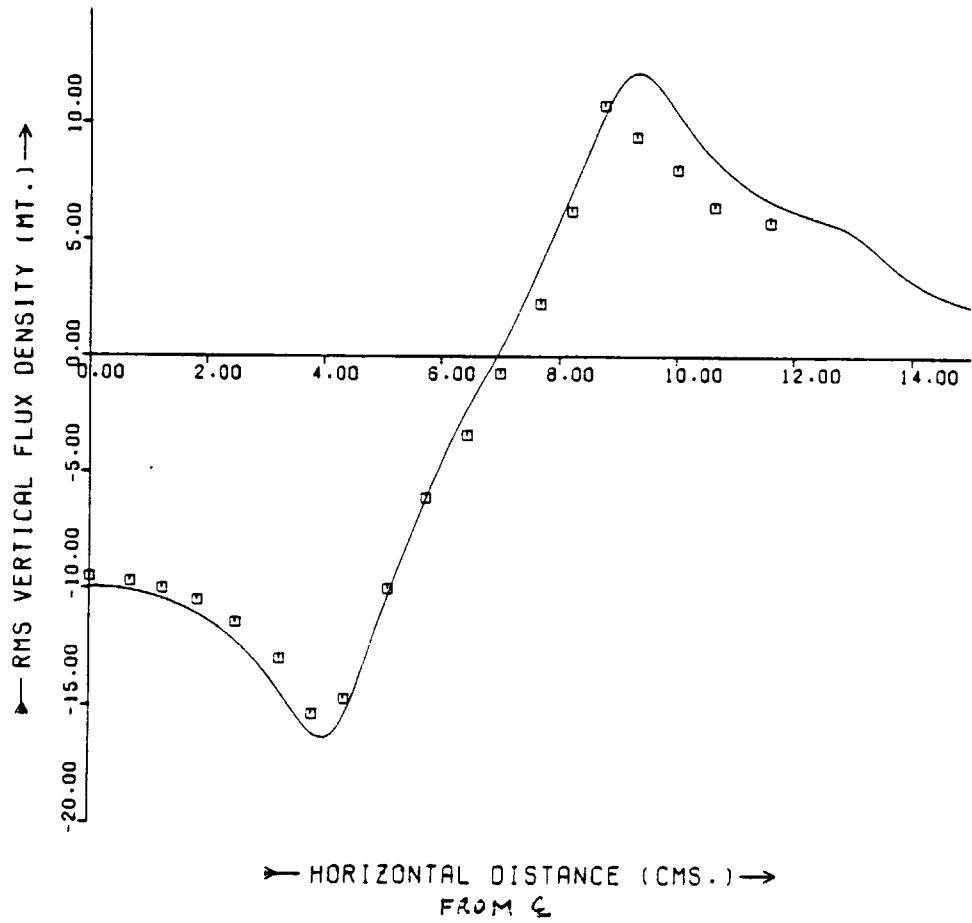


FIG.5.28 R.M.S. vertical flux density for FIG.5.24
along xx'

□ measured
— calculated

the same meshes are involved. At the centre of the machine the solutions given in figs.5.27 and 5.28 both predict the measured vertical flux density to within 5%.

5.6 AN ANALYSIS OF A XI-CORE MACHINE ON OPEN CIRCUIT

A Xi-core machine like the E-core machine is a transverse flux system and the geometry of the right half of the machine considered is given in fig.5.29. The analysis is two dimensional magnetostatic where the current density in the z-direction is given and for solution purposes the stator is formed from non conducting iron. Once again as for the E-core the iron is analysed and the relative permeability is set to 1000. Since the machine is symmetric only the right hand half is analysed and as for the E-core the boundary between the differential and integral domains was chosen to be the material surfaces. The measured results used for comparison were taken by Rajanathan^{5.8}. The measurements were taken with a current of 1408 Amps R.M.S. in the slot and at a height of 19mm above the stator.

5.6.1 Solutions for H_x

The conditions imposed on the problem were that on the centre line of the machine

$$H_y = \frac{\partial H_x}{\partial x} = 0 \quad (5.138)$$

Length of Stator = 457mm

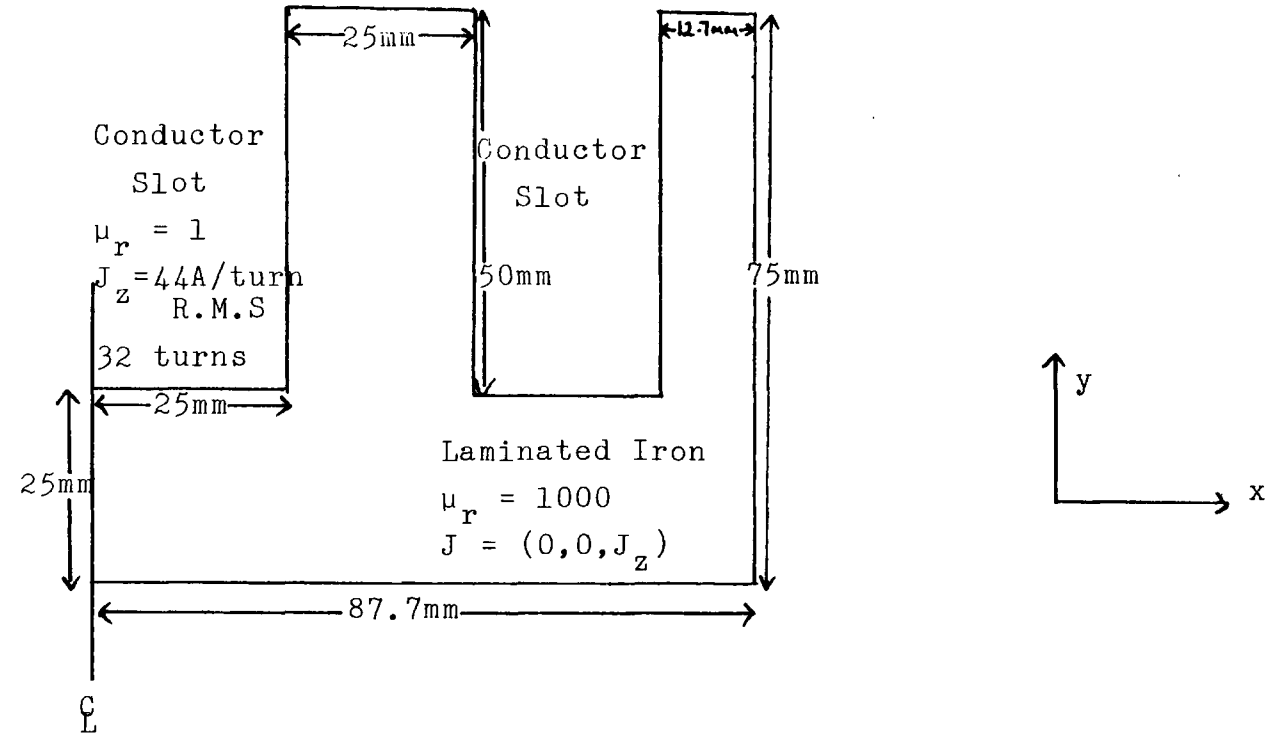


FIG.5.29

Xi-Core on Open Circuit

and at infinity,

$$H_x = H_y = 0 \quad (5.139)$$

The method of solution is that as used to calculate H_y for the E-core problem adapted for H_x . Fig.5.30 gives the mesh used for the first solution and fig.5.31 gives the result obtained. This result is very disappointing since at the centre of the machine the calculated result is 23% lower than the measured. The author has used a number of meshes for solution of this particular problem and found the results to be similar to those given in fig.5.31. Fig.5.32 shows one of the other meshes used where the iron has been refined and fig.5.33 shows the attained result. Perhaps the most disappointing feature of these results is that the curve obtained seems to possess the wrong nature. The measured results give the peak value of the horizontal flux density at the centre of the machine as 60% higher in magnitude than the peak value over the outer conductor slot, whereas the calculated results show them almost equal. Since the results were poor the author examined the problem depicted by the meshes shown in figs.5.30 and 5.32 where the iron was treated as air and found that the results compared favourably with the analytic result. This implies that assuming the measured results can be relied on the treatment of the iron is probably causing the problem.

The author suggests that initially four steps should be

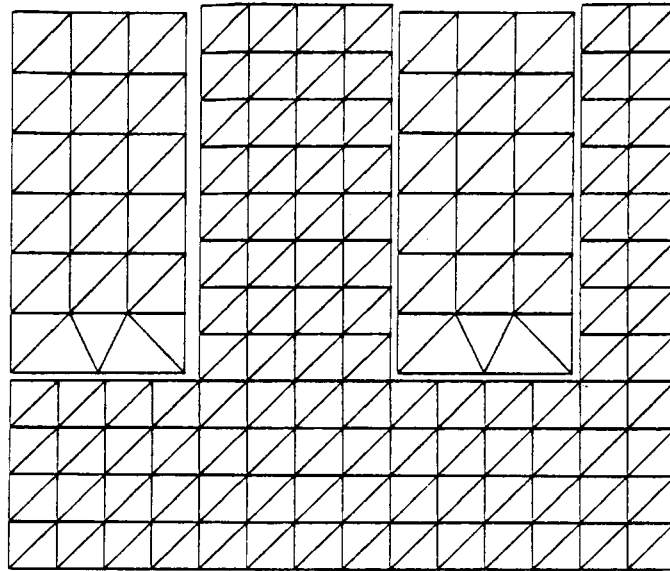
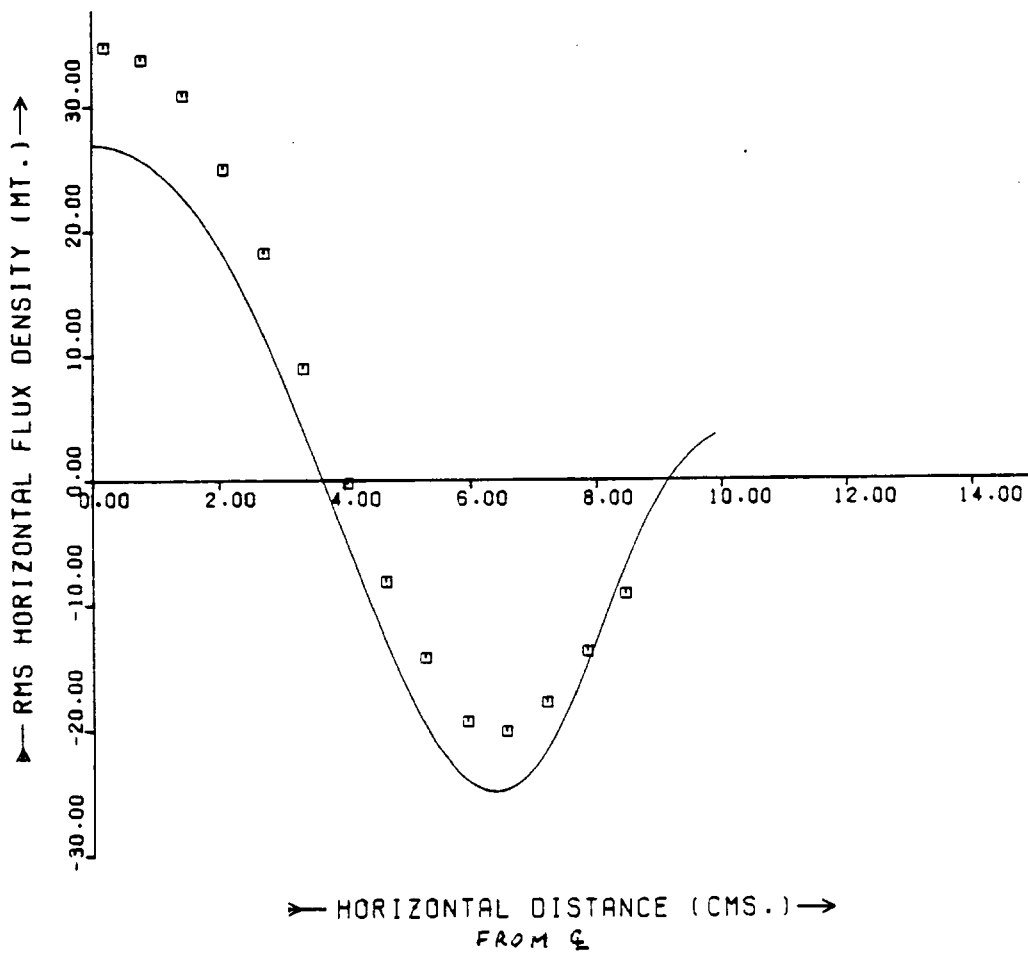


FIG.5.30 finite element mesh of right half of Xi-core



■ measured
 — calculated

FIG.5.31 R.M.S horizontal flux density
for FIG.5.30 along line xx'

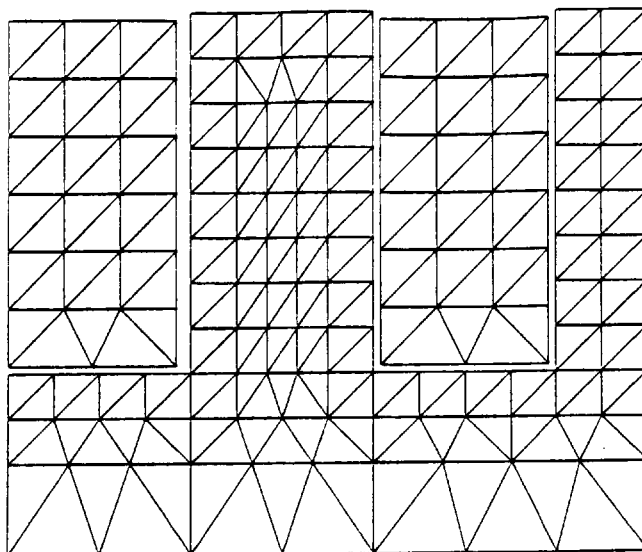
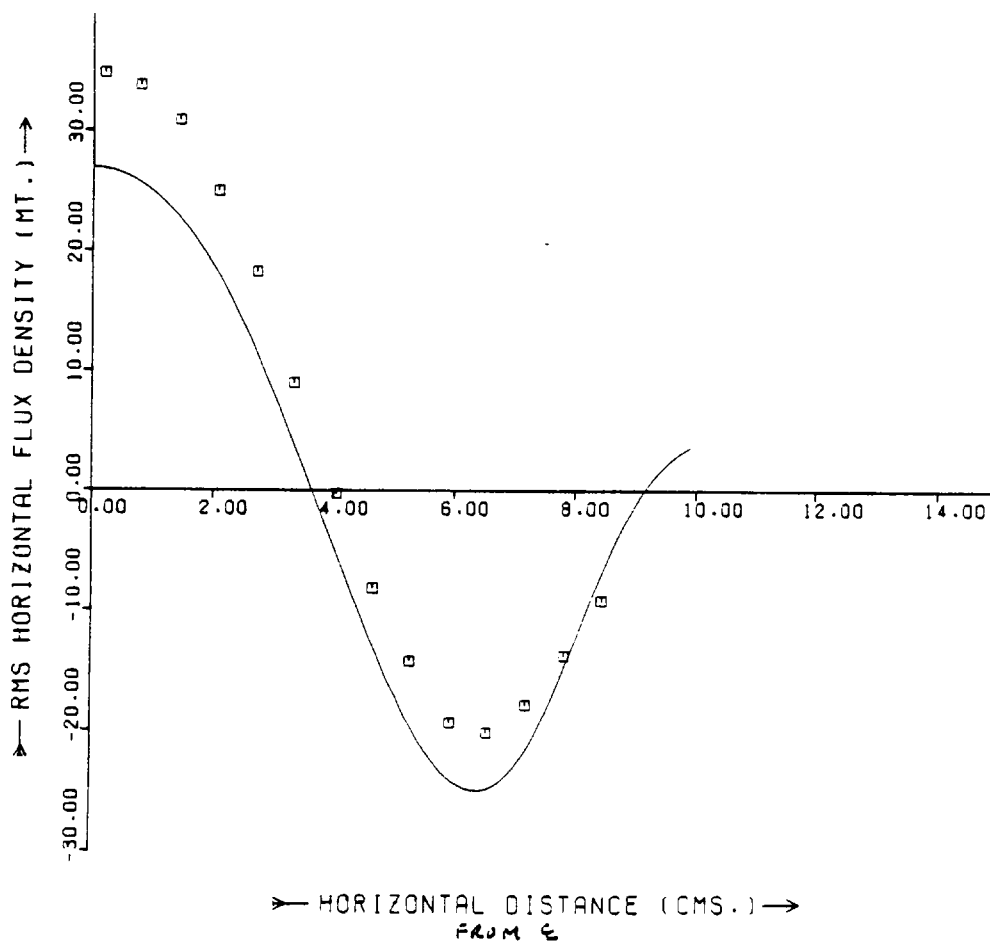


FIG.5.32 finite element mesh of right half of Xi-core



B measured
 / calculated

FIG.5.33 R.M.S horizontal flux density
 for FIG.5.32 along line xx'

taken to attempt to resolve this situation,

- 1) seek corroborative measured results,
- 2) investigate the results for finer meshes,
- 3) enforce the iron as a boundary condition,
- 4) consider the alternative representations of
$$\int_{\Gamma} W \frac{\partial \underline{H}}{\partial n} d\Gamma \text{ at corners.}$$

The first two suggested steps above are self explanatory but the other two, and in particular the last point, could benefit from a few more words of explanation.

With reference to point three above it should be recalled that the results presented in this chapter for both the E-core and Xi-core machines have included the iron in the analysis by regarding it as non-conducting and of high permeability. It is possible however to regard the iron as presenting a boundary condition by regarding its permeability as infinite; the condition imposed in this case is zero tangential $\underline{H}$.

The last point above is perhaps somewhat obscure, but the term given arises from the standard Weighted Residual approximation of the vector Laplacian for $\underline{H}$ (see section 3.3), where W is the weighting function and Γ the perimeter of the region where the Laplacian equation for $\underline{H}$ holds. The formulation as used by the author consistently in this thesis is to evaluate $\int_{\Gamma} W \frac{\partial \underline{H}}{\partial n} d\Gamma$ and then to impose the relevant boundary conditions on $\underline{H}$. It is possible though to consider involving the

boundary conditions in the integral before evaluation and this slightly modifies the resulting equation at corners, but not elsewhere.

The author has done some initial work on implementing points two, three and four above but at this stage the results are inconclusive and it would be unwise to pre-empt the results of further work.

5.6.2 Solutions for H_y

The conditions imposed on the problem were those given in (5.138) and (5.139). The method of solution is that as used for the solution of H_x in the E-core problem, adapted for H_y . Two solutions are presented, figs.5.34 and 5.35, consistent with the meshes given in figs.5.30 and 5.32. The results show a remarkable agreement with the measured results particularly in view of the poor results achieved for H_x .

5.7 CONCLUSIONS

It has been shown that the interface conditions for H at a two dimensional interface are straightforward to implement, have introduced no numerical instability and allow the construction of reasonable results. However the two natural extensions of this work, (i) to variable μ_r in two dimensional situations and (ii) to

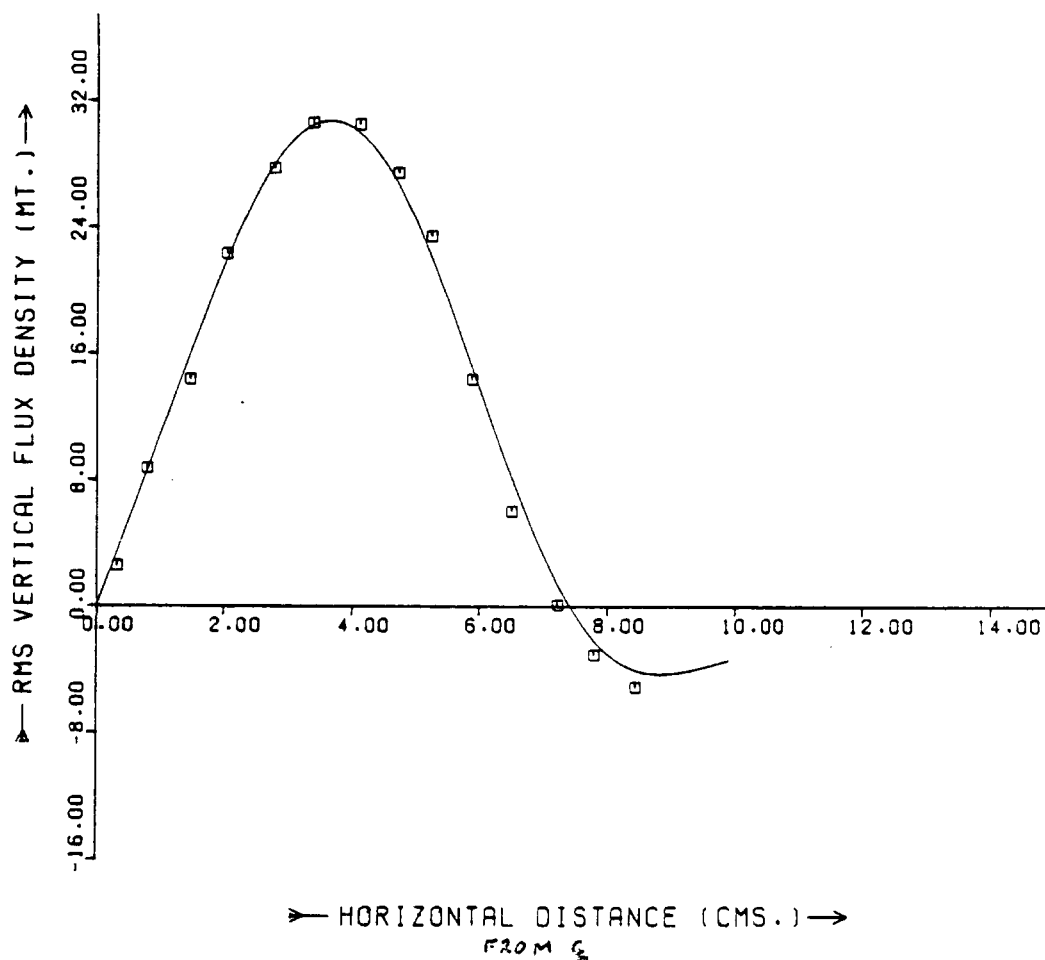


FIG.5.34 R.M.S vertical flux density for FIG.5.30
along line xx'

□ measured

— calculated

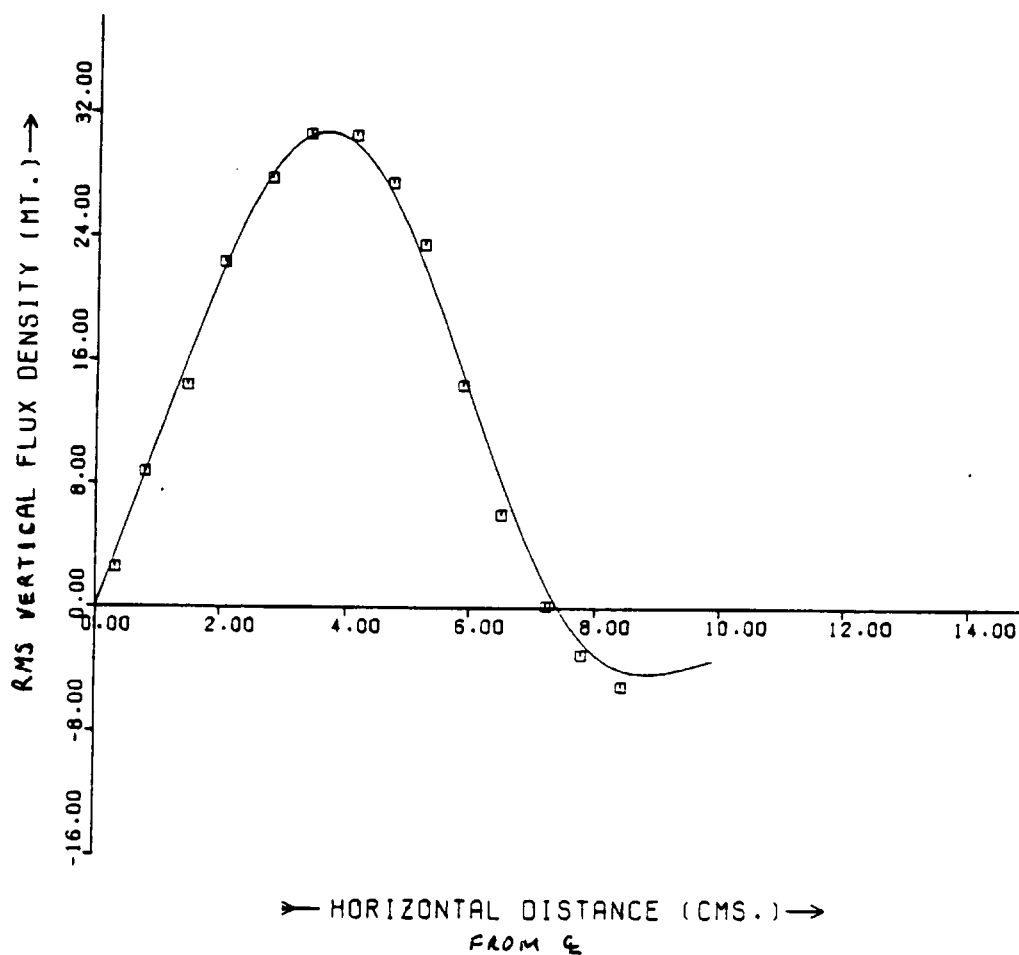


FIG.5.35 R.M.S Flux density for FIG.5.32 along xx'

□ measured
— calculated

three dimensional problems, do not look promising.

The former, although introducing no essential new difficulties, produces a field within permeable materials such that the $\underline{H}$ components are linked by the formulation within and on the surface. This means that the method is more demanding computationally than $\underline{A}$ or $(\underline{T}, \Omega)$ and therefore could hardly be seen as a preferred method. The latter, although computationally comparable to $\underline{A}$ or $(\underline{T}, \Omega)$, produces problems at edges and corners where in general the application of the interface conditions for $\underline{H}$ looks far from trivial. However for cuboid type geometries the analysis given at the beginning of the chapter on edge and corner conditions looks distinctly viable and in the authors opinion warrants investigation for both constant and varying μ_r .

The programs which have produced the results in this chapter were written with functionality in mind rather than efficiency but typical computer requirements for the solutions presented were 0.25-0.5 Mbytes of computer storage and 100-200 seconds of central processing time (on a Prime 750) where each real variable was allocated eight bytes. The majority of the processing time is used in evaluating the coefficients of the matrices representing the Greens Integral equation and the storage requirements are large because of the fully populated matrices which ensue. However the author believes that with some ramifications to the computer code both the storage and time required can be reduced

and thus for problems of the type analysed in this chapter the method can be made more comparable certainly in terms of time with formulations based on $\underline{A}$ or $(\underline{T}, \Omega)$.

Having achieved some good results analysing the iron it seems reasonable to conclude that to treat the iron as a boundary condition ($H_t = 0$) has much to commend it; first of all if the full solution has worked correctly imposing the boundary condition should hardly affect the result in the air, but secondly, and more importantly, the computer resources required are reduced. The author has carried out some preliminary tests which tend to support this conclusion although it does appear that the field at the corners needs special attention.

As was mentioned in the chapter the linking of $\underline{H}$ components at permeable corners meant that the conductor had to be modelled separately in order that one component at a time could be solved. This introduced an air gap between the conductor and the iron. The author has investigated the effect of several different air gap sizes and has found that even for a 1mm air gap no adverse effects were generated.

CHAPTER 6

EDDY CURRENTS

6.1 INTRODUCTION

Hitherto in this thesis the current density, $\underline{J}$, has always been given and the fields solved have been independent of time. For many practical problems though the dependence of the field on time and the induced current distribution is important. In this chapter the limitations placed on the formulation thus far, that $\underline{J}$ be given and the field be independent of time, will be relaxed so that no longer will $\underline{J}$ need to be given throughout the whole solution domain and the field varies sinusoidally in time.

In keeping with the majority of the work presented in the thesis thus far only two dimensional situations are discussed and the relative permeability and electrical conductivity are assumed constant over each material. Within this confine there are two important classes of problem that arise, transverse electric and transverse magnetic. In the former there are two components of $\underline{J}$ and one of $\underline{H}$ and in the latter two components of $\underline{H}$ and one of $\underline{J}$. Both types of problem will be examined in this chapter and some results presented.

6.2 THE GOVERNING EQUATION FOR $\underline{H}$ IN A TRANSVERSE ELECTRIC PROBLEM

The equations to be solved are,

$$\underline{\nabla} \times \underline{H} = \underline{J} \quad (6.1)$$

$$\underline{\nabla} \cdot \underline{B} = 0 \quad (6.2)$$

$$\underline{\nabla} \times \underline{E} = - \dot{\underline{B}} \quad (6.3)$$

where the dot signifies differentiation with respect to time. In transverse electric problems there are two components of $\underline{J}$ and one of $\underline{H}$ and without loss of generality the following configuration is assumed,

$$\underline{J} = (J_x, J_y, 0) \quad (6.4)$$

$$\underline{H} = (0, 0, H_z) \quad (6.5)$$

The vector $\underline{H}$ now looks like a scalar and this is the reason why it is chosen as the function for solution. Equations (6.1) to (6.3) produce, using the standard construction, the following equation for H_z ,

$$\nabla^2 H_z = k_1 \frac{\partial H_z}{\partial t} \quad (6.6)$$

where both the relative permeability and conductivity have been assumed constant and k_1 is given by

$$k_1 = \mu_0 \mu_r \sigma \quad (6.7)$$

where μ_0 is the primary constant. Since in this chapter it is assumed that the field quantities are sinusoidal in time, H_z may be written,

$$H_z(x, y, z, t) = H_z(x, y, z) e^{i\omega t} \quad (6.8)$$

where i is the square root of -1 and ω the frequency.

Bringing together equations (6.6) and (6.8) gives the Helmholtz equation for H_z ,

$$\nabla^2 H_z = k H_z \quad (6.9)$$

where,

$$k = i\omega\mu_0\mu_r\sigma = \frac{i}{d^2} \quad (6.10)$$

d the depth of penetration. Equation (6.9) is the governing equation for H_z .

6.3 EQUATIONS FOR DEALING WITH HOLES

In certain types of axially excited problems there are holes in the materials under consideration e.g. fig.6.3. It may be possible to ignore these holes because it is adjudged that their effect is minimal, but there are holes that have an important effect on the field. With reference to fig.6.3 the holes in the body of the lamination can be ignored whereas those at the base of the left hand teeth have an important effect on the current flow in that region. It is therefore necessary to develop a technique to model adequately such holes.

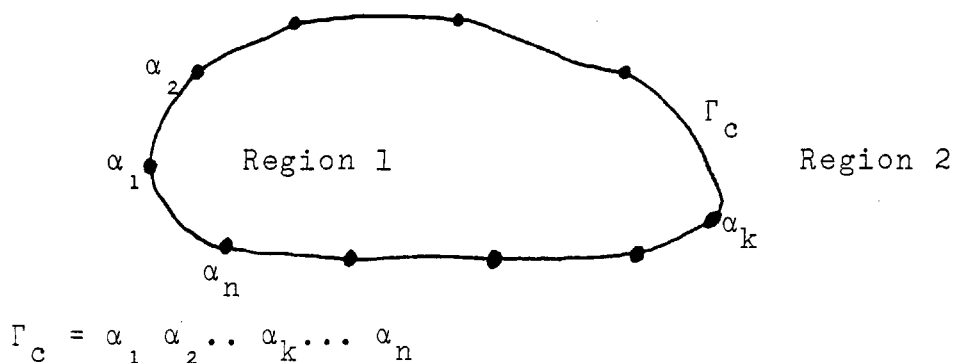


FIG.6.1

Consider fig.6.1 which represents a discretised two dimensional hole where region 2 is a current carrying region and region 1 is air. The analysis in section 6.2 holds for the situation depicted in fig.6.1 except that since there is no current in region 1 the governing equation for H_z reduces to,

$$\nabla^2 H_z^1 = 0 \quad (6.11)$$

where the 1 indicates the region from which the quantity is drawn. The condition to apply on Γ_c is that the normal component of $\underline{J}$ is zero. By analogy with zero normal $\underline{B}$ giving a zero tangential derivative of $\underline{A}$, the magnetic vector potential, zero normal $\underline{J}$ gives

$$\frac{\partial H_z}{\partial t} = 0 \quad \text{on } \Gamma_c \quad (6.12)$$

where $\frac{\partial}{\partial t}$ is a derivative in a tangential direction, which gives H_z constant on Γ_c . Since (6.11) is the governing equation in the hole, H_z is constant over region 1, the hole. However in the hole the following equation holds,

$$\underline{\nabla} \times \underline{E} = -i\mu_0 \mu_r \omega \underline{H} \quad (6.13)$$

where a sinusoidal variation in time has been incorporated. Applying Stokes theorem over the hole and remembering μ_r is unity gives,

$$\oint_{\Gamma_c} \underline{E} \cdot d\underline{\ell} = -i\mu_0 \omega H_z^c \times (\text{area of hole} = A_h) \quad (6.14)$$

since H_z is constant over the hole and where H_z^c is that constant. Now,

$$\underline{E} \cdot d\underline{\ell} = E_t d\Gamma \quad (6.15)$$

where E_t is the tangential component of E and since E_t is continuous across Γ_c equation (6.14) may be written, after multiplying by σ_2 (the conductivity of region 2),

$$\oint_{\Gamma_c} J_t^2 d\Gamma = \int_{\Gamma_c} \frac{\partial H_z^2}{\partial n} d\Gamma = -i\mu_0 \sigma_2 \omega H_z^C A_h \quad (6.16)$$

where 2 indicates the region the quantities are drawn from and,

$$J_t^2 = \frac{\partial H_z^2}{\partial n} \quad (6.17)$$

$\frac{\partial}{\partial n}$ the outward normal from region 2.

The remaining question is how to implement equation (6.16). The Weighted Residual form of the governing equation for H_z in region 2 is,

$$\int_{\Omega_2} \nabla W \cdot \nabla H_z d\Omega_2 + k \int_{\Omega_2} W H_z d\Omega_2 - \int_{\Gamma_2} W \frac{\partial H_z}{\partial n} d\Gamma_2 = 0 \quad (6.18)$$

where Ω_2 is the area of region 2 and Γ_2 the boundary perimeter of Ω_2 and W the weighting function. Moreover Γ_c is part of Γ_2 and if W is chosen so that (6.18) becomes the Galerkin formulation then

$$\sum_{i=1}^n \int_{\Gamma_c} W_{\alpha_i} \frac{\partial H_z^2}{\partial n} d\Gamma_c = \int_{\Gamma_c} \frac{\partial H_z^2}{\partial n} d\Gamma_c \quad (6.19)$$

where W_{α_i} is the weighting function associated with node i on Γ_c . Therefore if the approximating equations for each node on Γ_c are added together (6.19) ensues

and can be eliminated using (6.16) and since H_z is constant on Γ_c the nodal values equate.

6.4 A TRANSVERSE ELECTRIC PROBLEM

The problem in hand was to analyse a stack of laminations under an axial flux excitation. Fig.6.2 shows a schematic diagram of the form of the rig with the lamination stack surrounded by two circular sixty turn coils of mean diameter 82.5cms. Fig.6.3 shows the detail of one of the seventy seven laminations in the stack. $\underline{E}$ probes were placed on one of the laminations close to the centre of the stack to measure $\underline{E}$ at the edge of the lamination, it was then hoped to make comparisons between the measured $\underline{E}$ values and the calculated $\frac{\partial H}{\partial n}$ values ($\frac{\partial}{\partial n}$ the outward normal derivative). Fig.6.4 shows the positioning of the thirteen $\underline{E}$ probes used in this experiment. The measurements were made by Carpenter^{6.1} and each of the three sets given is for a different stacking factor which gives a different effective depth of penetration in each case. The three stacking factors were 0.84, 0.605, and 0.387. The iron had a resistivity of $50 \times 10^{-8} \Omega m$ and assuming a relative permeability of 100 this gave the respective depths of penetration as 14.6mm, 22.5mm and 28mm. The equations used for these calculations were^{6.2},

$$\mu_{res} = \mu_r / (k_s + \mu_r (1 - k_s)) \quad (6.20)$$

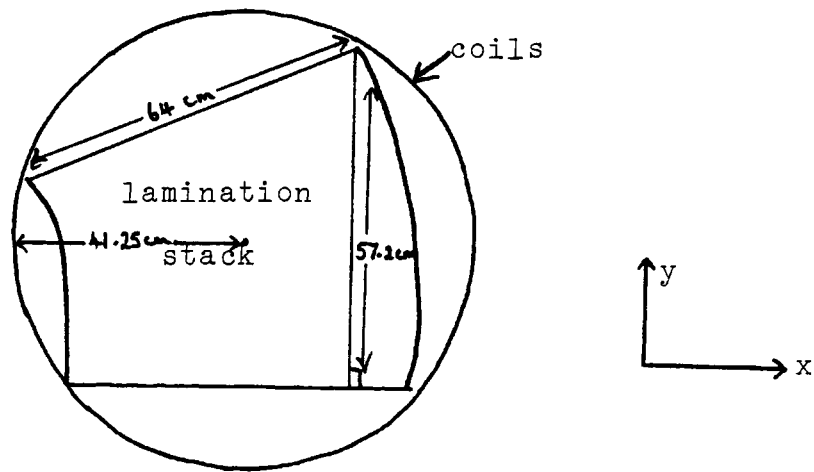
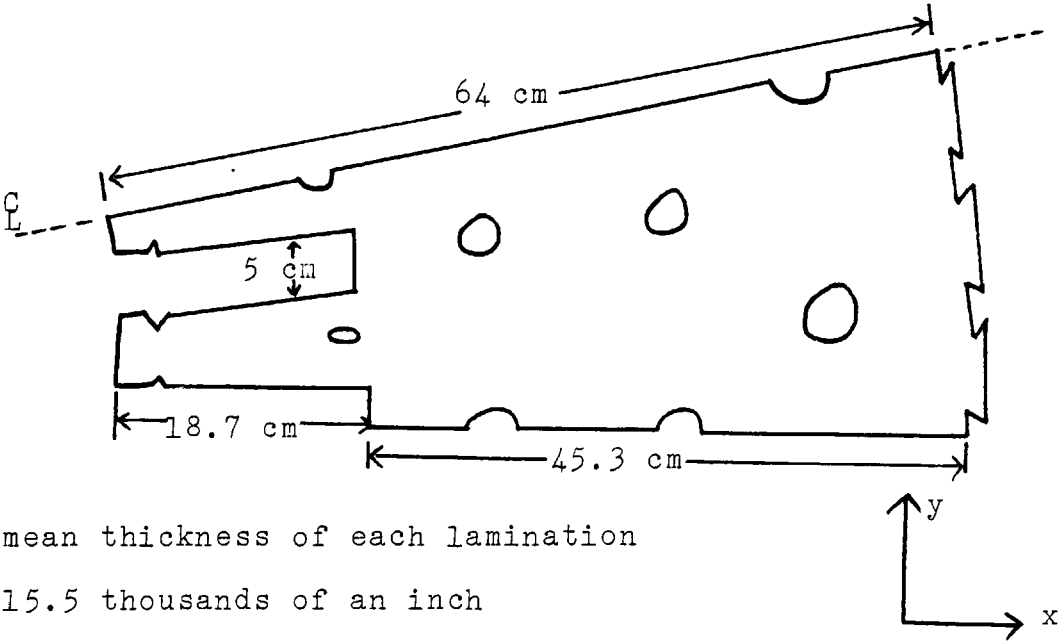


FIG.6.2 schematic diagram of lamination rig



mean thickness of each lamination
15.5 thousands of an inch

FIG.6.3 detail of one half of one lamination

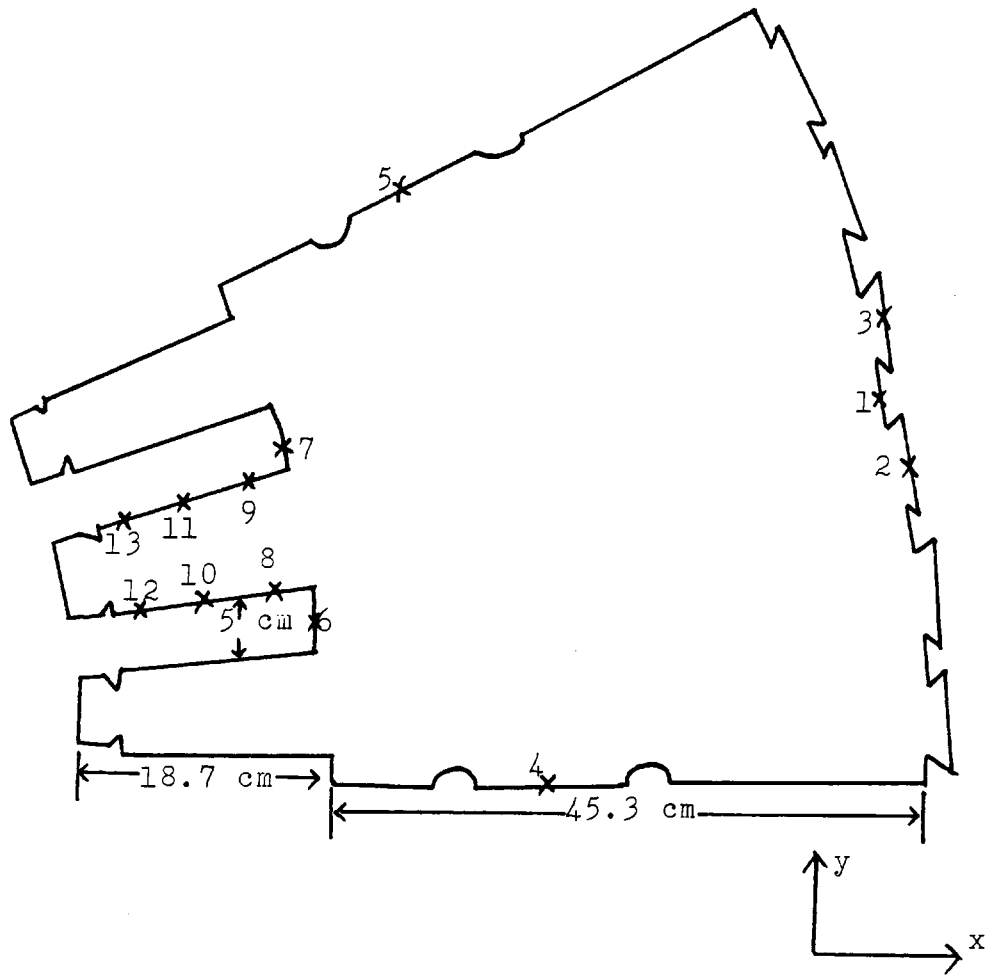


FIG.6.4 E probe positions

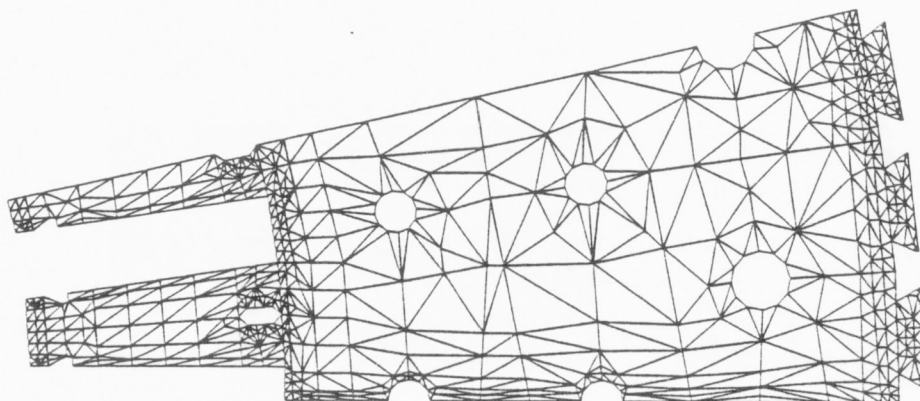


FIG. 6.5 lamination mesh

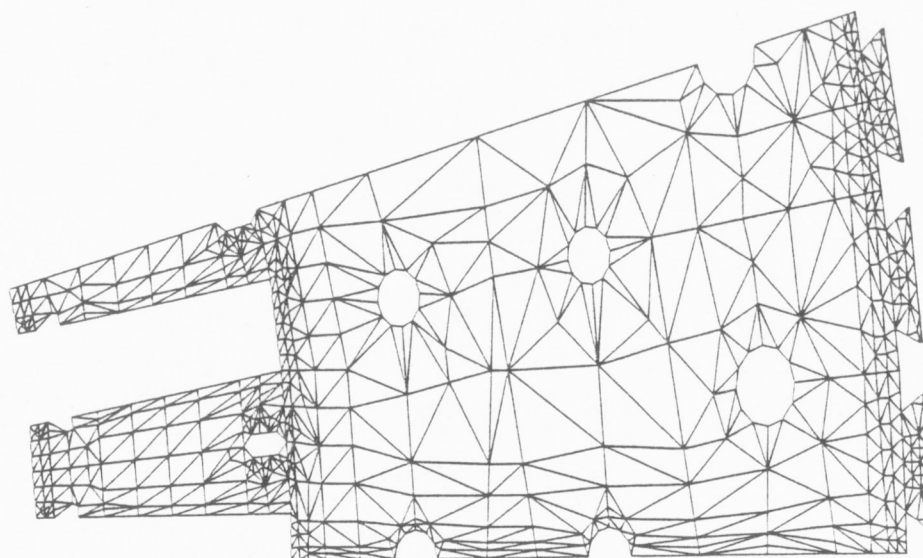


FIG. 6.6 expanded lamination mesh

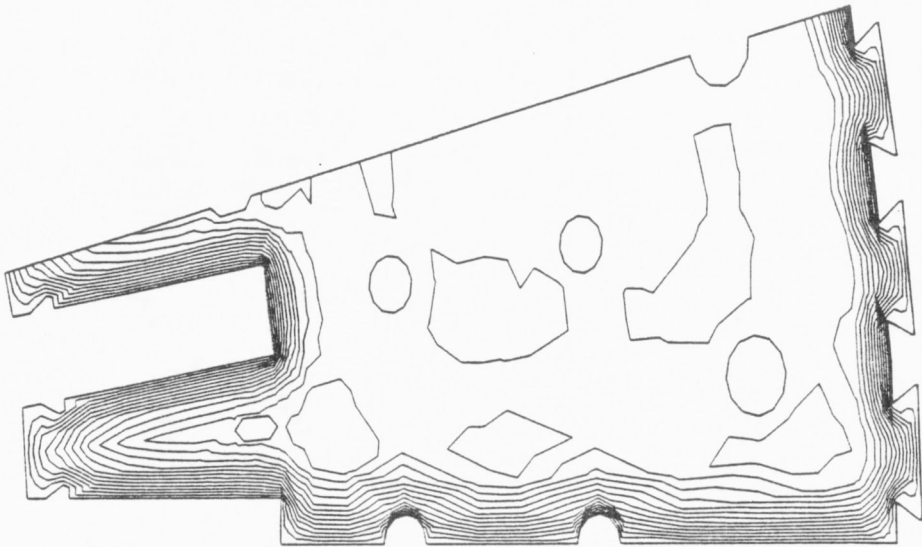


FIG.6.7 lines of current flow d=14.6mm

Search Coil	Measured		Calculated	
Fig.6.4	Tangential <u>E</u>		Tangential <u>E</u>	
	Magnitude	Phase	Magnitude	Phase
12	4.6	-21°	3.7	-49°
10	4.6	-24°	3.8	-40°
8	4.5	-28°	3.5	-38°
6	4.4	-32°	3.3	-35°
4	4.85	-50°	3.5	-44°
2	5.0	-22°	3.3	-36°
1	5.4	-35°	3.2	-49°

TABLE 6.1 Calculated and Measured Results for
 Lamination d=14.6mm

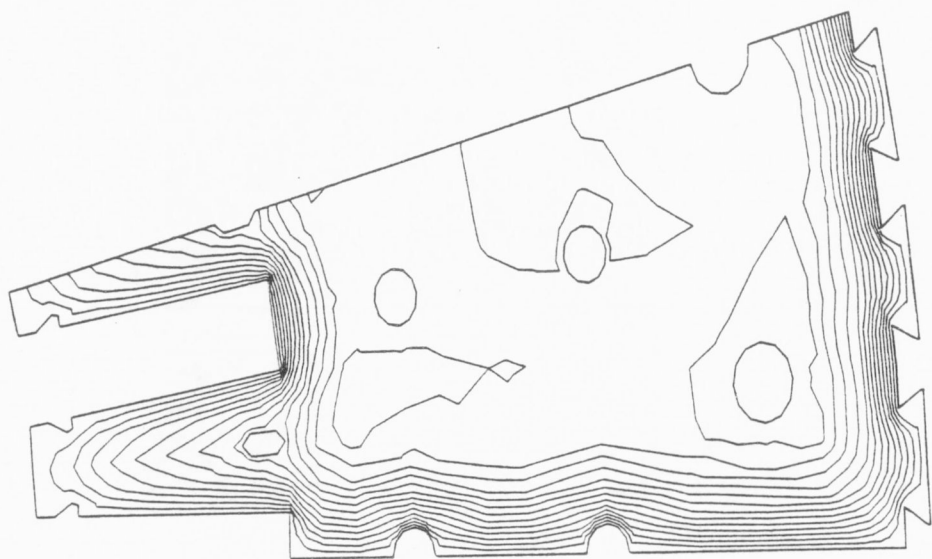


FIG.6.8 lines of current flow d=22.5mm

Search Coil	Measured		Calculated	
Fig.6.4	Tangential <u>E</u>		Tangential <u>E</u>	
	Magnitude	Phase	Magnitude	Phase
12	3.0	-13°	2.2	-66°
10	3.35	-18°	2.5	-57°
8	3.6	-29°	2.4	-49°
6	3.95	-45°	2.4	-35°
4	3.6	-38°	2.2	-46°
2	4.1	-27°	2.3	-38°
1	4.3	-43°	1.9	-54°

TABLE 6.2 Calculated and Measured Results for
 lamination d=22.5mm

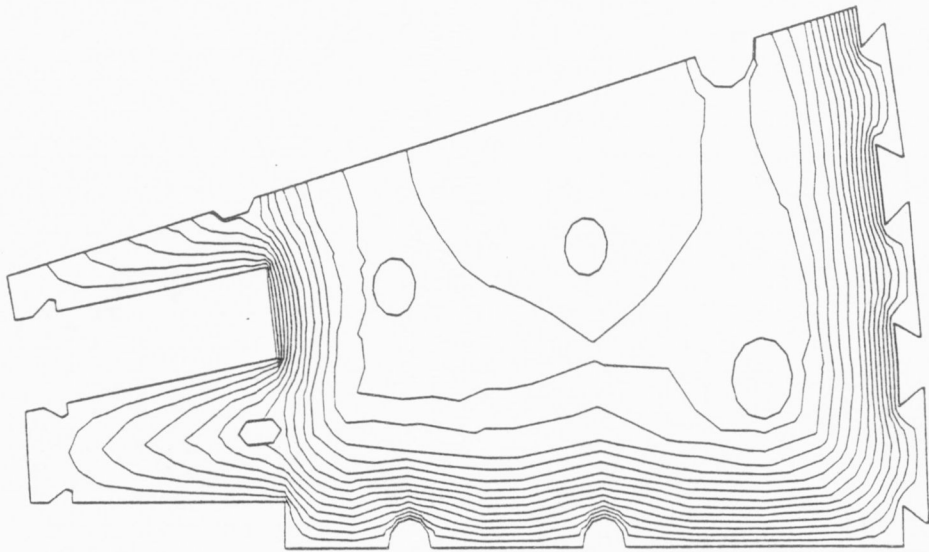


FIG.6.9 lines of current flow d=28mm

Search Coil Fig.6.4	Measured		Calculated	
	Tangential <u>E</u>		Tangential <u>E</u>	
	Magnitude	Phase	Magnitude	Phase
12	2.3	-8°	1.5	-73°
10	2.5	-14°	1.8	-66°
8	3.1	-26°	1.9	-55°
6	4.3	-45°	2.1	-35°
4	3.6	-40°	1.7	-46°
2	2.45	-28°	1.9	-39.5°
1	4.5	-45°	1.4	-55°

TABLE 6.3 Calculated and Measured Results for
Lamination d=28mm

and,

$$d^2 = 1/(\sigma\omega\mu_0\mu_{res}) \quad (6.21)$$

where μ_{res} is the resulting relative permeability, k_s the stacking factor and d the depth of penetration.

6.4.1 The Calculated Results

The mesh used for solution is shown in fig.6.5 and it shows concentrations of elements where it was adjudged that there would be maximum variation in Hz. The mean depth of edge element into the lamination is about 9mm which compares favourably with the 28mm and 22.5mm depths of penetration although not so well with the 14.6mm.

The results were calculated by assuming a constant Hz distribution on the periphery of the lamination and by solving directly the Galerkin approximation, for linear elements, to equation (6.18) the Weighted Residual form of equation (6.6) the governing equation for Hz, where the holes were treated as discussed in section 6.3.

6.4.2 Comparison Between Calculated and Measured Results

In order that more detail could be seen on the plots of current flow the plots were distorted by a factor of about 1.2 in the vertical direction. Fig.6.6 shows the expanded mesh on which the three plots (figs.6.7, 6.8,

and 6.9) are based. The figures 6.7 to 6.9 are consistent with what was expected; the smaller d , the depth of penetration, the more closely the current hugs the periphery. However the comparisons between the calculated and measured values of $\underline{E}$ given in tables 6.1 to 6.3 are not as good as was hoped for. It was anticipated that the results would be slightly affected by the decision to impose a uniform $\underline{H}$ on the periphery of the lamination but it was not envisaged that such a large discrepancy would be introduced. The measurements themselves are subject to certain inaccuracies but as the author does not know the measurement technique this is impossible to assess.

The nature of the three plots (figures 6.7 to 6.9) indicate that the method has been implemented correctly and that therefore the poor comparisons shown in tables 6.1 to 6.3 are the cumulative effect of errors produced by the mesh, the differentiation to produce $\underline{E}$, the two dimensional assumptions and the measured results.

6.5 TRANSVERSE MAGNETIC

The previous sections in this chapter have considered a two dimensional transverse electric problem where there are two components of $\underline{J}$ and one of $\underline{H}$, the concern of the remainder of this chapter is the consideration of two dimensional transverse magnetic problems where there are two components of $\underline{H}$ and one of $\underline{J}$.

The conditions imposed are that both μ_r (the relative permeability) and σ (the electrical conductivity) are constant over each material and that the variation of each field with time is sinusoidal with a frequency of fifty hertz. Implementing these restrictions reduces Maxwells equations to the following, over any one material,

$$\underline{\nabla} \times \underline{H} = \underline{J} \quad (6.22)$$

$$\underline{\nabla} \cdot \underline{H} = 0 \quad (6.23)$$

$$\underline{\nabla} \times \underline{E} = -i\omega\mu_0\mu_r\underline{H} \quad (6.24)$$

$$\underline{\nabla} \cdot \underline{J} = 0 \quad (6.25)$$

where i is the square root of minus one, ω the frequency and μ_0 the primary constant. For regions where $\underline{J}$ is known the above four equations contract to produce the vector Laplacian for $\underline{H}$ and the analysis of the earlier chapters is sufficient. However for regions where $\underline{J}$ is not known the above four equations give, using the standard construction,

$$\nabla^2 \underline{H} = k\underline{H} \quad (6.26)$$

as the governing equation for $\underline{H}$ and

$$\nabla^2 \underline{J} = k\underline{J} \quad (6.27)$$

as the governing equation for $\underline{J}$, where both equations hold within the region of interest only and

$$k = i\sigma\omega\mu_0\mu_r \quad (6.28)$$

Equation (6.26) recognises that since $\underline{J}$ is unknown there are second differential sources of $\underline{H}$ throughout the whole conductor but these have been eliminated in terms of $\underline{H}$. However the $\underline{H}$ sources arising from the

conductor surface due to the change in σ across the surface still exist and require elimination in terms of $\underline{H}$. As was discussed in chapter three the discontinuity in σ gives rise to a discontinuity in the normal derivative of $\underline{H}$ which explicitly introduces surface $\underline{J}$ terms. In magnetostatic problems this explicit appearance of $\underline{J}$ is no problem but in regions where $\underline{J}$ is unknown it requires removal in terms of $\underline{H}$.

6.6 ELIMINATING SURFACE $\underline{J}$ IN TERMS OF $\underline{H}$

As has just been stated the unknown surface $\underline{J}$ terms arise from a substitution for $\frac{\partial \underline{H}}{\partial n}$ which utilises (6.22). It is worth pointing out at this stage that since as far as $\underline{J}$ is concerned the conductor surface is not crossed the surface $\underline{J}$ is uniquely defined as that value at the discontinuity approached from within the conductor. Consider fig.6.10 which shows a graph of J_z along a straight line

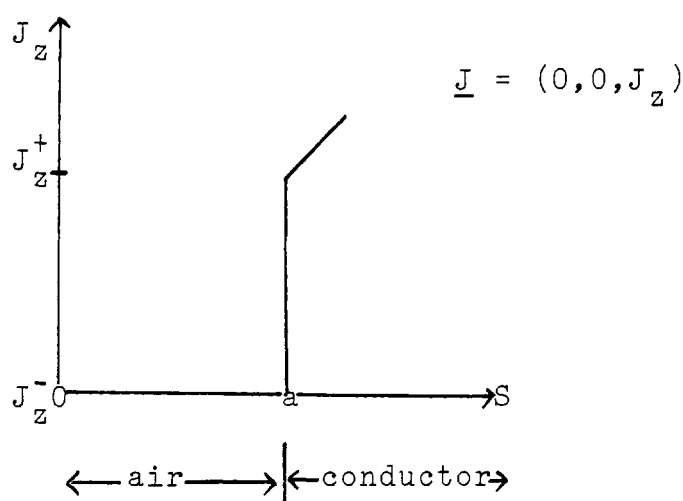


FIG.6.10

cutting a typical two dimensional conductor at a , the conductor surrounded by air. At the point, a , J_z has two values which can be represented by J_z^- and J_z^+ , where the signs indicate the direction of approach to a , and therefore for the application considered the value of J_z on the surface is considered as J_z^+ .

Since the surface $\underline{J}$ terms arise from (6.22) it is obvious that they must be eliminated using (6.24), (6.25) or both. In fact neither (6.24) or (6.25) is sufficient to define $\underline{J}$ uniquely, and therefore the two equations must be combined to produce a unique $\underline{J}$. The standard construction for the governing equation for $\underline{J}$ (6.27) in the conductor suffices for this purpose as will be shown. This raises two questions which require attention,

- 1) how can a relationship between $\underline{J}$ and $\underline{H}$ be established from (6.27)?
- 2) even if a relationship between $\underline{J}$ and $\underline{H}$ can be established is $\underline{J}$ uniquely defined?

6.6.1 A Relationship Between $\underline{J}$ and $\underline{H}$

By way of introduction it is worth noting that the formulation now involves the solution of $\underline{J}$, $(0,0,J_z)$ and that therefore the $\underline{A}$ formulation for transverse magnetic situations looks distinctly preferable since the $\underline{H}$ formulation now involves three functions.

The experience gained with the governing equations for $\underline{H}$, from which $\underline{J}$ explicitly arose, prompted an examination of the Weighted Residual form of (6.27) for J_z ,

$$\begin{aligned} - \int_{\Omega} W \nabla^2 J_z d\Omega &= \int_{\Omega} \underline{\nabla} W \cdot \underline{\nabla} J_z d\Omega - \int_{\Gamma} W \frac{\partial J_z}{\partial n} d\Gamma \\ &= - \int_{\Omega} W k J_z d\Omega \end{aligned} \quad (6.29)$$

where Ω is the region in which (6.27) holds, Γ the bounding surface of Ω , W the chosen weighting function and k as given by (6.28). As with the $\underline{H}$ formulation the outward normal derivative of the vector under consideration has arisen, namely $\frac{\partial J_z}{\partial n}$. It is possible at a first glance to think this unhelpful but a moments careful consideration shows that this is exactly what is required.

Consider fig. 6.11,

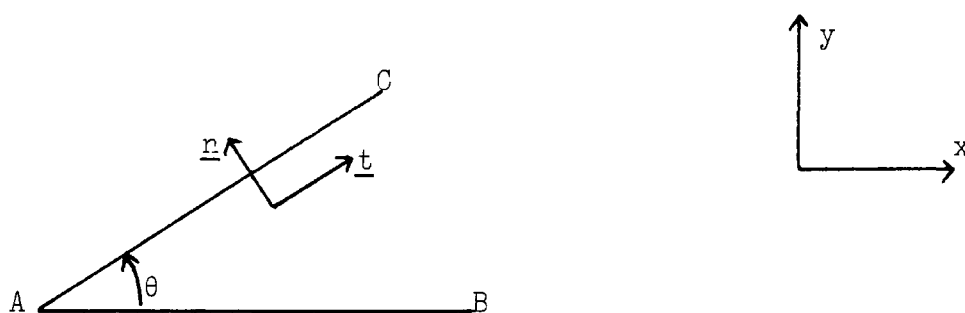


FIG.6.11

which shows a line segment AC with a defined normal $\underline{n}$

such that,

$$\frac{1}{r} \frac{\partial}{\partial \theta} = \frac{\partial}{\partial n} \quad (6.30)$$

In terms of $\underline{E}$ this gives,

$$\frac{\partial \underline{E}}{\partial n} \underline{z} = \frac{1}{r} \frac{\partial \underline{E}}{\partial \theta} \underline{z} = \frac{1}{r} \frac{\partial \underline{E}}{\partial x} \underline{z} \frac{dx}{d\theta} + \frac{1}{r} \frac{\partial \underline{E}}{\partial y} \underline{z} \frac{dy}{d\theta} \quad (6.31)$$

which reduces to,

$$\frac{\partial \underline{E}}{\partial n} \underline{z} = \frac{\partial \underline{E}}{\partial y} \underline{z} \cos \theta - \frac{\partial \underline{E}}{\partial x} \underline{z} \sin \theta \quad (6.32)$$

Moreover equation (6.24) gives,

$$\frac{\partial \underline{E}}{\partial y} \underline{z} = - \frac{kHx}{\sigma} \quad (6.33)$$

and,

$$\frac{\partial \underline{E}}{\partial n} \underline{z} = \frac{kHy}{\sigma} \quad (6.34)$$

so that equation (6.32) reduces to,

$$\frac{\partial \underline{E}}{\partial n} \underline{z} = - \frac{k}{\sigma} (Hx \cos \theta + Hy \sin \theta) = - \frac{k}{\sigma} Ht \quad (6.35)$$

where Ht is the tangential $\underline{H}$ component consistent with fig.6.10.

The equation (6.35) is generally true since (6.24) holds throughout the whole solution domain and $\frac{k}{\sigma}$ is independent of σ . However the equation (6.29) requires $\frac{\partial \underline{J}}{\partial n} \underline{z}$ on the surface and similarly to $\underline{J}_z$ it has two distinct values, $\frac{\partial \underline{J}}{\partial n} \underline{z}^-$ and $\frac{\partial \underline{J}}{\partial n} \underline{z}^+$, and if the value required is defined as that approached from within the conductor, then,

$$\frac{\partial \underline{J}}{\partial n} \underline{z} = -kHt \quad (6.36)$$

Therefore the Weighted Residual form of the governing equation for $\underline{J}$ produces a matrix equation of the form,

$$C \underline{J}_z = D \underline{H}_x^b + F \underline{H}_y^b \quad (6.37)$$

where $\underline{J}_z$ is the vector of nodal J_z values and $\underline{H}_x^b$ and $\underline{H}_y^b$ are the vectors of surface H components. Equation (6.37) will permit the substitution for surface J_z in terms of H as long as C is invertible.

6.6.2 Is J Uniquely Defined?

The question posed at the head of this section might also be rephrased as, does equation (6.37) require the addition of extra conditions? One that comes to mind is the fact that for any two dimensional region carrying eddy currents the enclosed current in that region must be zero.

Continuing in the standard way, two non trivial solutions for J_z (using the configuration of the last section) J_1 and J_2 are assumed such that,

$$J_3 = J_1 - J_2 \quad (6.38)$$

is a third non trivial solution of (6.27). The following vector identity is used,

$$-\nabla \cdot \nabla^2 \psi \equiv \nabla \cdot \nabla \psi - \nabla \cdot (\nabla \psi) \quad (6.39)$$

where,

$$\psi = J_3 \quad (6.40)$$

and the resulting equation is integrated over Ω the region in which (6.27) holds, to give using Gauss' divergence theorem,

$$\begin{aligned}
 - \int_{\Omega} J_3 \nabla^2 J_3 \, d\Omega &= \int_{\Omega} (\nabla \underline{J}_3 \cdot \nabla \underline{J}_3) \, d\Omega - \int_{\Gamma} J_3 \frac{\partial J_3}{\partial n} \, d\Gamma \\
 &= - \int_{\Omega} J_3 k J_3 \, d\Omega
 \end{aligned} \tag{6.41}$$

where Γ is the bounding surface of Ω . Equation (6.41) gives,

$$\int_{\Omega} (\nabla \underline{J}_3)^2 \, d\Omega + \int_{\Omega} k J_3^2 \, d\Omega = \int_{\Gamma} J_3 \frac{\partial J_3}{\partial n} \, d\Gamma \tag{6.42}$$

However for a unique $\underline{H}$, in particular H_t , $\frac{\partial J_3}{\partial n}$ is identically zero since equation (6.36) holds, thus giving,

$$\int_{\Omega} (\nabla \underline{J}_3)^2 \, d\Omega + \int_{\Omega} k J_3^2 \, d\Omega = 0 \tag{6.43}$$

The above equation represents a sum of squared quantities giving zero and thus it would be attractive to conclude,

$$J_3 \equiv 0 \tag{6.44}$$

giving a unique $\underline{J}$. However equation (6.43) represents a sum of squared complex quantities which can obviously be zero without of necessity implying that the individual quantities must each be zero. Thus a slight modification is required to produce (6.44). The governing equation for J_3 may be written,

$$\nabla^2 (J_R + i J_I) = i k, (J_R + i J_I) \tag{6.45}$$

where i is the square root of minus one,

$$k = i k_1 \tag{6.46}$$

and,

$$J_3 = J_R + i J_I \tag{6.47}$$

where J_R and J_I are non complex functions. Equation (6.45) gives,

$$\nabla^2 J_R = -k_1 J_I \quad (6.48)$$

and,

$$\nabla^2 J_I = k_1 J_R \quad (6.49)$$

on separating real and imaginary parts. If the earlier argument is now applied to these two equations the results are,

$$\int_{\Omega} (\nabla J_R)^2 d\Omega = \int_{\Omega} J_R J_I k_1 d\Omega \quad (6.50)$$

and,

$$\int_{\Omega} (\nabla J_I)^2 d\Omega = - \int_{\Omega} J_R J_I k_1 d\Omega \quad (6.51)$$

which sum to give,

$$\int_{\Omega} (\nabla J_I)^2 d\Omega + \int_{\Omega} (\nabla J_R)^2 d\Omega = 0 \quad (6.52)$$

which gives,

$$J_R = J_I = \text{a constant} \quad (6.53)$$

Moreover replacing (6.53) in equations (6.48) and (6.49) shows that,

$$J_3 \equiv 0 \quad (6.54)$$

which implies that as long as $\underline{H}$ is uniquely defined so is $\underline{J}$.

6.7 CONDITIONS FOR A UNIQUE $\underline{H}$

In this chapter eddy current situations are examined and it is the concern in this section to derive sufficient conditions to determine $\underline{H}$ uniquely for two dimensional transverse magnetic problems.

Recalling that the assumption has been made that the field quantities vary sinusoidally with time the two governing equations for $\underline{H}$ may be written,

$$\underline{\nabla} \times \underline{\nabla} \times \underline{H} = -k\underline{H} \quad (6.55)$$

and,

$$\mu_0 \mu_r \underline{\nabla} \cdot \underline{H} = 0 \quad (6.56)$$

where k is given by (6.28). When k is zero equations (6.55) and (6.56) represent the same conditions as discussed in chapter five and therefore it is sufficient to show that when k is non zero the discussion in chapter five can still be applied.

Consider two distinct solutions to equations (6.55) and (6.56) $\underline{H}_1$ and $\underline{H}_2$ over a region Ω where k is non zero. These two distinct solutions permit a third non trivial solution to be constructed, namely $\underline{H}_3$ where,

$$\underline{H}_3 = \underline{H}_1 - \underline{H}_2 \quad (6.57)$$

However since $\underline{H}_3$ is a complex quantity it will be separated thus,

$$\underline{H}_3 = \underline{H}_R + i \underline{H}_I \quad (6.58)$$

so that considering (6.55) the following two equations

ensue,

$$\underline{\nabla} \times \underline{\nabla} \times \underline{H}_R = k_1 \underline{H}_I \quad (6.59)$$

and,

$$\underline{\nabla} \times \underline{\nabla} \times \underline{H}_I = -k_1 \underline{H}_R \quad (6.60)$$

Consider the vector identity,

$$\underline{\nabla} \cdot (\underline{P} \times \underline{Q}) \equiv \underline{Q} \cdot \underline{\nabla} \times \underline{P} - \underline{P} \cdot \underline{\nabla} \times \underline{Q} \quad (6.61)$$

where,

$$\underline{P} = \underline{H}_R \quad (6.62)$$

and,

$$\underline{Q} = \underline{\nabla} \times \underline{H}_R \quad (6.63)$$

integrated over Ω to give, using Gauss' divergence theorem,

$$\begin{aligned} \int_{\Gamma} \underline{n} \cdot (\underline{H}_R \times (\underline{\nabla} \times \underline{H}_R)) \, d\Gamma &= \int_{\Omega} (\underline{\nabla} \times \underline{H}_R \cdot \underline{\nabla} \times \underline{H}_R) \, d\Omega \\ &\quad - \int_{\Omega} \underline{H}_R \cdot \underline{\nabla} \times \underline{\nabla} \times \underline{H}_R \, d\Omega \end{aligned} \quad (6.64)$$

where Γ is the bounding surface of Ω . For a unique $\underline{J}$ the above equation reduces to,

$$\int_{\Omega} (\underline{\nabla} \times \underline{H}_R)^2 \, d\Omega - \int_{\Omega} k_1 \underline{H}_R \cdot \underline{H}_I \, d\Omega = 0 \quad (6.65)$$

and similarly for $\underline{H}_I$,

$$\int_{\Omega} (\underline{\nabla} \times \underline{H}_I)^2 \, d\Omega + \int_{\Omega} k_1 \underline{H}_I \cdot \underline{H}_R \, d\Omega = 0 \quad (6.66)$$

The above two equations give,

$$\underline{\nabla} \times \underline{H}_3 = 0 \quad (6.67)$$

At this stage the argument in chapter five can be applied, and $\underline{H}$ is unique as long as the standard interface conditions are applied along with $\underline{H}$ zero at infinity.

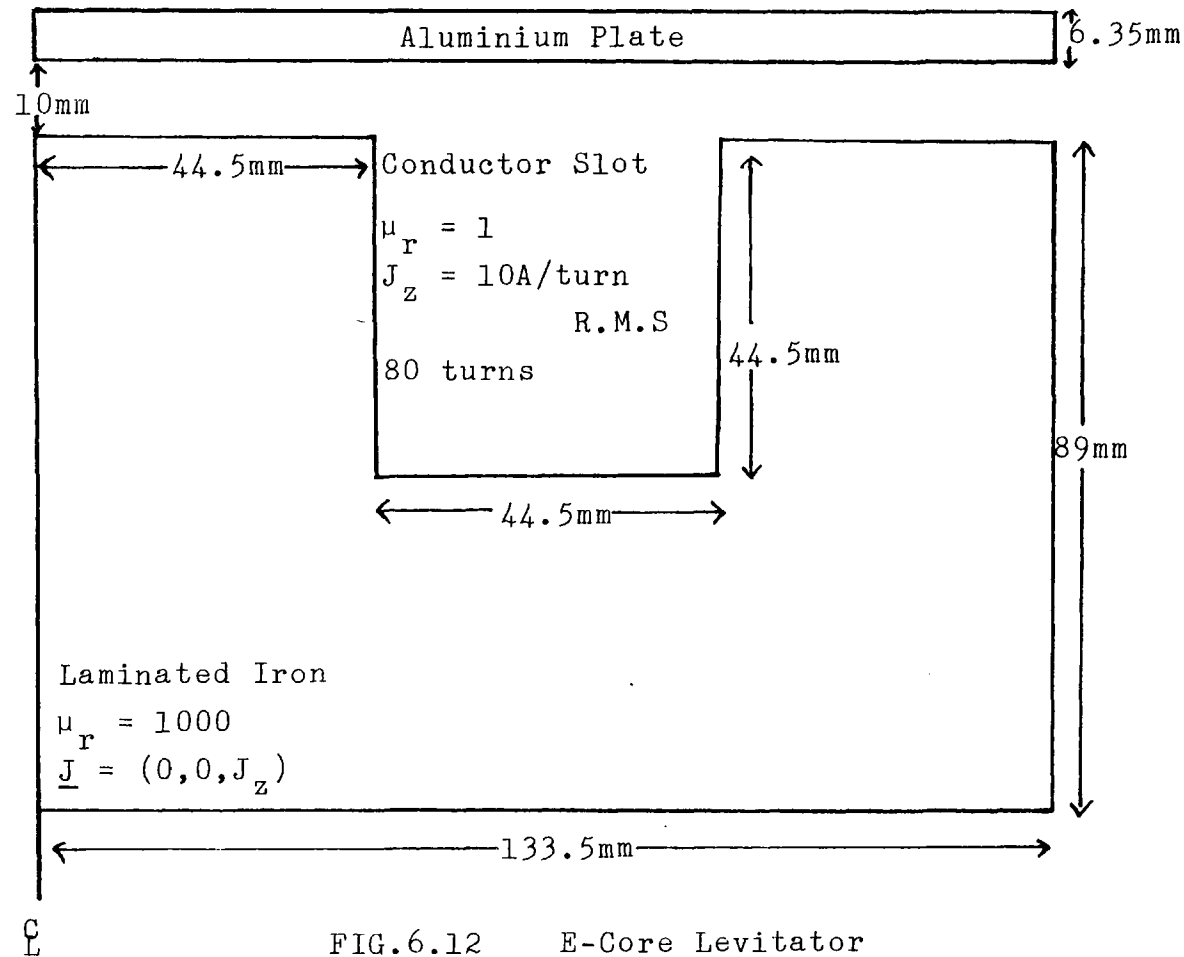
6.8 AN E-CORE LEVITATOR

Having discussed the approach for eddy current problems the E-core machine examined on open circuit in chapter five was used again but this time with a conducting secondary in order to validate the theory presented above. Fig.6.12 shows the geometry of the machine used and the only change from that given in fig.5.19 is the introduction of the conducting aluminium plate at a height of 10mm above the stator iron. The stator iron was again included in the analysis and was given a relative permeability of 1000. It was decided to include the iron so that the performance of the interface conditions could be assessed in this more computationally demanding situation. Similarly to the magnetostatic analysis carried out in chapter five the material surfaces were chosen as the boundaries between the differential and integral domains, since this permits the independent solution of the $\underline{H}$ components.

The calculated results are compared with the measured results presented by Wyatt^{6.3}. The results were measured at a height of 6mm above the stator iron using a Hall probe connected to a 'time average' meter. The

Length of Stator = 290mm

Conductivity of Plate = $1/2.9 \times 10^8 \text{ Sm}^{-1}$



results presented are for a symmetrically placed plate.

The calculation methods are the same as used for the E-core on open circuit discussed in chapter five with the addition of the analysis discussed earlier in this chapter for the conducting plate. Thus the vector Laplacian for $\underline{H}$ is solved in the iron, the air and the slot conductor and,

$$\nabla^2 \underline{H} = k \underline{H} \quad (6.68)$$

and,

$$\nabla^2 \underline{J} = k \underline{J} \quad (6.69)$$

in the conducting secondary.

6.8.1 Results for H_x

The conditions imposed on the problem were that on the centre line of the machine,

$$H_x = \frac{\partial H_y}{\partial n} = 0 \quad (6.70)$$

$$J_{\text{plate}} = 0 \quad (6.71)$$

and at infinity,

$$H_x = H_y = 0 \quad (6.72)$$

It is interesting to note in relation to this problem that the line integral of $\underline{H} \cdot d\underline{\ell}$ around the plate is zero automatically because of the symmetry of the machine.

Fig.6.13 shows the mesh for the first solution presented, a regular mesh not exploiting any pre-conceived ideas about the form of the solution. Also notice the odd

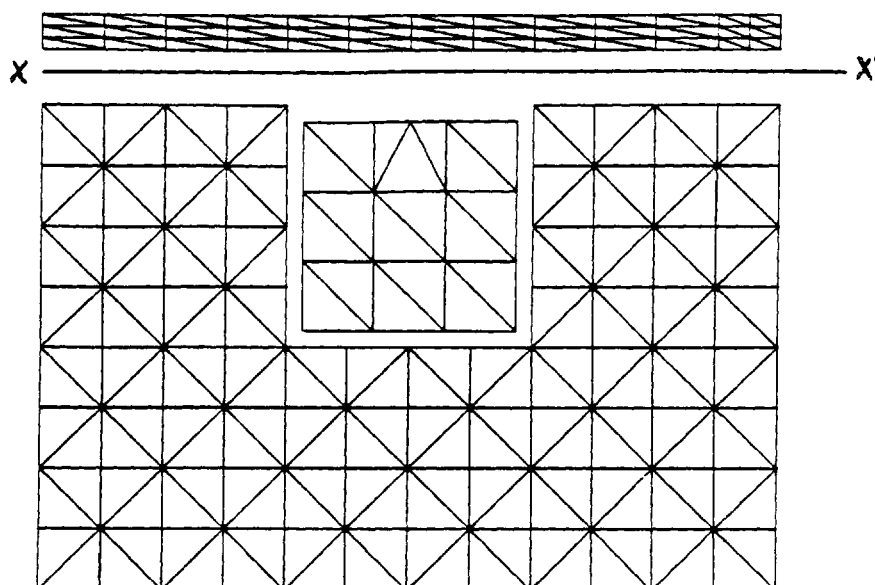
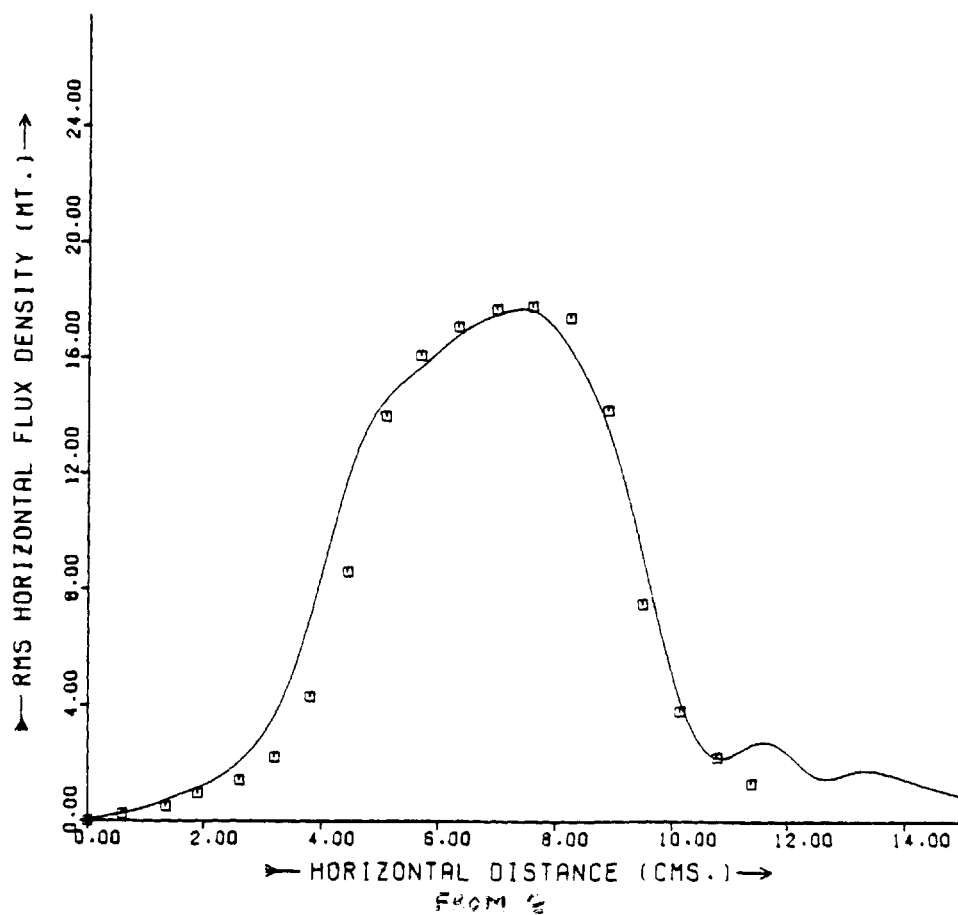


FIG.6.13 finite element mesh of right half of E-core levitator



□ measured FIG.6.14 R.M.S horizontal flux density
 — calculated for FIG.6.13 along line xx'

numbered boundary noded conductor which, as was discussed in chapter four, permits a unique solution. Fig.6.14 gives the comparison between the calculated and measured horizontal flux densities.

Fig.6.15 shows a second mesh used where the number of nodes in the iron has been reduced by assuming that the regions of interest are the top surfaces of the iron and the slot, with particular attention shown to the corners. The remainder of the mesh has been constructed by grading out from these surfaces. Fig.6.16 shows the comparison between the measured and calculated horizontal flux densities.

Since the conductor in the slot was easily adjustable it was reduced in size to see if it had any marked effect on the results. Fig.6.17 shows essentially the same mesh as fig.6.15 but with a smaller conductor. To maintain the current in the slot the current density was suitably adjusted. The conductor in fig.6.15 is 3mm from the iron on three sides whereas the distance is 10mm in fig.6.17. Fig.6.18 shows the result obtained and it is essentially the same as that given in fig. 6.16.

The calculated B_x values have both a real and an imaginary component and fig.6.19 shows the two components that comprise the calculated B_x given in fig.6.16.

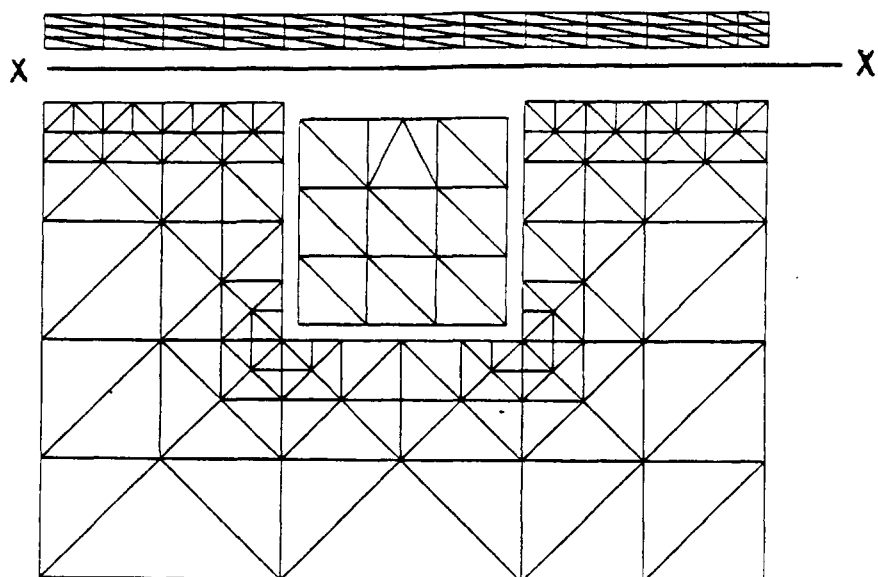
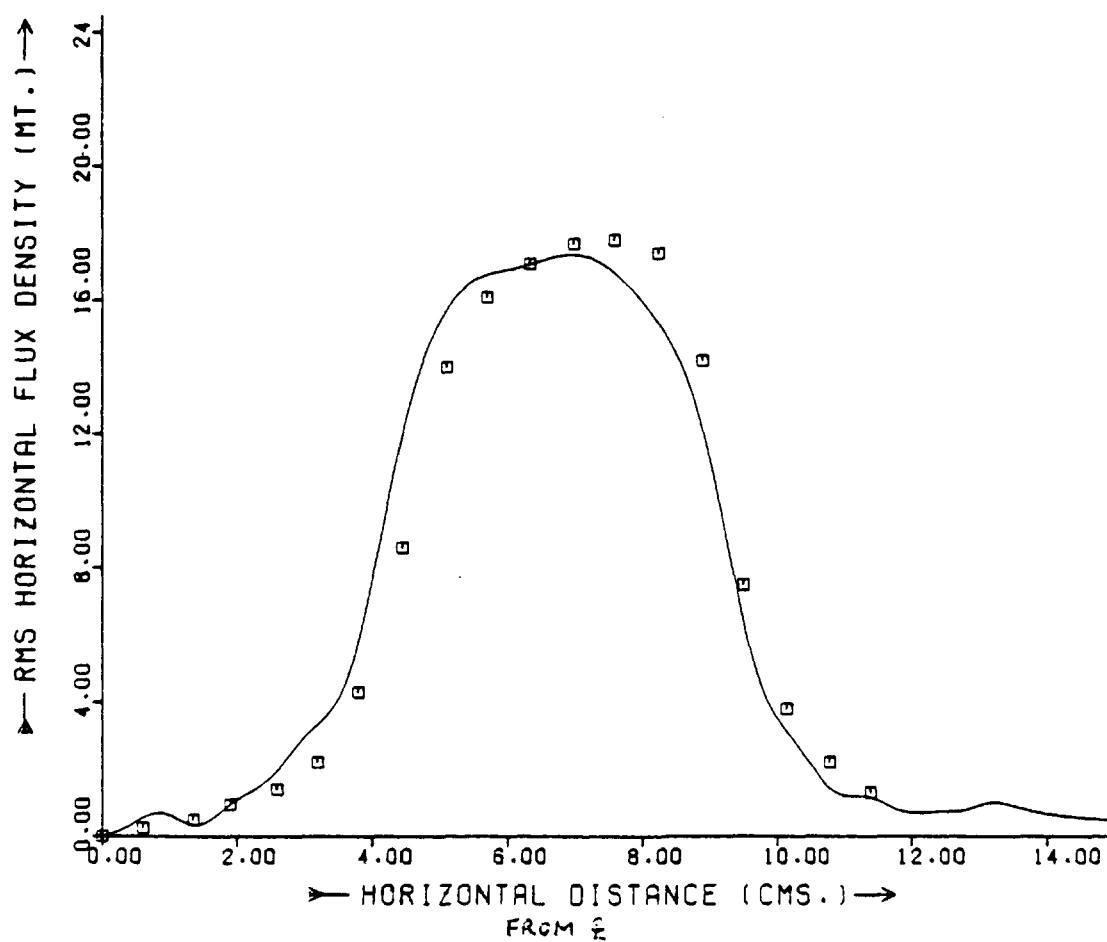


FIG.6.15 finite element mesh of right half of E-core levitator



□ measured FIG.6.16 R.M.S horizontal flux density
 — calculated for FIG.6.15 along line xx'

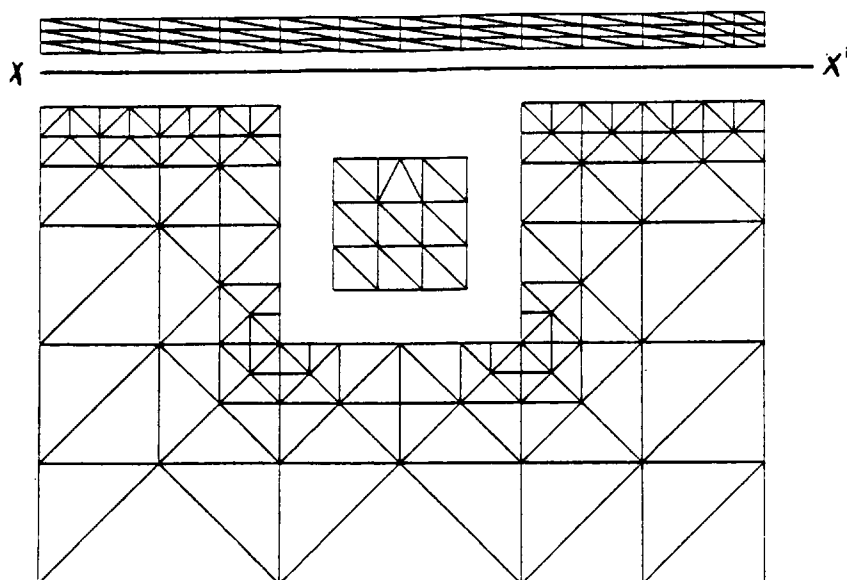
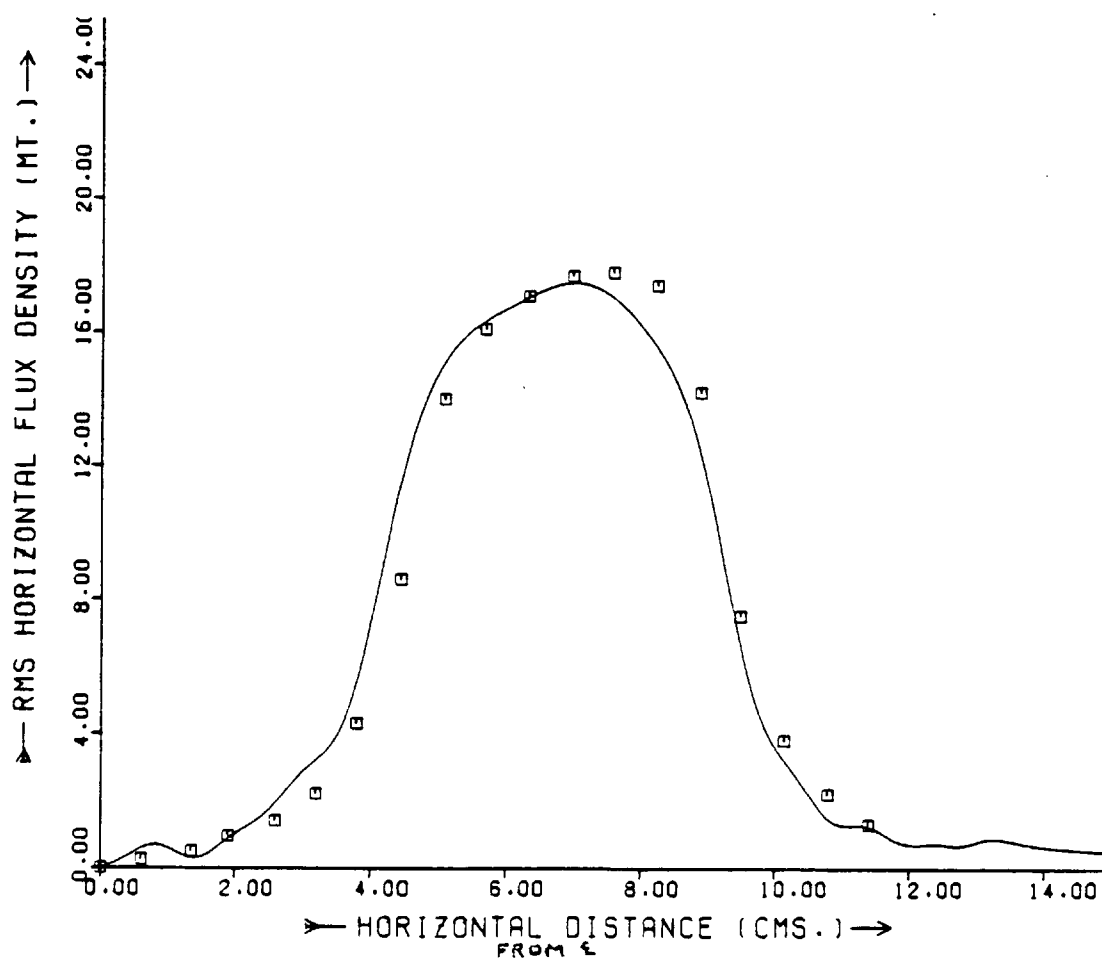


FIG.6.17 finite element mesh of right half of E-core levitator



■ measured FIG.6.18 R.M.S horizontal flux density
 — calculated for FIG.6.17 along line xx'

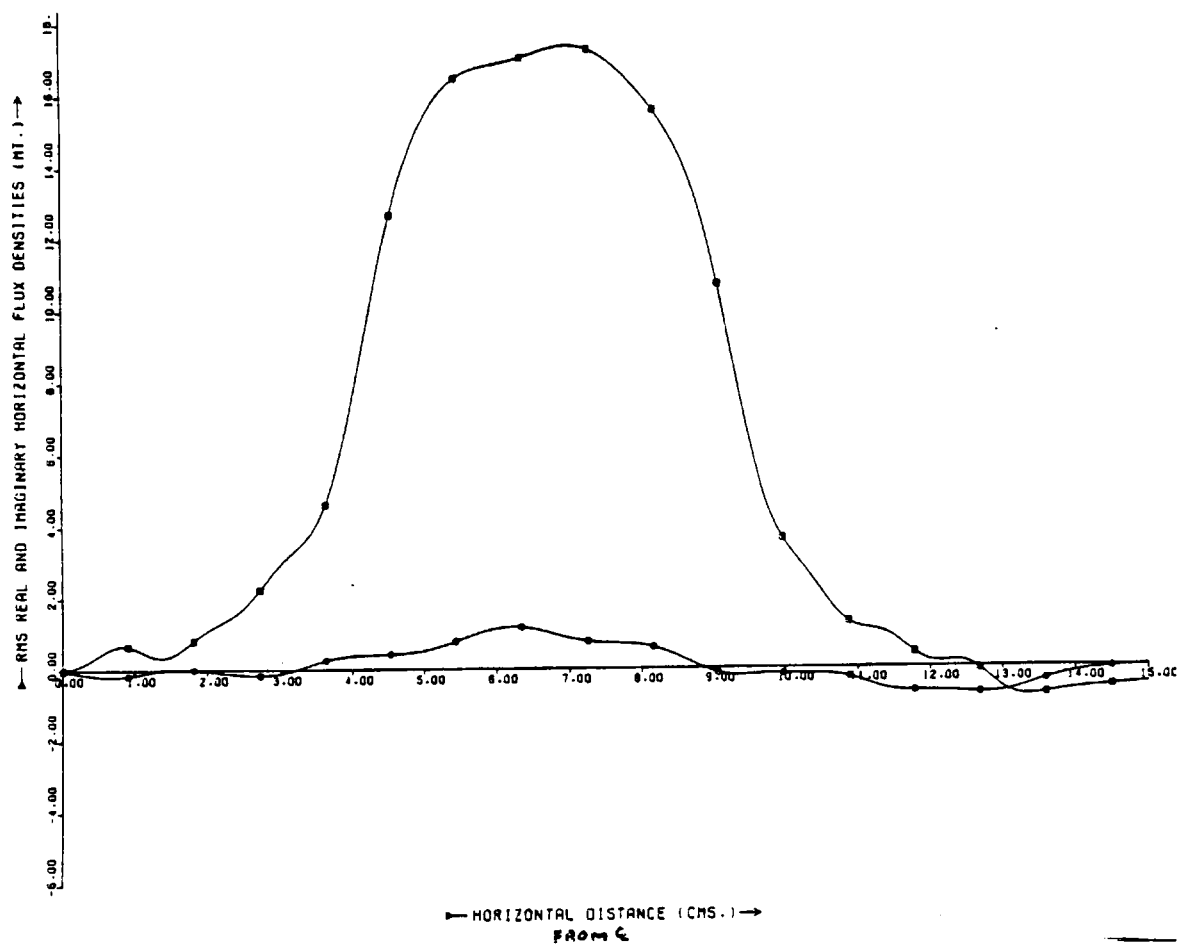


FIG.6.19 R.M.S real and imaginary horizontal flux densities for FIG.6.16 along line xx'

□ real

○ imaginary

The important feature is that it shows, as was expected, the result is dominated by the real component.

6.8.2 Results for H_y

The conditions imposed on the problem are those given in (6.70), (6.71) and (6.72). The results presented are for the same meshes that were used for the solution of H_x .

Fig.6.20 shows the results achieved for the mesh in fig. 6.13. The value at the centre of the machine on the y axis is within half of one percent of the measured value but the peak values occurring approximately over the slot corners seem to have been rounded off. This prompted the refining of this part of the mesh as shown in figs. 6.15 and 6.17.

Figs.6.21 and 6.22 give the results respectively for the meshes given in figs.6.15 and 6.17. Once again it can be seen that the adjustment of the conductor size has hardly affected the result. The altering of the mesh and in particular the refining of the slot corners has produced better peak comparisons.

Fig.6.23 shows the real and imaginary components of B_y for the calculated B_y presented in fig.6.21. It is interesting to note that in contrast to H_x the imaginary

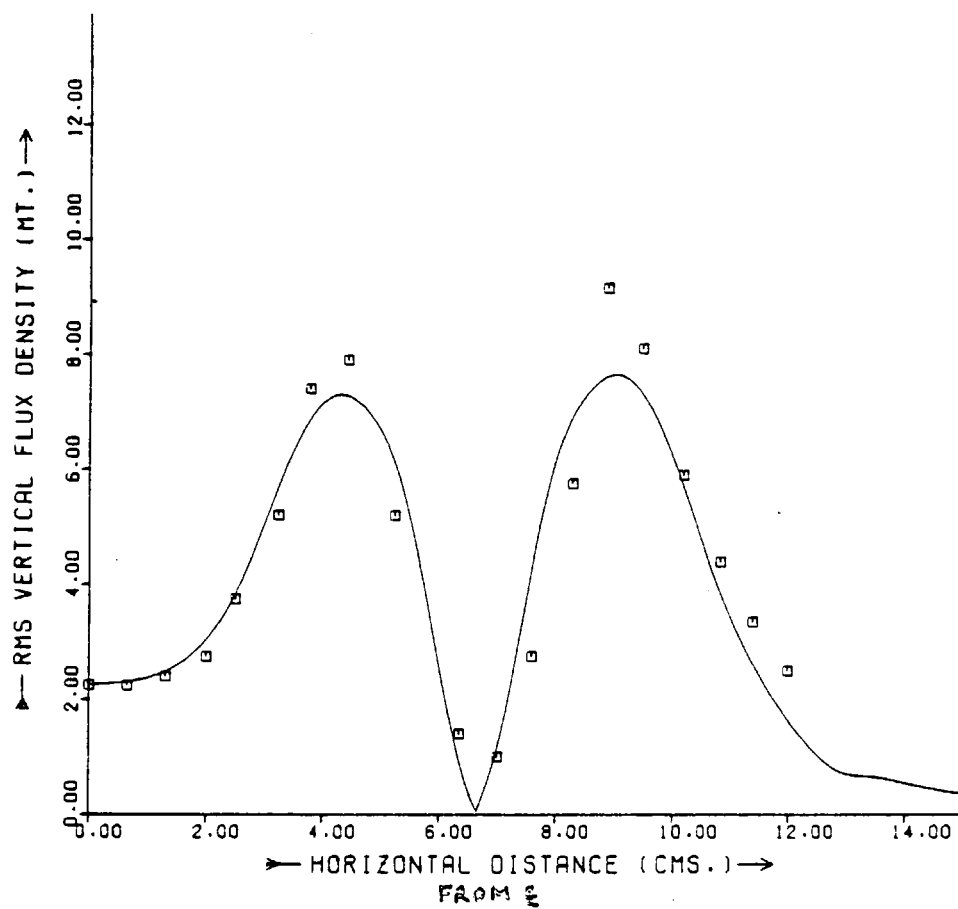


FIG.6.20 R.M.S vertical flux density for FIG.6.13
along line xx'

□ measured
— calculated

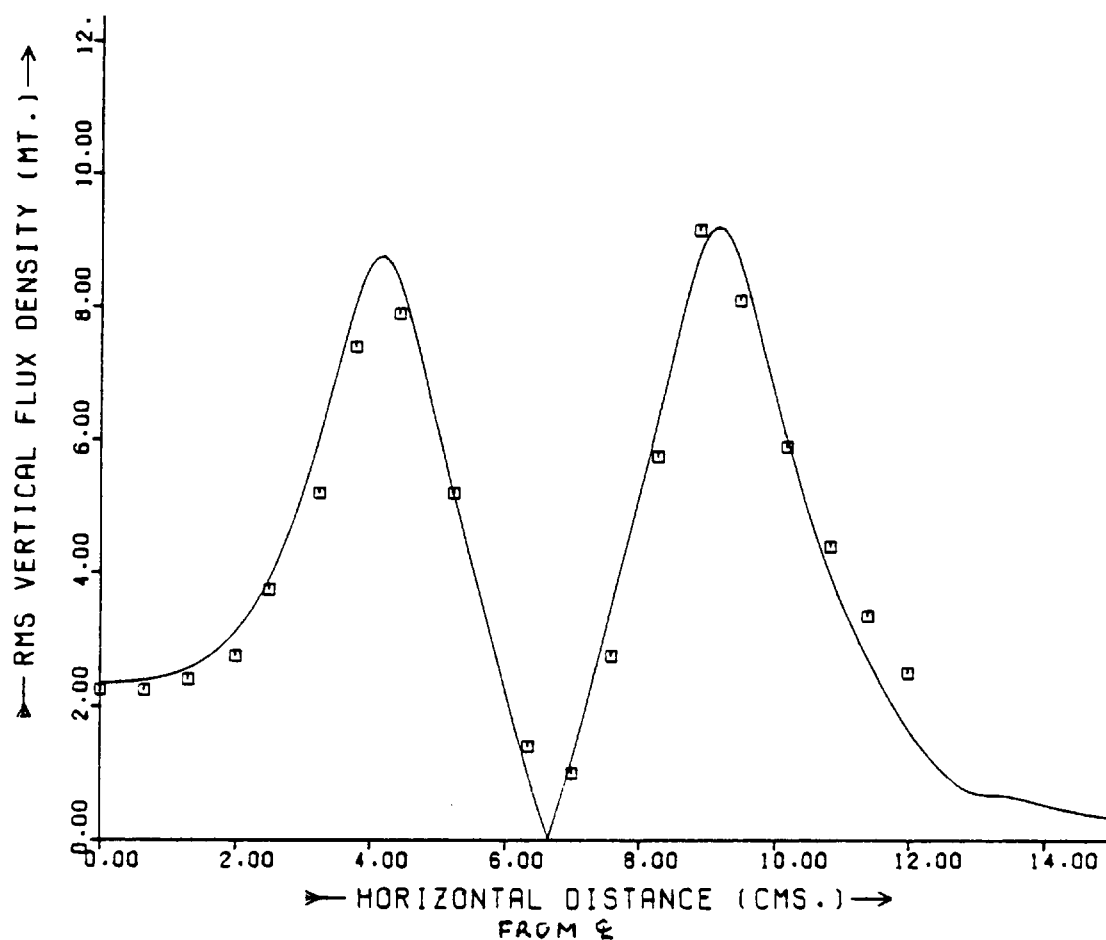


FIG.6.21 R.M.S vertical flux density for FIG.6.15
along line xx'

□ measured
— calculated

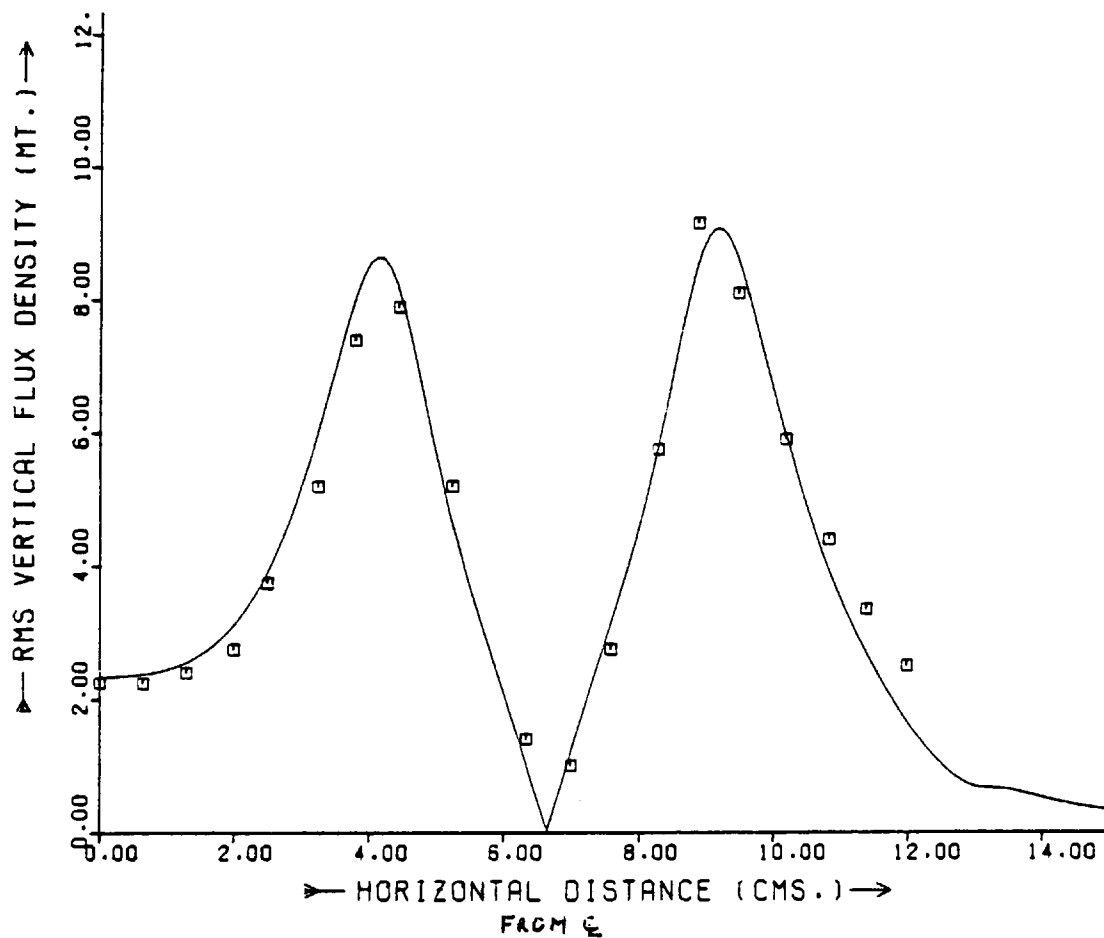


FIG.6.22 R.M.S vertical flux density for FIG.6.16
along line xx'

□ measured

— calculated

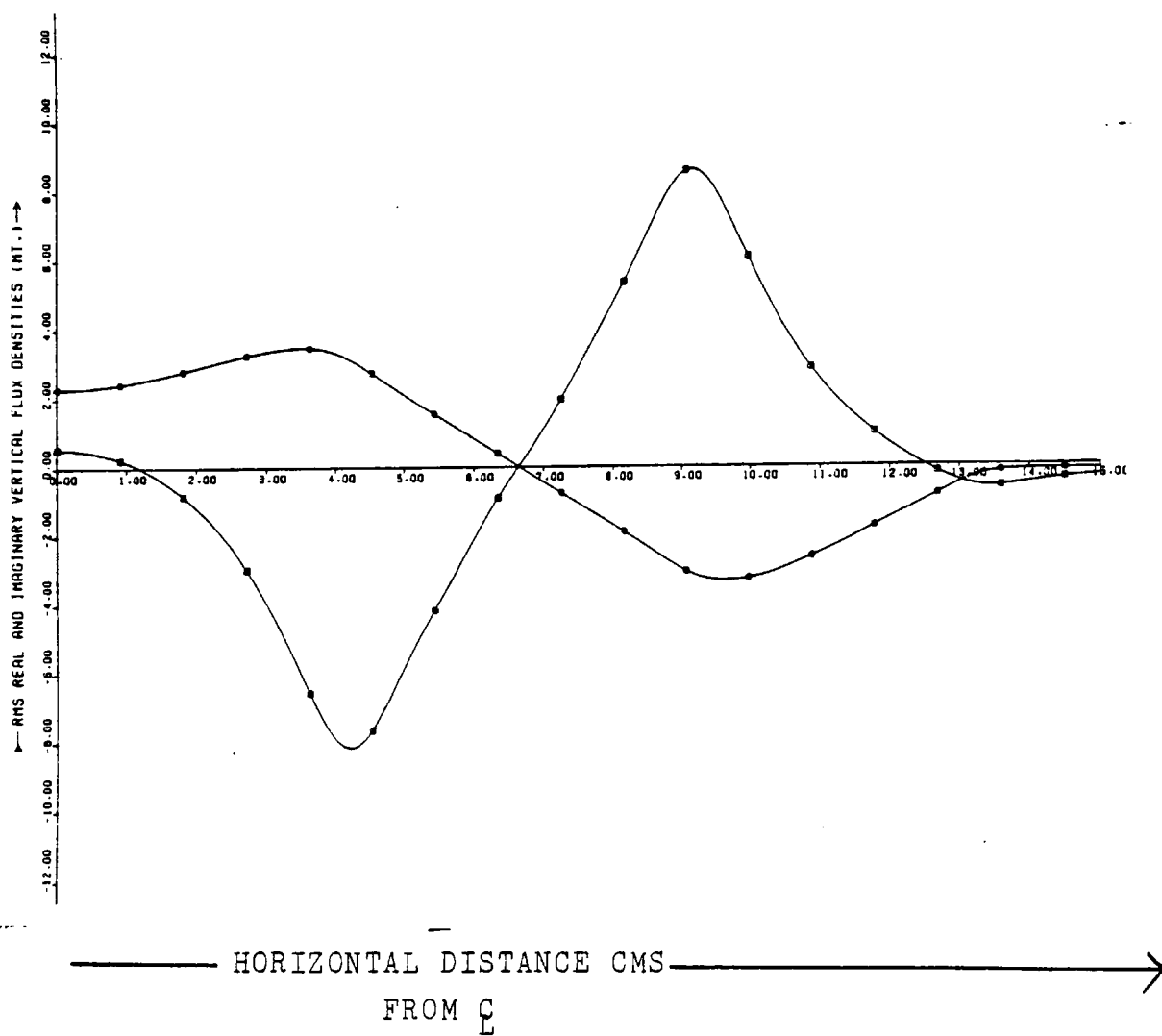


FIG.6.23 R.M.S real and imaginary vertical flux densities for FIG.6.21 along line xx'

□ real

○ imaginary

component of B_y plays an important part at the centre of the machine where it is the dominant component.

Although there is no experimental data presented for either figs.6.19 or 6.23 both give an indication that the method appears to be modelling the situation correctly.

6.9 A Xi-CORE LEVITATOR

The final machine analysed was a Xi-core, once again a transverse flux machine like the E-core. Fig.6.24 shows the geometry of the machine used and the only change from that given in fig.5.29 is the introduction of the conducting aluminium plate at a height of 19mm above the stator iron. The stator iron was again included in the analysis and was given a relative permeability of 1000. As before the boundary between the differential and integral domains was chosen to be the material surfaces.

The calculated results are compared with the measured results presented by Rajanathan^{6.4}. The results used for comparison were measured at a height of 12.7mm above the stator iron. Again as for the results given by Wyatt for the E-core a Hall probe was used to measure them. The results presented are for a symmetrically placed plate.

Length of Stator = 457mm

Conductivity of Plate = $1/2.9 \times 10^8 \text{ Sm}^{-1}$

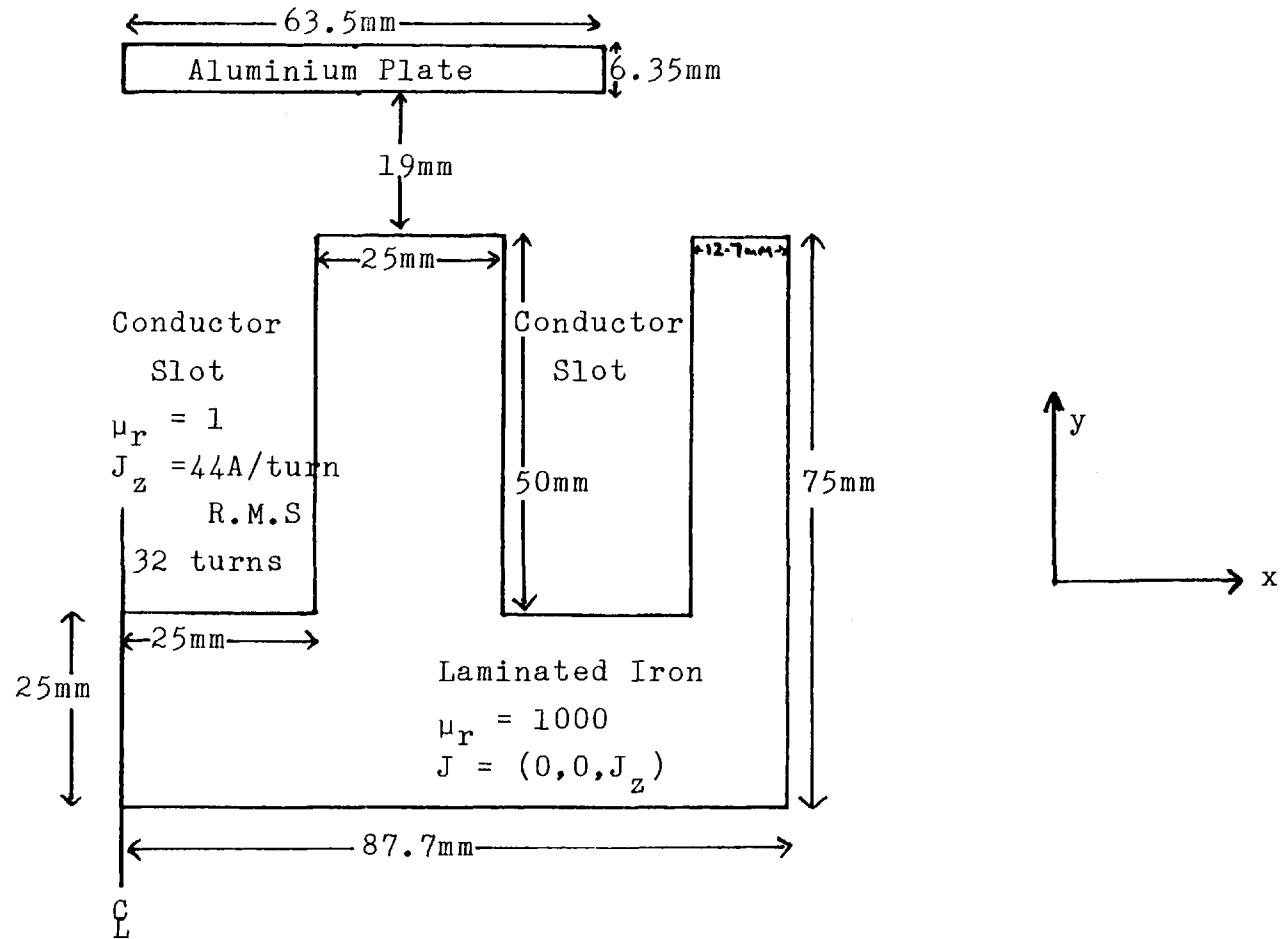


FIG.6.24 Xi-Core Levitator

The calculation methods are the same as used for the Xi-core on open circuit discussed in chapter five with the addition of the analysis discussed earlier in this chapter for the conducting plate together with the point discussed in appendix D. The equations solved in each region are as set out in section 6.8.

The Xi-core configuration (fig.6.24) is such that the line integral of $\underline{H} \cdot d\underline{l}$ around the plate is not automatically zero, and the author found that in order to obtain reasonable results it was necessary to impose this condition. This posed its own problems since the equation set out was already determined before the imposition of this extra condition. This means that the results vary depending which equation is replaced. This point needs further clarification but there has not been time to do this. Appendix D sets out how the author imposed this extra condition.

6.9.1 Results for Hx

The conditions imposed on the problem were that on the centre line of the machine,

$$H_y = \frac{\partial H_x}{\partial x} = 0 \quad (6.73)$$

$$\frac{\partial J}{\partial x}_{\text{plate}} = 0 \quad (6.74)$$

and at infinity,

$$H_x = H_y = 0 \quad (6.75)$$

Fig.6.25 shows the mesh used for the first solution presented. Once again a regular mesh was chosen in conjunction with odd numbered boundary noded conductors. Fig.6.26 gives the comparison between the measured and calculated horizontal flux densities. The result is very poor towards the centre of the machine but the right hand 'hump' is closer to the measured result. The result gained for H_x is not suprising since the magnetostatic results for H_x were poor.

Fig.6.27 shows the second mesh used, a much finer mesh than that given in fig.6.25. Fig.6.28 gives the comparison between the measured and calculated horizontal flux densities. This result is the best gained so far by the author for this particular problem.

6.9.2 Results for H_y

The conditions imposed on the problem are those given in (6.73), (6.74) and (6.75). The results presented are for the same meshes that were used for the solution of H_x .

Fig.6.29 shows the result achieved for the mesh in fig. 6.25. The calculated values are somewhat disappointing to the right of the peak but it was hoped that the finer mesh would give an improved result.

Fig.6.30 shows the result achieved for the finer mesh given in fig.6.27 and the result is suprisingly similar

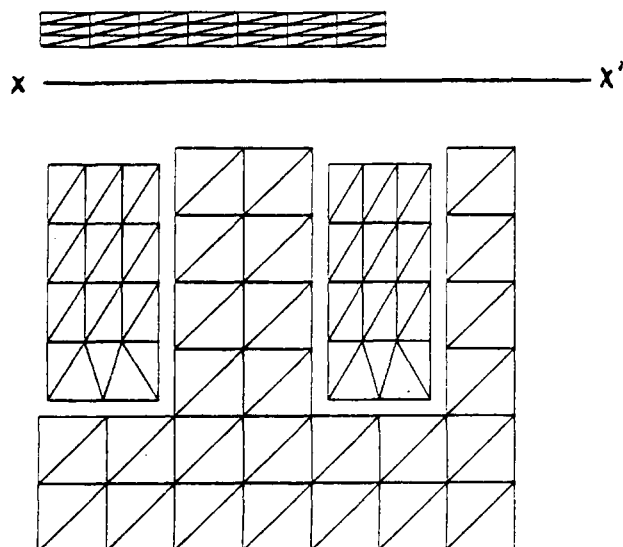
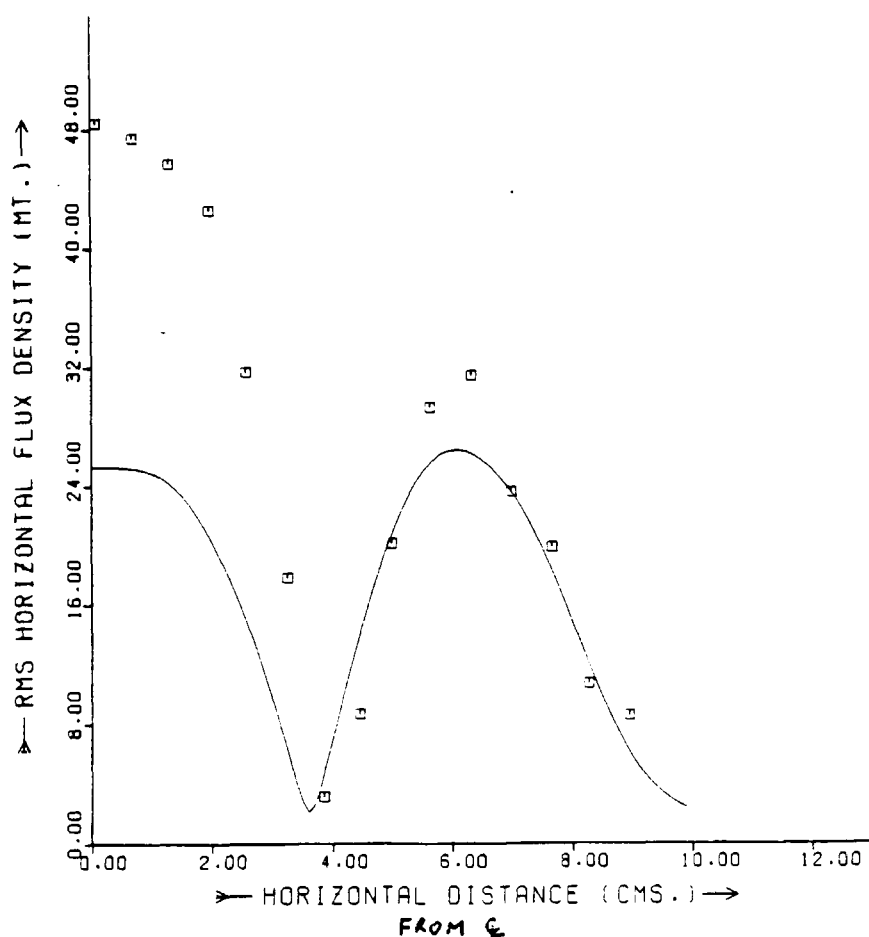


FIG.6.25 finite element mesh of right half of Xi-core levitator



■ measured
 — calculated

FIG.6.26 R.M.S horizontal flux density for FIG.6.25 along line xx'

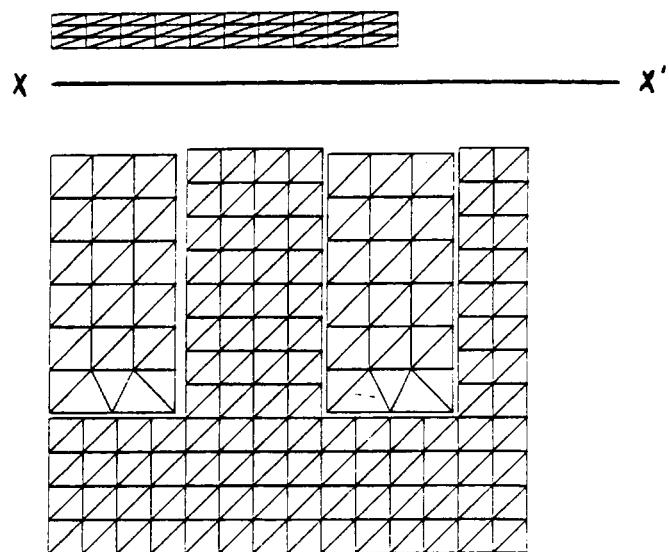
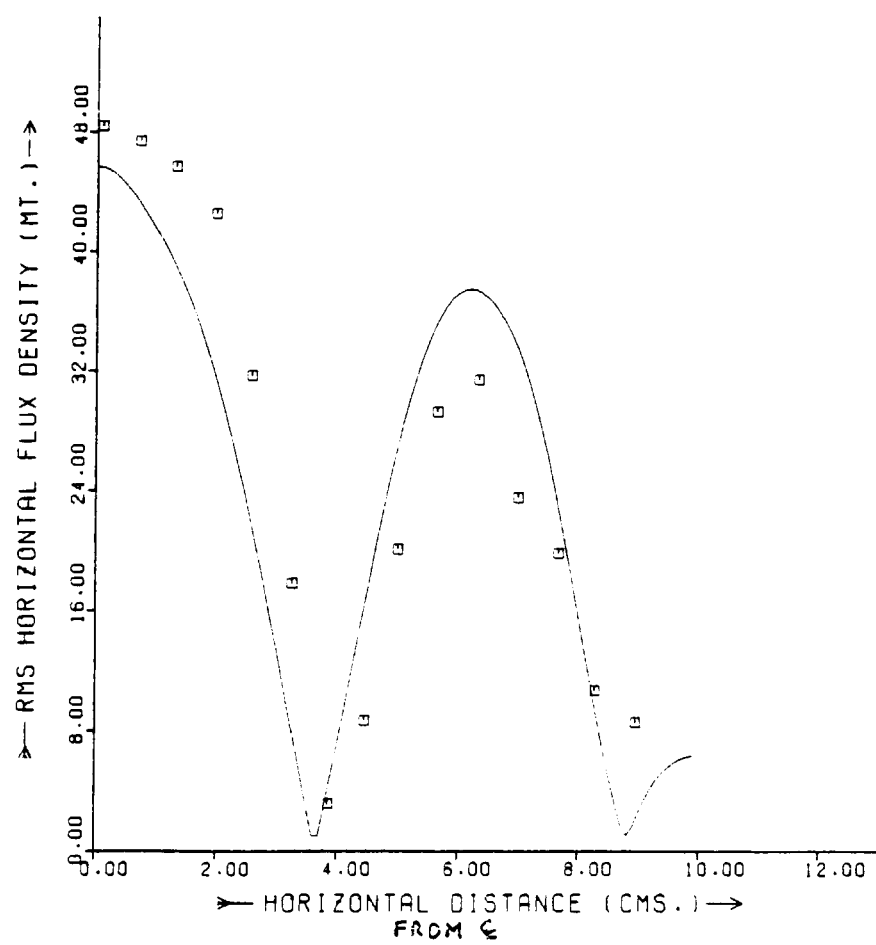


FIG.6.27 finite element mesh of right half of Xi-core levitator



□ measured
— calculated

FIG.6.28 R.M.S horizontal flux density
for FIG.6.27 along line xx'

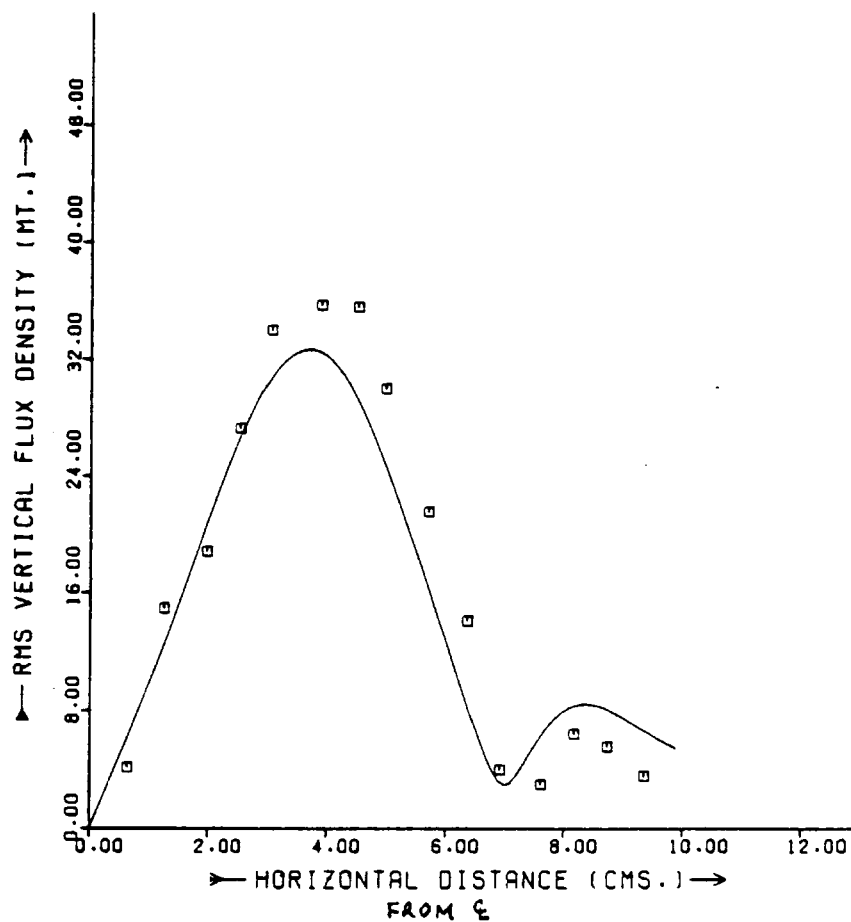


FIG.6.29 R.M.S vertical flux density for FIG.6.25
along line xx'

□ measured

— calculated

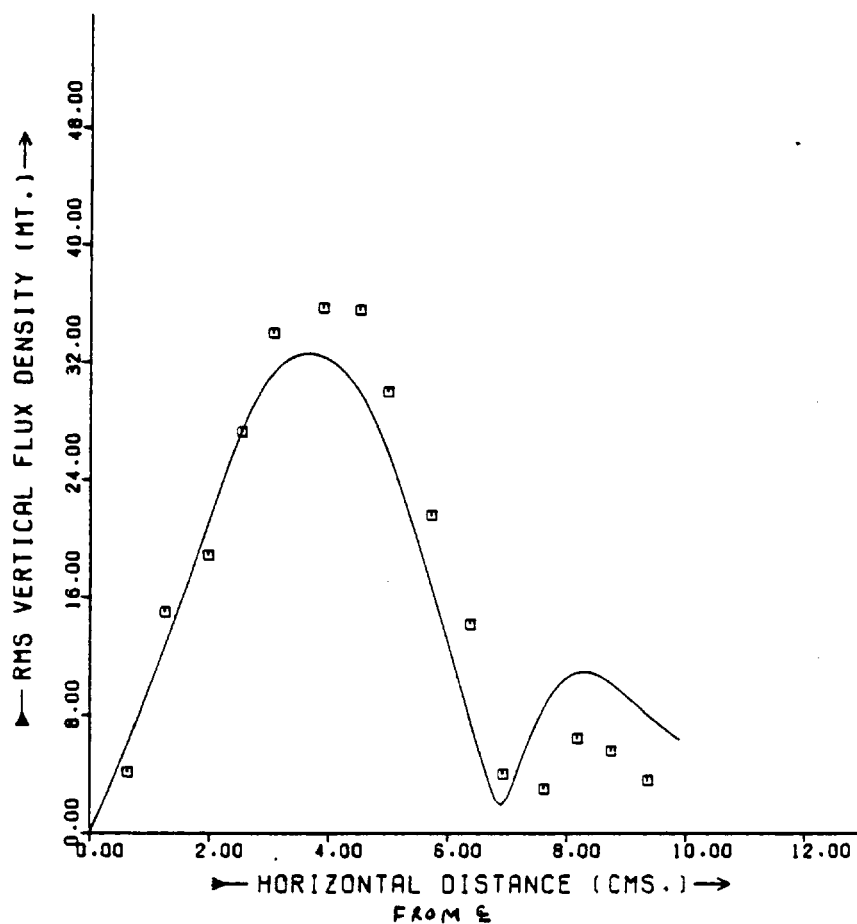


FIG.6.30 R.M.S vertical flux density for FIG.6.27
along line xx'

□ measured
— calculated

to that given in fig.6.29. This is an encouragement that the method is working correctly and is in direct contrast to the vastly different solutions gained for the same two meshes for B_x .

6.10 CONCLUSIONS

The chapter begins by discussing a two dimensional transverse electric problem of a stack of laminations under an axial excitation. The problem is a boundary value one since either H_z or $\frac{\partial H_z}{\partial n}$ are given on the boundary. This places it in direct contrast to everything else presented in this thesis which is concerned with an application of the hybrid technique. Although the comparison of results is very poor there are two conclusions that can be drawn from this example. Firstly the simplest formulation for any such problem must be in terms of $\underline{H}$ since only one function is required. The second point related to the first is that only boundary value problems can be analysed in this way. This is because at interfaces where on one side only,

$$\underline{\nabla} \times \underline{H} = 0 \quad (6.76)$$

as relationship for $\frac{\partial H}{\partial n}$ across the interface can be established since

$$\frac{\partial H_z}{\partial n} = 0 \quad (6.77)$$

on one side, and thus the relationship needs to be constructed explicitly via the $\underline{E}$ field which complicates the formulation. This implies that the hybrid

method is not viable for $\underline{H}$ transverse electric problems for either unbounded or bounded regions.

The remainder of the chapter is concerned with the extension of the hybrid formulation for transverse magnetic situations to eddy current problems. The results achieved indicate that the method works, since although the magnitude comparisons could be improved the shape of the curves obtained is generically correct. It is also reasonable to conclude that for finer meshes the results would improve within the limits of the two dimensional assumption inaccuracies, and the accuracy of the measurements.

Although the technique has been shown to work the fact that it is a three function analysis, namely (H_x, H_y, J_z) , is disappointing. However this needs to be put into perspective and it should be pointed out that essentially H_x or H_y is representative of $\frac{\partial H_y}{\partial n}$ or $\frac{\partial H_x}{\partial n}$ respectively and therefore the formulation is comparable function wise with $(\underline{T}, \Omega)$ which for a similar problem essentially has three functions also, namely $(\underline{T}, \ell, \frac{\partial \Omega}{\partial n})$. It so happens that $\frac{\partial \Omega}{\partial n}$ for a standard finite element approximation can be very easily eliminated^{6.5}, however if a similar hybrid technique was applied to $(\underline{T}, \Omega)$ the easy elimination of $\frac{\partial \Omega}{\partial n}$ would be replaced by a more difficult explicit elimination. Similar sentiments could be expressed about $\underline{A}$ which in its full regalia could also

be regarded as a three function solution, namely $(A_z, \frac{\partial A_z}{\partial n_z}, \emptyset)$. The author believes that the $\underline{H}$ formulation as described for two dimensional eddy current analysis requires more computing resources than $\underline{A}$ or $(\underline{T}, \Omega)$. Typical bounds for the solutions presented for the E-core and Xi-core machines are 1-2 Mbytes of storage and 600-1800 seconds (on a Prime 750) of central processing time. However the method only requires a little tuning before, the author believes, favourable comparisons with similar formulations based on $\underline{A}$ and $(\underline{T}, \Omega)$ could be achieved.

Within the restrictions discussed in section 5.7 the method can be extended to three dimensions and this study has served as a useful preparation for that step. However it does appear that in moving to a three dimensional geometry the formulation requires six functions in the analysis. This must be viewed as disappointing but it might be better than initially anticipated since two of the functions are confined to surfaces.

The Xi-core machine analysis raised this issue of the necessity to impose,

$$\int_S \underline{J} \cdot \underline{dS} = 0 \quad (6.78)$$

over the plate in order to produce reasonable results. This seems to conflict with the analysis given in sections 6.6.2 and 6.7 but the author believes that in

principle the imposition of (6.78) is unnecessary, although in practice it may be needed because of inaccuracies introduced by the discrete representation.

Finally a hybrid technique for $\underline{H}$ has been made to work in eddy current situations and this must be seen as a major achievement of the project. The principle of the hybrid technique of differential regions and integral regions is alone responsible for an independent formulation for each component of $\underline{H}$. A standard finite element mesh would explicitly require the two components of $\underline{H}$ to be solved simultaneously. Although to solve independently for each component of $\underline{H}$ is an advantage, the disadvantage is that every material part of the configuration under analysis must in general have an air gap between it and every other part. This adversely affects the formulation in two ways, firstly the fully populated matrix representing the integral domain is considerably enlarged and hence a consequent increase in computer time ensues, and secondly the possibility of small air gaps arises. Although to its credit the method has exhibited, thus far, no ill effects due to reasonably small air gaps a great deal more work is required to substantiate the stability thus far observed. This curious feature of the formulation, however, suggests that it should be tried with both $\underline{A}$ and $(\underline{T}, \Omega)$ where this problem would not arise.

Overall at this time the author would not recommend the H formulation but for further investigation. In contrast the author would recommend the hybrid technique however constructed, (i.e. differential-integral^{6.6}, differential-infinite elements^{6.7}, differential-ballooning^{6.8}), as a useful approach for unbounded field problems and perhaps for bounded regions as well when the formulation becomes differential-integral. Moreover with reference to this last point the analysis given in section 4.5 must be borne in mind and particularly the possibility of a non unique solution.

CHAPTER 7

DISCUSSION, CONCLUSIONS AND FURTHER WORK

7.1 VARIATIONS ON AN INTEGRAL THEME

It is the authors intention in this chapter to bring out certain aspects of the work which suggest worthwhile avenues of investigation.

In chapter three the author derived an analytic solution for two dimensional infinitely long bus bar problems in terms of $\underline{H}$. This approach opens up the possibility of boundary integral solutions in terms of $\underline{H}$ for magneto-static problems, for linear magnetic materials or alternatively a similar hybrid technique to that described in this thesis where the conductors are included in the exterior integral approximation. The author believes that the implementation of this last point is a necessity for the hybrid technique presented in this thesis. It would reduce the size of the matrices representing the total and integral solution domains thus giving a reduction in computation time. This could either be gladly accepted or put to good use by improving the discretisation elsewhere. This implementation would also remove discrete errors introduced by the conductor mesh. It is also possible to derive a boundary integral solution for the Helmholtz equation^{7.4} and thus sinusoidally varying fields over linear magnetic materials can

be represented by a boundary integral equation also. Thus it is possible to solve all of the problems in this thesis by suitable boundary equation techniques. Since the calculation of magnetic and electromagnetic fields relies on a tool box of approaches, it seems to the author that the implementation of some of the above ideas can be nothing but useful, and thus they warrant investigation.

7.2 A HINDSIGHT APPRECIATION OF THE FORMULATION

In chapter four the author discusses the difficulty of determining the matrix equations defining the problem and certain situations arise which are declared to be beyond the scope of the present method e.g. more than one even numbered boundary noded shape. It was said at the time that the difficulty expressed itself in that it was impossible to uniquely define those terms arising from the normal derivative of the component under consideration. With hindsight the author now sees that the values associated with these terms are irrelevant since they express nothing more than the linking between the differential and integral regions. This has implications for the technique presented in this thesis on these counts. Firstly, the possibility should be investigated of eliminating these terms rather than solving for them explicitly. The problem in doing this is that neither matrix (differential or integral) involving these terms

is invertible nor in general can one be constructed. Secondly, the supplementary formulation discussed in chapter four becomes unnecessary even at magnetic and eddy current interfaces, since the terms only express the linking characteristics of the two approximations. Thirdly, the Xi-core results which are so poor might be attributable to a linking of the two approximations which is but one of many that might be constructed from the equations. These points all require further investigation and depending on the answers a better and more reliable technique might well emerge.

7.3 COMPUTATIONAL ENHANCEMENTS

In chapters five and six it was suggested that the method could be 'tuned' in order to give better comparisons with other formulations with regards to computing resources required. The point made in section 7.2 regarding the supplementary formulation will certainly help to reduce time and storage in those problems using that formulation. However there are other possible areas of investigation that might prove useful. At present the author has been using a direct solution technique based on a symmetric profile routine for asymmetric matrices given by Zienkiewicz^{7.1}, adapted for complex numbers. If however the matrices representing the problem could be made symmetric the reduction in computer storage would not be insignificant. Moreover the

recent^{7.2} advances in weighting functions for the vector potential formulation in order to produce a symmetric matrix encourage the author to think that it is possible. A second interesting feature of recent developments in finite element techniques that might help reduce computation time is the idea of reduced orders of numerical integration^{7.3}. The author hitherto has constructed all the integral coefficients by analytic formulae, but apparently less accurate numerical evaluation of integral coefficients may give better results as well as reduce computation time involved.^{7.1}

7.4 GOING THREE DIMENSIONAL?

In chapter five the author stated that the extension of the technique into three dimensions was probably only viable for cuboid type geometries and in chapter six apprehensions were also expressed as to whether a six function solution had anything to offer. Those points pose questions which require answers; is a cuboid type geometry the only viable three dimensional configuration? How important are the H interface conditions at corners? Are there good reasons for having a six function three dimensional analysis? It may be that WE THINK WE KNOW the answers to these questions but that is exactly the point, often what WE THINK is in marked contrast to later when WE KNOW.

7.5 A FOND FAREWELL

The study has on the whole been enjoyable but like anything worthwhile it has contained its disappointments. However it would be fair to say that the disappointments have been balanced by the successes. The two major disappointments have been the failure to find general interface conditions for $\underline{H}$ at three dimensional interfaces and the prospect of a six function three dimensional analysis. Both points evoke the feeling that there must be an easier way! The two major successes have been the derivation of the general two dimensional interface conditions for $\underline{H}$ and the successful implementation of the hybrid philosophy for a limited class of two dimensional problems.

A new formulation has been examined and it has been shown to work albeit in a limited sense and to be worthy of further investigation.

APPENDIX ATHE SIMPLE LAYER ALTERNATIVE FOR H_y

Without loss of generality consider H_x to be defined as,

$$H_x = \int_{\Gamma} \frac{1}{2} \ln \left((x-x_1)^2 + (y-y_1)^2 \right) \sigma_x \, d\Gamma \quad (A.1)$$

where Γ is the surface of sources and (x_1, y_1) is the source point. The governing equations for $\underline{H}$ in a two dimensional region, R , are

$$\underline{\nabla} \cdot \underline{H} = 0 \quad (A.2)$$

and

$$\underline{\nabla} \times \underline{H} = 0 \quad (A.3)$$

such that they may be reduced to,

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0 \quad (A.4)$$

and

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0 \quad (A.5)$$

respectively. Bringing together equations (A.1) and (A.4) gives,

$$\frac{\partial H_y}{\partial y} = - \int_{\Gamma} \frac{(x-x_1)}{(x-x_1)^2 + (y-y_1)^2} \sigma_x \, d\Gamma \quad (A.6)$$

which may be integrated to give,

$$H_y = - \int_{\Gamma} \left\{ \tan^{-1} \left(\frac{y-y_1}{x-x_1} \right) + f_1(x) \right\} \sigma_x \, d\Gamma \quad (A.7)$$

Proceeding in a similar manner with equations (A.1) and (A.5) produces,

$$H_y = \int_{\Gamma} \left\{ \tan^{-1} \left(\frac{x-x_1}{y-y_1} \right) + f_2(y) \right\} \sigma_x d\Gamma \quad (A.8)$$

Equations (A.7) and (A.8) need to be reconciled but firstly another result is required. Consider

$$\tan \beta = \frac{x}{y} \quad (A.9)$$

and

$$\tan \alpha = \frac{y}{x} \quad (A.10)$$

then

$$\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \quad (A.11)$$

reduces to,

$$\tan (\alpha + \beta) = \frac{\frac{y}{x} + \frac{x}{y}}{1 - 1} = \pm \infty \quad (A.12)$$

which in turn gives,

$$\alpha + \beta = \tan^{-1} (\pm \infty) = \pm \frac{\pi}{2} \quad (A.13)$$

such that

$$\tan^{-1} \frac{y}{x} + \tan^{-1} \frac{x}{y} = \pm \frac{\pi}{2} \quad (A.14)$$

Since equations (A.7) and (A.8) are required to hold for any Γ , comparison of the integrals gives

$$f_1(x) = k_1 \quad \text{a constant} \quad (A.15)$$

and

$$f_2(y) = k_2 \quad \text{a constant} \quad (A.16)$$

such that,

$$k_1 + k_2 = \pm \frac{\pi}{2} \quad (\text{A.17})$$

For exterior regions only if the point (x,y) is such that,

$$x-x_1 \approx y-y_1 \quad (\text{A.18})$$

since

$$r = (x-x_1)^2 + (y-y_1)^2 \quad (\text{A.19})$$

is very large then

$$H_y = 0 \quad (\text{A.20})$$

and therefore

$$k_1 = k_2 = \frac{\pi}{4} \quad (\text{A.21})$$

such that in general

$$k_1 = k_2 = \pm \frac{\pi}{4} \quad (\text{A.22})$$

This means that H_y may be written,

$$H_y = - \int_{\Gamma} \left\{ \tan^{-1} \left(\frac{y-y_1}{x-x_1} \right) \pm \frac{\pi}{4} \right\} \sigma_x d\Gamma \quad (\text{A.23})$$

for infinite domains.

It is also interesting to note that as H_y approaches Γ there is no discontinuity produced. Consider fig.A.1.

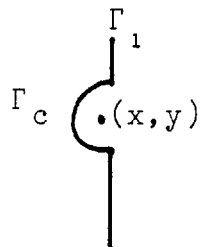


FIG.A.1

where the path Γ has been deformed by a semi-circle centred at (x, y) of radius ε . Equation (A.23) becomes,

$$H_y(x, y) = - \int_{\Gamma = \Gamma_1 + \Gamma_c} \left\{ \tan^{-1} \left(\frac{y - y_1}{x - x_1} \right) \pm \frac{\pi}{4} \right\} \sigma_x d\Gamma \quad (\text{A.24})$$

where as $\varepsilon \rightarrow 0$ $\Gamma_1 \rightarrow \Gamma$ and the integral over Γ_c may be written

$$I_{\Gamma_c} = - \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \left\{ \tan^{-1} \left(\frac{y - y_1}{x - x_1} \right) \pm \frac{\pi}{4} \right\} \sigma_x \varepsilon d\theta \quad (\text{A.25})$$

where,

$$y - y_1 = \varepsilon \sin \theta \quad (\text{A.26})$$

and

$$x - x_1 = \varepsilon \cos \theta \quad (\text{A.27})$$

such that equation (A.25) may be reexpressed,

$$I_{\Gamma_c} = - \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \left\{ \tan^{-1} \left(\frac{\varepsilon \sin \theta}{\varepsilon \cos \theta} \right) \pm \frac{\pi}{4} \right\} \left\{ \sigma_x(x, y) + O(\varepsilon) \right\} \varepsilon d\theta \quad (\text{A.28})$$

which reduces to

$$I_{\Gamma_c} = O(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \quad (\text{A.29})$$

and thus equation (A.23) holds for points both in R and on ∂R , the bounding surface of R .

APPENDIX B

INFINITE DERIVATIVE OF H

In chapter three the H_x field for a general shaped infinitely long conductor is,

$$2\pi H_x = \int_{\Gamma} J_z G \cos \theta \, d\Gamma \quad (B.1)$$

where J_z is the known current density, G is the two dimensional Greens function and $\cos \theta$ is defined such that,

$$\frac{\partial H_x}{\partial n} = \frac{\partial H_y}{\partial t} - J_z \cos \theta \quad (B.2)$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative and $\frac{\partial}{\partial t}$ is the consistent tangential derivative. Consider H_x at a point (x, y_1) ,

$$2\pi H_x(x, y_1) = J_z \int_{\Gamma} \ln \left[(x-x_1)^2 + (y-y_1)^2 \right] \cos \theta \, d\Gamma \quad (B.3)$$

such that,

$$2\pi \frac{\partial H_x(x, y_1)}{\partial x_1} = J_z \int_{\Gamma} \frac{2(x_1 - x)}{(x-x_1)^2 + (y-y_1)^2} \cos \theta \, d\Gamma \quad (B.4)$$

With reference to fig.B.1

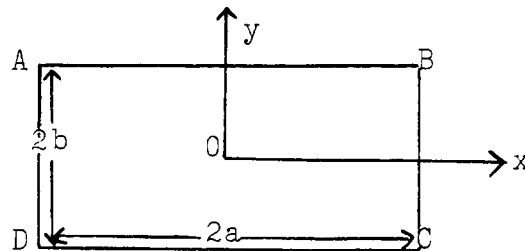


FIG.B.1

it can be seen that,

1) on the line segment AB $\frac{\partial H_x}{\partial n} = \frac{\partial H_x}{\partial y} = \frac{\partial H_y}{\partial x} - J_z$

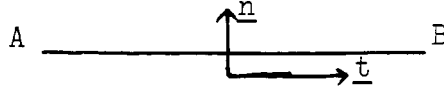


FIG.B.2

$$\therefore \theta = 0 \quad (B.5)$$

2) on the line segment BC $\frac{\partial H_x}{\partial n} = \frac{\partial H_x}{\partial x} = - \frac{\partial H_y}{\partial y}$

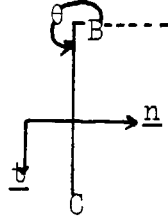


FIG.B.3

$$\therefore \theta = 3\pi/2 \quad (B.6)$$

3) on the line segment DC $\frac{\partial H_x}{\partial n} = -\frac{\partial H_x}{\partial y} = -\frac{\partial H_y}{\partial x} + J_z$

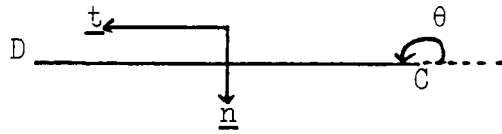


FIG.B.4

$$\therefore \theta = \pi \quad (B.7)$$

4) on the line segment AD $\frac{\partial H_x}{\partial n} = -\frac{\partial H_x}{\partial x} = \frac{\partial H_y}{\partial y}$

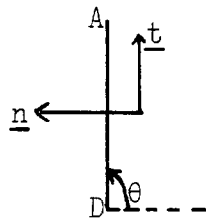


FIG.B.5

$$\therefore \theta = \frac{\pi}{2} \quad (B.8)$$

The above conditions imposed on (B.4) give,

$$2\pi \frac{\partial Hx}{\partial x_1}(x_1, y_1) = J_z \int_a^a \frac{2(x_1 - x)}{(x_1 - x)^2 + (y_1 - b)^2} dx$$

$$- J_z \int_a^a \frac{2(x_1 - x)}{(x_1 - x)^2 + (y_1 + b)^2} dx \quad (B.9)$$

which integrates to,

$$2\pi \frac{\partial Hx}{\partial x_1}(x_1, y_1) = J_z \left\{ -\ln \left[(x - x_1)^2 + (y_1 - b)^2 \right] \Big|_a^a \right.$$

$$\left. + \ln \left[(x - x_1)^2 + (y_1 + b)^2 \right] \Big|_{-a}^a \right\} \quad (B.10)$$

which gives,

$$2\pi \frac{\partial Hx}{\partial x_1}(x_1, y_1) = J_z \left[\ln \left((a + x_1)^2 + (y_1 - b)^2 \right) \right.$$

$$- \ln \left((a - x_1)^2 + (y_1 - b)^2 \right) + \ln \left((a - x_1)^2 + (y_1 + b)^2 \right)$$

$$\left. - \ln \left((a + x_1)^2 + (y_1 + b)^2 \right) \right] \quad (B.11)$$

From equation (B.11) it can be seen that $\frac{\partial Hx}{\partial x}$ becomes infinite at any corner point. For example consider the point (a,b) then (B.11) gives

$$2\pi \frac{\partial Hx}{\partial x_1}(a, b) = J_z \left[\ln(2a)^2 - \ln(0) + \ln(2b)^2 \right.$$

$$\left. + \ln \left((2a)^2 + (2b)^2 \right) \right] \quad (B.12)$$

where as long as J_z , a and b are finite and not zero,

$$\frac{2\pi}{J_z} \frac{\partial H_x(a,b)}{\partial x_1} = K - \ln(0) \quad (\text{B.13})$$

where K is a constant and if,

$$q = \ln(0) \quad (\text{B.14})$$

then,

$$e^q = 0 \quad (\text{B.15})$$

gives,

$$q = -\infty \quad (\text{B.16})$$

APPENDIX CLINEAR SHAPE FUNCTIONS

Consider a general triangle fig.C.1

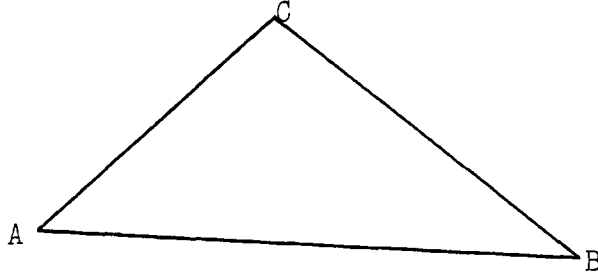


FIG.C.1

and a scalar function ϕ such that,

$$\left. \begin{aligned} \phi &= \phi_A & \text{at } A \\ \phi &= \phi_B & \text{at } B \\ \phi &= \phi_C & \text{at } C \end{aligned} \right\} \quad (C.1)$$

A linear approximation to ϕ over ABC produces

$$\phi = N_A(x,y)\phi_A + N_B(x,y)\phi_B + N_C(x,y)\phi_C \quad (C.2)$$

such that

$$N_A(x,y) = \begin{cases} 1 & \text{at } A \\ 0 & \text{at } B, C \text{ and outside } ABC \end{cases} \quad (C.3)$$

and similarly for $N_B(x,y)$ and $N_C(x,y)$. More generally over ABC the functions $N_i(x,y)$ may be written as,

$$N_i(x,y) = a_i x + b_i y + c_i \quad i \in (A, B, C) \quad (C.4)$$

Equations (C.3) and (C.4) combine to give a_i , b_i , c_i by solution of,

$$D \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \underline{R}_i \quad i \in (A, B, C) \quad (C.5)$$

where

$$D = \begin{bmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{bmatrix} \quad (C.6)$$

and

$$\underline{R}_i = \begin{bmatrix} \delta_{A,i} \\ \delta_{B,i} \\ \delta_{C,i} \end{bmatrix} \quad i \in (A, B, C) \quad (C.7)$$

such that,

$$\delta_{j,i} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \quad (C.8)$$

It is worth noting that for triangles the inversion of D will always exist^{C.1}.

The functions $N_i(x,y)$ are known as shape functions or basis functions and may be of any order^{C.2} but those derived above are linear shape or basis functions.

APPENDIX D

$$\underline{\text{APPLICATION OF}} \int_S \underline{J} \cdot d\underline{S} = 0$$

OVER A TWO DIMENSIONAL EDDY CURRENT REGION

Consider a two dimensional eddy current region, Ω , with constant relative permeability, μ_r , and electrical conductivity, σ , carrying a sinusoidally varying current at 50 hertz perpendicular to the two dimensional plane. Assuming that,

$$\underline{J} = (0, 0, J_z) \cdot e^{i\omega t} \quad (D.1)$$

the governing equation for $\underline{J}$ reduces to,

$$\nabla^2 J_z = k J_z \quad (D.2)$$

where

$$k = i\omega\mu_r\mu_0\sigma \quad (D.3)$$

where i is the square root of minus one. The Weighted Residual form of (D.2) is,

$$\begin{aligned} - \int_{\Omega} W \nabla^2 J_z d\Omega &= \int_{\Omega} \underline{\nabla} W \cdot \underline{\nabla} J_z d\Omega - \int_{\Gamma} W \frac{\partial J_z}{\partial n} d\Gamma \\ &= - \int_{\Omega} W k J_z d\Omega \end{aligned} \quad (D.4)$$

after an integration by parts^{D.1}, where W is the chosen weighting function and Γ is the perimeter of Ω . If (D.4) is evaluated using the Galerkin approach for linear elements and the equations summed, the following may be

written,

$$\begin{aligned} \sum_{e=1}^M \sum_{i=1}^3 \int_{\Omega} \nabla N_i^e \cdot \nabla J_z \, d\Omega - \sum_{e=1}^M \sum_{i=1}^3 \int_{\Gamma} N_i^e \frac{\partial J_z}{\partial n} \, d\Gamma \\ = - \sum_{e=1}^M \sum_{i=1}^3 \int_{\Omega} N_i^e k J_z \, d\Omega \end{aligned} \quad (D.5)$$

where N_i^e is the shape function for node i from element e where the region Ω has been divided into M triangles. The nature of the integrals is such that the summations may be taken under the integral signs and since^{D.1},

$$\sum_{i=1}^3 N_i^e = 1 \quad 1 \leq e \leq M \quad (D.6)$$

equation (D.5) reduces to,

$$\int_{\Gamma} \frac{\partial J_z}{\partial n} \, d\Gamma = k \int_{\Omega} J_z \, d\Omega \quad (D.7)$$

It was shown in chapter six that,

$$\frac{\partial J_z}{\partial n} = - k H t \quad (D.8)$$

where n and t are the normal and tangential directions defined in fig.6.11. It is now easily seen that (D.7) is equivalent to,

$$\int_{\Gamma} \underline{H} \cdot \underline{dl} = \int_{\Omega} J_z \, d\Omega \quad (D.9)$$

the integral form of Amperes Law.

The authors concern is to enforce,

$$\int_{\Omega} J_z \, d\Omega = 0 \quad (D.10)$$

and the above derivation of (D.9) shows that for the Galerkin representation of (D.4) all that is required is the replacing of one equation by the sum of all the equations. The terms in $\underline{H}$ representing $\int_{\Gamma} \underline{H} \cdot d\underline{\ell}$ are then explicitly replaced by zero enforcing (D.10).

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