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SOME PROBLEMS IN SOUND WAVES AND POTENTIAL THEORY USING THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

by

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ABSTRACT

In this thesis, which consists of three parts, some problems in sound waves and potential theory are discussed using the method of matched asymptotic expansions. The problem of diffraction of long plane waves by a cylindrically - tipped, infinite wedge and a related problem are discussed in the first part. The problem of scattering of long plane waves by a circular disc of finite thickness, for normal incidence, is considered in the second part, obtaining the solution in two regions, in the form of expansions of the small parameter ϵ = thickness / diameter of the disc. In the last part, the two-dimensional potential problem of two equal infinite strips of unit potential placed close to each other is considered, the solution being obtained in three regions, in this case, in the form of expansions of the small parameter ϵ = separation / breadth of strips.

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INTRODUCTION

The problems discussed in this thesis are in sound waves and potential theory, and the method used in solving these problems is that of matched asymptotic expansions.

In Part 1, the problem of diffraction of long plane waves by a hard, cylindrically - tipped, infinite wedge is considered. Here, the method (Crighton and Leppington, 1971) is essentially the matching of two forms of solutions in two overlapping regions called 'outer' and 'inner'. In the outer region, the body is treated as a sharp wedge, and a known solution (Carslaw,1919) of the sharp wedge problem is expressed in asymptotic form, for points closer than a wave length from the tip. In the inner region, the potential satisfies an incompressible flow problem with hard boundary condition on the body, its behaviour at infinity being the asymptotic form of the outer potential obtained for the sharp wedge. A conformal transformation enables us to solve this incompressible flow problem for the inner potential. As an example of other problems to which the same method could be applied, a semi-infinite slab with a particular cross-section is considered in the last Section.

In Part 2, the problem of scattering of long plane waves by a circular disc of small but finite thickness is considered, when the incidence is normal. First, the velocity potential is expanded for low wave-number k, and problems are formulated for successive approximations.

The first problem is trivial, and the second one reduces to an incompressible flow problem. To solve the latter in detail, the flow region is divided into two parts called 'outer' and 'inner'. The problems for potentials in these two regions are formulated, and after a preliminary investigation certain forms of expansions are assumed for the potentials in the two regions. Problems are then formulated and solved for the constituent potentials in these expansions, in the form of asymptotic expansions in cases where exact solutions are not possible. The outer and inner expansions so found are matched against each other, to determine unknown constants. The matching process (Crighton and Leppington, to appear) used here requires that the inner (and outer) expansion be taken up to a prescribed <u>order</u> ϵ^n with respect to the small parameter; this contrasts with the procedure of Van Dyke (1964) who deals with a prescribed <u>number</u> of terms. This means that in our case, all terms like $\epsilon^{3/2} \log \epsilon$ and $\epsilon^{3/2}$ must be included together. The expansions for the outer and inner potentials are used, in Section 5, to determine the far field up to order $k^2 \epsilon$. In the last Section, it is seen that this same method can be applied to several other problems, provided a suitable conformal transformation is available for solving the inner region problem.

In Part 3, the two-dimensional potential problem due to two equal parallel infinite strips charged to unit potential each and placed close to each other is considered.

In this case, following Susann Shaw (1970), the 2-D space is divided into three regions called 'exterior', 'edge' and 'interior'. Problems for potentials in these three regions are formulated, and after a preliminary investigation certain forms of expansions are assumed for the three potentials. Problems are then formulated and solved for the constituent potentials in these expansions. To determine unknown constants, the exterior expansion is matched against the edge expansion, using the same matching principle as in Part 2. The corresponding three-dimensional problem of two equal discs charged to unit potential each and placed close to each other is mentioned briefly in Sec.5, and in this case it is seen that the method breaks down.

PART 1

DIFFRACTION OF LONG PLANE WAVES BY A CYLINDRICALLY-TIPPED, INFINITE WEDGE

1. Introduction and Formulation

The problem of diffraction of long plane waves by a hard, infinite wedge with a cylindrical tip is considered here, in the case when the exterior angle of the wedge is $p\pi/q$, p and q being positive integers such that $1 < p/q \leq 2$. On a normal cross-section, polar coordinates r, θ are chosen so that the lines $\theta = 0$ and $\theta = p\pi/q$, $r \geq a$ correspond to the faces of the wedge, while the arc $r = a$, $0 \leq \theta \leq p\pi/q$ corresponds to the cylindrical tip. (See Fig.1).

The direction of the incident plane wave $e^{i(k\underline{n}.\underline{r} - \omega t)}$ is taken to be $\theta = \theta'$, where ω/k is the velocity of propagation, so that $\underline{n} = \underline{I}\cos\theta' + \underline{J}\sin\theta'$, $\underline{I}$ and $\underline{J}$ being unit vectors along the lines $\theta = 0$ (x-axis) and $\theta = \pi/2$ (y-axis) respectively. (See Fig.1). Since we are considering waves of very large length compared to the radius of the tip, it is assumed that $ka \ll 1$. The total velocity potential at any point (r,θ) is denoted by $\Phi(r,\theta)\, e^{-i\omega t}$, and the time factor $e^{-i\omega t}$ is suppressed throughout, its presence being understood. The potential $\Phi(r,\theta)$ satisfies

$$\left.\begin{aligned} &\text{(i)} \quad (\nabla^2 + k^2)\Phi = 0, \quad \text{when} \quad r > a,\ 0 < \theta < p\pi/q \\ &\text{(ii)} \quad \frac{\partial \Phi}{\partial n} = 0, \quad \text{on the boundary.} \end{aligned}\right\} \qquad (1.1)$$

Following Crighton and Leppington (1971), the flow region $r > a$, $0 < \theta < p\pi/q$ is divided into two parts as follows:

I Outer Region:- points such that $r \gg a$

II Inner Region:- points such that $(a \lesssim)\ r \ll 2\pi/k$, where $2\pi/k$ is the wave length λ.

Since $ka \ll 1$, the two regions I and II overlap for points such that $a \ll r \ll \lambda$.

In the outer region, the effect of the cylindrical tip is neglected, treating the solution Φ^I of (1.1) to be the same as that due to a sharp wedge. This sharp wedge solution is known (Carslaw,1919), and is expressed in asymptotic form, for small r ($\ll \lambda$).

In the inner region, the equation (1.1)(i) reduces to Laplace's equation so that the potential Φ^{II} satisfies an incompressible flow problem, with hard boundary condition (1.1)(ii) on the body, and the condition at infinity:

$$\Phi^{II} \sim \Phi^{I} \text{ (in asymptotic form, for small } r\text{), as } \frac{r}{a} \to \infty,$$

which results from the consideration of Φ in the common region $a \ll r \ll \lambda$. This problem for the inner potential is solved by the method of conformal transformations.

The method used here gives the approximate solution in a simple form, which agrees with the first two terms of the general solution (Tyras,1969). Also, it can be applied to other similar problems as is illustrated by the example given in Section 4.

Fig.1

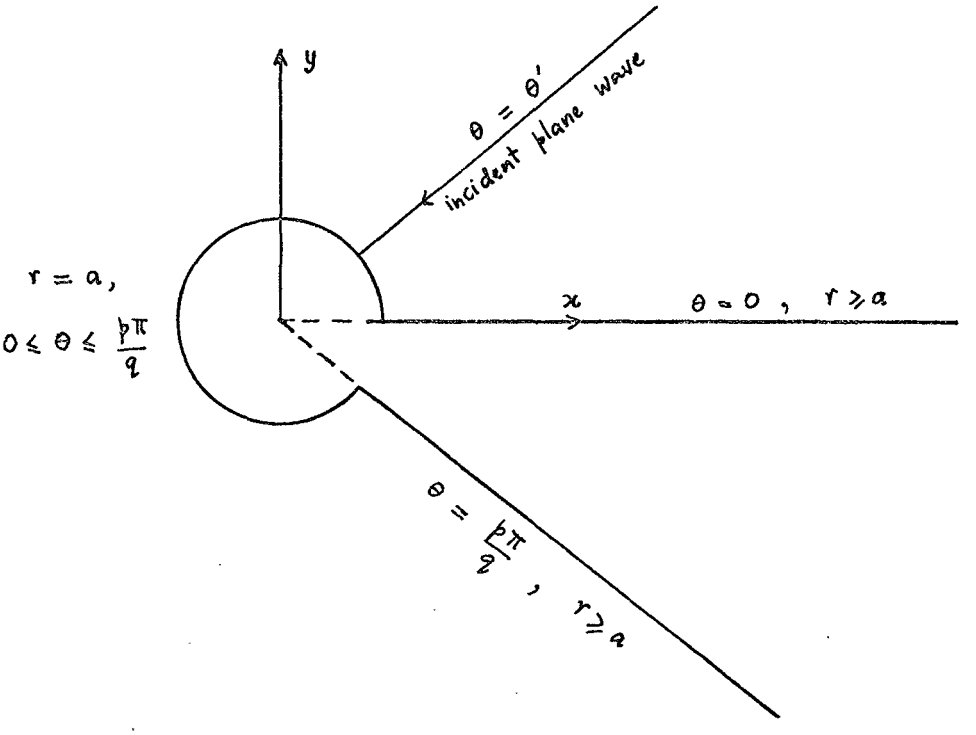
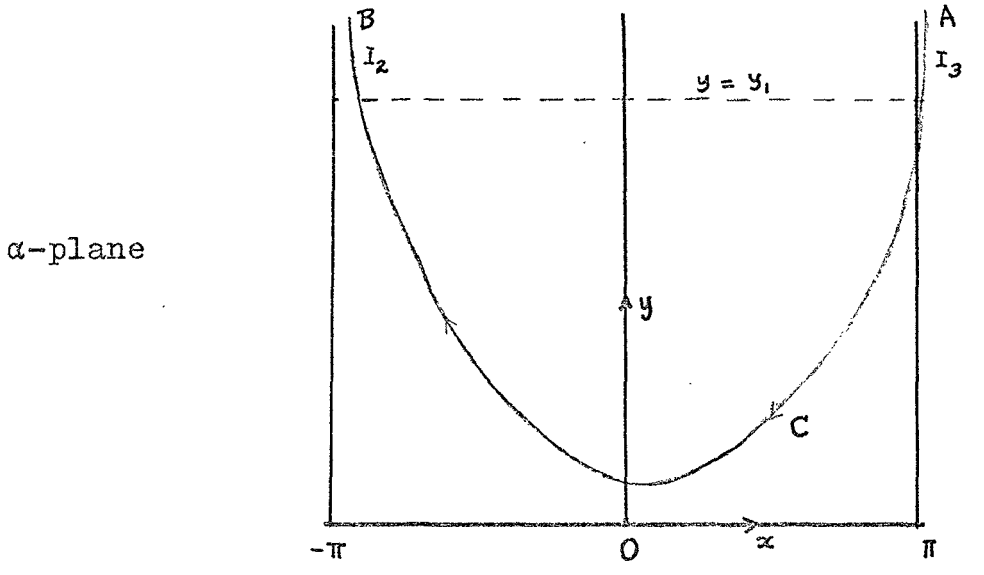

y
θ = θ'
incident plane wave
r = a,
0 ≤ θ ≤ pπ/q
x
θ = 0 , r ≥ a
θ = pπ/q , r ≥ a

Fig.2


α-plane
B
I2
y = y1
A
I3
y
C
-π
0
x
π

2. Outer Region

Let the potential in this region be denoted by Φ^I. By definition in section 1, distances from the centre of the tip, of points here are large compared with a, so that there is little effect of the tip. Thus, in region I, the solution Φ^I of (1.1) would be the same as that for the sharp wedge: $0 \leqslant \theta \leqslant p\pi/q$. The solution to this sharp wedge problem is known, and the particular form we quote here is due to Carslaw (1919):

$$\Phi^I = \frac{iq}{2p\pi} \int_C e^{ikr\cos\alpha} \left\{ \frac{\sin\frac{q\alpha}{p}}{\cos\frac{q\alpha}{p} - \cos\frac{q}{p}(\theta-\theta')} + \frac{\sin\frac{q\alpha}{p}}{\cos\frac{q\alpha}{p} - \cos\frac{q}{p}(\theta+\theta')} \right\} d\alpha, \quad (2.1)$$

where C is a contour in the α-plane, not cutting the real axis and asymptotic to the lines $\alpha = \pm\pi$, on the right hand side. (See Fig.2).

Using (2.1), an asymptotic expression for Φ^I, for small r ($\ll \lambda$), can be obtained as follows:

As $kr \to 0$, it is seen that the contour integral in (2.1) converges to a constant (see 2.6 → 2.7). To find r, θ dependence of Φ^I in this limit, (2.1) is differentiated partially with respect to r, with the result:

$$\frac{\partial\Phi^I}{\partial r} = -\frac{qk}{2\pi p} \int_C e^{ikr\cos\alpha} \cdot \cos\alpha \left\{ \frac{\sin\frac{q\alpha}{p}}{\cos\frac{q\alpha}{p} - \cos\frac{q}{p}(\theta-\theta')} + \frac{\sin\frac{q\alpha}{p}}{\cos\frac{q\alpha}{p} - \cos\frac{q}{p}(\theta+\theta')} \right\} d\alpha. \quad (2.2)$$

We want to estimate $\frac{\partial\Phi^I}{\partial r}$, as $kr \to 0$. The integral in (2.2) diverges at infinity in the α-plane, when $kr = 0$. Hence, the main contribution to $\frac{\partial\Phi^I}{\partial r}$ comes from the two parts of C which are far away from the real axis in the α-plane, and gives an approximate estimate for it. That is, if

$$I_1 = \int_C e^{ikr\cos\alpha} \cdot \frac{\cos\alpha \sin\frac{q\alpha}{p} \, d\alpha}{\cos\frac{q\alpha}{p} - \cos\frac{q}{p}(\theta-\theta')} ,$$

then $I_1 \sim I_2 + I_3$, (2.3)

where

on I_2: $\alpha = -\pi + \epsilon + iy$, $y_1 < y < \infty$,

and on I_3: $\alpha = \pi + \epsilon + iy$, $y_1 < y < \infty$,

ϵ being a small positive number and y_1 some positive number. Since y_1 can be taken as large as we please, integrands in I_2 and I_3 can be replaced by their asymptotic forms. Now,

$$\cos\alpha = \cosh(y - i\epsilon \pm i\pi) \sim \frac{1}{2}\exp(y - i\epsilon \pm i\pi) = -\frac{1}{2}e^{y-i\epsilon},$$

$$\cos\frac{q\alpha}{p} \sim \frac{1}{2}\exp\left\{\frac{q}{p}(y - i\epsilon \pm i\pi)\right\} \quad \text{and}$$

$$\sin\frac{q\alpha}{p} = i\sinh\left\{\frac{q}{p}(y - i\epsilon \pm i\pi)\right\} \sim \frac{i}{2}\exp\left\{\frac{q}{p}(y - i\epsilon \pm i\pi)\right\}, \quad \text{as } y \to \infty,$$

the + or - sign being chosen according as α is on I_2 or I_3 .

$$\therefore I_2 \sim \int_{y_1}^{\infty} e^{-\frac{ikr}{2}} \cdot \frac{e^{y-i\epsilon}\left(-\frac{1}{2}\right)\exp(y-i\epsilon)\cdot\frac{i}{2}\exp\left\{\frac{q}{p}(y-i\epsilon+i\pi)\right\}}{\frac{1}{2}\exp\left\{\frac{q}{p}(y-i\epsilon+i\pi)\right\} - \cos\frac{q}{p}(\theta-\theta')} \, (i\,dy)$$

$$= \frac{1}{2}\int_{y_1}^{\infty} e^{-\frac{ikr}{2}} \cdot \frac{e^{y-i\epsilon}\exp(y-i\epsilon)\, dy}{1 - 2\cos\frac{q}{p}(\theta-\theta')\cdot\exp\left\{-\frac{q}{p}(y-i\epsilon+i\pi)\right\}} ,$$

$$\text{and} \quad I_3 \sim -\frac{1}{2}\int_{y_1}^{\infty} e^{-\frac{ikr}{2}} \cdot e^{y-i\epsilon} \cdot \frac{\exp(y-i\epsilon)\, dy}{1 - 2\cos\frac{q}{p}(\theta-\theta') \cdot \exp\left\{-\frac{q}{p}(y-i\epsilon-i\pi)\right\}}$$

Making substitutions

$$X = e^y, \quad \frac{kr}{2} = R$$

and writing

$$e^{-i\epsilon} \simeq 1 - i\delta, \quad \text{where } \delta \text{ is a small number,}$$

we get

$$I_2 \sim \frac{1}{2}\int_{e^{y_1}}^{\infty} e^{-iRX(1-i\delta)} \cdot \frac{dX}{1 - 2\cos\frac{q}{p}(\theta-\theta') \cdot e^{-iq\pi/p} \cdot X^{-q/p}} \quad , \text{ and}$$

$$I_3 \sim -\frac{1}{2}\int_{e^{y_1}}^{\infty} e^{-iRX(1-i\delta)} \cdot \frac{dX}{1 - 2\cos\frac{q}{p}(\theta-\theta') \cdot e^{iq\pi/p} \cdot X^{-q/p}} \, .$$

Expanding the integrand in ascending powers of $X^{-q/p}$, we get

$$I_2 \sim \frac{1}{2}\int_{e^{y_1}}^{\infty} e^{-iRX(1-i\delta)} \cdot \left[1 + 2\cos\frac{q}{p}(\theta-\theta') \cdot e^{-iq\pi/p} \cdot X^{-q/p} + O(X^{-2q/p})\right] dX$$

$$\sim \frac{1}{2}\left[\frac{e^{-iRX(1-i\delta)}}{-iR(1-i\delta)}\right]_{e^{y_1}}^{\infty} + \cos\frac{q}{p}(\theta-\theta')\, e^{-iq\pi/p}\int_{e^{y_1}}^{\infty} X^{-q/p} \cdot e^{-iRX(1-i\delta)}\, dX$$

[Since $X = e^y$ is large when $y_1 < y < \infty$ other terms are neglected.]

Similarly,

$$I_3 \sim -\frac{1}{2}\left[\frac{e^{-iRX(1-i\delta)}}{-iR(1-i\delta)}\right]_{e^{y_1}}^{\infty} - \cos\frac{q}{p}(\theta-\theta')\, e^{iq\pi/p}\int_{e^{y_1}}^{\infty} X^{-q/p} \cdot e^{-iRX(1-i\delta)}\, dX$$

Substituting these two expressions for I_2 and I_3 in (2.3),

we get

$$I_1 \sim \cos\frac{q}{p}(\theta-\theta')\left\{-2i\sin\left(\frac{q\pi}{p}\right)\right\}\int_{e^{y_1}}^{\infty} X^{-q/p}\cdot e^{-iRX(1-i\delta)}\cdot dX \ .$$

Putting $RX = t$,

$$\int_{e^{y_1}}^{\infty} X^{-q/p}\cdot e^{-iRX(1-i\delta)}\,dX = R^{-1+q/p}\int_{Re^{y_1}}^{\infty} t^{-q/p}\cdot e^{-it(1-i\delta)}\cdot dt$$

If we consider values of r such that $r \ll \lambda$, wave length, then $kr \ll 1$, and therefore since $R = kr/2$, $R\,e^{y_1}$ can be replaced by zero.

Let $s = t\,e^{i(\frac{\pi}{2}-\epsilon)}$. Then $t = s\,e^{-i(\frac{\pi}{2}-\epsilon)} \simeq -is$, and

$$\int_0^{\infty} t^{-q/p}\,e^{-it(1-i\delta)}\,dt \sim \int_0^{\infty} s^{-q/p}\,e^{\frac{iq}{p}(\frac{\pi}{2}-\epsilon)}\cdot e^{-s}\cdot e^{-i(\frac{\pi}{2}-\epsilon)}\,ds$$

$$(\text{as } \epsilon\to 0) \qquad \sim e^{\frac{i\pi}{2}(\frac{q}{p}-1)}\,\Gamma\left(1-\frac{q}{p}\right) = -i\,e^{\frac{iq\pi}{2p}}\,\Gamma\left(1-\frac{q}{p}\right) .$$

Hence

$$I_1 \sim -2\,e^{\frac{iq\pi}{2p}}\,\Gamma\left(1-\frac{q}{p}\right)\sin\left(\frac{q\pi}{p}\right)\cdot\left(\frac{kr}{2}\right)^{-1+\frac{q}{p}}\cos\frac{q}{p}(\theta-\theta') \qquad (2.4)$$

Similarly if

$$I_1' = \int_C e^{ikr\cos\alpha}\,\frac{\cos\alpha\cdot\sin\frac{q\alpha}{p}\,d\alpha}{\cos\frac{q\alpha}{p}-\cos\frac{q}{p}(\theta+\theta')}\ , \text{ then}$$

replacing $(\theta-\theta')$ in (2.4) by $(\theta+\theta')$, we get

$$I_1' \sim -2\,e^{\frac{iq\pi}{2p}}\,\Gamma\left(1-\frac{q}{p}\right)\sin\left(\frac{q\pi}{p}\right)\cdot\left(\frac{kr}{2}\right)^{-1+\frac{q}{p}}\cos\frac{q}{p}(\theta+\theta') \qquad (2.5)$$

By (2.2)

$$\frac{\partial \Phi^I}{\partial r} = -\frac{qk}{2p\pi}\left(I_1 + I_1'\right)$$

Using the expressions (2.4) and (2.5), and simplifying

$$\frac{\partial \Phi^I}{\partial r} \sim A r^{-1+\frac{q}{p}} \cdot \cos\frac{q\theta}{p} \quad , \text{ where}$$

$$A = \frac{4q}{p} \frac{\left(\frac{k}{2}\right)^{q/p}}{\Gamma(q/p)} e^{\frac{iq\pi}{2p}} \cos\frac{q\theta'}{p} .$$

Integrating this expression for $\frac{\partial \Phi^I}{\partial r}$, with respect to r, we get

$$\Phi^I \sim \frac{Ap}{q} r^{q/p} \cos\frac{q\theta}{p} + f(\theta) , \qquad (2.6)$$

where $f(\theta)$ is an arbitrary function of θ.

It is easily seen that $f(\theta) = (\Phi^I)_{r=0}$.

Evaluation of $(\Phi^I)_{r=0}$

Putting r = 0 in (2.1), we get

$$(\Phi^I)_{r=0} = \frac{iq}{2p\pi} \int_C \left\{ \frac{\sin\frac{q\alpha}{p}}{\cos\frac{q\alpha}{p} - \cos\frac{q}{p}(\theta-\theta')} + \frac{\sin\frac{q\alpha}{p}}{\cos\frac{q\alpha}{p} - \cos\frac{q}{p}(\theta+\theta')} \right\} d\alpha .$$

Let $A \equiv \pi + \epsilon + iy$, $B \equiv -\pi + \epsilon + iy$ be two points (Fig.2) on the contour C. Then

$$\int_C = \lim_{\substack{\epsilon \to 0 \\ y \to \infty}} \int_A^B ,$$

$$(\Phi^I)_{r=0} = -\frac{i}{2\pi} \lim_{\substack{\epsilon \to 0 \\ y \to \infty}} \left[\log\left\{\cos\frac{q\alpha}{p} - \cos\frac{q}{p}(\theta-\theta')\right\} + \log\left\{\cos\frac{q\alpha}{p} - \cos\frac{q}{p}(\theta+\theta')\right\} \right]_A^B$$

Substituting the coordinates of A and B, and taking limits $\epsilon \to 0$, $y \to \infty$, in any order, we get

$$(\Phi^I)_{r=0} = -\frac{i}{2\pi} \cdot 2 \log\left(e^{i.2q\pi/p}\right) = \frac{2q}{p} .$$

Hence $f(\theta) = 2q/p$, the constant we lost in getting (2.2) from (2.1). Substituting this result in (2.6), we get

$$\left.\begin{aligned} \Phi^{I}(r,\theta) &\sim A_1 r^{q/p} \cos\frac{q\theta}{p} + \frac{2q}{p}, \quad r \ll \lambda, \\ \text{where} \quad A_1 &= \frac{A_p}{q} = \frac{4\left(\frac{k}{2}\right)^{q/p}}{\Gamma(q/p)} \cdot e^{\frac{iq\pi}{2p}} \cdot \cos\frac{q\theta'}{p}. \end{aligned}\right\} \tag{2.7}$$

(2.7) is the outer potential Φ^{I} [i.e. the solution of (1.1) in the outer region I], expressed in asymptotic form, for points closer than a wave length from the tip.

3. Inner Region

Points in this region are very close to the tip on a wave length scale, and therefore $\frac{1}{r} \gg k$ so that the equation (1.1)i:

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + k^2\Phi = 0$$

approximately becomes

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0.$$

Thus, if the potential in this region is denoted by Φ^{II}, then Φ^{II} satisfies Laplace's equation: $\nabla^2\Phi^{II} = 0$.

However, Φ^{II} still satisfies (1.1)ii, the vanishing of normal derivative on the boundary, since we are considering a hard body.

By the assumption $ka \ll 1$ (sec.1), there is a region of space such that $a \ll r \ll \lambda$, common to both outer region I and inner region II. In this common region, the two

potentials Φ^{I} and Φ^{II} must match with each other. We have found an expression for the outer potential Φ^{I} (2.7), for small r ($\ll \lambda$). The condition at infinity for the inner potential Φ^{II} must therefore be

$$\Phi^{II} \sim \Phi^{I}(r,\theta), \text{ (of 2.7)}, \quad \text{as } \frac{r}{a} \to \infty.$$

Thus the problem for the inner potential Φ^{II} is

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2 \Phi^{II} = 0, \text{ when } r > a, \ 0 < \theta < p\pi/q \\ &\text{(ii)} \quad \frac{\partial \Phi^{II}}{\partial n} = 0, \text{ on the boundary: } \theta = 0, \ r \geqslant a; \\ &\qquad\qquad r = a, \ 0 \leqslant \theta \leqslant p\pi/q; \quad \theta = p\pi/q, \ r \geqslant a \\ &\text{(iii)} \quad \Phi^{II} \sim \Phi^{I}(r,\theta), \text{ as } \frac{r}{a} \to \infty. \end{aligned}\right\} \quad (3.1)$$

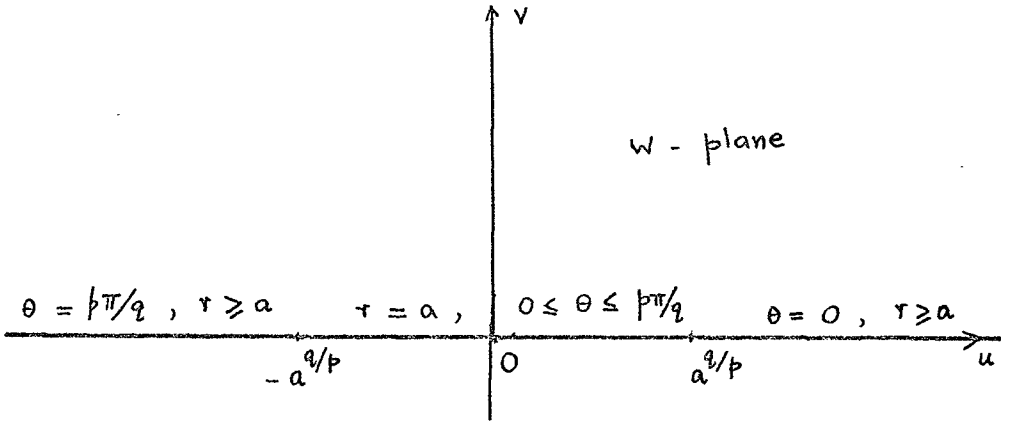
The incompressible flow problem (3.1) can be solved by making use of the conformal transformation

$$w = \frac{1}{2}\left[z^{q/p} + \frac{a^{2q/p}}{z^{q/p}} \right], \qquad (3.2)$$

from $z \equiv (r,\theta)$ plane (Fig.1) to w-plane (Fig.3). If $w = u + iv$, writing $z = r e^{i\theta}$ in (3.2) gives

$$\left.\begin{aligned} u &= \frac{1}{2}\left[r^{q/p} + \frac{a^{2q/p}}{r^{q/p}} \right] \cos\frac{q\theta}{p} \\ v &= \frac{1}{2}\left[r^{q/p} - \frac{a^{2q/p}}{r^{q/p}} \right] \sin\frac{q\theta}{p}. \end{aligned}\right\} \quad (3.3)$$

Fig.3



From (3.3), it is seen that the transformation (3.2) maps the region outside the wedge ($r \geqslant a$, $0 \leqslant \theta \leqslant p\pi/q$) into the upper half of w-plane, while the boundary of the wedge ($\theta = 0$, $r \geqslant a$; $r = a$, $0 \leqslant \theta \leqslant p\pi/q$; $\theta = p\pi/q$, $r \geqslant a$) is mapped on to the real u-axis. (See Fig.3.)

By (3.2), the approximate transformation, for large $|z|$ (i.e. $r \gg a$), is

$$w \sim \frac{1}{2} z^{q/p} \tag{3.4}$$

By (3.1), (3.2), (2.7) and (3.4), the problem for Φ^{II} in the w-plane is

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2 \Phi^{II} = 0, \text{ where } \nabla^2 = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \\ &\text{(ii)} \quad \frac{\partial \Phi^{II}}{\partial v} = 0, \text{ when } v = 0 \\ &\text{(iii)} \quad \Phi^{II} \sim 2A_1 \mathrm{Re}(w) + 2q/p, \text{ as } w \to \infty. \end{aligned}\right\} \tag{3.5}$$

Clearly, the solution of (3.5) is

$$\Phi^{II} = 2A_1 u + 2q/p .$$

By (3.3), it follows that, in the z-plane,

$$\left.\begin{aligned} \Phi^{II}(r,\theta) &= A_1\left[r^{q/p} + \frac{a^{2q/p}}{r^{q/p}} \right] \cos\frac{q\theta}{p} + \frac{2q}{p}\,, \\ & A_1 \text{ being given by (2.7).} \end{aligned}\right\} \qquad (3.6)$$

Thus, (3.6) gives the approximate inner potential Φ^{II} which matches with the outer potential Φ^{I} (2.7), in the common region $a \ll r \ll \lambda$. It can be shown that the result (3.6) agrees with the first two terms of the general solution (Tyras, 1969).

4. A related problem

As an example of other problems to which this method could be employed, consider the problem of diffraction of long plane waves by a semi-infinite slab of cross-section A B C B′A′, with AB parallel to A′B′ and at a distance d apart, $\hat{ABC} = \hat{A'B'C} = \pi/4$, faces BA, B′A′ extending to infinity. (See Fig.4).

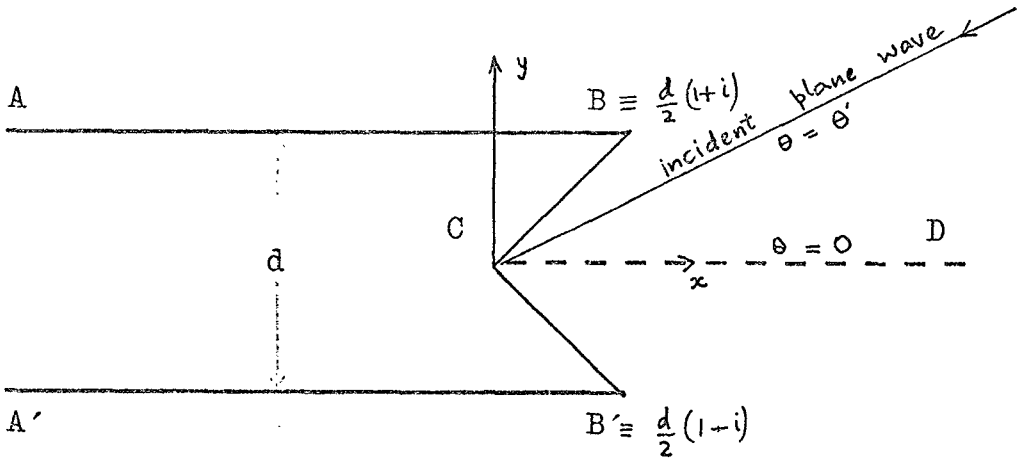
Define a system of polar coordinates r, θ so that C is the origin and CD is the line θ = 0. Let the incident plane wave (same as in the previous problem) make an angle θ′ with CD. Also, in this case, let us assume that $kd \ll 1$. As in section 1, divide the flow region into two parts so that

in the outer region I : $r \gg d$, and

in the inner region II: $r \ll \lambda$.

Since $kd \ll 1$, the two regions overlap for points such that $d \ll r \ll \lambda$.

Fig.4



In region I, distances of points from the edge are very large compared with the thickness of the slab. Therefore the potential Φ_I here would be the same as that for a semi-infinite plate, which is a special case of the sharp wedge considered in section 2. In (2.7) putting $p = 2$ and $q = 1$, and replacing θ and θ' by $\pi + \theta$ and $\pi + \theta'$ respectively we get the asymptotic form of Φ_I, for small r ($\ll \lambda$), as

$$\left.\begin{aligned} \Phi_I &\sim C_1\, r^{\frac{1}{2}} \sin\tfrac{\theta}{2}\ , \\ \text{where } C_1 &= \frac{4\left(\frac{k}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}\, e^{\frac{i\pi}{4}} \sin\tfrac{\theta'}{2}\ . \end{aligned}\right\} \qquad (4.1)$$

In region II, points are very close to the edge on a wave length scale, and therefore as in section 3, the potential Φ_{II} here satisfies Laplace's equation $\nabla^2\Phi_{II} = 0$. This incompressible flow problem is to be solved under 'hard' condition on the boundary of the slab, together with

the condition at infinity:

$$\Phi_{II} \sim \Phi_{I}\ (r,\theta) \quad (\text{ of } 4.1\ ), \quad \text{as} \quad \frac{r}{d} \to \infty,$$

arising from the consideration of the potential for points such that $d \ll r \ll \lambda$, common to regions I and II. Also, in this case, there is odd symmetry about the line CD so that $\Phi = 0$ on CD, and it is sufficient to consider the region of space above CD (x-axis). Thus, using (4.1), the problem for Φ_{II}, for $y > 0$, is

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2 \Phi_{II} = 0, \\ &\text{(ii)} \quad \frac{\partial \Phi_{II}}{\partial n} = 0, \text{ on AB and BC}, \\ &\text{(iii)} \quad \Phi_{II} = 0, \text{ on CD}, \\ &\text{(iv)} \quad \Phi_{II} \sim C_1 r^{\frac{1}{2}} \sin\frac{\theta}{2} = C_1 \operatorname{Re}(-i z^{\frac{1}{2}}) \ , \text{ as } \frac{r}{d} \to \infty. \end{aligned}\right\} \quad (4.2)$$

The region above A B C D in the z-plane (Fig.4) is mapped on to the upper half of w-plane by the conformal transformations:

$$z_1 = \frac{3\pi}{d} z \qquad (4.3)$$

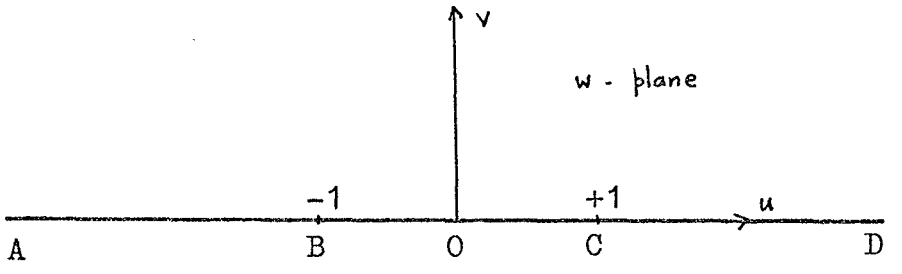
and

$$\left.\begin{aligned} &\frac{dz_1}{dw} = \left(\frac{w+1}{w-1}\right)^{3/4} \\ &\text{or} \quad z_1 = 8 \int_{\gamma}^{\infty} \frac{t^6\, dt}{(t^4-1)^2}, \quad \gamma = \left(\frac{w+1}{w-1}\right)^{1/4} \end{aligned}\right\} \quad (4.4)$$

(Kober, 1957). The points B and C are mapped onto -1 and

+1 respectively on the real u-axis (Fig.5).

Fig.5



By (4.4), for large $|w|$, $\frac{dz_1}{dw} \sim 1 \Rightarrow z_1 \sim w$.
Therefore at infinity,

$$w \sim \frac{3\pi}{d} z . \tag{4.5}$$

By (4.2) — (4.5), the problem for Φ_{II} in the w-plane is

$$\left.\begin{aligned}
&\text{(i)} \quad \nabla^2 \Phi_{II} = 0, \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \\
&\text{(ii)} \quad \frac{\partial \Phi_{II}}{\partial v} = 0, \quad \text{when} \quad v = 0, \ u < 1 \\
&\text{(iii)} \quad \Phi_{II} = 0, \quad \text{when} \quad v = 0, \ u > 1 \\
&\text{(iv)} \quad \Phi_{II} \sim C_1 \left(\frac{d}{3\pi}\right)^{\frac{1}{2}} \mathrm{Re}\left(-i w^{\frac{1}{2}}\right) \quad \text{at } \infty .
\end{aligned}\right\} \tag{4.6}$$

By shifting the origin to C $(\equiv +1)$ without rotation and considering polar coordinates R, β with respect to C and C D, it can be shown that the solution of (4.6) is

$$\Phi_{II} = C_1 \left(\frac{d}{3\pi}\right)^{\frac{1}{2}} R^{\frac{1}{2}} \sin\frac{\beta}{2} = C_1 \left(\frac{d}{3\pi}\right)^{\frac{1}{2}} \mathrm{Re}\left[-i(w-1)^{\frac{1}{2}}\right] . \tag{4.7}$$

Since w is not expressible explicitly in terms of z [see (4.3) and (4.4)], Φ_{II} does not take an explicit form in the z-plane. However, near the point $B \equiv \frac{d}{2}(1+i)$, an approximate transformation and hence an approximate expression for Φ_{II} can be worked out as follows:

Combining (4.3) and (4.4), we get

$$\left(\frac{3\pi}{d}\right) z = 8 \int_{\zeta}^{\infty} \frac{t^6\, dt}{(t^4-1)^2}, \quad \zeta = \left(\frac{w+1}{w-1}\right)^{\frac{1}{4}}. \tag{4.8}$$

When $w = -1$, $\zeta = 0$ and $z = \frac{d}{2}(1+i) = z_0$ say, so that

$$\left(\frac{3\pi}{d}\right) z_0 = 8 \int_{0}^{\infty} \frac{t^6\, dt}{(t^4-1)^2}.$$

Near $B \equiv z_0$, put $w = -1+\delta$, where δ is small. Then

$$\zeta = \left(\frac{\delta}{-2+\delta}\right)^{\frac{1}{4}} \simeq \pm\, e^{\pm \frac{i\pi}{4}} \left(\frac{\delta}{2}\right)^{\frac{1}{4}}, \text{ also small.}$$

Now, $$\int_{\zeta}^{\infty} \frac{t^6\, dt}{(t^4-1)^2} = \int_{0}^{\infty} \frac{t^6\, dt}{(t^4-1)^2} - \int_{0}^{\zeta} \frac{t^6\, dt}{(t^4-1)^2}$$

and $$\int_{0}^{\zeta} \frac{t^6\, dt}{(t^4-1)^2} \simeq \frac{\zeta^7}{7} = \pm\, \frac{e^{\pm \frac{7\pi i}{4}}}{7} \left(\frac{\delta}{2}\right)^{\frac{7}{4}}.$$

Substituting these results back in (4.8), we get

$$\frac{3\pi}{d}\left(\frac{z-z_0}{8}\right) \simeq \mp\, \frac{e^{\pm \frac{7\pi i}{4}}}{7} \left(\frac{\delta}{2}\right)^{\frac{7}{4}} = \mp\, \frac{e^{\mp \frac{\pi i}{4}}}{7} \left(\frac{\delta}{2}\right)^{\frac{7}{4}},$$

$$\therefore \quad \left(\frac{\delta}{2}\right)^{\frac{7}{4}} \simeq \frac{7}{8}\left(\frac{3\pi}{d}\right)(z-z_0)\, e^{\frac{n\pi i}{4}}, \quad n = 5, 3, 1, -1.$$

$$\text{i.e.} \quad \frac{\delta}{z} = \left\{ \frac{7}{8} \left(\frac{3\pi}{d} \right) (z - z_0) \right\}^{\frac{4}{7}} e^{\frac{n\pi i}{7}}$$

Since $\delta = w+1$, the approximate transformation equation near B $(z \equiv z_0)$ is

$$w = 2 \left\{ \frac{7}{8} \left(\frac{3\pi}{d} \right) (z - z_0) \right\}^{\frac{4}{7}} e^{\frac{n\pi i}{7}} - 1, \quad n = 5, 3, 1, -1. \qquad (4.9)$$

[Suitable n is to be chosen.]

Approximate estimate for Φ_{II}, near B

By (4.7) and (4.9), near $z \equiv z_0$,

$$\Phi_{II} \sim C_1 \left(\frac{d}{3\pi} \right)^{\frac{1}{2}} \mathrm{Re} \left[-i \cdot \sqrt{2}\, i \left\{ 1 - \left(\frac{7}{8} \right)^{\frac{4}{7}} \left(\frac{3\pi}{d} \right)^{\frac{4}{7}} (z - z_0)^{\frac{4}{7}} e^{\frac{n\pi i}{7}} \right\}^{\frac{1}{2}} \right].$$

Writing $z - z_0 = \rho e^{i\alpha}$ and expanding for small ρ, we get

$$\Phi_{II} \sim \sqrt{2}\, C_1 \left(\frac{d}{3\pi} \right)^{\frac{1}{2}} \left[1 - \frac{1}{2} \left(\frac{7}{8} \right)^{\frac{4}{7}} \left(\frac{3\pi}{d} \right)^{\frac{4}{7}} \rho^{\frac{4}{7}} \cos \frac{4\alpha + n\pi}{7} + O(\rho^{\frac{8}{7}}) \right],$$

$$n = 5, 3, 1, -1.$$

Since $\frac{\partial \Phi_{II}}{\partial \alpha} = 0$, when $\alpha = \pi, -3\pi/4$ (see Fig.4), the only value n can take is 3. Therefore, near B $\left[z \equiv z_0 \equiv \frac{d}{2}(1+i) \right]$

$$\Phi_{II} \sim \sqrt{2}\, C_1 \left(\frac{d}{3\pi} \right)^{\frac{1}{2}} \left[1 - \frac{1}{2} \left(\frac{7}{8} \right)^{\frac{4}{7}} \left(\frac{3\pi}{d} \right)^{\frac{4}{7}} |z - z_0|^{\frac{4}{7}} \cos \frac{4\alpha + 3\pi}{7} \right], \qquad (4.10)$$

C_1 being given by (4.1).

PART 2

SCATTERING OF LONG PLANE WAVES BY A CIRCULAR DISC OF FINITE THICKNESS

1. Introduction

The problem discussed here is that of the scattering of long plane waves by a circular disc of small but finite thickness, when the incidence is normal. It is assumed that the wavelength is much larger than the radius, taken to be unity, which in turn is much larger than the thickness, taken to be ϵ. In the low wave-number expansion for the velocity potential, the first term is trivial, and the second one satisfies an incompressible flow problem. To solve this problem in detail, the flow region is divided into two parts as follows:

(i) Outer Region:— consisting of points in the fluid 'far away' (i.e. at distances much larger than ϵ) from the edge of the disc,

(ii) Inner region:— consisting of points in the fluid 'close to' (i.e. at distances much less than a radius from) the edge of the disc.

Outer and inner coordinate systems based on the edge are defined accordingly and it is seen that the two regions overlap, a fact made use of in the boundary condition at infinity for the inner solution. As the problem reduces to a two-dimensional one in the inner region, the method of conformal transformations is used there. As suggested by

the behaviour of the first approximations (sec.3), solutions are found in the form of expansions up to order ϵ and $\epsilon^{3/2}$ respectively for the outer and inner potentials. These expansions are matched against each other in section 5, by the general matching principle outlined below:

Suppose $f(\delta,\epsilon)$ is a function of the variable δ and the parameter ϵ. Writing $\delta = \epsilon R$, let $g(R,\epsilon) = f(\delta,\epsilon)$. The variables δ and R are called outer and inner variables respectively. When f and g are expanded asymptotically for small ϵ, the resulting expansions are called outer and inner expansions respectively. Let $f_n(\delta,\epsilon)$ be the outer expansion to order ϵ^n (i.e. including ϵ^n, $\epsilon^n \log\epsilon$, $\epsilon^n(\log\epsilon)^2 \ldots\ldots$). Write this in terms of inner variable R, and then expand to order ϵ^m (i.e. including ϵ^m, $\epsilon^m \log\epsilon$, $\epsilon^m(\log\epsilon)^2 \ldots\ldots$) with the result, say $f_{n,m}(R,\epsilon)$. Similarly if $g_m(R,\epsilon)$ denotes the inner expansion to order ϵ^m, writing it in terms of outer variable δ and expanding to order ϵ^n, one arrives at $g_{m,n}(\delta,\epsilon)$ say. According to the matching principle (Crighton and Leppington, to appear) adopted here, these two expansions are identical (cf. Van Dyke,1964).

i.e. $f_{n,m}(R,\epsilon) = g_{m,n}(\delta,\epsilon)$.

In the present case, the outer expansion is up to order ϵ and the inner expansion is up to order $\epsilon^{3/2}$, and the matching procedure enables us to evaluate unknown constants.

Even though the third term in the low wave-number expansion for the velocity potential is not found explicitly, it is seen (sec.6) that its contribution to

the 'far field' is zero. Thus we are able to determine the far field to a higher order in k. In section 7, several other problems for which this method could be employed are indicated, the outer solution being the same as that for the disc problem.

2. Formulation of the problem

Define a system of cylindrical polar coordinates (r, λ, z) so that the disc occupies the region $r \leqslant 1$, $|z| \leqslant \epsilon$, all λ. Let the incident plane wave be represented by $e^{i(kz-\omega t)}$, where the non-dimensional wave-number $k \ll 1$ and ω/k is the velocity of propagation. Since there is symmetry about the z-axis, the total velocity potential is independent of λ and can be denoted by $\Phi(r,z)e^{-i\omega t}$. The time factor $e^{-i\omega t}$ is suppressed throughout, its presence being understood. The potential $\Phi(r,z)$ satisfies

$$\left.\begin{aligned}
&\text{(i)}\ (\nabla^2 + k^2)\Phi = 0, \text{ where } \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \\
&\text{(ii)a.}\quad \frac{\partial \Phi}{\partial z} = 0, \text{ when } z = \pm\epsilon,\ r \leqslant 1 \\
&\qquad\text{b.}\quad \frac{\partial \Phi}{\partial r} = 0, \text{ when } r = 1,\ |z| \leqslant \epsilon \\
&\text{(iii)}\ \Phi(\underline{y}) = e^{ikz} - \int_S \Phi(\underline{x}) \frac{\partial G}{\partial n}\, d\underline{x}\ , \text{ where } G = -\frac{e^{ik|\underline{x}-\underline{y}|}}{4\pi|\underline{x}-\underline{y}|},
\end{aligned}\right\} (2.1)$$

when the point $\underline{y}$ lies outside S, n being the outward normal from the surface S of the disc. Let $|\underline{x} - \underline{y}|$ be denoted by p.

To determine successive low wave-number approximations $\Phi_0, \Phi_1, \Phi_2, \cdots\cdots$ for Φ one expands (Morse and Feshbach, 1953) e^{ikz}, $\frac{\partial G}{\partial n}$ and Φ as series of the form

$$\Phi = \Phi_0 + (ik)\Phi_1 + \frac{(ik)^2}{2!}\Phi_2 + \cdots\cdots \tag{2.2}$$

substitutes in equations (2.1) and equates powers k. The resulting equations are

$$\left.\begin{aligned}
&\text{(i)} \quad \nabla^2\Phi_0 = \nabla^2\Phi_1 = 0, \quad \nabla^2\Phi_2 = 2\Phi_0, \cdots \\
&\text{(ii)a.} \quad \frac{\partial\Phi_0}{\partial z} = \frac{\partial\Phi_1}{\partial z} = \frac{\partial\Phi_2}{\partial z} = \cdots\cdots = 0, \quad z = \pm\epsilon, r \le 1 \\
&\quad\ \text{b.} \quad \frac{\partial\Phi_0}{\partial r} = \frac{\partial\Phi_1}{\partial r} = \frac{\partial\Phi_2}{\partial r} = \cdots\cdots = 0, \quad r = 1, |z| \le \epsilon \\
&\text{(iii)} \quad \Phi_0 = 1 - \frac{1}{4\pi}\int_S \frac{1}{\rho^2}\frac{\partial\rho}{\partial n}\Phi_0(\underline{x})\,d\underline{x}, \\
&\qquad\quad \Phi_1 = z - \frac{1}{4\pi}\int_S \frac{1}{\rho^2}\frac{\partial\rho}{\partial n}\Phi_1(\underline{x})\,d\underline{x}, \\
&\qquad\quad \Phi_2 = z^2 - \frac{1}{4\pi}\int_S \left\{\frac{\Phi_2(\underline{x})}{\rho^2} - \Phi_0(\underline{x})\right\}\frac{\partial\rho}{\partial n}\,d\underline{x}, \cdots
\end{aligned}\right\} \tag{2.3}$$

(2.3)(iii) $\Rightarrow$ $\Phi_0 \sim 1$,

$\Phi_1 \sim z$,

$\Phi_2 \sim z^2$, as $p \to \infty$.

It is easily seen that $\Phi_0 \equiv 1$.

3. The incompressible flow potential Φ_1

From (2.3) and the above behaviour of Φ_1 at infinity, the problem for Φ_1 is

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2 \Phi_1 = 0 \\ &\text{(ii)a.} \quad \frac{\partial \Phi_1}{\partial z} = 0, \quad \text{when} \quad z = \pm\epsilon, \ r < 1 \\ &\quad\ \text{b.} \quad \frac{\partial \Phi_1}{\partial r} = 0, \quad \text{when} \quad r = 1, \ |z| \leq \epsilon \\ &\text{(iii)} \quad \Phi_1 \sim z \quad \text{at} \quad \infty. \end{aligned}\right\} \tag{3.1}$$

(3.1) is equivalent to an incompressible flow problem with the disc placed normally against a uniform stream of unit velocity. This problem is tackled by dividing the flow region into two overlapping parts, as explained in section 1.

In the outer region, write

$$\Phi_1 = z + \phi , \tag{3.2}$$

where $\phi(r,z)$ is the scattered potential there. Now, the boundary condition (3.1)(ii)b, being essentially a condition at the edge, is dropped altogether. Also since Φ_1 is an odd function with respect to z, so is ϕ by (3.2), and hence

$$\phi(r,-z) = \phi(r,z) \implies \phi(r,0) = 0,$$

and it is sufficient to consider $z > 0$ alone.

Thus, the problem for ϕ, by (3.1) and (3.2) is

$$\left.\begin{aligned} &\text{(i)} && \nabla^2 \phi = 0 \\ &\text{(ii)} && \frac{\partial \phi(r,\epsilon)}{\partial z} = -1, \quad \text{when} \quad r < 1 \\ &\text{(iii)} && \phi(r,0) = 0, \quad \text{when} \quad r > 1. \end{aligned}\right\} \tag{3.3}$$

First outer approximation

As a first approximation to the problem (3.3) replace ϵ by zero and let $\phi_0(r,z)$ be the corresponding ϕ. The solution to this thin disc problem (Sneddon, 1966) is, for points on the disc,

$$\left.\begin{aligned} \phi(r,o) &= -\int_r^1 \frac{S(t)\,dt}{(t^2-r^2)^{\frac{1}{2}}}, \quad \text{where} \\ t\,S(t) &= \frac{2}{\pi}\,\frac{d}{dt}\int_o^t \frac{\rho\,d\rho}{(t^2-\rho^2)^{\frac{1}{2}}} \cdot \int_o^\rho u\,V(u)\,du, \quad V(u) = \left(\frac{\partial \phi}{\partial z}\right)_{disc}. \end{aligned}\right\} \tag{3.4}$$

In this case, by (3.3)(ii) with $\epsilon = o$, $V(u) = \left(\frac{\partial \phi_o}{\partial z}\right)_{disc} = -1$, and it can be shown that $S(t) = -\frac{2}{\pi}\,t$ and

$$\phi_o(r,o) = \frac{2}{\pi}\,(1-r^2)^{\frac{1}{2}}, \quad r < 1. \tag{3.5}$$

Because of axial symmetry, to find $\phi_o(r,z)$ using (3.5), Hankel Transforms can be used. Write

$$\tilde{\phi}_o(s,z) = \int_o^\infty r\,J_o(sr)\,\phi_o(r,z)\,dr,$$

whence

$$\phi_0(r,z) = \int_0^\infty s\, J_0(rs)\, \tilde{\phi}_0(s,z)\, ds .$$

Since ϕ_0 satisfies Laplace's equation,

$\tilde{\phi}_0 = A(s)\, e^{-sz}$, $z > 0$, where $A(s)$ is a function of s.

Using (3.5) and the transforms, we get

$$A(s) = \frac{2}{\pi} \int_0^1 r\, J_0(sr).(1-r^2)^{\frac{1}{2}}\, dr$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} . \frac{J_{3/2}(s)}{s^{3/2}} \quad \text{(Watson, 1944), and}$$

$$\phi_0(r,z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{J_0(rs).J_{3/2}(s)}{s^{1/2}}\, e^{-sz}\, ds , \quad z > 0 . \qquad (3.6)$$

For small z, using Taylor's theorem,

$$\phi_0(r,z) = \phi_0(r,0) + z.\frac{\partial \phi_0}{\partial z} + O(z^2)$$

$$\simeq \phi_0(r,0) - z \ , [\text{by (3.3)(ii) with } \epsilon = 0] . \qquad (3.7)$$

Thus, if $\Phi_1^\circ(r,z)$ denotes the first approximation of the total potential Φ_1, then by (3.2), (3.5) and (3.7), for small z,

$$\Phi_1^\circ(r,z) \simeq \phi_0(r,0) = \frac{2}{\pi}(1-r^2)^{\frac{1}{2}}, \quad r<1. \qquad (3.8)$$

Behaviour of Φ_1° near edge

To investigate more closely the behaviour of Φ_1° near the edge of the disc, define a new coordinate system based at a point $r = 1$, $z = 0$, any λ, on the edge, by

the relations

$$\left.\begin{aligned} r &= 1 + \delta\cos\theta \\ z &= \delta\sin\theta \end{aligned}\right\}, \quad \epsilon \ll \delta \ll 1. \quad \text{(See Fig.1.)} \tag{3.9}$$

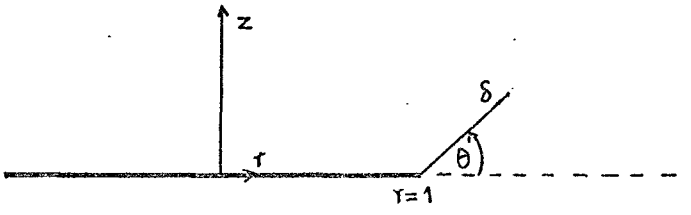
It is clear that on the upper surface of the disc $\theta = \pi$ and therefore by (3.8) and (3.9)

$$\Phi_1^0(1-\delta, 0) = \frac{2}{\pi}\cdot 2^{\frac{1}{2}}\,\delta^{\frac{1}{2}} + O(\delta^{\frac{3}{2}}).$$

Using this result and the fact that near edge Φ_1^0 satisfies (i) and (ii)a of (3.1) with $\epsilon = 0$, expressed in coordinates (δ,θ), we get

$$\Phi_1^0(\delta, \theta) = \frac{2}{\pi}\cdot 2^{\frac{1}{2}}\,\delta^{\frac{1}{2}}\sin\frac{\theta}{2}, \quad \delta \to 0. \tag{3.10}$$

Fig.1



Inner coordinates

As the point of observation approaches the edge, the thickness of the disc becomes more and more important, and once it is inside the inner region (sec.1) the effect of the thickness has to be taken into account. Thus the boundary condition for Φ_1 in (3.1)(ii)b, which was dropped in the outer region, has to be re-introduced here. Since typical distances in this region are of order ϵ, we are

led to define 'inner coordinates' (X,Y) and (R,Θ) by the relations

$$\left.\begin{aligned} r &= 1 + \epsilon X \\ z &= \epsilon Y \end{aligned}\right\}, \qquad \left.\begin{aligned} X &= R\cos\Theta \\ Y &= R\sin\Theta \end{aligned}\right\}, \tag{3.11}$$

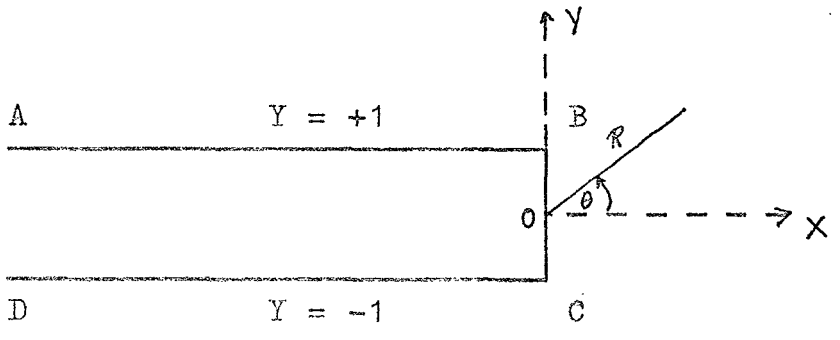
again based at a point ($r = 1$, $z = 0$, any λ) on the edge. If the coordinate systems (3.9) and (3.11) are based at the same point (same λ), then the relationship between δ and R is

$$\epsilon R = \delta.$$

δ and R will be called outer and inner variables respectively, near edge, Θ being the same in both cases.

On this new scale (X,Y) the edge geometry is magnified by a factor $\frac{1}{\epsilon}$ ($\equiv$ the radius of the disc). Since $\epsilon \ll 1$, the radius is so large that the curvature can be neglected as long as X and Y are finite. In other words, for all points in the inner region the body, in effect, is a semi-infinite slab whose cross-section is A B C D (Fig.2), given by $-1 \leqslant Y \leqslant 1$, $X \leqslant 0$; and the problem becomes a two-dimensional one.

Fig.2



Let ψ denote the total velocity potential in the inner region. By definitions of outer and inner regions (sec.1) it is clear that the two regions overlap, and therefore the condition at infinity for this inner potential must be:

$$\psi \sim \Phi_1 \quad \text{(outer potential)}, \quad \text{as} \quad R \to \infty.$$

The equations (i) and (ii) in (3.1) with Φ_1 replaced by ψ, when written in terms of (X,Y), and the above condition at infinity yield a 2-D problem for ψ:

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2\psi + \epsilon \frac{\partial \psi}{\partial X} + O(\epsilon^2) = 0, \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \\ &\text{(ii)} \quad \frac{\partial \psi}{\partial n} = 0, \quad \text{on AB, BC, CD} \\ &\text{(iii)} \quad \psi \sim \Phi_1(R,\theta), \quad \text{as} \quad R \to \infty. \end{aligned}\right\} \quad (3.12)$$

Since $\delta = \epsilon R$, expression (3.10) for the first approximation of the outer potential can be re-written as

$$\Phi_1^0(R,\theta) = \frac{2}{\pi} \cdot 2^{\frac{1}{2}} \epsilon^{\frac{1}{2}} R^{\frac{1}{2}} \sin\frac{\theta}{2}, \quad (3.13)$$

As a first approximation to problem (3.12), consider ψ satisfying $\nabla^2\psi = 0$ and $\psi \sim \Phi_1^0(R,\theta)$ (of 3.13) at infinity, instead of (i) and (iii). If we write this $\psi = \epsilon^{\frac{1}{2}}\psi_0$ the problem for ψ_0 is

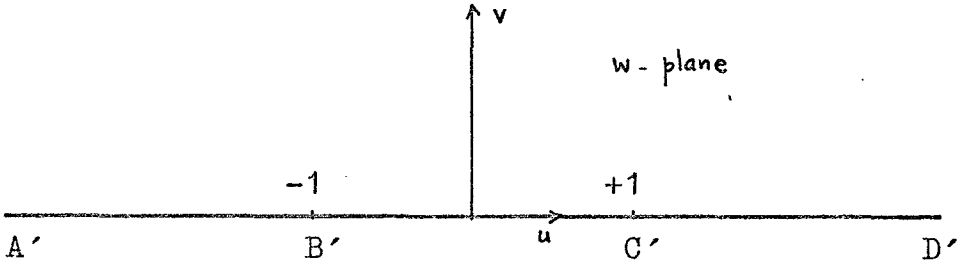
$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2\psi_0 = 0 \\ &\text{(ii)} \quad \frac{\partial \psi_0}{\partial n} = 0, \quad \text{on AB, BC, CD} \\ &\text{(iii)} \quad \psi_0 \sim \frac{2}{\pi} \cdot 2^{\frac{1}{2}} \mathcal{R}e\left(-iZ^{\frac{1}{2}}\right) \quad \text{at } \infty, \quad \text{where} \quad Z = Re^{i\theta}. \end{aligned}\right\} \quad (3.14)$$

The problem (3.14) is most easily solved by making use of the conformal transformation (Kober, 1957):

$$\left.\begin{aligned} \frac{dZ}{dw} &= -\frac{4}{\pi}(w^2-1)^{\frac{1}{2}} \\ \text{or } -\frac{\pi}{2}(Z+i) &= w(w^2-1)^{\frac{1}{2}} - \log\left\{w+(w^2-1)^{\frac{1}{2}}\right\}, \end{aligned}\right\} \quad (3.15)$$

which maps the region outside the strip A B C D into the upper half of w-plane (Fig.3), the points B and C being mapped onto -1 and +1 respectively.

Fig.3



For large $|Z|$ (large $|w|$), $-\frac{\pi}{2}Z \sim w^2 \Rightarrow w \sim i\left(\frac{\pi Z}{2}\right)^{\frac{1}{2}}$, and therefore the problem for ψ_0 in the w-plane is

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2\psi_0 = 0, \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \\ &\text{(ii)} \quad \frac{\partial\psi_0}{\partial v} = 0, \quad \text{when } v = 0 \\ &\text{(iii)} \quad \psi_0 \sim \frac{2}{\pi}\cdot 2^{\frac{1}{2}}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \mathrm{Re}(-w) \quad \text{at } \infty. \end{aligned}\right\} \quad (3.16)$$

Clearly the solution of (3.16) is

$$\psi_0 = \frac{4}{\pi^{3/2}}\,\mathrm{Re}(-w). \quad (3.17)$$

Since w cannot be expressed explicitly in terms of Z, ψ_0 does not take an explicit form in the Z-plane. However, for large $|Z|$, it can be shown that

$$w \sim i\left(\frac{\pi Z}{2}\right)^{\frac{1}{2}} + \frac{1+\log(2\pi Z)}{4i\left(\frac{\pi Z}{2}\right)^{\frac{1}{2}}} + O\left[\frac{(\log Z)^2}{Z^{3/2}}\right]. \qquad (3.18)$$

This expansion enables us to investigate more closely the behaviour of ψ_0 (or $\epsilon^{\frac{1}{2}}\psi_0$), when $|Z|$ is large. Since $Z = R\,e^{i\theta}$, from (3.17) and (3.18) we get

$$\psi_0 \sim \frac{2^{3/2}}{\pi}\left[R^{\frac{1}{2}}\sin\frac{\theta}{2} + \frac{1}{2\pi}R^{-\frac{1}{2}}\left\{(1+\log 2\pi R)\sin\frac{\theta}{2} - \theta\cos\frac{\theta}{2}\right\}\right], \quad R\to\infty.$$

Since $\epsilon R = \delta$, it follows that

$$\epsilon^{\frac{1}{2}}\psi_0 \sim \frac{2^{3/2}}{\pi}\left[\delta^{\frac{1}{2}}\sin\frac{\theta}{2} - (\epsilon\log\epsilon)\cdot\frac{1}{2\pi}\delta^{-\frac{1}{2}}\sin\frac{\theta}{2}\right.$$

$$\left. + \epsilon\cdot\frac{1}{2\pi}\delta^{-\frac{1}{2}}\left\{(1+\log 2\pi\delta)\sin\frac{\theta}{2} - \theta\cos\frac{\theta}{2}\right\}\right]. \qquad (3.19)$$

Expansion (3.19) is the first approximation ($\epsilon^{\frac{1}{2}}\psi_0$) of the inner potential ψ, expressed in outer coordinates (δ,θ) and expanded for small ϵ. This, therefore gives a clue to the behaviour of the total outer potential Φ_1 near edge.

4. (a) Outer Expansion

The first term of (3.19) is, as expected, $\Phi_1^0(\delta,\theta)$ (of 3.10). Further we find that the second and third terms are of order $\epsilon\log\epsilon$ and ϵ respectively. This suggests that for a solution of the problem (3.3) up to order ϵ, it is appropriate to try an expansion of the form

$$\phi \sim \phi_0 + (\epsilon\log\epsilon)\,\phi_1 + \epsilon\phi_2 , \qquad (4.1)$$

for the scattered potential ϕ, the total potential being given by (3.2).

Substituting the expansion (4.1) for ϕ in equations (3.3), expanding $\phi_z(r,\epsilon)$ using Taylor's theorem and equating corresponding order terms, we get

$$\left.\begin{aligned} &\text{(i)}\quad \nabla^2\phi_0 = \nabla^2\phi_1 = \nabla^2\phi_2 = 0 \\ &\text{(ii)}\quad \phi_{0z}(r,0) = -1\,,\ \phi_{1z}(r,0) = 0\,,\ \phi_{2z}(r,0) = -\phi_{0zz}(r,0)\ ;\ r<1 \\ &\text{(iii)}\quad \phi_0(r,0) = \phi_1(r,0) = \phi_2(r,0) = 0\ ;\ r>1\,. \end{aligned}\right\} \qquad (4.2)$$

The problem for ϕ_0 is the same as our first approximation to ϕ in section 3; its solution is given by (3.6).

Since the $\epsilon\log\epsilon$ term in (3.19) behaves like $\delta^{-1/2}$ on the disc, it is reasonable to expect $\phi_1(r,0) \sim \delta^{-1/2}$ near the edge of the disc. Thus we are led to suppose that

$$\phi_1(r,0) = \frac{A}{(1-r^2)^{1/2}}\ ;\quad r<1\,, \qquad (4.3)$$

where A is a constant to be determined. Whether this is actually the case is to be seen, after finding $\phi_1(r,z)$

on this supposition and showing that it satisfies the boundary condition (ii) for ϕ_1 in (4.2). Procedure is the same as for finding ϕ_0 [see (3.5) → (3.6)], and in this case

$$A(s) = A\int_0^1 \frac{J_0(sr).r}{(1-r^2)^{1/2}}\,dr$$

$$= A\,\frac{\sin s}{s} \qquad \text{(Watson, 1944), and}$$

$$\phi_1(r,z) = A\int_0^\infty J_0(rs)\,\sin s\; e^{-sz}\,ds\,, \qquad z>0\,. \tag{4.4}$$

Using properties of Bessel Functions (Watson, 1944) it can be verified that $\phi_{1z}(r,0) = 0$. Thus (4.4) is a solution of the problem for ϕ_1 in (4.2).

The problem for ϕ_2

Consider the function $\hat{\phi}(r,z)$ satisfying

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2\hat{\phi} = 0 \\ &\text{(ii)} \quad \hat{\phi}_z(r,0) = -\phi_0(r,0) \quad , \text{ when } r<1 \\ &\text{(iii)} \quad \hat{\phi}(r,0) = 0\,, \text{ when } r>1. \end{aligned}\right\} \tag{4.5}$$

The problem (4.5) for $\hat{\phi}$ is equivalent to the problem for ϕ_2 in (4.2), if we write $\phi_2 = \hat{\phi}_{zz}$; for

$$\text{(i)} \quad \nabla^2\hat{\phi} = 0 \Rightarrow \nabla^2\phi_2 = \nabla^2(\hat{\phi}_{zz}) = \frac{\partial^2}{\partial z^2}(\nabla^2\hat{\phi}) = 0$$

$$\text{(ii)} \quad \phi_{2z}(r,z) = \frac{\partial}{\partial z}(\hat{\phi}_{zz}) = \frac{\partial}{\partial z}\left[-\frac{\partial^2\hat{\phi}}{\partial r^2} - \frac{1}{r}\frac{\partial\hat{\phi}}{\partial r}\right] = -\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\hat{\phi}_z\,,$$

and in particular when $z = 0$, $r < 1$

$$\hat{\phi}_z(r,0) = -\phi_0(r,0) \Rightarrow \phi_{2z}(r,0) = -\phi_{0zz}(r,0)$$

(iii) $$\phi_2(r,z) = \hat{\phi}_{zz} = -\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\hat{\phi},$$

and in particular when $z = 0$, $r > 1$

$$\hat{\phi}(r,0) = 0 \Rightarrow \phi_2(r,0) = 0.$$

The solution of (4.5), like that of (3.3) with $\epsilon = 0$, is given by (3.4), $V(u)$ in this case being

$$(\hat{\phi}_z)_{disc} = -\phi_0 = -\frac{2}{\pi}(1-u^2)^{\frac{1}{2}}, \quad \text{by (4.5) and (3.5).}$$

By successive integration it can be shown that

$$S(t) = -\frac{1}{\pi^2}\left[2t + (1-t^2)\log\left\{\frac{1+t}{1-t}\right\}\right], \quad \text{and}$$

$$\hat{\phi}(r,0) = \frac{1}{\pi}\phi_0(r,0) + \frac{1}{\pi^2}\hat{\hat{\phi}}(r,0), \quad \text{where}$$

$$\hat{\hat{\phi}}(r,0) = \int_r^1 \frac{(1-t^2)}{(t^2-r^2)^{\frac{1}{2}}}\log\left\{\frac{1+t}{1-t}\right\}dt. \tag{4.6}$$

Since $\phi_2 = \hat{\phi}_{zz}$, it follows that

$$\phi_2(r,0) = \frac{1}{\pi}\phi_{0zz}(r,0) - \frac{1}{\pi^2}\left\{\hat{\hat{\phi}}_{rr}(r,0) + \frac{1}{r}\hat{\hat{\phi}}_r(r,0)\right\} \tag{4.7}$$

is a solution of the problem for ϕ_2, for points on disc.

Eigen Solutions for ϕ_2

Any solutions of the 'reduced' equations:

$$\left.\begin{aligned} &(i) && \nabla^2 \phi_2 = 0 \\ &(ii) && \phi_{2z}(r,0) = 0, \quad \text{when} \quad r<1 \\ &(iii) && \phi_2(r,0) = 0, \quad \text{when} \quad r>1 \end{aligned}\right\} \qquad (4.8)$$

have to be added to (4.7).

Consider the function $f(r,z) = \phi_{ozz}(r,z)$, where ϕ_o is the potential given by (3.6).

$$(i) \quad \nabla^2 f = \nabla^2(\phi_{ozz}) = \frac{\partial^2}{\partial z^2}(\nabla^2 \phi_o) = 0$$

$$(ii) \quad f_z(r,z) = \frac{\partial}{\partial z}(\phi_{ozz}) = \frac{\partial}{\partial z}\left[-\frac{\partial^2 \phi_o}{\partial r^2} - \frac{1}{r}\frac{\partial \phi_o}{\partial r}\right] = -\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\phi_{oz}$$

and when $z = o$, $r<1$: $\phi_{oz}(r,o) = -1 \Rightarrow f_z(r,o) = 0$

$$(iii) \quad f(r,z) = \phi_{ozz} = -\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\phi_o$$

and when $z = o$, $r>1$: $\phi_o(r,o) = 0 \Rightarrow f(r,o) = 0$.

Thus, $f(r,o) = \phi_{ozz}(r,o)$ is a solution of (4.8), for points on the disc. It also appears in the expression (4.7) for $\phi_2(r,o)$. However, its behaviour near the edge is $\sim \delta^{-3/2}$, incompatible with the fact that $\in \phi_2$, being the third term in the expansion (4.1) for ϕ, should be much less than $\phi_o \sim \delta^{1/2}$. Therefore we remove this $\delta^{-3/2}$ singularity by adding the eigen solution $-\frac{1}{\pi}\phi_{ozz}(r,o)$ to the expression (4.7). [As will be shown later (4.14), near edge,

the leading term in $\hat{\hat{\phi}}_{rr}(r,o)$ {of (4.7)} is of order $\delta^{-\frac{1}{2}} \log \delta$, which is admissible for ϕ_2, since $\epsilon \ll \delta$.]

Since the set (4.8) is similar to the problem for ϕ_1 in (4.2), $\phi_1(r,o)$ (4.3) is itself a solution of (4.8). Also $\phi_1(r,o) \sim \delta^{-\frac{1}{2}}$ near edge, and is admissible for ϕ_2. So we add $C\,\phi_1(r,o)$ to (4.7), where C is a constant to be determined.

From (4.7) and above, the 'corrected' ϕ_2 is

$$\phi_2(r,o) = -\frac{1}{\pi^2}\left\{ \hat{\hat{\phi}}_{rr}(r,o) + \frac{1}{r}\hat{\hat{\phi}}_r(r,o) \right\} + C\,\phi_1(r,o), \qquad (4.9)$$

$\hat{\hat{\phi}}(r,o)$ being given by (4.6).

Expansions for ϕ_o, ϕ_1, ϕ_2 (near edge)

It is difficult to deduce a general expression for $\phi_2(r,z)$ from (4.9), as $\hat{\hat{\phi}}$ occurs in the form of an integral (4.6). Near the edge of the disc, however, it is possible to evaluate $\hat{\hat{\phi}}(r,o)$ and hence $\phi_2(r,o)$ in the form of an expansion, for small δ.

By (3.5) and (4.3), this kind of expansions for $\phi_o(r,o)$ and $\phi_1(r,o)$ are :

$$\phi_o(r,o) \sim \frac{2}{\pi}\cdot 2^{\frac{1}{2}}\left\{ \delta^{\frac{1}{2}} - \frac{1}{4}\delta^{3/2} + O(\delta^{5/2}) \right\}, \quad \delta \to o \qquad (4.10)$$

$$\phi_1(r,o) \sim \frac{A}{2}\cdot 2^{\frac{1}{2}}\left\{ \delta^{-\frac{1}{2}} + \frac{1}{4}\delta^{\frac{1}{2}} + O(\delta^{3/2}) \right\}, \quad \delta \to o\,. \qquad (4.11)$$

Making the substitution $t = 1 - \delta p$, $\delta = 1 - r$, in (4.6) and expanding for small δ, we get

$$\hat{\hat{\phi}}(r,0) \sim -2^{1/2}(\delta^{3/2}\log\delta)\int_0^1 \frac{p\,dp}{(1-p)^{1/2}} + 2^{1/2}(\delta^{3/2})\int_0^1 \frac{p\log(\frac{2}{p})\,dp}{(1-p)^{1/2}}$$

$$-2^{1/2}(\delta^{5/2}\log\delta)\int_0^1 \tfrac{1}{4}p(1-p)^{1/2}\,dp$$

$$+2^{1/2}(\delta^{5/2})\int_0^1 \left\{\tfrac{1}{4}p(1-p)^{1/2}\log(\tfrac{2}{p}) - \tfrac{1}{2}\cdot\frac{p^2}{(1-p)^{1/2}}\right\}dp$$

$$+O(\delta^{7/2}\log\delta), \quad \delta\to 0.$$

Evaluating the integrals involved, we get

$$\hat{\hat{\phi}}(r,0) \sim -\tfrac{4}{3}\cdot 2^{1/2}\delta^{3/2}\log\delta + \tfrac{4}{3}2^{1/2}(\tfrac{5}{3}-\log 2)\delta^{3/2} - \tfrac{1}{15}\cdot 2^{1/2}\delta^{5/2}\log\delta$$

$$-\tfrac{1}{15}\cdot 2^{1/2}(\tfrac{89}{15}+\log 2)\delta^{5/2} + O(\delta^{7/2}\log\delta). \qquad (4.12)$$

Hence

$$\hat{\hat{\phi}}_r(r,0) \sim 2^{1/2}\Big[2\delta^{1/2}\log\delta - 2(1-\log 2)\delta^{1/2} + \tfrac{1}{6}\delta^{3/2}\log\delta$$

$$+\tfrac{1}{6}(\tfrac{19}{3}+\log 2)\delta^{3/2} + O(\delta^{5/2}\log\delta)\Big] \qquad (4.13)$$

and

$$\hat{\hat{\phi}}_{rr}(r,0) \sim -2^{1/2}\Big[\delta^{-1/2}\log\delta + (1+\log 2)\delta^{-1/2} + \tfrac{1}{4}\delta^{1/2}\log\delta$$

$$+\tfrac{1}{4}(7+\log 2)\delta^{1/2} + O(\delta^{3/2}\log\delta)\Big]. \qquad (4.14)$$

Substituting the expansions (4.11), (4.13) and (4.14) in the expression (4.9), we get

$$\phi_2(r,0) \sim \frac{2}{\pi^2}\left[\delta^{-\frac{1}{2}}\log\delta + (C'+1+\log 2)\delta^{-\frac{1}{2}} - \frac{7}{4}\delta^{\frac{1}{2}}\log\delta\right.$$

$$\left. + \frac{1}{4}(15+C'-7\log 2)\delta^{\frac{1}{2}} + O(\delta^{\frac{3}{2}}\log\delta)\right], \quad \delta\to 0, \quad (4.15)$$

where $\frac{2}{\pi^2}C' = AC$.

The expansions (4.10), (4.11) and (4.15) for potentials on the disc serve as a guide to finding an expansion for $\phi(r,z)$ near edge or $\phi(\delta,\theta)$. When written in terms of coordinates (δ,θ) (3.9) the equation (3.3)(i) becomes

$$\left[\frac{\partial^2}{\partial\delta^2} + \frac{1}{\delta}\cdot\frac{\partial}{\partial\delta} + \frac{1}{\delta^2}\cdot\frac{\partial^2}{\partial\theta^2} + \left(\cos\theta\frac{\partial}{\partial\delta} - \frac{\sin\theta}{\delta}\frac{\partial}{\partial\theta}\right) + O(\delta)\right]\phi = 0, \quad \delta\to 0, \quad (4.16)$$

which is the modified equation now satsfied by ϕ_0, ϕ_1 and ϕ_2. Remembering (3.7), it is reasonable to try solutions of (4.16) of the form

$$\phi_0 = \frac{2}{\pi}\cdot 2^{\frac{1}{2}}\left\{\delta^{\frac{1}{2}}f_1(\theta) - \delta\sin\theta + \delta^{\frac{3}{2}}f_2(\theta)\right\}$$

$$\phi_1 = \frac{A}{2}\cdot 2^{\frac{1}{2}}\left\{\delta^{-\frac{1}{2}}f_3(\theta) + \delta^{\frac{1}{2}}f_4(\theta)\right\}$$

$$\phi_2 = \frac{2^{\frac{1}{2}}}{\pi^2}\left[\delta^{-\frac{1}{2}}\left\{(\log\delta)f_5(\theta) + f_6(\theta)\right\} + \delta^{\frac{1}{2}}\left\{(\log\delta)f_7(\theta) + f_8(\theta)\right\}\right],$$

because, on the disc they are given by (4.10), (4.11) and (4.15) respectively. The result is a set of simple differential equations for $f_i(\theta)$, $i = 1,2,\cdots 8$. Also, since $\phi = 0$ when $\theta = 0$ [see (3.3)(iii)] $f_i(0) = 0$, and

using (4.10), (4.11) and (4.15) $f_i(\pi)$, $i = 1,2,\cdots 8$ are:

$1,\ -\frac{1}{4},\ 1,\ \frac{1}{4},\ 1,\ C'+1+\log 2,\ -\frac{7}{4},\ \frac{1}{4}(15+C'-7\log 2)$

respectively. These differential equations and boundary conditions yield:

$$f_1(\theta) = \sin\tfrac{\theta}{2}\ ,\quad f_2(\theta) = \tfrac{1}{2}\sin\tfrac{3\theta}{2} + \tfrac{1}{4}\sin\tfrac{\theta}{2}\ ,$$

$$f_3(\theta) = \sin\tfrac{\theta}{2}\ ,\quad f_4(\theta) = -\tfrac{1}{4}\sin\tfrac{3\theta}{2}\ ,$$

$$f_5(\theta) = \sin\tfrac{\theta}{2}\ ,\quad f_6(\theta) = (C'+1+\log 2)\sin\tfrac{\theta}{2} - \theta\cos\tfrac{\theta}{2}\ ,$$

$$f_7(\theta) = -2\sin\tfrac{\theta}{2} - \tfrac{1}{4}\sin\tfrac{3\theta}{2}\ ,$$

$$f_8(\theta) = (\tfrac{7}{2} - 2\log 2)\sin\tfrac{\theta}{2} - 2\theta\cos\tfrac{\theta}{2} - \tfrac{1}{4}(C'+1+\log 2)\sin\tfrac{3\theta}{2} + \tfrac{1}{4}\theta\cos\tfrac{3\theta}{2}.$$

Thus, we are able to find near edge expansions for ϕ_0, ϕ_1, ϕ_2, and hence using (4.1), an expansion for $\phi(\delta,\theta)$. By (3.2), it follows that the near edge expansion for the total outer potential Φ_1 is

$$\Phi_1(\delta,\theta) = \frac{2}{\pi}\cdot 2^{1/2}\left[\delta^{1/2}\sin\tfrac{\theta}{2} + \delta^{3/2}\left(\tfrac{1}{2}\sin\tfrac{3\theta}{2} + \tfrac{1}{4}\sin\tfrac{\theta}{2}\right)\right]$$

$$+(\epsilon\log\epsilon)\frac{A}{2}\cdot 2^{1/2}\left[\delta^{-1/2}\sin\tfrac{\theta}{2} - \tfrac{1}{4}\delta^{1/2}\sin\tfrac{3\theta}{2}\right]$$

$$+(\epsilon)\cdot\frac{2}{\pi^2}^{1/2}\left[\delta^{-1/2}\left\{\log\delta\,\sin\tfrac{\theta}{2} + (C'+1+\log 2)\sin\tfrac{\theta}{2} - \theta\cos\tfrac{\theta}{2}\right\}\right.$$

$$+\delta^{1/2}\left\{(-\log\delta)\left(2\sin\tfrac{\theta}{2} + \tfrac{1}{4}\sin\tfrac{3\theta}{2}\right) + \left(\tfrac{7}{2} - 2\log 2\right)\sin\tfrac{\theta}{2} - 2\theta\cos\tfrac{\theta}{2}\right.$$

$$\left.\left. - \tfrac{1}{4}(C'+1+\log 2)\sin\tfrac{3\theta}{2} + \tfrac{1}{4}\theta\cos\tfrac{3\theta}{2}\right\}\right], \quad \delta \to 0. \qquad (4.17)$$

(4.17) is the outer potential Φ_1 in terms of outer coordinates (δ,θ), expanded (for small δ) to order ϵ. When it is expressed in inner coordinates (R,θ) using $\delta = \epsilon R$, we find that the leading term is

$$\left(\frac{A}{2} + \frac{1}{\pi^2}\right) 2^{1/2} (\epsilon^{1/2} \log \epsilon) R^{-1/2} \sin\frac{\theta}{2} .$$

This contradicts the fact that the leading term should be

$$\Phi_1^0(R,\theta) = \frac{2}{\pi} \cdot 2^{1/2} \epsilon^{1/2} R^{1/2} \sin\frac{\theta}{2} \quad (\text{of } 3.13),$$

unless we choose the value of the constant A so as to cancel off the $\epsilon^{1/2}\log\epsilon$ term. Taking $A = -\frac{2}{\pi^2}$ and re-arranging terms, we get

$$\begin{aligned}\Phi_1(R,\theta) = {} & \frac{2^{3/2}}{\pi}(\epsilon^{1/2})\left[R^{1/2}\sin\frac{\theta}{2} + \frac{R^{-1/2}}{2\pi}\left\{\log R \sin\frac{\theta}{2} - \theta\cos\frac{\theta}{2} + (C'+1+\log 2)\sin\frac{\theta}{2}\right\}\right] \\ & - (\epsilon^{3/2}\log\epsilon)\,\frac{2^{3/2}}{\pi^2} R^{1/2}\sin\frac{\theta}{2} \\ & + \epsilon^{3/2}\left[\frac{2^{1/2}}{\pi}\left\{R^{3/2}\sin\frac{3\theta}{2} - \frac{2}{\pi}R^{1/2}\left(\log R\sin\frac{\theta}{2} + \theta\cos\frac{\theta}{2}\right) + \frac{R^{1/2}}{\pi}\left(\frac{7}{2} - 2\log 2\right)\sin\frac{\theta}{2}\right\}\right. \\ & \left. + \frac{2^{1/2}}{2\pi}\left\{R^{3/2}\sin\frac{\theta}{2} - \frac{R^{1/2}}{2\pi}\left(\log R\sin\frac{3\theta}{2} - \theta\cos\frac{3\theta}{2}\right) - \frac{R^{1/2}}{2\pi}(C'+1+\log 2)\sin\frac{3\theta}{2}\right\}\right] . \end{aligned} \tag{4.18}$$

The constant C' will be determined in Section 5, by matching (4.18) with the inner expansion.

4. (b) Inner Expansion

To solve the problem (3.12) up to a higher order in ϵ, we can now use $\Phi_1(R,\theta)$ (of 4.18) in the boundary condition (iii). This suggests an expansion of the form

$$\psi \sim \epsilon^{1/2}\psi_0 + (\epsilon^{3/2}\log\epsilon)\,\psi_1 + \epsilon^{3/2}\psi_2 \qquad (4.19)$$

for the inner potential ψ. Substituting (4.19) and (4.18) in (3.12) and equating corresponding order terms, we get

$$\left.\begin{aligned}
&\text{(i)}\quad \nabla^2\psi_0 = \nabla^2\psi_1 = 0,\ \nabla^2\psi_2 + \frac{\partial\psi_0}{\partial X} = 0\\
&\text{(ii)}\quad \frac{\partial\psi_0}{\partial n} = \frac{\partial\psi_1}{\partial n} = \frac{\partial\psi_2}{\partial n} = 0, \quad \text{on AB, BC, CD}\\
&\text{(iii)}\ \psi_0 \sim \frac{2^{3/2}}{\pi}\left[R^{1/2}\sin\frac{\theta}{2} + \frac{R^{-1/2}}{2\pi}\left\{\log R\sin\frac{\theta}{2} - \theta\cos\frac{\theta}{2} + (C'+1+\log 2)\sin\frac{\theta}{2}\right\}\right],\\
&\qquad \psi_1 \sim -\frac{2^{3/2}}{\pi^2}R^{1/2}\sin\frac{\theta}{2},\\
&\qquad \psi_2 \sim \frac{2^{1/2}}{\pi}\left\{R^{3/2}\sin\frac{3\theta}{2} - \frac{2}{\pi}R^{1/2}\left(\log R\sin\frac{\theta}{2} + \theta\cos\frac{\theta}{2}\right) + \frac{R^{1/2}}{\pi}\left(\frac{7}{2} - 2\log 2\right)\sin\frac{\theta}{2}\right\}\\
&\qquad\quad + \frac{2^{1/2}}{2\pi}\left\{R^{3/2}\sin\frac{3\theta}{2} - \frac{R^{1/2}}{2\pi}\left(\log R\sin\frac{3\theta}{2} - \theta\cos\frac{3\theta}{2}\right) - \frac{R^{1/2}}{2\pi}(C'+1+\log 2)\sin\frac{3\theta}{2}\right\},\ \text{as } R\to\infty.
\end{aligned}\right\} \qquad (4.20)$$

The problem for ψ_0 in (4.20) is the same as (3.14), and its solution is given by (3.17). The problem for ψ_1 is also the same as (3.14), apart from a constant factor $\left(-\frac{1}{\pi}\right)$; hence its solution is

$$\psi_1 = \frac{4}{\pi^{5/2}}\,Re(w). \qquad (4.21)$$

Making use of ψ_0(3.17) and the transformation equations (3.15) and (3.18), the problem for ψ_2 can be transformed to one in the w-plane:

$$\left.\begin{aligned}
&\text{(i)}\quad \nabla_{uv}^2 \psi_2 = -\frac{16}{\pi^{5/2}} \mathrm{Re}\left\{(w^2-1)^{1/2}\right\}, \quad \text{where } \nabla_{uv}^2 = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \\
&\text{(ii)}\quad \frac{\partial \psi_2}{\partial v} = 0 \quad , \quad \text{when } v = 0 \\
&\text{(iii)}\quad \psi_2 \sim \frac{4}{\pi^{5/2}} \mathrm{Re}(w^3) + \frac{2}{\pi^{5/2}}\left(5 + \log\frac{\pi^2}{2}\right)\mathrm{Re}(-w) \\
&\qquad\qquad + \frac{1}{\pi^{5/2}} \mathrm{Re}\left\{w(\log w^2 - i\pi)\right\} + \frac{1}{\pi^{3/2}} \mathrm{Re}(\bar{w} Z), \quad \text{at } \infty.
\end{aligned}\right\} \quad (4.22)$$

First, to find a particular solution of (4.22)(i), write

$$\left.\begin{aligned}
&\psi_2 = \psi_p + \hat{\psi} \quad , \quad \text{where } \nabla_{uv}^2 \psi_p = -\frac{16}{\pi^{5/2}} \mathrm{Re}\left\{(w^2-1)^{1/2}\right\}. \\
&\text{Since } \nabla_{uv}^2 = 4\frac{\partial^2}{\partial \bar{w} \partial w}, \ \psi_p = \frac{1}{\pi^{3/2}} \mathrm{Re}(\bar{w} Z).
\end{aligned}\right\} \quad (4.23)$$

Using (4.23) and (3.15), we get

$$\left.\begin{aligned}
&\left(\frac{\partial \psi_p}{\partial v}\right)_{v=0} = F_1(u) \quad , \quad \text{where} \\
&F_1(u) = \frac{1}{\pi^{3/2}}, \quad u \le -1 \\
&\qquad = \frac{1}{\pi^{3/2}}\left[\frac{2}{\pi} u (1-u^2)^{1/2} + \frac{2}{\pi}\cos^{-1} u - 1\right], \quad -1 \le u \le 1 \\
&\qquad = -\frac{1}{\pi^{3/2}}, \quad u \ge 1.
\end{aligned}\right\} \quad (4.24)$$

By (4.22), (4.23) and (4.24), the problem for $\hat{\psi}$ is

$$\left.\begin{aligned} &\text{(i)}\quad \nabla^2 \hat{\psi} = 0 \\ &\text{(ii)}\quad \frac{\partial \hat{\psi}}{\partial v} = -F_1(u), \quad \text{when } v = 0 \\ &\text{(iii)}\quad \hat{\psi} \sim \frac{4}{\pi^{5/2}} \mathrm{Re}(w^3) + \frac{2}{\pi^{5/2}}\left(5 + \log\frac{\pi^2}{2}\right)\mathrm{Re}(-w) \\ &\qquad\qquad + \frac{1}{\pi^{5/2}} \mathrm{Re}\left\{w(\log w^2 - i\pi)\right\}, \quad \text{at } \infty. \end{aligned}\right\} \tag{4.25}$$

Next, to find a particular solution of equations (4.25) (i) and (ii), write

$$\left.\begin{aligned} &\hat{\psi} = \hat{\psi}_p + \hat{\hat{\psi}}, \quad \text{where} \\ &\nabla^2 \hat{\psi}_p = 0 \quad \text{and} \quad \left(\frac{\partial \hat{\psi}_p}{\partial v}\right)_{v=0} = -F_1(u). \end{aligned}\right\} \tag{4.26}$$

Since $F_1(u)$ has a constant $\left(\mp \frac{1}{\pi^{3/2}}\right)$ behaviour as $u \to \pm\infty$, define a new function

$$F_2(u) = -\frac{2}{\pi^{5/2}}\left[\tan^{-1}\left(\frac{u}{\alpha}\right) + \frac{\alpha u}{u^2+\alpha^2}\right], (\alpha \text{ a real number}), \tag{4.27}$$

which has the same constant behaviour as $u \to \pm\infty$, so that $F_1(u) - F_2(u) \to 0$ as $u \to \pm\infty$.

$$\left.\begin{aligned} &\text{Let}\quad \hat{\psi}_p = \psi_{p1} + \psi_{p2}, \quad \text{where} \\ &\nabla^2 \psi_{p1} = 0 \quad \text{and} \quad \left(\frac{\partial \psi_{p1}}{\partial v}\right)_{v=0} = -F_2(u). \\ &\text{Now,}\qquad \psi_{p1} = \frac{1}{\pi^{5/2}} \mathrm{Re}\left[w\left\{\log(w+i\alpha)^2 - i\pi\right\}\right]. \end{aligned}\right\} \tag{4.28}$$

By (4.26) and (4.28), the problem for ψ_{p2} is

$$\nabla^2 \psi_{p2} = 0 \ , \quad \left(\frac{\partial \psi_{p2}}{\partial v}\right)_{v=0} = -F_1(u) + F_2(u).$$

It is difficult to solve this problem near origin. However, for large $|w|$: $\psi_{p2} \sim 0$, <u>for</u>

(4.24) and (4.27) $\Rightarrow$ $F_1(u) - F_2(u) \sim \frac{1}{u^3}$ for $|u| > 1$, and as $u \to \pm\infty$, $\frac{1}{u^3} \to 0$ fast enough to ensure that $\psi_{p2} \sim 0$.

Thus, by (4.28)

$$\left.\begin{aligned} \hat{\psi}_p &\sim \frac{1}{\pi^{5/2}} \mathrm{Re}\left[w\left\{\log(w+i\alpha)^2 - i\pi\right\}\right] \quad , \text{ for large } |w| \\ &\to \frac{1}{\pi^{5/2}} \mathrm{Re}\left[w\left\{\log w^2 - i\pi\right\}\right] \ , \ |w| \to \infty. \end{aligned}\right\} \quad (4.29)$$

By (4.26), (4.25) and (4.29), the problem for $\hat{\hat{\psi}}$ is

(i) $\nabla^2 \hat{\hat{\psi}} = 0$

(ii) $\frac{\partial \hat{\hat{\psi}}}{\partial v} = 0$, when $v = 0$

(iii) $\hat{\hat{\psi}} \sim \frac{4}{\pi^{5/2}} \mathrm{Re}(w^3) + \frac{2}{\pi^{5/2}}\left(5 + \log\frac{\pi^2}{2}\right) \mathrm{Re}(-w)$, $|w| \to \infty$.

Solution is

$$\hat{\hat{\psi}} = \frac{4}{\pi^{5/2}} \mathrm{Re}\,(w^3) + \frac{2}{\pi^{5/2}}\left(5 + \log\frac{\pi^2}{2}\right) \mathrm{Re}\,(-w). \quad (4.30)$$

By (4.23), (4.26), (4.29) and (4.30), for large $|w|$:

$$\psi_2 \sim \frac{1}{\pi^{3/2}} \mathrm{Re}\,(\bar{w} Z) + \frac{1}{\pi^{5/2}} \mathrm{Re}\left[w\left\{\log(w+i\alpha)^2 - i\pi\right\}\right]$$

$$+\frac{4}{\pi^{5/2}} \mathrm{Re}\,(w^3) + \frac{2}{\pi^{5/2}}\left(5 + \log\frac{\pi^2}{2}\right) \mathrm{Re}\,(-w). \quad (4.31)$$

Expanding (3.17), (4.21) and (4.31) for large $|Z|$ using (3.18), writing $Z = R e^{i\theta}$ and substituting in (4.19), we get

$$\psi(R,\theta) \sim \epsilon^{1/2}\,\frac{2^{3/2}}{\pi}\left[R^{1/2}\sin\frac{\theta}{2} + \frac{R^{-1/2}}{2\pi}\left\{(\log R)\sin\frac{\theta}{2} - \theta\cos\frac{\theta}{2} + (1+\log 2\pi)\sin\frac{\theta}{2}\right\}\right]$$

$$+(\epsilon^{3/2}\log\epsilon)\left(-\frac{2^{3/2}}{\pi^2}\right)R^{1/2}\sin\frac{\theta}{2}$$

$$+\epsilon^{3/2}\left[\frac{2^{1/2}}{2\pi}\left\{R^{3/2}\sin\frac{\theta}{2} - \frac{R^{1/2}}{2\pi}\left(\log R\sin\frac{3\theta}{2} - \theta\cos\frac{3\theta}{2}\right) - \frac{R^{1/2}}{2\pi}(1+\log 2\pi)\sin\frac{3\theta}{2}\right\}\right.$$

$$\left.+\frac{2^{1/2}}{\pi}\left\{R^{3/2}\sin\frac{3\theta}{2} - \frac{2}{\pi}R^{1/2}\left(\log R\sin\frac{\theta}{2} + \theta\cos\frac{\theta}{2}\right) + \frac{R^{1/2}}{\pi}\left(\frac{7}{2} - 2\log 2\right)\sin\frac{\theta}{2}\right\}\right], \quad R\to\infty. \tag{4.32}$$

This is the inner potential ψ, in terms of inner coordinates (R,θ), expanded (for large R) to order $\epsilon^{3/2}$.

5. Matching of outer and inner expansions

When (4.32) is witten in terms of outer coordinates (δ,θ) and re-arranged, we get

$$\psi(\delta,\theta) \sim \frac{2^{3/2}}{\pi}\left[\delta^{1/2}\sin\frac{\theta}{2} + \delta^{3/2}\left(\frac{1}{2}\sin\frac{3\theta}{2} + \frac{1}{4}\sin\frac{\theta}{2}\right)\right]$$

$$- \frac{2^{1/2}}{\pi^2}(\epsilon\log\epsilon)\left[\delta^{-1/2}\sin\frac{\theta}{2} - \frac{1}{4}\delta^{1/2}\sin\frac{3\theta}{2}\right]$$

$$+ \frac{2^{1/2}}{\pi^2}(\epsilon)\left[\delta^{-1/2}\left\{\left(\log\delta\sin\frac{\theta}{2} - \theta\cos\frac{\theta}{2}\right) + \left(1+\log 2\pi\right)\sin\frac{\theta}{2}\right\}\right.$$

$$+ \delta^{1/2}\left\{(-2)\left(\log\delta\sin\frac{\theta}{2} + \theta\cos\frac{\theta}{2}\right) - \frac{1}{4}\left(\log\delta\sin\frac{3\theta}{2} - \theta\cos\frac{3\theta}{2}\right)\right.$$

$$\left.\left. + \left(\frac{7}{2} - 2\log 2\right)\sin\frac{\theta}{2} - \frac{1}{4}(1+\log 2\pi)\sin\frac{3\theta}{2}\right\}\right] . \qquad (5.1)$$

Thus (5.1) gives the inner expansion to order $\epsilon^{3/2}$, expressed in outer coordinates (δ,θ) and expanded to order ϵ. Also (4.18) gives the outer expansion to order ϵ, expressed in inner coordinates (R,θ) and expanded to order $\epsilon^{3/2}$. By matching principle of Section 1, these two expansions must be identical. Comparing (5.1) with (4.18) using the identity $\delta = \epsilon R$, it is seen that the constant $C' = \log\pi$.

6. The far field potential

By (2.1)(iii), the scattered potential

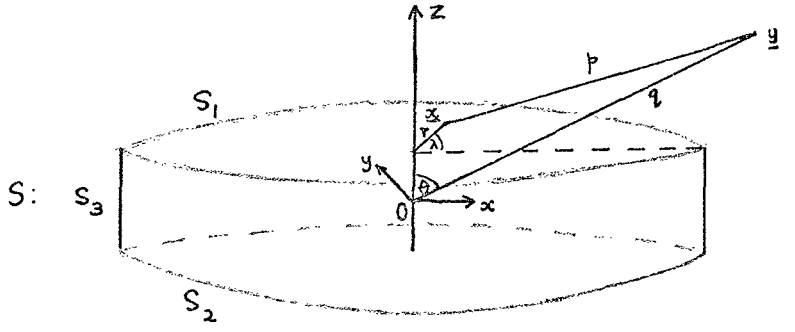
$$\Phi_s(\underline{y}) = \frac{1}{4\pi}\int_S \left(\frac{ik}{\rho} - \frac{1}{\rho^2}\right) e^{ik\rho}\,\frac{\partial \rho}{\partial n}\,\Phi(\underline{x})\,d\underline{x}\,, \quad \rho = |\underline{x}-\underline{y}|. \tag{6.1}$$

Let $\underline{x} = (r\cos\lambda,\ r\sin\lambda,\ z)$

$\underline{y} = (q\sin\theta,\ 0,\ q\cos\theta)$ (see Fig.4),

with respect to a rectangular Cartesian coordinate system (x,y,z) based at the centre of the disc.

Fig.4



Then $\rho^2 = q^2 + r^2 - 2qr\sin\theta\cos\lambda - 2qz\cos\theta + z^2$,

$$\frac{\partial \rho}{\partial z} = \frac{-q\cos\theta + z}{\rho} \qquad \text{and} \qquad \frac{\partial \rho}{\partial r} = \frac{r - q\sin\theta\cos\lambda}{\rho}.$$

It follows that, when p and q are large,

$$\left.\begin{aligned} &p \sim q - (r\sin\theta\cos\lambda + z\cos\theta) \\ &\left[S_1\colon z = \epsilon,\ r\le 1,\quad S_2\colon z = -\epsilon,\ r\le 1,\quad S_3\colon r = 1,\ |z| \le \epsilon\right], \text{ and} \\ &\frac{\partial p}{\partial n} \sim -\cos\theta,\ \cos\theta,\ -\sin\theta\cos\lambda, \text{ on } S_1, S_2, S_3 \text{ resp.} \end{aligned}\right\} \tag{6.2}$$

By (6.1) and (6.2), for large q

$$\left.\begin{aligned} \Phi_s &\sim f(\theta)\ \frac{e^{ikq}}{q} \quad , \quad \text{where} \\ f(\theta) &= \frac{ik}{4\pi}\int_S e^{-ik(r\sin\theta\cos\lambda + z\cos\theta)}\ \frac{\partial \phi}{\partial n}\ \Phi\, ds\ . \end{aligned}\right\} \tag{6.3}$$

Expanding $e^{-ik(r\sin\theta\cos\lambda + z\cos\theta)}$ and Φ in the form of the series (2.2), substituting in (6.3) and arranging terms in ascending order of (ik), we get

$$f(\theta) = \frac{ik}{4\pi}\Big[\int_S \Phi_0 \frac{\partial \phi}{\partial n}\, ds + (ik)\int_S \Big\{\Phi_1 - \Phi_0(r\sin\theta\cos\lambda + z\cos\theta)\Big\}\frac{\partial \phi}{\partial n}\, ds$$

$$+\frac{(ik)^2}{2}\int_S \Big\{\Phi_2 - 2\Phi_1(r\sin\theta\cos\lambda + z\cos\theta) + \Phi_0(r\sin\theta\cos\lambda + z\cos\theta)^2\Big\}\frac{\partial \phi}{\partial n}\, ds + O(k^3)\Big]$$

Making use of the results (6.2), the fact that $\Phi_0 \equiv 1$ and the odd symmetry of Φ_1 and the even symmetry of Φ_2 about the plane $z = o$, it can be shown that the first and the last integrals vanish, and that

$$f(\theta) = k^2\left[(\cos\theta)\int_0^1 r\,\Phi_1(r,\epsilon)\,dr - \tfrac{1}{2}\epsilon\right] + O(k^4)\ . \tag{6.4}$$

To evaluate the integral appearing in (6.4), its range is divided into two parts, as there are two different expressions for the potential Φ_1 in the outer and the inner regions. Let $r = \alpha$, $\epsilon \ll \alpha \ll 1$, be the dividing point.

By (3.2), in the outer region

$$\Phi_1(r,\epsilon) = \epsilon + \phi(r,\epsilon) = \phi(r,o) + O(\epsilon^2)\ , \quad \because\ \phi_z(r,o) = -1 + O(\epsilon).$$

In the inner region $\Phi_1 = \Psi$.

Thus $$\int_0^1 r\,\Phi_1(r,\epsilon)\,dr = \int_0^{1-\alpha} r\,\phi(r,0)\,dr + \int_{1-\alpha}^1 r\,\psi\,dr. \qquad (6.5)$$

Using (3.5), (4.3), (4.9) and the expansion (4.1):

$$\int_0^{1-\alpha} r\,\phi(r,0)\,dr = \frac{2}{3\pi} - (\epsilon\log\epsilon)\frac{2}{\pi^2} + (\epsilon)\frac{\log\pi^2}{\pi^2}$$

$$-\frac{2}{3\pi}(2\alpha)^{3/2} + \frac{2^{1/2}}{\pi}\alpha^{5/2} + O(\alpha^{7/2}) + O\left[(\epsilon\log\epsilon)\alpha^{1/2}\right] \qquad (6.6)$$

Using (4.19) and the inner coordinates (X,Y) (of 3.11):

$$\int_{1-\alpha}^1 r\,\psi\,dr = \epsilon^{3/2}\int_{-\alpha/\epsilon}^0 \psi_0(X,1)\,dX + (\epsilon^{5/2}\log\epsilon)\int_{-\alpha/\epsilon}^0 \psi_1(X,1)\,dX$$

$$+\epsilon^{5/2}\int_{-\alpha/\epsilon}^0 \left\{\psi_2(X,1) + X\,\psi_0(X,1)\right\}dX .$$

It is easier to evaluate these integrals in the w-plane, as ψ_0, ψ_1, ψ_2 are known explicitly there. For points on A B (Fig.2) $Z = X + i, (X < 0)$ and for corresponding points on A′ B′ (Fig.3) $w = -\Omega, (\Omega > 1)$. If we make these substitutions for Z and w, the transformation (3.15) becomes

$$\left.\begin{aligned} \frac{dX}{d\Omega} &= -\frac{4}{\pi}(\Omega^2-1)^{1/2} \\ \text{or}\quad X &= -\frac{2}{\pi}\left[\Omega(\Omega^2-1)^{1/2} - \log\left\{\Omega + (\Omega^2-1)^{1/2}\right\}\right]. \end{aligned}\right\} \qquad (6.7)$$

In particular, when $X = -\alpha/\epsilon$, the corresponding $\Omega = \Omega_1$ is given by $\Omega_1^2 \simeq \frac{\pi\alpha}{2\epsilon}$.

Using (3.17), (4.21), (4.31) and (6.7), we get

$$\epsilon^{3/2}\int_{-\alpha/\epsilon}^{0}\psi_0\,dX = \epsilon^{3/2}\int_{\Omega_1}^{1}\psi_0(\Omega)\frac{dX}{d\Omega}\,d\Omega \simeq \frac{2^{5/2}}{3\pi}\alpha^{3/2}\left(1-\frac{2\epsilon}{\pi\alpha}\right)^{3/2},$$

$$(\epsilon^{5/2}\log\epsilon)\int_{-\alpha/\epsilon}^{0}\psi_1\,dX = (\epsilon^{5/2}\log\epsilon)\int_{\Omega_1}^{1}\psi_1(\Omega)\frac{dX}{d\Omega}\,d\Omega \simeq -(\epsilon\log\epsilon)\frac{2^{5/2}}{3\pi^2}\alpha^{3/2}\left(1-\frac{2\epsilon}{\pi\alpha}\right)^{3/2},$$

$$\epsilon^{5/2}\int_{-\alpha/\epsilon}^{0}\left\{X\psi_0+\psi_2\right\}dX = \epsilon^{5/2}\int_{\Omega_1}^{1}\left\{X(\Omega)\,\psi_0(\Omega)+\psi_2(\Omega)\right\}\frac{dX}{d\Omega}\,d\Omega$$

$$\simeq -\frac{2^{1/2}}{\pi}\alpha^{5/2}+O\left[(\epsilon\log\epsilon)\,\alpha^{3/2}\right].$$

Hence

$$\int_{1-\alpha}^{1} r\psi\,dr \simeq \frac{2^{5/2}}{3\pi}\alpha^{3/2}-\frac{2^{1/2}}{\pi}\alpha^{5/2}+O\left[(\epsilon\log\epsilon)\,\alpha^{3/2}\right]. \tag{6.8}$$

Substituting (6.6) and (6.8) in (6.5), we get

$$\int_0^1 r\Phi_1(r,\epsilon)\,dr \simeq \frac{2}{3\pi}-(\epsilon\log\epsilon)\frac{2}{\pi^2}+(\epsilon)\frac{\log\pi^2}{\pi^2}+O\left[(\epsilon\log\epsilon)\alpha^{1/2}\right]+O(\alpha^{3/2}) \tag{6.9}$$

In the error, one leading term: $\epsilon|\log\epsilon|\,\alpha^{1/2} \ll \epsilon$, $\because \alpha \ll 1$. The other leading term: $\alpha^{3/2} \ll \epsilon$, only if $\alpha \ll \epsilon^{2/3}$. Since α is any number such that $\epsilon \ll \alpha \ll 1$, it may be so chosen that this condition is satsfied, and then the error becomes much smaller than the three terms independent of α.

Since the thrust on the disc is proportional to $\int_0^1 r\Phi_1\,dr$, (6.9) gives a measure of this thrust.

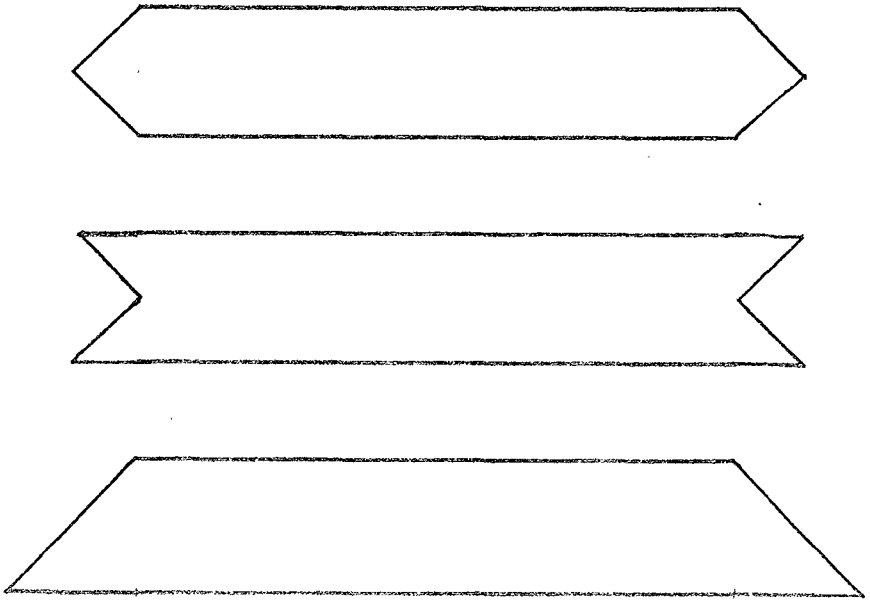
Substituting (6.9) in (6.4) and then (6.4) in (6.3), we get the far field scattered potential as

$$\left.\begin{aligned}&\Phi_s \sim \frac{e^{ikq}}{q}f(\theta), \quad kq \gg 1,\\ &\text{where}\quad f(\theta) = k^2\left[\left\{\frac{2}{3\pi}-(\epsilon\log\epsilon)\frac{2}{\pi^2}+(\epsilon)\frac{\log\pi^2}{\pi^2}\right\}\cos\theta-\frac{1}{2}\epsilon\right].\end{aligned}\right\} \tag{6.10}$$

7. Conclusion

In finding the outer potential in Section 4. (a), we do not pay any attention to the shape of the edge of the disc, except for assuming that the body is symmetrical about the z-axis and looks like a circular disc in the outer region. In other words, the outer expansion is valid for any symmetrical, circular disc - shaped body provided its variation, at the edge, from the circular disc is of order ϵ. For example, disc - shaped bodies with cross-sections seen in Fig.5 come under this category. Thus, the method used for the circular disc could be applied to several similar problems. However, only a few of these may be solved due to difficulty in finding a suitable conformal transformation in the inner region.

Fig.5



PART 3

TWO INFINITE STRIPS CHARGED TO UNIT POTENTIAL EACH AND PLACED CLOSE TO EACH OTHER

1. Introduction

The two-dimensional potential problem due to two equal parallel infinite strips charged to unit potential each and placed close to each other, the ratio of separation to breadth being ϵ ($\ll 1$), is here considered. Following Susann Shaw (1970), who treated the two disc problem, the 2-D space in this case is divided into three regions as follows:

I Exterior Region:- consisting of points 'far away' (i.e. at distances much larger than the separating distance) from the two edges and not in between the strips.

II Edge Region:- consisting of points 'close to' (i.e. at distances much less than half the breadth from) the two edges.

III Interior Region:- consisting of points far away (as in I) from the two edges and in between the strips.

Denoting the potentials in regions I, II and III by ϕ^{I}, ϕ^{II} and ϕ^{III} respectively, problems for these potentials are formulated, using coordinate systems defined appropriately as:

(i) Exterior and Interior coordinate systems based at the centre,

(ii) 'Outer edge' and 'Inner edge' coordinate systems based at the right hand side edge.

The fact that the region II overlaps with the near edge limits of regions I and III is made use of in the boundary condition at infinity for ϕ^{II}, which is obtained easily by the method of conformal transformations. As suggested by the behaviour of first approximations, ϕ^{I}, ϕ^{II} and ϕ^{III} are found in the form of expansions up to order ϵ, $\epsilon^{3/2}$ and $\epsilon^{3/2} e^{-\frac{\pi}{2\epsilon}}$ respectively. It is seen that an additional small factor $e^{-\frac{\pi}{2\epsilon}}$ (a function of ϵ) has to be introduced in the interior potential ϕ^{III} for the correct matching, near edge, with the edge potential ϕ^{II}. The unknown constants appearing in ϕ^{I} and ϕ^{III} are determined by matching expansions for ϕ^{I} and ϕ^{II} against each other, using the same matching principle as in part 2.

In the appendix, the 'correct' solution for a strip of very large but finite length is discussed.

2. Formulation and Approximate Solutions

For convenience of analysis choose the unit of length so that the breadth of the strips is 2 and their separation is 2ϵ. Define a Cartesian coordinate system (x,y), called 'exterior', so that the strips correspond to the line segments $y = \pm\epsilon$, $|x| \leq 1$. Then $\Phi(x,y)$, the potential at any point satisfies

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2\Phi = 0, \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\ &\text{(ii)} \quad \Phi(x, \pm\epsilon) = 1, \quad \text{when} \quad |x| < 1. \end{aligned}\right\} \qquad (2.1)$$

Since this is a 2-D problem, there is no solution of (2.1) which tends to zero at infinity. We have to take

$$\Phi \sim A \log r + B \quad (r^2 = x^2 + y^2) \quad \text{at} \quad \infty,$$

where A and B are constants to be determined. With this behaviour at ∞, the total charge $= -A/2$ (see Sec.5).

Because of $\log r$ behaviour at infinity, the trivial solution $\phi \equiv 1$ satisfies all the conditions (2.1). The problem is therefore not uniquely posed, for if ϕ is a solution, then so is $\Phi = \alpha\phi + (1 - \alpha)$. Thus we are free to choose A. So we take $A = 1$, find ϕ satisfying (2.1) and the condition $\phi \sim \log r + B$ at ∞, and get the general solution Φ by using $\Phi = \alpha\phi + (1-\alpha)$, where α is the scaling factor appropriate for two strips.

For a discussion as to which is the 'correct' solution for a strip of very large but finite length, see appendix.

By (2.1) and the above behaviour at infinity of the potential, the problem for the potential $\phi^I(x,y)$ in the exterior region satisfies

$$\left.\begin{aligned}
&\text{(i)} \quad \nabla^2 \phi^I = 0 \\
&\text{(ii)} \quad \phi^I(x,\pm\epsilon) = 1, \quad \text{when } |x|<1 \\
&\text{(iii)} \quad \frac{\partial \phi^I}{\partial y}(x,0) = 0, \quad \text{when } |x|>1 \\
&\qquad [\text{By even symmetry of } \phi^I \text{ about the x-axis}] \\
&\text{(iv)} \quad \phi^I \sim \log r + B \quad \text{at } \infty, \text{ where} \\
&\qquad B \text{ is a constant which may depend on } \epsilon.
\end{aligned}\right\} \quad (2.2)$$

Exterior Region: First Approximation

As a first approximation to the problem (2.2), replace ϵ by zero. Then (2.2) reduces to a problem due to a single strip and the potential, denoted by ϕ_o^I, now satisfies

$$\left.\begin{aligned}
&\text{(i)} \quad \nabla^2 \phi_o^I = 0 \\
&\text{(ii)} \quad \phi_o^I(x,0) = 1, \quad \text{when } |x|<1 \\
&\text{(iii)} \quad \frac{\partial \phi_o^I}{\partial y}(x,0) = 0, \quad \text{when } |x|>1 \\
&\text{(iv)} \quad \phi_o^I \sim \log r + B_o \text{ at } \infty, \text{ where } B_o \text{ is a const.}
\end{aligned}\right\} \quad (2.3)$$

An integral equation for ϕ_o^I can be obtained as

$$\phi_o^I(\underline{\xi}) = \int_\Gamma \left(G \frac{\partial \phi_o^I}{\partial n} - \phi_o^I \frac{\partial G}{\partial n} \right) d\underline{\xi}' , \quad (2.4)$$

where G is a suitable Green's function in 2-D,

$\Gamma\ (= \Gamma_1 + \Gamma_2 + \Gamma_3)$ is the contour shown in Fig.1,

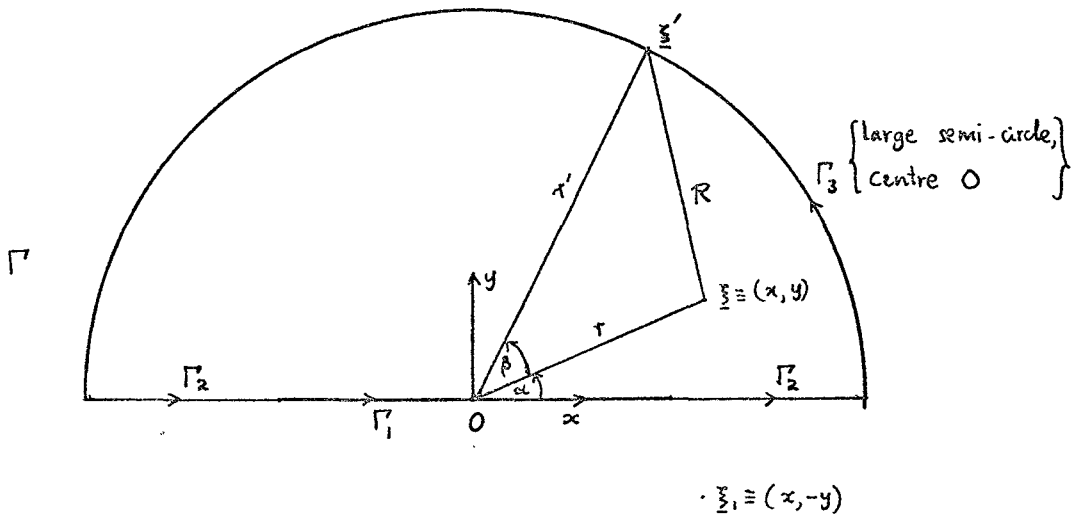
$\underline{\xi} \equiv (x,y)$ is any point in the exterior region and

$\underline{\xi}' \equiv (x',y')$ is any point on Γ.

Choose $G = \frac{1}{2\pi} \log R + \frac{1}{2\pi} \log R_1$,

where $R = |\underline{\xi} - \underline{\xi}'|$ and $R_1 = |\underline{\xi}_1 - \underline{\xi}'|$, $\underline{\xi}_1 \equiv (x,-y)$.

Fig.1



When $\underline{\xi}'$ is on Γ_1 or Γ_2,

$y' = 0$, $R = R_1$ and $\frac{\partial G}{\partial n} = \frac{\partial G}{\partial y'} = 0$.

$$\therefore \int_{\Gamma_1} = \frac{1}{\pi} \int_{-1}^{1} \log R \cdot \frac{\partial \phi_0^r}{\partial y}(r',0)\, dr' \quad , \text{ where } R^2 = (x-r')^2 + y^2, \qquad (2.5)$$

and [using (2.3)(iii) also] $$\int_{\Gamma_2} = 0 \ . \qquad (2.6)$$

When ξ' is on Γ_3',

by (2.3)(iv) $\phi_0^I(\xi') \sim \log r' + B_0$, and $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r'}$.

$$\therefore \int_{\Gamma_3'} = \int_{\Gamma_3'} \left\{ \frac{1}{2\pi} \log R + \frac{1}{2\pi} \log R_1 \right\} \left(-\frac{1}{r'}\right) d\xi'$$

$$- \int_{\Gamma_3'} \left(\log r' + B_0\right) \left\{ \frac{1}{2\pi R} \left(-\frac{\partial R}{\partial r'}\right) + \frac{1}{2\pi R_1} \left(-\frac{\partial R_1}{\partial r'}\right) \right\} d\xi'$$

$$\simeq -\frac{1}{2\pi} \int_{-\alpha}^{\pi-\alpha} \left\{ \log(r' - r\cos\beta) + \log\left[r' - r\cos(\beta+2\alpha)\right] \right\} d\beta$$

$$+ \frac{1}{2\pi} \int_{-\alpha}^{\pi-\alpha} \left(\log r' + B_0\right) \left\{ \frac{r' - r\cos\beta}{r^2 + r'^2 - 2rr'\cos\beta} + \frac{r' - r\cos(\beta+2\alpha)}{r^2 + r'^2 - 2rr'\cos(\beta+2\alpha)} \right\} r' d\beta$$

[by geometry of Fig.1].

Expanding integrands for large r', we get

$$\int_{\Gamma_3'} \simeq -\frac{1}{2\pi} \int_{-\alpha}^{\pi-\alpha} \left\{ 2\log r' - \frac{r\cos\beta}{r'} - \frac{r\cos(\beta+2\alpha)}{r'} + O\left(\frac{1}{r'^2}\right) \right\} d\beta$$

$$+ \frac{1}{2\pi} \int_{-\alpha}^{\pi-\alpha} \left\{ 2\log r' + 2B_0 + \frac{\log r'}{r'} \cdot r\cos\beta + \frac{\log r'}{r'} \cdot r\cos(\beta+2\alpha) \right.$$

$$\left. + \frac{B_0}{r'} \cdot r\cos\beta + \frac{B_0}{r'} \cdot r\cos(\beta+2\alpha) + O\left(\frac{\log r'}{r'^2}\right) \right\} d\beta$$

Therefore, as $r' \to \infty$, $\int_{\Gamma_3'} \to B_0$. (2.7)

By (2.4) — (2.7)

$$\left.\begin{aligned} \phi_0^I(\xi) &= \frac{1}{\pi} \int_{-1}^{1} (\log R)\, V(t')\, dt' + B_0, \\ \text{where } R^2 &= (x-t')^2 + y^2 \quad \text{and} \quad V(x) = \left.\frac{\partial \phi_0^I(x,0)}{\partial y}\right|_{|x|<1}. \end{aligned}\right\} \quad (2.8)$$

In particular,

(a) when ξ is on the strip, by (2.3)(ii),

$$1 = \frac{1}{\pi}\int_{-1}^{1} \log|x-r'| \cdot V(r')\, dr' + B_0 .$$

Therefore, by Cooke (1970),

$$V(r') = \frac{B_0 - 1}{\log 2} \cdot \frac{1}{(1-r'^2)^{1/2}} .$$

(b) When $\xi \to \infty$, $R \sim r$ so that

$$\phi_o^I \sim (\log r) \cdot \frac{1}{\pi}\int_{-1}^{1} V(r')\, dr' + B_0 .$$

Substituting the above expression for $V(r')$ here and comparing the result with (2.3)(iv), we get

$$1 = \frac{B_0 - 1}{\log 2} .$$

Thus,

$$V(x) = \frac{1}{(1-x^2)^{1/2}} = \phi_{oy}^I(x,0) . \qquad (2.9)$$

(2.8) and (2.9) give the solution to the problem (2.3).

Behaviour of ϕ_o^I near edge

Define a coordinate system (δ,θ) based at the point $(x=1, y=0)$ on the right hand side edge such that

$$\left.\begin{aligned} x &= 1 + \delta\cos\theta \\ y &= \delta\sin\theta \end{aligned}\right\}, \quad \epsilon \ll \delta \ll 1. \qquad (2.10)$$

This system will be called 'outer edge coordinates'. On the upper side of the strip $\theta = \pi$ and therefore, by (2.9) and (2.10),

$$V(1-\delta) = \frac{2^{1/2}}{2}\left\{ \delta^{-1/2} + \frac{1}{4}\delta^{1/2} + \frac{3}{32}\delta^{3/2} + O(\delta^{5/2}) \right\}.$$

Using this result and the fact that near edge ϕ_o^{I} satisfies (i), (ii) and (iii) in (2.3) expressed in coordinates (δ,θ), we get

$$\phi_o^{I}(\delta,\theta) = 1 + 2^{1/2}\left\{ \delta^{1/2}\cos\frac{\theta}{2} - \frac{1}{12}\delta^{3/2}\cos\frac{3\theta}{2} + \frac{3}{160}\delta^{5/2}\cos\frac{5\theta}{2} + O(\delta^{7/2})\right\}, \delta\to 0. \quad (2.11)$$

Edge Region

Define a system of 'inner edge' coordinates (X,Y) or (R,θ) based at the same point as (2.10), such that

$$\left.\begin{aligned} x &= 1 + \epsilon X \\ y &= \epsilon Y \end{aligned}\right\}, \qquad \left.\begin{aligned} X &= R\cos\theta \\ Y &= R\sin\theta \end{aligned}\right\}. \quad (2.12)$$

By (2.10) and (2.12), the relationship between δ and R is

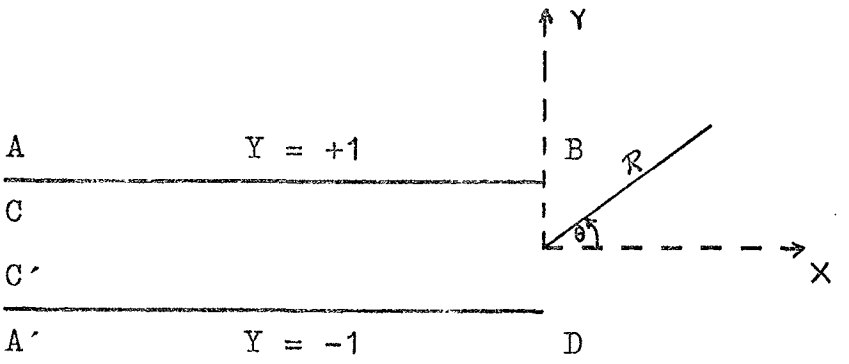
$$\epsilon R = \delta. \quad (2.13)$$

On the (X,Y) scale the separation between the strips is 2, and the distance to the centre $(x = o,\ y = o)$ from the origin $(X = 0,\ Y = 0)$ is $\frac{1}{\epsilon}$. Since $\epsilon \ll 1$, we can regard the body in the edge region as two semi-infinite planes of cross-section A B C, C′D A′ (Fig.2) given by $Y = \pm 1,\ X \leqslant 0$.

From the definition of the three regions (Sec.1), it is clear that the infinity of region II overlaps with the near edge limits of I and III, and therefore the condition at infinity for the egde potential ϕ^{II} must be

$$\phi^{II} \sim \left.\begin{aligned} &(a)\ \phi^{I} \text{ towards exterior} \\ &(b)\ \phi^{III} \text{ towards interior} \end{aligned}\right\}. \quad (2.14)$$

Fig.2



Expressing (2.1) in terms of inner edge coordinates (X,Y) (of 2.12) and using (2.14), the problem for ϕ^{II} becomes

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2 \phi^{II} = 0, \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \\ &\text{(ii)} \quad \phi^{II}(X, \pm 1) = 1, \quad \text{when} \quad X \leqslant 0 \\ &\text{(iii)a.} \quad \phi^{II} \sim \phi^{I}(R, \theta), \quad \text{as} \quad R \to \infty, \\ &\qquad \text{b.} \quad \phi^{II} \sim \phi^{III}(X, Y), \quad \text{as} \quad X \to -\infty. \end{aligned}\right\} \tag{2.15}$$

Interior Region

Define a system of 'interior' coordinates (X′,Y′), based at the same origin as (x,y), by the relation

$$\left.\begin{aligned} x &= \epsilon X' \\ y &= \epsilon Y' \end{aligned}\right\}. \tag{2.16}$$

On the (X′,Y′) scale also the separation between the strips is 2, and the distance from the centre to either edge is $\frac{1}{\epsilon}$. So we can regard the body in the interior region

as two parallel infinite planes represented by the two lines $Y' = \pm 1$. Here, the condition at infinity for the potential ϕ^{III} is

$$\phi^{III} \sim \phi^{II}, \text{ towards the edge.} \tag{2.17}$$

Expressing (2.1) in terms of interior coordinates (X',Y') (2.16) and using (2.17), the problem for ϕ^{III} becomes

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2 \phi^{III} = 0, \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial X'^2} + \frac{\partial^2}{\partial Y'^2} \\ &\text{(ii)} \quad \phi^{III}(X, \pm 1) = 1, \quad \text{for all } X \\ &\text{(iii)} \quad \phi^{III} \sim \phi^{II}(X', Y'), \quad \text{as } X' \to +\infty, \\ &\qquad [\text{concentrating our attention on the side } x>0, \\ &\qquad \text{the opposite side being similar}]. \end{aligned}\right\} \tag{2.18}$$

The general solution of (2.18), with symmetry about the X'- axis is of the form

$$\phi^{III} = 1 + \sum_{n=0}^{N} a_n \cosh(2n+1)\tfrac{\pi}{2} X' \cdot \cos(2n+1)\tfrac{\pi}{2} Y', \tag{2.19}$$

where the constants a_n (which may depend on ϵ) have to be chosen so as to satisfy the last condition, N being some positive number.

For points common to both interior and edge regions, the relation between interior coordinates and inner edge coordinates, by (2.12) and (2.16), is

$$\left.\begin{aligned} X' &= \frac{1}{\epsilon} + X \\ Y' &= Y \end{aligned}\right\}. \tag{2.20}$$

By (2.19) and (2.20), near the edge

$$\phi^{III}(X,Y) \sim 1 + \tfrac{1}{2}\sum_{n=0}^{N} a_n e^{\frac{\pi}{2}(2n+1)(\frac{1}{\epsilon}+X)} \cdot \cos(2n+1)\tfrac{\pi}{2}Y \,. \qquad (2.21)$$

Edge Region: First Approximation

Expressing the expansion (2.11) for ϕ_o^I in inner edge coordinates (R,θ) using (2.13) and taking the leading term,

$$\phi_o^I(R,\theta) = 1 + \epsilon^{1/2}\cdot 2^{1/2} R^{1/2}\cos\frac{\theta}{2} + O(\epsilon^{3/2}) \,. \qquad (2.22)$$

(2.22) suggests the form $1 + \epsilon^{1/2}\phi_o^{II}$ for the first approximation to the potential ϕ^{II} in the edge region. If this form is substituted in (2.15), together with (2.21) and (2.22), then the problem for ϕ_o^{II} is

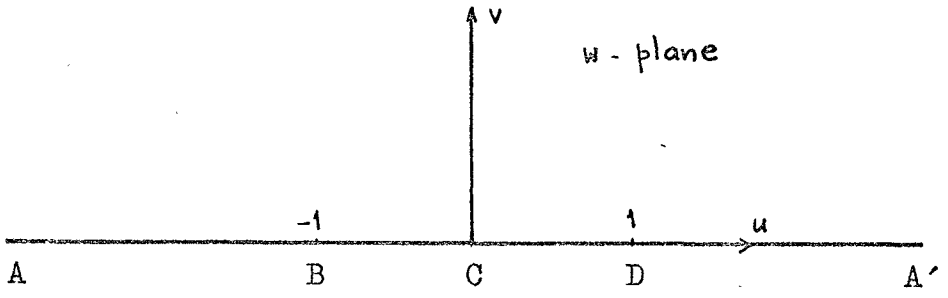
$$\left.\begin{aligned}
&\text{(i)} \quad \nabla^2 \phi_o^{II} = 0 \\
&\text{(ii)} \quad \phi_o^{II}(X,\pm 1) = 0, \quad \text{when} \quad X \le 0 \\
&\text{(iii)a.} \quad \phi_o^{II} \sim 2^{1/2}\,\mathrm{Re}(Z^{1/2}), \quad \text{as} \quad R\to\infty, \quad \text{where} \quad Z = R\,e^{i\theta} \\
&\quad\ \text{b.} \quad \phi_o^{II} \sim \tfrac{1}{2}\epsilon^{-1/2}\sum_{n=0}^{N} a_n e^{\frac{\pi}{2}(2n+1)(\frac{1}{\epsilon}+X)} \cdot \cos(2n+1)\tfrac{\pi}{2}Y, \quad \text{as} \quad X\to-\infty
\end{aligned}\right\} \qquad (2.23)$$

This problem is easily solved by making use of the conformal transformation (Kober, 1957):

$$\left.\begin{aligned}
&\frac{dZ}{dw} = -\frac{2}{\pi}\left(\frac{w^2-1}{w}\right) \\
&\text{or} \quad \tfrac{\pi}{2}(Z+i) = \log w - \frac{w^2-1}{2}
\end{aligned}\right\}, \qquad (2.24)$$

which maps the whole of Z-plane, excepting the lines A B C, C′D A′, into the upper half of w-plane (Fig.3), the points B, C, D being mapped on to -1, 0, +1 respectively.

Fig.3



First approximations to the transformation equation (2.24) are:

$$\left.\begin{aligned} &\text{(i) for large } |w|, \quad -\pi Z \sim w^2 \Rightarrow w \sim i(\pi Z)^{1/2} \\ &\text{(ii) for small } |w|, \quad \tfrac{\pi}{2}(Z+i) - \tfrac{1}{2} \sim \log w \Rightarrow w \sim e^{Z_1}, \\ &\qquad\qquad \text{where} \quad Z_1 = \tfrac{\pi}{2}(Z+i) - \tfrac{1}{2}. \end{aligned}\right\} \quad (2.25)$$

By (2.23), (2.24) and (2.25), the problem for ϕ_o^{II} in the w-plane is

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2 \phi_o^{II} = 0, \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \\ &\text{(ii)} \quad \phi_o^{II} = 0, \quad \text{when} \quad v = 0 \\ &\text{(iii)a.} \quad \phi_o^{II} \sim \left(\tfrac{2}{\pi}\right)^{1/2} \mathrm{Re}\,(-iw) \quad , \quad \text{as} \quad |w| \to \infty \\ &\qquad \text{b.} \quad \phi_o^{II} \sim \tfrac{1}{2}\epsilon^{-1/2} \sum_{n=0}^{N} a_n e^{\frac{\pi}{2\epsilon}(2n+1)} \cdot e^{\frac{2n+1}{2}} \mathrm{Re}\,(-iw)^{2n+1}, \quad \text{as} \quad |w| \to 0. \end{aligned}\right\} \quad (2.26)$$

Since ϕ_0^{II} satisfies both (iii)a and (iii)b in (2.26), the only value n can take in o, and a_0 should contain a factor $\epsilon^{1/2} e^{-\pi/2\epsilon}$ (a function of ϵ). Thus the solution of (2.26) is

$$\phi_0^{II} = \left(\frac{2}{\pi}\right)^{1/2} Re(-iw), \tag{2.27}$$

and a_n are given by

$$a_0 = \frac{2^{3/2}}{\pi^{1/2}} e^{-1/2} \left(\epsilon^{1/2} e^{-\pi/2\epsilon}\right) \quad \text{and} \quad a_n = 0, \; n \geq 1. \tag{2.28}$$

Interior Region: First Approximation

Substituting (2.28) in (2.19) we immediately get the first approximation of the interior potential ϕ^{III} in the form $1 + \epsilon^{1/2} e^{-\pi/2\epsilon} \phi_0^{III}$:

$$\phi^{III} \sim 1 + \left(\epsilon^{1/2} e^{-\pi/2\epsilon}\right) \frac{2^{3/2}}{\pi^{1/2}} e^{-1/2} \cosh \tfrac{\pi}{2} X' \cos \tfrac{\pi}{2} Y'. \tag{2.29}$$

Behaviour of ϕ_0^{II} towards exterior

Since it is not possible to express w explicitly in terms of Z (see 2.24), ϕ_0^{II} (2.27) does not take an explicit form in the Z-plane. However, an asymptotic expansion for ϕ_0^{II}, for large $|Z|$, can be found as follows:

Substituting (2.25)(i) in the $\log w$ term in (2.24), one finds that the second approximation to the transformation equation (2.24), for large $|w|$ is

$$w^2 \sim -\pi Z + \log(\pi Z) + 1, \tag{2.30}$$

and expanding this for large $|Z|$,

$$w \sim i(\pi Z)^{1/2} - \frac{i}{2(\pi Z)^{1/2}} \left\{\log(\pi Z) + 1\right\} + O\left\{\frac{(\log Z)^2}{Z^{3/2}}\right\}, \quad |Z| \to \infty. \tag{2.31}$$

Since $Z = R\,e^{i\Theta}$, by (2.27) and (2.31),

$$\phi_0^{II} = 2^{1/2}\left[R^{1/2}\cos\frac{\theta}{2} - \frac{1}{2\pi}R^{-1/2}\left\{(1+\log\pi R)\cos\frac{\theta}{2} + \theta\sin\frac{\theta}{2}\right\} + O\left\{\frac{(\log R)^2}{R^{3/2}}\right\}\right], \quad R\to\infty.$$

By (2.13), it follows that

$$1 + \epsilon^{1/2}\phi_0^{II} \sim 1 + 2^{1/2}\left[\delta^{1/2}\cos\frac{\theta}{2} + (\epsilon\log\epsilon)\cdot\frac{1}{2\pi}\delta^{-1/2}\cos\frac{\theta}{2} - \epsilon\cdot\frac{1}{2\pi}\delta^{-1/2}\left\{(1+\log\pi\delta)\cos\frac{\theta}{2} + \theta\sin\frac{\theta}{2}\right\}\right] \tag{2.32}$$

(2.32) is the first approximation of the edge potential, expressed in outer edge coordinates (δ,θ) and expanded for small ϵ. Therefore, this should give some indication as to how the exterior potential ϕ^I behaves near the edge.

3. (a) Exterior Region: Higher Approximations

In the expansion (2.32), the first term, apart from 1, is the same as the first term of ϕ_0^I (2.11), as expected. Further, it is seen that the second and the third terms are of order $\epsilon\log\epsilon$, ϵ respectively. This suggests that for a solution of the problem (2.2), up to order ϵ, it is appropriate to try an expansion of the form

$$\phi^I = \phi_0^I + (\epsilon\log\epsilon)\phi_1^I + \epsilon\phi_2^I . \tag{3.1}$$

Substituting the expansion (3.1) for ϕ^I in (2.2), taking $B = B_0 + (\epsilon\log\epsilon)B_1 + \epsilon B_2$ [i.e. in the same form as (3.1)], where B_0, B_1, B_2 are arbitrary constants, expanding $\phi^I(x,\pm\epsilon)$ using Taylor's theorem and equating corresponding order terms, we get

$$\left.\begin{aligned}
&\text{(i)}\quad \nabla^2\phi_0^I = \nabla^2\phi_1^I = \nabla^2\phi_2^I = 0\\
&\text{(ii)}\quad \phi_0^I(x,0)=1\ ,\ \phi_1^I(x,0)=0\ ,\ \phi_2^I(x,0) = \mp\phi_{0y}^I(x,0)\ ,\ |x|<1\\
&\text{(iii)}\quad \phi_{0y}^I(x,0) = \phi_{1y}^I(x,0) = \phi_{2y}^I(x,0) = 0\ ,\ \text{when}\ |x|>1\\
&\text{(iv)}\quad \phi_0^I \sim \log r + B_0\ ,\ \phi_1^I \sim B_1\ ,\ \phi_2^I \sim B_2\ \text{at}\ \infty .
\end{aligned}\right\} \quad (3.2)$$

Because of symmetry about the x-axis $\phi_i^I(x,-y) = \phi_i^I(x,y)$, $i=0,1,2$; and it is sufficient to consider $y>0$ alone.

The problem for ϕ_0^I is the same as (2.3) and its solution is given by (2.8) and (2.9), asymptotic form of the solution near edge being (2.11).

To solve the problem for ϕ_1^I, consider the function $\phi(x,y)$ satisfying

$$\left.\begin{aligned}
&\text{(i)}\quad \nabla^2\phi = 0\\
&\text{(ii)}\quad \phi(x,0) = 1,\ \text{when}\ |x|<1\\
&\text{(iii)}\quad \phi_y(x,0) = 0,\ \text{when}\ |x|>1\\
&\text{(iv)}\quad \phi \sim 0\ \text{at}\ \infty.
\end{aligned}\right\} \quad (3.3)$$

The problem (3.3) has a solution of the form

$$\phi(x,y) = \int_0^\infty u^{-1} A(u)\, e^{-yu} \cos(xu)\, du\ ,\ y>0\ , \quad (3.4)$$

provided the function $A(u)$ satisfies Dual Integral equations

$$\left.\begin{aligned}
&\int_0^\infty u^{-1} A(u) \cos(xu)\, du = 1\ ,\ |x|<1\\
&\int_0^\infty A(u) \cos(xu)\, du = 0\ ,\ |x|>1
\end{aligned}\right\} . \quad (3.5)$$

Solution of (3.5) is (Sneddon, 1966), $A(u) = u\,J_1(u)$. Substituting this $A(u)$ back in (3.4), the solution of (3.3) is obtained. Comparing the problem (3.3) with the problem for ϕ_1^I in (3.2), it follows that

$$\phi_1^I(x,y) = B_1 - B_1\int_0^\infty J_1(u)\,e^{-yu}\cos(xu)\,du, \quad y>0, \; B_1 \text{ arbitrary.} \tag{3.6}$$

In particular, using properties of Bessel functions (Watson, 1944),

$$\left.\begin{aligned} \phi_1^I(x,0) &= B_1 + \frac{B_1}{\sqrt{(x^2-1)}\left\{x+\sqrt{(x^2-1)}\right\}}, \quad |x|>1 \\ \text{and} \quad \frac{\partial\phi_1^I}{\partial y}(x,0) &= \frac{B_1}{(1-x^2)^{3/2}}, \quad |x|<1. \end{aligned}\right\} \tag{3.7}$$

Near edge expansions for (3.7) [putting $\theta = 0, \pi$ respectively in (2.10) and substituting] are :

$$\left.\begin{aligned} \phi_1^I(1+\delta,0) &= \frac{B_1}{2}2^{1/2}\left[\delta^{-1/2} + \tfrac{3}{4}\delta^{1/2} + O(\delta)\right], \\ \frac{\partial\phi_1^I}{\partial y}(1-\delta,0) &= \frac{B_1}{2^{3/2}}\left[\delta^{-3/2} + \tfrac{3}{4}\delta^{-1/2} + O(\delta^{1/2})\right]. \end{aligned}\right\} \tag{3.8}$$

Using (3.8) and the fact that near edge ϕ_1^I satisfies (i), (ii) and (iii) in (3.2), expressed in coordinates (δ,θ), we get

$$\phi_1^I(\delta,\theta) = \frac{B_1}{2}2^{1/2}\left[\delta^{-1/2}\cos\tfrac{\theta}{2} + \tfrac{3}{4}\delta^{1/2}\cos\tfrac{\theta}{2} + O(\delta)\right]. \tag{3.9}$$

By (3.2) and (2.9), the problem for ϕ_2^I, for $y>0$, is

$$\left.\begin{aligned}
&\text{(i)} \quad \nabla^2 \phi_2^I = 0 \\
&\text{(ii)} \quad \phi_2^I(x,0) = -\frac{1}{(1-x^2)^{1/2}}, \quad \text{when } |x|<1 \\
&\text{(iii)} \quad \phi_{2y}^I(x,0) = 0, \quad \text{when } |x|>1 \\
&\text{(iv)} \quad \phi_2^I \sim B_2 \quad \text{at } \infty.
\end{aligned}\right\} \tag{3.10}$$

To solve (3.10), consider the function $\psi(x,y)$ satisfying

$$\left.\begin{aligned}
&\text{(i)} \quad \nabla^2 \psi = 0 \\
&\text{(ii)} \quad \psi(x,0) = f(x), \quad |x|<1, \quad \text{where } f(x) = x\sin^{-1}x + (1-x^2)^{1/2} \\
&\text{(iii)} \quad \psi_y(x,0) = 0, \quad \text{when } |x|>1 \\
&\text{(iv)} \quad \psi \sim C\log r + D \quad \text{at } \infty, \\
&\qquad \text{where } C \text{ and } D \text{ are arbitrary constants.}
\end{aligned}\right\} \tag{3.11}$$

The problem (3.11) is equivalent to (3.10), if we write

$\psi_{yy} = \phi_2^I$, for

(i) $\nabla^2\psi = 0 \Rightarrow \nabla^2 \phi_2^I = \nabla^2(\psi_{yy}) = \frac{\partial^2}{\partial y^2}(\nabla^2\psi) = 0.$

(ii) $\phi_2^I(x,y) = \psi_{yy} = -\psi_{xx} = -\frac{\partial^2}{\partial x^2}\{\psi(x,y)\},$

and when $y = 0$, $|x|<1$: $\phi_2^I(x,0) = -f''(x) = -\frac{1}{(1-x^2)^{1/2}}$

(iii) $\phi_{2y}^I(x,y) = \psi_{yyy} = \frac{\partial}{\partial y}\{-\psi_{xx}\} = -\frac{\partial^2}{\partial x^2}\{\psi_y(x,y)\},$

and when $y = 0$, $|x|>1$: $\phi_{2y}^I(x,0) = 0.$

(iv) $\psi \sim C\log r + D$ at $\infty \Rightarrow \phi_2^I \sim 0$ at ∞.

Since this is the only meaningful behaviour at infinity we can expect for ψ, in (3.10)(iv) let us take $B_2 = 0$.

To solve (3.11) for ψ a procedure similar to that used for ϕ_0^I [see (2.3)—(2.8)] is followed, with the result:

$$\left.\begin{aligned} \psi(\underline{\xi}) &= \frac{1}{\pi}\int_{-1}^{1} (\log R).\, U(r')\, dr' \;+\; D\,, \\ \text{where}\quad R^2 &= (x-r')^2 + y^2 \quad\text{and}\quad U(x) = \frac{\partial \psi}{\partial y}(x,0)\Big|_{|x|<1} \end{aligned}\right\} \qquad (3.12)$$

In particular, when $\underline{\xi} \equiv (x,o)$, by (3.11)(ii),

$$f(x) = \frac{1}{\pi}\int_{-1}^{1} \log|x-r'|\,.\,U(r')\,dr' + D\,.$$

Therefore, by Cooke (1970),

$$U(r') = \frac{1}{\pi(1-r'^2)^{1/2}}\cdot\int_{-1}^{1}\frac{(1-x^2)^{1/2} f'(x)}{x-r'}\,dx + \frac{1}{\pi(1-r'^2)^{1/2}}\cdot\frac{1}{(-\log 2)}\cdot\int_{-1}^{1}\frac{f(x)-D}{(1-x^2)^{1/2}}\,dx$$

$$= \frac{1}{\pi(1-r'^2)^{1/2}}\cdot\int_{-1}^{1}\frac{(1-x^2)^{1/2}\sin^{-1}x}{x-r'}\,dx - \frac{1}{\log 2}\left(\frac{4}{\pi}-D\right)\frac{1}{(1-r'^2)^{1/2}}\,.$$

Near edge, writing $r' = 1-\delta$,

$$U(1-\delta) = -\frac{(1-\delta/2)^{-1/2}}{\pi(2\delta)^{1/2}}\cdot\int_{-1}^{1}\frac{(1-x^2)^{1/2}\sin^{-1}x}{1-x-\delta}\,dx - \frac{4/\pi - D}{\log 2}\cdot\frac{(1-\delta/2)^{-1/2}}{(2\delta)^{1/2}}\,. \qquad (3.13)$$

An approximate estimate for the integral appearing in (3.13) can be found in the form of an expansion of δ, as follows:

$$\left.\begin{aligned} \int_{-1}^{1}\frac{(1-x^2)^{1/2}\sin^{-1}x}{1-x-\delta}\,dx &= \int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{1/2}\sin^{-1}x\,dx \;+\; \delta I_1\,, \\ \text{where}\quad I_1 &= \int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{1/2}\cdot\frac{\sin^{-1}x}{1-x-\delta}\,dx \quad \cdots\cdots \text{[Divergent as } \delta\to 0\text{]}. \end{aligned}\right\} \qquad (3.14)$$

Now, $I_1 = \int_0^2 \frac{(2-\rho)^{1/2} \sin^{-1}(1-\rho)}{\rho^{1/2}(\rho-\delta)}\, d\rho$.

Using the expansion for $\sin^{-1}(1-\rho)$ (Abramowitz and Stegun, 1965), we get

$$(2-\rho)^{1/2} \sin^{-1}(1-\rho) = 2^{1/2}\cdot\frac{\pi}{2} - 2\rho^{1/2} - 2^{1/2}\cdot\frac{\pi}{8}\rho + \frac{\rho^{3/2}}{3} - 2^{1/2}\cdot\frac{11}{64}\rho^2 + O(\rho^{5/2}),\ |\rho|<2. \quad (3.15)$$

Hence

$$I_1 = 2^{1/2}\cdot\frac{\pi}{2}\int_0^2 \frac{d\rho}{\rho^{1/2}(\rho-\delta)} - 2\int_0^2 \frac{\rho^{1/2}\, d\rho}{\rho^{1/2}(\rho-\delta)} + R_1 ,$$

$$\text{where } R_1 = \int_0^2 \frac{N_1(\rho)\, d\rho}{\rho^{1/2}(\rho-\delta)},\ N_1(\rho) = (2-\rho)^{1/2}\sin^{-1}(1-\rho) - 2^{1/2}\cdot\frac{\pi}{2} + 2\rho^{1/2}. \quad (3.16)$$

$$\therefore I_1 = -\pi\left\{1 + \frac{\delta}{6} + O(\delta^2)\right\} - 2\left\{-\log\delta + \log 2 - \frac{\delta}{2} + O(\delta^2)\right\} + R_1. \quad (3.17)$$

$$\text{Now } R_1 = \int_0^2 \frac{N_1(\rho)}{\rho^{3/2}}\, d\rho + \delta\int_0^2 \frac{N_1(\rho)\, d\rho}{\rho^{3/2}(\rho-\delta)}$$

$$= K_1 + \delta I_2 , \text{ where constant } K_1 = \int_0^2 \frac{N_1(\rho)}{\rho^{3/2}}\, d\rho . \quad (3.18)$$

Using (3.15), (3.16) and (3.18), we get

$$I_2 = -2^{1/2}\cdot\frac{\pi}{8}\int_0^2 \frac{\rho\, d\rho}{\rho^{3/2}(\rho-\delta)} + \frac{1}{3}\int_0^2 \frac{\rho^{3/2}\, d\rho}{\rho^{3/2}(\rho-\delta)} + R_2 ,$$

$$\text{where } R_2 = \int_0^2 \frac{N_2(\rho)\, d\rho}{\rho^{3/2}(\rho-\delta)},\ N_2(\rho) = N_1(\rho) + 2^{1/2}\cdot\frac{\pi}{8}\rho - \frac{1}{3}\rho^{3/2}. \quad (3.19)$$

$$\left.\begin{aligned} &\therefore I_2 \simeq \frac{\pi}{4}\left\{1 + O(\delta)\right\} + \frac{1}{3}\left\{-\log\delta + \log 2 + O(\delta)\right\} + K_2 , \\ &\text{where constant } K_2 = \int_0^2 \frac{N_2(\rho)}{\rho^{5/2}}\, d\rho . \end{aligned}\right\} \quad (3.20)$$

By (3.14), (3.17), (3.18) and (3.20), we get

$$\left.\begin{aligned} \int_{-1}^{1} \frac{(1-x^2)^{1/2}\sin^{-1}x}{1-x-\delta}\,dx &= 2 + 2\delta\log\delta + K_3\delta - \tfrac{1}{3}\delta^2\log\delta + K_4\delta^2 + O(\delta^3\log\delta), \\ \text{where } K_3 &= K_1 - \pi - 2\log 2\,, \quad K_4 = K_2 + 1 + \frac{\pi}{12} + \frac{1}{3}\log 2\,. \end{aligned}\right\} \tag{3.21}$$

Substituting (3.21) in (3.13), expanding for small δ and simplifying:

$$U(1-\delta) = C_1\delta^{-1/2} + C_2\delta^{1/2}\log\delta + C_3\delta^{1/2} + C_4\delta^{3/2}\log\delta + C_5\delta^{3/2} + O(\delta^{5/2}\log\delta), \tag{3.22}$$

$$\left.\begin{aligned} \text{where } C_1 &= -\frac{2^{1/2}}{\pi}\left\{1 + \frac{2}{\log 2} - \frac{\pi}{2}D\right\}, \\ C_2 &= -\frac{2^{1/2}}{\pi}\,, \\ C_3 &= -\frac{2^{1/2}}{\pi}\left\{\frac{1}{2}K_3 + \frac{1}{4}\left(1 + \frac{2}{\log 2} - \frac{\pi}{2}D\right)\right\} = -\frac{2^{1/2}}{2\pi}K_3 + \frac{1}{4}C_1\,, \\ C_4 &= -\frac{2^{1/2}}{12\pi}\,, \\ \text{and}\quad C_5 &= -\frac{2^{1/2}}{\pi}\left\{\frac{1}{2}K_4 + \frac{1}{8}K_3 + \frac{3}{32}\left(1 + \frac{2}{\log 2} - \frac{\pi}{2}D\right)\right\} = -\frac{2^{1/2}}{\pi}\left(\frac{K_4}{2} + \frac{K_3}{8}\right) + \frac{3}{32}C_1\,. \end{aligned}\right\} \tag{3.23}$$

Also, by (3.11)(ii)

$$\psi(x,0) = x\sin^{-1}x + (1-x^2)^{1/2}, \quad |x| < 1,$$

and expanding this for small δ (near edge):

$$\psi(1-\delta,0) = \frac{\pi}{2} - \frac{\pi}{2}\delta + \frac{2^{3/2}}{3}\delta^{3/2} + \frac{2^{1/2}}{30}\delta^{5/2} + O(\delta^{7/2}). \tag{3.24}$$

Using (3.22), (3.24) and the fact that near edge ψ satisfies (i) and (iii) in (3.11), expressed in (δ,θ), we get

$$\psi(\delta,\theta) = \frac{\pi}{2} + \frac{\pi}{2}\delta\cos\theta + \frac{2^{3/2}}{3\pi}\delta^{3/2}\left\{\log\delta\cos\frac{3\theta}{2} - \theta\sin\frac{3\theta}{2}\right\}$$

$$- \frac{2^{1/2}}{30\pi}\delta^{5/2}\left\{\log\delta\cos\frac{5\theta}{2} - \theta\sin\frac{5\theta}{2}\right\}$$

$$+ P\,\delta^{1/2}\cos\frac{\theta}{2} + Q\,\delta^{3/2}\cos\frac{3\theta}{2} + R\,\delta^{5/2}\cos\frac{5\theta}{2}\ , \qquad (3.25)$$

where P, Q, R are constants to be determined.
[The last three terms in (3.25) are the only possible eigen solutions for ψ.]

Comparing $\frac{\partial\psi}{\partial y}(1-\delta,0) = -\frac{1}{\delta}\frac{\partial\psi}{\partial\theta}\Big|_{\theta=\pi}$ from (3.25) with (3.22):

$$P = 2C_1\ ,\quad Q = -\frac{2}{3}\left(C_3 + \frac{2^{3/2}}{3\pi}\right)\ ,\quad R = \frac{2}{5}\left(C_5 + \frac{2^{1/2}}{30\pi}\right).$$

Substituting these values of P, Q, R back in (3.25) and differentiating with respect to y using (2.10), we get

$$\psi_y(\delta,\theta) = C_1\delta^{-1/2}\sin\frac{\theta}{2} + C_2\delta^{1/2}\left\{\log\delta\sin\frac{\theta}{2} + \theta\cos\frac{\theta}{2}\right\} + C_3\delta^{1/2}\sin\frac{\theta}{2}$$

$$- C_4\delta^{3/2}\left\{\log\delta\sin\frac{3\theta}{2} + \theta\cos\frac{3\theta}{2}\right\} - C_5\delta^{3/2}\sin\frac{3\theta}{2}\ ; \qquad (3.26)$$

and differentiating again with respect to y,

$$\psi_{yy}(\delta,\theta) = \frac{1}{2}C_1\delta^{-3/2}\cos\frac{3\theta}{2} + \frac{1}{2}C_2\delta^{-1/2}\left\{\log\delta\cos\frac{\theta}{2} + \theta\sin\frac{\theta}{2}\right\}$$

$$+ \left(C_2 + \tfrac{1}{2}C_3\right)\delta^{-1/2}\cos\frac{\theta}{2} - \frac{3}{2}C_4\delta^{1/2}\left\{\log\delta\cos\frac{\theta}{2} - \theta\sin\frac{\theta}{2}\right\}$$

$$- \left(C_4 + \tfrac{3}{2}C_5\right)\delta^{1/2}\cos\frac{\theta}{2} + O(\delta^{3/2}\log\delta). \qquad (3.27)$$

Since $\psi_{yy} = \phi_2^I$ it follows that, for points near edge, (3.27) is a solution of (3.10).

Eigen Solutions for ϕ_2^I

Any solutions of the reduced equations (see 3.10):

$$\left.\begin{aligned} &\text{(i)} \quad \nabla^2\phi_2^I = 0 \\ &\text{(ii)} \quad \phi_2^I(x,0) = 0, \quad \text{when} \quad |x|<1 \\ &\text{(iii)} \quad \phi_{2y}^I(x,0) = 0, \quad \text{when} \quad |x|>1 \end{aligned}\right\} \qquad (3.28)$$

have to be added to (3.27).

Consider the function $\phi_E(x,y) = \phi_{0yy}^I(x,y)$, where ϕ_0^I is the potential given by (2.8).

$$\text{(i)} \quad \nabla^2\phi_E = \nabla^2\left(\phi_{0yy}^I\right) = \frac{\partial^2}{\partial y^2}\left(\nabla^2\phi_0^I\right) = 0.$$

$$\text{(ii)} \quad \phi_E(x,y) = \frac{\partial^2}{\partial y^2}\left\{\phi_0^I(x,y)\right\} = -\frac{\partial^2}{\partial x^2}\left\{\phi_0^I(x,y)\right\},$$

and when $y = 0$, $|x|<1$: $\phi_0^I(x,0) = 1 \Rightarrow \phi_E(x,0) = 0.$

$$\text{(iii)} \quad \phi_{Ey}(x,y) = \frac{\partial}{\partial y}\left\{\phi_{0yy}^I(x,y)\right\} = \frac{\partial}{\partial y}\left\{-\phi_{0xx}^I(x,y)\right\} = -\frac{\partial^2}{\partial x^2}\left\{\phi_{0y}^I(x,y)\right\},$$

and when $y = 0$, $|x|>1$: $\phi_{0y}^I(x,0) = 0 \Rightarrow \phi_{Ey}(x,0) = 0.$

Thus, $\phi_E = \phi_{0yy}^I$ is a solution of (3.28).

By differentiating (2.11) twice with respect to y, the near edge expansion for ϕ_E is

$$\phi_{0yy}^I(\delta,\theta) = \frac{2^{1/2}}{4}\left\{\delta^{-3/2}\cos\frac{3\theta}{2} + \frac{1}{4}\delta^{-1/2}\cos\frac{\theta}{2} - \frac{9}{32}\delta^{1/2}\cos\frac{\theta}{2} + O(\delta^{3/2})\right\}. \qquad (3.29)$$

Since the problem for ϕ_1^I in (3.2) is similar to (3.28), ϕ_1^I itself is a solution of (3.28); and $\phi_1^I(\delta,\theta)$, the near edge expansion of ϕ_1^I is given by (3.9).

Adding arbitrary multiples of asymptotic expansions of eigen solutions ϕ^I_{oyy} (3.29) and ϕ^I_1 (3.9) to the expansion for ψ_{yy} (3.27), we get

$$\phi^I_2(\delta,\theta) = \frac{C_1}{2}\delta^{-3/2}\cos\frac{3\theta}{2} + \frac{C_2}{2}\delta^{-1/2}\left\{\log\delta\cos\frac{\theta}{2} + \theta\sin\frac{\theta}{2}\right\} + \left(C_2 + \frac{C_3}{2}\right)\delta^{-1/2}\cos\frac{\theta}{2}$$

$$- \frac{3}{2}C_4\delta^{1/2}\left\{\log\delta\cos\frac{\theta}{2} - \theta\sin\frac{\theta}{2}\right\} - \left(C_4 + \frac{3}{2}C_5\right)\delta^{1/2}\cos\frac{\theta}{2}$$

$$+ E\left\{\delta^{-3/2}\cos\frac{3\theta}{2} + \frac{1}{4}\delta^{-1/2}\cos\frac{\theta}{2} - \frac{9}{32}\delta^{1/2}\cos\frac{\theta}{2}\right\}$$

$$+ F\left\{\delta^{-1/2}\cos\frac{\theta}{2} + \frac{3}{4}\delta^{1/2}\cos\frac{\theta}{2}\right\} + O\left(\delta^{3/2}\log\delta\right), \qquad (3.30)$$

where E and F are arbitrary constants.

Now $\epsilon\phi^I_2$, being the third term in the expansion (3.1) for ϕ^I, must be much less than $\phi^I_0 \sim \delta^{1/2}$, near edge. Therefore, the near edge behaviour of ϕ^I_2 cannot be as singular as $\delta^{-3/2}$. This singularity is therefore removed by choosing the value of the constant $E = -\frac{1}{2}C_1$. The coefficients of (3.30) are then simplified using (3.23), with the result:

$$\left.\begin{aligned}\phi^I_2(\delta,\theta) = &-\frac{2^{1/2}}{2\pi}\delta^{-1/2}\left[\left\{\log\delta\cos\frac{\theta}{2} + \theta\sin\frac{\theta}{2}\right\} + G\cos\frac{\theta}{2}\right] \\ &+ \frac{2^{1/2}}{8\pi}\delta^{1/2}\left[\left\{\log\delta\cos\frac{\theta}{2} - \theta\sin\frac{\theta}{2}\right\} + H\cos\frac{\theta}{2}\right] + O\left(\delta^{3/2}\log\delta\right),\end{aligned}\right\} \qquad (3.31)$$

where $G = 2 + \frac{1}{2}K_3 - 2^{1/4}\pi.F$

and $H = \frac{2}{3} + 6\left(K_4 + \frac{1}{4}K_3\right) + 2^{1/2}3\pi.F$.

[The value of the constant F is to be determined in Sec.4].

Substituting (2.11), (3.9) and (3.31) in (3.1), we get

$$\phi^I(\delta,\theta) = 1 + \epsilon^{1/2}\left\{\delta^{1/2}\cos\frac{\theta}{2} - \frac{i}{12}\delta^{3/2}\cos\frac{3\theta}{2} + O(\delta^{5/2})\right\}$$

$$+ (\epsilon\log\epsilon)\,J.\left\{\delta^{-1/2}\cos\frac{\theta}{2} + \frac{3}{4}\delta^{1/2}\cos\frac{\theta}{2} + O(\delta^{3/2})\right\}$$

$$+ \epsilon\left[-\frac{2^{1/2}}{2\pi}\delta^{-1/2}\left\{\left(\log\delta\cos\frac{\theta}{2} + \theta\sin\frac{\theta}{2}\right) + G\cos\frac{\theta}{2}\right\}\right.$$

$$\left. + \frac{2^{1/2}}{8\pi}\delta^{1/2}\left\{\left(\log\delta\cos\frac{\theta}{2} - \theta\sin\frac{\theta}{2}\right) + H\cos\frac{\theta}{2}\right\} + O(\delta^{3/2}\log\delta)\right], \qquad (3.32)$$

where J is an arbitrary constant.

(3.32) is the exterior potential ϕ^I near edge, in terms of outer edge coordinates (δ,θ), expanded (for small δ) to order ϵ. When it is expressed in inner edge coordinates (R,θ) using (2.13), it is found that the leading term (apart from 1) is

$$(\epsilon^{1/2}\log\epsilon)\left(J - \frac{2^{1/2}}{2\pi}\right)R^{-1/2}\cos\frac{\theta}{2}.$$

This contradicts the fact that the leading term should be $\phi_0^I(R,\theta)$ (2.22), unless the value of the constant J is so chosen as to cancel off the $\epsilon^{1/2}\log\epsilon$ term. Taking $J = \frac{2^{1/2}}{2\pi}$ and re-arranging terms, we get

$$\phi^I(R,\theta) = 1 + \epsilon^{1/2}2^{1/2}\left[R^{1/2}\cos\frac{\theta}{2} - \frac{R^{-1/2}}{2\pi}\left\{\left(\log R\cos\frac{\theta}{2} + \theta\sin\frac{\theta}{2}\right) + G\cos\frac{\theta}{2}\right\}\right]$$

$$+ (\epsilon^{3/2}\log\epsilon)\frac{2^{1/2}}{2\pi}R^{1/2}\cos\frac{\theta}{2}$$

$$+ \epsilon^{3/2}\left(-\frac{2^{1/2}}{12}\right)\left[R^{3/2}\cos\frac{3\theta}{2} - \frac{3}{2\pi}R^{1/2}\left\{\left(\log R\cos\frac{\theta}{2} - \theta\sin\frac{\theta}{2}\right) + H\cos\frac{\theta}{2}\right\}\right]$$

$$+ O(\epsilon^{5/2}\log\epsilon). \qquad (3.33)$$

The only unknown constant F in G and H [see (3.31)] will be determined, in Sec.4, by matching (3.33) with the edge expansion.

3. (b) Edge Region and Interior Region:

Higher Approximations

To solve (2.15) for ϕ^{II} up to a higher order in ϵ, we can now use $\phi^{I}(R,\theta)$ (3.33) in the boundary condition (iii)a. This suggests an expansion of the form

$$\phi^{II} = 1 + \epsilon^{1/2}\phi_0^{II} + (\epsilon^{3/2}\log\epsilon)\phi_1^{II} + \epsilon^{3/2}\phi_2^{II}, \qquad (3.34)$$

for the edge potential ϕ^{II} (up to order $\epsilon^{3/2}$).

In Section 2, we found that by taking the first approximation of ϕ^{II} in the form $1 + \epsilon^{1/2}\phi_0^{II}$, the corresponding ϕ^{III} is of the form $1 + \epsilon^{1/2} e^{\frac{-\pi}{2\epsilon}}\phi_0^{III}$ (2.29). That is, the interior potential should contain an additional factor $e^{\frac{-\pi}{2\epsilon}}$, for it to match with the edge potential. Once fixed at $e^{\frac{-\pi}{2\epsilon}}$ (for the first approximation), this factor must be the same for higher approximations ϕ_1^{III}, ϕ_2^{III} also. Therefore to match against the expansion (3.34), we should take ϕ^{III} in the form

$$\phi^{III} = 1 + \left\{\epsilon^{1/2}.a + (\epsilon^{3/2}\log\epsilon)b + \epsilon^{3/2}.c\right\} e^{\frac{-\pi}{2\epsilon}} \cosh\tfrac{\pi}{2}X' \cos\tfrac{\pi}{2}Y', \qquad (3.35)$$

where constants a, b, c (independent of ϵ) are to be evaluated later.

[(3.35) is equivalent to (2.19) with only the first term and appropriate a_0.]

By (2.20) and (3.35), towards the edge

$$\phi^{III}(X,Y) = 1 + \frac{1}{2}\left\{\epsilon^{1/2}.a + (\epsilon^{3/2}\log\epsilon)\,b + \epsilon^{3/2}.c\right\} e^{\frac{\pi}{2}X}\cos\frac{\pi}{2}Y. \qquad (3.36)$$

Substituting (3.34), (3.33) and (3.36) in (2.15) and equating corresponding order terms, we get

$$\left.\begin{aligned}
&\text{(i)}\quad \nabla^2\phi_0^{II} = \nabla^2\phi_1^{II} = \nabla^2\phi_2^{II} = 0\\
&\text{(ii)}\quad \phi_0^{II}(X,\pm1) = \phi_1^{II}(X,\pm1) = \phi_2^{II}(X,\pm1) = 0, \qquad \text{when } X\le 0\\
&\text{(iii)a.}\quad \phi_0^{II} \sim 2^{1/2}\left[R^{1/2}\cos\tfrac{\theta}{2} - \frac{R^{-1/2}}{2\pi}\left\{\left(\log R\cos\tfrac{\theta}{2} + \theta\sin\tfrac{\theta}{2}\right) + G\cos\tfrac{\theta}{2}\right\}\right]\\
&\qquad\qquad \phi_1^{II} \sim \frac{2^{1/2}}{2\pi}R^{1/2}\cos\tfrac{\theta}{2}\\
&\qquad\qquad \phi_2^{II} \sim -\frac{2^{1/2}}{12}\left[R^{3/2}\cos\tfrac{3\theta}{2} - \frac{3}{2\pi}R^{1/2}\left\{\left(\log R\cos\tfrac{\theta}{2} - \theta\sin\tfrac{\theta}{2}\right) + H\cos\tfrac{\theta}{2}\right\}\right]\\
&\qquad\qquad\qquad \text{as } R\to\infty,\\
&\text{b.}\quad \phi_0^{II} \sim \tfrac{1}{2}a\,e^{\frac{\pi}{2}X}\cos\tfrac{\pi}{2}Y\\
&\qquad\qquad \phi_1^{II} \sim \tfrac{1}{2}b\,e^{\frac{\pi}{2}X}\cos\tfrac{\pi}{2}Y\\
&\qquad\qquad \phi_2^{II} \sim \tfrac{1}{2}c\,e^{\frac{\pi}{2}X}\cos\tfrac{\pi}{2}Y\\
&\qquad\qquad\qquad \text{as } X\to-\infty.
\end{aligned}\right\} \qquad (3.37)$$

The problem for ϕ_0^{II} in (3.37) is the same as (2.23) with $a_0 = a\,\epsilon^{1/2}e^{-\frac{\pi}{2\epsilon}}$ and $a_n = 0$, $n\ge 1$. Its solution is given by (2.27), and the value of a is, by (2.28),

$$a = \frac{2^{3/2}}{\pi^{1/2}}\,e^{-1/2}. \qquad (3.38)$$

The problem for ϕ_1^{II} in (3.37) is the same as that for ϕ_0^{II}, with an additional constant factor $\left(\frac{1}{2\pi}\right)$. Its solution is therefore

$$\phi_1^{II} = \frac{2^{1/2}}{2\pi^{3/2}} Re(-i\omega), \tag{3.39}$$

and the value of b is

$$b = \frac{2^{1/2}}{\pi^{3/2}} e^{-1/2}. \tag{3.40}$$

Using (2.24), (2.30) and (2.31), the problem for ϕ_2^{II} (of 3.37) in the w-plane is

$$\left.\begin{aligned}
&\text{(i)} \quad \nabla^2 \phi_2^{II} = 0 \\
&\text{(ii)} \quad \phi_2^{II} = 0, \quad \text{when} \quad v = 0 \\
&\text{(iii)a.} \quad \phi_2^{II} \sim -\frac{2^{1/2}}{12\pi^{3/2}} Re(i\omega^3) + \frac{2^{1/2}}{8\pi^{3/2}}\left\{H - (1+\log\pi)\right\} Re(-i\omega), \quad \text{as} \quad |w| \to \infty \\
&\quad\ \text{b.} \quad \phi_2^{II} \sim \tfrac{1}{2} c \cdot e^{1/2} Re(-i\omega), \qquad \text{as} \quad |w| \to 0.
\end{aligned}\right\} \tag{3.41}$$

Solution of (3.41) is

$$\left.\begin{aligned}
&\phi_2^{II} = -\frac{2^{1/2}}{12\pi^{3/2}} Re(i\omega^3) + \frac{2^{1/2}}{8\pi^{3/2}} K\, Re(-i\omega), \\
&\text{where} \quad K = H - 1 - \log\pi,
\end{aligned}\right\} \tag{3.42}$$

and $$c = \frac{2^{1/2}}{4\pi^{3/2}} K e^{-1/2}. \tag{3.43}$$

[It should be noted that, although $Re(iw^3)$ is the dominating term in ϕ_2^{II} (3.42) for large $|w|$, it has little effect towards the interior.]

Substituting (3.38), (3.40) and (3.43) in (3.35):

$$\phi^{III} = 1 + e^{-\left(\frac{\pi}{2\epsilon}+\frac{1}{2}\right)} \left\{ \frac{2^{3/2}}{\pi^{1/2}} \epsilon^{1/2} + \frac{2^{1/2}}{\pi^{3/2}} \epsilon^{3/2} \log\epsilon + \frac{2^{1/2} K}{4\pi^{3/2}} \epsilon^{3/2} \right\} \cosh\frac{\pi}{2} X' \cos\frac{\pi}{2} Y'. \tag{3.44}$$

Expanding (2.27), (3.39) and (3.42) for large $|Z|$ using (2.30) and (2.31), writing $Z = R\,e^{i\Theta}$ and substituting in (3.34), we get

$$\phi^{II}(R,\Theta) = 1 + \epsilon^{1/2}.2^{1/2}\left[R^{1/2}\cos\tfrac{\Theta}{2} - \frac{R^{-1/2}}{2\pi}\left\{\left(\log R\cos\tfrac{\Theta}{2} + \Theta\sin\tfrac{\Theta}{2}\right) + (1+\log\pi)\cos\tfrac{\Theta}{2}\right\}\right]$$

$$+ \left(\epsilon^{3/2}\log\epsilon\right)\frac{2^{1/2}}{2\pi}R^{1/2}\cos\tfrac{\Theta}{2}$$

$$+ \epsilon^{3/2}\left[-\frac{2^{1/2}}{12}\left\{R^{3/2}\cos\tfrac{3\Theta}{2} - \frac{3}{2\pi}R^{1/2}\left(\log R\cos\tfrac{\Theta}{2} - \Theta\sin\tfrac{\Theta}{2}\right) - \frac{3}{2\pi}(1+\log\pi)R^{1/2}\cos\tfrac{\Theta}{2}\right\}\right.$$

$$\left. + \frac{2^{1/2}}{8\pi}K.R^{1/2}\cos\tfrac{\Theta}{2}\right], \quad R \to \infty. \qquad (3.45)$$

This is the edge potential ϕ^{II}, in terms of inner edge coordinates (R,Θ), expanded (for large R) to order $\epsilon^{3/2}$.

4. Matching

When (3.45) is written in terms of outer edge coordinates (δ,Θ) using (2.13) and re-arranged, the result is

$$\phi^{II}(\delta,\theta) = 1 + 2^{1/2}\left[\delta^{1/2}\cos\tfrac{\theta}{2} - \frac{1}{12}\delta^{3/2}\cos\tfrac{3\theta}{2}\right]$$

$$+ (\epsilon\log\epsilon)\frac{2^{1/2}}{2\pi}\left[\delta^{-1/2}\cos\tfrac{\theta}{2} + \frac{3}{4}\delta^{1/2}\cos\tfrac{\theta}{2}\right]$$

$$+ \epsilon\left[-\frac{2^{1/2}}{2\pi}\delta^{-1/2}\left\{\left(\log\delta\cos\tfrac{\theta}{2} + \theta\sin\tfrac{\theta}{2}\right) + \left(1+\log\pi\right)\cos\tfrac{\theta}{2}\right\}\right.$$

$$\left. + \frac{2^{1/2}}{8\pi}\delta^{1/2}\left\{\left(\log\delta\cos\tfrac{\theta}{2} - \theta\sin\tfrac{\theta}{2}\right) + \left(K+1+\log\pi\right)\cos\tfrac{\theta}{2}\right\}\right]. \qquad (4.1)$$

(4.1) gives the expansion for the edge potential to order $\epsilon^{3/2}$, expressed in outer edge coordinates (δ,θ) and expanded to order ϵ. Also, (3.33) gives the expansion for the exterior potential to order ϵ, expressed in inner edge coordinates (R,θ) and expanded to order $\epsilon^{3/2}$. By the matching principle (of Section 1, Part 2), these two expansions must be identical. Comparing (3.33) with (4.1) using (2.13), it is seen that the constant $G = 1 + \log \pi$. Substituting this value of G in (3.31)

$$F = \frac{2^{1/2}}{2\pi}\left(\frac{1}{2}K_3 + 1 - \log\pi\right) \quad , \text{ and hence}$$

$$H = \frac{11}{3} + 6K_4 + 3K_3 - 3\log\pi$$

$$= \frac{29}{3} - \frac{5\pi}{2} + 6K_2 + 3K_1 - 3\log\pi - 4\log 2, \quad \{\text{by (3.21)}\}.$$

Hence by (3.42),

$$K = \frac{26}{3} - \frac{5\pi}{2} - 4\log 2\pi + 6K_2 + 3K_1 .$$

When the value of this constant K is substituted in (3.44), the expansion for the interior potential ϕ^{III} is determined completely.

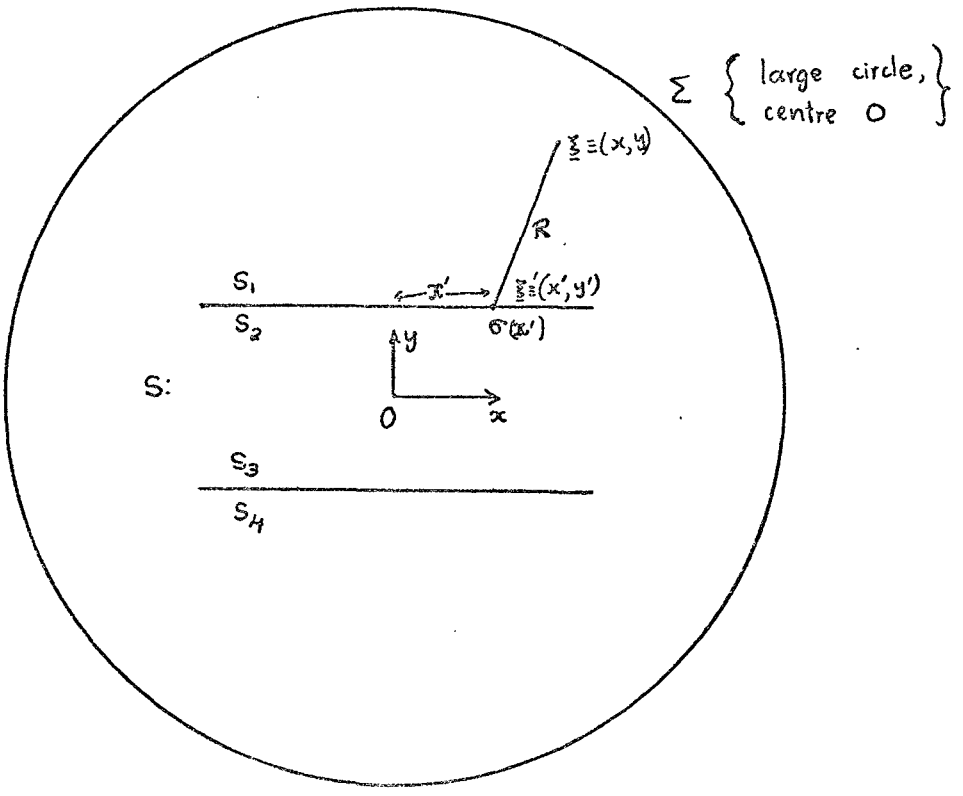
5. Concluding Remarks

A noteworthy feature in the matching of interior and edge potentials is the appearance of the small factor $e^{-\frac{\pi}{2\epsilon}}$ in the interior potential ϕ^{III} (3.44). This shows that when X' is small (near the centre) ϕ^{III} is nearly 1, all the constituent potentials of ϕ^{III} being very small. Towards the edge, however, as $X' \to \infty$ the constituent potentials contribute appreciable amounts comparable with the corresponding components of the edge potential.

Considering the two strips as a single body, an integral equation for the potential $\phi(x,y)$ can be obtained as:

$$\phi(\underline{\xi}) = \int_{S+\Sigma} \left(G \frac{\partial \phi}{\partial n} - \phi \frac{\partial G}{\partial n} \right) d\underline{\xi}' \quad , \text{ where } \quad G = \frac{1}{2\pi} \log R \, , \quad R = |\underline{\xi} - \underline{\xi}'| .$$

[See Fig.4]:



Using $\phi = 1$ and $\frac{\partial \phi}{\partial n} = -4\pi\sigma$ on the surface S of the strips, where σ is the charge density, and assuming the behaviour $\phi \sim \log r + B$ at ∞, it can be shown that

$$\phi(\underline{x}) = -4(\log r)\int_{-1}^{1} \sigma(x')\, dx' + B .$$

It follows that,

the total charge on the two strips $= 2\int_{-1}^{1} \sigma(x')\, dx' = -\frac{1}{2}$.

The total charge can also be worked out using potentials ϕ^{I}, ϕ^{II}, ϕ^{III} in the three regions discussed, and the result agrees with the above value.

The corresponding 3-D problem

The problem of two equal co-axial circular discs charged to unit potential each and placed close to each other, the ratio of separation to diameter being ϵ ($\ll 1$), was considered; and it was found that, near edge,

(i) exterior potential (first approximation) behaves like $-\frac{4}{\pi}\epsilon^{\frac{1}{2}} R^{\frac{1}{2}} \cos\frac{\theta}{2}$, when expressed in inner edge coordinates (R,θ) {defined as in (2.12)},

(ii) interior potential behaves like

(constant).$e^{\lambda X}.J_0(\lambda Y)$, where $J_0(\lambda) = 0$, when expressed in inner edge coordinates (X,Y) {defined as in (2.12)}.

It is not possible to find a suitable edge potential which satisfies both conditions (i) and (ii), so we cannot proceed further, with the present method.

APPENDIX

'CORRECT' SOLUTION FOR A STRIP OF VERY LARGE BUT FINITE LENGTH

For any two-dimensional body S charged to unit potential, the behaviour of the potential Φ at infinity (since $\Phi \not\to 0$ at ∞) has to be taken as $\Phi \sim A \log r + B$, where r is the distance from a convenient point and A and B are constants. Because of $\log r$ behaviour at ∞, there is a trivial solution $\phi = 1$ to this 2-D problem. Also, if ϕ is a solution, then so is $\Phi = \alpha\phi + (1-\alpha)$, so that the problem is not uniquely posed. Thus, the constant A is arbitrary and its value can be freely chosen. We take $A = 1$, find the solution of the 2-D problem with the behaviour $\phi \sim \log r + B$ at ∞. To get the general solution Φ for the given problem, we then use $\Phi = \alpha\phi + (1-\alpha)$, where α is the appropriate scaling constant for the given body S.

One might intuitively suppose that if the body is sufficiently long the solution would become independent of its precise length and shape. However the constant will depend, in general, on the precise geometry of the body.

To illustrate these remarks with a specific example, the scaling constant is now calculated for the case of a single long ellipse of semi-major and minor axes a and b, with $a \gg b$.

Using the result (Sneddon, 1966)

$$\left.\frac{\partial \Phi}{\partial z}\right|_{body} = C\left\{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right\}^{-1/2}, \qquad C = \text{constant},$$

for a thin ellipse bounded by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$,

charged to unit potential, it can be shown that

$$Cb = \frac{1}{K(k)}, \quad \text{where} \quad K(k) = \int_0^{\pi/2} \frac{d\theta}{\left\{1 - k^2 \sin^2\theta\right\}^{1/2}}, \quad k^2 = 1 - \frac{b^2}{a^2}.$$

By Whittaker and Watson (1927),

$$K(k) = \frac{\pi}{2} F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\right) \to \log\left(\frac{a}{b}\right), \qquad \text{as} \quad k^2 \to 1 - 0.$$

$$\therefore \quad Cb \to \frac{1}{\log\left(\frac{a}{b}\right)}, \qquad \text{as} \quad \frac{a}{b} \to \infty,$$

and in this limit

$$\left.\frac{\partial \Phi}{\partial z}\right|_{body} \to \frac{Cb}{(b^2 - y^2)^{1/2}}.$$

Also, for a single strip of breadth 2, by first paragraph,

$$\Phi = \alpha\, \phi_o^I + (1 - \alpha),$$

where ϕ_o^I is the potential given by (2.8) and (2.9), and α is the correct scaling constant for a single strip.

$$\therefore \quad \left.\frac{\partial \Phi}{\partial z}\right|_{strip} = \frac{\alpha}{(1 - y^2)^{1/2}}.$$

It follows that, since the ellipse in the limit $\frac{a}{b} \to \infty$ corresponds to a single strip if our attention is concentrated on the normal plane through its minor axis, the required scaling constant α for a single strip of breadth 2 is

$$\frac{1}{\log\left(\frac{a}{b}\right)} .$$

Thus, in this case α depends on the ratio a/b, however large this ratio may be. Probably, for a different shape the scaling factor is different. The explanation for this dependence of α on the length and the shape of the strip is, no matter how long the strip, it is not the same as the two-dimensional problem, and for a long strip its precise shape and precise length do affect the charge distribution near middle.

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