

SENSITIVITY ANALYSIS AND OPTIMISATION TECHNIQUES
IN THE TOLERANCING AND DESIGN OF ELECTRICAL NETWORKS

by

Dalip Sud

Thesis submitted for the degree of
Doctor of Philosophy in the Faculty
of Engineering, University of London.

Department of Electrical Engineering,
Imperial College of Science & Technology,
University of London.

January 1975

ABSTRACT

Generalised proofs of the necessary and sufficient conditions for the existence of higher order sensitivity invariants in linear systems in general and nonlinear circuits have been found. This general approach has led to the discovery of invariants for signal flow graphs. These results have not only proved useful in checking numerical algorithms generating higher order sensitivities but also in getting a deeper insight into the tolerance assignment problem in electrical networks.

Convergence criteria for the multivariate Taylor expansion of network functions have been found. A truncated Taylor expansion approximation has been used to evaluate the circuit response changes due to circuit element variations for both reciprocal and nonreciprocal networks. Statistical tests have been used to compare the approximate response with the corresponding exact calculations. On the basis of these results a fast and efficient, though approximate, Monte Carlo Circuit Analysis programme has been implemented.

A completely automated tolerance assignment procedure using Nonlinear Programming has been developed and implemented. Once the network response specifications have been specified the multi-dimensional Euclidian component space is systematically searched to determine the maximum allowable component deviations above and below the nominal value. The permissible tolerance can then be chosen. The importance and necessity of strictly controlling the sign of the component deviations is fully discussed. The information obtained in searching

the feasible space during the tolerance assignment procedure is used to design centre circuits. Hence the component tolerance may be widened. The necessity of using a Taylor expansion of modulus terms is discussed when tolerances for homogeneous network functions are being found.

A direct method has been discovered for the fast computation of differential network response sensitivity to circuit elements after simultaneous large changes to one or more of these elements. Unlike the adjoint network approach, this method does not require the solution of the full nodal admittance matrix equations if not all the elements are subject to large change. The efficiency of the method makes it particularly suitable for dynamic exploration of circuit behaviour due to parameter variation in an interactive graphics environment, as well as in a wide class of circuit optimisation procedures. The method has been successfully implemented on a PDP-15 with a graphics processor to display sensitivities for large changes to selected elements.

A method for computing tolerance regions in the admittance or impedance planes of network branches for specified response changes over a frequency range has been found. Using this method the effect on the branch admittance or impedance for different response specifications can be easily found.

Statement of Originality

In this thesis all the reported work which is not specifically ascribed to others, usually by quoting the appropriate references, is original to the best of the author's knowledge.

The major original contributions are listed below:

Chapter 2

The development of higher order sensitivity invariant relationships and the sensitivity invariants for signal flow graphs [105].

Chapter 3

Testing the adequacy of the third-order Taylor series approximation to the voltage response and demonstrating that such an approximation is computationally faster than the fastest method for an exact calculation [53].

Chapter 4

The use of a complex Taylor series for voltage response calculation in a tolerance assignment algorithm. The characterisation of the region of worst case component deviations in a multi-dimensional component space based on the results of the sensitivity invariant relationships. The use of the 'modulus' Taylor series for tolerancing networks with homogeneous network functions. The development of an automated tolerance assignment procedure for tolerancing reciprocal and non-reciprocal networks. A procedure for design centering the components of a circuit is described [61].

Chapter 5

The discovery of a method of computing first- and higher-order differential sensitivities when one or more components of a circuit are

changed from their nominal value by finite amounts, without the need to solve repeatedly the full set of circuit equations as evinced by the adjoint method. The implementation of this method within an interactive graphic circuit-design facility [106].

Chapter 6

A method for computing the change in admittance values of circuit branches for specified alteration in response voltage at a number of frequencies.

ACKNOWLEDGEMENTS

The research work presented in this thesis was carried out during the tenure of a S.R.C. Research Studentship, held at the Electrical Engineering Department, Imperial College, London (1971-1974).

I would like to express my sincere thanks to my supervisor, Dr. Robert Spence, for his support and advice throughout the course of the research, and also to Professor E.C. Cherry for allowing me access to the facilities of the Communication Section of the Department. I would also like to acknowledge the advice and friendship of my colleagues which has helped to sustain me throughout this period. I have appreciated in particular the criticism of Dr. W. Saraga, Stuart Lawson, Jadwiga Dunn and Dr. D. Agnew. Finally, many thanks are due to Christine Lourdan for typing the final draft of the thesis and my papers.

गायत्री महामन्त्र
ओ३म् भूर्भुवः स्वः तत्सवितुर्वरेण्यं भर्गो
देवस्य धियो नः प्रचोदयात् ।

Gaytri:

'Let us remember our beloved God, who is the creator and protector of everything, the destroyer of misery, the embodiment of bliss, the light, the embodiment of the pure mind, the one who persuades our faculties to do good deeds.'

To my father and mother

Definitions

$\underline{A}$	reduced incidence matrix
$\underline{A}^T$	transpose of $\underline{A}$
a_{ij}	element in row i and column j of the matrix $\underline{A}$
$\underline{B}_t$	simulation current source vector
$C(\underline{\Delta})$	cost function
δ_{ij}	Kronecker delta function, $\delta_{ij}=1$ when $i=j$; $\delta_{ij}=0$ when $i \neq j$
F, f	a circuit response function
g_m	mutual admittance of a voltage controlled current source
$\underline{I}_n$	nodal current source vector
$\delta \underline{I}_n$	change in the nodal current vector
$\underline{O}$	null matrix
m	number of variable branches in a circuit
n	number of nodes in a circuit
n_b	number of circuit branches
$S_{x_i}^F$	normalised sensitivity of F with respect to x_i
$\underline{T}$	transmission matrix of a signal flow graph
$\underline{V}_b$	branch voltage vector
$\underline{V}_n$	nodal voltage vector
$\delta \underline{V}_n$	change in the nodal voltage vector
ω	frequency in radians
$\underline{x}$	vector of variable network parameters (components)
$\Delta_i = \frac{\Delta x_i}{x_i}$	fractional change in a network parameter i
$\underline{Y}_b$	branch admittance matrix
$\underline{Y}_n$	nodal admittance matrix
$\Delta \underline{Y}_n$	change in the nodal admittance matrix
$\underline{Z}_n$	impedance matrix

C O N T E N T S

	Page
 CHAPTER 1 - <u>COMPUTER AIDED CIRCUIT DESIGN</u>	
1.1 Introduction	1
1.2 The circuit design process	1
1.3 Basic processes in CACD	5
1.4 Evolution of the research project	8
1.5 Transistor model	9
1.6 Computers	10
 CHAPTER 2 - <u>SUMMED-SENSITIVITY INVARIANTS OF ELECTRIC CIRCUITS</u>	
2.1 Introduction	13
2.2 Circuit function	15
2.3 Homogeneity	18
2.4 The homogeneity of circuit functions	19
2.5 Summed sensitivity invariants	22
2.5.1 R, L, C circuits	22
2.5.2 Reciprocal parameters	23
2.5.3 The effect of frequency	23
2.6 Nonlinear circuits	25
2.7 Higher order sensitivities	27
2.8 Component parameters	28
2.9 Scattering parameter sensitivity	31
2.10 Signal flow graphs	31
2.11 Applications of sensitivity invariants	34
2.12 Conclusions	36
 CHAPTER 3 - <u>CIRCUIT RESPONSE CALCULATION UNDER PARAMETER VARIATION</u>	
3.1 Introduction	37
3.2 Different methods of response calculation	38
3.2a Exact methods	39
3.2b Approximate methods	40
3.3 Taylor expansion of a complex response function	40
3.3a Convergence proofs for the Taylor expansion	41
3.3b Comparison test of convergence	45
3.3c Choice of truncation point	46

3.4	Accuracy of a truncated Taylor expansion compared to an exact calculation	48
3.4a	Correlation coefficient	50
3.4b	Histograms	50
3.4c	Analysis of numerical results	51
1)	Active circuits	52
2)	Passive circuits	53
3.4d	Histograms for non-Gaussian distribution	55
3.4e	General remarks on numerical results	55
3.5	Monte Carlo circuit analysis	56
3.6	Conclusions	57
	Diagrams	59

CHAPTER 4 - TOLERANCE ASSIGNMENT AND DESIGN CENTERING

4.1	Introduction	70
4.2	Earlier tolerance assignment algorithms	75
4.3	Choice of examples	79
4.4	The problem	79
4.5	Constraint evaluation	83
4.6	Selection of sign of component deviations	84
4.7	Treatment of homogeneous network functions	89
4.8	The choice of a nonlinear programming algorithm	90
a)	Penalty function methods	90
b)	The flexible tolerance algorithm	93
4.9	Design centering	94
4.10	Measurement of the success of the algorithm	102
4.11	Preferred values tolerances	105
4.12	Conclusions	107

CHAPTER 5 - DIFFERENTIAL SENSITIVITY AFTER SIMULTANEOUS LARGE CHANGES IN CIRCUIT ELEMENTS

5.1	Introduction	109
5.2	Small change sensitivity	110
5.3	Large change sensitivity for a single element	111
5.4	Differential sensitivity after large changes in one parameter	113
5.5	Extension of the method for simultaneous large changes in more than one parameter	115
5.6	Computational efficiency	119
5.7	Breakdown of computational cost	122
5.8	Numerical accuracy	124
5.9	Interactive graphics in circuit design	125
5.10	Dynamic display of differential sensitivity	127
5.11	Example	130
5.12	Conclusions	134

CHAPTER 6 - ADMITTANCE TOLERANCE REGIONS FOR MULTIPLY DEFINED VOLTAGE RESPONSE CHANGES

6.1	Introduction	137
6.2	Large change sensitivity model	139
6.3	To find admittance changes given response tolerances	142
6.4	Lineo-linear transformations	144
6.5	Tolerance regions	147
6.6	Conclusions	151

CHAPTER 7 - EPILOGUE

7.1	Summary of the contents of thesis	153
7.2	Conclusions and suggestions for future research	155

APPENDIX I - DERIVATIVE SENSITIVITY CALCULATIONS

I.1	Introduction	160
I.2	Derivation of nodal sensitivities	160
I.3	Sensitivities for the voltage gain	162
I.4	Sensitivities with respect to components	163
I.5	Implementation of the sensitivity analysis program	165
I.6	Computational speed enhancement	166
I.7	Accuracy of the program	168
I.8	Conclusions	169

APPENDIX II - NONLINEAR PROGRAMMING AND OPTIMISATION

II.1	Introduction	171
II.2	Notation and terminology	171
II.3	Minimum of a function	173
II.4	Minimum of a constrained function	174
II.5	Convexity	175
II.6	Unconstrained minimisation procedures	177
	Derivative methods	177
	Direct search methods	179
II.7	Constrained minimisation procedures	181
	Penalty function methods	182
	The flexible tolerance method	185
II.8	Conclusions	187

APPENDIX III - <u>TABLES OF RESULTS FOR CHAPTER 3</u>	188
---	-----

REFERENCES	218
------------	-----

CHAPTER 1

COMPUTER AIDED CIRCUIT DESIGN

1.1 Introduction

Circuit design like any creative process is difficult to identify and categorise, except in the broadest of terms. Though standard procedures can be used in many cases [1 - 3], problems often arise requiring original and perhaps novel solutions. Herein lies the basic difficulty in providing a completely automated synthesis procedure for circuits. Nevertheless, it is possible to provide the designer with tools, either test bench equipment or computer aids to explore the behaviour of a circuit. The information so gained will hopefully lead the designer to make suitable modifications which will improve the circuit. Though the term 'improvement' is necessarily vague, at any particular time the designer will have an intermediate aim or goal which he is trying to achieve. If some of these intermediate aims in the complete design process can be identified then it might be possible to provide specific aids to help guide the designer towards his final objective.

1.2 The Circuit Design Process

To obtain an understanding of the circuit design process, an attempt has been made in Fig.1.1 to reduce this into a sequence of logical steps at a macro-level. Each logical step can be regarded as a sub-process, which can and usually is subdivided into more

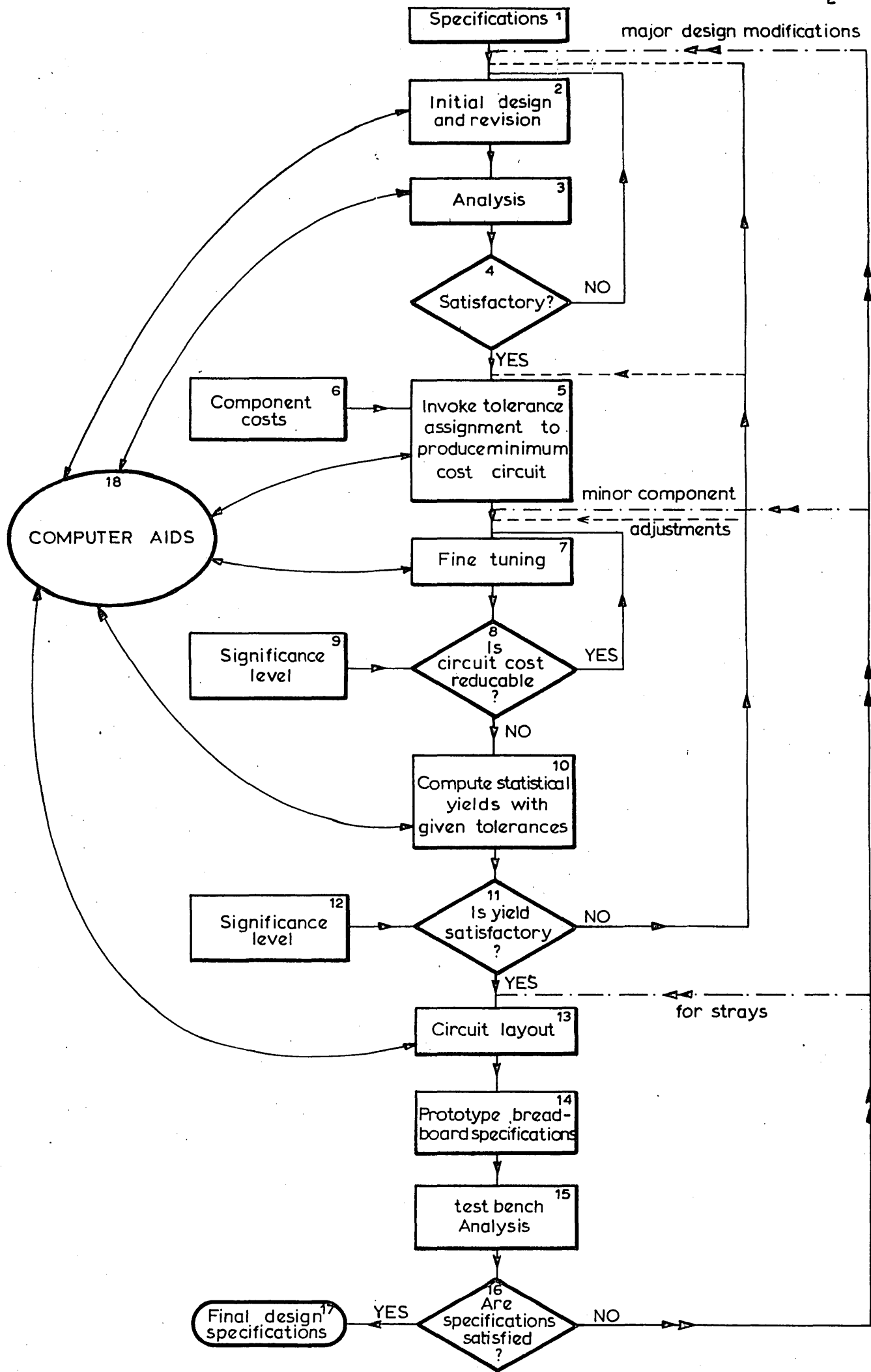


FIG. 1.1: Simplified flow chart of a circuit design process at a macro-level

basic processes. The designer controls the flow from one sub-process to the next using any 'hard' information or intuition. Therefore, each sub-process must provide him with information, more on demand, to enable him to direct operations with confidence. At any intermediate point between the specifications (1) and the final design (17) there may be considerable back tracking. The basic-processes are identifiable in any design aid, be they test bench equipment, tables of circuit and component values, or computer simulation programs. Each sub-process calls upon the appropriate basic processes to obtain specific results. Therefore, all basic processes within each sub-process must be highly automated requiring little or no designer participation, thus leaving him free to concentrate on the design itself. However, there is a need to provide sufficient entry points to the basic processes allowing manipulation under designer control.

The designer is normally supplied with some initial response specification (1) which may be for example amplitude, phase, input impedance as a function of frequency or time. He is then asked to produce a circuit which will meet the given specification. After satisfying himself that the initial specifications are reasonable, he proceeds to look for an initial design. From past experience he may know of circuits which are likely to give the required response. An initial analysis may indicate a need to relax or tighten some of the specifications to overcome any envisaged difficulties. The designer now commences to make modifications and refinements (2), (3) and (4) using any available aids to help him.

Traditionally a paper design would be followed by a breadboard circuit, which would then be changed directly until the specifications

were met. If the specifications could not be met for any reason then either the specifications would have to be altered or greater effort expended in the attempt to obtain the desired solution. The advent of large computers [4] and the parallel growth in complex circuitry and semiconductor technology [5] has made it necessary to develop and use computer-aided circuit design (CACD) aids [73] to help circuit designers and manufacturers. The need for CACD becomes more apparent for integrated circuit design [7] where simulation using discrete components may prove to be unrealistic at best if not impossible in practice. It has become common practice to simulate circuit designs, both breadboard and integrated, on digital computers [73]. A set of analyses programs (18) are then used by the designer to obtain information about the behaviour of the design. A computer simulation allows relatively easy circuit modification and retesting when compared to the traditional breadboard.

Once a satisfactory design has been achieved, the need to know manufacturing costs of the circuit is of great commercial significance [57]. To achieve a minimum cost design, in terms of the component tolerances, which is commensurate with minimal specification failure of the manufactured circuit, the tolerance assignment process is invoked (5), together with a fine tuning algorithm (7) for increasing the manufacturing yield. The yield is determined (10) from a statistical analysis [6]. At this point the designer may feel that the design is too expensive and go back to make further design amendments. If all proves satisfactory the designer can ask for the design to be laid out (12), [76] ready for a prototype construction. However, there will almost certainly be a need to

identify and remove the effect of any strays due to cross-coupling between different parts of the circuit. Only a modest attempt [7] has yet been made to devise numerical techniques to discover the occurrence of strays. Much work still needs to be done in this area. It is still more common in this respect to construct the circuit and perform test bench analysis (15). The designer may find that he has to modify his design, resulting in a new layout, minor tuning adjustments or even major design modifications. When the circuit has passed successfully through all these stages of design to the designer's satisfaction the final design specifications (17) can be sent for actual production of the circuit. It is usual practice to sample the production and test for units for correct functioning.

1.3 Basic Processes In CACD

Underlying the computer-aided design process as described above is the availability of a comprehensive suite of circuit design [8] and analysis procedures. Some of the most important basic processes are identified in Fig.1.2. These processes will be called upon by the designer at various times to perform specific tasks as indicated in Fig.1.1. The continuing importance [75] of the circuit design process of Fig.1.1 will not form the basic subject matter of this thesis. The scope of this thesis is confined to illuminating some of the basic processes as identified in Fig.1.2.

In any design facility adequate provision for models (8) used in the simulation of the devices used in circuit manufacture must be made. These models must also be able to describe the behaviour of the device embedded in its circuit environment. Work in this

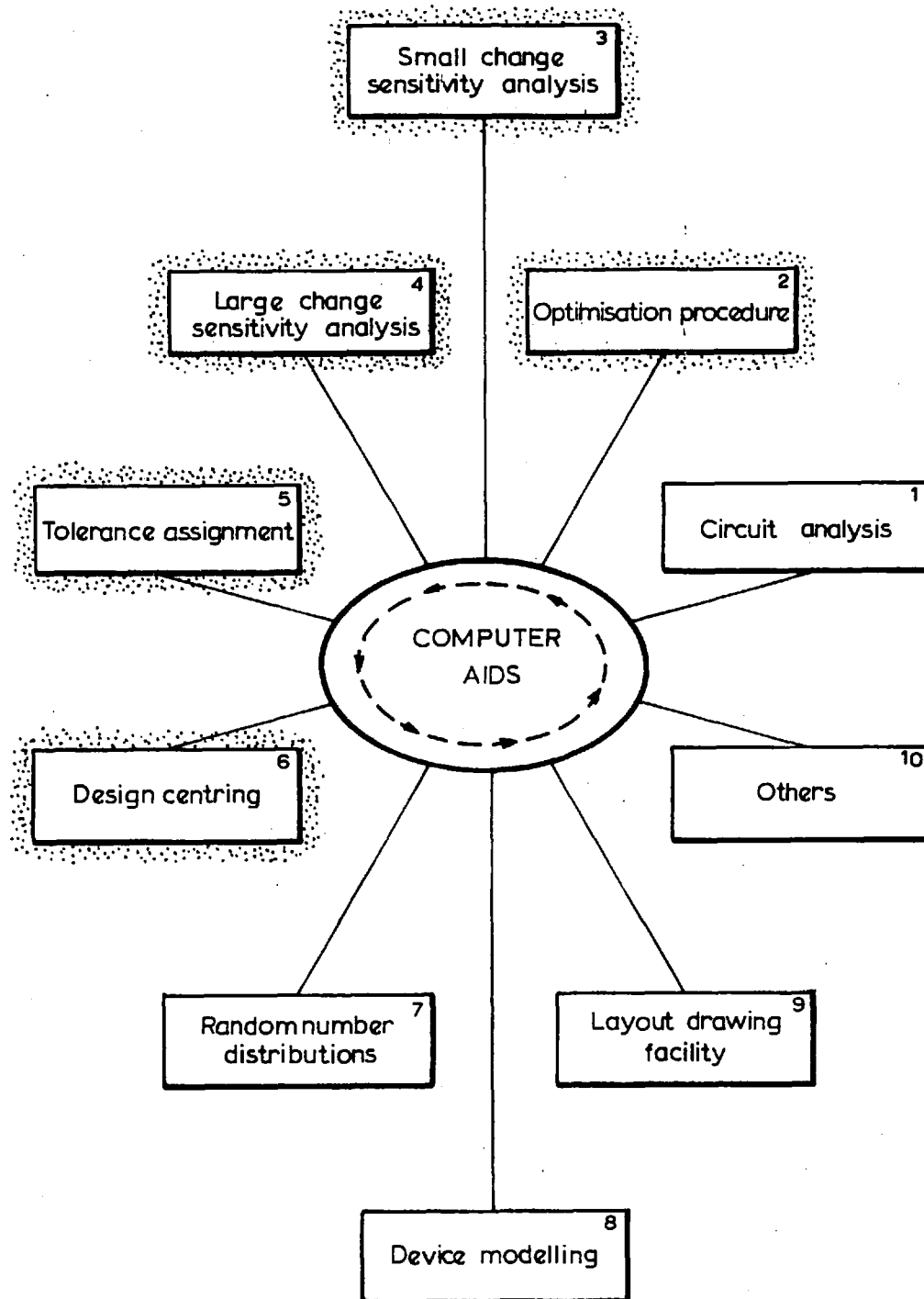


FIG.1.2: The principal computer aids in a circuit design facility.

area has been done by many researchers including Villalaz [9]. Once suitable device models are available a circuit can be simulated on a digital computer.

The topological circuit description most suitable for input to a computer is based on nodal analysis [32]. A set of circuit analysis programs (1) can now produce any desired response measure [73]. Both exact and approximate [53] (see Chapter 3) methods of response calculation have a role to play in circuit analysis. If a given circuit has to be modified, this can be done either independently by the designer or under the guidance of a suitable optimisation procedure (2), (see Appendix 2). For optimisation [92] as well as for exploration of a design with respect to circuit component variations efficient methods are required for both small change (3) (differential) and large change (4) sensitivities [10] (see Chapter 5 and 6). In recent years the basic processes of tolerance assignment (5), [56,61] and design centering (6) [61] have also been developed and investigated (see Chapter 4). The use of different random number distributions (7) in statistical circuit analysis and design [6,55] is a considerable problem. This is primarily due to the lack of predictable data from manufacturers for component value distributions. The aforesaid processes will receive greater attention in the remaining Chapters of this thesis.

Over many years much work has been done in designing and implementing facilities (9) for circuit layout [76]. These have not only been based on planar graph theory [77] but also on empirical evidence of how draughtsmen perform the work. The principal object here is usually to fit a circuit in the smallest possible area, while maintaining connection distances to a minimum subject to no overlapping

of connections if possible. Both this logic process and that of device modelling will not be considered further in this thesis.

Finally, the box marked (10) refers to those processes which are as yet either insufficiently tested or developed to provide a reliable computer aid. Some like stray analysis [75] are now beginning to receive some attention. Other possible areas of potential investigation and development are briefly indicated in Chapters 6 and 7.

1.4 Evolution of the Research Project

The main aim of the research reported in this thesis is not only to develop and enhance the theory and numerical techniques of some of the major basic processes in CACD but also to test those with suitable circuit examples. The principal goal of the research was to investigate the tolerance assignment problem and devise feasible solutions for it (see Chapter 4). As the investigation progressed cognizance of different fields in circuit theory became apparent; in particular, sensitivity analysis, (see Appendix 1 and Chapter 5) and the fundamental properties of sensitivity invariants in electrical circuits (see Chapter 2). The need for extremely fast response calculations in tolerance assignment lead to investigations of the complex Taylor approximation (see Chapter 3). In taking adequate account of the effect of simultaneous large element changes on the differential sensitivity, the discovery of the "Large change/Small change" algorithm of Chapter 5 was realised. Having devised and implemented a tolerance assignment and design centering algorithm (see Chapter 4) using nonlinear programming (see Appendix 2), could we discover an optimisation-free algorithm for tolerance assignment?

To answer this question, a preliminary study of complex function analysis was undertaken with particular emphasis on conformal mappings [80] (see Chapter 6). This study led to the discovery of a method for obtaining tolerance regions in the admittance plane for voltage changes at one or more frequencies. The initial results of this investigation are encouraging, though more work remains to be done to verify the applicability of this method for practical circuit design. The potential application of the Large change/Small change method of Chapter 5 to tolerance assignment is discussed briefly in Chapter 7. Several of the above mentioned fields are identified in Fig.1.2 in bold outline.

Throughout the reported research, the theory and results are confined to lumped, linear, time invariant networks, (both passive and active) analysed in the frequency domain. The dominant measure of the response used is that of output voltage due to a constant current excitation. In many cases, the theoretical ideas developed in this thesis apply to linear systems in general and can be extended to time domain analysis also.

1.5 Transistor Models

To simulate electrical circuits, the equivalent circuit type model has been chosen [9]. The primary reason for doing this is that the complexity of these models can be suitably altered by eliminating or adding elements to achieve the desired effect. The active devices, namely transistors, used in the different examples to follow are modelled using the hybrid- π circuit as shown in Fig.1.3. The component values in this model are calculated [8]

from the manufacturer's technical data of h-parameters for the device in question.

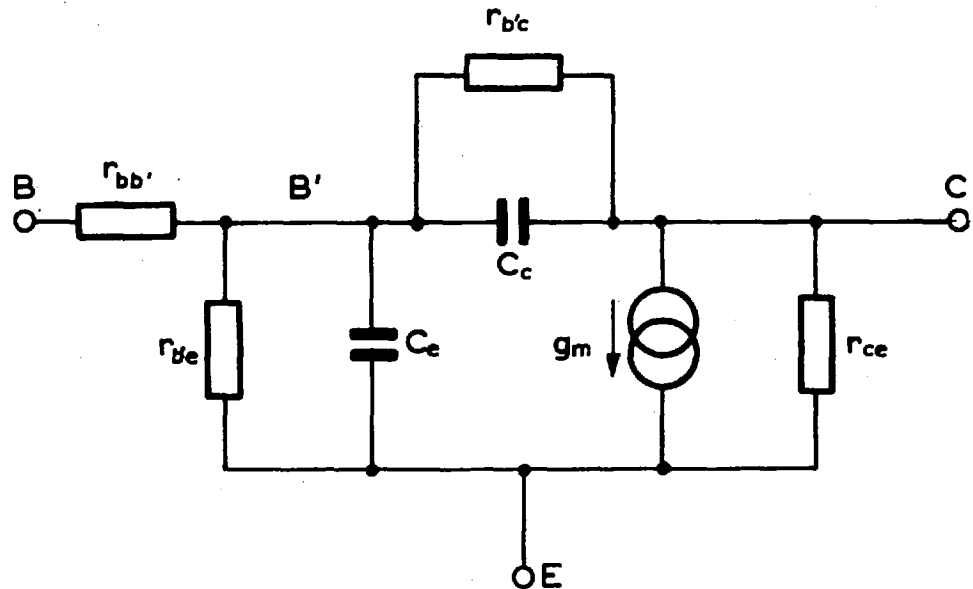


FIG.1.3: Hybrid- π transistor model in the common emitter configuration

Typical component values in this model at frequencies below 1.6 MHz for a transistor at room temperature are:

$$\begin{array}{lll}
 g_m = 50 \text{ mA/V} & r_{bb'} = 100\Omega & r_{b'e} = 1 \text{ K}\Omega \\
 r_{b'c} = 4 \text{ M}\Omega & r_{ce} = 80 \text{ K}\Omega & C_c = 3 \text{ pF} \quad C_e = 100 \text{ pF}
 \end{array}$$

1.6 Computers

Any discussion of CACD must start with some background considerations of the capabilities of different computers. Three basic computer groups can be identified: desk calculator, minicomputers and large multiprocessing machines. The desk calculator restricts the user to a fairly limited set of calculations, and when used iteratively is prone to human errors, such as occur in keying in the

correct information. Though it is useful, it can only play a very limited role in a CACD facility.

The range of minicomputers is large though they do present certain common characteristics that are significant in a design facility. In most cases their accuracy is limited to 7 - 10 decimal places, added to this most of them have a limited basic instruction set. The addressing and arithmetic capabilities can often be limited when compared with the need to do a large amount of arithmetic in most circuit analysis algorithms. The machines are relatively expensive, however, and often have dedicated graphics facilities. While the former mitigates against their use in analysis requiring optimisation procedures, the latter can provide a powerful interface for the designer to interact [7] with both the sub and basic design processes identified earlier.

The large multiprocessing machines like the CDC 6000 series are increasingly being made available for use in an interactive environment as well as in the earlier batch mode. These can easily give accuracies of more than 10 significant digits in single precision and are often specifically designed to perform arithmetic at high speed. Another significant feature of these is the large amount of core space available for both programs and data.

Numerical procedures have been implemented on both mini- and multiprocessing machines. However, a comparison of efficiency amongst different numerical procedures as implemented on different machines poses many problems. A simple comparison of C times can hardly be regarded as adequate. Its dependence not only on the machine but also on the mode of programming and compilation cannot be overlooked lightly. There have been attempts [100] to run stan-

dard problems on different computers but even these comparisons can only serve as a rough guide. Therefore, no attempt has been made to give precise times for the different computations reported in this thesis. Where relevant, the number of operations for different procedures are calculated and compared. An operation is defined as one complex multiplication or division plus an addition or subtraction.

CHAPTER 2

SUMMED-SENSITIVITY INVARIANTS OF ELECTRIC CIRCUITS

2.1 Introduction

In an electrical circuit, a change in the value of a component parameter, whether intentional or parasitic, will normally cause a change in overall circuit behaviour. In other words, circuit behaviour is sensitive to variation in component parameter values. Changes in these values may be large or small; in each case an appropriate measure of sensitivity [11] describes their effect on a function (e.g. voltage response) of the circuit.

One measure of sensitivity describes the effect, on a circuit function, of infinitesimally small changes in component parameter value. If ΔF is the change in a circuit function F due to a change Δx in the value of a component parameter x , then the normalized sensitivity of F with respect to x , denoted by S_x^F , is defined by the relation

$$S_x^F = \left. \frac{\Delta F/F_o}{\Delta x/x_o} \right|_{\Delta x \rightarrow 0} \quad (2.1)$$

where the subscript 'o' refers to a nominal value. Though the circuit function F may be related to the time- or frequency-domain behaviour of the circuit, and could be any of a wide variety of circuit attributes such as voltage ratio, self-impedance or rise-time, in this chapter only the frequency-domain behaviour is

considered. Similarly, a wide choice exists for the component parameter x ; it may be a resistance, an admittance or perhaps the constant of exponentiation in a diode relation. The superscript (F) and subscript (x) may be omitted from S if they are apparent from the context. An un-normalized sensitivity may be defined by omitting the nominal values of F and x from the definition in equation (2.1); similarly, a semi-normalized sensitivity is defined by omitting only the nominal value of F .

Almost six years ago an interesting property of component sensitivities* was remarked upon [12]. It was found that for the voltage gain of any linear circuit, the sum of all the normalized component sensitivities is a constant which can be established without reference to the components or to the manner in which they are connected. More formally, it was found that

$$\sum_{\text{all } i} S_{x_i}^F = K \quad (2.2)$$

where K is a constant and i ranges over all the component parameters. Not surprisingly, K is referred to as a summed-sensitivity invariant; its value is determined not by the circuit but by the dimensions of F and x . Summed sensitivity invariance is clearly of great theoretical and practical interest, not least because it is obviously futile to test alternative circuits and component values in order to minimize the sum of the component sensitivities. Furthermore, as a measure of sensitivity forms a basic part of the reported research work in this thesis, it is useful to develop more fully some of the existing work on sensitivity invariants [13]. These not only have important

* We refer to $S_{x_i}^F$ as a component sensitivity since it is defined for a particular component parameter.

significance in the tolerance assignment problem (see Chapter 4) but also practical significance in testing the sensitivity programs developed (see Appendix 1).

In recent years much attention [14-26] has been focussed upon the theory of summed sensitivity invariants, with the result that the types of circuit and the circuit functions that the theory encompassed have been considerably extended. One purpose of this Chapter is therefore to consolidate the more significant of these results, but without the mathematical complexity that has often characterized their derivation. The principal means by which this consolidation is achieved not only helps to clarify the fundamental basis of the invariant property, but also leads to the derivation of new higher order invariants and those for signal flow graph networks. The fundamental property of summed-sensitivity invariance is, in fact, that property of electrical circuits (shared by many other physical systems) known as homogeneity. It is this property, first pointed out about ten years ago [13], that is the underlying theme of this Chapter.

2.2 Circuit Function

The fundamental property of homogeneity is perhaps best introduced by means of a simple example such as the circuit of Fig.2.1a.

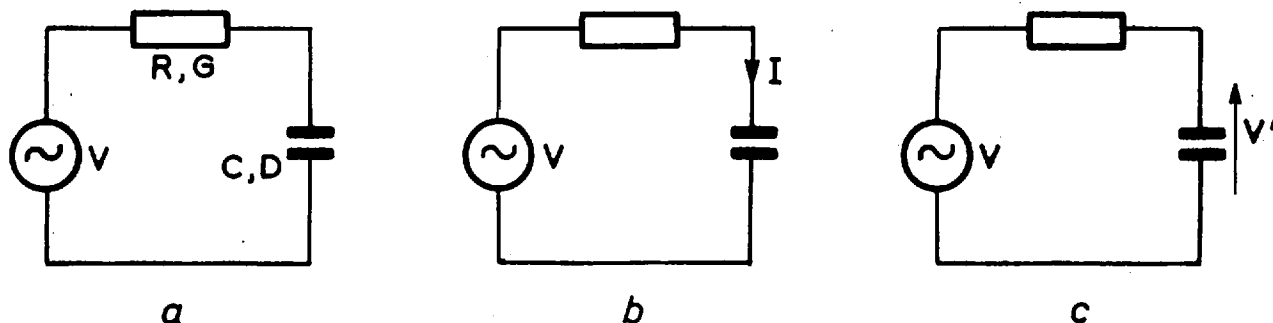


FIG. 2.1

It is vital to the discussion to acknowledge that the linear resistor can be characterized either by its resistance R or its reciprocal, the conductance G ; similarly, that either the capacitance C or its reciprocal, the elastance D , are alternative parameters describing the linear capacitor. Let us suppose that the circuit function of interest is a transfer admittance $Y = I/V$ (Fig.2.1b) at a frequency ω . Depending upon the parameter pair chosen to describe the components of the circuit, there are four equivalent expressions for Y in terms of the component parameters:

$$Y = \frac{j\omega C}{1+j\omega C/G} = \frac{j\omega}{D+j\omega R} = \frac{j\omega C}{1+j\omega CR} = \frac{j\omega G}{GD+j\omega} \quad (2.3)$$

From equation (2.3) we see that if both G and C are (say) doubled in value, then Y is also doubled in value. More generally, if both G and C are multiplied by a constant λ , then Y is also multiplied by λ . A similar property of proportionality applies to the parameter set $\{R, D\}$, but not to the parameter sets $\{R, C\}$ and $\{G, D\}$. More formally, the circuit function Y is said to be homogeneous in the parameter sets $\{R, D\}$ and $\{G, C\}$, but not in the other parameter sets. Furthermore, the homogeneity of the function Y is said to be of order one in the parameter set $\{G, C\}$ because the multiplication of both G and C by a factor λ causes Y to be multiplied by λ raised to the power one. In contrast, examination of (2.3) shows that the homogeneity of Y with respect to the parameter set $\{R, D\}$ is of order minus one. However, the order of homogeneity is not completely determined by the parameter set; examination of the alternative expressions for a voltage gain (V'/V) of the same circuit (see Fig.2.1c).

$$\frac{V'}{V} = \frac{1}{j\omega CR+1} = \frac{G}{j\omega C+G} = \frac{D}{j\omega R+D} = \frac{GD}{j\omega+GD} \quad (2.4)$$

shows that whereas the function is again homogeneous with respect to the parameter sets $\{G, C\}$ and $\{R, D\}$, the order of homogeneity in each case is now zero. For example, if both R and D are multiplied by a factor λ , the circuit function is multiplied by λ raised to the power zero; i.e., it is left unchanged.

It is not difficult to appreciate from the above example that there will be other groupings of circuit functions and component parameters to which the property of homogeneity applies, to various orders. In anticipation of a later result, some groupings which apply to the frequency domain behaviour of linear circuits are presented in Table 2.1, whose entries indicate the order of homogeneity.

Circuit function \ Parameter set	Parameter set	
	G, C, L^{-1}, g_m	R, C^{-1}, L, r_m
Voltage or Current Ratio	0	0
Self- or Transfer-Impedance	-1	1
Self- or Transfer-Admittance	1	-1

TABLE 2.1

The order of homogeneity of various linear circuit functions with respect to two parameter sets.

The important point to note is that whatever circuit function is under consideration, and whatever components are present within the circuit (i.e. linear resistors (R), inductors (L), capacitors (C) and voltage-controlled current sources (g_m) or current-controlled voltage sources (r_m), it is always possible to identify at least two parameter sets with respect to which the function is homogeneous. One parameter set which is often employed in this chapter is $\{G, L^{-1}, C, g_m\}$ and will be called the 'pseudo-admittance set'.

2.3 Homogeneity

The fact that a function is homogeneous with respect to a set of parameters has implications for the associated sensitivities. This fact, as well as additional insight into sensitivity invariants, emerges from the following definition of homogeneity and statement of Euler's Theorem:

Definition

A function $F(x_1, x_2, x_3, \dots, x_n)$ is said to be homogeneous with respect to the parameters set $\{x_1, x_2, \dots, x_n\}$ if and only if

$$F(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^K F(x_1, x_2, \dots, x_n) \quad (2.5)$$

where λ is an arbitrary constant and K is the order of homogeneity.

Euler's Theorem

If a function is homogeneous and of order K , then*

$$\frac{\partial F}{\partial x_1} x_1 + \frac{\partial F}{\partial x_2} x_2 + \dots + \frac{\partial F}{\partial x_n} x_n = K F(x_1, x_2, \dots, x_n) \quad (2.6)$$

*Throughout this chapter the normal conditions for differentiability are assumed, namely, that the function and its partial derivatives are continuous and exist for all combinations of parameter values in a given range of these values.

The significance of the above definition and theorem lies in the fact that the left-hand side of (2.6) can be identified as the sum of the semi-normalized sensitivities of F with respect to the parameters x . Thus, if both sides are divided by F , the sum of the normalized sensitivities of F with respect to the parameters $x_1, x_2, x_3, \dots, x_n$ is seen to be a constant, K , which is the order of homogeneity. Thus, for the transfer admittance defined in Fig. 2.1b and the parameter set $\{G, C\}$, for the combination of which the order of homogeneity is one, application of Euler's Theorem yields the relation

$$\frac{\partial Y}{\partial G} G + \frac{\partial Y}{\partial C} C = Y$$

In other words

$$S_G^Y + S_C^Y = 1$$

or

$$\sum S^Y = 1$$

all conductance
and capacitance

2.4 The Homogeneity of Circuit Functions

So far, the property of homogeneity has been introduced by means of an example based on a simple circuit containing one resistor and one capacitor. For the property of homogeneity to be significant it must be shown that a very wide range of circuits and circuit functions possesses this property. Consider a circuit composed of any number of interconnected branches of the form shown in Fig.2.2. Clearly, any circuit composed of linear resistors,

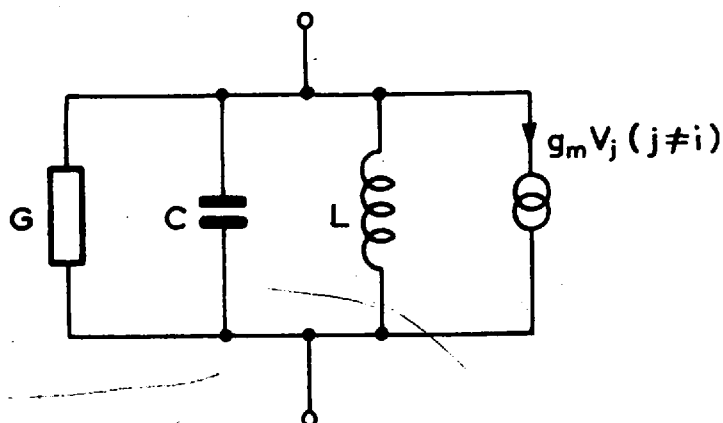


FIG. 2-2
A representative branch (i) of a circuit

capacitors, inductors and voltage-controlled current sources can be modelled in this way. If the circuit is excited at its nodes by current sources, its behaviour is described by the nodal matrix equation

$$\underline{I} = \underline{Y} \underline{V} \quad (2.7)$$

where $\underline{I}$ and $\underline{V}$ are, respectively, the vectors of nodal current excitation and nodal voltage response, and $\underline{Y}$ is the nodal admittance matrix appropriate to the frequency of interest. The nodal voltage response can be expressed in terms of the nodal current excitation by

$$\underline{V} = \underline{Z} \underline{I} \quad (2.8)$$

where $\underline{Z}$, the nodal impedance matrix, is the inverse of $\underline{Y}$. Thus, any element of $\underline{Z}$ can be expressed as

$$Z_{ji} = \frac{\Delta_{ij}}{\Delta} \quad (2.9)$$

where Δ is the determinant, and Δ_{ij} the appropriate cofactor, of

the nodal admittance matrix $\underline{Y}$.

Examination of Fig.2.2 shows that each non-zero element of the nodal admittance matrix has the form

$$G + j(\omega C - \frac{L^{-1}}{\omega}) + g_m \quad (2.10)$$

where each G , C , L^{-1} and g_m may be the result of more than one branch connected in parallel; the sign is governed in the usual way by the element's position in the matrix. It is easy to see that if every G , C , L^{-1} and g_m is multiplied by λ , then every element of $\underline{Y}$ will also be multiplied by λ . Now the determinant of a matrix is defined [28] as the sum of the $n!$ products each of n elements chosen in such a way that each (unrepeated) term in the sum comprises one element from each row of the matrix and one from each column; this apart from a sign, which is immaterial to the discussion. From this definition it follows that if each element of $\underline{Y}$ is multiplied by the factor λ , then the determinant Δ of $\underline{Y}$ will be multiplied by λ^n if $\underline{Y}$ is an $(n \times n)$ matrix. Similarly, it follows that any first cofactor of the $(n \times n)$ matrix $\underline{Y}$ will be multiplied by λ^{n-1} . Thus, any self- or transfer-impedance (i.e. any element of $\underline{Z}$) will be multiplied by λ^{-1} . We conclude, therefore, that the self or transfer-impedance of any circuit containing branches of the form shown in Fig.2.2 is a homogeneous function, of order minus one, in the parameters G , C , L^{-1} and g_m .

By similar reasoning, it is a straightforward matter to establish that transfer admittances are homogeneous of order minus one in the parameter set $\{R, L, C^{-1}, r_m\}$, where r_m is the mutual resistance of a current-controlled voltage source, and that current

and voltage ratios are homogeneous functions of order zero in both parameter sets. In this way, the properties summarized in Table 2.1 may be derived.

Since homogeneity has been shown to be a sufficient condition for the existence of a summed sensitivity invariant, one might ask if it is also a necessary condition. In other words, if a non-homogeneous function is proposed, can a summed sensitivity invariant be found for the specified parameter set? The answer is 'no'; there is a proof, due to Euler [29] that any function which satisfies the Euler relation is homogeneous. Nevertheless, a function which is inhomogeneous can often be made homogeneous by a simple transformation applied to one or more of the parameters, permitting a summed sensitivity invariant to be obtained: examples in sections 2.5.2 and 8 illustrate this fact.

2.5 Summed-sensitivity Invariants

The importance of the property of homogeneity can be demonstrated by some examples chosen to illustrate various aspects of summed-sensitivity invariants.

2.5.1 R, L, C Circuits

Consider a circuit composed of linear resistors, inductors, and capacitors. Since any self- or transfer-admittance Y of the circuit is homogeneous of order minus one in the parameter set $\{R, D, L\}$, it follows from Euler's Theorem that

$$\sum_{\text{all } R} S_R^Y + \sum_{\text{all } D} S_D^Y + \sum_{\text{all } L} S_L^Y = -1 \quad (2.11)$$

Note that the current and voltage defining the admittance Y can be chosen arbitrarily from all the currents and voltages that can be identified in the circuit. If it is a circuit impedance Z that is of interest, then Z replaces Y on the left-hand side of (2.11) and, because Z is homogeneous of order one in the parameter set $\{R, D, L\}$, the constant on the right-hand side of (2.11) is now $+1$.

2.5.2 Reciprocal Parameters

It may be more convenient for the capacitance of a capacitor, rather than its elastance, to appear in a summed-sensitivity invariant such as (2.11). Because of the mutually reciprocal relation between C and D , $C^2 \partial Y / \partial C = - \partial Y / \partial D$, so that (2.11) becomes

$$\sum_{\text{all } R} S_R^Y - \sum_{\text{all } C} S_C^Y + \sum_{\text{all } L} S_L^Y = -1 \quad (2.12)$$

Thus if any parameter in a set is replaced by its reciprocal, the corresponding term in the sensitivity summation must be reversed in sign [30]. In this way, an extensive range of summed-sensitivity invariants may be generated for parameter sets for which the circuit function is not homogeneous; equation (2.12) is an example for the set $\{R, L, C\}$. Equation (2.12) is, in fact, a simple example of the use of the chain rule of differentiation.

2.5.3 The Effect of Frequency

Given that the major functions of a linear circuit are homogeneous in the parameter set $\{G, C, L^{-1}, g_m\}$, it follows that they are also homogeneous, of the same order, in the admittance set

$\{G, (\omega C), (\omega L)^{-1}, g_m\}$, thereby allowing summed-sensitivity invariants of the form

$$\sum_{\text{all } G} S_G^Z + \sum_{\text{all } C} S_{B_C}^Z + \sum_{\text{all } L} S_{B_L}^Z + \sum_{\text{all } g_m} S_{g_m}^Z = 1 \quad (2.13)$$

to be generated, where B_C and B_L are the susceptance of capacitors and inductors, respectively. But useful invariants can also be generated which involve frequency explicitly, as an independent variable [21]. Thus for the circuit of Fig.1, whose transfer admittance can be expressed as

$$Y = \frac{j\omega C}{1 + j\omega C/G} \quad (2.14)$$

it is clear that if the value of G remains constant while both C and $(\omega)^{-1}$ are multiplied by a constant λ , then the value of Y is multiplied by λ^0 ; i.e., it remains unchanged. Since we have shown that Y is homogeneous of order zero in the parameter set $\{C, (\omega)^{-1}\}$; it follows that

$$\frac{\partial Y}{\partial C} \frac{C}{Y} + \frac{\partial Y}{\partial (\omega^{-1})} \frac{(\omega^{-1})}{Y} = 0 \quad (2.15)$$

or, recalling the rule to be followed if a parameter is replaced by its reciprocal,

$$S_C^Y = S_\omega^Y \quad (2.16)$$

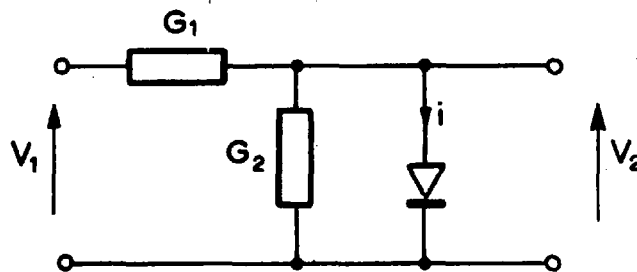
Here we have a relation between the sensitivity of a circuit function to a particular component, and the (normalized) rate of change of that function with respect to frequency. This is a simple example of a more general property of linear circuits [21]. Some interesting

relations between sensitivity invariants for single component sets and the normalized sensitivity of the circuit function with respect to the complex frequency have been developed in reference [33].

2.6 Nonlinear Circuits

The property of homogeneity, and the existence of summed-sensitivity invariants, is not associated exclusively with linear circuits. For example, in the circuit of Fig.2.3, application of Kirchhoff's Current Law yields the relation

$$G_1 V_1 = G_2 V_2 + a(\exp(\beta V_2) - 1) \quad (2.17)$$



$$i = a (\exp (\beta V_2) - 1)$$

FIG. 2.3

A non-linear diode circuit

With a view to finding a summed-sensitivity invariant associated with the scalar voltage V_2 we seek a homogeneous relation for V_2 with respect to a parameter set. More than one set can be found. One is the set $\{R_1, R_2, \beta^{-1}, V_1\}$, where $R_1 = G_1^{-1}$ and $R_2 = G_2^{-1}$; a set which involves the exponentiation parameter (β). To show that this is a set in which V_2 is homogeneous, (2.17) is first written as:

$$V_2 = \frac{R_2}{R_1 + R_2} V_1 - \frac{R_1 R_2}{R_1 + R_2} a \left[\exp \left(\frac{V_2}{\beta^{-1}} \right) - 1 \right] \quad (2.18)$$

We test for homogeneity first by multiplying each member of the set by λ , and finding the new value of V_2 , denoted V'_2

$$V'_2(\lambda R_1, \lambda R_2, \lambda \beta^{-1}, \lambda V_1) = \lambda \frac{R_2 V_1}{R_1 + R_2} - \lambda \frac{R_1 R_2 a}{R_1 + R_2} \left[\exp \left(\frac{V'_2}{\lambda \beta^{-1}} \right) - 1 \right] \quad (2.19)$$

However, if it is found that

$$V'_2(\lambda R_1, \lambda R_2, \lambda \beta^{-1}, \lambda V_1) = \lambda^K V_2 \quad (2.20)$$

then the function V_2 is homogeneous of order K in the proposed parameter set. Since equation (2.19) does satisfy (2.20) with $K = 1$, we conclude that V_2 is homogeneous of order unity in the parameter set $\{R_1, R_2, \beta^{-1}, V_1\}$. It follows immediately that

$$s_{R_1}^{V_2} + s_{R_2}^{V_2} + s_{\beta^{-1}}^{V_2} + s_{V_1}^{V_2} = 1 \quad (2.21)$$

There are other parameter sets that can be identified from equation (2.17). One such is $\{G_1, G_2, a\}$. This set can be verified, as above, to be homogeneous of order zero, yielding the relation [26].

$$s_{G_1}^{V_2} + s_{G_2}^{V_2} + s_a^{V_2} = 0 \quad (2.22)$$

2.7 Higher Order Sensitivities

The conventional measure of sensitivity defined in (2.1) is called a first-order sensitivity. Since they are of practical importance, second-, third- and higher-order sensitivities have also been defined. The question naturally arises as to whether these

higher-order sensitivities possess invariance properties.

That they do is easily demonstrated. Consider again a circuit function F which is homogeneous of order K in the parameter set $\{x_i\}$; so that equation (2.4) is relevant. Differentiation of (2.4) with respect to λ yields

$$\sum_{\text{all } i} \frac{\partial F}{\partial \lambda x_i} x_i = K \lambda^{K-1} F \quad (2.23)$$

A first-order sensitivity invariant, such as is expressed by equation (2.11), is obtained by setting λ to unity. The second-order sensitivity of F with respect to the x_i can be obtained by differentiation of (2.23) with respect to λ to yield

$$\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 F}{\partial (\lambda x_i) \partial (\lambda x_j)} x_i x_j = K(K-1) \lambda^{K-2} F \quad (2.24)$$

If λ is now set to unity, equation (2.2) becomes

$$\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 F}{\partial x_i \partial x_j} \frac{x_i x_j}{F} = K(K-1) \quad (2.28)$$

an expression whose left-hand side is the summation of all the normalized second-order sensitivities; n is the number of component parameters.

Summed-sensitivity invariants for higher-order sensitivities may similarly be derived by successive differentiation of (2.23) before setting λ to unity. The results for the three values of K of interest are easily deduced from equation (2.24): (a) if $K=0$, all higher-order summed-sensitivity invariants are zero, (b) if $K=1$, all are zero except for the first-order sensitivities, when the invariant is unity, (c) if $K = -1$, the invariant is $(-1)^{2t-1} t!$

where t is the order of sensitivity.

Relationships similar to equation (2.12) of Section 2.5.2 for reciprocal parameters like capacitances can be also developed for higher-order sensitivities. Thus:

$$\frac{\partial^2 F}{\partial D_j \partial D_i} = \frac{\partial^2 F}{\partial C_j \partial C_i} C_i^2 C_j^2 + C_j^2 \frac{\partial C_i^2}{\partial C_j} \frac{\partial F}{\partial C_i}$$

If it is assumed that D_j and D_i are independent then

$$\frac{\partial^2 F}{\partial D_j \partial D_i} = \frac{\partial^2 F}{\partial C_i \partial C_j} C_i^2 C_j^2 + 2 C_i^3 \frac{\partial F}{\partial C_i} \delta_{ij}$$

The second order invariant relationship for the voltage gain of an $\{R, D\}$ network is obtained from equation (2.25)

$$\sum_i \sum_j \frac{\partial^2 F}{\partial R_i \partial R_j} \frac{R_i R_j}{F} + \sum_i \sum_j \frac{\partial^2 F}{\partial D_i \partial D_j} \frac{D_i D_j}{F} = 0 \quad (2.26)$$

However, for an $\{R, C\}$ network this becomes

$$\sum_i \sum_j \frac{\partial^2 F}{\partial R_i \partial R_j} \frac{R_i R_j}{F} + \sum_i \sum_j \frac{\partial^2 F}{\partial C_i \partial C_j} \frac{C_i C_j}{F} + 2 \sum_i \frac{\partial F}{\partial C_i} \frac{C_i}{F} = 0 \quad (2.27)$$

Equation (2.27) cannot be strictly regarded as a second order invariant because of the existence of the first order terms.

Similarly, all higher order relationships for reciprocal parameters will have lower order terms and consequently are not invariants of a given order.

2.8 Component Parameters

Our previous discussion has been confined to the effect, upon

a circuit function, of small changes in linear 2-terminal components and two types of controlled source. The question naturally arises 'Are components such as gyrators, coupled inductors, ideal transformers and linear models of transistors also described by parameters that can be included in a set for which a circuit function is homogeneous?' In answering this question we shall illustrate an alternative method for examining the homogeneity of a circuit function with respect to a parameter set.

The nodal admittance* matrix $\underline{Y}$ of a linear circuit is related to the branch-node incidence matrix $\underline{A}$ and the branch admittance matrix $\underline{YB}$ by the expression [31]

$$\underline{Y} = \underline{A}^t \underline{YB} \underline{A} \quad (2.29)$$

Since the elements of $\underline{A}$ have the value 1, -1 or 0, pre- and post-multiplication of $\underline{YB}$ by $\underline{A}$ and $\underline{A}^t$ respectively can only lead to the additive modification of $\underline{Y}$ by elements of $\underline{YB}$. Thus, if the elements of $\underline{YB}$ are homogeneous (of the same order) with respect to a given parameter set, then so also will be the elements of $\underline{Y}$. The question posed above is therefore equivalent to asking whether the connection of a new linear component to an existing circuit results in the parameters of that component appearing directly (or multiplied by a constant) in the original or extended $\underline{YB}$ matrix.

Since the gyrator, as well as any other 3-terminal linear component described by a 2 by 2 admittance matrix, adds the same admittance parameters to new or existing elements of the branch admittance matrix,

*An analogous development can be conducted in terms of the branch impedance matrix.

these parameters can be included in the homogeneous set already containing the pseudo-admittance parameters. For example, for a circuit containing linear resistance (R), inductance (L), capacitance (C), a gyrator of gyration conductance G and a 3-terminal terminal transistor described at the frequency of interest by the admittance parameters y_{11} , y_{12} , y_{21} and y_{22} , a valid parameter set, with respect to which any circuit function is homogeneous, is the set

$$\{R^{-1}, C, L^{-1}, G, y_{11}, y_{12}, y_{21}, y_{22}\} \quad (2.30)$$

For a pair of coupled inductors with non-ideal coupling ($L_1 L_2 - M^2 \neq 0$), inversion of the appropriate 2-branch impedance matrix to obtain the admittance matrix.

$$\frac{1}{j\omega(L_1 L_2 - M^2)} \begin{bmatrix} L_2 & -M \\ -M & L_1 \end{bmatrix}$$

shows that the reciprocal inductances L_1^{-1} , L_2^{-1} and M^{-1} can also be included in the above set.

An ideal transformer is non-energetic and incapable of description by an admittance matrix of finite-valued elements: rather, it imposes constraints of a topological nature upon any circuit to which it is connected. The manner in which the connection of an ideal transformer modifies the admittance of a linear circuit [34] is such that its turns ratio cannot in general be included in a homogenous parameter set.

2.9 Scattering Parameter Sensitivity

It can easily be demonstrated that the scattering matrix $\underline{S}$ possesses sensitivity invariants [27]. Briefly, the basis-free normalized scattering matrix for a linear time-invariant circuit with resistive reference is given [32] by:

$$\underline{S} = \underline{I} - \underline{r}^{\frac{1}{2}} (\underline{Z} + \underline{r})^{-1} \underline{r}^{\frac{1}{2}} \quad (2.31)$$

where $\underline{r} = \text{diag}\{r_i\}$; r_i are the reference resistors and $\underline{Z}$ is the impedance matrix for the network. The reference resistors are arbitrary constants in the formulation of $\underline{S}$. It is only meaningful to seek sensitivities with respect to the "reference resistors" in an actual example, where each r_i takes on a specific value (e.g. load or source resistance).

As has previously been proved, both $\underline{Z}$ and its inverse are homogeneous matrices. It can easily be seen that $\underline{r}$ is homogeneous of degree one, just as $\underline{Z}$. It follows that $(\underline{Z} + \underline{r})$ is homogeneous of the same order. Therefore it can be stated directly that $\underline{S}$ must possess invariants in terms of the parameters of $\underline{r}$, $\underline{Z}$ and/or $\underline{r}^{-1}$, $\underline{Y}$. In terms of the impedance parameter set used in the previous section it can be seen from equation (2.31) that $\underline{S}$ is homogeneous of order zero. Hence,

$$\sum_{i=1}^n x_i \frac{\partial \underline{S}}{\partial x_i} = 0 \quad (2.32)$$

for all x_i belonging to the set $\{\underline{Z}, \underline{r}\}$ or $\{\underline{Y}, \underline{r}^{-1}\}$.

2.10 Signal Flow Graph

Summed-sensitivity invariants can be developed for other systems

just as for electrical networks. Thus the sensitivity in signal flow graphs [36] is defined for the gain function G as,

$$S_t^G = \frac{\Delta}{\frac{\partial G}{\partial t}} \frac{t}{G} \quad (2.33)$$

where S_t^G is said to be the normalized sensitivity of a signal flow graph having a gain G the source and some defined output node, with respect to some branch transmission t . If it is possible to show that G is a homogeneous function of a given order then as before invariants can be found. To show that G is homogeneous consider the formal algebraic description [36] of an n node signal flow graph:

$$\left. \begin{aligned} x_1 &= t_{11}x_1 + t_{12}x_2 + \dots + t_{1n}x_n + t_{10}x_0 \\ x_2 &= t_{21}x_1 + t_{22}x_2 + \dots + t_{2n}x_n + t_{20}x_0 \\ &\vdots \\ x_n &= t_{n1}x_1 + t_{n2}x_2 + \dots + t_{nn}x_n + t_{n0}x_0 \end{aligned} \right\} \quad (2.34)$$

where $x_0, \dots, x_n$ characterise the nodal variables and the t 's are the transmissions or branch variables. The set of equations (2.34) can be rewritten replacing $\{t_{ii} = \alpha_{ii} + \beta_{ii}\}; i = 1, 2, \dots, n$

$$\left. \begin{aligned} x_1 &= (\alpha_{11} + \beta_{11})x_1 + t_{12}x_2 + \dots + t_{1n}x_n + t_{10}x_0 \\ x_2 &= t_{21}x_1 + (\alpha_{22} + \beta_{22})x_2 + \dots + t_{2n}x_n + t_{20}x_0 \\ &\vdots \\ x_n &= t_{n1}x_1 + t_{n2}x_2 + \dots + (\alpha_{nn} + \beta_{nn})x_n + t_{n0}x_0 \end{aligned} \right\} \quad (2.35)$$

Defining $\{\phi_{ii} = (1 - \beta_{ii})\}$; $i = 1, 2, \dots, n$; equation (2.35) can be rewritten as:

$$\left. \begin{aligned} \phi_{11}x_1 &= \alpha_{11}x_1 + t_{12}x_2 + \dots + t_{1n}x_n + t_{10}x_0 \\ \phi_{22}x_2 &= t_{21}x_1 + \alpha_{22}x_2 + \dots + t_{2n}x_n + t_{20}x_0 \\ &\vdots \\ \phi_{nn}x_n &= t_{n1}x_1 + t_{n2}x_2 + \dots + \alpha_{nn}x_n + t_{n0}x_0 \end{aligned} \right\} \quad (2.36)$$

The set of equations (2.36) can be written more concisely in matrix form:

$$(\underline{\phi} - \underline{T}) \underline{x} = x_0 \underline{T}_0 \quad (2.37)$$

Following the arguments of Section 2.4 it can be seen directly from equations (2.36) that $(\underline{\phi} - \underline{T})$ is homogeneous of order one, in the parameter set $\Omega \in \{t_{ij}, \phi_{ii}, \alpha_{ii}\}$ ($t \neq j$) $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$. Similarly the vector $\underline{T}_0$ is homogeneous of order one in terms of $\{t_{i0}\}$, $i = 1, 2, \dots, n$. Solving equation (2.37) to find the gain between the input node x_0 and some output node q :

$$\underline{x} = x_0 (\underline{\phi} - \underline{T})^{-1} \underline{T}_0 \quad (2.38)$$

Therefore the gain function $G = \frac{x_q}{x_0}$ in terms of the parameters of the set Ω and $\{t_{i0}\}$, $i = 1, 2, \dots, n$, must be homogeneous of order zero. Invoking the Euler theorem Section 2.3

$$\left[\frac{\partial G}{\partial t_{ij}} t_{ij} \right]_{i \neq j} + \frac{\partial G}{\partial \alpha_{ii}} \alpha_{ii} + \frac{\partial G}{\partial \phi_{ii}} \phi_{ii} = 0 \quad (2.39)$$

for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$

Evaluating the differentials of equation (2.39) with $\alpha_{ii}=t_{ii}$ and $\phi_{ii}=1$ for $i = 1, 2, \dots, n$, yields

$$\left[\frac{\partial G}{\partial t_{ij}} \right]_{t_{ij}=t_{ij}}^{t_{ij}} + \frac{\partial G}{\partial \alpha_{ii}} \bigg|_{\alpha_{ii}=t_{ii}}^{\alpha_{ii}} + \frac{\partial G}{\partial \phi_{ii}} \bigg|_{\phi_{ii}=1}^{\phi_{ii}} = 0 \quad (2.40)$$

for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$.

On dividing equation (2.40) by G , the first two sets of terms can be identified with the sensitivities of the SFG gain with respect to all the branch transmissions. The third set of terms in (2.40) can be interpreted as the sensitivities of the unity gain self loops; the latter being an interpretation due to Coats [37]; associated with the set of SFG equations (2.36). Equation (2.40) can now be written in the more familiar summed sensitivity invariant form:

$$\sum_{\substack{\text{all branch} \\ \text{transmissions}}} \frac{\partial G}{\partial t_{ij}} \frac{t_{ij}}{G} + \sum_{\substack{\text{all self} \\ \text{loops}}} \frac{\partial G}{\partial \phi_{ii}} \frac{\phi_{ii}}{G} = 0 \quad (2.41)$$

It can be seen that in the above invariant relationship the additional terms of the second summation are not associated with any physical transmissions. However, without the inclusion of these terms an invariant relationship cannot be found in general as with electrical networks. Higher-order invariants can be simply obtained as in Section 2.7.

2.11 Applications of Sensitivity Invariants

The existence of sensitivity invariants has been developed so far without reference to any uses. However, they are of con-

siderable importance, not only theoretically but also from the practical viewpoint. Thus invariants and relationships based on them, e.g. equation (2.23) have been used to provide an independent check on the differential sensitivity calculation program (see Appendix 1). Although the computer calculated sensitivities were checked against analytically calculated sensitivities for simple circuit examples, a much broader class of networks could be used when checking the summed sensitivities against the invariant relationships.

Sensitivity invariants were first encountered by Leeds and Ugron [12] whilst developing their sensitivity minimisation procedures first proposed by Schofler [35]. As the algebraic sum of first order sensitivities can often be constant and independent of the parameter changes the need to use the sum of the squared sensitivities is clearly recognised. Furthermore, relationships based on invariants can be used as auxiliary conditions in optimisation procedures, such as the use of Lagrange multipliers. One interesting feature of finding minimum sensitivity networks occurs in signal flow graphs. Firstly, equation (2.40) can be used as an auxiliary condition directly or by weighting the self-loop sensitivities to be zero. Secondly, investigations could be carried out in which the sensitivities of the branch transmissions are reduced while those of the self loops increased, because the latter are purely notional and an increase in their sensitivity may not be important. These considerations have not been pursued here because they fall slightly outside the field of research reported in this thesis.

In the tolerance assignment problem, discussed more fully in Chapter 4, regions in an m dimensional Euclidian component

space E^m have been identified in which the worst case component deviations are likely to occur. The worst case deviations are those for which the circuit response changes by a maximum amount. Thus for smaller deviations the response change will also be smaller. Furthermore, it will be shown that it may not be possible to search more than two of the 2^m sectors of E^m .

2.12 Conclusions

The most important point that has been established is that summed sensitivity invariants exist only because circuit functions are homogeneous. Homogeneity leads directly to invariants for any order of differentiation and, most importantly, indicates how different parameter sets can be chosen in a fairly straightforward manner to yield homogeneous functions. This approach shows how invariants can be developed for both linear and nonlinear circuits. Similar analytical methods can be applied to other systems, as shown by the development of invariants for signal flow graphs. Furthermore, the relationships derived in this chapter have not only been used to check the sensitivity programs but also have led to a deeper insight into the tolerance assignment problem.

CHAPTER 3

CIRCUIT RESPONSE CALCULATION UNDER PARAMETER VARIATION

3.1 Introduction

Electric circuits are designed to modify a known signal in a specified manner into another form. The modified signal is often termed the response of the circuit. This response may be the output voltage, phase or some other property. In trying to study circuit behaviour it is therefore necessary to find the change in response for changes in the component values of the circuit. When circuits are simulated on a computer either an exact or an approximate numerical method can be employed to compute the response. The choice between the two methods is governed by the trade-off between the accuracy and computational cost amongst the many different numerical procedures that are available. In most simple circuits of around half a dozen nodes an exact calculation (see Section 3.2a) is often the fastest and certainly accurate to within the limits of the computers accuracy. However, for moderate size networks of up to forty nodes this approach is likely to be costly and it may be more advisable to seek an approximation [38,39,40] to the exact response change. Naturally, the use of a given approximation will depend upon how closely it corresponds to the exact response change. For most large networks, that is networks with more than forty nodes, sparse matrix techniques [41] can be very efficient computationally.

Notwithstanding the many different methods of response calculation,

a major difficulty exists in defining suitable criterion for the comparison of these methods. In the past the usual approach has been to compute the number of multiplications (or divisions) required for each method. This was based on the fact that multiplications on a computer were perhaps twenty times more expensive than additions or subtractions. Thus providing all methods were of similar accuracy, the method with the least number of multiplications could be regarded as the "best". With improvements in computer hardware, multiplication times [42] have come down to as little as ten times those of additions. Thus the old measure of operations may require modifications, especially when applied to sophisticated indexing schemes used in sparse matrix techniques.

Judging the goodness of an approximation to an exact response calculation poses even greater problems. As the accuracy of an approximation method may vary from circuit type to circuit type and the choice of frequency points of analysis, statistical techniques have been employed (see Section 3.4) to make comparisons for the relative accuracy of responses from different methods.

In this chapter a Taylor expansion to the response function for large perturbations in the circuit elements will be developed. Some theoretical aspects of the Taylor series will be highlighted and a convergence criterion derived. A comparison will be made between the most efficient exact method [43] and the truncated Taylor expansion with a representation range of circuit types. Thus it is hoped to show the feasibility of the Taylor approximation.

3.2 Different Methods of Response Calculations

One of the most critical aspects of making a numerical calculation

of a response function is to take cognizance of the poles of the function. This difficulty exists for both the exact and approximate methods. When making an exact calculation using the the admittance matrix, the latter may become singular close to a pole, and hence a large error may occur. Approximate methods using derivative information suffer because the derivatives are changing rapidly in the vicinity of a pole. The change in derivatives for changes in component values can now be investigated easily (see Chapter 5).

3.2a Exact Methods

Undoubtedly the most well known and often used method is that of Gaussian elimination [44], or some variant of this. Thus for an n node network the number of operations (one operation consists of one multiplication and one addition or subtraction) needed to compute the response using standard Gaussian elimination is $(\frac{1}{3}n^3 + n^2 - \frac{1}{3}n)$. However, if only a subset of the circuit components are liable to vary at any one time, then providing these components are incident on m nodes other than the reference, where $m \leq n$, it is possible to reduce the number of operations to $(\frac{1}{3}m^3 + m^2 - \frac{1}{3}m)$ using the substitution current source technique, see Chapter 5. Recent advances in the use of sparse matrix techniques [45], where the banded nature of the nodal admittance matrix are usefully exploited, indicate that it may be possible to reduce the number of operations to less than 5% of simple Gaussian elimination. However, the amount of programming effort, computer time and space required to set up the sparse matrix tableau would not really justify their use for networks of less than forty or fifty nodes. Finally, it is worth noting the developments in algebraic expression evaluation languages [46]. Though there has

been some interest in using them for circuit response calculations [47] it is generally true that the computer time/space needed to generate the transfer function for circuits as little as ten nodes can be very large. Therefore, unless the circuit response needs to be calculated many hundreds of times, these techniques may prove to be impractical.

3.2b Approximate Methods

Of the approximate methods the three best known are the use of the multivariate Taylor and Moment series [38,39]. A third method employs orthogonal functions [40]. The last two mentioned methods have been used to determine the response distribution of the network from a knowledge of the component distribution functions. However, component distributions are well known to be badly skewed and/or being bimodal. Therefore, they are difficult to represent with simple mathematical functions, which makes them undesirable for use in predicting the response distributions. Matrix inversion by iteration [44] is also available but it is difficult to determine the cost with any precision because the convergence depends upon the matrix itself, and in particular its eigen values. Thus the more direct approach of a Taylor expansion of the response function appears to be attractive, providing the necessary differential coefficients can be easily calculated and convergence conditions derived. The former requirement is satisfied by the recent method [58] of calculating the network sensitivities while the latter are derived in Section 3.3.

3.3 Taylor Expansion of a Complex Response Function

Assume that the voltage response function $F(\underline{x}^0, \omega_1)$ satisfies the

usual mathematical conditions of continuity and differentiability in the neighbourhood of $\underline{x}^0$. If the change in response due to changes in m components of a circuit, which may have more than m components, is required, then this can be represented by:

$$F(\underline{x}^0 + \underline{\Delta x}) = F(\underline{x}^0) + \sum_{i=1}^m S_{x_i}^F \frac{\Delta x_i}{x_i} + \frac{1}{2!} \sum_{i=1}^m \sum_{j=1}^m S_{x_i x_j}^F \frac{\Delta x_i}{x_i} \frac{\Delta x_j}{x_j} + \dots \quad (3.1)$$

where $S_{x_i}^F$ etc. are the first, second and higher order sensitivities and $x_i \in \{m \text{ circuit components liable to changes } \Delta x_i\}$.

The attraction in using a truncated Taylor series expansion is, primarily, the efficiency with which the sensitivities can now be derived (see Appendix 1).

3.3a Convergence Proof for the Taylor Expansion

If changes in only one element x are considered it is possible to show from the results of reference [48] that the series (3.1) will only converge for $|\Delta x \frac{D'}{D}| < 1$; where $F = \frac{N}{D}$, D' is the first differential with respect to x of the denominator polynomial D and N is the numerator polynomial. However, no such simple expression exists for the case of simultaneous changes in more than one component. To study the more general case of several variable components we start by considering the matrix equation describing a circuit. (The following mathematical proof can be skipped without any loss in continuity. A more useful convergence test is given in Section 3.3b.)

$$\underline{V}_n = \underline{Y}_n^{-1} \underline{I}_n$$

When some of the component values change, the admittance matrix $\underline{Y}_n$ also changes. Thus:

$$\underline{Y}_n \rightarrow \underline{Y}_n + \underline{\delta Y}_n$$

If the source currents given by $\underline{I}_n$ remain the same as before, the change in the voltage vector $\underline{V}_n$ for the change $\underline{\delta Y}_n$ is:

$$\begin{aligned}\underline{V}_n &\rightarrow \underline{V}_n + \underline{\delta V}_n \\ \underline{V}_n + \underline{\delta V}_n &= (\underline{Y}_n + \underline{\delta Y}_n)^{-1} \underline{I}_n\end{aligned}\quad (3.2)$$

Expanding equation (3.2) in a power series yields:

$$\underline{V}_n + \underline{\delta V}_n = \left[1 - \underline{Y}_n^{-1} \underline{\delta Y}_n + (\underline{Y}_n^{-1} \underline{\delta Y}_n)^2 - \dots \right] \underline{Y}_n^{-1} \underline{I}_n \quad (3.3)$$

where the criterion for convergence is that $\underline{Y}_n^{-1} \underline{\delta Y}_n$ be a convergent matrix. In other words:

$$(\underline{Y}_n^{-1} \underline{\delta Y}_n)^t \rightarrow \underline{0} \quad \text{as } t \rightarrow \infty \quad (3.4)$$

Equation (3.3) is in fact the Taylor expansion of $\underline{V}_n + \underline{\delta V}_n$ about $\underline{V}_n$. Therefore, once the output node of the circuit is identified with a node-number, selection of the corresponding row from $\underline{V}_n + \underline{\delta V}_n$ yields the new output voltage F .

To prove the validity of the condition (3.4), let $\underline{U}$ be a matrix such that $\underline{D} = \underline{U} \underline{Y}_n \underline{U}^{-1}$ is a diagonal matrix, and $\underline{V}_n = \underline{U}^{-1} \underline{V}'_n$.

Multiplying the nodal matrix equation by $\underline{U}$ yields:

$$\underline{U} \underline{Y}_n \underline{V}_n = \underline{U} \underline{I}_n \quad (3.5)$$

$$\text{and} \quad \underline{U} \underline{Y}_n \underline{U}^{-1} \underline{V}'_n = \underline{U} \underline{I}_n \quad (3.6)$$

$$\underline{V}'_n = \underline{U} \underline{Y}_n^{-1} \underline{U}^{-1} \underline{U} \underline{I}_n \quad (3.7)$$

$$= \underline{D} \underline{U} \underline{I}_n \quad (3.8)$$

$$\underline{V}_n = \underline{U}^{-1} \underline{D} \underline{U} \underline{I}_n \quad (3.9)$$

Similarly for equation (3.2)

$$\underline{V}_n + \underline{\delta V}_n = \underline{U}^{-1} (\underline{D} + \underline{U} \underline{\delta Y}_n \underline{U}^{-1})^{-1} \underline{U} \underline{I}_n \quad (3.10)$$

$$= \underline{U}^{-1} [\underline{D}(1 + \underline{D}^{-1} \underline{U} \underline{\delta Y}_n \underline{U}^{-1})]^{-1} \underline{U} \underline{I}_n \quad (3.11)$$

$$= \underline{U}^{-1} [1 + \underline{D}^{-1} \underline{U} \underline{\delta Y}_n \underline{U}^{-1}]^{-1} \underline{D}^{-1} \underline{U} \underline{I}_n \quad (3.12)$$

$$\text{Let } \underline{\delta B} = \underline{U} \underline{\delta Y}_n \underline{U}^{-1} \quad (3.13)$$

$$\text{Then } \underline{V}_n + \underline{\delta V}_n = \underline{U}^{-1} [1 + \underline{D}^{-1} \underline{\delta B}]^{-1} \underline{D}^{-1} \underline{U} \underline{I}_n \quad (3.14)$$

$$= \underline{U}^{-1} [1 - \underline{D}^{-1} \underline{\delta B} + (\underline{D}^{-1} \underline{\delta B})^2 + \dots] \underline{V}'_n \quad (3.15)$$

The criterion for the convergence of equation (3.15) is:

$$(\underline{D}^{-1} \underline{\delta B})^t \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad (3.16)$$

$$\text{Let } \underline{C} = \underline{D}^{-1} \underline{\delta B} \quad (3.17)$$

The maximum magnitude of any element of $\underline{C}$ is given by:

$$|C_{ij}|_{\max} = \frac{|\delta b_{ij}|_{\max}}{|\lambda|_{\min}} \quad (3.18)$$

where $|\delta b_{ij}|_{\max}$ is the element of maximum magnitude in $\underline{\delta B}$ and $|\lambda|_{\min}$ is the minimum magnitude eigen value of $\underline{D}$. A new matrix $\underline{S}$ is now defined, each of whose elements are $|C_{ij}|_{\max}$.

$$\text{Then, } S_{ij} \geq |C_{ij}| \quad \text{for all } i \text{ and } j \quad (3.19)$$

Thus if a condition can be found for

$$\underline{S}^t \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad (3.20)$$

$$\text{Then } \underline{C}^t \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad (3.21)$$

$$\underline{S}^t = [|C_{ij}|_{\max} \underline{S}]^t \quad (3.22)$$

where $\underline{S}'$ has all elements of unity value.

$$\underline{S}^t = \left| c_{ij} \right|_{\max}^t \underline{S}'^t \quad (3.23)$$

Lemma

$$\underline{S}'^2 = n \underline{S}' \quad (1)$$

where n is the dimension of the square matrix $\underline{S}'$

$$\begin{aligned} \underline{S}'^3 &= \underline{S}'^2 \underline{S}' \\ &= n \underline{S}'^2 \\ &= n^2 \underline{S}' \end{aligned} \quad (2)$$

$$\therefore \underline{S}'^t = n^{t-1} \underline{S}' \quad (3)$$

It is easy to prove the validity of (3) by induction on t .

$$\begin{aligned} \text{Hence, } \underline{S}^t &= \left| c_{ij} \right|_{\max} n^{t-1} \underline{S}' \\ &= \left[\left| c_{ij} \right|_{\max} n \right]^t \frac{1}{n} \underline{S}' \end{aligned} \quad (3.24)$$

Since the positive constant $n \geq 1$; it follows that

$$\left| c_{ij} \right|_{\max} n < 1 \quad (3.25)$$

$$\text{for } \underline{S}^t \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

Hence a sufficient condition for (3.21) to be valid is:

$$\frac{\left| \delta_{ij}^b \right|_{\max}}{\left| \lambda \right|_{\min}} n < 1 \quad (3.26)$$

$$\text{or } \left| \delta_{ij}^b \right|_{\max} < \frac{\left| \lambda \right|_{\min}}{n} \quad (3.27)$$

In most cases the inequality (3.27) is likely to be stronger than necessary for the convergence of $\underline{C}$ because it is based on the convergence of $\underline{S}$ which is a full matrix of the maximum modulus term of $\underline{C}$.

One immediate problem in using the inequality (3.27) is the need to calculate $|\delta b_{ij}|_{\max}$ and $|\lambda|_{\min}$ at every frequency point. As the computation cost for making the necessary calculations is very high this condition would not be used in practice. Nevertheless, it does highlight certain interesting properties, namely that the position of the closest pole to the frequency of analysis will have a strong effect on the convergence. Thus the closer we are to a pole the smaller the component deviations to guarantee convergence.

3.3b Comparison Test of Convergence

The basis of the comparison test is to find a convergent series such that term by term each term in the series (3.1) is less than or equal to the terms of the known convergent series. Typical series used in such tests are the exponential and geometric series for example. For the purposes of the test we rewrite the series (3.11) in terms of two new variables c and a , which are defined as follows:

$$c = \text{one of } \left\{ |S_{x_i}^F|_{\max} \text{ or } |S_{x_i x_j}^F|_{\max} \text{ or } \dots \right\}^*$$

$$a = \text{maximum of } \left\{ \left| \frac{\Delta x_i}{x_i} \right| \right\}$$

The series (3.1) can be written as:

$$|F(\underline{x}^0 + \underline{\Delta x})| \leq X = |F(\underline{x}^0)| + c m a + c m^2 a^2 + c m^3 a^3 + \dots + \quad (3.28)$$

$$= |F(\underline{x}^0)| + c m a (1 - m a) \quad (3.29)$$

$$\text{for } |m a| < 1 \quad (3.30)$$

*We use the modulus of the complex sensitivities because

$$S_{x_i}^F \leq |S_{x_i}^F| \text{ etc.}$$

The series (3.28) can only be summed under the simplifying assumption

that
$$\left| \frac{S_{x_i x_j \dots x_t}^F}{t!} \right| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

That is, the higher order sensitivities are bounded and only increase at the same rate as the factorial. It is possible to devise circuit examples (with frequency points chosen close to a pole) whose sensitivities become unbounded with increasing order of the differentials. However, in practice the simple condition $|ma| < 1$ has proved to be adequate (see Section 3.4). An interesting point to note is that the sensitivities only become unbounded close to a pole and this is also the region in which exact methods become inaccurate because the admittance matrix is becoming singular.

The condition (3.30) indicates that the upper limit on the allowed fractional changes in each component is only dependent upon m , the number of variable elements. Thus for $m = 10$, a 10% change in each element could be tolerated. With this condition rough guides to the permitted component variations can be easily obtained unlike the case with the previous condition (3.27).

3.3c Choice of Truncation Point

The computational cost in terms of the number of operations for an exact single large change sensitivity type analysis (see Section 5.6) once the appropriate impedance matrix has been calculated is:

$$C_{\text{exact}} = \frac{m^3}{3} + m^2 + \frac{5}{3}m \quad (3.31)$$

It can be seen in Fig. 3.6 that C_{exact} increases rapidly with m .

To justify the use of the Taylor series, its cost must be less than C_{exact} . Equation (3.1) shows that the number of terms in the series increases by multiples of m for successive orders. Consequently the computational cost for the Taylor series also increases by multiples

of m . To keep the computational cost below C_{exact} only a limited number of terms of the Taylor series can be used. The choice of the truncation point must also take account of the accuracy of the resulting approximation. Therefore, it is necessary to investigate the change in accuracy for truncation at successive orders. Statistical methods are employed (see Section 3.4b) to compare the response approximation using the Taylor series truncated at different orders with the corresponding exact calculation. It will then be possible not only to judge the accuracy of the truncated Taylor series but also to find out the best point at which to truncate. No one circuit is felt to be adequate for such a test. Therefore, a range of circuit types as shown in Fig.3.1 to Fig.3.5 have been used.

As a preliminary to proceeding with these statistical tests it is useful to make some numerical comparison between the exact method cost and the cost of the truncated Taylor series. For this purpose all terms up to the second order and a further m third order terms are chosen for the truncated series approximation. The cost of such a calculation is:

$$C_{\text{Taylor}} = 2m(m + 2) \quad (3.32)$$

This cost is plotted in Fig.3.6, and the break-even point with respect to an exact calculation is calculated as $m = 4.5414$. Thus for five or more variable components this approximation becomes considerably cheaper than the exact one. The cost of computing the sensitivities adds a negligible overhead to the cost of the Taylor series, because both the sensitivity calculation and the large change sensitivity method of exact calculation uses the impedance matrix, and the cost of obtaining the impedance matrix of a circuit is much higher than the other calculations.

3.4 Accuracy of a Truncated Taylor Expansion Compared to an Exact Calculation

As each different circuit has a unique circuit function it is insufficient to test just one or two simple examples. For instance, while filters tend to have poles and zeros close together in the region of interest along the frequency axis, in general purpose amplifiers, poles are often well separated. As we are interested in studying the effect on the response for simultaneous large changes in many components of the circuits under test, we employ statistical methods, often termed Monte Carlo analysis. The changes in component values for such an analysis requires a series of random deviates. The choice of the distribution function from which the random numbers are chosen presents some difficulties. Should this distribution function resemble the probable distributions of circuit components generally available to the manufacturer or should a standard Gaussian distribution be used? While the use of the latter is straightforward, the former requires a considerable effort if a representative number of distributions are to be chosen. Many manufacturers prefer the use of a uniform distribution [49] or some bimodal distribution as representative of the component batches available. We have employed two distributions, shown in Fig.3.7 and Fig.3.8, from which random deviates are chosen.

To perform the different tests each selected component of a circuit is assigned its own distribution. The width of each distribution is suitably scaled so that the range of random numbers generated fall within the tolerance limits of each of the components. A series of pseudo random circuits, having component deviations within the specified tolerances on each of the components, can be generated by selecting pseudo random numbers from the relevant distributions. For

these circuits the response at the output nodes at a given frequency is calculated using an exact method. The same circuits are used in calculating the response with the truncated Taylor series. Many trials are performed, histograms plotted, means, standard deviations, variances and correlation coefficients found for all the circuits of Fig.3.1 to Fig.3.5.

One of the important aspects of the series approximation which requires testing is the effect of truncation at different orders. To isolate this effect it is necessary to use identical random number sequences. To achieve this all that is necessary is to use the same seed to start the pseudo random number generator. Finally, component tolerances of 10% and 20% have been used for each of the circuits under test.

Before performing any statistical tests we must ascertain the number of random trial circuits that will be needed to give reasonable statistical information. This can be done with the aid of the t-test [50]. We compute the mean response for fifty "random circuits", Fig.3.1, at a frequency of 100 Hz say. Under the assumption that the nominal response is the true mean, a t value is computed using the information given in Table 3.1. It was found that $P_{50}(|t| \leq 0.4168) = 66.46\%$. The experiment was repeated for a hundred trial circuits and under the same assumption it was found that $P_{100}(|t| \leq 0.4253) = 66.64\%$. Doubling the number of trials from fifty to one hundred has produced a very small change in the t value. Therefore, we can safely assume that fifty trial circuits will be sufficient to give a reasonable statistical sample.

No. of trials	Mean Difference $\bar{x} - \mu$	Standard Error $\frac{s}{\sqrt{n}}$
50	-0.480×10^{-4}	0.144×10^{-3}
100	-0.966×10^{-4}	0.227×10^{-3}

Table 3.1

3.4a Correlation Coefficient

Many different correlation coefficient have been defined [50] to show the degree of correlation between two sets of readings $\{x\}$ and $\{y\}$. Here the Pearson or linear correlation coefficient ρ is used.

$$\rho = \frac{1}{n\sigma_x\sigma_y} \left\{ \sum x y - \frac{\sum x \sum y}{n} \right\} \quad (3.33)$$

It can be shown that the expansion for ρ always lies between +1 and -1. When $\rho = -1$ the sets of values x and y are perfectly uncorrelated. When $\rho = +1$ there is perfect correlation. Therefore, the closer ρ is to unity the greater the degree of correlation. However, a high value of ρ can sometimes be obtained by pure chance. Hence, it is desirable to back up the values of ρ with other significant data.

3.4b Histograms

The information from which the statistical results are calculated can be represented pictorially in histograms. Thus the abscissa along

which the circuit response is plotted (see Fig.3.9) is divided into a known number of intervals. The response for many random trials of a circuit is computed at a given frequency and slotted into the correct interval and counted. Once all the trials are complete the normalised probability distribution is computed and the histograms plotted. In all the histograms presented in this thesis the interval for the response is computed by dividing the range, fifty per cent above and below the nominal response, into fifty units.

3.4c Analysis of Numerical Results

Random numbers from a Gaussian distribution, centered on the mean of the component values, with standard deviations $\sigma = \frac{1}{3}$ tolerance, have been used for the result in all the tables shown in Appendix III of the thesis. In each table the second column shows the nominal output voltage at each of the selected frequencies. The third column shows the mean difference* $E(FE - FT)$ between the exact (FE) and the truncated Taylor expansion (FT) response calculations. The fourth and fifth columns show the standard deviation and variance of $E(FE - FT)$. The remaining two columns give the correlation and mean ratio of the Taylor approximation and the exact response.

To obtain a relative figure of merit for the accuracy of the response at any one frequency consider line two in Table 1.i.(10). This table shows results for truncation at the first order terms of the complex Taylor series with 10% tolerances on all the variable components of the circuit in Fig.3.1. The nominal output voltage magnitude at 500 Hz is 0.57107 volts, while $E(FE - FT) = 4.691 \times 10^{-4}$ with a standard deviation of 1.327×10^{-3} and a variance of 1.762×10^{-6} .

* $E(x)$ denotes the expected value of the statistical variable x .

Relative to the output voltage, $E(FE - FT)$ is very small just as the dispersion given by the standard deviation. The high correlation coefficient of 0.9956 between FT and FE shows that the distribution of the complex Taylor approximation tracks very closely the exact response distribution. Further credence is lent to the previous interpretation by the mean ratio $(FT:FE)$ of 0.97% which is very close to unity. Traversing the table vertically shows the accuracy of the approximation over the frequency range. To obtain comparisons amongst the first-, second- and third-order response approximations for the circuit in Fig.3.a*, it is necessary to see Tables a.i.(10), a.ii.(10) and a.iii.(10). The tables corresponding to 20% component tolerances for the same circuit are labelled a.i.(20), a.ii.(20) and a.iii.(20) respectively.

1) Active Circuits

A comparison of Tables 1.i.(10), 1.ii.(10) and 1.iii.(10) for the circuit in Fig.3.1, indicates that the approximation (3.1) to the response improves with successive higher orders. In this example the number of variable components is eleven. The interesting feature is that the approximation at the first-order itself appears to be quite good with the mean ratio $FT:FE$ almost unity at all the frequencies. At the 20% component tolerance level, see Tables 1.i.(20), 1.ii.(20) and 1.iii.(20), the approximation shows signs of deterioration with the mean difference becoming larger than at the 10% tolerance level. The convergence criterion (3.30) is now becoming violated. Nevertheless, there is an improvement with increasing orders of truncation. The mean ratio $FT:FE$ at the third-order is about 0.78, averaged over all the tested frequencies while the correlation coefficients are about 0.8.

*The parameter "a" takes on values between 1 and 5 corresponding to the circuit figure number.

We repeat the same tests for a response approximation obtained from "modulus" Taylor series as shown in equation (3.34).

$$|F(\underline{x}^0 + \underline{\Delta x})| \leq |F(\underline{x}^0)| + \sum_{i=1}^m \left| S_{x_i}^F \frac{\Delta x_i}{x_i} \right| + \frac{1}{2!} \sum_{i=1}^m \sum_{j=1}^m \left| S_{x_i x_j}^F \frac{\Delta x_i}{x_i} \frac{\Delta x_j}{x_j} \right| + \dots \quad (3.34)$$

The results shown in Tables M1.i.(10) to M1.iii.(20) indicate that the modulus series response is always greater than the corresponding exact response. The mean ratio FT:FE often approaches 2. Even though the mean ratio becomes smaller at the 20% component tolerance level the correlation is very poor.

The results of another active circuit, Fig.3.2, tested, seem to be in close agreement with those of the circuit in Fig.3.1. The results for this circuit appear in Tables 2.i.(10) to M2.ii.(20).

2) Passive Circuits

From the table of results for circuit in Fig.3.3 there is no doubt that the approximation (3.1) to the exact response is very good at any tested order of truncation at all the frequencies at the 10% component tolerance level. At the 20% component tolerance level there is a deterioration but it can still be regarded as adequate at the higher orders of truncation where the mean ratio FT:FE are unity if not close to unity.

However, the approximation with the modulus series overestimates the exact response by as much as a factor of three at the 10% component tolerance level and by more than four at the 20% component tolerance level. There is even a negative correlation coefficient at a frequency of 10 KHz at the 20% component tolerance level in Table M3.i.(20).

The results for the band pass filter shown in Fig.3.4 are out of step with all the experiments done. The mean differences $E(FE - FT)$

*The prefix "M" stands for the modulus.

are very high at all the frequencies when compared to the nominal response in all the Tables of this circuit for the complex series (3.1). While the correlation coefficient is very high, approaching unity, at both the 10% and 20% component tolerance level for all orders of truncation the mean ratio $FT:FE$ is around the .35 to .37 mark at all the frequencies. The consistency of all the results is striking. The large underestimation of the response at frequency points close to the band edges was thought to have an explanation. At such points in the frequency plane the radius of curvature of the response function in the component space is expected to be small because the radius of curvature in the frequency domain is small, i.e. a pole in the frequency domain remains a pole in the component space also. As the Taylor series can only describe the local behaviour of a function away from any pole, the series predicted response deviations do not match the exact response. However, it is not possible to explain the results with similar arguments at the frequencies of 250 Hz and 1000 Hz, which are well away from the band edges. All that can be concluded on the basis of the results is that the series converges slowly in this example. Nevertheless, the consistency of the mean ratio $FT:FE$ together with the high correlation coefficients suggests that multiplication of the response approximation by a factor of 1.5 at all the frequencies for any or the order of truncation would lead to a result very close to that of the exact calculation.

The response approximation with the modulus Taylor series appear to be much better. Note the high mean ratios $FT:FE$. However, the low correlation coefficients at all the orders of truncation mitigate against using this approximation.

For the resistive circuit of Fig.3.5 with nine variable components,

the response approximation with the complex series is very accurate compared to the exact calculation at all orders of truncation, at the 10% component tolerance level. There is a marked deterioration in the approximation at the 20% component tolerance level, even at the third order of truncation.

As with the earlier circuits of Figs.3.1, 3.2 and 3.3 the modulus series not only yields a large overestimate of the response but also lower correlation coefficients.

3.4d Histograms for a Non-Gaussian Distribution

To investigate the effect of using non-Gaussian random number distributions on the accuracy of the Taylor approximation over an exact calculation, the bimodal distribution of Fig.3.8 is used. This distribution is obtained by dropping to zero the probability for a standard Gaussian distribution at $|\sigma|$ on either side of the nominal value. The voltage amplitude response histograms of circuits having component distributions of the above type are shown in Fig.3.9 to Fig.3.14. These histograms confirm results shown in the tables of the corresponding circuits, though naturally the differences between exact and approximate means are now slightly worse. The increase in the standard deviations when the component tolerances change from 10% to 20% can be seen clearly.

3.4e General Remarks on Numerical Results

The evidence from the circuits tested suggests that the complex truncated Taylor expansion can give an adequate measure of the circuit response at the second or third order of truncation. In the majority of cases the response measure seems to be an underestimate of the exact calculation. Nevertheless, the correlation coefficient between the approximation and exact calculation are high, particularly when

the range of component variation is limited to a region around the inequality $\left| \frac{\Delta x_i}{x_i} \right| < \frac{1}{m}$. The consistency of the mean ratio FT:FE for each of the circuits suggests that a correction factor, which is the inverse of this ratio, may be calculated from a limited number of exact and approximation trials, say 10, for any circuit under test. Subsequently this correction factor could be used with the approximation to yield response calculations closer to the exact value. Naturally, this will be of importance only when large numbers of response calculations are necessary, as in Monte Carlo circuit analysis.

3.5 Monte Carlo Circuit Analysis

Traditionally, statistical circuit analysis [52] as described in the flow chart, Fig.3.15, has been confined to the final stages of circuit design to test the likely yields of circuits meeting the design requirements when manufactured from components of specified tolerances (see Chapter 4). The high computational cost of such analyses has limited their use in the intermediate stages of the circuit design process. For instance, the designer might wish to choose between slightly different circuit configurations with different tolerance components in the middle of a design and therefore needs to have some information of the different response distributions. In such circumstances rough estimates may be sufficient. Hence, for some years much effort has been devoted to finding methods incurring lower costs than directly solving the circuit equations.

At first it was hoped that the use of first-order differential sensitivity [51] might be good enough in making an approximation to the response changes. However, first order failed to account

adequately for the nonlinear nature of the circuit response function. In contrast our experience of using a Taylor series with second- and some third-order terms indicates that in most cases an adequate approximation to the exact response can be made (see Section 3.4d). As the cost of using this approximation is much lower than exact methods (see Section 3.4) the truncated Taylor series could be incorporated in an "approximate" Monte Carlo analysis programme [53]. Such programmes could find particular use on mini-computers (interactive environment) where the speed of a calculation is of great importance.

One significant problem that exists in all Monte Carlo circuit analysis is determining the number of trial circuits that should be used in the analysis. The experience of some industrial concerns [49] is that 30 - 50 trials are often sufficient for most statistical purposes. This is confirmed by our own experience as indicated by results of the t-test in Section 3.4.

3.6 Conclusions

It has been shown that a truncated complex Taylor series can be used in making an approximation to the circuit response when the component values are perturbed from their nominal value. In most cases response approximation has been shown to be reasonably accurate when compared to corresponding exact calculations for a variety of circuit types. As the different coefficients of the truncated Taylor series need only be evaluated once, the approximation is considerably faster than any exact method providing the circuit size lies within 5 - 40 nodes. When the circuit size increases beyond forty nodes, sparse matrix techniques become more economic. Empirical evidence suggests that if the component deviations are restricted according

to the inequality (3.3a), the Taylor series is quite likely to converge. Experimental evidence also indicates that the modulus Taylor series should be used with care, because there is little or no correlation between the response computed with the modulus series and exact calculation.

As the complex Taylor series approximation has been found to be reasonably accurate in computing the change in circuit response this series will be employed in an efficient tolerance assignment procedure to be described in the next chapter. The series approximation could also be used in an approximate Monte Carlo circuit analysis program.

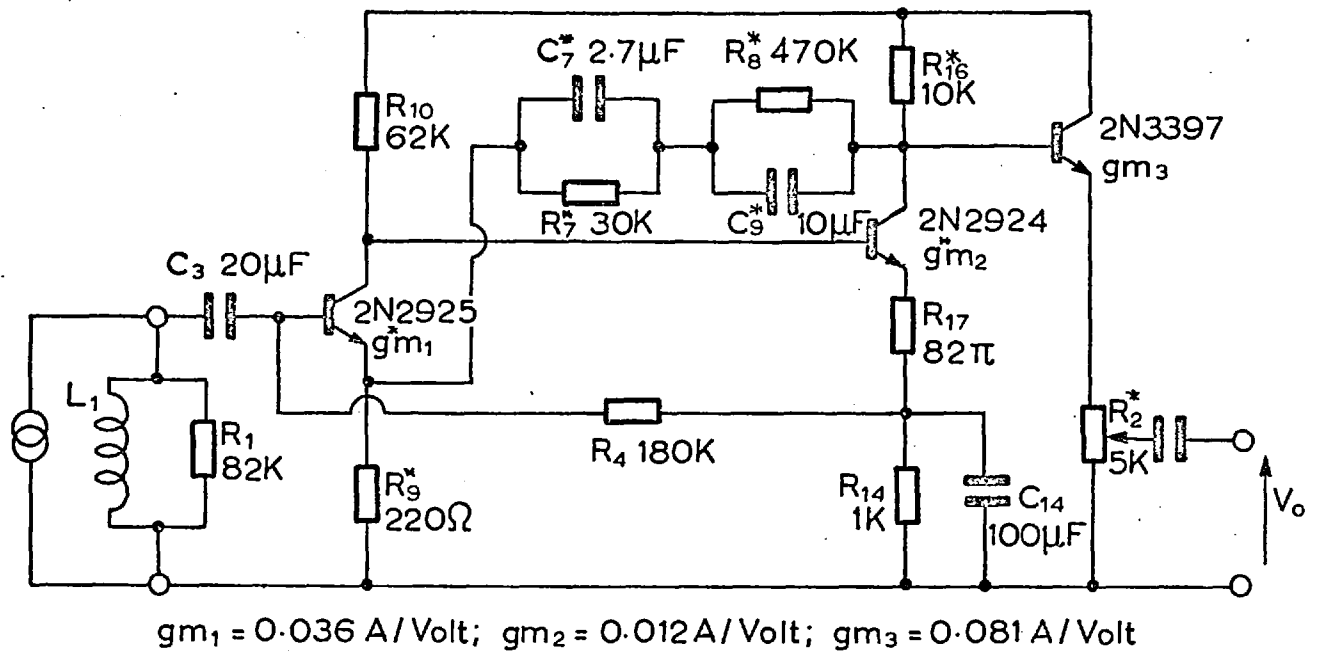


FIG. 3.1: Equaliser preamplifier.
(* identifies the variable components)

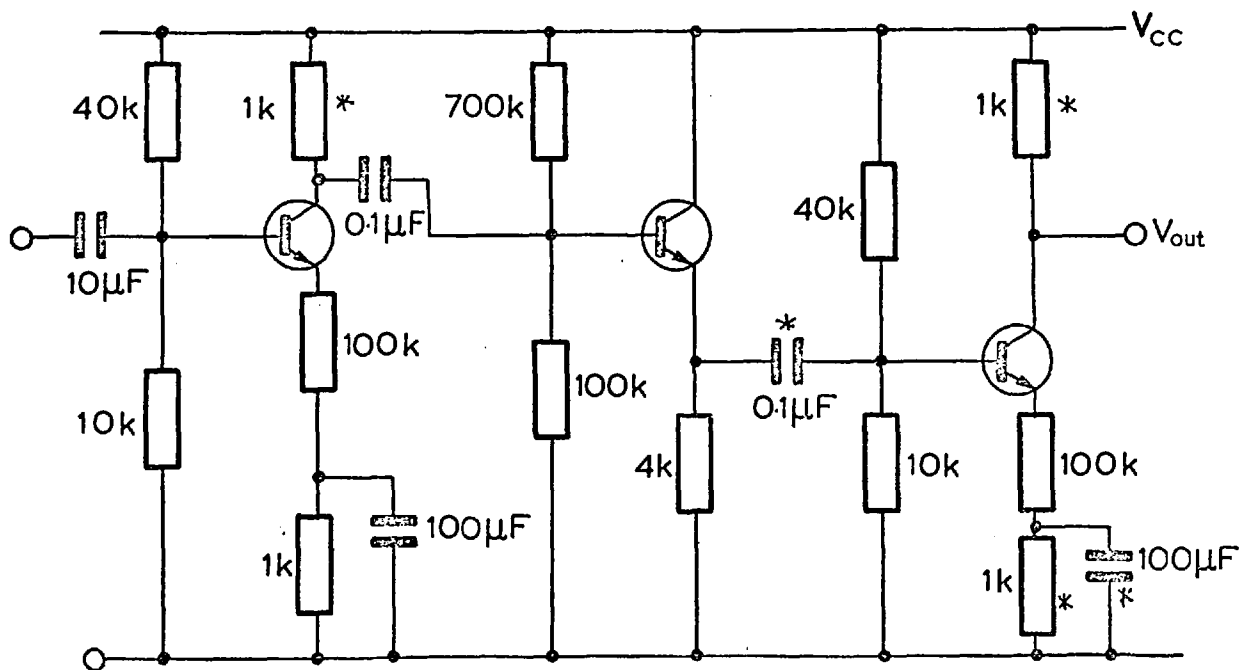


FIG. 3.2
Transistor amplifier

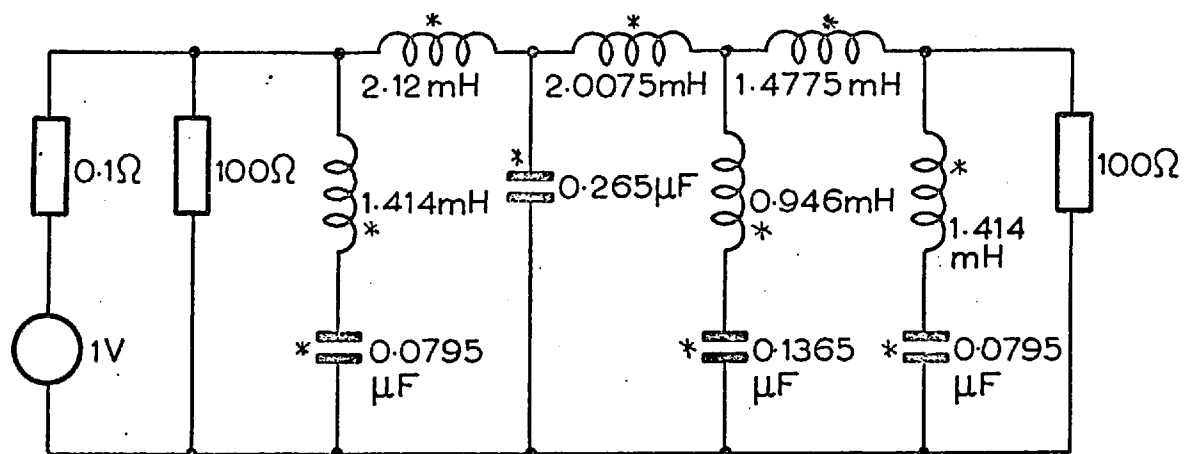


FIG. 3-3: Low Pass Filter

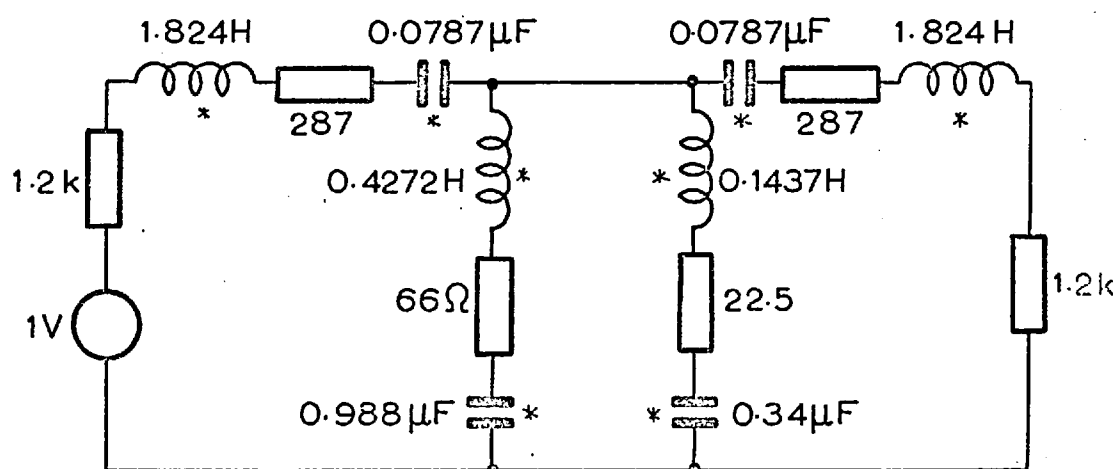


FIG. 3-4: Band Pass Filter

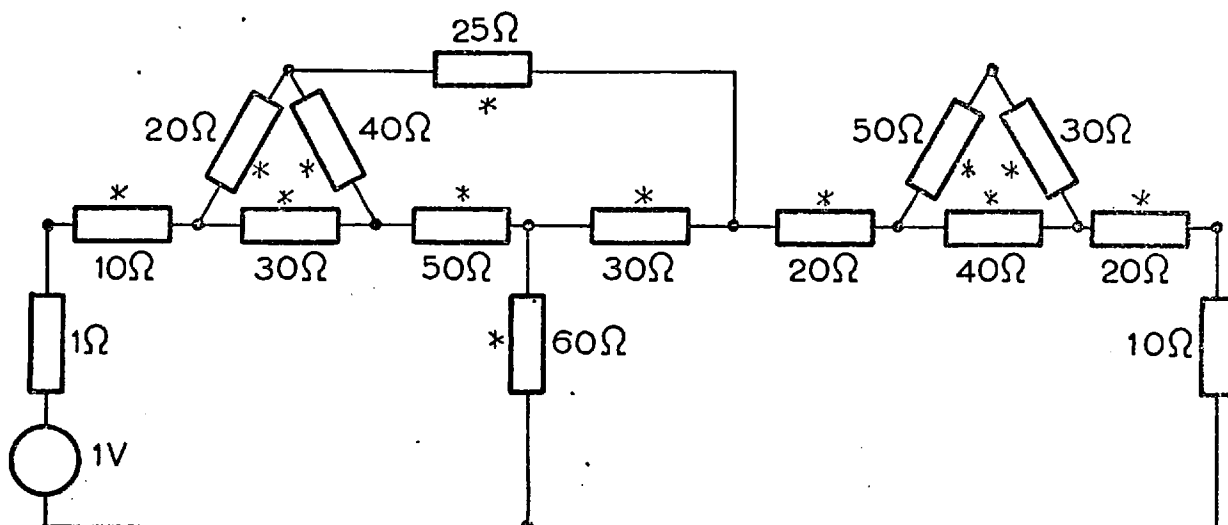


FIG. 3-5: Resistive Network

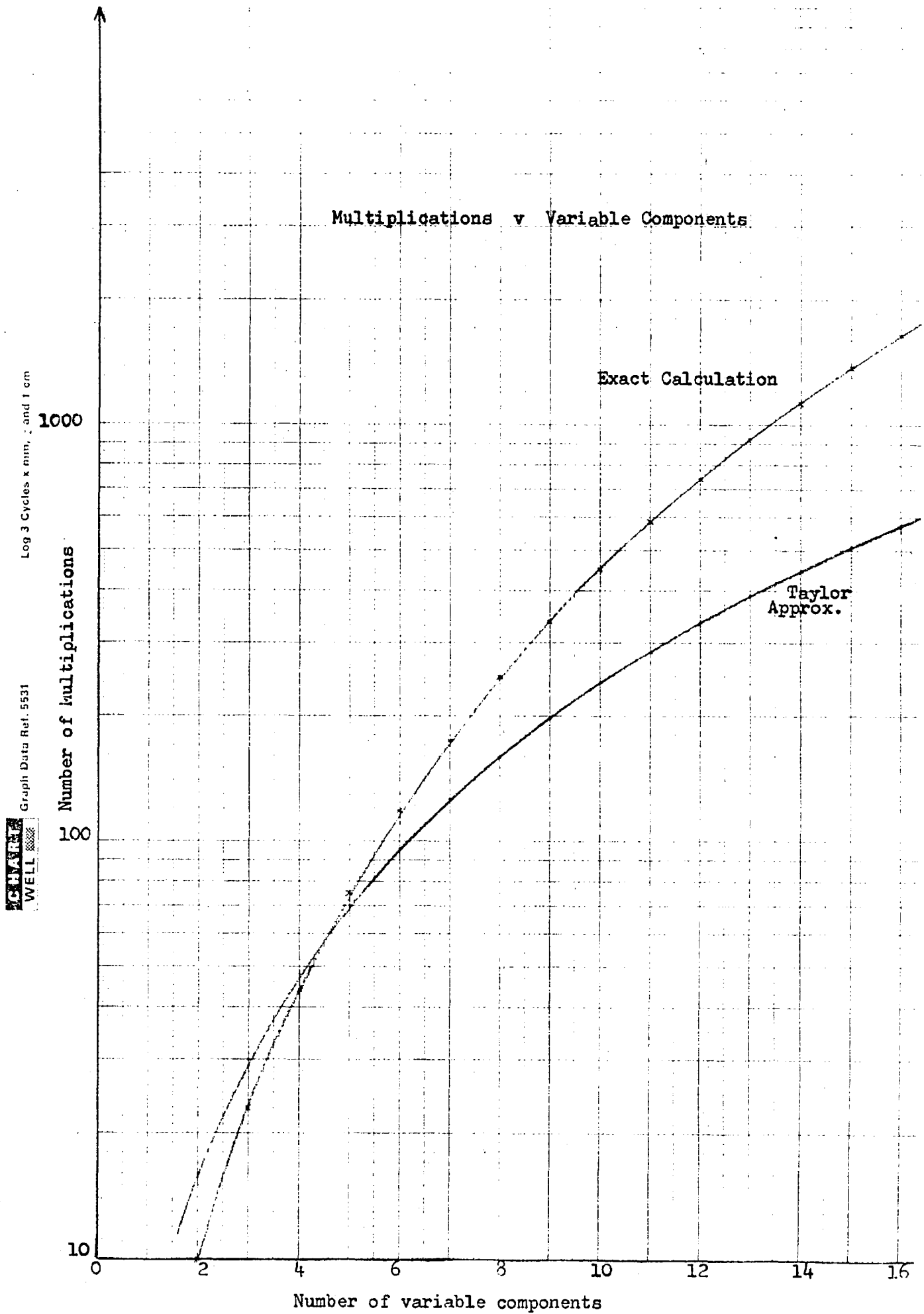


Fig. 3.6 A cost comparison of exact and Taylor response calculations

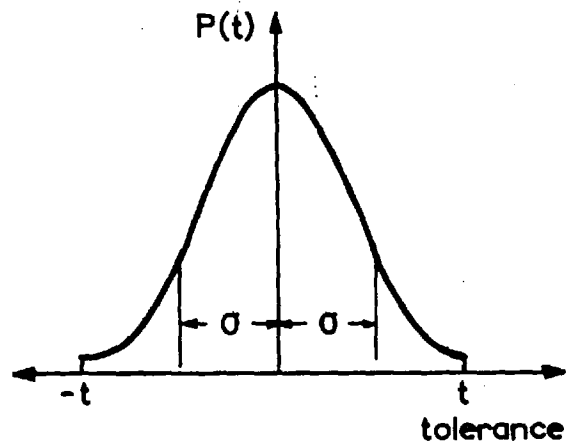


FIG. 3-7: Gaussian distribution for random number generator.

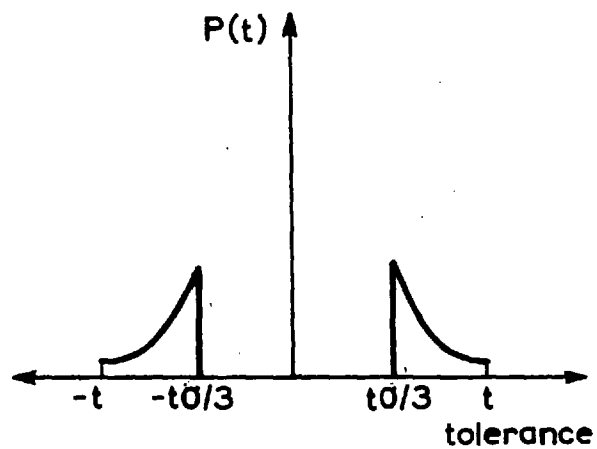


FIG. 3-8: Bimodal distribution for random number generator.

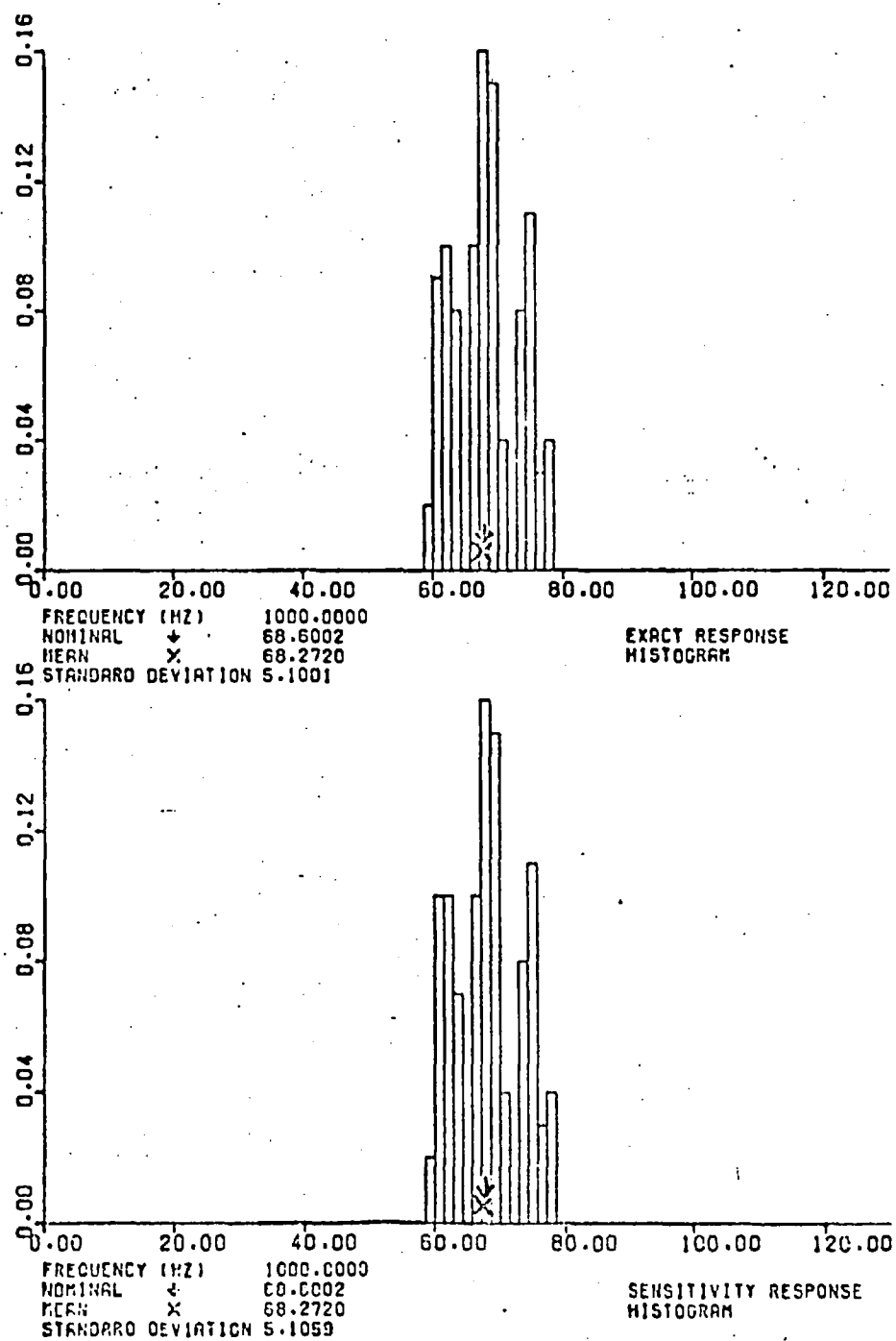


Fig.3.9 Plots for the three transistor amplifier of Fig.3.2 for truncation at the third order sensitivity terms. Component tolerances 10%

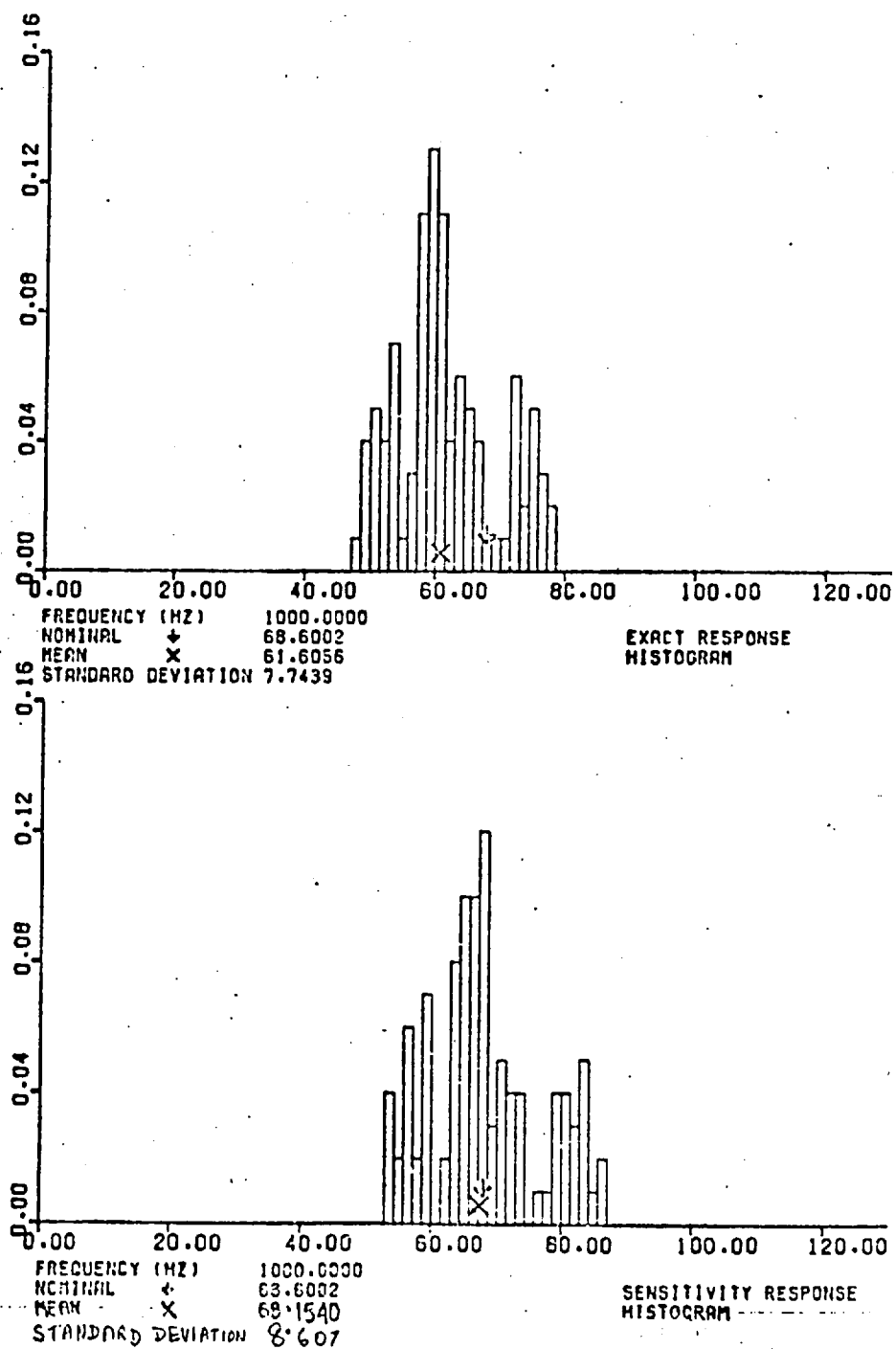


Fig. 3.10 Plots for the three transistor amplifier of Fig.3.2 for truncation at the third order sensitivity terms. Component tolerances of 20%

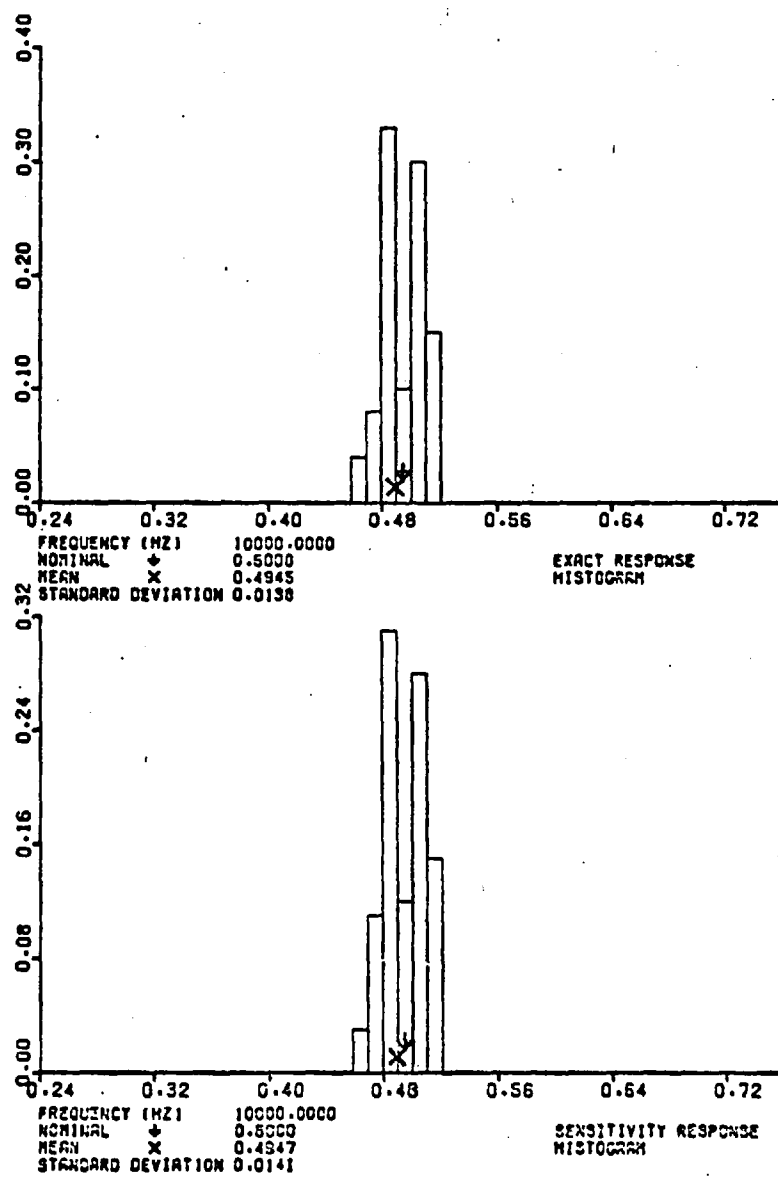


Fig.3.11 Plots for the low pass filter of Fig.3.3 for truncation at the third order sensitivity terms. Component tolerances of 10%

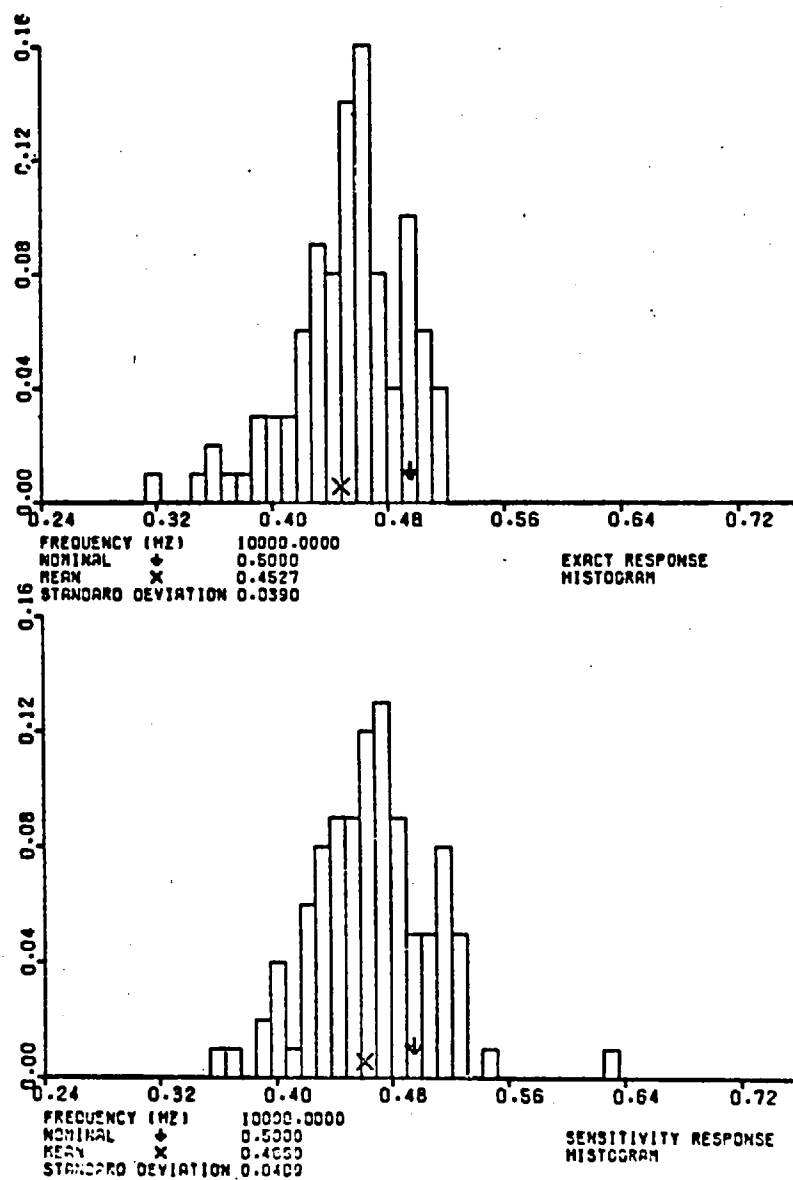


Fig.3.12 Plots for the low pass filter of Fig.3.3 for truncation at the third order sensitivity terms. Component tolerances of 20%

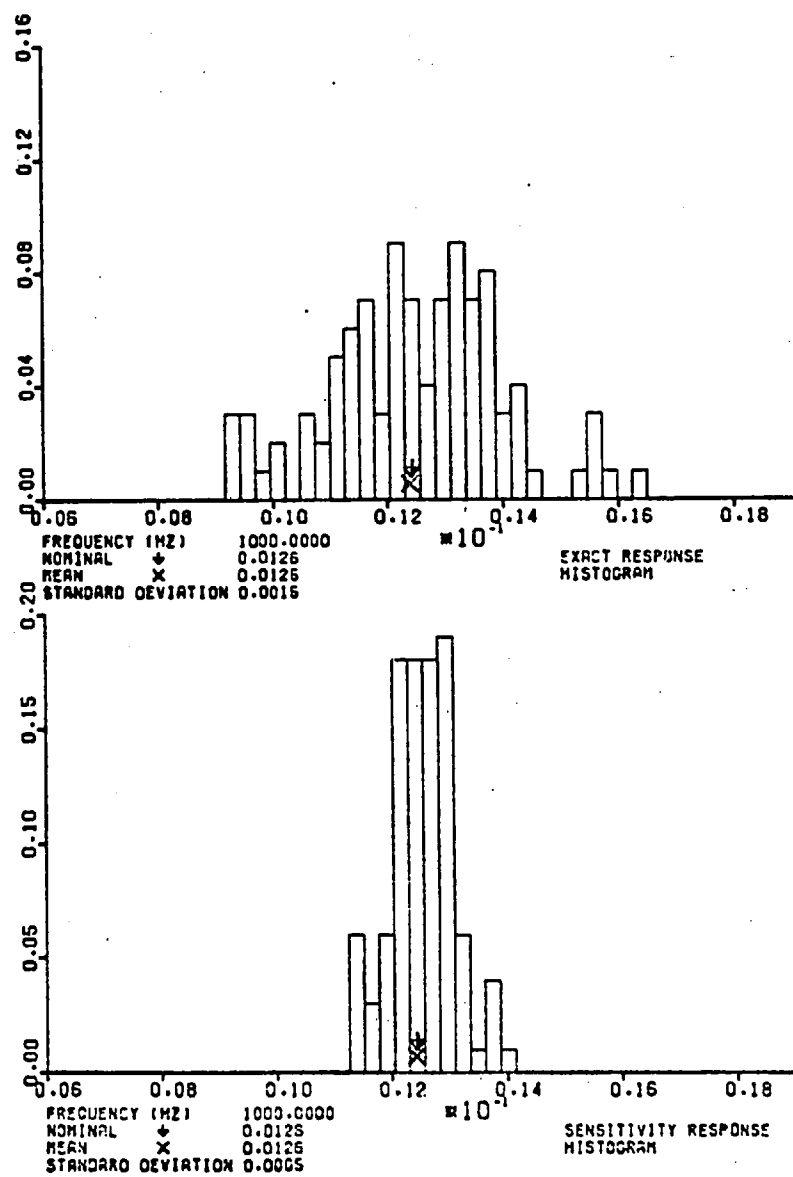


Fig.3.13 Plots for the band pass filter of Fig.3.4 for truncation at the third order sensitivity terms. Component tolerances of 10%

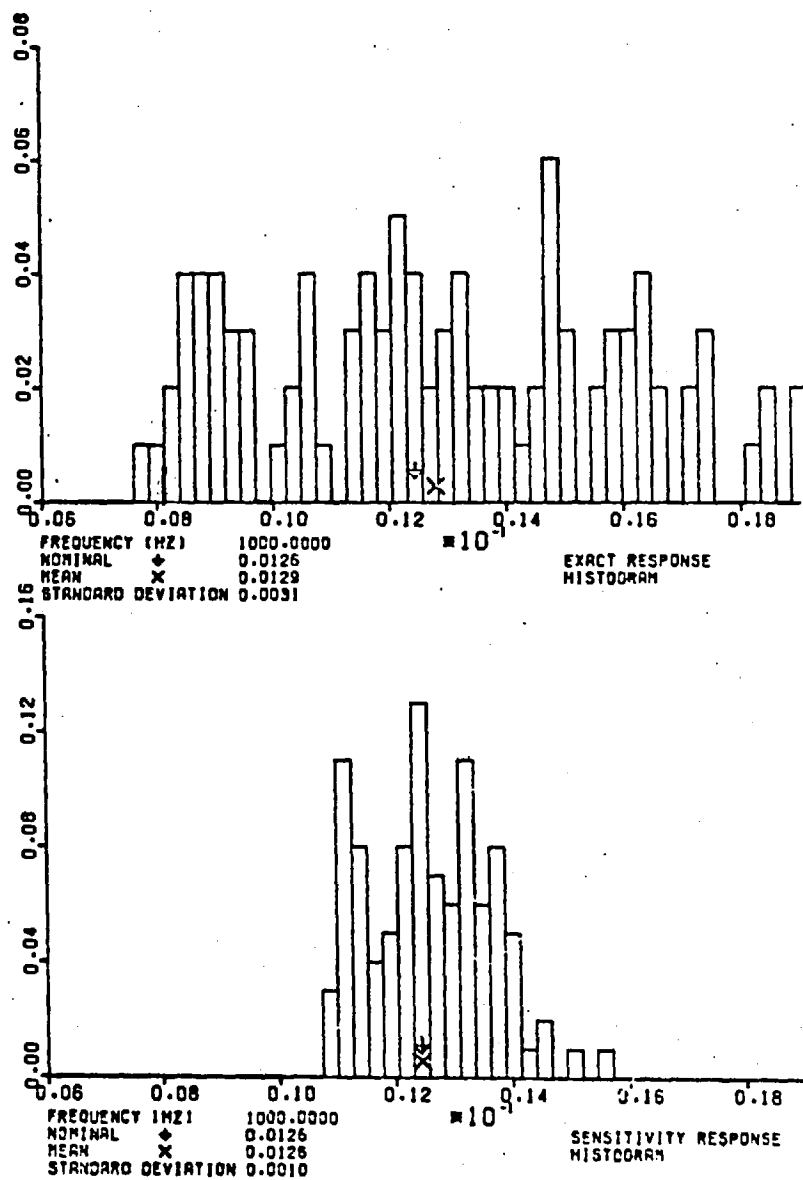


Fig.3.14 Plots for the band pass filter of Fig.3.4 for truncation at the third order sensitivity terms. Component tolerances of 20%

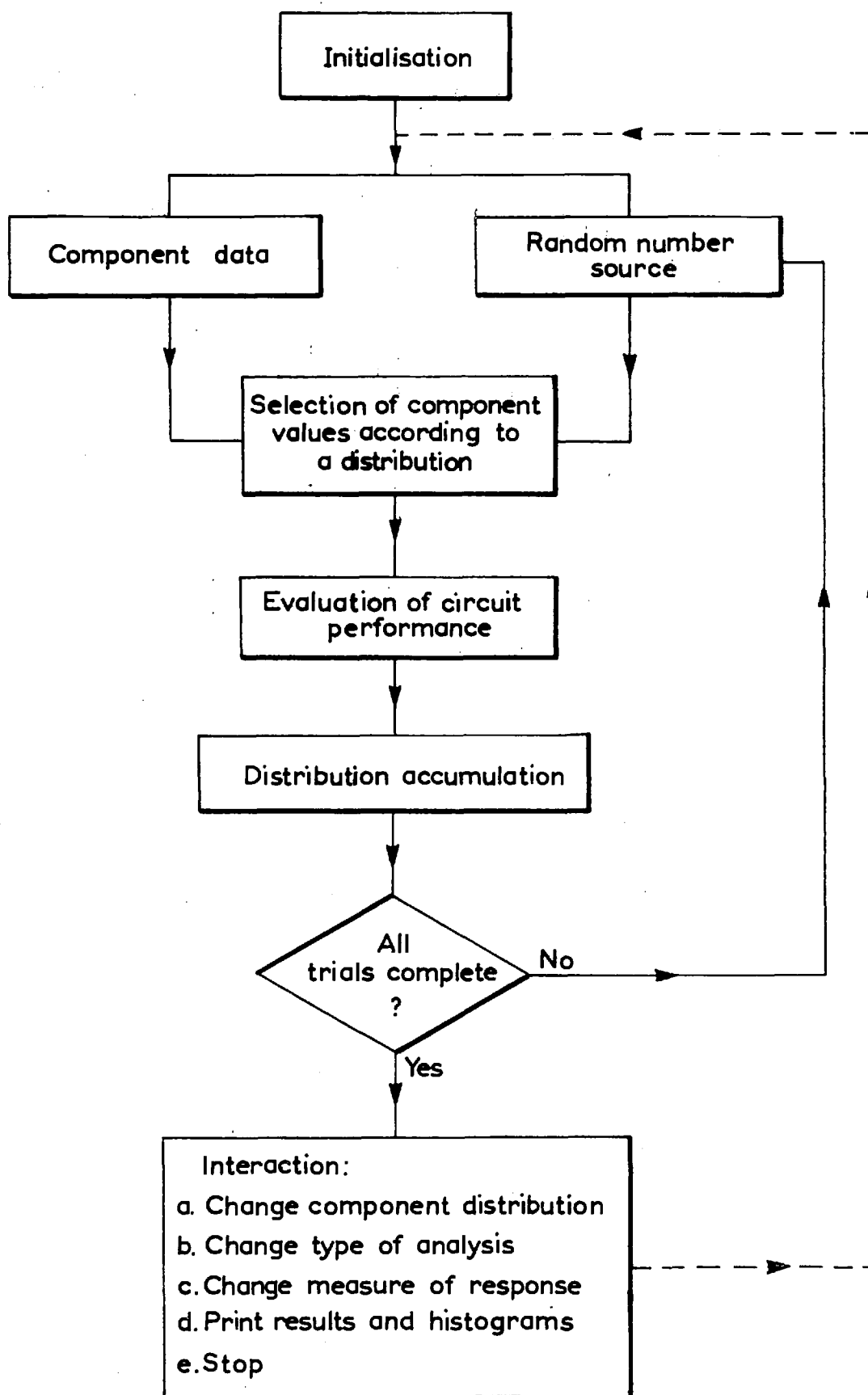


FIG. 3-15: Flow chart of a Monte Carlo Circuit Analysis Procedure

CHAPTER 4

TOLERANCE ASSIGNMENT AND DESIGN CENTERING

4.1 Introduction

When a circuit design, which meets the given specifications, has been achieved, it is desirable to know the maximum tolerances on each of its components, subject to the response remaining within the specifications. The interest in finding the maximum tolerances can be seen from a typical cost/tolerance relationship of a manufactured component shown in Fig.4.1. The wider the tolerance of each component

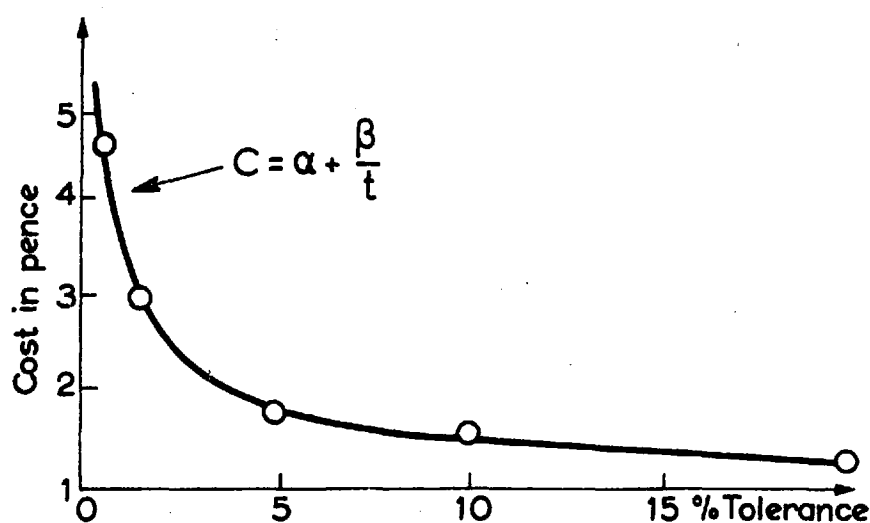


FIG.4.1: Typical cost vs. tolerance relationship of a manufactured component. The equation $C = 1.05 + \frac{3.5}{t}$ gives the least squares fit to the data of a $1\mu F$ metalised polyester capacitor [64]

in the circuit, the smaller the unit cost. As more and more circuits are being designed for increasingly larger production runs, the need for tolerance assignment to become an integral part of the circuit design process has been recognised [55,57,61,63]. It may also be economically attractive to manufacture circuits with very wide tolerances and to discard a small fraction of the final units which fail to meet the design specifications.

The tolerance of a component is the maximum percentage deviation of its value from the nominal x_i^0 . Thus for an absolute deviation Δx_i the tolerance t_i is:

$$t_i = \frac{|\Delta x_i|}{x_{i_0}} \times 100$$

Any method capable of discovering wide component tolerances subject to the circuit response specifications being met is termed a tolerance assignment procedure. This procedure is illustrated in the block diagram of Fig.4.2. Starting from an initial circuit design (1) which meets all the design specifications an algorithm (4) is used to provide a set of component deviations, which are related to the tolerances, for each of the components liable to change. A circuit analysis (5) allows the computation of the new circuit response which is compared (8) with the given specifications (2). From the set of component deviations provided by (4) the tolerances can be found. The total component cost is now computed (6) using the cost/tolerance relationship for each of the circuit components. A test (8) is now made to see if the response constraints are satisfied and that there has been a reduction in the component cost over the previous stage. A failure in either of these tests leads directly

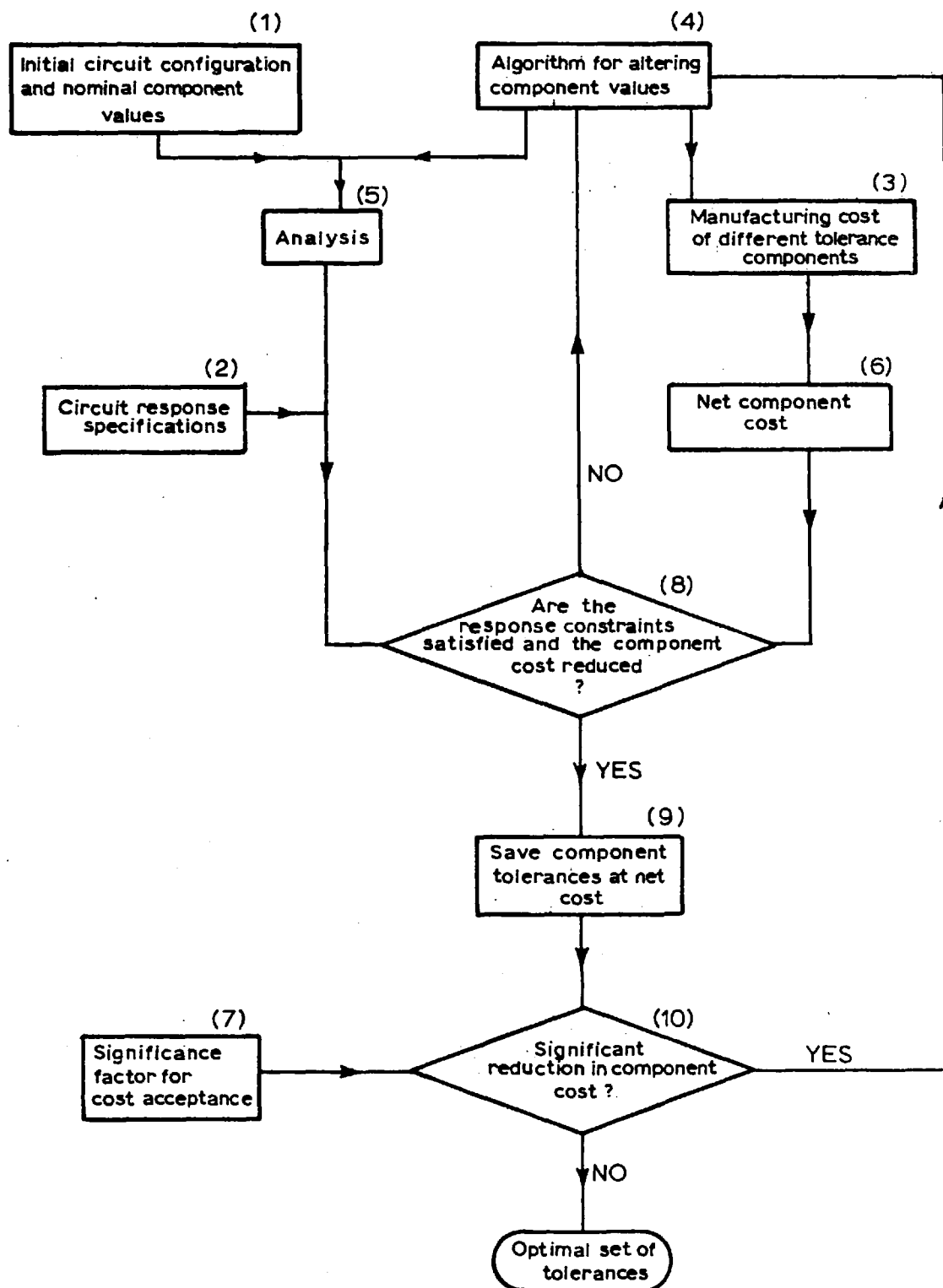


FIG. 4-2: Flow chart outlining a tolerance assignment procedure

back to (4). When both the response constraints are satisfied and the component cost is reduced, then a feasible set of component tolerances (9) have been found. If a significant reduction in the component cost over an earlier cost has been achieved (10) then a loop back to (4) is made to see if a further reduction in the cost can be made for another set of component deviations. When no further significant reduction in the cost (10) can be achieved with the algorithm an optional set of component tolerances is deemed to have been found.

Most of the individual blocks in Fig.4.2 present few computational difficulties except the strategy for the algorithm altering the component values (4). A further difficulty arises from the large number of times the different loops have to be repeated in arriving at the optimal set of tolerances. Therefore, if the tolerancing procedure is to be economically viable, computational efficiency is of paramount importance.

In addition to the tolerance assignment procedure just described it would be useful to investigate the effect of altering the nominal component value of a circuit, if this resulted in wider component tolerances. Any design margins in the circuit response would then be more fully exploited. Any process capable of making the aforesaid alterations to the nominal component values while satisfying the original response constraints is termed "design centering", and is illustrated in the block diagram of Fig.4.3.

In this chapter, after a brief discussion of some of the earlier tolerance assignment algorithms, a new strategy for both tolerance assignment and design centering will be described, and illustrated

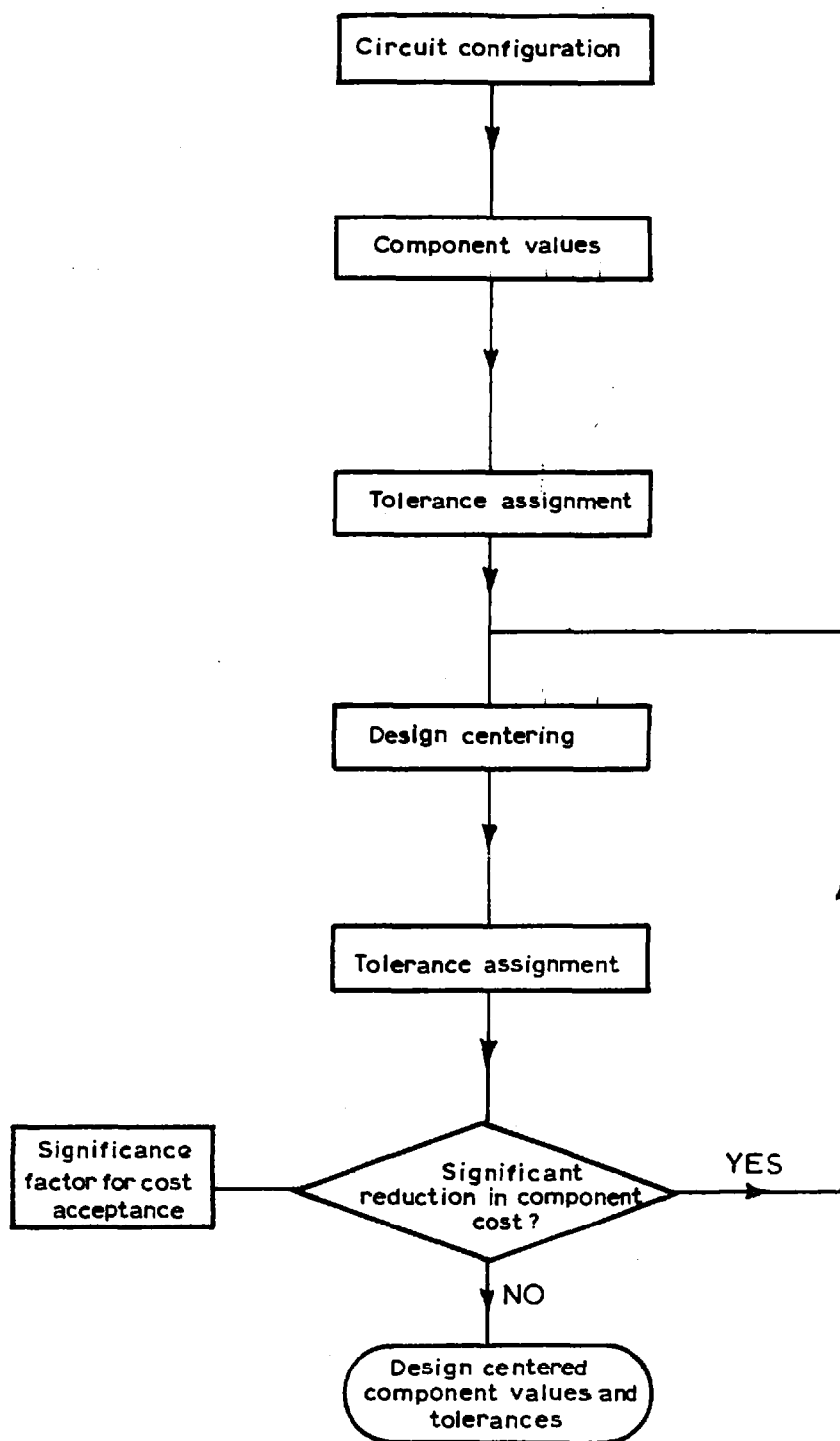


FIG.4-3: Flow chart of a design centering process

with appropriate examples. The measure of circuit performance employed in this chapter is the magnitude of the output voltage for a given input voltage unless stated otherwise.

4.2 Earlier Tolerance Assignment Algorithms

There is no known closed form solution to the tolerance assignment problem. However, three algorithms based on nonlinear programming have been proposed recently, in an attempt to discover the widest tolerances. One [54] is based on the concept of statistical moments, which are parameters describing a distribution of values. Given constraints on the second moment of the circuit response, the maximum possible central moments of the component distributions are found. These moments are then related to component tolerances. The translation from response constraints to response moments and from component moments to component tolerances poses immediate difficulties, for both require a knowledge of the relevant probability functions which are usually not available and expensive to determine. Furthermore, the method assumed that the expected mean value of the component distribution is the same as the starting nominal value, thus precluding any design centering. (See Section 4.9) Another algorithm [55] using a multivariate Taylor series approximation to the response has some similarities to the approach described in this thesis. However, there are significant technical and computational differences which are referred to as they arise. A third method [56] based on the concept of a 'floating polytope' is associated with a computational effort approximately proportional to 2^m , where m is the number of components. Furthermore, there remain certain unresolved problems concerning the assumption of convexity of the feasible component space. A totally different approach [57] based on

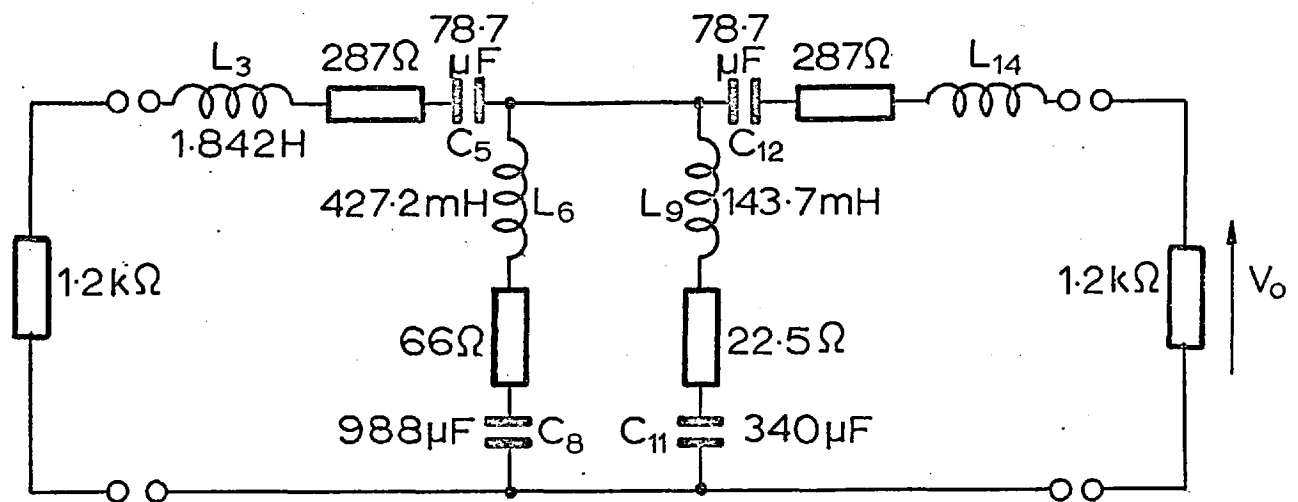


FIG.4.4: Band Pass Filter; having insertion loss specifications of:
 Response $< 3\text{dB}$ in pass band between $f = 360\text{--}490\text{ Hz}$
 Response $> 35\text{dB}$ in stop bands between $f = 0\text{--}240\text{ Hz}$ and $700\text{--}1000\text{ Hz}$

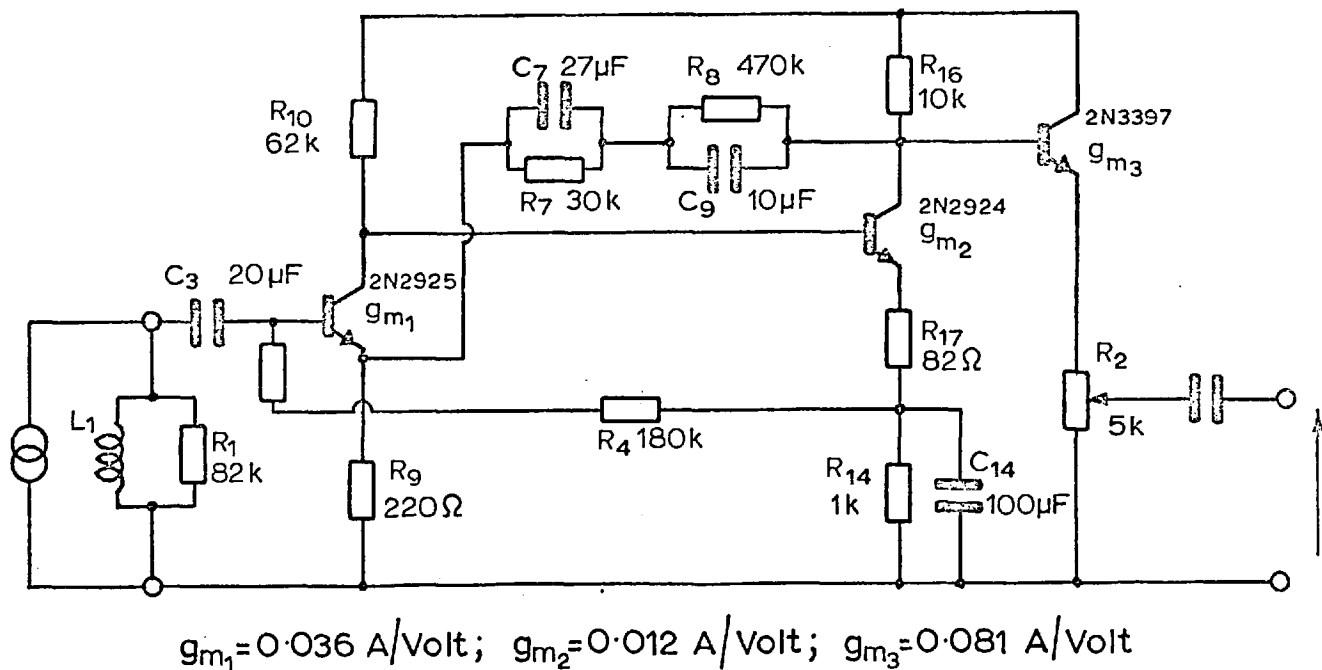


FIG.4.6: Equalizer preamplifier

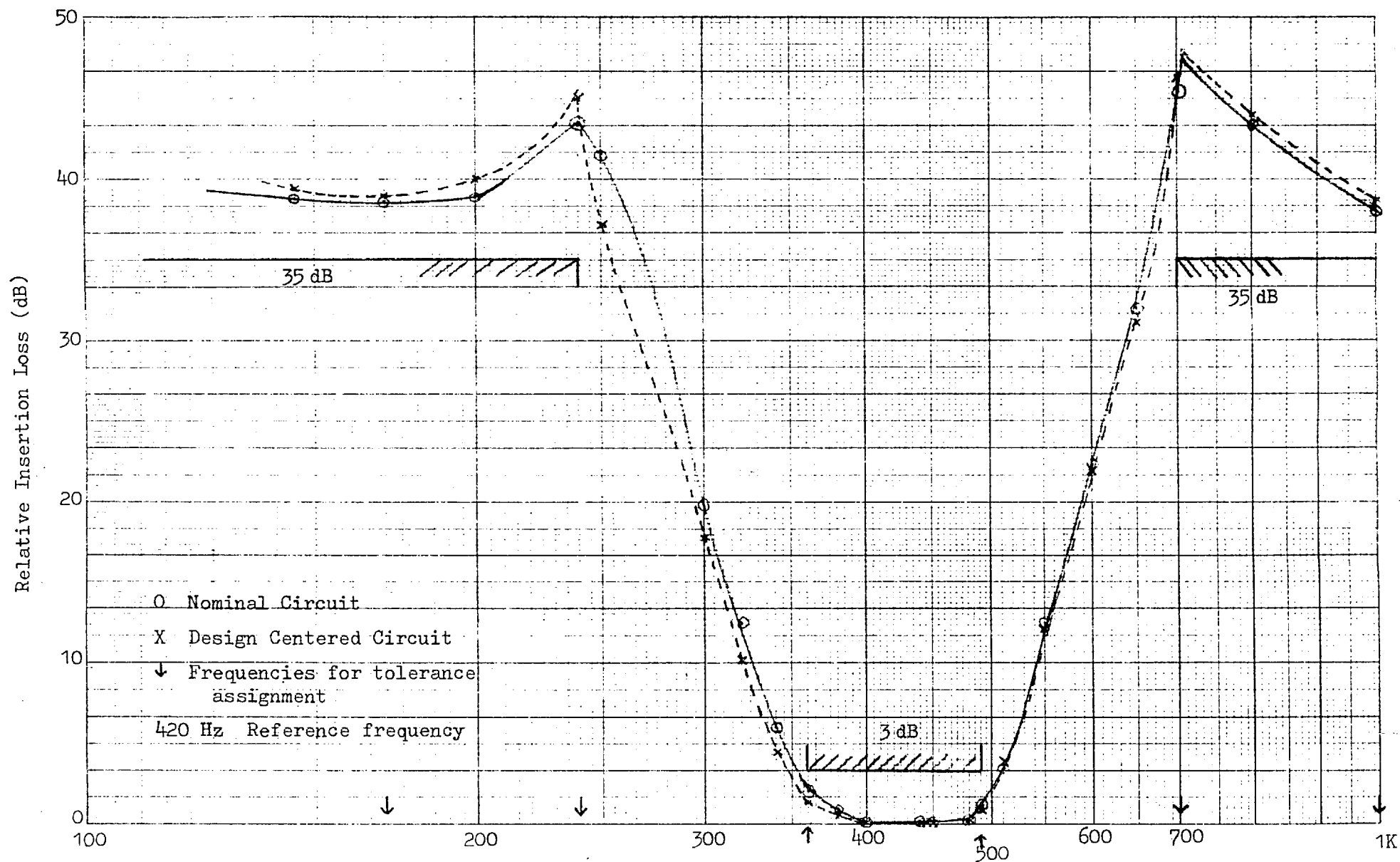
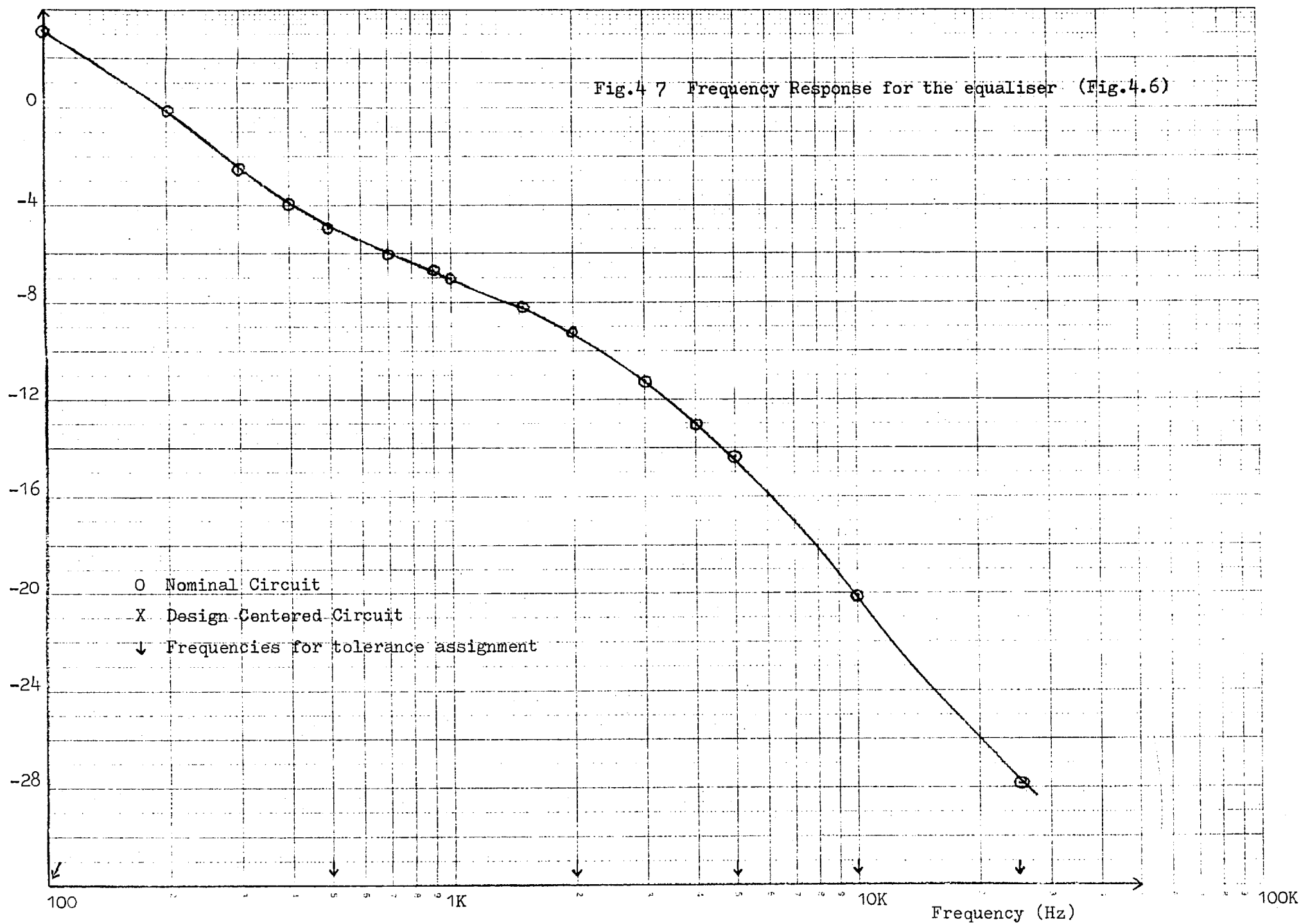


Fig.4.5 Frequency response for the band-pass filter (Fig.4.4)

Frequency (Hz)

Output voltage (dB)



performance contours is computationally expensive and only deals with pairwise parameter interaction.

4.3 Choice of Examples

In anticipation of the forthcoming discussion the two principal circuits, employed to test the proposed tolerance assignment and design centering algorithms, are presented here. The motivation for choosing the band pass filter shown in Fig.4.4 is the ability to compare results with those of other authors. The second circuit (Fig.4.6), an equaliser preamplifier for a magnetic cartridge pick-up [65], has been chosen to demonstrate the application of the tolerance assignment algorithm to active and nonreciprocal circuits.

The design specifications for the circuit in Fig.4.4 are shown in Fig.4.5 in the form of an insertion loss characteristic having a reference level of zero at 490 Hz. The gain characteristic of the equaliser is shown in Fig.4.7, and has a reference level of unity at 1KHz. For both circuits six frequency points as marked were chosen for the purposes of tolerance assignment and design centering. The specifications for the band pass filter demanded that the stop band loss should be above 35 dB while the pass band loss be below 3 dB. For the equaliser circuit the maximum allowed deviation from the nominal response was taken to be $\pm 5\%$ of the voltage magnitude.

4.4 The Problem

A tolerance is a measure of the maximum component variation. Therefore, if the maximum permissible fractional deviation Δx_i of the value of a component of nominal value x_i^0 can be found, then the tolerances t_i can be determined. For a given circuit, m of whose

components are liable to change, let us define a permissible region R of component variation in an m dimensional Euclidian component space E^m . The boundaries of this region are defined by the amplitude* response constraints in the frequency domain.

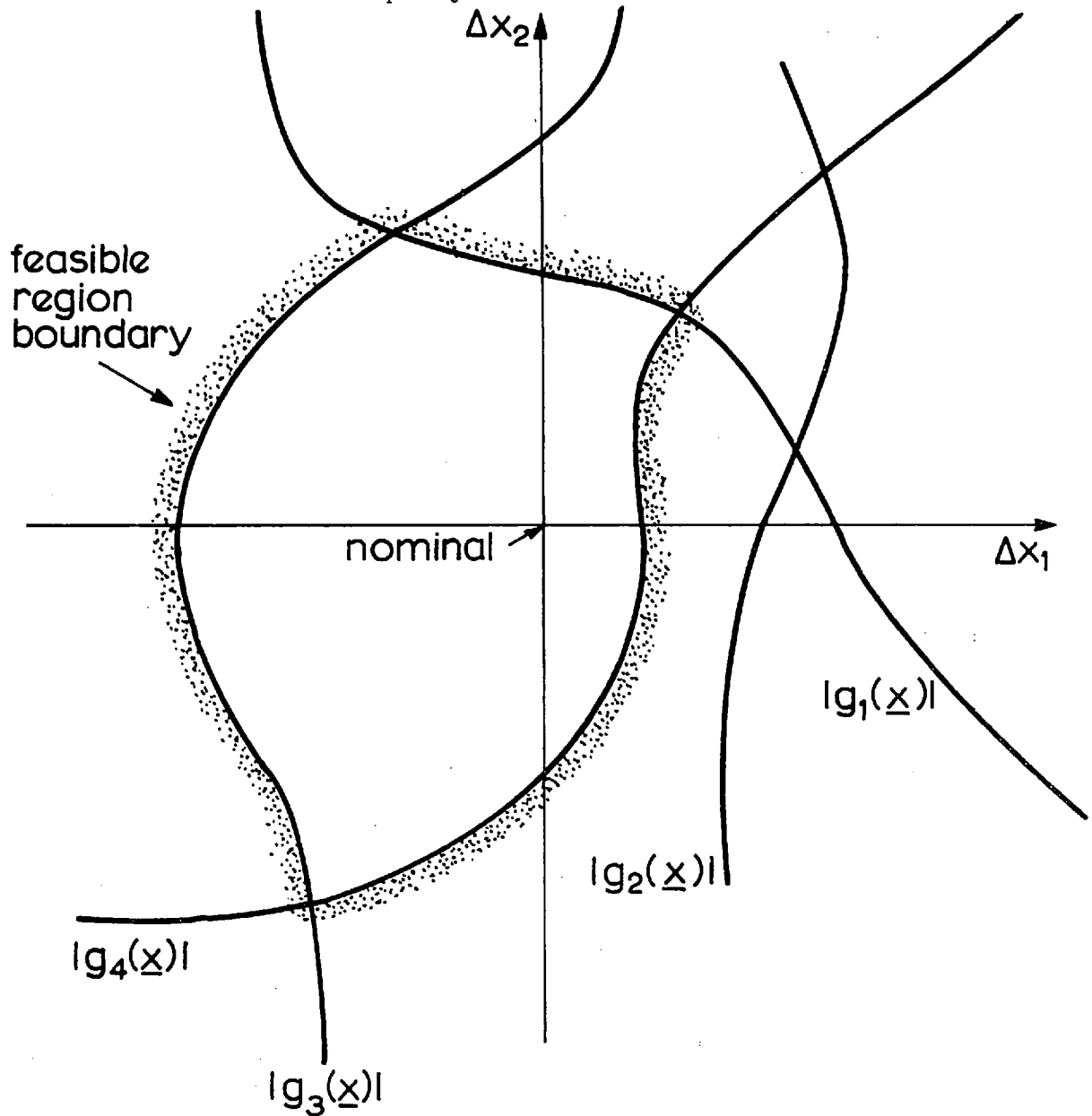


FIG. 4.8: A feasible region of a two component space example.

*Though the amplitude response in the frequency domain is used here and in the subsequent discussion, the tolerance assignment procedure discussed here could be adapted for any desired measure of response and any domain.

$$p_i(\underline{x}, \omega_i) = \left| \rho(\omega_i) - f(\underline{x}, \omega_i) \right| \geq 0, i = 1, 2, \dots, k \quad (4.1)$$

$$g_i(\underline{x}, \omega_i) = \left| f(\underline{x}, \omega_i) - \rho(\omega_i) \right| \geq 0, i = k+1, \dots, l \quad (4.2)$$

where $\rho(\omega_i); i=1, \dots, k$ are the k upper limits of the response specifications and $\rho(\omega_i); i=k+1, \dots, l$ are the $(l - k)$ lower limits at a total of l frequency points ω_i . The feasible region R is now defined as:

$$R \triangleq \left\{ \underline{x} \mid |g(\underline{x}, \omega_i)| \geq 0, i = 1, 2, \dots, k, \dots, k, \dots, l \right\} \quad (4.3)$$

where $\underline{x} = \underline{x}^{OT} (U + \text{diag.} \underline{\Delta})$, U is the unit matrix and $\underline{\Delta}$ are the fractional changes with $\Delta_i = \frac{\Delta x_i}{x_i^0}$. As an example consider the two-variable component spaces in Fig. 4.8 in which four frequency constraints are shown. Once the boundaries of R have been categorised the maximum permissible tolerances can be found. The primary difficulty in a direct evaluation is the massive amount of computation necessary to locate all the nonlinear constraint boundaries in all but the simplest of examples.

However, the tolerance assignment problem can be transformed into a classic nonlinear programming problem, once a suitable objective function incorporating the deviations $\underline{\Delta}$ and the constraints (4.1) and (4.2) is defined. The objective function C :

$$C(\underline{\Delta}) = \sum_{i=1}^m \alpha_i + \frac{\beta_i}{\Delta_i} \quad (4.4)$$

reflects the typical cost/tolerance relationship of the form $c(t) = \alpha + \beta/t$ shown in Fig. 4.1. This is also an appropriate function to be minimised in seeking the maximum tolerances associated with minimum circuit cost. The coefficients α and β can be obtained from a curve fit to the relevant data*. Contours of an objective function

*In all the examples of tolerancing and design centering presented here, the coefficients α and β have been chosen to be zero and unity respectively.

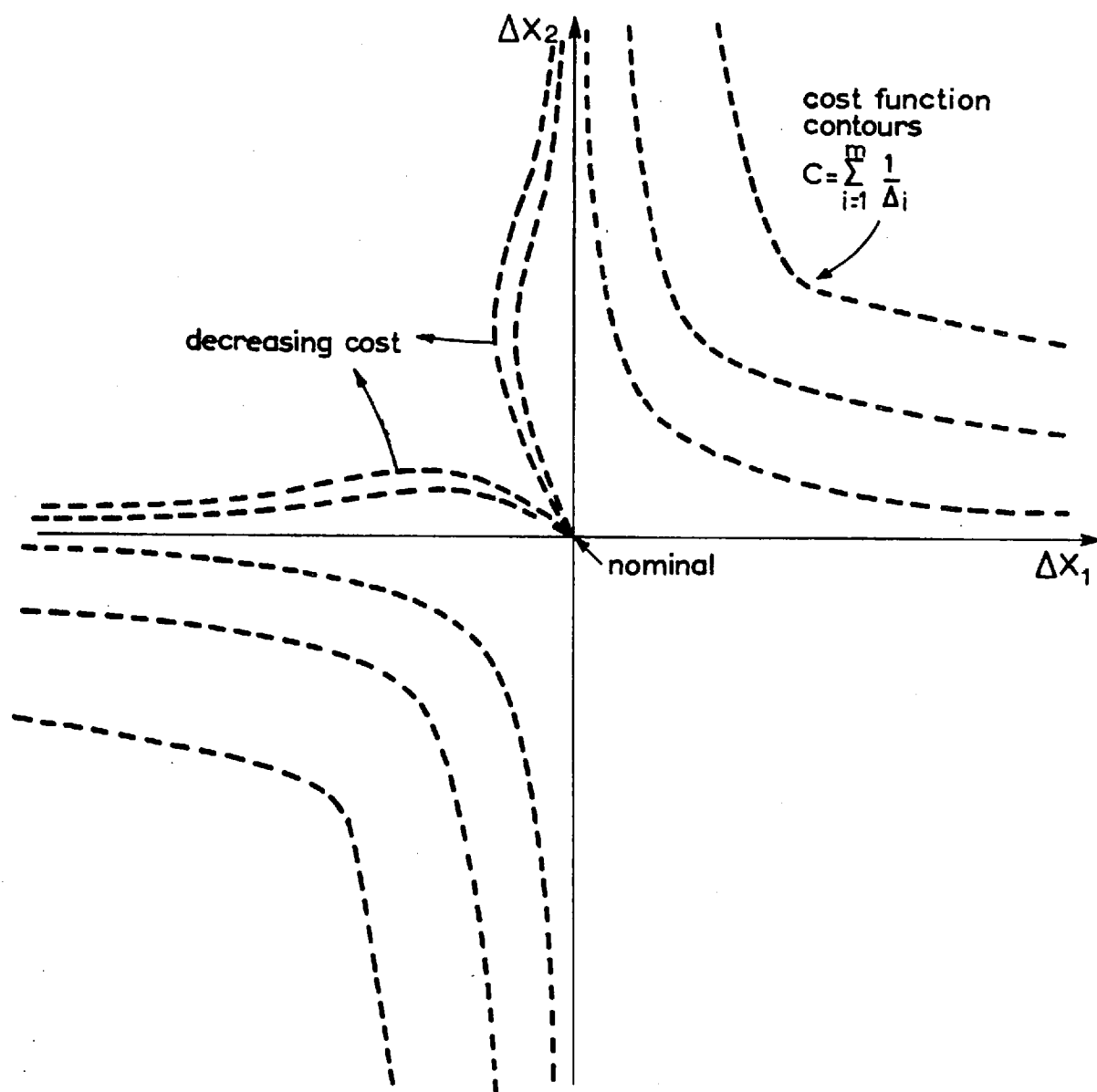


FIG. 4-9: Cost function of a two component example.

similar to (4.4) are shown for a two-parameter example in Fig.4.9. The contours in the second and fourth quadrants show the effect of cancellation of the partial costs when both positive and negative deviations occur. To prevent such cancellation the objective function (4.4) is modified by introducing the modulus of the deviations.

$$C(\underline{\Delta}) = \sum_{i=1}^m \alpha_i + \frac{\beta_i}{|\Delta_i|} \quad (4.5)$$

The objective function contours are now identical in all quadrants and take the form shown in the first quadrant of Fig.4.9. The problem can now be stated as minimising the objective function (4.5) subject to the nonlinear constraints (4.1) and (4.2).

4.5 Constraint Evaluation

To find out when one or more of the constraints have been violated, it is necessary to compute the response for any set of fractional deviations $\underline{\Delta}$, and then test the inequalities (4.1) and (4.2). An exact calculation of the response can be made [43], but as this computation has to be repeated for every point along the search path, the resulting computation can become prohibitively expensive. Instead we propose the use of a truncated complex Taylor expansion to the response. The efficiency and accuracy of this truncated series compared to an exact calculation has been discussed in Chapter 3.

Assume that the voltage gain function $f(\omega_1, \underline{x}^0)$ satisfies the usual mathematical conditions of continuity and differentiability in the neighbourhood of the nominal component values $\underline{x}^0$. The function may then be expanded as a complex Taylor series.

$$f(\omega_i, \underline{x}) = f(\omega_i, \underline{x}^0) + \sum_{j=1}^m S_{x_j}^f \Delta_j + \frac{1}{2!} \sum_{j=1}^m \sum_{k=1}^m S_{x_j x_k}^f \Delta_j \Delta_k + \dots \quad (4.6)$$

where $S_{x_j}^f = \frac{\partial f}{\partial x_j} x_j$ etc. and $\underline{x} = \underline{x}^{0T} (\underline{u} + \text{diag.} \underline{\Delta})$. In Chapter 3 conditions under which (4.6) converges have been derived and discussed. Furthermore, empirical experience with a variety of circuits has shown that a response approximation using the truncated complex Taylor expansion including all terms up to the second order, plus a further n third order 'self' terms, is not only faster but of adequate accuracy when compared with exact methods, for deviations of up to about 10%.

Pinel and Roberts [55] have used a similar Taylor expansion for their tolerance assignment procedure. However, they employed perturbation methods to produce a curve fit for the change in the amplitude response in a multi-dimensional component space for different component changes. The sensitivities were then computed using some differencing procedure. To achieve an adequate curve fit using perturbation methods requires many circuit analyses and is therefore computationally expensive. If insufficient perturbations are performed the amplitude response function may have only a limited resemblance to the actual response. The use of differencing methods to obtain the sensitivities is likely to introduce secondary errors into (4.6) when the sensitivities themselves are small, as is usually the case in the pass band of passive filters [60].

4.6 Selection of Sign of Component Division

Having chosen one of the many nonlinear optimisation procedures, [100] is it possible to accelerate the optimisation towards the optimal solution? Pinel and Roberts [55] state that they used the

adjoint network to obtain the first order component sensitivities, and that they forced the optimisation procedure to the worst case using the negative of the sign of linear sensitivities for each of the component deviations. Since sensitivities are complex entities it is not clear how the sign was ascribed to the deviations. Furthermore, as the following discussion shows, all the component signs must be restricted, in the region of search, if the optimisation is to yield meaningful results.

Consider a two component example of an RC circuit (Fig.4.10).

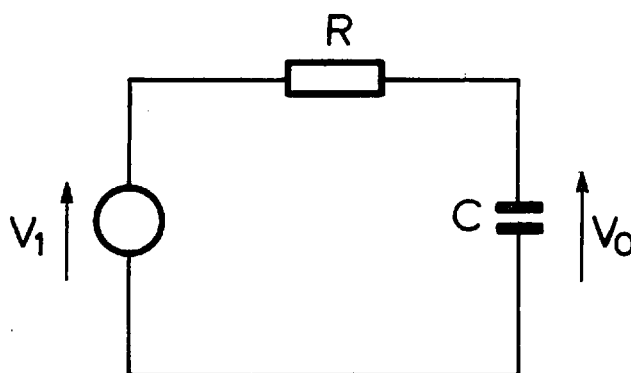


FIG. 4.10: A two component circuit example.

From the existence of summed sensitivity invariants [105] in electrical systems (see Chapter 2) the following relationship is obtained:

$$\sum_{\text{all } R} s_R^f - \sum_{\text{all } C} s_C^f = 0 \quad (4.7)$$

If both components are liable to change, then the response contour, for deviations from the nominal, can be indicated on Fig.4.11. From our experience, the first order terms in the Taylor expansion make a

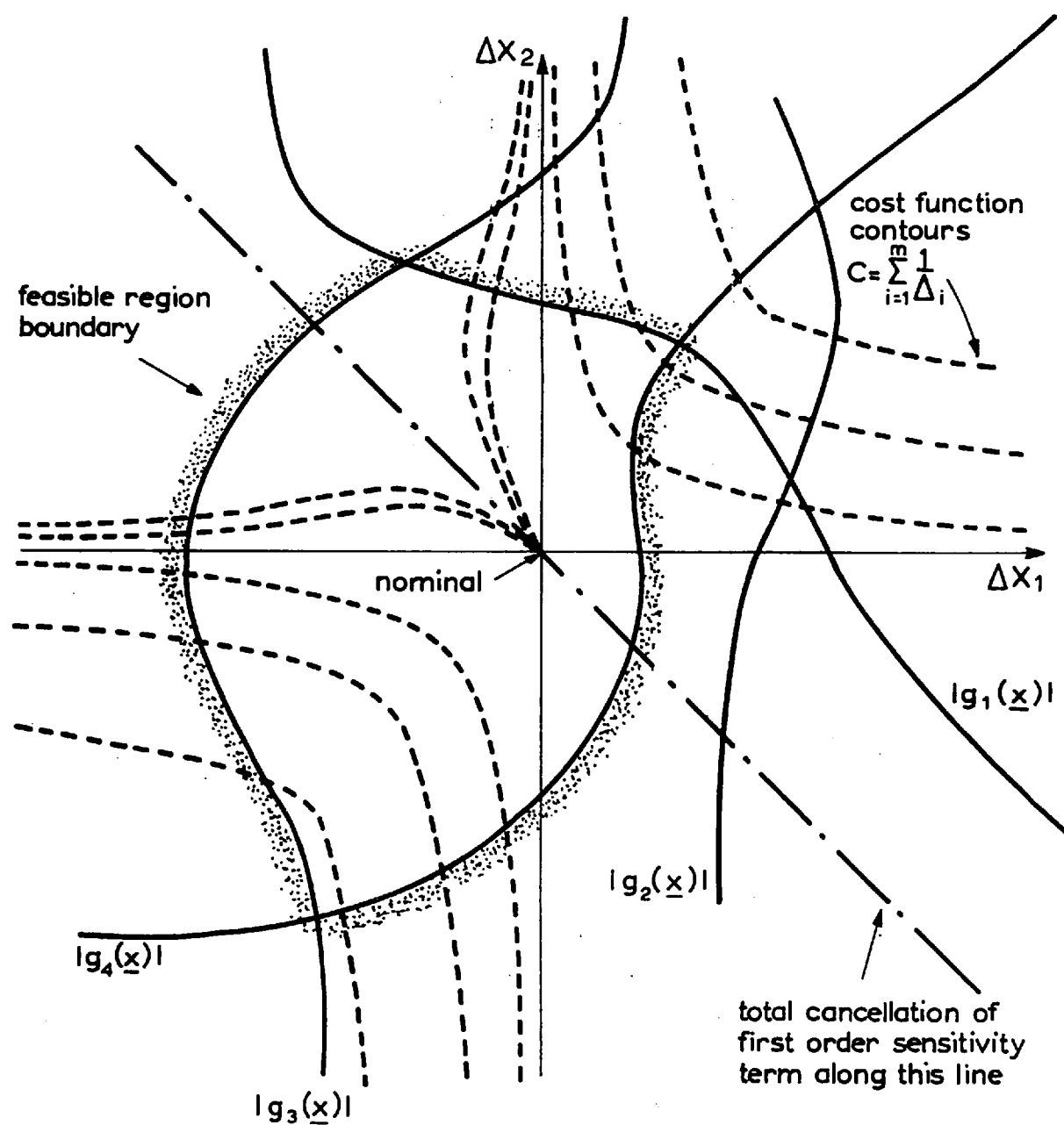


FIG. 4-11: A two component space example with four frequency constraints $|g_i(\underline{x})|$ and contours for the objective function

large contribution to the total response change. It can be seen from the invariant relationship that if the component deviations are equal and opposite in sign, then total cancellation of the first order terms of the series (4.6) will occur for an RC-network. In fact, any deviations restricted to the second and fourth quadrants will result in the partial cancellation of first order sensitivities and therefore minimum response change. Thus, any optimisation procedure searching in the second and fourth quadrants will automatically tend to adjust all the deviation magnitudes such that the constraints given by equations (4.1) and (4.2) are always satisfied, while the objective function is minimised. Hence, large but unrealistic deviations would be predicted and in practice these would yield tolerances which are too large. Similar arguments hold for larger circuits having invariant relationships of the type (4.7), with m variable components where $(2^m - 2)$ of the 2^m octants* in the m -dimensional Euclidian space cannot be searched.

To test that this hypothesis has some justification in practice note the results in Table 4.1 for the circuit in Fig.4.4. The tolerances in the tables were obtained using a nonlinear programming algorithm incorporating penalty functions, which is discussed in greater detail in Section 4.8a - it is sufficient here to be aware that a search for tolerances is possible. The first row shows tolerances obtained when no sign restriction was placed on the deviations. The predicted tolerances are very large as expected and a Monte Carlo analysis (see Section 4.10) using these tolerances indicates a large number of failures. The result of another test in which the starting signs of each of the deviations were chosen randomly again

*For a two dimensional space there are four octants, usually termed quadrants.

yields large tolerances and low success rates, giving further credence to the justification of our hypothesis.

	L_3	C_5	L_6	C_8	L_9	C_{11}	C_{12}	L_{14}		
Component % Tolerance	1.824H	.0787μ	.4272H	.988μ	.1437H	.34μ	.0787μ	1.824H	Cost = $\sum \frac{1}{ \Delta_i }$	Monte Carlo Success rate
No sign restriction $\underline{t}$	36.91	-44.92	-43.77	88.63	-82.78	40.95	31.74	37.67	.1653	186/600
Random starting sign $\underline{t}$	36.67	14.36	-19.71	-11.78	27.30	-27.39	11.83	47.78	.4111	346/600
Modulus Taylor series $\underline{t}$	1.733	1.485	1.128	1.653	3.668	1.801	1.485	1.753	4.8200	600/600

Table 4.1 Percentage tolerances of components of the band pass filter, Fig.4.4 170 Hz - 1000 Hz

To avoid the sensitivity cancellation problem, one approach would be to use a Taylor series with modulus sensitivity terms.

$$|f(\omega_i, \underline{x})| \leq |f(\omega_i, \underline{x}^0)| + \sum_{j=1}^m |S_{x_j}^f| \Delta_j + \frac{1}{2!} \sum_{j=1}^m \sum_{k=1}^m |S_{x_j x_k}^f| \Delta_j \Delta_k + \dots \quad (4.8)$$

The tolerances obtained when using this series to evaluate the constraints are shown in the third row. Though they yield a 100% success rate the tolerance cost is rather high when compared to the earlier costs. A more acceptable solution with a lower cost probably lies between the above extremes, and is more likely to be found using the complex Taylor series with certain sign restrictions on the elements of $\underline{\Delta}$.

Therefore it is suggested that a first search in the feasible component space is made by restricting all deviations to be positive, denoted by $\underline{\Delta}^+$. The second search is made with all deviations constrained to be negative, denoted by $\underline{\Delta}^-$. The final selection of tolerances is made by selecting the minimum corresponding modulus terms from the two vectors $\underline{\Delta}^+$ and $\underline{\Delta}^-$.

4.7 Treatment of Homogeneous Network Functions

Difficulties can arise with networks which have homogeneous response functions of order zero. Purely resistive networks, and RL networks, are examples. Furthermore, certain amplifier circuits may have been designed to be essentially resistive over some range of frequencies. As an example of homogeneous network functions, consider the summed sensitivity invariant for the voltage gain of a purely resistive network.

$$\sum_{\text{all } R} S_R^f = 0 \quad (4.9)$$

A comparison of equations (4.6) and (4.9) indicates that if all the resistors change by the same amount then the net contribution from the first order sensitivity terms in (4.6) will be zero. In fact, from the results of Chapter 2, Section 2.7 the contribution from all higher order terms can also be shown to be zero. Hence, if the initial design satisfies the constraints (4.1) and (4.2) then any optimisation procedure is more likely to make large and identical deviations in the components while still satisfying the constraints. That this is indeed the case can be seen from the results of Table 4.2, row 1 for tolerancing the circuit in Fig.4.12. This cancellation of the sensitivities can only be avoided if the modulus Taylor expansion (4.8) is used. The tolerances obtained when using such a series are shown in row 3 of Table 4.2.

Component % Tolerance	R ₂	R ₃	R ₄	R ₅	R ₆	R ₇	R ₈	R ₉	Cost	Monte Carlo success rate
No sign restriction	5.146	99.97	100.0	100.0	80.87	99.99	80.32	99.98	0.094	0/100
Random starting sign	4.139	44.85	14.59	4.999	10.75	23.55	9.336	100.0	0.785	43/100
Modulus Series	2.875	27.57	9.926	3.578	6.517	14.32	5.955	12.22	1.237	100/100

Table 4.2 Percentage tolerances of components of the circuit,

Fig.4.12 for $\pm 5\%$ variation on the output voltage

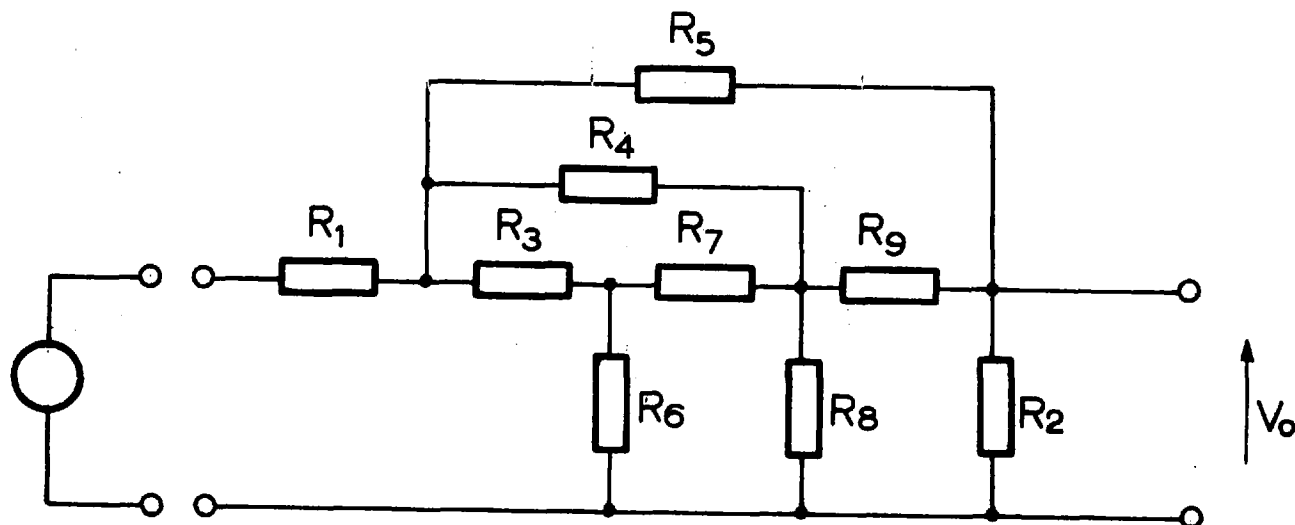


FIG. 4.12: Resistive network

Therefore for all network response functions whose summed sensitivity invariants are known constants, the modulus series should be used. It has been shown in Chapter 2 how such circuit response functions may be identified.

4.8 The Choice of a Nonlinear Programming Algorithm

A detailed discussion of unconstrained optimisation methods and nonlinear programming techniques is given in Appendix II. In this section only the two methods of nonlinear programming actually used in the tolerance assignment problem will be discussed.

(a) Penalty Function Methods

In these methods the nonlinear constrained optimisation problem as defined in Section 4.4 is transformed into a series of unconstrained minimisation problems. One form, suggested by Fiacco

and McCormick [84], utilises a new function called the penalty function

$$P(r, \underline{\Delta}) = C(\underline{\Delta}) + r \sum_{i=1}^l \frac{1}{g_i(\underline{x}, \omega_i)} \quad (4.10)$$

Starting from an interior point in the feasible space, and with $r = 1$, the unconstrained minimum of P is found. The parameter r is now reduced by a factor of 0.1 and the unconstrained minimisation of P repeated. This constitutes one iteration. The procedure is repeated until a specified convergence criterion for P is met at which point the optimisation terminates, and the optimal vector $\underline{\Delta}$ is found. However, it has been noted in practice that due to the necessity of taking discrete steps during the minimisation, the constraints of equations (4.1) and (4.2) can sometimes be violated. To force the function P back into the feasible region a further penalty has to be added to equation (4.10) which now reads:

$$P(r, \underline{\Delta}) = C(\underline{\Delta}) + r \sum_{i=1}^l \frac{1}{g_i} + \sum_{k=1}^l t_k g_k \quad (4.11)$$

where $t_k = 0 \quad \forall k \in \{1, 2, \dots, l\}; g_k \geq 0\}$

and $t_k = 1 \quad \forall k \in \{1, 2, \dots, l\}; g_k < 0\}$

Equation (4.11) is now minimised using the Powell Conjugate directions method [91], starting with $\underline{\Delta} = \underline{\Delta}^+$, (all deviations constrained to be positive). The full minimisation procedure is repeated with $\underline{\Delta} = \underline{\Delta}^-$ (all deviations constrained to be negative). Little or no extra computation has to be performed in constraining the deviations because of the nature of the objective function (4.4) which approaches ∞ at the origin from above and below.

The penalty function approach has been employed in compiling all

the Tables of results except for Table 4.5, which was obtained using the Flexible Tolerance algorithm [100] (see Appendix II and Section 4.8b) of nonlinear programming.

$\underline{\Delta}^+$	4.903	5.752	5.577	11.14	4.489	3.039	5.755	4.863	1.572	574/600
$\underline{\Delta}^-$	-4.061	-3.401	-2.200	-3.178	-8.566	-4.299	-3.401	-4.031	2.204	593/600
$\underline{t}^*$	4.061	3.401	2.200	3.178	4.489	3.309	3.401	4.021	2.397	591/600
Pinel et al [55]	3.32	2.41	1.14	1.89	7.80	2.07	3.30	2.41	3.4514	600/600

Table 4.3 Percentage tolerances of components of the band pass filter Fig.4.4

	R_2	R_7	C_7	R_8	C_8	R_9	R_{10}	R_{16}	ε_{m1}	ε_{m2}		
Component % Tolerance	5k	30k	2.7nF	470k	0.1nF	220 Ω	62k	10k	0.038	0.012	Cost	
$\underline{\Delta}^+$	11.82	16.43	2.353	20.47	6.669	2.245	23.22	19.39	11.93	20.76	1.467	368/480
$\underline{\Delta}^-$	-9.429	-15.03	-2.309	-13.26	-5.087	-2.175	-19.65	-13.85	-8.592	-14.22	1.669	378/480
$\underline{t}^*$	9.429	15.03	2.309	13.26	5.087	2.175	19.65	13.85	8.592	14.22	1.699	378/480

Table 4.4 Percentage tolerances of components of equaliser, Fig.4.6 with 5% tolerance on output voltage magnitude 100Hz - 25kHz

Rows 1 - 3 of Table 4.3 show the result of tolerance assignment on the band pass filter, Fig.4.4, with a near 100% yield. For comparison the (100% yield) tolerances obtained by Pinel at a much higher cost are shown in row 4. The high Monte Carlo success rates for the tolerances in rows 1 - 3 constitute an acceptable engineering solution. Integral values of the tolerances can be found by appropriate rounding of the continuous values and will be discussed in Section 4.11.

The tolerances assigned to the active circuit, Fig. 4.7, are shown in Table 4.4. In this case the Monte Carlo success rates of about 81% are lower than in the previous example. The response distribution shown in Fig. 4.16 is at the frequency at which the

greatest number of failures were encountered. One possible explanation of this low yield of 81% will be discussed in Section 4.10.

(b) The Flexible Tolerance Algorithm

This is a recently proposed direct search nonlinear programming algorithm [100]. It attempts to find the minimum of the objective function (4.4) subject to the constraints (4.1) and (4.2) using a variable or flexible simplex (polyhedron), whose vertices can lie in either the feasible or infeasible region, to search the component space for the optimal $\underline{\Delta}$. Unlike the penalty function method no attempt is made to solve a sequence of unconstrained minimisation problems. Therefore, it was hoped that a faster solution could be made.

The results using the Flexible Tolerance Algorithm of nonlinear programming for the circuit in Fig.4.4 are shown in Table 4.5.

$\underline{t}$	3.462	3.961	2.180	3.470	3.016	3.539	3.470	4.701	2.4029	299/300
-----------------	-------	-------	-------	-------	-------	-------	-------	-------	--------	---------

Table 4.5 Percentage tolerances of components of the band pass filter, Fig.4.4

However, contrary to expectation experience shows that the penalty function approach is faster than the direct search. Thus each iteration with the penalty function took 8 seconds of CP-time, and there was little improvement after the second iteration, as seen in Table 4.6. The identical problem with the direct search algorithm took 49.4 seconds of CP-time. Furthermore, because the flexible polyhedron uses feasible and nonfeasible points, it sometimes fails to find a feasible optimum as the polyhedron shrinks near the constraint boundary. No such difficulties were encountered with the penalty function method.

1st Iteration	4.724	4.096	4.409	15.00	3.510	2.632	4.096	4.474
2nd Iteration	4.711	5.333	5.298	11.34	4.230	2.949	5.333	4.711
6th Iteration	4.903	5.759	5.577	11.14	4.489	3.039	5.755	4.863

Table 4.6 Percentage tolerances of components of the band pass filter, Fig.4.4

As stated earlier the change in the tolerances at the second and sixth iterations of the penalty function method is very small. This can be understood from the two variable examples of Fig.4.11. It can be seen that the objective function (4.5) has an unconstrained minimum at infinite values of the deviations in the 1st and 3rd quadrants, and it is only the constraints (4.1) and (4.2) which prevent the search path from rapidly going towards infinity. The search path therefore quickly arrives at the constraint boundary almost at the first iteration. Thereafter much time is spent in moving to the final optimal point in very slow steps. Hence, it would be prudent to stop the optimisation after two or three iterations rather than waste time in finding the exact optimum.

4.9 Design Centering

A design centering procedure is one which determines the changes necessary in the nominal component values of a circuit such that the component tolerances are widened while the response distribution of the circuit remains within the originally specified limits. Thus design centering in the component space can be viewed as finding the centroid of the m -dimensional feasible region, providing it is convex, as shown in Fig.4.11 for a two component example. Finding the centroid in an m -dimensional feasible space requires a knowledge of the space boundaries, and is therefore a computationally impossible task for all but the most simplest networks. However, a simple intuitive approach which naturally follows from having the positive and negative

deviations $\underline{\Delta}^+$ and $\underline{\Delta}^-$ is to choose changes in the nominal value for each of the components of the circuit corresponding to respective elements of the vector $\underline{\delta}$.

$$\underline{\delta} = \frac{1}{2} (\underline{\Delta}^+ - \underline{\Delta}^-) \quad (4.12)$$

Thus the nominal value of each element of $\underline{x}^0$ is moved to lie midway between its maximum and minimum deviations. The tolerance assignment procedure as described in Section 4.8 is now repeated with a view to reducing the component cost, i.e. wider tolerances are obtained. Another $\underline{\delta}$ can now be computed followed by another tolerance assignment. In fact, this can form the basis of an iterative design centering procedure. However, the computational cost of each design centering iteration is high because of the need of a full tolerance assignment. Therefore, design centering must be achieved after one or at most two iterations, if this method is to be regarded as a practical proposition.

	1.833H	0.08027 μ	.4306H	1.08 μ	0.1401H	.3383 μ	.07929 μ	1.833H		
$\underline{t}^*$	4.68	5.61	3.58	5.38	4.33	2.91	5.36	4.56	1.844	585/600

Table 4.7 Percentage tolerances of components of the design centered band pass filter, Fig.4.4

	5.04k	30.21k	2.7nF	486.9k	0.1nF	220	62.5k	10.27k	38.6mA/V	12.39mA/V		
$\underline{t}^*$	9.631	15.01	2.312	13.56	5.122	2.166	20.40	14.34	8.92	14.85	1.652	400/480

Table 4.8 Percentage tolerances of components of design centered equaliser, Fig.4.6

In Tables 4.7 and 4.8 results are presented for a single iteration through the design centering procedure applied to the circuits of Fig.4.4 and Fig.4.6 respectively. The changes in the component values of the two circuits were obtained from the $\underline{\Delta}^+$ and $\underline{\Delta}^-$ results of Table 4.3 and 4.4 respectively. A comparison of costs from Tables 4.3 and 4.7

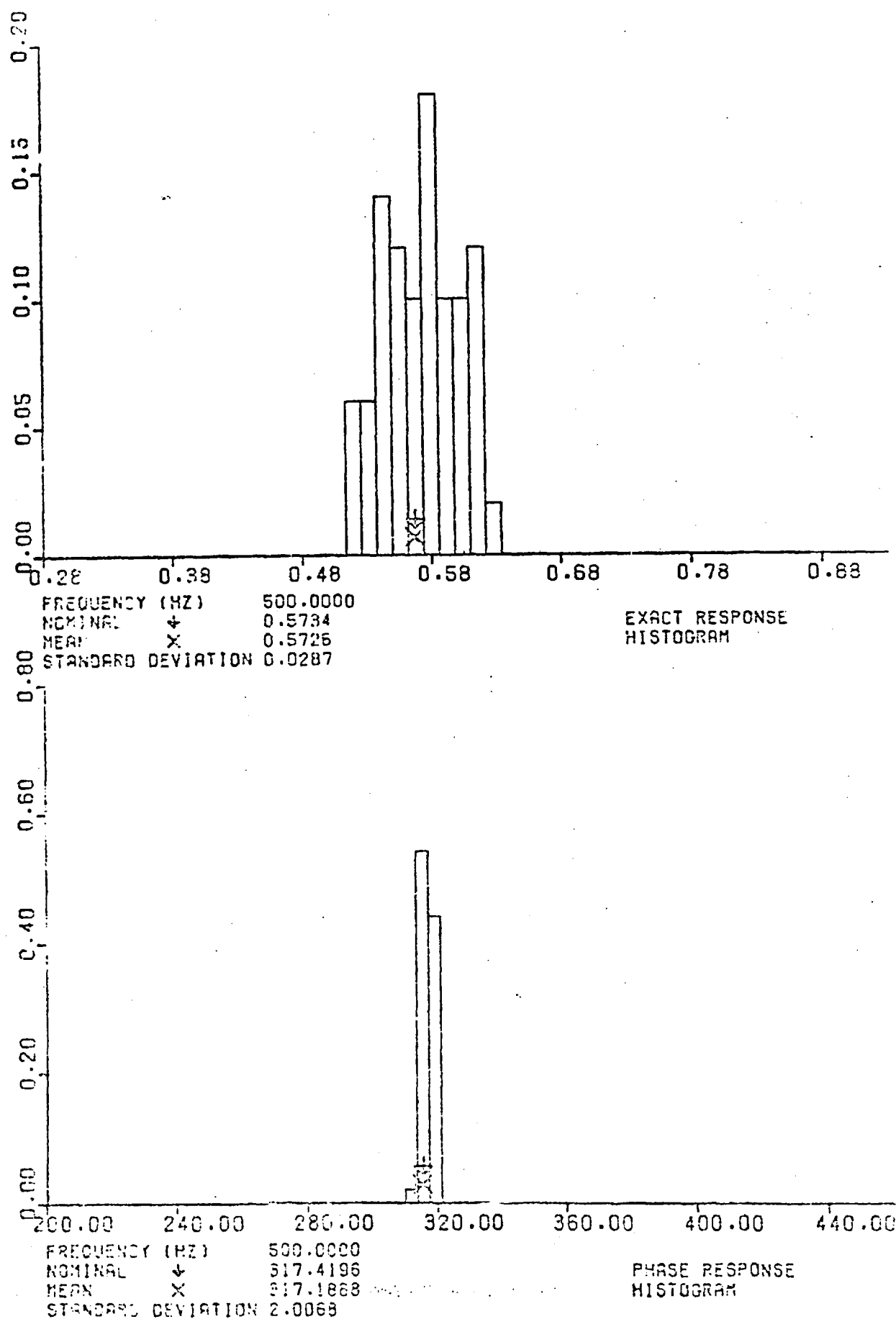


Fig.4.16 Response histogram of equaliser with design centered component values

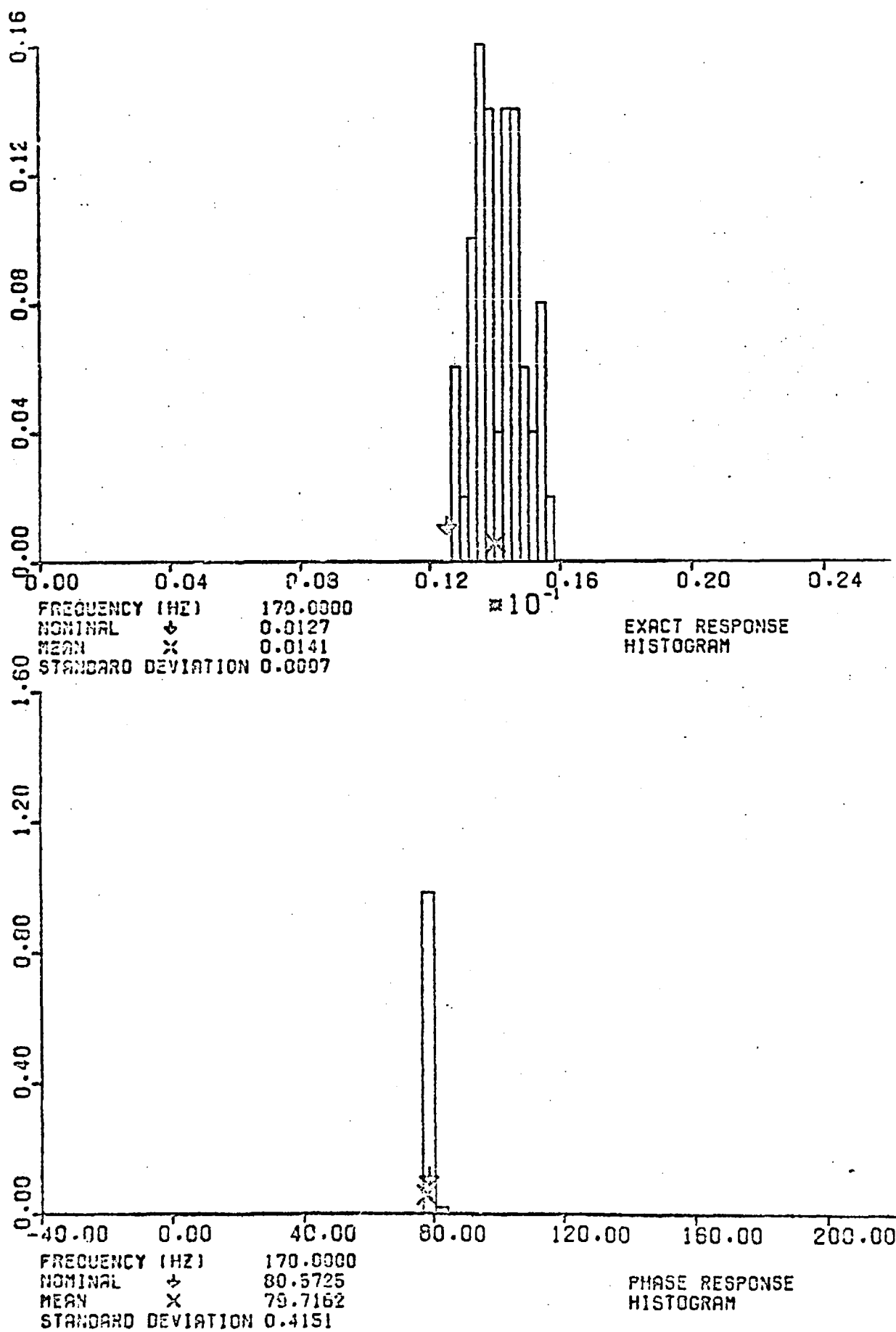


Fig.4.13 Response histogram of band pass filter with nominal component values

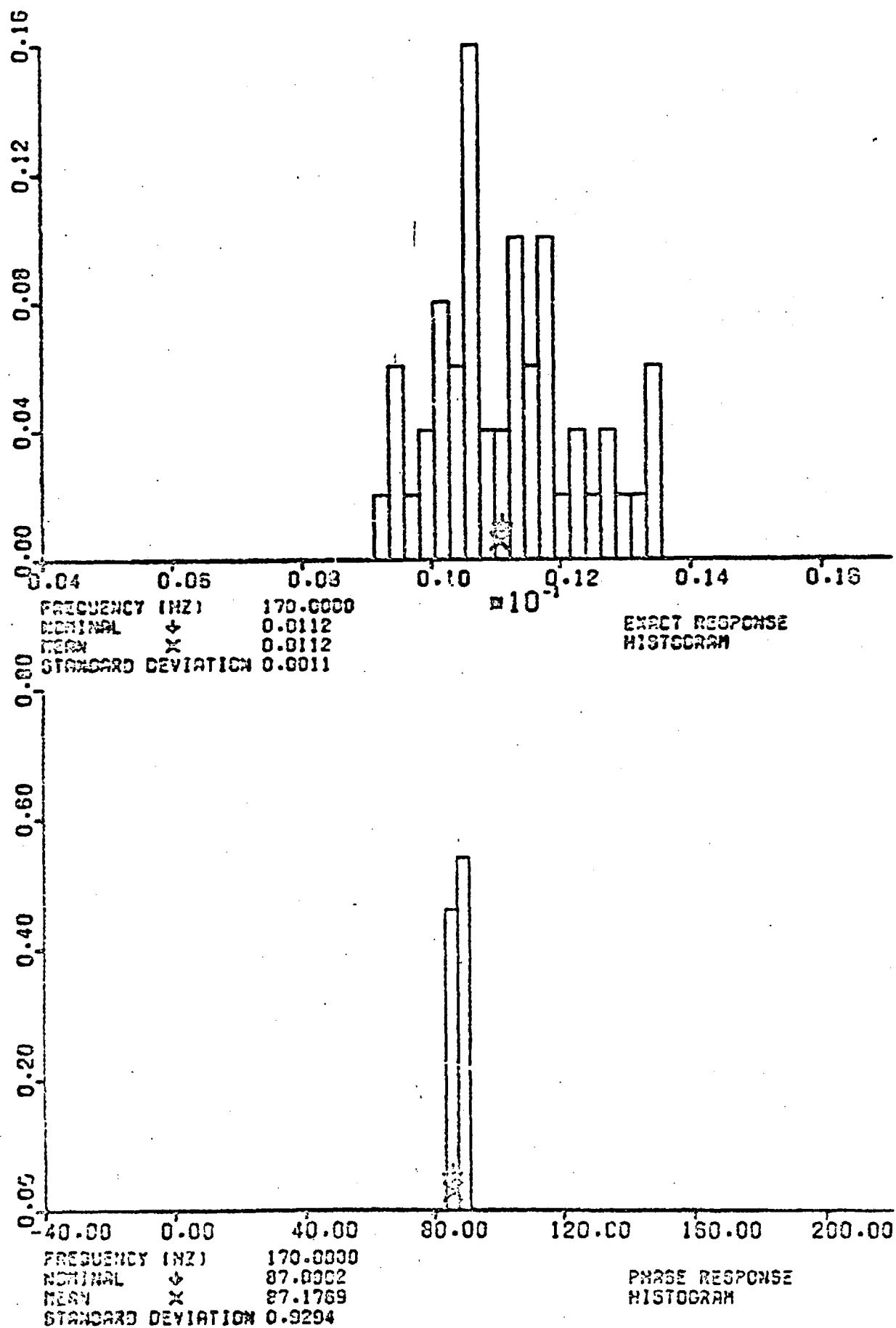
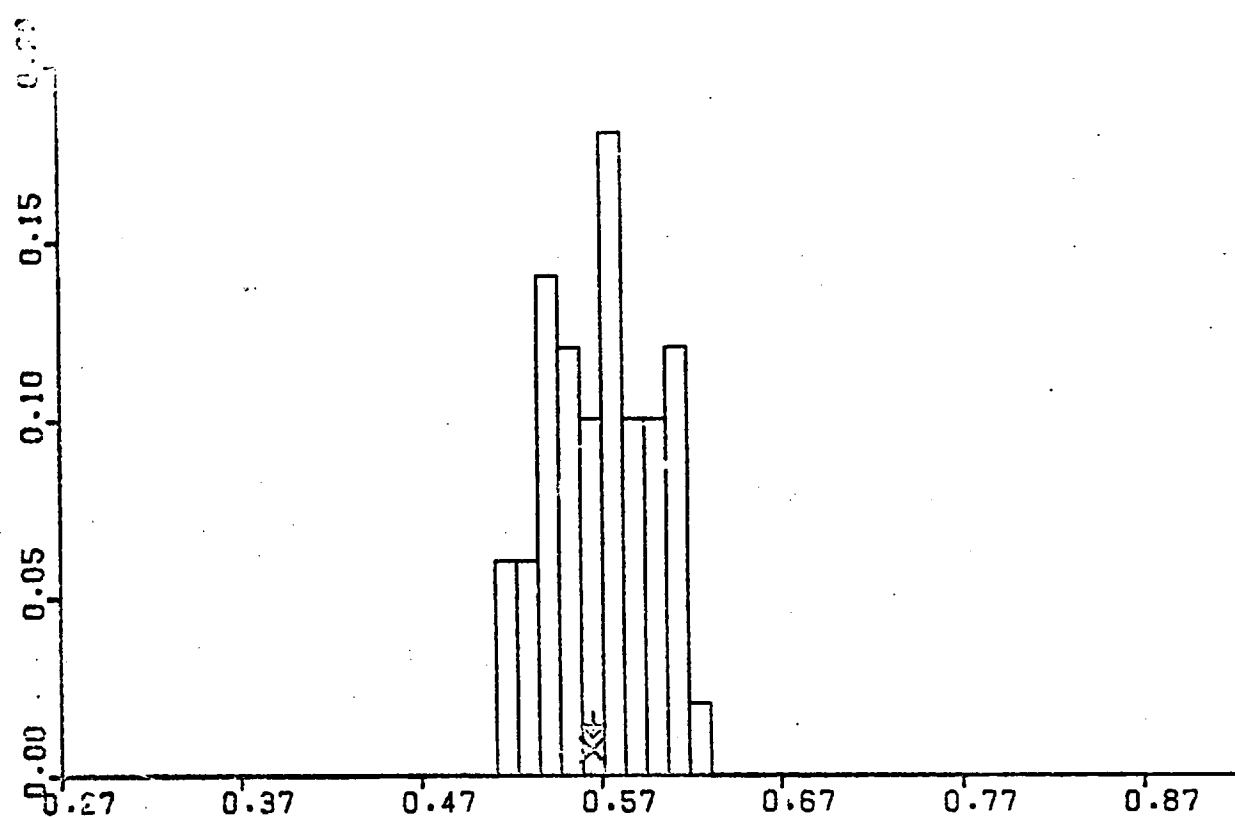
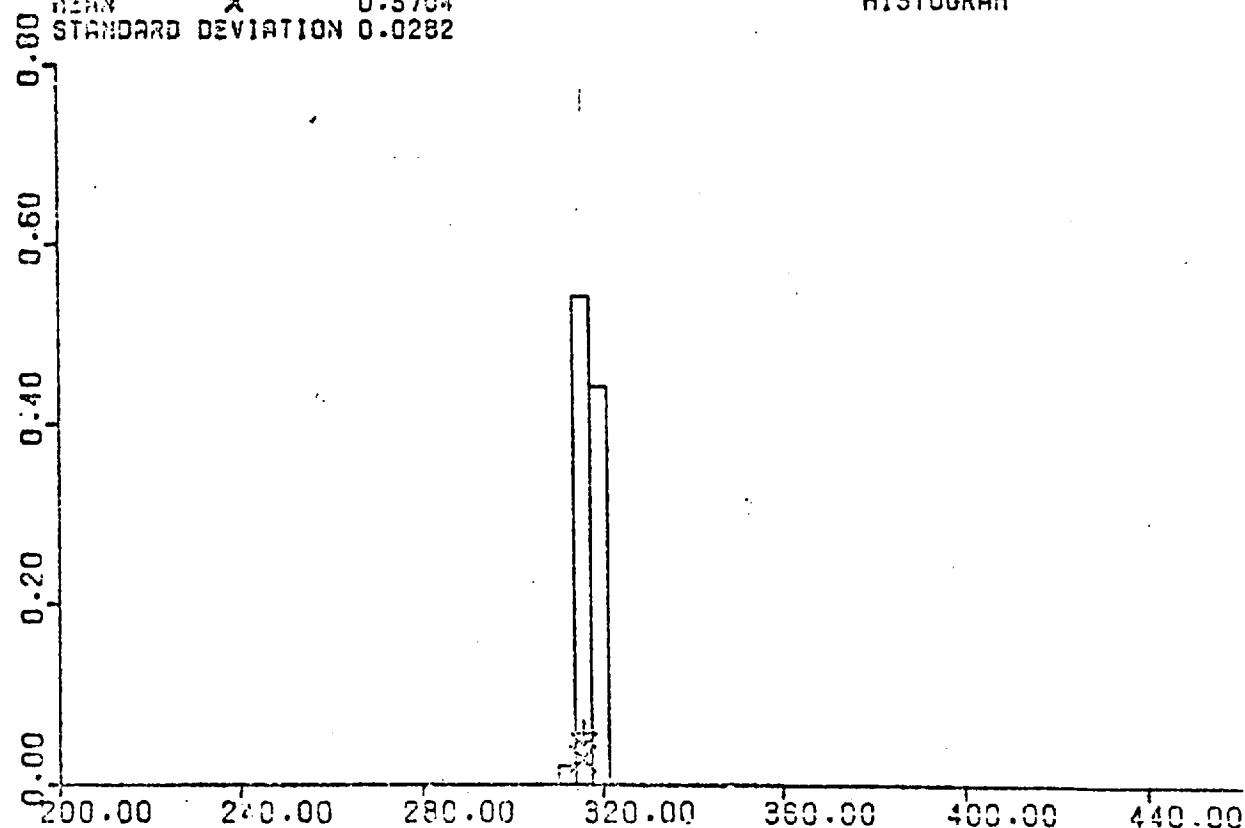


Fig.4.14 Response histogram of band pass filter with design centered component values



FREQUENCY (HZ) 500.0000
 NOMINAL + 0.5711
 MEAN X 0.5704
 STANDARD DEVIATION 0.0282

EXACT RESPONSE
 HISTOGRAM



FREQUENCY (HZ) 500.0000
 NOMINAL + 317.5780
 MEAN X 317.3433
 STANDARD DEVIATION 2.0160

PHASE RESPONSE
 HISTOGRAM

Fig.4.15 Response histogram of equaliser with nominal component values

shows that for the band pass filter a reduction of 23% on the original cost has been achieved without any marked deterioration in the Monte Carlo success rate. A similar comparison of costs between Tables 4.4 and 4.8 shows for the equaliser a reduction of 1.02% only, but there is an increase in the Monte Carlo test yield from 81% to 83%. The low value of the cost reduction seems to suggest that the active equaliser is already a good centered design while there is a considerable margin of improvement in the passive filter design.

To explain the above mentioned results consider the effect of design centering on the response distribution. If the response distribution is symmetrical with the distribution mean and the nominal response differing by a large amount then it can be said that the design is not centered. As an example note the distribution in Fig.4.13 for the circuit in Fig.4.4 having tolerances $\underline{t}^*$ chosen from Table 4.3. The effect of any design centering procedure then should be to shift the nominal and the distribution mean responses closer together, i.e. with the distribution centered on the new nominal response or vice-versa. This is precisely what has happened for the design centered passive filter having tolerances $\underline{t}^*$ chosen from Table 4.7 as shown in Fig.4.14. Earlier it was postulated that there was a limited margin for design centering in the active equaliser, that this is indeed the case can be seen from a comparison of the distributions shown in Figs. 4.15 and 4.16 for the original and design centered circuits respectively.

The effect of design centering on an originally skewed or multimodal response distribution is very difficult to discuss because of its dependence upon the exact nature of the distribution. Furthermore, since no badly skewed or non-uniform distributions have been found in all the circuit examples that have been dealt with, only some hypo-

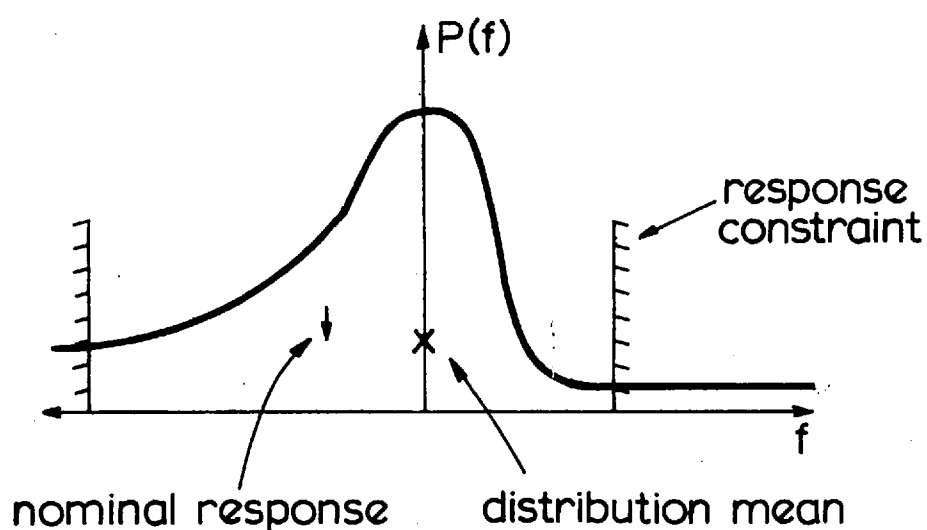


FIG. 4.17:

Skewed circuit response distribution at a given frequency

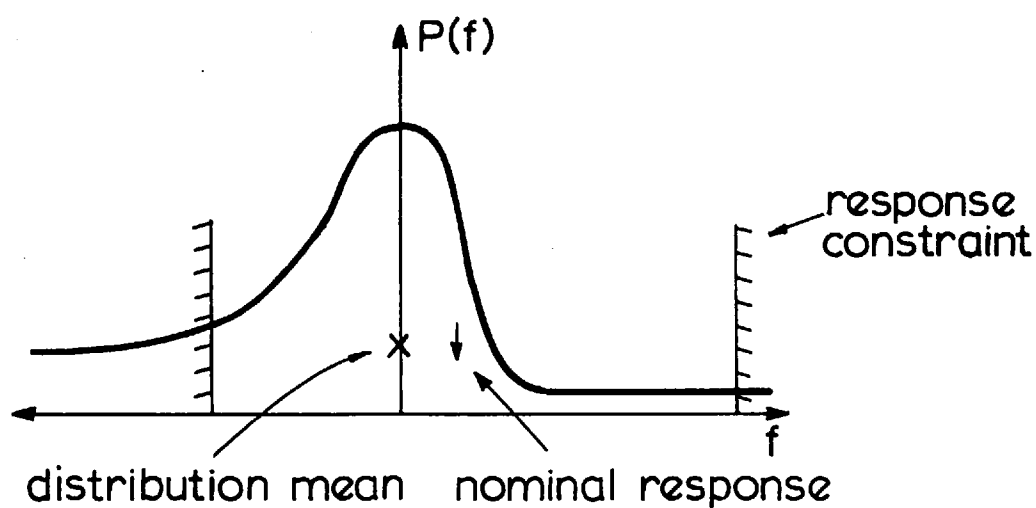


FIG. 4.18:

Skewed circuit response distribution at a given frequency

thetical cases can be considered. Thus for the distribution of Fig.4.17 design centering should ideally move the nominal and the mean closer together without making them coincide. For the distribution in Fig.4.18 design centering should interchange the positions of the mean and nominal responses.

Another consequence of design centering a network for one type of response, say the voltage gain, is that some other aspects of the design like the phase or input impedance may be altered beyond their original specifications. The latter can be taken care of easily by incorporating them as additional response constraints in an identical fashion to the voltage gain constraints (4.1) and (4.2), in the original tolerance assignment problem. Thus when the design centering is performed the effect of the additional constraints will be automatically accounted for.

4.10 Measurement of the Success of the Algorithm

The method of tolerance assignment described in this Chapter is, like all earlier methods, only capable of finding a local optimum for the component deviations from which the tolerances are obtained. If the feasible region is convex then the local optimum is also the global optimum. However, it is usually impossible to determine whether a region is convex or not and hence the tolerances found may not be the widest possible. Furthermore, though the tolerances found should guarantee 100% circuit yields, since they have been found subject to the non-violation of the given constraints, there is no way of knowing if the optimisation stopped on the most binding constraint see Fig.4.19. One way out of this difficulty would be to ensure that, once an optimum point has been found on some constraint boundary, the optimisation

procedure forces this point to move along the constraint boundaries until the most binding constraint, subject to the objective function (4.5) not increasing too much, is found. This is illustrated for a two component example in Fig. 4.19, where A is the optimal point found by the original algorithm and B is the point obtained during the forced move to the most binding constraint.

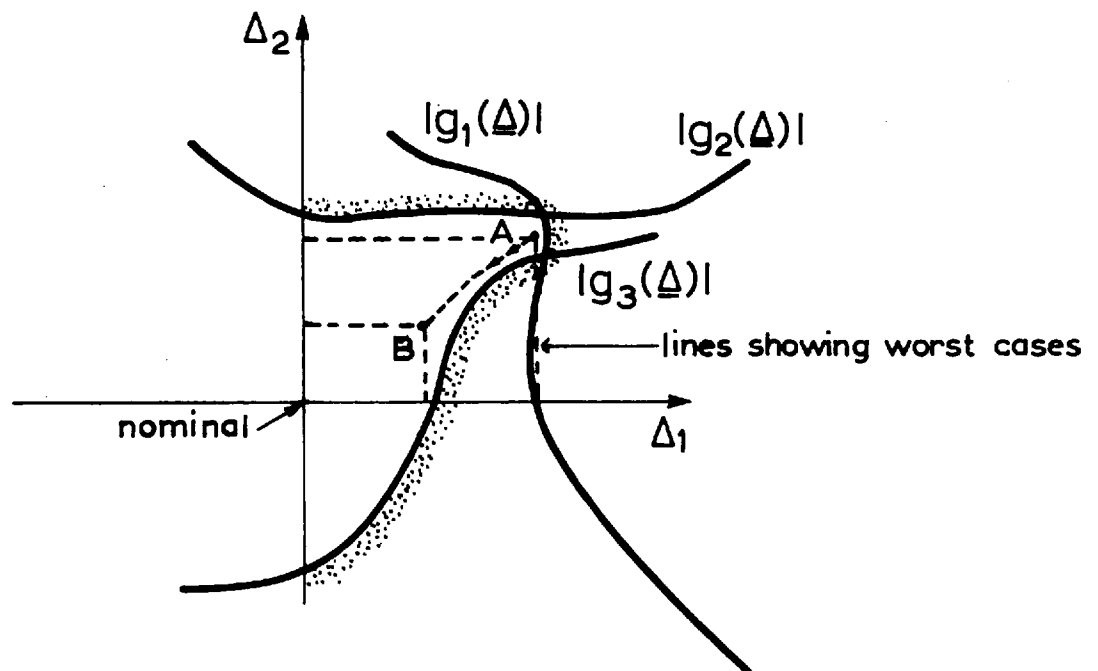


FIG.4.19: Two component example of a non-convex constrained region, showing the forced move from the optimal point, A, to a more acceptable point, B, for the component deviations.

However, it is well known that all nonlinear programming methods developed so far have considerable problems in moving along constraints because the repeated traversal across the latter results in small step lengths being taken along a search direction. Therefore, it would appear that the computational effort needed to accomplish the aforesaid

task is unrealistic at present. The success of any tolerance assignment algorithm must, therefore, be measured by performing a Monte Carlo circuit analysis using the found tolerance sets and computing the number of successful circuits produced.

A significant problem in Monte Carlo circuit analysis, as discussed in Chapter 3, Section 3.5 is the choice of distribution of component values when testing the acceptability of the predicted tolerances. Pinel et al [55] and Bandler et al [56] have indicated a preference for using a bimodal uniform distribution with a width factor $a = 0.95t^*$ as shown in Fig.4.20.

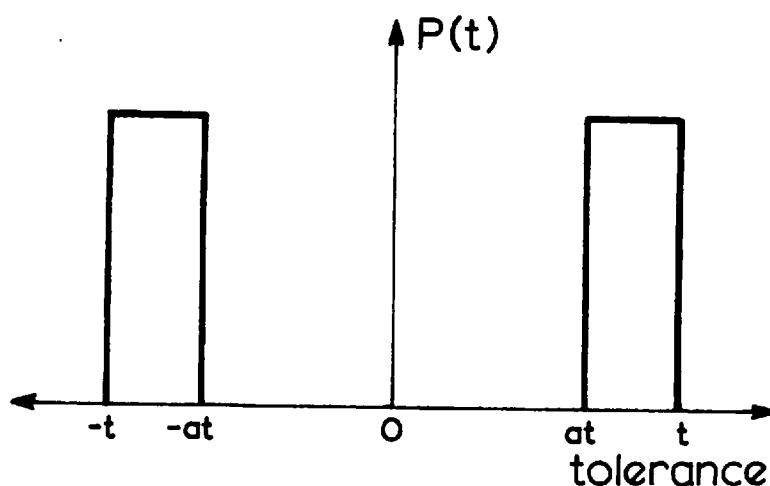


FIG. 4.20: A bimodal distribution of component tolerance values within $\pm t$.

Consider the two component examples shown in Fig.4.19 and assume that the tolerances are determined from the deviations given by point A. Deviations in other quadrants have not been considered here because they would make the present discussion more involved. If the circuit is now tested for relative component deviations within a circle of radius 0.05 centered on A, as would be the case when $a = 0.95$ in Fig. 4.20, there is a higher chance of a circuit response falling inside the feasible region than if a broader $a = 1.0$ or a = 0.5.

The component deviation circle would then have a radius of unity or 0.5. Therefore, for all the Monte Carlo tests carried out for the tolerance sets presented in this Chapter, the distribution width factor $a = 0.5$.

4.11 Preferred Value Tolerances

So far we have only obtained continuous value component tolerances. However, in practice components are usually marketed with preferred value tolerances of $\frac{1}{2}$, 1, 2, 5 and 10 per cent. To simplify the present discussion we shall only consider integer tolerances and not the preferred values. It is, of course, possible to design the tolerance assignment algorithm so that the Δ 's can only take on integral values. This is a form of integer programming. Though this approach appears to be attractive, some problems with nonlinear integer programming can be envisaged. Thus if the situation arises in which some of the circuit components have low tolerances, (e.g. the order of 1% to 2%) then the optimisation routine may waste a considerable time in crossing and recrossing the constraint boundaries as discussed in Section 4.10. As a consequence there is a rapid slowing down of the search, and the procedure may come to a premature halt without much movement from the nominal starting point. Furthermore, design centering information will become difficult to obtain unless the difference between the positive and negative deviations on each side of the nominal component is at least 1%. Therefore, it is felt that continuous values for the deviations must be found, as described. These can then be adjusted downwards to the nearest integer value.

In Table 4.9 we show integer value tolerance obtained from

	L_3	C_5	L_6	C_8	L_9	C_{11}	C_{12}	L_{14}	Cost	Monte Carlo Success rate
Nominal Circuit $\underline{t}^*$	4	3	2	3	4	3	3	4	2.5833	480/480
Design Centered $\underline{t}^*$	4	6	3	5	4	3	5	5	1.8833	479/480
Pinel et al [55]	3	3	1	2	5	2	3	3	3.533	480/480
Karafin [57]	3	5	2	2	3	5	5	3	2.599	476/480

Table 4.9 Integral percentage tolerances of components of the band pass filter Fig.4.4

those of Tables 4.3 and 4.7 for the band pass filter of Fig.4.4. The tolerance cost and Monte Carlo success rates are also shown. For comparison this Table shows the results obtained by Karafin* [57] and Pinel et al [55] for the nominal circuit. The tolerance cost obtained with the method described in this Chapter is the lowest. Karafin's tolerance costs are almost as low but his tolerance result in a few failures. Also Karafin took 20 minutes of CUP-time on a CDC 3500 computer to obtain his results, whereas those of row 1 in Table 4.9 took about 32 seconds of CPU-time on a CDC 6400. The aforesaid CPU-time is biased in our favour, because Karafin considered 30 constrained frequencies, while we have only considered 6 constraint frequencies. However, as he had to build performance contour maps, i.e. feasible region boundaries, for all pairs of components, 30 constraint frequencies were probably necessary to obtain a reasonably accurate curve fit to the points on the boundary. Pinel et al only required 21 seconds of CPU-time on an IBM 360/67 computer to obtain their tolerance values with 6 constraint frequency points. However, not only is their tolerance cost 36.7% higher than ours, but their method is fundamentally incapable of design centering. Our design

The tolerance cost set in [57] was a misprint, and should read as shown in Table 4.9.

centering took a further 32 seconds of CPU-time but reduced the tolerance cost to 1.8833, which is 87.6% less than that of Pinel et al and 38% less than that of Karafin.

	R_2	R_7	C_7	R_8	C_8	R_9	R_{10}	R_{16}	ϵ_{m1}	ϵ_{m2}	Cost	Monte Carlo Success rate
Nominal Circuit $\underline{t}^*$	9	15	2	13	5	2	20	14	9	14	1.7596	394/480
Design Centered $\underline{t}^*$	10	15	2	13	5	2	20	14	9	15	1.7428	393/480

Table 4.10 Integral percentage tolerances of components of equaliser, Fig.4.6

Integer value tolerances for the active equaliser circuit of Fig.4.6 are shown in Table 4.10 and were obtained by appropriate rounding of tolerances in Tables 4.4 and 4.8. The preferred value tolerance cost is slightly higher than the continuous value tolerances. There is no significant differences in the Monte Carlo success rates, which indicate yields of around 80% as before.

4.12 Conclusions

An automatic procedure capable of finding component tolerances commensurate with a minimum cost circuit has been described and implemented on a computer. This procedure can not only give sets of component tolerances that are a prerequisite for high manufacturing yields but also be used in design centering. The use of a complex Taylor series, to evaluate the response changes, easily accounts for the complex nature of the voltage response. The proposed method attempts to find the 'worst case' tolerances using two independent sets of deviations above and below the nominal value. The two last items either appear to have received little attention or have been computationally expensive. The concept of design centering and its effects have been described. It has also been demonstrated that design centering can

lead to a considerable reduction in cost under suitable circumstances. The tolerance assignment and design centering procedures have been tested with different circuit types and the adequacy of the tolerances obtained has been verified using Monte Carlo circuit analysis. Experience of using two different nonlinear programming methods for tolerance assignment indicates that the penalty function approach is superior in some respects to the Flexible Tolerance method. The difficulty associated with obtaining the globally optimum set of tolerances has been discussed. As a consequence there is a need to perform Monte Carlo circuit analysis to test the acceptability of the tolerances. Finally, it has been shown how continuous value tolerances can be transformed into integer value tolerances, from which the preferred values can be selected.

CHAPTER 5

DIFFERENTIAL SENSITIVITY AFTER SIMULTANEOUS LARGE CHANGES IN CIRCUIT ELEMENTS

5.1 Introduction

During the last ten years efficient methods for the evaluation of differential and large change sensitivities have been developed [58,43], and used in many areas of circuit design and analysis. This has led to advancement of techniques for circuit optimisations, sensitivity minimisation and tolerance assignment among others. However, some significant gaps still exist in the field of sensitivity analysis. One is the need to know the change in differential sensitivities when large changes are made in circuit elements. Using the adjoint network approach [62], this can be achieved by making a complete new analysis for each set of large changes to some of the circuit elements, but requires some considerable computer effort and time. The time factors take on an added significance when dealing with very large circuits in which only a few elements are required to change by large amounts.

The need to solve large sets of linear equations repeatedly has resulted in numerous sparse matrix techniques, to reduce the solution time in computers. However, all of these techniques require much programming effort and large amounts of computer storage and time to set up. Therefore, they are unsuitable for use in minicomputers which have been demonstrated to have considerable use in circuit

design [69,70].

In this chapter an efficient algorithm is presented, which not only solves the problem of finding the differential sensitivities but also yields the change in response due to large changes in one or more elements of the circuit with a minimal amount of computer effort. Only one nodal admittance matrix inversion is required at the specified frequency of analysis if only one element [104] is undergoing large change. If t of the circuit elements are subjected to large changes, then a further Gaussian elimination of up to a maximum of a $(t \times t)$ matrix is needed for each set of element changes. Significant advantages over the adjoint network approach can be achieved as most practical circuits do not have fully connected graphs. The total number of branches rarely exceeds the total number of nodes by 50%, and when only some of the branches are varying the cost of solving a $(t \times t)$ matrix is considerably less than the two solutions required by the adjoint network method for nonreciprocal networks.

5.2 Small Change Sensitivity

Consider a linear time-invariant network as shown in Fig. 5.1.

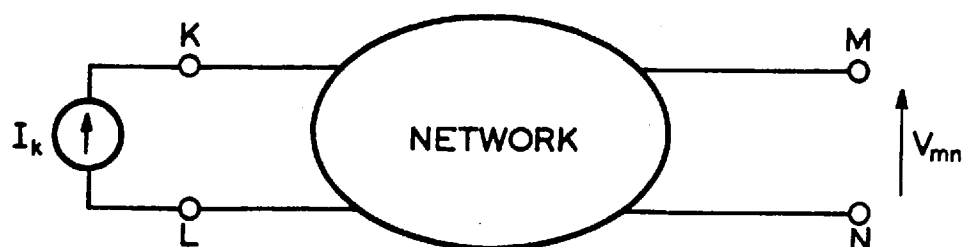


FIG. 5.1: Linear time invariant network.

The response of the network, for example the output voltage, can be easily found from the usual nodal matrix equation (5.1)

$$\underline{V}_n = \underline{Y}_n^{-1} \underline{I}_n \quad (5.1)$$

The nodal voltage sensitivities with respect to an element Y_K of the network can be found [58] by differentiating equation (5.1),

$$\begin{aligned} \frac{\partial \underline{V}_n}{\partial Y_K} &= - \underline{Y}_n^{-1} \frac{\partial \underline{Y}_n}{\partial Y_K} \underline{Y}_n^{-1} \underline{I}_n \\ &= - \underline{Y}_n^{-1} \underline{A} \underline{\delta}_{KK} \underline{v}_b \end{aligned} \quad (5.2)$$

where $\underline{\delta}_{KK}$ is a matrix with all elements zero except δ_{KK} which is unity, $\underline{A}$ is the reduced incidence matrix and $\underline{v}_b$ is the vector of branch voltages. If Y_K is a controlled source, then $\underline{\delta}_{KK}$ is replaced by $\underline{\delta}_{RS}$ with all elements zero except δ_{rs} which is unity. Equation (5.2) allows the calculation of all first order sensitivities with a single matrix inversion of $\underline{Y}_n$. To obtain higher-order sensitivities it is necessary only to differentiate (5.2) as many times as required. (See Appendix I for details.)

5.3 Large Change Sensitivities

If the exact effect on the response of a network for a large change in an element, is desired, then the large-change sensitivity model can be used [43]. Current sources are used to simulate the effect of one or more element changes. The values of these current sources can be determined entirely from the elements of $\underline{Y}_n^{-1}$ and the changes themselves. For simplicity only one variable component will be considered.

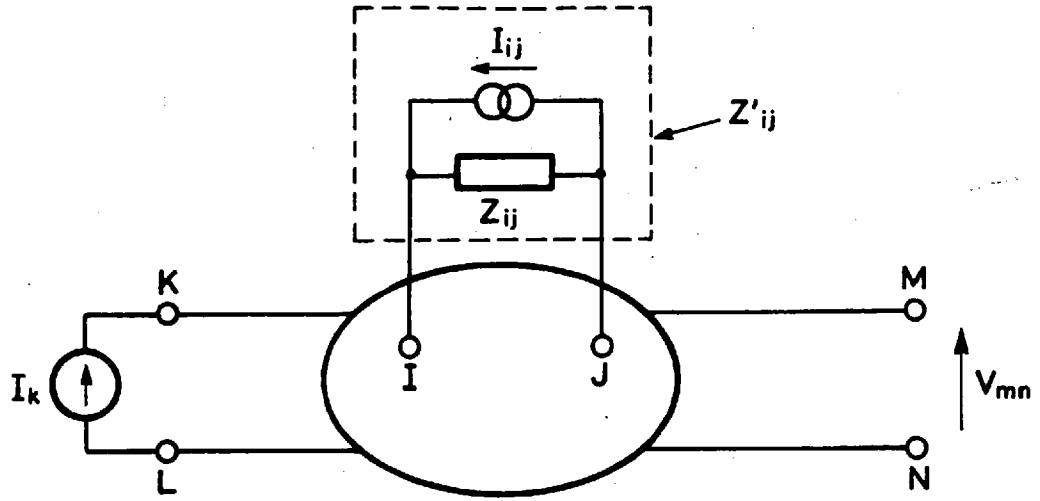


FIG. 5.2: Network with one variable branch exposed

Consider Z_{ij} between nodes I and J in Fig.5.2. Then Z'_{ij} is the new value of the component across I and J. It can be easily shown [43] that the current source required to simulate the change $(Z'_{ij} - Z_{ij})$ has a value given by (5.3). (See Section 5.3.)

$$I_{ij} = \frac{C_1(Z'_{ij} - Z_{ij})}{C_2(Z_{ij} - Z'_{ij}) + Z_{ij}Z'_{ij}} \quad (5.3)$$

$$\text{where } C_1 = \sum_K (z_{ik} - z_{jk} - z_{jl} + z_{il}) I_K \quad (5.4)$$

$$C_2 = (z_{ii} - z_{ij} - z_{ji} + z_{jj}) \quad (5.5)$$

all z 's being elements of $\underline{Y}_n^{-1}$ and I_K are the independent sources.

Therefore I_{ij} can be easily calculated once the change in the component value $(Z'_{ij} - Z_{ij})$ is specified. The new response V'_{mn} can now be calculated by augmenting the original nodal current vector $\underline{I}_n$ with a simulation current source vector $\underline{I}'$ and substitu-

ting in equation (5.1) to give (5.6)

$$\underline{I}_n = [\underline{I}_1 \ \underline{I}_2 \ \dots \dots \dots \underline{I}_n]$$

$$\underline{I}' = \begin{bmatrix} 0 & 0 & \dots & \underset{\substack{\text{ith} \\ \text{element}}}{I_{ij}} & 0 & 0 & \dots & -\underset{\substack{\text{jth} \\ \text{element}}}{I_{ij}} & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{V}'_n = \underline{Y}_n^{-1} (\underline{I}_n + \underline{I}') \quad (5.6)$$

5.4 Differential Sensitivity After Large Changes in One Parameter

To find the new differential sensitivities it is necessary to differentiate (5.6) with respect to Y_K .

$$\frac{\partial \underline{V}'_n}{\partial Y_K} = -\underline{Y}_n^{-1} \underline{A} \frac{\partial \underline{Y}_b}{\partial Y_K} \underline{A}^t \underline{Y}_n^{-1} (\underline{I}_n + \underline{I}') + \underline{Y}_n^{-1} \frac{\partial (\underline{I}_n + \underline{I}')}{\partial Y_K} \quad (5.7)$$

But $\frac{\partial \underline{I}_n}{\partial Y_K} = 0$ because all source currents are constant.

However $\frac{\partial \underline{I}'}{\partial Y_K} \neq 0$ because I_{ij} is not independent of Y_K . The only two non-zero elements of $\underline{I}'$ are I_{ij} . Hence, the only requirement is the evaluation of $\frac{\partial I_{ij}}{\partial Y_K}$ to obtain $\frac{\partial \underline{I}'}{\partial Y_K}$.

In terms of admittance change $(Y'_{ij} - Y_{ij}) = \delta Y_{ij}$; equation (5.3) can be rewritten as (5.8).

$$I_{ij} = -\frac{C_1 \delta Y_{ij}}{1 + C_2 \delta Y_{ij}} \quad (5.8)$$

$$\text{so, } \frac{\partial I_{ij}}{\partial Y_K} = -\frac{\delta Y_{ij}}{1 + C_2 \delta Y_{ij}} \frac{\partial C_1}{\partial Y_K} + \frac{C_1 \delta Y_{ij}^2}{(1 + C_2 \delta Y_{ij})^2} \frac{\partial C_2}{\partial Y_K}$$

$$= \frac{I_{ij}}{C_1} \left[\frac{\partial C_1}{\partial Y_K} + I_{ij} \frac{\partial C_2}{\partial Y_K} \right] \quad (5.9)$$

where $\frac{\partial \delta Y}{\partial Y_K} I_{ij} = 0$.

To evaluate the two differentials $\frac{\partial C_1}{\partial Y_K}$ and $\frac{\partial C_2}{\partial Y_K}$, we use the equation of the voltage across I and J viz:

$$V'_{ij} = V'_i - V'_j \quad (5.10)$$

$$= \sum_K (z_{ik} - z_{jk} - z_{jl} + z_{jk}) I_K + (z_{ii} - z_{ij} - z_{ji} + z_{jj}) I_{ij}$$

$$\text{Hence, } \frac{\partial V'_{ij}}{\partial Y_K} = \frac{\partial V'_i}{\partial Y_K} - \frac{\partial V'_j}{\partial Y_K} = \frac{\partial C_1}{\partial Y_K} + \frac{\partial C_2}{\partial Y_K} I_{ij} + C_2 \frac{\partial I_{ij}}{\partial Y_K} \quad (5.11)$$

$\frac{\partial V'_i}{\partial Y_K}$ and $\frac{\partial V'_j}{\partial Y_K}$ are the nodal voltage sensitivities at nodes I and J.

The port parameters C_1 and C_2 are independent of the simulation current source I_{ij} . The differentials of C_1 and C_2 with respect to Y_K , representing the port impedance sensitivities, can be evaluated therefore, from equation (5.7) and (5.11) while setting the terms $\frac{\partial I_{ij}}{\partial Y_K}$ to zero. Thus letting the simulation source current $I_{ij} = 0$ in the sensitivity calculation using equation (5.7) gives with appropriate substitution in equation (5.11)

$$\frac{\partial V'_i}{\partial Y_K} - \frac{\partial V'_j}{\partial Y_K} = \frac{\partial C_1}{\partial Y_K} \quad (5.12)$$

Similarly setting $I_K = 0$ (i.e. all the original sources) and the simulation current source $I_{ij} = 1$ gives:

$$\frac{\partial V'_i}{\partial Y_K} - \frac{\partial V'_j}{\partial Y_K} = \frac{\partial C_2}{\partial Y_K} \quad (5.13)$$

For a given circuit, (5.12) and (5.13) need only be evaluated once. Thereafter only simple multiplications are required to evaluate $\underline{I}'$

for different changes in the element Z_{ij} , and the corresponding first-order differential sensitivities can be easily found from equation (5.7). Furthermore, equation (5.7) can be easily extended to yield higher-order sensitivities as desired.

5.5 Extension of the Method for Simultaneous Changes in More Than One Parameter

When two or more components, which are connected across t branches, are liable to large changes, then these component changes can be simulated by t simulation current sources (Fig. 5.3) in a similar fashion to the one component variation case. The voltages V'_{ij_t} across each of the t variable branches (or ports) can be obtained as products of the source and simulation currents with the relevant port impedances, just as in equation (5.10) for a single variable branch, viz:

$$\left. \begin{aligned} V'_{i_1 j_1} &= C_{10} + C_{11} \beta_1 + C_{12} \beta_2 + \dots + C_{1t} \beta_t \\ V'_{i_2 j_2} &= C_{20} + C_{21} \beta_1 + C_{22} \beta_2 + \dots + C_{2t} \beta_t \\ &\vdots \\ V'_{i_t j_t} &= C_{t0} + C_{t1} \beta_1 + C_{t2} \beta_2 + \dots + C_{tt} \beta_t \end{aligned} \right\} \quad (5.14)$$

where $C_{10}, C_{20}, \dots, C_{t0}$ are terms equivalent to C_1 of equation (5.4) associated with the transfer impedance between the t variable component ports and the source ports; C_{pp} , $p = 1, \dots, t$, are the input impedances at the t variable ports; C_{pq} , $p = 1, \dots, t$, $q = 1, 2, \dots, t$, $p \neq q$, are the transfer impedances between ports p and q respectively; and $\beta_1, \dots, \beta_t$ are the simulation source currents, identical to the previous I_{ij} , across each of the variable ports. Writing equation

(5.14) in matrix form yields:

$$\underline{V}'_t = \underline{\delta}_t + \underline{X}_t \underline{\beta}_t \quad (5.15)$$

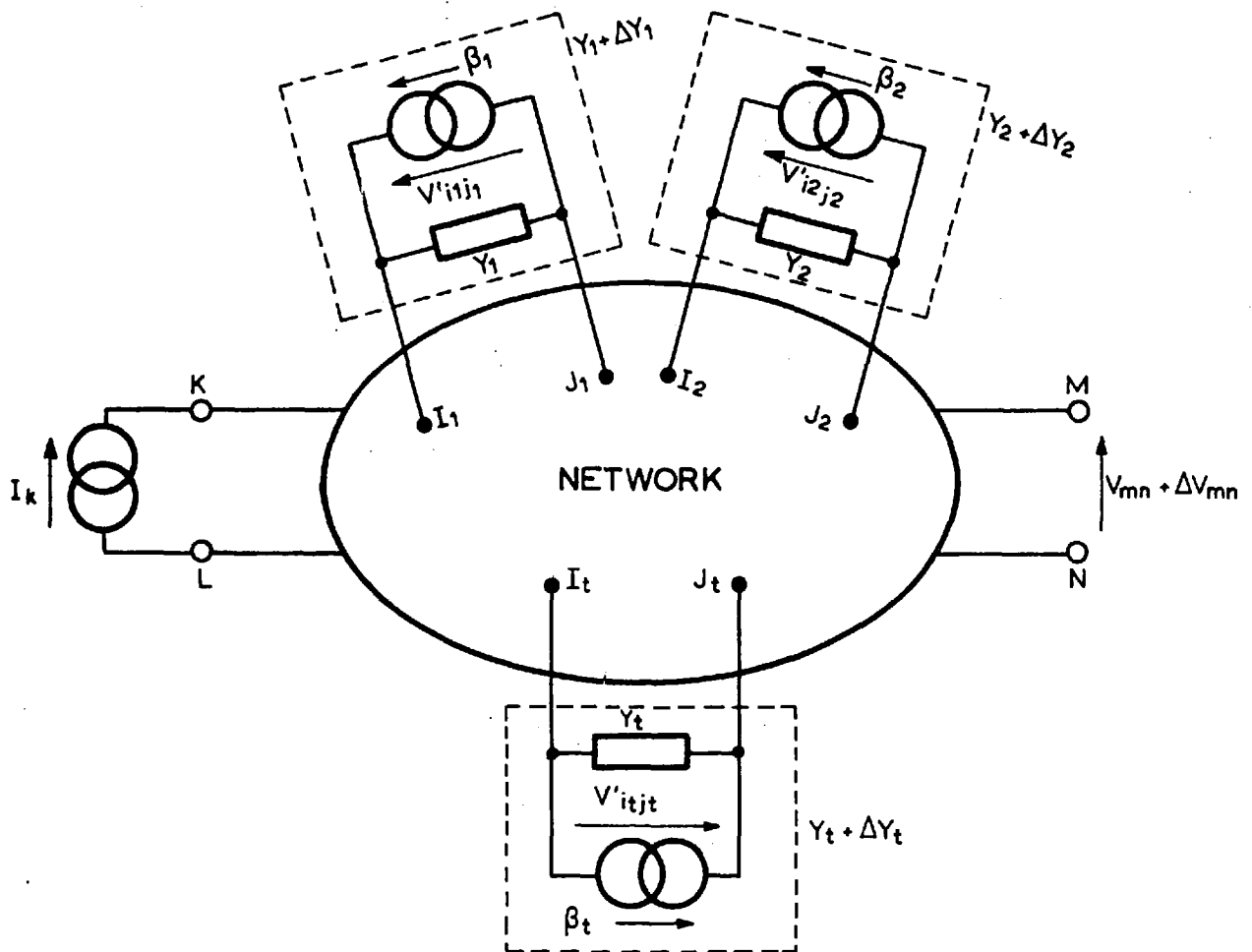


FIG. 5.3: Network with t variable branches, of which 3 are shown explicitly.

The relationship between the individual element admittance changes and the voltage across each element is given by:

$$- \begin{bmatrix} \Delta Y_1 & 0 & 0 & \dots & 0 \\ 0 & \Delta Y_2 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & \Delta Y_t \end{bmatrix} \begin{bmatrix} v'_{i_1 j_1} \\ v'_{i_2 j_2} \\ \vdots \\ v'_{i_t j_t} \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_t \end{bmatrix} \quad (5.16)$$

$$\text{or} \quad - \underline{\Delta Y}_t \underline{v}'_t = \underline{\beta}_t \quad (5.17)$$

Multiplying equation (5.15) by $\underline{\Delta Y}_t$ and substituting for $\underline{\Delta Y}_t \underline{v}'_t$ from equation (5.17) yields

$$\underline{\beta}_t = - \underline{\Delta Y}_t \underline{y}_t - \underline{\Delta Y}_t \underline{X}_t \underline{\beta}_t \quad (5.18)$$

$$\text{or} \quad \underline{\beta}_t = - (\underline{U} + \underline{\Delta Y}_t \underline{X}_t)^{-1} \underline{\Delta Y}_t \underline{y}_t \quad (5.19)$$

where $\underline{U}$ is a unit matrix. Equation (5.19) is in fact the more generalised matrix form of equation (5.8) for a single variable. Note that the calculation of the simulation current vector $\underline{\beta}_t$ only requires a knowledge of the admittance changes across each of the t variable branches, the vector $\underline{y}_t$ and matrix $\underline{X}_t$. The elements of $\underline{y}_t$ and $\underline{X}_t$ can be obtained from the elements of the impedance matrix of the original network as before. However, while equation (5.8) yields a simple formula for the evaluation of the simulation current source, the evaluation of t current sources $\underline{\beta}_t$, where $t > 1$, requires a matrix equation solution with the $(t \times t)$ matrix $(\underline{U} + \underline{\Delta Y}_t \underline{X}_t)$. Once the t simulation current sources have been evaluated using (5.19) a new nodal simulation current source vector $\underline{I}'$ can be formed,

sensitivities at each variable port is computed by setting all source currents to zero, ($I_K=0$) and each simulation port current, ($\beta_1 \dots \beta_t$) is in turn set to unity (the remaining β 's being zero). Having computed $\frac{\partial \underline{\beta}_t}{\partial Y_K}$ from equation (5.21), $\frac{\partial \underline{I}'}{\partial Y_K}$ can now be formed and substituted in equation (5.7) to yield the differential sensitivities.

The usefulness of equation (5.21) lies in the fact that the two differentials $\frac{\partial \underline{\delta}_t}{\partial Y_K}$ and $\frac{\partial \underline{X}_t}{\partial Y_K}$ need only be computed once for any different set of simultaneous component changes $\underline{\Delta Y}_t$. Furthermore, the differentials $\frac{\partial \underline{X}_t}{\partial Y_K}$, $\frac{\partial \underline{\delta}_t}{\partial Y_K}$ and $\underline{X}_t$ are computed directly from the elements of the original network impedance matrix. Therefore, to compute the differential sensitivity when elements across t branches of the original network are changed by large and finite amounts only a single matrix equation solution with the $(t \times t)$ matrix $(\underline{U} + \underline{\Delta Y}_t \underline{X}_t)$ is necessary to obtain the $\underline{\beta}_t$ vector using equation (5.19). Thereafter, the output voltage differential sensitivities can be found with simple matrix multiplications using equations (5.21) and (5.7). It is also worth noting that a knowledge of $\underline{\beta}_t$ also allows the computation of the exact response change via (5.6).

5.6 Computational Efficiency

In most numerical procedures involving matrices the efficiency of any set of computations depends to some extent on the effort spent in programming. Often there are trade-offs between the amount of computer storage used and the number of operations performed. Hence, if the intermediate results which are common to other calculations can be stored then the number of subsequent operations can be reduced. The programming effort and storage requirements for the present method

are minimal. A comparison of the number of operations* between the adjoint method [62] using LU factorisation [68] and the present method of 'Large Change/Small Change' of computing the output nodal voltage sensitivity with respect to all the branches of a non-reciprocal network are shown in Table 5.1.

Method	Fixed Operations	Operation for each new set of sensitivities
Adjoint with LU factorisation	Nil	$\frac{1}{3}n^3 + 2n^2 - \frac{1}{3}n + 2m^2$
Large Change/Small Change	$n^3 + 4mt(t+1)$	$\frac{1}{3}t^3 + 2t^2 + \frac{5}{3}t + 2m(t^2 + t + 1)$

n is the number of nodes of the original network

t is the number of ports across which there are variable branches

m is the number of branches with respect to which sensitivities are found

Table 5.1: Comparison of costs of different methods of sensitivity calculation

Using standard methods, a complete new analysis is required each time any element is changed by large amounts. With the 'Large Change/Small Change' method very few operations are necessary once the relevant coefficients have been computed. Thus the gain in efficiency is most striking when a few elements of a very large network undergo finite changes as can be seen from Fig. 5.4. It also indicates that for a thirty two node network up to thirty branches could be changed simultaneously using the proposed technique without exceeding the cost of sensitivity analysis with the adjoint

*One operation consists of either one multiplication or division plus an addition or subtraction.

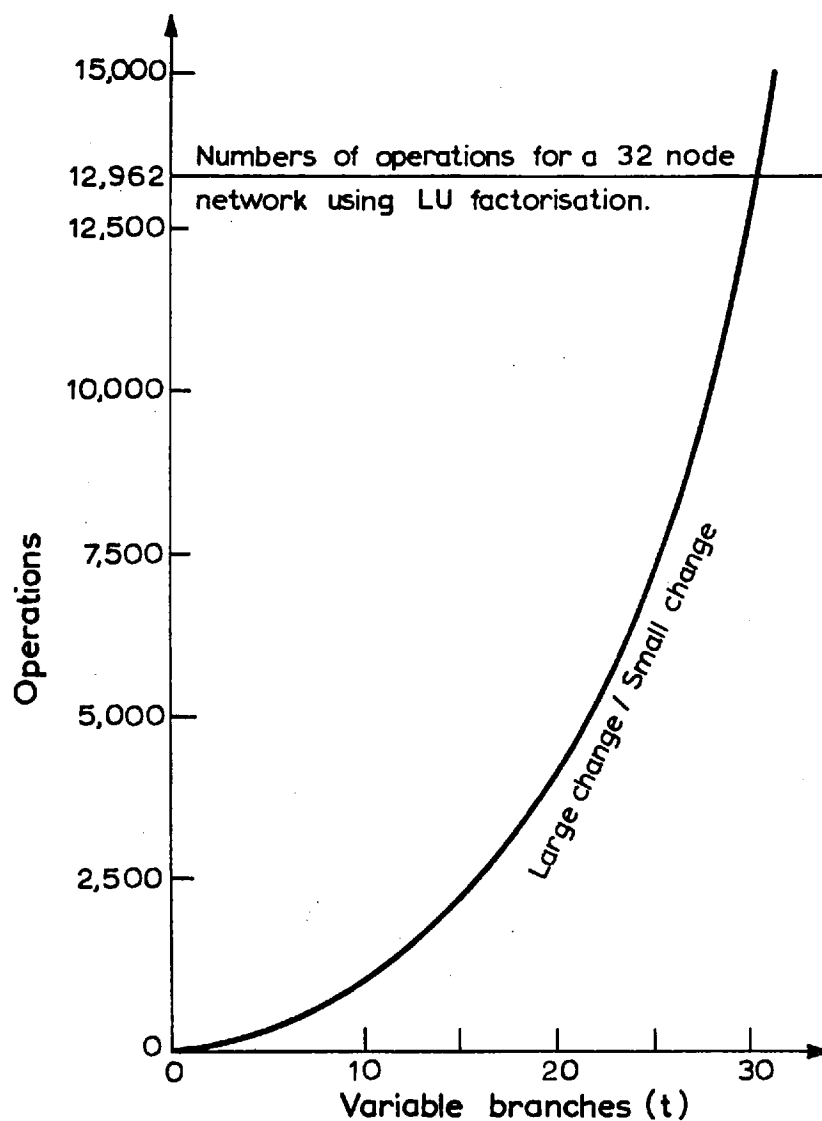


FIG.5.4: Comparison between the number of operations using the Adjoint and the Large change/Small change methods for computing one sensitivity, i.e. for $m=1$ in Table 1.

method using LU factorisation.

5.7 Breakdown of the Computational Cost

The total computational cost, in terms of the number of operations, for the Large change/Small change sensitivity calculation has been given in Table 5.1. Here we examine the detailed breakdown of the computational costs, an understanding of which will enable an efficient programming strategy to be adopted. This is of particular importance in an interactive graphics environment, where the speed of a particular set of computations will have an important effect on the attitudes of a user of the facility (see Section 5.9).

In Table 5.2 the fixed overheads to compute $\frac{\partial \underline{x}_t}{\partial \underline{y}_K}$ and $\frac{\partial \underline{y}_t}{\partial \underline{y}_K}$ are tabulated.

Parts of the Large change/Small change calculation	No. of operations
The inverse of $\underline{Y}_n$ is	n^3
For each of the t^2 elements of $\frac{\partial \underline{x}_t}{\partial \underline{y}_K}$	4
For each of the t elements of $\frac{\partial \underline{y}_t}{\partial \underline{y}_K}$	4
Total for m branches	$4m(t^2+t)+n^3$

Table 5.2: Numbers of multiplications the fixed overheads

Tables 5.3.i, 5.3.ii and 5.3.iii show the number of operations, termed variable operations, needed to compute the sensitivity of output nodal voltage to the m circuit elements when t elements undergo

simultaneous large changes. Thus Table 5.3.i gives the number of operations needed to compute the simulation current source vector $\underline{\beta}_t$, using equation (5.19).

	No. of operations
To obtain upper triangular of $(\underline{U} + \underline{\Delta Y}_t \underline{X}_t)$	$\sum_{i=1}^t i^2 + t$
To obtain $\underline{\Delta Y}_t \underline{\delta}_t$	t
Back substitution to get $\underline{\beta}_t$	$\sum_{i=1}^t i$
Total to get $\underline{\beta}_t$	$\frac{1}{3}t^3 + t^2 + \frac{5}{3}t$

Table 5.3.i: To obtain $\underline{\beta}_t$ using equation (5.19)

The computation of $\frac{\partial \underline{\beta}_t}{\partial Y_K}$ via equation (5.21) also involves the matrix $(\underline{U} + \underline{\Delta Y}_t \underline{X}_t)$ just as for computing $\underline{\beta}_t$. Therefore, storing the upper triangular form of $(\underline{U} + \underline{\Delta Y}_t \underline{X}_t)$ together with the pivots used in obtaining it will result in a saving of $\frac{1}{6}t(2t+1)(t+1)$ operations. The resulting operations for obtaining $\frac{\partial \underline{\beta}_t}{\partial Y_K}$ are shown in Table 5.3.ii.

	No. of operations
To compute $\underline{\Delta Y}_t \left(\frac{\partial \underline{X}_t}{\partial Y_K} \underline{\beta}_t - \frac{\partial \underline{\delta}_t}{\partial Y_K} \right)$	$t^2 + t$
Dividing the r.h.s. column vector in pivots by	$\sum_{i=1}^t i$
Back substitution to obtain $\frac{\partial \underline{\beta}_t}{\partial Y_K}$	$\sum_{i=1}^t i$
Total for getting $\frac{\partial \underline{\beta}_t}{\partial Y_K}$	$2t^2$
Total for m elements Y_K	$2mt^2$

Table 5.3.ii: To obtain $\frac{\partial \underline{\beta}_t}{\partial Y_K}$ using equation (5.21)

Having obtained all the different coefficients it is only necessary to solve the relevant parts of equation (5.7) to obtain the required differential sensitivities. The cost of these operations is tabulated in Table 5.3.iii.

	No. of operations
To compute $\underline{\alpha} = \underline{Y}_n^{-1} \underline{I}'$	$\sim t^2^*$
To compute $\underline{Y}_n^{-1} \underline{A} \frac{\partial \underline{Y}_b}{\partial Y_K} \underline{A}^t \underline{\alpha}$	2
To compute $\underline{Y}_n^{-1} \frac{\partial \underline{I}'}{\partial Y_K}$ for one nodal voltage sensitivity	2t
Total for each differential $\frac{\partial V_o}{\partial Y_K}$	2(t+1)
Total for m differential sensitivities	2m(t+1)

Table 5.3.iii: To obtain the sensitivities of one nodal voltage V_o using equation (5.7)

5.8 Numerical Accuracy

The accuracy of a numerical value produced in a computer depends to a large extent upon the numbers of operations needed to obtain it. This arises from the rounding or truncation that is performed after each arithmetic operation. As the total number of operations performed in obtaining the sensitivities are much fewer using the method described in this paper than other standard ones, the accuracy of the sensitivities should be much improved. The accuracy also depends

*Judicious programming makes for such a small number of operations when multiplications with unity are excluded. See Appendix I for details of sensitivity calculation.

upon the precision with which the simulation current sources and their differential coefficients have been calculated. As these coefficients will usually be few in number and used many times, it may be desirable to compute them using double precision arithmetic. Experience with the Large Change/Small Change method suggests its accuracy is comparable with standard methods, as can be seen from the results of Table 5.4 of Section 5.9.

5.9 Interactive Graphics In Circuit Design

The role of interactive graphics in computer-aided circuit design has come under increasing scrutiny in recent years [69,70,75]. The early development of circuit drawing programs [71] did little to encourage circuit designers to seek the aid of expensive interactive graphics facilities. The primary faults did not lie with the drawing programs, but with a limited understanding of the circuit design process itself. This design process is unlike the architectural drawing processes which the early graphical systems usually provided. In circuit design there is not only a need of a wide variety of analysis programs, but also a comprehensive system which allows the circuit designer to call up any analysis procedure on demand and without excessive effort on his part. The designer should be thinking of the design problem in hand and not of a complicated sequence of commands needed to call up the required programs.

One of the principal aim of an interactive computer graphics facility must be to allow a designer to confirm or to verify that the performance of his design lies within specifications. However, in the early stages of a circuit design, which is largely exploratory,

the designer often only requires a "ball park" estimate of many circuit properties rather than a precise knowledge of just a few. To obtain this "feel" for the design the exploration will often be incremental in a few parameters of the circuit. To prevent psychological closure, the effect of an incremental change must be displayed concurrently, or within an interval of few seconds, with the change [72]. This demands that the numerical procedures determining the effect of incremental changes must be fast if real time dynamic performance is to be achieved. Otherwise, all results must be precalculated and stored, ready for display on demand. The light pen on a graphics system provides the designer with a tool to effect changes in parameter values. These and other aspects of the design process have been incorporated in a circuit design facility on the PDP-15 with a VT-15 graphics processor by Apperley, Spence and Dunn at Imperial College.

The "Large Change/Small Change" algorithm developed in this chapter is an ideal candidate for incorporation into an interactive design facility. Since this numerical procedure can quickly compute a variety of circuit properties, from sensitivities to voltage response, for incremental variation in one or more circuit components, the designer can be provided with information almost concurrently with the initiation of the component changes.

The above procedure has been programmed on the PDP-15 and incorporated with the display system of Apperley [75], to provide a dynamic display of the normalised sensitivities when any selected component of the circuit under test changes by incremental amounts.

5.10 Dynamic Display of Differential Sensitivity

The first step in any circuit exploration is to "input" the schematic diagram into the computer. The ECAP [73] approach using an ordered set of numbers to represent the circuit graph and the respective component values is unsatisfactory, not only because it is prone to human error but also because it has the disadvantage of transforming the schematic diagram into a form far removed from the engineer's experience. Graphical input, with the aid of a light pen and a suitable menu of light buttons (Fig. 5.5) can provide a facility closely simulating pencil and paper drawings [75]. This is closer to the engineer's normal experience.

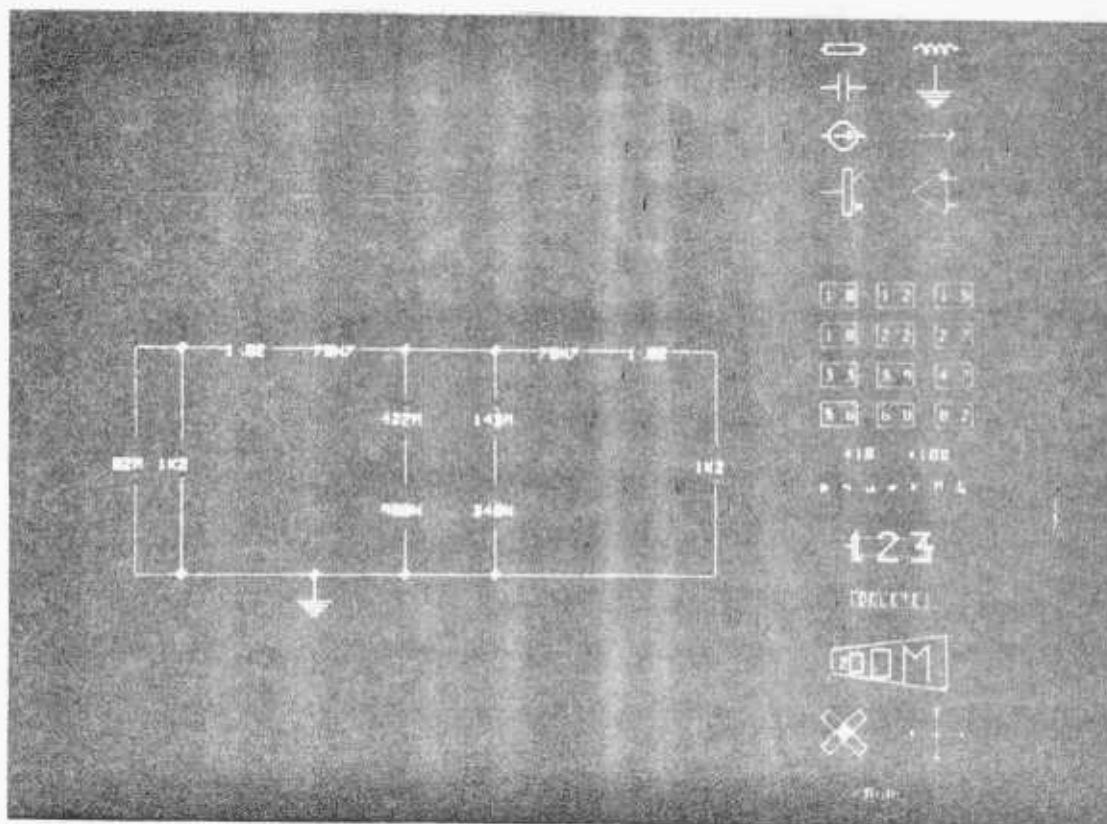


Fig.5.5 Graphic facility in circuit input mode, with menu of numerical and drawing light buttons as well as the analysis command

The circuit shown in Fig. 5.7 is drawn using the menu shown in Fig. 5.5. The numerical values of the components can be assigned by making light pen hits on the relevant values followed by a hit on the component. More accurate values can be input via the teletype. A scratch pad facility on the screen is provided by the delete button which allows the removal of one or more components of the drawn circuit. Hitting the light button above the "delete" allows the alternate display of the schematic diagram with or without the numerical values, compare Fig. 5.5 and Fig. 5.6. The  symbol allows the user to clear the screen and start again, while the  symbol allows the positioning of the circuit anywhere on the drawing area of the screen.

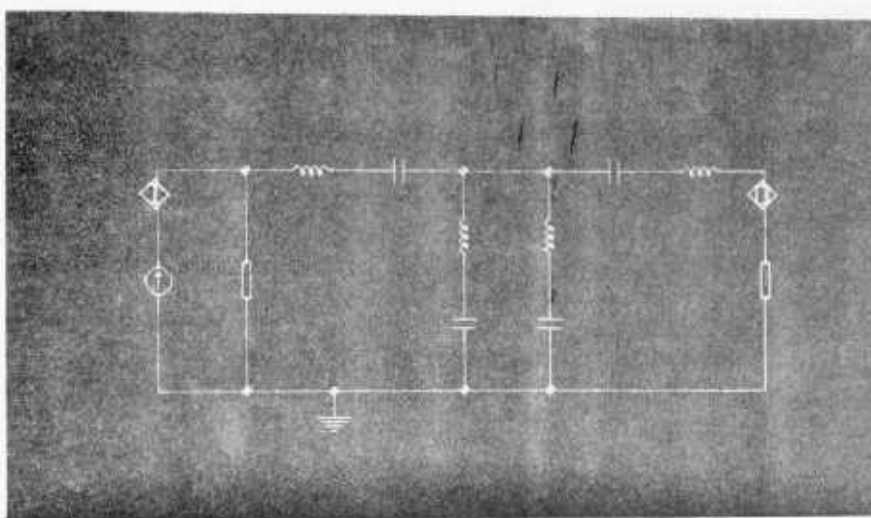


Fig.5.6 Display of the schematic circuit diagram on the screen

Once the circuit has been input satisfactorily, the designer can call up different analysis options by hitting the analysis - ANAL button at the bottom of the menu (Fig. 5.5). At present, hitting this button initiates the Large Change/Small Change sensitivity calculation option. The designer is immediately asked to indicate the frequency at which the circuit is to be analysed. The designer

then indicates the branch which he wishes to change by large incremental amounts. The range of component variation is pre-set to $\pm 50\%$ of the impedance value, but can be changed. After a time lapse needed for inverting the nodal admittance matrix and calculating relevant coefficients of the circuit on the screen, the computer is ready to display the normalised sensitivities of the voltage response to the components.

The sensitivity display is activated by hitting one of the components of the circuit on the screen. A light potentiometer appears on the right hand side of the screen (Fig.5.10). The arrowed positions indicate the numerical value of the component undergoing large change. The corresponding component sensitivities can be selectively displayed for one or more component with the aid of "circles" whose radius corresponds to the magnitude of the appropriate sensitivity. The large changes can be cycled through incrementally at short intervals and the corresponding change in individual component sensitivity can be dynamically displayed. At the end of each cycle the designer is presented with various options. He may select a different element for large change, change the frequency or printing out numerical values of the sensitivities, the output voltage and the phase response for each of the large changes.

To facilitate dynamic exploration of the circuit, controls must be provided to allow selective investigation of the available information. From a psychological viewpoint, the number of items that a user can keep track of simultaneously is limited [74]. Therefore, the designer is permitted to switch on or off as many of the sensitivity circles as desired by merely pointing to the relevant component with the light pen.

The present implementation allows the designer to explore rapidly the sensitivity and response change behaviour of a circuit due to incremental changes in its components values. Any correlation that might exist between two or more components will be indicated by changes in their sensitivities in a related manner. Thus, the sensitivities of a group of components might increase in unison, or one group of sensitivities increases while another may decrease.

5.11 Example

It is difficult to illustrate the dynamic nature of the Large Change/Small Change technique as implemented on a graphics facility. Still photographs can only hint at the power of the present techniques for design exploration. The circuit example of a band pass filter shown in Fig. 5.7 has been chosen for illustration.

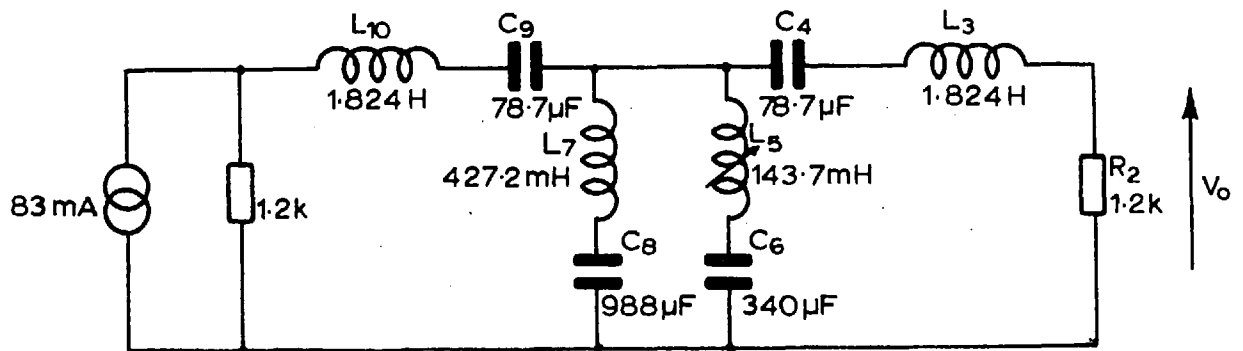


FIG 5.7: Band pass filter with L_5 changing by large amounts

This circuit is analysed at a frequency of 690 Hz, close to the resonance frequency of the $L_5 C_6$ combination. The inductor L_5

is changed from $0.5L_5$ to $1.5L_5$ in twenty one steps each displayed by an arrowed position on the light potentiometer. The magnitude of the first order sensitivity is given by a circle of the corresponding radius centred on the component. The changes in sensitivities, as the inductor changes its value, are shown in Figs. 5.8-5.10. The corresponding numerical values for the sensitivities and the voltage response are given in Table 5.4, using the Large Change/Small Change method of this paper and a direct calculation with standard techniques using a new admittance matrix for each set of large changes in L_5 .

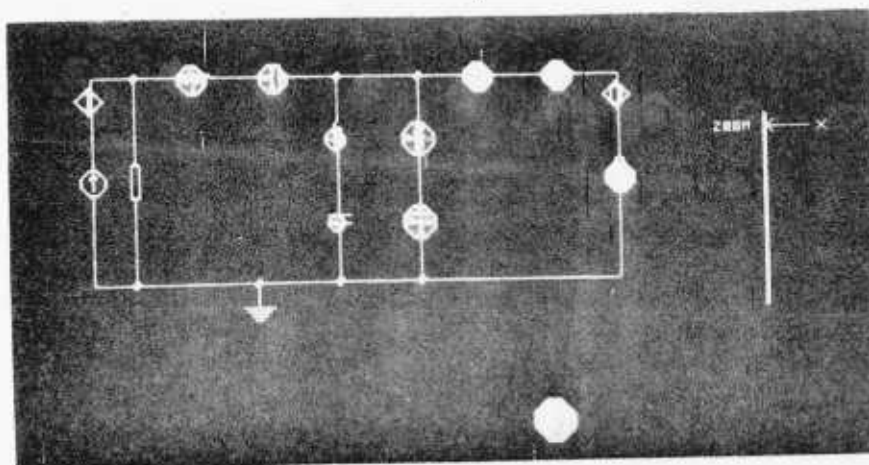


Fig.5.8 Sensitivities for $L_5 = 208\text{mH}$

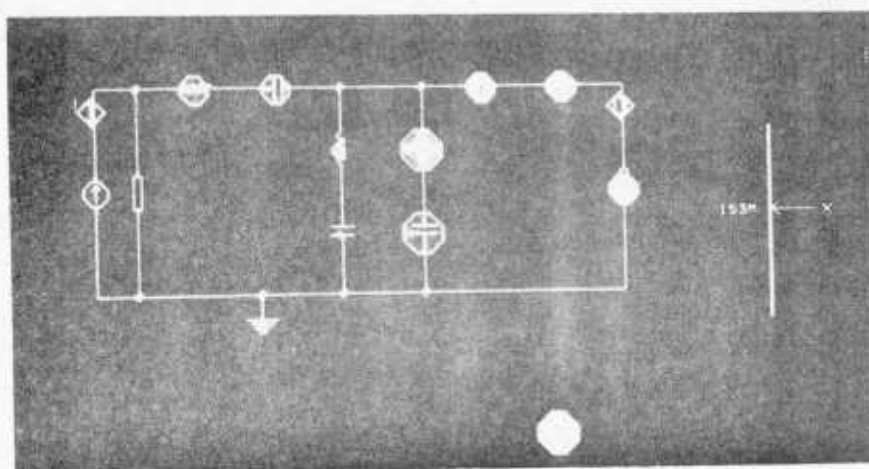


Fig.5.9 Sensitivities for $L_5 = 153\text{mH}$

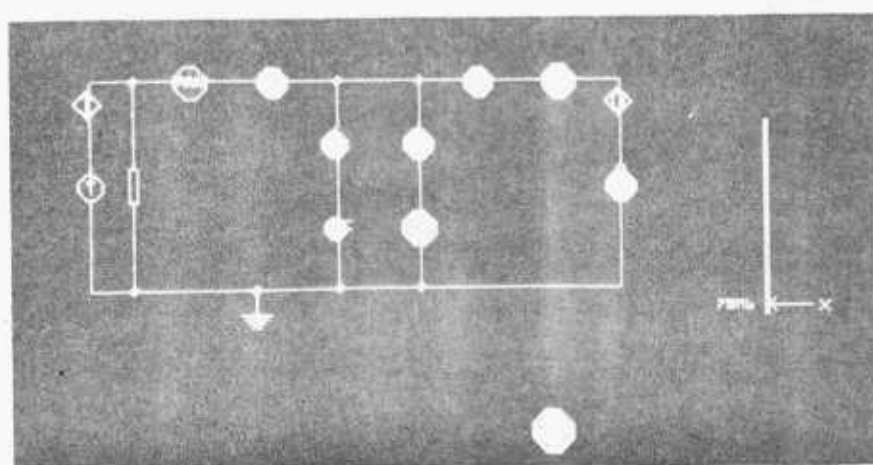


Fig.5.10 Sensitivities for $L_5 = 78.6\text{mH}$

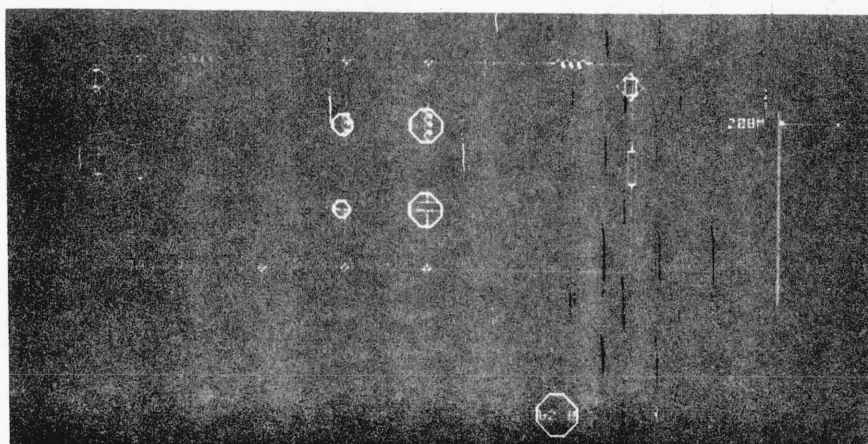


Fig.5.11 Sensitivities for $L_5 = 208\text{mH}$

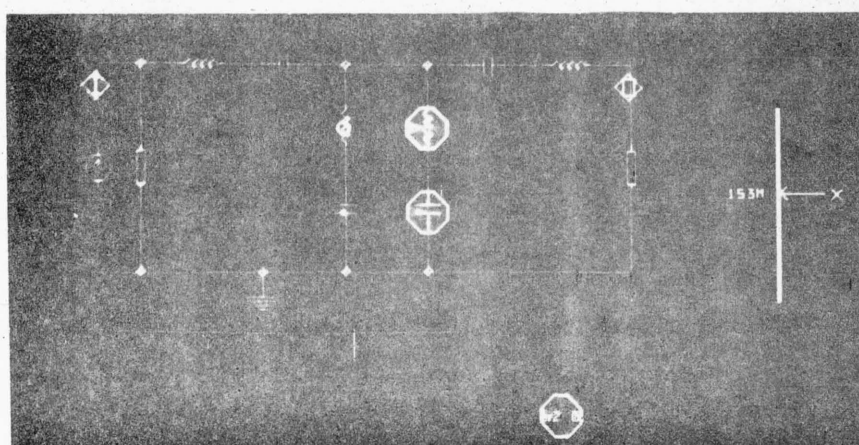


Fig.5.12 Sensitivities for $L_5 = 153\text{mH}$

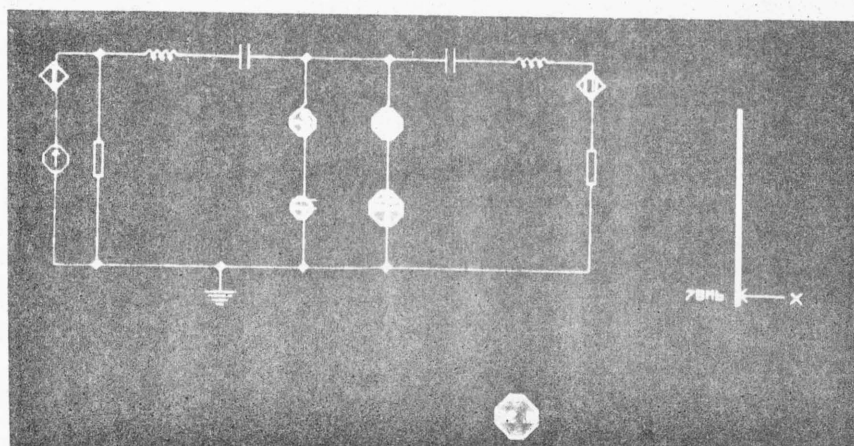


Fig.5.13 Sensitivities for $L_5 = 78.6\text{mH}$

A comparison of the results in Table 5.4 shows up no discrepancies in the numerical values calculated using the two methods. However, while the cost of the direct calculation for the twenty one large changes in L_5 was (21×212) operations, the total number of operations required using the Large Change/Small Change algorithm was (21×10) operations.

L_5	Large Change/Small Change		Direct Calculation	
	$ s_{C_6}^{V_o} $	$ V_o $	$ s_{C_6}^{V_o} $	$ V_o $
208.71 mH	3028	2.287 2 V	3028	2.287 2 V
153.96 mH	62840	0.049 714 V	62840	0.049714 V
18.693 mH	2443	0.8320 V	2443	0.8320 V

Table 5.4: Sensitivities and output voltage magnitudes from the Large Change/Small Change and direct calculations.

In Figs.5.11-5.13 only four elements of the circuit have been activated for sensitivity display. These clearly show that the sensitivities of L_5 and C_6 are strongly related in a positive sense. That is, they track each other. However, these sensitivities are negatively related with those of L_7 and C_8 . For, while the sensitivities of L_5 and C_6 increase rapidly and then decrease, those of L_7 and C_8 reach a minimum when the former obtain their highest values. In large networks such effects would be both difficult and expensive to investigate without the aid of an interactive graphics system and the Large Change/Small Change algorithm.

5.12 Conclusion

A numerical procedure for obtaining the exact differential

sensitivities of a circuit when one of its components is changed by large incremental amounts has been described. This procedure has been further enhanced to compute the exact sensitivities when two or more components simultaneously undergo large changes. The efficiency of this technique over that of the previously available method of the adjoint network, when not all components of the circuit are undergoing simultaneous large changes has been clearly demonstrated. The resulting reduction in the number of operations to compute not only the sensitivities but the voltage gain will result in greater numerical accuracy. The simplicity of the proposed method to some extent obviates the need for employing expensive sparse matrix techniques in many areas of circuit design where only a small subset of the total number of circuit elements are liable to large variations e.g. circuit optimisation using gradient methods.

The described method has been successfully employed in an interactive graphics facility for computer aided circuit design. The differential sensitivities of one or more components of a circuit can be dynamically displayed when one element undergoes large incremental changes. The ability to explore the sensitivity behaviour of the circuit for large changes in a component, without the long delays necessary with earlier methods, gives a powerful exploratory "tool" to the designer. Furthermore, graphical implementation of the method for computing the differential sensitivities for simultaneous large changes in more than one component (see Section 4.4) can be used to tune a circuit dynamically for minimum sensitivity. The voltage magnitude or the phase response of the circuit can be monitored simultaneously. The immediacy of the method is a valuable aid in circuit optimisation, which hitherto has depended on batch processing

using large multiprogrammed machines. The designer obtains additional insight into the behaviour of his circuit, thus increasing the effectiveness and efficiency of the design process.

CHAPTER 6

ADMITTANCE TOLERANCE REGIONS FOR MULTIPLY-DEFINED VOLTAGE RESPONSE CHANGES

6.1 Introduction

This chapter constitutes the preliminary theoretical investigation of directly transforming voltage response changes to circuit parameter changes in circuit theory. Some of the original ideas of Villalaz [68] have been developed extensively to form a basis for future experimental investigation.

In many areas of circuit design some of which have been discussed in earlier chapters, there is considerable interest in computing the effect of simultaneous component changes on the response. Many algorithms have been developed over the years to compute the change in circuit response and other properties such as sensitivity, in order to gain insight into circuit behaviour. However, there are areas of circuit design in which there is a need to know the permitted changes in network branch values for specified tolerances in the circuit response. One such field is tolerance assignment. Another is connected with circuit design, where it is desirable to know what changes should be made to existing component values to force the circuit response to lie within some defined limits.

The difficulty in transforming a response change at any frequency to component or branch changes results from having a single equation for the voltage response in many component variables. Therefore, the traditional approach has been to make changes in the element

values guided by some optimisation procedure and compute the response, with the hope that this lies within the error criterion. From the acquisition of data from this iterative analysis some insight into the allowed parameter variations of the circuit is gained.

A first though computationally expensive approach to systematically examine the feasible component space (see Section 4.3 of Chapter 4) was made by Butler [78], by evaluating performance contours for pairs of elements. These performance contours are bounded regions in the component space, with the boundary being defined by the maximum permitted response error at a number of frequencies. Though a more efficient algorithm for doing the same has been described by Villalaz [79], the computation of the allowed regions is still very expensive for more than a few elements undergoing simultaneous changes. Added to this is the difficulty of adequately representing multi-dimensional performance contours when three or more parameters vary simultaneously.

In this chapter an efficient method is developed to directly compute the changes in branch admittance values and branch admittance tolerance regions for any set of specified changes in the nominal response over a set of frequencies, without the need of repeated circuit analysis. It is supposed that the allowed voltage response error vector $\underline{\Delta V}_{mn}^{(t)}$ of a network in Fig.6.1 at t or more frequencies is defined, and the allowable simultaneous variations in the branch admittances of this network are desired. As voltage changes are to be transformed into corresponding admittance changes no direct control can be exercised to keep these admittances wholly real or imaginary. Usually each admittance change would be expected to have a real and imaginary part. Thus a network branch which originally had a resistor only would acquire a resistive and reactive part

after analysis. A similar result holds for an originally reactive branch. In spite of this difficulty the uses of this type of transformation have been clearly demonstrated, under some simplifying assumptions, by Villalaz [43,68,82].

To obtain a clear understanding of the proposed solution to transforming voltage response changes to branch admittance changes, it is necessary to consider first the generalised large change sensitivity method developed in Section 5.5 of Chapter 5. At that time our interest lay in obtaining the response change in an efficient manner when a given number of network elements were changed simultaneously.

6.2 The Large Change Sensitivity Model

As in the previous chapter consider a current excited network of n nodes and b branches. Using nodal analysis all the nodal voltages can be obtained from the well known impedance matrix equation

$$\underline{V}_n = \underline{Y}_n^{-1} \underline{I}_n \quad (6.1)$$

where $\underline{I}_n$ is a vector of source currents, $\underline{V}_n$ is the corresponding vector of nodal voltages and $\underline{Y}_n$ is the nodal admittance matrix. As a first step we identify t variable branches of interest from the network Fig.6.1 by making solder connections [43] and attach t current sources, defined by the elements of a vector $\underline{\beta}_t$, across each of these branches. By taking on suitable values these current sources can simulate the admittance changes, given by the t element of a diagonal matrix $\underline{\Delta Y}_t$, in each of the identified t branches. Ohm's law yields a relationship between the t variable branch voltages $\underline{V}'_t$, the branch currents $\underline{\beta}_t$ and admittances $\underline{\Delta Y}_t$.

$$-\underline{\Delta Y}_t \underline{V}'_t = \underline{\beta}_t \quad (6.2)$$

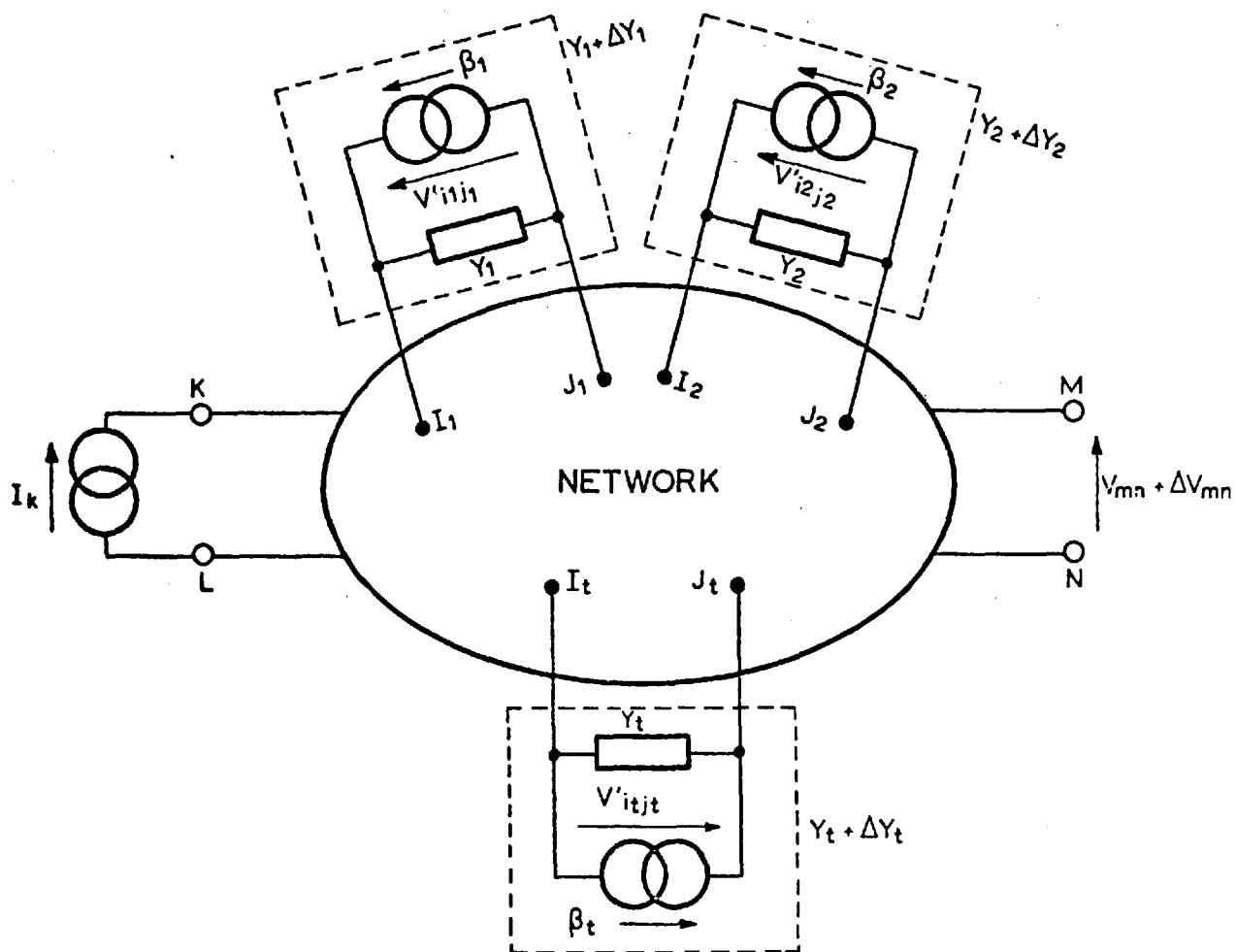


FIG. 6.1: Network with t variable branches, of which 3 are shown explicitly.

Using equation (6.2) with other properties of the know network a unique relationship between the changes $\Delta \underline{Y}_t$ and the simulation current vector $\underline{\beta}_t$ has been derived in terms of elements of the inverse nodal

admittance matrix, in Chapter 5, Section 5.5.

$$\underline{\beta}_t = - (\underline{U} + \underline{\Delta Y}_t \underline{X}_t)^{-1} \underline{\Delta Y}_t \underline{y}_t \quad (6.3)$$

The modified circuit response with the addition of the simulation current sources to the original network is identical to that which would have been obtained if the changes $\underline{\Delta Y}_t$ had actually been made to the relevant branches. To obtain the new nodal voltage vector it is first necessary to transform the branch current vector $\underline{\beta}_t$ into a nodal simulation current vector $\underline{\beta}_n$. $\underline{\beta}_n$ is obtained by expanding $\underline{\beta}_t$ with the insertion of zero value current sources for all the (b-t) non-variable branches. Then pre-multiplication by the reduced incidence matrix yields $\underline{\beta}_n$ (see Section 5.5 of Chapter 5 also). The original nodal current vector $\underline{I}_n$ is now augmented by $\underline{\beta}_n$.

$$\underline{I}'_n = \underline{I}_n + \underline{\beta}_n \quad (6.4)$$

The new set of nodal voltages $\underline{V}'_n$ can now be evaluated by substituting (6.4) into the nodal voltage equation viz.

$$\begin{aligned} \underline{V}'_n &= \underline{Y}_n^{-1} \underline{I}'_n \\ &= \underline{Y}_n^{-1} (\underline{I}_n + \underline{\beta}_n) \\ &= \underline{V}_n + \underline{Y}_n^{-1} \underline{\beta}_n \end{aligned} \quad (6.5)$$

The change in the original nodal voltage vector $\underline{\Delta V}_n$ is therefore:

$$\underline{\Delta V}_n = \underline{Y}_n^{-1} \underline{\beta}_n \quad (6.6)$$

If it is assumed that there is only one non-zero simulation current source β_1 across the nodes I_1 and J_1 , then the change in the output voltage ΔV_{mn} is given by:

$$\begin{aligned}
 \Delta V_{mn} &= (z_{mi_1} - z_{mj_1} - z_{nj_1} + z_{ni_1}) \beta_1 \\
 &= D_1 \beta_1
 \end{aligned} \tag{6.7}$$

where D_1 is the transfer impedance between the output port MN and the simulation port $I_1 J_1$. Similarly for t simulation current sources the change in the output voltage becomes:

$$\begin{aligned}
 \Delta V_{mn} &= D_1 \beta_1 + D_2 \beta_2 + \dots + D_t \beta_t \\
 &= [D_1 \ D_2 \ \dots \ D_t] \beta_t \\
 &= \underline{D}^T \beta_t
 \end{aligned} \tag{6.8}$$

Finally substituting for β_t from equation (6.3) in equation (6.8) yields an equation for the change in the voltage at the output nodes in terms of the branch admittance changes $\underline{\Delta Y}_t$ and the elements of the original nodal impedance matrix.

$$\Delta V_{mn} = - \underline{D}^T (\underline{U} + \underline{\Delta Y}_t \underline{X}_t)^{-1} \underline{\Delta Y}_t \underline{X}_t \tag{6.9}$$

Therefore to evaluate the change in the response voltage ΔV_{mn} due to the element changes given by $\underline{\Delta Y}_t$ it is only necessary to solve a matrix equation with the $(t \times t)$ matrix $(\underline{U} + \underline{\Delta Y}_t \underline{X}_t)$. However, if the interest lies in finding a unique and non-trivial value of $\underline{\Delta Y}_t$ given ΔV_{mn} , then this is impossible. For equation (6.9) is then a single equation in t unknowns.

6.3 To Find Admittance Changes Given Response Tolerances

It would appear to be impossible to find t linearly independent equations in ΔY 's similar to equation (6.9). However, it is possible to obtain t linearly independent equations in β 's similar to equation

(6.8). All that is needed is to specify the response changes $\Delta V_{mn}^{(t)}$ from the nominal at t distinct frequency points and write the corresponding equations similar to (6.8).

$$\left. \begin{aligned} \Delta V_{mn}^{(1)} &= D_1^{(1)} \beta_1 + D_2^{(1)} \beta_2 + \dots + D_t^{(1)} \beta_t \\ \Delta V_{mn}^{(2)} &= D_1^{(2)} \beta_1 + D_2^{(2)} \beta_2 + \dots + D_t^{(2)} \beta_t \\ &\vdots \\ \Delta V_{mn}^{(t)} &= D_1^{(t)} \beta_1 + D_2^{(t)} \beta_2 + \dots + D_t^{(t)} \beta_t \end{aligned} \right\} \quad (6.10)$$

where $D_s^{(f)*}$ is the transfer impedance between the output port MN and the selected port s at a frequency f . Note that the number of frequencies selected is equal to the number of variable branches. All the equations in (6.10) are linearly independent only if the network is frequency dependent. Thus purely resistive networks are not amenable to this type of analysis. Equation (6.10) can be written more compactly in matrix notation:

$$\underline{\Delta V}_{mn}^{(t)} = \underline{\Gamma} \underline{\beta}_t \quad (6.11)$$

where $\underline{\Gamma}$ is a $(t \times t)$ square matrix of transfer impedances $D_s^{(f)}$. A unique vector of simulation currents $\underline{\beta}_t$, commensurate with the allowable response changes $\underline{\Delta V}_{mn}^{(t)}$ can be easily evaluated now. However, our interest lies in finding the admittance change vector $\underline{\Delta Y}_t$ uniquely. At each frequency f the relationship between $\underline{\beta}_t$ and the changes $\underline{\Delta Y}_t^{(f)}$ are given by equation (6.3) viz.

$$\underline{\beta}_t = - (\underline{U} + \underline{\Delta Y}_t^{(f)} \underline{X}_t^{(f)})^{-1} \underline{\Delta Y}_t^{(f)} \underline{\gamma}_t^{(f)} \quad (6.12)$$

substituting for $\underline{\beta}_t$ from (6.11) in equation (6.12) yields

$$\underline{\Delta V}_{mn}^{(t)} = - \underline{\Gamma} (\underline{U} + \underline{\Delta Y}_t^{(f)} \underline{X}_t^{(f)})^{-1} \underline{\Delta Y}_t^{(f)} \underline{\gamma}_t^{(f)} \quad (6.13)$$

*The superscript 'f' has been introduced to denote frequency dependence.

Equation (6.13) can be expressed in terms of $\underline{\Delta Y}_t^{(f)}$:

$$\underline{\Delta Y}_t^{(f)} (\underline{X}_t^{(f)} \underline{T} \underline{\Delta V}_{mn}^{(t)} + \underline{Y}_t^{(f)}) = \underline{T}^{-1} \underline{\Delta V}_{mn}^{(t)} \quad (6.14)$$

Thus t different diagonal matrices $\underline{\Delta Y}_t^{(f)}$ corresponding to each of the t frequencies can be obtained by making the appropriate substitutions for the frequency dependent matrices $\underline{X}_t^{(f)}$ and $\underline{Y}_t^{(f)}$ in (6.14). The matrices $\underline{T}$, $\underline{X}_t^{(f)}$ and $\underline{Y}_t^{(f)}$ consist entirely of elements selected from the inverse nodal admittance matrices of the original network at the t selected frequencies, while $\underline{\Delta V}_{mn}^{(t)}$ is specified by the circuit designer.

6.4 Lineo-Linear Transformations

In this section some of the basic properties of lineo-linear transformations will be outlined briefly. The theory and rigorous mathematical proofs of such transformations can be found in reference [80]. The application of this theory to equations (6.13) and (6.14) allows conditions to be found under which these equations constitute conformal mappings. As a consequence a considerable saving in computational effort for obtaining regular tolerance regions is possible.

In the theory of functions of a single complex variable it proves useful to have bounded regions of variation of the independent variable which are changed by homographic substitutions into regions that are similarly bounded. Thus a circle or a straight line under a transformation or a set of transformation of the type:

$$\omega = \frac{az + b}{cz + d} \quad (6.15)$$

remains a circle or sometimes becomes a straight line*, providing the Jacobian, $J \frac{\partial(\omega)}{\partial(z)}$, is non-zero. Equation (6.15) is often called the

*A straight line is a circle of infinite radius.

bilinear transformation of a complex variable z into ω . The coefficients a , b , c and d may be real or complex. Under such transformations all boundary curves of order not higher than two, conserve their general character. Hence, the terms conformal or isogonal mappings.

An isogonal mapping, a term first introduced by C.G. Gauss [81], is one which preserves the sense as well as the magnitude of the angles. This geometrical interpretation is derived from the consideration of isoscles triangles in classical Euclidian geometry.

Can the well-known properties of the bilinear transformation be ascribed to functions of two or more complex variables? Forsythe [80] has shown that given frontiers, in a two variable z, z' field, when changed by a lineo-linear transformation of the type (6.16) results in conformal mappings in two planes given by ω and ω' .

$$\frac{\omega}{a'z + b'z' + c} = \frac{\omega'}{a''z + b''z' + c''} = \frac{1}{a''z + b''z' + c''} \quad (6.16)$$

Unfortunately equations (6.13) and (6.14) usually do not conform to a lineo-linear transformation of the type (6.16) even when $t = 2$. However, extracting one of the constituent equations from (6.14) allows conditions to be found under which each of the equations in (6.14) are bilinear. To simplify the notation we rewrite equation (6.14) as:

$$\underline{\Delta Y}_t^{(f)} (\underline{W}_t^{(f)} \underline{V} + \underline{Y}_t) = \underline{\alpha} \underline{V} \quad (6.17)$$

where $\underline{W}_t^{(f)} = \underline{X}_t^{(f)} \underline{T}$; $\underline{V} = \underline{V}_{mn}$ and $\underline{\alpha} = -\underline{T}^{-1}$.

Extraction of the p^{th} row from equation (6.17) yields:

$$\Delta Y_p (\omega_{p1} V_1 + \omega_{p2} V_2 + \dots + \omega_{pt} V_t + Y_p) = \alpha_{p1} V_1 + \alpha_{p2} V_2 + \dots + \alpha_{pt} V_t \quad (6.18)$$

If $(t - 1)$ of the voltage changes $V_1, V_2, \dots, V_t$ in the response at different frequencies are set to constant values, which may be different,

then equation (6.18) becomes an equation in two variables ΔY_p and the non constant response change V_q .

$$\Delta Y_p (\omega_{pq} V_q + \xi_1) = \alpha_{pq} + \xi_2 \quad (6.19)$$

$$\text{where } \xi_1 = \sum_{\substack{s=1 \\ s \neq q}}^t \omega_{ps} V_s + \gamma_p \quad \text{and} \quad \xi_2 = \sum_{\substack{s=1 \\ s \neq q}}^t \alpha_{ps} V_s$$

Equation (6.19) is in fact a bilinear transformation similar to (6.15). Similarly every constituent equation of the matrix equation (6.14) can be made to have a bilinear form by holding $(t - 1)$ of the variable voltages in ΔV_{mn} constant. This is the most generalised form of the bilinear transformation from the voltage plane at any frequency to an admittance plane at any frequency. By taking different sets of values of the $(t - 1)$ voltage changes, overlapping regions in the admittance plane can be plotted. The computational task is minimal because only three points on a circle in the V_q plane are necessary to obtain a corresponding circle in ΔY_p plane.

Using similar arguments it is possible to show that admittance regions can be transformed into voltage regions at different frequencies using equation (6.13) viz:

$$(\underline{U} + \underline{\Delta Y}_t^{(f)} \underline{X}_t^{(f)}) \underline{T}^{-1} \underline{\Delta V}_{mn} = - \underline{\Delta Y}_t^{(f)} \underline{\gamma}_t^{(f)} \quad (6.20)$$

We simplify the notation in (6.20) by making the following substitutions for the constant matrices:

$$\underline{W}'^{(f)} = \underline{X}_t^{(f)} \underline{T}^{-1} ; \underline{\alpha}' = \underline{T}^{-1} \quad \text{and} \quad \underline{V} = \underline{\Delta V}_{mn}.$$

$$\text{Then } (\underline{\alpha}' + \underline{\Delta Y}_t^{(f)} \underline{W}'^{(f)}) \underline{V} = - \underline{\Delta Y}_t^{(f)} \underline{\gamma}_t^{(f)} \quad (6.21)$$

As before the p^{th} constituent equation in the matrix equation (6.21)

can be written as:

$$\alpha'_{p1}V_1 + \alpha'_{p2}V_2 + \dots + \alpha'_{pt}V_t + \Delta Y_p(\omega'_{p1}V_1 + \omega'_{p2}V_2 + \dots + \omega'_{pt}V_t) = \Delta Y_p \gamma_p \quad (6.22)$$

Setting all voltages other than V_q to be constant yields:

$$\xi_3 + \alpha'_{pq}V_q + \Delta Y_p(\xi_4 + \omega'_{pq}V_q) = -\Delta Y_p \gamma_p \quad (6.23)$$

$$\text{where } \xi_3 = \sum_{\substack{s=1 \\ s \neq q}}^t \alpha'_{ps} V_s \quad \text{and} \quad \xi_4 = \sum_{\substack{s=1 \\ s \neq q}}^t \omega'_{ps} V_s$$

We can now rewrite (6.23) in the more familiar bilinear form:

$$V_q = \frac{-\Delta Y_p(\gamma_p + \xi_4) - \xi_3}{\alpha'_{pq} + \Delta Y_p \omega'_{pq}} \quad (6.24)$$

Equation (6.24) constitutes a conformal mapping from the admittance region of the p^{th} circuit branch to the voltage region at the q^{th} frequency V_q , for constant voltage variations at $(t-1)$ of the specified frequencies.

6.5 Tolerance Regions

It has been shown that transformation of a performance specification into a component tolerance region constitutes a conformal mapping. Such mappings have been used in transforming a circular tolerance region for a response voltage variation at one frequency into an admittance region for a selected branch of a network by Villalaz [63]. However, he was unable to transform the simultaneous effect of allowed voltage performance changes and a performance region at different frequencies on more than one circuit branch. Therefore, the corresponding component tolerance regions could not be found in the multi-frequency case.

For the permitted output voltage variation, the tolerance region in the admittance plane of each circuit branch can be plotted at any of the frequencies of interest using equation (6.19). The important factor here is that coefficients need only be calculated once using the matrix equation (6.17) and any subsequent calculations due to different response changes require simple multiplications only. Thus the effect of very complicated sets of voltage variations can be investigated with a minimum of computational effort.

The above can be illustrated by a simple example concerned with a network element and its voltage performance tolerance. First it is assumed that a circuit has been proposed and it is required to find the allowed tolerance of its elements given the tolerance on the network performance. For simplicity we shall assume that voltage tolerance regions are given at two frequencies f_1 and f_2 , i.e. $t = 2$. In view of the bilinear nature of the relation between the vector* of changes ΔV_{mn} in the response voltages, at the different frequencies, and the resulting change in the t branch admittances ΔY_t , considerable saving in computer effort is possible if the voltage tolerance region $V_o^{f_1}$, at one of the frequencies, is a regular closed region of second order [80] such as a circle. At the other frequency the response can change by an arbitrary but fixed amount. However, it is convenient to choose a circular voltage tolerance region $V_o^{f_2}$ at the second frequency also (Fig.6.2a). Then selecting a point on the circle $V_o^{f_2}$, the voltage tolerance region $V_o^{f_1}$ is transformed into the corresponding tolerance region Y_A^1 in the admittance plane, at frequency f_1 for the circuit branch A, see Fig.6.2b. From the theory of conformal mappings (see Section 6.4), the admittance plane tolerance region will be

*Note: only one of these voltages is a true variable at any one time. The other voltage changes are considered to be constant.

circular at each of the t frequencies, providing $V_o^{f_1}$ is circular. In general the centre M of the admittance circle Y_A^1 for one branch (Fig.6.2b) will not be coincident with nominal value N' of the admittance A . The inside of a tolerance region in the voltage plane will transform into either the inside or the outside of the region in the admittance plane depending on whether the nominal admittance lies inside or outside the region.

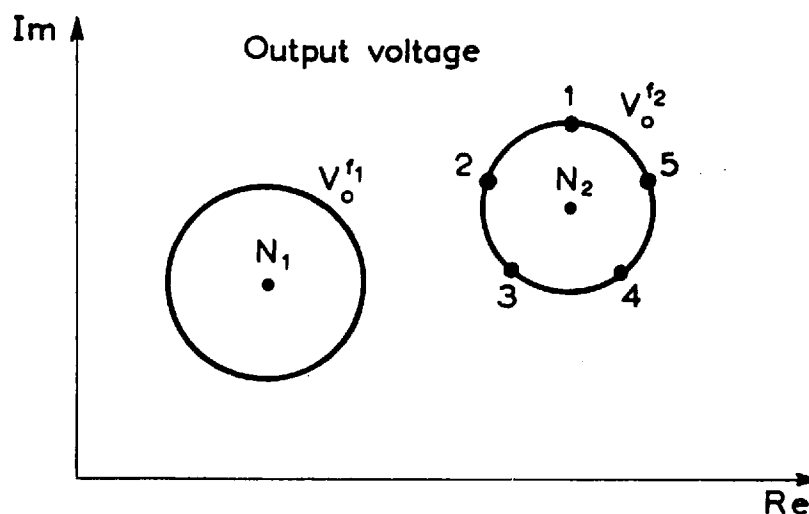


FIG.6-2a: Two nominal output voltages N_1 and N_2 at frequencies f_1 and f_2 are shown on the complex voltage plane together with two tolerance circles. The voltage region $V_o^{f_1}$ will be transformed into an admittance circle for each identified point on $V_o^{f_2}$.

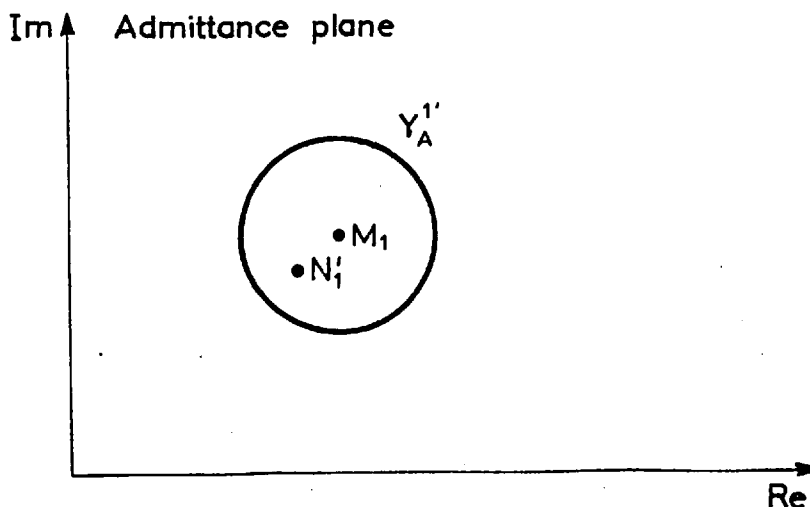


FIG.6-2b: Admittance region for branch A of the circuit at frequency f_1 corresponding to the voltage tolerance region $V_o^{f_1}$ and voltage $V_o^{f_2}$ at point 1 in FIG. 6-2a. The nominal centre point N_1 in the voltage plane transforms to N_1' in the admittance plane.

We now select another point on $V_o^{f_2}$ and compute a second admittance tolerance region Y_A^2 for the branch A at frequency f_1 . In this fashion we can traverse the boundary of $V_o^{f_2}$ and find successive tolerance regions for the branch A. The intersection of all these admittance tolerance regions, Y_A^i , $i = 1, 2, \dots, n$, then gives the actual admittance region at frequency f_1 for the circuit branch A commensurate with the two voltage tolerance regions $V_o^{f_1}$ and $V_o^{f_2}$. Thus the hatched area in Fig.6.2c defines the maximum range of admittance variation for the branch A at frequency f_1 providing the voltage variation at frequencies f_1 and f_2 is confined to the circular regions shown in Fig.6.2a. In a similar fashion a tolerance region could have been found at frequency f_2 .

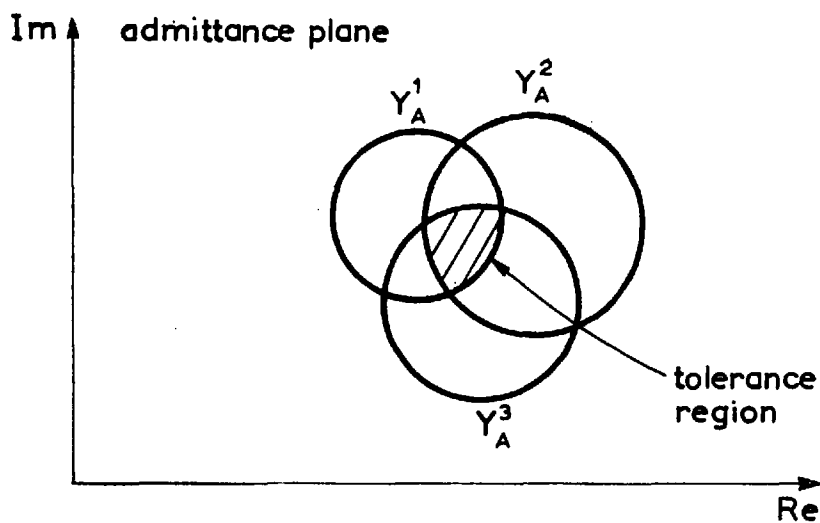


FIG.6.2c: The shaded area gives the tolerance region for branch A from the intersection of the three tolerance circles corresponding to $V_o^{f_1}$ and three of the five points on the circumference of $V_o^{f_2}$ as shown in FIG.6.2a.

The usefulness of such tolerance regions in tolerance assignment can be seen when it is remembered that complex admittance values can be identified with actual components by considering the real and imaginary parts separately. Thus the reciprocal of the real part of an admittance is identified with a resistance while dividing the imaginary part with the frequency in radians yields a reactance.

If the tolerance region of a set of admittances, computed for complex voltage response regions as described above, cuts, for example, the real zero axis of the admittance plane, the imaginary part of the particular admittance could be set to zero - this could indicate the possibility of removing an inductance or capacitance from the circuit. Similar considerations can be made for the case where the tolerance region intersects the imaginary zero axis of the admittance plane, or includes the origin.

P.A. Villalaz [68] has used a particular case of the generalised theory presented in this chapter to compute admittance tolerance circle for a voltage tolerance circle at one frequency only. He has used these admittance tolerance circles with great effect in model pessimization [82] and performance contour evaluation [79]. Whereas Villalaz could never compute the admittance tolerance region of a branch for voltage tolerance regions at more than one frequency, this can now be done easily to yield a more comprehensive tolerance region. Thus the present theory could be used to enhance the model pessimization and performance contour evaluation procedures.

6.6 Conclusion

In this chapter it has been shown that transformation of a complex voltage tolerance regions into corresponding admittance regions at one

or more frequencies constitute conformal mappings. The inverse mapping of an admittance region into a voltage region also constitutes a conformal mapping. A knowledge of the properties of conformal mapping leads to considerable saving in computational effort, for it is known that closed regions (of second order) such as circles transform into circles.

A procedure has been described for obtaining admittance tolerance regions for specified branches of a circuit for voltage tolerance regions at one or more frequencies. This constitutes a significant advance on the earlier work of Villalaz. The potential scope of using the aforesaid admittance tolerance regions in tolerance assignment, model pessimization and performance contour evaluation has been briefly mentioned.

CHAPTER 7

Epilogue

7.1 Summary of the Contents of the Thesis

The first Chapter introduced the subject of computer aided circuit design. Having identified a sequence of steps in the complete design process specific numerical procedures, suitable for computer implementation, are introduced. A brief introduction to the main characteristics of both mini- and large multi-programmed computers is given.

Chapter 2 deals with the theory of sensitivity invariants in electrical systems. The homogeneous nature of network functions is described, and using this property it is shown how all the different invariant formulae can be derived for linear systems, e.g. signal flow graphs. It is also shown how these results carry over to nonlinear circuits also. The chapter ends with a discussion of possible uses of such relationships, in particular their use in testing sensitivity programs and the tolerance assignment problem. The latter is discussed in greater detail in Chapter 4.

The next chapter is essentially a preparation for Chapter 4. The different methods of computing the circuit response are briefly described. There follows a full discussion of the uses of a multivariate truncated Taylor approximation to the circuit response function. Convergence criteria for this series are derived and are tested using Monte Carlo analyses on different circuits. The relative advantages of speed of the approximation over exact methods is outlined. Finally,

there is a brief discussion of the use of this series in an inexpensive Monte Carlo circuit analysis programme.

In Chapter 4 the problem of tolerance assignment and design centering is taken up. It is shown how tolerancing can be defined as a nonlinear programming problem, where the calculation of the circuit response due to parameter changes is of major importance. Hence the need for using a Taylor series. The theory of sensitivity invariants developed in Chapter 2 is invoked to show how the region of search of the feasible component space must be restricted during tolerance assignment. The uses of a Taylor series with modulus sensitivity terms for homogeneous network functions is discussed with an example. The concept of design centering is introduced and its consequences are demonstrated with suitable examples. This chapter is concluded with a comparison of results with two earlier authors for integral component tolerances.

Chapter 5 introduces a new concept of Large change/Small change sensitivity. A numerical algorithm is described for computing the differential sensitivity after simultaneous large changes in one or more components of a circuit. Its efficiency over standard methods of differential sensitivity calculation is demonstrated when not all the components of a circuit undergo large changes. The implementation of this technique on an interactive graphics facility, for dynamically exploring the sensitivity behaviour of circuits is described with the aid of a specific example. The uses of interactive graphics in circuit design is also briefly discussed.

In Chapter 6 we introduce the concept of Large change sensitivity first described in Chapter 5. The theory is then developed to yield admittance changes for multiple-defined voltage response changes at

different frequencies. Next we introduce the theory of conformal mappings for functions of a complex variable. We then derive conditions under which a voltage tolerance region at one frequency and finite voltage changes at other frequencies, can be transformed conformally into admittance tolerance regions for one or more circuit branches at the frequencies of analysis.

Appendix I discusses the implementation of a program for computing first-, second- and third-order differential sensitivities. Alternative programming strategies are discussed. The correct functioning of the programme is demonstrated with the use of the invariant relationships of Chapter 2.

Appendix II introduces the basic concepts of optimisation, both unconstrained and constrained. The terminology common in optimisation is defined together with a brief description of the principal methods. The penalty function and flexible tolerance algorithms of nonlinear programming are described in some detail because of their use in the tolerance assignment procedure of Chapter 4.

7.2 Conclusions and Suggestions For Future Research

In the research reported in this thesis an attempt has been made to identify numerical procedures which are important in any computer aided circuit design facility (Chapter 1). Some of these procedures have been discussed and developed in detail, not only from a purely theoretical point of view but also from the practical engineering aspect. Some numerical procedures, like tolerance assignment, are primarily suited for batch operation in large multi-processing machines while others, like the large change/small change sensitivity algorithm

are suitable for the increasingly popular mini-computers.

In Chapter 2 the theory and existence of sensitivity invariants in electrical systems was discussed. It was shown how the many different invariant relationships stem from the existence of homogeneous network functions including invariants for signal flow graphs. The invariant relationship for the latter contains terms for the "self-loop" sensitivities (see Section 2.10). These sensitivity terms raise some interesting questions, because the "self-loop" transmissions are purely notional. Therefore, it would be interesting to find out how significant are the "self-loop" sensitivities when compared to actual branch transmission sensitivities. Furthermore, is it possible to minimise the actual branch transmission sensitivities at the expense of increasing the "self-loop" sensitivities? If the last mentioned sensitivity minimisation was performed, what effect would it have on the actual branch transmission, that is, do transmissions remain positive or do they become negative?

The scope for future work on the Taylor series is more difficult to forecast. There might be some scope for a mathematician to obtain more useful convergence criteria for the series. However, it would probably be more fruitful to accumulate data on the relative accuracy of the third-order Taylor approximation when compared to an exact calculation for a greater number of circuits than has been done in this thesis.

The work on tolerance assignment and design centering as reported in Chapter 4 led to some interesting ideas, particularly with regard to the characterisation of the feasible region for component variation. More theoretical and experimental evidence would be useful to confirm that the worst case for component deviations lies in the first and third

quadrants (for the two variable example). If the worst case lies elsewhere then a new tolerance assignment procedure must be found as the proposed method cannot deal with deviations in the second and fourth quadrant (two variable example). Some preliminary evidence of three-dimensional performance contours [104] indicates that the feasible region for two of its variables takes the form shown in Fig.7.1 for the circuit examples tried. In all such situations the worst case would lie in regions where the deviations are either all positive or all negative. This is in concurrence with the findings as reported in Chapter 4. Further experiments could also be tried in tolerancing circuits for specified variation in the input or output impedance.

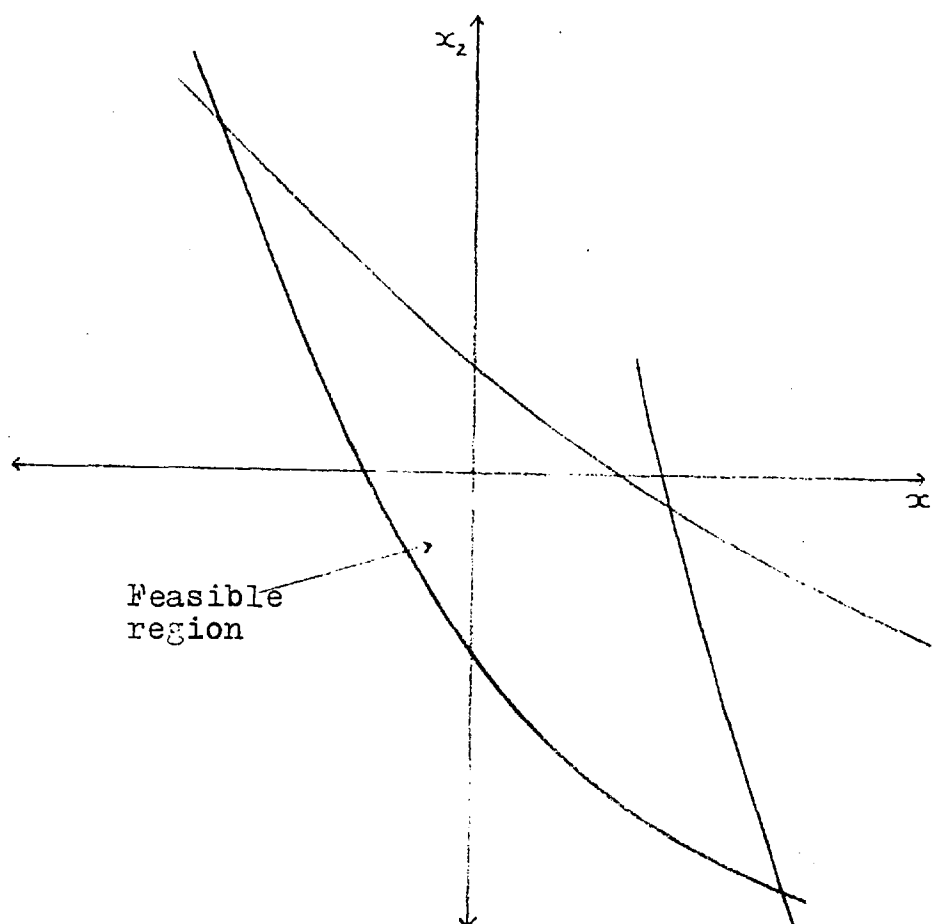


Fig. 7.1 A plane through a three dimensional performance contour

The theory of large change/small change sensitivity developed in Chapter 5 has the largest potential for incorporation not only in existing algorithms using component sensitivities but also in newer fields such as CACD in a graphics environment. Work is now in progress to incorporate the large change/small change algorithm in a circuit optimisation program using a gradient method of optimisation. There will be a considerable saving in computation effort when only a small subset of circuit components is being optimised. The role of computer graphics has already been discussed briefly in Section 5.9. However, there is a great need for evaluating the usefulness of the computer aids provided. For example, there is a need to know which symbols on a menu allow the designer to learn more rapidly. To isolate the effect on learning of different symbols from the background numerical calculations, the latter must be very fast, preferably a small fraction of a second. This can now be achieved for sensitivity calculations.

The large change/small change method of sensitivity calculation could be employed to discover quickly the maximum sensitivities for a given range of component value variation. Each of the maximum sensitivities could then form the differential coefficients of the Taylor series of Chapter 3. This series could then be said to give the "worst-case" response approximation over the known range of component variation.

Finally in Chapter 6 a numerical procedure for obtaining admittance changes and conformal mappings for known voltage response variation over a range of frequencies has been developed. The concept of tolerance regions has also been discussed. However, there is now a need for experimental evidence to establish the usefulness or otherwise of the proposed theory. Functions have been written in APL to compute admittance

tolerance regions for specified voltage variation. However, no clear interpretation of these tolerance regions has yet been found. This work needs to be extended and studied at greater length than was possible within the scope of the present thesis.

APPENDIX I

DERIVATIVE SENSITIVITY CALCULATION

I.1 Introduction

Several analytical methods are available for calculating the first order derivative sensitivity of a circuit response function. (A comprehensive list of references is to be found in a paper by S. Parker [59].) One of the best known and popular methods is due to Director and Rohrer [62], using the adjoint network approach. This method has been extended to yield higher order derivatives [54]. An earlier method [58] for computing higher order sensitivities is described below. This has been chosen primarily because only a single matrix inversion is required for obtaining first- and higher-order differential sensitivities, thus making it superior to the adjoint network approach where a larger number of network solutions are necessary [54].

As described in Chapter 3 sensitivity formulae are derived for terms up to the third order only. Also the component set has been restricted to $\{R, C, L, g_m\}$.

I.2 Derivation of Nodal Voltage Sensitivities

The usual matrix equation describing the current-voltage relationship in an n node circuit is shown in equation (1).

$$\underline{Y}_n \underline{V}_n = \underline{I}_n \quad (1)$$

$$\underline{A}^T \underline{V}_n = \underline{V}_b \quad (2)$$

$$\underline{Y}_n = \underline{A} \underline{Y}_b \underline{A}^T \quad (3)$$

$$\underline{V}_n = \underline{Y}_n^{-1} \underline{I}_n \quad (4)$$

Differentiating the matrix equation (4) with respect to a branch admittance y_k , where k is the appropriate branch number, yields

$$\frac{\partial \underline{V}_n}{\partial y_k} = - \underline{Y}_n^{-1} \frac{\partial \underline{Y}_n}{\partial y_k} \underline{V}_n \quad (5)$$

Using the relationships given by equations (2) and (3) gives:

$$\frac{\partial \underline{V}_n}{\partial y_k} = - \underline{Y}_n^{-1} \underline{A} \frac{\partial \underline{V}_b}{\partial y_k} \underline{V}_b \quad (6)$$

$$= - \underline{Y}_n^{-1} \underline{A} \underline{\delta}_{KK} \underline{V}_b \quad (7)$$

where $\underline{\delta}_{KK}$ is an $(n_b \times n_b)$ matrix with all elements zero except δ_{KK} , which is unity. Differentiating equation (7) yields the second derivative of the nodal voltage vector with respect to y_j and y_k .

$$\frac{\partial^2 \underline{V}_n}{\partial y_j \partial y_k} = - \underline{Y}_n^{-1} \underline{A} (\underline{\delta}_{KK} \frac{\partial \underline{V}_b}{\partial y_j} + \underline{\delta}_{JJ} \frac{\partial \underline{V}_b}{\partial y_k}) \quad (8)$$

where $\frac{\partial \underline{V}_b}{\partial y_j} = \underline{A}^T \frac{\partial \underline{V}_n}{\partial y_j}$.

Repeated differentiating of equation (7) allows the calculation of differentials of any order, providing these exist. Thus the third-order differentials are:

$$\frac{\partial^3 \underline{V}_n}{\partial y_i \partial y_j \partial y_k} = - \underline{Y}_n^{-1} \underline{A} (\underline{\delta}_{KK} \frac{\partial^2 \underline{V}_b}{\partial y_i \partial y_j} + \underline{\delta}_{JJ} \frac{\partial^2 \underline{V}_b}{\partial y_j \partial y_k} + \underline{\delta}_{II} \frac{\partial^2 \underline{V}_b}{\partial y_i \partial y_k}) \quad (9)$$

where $\frac{\partial^2 \underline{V}_b}{\partial y_i \partial y_j} = \underline{A}^T \frac{\partial^2 \underline{V}_n}{\partial y_i \partial y_j}$ etc..

To calculate the sensitivity with respect to a voltage controlled current source, the only change necessary in equations (7), (8) and (9) is to replace the $\underline{\delta}_{KK}$, $\underline{\delta}_{JJ}$ and $\underline{\delta}_{II}$ terms by terms like $\underline{\delta}_{KM}$ where K is

associated with the branch that contains the controlled source and M with the controlling voltage branch.

1.3 Sensitivities For Voltage Gain

From the above expressions, sensitivities for specific nodal or branch voltages can be found by choosing the appropriate terms from the sensitivity vectors. The sensitivities for voltage gain between two nodes or two ports can be found from the above-mentioned sensitivities. Thus consider the voltage gain $F = V_2/V_1$, where V_2 and V_1 are nodal voltages with respect to a common reference node. The first order sensitivity expression now becomes:

$$\frac{\partial F}{\partial y_k} = \frac{1}{V_1} \frac{\partial V_2}{\partial y_k} - \frac{V_2}{V_1^2} \frac{\partial V_1}{\partial y_k} \quad (11)$$

$$= \frac{V_2}{V_1} \left[\frac{1}{V_2} \frac{\partial V_2}{\partial y_k} - \frac{1}{V_1} \frac{\partial V_1}{\partial y_k} \right] \quad (12)$$

$$\frac{\partial F}{\partial y_k} \frac{y_k}{F} = y_k \left[\frac{1}{V_2} \frac{\partial V_2}{\partial y_k} - \frac{1}{V_1} \frac{\partial V_1}{\partial y_k} \right] \quad (13)$$

The corresponding second and third order sensitivity formulae are more complicated and are given in equations (14) and (15).

$$\begin{aligned} \frac{\partial^2 F}{\partial y_j \partial y_k} \frac{y_i y_k}{F} &= y_j y_k \left[\frac{1}{V_2} \frac{\partial^2 V_2}{\partial y_j \partial y_k} - \frac{1}{V_1} \frac{\partial^2 V_1}{\partial y_j \partial y_k} \right. \\ &\quad \left. - \frac{1}{V_1 V_2} \left(\frac{\partial V_1}{\partial y_j} \frac{\partial V_2}{\partial y_k} + \frac{\partial V_2}{\partial y_j} \frac{\partial V_1}{\partial y_k} \right) + 2 \frac{1}{V_1^2} \frac{\partial V_1}{\partial y_j} \frac{\partial V_1}{\partial y_k} \right] \quad (14) \end{aligned}$$

$$\begin{aligned} \frac{\partial^3 F}{\partial y_i \partial y_j \partial y_k} \frac{y_i y_j y_k}{F} &= y_i y_j y_k \left\{ \frac{1}{V} \frac{\partial^3 V_2}{\partial y_i \partial y_j \partial y_k} - \frac{1}{V_1} \frac{\partial^3 V_1}{\partial y_i \partial y_j \partial y_k} - \frac{6}{V_1^3} \frac{\partial V_1}{\partial y_i} \frac{\partial V_1}{\partial y_j} \frac{\partial V_1}{\partial y_k} \right. \\ &\quad - \frac{1}{V_1 V_2} \left[\frac{\partial V_1}{\partial y_k} \frac{\partial^2 V_2}{\partial y_i \partial y_j} + \frac{\partial V_2}{\partial y_k} \frac{\partial^2 V_1}{\partial y_i \partial y_j} + \frac{\partial V_2}{\partial y_j} \frac{\partial^2 V_1}{\partial y_k \partial y_i} + \frac{\partial V_1}{\partial y_i} \frac{\partial^2 V_2}{\partial y_k \partial y_j} + \frac{\partial V_1}{\partial y_i} \frac{\partial^2 V_2}{\partial y_j \partial y_k} \right. \\ &\quad \left. + \frac{\partial V_2}{\partial y_i} \frac{\partial^2 V_1}{\partial y_j \partial y_k} \right] + \frac{2}{V_1^2} \left[\frac{\partial V_1}{\partial y_i} \frac{\partial^2 V_1}{\partial y_j \partial y_k} + \frac{\partial V_1}{\partial y_j} \frac{\partial^2 V_1}{\partial y_i \partial y_k} + \frac{\partial V_1}{\partial y_k} \frac{\partial^2 V_1}{\partial y_i \partial y_j} \right] \\ &\quad \left. + \frac{2}{V_1^2 V_2} \left[\frac{\partial V_1}{\partial y_i} \frac{\partial V_1}{\partial y_j} \frac{\partial V_2}{\partial y_k} + \frac{\partial V_1}{\partial y_i} \frac{\partial V_1}{\partial y_k} \frac{\partial V_2}{\partial y_j} + \frac{\partial V_1}{\partial y_j} \frac{\partial V_1}{\partial y_k} \frac{\partial V_2}{\partial y_i} \right] \right\} \quad (15) \end{aligned}$$

The sensitivities derived so far are with respect to admittances. More commonly, however, sensitivities with respect to specific components $x \in \{R, L, C, g_m\}$ are desired. These can be calculated from the admittance sensitivities. The two step procedure is chosen for the following reasons:

a) Direct differentiation of the admittance matrix with respect to a component x requires knowledge of the algebraic form of each non-zero element of $\underline{Y}_0$ in equation (3) to generate the correct $\underline{\delta}$ matrix. However, differentiation with respect to an admittance results in very simple $\underline{\delta}$ matrices. Therefore, the computational cost is negligible when compared with the information needed for the direct calculation.

b) The programming task is made easier because simple indexing operations of the relevant matrices are required. The programme becomes naturally modular leading to easier debugging and formation of overlay structures. The latter is important in reducing the overall core space needed for running programmes.

c) Using the two step procedure, considerable flexibility is retained for future modification of the programmes to an extended component set from the present $\{R, L, C, g_m\}$.

I.4 Sensitivities With Respect To Components

To derive the necessary expressions for component sensitivities, consider a general network function ϕ in terms of admittance parameters y_i .

$$\phi = \phi(y_1, y_2, \dots, y_n) \quad (16)$$

As y_i can be defined in terms of R_i, C_i, L_i, g_{m_i} , the chain rule can

be employed to calculate the component sensitivities. Thus the most general admittance expression is given by (17).

$$y_i = \frac{1}{R_i} + j(\omega C_i - \frac{1}{\omega L_i}) + g_{m_i} \quad (17)$$

If across a particular branch only some of the components are present, it is only necessary to drop the non-existent component terms from (17). Differentiating (16) with respect to x_i yields:

$$\frac{\partial \phi}{\partial x_i} = \frac{\partial \phi}{\partial y_i} \frac{\partial y_i}{\partial x_i} \quad (18)$$

The second and third differentials are similarly given by:

$$\frac{\partial^2 \phi}{\partial x_j \partial x_i} = \frac{\partial^2 \phi}{\partial y_j \partial y_i} \frac{\partial y_j}{\partial x_j} \frac{\partial y_i}{\partial x_i} + \frac{\partial \phi}{\partial y_i} \frac{\partial^2 y_i}{\partial x_j \partial x_i} \delta_{ij} \quad (19)$$

$$\begin{aligned} \text{and } \frac{\partial^3 \phi}{\partial x_k \partial x_j \partial x_i} &= \frac{\partial^3 \phi}{\partial y_k \partial y_j \partial y_i} \frac{\partial y_j}{\partial x_i} \frac{\partial y_i}{\partial x_i} + \frac{\partial^2 \phi}{\partial y_j \partial y_i} \frac{\partial^2 y_j}{\partial x_k \partial x_j} \frac{\partial y_i}{\partial y_i} \delta_{kj} \\ &+ \frac{\partial^2 \phi}{\partial y_j \partial y_i} \frac{\partial y_j}{\partial x_j} \frac{\partial^2 y_i}{\partial x_k \partial x_i} \delta_{kj} \\ &+ \frac{\partial^2 \phi}{\partial y_k \partial y_i} \frac{\partial y_k}{\partial x_k} \frac{\partial^2 y_i}{\partial x_j \partial x_i} \delta_{ij} + \frac{\partial \phi}{\partial y_i} \frac{\partial^3 y_i}{\partial x_k \partial x_j \partial x_i} \delta_{ijk} \end{aligned} \quad (20)$$

where δ_{ij} , δ_{ki} , δ_{kj} and δ_{ijk} are unity when $i = j$, $k = i$, $k = j$ and $i = j = k$, otherwise zero. The Kroneker delta functions have been employed to account for the normal case when all the components are independent of each other. If, however, some components are related, e.g. inductor with its self resistance then these can be accounted for by dropping the δ functions from the corresponding equations. For the present all components are considered independent of each other. The differentials of y with respect to x can therefore be easily computed from (17) and the corresponding substitutions made in equations (18), (19) and (20).

It is evident from the above sensitivity formulae that if m components of a circuit are being considered, there is an m -vector of first order terms, an $(m \times m)$ matrix of second order and an $(m \times m \times m)$ matrix of third order terms. Thus the total number of terms required to be calculated and stored for each successive order, increases by multiples of m .

I.5 Implementation of the Sensitivity Analysis Program

The numerical formulae for calculating sensitivities of different voltage response functions with respect to various parameters are incorporated into a program capable of computing the necessary results from the circuit information. This information includes the circuit topology, the component values, the frequency of analysis, as well as a list of branch elements for which the sensitivities are to be found. The computational and storage requirements are such (see Chapter 3) that sensitivities up to second order and only the third order self terms are to be computed and stored.

A basic objective was that the final program should be capable of working within the lowest computer job priority, and storage requirements should allow for the maximum number of program runs. Furthermore, provision had to be made for future extensions that would be required to incorporate the tolerance assignment and Monte Carlo analysis programs. Within these computer limitations the program had to be capable of analysing realistic circuits. It has been found that circuits of 16 nodes (including the ground node), with a maximum of four controlled sources, could be analysed at fifteen frequencies while satisfying the above requirements.

The circuits are analysed using nodal analysis in preference to

mesh analysis, because the latter requires expensive search algorithms to identify loops within a circuit. Within the nodal analysis scheme employed, each branch of the circuit can have up to four different types of components in parallel across any pair of nodes (Fig.1). Therefore, it is possible to find sensitivities for a maximum of (15×4) components. However, due to storage requirement considerations, as well as results from Chapter 3, the number of components for which sensitivities can be found has been limited to 16.

I.6 Computational Speed Enhancement

In calculating sensitivities different programming strategies are available. Clearly the most desirable is that needing the least number of multiplications, indexing and testing. Added to these, both the storage requirements and programming effort should be minimal. Two alternative approaches are outlined; one due to Goddard and Spence [58] and one developed below. A single nodal admittance matrix inversion at each frequency is common to both and its cost, therefore, is not considered. A scheme using LU-factorisation [68] is considered unsuitable because it is only less expensive than the method using matrix inversion, for first-order sensitivities. For higher orders a further $2n_p$ network solutions are necessary [54] for n_p variable elements. The comparison of costs is based on the total number of complex multiplications or divisions plus an equivalent number of additions or subtractions.

Method (a)

In this method of Goddard and Spence [58] the appropriate elements of $\underline{Y}_n^{-1}$ are specifically indexed to compute the sensitivities.

of the n_b specified components. The only indexing information needed is a knowledge of the nodes to which the components are connected.

The computation cost for each order is given in Table 1.

	Operations
First order	$4 n_b$
Second order	$24n_b^2$
Third order (self terms only)	$52n_b$
Total cost	$56n_b + 24n_b^2$

Table 1 Cost in terms of the number of operation for method (a)

Method (b)

This method is based on computing an intermediate result which leads to an overall reduction in the number of multiplications at the expense of using $n_b(n + n_b)$ extra storage locations. From the equations (7), (8) and (9) it can be seen that if $\underline{C} = \underline{A}^T \underline{Y}_n^{-1} \underline{A}$ is computed, the total number of matrices used in calculating the branch voltage sensitivities is effectively reduced. If the $(n_b \times n_b)$ matrix $\underline{C}$ is stored in locations occupied by $\underline{Y}_n^{-1}$ then the aforesaid storage requirements, which include the storage of $\underline{A}$, can be reduced appropriately. The computational cost for the different operations is in Table 2.

	Operations
Compute <u>C</u>	$2n_b n$
First order (can be reduced to $2n_b$ at the expense of increasing cost for subsequent orders)	n_b^2
Second order (if all first order sensitivities had not been computed then this cost would be $4n_b^2$)	$4n_b^2$
Third order	$5n_b$
Total cost	$2n_b n + 5n_b + 5n_b^2$ $5n_b + 7n_b^2$

Table 2 Cost in terms of the number of operations for Method (b)

A direct comparison between the costs of methods (a) and (b) shows the second to be far superior in terms of the number of multiplications and additions in spite of the additional storage requirements.

I.7 Accuracy of the Program

To check that the program is functioning correctly it is only necessary to invoke the sensitivity invariant relationships, derived in Chapter 2. Thus the relationships given by equations (2.23) and (2.24) are used with the circuits of Figs.1 and 2.

Circuit	Frequency	First Order	Second Order
Fig. 1	170 Hz	4.23×10^{-16}	1.99×10^{-15}
Fig. 2	159 Hz	8.882×10^{-16}	0.0

Table 3

The results shown in Table 3 of the absolute of the sum of sensitivities are consistent with the theory and within the 16 digit accuracy of the computer arithmetic. Therefore, it can be assumed that the sensitivities are reasonably accurate. Nevertheless, it is impossible to state the degree of accuracy, which can only be determined by comparison with analytically derived results. As discussed in Chapter 3, there are no universally agreed circuits which can be used in such tests. In any case, the analytical derivation of sensitivities for reasonably complicated circuits can itself pose serious algebraic problems, whilst those for simple two or three node circuits may not be an adequate guide.

Conclusions

A detailed derivation of the sensitivity formulae has been made and factors influencing their implementation in computer programs discussed. It has been shown how sensitivities for correlated components may be obtained. Finally, a program for computing first-, second- and third-order (self terms only) sensitivities for uncorrelated $\{R, C, L, g_m\}$ networks has been implemented in FORTRAN and tested using sensitivity invariants.

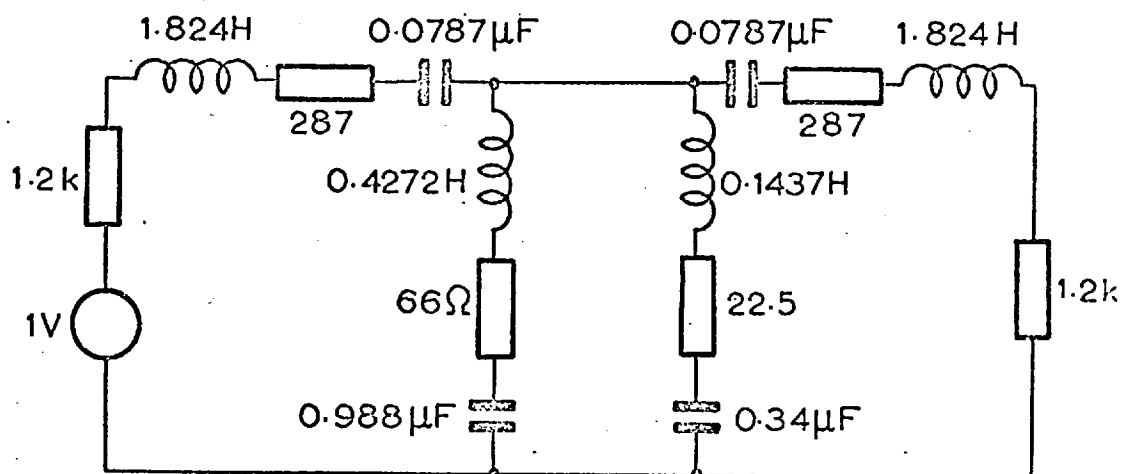


FIG. 1 : Band Pass Filter

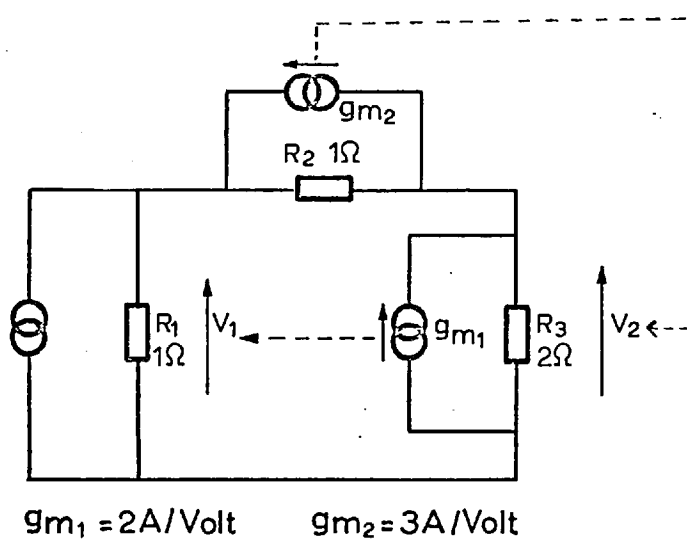


FIG.2: Active circuit

APPENDIX II

NONLINEAR PROGRAMMING

II.1 Introduction

In many mathematically formulated problems of the physical, engineering, management and social sciences there is interest in finding the maximum or minimum (optimal) values of a function. The application of differential calculus enables some of these problems to be solved analytically. When functions are subject to equality constraints, Lagrange multipliers can be employed. For linear functions subject to linear equality and/or inequality constraints, linear programming (LP) can usually be employed. However, many problems that are of interest can be adequately described by nonlinear equations only. In most such cases, analytical methods are of little use because of the need to solve sets of nonlinear equations. Therefore, there is much interest in devising numerical techniques, often called nonlinear programming or optimization methods, to discover the stationary values of nonlinear functions of several variables.

It is beyond the scope of this Appendix to cover the wide discipline of nonlinear programming. Only a brief survey of the salient features and underlying principles of the field, without rigorous mathematical proof is presented here.

II.2 Notation and Terminology

The general nonlinear programming problem can be summarised as:

$$\text{Minimise}^+ : f(\underline{x}) \quad \underline{x} \in E^n \quad (1)$$

subject to r linear and/or nonlinear equality constraints

$$h_j(\underline{x}) = 0 \quad j = 1, 2, \dots, r \quad (2)$$

and $(r - s)$ linear and/or nonlinear inequality constraints

$$g_j(\underline{x}) \geq 0 \quad j = r + 1, \dots, s \quad (3)$$

The domain of $\underline{x}$ for which all the constraints, given by (2) and (3), are satisfied is called the feasible region R .

$$R \in \{ \underline{x} \mid h_j(\underline{x}) = 0; g_j(\underline{x}) \geq 0 \text{ for all } j \} \quad (4)$$

All points not in R are termed nonfeasible, their domain being the nonfeasible region. The function to be minimised is termed the objective function and the optimal point is defined by the column vector $\underline{x}^* = [x_1^*, x_2^*, \dots, x_n^*]^T$ of the continuous independent variable in an n dimensional Euclidian space E^n .

In Fig.1 a two dimensional optimization problem with a simply connected feasible region bounded by three inequality constraints is described. If the feasible region were split into two or more disconnected parts it would be termed a non-simply connected region. When the equality constraint $h_1(\underline{x})$ is included the feasible region contracts to points along $h_1(\underline{x})$ inside R .

The effectiveness of a minimisation algorithm depends upon the amount of information available about the objective function and the constraints. In Fig.1 the unconstrained minimum of $f(\underline{x})$ lies outside the feasible region of the constrained problem. The minimum can be found sometimes by performing a random search. However, efficient numerical algorithms carry out a directed search for a minimum, the

⁺Maximising a function $f(\underline{x})$ is mathematically identical to minimising $-f(\underline{x})$.

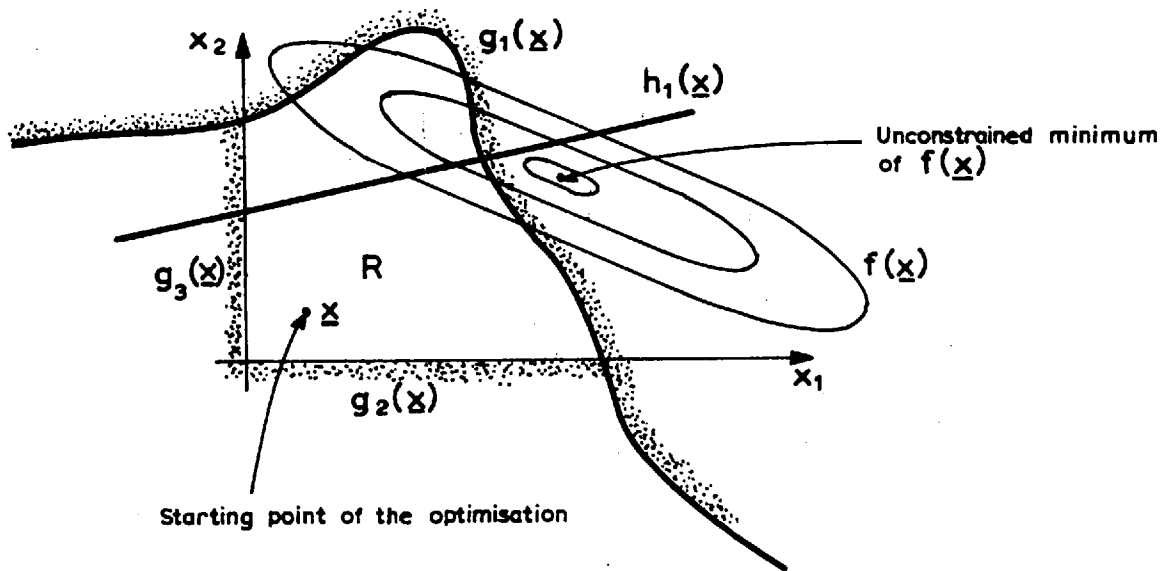


FIG.1: Example of a two variable constrained optimisation problem.

distinction between algorithms being the manner in which the search is directed.

II.3 Minimum of a Function

The necessary condition satisfied at a minimum of a continuously differentiable function $f(\underline{x})$ of n variables $\underline{x}$ is:

$$\nabla f(\underline{x}^*) = \underline{0} \quad (5)$$

The sufficient condition for $\underline{x}^*$ to be a minimum is that $\underline{H}$ the Hessian matrix ($n \times n$ square matrix of the second partial derivatives) of $f(\underline{x})$ be positive definite

$$\underline{\Delta x}^T \underline{H} \underline{\Delta x} > \underline{0} \quad (6)$$

where $\underline{\Delta x}$ is any set of deviations from $\underline{x}^*$.

A multi-variable function may and usually does have more than one minimum value (Fig.2). Each minimum value is called a local minimum and the lower value over the entire feasible region is called the

the global minimum. Mathematically it is impossible to distinguish between a local and global minimum. Therefore, most algorithms are only capable of finding local minima. Practical algorithms for finding global minima for general functions have yet to be devised.

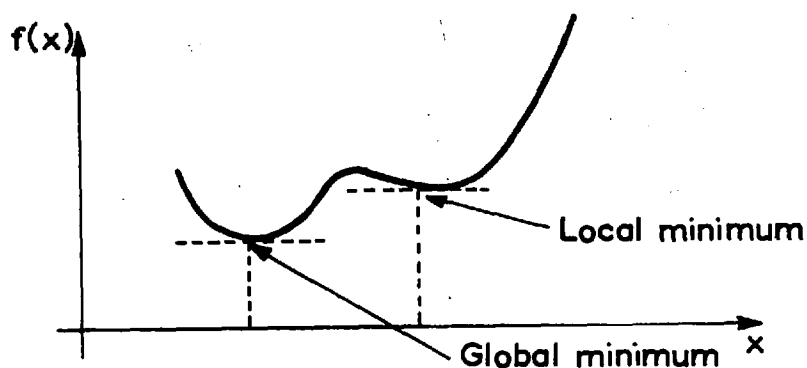


FIG.2: Example of a single variable function $f(x)$.

II.4 Minimum of a Constrained Function

A constrained local minimum $\underline{x}^*$ can be conceptually described as that point on $f(\underline{x})$ from which no decrease in the function value is possible along any smooth arc without leaving the feasible region R .

The conditions for the existence of a minimum of $f(\underline{x})$ subject to equality constraints $h_j(\underline{x})$ have been derived by Lagrange. A new function called the Lagrangian is formed from $f(\underline{x})$ and $h_j(\underline{x})$:

$$L(\underline{x}, \lambda_j) = f(\underline{x}) - \sum_{j=1}^r \lambda_j h_j(\underline{x}) \quad (7)$$

where λ_j 's are the Lagrange multipliers. Differentiating $L(\underline{x}, \lambda_j)$ with respect to $x_1, \dots, x_n$ and $\lambda_1, \dots, \lambda_r$ provides sufficient information to determine the constrained minimum. Kuhn and Tucker have extended this method to incorporate inequality constraints with the introduction of slack variables C_j which convert the former into

equality constraints. The new Lagrangian now reads:

$$L(\underline{x}, C_j, \lambda_j) = f(\underline{x}) - \sum_{j=r+1}^s \lambda_j (g_j(\underline{x}) - C_j) \quad (8)$$

Using equation (8) necessary and sufficient conditions for the existence of an extremum for the general constraining optimization problem (1), (2) and (3) have been obtained by Kuhn, Tucker, Fiacco and McCormick [83,84] amongst others. A detailed account of the complex mathematical proofs is to be found in a text by Fiacco and McCormick [84]

II.5 Convexity

A function is said to be convex in R if the Hessian matrix is positive definite over all R , i.e., if any straight line is drawn between two points of the function the line does not intercept the surface defined by the function. Thus in a single variable function example Fig. 3 represents a convex function while Fig.4 does not.

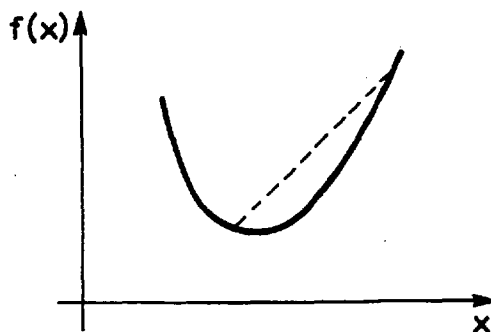


FIG.3: Convex function

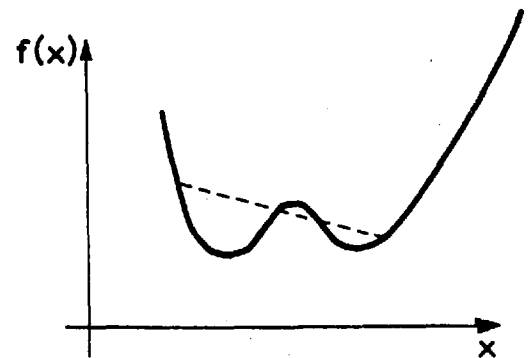


FIG.4: Nonconvex function

The convexity of a function is sufficient to ensure that a unique minimum exists. However, it is not a necessary condition as can be seen from the non convex curve in Fig.5.

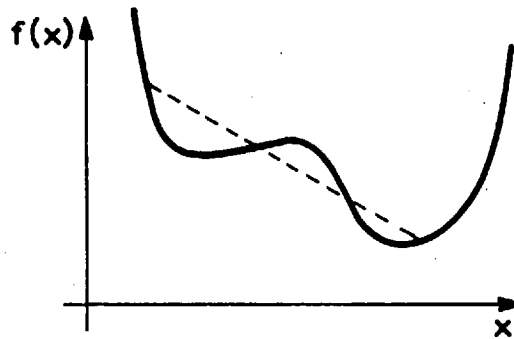


FIG. 5: Non-convex function possessing an unique minimum.

II.6 Unconstrained Minimisation Procedures

The procedures available for unconstrained minimisation can be broadly classified into two groups, gradient or derivative and direct or pattern search methods.

a) Derivative Methods

Given a continuously differentiable objective function $f(\underline{x})$ the simplest strategy for selecting a direction $\underline{d}^{i+1}$ for movement from the current position $\underline{x}^{i-1}$ is to choose the negative of the gradient vector

$$\underline{d}^{i+1} = - \nabla f(\underline{x}^i) \quad (9)$$

This is the first order steepest descent method, and produces good results when $\underline{x}$ is far from $\underline{x}^*$. However, the rate of convergence can become slow, close to the minimum, where the first derivative changes by small amounts. A second failing of this method is that

in choosing each new direction no account is taken of past iterations. To improve upon these, a second order method based on assuming $f(\underline{x})$ to be quadratic has been developed.

The change in the function value of a change $\underline{\Delta x}^i$ in the independent variables is approximated by a second order Taylor expansion about $\underline{x}^i$

$$f(\underline{x}^{i+1}) = f(\underline{x}^i) + \nabla^T f(\underline{x}^i) \underline{\Delta x}^i + \frac{1}{2} (\underline{\Delta x}^i)^T \nabla^2 f(\underline{x}^i) \underline{\Delta x}^i \quad (10)$$

The minimum of $f(\underline{x})$ in the direction $\underline{\Delta x}^i$ is obtained by differentiating $f(\underline{x}^{i+1})$ with respect to the components of $\underline{\Delta x}^i$ and equating the resulting expression to zero to yield,

$$\underline{\Delta x}^i = -[\nabla^2 f(\underline{x}^i)]^{-1} \nabla^T f(\underline{x}^i) \quad (11)$$

where $[\nabla^2 f(\underline{x}^i)]^{-1}$ is the inverse Hessian $\underline{H}$ of $f(\underline{x}^i)$. It should be noted that now both the direction and step length are specified at each iteration. Convergence to a minimum is guaranteed if the Hessian matrix remains positive definite at each iteration.

A second class of minimisation algorithms, based on the concept of conjugate directions, provide rapid convergence by making use of the quadratic behaviour of functions near their minima. Thus:

$$f(\underline{x}) \simeq a + \underline{b}^T \underline{x} + \frac{1}{2} \underline{x}^T \underline{H} \underline{x} \quad (12)$$

the first three terms of the n-dimensional Taylor series, and $\underline{H}$ is positive definite. The gradient of $f(\underline{x})$ is:

$$\nabla f(\underline{x}) \simeq \underline{b} + \underline{H} \underline{x} \quad (13)$$

If $f(\underline{x})$ is exactly quadratic and n successive linear searches are

conducted in direction $\underline{d}^i$ such that:

$$(\underline{d}^i)^T \underline{H} \underline{d}^j = 0 \quad i \neq j \quad (14)$$

$$(\underline{d}^i)^T \underline{H} \underline{d}^j \geq 0 \quad i = j \quad (15)$$

Then it can be shown that the minimum will be found on the n^{th} search [85], and the directions satisfying the conditions (14) and (15) are said to be conjugate. The method is quadratically convergent.

A quadratic approximation is accurate close to a minimum and therefore the use of conjugate directions will lead to rapid convergence to the minimum. In practice they are also found to be very efficient when the starting point is far from the minimum. Hence the popularity of algorithms using conjugate directions.

A third class of methods known as variable metric, or quasi-Newton [86], approximate the Hessian matrix or its inverse from first order derivations of the objective function. Many of these methods employ conjugate directions. The new $\underline{x}$ vector is computed from the preceding stage by:

$$\underline{x}^{i+1} = \underline{x}^i - \lambda^{*i} \gamma(\underline{x}^i) \nabla f(\underline{x}^i) \quad (16)$$

where λ^{*i} is some scalar, $\gamma(\underline{x}^i)$ is the direction matrix and represents the approximation to $\underline{H}^{-1}$. In the earlier second order method it was necessary to compute the second differentials of $f(\underline{x})$ and invert the Hessian. In the variable matrix projection method relations are used to compute $\gamma(\underline{x}^i)$ without performing either of the aforesaid operations.

$$\underline{H}^{-1}(\underline{x}^{i+1}) \simeq \omega(\underline{x}^{i+1}) = \omega \gamma(\underline{x}^i) + \Delta \gamma(\underline{x}^i) \quad (17)$$

where ω is some scalar. The selection of $\Delta \gamma(\underline{x}^i)$ determines the type

of variable metric method [87 - 90] . Convergence is guaranteed by forcing $\omega \mathcal{V}(\underline{x}^{i+1})$ to be positive definite.

b) Direct Search Methods

These are derivative free methods, the search directions being solely determined from successive objective function evaluations. Though derivative methods usually converge to a minimum faster than search methods, in practice the former have two main handicaps. First, in problems with a modestly large number of variables it is laborious or impossible to provide analytical functions or numerical procedures for the first and/or second derivatives of the objective function. Secondly, the preparation times for derivative methods are longer, thus increasing the chance of human error. Furthermore, search methods do not require that the objective function or its derivatives be continuous, each move being made when $f(\underline{x}^{i+1}) < f(\underline{x}^i)$. Though many search methods have been proposed, only two will be briefly described in this section.

One of the most popular methods in this category, developed by Powell [91] , assumes the objective function to display quadratic behaviour near its minimum and have a positive definite Hessian. The minimum is located by successive unidimensional searches along conjugate directions $\underline{d}^i$ from an initial point $\underline{x}_0^i$. The principle of the method is based on the idea that if the minimum of a quadratic function $f(\underline{x})$ is found along each of the p , ($p < n$) conjugate directions at each stage i of the search, and if a step is accordingly made in each direction, then the overall step from the start to the p^{th} step is conjugate to all the p search directions. The transition from the point $\underline{x}_0^i$ to the point $\underline{x}_m^i$ is given by:

$$\underline{x}_m^i = \underline{x}_0^i + \sum_{j=1}^{m-1} \lambda_j^i \underline{d}_j^i \quad (18)$$

The significant features of the method can be discerned from the two dimensional function example of Fig.6, where four steps $\underline{x}_0^o$, $\underline{x}_1^o$, $\underline{x}_2^o$ and $\underline{x}_3^o$ are taken along the two directions $\underline{d}_1^o$ and $\underline{d}_2^o$.

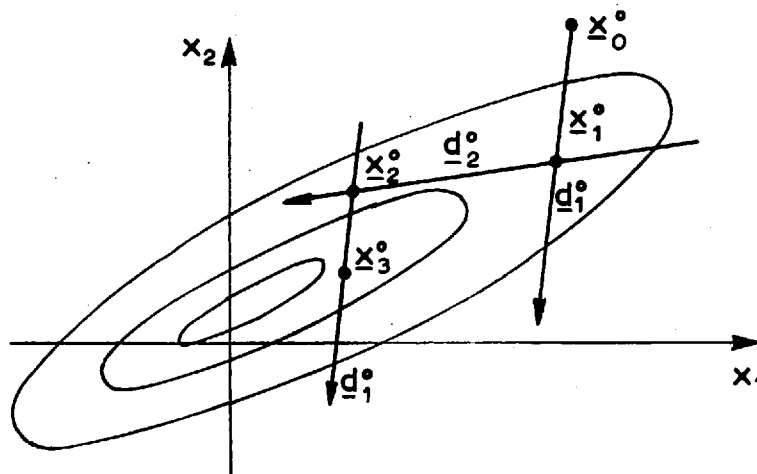


FIG.6: Powell's method of search along conjugate directions $\underline{d}_1^o$ and $\underline{d}_2^o$

It has been shown by Zangwill [93] that a slightly modified Powell's procedure will always converge to a minimum if $f(\underline{x})$ is convex.

Another method termed the flexible polyhedron search is based on the regular Simplex* search of Spendly, Hext and Himsworth [94], which by itself is deficient in that no acceleration of the search is possible and that the Simplex becomes small when searching in curving valleys or ridges. One method of overcoming these difficulties [95] is to allow the Simplex to change its shape, hence the term "flexible polyhedron". To minimise a function of n independent variables $(n + 1)$ vertices on $f(\underline{x})$ are used to form a polyhedron. The vertex point in E^n which yields the highest value of $f(\underline{x})$ is projected through the centroid of the remaining vertices. Lower values of the objective function are found by successively replacing the point

*Regular polyhedrons in E^n are termed Simplexes. Thus an equilateral triangle is a simplex in E^2 .

with the highest value of $f(\underline{x})$ by better (lower) points until the minimum is found. Fig.7 illustrates a single step for a two variable function.

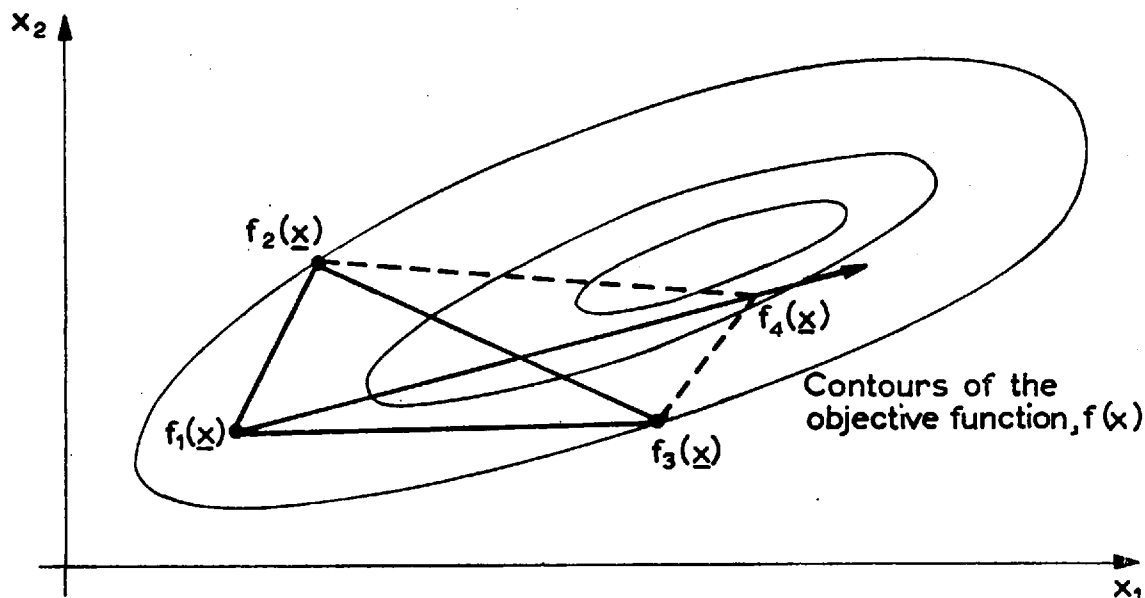


FIG.7: Example of the "flexible polyhedron search."

II.7 Constrained Minimisation Procedures

This branch of nonlinear programming is less well developed than unconstrained minimisation because of the need to test for the non-violation of constraints during the optimisation, together with the necessity of suitably altering the search direction when one or more of these constraint boundaries are violated. To date three principal approaches have been considered:

- i) To linearise all the nonlinear functions by making a first-order Taylor expansion about $\underline{x}$; then employing either the well known linear programming [96] or gradient projection methods [97] amongst others.

- ii) Transforming the constrained nonlinear problem into a sequence of unconstrained problems by using the penalty or barrier function approach [98].
- iii) The use of variable tolerances on the constraints to accommodate both feasible and nonfeasible $\underline{x}$ vectors during the minimisation of the objective function [99] .

The last two techniques have been used in the solution of the tolerance assignment problem discussed in Chapter 4 of the thesis. These will be outlined in broad detail and without mathematical rigour*.

a) Penalty Function Method

The principle behind all penalty function methods is to add one or more of the constraints to the objective function such that a continuous barrier is formed near the old constraint boundaries. The resulting function, equation (19), is then minimised using a conventional unconstrained optimisation procedure.

$$P(\underline{x}^i, \rho^i) = f(\underline{x}^i) + \sum_{j=1}^r \rho_j^i H h_j(\underline{x}^i) + \sum_{j=r+1}^s \rho_j^i G g_j(\underline{x}^i) \quad (19)$$

The generalised augmented function $P(\underline{x}^i, \rho^i)$ is often termed the "penalty function". The parameters ρ^i are suitably chosen weight factors with $\rho_j^i \geq 0$, H and G are functionals of $h_j(\underline{x})$ and $g_j(\underline{x})$ respectively and i is the number of complete stages in the numerical search. The choice of the functionals is such that:

$$\lim_{i \rightarrow \infty} |P(\underline{x}^i, \rho^i) - f(\underline{x}^i)| = 0 \quad (20)$$

$$\lim_{i \rightarrow \infty} \sum_{j=1}^r \rho_j^i H h_j(\underline{x}^i) = 0 \quad (21)$$

*For more details of these methods see references [84,100]

$$\lim_{i \rightarrow \infty} \sum_{j=r+1}^s \rho_j^i G_j(\underline{x}^i) = 0 \quad (22)$$

Penalty function methods are characterized by the location of the starting point which may be within or outside the feasible region yielding interior point and exterior point methods respectively. A third class of methods termed mixed methods are a combination of the earlier two. In interior point methods the aim is to keep away from and inside the constraint boundary by the suitable addition of penalties, while in the exterior point methods a sequence of nonfeasible points are generated which in the limit yield a feasible or near feasible solution. In the mixed methods some of the constraints are satisfied and not others, but at the solution all are satisfied within some specified tolerance.

For all categories of methods the usual choice of the equality constraint functional is: $H[h_j(\underline{x}^i)] = h_j^2(\underline{x}^i)$; and ρ_j^i is a sequence of increasing numbers. The inequality functional is, however, different for each category and must satisfy the following criteria:

- 1) $G_1[g_j(\underline{x}^i)] \rightarrow +\infty$ as $g_j(\underline{x}^i) \rightarrow 0$, which requires that $\underline{x}^i$ always remain an interior point, $\{\underline{x}^i \mid g_j(\underline{x}^i) > 0, j=r+1, \dots, s\}$ and $\rho_j^i \rightarrow 0$. Typical choices of G_1 are the logarithmic function [101], $G_1[g_j(\underline{x}^i)] = \ln g_j(\underline{x}^i)$ shown in Fig.8 and the inverse modulus function [84], $G_1[g_j(\underline{x}^i)] = 1/|g_j(\underline{x}^i)|$ shown in Fig.8.
- 2) $G_2[g_j(\underline{x}^i)] \rightarrow 0$ as $g_j(\underline{x}^i) \rightarrow 0$ and $\rho_j^i \rightarrow +\infty$. In this case exterior points only are accepted, $\{\underline{x}^i \mid g_j(\underline{x}^i) < 0; j = r+1, \dots, s\}$. A typical choice of functional [102]

is $G_2[g_j(\underline{x}^i)] = g_j^2(\underline{x}^i)$, shown in Fig.9.

- 3) $G_3[g_j(\underline{x}^i)] > 0$ for $g_j(\underline{x}^i) < 0$ and $G_4[g_j(\underline{x}^i)] = 0$ for $g_j(\underline{x}^i) \geq 0$. Here constraint satisfaction is sought at the solution only. Therefore, functionals of category(1) can be chosen with a suitable choice of ρ_j^i ; i.e. $\rho_j^i > 0$ for G_3 and $\rho_j^i = 0$ for G_4 .

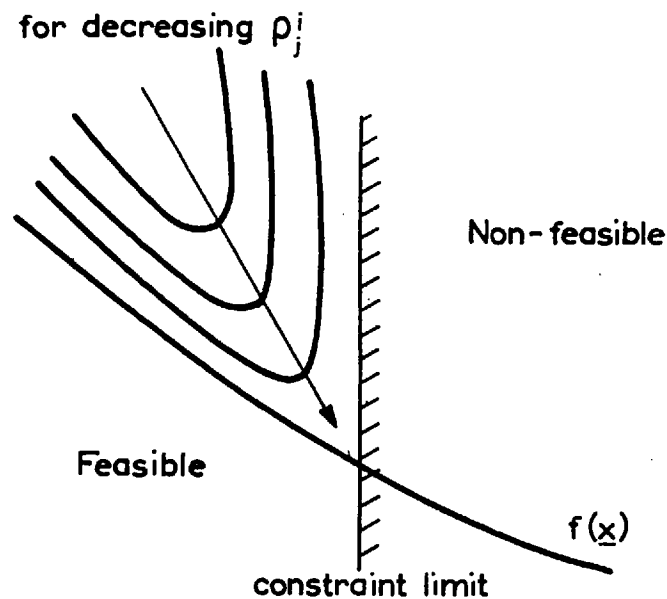


FIG.8: Logarithmic and inverse modulus barrier function

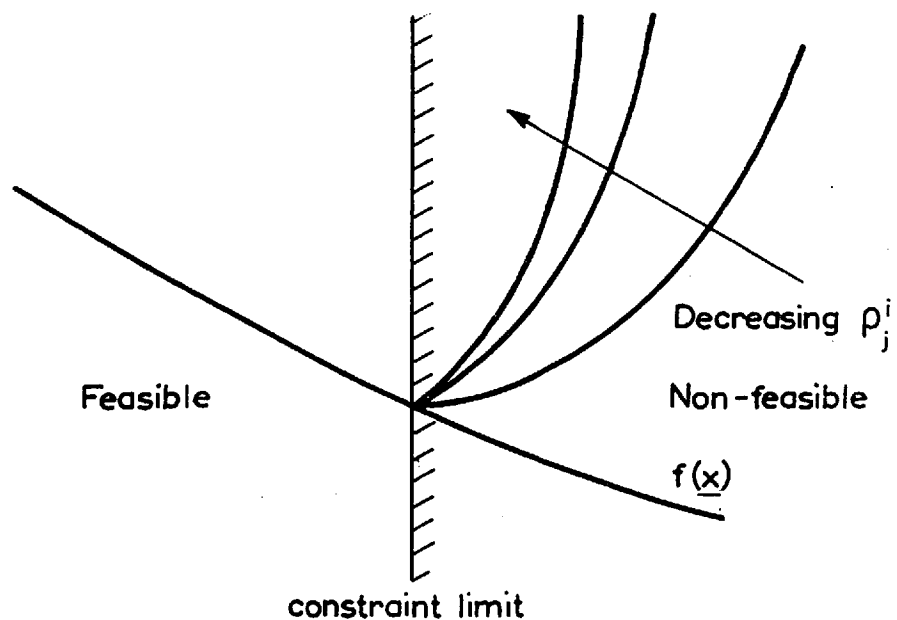


FIG.9: Quadratic loss function

The determination of values for the weight factor ρ_j^i poses some difficulties. Empirical evidence from several researchers [103] suggests that a reasonable choice for a sequence of ρ_j^i 's when $\lim \rho_j^i \rightarrow 0$ is to have $\rho_j^i = 1$ followed with a reduction by a factor of 0.1 at each stage i . Conversely when $\lim \rho_j^i \rightarrow \infty$; $\rho_j^i = 1$ and increase ρ_j^i by a factor of 10 at each stage i .

b) The Flexible Tolerance Method

In this method [99] the value of the objective function is improved using information from both feasible and some nonfeasible points, termed near-feasible points. This is achieved by allowing the constraint boundaries given by the functional $T[x]$ to have a variable tolerance Φ , which is gradually tightened at each stage i of the search. The basic problem defined by (1), (2) and (3) becomes:

$$\begin{aligned} &\text{Minimise } f(\underline{x}) \quad \underline{x} \in E^n \\ &\text{Subject to } \Phi^i - T[\underline{x}] \geq 0 \end{aligned}$$

As with many other constrained optimisation procedures a series of unconstrained searches are required in the algorithm. The tolerance criterion Φ is selected to be a positive decreasing function of the vertices of a flexible polyhedron in E^n (see Section 11.6b)

$$\Phi^i = \Phi^i(\underline{x}_1^i, \underline{x}_2^i, \dots, \underline{x}_{t+1}^i, \underline{x}_{t+2}^i)$$

A definition of Φ suggested by Paviani and Himmelblau [99] is:

$$\Phi^i = \min \left\{ \Phi^{i-1}, \frac{r+1}{t+1} \sum_{j=1}^{t+1} \|\underline{x}_j^i - \underline{x}_{t+2}^i\| \right\} = 2^{-(i+1)} \tau$$

where τ = size of initial polyhedron

r = number of equality constraints

$\underline{x}_j^i$ = j^{th} vertex of the polyhedron in E^n

t = $(n-r)$ = number of degrees of freedom of $f(\underline{x})$

($\because$ one degree of freedom is reduced for each equality constraint)

$\underline{x}_{t+2}^i$ = vertex of polyhedron with $n = r$

i = number of complete stages in the search

As the search proceeds the polyhedron shrinks to zero as does the tolerance

$$\lim_{\underline{x} \rightarrow \underline{x}^*} \Phi^i = 0$$

The functional $T[\underline{x}]$ of the constraints is defined only for the violated equality and/or inequality constraints:

$$T[\underline{x}] = + \left[\sum_{j=1}^r h_j^2(\underline{x}) + \sum_{j=r+1}^s U_j g_j^2(\underline{x}) \right]^{\frac{1}{2}}$$

where $U_j = 0$ for $g_j(\underline{x}) \geq 0$

and $U_j = 1$ for $g_j(\underline{x}) < 0$

The objective function and $T[\underline{x}]$ are minimised while trying to reduce the value of Φ^i thus tightening the region of near feasibility. Moves are only selected if $f(\underline{x}^{i+1}) < f(\underline{x}^i)$. If no such move along any of the directions is possible through the centroid of the polyhedron, the value of Φ^i reduces because of the contraction of the polyhedron, and the search terminates when $\Phi^i \leq \epsilon$; where ϵ is a suitable termination criterion. Greater details of the strategy of the flexible tolerance algorithm are to be found in references [99,100].

Conclusions

It is clear from the above that most nonlinear programming algorithms rely on the assumptions that functions are quadratic and convex in the domain of search. However, in most problems it is difficult if not impossible to prove the validity of these assumptions for the functions being optimised. In fact, there may be reasonable grounds to believe that the assumptions may be invalid, see Chapter 4 Section 4.7. In spite of this, many of these methods have proved to be successful in solving a variety of problems.

APPENDIX III

In this appendix the tables relating to the statistical tests performed on the truncated Taylor approximation presented in Chapter 3 are given. The contents of the tables are discussed in Section 3.4c of Chapter 3.

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.000E+02	1.441E+00	-1.452E-04	2.418E-03	5.761E-06	9.972E-01	1.003E+00
5.000E+02	5.711E-01	-4.631E-04	1.327E-03	1.762E-06	9.956E-01	9.770E-01
2.000E+03	3.420E-01	-3.824E-05	5.116E-04	2.516E-07	9.967E-01	9.973E-01
5.000E+03	1.846E-01	-1.231E-05	3.236E-04	1.047E-07	9.944E-01	1.002E+00
1.000E+04	9.332E-02	-9.030E-06	2.233E-04	4.986E-08	9.951E-01	9.973E-01
2.500E+04	4.132E-02	-1.274E-05	1.315E-04	1.790E-08	9.921E-01	1.005E+00

Table 1.i.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.000E+02	1.441E+00	3.262E-02	3.040E-02	1.561E-03	7.612E-01	7.027E-01
5.000E+02	5.711E-01	5.331E-03	3.439E-02	1.162E-03	3.484E-01	8.896E-01
2.000E+03	3.420E-01	6.121E-03	1.541E-02	2.693E-04	7.205E-01	7.903E-01
5.000E+03	1.846E-01	4.077E-03	2.521E-03	6.065E-05	6.773E-01	7.308E-01
1.000E+04	9.332E-02	2.203E-03	5.243E-03	2.748E-05	7.111E-01	7.434E-01
2.500E+04	4.132E-02	1.181E-03	2.241E-03	5.024E-06	5.935E-01	6.953E-01

Table 1.i.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.000E+02	1.441E+00	2.822E-05	1.674E-04	2.574E-08	1.000E+00	9.994E-01
5.000E+02	5.711E-01	5.044E-05	9.804E-05	9.611E-09	1.000E+00	9.975E-01
2.000E+03	3.427E-01	9.712E-05	7.118E-05	5.111E-10	1.000E+00	9.991E-01
5.000E+03	1.316E-01	1.053E-05	1.032E-05	2.325E-10	1.000E+00	9.980E-01
1.000E+04	5.833E-02	6.731E-06	9.334E-06	9.860E-11	1.000E+00	9.975E-01
2.500E+04	4.032E-02	4.032E-06	5.024E-06	7.000E-11	1.000E+00	9.969E-01

Table 1.ii.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.000E+02	1.441E+00	3.552E-02	3.587E-02	1.287E-03	8.223E-01	7.265E-01
5.000E+02	5.711E-01	6.500E-03	3.030E-02	9.162E-04	4.861E-01	8.652E-01
2.000E+03	3.427E-01	5.711E-03	1.607E-02	2.211E-04	7.761E-01	8.047E-01
5.000E+03	1.316E-01	4.721E-03	7.032E-03	6.340E-05	7.902E-01	7.447E-01
1.000E+04	5.833E-02	2.046E-03	4.735E-03	1.875E-05	6.175E-01	7.617E-01
2.500E+04	4.032E-02	1.076E-03	1.037E-03	3.374E-06	7.317E-01	6.968E-01

Table 1.ii.(20)

TOTAL N = 10

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)

FOR TRUNCATION AT THE THIRD ORDER (SELF) TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEANS	STANDARD DEVIATIONS	VARIANCES	CORRELATION		MEAN RATIO FM / FS
					FM	V FS	
1.000E+02	1.441E-00	-3.534E-05	6.228E-15	3.267E-09	1.000E+00		1.001E+00
5.000E+02	5.711E-01	-6.779E-07	1.835E-05	7.767E-10	1.000E+00		1.000E+00
2.000E+03	3.420E-01	-9.626E-06	2.226E-05	4.256E-10	1.000E+00		1.001E+00
5.000E+03	1.846E-01	-4.134E-07	7.371E-06	2.855E-11	1.000E+00		1.000E+00
1.000E+04	9.858E-02	1.534E-06	4.437E-06	1.968E-11	1.000E+00		9.995E-01
2.500E+04	4.032E-02	9.303E-07	2.510E-06	6.740E-12	1.000E+00		9.994E-01

Table 1.iii.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)

FOR TRUNCATION AT THE THIRD ORDER (SELF) TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEANS	STANDARD DEVIATIONS	VARIANCES	CORRELATION		MEAN RATIO FM / FS
					FM	V FS	
1.000E+02	1.441E-00	3.513E-12	3.520E-22	1.239E-03	8.226E-01		7.235E-01
5.000E+02	5.711E-01	6.140E-07	5.045E-02	9.272E-04	4.841E-01		6.728E-01
2.000E+03	3.420E-01	5.562E-03	1.477E-02	2.131E-04	7.806E-01		8.095E-01
5.000E+03	1.846E-01	4.632E-03	7.063E-03	6.741E-05	7.898E-01		7.511E-01
1.000E+04	9.858E-02	1.980E-03	4.320E-03	1.273E-05	8.171E-01		7.694E-01
2.500E+04	4.032E-02	1.348E-03	1.045E-03	3.693E-06	7.309E-01		7.047E-01

Table 1.iii.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.000E+02	1.441E+00	-5.137E-02	2.514E-02	6.268E-04	6.812E-01	2.230E+00
5.000E+02	5.711E-01	-2.037E-02	1.226E-02	1.504E-04	5.685E-01	1.997E+00
2.000E+03	3.420E-01	-1.218E-02	7.575E-03	5.730E-05	5.329E-01	2.088E+00
5.000E+03	1.846E-01	-5.128E-03	3.618E-03	1.301E-05	5.612E-01	1.945E+00
1.000E+04	9.358E-02	-1.953E-03	1.751E-03	3.068E-06	6.692E-01	1.577E+00
2.500E+04	4.032E-02	-6.104E-04	6.731E-04	4.530E-07	7.726E-01	1.384E+00

Table M.l.i.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.000E+02	1.441E+00	-1.006E-01	6.707E-02	4.498E-03	4.786E-01	1.774E+00
5.000E+02	5.711E-01	-3.338E-02	3.718E-02	1.382E-03	2.738E-01	1.691E+00
2.000E+03	3.421E-01	-1.665E-02	1.231E-02	3.724E-04	5.813E-01	1.570E+00
5.000E+03	1.846E-01	-4.472E-03	1.162E-02	1.127E-04	5.823E-01	1.242E+00
1.000E+04	9.358E-02	-2.390E-03	5.629E-03	3.160E-05	6.570E-01	1.278E+00
2.500E+04	4.032E-02	-7.137E-04	2.637E-03	6.953E-06	4.567E-01	1.201E+00

Table M.l.i.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.000E+02	1.441E+00	-6.635E-12	2.754E-02	7.584E-04	6.796E-01	2.399E+00
5.000E+02	5.711E-01	-2.324E-02	1.715E-02	1.720E-04	5.684E-01	2.138E+00
2.000E+03	3.420E-01	-1.383E-02	3.608E-03	7.060E-05	5.294E-01	2.236E+00
5.000E+03	1.946E-01	-5.730E-03	7.276E-03	1.581E-05	5.566E-01	2.167E+00
1.000E+04	9.336E-02	-2.258E-03	1.836E-03	3.371E-06	6.679E-01	1.667E+00
2.500E+04	4.032E-02	-7.333E-04	7.107E-04	5.040E-07	7.716E-01	1.465E+00

Table M.1.ii.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.000E+02	1.441E+00	-1.375E-11	8.017E-02	6.421E-03	4.761E-01	2.059E+00
5.000E+02	5.711E-01	-4.511E-02	4.353E-02	1.650E-03	2.768E-01	1.934E+00
2.000E+03	3.420E-01	-2.301E-02	2.000E-02	4.002E-04	5.945E-01	1.788E+00
5.000E+03	1.946E-01	-7.537E-03	1.786E-02	1.281E-04	5.940E-01	1.408E+00
1.000E+04	9.336E-02	-3.700E-03	5.840E-03	3.410E-05	6.550E-01	1.444E+00
2.500E+04	4.032E-02	-1.223E-03	2.841E-03	8.072E-06	4.685E-01	1.345E+00

Table M.1.ii.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE THIRD ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.000E+02	1.441E+00	-6.648E-02	2.750E-12	7.614E-04	6.800E-01	2.401E+00
5.000E+02	5.711E-01	-2.371E-02	1.311E-02	1.737E-04	5.692E-01	2.141E+00
2.000E+03	3.420E-01	-1.357E-02	8.436E-03	7.116E-05	5.292E-01	2.239E+00
5.000E+03	1.346E-01	-5.809E-03	7.238E-03	1.59E-05	5.592E-01	2.079E+00
1.000E+04	9.350E-02	-2.267E-03	1.830E-03	3.38E-06	6.680E-01	1.679E+00
2.500E+04	4.832E-02	-7.634E-04	7.132E-04	5.037E-07	7.716E-01	1.468E+00

Table M.1.iii.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE THIRD ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.000E+02	1.441E+00	-1.366E-01	8.068E-02	6.508E-03	4.763E-01	2.967E+00
5.000E+02	5.711E-01	-4.567E-02	4.133E-02	1.667E-03	2.783E-01	1.946E+00
2.000E+03	3.420E-01	-2.330E-02	2.874E-02	4.017E-04	5.973E-01	1.798E+00
5.000E+03	1.346E-01	-7.713E-03	1.100E-02	1.20E-04	5.952E-01	1.417E+00
1.000E+04	9.350E-02	-3.279E-03	5.867E-03	3.44E-05	6.551E-01	1.455E+00
2.500E+04	4.832E-02	-1.260E-03	2.268E-03	5.16E-06	4.712E-01	1.355E+00

Table M.1.iii.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)

FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEANS	STANDARD DEVIATIONS	VARIANCES	CORRELATION		MEAN RATIO FM / FS
					FM	V FS	
5.000E+02	5.788E+01	1.113E-12	6.725E-12	4.523E-13	9.991E-11	9.991E-11	
1.000E+03	6.860E+01	2.751E-12	1.261E-11	1.349E-12	9.973E-11	9.893E-11	
1.500E+03	7.131E+01	3.910E-12	7.467E-12	5.417E-13	9.954E-11	9.894E-11	

Table 2.i.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)

FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEANS	STANDARD DEVIATIONS	VARIANCES	CORRELATION		MEAN RATIO FM / FS
					FM	V FS	
5.000E+02	5.788E+01	1.407E-12	7.055E-12	4.936E-13	4.395E-11	7.499E-11	
1.000E+03	6.860E+01	9.344E-12	3.710E-11	1.458E-11	5.797E-11	8.844E-11	
1.500E+03	7.131E+01	4.980E-12	5.735E-12	1.429E-11	6.331E-11	9.249E-11	

Table 2.i.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEANS	STANDARD DEVIATIONS	VARIANCES	CORRELATION			MEAN RATIO	
					FM	V	FS	FM / FS	FM / FS
5.000E+02	5.788E+01	-1.971E-03	4.040E-03	1.640E-05	1.000E+00			1.001E+00	
1.000E+03	6.860E+01	-5.476E-01	7.051E-03	2.710E-06	1.000E+00			1.000E+00	
1.500E+03	7.131E+01	-4.764E-04	1.116E-03	1.246E-06	1.000E+00			1.000E+00	

Table 2.ii.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEANS	STANDARD DEVIATIONS	VARIANCES	CORRELATION			MEAN RATIO	
					FM	V	FS	FM / FS	FM / FS
5.000E+02	5.788E+01	1.483E+00	7.270E+00	1.173E+01	4.247E-01			7.476E-01	
1.000E+03	6.860E+01	9.732E-01	7.826E+00	1.464E+01	5.588E-01			8.646E-01	
1.500E+03	7.131E+01	4.749E-01	3.787E+00	1.434E+01	6.299E-01			9.258E-01	

Table 2.ii.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
 FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEANS	STANDARD DEVIATIONS	VARIANCES	CORRELATION FM V FS	MEAN RATIO FM / FS
5.000E+02	5.788E+01	-2.487E+00	1.601E+00	2.565E+00	5.766E-01	2.091E+00
1.000E+03	6.860E+01	-2.555E+00	2.119E+00	4.790E+00	3.756E-01	1.939E+00
1.500E+03	7.131E+01	-1.701E+00	1.727E+00	2.982E+00	5.307E-01	1.760E+00

Table M.2.i.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
 FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEANS	STANDARD DEVIATIONS	VARIANCES	CORRELATION FM V FS	MEAN RATIO FM / FS
5.000E+02	5.788E+01	-3.225E+00	4.004E+00	1.603E+01	1.064E-01	1.548E+00
1.000E+03	6.860E+01	-2.720E+00	4.953E+00	2.454E+01	2.462E-01	1.408E+00
1.500E+03	7.131E+01	-3.376E+00	5.077E+00	2.577E+01	4.113E-01	1.710E+00

Table M.2.i.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
 FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEANS	STANDARD DEVIATIONS	VARIANCES	CORRELATION			MEAN RATIO	
					FM	V	FS	FM / FS	
5.000E+02	5.788E+01	-2.716E+00	1.630E+00	2.355E+00	5.727E-01			2.191E+00	Table M.2.ii.(10)
1.000E+03	6.860E+01	-2.752E+00	2.322E+00	5.391E+00	3.684E-01			2.065E+00	
1.500E+03	7.131E+01	-2.048E+00	1.730E+00	3.203E+00	6.247E-01			1.815E+00	

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
 FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEANS	STANDARD DEVIATIONS	VARIANCES	CORRELATION			MEAN RATIO	
					FM	V	FS	FM / FS	
5.000E+02	5.788E+01	-4.001E+00	4.413E+00	1.247E+01	1.800E-01			1.681E+00	Table M.2.ii.(20)
1.000E+03	6.860E+01	-3.376E+00	5.138E+00	2.662E+01	2.413E-01			1.506E+00	
1.500E+03	7.131E+01	-4.504E+00	5.383E+00	2.897E+01	4.021E-01			1.825E+00	

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
4.000E+03	5.000E-01	3.735E-05	2.210E-04	5.467E-08	9.990E-01	9.970E-01
6.000E+03	4.939E-01	-5.562E-15	3.271E-04	1.071E-07	9.992E-01	1.004E+00
8.000E+03	5.000E-01	-1.333E-04	8.610E-04	7.413E-07	9.987E-01	1.006E+00
1.000E+04	5.000E-01	4.737E-04	4.423E-03	1.956E-05	9.690E-01	9.900E-01
1.400E+04	3.545E-05	-2.107E-05	1.130E-04	1.392E-08	8.963E-01	1.074E+00
1.500E+04	1.901E-03	3.541E-06	3.990E-05	1.592E-09	3.758E-01	9.866E-01
2.500E+04	3.174E-03	-1.375E-16	1.717E-05	1.034E-10	9.954E-01	1.006E+00
4.000E+04	1.313E-03	-7.182E-17	6.424E-06	4.210E-11	9.924E-01	1.009E+00

Table 3.i.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
4.000E+03	5.000E-01	3.129E-03	9.731E-03	9.507E-05	6.874E-01	8.751E-01
6.000E+03	4.939E-01	1.056E-02	1.751E-02	3.103E-04	7.569E-01	7.300E-01
8.000E+03	5.000E-01	1.540E-12	3.339E-02	9.232E-04	6.620E-01	7.532E-01
1.000E+04	5.000E-01	1.834E-12	5.550E-02	3.191E-03	4.469E-01	7.951E-01
1.400E+04	3.545E-05	-3.970E-15	7.133E-04	5.088E-07	4.071E-01	1.065E+00
1.500E+04	1.901E-03	-2.739E-05	1.810E-04	3.276E-08	8.798E-01	1.005E+00
2.500E+04	3.174E-03	5.769E-15	1.660E-04	2.763E-08	7.467E-01	8.660E-01
4.000E+04	1.313E-03	-4.130E-16	6.237E-05	3.891E-09	7.428E-01	1.032E+00

Table 3.i.(20)

THE STATISTICAL RESULTS ARE FOR (EX107) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
4.000E+03	5.000E-01	-7.607E-17	1.056E-05	1.114E-10	1.000E+00	1.000E+00
6.000E+03	4.000E-01	-1.131E-05	2.751E-05	5.529E-10	1.000E+00	1.001E+00
8.000E+03	5.000E-01	-5.022E-05	1.042E-04	1.085E-08	1.000E+00	1.002E+00
1.000E+04	5.000E-01	-2.644E-14	5.515E-14	3.031E-07	9.998E-01	1.006E+00
1.400E+04	3.545E-05	-6.630E-05	3.742E-05	1.117E-09	9.922E-01	1.123E+00
1.800E+04	1.071E-03	-9.53E-17	7.548E-06	5.690E-11	9.991E-01	1.003E+00
2.500E+04	3.174E-03	5.213E-02	2.731E-07	3.464E-13	1.000E+00	9.996E-01
4.000E+04	1.313E-07	2.327E-07	5.454E-07	2.974E-13	1.000E+00	9.969E-01

Table 3.ii.(10)

THE STATISTICAL RESULTS ARE FOR (EX107) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
4.000E+03	5.000E-01	3.002E-03	9.812E-03	9.620E-05	6.867E-01	8.802E-01
6.000E+03	4.000E-01	1.018E-02	1.740E-02	3.026E-04	7.573E-01	7.397E-01
8.000E+03	5.000E-01	1.442E-02	2.541E-02	8.074E-04	7.049E-01	7.690E-01
1.000E+04	5.000E-01	1.610E-12	5.118E-02	2.600E-03	5.551E-01	6.348E-01
1.400E+04	3.545E-05	-1.223E-14	7.756E-04	1.415E-07	8.751E-01	1.222E+00
1.800E+04	1.071E-03	-1.135E-06	9.330E-05	9.861E-09	9.587E-01	1.002E+00
2.500E+04	3.174E-03	4.643E-15	1.635E-04	2.871E-08	7.665E-01	6.921E-01
4.000E+04	1.313E-07	-7.470E-06	6.722E-05	4.791E-09	6.664E-01	1.159E+00

Table 3.ii.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)

FOR TRUNCATION AT THE THIRD ORDER (SELF) TERMS OF THE COMPLEX SERIES.

FREQUENCY	NOMINAL RESPONSE	STANDARD			CORRELATION		MEAN RATIO FM / FS
		MEANS	DEVIATIONS	VARIANCES	FM	V FS	
4.000E+03	5.000E-01	-3.353E-02	5.071E-02	6.357E-04	1.000E+00	1.000E+00	1.000E+00
6.000E+03	4.999E-01	-9.565E-02	2.200E-02	5.196E-04	1.000E+00	1.001E+00	1.001E+00
8.000E+03	5.000E-01	-2.722E-02	2.722E-02	7.307E-04	1.000E+00	1.001E+00	1.001E+00
1.000E+04	5.000E-01	-1.144E-02	5.966E-04	3.557E-07	9.998E-01	1.002E+00	1.002E+00
1.400E+04	3.545E-05	-9.556E-06	3.326E-05	1.153E-09	9.921E-01	1.033E+00	1.033E+00
1.800E+04	1.901E-03	-1.318E-06	7.713E-06	5.957E-11	9.991E-01	1.005E+00	1.005E+00
2.500E+04	3.174E-03	-2.991E-07	7.674E-07	5.839E-13	1.000E+00	1.001E+00	1.001E+00
4.000E+04	1.313E-03	9.854E-08	2.533E-07	6.414E-14	1.000E+00	9.987E-01	9.987E-01

Table 3.iii.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)

FOR TRUNCATION AT THE THIRD ORDER (SELF) TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	STANDARD			CORRELATION		MEAN RATIO FM / FS
		MEANS	DEVIATIONS	VARIANCES	FM	V FS	
4.000E+03	5.000E-01	2.930E-03	9.806E-07	9.616E-05	6.872E-01	8.807E-01	8.807E-01
6.000E+03	4.999E-01	1.922E-02	1.740E-02	3.128E-04	7.572E-01	7.385E-01	7.385E-01
8.000E+03	5.000E-01	1.460E-02	2.831E-02	8.015E-04	7.059E-01	7.661E-01	7.661E-01
1.000E+04	5.000E-01	1.554E-02	5.055E-02	2.555E-03	5.622E-01	8.224E-01	8.224E-01
1.400E+04	3.545E-05	-1.427E-04	3.670E-04	1.498E-07	8.657E-01	1.259E+00	1.259E+00
1.800E+04	1.901E-03	-8.504E-06	1.032E-04	1.033E-08	9.597E-01	1.015E+00	1.015E+00
2.500E+04	3.174E-03	4.341E-05	1.701E-04	2.894E-08	7.679E-01	8.991E-01	8.991E-01
4.000E+04	1.313E-03	-8.214E-06	6.976E-05	4.867E-09	5.839E-01	1.065E+00	1.065E+00

Table 3.iii.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
4.000E+03	5.000E-01	-1.453E-12	5.630E-03	3.227E-05	7.439E-01	2.162E+00
6.000E+03	4.979E-01	-2.505E-02	9.429E-03	8.891E-05	5.154E-01	2.639E+00
8.000E+03	5.010E-01	-4.250E-02	1.767E-02	3.122E-04	4.027E-01	2.804E+00
1.000E+04	5.100E-01	-7.738E-02	3.131E-02	9.802E-04	4.427E-01	2.618E+00
1.400E+04	3.545E-01	-1.000E-04	2.864E-04	6.872E-08	5.301E-01	1.699E+00
1.800E+04	1.901E-03	-4.055E-14	2.744E-04	5.492E-08	3.532E-01	2.668E+00
2.500E+04	3.174E-03	-4.100E-04	1.534E-04	2.509E-08	2.875E-01	2.904E+00
4.000E+04	1.313E-03	-1.768E-14	6.373E-05	4.061E-09	2.196E-01	2.799E+00

Table M.3.i.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
4.000E+03	5.100E-01	-2.870E-02	1.524E-02	2.323E-04	3.493E-01	2.146E+00
6.000E+03	4.939E-01	-3.738E-02	2.749E-02	5.056E-04	6.115E-01	1.969E+00
8.000E+03	5.010E-01	-7.343E-02	4.028E-02	1.622E-03	3.284E-01	2.176E+00
1.000E+04	5.000E-01	-1.763E-01	9.598E-02	8.946E-03	-1.269E-01	3.002E+00
1.400E+04	3.545E-01	-2.083E-14	7.755E-04	6.014E-07	2.538E-01	1.524E+00
1.800E+04	1.901E-03	-1.000E-03	4.751E-04	1.641E-07	4.188E-01	2.764E+00
2.500E+04	3.174E-03	-8.373E-04	3.660E-04	1.334E-07	1.027E-01	2.947E+00
4.000E+04	1.313E-03	-2.721E-04	1.164E-04	1.356E-08	2.077E-01	3.138E+00

Table M.3.i.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
4.000E+03	5.000E-01	-1.557E-12	6.150E-03	3.821E-05	7.421E-01	2.254E+00
6.000E+03	4.930E-01	-2.731E-02	1.116E-02	1.032E-04	5.190E-01	2.799E+00
8.000E+03	5.000E-01	-4.622E-02	1.246E-02	3.785E-04	3.958E-01	3.083E+00
1.000E+04	5.000E-01	-1.115E-01	3.875E-02	1.502E-03	4.565E-01	3.113E+00
1.400E+04	3.845E-01	-7.275E-04	6.323E-04	2.493E-07	5.015E-01	3.54E+00
1.800E+04	1.901E-03	-6.744E-04	3.247E-04	1.055E-07	3.461E-01	3.312E+00
2.500E+04	3.174E-03	-4.039E-04	1.215E-04	3.629E-08	2.811E-01	3.288E+00
4.000E+04	1.313E-03	-1.516E-04	7.146E-05	5.107E-09	2.198E-01	3.039E+00

Table M.3.ii.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
4.000E+03	5.000E-01	-3.311E-02	1.834E-02	2.837E-04	3.501E-01	2.322E+00
6.000E+03	4.930E-01	-4.719E-02	2.686E-02	6.178E-04	6.125E-01	2.205E+00
8.000E+03	5.000E-01	-1.022E-01	4.700E-02	2.209E-03	3.300E-01	2.638E+00
1.000E+04	5.000E-01	-2.004E-01	1.740E-01	1.817E-02	-1.143E-01	4.185E+00
1.400E+04	3.545E-05	-1.081E-03	1.736E-03	1.948E-06	2.563E-01	4.596E+00
1.800E+04	1.901E-03	-1.761E-03	7.191E-04	5.171E-07	4.376E-01	4.086E+00
2.500E+04	3.174E-03	-1.178E-03	5.779E-04	2.581E-07	9.686E-02	3.738E+00
4.000E+04	1.313E-03	-3.513E-04	1.446E-04	2.091E-08	2.132E-01	3.760E+00

Table M.3.ii.(20)

THE STATISTICAL RESULTS ARE FOR (EX10T) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE THIRD ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
4.000E+03	5.000E-01	-1.550E-12	6.476E-03	5.826E-05	7.421E-01	2.255E+00
6.000E+03	4.000E-01	-2.753E-12	1.018E-02	1.038E-04	5.191E-01	2.801E+00
8.000E+03	5.000E-01	-4.028E-12	1.097E-02	3.791E-04	3.960E-01	3.091E+00
1.000E+04	5.000E-01	-1.010E-01	3.037E-02	1.518E-03	4.511E-01	3.123E+00
1.400E+04	3.545E-03	-7.315E-14	5.023E-04	2.523E-07	5.017E-01	3.550E+00
1.800E+04	1.001E-03	-6.776E-04	2.272E-04	1.071E-07	3.452E-01	3.324E+00
2.500E+04	3.174E-03	-4.032E-04	1.011E-04	3.653E-08	2.013E-01	3.234E+00
4.000E+04	1.313E-03	-1.600E-04	7.162E-05	5.130E-09	2.210E-01	3.104E+00

Table M.3.iii.(10)

THE STATISTICAL RESULTS ARE FOR (EX10T) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE THIRD ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
4.000E+03	5.000E-01	-3.315E-12	1.686E-02	2.844E-04	3.505E-01	2.324E+00
6.000E+03	4.000E-01	-4.721E-12	2.439E-02	5.195E-04	6.131E-01	2.208E+00
8.000E+03	5.000E-01	-1.025E-01	4.797E-02	2.235E-03	3.300E-01	2.647E+00
1.000E+04	5.000E-01	-2.040E-01	1.764E-01	1.861E-02	-1.136E-01	4.226E+00
1.400E+04	3.545E-03	-2.008E-03	1.417E-03	2.007E-06	2.518E-01	4.651E+00
1.800E+04	1.001E-03	-1.704E-03	7.320E-04	5.371E-07	4.371E-01	4.126E+00
2.500E+04	3.174E-03	-1.109E-03	5.130E-04	2.641E-07	9.599E-02	3.763E+00
4.000E+04	1.313E-03	-3.536E-04	1.458E-04	2.126E-08	2.129E-01	3.778E+00

Table M.3.iii.(20)

The results presented in the next six tables refer to the band pass filter of Fig.3.4 in Chapter 3. This was the only analysed circuit in which the mean ratio between the Taylor and exact responses had very low values. These numerical results are discussed more fully in Section 3.4c(2) of Chapter 3.

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.700E+02	1.236E-02	4.386E-04	3.439E-04	1.183E-07	9.967E-01	3.639E-01
2.400E+02	6.679E-03	1.637E-03	9.392E-04	3.921E-07	9.864E-01	3.641E-01
7.000E+02	5.182E-03	1.545E-03	1.283E-03	1.647E-06	9.925E-01	3.626E-01
1.000E+03	1.257E-02	4.810E-04	3.396E-04	1.153E-07	9.946E-01	3.552E-01
3.600E+02	7.845E-01	7.238E-02	4.581E-02	2.199E-03	9.925E-01	3.617E-01
4.900E+02	8.681E-01	9.961E-02	6.594E-02	4.347E-03	9.875E-01	3.679E-01

Table 4.i.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.700E+02	1.266E-02	1.383E-03	1.279E-03	1.637E-06	7.595E-01	2.732E-01
2.400E+02	6.679E-03	3.854E-03	2.625E-03	6.892E-06	9.463E-01	3.287E-01
7.000E+02	5.182E-03	4.857E-03	4.711E-03	2.213E-05	8.593E-01	3.153E-01
1.000E+03	1.257E-02	1.269E-03	6.383E-04	4.737E-07	9.548E-01	3.472E-01
3.600E+02	7.845E-01	1.396E-01	8.034E-02	6.454E-03	9.563E-01	3.835E-01
4.900E+02	8.681E-01	1.670E-01	1.017E-01	1.034E-02	9.108E-01	3.995E-01

Table 4.i.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.700E+02	1.256E-02	4.711E-04	3.440E-04	1.157E-07	1.000E+00	3.605E-01
2.400E+02	6.679E-03	1.641E-03	9.286E-04	3.623E-07	1.000E+00	3.536E-01
7.000E+02	5.182E-03	1.653E-03	1.730E-03	1.768E-06	9.999E-01	3.574E-01
1.000E+03	1.257E-02	4.724E-04	3.336E-04	1.113E-07	1.000E+00	3.531E-01
3.600E+02	7.845E-01	7.111E-02	4.434E-02	1.966E-03	9.993E-01	3.738E-01
4.900E+02	8.681E-01	9.737E-02	6.110E-02	3.741E-03	9.953E-01	3.809E-01

Table 4.ii.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.700E+02	1.256E-02	1.873E-03	1.230E-03	1.514E-06	8.161E-01	2.770E-01
2.400E+02	6.679E-03	3.853E-03	2.517E-03	6.263E-06	9.864E-01	3.215E-01
7.000E+02	5.182E-03	4.846E-03	4.337E-03	1.881E-05	9.662E-01	3.158E-01
1.000E+03	1.257E-02	1.087E-02	6.805E-04	4.631E-07	9.786E-01	3.386E-01
3.600E+02	7.845E-01	1.726E-01	7.735E-02	5.336E-03	9.639E-01	4.157E-01
4.900E+02	8.681E-01	1.643E-01	7.718E-02	5.957E-03	9.567E-01	4.628E-01

Table 4.ii.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
 FOR FOUNDATION AT THE THIRD ORDER (SELF) TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT / V FE	MEAN RATIO FT / FE
1.700E+02	1.256E-02	4.710E-04	3.44E-04	1.187E-07	1.000E+00	3.606E-01
2.400E+02	6.670E-03	1.041E-03	3.232E-04	3.615E-07	1.000E+00	3.537E-01
7.000E+02	5.132E-03	1.078E-03	1.307E-03	1.761E-06	0.938E-01	3.535E-01
1.000E+03	1.257E-02	4.774E-04	3.332E-04	1.111E-07	1.000E+00	3.600E-01
3.000E+02	7.890E-03	7.887E-03	4.433E-02	2.011E-03	0.903E-01	3.551E-01
4.000E+02	6.641E-03	5.743E-02	6.136E-02	3.531E-03	0.571E-01	3.754E-01

Table 4.iii.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
 FOR FOUNDATION AT THE THIRD ORDER (SELF) TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT / V FE	MEAN RATIO FT / FE
1.700E+02	1.256E-02	1.471E-03	1.231E-03	1.514E-06	0.153E-01	2.775E-01
2.400E+02	6.670E-03	3.846E-03	3.514E-03	6.201E-06	0.303E-01	3.222E-01
7.000E+02	5.132E-03	4.112E-03	0.727E-03	1.001E-05	0.525E-01	3.221E-01
1.000E+03	1.257E-02	1.891E-03	6.737E-04	4.604E-07	0.794E-01	3.400E-01
3.000E+02	7.890E-03	1.770E-01	3.530E-02	5.651E-03	0.667E-01	3.930E-01
4.000E+02	6.641E-03	1.737E-01	0.27E-02	6.731E-03	0.466E-01	4.329E-01

Table 4.iii.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FL	MEAN RATIO FT / FE
1.700E+02	1.266E-02	1.458E-04	5.168E-04	2.671E-07	2.925E-01	8.521E-01
2.400E+02	6.679E-03	1.175E-03	1.227E-03	1.505E-06	5.711E-01	5.416E-01
7.000E+02	5.182E-03	9.714E-04	1.760E-03	3.097E-06	5.493E-01	6.235E-01
1.000E+03	1.257E-02	1.816E-04	4.533E-04	2.155E-07	5.243E-01	7.566E-01
3.600E+02	7.845E-01	9.661E-03	6.790E-02	4.610E-03	3.455E-01	9.148E-01
4.900E+02	6.631E-01	1.547E-02	9.650E-02	9.313E-03	2.927E-01	8.955E-01

Table M.4.i.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.700E+02	1.266E-02	1.238E-03	1.529E-03	2.337E-06	2.959E-01	5.222E-01
2.400E+02	6.679E-03	2.767E-03	3.131E-03	9.804E-06	6.344E-01	5.124E-01
7.000E+02	5.182E-03	3.755E-03	5.321E-03	2.832E-05	5.641E-01	4.769E-01
1.000E+03	1.257E-02	5.152E-04	9.381E-04	8.246E-07	5.278E-01	6.850E-01
3.600E+02	7.845E-01	1.235E-02	1.377E-01	1.897E-02	2.121E-01	9.429E-01
4.900E+02	6.631E-01	3.225E-02	1.480E-01	2.189E-02	5.420E-01	8.947E-01

Table M.4.i.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT / V FE	MEAN RATIO FT / FE
1.700E+02	1.236E-12	1.075E-14	5.215E-14	2.710E-07	2.545E-01	8.548E-01
2.400E+02	6.670E-03	9.666E-04	1.198E-03	1.435E-06	5.791E-01	6.111E-01
7.000E+02	5.132E-03	7.023E-04	1.753E-03	3.073E-06	5.337E-01	7.251E-01
1.000E+03	1.257E-12	1.710E-14	4.511E-14	2.031E-07	5.106E-01	6.231E-01
3.600E+02	7.845E-11	-1.004E-12	7.128E-02	5.081E-03	3.473E-01	1.175E+00
4.900E+02	8.511E-11	-3.753E-12	9.035E-12	9.970E-03	2.972E-01	1.213E+00

Table M.4.ii.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT / V FE	MEAN RATIO FT / FE
1.700E+02	1.256E-02	1.047E-03	1.531E-03	2.343E-06	2.934E-01	5.960E-01
2.400E+02	6.670E-03	2.032E-03	2.035E-03	6.973E-06	6.331E-01	6.419E-01
7.000E+02	5.132E-03	2.525E-03	5.237E-03	2.743E-05	5.230E-01	6.436E-01
1.000E+03	1.257E-12	3.710E-14	9.035E-04	8.031E-07	5.139E-01	7.981E-01
3.600E+02	7.845E-11	-1.105E-11	1.044E-01	3.775E-02	1.824E-01	1.524E+00
4.900E+02	8.511E-11	-1.627E-11	1.817E-01	3.301E-02	5.554E-01	1.531E+00

Table M.4.ii.(20)

THE STATISTICAL RESULTS ARE FOR (EXIST) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE THIRD ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.780E+02	1.266E-02	1.174E-04	5.215E-04	2.711E-07	2.646E-01	6.542E-01
2.480E+02	6.679E-03	9.868E-04	1.138E-03	1.435E-06	5.792E-01	6.114E-01
7.000E+02	5.112E-03	7.043E-04	1.753E-03	3.073E-06	5.334E-01	7.268E-01
1.000E+03	1.257E-02	1.307E-04	4.513E-04	2.036E-07	5.095E-01	8.247E-01
3.600E+02	7.347E-01	-2.193E-02	7.129E-02	5.082E-03	3.523E-01	1.105E+00
4.900E+02	8.611E-04	-7.577E-02	1.001E-01	1.001E-02	3.013E-01	1.227E+00

Table M.4.iii.(10)

THE STATISTICAL RESULTS ARE FOR (EXIST) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE THIRD ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.780E+02	1.266E-02	1.045E-03	1.531E-03	2.343E-06	2.934E-01	5.967E-01
2.480E+02	6.679E-03	2.026E-03	2.334E-03	3.965E-06	6.332E-01	6.430E-01
7.000E+02	5.112E-03	2.478E-03	5.250E-03	2.763E-05	5.137E-01	6.513E-01
1.000E+03	1.257E-02	7.230E-04	2.932E-04	3.080E-07	5.135E-01	8.023E-01
3.600E+02	7.347E-01	-1.213E-01	2.635E-01	4.141E-02	1.762E-01	1.579E+00
4.900E+02	8.611E-04	-1.032E-01	1.017E-01	5.635E-02	5.714E-01	1.533E+00

Table M.4.iii.(20)

THE STATISTICAL RESULTS ARE FOR (EXIST) - (TAYLOR APPROXIMATION)
 FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE	
1.530E+02	6.321E-02	1.151E-16	3.653E-05	1.321E-09	9.996E-01	9.994E-01	Table 5.i.(10)

THE STATISTICAL RESULTS ARE FOR (EXIST) - (TAYLOR APPROXIMATION)
 FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE	
1.530E+02	6.321E-02	3.567E-14	1.744E-03	3.041E-06	7.403E-01	9.003E-01	Table 5.i.(20)

THE STATISTICAL RESULTS ARE FOR (EXIST) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.590E+02	6.321E-12	6.022E-17	8.318E-07	7.953E-13	1.000E+00	9.997E-01

Table 5.ii.(10)

THE STATISTICAL RESULTS ARE FOR (EXIST) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.590E+02	6.321E-12	3.272E-14	1.671E-03	2.861E-06	7.593E-01	9.066E-01

Table 5.ii.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE THIRD ORDER (SELE) TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT / V FE	MEAN RATIO FT / FE
1.590E+02	6.32E-02	4.135E-17	6.644E-17	4.411E-13	1.100E+00	9.997E-01

Table 5.iii.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE THIRD ORDER (SELE) TERMS OF THE COMPLEX SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT / V FE	MEAN RATIO FT / FE
1.590E+02	6.321E-02	3.266E-14	1.471E-03	2.054E-06	7.994E-01	9.997E-01

Table 5.iii.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.530E+02	6.321E-12	-1.751E-13	8.256E-14	6.491E-07	7.751E-01	1.760E+00

Table M.5.i.(10)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE FIRST ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.530E+02	6.321E-12	-2.655E-13	2.459E-13	6.045E-06	4.845E-01	1.750E+00

Table M.5.i.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.590E+02	6.321E-02	-1.441E-03	8.142E-04	6.630E-07	7.743E-01	1.610E+00

Table M.5.ii.(10)

TOTAL NUMBER OF APPROXIMATIONS GREATER THAN EXACT VALUES IS 43 OUT OF 50 TRIALS

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE SECOND ORDER TERMS OF THE MODULUS SERIES

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE
1.590E+02	6.321E-02	-3.045E-03	2.553E-03	6.510E-06	4.823E-01	1.652E+00

Table M.5.ii.(20)

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE THIRD ORDER TERMS OF THE MODULUS SERISE

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE	
1.530E+02	6.321E-02	-1.441E-03	2.142E-04	6.629E-07	7.743E-01	1.810E+00	Table M.5.iii.(10)

TOTAL NUMBER OF APPROXIMATIONS GREATER THAN EXACT VALUES IS 43 OUT OF 50 TRIALS

THE STATISTICAL RESULTS ARE FOR (EXACT) - (TAYLOR APPROXIMATION)
FOR TRUNCATION AT THE THIRD ORDER TERMS OF THE MODULUS SERISE

FREQUENCY	NOMINAL RESPONSE	MEAN DIFFERENCE	STANDARD DEVIATIONS	VARIANCES	CORRELATION FT V FE	MEAN RATIO FT / FE	
1.530E+02	6.321E-02	-3.452E-03	2.553E-03	6.519E-06	4.822E-01	1.853E+00	Table M.5.iii.(20)

REFERENCES

1. BRUNE, O., 'Synthesis of a finite two-terminal network whose driving point impedance is a prescribed function of frequency', J. Maths Phys., Vol.10, pp.191-235, 1931.
2. DARLINGTON, S., 'Synthesis of reactance 4-poles which produce prescribed insertion loss characteristics', J. Maths. Phys., Vol.18, pp.257-353, 1939.
3. WILDER, R.J., 'Some circuit design techniques for linear integrated circuits', IEEE Trans. on Circuit Theory, Vol.CT-12, pp.586-590, 1965.
4. HOLLINGDALE, S.H. and TOOTHILL, G.C., 'Electronic Computers', Penguin Books Ltd., London, 1966.
5. WARNER, R. and FORDENWATT, J., Editors, 'Integrated Circuits: Design Principles and Fabrication', McGraw-Hill, New York, 1965.
6. BALABAN, P., KARAFIN, B.J. and SNYDER, D.B., 'A Monte Carlo analysis of integrated single substrate RC-oscillators', B.S.T.J., Vol.50, No.4, pp.1263-1291, 1971.
7. HERSKOWITZ, G.J., 'Computer-aided integrated circuit design', McGraw-Hill, New York, 1968.
8. MILLMAN, J. and HALKIAS, C.C., 'Electronic devices and circuits', McGraw-Hill, New York, 1967.
9. VILLALAZ, P.A., 'Theory and techniques of network sensitivity applied to electrical device modelling and circuit design', Ph.D. dissertation, London University, 1972.
10. GODDARD, P.J., 'The calculation and reduction of electrical network sensitivity', Ph.D. dissertation, London University, 1971.
11. BODE, H.W., 'Network analysis and feedback amplifier design', Princeton, N.J.: Van Nostrand, 1945.
12. LEEDS, J.V.JR. and UGRON, G.I., 'Simplified multiple parameter sensitivity and continuously equivalent networks', IEEE Trans. Circuit Theory, Vol. CT-14, pp.188-191, June 1967,
13. BELOVE, C., 'Sensitivity sums for homogeneous functions', IEEE Trans. Circuit Theory, Vol.CT-11, p.171, 1964.
14. KUH, E.S. and LAU, C.G., 'Sensitivity invariants of continuously equivalent networks', IEEE Trans. Circuit Theory, Vol.CT-15, pp.175-177, 1968.

15. ROSKA, T., 'Summed sensitivity invariants and their generation',
Electronics Letters, 1968, 4, pp.281-282.
16. HOLT, A.J.G. and FIDLER, J.K., 'Summed sensitivity of active networks',
Electronic Letters, 1968, 4, pp.385-386.
17. HOLT, A.J.G. and FIDLER, J.K., 'Sensitivity invariants of network
functions', Electronic Letters, 1968, 4, pp.85-87.
18. HOLT, A.J.G. and LEE, M.R., 'Summed sensitivity of active RC-networks',
Electronic Letters, 4, No.5, March 1968.
19. GEHER, K. and SOLYMOSI, J., 'Calculation of higher order sensitivities
and higher order sensitivity invariants', Proc. IEEE Int. Symp.
Network Theory, pp.7-8, 1971.
20. BUTLER, W.J. and HAYKIN, S.S., 'Multiparameter sensitivity problems
in network theory', Proc. IEE, Vol.117, No.12, pp.2228-2235,
December 1970.
21. GEHER, K., 'Theory of network tolerances', Akademiai Press, Budapest,
1971, pp.48-57.
22. PARKER, S.R., 'Sensitivity: old questions, some new answers', IEEE
Trans. Circuit Theory (Special Issue on Computer-aided Circuit
Design), Vol. CT-18, pp.27-35, January 1971.
23. SWAMY, M.N.S., BHUSHAN, C. and THULASIRAMAN, K., 'Bounds on the sum
of element sensitivity magnitudes of network functions', Proc.
IEEE Int. Symposium Circuit Theory, 1972, pp.21-25.
24. SWAMY, M.N.S., BHUSHAN, C. and THULASIRAMAN, K., 'Sensitivity invariants
for lumped-distributed networks', Electronics Letters, Vol.8,
No.2, pp.26-27. 1972.
25. SWAMY, M.N.S., BHUSHAN, C. and THULASIRAMAN, K., 'Sensitivity
invariants for linear time-invariant networks', IEEE Trans.
Circuit Theory, Vol. CT-20, pp.21-24, 1973.
26. SWAMY, M.N.S., BHUSHAN, C. and THULASIRAMAN, K., 'Sensitivity
invariants for non-linear networks', IEEE Trans. Circuit Theory
Vol. CT-19, pp.599-606, November 1972.
27. SABALTASH, M. and SEVIORA, R., 'Sensitivity invariants for scattering
matrices', IEEE Trans. Circuit Theory Vol. CT-18, pp.282-284,
March 1971.
28. BRAAE, R., 'Matrix algebra', Pitman, London 1963, p.18.
29. FIKHTANGOLTS, G.M., 'The fundamentals of mathematical analysis',
Pergamon Press, London 1965, Vol. 1, p.275.
30. BLOSTEIN, M.L., 'Some bounds on sensitivity in RLC networks', Proc.
of the 1st Allerton Conference, pp.488-501, 1963.

31. DESOER, C.A. and KUH, E.S., 'Basic Circuit Theory', McGraw-Hill, pp.419-468, 1969.
32. KUH, E.S. and ROHRER, R.A., 'Theory of linear active networks', San Francisco, California: Holden-Day, 1967, pp.265-278.
33. NEIRYNCK, J., 'Two-parameter dimensional analysis applied to the derivation of sensitivity invariants', Electronic Letters, 8, 1972, pp.436-437.
34. SPENCE, R., 'Linear Active Networks', J. Wiley, 1971, pp.66-67.
35. SCHOEFFLER, J.D., 'The synthesis of minimum sensitivity networks', IEEE Trans. Circuit Theory, Vol. CT-11, June 1964, pp.271-276.
36. ROBICHAUD, P.A., BOISVERT, M. and ROBERT, J., 'Signal Flow Graphs and Applications', Prentice-Hall, Englewood Cliffs, 1962.
37. COATES, C.L., 'Flow graph solutions of linear algebraic equations', Inst. Radio Engineers, Trans. CT-6, pp.170-187, 1959.
38. ROE, P. and TRIBUS, M., 'Prediction of response of linear systems with uncertain parameters', Proc. of Int. Conf. on Microwaves, Circuit Theory and Information Theory (Part 2) Tokyo, pp.35-36 1964.
39. ROE, P.H. and SETH, A.K., 'Approximation of network response statistical moments in terms of component moments' 14th Mid-West Symp. on Circuit Theory, Denver, pp.15.1-1 to 15.1-13, 1971.
40. GUTCHE, H., 'The calculation of probability densities of system functions of electrical networks by expansion in terms of Hermite and Laguerre polynomials with use of approximations of second order', IEEE Int. Symp. on Network Theory, London, pp.109-110, 1971.
41. TINNEY, W.F. and WALKER, J.W., 'Direct solution of sparse matrix equations by optimally ordered triangular factorisation', Proc. IEEE, Vol.55, pp.1801-1809, 1967.
42. Control Data 6000 Series Computer Systems, Kronos 2.0 Instant Manual pp.5.19-5.20, 1971.
43. VILLALAZ, P.A., GODDARD, P.J. and SPENCE, R., 'Method for the efficient computation of large-change sensitivity for linear non-reciprocal networks', Electronics Letters, Vol. 7, No. 4, pp.112-113, 1971.
44. GOODWIN, E.T. et al, 'Modern Computing Methods', H.M.S.O. London pp.1-21, 1961.
45. CHEN, R.M.M., 'Solving a class of large sparse linear systems of equations by partitioning', Proc. of 1973 IEEE Int. Conf. on Circuit Theory, Toronto, pp.223-226.

46. BOURNE, S., 'Cambridge University Algebra System Manual', 1970.
47. DOWNS, T., 'Symbolic evaluation of transmittances from the nodal matrix', *Electronic Letters*, No.5, pp.379-380, 1969.
48. FIDLER, J.K. and NIGHTINGALE, C., 'Differential incremental sensitivity relationships', *Electronics Letters*, Vol.8, No.25, pp.626-627, 1972.
49. BOYCE, A.H., Marconi Research Laboratories, Chelmsford, England, Private Communication.
50. EDWARDS, A.L., 'Experimental Design in Psychological Research', Third Edition, Holt, Rinehart and Winston, London 1967.
51. TEMES, G.C., EBERS, R.M. and GADENZ, R.N., 'Some novel aspects of the adjoint network concept in frequency-domain analysis and optimisation', *IEEE Int. Symp. on Electric Network Theory*, London, pp.111-112, 1971.
52. BOHLING, D.H. and O'NEILL, L.A., 'An interactive computer approach to tolerance analysis', *IEEE Trans. on Computers*, Vol.CT-19, pp.10-16, 1970.
53. SUD, D., 'Inexpensive Monte Carlo analysis', *Colloquium Digest on Computer-Aided Circuit Design*, London IEE, March 1974, pp.1-2.
54. SETH, A.K. and ROE, P.H., 'Selection of component tolerances for optimum circuit reproducibility', *IEEE Int. Symp. on Electrical Network Theory*, London 1971, pp.105-106.
55. PINEL, J.F. and ROBERTS, K.A., 'Tolerance assignment in linear networks using nonlinear programming', *IEEE Trans. Circuit Theory*, Vol. CT-19, pp.475-579, 1972.
56. BANDLER, J.W. and LIU, P.C., 'Automated network design with optimal tolerances', *Proc. IEEE Int. Symp. on Circuit Theory*, Toronto, 1973, pp.181-184.
57. KARAFIN, B.J., 'The optimal assignment of component tolerances for electrical networks', *B.S.T.J.*, Vol. 50, No.4, pp.1225-1242, 1971.
58. GODDARD, P.J. and SPENCE, R., 'Efficient method for the calculation of first- and second-order sensitivities', *Electronic Letters*, Vol.5, No.16, pp.351-352, 1969.
59. PARKER, S.R., 'Sensitivity: old questions, some new answers', *IEEE Trans. Circuit Theory*, Vol.CT-18, pp.27-35, 1971.
60. ORCHARD, H.J., 'Inductorless filters', *Electronic Letters*, Vol.2, No.6, pp.224-225, 1966.
61. SUD, D. and SPENCE, R., 'Component tolerance assignment and design centering', 1974 European Conf. on Circuit Theory and Design, IEE Conf. Pub. No.116, pp.165-170.

62. DIRECTOR, S.W. and ROHFER, R.A., 'The generalised adjoint network and network sensitivities', IEEE Trans. Circuit Theory, Vol.CT-16, pp.318-323, 1969.
63. PINEL, J.F., 'Active filter design for a tone receiver!', 1974 European Conf. on Circuit Theory and Design, IEE Conf. Pub. No.116, pp.342-347.
64. 'Electronic Component Catalogue', Conway Electronics Ltd., U.K. pp.8-9, 1973.
65. CLEARY, J.F., Editor, 'GE Transistor Manual', General Electric Co., New York, p.11, 1964.
66. VILLALAZ, P.A., GODDARD, P.J. and SPENCE, R., 'A new tool for predicting allowed spreads in network elements and parameters', NTZ, Vol.10, pp.526-528, 1971.
67. SPENCE, R. and CONSTANTINIDES, A.G., 'Large-change sensitivity of digital filters', Colloquium Digest on Digital and Distributed Filters, London 1973, IEE, 8/3.
68. DIRECTOR, S.W., 'LU-factorisation in network sensitivity computation', IEEE Trans. Circuit Theory, Vol.CT-18, pp.184-185, 1971.
69. FAZARINC, Z., 'Design-oriented CAD', IEEE Trans. Circuit Theory, Vol. CT-20, pp.673-682, 1973.
70. ABBAS, G. and NICKOLS, K., 'Compact transient analysis program', Electronic Letters, Vol.8, pp.3-4, 1972.
71. SPENCE, R. and DREW, A.J., 'Graphical exploration in electrical circuit and model design', Pre-seminar Digest, Second Man-Computer Communications Seminar, Interactive Graphics, Radio and E.E. Division, N.R.C., May-June 1971.
72. MARTIN, J., 'Design of man-computer dialogues', Prentice-Hall, Englewood Cliffs, 1973.
73. JENSEN, R.W. and LIEBERMAN, M.D., 'IBM Electronic Circuit Design Program (ECAP)', Prentice-Hall, Englewood Cliffs, 1968.
74. MILLER, G.A., 'The magic number seven, plus or minus two: some limits on our capacity for processing information', Psychological Review, Vol.63, pp.81-87, 1956.
75. APPERLEY, M.D. and SPENCE, R., 'On the use of interactive graphics in circuit design', IEEE Int. Symp. on Circuit Theory and Systems, San Francisco, pp.558-563, 1974.
76. JANE, R. and WILBER, D., 'Computer-aided design of printed circuit networks in computer-oriented circuit design', F.F. Kuo and W.G. Magnuson, Eds., Prentice-Hall, Englewood Cliffs, N.J. pp.497-521, 1969.

77. ENGL, N.L. and MLYNSKI, D.A., 'Topological theory of integrated circuits', Proc. NATO Advanced Study Institute on Network and Signal Theory, pp.163-174, 1972.
78. BUTLER, E.M., 'Realistic design using large change sensitivity and performance contours', IEEE Trans. Circuit Theory, Vol. CT-18, pp.58-66, 1971.
79. VILLALAZ, P.A., 'Method for the efficient calculation of performance contours', Electronic Letters, Vol.8, pp.575-577, 1972.
80. FORSYTHE, A.R., 'Theory of functions of two complex variables', Cambridge University Press, 1914.
81. BELL, E.T., 'Men of Mathematics', Pelican Books, 1953.
82. VILLALAZ, P.A. and SPENCE, R., 'Scheme for the elimination of redundant model complexity', Electronic Letters, Vol.8, No.2, pp.38-40, 1972.
83. KUHN, H.W. and TUCKER, A.W., 'NLP, Proc. of the Second Berkeley Symposium on Mathematics, Statistics and Probability', University of California Press, pp.481-482, 1951.
84. FIACCO, A. and McCORMICK, P.G., 'Nonlinear Programming: Sequential Unconstrained Minimisation Techniques', Wiley, New York, 1968.
85. FLETCHER, R. and REEVES, C.M., 'Function minimisation by conjugate gradients', Computer Journal, Vol.7, pp.149-154, July 1964.
86. FLETCHER, R. (ed.), 'Optimisation', Academic Press Inc. London 1969.
87. BROYDEN, C.G., 'Quasi-Newton methods and their application to function minimisation', Math. Comp. Vol.21, p.368, 1967.
88. FLETCHER, R. and POWELL, M.J.D., 'A rapidly convergent decent method for minimisation', Computer Journal, Vol. 6, pp.163-168, June 1963.
89. FLETCHER, R., 'A new approach to variable metric algorithms', Computer Journal, Vol.13, pp.317-322, 1970.
90. GILL, P.E. and MURRAY, W., 'Quasi-Newton methods for unconstrained optimisation', NPL Report, Maths. 97, April 1971.
91. POWELL, M.J.D., 'An efficient method for finding the minimum of a function of several variables without calculating the derivatives', Computer Journal, Vol.7, pp.155-162, 1964.
92. POWELL, M.J.D., 'An iterative method for finding the stationary values of a function of several variables', Computer Journal, Vol.5, No.2, pp.147-151, 1962.
93. ZANGWILL, W.I., 'Minimisation of a function without calculating derivatives', Computer Journal, Vol.10, pp.293-296, 1967.

94. SPENDLEY, W., HEXT, G.R. and HIMSWORTH, F.R., 'Sequential applications of simplex designs in optimisation and evolutionary operations', *Technometrics*, Vol.4, p.441, 1962.
95. NELDER, J.A. and MEAD, R., 'A simplex method for function minimisation', *Computer Journal*, Vol.1, pp.308-313, 1965.
96. VAJDA, S., 'Mathematical Programming', Addison Wesley, Mass., U.S.A., 1961.
97. MURTAGH, B.A. and SARGENT, R.W.H., 'A constrained minimisation method with quadratic convergence', in *Optimisation*, R. Fletcher (Ed.) Academic Press, London 1969.
98. CARRAL, C.W., 'The created response technique for optimising non-linear constrained systems', *Operations Research*, Vol.9, No.2, pp.169-184, 1961.
99. PAVIANI, D. and HIMMELBLAU, D.M., 'Constrained nonlinear optimisation by heuristic programming', *J. Oprs. Res. of America*, Vol.17, No.5, pp.872-882, 1969.
100. HIMMELBLAU, D.M., 'Applied Nonlinear Programming', McGraw-Hill, New York, 1972.
101. FRISH, R., 'The logarithmic potential method for solving linear programming problems', *Memorandum*, The University Institute of Economics, Oslo, 1955.
102. COURANT, R., 'Variational methods for the solution of problems of equilibrium and vibrations', *Bull. Am. Math. Soc.*, Vol.49, p.1, 1943.
103. BANDLER, J.W., 'Optimisation methods for computer-aided circuit design', *IEEE Trans. Microwaves*, Vol.MTT-17, pp.533-551, 1969.
104. LEUNG, K.H., Private communication, 1973.
105. SUD, D. and SPENCE, R., 'Sensitivity invariants in electric circuits', To appear in a forthcoming issue of NTZ.
106. SUD, D., 'Differential sensitivity after large changes in network elements', To appear in a forthcoming issue of the *Int.J. of Circuit Theory and Applications*.