

Interest-rate Models: an Extension to the Usage in the Energy Market and Pricing Exotic Energy Derivatives

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Abstract

In this thesis, we review various popular pricing models in the interest-rate market. Among these pricing models, we choose the LIBOR Market model (LMM) as the benchmark model. Based on market practice experience, we also develop a pricing model named the “Market volatility model”. By pricing vanilla interest-rate options such as interest-rate caps and swaptions, we compare the performance of our Market volatility model to that of the LMM. It is proved that the Market Volatility model produce comparable results to the LMM, while its computing efficiency largely exceeds that of the LMM.

Following the recent rapid development in the commodity market, in particular the energy market, we attempt to extend the use of our proposed Market volatility model from the interest-rate market to the energy market. We prove that the Market Volatility model is capable of pricing various energy derivative under the assumption of absence of the convenience yield. In addition, we propose a new type of exotic energy derivative which has a flexible option structure. This energy derivative is named as the Flex-Asian spread options (FASO). We give examples of different option structures within the FASO framework and use the Market volatility model to generate option prices and greeks for each structure.

Although the Market volatility model can be used to price various energy derivatives based on oil/gas contracts, it is not compatible with the structure of one of the most advanced derivatives in the energy market, the storage option. We modify the existing pricing model for storage options and use our own 3D-binomial tree approach to price gas storage contracts. By doing these, we improve the performance of the traditional storage model.

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1. Introduction

In 1973, Black and Scholes published their famous research paper which became a milestone of the modern financial theory. In this paper, Black and Scholes derived a closed-form pricing formula for pricing a European-style call option. Since then, research in the derivative pricing area has developed rapidly.

Among various financial markets, the interest-rate market is one of the most complicated markets. A good interest-rate pricing model has to model the “term-structure” of interest rates. Hence, instead of forecasting the performance at one particular time point of the interest-rate yield curve, we need to forecast the dynamic of the whole yield curve. The correct modeling of the associated volatility structure of this interest-rate yield curve will be a difficult task.

Ho (1995) concluded that there are three main types of models: modified Black-Scholes models, interest-rate models, and arbitrage-free models. In addition, interest-rate models and arbitrage-models can be sorted into the category of normal models and lognormal models. Unlike the equity-derivative field, we cannot specify any interest rate model as a "standard" model. Although a model may be sophisticated, it may not be the most appropriate one to use. For instance, traders require fast understanding of the formulation and efficiency for its implementation, so they usually prefer to use a simple bond model. In contrast, portfolio managers prefer an arbitrage-free model because the calibration instruments for the model are the market prices of benchmark bonds which can be easily observed. It is important to choose the appropriate model for different situations.

Although there is considered to be no general model for the interest-rate modelling sector, the LIBOR market models have become very popular. The reason for such popularity of market models is that they agree with the Black’s pricing formula under the assumption of log-normal distribution of the underlying forward rates. Using Black’s formula, a closed-form solution can be easily derived for the value of the product that we want to price. Before the introduction of market models, no interest-rate model is compatible with Black’s pricing formula for either caps or swaptions.

One short-coming of Black’s model is that it assumes the underlying volatility is constant over the life of the derivative, which is unaffected by changes in the price level of the underlying. This is obviously not a realistic assumption according to the market data. One solution is to use Stochastic Volatility models instead of the Black’s model. In Stochastic Volatility models, volatility of the underlying price follows a stochastic process, which allows us to model derivatives more accurately. Hagan et al. (2002) proposed a well-known Stochastic Volatility Model, named the SABR model.

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The SABR model soon becomes a popular pricing model for Vanilla options in the financial market. Similar to the LMM model, the calibration of the SABR model becomes very complicated when it comes to exotic-type options. In order to overcome the drawbacks of the LMM and SABR models, we propose a simple but intuitive model which is called the “Market Volatility model”, based on the interpolation of market quoted volatilities.

Like the interest-rate market, the commodity market is one of the oldest and most important financial markets. During year 2007-2009, the most recent financial crisis (the credit crisis) happened in all the world’s main economies, because of the collapse of the over-heated housing market which leads to subsequently high default rate in the credit derivative market. As a result, risk-averse investors sought a safe investment and switched from the credit market to the commodity market, more specifically, into the energy market. The trading volume in the energy market expanded dramatically during year 2007-2008. For example, data from the Commodity Futures Trading Commission (CFTC) showed that investment into commodities indices has jumped from less than \$10bn in 1998 to about \$235bn in mid-2008. As the energy/commodity market is one of the oldest and most important financial markets, it is essential for us to look into this market and research. Many energy derivatives have structures arising from the interest-rate (fixed-income) market. However, pricing activity in the energy market involves the effect of the so-called “convenience yield”, which is the extra cost caused by holding/storing a physical commodity. Moreover, a number of unique types of energy derivatives such as swing options and storage options exist only in this market due to the special characteristics of this market. Furthermore, the energy market is much more volatile than any other financial market. All these factors increase the difficulty of correctly modeling the price dynamics of an energy derivative.

Our work involves the following three contributions:

- We develop a robust model for use in both the interest-rate markets and the energy markets. This model is not covered in any literature, instead, it is developed based on market practice and trading experience.
- We propose a new type of option with a flexible structure, in order to capture the characteristics of the energy market.
- We modify the calibration procedure of the existing model for storage options to improve the model performance, moreover, we construct a new pricing approach for pricing storage options.

In Chapter 2 of this thesis, we review the two most popular term-structure models for pricing interest-rate derivatives, which are the LIBOR Market model (LMM) and the Stochastic- $\alpha\beta\rho$ (SABR) model. We then proposed a new pricing model which is referred to as the “Market volatility model”. The Market volatility model is an intuitive model arising from the real trading experience.

In Chapter 3, we test the performance of the Market volatility model for pricing vanilla interest-rate options such as interest-rate caps and swaptions. We use the prices output by the LMM for same

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set of interest-rate options as a benchmark. In addition, we use the Market volatility model to price exotic interest-rate derivatives such as CMS and CMS spread options.

In Chapter 4, we give a brief introduction to the two most important energy markets, the oil market and the natural gas market. By assuming the absence of the convenience yield, we extend the Market volatility model in Chapter 2 to be used in the energy markets. In Chapter 5, we use the Market volatility model to price a number of commonly-used energy derivatives, such as spread options, basket options and Asian options. We then propose a new type of energy derivative (the Flex-Asian spread options) with flexible option structure, which can be used to deal with the various tenor structures of a desired option. We showed four possible structures of the Flex-Asian spread option and we use the Market volatility model to generate option prices and greeks for each structure.

For the standard Asian-type options and their hybrids on oil/gas contracts, the Market volatility model works well. However, it is not compatible with the structure associated to one of the most advanced derivatives in the energy market, which is the storage option. In Chapter 6, we review various pricing methods for pricing storage options. We choose to use the “basket of spread options” technique to value storage options. From a base of the original method, we improve the model by modifying the volatility and correlation calibration procedures in order to account for the seasonality effect. In this way, the calibration results from the pricing model is improved. In addition, we develop our own 3D-binomial tree approach to price calendar spread options, instead of using the classical Kirk’s approximation. With our 3D-binomial tree approach, the presence of the skewness and fat-tail effect in the distribution of the storage prices can be captured in an exact way. At the end of Chapter 6, we use the 3D-binomial tree approach to price storage options based on the NYMEX data.

2. Interest-Rate Models

2.1. The LIBOR Market Model

Interest-rate modeling is a complicated topic, and it is widely regarded that a general pricing model does not exist. However, the LIBOR market model is commonly used in market practice today. The reason for the popularity of the LIBOR market model is that it is compatible with the Black's pricing formula (by assuming the underlying forward rates follow log-normal distributions), where none of the previous interest-rate model does. When using the Black's pricing formula, the price of a financial product can be derived easily as a closed-form solution. Therefore, closed-form solutions for prices of interest-rate caps and swaptions can be obtained. Since the caps and swaptions are two most important and actively-traded products in the interest-rate market, a model able to price caps and swaptions is desirable.

For pricing LIBOR-related derivatives in the real market, the log-normal LIBOR market models (LMM) are commonly used. Compared to other models, LMM has the following advantages:

- The underlying forward rates are assumed to be log-normally distributed, which excludes the probability of negative interest-rates.
- LMM uses market quoted rates for modeling purposes, which are easy to obtain.
- Bond market operates under the no-arbitrage assumption, which means that the same assumption is true for LMM

The log-normal market models (LMM) are based on the assumption that the underlying forward LIBOR/Swap rates are log-normally distributed. The traditional interest-rate models use continuous-compounding instantaneous rates as underlings, whereas the log-normal market models use simple-compounding market quoted rates as underlying. Market quoted rates are reset periodically. The most commonly used frequencies for resets are annually (reset every year), semi-annually (reset every six months) and quarterly (reset after every three months) . The time interval between the reset dates is called the *tenor*.

To start working with LMM, there are two main modeling approaches. The first approach is to derive market quoted rates from an underlying model. The BGM approach (by Brace, Gatarek and Musiela (1997)) is of this type. The BGM approach restricts the market quoted rate of a chosen tenor to be log-normal, in this way, the processes of market quoted rate are derived under

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the HJM framework which is defined in the next section. The imposed restrictions guarantee that caplets on rates of the chosen tenor is compatible with the Black's pricing formula. The second approach employ the "change of numeraire" method to find a martingale measure for the market quoted rates, so that the market rates can be modeled directly. The MR approach (by Musiela and Rutkowski (1997) or Rutkowski (1997)) is of this type. Same as the first approach, MR approach is built on the assumption of log-normally distributed underlying forward rates, which ensures that the model is compatible with the Black's formula.

For both approaches, the state variables can be expressed as a function of the corresponding volatilities or correlations. However, the combination of the volatilities or correlations varies with different choice of numeraires.

2.1.1. The Heath-Jarrow-Morton (HJM) framework

Heath, Jarrow and Morton (1992) proposed the HJM methodology, which combines several previous models into a joint term-structure modeling approach. The HJM approach is an important achievement in the interest-rate modeling area (under the no-arbitrage assumption), used for modeling instantaneous short rates and pricing interest-rate derivatives.

In the HJM approach, the instantaneous continuous-compounding forward rates $f(t, T)$ are specified to follow the process below

$$df(t, T) = \vec{a}(t, T) dt + \sigma(t, T) dW_P(t) \quad (2.1)$$

for any fixed maturity $T \leq \hat{T}$, with $\hat{T}$ as the maturity of the latest tenor that we considered. $\vec{a} \in \mathbb{R}$ and $\sigma \in \mathbb{R}^d$ are adapted stochastic processes. W is the d -dimensional standard Brownian motion with respect to the underlying real-world probability measure P . For every fixed $T \leq \hat{T}$ with $T \geq 0$, integrating both side of equation (2.1) from time 0 to t , we have

$$f(t, T) = f(0, T) + \int_0^t \vec{a}(s, T) ds + \int_0^t \sigma(s, T) dW(s) \quad (2.2)$$

Let us consider a unit zero-coupon bond maturing at time $T \leq \hat{T}$. The price of this bond at time $t \leq T$ is denoted by $B(t, T)$. $B(t, T)$ is governed by the following process

$$B(t, T) = e^{-\int_t^T f(t, s) ds} \quad (2.3)$$

By definition, the bond market is arbitrage-free. Therefore, we should be able to find a martingale measure. Under this martingale measure, the drift coefficient α of the instantaneous forward rate

2. Interest-Rate Models

is uniquely determined by the volatility coefficient and the market price of risk for interest-rate¹. We denote this martingale measure for the bond market by Q , and the corresponding Brownian motion by W_Q . Then,

$$\frac{dB(t, T)}{B(t, T)} = \bar{a}(t, T) dt + \bar{\sigma}(t, T) dW_Q(t) \quad (2.4)$$

where

$$\bar{a}(t, T) = f(t, t) \quad (2.5)$$

$$\bar{\sigma}(t, T) = - \int_t^T \sigma(t, s) ds \quad (2.6)$$

In the special case, volatility coefficient σ follows a deterministic function, so $\bar{\sigma}$ will also be deterministic. In this case, the pricing formula for interest-rate options will not depend on the choice of the risk premium.

Although the HJM approach appeared to be very successful theoretically, one drawback of this model is that it is based on arbitrage-free dynamics of the instantaneous continuous-compounding forward rates. By definition, the forward LIBOR rate over the future period $[T, T + \varepsilon]$ observed at time t satisfies the following relationship with bond prices:

$$1 + \varepsilon F(t, T) = \frac{B(t, T)}{B(t, T + \varepsilon)} \quad (2.7)$$

or equivalently,

$$F(t, T) = \frac{B(t, T) - B(t, T + \varepsilon)}{\varepsilon \cdot B(t, T + \varepsilon)} \quad (2.8)$$

Equation (2.8) shows that, if the volatility coefficient of bond prices follows a deterministic function, then the volatility of the forward LIBOR rate is not deterministic. We cannot derive a closed-form formula for caplet prices using the Black's formula under this framework. Hence, the HJM approach is not convenient in some circumstances.

As a result, an alternative term structure model (under the no-arbitrage dynamics) that is compatible with the Black's formula is desirable, which can be used for pricing standard interest-rate options. In the construction of this kind of models, the forward LIBORs will be modeled directly, instead of modeling the instantaneous rate.

¹as in reference [46]

2. Interest-Rate Models

2.1.2. Log-normal LIBOR Models

Following the conclusion in section 2.1.1, we describe several alternative approaches for modeling the forward LIBOR rates in this section, both in the continuous- and discrete-tenor setups. Before any further discussion, we introduce some notation.

Assume that we have a set of reset/settlement dates $\{T_i; i = 0, \dots, n\}$, with $0 < T_0 < T_1 < \dots < T_n = \hat{T}$. $\{T_i\}$ is referred to as the *tenor structure*. The length of time interval τ_i between two reset dates is $\tau_i = T_{i+1} - T_i$, for $i = 0, 1, 2, \dots, n-1$. Let $B(t, T_i)$ denotes the price at time t of a zero-coupon bond mature at time T_i , P denotes the real-world probability measure, and $P(T_i)$ denotes the forward martingale measure associated with the date T_i for $i = 0, 1, 2, \dots, n$. The corresponding d -dimensional Brownian motions are W_P and $W_P(T_i)$, respectively. Suppose we have a T_{i+1} -maturity zero-coupon bond with T_i as the settlement date, then the forward price f_B of this zero-coupon bond observed at time t is

$$f_B(t, T_{i+1}, T_i) = \frac{B(t, T_{i+1})}{B(t, T_i)}, \quad \forall t \in [0, T_i] \quad (2.9)$$

We will also introduce the definition of the Doléans exponential ζ , that is,

$$\zeta_t \left(\int_0^\bullet \eta_s dW_P(s) \right) = \exp \left(\int_0^t \eta_s dW_P(s) - \frac{1}{2} \int_0^t |\eta_s|^2 ds \right) \quad (2.10)$$

where “ $\bullet$ ” and “ $|\dots|$ ” denote the inner product and Euclidean norm in $\mathbb{R}^d$, respectively.

2.1.2.1. Determining the No-Arbitrage Drifts of Forward Rates

The LIBOR market model can be used to price certain type of derivatives given that particular measurability conditions are true. To price these products, a finite set of spanning forward rates and the present value of a chosen numeraire are required. When the forward rates follow the log-normal distribution simultaneously, their evolution could be expressed using a set of expectation vectors and the corresponding covariance matrices. To ensure this result, a necessary condition is that under the no-arbitrage assumption, drift coefficients of the forward rates must be a deterministic function of time.

Rebonato (2002) derived functional forms for the drift coefficients of the forward rates for varying numeraires. In general, Rebonato concluded that the drift coefficients are stochastic due to an embedded time dependence. In addition, the drift coefficients also depend explicitly on the set of the forward rates. In Rebonato’s method, the joint distribution of the forward rates is not conditionally log-normal. To solve this problem, Rebonato used the Runge-Kutta weak approximation method,

2. Interest-Rate Models

so the problem can be approximated into the evolution of log-normal variables. In this way, the results will remain to be valid.

Suppose that the instantaneous forward rate has the following dynamic under the real measure P :

$$df(t, T) = \vec{a}(t, T) dt + \sigma(t, T) dW_P(t), \quad t \leq T \quad (2.11)$$

with $W_P(t)$ as the d -dimensional standard independent Brownian motion; $\sigma(t, T)$ is a row vector of volatilities. From section 2.1.1, we know that the dynamic of the zero coupon bond price is:

$$\frac{dB(t, T)}{B(t, T)} = \bar{a}(t, T) dt + \bar{\sigma}(t, T) dW_P(t) \quad (2.12)$$

with $\bar{\sigma}(t, T)$ as the volatility row vector. From the HJM framework, we have that

$$\bar{a}(t, T) = f(t, T) - \int_t^T \vec{a}(t, u) du + \frac{1}{2} \int_t^T \sigma(t, u)^2 du \quad (2.13)$$

$$\bar{\sigma}(t, T) = - \int_t^T \sigma(t, u) du \quad (2.14)$$

Assume that we have a numeraire U , which is governed by the following process

$$\frac{dU}{U} = \mu_U dt + \sigma_U dW_P(t) \quad (2.15)$$

By definition, given some existing process g_t ($g_t \in \mathbb{R}^d$), if a measure Q satisfies

$$\kappa(T) = \frac{dQ}{dP} \Big|_{\mathcal{F}_t} = e^{-\int_t^T g_s dW_P(s) - \frac{1}{2} \int_t^T g_s^2 ds}, \quad P - a.s. \quad (2.16)$$

then Q is said to be an *equivalent measure* of P . $\kappa(T)$ is named the *Radon-Nikodym derivative*. g_t is referred to as the market price of risk. By the Girsanov's theorem, W_Q is a Brownian motion under the measure Q , and

$$W_Q(t) = W_P(t) + \int_0^t g_s ds \quad (2.17)$$

In order to guarantee the no-arbitrage dynamic of the bond market, we need to choose a suitable numeraire, under which a martingale measure will exist. For this purpose, we will test $\frac{B(t, T)}{U}$ to see if it is a martingale under measure Q .

By applying Ito's formula, we have

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$$\begin{aligned}
d\left(\frac{B(t, T)}{U}\right) &= \frac{1}{U}dB(t, T) - \frac{B(t, T)}{U^2}dU + \frac{1}{2}\left(\frac{2B(t, T)}{U^3} \cdot (dU)^2 - \frac{2}{U^2} \cdot dB(t, T) \cdot dU\right) \\
&= \frac{B(t, T)}{U} \cdot \left\{ [\bar{a}(t, T) - \mu_U + \sigma_U^2 - \bar{\sigma}(t, T) \cdot \sigma_U] dt + [\bar{\sigma}(t, T) - \sigma_U] dW_P(t) \right\} \\
&= \frac{B(t, T)}{U} \cdot \left\{ [\bar{a}(t, T) - \mu_U + (\sigma_U - g_s) \cdot (\bar{\sigma}(t, T) - \sigma_U)] dt + [\bar{\sigma}(t, T) - \sigma_U] dW_Q(t) \right\}
\end{aligned} \tag{2.18}$$

The no-arbitrage condition implies that

$$\bar{a}(t, T) = \mu_U + (\sigma_U - g_s) \cdot (\bar{\sigma}(t, T) - \sigma_U) \tag{2.19}$$

Under the no-arbitrage condition (2.19), $\frac{B(t, T)}{U}$ becomes a martingale under measure Q , governed by the following process

$$d\left(\frac{B(t, T)}{U}\right) = \frac{B(t, T)}{U} \cdot [\bar{\sigma}(t, T) - \sigma_U] dW_Q(t) \tag{2.20}$$

If the forward LIBOR rate $F(t, T)$ for a chosen tenor is a strictly positive semi-martingale, then $x_k = 1 + \tau \cdot F(t, T)$ is also a strictly positive semi-martingale. x_k follows the dynamic

$$\frac{dx_k}{x_k} = \mu_{x_k} dt + \sigma_{x_k}(t) dW_k(t) \tag{2.21}$$

By (2.7), we have

$$1 + \tau \cdot F(t, T) = \frac{B(t, T)}{B(t, T + \tau)} \tag{2.22}$$

Under the real measure P , the forward LIBOR rate $F(t, T_i) = F_i(t)$ has the following dynamic

$$\frac{dF_i(t)}{F_i(t)} = \mu_i dt + \sigma_i \cdot \tilde{b}_i(t) \cdot dW_P(t), \quad i = 0, \dots, n-1 \tag{2.23}$$

Let us chose the payoff U_i of the forward rate F_i as the numeraire. By (2.20),

$$d\frac{B(t, T_i)}{B(t, T_{i+1})} = [\bar{\sigma}(t, T_i) - \bar{\sigma}(t, T_{i+1})] dW_Q(t) \tag{2.24}$$

Define $\tau_i = T_{i+1} - T_i$. By applying the Ito's formula on both side of equation (2.22), we have

$$\begin{aligned}
\tau_i \cdot dF_i(t) &= d\left(\frac{B(t, T_i)}{B(t, T_{i+1})}\right) \\
&= \frac{B(t, T_i)}{B(t, T_{i+1})} [\bar{\sigma}(t, T_i) - \bar{\sigma}(t, T_{i+1})] dW_Q(t)
\end{aligned} \tag{2.25}$$

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$$\Rightarrow dF_i(t) = \frac{1 + \tau_i F_i(t)}{\tau_i} \cdot [\bar{\sigma}(t, T_i) - \bar{\sigma}(t, T_{i+1})] dW_Q(t) \quad (2.26)$$

Changing the measure will affect the the drift of the process, where the volatility of the process remains the same. By equating the volatility coefficients of equation (2.23) and (2.26), we have

$$\sigma_i \cdot \tilde{b}_i(t) \cdot F_i(t) = \frac{1 + \tau_i F_i(t)}{\tau_i} \cdot [\bar{\sigma}(t, T_i) - \bar{\sigma}(t, T_{i+1})] \quad (2.27)$$

$$\Rightarrow \bar{\sigma}(t, T_i) - \bar{\sigma}(t, T_{i+1}) = \frac{\sigma_i \cdot \tilde{b}_i(t) \cdot F_i(t) \cdot \tau_i}{1 + \tau_i F_i(t)} \quad (2.28)$$

If we chose $B(t, T_n)$ to be the numeraire, then

$$dF_i(t) = \frac{1 + \tau_i F_i(t)}{\tau_i} \cdot [\bar{\sigma}(t, T_i) - \bar{\sigma}(t, T_{i+1})] \cdot \{[\bar{\sigma}(t, T_n) - \bar{\sigma}(t, T_{i+1})] dt + dW_Q(t)\} \quad (2.29)$$

and by (2.28) we have

$$\bar{\sigma}(t, T_n) - \bar{\sigma}(t, T_{i+1}) = - \sum_{j=i+1}^{n-1} \frac{\sigma_j \cdot \tilde{b}_j(t) \cdot F_j(t) \cdot \tau_j}{1 + \tau_j F_j(t)} \quad (2.30)$$

Therefore, equation (2.29) becomes

$$dF_i(t) = \frac{1 + \tau_i F_i(t)}{\tau_i} \cdot \frac{\sigma_i \cdot \tilde{b}_i(t) \cdot F_i(t) \cdot \tau_i}{1 + \tau_i F_i(t)} \cdot \left(- \sum_{j=i+1}^{n-1} \frac{\sigma_j \cdot \tilde{b}_j(t) \cdot F_j(t) \cdot \tau_j}{1 + \tau_j F_j(t)} dt + dW_Q(t) \right) \quad (2.31)$$

$$\Rightarrow \frac{dF_i(t)}{F_i(t)} = \sigma_i \cdot \tilde{b}_i(t) \cdot \left(- \sum_{j=i+1}^{n-1} \frac{\sigma_j \cdot \tilde{b}_j(t) \cdot F_j(t) \cdot \tau_j}{1 + \tau_j F_j(t)} dt + dW_Q(t) \right) \quad (2.32)$$

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2.1.2.2. The Miltersen-Sandmann-Sondermann Approach

Miltersen et al. (1997) first attempted to construct a log-normal model of forward LIBOR rates. They start by assuming that the forward LIBOR rate $F(t, T)$ satisfies

$$dF(t, T) = \mu(t, T)dt + F(t, T)h(t, T)dW_P(t) \quad (2.33)$$

where $h(t, T)$ is the deterministic volatility function with values in $\mathbb{R}^d$. Suppose we have a zero-coupon bond mature at time $T + \tau$, with settlement date T . The forward price f_B of this zero-coupon bond observed at time t is

$$f_B(t, T + \tau, T) = \frac{B(t, T + \tau)}{B(t, T)} \quad (2.34)$$

Then f_B satisfies the following SDE under the forward measure $P(T)$:

$$\frac{df_B(t, T + \tau, T)}{f_B(t, T + \tau, T)} = -(1 - f_B(t, T + \tau, T))h(t, T)dW(t; T) \quad (2.35)$$

where $W(t; T) = W_Q(t) - \int_0^t \bar{\sigma}(s, T)ds$ for all $t \in [0, T]$.

Let $V_b = V_b(t, x)$ denote the functional form of the forward price of a put option on bond. V satisfied the partial differential equation

$$\frac{\partial V_b}{\partial t} + \frac{1}{2}|h(t, T)|^2 \cdot x^2 \cdot (1 - x)^2 \frac{\partial^2 V_b}{\partial x^2} = 0 \quad (2.36)$$

with the terminal condition $V_b(T, x) = \max(0, K - x)$. By solving the PDE (2.36), Miltersen et al. (1997) obtained the closed-form expression for the price of a bond option. Moreover, they derived a closed-form solution for the price of a caplet².

2.1.2.3. The Brace-Gatarek-Musiela Approach

In order to apply the Black's formula to LIBOR-related derivatives, researchers started to make attempt to model discretely compounded rates as log-normal processes directly. Based on the HJM approach, Brace, Gatarek, and Musiela (1997) derived the no-arbitrage conditions and the dynamics of the forward LIBOR rates, which is referred to as the Brace-Gatarek-Musiela (BGM) approach. Alternatively, it is known as the log-normal forward-LIBOR model (LGM). In the BGM approach, a whole yield curve is used, consistent with market pricing practice. This approach is compatible with the Black's formula, so the calibration to caps and swaptions is straight forward.

Given a family of bond prices $B(t, T)$ and the corresponding forward processes $f_B(t, T, \bar{U})$. In contrast to section 2.1.2.2, we now assume that τ denotes the length of the fixed accrual period, τ

²as in reference [46]

2. Interest-Rate Models

is strictly positive. Hence, the forward LIBOR rate $F(t, T)$ for the future date $T \leq \hat{T} - \tau$ observed at time $t \in [0, T]$ satisfies the following relationship

$$1 + \tau F(t, T) = f_B(t, T, T + \tau) = \frac{B(t, T)}{B(t, T + \tau)} \quad (2.37)$$

such that

$$F(0, T) = \tau^{-1} \left(\frac{B(0, T)}{B(0, T + \tau)} - 1 \right) \quad (2.38)$$

By applying Ito's formula to both sides of (2.37) and using equation (2.24), we obtain

$$dF(t, T) = \tau^{-1} \cdot f_B(t, T, T + \tau) \cdot (\bar{\sigma}(t, T) - \bar{\sigma}(t, T + \tau)) dW(T + \tau) \quad (2.39)$$

Let $\gamma(t, T, T + \tau) = \bar{\sigma}(t, T) - \bar{\sigma}(t, T + \tau)$,

$$dF(t, T) = \tau^{-1} (1 + \tau F(t, T)) \cdot \gamma(t, T, T + \tau) dW(t; T + \tau) \quad (2.40)$$

subject to the initial condition (2.38). Under the normal market condition, the forward LIBOR rates $F(t, T)$ are assumed to be strictly positive, then we could re-write equation (2.40) as:

$$dF(t, T) = F(t, T) \bar{\gamma}(t, T) dW(t; T + \tau) \quad (2.41)$$

where

$$\bar{\gamma}(t, T) = \frac{1 + \tau F(t, T)}{\tau F(t, T)} \cdot \gamma(t, T, T + \tau) \quad (2.42)$$

Therefore, $\bar{\gamma}(t, T)$ is random if $\gamma(t, T, T + \tau)$ is deterministic and vice versa.

Based on the set-up of the HJM model, Brace et al. (1997) built a model of log-normal forward LIBOR rates which relies on several assumptions. The BGM model uses the forward induction method instead of backward induction. To start with, Brace et al. assumed that for every $T \in [0, \hat{T}]$, the volatility $\bar{\sigma}(t, T)$ vanishes for every $t \in [(T - \tau) \vee 0, T]$. The forward LIBOR rate $F(t, T)$ is governed by the following process

$$dF(t, T) = \mu(t, T) dt + F(t, T) \bar{\gamma}(t, T) dW_P(t) \quad (2.43)$$

under the real-world measure P ; where $\bar{\gamma}$ is a deterministic function and μ is an un-specified parameter. Under the no-arbitrage condition, the instantaneous forward rate $f(t, T)$ is governed by the following process

2. Interest-Rate Models

$$df(t, T) = \sigma(t, T) \cdot v(t, T)dt + \sigma(t, T)dW_{P_t} \quad (2.44)$$

with $v(t, T) = -\bar{\sigma}(t, T) = \int_t^T \sigma(t, s)ds$. Using equation (2.42), we have that

$$\begin{aligned} v(t, T + \tau) - v(t, T) &= \int_T^{T+\tau} \sigma(t, s) ds \\ &= \frac{\tau F(t, T)}{1 + \tau F(t, T)} \cdot \bar{\gamma}(t, T) \end{aligned} \quad (2.45)$$

Aiming to find a representation of v in terms of forward LIBORs F , Brace et al. assumed that the following relationship holds:

$$v(t, T) = -\bar{\sigma}(t, T) = \sum_{m=1}^{\tau^{-1}(T-t)} \frac{\tau F(t, T - m\tau)}{1 + \tau F(t, T - m\tau)} \cdot \bar{\gamma}(t, T - m\tau) \quad (2.46)$$

By using forward induction, it was shown that there exists unique solutions to the processes governing the instantaneous forward rate $f(t, T)$ and the forward LIBOR $F(t, T)$ associated with v . To conclude, Brace et al. deduced that $F(t, T)$ satisfies

$$dF(t, T) = F(t, T) \cdot v(t, \hat{T} + \tau) \cdot \bar{\gamma}(t, T)dt + F(t, T)\bar{\gamma}(t, T)dW_P(t) \quad (2.47)$$

under the real-world probability measure $\hat{P}$.

2.1.2.4. The Musiela-Rutkowski Approach

Musiela and Rutkowski (1997) and Rutkowski (1997) proposed the Musiela-Rutkowski (MR) approach. Unlike the BGM approach, the Musiela-Rutkowski (MR) approach is not based on the HJM framework. The MR approach uses market quoted rates as inputs and models them directly. Underlying instantaneous rates become unrelated and are not required in the pricing procedure. We only need to use the "change of numeraire" method to find a martingale measure, so that the processes followed by the market quoted rates become martingales under it.

Assume that we have a set of reset/settlement dates $\{T_i; i = 0, \dots, n\}$, with $0 < T_0 < T_1 < \dots < T_n = \hat{T}$. $\{T_i\}$ is referred to as the *tenor structure*. Let $T_{-1} = 0$. The length τ_i between two reset dates is $\tau_i = T_{i+1} - T_i$, for $i = 0, 1, 2, \dots, n-1$. For every $i = 0, 1, \dots, n$,

$$T_i = \sum_{j=0}^i \tau_j \quad (2.48)$$

For $k = 0, 1, \dots, n$,

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$$\widehat{T}_k = \widehat{T} - \sum_{j=n-k+1}^n \tau_j = T_{n-k} \quad (2.49)$$

For any $i = 0, 1, \dots, n-1$, we define the forward LIBORs to be

$$F(t, T_i) = \frac{B(t, T_i) - B(t, T_{i+1})}{\tau_i B(t, T_{i+1})}, \quad \forall t \in [0, T_i] \quad (2.50)$$

For any $i = 0, 1, \dots, n$, a probability measure P_{T_i} (on $(\Omega, \mathcal{F}_{T_i})$) which is an equivalent measure to the real-world probability measure P is a forward probability measure for T_i , such that the forward LIBOR $F(\cdot, T_i)$ follows a (local) martingale under P_{T_i} for the date T_{i+1} . If the above process is strictly positive, then we could follow the standard arguments to prove the existence of the corresponding volatility process.

Assume that there exists an exogenous volatility process $\theta(\cdot, T_i)$ for the corresponding forward LIBOR $F(\cdot, T_i)$ for any date T_i . θ can either be a deterministic function of time or the underlying forward LIBOR rates (with values in $\mathbb{R}^d$), or it can follow a d -dimensional (adapted) stochastic process. The processes of the volatilities of forward LIBORs are assumed to be bounded for simplicity³.

Assume that for $i = 0, 1, \dots, n-1$, we have a set of bounded (adapted) volatility processes $\theta(\cdot, T_i)$ of the associated forward LIBORs $F(\cdot, T_i)$, and a fixed initial term structure of interest rates specified by a family of bond prices $B(0, T_i)$, where $B(0, T_i) > B(0, T_{i+1})$.

Using the above assumptions, we aim to construct a family of forward LIBORs $F(\cdot, T_i)$ such that

$$dF(t, T_i) = F(t, T_i) \theta(t, T_i) dW_{T_{i+1}}(t), \quad \forall t \in [0, T_i] \quad (2.51)$$

with

$$F(0, T_i) = \frac{B(0, T_i) - B(0, T_{i+1})}{\tau_i B(0, T_{i+1})} \quad (2.52)$$

for any $i = 0, 1, \dots, n-1$. $W(T_i)$ is a family of processes which follow the d -dimensional standard Brownian motion under the (mutually equivalent) probability measure P_{T_i} , for $i = 0, 1, \dots, n$.

We start the construction of the model based on backward induction. Let us define the forward LIBOR rate with respect to the longest maturity T_{n-1} , which is $F(\cdot, T_{n-1}) = F(\cdot, \widehat{T}_1)$. $F(\cdot, \widehat{T}_1)$ is governed by the following SDE under the real-world probability measure P

³as in reference [46]

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$$dF(t, \widehat{T}_1) = F(t, \widehat{T}_1)\theta(t, \widehat{T}_1)dW_P(t) \quad (2.53)$$

with the initial condition

$$F(0, \widehat{T}_1) = \frac{B(0, \widehat{T}_1) - B(0, \widehat{T})}{\tau_i B(0, \widehat{T})} \quad (2.54)$$

Using equation (2.10), we have that

$$F(t, \widehat{T}_1) = \frac{B(0, \widehat{T}_1) - B(0, \widehat{T})}{\tau_{n-1} B(0, \widehat{T})} \xi_t \left(\int_0^\bullet \theta(s, \widehat{T}_1) dW_s \right) \quad (2.55)$$

Since $B(0, \widehat{T}_1) > B(0, \widehat{T})$, it follows immediately that $F(\cdot, \widehat{T}_1)$ follows a strictly positive martingale under $P_{\widehat{T}} = P$. To define the forward LIBOR for the date $\widehat{T}_2$, we need to introduce the forward probability measure Q for date $\widehat{T}_1$. By definition, Q is an equivalent measure to P such that there exists processes

$$f_2(t, \widehat{T}_m) = \frac{B(t, \widehat{T}_m)}{B(t, \widehat{T}_1)} \quad (2.56)$$

which are Q -local martingales. $f_2(t, \widehat{T}_m)$ satisfies the following relationship

$$f_2(t, \widehat{T}_m) = \frac{f_1(t, \widehat{T}_m)}{1 + \tau_{n-1} F(t, \widehat{T}_1)} \quad (2.57)$$

Let us introduce a Lemma which is the direct result following the implication of Ito's Lemma:

Lemma. Let a and b be real-valued adapted processes, such that $da(t) = \alpha(t) dW(t)$, $db(t) = \beta(t) dW(t)$. Assume, in addition, that $b(t) > -1$ for every t and denote $s(t) = [1 + b(t)]^{-1}$. Then

$$\begin{aligned} d(s(t) \cdot a(t)) &= s(t) \cdot [\alpha(t) - s(t) \cdot a(t) \cdot \beta(t)] \cdot [dW(t) - s(t) \cdot \beta(t) \cdot dt] \\ ds(t) &= s(t)^2 \cdot \beta(t) \cdot [s(t) \cdot \beta(t) \cdot dt - dW(t)] \end{aligned} \quad (2.58)$$

It follows immediately from lemma (2.58) that

$$df_2(t, \widehat{T}_m) = u_m(t) \cdot \left(dW_P(t) - \frac{\tau_{n-1} F(t, \widehat{T}_1)}{1 + \tau_{n-1} F(t, \widehat{T}_1)} \theta(t, \widehat{T}_1) dt \right) \quad (2.59)$$

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for certain process $u_m(t)$. There exists a probability measure $W_{P_{\widehat{T}_1}}(t)$ such that for all $t \in [0, \widehat{T}_1]$,

$$W_{P_{\widehat{T}_1}}(t) = W_P(t) - \int_0^t \frac{\tau_{n-1}F(s, \widehat{T}_1)}{1 + \tau_{n-1}F(s, \widehat{T}_1)} \theta(s, \widehat{T}_1) ds = W_P(t) - \int_0^t \widehat{\theta}(s, \widehat{T}_1) ds \quad (2.60)$$

$W_{P_{\widehat{T}_1}}(t)$ is a standard Brownian motion. Results in equation (2.60) can be easily proved using Girsanov's theorem, by setting

$$\frac{dP_{\widehat{T}_1}}{dP} = \zeta_{\widehat{T}_1} \left(\int_0^\bullet \widehat{\theta}(s, \widehat{T}_1) dW_P(s) \right), \quad P - a.s. \quad (2.61)$$

Under the probability measure $P_{\widehat{T}_1}$, we specify the dynamics of the forward LIBOR for date T_2^* as

$$dF(t, \widehat{T}_1) = F(t, \widehat{T}_2) \theta(t, \widehat{T}_2) dW_{P_{\widehat{T}_1}}(t) \quad (2.62)$$

with the initial condition

$$F(t, \widehat{T}_2) = \frac{B(0, \widehat{T}_2) - B(0, \widehat{T}_1)}{\tau_{n-1}B(0, \widehat{T}_1)} \quad (2.63)$$

Assume that there exists processes $F(\cdot, \widehat{T}_1), \dots, F(\cdot, \widehat{T}_k)$, with the associated Brownian motion $W_{\widehat{T}_{k-1}}$ and the probability measures $P_{\widehat{T}_{k-1}}$. Following an easy calculation, it can be shown that

$$f_{k+1}(t, \widehat{T}_m) = \frac{B(t, \widehat{T}_m)}{B(t, \widehat{T}_k)} = \frac{f_k(t, \widehat{T}_m)}{1 + \tau_{n-k}F(t, \widehat{T}_k)} \quad (2.64)$$

By applying (2.58), we have that

$$W_{\widehat{T}_k}(t) = W_{\widehat{T}_{k-1}}(t) - \int_0^t \frac{\tau_{n-k}F(s, \widehat{T}_k)}{1 + \tau_{n-k}F(s, \widehat{T}_k)} \theta(s, \widehat{T}_k) ds, \quad \forall t \in [0, \widehat{T}_k] \quad (2.65)$$

The process $F(\cdot, \widehat{T}_{k+1})$ is the solution to the following SDE

$$dF(t, \widehat{T}_{k+1}) = F(t, \widehat{T}_{k+1}) \theta(t, \widehat{T}_{k+1}) dW_{\widehat{T}_k}(t) \quad (2.66)$$

such that

$$F(0, \widehat{T}_{k+1}) = \frac{B(0, \widehat{T}_{k+1}) - B(0, \widehat{T}_k)}{\tau_{n-k}B(0, \widehat{T}_k)} \quad (2.67)$$

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Based on the above arguments, it can be deduced that if $F(0, T_k)$ is log-normally distributed under the forward probability measure $P_{T_{k+1}}$ (for all $t \in [0, T_k]$), then the volatility coefficient $\theta(\cdot, T_k)$ is a deterministic function.

To conclude, the MR approach focuses on market quoted product prices and rates as inputs to the model, instantaneous rates are not required in the modeling approach. This setting is convenient and is widely employed in market practice, it is also the approach that we use for LIBOR market model pricing in this thesis.

2.1.3. Descriptions of the Forward Rate Dynamics of the LIBOR Market Model

The evolution of (discrete) forward rates is the core of the LIBOR market model. The forward LIBOR rates are expressed in terms of ratios of bond prices. In this thesis, we assume that bond price processes have a canonical tenor structure, that is, the time interval between any two successive reset dates has the same length. Given a forward rate, the time interval between its reset and settlement dates will be referred to as the tenor of the forward rate itself ⁴.

Rebonato (2002) showed that there are several ways to describe the dynamics of the forward LIBOR rates. In this chapter, we present three types of descriptions of the forward LIBORs, as proposed by Rebonato. In fact, the formulation of these three descriptions are equivalent. The choice of which description to use will depend on the context of problems in market practice. By introducing different descriptions of forward LIBORs, we attempt to decompose the volatility coefficients into a combination of two terms. The first term is defined in Rebonato's work as "...extracted from historical data, which is linked to the correlation among all the forward rates". The second term is a variable deduced from market quoted prices for products, i.e. caps and European swaptions. In contrast to the first term, the second term is related to one single forward rate at a time ⁵.

2.1.3.1. The First Description of Forward LIBOR Rates

Assume that given a time-dependent $n \times 1$ column vector of percentage increments of forward (LIBOR) rates $\frac{dF(t, T)}{F(t, T)}$, under the probability measure Q implied by the chosen numeraire, $\frac{dF(t, T)}{F(t, T)}$ is governed by the following process:

$$\frac{dF(t, T)}{F(t, T)} = \mu(F_i(t); t, T) dt + b(t, T) dW_Q(t) \quad (2.68)$$

with $dW_Q(t)$ as the $n \times 1$ column vector which contains correlated standard Brownian motions associated to measure Q . In equation (2.68), the drift coefficient, $\mu(F(t, T); t, T)$, is an $n \times 1$

⁴as in reference [43]

⁵as in reference [43]

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column vector which can depend both on time and on the evolution of the forward rates. $F_i(t) = F(t, T_i)$ as defined in section 2.1.2.1. $b(t, T) \in \mathbb{R}^d$ is an $n \times n$ diagonal matrix, with the i -th element equaled to the instantaneous (percentage) volatility b_i of the forward rate resetting at time T_i .

Note that b_i is the time-dependent instantaneous volatility of the forward rate resetting at time T_i . It is related to the corresponding implied (Black) volatility $\sigma_{Bl}(T_i)$ by the following relationship

$$\int_0^{T_i} b_i^2(s) ds = \sigma_{Bl}^2(T_i) \cdot T_i \quad (2.69)$$

Applying Ito's Lemma on SDE (2.68), we obtain

$$F_i(t) = F_i(0) \cdot \exp \left(\int_0^t \left(\mu(F_i(s); s, T_i) - \frac{1}{2} b_i^2(s) \right) ds + \int_0^t b_i(s) dW_{T_i}(s) \right) \quad (2.70)$$

which is the value of forward LIBOR rate F with maturity at date T_i , observed at time t .

The Brownian increments are correlated in such a way that

$$dW \cdot dW^T = \rho dt \quad (2.71)$$

and

$$\rho_{ij} dt = dW_{T_i} \cdot dW_{T_j} \quad (2.72)$$

where $\rho \in \mathbb{R}^d$ is the $n \times n$ symmetric instantaneous correlation matrix; M^T denotes the transpose of M .

2.1.3.2. The Second Description of Forward LIBOR Rates

The dynamics of the set of forward rates are similar to (2.68), such that

$$\frac{dF(t, T)}{F(t, T)} = \mu(F(t, T); t, T) dt + \sigma(t, T) dZ(t) \quad (2.73)$$

In (2.73), $\frac{dF(t, T)}{F(t, T)}$ and $\mu(F(t, T); t, T)$ have the same definition as in section 2.1.3.1, but the definition of the randomness part of the SDE is different. The volatility matrix $b(t, T)$ in section 2.1.3.1 is replaced by $\sigma(t, T)$, which is an $n \times k$ matrix ($k \leq n$) with values in $\mathbb{R}^d$. And the i, j -th element $\sigma_{ij}(t)$ is equal to the amount of effect of the j -th orthogonal Brownian motion on the i -th forward rate. Moreover, $dW_Q(t)$ is replaced by $dZ(t)$, which is a $k \times 1$ vector of (orthogonal) Wiener processes.

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Because of the orthogonality of the Wiener processes of $dZ(t)$, the $n \times n$ covariance matrix between the forward rates can be expressed as

$$\text{cov} \left[\frac{dF(t, T)}{F(t, T)} \right] = E \left[\frac{dF_i(t)}{F_i(t)} \cdot \frac{dF_i(k)}{F_i(k)} \right] = \sigma(t, T) \cdot \sigma(t, T)^T \quad (2.74)$$

with

$$\frac{dF_i(t)}{F_i(t)} = \mu(F_i(t); t, T_i) dt + \sum_{j=1}^k \sigma_{ij} dW_{T_j}(t) \quad (2.75)$$

The second approach is in fact equivalent to the first approach, with the elements of volatility matrices $b(t, T)$ and $\sigma(t, T)$ satisfying the following relationship:

$$b_i^2(t) = \sum_{j=1}^k \sigma_{ij}^2(t) \quad (2.76)$$

2.1.3.3. The Third Description of Forward LIBOR Rates

Using the dynamics in (2.75), if we divide and multiply each element σ_{ij} by the volatility of the i -th forward rate, we have:

$$\begin{aligned} \frac{dF_i(t)}{F_i(t)} &= \mu(F_i(t); t, T_i) dt + \sigma_i \sum_{j=1}^k \frac{\sigma_{ij}}{\sigma_i} dW_{T_j}(t) \\ &= \mu(F_i(t); t, T_i) dt + \sigma_i \sum_{j=1}^k \frac{\sigma_{ij}}{\sqrt{\sum_{j=1}^k \sigma_{ij}^2}} dW_{T_j}(t) \\ &= \mu(F_i(t); t, T_i) dt + \sigma_i \sum_{j=1}^k a_{ij} dW_{T_j}(t) \end{aligned} \quad (2.77)$$

with.

$$a_{ij} = \frac{\sigma_{ij}}{\sqrt{\sum_{j=1}^k \sigma_{ij}^2}} \quad (2.78)$$

Let A be the matrix of the collection of coefficients $\{a_{ij}\}$, then A and the correlation matrix ρ in equation (2.71) satisfies the following relationship:

$$A \cdot A^T = \rho \quad (2.79)$$

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2.1.3.4. Principal Component Analysis (PCA)

Suppose that we use the second description of forward rates as in section (2.1.3.2). Let $k = n$. Integrating both sides of (2.75), we have

$$F_i(t) = F_i(0) \cdot \exp \left(\int_0^t \left(\mu(F_i(s); s, T_i) - \frac{1}{2} \sigma_i^2(s) \right) ds + \int_0^t \sum_{j=1}^k \sigma_{ij}^2(s) dW_{T_j}(s) \right) \quad (2.80)$$

Recall that the correlation matrix ρ is a real symmetric matrix. In general market practice, ρ is positive finite with full rank. Therefore, there exists some matrices λ and G such that

$$\rho = G \cdot \lambda \cdot G^{-1} \quad (2.81)$$

where

$$\lambda = \begin{pmatrix} \lambda_1(t) & & & \\ & \lambda_2(t) & & \\ & & \ddots & \\ & & & \lambda_n(t) \end{pmatrix} \quad (2.82)$$

is the diagonal matrix of eigenvalues of ρ . The matrix G contains n associated independent eigenvectors:

$$G = \begin{pmatrix} g_{11}(t) & g_{12}(t) & \dots & g_{1n}(t) \\ g_{21}(t) & g_{22}(t) & \dots & g_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1}(t) & g_{n2}(t) & & g_{nn}(t) \end{pmatrix} \quad (2.83)$$

Hence, equation (2.80) can be expressed as

$$F_i(t) = F_i(0) \cdot \exp \left(\int_0^t \left(\mu(F_i(s); s, T_i) - \frac{1}{2} \sigma_i^2(s) \right) ds + \int_0^t \sum_{j=1}^k \sqrt{\lambda_j} \cdot g_{ij}(s) dW_{T_j}(s) \right) \quad (2.84)$$

For the third description of forward rates as in section 2.1.3.3, if we define

$$\sum_{j=1}^k a_{ij}^2 = 1 \quad (2.85)$$

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for $1 \leq i \leq n$, $1 \leq j \leq k$. Then we can derive a closed-form solution for caplet prices by using

$$F_i(t) = F_i(0) \cdot \exp \left(\int_0^t \left(\mu(F_i(s); s, T_i) - \frac{1}{2} \sigma_i^2(s) \right) ds + \int_0^t \sigma_i(s) \sum_{j=1}^k a_{ij}(s) dW_{T_j}(s) \right) \quad (2.86)$$

By equating (2.84) and (2.86), it can be deduced that the market caplets will be correctly priced if

$$\sum_{j=1}^k \lambda_j g_{ij}^2 = \sigma_i^2 \sum_{j=1}^k a_{ij}^2 \quad (2.87)$$

It is worth noting that using equations (2.84), (2.86) and (2.87) for caplet pricing is rather complicated. For simplicity, one may use the Black's formula to derive caplet prices. Nevertheless, equations (2.84), (2.86) and (2.87) will become useful when it comes to the pricing of exotic interest-rate products.

2.1.4. Finding the Suitable Functional Forms for Instantaneous Volatilities

To choose a suitable functional form for the volatilities, one must ensure that the term structure of volatilities is time-homogeneous. Rebonato (2002) proposed square-integrable instantaneous volatility functions, together with the restriction that the instantaneous volatilities depend on the time t of observation, and the maturity date T of the forward rate itself⁶. Hence, we start with the instantaneous volatility function as

$$\sigma_{in} = \sigma_{in}(t, T) \quad (2.88)$$

Assume that there exists an instantaneous volatility function capable of re-producing the current term of volatilities at any point in the future. Let us denote the market quoted (implied) volatility by $\sigma_{Bl}(T)$ for the forward rate with maturity date T . Then the term structure of (implied) volatilities and the instantaneous volatilities satisfies the following relationship:

$$\sigma_{Bl}^2(T_i) \cdot T_i = \int_0^{T_i} \sigma_{in}(s, T_i)^2 ds, \quad 1 \leq i \leq n \quad (2.89)$$

⁶as in reference [43]

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Consider a time τ such that $T_{m-1} \leq \tau \leq T_m$. We have

$$\sigma_{Bl}^2(\tau, T_i) \cdot (T_i - \tau) = \int_{\tau}^{T_i} \sigma_{in}(s, T_i)^2 ds, \quad 1 \leq i \leq n - m \quad (2.90)$$

To ensure that a given current term structure of volatilities is time-homogeneous, the following condition must be satisfied for any τ :

$$\int_0^T \sigma_{in}(s, T)^2 ds = \int_{\tau}^{T+\tau} \sigma_{in}(s, T + \tau)^2 ds \quad (2.91)$$

It can be easily proved that this condition will be fulfilled if

$$\sigma_{in}(t, T) = \sigma_{in}(t, T - t) \quad (2.92)$$

If condition (2.92) is true, then

$$\int_{\tau}^{T+\tau} \sigma_{in}(s, T + \tau)^2 ds = \int_{\tau}^{T+\tau} \sigma_{in}(T + \tau - s)^2 ds = \int_0^T \sigma_{in}(T - s)^2 ds \quad (2.93)$$

In order to price a current caplet which matures at $T + \tau$ years later, the following condition must be satisfied:

$$\int_0^{T+\tau} \sigma_{in}(s, T)^2 ds = \sigma_{Bl}^2(T + \tau) \cdot (T + \tau) \quad (2.94)$$

Thus, the time-homogeneity condition (2.88) can be expressed as

$$\sigma_{Bl}^2(T) \cdot T = \int_{\tau}^{T+\tau} \sigma_{in}(s, T + \tau)^2 ds = \int_0^T \sigma_{in}(T - s)^2 ds \quad (2.95)$$

Let us define the quantities $b^*(i, j) \equiv \int_{T_{i-1}}^{T_i} \sigma_{in}(s, T_j)^2 ds$, with i as the index of the price-sensitive event and j as the index of the forward rate⁷. We have

$$\sigma_{Bl}^2(T_j) \cdot T_j = \sum_{i=1}^j b^*(i, j) \quad (2.96)$$

We could arrange the quantities $b^*(i, j)$ into a matrix B such that

⁷ as in reference [43]

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$$B = \begin{pmatrix} b^*(1, 1) & & & \\ b^*(1, 2) & b^*(2, 2) & & \\ b^*(1, 3) & b^*(2, 3) & b^*(3, 3) & \\ \vdots & \vdots & \vdots & \ddots \\ b^*(1, j) & b^*(2, j) & \cdots & b^*(j, j) \end{pmatrix} \quad (2.97)$$

When the condition ((2.92)) is fulfilled, matrix B becomes

$$B = \begin{pmatrix} b^*(1, 1) & & & \\ b^*(1, 2) & b^*(1, 1) & & \\ b^*(1, 3) & b^*(1, 2) & b^*(1, 1) & \\ \vdots & \vdots & \vdots & \ddots \\ b^*(1, j) & b^*(1, j-1) & \cdots & b^*(1, 1) \end{pmatrix} \quad (2.98)$$

Rebonato (2002) showed that forward rates can be approximated by a set of (discrete) conditionally joint-Gaussian processes. In this way, we can use the LIBOR market model to price LIBOR-related derivatives easily. Recall that the covariance elements between two forward rates $F_i(t)$ and $F_j(t)$ are defined as

$$\int_{T_m}^{T_{m+1}} \sigma_i(s) \sigma_j(s) \rho_{ij}(s) ds \quad (2.99)$$

where σ_i and σ_j are instantaneous volatility functions of forward rates $F_i(t)$ and $F_j(t)$, respectively. $\rho_{ij}(t) = \rho(t, T_i, T_j)$ is the corresponding correlation coefficient. The covariance elements can be expressed in terms of instantaneous volatility and correlation functions. Therefore, it is very important to choose suitable functional forms for the volatilities and correlations. One essential criterion for the choice of such functional forms is that they should only contain a small number of free parameters.

Rebonato (2002) proposed that instantaneous volatilities can have a general functional form:

$$\sigma_{in}(t) = \alpha(T-t) \cdot \delta(t) \cdot \beta(T) \quad (2.100)$$

where $\alpha(T-t)$ is the time-homogeneous part, $\delta(t)$ is the time dependency and $\beta(T)$ is the forward rate dependency⁸. If the instantaneous volatility is purely time-homogeneous, then we have the matrix B in (2.98) as a result. If the instantaneous volatility function is in the form of (2.100), then we could re-arrange the matrix B . For simplicity, assume that $\delta(t) = \delta_m$ which is piecewise constant, for $T_{m-1} \leq t \leq T_m$, $m = 1, 2, \dots, j$, then

⁸as in reference [43]

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$$B = \begin{pmatrix} b^*(1, 1) \delta_1^2 \beta(T_1)^2 & & & & \\ b^*(1, 2) \delta_1^2 \beta(T_2)^2 & b^*(1, 1) \delta_2^2 \beta(T_2)^2 & & & \\ b^*(1, 3) \delta_1^2 \beta(T_3)^2 & b^*(1, 2) \delta_2^2 \beta(T_3)^2 & b^*(1, 1) \delta_3^2 \beta(T_3)^2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ b^*(1, j) \delta_1^2 \beta(T_j)^2 & b^*(1, j-1) \delta_2^2 \beta(T_j)^2 & \dots & & b^*(1, 1) \delta_j^2 \beta(T_j)^2 \end{pmatrix} \quad (2.101)$$

Similarly, the instantaneous volatility function in (2.99) is given by

$$\sigma_i(t) = \alpha(T_i - t) \cdot \delta(t) \cdot \beta(T_i), \quad i = 1, \dots, n-1 \quad (2.102)$$

Rebonato (2002) specified that in equation (2.102),

$$\alpha(T_i - t) = [a_1 + a_2 \cdot (T_i - t) \cdot e^{-a_2(T_i - t)} + a_4] \quad (2.103)$$

$$\delta(t) = e^{-\delta_0 t} \cdot [\delta_1 \cdot \sin(\hat{\beta}t + \delta_2) + \delta_3 \cdot \sin(2\hat{\beta}t + \delta_4) + \delta_5 \cdot \sin(3\hat{\beta}t + \delta_6)] + \delta_7 \quad (2.104)$$

$$\beta(T_i) = 1 + \tau_i \quad (2.105)$$

$$\hat{\beta} = \frac{\pi}{T_{max}} \beta(T) \quad (2.106)$$

2.1.5. Calibration of the Caplet and Swaption Volatilities in LMM

One of the most important tasks in the innovation of the log-normal LIBOR market models is to employ the Black's formula in the market practice of pricing caps and European swaptions. For a detailed description of the products themselves and the corresponding pricing formula, readers should refer to later sections 3.1.2 and 3.1.1. In this section, we only present the calibration procedure of the caplet and swaption volatilities.

For a cap, the log-normal LIBOR market model assumes that the underlying interest-rates of each caplets are log-normally distributed. For the case of a European swaption, it is assumed that the underlying swap rate is log-normal at the options maturity. Market practitioners usually assume that the dynamics of the underlying rates of the products are governed by processes which follows a geometric Brownian motion under some probability measure, say Q .

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2.1.5.1. Calibration for Caplet Volatilities

Before discussing the calibration of the caplet volatilities, we should briefly explain the concept of the displaced diffusion process of forward LIBORs. In this method, the forward LIBORs $F_i(t)$ are shifted by a constant displacement ω_i , so that $(F_i(t) + \omega_i)$ will be used as the new state variable. $(F_i(t) + \omega_i)$ are again log-normally distributed, so they have deterministic volatilities. The shifted forward LIBORs are governed by the following SDE:

$$\frac{d(F_i(t) + \omega_i)}{F_i(t) + \omega_i} = \mu^*(F_i(t), \omega_i; t, T_i) dt + \sigma_i^*(t) dW_{T_i}^*(t) \quad (2.107)$$

for $i = 1, \dots, n-1$. μ^* and σ_i^* are the drift and volatility coefficients of the shifted forward LIBORs, respectively. $W_{T_i}^*$ is a standard Brownian motion.

Given a cap which consists n caplets, we denote the value of the i -th caplet as Cpl_i . Assume that the cap has a canonical tenor structure and so does the underlying forward rates. For $i = 1, \dots, n$, let $\sigma_i(t) = \sigma(t, T_i)$ denotes the time- t instantaneous volatility of caplet price Cpl_i with maturity at time T_i . σ_i is calculated by stripping the market quoted (implied) Black volatility for the associated caplet. From (2.89), we have

$$\sigma_{Bl}^2(T_i) \cdot T_i = \int_0^{T_i} \sigma_i(s)^2 ds, \quad 1 \leq i \leq n \quad (2.108)$$

but now $\sigma_{Bl}(T_i)$ represents the shifted (implied) Black volatility of caplet price Cpl_i , which is calculated by multiplying the market quoted volatility for caplet Cpl_i with a coefficient of $\frac{F_i(0)}{F_i(0) + \omega_i}$, where ω_i is the coefficient of diffusion displacement. Then we will minimize the average squared-error between the model- and market quoted (implied)- volatilities:

$$\min \left\{ \frac{1}{n} \sum_{i=1}^n [\sigma_{Bl}^2(T_i) - \sigma_i]^2 \right\} \quad (2.109)$$

with n as the number of caps that we will calibrate to.

In this thesis, “cap” will refer to caps with canonical tenor structures. In this case, we have

$$\begin{aligned} \sigma_i &= \sqrt{\frac{1}{T_i} \sum_{j=1}^i b^*(j, i)} \\ &= \sqrt{\frac{1}{T_i} \sum_{j=1}^i \left\{ \left(\sigma_i^2 \cdot \frac{T_{j-1} + T_j}{2} \right) \cdot \tau_j \right\}} \end{aligned} \quad (2.110)$$

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such that

$$b^*(j, i) \equiv \int_{T_{j-1}}^{T_j} \sigma_i^2(s) ds, \quad \text{for } i = 1, \dots, n; j = 1, \dots, i \quad (2.111)$$

as in section 2.1.4.

2.1.5.2. Calibration for European Swaption Volatilities

Given an European swaption, the instantaneous volatility σ of the associated forward swap rate is calculated via stripping the market quoted (implied) Black volatility for this swaption. Let $Y(t, T_{k-1}, T_m)$ denotes the $(m - k)$ -period forward swap rate at time t , with the swaption maturity at time T_{k-1} and the underlying swap (e.g. payer swap) matures at T_m . By definition, the forward swap rate can be expressed in terms of bonds prices as:

$$Y(t, T_{k-1}, T_m) = \frac{B(t, T_{k-1}) - B(t, T_m)}{\sum_{i=k}^m \tau_i \cdot B(t, T_i)} \quad (2.112)$$

Let σ_{Bl} refer to the market quoted implied Black volatility. Then σ and σ_{Bl} satisfies the following relationship

$$\sigma_{Bl}(T_{k-1}, T_m) = \sqrt{\frac{1}{T_{k-1} - t} \int_t^{T_{k-1}} \sigma_P^2(s) ds} \quad (2.113)$$

with $\sigma_P(t)$ as the percentage instantaneous volatility which is given by:

$$\sigma_P^2(t) = \frac{1}{Y^2(t, T_{k-1}, T_m)} \cdot \sigma^2(Y(t, T_{k-1}, T_m)) \quad (2.114)$$

$$= \sum_{i=k}^m \sum_{j=k}^m \{\chi_{i-1, j-1}(t) \cdot \sigma_{i-1}(t) \cdot \sigma_{j-1}(t) \cdot \rho_{i-1, j-1}(t)\} \quad (2.115)$$

$$\chi_{i-1, j-1}(t) = \frac{\frac{\partial Y}{\partial(F_{i-1}(t) + \omega_{i-1})} \cdot (F_{i-1}(t) + \omega_{i-1}) \cdot \frac{\partial Y}{\partial(F_{j-1}(t) + \omega_{j-1})} \cdot (F_{j-1}(t) + \omega_{j-1})}{Y^2(t, T_{k-1}, T_m)} \quad (2.116)$$

$\sigma_P(t)$ is the weighted average of covariance elements of shifted forward LIBOR rates. $\rho_{i-1, j-1}(t)$ in (2.114) is the row $i - 1$, column $j - 1$ entry of the instantaneous correlation matrix for the associated swaption.

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2.1.6. Calibration for Instantaneous Correlations

In equation (2.115), $\rho_{i-1, j-1}(t)$ is the row $i-1$, column $j-1$ entry of the instantaneous correlation matrix for the associated swaption. In order to choose a suitable functional form for ρ , we must ensure that it is consistent with the property of correlations. First, we must have $\rho_{ij}(t) \in [-1, 1]$ for any i, j at time t . Second, we need to assume that the correlation function is integrable over any interval $[T_i, T_{i+1}]$. It is desirable for the instantaneous correlation function to be time-homogeneous. Hence, the following form of correlation functions was proposed:

$$\rho_{ij} = \rho(T_i - t, T_j - t) \quad (2.117)$$

Recall that, using the third description of forward rates, we have the following SDE:

$$\frac{dF_i(t)}{F_i(t)} = \mu(F_i(t); t, T_i) dt + \sigma_i \sum_{j=1}^k a_{ij} dW_{T_j}(t) \quad (2.118)$$

$$\sum_{j=1}^k a_{ij}^2 = 1 \quad (2.119)$$

We could re-write equation (2.118) as

$$\frac{dF(t, T)}{F(t, T)} = \mu(F(t, T); t, T) dt + \vec{B} \cdot A dW \quad (2.120)$$

with $\frac{dF(t, T)}{F(t, T)}$ and $\mu(F(t, T); t, T) dt$ as $n \times 1$ vectors containing the increments in the forward rates and corresponding drifts, respectively. dW is the $k \times 1$ vector of Brownian increments. $\vec{B}$ is the $n \times n$ diagonal matrix with instantaneous volatilities σ_i along its diagonal:

$$\vec{B} = \begin{pmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \ddots & \vdots \\ 0 & 0 & \sigma_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \sigma_n \end{pmatrix} \quad (2.121)$$

A is the $n \times k$ matrix formed by n row vectors a_i , $1 \leq i \leq n$; where each a_i contains k elements such that $a_i = \{a_{ij}; 1 \leq j \leq k\}$.

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$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix} \quad (2.122)$$

If the elements a_{ij} fulfill (2.119), the norm of all the row vectors a_i is equal to unity.

Recall that the correlation between two forward rates is $\rho_{ij} = dW_i dW_j$, and we have

$$A \cdot A^T = \rho \quad (2.123)$$

The elements a_{ij} of A can be expressed using angular coordinates:

$$b_{ij}(t) = \cos \kappa_{ij}(t) \cdot \prod_{l=1}^{j-1} \sin \kappa_{il}(t), \quad j = 1, \dots, u-1 \quad (2.124)$$

$$b_{ij}(t) = \prod_{l=1}^{j-1} \sin \kappa_{il}(t), \quad k = u \quad (2.125)$$

where $\{\kappa_{ij}\} \in \mathbb{R}^d$ is an arbitrary set of real numbers. u is the number of factors in the model. (2.124) and (2.125) guarantee that $\sum_{j=1}^k a_{ij}^2 = 1$, which is the essential condition for caplet pricing. For example, if we have a two-factor model, then $u = 2$ and equations (2.124) and (2.125) reduce to

$$a_{i1}(t) = \sin(\kappa_i) \quad (2.126)$$

$$a_{i2}(t) = \cos(\kappa_i) \quad (2.127)$$

From the fact that $\sin^2(x) + \cos^2(x) = 1$ for any x , it is obvious that $a_{i1}^2 + a_{i2}^2 = 1$ so it satisfies the constraint. This guarantees the correct caplet pricing using a two-factor LIBOR market model.

In calibration, we aim to fit the parameters so that a_{ij} s will have optimal value, while the values in the output correlation matrix are as close to values in the target correlation matrix as possible. To do this, we minimize the function λ such that

$$\lambda^2 = \sum_{i,j=1}^{n+\frac{n-1}{2}} \left(\rho_{ij}^{target} - \rho_{ij}^{model} \right)^2 \quad (2.128)$$

$$= \sum_{i,j=1}^{n+\frac{n-1}{2}} \left(\rho_{ij}^{target} - a_i a_j^T \right)^2 \quad (2.129)$$

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n is the number of forward rates that we need to model. ρ_{ij}^{target} is the (i, j) -th entry of the target correlation matrix (i.e. the market quoted correlation matrix) that we are calibrating to. ρ_{ij}^{model} is the (i, j) -th entry of the correlation matrix outputted by our model.

In practice, we can calibrate to the target correlation matrix using two different approaches. First, if we obtain an exogenous correlation matrix from the historical data, then we could use Principal Component Analysis as described in section 2.1.3.4, combined with an optimization procedure. However, in this approach, we need to re-run the Principal Component Analysis every time there is a shock to the correlation matrix, if we relax the assumption of the time-homogeneity of the correlation matrix. As a result, it will be hard to implement correlation-related sensitivity analysis, because we need to re-run the PCA many times.

Because of the drawback with the first approach, we employed a second approach. In this approach, we specify the functional forms for correlation, then fit the exogenous correlation matrix to those functional forms. Then we apply the the Principal Component Analysis and the optimization procedure on the fitted correlation matrix as in the first approach. From market practice, the following functional forms was suggested for correlations:

$$\rho_{ij}(t) = \exp \left[-u \cdot \hat{\alpha}(T_i - t, T_j - t) - v \cdot \hat{\beta}(T_i - t, T_j - t) \right]; \quad u, v \in [0, 4] \quad (2.130)$$

$$\rho_{ij}(t) = \exp \left[-|T_i - T_j| \cdot \left(u + v \cdot \left(1 - \frac{T_i - t + T_j - t}{2(\hat{T} - t)} \right) \right) \right]; \quad u, v \in [0, 0.25] \quad (2.131)$$

$$\rho_{ij}(t) = v + (1 - v) \cdot \exp \left[-u \cdot \left| \sqrt{T_i - t} - \sqrt{T_j - t} \right| \right]; \quad u, v \in [0, 1] \quad (2.132)$$

such that

$$\hat{\alpha} = \frac{\hat{\beta}}{y_1} \{ (\varphi_i + \varphi_j) \cdot [(\varphi_i + \varphi_j) + 3y_0] - \varphi_i \varphi_j + y_2 \} \quad (2.133)$$

$$\hat{\beta} = \frac{10 |\varphi_i - \varphi_j|}{y_0} \quad (2.134)$$

$$\varphi_i = \frac{T_i - t}{2(\hat{T} - t)} \cdot n, \quad \varphi_j = \frac{T_j - t}{2(\hat{T} - t)} \cdot n \quad (2.135)$$

$$y_0 = 1 - n, \quad y_1 = (n - 2)(n - 3), \quad y_2 = 2n^2 - n - 4 \quad (2.136)$$

with n as the number of forward rates in the model.

Users of the above model should choose a functional form for correlation from (2.130), (2.131) and (2.132). After that, a range of values for parameters u and v are inputted in order to generate

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different fitted correlation matrices. When the underlying currency for the model is US Dollar, users are suggested to use input values around 0.06 and 0.03 for u and v , respectively (given by market practice). In this case, the maximum input for u and v are around 0.08 and 0.07 respectively. These values arise from the daily observation of market practices. Input values exceeded this range may result in zero or negative correlation values. For the results of input and fitted correlation matrices, readers should refer to the appendix.

2.1.7. Incompatibility between the LSM and the LFM

Based on the LMM framework, we have the log-normal Forward Swap rate model (LSM) and the Log-normal Forward LIBOR rate model (LFM). Under both models, analytical pricing formula can be easily achieved so they have been very popular among practitioners. However, the LSM and LFM are in-compatible with each other in theory.

The underlying assumption of LSM is that the forward swap rates are log-normally distributed. Similarly, the underlying assumption of LFM is that the forward LIBOR rates are log-normally distributed. Recall that the $(m - k)$ -period swap rate is defined as

$$Y(t, T_{k-1}, T_m) = \frac{B(t, T_{k-1}) - B(t, T_m)}{\sum_{i=k}^m \tau_i \cdot B(t, T_i)} \quad (2.137)$$

where $B(t, T_i)$ is the time- t price of a zero-coupon bond maturing at T . We know that the following relationship holds between the bond price and the forward LIBOR rate:

$$F(t, T) = \frac{B(t, T) - B(t, T + \varepsilon)}{\varepsilon \cdot B(t, T + \varepsilon)} \quad (2.138)$$

with $F(t, T)$ as the forward LIBOR rate over the future period $[T, T + \varepsilon]$. Equation (2.138) implies that the volatility coefficient of bond prices and the volatility of forward LIBOR rates cannot be deterministic at the same time. Since the interest-rate yield curve is constructed using forward LIBOR rates (by changing the numeraire), we can express forward swap rates and forward LIBOR rates under the same measure. Therefore, we can choose to model the evolution of the swap rates under either the LSM or the LFM framework. Under the LSM framework, the swaption price can be computed via a Black-Scholes type formula, based on log-normally distributed swap rates. However, when we price a swaption under the LFM framework, there is no guarantee that log-normally distributed forward LIBOR rates will generate log-normally distributed swap rates. This is demonstrated clearly by equations (2.137) and (2.138). Hence, LSM and LFM are incompatible in theory. However, researchers such as Brace, Dun and Barton (1998) and Brigo and Mercurio (2006) argued that in practice it has been observed that the swap rates and the corresponding forward LIBOR rates are log-normally distributed most of the time.

2.2. The Stochastic- $\alpha\beta\rho$ (SABR) model

In the standard Black-Scholes-Merton setting, it is assumed that the volatility is a constant. However, this assumption is often violated in reality as options with different strike prices require different volatilities to match their market prices. In market practice, we call this effect “the volatility smile”. For the pricing and hedging activities, it is very important for practitioners to handle the volatility smiles in the correct way.

In order to incorporate the volatility smile effect into the pricing models, Hagan et al. (2002) proposed the Stochastic- $\alpha\beta\rho$ (SABR) model. They claimed that the dynamic behaviour of smile-and skews predicted by the SABR model tend to be consistent with those observed in the market. Another advantage of the SABR model is that analytical formulas for prices of vanilla interest-rate derivatives can be obtained easily on the base of the Black’s formula.

2.2.1. The Stochastic- β Model Framework

The SABR model is a two-factor stochastic volatility model, where the forward rate $\hat{F}$ and the volatility $\hat{\sigma}$ are governed by the following stochastic differential equations under the forward measure:

$$d\hat{F}(t) = \hat{\sigma} \cdot J(\hat{F}(t)) dW_1(t) \quad (2.139)$$

u

$$d\hat{\sigma}(t) = v \cdot \hat{\sigma}(t) dW_2(t) \quad (2.140)$$

$$dW_1 dW_2 = \rho dt \quad (2.141)$$

with $\hat{F}(0) = f$ and $\hat{\sigma}(0) = \sigma$. v is the volatility of volatility (vol-vol), and ρ is the correlation between $\hat{F}$ and $\hat{\sigma}$.

For the market practitioners, a common choice for $J(\hat{F})$ is to assume the functional form $J(\hat{F}) = \hat{F}^\beta$. Under this assumption, the prices of vanilla options will be functions of σ , v , ρ and β . We then calibrate these four parameters from market quoted data to ensure that the skew values outputted by our model will match the market quoted skew values.

With the choice of $\beta = 0$, the SABR model will become the stochastic Normal model. With the choice of $\beta = 1$, it will become the stochastic Log-normal model.

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2.2.1.1. Implied Normal Volatility

In the stochastic- β model, the implied volatility is:

$$\sigma_{imp} = \varepsilon \sigma \cdot (fK)^{\frac{\beta}{2}} \cdot \frac{\omega}{Y(\omega)} \cdot \frac{1 + \frac{1}{24} \ln^2 \left(\frac{f}{K} \right) + \frac{1}{1920} \ln^4 \left(\frac{f}{K} \right) + \dots}{1 + \left(\frac{-\beta \cdot (2-\beta) \cdot \sigma^2}{24 \cdot (fK)^{1-\beta}} + v^2 \cdot \frac{2-3\rho^2}{24} + \frac{\rho v \sigma \beta}{4 \cdot (fK)^{\frac{1-\beta}{2}}} \right) \varepsilon^2 \tau + \dots} \quad (2.142)$$

where

$$Y(\omega) = \int_0^\omega \frac{d\zeta}{\sqrt{1 - 2\rho\zeta + \zeta^2}} = \ln \left(\frac{\sqrt{\omega^2 - 2\rho\omega + 1} + \omega - \rho}{1 - \rho} \right) \quad (2.143)$$

$$\omega = \varepsilon v z = \frac{v}{\sigma} \cdot \frac{f - K}{J(f^*)} = \frac{v}{\sigma} \cdot \frac{f - K}{(f^*)^\beta} \quad (2.144)$$

with $f^* = \sqrt{fK}$.

2.2.1.2. Implied Black Volatility

The implied Black volatility is a function of the option strike K and the current forward price f :

$$\sigma_{Bl} = \frac{\sigma}{(fK)^{\frac{1-\beta}{2}}} \cdot \frac{\omega}{Y(\omega)} \cdot \frac{1 + \varepsilon^2 \tau \left[\frac{\sigma^2 (1-\beta)^2}{24 \cdot (fK)^{1-\beta}} + v^2 \cdot \frac{2-3\rho^2}{24} + \frac{\rho v \sigma \beta}{4 \cdot (fK)^{\frac{1-\beta}{2}}} \right] + \dots}{1 + \frac{(1-\beta)^2}{24} \ln^2 \left(\frac{f}{K} \right) + \frac{(1-\beta)^4}{1920} \ln^4 \left(\frac{f}{K} \right) + \dots} \quad (2.145)$$

When the option is at-the-money, that is, $K = f$, the implied Black volatility in (2.145) becomes

$$\sigma_{ATM} = \frac{\sigma}{f^{1-\beta}} \cdot \left[1 + \varepsilon^2 \tau \left(\frac{\sigma^2 (1-\beta)^2}{24 \cdot f^{2(1-\beta)}} + v^2 \cdot \frac{2-3\rho^2}{24} + \frac{\rho v \sigma \beta}{4 \cdot f^{1-\beta}} \right) + \dots \right] \quad (2.146)$$

Assume that we have found the parameters v and ρ , the equation (2.146) can be inverted to

$$\frac{\sigma^3 (1-\beta)^2 \tau}{24 \cdot f^{2(1-\beta)}} + \frac{\rho v \beta \tau}{4 \cdot f^{1-\beta}} \cdot \sigma^2 + \sigma \left(1 + \frac{2-3\rho^2}{24} \cdot v^2 \tau \right) - \sigma_{ATM} \cdot f^{1-\beta} = 0 \quad (2.147)$$

By solving the cubic equation (2.147), we will be able to find the solution of σ as the smallest positive root of the equation.

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2.2.2. Pricing Formula for a Vanilla Call Option

Let us consider a vanilla call option with option expiration T and a fixed strike price K . Under the setting described in section 2.2.1, the value of this call option (before discounting) at current time t is

$$V_{call}(t, f, \sigma) = \int_{-\infty}^{\infty} \int_K^{\infty} (F - K) \cdot p(t, f, \sigma; T, F, U) dF dU \quad (2.148)$$

given that $\hat{F}(0) = f$ and $\hat{\sigma}(0) = \sigma$ and $p(t, f, \sigma; T, F, U)$ is the associated probability density function.

By applying the asymptotic expansion with ε as some positive arbitrary small number, we have

$$V_{call}(t, f, \sigma) = (f - K)^+ + \frac{|f - K|}{4\sqrt{\pi}} \int_{\frac{\infty}{2\tau} - \varepsilon^2 \chi}^{\infty} \frac{e^{-\hat{k}}}{\hat{k}^{\frac{3}{2}}} d\hat{k} \quad (2.149)$$

where

$$\hat{k} = k - \varepsilon^2 \chi \quad (2.150)$$

$$k = \frac{y^2}{2\tau} > 0 \quad (2.151)$$

$$y = \frac{1}{\varepsilon v} \int_0^{\varepsilon v z} \frac{d\zeta}{\sqrt{1 - 2\rho\zeta + \zeta^2}} \quad (2.152)$$

2.3. The CIR-Jump model

The jump-diffusion models have attracted attention from researchers over the past years. In general, a jump-diffusion process is defined as

$$\frac{dS(t)}{dS(t)} = f(t, t) dt + \sigma dW_Q(t) + \varphi(t) \cdot dJ(t) \quad (2.153)$$

where J is a simple Poisson jump process with jump rate λ . The jump process J is driven by a collection of time-homogeneous Poisson processes and will be exponentially distributed. The Poisson process shares some of the properties with the Brownian motion process. For example, both processes are special cases of processes formed by stationary independent increments. As the intensity of the time-homogeneous Poisson processes increase, the frequency of the jumps will increase. And as the mean of the exponential distribution of J increases, the jump sizes will increase. The shock dJ is always finite.

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Vasicek (1977) proposed a model for the evolution of interest rates. In his model, the dynamic of the short-term rates are assumed to follow the Ornstein-Uhlenbeck process:

$$df(t, t) = \alpha [\mu - f(t, t)] dt + \sigma dW_Q(t) \quad (2.154)$$

with α, μ, σ as positive constants and $W_Q(t)$ is the risk-neutral measure. Cox, Ingersoll and Ross (1985) modified this model by introducing an extra term in the diffusion coefficient:

$$df(t, t) = k [\theta - f(t, t)] dt + \sigma \sqrt{f(t, t)} dW_Q(t) \quad (2.155)$$

with $f(0) = f_0$, and f_0, k, θ, σ are positive constants. A restriction $\sigma < \sqrt{2k\theta}$ must be imposed so that the short rate $f(t, t)$ in equation (2.155) is guaranteed positive.

In the Cox-Ingersoll-Ross (CIR) model, the time- t price of a zero-coupon bond with maturity T has the following definition:

$$B(t, T) = G(t, T) e^{-H(t, T)f(t, t)} \quad (2.156)$$

such that

$$G(t, T) = \left\{ \frac{2g \cdot e^{\frac{(g+h)(T-t)}{2}}}{2g + (g+h) [e^{g(T-t)} - 1]} \right\}^{\frac{2h\beta}{\sigma^2}} \quad (2.157)$$

$$H(t, T) = \frac{2 [e^{g(T-t)} - 1]}{2g + (g+h) [e^{g(T-t)} - 1]} \quad (2.158)$$

with $g = \sqrt{2\sigma^2 + h^2}$. Using the Ito's formula, we can derived the the SDE of bond prices from equation 2.155 and 2.156, under the risk-neutral measure Q :

$$\frac{dB(t, T)}{B(t, T)} = f(t, t) dt - H(t, T) \cdot \sigma \sqrt{f(t, t)} dW_Q(t) \quad (2.159)$$

Brigo and El-Bachir (2005) suggested that the CIR model can be extended to a jump-diffusion model, by introducing a jump component. Apart from the Brownian shocks, jumpy shocks corresponding to a compound Poisson process have been added to the model. In this way, the SDE in equation 2.155 becomes

$$df(t, t) = k [\theta - f(t, t)] dt + \sigma \sqrt{f(t, t)} dW_Q(t) + dJ(t) \quad (2.160)$$

with J denoting a pure jump process. To keep the consistency between the ‘‘CIR-jump’’ model and the interest-rate market, a condition such that the jumps must be positive need to be imposed.

2.4. The Market Volatility Model

Let us consider a set of maturity dates $\{T_i\}_{i=0,\dots,n}$. Let $t = 0$ be the current time. At time t , suppose we have a forward rate $F(t, T_i, T_{i-1})$ which matures at time T_{i-1} , with settlement date T_i . The corresponding year fraction is $\tau_i = T_i - T_{i-1}$, for $i > 0$. $\{\tau_i\}_{i=0,\dots,n}$ is the collection of year fractions associated with $\{T_i\}_{i=0,\dots,n}$. We define $T_{-1} = 0$, and τ_0 is the year fraction from T_0 to the settlement date.

Denote $F(t, T_i, T_{i-1})$ by $F_i(t)$, $i = 1, \dots, n$. At time T_{i-1} , $F_i(T_{i-1})$ will become the simply-compounded spot rate $L(T_{i-1}, T_i)$. Let $B(\cdot, T_i)$ be the numeraire of the forward rate matures at T_i , i.e. $B(\cdot, T_i)$ is the price of the bond which matures at T_i . The probability measure Q^i associated with the numeraire $B(\cdot, T_i)$ is the forward measure for the maturity T_i . By definition, we have

$$F_i(t) B(t, T_i) = \frac{[B(t, T_{i-1}) - B(t, T_i)]}{\tau_i} \quad (2.161)$$

under simple compounding. Hence, $F_i(t) B(t, T_i)$ is the price of a tradable asset. Under the numeraire $B(t, T_i)$, the price $F_i(t) B(t, T_i)$ of the tradable asset is simply $F_i(t)$ itself. By definition, $F_i(t)$ is a martingale under the measure associated to this numeraire, so $F_i(t)$ is a martingale under Q . It follows that if $F_i(t)$ is modeled according to a diffusion process, it needs to be driftless under Q .⁹

Assume that $F_i(t)$ is governed by the following driftless process under Q :

$$dF_i(t) = \sigma_i(t) F_i(t) dW_i(t), \quad t \leq T_{i-1} \quad (2.162)$$

with $dW_i(t)$ as the $n \times 1$ column vector which contains correlated standard Brownian motions associated to measure Q . $\sigma_i(t)$ is the instantaneous volatility of the forward LIBOR rate $F_i(t)$ at time t .

Normally, we assume that the instantaneous volatilities are piecewise-constant:

$$\sigma_i(t) = \sigma_{i,s(t)}, \quad t > 0 \quad (2.163)$$

where $s(t)$ is the index number such that all previous forward rates $F_{s(t)-1}, F_{s(t)-2}, \dots, F_0$ have expired at time t , and $F_{s(t)}$ will be the first existing forward rate at time t .

In general, we set

⁹as in reference [11]

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$$\begin{cases} s(t) = j & \text{for } T_{j-2} < t \leq T_{j-1}, j \geq 1 \\ s(t) = 1 & \text{for } t = 0 \end{cases} \quad (2.164)$$

When we price exotic-type options, we normally obtain the implied volatility surface of a comparable product in the market, then we calibrate the volatility surface of our desired exotic option to that of the reference product. The calibration procedure is very time-consuming. In the real market, volatilities of many liquid interest-rate options are quoted in the market, i.e. for caps and European swaptions. The daily trading volumes of these interest-rate options are very high and so the corresponding market is very liquid. Hence, the market quoted volatilities of these liquid interest-rate options can be regarded as a benchmark, as many physical trades happen in the market are based on this information. In this case, we can assume that market quoted volatilities represent the true characteristic of our desired options, and use them directly for pricing. To do this, we select a market quoted volatility matrix with respect to different option maturity dates and strike prices, then we use interpolation to extend this volatility matrix to match the option structure of our desired option. In this way, equation (2.163) becomes

$$\sigma_i(t) = \sigma_i, \quad t > 0 \quad (2.165)$$

where σ_i is the interpolated volatility (from the market quoted volatility matrix) of the forward rate $F_i(t)$ matures at T_i , observed at time t . σ_i is then plugged into the standard Black's type pricing formula to achieve the option's price. In contrast to the calibration of the volatility surface, the interpolation of the market quoted volatility surface is easy to implement in the computing procedure and hence is extremely time-efficient. This model arises from trading experiences, and we named it the *Market Volatility model*. It is commonly employed in the market practice when dealing with liquid interest-rate options.

Caps and European swaptions are probably two of the most popular interest-rate options in the market. We applied both the LMM and the Market Volatility model in the pricing of Caps and Swaptions. As shown in section 3.3, the two models produce comparable results. In addition, the output prices display the same upward sloping/humped trend. Therefore, we conclude that the Market Volatility model outperforms the LMM via both the simplicity and the efficiency of computation.

In the following section, we will explain the interpolation method used for Caps and European swaptions in more detail.

2.4.1. Interpolation of the Piecewise-Constant Volatilities of Forward Rates

2.4.1.1. The "Stripping" Method for the Caplet Volatilities

σ_i in the standard Black's pricing formula, i.e. as in equations (3.6) and (3.7), denotes the market quoted implied Black volatility for the related caplet. Market quoted volatilities are obtained for

2. Interest-Rate Models

caps formed by one year caplets, with the maturity of the cap varying from 1 year up to 30 years. In order to have an implied volatility surface incorporating the smile effect, we obtain the market quoted cap volatilities with respect to at-the-money (ATM) strike price, and strike prices from 0.01 up to 0.09. The result volatility matrix is as shown in figure 2.1:

USD Cap Volatility											
ATM		1.00%	2.00%	3.00%	4.00%	5.00%	6.00%	7.00%	8.00%	9.00%	
10.4	1Y	30.8	23.8	18.0	14.6	11.7	10.1	10.3	10.1	13.6	
13.1	2Y	31.8	27.6	22.1	18.2	14.6	12.8	12.7	13.3	15.3	
14.4	3Y	33.4	28.9	23.5	19.4	15.8	14.0	13.8	14.5	15.8	
15.1	4Y	33.9	29.5	23.9	19.8	16.5	14.6	14.3	14.8	15.8	
15.4	5Y	35.1	29.8	24.1	20.1	16.9	15.0	14.7	14.9	15.7	
15.7	6Y	35.8	29.7	24.1	20.3	17.2	15.3	14.9	15.0	15.5	
15.8	7Y	35.9	29.5	24.1	20.3	17.4	15.4	14.9	15.0	15.3	
15.9	8Y	35.9	29.4	24.0	20.2	17.4	15.5	14.9	14.9	15.1	
15.9	9Y	35.8	29.2	23.9	20.2	17.5	15.5	14.8	14.8	14.9	
15.8	10Y	35.7	29.0	23.6	20.1	17.4	15.4	14.8	14.7	14.7	
15.5	15Y	34.6	28.5	22.6	19.2	16.9	15.0	14.3	14.2	13.9	
14.9	20Y	33.9	28.1	21.9	18.5	16.4	14.5	13.8	13.6	13.2	
14.3	30Y										

Figure 2.1.: Market quoted Cap volatility matrix (in USD)

After we set up the cap volatility matrix, the caplet volatilities are stripped from the corresponding market quoted cap volatilities. We assumed that the caplet volatilities are piecewise constant when stripping the caplet volatilities. For example, we have a cap on a semi-annually (interest-rate) yield curve, in this case each one year period of the cap contains two 6-month caplets and they have the same volatility.

First of all, we calculate the 1 year, 2 year,... , 9 year cap prices using flat volatilities. For example, a 1-year cap is formed by two 6-month caplets, both with volatilities equal to the market quoted volatility for a 1-year cap. A 2-year cap is formed by four 6-month caplets, with each caplet volatility equal to the market quoted volatility for a 2-year cap. Using the similar setting, we calculate the cap prices up to 9 years. That is,

$$\begin{aligned}
 V_{cap}(1Y) &= \sum_{i=1}^2 V_{cpl_i}(\sigma_{flat}(1Y)), \\
 V_{cap}(2Y) &= \sum_{i=1}^4 V_{cpl_i}(\sigma_{flat}(2Y)), \\
 &\vdots
 \end{aligned}$$

with $V_{cap}(T)$ as the price of the cap with maturity T , $\sigma_{flat}(T)$ is the market quoted volatility for the cap with maturity T , and V_{cpl_i} is the price of the i th caplet of the associated cap.

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Secondly, we “strip” the caplet volatilities from the cap prices calculated in the previous step. This is implemented via the bootstrap method. To start with, we have that the difference between the value of a 2- year cap and the value of a 1-year cap equals to the sum of the two 6-month caplets start from the 1 year point. By using the induction method, we have

$$\begin{aligned}
 V_{cap}(1Y) &= \sum_{i=1}^2 V_{cpl_i}(\sigma_{flat}(1Y)) = \sum_{i=1}^2 V_{cpl_i}(\sigma_i), \\
 V_{cap}(2Y) &= \sum_{i=1}^4 V_{cpl_i}(\sigma_{flat}(2Y)) = \sum_{i=1}^4 V_{cpl_i}(\sigma_i) = \sum_{i=1}^2 V_{cpl_i}(\sigma_{flat}(1Y)) + \sum_{i=3}^4 V_{cpl_i}(\sigma_i), \\
 &\vdots \\
 V_{cap}(nY) &= \sum_{i=1}^{2n} V_{cpl_i}(\sigma_{flat}(nY)) = \sum_{i=1}^{2n} V_{cpl_i}(\sigma_i) \\
 &= \sum_{i=1}^{2n-2} V_{cpl_i}(\sigma_{flat}(nY)) + \sum_{i=2n-1}^{2n} V_{cpl_i}(\sigma_i)
 \end{aligned} \tag{2.166}$$

σ_i in (2.166) is the volatility of the i th caplets of the cap. Using the induction method above together with formula (2.166), we can obtain a full range of caplet volatilities from market quoted cap volatilities. .

2.4.1.2. The Bi-linear Interpolation for the Swaption Volatilities

σ_i in the standard Black’s pricing formula, i.e. as in equations (3.13) and (3.14), denotes the market quoted implied Black volatility for the related swaption volatility cube. Note that we have a 3-dimensional swaption volatility cube rather than a 2-dimensional swaption volatility matrix. The result swaption volatilities are generated by two steps.

First of all, we obtain the market quoted volatility matrix of swaptions with the maturity of the swaption varying from 1 month up to 30 years, and the length of the underlying swap varying from 1 year up to 30 years. The result volatility matrix is as shown in figure 2.2:

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USD Swap Volatility												
Expiry/Tenor	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y
1M	9.6	11.6	11.9	12.0	12.1	12.1	12.0	12.0	11.9	11.9	11.7	11.7
3M	10.8	12.6	12.9	13.0	13.1	13.0	12.9	12.9	12.8	12.7	12.7	12.6
6M	12.2	13.4	13.7	13.7	13.7	13.6	13.5	13.4	13.4	13.3	13.1	12.9
1Y	14.3	14.6	14.6	14.6	14.5	14.4	14.3	14.2	14.1	14.1	13.8	13.4
2Y	15.6	15.6	15.6	15.5	15.4	15.3	15.1	15.0	14.9	14.8	14.2	13.8
3Y	16.1	16.1	16.0	15.8	15.7	15.6	15.4	15.3	15.2	15.0	14.3	13.8
4Y	16.3	16.2	16.0	15.9	15.7	15.5	15.4	15.2	15.1	15.0	14.1	13.6
5Y	16.4	16.2	16.0	15.8	15.5	15.4	15.2	15.1	15.0	14.9	13.9	13.4
7Y	15.9	15.6	15.4	15.3	15.1	14.9	14.7	14.6	14.4	14.3	13.2	12.7
10Y	14.9	14.7	14.5	14.4	14.2	14.0	13.9	13.7	13.6	13.4	12.4	11.9
15Y	13.5	13.3	13.2	13.1	13.0	12.8	12.7	12.5	12.4	12.3	11.4	10.8
20Y	12.6	12.5	12.4	12.2	12.1	12.0	11.9	11.7	11.6	11.5	10.6	10.1

Figure 2.2.: Market quoted Swaption volatility matrix (in USD)

In figure 2.2, the “Expiry” column contains the expiry dates of the swaption, the “Tenor” row contains the lengths of the underlying swaps of the swaption. Given the market quoted swaption volatility matrix with ATM strike prices at certain maturity dates, we use bilinear interpolation in the direction of both “Expiry” and “Tenor” of the swaption, then we will obtain a full swaption volatility matrix for our desired tenor structure.

To use bilinear interpolation to find the volatility of a swaption, we perform linear interpolation first in the direction of swaption “Expiry”, then in the direction of swaption “Tenor”. For example, let $\sigma_{i,j}$ denotes the volatility for the swaption with option expiry at time T_i on a swap with length $\tau_j = T_j - T_0$, $T_0 = 0$. If $0 < i < \dots < k < n$, then the linear interpolation implies that

$$\sigma_{i,j} = \sigma_{0,j} + [\sigma_{k,j} - \sigma_{0,j}] \cdot \frac{T_i - T_0}{T_k - T_0} \quad (2.167)$$

Similarly, if $0 < j < \dots < l < n$, then the linear interpolation implies that

$$\sigma_{i,j} = \sigma_{i,0} + [\sigma_{i,l} - \sigma_{i,0}] \cdot \frac{T_j - T_0}{T_l - T_0} \quad (2.168)$$

Secondly, if the strike price of the desired swaption is not ATM, we need to add the corresponding skew value to the swaption volatility generated in the first step. Swaption skew values $\eta_{i,j}$ are quoted with respect to the option strike skewness equal to ± 200 , ± 100 , ± 50 , ± 25 basis points. The result matrix is as shown in figure 2.3:

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	-200bp	-100bp	-50bp	-25bp	ATM	25bp	50bp	100bp	200bp
3M2Y	+11.85	+4.67	+1.78	+0.69	0	-0.17	+0.14	+1.48	+4.71
3M5Y	+7.68	+2.73	+0.97	+0.37	0	-0.12	-0.01	+0.65	+2.63
3M10Y	+6.07	+1.99	+0.65	+0.22	0	-0.03	+0.11	+0.72	+2.44
3M20Y	+4.73	+1.50	+0.49	+0.18	0	-0.05	+0.01	+0.38	+1.58
3M30Y	+3.95	+1.18	+0.37	+0.13	0	-0.02	+0.04	+0.37	+1.41
1Y2Y	+7.51	+2.86	+1.16	+0.50	0	-0.33	-0.50	-0.43	+0.65
1Y5Y	+6.77	+2.49	+0.98	+0.42	0	-0.27	-0.39	-0.28	+0.72
1Y10Y	+5.47	+1.93	+0.73	+0.31	0	-0.18	-0.25	-0.11	+0.80
1Y20Y	+4.59	+1.63	+0.64	+0.27	0	-0.18	-0.28	-0.25	+0.34
1Y30Y	+4.08	+1.45	+0.57	+0.24	0	-0.17	-0.26	-0.27	+0.18
5Y2Y	+4.66	+1.93	+0.87	+0.41	0	-0.36	-0.68	-1.18	-1.71
5Y5Y	+4.77	+1.97	+0.89	+0.42	0	-0.37	-0.69	-1.18	-1.66
5Y10Y	+4.47	+1.84	+0.83	+0.39	0	-0.34	-0.64	-1.09	-1.51
5Y20Y	+4.15	+1.68	+0.75	+0.35	0	-0.30	-0.56	-0.93	-1.22
5Y30Y	+4.10	+1.65	+0.73	+0.34	0	-0.29	-0.53	-0.88	-1.11
10Y2Y	+3.90	+1.62	+0.73	+0.35	0	-0.31	-0.59	-1.03	-1.55
10Y5Y	+4.01	+1.67	+0.76	+0.36	0	-0.32	-0.60	-1.05	-1.53
10Y10Y	+3.90	+1.62	+0.73	+0.35	0	-0.31	-0.58	-1.01	-1.49
10Y20Y	+3.65	+1.50	+0.67	+0.32	0	-0.28	-0.52	-0.88	-1.22
10Y30Y	+3.57	+1.46	+0.65	+0.31	0	-0.27	-0.49	-0.84	-1.14
20Y2Y	+3.53	+1.47	+0.66	+0.32	0	-0.28	-0.53	-0.94	-1.42
20Y5Y	+3.67	+1.52	+0.68	+0.32	0	-0.29	-0.54	-0.94	-1.37
20Y10Y	+3.87	+1.59	+0.71	+0.33	0	-0.29	-0.54	-0.90	-1.21
20Y20Y	+3.67	+1.50	+0.67	+0.31	0	-0.27	-0.50	-0.84	-1.09
20Y30Y	+3.60	+1.47	+0.65	+0.31	0	-0.26	-0.48	-0.80	-1.01
30Y2Y	+3.42	+1.41	+0.64	+0.30	0	-0.27	-0.51	-0.88	-1.31
30Y5Y	+3.54	+1.47	+0.66	+0.31	0	-0.28	-0.52	-0.90	-1.28
30Y10Y	+3.81	+1.58	+0.71	+0.33	0	-0.29	-0.54	-0.90	-1.18
30Y20Y	+3.89	+1.61	+0.72	+0.34	0	-0.29	-0.54	-0.91	-1.19
30Y30Y	+3.95	+1.63	+0.73	+0.34	0	-0.30	-0.55	-0.91	-1.18

Figure 2.3.: Market quoted Swaption Skew matrix (in USD)

Similarly as in the first step, we use bilinear interpolation to extend the market quoted skew value grid. Using the resulting volatility $\sigma_{i,j} + \eta_{i,j}$ together with the Black's formula, we can obtain the price of our desired swaption.

2.4.2. The link between the Market Volatility Model and the Implied Trinomial Tree Method

Derman, Kani, and Chriss (1996) proposed the so-called “Implied trinomial tree” method to account for the volatility smile effect in the equity market and price equity options. This method is an extension of the traditional trinomial tree for pricing options, with each node of the tree denoting a (free) choice of state prices. The intuition behind the “implied trinomial tree” method is similar as for the Market volatility model, since both models derive volatility smile surfaces from market quotes of “standard” options actively traded in the market.

In the Black-Scholes model, volatility is assumed to be constant through the option's price, regardless of price changes in the underlying. This is obviously not so realistic. In contrast to the

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Black-Scholes model, the Implied trinomial tree method assumed that the local volatilities of the spot price processes is a function of the price themselves and time. Hence, the spot price processes are governed by the following SDE:

$$\frac{dS(t)}{S(t)} = f(t, t) dt + \sigma(t, S(t)) dW_P(t) \quad (2.169)$$

where $\sigma(t, S(t))$ is the local volatility which is a function of time t and the spot stock price $S(t)$ at t . r is the risk-free interest rate and W_P is the corresponding Brownian motion. The Implied trinomial tree method is based on the risk-neutral valuation approach. At each node, the expected value of stock prices must equal to the known forward price of that node, according to the no-arbitrage pricing theory. By interpolating the smile surface for different strike prices and time points corresponding to the nodes of the implied trinomial tree, we could use this tree to price call/put option on stock prices.

In practice, a set of market quotes of actively traded options (with respect to different level of strike prices and option expiration) on a desired stock are obtained, and practitioner calculates the implied volatilities from these quotes. For any point without a quote, interpolation and extrapolation methods are used to obtain the implied volatility from existing quotes. The prices of the corresponding equity options are then derived using these implied volatilities from the market. Hence, the volatility smile effect is embedded in the Implied trinomial tree method.

Derman and Kani (1994) showed that the local volatility function $\sigma(t, S(t))$ in equation (2.169) can be extracted directly from market quotes of actively-traded options¹⁰. Hence, the theory behind the Implied trinomial tree method is same as the one backing the Market Volatility model: with enough liquidity in the market, the market quotes of actively-traded options is efficient enough for interpreting the current market conditions. Therefore, a possible future research direction is to apply the Market volatility model in the equity market for pricing equity options. This is beyond the scope of this thesis and will not be discussed any further here.

2.5. Summary

Modeling interest-rate is a challenging task, because it involves modeling the term structure of the whole interest-rate yield curve. In this chapter, we review two of the most popular interest-rate models, the LMM model and the SABR model.

The LMM model is widely used in the interest-rate market since it is compatible with the Black's analytical pricing formula. The centre of the LMM model is modeling the evolution of forward interest-rates. Rebonato (2002) showed that there are several ways to describe the dynamics of

¹⁰As seen in [19]

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the forward LIBOR rates. In section 2.1, we review three equivalent descriptions of forward LIBORs, following by the discussion regarding the calibration of volatilities for interest rate caps and European swaptions.

Hagan et al. (2002) proposed the SABR model. It soon became popular among market practitioners. Under the SABR model framework, the implied Black volatility can be obtained via an analytical formula, which hugely increase the convenience for model implementation. In section 2.2, we review the framework of the SABR model and the associated volatilities functional forms.

Both LMM model and the SABR model have their drawbacks. The calibration procedure of the LMM model involves complex computing procedures. Although both models are widely used for pricing vanilla interest-rate derivatives, the models themselves must be modified if one wants to price exotic interest-rate derivatives. Moreover, we normally obtain the implied volatility surface of a reference product when we want to price exotic options, then we calibrate the volatility surface of our desired exotic option to that of the reference product. On one hand, the calibration procedures for both models are very time-consuming. On the other hand, the liquidity of the implied volatilities data of a reference product cannot be guaranteed, which gives invalid calibration results.

In order to overcome the drawback of the existing interest-rate models, we proposed a new type of interest-rate model, which we name as the “Market Volatility model”. The Market Volatility model arises from the real trading experience. This model interpolates the implied volatilities directly from the market quoted data and use the interpolated implied volatility surface as the input for pricing. No calibration is involved and so this model is computationally robust. In section 2.4, we discuss about the framework of the Market Volatility model and we go into the details of how the cap and swaption volatilities are computed via interpolation. In the next chapter, we will use the Market Volatility model to price several vanilla- and exotic- type interest-rate derivatives.

3. Pricing Interest-rate Options

The interest-rate market is one of the most active and important in finance. Many interest-rate derivatives were developed to users' requirement. Interest-rate options are financial instruments with payoffs dependent on the behaviour of interest rates. The pricing of interest-rate options normally depends on the behaviour of the whole zero-coupon yield curve. Therefore, a good model for the pricing of such options should capture the term-structure characteristic of the yield curve.

3.1. European-style Interest-Rate Options

Caps and European swaptions are two of the most popular interest-rate options available in the market. A log-normal model of LIBORs is very attractive, because it allows us to price caps and swaptions using Black's formula. In the standard pricing procedure, we assume that the underlying forward LIBOR rate follows a geometric Brownian motion under the market probability measure, P . For pricing a cap, it is assumed that the underlying forward LIBOR rate governed by each corresponding caplet has a log-normal distribution. For pricing an European swaption, it is assumed that the underlying swap rate is log-normally distributed on the option expiry date. Under these assumptions, one can develop the analytical pricing formula for caps and swaptions.

3.1.1. Interest-Rate Caps/Floors

Buying an interest-rate cap on a floating-rate note can prevent the user from losing money when the floating interest-rate rise above a certain level, this level is referred to as the *cap rate*. An interest-rate cap is formed by n call options on the underlying LIBOR rate, the frequency of the call options are same as the cap's tenor structure, and each underlying call option referred to as a *caplet* of the cap. Assume that we have a cap with maturity date T and a fixed cap rate K , that is, the strike price of each caplet in this cap is K . Suppose we have a set of reset/settlement dates $\{T_i; i = 1, \dots, n\}$ with $T_0 = 0$ and $T_{n+1} = T$. The length of time interval τ_i between two reset dates is $\tau_i = T_{i+1} - T_i$ for $i = 1, \dots, n$. Let L_i refer to the forward LIBOR rate for the period between time T_i and T_{i+1} , observed at time T_i for $i = 1, \dots, n$.

For $i = 1, \dots, n$, the payoff of the cap at time T_{i+1} is

3. Pricing Interest-rate Options

$$\tau_i \cdot \max (L (T_i, T_i) - K, 0) \quad (3.1)$$

By assuming that the underlying forward interest rates are log-normally distributed with zero drift rate, we could price a cap using a Black's type formula.

Assume that the forward LIBOR $L (t, T)$ is governed by the GBM process under the real-world probability measure P . Let Q be the risk-neutral measure, then

$$dL (t, T) = \sigma L (t, T) dW_Q (t) \quad (3.2)$$

It follows that the time- t price of a caplet with expiry date T and strike price K is

$$V_{cpl_i} (t) = \tau_i \cdot B (t, T + \tau_i) \cdot E_Q \left((L (T_i, T_i) - K)^+ \mid \mathcal{F}_t \right) \quad (3.3)$$

The price of a cap settled in arrears with reset dates $\{T_i\}$ is

$$V_{cap} (t) = \sum_{i=1}^n V_{cpl_i} (t) \quad (3.4)$$

$$= \sum_{i=1}^n \tau_i \cdot B (t, T_{i+1}) \cdot (L (t, T_i) \cdot N (d_1) - K \cdot N (d_2)) \quad (3.5)$$

with

$$d_1 = \frac{\ln (L (t, T_i) / K) + \frac{1}{2} \sigma_i^2 (T_i - t)}{\sigma_i \sqrt{T_i - t}} \quad (3.6)$$

$$d_2 = d_1 - \sigma_i \sqrt{T_i - t} \quad (3.7)$$

for every $i = 0, 1, \dots, n - 1$.

Interest-rate floors are defined in a similar way as the caps, except that an interest-rate floor is formed by n put options on the underlying LIBOR rate instead of call options, with each underlying put option referred to as a *floorlet* of the floor. Buying an interest-rate floor on a floating-rate note can prevent the user from losing money when the floating interest-rate fall below a certain level. Similar to the caps, a floor provides a payoff at time T_{i+1} which is

$$\tau_i \cdot \max (K - L (T_i, T_i), 0) \quad (3.8)$$

3. Pricing Interest-rate Options

for $i = 1, \dots, n$. The value of a floor is the sum of values of all associated floorlets. Since the value of cap/floor are the sum of value of interest-rate call/put options, a “cap-floor parity” relationship exists due to the fact of the put-call parity relationship. Using the cap-floor parity relationship together with equations (3.3), (3.6) and (3.7), the value of a floor can be derived:

$$V_{floor}(t) = V_{cap}(t) - \sum_{i=0}^{n-1} (B(t, T_i) - \tilde{\tau}_i \cdot B(t, T_{i+1})) \quad (3.9)$$

such that $\tilde{\tau}_i = 1 + K \cdot \tau_i$. It can be proved that the valuation formula obtained for caps and floors in the log-normal forward-LIBOR model agree with market practice.

3.1.2. European Swaptions

European swaptions are vanilla options on interest-rate swaps. Consequently, the holder of the swaption has the right to buy/sell a certain interest rate swap at a certain time in the future. As with the interest-rate swaps, the swaption holder can convert the floating interest rate payment into a fixed interest rate payments and vice-versa, and they have the additional optionality which guarantees that the fixed/floating rate of interest they will pay on a loan at some future time will not exceed a certain level. Because of the popularity of the European swaptions, a liquid swaption market (on the LIBOR rates) exists in all major currencies of the world.

The holder of an interest-rate swap will pay/receive a fixed rate in exchange of receiving/paying LIBOR rates, this fixed rate is referred to as the *swap rate*. A *payer swaption* allows the option holder to enter an interest-rate swap during the life of the swaption, in order to pay a fixed rate and receive the LIBOR rate in return. The *receiver swaption* works in the reverse way. In the market practice of pricing an European swaption, we usually assume that the underlying swap rate is log-normally distributed at the maturity of the option.

Let us consider a payer swaption with maturity T on an n -year underlying interest-rate swap, where the swaption holder pay a fixed rate K in exchange of the LIBOR rate. The length of time interval between two reset dates is defined as $\tau_i = T_{i+1} - T_i$ for $i = 1, \dots, n$. We define $T_0 = 0$ and $T_{n+1} = T$. Let $B(t, T)$ denotes the time- t price of a zero-coupon bond paying 1 unite of corresponding currency at time T . The forward swap rate $s(t)$ at time t is defined as

$$s(t) = \frac{B(t, T_0) - B(t, T_n)}{\sum_{i=1}^{n-1} \tau_i \cdot B(t, T_{i+1})} \quad (3.10)$$

where the values of zero-coupon bonds can be obtained by bootstrapping the yield curve.

Suppose that the compounding frequency of the underlying interest rate of the swap is m times per annum. Then the value of the payer swaption can be obtained via the Black’s type formula:

3. Pricing Interest-rate Options

$$V_{payer\ swaption} = \tilde{B} \cdot [s(T_0) \cdot N(d_1) - K \cdot N(d_2)] \quad (3.11)$$

such that

$$\tilde{B} = \frac{1}{m} \sum_{i=1}^{mn} B(T_0, T_i) \quad (3.12)$$

$$d_1 = \frac{\ln(s(T_0)/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad (3.13)$$

$$d_2 = \frac{\ln(s(T_0)/K) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \quad (3.14)$$

From the above equations, we can see that a payer swaption can be regarded as a call option on a swap. Similarly, a receiver swaption is a put option on a swap. The valuation formula of a receiver swaption is

$$V_{receiver\ swaption} = \tilde{B} \cdot [K \cdot N(-d_2) - s(T_0) \cdot N(-d_1)] \quad (3.15)$$

with $\tilde{B}$, d_1 , d_2 as defined in equations (3.12), (3.13) and (3.14).

3.2. Exotic Interest-Rate Options

3.2.1. Constant Maturity Swaps

A constant maturity swap (CMS) is a specific type of interest rate swap, where the effective floating rate is the swap rate for a swap of certain length. Hence, the reset frequency of the swap rate of the reference swap may be different from those of the CMS. Suppose we have a set of reset/settlement dates $\{T_i; i = 1, \dots, n\}$ with $T_0 = 0$ and $T_{n+1} = T$. The length of time interval τ_i between two reset dates is $\tau_i = T_{i+1} - T_i$ for $i = 1, \dots, n$. The payment on the settlement date T_{i+1} is equal to the swap rate observed on the preceding reset date T_i . Let $\hat{s}(T_i)$ be the swap rate observed at time T_i . For example, $\hat{s}(T_i)$ is equal to the two-year swap rate observed in the market at the current time. Then the floating payment at T_{i+1} will be

$$\tau_i \cdot \hat{s}(T_i) \quad (3.16)$$

As the normal swaps, the CMS can be valued as a series of cash flows associated to the discounted forward rates. However, the forward interest rates need to be adjusted according to the time lag

3. Pricing Interest-rate Options

between the fixing date and the settlement date for the observed interest rate. In market practice, this adjustment is known as the *convexity adjustment*.

Given the matrix of market quoted volatility's of swaps with at-the-money strike prices corresponding to certain maturity dates, we apply bilinear interpolation in the directions of terms of CMS and tenors of the underlying swaps, in order to obtain a complete matrix of implied volatilities for our desired term structure. The full approach was described in section 2.4.1.2.

In our work of pricing CMS, we consider CMS with interest rate payments settled in arrears. The underlying CMS rate are adjusted using the method proposed by Pelsser (2001). For derivative contracts with a payoff based on an interest rate of a single currency, the numeraire $B^*(T_i)$ which generates a martingale measure for the interest rate at time t is always a portfolio of discount bonds of the form

$$\sum_i^{n-1} \tau_i \cdot B(t, T_{i+1}) \quad (3.17)$$

Suppose that $s(T_i)$ is the forward value of the swap rate $\hat{s}(T_i)$. At time T_m such that $i \leq m \leq n+1$, we have the following approximation from Pelsser's method:

$$E^{P_{T_m}}(s(T_i)) = s(T_0) \cdot \left[\frac{\alpha + \beta \cdot s(T_0) \cdot e^{\sigma_y^2 T_i}}{\alpha + \beta \cdot s(T_0)} \right] \quad (3.18)$$

with

$$\alpha = \frac{1}{\sum_{i=1}^m \tau_{i-1}} \quad (3.19)$$

$$\beta = \frac{1}{s(T_0)} \cdot \left[\frac{B(T_0, T_m)}{B^*(T_0)} - \alpha \right] \quad (3.20)$$

and P_{T_m} is the probability measure associated to reset date T_m .

As defined in equation (3.10), the forward swap rate at time t is

$$s(t) = \frac{B(t, T_0) - B(t, T)}{B^*(t)} \quad (3.21)$$

In a CMS, the maturity date T_0 of the zero-coupon bond price $B(t, T_0)$ is in fact the starting date of the reference swap, and the maturity date T_n of the zero-coupon bond price $B(t, T)$ is equal to the last payment date of the reference swap. Using the adjusted forward swap rate generated by equation (3.18), we can calculate the time-0 value of a CMS.

3. Pricing Interest-rate Options

3.2.2. CMS Spread Options

Assume that we have two different underlying assets, S_1 and S_2 , with forward values denoted by $F_1(t)$ and $F_2(t)$ respectively. A spread option is an European option written on the difference between the payoff of S_1 and S_2 on the option maturity date. A CMS spread option is a special type of spread options, with the underlying $F_1(t)$ and $F_2(t)$ being the convexity-adjusted forward swap (CMS) rates. The payoff of a CMS spread option at maturity T is

$$\max(F_1(T) - F_2(T) - K, 0) \quad (3.22)$$

For a two-asset CMS spread call option, the payoff function is given as in equation (3.22). Similarly, the payoff for a two-asset CMS spread put option is

$$\max(K - F_1(T) + F_2(T), 0) \quad (3.23)$$

Assume that the CMS spread $S = F_1(T) - F_2(T)$ is a normally distributed random variable with volatility:

$$\sigma_S^2(F_1, F_2, \rho_{12}) = \sigma_1^2 F_1^2 + \sigma_2^2 F_2^2 - 2\rho_{12}\sigma_1\sigma_2 F_1 F_2 \quad (3.24)$$

where σ_1 and σ_2 are the log-normal volatilities of S_1 and S_2 respectively, and ρ_{12} is their historical correlation.

The correlation between two swap rates are calculated via the the following formula:

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2} \quad (3.25)$$

where σ_1 and σ_2 are the average volatilities of historical (monthly) log-normal returns of S_1 and S_2 respectively, and σ_{12} is the average covariance. In market practice, we observed that if the sampling frequency increase from monthly to weekly, or even daily, the result will not change significantly.

Therefore, the pricing formula for the value of a call option on the CMS spread is:

$$V_{call} = B(0, T) \cdot \left[(F_1(T) - F_2(T) + m - K) \cdot N(h) + \sigma_S \sqrt{T} \cdot n(h) \right] \quad (3.26)$$

such that

$$h = (S_1(T) - S_2(T) + m - K) / \sigma_S \sqrt{T} \quad (3.27)$$

where m is the spread margin, K is the option strike price, T is the time-to-fixing for the underlying CMS rates, and $B(0, T)$ is the zero-coupon bond price. $N(h)$ is the cumulative distribution function for h , and $n(h)$ is the associated probability mass function for h .

3. Pricing Interest-rate Options

3.3. Performance of LMM v.s. the Market Volatility Model

From the test results presented in sections 3.3.1 and 3.3.2 below, we show that using the same set of input market data, LMM and the Market Volatility model generates comparable option prices with same price trends. In contrast to the LMM that requires complicated and time-consuming calibration procedure in the implementation, the Market Volatility model is a robust method which is easy to implement and it produces option prices with acceptable accuracy. Therefore, we conclude that the Market Volatility model is a more efficient model for option pricing.

3.3.1. European Swaptions

In order to test the performances of the LMM and the Market Volatility model, we use both models to price European swaptions where the lives of the swaptions vary from three months up to five years, with the length of the underlying swap varies from three months up to five years. The market data used in our tests are obtained from the data quoted in the US market, on 3rd July 2006 on Bloomberg.

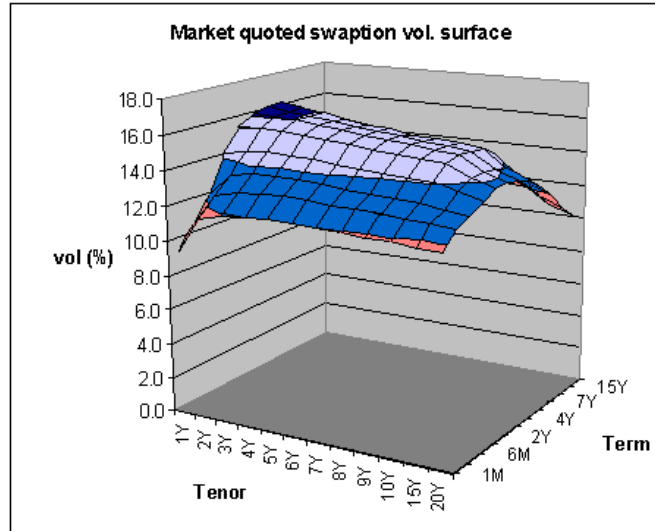


Figure 3.1.: Market quoted swaption volatility surface with respect to different option terms/tenors

In our tests, we price European swaptions with the quarterly LIBOR-rate curve as the underlying. Prices for European swaptions are obtained when the at-the-money strike prices, strike prices with -25/+25 basis points skewness, strike prices with -30/+30 basis points skewness, and strike prices with -200/+200 basis points skewness are used respectively. Figures 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8 and 3.9 below presents part of our test results. For the full set of test results, readers should refer to appendix A.1 of this thesis.

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From figures 3.2, 3.3, 3.4 and 3.5, we conclude that as the swaption strike prices increase with respect to the skewness, the absolute differences between the swaption volatilities calculated by LMM and the Market Volatility model tend to increase as well. The absolute differences for the short-term volatilities (from three month up to one year) tend to be larger, because we assumed a flat volatility structure for swaption with maturities between this period.

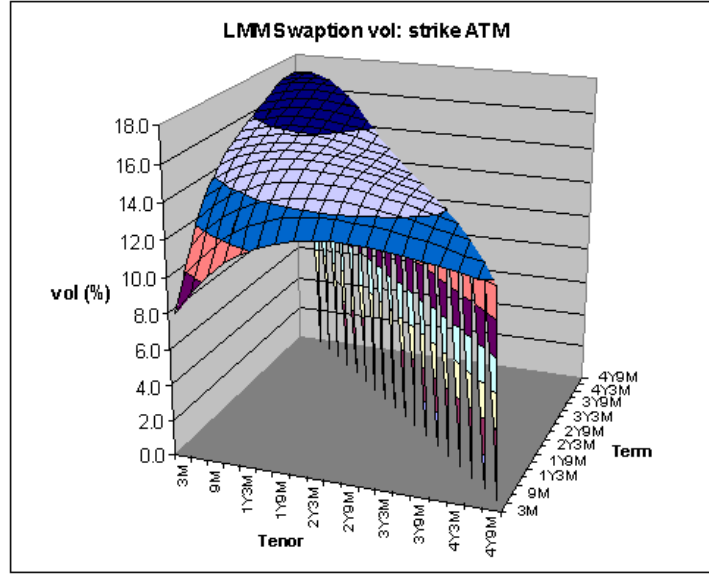


Figure 3.2.: Swaption volatilities with respect to the at-the-money strike prices, calculated by the LMM model

3. Pricing Interest-rate Options

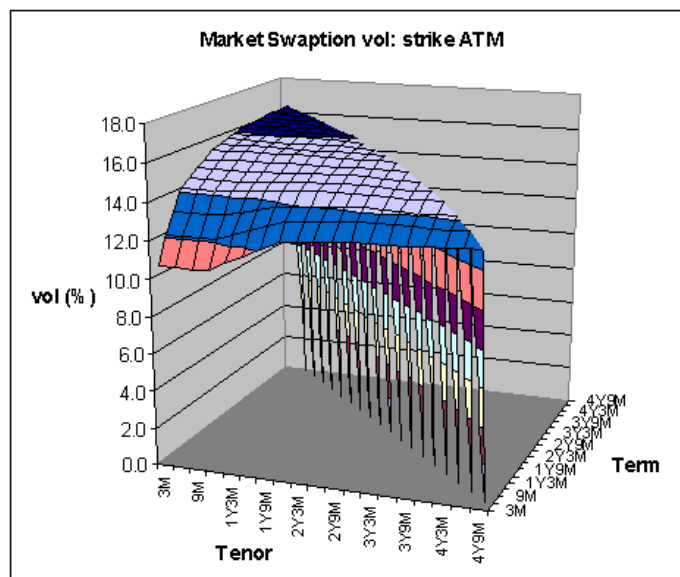


Figure 3.3.: Swaption volatilities with respect to the at-the-money strike prices, calculated by the Market Volatility model

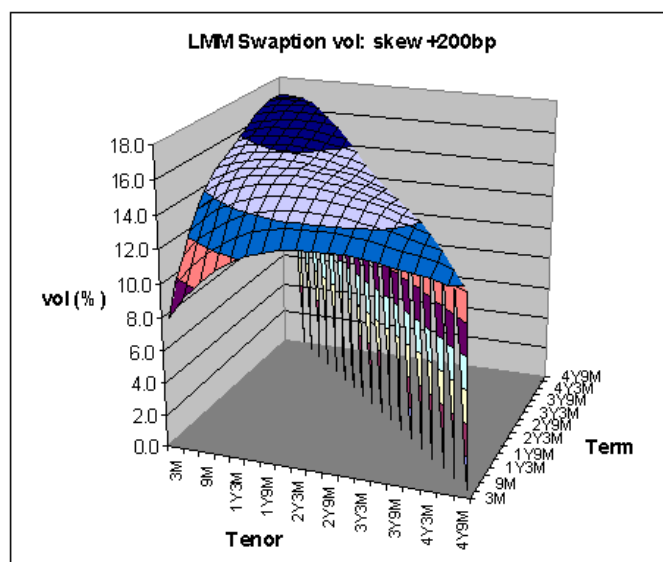


Figure 3.4.: Swaption volatilities with respect to strike prices with skewness +200 bp, calculated by the LMM model

3. Pricing Interest-rate Options

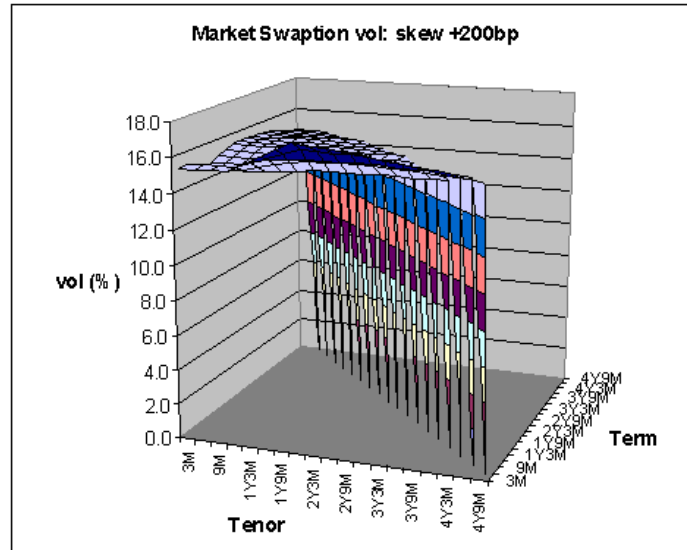


Figure 3.5.: Swaption volatilities with respect to strike prices with skewness +200 bp, calculated by the Market Volatility model

Figures 3.6, 3.7, 3.8 and 3.9 show that the absolute differences between the swaption prices and calculated by LMM and the Market Volatility model tend to increase with respect to the skewness of the swaption strike prices. In addition, the swaption prices outputted from the two models follow the same trend.

3. Pricing Interest-rate Options

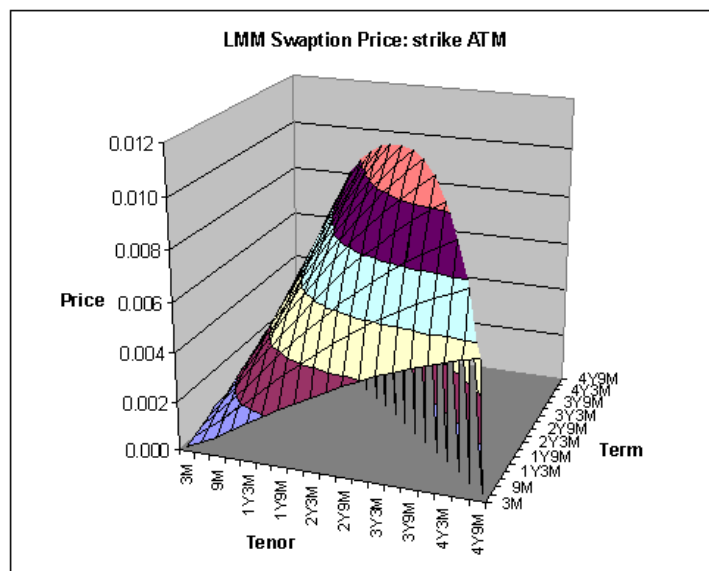


Figure 3.6.: Swaption prices with respect to the at-the-money strike prices, calculated by the LMM model

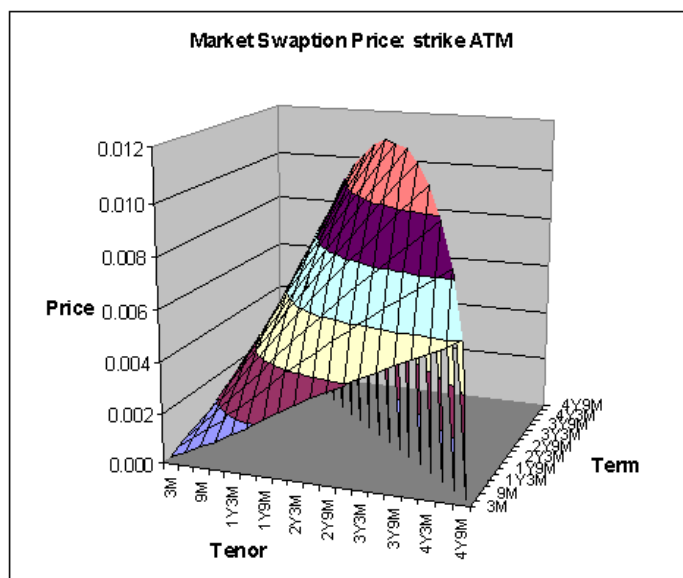


Figure 3.7.: Swaption prices with respect to the at-the-money strike prices, calculated by the Market Volatility model

3. Pricing Interest-rate Options

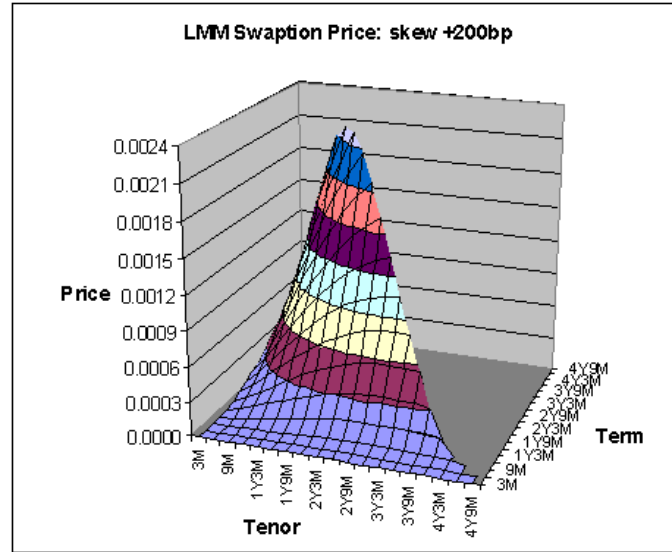


Figure 3.8.: Swaption prices with respect to strike prices with skewness +200 bp, calculated by the LMM model

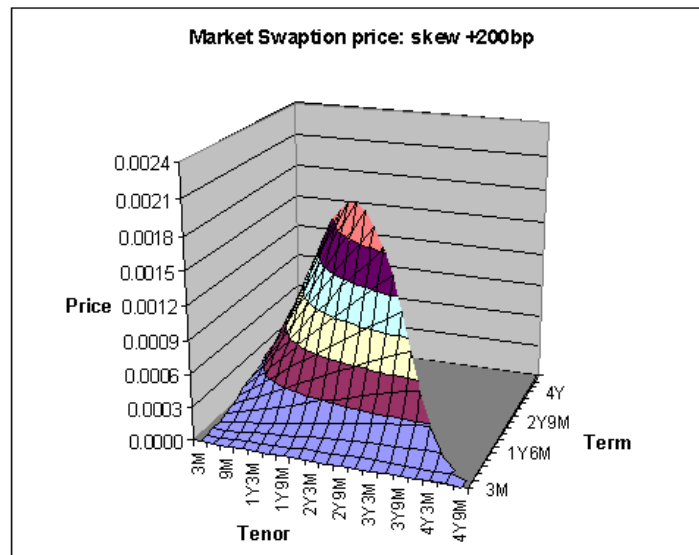


Figure 3.9.: Swaption prices with respect to strike prices with skewness +200 bp, calculated by the Market Volatility model

In section 2.1.7, we discussed about the theoretical incompatibility between the LSM and LFM models. This incompatibility could lead to biased test results when pricing European swaptions.

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In order to check the validity of results generated by the Market Volatility model, we also compared the output European swaption prices to the prices generated by Monte-Carlo simulations. Monte-Carlo simulations were performed based on the Milstein scheme discretization, with number of simulations being 10,000 and the time step size being 0.001.

Figures 3.10 and 3.11 represent swaption prices calculated by Monte-Carlo simulations. By comparing the results to those presented in figures 3.7 and 3.9, we conclude that swaption prices generated by Monte-Carlo simulations and the Market Volatility model both tend to increase with respect to the skewness of the swaption strike prices. In addition, the swaption prices generated by both models obtain same shapes and trend, which again proved the validity of test results generated by the Market Volatility model.

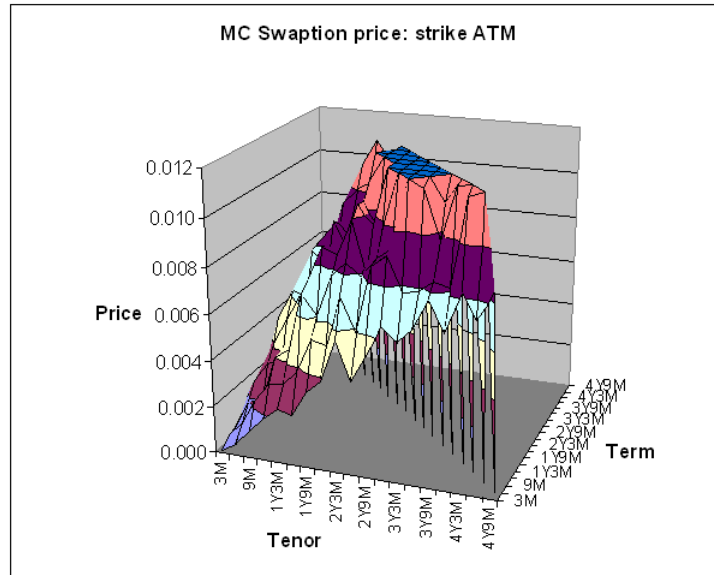


Figure 3.10.: Swaption prices with respect to the at-the-money strike prices, calculated by Monte-Carlo simulations

3. Pricing Interest-rate Options

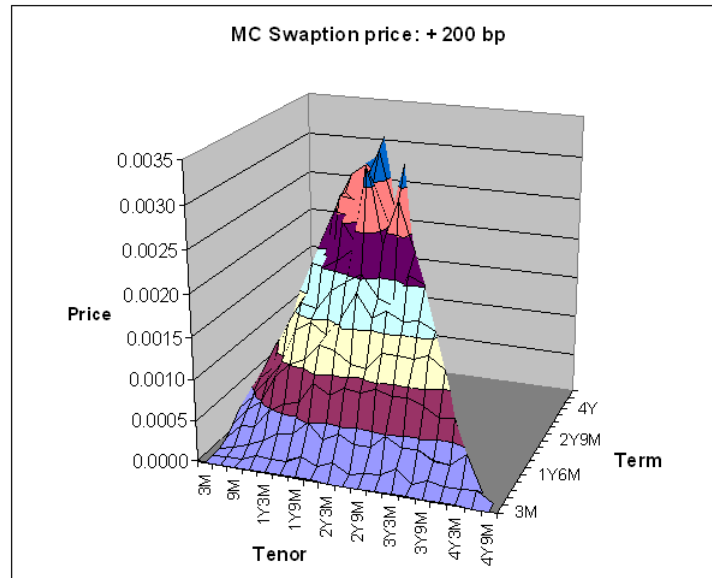


Figure 3.11.: Swaption prices with respect to strike prices with skewness +200 bp, calculated by Monte-Carlo simulations

3.3.2. Caps

In order to test the performances of the LMM and the Market Volatility model, we use both models to price interest-rate caps with maturities vary from three months up to five years. The market data used in our tests are obtained from the data quoted in the US market, on 3rd July 2006 on Bloomberg.

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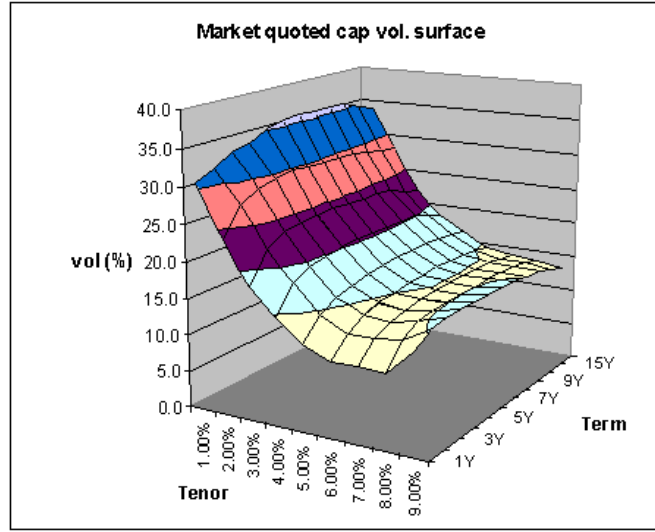


Figure 3.12.: Market quoted cap volatility surface with respect to different option terms/tenors

In our tests, we price interest-rate caps with the quarterly LIBOR-rate curve as the underlying. Prices for caps are obtained when the at-the-money cap rates, cap rate 0.03, cap rate 0.05, and cap rate 0.075 are used respectively. Figures 3.13, 3.14, 3.15 and 3.16 below presents part of our test results. For the full set of test results, readers should refer to appendix A.2 of this thesis.

From figures 3.13 and 3.14, we conclude that as the cap rates decrease with respect to the skewness, the absolute differences between the caplet volatilities calculated by LMM and the Market Volatility model tend to increase. The absolute differences for the short-term volatilities (from three month up to one year) tend to be larger, because we assumed a flat volatility structure for caps with maturities between this period.

3. Pricing Interest-rate Options

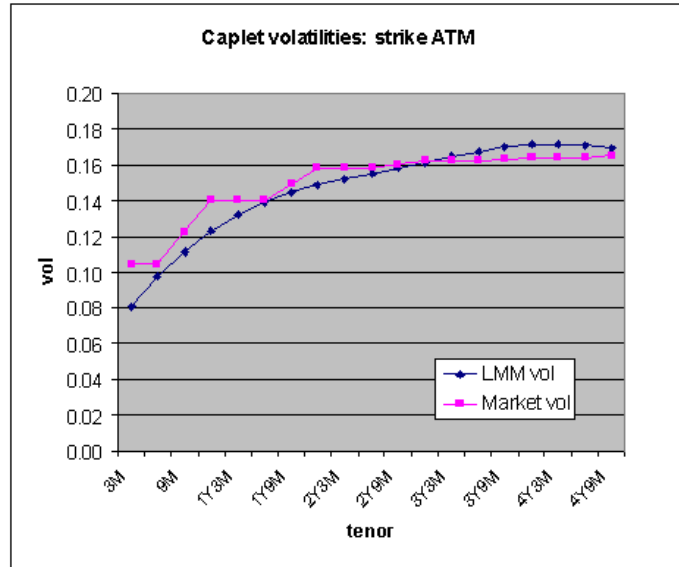


Figure 3.13.: Caplet volatilities with respect to the at-the-money strike prices, calculated by the LMM and the Market Volatility model respectively

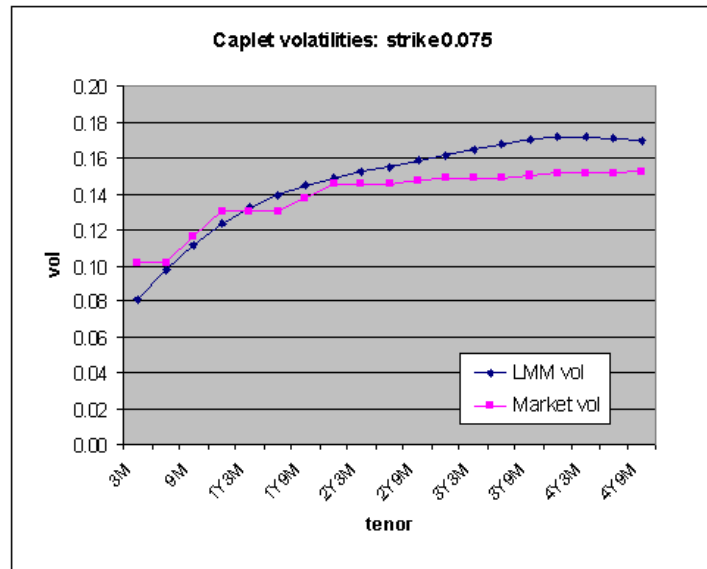


Figure 3.14.: Caplet volatilities with respect to strike prices with skewness around +200 bp, calculated by the LMM and the Market Volatility model respectively

Figures 3.15 and 3.16 show that the absolute differences between the cap prices and calculated by

3. Pricing Interest-rate Options

LMM and the Market Volatility model tend to increase with respect to the skewness of the swaption strike prices. In addition, the cap prices outputted from the two models follow the same trend.

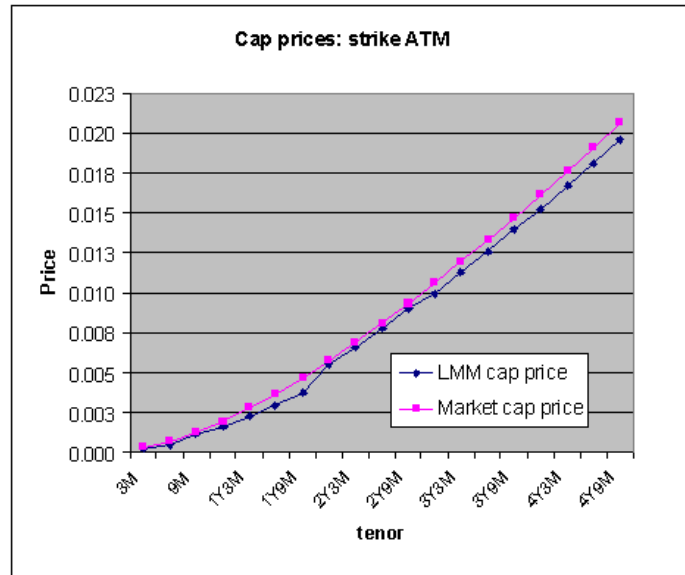


Figure 3.15.: Cap prices with respect to the at-the-money strike prices, calculated by the LMM and the Market Volatility model respectively

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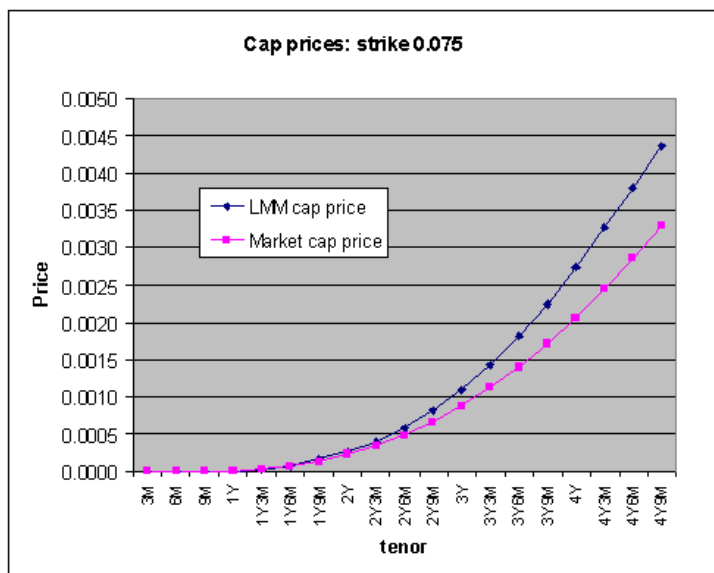


Figure 3.16.: Cap prices with respect to strike prices with skewness around +200 bp, calculated by the LMM and the Market Volatility model respectively

3.4. Pricing Exotic Interest-Rate Options Using the Market Volatility Model

Following the conclusion in section 3.3, we use the Market Volatility model to price CMS swaps and CMS spread options, as described in sections 3.2.1 and 3.2.2. The market data used in our tests are obtained from the data quoted in the JPY market, on 13th January 2007 on Bloomberg. The spread margin is set to be 0.

For pricing CMS swaps, we value the CMS rates of a 10-years CMS on a 10-years swap, a 10-years CMS on a 5-years swap, and a 2-years CMS on a 5-years swap, respectively. We use the annually compounding yield curve and the notional principal is 10,000 JPY. The test results generated by the Market Volatility model were compared to the over-the-counter (OTC) prices quote, and were proved to replicate OTC prices with an acceptable accuracy. Part of the test results were shown in tables B.1, B.2 and B.4 in appendix B.1.

Figures 3.17 and 3.18 below show the swap rates and prices for a 10-years CMS on a 10-years swap, generated by the Market Volatility model and from the OTC market quotes, respectively.

3. Pricing Interest-rate Options

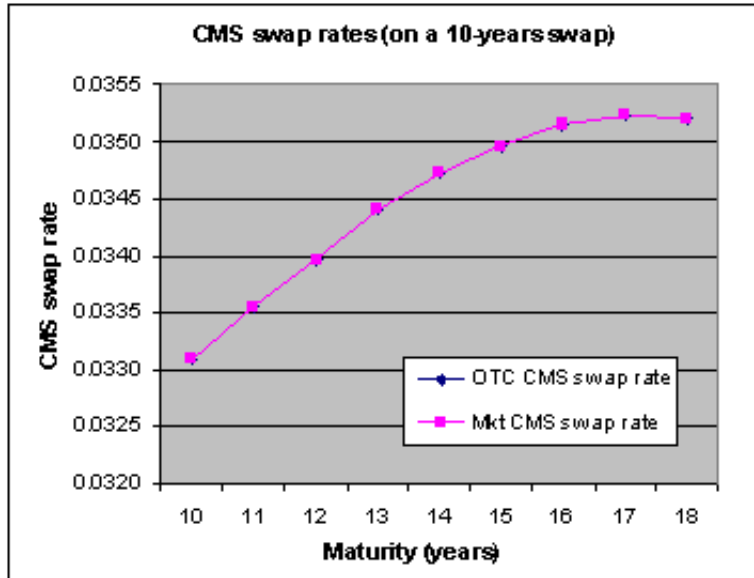


Figure 3.17.: CMS swap rates from OTC quotes and the Market Volatility model respectively

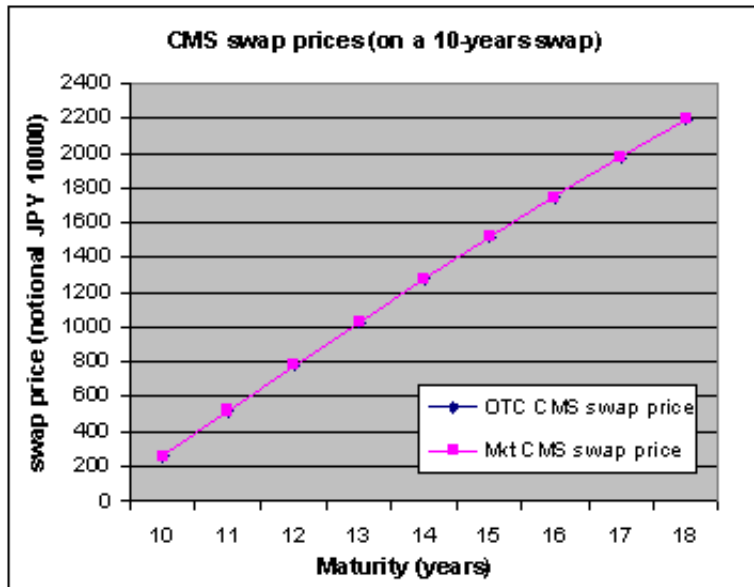


Figure 3.18.: CMS swap prices from OTC quotes and the Market Volatility model respectively

Figures 3.19 and 3.20 below show the swap rates and prices for a 2-years CMS on a 5-years swap, generated by the Market Volatility model and from the OTC market quotes, respectively.

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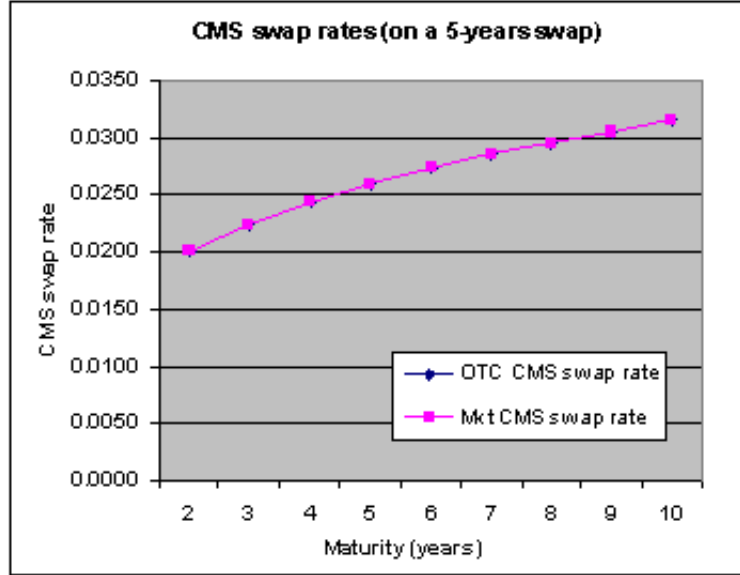


Figure 3.19.: CMS swap rates from OTC quotes and the Market Volatility model respectively

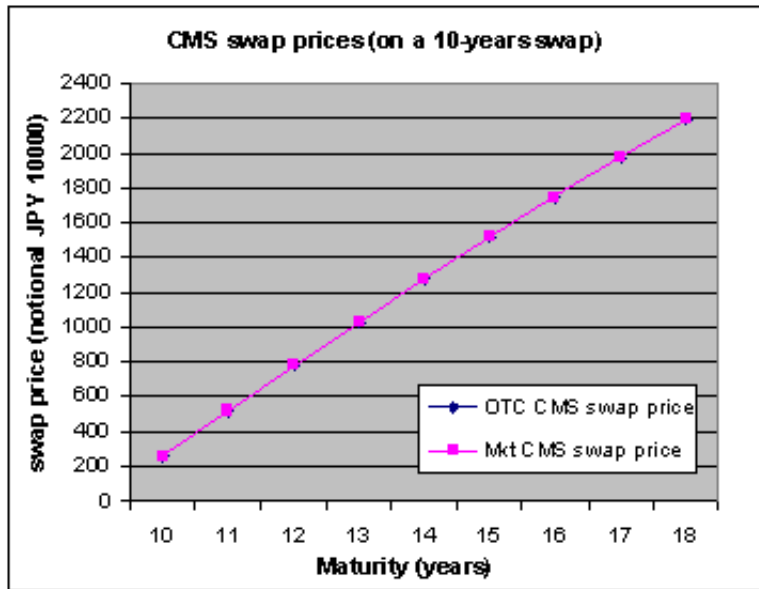


Figure 3.20.: CMS swap prices from OTC quotes and the Market Volatility model respectively

For both CMS swap rates and prices, the relative differences between results generated by the Market Volatility model and the OTC market quotes are within 10^{-5} . We conclude that the

3. Pricing Interest-rate Options

Market Volatility model values CMS rates and the corresponding convexity adjustments correctly, since the differences are insignificant.

Using the annually compounding yield curve and a notional principal equals to 10,000 JPY, we value a CMS spread option on the spread of a 10-years swap and a 2-years swap. This CMS spread options is referred to as the 2Y-10Y CSO. Test results are shown in Appendix B.2 for a 10-years CMS 10Y-2Y spread option with different strike prices. Figures 3.21 and 3.22 below show the volatilities and prices of the 2Y-10Y CSO, generated by the Market Volatility model and from the OTC market quotes, respectively.

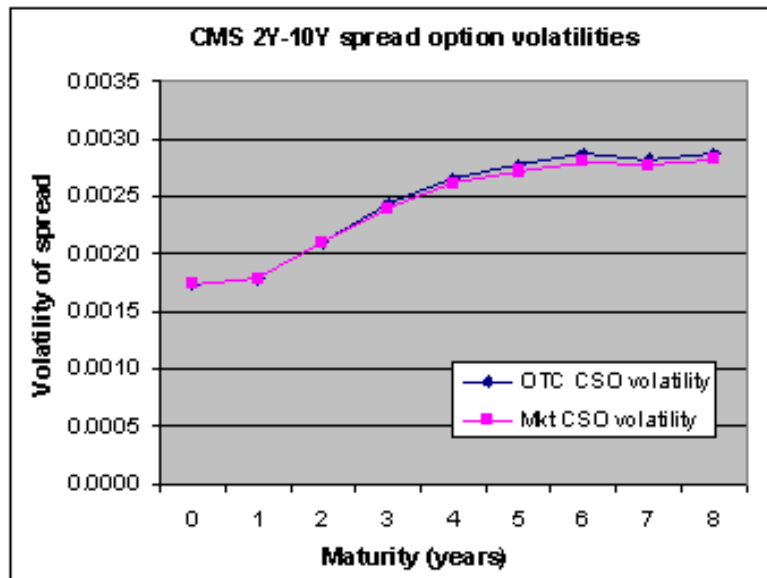


Figure 3.21.: CSO volatilities for spread between 2Y and 10Y CMS swap rates

3. Pricing Interest-rate Options

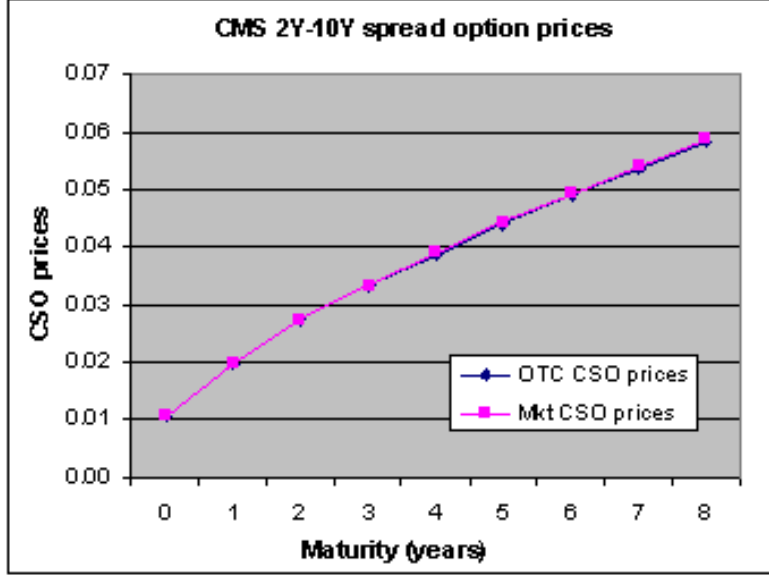


Figure 3.22.: CSO prices for spread between 2Y and 10Y CMS swap rates

For both 2Y-10Y CSO volatilities and prices, the relative differences between results generated by the Market Volatility model and the OTC market quotes are within 10^{-2} . We conclude that the Market Volatility model values 2Y-10Y CSOs correctly, since the differences are acceptable.

3.5. Summary

In this chapter, we review various existing interest-rate derivatives in the market. We then test the model performance of the Market Volatility model proposed in the previous section 2.4. In addition, we prove that the Market Volatility model is capable of pricing both vanilla- and exotic-type interest-rate derivatives.

In section 3.1, we review the structures of two most important and commonly-used Vanilla interest-rate derivatives, caps and European swaptions. In section 3.2, we review the structure of two Exotic interest-rate derivatives, CMS swap and CMS spread option. In order to test the pricing performance of the Market Volatility model, we used this model to compute the volatility surfaces with the smile effect for interest-rate caps and swaptions. The result volatility surfaces are then used to price interest-rate caps and swaptions via the Black's formula. The output volatilities and prices for interest-rate caps and swaptions are compared to results generated by the LMM model for the same products. In addition, Monte-Carlo simulations were used to generate European swaption prices and the results are compared to those of the Market volatility model. The corresponding

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test results are shown in section 3.3. We conclude that the Market Volatility model produced comparable results to those of the LMM model. Moreover, the Market Volatility model can be used to price exotic interest-rate derivatives, due to its assumption of intuitive structure of volatilities. In contrast, the LMM model cannot be simply used to price exotic interest-rate derivatives without model modification. Here we use the Market Volatility model to price the CMS spread option (option on the spread between 2-years and 1-years CMS swap rates) and the results were shown in section 3.4. By comparing test results generated by the Market Volatility model to the over-the-counter quotes, we conclude that the Market Volatility model produces comparable results.

4. From the Interest-Rate Market to the Energy Market

4.1. Introduction of the Energy Market

The energy markets involve different types of commodities, in particular fuels. Fuel commodities include oil, natural gas, coal and the corresponding refined products. Electricity, emission, outage insurance are also included in the category of the energy market.

In this thesis, we focus on oil and natural gas markets, the two main markets in the category of fuels. Since oil and natural gas are physical commodities, the activities involving oil and natural gas normally include the production, distribution and consumption process.

4.1.1. The Oil Market

In the past decades, the oil market has developed rapidly to become the largest commodity market in the world. Refined products such as heating oil and gasoline are traded in this market as well. In this thesis, the oil market that we focused on is the crude oil market.

As any other commodity, the price of crude oil is affected by the supply-demand relationship. Shortage in supply will result in the rise of the oil price, and oversupply will result in the fall of the oil price. In order to effectively control the oil prices, OPEC was formed in 1960 by the five main oil-producing countries: Saudi Arabia, Iran, Iraq, Kuwait and Venezuela. In 1970s, another six oil-exporting countries also joined the OPEC.

Between year 1950 and 1960, the price of crude oil remained around \$13 per barrel. In the late 1970s, the outbreak of the Iran/Iraq war led to a sharp fall of the daily production of the crude oil. As a result, the oil price rose from \$14 to \$35 per barrel at the start of 1980s. The rapid increase in oil price helped to create a revolution of the technologies used in the energy-saving section. Consumers started to apply better insulation to their homes, and more energy efficient power plants were built. Between year 1982 to 1985, OPEC attempted to restrict the oil production outputs so to control the price level. However, various members of OPEC frequently broke the rule and over-produced. In late 1985, the attempt of stabilising oil prices by OPEC finally collapsed.

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Subsequently, an active forward and spot market come into existence, which has trading horizons extending to over 10 years forward.

There exists a positive relationship between the crude oil price and the health of the world economy. In the short term, the weakness of the world economy will cause the weakness of the world's main currencies, which will result in a rise of the oil price. However, in the long-term effect, industrial activities will reduce largely when the world economy enters into a recession. As a result, the demand of oil will be weakened. The above effects were demonstrated well by the movements of the oil price through year 2007 to 2009, when the recent financial crisis (the credit crisis) happened in all the world's main economies caused by the collapse of the over-heated housing market. During the credit crisis, crude oil price was driven high, both by the tight supply-demand relationship in the market and the market speculation on a rising trend of the price of crude oil. The weakness of the US dollar during this period also contributed to the rise of oil price. The price of crude oil exceeded \$140 per barrel at its peak in June 2008, before dropping sharply and fluctuating between \$50 - \$70 per barrel in 2009.

Different grades are defined for the crude oil, with respect to the percentage of the gravity and sulfur content. For example, the most expensive type of crude oil is termed as the "sweet" crude oil. The term "sweet" means that the sulfur content in the crude oil is below 0.5%. Crude with high sulfur content is termed as "sour". For the world benchmark, the Brent crude oil is used. According to historical data, up to two-thirds of the world supply is priced with the Brent as the reference. For the United States, the West Texas Intermediate (WTI) crude oil is set to be the benchmark.

The main forward market in the United States is the New York Mercantile Exchange(NYMEX), with the underlying prices usually base on the price of the light sweet crude oil. The scale of the spot market is much smaller than the forward market.

4.1.2. The Natural Gas Market

The user of the gas market involves the industrial, commercial and residential consumers. The market price of natural gas is comprised of two factors, which are the "original" price of the underlying gas itself and the transportation cost of the gas.

Through the 1970s to 1980s period, the natural gas market underwent dramatic changes. In the past, both price of underlying gas and the corresponding transportation cost are controlled by the federal regulation. Nowadays, the price of underlying gas is no longer regulated, so it fluctuates with respect to the supply/demand relationship. In addition, the cost of the transportation services and the associated sales functions are required to be separated for all pipelines. As a result, the market price of natural gas is un-regulated. Despite the un-regulated market, the price of natural

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gas between the late 1980s to late 1990s actually shows a downward trend. This is mainly due to the declining cost of the transmission and distribution.

The market price of natural gas is affected heavily by seasonality. Usually, the residential consumption of natural gas will peak in the winter season and the consumption is relatively low in summer. This is due to the high demand of the home heating during winter. However, the demand of natural gas may also grow in summer, because the the demand for air-conditioning is high during summer. Subsequently, natural gas will be consumed by the gas-powered electrical generation. To give an example, from year 1997 to 1998, the winter season is warmer than normal starting from November 1997, which continued through out the entire heating season (from October 1997 to March 1998). In addition, this warmer weather is experienced for all regions of United States. Because of the lower demand of natural gas, the storage level of gas was sufficient enough for the nation's supply throughout the eating season. Hence, the price of natural gas experienced a sharp decline compared to the previous heating season. Figures 4.1 and 4.2 below give a detailed demonstration for the natural gas price movement in our example.

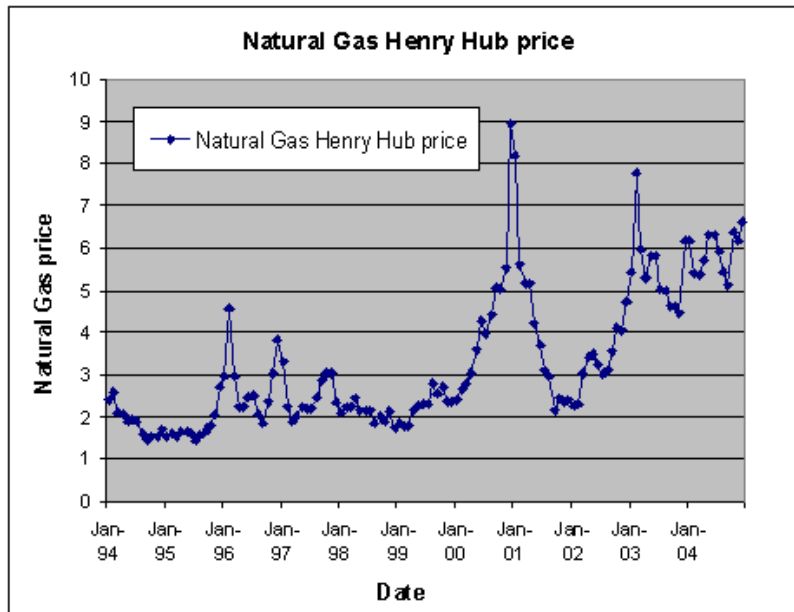


Figure 4.1.: Historical prices (Henry Hub) for Natural gas between year 1994 - 2004

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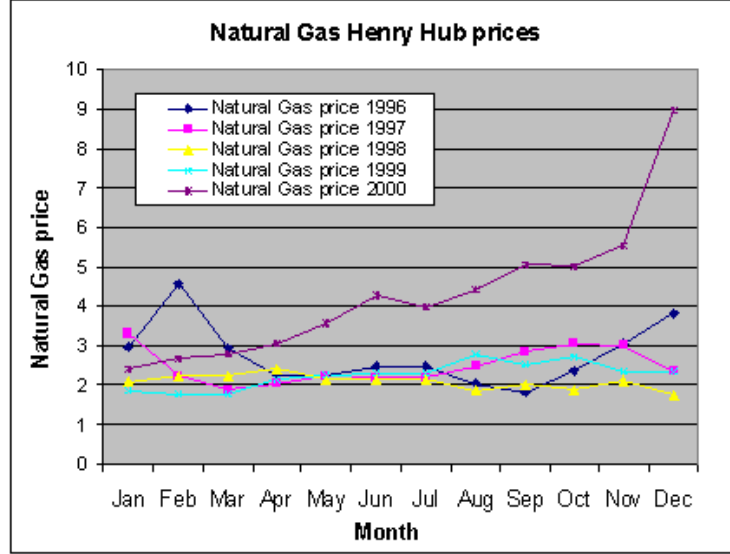


Figure 4.2.: Historical prices (Henry Hub) for Natural gas for year 1996 - 2000

The physical trade/transactions in the natural gas market occurs on a daily or monthly basis. The monthly transactions usually occur during the so-called *bid-week*, which is the last week of the month preceding the contract month¹. The benchmark price in the monthly market is referred to as the *index price*. The index price is not the price of a single contract, instead, it is the average of the transaction prices of the monthly contracts. By using an average price, the speculation activities on large price movements caused by temporary market conditions (which is likely to happen in the volatile commodity markets) could be avoided.

4.2. The Spot-Forward Relationship in the Energy/Commodity Market

In the general market practice of asset price, for example, when the underlying asset is the price of a stock, we assume that the asset price follows the GBM process. Given a unit of stock with spot price $S(t)$ at time t , then $S(t)$ is governed by the following SDE:

$$dS(t) = \mu S(t) dt + \sigma S(t) dW_P(t) \quad (4.1)$$

where μ is the drift rate and σ is the associated volatility. Both μ and σ are constants.

¹as in reference [21]

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For the equity market, it is common that the stock holder will receive dividend of the stock at a pre-agreed frequency. Suppose that we have a forward contract on the stock that we specified above, maturing at time T . Let us denote the forward price of the stock by $F(t, T)$. When the stock pays a constant dividend yield d , we have the following relationship between the spot price and the forward price of the stock:

$$F(t, T) = S(t) e^{(r-d)(T-t)} \quad (4.2)$$

where r is the risk-free interest rate. The relationship in equation (4.2) can be proved using the no-arbitrage argument. In the case of non-dividend paying stock, equation (4.2) will simply become

$$F(t, T) = S(t) e^{r(T-t)} \quad (4.3)$$

Equation (4.2) works well in the equity markets or the interest-rate markets, however, it doesn't always hold for the commodity market. In the general no-arbitrage argument for pricing forward contracts, we assume that we will be able to short or borrow the underlying asset without cost. For the physical commodities such as oil and gas, this is not always true. There is likely to be a cost for storing these commodities, and the amount of storage cost could be high. In the presence of the storage cost, we can only achieve an inequality via the no-arbitrage argument:

$$F(t, T) \leq (S(t) + c) e^{r(T-t)} \quad (4.4)$$

The equation (4.4) can be proved using the no-arbitrage argument. To start with, assume that

$$F(t, T) > (S(t) + c) e_t^{r(T-t)} \quad (4.5)$$

Using equation (4.5), we can create the following arbitrage strategy:

At time t	At maturity T
Borrow $S(t) + c$ of cash at the risk-free rate; buy the commodity in the spot market and pay storage costs; store the commodity until the maturity of the forward contract	Return $-(S(t) + c) e^{r(T-t)}$ for borrowed cash and the interest
Short the forward contract at $F(t, T)$	Exercise the forward contract (sell the commodity) to receive $F(t, T)$
	Generate a risk-less profit $F(t, T) - (S(t) + c) e_t^{r(T-t)} > 0$

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Under the no-arbitrage condition, market will automatically self-adjust in a short time, in order to eliminate existing arbitrage opportunities. Hence, the relationship in equation (4.5) which generates a profit through arbitrage will not hold long. More specifically, it will not hold until the maturity of the forward contract. Therefore, the relationship in equation (4.4) must be true.

It is common that a commodity is hold not for the investment purpose, but for consumption. In this case, the holder of the commodity would not wish to sell the commodity, if they predict that there will be a shortage in supply during the period between the sale and regaining the ownership. Hence, an additional premium need to be paid so the commodity holder will agree to sell for the profit. Let us denote this additional premium by c_p . Using the same method as above, we assume that

$$(S(t) + c) e^{r(T-t)} - F(t, T) > c_p \quad (4.6)$$

Based on the assumption in equation (4.6), the holder of commodity can create the following arbitrage strategy:

At time t	At maturity T
Sell the commodity to receive $S(t)$ and the released the storage cost c ; invest $S(t) + c$ at the risk-free rate	Close the bank account and cash in $(S(t) + c) e^{r(T-t)}$
Long the forward contract at $F(t, T)$	Exercise the forward contract (buy the commodity): $-F(t, T)$
	Generate a risk-less profit $(S(t) + c) e^{r(T-t)} - F(t, T) > c_p$

With the guarantee from the profit generated by the above strategy, the holder of commodity will be willing to sell. Since the market tends to no-arbitrage, the relationship in equation (4.6) will quickly re-balance and become

$$0 < (S(t) + c) e^{r(T-t)} - F(t, T) \leq c_p \quad (4.7)$$

From relationships in (4.4) and (4.7), we can deduce that

$$F(t, T) = S(t) e_t^{(r+c_y)(T-t)} \quad (4.8)$$

with $c_y = c^* - c_p$, where c^* is the scaled storage cost and c_p is the benefit from the (physical) commodity. c_y is normally referred to as the *convenience yield*. When the convenience yield is constant, we can employ the risk-neutral valuation method by assuming that the price of the underlying commodity is governed by the following process

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$$dS(t) = (r - c_y) S(t) dt + \sigma S(t) dW_P(t) \quad (4.9)$$

given the relationship in (4.8).

4.3. Common Pricing Models in the Energy Market

4.3.1. The Bachelier Model

In 1900, Bachelier proposed a pricing model based on arithmetic Brownian motion processes. Under this model, the spot price process S satisfies the following stochastic differential equation:

$$dS(t) = \mu dt + \sigma dW_P(t) \quad (4.10)$$

where μ and σ are the drift rate and volatility coefficient respectively, with σ being strictly positive. $W_P(t)$ is the Brownian motion at time t under the real-world probability measure P . It can be easily deduced from the properties that equation (4.10) implies that $E(dS(t)) = \mu dt$. (4.10) also implies that the price changes $dS(t)$ are independent and stationary, which are desirable properties from the modeling aspect. Given $S(t + \varepsilon)$ as the price change over the time period $[t, t + \varepsilon]$ of a stock/ commodity, one drawback of Bachelier's model is that $S(t + \varepsilon)$ may take negative values. This is inconsistent with the real market condition. For example, if S is the spot price of crude oil or natural gas, then S will obviously be positive.

4.3.2. The Black-Scholes Model

In order to overcome the drawback of the Bachelier's model, a modified approach was proposed in 1965 by Paul Samuelson. Instead of assuming the stock price follows an arithmetic Brownian motion, Samuelson assumed that the return of the stock follows an arithmetic Brownian motion. In this way, the possibility of negative stock price is omitted. Based on this assumption, Black and Scholes (1973) proposed their famous model which leads to an analytical pricing formula for vanilla call options:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW_P(t) \quad (4.11)$$

with μ , σ and $W_P(t)$ as defined in previous section 4.3.1.

In the Black-Scholes model, returns are normally distributed. In addition, the Black-Scholes model assumes that the stock price grows on average over time. This assumption is consistent with the nature of many energy commodities. Many commodities indexes are given as arithmetic averages

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of prices of the underlying commodity over a certain period, i.e. a month, in order to rule out speculation in the market. Since the index prices are quoted as arithmetic averages, it is impossible for a single market player to manipulate the market by trading large volumes of commodities. In addition, analytical pricing formula can be easily achieved using the Black-Scholes model. Therefore, this model is widely employed in pricing energy derivatives, especially in the oil market.

4.3.3. The Mean-Reverting Model

Vasicek (1977) proposed a model for the evolution of interest rates. In his model, the dynamic of the short-term rates are assumed to follow the Ornstein-Uhlenbeck process:

$$df(t) = \alpha [\mu - f(t, t)] dt + \sigma dW_Q(t) \quad (4.12)$$

with α , μ , σ as positive constants and $W_Q(t)$ is the risk-neutral measure. For $f(t, t) < \mu$, the expected change of the short-term rate is positive. For $f(t, t) > \mu$, the expected change is negative and this creates a property of mean-reversion towards μ . This feature of the mean-reverting model makes it suitable for the energy market. Because one trend exhibited by commodity prices is that they tend to mean-revert to a level in short term. One of the popular mean-reverting model used in the energy is

$$\frac{dS(t)}{S(t)} = \alpha [\mu - \ln S(t)] dt + \sigma dW_Q(t) \quad (4.13)$$

with α as the mean-reversion coefficient, μ , σ as defined in section 4.3.1. The mean-reverting models are usually employed in modeling spot price processes as for pricing daily options, because they capture the mean-reverting property in the spot market.

4.3.4. The Jump-Diffusion Model

Although the geometric Brownian motion has been a benchmark model for pricing energy options, the “jump-diffusion” models has attracted interests as they can account for the unexpected large-scale movements in spot prices. Merton (1976) proposed the well-known jump-diffusion model by adding a “jump” component to the diffusion term when modeling spot prices:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW_Q(t) + \varphi(t) dJ(t) \quad (4.14)$$

with μ , σ and $W_Q(t)$ as defined in previous section 4.3.1. $\{\varphi(t)\}$ are normally distributed random variables which represents the magnitude of “jumps”, so it can take either positive or negative values. $J(t)$ is a Poisson jump process.

²As seen in reference [22]

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4.3.5. The Stochastic Volatility Model

Based on Heston's stochastic volatility model, Eydeland and Geman (1998) proposed an extension of the model which allows potential state variables such as the convenience yield. The result is a two factor mean-reverting model:

$$\frac{dS(t)}{S(t)} = \alpha [\mu - \ln S(t)] dt + \sigma(t) dW_1(t) \quad (4.15)$$

$$d\bar{\sigma}(t) = \beta [\theta - \bar{\sigma}(t)] + \gamma \sqrt{\bar{\sigma}(t)} dW_2(t) \quad (4.16)$$

such that

$$\bar{\sigma}(t) = \sigma^2(t) \quad (4.17)$$

$$dW_1(t) \cdot dW_2(t) = \rho dt \quad (4.18)$$

with $\alpha, \beta, \theta, \gamma$ as positive constants.

4.4. The Market Volatility Model for the Energy Market

In practice, we price energy derivatives (i.e. spread options, basket options, Asian options) on oil or gas future contracts based on the assumption of absence of the convenience yield. Because the future market has high liquidity, the trades can be done in a short time and so the effect of storage cost could be ignored. Under this assumption, the relationship in equation (4.8) will become the relationship in equation (4.3), which is basically the spot-forward relationship for the interest-rate market. In this case, the Market Volatility model that we described in section 2.4 could be applied in the pricing of commodity derivatives, but with the underlying changed from forward rates to forward prices of commodities.

Under the assumption of absence of the convenience yield, the spot price process described in equation (4.9) will become

$$dS(t) = \mu S(t) dt + \sigma S(t) dW_P(t) \quad (4.19)$$

which is the general GBM process.

Applying the Girsanov theorem to equation (4.19), we will obtain the driftless stochastic process for the forward prices of the underlying commodities:

$$dF(t, T) = \sigma F(t, T) dW_Q(t) \quad (4.20)$$

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where Q is the risk-neutral probability measure.

Based on the process in (4.20) with the method for interpolating market quoted volatilities as described in section 2.4, we will be able to price various energy derivatives via numerical computing procedures, which is theoretically simple to implement but is a robust and powerful method. In this way, we extend the Market Volatility model to the energy market. In the later chapter, we propose a new type of exotic energy derivative, which is named as the *Flex-Asian spread option* (FASO). We then use the methodology of the Market Volatility model to price the FASO. For the detail of how this was done, readers should refer to the section 5.4.

4.5. Summary

In chapter 3, we showed that the Market Volatility model can be used to price various vanilla- and exotic-type of interest-rate derivatives. Following the analysis of the Market Volatility model in the previous chapters, in this chapter we look into a new market that the Market Volatility model can be applied to.

In section 4.1, we give a brief specification of the world's two main fuel markets: the oil market and the natural gas market. In section 4.2, we review the difference between the spot-forward relationships for the interest-rate market and the commodity/energy market. We showed that the two different spot-forward relationships become the same under the assumption of the absence of the convenience yield. Under this circumstances, the Market Volatility model can be applied to the commodity market, and we can price commodity derivatives with the Market Volatility model using same numerical techniques that we use for pricing interest-rate derivatives. In the next chapter, we propose a new type of exotic energy derivative and we will use the Market Volatility model to price this derivative.

5. Pricing Exotic Energy Options

5.1. Spread Options

As in interest-rate markets, the spread option is one of the most popular derivatives in the commodity markets. The spread between the payoffs of two different underlyings is largely traded in the energy commodity markets. Theoretically, we can use a spread option on all the fundamental physical assets in the energy market, such as oil refiners, storage facilities, etc. Spread options can also be employed in constructing a portfolio of energy derivatives.

A spread option between two commodities S_1 and S_2 is an European option written on the spread between the payoffs. Let $F_1(t)$ and $F_2(t)$ be the forward values of S_1 and S_2 . In the energy market, the spread $F_1 - F_2$ will normally be the spread between a oil/gas input and the output will either be electricity or a refined product¹. The payoff of a spread call option at option maturity T is

$$\max(F_1(T) - F_2(T) - K, 0) \quad (5.1)$$

As described in section 3.2.2, by letting spread $S = F_1(T) - F_2(T)$, the pricing formula for the value of a call spread option is

$$V_{call} = B(0, T) \cdot \left[(F_1(T) - F_2(T) + m - K) \cdot N(h) + \sigma_S \sqrt{T} \cdot n(h) \right] \quad (5.2)$$

such that

$$h = (S_1(T) - S_2(T) + m - K) / \sigma_S \sqrt{T} \quad (5.3)$$

where m is the spread margin, K is the option strike price, T is the time-to-fixing for the underlying CMS rates, and $B(0, T)$ is the zero-coupon bond price. $N(h)$ is the cumulative distribution function for h , and $n(h)$ is the associated probability mass function for h . σ_S is the volatility of the spread given by equations (3.24) and (3.25).

¹as in reference [21]

5.2. Basket Options

A basket option is an European option written on the payoff of the combined value of two or more underlying assets. The spread option that we described in section 5.1 can be considered as a basket option of two underlying assets. Basket options play an important role in the portfolio management area. The option user can buy a basket option in order to hedge the risk arising from the price movements of the assets in the basket, which will cost less than hedging the risk using a basket of single options on each asset.

Let us consider a basket of n underlying assets, such that the asset prices are not perfectly correlated. The forward price of the i th asset in the basket at maturity T is $F_i(T)$. Then the payoff of a basket call option at maturity day T is

$$\max \left(\sum_{i=1}^n w_i F_i(T) - K, 0 \right) \quad (5.4)$$

with K as the strike price of the option and w_i as the weight assigned to the i th underlying commodity in the basket.

In the market of pricing a basket option, it is normally assumed that the price of each underlying asset is log-normally distributed. However, the distribution of a sum of correlated log-normal random variables will not be log-normal. As a result, numerical methods must be used to price basket options.

5.3. Asian Options

The market of Asian options developed rapidly, and it is now one of the most commonly traded derivatives in the energy market. That is because many energy indices quoted forward asset prices as the arithmetic averages of the spot price. For example, in the oil market, it usually takes a long time for the transportation and delivery of the crude oil. During the long delivery period, market conditions can be extremely volatile. By quoting the arithmetic average of oil prices on the index, the effect of a spike of the oil price which happened in a short period can be reduced and speculation activities due to the sharp movements in prices can be avoided. Therefore, most oil options tend to be Asian options.

There are two types of Asian options: the average-strike Asian option and the average-price Asian option. The latter one is more popular among users, and it is the type of Asian option that we chose to use. The payoff of the average-price Asian option is the spread between the option strike price and the arithmetic average of forward prices of the underlying asset over a certain time period.

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Given a set of reset dates $\{T_i; i = 1, \dots, n\}$ with $T_0 = 0$ and $T_{n+1} = T$. Let the forward price associated to the reset date T_i be $F(T_i)$. Given a pre-specified option strike price K , the payoff of an average-price Asian call option is

$$\max(F^*(T) - K, 0) \quad (5.5)$$

such that

$$F^*(T) = \frac{\sum_{i=1}^n F(T_i)}{n} \quad (5.6)$$

Similarly, the payoff of an average-price Asian put option is

$$\max(K - F^*(T), 0) \quad (5.7)$$

with $F^*(T)$ as defined in equation (5.6).

Under the standard assumptions for asset price dynamics, there is no closed-form representation of the risk-neutral probability distribution of the arithmetic average of asset prices. Hence, numerical methods must be used when pricing (arithmetic) average-price Asian options.

5.4. Flex-Asian Spread Options (FASO)

According to the trader's need, we proposed a new type of energy derivatives, which has a flexible option payoff structure and is compatible for pricing many types of structured energy products. We name this energy derivative as the *Flex-Asian spread option* (FASO).

The terminal pay-off function of the FASO is defined by the user according to user's need. The payoff function is specified as either $\max[L_1(T), L_2(T), \dots, L_n(T)]$ or $\min[L_1(T), L_2(T), \dots, L_n(T)]$, where $L_i(T)$, $i = 1, \dots, n$ are the values generated from the n payoff scenarios at option expiry time T . L_i is the i th payoff scenario of the option, and it is a user-defined linear combination of payoffs of the underlying assets. Each L_i is specified through the setting of the price averaging windows and forward prices of the underlying assets, together with the fixed strike price associated to this payoff scenario. By specifying the weighting coefficients featuring in the L_i , and by varying the number of payoff scenarios, one has significant flexibility in matching a terminal pay-off function to that embedded in the contractual terms of a physical supply deal.

The FASO can handle options with up to 50 payoff scenarios. For simplicity, here we concentrate on FASO which is based on the different combinations of payoffs of six underlying assets. In order to demonstrate the flexibility of the FASO, we will price four different payoff structures of FASO

5. Pricing Exotic Energy Options

in the rest of this section, which we simply name as the FASO type-1, type-2, type-3 and type-4 options. The payoff structures related to different option configurations will be shown in the following sections.

5.4.1. The FASO Type-1 Option

This type of Option involves 6 different underlying commodities and three payoff scenarios, with the option expiry date lies before all the price averaging windows. Let $F_i(T)$ be the forward price

for the i th underlying commodity of the option at maturity date T , and w_{ji} be the weight assigned to the j th payoff scenario of the i th commodity in the option. The option payoff structure is shown as in figure 5.1 below:

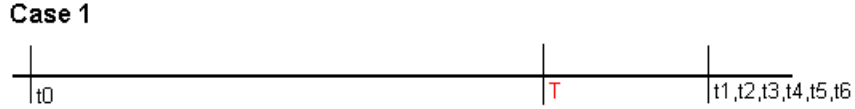


Figure 5.1.: Payoff structure of FASO type 1

The payoff of the FASO type-1 option is governed by the following function:

$$V(T) = \max[\phi((w_{11}F_1(T) - w_{12}F_2(T)) - (w_{13}F_3(T) - w_{14}F_4(T)) - K_1; (w_{21}F_1(T) - w_{22}F_2(T)) - (w_{25}F_5(T) - w_{26}F_6(T)) - K_2; 0] \quad (5.8)$$

where $\phi = 1$ for a call and -1 for a put option. K_1 and K_2 represent the strike prices of the option for the first and second payoff scenarios, respectively.

Since the option expiry date is set prior to the start of all price averaging windows, it is effectively a hybrid of a basket option on six future contracts, so there is no price averaging features in the terminal payoff.

5.4.2. The FASO Type-2 Option

This type of option involves 3 different underlying commodities and three payoff scenarios, with the option expiry date lies in the first price averaging window and before the other two price averaging windows.

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Let $F_i(T)$ be the forward price for the i th underlying commodity of the option at option maturity date T . If the expiry date is set to lie in one of the pricing windows, then the weighting used to calculate the average price will be modified for any price averaging window in which the option expiry lies. The option payoff structure is as shown in figure 5.2 below:

Case 2

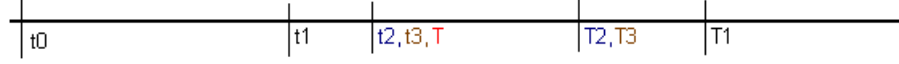


Figure 5.2.: Payoff structure of FASO type 2

The payoff of the option is governed by the following function:

$$V(T) = \max\left[\phi\left(\frac{1}{D_1} \sum_{t_i=t_1}^{T_1} F_1(t_i) + \left(\frac{D_1 - d_1}{D_1}\right) F_1(T) - F_2(T) - K_1; \frac{1}{D_1} \sum_{t_i=t_1}^{T_1} F_1(t_i) + \left(\frac{D_1 - d_1}{D_1}\right) F_1(T) - F_3(T) - K_2; 0\right]\right] \quad (5.9)$$

where $\phi = 1$ for a call and -1 for a put option. K_1 and K_2 represent the strike prices of the option for the first and second payoff scenarios, respectively. T_1 is the final fixing date of the pricing window of the first commodity. D_1 is the total number of days in the pricing window of the first commodity, and d_1 is the number of days between T and T_1 . $F_1(t_i)$ is the forward price for the first commodity evaluated at days $t_i \in [t_0, T_1]$.

This option is effectively a hybrid of an Asian spread option on three future contracts.

5.4.3. The FASO Type-3 Option

This type of option involves 3 different underlying commodities and three payoff scenarios, with the option expiry date lies after all the price averaging windows. In this case, the option expiry date is set to be the longest final fixing of the price averaging windows.

Let $F_i(T)$ be the forward price for the i th underlying commodity of the option at option maturity date T . The option payoff structure is as shown in figure 5.3 below:

5. Pricing Exotic Energy Options

Case 3

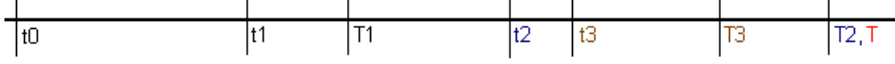


Figure 5.3.: Payoff structure of FASO type 3

The payoff of the option is governed by the following function:

$$V(T) = \max\left[\phi\left(\frac{1}{D_1} \sum_{t_i=t_1}^{T_1} F_1(t_i) - \frac{1}{D_2} \sum_{t_j=t_2}^{T_2} F_2(t_j) - K_1; \frac{1}{D_1} \sum_{t_i=t_1}^{T_1} F_1(t_i) - \frac{1}{D_3} \sum_{t_l=t_3}^{T_3} F_3(t_l) - K_2; 0\right)\right] \quad (5.10)$$

where $\phi = 1$ for a call and -1 for a put option. K_1 and K_2 represent the strike prices of the option for the first and second payoff scenarios, respectively. T_i is the final fixing date of the pricing window of the i th commodity. D_i represent the total number of days in the i th price averaging window. $F_1(t_i)$, $F_2(t_j)$, $F_3(t_l)$ are the forward prices of the first, second and third commodities, evaluated at days $t_i \in [t_0, T_1]$, $t_j \in [t_0, T_2]$, $t_l \in [t_0, T_3]$, respectively.

This option is effectively a hybrid of an Asian spread option on three futures contracts.

5.4.4. The FASO Type-4 Option

This type of option involves 3 prices during different pricing periods of one underlying commodity and three payoff scenarios, with the option expiry date lies in the first price averaging window, after the second price averaging window and before the third price averaging window.

Similar to case 2, if the expiry date is set to lie in one of the pricing windows, then the weighting used to calculate the average price will be modified for any price averaging window in which the option expiry lies. Let $F_i(T)$ be the forward price for the i th underlying commodity of the option at option maturity date T . The option payoff structure is as shown in figure 5.4.4 below:

Case 4

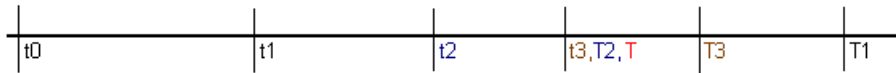


Figure 5.4.: Payoff structure of FASO type 4

5. Pricing Exotic Energy Options

The payoff of the option is governed by the following function:

$$V_T = \max[\phi(\frac{1}{D_1} \sum_{t_i=t_1}^{T_1} F_1(t_i) + \left(\frac{D_1-d_1}{D_1}\right) F_1(T) - \frac{1}{D_2} \sum_{t_j=t_2}^{T_2} F_2(t_j) - K_1; \frac{1}{D_1} \sum_{t_i=t_1}^{T_1} F_1(t_i) + \left(\frac{D_1-d_1}{D_1}\right) F_1(T) - F_3(T) - K_2; 0] \quad (5.11)$$

where $\phi = 1$ for a call and -1 for a put option. K_1 and K_2 represent the strike prices of the option for the first and second payoff scenarios, respectively. T_i is the final fixing date of the pricing window of the i th commodity. D_1 is the total number of days in the pricing window of the first commodity, and d_1 is the number of days between T and T_1 . $F_1(t_i)$ and $F_2(t_j)$ are the forward prices of the first and second commodities, evaluated at days $t_i \in [t_0, T_1]$, $t_j \in [t_0, T_2]$, respectively.

This option is effectively a hybrid of an Asian spread option on three futures contracts.

5.5. Test Scenarios for FASO

The structure of the FASO that we proposed in section 5.4 has great flexibility, hence, it is very useful when applying to the real physical trade according to user's varying need. In this section, we price the FASO type-1, 2, 3 and 4 options with different inputs.

5.5.1. Moneyness Definition

Let $F_i(T)$ be the forward price for the i th underlying commodity of the option observed at the starting time t , and w_{ji} be the corresponding weight for the i th underlying commodity associated with the j th pay-off scenario of the option. We define the overall moneyness associated with the j th pay-off scenario of the FASO as

$$\chi_j = \frac{1}{K_j} \sum_{i=1}^N w_{ji} F_i(t) \quad (5.12)$$

with K_j as the constant strike price associated with the j th pay-off scenario of the option. We test across different scenarios for different levels of χ_j , by varying the forward prices while using a constant combination of weights of the underlings.

For the FASO type-1 option, the strike prices and the weights used are shown in table 5.1 below. Table 5.5.1 shows the forward prices used for each level of moneyness.

5. Pricing Exotic Energy Options

K_1	500					
w_{1j}	1	-1	-1	1	0	0
K_2	300					
w_{2j}	1	-1	0	0	-1	0

Table 5.1.: Strike prices and weights for the FASO type-1 option

χ_1, χ_2	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
$F_1(t)$	100	150	150	200	300	250	450	400	500	650
$F_2(t)$	50	50	50	100	100	150	150	100	100	150
$F_3(t)$	50	50	50	100	100	150	50	100	50	100
$F_4(t)$	100	150	250	400	400	650	450	600	550	600
$F_5(t)$	90	60	70	60	100	40	80	70	60	200
$F_6(t)$	100	80	150	200	200	300	200	250	200	300

Table 5.2.: Starting forward prices F_j with respect to moneyness χ_1 and χ_2 for the FASO type-1 option

For the FASO type-2, 3 and 4 options, the strike prices and the weights used are shown in table 5.3 below. Table 5.4 shows the forward prices used for each level of moneyness.

K_1	50		
w_{1j}	1	-1	0
K_2	30		
w_{2j}	1	0	-1

Table 5.3.: Strike prices and weights for the FASO type-2, 3 and 4 options

χ_1, χ_2	0.2	0.6	1	1.4	1.8
$F_1(t)$	100	100	100	100	100
$F_2(t)$	90	70	50	30	10
$F_3(t)$	94	82	70	58	46

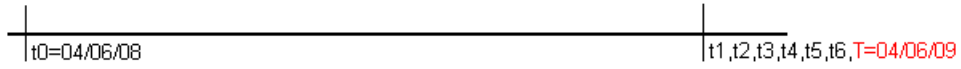
Table 5.4.: Starting forward prices F_j with respect to moneyness χ_1 and χ_2 for the FASO type-2, 3 and 4 options

Note that the forward prices in tables 5.5.1 and 5.4 are synthetic data chosen for the testing purposes. We could also extract an oil/gas forward price curve from the market quoted data for our inputs, but it has no impact on the methodology of our test scenarios.

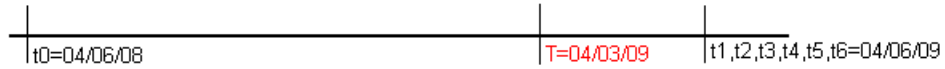
5.5.2. Structures of the Pricing Windows in FASO

For each type of FASOs described in section 5.4, we consider different test scenarios with respect to different pricing window structures of the option. The option structures for different scenarios are shown in the figures below:

Case 1.1



Case 1.2



Case 1.3

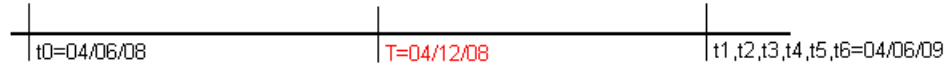
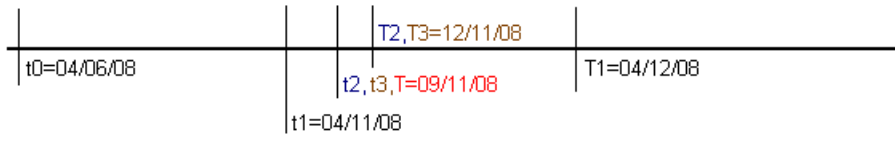


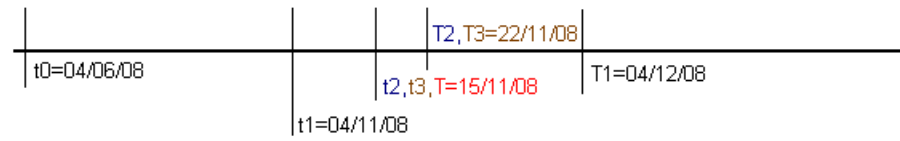
Figure 5.5.: Test scenarios for the FASO type-1 option

5. Pricing Exotic Energy Options

Case 2.1



Case 2.2



Case 2.3

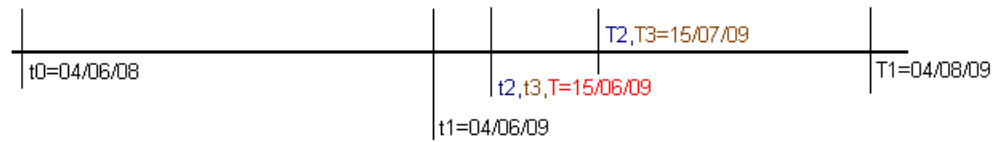
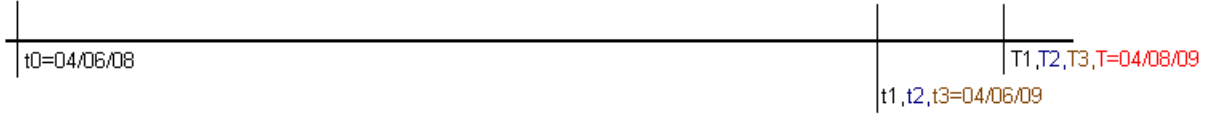


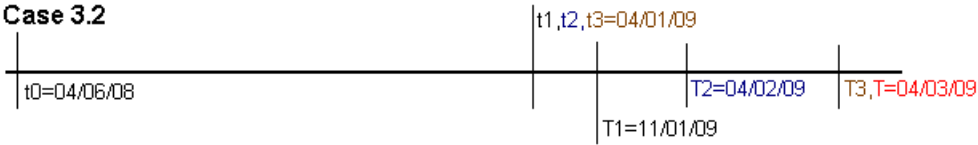
Figure 5.6.: Test scenarios for the FASO type-2 option

5. Pricing Exotic Energy Options

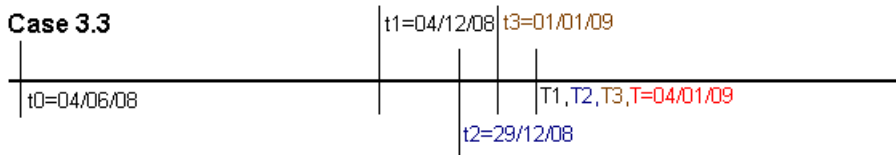
Case 3.1



Case 3.2



Case 3.3



Case 3.4

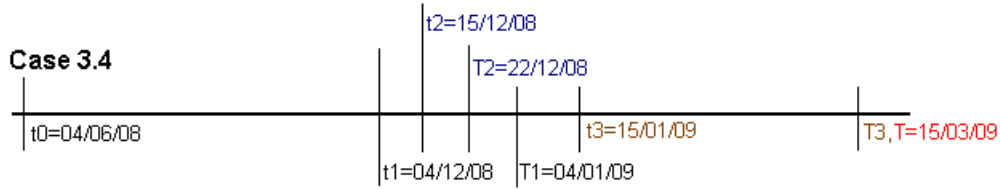
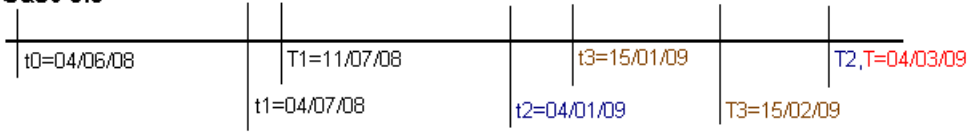


Figure 5.7.: First set of test scenarios for the FASO type-3 option

5. Pricing Exotic Energy Options

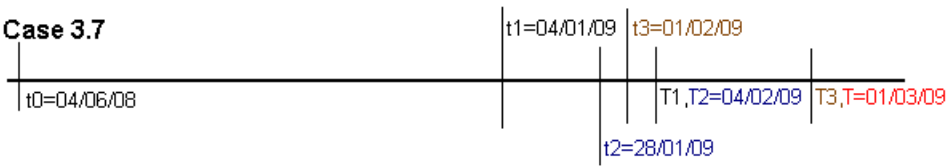
Case 3.5



Case 3.6



Case 3.7



Case 3.8

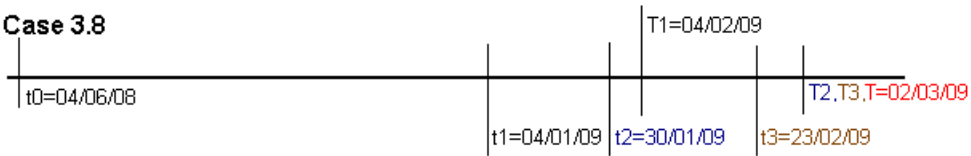


Figure 5.8.: Second set of test scenarios for the FASO type-3 option

5. Pricing Exotic Energy Options

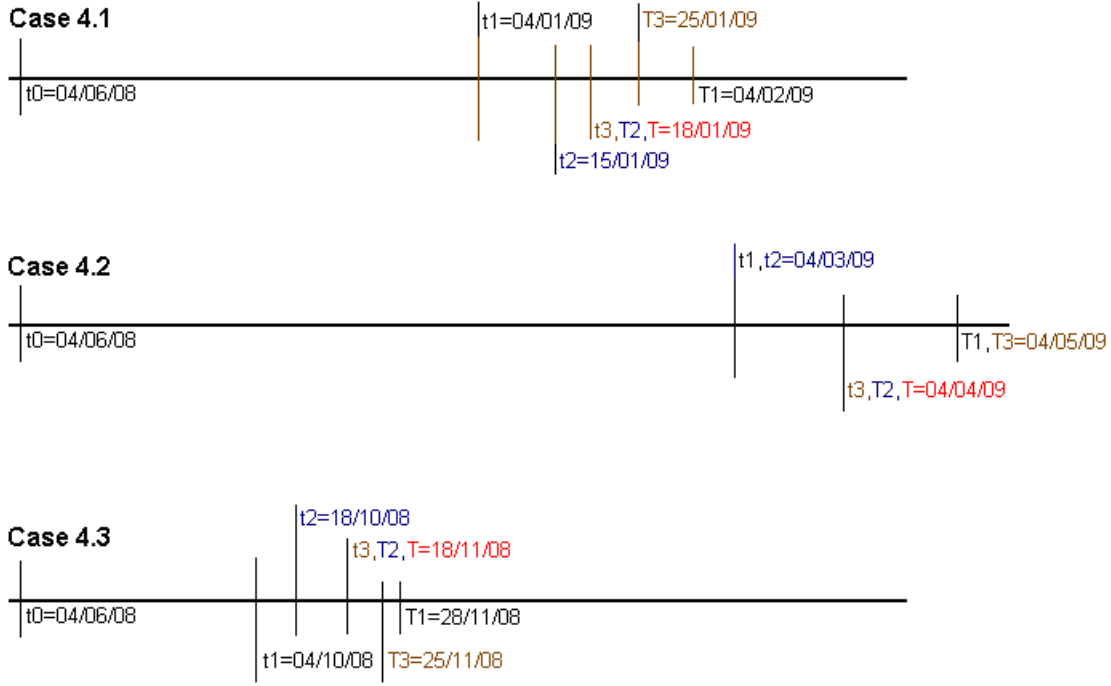


Figure 5.9.: Test scenarios for the FASO type-4 option

5.5.3. Correlation Matrices

For each test scenario described in section 5.5.2, we further designed three main scenarios by considering different correlation matrices. Two scenarios represent the extreme cases, i.e. maximum and minimum level of correlation, while the third one represents a mixed scenario which encompasses both positive and negative correlations.

For option type 1, the correlation matrices used for these three cases are shown below in Table 5.5:

5. Pricing Exotic Energy Options

minimum						maximum						mixed					
1	0.01	0.01	0.01	0.01	0.01	1	0.999	0.999	0.999	0.999	0.999	1	0.5	-0.5	-0.5	-0.5	0.5
0.01	1	0.01	0.01	0.01	0.01	0.999	1	0.999	0.999	0.999	0.99	0.5	1	-0.5	-0.5	-0.5	0.5
0.01	0.01	1	0.01	0.01	0.01	0.999	0.999	1	0.999	0.999	0.999	-0.5	-0.5	1	0.5	0.5	-0.5
0.01	0.01	0.01	1	0.01	0.01	0.999	0.999	0.999	1	0.999	0.999	-0.5	-0.5	0.5	1	0.5	-0.5
0.01	0.01	0.01	0.01	1	0.01	0.999	0.999	0.999	0.999	1	0.999	-0.5	-0.5	0.5	0.5	1	-0.5
0.01	0.01	0.01	0.01	0.01	1	0.999	0.999	0.999	0.999	0.999	1	0.5	0.5	-0.5	-0.5	-0.5	1

Table 5.5.: Correlation matrices used for the FASO type-1 option

The correlation matrices used for these three cases are shown below in Table 5.6:

minimum			maximum			mixed		
1	0.01	0.01	1	0.999	0.999	1	0.5	-0.5
0.01	1	0.01	0.999	1	0.999	0.5	1	-0.5
0.01	0.01	1	0.999	0.999	1	-0.5	-0.5	1

Table 5.6.: Correlation matrices used for the FASO type-2, 3 and 4 options

5.5.4. Volatilities

For each configuration considered (as described in section 5.5.1, 5.5.2 and 5.5.3), the values of both call and put FASOs were calculated with $\tilde{\sigma}$ varying from 0.1 to 1. The values of $\tilde{\sigma}$ represent the volatility of all of the commodities underlying the option. For instance, $\tilde{\sigma} = 0.1$ implies that $\sigma = \{0.1, 0.1, 0.1, 0.1, 0.1, 0.1\}$ for the FASO type-1 option and $\sigma = \{0.1, 0.1, 0.1\}$ for FASO type-2, 3 and 4 options.

For the same reason stated in section 5.5.1, we use synthetic data for the input implied volatilities. In the market practice of pricing, we will obtain an implied volatility matrix and use interpolation to extend it to a full implied volatility surface. The method that we employed in the real market is in fact using the same methodology as in the market volatility model described in section 2.4.

5.5.5. Test Results

Using the settings in section 5.5.1 to section 5.5.4, we price the FASO type-1, 2, 3 and 4 options. Part of the corresponding results are shown in the figures in this section. For further test results of option prices and sensitivities, users should refer to Appendix C of this thesis.

5. Pricing Exotic Energy Options

Since FASO is a newly proposed structured product, it is entirely traded over-the-counter. There is no market quoted prices available for comparison. However, we conclude that our FASO output prices are reasonable due to the trader's comments. We draw to the further conclusion that the market volatility model could be used in the energy markets, as well as in the interest-rate markets.

The prices of FASO type-1 call options with respect to different correlation scenarios are shown in figures 5.10 and 5.11. The corresponding pricing window structure is as shown in case 1.1 in figure 5.5 in section 5.5.2. The following three trends are shown in the prices movements. First, the call option prices tend to increase as the option moneyness increases. Second, the call option prices tend to increase as the input volatilities increase. Third, the call option prices tend to decrease as the input correlations increase. As for the FASO type-1 put options, the prices tend to decrease as the option moneyness and correlations increase and the prices tend to increase as the option volatilties increase.

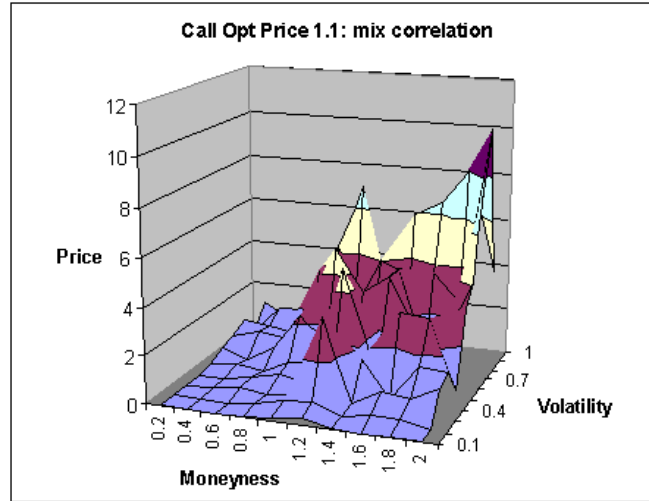


Figure 5.10.: FASO type-1 call option; Case 1.1 with the mixed correlation matrix

5. Pricing Exotic Energy Options

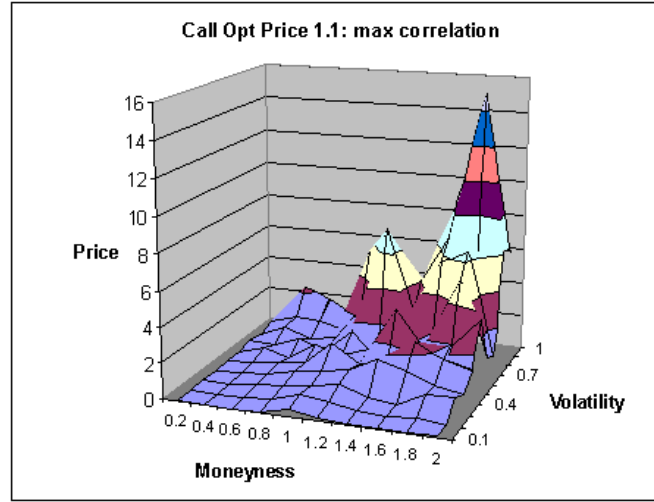


Figure 5.11.: FASO type-1 call option; Case 1.1 with the maximum correlation matrix

The prices of FASO type-2 call options with respect to different correlation scenarios are shown in figures 5.12 and 5.13. The corresponding pricing window structure is as shown in case 2.3 in figure 5.6 in section 5.5.2. The following three trends are shown in the prices movements. First, the call option prices tend to increase as the option moneyness increases. Second, the call option prices tend to increase as the input volatilities increase. Third, the call option prices tend to decrease as the input correlations increase. As for the FASO type-2 put options, the prices tend to decrease as the option moneyness and correlations increase and the prices tend to increase as the option volatilties increase.

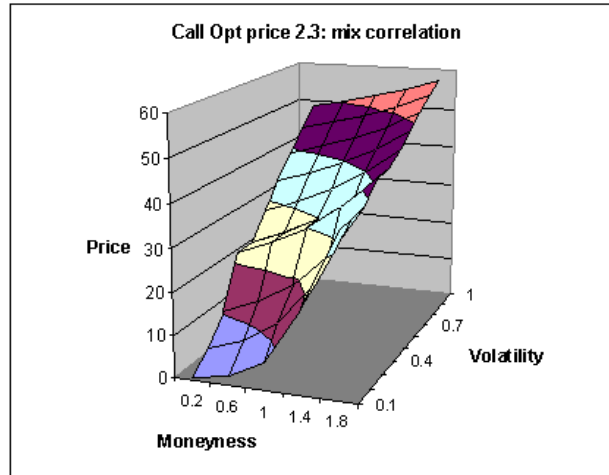


Figure 5.12.: FASO type-2 call option; Case 2.3 with the mixed correlation matrix

5. Pricing Exotic Energy Options

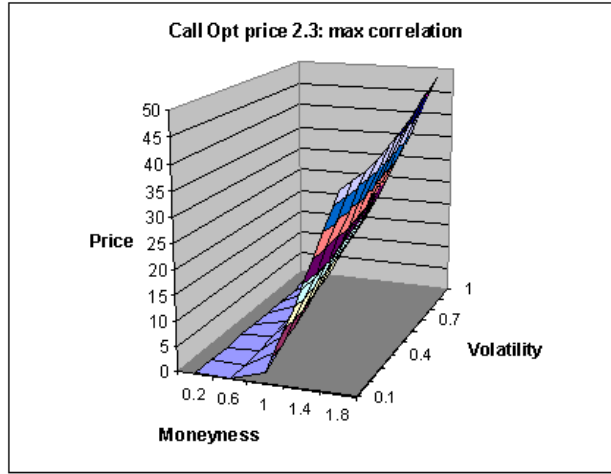


Figure 5.13.: FASO type-2 option; Case 2.3 with the maximum correlation matrix

The prices of FASO type-3 call options with respect to different correlation scenarios are shown in figures 5.14 and 5.15. The corresponding pricing window structure is as shown in case 3.6 in figure 5.8 in section 5.5.2. The following three trends are shown in the prices movements. First, the call option prices tend to increase as the option moneyness increases. Second, the call option prices tend to increase as the input volatilities increase. Third, the call option prices tend to decrease as the input correlations increase. As for the FASO type-3 put options, the prices tend to decrease as the option moneyness and correlations increase and the prices tend to increase as the option volatilities increase.

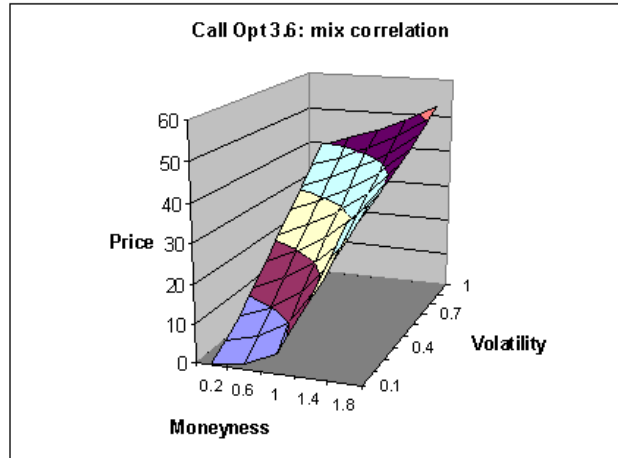


Figure 5.14.: FASO type-3 option; Case 3.6 with the mixed correlation matrix

5. Pricing Exotic Energy Options

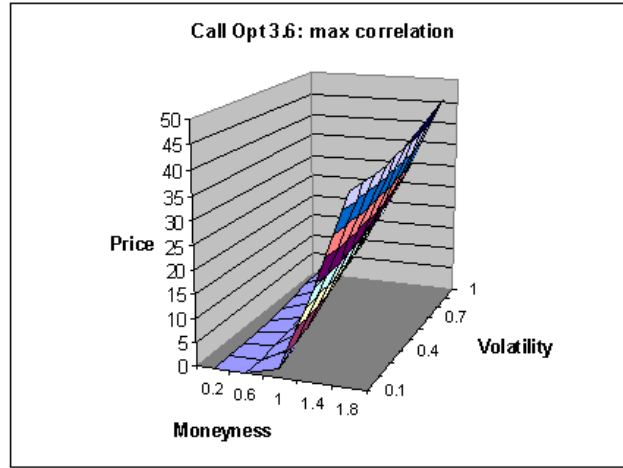


Figure 5.15.: FASO type-3 call option; Case 3.6 with the maximum correlation matrix

The prices of FASO type-4 call options with respect to different correlation scenarios are shown in figures 5.16 and 5.17. The corresponding pricing window structure is as shown in case 3.6 in figure 5.9 in section 5.5.2. The following three trends are shown in the prices movements. First, the call option prices tend to increase as the option moneyness increases. Second, the call option prices tend to increase as the input volatilities increase. Third, the call option prices tend to decrease as the input correlations increase. As for the FASO type-4 put options, the prices tend to decrease as the option moneyness and correlations increase and the prices tend to increase as the option volatilities increase.

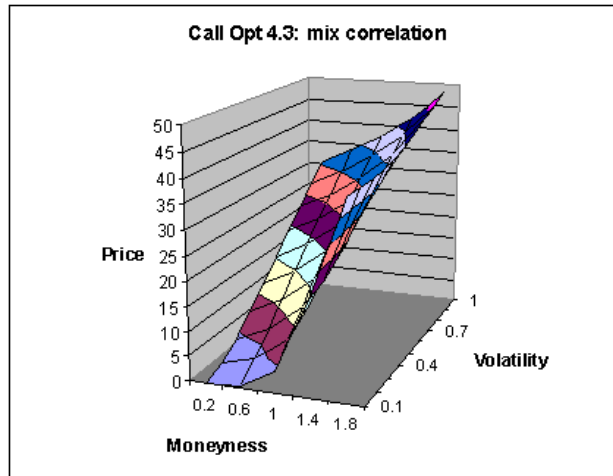


Figure 5.16.: FASO type-4 call option; Case 4.3 with the mixed correlation matrix

5. Pricing Exotic Energy Options

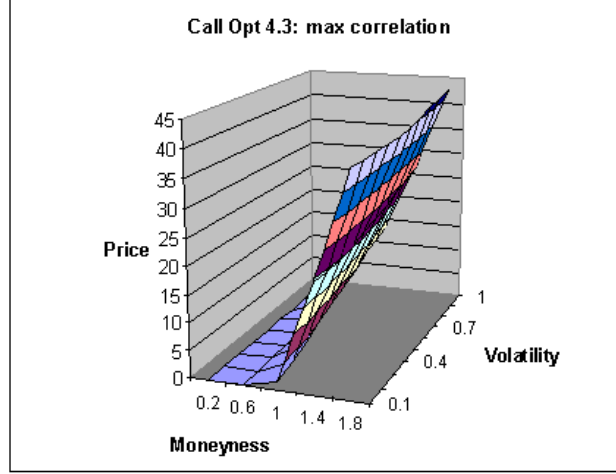


Figure 5.17.: FASO type-4 call option; Case 4.3 with the maximum correlation matrix

As shown from the results above, the FASO call option prices tend to increase as the option moneyness and volatilities increase, and the call option prices tend to decrease as the input correlations increase. Similarly, the FASO put option prices tend to decrease as the option moneyness and correlations increase, and the call option prices tend to increase as the input volatilities increase. We perform tests for all pricing window structures shown in section 5.5.2, and we find that the same trends display in all corresponding FASO option prices. Therefore, we conclude that our model works correctly.

In addition to the results presented in this section, we also work out the prices of call and put options based on the minimum payoff from the same pay-off structure rather than the maximum payoffs. For example, in equation (5.8) we now use

$$V(T) = \min [\phi((w_{11}F_1(T) - w_{12}F_2(T)) - (w_{13}F_3(T) - w_{14}F_4(T)) - K_1; (w_{21}F_1(T) - w_{22}F_2(T)) - (w_{25}F_5(T) - w_{26}F_6(T)) - K_2; 0)] \quad (5.13)$$

We perform tests on all test scenarios to price call and put options based on the minimum payoff, and found it produces comparable results.

5.6. Summary

Following the discussion in chapter 4, we propose a new type of exotic energy derivatives in this chapter, and we use our proposed Market Volatility model to price it.

5. Pricing Exotic Energy Options

In section 5.2 and 5.3, we review two of the most commonly used energy derivatives, which are the basket options and the Asian options. Basing on the real trading experiences, we proposed a new type of exotic energy derivative, which we name as the “Flex-Asian spread option” (FASO). FASO is a hybrid of basket- and Asian-type options. It has a flexible pay-off structure that can be set to completely match the model user’s desired pay-off function. A detail specification of the structure of FASO is given in section 5.4. We then use synthetic data as input to the Market Volatility model for pricing FASO under various test scenarios. A detailed analysis of different test scenarios and the corresponding test results are given in section 5.5 and 5.5.5.

From our test results, we conclude that the FASO call option prices tend to increase as the option moneyness and volatilities increase, and the call option prices tend to decrease as the input correlations increase. Similarly, the FASO put option prices tend to decrease as the option moneyness and correlations increase, and the call option prices tend to increase as the input volatilities increase. The same trends display in all corresponding FASO option prices, which prove the validity of our model.

6. The Storage Model

6.1. The Intuition Behind the Development of Storage Option Pricing Models

In Section 4.1.2, we gave a brief overview of the natural gas market. Natural gas is one of the fastest growing energy commodity in the past decade. Because of the reduced reserve amount in oil, the supply-demand relationship tighten and oil prices were pushed to historical high. In 2008, oil price once roared to near \$140 per barrel. Because of the reducing amount of world reserve of crude oil and the increasing production cost, major oil companies started new projects targeting gas instead of oil. In contrast to oil, the world reserve of gas remains remarkably promising. In the past 20 years, the reserve of gas discovered has doubled in the amount. In March 2009, statistics showed that the world reserve natural gas was around 178 trillion m^3 . Nowadays, gas is accounted for 25% of the total energy balance and this ratio will keep increasing in future.

The price of natural gas has also risen over the past years. At Henry Hub, price increased from \$2.76 per million British thermal units in the third quarter of 2001 to a peaking level in summer 2008, where the price of gas was over \$12. The volatility associated to prices of gas is greater than the volatility associated to prices of oil.

Storage has been a crucial part for managing the relationship between supply and demand in natural gas markets. As a consequence, storage option plays an important role in the natural gas market, as significant value can be generated during the storage re-balancing process. Before going into more details of the valuation of a gas storage facility, we will give some explanation of the underground natural gas storage. Three main types of underground natural gas storage exist: depleted natural gas fields; aquifers, where water can go up or down depending on the type of gas injected¹; Salt caverns, where a storage cavern is created by created by drilling into salt layers and injecting water to dissolve the salt, natural gas are then pumped in to force out the water.

For the base load purpose, the depleted fields are most commonly encountered as storage reservoirs. The injection season for gas is usually from March/April until October/November, when the price of gas is relatively low. And the gas is usually withdrawn from the storage during winter seasons, when the demand of gas is high due to the heating purpose and so the price of gas will be relatively high. Occasionally, there can be an upward spike in prices during summer and injections can also be

¹As seen in reference [22]

6. The Storage Model

suspended. Instead of injecting into the storage, gas can be diverted to the spot market in order to be benefited by the spike in prices. In general, this kind of storage reservoirs take advantage of the spread between the summer and winter gas prices. For hedging purpose, injection and withdrawal schemes are usually associated with the purchase/ sale of gas in the future and forward markets, in order to lock in a profit. The flexible time hedging decisions and dynamic injection/ withdrawal scheme generate (embedded) optionality for the storage device, which can yield an extra value.

The storage facility provides an inventory of energy commodity. Hence, it provides insurance against any unpredicted variation in supply/ demand in the market. The value of a gas storage is the discounted value of total expected revenues under an optimal strategy, considering all possible future price process for gas. Storage is particularly useful in the natural gas market, as the price of natural gas is largely effected by seasonality, and a storage facility allows the user to shift the production capacity of gas (by storing gas in one period and delivering it in a later period). The seasonal shapes presented in the forward curve of gas prices have a large impact on the value of a storage facility.

In a re-regulated market, the owners of storage facilities run them on a commercial basis, by selling capacity to marketers. In September 2004, the journal RISK published a study. The study showed that around 70% of natural gas distribution companies in the US use storage as a hedge. In Europe, many countries which are net importers of natural gas also have strategic storage. For example, France's gas storage represents 24% of the annual demand and Germany's gas storage represents 21% of the annual demand. In Italy, gas storage represents 23% of the annual demand and this ratio is less than 4% in the UK².

Although we could use the Market Volatility model to value a number of energy derivatives, i.e. the FASOs as described in section 5.4, this model is not compatible with more advanced structures associated with certain types of energy derivatives, i.e. storage options. Following the discussion in this section, we see that the storage option is one of the most popular energy derivatives, and it has been used in the gas markets extensively. Therefore, we need to introduce the "Storage models" which can be used to price such an important type of option.

6.2. Various Methods for Valuing a Storage Contract

The structure of storage options is very complicated, hence, their valuation is a challenging task.

The storage options is particularly desirable within the natural gas market. Because prices of the natural gas tend to be low in the summer and high in the winter. Unlike the electricity prices, where large spikes can occur during the summer peak-time, natural gas prices are relatively more

²Quoted from reference [22]

6. The Storage Model

“stable” during the peak- and non-peak period. In the summer peak-period, natural gas prices are normally 25% to 30% higher. The reason for that is the storage facility for gas is relatively cheaper, so the cost for storing the gas produced in summer period into the winter period is relatively low. These fast-turning storage facilities allow the users to manage the production capacity in order to supply gas according to demand fluctuations. Moreover, the fluctuation of demand for gas can be highly volatile. We have to respond to the changing market conditions immediately to meet the demand, otherwise, all the gas that we stored from the summer period will have little value.

From the reasons above, we can see that storage plays a major role in the fuel market (i.e. natural gas), due to its availability to manage the supply-demand relationship.

The existing quantitative finance literature covers a small number of pricing techniques of (natural gas) storage options, however, none of these models is regarded as a standard pricing model. Most of the existing pricing techniques price the storage options using the real options dynamic.

Based on the current static view of the forward curve, the simplest trading strategy for a storage asset is to ensure that the storage operator would satisfy inventory requirements for the duration of the storage contract by buying and selling (forward contracts) at fixed prices. This trading strategy is set on the valuation date, on the base of attempt to maximise the profit generated by the strategy. This maximal profit is referred to as the *intrinsic value* of storage. The intrinsic value is the value that could be locked-in on the valuation date.

As the intrinsic value is developed from a static view of the forward price structure, it is independent of any modeling assumptions. Instead, it is based on the general trading rule of “buy low, sell high”. When the price of the storage asset moves through time, one can re-balance the trading positions to create additional profit, since option value is embedded in the asset. The option value arises during this procedure is referred to as the *Extrinsic value* of storage. The extrinsic value is dependent on the representation of price dynamics used (e.g. single-factor vs. multi-factor diffusion). Therefore, this representation exhibits model risk.

The total value of the storage contract, which is the sum of the intrinsic and extrinsic value, is called the *Option Premium*. It equals to the profit that the contract holder can expect to lock-in by applying dynamic-hedging to the contract.

In the following sections, we discuss about different models which can be used to calculate the premium for storage contracts.

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6.2.1. The Rolling Intrinsic Method

The Rolling Intrinsic (RI) method is a widely-known pricing method in the energy market. It is based on the intrinsic value of the price dynamics of the storage asset. In order to illustrate this idea more precisely, Let us look at the storage asset before the starting date of the storage period, for every single day of the storage period users can buy/sell swap or future contracts with the available storage amount on the forward market. In this way, by entering long positions in cheap periods and storing to enter short positions in subsequent more expensive periods, once can lock-in a profit instantaneously. However, this profit is based on the static view of the storage value, so it does not take the future potential upside of prices into account.

In order to incorporate the full optionality of the storage option, one must take the random price dynamics into account. In the RI pricing method, we apply shocks to the forward curve every day until the maturity of the storage contract. Assume that a full initial hedge has been placed on day one to lock-in existing spreads between forward contracts with different maturities, one may find it profitable to roll the intrinsic hedge on each subsequent simulated day. Given that cashing-in from closing-out existing positions and the corresponding change in the value of the intrinsic hedge have a net effect to increase the overall option value, the intrinsic hedge may be adjusted. For each simulated day, the cumulative cashflows from rolling the intrinsic hedge will be calculated. At the maturity date of the storage contract, the final cumulative cashflow is recorded and it represents the profit obtained over the period, based on the chosen set of scenarios of forward curves. By iterating through this process a large number of times, the expectation of the final cumulative cashflow can be calculated. Discount the expected value of the final cumulative cashflow back to the valuation date, we then have an approximation for the premium of the storage contract.

As the embedded trading strategy of the RI method can be easily understood by practitioners with little or no quantitative background, it is popular among trading professionals. In the market practice, RI method is commonly based on a sub-optimal trading strategy, such that investment decisions are made without accounting for price curve dynamics. An optimal decision should consider all future possible paths of the forward price. Therefore, the average profit generated through the sub-optimal strategy underestimates the asset premium.

The frequency of the (natural gas) forward curve plays an important role, as it decides how the forward curve rolls through time. Changing from daily frequency to weekly frequency will result in a totally different forward curve. Hence, we need to be careful with our modeling assumptions when moving from one type of underlying contract to another. This problem also lies in the Trinomial Tree Forest method, which will be discussed in section 6.2.2.

In summary, the RI method is in fact a kind of Monte-Carlo method. The pricing is based on optimising the trading strategy of the storage asset based on an instantaneous view of the forward market. The price simulation process is crucial for the valuation, especially for natural gas, as it exhibits strong seasonal patterns in asset prices and volatilities.

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6.2.2. The Trinomial Tree Forest Method

The Trinomial Tree Forest (TTF) method is a stochastic dynamic programming approach applied in pricing storage contracts. The pricing methodology is intuitive but the implementation is complicate. Generally, the TTF method is implemented via the following three sequential steps:

1. Build a trinomial tree to model the spot and matching forward price structures
2. Discretise storage volume and find the attainable storage levels
3. Applying the backward deduction to the storage price at every attainable volume

The attainable volume levels can be found based on the operational constraints embedded in the storage contract (e.g. injection/withdrawal rates, minimum /maximum capacity allowed, etc) ³. The storage level of natural gas is the state variable of the optimisation process. Given the space of possible future spot prices, the target of the optimisation is to find the optimal level of gas in storage as a function of time. It starts from selecting the attainable volume nodes on the maturity date of storage. Generally, the storage capacity need to be empty on the hand-over day, so there will only be one attainable volume node. By looking at the adjacent volume nodes backwards, the storage value is derived via a local optimisation procedure regarding both the spot transactions and future storage value. The value of the storage contract is calculated using the initial storage level of (natural) gas at the first node in the tree.

The TTF model is based on discretised representations of:

1. The space of attainable volume nodes
2. The space of future price levels

In contrast to the RI method, the TTF method is based on a (theoretically) optimal investment strategy. Given that the correct assumptions were made, the output price from the TTF method is the non-biased premium of the storage contract. However, there have been questions regarding the stability and systematic convergence qualities of the TTF method. In addition, this method is straight-forward in theory but is rather complicate to apply in the practice. Hence, less market practitioner uses the TTF method in comparison to the RI method.

6.2.3. The Basket of Spread Options Method

At a certain time point during the storage period, the storage value of a storage contract can be regarded as equivalent to a basket of calendar spread call options⁴. The option strike prices are appropriately adjusted to include any injection/withdrawal costs. By considering the constraints such as inventory requirements and injection/withdrawal rates, the user of the basket of spread

³as in reference [44]

⁴The definition of a calendar spread option is as stated in section 5.1

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options (BSO) method will solve an optimisation problem in order to find a suitable selection of spread options in the basket.

Suppose there are n months within a storage contract, and there is a traded forward contract for each month, then there will be $\frac{n(n-1)}{2}$ possible choices of calendar spread options. In theory, there should be $n(n-1)$ possible calendar spread options, with $\frac{n(n-1)}{2}$ call options and $\frac{n(n-1)}{2}$ options. However, the injection into the storage has to occur before withdrawal. Hence, only call options are considered in the BSO method. After the calendar spread options are priced, an optimisation program is used to choose the optimal basket of calendar spread options subject to the physical constraints. The value of this optimal basket of spread call options replicates the value of the storage contract. Then we can obtain the value of the storage contract from the profit that we received by selling the optimal basket of spread call options.

In the real market, (calendar) spread options of natural gas are mostly traded over-the-counter. In market practice, the value of the optimal basket of (calendar) call options is often replicated by taking delta hedging positions in the underlying forward contracts.

Since the BSO method contains the pricing issue of spread options, it involves the calibration issue depending on the type of pricing models that we used. Volatilities and correlations of the underlying forward contracts (in the spread options) are calibrated to the historical market data. The over-the-counter market type of spread options makes the calibration task difficult. During the calibration procedure, we also need to consider the term structure characteristics (i.e. seasonality) of volatilities and correlations. Otherwise, the result of our calibration will lead to a biased value of storage contracts and increase risks for the trading positions.

The premium of the basket of calendar spread call options is considered an approximation to the storage value obtained by using the RI method. According to Eydeland and Wolyniec [2003], the approximation becomes exact in the absence of injection/withdrawal costs. In the presence of injection/withdrawal costs, the basket of spread options can provide a sufficiently tight lower bound for the rolling intrinsic value. Compared to the RI or TTF method, the BSO is much more computationally efficient.

6.2.3.1. The Rolling Basket of Spread Options Method

The rolling basket of spread options (RBSO) method is an extension of the basket of spread options method. This approach involves rolling (i.e. switching) into a new optimal basket of spread call options if it is more profitable. While we are hedging the existing basket discretely under changing market conditions, a new basket of calendar spread call options with a higher value may become available. If the updated value of the existing basket of spread call options plus the cumulative

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marking-to-market cashflows from re-hedging is less than the value of the new basket of spread call options plus any re-balancing costs⁵, it will be optimal to roll into the new basket.

Empirical results published by FEA indicate that, in general, the storage value obtained from the RBSO method is equivalent to the result obtained from the RI method.

6.2.4. The Least Squares Monte-Carlo Method

The least squares Monte-Carlo (LSM) method is based on the mathematical concept of conditional expectation. First of all, a large number of price scenarios were generated over the life of the contract through simulations. Then the least squares regression was applied via a local optimisation algorithm through a backward procedure, in order to work out the local value of the storage asset at each attainable volume level. Longstaff and Schwartz [2001] originally applied the Least Squares Monte Carlo (LSM) algorithm to the valuation of American-Bermudan options. Subsequently, it has been extended and applied as a method for pricing storage contracts.

Let us look at the original application of the LSM approach for the valuation of an American-style option. Recall that the holder of an American option may exercise at any time prior to the option maturity. At each time step, the decision whether to exercise the American option or not depends on the value of continuing without exercising (the continuation value) against that of exercising immediately (the intrinsic value).

At the beginning of the LSM method, the time to maturity of the desired option is discretised into a number of potential exercise time steps. At each of these steps, several paths representing the underlying asset price evolution are simulated via Monte-Carlo simulation. The continuation value, which is the risk-neutral conditional expectation of the discounted future cashflows generated by the option, is calculated via least squares regression at each time step. In particular, the simulated future cashflows at each time step are regressed on the simulated underlying asset prices of that time step, using a pre-specified functional form.

The initial step of the LSM algorithm is to define the cashflows at maturity of the option. By working backwards, the recursive algorithm estimates the conditional expectations and provides the optimal decision at each discretised time step. The reason for having a backward procedure is that at each time step the decision to exercise could change all the subsequent cashflows. For example, if at any time step the optimal decision would be exercising the option, then the future cashflows would become zero because we no longer hold the option. These updated cashflows will be used in subsequent least squares regressions. As a final step, the cash flow at the exercise time

⁵Re-balancing costs represent the profit or loss arisen from the re-balancing of the existing delta positions, which hedges the existing basket of spread call options at the new market prices.

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step within each simulated path is discounted appropriately back to time 0. The average of all these discounted cashflows is calculated and this average represents the value of the option.

The application of the LSM algorithm to the pricing of swing options and gas storage facilities is more complicated than the pricing of American-style options, this is because one is faced with multiple exercise-right options. As a consequence the LSM algorithm needs to be extended. This is done by adding an extra dimension to the otherwise two dimensional⁶ LSM matrix. In the case of swing options the additional dimension is the number of the remaining exercise rights. For storage valuations, it is the inventory level of the facility. The additional complexity lies in the fact that at each time step we have to make the optimal decision by comparing various continuation values (along with any immediate cashflows) corresponding to each of the possible values of the additional dimension⁷.

A storage facility is similar to having an up and down swing option for every day of the contract. On any day we may buy gas and inject it into storage, or withdraw gas and sell it into the market. The injection and withdrawal restrictions are governed by ratchets which are unique choices of inventory level dependent to each type of storage facility. These ratchets merely represent the reality that as the inventory level increases it becomes harder to inject gas, but as the inventory level decreases it becomes harder to withdraw gas. Our possible actions may also be constrained by maximum/minimum inventory levels that apply through the duration of a storage contract, as well as initial and final conditions associated to the inventory level in which the facility is obtained and must be returned. If any of these constraints were broken, penalties may occur dependent on the inventory level and gas spot price. The type of facility may also incur fixed injection and withdrawal costs which cover the additional energy required to operate in injection and withdrawal modes. In addition to the physical constraints, we are also governed by market factors such as bid-ask spreads, transaction costs and discount rates.

In order to value a gas storage facility using the LSM algorithm, the LSM matrix must include a third dimension consisting of the volume of gas in the storage facility. By working recursively from maturity back to time 0, for each possible volume level H at each discretised time step and along each simulated path of the underlying gas price, we have to compare the sum of the continuation value to any immediate cashflows incurred from the following three possibilities :

1. Buy volume h_u of gas to inject into the storage; continue with inventory level $H + h_u$
2. Remain at the current volume of gas; continue at inventory level H
3. Withdraw volume h_d of gas from the storage to sell; continue at inventory level $H - h_d$

⁶The original application of the LSM algorithm by Longstaff and Schwartz (2001) involves a two-dimensional LSM matrix, with the dimensions equal to the number of simulated paths for the underlying asset price and the number of discretised exercise time steps considered respectively.

⁷as in reference [44]

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The choices of h_u and h_d are restricted by physical constraints such as ratchets, maximum/minimum inventory levels, initial/terminal constraints, etc. Continuation values are calculated using least squares regressions. At each time step, the optimal decision is made, then the future cashflows will be modified according to that optimal decision. At time 0, the value of the storage facility is calculated as the average of the sums of the discounted cashflows from the final cashflow matrix corresponding to the initial inventory level.

The LSM method is effectively a Monte-Carlo approach. Hence, it offers great flexibility in the choice of the underlying price process. It also allows the distribution of the storage premium to be obtained, and consequently a measure of the pricing error can be estimated. The difficulty of calibration of the LSM method can vary greatly, depending on the chosen price process. Similar to the TTF method, this valuation technique is based on a theoretically optimal investment strategy, so theoretically the model does not undervalue the storage contract.

Although the TTF method is more computationally efficient than the LSM method, it only allows the underlying price process to be governed by GBM or single factor mean reversion. In contrast, The LSM algorithm can be applied to price paths with respect to any underlying price process. It could also be extended to problems based on multiple commodity price simulations.

6.3. The Storage Model Methodology

In this section, we will introduce the storage model, which is a comprehensive model for the purpose of pricing and hedging physical storage contracts with constraints. A physical storage contract is specified through user inputs, then the storage model takes the user inputs and returns the gas flow schedules for both intrinsic and extrinsic cases, together with the associated greeks' exposure.

Our storage model arises from the pricing method described in the previous section 6.2.3, which maximises the value of a storage contract by picking up a basket of profitable calendar spread options during the contract term. Hence, the core of the storage model is a calendar spread option pricing model.

In order to price a calendar spread option properly, the average volatility of a given forward curve over the life of the calendar spread option is necessary. The correlation between the two legs of the calendar spread option also has an impact on the price of the option. After pricing the calendar spread options, a linear programming problem subject to physical constraints is used to find the most profitable basket of calendar spread options and the optimal flow schedule.

6.3.1. Brief Introduction of the Calibration Process

The figure 6.1 below shows the basic steps of calibration procedure:

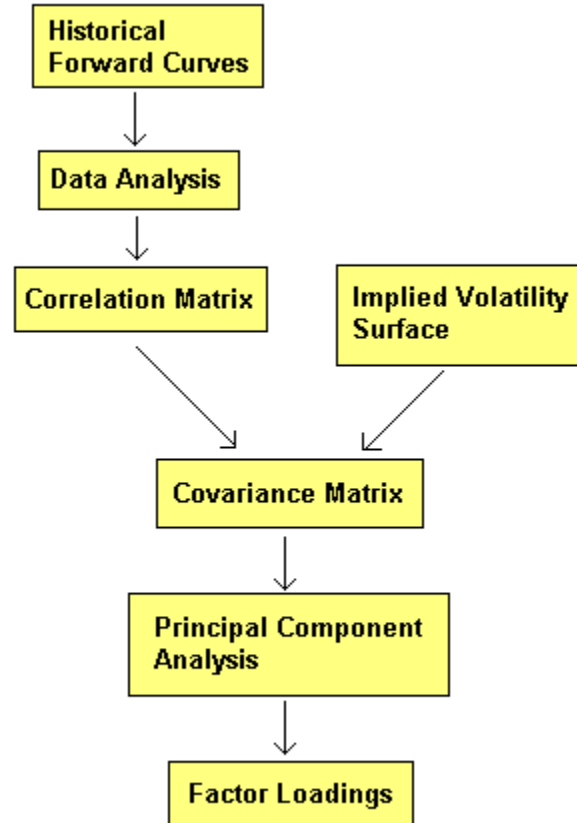


Figure 6.1.: Principal Components calibration process

First of all, the forward curve structure is simulated using Principal Component Analysis (PCA). The idea is to simulate the curve according to the historical characteristics observed during the current calendar month. For example, on December 10th, the forward curves are simulated to reflect price movements during previous December months. A study of the historical data will describe the correlation matrix between the monthly swaps. The full variance/covariance matrix will be derived from the implied volatility structure.

To simulate the price processes, we need the volatility and correlation parameters. These simulation parameters are user inputs which need to be calibrated independently. In the following sections,

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we will give more detail for the calibration procedure.

6.3.2. Modeling volatilities

For calibrating volatilities used for storage option valuation, we could choose to use two different approaches. In the first approach, we assume that the instantaneous volatility has an exponential decay functional form

$$\sigma_{in}(t, T) = \sigma_0 + \sigma_1 e^{\beta(T-t)} \quad (6.1)$$

with σ_0 as the long-term volatilities and β as the decay rate. Recall that, from equation (2.90) we have that the instantaneous volatilities and the implied Black's volatilities satisfies the following relationship

$$\sigma_{Bl}(t, T) = \sqrt{\frac{1}{T-t} \int_t^T \sigma_{in}^2(s, T) ds} \quad (6.2)$$

Given a forward contract with price $F(0, T)$, the forward volatility of this contract over a reference period $[T_0, T_1]$ where $0 \leq T_0 \leq T_1 \leq T$ is given by the following formulation:

$$\sigma^{F(0,T)}(T_0, T_1) = \sqrt{\sigma_0^2 - \frac{2\sigma_0\sigma_1 [e^{\beta(T-T_1)} - e^{\beta(T-T_0)}]}{\beta(T_1 - T_0)} - \frac{\sigma_1^2 [e^{2\beta(T-T_1)} - e^{2\beta(T-T_0)}]}{2\beta(T_1 - T_0)}} \quad (6.3)$$

The parameters σ_0 and β are calibrated from historic data. After fixing the values of σ_0 and β , we will able to find σ_1 by solving the quadratic equation:

$$A\sigma_1^2 + B\sigma_1 + C = 0 \quad (6.4)$$

where

$$A = \frac{1 - e^{-2\beta(T-t)}}{2\beta} \quad (6.5)$$

$$B = 2\sigma_0 \cdot \frac{1 - e^{-\beta(T-t)}}{\beta} \quad (6.6)$$

$$C = \left[\sigma_0^2 - \sigma^{F(0,T)}(T_0, T_1)^2 \right] \cdot (T - t) \quad (6.7)$$

Hence, we have

$$\sigma_1 = \frac{-2\sigma_0 (1 - e^{\beta T}) - \beta T \sqrt{\left(-2\sigma_0 \cdot \frac{1 - e^{\beta T}}{\beta T}\right)^2 + 2 \cdot \frac{1 - e^{2\beta T}}{\beta T} \left[\sigma_0^2 - \sigma^{F(0,T)}(T_0, T_1)^2\right]}}{1 - e^{2\beta T}} \quad (6.8)$$

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The first approach is based on the assumption that local volatilities take the general functional form, however, it failed to capture the seasonal effects which could cause large spikes in the price movements of our desired commodities. Hence, we propose a second approach for calibration of volatilities, which takes the seasonal effect into account by adding in an additional calibration parameter in order to capture this characteristic.

Again, let us assume that the instantaneous volatility has an exponential decay functional form

$$\sigma_{in}(t, T) = a + [b + c(T - t)] e^{v(T) - d(T-t)} \quad (6.9)$$

where $\{a, b, c, d\}$ are estimated from historical forward prices, and $v(T)$ is the deterministic seasonal premium.

To relate the parameter T (the expiry date) to M (calendar month), the following rule is applied: if T is in a particular calendar month M ($M = 1, 2, \dots, 12$), then $v(T) = v(M)$. For instance, if a futures contract expires on 20 March 2009, then the seasonal premium for the contract always equals $v(3)$, regardless of the current date. Note that under this setting the same seasonal premium will be applicable to contracts expiring in different years. That is, the seasonal premium $\{v(M)\}_{M=1, 2, \dots, 12}$ represents the collection of long-term average premium (expressed in %) on futures expiring in the calendar month M ($M = 1, 2, \dots, 12$) with respect to the average forward price.

Suppose that we have a historical dataset of n daily forward curves, $\{F(t, 1), F(t, 2), \dots, F(t, 12)\}_{t=1, \dots, n}$, with maturities of up to one year. The least squares estimator for the maturity month effect $v(T)$ is the average deviation of the log-futures price from these daily averages:

$$\hat{v}(T) = \frac{1}{n} \sum_{t=1}^n \left(\ln F(t, T) - \ln \bar{F}(t) \right) \quad (6.10)$$

where $\bar{F}(t)$ is the geometric average of the current forward prices

$$\bar{F}(t) = \sqrt[N]{\prod_{T=1}^N F(t, T)} \quad (6.11)$$

or, equivalently,

$$\ln \bar{F}(t) = \frac{1}{N} \sum_{T=1}^N \ln F(t, T) \quad (6.12)$$

where N is the most distant liquid maturity. We assume that N is a multiple of 12, i.e., we include maturities up to a whole year or an integer number of years. In addition, we require that $\sum_{T=1}^{12} v(T) = 0$. This is a reasonable assumption, since $v(T)$ is defined with respect to the average price level.

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The nearby futures price (i.e., the one that expires in the current month) are often unstable due to expiration effects, and we chose to exclude them from the estimation procedure. If we have futures prices for expiry dates beyond 1 year, we can estimate $\ln \bar{F}(t)$ by $\frac{1}{13} \sum_{T=2}^{13} \ln F(t, T)$. If fewer than 13 consecutive expiry months are available, then the following procedure can be employed.

We obtain estimates of the differences $v(M) - v(L)$ for all possible combinations of months (M, L) , by averaging $\{\ln F(t, M) - \ln F(t, L)\}$ over the entire historical dataset. In this way we obtain the matrix of difference estimates:

$\widehat{v(1) - v(2)}$				
$\widehat{v(1) - v(3)}$	$\widehat{v(2) - v(3)}$			
$\widehat{v(1) - v(4)}$	$\widehat{v(2) - v(4)}$	$\widehat{v(3) - v(4)}$		
$\vdots$	$\vdots$	$\vdots$		
$\widehat{v(1) - v(12)}$	$\widehat{v(2) - v(12)}$	$\widehat{v(3) - v(12)}$	$\cdots$	$\widehat{v(11) - v(12)}$

The individual estimates for $v(M)$ can be obtained by adding up the columns of the above matrix and using the restriction $\sum_{M=1}^{12} v(M) = 0$. Denoting the sum of the first column by $\sum_1$, we observe that

$$\sum_1 = 11 \cdot v(1) - \sum_{M=2}^{12} v(M) = 12 \cdot v(1) \quad (6.13)$$

So from the first column $\sum_1$ we obtain the estimate for $v(1)$:

$$\hat{v}(1) = \frac{\sum_1}{12} \quad (6.14)$$

Let $\sum_2$ denote the sum of the second column. Then we have

$$\sum_2 = 10 \cdot v(2) - \sum_{M=3}^{12} v(M) = 11 \cdot v(2) + v(1) \quad (6.15)$$

It follows that the estimate for $v(2)$ is

$$\hat{v}(2) = \frac{\sum_2 - \hat{v}(1)}{11} \quad (6.16)$$

Continuing with this procedure, we find the estimates for all $v(M)$, $M = 1, 2, \dots, 12$.

The parameters $\{a, b, c, d\}$ are estimated first by means of the least squares procedure

$$\min_{\{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}} \frac{1}{N} \sum_{i=1}^N [\sigma_{loc} - \sigma_{in}(t, T)]^2 \quad (6.17)$$

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where σ_{loc} is calibrated from historical forward prices and $\sigma_{in}(t, T)$ is given in equation (6.9). Again, for an assumed local volatility function $\sigma_{in}(t, T)$, the relationship with implied volatility $\sigma_{Bl}(0, T)$ is given by

$$\sigma_{Bl}(0, T) = \sqrt{\frac{1}{T} \int_0^T \sigma_{in}^2(s, T) ds} \quad (6.18)$$

With the procedure described in equation (6.17), a common method is then to choose three out of four estimates $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$ together with implied volatility and back-substitution to derive the last estimate. However, two problems occur with this technique:

1. As can be seen from equation (6.18), $\sigma_{in}(t, T)$ is a quadratic function of implied volatility thus having two roots, it is not clear which root should be chosen, and if so when and why.
2. The estimates $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$ including the one obtained from using implied volatility, may not be the optimal solution of minimisation problem in (6.17).

A modified approach is suggested as the following: Applying the least squares procedure twice, we first solve the estimates $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\}$ for the minimisation problem in (6.17). Then noticing that the minimum is achieved when $\sigma_{loc} = \sigma_{in}(t, T)$, we write

$$\sigma_{loc} = \tilde{a} + (\tilde{b} + \tilde{c}(T - t)) e^{v(T) - \tilde{d}(T - t)} \quad (6.19)$$

and derive

$$\tilde{b} = (\sigma_{loc} - \tilde{a}) e^{-(v(T) - \tilde{d}(T - t))} - \tilde{c}(T - t) \quad (6.20)$$

Now, by fixing the estimates $\tilde{a}, \tilde{d}$, the above expression implies that $\tilde{b}$ is a function of $\tilde{c}$. Let $\hat{b}$ and $\hat{c}$ denote the estimates of the estimates $\tilde{b}$ and $\tilde{c}$ respectively. Equation (6.20) can be rewritten as

$$\hat{b} = (\sigma_{loc} - \tilde{a}) e^{-(v(T) - \tilde{d}(T - t))} - \hat{c}(T - t) \quad (6.21)$$

Substituting this back into (6.9), we solve

$$\min_{\hat{c}} \frac{1}{N} \sum_{i=1}^N [\sigma_{imp} - \sigma_{Bl}(0, T)]^2 \quad (6.22)$$

where σ_{imp} denotes the market implied volatilities and $\sigma_{Bl}(0, T)$ is given by equation (6.18). After the estimate $\hat{c}$ is calculated, $\hat{b}$ is directly followed by (6.21).

We use historical datasets of NYMEX futures prices for natural gas (01/06-07/07). The estimated seasonal premiums are depicted in figure 6.2. As expected, futures expiring in winter have high premiums and summer futures are at a discount. Figures 6.3 and 6.4 further show comparisons of historical and analytical local and implied volatilities respectively.

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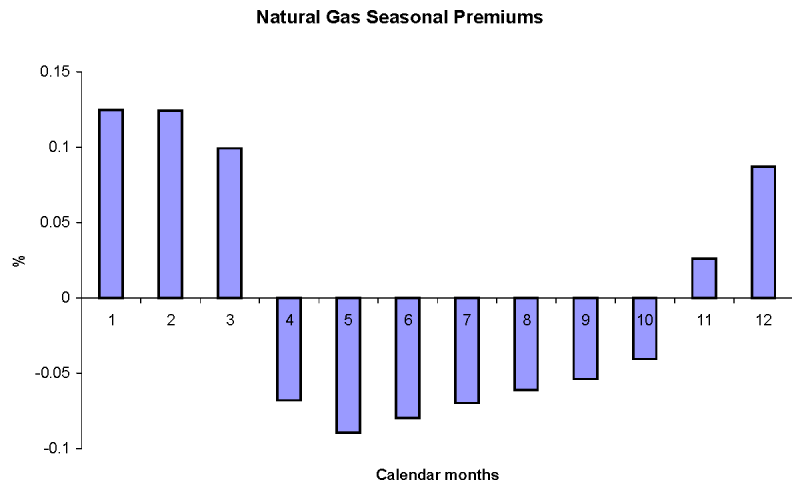


Figure 6.2.: Natural Gas seasonal premiums

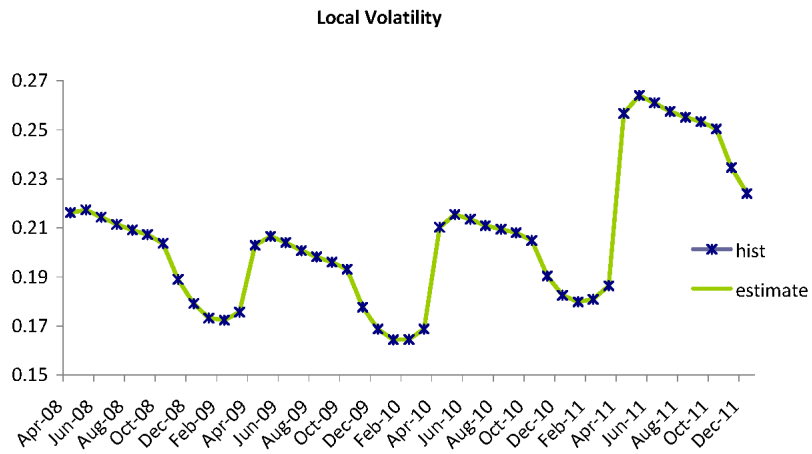


Figure 6.3.: Local volatilities

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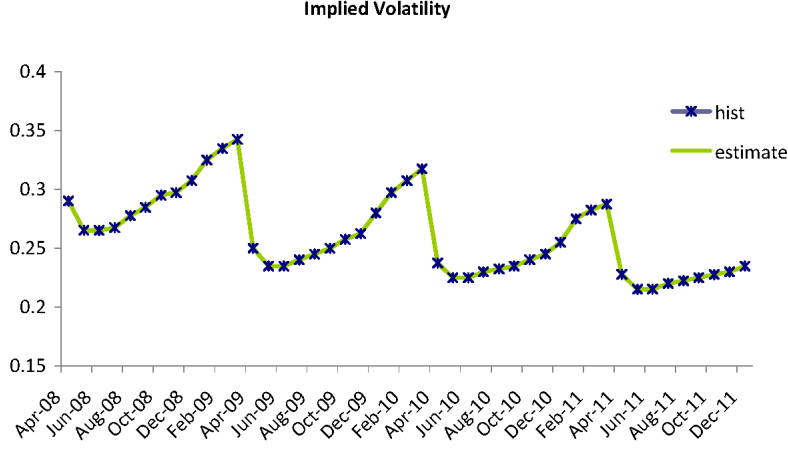


Figure 6.4.: Implied volatilities

By including the parameter of the seasonal effect in the calibration, we conclude that the estimated volatilities and the historical volatilities fitted perfectly.

6.3.3. Modeling correlation

The correlation between two calendar months, which is used to price a calendar spread option is calculated based on historical price data and the volatility model.

6.3.3.1. Local Correlation

The local correlation between two forward contracts $F(0, T_1)$ and $F(0, T_2)$, $0 \leq T_1 \leq T_2 < T$, is assumed to have a function form as follows:

$$\rho_L^{[T_1, T_2]} = \rho_0 + a(T_2 - T_1)^2 + b|T_2 - T_1| + c|T_2^* - T_1^*| \quad (6.23)$$

where T_1^* and T_2^* are the calendar years such that T_1 and T_2 belong to, respectively. ρ_0 and $\{a, b, c\}$ need to be estimated from historical forward prices. In the standard approach, we divide the calendar months into two groups: summer months and winter months, in order to capture the seasonality effect. Hence, the calendar months April to October will be referred to as the summer months, and the calendar months November to March will be referred to as the winter months. In this way, we will have three groups of data for estimating the seasonal correlation, and ρ_0 will have

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different values ρ_{ss} , ρ_{sw} , ρ_{ww} associated to different seasonal groups of data, as show in the table below:

Seasonal Correlations		
Calendar months	<i>Apr. - Oct.</i>	<i>Nov. - Mar.</i>
<i>Apr. - Oct.</i>	ρ_{ss}	ρ_{sw}
<i>Nov. - Mar.</i>	ρ_{sw}	ρ_{ww}

In the above table, ρ_{ss} is the estimate of ρ_0 with respect to the summer-to-summer data, where both T_1 and T_2 are summer months. ρ_{sw} is the estimate of ρ_0 with respect to the summer-to-winter data, one of T_1 and T_2 is a summer month and the other is a winter month. ρ_{ww} is the estimate of ρ_0 with respect to the winter-to-winter data, where both T_1 and T_2 are winter months.

In order to improve the model performance, we use a different approach to capture the seasonality effect presented in the historical data. This will be discussed in the later section 6.3.3.4.

6.3.3.2. Term Correlation

The term correlation over the period $[t, T]$ between two forward contracts $F(0, T_1)$ and $F(0, T_2)$, $0 \leq T \leq T_1 \leq T_2$, is computed according to the following formula:

$$\rho^{[T_1, T_2]}(t, T) = \frac{\int_t^T \sigma^{F(0, T_1)}(s) \cdot \sigma^{F(0, T_2)}(s) \cdot \rho_L^{[T_1, T_2]} ds}{\sigma^{F(0, T_1)}(t, T) \cdot \sigma^{F(0, T_2)}(t, T) \cdot (T - t)} \quad (6.24)$$

where $\sigma^{F(0, T_1)}(t, T)$ and $\sigma^{F(0, T_2)}(t, T)$ are given by equation (6.3). The function $\sigma^{F(0, T_i)}(\bullet)$ in the above equation represents the local volatility defined by,

$$\sigma^{F(0, T_i)}(s) = \sigma_0^{T_i} + \sigma_1^{T_i} e^{\beta^{T_i}(T_i - s)} \quad (6.25)$$

whereas $\sigma_0^{T_i}$ and β^{T_i} are user's inputs and $\sigma_1^{T_i}$ is calculated by equation (6.8).

6.3.3.3. Cash Correlation

While term correlation is enough for future-to-future calendar spread option pricing, cash correlation is needed to price cash-to-future calendar spread options. Cash correlation is defined as the correlation between spot price of a front month and forward price of a back month. We assume that the cash correlation between a front month T_1 and a back month T_2 has the following function form:

$$\rho_C^{[T_1, T_2]} = ae^{-b(T_2 - T_1 - T_c)} - d(T_2 - T_1 - T_c) + c \quad (6.26)$$

where $\{a, b, c, d\}$ are estimated from historical spot and forward prices, and $T_c = 15/365.25$.

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6.3.3.4. Correlation Analysis

We estimate the correlations from historical spot and forward prices and parameterises them. The functions implemented are given as in equations (6.23) and (6.26) for pricing future-to-future and cash-to-future calendar spread options respectively. A calibration procedure is used to find the correlation parameters that minimise the average squared discrepancy between the model and estimated correlations:

$$\frac{1}{N} \sum_{i=1}^N \left(\rho_L^{mod} - \rho_L^{est} \right)^2 \quad (6.27)$$

As shown in previous section 6.3.3.1, the normal calibration of correlation models the base rho ρ_0 associated with pairs of different months, i.e. summer-to-summer, summer-to-winter and winter-to-winter months pairs, in order to reflect the seasonality

A benchmark test against our correlation model was carried out by using real data from the NYMEX natural gas price database. From the historical data, we observed that the gas prices changed during the so-called “shoulder months” of November and December of the year. In order to capture this characteristic, we estimated ρ_0 corresponding to months pairs rather than seasonal pairs as described in section 6.3.3.1. Hence, we divide the historical data into four groups associated to different calendar months, which are April-October, November, December, January-March respectively.

Figures 6.5 to 6.7 below show the local correlations calibrated from historical data, the seasonal groups of data and the monthly groups of data. Tables D.6 to D.10 in appendix D.3.1 contain the full set of test results of local correlations calibrated from different set of data and their absolute differences. Results were confirmed to match for both groups of data. Although both selection of groups of data for calibration seem to work well, we conclude that using the monthly groups of data gives a better calibration result.

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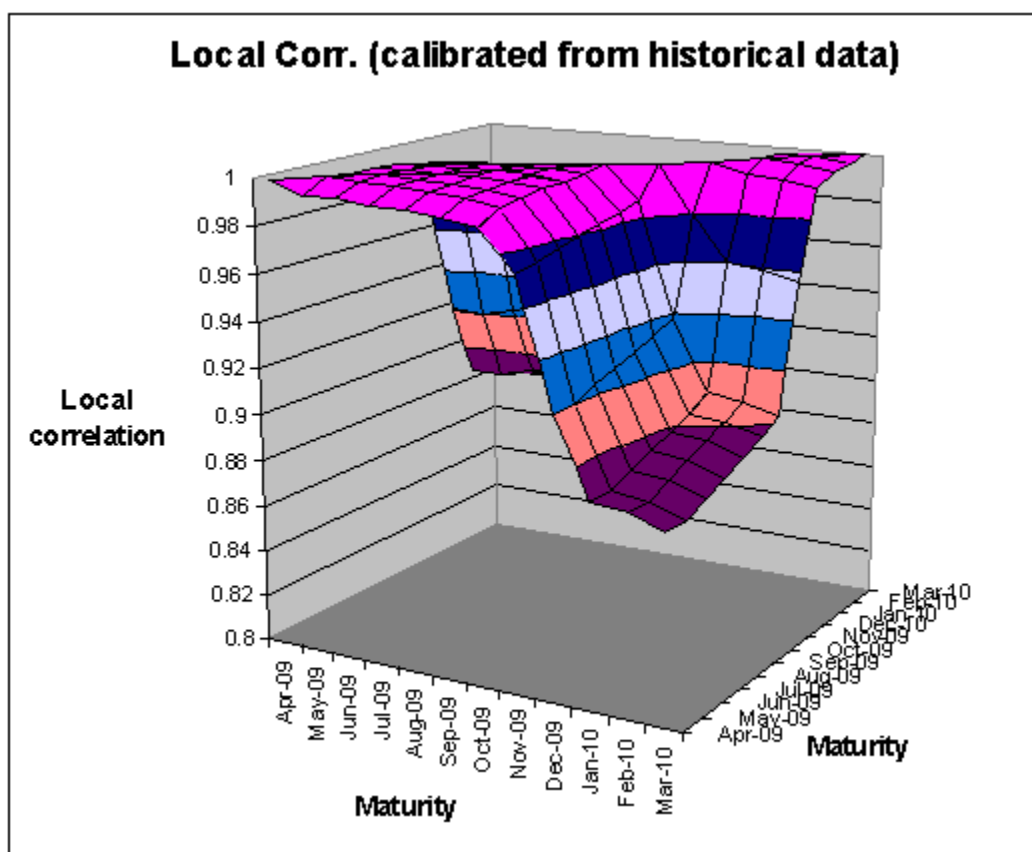


Figure 6.5.: Local correlations calibrated from historical data

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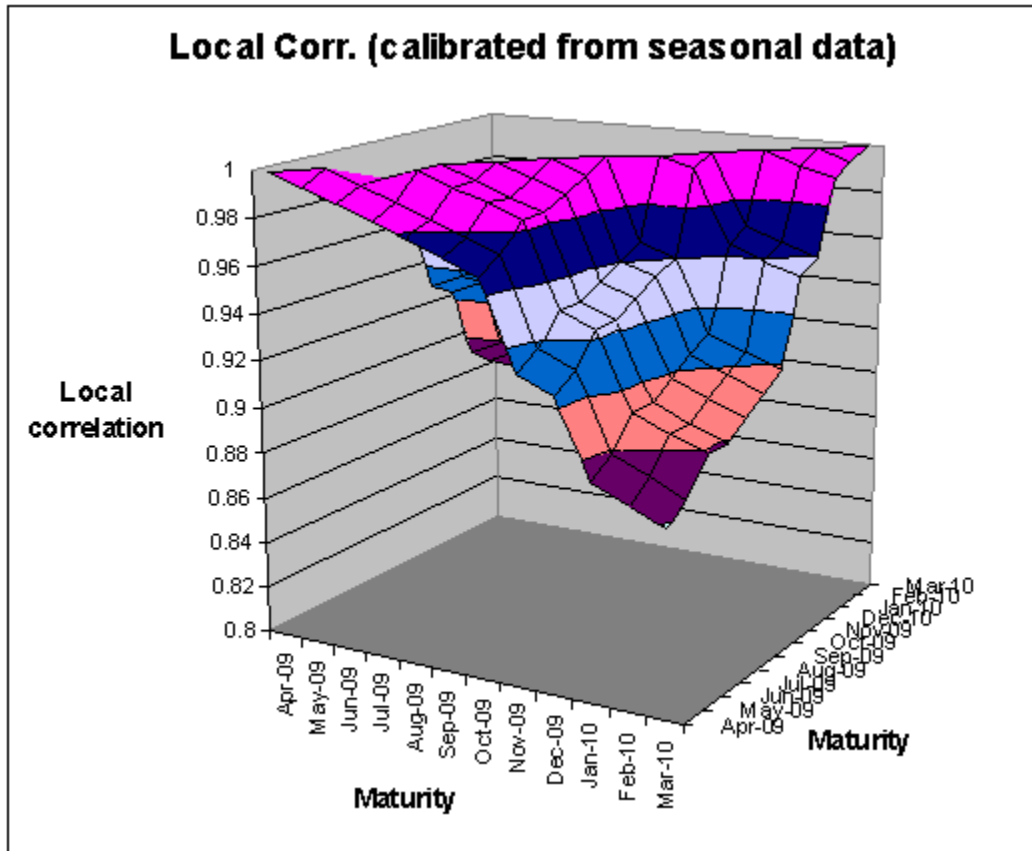


Figure 6.6.: Local correlations calibrated from seasonal data

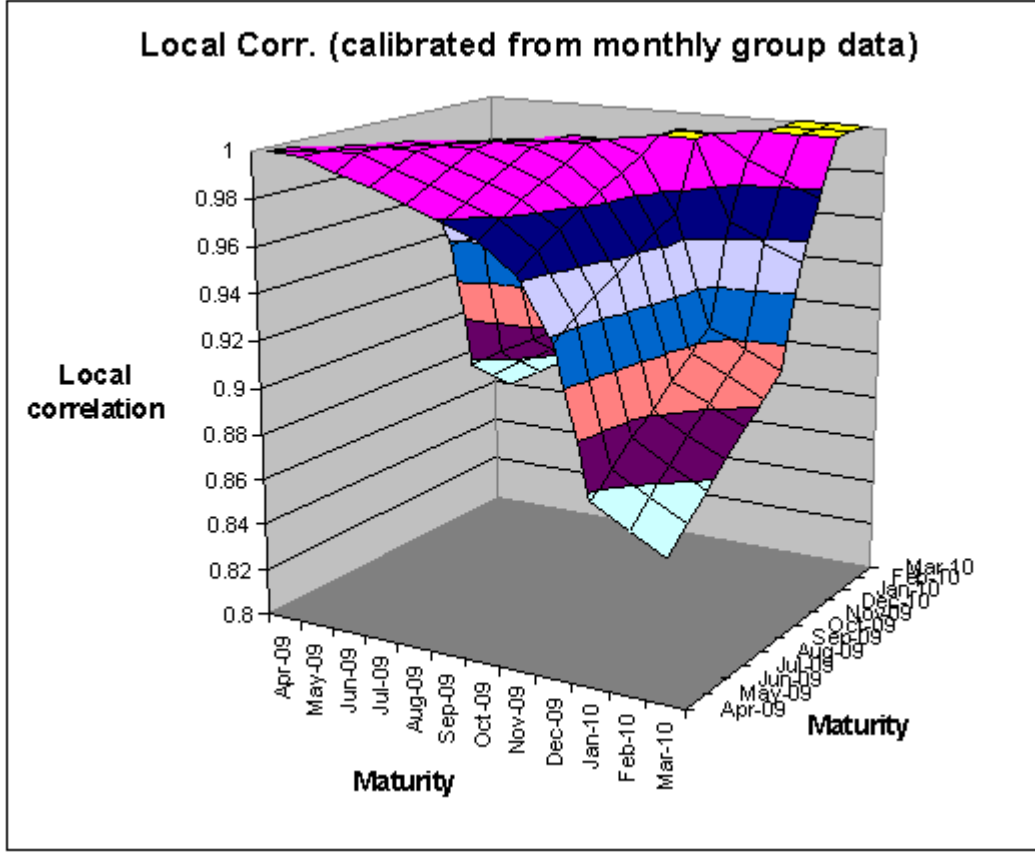


Figure 6.7.: Local correlations calibrated from monthly group data

6.3.4. Calendar Spread Option Pricing Model

As discussed in the previous section, the core of our storage model is a calendar spread option pricing model. For pricing the calendar spread options, we choose to use the well-known Kirk's approximation.

Suppose that we are given a calendar spread call option with option expiration T and a fixed strike K , on the spread between two underlying forward contracts $F(0, T_1)$ and $F(0, T_2)$ such that $0 \leq T \leq T_1 \leq T_2$. Then the payoff function of the spread option at T is

$$\max [F(T, T_2) - F(T, T_1) - K, 0] \quad (6.28)$$

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By Kirk's approximation, the option price is given by the following:

$$V_{call} = e^{-rT} [F(0, T_2) \cdot N(d_1) - (F(0, T_1) + K) \cdot N(d_2)] \quad (6.29)$$

with

$$d_1 = \frac{\ln\left(\frac{F(0, T_2)}{F(0, T_1) + K}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad (6.30)$$

$$d_2 = d_1 - \sigma\sqrt{T} \quad (6.31)$$

$$\sigma = \sqrt{\sigma^{F(0, T_2)}(0, T)^2 + \left(\sigma^{F(0, T_1)}(0, T) \cdot \frac{F(0, T_1)}{F(0, T_1) + K}\right)^2 - 2\rho \cdot \sigma^{F(0, T_1)}(0, T) \cdot \sigma^{F(0, T_2)}(0, T) \cdot \frac{F(0, T_1)}{F(0, T_1) + K}} \quad (6.32)$$

whereas ρ is the correlation between two assets. When pricing a future-to-future calendar spread option, ρ should be the term correlation. Otherwise, when pricing a cash-to-future calendar spread option, ρ should be the blended correlation. Similarly, the front month volatility should be blended too when pricing a cash-to-future option.

6.3.5. The Analytic Formulas of Spread Option Greeks for the Storage Model

Let F_1 denote the price of a forward contract $F_1(0, T_1)$ with maturity T_1 , and r_1 is the associated domestic risk-free interest rate. Similarly, let F_2 denote the price of a forward contract $F_2(0, T_2)$ with $T_1 \leq T_2$, and r_2 is the domestic risk-free interest rate associated to F_2 . Then Kirk's approximation of a spread call option gives that

$$V_{call} = e^{-r_1 T_1} \left[F_2 \cdot e^{-r_2 T_2 + r_1 T_1} \cdot N(d_1) - (F_1 + K) \cdot N(d_2) \right] \quad (6.33)$$

where

$$d_1 = \frac{\ln\left(\frac{F_2 e^{-r_2 T_2 + r_1 T_1}}{F_1 + K}\right) + \frac{1}{2}\sigma^2 T_1}{\sigma\sqrt{T_1}} \quad (6.34)$$

$$d_2 = d_1 - \sigma\sqrt{T_1} \quad (6.35)$$

$$\sigma = \sqrt{\sigma_2^2 + \left(\sigma_1 \cdot \frac{F_1}{F_1 + K}\right)^2 - 2\rho\sigma_1\sigma_2 \cdot \frac{F_1}{F_1 + K}} \quad (6.36)$$

An important feature to note about storage is that the solution to optimisation problem will always be at the upper or lower boundary constraints, that is, a “bang-bang” optimal control.

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This is because a rational trader will always purchase the maximum allowable volume and sell the maximum allowable volume at the most economical prices.

$$\sigma_1 = \begin{cases} \sigma_{imp}, & \text{if futs-to-futs} \\ f\left(\sigma_{imp}^{fr}, \sigma_{spot}\right), & \text{if cash-to-futs} \end{cases} \quad (6.37)$$

$$\sigma_2 = \begin{cases} f\left(\sigma_{imp}\right), & \text{if futs-to-futs} \\ f\left(\sigma_{imp}^{bk}\right), & \text{if cash-to-futs} \end{cases} \quad (6.38)$$

Using the above equations, we can compute the following derivatives:

$$\frac{\partial V_{call}}{\partial \sigma} = F_2 \cdot e^{-r_2 T_2} \cdot n(d_1) \cdot \sqrt{T_1} \quad (6.39)$$

$$\frac{\partial \sigma}{\partial F_1} = \frac{1}{\sigma} \cdot \frac{K}{(F_1 + K)^2} \cdot \left(\sigma_1^2 \cdot \frac{F_1}{F_1 + K} - \rho \sigma_1 \sigma_2 \right) \quad (6.40)$$

$$\frac{\partial^2 \sigma}{\partial F_1^2} = \frac{1}{\sigma} \cdot \frac{K}{(F_1 + K)^3} \cdot \left(\sigma_1^2 \cdot \frac{K - 2F_1}{F_1 + K} + 2\rho \sigma_1 \sigma_2 - \frac{\sigma_1^2 F_1 - \rho \sigma_1 \sigma_2 (F_1 + K)}{\sigma} \cdot \frac{\partial \sigma}{\partial F_1} \right) \quad (6.41)$$

$$\frac{\partial \sigma}{\partial \sigma_1} = \frac{1}{\sigma} \cdot \frac{F_1}{F_1 + K} \left(\sigma_1 \frac{F_1}{F_1 + K} - \rho \sigma_2 \right) \quad (6.42)$$

$$\frac{\partial \sigma}{\partial \sigma_2} = \frac{1}{\sigma} \left(\sigma_2 - \rho \sigma_1 \frac{F_1}{F_1 + K} \right) \quad (6.43)$$

$$\frac{\partial \sigma}{\partial \rho} = -\frac{\sigma_1 \sigma_2}{\sigma} \cdot \frac{F_1}{F_1 + K} \quad (6.44)$$

$$\frac{\partial d_2}{\partial F_1} = -\frac{1}{\sigma} \left(d_1 \frac{\partial \sigma}{\partial F_1} + \frac{1}{(F_1 + K) \sqrt{T_1}} \right) \quad (6.45)$$

Let $n(\cdot)$ denote the normal density function. Notice that

$$n(d_2) = n(d_1) \cdot \frac{F_2 e^{-r_2 T_2 + r_1 T_1}}{F_1 + K} \quad (6.46)$$

Then the Greeks are calculated as below:

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- Delta 1:

$$\frac{\partial V_{call}}{\partial F_1} = \frac{\partial V_{call}}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial F_1} - e^{-r_1 T_1} \cdot N(d_2) \quad (6.47)$$

- Delta 2:

$$\frac{\partial V_{call}}{\partial F_2} = e^{-r_2 T_2} \cdot N(d_1) \quad (6.48)$$

- Gamma 1:

$$\frac{\partial^2 V_{call}}{\partial F_1^2} = e^{-r_1 T_1} \cdot n(d_2) \cdot \left[(F_1 + K) \cdot \sqrt{T_1} \cdot \left(\frac{\partial^2 V_{call}}{\partial F_1^2} - d_2 \cdot \frac{\partial \sigma}{\partial F_1} \cdot \frac{\partial d_2}{\partial F_1} \right) + \frac{\partial \sigma}{\partial F_1} \cdot \sqrt{T_1} - \frac{\partial d_2}{\partial F_1} \right] \quad (6.49)$$

- Gamma 2:

$$\frac{\partial^2 V_{call}}{\partial F_2^2} = e^{-r_2 T_2} \cdot n(d_1) \cdot \frac{1}{\sigma \sqrt{T} \cdot F_2} \quad (6.50)$$

- Future-to-Future Vega:

$$\frac{\partial V_{call}}{\partial \sigma_1} = \frac{\partial V_{call}}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial \sigma_1} \quad (6.51)$$

$$\frac{\partial V_{call}}{\partial \sigma_{imp}} = \frac{\partial V_{call}}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial \sigma_2} \cdot \frac{\partial \sigma_2}{\partial \sigma_{imp}} \quad (6.52)$$

- Cash-to-Future Vega:

$$\frac{\partial V_{call}}{\partial \sigma_{imp}^{fr}} = \frac{\partial V_{call}}{\partial \sigma} \left(\frac{\partial \sigma}{\partial \sigma_1} \cdot \frac{\partial \sigma_1}{\partial \sigma_{imp}^{fr}} + \frac{\partial \sigma}{\partial \rho} \cdot \frac{\partial \rho}{\partial \sigma_{imp}^{fr}} \right) \quad (6.53)$$

$$\frac{\partial V_{call}}{\partial \sigma_{imp}^{bk}} = \frac{\partial V_{call}}{\partial \sigma} \left(\frac{\partial \sigma}{\partial \sigma_2} \cdot \frac{\partial \sigma_2}{\partial \sigma_{imp}^{bk}} + \frac{\partial \sigma}{\partial \rho} \cdot \frac{\partial \rho}{\partial \sigma_{imp}^{bk}} \right) \quad (6.54)$$

$$\frac{\partial V_{call}}{\partial \sigma_{spot}} = \frac{\partial V_{call}}{\partial \sigma} \left(\frac{\partial \sigma}{\partial \sigma_1} \cdot \frac{\partial \sigma_1}{\partial \sigma_{spot}} + \frac{\partial \sigma}{\partial \rho} \cdot \frac{\partial \rho}{\partial \sigma_{spot}} \right) \quad (6.55)$$

6.3.6. Linear Programming Problem

Owning a natural gas storage facility is equivalent to owning a portfolio of calendar spread options subject to physical and contractual constraints. To maximise the value of a storage facility, one need select the most profitable calendar spread options, subject to the constraints, to construct the portfolio. Therefore, after having all calendar spread option prices, the storage valuation problem

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becomes a linear programming problem. An optimisation program will be used to choose the optimal basket of calendar spread options subject to the physical constraints.

Denote the initial gas storage level by P_0 and the number of contract months by N . Let V_{ij} be the price of the calendar spread option and V_{0j} be the spread between month j and the average month commodity cost of initial underground gas. The optimisation problem is formulated as follows:

$$\max_{V_{i,j}} \left(\sum_{i,j} V_{ij} p_{ij} + \sum_j V_{0j} p_{0j} \right) \quad (6.56)$$

subject to

$$P_i = P_{i-1} + \sum_j p_{ij} - \sum_j p_{ji} - p_{0i}, \quad \forall i = 1, \dots, N \quad (6.57)$$

$$P_0 = \sum_i p_{0i} \quad (6.58)$$

$$P_{i,\min} \leq P_i \leq P_{i,\max}, \quad \forall i = 1, \dots, N \quad (6.59)$$

$$p_{ij} \geq 0 \quad (6.60)$$

$$P_i \geq 0 \quad (6.61)$$

where decision variable p_{ij} represents the amount of gas injected in month i and withdrew in month j , p_{0j} denotes the initial underground gas withdrew in month j , and P_i is gas storage level at the end of month i .

6.4. Storage values

Commodity storage has two value components: intrinsic value and extrinsic value. The intrinsic value is determined by seasonal price spreads of the forward curve and risk-free interest rates. The extrinsic value is captured from the price volatility through dynamically optimising injection and withdrawal operations during the lifetime of a storage contract. A more detailed discussion was shown in section 6.2.

When pricing the storage options, we used four sets of prices from the NYMEX natural gas price database, corresponding to the information available at the close of trading on the following four days: 01/03/2006 (Spring), 01/06/2006 (Summer), 01/09/2006 (Fall), and 01/12/2006 (Winter) respectively. The choice of these dates allowed us to look into cases with trading information reflecting the seasonality characteristics of the year. For each selected trading day, we considered the Henry Hub spot price and futures prices corresponding to the first 12 maturities (recall that

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Henry Hub is the delivery location of the NYMEX natural gas futures contract). This gave us four forward curves, each consisting of 13 prices, as shown in figures 6.8 and 6.9 below:

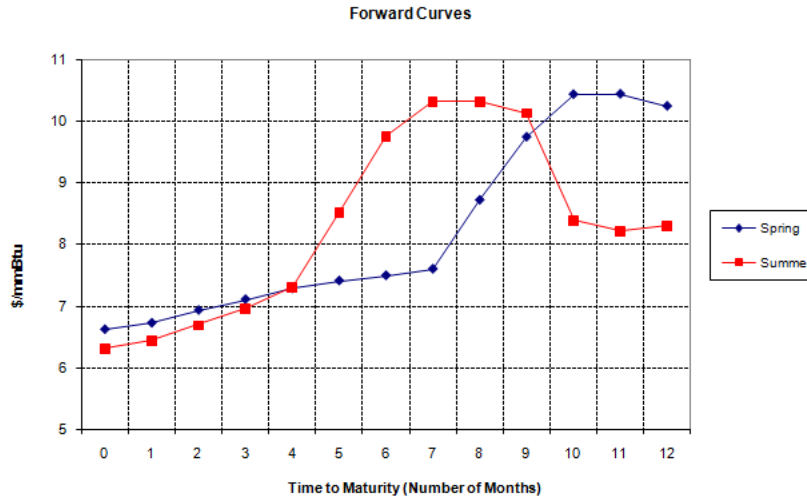


Figure 6.8.: NYMEX natural gas forward curve on 01/03/2006 (Spring) and 01/06/2006 (Summer)

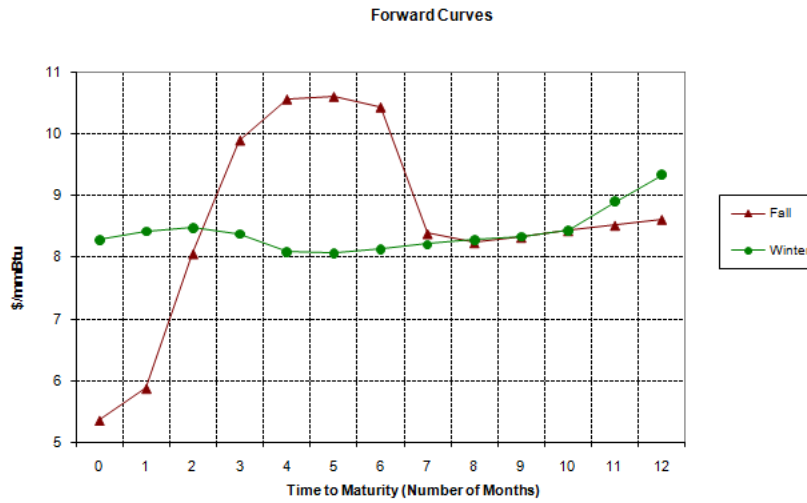


Figure 6.9.: NYMEX natural gas forward curve on 01/09/2006 (Fall) and 01/12/2006 (Winter)

The market implied volatilities of the 12 future prices on each of the four trading days considered were calibrated using NYMEX quoted prices of call options on natural gas futures, through the

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second approach for volatility calibration described in section 6.3.2. Figures 6.10 and 6.11 illustrate the implied volatilities on each of the four trading days:

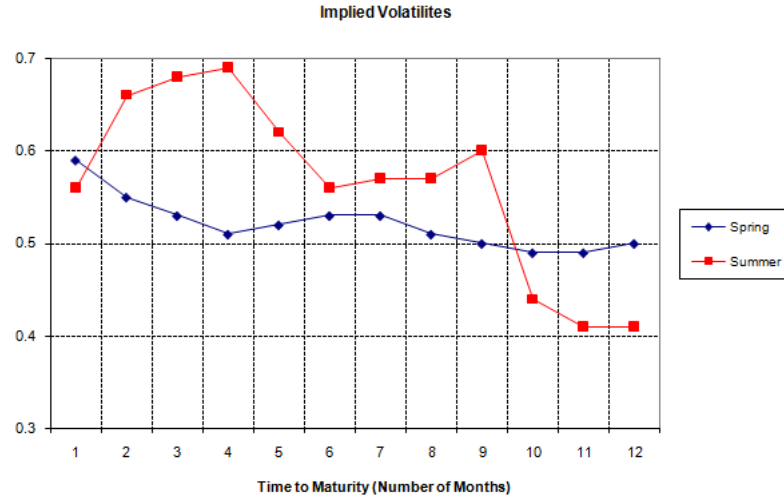


Figure 6.10.: NYMEX natural gas implied volatilities on 01/03/2006 (Spring) and 01/06/2006 (Summer)

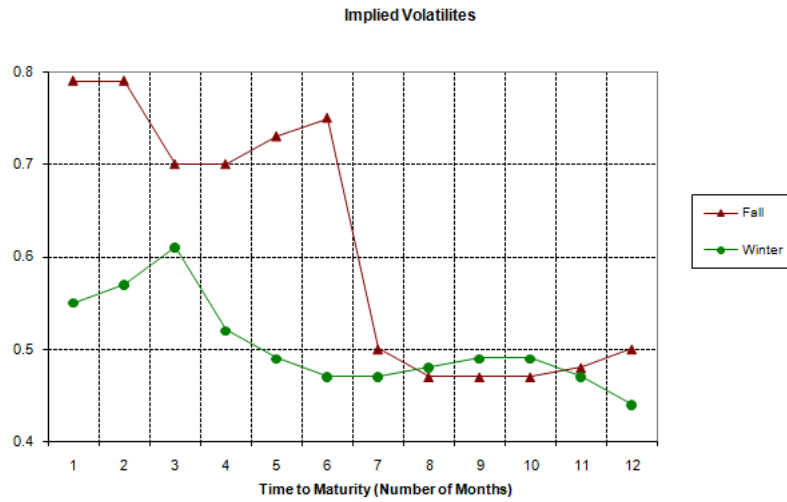


Figure 6.11.: NYMEX natural gas implied volatilities on 01/09/2006 (Fall) and 01/12/2006 (Winter)

From the above figures, we can see that the implied volatilities tend to decrease when the option

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maturity increases. The patterns of implied volatility also exhibit seasonality effect, as the forward curves.

Using the method described in subsection 6.3.3.4, we employed market data to imply our local correlations between futures prices for different maturities. Tables D.11 to D.14 represent the local correlation matrices for four forward curves respectively. The minimum inventory level is set to be zero and we employ the static injection/withdraw schemes, that is, the daily injection/ withdraw capacities are constant. For the cash correlations, correlation matrices were constructed in a similar way by first calibrating the parameters $\{a, b, c, d\}$ from historical data and then calculating from equation (6.3.3.3) for different maturities. Table 6.1 below summarises them for 12 maturities.

Correlations	2M	3M	4M	5M	6M	7M	8M	9M	10M	11M	12M
1M	0.919322	0.803452	0.732284	0.683158	0.648359	0.622545	0.600556	0.581908	0.564208	0.547428	0.532732
2M		0.914639	0.803452	0.730383	0.681854	0.648359	0.621775	0.600556	0.581317	0.563655	0.548491
3M			0.919322	0.803452	0.730383	0.683158	0.648359	0.622545	0.600556	0.581317	0.565317
4M				0.914639	0.800553	0.730383	0.681854	0.648359	0.621775	0.599901	0.582501
5M					0.914639	0.803452	0.730383	0.683158	0.648359	0.621775	0.601875
6M						0.919322	0.803452	0.732284	0.683158	0.648359	0.624100
7M							0.914639	0.803452	0.730383	0.681854	0.650307
8M								0.919322	0.803452	0.730383	0.685810
9M									0.914639	0.800553	0.734210
10M										0.914639	0.809382
11M											0.928917

Table 6.1.: Local correlation corresponding to the Spring forward curve

Because of the simplicity of the implementation of the Kirk's approximation, it is often used for pricing storage options. Since the Kirk's formula only provides an approximation to spread option values, the true option values could be under-estimated by using this method. Moreover, the Kirk's formula only works under the assumption of log-normally distributed storage option prices. Hence, if we use Kirk's approximation to price advanced exotic options on storage (such as compound options on storage), errors will occur in the output prices because the distribution of the storage prices often exhibit the skewness and fat-tail effect. Aiming to overcome the drawbacks of the Kirk's formula, we build a three-dimensional binomial tree to price storage options, where each node in the tree leads to four possible future outcomes. For how the 3D-binomial tree works for pricing, it is best to be understood through the structure shown in figure 6.12 below, which depicts a 3D-binomial tree model for two asset options:

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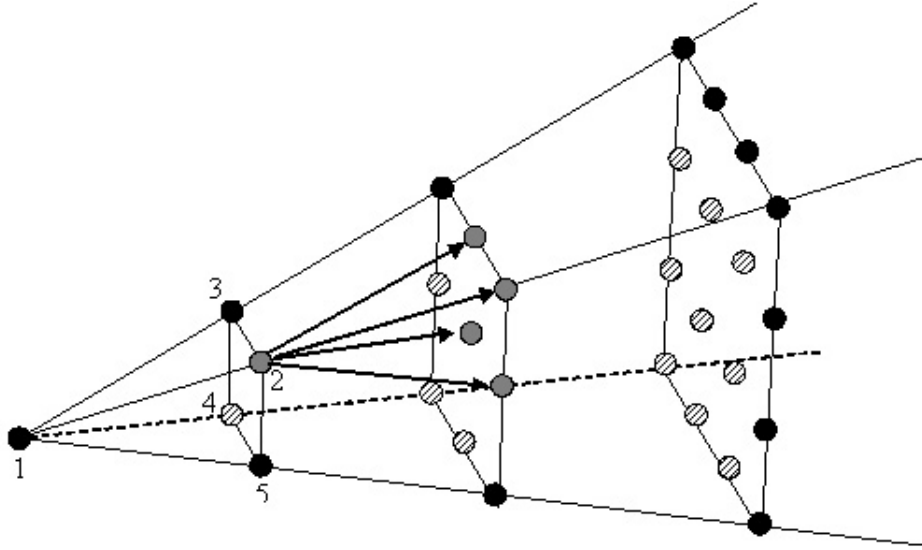


Figure 6.12.: 3D binomial tree structure

In our work, results generated by Kirk's approximation was compared with those generated by the 3D-binomial tree approach. The option prices computed from Kirk's closed formula and 3D-binomial tree are shown in tables D.1 and D.2. 500 time steps were used in the tree method and the run time was about 17 seconds on average. A comparison of results obtained by the Kirk's approximation and the 3D-binomial tree method, tabulated in table D.3, suggests that the order of the absolute difference was less than $1.0E-04$.

Using the forward curves and implied volatilities as in figure 6.8 to 6.11, we calculated the values of 12-months storage options using both the Kirk's approximation and the 3D-binomial tree, with the underlying as future-to-future or cash-to-future spread options. The results are shown as in table 6.2 below:

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Storage Value			
Cases	Kirk's Approx.	3D-binomial Tree	Intrinsic Value
Spring-FF	3254166.73	3254138.96	3120743.37
Spring-CF	4104141.60	4104296.66	3120743.37
Summer-FF	3722597.92	3722588.64	3348348.62
Summer-CF	4575197.17	4575429.05	3348348.62
Fall-FF	2959166.27	2959188.50	2441046.52
Fall-CF	4271951.12	4272230.03	2441046.52
Winter-FF	1019289.51	1019345.36	649355.73
Winter-CF	2223464.62	2223836.78	649355.73

Table 6.2.: Values of storage options

The maximum relative difference between storage values generated by Kirk's approximation and the 3D-binomial tree is $1.7\text{E-}04$ (for constant daily injection). Therefore, we conclude that the 3D-binomial tree method works well for pricing storage options.

6.5. Summary

The storage option is one of the most popular product in the natural gas market, because it accounts for the seasonality effect on the market price of natural gas. In the previous chapters, we proposed the Market volatility model and apply it in the pricing of various interest-rate and commodity derivatives, however, this model is not compatible with the structure of the storage options. In this chapter, we reviewed pricing methods for storage options and we chose to use the "basket of spread options" pricing method.

On the base of the "basket of spread options" pricing method, we made three improvements on the base of the traditional pricing model. First, we modified the local volatility functional form in order to capture the seasonality effect. To do this, we add an extra parameter (which is generated from seasonal historical data) to our assumed functional form for local volatilities. Second, we calibrate the correlations using four groups of market data, each associated to a different season of a year. In this way, we increase the effectiveness of our calibration of correlations. Third, we built a 3D-binomial tree to price the spread options in the storage option, rather than employing the Kirk's formula in calculation. We finished this chapter by pricing storage options using real market data.

7. Conclusion

7.1. Contributions

The following three contributions were made in this thesis. First of all, we developed a robust model for use in both the interest-rate markets and the energy markets. This model is not covered in any literature, instead, it arises from market practice and trading experience. Second, we proposed a new type of option with a flexible structure, to capture the characteristics of the energy market. Third, we modified the existing pricing model for storage options to improve the model performance.

Modeling interest-rate is a challenging task, because it involves modeling the term structure of the whole interest-rate yield curve. In Chapter 2 of this thesis, we reviewed various popular pricing models in the interest-rate market. Among these pricing models, the LIBOR Market model (LMM) is most highly regarded, so we used it as a benchmark model. The popularity of the LMM model comes from its compatibility with the Black's analytical pricing formula. The centre of the LMM model is modeling the evolution of forward interest-rates. However, the LMM model has its own drawback, which is caused by the complex computing procedures introduced by the calibration procedure of this model. Although the LMM model is widely used for pricing vanilla interest-rate derivatives, it must be modified if one wants to price exotic interest-rate derivatives.

In order to overcome the drawback of the existing interest-rate models, we proposed a new type of interest-rate model in Section 2.4. We name this model as the "Market Volatility model". From market practice, it was observed that if the market is liquid enough for a product then we could obtain the corresponding implied volatilities from the market quotes, and a full volatility surface could be obtained through interpolation in order to account for the smile effect. Based on this observation, we introduce an intuitive assumption of our Market Volatility model: with enough liquidity in the market, the market quoted volatilities are effective enough for interpreting the market condition. In this way, we do not need to do any calibration to find the implied volatilities. Instead, we obtain the representative data based on our current view of the market, then we use bilinear interpolation to compute the implied volatilities associated with every desired option maturity. The skew values are obtained in a similar way. In Section 2.4, we discuss about the framework of the Market Volatility model and we go into the details of how the cap and swaption volatilities are computed via interpolation. The core model of the Market volatility model is based on the standard Black's analytical formula, which gives robust computing efficiency of this model.

In Chapter 3, we test the performance of the Market Volatility model proposed in the previous Section 2.4. In order to test the pricing performance of the Market Volatility model, we used this

7. Conclusion

model to compute the volatility surfaces with the smile effect for interest-rate caps and swaptions. The resulting volatility surfaces are then used to price interest-rate caps and swaptions via Black's formula. The output volatilities and prices for interest-rate caps and swaptions are compared to results generated by the LMM model for the same products. The corresponding test results were shown in Section 3.3. It is proved that the Market Volatility model produced comparable results to those of the LMM model, while its computing efficiency largely exceeds that of the LMM. Moreover, the Market Volatility model can be used to price exotic interest-rate derivatives such as CMS spread options, due to its assumption of intuitive structure of volatilities. In contrast, the LMM model cannot be simply used to price exotic interest-rate derivatives without modification.

In Chapter 4, we look into a new market that the Market Volatility model can be applied to, which is the energy market. Following the credit crisis during year 2007-2009, much capital escaped from the credit market and flowed into the commodity market. Under this influence, the commodity market developed rapidly, and so did the energy derivatives market. Traditionally, many energy derivatives have structures arisen from the interest-rate (fixed-income) market. The only difference between the energy market and the interest-rate market is that the spot-forward relationship is no longer an equality, because of the presence of the convenience yield (recall that the convenience yield is the extra cost caused by holding/storing a physical commodity). However, in practice, we often price energy derivatives based on the assumption of absence of the convenience yield. Under these circumstances, the spot-forward relationship in the commodity market will be the same as in the interest-rate market. Therefore, we can extend the use of our proposed Market volatility model from the interest-rate market to the commodity market. The Market Volatility model can be applied to the commodity market, and we can price commodity derivatives with this model using same numerical techniques that we use for pricing interest-rate derivatives. In this thesis, we focused on the category of energy markets.

In Chapter 5, we apply the Market volatility model to price various energy derivatives, i.e. spread options, basket options, Asian options, etc (under the assumption of the absence of convenience yield). Moreover, we proposed a new type of exotic energy derivative with flexible option structure, based on the real trading experiences. We named this energy derivative as the Flex-Asian spread options (FASO), which is a hybrid of basket- and Asian-type options. The FASO has a flexible pay-off structure that can be set to completely match the model user's desired pay-off function. It is particularly useful in the energy market, as many energy derivatives are traded over-the-counter and we often face non-standard tenor structures in the real physical trade. In Section 5.4, we give examples of different option structures of the FASO. We then use synthetic data as input to the Market Volatility model for pricing the FASO under various test scenarios. A detailed analysis of different test scenarios and the corresponding test results are given in Section 5.5.

Although the Market volatility model could be used to price various energy derivatives based on oil/gas contracts, it is not compatible with the structure associated to one of the most advanced derivatives in the energy market, which is the storage option. Storage options are heavily used in the natural gas market, because the market price of natural gas is affected by the seasonality

7. Conclusion

factor. As a result, natural gas is often stored during the “low-price” months and is delivered in the relative “high-price” months. In Chapter 6, we reviewed various pricing methods for storage options and chose to use the “basket of spread options” pricing method as described in Section 6.2.3, and we made three improvements on the base of the traditional pricing model.

When calibrating the volatilities of storage options, we normally assume that the instantaneous volatilities follow the exponential decay function as in equation (6.1) in Section 6.3.2. One drawback of this assumed functional form is that it failed to capture the seasonality effect, which has large impact on the natural gas prices. Failure to capture the seasonality effect will lead to biased or even invalid calibration results. In order to over this drawback, we add in an additional parameter to the calibration function. By doing this, we increase the effectiveness of our calibration of volatilities significantly.

Similar to volatilities, if we just simply calibrate the correlations of storage options, we will fail to capture the seasonality effect. For the calibration of correlations, we divided the market data into four groups associated to different seasonal effects, and we calibrate four times using each group of data respectively in order to generate appropriate local correlations with respect to seasonality.

Finally, we built a 3D-binomial tree to price the spread options in the storage option. Kirk’s formula is commonly employed in pricing spread options, however, it generates an approximate real option value which could introduce a risk of underestimating the true option value. In contrast to the Kirk’s approximation for pricing spread options, our 3D-binomial tree method improved the accuracy of pricing. Moreover, one must apply the Kirk’s formula under the assumption of log-normally distributed storage option prices. Such restriction does not exist when we use the 3D-binomial tree method for pricing. Consequently, it is more suitable to use the 3D-binomial tree method when we attempt to price more advanced derivatives, i.e. the compound option on storage. If we use Kirk’s approximation to price advanced exotic options on storage, errors will occur in the output prices because the distribution of the storage prices often exhibit the skewness and fat-tail effect.

Based on the discussion in Chapter 6, we finished this thesis by pricing storage options using real market data. For the detailed implementation, readers should refer to Section 6.3.

7.2. Future Research

Our research is confined by the scope of this thesis, however, there are possibilities of further research.

In this thesis, we proposed the new Market Volatility model, and we showed how it can be used to price various interest-rate and commodity derivatives, using the interpolated volatility surface as

7. Conclusion

input. We could research into further depth of the construction of the implied volatility surface in the Market volatility model, which will lead to higher accuracy of the model. This could be done in two ways. First, we could use empirical studies to improve the efficiency of the input data when choosing market quoted implied volatility matrices. Secondly, we could employ an interpolation method other than the bilinear interpolation one used in our model. As shown in Section 2.4, we employed the bilinear interpolation when extending the implied volatility matrices. To do this, we select a market quoted volatility matrix with respect to different option maturity dates and strike prices, then we use bilinear interpolation to extend this volatility matrix to match our desired option structure. For the illiquid options with extremely short or long maturities, we assume they have a flat volatility structure. For future research, we could use interpolation methods such as the cubic spline interpolation to improve the goodness of fit between our desired volatility surface and the market quoted implied volatility surface.

In addition, the bilinear interpolation in the Market Volatility model is performed under the assumption of piece-wise constant volatilities. We could progress further by relaxing this assumption. For example, we could assume a linear functional form for the relationship between volatilities of different maturity dates, which could help to prove the model performance.

In Chapter 5 of this thesis, we propose a new type of exotic energy derivative with flexible option structure, named as the Flex-Asian spread options (FASO). The FASO is designed according to trader's need. It allows one to match the option's terminal pay-off function exactly to the contractual terms of a physical supply deal. By varying the terminal pay-off function of the option, the FASO becomes a spread option, a basket option or an Asian option, with respect to the option user's setting. The current scope of our FASO option can handle options with up to 50 payoff scenarios related to different underlying commodities. In our work, we design four FASO option structures and we price them using the Market Volatility model and synthetic data. Our first FASO option is a hybrid of an oil basket option on six future contracts. The second, third and fourth FASO options are hybrids of oil Asian spread option on three future contracts, with the option maturity dates sit in different pricing windows. An extension of the use of the FASO is to apply this type of option in other markets such as the transportation and transmission markets, as the spread option structure plays an important role in both markets. The transportation market often involves transporting natural gas or power fuel capacity from one point in the system to another. If the points are liquid, then the one-directional capacity flow between any two points can be represented by a combination of spread options. If gas or power fuel is allowed to flow in two directions, then there will be two spread options corresponded to this right at each point. This combination of spread options can be matched well by using our FASO option structures. Similar dynamics operate in the emission market except that no exact route could be determined when translating power from one location to another. As a result, the strike prices of the associated spread options does not have a standard structure. Instead, they must be structured to account for the charges generated by translating power along the path.

As for the storage model (basket of calendar spread option method) shown in Chapter 6, we could

7. Conclusion

apply the pricing model discussed in Section 6.3 to more complicated situations, i.e. underlying storage contracts have ratcheted injection/ withdraw schedule. By saying ratcheted injection/withdraw schedule, we mean that the injection/ withdraw capacities are dependent on the storage volume and time. In addition, we could use the storage prices outputted from our model to price more advanced exotic options. For example, the compound options on storage, which are options on storage options, could be priced on the base of the 3D-binomial tree method. We could also easily extend the use of our storage model from the gas markets to oil markets.

A. Test Results for the LMM and the Market Volatility Models

A.1. European Swaptions

A.1.1. Swaption Volatilities

A.1.1.1. Test Results from the LMM model

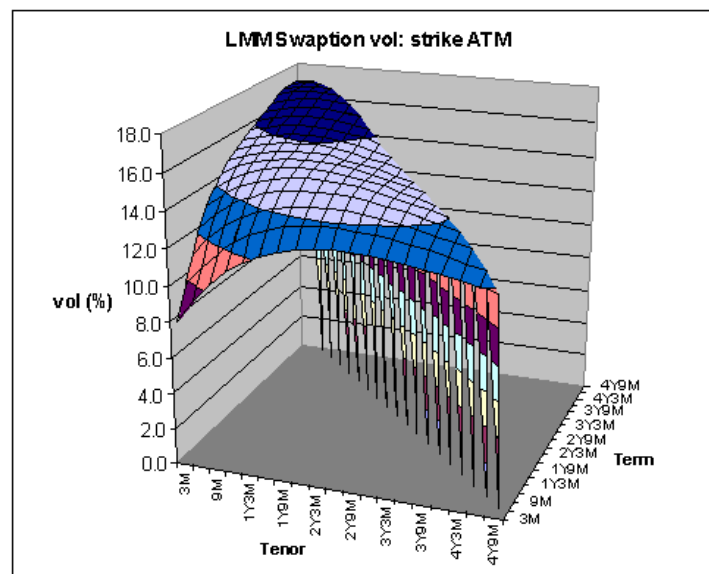


Figure A.1.: Swaption volatilities with respect to the at-the-money strike prices, calculated by the LMM model

A. Test Results for the LMM and the Market Volatility Models

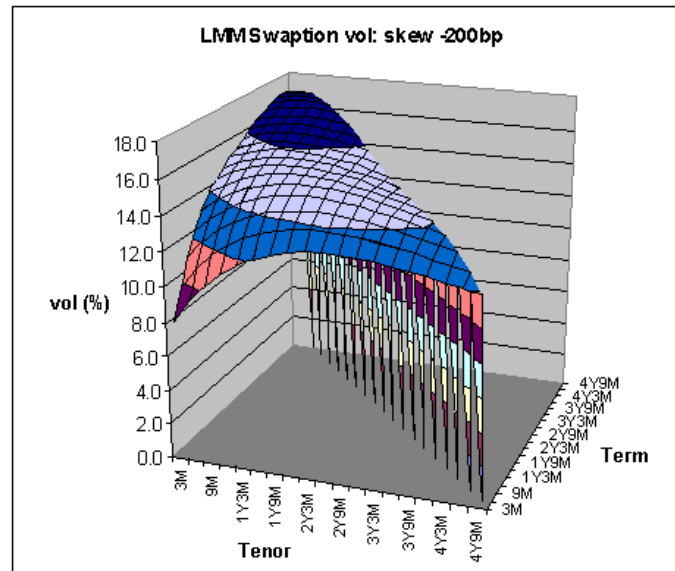


Figure A.2.: Swaption volatilities with respect to strike prices with skewness -200 bp, calculated by the LMM model

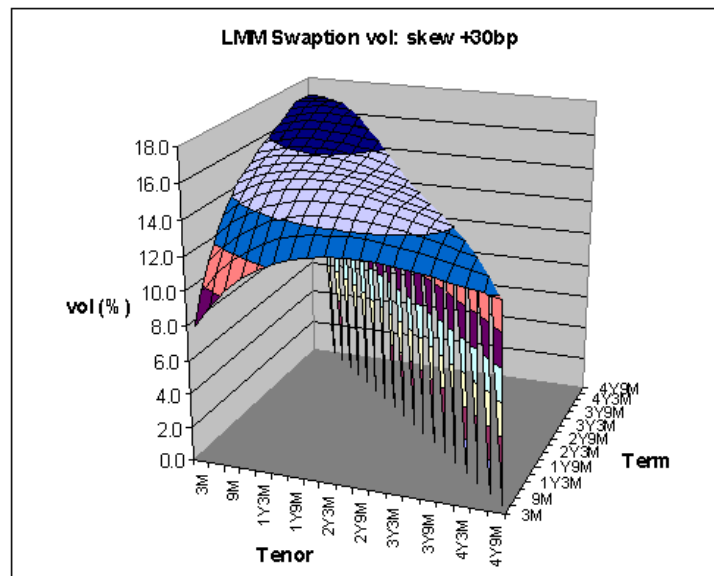


Figure A.3.: Swaption volatilities with respect to strike prices with skewness +30 bp, calculated by the LMM model

A. Test Results for the LMM and the Market Volatility Models

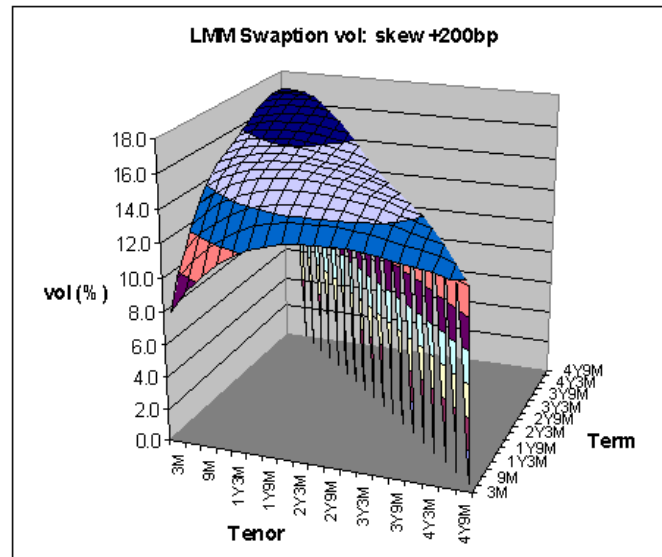


Figure A.4.: Swaption volatilities with respect to strike prices with skewness +200 bp, calculated by the LMM model

A.1.1.2. Test Results from the Market Volatility model

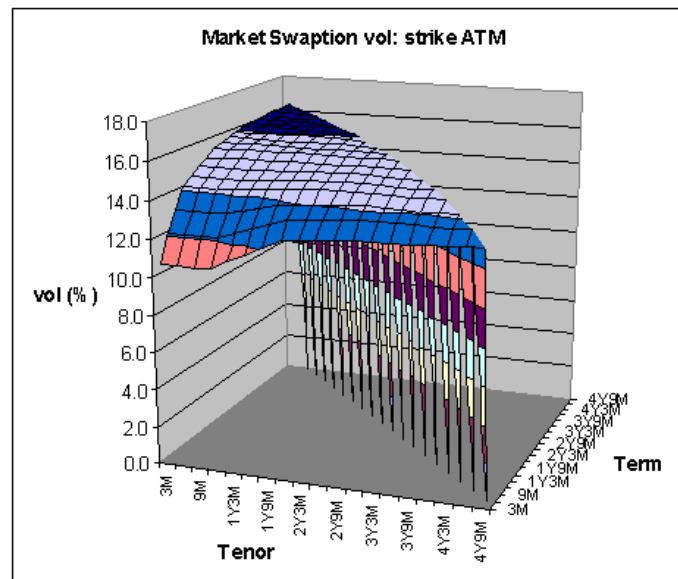


Figure A.5.: Swaption volatilities with respect to the at-the-money strike prices, calculated by the Market Volatility model

A. Test Results for the LMM and the Market Volatility Models

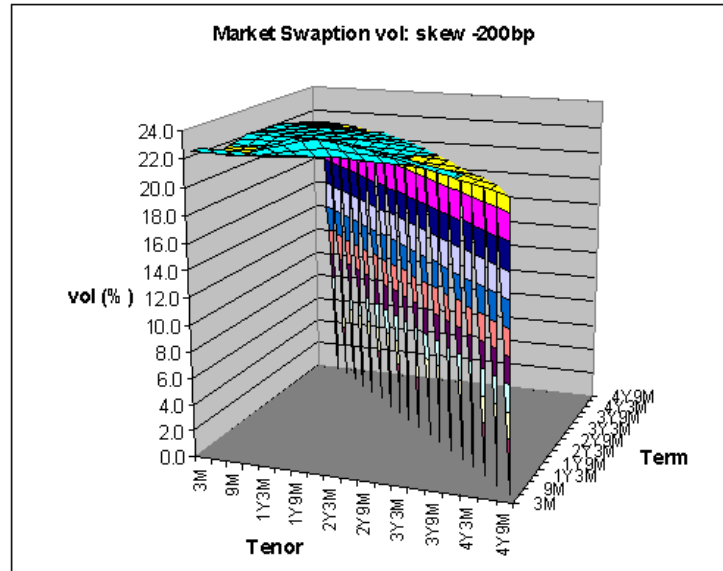


Figure A.6.: Swaption volatilities with respect to strike prices with skewness -200 bp, calculated by the Market Volatility model

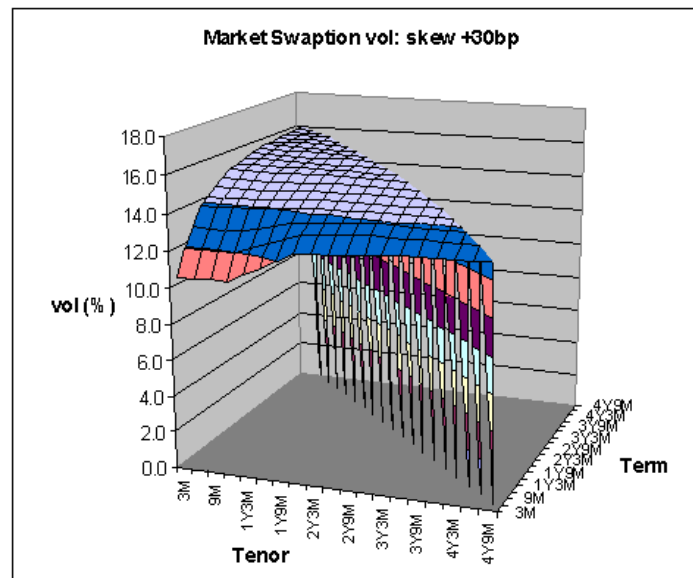


Figure A.7.: Swaption volatilities with respect to strike prices with skewness +30 bp, calculated by the Market Volatility model

A. Test Results for the LMM and the Market Volatility Models

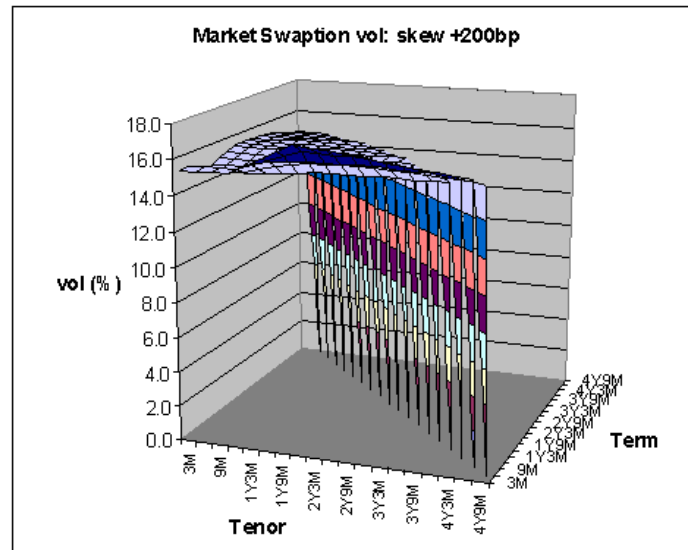


Figure A.8.: Swaption volatilities with respect to strike prices with skewness +200 bp, calculated by the Market Volatility model

A. Test Results for the LMM and the Market Volatility Models

A.1.2. Swaption prices

A.1.2.1. Test Results from the LMM model

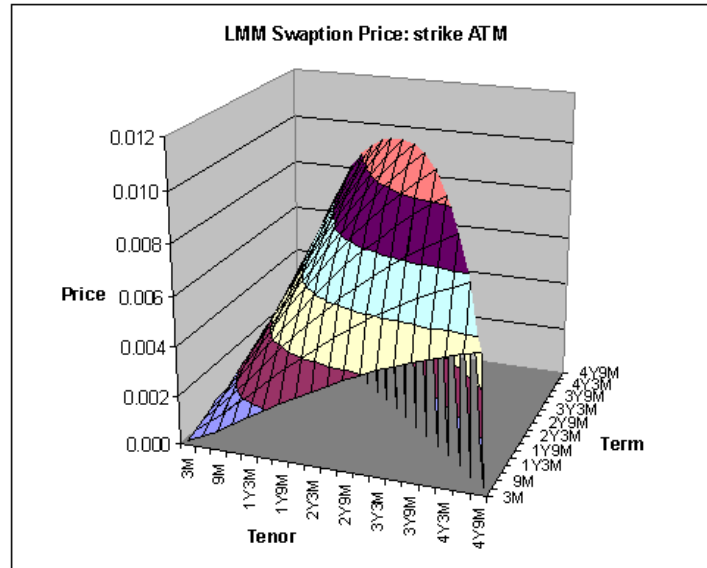


Figure A.9.: Swaption prices with respect to the at-the-money strike prices, calculated by the LMM model

A. Test Results for the LMM and the Market Volatility Models

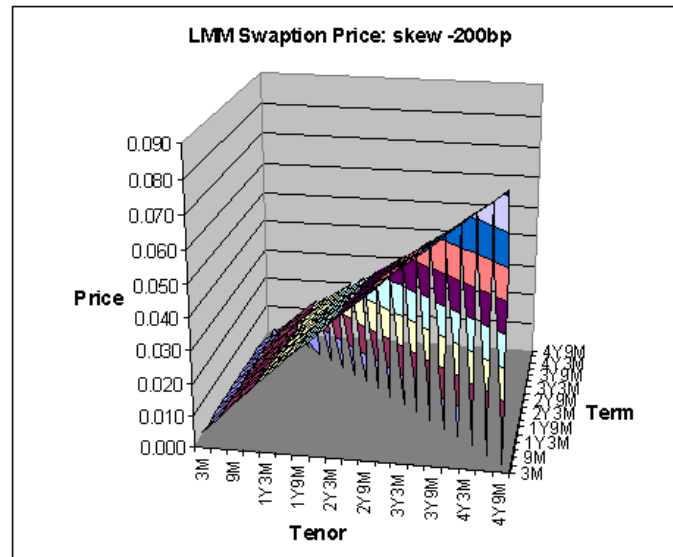


Figure A.10.: Swaption prices with respect to strike prices with skewness -200 bp, calculated by the LMM model

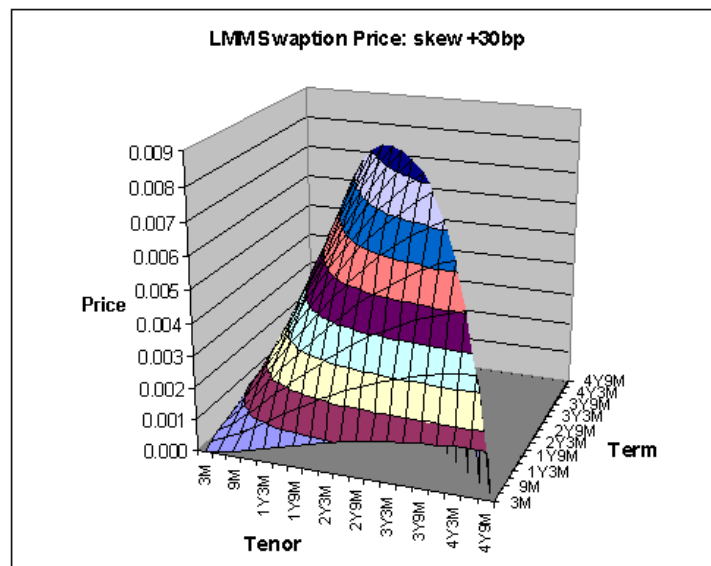


Figure A.11.: Swaption prices with respect to strike prices with skewness +30 bp, calculated by the LMM model

A. Test Results for the LMM and the Market Volatility Models

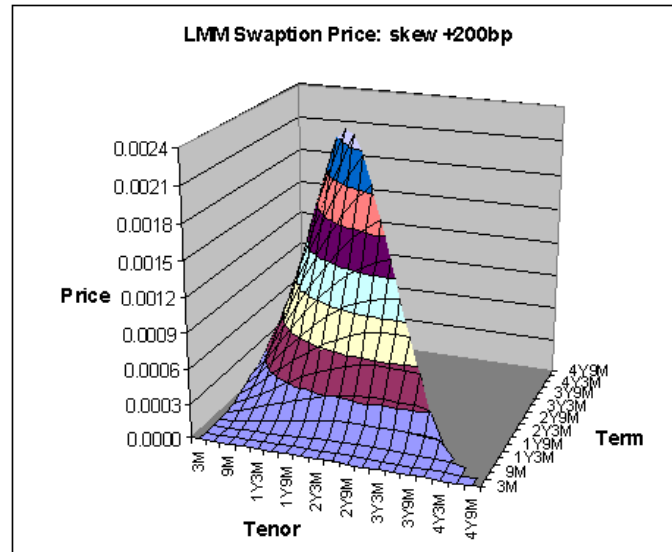


Figure A.12.: Swaption prices with respect to strike prices with skewness +200 bp, calculated by the LMM model

A.1.2.2. Test Results from the Market Volatility model

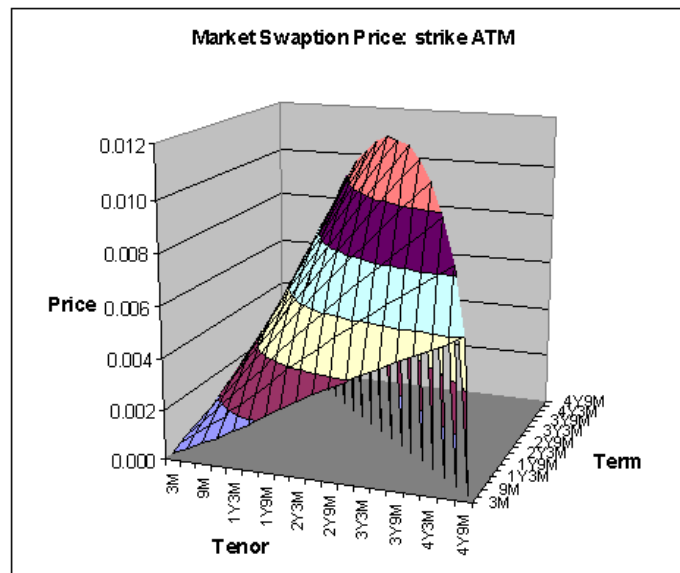


Figure A.13.: Swaption prices with respect to the at-the-money strike prices, calculated by the Market Volatility model

A. Test Results for the LMM and the Market Volatility Models

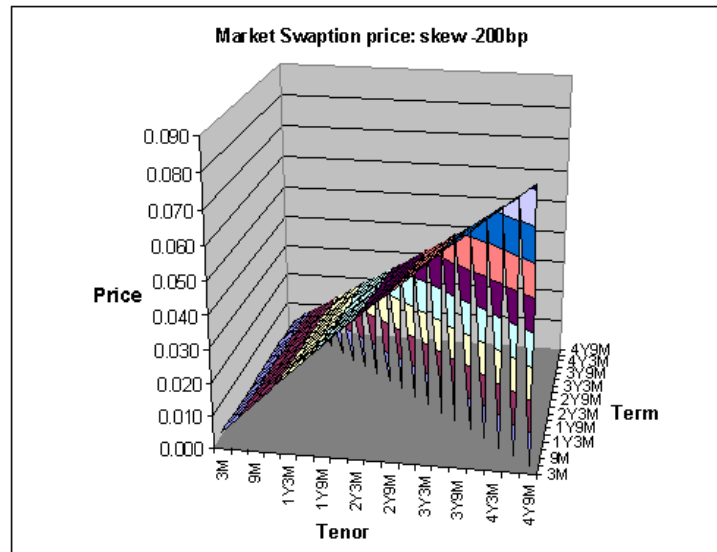


Figure A.14.: Swaption prices with respect to strike prices with skewness -200 bp, calculated by the Market Volatility model

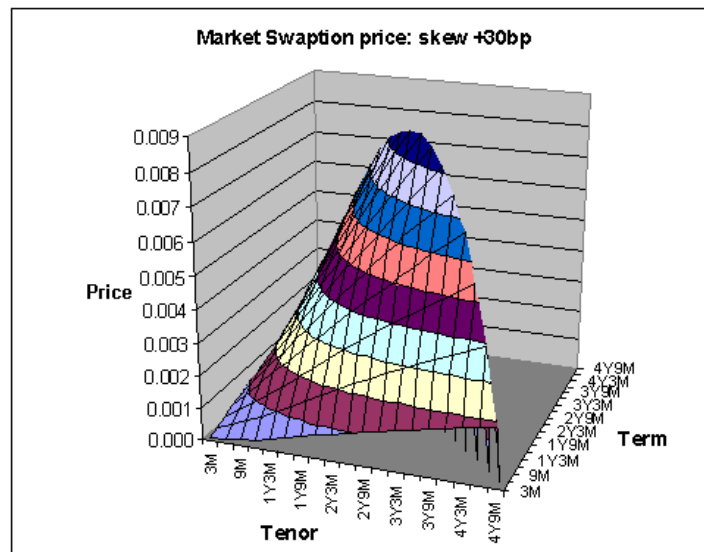


Figure A.15.: Swaption prices with respect to strike prices with skewness +30 bp, calculated by the Market Volatility model

A. Test Results for the LMM and the Market Volatility Models

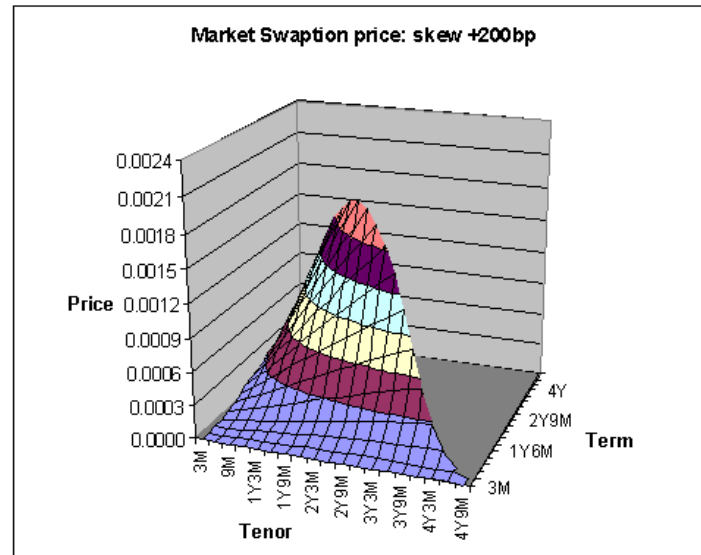


Figure A.16.: Swaption prices with respect to strike prices with skewness +200 bp, calculated by the Market Volatility model

A.1.2.3. Test Results from Monte-Carlo Simulations

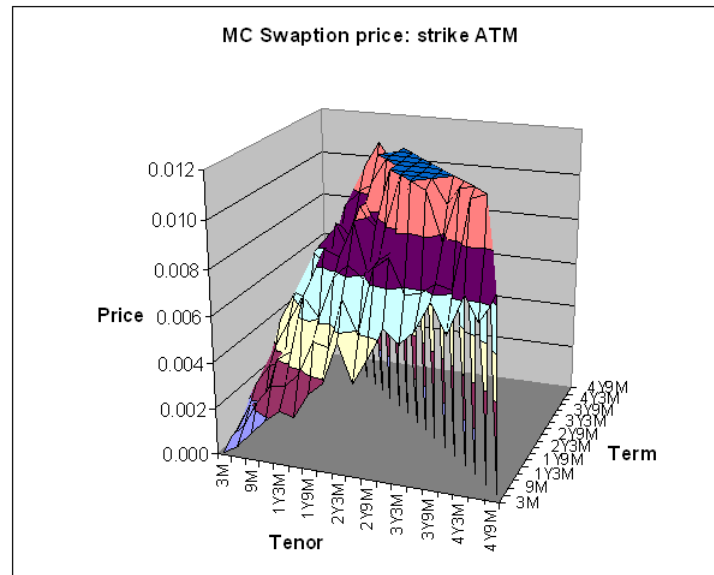


Figure A.17.: Swaption prices with respect to the at-the-money strike prices, calculated by Monte-Carlo simulations

A. Test Results for the LMM and the Market Volatility Models

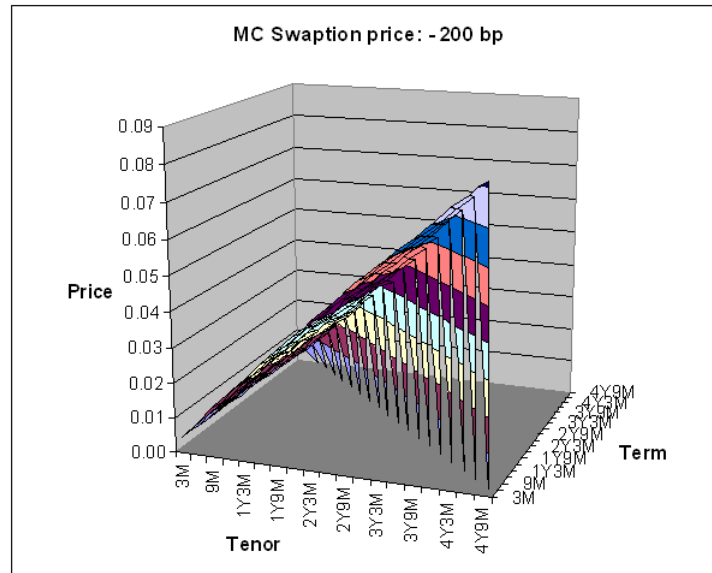


Figure A.18.: Swaption prices with respect to strike prices with skewness -200 bp, calculated by Monte-Carlo simulations

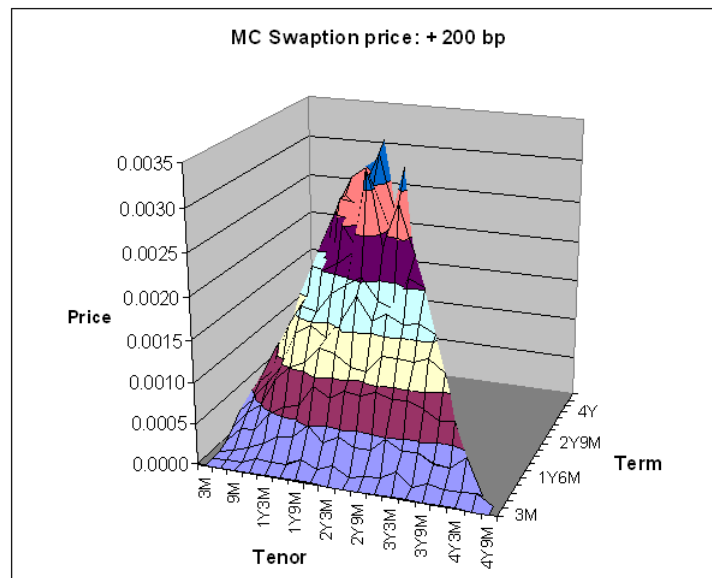


Figure A.19.: Swaption prices with respect to strike prices with skewness +200 bp, calculated by Monte-Carlo simulations

A.2. Caps

A.2.1. Caplet Volatilities (LMM and Market Volatility models)

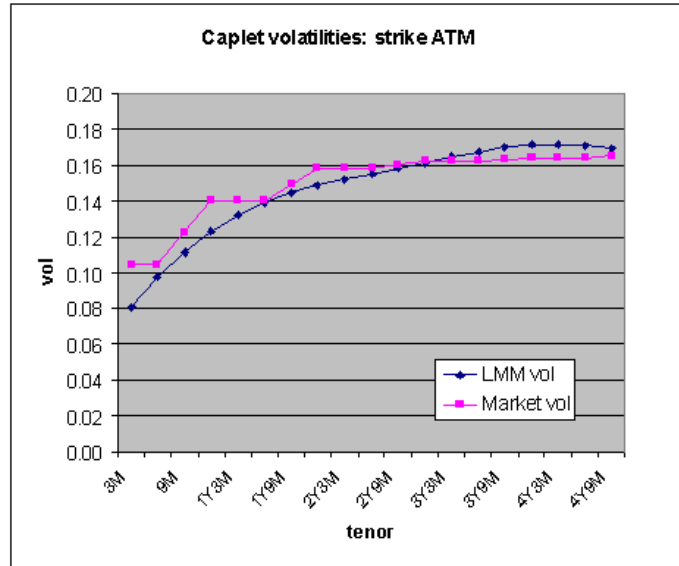


Figure A.20.: Caplet volatilities with respect to the at-the-money strike prices, calculated by the LMM and the Market Volatility model respectively

A. Test Results for the LMM and the Market Volatility Models

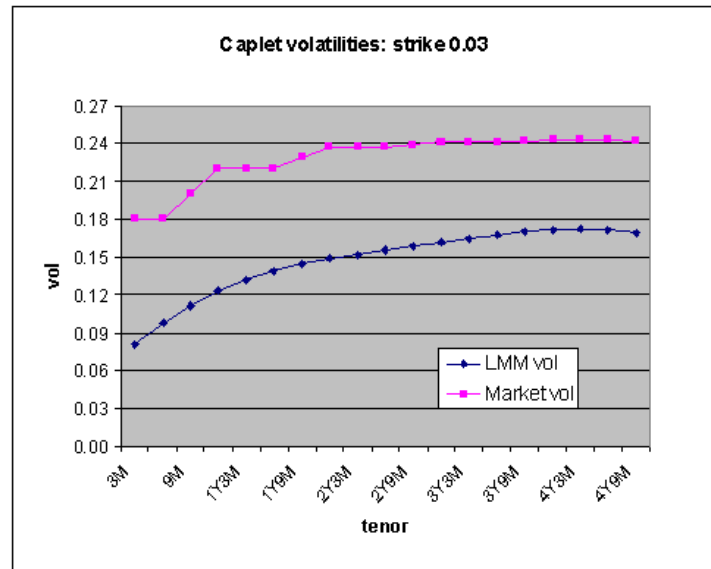


Figure A.21.: Caplet volatilities with respect to strike prices with skewness around -200 bp, calculated by the LMM and the Market Volatility model respectively

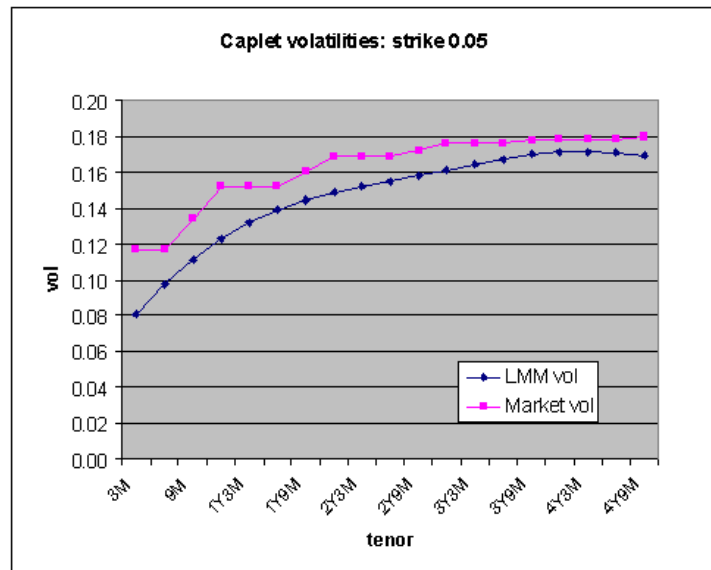


Figure A.22.: Caplet volatilities with respect to strike prices with skewness around -50 bp, calculated by the LMM and the Market Volatility model respectively

A. Test Results for the LMM and the Market Volatility Models

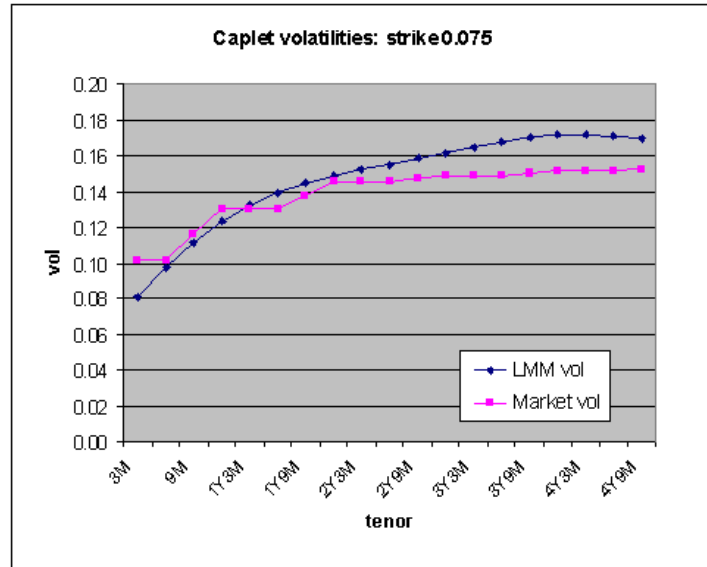


Figure A.23.: Caplet volatilities with respect to strike prices with skewness around +200 bp, calculated by the LMM and the Market Volatility model respectively

A.2.2. Cap prices (LMM and Market Volatility models)

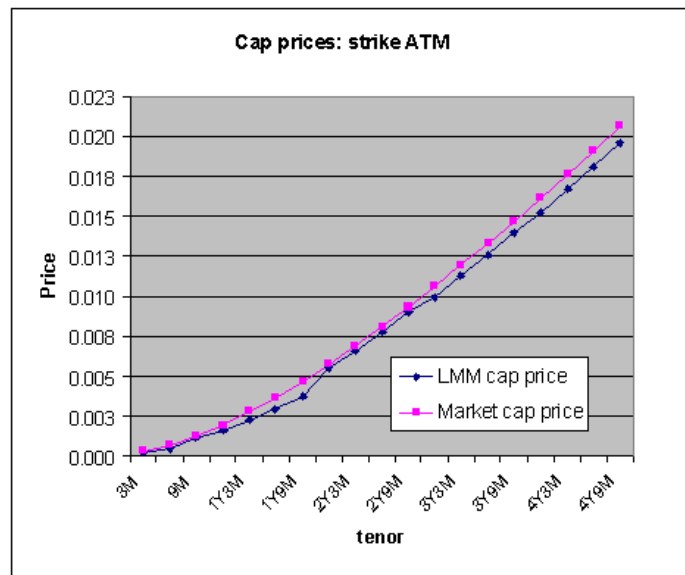


Figure A.24.: Cap prices with respect to the at-the-money strike prices, calculated by the LMM and the Market Volatility model respectively

A. Test Results for the LMM and the Market Volatility Models

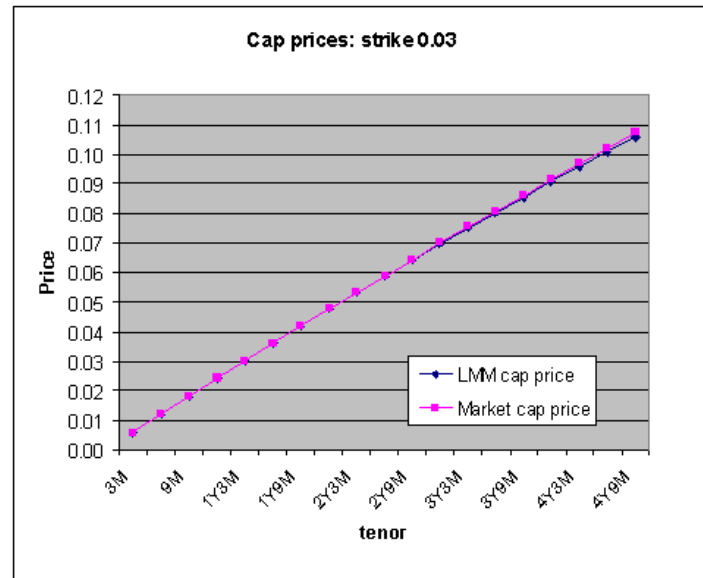


Figure A.25.: Cap prices with respect to strike prices with skewness around -200 bp, calculated by the LMM and the Market Volatility model respectively

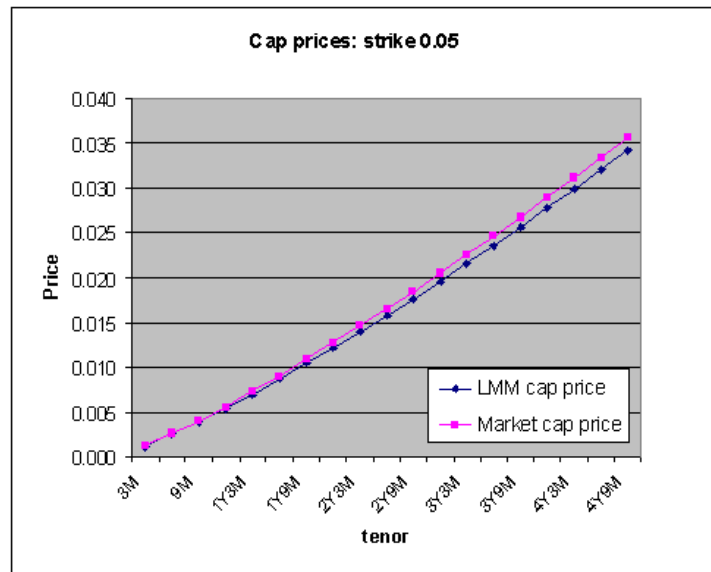


Figure A.26.: Cap prices with respect to strike prices with skewness around -50 bp, calculated by the LMM and the Market Volatility model respectively

A. Test Results for the LMM and the Market Volatility Models

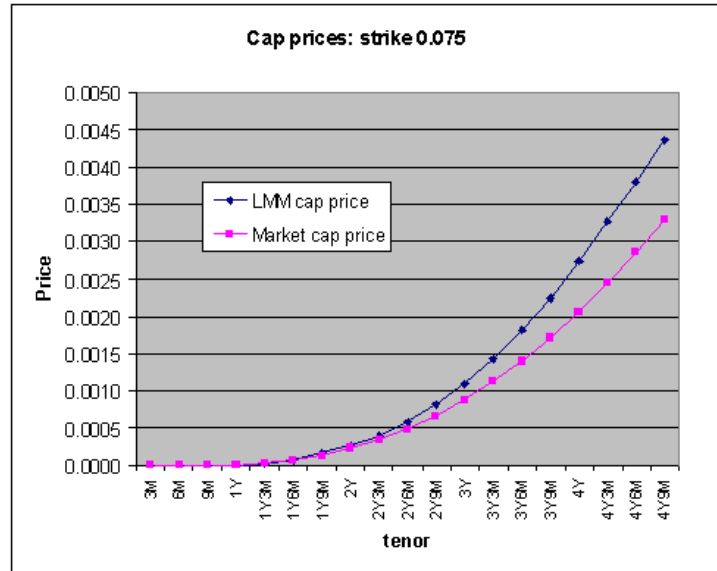


Figure A.27.: Cap prices with respect to strike prices with skewness around +200 bp, calculated by the LMM and the Market Volatility model respectively

B. Test Results for CMS Swaps and CMS 2Y-10Y Spread Options

B.1. CMS Swaps

CMS swap				
Term	Discount factor	CMS swap rate	Convexity adjustment	CMS swap price
2	0.9685	0.02009438	0.00017207	194.61235558
3	0.9515	0.02245088	0.00026292	408.22904480
4	0.9321	0.02435603	0.00036370	635.26253758
5	0.9116	0.02591245	0.00045620	872.11857427
6	0.8899	0.02733750	0.00055060	1115.39460372
7	0.8674	0.02853615	0.00062585	1362.92425839
8	0.8448	0.02947014	0.00071395	1611.87613087
9	0.8217	0.03047183	0.00079327	1862.93789140
10	0.7984	0.03148126	0.00086311	2114.28629436

Table B.1.: Prices for a CMS swap starting in 2 years on a 5 year swap, with the notional principal 10000 JPY

CMS swap				
Term	Discount factor	CMS swap rate	Convexity adjustment	CMS swap price
10	0.7984	0.03309163	0.00155164	264.20576282
11	0.7752	0.03355202	0.00170052	524.29958121
12	0.7525	0.03397073	0.00184445	779.94593015
13	0.7296	0.03441296	0.00197045	1031.69653950
14	0.7064	0.03473135	0.00206964	1277.04317371
15	0.6853	0.03495839	0.00217952	1516.60685124
16	0.6642	0.03515100	0.00229414	1750.07891089
17	0.6431	0.03523312	0.00238016	1977.29292148
18	0.6222	0.03520151	0.00243820	2196.32249128

Table B.2.: Prices for a CMS swap starting in 10 years on a 10 year swap, with the notional principal 10000 JPY

B. Test Results for CMS Swaps and CMS 2Y-10Y Spread Options

CMS swap				
Term	Discount factor	CMS swap rate	Convexity adjustment	CMS swap price
10	0.7984	0.03148126	0.00086311	251.34840296
11	0.7752	0.03192168	0.00093722	498.80388471
12	0.7525	0.03236469	0.00100929	742.36405589
13	0.7296	0.03294717	0.00106702	983.39156144
14	0.7064	0.03340300	0.00110012	1219.35460727
15	0.6853	0.03377708	0.00115879	1450.82297580
16	0.6642	0.03408484	0.00121730	1677.21365717
17	0.6431	0.03428332	0.00125270	1898.30253336
18	0.6222	0.03436794	0.00126420	2112.14548905

Table B.3.: Prices for a CMS swap starting in 10 years on a 5 year swap, with the notional principal 10000 JPY

B.2. CMS 2Y-10Y Spread Options

CMS 2Y-10Y spread option					
Term	Discount factor	Convexity adj. 10Y	Convexity adj. 2Y	Vol spread	Option price
0	0.9933	0.00000052	0.00000005	0.00172852	0.01066304
1	0.9826	0.00013142	0.00001961	0.00178373	0.01969232
2	0.9684	0.00027834	0.00005226	0.00209305	0.02721214
3	0.9514	0.00043598	0.00009023	0.00240271	0.03342698
4	0.9320	0.00059907	0.00012946	0.00261230	0.03902246
5	0.9114	0.00076921	0.00016916	0.00272174	0.04436446
6	0.8898	0.00092540	0.00020199	0.00280322	0.04931967
7	0.8673	0.00106305	0.00021815	0.00275855	0.05406113
8	0.8446	0.00123346	0.00025075	0.00280995	0.05874349

Table B.4.: Prices for a CMS swap starting in 2 years on a 5 year swap, with the notional principal 10000 JPY

C. Test Results for the FASO with the Market Volatility Models

C.1. The FASO Type-1 Option - Case 1.1

C.1.1. Prices

C.1.1.1. Prices for Mixed Correlations

χ_1, χ_2										
$\tilde{\sigma}$	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.1	0.0000	0.0000	0.0002	0.7221	26.0432	109.6253	192.3608	286.6608	380.5408	475.7300
0.2	0.0000	0.0054	0.6908	11.2098	52.3978	138.2389	206.3123	300.6750	386.5438	484.2693
0.3	0.0351	0.3794	4.8247	30.1639	79.0592	169.7450	225.8367	321.9509	400.7032	503.3148
0.4	0.4715	2.1427	13.2028	52.5897	105.9885	202.7901	248.4219	345.9564	418.7779	527.2035
0.5	1.9330	5.7649	25.0977	76.6367	133.1270	236.7016	273.0371	371.6813	438.9152	553.7378
0.6	4.6545	11.2520	39.4619	101.5527	160.4090	271.1149	298.9770	398.7551	460.6273	582.2392
0.7	8.6720	18.4385	55.4713	126.9612	187.7490	305.7901	325.7793	426.8206	483.6284	612.2935
0.8	13.8949	27.0726	72.5522	152.6226	215.0691	340.5803	353.1432	455.5478	507.6688	643.5567
0.9	20.1850	36.9088	90.3158	178.3631	242.2856	375.2765	380.8157	484.7058	532.4539	675.6131
1	27.4295	47.6529	108.4712	204.0408	269.3149	409.7120	408.6071	514.0528	557.7521	708.1595

Table C.1.: Premium for different levels of moneyness and volatility for FASO type-1 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2										
$\tilde{\sigma}$	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.1	380.4448	285.3355	190.4686	101.5816	25.7695	1.8987	0.1341	0.0006	0.0000	0.0005
0.2	380.4640	285.4663	195.9394	119.0444	51.3405	19.8202	5.6520	1.1764	0.2713	1.6999
0.3	380.8082	286.9773	206.0542	138.7790	76.7995	45.8500	19.6174	7.8541	3.4474	11.0905
0.4	382.3957	290.6379	218.0838	159.3772	102.1982	74.0351	39.0960	20.9003	11.3894	27.6692
0.5	385.5048	296.0429	231.1193	180.3615	127.5593	102.7870	61.7102	38.9088	23.8960	49.3721
0.6	389.9373	302.8713	244.8663	201.5207	152.8882	131.6561	86.1343	60.2531	40.1477	74.8457
0.7	395.3849	310.9673	259.1689	222.7471	178.1485	160.4444	111.5702	83.7665	59.2222	103.0055
0.8	401.7081	320.1376	273.8835	243.9715	203.2996	189.0256	137.5654	108.6866	80.3515	133.0391
0.9	408.8008	330.1856	288.8774	265.1357	228.2861	217.3549	163.8238	134.4978	102.9287	164.3241
1	416.5660	340.9270	304.0250	286.1681	253.0507	245.3481	190.1492	160.8312	126.4342	196.3827

Table C.2.: Premium for different levels of moneyness and volatility for FASO type-1 put option

C. Test Results for the FASO with the Market Volatility Models

C.1.1.2. Prices for Maximum Correlations

χ_1, χ_2										
$\tilde{\sigma}$	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.1	0.0000	0.0000	0.0000	0.1945	19.0166	95.8162	190.1906	285.2907	380.3965	475.5012
0.2	0.0000	0.0000	0.1182	5.6769	37.9579	105.4122	192.2055	285.5535	380.3202	475.3635
0.3	0.0000	0.0260	2.0010	16.9053	56.7719	121.1841	200.8937	289.4511	381.8231	475.8210
0.4	0.0002	0.4377	6.9139	30.5384	75.4059	139.0694	214.5770	298.7142	387.5273	479.2024
0.5	0.0194	1.9407	14.3511	45.1590	93.8109	157.7059	230.8601	311.9665	397.6436	486.6928
0.6	0.1543	4.8149	23.4631	60.1623	111.9285	176.5666	248.4829	327.6506	411.0191	497.8275
0.7	0.5841	8.9705	33.5913	75.2440	129.7060	195.3735	266.7131	344.7364	426.5015	511.6669
0.8	1.4781	14.1826	44.3162	90.2273	147.0972	213.9419	285.1353	362.5982	443.2849	527.2650
0.9	2.8936	20.1998	55.3517	104.9969	164.0609	232.1748	303.4945	380.7525	460.8157	543.9914
1	4.8268	26.7921	66.4943	119.4663	180.5617	249.9881	321.6057	398.9389	478.6332	561.3595

Table C.3.: Premium for different levels of moneyness and volatility for FASO type-1 call option

χ_1, χ_2										
$\tilde{\sigma}$	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.1	380.4450	285.3398	190.2347	95.3301	19.0387	0.7176	0.0050	0.0000	0.0000	0.0000
0.2	380.4613	285.3700	190.4013	100.8922	38.0464	10.3564	2.1167	0.3754	0.0536	0.0081
0.3	380.4886	285.4474	192.3680	112.2130	56.9738	26.2489	10.9717	4.4466	1.7700	0.7068
0.4	380.5279	285.9353	197.3977	125.9669	75.7663	44.2952	24.8942	13.9704	7.7782	4.4260
0.5	380.5993	287.5412	204.9589	140.7394	94.3689	63.1525	41.5183	27.5701	18.2943	12.3668
0.6	380.8077	290.5234	214.2246	155.9325	112.7369	82.2795	59.5329	43.7099	32.1720	24.0747
0.7	381.3268	294.8287	224.5360	171.2468	130.8356	101.4169	78.2300	61.3342	48.2951	38.5999
0.8	382.3154	300.2144	235.4740	186.5089	148.6174	120.4034	97.2400	79.8060	65.8163	55.0660
0.9	383.8476	306.4202	246.7614	201.6084	166.0468	139.1191	116.2872	98.7176	84.1977	72.7689
1	385.9030	313.2282	258.2007	216.4660	183.0875	157.4706	135.2103	117.7679	103.0309	91.2405

Table C.4.: Premium for different levels of moneyness and volatility for FASO type-1 put option

C. Test Results for the FASO with the Market Volatility Models

C.1.1.3. Prices for Minimum Correlations

χ_1, χ_2										
$\tilde{\sigma}$	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.1	0.0000	0.0000	0.0000	0.7781	26.1839	107.7112	191.0673	285.9099	380.4273	475.5287
0.2	0.0000	0.0006	0.2813	12.5771	52.6178	136.1518	201.2382	295.8378	383.3009	479.0286
0.3	0.0029	0.1643	3.5300	32.5781	79.3519	168.7430	219.7468	313.8753	393.0821	491.2042
0.4	0.1300	1.3872	11.4603	55.6800	106.4213	203.1854	242.9745	336.4353	408.1544	510.2407
0.5	0.8694	4.6068	23.1024	80.3105	133.8306	238.6943	268.8227	362.1022	427.0960	534.4904
0.6	2.7261	10.0550	37.2992	105.8529	161.5595	274.9070	296.2639	389.9249	448.9778	562.6970
0.7	6.0056	17.5085	53.2367	131.9934	189.5583	311.5663	324.7732	419.3018	473.0591	593.8332
0.8	10.7716	26.6110	70.3671	158.5345	217.7767	348.5321	354.0283	449.7773	498.7182	627.0415
0.9	16.9438	37.0163	88.3273	185.3267	246.1288	385.6449	383.7262	480.9976	525.5472	661.7320
1	24.3658	48.4205	106.8535	212.2442	274.5251	422.7278	413.6406	512.7080	553.1887	697.4009

Table C.5.: Premium for different levels of moneyness and volatility for FASO type-1 call option

χ_1, χ_2										
$\tilde{\sigma}$	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.1	380.4489	285.3412	190.3000	100.1559	25.9994	3.1601	0.0964	0.0001	0.0000	0.0000
0.2	380.4771	285.4198	193.5168	116.5760	51.8920	24.9546	5.8776	0.8382	0.1325	0.5438
0.3	380.6054	286.3274	201.2255	136.6563	77.7513	53.8264	21.5717	8.0104	2.7864	6.1291
0.4	381.3498	289.0809	211.6574	158.1817	103.6371	84.5104	42.9240	22.9773	11.1925	19.7725
0.5	383.2801	293.8915	223.9072	180.3528	129.5906	115.6914	67.1822	43.3151	25.1492	40.2626
0.6	386.5971	300.6040	237.4000	202.8497	155.6306	146.9717	93.0061	66.8542	43.2739	65.8257
0.7	391.2733	308.9115	251.8138	225.5198	181.7493	178.1391	119.7297	92.4470	64.3020	94.9516
0.8	397.2181	318.5542	266.9052	248.2578	207.9195	209.0973	146.9570	119.3072	87.3546	126.5431
0.9	404.3195	329.2464	282.4786	270.9974	234.1091	239.7580	174.4722	146.9497	111.7013	159.8309
1	412.3922	340.7437	298.3877	293.6645	260.2556	270.0715	202.1142	175.0634	136.8677	194.2507

Table C.6.: Premium for different levels of moneyness and volatility for FASO type-1 put option

C. Test Results for the FASO with the Market Volatility Models

C.1.2. Deltas of the FASO Type-1 Option - Case 1.1

C.1.2.1. Deltas for Mixed Correlations

χ_1, χ_2											
$\tilde{\sigma}$		0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.1	Δ_1	0.0000	0.0000	0.0000	0.0532	0.7000	0.9498	0.9511	0.9511	0.9511	0.9511
	Δ_2	0.0000	0.0000	0.0000	-0.0486	-0.6820	-0.9496	-0.9511	-0.9511	-0.9511	-0.9511
	Δ_3	0.0000	0.0000	0.0000	-0.0054	-0.3292	-0.6673	-0.8740	-0.9081	-0.9459	-0.9444
	Δ_4	0.0000	0.0000	0.0000	0.0065	0.3512	0.6839	0.8795	0.9119	0.9464	0.9450
	Δ_5	0.0000	0.0000	0.0000	-0.0355	-0.3341	-0.2783	-0.0756	-0.0420	-0.0051	-0.0064
	Δ_6	0.0000	0.0000	0.0000	0.0485	0.3885	0.3226	0.0978	0.0559	0.0076	0.0097
0.5	Δ_1	0.0603	0.1411	0.3255	0.5576	0.7215	0.8240	0.8782	0.9252	0.9399	0.9395
	Δ_2	-0.0399	-0.0938	-0.2587	-0.4620	-0.6336	-0.7561	-0.8185	-0.9049	-0.9293	-0.9283
	Δ_3	-0.0001	-0.0044	-0.0739	-0.2356	-0.3228	-0.4686	-0.5581	-0.6142	-0.6934	-0.6912
	Δ_4	0.0003	0.0109	0.1279	0.3455	0.4433	0.5853	0.6482	0.7027	0.7595	0.7560
	Δ_5	-0.0114	-0.0330	-0.0970	-0.1726	-0.2298	-0.2678	-0.2448	-0.2598	-0.2200	-0.2057
	Δ_6	0.0624	0.1222	0.2890	0.4120	0.4958	0.5279	0.5055	0.5170	0.4652	0.4680
1	Δ_1	0.3261	0.4336	0.5545	0.6527	0.7373	0.7643	0.8125	0.8549	0.8766	0.8825
	Δ_2	-0.1823	-0.2457	-0.3816	-0.4482	-0.5420	-0.5692	-0.6095	-0.7159	-0.7453	-0.7545
	Δ_3	-0.0240	-0.0695	-0.1805	-0.2706	-0.2954	-0.3934	-0.4332	-0.4632	-0.5327	-0.5059
	Δ_4	0.0903	0.2022	0.3860	0.5224	0.5533	0.6508	0.6564	0.7019	0.7339	0.7277
	Δ_5	-0.0290	-0.0456	-0.0762	-0.1045	-0.1241	-0.1480	-0.1462	-0.1644	-0.1634	-0.1496
	Δ_6	0.3345	0.3751	0.5182	0.5714	0.6164	0.6391	0.6342	0.6741	0.6646	0.6756

Table C.7.: Deltas for different levels of moneyness and volatility for FASO type-1 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2											
$\tilde{\sigma}$		0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.1	Δ_1	-0.9511	-0.9511	-0.9511	-0.9510	-0.6792	-0.0970	-0.0068	-0.0001	0.0000	0.0000
	Δ_2	0.9511	0.9511	0.9511	0.9511	0.6985	0.1062	0.0079	0.0001	0.0000	0.0000
	Δ_3	0.9511	0.9511	0.9317	0.7330	0.3309	0.0336	0.0000	0.0000	0.0000	0.0000
	Δ_4	-0.9511	-0.9511	-0.9295	-0.7183	-0.3100	-0.0289	0.0000	0.0000	0.0000	0.0000
	Δ_5	0.0000	0.0000	0.0203	0.2225	0.3878	0.0797	0.0104	0.0001	0.0000	0.0001
	Δ_6	0.0000	0.0000	-0.0143	-0.1857	-0.3335	-0.0608	-0.0072	-0.0001	0.0000	0.0000
0.5	Δ_1	-0.9497	-0.9392	-0.9217	-0.8295	-0.6030	-0.5693	-0.3038	-0.2165	-0.1265	-0.1473
	Δ_2	0.9521	0.9478	0.9393	0.8860	0.7168	0.6858	0.4360	0.3150	0.1995	0.2164
	Δ_3	0.8514	0.7835	0.5910	0.4845	0.3443	0.3097	0.1276	0.0970	0.0350	0.0260
	Δ_4	-0.8077	-0.7098	-0.5008	-0.3739	-0.2457	-0.2066	-0.0830	-0.0531	-0.0187	-0.0132
	Δ_5	0.1405	0.2041	0.3908	0.4575	0.4892	0.4352	0.4355	0.3660	0.3055	0.3717
	Δ_6	-0.0344	-0.0680	-0.1687	-0.2203	-0.2292	-0.1951	-0.1946	-0.1438	-0.1157	-0.1433
1	Δ_1	-0.8575	-0.7532	-0.7271	-0.6456	-0.4857	-0.5397	-0.3367	-0.3107	-0.2330	-0.2192
	Δ_2	0.9334	0.8890	0.8781	0.8474	0.7414	0.7909	0.6300	0.5813	0.4922	0.4735
	Δ_3	0.7040	0.6515	0.5087	0.4784	0.4070	0.4120	0.2564	0.2822	0.1904	0.1756
	Δ_4	-0.5212	-0.4316	-0.3013	-0.2416	-0.1910	-0.1761	-0.1149	-0.1063	-0.0734	-0.0616
	Δ_5	0.4142	0.4276	0.5414	0.5506	0.5952	0.5483	0.6096	0.5840	0.5860	0.6369
	Δ_6	-0.0477	-0.0654	-0.1031	-0.1185	-0.1253	-0.1171	-0.1285	-0.1169	-0.1195	-0.1283

Table C.8.: Deltas for different levels of moneyness and volatility for FASO type-1 put option

C. Test Results for the FASO with the Market Volatility Models

C.1.2.2. Deltas for Maximum Correlations

χ_1, χ_2											
$\tilde{\sigma}$		0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.1	Δ_1	0.0000	0.0000	0.0000	0.0157	0.5121	0.9263	0.9508	0.9511	0.9511	0.9511
	Δ_2	0.0000	0.0000	0.0000	-0.0157	-0.5117	-0.9262	-0.9508	-0.9511	-0.9511	-0.9511
	Δ_3	0.0000	0.0000	0.0000	-0.0120	-0.4662	-0.9136	-0.9507	-0.9511	-0.9511	-0.9511
	Δ_4	0.0000	0.0000	0.0000	0.0121	0.4669	0.9138	0.9507	0.9511	0.9511	0.9511
	Δ_5	0.0000	0.0000	0.0000	-0.0037	-0.0453	-0.0126	-0.0001	0.0000	0.0000	0.0000
	Δ_6	0.0000	0.0000	0.0000	0.0037	0.0458	0.0127	0.0001	0.0000	0.0000	0.0000
0.5	Δ_1	0.0019	0.0584	0.2220	0.4210	0.5848	0.7103	0.7886	0.8452	0.8809	0.9051
	Δ_2	-0.0018	-0.0579	-0.2209	-0.4194	-0.5831	-0.7092	-0.7875	-0.8446	-0.8806	-0.9048
	Δ_3	-0.0011	-0.0456	-0.1842	-0.3654	-0.5356	-0.6551	-0.7630	-0.8230	-0.8680	-0.8938
	Δ_4	0.0011	0.0465	0.1864	0.3694	0.5388	0.6592	0.7645	0.8244	0.8687	0.8944
	Δ_5	-0.0007	-0.0118	-0.0359	-0.0530	-0.0464	-0.0533	-0.0242	-0.0212	-0.0124	-0.0106
	Δ_6	0.0008	0.0126	0.0381	0.0557	0.0489	0.0556	0.0253	0.0221	0.0128	0.0112
1	Δ_1	0.1441	0.3310	0.4837	0.5927	0.6675	0.7256	0.7632	0.7965	0.8208	0.8414
	Δ_2	-0.1411	-0.3275	-0.4818	-0.5904	-0.6653	-0.7242	-0.7614	-0.7958	-0.8202	-0.8406
	Δ_3	-0.0941	-0.2803	-0.4233	-0.5266	-0.6153	-0.6640	-0.7324	-0.7655	-0.7993	-0.8191
	Δ_4	0.0988	0.2866	0.4304	0.5361	0.6221	0.6731	0.7359	0.7696	0.8017	0.8214
	Δ_5	-0.0432	-0.0437	-0.0550	-0.0606	-0.0466	-0.0577	-0.0279	-0.0287	-0.0198	-0.0198
	Δ_6	0.0501	0.0492	0.0624	0.0672	0.0522	0.0632	0.0307	0.0314	0.0214	0.0222

Table C.9.: Deltas for different levels of moneyness and volatility for FASO type-1 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2											
$\tilde{\sigma}$		0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.1	Δ_1	-0.9511	-0.9511	-0.9511	-0.9390	-0.4748	-0.0322	-0.0003	0.0000	0.0000	0.0000
	Δ_2	0.9511	0.9511	0.9511	0.9391	0.4752	0.0322	0.0003	0.0000	0.0000	0.0000
	Δ_3	0.9511	0.9511	0.9511	0.9328	0.4305	0.0249	0.0002	0.0000	0.0000	0.0000
	Δ_4	-0.9511	-0.9511	-0.9511	-0.9326	-0.4299	-0.0248	-0.0002	0.0000	0.0000	0.0000
	Δ_5	0.0000	0.0000	0.0000	0.0063	0.0449	0.0073	0.0001	0.0000	0.0000	0.0000
	Δ_6	0.0000	0.0000	0.0000	-0.0063	-0.0444	-0.0072	-0.0001	0.0000	0.0000	0.0000
0.5	Δ_1	-0.9487	-0.9019	-0.7588	-0.5705	-0.3990	-0.2762	-0.1800	-0.1199	-0.0782	-0.0524
	Δ_2	0.9490	0.9026	0.7600	0.5724	0.4011	0.2777	0.1816	0.1207	0.0788	0.0530
	Δ_3	0.9476	0.8886	0.7225	0.5237	0.3603	0.2381	0.1586	0.1041	0.0679	0.0438
	Δ_4	-0.9474	-0.8874	-0.7199	-0.5198	-0.3574	-0.2349	-0.1572	-0.1029	-0.0672	-0.0432
	Δ_5	0.0013	0.0145	0.0383	0.0494	0.0416	0.0399	0.0230	0.0169	0.0110	0.0094
	Δ_6	-0.0012	-0.0136	-0.0361	-0.0471	-0.0395	-0.0383	-0.0220	-0.0161	-0.0106	-0.0089
1	Δ_1	-0.8417	-0.6511	-0.5023	-0.3940	-0.3106	-0.2546	-0.2027	-0.1681	-0.1390	-0.1188
	Δ_2	0.8459	0.6557	0.5059	0.3980	0.3143	0.2576	0.2062	0.1703	0.1410	0.1211
	Δ_3	0.7953	0.6124	0.4606	0.3541	0.2793	0.2219	0.1814	0.1493	0.1249	0.1035
	Δ_4	-0.7896	-0.6063	-0.4544	-0.3469	-0.2740	-0.2160	-0.1785	-0.1462	-0.1227	-0.1016
	Δ_5	0.0545	0.0460	0.0471	0.0452	0.0366	0.0360	0.0248	0.0214	0.0163	0.0181
	Δ_6	-0.0471	-0.0409	-0.0419	-0.0409	-0.0328	-0.0331	-0.0227	-0.0197	-0.0151	-0.0163

Table C.10.: Deltas for different levels of moneyness and volatility for FASO type-1 put option

C. Test Results for the FASO with the Market Volatility Models

C.1.2.3. Deltas for Minimum Correlations

χ_1, χ_2											
$\tilde{\sigma}$		0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.1	Δ_1	0.0000	0.0000	0.0000	0.0516	0.6480	0.9398	0.9510	0.9511	0.9511	0.9511
	Δ_2	0.0000	0.0000	0.0000	-0.0434	-0.6145	-0.9369	-0.9510	-0.9511	-0.9511	-0.9511
	Δ_3	0.0000	0.0000	0.0000	-0.0181	-0.3353	-0.6539	-0.9026	-0.9226	-0.9498	-0.9499
	Δ_4	0.0000	0.0000	0.0000	0.0234	0.3736	0.6897	0.9114	0.9286	0.9501	0.9502
	Δ_5	0.0000	0.0000	0.0000	-0.0258	-0.2718	-0.2750	-0.0456	-0.0266	-0.0013	-0.0011
	Δ_6	0.0000	0.0000	0.0000	0.0314	0.2992	0.2924	0.0522	0.0305	0.0015	0.0015
0.5	Δ_1	0.0307	0.1192	0.2978	0.5422	0.6893	0.7837	0.8352	0.8952	0.9178	0.9175
	Δ_2	-0.0114	-0.0488	-0.1813	-0.3837	-0.5242	-0.6549	-0.7000	-0.8228	-0.8629	-0.8598
	Δ_3	-0.0005	-0.0105	-0.0726	-0.1913	-0.2651	-0.3551	-0.4453	-0.4979	-0.5932	-0.5924
	Δ_4	0.0027	0.0350	0.1709	0.3681	0.4587	0.5647	0.6271	0.6888	0.7544	0.7512
	Δ_5	-0.0099	-0.0363	-0.1014	-0.1925	-0.2326	-0.3028	-0.2514	-0.2880	-0.2439	-0.2064
	Δ_6	0.0323	0.0709	0.2103	0.3322	0.3834	0.4415	0.3870	0.4187	0.3593	0.3587
1	Δ_1	0.2846	0.4280	0.5379	0.6512	0.7269	0.7582	0.8019	0.8358	0.8577	0.8624
	Δ_2	-0.0800	-0.1374	-0.2490	-0.3309	-0.3923	-0.4486	-0.4581	-0.5605	-0.5857	-0.5837
	Δ_3	-0.0160	-0.0460	-0.1049	-0.1580	-0.1819	-0.2264	-0.2670	-0.2866	-0.3399	-0.3265
	Δ_4	0.1321	0.2560	0.4143	0.5320	0.5651	0.6326	0.6386	0.6855	0.7134	0.7061
	Δ_5	-0.0519	-0.0790	-0.1231	-0.1695	-0.1738	-0.2426	-0.2024	-0.2346	-0.2295	-0.1823
	Δ_6	0.2685	0.2846	0.4322	0.4911	0.5064	0.5583	0.5159	0.5556	0.5298	0.5413

Table C.11.: Deltas for different levels of moneyness and volatility for FASO type-1 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2											
$\tilde{\sigma}$		0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.1	Δ_1	-0.9511	-0.9511	-0.9511	-0.9478	-0.6190	-0.1195	-0.0054	0.0000	0.0000	0.0000
	Δ_2	0.9511	0.9511	0.9511	0.9487	0.6532	0.1375	0.0075	0.0000	0.0000	0.0000
	Δ_3	0.9511	0.9511	0.9455	0.7737	0.3725	0.0724	0.0003	0.0000	0.0000	0.0000
	Δ_4	-0.9511	-0.9511	-0.9436	-0.7448	-0.3343	-0.0585	-0.0002	0.0000	0.0000	0.0000
	Δ_5	0.0000	0.0000	0.0062	0.1831	0.2898	0.0619	0.0068	0.0000	0.0000	0.0000
	Δ_6	0.0000	0.0000	-0.0053	-0.1694	-0.2632	-0.0528	-0.0058	0.0000	0.0000	0.0000
0.5	Δ_1	-0.9481	-0.9201	-0.8723	-0.7349	-0.5456	-0.4918	-0.2898	-0.2191	-0.1310	-0.1460
	Δ_2	0.9520	0.9438	0.9224	0.8493	0.7241	0.6683	0.5049	0.3952	0.2848	0.3041
	Δ_3	0.9158	0.8610	0.7129	0.5871	0.4579	0.4145	0.2257	0.2041	0.1065	0.0894
	Δ_4	-0.8617	-0.7433	-0.5336	-0.3772	-0.2676	-0.2188	-0.1155	-0.0900	-0.0454	-0.0353
	Δ_5	0.0828	0.1297	0.2738	0.3153	0.3348	0.2652	0.2764	0.2042	0.1687	0.2454
	Δ_6	-0.0316	-0.0754	-0.1705	-0.2052	-0.2018	-0.1601	-0.1688	-0.1115	-0.0970	-0.1272
1	Δ_1	-0.8231	-0.7026	-0.6595	-0.5624	-0.4382	-0.4544	-0.2978	-0.2874	-0.2183	-0.2188
	Δ_2	0.9418	0.9141	0.8955	0.8628	0.8036	0.8075	0.7223	0.6866	0.6284	0.6287
	Δ_3	0.8443	0.7918	0.7023	0.6550	0.5841	0.5868	0.4329	0.4604	0.3585	0.3456
	Δ_4	-0.5708	-0.4470	-0.3207	-0.2445	-0.2001	-0.1773	-0.1317	-0.1211	-0.0925	-0.0819
	Δ_5	0.3414	0.3185	0.3908	0.3750	0.4305	0.3241	0.3954	0.3589	0.3444	0.4628
	Δ_6	-0.0784	-0.1019	-0.1279	-0.1312	-0.1395	-0.1117	-0.1328	-0.1115	-0.1156	-0.1375

Table C.12.: Deltas for different levels of moneyness and volatility for FASO type-1 put option

C.2. The FASO Type-2 Option - Case 2.3

C.2.1. Premia

C.2.1.1. Premia for Mixed Correlations

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	0.5929	2.0264	6.4309	19.5714	37.9815
0.2	4.6471	7.5282	12.6612	22.4378	38.1275
0.3	10.2342	13.6502	18.6590	26.4985	39.0247
0.4	16.2865	19.7697	24.3962	30.9075	40.8142
0.5	22.3658	25.6895	29.8486	35.3299	43.2315
0.6	28.2683	31.3247	34.9989	39.6268	45.9972
0.7	33.8903	36.6320	39.8353	43.7375	48.9061
0.8	39.1735	41.5890	44.3524	47.6317	51.8354
0.9	44.0906	46.1873	48.5474	51.2889	54.7145
1	48.6311	50.4277	52.4240	54.7041	57.4916

Table C.13.: Premium for different levels of moneyness and volatility for FASO type-2 call option

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	39.3314	21.2411	6.5963	1.3958	0.1254
0.2	44.3813	27.0097	13.3196	5.7875	2.1430
0.3	50.9888	33.9965	20.1291	10.9233	5.5857
0.4	58.5156	41.4337	26.9828	16.3445	9.5277
0.5	66.5185	49.0460	33.8391	21.8777	13.6628
0.6	74.7430	56.7053	40.6538	27.4435	17.8614
0.7	83.0321	64.3279	47.3883	32.9823	22.0644
0.8	91.2779	71.8510	54.0073	38.4536	26.2347
0.9	99.4022	79.2245	60.4798	43.8215	30.3433
1	107.3445	86.4100	66.7809	49.0631	34.3638

Table C.14.: Premium for different levels of moneyness and volatility for FASO type-2 put option

C. Test Results for the FASO with the Market Volatility Models

C.2.1.2. Premia for Maximum Correlations

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	0.0000	0.0000	1.9170	18.9904	37.9820
0.2	0.0000	0.0158	3.8257	19.1993	37.9751
0.3	0.0000	0.2176	5.7200	20.0901	38.1305
0.4	0.0007	0.7383	7.5947	21.4867	38.7246
0.5	0.0097	1.5113	9.4446	23.1362	39.7659
0.6	0.0448	2.4504	11.2651	24.9183	41.1416
0.7	0.1340	3.4898	13.0519	26.7623	42.7206
0.8	0.2897	4.5880	14.8025	28.6282	44.4290
0.9	0.5177	5.7167	16.5133	30.4867	46.2158
1	0.8129	6.8567	18.1818	32.3240	48.0420

Table C.15.: Premium for different levels of moneyness and volatility for FASO type-2 call option

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	37.9876	18.9952	1.9197	0.0008	0.0000
0.2	37.9903	19.0168	3.8364	0.2207	0.0071
0.3	37.9949	19.2304	5.7446	1.1303	0.1859
0.4	38.0022	19.7648	7.6393	2.5536	0.8132
0.5	38.0206	20.5549	9.5151	4.2358	1.8970
0.6	38.0738	21.5149	11.3669	6.0606	3.3250
0.7	38.1839	22.5786	13.1910	7.9538	4.9645
0.8	38.3626	23.7033	14.9835	9.8748	6.7433
0.9	38.6107	24.8607	16.7390	11.7943	8.6080
1	38.9247	26.0319	18.4545	13.6975	10.5174

Table C.16.: Premium for different levels of moneyness and volatility for FASO type-2 put option

C. Test Results for the FASO with the Market Volatility Models

C.2.1.3. Premia for Minimum Correlations

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	0.2421	1.3227	6.0157	19.3460	37.9814
0.2	3.1827	6.1714	11.8244	21.9843	38.0612
0.3	8.0943	11.8425	17.4059	25.7879	38.8323
0.4	13.6602	17.5530	22.7436	29.8865	40.4871
0.5	19.3369	23.0968	27.8252	33.9815	42.7117
0.6	24.8956	28.3984	32.6430	37.9664	45.2485
0.7	30.2289	33.4230	37.1932	41.7930	47.9180
0.8	35.2859	38.1538	41.4747	45.4439	50.6178
0.9	40.0413	42.5863	45.4911	48.9061	53.2842
1	44.4875	46.7243	49.2479	52.1774	55.8782

Table C.17.: Premium for different levels of moneyness and volatility for FASO type-2 call option

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	38.6952	20.6255	6.1994	0.8202	0.0351
0.2	42.8323	26.1048	12.5559	4.4221	1.2047
0.3	49.1005	32.8320	19.0398	9.1187	3.7358
0.4	56.4071	40.0169	25.6187	14.2452	6.9535
0.5	64.2261	47.4220	32.2598	19.5782	10.5715
0.6	72.3177	54.9318	38.9294	25.0241	14.4281
0.7	80.5428	62.4752	45.5927	30.5214	18.4294
0.8	88.8019	69.9976	52.2177	36.0232	22.5184
0.9	97.0247	77.4535	58.7733	41.4902	26.6487
1	105.1548	84.8037	65.2337	46.8976	30.7818

Table C.18.: Premium for different levels of moneyness and volatility for FASO type-2 put option

C. Test Results for the FASO with the Market Volatility Models

C.2.2. Deltas

C.2.2.1. Deltas for Mixed Correlations

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	0.0890	0.2424	0.6134	0.9450	0.9496
	Δ_2	0.0000	-0.0025	-0.1804	-0.7774	-0.9493
	Δ_3	-0.0643	-0.1893	-0.3514	-0.1271	-0.0002
0.3	Δ_1	0.4127	0.5197	0.6568	0.8174	0.9257
	Δ_2	-0.0173	-0.0497	-0.1349	-0.3439	-0.7213
	Δ_3	-0.2087	-0.2571	-0.2880	-0.2486	-0.0853
0.5	Δ_1	0.5633	0.6262	0.6978	0.7794	0.8643
	Δ_2	-0.0327	-0.0553	-0.0978	-0.1879	-0.4157
	Δ_3	-0.1943	-0.2157	-0.2301	-0.2200	-0.1393
0.7	Δ_1	0.6543	0.6921	0.7338	0.7813	0.8343
	Δ_2	-0.0325	-0.0455	-0.0678	-0.1123	-0.2326
	Δ_3	-0.1583	-0.1702	-0.1788	-0.1776	-0.1405
1	Δ_1	0.7409	0.7600	0.7798	0.8023	0.8278
	Δ_2	-0.0222	-0.0278	-0.0362	-0.0526	-0.0961
	Δ_3	-0.1054	-0.1113	-0.1162	-0.1184	-0.1093

Table C.19.: Deltas for different levels of moneyness and volatility for FASO type-2 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	-0.9496	-0.9320	-0.5678	-0.1637	-0.0221
	Δ_2	0.8148	0.7019	0.2338	0.0002	0.0000
	Δ_3	0.1761	0.2894	0.4184	0.2094	0.0330
0.3	Δ_1	-0.8498	-0.7222	-0.5193	-0.3116	-0.1833
	Δ_2	0.6342	0.5051	0.2927	0.0705	0.0008
	Δ_3	0.4392	0.4703	0.4864	0.4562	0.3469
0.5	Δ_1	-0.7115	-0.6054	-0.4706	-0.3258	-0.2167
	Δ_2	0.5911	0.4960	0.3554	0.1727	0.0258
	Δ_3	0.5403	0.5494	0.5541	0.5481	0.5104
0.7	Δ_1	-0.6072	-0.5223	-0.4214	-0.3122	-0.2137
	Δ_2	0.6004	0.5266	0.4188	0.2687	0.0863
	Δ_3	0.6142	0.6169	0.6189	0.6185	0.6086
1	Δ_1	-0.4857	-0.4248	-0.3529	-0.2745	-0.1928
	Δ_2	0.6464	0.5940	0.5152	0.3993	0.2063
	Δ_3	0.7080	0.7074	0.7073	0.7076	0.7076

Table C.20.: Deltas for different levels of moneyness and volatility for FASO type-2 put option

C. Test Results for the FASO with the Market Volatility Models

C.2.2.2. Deltas for Maximum Correlations

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	0.0000	0.0000	0.5074	0.9493	0.9496
	Δ_2	0.0000	0.0000	-0.4725	-0.9492	-0.9496
	Δ_3	0.0000	0.0000	-0.0347	0.0000	0.0000
0.3	Δ_1	0.0000	0.0673	0.5461	0.8541	0.9330
	Δ_2	0.0000	-0.0486	-0.5103	-0.8437	-0.9312
	Δ_3	0.0000	-0.0183	-0.0352	-0.0104	-0.0018
0.5	Δ_1	0.0060	0.2303	0.5823	0.7836	0.8754
	Δ_2	-0.0013	-0.1800	-0.5470	-0.7680	-0.8701
	Δ_3	-0.0045	-0.0486	-0.0344	-0.0154	-0.0054
0.7	Δ_1	0.0484	0.3576	0.6164	0.7587	0.8379
	Δ_2	-0.0205	-0.2921	-0.5819	-0.7434	-0.8304
	Δ_3	-0.0262	-0.0625	-0.0334	-0.0152	-0.0077
1	Δ_1	0.1750	0.4923	0.6644	0.7601	0.8157
	Δ_2	-0.0860	-0.4166	-0.6307	-0.7443	-0.8082
	Δ_3	-0.0819	-0.0708	-0.0322	-0.0156	-0.0078

Table C.21.: Deltas for different levels of moneyness and volatility for FASO type-2 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	-0.9496	-0.9496	-0.4692	-0.0004	0.0000
	Δ_2	0.9496	0.9496	0.4342	0.0003	0.0000
	Δ_3	0.0000	0.0000	0.0352	0.0000	0.0000
0.3	Δ_1	-0.9492	-0.8988	-0.4318	-0.1031	-0.0176
	Δ_2	0.9492	0.8765	0.3970	0.0934	0.0160
	Δ_3	0.0000	0.0228	0.0354	0.0098	0.0016
0.5	Δ_1	-0.9465	-0.7589	-0.3944	-0.1764	-0.0772
	Δ_2	0.9394	0.7067	0.3605	0.1621	0.0719
	Δ_3	0.0075	0.0542	0.0349	0.0146	0.0053
0.7	Δ_1	-0.9257	-0.6421	-0.3589	-0.2005	-0.1153
	Δ_2	0.8859	0.5764	0.3251	0.1850	0.1081
	Δ_3	0.0422	0.0691	0.0351	0.0160	0.0072
1	Δ_1	-0.8476	-0.5125	-0.3052	-0.1969	-0.1351
	Δ_2	0.7500	0.4428	0.2752	0.1821	0.1272
	Δ_3	0.1059	0.0750	0.0318	0.0154	0.0080

Table C.22.: Deltas for different levels of moneyness and volatility for FASO type-2 put option

C. Test Results for the FASO with the Market Volatility Models

C.2.2.3. Deltas for Minimum Correlations

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	0.0486	0.2042	0.6112	0.9360	0.9496
	Δ_2	-0.0006	-0.0188	-0.2478	-0.8192	-0.9496
	Δ_3	-0.0353	-0.1452	-0.2860	-0.0881	0.0000
0.3	Δ_1	0.3905	0.5128	0.6542	0.8022	0.9186
	Δ_2	-0.0520	-0.0994	-0.1966	-0.3948	-0.7652
	Δ_3	-0.1684	-0.2123	-0.2364	-0.2010	-0.0620
0.5	Δ_1	0.5499	0.6174	0.6934	0.7740	0.8607
	Δ_2	-0.0709	-0.1000	-0.1520	-0.2476	-0.4841
	Δ_3	-0.1629	-0.1819	-0.1919	-0.1810	-0.1096
0.7	Δ_1	0.6397	0.6821	0.7282	0.7781	0.8339
	Δ_2	-0.0659	-0.0843	-0.1146	-0.1689	-0.3091
	Δ_3	-0.1368	-0.1475	-0.1535	-0.1497	-0.1139
1	Δ_1	0.7250	0.7475	0.7720	0.7969	0.8282
	Δ_2	-0.0478	-0.0571	-0.0710	-0.0963	-0.1668
	Δ_3	-0.0968	-0.1023	-0.1068	-0.1057	-0.0932

Table C.23.: Deltas for different levels of moneyness and volatility for FASO type-2 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	-0.9478	-0.9048	-0.5648	-0.1274	-0.0083
	Δ_2	0.8535	0.7236	0.3060	0.0049	0.0000
	Δ_3	0.1208	0.2316	0.3390	0.1546	0.0120
0.3	Δ_1	-0.7904	-0.6788	-0.5162	-0.3226	-0.1628
	Δ_2	0.6362	0.5320	0.3677	0.1541	0.0111
	Δ_3	0.3639	0.3850	0.3964	0.3752	0.2773
0.5	Δ_1	-0.6517	-0.5668	-0.4663	-0.3447	-0.2233
	Δ_2	0.6038	0.5323	0.4330	0.2809	0.0872
	Δ_3	0.4471	0.4526	0.4546	0.4518	0.4257
0.7	Δ_1	-0.5508	-0.4867	-0.4154	-0.3304	-0.2351
	Δ_2	0.6196	0.5685	0.4972	0.3835	0.1929
	Δ_3	0.5136	0.5136	0.5141	0.5146	0.5109
1	Δ_1	-0.4318	-0.3885	-0.3424	-0.2854	-0.2208
	Δ_2	0.6716	0.6393	0.5928	0.5128	0.3527
	Δ_3	0.6041	0.6012	0.6003	0.6002	0.6059

Table C.24.: Deltas for different levels of moneyness and volatility for FASO type-2 put option

C.3. The FASO Type-3 Option - Case 3.6

C.3.1. Premia

C.3.1.1. Premia for Mixed correlations

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	0.1997	1.1297	5.1659	19.5372	38.5014
0.2	2.6729	5.1260	10.2171	21.3718	38.5514
0.3	6.6981	9.8946	15.1366	24.2941	38.9537
0.4	11.3455	14.8430	19.9091	27.6891	39.9253
0.5	16.2247	19.7691	24.5214	31.2453	41.4386
0.6	21.1389	24.5801	28.9616	34.8214	43.3485
0.7	25.9751	29.2293	33.2203	38.3419	45.5247
0.8	30.6710	33.6895	37.2915	41.7665	47.8512
0.9	35.1836	37.9448	41.1703	45.0762	50.2402
1	39.4883	41.9861	44.8533	48.2574	52.6419

Table C.25.: Premium for different levels of moneyness and volatility for FASO type-3 call option

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	39.1741	20.6470	5.2617	0.7136	0.0240
0.2	42.6599	24.7448	10.5998	3.8138	1.0257
0.3	47.3931	29.9485	15.9935	7.6673	3.3220
0.4	52.9037	35.6363	21.4211	11.8242	6.1909
0.5	58.9060	41.5451	26.8608	16.1306	9.3125
0.6	65.1950	47.5557	32.2902	20.5094	12.5537
0.7	71.6326	53.5978	37.6883	24.9090	15.8505
0.8	78.1262	59.6194	43.0329	29.2960	19.1608
0.9	84.6103	65.5833	48.3039	33.6445	22.4608
1	91.0362	71.4593	53.4822	37.9331	25.7310

Table C.26.: Premium for different levels of moneyness and volatility for FASO type-3 put option

C. Test Results for the FASO with the Market Volatility Models

C.3.1.2. Premia for Maximum Correlations

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	0.0000	0.0000	1.7151	19.2512	38.5012
0.2	0.0001	0.0341	3.4243	19.3400	38.5052
0.3	0.0079	0.3106	5.1264	19.8595	38.5519
0.4	0.0749	0.9079	6.8200	20.8158	38.8073
0.5	0.2789	1.7508	8.5029	22.0539	39.3660
0.6	0.6408	2.7627	10.1736	23.4584	40.2128
0.7	1.1521	3.8891	11.8318	24.9660	41.2901
0.8	1.7867	5.0941	13.4764	26.5322	42.5378
0.9	2.5274	6.3513	15.1054	28.1275	43.9088
1	3.3580	7.6443	16.7177	29.7377	45.3682

Table C.27.: Premium for different levels of moneyness and volatility for FASO type-3 call option

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	38.4991	19.2491	1.7200	0.0000	0.0000
0.2	38.4965	19.3122	3.4438	0.0740	0.0005
0.3	38.5254	19.6476	5.1705	0.5739	0.0370
0.4	38.6735	20.3011	6.8995	1.5272	0.2710
0.5	38.9758	21.2043	8.6305	2.7764	0.8137
0.6	39.4178	22.2871	10.3620	4.2031	1.6498
0.7	39.9807	23.4949	12.0924	5.7390	2.7202
0.8	40.6527	24.7927	13.8216	7.3441	3.9686
0.9	41.4251	26.1548	15.5489	8.9938	5.3394
1	42.2896	27.5648	17.2737	10.6731	6.7929

Table C.28.: Premium for different levels of moneyness and volatility for FASO type-3 put option

C. Test Results for the FASO with the Market Volatility Models

C.3.1.3. Premia for Minimum Correlations

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	0.0573	0.6496	4.8509	19.3945	38.5013
0.2	1.6272	4.0214	9.5812	20.9956	38.5242
0.3	4.9842	8.4133	14.1796	23.7564	38.8176
0.4	9.1832	13.0392	18.6373	26.9397	39.6909
0.5	13.7109	17.6596	22.9462	30.2578	41.0998
0.6	18.3203	22.1837	27.1005	33.5858	42.8785
0.7	22.8891	26.5691	31.0952	36.8622	44.8925
0.8	27.3487	30.7936	34.9289	40.0599	47.0341
0.9	31.6585	34.8452	38.6012	43.1645	49.2354
1	35.7970	38.7177	42.1109	46.1663	51.4579

Table C.29.: Premium for different levels of moneyness and volatility for FASO type-3 call option

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	38.8095	20.1666	4.9592	0.3650	0.0032
0.2	41.3968	23.9529	10.0127	2.7295	0.4932
0.3	45.7170	28.9768	15.1458	6.1420	2.0353
0.4	51.0266	34.4785	20.3429	10.0067	4.2172
0.5	56.8837	40.2149	25.5876	14.1002	6.7821
0.6	63.0586	46.0805	30.8626	18.3232	9.5971
0.7	69.4197	52.0154	36.1509	22.6180	12.5807
0.8	75.8849	57.9792	41.4350	26.9477	15.6797
0.9	82.3922	63.9405	46.6985	31.2888	18.8483
1	88.9002	69.8719	51.9243	35.6199	22.0560

Table C.30.: Premium for different levels of moneyness and volatility for FASO type-3 put option

C. Test Results for the FASO with the Market Volatility Models

C.3.2. Deltas

C.3.2.1. Deltas for Mixed Correlations

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	0.0418	0.1828	0.6187	0.9620	0.9625
	Δ_2	0.0000	-0.0004	-0.1921	-0.8530	-0.9625
	Δ_3	-0.0309	-0.1484	-0.3608	-0.0851	0.0000
0.3	Δ_1	0.3429	0.4725	0.6556	0.8645	0.9557
	Δ_2	-0.0090	-0.0413	-0.1543	-0.4420	-0.8354
	Δ_3	-0.1962	-0.2647	-0.3100	-0.2478	-0.0550
0.5	Δ_1	0.5043	0.5885	0.6902	0.8074	0.9103
	Δ_2	-0.0285	-0.0586	-0.1211	-0.2607	-0.5661
	Δ_3	-0.2098	-0.2410	-0.2620	-0.2419	-0.1240
0.7	Δ_1	0.6035	0.6592	0.7229	0.7956	0.8699
	Δ_2	-0.0357	-0.0564	-0.0933	-0.1691	-0.3585
	Δ_3	-0.1871	-0.2046	-0.2174	-0.2115	-0.1467
1	Δ_1	0.7008	0.7312	0.7665	0.8045	0.8481
	Δ_2	-0.0317	-0.0422	-0.0597	-0.0930	-0.1827
	Δ_3	-0.1425	-0.1513	-0.1593	-0.1596	-0.1349

Table C.31.: Deltas for different levels of moneyness and volatility for FASO type-3 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	-0.9625	-0.9588	-0.5815	-0.1181	-0.0061
	Δ_2	0.8699	0.7600	0.2345	0.0000	0.0000
	Δ_3	0.1179	0.2404	0.4145	0.1474	0.0089
0.3	Δ_1	-0.9112	-0.7905	-0.5433	-0.2944	-0.1539
	Δ_2	0.6791	0.5441	0.2809	0.0413	0.0000
	Δ_3	0.3964	0.4415	0.4688	0.4148	0.2702
0.5	Δ_1	-0.7916	-0.6727	-0.5042	-0.3280	-0.2063
	Δ_2	0.6151	0.5040	0.3298	0.1261	0.0080
	Δ_3	0.4974	0.5145	0.5225	0.5057	0.4372
0.7	Δ_1	-0.6942	-0.5910	-0.4660	-0.3271	-0.2182
	Δ_2	0.6032	0.5115	0.3807	0.2053	0.0426
	Δ_3	0.5644	0.5715	0.5750	0.5688	0.5403
1	Δ_1	-0.5790	-0.5010	-0.4083	-0.3050	-0.2096
	Δ_2	0.6223	0.5553	0.4557	0.3141	0.1269
	Δ_3	0.6464	0.6475	0.6486	0.6467	0.6395

Table C.32.: Deltas for different levels of moneyness and volatility for FASO type-3 put option

C. Test Results for the FASO with the Market Volatility Models

C.3.2.2. Deltas for Maximum Correlations

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	0.0000	0.0001	0.5816	0.9625	0.9625
	Δ_2	0.0000	0.0000	-0.3683	-0.9624	-0.9625
	Δ_3	0.0000	-0.0001	-0.2081	-0.0001	0.0000
0.3	Δ_1	0.0030	0.0902	0.6055	0.9185	0.9594
	Δ_2	0.0000	-0.0086	-0.3877	-0.8530	-0.9535
	Δ_3	-0.0025	-0.0736	-0.2022	-0.0607	-0.0054
0.5	Δ_1	0.0550	0.2512	0.6302	0.8494	0.9280
	Δ_2	-0.0002	-0.0620	-0.4088	-0.7414	-0.8954
	Δ_3	-0.0447	-0.1643	-0.1958	-0.0953	-0.0288
0.7	Δ_1	0.1386	0.3708	0.6543	0.8141	0.8915
	Δ_2	-0.0018	-0.1224	-0.4258	-0.6898	-0.8395
	Δ_3	-0.1080	-0.2071	-0.1928	-0.1045	-0.0440
1	Δ_1	0.2569	0.4919	0.6903	0.8000	0.8595
	Δ_2	-0.0140	-0.1917	-0.4480	-0.6651	-0.7952
	Δ_3	-0.1827	-0.2352	-0.1905	-0.1050	-0.0506

Table C.33.: Deltas for different levels of moneyness and volatility for FASO type-3 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	-0.9625	-0.9625	-0.5589	0.0000	0.0000
	Δ_2	0.9625	0.9623	0.3474	0.0000	0.0000
	Δ_3	0.0000	0.0002	0.2169	0.0000	0.0000
0.3	Δ_1	-0.9626	-0.9473	-0.5358	-0.0894	-0.0063
	Δ_2	0.9494	0.8332	0.3252	0.0422	0.0033
	Δ_3	0.0149	0.1234	0.2273	0.0516	0.0033
0.5	Δ_1	-0.9620	-0.8678	-0.5139	-0.1956	-0.0569
	Δ_2	0.8723	0.6569	0.3036	0.1074	0.0340
	Δ_3	0.1035	0.2371	0.2395	0.1017	0.0262
0.7	Δ_1	-0.9519	-0.7852	-0.4916	-0.2464	-0.1096
	Δ_2	0.7859	0.5367	0.2807	0.1390	0.0695
	Δ_3	0.1967	0.2922	0.2534	0.1320	0.0484
1	Δ_1	-0.8991	-0.6882	-0.4613	-0.2766	-0.1540
	Δ_2	0.6622	0.4202	0.2488	0.1551	0.0993
	Δ_3	0.2973	0.3385	0.2776	0.1639	0.0728

Table C.34.: Deltas for different levels of moneyness and volatility for FASO type-3 put option

C. Test Results for the FASO with the Market Volatility Models

C.3.2.3. Deltas for Minimum Correlations

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	0.0171	0.1377	0.6160	0.9596	0.9625
	Δ_2	0.0000	-0.0079	-0.2584	-0.8927	-0.9625
	Δ_3	-0.0128	-0.1056	-0.2949	-0.0527	0.0000
0.3	Δ_1	0.3140	0.4641	0.6518	0.8458	0.9516
	Δ_2	-0.0377	-0.0907	-0.2162	-0.4857	-0.8711
	Δ_3	-0.1546	-0.2161	-0.2539	-0.2013	-0.0368
0.5	Δ_1	0.4883	0.5813	0.6857	0.7981	0.9037
	Δ_2	-0.0675	-0.1063	-0.1768	-0.3144	-0.6173
	Δ_3	-0.1717	-0.2003	-0.2173	-0.2003	-0.0996
0.7	Δ_1	0.5897	0.6503	0.7173	0.7901	0.8674
	Δ_2	-0.0734	-0.0986	-0.1433	-0.2253	-0.4277
	Δ_3	-0.1562	-0.1733	-0.1830	-0.1764	-0.1181
1	Δ_1	0.6860	0.7204	0.7588	0.8001	0.8486
	Δ_2	-0.0622	-0.0766	-0.1010	-0.1425	-0.2557
	Δ_3	-0.1242	-0.1329	-0.1383	-0.1375	-0.1124

Table C.35.: Deltas for different levels of moneyness and volatility for FASO type-3 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	-0.9624	-0.9451	-0.5784	-0.0806	-0.0013
	Δ_2	0.9046	0.7911	0.3050	0.0006	0.0000
	Δ_3	0.0722	0.1865	0.3379	0.0981	0.0018
0.3	Δ_1	-0.8623	-0.7445	-0.5393	-0.2966	-0.1252
	Δ_2	0.6829	0.5613	0.3546	0.1102	0.0020
	Δ_3	0.3306	0.3666	0.3824	0.3390	0.2047
0.5	Δ_1	-0.7315	-0.6327	-0.5000	-0.3441	-0.2001
	Δ_2	0.6196	0.5345	0.4065	0.2278	0.0442
	Δ_3	0.4169	0.4268	0.4284	0.4147	0.3597
0.7	Δ_1	-0.6346	-0.5563	-0.4596	-0.3449	-0.2298
	Δ_2	0.6143	0.5513	0.4584	0.3195	0.1224
	Δ_3	0.4741	0.4758	0.4750	0.4677	0.4501
1	Δ_1	-0.5235	-0.4648	-0.3999	-0.3204	-0.2337
	Δ_2	0.6447	0.6002	0.5359	0.4349	0.2531
	Δ_3	0.5469	0.5442	0.5438	0.5396	0.5380

Table C.36.: Deltas for different levels of moneyness and volatility for FASO type-3 put option

C.4. The FASO Type-4 Option - Case 4.3

C.4.1. Premia

C.4.1.1. Premia for Mixed Correlations

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	0.0461	0.5788	4.1722	19.6553	39.0885
0.2	1.4239	3.3976	8.2681	20.7663	39.0907
0.3	4.2050	7.0152	12.2791	22.7450	39.2219
0.4	7.6401	10.9057	16.1966	25.2301	39.6540
0.5	11.3849	14.8734	20.0135	27.9621	40.4557
0.6	15.2709	18.8252	23.7233	30.7940	41.6017
0.7	19.1994	22.7103	27.3200	33.6446	43.0194
0.8	23.1021	26.4984	30.7982	36.4745	44.6323
0.9	26.9346	30.1716	34.1541	39.2572	46.3794
1	30.6680	33.7180	37.3850	41.9749	48.2094

Table C.37.: Premium for different levels of moneyness and volatility for FASO type-4 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	39.3687	20.3600	4.2391	0.3269	0.0021
0.2	41.7011	23.2709	8.5352	2.4521	0.4308
0.3	45.1420	27.1292	12.8773	5.3451	1.8561
0.4	49.2001	31.5018	17.2545	8.5402	3.8947
0.5	53.7219	36.1436	21.6555	11.9089	6.2487
0.6	58.5645	40.9312	26.0689	15.3818	8.7588
0.7	63.6220	45.7974	30.4831	18.9112	11.3531
0.8	68.8172	50.7039	34.8871	22.4725	13.9969
0.9	74.0854	55.6205	39.2689	26.0436	16.6695
1	79.3843	60.5233	43.6171	29.6074	19.3539

Table C.38.: Premium for different levels of moneyness and volatility for FASO type-4 put option

C. Test Results for the FASO with the Market Volatility Models

C.4.1.2. Premia for Maximum Correlations

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	0.0000	0.0000	1.3296	19.5439	39.0886
0.2	0.0000	0.0005	2.6596	19.5606	39.0846
0.3	0.0000	0.0309	3.9892	19.7632	39.0829
0.4	0.0008	0.1941	5.3173	20.2667	39.1344
0.5	0.0113	0.5502	6.6429	21.0216	39.3266
0.6	0.0540	1.0824	7.9652	21.9462	39.7042
0.7	0.1529	1.7536	9.2830	22.9906	40.2713
0.8	0.3216	2.5303	10.5957	24.1153	40.9957
0.9	0.5645	3.3866	11.9021	25.2974	41.8408
1	0.8801	4.3016	13.2009	26.5196	42.7867

Table C.39.: Premium for different levels of moneyness and volatility for FASO type-4 call option

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	39.0901	19.5455	1.3282	0.0000	0.0000
0.2	39.0907	19.5729	2.6543	0.0083	0.0000
0.3	39.1014	19.7650	3.9773	0.1591	0.0030
0.4	39.1781	20.1557	5.2962	0.6099	0.0494
0.5	39.3701	20.7041	6.6109	1.3315	0.2275
0.6	39.6786	21.3773	7.9204	2.2376	0.5904
0.7	40.0872	22.1474	9.2246	3.2716	1.1447
0.8	40.5732	22.9913	10.5231	4.3886	1.8625
0.9	41.1204	23.8904	11.8159	5.5596	2.7122
1	41.7166	24.8334	13.1022	6.7683	3.6658

Table C.40.: Premium for different levels of moneyness and volatility for FASO type-4 put option

C. Test Results for the FASO with the Market Volatility Models

C.4.1.3. Premia for Minimum Correlations

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	0.0074	0.2738	3.9119	19.5860	39.0885
0.2	0.7174	2.4509	7.7429	20.4413	39.0859
0.3	2.8019	5.6934	11.4875	22.2902	39.1536
0.4	5.7396	9.2941	15.1408	24.6373	39.4961
0.5	9.1281	13.0000	18.6983	27.1912	40.2152
0.6	12.7181	16.6999	22.1565	29.8224	41.2728
0.7	16.3857	20.3428	25.5124	32.4688	42.5844
0.8	20.0541	23.9034	28.7637	35.0960	44.0752
0.9	23.6757	27.3669	31.9085	37.6799	45.6844
1	27.2195	30.7228	34.9461	40.2098	47.3643

Table C.41.: Premium for different levels of moneyness and volatility for FASO type-4 call option

χ_1, χ_2					
$\tilde{\sigma}$	0.2	0.6	1	1.4	1.8
0.1	39.1917	20.0111	3.9865	0.1327	0.0001
0.2	40.7172	22.5371	8.0408	1.5997	0.1576
0.3	43.6208	26.2446	12.1556	4.0121	1.0032
0.4	47.4274	30.4707	16.3233	6.8898	2.4315
0.5	51.7987	34.9553	20.5356	10.0300	4.2250
0.6	56.5345	39.5951	24.7842	13.3242	6.2675
0.7	61.4991	44.3334	29.0604	16.7148	8.4819
0.8	66.6130	49.1364	33.3558	20.1730	10.8272
0.9	71.8286	53.9795	37.6615	23.6759	13.2731
1	77.1102	58.8443	41.9686	27.2048	15.7952

Table C.42.: Premium for different levels of moneyness and volatility for FASO type-4 put option

C. Test Results for the FASO with the Market Volatility Models

C.4.2. Deltas

C.4.2.1. Deltas for Mixed Correlations

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	0.0150	0.1261	0.6278	0.9772	0.9772
	Δ_2	0.0000	0.0000	-0.1997	-0.9171	-0.9772
	Δ_3	-0.0113	-0.1046	-0.3748	-0.0480	0.0000
0.3	Δ_1	0.2742	0.4160	0.6561	0.9137	0.9759
	Δ_2	-0.0031	-0.0268	-0.1679	-0.5484	-0.9175
	Δ_3	-0.1722	-0.2619	-0.3323	-0.2351	-0.0289
0.5	Δ_1	0.4341	0.5436	0.6840	0.8425	0.9509
	Δ_2	-0.0182	-0.0521	-0.1397	-0.3415	-0.7116
	Δ_3	-0.2125	-0.2611	-0.2918	-0.2566	-0.0983
0.7	Δ_1	0.5394	0.6182	0.7102	0.8141	0.9112
	Δ_2	-0.0305	-0.0588	-0.1149	-0.2338	-0.5066
	Δ_3	-0.2065	-0.2348	-0.2530	-0.2382	-0.1375
1	Δ_1	0.6438	0.6921	0.7469	0.8060	0.8717
	Δ_2	-0.0344	-0.0521	-0.0828	-0.1410	-0.2939
	Δ_3	-0.1755	-0.1901	-0.2011	-0.1982	-0.1501

Table C.43.: Deltas for different levels of moneyness and volatility for FASO type-4 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	-0.9772	-0.9769	-0.5984	-0.0744	-0.0007
	Δ_2	0.9223	0.8185	0.2349	0.0000	0.0000
	Δ_3	0.0685	0.1869	0.4180	0.0912	0.0011
0.3	Δ_1	-0.9574	-0.8579	-0.5673	-0.2687	-0.1193
	Δ_2	0.7262	0.5904	0.2716	0.0166	0.0000
	Δ_3	0.3520	0.4173	0.4620	0.3733	0.1979
0.5	Δ_1	-0.8676	-0.7395	-0.5357	-0.3226	-0.1913
	Δ_2	0.6497	0.5222	0.3101	0.0795	0.0012
	Δ_3	0.4630	0.4918	0.5063	0.4772	0.3710
0.7	Δ_1	-0.7748	-0.6614	-0.5046	-0.3367	-0.2141
	Δ_2	0.6173	0.5133	0.3503	0.1487	0.0130
	Δ_3	0.5296	0.5433	0.5505	0.5386	0.4801
1	Δ_1	-0.6658	-0.5723	-0.4573	-0.3309	-0.2215
	Δ_2	0.6147	0.5332	0.4121	0.2447	0.0640
	Δ_3	0.6068	0.6108	0.6136	0.6111	0.5917

Table C.44.: Deltas for different levels of moneyness and volatility for FASO type-4 put option

C. Test Results for the FASO with the Market Volatility Models

C.4.2.2. Deltas for Maximum Correlations

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	0.0000	0.0000	0.5922	0.9772	0.9772
	Δ_2	0.0000	0.0000	-0.3913	-0.9772	-0.9772
	Δ_3	0.0000	0.0000	-0.1974	0.0000	0.0000
0.3	Δ_1	0.0000	0.0178	0.6116	0.9597	0.9768
	Δ_2	0.0000	-0.0025	-0.4065	-0.9108	-0.9753
	Δ_3	0.0000	-0.0141	-0.1945	-0.0457	-0.0014
0.5	Δ_1	0.0046	0.1396	0.6318	0.8997	0.9627
	Δ_2	0.0000	-0.0444	-0.4220	-0.8020	-0.9421
	Δ_3	-0.0039	-0.0863	-0.1919	-0.0882	-0.0185
0.7	Δ_1	0.0366	0.2735	0.6520	0.8551	0.9319
	Δ_2	-0.0004	-0.1096	-0.4360	-0.7358	-0.8892
	Δ_3	-0.0300	-0.1446	-0.1904	-0.1036	-0.0371
1	Δ_1	0.1253	0.4213	0.6815	0.8193	0.8923
	Δ_2	-0.0100	-0.1927	-0.4554	-0.6857	-0.8343
	Δ_3	-0.0928	-0.1923	-0.1884	-0.1093	-0.0478

Table C.45.: Deltas for different levels of moneyness and volatility for FASO type-4 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	-0.9772	-0.9772	-0.5726	0.0000	0.0000
	Δ_2	0.9772	0.9772	0.3760	0.0000	0.0000
	Δ_3	0.0000	0.0000	0.2001	0.0000	0.0000
0.3	Δ_1	-0.9771	-0.9759	-0.5525	-0.0350	-0.0007
	Δ_2	0.9718	0.8923	0.3594	0.0224	0.0005
	Δ_3	0.0060	0.0902	0.2035	0.0131	0.0002
0.5	Δ_1	-0.9769	-0.9491	-0.5338	-0.1317	-0.0219
	Δ_2	0.9149	0.7566	0.3429	0.0865	0.0168
	Δ_3	0.0714	0.2124	0.2085	0.0490	0.0053
0.7	Δ_1	-0.9764	-0.8947	-0.5159	-0.1953	-0.0643
	Δ_2	0.8398	0.6500	0.3261	0.1300	0.0491
	Δ_3	0.1603	0.2775	0.2150	0.0731	0.0163
1	Δ_1	-0.9720	-0.8140	-0.4912	-0.2397	-0.1169
	Δ_2	0.7542	0.5399	0.3016	0.1605	0.0885
	Δ_3	0.2639	0.3252	0.2267	0.0934	0.0322

Table C.46.: Deltas for different levels of moneyness and volatility for FASO type-4 put option

C. Test Results for the FASO with the Market Volatility Models

C.4.2.3. Deltas for Minimum Correlations

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	0.0034	0.0808	0.6261	0.9769	0.9772
	Δ_2	0.0000	-0.0014	-0.2663	-0.9474	-0.9772
	Δ_3	-0.0026	-0.0667	-0.3091	-0.0240	0.0000
0.3	Δ_1	0.2338	0.4061	0.6545	0.8917	0.9748
	Δ_2	-0.0200	-0.0713	-0.2309	-0.5806	-0.9426
	Δ_3	-0.1335	-0.2162	-0.2755	-0.1945	-0.0167
0.5	Δ_1	0.4157	0.5396	0.6809	0.8301	0.9439
	Δ_2	-0.0541	-0.1033	-0.1982	-0.3920	-0.7565
	Δ_3	-0.1746	-0.2174	-0.2433	-0.2132	-0.0759
0.7	Δ_1	0.5266	0.6113	0.7064	0.8065	0.9057
	Δ_2	-0.0698	-0.1055	-0.1686	-0.2891	-0.5651
	Δ_3	-0.1725	-0.1978	-0.2136	-0.1993	-0.1123
1	Δ_1	0.6309	0.6837	0.7417	0.8031	0.8714
	Δ_2	-0.0704	-0.0930	-0.1304	-0.1969	-0.3680
	Δ_3	-0.1501	-0.1641	-0.1728	-0.1693	-0.1238

Table C.47.: Deltas for different levels of moneyness and volatility for FASO type-4 call option

C. Test Results for the FASO with the Market Volatility Models

χ_1, χ_2						
$\tilde{\sigma}$		0.2	0.6	1	1.4	1.8
0.1	Δ_1	-0.9772	-0.9726	-0.5966	-0.0425	0.0000
	Δ_2	0.9502	0.8567	0.3039	0.0000	0.0000
	Δ_3	0.0332	0.1356	0.3446	0.0512	0.0001
0.3	Δ_1	-0.9226	-0.8087	-0.5659	-0.2626	-0.0869
	Δ_2	0.7354	0.5990	0.3437	0.0637	0.0000
	Δ_3	0.2936	0.3501	0.3810	0.3078	0.1366
0.5	Δ_1	-0.8077	-0.6989	-0.5346	-0.3372	-0.1727
	Δ_2	0.6487	0.5464	0.3851	0.1706	0.0141
	Δ_3	0.3904	0.4118	0.4185	0.3939	0.2992
0.7	Δ_1	-0.7167	-0.6251	-0.5018	-0.3562	-0.2154
	Δ_2	0.6233	0.5456	0.4272	0.2545	0.0589
	Δ_3	0.4480	0.4553	0.4559	0.4464	0.4013
1	Δ_1	-0.6109	-0.5400	-0.4523	-0.3508	-0.2408
	Δ_2	0.6306	0.5746	0.4908	0.3613	0.1608
	Δ_3	0.5142	0.5144	0.5124	0.5101	0.4983

Table C.48.: Deltas for different levels of moneyness and volatility for FASO type-4 put option

D. Test Results for the Storage Model

D.1. Prices of Spread Options

Prices	May-09	Jun-09	Jul-09	Aug-09	Sep-09	Oct-09	Nov-09	Dec-09	Jan-10	Feb-10	Mar-10
Apr-09	0.129007	0.249033	0.361247	0.450528	0.484423	0.557317	0.889862	1.216017	1.432563	1.417423	1.243483
May-09		0.180902	0.302261	0.397917	0.438795	0.517449	0.850490	1.172885	1.391816	1.376867	1.204225
Jun-09			0.179239	0.282600	0.333631	0.417480	0.751840	1.062824	1.286020	1.272929	1.107068
Jul-09				0.167953	0.232472	0.321516	0.661990	0.962123	1.190450	1.179537	1.020950
Aug-09					0.138922	0.236249	0.595896	0.890344	1.124877	1.116137	0.963566
Sep-09						0.172103	0.581877	0.883253	1.124482	1.117402	0.966875
Oct-09							0.540869	0.844165	1.092219	1.087545	0.941799
Nov-09								0.411534	0.729400	0.732398	0.603151
Dec-10									0.447032	0.457273	0.363031
Jan-10										0.128217	0.096562
Feb-10											0.053705

Table D.1.: Prices of futs-to-futs Calendar spread options computed using the Kirk's approximation

Prices	May-09	Jun-09	Jul-09	Aug-09	Sep-09	Oct-09	Nov-09	Dec-09	Jan-10	Feb-10	Mar-10
Apr-09	0.129022	0.249048	0.361260	0.450539	0.484434	0.557324	0.889864	1.215995	1.432462	1.417469	1.243562
May-09		0.180914	0.302270	0.397924	0.438802	0.517451	0.850490	1.172788	1.391786	1.376843	1.204218
Jun-09			0.179248	0.282611	0.333645	0.417483	0.751848	1.062803	1.285999	1.272914	1.107072
Jul-09				0.167966	0.232484	0.321520	0.662008	0.962157	1.190441	1.179533	1.020965
Aug-09					0.138932	0.236253	0.595923	0.890352	1.124950	1.116074	0.963591
Sep-09						0.172112	0.581918	0.883262	1.124490	1.117415	0.966910
Oct-09							0.540912	0.844181	1.092238	1.087569	0.941846
Nov-09								0.411532	0.729422	0.732424	0.603201
Dec-10									0.447075	0.457323	0.363090
Jan-10										0.128241	0.096584
Feb-10											0.053716

Table D.2.: Prices of futs-to-futs Calendar spread options computed using the 3D-binomial tree

D. Test Results for the Storage Model

Prices	May-09	Jun-09	Jul-09	Aug-09	Sep-09	Oct-09	Nov-09	Dec-09	Jan-10	Feb-10	Mar-10
Apr-09	0.000014	0.000015	0.000013	0.000011	0.000012	0.000007	0.000002	0.000022	0.000101	0.000045	0.000079
May-09		0.000012	0.000010	0.000007	0.000007	0.000002	0.000000	0.000097	0.000030	0.000025	0.000007
Jun-09			0.000010	0.000011	0.000015	0.000003	0.000009	0.000021	0.000020	0.000016	0.000004
Jul-09				0.000013	0.000011	0.000004	0.000018	0.000034	0.000009	0.000004	0.000016
Aug-09					0.000010	0.000004	0.000027	0.000008	0.000073	0.000064	0.000026
Sep-09						0.000008	0.000041	0.000009	0.000007	0.000013	0.000035
Oct-09							0.000042	0.000016	0.000019	0.000024	0.000047
Nov-09								0.000002	0.000023	0.000026	0.000050
Dec-10									0.000043	0.000050	0.000059
Jan-10										0.000024	0.000022
Feb-10											0.000010

Table D.3.: Absolute differences between the futs-to-futs Calendar spread options computed using the Kirk's approximation and the 3D-binomial tree method

D.2. NYMEX Natural Gas Forward Prices and Implied Volatilities

Months to Maturity	Spring	Summer	Fall	Winter
0	6.63	6.315	5.36	8.285
1	6.733	6.448	5.877	8.422
2	6.94	6.698	8.057	8.48
3	7.115	6.963	9.897	8.38
4	7.29	7.318	10.562	8.08
5	7.417	8.523	10.602	8.062
6	7.502	9.758	10.432	8.125
7	7.607	10.318	8.382	8.207
8	8.722	10.323	8.232	8.278
9	9.742	10.138	8.322	8.333
10	10.432	8.388	8.427	8.433
11	10.437	8.218	8.517	8.908
12	10.242	8.315	8.612	9.338

Table D.4.: NYMEX natural gas forward curves (\$/mmBtu)

D. Test Results for the Storage Model

Months to Maturity	Spring	Summer	Fall	Winter
1	0.59	0.56	0.79	0.55
2	0.55	0.66	0.79	0.57
3	0.53	0.68	0.7	0.61
4	0.51	0.69	0.7	0.52
5	0.52	0.62	0.73	0.49
6	0.53	0.56	0.75	0.47
7	0.53	0.57	0.5	0.47
8	0.51	0.57	0.47	0.48
9	0.5	0.6	0.47	0.49
10	0.49	0.44	0.47	0.49
11	0.49	0.41	0.48	0.47
12	0.5	0.41	0.5	0.44

Table D.5.: NYMEX natural gas implied volatilities

D.3. Local Correlations

D.3.1. Benchmark Test for the Calibration of Correlations

Correlations	Apr-09	May-09	Jun-09	Jul-09	Aug-09	Sep-09	Oct-09	Nov-09	Dec-09	Jan-10	Feb-10	Mar-10
Apr-09	1	0.995271	0.994932	0.994079	0.993314	0.992	0.990062	0.972943	0.920141	0.887152	0.885512	0.880342
May-09	0.995271	1	0.999534	0.998836	0.998113	0.996929	0.995086	0.975431	0.920262	0.886942	0.885299	0.88031
Jun-09	0.994932	0.999534	1	0.999552	0.998937	0.997891	0.996246	0.97769	0.923651	0.890949	0.889351	0.884492
Jul-09	0.994079	0.998836	0.999552	1	0.999639	0.99881	0.997376	0.979886	0.927058	0.894848	0.893302	0.888599
Aug-09	0.993314	0.998113	0.998937	0.999639	1	0.999332	0.9981416	0.981821	0.930369	0.898614	0.897087	0.892508
Sep-09	0.992	0.996929	0.997891	0.998806	0.999332	1	0.99931	0.984072	0.933838	0.902624	0.901164	0.896905
Oct-09	0.990062	0.995086	0.996246	0.997376	0.998142	0.99931	1	0.986612	0.937802	0.907318	0.905956	0.901801
Nov-09	0.972943	0.975431	0.97769	0.979886	0.981821	0.984072	0.986612	1	0.9795	0.96019	0.959289	0.955826
Dec-10	0.920141	0.920262	0.923651	0.927058	0.930369	0.933838	0.937802	0.9795	1	0.995068	0.994339	0.992112
Jan-10	0.887152	0.886942	0.89095	0.894848	0.898614	0.902624	0.907318	0.96019	0.995068	1	0.999338	0.997238
Feb-10	0.885512	0.8853	0.889351	0.893302	0.897087	0.901164	0.905956	0.959288	0.994339	0.999338	1	0.998646
Mar-10	0.880342	0.88031	0.884492	0.888599	0.892509	0.896905	0.9018	0.955826	0.992112	0.997238	0.998646	1

Table D.6.: Local correlations calibrated from historical data

D. Test Results for the Storage Model

Correlations	Apr-09	May-09	Jun-09	Jul-09	Aug-09	Sep-09	Oct-09	Nov-09	Dec-09	Jan-10	Feb-10	Mar-10
Apr-09	1	0.994701	0.989442	0.984128	0.978402	0.972438	0.966441	0.930726	0.924276	0.892141	0.885005	0.878355
May-09	0.994701	1	0.994535	0.989442	0.983947	0.978213	0.972438	0.936953	0.930726	0.898821	0.891914	0.885472
Jun-09	0.989442	0.994535	1	0.994701	0.989442	0.983947	0.978402	0.943154	0.937157	0.905489	0.898821	0.892593
Jul-09	0.984128	0.989442	0.994701	1	0.994535	0.989269	0.983947	0.948929	0.943154	0.911716	0.905278	0.899258
Aug-09	0.978402	0.983947	0.989442	0.994535	1	0.994535	0.989442	0.954662	0.949117	0.917918	0.911716	0.905911
Sep-09	0.972438	0.978213	0.983947	0.989269	0.994535	1	0.994701	0.960158	0.954843	0.923881	0.917918	0.912327
Oct-09	0.966441	0.972438	0.978402	0.983947	0.989442	0.994701	1	0.96525	0.960158	0.929426	0.923692	0.918309
Nov-09	0.930726	0.936953	0.943154	0.948929	0.954662	0.960158	0.96525	1	0.995138	0.964643	0.959147	0.953979
Dec-10	0.924276	0.930726	0.937157	0.943154	0.949117	0.954843	0.960158	0.995138	1	0.969735	0.964469	0.959509
Jan-10	0.892141	0.898821	0.905489	0.911716	0.917918	0.923881	0.929426	0.964643	0.969735	1	0.994972	0.990226
Feb-10	0.885005	0.891914	0.898821	0.905278	0.911716	0.917918	0.923692	0.959147	0.964469	0.994972	1	0.995469
Mar-10	0.878355	0.885472	0.892593	0.899258	0.905911	0.912327	0.918309	0.953979	0.959509	0.990226	0.995469	1

Table D.7.: Local correlations calibrated from seasonal groups of data

Correlations	Apr-09	May-09	Jun-09	Jul-09	Aug-09	Sep-09	Oct-09	Nov-09	Dec-09	Jan-10	Feb-10	Mar-10
Apr-09	1	0.999741	0.994854	0.989951	0.984631	0.979091	0.973558	0.96238	0.937812	0.876677	0.867809	0.859501
May-09	0.999741	1	0.999551	0.994853	0.989747	0.98442	0.979091	0.968126	0.943764	0.885087	0.876431	0.868315
Jun-09	0.994853	0.999551	1	0.999741	0.994854	0.989746	0.984631	0.973885	0.949736	0.893523	0.885087	0.87717
Jul-09	0.989951	0.994853	0.999741	1	0.999551	0.994656	0.989746	0.979214	0.95527	0.901515	0.893291	0.885565
Aug-09	0.984631	0.989746	0.994854	0.999551	1	0.999551	0.994854	0.984541	0.960809	0.909519	0.901515	0.893988
Sep-09	0.979091	0.98442	0.989747	0.994657	0.999551	1	0.999741	0.989648	0.966129	0.917303	0.909519	0.90219
Oct-09	0.973558	0.979091	0.984631	0.989747	0.994854	0.999741	1	0.994345	0.971032	0.924663	0.917092	0.909955
Nov-09	0.96238	0.968126	0.973885	0.979214	0.984541	0.989648	0.994345	1	1	0.966467	0.959115	0.952177
Dec-10	0.9378119	0.943764	0.949736	0.95527	0.960809	0.966129	0.971032	1	1	0.990482	0.983343	0.976597
Jan-10	0.876677	0.885087	0.893523	0.901515	0.909519	0.917303	0.924663	0.966467	0.990482	1	1	1
Feb-10	0.867809	0.876431	0.885087	0.893291	0.901515	0.909519	0.917092	0.959115	0.983343	1	1	1
Mar-10	0.859501	0.868315	0.87717	0.885565	0.893988	0.90219	0.909955	0.952177	0.976597	1	1	1

Table D.8.: Local correlations calibrated from monthly groups of data

D. Test Results for the Storage Model

Abs. Diff.	Apr-09	May-09	Jun-09	Jul-09	Aug-09	Sep-09	Oct-09	Nov-09	Dec-09	Jan-10	Feb-10	Mar-10
Apr-09	0	0.000573	0.005518	0.01001	0.015012	0.01972	0.023858	0.043391	0.004494	0.005624	0.000572	0.002258
May-09	0.000573	0	0.005001	0.009405	0.014193	0.018773	0.02276	0.039447	0.01137	0.013394	0.007472	0.005864
Jun-09	0.005518	0.005001	0	0.004853	0.009505	0.013974	0.017911	0.035324	0.014622	0.016319	0.010648	0.009159
Jul-09	0.01001	0.009405	0.004853	0	0.005106	0.009548	0.013464	0.031592	0.017362	0.01885	0.013406	0.011995
Aug-09	0.015012	0.014193	0.009505	0.005106	0	0.0048	0.008716	0.027661	0.020151	0.021482	0.016308	0.015017
Sep-09	0.01972	0.018773	0.013974	0.009548	0.0048	0	0.004612	0.024301	0.022493	0.02355	0.018591	0.017195
Oct-09	0.023858	0.02276	0.017911	0.013464	0.008716	0.004612	0	0.021652	0.023839	0.024367	0.019577	0.018306
Nov-09	0.043391	0.039447	0.035324	0.031592	0.027661	0.024301	0.021652	0	0.015965	0.004638	0.000147	0.001932
Dec-10	0.004494	0.01137	0.014622	0.017362	0.020151	0.022493	0.023839	0.015965	0	0.025458	0.03004	0.032862
Jan-10	0.005624	0.013394	0.016319	0.01885	0.021482	0.02355	0.024367	0.004638	0.025458	0	0.004369	0.007032
Feb-10	0.000572	0.007472	0.010648	0.013406	0.016308	0.018591	0.019577	0.000147	0.03004	0.004369	0	0.003182
Mar-10	0.002258	0.005864	0.009159	0.011995	0.015017	0.017195	0.018306	0.001932	0.032862	0.007032	0.003182	0

Table D.9.: Absolute differences between the local correlations calibrated from the seasonal groups of data and the historical data

Abs. Diff.	Apr-09	May-09	Jun-09	Jul-09	Aug-09	Sep-09	Oct-09	Nov-09	Dec-09	Jan-10	Feb-10	Mar-10
Apr-09	0	0.004491	0.000078	0.004153	0.008741	0.013013	0.01667	0.010857	0.019205	0.011807	0.019992	0.023674
May-09	0.004491	0	0.000017	0.003987	0.008382	0.012548	0.016073	0.007489	0.025538	0.002091	0.010017	0.013626
Jun-09	0.000078	0.000017	0	0.00019	0.004088	0.008162	0.011658	0.003892	0.028241	0.002889	0.004795	0.008279
Jul-09	0.004153	0.003987	0.00019	0	0.000089	0.004155	0.00765	0.000686	0.030431	0.00745	0.000013	0.003414
Aug-09	0.008741	0.008382	0.004088	0.000089	0	0.000219	0.003294	0.00277	0.032719	0.012135	0.004936	0.001657
Sep-09	0.013013	0.012548	0.008162	0.004155	0.000219	0	0.000431	0.005666	0.034579	0.016263	0.009271	0.005893
Oct-09	0.01667	0.016073	0.011658	0.00765	0.003294	0.000431	0	0.007837	0.035434	0.019118	0.012292	0.009042
Nov-09	0.010857	0.007489	0.003892	0.000686	0.00277	0.005666	0.007837	0	0.020929	0.006538	0.000181	0.003818
Dec-10	0.019205	0.025538	0.028241	0.030431	0.032719	0.034579	0.035434	0.020929	0	0.004608	0.011059	0.015638
Jan-10	0.011807	0.002091	0.002889	0.00745	0.012135	0.016263	0.019118	0.006538	0.004608	0	0.000662	0.002769
Feb-10	0.019992	0.010017	0.004795	0.000013	0.004936	0.009271	0.012292	0.000181	0.011059	0.000662	0	0.001355
Mar-10	0.023674	0.013626	0.008279	0.003414	0.001657	0.005893	0.009042	0.003818	0.015638	0.002769	0.001355	0

Table D.10.: Absolute differences between the local correlations calibrated from the monthly groups of data and the historical data

D. Test Results for the Storage Model

D.3.2. Local Correlations for Valuing Storage Contracts

Correlations	Apr-06	May-06	Jun-06	Jul-06	Aug-06	Sep-06	Oct-06	Nov-06	Dec-06	Jan-07	Feb-07	Mar-07
Apr-06	1	0.997307	0.996289	0.994895	0.993363	0.992041	0.990317	0.979053	0.962204	0.95051	0.948971	0.946238
May-06	0.997307	1	0.999523	0.998526	0.99734	0.996128	0.994577	0.983065	0.966074	0.954385	0.953258	0.951013
Jun-06	0.996289	0.999523	1	0.999611	0.998845	0.997906	0.996656	0.98643	0.970543	0.959255	0.958176	0.956017
Jul-06	0.994895	0.998526	0.999611	1	0.999623	0.998927	0.997958	0.988943	0.974059	0.963079	0.962026	0.959904
Aug-06	0.993363	0.99734	0.998845	0.999623	1	0.999611	0.998946	0.990854	0.976622	0.965795	0.964776	0.962724
Sep-06	0.992041	0.996128	0.997906	0.998927	0.999611	1	0.99969	0.992393	0.978751	0.967945	0.966892	0.964771
Oct-06	0.990317	0.994577	0.996656	0.997958	0.998946	0.99969	1	0.993609	0.980602	0.969956	0.968876	0.966809
Nov-06	0.979053	0.983065	0.98643	0.988943	0.990854	0.992393	0.993609	1	0.995816	0.989687	0.98864	0.986755
Dec-06	0.962204	0.966074	0.970543	0.974059	0.976622	0.978751	0.980602	0.995816	1	0.997599	0.996621	0.995011
Jan-07	0.95051	0.954385	0.959255	0.963079	0.965795	0.967945	0.969956	0.989687	0.997599	1	0.999379	0.997803
Feb-07	0.948971	0.953258	0.958176	0.962026	0.964776	0.966892	0.968876	0.98864	0.996621	0.999379	1	0.9991
Mar-07	0.946238	0.951013	0.956017	0.959904	0.962724	0.964771	0.966809	0.986755	0.995011	0.997803	0.9991	1

Table D.11.: Local correlation corresponding to the Spring forward curve

Correlations	Jul-06	Aug-06	Sep-06	Oct-06	Nov-06	Dec-06	Jan-07	Feb-07	Mar-07	Apr-07	May-07	Jun-07
Jul-06	1	0.999623	0.998927	0.997958	0.988943	0.974059	0.963079	0.962026	0.959904	0.926782	0.914751	0.912005
Aug-06	0.999623	1	0.999611	0.998946	0.990854	0.976622	0.965795	0.964776	0.962724	0.930088	0.918045	0.915172
Sep-06	0.998927	0.999611	1	0.99969	0.992393	0.978751	0.967945	0.966892	0.964771	0.93224	0.920291	0.917557
Oct-06	0.997958	0.998946	0.99969	1	0.993609	0.980602	0.969956	0.968876	0.966809	0.934567	0.922723	0.920044
Nov-06	0.988943	0.990854	0.992393	0.993609	1	0.995816	0.989687	0.98864	0.986755	0.954672	0.94294	0.940316
Dec-06	0.974059	0.976622	0.978751	0.980602	0.995816	1	0.997599	0.996621	0.995011	0.963375	0.952049	0.949429
Jan-07	0.963079	0.965795	0.967945	0.969956	0.989687	0.997599	1	0.999379	0.997803	0.966848	0.955706	0.953676
Feb-07	0.962026	0.964776	0.966892	0.968876	0.98864	0.996621	0.999379	1	0.9991	0.970032	0.959365	0.957452
Mar-07	0.959904	0.962724	0.964771	0.966809	0.986755	0.995011	0.997803	0.9991	1	0.973218	0.963082	0.961182
Apr-07	0.926782	0.930088	0.93224	0.934567	0.954672	0.963375	0.966848	0.970032	0.973218	1	0.996863	0.995918
May-07	0.914751	0.918045	0.920291	0.922723	0.94294	0.952049	0.955706	0.959365	0.963082	0.996863	1	0.999119
Jun-07	0.912005	0.915172	0.917557	0.920044	0.940316	0.949429	0.953676	0.957452	0.961182	0.995918	0.999119	1

Table D.12.: Local correlation corresponding to the Summer forward curve

D. Test Results for the Storage Model

Correlations	Oct-06	Nov-06	Dec-06	Jan-07	Feb-07	Mar-07	Apr-07	May-07	Jun-07	Jul-07	Aug-07	Sep-07
Oct-06	1	0.993609	0.980602	0.969956	0.968876	0.966809	0.934567	0.922723	0.920044	0.917543	0.914993	0.912594
Nov-06	0.993609	1	0.995816	0.989687	0.98864	0.986755	0.954672	0.94294	0.940316	0.937882	0.935561	0.933528
Dec-06	0.980602	0.995816	1	0.997599	0.996621	0.995011	0.963375	0.952049	0.949429	0.947153	0.945229	0.943478
Jan-07	0.969956	0.989687	0.997599	1	0.999379	0.997803	0.966848	0.955706	0.953676	0.951645	0.949953	0.94846
Feb-07	0.968876	0.98864	0.996621	0.999379	1	0.9991	0.970032	0.959365	0.957452	0.955502	0.953936	0.952485
Mar-07	0.966809	0.986755	0.995011	0.997803	0.9991	1	0.973218	0.963082	0.961182	0.959254	0.957788	0.956449
Apr-07	0.934567	0.954672	0.963375	0.966848	0.970032	0.973218	1	0.996863	0.995918	0.994468	0.99327	0.992717
May-07	0.922723	0.94294	0.952049	0.955706	0.959365	0.963082	0.996863	1	0.999119	0.997899	0.996793	0.996116
Jun-07	0.920044	0.940316	0.949429	0.953676	0.957452	0.961182	0.995918	0.999119	1	0.999511	0.998605	0.99802
Jul-07	0.917543	0.937882	0.947153	0.951645	0.955502	0.959254	0.994468	0.997899	0.999511	1	0.999361	0.998757
Aug-07	0.914993	0.935561	0.945229	0.949953	0.953936	0.957788	0.99327	0.996793	0.998605	0.999361	1	0.999521
Sep-07	0.912594	0.933528	0.943478	0.94846	0.952485	0.956449	0.992717	0.996116	0.99802	0.998757	0.999521	1

Table D.13.: Local correlation corresponding to the Fall forward curve

Correlations	Jan-07	Feb-07	Mar-07	Apr-07	May-07	Jun-07	Jul-07	Aug-07	Sep-07	Oct-07	Nov-07	Dec-07
Jan-07	1	0.999379	0.997803	0.966848	0.955706	0.953676	0.951645	0.949953	0.94846	0.946818	0.945153	0.941546
Feb-07	0.999379	1	0.9991	0.970032	0.959365	0.957452	0.955502	0.953936	0.952485	0.950751	0.949039	0.945467
Mar-07	0.997803	0.9991	1	0.973218	0.963082	0.961182	0.959254	0.957788	0.956449	0.954701	0.953008	0.949551
Apr-07	0.966848	0.970032	0.973218	1	0.996863	0.995918	0.994468	0.99327	0.992717	0.991276	0.989075	0.985182
May-07	0.955706	0.959365	0.963082	0.996863	1	0.999119	0.997899	0.996793	0.996116	0.994797	0.992966	0.9894
Jun-07	0.953676	0.957452	0.961182	0.995918	0.999119	1	0.999511	0.998605	0.99802	0.99695	0.994986	0.991215
Jul-07	0.951645	0.955502	0.959254	0.994468	0.997899	0.999511	1	0.999361	0.998757	0.997783	0.995852	0.992131
Aug-07	0.949953	0.953936	0.957788	0.99327	0.996793	0.998605	0.999361	1	0.999521	0.998766	0.996952	0.993316
Sep-07	0.94846	0.952485	0.956449	0.992717	0.996116	0.99802	0.998757	0.999521	1	0.999472	0.99779	0.994305
Oct-07	0.946818	0.950751	0.954701	0.991276	0.994797	0.99695	0.997783	0.998766	0.999472	1	0.998153	0.994749
Nov-07	0.945153	0.949039	0.953008	0.989075	0.992966	0.994986	0.995852	0.996952	0.99779	0.998153	1	0.998602
Dec-07	0.941546	0.945467	0.949551	0.985182	0.9894	0.991215	0.992131	0.993316	0.994305	0.994749	0.998602	1

Table D.14.: Local correlation corresponding to the Winter forward curve

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