Comment

Comment on 'Solvable model of bound states in the continuum (BIC) in one dimension' (Physica Scripta (2019) 94 105214)

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Abstract. We elucidate the status of the special scattering states found in this paper and explore further the relationships between scattering states and bound states of different non-analytic segments (depending on $|x|$) of the exponential potential.

Introduction

In the above paper[1] certain discrete states of the bottomless potential $V_1 \equiv V_0(1-e^{2|x|/a})$ were identified within the continuum of scattering states. At the time, the precise significance of these states was not clear, but it was noticed in numerical studies[2] that their energies corresponded to the bound-state energies of the related potential well $V_2 \equiv V_0(1-e^{-2|x|/a})$.

In this comment we establish the connection between these two sets of states, and show further that at these energies there occur special solutions for the bottomless analytic potential $V_3 \equiv V_0(1-e^{-2x/a})$, which agrees with V_1 on the left ($x<0$) and V_2 on the right. Similar relationships link special scattering states of $V_5 \equiv V_0(e^{-2|x|/a}-1)$ to the bound states of $V_4 \equiv V_0(e^{2|x|/a}-1)$ and special solutions of the analytic potential $V_6 \equiv V_0(e^{-2x/a}-1)$. Furthermore the relation between poles of the scattering amplitudes of V_1 and the bound states of V_4 established in Ref. [3] can be extended to a similar connection between the poles of the scattering amplitudes of V_5 and the bound states of V_2.

Finally we discuss the nature of the special scattering states of the bottomless potentials. They are normalizable, so in a sense can be regarded as unusual bound states, but their momentum expectation values diverge.

The exponential potential and its variants

Fig. 1 shows the various related potentials mentioned above.

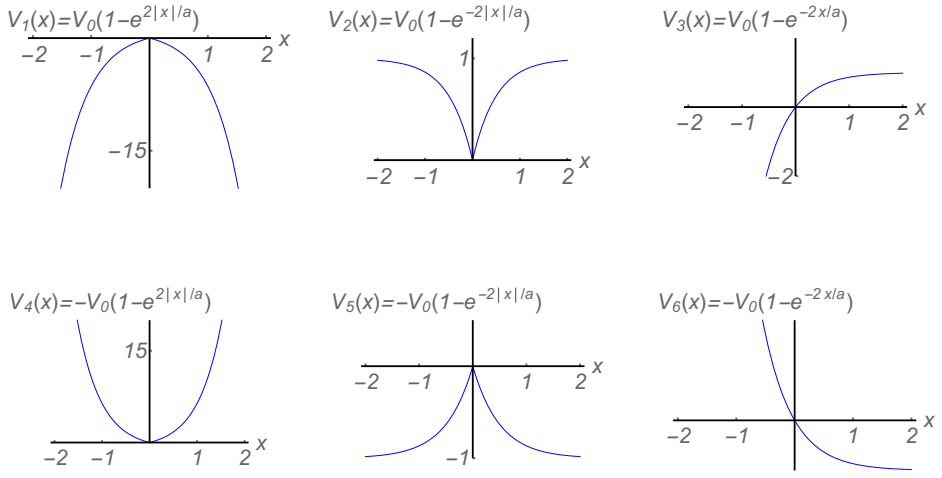


Figure 1. The various potentials associated with those of Ref. [1] and [3]. For ease of illustration we have taken $V_0 = a = 1$.

We begin by reviewing the results of Ref. [1] for the potential $V_1 \equiv V_0(1 - e^{2|x|/a})$. By defining $\kappa = \sqrt{V_0 - E}$ and $q = \sqrt{V_0}$ the Schrödinger equation, in units where $2m/\hbar^2 = 1$, can be written as

$$\psi'' + (q^2 e^{2|x|/a} - \kappa^2)\psi = 0. \tag{1}$$

This equation can be transformed into the Bessel equation, whose two independent solutions are $J_{\pm\kappa a}(z)$, where $z = qa\, e^{|x|/a}$. Their behaviour as $x \to \pm\infty$ is governed by the large-z behaviour of $J_\nu(z)$, Eq. (9.2.2) of Ref. [4], namely $J_\nu(z) \sim \sqrt{2/(\pi z)}\cos(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)$. That is, they fall off like the square root of an exponential, while oscillating more and more rapidly. Hence they are both square integrable, which is rather surprising from the classical point of view.

In addition to satisfying the Schrödinger equation for $x > 0$ and $x < 0$, the wave-functions $\psi(x)$ must connect smoothly at $x = 0$, with no discontinuity in $\psi'(x)$. This is achieved in the combinations

$$\begin{aligned} \psi_{\rm even}(x) &= J'_{-\kappa a}(qa)J_{\kappa a}(z) - J'_{\kappa a}(qa)J_{-\kappa a}(z), \\ \psi_{\rm odd}(x) &= \mathrm{sgn}(x)\,[J_{-\kappa a}(qa)J_{\kappa a}(z) - J_{\kappa a}(qa)J_{-\kappa a}(z).] \end{aligned} \tag{2}$$

The first has $\psi'(0) = 0$, the second $\psi(0) = 0$.

There is a continuum of such solutions. They are real and square integrable and can be considered as rather unorthodox bound states. However, scattering states can be constructed by taking appropriate linear combinations of $\psi_{\rm even}$ and $\psi_{\rm odd}$. Although

they fall off rapidly, the probability current is finite because the fall-off in ψ is exactly compensated by the derivative of its rapid oscillations.

In Ref. [1] the authors observed that within this continuum there are special discrete values of κ, hence E, for which the solution $\psi(x) \propto J_{\kappa a}(z)$ alone. These special values are given by the roots of $J'_{\kappa a}(qa) = 0$ for $\psi_{\rm even}$ and of $J_{\kappa a}(qa) = 0$ for $\psi_{\rm odd}$. They occur for $0 < E < V_0$. The wave-functions for the two lowest-energy states ψ_0 and ψ_1 are shown in Figs. 2(a), (b).

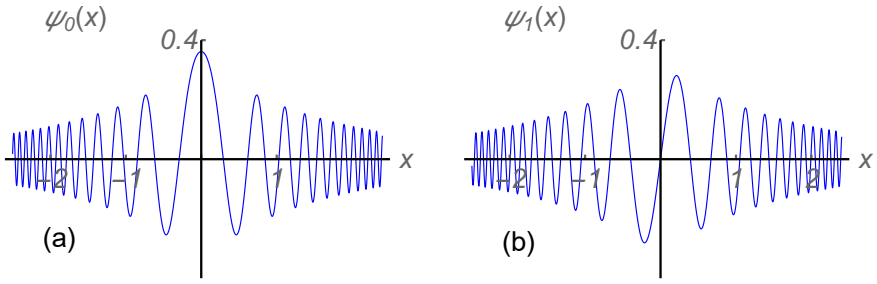


Figure 2. Wave-functions $\psi_0(x)$ and $\psi_1(x)$ of $V_1 \equiv V_0(1-e^{2|x|/a})$ for the lowest values of E where (a) $J'_{\kappa a}(qa) = 0$, (b) $J_{\kappa a}(qa) = 0$. In case (b) the sign of the wave-function has been reversed at $x = 0$ to avoid a cusp. In this figure and the following two figures we have taken $a = 1$, $V_0 = 50$, in which case $E_0 = 18.611$ and $E_1 = 37.263$.

In fact, these particular energies turn out[2] to be the energy levels of the associated confining potential $V_2 = V_0(1 - e^{-2|x|/a})$, shown in the upper panel of Fig. 1. The corresponding wave-functions, $J_{\kappa a}(qa\, e^{-|x|/a})$ and $\mathrm{sgn}(x)J_{\kappa a}(qa\, e^{-|x|/a})$ are shown in Figs. 3(a), (b) respectively. In this case their behaviour for large $|x|$ is governed by the small-z behaviour of the Bessel function, Eq. (9.1.7) of Ref. [4], namely $J_\nu(z) \sim (\frac{1}{2}z)^\nu/\Gamma(\nu+1)$. In this case the eigenvalue condition is that $J_{-\kappa a}(qa\, e^{-|x|/a})$ be excluded, because it blows up at infinity.

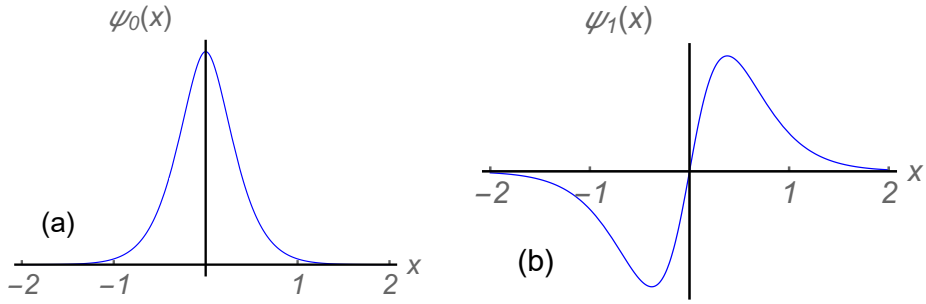


Figure 3. Eigenfunctions of $V_2 \equiv V_0(1 - e^{-2|x|/a})$ for the lowest values of E where (a) $J'_{\kappa a}(qa) = 0$, (b) $J_{\kappa a}(qa) = 0$. Again the sign of the wave-function has been reversed at $x = 0$ in case (b).

There are also other discrete energies, given by the roots of $J'_{-\kappa a}(qa) = 0$ and $J_{-\kappa a}(qa) = 0$, for which the wave functions are proportional instead to $J_{-\kappa a}(qa\, e^{-|x|/a})$, but these seem to have no physical significance.

We are now in a position to put these two pieces together by considering the third, analytic, potential $V_3 \equiv V_0(1 - e^{-2x/a})$ of Fig. 1. This is a smooth continuation from $V_2(x)$ for $x > 0$ to the original potential $V_1(x)$ for $x < 0$. Likewise, the eigenfunctions will be those of V_2 on the right, matching smoothly onto those of the original potential on the left, as shown in Figs. 4(a), (b). So we have an analytic potential that goes to $-\infty$ on the left, but nonetheless has genuine discrete eigenvalues and eigenfunctions.

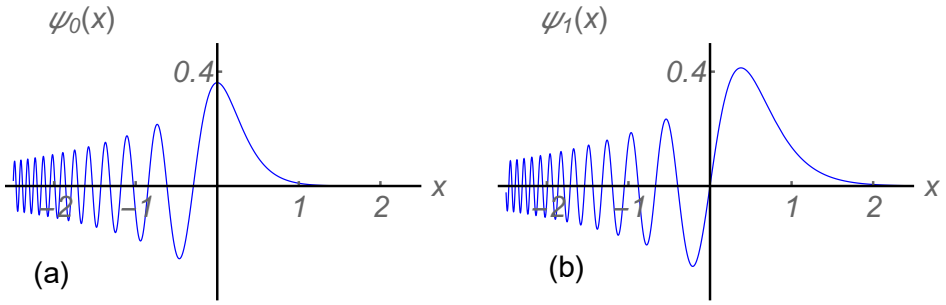


Figure 4. Eigenfunctions of $V_3 \equiv V_0(1-e^{-2x/a})$ for the lowest values of E where (a) $J'_{\kappa a}(qa) = 0$, (b) $J_{\kappa a}(qa) = 0$.

Thus we have connected the scattering properties of V_1 to the bound states of V_2. A further connection between the different potentials was noted in Ref. [3], namely that poles in the scattering amplitudes R and T for V_1 when the sign of V_0 is reversed correspond to the bound-state energies of V_4.

A similar analysis reveals that there are special scattering solutions of V_5 at the bound-state energies of V_4 and that the left and right segments of the two wave-functions can be combined to give overall solutions of the analytic potential V_6. And again, the poles in the scattering amplitudes for V_5 correspond to the bound-state energies of V_2. Full details are given in [5].

We have established the interconnections between the scattering and bound states of all the different non-analytic pieces of the analytic potentials $\pm V_0(1 - e^{2x/a})$. As mentioned above, the wave-functions at the special values of E for V_1 or V_3 are actually normalizable in spite of the fact that these potentials are both bottomless. They can be considered as static scattering states, or alternatively as rather unorthodox bound states. Because the wave-functions decrease exponentially, all their moments $\langle x^n \rangle$ are finite. However, because of the very rapid oscillations, the expectation values of moments of momenta are not. That is, for V_1 every x-derivative produces a factor of $dz/dx = \pm q e^{|x|/a}$. The expectation value $\langle p \rangle$ can be taken to be zero, being the integral of an odd function of x, but all higher moments diverge.

References

[1] Ahmed Z, Kumar S, Ghosh D and Goswami T 2019 *Phys. Script.* **94** 105214
[2] We are grateful to B. Rath for this remark.
[3] Ahmed Z, Kumar S, Ghosh D and Turumella N 2018 *Eur. J. Phys.* **39** 025404
[4] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (N. B. of Standards)
[5] Ahmed Z and Jones H F 2021 *arXiv*: 2102.06095