

A THESIS PRESENTED FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

A physical space analysis of axisymmetric solutions to the Teukolsky equation on the Kerr spacetime

Author

Georgios CHALIVOPULOS

Supervisor

Professor Gustav HOLZEGEL

Department of Mathematics
Imperial College London
180 Queen's Gate, London, SW7 2BZ

December 9, 2020

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Acknowledgements

I would like to thank Professor Gustav Holzegel for his patient supervision and guidance. I would also like to thank Gabriele Benomio for valuable comments and discussion during the drafting phase of this thesis.

Georgios Chalivopoulos

Abstract

In this thesis we prove an integrated local energy decay estimate for axisymmetric solutions to the Teukolsky equation on slowly rotating Kerr exterior backgrounds. We employ physical space transformation theory based on fixed-frequency theory of Chandrasekhar (as was introduced in [12]), resulting in a generalised Regge-Wheeler type equation. This allows us, under the assumption of axisymmetry, to use **entirely physical space methods**. This is convenient for potentially proving a non-linear stability result (restricted to axisymmetry) for slowly rotating Kerr spacetimes. We also prove an integrated local energy decay estimate for solutions to the homogeneous generalised Regge-Wheeler equation on Kerr exterior backgrounds for the full subextremal range of Kerr parameters.

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1 Introduction

An important open problem in general relativity is the stability of the Kerr family of black hole solutions of the Einstein vacuum equations

$$\text{Ric}(g) = 0. \quad (1.1)$$

The ultimate goal is to establish nonlinear stability of the Kerr family of spacetimes $(\mathcal{M}, g_{M,a})$ (which includes the Schwarzschild family $(\mathcal{M}, g_M)$ with $a = 0$) in the context of the initial value problem for (1.1). The analogous goal for the Minkowski space, the trivial solution of (1.1), was achieved in the monumental work of Christodoulou and Klainerman [10] in 1993. Since then, alternative proofs for the nonlinear stability of Minkowski space were found. For example, in 2010 Lindblad and Rodnianski gave an alternative proof utilising a harmonic gauge (see [27]).

An important intermediate step towards this goal is to prove linear stability, i.e. boundedness and decay statements for the linearisation of equation (1.1) around the Schwarzschild and Kerr solutions. In their paper [11], Dafermos, Holzegel and Rodnianski provided a complete study of the linear stability of the Schwarzschild solution. A key step was proving boundedness and decay of solutions of the so called spin ± 2 Teukolsky equation.

In their next paper [12], Dafermos, Holzegel, and Rodnianski extended their boundedness and decay results for the Teukolsky equation from the Schwarzschild case ($a = 0$) to the slowly rotating Kerr case, corresponding to the values of parameters $|a| \ll M$. One of the difficulties the authors faced passing from the $a = 0$ case to the $|a| > 0$ case was the fact that trapped null geodesics no longer approach a zero measure set $\{r = 3M\}$ in spacetime. Instead, trapping occurs in a range of values of r . This is an important obstacle because integrated local energy decay estimates (or "ILED" estimates for short) that do not lose derivatives must degenerate where trapping occurs (see [39] and also the classical [37]). The authors dealt with this obstacle by using **frequency localised analysis** in the region where trapping occurs.

In addition to [11] and [12], other important contributions towards stability of the Schwarzschild and Kerr solution have been carried out in the recent years. Namely, for Schwarzschild, see [25] and [13], and for Kerr, see [3], [24], [23].

In this thesis, we restrict to **axisymmetric solutions**. This results in the following simplifications in the analysis:

1. The usual issue of superradiance on Kerr is overcome easily (see discussion in the beginning of 1.5.1). We are therefore able to obtain a coercive energy on spacelike slices.
2. Trapped null geodesics asymptote to a single value of $r = r_{\text{trap}}$, akin to the Schwarzschild case where this value r_{trap} is equal to $3M$ (where M is the Schwarzschild parameter).
3. The Teukolsky equation and the associated transformation theory simplifies considerably, as certain terms vanish.

These simplifications allow us to give an alternative proof of boundedness and decay of axisymmetric solutions of the Teukolsky equation on Kerr, using **entirely physical space methods**. This in turn may allow one to directly employ the approach of [13] to prove a restricted (to axisymmetry) non-linear stability result for slowly rotating Kerr spacetimes.

1.1 Teukolsky and (generalised) Regge–Wheeler equations

The Teukolsky equation on a Kerr background is:

$$\begin{aligned} \mathfrak{T}^{[s]} \alpha^{[s]} := & \square_g \alpha^{[s]} + \frac{2s}{\rho^2} (r - M) \partial_r \alpha^{[s]} + \frac{2s}{\rho^2} \left(\frac{a(r - M)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right) \partial_\phi \alpha^{[s]} \\ & + \frac{2s}{\rho^2} \left(\frac{M(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right) \partial_t \alpha^{[s]} + \frac{1}{\rho^2} (s - s^2 \cot^2 \theta) \alpha^{[s]} = 0, \end{aligned} \quad (1.2)$$

where

$$\begin{aligned} \Delta &= r^2 + a^2 - 2Mr, \\ \rho^2 &= r^2 + a^2 \cos^2 \theta, \end{aligned}$$

$s \in \frac{1}{2}\mathbb{Z}$ is a spin parameter, and $g = g_{M,a}$ is the Kerr metric (see Section 2). For us, the values $s = \pm 2$ will be important, where $\alpha^{[\pm 2]}$ are spin ± 2 weighted complex scalar valued functions (see Section 2.2.1 of [12]).

To introduce the history of the Teukolsky equation, one must first step back and discuss briefly the history of the study of the equations of gravitational perturbations around Schwarzschild (meaning the linearisation of (1.1) around Schwarzschild) and Kerr families of solutions. This study for the Schwarzschild case was initiated by Regge and Wheeler in 1957 (see [34]) and continued in 1970 by Vishveshwara [43] and Zerilli [46]. An alternative approach using the Newman-Penrose formalism (see [31]) was introduced in 1973 by Bardeen and Press [6] for the Schwarzschild family, and extended to the Kerr family by Saul Teukolsky [42] in the same year. A major upshot of these works was the realization that various curvature components (with respect to a null frame) satisfy a decoupled wave equation - the now called Teukolsky equation.

In particular, Teukolsky identified two gauge invariant quantities that decouple from the system obtained by linearisation of (1.1) around the general Kerr solution. These quantities correspond to the extremal curvature components in the Newman-Penrose formalism and can be represented by (spin ± 2 weighted) complex functions $\alpha^{[\pm 2]}$ satisfying respectively the spin ± 2 Teukolsky equation (1.2).

Other values of spin s are also important.

1.1.1 The case $s = 0$ (the wave equation)

In the case $s = 0$, the equation (1.2) reduces to the homogeneous covariant wave equation on a fixed Kerr background

$$0 = \square_g \alpha := \nabla^\mu \nabla_\mu \alpha = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \alpha), \quad (1.3)$$

where $\alpha = \alpha^{[0]}$ is now a scalar. Equation (1.3) carries a significant amount of importance in its own right, as it can be viewed as a poor man's analogue for the more complicated problem of gravitational perturbations. A lot of work has been done on equation (1.3).

In 1987, Kay and Wald [26] proved boundedness for solutions of the scalar wave equation on Schwarzschild (i.e. (1.2) with $s = a = 0$) posed as a Cauchy problem with regular localised initial data. Already in this early work, non-trivial issues arise on the event horizon. This has to do with the fact that the Killing field ∂_t , which is timelike in the exterior of Schwarzschild, is in fact null on the event horizon, leading to a bounded energy that degenerates at the horizon. These issues have been understood in a more robust way using the red-shift energy identity (see [17] and also the notes [21]).

The $s = 0$ case has seen even more activity from the early 2000s onwards (see for example [4, 14, 5, 17, 18, 19, 35, 21, 2, 22]) that culminated in 2016 (see [22]) in boundedness and decay results for (1.3) in the full subextremal range of Kerr parameters $|a| < M$.

There are two main difficulties when passing from the $a = 0$ to the $|a| > 0$, namely superradiance and the fact that null geodesics no longer approach a single value of r (as is the case in the $a = 0$ case with $r = 3M$). The latter is an issue because integrated local energy decay estimates, an important step for proving decay of solutions, must necessarily degenerate at trapping (if one wants to avoid losing derivatives). On the other hand, superradiance (first discussed by Zeldovich in 1971 [45]), is the phenomenon that the energy flux at null infinity can be larger than the initial energy. This is due to the fact that there exist a region (so called ergoregion) in the black hole exterior (close to the event horizon) where the Killing field ∂_t is actually spacelike, and thus ∂_t no longer gives rise to a statement of boundedness of a non-negative energy. It should be noted that the issues of superradiance were overcome in the slowly rotating case $|a| \ll M$, exploiting the fact that the terms arising from ∂_t not being everywhere causal are small and can be absorbed by good terms one obtains from utilising the red-shift energy identity near the event horizon.

We now address the issue of trapped null geodesics. In the Schwarzschild case ($a = 0$), trapping of null geodesics occurs at the codimension-1 hypersurface $\{r = 3M\}$ in physical space called the photon sphere. The fact that this is a single value of r allowed for recovering an integrated local energy decay estimate (also called Morawetz estimate, named after Cathleen Morawetz [30] who proved such estimates in the context of scattering of waves by obstacles) by appealing to an energy identity associated with the multiplier vector field $f(r)\partial_r$, where the function f is chosen to vanish at $r = 3M$. As mentioned before, this estimate is necessarily degenerate at $r = 3M$.

In the slowly rotating case $|a| \ll M$, trapped null geodesics exist for a range of r -values in a neighbourhood of $r = 3M$. This makes it impossible (see [1]) to use a multiplier vector field in physical space to obtain an ILED estimate. This was overcome by constructing frequency-localised generalisations of physical space multipliers resulting in frequency localised Morawetz estimates ([18, 22]). For this one employs Carter's separation [8] of the equation (1.3), as discussed in detail in section 8.1 of [19]. Importantly, for fixed frequency, trapping does happen at a single value of r .

In the full subextremal case $|a| < M$ the issue of superradiance poses a greater difficulty since it can no longer be treated as a small error and is a priori coupled to the problem of trapped null geodesics. However, these issues were overcome in [22]. The fact that superradiant frequencies happen not to be trapped and the results of [40] allowed the authors of [22] to extend energy boundedness and ILED estimates from the $a = 0$ case to the whole subextremal case $|a| < M$.

We should note that in the special case of axisymmetric solutions, superradiance in fact does not play a role. While there exists an ergoregion in the underlying spacetime, the energy fluxes on the event horizon and at spacelike slices at later times can be shown to be nonnegative and one in fact can obtain an energy boundedness statement. Moreover, axisymmetric geodesics asymptote to a single value of $r = r_{\text{trap}}$ (which is now different from $3M$) which suggests that a Morawetz estimate can be obtained from a physical space vector field multiplier, akin to Schwarzschild.

1.1.2 The cases $s = \pm 1$

The cases of (1.2) with $s = \pm 1$ were studied in 2016 by Pasqualotto [33] as part of an alternative proof of boundedness and polynomial decay for the Maxwell equations on Schwarzschild (which was first proven in 2008 by Blue [7]). In 2017, Ma (see [28]) extended these results to the case of Maxwell equations on (exterior) Kerr (for range of parameters $|a| \ll M$), proving a Morawetz estimate for quantities (corresponding to extremal curvature components in the Newman-Penrose formalism) satisfying equation (1.2) with spin $s = \pm 1$.

1.1.3 The cases $s = \pm 2$

The cases $s = \pm 2$ (for $a = 0$) were studied by Dafermos, Holzegel and Rodnianski, as part of their study of linear stability of the Schwarzschild solution to gravitational perturbations [11]. The main difficulty of the $s = \pm 2$ case (as opposed to the $s = 0$ case, say) is that there does not exist, for equation (1.2), an obvious analogue for conserved (coercive) energy associated to the Killing field ∂_t . To overcome this problem, Dafermos, Holzegel, and Rodnianski employed particular transformations, as follows:

$$\begin{aligned}
P^{[+2]} &= \frac{-1}{2(r-2M)} \underline{L} \left(\frac{r^3}{r-2M} \underline{L} \left(\frac{(r-2M)^2}{r} \alpha^{[+2]} \right) \right), \\
P^{[-2]} &= \frac{-1}{2(r-2M)} L \left(\frac{r^3}{r-2M} L \left(r^{-3} \alpha^{[-2]} \right) \right),
\end{aligned} \tag{1.4}$$

where $L = \partial_t - \partial_{r^*}$ and $\underline{L} = \partial_t + \partial_{r^*}$ are elements of a null frame, and r^* is the Regge–Wheeler coordinate, and ∂_{r^*} is defined with respect to the $(t, r^*, \theta, \varphi)$ coordinates on Schwarzschild. These transformations can be viewed as physical space analogues of the fixed-frequency transformation theory introduced by Chandrasekhar [9] in the context of mode decoupled solutions.

The quantities

$$\begin{aligned}
\Psi^{[+2]} &= r^3 P^{[+2]}, \\
\Psi^{[-2]} &= r^3 P^{[-2]},
\end{aligned}$$

were then be shown to satisfy the Regge–Wheeler equation

$$L \underline{L} \Psi^{[\pm 2]} + \frac{\Omega^2}{r^2} \left(\overset{\circ}{\Delta}^{[\pm 2]} \pm 2 \right) \Psi^{[\pm 2]} + \Omega^2 \left(\frac{4}{r^2} - \frac{6M}{r^3} \right) \Psi^{[\pm 2]} = 0, \tag{1.5}$$

where

$$\Omega^2 = 1 - \frac{2M}{r},$$

and $\overset{\circ}{\Delta}^{[\pm 2]}$ is the spin- ± 2 -weighted Laplacian operator on the unit sphere

$$\overset{\circ}{\Delta}^{[\pm 2]} = -\frac{1}{\sin \theta} \partial_\theta \left(\sin \theta \partial_\theta \right) - \frac{1}{\sin^2 \theta} \partial_\varphi^2 - 2(\pm 2)i \frac{\cos \theta}{\sin^2 \theta} \partial_\varphi + 4 \cot^2 \theta \mp 2.$$

Now, equation (1.5) in fact has the usual good structure for energy estimates and thus boundedness and integrated local energy decay estimates were obtained for $\Psi^{[\pm 2]}$. Estimates for $\alpha^{[\pm 2]}$ were subsequently obtained via integrating the transformation relations (1.4) as transport equations and employing the Teukolsky equation itself as an elliptic relation to mitigate loss of derivatives. This led to integrated local energy decay and energy boundedness estimates for $\alpha^{[\pm 2]}$, and hence pointwise decay using the r^p energy hierarchy akin to [16].

In 2017 (see [12]), Dafermos, Holzegel, and Rodnianski extended their analysis of the $s = \pm 2$ Teukolsky equation from the Schwarzschild case ($a = 0$) to the slowly rotating Kerr case ($|a| \ll M$), proving boundedness and decay of solutions. The rough statement of the theorem they proved is as follows:

Theorem 1.6 (Dafermos, Holzegel, Rodnianski, 2017). *Fix $M > 0$. There exists $0 < a_0 = a_0(M) \ll M$ such that for all $|a| < a_0$ we have the following:*

Solutions $\alpha^{[\pm 2]}$ to the spin $s = \pm 2$ Teukolsky equation (1.2) on Kerr exterior spacetimes $(\mathcal{M}, g_{a,M})$ arising from regular localised initial data on a Cauchy hypersurface Σ_0 remain uniformly bounded and satisfy an r^p -weighted energy hierarchy and polynomial decay.

Passing from the $a = 0$ to the $a \ll M$ case, the authors faced difficulties analogous to the ones encountered in the $s = 0$ case of wave equation (1.3). The smallness of a helps overcoming superradiance and the obstructions to decay posed by trapped null geodesics were overcome by utilising frequency localised energy estimates akin to [18, 22].

The transformed quantities $P^{[\pm 2]}$ of [11] from the Schwarzschild case were generalised to the Kerr case as follows:

$$\begin{aligned}
P^{[+2]} &= -\frac{(r^2 + a^2)^{\frac{1}{2}}}{2\Delta} \underline{L} \left(\frac{(r^2 + a^2)^2}{\Delta} \underline{L} \left(\Delta^2 (r^2 + a^2)^{-\frac{3}{2}} \alpha^{[+2]} \right) \right), \\
P^{[-2]} &= -\frac{(r^2 + a^2)^{\frac{1}{2}}}{2\Delta} L \left(\frac{(r^2 + a^2)^2}{\Delta} L \left((r^2 + a^2)^{-\frac{3}{2}} \alpha^{[-2]} \right) \right).
\end{aligned} \tag{1.7}$$

The rescaled quantities $\Psi^{[\pm 2]} = (r^2 + a^2)^{\frac{3}{2}} P^{[\pm 2]}$ can then be shown to satisfy equations (analogous to (1.5)) of the form

$$\begin{aligned}\Re^{[+2]} \Psi^{[+2]} &= c_1^{[+2]}(r) \partial_\varphi \left(\underline{L} \alpha^{[+2]} \right) + c_2^{[+2]}(r) \underline{L} \alpha^{[+2]} + c_3^{[+2]}(r) \partial_\varphi \alpha^{[+2]} + c_4^{[+2]}(r) \alpha^{[+2]}, \\ \Re^{[-2]} \Psi^{[-2]} &= c_1^{[-2]}(r) \partial_\varphi \left(L \alpha^{[-2]} \right) + c_2^{[-2]}(r) L \alpha^{[-2]} + c_3^{[-2]}(r) \partial_\varphi \alpha^{[-2]} + c_4^{[-2]}(r) \alpha^{[-2]},\end{aligned}\tag{1.8}$$

where $\Re^{[\pm 2]}$ are second order differential operators generalising the Regge–Wheeler operators (cases ± 2 respectively) appearing on the left hand side of equation (1.5), and the weights $c_i^{[\pm 2]}(r)$ are proportional to a . Once again, these have good divergence properties that allow one to derive energy estimates. For the precise form of equations (1.8), see Proposition 4.8 and equation (4.9).

As opposed to equation (1.5), where one enjoyed total decoupling from the Teukolsky equation (1.2), equation (1.8) now has a nonzero right hand side. However, the right side is small in a since the weights $c_i^{[\pm 2]}$ are proportional to a . Moreover, one can integrate the relations (1.7) (and their commuted versions with Killing fields $\partial_t, \partial_\varphi$) in order to obtain transport estimates that allow one to estimate spacetime integrals of terms involving $\alpha^{[\pm]}$, $L\alpha^{[-2]}$, $\underline{L}\alpha^{[+2]}$ and their derivatives by spacetime integrals of terms involving $\Psi^{[\pm 2]}$.

Note that in the axisymmetric case, the right hand side of equation (1.2) is of even lower order, since the ∂_φ derivatives disappear. As we shall see, this eliminates the need for commuted versions of transport estimates.

In addition, in 2017, independently of [12], Ma [29] gave a proof of integrated local energy decay estimates (key ingredient to proving decay) for the $s = \pm 2$ Teukolsky equation (1.2) in the $|a| \ll M$ case. This work also used frequency localised analysis in a similar way as [12].

Mode stability for the Teukolsky equation was recently addressed in [36]. See also the classical [44].

Recently, the authors of [41] studied the boundedness and decay of solutions to the Teukolsky equation on full subextremal Kerr backgrounds. In contrast to this thesis, the authors of [41] used frequency space analysis. The physical space approach of this thesis may prove convenient for nonlinear applications.

1.2 The axisymmetric case

We now make the assumption of axisymmetry, i.e. that we have

$$\partial_\varphi \alpha^{[s]} = 0.$$

Thus the equation (1.2) becomes

$$\square_g \alpha^{[s]} + \frac{2s}{\rho^2} (r - M) \partial_r \alpha^{[s]} + \frac{2s}{\rho^2} \left(\frac{M(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right) \partial_t \alpha^{[s]} + \frac{1}{\rho^2} (s - s^2 \cot^2 \theta) \alpha^{[s]} = 0.\tag{1.9}$$

Given $\alpha^{[\pm 2]}$, recall we have the transformations (1.7). Let us also label the intermediate quantities $\psi^{[\pm 2]}$, i.e. we can write:

$$\begin{aligned}\psi^{[+2]} &:= -\frac{1}{2} \Delta^{-\frac{3}{2}} (r^2 + a^2)^2 \underline{L} \left(\Delta^2 (r^2 + a^2)^{-\frac{3}{2}} \alpha^{[+2]} \right), \\ \psi^{[-2]} &:= +\frac{1}{2} \Delta^{-\frac{3}{2}} (r^2 + a^2)^2 L \left((r^2 + a^2)^{-\frac{3}{2}} \alpha^{[-2]} \right), \\ \Psi^{[+2]} &:= \frac{(r^2 + a^2)^2}{\Delta} \underline{L} \left(\sqrt{\Delta} \psi^{[+2]} \right), \\ \Psi^{[-2]} &:= -\frac{(r^2 + a^2)^2}{\Delta} L \left(\sqrt{\Delta} \psi^{[-2]} \right),\end{aligned}\tag{1.10}$$

where

$$L = \partial_{r^*} + T + \frac{a}{r^2 + a^2} \Phi$$

and

$$\underline{L} = -\partial_{r^*} + T + \frac{a}{r^2 + a^2} \Phi.$$

Here, the vector fields are $T = \partial_t$ and $\Phi = \partial_\varphi$ (see Section 2).

Remark 1.11. Note that the assumption of axisymmetry easily implies that $\partial_\varphi \psi^{[\pm 2]} = 0$ and $\partial_\varphi \Psi^{[\pm 2]} = 0$.

Remark 1.12. We have (note, the prime "'" denotes derivative with respect to r^*)

$$(L - \underline{L})\Psi^{[\pm 2]} = 2 \left(\Psi^{[\pm 2]} \right)'$$

and, under the assumption of axisymmetry, also

$$(L + \underline{L})\Psi^{[\pm 2]} = 2T\Psi^{[\pm 2]}.$$

It can be then shown that the quantities $\Psi^{[\pm 2]}$ satisfy the inhomogeneous Regge–Wheeler type equation

$$\mathfrak{R}^{[\pm 2]}\Psi^{[\pm 2]} = -\frac{\rho^2}{\Delta} \left(\mathfrak{J}^{[\pm 2]} \right),$$

where the operator $\mathfrak{R}^{[s]}$ is called the generalised Regge–Wheeler operator and is defined as

$$\begin{aligned} \Delta \rho^{-2} \mathfrak{R}^{[s]} := & \frac{1}{2} (L\underline{L} + \underline{L}L) + w \left[\left(\overset{\circ}{\Delta}^{[s]} + s^2 + s \right) - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right] \\ & - w(2aT\Phi + a^2 \sin^2 \theta TT - 2isa \cos \theta T), \end{aligned}$$

and the terms on the right hand sides are of the form

$$\begin{aligned} -\frac{r^2 + a^2}{\Delta} \mathfrak{J}^{[+2]} &= a^2 h_1 \cdot \left(\sqrt{\Delta} \psi^{[+2]} \right) + a^2 h_2 \cdot \left(\alpha^{[+2]} \Delta^2 (r^2 + a^2)^{-\frac{3}{2}} \right), \\ -\frac{r^2 + a^2}{\Delta} \mathfrak{J}^{[-2]} &= a^2 h_1 \cdot \left(\sqrt{\Delta} \psi^{[-2]} \right) + a^2 h_2 \cdot \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right), \end{aligned}$$

where h_1, h_2 are functions satisfying the following:

There exist constants $c_1, C_1, c_2, C_2 > 0$ such that for all $r \geq r_+$:

$$\begin{aligned} c_1 &\leq |h_1(r)| \cdot r^3 \leq C_1, \\ c_2 &\leq |h_2(r)| \cdot r^2 \leq C_2. \end{aligned}$$

Hence, the Regge–Wheeler type equation we work with in the axisymmetric case is of the form

$$\begin{aligned} \frac{1}{2} (L\underline{L} + \underline{L}L) \Psi^{[s]} + w \left(\overset{\circ}{\Delta}^{[s]} + s \right) \Psi^{[s]} + w \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right) \Psi^{[s]} \\ - w(a^2 \sin^2 \theta TT - 2isa \cos \theta T) \Psi^{[s]} = -\mathfrak{J}^{[s]}. \end{aligned} \tag{1.13}$$

for $s = \pm 2$.

Remark 1.14. Note that in the case $a = 0$ (corresponding to the Schwarzschild background), the equation (1.13) simplifies. Firstly, the right hand side disappears, making the equation homogeneous in Ψ . Secondly, the problematic terms $w(a^2 \sin^2 \theta TT + 4ia \cos \theta T) \Psi$ containing the second derivative $TT\Psi$ disappear too. What is left is a principal part and a potential term. Thus the resultant equation in the $a = 0$ case demonstrably manifests good structure, permitting energy boundedness and a subsequent Morawetz estimate.

We are also interested in the homogeneous version of equation (1.13), where we keep $a > 0$ but set the right hand side to 0:

$$\begin{aligned} \frac{1}{2}(L\bar{L} + \bar{L}L)\Psi^{[s]} + w\left(\overset{\circ}{\Delta}^{[s]} + s\right)\Psi^{[s]} + w\left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2\Delta}{(r^2 + a^2)^2}\right)\Psi^{[s]} \\ - w(a^2 \sin^2 \theta TT - 2isa \cos \theta T)\Psi^{[s]} = 0. \end{aligned} \quad (1.15)$$

Note this is a linear equation purely in $\Psi^{[s]}$, with no coupling to lower order terms $\alpha^{[s]}$ and $\psi^{[s]}$. This equation can be thought of as an analogue of (1.3) as a "poor man's" model for the linearisation of (1.1), under the assumption of axisymmetry. We shall be able to prove a Morawetz estimate for equation (1.15) in the full subextremal range of parameters $|a| < M$, see Theorem 1.17 below.

1.3 Rough statements of main results

We state rough versions of main results, that are analogous to Theorem 1.6. For precise statements of main results, see Theorems 4.20 and 4.21.

Theorem 1.16. *Fix $M > 0$. There exists $0 < a_0 = a_0(M) < M$ such that for all $|a| < a_0$ the following holds:*

*Let $\alpha^{[-2]}$ be a solution to the spin $s = -2$ Teukolsky equation (1.9) on Kerr exterior spacetime $(\mathcal{M}, g_{a,M})$ arising from regular smooth localised **axisymmetric** initial data on a Cauchy hypersurface Σ_0 . Let $\psi^{[-2]}, \Psi^{[-2]}$ be as defined by transformations (4.7). Then we have an ILED estimate of the form*

$$\begin{aligned} \int_{\mathcal{R}(\tau_1, \tau_2)} \mathbb{S}[\Psi^{[-2]}, \psi^{[-2]}, \alpha^{[-2]}] dt dr d\omega + \int_{\Sigma_{\tau_2}} \mathbb{E}[\Psi^{[-2]}, \psi^{[-2]}, \alpha^{[-2]}] dr d\omega + \int_{\mathcal{H}^+(\tau_1, \tau_2)} |(L + \bar{L})\Psi^{[-2]}|^2 dV_{\mathcal{H}^+} \\ \leq C(M) \int_{\Sigma_{\tau_1}} \mathbb{E}[\Psi^{[-2]}, \psi^{[-2]}, \alpha^{[-2]}] dr d\omega, \end{aligned}$$

where $\mathbb{S}[\Psi^{[-2]}, \psi^{[-2]}, \alpha^{[-2]}]$ and $\mathbb{E}[\Psi^{[-2]}, \psi^{[-2]}, \alpha^{[-2]}]$ are coercive energy densities involving $\alpha^{[-2]}, \psi^{[-2]}, \Psi^{[-2]}$ and derivatives of $\Psi^{[-2]}$ (see Section 4.3 for definitions of energies).

Theorem 1.17. *Let $|a| < M$ and assume $s = \pm 2$. Let $\Psi^{[s]}$ be a solution to the spin s homogeneous generalised Regge–Wheeler equation (1.15) on Kerr exterior spacetime $(\mathcal{M}, g_{a,M})$ arising from regular smooth localised **axisymmetric** initial data on a Cauchy hypersurface Σ_0 . Then we have an ILED estimate of the form*

$$\begin{aligned} \int_{\mathcal{R}(\tau_1, \tau_2)} \mathbb{S}[\Psi^{[s]}] dt dr d\omega + \int_{\Sigma_{\tau_2}} \mathbb{E}[\Psi^{[s]}] dr d\omega + \int_{\mathcal{H}^+(\tau_1, \tau_2)} |(L + \bar{L})\Psi^{[s]}|^2 dV_{\mathcal{H}^+} \\ \leq C(M) \int_{\Sigma_{\tau_1}} \mathbb{E}[\Psi^{[s]}] dr d\omega, \end{aligned}$$

where $\mathbb{E}[\Psi^{[s]}], \mathbb{S}[\Psi^{[s]}]$ are coercive energy densities depending only on $\Psi^{[s]}$ and derivatives of $\Psi^{[s]}$.

Remark 1.18. Theorems 1.16 and 1.17 provide integrated local energy decay estimates for solutions of equations (1.9) and (1.15) respectively. This allows one to invoke the method (also based on purely physical space arguments) of Dafermos and Rodnianski (introduced in [16]) to obtain also decay statements for solutions of these equations.

Remark 1.19. The estimates in Theorems 1.16 and 1.17 degenerate at the event horizon (see also Theorem 4.20 and Theorem 4.21). One can, however, using a red-shift multiplier (see [17]), obtain versions of these estimates without this degeneration.

Remark 1.20. To prove Theorems 1.16 and 1.17, only physical space methods are used, which is possible due to the assumption of axisymmetry. Hence, Theorem 1.16 can be thought of as an analogue to Theorem 1.6 of [12], where we restrict to the axisymmetric case and provide a much easier proof. On the other hand, Theorem 1.17 allows us to extend to the whole subextremal range of parameters $a < M$, if we restrict to the homogeneous case.

Remark 1.21. One can also think of these theorems as respectively overcoming two main difficulties. Theorem 1.16 overcomes (in the small a regime) the issue of error terms coming from the right hand side of equation (1.13). On the other hand, Theorem 1.17 focuses on overcoming the difficulty arising from the presence of the term

$$-w \left(a^2 \sin^2 \theta TT - 2isa \cos \theta T \right) \Psi^{[s]},$$

which appears on the left hand side of equation (1.13). Theorem 1.17 also gives us a way to obtain extra positivity which one could potentially use to absorb error terms (coming from a right hand side of (1.13)) for larger values of a . The next step would be to see how much can be absorbed, and (in an ideal scenario) extend Theorem 1.16 to a result about the whole range $a < M$.

Remark 1.22. Regarding equation (1.15) as an analogue of the wave equation (1.3) as a poor man's linearisation of (1.1) (under axisymmetry), one can compare Theorem 1.17 to previous results for the wave equation. For the study of axisymmetric solutions to the wave equation on Kerr exterior backgrounds (using frequency decomposition), see [18]. The advantage of the setting of equation (1.15) is the presence of a Poincaré inequality for spin ± 2 complex valued functions - see (1.35) and also page 18 of [12].

Remark 1.23. An estimate similar to Theorem 1.16 may be proven for the $s = +2$ case using similar methods. We will just include the $s = -2$ case in this thesis. In the $s = +2$ case, the right hand side of equation (1.13) has a different form, containing terms involving $\alpha^{[+2]}$ and $\psi^{[+2]}$ instead of $\alpha^{[-2]}$ and $\psi^{[-2]}$, and energies exhibit different weights at the horizon and null infinity. Note, the quantities $\alpha^{[-2]}$ and $\alpha^{[+2]}$ are in fact non-trivially related to each other. For fixed frequency solutions, these relations are well known and often referred to as the Teukolsky-Starobinskii identities (see [38]). However, we shall not be using these relations in this thesis.

Remark 1.24. Theorem 1.17 in fact extends to the extremal Kerr case $|a| = M$, in which case the energy densities exhibit an additional degeneration at the event horizon (see Theorem 4.21 for details).

1.4 Notation for estimates

We define some shorthand notation that we use for estimates.

For two expression E_1 and E_2 , we write

$$E_1 \lesssim E_2$$

if there exists a constant $C > 0$, only (potentially) dependent on M (but not on a), such that

$$E_1 \leq C \cdot E_2.$$

Similarly, we write

$$E_1 \sim E_2$$

if there exist constants $c, C > 0$, only (potentially) dependent on M , such that

$$c|E_2| \leq |E_1| \leq C|E_2|.$$

For example, for a function $F = F(r)$, we will often write

$$F \sim \frac{1}{r^2}$$

if there exist constants $c, C > 0$, only (potentially) dependent on M , such that

$$\frac{c}{r^2} \leq |F(r)| \leq \frac{C}{r^2}$$

for all $r \geq r_+$ (here r_+ is the location of the event horizon, so the range $r \geq r_+$ corresponds to the exterior of the Kerr black hole - see Section 2 for detail).

1.5 Outline of the proofs of main results

In this section we outline rough sketch proofs of the main results. While this is not a precise treatment, it demonstrates the overall logic of the proof.

We choose the foliation by constant t^* (for definition of coordinates on Kerr, see Section 2) slices - for a value $\tau \geq 0$ define

$$\Sigma_\tau := \{t^* = \tau\}.$$

For fixed values $\tau_2 \geq \tau_1 \geq 0$, define the spacetime region

$$\mathcal{R}(\tau_1, \tau_2) := \bigcup_{\tau_1 \leq \tau \leq \tau_2} \Sigma_\tau.$$

For the reader familiar with the concept of Penrose diagram, the following picture in Figure 1 will be instructive.

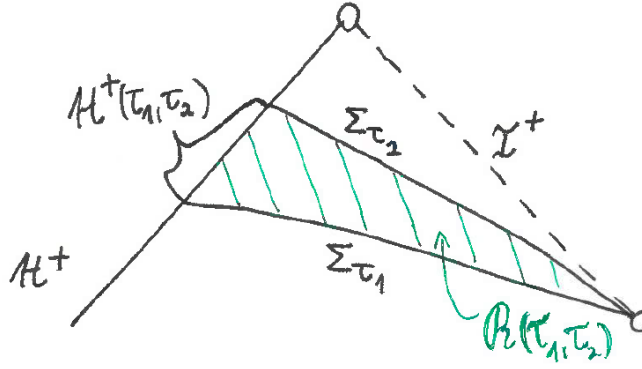


Figure 1: Penrose diagram, exterior of a Kerr black hole foliated by spacelike slices extending to the event horizon.

1.5.1 Energy boundedness (up to error)

We outline how we obtain an energy boundedness (up to error) estimate for equation (1.13). For details see Section 6.

First, we briefly revisit the issue of superradiance. As mentioned before, Kerr spacetimes exhibit the presence of an ergoregion - a region of spacetime where the vector field T fails to be causal, defined by the set $\{\Delta - a^2 \sin^2 \theta < 0\}$. Generally, this presents an obstacle for obtaining an energy boundedness estimate for a non-negative energy on spacelike slices (see section 2.2.1). However, under axisymmetry, despite the presence of an ergoregion in the underlying spacetime, one does enjoy an energy boundedness statement for a non-negative energy. We can easily illuminate this mechanism in the case of the wave equation (1.3), where one has the well-known energy-momentum tensor of the form

$$\mathbb{T}_{\mu\nu}[\alpha] = \partial_\mu \alpha \partial_\nu \alpha - \frac{1}{2} g_{\mu\nu} g^{\beta\gamma} \partial_\beta \alpha \partial_\gamma \alpha. \quad (1.25)$$

The energy flux on spacelike hypersurfaces Σ_τ is

$$\mathbb{T}[\alpha](T, n_\Sigma) = \mathbb{T}[\alpha]\left(\frac{1}{2}(L + \underline{L}), n_\Sigma\right) - \frac{a}{r^2 + a^2} \mathbb{T}[\alpha](\Phi, n_\Sigma). \quad (1.26)$$

The vector fields $\frac{1}{2}(L + \underline{L})$ and n_Σ are both future directed causal in the entire exterior of the black hole $\{r \geq r_+\}$, thus the first term on the right hand side of (1.26) is non-negative in the whole exterior. Moreover, under axisymmetry, the second term is zero, since $\Phi\alpha = 0$ and $g(\Phi, n_\Sigma) = 0$. Hence the non-negativity of the energy flux $\mathbb{T}[\alpha](T, n_\Sigma)$ follows immediately.

If we actually calculate g^{-1} , we can derive

$$\begin{aligned} \mathbb{T}[\alpha](T, n_\Sigma) = & \frac{1}{\sqrt{1 + \frac{2Mr}{\rho^2}}} \left[\frac{1}{2} \left(1 + \frac{2Mr}{\rho^2} \right) (\partial_{t^*} \alpha)^2 + \frac{1}{2\rho^2} (\partial_\theta \alpha)^2 \right. \\ & \left. + \frac{1}{2\rho^2} \left(a \sin \theta \partial_r \alpha + \frac{1}{\sin \theta} \Phi \alpha \right)^2 + \frac{\Delta - a^2 \sin^2 \theta}{2\rho^2} (\partial_r \alpha)^2 \right]. \end{aligned} \quad (1.27)$$

We can see the quantity $\Delta - a^2 \sin^2 \theta$ in the coefficient of $(\partial_r \alpha)^2$ is precisely the quantity that is negative in the ergoregion. Under axisymmetry, the $\Phi \alpha$ term vanishes, and hence the flux reduces to

$$\mathbb{T}[\alpha](T, n_\Sigma) = \frac{1}{\sqrt{1 + \frac{2Mr}{\rho^2}}} \left[\frac{1}{2} \left(1 + \frac{2Mr}{\rho^2} \right) (\partial_{t^*} \alpha)^2 + \frac{1}{2\rho^2} (\partial_\theta \alpha)^2 + \frac{\Delta}{2\rho^2} (\partial_r \alpha)^2 \right], \quad (1.28)$$

which is non-negative for all $r \geq r_+$.

The flux on a section of the event horizon $\mathcal{H}^+(\tau_1, \tau_2)$ can be shown to be non-negative using a similar argument. In fact, one computes

$$\mathbb{T}[\alpha](T, n_{\mathcal{H}^+}) = (\partial_{t^*} \alpha)^2 \quad (1.29)$$

on $\mathcal{H}^+$, for axisymmetric α .

The non-negativity of the above fluxes means that we indeed obtain an unconditional energy boundedness statement.

We return to addressing the matter of energy boundedness for equation (1.13). Multiplying (1.13) by $T\overline{\Psi^{[s]}}$ taking real parts and integrating, we obtain, after applying the Divergence theorem, an energy boundedness estimate of the form

$$\begin{aligned} & \int_{\Sigma_{\tau_2}} \mathbb{E}[\Psi^{[s]}] dr d\omega + \int_{\mathcal{H}^+(\tau_1, \tau_2)} |(L + \underline{L})\Psi^{[s]}|^2 dV_{\mathcal{H}^+} \\ & \lesssim \int_{\Sigma_{\tau_1}} \mathbb{E}[\Psi^{[s]}] dr d\omega + \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[s]} \cdot T\overline{\Psi^{[s]}} \right\} dt^* dr d\omega, \end{aligned} \quad (1.30)$$

where

$$\mathbb{E}[\Psi^{[s]}] \sim |L\Psi^{[s]}|^2 + \frac{r^2 + a^2}{\Delta} |\underline{L}\Psi^{[s]}|^2 + \frac{1}{r^2} \left| \overset{\circ}{\nabla}^{[s]} \Psi^{[s]} \right|^2 + \frac{1}{r^2} |\Psi^{[s]}|^2.$$

Remark 1.31. Note, we may rewrite the term

$$\frac{r^2 + a^2}{\Delta} |\underline{L}\Psi^{[s]}|^2 = (r^2 + a^2) \Delta \left| \Delta^{-1} \underline{L}\Psi^{[s]} \right|^2$$

and since $\Delta^{-1} \underline{L}$ has a finite limit on $\mathcal{H}^+$, we conclude that this term degenerates on the horizon.

Remark 1.32. We see that (1.30) is only an energy boundedness estimate up to an error. However, we can see that this error is proportional to a^2 . Note, in the case of the homogeneous equation (1.15) the error term is not present and we have an unconditional energy boundedness statement.

1.5.2 Provisional Morawetz estimate

We outline how we obtain a preliminary Morawetz estimate that we use to prove the main results. See Section 7 for more details.

We multiply equation (1.13) by $f\overline{\Psi^{[s]}}' + f'\overline{\Psi^{[s]}}$ (for an appropriate function $f = f(r)$), and, after taking real parts, integrating, and applying Divergence theorem, we obtain an estimate of the form

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ c_{(L-\underline{L})\Psi}^f(r) \left| (L-\underline{L})\Psi^{[s]} \right|^2 + c_{\overset{\circ}{\nabla}\Psi}^f(r) \left| \overset{\circ}{\nabla}\Psi^{[s]} \right|^2 + c_{\Psi}^f \left| \Psi^{[s]} \right|^2 \right\} dt^* dr d\omega \\ & \leq \int_{\Sigma_{\tau_1}} \text{Terms}_{\Sigma}^f \left[\Psi^{[s]} \right] dr d\omega - \int_{\Sigma_{\tau_2}} \text{Terms}_{\Sigma}^f \left[\Psi^{[s]} \right] dr d\omega + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \text{Terms}_{\mathcal{H}^+}^f \left[\Psi^{[s]} \right] dV_{\mathcal{H}^+} \\ & \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathcal{I}^{[s]} \cdot \left(f'\overline{\Psi^{[s]}} + 2f\overline{\Psi^{[s]}}' \right) \right\} dt^* dr d\omega, \end{aligned} \quad (1.33)$$

where the coefficients $c_{(L-\underline{L})\Psi}^f$, $c_{\overset{\circ}{\nabla}\Psi}^f$, c_{Ψ}^f are non-negative functions dependent only on r .

Note that in (1.33), we lack control on the $(L+\underline{L})\Psi^{[s]}$ derivative. In order to gain control on the $(L+\underline{L})\Psi^{[s]}$ derivative, we multiply equation (1.13) by $y\overline{\Psi^{[s]}}'$, for a suitable function $y = y(r)$. After taking real parts, integrating and applying Divergence theorem, we obtain the following estimate:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ c_{(L+\underline{L})\Psi}^y(r) \left| (L+\underline{L})\Psi^{[s]} \right|^2 + c_{(L-\underline{L})\Psi}^y(r) \left| (L-\underline{L})\Psi^{[s]} \right|^2 \right. \\ & \quad \left. + c_{\overset{\circ}{\nabla}\Psi}^y(r) \left| \overset{\circ}{\nabla}\Psi^{[s]} \right|^2 + c_{\Psi}^y(r) \left| \Psi^{[s]} \right|^2 \right\} dt^* dr d\omega \\ & \leq \int_{\Sigma_{\tau_1}} \text{Terms}_{\Sigma}^y \left[\Psi^{[s]} \right] dr d\omega - \int_{\Sigma_{\tau_2}} \text{Terms}_{\Sigma}^y \left[\Psi^{[s]} \right] dr d\omega + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \text{Terms}_{\mathcal{H}^+}^y \left[\Psi^{[s]} \right] dV_{\mathcal{H}^+} \\ & \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathcal{I}^{[s]} \cdot 2y\overline{\Psi^{[s]}}' \right\} dt^* dr d\omega. \end{aligned} \quad (1.34)$$

We now add (1.33) and (1.34). We also use the Poincaré inequality (see page 18 of [12])

$$\int_0^\pi \int_0^{2\pi} d\varphi d\theta \sin \theta \left| \overset{\circ}{\nabla}\Psi^{[s]} \right|^2 \geq 2 \int_0^\pi \int_0^{2\pi} d\varphi d\theta \sin \theta \left| \Psi^{[s]} \right|^2 \quad (1.35)$$

to obtain more positivity for the coefficient of the $\left| \Psi^{[s]} \right|^2$ term.

We obtain:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ c_{(L+\underline{L})\Psi}(a, r) \left| (L+\underline{L})\Psi^{[s]} \right|^2 + c_{(L-\underline{L})\Psi}(a, r) \left| (L-\underline{L})\Psi^{[s]} \right|^2 + c_{\Psi}(a, r) \left| \Psi^{[s]} \right|^2 \right\} dt^* dr d\omega \\ & \lesssim \int_{\Sigma_{\tau_1}} \left\{ \text{Terms}_{\Sigma}^f \left[\Psi^{[s]} \right] + \text{Terms}_{\Sigma}^y \left[\Psi^{[s]} \right] \right\} dr d\omega \\ & \quad - \int_{\Sigma_{\tau_2}} \left\{ \text{Terms}_{\Sigma}^f \left[\Psi^{[s]} \right] + \text{Terms}_{\Sigma}^y \left[\Psi^{[s]} \right] \right\} dr d\omega \\ & \quad + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left\{ \text{Terms}_{\mathcal{H}^+}^f \left[\Psi^{[s]} \right] + \text{Terms}_{\mathcal{H}^+}^y \left[\Psi^{[s]} \right] \right\} dV_{\mathcal{H}^+} \\ & \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathcal{I}^{[s]} \cdot 2y\overline{\Psi^{[s]}}' \right\} dt^* dr d\omega \\ & \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathcal{I}^{[s]} \cdot \left(f'\overline{\Psi^{[s]}} + 2f\overline{\Psi^{[s]}}' \right) \right\} dt^* dr d\omega. \end{aligned} \quad (1.36)$$

We now couple the (up to error) energy boundedness estimate (1.30) with (1.36), estimating boundary terms, and we obtain the provisional estimate:

$$\begin{aligned}
& \int_{\Sigma_{\tau_2}} \mathbb{E} \left[\Psi^{[s]} \right] dr d\omega + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| (L + \underline{L}) \Psi^{[s]} \right|^2 dV_{\mathcal{H}^+} \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \mathfrak{c}_{(L+\underline{L})\Psi}(a, r) \left| (L + \underline{L}) \Psi^{[s]} \right|^2 + \mathfrak{c}_{(L-\underline{L})\Psi}(a, r) \left| (L - \underline{L}) \Psi^{[s]} \right|^2 + \mathfrak{c}_{\Psi}(a, r) \left| \Psi^{[s]} \right|^2 \right\} dt^* dr d\omega \\
& \lesssim \int_{\Sigma_{\tau_1}} \mathbb{E} \left[\Psi^{[s]} \right] dr d\omega \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{J}^{[s]} \cdot 2y \overline{\Psi^{[s]}}' \right\} dt^* dr d\omega \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{J}^{[s]} \cdot \left(f' \overline{\Psi^{[s]}} + 2f \overline{\Psi^{[s]}}' \right) \right\} dt^* dr d\omega \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{J}^{[s]} \cdot T \overline{\Psi^{[s]}} \right\} dt^* dr d\omega.
\end{aligned} \tag{1.37}$$

1.5.3 Outline of the proof of Theorem 1.16

(For details see Section 9.2)

Set

$$s = -2.$$

Recall

$$-\frac{r^2 + a^2}{\Delta} \mathfrak{J}^{[-2]} = a^2 h_1 \cdot \left(\sqrt{\Delta} \psi^{[-2]} \right) + a^2 h_2 \cdot \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right).$$

Hence we may write (1.37) as

$$\begin{aligned}
& \int_{\Sigma_{\tau_2}} \mathbb{E} \left[\Psi^{[-2]} \right] dr d\omega + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| (L + \underline{L}) \Psi^{[-2]} \right|^2 dV_{\mathcal{H}^+} \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \frac{1}{r^4} (r - r_{\text{trap}})^2 \left| (L + \underline{L}) \Psi^{[-2]} \right|^2 + \frac{1}{r^2} \left| (L - \underline{L}) \Psi^{[-2]} \right|^2 + \frac{1}{r^3} \left| \Psi^{[-2]} \right|^2 \right\} dt^* dr d\omega \\
& \lesssim \int_{\Sigma_{\tau_1}} \mathbb{E} \left[\Psi^{[-2]} \right] dr d\omega \\
& + a^2 \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ \left(h_1 \cdot \sqrt{\Delta} \psi^{[-2]} + h_2 \cdot \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right) \cdot 2y \overline{\Psi^{[-2]}}' \right\} dt^* dr d\omega \tag{B} \\
& + a^2 \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ \left(h_1 \cdot \sqrt{\Delta} \psi^{[-2]} + h_2 \cdot \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right) \cdot \left(f' \overline{\Psi^{[-2]}} + 2f \overline{\Psi^{[-2]}}' \right) \right\} dt^* dr d\omega \tag{C} \\
& + a^2 \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ \left(h_1 \cdot \sqrt{\Delta} \psi^{[-2]} + h_2 \cdot \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right) \cdot T \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega. \tag{D}
\end{aligned} \tag{1.38}$$

Remark 1.39. For a small enough, we can bound the coefficients $\mathfrak{c}_{(L+\underline{L})\Psi}$, $\mathfrak{c}_{(L-\underline{L})\Psi}$, $\mathfrak{c}_{\Psi}$ from below by the explicit functions of r that we wrote in (1.38).

1.5.3.1 Transport estimates (See Section 8)

Note that on the right hand side of (1.38), there are spacetime error terms involving $\alpha^{[-2]}$ and $\psi^{[-2]}$. A priori, one can not absorb these by the left hand side of (1.38). We aim to bound these error terms by spacetime integrals involving $\Psi^{[-2]}$ only, which can afterwards be absorbed by the left hand side of (1.38), for a small enough.

To this end, we use transport estimates (see Section 8). We can interpret the relations (4.7) as transport equations:

$$\begin{aligned} L \left((r^2 + a^2)^{\frac{3}{2}} \alpha^{[-2]} \right) &= \frac{2\Delta^{\frac{3}{2}}}{(r^2 + a^2)^2} \psi^{[-2]} \\ L \left(\sqrt{\Delta} \psi^{[-2]} \right) &= - \frac{\Delta}{(r^2 + a^2)^2} \Psi^{[-2]} \end{aligned} \quad (1.40)$$

Integrating these equations (after multiplying by appropriate weights), we obtain the estimates:

$$\begin{aligned} &\int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{r^2} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dt^* dr d\omega + \int_{\Sigma_{\tau_2}} \frac{r^2 + a^2}{\Delta} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dr d\omega \\ &\lesssim \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{r^2}{(r^2 + a^2)^2} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega + \int_{\Sigma_{\tau_1}} \frac{r^2 + a^2}{\Delta} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dr d\omega \end{aligned} \quad (1.41)$$

and

$$\begin{aligned} &\int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{r^2} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega + \int_{\Sigma_{\tau_2}} \frac{r^2 + a^2}{\Delta r} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dr d\omega \\ &\lesssim \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{2}{(r^2 + a^2)^2} \left| \Psi^{[-2]} \right|^2 dt^* dr d\omega + \int_{\Sigma_{\tau_1}} \frac{r^2 + a^2}{\Delta r} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dr d\omega. \end{aligned} \quad (1.42)$$

We see that these estimates allow us to estimate spacetime integrals of terms involving $\alpha^{[-2]}$ and $\psi^{[-2]}$ by spacetime integrals of terms involving only $\Psi^{[-2]}$, which is what we need.

Note that the boundary terms in (1.41) and (1.42) come with a good sign. Below, we use these good boundary terms to absorb error boundary terms (for example ones coming from term (D) - see below).

1.5.3.2 Absorbing errors Let us now describe how we treat error terms by absorbing them by the left hand side of (1.38).

Terms (B) and (C) contain cross-terms of the form $\psi^{[-2]} \cdot \Psi^{[-2]}$, $\alpha^{[-2]} \cdot \Psi^{[-2]}$, $\psi^{[-2]} \cdot (L - \underline{L})\Psi^{[-2]}$, and $\alpha^{[-2]} \cdot (L - \underline{L})\Psi^{[-2]}$. To treat these terms, we can just use Cauchy-Schwarz inequality and then absorb (for a small enough) the resulting $\left| \Psi^{[-2]} \right|^2$ and $\left| (L - \underline{L})\Psi^{[-2]} \right|^2$ terms by the left hand side. Of course, this approach also produces terms that are spacetime integrals of the quantities $\left| \psi^{[-2]} \right|^2$ and $\left| \alpha^{[-2]} \right|^2$. We can estimate these by a term of the form $\int_{\mathcal{R}(\tau_1, \tau_2)} \left| \Psi^{[-2]} \right|^2$ by using transport estimates (1.41) and (1.42).

Finally, term (D) is the most problematic one. It contains cross terms of the form $\psi^{[-2]} \cdot (L + \underline{L})\Psi^{[-2]}$ and $\alpha^{[-2]} \cdot (L + \underline{L})\Psi^{[-2]}$. However, the naive approach of using Cauchy-Schwarz directly won't work due to the fact that the term $\left| (L + \underline{L})\Psi^{[-2]} \right|^2$ appears on the left hand side of (1.38) with a weight proportional to $(r - r_{\text{trap}})^2$. This degeneration prevents us from using the naive approach. Instead, we use the fact that, by definition of the vector field L , we have

$$T\Psi^{[-2]} = L\Psi^{[-2]} - \Psi^{[-2]'} = L\Psi^{[-2]} - \frac{1}{2}(L - \underline{L})\Psi^{[-2]},$$

where we used the fact that $\Phi\Psi^{[-2]} = 0$ (by axisymmetry).

The resulting cross terms of the form $\psi^{[-2]} \cdot (L - \underline{L})\Psi^{[-2]}$ and $\alpha^{[-2]} \cdot (L - \underline{L})\Psi^{[-2]}$ can be treated in the same way as the terms (B) and (C), since $(L - \underline{L})\Psi^{[-2]}$ is controlled non-degenerately near r_{trap} . For the other resulting cross terms, the ones of the form $\psi^{[-2]} \cdot L\Psi^{[-2]}$ and $\alpha^{[-2]} \cdot L\Psi^{[-2]}$, we use integration by parts to shift the L -derivative

from $\Psi^{[-2]}$ to $\alpha^{[-2]}$ and $\psi^{[-2]}$, and use the fact that by definition (up to weights), we have

$$\begin{aligned}\psi^{[-2]} &= L \alpha^{[-2]}, \\ \Psi^{[-2]} &= L \psi^{[-2]}.\end{aligned}$$

This approach produces spacetime terms of the form

$$\int_{\mathcal{R}(\tau_1, \tau_2)} w_{\psi^{[-2]}}(r) \left| \psi^{[-2]} \right|^2 dt^* dr d\omega$$

and

$$\int_{\mathcal{R}(\tau_1, \tau_2)} w_{\alpha^{[-2]}}(r) \left| \alpha^{[-2]} \right|^2 dt^* dr d\omega,$$

where $w_{\psi^{[-2]}}$ and $w_{\alpha^{[-2]}}$ are appropriate weights, and boundary terms that are all proportional to a . For small a , these boundary terms can be absorbed by the good boundary terms coming from the transport estimates (1.41) & (1.42) and also the ones coming from the (up to error) energy boundedness estimate (1.30).

Finally, we conclude that for a small enough, after coupling (1.38) with transport estimates (1.41) & (1.42), and absorbing all the error terms, we end up with a Morawetz estimate of the form:

$$\begin{aligned}& \int_{\Sigma_{\tau_2}} \mathbb{E} \left[\Psi^{[-2]} \right] dr d\omega + \int_{\Sigma_{\tau_2}} \frac{r^2 + a^2}{r\Delta} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dr d\omega + \int_{\Sigma_{\tau_2}} \frac{r^2 + a^2}{\Delta} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dr d\omega \\& + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| (L + \underline{L}) \Psi^{[-2]} \right|^2 dV_{\mathcal{H}^+} \\& + \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \frac{1}{r^4} (r - r_{\text{trap}})^2 \left| (L + \underline{L}) \Psi^{[-2]} \right|^2 + \frac{1}{r^2} \left| (L - \underline{L}) \Psi^{[-2]} \right|^2 + \frac{1}{r^3} \left| \Psi^{[-2]} \right|^2 \right\} dt^* dr d\omega \\& + \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{r^2} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt dr d\omega + \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{r^2} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dt^* dr d\omega \\& \lesssim \int_{\Sigma_{\tau_1}} \mathbb{E} \left[\Psi^{[-2]} \right] dr d\omega + \int_{\Sigma_{\tau_1}} \frac{r^2 + a^2}{r\Delta} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dr d\omega + \int_{\Sigma_{\tau_1}} \frac{r^2 + a^2}{\Delta} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dr d\omega.\end{aligned} \tag{1.43}$$

Remark 1.44. Note, after gaining control on terms $\left| (L + \underline{L}) \Psi^{[-2]} \right|^2$, $\left| (L - \underline{L}) \Psi^{[-2]} \right|^2$ and $\left| \Psi^{[-2]} \right|^2$, we may regain control on $\left| \overset{\circ}{\Psi}^{[-2]} \Psi^{[-2]} \right|^2$ by repeating the proof but this time, instead of using the Poincaré inequality (1.35), using the control given by (1.43).

1.5.4 Outline of the proof of Theorem 1.17

(For details see Section 9.1)

Assume

$$s = \pm 2.$$

For the homogeneous equation (1.15), (1.37) becomes

$$\begin{aligned}
& \int_{\Sigma_{\tau_2}} \mathbb{E} \left[\Psi^{[s]} \right] dr d\omega + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| (L + \underline{L}) \Psi^{[s]} \right|^2 dV_{\mathcal{H}^+} \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \mathfrak{c}_{(L+\underline{L})\Psi}(a, r) \left| (L + \underline{L}) \Psi^{[s]} \right|^2 + \mathfrak{c}_{(L-\underline{L})\Psi}(a, r) \left| (L - \underline{L}) \Psi^{[s]} \right|^2 + \mathfrak{c}_{\Psi}(a, r) \left| \Psi^{[s]} \right|^2 \right\} dt^* dr d\omega \\
& \lesssim \int_{\Sigma_{\tau_1}} \mathbb{E} \left[\Psi^{[s]} \right] dr d\omega.
\end{aligned} \tag{1.45}$$

We can show that

$$\mathfrak{c}_{(L+\underline{L})\Psi}(a, r) \gtrsim \frac{1}{r^4} (r - r_{\text{trap}})^2$$

for all $r \geq r_+$ and all $a \in [0, M]$.

However, the coefficient $\mathfrak{c}_{\Psi}(a, r)$ becomes negative in a region around r_{trap} for values of a beyond a certain threshold (this turns out to be around $a = 0.6M$). To obtain more positivity, we prove an estimate of the form (see Lemma 9.1)

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \mathfrak{c}_{(L-\underline{L})\Psi}(a, r) \left| (L - \underline{L}) \Psi^{[s]} \right|^2 dt^* dr d\omega \\
& \geq \int_{\mathcal{R}(\tau_1, \tau_2)} \mathfrak{c}_W(a, r) \left| \Psi^{[s]} \right|^2 dt^* dr d\omega - \int_{\Sigma_{\tau_2}} \mathfrak{c}_W^{\Sigma}(a, r) \left| \Psi^{[s]} \right|^2 dr d\omega + \int_{\Sigma_{\tau_1}} \mathfrak{c}_W^{\Sigma}(a, r) \left| \Psi^{[s]} \right|^2 dr d\omega,
\end{aligned} \tag{1.46}$$

where the coefficients $\mathfrak{c}_W$ and $\mathfrak{c}_W^{\Sigma}$ depend on a smooth weight function $W = W(r)$.

Combining (1.45) with (1.46), we obtain

$$\begin{aligned}
& \int_{\Sigma_{\tau_2}} \left[\mathbb{E} \left[\Psi^{[s]} \right] - \mathfrak{c}_W^{\Sigma}(a, r) \left| \Psi^{[s]} \right|^2 \right] dr d\omega + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| (L + \underline{L}) \Psi^{[s]} \right|^2 dV_{\mathcal{H}^+} \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \mathfrak{c}_{(L+\underline{L})\Psi}(a, r) \left| (L + \underline{L}) \Psi^{[s]} \right|^2 + \left(\mathfrak{c}_{\Psi}(a, r) + \mathfrak{c}_W(a, r) \right) \left| \Psi^{[s]} \right|^2 \right\} dt^* dr d\omega \\
& \lesssim \int_{\Sigma_{\tau_1}} \left[\mathbb{E} \left[\Psi^{[s]} \right] - \mathfrak{c}_W^{\Sigma}(a, r) \left| \Psi^{[s]} \right|^2 \right] dr d\omega.
\end{aligned} \tag{1.47}$$

It turns out that, for a suitable choice of weight function W , we have the bounds

$$\begin{aligned}
\mathbb{E} \left[\Psi^{[s]} \right] - \mathfrak{c}_W^{\Sigma}(a, r) \left| \Psi^{[s]} \right|^2 & \gtrsim \mathbb{E} \left[\Psi^{[s]} \right], \\
\mathfrak{c}_{\Psi}(a, r) + \mathfrak{c}_W(a, r) & \gtrsim \frac{r - r_+}{r^4},
\end{aligned}$$

for all $r \geq r_+$, for all $a \leq M$. We obtain the estimate

$$\begin{aligned}
& \int_{\Sigma_{\tau_2}} \mathbb{E} \left[\Psi^{[s]} \right] dr d\omega + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| (L + \underline{L}) \Psi^{[s]} \right|^2 dV_{\mathcal{H}^+} \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \frac{1}{r^4} (r - r_{\text{trap}})^2 \left| (L + \underline{L}) \Psi^{[s]} \right|^2 + \frac{r - r_+}{r^4} \left| \Psi^{[s]} \right|^2 \right\} dt^* dr d\omega \\
& \lesssim \int_{\Sigma_{\tau_1}} \mathbb{E} \left[\Psi^{[s]} \right] dr d\omega.
\end{aligned} \tag{1.48}$$

Having obtained control on the $\left| (L + \underline{L}) \Psi^{[s]} \right|^2$ and $\left| \Psi^{[s]} \right|^2$ terms in the spacetime integral, we can regain control on the $\left| (L - \underline{L}) \Psi^{[s]} \right|^2$ from (1.45). We then regain control on the $\left| \overset{\circ}{\Psi}^{[s]} \Psi^{[s]} \right|^2$ term as before (see Remark 1.44). Finally, we get an estimate of the form

$$\begin{aligned}
& \int_{\Sigma_{\tau_2}} \mathbb{E} \left[\Psi^{[s]} \right] dr d\omega + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| (L + \underline{L}) \Psi^{[s]} \right|^2 dV_{\mathcal{H}^+} \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \frac{1}{r^4} (r - r_{\text{trap}})^2 \left| (L + \underline{L}) \Psi^{[s]} \right|^2 + \frac{1}{r^2} \left| (L - \underline{L}) \Psi^{[s]} \right|^2 + \frac{r - r_+}{r^4} \left| \Psi^{[s]} \right|^2 \right. \\
& \quad \left. + \frac{r - r_+}{r^6} (r - r_{\text{trap}})^2 \left| \overset{\circ}{\nabla}^{[s]} \Psi^{[s]} \right|^2 \right\} dt^* dr d\omega \\
& \lesssim \int_{\Sigma_{\tau_1}} \mathbb{E} \left[\Psi^{[s]} \right] dr d\omega
\end{aligned} \tag{1.49}$$

for all $|a| \in [0, M]$.

Remark 1.50. Note the additional degeneration at the event horizon $\{r = r_+\}$. This has to be included if we want to include the extremal $a = M$ case.

1.6 Use of Mathematica software

Throughout the proof, we use Mathematica software to plot various coefficients of terms in energy expressions. This is done to verify positivity of long complicated expressions and to visualise behaviour of said expressions for different parameters. While the rigour of the proof suffers due this approach to some extent, it is a way to avoid writing out unnecessary long and complicated computations, that are not mathematically interesting .

2 Kerr spacetime

We describe the Kerr family of spacetimes, coordinates and important vector fields. The Kerr family of spacetimes are a two parameter family of solutions of (1.1) of the form $(\mathcal{M}, g_{M,a})$. These solutions serve as a model of rotating black holes with mass M and angular momentum a . For a detailed treatment of the geometry of Kerr spacetimes, see [32].

2.1 The Kerr spacetime, coordinates, and vector fields

2.1.1 The fixed manifold with boundary

Define

$$\mathcal{M} := \mathbb{R} \times \mathbb{R}^+ \times (0, \pi) \times (0, 2\pi)$$

where we call the respective coordinates $(t^*, y^*, \theta^*, \varphi^*)$. These are called fixed coordinates.

Define the set (called the event horizon)

$$\mathcal{H}^+ := \partial\mathcal{M} = \{y^* = 0\}.$$

Also, denote the vector fields (in fixed coordinates)

$$T := \partial_{t^*}$$

$$\Phi := \partial_{\varphi^*}$$

2.1.2 Kerr-star coordinates

Define the set

$$\mathcal{P} := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq |x_1| < x_2\}.$$

We can think of $\mathcal{P}$ as the parameter space where the parameters a, M live.

Define the smooth map

$$r : \mathcal{P} \times (0, \infty) \rightarrow \mathbb{R}$$

such that for all $(a, M) \in \mathcal{P}$

$$r(a, M, \cdot) : (0, \infty) \rightarrow (M + \sqrt{M^2 - a^2}, \infty)$$

is a diffeomorphism, and

$$r(a, M, y^*) = y^*$$

for all $y^* \in (3M, \infty)$.

Now for each fixed pair of parameters $(a, M) \in \mathcal{P}$ we have a coordinate system $(t^*, r = r(a, M, y^*), \theta^*, \varphi^*)$ on $\mathcal{M}$ called the Kerr-star coordinates.

Moreover, note that the vector fields T, Φ defined in fixed coordinates (still) correspond to the vector fields $\partial_{t^*}, \partial_{\varphi^*}$ in Kerr-star coordinates, i.e.

$$T = \partial_{t^*}$$

and

$$\Phi = \partial_{\varphi^*}$$

where the right hand sides are with respect to the Kerr star coordinates.

Define the smooth functions of the parameters

$$r_{\pm} : \mathcal{P} \rightarrow \mathbb{R}$$

by

$$r_{\pm}(a, M) := M \pm \sqrt{M^2 - a^2}.$$

Note that the event horizon $\mathcal{H}^+ = \{y^* = 0\}$ corresponds to $\{r = r_+\}$.

2.1.3 The coordinate r^*

Given parameters $|a| < M$, define the rescaled version of the r coordinate on $\{r > r_+\}$ by

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}, \quad r^*(3M) = 0,$$

where

$$\Delta = (r - r_+)(r - r_-) = r^2 + a^2 - 2Mr.$$

Note that $r^* \rightarrow -\infty$ as $r \rightarrow r_+$, and $r^* \rightarrow \infty$ as $r \rightarrow \infty$.

For a parameter R (thought of as an r value), we denote $R^* = r^*(R)$.

In addition, for any function $F = F(r)$, we use the notation F' to denote the derivative $\frac{\partial F}{\partial r^*}$.

2.1.4 Boyer-Lindquist (BL) coordinates

Let $(a, M) \in \mathcal{P}$ be fixed parameters and let $(t^*, r, \theta^*, \varphi^*)$ be an associated system of Kerr-star coordinates on $\mathcal{M}$. In the interior of $\mathcal{M}$ (i.e. for $r > r_+$), we define

$$\begin{aligned} t(t^*, r) &:= t^* - \bar{t}(r), \\ \varphi(\varphi^*, r) &:= \varphi^* - \bar{\varphi}(r) \mod 2\pi, \\ \theta &:= \theta^*, \end{aligned}$$

where

$$\begin{aligned} \frac{d\bar{t}}{dr} &= \frac{2Mr}{\Delta}, \\ \frac{d\bar{\varphi}}{dr} &= \frac{a}{\Delta}. \end{aligned}$$

The coordinates (t, r, θ, φ) are called Boyer-Lindquist coordinates and they map the interior of $\mathcal{M}$ to

$$(-\infty, \infty)_t \times (r_+, \infty)_r \times [0, \pi]_\theta \times [0, 2\pi)_\varphi$$

Remark 2.1. We can compute that

$$\begin{aligned} T &= \partial_t, \\ \Phi &= \partial_\varphi, \end{aligned}$$

where the right hand sides are with respect to the Boyer-Lindquist coordinates.

2.1.5 The Kerr metric in BL coordinates

Given parameters $(a, M) \in \mathcal{P}$ (i.e. $|a| < M$), we may define in Boyer-Lindquist coordinates the Kerr metric $g_{M,a}$ by

$$g_{M,a} = -\frac{\Delta}{\rho^2}(dt - a \sin^2 \theta d\varphi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2}(adt - (r^2 + a^2)d\varphi)^2, \quad (2.2)$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta.$$

Remark 2.3. Note that in the case $a = 0$ this reduces to the Schwarzschild metric, i.e. $g_{M,0} = g_M$.

In matrix form we have

$$g = \begin{bmatrix} \frac{2Mr - \rho^2}{\rho^2} & 0 & 0 & \frac{-2Mr a \sin^2 \theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ \frac{-2Mr a \sin^2 \theta}{\rho^2} & 0 & 0 & \frac{\sin^2 \theta}{\rho^2}((r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta) \end{bmatrix}.$$

We can compute

$$g^{-1} = \begin{bmatrix} \frac{-1}{\Delta\rho^2} ((r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta) & 0 & 0 & \frac{-2Mra}{\Delta\rho^2} \\ 0 & \frac{\Delta}{\rho^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ \frac{-2Mra}{\Delta\rho^2} & 0 & 0 & \frac{\rho^2 - 2Mr}{\Delta\rho^2 \sin^2 \theta} \end{bmatrix}.$$

2.1.6 Kerr metric in Kerr-star coordinates

We can compute

$$g = \begin{bmatrix} -\left(1 - \frac{2Mr}{\rho^2}\right) & \frac{2Mr}{\rho^2} & 0 & \frac{-2Mra \sin^2 \theta}{\rho^2} \\ \frac{2Mr}{\rho^2} & 1 + \frac{2Mr}{\rho^2} & 0 & \frac{-a \sin^2 \theta (2Mr + \rho^2)}{\rho^2} \\ 0 & 0 & \rho^2 & 0 \\ \frac{-2Mra \sin^2 \theta}{\rho^2} & \frac{-a \sin^2 \theta (2Mr + \rho^2)}{\rho^2} & 0 & \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta) \end{bmatrix}$$

and

$$g^{-1} = \begin{bmatrix} -\left(1 + \frac{2Mr}{\rho^2}\right) & \frac{2Mr}{\rho^2} & 0 & 0 \\ \frac{2Mr}{\rho^2} & \frac{\Delta}{\rho^2} & 0 & \frac{a}{\rho^2} \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ 0 & \frac{a}{\rho^2} & 0 & \frac{1}{\rho^2 \sin^2 \theta} \end{bmatrix}.$$

2.1.7 Vector fields

Firstly, note that from the form of the metric, it is immediate that the vector fields T and Φ are Killing.

Define the Killing field

$$K = T + \omega_+ \Phi,$$

where $\omega_+ := \frac{a}{2Mr_+}$. K is null on the horizon $\mathcal{H}^+$.

Define also the null vector fields:

$$L := \partial_{r^*} + T + \frac{a}{r^2 + a^2} \Phi,$$

$$\underline{L} := -\partial_{r^*} + T + \frac{a}{r^2 + a^2} \Phi,$$

where ∂_{r^*} is defined with respect to the $(t, r^*, \theta, \varphi)$ -coordinates (i.e. $\partial_{r^*} = \frac{\Delta}{r^2 + a^2} \partial_r$ in BL coordinates).

The vector field L extends smoothly to $\mathcal{H}^+$ to be parallel to the null generator K . $\underline{L}$ extends smoothly to $\mathcal{H}^+$ so as to vanish identically, while the quantity $\Delta^{-1} \underline{L}$ has a smooth nontrivial limit on $\mathcal{H}^+$.

Remark 2.4. Note, we assume axisymmetric solutions, therefore we have

$$\Phi\Psi = 0$$

for a solution Ψ . Therefore, for example, we may write

$$(L + \underline{L})\Psi = 2T\Psi$$

which would not be true in the non-axisymmetric case.

2.1.8 Rescaled coordinates

It is useful to introduce a rescaling of the r coordinate

$$\tilde{r} := \frac{r}{M}$$

and also

$$\tilde{a} := \frac{a}{M}.$$

Then

$$\begin{aligned}\tilde{a} &\in [0, 1], \\ \tilde{r} &\in [\tilde{r}_+, \infty),\end{aligned}$$

where

$$\tilde{r}_+ = \frac{r_+}{M} = \frac{1}{M} \left(M^2 + \sqrt{M^2 - a^2} \right) = 1 + \sqrt{1 - \tilde{a}^2}.$$

This is convenient because the quantities $\tilde{a}$ and $\tilde{r}$ are dimensionless, and expressions for quantities like $w = \frac{\Delta}{(r^2 + a^2)^2}$ become the following:

$$w = \frac{r^2 + a^2 - 2Mr}{(r^2 + a^2)^2} = \frac{(M\tilde{r})^2 + (M\tilde{a})^2 - 2M(M\tilde{r})}{\left((M\tilde{r})^2 + (M\tilde{a})^2 \right)^2} = \frac{1}{M^2} \frac{\tilde{r}^2 + \tilde{a}^2 - 2\tilde{r}}{(\tilde{r}^2 + \tilde{a}^2)^2}.$$

We can see this helps us see the M -dependency more clearly, pulling the factor $\frac{1}{M^2}$ out in front of the rest of the expression (which depends only on $\tilde{a}$).

Note that we have

$$\partial_r = \frac{1}{M} \partial_{\tilde{r}}$$

and

$$dr = M d\tilde{r}.$$

In the same way, we can introduce a rescaling of the t^* coordinate

$$\tilde{t}^* := \frac{t^*}{M}.$$

We have

$$\partial_{t^*} = \frac{1}{M} \partial_{\tilde{t}^*}$$

and

$$dt^* = M d\tilde{t}^*.$$

2.2 Foliation and volume form

2.2.1 Spacelike slices

For all values $\tau \in \mathbb{R}$ denote $\Sigma_\tau := \{t^* = \tau\}$. These constant t^* slices are spacelike and they are useful when examining behaviour near the horizon $\mathcal{H}^+$.

Denote the unit future normal of Σ_τ by n_Σ . We can compute that

$$\begin{aligned}n_\Sigma &= \frac{-1}{\sqrt{-g^{t^*t^*}}} \left(g^{t^*t^*} \partial_{t^*} + g^{t^*r} \partial_r \right) \\ &= \sqrt{1 + \frac{2Mr}{\rho^2}} \partial_{t^*} - \frac{\frac{2Mr}{\rho^2}}{\sqrt{1 + \frac{2Mr}{\rho^2}}} \partial_r\end{aligned}$$

and the volume form on these slices can be computed to be

$$dV_\Sigma = \sqrt{1 + \frac{2Mr}{\rho^2}} \rho^2 dr d\omega.$$

We also define for $\tau_1 \leq \tau_2$ the spacetime region

$$\mathcal{R}(\tau_1, \tau_2) := \bigcup_{\tau_1 \leq \tau \leq \tau_2} \Sigma_\tau.$$

Denote also

$$\mathcal{H}^+(\tau_1, \tau_2) := \mathcal{H}^+ \cap \mathcal{R}(\tau_1, \tau_2).$$

Define also the region

$$\mathcal{R}_0 := \{t^* \geq 0\} = \bigcup_{\tau \geq 0} \Sigma_\tau.$$

2.3 Trapping

We can compute that

$$\partial_r w = \frac{-2}{(r^2 + a^2)^3} (r^3 - 3Mr^2 + a^2r + a^2M) = \frac{1}{M^3} \frac{-2}{(\tilde{r}^2 + \tilde{a}^2)^3} (\tilde{r}^3 - 3\tilde{r}^2 + \tilde{a}^2\tilde{r} + \tilde{a}^2).$$

It can be shown that the polynomial $p(r) := r^3 - 3Mr^2 + a^2r + a^2M$ has a unique root in the interval $[r_+, \infty)$. We shall call this root r_{trap} . In the rescaled coordinates, we have

$$\tilde{r}_{\text{trap}} := \frac{r_{\text{trap}}}{M}.$$

We can readily see that $\tilde{r}_{\text{trap}}$ is also the unique root of the polynomial $p(\tilde{r}) := \tilde{r}^3 - 3\tilde{r}^2 + \tilde{a}^2\tilde{r} + \tilde{a}^2$ in the interval $[\tilde{r}_+, \infty)$. Hence it is clear that $\tilde{r}_{\text{trap}}$ does not depend on M , only on $\tilde{a}$.

We can also write

$$M^2(-\partial_{\tilde{r}} w) = \frac{2}{(\tilde{r}^2 + \tilde{a}^2)^3} (\tilde{r}^3 - 3\tilde{r}^2 + \tilde{a}^2\tilde{r} + \tilde{a}^2) = \frac{2\tilde{r}}{(\tilde{r}^2 + \tilde{a}^2)^3} \left(1 - \frac{\tilde{r}_{\text{trap}}}{\tilde{r}}\right) Q(\tilde{r})$$

which defines the quadratic polynomial Q .

It can be shown that the polynomial Q has a root $\tilde{r}_0 \in [0, 1]$, such that

$$\begin{aligned} \lim_{\tilde{a} \rightarrow 1} \tilde{r}_0 &= 1, \\ \lim_{\tilde{a} \rightarrow 0} \tilde{r}_0 &= 0. \end{aligned}$$

Note, $\tilde{r}_0 \leq 1 \leq 1 + \sqrt{1 - \tilde{a}^2} = \tilde{r}_+$ for all $\tilde{a} \in [0, 1]$.

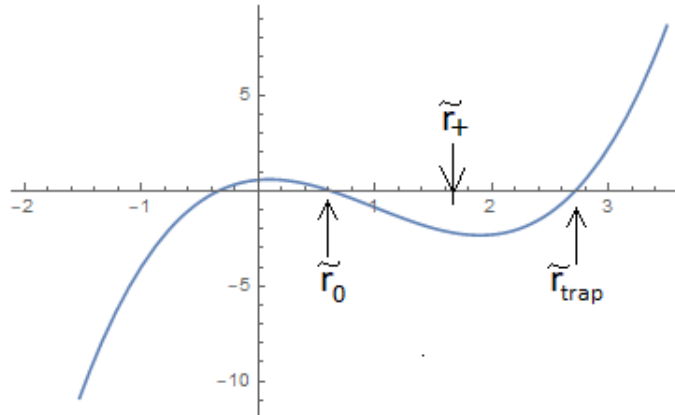


Figure 2: $\tilde{p}(\tilde{r})$ for $\tilde{a} = 0.75$. Note, here $\tilde{r}_+ \approx 1.66$.

2.4 Inner products

We compute useful inner products which we will need when applying Divergence Theorem in the spacetime region $\mathcal{R}(\tau_1, \tau_2)$. We have:

$$\begin{aligned}
g(L + \underline{L}, n_\Sigma) &= \frac{-2}{\sqrt{1 + \frac{2Mr}{\rho^2}}}, \\
g(L + \underline{L}, n_\Sigma) \frac{r^2 + a^2}{\Delta \rho^2} dV_\Sigma &= -2 \frac{r^2 + a^2}{\Delta} dr d\omega, \\
g(L - \underline{L}, n_\Sigma) &= \frac{-2}{\sqrt{1 + \frac{2Mr}{\rho^2}}} \frac{2Mr}{r^2 + a^2}, \\
g(L - \underline{L}, n_\Sigma) \frac{r^2 + a^2}{\Delta \rho^2} dV_\Sigma &= -\frac{4Mr}{\Delta} dr d\omega, \\
g(L + \underline{L}, n_{\mathcal{H}^+}) &= \frac{-\Delta \rho_+^2}{Mr_+(r^2 + a^2)}, \\
\frac{r^2 + a^2}{\Delta \rho^2} g(L + \underline{L}, n_{\mathcal{H}^+}) &= \frac{-\rho_+^2}{Mr_+ \rho^2}, \\
\frac{r^2 + a^2}{\Delta \rho^2} g(L - \underline{L}, n_{\mathcal{H}^+}) &= 0,
\end{aligned}$$

where we used the fact that $dV_\Sigma = \sqrt{1 + \frac{2Mr}{\rho^2}} \rho^2 dr d\omega$.

2.5 Useful covariant identities

The following Lemma will be useful whenever we apply the Divergence Theorem in later computations.

Lemma 2.5. *The vector fields L and $\underline{L}$ satisfy the covariant identities:*

$$\begin{aligned}
\nabla_\mu \left(\frac{r^2 + a^2}{\Delta \rho^2} \underline{L}^\mu \right) &= 0, \\
\nabla_\mu \left(\frac{r^2 + a^2}{\Delta \rho^2} L^\mu \right) &= 0.
\end{aligned}$$

Proof. We compute this in Boyer-Lindquist coordinates. Note that in these coordinates we have

$$L = \frac{\Delta}{r^2 + a^2} \partial_r + \partial_t + \frac{a}{r^2 + a^2} \partial_\varphi.$$

We compute

$$\begin{aligned}
& \nabla_\mu \left(\frac{1}{\rho^2} \frac{r^2 + a^2}{\Delta} L^\mu \right) \\
&= \nabla_\mu \left(\frac{1}{\rho^2} \frac{r^2 + a^2}{\Delta} \right) L^\mu + \frac{1}{\rho^2} \frac{r^2 + a^2}{\Delta} \nabla_\mu L^\mu \\
&= \partial_\mu \left(\frac{1}{\rho^2} \frac{r^2 + a^2}{\Delta} \right) L^\mu + \frac{1}{\rho^2} \frac{r^2 + a^2}{\Delta} \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} L^\mu) \\
&= \partial_r \left(\frac{1}{\rho^2} \frac{r^2 + a^2}{\Delta} \right) L^r + \frac{1}{\rho^2} \frac{r^2 + a^2}{\Delta} \frac{1}{\rho^2} \partial_r (\rho^2 L^r) \\
&= \partial_r \left(\frac{1}{\rho^2} \frac{r^2 + a^2}{\Delta} \right) \frac{\Delta}{r^2 + a^2} + \frac{1}{(\rho^2)^2} \frac{r^2 + a^2}{\Delta} \partial_r (\rho^2 \frac{\Delta}{r^2 + a^2}) \\
&= \partial_r \left(\frac{1}{\rho^2} \right) + \frac{1}{\rho^2} \partial_r \left(\frac{r^2 + a^2}{\Delta} \right) \frac{\Delta}{r^2 + a^2} + \frac{1}{(\rho^2)^2} \frac{r^2 + a^2}{\Delta} \left[\partial_r (\rho^2) \frac{\Delta}{r^2 + a^2} + \rho^2 \partial_r \left(\frac{\Delta}{r^2 + a^2} \right) \right] \\
&= \partial_r \left(\frac{1}{\rho^2} \right) + \frac{1}{\rho^2} \partial_r \left(\frac{r^2 + a^2}{\Delta} \right) \frac{\Delta}{r^2 + a^2} + \frac{1}{(\rho^2)^2} \partial_r (\rho^2) + \frac{1}{\rho^2} \partial_r \left(\frac{\Delta}{r^2 + a^2} \right) \frac{r^2 + a^2}{\Delta} \\
&= \frac{1}{\rho^2} \left[\partial_r \left(\frac{r^2 + a^2}{\Delta} \right) \frac{\Delta}{r^2 + a^2} + \partial_r \left(\frac{\Delta}{r^2 + a^2} \right) \frac{r^2 + a^2}{\Delta} \right] \\
&= \frac{1}{\rho^2} \partial_r \left[\frac{r^2 + a^2}{\Delta} \frac{\Delta}{r^2 + a^2} \right] \\
&= 0
\end{aligned}$$

Similar computation shows the $\underline{L}$ case.

□

3 Spin s -weighted complex functions

The Teukolsky equation concerns functions whose (θ, φ^*) dependence is that of a spin s -weighted function, for $s \in \frac{1}{2}\mathbb{Z}$.

In this section we briefly recall the notion of spin s -weighted complex functions and the spaces they live in. For more details see Section 2.2.1 of [12].

Fix $s \in \frac{1}{2}\mathbb{Z}$ (for the purposes of this thesis $s = \pm 2$).

Let (θ, φ^*) denote spherical coordinates on S^2 and define the vector fields (which are, note, dependent on s)

$$\begin{aligned}\tilde{Z}_1 &= -\sin \varphi^* \partial_\theta + \cos \varphi^* \left(-is \csc \theta - \cot \theta \partial_{\varphi^*} \right), \\ \tilde{Z}_2 &= -\cos \varphi^* \partial_\theta - \sin \varphi^* \left(-is \csc \theta - \cot \theta \partial_{\varphi^*} \right), \\ \tilde{Z}_3 &= \partial_{\varphi^*}.\end{aligned}$$

A complex-valued function f of (θ, φ^*) is a smooth spin s -weighted function on S^2 , i.e. $f \in \mathcal{S}_\infty^{[s]}$, if for any $k_1, k_2, k_3 \in \mathbb{N} \cup \{0\}$,

$$(\tilde{Z}_1)^{k_1} (\tilde{Z}_2)^{k_2} (\tilde{Z}_3)^{k_3} f$$

is a function of the coordinates (θ, φ^*) which is smooth for $\theta \neq 0, \pi$ (i.e. away from the north and south pole of the coordinate system), and such that

$$e^{is\varphi} (\tilde{Z}_1)^{k_1} (\tilde{Z}_2)^{k_2} (\tilde{Z}_3)^{k_3} f$$

extends continuously to $\theta = 0$, and

$$e^{-is\varphi} (\tilde{Z}_1)^{k_1} (\tilde{Z}_2)^{k_2} (\tilde{Z}_3)^{k_3} f$$

extends continuously to $\theta = \pi$.

Fixing Kerr parameters (a, M) , let now f be a function of the Kerr-star coordinates $(t^*, r, \theta, \varphi^*)$. We say that f is a smooth spin s -weighted function on $\mathcal{R}$, i.e. $f \in \mathcal{S}_\infty^{[s]}(\mathcal{R})$, if for any $k_4, k_5 \in \mathbb{N} \cup \{0\}$,

$$(\partial_{t^*})^{k_4} (\partial_r)^{k_5} f \Big|_{(t^*, r) = (\bar{t}^*, \bar{r})} \in \mathcal{S}_\infty^{[s]}$$

for all $(\bar{t}^*, \bar{r}) \in \mathbb{R} \times [r_+, \infty)$.

Similarly, we define f to be a smooth spin s -weighted function on a slice Σ_τ , i.e. $f \in \mathcal{S}_\infty^{[s]}(\Sigma_\tau)$, if for any $k_4 \in \mathbb{N} \cup \{0\}$,

$$(\partial_r)^{k_4} f \Big|_{r=\bar{r}} \in \mathcal{S}_\infty^{[s]}$$

for all $\bar{r} \in [r_+, \infty)$.

Define, for $m \in \mathbb{N} \cup \{0\}$, the Sobolev space $^{[s]}H^m(\Sigma_\tau)$ to be the completion of $\mathcal{S}_\infty^{[s]}(\Sigma_\tau)$ with respect to the norm

$$\|f\|_{^{[s]}H^m(\Sigma_\tau)}^2 = \sum_{i=0}^m \sum_{k_1+k_2+k_3+k_4=i} \int_{\Sigma_\tau} \left| (\tilde{Z}_1)^{k_1} (\tilde{Z}_2)^{k_2} (\tilde{Z}_3)^{k_3} (\partial_r)^{k_4} f \right|^2 dV_\Sigma.$$

If U is an open subset of Σ_τ , we can define $\mathcal{S}_\infty^{[s]}(U)$ and $^{[s]}H^m(U)$ in the obvious way. This allows us to define the space

$$^{[s]}H_{\text{loc}}^m(\Sigma_\tau) := \left\{ f : \Sigma_\tau \rightarrow \mathbb{C} \mid f|_U \in ^{[s]}H^m(U) \ \forall U \Subset \Sigma_\tau \right\}.$$

4 Teukolsky and Regge–Wheeler equation under axisymmetry

The Teukolsky equation for general spin $s \in \frac{1}{2}\mathbb{Z}$ is

$$\begin{aligned} \mathfrak{T}^{[s]} \alpha^{[s]} &:= \square_g \alpha^{[s]} + \frac{2s}{\rho^2} (r - M) \partial_r \alpha^{[s]} + \frac{2s}{\rho^2} \left(\frac{a(r - M)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right) \partial_\varphi \alpha^{[s]} \\ &+ \frac{2s}{\rho^2} \left(\frac{M(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right) \partial_t \alpha^{[s]} + \frac{1}{\rho^2} (s - s^2 \cot^2 \theta) \alpha^{[s]} = 0. \end{aligned} \quad (4.1)$$

Definition 4.2. We call a sufficiently regular solution $\alpha^{[s]}$ of equation (4.1) axisymmetric if

$$\Phi \alpha^{[s]} = 0.$$

See also section 4.5 for relation to initial data.

Remark 4.3. Note that under axisymmetry the third term on the right hand side of equation (4.1) vanishes.

Let us restrict to the cases $s = \pm 2$. Define the rescaled quantities

$$\begin{aligned} \tilde{\alpha}^{[+2]} &:= \Delta^2 (r^2 + a^2)^{-\frac{3}{2}} \alpha^{[+2]}, \\ \tilde{\alpha}^{[-2]} &:= \Delta^{-2} (r^2 + a^2)^{-\frac{3}{2}} \alpha^{[-2]}. \end{aligned}$$

Define the modified Teukolsky operator $\tilde{\mathfrak{T}}^{[s]}$ by

$$\begin{aligned} \Delta \rho^{-2} \tilde{\mathfrak{T}}^{[s]} &= \frac{1}{2} (L \underline{L} + \underline{L} L) + w \left(\mathring{\Delta}^{[s]} + s - 3 \frac{a^4 + a^2 r^2 - 2Mr^3}{(r^2 + a^2)^2} + 2 \right) \\ &- w (2aT\Phi + a^2 \sin^2 \theta TT - 2isa \cos \theta T) + \mathfrak{t}^{[s]}, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{t}^{[+2]} &= -2 \frac{w'}{w} \underline{L} - 8aw \frac{r}{r^2 + a^2} \Phi, \\ \mathfrak{t}^{[-2]} &= +2 \frac{w'}{w} L + 8aw \frac{r}{r^2 + a^2} \Phi, \end{aligned}$$

with

$$w = \frac{\Delta}{(r^2 + a^2)^2}.$$

We can check that for $s = +2$ the equation (4.1) can be rewritten as

$$\tilde{\mathfrak{T}}^{[+2]} \tilde{\alpha}^{[+2]} = 0 \quad (4.4)$$

and for $s = -2$ the equation (4.1) can be rewritten as

$$\tilde{\mathfrak{T}}^{[-2]} \left(\Delta^2 \tilde{\alpha}^{[-2]} \right) = 0. \quad (4.5)$$

We may combine equations (4.4) and (4.5) as follows:

$$\tilde{\mathfrak{T}}^{[s]} \left(\left(s \left(1 + \Delta^2 \right) + 2 \left(1 - \Delta^2 \right) \right) \tilde{\alpha}^{[s]} \right) = 0. \quad (4.6)$$

Define further, given functions $\alpha^{[\pm 2]}$, the quantities

$$\begin{aligned} \psi^{[+2]} &:= -\frac{1}{2} \Delta^{-\frac{3}{2}} (r^2 + a^2)^2 \underline{L}^\mu \nabla_\mu \left(\Delta^2 (r^2 + a^2)^{-\frac{3}{2}} \alpha^{[+2]} \right), \\ \psi^{[-2]} &:= +\frac{1}{2} \Delta^{-\frac{3}{2}} (r^2 + a^2)^2 L^\mu \nabla_\mu \left((r^2 + a^2)^{-\frac{3}{2}} \alpha^{[-2]} \right), \\ \Psi^{[+2]} &:= \frac{(r^2 + a^2)^2}{\Delta} \underline{L}^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[+2]} \right), \\ \Psi^{[-2]} &:= -\frac{(r^2 + a^2)^2}{\Delta} L^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[-2]} \right). \end{aligned} \quad (4.7)$$

We now cite a proposition from [12] (Proposition 3.2.1 on page 21).

Proposition 4.8. *If $\tilde{\alpha}^{[\pm 2]}$ satisfy the inhomogeneous equations*

$$\tilde{\mathfrak{Z}}^{[+2]}\tilde{\alpha}^{[+2]} = F^{[+2]}$$

and

$$\tilde{\mathfrak{Z}}^{[-2]} \left(\Delta^2 \tilde{\alpha}^{[-2]} \right) = \Delta^2 F^{[-2]}$$

then the quantities $\Psi^{[\pm 2]}$ satisfy the equation

$$\mathfrak{R}^{[\pm 2]}\Psi^{[\pm 2]} = -\frac{\rho^2}{\Delta} \left(\mathfrak{J}^{[\pm 2]} + \mathfrak{G}^{[\pm 2]} \right)$$

where the operator $\mathfrak{R}^{[s]}$ is called the generalised Regge–Wheeler operator and is defined as

$$\begin{aligned} \Delta \rho^{-2} \mathfrak{R}^{[s]} := & \frac{1}{2} (L\bar{L} + \bar{L}L) + w \left[\left(\mathring{\Delta}^{[s]} + s^2 + s \right) - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right] \\ & - w(2aT\Phi + a^2 \sin^2 \theta TT - 2isa \cos \theta T), \end{aligned}$$

and the symbols on the right hand side are defined as follows:

$$\begin{aligned} \mathfrak{J}^{[+2]} := & w \left[\frac{-8r^2 + 8a^2}{r^2 + a^2} a\Phi - 20a^2 \frac{r^3 - 3Mr^2 + ra^2 + Ma^2}{(r^2 + a^2)^2} \right] \left(\sqrt{\Delta} \psi^{[+2]} \right) \\ & + a^2 w \left[-12 \frac{r}{r^2 + a^2} a\Phi + 3 \left(\frac{r^4 - a^4 + 10Mr^3 - 6Ma^2 r}{(r^2 + a^2)^2} \right) \right] \left(\alpha^{[+2]} \Delta^2 (r^2 + a^2)^{-\frac{3}{2}} \right), \end{aligned}$$

$$\mathfrak{G}^{[+2]} := \frac{1}{2} \bar{L} \left(\frac{(r^2 + a^2)^2}{\Delta} \bar{L} \left(\frac{\Delta}{w\rho^2} F^{[+2]} \right) \right),$$

and

$$\begin{aligned} \mathfrak{J}^{[-2]} := & w \left[\frac{8r^2 - 8a^2}{r^2 + a^2} a\Phi - 20a^2 \frac{r^3 - 3Mr^2 + ra^2 + Ma^2}{(r^2 + a^2)^2} \right] \left(\sqrt{\Delta} \psi^{[-2]} \right) \\ & + a^2 w \left[+12 \frac{r}{r^2 + a^2} a\Phi + 3 \left(\frac{r^4 - a^4 + 10Mr^3 - 6Ma^2 r}{(r^2 + a^2)^2} \right) \right] \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right), \end{aligned}$$

$$\mathfrak{G}^{[-2]} := \frac{1}{2} L \left(\frac{(r^2 + a^2)^2}{\Delta} L \left(\frac{\Delta^3}{w\rho^2} F^{[-2]} \right) \right).$$

Proof. This can be proven by direct computation. See Appendix of [12]. □

In light of (4.4) and (4.5), we can use Proposition 4.8 to conclude that

$$\Delta \rho^{-2} \mathfrak{R}^{[s]} \Psi^{[s]} = -\mathfrak{J}^{[s]}$$

where $s = \pm 2$. Expanding, the equation becomes

$$\begin{aligned} \frac{1}{2} (L\bar{L} + \bar{L}L) \Psi^{[s]} + w \left(\mathring{\Delta}^{[s]} + s \right) \Psi^{[s]} + w \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right) \Psi^{[s]} \\ - w(2aT\Phi + a^2 \sin^2 \theta TT - 2isa \cos \theta T) \Psi^{[s]} = -\mathfrak{J}^{[s]}. \end{aligned} \tag{4.9}$$

Remark 4.10. Note, under the assumption of axisymmetry, the term involving Φ vanishes.

Hence, using the above transformations, the inhomogeneous (generalised) Regge–Wheeler equation was obtained from the Teukolsky equation.

4.1 Homogeneous (generalised) Regge–Wheeler equation

We are also interested in the homogeneous version of equation (4.9), namely:

$$\begin{aligned} \frac{1}{2}(L\underline{L} + \underline{L}L)\Psi^{[s]} + w \left(\overset{\circ}{\Delta}^{[s]} + s \right) \Psi^{[s]} + w \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2\Delta}{(r^2 + a^2)^2} \right) \Psi^{[s]} \\ - w(2aT\Phi + a^2 \sin^2 \theta TT - 2isa \cos \theta T) \Psi^{[s]} = 0. \end{aligned} \quad (4.11)$$

Remark 4.12. Note, equation (4.11) can also be viewed as a standalone equation for spin s -weighted functions $\Psi^{[s]}$ (completely decoupled from $\alpha^{[s]}$ and $\psi^{[s]}$). Note also that for axisymmetric solutions the term involving Φ vanishes.

4.2 Recurring quantities

We define some quantities that will recur, so it is useful to have a name for them.

$$\begin{aligned} \tilde{\Delta} &:= \frac{1}{M^2} \Delta = \tilde{r}^2 + \tilde{a}^2 - 2\tilde{r} \\ D &:= \frac{\Delta}{r^2 + a^2} = \frac{\tilde{r}^2 + \tilde{a}^2 - 2\tilde{r}}{\tilde{r}^2 + \tilde{a}^2} = \frac{\tilde{\Delta}}{\tilde{r}^2 + \tilde{a}^2} \\ w &:= \frac{\Delta}{(r^2 + a^2)^2} = \frac{1}{M^2} \frac{\tilde{\Delta}}{(\tilde{r}^2 + \tilde{a}^2)^2} = \frac{1}{M^2} \tilde{w} \\ P &:= 4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2\Delta}{(r^2 + a^2)^2} = 4 - \frac{6\tilde{r}}{\tilde{r}^2 + \tilde{a}^2} \frac{\tilde{r}^2 - \tilde{a}^2}{\tilde{r}^2 + \tilde{a}^2} - \frac{7\tilde{a}^2(\tilde{r}^2 + \tilde{a}^2 - 2\tilde{r})}{(\tilde{r}^2 + \tilde{a}^2)^2} \\ h_1 &:= \frac{20a^2}{(r^2 + a^2)^3} (r^3 - 3Mr^2 + a^2r + a^2) = \frac{20\tilde{a}^2}{M(\tilde{r}^2 + \tilde{a}^2)^3} (\tilde{r}^3 - 3\tilde{r}^2 + \tilde{a}^2\tilde{r} + \tilde{a}^2) =: \frac{1}{M} \tilde{h}_1 \\ h_2 &:= \frac{-3a^2}{(r^2 + a^2)^3} (r^4 + 10Mr^3 - 6Ma^2r - a^4) = \frac{-3\tilde{a}^2}{(\tilde{r}^2 + \tilde{a}^2)^3} (\tilde{r}^4 + 10\tilde{r}^3 - 6\tilde{a}^2\tilde{r} - \tilde{a}^4) =: \tilde{h}_2 \end{aligned}$$

4.3 Energies

We define the energy

$$\begin{aligned} \mathbb{E}_a[\Psi^{[s]}] &:= \frac{\rho^2}{8(r^2 + a^2)} \left| (L + \underline{L})\Psi^{[s]} \right|^2 + \frac{1}{8} \left| (L - \underline{L})\Psi^{[s]} \right|^2 + \frac{Mr}{\Delta} \left| \underline{L}\Psi^{[s]} \right|^2 \\ &\quad + \frac{1}{2(r^2 + a^2)} \left| \overset{\circ}{\nabla}^{[s]} \Psi^{[s]} \right|^2 + \frac{1}{2(r^2 + a^2)} P \left| \Psi^{[s]} \right|^2, \end{aligned} \quad (4.13)$$

and the equivalent energy expressed in rescaled coordinates

$$\tilde{\mathbb{E}}_{\tilde{a}}[\Psi^{[s]}] := \left| M(L + \underline{L})\Psi^{[s]} \right|^2 + \left| M(L - \underline{L})\Psi^{[s]} \right|^2 + \frac{\tilde{r}}{\tilde{\Delta}} \left| M\underline{L}\Psi^{[s]} \right|^2 + \frac{1}{\tilde{r}^2} \left| \Psi^{[s]} \right|^2 + \frac{1}{\tilde{r}^2} \left| \overset{\circ}{\nabla}^{[s]} \Psi^{[s]} \right|^2. \quad (4.14)$$

Note that

$$\begin{aligned} M(L + \underline{L})\Psi^{[s]} &= 2\partial_{\tilde{r}^*} \Psi^{[s]}, \\ M(L - \underline{L})\Psi^{[s]} &= 2\frac{\tilde{\Delta}}{\tilde{r}^2 + \tilde{a}^2} \partial_{\tilde{r}} \Psi^{[s]}, \end{aligned}$$

so this energy is in fact independent of M .

Remark 4.15. We can check that for all r

$$\begin{aligned} \frac{1}{2} &\leq \frac{\rho^2}{r^2 + a^2} \leq 1, \\ \frac{1}{2} &\leq P \leq 4, \end{aligned}$$

so these energies are indeed coercive, and we can derive the inequalities

$$\frac{1}{16M^2} \tilde{\mathbb{E}}_{\bar{a}} [\Psi^{[s]}] \leq \mathbb{E}_a [\Psi^{[s]}] \leq \frac{2}{M^2} \tilde{\mathbb{E}}_{\bar{a}} [\Psi^{[s]}]. \quad (4.16)$$

Hence the energies are indeed equivalent.

Remark 4.17. Note that since $\Delta^{-1}\underline{L}$ has a smooth nontrivial limit on $\mathcal{H}^+$, we can write

$$\frac{8Mr}{\Delta} |\underline{L}\Psi|^2 = 8Mr\Delta |\Delta^{-1}\underline{L}\Psi|^2$$

and conclude that $\frac{8Mr}{\Delta} |\underline{L}\Psi|^2$ degenerates on $\mathcal{H}^+$.

4.4 Reduction to case $a > 0$

A priori, the parameter a is allowed to be negative, it only needs to satisfy the bound $|a| \leq M$. We argue that for $a \neq 0$ we can assume $a > 0$ without loss of generality.

Assume $a < 0$. Let

$$\bar{a} := -a > 0.$$

Let us make (using the BL coordinates) the following coordinate transformations:

$$\begin{aligned} \varphi &\rightarrow -\varphi =: \bar{\varphi}, \\ \theta &\rightarrow \pi - \theta =: \bar{\theta}. \end{aligned} \quad (4.18)$$

Then we can show that:

1. The form of the Kerr metric remains unchanged. In particular we can write $g_{M,a}$ (see (2.2)) in coordinates $(t, r, \bar{\theta}, \bar{\varphi})$ as follows:

$$g_{M,a} = -\frac{\Delta}{\rho^2} \left(dt - \bar{a} \sin^2 \bar{\theta} d\bar{\varphi} \right)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\bar{\theta}^2 + \frac{\sin^2 \bar{\theta}}{\rho^2} \left(\bar{a} dt - (r^2 + \bar{a}^2) d\bar{\varphi} \right)^2$$

where

$$\Delta = r^2 + \bar{a}^2 - 2Mr$$

and

$$\rho^2 = r^2 + \bar{a}^2 \cos^2 \bar{\theta}.$$

Hence we reduced the metric to the form (2.2) where the parameter $\bar{a}$ is positive.

2. The form of the Teukolsky operator $\mathfrak{T}^{[s]}$ (see (1.2)) remains unchanged. In particular, we may write it with respect to the coordinates $(t, r, \bar{\theta}, \bar{\varphi})$ as follows:

$$\begin{aligned} \mathfrak{T}^{[s]} &= \square_g + \frac{2s}{\rho^2} (r - M) \partial_r + \frac{2s}{\rho^2} \left(\frac{\bar{a}(r - M)}{\Delta} + i \frac{\cos \bar{\theta}}{\sin^2 \bar{\theta}} \right) \partial_{\bar{\varphi}} \\ &\quad + \frac{2s}{\rho^2} \left(\frac{M(r^2 - \bar{a}^2)}{\Delta} - r - i \bar{a} \cos \bar{\theta} \right) \partial_t + \frac{1}{\rho^2} \left(s - s^2 \cot^2 \bar{\theta} \right). \end{aligned}$$

Hence we reduced the Teukolsky operator to the form (1.2) where the parameter $\bar{a}$ is positive.

3. Even though generally the spin-weighted functions $\alpha^{[\pm 2]}$ are sensitive to coordinate transformations (because their definition depends on a frame on $\mathbb{S}^2$ induced by the angular coordinates), they remain unchanged under the transformation (4.18).

Hence, we can reduce the case of $a < 0$ to the case of $a > 0$.

4.5 Well-posedness

We quote a well-posedness result from [12] (see page 19 within).

Proposition 4.19. *Consider an exterior Kerr spacetime foliated by spacelike slices as in 2.2.1. Recall $\Sigma_0 = \{t^* = 0\}$ and $\mathcal{R}_0 = \{t^* \geq 0\}$. Assume $s = \pm 2$.*

Assume we have a pair of spin s -weighted complex valued functions $(\tilde{\alpha}_0^{[s]}, \tilde{\alpha}_1^{[s]}) \in {}^{[s]}H_{loc}^j(\Sigma_0) \times {}^{[s]}H_{loc}^{j-1}(\Sigma_0)$, where $j \geq 1$. Then there exists a unique complex valued $\tilde{\alpha}^{[s]}$ on $\mathcal{R}_0$ satisfying equation (4.6) (equivalently, $\alpha^{[s]}$ satisfying equation (4.1)) such that

$$\begin{aligned} \tilde{\alpha}^{[s]} &\in {}^{[s]}H_{loc}^j(\Sigma_\tau) \quad \forall \tau \geq 0, \\ n_{\Sigma_\tau} \tilde{\alpha}^{[s]} &\in {}^{[s]}H_{loc}^{j-1}(\Sigma_\tau) \quad \forall \tau \geq 0, \\ \tilde{\alpha}^{[s]}|_{\Sigma_0} &= \tilde{\alpha}_0^{[s]}, \\ (n_{\Sigma_0} \tilde{\alpha}^{[s]})|_{\Sigma_0} &= \tilde{\alpha}_1^{[s]}. \end{aligned}$$

Moreover, letting $j \geq 2$, for axisymmetric initial data, i.e. $(\tilde{\alpha}_0^{[s]}, \tilde{\alpha}_1^{[s]}) \in {}^{[s]}H_{loc}^j(\Sigma_0) \times {}^{[s]}H_{loc}^{j-1}(\Sigma_0)$ satisfying $\Phi \tilde{\alpha}_0^{[s]} = \Phi \tilde{\alpha}_1^{[s]} = 0$, the solution $\tilde{\alpha}^{[s]}$ is axisymmetric, i.e. $\Phi \tilde{\alpha}^{[s]} = 0$ on $\mathcal{R}_0$.

An analogous result holds for solutions $\Psi^{[s]}$ of equation (4.11).

4.6 Main results

We prove the following:

Theorem 4.20. *Fix $M > 0$. There exists $\tilde{a}_0 = \tilde{a}_0(M) \in [0, 1]$ and a constant $C = C(M) > 0$, such that for all $\tilde{a} \in [0, \tilde{a}_0]$ the following holds:*

*Let $\tilde{\alpha}^{[s]}$ be a smooth solution of the spin $s = -2$ Teukolsky equation (4.5) on a Kerr exterior background with parameters $(a, M) = (\tilde{a}M, M)$, arising from **axisymmetric** initial data as in Proposition 4.19. Set*

$$\alpha^{[-2]} = \Delta^2 \cdot \tilde{\alpha}^{[-2]}$$

Let $\psi^{[-2]}$ and $\Psi^{[-2]}$ be as defined by the transformations (4.7).

Then we have the Morawetz estimate

$$\begin{aligned} &\int_{\mathcal{R}(\tau_1, \tau_2)} \left[\frac{1}{\tilde{r}^4} (\tilde{r} - \tilde{r}_{trap})^2 \left| M(L + \underline{L}) \Psi^{[-2]} \right|^2 + \frac{1}{\tilde{r}^2} \left| M(L - \underline{L}) \Psi^{[-2]} \right|^2 \right. \\ &\quad \left. + \frac{1}{\tilde{r}^5} (\tilde{r} - \tilde{r}_{trap})^2 \left| \overset{\circ}{\nabla}^{[-2]} \Psi^{[-2]} \right|^2 + \frac{1}{\tilde{r}^3} \left| \Psi^{[-2]} \right|^2 \right] d\tilde{r}^* d\tilde{r} d\omega \\ &+ \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{D^2}{\tilde{r}^2} \left| \frac{\psi^{[-2]}(\tilde{r}^2 + \tilde{a}^2)}{\sqrt{\tilde{\Delta}}} \right|^2 d\tilde{r}^* d\tilde{r} d\omega + \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{D^4}{\tilde{r}^2} \left| \frac{\alpha^{[-2]} \sqrt{\tilde{r}^2 + \tilde{a}^2}}{\tilde{\Delta}^2} \right|^2 d\tilde{r}^* d\tilde{r} d\omega \\ &+ \int_{\Sigma_{\tau_2}} \tilde{\mathbb{E}}_{\tilde{a}}[\Psi^{[-2]}] d\tilde{r} d\omega + \int_{\Sigma_{\tau_2}} \frac{D}{\tilde{r}} \left| \frac{\psi^{[-2]}(\tilde{r}^2 + \tilde{a}^2)}{\sqrt{\tilde{\Delta}}} \right|^2 d\tilde{r} d\omega + \int_{\Sigma_{\tau_2}} D^3 \left| \frac{\alpha^{[-2]} \sqrt{\tilde{r}^2 + \tilde{a}^2}}{\tilde{\Delta}^2} \right|^2 d\tilde{r} d\omega \\ &+ \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| M(L + \underline{L}) \Psi^{[-2]} \right|^2 dV_{\mathcal{H}^+} \\ &\leq C(M) \cdot \left[\int_{\Sigma_{\tau_1}} \tilde{\mathbb{E}}_{\tilde{a}}[\Psi^{[-2]}] d\tilde{r} d\omega + \int_{\Sigma_{\tau_1}} \frac{D}{\tilde{r}} \left| \frac{\psi^{[-2]}(\tilde{r}^2 + \tilde{a}^2)}{\sqrt{\tilde{\Delta}}} \right|^2 d\tilde{r} d\omega + \int_{\Sigma_{\tau_1}} D^3 \left| \frac{\alpha^{[-2]} \sqrt{\tilde{r}^2 + \tilde{a}^2}}{\tilde{\Delta}^2} \right|^2 d\tilde{r} d\omega \right], \end{aligned}$$

for all $\tau_2 \geq \tau_1 \geq 0$.

Theorem 4.21. *Fix $M > 0$ and let $s = \pm 2$. There exists a constant $C = C(M) > 0$ such that for all $\tilde{a} \in [0, 1]$ the following holds:*

For any smooth solution $\Psi^{[s]}$ of the homogeneous spin- s generalised Regge–Wheeler equation (4.11) on a Kerr exterior background with parameters $(a, M) = (\tilde{a}M, M)$, arising from **axisymmetric** initial data as in Proposition 4.19, we have the Morawetz estimate

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \left[\frac{1}{\tilde{r}^4} (\tilde{r} - \tilde{r}_{trap})^2 \cdot \left| M(L + \underline{L})\Psi^{[s]} \right|^2 + \frac{1}{\tilde{r}^2} \cdot \left| M(L - \underline{L})\Psi^{[s]} \right|^2 + \left(b(\tilde{a}) + \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}} \right) \frac{1}{\tilde{r}^3} \cdot \left| \Psi^{[s]} \right|^2 \right. \\
& \quad \left. + \left(b(\tilde{a}) + \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}} \right) \frac{1}{\tilde{r}^5} (\tilde{r} - \tilde{r}_{trap})^2 \cdot \left| \overset{\circ}{\nabla}^{[s]} \Psi^{[s]} \right|^2 \right] d\tilde{t}^* d\tilde{r} d\omega \\
& + \int_{\Sigma_{\tau_2}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| M(L + \underline{L})\Psi^{[s]} \right|^2 dV_{\mathcal{H}^+} \\
& \leq C(M) \cdot \int_{\Sigma_{\tau_1}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega
\end{aligned}$$

for all $\tau_2 \geq \tau_1 \geq 0$, where $b = b(\tilde{a})$ is a non-negative constant satisfying $b(\tilde{a}) \searrow 0$ as $\tilde{a} \nearrow 1$.

Note that in the extremal $\tilde{a} = 1$ case, $b(\tilde{a}) = 0$, and hence the coefficients of $\left| \Psi^{[s]} \right|^2$ and $\left| \overset{\circ}{\nabla}^{[s]} \Psi^{[s]} \right|^2$ exhibit an additional degeneration at $\tilde{r} = \tilde{r}_+$.

5 Multiplier Identities for the generalised Regge–Wheeler equation

In this section we derive multiplier identities for the generalised Regge–Wheeler equation (4.9). These estimates will be useful to prove energy boundedness and Morawetz estimates. In this section, Ψ will be written instead of $\Psi^{[s]}$ to simplify the notation.

5.1 T-identity

Multiplying equation (4.9) by $T\bar{\Psi}$, taking real part we get

$$\begin{aligned} \operatorname{Re} \left\{ -\mathcal{J}^{[s]} \cdot T\bar{\Psi} \right\} &= \frac{1}{2} \operatorname{Re} \left\{ (L\underline{L} + \underline{L}L) \Psi T\bar{\Psi} \right\} + \operatorname{Re} \left\{ w \left(\overset{\circ}{\Delta}^{[s]} + s \right) \Psi T\bar{\Psi} \right\} \\ &\quad + \operatorname{Re} \left\{ w \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2\Delta}{(r^2 + a^2)^2} \right) \Psi T\bar{\Psi} \right\} \\ &\quad - \operatorname{Re} \left\{ wa^2 \sin^2 \theta (TT\Psi)(T\bar{\Psi}) \right\} \\ &\quad + \operatorname{Re} \left\{ w2isa \cos \theta (T\Psi)(T\bar{\Psi}) \right\}. \end{aligned}$$

We can compute the terms:

$$\begin{aligned} \frac{1}{2} \operatorname{Re} \left\{ (L\underline{L} + \underline{L}L) \Psi T\bar{\Psi} \right\} &= \frac{1}{16} (L + \underline{L}) \left\{ |(L + \underline{L})\Psi|^2 + |(L - \underline{L})\Psi|^2 \right\} \\ &\quad - \frac{1}{4} (L - \underline{L}) \operatorname{Re} \left\{ (L - \underline{L}) \Psi T\bar{\Psi} \right\}, \\ \int \operatorname{Re} \left\{ w \left(\overset{\circ}{\Delta}^{[s]} + s \right) \Psi T\bar{\Psi} \right\} \sin \theta d\theta d\varphi &= \int (L + \underline{L}) \left\{ \frac{1}{4} w |\overset{\circ}{\nabla}^{[s]} \Psi|^2 \right\} \sin \theta d\theta d\varphi, \\ \operatorname{Re} \left\{ w \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2\Delta}{(r^2 + a^2)^2} \right) \Psi T\bar{\Psi} \right\} &= \frac{1}{4} (L + \underline{L}) \left\{ w \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2\Delta}{(r^2 + a^2)^2} \right) |\Psi|^2 \right\}, \\ -\operatorname{Re} \left\{ wa^2 \sin^2 \theta (TT\Psi)(T\bar{\Psi}) \right\} &= (L + \underline{L}) \left\{ -\frac{1}{4} wa^2 \sin^2 \theta |T\Psi|^2 \right\}. \end{aligned}$$

Thus, after integrating with respect to $\sin \theta d\theta d\varphi$, we obtain

$$\int (L + \underline{L}) \left\{ F_{L+\underline{L}}^T \right\} \sin \theta d\theta d\varphi + \int (L - \underline{L}) \left\{ F_{L-\underline{L}}^T \right\} \sin \theta d\theta d\varphi = - \int \operatorname{Re} \left\{ \mathcal{J}^{[s]} T\bar{\Psi} \right\} \sin \theta d\theta d\varphi \quad (5.1)$$

where

$$\begin{aligned} F_{L+\underline{L}}^T &= \frac{1}{16} \left[\left(1 - wa^2 \sin^2 \theta \right) |(L + \underline{L})\Psi|^2 + |(L - \underline{L})\Psi|^2 \right. \\ &\quad \left. + 4w \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2\Delta}{(r^2 + a^2)^2} \right) |\Psi|^2 + 4w |\overset{\circ}{\nabla}^{[s]} \Psi|^2 \right] \end{aligned}$$

and

$$F_{L-\underline{L}}^T = -\frac{1}{8} \operatorname{Re} \left\{ (L - \underline{L})\Psi \cdot (L + \underline{L})\bar{\Psi} \right\}.$$

5.2 y-identity

Multiplying equation (4.9) by the first order term $y(L - \underline{L})\bar{\Psi}$, where y is a smooth radial function, and taking real parts, we obtain

$$\begin{aligned}
\operatorname{Re} \left\{ -\mathfrak{I}^{[s]} \cdot y(L - \underline{L}) \bar{\Psi} \right\} &= \frac{1}{2} \operatorname{Re} \left\{ (L\underline{L} + \underline{L}L) \Psi \cdot y(L - \underline{L}) \bar{\Psi} \right\} \\
&\quad + \operatorname{Re} \left\{ w \left(\mathring{\Delta}^{[s]} + s \right) \Psi \cdot y(L - \underline{L}) \bar{\Psi} \right\} \\
&\quad + \operatorname{Re} \left\{ w \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right) \Psi \cdot y(L - \underline{L}) \bar{\Psi} \right\} \\
&\quad - \operatorname{Re} \left\{ wa^2 \sin^2 \theta (TT\Psi) \cdot y(L - \underline{L}) \bar{\Psi} \right\} \\
&\quad + \operatorname{Re} \left\{ w2isa \cos \theta (T\Psi) \cdot y(L - \underline{L}) \bar{\Psi} \right\}.
\end{aligned}$$

We can compute the terms:

$$\begin{aligned}
&\operatorname{Re} \left\{ \frac{1}{2} (L\underline{L} + \underline{L}L) \Psi \cdot y(L - \underline{L}) \bar{\Psi} \right\} \\
&= (L + \underline{L}) \left(\frac{y}{4} \operatorname{Re} \left\{ (L + \underline{L}) \Psi (L - \underline{L}) \bar{\Psi} \right\} \right) - (L - \underline{L}) \left(\frac{y}{8} |(L + \underline{L}) \Psi|^2 + \frac{y}{8} |(L - \underline{L}) \Psi|^2 \right) \\
&\quad + \frac{y'}{4} \left(|(L + \underline{L}) \Psi|^2 + |(L - \underline{L}) \Psi|^2 \right), \\
&\int \operatorname{Re} \left\{ w \left(\mathring{\Delta}^{[s]} + s \right) \Psi \cdot y(L - \underline{L}) \bar{\Psi} \right\} \sin \theta d\theta d\varphi \\
&= \int (L - \underline{L}) \left\{ \frac{1}{2} wy \left| \mathring{\nabla}^{[s]} \Psi \right|^2 \right\} \sin \theta d\theta d\varphi - \int (wy)' \left| \mathring{\nabla}^{[s]} \Psi \right|^2 \sin \theta d\theta d\varphi, \\
&\operatorname{Re} \left\{ w \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right) \Psi \cdot y(L - \underline{L}) \bar{\Psi} \right\} \\
&= (L - \underline{L}) \left\{ \frac{1}{2} wy \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right) |\Psi|^2 \right\} \\
&\quad - \left(wy \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right) \right)' |\Psi|^2, \\
&-\operatorname{Re} \left\{ wa^2 \sin^2 \theta (TT\Psi) \cdot y(L - \underline{L}) \bar{\Psi} \right\} \\
&= (L + \underline{L}) \left\{ -\frac{1}{4} wya^2 \sin^2 \theta \operatorname{Re} \left\{ (L + \underline{L}) \Psi (L - \underline{L}) \bar{\Psi} \right\} \right\} + (L - \underline{L}) \left\{ \frac{1}{8} wya^2 \sin^2 \theta |(L + \underline{L}) \Psi|^2 \right\} \\
&\quad - \frac{1}{4} (wy)' a^2 \sin^2 \theta |(L + \underline{L}) \Psi|^2, \\
&\operatorname{Re} \left\{ w2isa \cos \theta (T\Psi) \cdot y(L - \underline{L}) \bar{\Psi} \right\} \\
&= (L - \underline{L}) \left\{ -wysa \cos \theta \operatorname{Im} \left\{ T\Psi \bar{\Psi} \right\} \right\} + 2sa \cos \theta (wy)' \operatorname{Im} \left\{ T\Psi \bar{\Psi} \right\} \\
&\quad + (L + \underline{L}) \left\{ wysa \cos \theta \operatorname{Im} \left\{ \Psi' \bar{\Psi} \right\} \right\},
\end{aligned}$$

where $y' = \frac{1}{2} (L - \underline{L}) y$.

After integrating with respect to $\sin \theta d\theta d\varphi$, we obtain:

$$\begin{aligned}
&\int (L + \underline{L}) \left\{ F_{L+\underline{L}}^y \right\} \sin \theta d\theta d\varphi + \int (L - \underline{L}) \left\{ F_{L-\underline{L}}^y \right\} \sin \theta d\theta d\varphi + \int I^y \sin \theta d\theta d\varphi \\
&= \int \operatorname{Re} \left\{ -\mathfrak{I}^{[s]} \cdot y(L - \underline{L}) \bar{\Psi} \right\} \sin \theta d\theta d\varphi
\end{aligned} \tag{5.2}$$

where

$$F_{L+\underline{L}}^y = \frac{y}{4} \left(1 - wa^2 \sin^2 \theta \right) \operatorname{Re} \left\{ (L + \underline{L}) \Psi (L - \underline{L}) \bar{\Psi} \right\} - wysa \cos \theta \operatorname{Im} \left(\Psi \bar{\Psi}' \right)$$

and

$$F_{L-\underline{L}}^y = -\frac{y}{8}|(L+\underline{L})\Psi|^2 - \frac{y}{8}|(L-\underline{L})\Psi|^2 + \frac{1}{2}wy \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 + \frac{1}{2}wy \left(4 - \frac{6Mr}{r^2+a^2} \frac{r^2-a^2}{r^2+a^2} - \frac{7a^2\Delta}{(r^2+a^2)^2} \right) |\Psi|^2 \\ + \frac{1}{8}wya^2 \sin^2 \theta |(L+\underline{L})\Psi|^2 + wysa \cos \theta \operatorname{Im} (\Psi T \bar{\Psi})$$

and

$$I^y = \frac{y'}{4}|(L+\underline{L})\Psi|^2 + \frac{y'}{4}|(L-\underline{L})\Psi|^2 - (wy)' \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 - \left(wy \left(4 - \frac{6Mr}{r^2+a^2} \frac{r^2-a^2}{r^2+a^2} - \frac{7a^2\Delta}{(r^2+a^2)^2} \right) \right)' |\Psi|^2 \\ - \frac{1}{4}(wy)' a^2 \sin^2 \theta |(L+\underline{L})\Psi|^2 - 2sa \cos \theta (wy)' \operatorname{Im} (\Psi T \bar{\Psi}).$$

5.3 h-identity

Multiplying equation (4.9) by the zero order multiplier $h\bar{\Psi}$, where h is a smooth radial function, and taking real parts, we obtain

$$\operatorname{Re} \left\{ -\mathcal{I}^{[s]} \cdot h\bar{\Psi} \right\} = \frac{1}{2} \operatorname{Re} \left\{ (L\underline{L} + \underline{L}L) \Psi \cdot h\bar{\Psi} \right\} \\ + \operatorname{Re} \left\{ w \left(\overset{\circ}{\Delta}^{[s]} + s \right) \Psi \cdot h\bar{\Psi} \right\} \\ + \operatorname{Re} \left\{ w \left(4 - \frac{6Mr}{r^2+a^2} \frac{r^2-a^2}{r^2+a^2} - \frac{7a^2\Delta}{(r^2+a^2)^2} \right) \Psi \cdot h\bar{\Psi} \right\} \\ - \operatorname{Re} \left\{ wa^2 \sin^2 \theta (TT\Psi) \cdot h\bar{\Psi} \right\} \\ + \operatorname{Re} \left\{ w2isa \cos \theta (T\Psi) \cdot h\bar{\Psi} \right\}.$$

We can compute the terms:

$$\operatorname{Re} \left\{ \frac{1}{2} (L\underline{L} + \underline{L}L) \Psi \cdot h\bar{\Psi} \right\} \\ = \frac{1}{4} (L+\underline{L}) \operatorname{Re} \left\{ (L+\underline{L})\Psi \cdot h\bar{\Psi} \right\} - \frac{1}{4} (L-\underline{L}) \left\{ \operatorname{Re} \left\{ (L-\underline{L})\Psi \cdot h\bar{\Psi} \right\} - h' |\Psi|^2 \right\} \\ + \frac{h}{4} \left[|(L-\underline{L})\Psi|^2 - |(L+\underline{L})\Psi|^2 \right] - \frac{h''}{2} |\Psi|^2, \\ \int \operatorname{Re} \left\{ w \left(\overset{\circ}{\Delta}^{[s]} + s \right) \Psi \cdot h\bar{\Psi} \right\} \sin \theta d\theta d\varphi \\ = \int hw \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \sin \theta d\theta d\varphi, \\ \operatorname{Re} \left\{ w \left(4 - \frac{6Mr}{r^2+a^2} \frac{r^2-a^2}{r^2+a^2} - \frac{7a^2\Delta}{(r^2+a^2)^2} \right) \Psi \cdot h\bar{\Psi} \right\} \\ = hw \left(4 - \frac{6Mr}{r^2+a^2} \frac{r^2-a^2}{r^2+a^2} - \frac{7a^2\Delta}{(r^2+a^2)^2} \right) |\Psi|^2, \\ - \operatorname{Re} \left\{ wa^2 \sin^2 \theta (TT\Psi) \cdot h\bar{\Psi} \right\} \\ = \frac{1}{4} (L+\underline{L}) \left\{ -hwa^2 \sin^2 \theta \operatorname{Re} \left\{ (L+\underline{L})\Psi \cdot \bar{\Psi} \right\} \right\} + \frac{1}{4} hwa^2 \sin^2 \theta |(L+\underline{L})\Psi|^2, \\ \operatorname{Re} \left\{ w2isa \cos \theta (T\Psi) \cdot h\bar{\Psi} \right\} \\ = -2hwsa \cos \theta \operatorname{Im} \left\{ T\Psi \cdot \bar{\Psi} \right\}.$$

After integrating with respect to $\sin \theta d\theta d\varphi$, we obtain

$$\begin{aligned} & \int (L + \underline{L}) \left\{ F_{L+\underline{L}}^h \right\} \sin \theta d\theta d\varphi + \int (L - \underline{L}) \left\{ F_{L-\underline{L}}^h \right\} \sin \theta d\theta d\varphi + \int I^h \sin \theta d\theta d\varphi \\ &= \int \operatorname{Re} \left\{ -\mathfrak{I}^{[s]} \cdot h \bar{\Psi} \right\} \sin \theta d\theta d\varphi \end{aligned} \quad (5.3)$$

where

$$F_{L+\underline{L}}^h = \frac{h}{4} \left(1 - wa^2 \sin^2 \theta \right) \operatorname{Re} \left\{ (L + \underline{L}) \Psi \cdot \bar{\Psi} \right\}$$

and

$$F_{L-\underline{L}}^h = -\frac{h}{4} \operatorname{Re} \left\{ (L - \underline{L}) \Psi \cdot \bar{\Psi} \right\} + \frac{h'}{4} |\Psi|^2$$

and

$$\begin{aligned} I^h &= \frac{h}{4} |(L - \underline{L}) \Psi|^2 - \frac{h}{4} |(L + \underline{L}) \Psi|^2 + hw \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \\ &+ \left[hw \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right) - \frac{h''}{2} \right] |\Psi|^2 \\ &+ \frac{h}{4} wa^2 \sin^2 \theta |(L + \underline{L}) \Psi|^2 + 2hwsa \cos \theta \operatorname{Im} \left\{ \Psi T \bar{\Psi} \right\}. \end{aligned}$$

5.4 f-identity

Adding the y -identity (5.2) and the h -identity (5.3), we get

$$\begin{aligned} & \int \operatorname{Re} \left\{ -\mathfrak{I}^{[s]} \cdot \left(h \bar{\Psi} + 2y \bar{\Psi}' \right) \right\} \sin \theta d\theta d\varphi \\ &= \int (L + \underline{L}) \left\{ \frac{h}{4} \left(1 - wa^2 \sin^2 \theta \right) \operatorname{Re} \left\{ (L + \underline{L}) \Psi \cdot \bar{\Psi} \right\} + \frac{y}{4} \left(1 - wa^2 \sin^2 \theta \right) \operatorname{Re} \left\{ (L + \underline{L}) \Psi \cdot (L - \underline{L}) \bar{\Psi} \right\} \right. \\ &\quad \left. - wysa \cos \theta \operatorname{Im} \left(\Psi \cdot \bar{\Psi}' \right) \right\} \sin \theta d\theta d\varphi \\ &+ \int (L - \underline{L}) \left\{ -\frac{h}{4} \operatorname{Re} \left\{ (L - \underline{L}) \Psi \cdot \bar{\Psi} \right\} - \frac{y}{8} \left(1 - wa^2 \sin^2 \theta \right) |(L + \underline{L}) \Psi|^2 - \frac{y}{8} |(L - \underline{L}) \Psi|^2 + \frac{1}{2} wy \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \right. \\ &\quad \left. + \left[\frac{1}{2} wy \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right) + \frac{h'}{2} \right] |\Psi|^2 \right. \\ &\quad \left. + wysa \cos \theta \operatorname{Im} \left(\Psi \cdot T \bar{\Psi} \right) \right\} \sin \theta d\theta d\varphi \\ &+ \int \left\{ \left(\frac{h}{4} + \frac{y'}{4} \right) |(L - \underline{L}) \Psi|^2 + \left(\frac{1}{4} (y' - h) \left(1 - wa^2 \sin^2 \theta \right) - \frac{1}{4} w' y a^2 \sin^2 \theta \right) |(L + \underline{L}) \Psi|^2 \right. \\ &\quad \left. + \left(w(h - y') - w' y \right) \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \right. \\ &\quad \left. + \left[(h - y') w \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right) - y \left(w \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right) \right)' - \frac{h''}{2} \right] |\Psi|^2 \right. \\ &\quad \left. + 2sa \cos \theta \left(w(h - y') - w' y \right) \operatorname{Im} \left(\Psi \cdot T \bar{\Psi} \right) \right\} \sin \theta d\theta d\varphi. \end{aligned}$$

Hence, choosing $y = f$ and $h = f'$, where f is a smooth radial function, certain terms cancel and we obtain

$$\begin{aligned} & \int (L + \underline{L}) \left\{ F_{L+\underline{L}}^f \right\} \sin \theta d\theta d\varphi + \int (L - \underline{L}) \left\{ F_{L-\underline{L}}^f \right\} \sin \theta d\theta d\varphi + \int I^f \sin \theta d\theta d\varphi \\ &= \int \operatorname{Re} \left\{ -\mathfrak{I}^{[s]} \cdot \left(f' \bar{\Psi} + 2f \bar{\Psi}' \right) \right\} \sin \theta d\theta d\varphi \end{aligned} \quad (5.4)$$

where

$$F_{L+\underline{L}}^f = \frac{f'}{4} \left(1 - wa^2 \sin^2 \theta \right) \text{Re} \{ (L + \underline{L}) \Psi \cdot \bar{\Psi} \} + \frac{f}{4} \left(1 - wa^2 \sin^2 \theta \right) \text{Re} \{ (L + \underline{L}) \Psi \cdot (L - \underline{L}) \bar{\Psi} \} \\ - wfsa \cos \theta \text{Im} \left(\Psi \cdot \bar{\Psi}' \right)$$

and

$$F_{L-\underline{L}}^f = -\frac{f'}{4} \text{Re} \{ (L - \underline{L}) \Psi \cdot \bar{\Psi} \} - \frac{f}{8} \left(1 - wa^2 \sin^2 \theta \right) |(L + \underline{L}) \Psi|^2 - \frac{f}{8} |(L - \underline{L}) \Psi|^2 \\ + \frac{1}{2} wf \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 + \left[\frac{1}{2} wf \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right) + \frac{f''}{2} \right] |\Psi|^2 \\ + wfsa \cos \theta \text{Im} \left(\Psi \cdot T \bar{\Psi} \right)$$

and

$$I^f = \frac{f'}{2} |(L - \underline{L}) \Psi|^2 - \frac{1}{4} w' f a^2 \sin^2 \theta |(L + \underline{L}) \Psi|^2 - w' f \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \\ - \left[f \left(w \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2} \right) \right)' + \frac{f'''}{2} \right] |\Psi|^2 \\ - 2sa \cos \theta w' f \text{Im} \{ \Psi \cdot T \bar{\Psi} \}.$$

6 Energy Boundedness for constant t^* slices (up to error)

In this section we prove a statement which can be thought of as an energy boundedness statement for solutions of equation (4.9) up to an error, for the foliation by the spacelike slices Σ_τ . To this end, we use the T-identity (5.1). The error comes from the right hand side of the equation (4.9) and is proportional to the parameter a .

Note that the non-negativity of the energy we obtain this way is due to the axisymmetry assumption. See the discussion about superradiance in the beginning of 1.5.1.

Note that the usual approach to proving an ILED (for example in the case of the wave equation) is to first show an a priori energy boundedness estimate and then use it when proving the ILED estimate. Due to the error terms on the right hand side of (4.9), we have to instead prove both energy boundedness and ILED "concurrently", coupling them to each other and absorbing all error terms by good terms at the end (after coupling also with the transport estimates).

For the homogeneous generalised Regge–Wheeler equation (4.11), the right hand side is not present, so we in fact obtain full energy boundedness of solutions.

Proposition 6.1. *Let $s = \pm 2$ and $j \geq 2$.*

Let $(\tilde{\alpha}_0^{[s]}, \tilde{\alpha}_1^{[s]}) \in {}^{[s]}H_{loc}^j(\Sigma_0) \times {}^{[s]}H_{loc}^{j-1}(\Sigma_0)$ be an axisymmetric initial data set. Let $\tilde{\alpha}^{[s]}$ be a solution of the Teukolsky equation (4.6) as arising from the well-posedness proposition 4.19. Let $\psi^{[s]}, \Psi^{[s]}$ be as defined by transformations (4.7). We have the identity

$$\begin{aligned} & \int_{\Sigma_{\tau_2}} \mathbb{E}_a[\Psi^{[s]}] dr d\omega + \frac{1}{16Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} |(L + \underline{L})\Psi^{[s]}|^2 dV_{\mathcal{H}^+} \\ &= \int_{\Sigma_{\tau_1}} \mathbb{E}_a[\Psi^{[s]}] dr d\omega + \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[s]} \cdot T\overline{\Psi^{[s]}} \right\} dt^* dr d\omega. \end{aligned} \quad (6.2)$$

Let $(\Psi_0^{[s]}, \Psi_1^{[s]}) \in {}^{[s]}H_{loc}^j(\Sigma_0) \times {}^{[s]}H_{loc}^{j-1}(\Sigma_0)$ be an axisymmetric data set and $\Psi^{[s]}$ a solution of the equation (4.11) as arising from the well-posedness proposition 4.19. We have the identity

$$\begin{aligned} & \int_{\Sigma_{\tau_2}} \mathbb{E}_a[\Psi^{[s]}] dr d\omega + \frac{1}{16Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} |(L + \underline{L})\Psi^{[s]}|^2 dV_{\mathcal{H}^+} \\ &= \int_{\Sigma_{\tau_1}} \mathbb{E}_a[\Psi^{[s]}] dr d\omega. \end{aligned} \quad (6.3)$$

Proof. Throughout this proof, we shall write Ψ instead of $\Psi^{[s]}$ to make the notation less cluttered.

Multiplying (5.1) by $\frac{r^2 + a^2}{\Delta}$ and integrating with respect to $dt^* dr$, we obtain for a spacetime region $\mathcal{M}$:

$$\begin{aligned} & \int \operatorname{Re} \left\{ -\mathfrak{I}^{[s]} \cdot T\overline{\Psi} \right\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega \\ &= \int (L + \underline{L}) \left\{ F_{L+\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega + \int (L - \underline{L}) \left\{ F_{L-\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega \\ &= \int_{\mathcal{M}} (L + \underline{L}) \left\{ F_{L+\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta \rho^2} dV + \int_{\mathcal{M}} (L - \underline{L}) \left\{ F_{L-\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta \rho^2} dV \\ &= \int_{\mathcal{M}} \nabla_\mu \left\{ \left\{ F_{L+\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta \rho^2} (L + \underline{L})^\mu \right\} dV + \int_{\mathcal{M}} \nabla_\mu \left\{ \left\{ F_{L-\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta \rho^2} (L - \underline{L})^\mu \right\} dV \\ &= \int_{\partial \mathcal{M}} \left\{ F_{L+\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta \rho^2} g(L + \underline{L}, n_{\partial \mathcal{M}}) dV_{\partial \mathcal{M}} + \int_{\partial \mathcal{M}} \left\{ F_{L-\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta \rho^2} g(L - \underline{L}, n_{\partial \mathcal{M}}) dV_{\partial \mathcal{M}} \end{aligned}$$

where the penultimate equality follows from the chain rule and the covariant identities lemma 2.5, and the last equality follows from the divergence theorem.

Hence, choosing $\mathcal{M} = \mathcal{R}(\tau_1, \tau_2)$ (with $\tau_2 \geq \tau_1$), we get

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\mathfrak{I}^{[s]} \cdot T\bar{\Psi} \right\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega \\
&= \int_{\Sigma_{\tau_2}} \left\{ F_{L+\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta \rho^2} g(L + \underline{L}, -n_\Sigma) dV_\Sigma + \int_{\Sigma_{\tau_1}} \left\{ F_{L+\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta \rho^2} g(L + \underline{L}, n_\Sigma) dV_\Sigma \\
&\quad + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left\{ F_{L+\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta \rho^2} g(L + \underline{L}, -n_{\mathcal{H}^+}) dV_{\mathcal{H}^+} \\
&\quad + \int_{\Sigma_{\tau_2}} \left\{ F_{L-\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta \rho^2} g(L - \underline{L}, -n_\Sigma) dV_\Sigma + \int_{\Sigma_{\tau_1}} \left\{ F_{L-\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta \rho^2} g(L - \underline{L}, n_\Sigma) dV_\Sigma \\
&\quad + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left\{ F_{L-\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta \rho^2} g(L - \underline{L}, -n_{\mathcal{H}^+}) dV_{\mathcal{H}^+}.
\end{aligned} \tag{6.4}$$

Substituting inner products (see section (2.4)) into (6.4), we obtain

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\mathfrak{I}^{[s]} \cdot T\bar{\Psi} \right\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega \\
&= 2 \int_{\Sigma_{\tau_2}} \left[\left\{ F_{L+\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta} + \left\{ F_{L-\underline{L}}^T \right\} \frac{2Mr}{\Delta} \right] dr d\omega - 2 \int_{\Sigma_{\tau_1}} \left[\left\{ F_{L+\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta} + \left\{ F_{L-\underline{L}}^T \right\} \frac{2Mr}{\Delta} \right] dr d\omega \\
&\quad + \frac{1}{Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left\{ F_{L+\underline{L}}^T \right\} dV_{\mathcal{H}^+}.
\end{aligned} \tag{6.5}$$

We can compute the integrands to be

$$\begin{aligned}
& \left\{ F_{L+\underline{L}}^T \right\} \frac{r^2 + a^2}{\Delta} + \left\{ F_{L-\underline{L}}^T \right\} \frac{2Mr}{\Delta} \\
&= \frac{1}{16} \left[\frac{\rho^2}{r^2 + a^2} |(L + \underline{L})\Psi|^2 + |(L - \underline{L})\Psi|^2 + \frac{4}{r^2 + a^2} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 + \frac{8Mr}{\Delta} |\underline{L}\Psi|^2 \right. \\
&\quad \left. + \frac{4}{r^2 + a^2} \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2\Delta}{(r^2 + a^2)^2} \right) |\Psi|^2 \right]
\end{aligned}$$

and

$$\frac{1}{Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left\{ F_{L+\underline{L}}^T \right\} dV_{\mathcal{H}^+} = \frac{1}{16Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left[|(L + \underline{L})\Psi|^2 + |(L - \underline{L})\Psi|^2 \right] dV_{\mathcal{H}^+}.$$

Hence, (6.5) becomes

$$\begin{aligned}
& \int_{\Sigma_{\tau_2}} \mathbb{E}_a \left[\Psi^{[s]} \right] dr d\omega + \frac{1}{16Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| (L + \underline{L})\Psi^{[s]} \right|^2 dV_{\mathcal{H}^+} \\
&= \int_{\Sigma_{\tau_1}} \mathbb{E}_a \left[\Psi^{[s]} \right] dr d\omega + \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[s]} \cdot T\bar{\Psi}^{[s]} \right\} dt^* dr d\omega.
\end{aligned} \tag{6.6}$$

For the homogeneous case, the same computation follows, the only difference being that the error term is not present, and we obtain a closed energy boundedness statement. $\square$

7 Preliminary ILED estimate for constant t^* slices

In this section we prove an estimate that includes spacetime terms. We will later use this estimate in Section 9 to prove main results.

Proposition 7.1. *Let*

$$\begin{aligned} f &= \left(1 - \frac{r_{\text{trap}}}{r}\right) \left(1 + \frac{M}{r}\right), \\ y &= \left(1 - \frac{r_{\text{trap}}}{r}\right)^2 f. \end{aligned} \tag{7.2}$$

Let $s = \pm 2$ and $j \geq 2$.

Let $(\tilde{\alpha}_0^{[s]}, \tilde{\alpha}_1^{[s]}) \in {}^{[s]}H_{\text{loc}}^j(\Sigma_0) \times {}^{[s]}H_{\text{loc}}^{j-1}(\Sigma_0)$ be an axisymmetric initial data set. Let $\tilde{\alpha}^{[s]}$ be a solution of the Teukolsky equation (4.6) as arising from the well-posedness proposition 4.19. Let $\psi^{[s]}, \Psi^{[s]}$ be as defined by transformations (4.7). We have the estimate:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \text{ST}[\Psi^{[s]}] dt^* dr d\omega + C_{EB} \int_{\Sigma_{\tau_2}} \mathbb{E}_a[\Psi^{[s]}] dr d\omega + \frac{\tilde{C}}{16Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} |(L + \underline{L})\Psi^{[s]}|^2 dV_{\mathcal{H}^+} \\ & \leq (2\tilde{C} - C_{EB}) \int_{\Sigma_{\tau_1}} \mathbb{E}_a[\Psi^{[s]}] dr d\omega \\ & \quad + \tilde{C} \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{J}^{[s]} \cdot T\overline{\Psi^{[s]}} \right\} dt^* dr d\omega \\ & \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{J}^{[s]} \cdot (f' \overline{\Psi^{[s]}} + 2f \overline{\Psi^{[s]}'}) \right\} dt^* dr d\omega \\ & \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{J}^{[s]} \cdot 2y \overline{\Psi^{[s]}'}) \right\} dt^* dr d\omega \end{aligned} \tag{7.3}$$

where

$$\begin{aligned} \text{ST}[\Psi^{[s]}] &= \left[\frac{\partial_r y}{4} + \left((-\partial_r w) f - \partial_r(wy) \right) \left(\frac{a^2 \sin^2 \theta}{4} - aM \cos^2 \theta \right) \right] \cdot |(L + \underline{L})\Psi^{[s]}|^2 \\ & \quad + \left(\frac{\partial_r f}{2} + \frac{\partial_r y}{4} \right) \cdot |(L - \underline{L})\Psi^{[s]}|^2 \\ & \quad - \left[f \partial_r(wP) + \frac{\partial_r f''}{2} + \partial_r(wyP) + \left(2 - \frac{a}{M} \right) \left((\partial_r w) f + \partial_r(wy) \right) \right] \cdot |\Psi^{[s]}|^2. \end{aligned}$$

Let $(\Psi_0^{[s]}, \Psi_1^{[s]}) \in {}^{[s]}H_{\text{loc}}^j(\Sigma_0) \times {}^{[s]}H_{\text{loc}}^{j-1}(\Sigma_0)$ be an axisymmetric data set and $\Psi^{[s]}$ a solution of the equation (4.11) as arising from the well-posedness proposition 4.19. For such $\Psi^{[s]}$, we get an estimate of the same form as (7.3) but without the terms involving $\mathfrak{J}^{[s]}$ on the right hand side.

Proof. Note, we will often write Ψ instead of $\Psi^{[s]}$ throughout the proof, in order to make the notation simpler.

Assume the choices

$$\begin{aligned} f &= \left(1 - \frac{r_{\text{trap}}}{r}\right) \left(1 + \frac{M}{r}\right), \\ y &= \left(1 - \frac{r_{\text{trap}}}{r}\right)^2 f \end{aligned} \tag{7.4}$$

in the f -identity (5.4) and y -identity (5.2).

Multiplying f -identity (5.4) by $\frac{r^2+a^2}{\Delta}$ and integrating with respect to $dt^* dr$, we obtain:

$$\begin{aligned}
& \int \operatorname{Re} \left\{ -\mathfrak{I}^{[s]} \cdot \left(f' \bar{\Psi} + 2f \bar{\Psi}' \right) \right\} \frac{r^2+a^2}{\Delta} dt^* dr d\omega \\
&= \int (L + \underline{L}) \left\{ F_{L+\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta} dt^* dr d\omega + \int (L - \underline{L}) \left\{ F_{L-\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta} dt^* dr d\omega \\
&\quad + \int \left\{ If \right\} \frac{r^2+a^2}{\Delta} dt^* dr d\omega \\
&= \int_{\mathcal{M}} (L + \underline{L}) \left\{ F_{L+\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta \rho^2} dV + \int_{\mathcal{M}} (L - \underline{L}) \left\{ F_{L-\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta \rho^2} dV \\
&\quad + \int \left\{ If \right\} \frac{r^2+a^2}{\Delta} dt^* dr d\omega \\
&= \int_{\mathcal{M}} \nabla_\mu \left\{ \left\{ F_{L+\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta \rho^2} (L + \underline{L})^\mu \right\} dV + \int_{\mathcal{M}} \nabla_\mu \left\{ \left\{ F_{L-\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta \rho^2} (L - \underline{L})^\mu \right\} dV \\
&\quad + \int \left\{ If \right\} \frac{r^2+a^2}{\Delta} dt^* dr d\omega \\
&= \int_{\partial \mathcal{M}} \left\{ F_{L+\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta \rho^2} g(L + \underline{L}, n_{\partial \mathcal{M}}) dV_{\partial \mathcal{M}} + \int_{\partial \mathcal{M}} \left\{ F_{L-\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta \rho^2} g(L - \underline{L}, n_{\partial \mathcal{M}}) dV_{\partial \mathcal{M}} \\
&\quad + \int \left\{ If \right\} \frac{r^2+a^2}{\Delta} dt^* dr d\omega
\end{aligned}$$

where the penultimate equality follows from the chain rule and the covariant identities of Lemma 2.5, and the last equality follows from the divergence theorem.

Hence, choosing $\mathcal{M} = \mathcal{R}(\tau_1, \tau_2)$ (with $\tau_2 \geq \tau_1$), we get:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\mathfrak{I}^{[s]} \cdot \left(f' \bar{\Psi} + 2f \bar{\Psi}' \right) \right\} \frac{r^2+a^2}{\Delta} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ If \right\} \frac{r^2+a^2}{\Delta} dt^* dr d\omega \\
&\quad + \int_{\Sigma_{\tau_2}} \left\{ F_{L+\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta \rho^2} g(L + \underline{L}, -n_\Sigma) dV_\Sigma + \int_{\Sigma_{\tau_1}} \left\{ F_{L+\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta \rho^2} g(L + \underline{L}, n_\Sigma) dV_\Sigma \\
&\quad + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left\{ F_{L+\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta \rho^2} g(L + \underline{L}, -n_{\mathcal{H}^+}) dV_{\mathcal{H}^+} \\
&\quad + \int_{\Sigma_{\tau_2}} \left\{ F_{L-\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta \rho^2} g(L - \underline{L}, -n_\Sigma) dV_\Sigma + \int_{\Sigma_{\tau_1}} \left\{ F_{L-\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta \rho^2} g(L - \underline{L}, n_\Sigma) dV_\Sigma \\
&\quad + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left\{ F_{L-\underline{L}}^f \right\} \frac{r^2+a^2}{\Delta \rho^2} g(L - \underline{L}, -n_{\mathcal{H}^+}) dV_{\mathcal{H}^+}.
\end{aligned}$$

Substituting the inner products, we obtain:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ I^f \right\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega \\
& + 2 \int_{\Sigma_{\tau_2}} \left[\left\{ F_{L+\underline{L}}^f \right\} \frac{r^2 + a^2}{\Delta} + \left\{ F_{L-\underline{L}}^f \right\} \frac{2Mr}{\Delta} \right] dr d\omega \\
& + \frac{1}{Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left\{ F_{L+\underline{L}}^f \right\} dV_{\mathcal{H}^+} \\
& = 2 \int_{\Sigma_{\tau_1}} \left[\left\{ F_{L+\underline{L}}^f \right\} \frac{r^2 + a^2}{\Delta} + \left\{ F_{L-\underline{L}}^f \right\} \frac{2Mr}{\Delta} \right] dr d\omega \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\mathfrak{I}^{[s]} \cdot \left(f' \bar{\Psi} + 2f \bar{\Psi}' \right) \right\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega.
\end{aligned} \tag{7.5}$$

Remark 7.6. Since r_{trap} is the root of $\partial_r w$ in the region $\{r \geq r_+\}$, note that our choice of f makes it so that the coefficients $-w'f$ that occur in I^f are non-negative.

We then get from (7.5) the following:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \frac{\partial_r f}{2} |(L - \underline{L})\Psi|^2 + \frac{1}{4} (-\partial_r w) f a^2 \sin^2 \theta |(L + \underline{L})\Psi|^2 + (-\partial_r w) f \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \right. \\
& \quad \left. - \left[f \partial_r (wP) + \frac{\partial_r f''}{2} \right] |\Psi|^2 + 2sa \cos \theta (-\partial_r w) f \operatorname{Im} (\Psi \cdot T\bar{\Psi}) \right\} dt^* dr d\omega \\
& = 2 \int_{\Sigma_{\tau_1}} \left[\left\{ F_{L+\underline{L}}^f \right\} \frac{r^2 + a^2}{\Delta} + \left\{ F_{L-\underline{L}}^f \right\} \frac{2Mr}{\Delta} \right] dr d\omega - 2 \int_{\Sigma_{\tau_2}} \left[\left\{ F_{L+\underline{L}}^f \right\} \frac{r^2 + a^2}{\Delta} + \left\{ F_{L-\underline{L}}^f \right\} \frac{2Mr}{\Delta} \right] dr d\omega \\
& \quad - \frac{1}{Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left\{ F_{L+\underline{L}}^f \right\} dV_{\mathcal{H}^+} \\
& \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[s]} \cdot \left(f' \bar{\Psi} + 2f \bar{\Psi}' \right) \right\} dt^* dr d\omega.
\end{aligned} \tag{7.7}$$

Remark 7.8. Note that in the above, we don't have good control on the $|(L + \underline{L})\Psi|^2$ term. We only have a term proportional to the parameter a in the spacetime integral. To get control on $|(L + \underline{L})\Psi|^2$ we use the y -identity (5.2).

Multiplying the y -identity (5.2) by $\frac{r^2 + a^2}{\Delta}$ and integrate with respect to $dt^* dr$, we obtain in a spacetime region $\mathcal{M}$:

$$\begin{aligned}
& \int \operatorname{Re} \left\{ -\mathfrak{I}^{[s]} \cdot 2y \bar{\Psi}' \right\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega \\
& = \int (L + \underline{L}) \left\{ F_{L+\underline{L}}^y \right\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega + \int (L - \underline{L}) \left\{ F_{L-\underline{L}}^y \right\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega + \int \{I^y\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega \\
& = \int_{\mathcal{M}} \nabla_\mu \left[\left\{ F_{L+\underline{L}}^y \right\} \frac{r^2 + a^2}{\Delta \rho^2} (L + \underline{L})^\mu \right] dV + \int_{\mathcal{M}} \nabla_\mu \left[\left\{ F_{L-\underline{L}}^y \right\} \frac{r^2 + a^2}{\Delta \rho^2} (L - \underline{L})^\mu \right] dV + \int \{I^y\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega \\
& = \int_{\partial \mathcal{M}} \left\{ F_{L+\underline{L}}^y \right\} \frac{r^2 + a^2}{\Delta \rho^2} g(L + \underline{L}, n_{\partial \mathcal{M}}) dV_{\partial \mathcal{M}} + \int_{\partial \mathcal{M}} \left\{ F_{L-\underline{L}}^y \right\} \frac{r^2 + a^2}{\Delta \rho^2} g(L - \underline{L}, n_{\partial \mathcal{M}}) dV_{\partial \mathcal{M}} + \int \{I^y\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega
\end{aligned}$$

where again, we used chain rule, Lemma (2.5), and the divergence theorem.

Choosing $\mathcal{M} = \mathcal{R}(\tau_1, \tau_2)$ and substituting in formulas for the relevant inner products and volume forms, we obtain:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \{I^y\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega \\
&= -2 \int_{\Sigma_{\tau_2}} \left[\frac{r^2 + a^2}{\Delta} \{F_{L+\underline{L}}^y\} + \frac{2Mr}{\Delta} \{F_{L-\underline{L}}^y\} \right] dr d\omega \\
&+ 2 \int_{\Sigma_{\tau_1}} \left[\frac{r^2 + a^2}{\Delta} \{F_{L+\underline{L}}^y\} + \frac{2Mr}{\Delta} \{F_{L-\underline{L}}^y\} \right] dr d\omega \\
&- \int_{\mathcal{H}^+(\tau_1, \tau_2)} \frac{1}{Mr_+} \{F_{L+\underline{L}}^y\} dV_{\mathcal{H}^+} \\
&+ \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ -\mathfrak{I}^{[s]} \cdot 2y\bar{\Psi}' \right\} \frac{r^2 + a^2}{\Delta} dt^* dr d\omega.
\end{aligned}$$

Substituting for I^y , we can compute this to be

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \frac{\partial_r y}{4} |(L + \underline{L})\Psi|^2 + \frac{\partial_r y}{4} |(L - \underline{L})\Psi|^2 - \partial_r(wy) \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 - \partial_r(wyP) |\Psi|^2 \right. \\
& \quad \left. - \frac{1}{4} \partial_r(wy) a^2 \sin^2 \theta |(L + \underline{L})\Psi|^2 - 2sa \cos \theta \partial_r(wy) \text{Im}(\Psi \cdot T\bar{\Psi}) \right\} dt^* dr d\omega \\
&= 2 \int_{\Sigma_{\tau_1}} \left[\{F_{L+\underline{L}}^y\} \frac{r^2 + a^2}{\Delta} + \{F_{L-\underline{L}}^y\} \frac{2Mr}{\Delta} \right] dr d\omega - 2 \int_{\Sigma_{\tau_2}} \left[\{F_{L+\underline{L}}^y\} \frac{r^2 + a^2}{\Delta} + \{F_{L-\underline{L}}^y\} \frac{2Mr}{\Delta} \right] dr d\omega \\
& \quad - \frac{1}{Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} \{F_{L+\underline{L}}^y\} dV_{\mathcal{H}^+} \\
&+ \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[s]} \cdot 2y\bar{\Psi}' \right\} dt^* dr d\omega.
\end{aligned} \tag{7.9}$$

We now add (7.7) and (7.9). We obtain:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \left[\frac{\partial_r y}{4} + \frac{1}{4} \left((-\partial_r w) f - \partial_r(wy) \right) a^2 \sin^2 \theta \right] |(L + \underline{L})\Psi|^2 + \left(\frac{\partial_r f}{2} + \frac{\partial_r y}{4} \right) |(L - \underline{L})\Psi|^2 \right. \\
& \quad + \left((-\partial_r w) f - \partial_r(wy) \right) \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 - \left[f \partial_r(wP) + \frac{\partial_r f''}{2} + \partial_r(wyP) \right] |\Psi|^2 \\
& \quad \left. + 2sa \cos \theta \left((-\partial_r w) f - \partial_r(wy) \right) \text{Im} \{ \Psi \cdot T\bar{\Psi} \} \right\} dt^* dr d\omega \\
&= \int_{\Sigma_{\tau_1}} 2 \left[\frac{r^2 + a^2}{\Delta} \{F_{L+\underline{L}}^f + F_{L+\underline{L}}^y\} + \frac{2Mr}{\Delta} \{F_{L-\underline{L}}^f + F_{L-\underline{L}}^y\} \right] dr d\omega \\
& \quad - \int_{\Sigma_{\tau_2}} 2 \left[\frac{r^2 + a^2}{\Delta} \{F_{L+\underline{L}}^f + F_{L+\underline{L}}^y\} + \frac{2Mr}{\Delta} \{F_{L-\underline{L}}^f + F_{L-\underline{L}}^y\} \right] dr d\omega \\
& \quad - \frac{1}{Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} \{F_{L+\underline{L}}^f + F_{L+\underline{L}}^y\} dV_{\mathcal{H}^+} \\
&+ \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[s]} \cdot (f'\bar{\Psi} + 2f\bar{\Psi}') \right\} dt^* dr d\omega \\
&+ \int_{\mathcal{R}(\tau_1, \tau_2)} \text{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[s]} \cdot 2y\bar{\Psi}' \right\} dt^* dr d\omega.
\end{aligned} \tag{7.10}$$

For the cross term

$$2sa \cos \theta \left((-\partial_r w) f - \partial_r (wy) \right) \text{Im} \{ \Psi \cdot T \bar{\Psi} \}$$

on the left hand side of the inequality (7.10), we can complete the square and obtain:

$$\begin{aligned} & 2sa \cos \theta \left((-\partial_r w) f - \partial_r (wy) \right) \text{Im} \{ \Psi \cdot T \bar{\Psi} \} \\ &= \left((-\partial_r w) f - \partial_r (wy) \right) \cdot 2 \text{Im} \left\{ \frac{\sqrt{a}}{\sqrt{M}} \Psi \cdot \sqrt{aM} s \cos \theta T \bar{\Psi} \right\} \\ &= \left((-\partial_r w) f - \partial_r (wy) \right) \cdot \left[\left| \sqrt{aM} s \cos \theta T \Psi - i \frac{\sqrt{a}}{\sqrt{M}} \Psi \right|^2 - \frac{a}{M} |\Psi|^2 - aM s^2 \cos^2 \theta |T \Psi|^2 \right] \\ &= \left((-\partial_r w) f - \partial_r (wy) \right) \cdot \left[\left| \sqrt{aM} s \cos \theta T \Psi - i \frac{\sqrt{a}}{\sqrt{M}} \Psi \right|^2 - \frac{a}{M} |\Psi|^2 - aM \cos^2 \theta |(L + \underline{L}) \Psi|^2 \right] \\ &\geq \left((-\partial_r w) f - \partial_r (wy) \right) \cdot \left[-\frac{a}{M} |\Psi|^2 - aM \cos^2 \theta |(L + \underline{L}) \Psi|^2 \right] \end{aligned} \quad (7.11)$$

where the inequality follows from the fact that

$$(-\partial_r w) f - \partial_r (wy) \geq 0.$$

Moreover, from the Poincaré inequality (see page 18 of [12])

$$\int_0^\pi \int_0^{2\pi} d\varphi d\theta \sin \theta \left| \overset{\circ}{\nabla}^{[\pm 2]} \Psi \right|^2 \geq 2 \int_0^\pi \int_0^{2\pi} d\varphi d\theta \sin \theta |\Psi|^2 \quad (7.12)$$

we get, after multiplying by $\left((-\partial_r w) f - \partial_r (wy) \right)$ and integrating in t^* and r , the estimate

$$\int_{\mathcal{R}(\tau_1, \tau_2)} \left((-\partial_r w) f - \partial_r (wy) \right) \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 dt^* dr d\omega \geq \int_{\mathcal{R}(\tau_1, \tau_2)} 2 \left((-\partial_r w) f - \partial_r (wy) \right) |\Psi|^2 dt^* dr d\omega. \quad (7.13)$$

Finally, for the right hand side of inequality (7.10), we have that

$$\int_{\mathcal{H}^+(\tau_1, \tau_2)} \left\{ F_{L+\underline{L}}^f + F_{L+\underline{L}}^y \right\} dV_{\mathcal{H}^+} = 0 \quad (7.14)$$

and

$$\left| \frac{r^2 + a^2}{\Delta} \left\{ F_{L+\underline{L}}^f + F_{L+\underline{L}}^y \right\} + \frac{2Mr}{\Delta} \left\{ F_{L-\underline{L}}^f + F_{L-\underline{L}}^y \right\} \right| \leq C \cdot \mathbb{E}_a \left[\Psi^{[s]} \right], \quad (7.15)$$

where

$$C = C(M) = \max \left\{ 18\sqrt{2} + \frac{8(2 + \sqrt{2})}{M}, 24 + 20\sqrt{2} + 8(2 + 7\sqrt{2})M \right\}. \quad (7.16)$$

For details, see Appendix B.

Combining (7.10) with (7.11), (7.13), and (7.14) and (7.15), we obtain

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \mathbf{ST} \left[\Psi^{[s]} \right] dt^* dr d\omega - 2C \int_{\Sigma_{\tau_2}} \mathbb{E}_a \left[\Psi^{[s]} \right] dr d\omega \\
& \leq 2C \int_{\Sigma_{\tau_1}} \mathbb{E}_a \left[\Psi^{[s]} \right] dr d\omega \\
& \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[s]} \cdot \left(f' \overline{\Psi^{[s]}} + 2f \overline{\Psi^{[s]}}' \right) \right\} dt^* dr d\omega \\
& \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[s]} \cdot 2y \overline{\Psi^{[s]}}' \right\} dt^* dr d\omega
\end{aligned} \tag{7.17}$$

where

$$\begin{aligned}
\mathbf{ST} \left[\Psi^{[s]} \right] &= \left[\frac{\partial_r y}{4} + \left((-\partial_r w) f - \partial_r (wy) \right) \left(\frac{a^2 \sin^2 \theta}{4} - aM \cos^2 \theta \right) \right] \cdot \left| (L + \underline{L}) \Psi^{[s]} \right|^2 \\
& \quad + \left(\frac{\partial_r f}{2} + \frac{\partial_r y}{4} \right) \cdot \left| (L - \underline{L}) \Psi^{[s]} \right|^2 \\
& \quad - \left[f \partial_r (wP) + \frac{\partial_r f''}{2} + \partial_r (wyP) + \left(2 - \frac{a}{M} \right) \left((\partial_r w) f + \partial_r (wy) \right) \right] \cdot \left| \Psi^{[s]} \right|^2.
\end{aligned}$$

Adding a $\tilde{C} := (2C + C_{\text{EB}})$ -multiple (for a positive constant C_{EB}) of energy boundedness (up to error) (6.6) to (7.17), we obtain

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \mathbf{ST} \left[\Psi^{[s]} \right] dt^* dr d\omega + C_{\text{EB}} \int_{\Sigma_{\tau_2}} \mathbb{E}_a \left[\Psi^{[s]} \right] dr d\omega + \frac{\tilde{C}}{16Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| (L + \underline{L}) \Psi^{[s]} \right|^2 dV_{\mathcal{H}^+} \\
& \leq \left(2\tilde{C} - C_{\text{EB}} \right) \int_{\Sigma_{\tau_1}} \mathbb{E}_a \left[\Psi^{[s]} \right] dr d\omega \\
& \quad + \tilde{C} \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[s]} \cdot T \overline{\Psi^{[s]}} \right\} dt^* dr d\omega \\
& \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[s]} \cdot \left(f' \overline{\Psi^{[s]}} + 2f \overline{\Psi^{[s]}}' \right) \right\} dt^* dr d\omega \\
& \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[s]} \cdot 2y \overline{\Psi^{[s]}}' \right\} dt^* dr d\omega.
\end{aligned} \tag{7.18}$$

□

8 Transport Estimates

The results of this section are used in the proof of Theorem 4.20 in section 9.2. Throughout this section, we assume that the quantities $\alpha^{[-2]}$, $\psi^{[-2]}$, and $\Psi^{[-2]}$ satisfy the assumptions of Theorem 4.20.

We prove estimates of the form

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} c_1(r) \cdot \left| \alpha^{[-2]} \right|^2 dt^* dr d\omega + \int_{\Sigma_{\tau_2}} b_1(r) \cdot \left| \alpha^{[-2]} \right|^2 dr d\omega \\ & \leq \int_{\mathcal{R}(\tau_1, \tau_2)} c_2(r) \left| \psi^{[-2]} \right|^2 dt^* dr d\omega + \int_{\Sigma_{\tau_1}} b_1(r) \cdot \left| \alpha^{[-2]} \right|^2 dr d\omega \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} c_3(r) \left| \psi^{[-2]} \right|^2 dt^* dr d\omega + \int_{\Sigma_{\tau_2}} b_2(r) \cdot \left| \psi^{[-2]} \right|^2 dr d\omega \\ & \leq \int_{\mathcal{R}(\tau_1, \tau_2)} c_4(r) \left| \Psi^{[-2]} \right|^2 dt^* dr d\omega + \int_{\Sigma_{\tau_1}} b_2(r) \cdot \left| \psi^{[-2]} \right|^2 dr d\omega \end{aligned}$$

where $c_1, c_2, c_3, c_4, b_1, b_2$ are suitable non-negative weights. Note, the boundary terms appear with a good sign.

Lemma 8.1. *For all $n, \eta \geq 0$, we have*

$$\begin{aligned} & \frac{2\Delta^{\frac{3}{2}}}{(r^2 + a^2)^2} \left(\frac{\Delta}{r^2 + a^2} \right)^{-n} \frac{1}{\rho^2} \left(1 + \frac{1}{r^\eta} \right) \left[\psi^{[-2]} \cdot \overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} + \overline{\psi^{[-2]}} \cdot \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right] \\ & = \nabla_\mu \left[\left(\frac{\Delta}{r^2 + a^2} \right)^{-n} \frac{1}{\rho^2} \left(1 + \frac{1}{r^\eta} \right) \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] \\ & + \left(\frac{\Delta}{r^2 + a^2} \right)^{-n} \frac{1}{\rho^2} \left[\frac{2M(n-1)(r^2 - a^2)}{(r^2 + a^2)^2} \left(1 + \frac{1}{r^\eta} \right) + \frac{\Delta}{r^2 + a^2} \cdot \frac{\eta}{r^{1+\eta}} \right] \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \end{aligned}$$

Proof. Fix $n, \eta \geq 0$.

By definition we have

$$\frac{2\Delta^{\frac{3}{2}} \psi^{[-2]}}{(r^2 + a^2)^2} = L^\mu \nabla_\mu \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right).$$

Thus, using chain rule, we derive:

$$\begin{aligned} & \frac{2\Delta^{\frac{3}{2}}}{(r^2 + a^2)^2} \left(\frac{\Delta}{r^2 + a^2} \right)^{-n} \frac{1}{\rho^2} \left(1 + \frac{1}{r^\eta} \right) \left[\psi^{[-2]} \cdot \overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} + \overline{\psi^{[-2]}} \cdot \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right] \\ & = \left(\frac{\Delta}{r^2 + a^2} \right)^{-n} \frac{1}{\rho^2} \left(1 + \frac{1}{r^\eta} \right) L^\mu \nabla_\mu \left[\left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \right] \\ & = \nabla_\mu \left[\left(\frac{\Delta}{r^2 + a^2} \right)^{-n} \frac{1}{\rho^2} \left(1 + \frac{1}{r^\eta} \right) \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] \\ & - \nabla_\mu \left[\left(\frac{\Delta}{r^2 + a^2} \right)^{-n} \frac{1}{\rho^2} \left(1 + \frac{1}{r^\eta} \right) L^\mu \right] \cdot \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2. \end{aligned}$$

Computing the last term using a covariant identity from Lemma 2.5, we obtain the result. □

Lemma 8.2. For all $\eta \geq 0$ we have:

$$\begin{aligned} & \nabla_\mu \left[\frac{r^2 + a^2}{\Delta \rho^2} \left(1 + \frac{1}{r^\eta} \right) \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] + \frac{1}{2} \frac{\eta}{r^{1+\eta} \rho^2} \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 \\ & \leq \frac{8}{\eta} \left(1 + \frac{1}{r_+^\eta} \right)^2 \frac{r^{1+\eta}}{(r^2 + a^2)^2} \frac{1}{\rho^2} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \end{aligned}$$

Proof. By Lemma 8.1, choosing $n = 1$, we have for all $\eta \geq 0$:

$$\begin{aligned} & \nabla_\mu \left[\frac{r^2 + a^2}{\Delta \rho^2} \left(1 + \frac{1}{r^\eta} \right) \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] + \frac{\eta}{r^{1+\eta} \rho^2} \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 \\ & = \frac{2\Delta^{\frac{3}{2}}}{(r^2 + a^2)^2} \left(\frac{\Delta}{r^2 + a^2} \right)^{-1} \frac{1}{\rho^2} \left(1 + \frac{1}{r^\eta} \right) \cdot 2\text{Re} \left\{ \overline{\psi^{[-2]}} \cdot \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right\} \\ & \leq \frac{2\sqrt{\Delta}}{(r^2 + a^2)\rho^2} \left(1 + \frac{1}{r^\eta} \right) W \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 + \frac{1}{W} \frac{2}{\sqrt{\Delta}(r^2 + a^2)\rho^2} \left(1 + \frac{1}{r^\eta} \right) \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \\ & = \frac{1}{2} \frac{\eta}{r^{1+\eta} \rho^2} \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 + \frac{8}{\eta} \left(1 + \frac{1}{r^\eta} \right)^2 \frac{r^{1+\eta}}{(r^2 + a^2)^2} \frac{1}{\rho^2} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \\ & \leq \frac{1}{2} \frac{\eta}{r^{1+\eta} \rho^2} \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 + \frac{8}{\eta} \left(1 + \frac{1}{r_+^\eta} \right)^2 \frac{r^{1+\eta}}{(r^2 + a^2)^2} \frac{1}{\rho^2} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2, \end{aligned}$$

where we used the Young's inequality with weight $W = \frac{\eta(r^2 + a^2)}{4r\sqrt{\Delta}(1+r^\eta)}$.

Subtracting $\frac{1}{2} \frac{\eta}{r^{1+\eta} \rho^2} \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2$ from both sides gives the result. □

Proposition 8.3. For the constant t^* foliation we have the estimate

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{2} \frac{\eta}{r^{1+\eta}} \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 dt^* dr d\omega \\ & + \int_{\Sigma_{\tau_2}} \frac{r^2 + a^2}{\Delta} \left(1 + \frac{1}{r^\eta} \right) \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 dr d\omega \\ & \leq \frac{8}{\eta} \left(1 + \frac{1}{r_+^\eta} \right)^2 \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{r^{1+\eta}}{(r^2 + a^2)^2} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega \\ & + \int_{\Sigma_{\tau_1}} \frac{r^2 + a^2}{\Delta} \left(1 + \frac{1}{r^\eta} \right) \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 dr d\omega \end{aligned}$$

for all $\eta > 0$.

Proof. Integrate the inequality from Lemma 8.2 in the region $\mathcal{R}(\tau_1, \tau_2)$, use Divergence Theorem, and substitute in inner products. Note, boundary term on $\mathcal{H}^+$ is zero since $\alpha^{[-2]}$ vanishes on $\mathcal{H}^+$ by assumption (in particular, $\alpha^{[-2]}$ is proportional to Δ^2). □

Proposition 8.4. For constant t^* foliation we have the estimate

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{2} \frac{1}{r^2} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega \\ & + \int_{\Sigma_{\tau_2}} \frac{r^2 + a^2}{\Delta r} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dr d\omega \\ & \leq \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{2}{(r^2 + a^2)^2} \left| \Psi^{[-2]} \right|^2 dt^* dr d\omega \\ & + \int_{\Sigma_{\tau_1}} \frac{r^2 + a^2}{\Delta r} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dr d\omega \end{aligned}$$

Proof. Similar to proof of Proposition 8.3. Note, boundary term on $\mathcal{H}^+$ is zero since $\sqrt{\Delta}\psi^{[-2]}$ vanishes on $\mathcal{H}^+$ (in particular, $\sqrt{\Delta}\psi^{[-2]}$ it is proportional to Δ).

□

9 Proof of main results

9.1 The homogeneous generalised Regge–Wheeler equation

In this section we prove Theorem 4.21.

First we prove an inequality which will be useful.

Lemma 9.1. *Let*

$$f = \left(1 - \frac{r_{\text{trap}}}{r}\right) \left(1 + \frac{M}{r}\right),$$

$$y = \left(1 - \frac{r_{\text{trap}}}{r}\right)^2 f,$$

and assume $0 \leq \tau_1 \leq \tau_2$. Assume $\Psi = \Psi^{[s]}$ as in Theorem 4.21.

For any smooth weight function W such that W and $\partial_r W$ are bounded in $[r_+, \infty)$, we have:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} G |(L - \underline{L})\Psi|^2 dt^* dr d\omega \\ & \geq \int_{\mathcal{R}(\tau_1, \tau_2)} \left[2\partial_r \left(\frac{\Delta}{r^2 + a^2} GW \right) - G W^2 \right] \cdot |\Psi|^2 dt^* dr d\omega \\ & \quad - \int_{\Sigma_{\tau_2}} GW \frac{4Mr}{r^2 + a^2} |\Psi|^2 dr d\omega + \int_{\Sigma_{\tau_1}} GW \frac{4Mr}{r^2 + a^2} |\Psi|^2 dr d\omega \end{aligned}$$

where $G = \left(\frac{\partial_r f}{2} + \frac{\partial_r y}{4} \right)$ and we wrote $\Psi = \Psi^{[s]}$ for brevity.

Proof. Note that we have

$$\int_{\mathcal{R}(\tau_1, \tau_2)} G |(L - \underline{L})\Psi + W\Psi|^2 dt^* dr d\omega \geq 0$$

since $G \geq 0$ (as f and y are non-decreasing). Hence we can derive:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} G |(L - \underline{L})\Psi|^2 dt^* dr d\omega \\ & = \int_{\mathcal{R}(\tau_1, \tau_2)} G \left[|(L - \underline{L})\Psi + W\Psi|^2 - W^2 |\Psi|^2 - W \cdot (L - \underline{L}) (|\Psi|^2) \right] dt^* dr d\omega \\ & \geq \int_{\mathcal{R}(\tau_1, \tau_2)} \left[-GW^2 |\Psi|^2 - GW \cdot (L - \underline{L}) (|\Psi|^2) \right] dt^* dr d\omega \\ & = \int_{\mathcal{R}(\tau_1, \tau_2)} \left[-GW^2 |\Psi|^2 + (L - \underline{L}) (GW) \cdot |\Psi|^2 - (L - \underline{L}) (GW |\Psi|^2) \right] dt^* dr d\omega. \end{aligned}$$

Thus we get

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} G |(L - \underline{L})\Psi|^2 dt^* dr d\omega \\ & \geq \int_{\mathcal{R}(\tau_1, \tau_2)} \left[(L - \underline{L}) (GW) - GW^2 \right] \cdot |\Psi|^2 dt^* dr d\omega + \int_{\mathcal{R}(\tau_1, \tau_2)} (L - \underline{L}) \left[-GW |\Psi|^2 \right] dt^* dr d\omega. \end{aligned} \tag{9.2}$$

For the second term on the right hand side of (9.2), we have:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} (L - \underline{L}) \left[-GW|\Psi|^2 \right] dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{\rho^2} (L - \underline{L})^\mu \nabla_\mu \left[GW|\Psi|^2 \right] \rho^2 dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[GW \frac{1}{\rho^2} |\Psi|^2 (L - \underline{L})^\mu \right] dV + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left(\frac{1}{\rho^2} (L - \underline{L})^\mu \right) \cdot GW|\Psi|^2 \rho^2 dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[GW \frac{1}{\rho^2} |\Psi|^2 (L - \underline{L})^\mu \right] dV + \int_{\mathcal{R}(\tau_1, \tau_2)} 2\partial_r \left(\frac{\Delta}{r^2 + a^2} \right) GW|\Psi|^2 dt^* dr d\omega,
\end{aligned}$$

where the last equality follows from the fact that we have (by covariant identities 2.5):

$$\begin{aligned}
\nabla_\mu \left(\frac{1}{\rho^2} (L - \underline{L})^\mu \right) &= \nabla_\mu \left(\frac{r^2 + a^2}{\Delta \rho^2} (L - \underline{L})^\mu \cdot \frac{\Delta}{r^2 + a^2} \right) = \frac{r^2 + a^2}{\Delta \rho^2} (L - \underline{L})^\mu \nabla_\mu \left(\frac{\Delta}{r^2 + a^2} \right) \\
&= \frac{r^2 + a^2}{\Delta \rho^2} (L - \underline{L})^{r*} \nabla_{r^*} \left(\frac{\Delta}{r^2 + a^2} \right) = \frac{r^2 + a^2}{\Delta \rho^2} 2\partial_{r^*} \left(\frac{\Delta}{r^2 + a^2} \right) = \frac{2}{\rho^2} \partial_r \left(\frac{\Delta}{r^2 + a^2} \right).
\end{aligned}$$

Using Divergence Theorem, we can further derive

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} (L - \underline{L}) \left[-GW|\Psi|^2 \right] dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} 2\partial_r \left(\frac{\Delta}{r^2 + a^2} \right) GW|\Psi|^2 dt^* dr d\omega \\
&\quad - \int_{\Sigma_{\tau_2}} GW \frac{1}{\rho^2} |\Psi|^2 g(L - \underline{L}, -n_\Sigma) dV_\Sigma - \int_{\Sigma_{\tau_1}} GW \frac{1}{\rho^2} |\Psi|^2 g(L - \underline{L}, n_\Sigma) dV_\Sigma \\
&\quad - \int_{\mathcal{H}^+(\tau_1, \tau_2)} GW \frac{1}{\rho^2} |\Psi|^2 g(L - \underline{L}, -n_{\mathcal{H}^+}) dV_{\mathcal{H}^+},
\end{aligned}$$

which, after substituting in inner products and volume forms, becomes

$$\begin{aligned}
\int_{\mathcal{R}(\tau_1, \tau_2)} (L - \underline{L}) \left[-GW|\Psi|^2 \right] dt^* dr d\omega &= \int_{\mathcal{R}(\tau_1, \tau_2)} 2\partial_r \left(\frac{\Delta}{r^2 + a^2} \right) GW|\Psi|^2 dt^* dr d\omega \\
&\quad - \int_{\Sigma_{\tau_2}} GW \frac{4Mr}{r^2 + a^2} |\Psi|^2 dr d\omega + \int_{\Sigma_{\tau_1}} GW \frac{4Mr}{r^2 + a^2} |\Psi|^2 dr d\omega.
\end{aligned} \tag{9.3}$$

Combining (9.2) and (9.3), we obtain

$$\begin{aligned}
\int_{\mathcal{R}(\tau_1, \tau_2)} G|(L - \underline{L})\Psi|^2 dt^* dr d\omega &\geq \int_{\mathcal{R}(\tau_1, \tau_2)} \left[(L - \underline{L}) (GW) - GW^2 + 2\partial_r \left(\frac{\Delta}{r^2 + a^2} \right) GW \right] \cdot |\Psi|^2 dt^* dr d\omega \\
&\quad - \int_{\Sigma_{\tau_2}} GW \frac{4Mr}{r^2 + a^2} |\Psi|^2 dr d\omega + \int_{\Sigma_{\tau_1}} GW \frac{4Mr}{r^2 + a^2} |\Psi|^2 dr d\omega.
\end{aligned} \tag{9.4}$$

Since

$$\begin{aligned}
(L - \underline{L}) (GW) + 2\partial_r \left(\frac{\Delta}{r^2 + a^2} \right) GW &= 2\partial_{r^*} (GW) + 2\partial_r \left(\frac{\Delta}{r^2 + a^2} \right) GW \\
&= 2 \frac{\Delta}{r^2 + a^2} \partial_r (GW) + 2\partial_r \left(\frac{\Delta}{r^2 + a^2} \right) GW \\
&= 2\partial_r \left(\frac{\Delta}{r^2 + a^2} GW \right),
\end{aligned}$$

(9.4) is what we wanted to show. □

We now wish to combine Lemma 9.1 with the estimate (7.3) of Proposition 7.1. This is a way of obtaining more positivity for the coefficient of $|\Psi|^2$, using the $|(L - \underline{L})\Psi|^2$ term. This extra positivity is necessary, as we shall see shortly.

By Proposition 7.1, we have:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \left[\frac{\partial_r y}{4} + \left((-\partial_r w) f - \partial_r (wy) \right) \left(\frac{a^2 \sin^2 \theta}{4} - aM \cos^2 \theta \right) \right] \cdot |(L + \underline{L})\Psi|^2 \right. \\
& \quad + \left(\frac{\partial_r f}{2} + \frac{\partial_r y}{4} \right) \cdot |(L - \underline{L})\Psi|^2 \\
& \quad \left. - \left[f \partial_r (wP) + \partial_r (wyP) + \frac{\partial_r f''}{2} + \left(2 - \frac{a}{M} \right) \left((\partial_r w) f + \partial_r (wy) \right) \right] \cdot |\Psi|^2 \right\} dt^* dr d\omega \\
& + C_{\text{EB}} \int_{\Sigma_{\tau_2}} \mathbb{E}_a [\Psi^{[s]}] dr d\omega + \frac{\tilde{C}}{16Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} |(L + \underline{L})\Psi|^2 dV_{\mathcal{H}^+} \\
& \leq (2\tilde{C} - C_{\text{EB}}) \int_{\Sigma_{\tau_1}} \mathbb{E}_a [\Psi^{[s]}] dr d\omega.
\end{aligned} \tag{9.5}$$

Remark 9.6. We note that the error terms generated by the right hand side of the equation are not present, since we are in the homogeneous case.

Combining (9.5) with Lemma 9.1, we obtain

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \left[\frac{\partial_r y}{4} + \left((-\partial_r w) f - \partial_r (wy) \right) \left(\frac{a^2 \sin^2 \theta}{4} - aM \cos^2 \theta \right) \right] \cdot |(L + \underline{L})\Psi|^2 \right. \\
& \quad - \left[f \partial_r (wP) + \partial_r (wyP) + \frac{\partial_r f''}{2} + \left(2 - \frac{a}{M} \right) \left((\partial_r w) f + \partial_r (wy) \right) \right. \\
& \quad \left. \left. - 2 \partial_r (DGW) + GW^2 \right] \cdot |\Psi|^2 \right\} dt^* dr d\omega \\
& + C_{\text{EB}} \int_{\Sigma_{\tau_2}} \mathbb{E}_a [\Psi^{[s]}] dr d\omega - \int_{\Sigma_{\tau_2}} GW \frac{4Mr}{r^2 + a^2} |\Psi|^2 dr d\omega + \frac{\tilde{C}}{16Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} |(L + \underline{L})\Psi|^2 dV_{\mathcal{H}^+} \\
& \leq (2\tilde{C} - C_{\text{EB}}) \int_{\Sigma_{\tau_1}} \mathbb{E}_a [\Psi^{[s]}] dr d\omega - \int_{\Sigma_{\tau_1}} GW \frac{4Mr}{r^2 + a^2} |\Psi|^2 dr d\omega,
\end{aligned} \tag{9.7}$$

where

$$G = \left(\frac{\partial_r f}{2} + \frac{\partial_r y}{4} \right).$$

Note, we have the freedom to choose the weight W .

The goal now is to achieve non-negativity for the coefficients of $|(L + \underline{L})\Psi|^2$ and $|\Psi|^2$ in the spacetime integral on the left hand side of (9.7).

For the coefficient of $|(L + \underline{L})\Psi|^2$, we can derive

$$\begin{aligned}
& \text{coeff. of } |(L + \underline{L})\Psi|^2 \\
&= \frac{1}{M} \left[\frac{\partial_{\tilde{r}} y}{4} + M^2 \left((-\partial_{\tilde{r}} w) f - \partial_{\tilde{r}}(wy) \right) \left(\frac{\tilde{a}^2 \sin^2 \theta}{4} - \tilde{a} \cos^2 \theta \right) \right] \\
&\geq \frac{1}{M} \left[\frac{\partial_{\tilde{r}} y}{4} - \tilde{a} M^2 \left((-\partial_{\tilde{r}} w) f - \partial_{\tilde{r}}(wy) \right) \right] \\
&= \frac{1}{M} \left(1 - \frac{\tilde{r}_{\text{trap}}}{\tilde{r}} \right)^2 \left\{ \frac{1}{4} \left(\frac{3\tilde{r}_{\text{trap}} - 1}{\tilde{r}^2} + \frac{4\tilde{r}_{\text{trap}}}{\tilde{r}^3} \right) \right. \\
&\quad \left. - \tilde{a} \left[\frac{2\tilde{r}Q(\tilde{r})}{(\tilde{r}^2 + \tilde{a}^2)^3} \left(1 + \frac{1}{\tilde{r}} \right) \left(1 + \left(1 - \frac{\tilde{r}_{\text{trap}}}{\tilde{r}} \right)^2 \right) - (M^2 w) \left(\frac{3\tilde{r}_{\text{trap}} - 1}{\tilde{r}^2} + \frac{4\tilde{r}_{\text{trap}}}{\tilde{r}^3} \right) \right] \right\} \\
&=: \frac{1}{M} \left(1 - \frac{\tilde{r}_{\text{trap}}}{\tilde{r}} \right)^2 \left\{ F_{\tilde{a}}(\tilde{r}) \right\},
\end{aligned}$$

where $Q = Q(\tilde{r})$ is a quadratic polynomial satisfying the relation

$$\begin{aligned}
M^2(-\partial_{\tilde{r}} w) &= -\partial_{\tilde{r}} \left(\frac{\tilde{r}^2 + \tilde{a}^2 - 2\tilde{r}}{(\tilde{r}^2 + \tilde{a}^2)^2} \right) = \frac{2}{(\tilde{r}^2 + \tilde{a}^2)^3} (\tilde{r}^3 - 3\tilde{r}^2 + \tilde{a}^2\tilde{r} + \tilde{a}^2) \\
&= \frac{2\tilde{r}}{(\tilde{r}^2 + \tilde{a}^2)^3} \left(1 - \frac{\tilde{r}_{\text{trap}}}{\tilde{r}} \right) Q(\tilde{r}).
\end{aligned}$$

Recall, $\tilde{r}_{\text{trap}}$ is defined as the root of the polynomial $\tilde{r}^3 - 3\tilde{r}^2 + \tilde{a}^2\tilde{r} + \tilde{a}^2$ in the interval $[\tilde{r}_+, \infty)$.

We can verify the non-negativity of $F_{\tilde{a}}(\tilde{r})$ for the entire range $\tilde{a} \in [0, 1]$ using Mathematica (for the code see Appendix D). We can produce plots for multiple values of $\tilde{a}$:

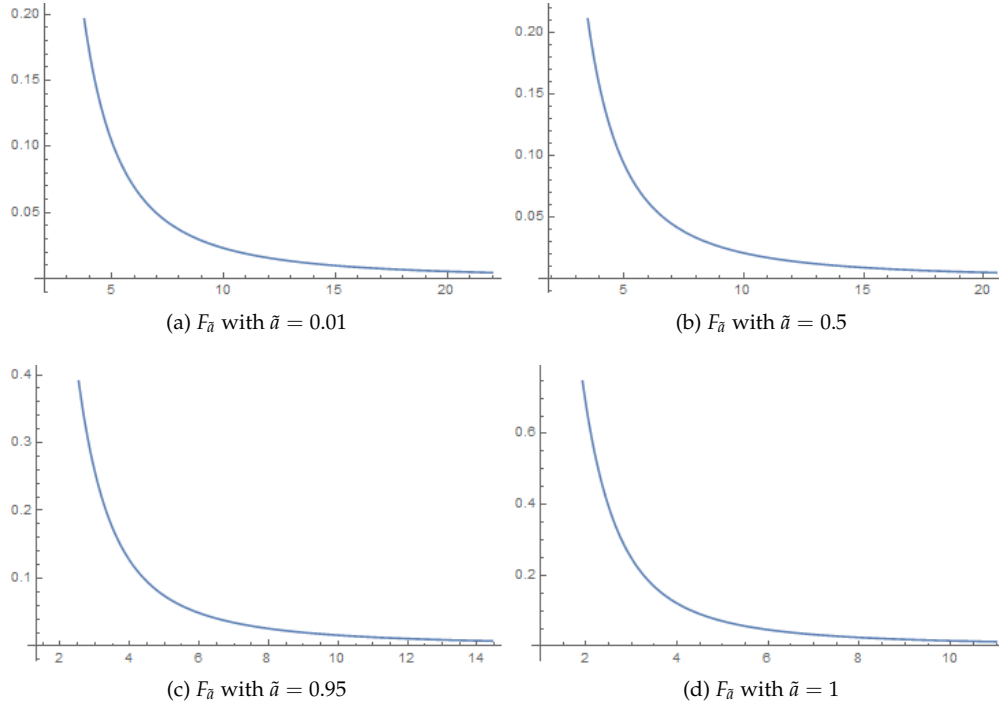


Figure 3: Behaviour of $F_{\tilde{a}}(\tilde{r})$ for different values of $\tilde{a} \in [0, 1]$ and $\tilde{r} \in [\tilde{r}_+, \infty)$.

Remark 9.8. Note that $F_{\tilde{a}}$ is the non-degenerate part of the coefficient of $|(L + \underline{L})\Psi|^2$. Of course, the coefficient vanishes at $\tilde{r}_{\text{trap}}$ quadratically.

Since $F_{\tilde{a}}$ decays like $\frac{1}{\tilde{r}^2}$, we may conclude that

$$\text{coeff. of } |(L + \underline{L})\Psi|^2 \gtrsim \frac{1}{M} \frac{1}{\tilde{r}^4} (\tilde{r} - \tilde{r}_{\text{trap}})^2 \quad (9.9)$$

for all $\tilde{a} \in [0, 1]$.

Let us turn our attention to the coefficient of $|\Psi|^2$ in (9.7). We have

$$\begin{aligned} & \text{coeff. of } |\Psi|^2 \\ &= - \left[f \partial_r (wP) + \partial_r (wyP) + \frac{\partial_r f''}{2} + \left(2 - \frac{a}{M}\right) \left((\partial_r w)f + \partial_r (wy) \right) \right] \\ & \quad + 2 \partial_r \left(\frac{\Delta}{r^2 + a^2} GW \right) - GW^2 \\ &= \frac{1}{M^3} \left\{ - \left[f \partial_{\tilde{r}} (M^2 wP) + \partial_{\tilde{r}} (M^2 wyP) + \frac{1}{2} \partial_{\tilde{r}} \left(D \partial_{\tilde{r}} (D \partial_{\tilde{r}} f) \right) + (2 - \tilde{a}) M^2 \left((\partial_{\tilde{r}} w)f + \partial_{\tilde{r}} (wy) \right) \right] \right. \\ & \quad \left. + 2 \partial_{\tilde{r}} \left(D \left(\frac{\partial_{\tilde{r}} f}{2} + \frac{\partial_{\tilde{r}} y}{4} \right) MW \right) - \left(\frac{\partial_{\tilde{r}} f}{2} + \frac{\partial_{\tilde{r}} y}{4} \right) (MW)^2 \right\} \\ &= \frac{1}{M^3} \left\{ \mathfrak{A}_{\tilde{a}}(\tilde{r}) + \mathfrak{B}_{\tilde{a}}(\tilde{r}) \right\} \end{aligned} \quad (9.10)$$

where

$$\mathfrak{A}_{\tilde{a}}(\tilde{r}) = (2 - \tilde{a}) \left((-\partial_{\tilde{r}} \tilde{w})f - \partial_{\tilde{r}} (\tilde{w}y) \right) - f \partial_{\tilde{r}} (\tilde{w}P) - \partial_{\tilde{r}} (\tilde{w}yP) - \frac{1}{2} \partial_{\tilde{r}} \left(D \partial_{\tilde{r}} (D \partial_{\tilde{r}} f) \right)$$

and

$$\mathfrak{B}_{\tilde{a}}(\tilde{r}) = 2 \partial_{\tilde{r}} (D \tilde{G} \tilde{W}) - \tilde{G} \tilde{W}^2,$$

where

$$\begin{aligned} \tilde{G} &= \left(\frac{\partial_{\tilde{r}} f}{2} + \frac{\partial_{\tilde{r}} y}{4} \right), \\ \tilde{W} &= MW. \end{aligned}$$

Note, one can think of $\mathfrak{A}_{\tilde{a}}$ as the $\tilde{W}$ -independent part of the coefficient of $|\Psi|^2$, and $\mathfrak{B}_{\tilde{a}}$ as the $\tilde{W}$ -dependent part. Note also, we still have freedom of choice of $\tilde{W}$. Figure 4 shows that the choice $\tilde{W} = 0$ is not enough to obtain non-negativity of the coefficient of $|\Psi|^2$ for the full range $\tilde{a} \leq 1$. Our goal now is to examine closely the failure of non-negativity of $\mathfrak{A}_{\tilde{a}}$ and subsequently make a suitable choice of $\tilde{W}$ so that the coefficient of $|\Psi|^2$ is non-negative for all $\tilde{a} \leq 1$.

Using Mathematica (for the code, see Appendix D), we can plot:

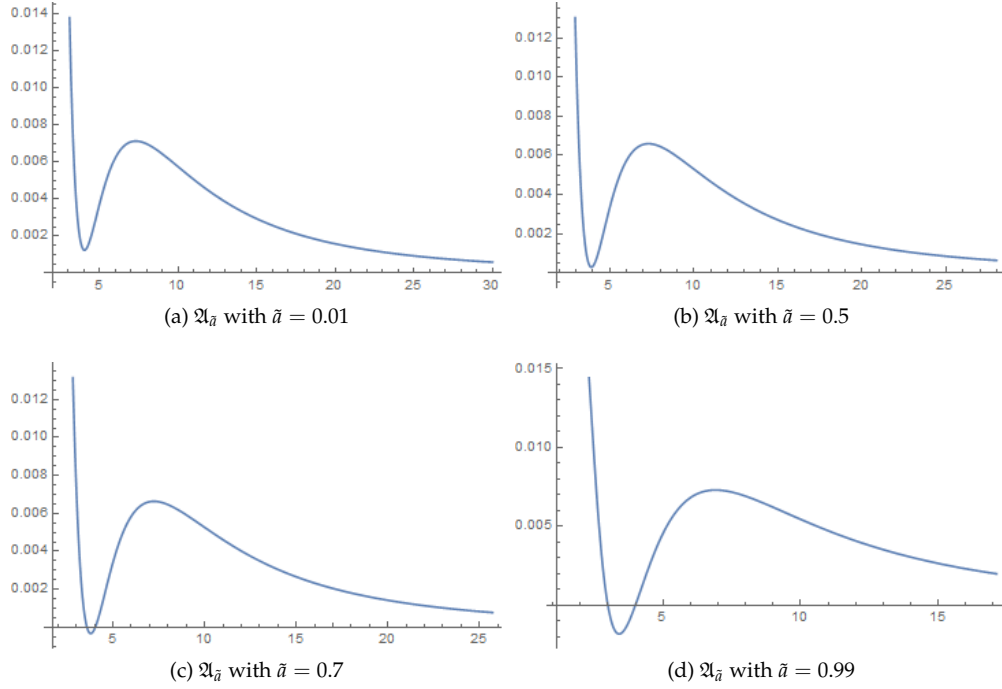


Figure 4: Behaviour of $\mathfrak{A}_{\tilde{a}}$ for different values of $\tilde{a} \in [0, 1)$.

We can observe that as $\tilde{a}$ increases towards 1, the value of $\mathfrak{A}_{\tilde{a}}$ dips below 0 in a small neighbourhood. This can be seen directly since the non-negative term

$$(2 - \tilde{a}) \left((-\partial_{\tilde{r}} \tilde{w}) f - \partial_{\tilde{r}} (\tilde{w} y) \right)$$

becomes smaller as $\tilde{a}$ increases. It is also easy to see that this region $\mathfrak{A}_{\tilde{a}}$ dips below 0 is close to $\tilde{r}_{\text{trap}}$ because the term $((-\partial_{\tilde{r}} \tilde{w}) f - \partial_{\tilde{r}} (\tilde{w} y))$ has a local minimum at $\tilde{r}_{\text{trap}}$, namely it is 0 at this point.

Next we examine the behaviour of $\mathfrak{A}_{\tilde{a}}$ near the horizon $\mathcal{H}^+$ as $\tilde{a}$ varies. We can rewrite $\mathfrak{A}_{\tilde{a}}$ as

$$\begin{aligned} \mathfrak{A}_{\tilde{a}}(\tilde{r}) = & (-\partial_{\tilde{r}} \tilde{w})(f + y)(P + 2 - \tilde{a}) - \tilde{w} \left(\partial_{\tilde{r}}((P + 2 - \tilde{a})y) + f \partial_{\tilde{r}} P \right) \\ & - \frac{1}{2} (\partial_{\tilde{r}} D)^2 (\partial_{\tilde{r}} f) - \frac{1}{2} D \left(\partial_{\tilde{r}}^2 D \right) (\partial_{\tilde{r}} f) - \frac{3}{2} D (\partial_{\tilde{r}} D) \left(\partial_{\tilde{r}}^2 f \right) - \frac{1}{2} D^2 \left(\partial_{\tilde{r}}^3 f \right). \end{aligned}$$

The behaviour near the horizon of the terms above is as follows:

$$\begin{aligned} \tilde{\Delta} &\sim (\tilde{r} - \tilde{r}_+)(\tilde{r} - \tilde{r}_-), \\ D &\sim (\tilde{r} - \tilde{r}_+)(\tilde{r} - \tilde{r}_-), \\ \partial_{\tilde{r}} D &\sim (\tilde{r} - \tilde{a}), \\ \tilde{w} &\sim (\tilde{r} - \tilde{r}_+)(\tilde{r} - \tilde{r}_-), \\ (-\partial_{\tilde{r}} \tilde{w}) &\sim (\tilde{r} - \tilde{r}_0). \end{aligned}$$

Since

$$\lim_{\tilde{a} \rightarrow 1} \tilde{r}_+ = \lim_{\tilde{a} \rightarrow 1} \tilde{r}_- = \lim_{\tilde{a} \rightarrow 1} \tilde{r}_0 = 1,$$

in the limit, all these terms degenerate at the horizon.

Using Mathematica, we may plot:

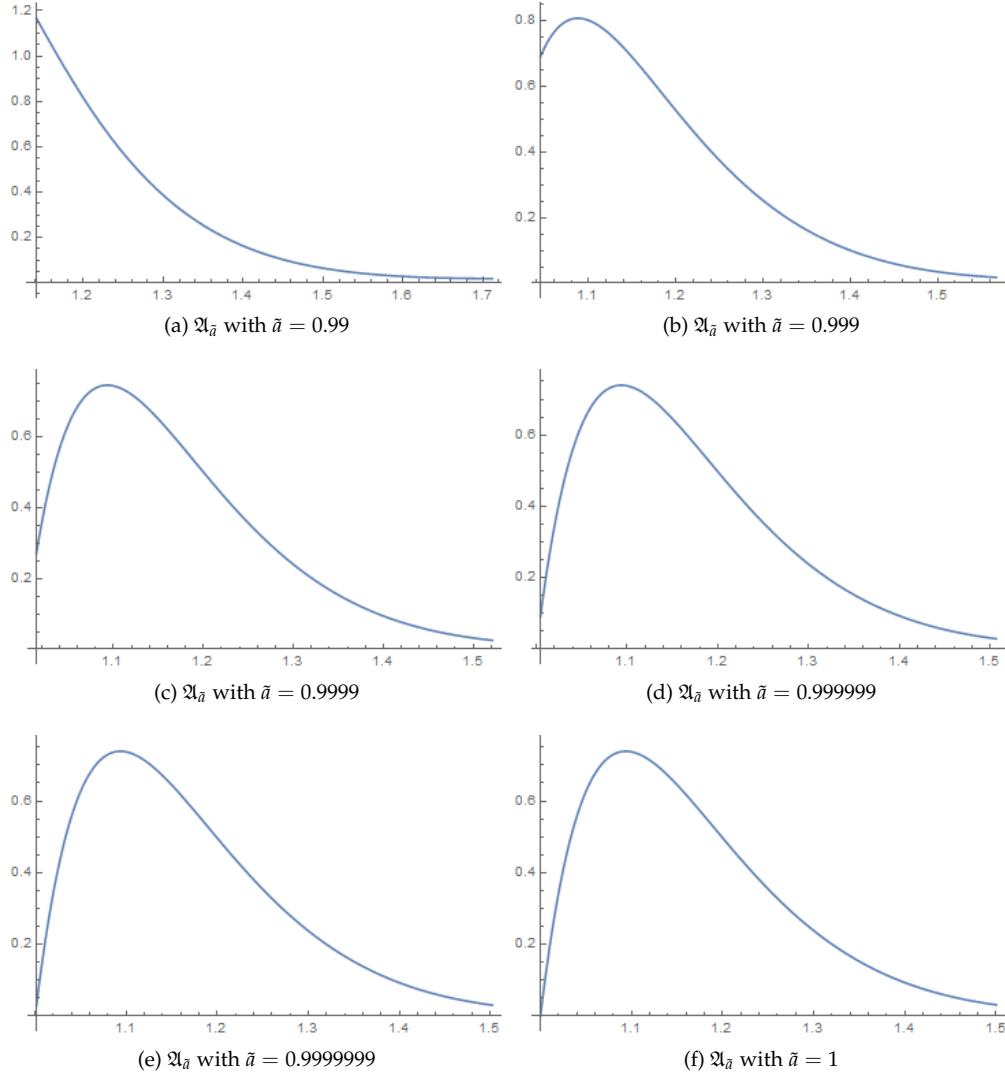


Figure 5: Behaviour of $\mathfrak{A}_{\tilde{a}}(\tilde{r})$ in a region close to the event horizon, as $\tilde{a} \rightarrow 1$.

Having examined the failure of non-negativity of $\mathfrak{A}_{\tilde{a}}$, we collect a list of properties that $\tilde{W}$ has to satisfy in order to remedy the situation.

We need the following properties for the choice of $\tilde{W}$:

1. We need to make the term $\mathfrak{B}_{\tilde{a}}$ positive in the neighbourhood of $\tilde{r}_{\text{trap}}$ where $\mathfrak{A}_{\tilde{a}}$ dips below 0. For this we need:

- $|\tilde{W}|$ small - this ensures that the negative term

$$-\tilde{G}\tilde{W}^2$$

is small, so that we don't introduce significantly more negativity.

- $\partial_{\tilde{r}}\tilde{W}$ is positive and large in the neighbourhood where $\mathfrak{A}_{\tilde{a}}$ is negative (close to $\tilde{r}_{\text{trap}}$). This ensures that the term

$$2\partial_{\tilde{r}}(D\tilde{G}\tilde{W})$$

is positive in this neighbourhood.

2. We need to make sure that $\tilde{W}$ decays fast enough as $\tilde{r}$ tends to infinity. Note that $\mathfrak{A}_{\tilde{a}} \sim \frac{1}{\tilde{r}^3}$ and $\tilde{G} \sim \frac{1}{\tilde{r}^2}$ for large $\tilde{r}$. We want to avoid a situation where

$$-\tilde{G}\tilde{W}^2$$

decays more slowly than $\mathfrak{A}_{\tilde{a}}$, causing $\mathfrak{A}_{\tilde{a}} + \mathfrak{B}_{\tilde{a}}$ to dip below 0 for large $\tilde{r}$. For example, making sure that

$$\tilde{W} \sim \frac{1}{\tilde{r}}$$

for large $\tilde{r}$, will do the trick.

3. We need to ensure that $\mathfrak{B}_{\tilde{a}}$ behaves appropriately near the event horizon. Note we may derive:

$$\begin{aligned} \mathfrak{B}_{\tilde{a}}(\tilde{r}) &= \frac{4(\tilde{r} - \tilde{a})(\tilde{r} + \tilde{a})}{(\tilde{r}^2 + \tilde{a}^2)^2} \tilde{G}\tilde{W} + \frac{2(\tilde{r} - \tilde{r}_+)(\tilde{r} - \tilde{r}_-)}{\tilde{r}^2 + \tilde{a}^2} \partial_{\tilde{r}} (\tilde{G}\tilde{W}) - \tilde{G}\tilde{W}^2 \\ &= (\tilde{r} - \tilde{r}_+)^2 \left[\frac{\tilde{r} - \tilde{a}}{\tilde{r} - \tilde{r}_+} \left(\frac{\tilde{W}}{\tilde{r} - \tilde{r}_+} \right) \frac{4(\tilde{r} + \tilde{a})}{(\tilde{r}^2 + \tilde{a}^2)^2} \tilde{G} + \frac{\tilde{r} - \tilde{r}_-}{\tilde{r} - \tilde{r}_+} \frac{2}{\tilde{r}^2 + \tilde{a}^2} \partial_{\tilde{r}} \left(\tilde{G} \left(\frac{\tilde{W}}{\tilde{r} - \tilde{r}_+} \right) (\tilde{r} - \tilde{r}_+) \right) - \tilde{G} \left(\frac{\tilde{W}}{\tilde{r} - \tilde{r}_+} \right)^2 \right] \end{aligned}$$

So if we choose $\tilde{W}$ such that $\tilde{W} \sim (\tilde{r} - \tilde{r}_+)$, then $\mathfrak{B}_{\tilde{a}} \sim (\tilde{r} - \tilde{r}_+)^2$ near the horizon, and the expression in the square brackets has a finite limit as $\tilde{a} \rightarrow 1$ at the horizon.

We can make the choice:

$$\tilde{W} := \frac{1}{20} \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}^2} \arctan \left(4(\tilde{r} - 1.5 \tilde{r}_{\text{trap}}) \right) \quad (9.11)$$

We can plot:

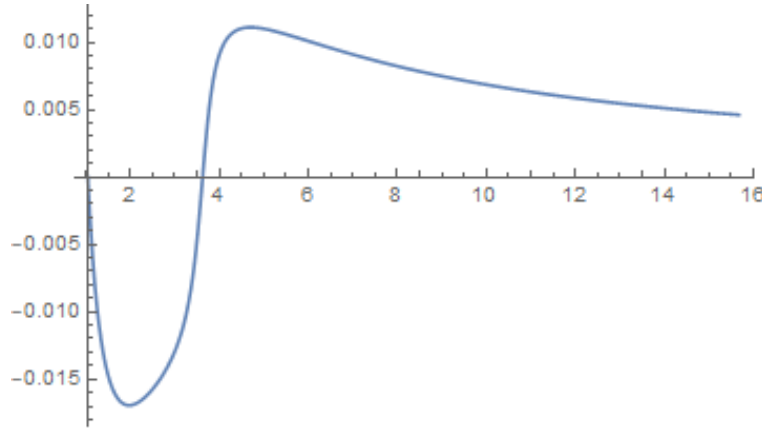


Figure 6: Plot of $\tilde{W}(\tilde{r})$, for the choice of $\tilde{W}$ (9.11).

We can summarize our numerical experiments in the following plots, where we plot $\mathfrak{A}_{\tilde{a}}$ and $\mathfrak{B}_{\tilde{a}}$ for different values of $\tilde{a} \in [0, 1]$:

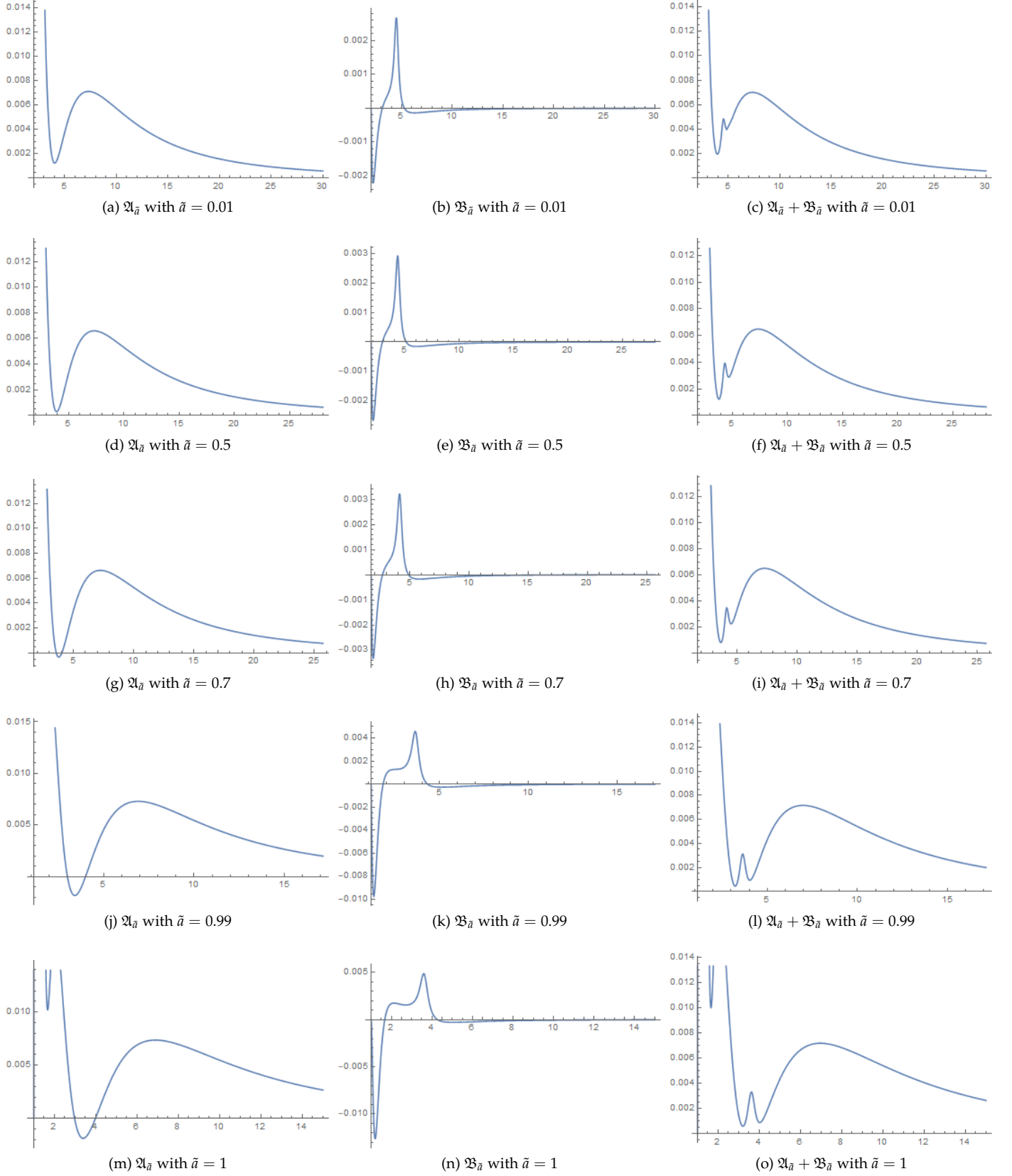


Figure 7: This figure demonstrates how the choice our $\tilde{W}$ makes the coefficient of $|\Psi|^2$ non-negative for all $\tilde{a} \in [0, 1]$ (for $\tilde{a} = 1$ note the degeneration at $\tilde{r} = \tilde{r}_+$).

Having made an appropriate choice of $\tilde{W}$, we can conclude that

$$\text{coeff. of } |\Psi|^2 \gtrsim \frac{1}{M^3} \left(b(\tilde{a}) + \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}} \right) \frac{1}{\tilde{r}^3} \quad (9.12)$$

for all $\tilde{r} \in [\tilde{r}_+, \infty)$ and all $\tilde{a} \in [0, 1]$, where $b(\tilde{a}) \searrow 0$ as $\tilde{a} \nearrow 1$.

Moreover, for boundary terms we have:

$$\begin{aligned} & C_{\text{EB}} \int_{\Sigma_{\tau_2}} \mathbb{E}_a \left[\Psi^{[s]} \right] dr d\omega - \int_{\Sigma_{\tau_2}} GW \frac{4Mr}{r^2 + a^2} |\Psi|^2 dr d\omega \\ &= C_{\text{EB}} \int_{\Sigma_{\tau_2}} \left[\frac{\rho^2}{8(r^2 + a^2)} |(L + \underline{L})\Psi|^2 + \frac{1}{8} |(L - \underline{L})\Psi|^2 + \frac{Mr}{\Delta} |\underline{L}\Psi|^2 + \frac{1}{2(r^2 + a^2)} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \right] dr d\omega \\ &+ \int_{\Sigma_{\tau_2}} \left[\frac{C_{\text{EB}}}{2(r^2 + a^2)} P - GW \frac{4Mr}{r^2 + a^2} \right] \cdot |\Psi|^2 dr d\omega \\ &\geq \frac{C_{\text{EB}}}{16M} \int_{\Sigma_{\tau_2}} \left[|M(L + \underline{L})\Psi|^2 + |(L - \underline{L})\Psi|^2 + \frac{\tilde{r}}{\tilde{\Delta}} |M\underline{L}\Psi|^2 + \frac{1}{\tilde{r}^2} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \right] d\tilde{r} d\omega \\ &+ \frac{1}{4M} \int_{\Sigma_{\tau_2}} \left[\frac{C_{\text{EB}}}{\tilde{r}^2 + \tilde{a}^2} - \frac{4}{5} \tilde{G} \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}(\tilde{r}^2 + \tilde{a}^2)} \arctan \left(4(\tilde{r} - 1.5\tilde{r}_{\text{trap}}) \right) \right] \cdot |\Psi|^2 d\tilde{r} d\omega. \end{aligned}$$

Since we have

$$\tilde{G} \sim \frac{1}{\tilde{r}^2},$$

we can estimate (for C_{EB} sufficiently big)

$$\frac{C_{\text{EB}}}{\tilde{r}^2 + \tilde{a}^2} - \frac{4}{5} \tilde{G} \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}(\tilde{r}^2 + \tilde{a}^2)} \arctan \left(4(\tilde{r} - 1.5\tilde{r}_{\text{trap}}) \right) \gtrsim \frac{1}{\tilde{r}^2}$$

for all $\tilde{a} \in [0, 1]$. Hence,

$$C_{\text{EB}} \int_{\Sigma_{\tau_2}} \mathbb{E}_a \left[\Psi^{[s]} \right] dr d\omega - \int_{\Sigma_{\tau_2}} GW \frac{4Mr}{r^2 + a^2} |\Psi|^2 dr d\omega \gtrsim \frac{1}{M} \int_{\Sigma_{\tau_2}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega, \quad (9.13)$$

where, recall,

$$\tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] = |M(L + \underline{L})\Psi^{[s]}|^2 + |M(L - \underline{L})\Psi^{[s]}|^2 + \frac{\tilde{r}}{\tilde{\Delta}} |M\underline{L}\Psi^{[s]}|^2 + \frac{1}{\tilde{r}^2} |\Psi^{[s]}|^2 + \frac{1}{\tilde{r}^2} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2.$$

Similarly,

$$\begin{aligned} & (2\tilde{C} - C_{\text{EB}}) \int_{\Sigma_{\tau_1}} \mathbb{E}_a \left[\Psi^{[s]} \right] dr d\omega - \int_{\Sigma_{\tau_1}} GW \frac{4Mr}{r^2 + a^2} |\Psi|^2 dr d\omega \\ &\lesssim (4C + C_{\text{EB}}) \frac{1}{M} \int_{\Sigma_{\tau_1}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega. \end{aligned} \quad (9.14)$$

Combining (9.7) with (9.9), (9.12), (9.13) and (9.14), we obtain

$$\begin{aligned} & \frac{1}{M} \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \frac{1}{\tilde{r}^4} (\tilde{r} - \tilde{r}_{\text{trap}})^2 \cdot |M(L + \underline{L})\Psi|^2 + \left(b(\tilde{a}) + \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}} \right) \frac{1}{\tilde{r}^3} \cdot |\Psi|^2 \right\} d\tilde{t}^* d\tilde{r} d\omega \\ &+ \frac{1}{M} \int_{\Sigma_{\tau_2}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega + \frac{2C + C_{\text{EB}}}{M^4} \int_{\mathcal{H}^+(\tau_1, \tau_2)} |M(L + \underline{L})\Psi|^2 dV_{\mathcal{H}^+} \\ &\lesssim \frac{4C + C_{\text{EB}}}{M} \int_{\Sigma_{\tau_1}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega \end{aligned} \quad (9.15)$$

for all $\tilde{a} \in [0, 1]$.

At the moment, we are lacking control on $(L - \underline{L})\Psi$ and $\nabla^{\circ[s]}\Psi$. To gain control on $(L - \underline{L})\Psi$, we go back to (9.5), from which we get:

$$\begin{aligned}
& \frac{1}{M} \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{\tilde{r}^2} \cdot \left| M(L - \underline{L})\Psi \right|^2 d\tilde{t}^* d\tilde{r} d\omega \\
& \lesssim \int_{\mathcal{R}(\tau_1, \tau_2)} \left(\frac{\partial_r f}{2} + \frac{\partial_r y}{4} \right) \cdot \left| (L - \underline{L})\Psi \right|^2 dt^* dr d\omega \\
& \leq \left(2\tilde{C} - C_{\text{EB}} \right) \int_{\Sigma_{\tau_1}} \mathbb{E}_a \left[\Psi^{[s]} \right] dr d\omega \\
& \quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \left[\frac{\partial_r y}{4} + \left((-\partial_r w) f - \partial_r (wy) \right) \left(\frac{a^2 \sin^2 \theta}{4} - aM \cos^2 \theta \right) \right] \cdot \left| (L + \underline{L})\Psi \right|^2 dt^* dr d\omega \\
& \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \left[f \partial_r (wP) + \partial_r (wyP) + \frac{\partial_r f''}{2} + \left(2 - \frac{a}{M} \right) \left((\partial_r w) f + \partial_r (wy) \right) \right] \cdot \left| \Psi \right|^2 dt^* dr d\omega \\
& \lesssim \frac{4C + C_{\text{EB}}}{M} \int_{\Sigma_{\tau_1}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega \\
& \quad + \frac{1}{M} \int_{\mathcal{R}(\tau_1, \tau_2)} \left| \frac{\partial_{\tilde{r}} y}{4} + M^2 \left((-\partial_{\tilde{r}} w) f - \partial_{\tilde{r}} (wy) \right) \left(\frac{\tilde{a}^2 \sin^2 \theta}{4} - \tilde{a} \cos^2 \theta \right) \right| \cdot \left| M(L + \underline{L})\Psi \right|^2 d\tilde{t}^* d\tilde{r} d\omega \quad (9.16) \\
& \quad + \frac{1}{M} \int_{\mathcal{R}(\tau_1, \tau_2)} \left| f \partial_{\tilde{r}} (M^2 wP) + \partial_{\tilde{r}} (M^2 wyP) + \frac{1}{2} \partial_{\tilde{r}} \left(D \partial_{\tilde{r}} (D \partial_{\tilde{r}} f) \right) \right. \\
& \quad \quad \left. + \left(2 - \tilde{a} \right) M^2 \left((\partial_{\tilde{r}} w) f + \partial_{\tilde{r}} (wy) \right) \right| \cdot \left| \Psi \right|^2 d\tilde{t}^* d\tilde{r} d\omega \\
& \lesssim \frac{4C + C_{\text{EB}}}{M} \int_{\Sigma_{\tau_1}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega \\
& \quad + \frac{1}{M} \int_{\mathcal{R}(\tau_1, \tau_2)} \left[\frac{1}{\tilde{r}^4} (\tilde{r} - \tilde{r}_{\text{trap}})^2 \cdot \left| M(L + \underline{L})\Psi \right|^2 + \left(b(\tilde{a}) + \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}} \right) \frac{1}{\tilde{r}^3} \cdot \left| \Psi \right|^2 \right] d\tilde{t}^* d\tilde{r} d\omega \\
& \lesssim \frac{4C + C_{\text{EB}}}{M} \int_{\Sigma_{\tau_1}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega,
\end{aligned}$$

where the last inequality follows from (9.15).

Hence, combining this with (9.15), we get

$$\begin{aligned}
& \frac{1}{M} \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \frac{1}{\tilde{r}^4} (\tilde{r} - \tilde{r}_{\text{trap}})^2 \cdot \left| M(L + \underline{L})\Psi \right|^2 + \frac{1}{\tilde{r}^2} \cdot \left| M(L - \underline{L})\Psi \right|^2 + \left(b(\tilde{a}) + \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}} \right) \frac{1}{\tilde{r}^3} \cdot \left| \Psi \right|^2 \right\} d\tilde{t}^* d\tilde{r} d\omega \\
& \quad + \frac{1}{M} \int_{\Sigma_{\tau_2}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega + \frac{2C + C_{\text{EB}}}{M^4} \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| M(L + \underline{L})\Psi \right|^2 dV_{\mathcal{H}^+} \\
& \lesssim \frac{4C + C_{\text{EB}}}{M} \int_{\Sigma_{\tau_1}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega \quad (9.17)
\end{aligned}$$

for all $\tilde{a} \in [0, 1]$.

To gain control on the $\nabla^{\circ[s]}\Psi$ term, we go back to (7.18) with the exception that we don't use the Poincaré inequality (7.12). We get

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \left((-\partial_r w) f - \partial_r (wy) \right) \cdot \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 dt^* dr d\omega \\
& \leq \left(2\tilde{C} - C_{\text{EB}} \right) \int_{\Sigma_{\tau_1}} \mathbb{E}_a \left[\Psi^{[s]} \right] dr d\omega \\
& \quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \left[\frac{\partial_r y}{4} + \left((-\partial_r w) f - \partial_r (wy) \right) \left(\frac{a^2 \sin^2 \theta}{4} - aM \cos^2 \theta \right) \right] \cdot \left| (L + \underline{L}) \Psi \right|^2 dt^* dr d\omega \\
& \quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \left(\frac{\partial_r f}{2} + \frac{\partial_r y}{4} \right) \cdot \left| (L - \underline{L}) \Psi \right|^2 dt^* dr d\omega \\
& \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \left[f \partial_r (wP) + \partial_r (wyP) + \frac{\partial_r f''}{2} - \frac{a}{M} \left((\partial_r w) f + \partial_r (wy) \right) \right] \cdot \left| \Psi \right|^2 dt^* dr d\omega.
\end{aligned} \tag{9.18}$$

By (9.17), we already can control the spacetime terms on the right hand side of (9.18) by the energy on Σ_{τ_1} . Hence, a similar computation to (9.16) yields the estimate

$$\frac{1}{M} \int_{\mathcal{R}(\tau_1, \tau_2)} \left(b(\tilde{a}) + \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}} \right) \frac{1}{\tilde{r}^5} (\tilde{r} - \tilde{r}_{\text{trap}})^2 \cdot \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 d\tilde{t}^* d\tilde{r} d\omega \lesssim \frac{4C + C_{\text{EB}}}{M} \int_{\Sigma_{\tau_1}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega, \tag{9.19}$$

also in light of the fact that

$$(-\partial_{\tilde{r}} \tilde{w}) f - \partial_{\tilde{r}} (\tilde{w} y) \gtrsim \left(b(\tilde{a}) + \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}} \right) \frac{1}{\tilde{r}^5} (\tilde{r} - \tilde{r}_{\text{trap}})^2 \quad \forall \tilde{a} \in [0, 1]. \tag{9.20}$$

Combining (9.17) and (9.19), we obtain for all $\tilde{a} \in [0, 1]$:

$$\begin{aligned}
& \frac{1}{M} \int_{\mathcal{R}(\tau_1, \tau_2)} \left\{ \frac{1}{\tilde{r}^4} (\tilde{r} - \tilde{r}_{\text{trap}})^2 \cdot \left| M(L + \underline{L}) \Psi \right|^2 + \frac{1}{\tilde{r}^2} \cdot \left| M(L - \underline{L}) \Psi \right|^2 + \left(b(\tilde{a}) + \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}} \right) \frac{1}{\tilde{r}^3} \cdot \left| \Psi \right|^2 \right. \\
& \quad \left. + \left(b(\tilde{a}) + \frac{\tilde{r} - \tilde{r}_+}{\tilde{r}} \right) \frac{1}{\tilde{r}^5} (\tilde{r} - \tilde{r}_{\text{trap}})^2 \cdot \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \right\} d\tilde{t}^* d\tilde{r} d\omega \\
& \quad + \frac{1}{M} \int_{\Sigma_{\tau_2}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega + \frac{2C + C_{\text{EB}}}{M^4} \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| M(L + \underline{L}) \Psi \right|^2 dV_{\mathcal{H}^+} \\
& \lesssim \frac{4C + C_{\text{EB}}}{M} \int_{\Sigma_{\tau_1}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[s]} \right] d\tilde{r} d\omega.
\end{aligned} \tag{9.21}$$

This concludes the proof of Theorem 4.21.

9.2 The generalised Regge–Wheeler equation for $s = -2$

In this section we prove Theorem 4.20.

Set

$$s = -2.$$

We write

$$-\frac{r^2 + a^2}{\Delta} \mathcal{I}^{[-2]} = a^2 h_1 \cdot \left(\sqrt{\Delta} \psi^{[-2]} \right) + a^2 h_2 \cdot \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right),$$

where

$$h_1 = \frac{20}{(r^2 + a^2)^3} \left(r^3 - 3Mr^2 + a^2 r + a^2 M \right) = \frac{20}{M^3 (\tilde{r}^2 + \tilde{a}^2)^3} \left(\tilde{r}^3 - 3\tilde{r}^2 + \tilde{a}^2 \tilde{r} + \tilde{a}^2 \right) = \frac{1}{M^3} \tilde{h}_1$$

and

$$h_2 = \frac{-3}{(r^2 + a^2)^3} (r^4 + 10Mr^3 - 6Ma^2r - a^4) = \frac{-3}{M^2(\tilde{r}^2 + \tilde{a}^2)^3} (\tilde{r}^4 + 10\tilde{r}^3 - 6\tilde{a}^2\tilde{r} - \tilde{a}^4) = \frac{1}{M^2} \tilde{h}_2.$$

Firstly, we prove a lemma which classifies the types of error terms coming from the right hand side of equation (4.1).

Lemma 9.22. *For $\Psi^{[-2]}$ as in Theorem (4.20), we have the estimate*

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \mathbf{ST} [\Psi^{[-2]}] dt^* dr d\omega + C_{EB} \int_{\Sigma_{\tau_2}} \mathbb{E}_a [\Psi^{[-2]}] dr d\omega + \frac{\tilde{C}}{16Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} |(L + \underline{L})\Psi^{[-2]}|^2 dV_{\mathcal{H}^+} \\ & \leq (2\tilde{C} - C_{EB}) \int_{\Sigma_{\tau_1}} \mathbb{E}_a [\Psi^{[-2]}] dr d\omega \\ & \quad + a^2 (\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3) \end{aligned} \tag{9.23}$$

where

$$H_1 = \tilde{C} \cdot h_1, \quad H_2 = \tilde{C} \cdot h_2.$$

Here, $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ are error terms of three different types:

- $\mathcal{E}_1$ are spacetime error terms that contain $\Psi^{[-2]}$.
- $\mathcal{E}_2$ are spacetime error terms containing only $\alpha^{[-2]}$ and $\psi^{[-2]}$, and not $\Psi^{[-2]}$.
- $\mathcal{E}_3$ are boundary error terms.

In particular,

$$\begin{aligned} \mathcal{E}_1 &= \int_{\mathcal{R}(\tau_1, \tau_2)} \left(2(f + y)h_1 - H_1 \right) \cdot \text{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}}' \right\} dt^* dr d\omega \\ & \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \left(2(f + y)h_2 - H_2 \right) \cdot \text{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}}' \right\} dt^* dr d\omega \\ & \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{\Delta H_1}{(r^2 + a^2)^2} |\Psi^{[-2]}|^2 dt^* dr d\omega, \\ \mathcal{E}_2 &= \int_{\mathcal{R}(\tau_1, \tau_2)} \left[\partial_r \left(\frac{r^2 + a^2}{2} \left(h_1 f' - \partial_r \left(\frac{\Delta H_1}{r^2 + a^2} \right) \right) \right) + 2h_2 f' - 3\partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) \right] \cdot |\sqrt{\Delta} \psi^{[-2]}|^2 dt^* dr d\omega \\ & \quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left[\frac{r^2 + a^2}{4} \partial_r \left((r^2 + a^2) \left(\partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) - h_2 f' \right) \right) \right] \cdot |\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}}|^2 dt^* dr d\omega, \end{aligned}$$

$$\begin{aligned}
\mathcal{E}_3 = & \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{(r^2 + a^2)^2}{2\Delta\rho^2} \left(\partial_r \left(\frac{\Delta H_1}{r^2 + a^2} \right) + \frac{2\Delta H_2}{(r^2 + a^2)^2} - h_1 f' \right) \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 L^\mu \right] dV \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{(r^2 + a^2)^2}{4\Delta\rho^2} \partial_r \left((r^2 + a^2) \left(h_2 f' - \partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) \right) \right) \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] dV \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{(r^2 + a^2)^2}{\Delta\rho^2} \left(\partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) - h_2 f' \right) \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta} \psi^{[-2]} \right\} L^\mu \right] dV \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{H_1}{\rho^2} \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} L^\mu \right] dV \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{H_2}{\rho^2} \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} L^\mu \right] dV.
\end{aligned}$$

Proof. We expand the error terms on the right hand side of (7.18). For the computations, see Appendix C. $\square$

We shall treat the different types of error terms differently.

For the error terms $\mathcal{E}_1$, we use Young's inequality to obtain the estimate

$$\begin{aligned}
\mathcal{E}_1 \leq & \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{2} \left| H_1 - 2(f + y)h_1 \right| \cdot \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{2} \left| H_2 - 2(f + y)h_2 \right| \cdot \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dt^* dr d\omega \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{8} \left(\left| H_1 - 2(f + y)h_1 \right| + \left| H_2 - 2(f + y)h_2 \right| \right) \cdot \left| (L - \underline{L}) \Psi^{[-2]} \right|^2 dt^* dr d\omega \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{\Delta H_1}{(r^2 + a^2)^2} \left| \Psi^{[-2]} \right|^2 dt^* dr d\omega.
\end{aligned} \tag{9.24}$$

The error terms $\mathcal{E}_2$ we leave in their current form.

For the boundary error terms $\mathcal{E}_3$, we apply divergence theorem in the spacetime region $\mathcal{R}(\tau_1, \tau_2)$ to produce error terms on Σ_{τ_1} , Σ_{τ_2} and $\mathcal{H}^+(\tau_1, \tau_2)$. We obtain:

$$\begin{aligned}
\mathcal{E}_3 = & \int_{\Sigma_{\tau_2}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \cdot \mathcal{E}_3(\Sigma) dr d\omega - \int_{\Sigma_{\tau_1}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \cdot \mathcal{E}_3(\Sigma) dr d\omega \\
& + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \mathcal{E}_3(\mathcal{H}^+) dV_{\mathcal{H}^+}
\end{aligned} \tag{9.25}$$

where

$$\begin{aligned}
\mathcal{E}_3(\Sigma) := & \left(\partial_r \left(\frac{\Delta H_1}{r^2 + a^2} \right) + \frac{2\Delta H_2}{(r^2 + a^2)^2} - h_1 f' \right) \frac{(r^2 + a^2)^2}{2\Delta} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \\
& + \partial_r \left((r^2 + a^2) \left(h_2 f' - \partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) \right) \right) \frac{(r^2 + a^2)^2}{4\Delta} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \\
& + \left(\partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) - h_2 f' \right) \frac{(r^2 + a^2)^2}{\Delta} \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta} \psi^{[-2]} \right\} \\
& + H_1 \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} + H_2 \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_3(\mathcal{H}^+) &:= \frac{1}{2} \left(\partial_r \left(\frac{\Delta H_1}{r^2 + a^2} \right) + \frac{2\Delta H_2}{(r^2 + a^2)^2} - h_1 f' \right) \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \\
&+ \frac{1}{4} \partial_r \left((r^2 + a^2) \left(h_2 f' - \partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) \right) \right) \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \\
&+ \left(\partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) - h_2 f' \right) \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta} \overline{\psi^{[-2]}} \right\} \\
&+ \frac{\Delta H_1}{(r^2 + a^2)^2} \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} + \frac{\Delta H_2}{(r^2 + a^2)^2} \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\}.
\end{aligned}$$

Note that, in fact,

$$\int_{\mathcal{H}^+(\tau_1, \tau_2)} \mathcal{E}_3(\mathcal{H}^+) dV_{\mathcal{H}^+} = 0, \tag{9.26}$$

since $\alpha^{[-2]}$ is proportional to Δ^2 by assumption (and hence $\sqrt{\Delta} \psi^{[-2]}$ is proportional to Δ , by definition (4.7)).

We can estimate:

$$\begin{aligned}
\left| \mathcal{E}_3(\Sigma) \right| &\leq \left[\left| \partial_r \left(\frac{\Delta H_1}{r^2 + a^2} \right) + \frac{2\Delta H_2}{(r^2 + a^2)^2} - h_1 f' \right| + \left| \partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) - h_2 f' \right| \right. \\
&\quad \left. + \frac{\Delta |H_1|}{(r^2 + a^2)^2} \right] \frac{(r^2 + a^2)^2}{2\Delta} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \\
&+ \left[\frac{1}{2} \left| \partial_r \left((r^2 + a^2) \left(h_2 f' - \partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) \right) \right) \right| + \left| \partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) - h_2 f' \right| \right. \\
&\quad \left. + \frac{\Delta |H_2|}{(r^2 + a^2)^2} \right] \frac{(r^2 + a^2)^2}{2\Delta} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \\
&+ \frac{1}{2} (|H_1| + |H_2|) \left| \Psi^{[-2]} \right|^2 \\
&\leq \left[\left| \partial_r (DH_1) \right| + \frac{2}{r^2 + a^2} D|H_2| + |h_1| f' \right. \\
&\quad \left. + \left| \partial_r (DH_2) \right| + |h_2| f' + \frac{1}{r^2 + a^2} D|H_1| \right] \frac{(r^2 + a^2)^2}{2\Delta} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \\
&+ \left[\frac{1}{2} \left| \partial_r \left((r^2 + a^2) h_2 f' \right) \right| + \frac{1}{2} \left| \partial_r \left((r^2 + a^2) \partial_r (DH_2) \right) \right| \right. \\
&\quad \left. + \left| \partial_r (DH_2) \right| + |h_2| f' + \frac{1}{r^2 + a^2} D|H_2| \right] \frac{(r^2 + a^2)^2}{2\Delta} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \\
&+ \frac{1}{2} (|H_1| + |H_2|) \left| \Psi^{[-2]} \right|^2
\end{aligned}$$

where we used Young's inequality and collected terms.

Remark 9.27. Recall, $D = \frac{\Delta}{r^2 + a^2} = \frac{r^2 + a^2 - 2Mr}{r^2 + a^2} = \frac{\tilde{r}^2 + \tilde{a}^2 - 2\tilde{r}}{\tilde{r}^2 + \tilde{a}^2} = \frac{\tilde{\Delta}}{\tilde{r}^2 + \tilde{a}^2}$.

Hence we have

$$\begin{aligned} \left| \mathcal{E}_3(\Sigma) \right| &\leq \mathcal{E}_3\left(\Sigma, \psi^{[-2]}\right) \frac{(r^2 + a^2)^2}{2\Delta} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 + \mathcal{E}_3\left(\Sigma, \alpha^{[-2]}\right) \frac{(r^2 + a^2)^2}{2\Delta} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \\ &\quad + \frac{1}{2} \left(|H_1| + |H_2| \right) \left| \Psi^{[-2]} \right|^2, \end{aligned} \quad (9.28)$$

where we define

$$\begin{aligned} \mathcal{E}_3\left(\Sigma, \psi^{[-2]}\right) &:= \left| \partial_r (DH_1) \right| + \frac{2}{r^2 + a^2} D|H_2| + |h_1|f' + \left| \partial_r (DH_2) \right| + |h_2|f' + \frac{1}{r^2 + a^2} D|H_1| \\ &= \frac{\tilde{C}}{M^4} \left| \partial_{\tilde{r}} (D\tilde{h}_1) \right| + \frac{2\tilde{C}}{M^4} \frac{1}{\tilde{r}^2 + \tilde{a}^2} D|\tilde{h}_2| + \frac{1}{M^4} D|\tilde{h}_1| \partial_{\tilde{r}} f + \frac{\tilde{C}}{M^3} \left| \partial_{\tilde{r}} (D\tilde{h}_2) \right| \\ &\quad + \frac{1}{M^3} D|\tilde{h}_2| \partial_{\tilde{r}} f + \frac{\tilde{C}}{M^5} \frac{1}{\tilde{r}^2 + \tilde{a}^2} D|\tilde{h}_1| \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_3\left(\Sigma, \alpha^{[-2]}\right) &:= \frac{1}{2} \left| \partial_r \left((r^2 + a^2) h_2 f' \right) \right| + \frac{1}{2} \left| \partial_r \left((r^2 + a^2) \partial_r (DH_2) \right) \right| + \left| \partial_r (DH_2) \right| + |h_2|f' + \frac{1}{r^2 + a^2} D|H_2| \\ &= \frac{1}{2M^2} \left| \partial_{\tilde{r}} \left((\tilde{r}^2 + \tilde{a}^2) \tilde{h}_2 D \partial_{\tilde{r}} f \right) \right| + \frac{\tilde{C}}{2M^2} \left| \partial_{\tilde{r}} \left((\tilde{r}^2 + \tilde{a}^2) \partial_{\tilde{r}} (D\tilde{h}_2) \right) \right| + \frac{\tilde{C}}{M^3} \left| \partial_{\tilde{r}} (D\tilde{h}_2) \right| \\ &\quad + \frac{1}{M^3} D|\tilde{h}_2| \partial_{\tilde{r}} f + \frac{\tilde{C}}{M^4} \frac{1}{\tilde{r}^2 + \tilde{a}^2} D|\tilde{h}_2|. \end{aligned}$$

Combining (9.25) with (9.26) and (9.28), we estimate

$$\begin{aligned} \mathcal{E}_3 &\leq \int_{\Sigma_{\tau_2}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \frac{(r^2 + a^2)^2}{2\Delta} \left[\mathcal{E}_3\left(\Sigma, \psi^{[-2]}\right) \cdot \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 + \mathcal{E}_3\left(\Sigma, \alpha^{[-2]}\right) \cdot \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \right] dr d\omega \\ &\quad + \int_{\Sigma_{\tau_2}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \frac{1}{2} \left(|H_1| + |H_2| \right) \cdot \left| \Psi^{[-2]} \right|^2 dr d\omega \\ &\quad + \int_{\Sigma_{\tau_1}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \frac{(r^2 + a^2)^2}{2\Delta} \left[\mathcal{E}_3\left(\Sigma, \psi^{[-2]}\right) \cdot \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 + \mathcal{E}_3\left(\Sigma, \alpha^{[-2]}\right) \cdot \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \right] dr d\omega \\ &\quad + \int_{\Sigma_{\tau_1}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \frac{1}{2} \left(|H_1| + |H_2| \right) \cdot \left| \Psi^{[-2]} \right|^2 dr d\omega. \end{aligned} \quad (9.29)$$

Combining (9.23) with (9.24) and (9.29), we obtain

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \text{ST} \left[\Psi^{[-2]} \right] dt^* dr d\omega \\
& - a^2 \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{8} \left(|H_1 - 2(f+y)h_1| + |H_2 - 2(f+y)h_2| \right) \cdot |(L - \underline{L})\Psi^{[-2]}|^2 dt^* dr d\omega \\
& - a^2 \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{\Delta H_1}{(r^2 + a^2)^2} |\Psi^{[-2]}|^2 dt^* dr d\omega \\
& - a^2 \int_{\mathcal{R}(\tau_1, \tau_2)} \left[\partial_r \left(\frac{r^2 + a^2}{2} \left(h_1 f' - \partial_r \left(\frac{\Delta H_1}{r^2 + a^2} \right) \right) \right) + 2h_2 f' \right. \\
& \quad \left. - 3\partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) + \frac{1}{2} |H_1 - 2(f+y)h_1| \right] \cdot |\sqrt{\Delta} \psi^{[-2]}|^2 dt^* dr d\omega \\
& - a^2 \int_{\mathcal{R}(\tau_1, \tau_2)} \left[\partial_r \left(\frac{r^2 + a^2}{4} \partial_r \left((r^2 + a^2) \left(\partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) - h_2 f' \right) \right) \right) \right. \\
& \quad \left. + \frac{1}{2} |H_2 - 2(f+y)h_2| \right] \cdot |\alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}}|^2 dt^* dr d\omega \\
& + C_{\text{EB}} \int_{\Sigma_{\tau_2}} \mathbb{E}_a \left[\Psi^{[-2]} \right] dr d\omega - a^2 \int_{\Sigma_{\tau_2}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \frac{1}{2} (|H_1| + |H_2|) \cdot |\Psi^{[-2]}|^2 dr d\omega \\
& - a^2 \int_{\Sigma_{\tau_2}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \frac{(r^2 + a^2)^2}{2\Delta} \mathcal{E}_3(\Sigma, \psi^{[-2]}) \cdot |\sqrt{\Delta} \psi^{[-2]}|^2 dr d\omega \\
& - a^2 \int_{\Sigma_{\tau_2}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \frac{(r^2 + a^2)^2}{2\Delta} \mathcal{E}_3(\Sigma, \alpha^{[-2]}) \cdot |\alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}}|^2 dr d\omega \\
& + \frac{\tilde{C}}{16Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} |(L + \underline{L})\Psi^{[-2]}|^2 dV_{\mathcal{H}^+} \\
& \leq (2\tilde{C} - C_{\text{EB}}) \int_{\Sigma_{\tau_1}} \mathbb{E}_a \left[\Psi^{[-2]} \right] dr d\omega \\
& + a^2 \int_{\Sigma_{\tau_1}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \frac{1}{2} (|H_1| + |H_2|) \cdot |\Psi^{[-2]}|^2 dr d\omega \\
& + a^2 \int_{\Sigma_{\tau_1}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \frac{(r^2 + a^2)^2}{2\Delta} \left[\mathcal{E}_3(\Sigma, \psi^{[-2]}) \cdot |\sqrt{\Delta} \psi^{[-2]}|^2 \right. \\
& \quad \left. + \mathcal{E}_3(\Sigma, \alpha^{[-2]}) \cdot |\alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}}|^2 \right] dr d\omega.
\end{aligned} \tag{9.30}$$

Note that some of the terms in (9.30) proportional to a^2 , namely the ones involving only $\Psi^{[-2]}$ (on line 3 and line 8 of (9.30)), can be absorbed straight away for a small enough (provided we show suitable behaviour of the coefficients at the event horizon and at infinity). To control terms involving $\alpha^{[-2]}$ and $\psi^{[-2]}$, we wish to add to (9.30) multiples of transport estimates (see Section 8). These transport estimates also contain boundary terms (on Σ_{τ_1} and Σ_{τ_2}) which have a good sign and will thus absorb the boundary error terms (that are, note, also proportional to a^2) involving $\alpha^{[-2]}$ and $\psi^{[-2]}$.

From Propositions 8.3 and 8.4, we get the estimates

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{C_{\text{tr}, \alpha^{[-2]}}}{2r^2} \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 dt^* dr d\omega \\
& + C_{\text{tr}, \alpha^{[-2]}} \int_{\Sigma_{\tau_2}} \frac{r^2 + a^2}{\Delta} \left(1 + \frac{1}{r} \right) \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 dr d\omega \\
& \leq C_{\text{tr}, \alpha^{[-2]}} 8 \left(1 + \frac{1}{r_+} \right)^2 \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{r^2}{(r^2 + a^2)^2} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega \\
& + C_{\text{tr}, \alpha^{[-2]}} \int_{\Sigma_{\tau_1}} \frac{r^2 + a^2}{\Delta} \left(1 + \frac{1}{r} \right) \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 dr d\omega
\end{aligned} \tag{9.31}$$

and

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{C_{\text{tr}, \psi^{[-2]}}}{2r^2} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega \\
& + C_{\text{tr}, \psi^{[-2]}} \int_{\Sigma_{\tau_2}} \frac{r^2 + a^2}{\Delta r} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dr d\omega \\
& \leq \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{2C_{\text{tr}, \psi^{[-2]}}}{(r^2 + a^2)^2} \left| \Psi^{[-2]} \right|^2 dt^* dr d\omega \\
& + C_{\text{tr}, \psi^{[-2]}} \int_{\Sigma_{\tau_1}} \frac{r^2 + a^2}{\Delta r} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dr d\omega,
\end{aligned} \tag{9.32}$$

where $C_{\text{tr}, \alpha^{[-2]}}$ and $C_{\text{tr}, \psi^{[-2]}}$ are non-negative constants, depending possibly on M .

Adding (9.31) and (9.32) to (9.30), we obtain

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \left[\text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), (L + \underline{L}) \Psi^{[-2]} \right) \cdot \left| (L + \underline{L}) \Psi^{[-2]} \right|^2 + \text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), (L - \underline{L}) \Psi^{[-2]} \right) \cdot \left| (L - \underline{L}) \Psi^{[-2]} \right|^2 \right. \\
& + \text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \Psi^{[-2]} \right) \cdot \left| \Psi^{[-2]} \right|^2 + \text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \psi^{[-2]} \right) \cdot \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \\
& \left. + \text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \alpha^{[-2]} \right) \cdot \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 \right] dt^* dr d\omega \\
& + \int_{\Sigma_{\tau_2}} \mathbb{B}_{\tilde{a}}^{\tau_2} \left[\Psi^{[-2]} \right] dr d\omega \\
& + \int_{\Sigma_{\tau_2}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) D^{-1} \text{cf}_{\tilde{a}} \left(\Sigma_{\tau_2}, \psi^{[-2]} \right) \cdot \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dr d\omega \\
& + \int_{\Sigma_{\tau_2}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) D^{-1} \text{cf}_{\tilde{a}} \left(\Sigma_{\tau_2}, \alpha^{[-2]} \right) \cdot \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 dr d\omega \\
& + \frac{\tilde{C}}{16Mr_+} \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| (L + \underline{L}) \Psi^{[-2]} \right|^2 dV_{\mathcal{H}^+} \\
& \leq \int_{\Sigma_{\tau_1}} \mathbb{B}_{\tilde{a}}^{\tau_1} \left[\Psi^{[-2]} \right] dr d\omega \\
& + \int_{\Sigma_{\tau_1}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) D^{-1} \text{cf}_{\tilde{a}} \left(\Sigma_{\tau_1}, \psi^{[-2]} \right) \cdot \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dr d\omega \\
& + \int_{\Sigma_{\tau_1}} \left(\frac{2Mr}{r^2 + a^2} + 1 \right) D^{-1} \text{cf}_{\tilde{a}} \left(\Sigma_{\tau_1}, \alpha^{[-2]} \right) \cdot \left| \alpha^{[-2]}(r^2 + a^2)^{-\frac{3}{2}} \right|^2 dr d\omega,
\end{aligned} \tag{9.33}$$

where

$$\begin{aligned}\text{cf}_{\tilde{a}}\left(\mathcal{R}(\tau_1, \tau_2), (L + \underline{L})\Psi^{[-2]}\right) &:= \frac{\partial_r y}{4} + \left((- \partial_r w) f - \partial_r (wy)\right) \left(\frac{a^2 \sin^2 \theta}{4} - aM \cos^2 \theta\right) \\ &= \frac{1}{M} \left[\frac{\partial_{\tilde{r}} y}{4} + \left((- \partial_{\tilde{r}} \tilde{w}) f - \partial_{\tilde{r}} (\tilde{w}y)\right) \left(\frac{\tilde{a}^2 \sin^2 \theta}{4} - \tilde{a} \cos^2 \theta\right) \right],\end{aligned}$$

$$\begin{aligned}\text{cf}_{\tilde{a}}\left(\mathcal{R}(\tau_1, \tau_2), (L - \underline{L})\Psi^{[-2]}\right) &:= \left(\frac{\partial_r f}{2} + \frac{\partial_r y}{4}\right) - \frac{a^2}{8} \left(|H_1 - 2(f+y)h_1| + |H_2 - 2(f+y)h_2|\right) \\ &= \frac{1}{M} \left[\left(\frac{\partial_{\tilde{r}} f}{2} + \frac{\partial_{\tilde{r}} y}{4}\right) - \frac{\tilde{a}^2}{8} |\tilde{C} - 2(f+y)| \left(|\tilde{h}_1| + M|\tilde{h}_2|\right) \right],\end{aligned}$$

$$\begin{aligned}\text{cf}_{\tilde{a}}\left(\mathcal{R}(\tau_1, \tau_2), \Psi^{[-2]}\right) &:= \left(2 - \frac{a}{M}\right) \left((- \partial_r w) f - \partial_r (wy)\right) - f \partial_r (wP) - \partial_r (wyP) - \frac{\partial_r f''}{2} - a^2 \frac{\Delta H_1}{(r^2 + a^2)^2} - \frac{2C_{\text{tr}, \psi^{[-2]}}}{(r^2 + a^2)^2} \\ &= \frac{1}{M^3} \left[\left(2 - \tilde{a}\right) \left((- \partial_{\tilde{r}} \tilde{w}) f - \partial_{\tilde{r}} (\tilde{w}y)\right) - f \partial_{\tilde{r}} (\tilde{w}P) - \partial_{\tilde{r}} (\tilde{w}yP) - \frac{1}{2} \partial_{\tilde{r}} \left(D \partial_{\tilde{r}} (D \partial_{\tilde{r}} f)\right) \right. \\ &\quad \left. - \tilde{a}^2 \tilde{C} D \frac{1}{\tilde{r}^2 + \tilde{a}^2} \tilde{h}_1 - \frac{2C_{\text{tr}, \psi^{[-2]}}}{M} \frac{1}{(\tilde{r}^2 + \tilde{a}^2)^2} \right],\end{aligned}$$

$$\begin{aligned}\text{cf}_{\tilde{a}}\left(\mathcal{R}(\tau_1, \tau_2), \psi^{[-2]}\right) &:= \frac{C_{\text{tr}, \psi^{[-2]}}}{2r^2} \\ &\quad - a^2 \left[\partial_r \left(\frac{r^2 + a^2}{2} \left(h_1 f' - \partial_r \left(\frac{\Delta H_1}{r^2 + a^2} \right) \right) \right) + 2h_2 f' - 3\partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) + \frac{1}{2} |H_1 - 2(f+y)h_1| \right] \\ &\quad - 8C_{\text{tr}, \alpha^{[-2]}} \left(1 + \frac{1}{r_+} \right)^2 \frac{r^2}{(r^2 + a^2)^2} \\ &= \frac{C_{\text{tr}, \psi^{[-2]}}}{2M^2 \tilde{r}^2} \\ &\quad - \tilde{a}^2 \left[\frac{1}{M} \partial_{\tilde{r}} \left(\frac{\tilde{r}^2 + \tilde{a}^2}{2} \left(\tilde{h}_1 D \partial_{\tilde{r}} f - \tilde{C} \partial_{\tilde{r}} (D \tilde{h}_1) \right) \right) + \frac{2}{M} \tilde{h}_2 D \partial_{\tilde{r}} f - \frac{3\tilde{C}}{M} \partial_{\tilde{r}} (D \tilde{h}_2) \right. \\ &\quad \left. + \frac{1}{2M} |\tilde{C} - 2(f+y)| \cdot |\tilde{h}_1| \right] \\ &\quad - \frac{8C_{\text{tr}, \alpha^{[-2]}}}{M^2} \left(1 + \frac{1}{M\tilde{r}_+} \right)^2 \frac{\tilde{r}^2}{(\tilde{r}^2 + \tilde{a}^2)^2},\end{aligned}$$

$$\begin{aligned}\text{cf}_{\tilde{a}}\left(\mathcal{R}(\tau_1, \tau_2), \alpha^{[-2]}\right) &:= \frac{C_{\text{tr}, \alpha^{[-2]}}}{2r^2} - a^2 \left[\partial_r \left(\frac{r^2 + a^2}{4} \partial_r \left((r^2 + a^2) \left(\partial_r \left(\frac{\Delta H_2}{r^2 + a^2} \right) - h_2 f' \right) \right) \right) + \frac{1}{2} |H_2 - 2(f+y)h_2| \right] \\ &= \frac{C_{\text{tr}, \alpha^{[-2]}}}{2M^2 \tilde{r}^2} - \tilde{a}^2 \left[M \partial_{\tilde{r}} \left(\frac{\tilde{r}^2 + \tilde{a}^2}{4} \partial_{\tilde{r}} \left((\tilde{r}^2 + \tilde{a}^2) \left(\tilde{C} \partial_{\tilde{r}} (D \tilde{h}_2) - D \tilde{h}_2 \partial_{\tilde{r}} f \right) \right) \right) + \frac{1}{2} |\tilde{C} - 2(f+y)| \cdot |\tilde{h}_2| \right],\end{aligned}$$

$$\begin{aligned}\mathbb{B}_{\tilde{a}}^{\tau_2} \left[\Psi^{[-2]} \right] &:= C_{\text{EB}} \cdot \mathbb{E}_a \left[\Psi^{[-2]} \right] - a^2 \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \frac{1}{2} \left(|H_1| + |H_2| \right) \cdot \left| \Psi^{[-2]} \right|^2 \\ &= C_{\text{EB}} \cdot \mathbb{E}_a \left[\Psi^{[-2]} \right] - \tilde{a}^2 \frac{\tilde{C}}{2} \left(\frac{2\tilde{r}}{\tilde{r}^2 + \tilde{a}^2} + 1 \right) \left(\frac{1}{M} |\tilde{h}_1| + |\tilde{h}_2| \right) \cdot \left| \Psi^{[-2]} \right|^2,\end{aligned}$$

$$\begin{aligned}
\text{cf}_{\tilde{a}} \left(\Sigma_{\tau_2}, \psi^{[-2]} \right) &:= \frac{C_{\text{tr}, \psi^{[-2]}}}{r} - a^2 \frac{r^2 + a^2}{2} \mathcal{E}_3 \left(\Sigma, \psi^{[-2]} \right) \\
&= \frac{C_{\text{tr}, \psi^{[-2]}}}{M\tilde{r}} - \tilde{a}^2 \frac{\tilde{r}^2 + \tilde{a}^2}{2} M^4 \mathcal{E}_3 \left(\Sigma, \psi^{[-2]} \right),
\end{aligned}$$

$$\begin{aligned}
\text{cf}_{\tilde{a}} \left(\Sigma_{\tau_2}, \alpha^{[-2]} \right) &:= C_{\text{tr}, \alpha^{[-2]}} \left(1 + \frac{1}{r} \right) - a^2 \frac{r^2 + a^2}{2} \mathcal{E}_3 \left(\Sigma, \alpha^{[-2]} \right) \\
&= C_{\text{tr}, \alpha^{[-2]}} \left(1 + \frac{1}{M\tilde{r}} \right) - \tilde{a}^2 \frac{\tilde{r}^2 + \tilde{a}^2}{2} M^4 \mathcal{E}_3 \left(\Sigma, \alpha^{[-2]} \right),
\end{aligned}$$

$$\begin{aligned}
\mathbb{B}_{\tilde{a}}^{\tau_1} \left[\Psi^{[-2]} \right] &:= \left(2\tilde{C} - C_{\text{EB}} \right) \cdot \mathbb{E}_a \left[\Psi^{[-2]} \right] + a^2 \left(\frac{2Mr}{r^2 + a^2} + 1 \right) \frac{1}{2} \left(|H_1| + |H_2| \right) \cdot \left| \Psi^{[-2]} \right|^2 \\
&= \left(2\tilde{C} - C_{\text{EB}} \right) \cdot \mathbb{E}_a \left[\Psi^{[-2]} \right] + \tilde{a}^2 \frac{\tilde{C}}{2} \left(\frac{2\tilde{r}}{\tilde{r}^2 + \tilde{a}^2} + 1 \right) \left(\frac{1}{M} |\tilde{h}_1| + |\tilde{h}_2| \right) \cdot \left| \Psi^{[-2]} \right|^2,
\end{aligned}$$

$$\begin{aligned}
\text{cf}_{\tilde{a}} \left(\Sigma_{\tau_1}, \psi^{[-2]} \right) &:= \frac{C_{\text{tr}, \psi^{[-2]}}}{r} + a^2 \frac{r^2 + a^2}{2} \mathcal{E}_3 \left(\Sigma, \psi^{[-2]} \right) \\
&= \frac{C_{\text{tr}, \psi^{[-2]}}}{M\tilde{r}} + \tilde{a}^2 \frac{\tilde{r}^2 + \tilde{a}^2}{2} M^4 \mathcal{E}_3 \left(\Sigma, \psi^{[-2]} \right),
\end{aligned}$$

$$\begin{aligned}
\text{cf}_{\tilde{a}} \left(\Sigma_{\tau_1}, \alpha^{[-2]} \right) &:= C_{\text{tr}, \alpha^{[-2]}} \left(1 + \frac{1}{r} \right) + a^2 \frac{r^2 + a^2}{2} \mathcal{E}_3 \left(\Sigma, \alpha^{[-2]} \right) \\
&= C_{\text{tr}, \alpha^{[-2]}} \left(1 + \frac{1}{M\tilde{r}} \right) + \tilde{a}^2 \frac{\tilde{r}^2 + \tilde{a}^2}{2} M^4 \mathcal{E}_3 \left(\Sigma, \alpha^{[-2]} \right).
\end{aligned}$$

We can estimate the following:

$$\begin{aligned}
\text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), (L + \underline{L}) \Psi^{[-2]} \right) &\geq \frac{1}{M} \left\{ \frac{\partial_{\tilde{r}} y}{4} + \left((-\partial_{\tilde{r}} \tilde{w}) f - \partial_{\tilde{r}} (\tilde{w} y) \right) \left(\frac{\tilde{a}^2 \sin^2 \theta}{4} - \tilde{a} \cos^2 \theta \right) \right\}, \\
\text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), (L - \underline{L}) \Psi^{[-2]} \right) &\geq \frac{1}{M} \left\{ \left(\frac{\partial_{\tilde{r}} f}{2} + \frac{\partial_{\tilde{r}} y}{4} \right) - \frac{1}{8} \left(\tilde{C} - 2(f + y)(\tilde{r}_+) \right) \max \{1, M\} \cdot \tilde{a}^2 \left(\frac{3}{\tilde{r}^2} + |o| \left(\frac{1}{\tilde{r}^3} \right) \right) \right\}, \\
\text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \Psi^{[-2]} \right) &\geq \frac{1}{M^3} \left\{ (2 - \tilde{a}) \left((-\partial_{\tilde{r}} \tilde{w}) f - \partial_{\tilde{r}} (\tilde{w} y) \right) - f \partial_{\tilde{r}} (\tilde{w} P) - \partial_{\tilde{r}} (\tilde{w} y P) - \frac{1}{2} \partial_{\tilde{r}} \left(D \partial_{\tilde{r}} (D \partial_{\tilde{r}} f) \right) \right. \\
&\quad \left. - \tilde{C} \cdot \tilde{a}^2 \left(\frac{20}{\tilde{r}^5} + |o| \left(\frac{1}{\tilde{r}^6} \right) \right) - \frac{2C_{\text{tr}, \psi^{[-2]}}}{M} \frac{1}{\tilde{r}^4} \right\}, \\
\text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \psi^{[-2]} \right) &\geq \frac{1}{M^2} \left\{ \frac{1}{2} \left[C_{\text{tr}, \psi^{[-2]}} - 16 C_{\text{tr}, \alpha^{[-2]}} \left(1 + \frac{1}{M} \right)^2 \right] \frac{1}{\tilde{r}^2} - 3\tilde{C}M \cdot \tilde{a}^2 \left(\frac{146}{\tilde{r}^3} + |o| \left(\frac{1}{\tilde{r}^4} \right) \right) \right\}, \\
\text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \alpha^{[-2]} \right) &\geq \frac{1}{M^2} \left\{ \frac{C_{\text{tr}, \alpha^{[-2]}}}{2\tilde{r}^2} \right. \\
&\quad \left. - \max \left\{ \frac{\tilde{C}M^3}{4}, \frac{M^2}{2} \left(\tilde{C} - 2(f + y)(r_+) \right) \right\} \cdot \tilde{a}^2 \left(\frac{147 + 6(\tilde{r}_{\text{trap}} - 1)}{\tilde{r}^2} + |o| \left(\frac{1}{\tilde{r}^3} \right) \right) \right\}, \\
\text{cf}_{\tilde{a}} \left(\Sigma_{\tau_2}, \psi^{[-2]} \right) &\geq \frac{1}{M} \left\{ \frac{C_{\text{tr}, \psi^{[-2]}}}{\tilde{r}} - \max \left\{ \frac{\tilde{C}}{2}, \tilde{C}M, \frac{\tilde{C}M^2}{2} \right\} \cdot \tilde{a}^2 \left(\frac{6}{\tilde{r}} + |o| \left(\frac{1}{\tilde{r}^2} \right) \right) \right\}, \\
\text{cf}_{\tilde{a}} \left(\Sigma_{\tau_2}, \alpha^{[-2]} \right) &\geq C_{\text{tr}, \alpha^{[-2]}} - \max \left\{ \frac{\tilde{C}}{2}, \frac{\tilde{C}M}{2}, \frac{\tilde{C}M^2}{4} \right\} \cdot \tilde{a}^2 \left(6 + |o| \left(\frac{1}{\tilde{r}} \right) \right), \\
\mathbb{B}_{\tilde{a}}^{\tau_2} \left[\Psi^{[-2]} \right] &\geq \frac{C_{\text{EB}}}{16M^2} \left[\left| M(L + \underline{L}) \Psi^{[-2]} \right|^2 + \left| M(L - \underline{L}) \Psi^{[-2]} \right|^2 + \frac{\tilde{r}}{\tilde{\Delta}} \left| M \underline{L} \Psi^{[-2]} \right|^2 + \frac{1}{\tilde{r}^2} \left| \overset{\circ}{\nabla}^{[-2]} \Psi^{[-2]} \right|^2 \right] \\
&\quad + \frac{1}{16M^2} \left[\frac{C_{\text{EB}}}{\tilde{r}^2} - 16 \tilde{C} \max \{M, M^2\} \cdot \tilde{a}^2 \left(\frac{3}{\tilde{r}^2} + |o| \left(\frac{1}{\tilde{r}^3} \right) \right) \right] \cdot \left| \Psi^{[-2]} \right|^2.
\end{aligned}$$

We can see that we have several inequalities of the form

$$F_{\tilde{a}}(\tilde{r}) \geq C_1(M) \left[\frac{C_2}{\tilde{r}^n} - C_3(M) \cdot \tilde{a}^2 \left(\frac{C_4}{\tilde{r}^m} + |o| \left(\frac{1}{\tilde{r}^{m+1}} \right) \right) \right],$$

where $n, m \in \mathbb{N}$ with $n \leq m$ and the C_i are constants some of which are dependent on M . The notation

$$|o| \left(\frac{1}{r^k} \right)$$

(with $k \in \mathbb{N}$) stands for any non-negative function $G = G(\tilde{r})$ (not to be confused with the notation from Lemma 9.1), defined for $\tilde{r} \geq \tilde{r}_+$, that satisfies

$$\lim_{\tilde{r} \rightarrow \infty} \tilde{r}^k G(\tilde{r}) = \text{constant}.$$

The fact that $n \geq m$ in these relations will allow us to absorb the negative term by the positive one, provided $\tilde{a}$ is small enough.

We can compute that for all $\tilde{a} \in [0, 1]$

$$\mathfrak{C}_{\tilde{a}}(\tilde{r}) := \left(2 - \tilde{a} \right) \left((-\partial_{\tilde{r}} \tilde{w}) f - \partial_{\tilde{r}} (\tilde{w} y) \right) - f \partial_{\tilde{r}} (\tilde{w} P) - \partial_{\tilde{r}} (\tilde{w} y P) - \frac{1}{2} \partial_{\tilde{r}} \left(D \partial_{\tilde{r}} (D \partial_{\tilde{r}} f) \right) \sim \frac{1}{\tilde{r}^3}$$

for large $\tilde{r}$. In fact, there exists a constants $b, B > 0$ such that

$$\frac{b}{\tilde{r}^3} \leq \mathfrak{C}_{\tilde{a}}(\tilde{r}) \leq \frac{B}{\tilde{r}^3} \quad \forall \tilde{r} \geq \tilde{r}_+$$

for all $\tilde{a} \leq 0.5$. The positivity of b can be verified numerically, see Figure 4 and also Figure 8 below.

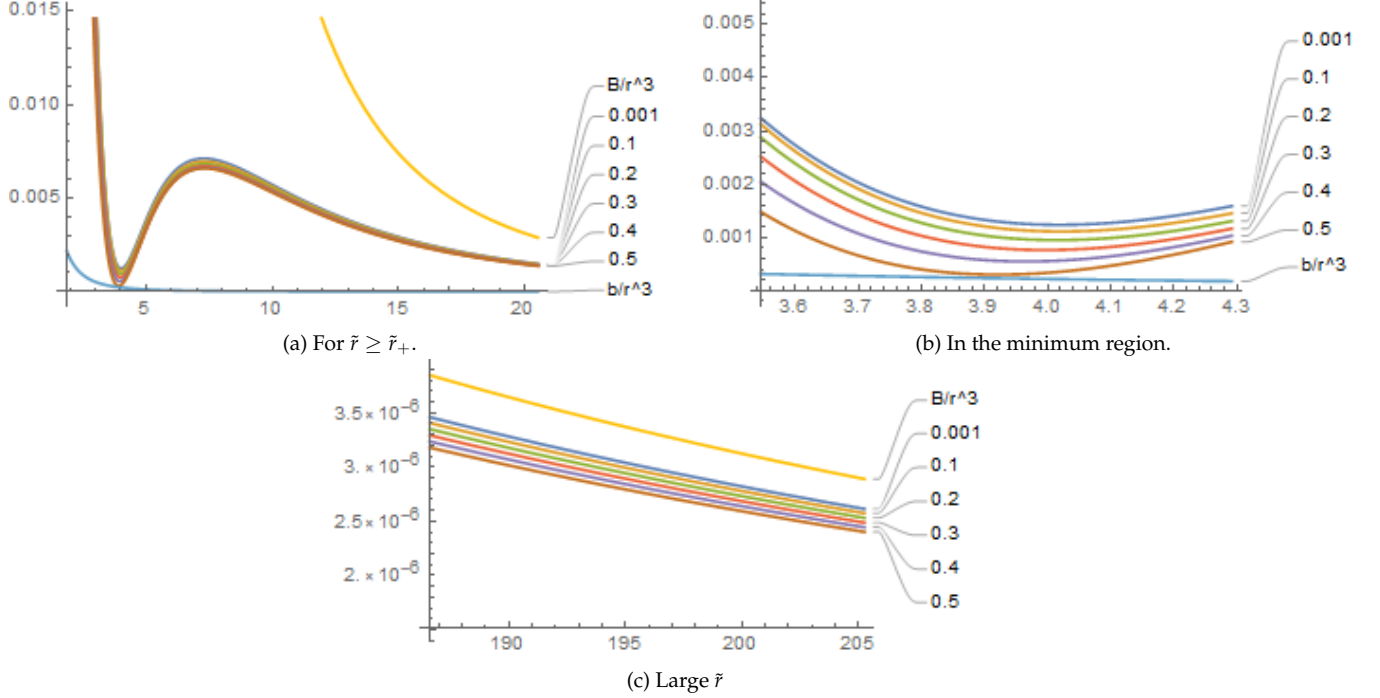


Figure 8: Behaviour of $\mathfrak{C}_{\tilde{a}}$ for different values of $\tilde{a} \leq 0.5$.

Hence, assuming $\tilde{a} \leq 0.5$, we have that

$$\text{cf}_{\tilde{a}}(\mathcal{R}(\tau_1, \tau_2), \Psi^{[-2]}) \geq \frac{1}{M^3} \left[\frac{b}{\tilde{r}^3} - \tilde{C} \cdot \tilde{a}^2 \left(\frac{20}{\tilde{r}^5} + |o| \left(\frac{1}{\tilde{r}^6} \right) \right) - \frac{2C_{\text{tr}, \psi^{[-2]}}}{M} \cdot \frac{1}{\tilde{r}^4} \right].$$

Choose

$$\begin{aligned} C_{\text{tr}, \psi^{[-2]}} &= M \cdot C'_{\text{tr}, \psi^{[-2]}}, \\ C_{\text{tr}, \alpha^{[-2]}} &= M \left(1 + \frac{1}{M} \right)^{-2} \cdot C'_{\text{tr}, \alpha^{[-2]}}, \end{aligned} \tag{9.34}$$

where we can choose constants $C'_{\text{tr}, \psi^{[-2]}} > 0$ and $C'_{\text{tr}, \alpha^{[-2]}} > 0$ independent of M .

Choose $C'_{\text{tr}, \psi^{[-2]}}$ small enough so that

$$2C'_{\text{tr}, \psi^{[-2]}} \cdot \frac{1}{\tilde{r}^4} \leq \frac{b}{4\tilde{r}^3}.$$

Also, there exist $\tilde{a}_1 = \tilde{a}_1(M) > 0$ such that for all $\tilde{a} \leq \tilde{a}_1$, we have

$$\tilde{C} \cdot \tilde{a}^2 \left(\frac{20}{\tilde{r}^5} + |o| \left(\frac{1}{\tilde{r}^6} \right) \right) \leq \frac{b}{4\tilde{r}^3}.$$

Hence for all $\tilde{a} \geq \tilde{a}_1$, we have

$$\text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \Psi^{[-2]} \right) \geq \frac{1}{M^3} \frac{b}{2\tilde{r}^3} \gtrsim \frac{1}{M^3} \frac{1}{\tilde{r}^3}. \quad (9.35)$$

Also, there exists $\tilde{a}_2 = \tilde{a}_2(M) > 0$ such that for all $\tilde{a} \geq \tilde{a}_2$, we have

$$\frac{1}{8} \left(\tilde{C} - 2(f+y)(r_+) \right) \max \{1, M\} \cdot \tilde{a}^2 \left(\frac{3}{\tilde{r}^2} + |o| \left(\frac{1}{\tilde{r}^3} \right) \right) \leq \frac{1}{2} \left(\frac{\partial_{\tilde{r}} f}{2} + \frac{\partial_{\tilde{r}} y}{4} \right).$$

Hence for all $\tilde{a} \geq \tilde{a}_2$:

$$\text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), (L - \underline{L}) \Psi^{[-2]} \right) \geq \frac{1}{2M} \left(\frac{\partial_{\tilde{r}} f}{2} + \frac{\partial_{\tilde{r}} y}{4} \right) \gtrsim \frac{1}{M} \frac{1}{\tilde{r}^2}. \quad (9.36)$$

In addition,

$$\text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), (L + \underline{L}) \Psi^{[-2]} \right) \sim \frac{1}{M} \frac{1}{\tilde{r}^4} (\tilde{r} - \tilde{r}_{\text{trap}})^2 \quad (9.37)$$

uniformly in $\tilde{a}$.

We also have:

$$\begin{aligned} \text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \psi^{[-2]} \right) &\geq \frac{1}{M} \left\{ \frac{1}{2} \left[C'_{\text{tr}, \psi^{[-2]}} - 16 C'_{\text{tr}, \alpha^{[-2]}} \right] \frac{1}{\tilde{r}^2} - 3\tilde{C} \cdot \tilde{a}^2 \left(\frac{146}{\tilde{r}^3} + |o| \left(\frac{1}{\tilde{r}^4} \right) \right) \right\} \\ &= \frac{1}{M} \left\{ \frac{C'_{\text{tr}, \psi^{[-2]}}}{4\tilde{r}^2} - 3\tilde{C} \cdot \tilde{a}^2 \left(\frac{146}{\tilde{r}^3} + |o| \left(\frac{1}{\tilde{r}^4} \right) \right) \right\}, \end{aligned}$$

where we chose

$$C'_{\text{tr}, \alpha^{[-2]}} = \frac{1}{32} C'_{\text{tr}, \psi^{[-2]}}.$$

There exists $\tilde{a}_3 = \tilde{a}_3(M) > 0$ such that for all $\tilde{a} \leq \tilde{a}_3$, we have

$$3\tilde{C} \cdot \tilde{a}^2 \left(\frac{146}{\tilde{r}^3} + |o| \left(\frac{1}{\tilde{r}^4} \right) \right) \leq \frac{C'_{\text{tr}, \psi^{[-2]}}}{8\tilde{r}^2}.$$

Hence,

$$\text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \psi^{[-2]} \right) \geq \frac{1}{M} \frac{C'_{\text{tr}, \psi^{[-2]}}}{8\tilde{r}^2} \gtrsim \frac{1}{M} \frac{1}{\tilde{r}^2} \quad (9.38)$$

for all $\tilde{a} \leq \tilde{a}_3$.

We also have:

$$\begin{aligned} &\text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \alpha^{[-2]} \right) \\ &\geq \frac{1}{M^2} \left\{ \frac{M}{2} \left(1 + \frac{1}{M} \right)^{-2} \cdot \frac{C'_{\text{tr}, \alpha^{[-2]}}}{\tilde{r}^2} - \max \left\{ \frac{\tilde{C}M^3}{4}, \frac{M^2}{2} \left(\tilde{C} - 2(f+y)(r_+) \right) \right\} \cdot \tilde{a}^2 \left(\frac{147 + 6(\tilde{r}_{\text{trap}} - 1)}{\tilde{r}^2} + |o| \left(\frac{1}{\tilde{r}^3} \right) \right) \right\} \\ &= \frac{1}{2M} \left(1 + \frac{1}{M} \right)^{-2} \left\{ \frac{C'_{\text{tr}, \alpha^{[-2]}}}{\tilde{r}^2} - M \left(1 + \frac{1}{M} \right)^2 \max \left\{ \frac{\tilde{C}M}{2}, \tilde{C} - 2(f+y)(r_+) \right\} \cdot \tilde{a}^2 \left(\frac{147 + 6(\tilde{r}_{\text{trap}} - 1)}{\tilde{r}^2} + |o| \left(\frac{1}{\tilde{r}^3} \right) \right) \right\}. \end{aligned}$$

There exists $\tilde{a}_4 = \tilde{a}_4(M) > 0$ such that for all $\tilde{a} \leq \tilde{a}_4$, we have

$$M \left(1 + \frac{1}{M}\right)^2 \max \left\{ \frac{\tilde{C}M}{2}, \tilde{C} - 2(f+y)(r_+) \right\} \cdot \tilde{a}^2 \left(\frac{147 + 6(\tilde{r}_{\text{trap}} - 1)}{\tilde{r}^2} + |o| \left(\frac{1}{\tilde{r}^3} \right) \right) \leq \frac{C'_{\text{tr}, \alpha^{[-2]}}}{2\tilde{r}^2}.$$

Hence,

$$\text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \alpha^{[-2]} \right) \geq \frac{1}{2M} \left(1 + \frac{1}{M}\right)^{-2} \frac{C'_{\text{tr}, \alpha^{[-2]}}}{2\tilde{r}^2} = \frac{1}{M} \left(1 + \frac{1}{M}\right)^{-2} \frac{C'_{\text{tr}, \psi^{[-2]}}}{128\tilde{r}^2} \gtrsim \frac{1}{M} \left(1 + \frac{1}{M}\right)^{-2} \frac{1}{\tilde{r}^2} \quad (9.39)$$

for all $\tilde{a} \leq \tilde{a}_4$.

We also have:

$$\text{cf}_{\tilde{a}} \left(\Sigma_{\tau_2}, \psi^{[-2]} \right) \geq \frac{C'_{\text{tr}, \psi^{[-2]}}}{\tilde{r}} - \max \left\{ \frac{\tilde{C}}{2M}, \tilde{C}, \frac{\tilde{C}M}{2} \right\} \cdot \tilde{a}^2 \left(\frac{6}{\tilde{r}} + |o| \left(\frac{1}{\tilde{r}^2} \right) \right).$$

There exists $\tilde{a}_5 = \tilde{a}_5(M) > 0$ such that for all $\tilde{a} \leq \tilde{a}_5$, we have

$$\max \left\{ \frac{\tilde{C}}{2M}, \tilde{C}, \frac{\tilde{C}M}{2} \right\} \cdot \tilde{a}^2 \left(\frac{6}{\tilde{r}} + |o| \left(\frac{1}{\tilde{r}^2} \right) \right) \leq \frac{C'_{\text{tr}, \psi^{[-2]}}}{2\tilde{r}}.$$

Hence,

$$\text{cf}_{\tilde{a}} \left(\Sigma_{\tau_2}, \psi^{[-2]} \right) \geq \frac{C'_{\text{tr}, \psi^{[-2]}}}{2\tilde{r}} \gtrsim \frac{1}{\tilde{r}} \quad (9.40)$$

for all $\tilde{a} \leq \tilde{a}_5$.

We also have:

$$\text{cf}_{\tilde{a}} \left(\Sigma_{\tau_2}, \alpha^{[-2]} \right) \geq M \left(1 + \frac{1}{M}\right)^{-2} \left\{ C'_{\text{tr}, \alpha^{[-2]}} - \frac{1}{M} \left(1 + \frac{1}{M}\right)^2 \max \left\{ \frac{\tilde{C}}{2}, \frac{\tilde{C}M}{2}, \frac{\tilde{C}M^2}{4} \right\} \cdot \tilde{a}^2 \left(6 + |o| \left(\frac{1}{\tilde{r}} \right) \right) \right\}.$$

There exists $\tilde{a}_6 = \tilde{a}_6(M) > 0$ such that for all $\tilde{a} \leq \tilde{a}_6$, we have

$$\frac{1}{M} \left(1 + \frac{1}{M}\right)^2 \max \left\{ \frac{\tilde{C}}{2}, \frac{\tilde{C}M}{2}, \frac{\tilde{C}M^2}{4} \right\} \cdot \tilde{a}^2 \left(6 + |o| \left(\frac{1}{\tilde{r}} \right) \right) \leq \frac{C'_{\text{tr}, \alpha^{[-2]}}}{2}.$$

Hence,

$$\text{cf}_{\tilde{a}} \left(\Sigma_{\tau_2}, \alpha^{[-2]} \right) \geq M \left(1 + \frac{1}{M}\right)^{-2} \frac{C'_{\text{tr}, \alpha^{[-2]}}}{2} = M \left(1 + \frac{1}{M}\right)^{-2} \frac{C'_{\text{tr}, \psi^{[-2]}}}{64} \gtrsim M \left(1 + \frac{1}{M}\right)^{-2} \quad (9.41)$$

for all $\tilde{a} \leq \tilde{a}_6$.

We also have:

$$\begin{aligned} \mathbb{B}_{\tilde{a}}^{\tau_2} \left[\Psi^{[-2]} \right] &\geq \frac{C_{\text{EB}}}{16M^2} \left[\left| M(L + \underline{L}) \Psi^{[-2]} \right|^2 + \left| M(L - \underline{L}) \Psi^{[-2]} \right|^2 + \frac{\tilde{r}}{\tilde{\Delta}} \left| M \underline{L} \Psi^{[-2]} \right|^2 + \frac{1}{\tilde{r}^2} \left| \overset{\circ}{\nabla}^{[-2]} \Psi^{[-2]} \right|^2 \right] \\ &\quad + \frac{1}{16M^2} \left[\frac{C_{\text{EB}}}{\tilde{r}^2} - 16 \tilde{C} \max \left\{ M, M^2 \right\} \cdot \tilde{a}^2 \left(\frac{3}{\tilde{r}^2} + |o| \left(\frac{1}{\tilde{r}^3} \right) \right) \right] \cdot \left| \Psi^{[-2]} \right|^2. \end{aligned}$$

There exists $\tilde{a}_7 = \tilde{a}_7(M) > 0$ such that for all $\tilde{a} \leq \tilde{a}_7$, we have

$$16 \tilde{C} \max \left\{ M, M^2 \right\} \cdot \tilde{a}^2 \left(\frac{3}{\tilde{r}^2} + |o| \left(\frac{1}{\tilde{r}^3} \right) \right) \leq \frac{C_{\text{EB}}}{2\tilde{r}^2}.$$

Hence, for all $\tilde{a} \leq \tilde{a}_7$, we have:

$$\begin{aligned}
& \mathbb{B}_{\tilde{a}}^{\tau_2} [\Psi^{[-2]}] \\
& \geq \frac{C_{\text{EB}}}{16M^2} \left[\left| M(L + \underline{L})\Psi^{[-2]} \right|^2 + \left| M(L - \underline{L})\Psi^{[-2]} \right|^2 + \frac{\tilde{r}}{\tilde{\Delta}} \left| M\underline{L}\Psi^{[-2]} \right|^2 + \frac{1}{\tilde{r}^2} \left| \overset{\circ}{\nabla}^{[-2]} \Psi^{[-2]} \right|^2 \right] + \frac{C_{\text{EB}}}{32M^2} \frac{1}{\tilde{r}^2} \left| \Psi^{[-2]} \right|^2 \\
& \geq \frac{C_{\text{EB}}}{32M^2} \left[\left| M(L + \underline{L})\Psi^{[-2]} \right|^2 + \left| M(L - \underline{L})\Psi^{[-2]} \right|^2 + \frac{\tilde{r}}{\tilde{\Delta}} \left| M\underline{L}\Psi^{[-2]} \right|^2 + \frac{1}{\tilde{r}^2} \left| \overset{\circ}{\nabla}^{[-2]} \Psi^{[-2]} \right|^2 + \frac{1}{\tilde{r}^2} \left| \Psi^{[-2]} \right|^2 \right] \\
& \gtrsim \frac{1}{M^2} \left[\left| M(L + \underline{L})\Psi^{[-2]} \right|^2 + \left| M(L - \underline{L})\Psi^{[-2]} \right|^2 + \frac{\tilde{r}}{\tilde{\Delta}} \left| M\underline{L}\Psi^{[-2]} \right|^2 + \frac{1}{\tilde{r}^2} \left| \overset{\circ}{\nabla}^{[-2]} \Psi^{[-2]} \right|^2 + \frac{1}{\tilde{r}^2} \left| \Psi^{[-2]} \right|^2 \right] \\
& = \frac{1}{M^2} \tilde{\mathbb{E}}_{\tilde{a}} [\Psi^{[-2]}].
\end{aligned} \tag{9.42}$$

Moreover, for the terms on the right hand side of (9.33), we have:

$$\begin{aligned}
& \text{cf}_{\tilde{a}} \left(\Sigma_{\tau_1}, \alpha^{[-2]} \right) \lesssim \max \left\{ 1, M^2 \right\} \left(1 + \tilde{C} \right), \\
& \text{cf}_{\tilde{a}} \left(\Sigma_{\tau_1}, \psi^{[-2]} \right) \lesssim \max \left\{ C'_{\text{tr}, \psi^{[-2]}}, \frac{\tilde{C}}{2M'}, \tilde{C}, \frac{\tilde{C}M}{2} \right\} \cdot \frac{1}{\tilde{r}}, \\
& \mathbb{B}_{\tilde{a}}^{\tau_1} [\Psi^{[-2]}] \lesssim \max \left\{ \tilde{C}, \frac{\tilde{C}}{M'}, \frac{2(2\tilde{C} - C_{\text{EB}})}{M^2} \right\} \cdot \tilde{\mathbb{E}}_{\tilde{a}} [\Psi^{[-2]}],
\end{aligned} \tag{9.43}$$

for all $\tilde{a} \in [0, 1]$.

In summary, we have:

$$\begin{aligned}
& \text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), (L + \underline{L})\Psi^{[-2]} \right) \gtrsim \frac{1}{M} \frac{1}{\tilde{r}^4} (\tilde{r} - \tilde{r}_{\text{trap}})^2, \\
& \text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), (L - \underline{L})\Psi^{[-2]} \right) \gtrsim \frac{1}{M} \frac{1}{\tilde{r}^2}, \\
& \text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \Psi^{[-2]} \right) \gtrsim \frac{1}{M^3} \frac{1}{\tilde{r}^3}, \\
& \text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \psi^{[-2]} \right) \gtrsim \frac{1}{M} \frac{1}{\tilde{r}^2}, \\
& \text{cf}_{\tilde{a}} \left(\mathcal{R}(\tau_1, \tau_2), \alpha^{[-2]} \right) \gtrsim \frac{1}{M} \left(1 + \frac{1}{M} \right)^{-2} \frac{1}{\tilde{r}^2}, \\
& \text{cf}_{\tilde{a}} \left(\Sigma_{\tau_2}, \psi^{[-2]} \right) \gtrsim \frac{1}{\tilde{r}}, \\
& \text{cf}_{\tilde{a}} \left(\Sigma_{\tau_2}, \alpha^{[-2]} \right) \gtrsim M \left(1 + \frac{1}{M} \right)^{-2}, \\
& \mathbb{B}_{\tilde{a}}^{\tau_2} [\Psi^{[-2]}] \gtrsim \frac{1}{M^2} \tilde{\mathbb{E}}_{\tilde{a}} [\Psi^{[-2]}],
\end{aligned} \tag{9.44}$$

for all $\tilde{a} \leq \min \left\{ 0.5, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5, \tilde{a}_6, \tilde{a}_7 \right\} =: a_0$,

and also:

$$\begin{aligned}
& \text{cf}_{\tilde{a}} \left(\Sigma_{\tau_1}, \psi^{[-2]} \right) \lesssim \max \left\{ \frac{\tilde{C}}{M}, \tilde{C}, \tilde{C}M \right\} \frac{1}{\tilde{r}}, \\
& \text{cf}_{\tilde{a}} \left(\Sigma_{\tau_1}, \alpha^{[-2]} \right) \lesssim \max \left\{ 1, M^2 \right\} \left(1 + \tilde{C} \right), \\
& \mathbb{B}_{\tilde{a}}^{\tau_1} [\Psi^{[-2]}] \lesssim \max \left\{ \tilde{C}, \frac{\tilde{C}}{M}, \frac{2\tilde{C} - C_{\text{EB}}}{M^2} \right\} \tilde{\mathbb{E}}_{\tilde{a}} [\Psi^{[-2]}],
\end{aligned} \tag{9.45}$$

for all $\tilde{a} \in [0, 1]$.

Remark 9.46. Here, all the underlying constants in all the $\lesssim$ signs are actually independent of M , with all the dependencies on M explicitly stated.

Hence, combining (9.44), (9.45) with (9.33), we obtain for all $\tilde{a} \leq \tilde{a}_0$:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \left[\frac{1}{\tilde{r}^4} (\tilde{r} - \tilde{r}_{\text{trap}})^2 \left| M(L + \underline{L}) \Psi^{[-2]} \right|^2 + \frac{1}{\tilde{r}^2} \left| M(L - \underline{L}) \Psi^{[-2]} \right|^2 + \frac{1}{\tilde{r}^3} \left| \Psi^{[-2]} \right|^2 \right] d\tilde{t}^* d\tilde{r} d\omega \\
& + \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{D^2}{\tilde{r}^2} \left| \frac{\psi^{[-2]}(\tilde{r}^2 + \tilde{a}^2)}{\sqrt{\tilde{\Delta}}} \right|^2 d\tilde{t}^* d\tilde{r} d\omega + \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{D^4}{\tilde{r}^2} \left| \frac{\alpha^{[-2]} \sqrt{\tilde{r}^2 + \tilde{a}^2}}{\tilde{\Delta}^2} \right|^2 d\tilde{t}^* d\tilde{r} d\omega \\
& + \int_{\Sigma_{\tau_2}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[-2]} \right] d\tilde{r} d\omega + \int_{\Sigma_{\tau_2}} \frac{D}{\tilde{r}} \left| \frac{\psi^{[-2]}(\tilde{r}^2 + \tilde{a}^2)}{\sqrt{\tilde{\Delta}}} \right|^2 d\tilde{r} d\omega + \int_{\Sigma_{\tau_2}} D^3 \left| \frac{\alpha^{[-2]} \sqrt{\tilde{r}^2 + \tilde{a}^2}}{\tilde{\Delta}^2} \right|^2 d\tilde{r} d\omega \\
& + \int_{\mathcal{H}^+(\tau_1, \tau_2)} \left| M(L + \underline{L}) \Psi^{[-2]} \right|^2 dV_{\mathcal{H}^+} \\
& \lesssim \text{Const}(M) \cdot \left[\int_{\Sigma_{\tau_1}} \tilde{\mathbb{E}}_{\tilde{a}} \left[\Psi^{[-2]} \right] d\tilde{r} d\omega + \int_{\Sigma_{\tau_1}} \frac{D}{\tilde{r}} \left| \frac{\psi^{[-2]}(\tilde{r}^2 + \tilde{a}^2)}{\sqrt{\tilde{\Delta}}} \right|^2 d\tilde{r} d\omega + \int_{\Sigma_{\tau_1}} D^3 \left| \frac{\alpha^{[-2]} \sqrt{\tilde{r}^2 + \tilde{a}^2}}{\tilde{\Delta}^2} \right|^2 d\tilde{r} d\omega \right],
\end{aligned} \tag{9.47}$$

where

$$\text{Const}(M) = \frac{\max \left\{ \frac{2\tilde{C} - C_{\text{EB}}}{M}, \tilde{C}, M\tilde{C}, M^2\tilde{C}, M^3\tilde{C}, M^4\tilde{C}, \frac{\tilde{C}+1}{M^3}, \frac{\tilde{C}+1}{M^5} \right\}}{\min \left\{ \frac{1}{M}, \frac{\tilde{C}}{M^4}, \frac{1}{M^4} \left(1 + \frac{1}{M} \right)^{-2}, \frac{1}{M^5} \left(1 + \frac{1}{M} \right)^{-2}, M^3 \right\}}$$

and the underlying constant in $\lesssim$ is independent of M .

Note that we do not control the term $\left| \overset{\circ}{\nabla}^{[-2]} \Psi^{[-2]} \right|^2$ in the above. However, since we now have control on all the other terms, we may re-do the computation starting from (7.10), this time not using the Poincare inequality (7.12). Recall the reason for using it was to gain additional positivity for the coefficient of $\left| \Psi^{[-2]} \right|^2$ in the spacetime integral. Having obtained the estimate (9.47), this extra positivity is not necessary.

This concludes the proof of Theorem (4.20).

A Some useful bounds

We have the following bounds in the black hole exterior (i.e. $r \geq r_+$).

•

$$M \leq r_+ \leq 2M$$

•

$$(1 + \sqrt{2}) M \leq r_{\text{trap}} \leq 3M$$

•

$$\frac{3}{2} \leq \frac{r_{\text{trap}}}{r_+} \leq 1 + \sqrt{2}$$

•

$$-2\sqrt{2} \leq f(r_+) \leq -\frac{3}{4}$$

•

$$-\sqrt{2} \leq 1 - \frac{r_{\text{trap}}}{r_+} \leq -\frac{1}{2}$$

•

$$\frac{1}{4} \leq \left(1 - \frac{r_{\text{trap}}}{r_+}\right)^2 \leq 2$$

•

$$\left(1 - \frac{r_{\text{trap}}}{r}\right)^2 \leq 2$$

•

$$|f| \leq \max\{|f(r_+)|, 1\} \leq 2\sqrt{2}$$

•

$$|y| \leq \left(1 - \frac{r_{\text{trap}}}{r}\right)^2 |f| \leq 2|f| \leq 4\sqrt{2}$$

•

$$|f + y| \leq 6\sqrt{2}$$

•

$$\frac{1}{2} \leq \left(4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2\Delta}{(r^2 + a^2)^2}\right) \leq 4$$

•

$$\frac{1}{2} \leq 1 - wa^2 \sin^2 \theta \leq 1 + \frac{a^2}{2Mr_+} \leq \frac{3}{2}$$

•

$$1 \leq \frac{r^2 + a^2}{\rho^2} \leq 1 + \frac{a^2}{r_+^2}$$

•

$$\frac{2Mr}{\Delta} w = \frac{2Mr}{(r^2 + a^2)^2} = \left(\frac{2Mr}{r^2 + a^2}\right) \frac{1}{r^2 + a^2} \leq \frac{1}{r^2 + a^2} \leq \frac{1}{r_+^2 + a^2} = \frac{1}{2Mr_+}$$

B Estimating boundary terms

Here we show how we can obtain the bound

$$\begin{aligned} & \frac{r^2 + a^2}{\Delta} \left\{ F_{L+\underline{L}}^f + F_{L+\underline{L}}^y \right\} + \frac{2Mr}{\Delta} \left\{ F_{L-\underline{L}}^f + F_{L-\underline{L}}^y \right\} \\ & \leq \left| \frac{r^2 + a^2}{\Delta} \left\{ F_{L+\underline{L}}^f + F_{L+\underline{L}}^y \right\} + \frac{2Mr}{\Delta} \left\{ F_{L-\underline{L}}^f + F_{L-\underline{L}}^y \right\} \right| \\ & \leq C \cdot \mathbb{E} \mathbb{B}[\Psi] \end{aligned}$$

where $C = C(M)$ will be computed explicitly.

We can compute

$$\begin{aligned} & \frac{r^2 + a^2}{\Delta} \left\{ F_{L+\underline{L}}^f + F_{L+\underline{L}}^y \right\} + \frac{2Mr}{\Delta} \left\{ F_{L-\underline{L}}^f + F_{L-\underline{L}}^y \right\} \\ & = \frac{f'}{4} \left(1 - wa^2 \sin^2 \theta \right) \operatorname{Re} \left\{ (L + \underline{L}) \Psi \cdot \bar{\Psi} \right\} + \frac{f+y}{4} \left(1 - wa^2 \sin^2 \theta \right) \operatorname{Re} \left\{ (L + \underline{L}) \Psi \cdot (L - \underline{L}) \bar{\Psi} \right\} \\ & \quad - w(f+y)sa \cos \theta \operatorname{Im} \left\{ \Psi \cdot \bar{\Psi}' \right\} \\ & + \frac{2Mr}{\Delta} \left[-\frac{f+y}{2} \left(1 - wa^2 \sin^2 \theta \right) |\underline{L}\Psi|^2 - \frac{f+y}{8} wa^2 \sin^2 \theta |(L - \underline{L})\Psi|^2 + w(f+y)sa \cos \theta \cdot \operatorname{Im} \left\{ \Psi \cdot \underline{L}\bar{\Psi} \right\} \right. \\ & \quad \left. + \frac{1}{2} w(f+y) \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 + \left(\frac{f+y}{2} wP + \frac{f''}{4} \right) |\Psi|^2 + \frac{f'}{2} \operatorname{Re} \left\{ \underline{L}\Psi \cdot \bar{\Psi} \right\} - \frac{f'}{4} wa^2 \sin^2 \theta \operatorname{Re} \left\{ (L + \underline{L}) \Psi \cdot \bar{\Psi} \right\} \right] \end{aligned}$$

where

$$P := 4 - \frac{6Mr}{r^2 + a^2} \frac{r^2 - a^2}{r^2 + a^2} - \frac{7a^2 \Delta}{(r^2 + a^2)^2}$$

We may estimate the individual terms as follows:

•

$$\begin{aligned} (1) & := \frac{f'}{4} \left(1 - wa^2 \sin^2 \theta \right) \operatorname{Re} \left\{ (L + \underline{L}) \Psi \cdot \bar{\Psi} \right\} \leq |(1)| \\ & \leq \frac{f'}{8} \left(1 + \frac{a^2}{2Mr_+} \right) \left(|(L + \underline{L})\Psi|^2 + |\Psi|^2 \right) \\ & \leq \left(\max_{r \geq r_+} f'(r) \right) \left(1 + \frac{a^2}{2Mr_+} \right) \left(1 + \frac{a^2}{r_+^2} \right) \cdot \left(\frac{\rho^2}{8(r^2 + a^2)} |(L + \underline{L})\Psi|^2 \right) \\ & \quad + \left(1 + \frac{a^2}{2Mr_+} \right) \sup_{r \geq r_+} \left(\frac{f'(r)}{\frac{4}{r^2 + a^2} P} \right) \cdot \left(\frac{1}{2(r^2 + a^2)} P |\Psi|^2 \right) \end{aligned}$$

•

$$\begin{aligned} (2) & := \frac{f+y}{4} \left(1 - wa^2 \sin^2 \theta \right) \operatorname{Re} \left\{ (L + \underline{L}) \Psi \cdot (L - \underline{L}) \bar{\Psi} \right\} \leq |(2)| \\ & = \frac{1}{4} |f+y| \left(1 - wa^2 \sin^2 \theta \right) \left| \operatorname{Re} \left\{ (L + \underline{L}) \Psi \cdot (L - \underline{L}) \bar{\Psi} \right\} \right| \\ & \leq \frac{1}{8} |f+y| \left(1 + \frac{a^2}{2Mr_+} \right) \left(|(L + \underline{L})\Psi|^2 + |(L - \underline{L})\Psi|^2 \right) \\ & \leq |f+y| \left(1 + \frac{a^2}{2Mr_+} \right) \left(1 + \frac{a^2}{r_+^2} \right) \cdot \left(\frac{\rho^2}{8(r^2 + a^2)} |(L + \underline{L})\Psi|^2 \right) \\ & \quad + |f+y| \left(1 + \frac{a^2}{2Mr_+} \right) \cdot \left(\frac{1}{8} |(L - \underline{L})\Psi|^2 \right) \end{aligned}$$

$$\begin{aligned}
(3) &:= -w(f+y)sa \cos \theta \operatorname{Im} \left\{ \Psi \cdot \bar{\Psi}' \right\} \leq |(3)| = |f+y|aw|s \cos \theta| \left| \operatorname{Im} \left\{ \Psi \cdot \bar{\Psi}' \right\} \right| \\
&\leq |f+y|aw 2 \left| \operatorname{Im} \left\{ \Psi \cdot \bar{\Psi}' \right\} \right| = |f+y|aw 2 \left| \operatorname{Im} \left\{ \Psi \cdot \frac{1}{2}(L - \underline{L})\bar{\Psi} \right\} \right| \\
&\leq |f+y|aw \left(|\Psi|^2 + \frac{1}{4}|(L - \underline{L})\Psi|^2 \right) \\
&= |f+y|a \frac{\Delta}{r^2+a^2} \frac{1}{r^2+a^2} |\Psi|^2 + \frac{1}{4}|f+y|aw|(L - \underline{L})\Psi|^2 \\
&\leq |f+y|a \sup_{r \geq r_+} \left(\frac{\frac{\Delta}{r^2+a^2}}{P} \right) P \frac{1}{r^2+a^2} |\Psi|^2 + \frac{1}{4}|f+y|a \left(\max_{r \geq r_+} w(r) \right) |(L - \underline{L})\Psi|^2 \\
&= 2|f+y|a \sup_{r \geq r_+} \left(\frac{\frac{\Delta}{r^2+a^2}}{P} \right) \cdot \left(\frac{1}{8} \frac{4}{r^2+a^2} P |\Psi|^2 \right) + 2|f+y|a \left(\max_{r \geq r_+} w(r) \right) \cdot \left(\frac{1}{8} |(L - \underline{L})\Psi|^2 \right)
\end{aligned}$$

$$\begin{aligned}
(4) &:= -\frac{2Mr}{\Delta} \frac{f+y}{2} (1 - wa^2 \sin^2 \theta) |\underline{L}\Psi|^2 \leq |(4)| \\
&\leq |f+y| \left(1 + \frac{a^2}{2Mr_+} \right) \cdot \left(\frac{Mr}{\Delta} |\underline{L}\Psi|^2 \right)
\end{aligned}$$

$$\begin{aligned}
(5) &:= -\frac{2Mr}{\Delta} \frac{f+y}{8} wa^2 \sin^2 \theta |(L - \underline{L})\Psi|^2 \leq |(5)| = \frac{|f+y|}{8} \frac{2Mr}{\Delta} wa^2 \sin^2 \theta |(L - \underline{L})\Psi|^2 \\
&\leq \frac{|f+y|}{8} \left(\frac{2Mr}{\Delta} w \right) a^2 |(L - \underline{L})\Psi|^2 \\
&\leq \frac{|f+y|}{8} \left(\frac{1}{2Mr_+} \right) a^2 |(L - \underline{L})\Psi|^2 \\
&= |f+y| \frac{a^2}{2Mr_+} \cdot \left(\frac{1}{8} |(L - \underline{L})\Psi|^2 \right)
\end{aligned}$$

$$\begin{aligned}
(6) &:= \frac{2Mr}{\Delta} w(f+y)sa \cos \theta \operatorname{Im} \left\{ \Psi \cdot \underline{L}\bar{\Psi} \right\} \leq |(6)| = \frac{2Mr}{\Delta} wa|f+y||s| |\cos \theta| \cdot \left| \operatorname{Im} \left\{ \Psi \cdot \underline{L}\bar{\Psi} \right\} \right| \\
&\leq \frac{2Mr}{\Delta} wa|f+y| 2 \left| \operatorname{Im} \left\{ \Psi \cdot \underline{L}\bar{\Psi} \right\} \right| \\
&\leq a|f+y| \left(\frac{2Mr}{\Delta} w \right) |\Psi|^2 + 2a|f+y|w \frac{Mr}{\Delta} |\underline{L}\Psi|^2 \\
&= a|f+y| \frac{2Mr}{(r^2+a^2)^2} |\Psi|^2 + 2a|f+y|w \frac{Mr}{\Delta} |\underline{L}\Psi|^2 \\
&\leq a|f+y| \sup_{r \geq r_+} \left(\frac{\frac{2Mr}{(r^2+a^2)^2}}{\frac{4}{r^2+a^2} P} \right) \frac{4}{r^2+a^2} P |\Psi|^2 + 2a|f+y| \left(\max_{r \geq r_+} w(r) \right) \frac{Mr}{\Delta} |\underline{L}\Psi|^2 \\
&= \frac{a|f+y|}{2} \sup_{r \geq r_+} \left(\frac{1}{\frac{r^2+a^2}{Mr} P} \right) \frac{4}{r^2+a^2} P |\Psi|^2 + 2a|f+y| \left(\max_{r \geq r_+} w(r) \right) \frac{Mr}{\Delta} |\underline{L}\Psi|^2 \\
&\leq 4a|f+y| \sup_{r \geq r_+} \left(\frac{1}{\frac{r^2+a^2}{Mr} P} \right) \cdot \left(\frac{4}{8(r^2+a^2)} P |\Psi|^2 \right) + 2a|f+y| \left(\max_{r \geq r_+} w(r) \right) \cdot \left(\frac{Mr}{\Delta} |\underline{L}\Psi|^2 \right)
\end{aligned}$$

$$\begin{aligned}
(7) &:= \frac{2Mr}{\Delta} \frac{f+y}{2} w \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \leq |(7)| = \frac{1}{2} |f+y| \frac{2Mr}{\Delta} w \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \\
&= \frac{1}{2} |f+y| \frac{2Mr}{r^2+a^2} \frac{1}{r^2+a^2} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \\
&\leq \frac{1}{2} |f+y| \frac{1}{r^2+a^2} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \\
&= |f+y| \cdot \left(\frac{1}{2(r^2+a^2)} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \right)
\end{aligned}$$

$$\begin{aligned}
(8) &:= \frac{2Mr}{\Delta} \left(\frac{f+y}{2} wP + \frac{f''}{4} \right) |\Psi|^2 \leq |(8)| = \left| \frac{2Mr}{\Delta} \left(\frac{f+y}{2} wP + \frac{f''}{4} \right) \right| |\Psi|^2 \\
&\leq \sup_{r \geq r_+} \left(\frac{\left| \frac{2Mr}{\Delta} \left(\frac{f+y}{2} wP + \frac{f''}{4} \right) \right|}{\frac{1}{2(r^2+a^2)} P} \right) \cdot \frac{1}{2(r^2+a^2)} P |\Psi|^2 \\
&= \sup_{r \geq r_+} \left| \frac{\frac{2Mr}{\Delta} \left(\frac{f+y}{2} wP + \frac{f''}{4} \right)}{\frac{1}{2(r^2+a^2)} P} \right| \cdot \left(\frac{1}{2(r^2+a^2)} P |\Psi|^2 \right)
\end{aligned}$$

$$\begin{aligned}
(9) &:= \frac{2Mr}{\Delta} \frac{f'}{2} \operatorname{Re} \{ \underline{L} \Psi \cdot \bar{\Psi} \} \leq |(9)| = \frac{2Mr}{\Delta} \frac{f'}{2} |\operatorname{Re} \{ \underline{L} \Psi \cdot \bar{\Psi} \}| \\
&\leq \frac{f'}{4} \frac{2Mr}{\Delta} |\underline{L} \Psi|^2 + \frac{f'}{4} \frac{2Mr}{\Delta} |\Psi|^2 \\
&\leq \frac{1}{2} \left(\max_{r \geq r_+} f'(r) \right) \cdot \frac{Mr}{\Delta} |\underline{L} \Psi|^2 + \sup_{r \geq r_+} \left(\frac{\frac{f'}{4} \frac{2Mr}{\Delta}}{\frac{1}{2(r^2+a^2)} P} \right) \cdot \frac{1}{2(r^2+a^2)} P |\Psi|^2 \\
&\leq \frac{1}{2} \left(\max_{r \geq r_+} f'(r) \right) \cdot \left(\frac{Mr}{\Delta} |\underline{L} \Psi|^2 \right) + \sup_{r \geq r_+} \left(\frac{Mr \partial_r f}{P} \right) \cdot \left(\frac{1}{2(r^2+a^2)} P |\Psi|^2 \right)
\end{aligned}$$

$$\begin{aligned}
(10) &:= -\frac{2Mr}{\Delta} \frac{f'}{4} w a^2 \sin^2 \theta \operatorname{Re} \{ (L + \underline{L}) \Psi \cdot \bar{\Psi} \} \leq |(10)| = \frac{Mr}{\Delta} \frac{f'}{4} w a^2 \sin^2 \theta 2 |\operatorname{Re} \{ (L + \underline{L}) \Psi \cdot \bar{\Psi} \}| \\
&\leq \frac{f'}{4} \frac{Mr}{\Delta} w a^2 |(L + \underline{L}) \Psi|^2 + a^2 \frac{Mr}{\Delta} \frac{f'}{4} w |\Psi|^2 \\
&= 2a^2 \frac{Mr f'}{(r^2+a^2) \rho^2} \cdot \left(\frac{\rho^2}{8(r^2+a^2)} |(L + \underline{L}) \Psi|^2 \right) + a^2 \frac{Mr}{\Delta} \frac{f'}{4} w |\Psi|^2 \\
&\leq 2a^2 \frac{M f'}{r(r^2+a^2)} \cdot \left(\frac{\rho^2}{8(r^2+a^2)} |(L + \underline{L}) \Psi|^2 \right) + a^2 \frac{Mr}{\Delta} \frac{f'}{4} w |\Psi|^2 \\
&\leq 2a^2 \sup_{r \geq r_+} \left(\frac{M f'}{r(r^2+a^2)} \right) \cdot \left(\frac{\rho^2}{8(r^2+a^2)} |(L + \underline{L}) \Psi|^2 \right) + a^2 \sup_{r \geq r_+} \left(\frac{\frac{Mr}{\Delta} \frac{f'}{4} w}{\frac{1}{2(r^2+a^2)} P} \right) \cdot \left(\frac{1}{2(r^2+a^2)} P |\Psi|^2 \right)
\end{aligned}$$

Hence we can estimate:

$$\begin{aligned}
& \left| \frac{r^2 + a^2}{\Delta} \{F_{L+\underline{L}}^f + F_{L+\underline{L}}^y\} + \frac{2Mr}{\Delta} \{F_{L-\underline{L}}^f + F_{L-\underline{L}}^y\} \right| = \left| (1) + (2) + (3) + (4) + (5) + (6) + (7) + (8) + (9) + (10) \right| \\
& \leq \left(\max_{r \geq r_+} f'(r) \right) \left(1 + \frac{a^2}{2Mr_+} \right) \left(1 + \frac{a^2}{r_+^2} \right) \cdot \left(\frac{\rho^2}{8(r^2 + a^2)} |(L + \underline{L})\Psi|^2 \right) + \left(1 + \frac{a^2}{2Mr_+} \right) \sup_{r \geq r_+} \left(\frac{f'(r)}{\frac{4}{r^2 + a^2} P} \right) \cdot \left(\frac{1}{2(r^2 + a^2)} P |\Psi|^2 \right) \\
& + |f + y| \left(1 + \frac{a^2}{2Mr_+} \right) \left(1 + \frac{a^2}{r_+^2} \right) \cdot \left(\frac{\rho^2}{8(r^2 + a^2)} |(L + \underline{L})\Psi|^2 \right) + |f + y| \left(1 + \frac{a^2}{2Mr_+} \right) \cdot \left(\frac{1}{8} |(L - \underline{L})\Psi|^2 \right) \\
& + 2|f + y|a \sup_{r \geq r_+} \left(\frac{\frac{\Delta}{r^2 + a^2}}{P} \right) \cdot \left(\frac{1}{2(r^2 + a^2)} P |\Psi|^2 \right) + 2|f + y|a \left(\max_{r \geq r_+} w(r) \right) \cdot \left(\frac{1}{8} |(L - \underline{L})\Psi|^2 \right) \\
& + |f + y| \left(1 + \frac{a^2}{2Mr_+} \right) \cdot \left(\frac{Mr}{\Delta} |\underline{L}\Psi|^2 \right) \\
& + |f + y| \frac{a^2}{2Mr_+} \cdot \left(\frac{1}{8} |(L - \underline{L})\Psi|^2 \right) \\
& + 4a|f + y| \sup_{r \geq r_+} \left(\frac{1}{\frac{r^2 + a^2}{Mr} P} \right) \cdot \left(\frac{1}{2(r^2 + a^2)} P |\Psi|^2 \right) + 2a|f + y| \left(\max_{r \geq r_+} w(r) \right) \cdot \left(\frac{Mr}{\Delta} |\underline{L}\Psi|^2 \right) \\
& + |f + y| \cdot \left(\frac{1}{2(r^2 + a^2)} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \right) \\
& + \sup_{r \geq r_+} \left| \frac{\frac{2Mr}{\Delta} \left(\frac{f+y}{2} wP + \frac{f''}{4} \right)}{\frac{1}{2(r^2 + a^2)} P} \right| \cdot \left(\frac{1}{2(r^2 + a^2)} P |\Psi|^2 \right) \\
& + \frac{1}{2} \left(\max_{r \geq r_+} f'(r) \right) \cdot \left(\frac{Mr}{\Delta} |\underline{L}\Psi|^2 \right) + \sup_{r \geq r_+} \left(\frac{Mr \partial_r f}{P} \right) \cdot \left(\frac{1}{2(r^2 + a^2)} P |\Psi|^2 \right) \\
& + 2a^2 \sup_{r \geq r_+} \left(\frac{Mf'}{r(r^2 + a^2)} \right) \cdot \left(\frac{\rho^2}{8(r^2 + a^2)} |(L + \underline{L})\Psi|^2 \right) + a^2 \sup_{r \geq r_+} \left(\frac{\frac{Mr}{\Delta} \frac{f'}{4} w}{\frac{1}{2(r^2 + a^2)} P} \right) \cdot \left(\frac{1}{2(r^2 + a^2)} P |\Psi|^2 \right)
\end{aligned}$$

Collecting terms, we write:

$$\begin{aligned}
& \left| \frac{r^2 + a^2}{\Delta} \{F_{L+\underline{L}}^f + F_{L+\underline{L}}^y\} + \frac{2Mr}{\Delta} \{F_{L-\underline{L}}^f + F_{L-\underline{L}}^y\} \right| \\
& \leq \left[\left(\max_{r \geq r_+} f'(r) \right) \left(1 + \frac{a^2}{2Mr_+} \right) \left(1 + \frac{a^2}{r_+^2} \right) + |f + y| \left(1 + \frac{a^2}{2Mr_+} \right) \left(1 + \frac{a^2}{r_+^2} \right) \right. \\
& \quad \left. + 2a^2 \sup_{r \geq r_+} \left(\frac{Mf'}{r(r^2 + a^2)} \right) \right] \cdot \frac{\rho^2}{8(r^2 + a^2)} |(L + \underline{L})\Psi|^2 \\
& \quad + |f + y| \left[\left(1 + \frac{a^2}{2Mr_+} \right) + 2a \left(\max_{r \geq r_+} w(r) \right) + \frac{a^2}{2Mr_+} \right] \cdot \frac{1}{8} |(L - \underline{L})\Psi|^2 \\
& \quad + \left[|f + y| \left(1 + \frac{a^2}{2Mr_+} \right) + 2|f + y|a \left(\max_{r \geq r_+} w(r) \right) + \frac{1}{2} \left(\max_{r \geq r_+} f'(r) \right) \right] \cdot \frac{Mr}{\Delta} |\underline{L}\Psi|^2 \\
& \quad + |f + y| \cdot \frac{1}{2(r^2 + a^2)} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \\
& \quad + \left[\left(1 + \frac{a^2}{2Mr_+} \right) \sup_{r \geq r_+} \left(\frac{f'(r)}{\frac{4}{r^2 + a^2} P} \right) + 2|f + y|a \sup_{r \geq r_+} \left(\frac{\frac{\Delta}{r^2 + a^2}}{P} \right) + 4|f + y|a \sup_{r \geq r_+} \left(\frac{1}{\frac{r^2 + a^2}{Mr} P} \right) \right. \\
& \quad \left. + \sup_{r \geq r_+} \left| \frac{\frac{2Mr}{\Delta} \left(\frac{f+y}{2} wP + \frac{f''}{4} \right)}{\frac{1}{2(r^2 + a^2)} P} \right| + \sup_{r \geq r_+} \left(\frac{Mr \partial_r f}{P} \right) + a^2 \sup_{r \geq r_+} \left(\frac{\frac{Mr}{\Delta} \frac{f'}{4} w}{\frac{1}{2(r^2 + a^2)} P} \right) \right] \cdot \frac{1}{2(r^2 + a^2)} P |\Psi|^2 \\
& =: [A] \cdot \frac{\rho^2}{8(r^2 + a^2)} |(L + \underline{L})\Psi|^2 + [B] \cdot \frac{1}{8} |(L - \underline{L})\Psi|^2 + [C] \cdot \frac{Mr}{\Delta} |\underline{L}\Psi|^2 \\
& \quad + |f + y| \cdot \frac{1}{2(r^2 + a^2)} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 + [D] \cdot \frac{1}{2(r^2 + a^2)} P |\Psi|^2
\end{aligned} \tag{B.1}$$

We can derive the bounds:

- $1 + \frac{a^2}{2Mr_+} \leq \frac{3}{2}$
- $1 + \frac{a^2}{r_+^2} \leq 2$
- $\left(\max_{r \geq r_+} f'(r) \right) \leq \frac{1}{M\tilde{r}_+^2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right)$
- $\left(\max_{r \geq r_+} w(r) \right) \leq \frac{1}{2M^2}$
- $\frac{a^2}{Mr_+} \leq 1$
- $\sup_{r \geq r_+} \left(\frac{Mf'}{r(r^2 + a^2)} \right) \leq \frac{1}{M^3} \frac{1}{2\tilde{r}_+^4} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right)$
- $\sup_{r \geq r_+} \left(\frac{f'(r)}{\frac{4}{r^2 + a^2} P} \right) \leq M \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right)$
- $\sup_{r \geq r_+} \left(\frac{\frac{\Delta}{r^2 + a^2}}{P} \right) \leq 2$
- $\sup_{r \geq r_+} \left(\frac{1}{\frac{r^2 + a^2}{Mr} P} \right) \leq 1$
- $\sup_{r \geq r_+} \left| \frac{\frac{2Mr}{\Delta} \left(\frac{f+y}{2} wP + \frac{f''}{4} \right)}{\frac{1}{2(r^2 + a^2)} P} \right| \leq 6\sqrt{2} + \frac{1}{\tilde{r}_+} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) + \frac{4}{\tilde{r}_+^2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{3\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right)$

- $\sup_{r \geq r_+} \left(\frac{Mr \partial_r f}{P} \right) \leq \frac{2M}{\tilde{r}_+} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right)$
- $\sup_{r \geq r_+} \left(\frac{\frac{Mr}{\frac{1}{4}} \frac{f' w}{2(r^2+a^2)} P}{P} \right) \leq \frac{1}{2M\tilde{r}_+^2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right)$

Using these bounds, we may estimate:

- $$\begin{aligned}
 [A] &= \left(|f+y| + \max_{r \geq r_+} f'(r) \right) \left(1 + \frac{a^2}{2Mr_+} \right) \left(1 + \frac{a^2}{r_+^2} \right) + 2a^2 \sup_{r \geq r_+} \left(\frac{Mf'}{r(r^2+a^2)} \right) \\
 &\leq \left(6\sqrt{2} + \frac{1}{M\tilde{r}_+^2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \right) \cdot \frac{3}{2} \cdot 2 + 2a^2 \frac{1}{M^3} \frac{1}{2\tilde{r}_+^4} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \\
 &\leq 3 \left(6\sqrt{2} + \frac{1}{M\tilde{r}_+^2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \right) + \frac{1}{M\tilde{r}_+^4} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \\
 &= 18\sqrt{2} + \left(\frac{3}{M\tilde{r}_+^2} + \frac{1}{M\tilde{r}_+^4} \right) \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \\
 &\leq 18\sqrt{2} + \frac{4}{M} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right)
 \end{aligned}$$
- $$\begin{aligned}
 [B] &= |f+y| \left[\left(1 + \frac{a^2}{2Mr_+} \right) + 2a \left(\max_{r \geq r_+} w(r) \right) + \frac{a^2}{2Mr_+} \right] \\
 &\leq 6\sqrt{2} \left(\frac{3}{2} + \frac{a}{M^2} + \frac{1}{2} \right) = 6\sqrt{2} \left(2 + \frac{1}{M} \frac{a}{M} \right) \\
 &\leq 6\sqrt{2} \left(2 + \frac{1}{M} \right) = 12\sqrt{2} + \frac{6\sqrt{2}}{M}
 \end{aligned}$$
- $$\begin{aligned}
 [C] &= |f+y| \left(\left(1 + \frac{a^2}{2Mr_+} \right) + 2a \left(\max_{r \geq r_+} w(r) \right) \right) + \frac{1}{2} \left(\max_{r \geq r_+} f'(r) \right) \\
 &\leq 6\sqrt{2} \left(\frac{3}{2} + \frac{a}{M^2} \right) + \frac{1}{2M\tilde{r}_+^2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \\
 &= 9\sqrt{2} + \frac{6\sqrt{2}}{M} \left(\frac{a}{M} \right) + \frac{1}{2M} \frac{1}{\tilde{r}_+^2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \\
 &\leq 9\sqrt{2} + \frac{6\sqrt{2}}{M} + \frac{1}{2M} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right)
 \end{aligned}$$

$$\begin{aligned}
[D] &= \left(1 + \frac{a^2}{2Mr_+}\right) \sup_{r \geq r_+} \left(\frac{f'(r)}{\frac{4}{r^2+a^2}P} \right) + 2|f+y|a \sup_{r \geq r_+} \left(\frac{\frac{\Delta}{r^2+a^2}}{P} \right) + 4|f+y|a \sup_{r \geq r_+} \left(\frac{1}{\frac{r^2+a^2}{Mr}P} \right) \\
&\quad + \sup_{r \geq r_+} \left| \frac{\frac{2Mr}{\Delta} \left(\frac{f+y}{2}wP + \frac{f''}{4} \right)}{\frac{1}{2(r^2+a^2)}P} \right| + \sup_{r \geq r_+} \left(\frac{Mr \partial_r f}{P} \right) + a^2 \sup_{r \geq r_+} \left(\frac{\frac{Mr}{\Delta} \frac{f'}{4}w}{\frac{1}{2(r^2+a^2)}P} \right) \\
&\leq \frac{3}{2} \cdot M \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) + 2 \cdot 6\sqrt{2} \cdot a \cdot 2 + 4 \cdot 6\sqrt{2} \cdot a \cdot 1 \\
&\quad + 6\sqrt{2} + \frac{1}{\tilde{r}_+} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) + \frac{4}{\tilde{r}_+^2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{3\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \\
&\quad + \frac{2M}{\tilde{r}_+} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) + a^2 \frac{1}{2M\tilde{r}_+^2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \\
&\leq \frac{3M}{2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) + 48\sqrt{2}a \\
&\quad + 6\sqrt{2} + \frac{1}{\tilde{r}_+} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) + \frac{4}{\tilde{r}_+^2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{3\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \\
&\quad + \frac{2M}{\tilde{r}_+} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) + \frac{M}{2\tilde{r}_+^2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \\
&= 6\sqrt{2} + 48\sqrt{2}a + \left(\frac{3M}{2} + \frac{1}{\tilde{r}_+} + \frac{2M}{\tilde{r}_+} + \frac{M}{2\tilde{r}_+^2} \right) \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) + \frac{4}{\tilde{r}_+^2} \left(\tilde{r}_{\text{trap}} - 1 + \frac{3\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \\
&\leq 6\sqrt{2} + 48\sqrt{2}M + \left(\frac{3M}{2} + 1 + 2M + \frac{M}{2} \right) \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) + 4 \left(\tilde{r}_{\text{trap}} - 1 + \frac{3\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \\
&= 6\sqrt{2} + 48\sqrt{2}M + (1+4M) \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) + 4 \left(\tilde{r}_{\text{trap}} - 1 + \frac{3\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right)
\end{aligned}$$

Hence, combining the estimates for $[A]$, $[B]$, $[C]$, $[D]$ with (B.1), we obtain:

$$\begin{aligned}
&\left| \frac{r^2+a^2}{\Delta} \left\{ F_{L+\underline{L}}^f + F_{L+\underline{L}}^y \right\} + \frac{2Mr}{\Delta} \left\{ F_{L-\underline{L}}^f + F_{L-\underline{L}}^y \right\} \right| \\
&\leq [A] \cdot \frac{\rho^2}{8(r^2+a^2)} |(L+\underline{L})\Psi|^2 + [B] \cdot \frac{1}{8} |(L-\underline{L})\Psi|^2 + [C] \cdot \frac{Mr}{\Delta} |\underline{L}\Psi|^2 \\
&\quad + |f+y| \cdot \frac{1}{2(r^2+a^2)} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 + [D] \cdot \frac{1}{2(r^2+a^2)} P |\Psi|^2 \\
&\leq \left[18\sqrt{2} + \frac{4}{M} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \right] \cdot \frac{\rho^2}{8(r^2+a^2)} |(L+\underline{L})\Psi|^2 + \left[12\sqrt{2} + \frac{6\sqrt{2}}{M} \right] \cdot \frac{1}{8} |(L-\underline{L})\Psi|^2 \\
&\quad + \left[9\sqrt{2} + \frac{6\sqrt{2}}{M} + \frac{1}{2M} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \right] \cdot \frac{Mr}{\Delta} |\underline{L}\Psi|^2 + 6\sqrt{2} \cdot \frac{1}{2(r^2+a^2)} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 \\
&\quad + \left[6\sqrt{2} + 48\sqrt{2}M + (1+4M) \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) + 4 \left(\tilde{r}_{\text{trap}} - 1 + \frac{3\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \right] \cdot \frac{1}{2(r^2+a^2)} P |\Psi|^2 \\
&\leq C(M, a) \cdot \left[\frac{\rho^2}{8(r^2+a^2)} |(L+\underline{L})\Psi|^2 + \frac{1}{8} |(L-\underline{L})\Psi|^2 + \frac{Mr}{\Delta} |\underline{L}\Psi|^2 + \frac{1}{2(r^2+a^2)} \left| \overset{\circ}{\nabla}^{[s]} \Psi \right|^2 + \frac{1}{2(r^2+a^2)} P |\Psi|^2 \right] \\
&= C(M, a) \cdot \mathbb{E}\mathbb{B}[\Psi]
\end{aligned}$$

where

$$C(M, a) = \max \left\{ 18\sqrt{2} + \frac{4}{M} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right), 12\sqrt{2} + \frac{6\sqrt{2}}{M}, 9\sqrt{2} + \frac{6\sqrt{2}}{M} + \frac{1}{2M} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right), \right. \\ \left. 6\sqrt{2}, 6\sqrt{2} + 48\sqrt{2}M + (1 + 4M) \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) + 4 \left(\tilde{r}_{\text{trap}} - 1 + \frac{3\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \right\}$$

Since

$$18\sqrt{2} + \frac{4}{M} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) > 12\sqrt{2} + \frac{6\sqrt{2}}{M} > 6\sqrt{2}$$

and

$$18\sqrt{2} + \frac{4}{M} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) > 9\sqrt{2} + \frac{6\sqrt{2}}{M} + \frac{1}{2M} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right)$$

for all M and a , we have that

$$C(M, a) = \max \left\{ 18\sqrt{2} + \frac{4}{M} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right), 6\sqrt{2} + 48\sqrt{2}M + (1 + 4M) \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \right\}$$

which we may bound (for all a) as follows:

$$C(M, a) = \max \left\{ 18\sqrt{2} + \frac{4}{M} \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right), 6\sqrt{2} + 48\sqrt{2}M + (1 + 4M) \left(\tilde{r}_{\text{trap}} - 1 + \frac{2\tilde{r}_{\text{trap}}}{\tilde{r}_+} \right) \right\} \\ \leq \max \left\{ 18\sqrt{2} + \frac{8(2 + \sqrt{2})}{M}, 24 + 20\sqrt{2} + 8(2 + 7\sqrt{2})M \right\} \\ =: C(M)$$

In summary, we have derived the estimate:

$$\left| \frac{r^2 + a^2}{\Delta} \left\{ F_{L+\underline{L}}^f + F_{L+\underline{L}}^y \right\} + \frac{2Mr}{\Delta} \left\{ F_{L-\underline{L}}^f + F_{L-\underline{L}}^y \right\} \right| \leq C(M) \cdot \mathbb{E} \mathbb{B} [\Psi] \quad (\text{B.2})$$

where

$$C(M) = \max \left\{ 18\sqrt{2} + \frac{8(2 + \sqrt{2})}{M}, 24 + 20\sqrt{2} + 8(2 + 7\sqrt{2})M \right\}$$

C Expanding error terms

C.1 Error terms involving $\Psi^{[-2]}$

We have:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathcal{J}^{[-2]} \cdot T\overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ \left[h_1 \cdot \left(\sqrt{\Delta} \psi^{[-2]} \right) + h_2 \cdot \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right) \right] \cdot f' \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} h_1 f' \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega + \int_{\mathcal{R}(\tau_1, \tau_2)} h_2 f' \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega
\end{aligned} \tag{C.1}$$

We now use the definition of $\Psi^{[-2]}$, which gives us:

$$\overline{\Psi^{[-2]}} = -\frac{(r^2 + a^2)^2}{\Delta} L^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[-2]} \right)$$

to derive:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} h_1 f' \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} h_1 f' \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \left(-\frac{(r^2 + a^2)^2}{\Delta} L^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[-2]} \right) \right) \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} -\frac{(r^2 + a^2)^2}{\Delta} h_1 f' \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot L^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[-2]} \right) \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} -\frac{(r^2 + a^2)^2}{2\Delta} h_1 f' \left[\sqrt{\Delta} \psi^{[-2]} \cdot L^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[-2]} \right) + \sqrt{\Delta} \psi^{[-2]} \cdot L^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[-2]} \right) \right] dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} -\frac{(r^2 + a^2)^2}{2\Delta \rho^2} h_1 f' L^\mu \nabla_\mu \left[\left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \right] \rho^2 dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{(r^2 + a^2)^2}{2\Delta \rho^2} h_1 f' \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 L^\mu \right] dV + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left(\frac{(r^2 + a^2)^2}{2\Delta \rho^2} h_1 f' L^\mu \right) \cdot \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \rho^2 dt^* dr d\omega
\end{aligned}$$

By covariant identities 2.5, we have

$$\begin{aligned}
\nabla_\mu \left(\frac{(r^2 + a^2)^2}{2\Delta \rho^2} h_1 f' L^\mu \right) &= \nabla_\mu \left(\frac{r^2 + a^2}{\Delta \rho^2} L^\mu \cdot \frac{r^2 + a^2}{2} h_1 f' \right) \\
&= \frac{r^2 + a^2}{\Delta \rho^2} L^\mu \nabla_\mu \left(\frac{r^2 + a^2}{2} h_1 f' \right) \\
&= \frac{r^2 + a^2}{\Delta \rho^2} L^{r^*} \nabla_{r^*} \left(\frac{r^2 + a^2}{2} h_1 f' \right) \\
&= \frac{r^2 + a^2}{\Delta \rho^2} \partial_{r^*} \left(\frac{r^2 + a^2}{2} h_1 f' \right) \\
&= \frac{1}{\rho^2} \partial_r \left(\frac{r^2 + a^2}{2} h_1 f' \right)
\end{aligned}$$

Thus we derived:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} h_1 f' \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{(r^2 + a^2)^2}{2\Delta \rho^2} h_1 f' \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 L^\mu \right] dV \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{r^2 + a^2}{2} h_1 f' \right) \cdot \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega
\end{aligned} \tag{C.2}$$

Moreover,

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} h_2 f' \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} - \frac{(r^2 + a^2)^2}{\Delta} h_2 f' \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot L^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[-2]} \right) \right\} dt^* dr d\omega
\end{aligned}$$

where we have

$$\begin{aligned}
& \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot L^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[-2]} \right) \right\} \\
&= \frac{1}{2} \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot L^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[-2]} \right) + \overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \cdot L^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[-2]} \right) \right) \\
&= \left[L^\mu \nabla_\mu \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta} \psi^{[-2]} \right) - L^\mu \nabla_\mu \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right) \cdot \sqrt{\Delta} \psi^{[-2]} \right. \\
&\quad \left. + L^\mu \nabla_\mu \left(\overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta} \psi^{[-2]} \right) - L^\mu \nabla_\mu \left(\overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \right) \cdot \sqrt{\Delta} \psi^{[-2]} \right] \\
&= L^\mu \nabla_\mu \left[\operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta} \psi^{[-2]} \right\} \right] - \frac{2\Delta}{(r^2 + a^2)^2} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2
\end{aligned}$$

where we used the definition of $\psi^{[-2]}$:

$$\sqrt{\Delta} \psi^{[-2]} = \frac{(r^2 + a^2)^2}{2\Delta} L^\mu \nabla_\mu \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right)$$

Hence we have:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} h_2 f' \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} - \frac{(r^2 + a^2)^2}{\Delta \rho^2} h_2 f' L^\mu \nabla_\mu \left[\operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta} \psi^{[-2]} \right\} \right] dV \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} 2h_2 f' \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[- \frac{(r^2 + a^2)^2}{\Delta \rho^2} h_2 f' \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta} \psi^{[-2]} \right\} L^\mu \right] dV \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left(\frac{(r^2 + a^2)^2}{\Delta \rho^2} h_2 f' L^\mu \right) \cdot \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta} \psi^{[-2]} \right\} \rho^2 dt^* dr d\omega \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} 2h_2 f' \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[h_2 f' \frac{(r^2 + a^2)^2}{\Delta \rho^2} \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta} \psi^{[-2]} \right\} L^\mu \right] dV + \int_{\mathcal{R}(\tau_1, \tau_2)} 2h_2 f' \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left((r^2 + a^2) h_2 f' \right) \cdot \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta} \psi^{[-2]} \right\} dt^* dr d\omega
\end{aligned}$$

For the last term above we have:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left((r^2 + a^2) h_2 f' \right) \cdot \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left((r^2 + a^2) h_2 f' \right) \cdot \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \frac{(r^2 + a^2)^2}{2\Delta} L^\mu \nabla_\mu \left(\overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \right) \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left((r^2 + a^2) h_2 f' \right) \frac{(r^2 + a^2)^2}{4\Delta \rho^2} L^\mu \nabla_\mu \left[\left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \right] \rho^2 dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left((r^2 + a^2) h_2 f' \right) \frac{(r^2 + a^2)^2}{4\Delta \rho^2} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left(\frac{r^2 + a^2}{\Delta \rho^2} L^\mu \cdot \frac{r^2 + a^2}{4} \partial_r \left((r^2 + a^2) h_2 f' \right) \right) \cdot \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \rho^2 dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left((r^2 + a^2) h_2 f' \right) \frac{(r^2 + a^2)^2}{4\Delta \rho^2} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{r^2 + a^2}{4} \partial_r \left((r^2 + a^2) h_2 f' \right) \right) \cdot \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dt^* dr d\omega
\end{aligned}$$

Thus we have:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} h_2 f' \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[h_2 f' \frac{(r^2 + a^2)^2}{\Delta \rho^2} \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} L^\mu \right] dV \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} 2h_2 f' \left| \sqrt{\Delta \psi^{[-2]}} \right|^2 dt^* dr d\omega \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left((r^2 + a^2) h_2 f' \right) \frac{(r^2 + a^2)^2}{4\Delta \rho^2} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{r^2 + a^2}{4} \partial_r \left((r^2 + a^2) h_2 f' \right) \right) \cdot \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dt^* dr d\omega
\end{aligned} \tag{C.3}$$

Now, combining (C.1) with (C.2) and (C.3), we obtain:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathcal{I}^{[-2]} \cdot f' \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \left(\partial_r \left(\frac{r^2 + a^2}{2} h_1 f' \right) + 2h_2 f' \right) \left| \sqrt{\Delta \psi^{[-2]}} \right|^2 dt^* dr d\omega \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{r^2 + a^2}{4} \partial_r \left((r^2 + a^2) h_2 f' \right) \right) \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dt^* dr d\omega \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[h_1 f' \frac{(r^2 + a^2)^2}{2\Delta \rho^2} \left| \sqrt{\Delta \psi^{[-2]}} \right|^2 L^\mu \right] dV \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left((r^2 + a^2) h_2 f' \right) \frac{(r^2 + a^2)^2}{4\Delta \rho^2} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[h_2 f' \frac{(r^2 + a^2)^2}{\Delta \rho^2} \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} L^\mu \right] dV
\end{aligned} \tag{C.4}$$

C.2 Error terms involving $T\Psi^{[-2]}$

Here we use a trick, utilising axisymmetry $\Phi\Psi^{[-2]} = 0$. We have:

$$2T\Psi^{[-2]} = (L + \underline{L})\Psi^{[-2]}$$

Since, we also have

$$2\Psi^{[-2]'} = (L - \underline{L})\Psi^{[-2]}$$

we can add the above equalities and obtain:

$$T\Psi^{[-2]} = L\Psi^{[-2]} - \Psi^{[-2]'} \quad (\text{C.5})$$

In light of (C.5), we can write:

$$\begin{aligned} & \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathfrak{I}^{[-2]} \cdot T\overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\ &= \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ \left[h_1 \cdot \left(\sqrt{\Delta} \psi^{[-2]} \right) + h_2 \cdot \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right) \right] \cdot \left(L\overline{\Psi^{[-2]}} - \overline{\Psi^{[-2]'}} \right) \right\} dt^* dr d\omega \\ &= \int_{\mathcal{R}(\tau_1, \tau_2)} h_1 \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot L\overline{\Psi^{[-2]}} \right\} dt^* dr d\omega + \int_{\mathcal{R}(\tau_1, \tau_2)} h_2 \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot L\overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\ &\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} h_1 \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]'}} \right\} dt^* dr d\omega - \int_{\mathcal{R}(\tau_1, \tau_2)} h_2 \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]'}} \right\} dt^* dr d\omega \end{aligned} \quad (\text{C.6})$$

Let us label some terms from above:

$$\begin{aligned} A &:= \int_{\mathcal{R}(\tau_1, \tau_2)} h_1 \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot L\overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\ B &:= \int_{\mathcal{R}(\tau_1, \tau_2)} h_2 \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot L\overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \end{aligned}$$

Remark C.7. We leave the other two terms

$$\begin{aligned} & - \int_{\mathcal{R}(\tau_1, \tau_2)} h_1 \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]'}} \right\} dt^* dr d\omega \\ & - \int_{\mathcal{R}(\tau_1, \tau_2)} h_2 \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]'}} \right\} dt^* dr d\omega \end{aligned}$$

in their current form. These are the terms we ultimately have to treat by using Young's inequality.

C.2.1 The term involving $\psi^{[-2]}$

We have:

$$\begin{aligned} \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot L\overline{\Psi^{[-2]}} \right\} &= \frac{1}{2} \left[\sqrt{\Delta} \psi^{[-2]} \cdot L\overline{\Psi^{[-2]}} + \sqrt{\Delta} \overline{\psi^{[-2]}} \cdot L\Psi^{[-2]} \right] \\ &= \frac{1}{2} \left[L \left(\sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} + \sqrt{\Delta} \overline{\psi^{[-2]}} \cdot \Psi^{[-2]} \right) - L \left(\sqrt{\Delta} \psi^{[-2]} \right) \cdot \overline{\Psi^{[-2]}} - L \left(\sqrt{\Delta} \overline{\psi^{[-2]}} \right) \cdot \Psi^{[-2]} \right] \\ &= L \left(\operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} \right) + \frac{\Delta}{(r^2 + a^2)^2} \left| \Psi^{[-2]} \right|^2 \end{aligned}$$

where the last equality follows by using the relation

$$-\frac{\Delta}{(r^2 + a^2)^2} \Psi^{[-2]} = L \left(\sqrt{\Delta} \psi^{[-2]} \right)$$

Hence

$$\begin{aligned}
A &= \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{\Delta h_1}{(r^2 + a^2)^2} \left| \Psi^{[-2]} \right|^2 dt^* dr d\omega + \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{h_1}{\rho^2} L^\mu \nabla_\mu \left(\text{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} \right) \rho^2 dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{\Delta h_1}{(r^2 + a^2)^2} \left| \Psi^{[-2]} \right|^2 dt^* dr d\omega + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{h_1}{\rho^2} \text{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left(\frac{h_1}{\rho^2} L^\mu \right) \cdot \text{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} \rho^2 dt^* dr d\omega
\end{aligned}$$

In light of the covariant identities 2.5, we have:

$$\nabla_\mu \left(\frac{h_1}{\rho^2} L^\mu \right) = \frac{1}{\rho^2} \partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right)$$

Hence we have

$$\begin{aligned}
A &= \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{\Delta h_1}{(r^2 + a^2)^2} \left| \Psi^{[-2]} \right|^2 dt^* dr d\omega + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{h_1}{\rho^2} \text{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right) \cdot \text{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega
\end{aligned} \tag{C.8}$$

Using the relation

$$\Psi^{[-2]} = -\frac{(r^2 + a^2)^2}{\Delta} L^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[-2]} \right)$$

we may derive:

$$\begin{aligned}
&- \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right) \cdot \text{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right) \frac{(r^2 + a^2)^2}{\Delta} \text{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot L^\mu \nabla_\mu \left(\sqrt{\Delta} \psi^{[-2]} \right) \right\} \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right) \frac{(r^2 + a^2)^2}{2\Delta\rho^2} L^\mu \nabla_\mu \left[\left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \right] \rho^2 dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{(r^2 + a^2)^2}{2\Delta\rho^2} \partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right) \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left(\frac{(r^2 + a^2)^2}{2\Delta\rho^2} \partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right) L^\mu \right) \cdot \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \rho^2 dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{(r^2 + a^2)^2}{2\Delta\rho^2} \partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right) \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{r^2 + a^2}{2} \partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right) \right) \cdot \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega
\end{aligned}$$

Hence term A becomes:

$$\begin{aligned}
A &= \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{\Delta h_1}{(r^2 + a^2)^2} \left| \Psi^{[-2]} \right|^2 dt^* dr d\omega - \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{r^2 + a^2}{2} \partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right) \right) \cdot \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{h_1}{\rho^2} \text{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} L^\mu \right] dV \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{(r^2 + a^2)^2}{2\Delta\rho^2} \partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right) \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 L^\mu \right] dV
\end{aligned} \tag{C.9}$$

C.2.2 The term involving $\alpha^{[-2]}$

We have:

$$\begin{aligned}
& \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot L \overline{\Psi^{[-2]}} \right\} = \frac{1}{2} \left[\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot L \overline{\Psi^{[-2]}} + \overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \cdot L \Psi^{[-2]} \right] \\
&= \frac{1}{2} \left[L \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} + \overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \cdot \Psi^{[-2]} \right) \right. \\
&\quad \left. - L \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right) \cdot \overline{\Psi^{[-2]}} - L \left(\overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \right) \cdot \Psi^{[-2]} \right] \\
&= L \left(\operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} \right) - \frac{\Delta}{(r^2 + a^2)^2} \left(\sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} + \sqrt{\Delta} \overline{\psi^{[-2]}} \cdot \Psi^{[-2]} \right) \\
&= L \left(\operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} \right) + \sqrt{\Delta} \psi^{[-2]} \cdot L \left(\sqrt{\Delta} \overline{\psi^{[-2]}} \right) + \sqrt{\Delta} \overline{\psi^{[-2]}} \cdot L \left(\sqrt{\Delta} \psi^{[-2]} \right) \\
&= L \left(\operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} + \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 \right)
\end{aligned}$$

where we used the relations

$$\begin{aligned}
\frac{2\Delta}{(r^2 + a^2)^2} \left(\sqrt{\Delta} \psi^{[-2]} \right) &= L \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right) \\
-\frac{\Delta}{(r^2 + a^2)^2} \Psi^{[-2]} &= L \left(\sqrt{\Delta} \psi^{[-2]} \right)
\end{aligned}$$

Hence:

$$\begin{aligned}
B &= \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{h_2}{\rho^2} L^\mu \nabla_\mu \left(\left| \sqrt{\Delta} \psi^{[-2]} \right|^2 + \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} \right) \rho^2 dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{h_2}{\rho^2} \left(\left| \sqrt{\Delta} \psi^{[-2]} \right|^2 + \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} \right) L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left(\frac{h_2}{\rho^2} L^\mu \right) \cdot \left(\left| \sqrt{\Delta} \psi^{[-2]} \right|^2 + \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} \right) \rho^2 dt^* dr d\omega
\end{aligned}$$

Since

$$\nabla_\mu \left(\frac{h_2}{\rho^2} L^\mu \right) = \nabla_\mu \left(\frac{r^2 + a^2}{\Delta \rho^2} L^\mu \cdot \frac{\Delta h_2}{r^2 + a^2} \right) = \frac{1}{\rho^2} \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right)$$

by covariant identities 2.5, we obtain:

$$\begin{aligned}
B &= \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{h_2}{\rho^2} \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 L^\mu \right] dV + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{h_2}{\rho^2} \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \cdot \left| \sqrt{\Delta} \psi^{[-2]} \right|^2 dt^* dr d\omega \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \cdot \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \quad \left(=: X \right)
\end{aligned} \tag{C.10}$$

The last term,

$$X = - \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \cdot \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} dt^* dr d\omega$$

can be expanded further. We have

$$\begin{aligned}
X &= - \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \cdot \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \left(-\frac{(r^2 + a^2)^2}{\Delta} L \left(\sqrt{\Delta} \psi^{[-2]} \right) \right) \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \frac{(r^2 + a^2)^2}{\Delta} \cdot \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot L \left(\sqrt{\Delta} \psi^{[-2]} \right) \right\} dt^* dr d\omega
\end{aligned}$$

and

$$\begin{aligned}
& \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot L \left(\sqrt{\Delta \psi^{[-2]}} \right) \right\} \\
&= \frac{1}{2} \left[\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot L \left(\sqrt{\Delta \psi^{[-2]}} \right) + \overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \cdot L \left(\sqrt{\Delta \psi^{[-2]}} \right) \right] \\
&= \frac{1}{2} \left[L \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} + \overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right) \right. \\
&\quad \left. - L \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right) \cdot \sqrt{\Delta \psi^{[-2]}} - L \left(\overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \right) \cdot \sqrt{\Delta \psi^{[-2]}} \right] \\
&= L \left(\operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} \right) \\
&\quad - \frac{1}{2} \left[L \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right) \cdot \sqrt{\Delta \psi^{[-2]}} + L \left(\overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \right) \cdot \sqrt{\Delta \psi^{[-2]}} \right] \\
&= L \left(\operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} \right) \\
&\quad - \frac{1}{2} \left[\frac{2\Delta}{(r^2 + a^2)^2} \sqrt{\Delta \psi^{[-2]}} \cdot \sqrt{\Delta \psi^{[-2]}} + \frac{2\Delta}{(r^2 + a^2)^2} \sqrt{\Delta \psi^{[-2]}} \cdot \sqrt{\Delta \psi^{[-2]}} \right] \\
&= L \left(\operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} \right) - \frac{2\Delta}{(r^2 + a^2)^2} \left| \sqrt{\Delta \psi^{[-2]}} \right|^2
\end{aligned}$$

where we used the relation

$$L \left(\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right) = \frac{2\Delta}{(r^2 + a^2)^2} \left(\sqrt{\Delta \psi^{[-2]}} \right)$$

Hence

$$\begin{aligned}
X &= \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \frac{(r^2 + a^2)^2}{\Delta \rho^2} L^\mu \nabla_\mu \left(\operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} \right) \rho^2 dt^* dr d\omega \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \frac{(r^2 + a^2)^2}{\Delta} \left(-\frac{2\Delta}{(r^2 + a^2)^2} \left| \sqrt{\Delta \psi^{[-2]}} \right|^2 \right) dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} 2 \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \cdot \left| \sqrt{\Delta \psi^{[-2]}} \right|^2 dt^* dr d\omega \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \frac{(r^2 + a^2)^2}{\Delta \rho^2} \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left(\partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \frac{(r^2 + a^2)^2}{\Delta \rho^2} L^\mu \right) \cdot \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} \rho^2 dt^* dr d\omega \quad \left(=: Y \right)
\end{aligned}$$

The last term Y can be written as

$$\begin{aligned}
Y &= - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left(\partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \frac{(r^2 + a^2)^2}{\Delta \rho^2} L^\mu \right) \cdot \text{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} \rho^2 dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left(\frac{r^2 + a^2}{\Delta \rho^2} L^\mu \cdot (r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \cdot \text{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} \rho^2 dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{\rho^2} \partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \cdot \text{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} \rho^2 dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{\rho^2} \partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \cdot \text{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \frac{(r^2 + a^2)^2}{2\Delta} L \left(\overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \right) \right\} \rho^2 dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{1}{\rho^2} \partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \frac{(r^2 + a^2)^2}{2\Delta} \cdot \text{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot L \left(\overline{\alpha^{[-2]}} (r^2 + a^2)^{-\frac{3}{2}} \right) \right\} \rho^2 dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \frac{(r^2 + a^2)^2}{4\Delta \rho^2} \cdot L^\mu \nabla_\mu \left[\left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \right] \rho^2 dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \frac{(r^2 + a^2)^2}{4\Delta \rho^2} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] dV \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left(\frac{(r^2 + a^2)^2}{4\Delta \rho^2} \partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) L^\mu \right) \cdot \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 \rho^2 dt^* dr d\omega \\
&= - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \frac{(r^2 + a^2)^2}{4\Delta \rho^2} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] dV \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{r^2 + a^2}{4} \partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \right) \cdot \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dt^* dr d\omega
\end{aligned}$$

Hence we have:

$$\begin{aligned}
X &= - \int_{\mathcal{R}(\tau_1, \tau_2)} 2 \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \cdot \left| \sqrt{\Delta \psi^{[-2]}} \right|^2 dt^* dr d\omega \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{r^2 + a^2}{4} \partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \right) \cdot \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dt^* dr d\omega \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \frac{(r^2 + a^2)^2}{\Delta \rho^2} \text{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \frac{(r^2 + a^2)^2}{4\Delta \rho^2} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] dV
\end{aligned} \tag{C.11}$$

Combining (C.10) and (C.11), we obtain:

$$\begin{aligned}
B &= - \int_{\mathcal{R}(\tau_1, \tau_2)} 3 \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \cdot \left| \sqrt{\Delta \psi^{[-2]}} \right|^2 dt^* dr d\omega \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left(\frac{r^2 + a^2}{4} \partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \right) \cdot \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 dt^* dr d\omega \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{h_2}{\rho^2} \left| \sqrt{\Delta \psi^{[-2]}} \right|^2 L^\mu \right] dV + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{h_2}{\rho^2} \text{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} L^\mu \right] dV \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \frac{(r^2 + a^2)^2}{\Delta \rho^2} \text{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta \psi^{[-2]}} \right\} L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \frac{(r^2 + a^2)^2}{4\Delta \rho^2} \left| \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \right|^2 L^\mu \right] dV
\end{aligned} \tag{C.12}$$

C.2.3 Final expansion for $T\Psi^{[-2]}$ errors

Combining (C.6) with (C.9) and (C.12), we obtain:

$$\begin{aligned}
& \int_{\mathcal{R}(\tau_1, \tau_2)} \operatorname{Re} \left\{ -\frac{r^2 + a^2}{\Delta} \mathcal{I}^{[-2]} \cdot T\overline{\Psi^{[-2]}} \right\} dt^* dr d\omega \\
&= \int_{\mathcal{R}(\tau_1, \tau_2)} \frac{\Delta h_1}{(r^2 + a^2)^2} |\Psi^{[-2]}|^2 dt^* dr d\omega \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \left[\partial_r \left(\frac{r^2 + a^2}{2} \partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right) \right) + 3 \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right] \cdot |\sqrt{\Delta} \psi^{[-2]}|^2 dt^* dr d\omega \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \partial_r \left[\frac{r^2 + a^2}{4} \partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \right] \cdot |\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}}|^2 dt^* dr d\omega \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\left(\partial_r \left(\frac{\Delta h_1}{r^2 + a^2} \right) + \frac{2 \Delta h_2}{(r^2 + a^2)^2} \right) \frac{(r^2 + a^2)^2}{2 \Delta \rho^2} |\sqrt{\Delta} \psi^{[-2]}|^2 L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left((r^2 + a^2) \partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \right) \frac{(r^2 + a^2)^2}{4 \Delta \rho^2} |\alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}}|^2 L^\mu \right] dV \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{h_1}{\rho^2} \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}} \right\} L^\mu \right] dV \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\frac{h_2}{\rho^2} \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}} \right\} L^\mu \right] dV \\
&\quad + \int_{\mathcal{R}(\tau_1, \tau_2)} \nabla_\mu \left[\partial_r \left(\frac{\Delta h_2}{r^2 + a^2} \right) \frac{(r^2 + a^2)^2}{\Delta \rho^2} \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \sqrt{\Delta} \psi^{[-2]} \right\} L^\mu \right] dV \\
&\quad - \int_{\mathcal{R}(\tau_1, \tau_2)} h_1 \operatorname{Re} \left\{ \sqrt{\Delta} \psi^{[-2]} \cdot \overline{\Psi^{[-2]}}' \right\} dt^* dr d\omega - \int_{\mathcal{R}(\tau_1, \tau_2)} h_2 \operatorname{Re} \left\{ \alpha^{[-2]} (r^2 + a^2)^{-\frac{3}{2}} \cdot \overline{\Psi^{[-2]}}' \right\} dt^* dr d\omega
\end{aligned} \tag{C.13}$$

D Mathematica code - coefficients

Here we include Mathematica code that was used to examine the non-negativity of the coefficients of the $|(L + \underline{L})\Psi|^2$ and $|\Psi|^2$.

```
(* Kerr Parameters *)
M = 10;
a = 0.99*M;
aa = a/M;

(*
Notation:
- rr is \tilde{r} = \frac{r}{M}
- aa is \tilde{a} = \frac{a}{M}
- rrplus is r_{+}/M
- names of functions are written in double letters (e.g. ff rather than f) to
signify that they are written in the rr coordinate here.
*)

rrplus = 1 + Sqrt[1 - aa^2]; (* rrplus = rplus/M ≥ 1 *)
pp = rr^3 - 3*rr^2 + aa^2*rr + aa^2;
rrtrap = rr /. FindRoot[pp, {rr, 4}]; (* rrtrap = rtrap/M *)

ff = (1-rrtrap/rr)*(1+1/rr);
ffrr = D[ff, rr];
yy = (1-rrtrap/rr)^2 * ff;
yyrr = D[yy, rr];
DD = (rr^2+aa^2-2*rr)/(rr^2+aa^2);
PP = 4 - 6*rr/(rr^2+aa^2) * (rr^2-aa^2)/(rr^2+aa^2) - 7*aa^2*(rr^2+aa^2-2*rr)/(rr^2+aa^2)^2;
ww = 1/M^2 * ( rr^2 + aa^2 - 2*rr ) / ( rr^2 + aa^2 )^2;
wwrr = D[ww, rr];

(* coeff of |(L+\underline{L})\Psi|^2 *)
TPsiCoeff = D[yy, rr]/4 + aa * M^2 * ( D[ww, rr]*ff + D[ww*yy, rr] );
TPsiCoeffNondeg = TPsiCoeff / (1-rrtrap/rr)^2 * rr^2;
Plot[TPsiCoeffNondeg, {rr, 100*rrplus, 600*rrplus}];

(* coeff of |\Psi|^2 *)
fppTerm = 1/2 * D[DD*D[DD*D[ff, rr], rr], rr];
A1 = - ( ff*D[M^2*ww*PP, rr] + D[M^2*ww*yy*PP, rr] + fppTerm + (2-aa)*M^2*(D[ww, rr]*ff+D[ww*yy, rr]) );
GG = ( D[ff, rr]/2 + D[yy, rr]/4 );
WW = 1/M * 1/20 * 1/rr * ArcTan[4*(rr-1.55*rrtrap)];
Plot[M*WW, {rr, rrplus, 8*rrplus}];
A2 = 2 * D[DD*GG*M*WW, rr] - GG * (M*WW)^2;
Plot[DD*GG*M*WW, {rr, rrplus, 11*rrplus}];
Plot[A1, {rr, rrplus, 23*rrplus}];
Plot[A2, {rr, rrplus, 13*rrplus}];
Plot[A1+A2, {rr, rrplus, 23*rrplus}];
```

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