

A self-organizing network in the weak-coupling limit

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Abstract

We prove the existence of spatially localized ground states of the diffusive Haken model. This model describes a self-organizing network whose elements are arranged on a d -dimensional lattice with short-range diffusive coupling. The network evolves according to a competitive gradient dynamics in which the effects of diffusion are counteracted by a localizing potential that incorporates an additional global coupling term. In the absence of diffusive coupling, the ground states of the system are strictly localized, that is, only one lattice site is excited. For sufficiently small non-zero diffusive coupling α , it is shown analytically that localized ground states persist in the network with the excitations exponentially decaying in space. Numerical results establish that localization occurs for arbitrary values of α in one dimension but vanishes beyond a critical coupling $\alpha_c(d)$ when $d > 1$. The one-dimensional localized states are interpreted in terms of instanton solutions of a continuum version of the model.

I. INTRODUCTION

The study of spatially localized states in extended systems is an area of great current interest. Much of this interest has focused on spatially localized periodic oscillations in networks of coupled nonlinear oscillators - so-called *discrete breathers*. The existence of these discrete breathers has been demonstrated both numerically [1, 2] and analytically [3–5]. A number of applications of such localized structures have been proposed including DNA dynamics in molecular biology [6]. One method for proving the existence of discrete breathers in weakly coupled systems is based on the uniform continuation of periodic solutions constructed in the uncoupled limit [3, 7] (also known as the *anti-integrable* or *anti-continuum* limit). More generally, investigating how certain features of network dynamics persist from the uncoupled limit has become a powerful method for studying weakly coupled systems [8, 9]. Examples include the study of ground state systems in condensed matter physics [10–12] and multistability and wave propagation failure in networks of bistable elements [13, 14].

A distinct but very important example of localized structures in networks is found in neurobiology. In many regions of the cortex, groups of adjacent neurons appear to form higher functional units that serve to analyse some particular stimulus feature such as the orientation of an edge of an image or velocity of movement (see [15] and references therein). The association of groups of neurons with local properties of visual images varies in a regular way with the location of the neurons in the visual cortex. Thus a so-called *topographic map* of various features of an image is built up. Another example of such a phenomenon is found in the auditory cortex of the bat where a topographic map of signal amplitude versus pitch is formed. This generates a frequency spectrogram of an auditory signal. In addition, bats possess maps representing the time difference between two acoustic events, which plays an important role in the sonar orientation of the animal [16]. Topographic maps are not limited to sensory regions but also occur in the motor cortex. While sensory maps generate spatially localized regions of excitation whose location represents the signal features being analysed, motor maps create from spatially localized excitations an activity pattern in space and time that triggers a particular movement [17].

Neural network models of the formation of coherent structures in brain function generally involve two aspects: (i) a selection mechanism for determining the centre of a localized

excitation in response to an input, and (ii) an interaction mechanism that spreads the response over a neighbouring region of the network leading to the formation of a topographic map. A network typically learns the appropriate form of interaction by an adaptive process [19, 20]. The selection mechanism, on the other hand, usually involves a sequential search for the optimal response. Exploiting certain similarities between pattern formation in synergetic systems and pattern recognition, Haken [21] has constructed a simple neural network model that implements this selection process using a form of competitive gradient dynamics. (Such a model does not itself have a direct biological interpretation, but is rather a caricature of more realistic models). The ground states of the system consist of strictly localized states in which only one neuron is excited and the remainder quiescent. In other words, the network dynamically realises a *winner-take-all* strategy. An interesting extension of this type of network is the diffusive Haken model in which the introduction of a diffusive interaction between the neurons leads to a delocalization of the original model's ground states [22]. When there exists a balance between the effects of diffusion and localization it is possible to obtain new ground states that are localized excitations (or bubbles) distributed over many neurons. Such ground states are more robust than those of the simple Haken model and also better reflect the kind of coherent structures found in neurobiological systems.

One of the major claims of Ref. [22] was that a network arranged on a d -dimensional square lattice with standard nearest neighbour diffusive coupling possesses localized ground states when $d = 1$ but not for $d > 1$, even for arbitrarily small diffusive coupling. This claim was based on numerical simulations of the lattice model together with some variational calculations of a continuum model obtained in the limit of large diffusive coupling. In this paper we prove analytically that such a claim is false: *the diffusive Haken model supports localized ground states in any finite dimension provided that the diffusive coupling is sufficiently small*. (Such a result also holds for more general choices of coupling provided that the strength of coupling decays exponentially with distance on the lattice). Our approach is based on a uniform continuation from the case of zero diffusive coupling.

The structure of the paper is as follows. In section 2 we introduce the diffusive Haken model and describe the ground states of the system in the absence of diffusive coupling. The weak coupling limit is then analysed in section 3 expanding upon our previous work in Ref. [23]. Numerical results are presented in section 4, which establish that localization occurs for arbitrary values of α in one dimension but vanishes beyond a critical coupling

$\alpha_c(d)$ when $d > 1$. Finally, in section 5 we show how the one-dimensional localized states can be interpreted in terms of instanton solutions of a continuum version of the diffusive Haken model.

II. DIFFUSIVE HAKEN MODEL

Consider a network of N^d elements arranged on a d -dimensional lattice Γ . Denote the state of the i th element by $q_i \in \mathbf{R}$ with $i \in \Gamma$. In the diffusive Haken model, the state of the network at time t , $Q(t) = \{q_i(t), i \in \Gamma\}$, evolves according to the gradient dynamical rule [22]

$$\dot{q}_i = -\frac{\partial V}{\partial q_i} \quad (\text{II.1})$$

with a potential V of the form

$$V[Q, \alpha] = \frac{\alpha}{2} \sum_{\langle i, j \rangle} (q_j - q_i)^2 - \frac{1}{2} D[Q] + \frac{1}{2} D[Q]^2 - \frac{1}{4} \sum_i q_i^4 \quad (\text{II.2})$$

where $\langle i, j \rangle$ denotes summation over nearest neighbour pairs and

$$D[Q] = \sum_{i \in \Gamma} q_i^2 \quad (\text{II.3})$$

The first term on the right-hand side of equation (2.2) represents a nearest neighbour diffusive interaction with coupling constant α whilst the remaining terms correspond to the localizing potential introduced by Haken [21].

Substituting equation (2.2) into (2.1) gives the equation of motion

$$\dot{q}_i = \alpha \sum_{\langle j|i \rangle} [q_j - q_i] + (1 - 2D[q] + q_i^2) q_i \quad (\text{II.4})$$

where $\langle j|i \rangle$ denotes summation over all nearest neighbours j of i . Equation (2.4) is invariant under the transformation $Q \rightarrow -Q$. Moreover, $q_i(t) \geq 0$ for all $t > 0$ and $i \in \Gamma$ if $q_i(0) \geq 0$ for all $i \in \Gamma$. For suppose that $q_i(t) = 0$ and $q_j(t) \geq 0$ for all $j \neq i$. Setting $q_i = 0$ on the right-hand side of equation (2.4) shows that $\dot{q}_i(t) \geq 0$. That is, q_i cannot cross over to the negative real axis. Since we are interested in finite energy states of the network, we shall also assume that $D[Q] < \infty$. In other words, each state Q is taken to be square-summable. Hence we can restrict our discussion to solutions in the domain

$$\mathcal{H} = \{Q \in \mathbf{R}_+^M \mid |Q|_2 < \infty\}$$

where $M = N^d$ and $|Q|_2 = \sqrt{\sum_i q_i^2}$ is the l_2 norm on the vector space of network states $\mathcal{H}$.

The network converges to one of the stationary states of the potential V , that is, $\partial V / \partial q_i = 0$ for all $i \in \Gamma$. (Note that V is bounded from below). Such a state may be obtained by setting $\dot{q}_i = 0$ in equation (2.4), solving the resulting time-independent equation for fixed D and then determining D self-consistently using equation (2.3). This procedure can be carried out explicitly in the case of zero diffusive coupling ($\alpha = 0$) for which the lattice structure does not play a role. Equation (2.4) then becomes

$$\dot{q}_i = q_i^3 + (1 - 2D)q_i \quad (\text{II.5})$$

The equilibria of equation (2.5), which we denote by $\bar{Q}$, satisfy $\bar{q}_i = 0$ or $\bar{q}_i = \sqrt{2D - 1}$. Hence the stationary states can be divided into $M + 1$ classes determined by the number m of excited sites [22]. For a given m , $D = m/(2m - 1)$ and the corresponding potential is $V^{(m)} = -(m/4)(2m - 1)^{-1}$. We shall establish below that the stationary states $m = 1$ are stable, whereas all other stationary states are either unstable ($m = 0$) or saddle points ($m > 1$). For $m = 1$ there exists a single excited site, i_0 say, such that $\bar{q}_i = \delta_{i_0, i}$. Moreover $D = 1$ and $V^{(1)} = -1/4$. This is one of the N strictly localized ground states of the network. There are two homogeneous stationary states given by the *vacuum* state $m = 0$ and the *dissipative* state $m = M$. The former satisfies $q_i = 0$ for all i and $V^{(0)} = 0$ and the latter has $\bar{q}_i = 1/\sqrt{2M - 1}$ for all i and $V^{(M)} = -M/(8M - 4)$. In the large M limit the dissipative state becomes pointwise identical to the vacuum state but has lower energy, $V^{(\infty)} = -1/8$. Also note that for an infinite lattice the dissipative state is marginally stable.

In order to determine the stability of a stationary state $\bar{Q}$, we need to consider the eigenvalues λ of the Jacobian

$$J_{ij} = - \left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_{Q=\bar{Q}, \alpha=0} \quad (\text{II.6})$$

A stationary state will be stable if and only if all the eigenvalues are strictly negative. Substituting equation (2.2) into (2.6) gives

$$J_{ij} = -4\bar{q}_i \bar{q}_j, \quad i \neq j \quad (\text{II.7})$$

$$J_{ii} = -\bar{q}_i^2 + 1 - 2D \quad (\text{II.8})$$

In the case $m = 0$ the matrix J reduces to the identity matrix so that $\lambda = 1$ is M -fold degenerate and hence the vacuum state is unstable. On the other hand, when $m = 1$ the

matrix J is diagonal with an $(M-1)$ -fold degenerate eigenvalue $\lambda = -1$ and a non-degenerate eigenvalue $\lambda = -2$; singly localized states are thus stable. Finally, consider the case $m > 1$. By an appropriate re-ordering of the matrix J it is simple to show that the stability of the system will be determined by the solutions λ of the $m \times m$ determinant equation [21]

$$\begin{vmatrix} a - \lambda & 2a & 2a & \dots & 2a \\ 2a & a - \lambda & 2a & \dots & 2a \\ 2a & 2a & a - \lambda & \dots & 2a \\ \vdots & & & & \\ 2a & 2a & 2a & \dots & a - \lambda \end{vmatrix} = 0 \quad (\text{II.9})$$

where $a = -2/(2m-1)$. One finds that there is an $(m-1)$ -fold degenerate positive eigenvalue $\lambda = -a$ and a non-degenerate negative eigenvalue $\lambda = (2m-1)a$. (The remaining $M-m$ eigenvalues of J satisfy $\lambda = a/2$). Hence, multiply localized states are saddles.

We conclude that in the case of zero diffusive coupling the ground states of the system consist of strictly localized states in which only one site is excited and the remainder quiescent; the particular ground state selected depends on the initial data and/or additional applied inputs. In other words, the network dynamically realises a *winner-take-all* strategy. Now suppose that the diffusive interaction between neighbouring sites is switched on. This leads to a delocalization of the original ground states [22]. For sufficiently small coupling α , there exists a balance between the effects of diffusion and localization so that it is possible to obtain new ground states that are localized excitations (or lattice instantons) distributed over many lattice sites. A simple heuristic argument for the existence of localized ground states proceeds as follows [23]. We shall assume for concreteness that the network is infinite in extent. Suppose that the initial state of the network is one of the strictly localized states $q_i(0) = \delta_{i,i_0}$. Substitution into equation (2.2) then shows that the potential at $t = 0$ satisfies $V < -1/8$ for $\alpha < 1/8d$. Since $dV/dt \leq 0$ for all t the final state cannot be the dissipative state. Moreover, we expect this final state to be localized since $D < \infty$ for finite values of V so that the equilibrium lattice configuration must decay sufficiently fast with distance on the lattice from i_0 .

Although the above energy argument supports the existence of localized ground states, it is still necessary to establish their relationship to the ground states at $\alpha = 0$, their stability properties, how they decay with spatial location on the lattice and also the structural stability of the system. All of these issues can be tackled in terms of a uniform continuation

from the uncoupled limit as shown in section 3. The analytical results for small α will be supplemented by numerical simulations in section 4 where the behaviour of the system for larger values of the diffusive coupling will be studied. We shall find that localized states persist for all values of α in one dimension but vanish beyond a critical value $\alpha_c(d)$ for $d > 1$.

III. WEAK-COUPLING LIMIT

A. Existence of localized states

Stationary states of the diffusive Haken model satisfy

$$(1 - 2D)q_i + q_i^3 + \alpha \sum_{\langle j|i \rangle} (q_j - q_i) = 0 \quad (\text{III.1})$$

For the moment, we shall consider both α and D as free parameters; later on D will be determined self-consistently using equation (2.3). In terms of the network state $Q \in \mathcal{H}$, equation (3.1) can be rewritten in the form

$$F(Q, D) + \alpha K(Q) \equiv G(Q, \alpha, D) = 0 \quad (\text{III.2})$$

where

$$[F(Q, D)]_i = q_i^3 + (1 - 2D)q_i \quad (\text{III.3})$$

$$[K(Q)]_i = \sum_{\langle j|i \rangle} (q_j - q_i) \quad (\text{III.4})$$

The condition for an equilibrium is then

$$G(Q, \alpha, D) = 0, \quad G : \mathcal{H} \times \mathbf{R}^2 \rightarrow \mathcal{H} \quad (\text{III.5})$$

Equation (3.5) is formally very similar to the coupled system of bistable elements studied by Mackay and Sepulchre [13]; such a system would be obtained under the transformation $F(Q, D) \rightarrow -F(Q, D)$. The stability properties of the system for fixed D are very different, however, from the full Haken model with D determined self-consistently by equation (2.3). For a given $D = D_0$, $D_0 > 1/2$, and $\alpha = 0$ the equilibria of equation (3.5) satisfy $\bar{q}_i = 0$ or $\bar{q}_i = \pm\sqrt{2D_0 - 1}$ (if negative solutions are included). Denote the Jacobian $\partial G / \partial Q$ by δG . Since $[\delta G(\bar{Q}, 0, D_0)]_{ij} = \delta_{i,j} \lambda_i$ with $\lambda_i = -(2D_0 - 1)$ if $\bar{q}_i = 0$ and $\lambda_i = 2(2D_0 - 1)$ if $\bar{q}_i \neq 0$,

δG is invertible at the stationary point $(\bar{Q}, 0, D_0)$. Hence one can use the IFT to prove the existence of a local continuation of each $\bar{Q}$ for sufficiently small α .

Theorem 1: There exists $\alpha_0, \delta_0 > 0$ such that for $|\alpha| < \alpha_0$ and $|D - D_0| < \delta_0$ there is a locally unique continuation $Q(\alpha, D)$ with

$$G(Q(\alpha, D), \alpha, D) = 0 \quad (\text{III.6})$$

and $Q(0, D_0) = \bar{Q}$ where $\bar{Q}$ is a given equilibrium.

Differentiating equation (3.6) shows that

$$-\frac{\partial Q}{\partial \alpha} = [\delta G(Q, \alpha, D)]^{-1} K(Q), \quad \frac{\partial Q}{\partial D} = 2[\delta G(Q, \alpha, D)]^{-1} Q \quad (\text{III.7})$$

Hence the solution $Q(\alpha, D)$ is differentiable with respect to α and D as long as δG has a bounded inverse.

Recall that given a norm $|\cdot|$ on a vector space $\mathcal{H}$, a linear operator L acting on $\mathcal{H}$ is bounded if there exists some real constant c such that

$$|LQ| \leq c|Q| \quad \text{for all } Q \in \mathcal{H}$$

We can then define a corresponding norm $\|\cdot\|$ on the space of bounded linear operators $L : \mathcal{H} \rightarrow \mathcal{H}$ according to

$$\|L\| = \sup_{\substack{Q \in \mathcal{H} \\ |Q|=1}} |LQ| \quad (\text{III.8})$$

Later on it will be useful to consider both the l_2 norm introduced previously and the sup norm on $\mathcal{H}$ defined by

$$|Q|_\infty = \sup_{i \in \Gamma} |q_i|, \quad (\text{III.9})$$

We shall denote the corresponding norms on the space of bounded linear operators by $\|\cdot\|_2$ and $\|\cdot\|$ respectively.

We now substitute the solution $Q(\alpha, D)$ of (3.5) into the self-consistency condition (2.3) and apply the IFT a second time with $\bar{Q}$ taken to be a strictly localized state.

Theorem 2: There exists $\alpha_1 > 0$ such that for $|\alpha| < \alpha_1$ there is a local continuation $D(\alpha)$ and $Q(\alpha) = Q(\alpha, D(\alpha)) \in \mathcal{H}$ such that $G(Q(\alpha), \alpha, D(\alpha)) = 0$ and

$$\sum_i q_i^2(\alpha, D(\alpha)) = D(\alpha) \quad (\text{III.10})$$

with $D(0) = 1$ and $q_i(0, 1) = \delta_{i,i_0}$.

Differentiate equation (3.10) with respect to α :

$$\frac{dD}{d\alpha} = \frac{2\langle Q | \frac{\partial Q}{\partial \alpha} \rangle}{1 - 2\langle Q | \frac{\partial Q}{\partial D} \rangle} \quad (\text{III.11})$$

We have introduced an inner product on the vector space $\mathcal{H}$ according to

$$\langle Q | Q' \rangle = \sum_{i \in \Gamma} q_i q'_i \quad (\text{III.12})$$

such that $\mathcal{H}$ becomes a Hilbert space. Substituting equation (3.7) into (3.11) gives

$$\frac{dD}{d\alpha} = \frac{2\langle Q | \delta G^{-1} | K[Q] \rangle}{4\langle Q | \delta G^{-1} | Q \rangle - 1} \quad (\text{III.13})$$

At the stationary point $(Q, \alpha, D) = (\bar{Q}, 0, 1)$ we have

$$\frac{dD}{d\alpha} = -2d \quad (\text{III.14})$$

where d is the dimension of the lattice. It follows from equations (3.7), (3.13) and the relation

$$\frac{dQ}{d\alpha} = \frac{\partial Q}{\partial \alpha} + \frac{\partial Q}{\partial D} \frac{dD}{d\alpha} \quad (\text{III.15})$$

that the solution $Q(\alpha)$ remains differentiable provided that

(i) δG has a bounded inverse

(ii) $4\langle Q | \delta G^{-1} | Q \rangle > 1$

Condition (ii) can be written in a more useful form using some basic properties of bounded symmetric linear operators on the Hilbert space $\mathcal{H}$ [24]:

$$\|L\|_2 \leq \|L\| \quad (\text{III.16})$$

$$\inf_{Q \neq 0} \frac{|\langle Q | LQ \rangle|}{\langle Q | Q \rangle} = \frac{1}{\|L^{-1}\|_2} \quad (\text{III.17})$$

Setting $L = \delta G^{-1}$ we have

$$|\langle Q | \delta G^{-1} | Q \rangle| \geq \frac{\langle Q | Q \rangle}{\|\delta G\|} \quad (\text{III.18})$$

Hence, condition (ii) becomes

(ii) $\|\delta G\| < 4D$

We shall now use conditions (i) and (ii) to estimate a lower bound α_1 for the existence of local continuations $Q(\alpha)$ of the strictly localized states $\bar{Q}$ present in the case of zero coupling. We shall then determine whether or not these localized states are candidate ground states, that is, $V < -1/8$ in the case of an infinite network, and also whether or not they are stable.

B. Estimation of α_1

We first consider condition (i), namely, whether or not δG has a bounded inverse. For the moment assume that D is fixed with $1/2 < D < 1$. (Equation (3.14) shows that $D < 1$, at least for sufficiently small α . The lower bound for D occurs for the dissipative state). Suppose that each lattice site i is to be assigned a value $q_i(\alpha) \geq 0$ which is a continuation of the strictly localized state $\bar{q}_i = \delta_{i,i_0} \sqrt{2D-1}$. This can be achieved uniquely, by taking $q_i(\alpha)$ to be the middle root (if $i \neq i_0$) or the largest root (if $i = i_0$) of the equation $f(q_i) \equiv q_i^3 + (1-2D)q_i = -\alpha \sum_{\langle j|i \rangle} (q_j - q_i)$, provided that

$$\alpha \sup_{i \in \Gamma} \left| \sum_{\langle j|i \rangle} (q_j(\alpha) - q_i(\alpha)) \right| < |f(q_{\pm})| \quad (\text{III.19})$$

where $q_{\pm}$ are the critical points of f , that is, $f'(q_{\pm}) = 0$ and

$$q_{\pm} = \pm \sqrt{(2D-1)/3}, \quad f(q_{\pm}) = \mp 2 \left(\frac{2D-1}{3} \right)^{3/2} \quad (\text{III.20})$$

(see figure 1). It is easy to establish that $q_i(\alpha) \in [0, q_+)$ for all $i \neq i_0$ and $q_i(\alpha) \in (q_+, q_{\max})$ where $q_{\max}$ is the positive root of the equation $f(q) = f(q_-)$, that is, $q_{\max} = 2q_+$. This follows from figure 1 and the result $f(v) \leq 0$ and $f(w) \geq 0$ where $v = \inf_i q_i(\alpha)$ and $w = \sup_i q_i(\alpha)$. Hence, equation (3.19) holds if

$$\alpha < \frac{2D-1}{6d} \quad (\text{III.21})$$

Condition (3.21) ensures that δG is invertible. To show this, separate out the diagonal and off-diagonal parts of δG by writing $\delta G(\alpha) = \Lambda + \alpha L_1$ where

$$\Lambda_{ij} = (f'(q_i) - 2d\alpha)\delta_{ij}, \quad (L_1)_{ij} = \sum_{\langle k|i \rangle} \delta_{j,k} \quad (\text{III.22})$$

Then

$$\|\delta G(\alpha)^{-1}\| \leq \frac{1}{\|\Lambda^{-1}\|^{-1} - \alpha \|L_1\|} \quad (\text{III.23})$$

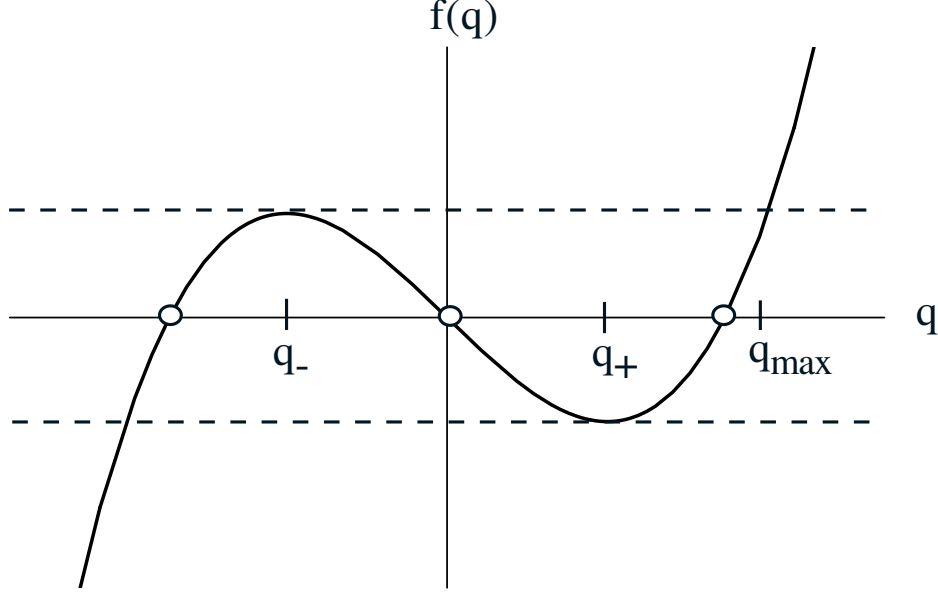


FIG. 1. Cubic nonlinearity $f(q) = q^3 + (1 - 2D)q$ for fixed D , $1/2 < D \leq 1$. Critical points are $q_{\pm} = \pm 2\sqrt{(2D - 1)/3}$ and $q_{\max} = 2q_+$.

provided that $\alpha \|L_1\| < \|\Lambda^{-1}\|^{-1}$. Since

$$\|L_1\| = \sup_{i \in \Gamma} \sum_{j \in \Gamma} |L_1|_{ij} = 2d, \quad \|\Lambda^{-1}\| = \sup_{i \in \Gamma} |f'(q_i) - 2d\alpha|^{-1}$$

it follows that $\delta G(Q(\alpha), \alpha)$ is invertible when

$$|-2d\alpha + f'(q_i(\alpha))| > 2d\alpha \quad (\text{III.24})$$

for all $i \in \Gamma$. Condition (3.24) is immediately seen to hold for all $i \neq i_0$ since $q_i(\alpha) \in [0, q_+)$ so that $f'(q_i(\alpha)) < 0$ (see figure 1). In the case $i = i_0$ we have $q_{i_0}(\alpha) \in (q_+, q_{\max})$ so that $f'(q_{i_0}(\alpha)) > f'(q_+) = 2(2D - 1)$. Hence, equation (3.24) holds for $i = i_0$ provided that $f'(q_+(\alpha)) > 4\alpha d$, which reduces to the condition $\alpha < (2D - 1)/2d$. We now recall that D is itself dependent on α . Using equation (3.14) we have that for $\alpha \ll 1$, $D(\alpha) \approx 1 - 2d\alpha$. Substituting this approximation into equation (3.21) then yields the result that δG is invertible when

$$\alpha < \alpha_1 = \frac{1}{10d} \quad (\text{III.25})$$

It is simple to show that condition (ii) is also satisfied when (3.25) holds. For

$$\begin{aligned}\|\delta G\| &\leq 4d\alpha + \sup_i (3q_i(\alpha)^2 + 1 - 2D) \\ &< 4d\alpha + 3q_{\max}^2 + 1 - 2D \\ &= 4d\alpha + 3(2D - 1)\end{aligned}$$

Condition (ii) requires that $\alpha < (3 - 2D)/4d$ which is a weaker condition than (3.21) (and hence (3.25)) since $1/2 < D < 1$. Note that our estimate (3.25) for the value of diffusive coupling below which localized states are proven to exist is close to the value $1/8d$ obtained using the energy argument at the end of section 2.

C. Energy and stability

The unique state $Q(\alpha)$ obtained as a local continuation of the strictly localized state $\bar{Q}$ (Theorem 2) is in fact a candidate ground state. For Taylor expand the energy function V of equation (2.2) to first order in α and use the fact that $\bar{Q}$ is a stationary point of V , $\partial V/\partial Q = 0$ at $Q = \bar{Q}$:

$$V[Q(\alpha), \alpha] \approx -\frac{1}{4} + \frac{\alpha}{2} \sum_{\langle i, j \rangle} (\bar{q}_j - \bar{q}_i)^2 = -\frac{1}{4} + d\alpha \quad (\text{III.26})$$

Equation (3.26) shows that (at least to this level of approximation) $V < -1/8$ if $\alpha < 1/4d$. Thus the localized state $Q(\alpha)$ with $\alpha < \alpha_1$ has lower energy than the dissipative state.

In order to investigate the linear stability of $Q(\alpha)$ we linearize equation (2.4) about $Q(\alpha)$. Setting $u_i = q_i - q_i(\alpha)$ and expanding to first order in u_i gives the linear equation

$$\dot{u}_i = \sum_{\langle j | i \rangle} [\delta G(\alpha)_{ij} - 4q_i(\alpha)q_j(\alpha)] u_j \quad (\text{III.27})$$

where we have set $G[Q(\alpha), \alpha, D(\alpha)] = G(\alpha)$. Hence, the condition for linear stability is that all eigenvalues of the matrix

$$J(\alpha) = \delta G(\alpha) - 4Q(\alpha) \otimes Q(\alpha) \quad (\text{III.28})$$

are bounded below zero. The matrix $J(0)$ is given by equations (2.7) and (2.8) with $\bar{q}_i = \delta_{i, i_0}$. Its spectrum consists of a single non-degenerate eigenvalue $\lambda = -2$ and a single degenerate eigenvalue $\lambda = -1$. Since the spectrum of $J(\alpha)$ is real, it follows that $Q(\alpha)$ will remain linearly stable as α increases from zero provided that $J(\alpha)$ remains invertible.

We shall now prove that conditions (i) and (ii) of section (3.1) ensure that $J(\alpha)$ is invertible. First write

$$\begin{aligned} J(\alpha)^{-1} &= [\delta G(\alpha) - 4Q(\alpha) \otimes Q(\alpha)]^{-1} \\ &= \delta G(\alpha)^{-1} T(\alpha)^{-1} \end{aligned} \quad (\text{III.29})$$

where

$$T(\alpha) = I - 4[Q(\alpha) \otimes Q(\alpha)]\delta G(\alpha)^{-1} \quad (\text{III.30})$$

and I is the identity matrix. Suppose that $T(\alpha)^{-1}$ exists and for an arbitrary $P \in \mathcal{H}$ consider the linear equation

$$R = T(\alpha)^{-1}P \quad (\text{III.31})$$

Equations (3.30) and (3.31) imply

$$P = T(\alpha)R = R - 4Q(\alpha)\langle Q(\alpha)|\delta G(\alpha)^{-1}|R\rangle \quad (\text{III.32})$$

Equation (3.32) shows that the unknown vector R must be of the form $R = P + \kappa Q$ for some real constant κ . Substituting this into (3.32) allows us to determine κ so that

$$\kappa = \frac{4\langle Q(\alpha)|\delta G(\alpha)^{-1}|P\rangle}{1 - 4\langle Q(\alpha)|\delta G(\alpha)^{-1}|Q(\alpha)\rangle} \quad (\text{III.33})$$

Since P is arbitrary, we conclude from (3.33) that $T(\alpha)$ and hence $J(\alpha)$ have bounded inverse provided that conditions (i) and (ii) hold. Finally, note that substituting equations (3.7) and (3.13) into (3.15) and comparing the result with equations (3.31)-(3.33) when $P = K[Q]$ yields

$$\frac{dQ}{d\alpha} = -J(\alpha)^{-1}K[Q(\alpha)] \quad (\text{III.34})$$

This equation also follows from applying the IFT directly to equation (3.1) after substituting for D using equation (2.3). One could then work backwards to derive conditions (i) and (ii).

D. Localization length

We shall now show that the continuation $Q(\alpha)$ of a strictly localized state is exponentially localized and determine an upper bound for its localization length. We shall follow the analysis of Baesens and Mackay [25], who treat the issue of exponentially localized network

states within a general framework. For convenience, we consider an infinite lattice, $\Gamma = \mathbf{R}^d$, and take the distance $d(i, j)$ between sites $i = (i_1, \dots, i_d)$ and $j = (j_1, \dots, j_d)$ to be $d(i, j) = \sum_{p=1}^d |i_p - j_p|$. Without loss of generality, the strictly localized state is assumed to be located at the origin of the lattice $i = 0$.

Suppose that $Q(\alpha)$ is exponentially localized about the origin. In other words,

$$|q_i(\alpha)| \leq C(\alpha) \lambda^{d(i,0)} \quad (\text{III.35})$$

for some $C(\alpha)$ and $0 < \lambda < 1$. Equation (3.35) holds for $\alpha = 0$ by construction; one needs to verify that the form of equation (3.35) is preserved as α increases from zero according to equation (3.34). From equations (3.29)-(3.34), we have

$$\left| \frac{dq_i}{d\alpha} \right| \leq \sum_{j \in \Gamma} |\delta G_{ij}^{-1}| |K[Q]_j + \kappa q_j| \quad (\text{III.36})$$

with κ given by equation (3.33) for $P = K[Q]$. Equation (3.35) implies that

$$|K[Q]_i + \kappa q_i| \leq \widehat{C}(\alpha) \lambda^{d(i,0)} \quad (\text{III.37})$$

where

$$\widehat{C}(\alpha) = C(\alpha) |d(\lambda + \lambda^{-1} - 2) + \kappa| \quad (\text{III.38})$$

From equations (3.36) and (3.37) and theorem 1 of [25], it can be shown that if the matrix δG is invertible then

$$\left| \frac{dq_i}{d\alpha} \right| \leq \widehat{C}(\alpha) \frac{z^{-d(i,0)}}{\|\delta G^{-1}\|^{-1} - \beta(z)} \quad (\text{III.39})$$

for any $z \in [1, 1/\lambda]$ such that $\beta(z) < \|\delta G^{-1}\|^{-1}$ where

$$\beta(z) = \sup_{i \in \Gamma} \sum_{j \in \Gamma} |\delta G_{ij}| (z^{d(i,j)} - 1) \quad (\text{III.40})$$

From the decomposition of δG given by equation (3.22), we see that

$$\beta(z) = 2d\alpha(z - 1) \quad (\text{III.41})$$

and from equations (3.23) and (3.24),

$$\|\delta G^{-1}\| \leq \frac{1}{K(\alpha)}, \quad K(\alpha) = \inf_{i \in \Gamma} |f'(q_i) - 2d\alpha| - 2d\alpha \quad (\text{III.42})$$

Substituting equations (3.41) and (3.42) into (3.39) then gives

$$\left| \frac{dq_i}{d\alpha} \right| \leq \frac{dC}{d\alpha} \lambda^{d(i,0)} \quad (\text{III.43})$$

for any $\lambda < 1$ such that $\beta(1/\lambda) < K(\alpha)$ and

$$\frac{dC}{d\alpha} = \frac{\hat{C}(\alpha)}{K(\alpha) - \beta(1/\lambda)} \quad (\text{III.44})$$

We can determine a lower bound λ^* for the exponential decay rate by solving $\beta(1/\lambda) = K(\alpha)$. Using equation (3.41)

$$\lambda^* = \left[1 + \frac{K(\alpha)}{2d\alpha} \right]^{-1} \quad (\text{III.45})$$

It follows that the state $Q(\alpha)$ decays at least as fast as λ^* , and hence an upper bound for the localization length is $1/\ln(1/\lambda^*)$. The amplitude $C(\alpha)$ can also be calculated, if needed, using equation (3.44). However, this is complicated by the fact that κ itself depends on $C(\alpha)$. Alternatively, we could use the fact that for small α , $D(\alpha) \approx 1 - 2d\alpha$. Now from equations (2.3) and (3.35)

$$D[Q] = \sum_{i \in \Gamma} q_i^2 \leq C(\alpha)^2 \sum_{i \in \Gamma} \lambda^{2d(i,0)} \quad (\text{III.46})$$

Furthermore,

$$\sum_{i \in \Gamma} \lambda^{2d(i,0)} = 1 + N(\lambda), \quad N(\lambda) = \sum_{n=1}^{\infty} \mathcal{N}(n) \lambda^{2n} \quad (\text{III.47})$$

where $\mathcal{N}(n)$ is the number of lattice sites at a distance of n from the central site. For example, $\mathcal{N}(n) = 2$ in one dimension and $\mathcal{N}(n) = 4n$ in two dimensions. Thus

$$N(\lambda) = \frac{\lambda^2}{1 - \lambda^2} \quad (d = 1), \quad N(\lambda) = \frac{4\lambda^2}{(1 - \lambda^2)^2} \quad (d = 2) \quad (\text{III.48})$$

It follows that for small α we can take

$$C(\alpha) \approx \sqrt{\frac{1 - 2d\alpha}{1 + N(\lambda)}} \quad (\text{III.49})$$

E. Structural stability

So far we have shown that for sufficiently small coupling α there exist stable localized states $Q(\alpha)$ of the diffusive Haken model that are local continuations of the global ground

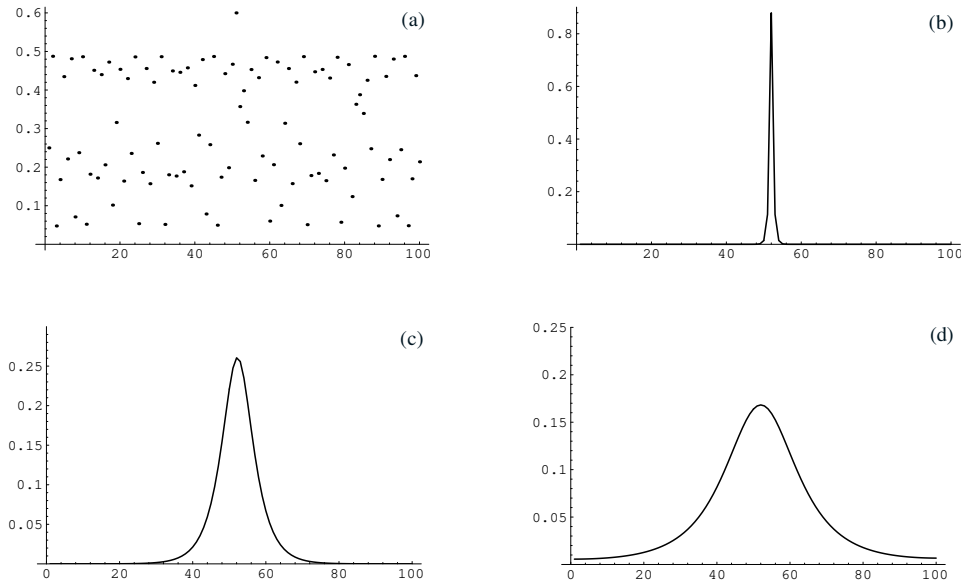


FIG. 2. Localized steady-state of the one-dimensional diffusive Haken model for a range of values of the diffusive coupling α : (a) initial data with lattice site at $i = 50$ the most excited, (b) final state when $\alpha = 0.1$, (c) final state when $\alpha = 0.5$, (d) final state when $\alpha = 1.0$.

states of the system with zero coupling. We would like to identify the states $Q(\alpha)$ as the ground states of the coupled system. However, there still remains an important question: are additional equilibria created on adding a small coupling? In other words, is the uncoupled system structurally stable? The creation of new equilibria could, in principle, introduce a new ground state of the system that is not a local continuation of one of the zero coupling ground states. Although we do not wish to explore this issue in any detail here, we expect the system to be structurally stable. This is certainly the case for a finite-dimensional network where one can appeal to the fact that the system is uniformly hyperbolic [9].

IV. NUMERICAL RESULTS

In order to explore the behaviour of the diffusive Haken model away from the weak coupling limit, it is necessary to consider numerical solutions of equation (2.3) and (2.4). We use a simple finite difference scheme, namely Euler's direct method. Note that applying nearest neighbour coupling to boundary sites in a finite network corresponds to a zero-flux condition (the analogue of a Neumann boundary condition in continuum models); our

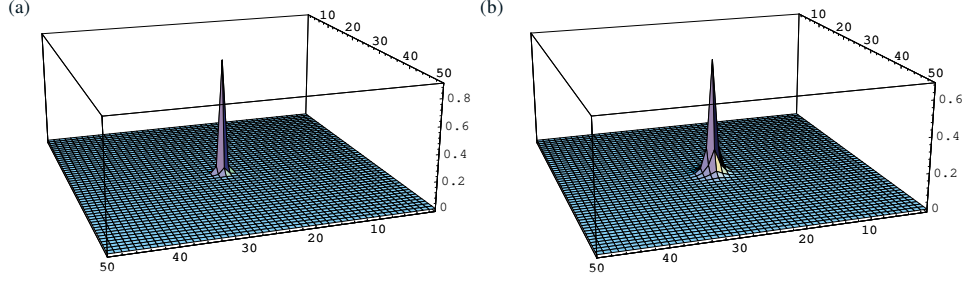


FIG. 3. Localized steady-state of the two-dimensional diffusive Haken model for (a) $\alpha = 0.05$ and (b) $\alpha = 0.097$. Initial data consists of a strictly localized state at the centre of the lattice.

results are not sensitive to the choice of boundary condition for sufficiently large N . Typical results for a one-dimensional network are shown in figure 2. The stationary state of a network of $N = 100$ lattice sites is displayed for a range of values of the diffusive coupling α , $\alpha = 0.1, 0.5, 1.0$. The initial data is scattered in the range $[0, 0.5]$ except for the lattice site at $i = 50$ which is chosen to have the value 0.6. It is clear from the figure that for each value of α the stationary solution is localized around $i = 50$. Moreover, one finds that the energy is less than the dissipative state, $V < -1/8$. The value of α for which localized states persist is only limited by the size of the network since increasing α increases the localization length so that boundary effects become more and more important.

The behaviour of the network as a function of α differs considerably in higher dimensions. This is illustrated in figure 3 for a two-dimensional network consisting of 50×50 lattice sites. The initial data is taken to be a strictly localized state at the centre of the lattice. One finds that a localized stationary state now only occurs for small values of α , $\alpha < \alpha_c$ where $\alpha_c \approx 0.098$. There is a sharp phase transition at α_c beyond which the network converges to the dissipative state. Another difference between one and two dimensions is that in the latter case localized ground states have relatively small basins of attraction whose size is very sensitive to the value of α . For example, scattered initial data along the lines of figure 2 only converges to a localized state for values of α well below α_c . We shall consider such issues in more detail elsewhere. It would appear, therefore, that in two or more dimensions the delocalizing effects of diffusion dominate.

V. CONTINUUM LIMIT

In section 3 we proved that for sufficiently small coupling α there exist localized ground states of the diffusive Haken model in any finite dimension d . We also showed numerically in section 4 that for $d > 1$ there exists a critical coupling $\alpha_c(d)$ beyond which localized ground states cease to exist. We shall now investigate to what extent these various results apply to a continuous version of the Haken model. Introduce the lattice spacing a and assume that the diffusive coupling constant scales as $1/a^2$. We thus consider $\alpha \rightarrow \alpha/a^2$ with α taken to be a -independent. Set $x = ai$ and write $q_i = q(x)$. Taking the continuum limit $a \rightarrow 0$ in equation (2.2) then yields the potential or “action”

$$V[q, \alpha] = \int_S dx \left[\frac{\alpha}{2} (\nabla q(x))^2 - \frac{q(x)^4}{4} - \frac{q(x)^2}{2} \right] + \frac{1}{2} \left[\int_S dx q(x)^2 \right]^2 \quad (\text{V.1})$$

where the integration domain is $S \subseteq \mathbf{R}^d$. The time-independent states of the continuum model are now given by stationary solutions of the potential V , and these satisfy the Euler-Lagrange equations

$$\nabla \cdot \frac{\delta V}{\delta \nabla q(x)} = \frac{\delta V}{\delta q(x)} \quad (\text{V.2})$$

where $\delta/\delta q(x)$ denotes a functional derivative. Equations (5.1) and (5.2) give

$$\alpha \nabla^2 q(x) = (2D[q] - 1)q(x) - q(x)^3 \quad (\text{V.3})$$

with

$$D[q] = \int_S dx q(x)^2 \quad (\text{V.4})$$

First, consider constant solutions. These are simply the continuous versions of the homogeneous states of the original discrete model (section 2). The vacuum state is given trivially by $q(x) = 0$, $D[q] = 0$ and $V[q, \alpha] = 0$. For a finite system with periodic boundary conditions, the dissipative state satisfies

$$q(x) = \sqrt{\frac{1}{2\Omega - 1}}, \quad D[q] = \frac{\Omega}{2\Omega - 1}, \quad V[q, \alpha] = -\frac{\Omega}{8\Omega - 1} \quad (\text{V.5})$$

As in the discrete model, we restrict ourselves to the case of positive solutions. Note that $\Omega = \int_S dx$ determines the size of the system and we assume $\Omega > 1/2$ so that the dissipative state exists. In the thermodynamic limit $\Omega \rightarrow \infty$, the dissipative state becomes pointwise identical to the vacuum state but has lower energy $V = -1/8$. It follows that the ground

states of an infinite system can only be spatially structured (eg. localized) if there exist states with energy $V < -1/8$.

A major difference between the continuum and discrete models is that in the absence of diffusion ($\alpha = 0$) the potential V of the former is unbounded from below. This is a consequence of the fact that the continuum model can have states localized to a region whose support has arbitrarily small measure, whereas in the discrete model the smallest localized region is a single lattice site. Consider, as an example, the state $q(x) = q_0$ for all $x \in S_0 \subset S$ and $q(x) = 0$ otherwise. Let $q_0 = 1/\sqrt{\Omega_0}$ where $\Omega_0 = \int_{S_0} dx$ so that $D[q] = 1$. Substitution into equation (5.1) with $\alpha = 0$ then gives $V = -1/4\Omega_0$ with $V \rightarrow -\infty$ as $\Omega_0 \rightarrow 0$. Thus for $\alpha = 0$ there do not exist lowest energy states (ground states) that are global minima of the potential V since V is unbounded from below.

One can use similar energy arguments to show that V remains unbounded for non-zero coupling α . The presence of a diffusion term introduces a smoothness constraint so that the previous choice of discontinuous test functions is no longer appropriate. Instead, we shall consider normalized Gaussians of the form $q(x) = q_0 e^{-\sigma^2|x|^2}$ where $|x|$ is the Euclidean norm on $\mathbf{R}^d$ and $1/\sigma^2$ determines the width of the Gaussian. Since q is radially symmetric, the potential (5.1) reduces to

$$V[q, \alpha] = \gamma_d \int_0^\infty dr r^{d-1} \left[\alpha \left(\frac{dq}{dr} \right)^2 - \frac{q(r)^4}{4} \right] - \frac{1}{2} D[q] + \frac{1}{2} D[q]^2 \quad (\text{V.6})$$

where

$$D[q] = \gamma_d \int_0^\infty dr r^{d-1} q(r)^2 \quad (\text{V.7})$$

and $\gamma_d = d\pi^{d/2}/\Gamma(1 + d/2)$ is the surface area of a unit sphere in d dimensions. We choose $q_0^2 = a(2\sigma^2/\pi)^{d/2}$ so that $D = a$. Substitution of the Gaussian function into (5.6) gives

$$V(\sigma) = -\frac{a}{2} + \frac{a^2}{2} - \frac{a^2}{4} \left(\frac{\sigma}{\sqrt{\pi}} \right)^d + \alpha \sigma^2 a \quad (\text{V.8})$$

It is clear from equation (5.8) that $V \rightarrow -\infty$ as $a \rightarrow \infty$ for arbitrary coupling α provided that $\sigma^d > 2\sqrt{\pi^d}$. (Note that such low energy states were not taken into account in the analysis of Ref. [22]).

Although global minima of the continuum model do not exist, one can still ask whether or not there exist local minima of the potential with energy less than the dissipative state. From our investigation of the discrete model in sections 3 and 4, we know that weakly

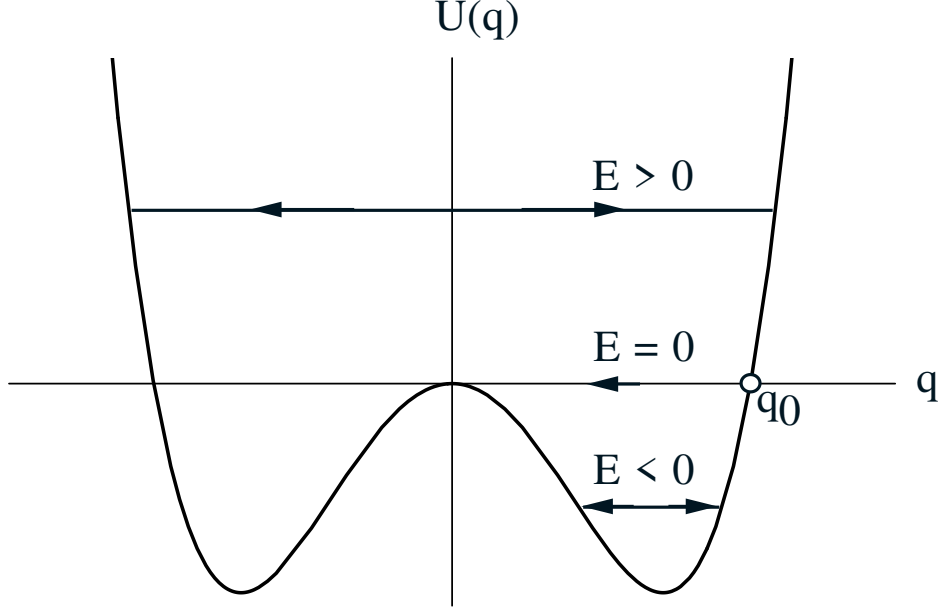


FIG. 4. Undamped motion of a particle in the auxiliary potential $U(q)$ for $1/2 < D \leq 1$: (a) periodic solution when $E < 0$ or $E > 0$, (b) instanton solution when $E = 0$.

localized stationary solutions distributed over many lattice sites exist in one-dimension but not in higher dimensions. This is a consequence of the fact that localized states persist for large α only when $d = 1$. Since weakly localized states should be reasonably well approximated by a continuum version of the model, we expect the latter to possess localized states corresponding to local minima of the potential (5.1) for $d = 1$ but not for $d > 1$. Following Ref. [22], we shall establish the existence of such states for arbitrarily large values of α in the one-dimensional continuum model ($d = 1$) by finding explicit instanton-like solutions to the Euler-Lagrange equation (5.3).

We shall proceed by solving the equation

$$\alpha \frac{d^2 q}{dx^2} = (2D - 1)q(x) - q(x)^3 \quad (\text{V.9})$$

for fixed D using standard phase-plane analysis with D then determined self-consistently using equation (5.4). Integrating equation (5.9) with D fixed gives the “energy” equation

$$\alpha \left(\frac{dq}{dx} \right)^2 + U(q(x)) = E \quad (\text{V.10})$$

where U is the auxiliary potential

$$U(q) = -\frac{2D - 1}{2}q^2 + \frac{q^4}{4} \quad (\text{V.11})$$

We require candidate finite energy solutions to be square-integrable $\int dx q(x)^2 < \infty$ so that $D[q] < \infty$. This implies that $q(x) \rightarrow 0$ as $x \rightarrow \pm\infty$; such finite energy configurations will be called instantons. Equation (5.10) formally describes an undamped particle of energy E moving in a potential U with q reinterpreted as a particle “position” and x as a continuous “time”. An instanton configuration then corresponds to a homoclinic orbit of the particle through the fixed point at $q = 0$. For $D < 1/2$ the potential $U(q)$ is monostable with a single minimum at $q = 0$. All non-homogeneous solutions are periodic functions of x and are thus not square-integrable. On the other-hand, when $D > 1/2$ the potential $U(q)$ is bistable and possesses an instanton solution when $E = 0$ (see figure 4). Setting $E = 0$ in equation (5.10) and integrating with respect to x yields the following analytical expression for the instanton (centred at $x = 0$):

$$q(x) = q_0 \left[\cosh \left(\frac{q_0 x}{\sqrt{2\alpha}} \right) \right]^{-1} \quad (\text{V.12})$$

The amplitude of the instanton is $q_0 = \sqrt{(2D - 1)2}$. Substituting equation (5.12) into (5.4) leads to the self-consistency condition for D

$$D = 4\sqrt{\alpha(2D - 1)} \quad (\text{V.13})$$

This has real solutions provided that $\alpha \geq \bar{\alpha} = \frac{1}{16}$. Keeping only the lower energy solution, we find that the amplitude of the instanton as a function of the coupling α is

$$q_0(\alpha) = \sqrt{2\alpha/\bar{\alpha}} \left(1 - \sqrt{1 - \bar{\alpha}/\alpha} \right) \quad (\text{V.14})$$

The associated potential is

$$V(\alpha) = -\sqrt{2\alpha}q_0 + 4\alpha q_0^2 - \frac{\sqrt{2\alpha}}{6}q_0^3 \quad (\text{V.15})$$

V is a monotonically increasing function of α with $V(\bar{\alpha}) = -1/6$ and $\lim_{\alpha \rightarrow \infty} V(\alpha) = -1/8$ [22]. We deduce that the one-dimensional continuum model has a localized stationary state of the form (5.14) for all values of the diffusive coupling α such that $\alpha \geq \bar{\alpha}$. We interpret these states as continuous versions of the weakly localized solutions found in the discrete one-dimensional model.

The analysis of the Euler-Lagrange equation (5.3) is more difficult in higher dimensions ($d > 1$). In particular, it does not appear possible to generate explicit solutions. However, there is good evidence to suggest that there are no stationary solutions of the continuous

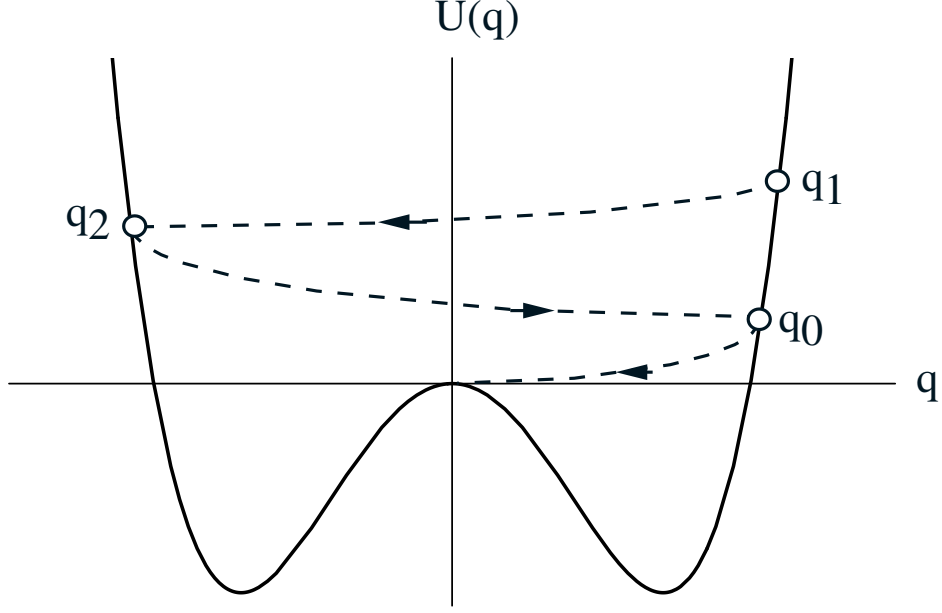


FIG. 5. Damped motion of a particle in the auxiliary potential $U(q)$ for $1/2 < D \leq 1$: instanton solutions for different starting points q_0, q_1, q_2 are shown. The lowest energy solution is the one with the minimum number of turning points.

model in more than one dimension with energy lower than the dissipative state. We shall show this for a restricted class of solutions (see Ref. [22] for more details). Given the various symmetries of the potential V , equation (5.1), a reasonable assumption concerning a candidate low energy state is that it has radial symmetry, $q(x) = q(r)$, where $r = |x|$ is the radial distance from the origin. Under such an assumption, one reduces the search to that of finding local minima of the potential (5.6). Such minima satisfy the Euler-Lagrange equation

$$\alpha \left[\frac{d^2 q}{dr^2} + \frac{d-1}{r} \frac{dq}{dr} \right] = -\frac{dU}{dq} \quad (\text{V.16})$$

Equation (5.18) formally describes the dynamics of a particle moving in a potential U in the presence of damping, with q once again interpreted as the particle position and r now acting as the time variable. Substituting the energy equation (5.11) into (5.18) shows that the energy monotonically decreases along a particle trajectory according to

$$\frac{dE}{dr} = -\alpha \frac{d-1}{r} \left(\frac{dq}{dr} \right)^2 \quad (\text{V.17})$$

The right-hand side vanishes in the one-dimensional case (zero damping). Since the energy of an instanton solution is expected to increase with the number of turning points or nodes (see figure 5), the search for low energy local minima reduces to the question whether or not the unique lowest order self-consistent instanton solution has energy lower than the dissipative state. Numerical calculations carried out in Ref. [22] show that such solutions in fact have energy higher than the dissipative state. This is consistent with our findings for the discrete model, namely that there are no weakly localized ground states (see figure 3).

VI. DISCUSSION

The analysis of the weak coupling limit (section 3) can be extended to more general choices of coupling than discussed in this paper. This includes the non-standard diffusive coupling of Ref. [22], for which equation (3.4) is replaced by

$$K[Q]_i = \sum_{\langle j|i \rangle} (q_j - q_i)^{2n+1} \quad (\text{VI.1})$$

for integer $n = 1, 2, \dots$. One can also consider long-range interactions provided that the strength of the coupling decays exponentially with distance on the lattice, for example

$$K[Q]_i = \sum_{j \in \Gamma} \gamma^{d(i,j)} (q_j - q_i) \quad (\text{VI.2})$$

such that $0 < \gamma < 1$. This raises an interesting question concerning the persistence of localized ground states away from the weak coupling limit. It has been established elsewhere that non-standard diffusion increases the tendency for localization with weakly localized ground states occurring in more than one dimension [22]. It would be very interesting to carry out a more detailed comparison of different forms of interaction to understand in more general terms the conditions necessary for localization. Another issue we hope to explore elsewhere is the dependence of the basin of attraction of a localized stationary state on the coupling.

Finally, the method of uniform continuation from the uncoupled limit (section 3) is becoming an increasingly powerful tool for studying a wide range of network dynamical systems. As mentioned in the introduction, an important application is the analysis of discrete breathers in systems of coupled nonlinear oscillators [3]. We briefly indicate here how the analysis of the diffusive Haken model for fixed D (Theorem 1), can be used to prove the

existence of discrete breathers for the discrete nonlinear Schrödinger equation (NLS) on a d -dimensional lattice:

$$i\frac{d\psi_i}{dt} = \psi_i|\psi_i|^2 + \alpha \sum_{\langle i,j \rangle} (\psi_j - \psi_i) \quad (\text{VI.3})$$

Following Mackay and Aubry [3], we change variables by going to a rotating frame,

$$\psi_i(t) = \phi_i(t)e^{-i\omega t} \quad (\text{VI.4})$$

so that

$$i\frac{d\phi_i}{dt} = -\omega\phi_i + \phi_i|\phi_i|^2 + \alpha \sum_{\langle j|i \rangle} (\phi_j - \phi_i) \quad (\text{VI.5})$$

Equation (6.5) may be formulated in terms of a Hamiltonian system by introducing the conjugate variables $p_i = \Im(\phi_i)$ and $q_i = \Re(\phi_i)$; the associated Hamiltonian is

$$H(q, p) = \frac{\omega}{2} \sum_{i \in \Gamma} (p_i^2 + q_i^2) + \frac{1}{4} \sum_{i \in \Gamma} (p_i^2 + q_i^2)^2 + \frac{\alpha}{2} \sum_{\langle i,j \rangle} (p_j - p_i)^2 + (q_j - q_i)^2 \quad (\text{VI.6})$$

Time-independent solutions of equations (6.5) in the case of zero coupling ($\alpha = 0$) satisfy $\phi_i = 0$ or $|\phi_i|^2 = \omega$. Suppose that we break the degeneracy on the circle by restricting attention to real-valued ϕ_i 's. Stationary solutions of (6.5) then satisfy

$$\phi_i^3 - \omega\phi_i + \alpha \sum_{\langle j|i \rangle} (\phi_j - \phi_i) = 0 \quad (\text{VI.7})$$

and are given by the configurations $\phi_i = 0$ or $\phi_i = \pm\sqrt{\omega}$. Equation is identical in form to equation (3.1) with $\omega = 2D - 1$ for constant D . Therefore we can immediately apply the implicit function theorem along similar lines to theorem 1 of section 3: all stationary solutions of the time-independent NLS at $\alpha = 0$ have locally unique continuations for sufficiently small α . Moving back to the original non-rotating frame shows that these solutions are time-periodic, spatially localized oscillations - discrete breathers. For a much more general discussion of discrete breathers in the weak coupling limit see [3, 7].

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