Continuous and discrete Clebsch variational principles

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Abstract

The Clebsch method provides a unifying approach for deriving variational principles for continuous and discrete dynamical systems where elements of a vector space are used to control dynamics on the cotangent bundle of a Lie group *via* a velocity map. This paper proves a reduction theorem which states that the canonical variables on the Lie group can be eliminated, if and only if the velocity map is a Lie algebra action, thereby producing the Euler-Poincaré (EP) equation for the vector space variables. In this case, the map from the canonical variables on the Lie group to the vector space is the standard momentum map defined using the diamond operator. We apply the Clebsch method in examples of the rotating rigid body and the incompressible Euler equations. Along the way, we explain how singular solutions of the EP equation for the diffeomorphism group (EPDiff) arise as momentum maps in the Clebsch approach. In the case of finite dimensional Lie groups, the Clebsch variational principle is discretised to produce a variational integrator for the dynamical system. We obtain a discrete map from which the variables on the cotangent bundle of a Lie group may be eliminated to produce a discrete EP equation for elements of the vector space. We give an integrator for the rotating rigid body as an example. We also briefly discuss how to discretise infinite-dimensional Clebsch systems, so as to produce conservative numerical methods for fluid dynamics.

1 Introduction

We are dealing with variational principles defined by an action (or cost, for the optimal control problem)

$$S = \int l[\boldsymbol{\xi}(t)]\, dt\,, \tag{1}$$

whose Lagrangian (or cost functional) $l : V \mapsto \mathbb{R}$ is defined on vectors $\boldsymbol{\xi}$ in a vector space V, subject to a condition imposed by a ***velocity map*** from the vector space V to the tangent space $T_{\boldsymbol{Q}}M$ of a manifold M at the point $\boldsymbol{Q}$,

$$\mathcal{L}_\xi : V \times M \mapsto T_Q M\,. \tag{2}$$

The velocity map $\mathcal{L}_\xi$ introduces the dynamics,

$$\dot{\boldsymbol{Q}}(t) = \mathcal{L}_\xi \boldsymbol{Q}(t), \tag{3}$$

where $\boldsymbol{\xi} \in V$ and $\dot{\boldsymbol{Q}} \in T_Q M$ is tangent to the curve $\boldsymbol{Q}(t)$ in the manifold M.

Such variational principles arise in two different contexts:

1. The ***optimal control*** context, in which one seeks solutions for $\boldsymbol{Q}(t)$ governed by the dynamics (3) that control the motion along a curve in an interval $0 \leq t \leq T$ so as to minimise the cost in (1) for a given cost functional $l[\boldsymbol{\xi}]$.

2. The ***Hamilton's principle*** context, in which stationarity $\delta S = 0$ of the action in (1) implies dynamical equations for $\boldsymbol{\xi}$ subject to the constraint imposed by the velocity map (3).

One approach that applies in both contexts was first introduced for ideal fluid dynamics in Serrin [Ser59] and in Seliger and Whitham [SW68]. This is the ***Clebsch approach*** for deriving variational principles for Eulerian fluid dynamics. A similar approach later emerged in the work of Bloch et al. [BCMR98] in the optimal control of rigid bodies. This approach enforces equation (3) through a Lagrange multiplier term in the action or cost. Doing so produces dynamical equations for $\boldsymbol{Q}$ and for the Lagrange multiplier $\boldsymbol{P}$ in terms of $\boldsymbol{\xi}$, together with a formula for $\boldsymbol{\xi}$ given in terms of $\boldsymbol{Q}$ and $\boldsymbol{P}$. The Lagrange multiplier $\boldsymbol{P}$ is also the canonically conjugate momentum for $\boldsymbol{Q}$ in the corresponding Hamiltonian formulation, and the formula for $\boldsymbol{\xi}$ in terms of $\boldsymbol{Q}$ and $\boldsymbol{P}$ has special significance in the Hamiltonian framework.

Such variational principles are said to be ***implicit Lagrangian systems*** and the subject has now reached a high state of mathematical development [YM06]. In this paper we take a "bare hands" approach to investigating this sort of problem.

Section 2 describes the conditions under which the coordinate $\boldsymbol{Q}$ and its canonical momentum $\boldsymbol{P}$ in T^*M may be eliminated from an implicit Lagrangian system in order to obtain a dynamical system for $\boldsymbol{\xi}$ only. Answering this question summons a Lie algebra structure on V. That is, $\boldsymbol{Q}$ and $\boldsymbol{P}$ may be eliminated if and only if $\mathcal{L}_{\boldsymbol{\xi}}$ corresponds to the action of some Lie algebra $\mathfrak{g}$ on M; that is $\mathcal{L}_{\boldsymbol{\xi}} : \mathfrak{g} \times M \to TM$ for $\boldsymbol{\xi} \in \mathfrak{g}$. In this case the dynamical system for $\boldsymbol{\xi}$ is always the Euler-Poincaré equation for V with the appropriate ad^* operator defined on the dual Lie algebra $\mathfrak{g}^*$ through the natural pairing induced by taking variational derivatives. The formula for $\boldsymbol{\xi}$ in terms of $\boldsymbol{Q}$ and $\boldsymbol{P}$ is then found *via* a cotangent-lift momentum map. This key result for the continuous case is stated and proved in Theorem 12.

In the case where V is the Lie algebra of vector fields $\mathfrak{X}(\mathbb{R}^n)$, there is a choice of Lie algebra actions on different spaces *e.g.* Lie derivatives, left-action on embedding space *etc.* In each case the resulting system for $\boldsymbol{\xi}$ is the Euler-Poincaré equation for the diffeomorphism group. This is the EPDiff equation.

Potential energy terms may also be introduced into the action in an implicit Lagrangian system by following standard procedure for Hamilton's principle.

This allows an extension of the Clebsch variational principle to obtain Euler-Poincaré equations with advected quantities [HMR98]. Many equations of fluid dynamics may be obtained this way. Section 3 discusses the standard example of the incompressible Euler equations and provides references for other examples. Again all calculations are performed explicitly.

Clebsch variational principles not only unify the subject, they also provide a systematic framework for deriving numerical integrators. A great deal of activity and rapid development of these variational integrators has recently transpired. See [BRM07] for an up-to-date survey of the subject, a bibliography and new results from the same viewpoint as the present paper.

Section 4 discusses the potential for constructing numerical integration methods by discretising the Clebsch variational approach in both space and time and deriving the resulting discrete equations of motion. The Clebsch approach provides a method of obtaining variational integrators [LMOW03] simply by discretising the Clebsch variational principle in both space and time. These integrators are symplectic and hence they fit into the backward-error analysis framework [LR05]. This means that they preserve the Hamiltonian within $\mathcal{O}(\Delta t^p)$ (p is the order of the method in time) and also preserve conservation laws associated with symmetries (provided that the symmetries are retained by the spatial discretisation). We show that for dynamics on finite-dimensional Lie groups, the discrete Clebsch variational principle results in equations where $\boldsymbol{Q}$ and $\boldsymbol{P}$ can once again be eliminated to produce a conservative integrator for the equation for $\boldsymbol{\xi}$ only. We give an example of a Clebsch integrator for the rigid body equation which after eliminating $\boldsymbol{Q}$ and $\boldsymbol{P}$ takes a particularly elegant form. The last section of the paper closes by discussing some possible directions for applying the Clebsch approach and other recently developed parallel approaches for the case of infinite-dimensional systems.

2 Clebsch variational principles

Definition 1 (Velocity map)**.** *Suppose V is a vector space and M is a manifold. For each $\boldsymbol{Q} \in M$ and $\boldsymbol{\xi} \in V$, we define the linear **velocity map** $\mathcal{L}_\xi : V \times M \mapsto T_Q M$.*

For a given element $\boldsymbol{\xi}$ of V, we use $\mathcal{L}$ to define velocity on $T_{\boldsymbol{Q}}M$ *via*

$$\dot{\boldsymbol{Q}} = \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{Q}, \qquad \boldsymbol{Q} \in M.$$

Definition 2 (Clebsch action principle)**.** *For a given functional $l : V \to \mathbb{R}$, the Clebsch action principle is*

$$\delta \int_{t_1}^{t_2} l[\boldsymbol{\xi}(t)] + \left\langle \boldsymbol{P}(t), \dot{\boldsymbol{Q}}(t) - \mathcal{L}_{\boldsymbol{\xi}(t)} \boldsymbol{Q} \right\rangle_{T^*M} \mathrm{d}\, t = 0, \tag{4}$$

where $\boldsymbol{P}$ is a Lagrange multiplier in $T^*_{\boldsymbol{Q}}M$ and $\langle \cdot, \cdot \rangle_{T^*M}$ is the standard inner product on T^*M.

Remark 3. *The solutions of this action principle minimise* $\int_{t_1}^{t_2} l(\boldsymbol{\xi})\,\mathrm{d}\,t$ *subject to the constraint that* $\boldsymbol{Q}$ *is directed by the velocity map* $\mathcal{L}$*. The second term in (4) imposes the definition of velocity by a Lagrange multiplier that will turn out to be the conjugate momentum in the Hamiltonian formulation. This will be shown in theorem 12.*

To write down the general solutions, we need to define the diamond operator.

Definition 4 (Diamond operator)**.** *Let* $\mathcal{L}$ *be a velocity map from* V *to* M *as defined above. The operator* $\diamond : T^*M \to V^*$ *satisfies*

$$\langle \boldsymbol{P} \diamond \boldsymbol{Q}, \boldsymbol{\xi} \rangle_V = -\langle \boldsymbol{P}, \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{Q} \rangle_{T^*M}.$$

where $\langle \cdot, \cdot \rangle_V$ *is the inner product on* $V \times V^*$*.*

Remark 5. *For the case where the velocity map is minus the Lie derivative, the diamond operation is the dual action of the Lie derivative.*

Remark 6. *Later we shall see that for the case where the velocity map is a Lie algebra action, the quantity* $-\boldsymbol{P} \diamond \boldsymbol{Q}$ *is a cotangent-lift momentum map.*

Lemma 7 (Clebsch equations)**.**
The optimising solutions for the action principle (4) satisfy:

$$\begin{aligned} \frac{\delta l}{\delta \boldsymbol{\xi}} &= -\boldsymbol{P} \diamond \boldsymbol{Q}, \\ \dot{\boldsymbol{Q}} &= \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{Q}, \\ \dot{\boldsymbol{P}} &= -\left(T_{\boldsymbol{Q}} \mathcal{L}_{\boldsymbol{\xi}}\right)^T \boldsymbol{P}, \end{aligned}$$

where one defines the variational derivative operations

$$\delta l = \left\langle \frac{\delta l}{\delta \boldsymbol{\xi}}, \delta \boldsymbol{\xi} \right\rangle_V, \quad \text{and} \quad \langle \boldsymbol{P}, T_{\boldsymbol{Q}} \mathcal{L}_{\boldsymbol{\xi}} \delta \boldsymbol{Q} \rangle = \langle (T_{\boldsymbol{Q}} \mathcal{L}_{\boldsymbol{\xi}})^T \boldsymbol{P}, \delta \boldsymbol{Q} \rangle.$$

Proof.

$$\begin{aligned} 0 &= \delta \int_{t_1}^{t_2} l[\boldsymbol{\xi}] + \left\langle \boldsymbol{P}, \dot{\boldsymbol{Q}} - \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{Q} \right\rangle_{T^*M} \mathrm{d}\,t, \\ &= \int_{t_1}^{t_2} \left\langle \frac{\delta l}{\delta \boldsymbol{\xi}}, \delta \boldsymbol{\xi} \right\rangle_V + \left\langle \delta \boldsymbol{P}, \dot{\boldsymbol{Q}} - \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{Q} \right\rangle_{T^*M} \\ &\quad + \left\langle \boldsymbol{P}, \delta \dot{\boldsymbol{Q}} - \left(T_{\boldsymbol{Q}} \mathcal{L}_{\boldsymbol{\xi}}\right) \cdot \delta \boldsymbol{Q} - \mathcal{L}_{\delta \boldsymbol{\xi}} \boldsymbol{Q} \right\rangle_{T^*M} \mathrm{d}\,t, \\ &= \left\langle \frac{\delta l}{\delta \boldsymbol{\xi}} + \boldsymbol{P} \diamond \boldsymbol{Q}, \delta \boldsymbol{\xi} \right\rangle_V + \left\langle \delta \boldsymbol{P}, \dot{\boldsymbol{Q}} - \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{Q} \right\rangle_{T^*M} \\ &\quad + \left\langle -\dot{\boldsymbol{P}} - \left(T_{\boldsymbol{Q}} \mathcal{L}_{\boldsymbol{\xi}}\right)^T \boldsymbol{P}, \delta \boldsymbol{Q} \right\rangle_{T^*M} \mathrm{d}\,t, \end{aligned}$$

and the result follows since $\delta \boldsymbol{\xi}$, $\delta \boldsymbol{P}$, and $\delta \boldsymbol{Q}$ are arbitrary. □ □

Lemma 8 (Legendre transform)**.** *Suppose l is chosen such that it is possible to solve for $\boldsymbol{\xi}$ from $\delta l/\delta\boldsymbol{\xi}$,* i.e. *there exists an operator G such that*

$$\boldsymbol{\xi} = G\frac{\delta l}{\delta \boldsymbol{\xi}}.$$

Then the Clebsch equations for $\boldsymbol{Q}$ and $\boldsymbol{P}$ are canonical with Hamiltonian given by

$$H = \langle \boldsymbol{P}, \mathcal{L}_{G(\boldsymbol{P}\diamond\boldsymbol{Q})}\boldsymbol{Q}\rangle - l\left[G\left(\boldsymbol{P}\diamond\boldsymbol{Q}\right)\right].$$

Proof. The result follows directly from calculating the canonical equations for this Hamiltonian. □ □

Example 9 (The rigid body: Bloch et al. 1998)**.** *For example, consider the rigid body in coordinates $(Q,\dot{Q}) \in TSO(3)$. The angular velocity Ω of the rigid body is defined as*

$$\dot{Q} = Q\Omega \quad \textit{with} \quad \Omega^T = -\,\Omega \in \mathfrak{so}(3).$$

The Clebsch variational principle for the rigid body is given by $\delta S = 0$ with action $S = \int L\,dt$ and implicit Lagrangian

$$L(\Omega, P, Q) = \frac{1}{2}\Omega\cdot I\Omega + P^T\cdot(\dot{Q} - Q\Omega)$$

for constant, symmetric I. The variations of L are given by:

$$\delta\int L\,dt = \int\left[\delta\Omega\cdot(I\Omega - Q^TP) + \delta P^T\cdot(\dot{Q}-Q\Omega) - \delta Q^T\cdot(\dot{P}-P\Omega)\right]dt.$$

Thus, stationarity of this implicit variational principle implies a set of rigid body equations which first appeared in the work of Bloch et al. [BCMR98] on optimal control of rigid bodies

$$I\Omega = -P\diamond Q = Q^TP\,,\quad \dot{Q} = \mathcal{L}_\Omega Q = Q\Omega\,,\quad \dot{P} = (T_Q\mathcal{L}_\Omega Q)^TP = P\Omega\,.$$

In this particular example, one can study the dynamics of Ω by calculating

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}I\Omega &= \dot{Q}^TP + Q^T\dot{P},\\ &= (Q\Omega)^TP + Q^TP\Omega,\\ &= \Omega^T(Q^TP) + (Q^TP)\Omega,\\ &= \Omega^T(I\Omega) + (I\Omega)\Omega = [I\Omega,\Omega].\end{aligned}$$

Thus, the Clebsch equations yield the dynamics for Ω by eliminating P and Q via *the derivative of $\delta l/\delta\xi$.*

We will determine when it is possible to obtain a dynamical system for $\boldsymbol{\xi}$ by eliminating the Clebsch variables. First we need to make two more definitions.

Definition 10 (Closure)**.** *Let V be a vector space with $\mathcal{L}$ being a velocity map $V \times M \to TM$. The velocity map $\mathcal{L}$ is said to be* ***closed*** *if, for every pair of vectors $\boldsymbol{u}, \boldsymbol{v} \in V$, there exists a third vector $\boldsymbol{w} \in V$ such that*

$$\mathcal{L}_{\boldsymbol{w}}\boldsymbol{Q} = ((T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{v}})\mathcal{L}_{\boldsymbol{u}} - (T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{u}})\mathcal{L}_{\boldsymbol{v}})\boldsymbol{Q},$$

where $T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{u}}$ is the tangent of $\mathcal{L}_{\boldsymbol{u}}$ evaluated at $\boldsymbol{Q}$. In that case, the velocity map induces an algebra structure on V with bracket

$$\mathcal{L}_{[\boldsymbol{u},\boldsymbol{v}]} = (T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{v}})\mathcal{L}_{\boldsymbol{u}} - (T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{u}})\mathcal{L}_{\boldsymbol{v}},$$

which satisfies the Jacobi identity.

Definition 11 (ad- and ad*-operators)**.** *Let $\mathcal{L}$ define an velocity map from V on M with induced bracket $[\cdot,\cdot] : V \times V \to V$. We define the* ad*-operator by*

$$\mathrm{ad}_{\boldsymbol{u}}\,\boldsymbol{v} = -[\boldsymbol{u}, \boldsymbol{v}].$$

For $\boldsymbol{m}$ in V^* we define the ad**-operator by*

$$\langle \mathrm{ad}^*_{\boldsymbol{u}}\,\boldsymbol{m}, \boldsymbol{v}\rangle_V = \langle \boldsymbol{m}, \mathrm{ad}_{\boldsymbol{u}}\,\boldsymbol{v}\rangle_V .$$

Theorem 12 (Elimination requires closure of the velocity map)**.**
Let V be a vector space, and let $\mathcal{L}$ be a velocity map from $V \times M \to TM$. Then the cotangent variables $(\boldsymbol{P}, \boldsymbol{Q}) \in T^*M$ may be eliminated from the Clebsch equation if and only if the image of $\mathcal{L}$ in TM is closed under the Lie bracket. Furthermore, when this closure condition holds:

1. *$\mathcal{L}$ induces a Lie algebra structure on V,*
2. *$-\boldsymbol{P} \diamond \boldsymbol{Q}$ is a cotangent-lift momentum map, and*
3. *$\boldsymbol{\xi}$ satisfies the Euler-Poincaré equation:*

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\delta l}{\delta \boldsymbol{\xi}} + \mathrm{ad}^*_{\boldsymbol{\xi}}\frac{\delta l}{\delta \boldsymbol{\xi}} = 0.$$

Proof. First suppose that $\mathcal{L}$ is closed. For any vector $\boldsymbol{w} \in V$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,t}\left\langle \frac{\delta l}{\delta \boldsymbol{\xi}}(\boldsymbol{\xi}), \boldsymbol{w}\right\rangle_V &= -\frac{\mathrm{d}}{\mathrm{d}\,t}\langle \boldsymbol{P}\diamond\boldsymbol{Q}, \boldsymbol{w}\rangle_V \\
[\text{Definition of } \diamond] &= \frac{\mathrm{d}}{\mathrm{d}\,t}\langle \boldsymbol{P}, \mathcal{L}_{\boldsymbol{w}}\boldsymbol{Q}\rangle_{T^*M} \\
[\boldsymbol{P} \text{ and } \boldsymbol{Q} \text{ equations}] &= \langle -(T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{\xi}})^T\boldsymbol{P}, \mathcal{L}_{\boldsymbol{w}}\boldsymbol{Q}\rangle_{T^*M} + \langle \boldsymbol{P}, (T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{w}})\mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{Q}\rangle_{T^*M} \\
&= \langle \boldsymbol{P}, (-(T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{\xi}})\mathcal{L}_{\boldsymbol{w}} + (T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{w}})\mathcal{L}_{\boldsymbol{\xi}})\,\boldsymbol{Q}\rangle_{T^*M} \\
[\text{closure}] &= \langle \boldsymbol{P}, -\mathcal{L}_{[\boldsymbol{\xi},\boldsymbol{w}]}\boldsymbol{Q}\rangle_{T^*M} \\
[\text{Definition of } \diamond] &= \langle \boldsymbol{P}\diamond\boldsymbol{Q}, [\boldsymbol{\xi}, \boldsymbol{w}]\rangle_V \\
&= -\left\langle \frac{\delta l}{\delta \boldsymbol{\xi}}(\boldsymbol{\xi}), \mathrm{ad}_{\boldsymbol{\xi}}\,\boldsymbol{w}\right\rangle_V , \\
&= -\left\langle \mathrm{ad}^*_{\boldsymbol{\xi}}\frac{\delta l}{\delta \boldsymbol{\xi}}(\boldsymbol{\xi}), \boldsymbol{w}\right\rangle_V ,
\end{aligned}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\delta l}{\delta \boldsymbol{\xi}} + \mathrm{ad}^*_{\boldsymbol{\xi}}\frac{\delta l}{\delta \boldsymbol{\xi}} = 0,$$

which is a closed system for $\boldsymbol{\xi}$. Hence, closure allows elimination of $\boldsymbol{P}$ and $\boldsymbol{Q}$.

Conversely, suppose that $\mathcal{L}$ is not closed. Then there exist $\boldsymbol{u}, \boldsymbol{v} \in V$ such that

$$(T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{v}})\mathcal{L}_{\boldsymbol{u}} - (T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{u}})\mathcal{L}_{\boldsymbol{v}} \neq \mathcal{L}_{\boldsymbol{w}}\,, \tag{5}$$

for any $\boldsymbol{w} \in V$. Now assume, aiming for a contradiction, that $\boldsymbol{P}$ and $\boldsymbol{Q}$ may be eliminated from the equations. For any $\boldsymbol{Q}$, we may find a $\boldsymbol{P}$ such that

$$\frac{\delta l}{\delta \boldsymbol{\xi}}(\boldsymbol{u}) := \frac{\delta l}{\delta \boldsymbol{\xi}}\Big|_{\boldsymbol{\xi}=\boldsymbol{u}} = \boldsymbol{P}\diamond\boldsymbol{Q}.$$

Then

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\delta l}{\delta \boldsymbol{u}}(\boldsymbol{u}), \boldsymbol{v}\right\rangle_V \;=\; \langle \boldsymbol{P}, (-(T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{u}})\mathcal{L}_{\boldsymbol{v}} + (T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{v}})\mathcal{L}_{\boldsymbol{u}})\,\boldsymbol{Q}\rangle. \tag{6}$$

In order for this to be consistent, the left-hand side must be linear in $\boldsymbol{\xi}$ and $\delta l/\delta\boldsymbol{\xi}$ (treating $\delta l/\delta\boldsymbol{\xi}$ as a separate variable), and hence we may write

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\delta l}{\delta \boldsymbol{\xi}}(\boldsymbol{u}) = \hat{\mathcal{L}}_{\boldsymbol{u}}\frac{\delta l}{\delta \boldsymbol{\xi}}(\boldsymbol{u}),$$

for some linear operator $\hat{\mathcal{L}}_{\boldsymbol{u}}$. We pair this with a vector, then use the Clebsch variational equation and the definition of diamond to write

$$\begin{aligned}
\left\langle \frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\delta l}{\delta \boldsymbol{\xi}}(\boldsymbol{u}), \boldsymbol{v}\right\rangle_V &= \left\langle \hat{\mathcal{L}}_{\boldsymbol{u}}\frac{\delta l}{\delta \boldsymbol{\xi}}(\boldsymbol{u}), \boldsymbol{v}\right\rangle_V \\
&= \left\langle \frac{\delta l}{\delta \boldsymbol{\xi}}(\boldsymbol{u}), \hat{\mathcal{L}}^*_{\boldsymbol{u}}\boldsymbol{v}\right\rangle_V \\
&= -\left\langle \boldsymbol{P}\diamond\boldsymbol{Q}, \hat{\mathcal{L}}^*_{\boldsymbol{u}}\boldsymbol{v}\right\rangle_V \\
&= \left\langle \boldsymbol{P}, \mathcal{L}_{\hat{\mathcal{L}}^*_{\boldsymbol{u}}\boldsymbol{v}}\boldsymbol{Q}\right\rangle_V.
\end{aligned}$$

Consequently, comparing with equation (6) yields

$$\mathcal{L}_{\hat{\mathcal{L}}^*_{\boldsymbol{u}}\boldsymbol{v}}\boldsymbol{Q} = [(T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{u}})\mathcal{L}_{\boldsymbol{v}} - (T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{v}})\mathcal{L}_{\boldsymbol{u}}]\boldsymbol{Q}$$

for any $\boldsymbol{Q}$, which contradicts the assumption (5). Hence, $\boldsymbol{P}$ and $\boldsymbol{Q}$ may be eliminated from the Clebsch equation if and only if the image of $\mathcal{L}$ in TM is closed under the Lie bracket.

The fact that $-\boldsymbol{P}\diamond\boldsymbol{Q}$ is a cotangent-lift momentum map comes straight from the definition: if $\mathcal{L}$ is closed, then V is a Lie algebra with bracket induced by $\mathcal{L}$, and the map $\mathcal{L}$ is a Lie algebra action on M. The Hamiltonian for the corresponding cotangent-lifted action is

$$h_{\boldsymbol{\xi}} = \langle \boldsymbol{P}, \mathcal{L}_{\boldsymbol{u}}\boldsymbol{Q}\rangle_{T^*M},$$

and the momentum map $\boldsymbol{J}$ for this action is defined by the relation

$$\langle \boldsymbol{J}, \boldsymbol{\xi} \rangle_V = \langle \boldsymbol{P}, \mathcal{L}_{\boldsymbol{u}} \boldsymbol{Q} \rangle_{T^*M}$$

which matches Definition 4, and hence

$$\boldsymbol{J} = -\boldsymbol{P} \diamond \boldsymbol{Q}.$$

□ □

Remark 13. *As a result of Theorem 12, solutions of the Clebsch equations may be composed by $\diamond$ into solutions of the following equation*

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \frac{\delta l}{\delta \boldsymbol{\xi}} + \mathrm{ad}^*_{\boldsymbol{\xi}} \frac{\delta l}{\delta \boldsymbol{\xi}} = 0,$$

whenever $\mathcal{L}$ is closed. This equation is called the Euler-Poincaré equation and it describes geodesic motion on the space V whenever l is a quadratic functional (metric).

To extend the scope of this framework it is useful to introduce potential energy terms to l in the action principle, *i.e.* let l be a function of the elements of M as well as V.

Definition 14 (Clebsch action principle with potential energy)**.**
For a given functional $l : V \times M \to \mathbb{R}$, the Clebsch action principle is

$$\delta \int_{t_1}^{t_2} l[\boldsymbol{\xi}(t), \boldsymbol{Q}(t)] + \left\langle \boldsymbol{P}(t), \dot{\boldsymbol{Q}}(t) - \mathcal{L}_{\boldsymbol{\xi}(t)} \boldsymbol{Q} \right\rangle_{T^*M} \mathrm{d}\, t = 0, \tag{7}$$

where $\boldsymbol{P}$ is a Lagrange multiplier in $T^*_{\boldsymbol{Q}} M$ and $\langle \cdot, \cdot \rangle_{T^*M}$ is the usual inner product on T^*M.

Lemma 15 (Clebsch equations with potential energy)**.** *The optimising solutions for the action principle (4) are:*

$$\begin{aligned} \frac{\delta l}{\delta \boldsymbol{\xi}} &= -\boldsymbol{P} \diamond \boldsymbol{Q}, \\ \dot{\boldsymbol{Q}} &= \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{Q}, \\ \dot{\boldsymbol{P}} &= -\left(T_{\boldsymbol{Q}} \mathcal{L}_{\boldsymbol{\xi}}\right)^T \boldsymbol{P} + \frac{\delta l}{\delta \boldsymbol{Q}}. \end{aligned}$$

Proof.

$$\begin{aligned}
0 &= \delta \int_{t_1}^{t_2} l[\boldsymbol{\xi}, \boldsymbol{Q}] + \left\langle \boldsymbol{P}, \dot{\boldsymbol{Q}} - \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{Q} \right\rangle_{T^*M} \mathrm{d}\, t, \\
&= \int_{t_1}^{t_2} \left\langle \frac{\delta l}{\delta \boldsymbol{\xi}}, \delta \boldsymbol{\xi} \right\rangle_V + \left\langle \frac{\delta l}{\delta \boldsymbol{Q}}, \delta \boldsymbol{Q} \right\rangle + \left\langle \delta \boldsymbol{P}, \dot{\boldsymbol{Q}} - \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{Q} \right\rangle_{T^*M} \\
&\quad + \left\langle \boldsymbol{P}, \delta \dot{\boldsymbol{Q}} - (T_{\boldsymbol{Q}} \mathcal{L}_{\boldsymbol{\xi}}) \cdot \delta \boldsymbol{Q} - \mathcal{L}_{\delta \boldsymbol{\xi}} \boldsymbol{Q} \right\rangle_{T^*M} \mathrm{d}\, t, \\
&= \left\langle \frac{\delta l}{\delta \boldsymbol{\xi}} + \boldsymbol{P} \diamond \boldsymbol{Q}, \delta \boldsymbol{\xi} \right\rangle_V + \left\langle \delta \boldsymbol{P}, \dot{\boldsymbol{Q}} - \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{Q} \right\rangle_{T^*M} \\
&\quad + \left\langle -\dot{\boldsymbol{P}} - (T_{\boldsymbol{Q}} \mathcal{L}_{\boldsymbol{\xi}})^T \boldsymbol{P} + \frac{\delta l}{\delta \boldsymbol{Q}}, \delta \boldsymbol{Q} \right\rangle_{T^*M} \mathrm{d}\, t,
\end{aligned}$$

and the result follows since $\delta\boldsymbol{\xi}$, $\delta\boldsymbol{P}$, and $\delta\boldsymbol{Q}$ are arbitrary. □ □

Theorem 16 (Elimination theorem with potential energy)**.** *Let V be a vector space, and let $\mathcal{L}$ be a velocity map from $V \to TM$. The cotangent variables $\boldsymbol{P}$ and $\boldsymbol{Q}$ may be eliminated from the Clebsch equation with potential energy, if and only if the image of $\mathcal{L}$ in TM is closed under the Lie bracket.*

Furthermore, when the condition holds, $\boldsymbol{\xi}$ and $\boldsymbol{Q}$ satisfy the Euler-Poincaré equation with advected quantities:

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \frac{\delta l}{\delta \boldsymbol{\xi}} + \mathrm{ad}^*_{\boldsymbol{\xi}} \frac{\delta l}{\delta \boldsymbol{\xi}} = -\frac{\delta l}{\delta \boldsymbol{Q}} \diamond \boldsymbol{Q}, \quad \boldsymbol{Q}_t = \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{Q}.$$

Proof. The proof follows the proof of Theorem 12, using

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\, t} \left\langle \frac{\delta l}{\delta \boldsymbol{\xi}}, \boldsymbol{w} \right\rangle_V &= -\frac{\mathrm{d}}{\mathrm{d}\, t} \left\langle \boldsymbol{P} \diamond \boldsymbol{Q}, \boldsymbol{w} \right\rangle_V, \\
&= \frac{\mathrm{d}}{\mathrm{d}\, t} \left\langle \boldsymbol{P}, \mathcal{L}_{\boldsymbol{w}} \boldsymbol{Q} \right\rangle_V \\
&= \left\langle -(T_{\boldsymbol{Q}} \mathcal{L}_{\boldsymbol{\xi}})^T \boldsymbol{P} + \frac{\delta l}{\delta \boldsymbol{Q}}, \mathcal{L}_{\boldsymbol{w}} \boldsymbol{Q} \right\rangle_{T^*M} + \left\langle \boldsymbol{P}, (T_{\boldsymbol{Q}} \mathcal{L}_{\boldsymbol{w}}) \mathcal{L}_{\boldsymbol{\xi}} \boldsymbol{Q} \right\rangle_{T^*M}, \\
&= \left\langle -\mathrm{ad}^*_{\boldsymbol{\xi}} \frac{\delta l}{\delta \boldsymbol{Q}} - \frac{\delta l}{\delta \boldsymbol{Q}} \diamond \boldsymbol{Q}, \boldsymbol{w} \right\rangle.
\end{aligned}$$

□ □

3 Examples

3.1 Singular solutions of EPDiff

When V is the space of vector fields $\mathfrak{X}(\mathbb{R}^n)$, under the conditions of Theorem 12 the Clebsch implicit variational principle yields the Euler-Poincaré equation

for diffeomorphisms (EPDiff):

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\delta l}{\delta \boldsymbol{u}} + \nabla\cdot(\boldsymbol{u}\frac{\delta l}{\delta \boldsymbol{u}}) + (\nabla \boldsymbol{u})^T\cdot\frac{\delta l}{\delta \boldsymbol{u}} = 0.$$

One possible way to form a Clebsch principle for EPDiff is to consider the left-action of vector fields on the space of embeddings $M = \mathrm{Emb}(S,\mathbb{R}^n)$ for some manifold S (such as the circle, or the sphere). For an embedding

$$\boldsymbol{Q} : S \to \mathbb{R}^n,$$

the velocity map is defined by composition of functions

$$\mathcal{L}_{\boldsymbol{u}}\boldsymbol{Q}(s) = \boldsymbol{u}(\boldsymbol{Q}(s)), \quad s \in S.$$

The Clebsch principle is then

$$\delta\int_{t_1}^{t_2} l[\boldsymbol{u}] + \langle \boldsymbol{P}, \dot{\boldsymbol{Q}} - \boldsymbol{u}(\boldsymbol{Q})\rangle_{T^*M}\,\mathrm{d}\,t = 0,$$

where the inner product in the second term is defined as

$$\int_S \boldsymbol{P}(s,t)\cdot\left(\dot{\boldsymbol{Q}}(s,t) - \boldsymbol{u}(\boldsymbol{Q}(s,t))\right)\mathrm{d}\,s.$$

The diamond operator ($\diamond$) is thus defined in this case by

$$\begin{aligned}
\langle(\boldsymbol{P}\diamond\boldsymbol{Q})\,,\boldsymbol{u}\rangle_V &= \langle \boldsymbol{P}, \mathcal{L}_{\boldsymbol{u}}\boldsymbol{Q}\rangle_{T^*M} \\
&= \int_S \boldsymbol{P}(s)\cdot\boldsymbol{u}(\boldsymbol{Q}(s))\,\mathrm{d}\,s, \\
&= \int_S \boldsymbol{P}(s)\int_M \delta(\boldsymbol{x}-\boldsymbol{Q}(s,t))\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\,V(\boldsymbol{x})\,\mathrm{d}\,s \\
&= \int_M\left(\int_S \boldsymbol{P}(s)\delta(\boldsymbol{x}-\boldsymbol{Q}(s,t))\,\mathrm{d}\,s\right)\cdot\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\,V(\boldsymbol{x}).
\end{aligned}$$

Consequently, one finds

$$\boldsymbol{P}\diamond\boldsymbol{Q} = \int_S \boldsymbol{P}(s)\delta(\boldsymbol{x}-\boldsymbol{Q}(s,t))\,\mathrm{d}\,s\,,$$

which is the singular solution ansatz of [HM04].

One then calculates the Clebsch equations as

$$\frac{\delta l}{\delta \boldsymbol{u}} = \boldsymbol{P}\diamond\boldsymbol{Q} = \int_S \boldsymbol{P}(s,t)\delta(\boldsymbol{x}-\boldsymbol{Q}(s,t))\,\mathrm{d}\,s, \tag{8}$$

$$\dot{\boldsymbol{Q}} = \mathcal{L}_{\boldsymbol{u}}\boldsymbol{Q} = \boldsymbol{u}(\boldsymbol{Q}), \tag{9}$$

$$\dot{\boldsymbol{P}} = -(T_{\boldsymbol{Q}}\mathcal{L}_{\boldsymbol{u}})^T\boldsymbol{P} = -\sum_k P_k\frac{\partial u^k}{\partial \boldsymbol{Q}} = -(\nabla\boldsymbol{u}(\boldsymbol{Q}))^T\cdot\boldsymbol{P}, \tag{10}$$

which we know to be canonically Hamiltonian from Lemma 8. Furthermore, we know that $\delta l/\delta \boldsymbol{u}$ satisfies the EPDiff equation from Theorem 12. To verify this statement, take the inner product with a test function and differentiate in time, as follows.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,t}\left\langle \frac{\delta l}{\delta \boldsymbol{u}}, \boldsymbol{w}\right\rangle_{\mathfrak{x}(M)} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\langle \boldsymbol{P}\diamond\boldsymbol{Q}, \boldsymbol{w}\rangle_{\mathfrak{x}(M)},\\
&= \frac{\mathrm{d}}{\mathrm{d}\,t}\int_M \left(\int_S \boldsymbol{P}(s,t)\delta(\boldsymbol{x}-\boldsymbol{Q}(s,t))\,\mathrm{d}\,s\right)\cdot\boldsymbol{w}(\boldsymbol{x})\,\mathrm{d}\,\mathrm{Vol}(\boldsymbol{x}),\\
&= \frac{\mathrm{d}}{\mathrm{d}\,t}\int_S \boldsymbol{P}\cdot\boldsymbol{w}(\boldsymbol{Q})\,\mathrm{d}\,s,\\
&= \int_S (\dot{\boldsymbol{P}}\cdot\boldsymbol{w}(\boldsymbol{Q}) + \boldsymbol{P}\cdot\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{Q}}\cdot\dot{\boldsymbol{Q}})\,\mathrm{d}\,s,\\
&= \int_S -\big((\nabla\boldsymbol{u})^T\cdot\boldsymbol{P}\big)\cdot\boldsymbol{w}(\boldsymbol{Q}) + \boldsymbol{P}\cdot\nabla\boldsymbol{w}(\boldsymbol{Q})\cdot\boldsymbol{u}(\boldsymbol{Q})\,\mathrm{d}\,s,\\
&= \int_M \Bigg(-\int_S \boldsymbol{P}\delta(\boldsymbol{x}-\boldsymbol{Q}(s))\,\mathrm{d}\,s\cdot\nabla\boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{w}(\boldsymbol{x}),\\
&\qquad + \int_S \boldsymbol{P}\delta(\boldsymbol{x}-\boldsymbol{Q}(s))\,\mathrm{d}\,s\cdot\nabla\boldsymbol{w}(\boldsymbol{x})\cdot\boldsymbol{u}(\boldsymbol{x})\Bigg)\,\mathrm{d}\,\mathrm{Vol}(\boldsymbol{x}),\\
&= -\left\langle (\nabla\boldsymbol{u})^T\cdot\frac{\delta l}{\delta\boldsymbol{u}}, \boldsymbol{w}\right\rangle_{\mathfrak{x}(M)} + \left\langle \frac{\delta l}{\delta\boldsymbol{u}}, \nabla\boldsymbol{w}\cdot\boldsymbol{u}\right\rangle_{\mathfrak{x}(M)}.
\end{aligned}$$

This shows that $\delta l/\delta\boldsymbol{u}$ satisfies the weak form of EPDiff:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left\langle \frac{\delta l}{\delta \boldsymbol{u}}, \boldsymbol{w}\right\rangle_{\mathfrak{x}(M)} + \left\langle (\nabla\boldsymbol{u})^T\cdot\frac{\delta l}{\delta\boldsymbol{u}}, \boldsymbol{w}\right\rangle_{\mathfrak{x}(M)} + \left\langle \frac{\delta l}{\delta\boldsymbol{u}}, \nabla\boldsymbol{w}\cdot\boldsymbol{u}\right\rangle_{\mathfrak{x}(M)} = 0.$$

Remark 17. *In consonance with Theorem 12 and as discussed in [HM04] the singular solution ansatz for EPDiff given by (8) is an equivariant momentum map arising from the cotangent lift of the action of vector fields corresponding to composition the left.*

3.2 Euler equations

The Lagrangian for the incompressible Euler equations is

$$l[\boldsymbol{u}, D] = \int_M \frac{D}{2}|\boldsymbol{u}|^2 + p(1-D)\,\mathrm{d}\,\mathrm{Vol}(\boldsymbol{x}),$$

where D is the density, and p is a Lagrange multiplier introduced to enforce the constraint that the fluid is incompressible. Not unexpectedly, the quantity p turns out to be the pressure.

There are several ways to write down a Clebsch variational principle for the incompressible Euler equations. However in order to obtain a set of variables which include all possible solutions one needs to include at least d scalar

Lagrange multipliers where d is the spatial dimension of M. If, for example, we only use one Lagrange multiplier then only the vorticity-free solutions are obtained, as first noticed by Lin (see [CM87] for discussion and references).

As described in [HK83], one possible way to construct such a Clebsch variational principle is to use the action of the diffeomorphisms on the Lagrangian labels, which satisfy

$$\ell^A_t = \mathcal{L}_{\boldsymbol{u}}\ell^A = -\,\boldsymbol{u}\cdot\nabla\ell^A\,, \quad \ell^A(\boldsymbol{x},0) = x^A, \quad A = 1,\dots,d.$$

The Clebsch variational principle is then

$$\begin{aligned}\delta\int_{t_1}^{t_2}\int_M \frac{D}{2}|\boldsymbol{u}|^2 - p(D-1)\,\mathrm{d}\,\mathrm{Vol}(\boldsymbol{x})&\\ + \int_M \boldsymbol{P}\cdot(\boldsymbol{\ell}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{\ell})\,\mathrm{d}\,\mathrm{Vol}(\boldsymbol{x})&\\ + \int_M \phi\cdot(D_t + \nabla\cdot(\boldsymbol{u}D))\,\mathrm{d}\,\mathrm{Vol}(\boldsymbol{x})\,\mathrm{d}\,t \quad &= \quad 0\,.\end{aligned}$$

Here we have also introduced an additional Lagrange multiplier ϕ to enforce the dynamics of the density D which means we have $d+1$ scalar Lagrange multipliers. The system could be reduced to d multipliers but this would unnecessarily complicate the exposition.

The Clebsch equations are:

$$\begin{aligned}\frac{\partial\boldsymbol{\ell}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\ell} &= 0,\\ \frac{\partial D}{\partial t} + \nabla\cdot(\boldsymbol{u}D) &= 0,\\ \frac{\partial\boldsymbol{P}}{\partial t} + \nabla\cdot(\boldsymbol{u}\boldsymbol{P}) &= 0,\\ \frac{\partial\phi}{\partial t} + \boldsymbol{u}\cdot\nabla\phi &= \frac{\delta l}{\delta D} = \frac{1}{2}|\boldsymbol{u}|^2 - p,\\ \frac{\delta l}{\delta\boldsymbol{u}} &= D\boldsymbol{u} = -(\nabla\boldsymbol{\ell})^T\cdot\boldsymbol{P} - D\nabla\phi,\end{aligned}$$

together with the constraint $D = 1$. After elimination one obtains

$$\begin{aligned}\frac{\partial(\boldsymbol{u}D)}{\partial t} + \boldsymbol{u}\cdot\nabla(\boldsymbol{u}D) + (\nabla\boldsymbol{u})^T\cdot\boldsymbol{u}D &= D\nabla\frac{1}{2}|\boldsymbol{u}|^2 - \nabla p,\\ \frac{\partial D}{\partial t} + \nabla\cdot(\boldsymbol{u}D) &= 0\,, \quad D = 1\end{aligned}$$

which becomes

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p, \quad \nabla\cdot\boldsymbol{u} = 0,$$

after substituting $D = 1$.

The Clebsch variational principle encompasses essentially all fluid theories. See [Hol02] for references and the example of a complex fluid.

4 Clebsch integrators

In this section we discuss the potential for constructing numerical integration methods by discretising the Clebsch variational principle in both space and time and thereby deriving the resulting discrete equations of motion. Any numerical method obtained this way will automatically be a variational integrator, and hence will inherit the accompanying conservative properties such as exact preservation of momenta where the discretisation preserves the symmetry of the continuous system, and the long-time approximate conservation of energy *via* backward-error analysis. See [LMOW03] for a review of variational integrator methods.

In the finite dimensional case one simply needs to follow the variational integrator programme by finding a map which approximates the time-derivative, and then substituting it into the Clebsch variational principle. In the following sections we show how to do this for the case of finite dimensional Lie groups such as $SO(3)$.

Definition 18. *For a manifold M, define the* derivative map $\phi_{\Delta t}$ *by*

$$\phi_{\Delta t} : M \times M \to TM, \qquad (\boldsymbol{Q}^{n+1}, \boldsymbol{Q}^n) \mapsto T_{\boldsymbol{Q}^{n+1}} M,$$

such that $\phi_{\Delta t}(\boldsymbol{Q}^{n+1}, \boldsymbol{Q}^n)$ is an approximation to $\dot{\boldsymbol{Q}}$ at $\boldsymbol{Q}^n$ [LMOW03].

Definition 19 (First-order discrete Clebsch action principle)**.**
For a given functional $l(\boldsymbol{\xi})$, the first-order discrete Clebsch action principle is

$$\delta F = \delta \sum_{n=0}^{N-1} \left(l(\boldsymbol{\xi}^n) + \langle \boldsymbol{P}^n, \phi_{\Delta t}(\boldsymbol{Q}^{n+1}, \boldsymbol{Q}^n) - L_{\boldsymbol{\xi}^n} \boldsymbol{Q}^n \rangle \right) = 0.$$

Lemma 20. *The first-order discrete action principle given in Definition 19 is optimised by $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{\xi}$ satisfying the* ***Discrete Clebsch equations***

$$\begin{aligned}
\frac{\delta l}{\delta \boldsymbol{\xi}}(\boldsymbol{\xi})^n + \boldsymbol{P}^n \diamond \boldsymbol{Q}^n &= 0, \\
\phi_{\Delta t}(\boldsymbol{Q}^{n+1}, \boldsymbol{Q}^n) &= L_{\boldsymbol{\xi}^n} \boldsymbol{Q}^n, \\
\left(D_1 \phi_{\Delta t}(\boldsymbol{Q}^n, \boldsymbol{Q}^{n-1})\right)^T \boldsymbol{P}^{n-1} + \left(D_2 \phi_{\Delta t}(\boldsymbol{Q}^{n+1}, \boldsymbol{Q}^n)\right)^T \boldsymbol{P}^n &= T_{\boldsymbol{Q}^n} \left(\mathcal{L}_{\boldsymbol{\xi}^n} \boldsymbol{Q}\right)^T \boldsymbol{P}^n.
\end{aligned}$$

Proof.

$$\begin{aligned}
\delta F = \sum_{n=0}^{N-1} \Bigg(& \left\langle \frac{\delta l}{\delta \boldsymbol{\xi}}(\boldsymbol{\xi})^n, \delta \boldsymbol{\xi}^n \right\rangle - \langle \boldsymbol{P}^n, L_{\delta \boldsymbol{\xi}^n} \boldsymbol{Q}^n \rangle \\
& + \langle \delta \boldsymbol{P}^n, \phi_{\Delta t}(\boldsymbol{Q}^{n+1}, \boldsymbol{Q}^n) - L_{\boldsymbol{\xi}^n} \boldsymbol{Q}^n \rangle \\
& + \Big\langle \boldsymbol{P}^n, D_1 \phi_{\Delta t}(\boldsymbol{Q}^{n+1}, \boldsymbol{Q}^n) \delta \boldsymbol{Q}^{n+1} + D_2 \phi_{\Delta t}(\boldsymbol{Q}^{n+1}, \boldsymbol{Q}^n) \delta \boldsymbol{Q}^n \\
& \quad - T_{\boldsymbol{Q}^n} \left(\mathcal{L}_{\boldsymbol{\xi}^n} \boldsymbol{Q}\right) \delta \boldsymbol{Q}^n \Big\rangle \Bigg),
\end{aligned}$$

and the result follows after renumbering indices. □ □

Remark 21. *When M is a vector space, one may choose*

$$\phi_{\Delta t}(\boldsymbol{Q}^{n+1}, \boldsymbol{Q}^n) = \frac{\boldsymbol{Q}^{n+1} - \boldsymbol{Q}^n}{\Delta t},$$

and the equations become

$$\begin{aligned} \frac{\delta l}{\delta \boldsymbol{\xi}}(\boldsymbol{\xi}^n) &= -\boldsymbol{P}^n \diamond \boldsymbol{Q}^n, \\ \boldsymbol{Q}^{n+1} &= \boldsymbol{Q}^n + \Delta t L_{\boldsymbol{\xi}^n} \boldsymbol{Q}^n, \\ \boldsymbol{P}^{n+1} &= \boldsymbol{P}^n - \Delta t \left(T_{\boldsymbol{Q}^{n+1}} \mathcal{L}_{\boldsymbol{\xi}^{n+1}} \boldsymbol{Q} \right)^T \boldsymbol{P}^{n+1}. \end{aligned}$$

Remark 22. *These equations give the first-order symplectic method, known in [LR05] as symplectic Euler-A, for the Hamiltonian system given in Lemma 8. The adjoint method to this, known in [LR05] as symplectic Euler-B, is obtained from the following discrete variational principle:*

$$\delta \sum_{n=1}^{N} \left(l(\boldsymbol{\xi}^n) + \langle \boldsymbol{P}^n, -\phi_{\Delta t}(\boldsymbol{Q}^{n-1}, \boldsymbol{Q}^n) - \mathcal{L}_{\boldsymbol{\xi}^n} \boldsymbol{Q}^n \rangle \right) = 0.$$

Remark 23. *Higher-order schemes may be obtained by replacing the first-order discretisation of the $\boldsymbol{Q}$-equation enforced by the Lagrange multiplier $\boldsymbol{P}$ by Munthe-Kaas methods [MK98] (Runge-Kutta methods on Lie groups).*

4.1 First-order integrators for matrix groups

In this section we show how to construct a derivative map for the case of matrix groups. For this entire section:

1. Q is a d-dimensional complex matrix.
2. The velocity map acts by matrix multiplication by X from the right

$$\mathcal{L}_{\mathrm{X}} \mathrm{Q} = \mathrm{QX}.$$

3. The inner product on TM is defined by the matrix trace using the transpose-conjugate operation ($\dagger$)

$$\langle \mathrm{P}, \dot{\mathrm{Q}} \rangle = \mathrm{Tr}\left(\mathrm{PQ}^\dagger\right) = \sum_{kl} P_{kl} \bar{Q}_{kl},$$

   where the overbar indicates complex conjugation.

4. The diamond operation is given by $\mathrm{P} \diamond \mathrm{Q} = -\mathrm{Q}^\dagger \mathrm{P}$ since

$$\begin{aligned} \langle \mathrm{P} \diamond \mathrm{Q}, \mathrm{X} \rangle &= -\langle \mathrm{P}, \mathcal{L}_{\mathrm{X}} \mathrm{Q} \rangle \\ &= -\mathrm{Tr}\left(\mathrm{P}(\mathrm{QX})^\dagger\right) \\ &= -\mathrm{Tr}\left(\mathrm{Q}^\dagger \mathrm{P} \mathrm{X}^\dagger\right) \\ &= -\langle \mathrm{Q}^\dagger \mathrm{P}, \mathrm{X} \rangle. \end{aligned}$$

We first define the exponential map when the velocity map $\mathcal{L}_{\mathrm{X}}$ is defined by right matrix multiplication.

Definition 24 (exponential map)**.** *The* exponential map *corresponding to the velocity map $\mathcal{L}_{\mathrm{X}}$ is the solution of the equation*

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\exp(t\mathrm{X}) = \mathcal{L}_{\mathrm{X}}\exp(t\mathrm{X}) := \exp(t\mathrm{X})\mathrm{X}, \quad \exp(0) = \mathrm{Id}\,.$$

We also define the logarithm map:

Definition 25 (logarithm map)**.** *Let $\mathcal{L}$ be a velocity map. The logarithm map is defined by*

$$\log(\exp(\mathrm{X})) = \mathrm{X},$$

for any vector element X *of V.*

For a pair of group elements Q_1, Q_2, we seek a vector X such that

$$\mathrm{Q}_2 = \mathrm{Q}_1\exp(\Delta t\mathrm{X}).$$

This allows one to use $\mathcal{L}_{\mathrm{X}}\mathrm{Q}_2$ as an approximation for the time derivative at Q_2, thereby motivating the following definition.

Definition 26 (Discrete approximation of time derivative)**.**
For two group elements Q_1, Q_2 an approximation to the time derivative at Q_1 of a solution which passes between Q_1 and Q_2 in time Δt is

$$\phi_{\Delta t}(\mathrm{Q}_2, \mathrm{Q}_1) = \mathrm{Q}_1\frac{1}{\Delta t}\overline{\log}\left(\mathrm{Q}_1^{-1}\mathrm{Q}_2\right),$$

where $\overline{\log} : \Omega \to V$ is given by

$$\overline{\log}(A) = \log(A) + \mathcal{O}((A-I)^p),$$

for some positive integer p.

Let us now construct the Clebsch integrator.

Definition 27 (First-order discretisation)**.** *We replace the time integral in the Clebsch variational principle by a sum and substitute our discrete approximation of the time derivative to get*

$$\delta\sum_{n=1}^{N}\left(l\left(\mathrm{X}^{n-1}\right) + \left\langle \mathrm{P}^{n-1}, \frac{1}{\Delta t}\mathrm{Q}^{n-1}\overline{\log}\left(\left(\mathrm{Q}^{n-1}\right)^{-1}\mathrm{Q}^{n}\right) - \mathrm{Q}^{n-1}\mathrm{X}^{n-1}\right\rangle\right) = 0.$$

Note that the inner product is taken at the point Q^{n-1}.

Theorem 28. *The Clebsch variational principle given in Definition 27 leads to the following first-order symplectic Euler discretisation:*

$$\begin{aligned}
\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n-1}) &= (\mathrm{Q}^{n-1})^\dagger \mathrm{P}^{n-1}, && (11)\\
\mathrm{Q}^n &= \mathrm{Q}^{n-1}\overline{\exp}(\Delta t \mathrm{X}^{n-1}), && (12)\\
\mathrm{P}^n &= \left((\mathrm{Q}^n)^\dagger\right)^{-1}\left(\left(T_{\overline{\exp}(\Delta t \mathrm{X}^n)}\overline{\log}\right)^\dagger\right)^{-1}\left(\overline{\exp}(\Delta t \mathrm{X}^{n-1})\right)^\dagger \\
&\qquad \left(T_{\overline{\exp}(\Delta t \mathrm{X}^{n-1})}\overline{\log}\right)^\dagger \left(\mathrm{Q}^{n-1}\right)^\dagger \mathrm{P}^{n-1}\left(\overline{\exp}(\Delta t \mathrm{X}^n)\right)^{-1}, && (13)
\end{aligned}$$

where $\overline{\exp}$ *is the inverse of the* $\overline{\log}$ *operation.*

Remark 29. *The discrete Clebsch momentum map takes the expected form.*

Proof. The variational principle becomes

$$\begin{aligned}
0 = \sum_{n=1}^{N}\Bigg(& \left\langle \frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n-1}) + (\mathrm{Q}^{n-1})^\dagger \mathrm{P}^{n-1}, \delta \mathrm{X}^{n-1}\right\rangle + \\
& \left\langle \delta \mathrm{P}^{n-1}, \frac{1}{\Delta t}\mathrm{Q}^{n-1}\overline{\log}\left(\left(\mathrm{Q}^{n-1}\right)^{-1}\mathrm{Q}^n\right) - \mathrm{Q}^{n-1}\mathrm{X}^{n-1}\right\rangle + \\
& \Bigg\langle \mathrm{P}^{n-1}, \frac{1}{\Delta t}\mathrm{Q}^{n-1}\left(T_{(\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^n}\overline{\log}\right)\left(\mathrm{Q}^{n-1}\right)^{-1} \\
& \qquad \left(\delta \mathrm{Q}^n - \delta \mathrm{Q}^{n-1}\left(\mathrm{Q}^{n-1}\right)^{-1}\mathrm{Q}^n\right) + \\
& \qquad \delta \mathrm{Q}^{n-1}\left(\frac{1}{\Delta t}\overline{\log}\left(\left(\mathrm{Q}^{n-1}\right)^{-1}\mathrm{Q}^n\right) - \mathrm{X}^{n-1}\right)\Bigg\rangle\Bigg).
\end{aligned}$$

The discrete Clebsch equations are then

$$\begin{aligned}
0 &= \frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n-1}) - (\mathrm{Q}^{n-1})^\dagger \mathrm{P}^{n-1}, && (14)\\
0 &= \mathrm{Q}^{n-1}\overline{\log}\left(\left(\mathrm{Q}^{n-1}\right)^{-1}\mathrm{Q}^n\right) - \Delta t \mathrm{Q}^{n-1}\mathrm{X}^{n-1}, && (15)\\
0 &= \left(\left(\mathrm{Q}^{n-1}\right)^{-1}\right)^\dagger \left(T_{(\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^n}\overline{\log}\right)^\dagger \left(\mathrm{Q}^{n-1}\right)^\dagger \mathrm{P}^{n-1} - \\
&\qquad \left((\mathrm{Q}^n)^{-1}\right)^\dagger \left(T_{(\mathrm{Q}^n)^{-1}\mathrm{Q}^{n+1}}\overline{\log}\right)^\dagger (\mathrm{Q}^n)^\dagger \\
&\qquad\qquad \mathrm{P}^n\left((\mathrm{Q}^n)^{-1}\mathrm{Q}^{n+1}\right) + && (16)\\
&\qquad \left(\frac{1}{\Delta t}\overline{\log}\left((\mathrm{Q}^n)^{-1}\mathrm{Q}^{n+1}\right)\mathrm{Q} - \mathrm{X}^n\right)^\dagger \mathrm{P}^n. && (17)
\end{aligned}$$

After rearrangement, equation (15) becomes

$$\mathrm{Q}^{n+1} = \mathrm{Q}^n\overline{\exp}(\Delta t \mathrm{X}^n),$$

and the last equation simplifies to

$$\begin{aligned} 0 &= \left(\left(\mathrm{Q}^{n-1}\right)^{-1}\mathrm{Q}^{n}\right)^{\dagger}\left(T_{\left(\mathrm{Q}^{n-1}\right)^{-1}\mathrm{Q}^{n}}\overline{\log}\right)^{\dagger}\left(\mathrm{Q}^{n-1}\right)^{\dagger}\mathrm{P}^{n-1} + \\ &\quad \left(T_{\left(\mathrm{Q}^{n}\right)^{-1}\mathrm{Q}^{n+1}}\overline{\log}\right)^{\dagger}\left(\mathrm{Q}^{n}\right)^{\dagger}\mathrm{P}^{n}\left(\left(\mathrm{Q}^{n}\right)^{-1}\mathrm{Q}^{n+1}\right), \\ &= \left(\overline{\exp}(\Delta t\mathrm{X}^{n-1})\right)^{\dagger}\left(T_{\overline{\exp}(\Delta t\mathrm{X}^{n-1})}\overline{\log}\right)^{\dagger}\left(\mathrm{Q}^{n-1}\right)^{\dagger}\mathrm{P}^{n-1} - \\ &\quad \left(T_{\overline{\exp}(\Delta t\mathrm{X}^{n})}\overline{\log}\right)^{\dagger}\left(\mathrm{Q}^{n}\right)^{\dagger}\mathrm{P}^{n}\left(\overline{\exp}(\Delta t\mathrm{X}^{n})\right), \end{aligned}$$

as required in the statement of Theorem 28. □ □

Corollary 30. P *and* Q *can be eliminated from the equations arising from the discrete variational principle in Definition 27 to obtain the equation*

$$\begin{aligned} 0 &= \left(\overline{\exp}(\Delta t\mathrm{X}^{n-1})\right)^{\dagger}\left(T_{\overline{\exp}(\Delta t\mathrm{X}^{n-1})}\overline{\log}\right)^{\dagger}\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n-1}) - \\ &\quad \left(T_{\overline{\exp}(\Delta t\mathrm{X}^{n})}\overline{\log}\right)^{\dagger}\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n})\left(\overline{\exp}(\Delta t\mathrm{X}^{n})\right). \end{aligned}$$

Proof. Substitute equation (11) into equation (12). □ □

Cayley transform methods In the following example, we derive an integrator for the rigid body equations by approximating the exponential map using the Cayley transform, which preserves the property of mapping from the Lie algebra into the group. This property of the Cayley transform has long been used for ensuring that numerical schemes preserve Lie group structure. In [AKW93], the Cayley transform was used to reconstruct the attitude of a rotating rigid body from numerical solutions of the body angular momentum equation obtained from the midpoint rule. It was noted that the conservation of the Casimir $\|\boldsymbol{m}\|^2$ (where $\boldsymbol{m}$ is the angular momentum) was necessary to obtain conservation of spatial angular momentum. [LS94] proposed to transform a Hamiltonian system on a Lie group onto the Lie algebra using the Cayley transform (rather than the exponential map), and integrating the resulting equation numerically. This approach was developed in [Ise01] which suggested that, rather than integrating the Lie algebra equation using a Runge-Kutta method (producing a Cayley Munthe-Kaas method [MK98]), one could obtain an efficient scheme by using a truncated Magnus expansion and, if a suitable numerical quadrature is used to approximate the integrals in the series, one obtains a time-reversible method of even order. In the example below, we embed the Cayley transform into the discrete Clebsch variational principle; higher-order schemes could be produced by using the methods of Munthe-Kaas and Iserles.

Example 31 (Rigid body integrator)**.** *For the case where V is $\mathfrak{so}(3)$ and acts on $SO(3)$, we may approximate the exponential map to first-order using the Cayley transform*

$$\overline{\exp}(\mathrm{X}) = \left(I + \frac{\mathrm{X}}{2}\right)\left(I - \frac{\mathrm{X}}{2}\right)^{-1}.$$

We obtain a corresponding approximation to the logarithm by taking the inverse:

$$\begin{aligned}
\overline{\log} A &= \overline{\exp}^{-1} A, \\
&= 2(A-I)(A+I)^{-1} = (A-I)\left(I + \frac{A-I}{2}\right)^{-1}, \\
&= (A-I) + \mathcal{O}\left((A-I)^2\right), \\
&= \overline{\log}(A) + \mathcal{O}\left((A-I)^2\right).
\end{aligned}$$

Our approximation to the time derivative is then

$$\begin{aligned}
\phi_{\Delta t}(\mathrm{Q}^{n+1}, \mathrm{Q}^n) &= \mathrm{Q}^n \overline{\log}((\mathrm{Q}^n)^{-1}\mathrm{Q}^{n+1}), \\
&= \left(\frac{\mathrm{Q}^{n+1} - \mathrm{Q}^n}{\Delta t}\right)\left(I + \frac{1}{2}\left((\mathrm{Q}^n)^{-1}\mathrm{Q}^{n+1} - I\right)\right)^{-1},
\end{aligned}$$

which is the usual linear difference with a projection applied to ensure that $\phi_{\Delta t}$ *is in* $\mathfrak{so}(3)$.

The Q- *and* X*-equations are then*

$$\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^n) = (\mathrm{Q}^n)^T \mathrm{P}^n, \tag{18}$$

$$\mathrm{Q}^{n+1}\left(I - \Delta t \frac{\mathrm{X}^n}{2}\right) = \mathrm{Q}^n\left(I + \Delta t \frac{\mathrm{X}^n}{2}\right). \tag{19}$$

The Q*-component of the variational principle, which gives rise to the* P*-equation, is*

$$\sum_{n=1}^{N} \left\langle \mathrm{P}^{n-1}, \mathrm{Q}^{n-1} \delta \overline{\log}\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^n\right)\right\rangle = 0.$$

Making use of the formula

$$\begin{aligned}
\delta\left(\overline{\log}\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^n\right)\right)\left(\frac{\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^n\right) + I}{2}\right) + & \\
\overline{\log}\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^n\right)\frac{\delta\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^n\right)}{2} &= \delta\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^n\right),
\end{aligned}$$

we have

$$\begin{aligned}
\delta\left(\overline{\log}\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^{n}\right)\right) &= \left(I-\frac{1}{2}\overline{\log}\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^{n}\right)\right) \\
&\qquad \delta\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^{n}\right)\left(\frac{\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^{n}\right)+I}{2}\right)^{-1}, \\
&= \left(\frac{\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^{n}\right)+I}{2}\right)^{-1} \\
&\qquad \delta\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^{n}\right)\left(\frac{\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^{n}\right)+I}{2}\right)^{-1}, \\
&= \left(\frac{\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^{n}\right)+I}{2}\right)^{-1}(\mathrm{Q}^{n-1})^{-1} \\
&\qquad \left(\delta\mathrm{Q}^{n}-\delta\mathrm{Q}^{n-1}(\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^{n}\right)\left(\frac{\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^{n}\right)+I}{2}\right)^{-1}.
\end{aligned}$$

Consequently the P*-equation in this formulation is*

$$\begin{aligned}
&\left((Q^{n-1})^{-1}\right)^{T}\left(\left(\frac{\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^{n}\right)+I}{2}\right)^{-1}\right)^{T} \\
&\qquad (\mathrm{Q}^{n-1})^{T}\mathrm{P}^{n-1}\left(\left(\frac{\left((\mathrm{Q}^{n-1})^{-1}\mathrm{Q}^{n}\right)+I}{2}\right)^{-1}\right)^{T} \\
= \;&\left((Q^{n})^{-1}\right)^{T}\left(\left(\frac{\left((\mathrm{Q}^{n})^{-1}\mathrm{Q}^{n+1}\right)+I}{2}\right)^{-1}\right)^{T} \\
&\qquad (\mathrm{Q}^{n})^{T}\mathrm{P}^{n}\left((\mathrm{Q}^{n})^{-1}\mathrm{Q}^{n+1}\right)^{T}\left(\left(\frac{\left((\mathrm{Q}^{n})^{-1}\mathrm{Q}^{n+1}\right)+I}{2}\right)^{-1}\right)^{T}.
\end{aligned}$$

Making use of equations (18-19) allows this to be rearranged into the more compact form,

$$\begin{aligned}
&\left(\overline{\exp}(\Delta t\mathrm{X}^{n-1})\right)^{T}\left(\left(\frac{I+\overline{\exp}(\Delta t\mathrm{X}^{n-1})}{2}\right)^{-1}\right)^{T} \\
&\qquad \frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n-1})\left(\left(\frac{I+\overline{\exp}(\Delta t\mathrm{X}^{n-1})}{2}\right)^{-1}\right)^{T} \\
= \;&\left(\left(\frac{I+\overline{\exp}(\Delta t\mathrm{X}^{n})}{2}\right)^{-1}\right)^{T}\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n})\overline{\exp}(\Delta t\mathrm{X}^{n})^{T}\left(\left(\frac{I+\overline{\exp}(\Delta t\mathrm{X}^{n})}{2}\right)^{-1}\right)^{T}
\end{aligned}$$

Finally, making use of the Cayley transform approximation,

$$\overline{\exp}(\Delta t\mathrm{X}) = \left(I + \Delta t\frac{\mathrm{X}}{2}\right)\left(I - \Delta t\frac{\mathrm{X}}{2}\right)^{-1},$$

and its consequence,

$$I + \overline{\exp}(\Delta t\mathrm{X}) = 2\left(I - \Delta t\frac{\mathrm{X}}{2}\right)^{-1},$$

we obtain the reduced equation for X*:*

$$\begin{aligned}&\left(I - \Delta t\frac{\mathrm{X}^{n-1}}{2}\right)\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n-1})\left(I + \Delta t\frac{\mathrm{X}^{n-1}}{2}\right)\\ =\;&\left(I + \frac{\Delta t\mathrm{X}^{n}}{2}\right)\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n})\left(I - \Delta t\frac{\mathrm{X}^{n}}{2}\right).\end{aligned}$$

This finally rearranges to become

$$\begin{aligned}\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^n) \;=\;& \frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n-1}) + \frac{\Delta t}{2}\left(\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n-1})\mathrm{X}^{n-1} + \frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n})\mathrm{X}^{n}\right)\\ &+\frac{\Delta t}{2}\left(\left(\mathrm{X}^{n-1}\right)^T\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n-1}) + \left(\mathrm{X}^{n}\right)^T\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n})\right),\\ &+\frac{(\Delta t)^2}{4}\left(\mathrm{X}^n\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n})\mathrm{X}^{n} - \mathrm{X}^{n-1}\frac{\delta l}{\delta \mathrm{X}}(\mathrm{X}^{n-1})\mathrm{X}^{n-1}\right)\end{aligned}$$

which is the discrete rigid body equation obtained from this choice of discrete Clebsch variational principle.

Remark 32. *The Clebsch integrator for the rigid body obtained using the Cayley transform approximation for* exp *is equivalent to the CAY-integrator obtained from the discrete Hamilton-Pontryagin principle in [BRM07]. In that case, the equations are obtained by extremising a functional on a Lie algebra (in this case the kinetic energy as a function of the body angular velocity* X*) subject to the constraint that* $\dot{\mathrm{Q}}\mathrm{Q}^{-1} = \mathrm{X}$*; in the case of the CAY-integrator the exponential map is again discretised using the Cayley transform. The Hamilton-Pontryagin principle provides an alternative viewpoint to the Clebsch principle; the extra feature in the Clebsch framework is the role of the* $\diamond$*-operator as a momentum map.*

A plot of the dynamics obtained from this discrete integrator is given in Figure 1.

5 Summary and outlook

This paper has discussed Clebsch variational principles from the point of view of a velocity map $\dot{\boldsymbol{Q}} = \mathcal{L}_{\boldsymbol{\xi}}\boldsymbol{Q}$ which allows the dynamics $\boldsymbol{Q}(t)$ on a manifold to

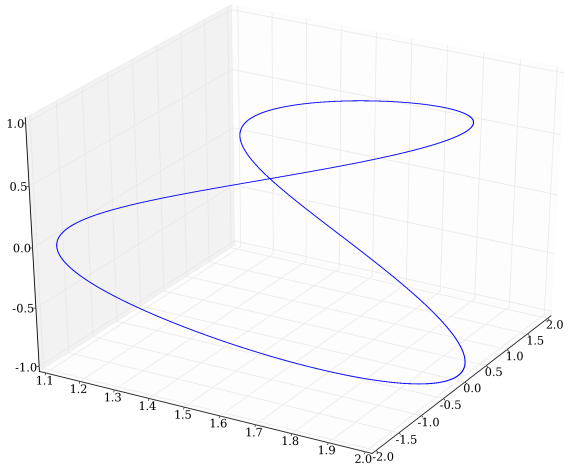


Figure 1: A plot showing numerical integration over 100 periodic orbits using the discrete Clebsch integrator for the rotating rigid body with moment of inertia eigenvalues (0.5,0.6,1) and $\Delta t = 0.1$. The good energy conservation and exact angular momentum conservation are illustrated through the persistence of the periodic orbit structure over this long time integration interval.

be controlled by a time-series $\boldsymbol{\xi}(t)$ of elements of a vector space, using Lagrange multipliers $\boldsymbol{P}(t)$. Theorem 12 shows that $\boldsymbol{Q}$ and $\boldsymbol{P}$ may be eliminated from the resulting Clebsch equations, if and only if the velocity map $\mathcal{L}_{\boldsymbol{\xi}}$ is a Lie algebra action on $\boldsymbol{Q} \in M$. The Clebsch framework for velocity maps thus has a clear connection with the theory of Euler-Poincaré reduction; namely, the equations obtained are the Euler-Poincaré equations on the dual of the Lie algebra. For the continuous time case where the velocity map is assumed to be a Lie derivative, this connection is not unexpected.

Examples in the paper included the finite-dimensional rigid body equation, and two infinite-dimensional examples: the singular solutions of the EPDiff equation and the incompressible Euler equation. In the EPDiff example, the Clebsch framework derives the singular solutions as a family of momentum maps.

Finally the paper showed how discrete Clebsch variational principles can be used to produce numerical methods for Clebsch equations. For the case of finite-dimensional Lie groups, in which the variational principle need only be discretised in time, one may again eliminate $\boldsymbol{Q}$ and $\boldsymbol{P}$ using the discrete approximation for the time derivative in Definition 26 to obtain a conservative numerical method in terms of $\boldsymbol{\xi}$ only. The example of discretisation for the rigid body resulted in the CAY integrator for the associated EP equation in [BRM07]. Possible extensions of this technique would be to obtain higher-order time-integration methods based on Runge-Kutta/Munthe-Kaas methods

[MK98] or Magnus methods [Ise01], and to apply the Clebsch integrator to systems with potentials such as the heavy top.

In the case of infinite-dimensional systems, it is necessary to discretise the variational principle in space as well as time. if one wishes to eliminate $\boldsymbol{Q}$ and $\boldsymbol{P}$ thereby obtaining a closed discrete equation for $\boldsymbol{\xi}$, Theorem (12) requires the spatial discretisation of the velocity map to remain a Lie algebra action on the discretised space. This is a key step for making future progress in applying the Clebsch method for discretising fluid dynamics and other infinite-dimensional evolutionary systems. In the special case where the Lagrangian is at most linear in space-time derivatives (without higher derivatives), then the resulting PDE is multisymplectic (see [BR01] and cited papers). Any discrete Clebsch variational principle leads to a multisymplectic Clebsch integrator. See [CHH07] for more details.

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