A Numerical Method for the Solution of Plane Elastic Problems

by

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ABSTRACT

A method is described for the numerical determination of the stresses and displacements in an elastic plate under conditions of generalised plane stress. The stresses or the displacements may be prescribed at all boundary points; alternatively the boundary conditions may be mixed.

The complex potentials of Muskhelishvili are approximated by linear combinations of elementary analytic functions. From each of these functions contributions to the boundary stresses or displacements can be calculated. The coefficient multipliers are chosen so that a suitably weighted sum of squares of residual stresses or displacements at some convenient set of points on the boundary is minimized. (The number of points exceeds the number of functions.) The method of solution implies the construction of an orthogonal set of vectors from the boundary contributions.

For general regions the approximations are rational functions. For simply connected regions, however, the complex potentials can in principle be approximated by polynomials, but an example shows that for non-convex domains polynomials are unsatisfactory and the use of functions with singularities outside the region can be advantageous. The presence of discontinuities in the boundary or the boundary values also demands special consideration.

Originality is claimed for extending the methods of Davis and Rabinowitz to realistic problems of generalised plane stress by means of complex potentials and showing that quite accurate solutions can be obtained. Although Mikhlin uses the method of least squares to determine approximations to the complex potentials, his discussion is mainly theoretical and his interest in numerical results marginal.

The application of Lanczos' σ - factors in Chapter 3 and the methods of removing discontinuities suggested in Chapter 5 are believed to be original, as are also the additional case of Williams' analysis and the technique in Chapter 6 of allowing a singularity of unknown magnitude to remove itself.

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CHAPTER 1

Introduction

§ 1. Solution of boundary value problems by the method of least squares.

In a series of articles, Davis and Rabinowitz (1956, 1957, 1961) have shown the possibility of computing numerical solutions of boundary value problems by means of least squares approximations. The two earlier articles were concerned with the Dirichlet problem for Laplace's equation in two dimensions whilst in the third, the work was extended to mixed boundary problems, problems in three dimensions and problems on the biharmonic equation in two dimensions.

To explain the method, consider the Dirichlet problem in which it is required to determine a function $u = u(x,y)$ which satisfies

$$\nabla^2 u = 0 \tag{1.1}$$

throughout some region R and which takes assigned values

$$u(P) = \gamma(P) \tag{1.2}$$

at all points P of C, the boundary of R.

The problem is solved by taking as an approximation to u a linear combination of functions $u_1, u_2, \ldots, u_n$ selected from a complete set of solutions of (1.1)

$$\tilde{u} = \sum_{j=1}^{n} a_j u_j ,$$

choosing the coefficients a_j so that

$$I = \sum_{i=1}^{N} w_i \left[\gamma(P_i) - \tilde{u}(P_i) \right]^2$$

is a minimum, $P_1, P_2, \ldots, P_i, \ldots, P_N$ $(N \geq n)$ being points of C and w_i a set of positive weights. The points P_i are suitably distributed

round C ; they might for example be equidistant in one of the co-ordinates. The weights w_i are usually chosen so that

$$\sum_{i=1}^{N} w_i v(P_i) \simeq \oint_C v(P)\,ds$$

for an arbitrary function v ; for example the w_i are often the mean of the lengths of the adjacent chords $P_{i-1}P_i$ and P_iP_{i+1} .

To determine the coefficients a_j , the values $u_j(P_i)$ $(i=1,2,\ldots,N)$ are taken to be the components of an N-dimensional vector f_j , there being n of these vectors. From them a set of <u>orthonormal</u> vectors φ_j is constructed by taking linear combinations. The i^{th} component of φ_j is the corresponding linear combination of the $u_j(P_i)$ and may be denoted $v_j(P_i)$. The vectors φ_j are orthonormal in the sense that the inner product

$$(\varphi_j, \varphi_k) \equiv \sum_{i=1}^{N} w_i v_j(P_i) v_k(P_i) = 0 \text{ , for } j \neq k ,$$

and the norm

$$\|\varphi_j\| = (\varphi_j, \varphi_j)^{\frac{1}{2}} = 1 .$$

If f denotes the vector with components $\gamma(P_i)$, the problem is reduced to the determination of coefficients b_j so that

$$I \equiv \|f - \sum_{j=1}^{n} a_j f_j\|^2 = \|f - \sum_{j=1}^{n} b_j \varphi_j\|^2$$

is a minimum and by a well known result the coefficients b_j are the Fourier coefficients (f, φ_j) .

The approximate solution of the boundary problem is thus obtained in the alternative form

$$\tilde{u} = \sum_{j=1}^{n} b_j v_j .$$

In the mixed boundary problem where

$$\nabla^2 u = 0 \text{ in } R$$

and (1.2) is replaced by the condition

$$\alpha(P)u(P) + \beta(P)\frac{\partial u}{\partial n}(P) = \gamma(P) \text{ on } C ,$$

α, β and γ being given functions, the coefficients a_j have now to be chosen to minimize

$$I = \sum_{i=1}^{N} w_i \left[\gamma(P_i) - \alpha(P_i)\tilde{u}(P_i) - \beta(P_i)\frac{\partial \tilde{u}}{\partial n}(P_i)\right]^2 .$$

Essentially, the only modification required is to make the components of the vector f_j the values

$$\alpha(P_i)u_j(P_i) + \beta(P_i)\frac{\partial u_j}{\partial n}(P_i), \quad i=1,2,\ldots,N .$$

These are then orthonormalized as before.

For the biharmonic equation

$$\nabla^4 U = 0 , \tag{1.3}$$

the boundary conditions considered by Davis and Rabinowitz are always of the form

$$U(P) = \gamma_1(P) , \frac{\partial U}{\partial n}(P) = \gamma_2(P) \text{ on } C , \tag{1.4}$$

where γ_1 and γ_2 are prescribed functions. In this case, they minimize

$$I = \sum_{i=1}^{N} w_i^{(1)} \left[\gamma_1(P_i) - \tilde{U}(P_i)\right]^2 + \sum_{i=1}^{N} w_i^{(2)} \left[\gamma_2(P_i) - \frac{\partial \tilde{U}}{\partial n}(P_i)\right]^2 .$$

taking $\tilde{U}$ to be a linear combination of solutions of (1.3). Details are not given, but it would be natural to assume that the vectors f_j are now made of dimension $2N$ by taking $U_j(P_i)$ and $\frac{\partial U_j}{\partial n}(P_i)$ together, with $w_i = w_i^{(1)}$ or $w_i^{(2)}$ as appropriate.

§ 2. Some problems in the theory of elasticity

Boundary problems for the 2 dimensional biharmonic equation arise in considering either the bending of a thin plate with transverse loading or the extension of a thin flat plate under conditions of generalised plane stress.

In the former, the displacement $w = w(x,y)$ of the middle surface of the plate in the transverse (z) direction satisfies

$$\nabla^4 w = -p(x,y) / D \ , \tag{1.5}$$

where $p(x,y)$ is the intensity of loading per unit area and D is the flexural rigidity. If the plate is clamped at its rim C, then

$$w = 0, \ \frac{\partial w}{\partial n} = 0 \quad \text{on} \quad C \ ,$$

and so if a particular solution w_0 of (1.5) can be found, the substituion.

$$w = U - w_0$$

gives $$\nabla^4 U = 0 \ ,$$

with $$U = w_0 \ , \ \frac{\partial U}{\partial n} = \frac{\partial w_0}{\partial n} \quad \text{on} \quad C \ ,$$

which is of the form (1.3), (1.4).

In the second problem, the faces of the plate, $z = \pm h$, are free from stress and any forces applied to the edges are parallel to the faces. Mean values of the stresses and displacements are used, denoted by

$$\breve{pq} = \frac{1}{2h} \int_{-h}^{h} \hat{pq} \, dz \ , \ \bar{u} = \frac{1}{2h} \int_{-h}^{h} u dz \ .$$

In the absence of body force, the equations of equilibrum become

$$\left.\begin{aligned} \frac{\partial \breve{xx}}{\partial x} + \frac{\partial \breve{xy}}{\partial y} = 0 \ , \\ \frac{\partial \breve{xy}}{\partial x} + \frac{\partial \breve{yy}}{\partial y} = 0 \ , \end{aligned}\right\} \tag{1.6}$$

on averaging, and the strain-stress relations become

$$\left.\begin{aligned} 2\mu \frac{\partial \bar{u}}{\partial x} = \breve{xx} - \eta \ (\breve{xx} + \breve{yy}) \ , \\ 2\mu \frac{\partial \bar{v}}{\partial y} = \breve{yy} - \eta \ (\breve{xx} + \breve{yy}) \ , \end{aligned}\right\} \tag{1.7}$$

$$\mu \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) = \breve{xy} \ , \tag{1.8}$$

where μ is the modulus of rigidity and η is the modified Poisson's ratio given by $\eta = \frac{\sigma}{1+\sigma}$.

Equations (1.6) are necessary and sufficient for the existence of a function $U(x,y)$, known as Airy's stress function, such that

$$\overset{\smile}{xx} = \frac{\partial^2 U}{\partial y^2} , \quad \overset{\smile}{yy} = \frac{\partial^2 U}{\partial x^2} , \quad \overset{\smile}{xy} = - \frac{\partial^2 U}{\partial x \partial y} . \tag{1.9}$$

Elimination of $\bar{u}$ and $\bar{v}$ from equations (1.7) and use of (1.8) shows that

$$\nabla^2 (\overset{\smile}{xx} + \overset{\smile}{yy}) = 0 \tag{1.10}$$

i.e. that

$$\nabla^4 U = 0 . \tag{1.11}$$

For the boundary conditions, suppose that the region R is simply connected and that the components $\overset{\smile}{xn}$ and $\overset{\smile}{yn}$ of the stress vector acting on an element ds of the boundary are known at all points. Then (see Muskhelishvili (1953) page 113),

$$\overset{\smile}{xn} = \frac{d}{ds} \left(\frac{\partial U}{\partial y}\right) , \quad \overset{\smile}{yn} = - \frac{d}{ds} \left(\frac{\partial U}{\partial x}\right) ,$$

and so integrating along an arc of the boundary,

$$\left.\begin{aligned} \frac{\partial U}{\partial x} &= - \int_0^s \overset{\smile}{yn}\, ds + \alpha = - Y + \alpha , \\ \frac{\partial U}{\partial y} &= \int_0^s \overset{\smile}{xn}\, ds + \beta = X + \beta , \end{aligned}\right\} \tag{1.12}$$

where X and Y are the components of the resultant force acting on the arc and α and β are constants. A further integration gives

$$U = \int_0^s (- Ydx + Xdy) + \alpha x + \beta y + \gamma , \tag{1.13}$$

where γ is a constant. Since the function $u = \alpha x + \beta y + \gamma$ gives rise to zero stresses, the constants α, β, γ may be assigned arbitrarily. Thus values of U and $\frac{\partial U}{\partial n}$ may be determined at all points by numerical integration if necessary.

If however the components u and v of the displacement are given on the boundary it is not possible to determine values for U and $\frac{\partial U}{\partial n}$. In fact the problem can no longer be formulated simply in terms of a single stress function U, integration of the equations (1.7), (1.8) requiring the introduction of an auxiliary harmonic function. Dropping the bars for convenience, the equations may be written

$$\left.\begin{aligned} 2\mu \frac{\partial u}{\partial x} &= -\frac{\partial^2 U}{\partial x^2} + (1-\eta)\nabla^2 U , \\ 2\mu \frac{\partial v}{\partial y} &= -\frac{\partial^2 U}{\partial y^2} + (1-\eta)\nabla^2 U , \end{aligned}\right\} \tag{1.14}$$

$$\frac{\partial}{\partial x}\left(2\mu v + \frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial y}\left(2\mu u + \frac{\partial U}{\partial x}\right) = 0 , \tag{1.15}$$

and are satisfied by taking

$$\left.\begin{aligned} 2\mu u + \frac{\partial U}{\partial x} &= \frac{\partial \Psi}{\partial x} , \\ 2\mu v + \frac{\partial U}{\partial y} &= \frac{\partial \Psi}{\partial y} , \end{aligned}\right\} \tag{1.16}$$

where

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{\partial^2 \Psi}{\partial y^2} = (1-\eta)\nabla^2 U . \tag{1.17}$$

It would be possible to apply least squares techniques to boundary conditions of the form (1.16) with u and v given functions but the need to use biharmonic functions U_j and harmonic functions Ψ_j related by (1.17) would be inconvenient.

A further difficulty arises when the region R is multiply connected and has $n+1$ boundaries C_i $(i = 0,1,\ldots,n)$. If the boundary tractions are given, the determination of U and $\frac{\partial U}{\partial n}$ from (1.12) and (1.13) introduces different constants α_i, β_i, γ_i for each C_i. Of these, the three constants α_0, β_0, γ_0 (say) on C_0 may be assigned arbitrarily, leaving $3n$ unknown constants in the boundary values and hence in the stress function $U(x,y)$. To determine them, Michell (1900) observes that whereas equations (1.7) imply that $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are single valued if the stresses are single valued, the same is not necessarily true of

$\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, u and v . Imposing the $3n$ conditions

$$\int_{c_i} d\left(\frac{\partial u}{\partial y}\right) = 0 \ , \ \int_{c_i} du = 0 \ , \ \int_{c_i} dv = 0 \ , \ i = 1,2,\ldots,n,$$

leads to $3n$ conditions on $U(x,y)$ from which the constants may be determined.

[The conditions $\int_{c_i} d\left(\frac{\partial v}{\partial x}\right) = 0$ and $\int_{c_i} d\left(\frac{\partial u}{\partial y}\right) = 0$ are equivalent.]

3. Complex Potentials

As an alternative to expressing the stresses and displacements in terms of a biharmonic function U and a related harmonic function Ψ , it is possible to express them in terms of two unrelated complex functions. This was done by Muskhelishvili in 1919 and is described in Chapter 5 of Muskhelishvili (1953) † .

Suppose initially that the region R is simply connected. Since $\nabla^2(\nabla^2 U) = 0$, it follows that $\nabla^2 U$ may be taken to be the real part of a function of the complex variable $z = x + iy$, regular in R :

$$\nabla^2 U = \mathrm{Re}\ f(z) \ .$$

Now let $$\varphi(z) = p + iq = \tfrac{1}{4}\int f(z)\, dz \ .$$

Then $$\nabla^2 U = 4\frac{\partial p}{\partial x} = 4\frac{\partial q}{\partial y} \ ,$$

and so equations (1.7) become

$$2\mu\frac{\partial u}{\partial x} = -\frac{\partial^2 U}{\partial x^2} + 4(1-\eta)\frac{\partial p}{\partial x} \ ,$$

$$2\mu\frac{\partial v}{\partial y} = -\frac{\partial^2 U}{\partial y^2} + 4(1-\eta)\frac{\partial q}{\partial y} \ .$$

† Muskhelishvili's book is referred to subsequently as M.

Integrating these and then resubstituting in equation (1.8) gives

$$2\mu u = -\frac{\partial U}{\partial x} + 4(1-\eta)p$$

$$2\mu v = -\frac{\partial U}{\partial y} + 4(1-\eta)q \ ,$$

where p and q are arbitrary merely to the extent of terms corresponding to rigid body displacements.

It may then be shown by differentiating that

$$U - px - qy$$

is a harmonic function and so may be taken to be the real part of another function, regular in R ,

$$U - px - qy = \mathrm{Re}\ \chi(z) \ .$$

Thus $$U = \mathrm{Re}\ [\bar{z}\,\varphi(z) + \chi(z)] \ ,$$

or $$2U = \bar{z}\,\varphi(z) + z\,\bar{\varphi}(\bar{z}) + \chi(z) + \bar{\chi}(\bar{z}) \ .$$

The stresses and displacements are then given by the formulae (M. pages 112 and 114)

$$\left.\begin{aligned} \breve{xx} + \breve{yy} &= 4\ \mathrm{Re}\ \varphi'(z) = 2[\varphi'(z) + \bar{\varphi}'(\bar{z})] \ , \\ \breve{yy} - \breve{xx} + 2i\breve{xy} &= 2[\bar{z}\varphi''(z) + \psi'(z)] \ , \end{aligned}\right\} \qquad (1.18)$$

$$2\mu(u + iv) = \varkappa\,\varphi(z) - z\,\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z}) \ , \qquad (1.19)$$

where $\varkappa = 3 - 4\eta$ and $\psi(z) = \chi'(z)$.

Formulae for the resultant force and couple acting on an arc of the boundary are also given by Muskhelishvili. Although their use would appear to be indicated when boundary stresses are given, there is in fact a far more effective formula, attributed by Muskhelishvili (M. page 147) to Kolosov:

Let N and T be the normal and tangential components of the stresses acting on the boundary C, the normal being the outward normal and the tangent pointing to the left of the normal.

Then $$\Phi(z) + \overline{\Phi}(\bar{z}) - e^{2i\alpha}\,[\bar{z}\,\Phi'(z) + \Psi(z)] = N - iT \quad \text{on} \quad C\,, \qquad (1.20)$$

where $\Phi(z) = \varphi'(z)$, $\Psi(z) = \psi'(z)$ and α is the angle between the normal and the x - axis.

It is observed by Muskhelishvili that for a given state of stress, $\psi(z)$ is uniquely defined whilst $\Phi(z)$, $\varphi(z)$, $\psi(z)$ are defined apart from terms iC, $iCz + \gamma$, γ' respectively, C being real, γ and γ' complex constants.

Conversely, a state of stress is not altered if one replaces

$$\varphi(z) \text{ by } \varphi(z) + iCz + \gamma\,,$$

$$\psi(z) \text{ by } \psi(z) + \gamma'\,.$$

However, substituting these in (1.19) shows that the displacements will be altered unless

$$C = 0 \quad \text{and} \quad \kappa\gamma - \bar{\gamma}' = 0\,.$$

When the origin lies within the region R occupied by the body and the stresses are given, Muskhelishvili chooses the three constants so that

$$\varphi(0) = 0\,, \quad \operatorname{Im}\varphi'(0) = 0\,, \quad \psi(0) = 0\,.$$

When the displacements are given, only one condition can be imposed and Muskhelishvili takes $\varphi(0) = 0$.

§ 4. Approximation of complex potentials using the method of least squares.

Suppose that the region R occupied by the plate is simply connected

and that the boundary of R is a Jordan curve C. Let the displacements u and v be specified at each point of C and let them be continuous on C. Then the problem reduces to finding functions $\varphi(z)$ and $\psi(z)$ which are regular in R and which satisfy on C the relation

$$\kappa\,\varphi(\zeta) - \zeta\,\bar{\varpi}'(\bar{\zeta}) - \bar{\psi}(\bar{\zeta}) = 2\mu(u + iv)\,. \tag{1.21}$$

Moreover it is possible to take $\varphi(0) = 0$.

A fundamental theorem proved by Walsh in 1926 and given in Walsh (1956) states that if the function $f(z)$ is regular within a region R bounded by a Jordan curve C, and continuous in $\bar{R}$ (the closure of R), then $f(z)$ can be uniformly approximated in $\bar{R}$ by a polynomial in z. The functions $1, z, z^2, \ldots$ thus form a complete set in the space of such functions $f(z)$.

It follows that $\varphi(z)$ and $\psi(z)$ can be approximated by polynomials

$$\varphi_n(z) \equiv \sum_{k=1}^{n} \alpha_k\, z^k\,,\quad \psi_n(z) \equiv \sum_{k=0}^{n} \beta_k\, z^k\,, \tag{1.22}$$

which for convenience will be written in the forms:

$$\left.\begin{aligned} \varphi_n(z) &= \sum_{r=1}^{n} (a_{4r-1} + ia_{4r})z^r\,, \\ \psi_n(z) &= \sum_{r=0}^{n} (a_{4r+1} + ia_{4r+2})z^r\,, \end{aligned}\right\} \tag{1.23}$$

where the a's are all real.

Let $z^r \equiv U_r + iV_r$ and $\bar{z}\,\dfrac{d}{dz}\,z^r \equiv U_r^* + iV_r^*$.

Then

$$\begin{aligned} \kappa\,\varphi_n(z) - z\bar{\varphi}_n'(\bar{z}) - \bar{\psi}_n(\bar{z}) = \kappa &\sum_{r=1}^{n} (a_{4r-1} + ia_{4r})\,(U_r + iV_r) \\ &- \sum_{r=1}^{n} (a_{4r-1} - ia_{4r})\,(U_r^* - iV_r^*) \\ &- \sum_{r=0}^{n} (a_{4r+1} - ia_{4r+2})\,(U_r - iV_r)\,, \end{aligned}$$

$$= \sum_{j=1}^{4n+3} a_j \, (F_j + iG_j) \, ,$$

where the F_j and G_j are functions of x and y given by

$$F_{4r+1} = -U_r \, , \quad G_{4r+1} = V_r \, , \quad (r = 0,1,2,\ldots,n)$$

$$F_{4r+2} = V_r \, , \quad G_{4r+2} = U_r \, , \quad (r = 0,1,2,\ldots,n)$$

$$F_{4r-1} = \varkappa U_r - U_r^* \, , \quad G_{4r-1} = \varkappa V_r + V_r^* \, , \quad (r = 1,2,\ldots,n)$$

$$F_{4r} = -\varkappa V_r + V_r^* \, , \quad G_{4r} = \varkappa U_r + U_r^* \, , \quad (r = 1,2,\ldots,n)$$

Denoting boundary values by $F_j(P_i)$ etc. as before, the least squares problem may be posed in the form: Required the coefficients a_j so that

$$I = \sum_{i=1}^{N} w_i \, [2\mu u(P_i) - \sum_{j=1}^{4n+2} a_j \, F_j(P_i)]^2$$

$$+ \sum_{i=1}^{N} w_i \, [2\mu v(P_i) - \sum_{j=1}^{4n+2} a_j \, G_j(P_i)]^2 \qquad (1.24)$$

shall be a minimum. (It is now necessary that $2N \geq 4n+2$). This may be solved in exactly the same manner as before, taking the vectors f_j to be of dimension $2N$ with components $F_j(P_i)$ and $G_j(P_i)$, $i = 1,2,\ldots,N$.

It may be noted that because the a_j are real, the right hand side of (1.24) can be written

$$\| 2\mu(u + iv) - \sum_{j=1}^{4n+2} a_j \, (F_j + iG_j) \|^2 \, ,$$

where the inner product of two complex valued boundary functions $f(P)$ and $g(P)$ is taken to be

$$(f,g) = \sum_{i=1}^{N} w_i \, f(P_i) \, \overline{g(P_i)} \qquad (1.25)$$

and

$$\|f\|^2 = (f,f) \, .$$

The summations in (1.24) could have used two different sets of weights but since each of u and v is a displacement, it is natural to make them the same in the two summations. There would seem to be less justification for doing so when approximating the boundary values U and $\frac{\partial U}{\partial n}$ which have quite different physical significances.

The application of the method of least squares to the present problem has been discussed analytically by Mikhlin (1949, 1964). Mikhlin writes

$$g(\zeta) \equiv \varkappa\, \varphi(\zeta) - \zeta\, \overline{\varphi'(\zeta)} - \overline{\psi(\zeta)}$$

and denotes by $g_{(k)}(\zeta)$ the functions $F_k + iG_k$ and the approximation to $g(\zeta)$ by

$$g_n(\zeta) = \sum_{k=1}^{4n+2} a_k\, g_{(k)}(\zeta)$$

although his ordering of the terms is different. The function which is minimized is

$$\|g_n - g\|^2 = (g_n - g,\ g_n - g)\ ,$$

where the inner product of two functions is defined by

$$(f,g) = \oint_C f(\zeta)\, \overline{g(\zeta)}\, ds\ . \tag{1.26}$$

Because the coefficients a_k are real,

$$\|g_n - g\|^2 = \sum_{j=1}^{4n+2} \sum_{k=1}^{4n+2} a_j\, a_k\, (g_{(j)},\ g_{(k)}) - 2 \sum_{k=1}^{4n+2} a_k\, \mathrm{Re}(g, g_{(k)}) + (g,g)$$

and the normal equations become

$$\sum_{k=1}^{4n+2} a_k\, \mathrm{Re}\,(g_{(k)},\ g_{(j)}) = \mathrm{Re}\,(g,\ g_{(j)}), \quad j = 1,2,\ldots,4n+2\ . \tag{1.27}$$

To demonstrate that this system is soluble, Mikhlin considers the homogeneous system

$$\sum_{k=1}^{4n+2} a_k\, \mathrm{Re}\,(g_{(k)},\ g_{(j)}) = 0\ , \qquad j = 1,2,\ldots,4n+2\ . \tag{1.28}$$

Multiplying by a_j and adding gives $\mathrm{Re}\,(g_n, g_n) = 0$.

But $\mathrm{Re}\,(g_n, g_n) = \mathrm{Re}\,\|g_n\|^2 = \|g_n\|^2$ and so $g_n = 0$.

But $$g_n(\zeta) = \varkappa\,\varphi_n(\zeta) - \zeta\,\bar{\varphi}_n'(\bar{\zeta}) - \bar{\psi}_n(\bar{\zeta})$$

and it follows from a known theorem on uniqueness of the solution of the elasticity problem that

$$\varphi_n(z) \equiv 0\,, \quad \psi_n(z) \equiv 0 \quad \text{in} \quad R\,.$$

These immediately give $a_k = 0$ for all k.

Thus (1.28) has only the trivial solution $a_k = 0$ and it follows that the non-homogeneous system (1.27) can always be solved.

It is further shown by Mikhlin that, provided the given function $g(\zeta)$ is sufficiently smooth,

$$\|\varphi_n - \varphi\| \to 0 \quad \text{when} \quad n \to \infty$$

and thence that $\varphi_n(z) \to \varphi(z)$, $\psi_n(z) \to \psi(z)$ uniformly within R.

If the finite summation (1.25) is used in place of the integral inner product (1.26), these conclusions will still follow by making $N \to \infty$ before $n \to \infty$, provided that

$$\sum_{i=1}^{N} w_i\, f(P_i)\, \overline{g(P_i)} \to \int_C f(\zeta)\, \overline{g(\zeta)}\, ds\,.$$

If, instead of the boundary displacements being given, the normal and tangential components N and T of the boundary tractions are given, the problem reduces to finding functions $\Phi(z)$ and $\Psi(z)$ which are regular in R and satisfy on C the relation (1.20):

$$\Phi(z) + \bar{\Phi}(\bar{z}) - e^{2i\alpha}\,[\bar{z}\,\Phi'(z) + \Psi(z)] = N - iT\,.$$

In this case it is possible to take $\mathrm{Im}\,\Phi(0) = 0$.

As before, polynomial approximations to $\Phi(z)$ and $\Psi(z)$ are taken, in the form:

$$\Phi_n(z) = \sum_{k=0}^{n} A_k z^k , \quad \Psi_n(z) = \sum_{k=0}^{n} B_k z^k ,$$

where A_0 is real, and for convenience these may be written:

$$\Phi_n(z) = a_0 + \sum_{r=1}^{n} (a_{4r-1} + ia_{4r}) z^r ,$$

$$\Psi_n(z) = \sum_{r=0}^{n} (a_{4r+1} + ia_{4r+2}) z^r .$$

Let $z^r \equiv U_r + iV_r$,

$$e^{2i\alpha} z^r \equiv U_r^* + iV_r^* ,$$

$$e^{2i\alpha} z \frac{d}{dz} z^r \equiv U_r^{**} + iV_r^{**} .$$

Then $\Phi_n(z) + \overline{\Phi_n(z)} - e^{2i\alpha} [\bar{z} \Phi_n'(z) + \Psi_n'(z)]$

$$= 2a_0 + 2\sum_{r=1}^{n} (a_{4r-1} U_r - a_{4r} V_r)$$

$$- \sum_{r=1}^{n} (a_{4r-1} + ia_{4r}) (U_r^{**} + iV_r^{**})$$

$$- \sum_{r=0}^{n} (a_{4r+1} + ia_{4r+2}) (U_r^* + iV_r^*) ,$$

$$\equiv \sum_{j=0}^{4n+2} a_j (F_j - iG_j) ,$$

where

$$F_0 = 2U_0 , \qquad G_0 = 0 ,$$

$$F_{4r+1} = - U_r^* , \quad G_{4r+1} = V_r^* , \qquad (r = 0,1,2,\ldots,n)$$

$$F_{4r+2} = V_r^* , \quad G_{4r+2} = U_r^* , \qquad (r = 0,1,2,\ldots,n)$$

$$F_{4r-1} = 2U_r - U_r^{**} , \quad G_{4r-1} = V_r^{**} , \qquad (r = 1,2,\ldots n)$$

$$F_{4r} = - 2V_r + V_r^{**} , \quad G_{4r} = U_r^{**} , \qquad (r = 1,2,\ldots n).$$

The least squares problem then becomes the minimization by choice of a_j of the expression

$$I = \sum_{i=1}^{N} w_i [N(P_i) - \sum_{j=0}^{4n+2} a_j F_j(P_i)]^2 + \sum_{i=1}^{N} w_i [T(P_i) - \sum_{j=0}^{4n+2} a_j G_j(P_i)]^2 \tag{1.29}$$

and may be dealt with as before.

§ 5. Methods for constructing an orthogonal set of vectors.

To construct an orthonormal set of vectors $\varphi_1, \varphi_2, \ldots, \varphi_n$ from a given set of linearly independent vectors $f_1, f_2, \ldots, f_n$, Davis and Rabinowitz use the well known Gram-Schmidt process in which linear combinations equivalent to

$$\begin{aligned} \varphi_1 &= a_{11} f_1 \\ \varphi_2 &= a_{21} f_1 + a_{22} f_2 \\ \varphi_3 &= a_{31} f_1 + a_{32} f_2 + a_{33} f_3 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{1.30}$$

are taken, namely

$$\varphi_1 = \psi_1 / \|\psi_1\| \quad , \quad \text{where} \quad \psi_1 = f_1 \ ,$$

$$\varphi_2 = \psi_2 / \|\psi_2\| \quad , \quad \text{where} \quad \psi_2 = f_2 - (f_2, \varphi_1)\, \varphi_1 \ ,$$

and generally

$$\varphi_k = \psi_k / \|\psi_k\| \quad , \quad \text{where} \quad \psi_k = f_k - \sum_{j=1}^{k-1} (f_k, \varphi_j)\, \varphi_j \quad . \tag{1.31}$$

For a given vector f, the linear combination

$$\tilde{f} = \sum_{k=1}^{n} b_k \varphi_k$$

which minimizes $\|f - \tilde{f}\|^2$

is $$\tilde{f} = \sum_{k=1}^{n} (f, \varphi_k)\, \varphi_k \ , \tag{1.32}$$

and the components $\delta_1, \delta_2, \ldots, \delta_N$ of the residual vector

$$\delta = f - \tilde{f} = f - \sum_{k=1}^{n} (f, \varphi_k)\, \varphi_k \tag{1.33}$$

can obviously be found by a further application of the formula (1.31).

Normally it is required to express $\tilde{f}$ in terms of the original vectors f_j. Substituting from (1.30) gives

$$\begin{aligned} \tilde{f} &= \sum_{k=1}^{n} (f, \varphi_k) \sum_{j=1}^{k} a_{kj} f_j \\ &= \sum_{j=1}^{n} a_j f_j \ , \end{aligned}$$

where $$a_j = \sum_{k=j}^{n} (f, \varphi_k)\, a_{kj} \quad . \tag{1.34}$$

Davis and Rabinowitz obtain the coefficients a_j and also the coefficients a_{ij} by augmenting each of the vectors f_j by the n-dimensional vector whose k^{th} component is δ_{jk} (Kronecker's delta) and augmenting the vector f by an n-dimensional zero vector. If the Gram-Schmidt process is applied to the augmented $N + n$ dimensional vectors, with the proviso that inner products are taken over the original N components only, the elements δ_{jk} are replaced by a_{jk} and the zero elements by $- a_j$.

Another device used by Davis and Rabinowitz consists of re-applying the orthogonalizing process to each of the vectors φ_k as they are computed, in an attempt to alleviate some of the effects of rounding errors. If the vectors $\varphi_1, \varphi_2, \ldots, \varphi_{k-1}$ are substantially orthonormal but the computed k^{th} vector $\bar{\varphi}_k$ although normalized is slightly non-orthogonal to the others, it is shown that

$$\varphi_k = \bar{\varphi}_k - \sum_{j=1}^{k-1} (\bar{\varphi}_k, \varphi_j)\, \varphi_j$$

is a better approximation to an orthonormal vector than $\bar{\varphi}_k$.

Almost all of Davis and Rabinowitz's computations were carried out in double precision floating point arithmetic. Single precision (36 bit) arithmetic was considered adequate only for small problems involving up to 10 - 12 functions.

Rice (1966) has shown that any non-orthogonality of the earlier vectors φ_j is passed on to the vector ψ_k in (1.31) and tends to be magnified by normalizing. For suppose that

$$(\varphi_i, \varphi_j) = \epsilon_{ij} \quad \text{for } i \neq j \ , \ i \leq k-1 \ , \ j \leq k-1 \ ,$$

and that $$(\varphi_i, \varphi_i) = 1 \ .$$

Let $$f_k = \sum_{i=1}^{k-1} \beta_i \varphi_i + \eta$$

where $(\eta, \varphi_i) = 0$ for $i = 1,2,\ldots,k-1$.

Then $$(f_k, \varphi_j) = \sum_{\substack{i=1 \\ i \neq j}}^{k-1} \beta_i \epsilon_{ij} + \beta_j ,$$

and so substituting in (1.31),

$$\psi_k = \eta - \sum_{j=1}^{k-1} \Big(\sum_{i \neq j} \beta_i \epsilon_{ij} \Big) \varphi_j .$$

Hence if $1<k$,

$$\epsilon_{k1} = (\varphi_k, \varphi_1) = - \sum_{j=1}^{k-1} \Big(\sum_{i \neq j} \beta_i \epsilon_{ij} \Big) \epsilon_{j1} / \|\psi_k\| ,$$

and since $\epsilon_{11} = 1$, this summation contains the terms

$$- \sum_{i \neq 1}^{k-1} \beta_i \epsilon_{i1} / \|\psi_k\| .$$

Rice's experience is that $\|\eta\|$ and hence $\|\psi_k\|$ become small for vectors f_k whose components are values of mathematical functions. This quickly causes ψ_k to become a linear combination of the φ_j $(j = 1,2,\ldots,k-1)$ and in particular $\epsilon_{k,k-1}$ tends to 1. A modification of the Gram-Schmidt process (1.30) is advocated, given by

$$\psi_k^{(1)} = f_k - (f_k, \varphi_1) \varphi_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\psi_k^{(j)} = \psi_k^{(j-1)} - (\psi_k^{(j-1)}, \varphi_j) \varphi_j$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\varphi_k = \psi_k^{(k-1)} / \|\psi_k^{(k-1)}\|$$

which ensures at least that $\epsilon_{k,k-1} = 0$.

Although elegant, the Davis - Rabinowitz scheme has the disadvantage that the vectors $\varphi_1, \varphi_2, \ldots, \varphi_n$ have actually to be computed and so unless the storage space is doubled, the vectors $f_1, f_2, \ldots, f_n$ must be overwritten. Quite apart from the amount of computation involved, this means that it is not possible to find the

residual vector $f - \sum_{j=1}^{n} a_j f_j$,

but only instead $f - \sum_{k=1}^{n} (f, \varphi_k) \varphi_k$,

which although theoretically the same, need not necessarily be so because of rounding errors.

An alternative procedure which uses precisely the same amount of storage and avoids computing new vectors can be based on a method described by Guest (1950) for curve fitting by least squares.

Given linearly independent vectors $f_1, f_2, \ldots, f_n$ as before, coefficients a_{kj} are found so that

$$\begin{aligned} \psi_1 &= f_1 \\ \psi_2 &= a_{21} f_1 + f_2 \\ \psi_3 &= a_{31} f_1 + a_{32} f_2 + f_3 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \psi_n &= a_{n1} f_1 + a_{n2} f_2 + \ldots + f_n \end{aligned} \tag{1.35}$$

form an orthogonal (but not usually normal) set of vectors.

To do this, consider the equivalent forms

$$\left.\begin{aligned} \psi_1 &= f_1 \\ \psi_2 &= c_{21} \psi_1 + f_2 \\ \psi_3 &= c_{31} \psi_1 + c_{32} \psi_2 + f_3 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \psi_n &= c_{n1} \psi_1 + c_{n2} \psi_2 + \ldots + f_n \end{aligned}\right\} \tag{1.36}$$

or $$\psi_k = \sum_{j=1}^{k-1} c_{kj} \psi_j + f_k \ .$$

The coefficients c_{kj} are readily found. For,

$$(\psi_k, \psi_r) = \sum_{j=1}^{k-1} c_{kj} (\psi_j, \psi_r) + (f_k, \psi_r) \ ,$$

and if $r < k$, this gives

$$0 = c_{kr} (\psi_r, \psi_r) + (f_k, \psi_r) \ ,$$

or $$c_{kr} = - (f_k, \psi_r) / (\psi_r, \psi_r) \ . \tag{1.37}$$

However, since it is not proposed to calculate the components of the vectors ψ_r , the inner products (f_k, ψ_r) must be found indirectly. Defining $a_{rr} = 1$, equations (1.35) can be written

$$\psi_r = \sum_{s=1}^{r} a_{rs} f_s , \tag{1.38}$$

and so

$$c_{kr} = - \sum_{s=1}^{r} a_{rs} (f_k, f_s) / (\psi_r, \psi_r) . \tag{1.39}$$

Then from (1.36) and (1.38),

$$\begin{aligned} \psi_k &= \sum_{j=1}^{k-1} c_{kj} \sum_{s=1}^{j} a_{js} f_s + f_k , \\ &= \sum_{r=1}^{k-1} a_{kr} f_r + f_k , \end{aligned}$$

where

$$a_{kr} = \sum_{s=r}^{k-1} c_{ks} a_{sr} . \tag{1.40}$$

Having found the coefficients a_{kr} , the inner product (ψ_k, ψ_k) may be found in readiness for the next stage. From (1.36) it follows that

$$\begin{aligned} (\psi_k, \psi_k) &= (f_k, \psi_k) \\ &= \sum_{r=1}^{k} a_{kr} (f_k, f_r) . \end{aligned} \tag{1.41}$$

For the approximation of an arbitrary vector f , let

$$f = \sum_{j=1}^{n} a_j f_j + \delta , \tag{1.42}$$

where δ is the residual vector. If the equivalent form

$$f = \sum_{j=1}^{n} c_j \psi_j + \delta \tag{1.43}$$

is taken, $\|\delta\|^2$ is least when the c_j are the Fourier coefficients, that is when

$$(f, \psi_r) = c_r (\psi_r, \psi_r)$$

or

$$c_r = (f, \psi_r) / (\psi_r, \psi_r) . \tag{1.44}$$

Since (1.44) is of the same form as (1.37) it follows that repeating steps (1.39) and (1.40) with $f_{n+1} = f$ will lead to $a_{n+1,r} = - a_r$.

In addition, since $\delta = - \sum_{j=1}^{n} c^j \psi^j + f$,

$\delta = -$

repeating (1.41) leads to

$$(\psi_{n+1}\,,\ \psi_{n+1}) = (\delta,\ \delta) = \|\delta\|^2 \ .$$

Comparison of this value of $\|\delta\|^2$ with that derived directly from (1.42) provides a useful numerical check on the orthogonality (or lack of orthogonality) of the vectors ψ_j since the former value has been derived on the assumption that the vector δ is orthogonal to each of the vector ψ_j .

The process described above is related to the solution of the corresponding <u>normal equations</u> by Choleski's method. If the coefficients a_j are to minimize

$$\|f - \sum_{j=1}^{n} a_j\ f_j\|^2 \ ,$$

differentiating w.r.t a_k and equating to zero gives

$$\sum_{j=1}^{n} (f_k\,,\ f_j)\ a_j = (f_k\,,\ f)\,,\quad k = 1,2,\ldots,n \ . \tag{1.44}$$

Let F and Ψ denote the $n \times N$ matrices whose <u>rows</u> are the components of the vectors f_j and ψ_j respectively and let A and C denote the lower triangular matrices

$$\begin{aligned} A_{ii} &= 1\ ,\quad A_{ij} = a_{ij} \quad i > j\ , \\ &\qquad\quad = 0 \quad i < j\ ; \\ C_{ii} &= 1\ ,\quad C_{ij} = -\,c_{ij}\, i > j\ , \\ &\qquad\quad = 0 \quad i < j\ ; \end{aligned}$$

The equations (1.35) and (1.36) may then be written

$$\Psi = AF\ ,\qquad F = C\Psi\ ,$$

whilst the matrix of coefficients in the normal equations (1.44) is FF^T .

Now $$FF^T = C\Psi\Psi^T C^T$$

and since $\Psi\Psi^T$ is a diagonal matrix it follows that $\Psi\Psi^T C^T$ is an upper triangular matrix. The matrix $A = C^{-1}$.

§ 6. Comparison of the arithmetical operations involved

Examination of the two orthogonalizing processes shows that $\frac{1}{2}n\,(n+1)$ inner products have to be computed in each case. At the k^{th} stage, the Gram - Schmidt process uses $k-1$ inner products (f_k, φ_j) together with (ψ_k, ψ_k) while the alternative process uses k inner products (f_k, f_j). If these are unweighted, the calculation of each requires N accumulations (multiplication followed by addition) giving altogether $\frac{1}{2}\,Nn\,(n+1)$ accumulations. If the inner products are weighted, it is only necessary to multiply the components of f_k by the corresponding weights before carrying out the k^{th} stage. This requires Nn multiplications and a further Nn multiplications are required in the Gram - Schmidt process to evaluate the inner products (ψ_k, ψ_k).

Elsewhere, there is a considerable difference between the number of operations in the two processes. Using augmented vectors, the calculation of ψ_k from (1.31) in the Gram - Schmidt process requires $(N+n)\times(k-1)$ accumulations and so all told

$$\tfrac{1}{2}\,(N+n)\,n\,(n-1) \quad \text{accumulations}$$

are needed. Normalizing the vectors ψ_k requires the calculation of n square roots, n reciprocals and $(N+n)n$ multiplications. In the alternative process the calculation of c_{kr}, a_{kr} and (ψ_k, ψ_k) from (1.39), (1.40) and (1.41) requires

$$\tfrac{1}{2}k(k-1) + \tfrac{1}{2}k(k-1) + k = k^2 \quad \text{accumulations}$$

at the k^{th} stage. Thus all told,

$$\tfrac{1}{3}n(n+\tfrac{1}{2})\,(n+1) \quad \text{accumulations}$$

are required. In addition, n reciprocals and $\frac{1}{2}n\,(n-1)$ multiplications are needed during the calculation of the coefficients c_{kr}.

Lastly, in finding the linear combination $\sum_{j=1}^{n} a_j\, f_j$ which approximates a given vector f, roughly the same number of operations is required in

each process, assuming that the residuals are also found. In both processes, n inner products (f, φ_k) or (f, f_k) are required and computing these needs Nn accumulations and possibly N multiplications. Computing the residuals δ_i and the coefficients a_j from (1.34) in the Davis - Rabinowitz scheme then requires $(N + n)n$ accumulations. Computing the coefficients c_r, a_r from (1.39), (1.40) in the alternative scheme and obtaining the residuals directly requires

$$\tfrac{1}{2}n(n + 1) + \tfrac{1}{2}n(n + 1) + Nn = (N + n + 1)n \text{ accumulations}$$

together with one reciprocal and n multiplications.

Because of the smaller number of accumulations and the advantage of preserving the original vectors, the alternative scheme was used in the computations described in the remaining chapters. Single precision arithmetic was always found sufficient.

CHAPTER 2

Simply Connected Regions

In this Chapter, the method described in Chapter 1 is applied to two problems in which the region occupied by the plate is simply connected and the boundary stresses are prescribed. In each case, the boundary stresses are symmetrical about the x- and y- axes and it follows that the internal stresses and displacements will also be symmetrical about Ox and Oy. Because of this, some simplification is possible, as is shown in the following section.

§ 1. Symmetry considerations

Suppose first that the stresses and displacements are symmetrical about the x- axis. Then if u and v are the components of the displacement at the point (x,y), the components at the point $(x,-y)$ will be u and $-v$.

But
$$2\mu(u + iv) = \varkappa\,\varphi(z) - z\,\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z}) , \tag{2.1}$$

and so on changing y into $-y$ it must follow that

$$2\mu(u - iv) = \varkappa\,\varphi(\bar{z}) - \bar{z}\,\bar{\varphi}'(z) - \bar{\psi}(z) .$$

On the other hand, taking the conjugate of (2.1),

$$2\mu(u - iv) = \varkappa\,\bar{\varphi}(\bar{z}) - \bar{z}\,\varphi'(z) - \psi(z) . \tag{2.2}$$

Comparing these two gives

$$\bar{\varphi}(z) = \varphi(z) , \quad \bar{\psi}(z) = \psi(z) , \tag{2.3}$$

and on differentiating with respect to z,

$$\bar{\Phi}(z) = \Phi(z) , \quad \bar{\Psi}(z) = \Psi(z) . \tag{2.4}$$

Obviously these relations should also be satisfied by functions used

to approximate $\varphi(z)$ and $\psi(z)$ or $\Phi(z)$ and $\Psi(z)$. In particular, for simply connected regions, the polynomials

$$\varphi_n(z) = \sum_{k=1}^{n} \alpha_k z^k, \quad \psi_n(z) = \sum_{k=0}^{n} \beta_k z^k,$$

$$\Phi_n(z) = \sum_{k=0}^{n} A_k z^k, \quad \Psi_n(z) = \sum_{k=0}^{n} \beta_k z^k,$$

of Chapter 1, § 4, should have all their coefficients <u>real</u>.

If the stresses and displacements are symmetrical about the y- axis, the components of the displacement at the point $(-x, y)$ should be $-u$ and v. Since $-x + iy = -\bar{z}$, this implies that

$$2\mu(-u + iv) = \kappa\,\varphi(-\bar{z}) - (-\bar{z})\,\bar{\varphi}'(-z) - \bar{\psi}(-z),$$

or $$2\mu(u - iv) = -\kappa\,\varphi(-\bar{z}) - \bar{z}\,\bar{\varphi}'(-z) + \bar{\psi}(-z).$$

Comparing this with (2.2) gives

$$\bar{\varphi}(-z) = -\varphi(z), \quad \bar{\psi}(-z) = -\psi(z),$$

or $$\bar{\varphi}(z) = -\varphi(-z), \quad \bar{\psi}(z) = -\psi(-z), \tag{2.5}$$

and on differentiating w.r.t. z

$$\bar{\Phi}(z) = \Phi(-z), \quad \bar{\Psi}(z) = \Psi(-z). \tag{2.6}$$

Lastly, if the stresses and displacements are symmetrical about both x- and y- axes, the relations (2.3) and (2.5) may be combined to give

$$\varphi(z) = -\varphi(-z), \quad \psi(z) = -\psi(-z). \tag{2.7}$$

Similarly, (2.4) and (2.6) give

$$\Phi(z) = \Phi(-z), \quad \Psi(z) = \Psi(-z). \tag{2.8}$$

In this case, for a simply connected region, the polynomials $\varphi_n(z)$ and $\psi_n(z)$ should contain only <u>odd</u> powers whilst the polynomials $\Phi_n(z)$ and $\Psi_n(z)$ should contain <u>even</u> powers only.

The existence of symmetry about one of the co-ordinate axes also enables one to use approximately half the number of boundary points that

would otherwise be taken. Suppose for example that the region is symmetrical about Ox. If N is even and $N_1 = \frac{1}{2}N + 1$, the boundary points $P_1, P_2, \ldots, P_N$ can be disposed with P_1 and P_{N_1} on Ox and the remaining points arranged in pairs so that P_i and P_j, where $j = 2N_1 - i$, are reflections in Ox. Then, if $w_i = w_j$ and the real functions f and g are such that $f(P_i) = f(P_j)$, $g(P_i) = g(P_j)$, the inner product

$$(f, g) \equiv \sum_{i=1}^{N} w_i \, f(P_i) \, g(P_i)$$

becomes $$(f, g) = 2 \sum_{i=1}^{N_1}{}'' \, w_i \, f(P_i) \, g(P_i)$$

where Σ'' indicates that the weights for $i = 1, N_1$ are to be halved. Thus only points on half the boundary have to be used.

Similarly, when symmetry exists about Ox and Oy it is sufficient to use points on a quarter of the boundary only. The inner product then becomes

$$(f, g) = 4 \sum_{i=1}^{N_1}{}'' \, w_i \, f(P_i) \, g(P_i)$$

where $N_1 = \frac{1}{4}N + 1$.

§ 2. Square plate with prescribed tractions over two opposite edges

For the first application, an example considered by Timoshenko and Goodier (1951) was taken. In this, a square plate bounded by the lines $x = \pm 1$, $y = \pm 1$ is subjected to the parabolic loading

$$\breve{xx} = T(1 - y^2), \quad \breve{xy} = 0$$

over the edges $x = \pm 1$, T being a constant, while the remaining edges are free from stress. The stresses at internal points are to be determined.

Forty-two points P_1, P_2, ..., P_{42} were taken on the quarter boundary ABC (Figure 1), P_1 coinciding with A, P_{42} with C and the remainder equally spaced at distance apart 0·05. The corner point B occurs twice (P_{21}, P_{22}) and is counted as a member of each of the sides AB, BC.

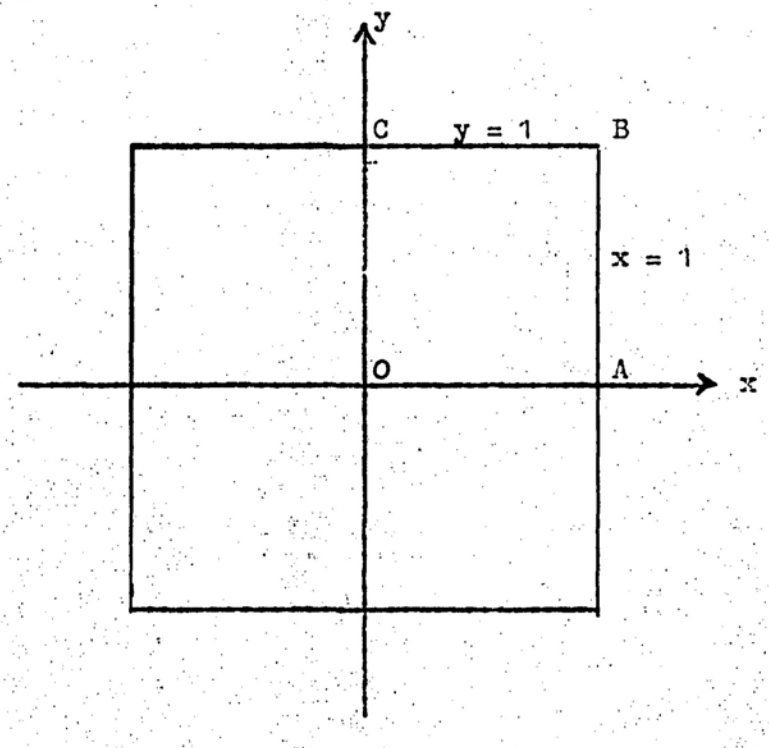


Fig. 1

Weights w_i were taken equal to unity except at the end points A and C and the double point B where the weights were taken equal to one half. For convenience, the constant T was made unity.

Because of the symmetry about Ox and Oy, the complex potentials $\Phi(z)$ and $\Psi(z)$ can be approximated by polynomials containing even powers only with real coefficients. These were

$$\left.\begin{aligned} \Phi_{2n}(z) &= \sum_{k=0}^{n} A_k \, (z/a)^{2k} \, , \\ \Psi_{2n}(z) &= \sum_{k=0}^{n} B_k \, (z/a)^{2k} \, , \end{aligned}\right\} \qquad (2.9)$$

where the factor $a = \sqrt{2}$ was introduced to keep $|z/a| \leq 1$ over the entire region.

Changing the notation slightly, let

$$\Phi_{2n}(z) = \sum_{r=0}^{n} a_{2r} \ (z/a)^{2r} \ ,$$

$$\Psi_{2n}(z) = \sum_{r=0}^{n} a_{2r+1} \ (z/a)^{2r} \ .$$

Then $$\bar{z} \ \Phi'_{2n}(z) = (z\bar{z}/a^2) \sum_{r=1}^{n} 2r \ a_{2r} \ (z/a)^{2r-2} \ ,$$

and so writing $$\rho^2 = z\bar{z}/a^2 \ ,$$

$$(z/a)^{2r} = U_{2r} + iV_{2r} \ ,$$

$$e^{2i\alpha} \ (z/a)^{2r} = U_{2r}^{*} + iV_{2r}^{*} \ ,$$

it follows that approximations $\tilde{N}$, $\tilde{T}$ to the normal and tangential boundary tractions are given by

$$\tilde{N} - i\tilde{T} \equiv \Phi_{2n}(z) + \bar{\Phi}_{2n}(\bar{z}) - e^{2i\alpha} \ [\bar{z} \ \Phi'_{2n}(z) + \Psi_{2n}(z)]$$

$$= 2\sum_{r=0}^{n} a_{2r} \ U_{2r} - 2\rho^2 \sum_{r=1}^{n} r.a_{2r} \ (U_{2r-2}^{*} + iV_{2r-2}^{*})$$

$$- \sum_{r=0}^{n} a_{2r+1} \ (U_{2r}^{*} + iV_{2r}) \ .$$

Hence $$\tilde{N} + i\tilde{T} = \sum_{j=0}^{2n+1} a_j \ (F_j + iG_j) \ ,$$

where $$F_{2r} = 2U_{2r} - 2r\rho^2 \ U_{2r-2}^{*}, \quad G_{2r} = 2r\rho^2 \ V_{2r-2}^{*} \ ,$$

$$F_{2r+1} = - U_{2r}^{*} \ , \quad G_{2r+1} = V_{2r}^{*} \ .$$

The coefficients a_j may then be determined in the usual way from the requirement that

$$I = \sum_{i=1}^{N} w_i \ \{N(P_i) - \tilde{N}(P_i)\}^2 + \sum_{i=1}^{N} w_i \ \{T(P_i) - \tilde{T}(P_i)\}^2$$

shall be a minimum. $(N = 42)$

The program was arranged to print the coefficients A_k and B_k in (2.9) together with the residuals at each boundary point:

$$\left.\begin{aligned} R_N(P_i) &\equiv N(P_i) - \tilde{N}(P_i) \ , \\ R_T(P_i) &\equiv T(P_i) - \tilde{T}(P_i) \ , \end{aligned}\right\} \qquad (2.10)$$

and the root mean squares of these:

$$\left.\begin{aligned} R_N(\text{rms}) &= \Big[\sum_i w_i \{R_N(P_i)\}^2 / \sum_i w_i\Big]^{\frac{1}{2}} , \\ R_{\tilde{T}}(\text{rms}) &= \Big[\sum_i w_i \{R_T(P_i)\}^2 / \sum_i w_i\Big]^{\frac{1}{2}} . \end{aligned}\right\} \qquad (2.11)$$

As a check against the presence of errors due to rounding in the orthogonalizing process, the root mean square of all the residuals,

$$E(\text{rms}) = \Big[\sum_i w_i \{R_N^2(P_i) + R_{\tilde{T}}^2(P_i)\} / 2\sum_i w_i\Big]^{\frac{1}{2}} ,$$

was computed directly and compared with the value $E'(\text{rms})$ obtained indirectly from the expansion of $\tilde{N} + i\tilde{T}$ in terms of orthogonal vectors.

In addition, the stresses were calculated at points comprising a square mesh of side $1/8$ using the formulae (1.18):

$$\widetilde{xx} + \widetilde{yy} = 2\,[\Phi(z) + \overline{\Phi}(\bar{z})] ,$$

$$\widetilde{yy} - \widetilde{xx} + 2i\widetilde{xy} = 2\,[\bar{z}\,\Phi'(z) + \Psi(z)] ,$$

taking $\Phi_{2n}(z)$, $\Psi_{2n}(z)$ as approximations to $\Phi(z)$, $\Psi(z)$.

Results obtained when terms up to $(z/a)^{20}$ and $(z/a)^{28}$ respectively are used in the approximating polynomials are shown in Table 1 ($n = 10$ and $n = 14$) . $R_N(\text{Max})$ and $R_{\tilde{T}}(\text{Max})$ are the greatest <u>absolute</u> values of the residuals $R_N(P_i)$ and $R_{\tilde{T}}(P_i)$ found at any of the 42 boundary points.

Table 1

n	$R_N(\text{Max})$	$R_{\tilde{T}}(\text{Max})$	$R_N(\text{rms})$	$R_{\tilde{T}}(\text{rms})$
10	$1{\cdot}64 \times 10^{-3}$	$5{\cdot}87 \times 10^{-4}$	$4{\cdot}36 \times 10^{-4}$	$3{\cdot}40 \times 10^{-4}$
14	$1{\cdot}94 \times 10^{-4}$	$2{\cdot}69 \times 10^{-4}$	$7{\cdot}20 \times 10^{-5}$	$1{\cdot}04 \times 10^{-4}$

In each case, the numerically greatest residuals are found at or near to the corner point B .

The agreement between E(rms) and E'(rms) was satisfactory:

For $n = 10$, $E = E' = 3{\cdot}909 \times 10^{-4}$, while for $n = 14$, $E = 8{\cdot}931 \times 10^{-5}$, $E' = 8{\cdot}938 \times 10^{-5}$.

The stresses $\breve{xx}$, $\breve{yy}$ and $\breve{xy}$ calculated in the two cases agreed to 3 decimal places except at points on $y = 1$ where differences of 1 or 2 units in the third place occurred in some values of $\breve{xx}$ and on $x = 1$ where similar differences occurred in some values of $\breve{xx}$ and $\breve{yy}$.

The coefficients A_k and B_k corresponding to $n = 14$ are given to 5 decimal places in Table 2, values less than 10^{-5} being omitted. The largest of the omitted coefficients is $B_{10} = -2{\cdot}09 \times 10^{-6}$. Stresses calculated from these coefficients are given in Figures 2 and 3. The entire calculation took less than 4 seconds on the "Atlas" computer including 1 second to compile the program which was written in EXCHLF.

Table 2

k	A_k	B_k
0	0·17953	- 0·50000
1	0·25000	- 0·52288
2	0·19357	
3		0·20311
4	- 0·00271	
5		0·00824
6	- 0·00513	
7		- 0·09680
8	0·01020	
9		- 0·14551
10	0·01636	
11		- 0·26658
12	0·02281	
13		0·00574
14	0·00511	

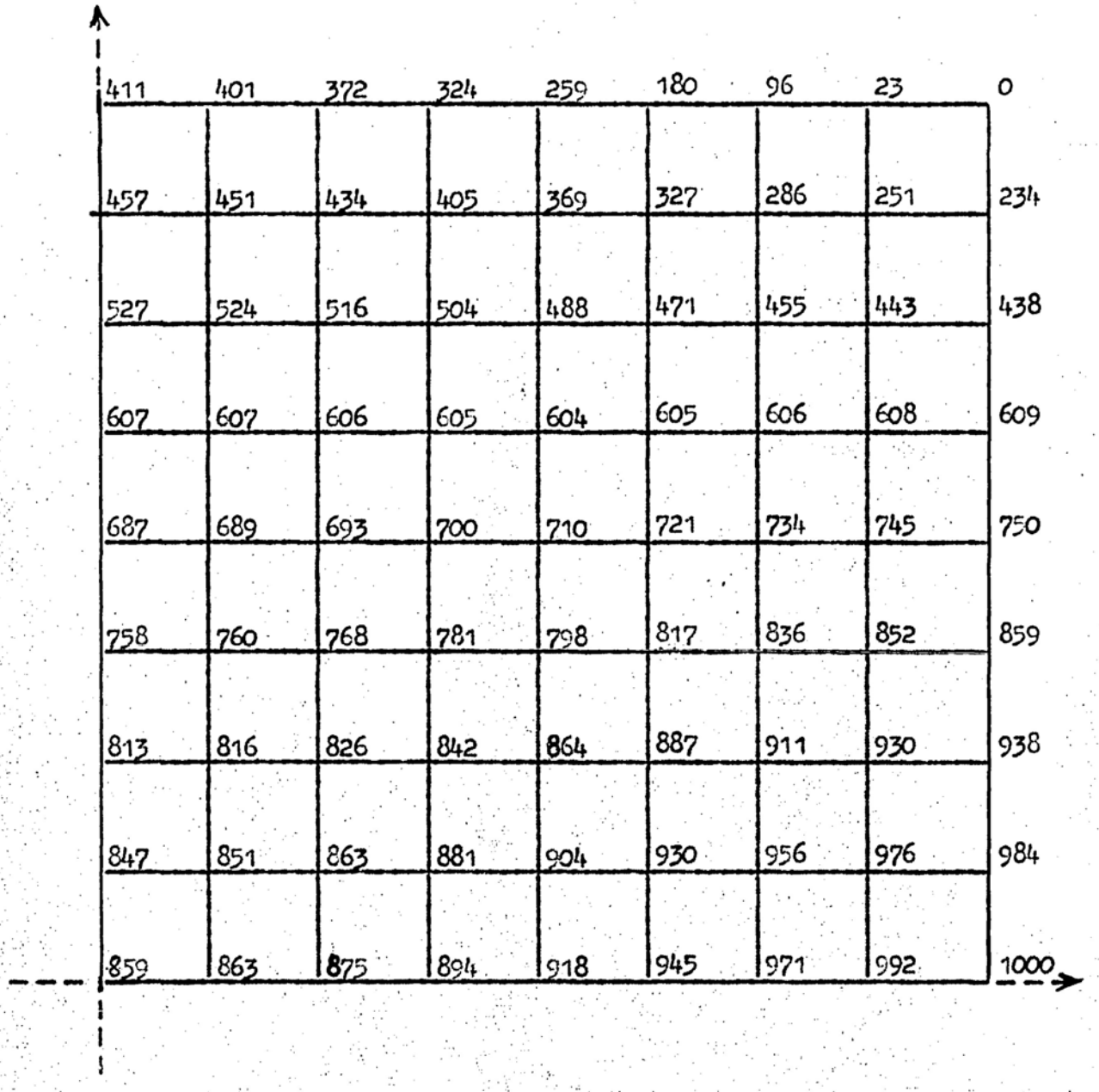
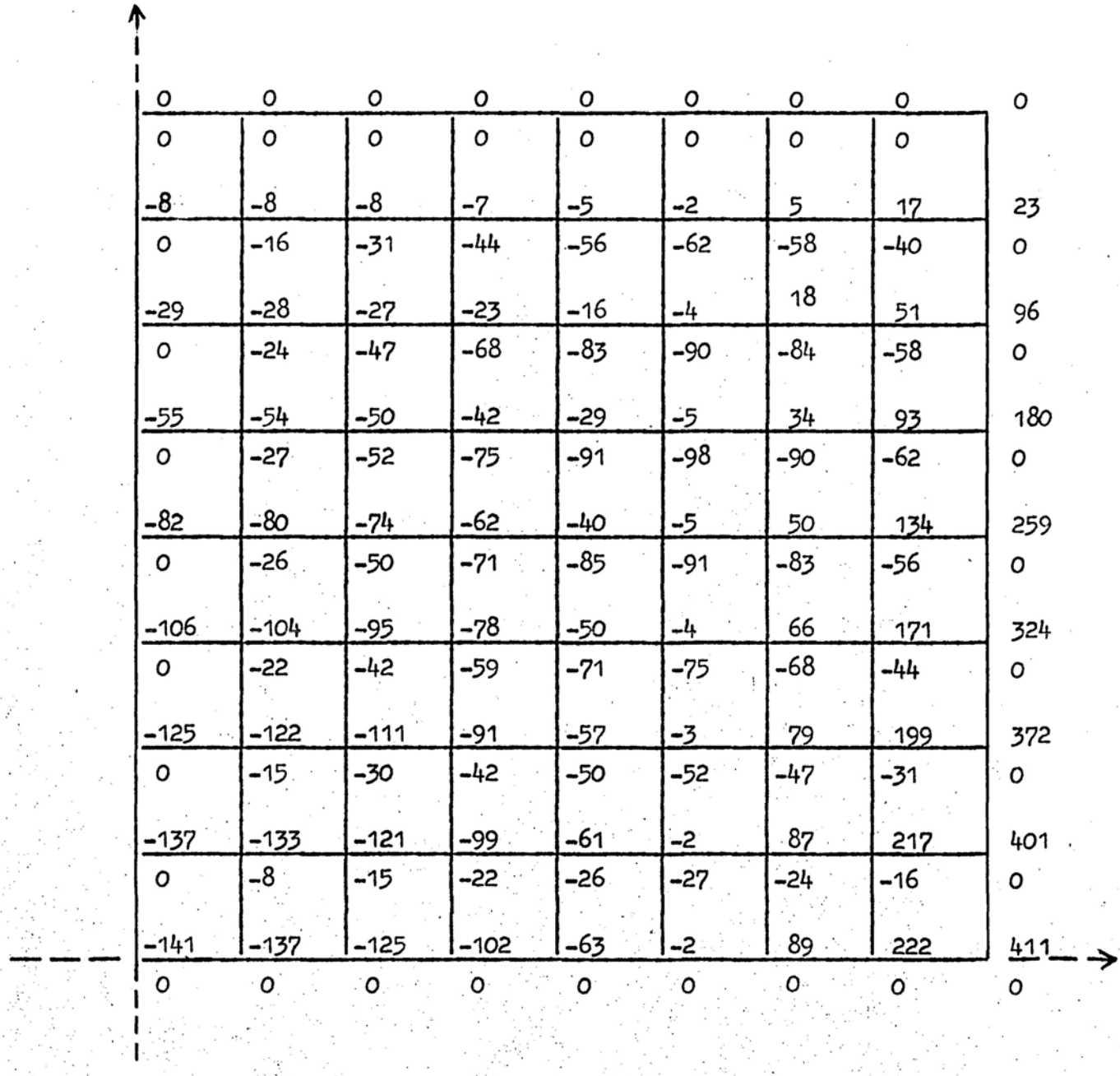


Fig. 2

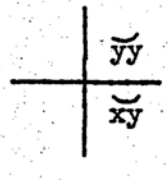
Stress distribution $\widetilde{xx}$ (in units 10^{-3} x T)



0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	
-8	-8	-8	-7	-5	-2	5	17	23
0	-16	-31	-44	-56	-62	-58	-40	0
-29	-28	-27	-23	-16	-4	18	51	96
0	-24	-47	-68	-83	-90	-84	-58	0
-55	-54	-50	-42	-29	-5	34	93	180
0	-27	-52	-75	-91	-98	-90	-62	0
-82	-80	-74	-62	-40	-5	50	134	259
0	-26	-50	-71	-85	-91	-83	-56	0
-106	-104	-95	-78	-50	-4	66	171	324
0	-22	-42	-59	-71	-75	-68	-44	0
-125	-122	-111	-91	-57	-3	79	199	372
0	-15	-30	-42	-50	-52	-47	-31	0
-137	-133	-121	-99	-61	-2	87	217	401
0	-8	-15	-22	-26	-27	-24	-16	0
-141	-137	-125	-102	-63	-2	89	222	411
0	0	0	0	0	0	0	0	0

Fig. 3

Stress distributions $\widetilde{yy}$ and $\widetilde{xy}$ (in units 10^{-3} x T)



The values of the coefficients A_1 and B_0 and the vanishing of the coefficients $A_3, A_5, \ldots, A_{13}, B_2, B_4, \ldots, B_{14}$, apart from the effects of rounding errors, are remarkable but might have been anticipated. In fact, the complex potentials

$$\Phi(z) = \tfrac{1}{4}\left(\frac{z}{a}\right)^2 = \tfrac{1}{8}z^2 \ , \quad \Psi(z) = -\tfrac{1}{2}$$

lead to stresses given by

$$\widetilde{xx} + \widetilde{yy} = 4\,\mathrm{Re}\,\Phi(z) = \tfrac{1}{2}(x^2 - y^2) \ ,$$

$$\widetilde{yy} - \widetilde{xx} + 2i\widetilde{xy} = 2[\bar{z}\,\Phi'(z) + \Psi(z)] = \tfrac{1}{2}(x^2 + y^2) - 1 \ ,$$

i.e. $$\widetilde{xx} = \tfrac{1}{2}(1 - y^2), \ \widetilde{yy} = -\tfrac{1}{2}(1 - x^2), \ \widetilde{xy} = 0 \ ,$$

and if these stresses are subtracted from those to be determined, a new stress problem is obtained in which the boundary conditions become

$$\widetilde{xx} = \tfrac{1}{2}(1 - y^2), \ \widetilde{xy} = 0 \quad \text{on} \quad x = \pm 1 \ ,$$

$$\widetilde{yy} = \tfrac{1}{2}(1 - x^2), \ \widetilde{xy} = 0 \quad \text{on} \quad y = \pm 1 \ .$$

Clearly this problem has eight fold symmetry, there being now the additional lines of symmetry $y = \pm x$. If the displacement at the point (x,y) has components u and v, the displacement at the point (y,x) must have components v and u. Hence if $\varphi(z)$ and $\psi(z)$ are the complex potentials, so that

$$2\mu(u + iv) = \varkappa\,\varphi(z) - z\,\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z}) \ ,$$

it must follow on replacing $z = x + iy$ by $y + ix = i\bar{z}$ that

$$2\mu(v + iu) = \varkappa\,\varphi(i\bar{z}) - i\bar{z}\,\bar{\varphi}'(-iz) - \bar{\psi}(-iz) \ .$$

Comparison of these shows that

$$\varphi(i\bar{z}) = i\bar{\varphi}(\bar{z}) \ , \quad \psi(i\bar{z}) = -i\bar{\psi}(\bar{z}) \ ,$$

and if these are combined with the equations (2.3):

$$\bar{\varphi}(z) = \varphi(z) \ , \quad \bar{\psi}(z) = \psi(z) \ ,$$

which are a consequence of the symmetry about Ox, it follows that

$$\varphi(iz) = i\varphi(z) \ , \quad \psi(iz) = -i\psi(z) \ ,$$

and then on differentiating that

$$\Phi(iz) = \Phi(z) \ , \quad \Psi(iz) = -\Psi(z) \ .$$

The first of these conditions explains the absence of terms in $(z/a)^2$, $(z/a)^6$, ... from $\Phi_{2n}(z)$ and the second the absence of terms in $(z/a)^4$, $(z/a)^8$, ... from $\Psi_{2n}(z)$.

To check the results obtained previously, the eight-fold symmetric problem was solved taking 33 points on AB alone, equally spaced. Weights were taken to be unity except at A and B where they were one half and the approximating polynomials were

$$\Phi_{4n}(z) = \sum_{k=0}^{n} A_k \ (z/a)^{4k} \ , \quad \Psi_{4n-2}(z) = \sum_{k=0}^{n} B_k \ (z/a)^{4k-2} \ .$$

Taking $n = 10$ gave the following values

R_N(Max)	R_T(Max)	R_N(rms)	R_T(rms)
$4{\cdot}25 \times 10^{-5}$	$1{\cdot}26 \times 10^{-4}$	$2{\cdot}07 \times 10^{-5}$	$3{\cdot}25 \times 10^{-5}$

and superimposing the stresses corresponding to the potentials $\Phi(z) = \frac{1}{4}(z/a)^2$, $\Psi(z) = -\frac{1}{2}$ produced stresses which agreed everywhere with those in Figures 2 and 3 to 3 decimal places. For the most part, the stresses agreed to 4 decimal places with occasional differences of 1 or 2 units in the 4^{th} decimal place and a difference of 3 units in 4 of the 243 values.

§ 3. Rectangular plate with two notches

The second problem taken was that of determining the stresses when the test piece shown in Figure 4 is subjected to a uniform tension over the sides $x = \pm 4$, the remaining portions of the boundary being free from applied stress. Stresses in this case have been calculated by relaxation methods in Southwell (1956) and more recently by finite

difference methods using an irregular grid in a paper by Griffin and Varga (1963).

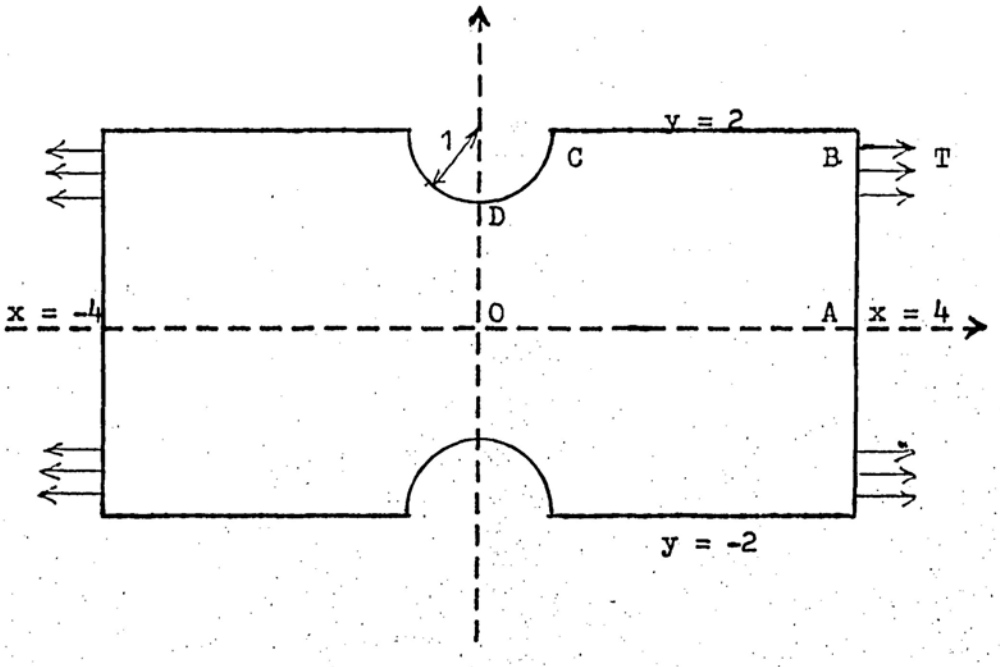


Fig. 4

There is again four fold symmetry and it is sufficient to consider the region in the first quadrant only. 55 points were taken, 17 points on AB equally spaced at interval 0·125, 25 points on BC equally spaced at interval 0·125 and 13 points on CD equally spaced at interval $7\frac{1}{2}^{\circ}$ i.e. at distance apart $25 \sin 3\frac{3}{4}^{\circ} \simeq 0{\cdot}131$. Each of the angular points B and C occurs twice. Weights equal to the mean of adjacent chords were used, those at the points A and D and each of the two points B and the two points C being halved.

In the first instance, the complex potentials $\Phi(z)$ and $\Psi(z)$ were approximated as before by the even polynomials

$$\Phi_{2n}(z) = \sum_{k=0}^{n} A_k \, (z/a)^{2k} \quad , \quad \Psi_{2n}(z) = \sum_{k=0}^{n} B_k \, (z/a)^{2k} \quad ,$$

with $a = 20$, the distance of B from the origin O . Taking $n = 18$ gave the results a) in Table 3. The residuals at most of the boundary points are of order 10^{-1} and the numerically largest ones, $R_N(\max) = 3{\cdot}56 \times 10^{-1}$, $R_T(\max) = 3{\cdot}62 \times 10^{-1}$ (taken on the notch at angles $22\frac{1}{2}^{\circ}$ and $55\frac{1}{2}^{\circ}$ respectively measured from C) are more than

1000 times greater than those in the previous problem where fewer terms were used. Moreover, some of the coefficients were very large, e.g. $B_{10} = 4{\cdot}266 \times 10^3$ and not surprisingly, the calculated stresses were absurdly wrong. Along the line $x = 0$, for example, the values of $\widetilde{xx}$ were practically constant at 0·69. Thus although Walsh's theorem is valid for this domain, the degrees of the polynomials required to approximate $\Phi(z)$ and $\Psi(z)$ at all usefully will need to be very great.

It is clear that the two notches are responsible for the difficulties encountered and consequently rather than increase the degrees of the polynomials, it would seem desirable to add terms better able to represent $\Phi(z)$ and $\Psi(z)$ in the neighbourhood of the notches. The obvious suggestion is to include equal numbers of terms of the form $A_k'(z - 2i)^{-k}$, $A_k''(z + 2i)^{-k}$, $k = 1,2,\ldots$, in the approximation to $\Phi(z)$ and similarly for the approximation to $\Psi(z)$. Since the symmetry condition $\bar{\Phi}(z) = \Psi(z)$ (2.4) must also be satisfied by approximations to $\Phi(z)$, one requires $A_k'' = \bar{A}_k'$. Likewise, for the condition $\Phi(-z) = \Phi(z)$ (2.8), one requires $A_k'' = (-1)^k A_k'$.

Hence to fulfill these conditions, the approximations adopted were

$$\left.\begin{aligned} \widetilde{\Phi}(z) &\equiv \sum_{k=0}^{n} A_k\,(z/a)^{2k} + \sum_{k=1}^{n'} A_k'\left[\left(\frac{i}{z - 2i}\right)^k + \left(\frac{-i}{z + 2i}\right)^k\right], \\ \widetilde{\Psi}(z) &\equiv \sum_{k=0}^{n} B_k\,(z/a)^{2k} + \sum_{k=1}^{n'} B_k'\left[\left(\frac{i}{z - 2i}\right)^k + \left(\frac{-i}{z + 2i}\right)^k\right]. \end{aligned}\right\} \qquad (2.12)$$

No scale factor was included in the terms of the second series since the greatest value of $|i/(z - 2i)|$ is unity, on the notch. The programming was straightforward and similar to that to be described in Chapter 3.

Entries b) to f) in Table 3 give the results obtained for various values of n and n', using a total of 19 approximating functions in each case. The introduction of descending powers of $z \pm 2i$ produces a marked improvement in the residuals, but as shown by f), these terms

should not be included at the expense of terms in the first series. The best results d) were obtained with roughly equal numbers of terms in the two series.

There is little point in recording any one set of coefficients A_k, A_k', B_k, B_k' because they varied considerably between the five cases. Despite this, the calculated stresses at some 60 chosen points were remarkably consistent, especially in cases c), d) and e) where most of the stresses agreed to 3 decimal places. The greatest differences occurred in the value of $\breve{xx}$ at the bottom of the notch; the values obtained in cases b), c), d), e), f) respectively were 3·240, 3·247, 3·248, 3·252, 3·259 .

Values of the stresses obtained in case d) are given to 2 decimal places in Figures 5 and 6. The values of $\breve{xx}$ are in fair agreement with those given by Griffin and Varga except that at the bottom of the notch their value is 3·15 .

However, if the five point Newton-Cotes formula is applied to the values of $\breve{xx}$ along the line $x = 0$ obtained in d):

y	0	0·25	0·5	0·75	1·0
$\breve{xx}$	1·606	1·653	1·823	2·229	3·248

then $\int_0^D \breve{xx}\, dy = \frac{180·078}{90} = 2·001$, which is in good agreement with $\int_A^B \breve{xx}\, dy$.

Table 4 gives some results obtained with 15 approximating functions of which the first 7 were invariably the powers $(z/a)^{2k}$, $k = 0(1)6$, but the remaining 8 were varied. In a) they are combinations of $(z \pm 2i)^{-k}$, $k = 1(1)8$; in b) terms $\log (z \pm 2i)$ are taken in place of $(z \pm 2i)^{-8}$; in c) terms $(z \pm 2i)^{-9}$ are taken in place of $(z \pm 2i)^{-1}$. Since the material does not enclose the points $z = \pm 2i$, the functions $\log (z \pm 2i)$ are regular in the domain of the plate and there is nothing to forbid their

inclusion. Conversely, if the material did enclose the points $z = \pm 2i$, the terms $(z \pm 2i)^{-1}$ would be excluded if the enclosing boundary were free from stress. But although d) and b) give better results than a), the improvement is not really great enough to draw any definite conclusions. There were no substantial alterations in the stresses and in particular b) and c) gave $\bar{\sigma}_{xx} = 3{\cdot}249$ and $3{\cdot}247$ respectively at the bottom of the notch.

Table 3

	No. of terms		Max^{n} Residuals R_N ($\times 10^{-3}$)			Max^{n} Residuals R_T ($\times 10^{-3}$)			E(rms) ($\times 10^{-3}$)
	First series	Second series	x = 4	y = 2	Notch	x = 4	y = 2	Notch	
a)	19	-	217	337	356	116	248	362	158
b)	13	6	3·57	5·11	4·91	2·03	10·0	7·20	2·42
c)	10	9	0·49	2·86	1·67	0·41	3·50	3·27	0·95
d)	9	10	0·27	2·18	2·58	1·45	1·45	1·71	0·66
e)	6	13	1·68	5·46	2·76	2·65	2·65	1·67	1·07
f)	4	15	11·3	21·4	5·37	5·50	8·81	4·38	4·51

Table 4

	Terms taken			Max^{n} Residuals R_N ($\times 10^{-3}$)			Max^{n} Residuals R_T ($\times 10^{-3}$)			E(rms) ($\times 10^{-3}$)
	First	Second series								
	k	log	k	x = 4	y = 2	Notch	x = 4	y = 2	Notch	
a)	0-6	No	1-8	2·59	5·03	7·36	0·86	6·31	8·74	2·58
b)	0-6	Yes	1-7	4·06	2·19	2·67	2·84	6·69	5·94	1·66
c)	0-6	No	2-9	2·40	3·80	2·83	0·87	4·42	4·92	1·47

Stress Distribution $\widetilde{xx}$

(in units of T).

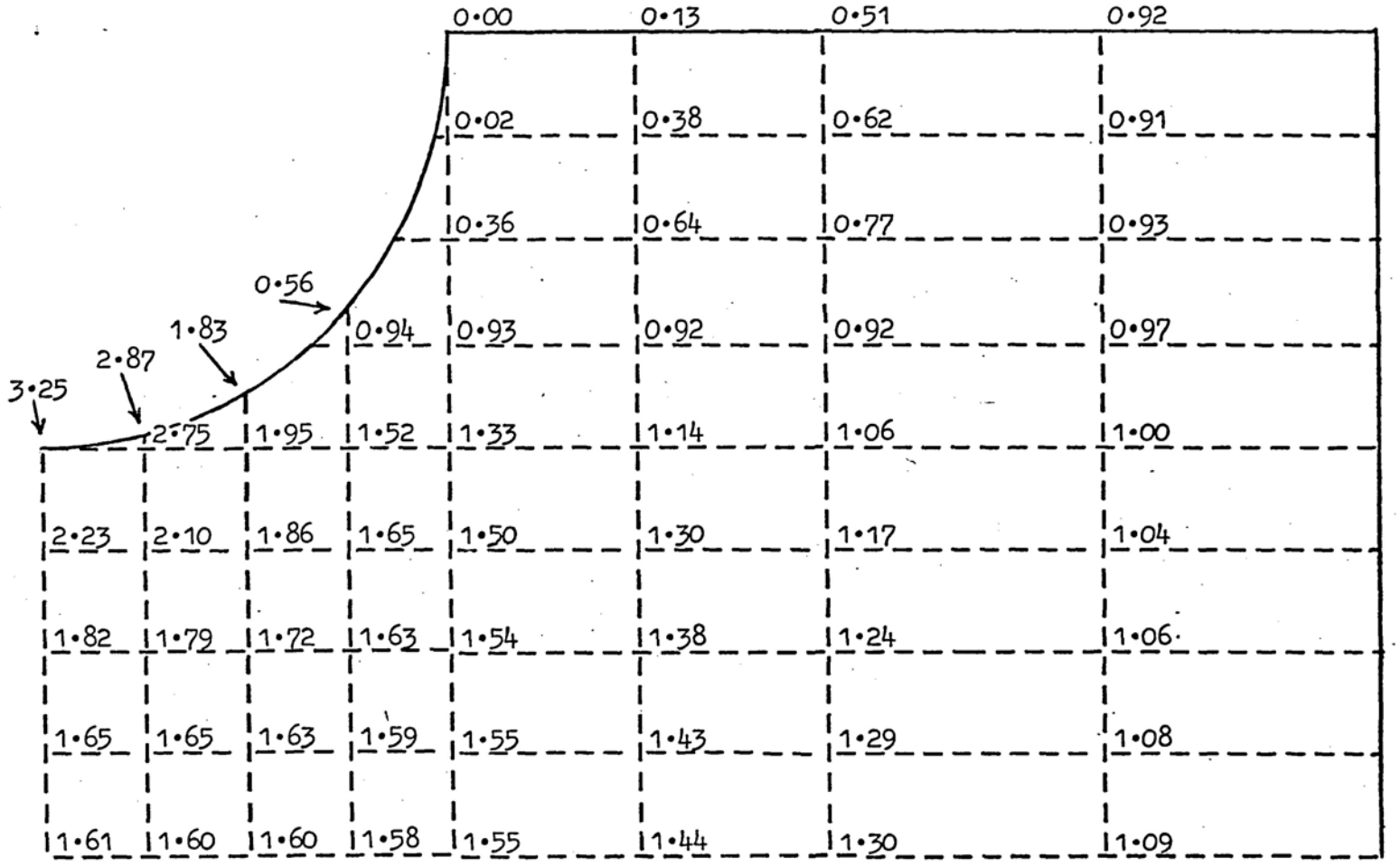


Figure 5

Stress Distributions $\widetilde{yy}$ and $\widetilde{xy}$

(in units of T)

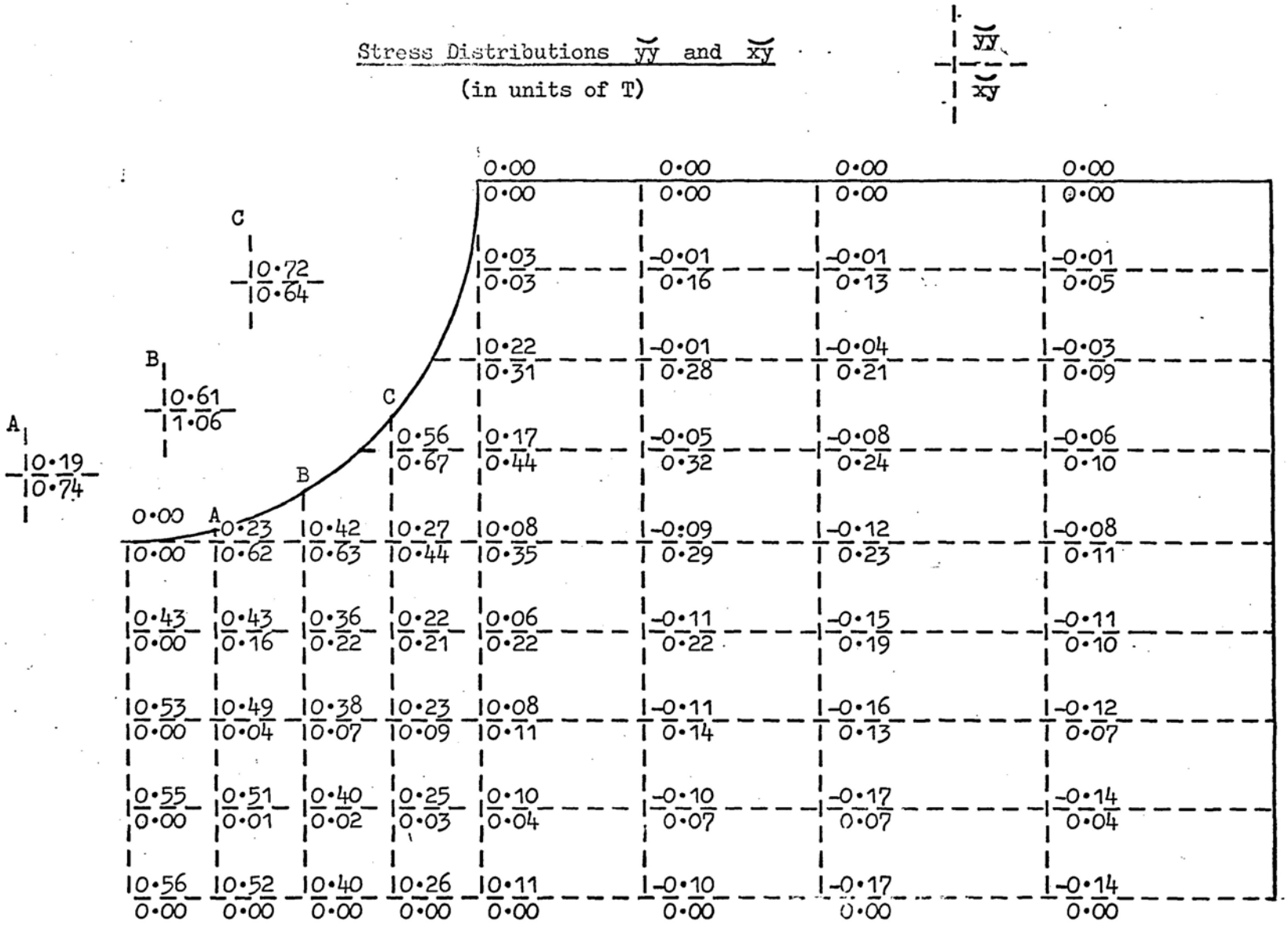


Figure 6.

CHAPTER 3

Multiply Connected and Infinite Regions

In this Chapter, using suitably chosen approximating functions, the method of Chapter 1 is applied to the determination of stresses in plates having one or more holes. In the first of the two problems considered, the boundary displacements are prescribed while in the second, the boundary stresses are prescribed.

The forms taken by the complex potentials in multiply connected and infinite regions have been given by Muskhelishvili (M. pp 120 - 128). A brief account of their derivation is given first.

§ 1. Finite multiply connected regions

Let the region S occupied by the body be bounded by several simple closed contours I_0, I_1, I_2, ..., I_n, the contour I_0 enclosing all the others.

The components of stress and displacement must be single valued throughout S, but the complex potentials $\varphi(z)$ and $\psi(z)$ although regular in any simply connected part of S, need not be single valued. For example, the formulae (1.18):

$$\widetilde{xx} + \widetilde{yy} = 4 \operatorname{Re} \Phi(z) ,$$

$$\widetilde{yy} - \widetilde{xx} + 2i\widetilde{xy} = 2[\bar{z} \Phi'(z) + \Psi(z)] ,$$

show that $\Psi(z)$ is regular in S but allow $\Phi(z)$ to be of the form

$$\Phi(z) = \sum_{j=1}^{n} A_j \log(z - z_j) + \Phi^*(z) ,$$

where the constants A_j are real, the points z_1, z_2, ..., z_n are arbitrary but inside I_1, I_2, ..., I_n respectively and $\Phi^*(z)$ is regular in S.

Allowing that $\phi^*(z)$ and $\Upsilon(z)$ may have poles within L_1, L_2, ..., L_n, integrating gives

$$\varphi(z) = z \sum_{j=1}^{n} A_j \log(z - z_j) + \sum_{j=1}^{n} \gamma_j \log(z - z_j) + \varphi^*(z) ,$$

$$\psi(z) = \sum_{j=1}^{n} \gamma_j' \log(z - z_j) + \psi^*(z) ,$$

where the constants γ_j, γ_j' are in general complex and $\varphi^*(z)$, $\psi^*(z)$ are regular in S.

However, substituting in the formula (1.19):

$$2\mu (u + iv) = \kappa \varphi(z) - z \overline{\varphi'(z)} - \overline{\psi(z)} ,$$

it is found that the displacements are not single valued unless

$$A_j = 0 \quad \text{and} \quad \kappa \gamma_j + \gamma_j' = 0 , \quad j = 1,2,\ldots,m .$$

Finally, Muskhelishvili shows that the constants γ_j, γ_j' can be expressed in terms of the components X_j, Y_j of the resultant of the external forces applied to the contour L_j, giving

$$\left.\begin{aligned} \varphi(z) &= -\frac{1}{2\pi(\kappa+1)} \sum_{j=1}^{n} (X_j + iY_j) \log(z - z_j) + \varphi^*(z) \\ \psi(z) &= \frac{\kappa}{2\pi(\kappa+1)} \sum_{j=1}^{n} (X_j - iY_j) \log(z - z_j) + \psi^*(z) \end{aligned}\right\} \qquad (3.1)$$

§ 2. Infinite regions

If the contour L_0 is moved entirely to infinity there is some simplification. Let L_R be a circle of radius R sufficiently large for all the contours L_j to lie inside L_R. Then writing $\log(z - z_j) = \log z + \log(1 - z_j/z)$ and expanding gives

$$\varphi(z) = -\frac{X + iY}{2\pi(\kappa+1)} \log z + \varphi^{**}(z) ,$$

$$\psi(z) = \frac{\kappa(X - iY)}{2\pi(\kappa+1)} \log z + \psi^{**}(z) ,$$

where $X = \sum_{j=1}^{n} X_j$, $Y = \sum_{j=1}^{n} Y_j$, and $\varphi^{**}(z)$, $\psi^{**}(z)$ are regular outside L_R except possibly at infinity, i.e. they possess Laurent

expansions

$$\varphi^{**}(z) = \sum_{-\infty}^{\infty} c_n z^n , \quad \psi^{**}(z) = \sum_{-\infty}^{\infty} c_n' z^n$$

converging uniformly in every finite region outside L_R.

If the further condition is imposed that the components of stress are to be bounded everywhere, including infinity, then for $n \geq 2$, $c_n = c_n' = 0$, leaving

$$\left.\begin{aligned} \varphi(z) &= -\frac{X + iY}{2\pi(\kappa+1)} \log z + \Gamma z + \varphi^0(z) \\ \psi(z) &= \frac{\kappa(X - iY)}{2\pi(\kappa+1)} \log z + \Gamma' z + \psi^0(z), \end{aligned}\right\} \tag{3.2}$$

where $\Gamma = B + iC$ and $\Gamma' = B' + iC'$ are constants and

$$\varphi^0(z) \equiv \sum_{n=0}^{\infty} a_n/z^n , \quad \psi^0(z) \equiv \sum_{n=0}^{\infty} a_n'/z^n$$

(say) are regular outside L_R, including the point at infinity.

The constants B, B' and C' have a simple physical significance. If N_1, N_2 are the principal stresses at infinity and β the angle made by the direction of N_1 with the x- axis,

$$\left.\begin{aligned} B &= \tfrac{1}{4}(N_1 + N_2) , \\ \text{and} \qquad B' + iC' &= -\tfrac{1}{2}(N_1 - N_2)e^{-2i\beta} . \end{aligned}\right\} \tag{3.3}$$

The constant C does not affect the stresses. It is related to the rotation of an infinitely remote part of the plane.

§ 3. Approximation of the complex potentials in multiply connected and infinite regions

In addition to the theorem on polynomial approximation quoted in § 4 of Chapter 1, Walsh (1956) has proved that if the function f(z) is regular in the region S bounded by a finite number of Jordan arcs $L_1, L_2, \ldots, L_n$, and continuous in $\overline{S}$ (the closure of S), then f(z)

can be uniformly approximated in $\overline{S}$ by a rational function of z. Moreover, if points $z_1, z_2, \ldots, z_m$ are chosen, separated respectively from the interior of S by the curves $L_1, L_2, \ldots, L_m$, this rational function can be chosen to have all its poles in these points z_j.

Thus immediately, the functions $\varphi^*(z)$ and $\psi^*(z)$ may be approximated by series of the form

$$\varphi_n^*(z) = \sum_{k=0}^{n} a_{0k} z^k + \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} (z - z_j)^{-k} ,$$

$$\psi_n^*(z) = \sum_{k=0}^{n} b_{0k} z^k + \sum_{j=1}^{n} \sum_{k=1}^{n} b_{jk} (z - z_j)^{-k} ,$$

while for the functions $\varphi^0(z)$, $\psi^0(z)$, the ascending powers are omitted.

Mikhlin (1949) has shown that when the boundary displacements or boundary stresses are given, the coefficients a_{jk}, b_{jk} may be determined by the method of least squares and has established the uniform convergence of $\varphi_n^*(z)$ and $\psi_n^*(z)$ to $\varphi^*(z)$ and $\psi^*(z)$ respectively when $n \to \infty$.

§ 4. Infinite plate with a circular hole, the boundary having prescribed displacements

Three problems were considered together. In each, the plate is free from stresses at infinity and the displacements on the circle are purely radial, the components v_r, v_θ in polar co-ordinates being

$$\text{(i)} \quad v_r = \varepsilon|\cos\theta| , \quad v_\theta = 0$$

$$\text{(ii)} \quad v_r = \varepsilon|\cos\theta|^2 , \quad v_\theta = 0$$

$$\text{(iii)} \quad v_r = \varepsilon|\cos\theta|^3 , \quad v_\theta = 0 .$$

It is required to find the stresses, in particular those at points on the boundary of the hole.

Since there are no stresses at infinity, the constants Γ, Γ' in (3.2) are zero. Also, because the displacements are symmetrical about Ox and Oy it follows that the external forces which will need to be applied to produce them will have zero resultant so that $X = Y = 0$. As a further consequence of symmetry, the complex potentials satisfy the conditions

$$\varphi(\bar{z}) = \varphi(z) , \qquad \psi(\bar{z}) = \psi(z) ,$$
$$\varphi(-z) = -\varphi(z) , \qquad \psi(-z) = -\psi(z) ,$$

and so may be uniformly approximated by polynomials in $1/z$ containing odd powers only with real coefficients:

$$\varphi_{2n-1}(z) = \sum_{k=1}^{n} \alpha_k / z^{2k-1} , \quad \psi_{2n-1}(z) = \sum_{k=1}^{n} \beta_k / z^{2k-1} ,$$

or with a change of notation,

$$\varphi_{2n-1}(z) = \sum_{r=1}^{n} a_{2r} / z^{2r-1} , \quad \psi_{2n-1}(z) = \sum_{r=1}^{n} a_{2r-1} / z^{2r-1} .$$

Writing $z\bar{z} = \rho^2$

and $1/z^{2r-1} = U_{2r-1} + iV_{2r-1}$,

one has $2\mu(\bar{u} + i\bar{v}) \equiv \kappa \varphi_{2n-1}(z) - z\bar{\varphi}'_{2n-1}(\bar{z}) - \bar{\psi}_{2n-1}(\bar{z})$

$$= \kappa \sum_{r=1}^{n} a_{2r} (U_{2r-1} + iV_{2r-1}) + \rho^2 \sum_{r=1}^{n} (2r-1)\, a_{2r} (U_{2r+1} - iV_{2r+1})$$
$$- \sum_{r=1}^{n} a_{2r-1} (U_{2r-1} - iV_{2r-1})$$
$$= \sum_{j=1}^{n} a_j (F_j + iG_j) ,$$

where $F_{2r-1} = -U_{2r-1}$, $G_{2r-1} = V_{2r-1}$,

$$F_{2r} = \kappa U_{2r-1} + (2r-1)\rho^2 U_{2r+1} , \quad G_{2r} = \kappa V_{2r-1} + (2r-1)\rho^2 V_{2r+1} .$$

The coefficients a_j may thus be determined in the usual way.

In the program used, the residual displacements at boundary points:

$$R_u(P_i) = 2\mu [u(P_i) - \bar{u}(P_i)] ,$$
$$R_v(P_i) = 2\mu [v(P_i) - \bar{v}(P_i)] ,$$

and their r.m.s values

$$R_u(\text{rms}) = \{\sum_{i=1}^{N} w_i \, R_u^2(P_i)/\sum_{i=1}^{N} w_i\}^{\frac{1}{2}} ,$$

$$R_v(\text{rms}) = \{\sum_{i=1}^{N} w_i \, R_v^2(P_i)/\sum_{i=1}^{N} w_i\}^{\frac{1}{2}} ,$$

were printed out, together with the coefficients

$$A_k = -(2k-1)\,\alpha_k \ , \quad B_k = -(2k-1)\,\beta_k \ ,$$

in the series $\Phi_{2n}(z) = \sum_{k=1}^{n} A_k/z^{2k}$, $\Psi_{2n}(z) = \sum_{k=1}^{n} B_k/z^{2k}$

where $\Phi_{2n}(z) \equiv \varphi'_{2n-1}(z)$, $\Psi_{2n}(z) \equiv \psi'_{2n-1}(z)$.

Using these coefficients, the stress combinations $\breve{xx} + \breve{yy}$, $\breve{yy} - \breve{xx} + 2i\breve{xy}$ can be found as before at any chosen points. From these, the principal stresses P, Q and the angle α made by the direction of P with Ox may be determined.

For if On, Os are new axes obtained by rotating Ox, Oy through an angle α , (Figure 1)

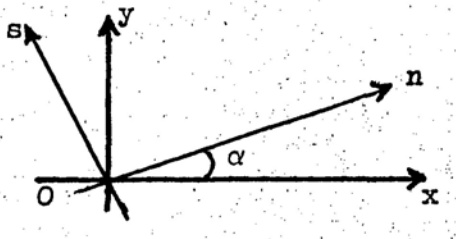


Figure 1

the new stress components are given by

$$\breve{nn} = \breve{xx}\cos^2\alpha + \breve{yy}\sin^2\alpha + 2\breve{xy}\sin\alpha\cos\alpha \ ,$$

$$\breve{ss} = \breve{xx}\sin^2\alpha + \breve{yy}\cos^2\alpha - 2\breve{xy}\sin\alpha\cos\alpha \ ,$$

$$\breve{ns} = -\breve{xx}\sin\alpha\cos\alpha + \breve{yy}\sin\alpha\cos\alpha + \breve{xy}(\cos^2\alpha - \sin^2\alpha) \ .$$

If On, Os are principal directions, the shear stress $\breve{ns} = 0$. This gives

$$\tan 2\alpha = \frac{2\breve{xy}}{\breve{xx}-\breve{yy}} \qquad (3.4)$$

and if the angle α is chosen so that $\cos 2\alpha$ has the same sign as

$\breve{xx} - \breve{yy}$ and $\sin 2\alpha$ has the same sign as $2\breve{xy}$, it is then easily shown that

$$\left.\begin{aligned} P = \breve{nn} &= \tfrac{1}{2}[\breve{xx} + \breve{yy} + \{(\breve{yy} - \breve{xx})^2 + (2\breve{xy})^2\}^{\frac{1}{2}}] \\ Q = \breve{ss} &= \tfrac{1}{2}[\breve{xx} + \breve{yy} - \{(\breve{yy} - \breve{xx})^2 + (2\breve{xy})^2\}^{\frac{1}{2}}], \end{aligned}\right\} \qquad (3.5)$$

where the positive square root is to be taken. The maximum shear stress at the point is $\frac{1}{2}(P - Q)$.

The boundary was taken to be the unit circle $|z| = 1$. Because of symmetry, only one quarter of the boundary was used, there being 45 equally spaced points P_t at $\theta = 1^\circ(2^\circ)89^\circ$ with weights $w_t = 1$. The principal stresses and the angle α were calculated at the boundary points $\theta = 0^\circ(1^\circ)90^\circ$. For convenience, ϵ was taken to be $1/2\mu$ and the modified Poisson's ratio η was made $0{\cdot}25$ giving $\varkappa = 2{\cdot}0$.

Initially, 10 terms were taken in each of the series so that $\varphi(z)$, $\psi(z)$ are approximated by terms up to and including $1/z^{19}$ and $\Phi(z)$, $\Psi(z)$ are approximated by terms up to $1/z^{30}$. Coefficients A_k, B_k obtained in this way are given in Table 1 for the three cases.

The coefficients in case (ii) agree apart from inevitable rounding errors with those of the analytical solution

$$\Phi(z) = -\frac{1}{4\mu z^2}, \quad \Psi(z) = \frac{1}{2z^3} + \frac{3(\varkappa-1)}{4\mu z^4},$$

which will be established in § 6.

The behaviour of the coefficients B_k in case (i) clearly indicates that the calculated stresses will not be reliable and repeating the computation with 15 terms and 22 terms in each of the series confirmed this. The stresses calculated at some representative points are given in Table 2 and are compared with the values obtained from the analytical solution

$$\Phi(z) = \frac{1}{\varkappa\pi i}\left[z \log \frac{z+i}{z-i} - 2i\right],$$

$$\Psi(z) = \frac{1}{\pi i}\left[\frac{1}{z^3} \log \frac{z+i}{z-i} + \frac{2i}{z^2}\right] - \frac{2}{\pi\mu} \frac{1}{z^2(z^2+1)},$$

which also will be established in § 6 . The difficulty in this case is due to the stresses tending (slowly) to infinity as $\theta \to 90^{\circ}$.

By contrast, in case (iii), increasing the number of terms to 15 produced changes of one unit only in the third place in 36 of the calculated stresses and increasing to 22 produced changes of one unit in 10 of the calculated stresses. Final values are shown in Table 3. They have been checked by re-running the program with boundary points at $\theta = 0^{\circ}(2^{\circ})90^{\circ}$.

Table 1

Coefficients A_k and B_k

Case (i)

k	A_k	B_k
1	-0·106087	0·636652
2	0·063613	0·318261
3	-0·045392	0·106022
4	0·035254	-0·190645
5	-0·028790	0·226632
6	0·024305	-0·246317
7	-0·021006	0·258515
8	0·018475	-0·266616
9	-0·016469	0·272198
10	0·001707	-0·276094

Case (ii)

k	A_k	B_k
1	-0·125000	0·500000
2	0·000000	0·375000

The remaining A_k, B_k were all very nearly zero, the numerically largest being $B_9 = -1{\cdot}8 \times 10^{-9}$.

Case (iii)

k	A_k	B_k
1	-0·127324	0·424413
2	-0·054567	0·381972
3	0·010105	-0·090946
4	-0·003858	0·042441
5	0·001908	-0·024803
6	-0·001088	0·016323
7	0·000681	-0·011574
8	-0·000455	0·008641
9	0·000319	-0·006700
10	-0·000027	0·005349

Table 2

Case (i) $2\mu v_r = |\cos\theta|$, $2\mu v_\theta = 0$.

	Computed P			
θ	n=10	n=15	n=22	True Value
0°	0·895	0·956	0·916	0·932
15°	0·904	0·921	0·911	0·917
30°	0·902	0·848	0·885	0·873
45°	0·840	0·789	0·818	0·802
60°	0·708	0·741	0·711	0·714
75°	0·570	0·688	0·614	0·648
80°	0·712	0·594	0·700	0·667
85°	0·970	0·931	0·797	0·796
90°	1·082	1·266	1·432	∞

	Computed Q			
θ	n=10	n=15	n=22	True Value
0°	-1·192	-1·212	-1·201	-1·205
15°	-1·173	-1·179	-1·178	-1·180
30°	-1·107	-1·098	-1·105	-1·105
45°	-0·984	-0·978	-0·976	-0·972
60°	-0·784	-0·769	-0·770	-0·761
75°	-0·449	-0·384	-0·429	-0·415
80°	-0·278	-0·274	-0·208	-0·239
85°	0·116	0·038	-0·038	0·004
90°	0·363	0·425	0·482	∞

	Computed α (degrees)			
θ	n=10	n=15	n=22	True Value
0°	-90·0	-90·0	-90·0	-90
15°	-69·3	-69·4	-69·6	-69·9
30°	-49·0	-49·5	-49·3	-49·6
45°	-29·0	-29·4	-28·9	-28·8
60°	8·0	-7·1	-7·4	-7·0
75°	17·3	20·2	18·3	18·9
80°	29·6	29·2	32·0	30·8
85°	53·2	49·3	46·4	48·3
90°	90·0	90·0	90·0	90

The numerically greatest residuals

$$R_u(\max) = \max_i |R_u(P_i)| \ , \quad R_v(\max) = \max_i |R_v(P_i)| \ ,$$

and the root mean squares $R_u(\text{rms})$, $R_v(\text{rms})$ may be compared for cases (i) and (iii) in Table 4.

Table 3

Case (iii) $2\mu v_r = |\cos\theta|^3$, $2\mu v_\theta = 0$.

θ	P	Q	α(degrees)
0°	0·826	-1·523	-90·0
5°	0·828	-1·508	-81·4
10°	0·835	-1·463	-72·7
15°	0·844	-1·391	-64·1
20°	0·855	-1·294	-55·5
25°	0·864	-1·177	-46·9
30°	0·868	-1·045	-38·3
35°	0·864	-0·904	-29·6
40°	0·850	-0·759	-21·0
45°	0·822	-0·616	-12·4
50°	0·780	-0·480	-3·8
55°	0·723	-0·356	4·8
60°	0·651	-0·246	13·5
65°	0·566	-0·153	22·2
70°	0·472	-0·079	31·1
75°	0·374	-0·022	40·5
80°	0·279	0·019	51·1
85°	0·199	0·045	65·7
90°	0·160	0·053	90·0

Table 4

Case (i) $2\mu v_r = |\cos\theta|$, $2\mu v_\theta = 0$

n	R_u (max)	R_v (max)	R_u (rms)	R_v (rms)
10	$2{\cdot}73 \times 10^{-3}$	$16{\cdot}7 \times 10^{-3}$	$1{\cdot}43 \times 10^{-3}$	$3{\cdot}54 \times 10^{-3}$
15	1·24	7·56	0·63	1·82
22	0·50	4·12	0·28	0·85

Case (iii) $2\mu v_r = |\cos\theta|^3$, $2\mu v_\theta = 0$

n	R_u (max)	R_v (max)	R_u (rms)	R_v (rms)
10	$4{\cdot}03 \times 10^{-5}$	$13{\cdot}4 \times 10^{-5}$	$2{\cdot}09 \times 10^{-5}$	$3{\cdot}48 \times 10^{-5}$
15	0·82	3·28	0·43	0·87
22	0·17	1·00	0·09	0·23

R_u (max) always occurred in the range 67° - 75° while R_v (max) occurred at 87° or 89° .

§ 5. Application of Lanczos' " σ - factors "

It is observed by Lanczos (1957) that if $f(x)$ has the Fourier expansion

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \, e^{ikx} \quad , \; -\pi \le x \le \pi \; ,$$

there is a serious loss of accuracy if one attempts to approximate the derivative $f'(x)$ by differentiating the truncated series

$$f_n(x) \equiv \sum_{k=-(n-1)}^{n-1} c_k \, e^{ikx} \; .$$

The reason is that the difference

$$\begin{aligned} \eta_n(x) &\equiv f(x) - f_n(x) \\ &= e^{inx} \sum_{k=0}^{\infty} c_{n+k} \cdot e^{ikx} + e^{-inx} \sum_{k=0}^{\infty} c_{-n-k} \, e^{-ikx} \; . \\ &= e^{inx} \, \rho_n(x) + e^{-inx} \, \rho_{-n}(x) \qquad \text{(say)} \end{aligned}$$

gives rise to terms $\pm in e^{\pm inx} \rho_{\pm n}(x)$ when differentiated.

To overcome the difficulty, Lanczos replaces the process of differentiation by one of differencing, taking

$$\mathfrak{D}_n \, f(x) \equiv \frac{f(x + \pi/m) - f(x - \pi/m)}{2\pi/m} \; .$$

Assuming $f(x)$ possesses the necessary continuous derivatives, Taylor's theorem gives

$$\mathfrak{D}_n \, f(x) = f'(x) + \tfrac{1}{6} \left(\tfrac{\pi}{m}\right)^2 f'''(x + \theta\pi/m) \; ,$$

where $-1 < \theta < 1$.

On the other hand,

$$\mathfrak{D}_m \, \eta_n(x) = - e^{inx} \mathfrak{D}_n \rho_n(x) - e^{-inx} \mathfrak{D}_n \rho_{-n}(x)$$

and the troublesome factor n is avoided.

In place of the terms ike^{ikx} in $f_n'(x)$, one now has

$$\mathfrak{D}_n\, e^{ikx} = \frac{e^{ik(x + \pi/m)} - e^{ik(x - \pi/m)}}{2\pi/m}$$

$$= i\,\sigma_k\, e^{ikx}$$

where $\sigma_k = \dfrac{\sin k\pi/m}{\pi/m}$, $\qquad k = 0, \pm1, \pm2, \ldots \pm(m-1)$.

[For convenience, Lanczos' notation has been altered at this point]

This device can obviously be extended to series of powers of the form

$$f_n(z) = \sum_{k=-(n-1)}^{n-1} c_k\, z^k .$$

For if $z = re^{i\theta}$ and $f(z)$ is regular,

$$\frac{\partial}{\partial\theta} f(z) = f'(z)\, ire^{i\theta}$$

so that $$f'(z) = \frac{1}{iz}\,\frac{\partial}{\partial\theta}\, f(re^{i\theta}) .$$

Hence it is natural to define

$$\mathfrak{D}_n\, f(z) = \frac{1}{iz}\,\frac{f(re^{i\overline{\theta + \pi/m}}) - f(re^{i\overline{\theta - \pi/m}})}{2\pi/m}$$

$$= \frac{f(ze^{i\pi/m}) - f(ze^{-i\pi/m})}{2\pi iz/m} . \qquad (3.6)$$

In particular, $\mathfrak{D}_n\, z^k = \dfrac{\sin k\pi/m}{\pi/m}\, z^{k-1} = \sigma_k\, z^{k-1}$.

The form of the quotient (3.6) suggests that a more natural definition would be

$$\mathfrak{D}_n\, f(z) = \frac{f(ze^{i\pi/m}) - f(ze^{-i\pi/m})}{ze^{i\pi/m} - ze^{-i\pi/m}}$$

$$= \frac{f(ze^{i\pi/m}) - f(ze^{-i\pi/m})}{2iz\,\sin(\pi/m)}$$

and it may again be shown that the difference between $\mathfrak{D}_n f(z)$ and $f'(z)$ is $O(\pi^2/m^2)$.

With this definition, $\mathfrak{D}_n\, z^k = \sigma_k\, z^{k-1}$ where now $\sigma_k = \dfrac{\sin(k\pi/m)}{\sin(\pi/m)}$.

With σ - factors of this form used in the program for the calculation of $\varphi'_{2n-1}(z)$, $\psi'_{2n-1}(z)$ and $\Psi'_{2n}(z)$, the stresses P,Q and angles α given in Tables 5, 6 and 7 respectively were obtained. These should be compared with Table 2. It is clear that this device is useful and gives remarkably good results for $0^\circ \leq \theta \leq 80^\circ$ but cannot be expected to deal with the stress singularity at $\theta = 90^\circ$.

Table 5

Case (i), using Lanczos' σ - factors

θ	Computed P n=10	n=15	n=22	True Value
0°	0·923	0·927	0·930	0·932
5°	0·920	0·926	0·928	0·930
10°	0·914	0·920	0·923	0·925
15°	0·908	0·912	0·915	0·917
20°	0·896	0·901	0·903	0·905
25°	0·880	0·886	0·889	0·891
30°	0·863	0·869	0·871	0·873
35°	0·843	0·847	0·850	0·852
40°	0·819	0·824	0·826	0·828
45°	0·792	0·798	0·800	0·802
50°	0·765	0·769	0·772	0·774
55°	0·737	0·740	0·742	0·744
60°	0·706	0·710	0·712	0·714
65°	0·675	0·681	0·683	0·685
70°	0·657	0·659	0·658	0·660
75°	0·687	0·642	0·649	0·648
80°	0·775	0·717	0·663	0·667
82°	0·815	0·796	0·715	0·696
85°	0·860	0·942	0·924	0·796
87°	0·875	1·024	1·109	0·954
88°	0·878	1·053	1·186	1·105
89°	0·879	1·070	1·237	1·394
90°	0·879	1·076	1·255	∞

Table 6

Case (i), using Lanczos' σ - factors

θ	Computed Q n=10	n=15	n=22	True Value
0°	-1·197	-1·201	-1·203	-1·205
5°	-1·194	-1·198	-1·200	-1·202
10°	-1·186	-1·191	-1·192	-1·194
15°	-1·172	-1·176	-1·179	-1·180
20°	-1·154	-1·158	-1·160	-1·161
25°	-1·129	-1·132	-1·34	-1·136
30°	-1·095	-1·101	-1·103	-1·105
35°	-1·059	-1·064	-1·066	-1·068
40°	-1·017	-1·019	-1·021	-1·023
45°	-0·961	-0·969	-0·969	-0·972
50°	-0·900	-0·906	-0·910	-0·912
55°	-0·836	-0·839	-0·839	-0·842
60°	-0·751	-0·754	-0·759	-0·761
65°	-0·643	-0·661	-0·664	-0·666
70°	-0·544	-0·545	-0·546	-0·553
75°	-0·431	-0·396	-0·413	-0·415
80°	-0·204	-0·255	-0·222	-0·239
82°	-0·076	-0·155	-0·155	-0·153
85°	0·123	0·062	0·010	0·004
87°	0·233	0·209	0·184	0·141
88°	0·272	0·265	0·264	0·232
89°	0·296	0·302	0·319	0·368
90°	0·305	0·315	0·339	∞

Table 7

Case (i), using Lanczos' σ - factors

θ	Computed α (degrees) n=10	n=15	n=22	True Value
0°	-90·0	-90·0	-90·0	-90
5°	-83·3	-83·3	-83·3	-83·3
10°	-76·6	-76·6	-76·6	-76·6
15°	-69·9	-69·9	-69·9	-69·9
20°	-63·2	-63·2	-63·1	-63·1
25°	-56·4	-56·4	-56·4	-56·4
30°	-49·7	-49·6	-49·6	-49·6
35°	-42·9	-42·8	-42·8	-42·7
40°	-36·0	-35·9	-35·9	-35·8
45°	-29·0	-28·9	-28·9	-28·8
50°	-22·1	-21·9	-21·8	-21·7
55°	-14·9	-14·6	-14·6	-14·5
60°	-7·3	-7·1	-7·0	-7·0
65°	0·2	0·5	0·7	0·9
70°	7·7	9·1	9·2	9·3
75°	17·4	17·7	18·6	18·9
80°	32·1	28·9	29·4	30·8
82°	39·9	36·3	34·2	36·8
85°	54·5	51·9	47·9	48·3
87°	67·0	65·5	62·5	58·5
88°	74·2	73·2	71·1	64·7
89°	81·9	81·5	80·4	71·8
90°	90·0	90·0	90·0	90

§ 6. Analytical Solutions

The radial and transverse components of the displacement, v_r, v_θ are given by

$$2\mu(v_r + iv_\theta) = e^{-i\theta}\left[\kappa\,\varphi(z) - z\bar{\varphi}'(\bar{z}) - \bar{\psi}(\bar{z})\right] \tag{3.7}$$

Let γ denote the unit circle. Then on γ, $z = e^{i\theta} = \sigma$ (say), $\bar{z} = 1/\sigma$ and $v_\theta = 0$ in each of the problems. Hence

$$2\mu v_r \sigma = \kappa\,\varphi(\sigma) - \sigma\,\bar{\varphi}'(1/\sigma) - \bar{\psi}(1/\sigma)\ . \tag{3.8}$$

The functions $\bar{\psi}(1/\zeta)$, $\zeta\,\bar{\varphi}'(1/\zeta)$ are regular for $|\zeta| < 1$, and so if the point z lies outside γ, Cauchy's theorem gives

$$\frac{1}{2\pi i}\int_\gamma \frac{2\mu v_r\,\sigma\,d\sigma}{\sigma - z} = \varkappa\,\frac{1}{2\pi i}\int_\gamma \frac{\varphi(\sigma)\,d\sigma}{\sigma - z}$$

$$= \varkappa\,[\varphi(\infty) - \varphi(z)]$$

$$= -\varkappa\,\varphi(z)\ . \tag{3.9}$$

Similarly, taking the conjugate of (3.8) gives

$$2\mu v_r/\sigma = \varkappa\,\bar{\varphi}(1/\sigma) - (1/\sigma)\,\varphi'(\sigma) - \psi(\sigma)\ , \tag{3.10}$$

and since $\bar{\varphi}(1/\zeta)$ is regular for $|\zeta| < 1$, it follows that

$$\frac{1}{2\pi i}\int_\gamma \frac{2\mu v_r\,d\sigma}{\sigma(\sigma - z)} = \psi(z) + \frac{1}{z}\,\varphi'(z)\ . \tag{3.11}$$

In case (ii), $2\mu v_r = \cos^2\theta = \frac{1}{4}(\sigma + 1/\sigma)^2$ so that

$$\varkappa\,\varphi(z) = -\frac{1}{2\pi i}\int_\gamma \tfrac{1}{4}\,\frac{(\sigma^2+1)^2}{\sigma(\sigma - z)}\,d\sigma\ .$$

Calculating the residue at the simple pole $\sigma = 0$ gives

$$\varphi(z) = 1/(4\varkappa z) \quad \text{or} \quad \Phi(z) = -1/(4\varkappa z^2)\ .$$

$$\text{Also,}\quad \psi(z) + \frac{1}{z}\,\varphi'(z) = \frac{1}{2\pi i}\int_\gamma \tfrac{1}{4}\,\frac{(\sigma^2+1)^2\,d\sigma}{\sigma^3(\sigma - z)}\ .$$

The residue at the triple pole $\sigma = 0$ is $-1/(2z) - 1/(4z^3)$, giving

$$\psi(z) = -\tfrac{1}{2}\cdot\frac{1}{z} - \tfrac{1}{4}\,\frac{\varkappa - 1}{\varkappa}\cdot\frac{1}{z^3}\ ,$$

$$\text{or}\quad \Psi(z) = \tfrac{1}{2}\cdot\frac{1}{z^2} + \tfrac{3}{4}\,\frac{\varkappa - 1}{\varkappa}\cdot\frac{1}{z^4}\ .$$

In case (i), $2\mu v_r = |\cos\theta|$, so that if γ_1, γ_2 denote the parts of γ on which $R(\sigma) > 0$, $R(\sigma) < 0$ respectively,

$$\varkappa\,\varphi(z) = -\frac{1}{2\pi i}\int_{\gamma_1} \tfrac{1}{2}\left(\sigma + \frac{1}{\sigma}\right)\frac{\sigma\,d\sigma}{\sigma - z} + \frac{1}{2\pi i}\int_{\gamma_2} \tfrac{1}{2}\left(\sigma + \frac{1}{\sigma}\right)\frac{\sigma\,d\sigma}{\sigma - z}\ .$$

Both paths of integration can be deformed into the segment of the imaginary axis joining the points $\pm i$ giving

$$\kappa\,\varphi(z) = -\frac{1}{2\pi i}\int_{-1}^{1}\frac{(\sigma^2+1)\,d\sigma}{\sigma-z}$$

$$= -\frac{1}{2\pi i}\int_{-1}^{1}\{\sigma + z + \frac{z^2+1}{\sigma-z}\}\,d\sigma$$

$$= \frac{1}{2\pi i}[(z^2+1)\log\frac{z+i}{z-i} - 2iz]$$

and $\kappa\,\psi(z) = \frac{1}{\pi i}[z\log\frac{z+i}{z-i} - 2i]$.

Evaluation of the corresponding integral for $\psi(z)$ can be avoided by observing that on equating the values of $2\mu v_r$ given by (3.8) and (3.10) and rearranging one has

$$\kappa(1/\sigma)\,\varphi(\sigma) + \varphi'(\sigma) + \sigma\,\psi(\sigma) \equiv \kappa\,\sigma\,\bar{\varphi}(1/\sigma) + \bar{\varphi}'(1/\sigma) + (1/\sigma)\,\bar{\psi}(1/\sigma).$$

But the functions $\varphi(z)$, $\psi(z)$ are regular outside $|z| = 1$ and possess expansions

$$\varphi(z) = \sum_{n=1}^{\infty} a_n/z^n \quad , \quad \psi(z) = \sum_{n=1}^{\infty} b_n/z^n \quad ,$$

and consequently the identity can only be satisfied if

$$\kappa\,\varphi(\sigma)/\sigma + \varphi'(\sigma) + \sigma\,\psi(\sigma) - b_1 = 0 .$$

This implies that

$$\psi(z) = b_1/z - \varphi'(z)/z - \kappa\,\varphi(z)/z^2. \qquad (3.12)$$

The constant b_1 can be found by integrating both sides of (3.10) round γ giving

$$b_1 = -\frac{1}{2\pi i}\int_{\gamma} 2\mu v_r\,\frac{d\sigma}{\sigma} = -\frac{1}{2\pi}\int_{0}^{2\pi} 2\mu v_r\,d\theta = -\frac{2}{\pi} .$$

Equation (3.12) then gives

$$\psi(z) = -\frac{1}{\kappa\pi i}[\log\frac{z+i}{z-i} - \frac{2i}{z}] - \frac{1}{2\pi i}[(1+\frac{1}{z^2})\log\frac{z+i}{z-i} + \frac{2i}{z}]$$

and $\Psi(z) = \frac{1}{\pi i}[\frac{1}{z^2}\log\frac{z+i}{z-i} + \frac{2i}{z^2}] - \frac{2}{\pi\kappa}\,\frac{1}{z^2(z^2+1)}$.

§ 7. Infinite plate with two equal unstressed circular holes and uniform tension at infinity

For an application of the method to a multiply connected region, the problem taken was the determination of the stresses in an infinite plate containing two equal circular holes with stress free boundaries, there being uniform tension at infinity (Figure 2).

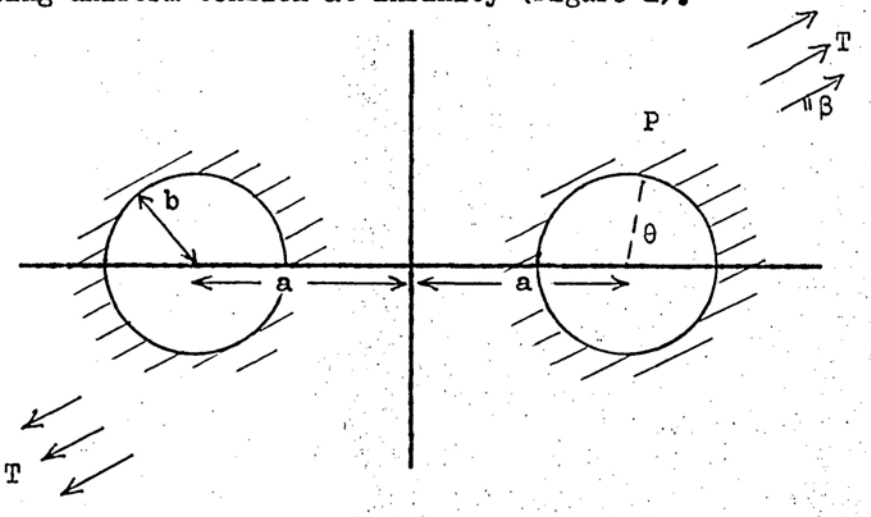


Figure 2

This problem has been solved analytically by Ling (1948) who expressed the Airy stress function as an infinite series of solutions of the biharmonic equation in bipolar co-ordinates ξ, η given by

$$x + iy = - a \coth \tfrac{1}{2}i (\xi + i\eta) .$$

Ling calculated the boundary hoop stress $\breve{\theta\theta}$ for several values of the ratio $\lambda = a/b$:

(i) at $\theta = \pm 90^{\circ}$ when the tension T is parallel to the line of centres;

(ii) at $\theta = 0^{\circ}$ and 180° when the tension T is perpendicular to the line of centres;

(iii) at $\theta = 0^{\circ}$ and 180° when there is "all round tension".

From (3.2), the complex potentials have the forms

$$\varphi(z) = \tfrac{1}{4}\ Tz + \varphi^{o}(z) ,$$

$$\psi(z) = -\tfrac{1}{2}\ Tze^{-2i\beta} + \psi^{o}(z) ,$$

where $\varphi^{o}(z)$ and $\psi^{o}(z)$ are regular everywhere outside the two circles including the point at infinity.

These potentials give rise to zero normal and tangential stresses over the boundary of each hole. Consequently, from the Kolosov formula (1.20) it follows that the potentials

$$\varphi^{o}(z) = \varphi(z) - \tfrac{1}{4}Tz ,$$

$$\psi^{o}(z) = \psi(z) + \tfrac{1}{2}Tze^{-2i\beta},$$

must produce normal and tangential stresses given by

$$N^{o} - iT^{o} = -\tfrac{1}{2}T - e^{2i\alpha} \cdot \tfrac{1}{2}Te^{-2i\beta} ,$$

α being the angle made by the inward normal with Ox .

Thus, dropping the superscript o , the problem is reduced to one of determining potentials $\varphi(z)$, $\psi(z)$ regular everywhere outside the two circles and corresponding to the boundary stresses N, T which in the two cases considered by Ling are given by

$$N - iT = -T\cos\alpha\ (\cos\alpha + i\sin\alpha) , \quad (\beta = 0)$$

$$= -T\sin\alpha\ (\sin\alpha - i\cos\alpha) . \quad (\beta = \pi/2)$$

By Walsh's theorem, the functions $\varphi(z)$, $\psi(z)$ can be uniformly approximated by rational functions of the form

$$\varphi_n(z) = \alpha_0 + \sum_{k=1}^{n} \alpha_k /(z-a)^k + \sum_{k=1}^{n} \alpha_k '/(z+a)^k ,$$

$$\psi_n(z) = \beta_0 + \sum_{k=1}^{n} \beta_k /(z-a)^k + \sum_{k=1}^{n} \beta_k '/(z+a)^k ,$$

and so the functions $\Phi(z) = \varphi'(z)$, $\Psi(z) = \psi'(z)$ can be uniformly approximated by the rational functions

$$\Phi_n(z) = \sum_{k=2}^{n+1} A_k/(z-a)^k + \sum_{k=3}^{n+1} A_k'/(z+a)^k ,$$

$$\Psi_n(z) = \sum_{k=2}^{n+1} B_k/(z-a)^k + \sum_{k=3}^{n+1} B_k'/(z+a)^k .$$

When $\beta = 0$ or $\pi/2$, the cases considered by Ling, there is four-fold symmetry. In these cases (the only cases treated here), the approximate potentials should be such that

$$\bar{\Phi}_n(z) = \Phi_n(z) , \quad \bar{\Psi}_n(z) = \Psi_n(z)$$

$$\Phi_n(-z) = \Phi_n(z) , \quad \Psi_n(-z) = \Psi_n(z) .$$

The first of these requires A_k, A'_k, B_k, B'_k to be real and the second that $A_k' = (-1)^k A_k$, $B_k' = (-1)^k B_k$.

Hence with a slight change of notation the approximating functions have the forms

$$\Phi_n(z) = \sum_{r=1}^{n} a_{2r-1} [1/(z-a)^{r+1} + (-1)^{r+1}/(z+a)^{r+1}]$$

$$\Psi_n(z) = \sum_{r=1}^{n} a_{2r} [1/(z-a)^{r+1} + (-1)^{r+1}/(z+a)^{r+1}]$$

Writing $1/(z-a)^{r+1} + (-1)^{r+1}/(z+a)^{r+1} \equiv U_r + iV_r$,

$$e^{2i\alpha} [1/(z-a)^{r+1} + (-1)^{r+1}/(z+a)^{r+1}] \equiv U_r^* + iV_r^* ,$$

$$\bar{z}e^{2i\alpha} [1/(z-a)^{r+2} + (-1)^{r+1}/(z+a)^{r+2}] \equiv U_r^{**} + iV_r^{**} ,$$

the Kolosov formula gives

$$\tilde{N} - i\tilde{T} \equiv \Phi_n(z) + \overline{\Phi_n(z)} - e^{2i\alpha} [\bar{z}\Phi_n'(z) + \Psi_n(z)]$$

$$= 2\sum_{r=1}^{n} a_{2r-1} U_r + \sum_{r=1}^{n} a_{2r-1} (r+1) (U_r^{**} + iV_r^{**})$$

$$- \sum_{r=1}^{n} a_{2r} (U_r^* + iV_r^*)$$

so that $$\tilde{N} + i\tilde{T} = \sum_{j=1}^{2n-1} a_j (F_j + iG_j) ,$$

where $F_{2r-1} = 2U_r + (r+1)U_r^{**}$, $G_{2r-1} = -(r+1)V_r^{**}$,

$F_{2r} = -U_r^*$, $G_{2r} = V_r^*$.

The coefficients a_j may be determined in the usual way.

Throughout the computations, the radius b of the two circles was taken to be unity as also was the tension T . Because of symmetry it is sufficient to consider the upper half of the right hand circular boundary only. 61 points P_i were taken corresponding to $\theta = 0^\circ(3^\circ)180^\circ$. Weights were unity except at the points $\theta = 0^\circ$, $\theta = 180^\circ$ where they were one half.

§ 8. Numerical results for $\lambda = 2$

For the case $\lambda = 2$, the coefficients A_k , B_k and the residuals $R_N(P_i)$, $R_T(P_i)$ (defined as before) were found using terms up to and including (i) $1/(z \pm a)^8$ (ii) $1/(z \pm a)^{10}$ (iii) $1/(z \pm a)^{12}$ in each of the functions $\Phi_n(z)$, $\Psi_n(z)$. These correspond to n = 7, 9 and 11 respectively. The numerically greatest residuals and the root mean square residuals are given below in Table 7. $R_N(\max)$ occurred each time at $\theta = 180^\circ$ and $R_T(\max)$ occurred at 168° or 171° irrespective of whether $\beta = 0$ or $\beta = \pi/2$.

Table 7

(i) $\beta = 0$ (Longitudinal tension)

	$R_N(\max)$	$R_T(\max)$	$R_N(\text{rms})$	$R_T(\text{rms})$
n=7	$76{\cdot}5 \times 10^{-5}$	131×10^{-5}	$45{\cdot}8 \times 10^{-5}$	$65{\cdot}2 \times 10^{-5}$
n=9	$7{\cdot}2 \times 10^{-5}$	$11{\cdot}1 \times 10^{-5}$	$4{\cdot}3 \times 10^{-5}$	$5{\cdot}6 \times 10^{-5}$
n=11	$0{\cdot}6 \times 10^{-5}$	$0{\cdot}8 \times 10^{-5}$	$0{\cdot}3 \times 10^{-5}$	$0{\cdot}4 \times 10^{-5}$

(ii) $\beta = \pi/2$ (Transverse tension)

	$R_N(\max)$	$R_T(\max)$	$R_N(\text{rms})$	$R_T(\text{rms})$
n=7	$58{\cdot}3 \times 10^{-5}$	$98{\cdot}3 \times 10^{-5}$	$33{\cdot}8 \times 10^{-5}$	$48{\cdot}4 \times 10^{-5}$
n=9	$6{\cdot}0 \times 10^{-5}$	$9{\cdot}2 \times 10^{-5}$	$3{\cdot}5 \times 10^{-5}$	$4{\cdot}7 \times 10^{-5}$
n=11	$0{\cdot}5 \times 10^{-5}$	$0{\cdot}7 \times 10^{-5}$	$0{\cdot}3 \times 10^{-5}$	$0{\cdot}4 \times 10^{-5}$

The stresses $\widetilde{xx}$, $\widetilde{yy}$ and $\widetilde{xy}$ were calculated at 89 points in the first quadrant in the neighbourhood of the hole. The points lay either on a grid with spacing 0·25b or else on the boundary of the hole at $\theta = 0^\circ(15^\circ)180^\circ$. Increasing the number of terms in each series from

7 to 9 produced changes in the third decimal place in 62 of the 267 stresses in the case $\beta = 0$ and in 42 of the 267 stresses in the case $\beta = \pi/2$. The changes were mostly of one unit. Taking 11 terms produced only 4 further alterations of one unit in the case $\beta = 0$ and 8 in the case $\beta = \pi/2$.

The coefficients A_k, B_k for n=11 are given below in Table 8 and stresses calculated from them are exhibited (to 2 decimal places) in Figures 3 and 4.

Table 8

(i) $\beta = 0$ (Longitudinal tension)

k	A_k	B_k
2	-0·430410	0·447734
3	-0·043798	-1·708787
4	0·019340	-1·558752
5	-0·007262	-0·018929
6	0·002475	0·023603
7	-0·000789	-0·013738
8	0·000239	0·006240
9	-0·000070	-0·002474
10	0·000020	0·000897
11	0·000000	-0·000304
12	0·000000	0·000211

(ii) $\beta = \pi/2$ (Transverse tension)

k	A_k	B_k
2	0·486638	0·560191
3	0·017575	1·931597
4	-0·010251	1·570933
5	0·004494	-0·013554
6	-0·001695	-0·005746
7	0·000582	0·006449
8	-0·000187	-0·003670
9	0·000057	0·001649
10	-0·000017	-0·000650
11	0·000000	0·000235
12	0·000000	-0·000181

When n=11, a lack of agreement between the root mean squares of the residuals calculated directly (E) and indirectly (E') became evident, the values being:

$$\beta = 0 \; ; \quad E(\text{rms}) = 3{\cdot}8 \times 10^{-6} \; , \; E'(\text{rms}) = 4{\cdot}8 \times 10^{-6} \; ,$$

$$\beta = \pi/2 \; ; \quad E(\text{rms}) = 3{\cdot}4 \times 10^{-6} \; , \; E'(\text{rms}) = 5{\cdot}0 \times 10^{-6} \; .$$

As explained in Chapter 1, if f is the vector of boundary values and δ the residual vector, then because δ is orthogonal to the vectors $\psi_1, \psi_2, \ldots, \psi_n$, the values $E \equiv (\delta,\delta)^{\frac{1}{2}}$ and $E' \equiv (f,\delta)^{\frac{1}{2}}$ should be equal. The ATLAS machine has a precision of 11-12 decimal digits when performing floating point calculations so that one could expect some of the calculated inner products (ψ_j, δ) to be of order 10^{-12} which is directly comparable with (δ,δ) here.

$\beta = 0$. Uniform Tension at Infinity parallel to Ox .

Stress distribution $\underset{\sim}{xx}$. $a/b = 2$

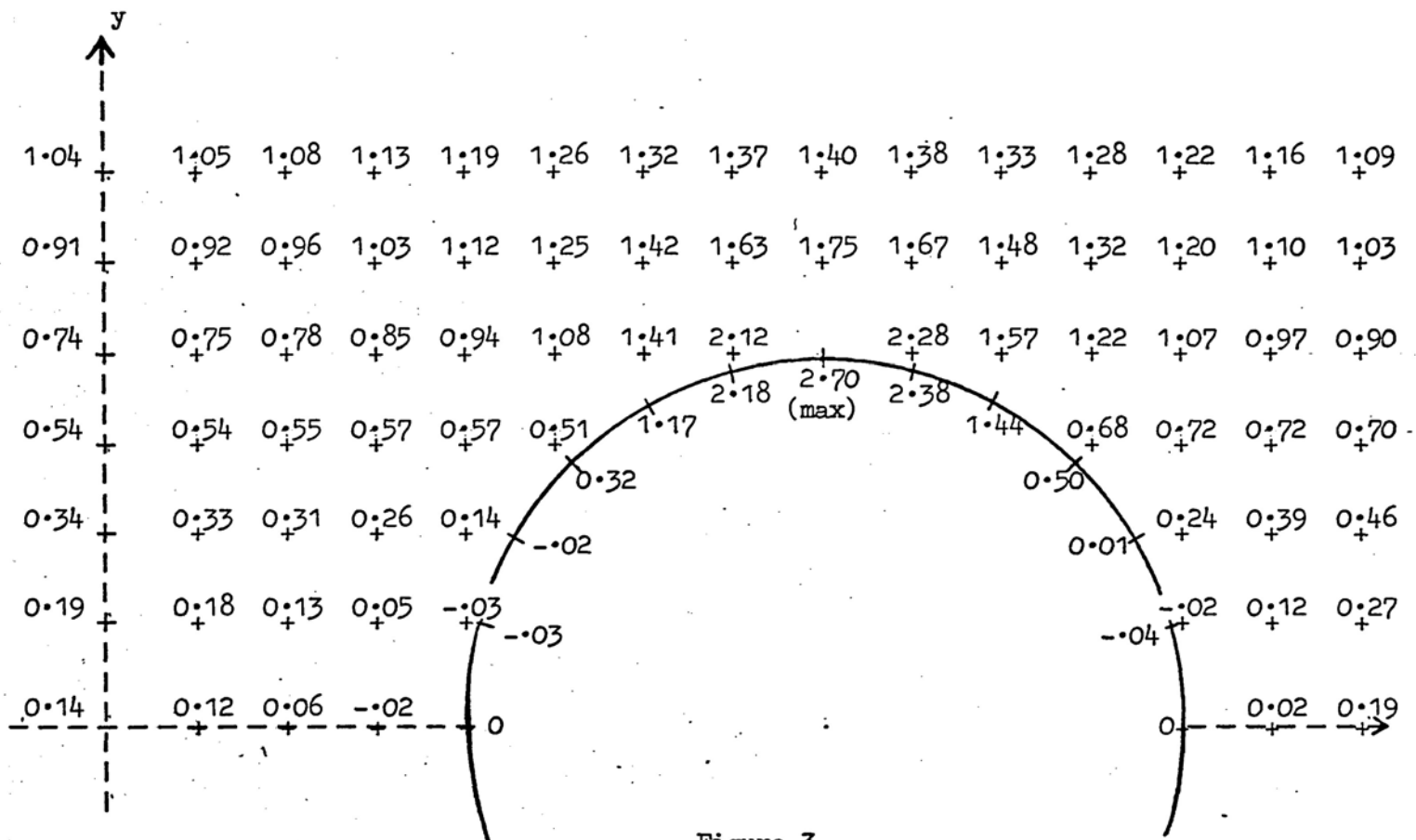


Figure 3

$\beta = \pi/2$. Uniform Tension at Infinity parallel to Oy .

Stress component $\breve{y}\,\breve{y}$. $a/b = 2$

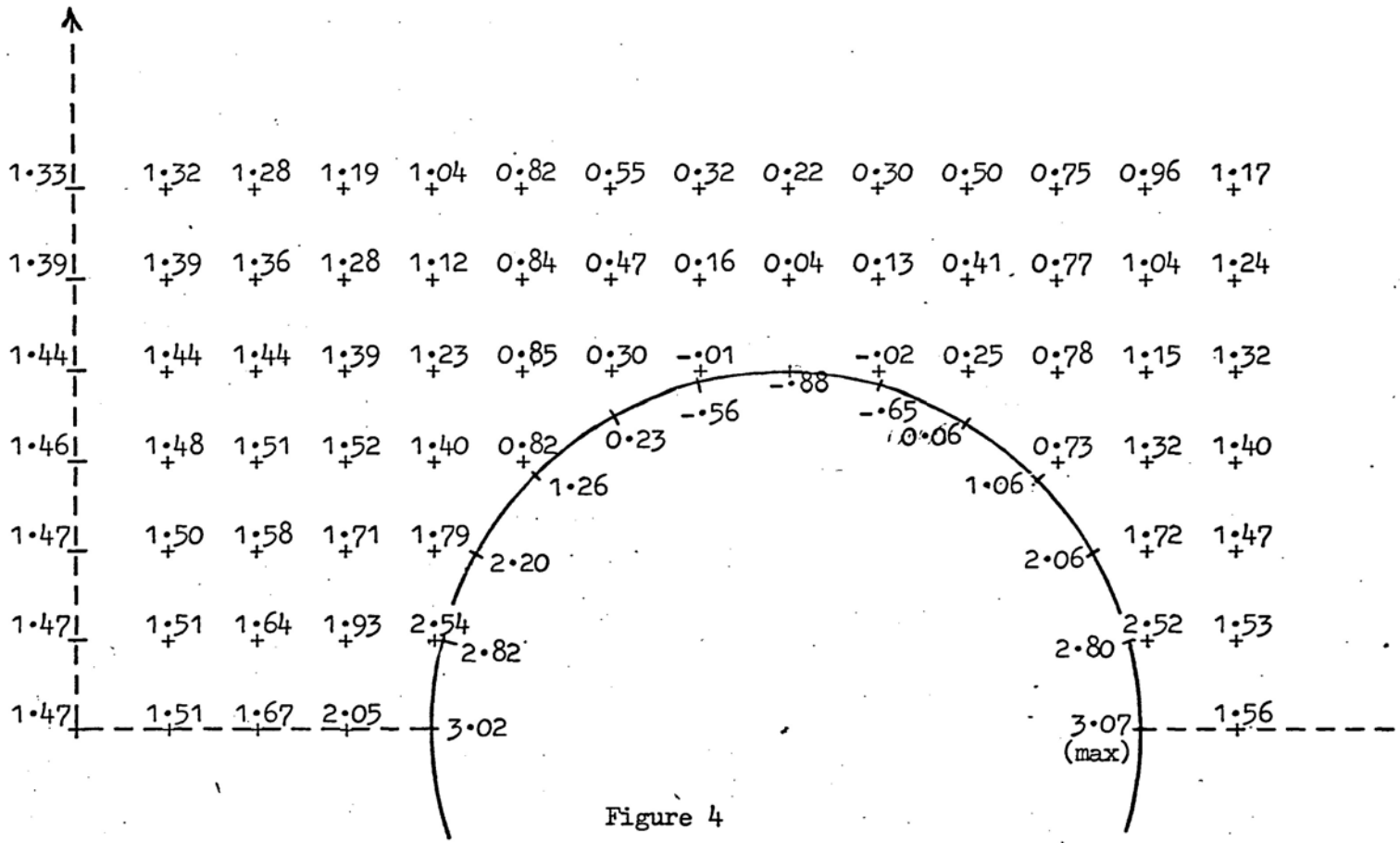


Figure 4

Nevertheless, as a check, the calculation for n=11 was repeated using a Gram-Schmidt procedure in which the vectors obtained were re-orthogonalized once. The coefficients A_k, B_k obtained differed from those in Table only in the 9^{th} decimal place. The residuals calculated from the Fourier expansion agreed to 3 significant figures with those calculated directly which in turn agreed with those obtained previously.

§ 9. Numerical results for $\lambda = 1{\cdot}5,\ 1{\cdot}2$

In a further series of calculations, still using 61 boundary points $\theta = 0^\circ\ (3^\circ)180^\circ$, the peripheral hoop stress $\breve{\theta\theta}$ was found to 3 decimal places at these points when $\lambda = 2{\cdot}0$, $1{\cdot}5$ and $1{\cdot}2$.

As before, when $\lambda=2{\cdot}0$, 11 terms in each series proved sufficient, 13 terms producing no changes in the hoop stress.

For $\lambda = 1{\cdot}5$, the numerically greatest residuals and root mean square residuals obtained with 9, 11 and 13 terms in each series were as follows:

Table 9

$\beta = 0$ (Longitudinal tension)

	R_N(max)	R_T(max)	R_N(rms)	R_T(rms)
n=9	$14{\cdot}1 \times 10^{-5}$	$17{\cdot}8 \times 10^{-5}$	$9{\cdot}0 \times 10^{-5}$	$10{\cdot}5 \times 10^{-5}$
n=11	$7{\cdot}1 \times 10^{-5}$	$8{\cdot}1 \times 10^{-5}$	$2{\cdot}6 \times 10^{-5}$	$3{\cdot}2 \times 10^{-5}$
n=13	$2{\cdot}3 \times 10^{-5}$	$2{\cdot}8 \times 10^{-5}$	$0{\cdot}9 \times 10^{-5}$	$1{\cdot}2 \times 10^{-5}$

$\beta = \pi/2$ (Transverse tension)

	R_N(max)	R_T(max)	R_N(rms)	R_T(rms)
n=9	$11{\cdot}5 \times 10^{-5}$	$15{\cdot}9 \times 10^{-5}$	$7{\cdot}3 \times 10^{-5}$	$9{\cdot}0 \times 10^{-5}$
n=11	$4{\cdot}5 \times 10^{-5}$	$5{\cdot}2 \times 10^{-5}$	$1{\cdot}6 \times 10^{-5}$	$2{\cdot}0 \times 10^{-5}$
n=13	$1{\cdot}6 \times 10^{-5}$	$2{\cdot}0 \times 10^{-5}$	$0{\cdot}7 \times 10^{-5}$	$0{\cdot}8 \times 10^{-5}$

Increasing the number of terms in each series from 9 to 11 produced alterations of 1 unit in the third decimal place in 14 of the 61 values of $\breve{\theta\theta}$ in the case $\beta = \pi/2$. Taking 13 terms produced 2 further alterations

in the values for $\beta = 0$ and 1 alteration for $\beta = \pi/2$.

For $\lambda = 1{\cdot}2$, rather more terms were needed for comparable accuracy. The numerically greatest residuals and root mean square residuals obtained using 11, 14, 17 and 19 terms in each series were as follows:

Table 10

$\beta = 0$ (Longitudinal tension)

	R_N(max)	R_T(max)	R_N(rms)	R_T(rms)
n=11	179 x 10^{-5}	217 x 10^{-5}	80·4 x 10^{-5}	101 x 10^{-5}
n=14	15·0 x 10^{-5}	18·2 x 10^{-5}	8·2 x 10^{-5}	9·8 x 10^{-5}
n=17	3·5 x 10^{-5}	3·1 x 10^{-5}	1·0 x 10^{-5}	1·0 x 10^{-5}
n=19	2·1 x 10^{-5}	2·0 x 10^{-5}	0·7 x 10^{-5}	0·8 x 10^{-5}

$\beta = \pi/2$ (Transverse tension)

	R_N(max)	R_T(max)	R_N(rms)	R_T(rms)
n=11	231 x 10^{-5}	299 x 10^{-5}	99·4 x 10^{-5}	127 x 10^{-5}
n=14	30·2 x 10^{-5}	39·7 x 10^{-5}	14·5 x 10^{-5}	17·6 x 10^{-5}
n=17	2·2 x 10^{-5}	2·6 x 10^{-5}	1·2 x 10^{-5}	1·3 x 10^{-5}
n=19	0·4 x 10^{-5}	0·3 x 10^{-5}	0·2 x 10^{-5}	0·1 x 10^{-5}

Here, 11 terms in each series were clearly insufficient. Increasing the number to 14 produced changes in nearly all the values of $\breve{\theta\theta}$, the largest change being 9 units in the third decimal place. Increasing the number of terms from 14 to 17 produced alterations of 1 unit in the third place in 22 of the 61 values in the case $\beta = 0$ and in 33 of the 61 values in the case $\beta = \pi/2$. Increasing the number of terms from 17 to 19 produced just one change of value in the case $\beta = \pi/2$.

Values of the hoop stress at $\theta = 0^\circ(15^\circ)180^\circ$ for $\lambda = 2{\cdot}0$, $1{\cdot}5$ and $1{\cdot}2$ are given in the following Table. The values at $\theta = 0^\circ, 90^\circ, 180^\circ$ agree with those of Ling in the cases $\lambda = 2{\cdot}0$ and $1{\cdot}5$.

Table 11

Peripheral Hoop Stress $\breve{0\theta}$

(i) $\beta = 0$ (Longitudinal tension)

θ	0°	15°	30°	45°	60°	75°	90°
λ=2·0	-0·908	-0·651	0·049	0·994	1·916	2·549	2·703
λ=1·5	-0·896	-0·639	0·059	1·000	1·909	2·516	2·623
λ=1·2	-0·905	-0·647	0·053	0·995	1·904	2·501	2·580

θ	105°	120°	135°	150°	165°	180°
λ=2·0	2·332	1·559	0·648	-0·095	-0·498	-0·609
λ=1·5	2·177	1·314	0·364	-0·259	-0·407	-0·376
λ=1·2	2·074	1·121	0·120	-0·390	-0·415	-0·428

(ii) $\beta = \pi/2$ (Transverse tension)

θ	0°	15°	30°	45°	60°	75°	90°
λ=2·0	3·066	2·797	2·063	1·062	0·065	-0·649	-0·881
λ=1·5	3·151	2·878	2·133	1·115	0·098	-0·639	-0·884
λ=1·2	3·298	3·020	2·261	1·261	0·162	-0·624	-0·924

θ	105°	120°	135°	150°	165°	180°
λ=2·0	-0·558	0·232	1·256	2·200	2·816	3·020
λ=1·5	-0·547	0·302	1·423	2·446	3·073	3·264
λ=1·2	-0·631	0·224	1·441	2·693	3·814	4·423

§ 10. Maximum Hoop Stress

When the tension is perpendicular to the line of centres, the maximum peripheral hoop stress necessarily occurs at $\theta = 0°$ or $\theta = 180°$. The earlier results thus yield the following table:

$\beta = \pi/2$ (Transverse tension)

λ	$\breve{0\theta}_{max}$	θ°
2·0	3·066	0
1·5	3·264	180°
1·2	4·423	180°

For tension parallel to the line of centres, Ling assumes that the greatest hoop stress will occur very close to $\theta = 90^\circ$ so that the value of $\breve{0\theta}$ at 90° can be taken as the maximum. Since the present method appeared to give quite accurate stresses, it was of interest to find the true maximum values.

There was no difficulty in locating the angle θ for which $\breve{\theta\theta}$ is greatest to the nearest degree. Subsequently, $\breve{\theta\theta}$ was calculated at an interval of $0{\cdot}1^\circ$ over small ranges and the results are given in Table 12. In these calculations, the values of n used for $\lambda = 2{\cdot}0,\ 1{\cdot}5,\ 1{\cdot}2$ were 11, 14, 17 respectively. The number of decimal places given was limited to 4 because the residual stresses were known to be of order 10^{-5} .

Taking the middle of the angles for which the hoop stress has its largest value gives the following table:

$\beta = 0$ (Longitudinal tension)

λ	$\breve{\theta\theta}_{max}$	θ°
2·0	2·715	86·8
1·5	2·650	85·3
1·2	2·619	84·6

Since these computations were carried out, Haddon (1967) has considered the problem of an infinite plate with two unequal circular holes, using a bilinear transformation to map the domain on to an annulus. Among other results, he finds the maximum hoop stresses and the angles at which they occur for the case of equal circles and $\beta = 0$. His results agree with those above.

Table 12

a) $\lambda = 2{\cdot}0$

Angle θ°	Hoop Stress $\breve{\theta\theta}$
86·0	2·7138
1	2·7140
2	2·7141
3	2·7143
4	2·7144
86·5	2·7145
6	2·7146
7	2·7146
8	2·7146
9	2·7146
87·0	2·7146
1	2·7145
2	2·7145
3	2·7144
4	2·7143
87·5	2·7141

b) $\lambda = 1{\cdot}5$

Angle θ°	Hoop Stress $\breve{\theta\theta}$
84·5	2·6490
6	2·6492
7	2·6494
8	2·6495
9	2·6497
85·0	2·6498
1	2·6499
2	2·6499
3	2·6500
4	2·6500
85·5	2·6500
6	2·6499
7	2·6499
8	2·6498
9	2·6497
86·0	2·6495

c) $\lambda = 1{\cdot}2$

Angle θ°	Hoop Stress $\breve{\theta\theta}$
84·0	2·6183
1	2·6185
2	2·6186
3	2·6187
4	2·6187
84·5	2·6188
6	2·6188
7	2·6188
8	2·6187
9	2·6186
85·0	2·6185
1	2·6184
2	2·6183
3	2·6181
4	2·6179
85·5	2·6177

CHAPTER 4

Mixed Boundary Conditions

In the problems considered previously, either the tractions or the displacements have been given at all points of the boundary. There is however no essential difficulty in applying the present method if the boundary conditions are mixed, i.e. the stresses are prescribed on some parts of the boundary and the displacements on the remaining parts; alternatively there may be parts of the boundary on which one component of the traction and one component of the displacement is prescribed. As an illustration, the following problem is considered in this chapter:

§ 1. Extension of an infinite plate with a rigid circular inclusion.

A rigid circular disc is inserted into a circular hole of the same radius in an infinite elastic plate. The plate is then extended by applying a unidirectional tension, causing the edge of the disc to lose contact with the boundary of the hole over two equal arcs DA, BC. (Figure 1)

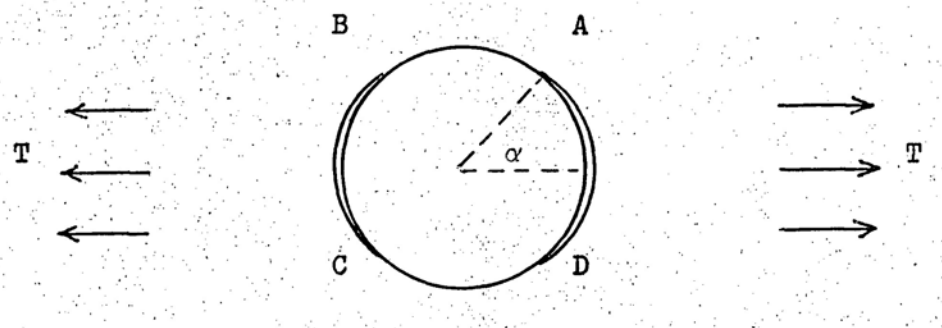


Figure 1

The angle α is not known in advance and its determination is an essential part of the problem.

If the included disc is assumed to be perfectly rough, the boundary conditions are

$$\breve{rr} = 0 \ , \ \breve{r\theta} = 0 \quad \text{along DA and BC} \ ,$$
$$v_r = 0 \ , \ v_\theta = 0 \quad \text{along AB and CD} \ .$$

If the included disc is assumed to be perfectly smooth, the boundary conditions become

$$\breve{rr} = 0 \ , \ \breve{r\theta} = 0 \quad \text{along DA and BC} \ ,$$
$$v_r = 0 \ , \ \breve{r\theta} = 0 \quad \text{along AB and CD} \ .$$

Furthermore, in both cases the solution has to satisfy the inequalities

$$\left.\begin{aligned} v_r &\geq 0 \quad \text{along DA and BC} \ , \\ \breve{rr} &\leq 0 \quad \text{along AB and CD} \ . \end{aligned}\right\} \tag{4.0}$$

It is these inequalities which provide the additional information for the determination of the angle α .

The case of the smooth disc was considered by Sheremetyev (1952) who constructed an integral equation for the boundary radial stress $\breve{rr}$. No attempt was made to solve the equation, but in a subsequent paper (Sheremetyev, 1960) he gives a lengthy analytical method for the related problem in which there are no stresses at infinity and a force is applied to the disc. No numerical results are given.

Without loss of generality, the radius of the hole may be taken to be unity. The boundary stresses and displacements are then given by

$$\begin{aligned} \breve{rr} - i\breve{r\theta} &= \Phi(z) + \overline{\Phi(z)} - e^{2i\theta} \left[\bar{z}\, \Phi'(z) + \Psi(z)\right] \\ &= \Phi(z) + \overline{\Phi(z)} - z\, \Phi'(z) - z^2\, \Psi(z) \ , \end{aligned} \tag{4.1}$$

$$\begin{aligned} 2\mu(v_r - iv_\theta) &= e^{i\theta} \left[\kappa\, \overline{\varphi(z)} - \bar{z}\, \varphi'(z) - \psi(z)\right] \\ &= \kappa\, \overline{\varphi(z)/z} - \varphi'(z) - z\, \psi(z) \ . \end{aligned} \tag{4.2}$$

As in Chapter 3, the complex potentials have the forms

$$\varphi(z) = \tfrac{1}{4} Tz + \varphi^0(z) ,$$

$$\psi(z) = -\tfrac{1}{2} Tz + \psi^0(z) ,$$

where $\varphi^0(z)$, $\psi^0(z)$ are regular for $|z|>1$ and vanish at infinity.

The potentials $\varphi(z) = \frac{1}{4} Tz$, $\psi(z) = -\frac{1}{2} Tz$ give rise to the boundary values

$$\widetilde{rr} = T\cos^2\theta , \quad \widetilde{r\theta} = -T\sin\theta\cos\theta ,$$

$$2\mu v_r = T(\cos^2\theta - \eta) , \quad 2\mu v_\theta = -T\sin\theta\cos\theta .$$

Hence in the case of the rough disc, the potentials $\varphi^0(z)$, $\psi^0(z)$ must make

$$\left.\begin{aligned} R^0 \equiv \widetilde{rr}^0 &= -T\cos^2\theta \\ S^0 \equiv \widetilde{r\theta}^0 &= -T\sin\theta\cos\theta \end{aligned}\right\} \text{ on DA, BC ,} \tag{4.3}$$

$$\left.\begin{aligned} 2\mu v_r{}^0 &= T(\eta - \cos^2\theta) \\ 2\mu v_\theta{}^0 &= T\sin\theta\cos\theta \end{aligned}\right\} \text{ on AB, CD .} \tag{4.4}$$

For the smooth disc, $2\mu v_\theta{}^0$ is replaced by $\widetilde{r\theta}^0$.

Because of symmetry about Ox and Oy , approximations to $\varphi^0(z)$, $\psi^0(z)$ may be taken as

$$\varphi_{2n-1}(z) = \sum_{k=1}^{n} \alpha_k / z^{2k-1} , \quad \psi_{2n-1}(z) = \sum_{k=1}^{n} \beta_k / z^{2k-1} ,$$

or with the usual change of notation,

$$\varphi_{2n-1}(z) = \sum_{s=1}^{n} a_{2s} / z^{2s-1} , \quad \psi_{2n-1}(z) = \sum_{s=1}^{n} a_{2s-1} / z^{2s-1} . \tag{4.5}$$

Corresponding approximations to $\Phi^0(z)$, $\Psi^0(z)$ are then

$$\Phi_{2n}(z) = -\sum_{s=1}^{n} (2s-1)\, a_{2s} / z^{2s} ,$$

$$\Psi_{2n}(z) = -\sum_{s=1}^{n} (2s-1)\, a_{2s-1} / z^{2s} . \tag{4.6}$$

[The superscript ^{0} is dropped henceforth.]

Writing $1/z^s = U_s + iV_s$, the approximate boundary displacements are given by

$$2\mu(\tilde{v}_r - i\tilde{v}_\theta) \equiv \kappa\,\bar{\varphi}_{2n-1}(\bar{z})/z - \varphi'_{2n-1}(z) - z\,\psi_{2n-1}(z)$$

$$= \kappa \sum_{s=1}^{n} a_{2s}\,(U_{2s} - iV_{2s}) + \sum_{s=1}^{n} (2s-1)\,a_{2s}\,(U_{2s} + iV_{2s})$$

$$- \sum_{s=1}^{n} a_{2s-1}\,(U_{2s-2} + iV_{2s-2}) ,$$

$$= \sum_{j=1}^{n} a_j\,(F_j - iG_j) \qquad \text{(say)} ,$$

where $F_{2s-1} = -U_{2s-2}$, $G_{2s-1} = V_{2s-2}$,

$F_{2s} = (\kappa + 2s - 1)U_{2s}$, $G_{2s} = (\kappa - 2s + 1)V_{2s}$.

Similarly, the approximate boundary stresses are given by

$$\tilde{R} - i\tilde{S} \equiv \varphi_{2n}(z) + \bar{\varphi}_{2n}(\bar{z}) - z\,\psi'_{2n}(z) - z^2\,\chi_{2n}(z)$$

$$= -2\sum_{s=1}^{n}(2s-1)a_{2s}U_{2s} - \sum_{s=1}^{n} 2s(2s-1)\,a_{2s}\,(U_{2s} + iV_{2s})$$

$$+ \sum_{s=1}^{n} (2s-1)\,a_{2s-1}\,(U_{2s-2} + iV_{2s-2})$$

$$= \sum_{j=1}^{n} a_j\,(F_j' - iG_j') \qquad \text{(say)} ,$$

where $F'_{2s-1} = (2s-1)U_{2s-2}$, $G'_{2s-1} = -(2s-1)V_{2s-2}$,

$F'_{2s} = -(2s-1)(2s+2)U_{2s}$, $G'_{2s} = (2s-1).2s\,V_{2s}$.

Let points P_i , $i = 1(1)N$, be taken on the quadrant of the boundary $0 \le \theta \le \pi/2$, the numbering to commence from the end $\theta = 0$. If the point A $(\theta = \alpha)$ lies between P_M and P_{M+1} , the least squares problem may be formulated as that of minimizing

$$I = \sum_{i=1}^{M} w_i\,\{[R(P_i) - \tilde{R}(P_i)]^2 + [S(P_i) - \tilde{S}(P_i)]^2\}$$

$$+ \sum_{i=M+1}^{N} w_i\,\{[2\mu v_r(P_i) - 2\mu\tilde{v}_r(P_i)]^2 + [2\mu v_\theta(P_i) - 2\mu\tilde{v}_\theta(P_i)]^2\}$$

in the case of the rough disc and

$$I = \sum_{i=1}^{M} w_i \{[R(P_i) - \tilde{R}(P_i)]^2 + [S(P_i) - \tilde{S}(P_i)]^2\}$$

$$+ \sum_{i=M+1}^{N} w_i \{[2\mu v_r(P_i) - 2\mu\tilde{v}_r(P_i)]^2 + [S(P_i) - \tilde{S}(P_i)]^2\}$$

in the case of the smooth disc. In these expressions, $R(P_i) = R^0(P_i) = T\cos^2\theta_i$, etc. and w_i are positive weights.

The coefficients a_j may thus be determined in the usual way for any chosen value of M (i.e. of α).

Intuitively, the angle α can be expected to lie within the range $60^0 < \alpha < 90^0$. For in the case of a plate with an unstressed circular hole (of unit radius) and subject to uni-directional tension at infinity, the complex potentials are known to be (M. page 201)

$$\varphi(z) = \tfrac{1}{4}Tz + \tfrac{1}{2}T/z ,$$

$$\psi(z) = -\tfrac{1}{2}Tz - \tfrac{1}{2}T/z + \tfrac{1}{2}T/z^3,$$

and the boundary displacements are given by

$$2\mu v_r = T(1 - \eta)(1 + 2\cos 2\theta) ,$$

$$2\mu v_\theta = -T(1 - \eta).2\sin 2\theta .$$

Thus v_r is positive when $0^0 \le \theta < 60^0$, negative when $60^0 < \theta \le 90^0$. Inserting the rigid disc prevents v_r from becoming negative.

§ 2. Numerical results

Initially, both the rough disc and the smooth disc were considered. Points P_i were taken corresponding to $\theta = 1^0(2^0)89^0$, giving $N = 45$, and M was made 30, 33, 36, 39, 42 in succession, these corresponding to $\alpha = 60^0, 66^0, 72^0, 78^0, 84^0$ (or rather α between 59^0 and 61^0, 65^0 and 67^0 etc.). 15 terms were taken in each of the series $\varphi_{2n-1}(z)$, $\psi_{2n-1}(z)$ i.e. up to and including $1/z^{29}$. In the case of the rough disc, the program was arranged to calculate the root mean squares of the residual radial and shear stresses over $0 \le \theta < \alpha$:

$$R_R(\text{rms}) = \{\sum_{i=1}^{M} w_i\ [R(P_i) - \tilde{R}(P_i)]^2 / \sum_{i=1}^{M} w_i\}^{\frac{1}{2}},$$

$$R_S(\text{rms}) = \{\sum_{i=1}^{M} w_i\ [S(P_i) - \tilde{S}(P_i)]^2 / \sum_{i=1}^{M} w_i\}^{\frac{1}{2}},$$

and of the residual radial and transverse displacements over $\alpha < \theta \leq 90^\circ$:

$$R_{v_r}(\text{rms}) = \{\sum_{M+1}^{N} w_i\ [2\mu v_r(P_i) - 2\mu\tilde{v}_r(P_i)]^2 / \sum_{M+1}^{N} w_i\}^{\frac{1}{2}},$$

$$R_{v_\theta}(\text{rms}) = \{\sum_{M+1}^{N} w_i\ [2\mu v_\theta(P_i) - 2\mu\tilde{v}_\theta(P_i)]^2 / \sum_{M+1}^{N} w_i\}^{\frac{1}{2}}.$$

The root mean squares of the residual stresses together:

$$R_T(\text{rms}) = \{\tfrac{1}{2}[R_R^2 + R_S^2]\}^{\frac{1}{2}}$$

and of the residual displacements together:

$$R_v(\text{rms}) = \{\tfrac{1}{2}[R_{v_r}^2 + R_{v_\theta}^2]\}^{\frac{1}{2}}$$

were also found.

In the case of the smooth disc, $R_R(\text{rms})$, $R_{v_r}(\text{rms})$ and $R_S(\text{rms})$ were again calculated, but this time with $R_S(\text{rms})$ calculated from the residual shear stresses over the entire quadrant $0 \leq \theta \leq 90^\circ$:

$$R_S(\text{rms}) = \{\sum_{i=1}^{N} w_i\ [S(P_i) - \tilde{S}(P_i)]^2 / \sum_{i=1}^{N} w_i\}^{\frac{1}{2}}.$$

The weights w_i were taken equal to unity and for convenience, the tension at infinity was taken as unity. The modified Poisson's ratio η had the value $0{\cdot}25$. The results for the rough disc are not at all promising. Table 1 does not reveal an angle α for which the boundary conditions are best satisfied and the graphs in Figure 2 show that for each choice of α there are ranges for which the inequalities (4.0) are not satisfied.

In contrast, for the smooth disc there is a clear indication in Table 2 that the boundary conditions are better satisfied for $\alpha = 66^\circ$ or 72° than for any other choice and the graphs in Figure 3 suggest that the true value of α lies somewhere between 66° and 72°. The problem of

the rough disc was therefore set aside and attention concentrated on locating the angle α more precisely for the smooth disc.

Table 1

Rough Disc $N = 45$, $n = 15$.

α	R_T(rms)	R_V(rms)
60°	$3{\cdot}61 \times 10^{-2}$	$7{\cdot}75 \times 10^{-2}$
66°	$3{\cdot}25$	$7{\cdot}84$
72°	$2{\cdot}65$	$8{\cdot}06$
78°	$1{\cdot}91$	$9{\cdot}17$
84°	$4{\cdot}27 \times 10^{-2}$	$8{\cdot}54 \times 10^{-2}$

Table 2

Smooth Disc $N = 45$, $n = 15$.

α	R_R(rms)	R_{Vr} (rms)	R_S(rms)
60°	$1{\cdot}14 \times 10^{-2}$	$4{\cdot}05 \times 10^{-2}$	$0{\cdot}09 \times 10^{-2}$
66°	$0{\cdot}25$	$1{\cdot}70$	$0{\cdot}08$
72°	$0{\cdot}48$	$1{\cdot}52$	$0{\cdot}04$
78°	$1{\cdot}24$	$6{\cdot}16$	$0{\cdot}34$
84°	$5{\cdot}65 \times 10^{-2}$	$9{\cdot}02 \times 10^{-2}$	$1{\cdot}88 \times 10^{-2}$

§ 3. The case of a smooth inclusion

The relative smallness of R_S(rms) for each of the angles α in Table 2 is noticeable and suggests that the boundary condition $\breve{r\theta} = 0$ is an easy one to satisfy. In fact it may be shown to imply a simple relation between the complex potentials. For if

$$\varphi(z) = \tfrac{1}{4} Tz + \sum_{k=1}^{\infty} \alpha_k / z^{2k-1} ,$$

$$\psi(z) = -\tfrac{1}{2} Tz + \sum_{k=1}^{\infty} \beta_k / z^{2k-1} ,$$

then at points of the boundary, the stress combination

$$\tfrac{1}{2}(\breve{\theta\theta} - \breve{rr} + 2i\breve{r\theta}) \equiv z\,\varphi'(z) + z^2\,\Psi(z)$$

$$= -\tfrac{1}{2} Tz^2 + \sum_{k=1}^{\infty} 2k(2k-1)\,\alpha_k / z^{2k} - \sum_{k=1}^{\infty} (2k-1)\,\beta_k / z^{2k-2} \qquad (4.7)$$

Figure 2

Rough Disc. N=45 , n=15

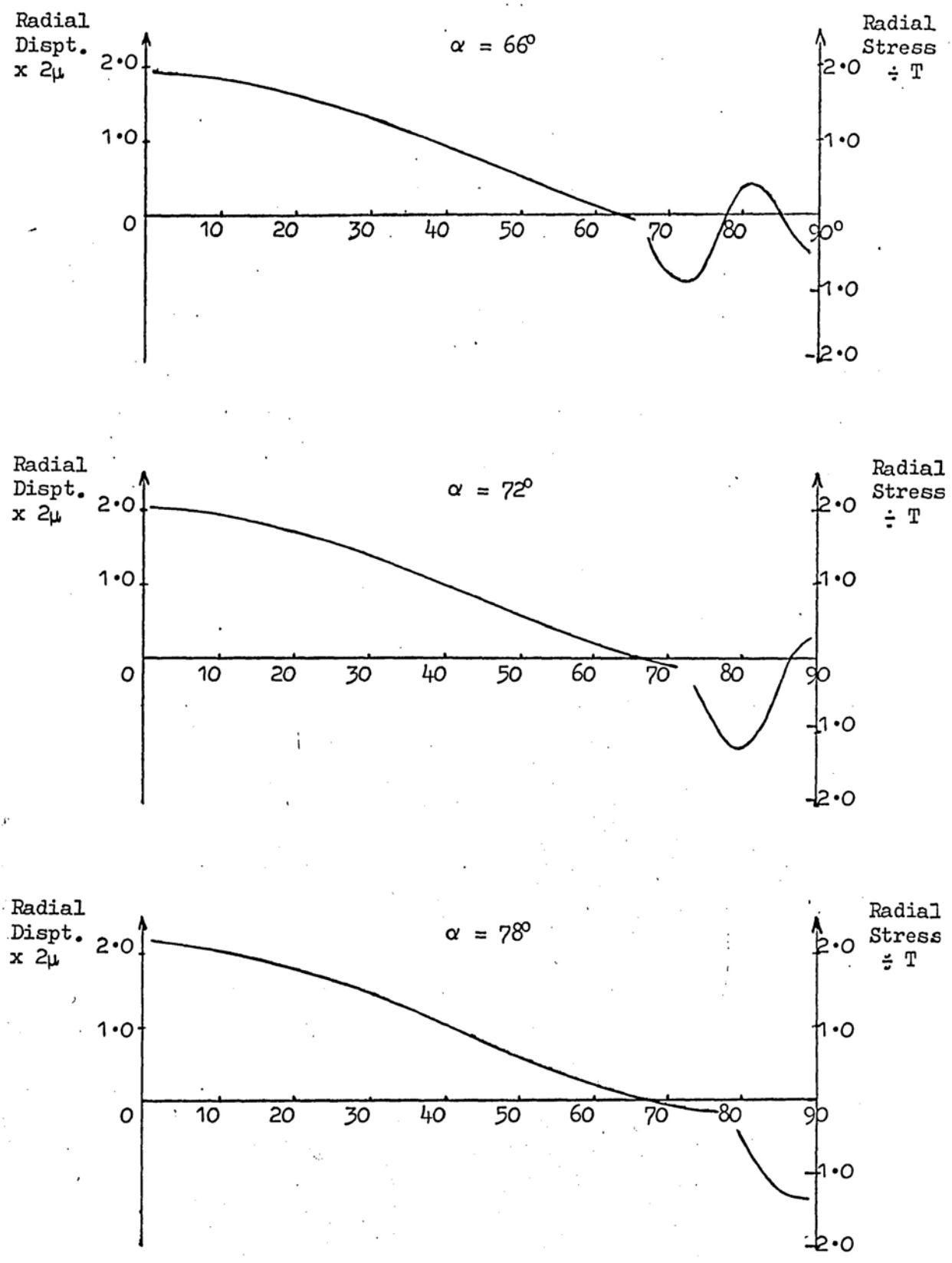
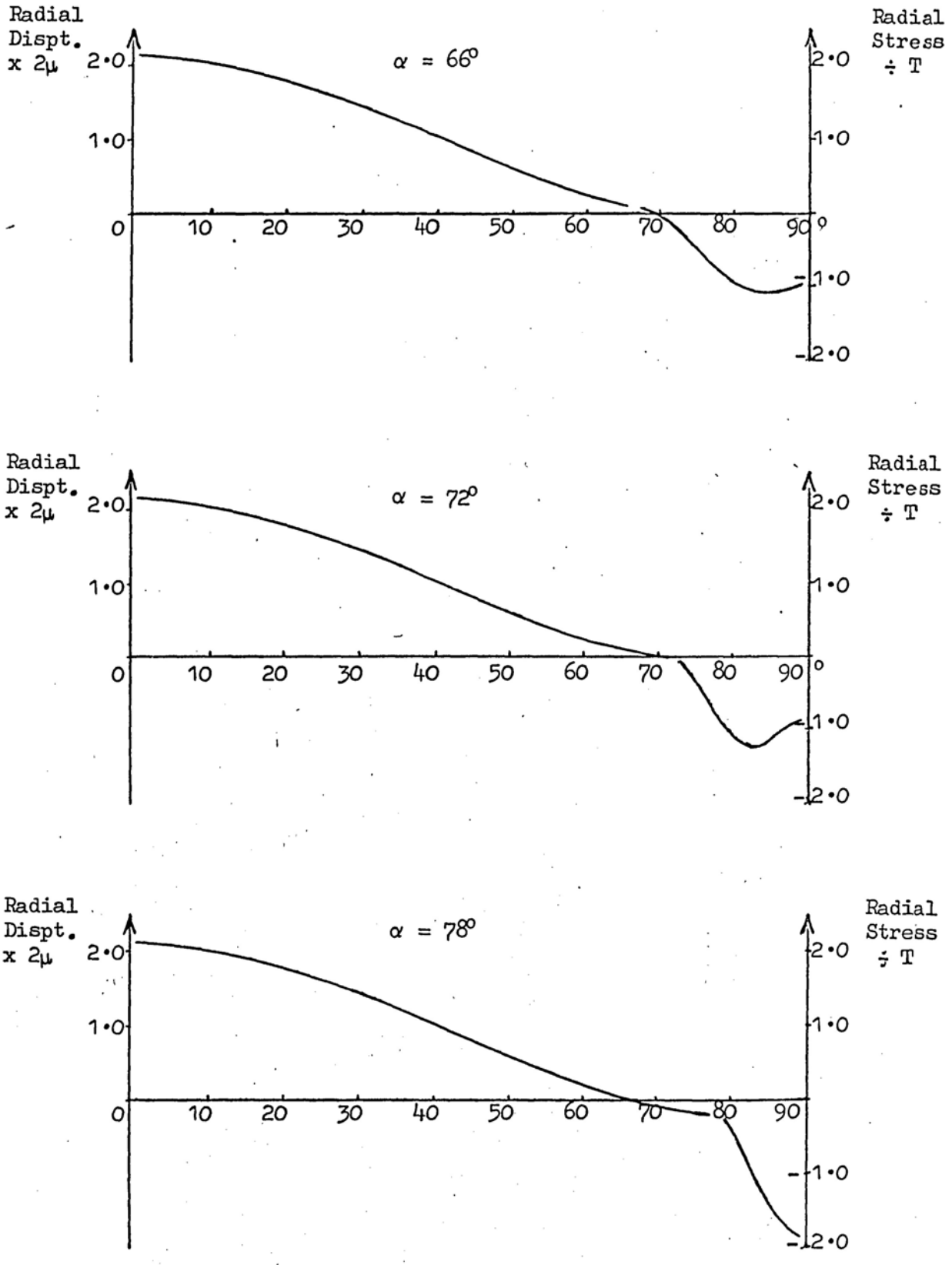


Figure 3

Smooth Disc N=45, n=15



$$= -\tfrac{1}{2}\,T(U_2 - iV_2) + \sum_{k=1}^{\infty} 2k(2k-1)\,\alpha_k\,(U_{2k} + iV_{2k})$$

$$- \beta_1 - \sum_{k=1}^{\infty} (2k+1)\,\beta_{k+1}\,(U_{2k} + iV_{2k}), \tag{4.8}$$

and so if $\breve{r\theta} = 0$ at all points then necessarily

$$\tfrac{1}{2}\,T + 2.1\,\alpha_1 - 3\beta_2 = 0\,,$$

$$2k(2k-1)\,\alpha_k - (2k+1)\,\beta_{k+1} = 0\,,\quad k \geq 2\,.$$

Using these in (4.7) gives the relation

$$z\,\Phi'(z) + z^2\,\Psi(z) = -\tfrac{1}{2}\,T\left(z^2 + \frac{1}{z^2}\right) - \beta_1\,, \tag{4.9}$$

which although established only for points on the boundary, holds throughout the region $|z| \geq 1$ by an appeal to the principle of analytical continuation. This relation was used by Sheremetyev (1952) in constructing his integral equation.

Equivalently, but for boundary points only, (4.8) becomes

$$\tfrac{1}{2}(\breve{\theta\theta} - \breve{rr}) = -\tfrac{1}{2}\,TU_2 - \beta_1 - \tfrac{1}{2}\,TU_2$$

$$= -\,TU_2 - \beta \tag{4.10}$$

and combining this with the formula

$$\tfrac{1}{2}(\breve{\theta\theta} + \breve{rr}) = \Phi(z) + \overline{\Phi(z)}$$

$$= \tfrac{1}{2}\,T - 2\sum_{k=1}^{\infty} (2k-1)\,\alpha_k\,U_{2k}$$

gives $$\breve{rr} = T(\tfrac{1}{2} + U_2) + \beta_1 - 2\sum_{k=1}^{\infty} (2k-1)\,\alpha_k\,U_{2k}\,. \tag{4.11}$$

The coefficients β_k can also be eliminated from the formula for the boundary displacements:

$$2\mu\,(v_r - iv_\theta) = \varkappa\,\overline{\varphi(z)/z} - \varphi'(z) - z\,\psi(z)$$

$$= \tfrac{1}{4}\,(\varkappa - 1)\,T + \tfrac{1}{2}\,Tz^2 + \varkappa\sum_{k=1}^{\infty} \alpha_k/z^{2k}$$

$$+ \sum_{k=1}^{\infty} (2k-1)\,\alpha_k/z^{2k} - \sum_{k=1}^{\infty} \beta_k/z^{2k-2}\,.$$

In particular, the radial component is given by

$$2\mu v_r = T(1-\eta) + \tfrac{1}{2} TU_3 + \sum_{k=1}^{\infty} (\alpha + 2k - 1)\, \alpha_k\, U_{3k}$$

$$- \beta_1 - \sum_{k=1}^{\infty} \beta_{k+1}\, U_{3k}$$

$$= T(\tfrac{1}{2} - \eta + \tfrac{1}{3} U_3) - \beta_1 + \sum_{k=1}^{\infty} (\alpha + \frac{2k-1}{2k+1})\, \alpha_k\, U_{3k} \,. \qquad (4.12)$$

Thus for the smooth disc a marked simplification is possible. The boundary condition $\tilde{rr} = 0$ can be taken care of automatically and the problem is reduced to the determination of one complex potential only (or one set of coefficients). If points P_i are taken as before and $R(=\tilde{rr})$ and $2\mu v_r$ are approximated by the finite sums:

$$\tilde{R} = T(\tfrac{1}{2} + U_3) + \beta_1 - 2 \sum_{k=1}^{n} (2k-1)\, \alpha_k\, U_{3k} \,, \qquad (4.13)$$

$$2\mu \tilde{v}_r = T(\tfrac{1}{2} - \eta + \tfrac{1}{3} U_3) - \beta_1 + \sum_{k=1}^{n} (\alpha + \frac{2k-1}{2k+1})\, \alpha_k\, U_{3k} \,, \qquad (4.14)$$

the coefficients $\beta_1, \alpha_1, \alpha_2, \ldots, \alpha_n$ can be found by minimizing

$$I = \sum_{i=1}^{M} w_i\, [\tilde{R}(P_i)]^2 + \sum_{i=M+1}^{N} w_i\, [2\mu \tilde{v}_r(P_i)]^2 \,.$$

This simplification makes it possible to double the number of boundary points and also the number of terms in the series approximating $\varphi(z)$ and $\psi(z)$ while using roughly the same amount of computing time as before.

§ 4. Numerical results

Two versions of a program incorporating these ideas were run. In the first, 90 boundary points $\theta = \frac{1}{2}^\circ (1^\circ) 89\frac{1}{2}^\circ$ were used, the points carrying equal weights. In the second, there were 91 points $\theta = 0^\circ (1^\circ) 90^\circ$ with the weights for 0° and 90° halved. The first program allows α to take integer values (in degrees) and the second allows α to take values midway between integers. In each case the radial displacement and radial stress were calculated at each of the boundary points together with the root mean squares:

$$R_R(\text{rms}) = \Big(\sum_{i=1}^{M} w_i \, [\tilde{R}(P_i)]^2 / \sum_{i=1}^{M} w_i \Big)^{\frac{1}{2}} ,$$

$$R_{V_r}(\text{rms}) = \Big(\sum_{i=M+1}^{N} w_i \, [2\mu\tilde{v}_r(P_i)]^2 / \sum_{i=M+1}^{N} w_i \Big)^{\frac{1}{2}} .$$

Table 3 (N = 90 , n = 21)

α	R_R(rms)	R_{V_r} (rms)
66°	$3{\cdot}58 \times 10^{-3}$	$13{\cdot}6 \times 10^{-3}$
67°	$2{\cdot}25$	$10{\cdot}8$
68°	$1{\cdot}04$	$7{\cdot}63$
69°	$0{\cdot}93$	$4{\cdot}51$
70°	$1{\cdot}89$	$3{\cdot}04$
71°	$2{\cdot}63$	$5{\cdot}98$
72°	$2{\cdot}90 \times 10^{-3}$	$9{\cdot}87 \times 10^{-3}$

Table 4 (N = 91 , n = 21)

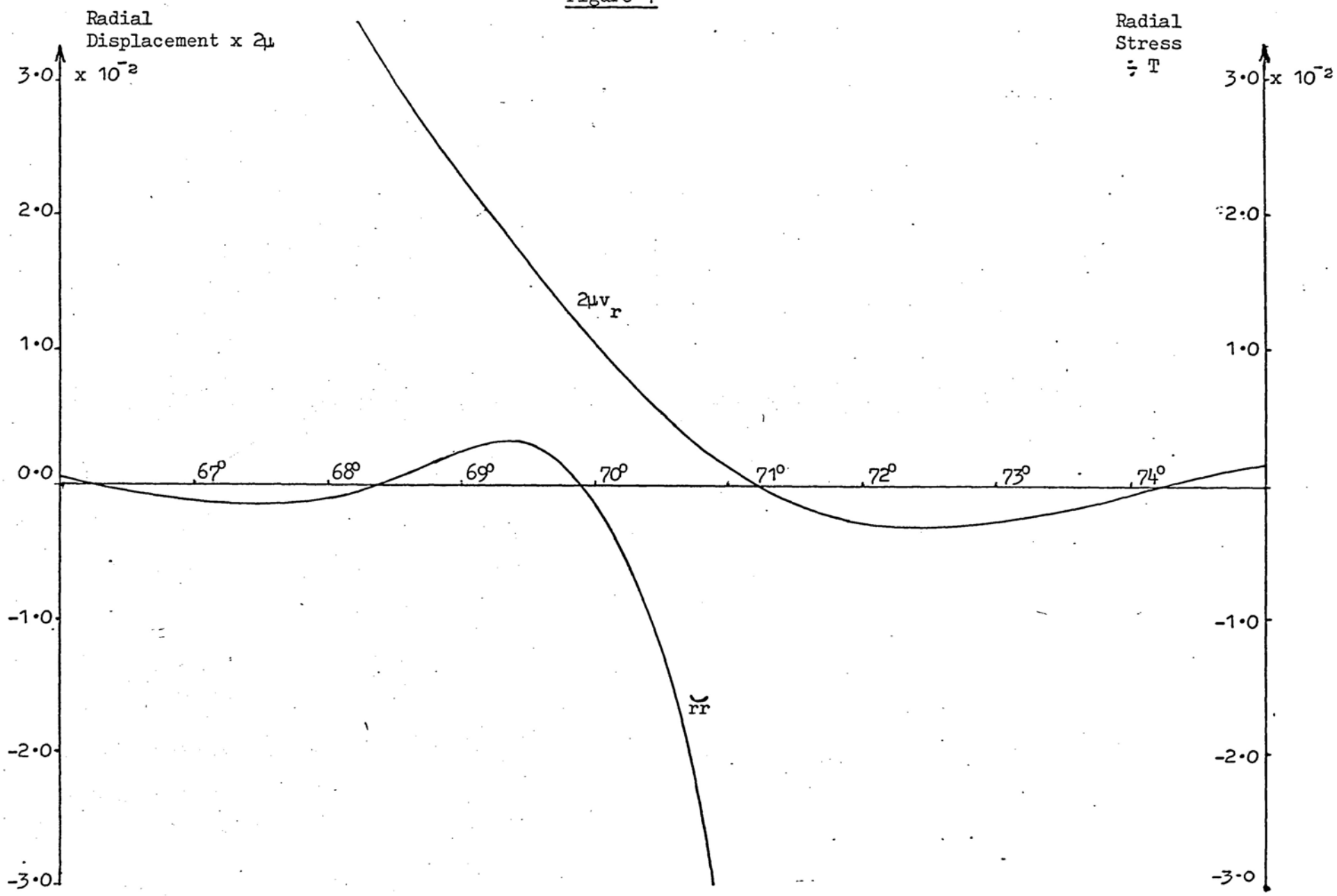
α	R_R(rms)	R_{V_r} (rms)
$66\frac{1}{2}^\circ$	$2{\cdot}96 \times 10^{-3}$	$12{\cdot}1 \times 10^{-3}$
$67\frac{1}{2}^\circ$	$1{\cdot}57$	$9{\cdot}15$
$68\frac{1}{2}^\circ$	$0{\cdot}78$	$5{\cdot}95$
$69\frac{1}{2}^\circ$	$1{\cdot}36$	$3{\cdot}29$
$70\frac{1}{2}^\circ$	$2{\cdot}33$	$4{\cdot}10$
$71\frac{1}{2}^\circ$	$2{\cdot}75 \times 10^{-3}$	$7{\cdot}98 \times 10^{-3}$

Table 5 (N = 90 or 91 , n = 35)

α	R_R(rms)	R_{V_r} (rms)
69°	$0{\cdot}41 \times 10^{-3}$	$3{\cdot}62 \times 10^{-3}$
$69\frac{1}{2}^\circ$	$0{\cdot}20$	$2{\cdot}52$
70°	$0{\cdot}23$	$1{\cdot}60$
$70\frac{1}{2}^\circ$	$0{\cdot}46$	$1{\cdot}27$
71°	$0{\cdot}75 \times 10^{-3}$	$1{\cdot}80 \times 10^{-3}$

In the first instance, the series for $\tilde{R}$ and $2\mu\tilde{v}_r$ were terminated after 21 terms, leaving 22 unknown coefficients, and the angle α was given the values $66^\circ\,(1^\circ)\,72^\circ$, $66\frac{1}{2}^\circ\,(1^\circ)\,71\frac{1}{2}^\circ$. The r.m.s. residuals found in this way (with $\eta = 0{\cdot}25$ as before) are given in Tables 3 and 4. Subsequently, the series for $\tilde{R}$ and $2\mu\tilde{v}_r$ were extended to 35 terms

Figure 4


Radial
Displacement x 2μ
x 10⁻²
3·0
2·0
1·0
0·0
-1·0
-2·0
-3·0
67°
68°
69°
70°
71°
72°
73°
74°
2μv_r
τ_rr
Radial
Stress
÷ T
x 10⁻²

(36 coefficients) with the narrowed range $69^\circ \leq \alpha \leq 71^\circ$. Results obtained in this way from the two programs are combined in Table 5.

Comparison of Table 3 with the earlier Table 2 shows that the residuals for $\alpha = 66^\circ$ and $\alpha = 72^\circ$ are very little altered by increasing the number of terms in in the series $\varphi_{2n-1}(z)$ and $\psi_{2n-1}(z)$ from 15 to 21. Increasing the number from 21 to 35 does produce a reduction in both R_R(rms) and R_{v_r}(rms) for values of α in the neighbourhood of 70°. There is however no clear pointer to any particular value of α being the correct one; the minimum values of R_R and R_{v_r} occur at different values of α. Moreover, it is not possible to decide the value of α from the behaviour of $\widetilde{rr}$ and $2\mu v_r$ near $\theta = \alpha$. Typical graphs for the range $66^\circ \leq \theta \leq 75^\circ$ are shown in Figure 4 ($\alpha = 70\frac{1}{2}^\circ$).

More serious, an oscillation superimposed on the values of $\widetilde{rr}$ for $\theta > \alpha$ was noticeable when $n = 21$ and more pronounced when $n = 35$. It seemed possible that this oscillation could be attributed to using $\varphi'_{2n-1}(z)$ as an approximation to $\varphi'(z)$ when $\varphi'_{2n-1}(z)$ is found by term by term differentiation. If this were so, the use of Lanczos' "σ - factors" might be beneficial.

§ 5. Application of Lanczos' "σ - factors"

It was shown in Chapter 3 that if the function $f(z)$ is approximated by a series of the form $f_n(z) \equiv \sum_{-(n-1)}^{n-1} c_k z^k$, then $f_n'(z)$ may not be a good approximation to $f'(z)$. In place of ckz^{k-1} one should use $\sigma_k z^{k-1}$ where

$$\sigma_k = \sin(\pi k/m)/(\pi/m) .$$

In the present problem, because of the relation (4.9), the boundary radial stress can be written

$$\widetilde{rr} = \tfrac{1}{2} T + \tfrac{1}{2} T(z^2 + 1/z^2) + \beta_1 - 2 \operatorname{Re} \varphi'(z) , \qquad (4.15)$$

where $z = e^{i\theta}$. Likewise, using (4.9) gives

$$\frac{d}{dz}\left[\frac{1}{z}\ \varphi'(z) + \psi(z)\right] = -\tfrac{1}{2}\,T(1 + 1/z^4) - \beta_1/z^3 - \varphi'(z)/z^3$$

and so the boundary displacements are given by

$$2\mu(v_r - iv_\theta) = \kappa\ \overline{\varphi(z)/z} + \tfrac{1}{2}\,T(z^3 - 1/3z^2) - \beta_1 + z\int \varphi'(z)\ dz/z^3 \ . \tag{4.16}$$

Formulae (4.13) and (4. 14) are approximations derived from (4.15) and (4.16) by taking $\varphi_{2n-1}(z)$ in place of $\varphi(z)$. Since $\varphi'(z)/z^3$ is integrated in (4.16), it was not considered necessary to apply σ - factors to (4.14). However, (4.13) might with advantage be replaced by

$$\widetilde{R} = T(\tfrac{1}{2} + U_2) + \beta_1 - 2\sum_{k=1}^{n} \alpha_k\ \sigma_k\ U_{2k}\ , \tag{4.17}$$

where in this case,

$$\sigma_k = \sin\frac{(2k-1)\pi}{(2n+1)} \Big/ \frac{\pi}{(2n+1)} \ .$$

Taking boundary points $\theta = \frac{1}{2}^\circ(1^\circ)89\frac{1}{2}^\circ$ or $0^\circ(1^\circ)90^\circ$ and using 35 terms of the series in (4.17), (4.14) gave the following results:

Table 6

($N = 90/91$, $n = 35$)

α	R_R(rms)	R_{v_r} (rms)
69°	$0{\cdot}48 \times 10^{-3}$	$3{\cdot}40 \times 10^{-3}$
$69\frac{1}{2}^\circ$	$0{\cdot}45$	$2{\cdot}10$
70°	$0{\cdot}74$	$1{\cdot}49$
$70\frac{1}{2}^\circ$	$1{\cdot}12 \times 10^{-3}$	$2{\cdot}29 \times 10^{-3}$

For $\theta > \alpha$, the radial stress was completely free from the superimposed oscillation noted before. Nevertheless, the angle α still cannot be decided with certainty. When $\alpha = 70^\circ$, for example, the behaviour of $\widetilde{rr}$ and $2\mu v_r$ over the range $66^\circ < \theta < 75^\circ$ is very similar to that shown in Figure 4. It is clear that a new approach to the problem of determining α is required and this forms the subject of Chapter 6.

CHAPTER 5

Discontinuous Boundary Values and Stress Singularities

In this Chapter, the effects of discontinuities in the boundary data are discussed. The situations considered are

(i) when there is an abrupt change at some boundary point in the applied normal or tangential stress,

(ii) when there is an abrupt change in the prescribed boundary displacements at some point,

(iii) when there is an abrupt change in direction of the boundary at some point.

In each of these cases there may be a complete loss of accuracy in the calculated stresses and displacements near the point in question whatever numerical method is used.

§ 1. Discontinuous boundary tractions

As an example, consider the case of a circular disc of radius unity subject to a uniform normal traction p over the arcs $-\alpha < \theta < \alpha$ and $\pi - \alpha < \theta < \pi + \alpha$, the rest of the boundary being free from stress. It will be shown later that the solution of this problem is given by the complex potentials

$$\left.\begin{aligned} \Phi(z) &= \frac{p}{2\pi i} \log \frac{(a-z)(a+z)}{(\bar{a}-z)(\bar{a}+z)} - \frac{p\alpha}{\pi}, \\ \Psi(z) &= \frac{p}{\pi i}\left[\frac{1}{a^2-z^2} - \frac{1}{\bar{a}^2-z^2}\right], \end{aligned}\right\} \tag{5.1}$$

where $a \equiv e^{i\alpha}$. Clearly, both $\Phi(z)$ and $\Psi(z)$ tend to infinity on approaching the four points on the boundary at which there is a discontinuity

in the applied tractions. The stresses, however, are everywhere finite. For example, in the neighbourhood of the point $z = a$,

$$\Phi(z) \sim \frac{p}{2\pi i} \log(a-z),$$

$$\Psi(z) \sim \frac{p}{2\pi i} \frac{1}{a(a-z)},$$

and so

$$\Phi(z) + \overline{\Phi(z)} \sim \frac{p}{\pi} \arg(a-z),$$

$$\bar{z}\, \Phi'(z) + \Psi(z) \sim \frac{p}{2\pi i} \frac{\bar{a}-\bar{z}}{a-z}.$$

which have finite limits.

Nevertheless, because the complex potentials cannot be approximated by polynomials in the neighbourhood of any of the points $z = \pm a, \pm \bar{a}$, an application of the method of Chapter 2 will yield stresses which are quite inaccurate in the region of the points of discontinuity.

To obtain accurate values for the stresses it is necessary to "subtract out the discontinuities" i.e. to reduce the problem to one in which the boundary tractions are continuous and to superpose a solution representing the effect of the discontinuities. Two methods for doing this will be described, both based on an elementary solution of the equations for generalised plane stress due to Michell (see M. page 377).

§ 2. Michell's radial stress system

The complex potentials

$$\Phi(z) = -\frac{F}{2\pi} \frac{1}{z} \quad , \quad \Psi(z) = \frac{\bar{F}}{2\pi} \frac{1}{z}$$

give rise to the state of stress:

$$\widetilde{rr} = -\frac{1}{\pi} \left(\frac{F}{z} + \frac{\bar{F}}{\bar{z}}\right), \quad \widetilde{r\theta} = 0 \, , \quad \widetilde{\theta\theta} = 0 \quad ,$$

which is purely radial.

If these are applied to any half plane which has the origin as a

boundary point, the boundary will be free from stress except at the origin where integration round a small semicircle of radius r (Figure 1) shows there is a concentrated force given by

$$-\int_{\alpha-\pi/2}^{\alpha+\pi/2} \overset{\smile}{rr}\, e^{i\theta}.rd\theta = \frac{1}{\pi}\int_{\alpha-\pi/2}^{\alpha+\pi/2} \left(\frac{F}{r} + \frac{\bar{F}}{r}\, e^{2i\theta}\right)rd\theta$$

$$= F .$$

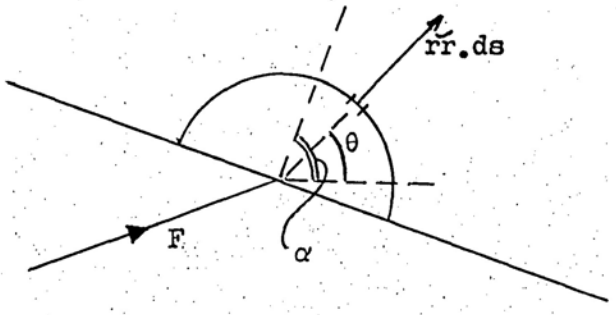


Figure 1

If the origin of coordinates is taken so that the force F acts at a point t on the boundary of a half plane, the formulae of transformation given by Muskhelishvili (M.page 135) show that the complex potentials become

$$\phi(z) = -\frac{F}{2\pi}\frac{1}{z-t} = \frac{F}{2\pi}\frac{1}{t-z} , \tag{5.2}$$

$$\psi(z) = \frac{\bar{F}}{2\pi}\frac{1}{z-t} = \bar{t}\,\frac{F}{2\pi}\frac{1}{(z-t)^2}$$

$$= -\frac{\bar{F}}{2\pi}\frac{1}{t-z} - \frac{F}{2\pi}\frac{\bar{t}}{(t-z)^2} . \tag{5.3}$$

§ 3. First Method

Suppose that at a point t on the boundary C of a region R there are normal and tangential stresses $N = N(t)$ and $T = T(t)$ respectively, the direction of N making an angle α with Ox.

Taking $F = (N+iT)e^{i\alpha}\,ds$,

$\bar{F} = (N-iT)e^{-i\alpha}\,ds$,

and writing $dt = ds e^{i(\pi/2+\alpha)} = i\,ds e^{i\alpha}$,

one is led to consider the complex potentials

$$\Phi(z) = \frac{1}{2\pi i}\int_C \frac{N+iT}{t-z}\,dt\ , \tag{5.4}$$

$$\Psi(z) = -\frac{1}{2\pi i}\int_C \frac{N-iT}{t-z}\,e^{-2i\alpha}\,dt - \frac{1}{2\pi i}\int \frac{N+iT}{(t-z)^2}\,\bar{t}\,dt$$

$$= \frac{1}{2\pi i}\int_C \frac{N-iT}{t-z}\,d\bar{t} - \frac{1}{2\pi i}\int_C \frac{N+iT}{(t-z)^2}\,\bar{t}\,dt\ . \tag{5.5}$$

(For simplicity, the region R will be assumed simply connected and the curve C will be supposed to be smooth).

If C is the boundary of a half plane, these potentials can be expected to solve the problem completely. In fact if C is the real axis, formulae (5.4) and (5.5) agree with formulae given by Muskhelishvili (M. page 393) when allowance is made for the reversed sense of description of the contour and the reversed direction of T . For an arbitrary curve C , the potentials lead to boundary stresses N'(t) and T'(t) which differ from N(t) and T(t) in general. These will now be found.

For brevity, write $N(t) + iT(t) = \Omega(t)$, so that

$$\Phi(z) = \frac{1}{2\pi i}\int_C \frac{\Omega(t)}{t-z}\,dt\ . \tag{5.6}$$

Let t_0 be a point on some part of C upon which $\Omega(t)$ satisfies the Holder condition

$$|\Omega(t_1) - \Omega(t_2)| \le A\,|t_1 - t_2|^{\mu},\ (A > 0\ ,\ 0 < \mu \le 1)$$

for any two points t_1 , t_2 . Then if $z \to t_0$ from inside R , it follows from the Plemelj formula (M. page 262) that

$$\Phi(t_0) = \tfrac{1}{2}\,\Omega(t_0) + \frac{1}{2\pi i}\int_C \frac{\Omega(t)}{t-t_0}\,dt$$

and hence that

$$\Phi(t_o) + \overline{\Phi(t_o)} = N(t_o) + \frac{1}{2\pi i}\int_c \left\{\frac{\Omega(t)}{t-t_o}\,dt - \frac{\bar\Omega(t)}{\bar t-\bar t_o}\,d\bar t\right\}, \qquad (5.7)$$

where the contour integrals are now Cauchy principal values. Further, from (5.6), $\Phi'(z) = \frac{1}{2\pi i}\int_c \frac{\Omega(t)}{(t-z)^2}\,dt$,

and so $\bar z\,\Phi'(z) + \Psi(z) = -\frac{1}{2\pi i}\int_c \Omega(t)\,\frac{\bar t-\bar z}{(t-z)^2}\,dt + \frac{1}{2\pi i}\int_c \bar\Omega(t)\,\frac{d\bar t}{t-z}$

$$= \frac{1}{2\pi i}\int_c \Omega(t)\,d\left(\frac{\bar t-\bar z}{t-z}\right) + \frac{1}{2\pi i}\int_c \frac{\Omega(t)-\bar\Omega(t)}{t-z}\,e^{-2i\alpha}\,dt.$$

Writing $t - z = re^{i\varphi}$, so that $d\left(\frac{\bar t-\bar z}{t-z}\right) = -2ie^{-2i\varphi}\,d\varphi$, it is seen that the first integral varies continuously with z and so applying the Plemelj formula to the second integral,

$$\bar t\,\Phi'(t_o) + \Psi(t_o) = \frac{1}{2\pi i}\int_c \Omega(t)\,d\left(\frac{\bar t-\bar t_o}{t-t_o}\right)$$

$$+ \tfrac{1}{2}[\Omega(t_o) - \bar\Omega(t_o)]e^{-2i\alpha_o} + \frac{1}{2\pi i}\int_c \frac{\Omega(t)-\bar\Omega(t)}{t-t_o}\,e^{-2i\alpha}\,dt$$

$$= iT(t_o)e^{-2i\alpha} - \frac{1}{2\pi i}\int_c \left(\frac{\bar t-\bar t_o}{t-t_o}\right)\left[\frac{\Omega(t)}{t-t_o}\,dt - \frac{\bar\Omega(t)}{\bar t-\bar t_o}\,d\bar t\right].$$

Finally, from the Kolosov formula,

$$N'(t_o) - iT'(t_o) = \Phi(t_o) + \overline{\Phi(t_o)} - e^{2i\alpha_o}\,[\bar t_o\,\Phi'(t_o) + \Psi(t_o)]$$

$$= N(t_o) - iT(t_o) + \frac{1}{2\pi i}\int_c \left[\frac{N+iT}{t-t_o}\,dt - \frac{N-iT}{\bar t-\bar t_o}\,d\bar t\right]$$

$$+ e^{2i\alpha_o}\,\frac{1}{2\pi i}\int_c \left(\frac{\bar t-\bar t_o}{t-t_o}\right)\left[\frac{N+iT}{t-t_o}\,dt - \frac{N-iT}{\bar t-\bar t_o}\,d\bar t\right]. \qquad (5.8)$$

This result can also be expressed in real form by putting $t - t_o = re^{i\varphi}$, $\frac{dt}{t-t_o} = \frac{dr}{r} + i\,d\varphi$, to give

$$N'(t_o) - iT'(t_o) = N(t_o) - iT(t_o) + \frac{1}{\pi}\int_c \left[1 + e^{-2i(\varphi-\alpha_o)}\right]\left(N d\varphi + T\frac{dr}{r}\right)$$

or $$N'(t_o) = N(t_o) + \frac{2}{\pi}\int_c \left(N d\varphi + T\frac{dr}{r}\right)\cos^2(\varphi - \alpha_o), \qquad (5.9)$$

$$T'(t_o) = T(t_o) + \frac{2}{\pi}\int_c \left(N d\varphi + T\frac{dr}{r}\right)\sin(\varphi - \alpha_o)\cos(\varphi - \alpha_o). \qquad (5.10)$$

In deriving these equations, it has been supposed that t_o is not a

point at which $N(t) + iT(t)$ is discontinuous for then the Cauchy principal values which have been used do not exist. However, in the real forms (5.9), (5.10), $\cos(\varphi - \alpha_0) \to 0$ when $r \to 0$. If the curve C has continuous curvature (in addition to being smooth), $\cos(\varphi - \alpha_0)/r$ is a continuous function of t; in particular,

$$\lim_{r \to 0} \frac{\cos(\varphi - \alpha_0)}{r} = -\frac{1}{2\rho_0} ,$$

where ρ_0 is the radius of curvature at the point t_0. It follows that if, for example, $N(t)$ and $T(t)$ are sectionally continuous functions of t, then the integrals in (5.9) and (5.10):

$$N''(t_0) \equiv \frac{2}{\pi} \int_C (N d\varphi + \frac{T dr}{r}) \cos^2(\varphi - \alpha_0) ,$$

$$T''(t_0) \equiv \frac{2}{\pi} \int_C (N d\varphi + \frac{T dr}{r}) \sin(\varphi - \alpha) \cos(\varphi - \alpha_0) ,$$

are continuous functions of t_0; the discontinuities are present only in $N(t_0)$ and $T(t_0)$.

The determination of the potentials corresponding to tractions $N''(t)$ and $T''(t)$ is thus a "smooth" problem, and since the potentials corresponding to $N'(t)$ and $T'(t)$ are known, those corresponding to $N = N' - N''$ and $T = T' - T''$ can be found.

As an example, let the boundary γ of a circular disc of unit radius be acted on by <u>any</u> sectionally continuous distribution of radial and transverse stresses having zero resultant force and zero resultant moment.

Using (5.8) with $\bar{t} = 1/t$, $\bar{t}_0 = 1/t_0$, gives

$$N'(t_0) - iT'(t_0) = N(t_0) - iT(t_0) + \frac{1}{2\pi i} \int_\gamma (1 - \frac{t_0}{t}) \frac{N+iT}{t-t_0} \, dt$$

$$- \frac{1}{2\pi i} \int_\gamma t_0^2 (1 - \frac{t}{t_0}) \frac{N-iT}{t-t_0} \, dt ,$$

$$= N(t_0) - iT(t_0) + \frac{1}{2\pi i} \int_\gamma (N + iT) \frac{dt}{t} + t_0 \frac{1}{2\pi i} \int_\gamma (N - iT) \, d\bar{t} .$$

Now $$\int_\gamma (N - iT) \, d\bar{t} = (-i) \int_\gamma (\tilde{n}\tilde{n} - i\tilde{n}\tilde{s}) e^{-i\alpha} \, ds$$

$$= (-i) \int_\gamma (\breve{nx} - i\breve{ny})\, ds \ ,$$

and this is zero because the applied stresses give zero resultant force. Also, $\frac{1}{i}\int_\gamma (N + iT)\frac{dt}{t} = \int_\gamma (\breve{nx} + i\breve{ny})\,\bar{t}\, ds$ and the <u>imaginary</u> part of this is zero since it represents the resultant moment.

Hence $N'(t_o) - iT'(t_o) = N(t_o) - iT(t_o) + \frac{1}{2\pi i}\int_\gamma (N + iT)\,\bar{t}\, dt$,

where the last term is a real constant, Equating real and imaginary parts and dropping the suffix o gives

$$N(t) = N'(t) - \frac{1}{2\pi i}\int_\gamma (N + iT)\,\bar{t}\, dt \ ,$$

$$T(t) = T'(t) \ .$$

Thus in this case removing the stresses corresponding to the potentials (5.4), (5.5) leaves the elementary problem of finding the stresses when the circular boundary is subjected to a uniform normal pressure

$$P = \frac{1}{2\pi i}\int_\gamma (N + iT)\,\bar{t}\, dt \ .$$

This problem is solved by the potentials

$$\Phi(z) = -\tfrac{1}{2}P \quad , \quad \Psi(z) = 0 \ ,$$

and so the given problem is solved by

$$\Phi(z) = \frac{1}{2\pi i}\int_\gamma \frac{N+iT}{t-z}\, dt - \frac{1}{4\pi i}\int_\gamma (N + iT)\,\bar{t}\, dt \ ,$$

$$\Psi(z) = \frac{1}{2\pi i}\int_\gamma \frac{N+iT}{t-z}\, d\bar{t} - \frac{1}{2\pi i}\int_\gamma \frac{N+iT}{(t-z)^2}\,\bar{t}\, dt \ ,$$

This solution can also be obtained by a limiting process from the solution for a circular disc under concentrated forces which is given by Muskhelishvili (M. page 332).

For the problem cited at the beginning of this chapter, $T(t) = 0$ so that

$$\Phi(z) = \frac{1}{2\pi i}\int_{\gamma} N(t)\left\{\frac{1}{t-z} - \tfrac{1}{2}\frac{1}{t}\right\} dt ,$$

$$\Psi(z) = -\frac{1}{2\pi i}\int_{\gamma} N(t)\left\{\frac{1}{(t-z)t^2} + \frac{1}{(t-z)^3 t}\right\} dt ,$$

$$= -\frac{1}{2\pi i}\frac{1}{z}\int_{\gamma} N(t)\left\{\frac{1}{(t-z)^2} - \frac{1}{t^2}\right\} dt .$$

Taking $N(t) = p$ over the arcs $-\alpha < \arg t < \alpha$, $\pi - \alpha < \arg t < \pi + \alpha$ and $N(t) = 0$ elsewhere gives

$$\Phi(z) = \frac{p}{2\pi i}\left\{\log\frac{(a-z)(a+z)}{(\bar{a}-z)(\bar{a}+z)} - \tfrac{1}{2}\log\frac{a^2}{\bar{a}^2}\right\} ,$$

$$\Psi(z) = \frac{p}{2\pi i}\cdot\frac{1}{z}\left\{\frac{1}{a-z} - \frac{1}{\bar{a}-z} - \frac{1}{a+z} + \frac{1}{\bar{a}+z}\right\} ,$$

which reduce to the results stated earlier. ($a = e^{i\alpha}$)

§ 4. Second Method

Although the evaluation of the integrals in (5.9) and (5.10) and the subsequent calculation of the stresses from the complex potentials (5.4) and (5.5) does not present an insurmountable numerical difficulty, it nevertheless seems desirable to describe an alternative method which does not involve the evaluation of integrals.

In this method, a tangent is drawn to the boundary curve at each point where there is a discontinuity in the applied normal traction or the applied tangential traction. Complex potentials are then taken for the problem of a half plane (with its boundary along the tangent) under a set of boundary tractions which possess the same type of discontinuity as the given tractions. The stresses to be "subtracted out" are calculated from these potentials.

To obtain the appropriate complex potentials, consider a half plane whose boundary has an outward normal making an angle α with Ox. Let the boundary be free from stress except for the segment A_1A_2 between the points $z = a_1$ and $z = a_2$ (Figure 2).

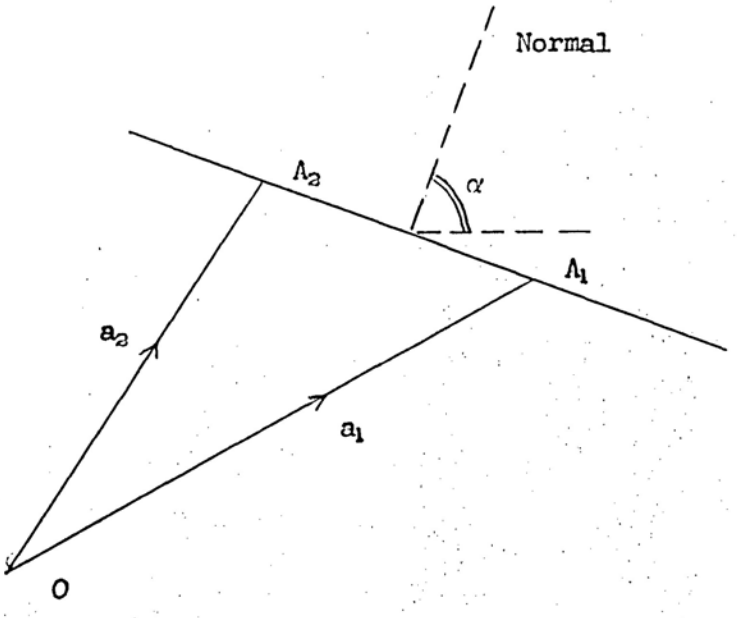


Figure 2

In the first instance, let the segment $A_1 A_2$ be subject to a uniform normal stress $\widetilde{nn} = N$. Then from (5.2) and (5.3), the complex potentials at any point z are given by

$$\Phi(z) = \frac{N}{2\pi i}\int_{a_1}^{a_2} \frac{dt}{t-z}$$

$$= \frac{N}{2\pi i}\left[\log(a_2-z) - \log(a_1-z)\right],$$

$$\Psi(z) = \frac{N}{2\pi i}\int_{a_1}^{a_2} \frac{d\bar{t}}{t-z} - \frac{\bar{t}\,dt}{(t-z)^2}$$

$$= \frac{N}{2\pi i}\int_{a_1}^{a_2} d\left(\frac{\bar{t}}{t-z}\right)$$

$$= \frac{N}{2\pi i}\left[\frac{\bar{a}_2}{a_2-z} - \frac{\bar{a}_1}{a_1-z}\right].$$

As A_1 is moved further away from A_2, both $a_1 - z$ and a_1 increasingly approximate to $|a_1|e^{i(\alpha-\pi/2)}$. Dropping the imaginary constant $\frac{N}{2\pi i}\log|a_1|$ from $\Phi(z)$, proceeding to the limit and writing $a_2 = a$ shows that the potentials

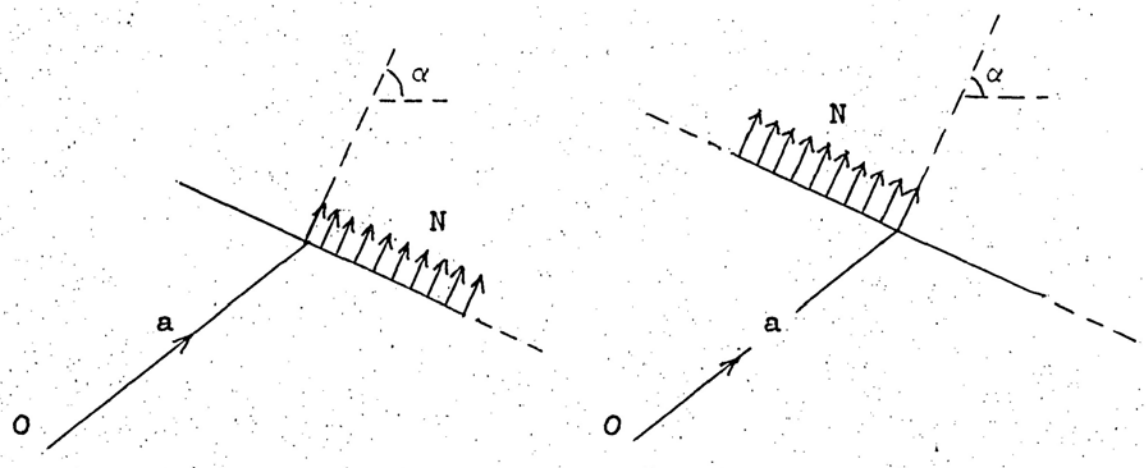
$$\left.\begin{aligned}\Phi(z) &= \frac{N}{2\pi i}\left[\log(a-z) - i(\alpha-\pi/2)\right],\\ \Psi(z) &= \frac{N}{2\pi i}\left[\frac{\bar{a}}{a-z} + e^{-2i\alpha}\right],\end{aligned}\right\} \qquad (5.11)$$

give uniform normal stress N over half the boundary as shown in Figure 3, case (i).

Similarly, moving A_2 to infinity and writing $a_1 = a$, one finds that the potentials

$$\left.\begin{aligned} \Phi(z) &= \frac{N}{2\pi i}\,[-\log(a-z) + i(\alpha+\pi/2)]\,, \\ \Psi(z) &= \frac{N}{2\pi i}\,[-\frac{\bar{a}}{a-z} - e^{-2i\alpha}]\,, \end{aligned}\right\} \qquad (5.12)$$

give uniform normal stress N over the other half of the boundary as shown in Figure 3, case (ii).



Case (i) Case (ii)

Figure 3

As an example, consider again the problem of a circular disc with uniform normal loading p over the arcs $-\alpha < \theta < \alpha$ and $\pi - \alpha < \theta < \pi + \alpha$, the remainder of the boundary being free from stress.

Adding together solutions corresponding to case (i) for the discontinuities at $\theta = \alpha$, $\theta = \pi + \alpha$ and solutions corresponding to case (ii) for the discontinuities at $\theta = -\alpha$, $\pi - \alpha$ gives

$$\Phi(z) = \frac{p}{2\pi i}\left[\log(a-z) + i\left(\frac{\pi}{2} - \alpha\right)\right.$$

$$- \log(\bar{a}-z) + i\left(\frac{\pi}{2} - \alpha\right)$$

$$+ \log(-a-z) + i\left(\frac{\pi}{2} - \pi - \alpha\right)$$

$$\left. - \log(-\bar{a}-z) + i\left(\frac{\pi}{2} + \pi - \alpha\right)\right]$$

$$= \frac{p}{2\pi i} \log \frac{(a-z)(a+z)}{(\bar{a}-z)(\bar{a}+z)} + p\left(1 - \frac{2\alpha}{\pi}\right), \qquad (5.13)$$

and $$\Psi(z) = \frac{p}{2\pi i}\left[\frac{\bar{a}}{a-z} + e^{-2i\alpha} - \frac{a}{\bar{a}-z} - e^{2i\alpha}\right.$$

$$\left. \frac{-\bar{a}}{(-a-z)} + e^{-2i(\pi+\alpha)} + \frac{a}{(-\bar{a}-z)} - e^{-2i(\pi-\alpha)}\right]$$

$$= \frac{p}{\pi i}\left[\frac{1}{a^2-z^2} - \frac{1}{\bar{a}^2-z^2}\right] - \frac{p}{\pi i}\left[e^{2i\alpha} - e^{-2i\alpha}\right]. \qquad (5.14)$$

Comparing these potentials with those of the analytical solution (5.1) it is seen they differ only by the (real) constant terms

$$\Phi(z) = -p\left(1 - \frac{2\alpha}{\pi}\right),$$

$$\Psi(z) = \frac{2p}{\pi} \sin 2\alpha .$$

Thus in this case, "subtracting out" the stresses corresponding to (5.11) and (5.12) leaves boundary stresses for which the application of the method of Chapter 2 would be trivial.

Turning now to the problem of discontinuities in the tangential stress, let the segment $A_1 A_2$ of Figure 2 be subject to a uniform tangential stress $\widetilde{ns} = T$. In this case the complex potentials become

$$\Phi(z) = \frac{T}{2\pi}\int_{a_1}^{a_2} \frac{dt}{t-z}$$

$$= \frac{T}{2\pi}\left[\log(a_2-z) - \log(a_1-z)\right],$$

$$\Psi(z) = -\frac{T}{2\pi}\int_{a_1}^{a_2} \frac{d\bar{t}}{t-z} - \frac{T}{2\pi}\int_{a_1}^{a_2} \frac{\bar{t}\,dt}{(t-z)^2}$$

$$= \frac{T}{\pi} e^{-2i\alpha}\int_{a_1}^{a_2} \frac{dt}{t-z} + \frac{T}{2\pi}\int_{a_1}^{a_2} d\left(\frac{\bar{t}}{t-z}\right)$$

$$= \frac{T}{\pi} e^{-2i\alpha}\left[\log(a_2-z) - \log(a_1-z)\right]$$

$$+ \frac{T}{2\pi}\left[\frac{\bar{a}_2}{a_2-z} - \frac{\bar{a}_1}{a_1-z}\right].$$

It is not possible in this case to make (say) $|a_3| \to \infty$ because of the presence of an infinite real constant.

However, consider the complex potentials

$$\Phi(z) = \frac{T}{2\pi}\,[C - \log(a-z)]\,,$$

$$\Psi(z) = \frac{T}{\pi}\,e^{-2i\alpha}\,[C - \log(a-z) + \frac{T}{2\pi}\,[-e^{-2i\alpha} - \frac{\bar{a}}{a-z}]\,,$$

where C is a (complex) constant and a has been written in place of a_1. If z is a boundary point of the half plane used before, so that $a - z = \rho e^{i(\alpha \pm \frac{\pi}{2})}$, then

$$\bar{z}\,\Phi'(z) + \Psi(z) \equiv -\frac{T}{2\pi}\,\frac{\bar{a}-\bar{z}}{a-z} - \frac{T}{2\pi}\,e^{-2i\alpha} + \frac{T}{\pi}\,e^{-2i\alpha}\,[C - \log(a-z)]$$

$$= \frac{T}{\pi}\,e^{-2i\alpha}\,[C - \log\rho - i(\alpha \pm \frac{\pi}{2})]\,.$$

Since also

$$\Phi(z) = \frac{T}{2\pi}\,[C - \log\rho - i(\alpha \pm \frac{\pi}{2})]\,,$$

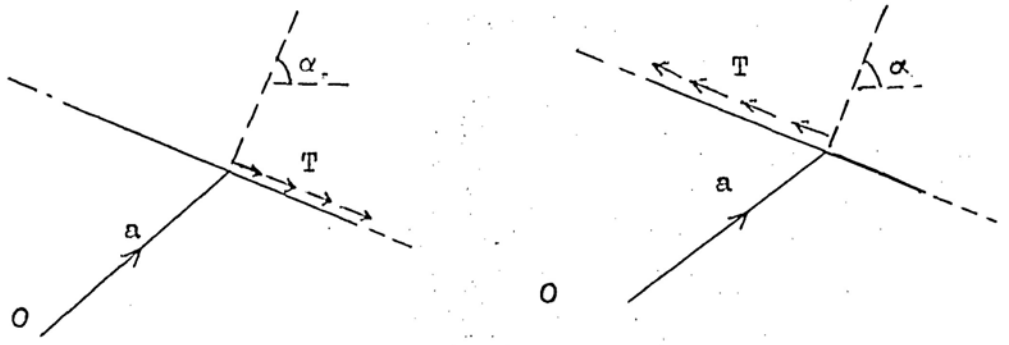
it follows from the Kolosov formula that the boundary tractions are given by

$$\breve{nn} - i\breve{ns} = \Phi(z) + \overline{\Phi(z)} - e^{2i\alpha}\,[\bar{z}\,\Phi'(z) + \Psi(z)]$$

$$= \frac{T}{2\pi}\,[\bar{C} - C + i(2\alpha \pm \pi)]\,.$$

Taking $\bar{C} - C = -2i\alpha + i\pi$, there is a uniform tangential stress $\breve{ns} = -T$ acting over half of the boundary as shown in Figure 4 case (i).

Alternatively, taking $\bar{C} - C = -2i\alpha - i\pi$, there is a uniform tangential stress $\breve{ns} = T$ acting over half of the boundary as shown in Figure 5, case (ii).



Case (i) Case (ii)

Figure 4

§ 5. Discontinuous boundary displacements

It does not seem possible to construct a solution which is the counterpart of Michell's for the problem of discontinuous boundary displacements. Nevertheless, complex potentials completely analogous to (5.4) and (5.5) have been given by Sherman and used by him to construct Fredholm integral equations. (see M. page 420)

Let $u = u(t)$, $v = v(t)$ be the component displacements at any point t of the curve C and let $w(t) \equiv 2\mu(u+iv)$. Following Sherman, consider the potentials

$$\varphi(z) = \frac{1}{2\pi i\mu} \int_C \frac{w(t)}{t-z}\, dt \,, \tag{5.15}$$

$$\psi(z) = -\frac{1}{2\pi i} \int_C \frac{w(t)}{t-z}\, dt + \frac{1}{2\pi i\mu} \int_C \frac{w(t)\,d\bar{t}}{t-z} - \frac{1}{2\pi i\mu} \int_C \frac{\bar{t}\,w(t)\,dt}{(t-z)^2} \,. \tag{5.16}$$

When C is the real axis, these potentials may be shown to solve the problem for the upper half plane (M. p 419). For an arbitrary curve C , the potentials lead to displacements $u'(t)$, $v'(t)$ in general different from $u(t)$, $v(t)$. Making z tend to some point t_0 of C as before and assuming $w(t)$ satisfies the Hölder condition, gives

$$\varkappa\, \varphi(t_0) = \tfrac{1}{2} w(t_0) + \frac{1}{2\pi i} \int_C \frac{w(t)}{t-t_0}\, dt \,.$$

Also, $\bar{z}\,\varphi'(z) + \psi(z) = \frac{1}{2\pi i\kappa}\int_c \frac{\bar{z}-\bar{t}}{(t-z)^2}\,w(t)\,dt - \frac{1}{2\pi i}\int_c \frac{\bar{w}(t)}{t-z}\,dt$

$$+ \frac{1}{2\pi i\kappa}\int_c \frac{w(t)}{t-z}\,d\bar{t}\ ,$$

$$= -\frac{1}{2\pi i}\int_c \frac{\bar{w}(t)}{t-z}\,dt + \frac{1}{2\pi i\kappa}\int_c w(t)\,d\left(\frac{\bar{t}-\bar{z}}{t-z}\right)\ ,$$

from which $\bar{t}_o\,\varphi'(t_o) + \psi(t_o) = -\frac{1}{2}\,\bar{w}(t_o) - \frac{1}{2\pi i}\int_c \frac{\bar{w}(t)}{t-t_o}\,dt$

$$+ \frac{1}{2\pi i\kappa}\int_c w(t)\,d\left(\frac{\bar{t}-\bar{t}_o}{t-t_o}\right)\ .$$

Hence $2\mu\,[u'(t_o) + iv'(t_o)] \equiv \kappa\,\varphi(t_o) - t_o\,\overline{\varphi'(t_o)} - \overline{\psi(t_o)}$

$$= w(t_o) + \frac{1}{2\pi i}\int_c w(t)\,d\log\frac{t-t_o}{\bar{t}-\bar{t}_o}$$

$$+ \frac{1}{2\pi i\kappa}\int_c \bar{w}(t)\,d\left(\frac{t-t_o}{\bar{t}-\bar{t}_o}\right)\ ,$$

or $u'(t_o) + iv'(t_o) = u(t_o) + iv(t_o) + \frac{1}{2\pi i}\int_c (u+iv)\,d\log\frac{t-t_o}{\bar{t}-\bar{t}_o}$

$$+ \frac{1}{2\pi i\kappa}\int_c (u-iv)\,d\left(\frac{t-t_o}{\bar{t}-\bar{t}_o}\right)\ . \qquad (5.17)$$

As before, real forms are obtained by writing $t - t_o = re^{i\varphi}$. This gives

$$u'(t_o) + iv'(t_o) = u(t_o) + iv(t_o) + \frac{1}{\pi}\int_c (u+iv)\,d\varphi + \frac{1}{\pi\kappa}\int_c (u-iv)e^{2i\varphi}\,d\varphi$$

and so $u'(t_o) = u(t_o) + \frac{1}{\pi\kappa}\int_c \{u(t)\,(\kappa + \cos 2\varphi) + v(t)\sin 2\varphi\}\,d\varphi, \qquad (5.18)$

$$v'(t_o) = v(t_o) + \frac{1}{\pi\kappa}\int_c \{u(t)\sin 2\varphi - v(t)\,(\kappa - \cos 2\varphi)\}\,d\varphi\ . \qquad (5.19)$$

Clearly neither of these integrals is singular; if $u(t)$ and $v(t)$ are sectionally continuous each of the integrals is a continuous function of t_o. Thus, as with equations (5.8) and (5.9), the solution of a discontinuous problem can be reduced to the solution of one in which the boundary displacements are continuous.

As an example, let the boundary γ of a unit circular hole in an infinite plate be given any set of <u>purely radial</u> displacements $v_r(t)$, so that

$$u(t) + iv(t) = v_r(t)\,e^{i\theta} = tv_r(t)\ ,$$

and suppose that $v_r(t)$ is sectionally continuous.

Since $\bar{t} = 1/t$, $\bar{t}_o = 1/t_o$, one has

$$(t-t_o)/(\bar{t}-\bar{t}_o) = - tt_o \quad ,$$

and because the region is exterior to γ , the contour C becomes the circle γ traversed clockwise. Hence, from (5.17),

$$u'(t_o) + iv'(t_o) = v_r(t_o)\, t_o - \frac{1}{2\pi i}\int_\gamma v_r(t)\, dt + \frac{t_o}{2\pi i\mu}\int_\gamma v_r(t)\,\frac{dt}{t} \quad .$$

The second term is a complex constant which represents a rigid body displacement and will be assumed to be zero. Writing

$$u'(t_o) + iv'(t_o) = [v_r'(t_o) + iv_\theta'(t_o)]\, t_o$$

then gives $v_r'(t_o) + iv_\theta'(t_o) = v_r(t_o) + \frac{1}{2\pi i\mu}\int_\gamma v_r(t)\,\frac{dt}{t}$,

and since $\frac{1}{2\pi i\mu}\int_\gamma v_r(t)\,\frac{dt}{t} = \frac{1}{2\pi\mu}\int_0^{2\pi} v_r\, d\theta$ is real, it follows that

$$v_r'(t_o) = v_r(t_o) + \frac{1}{2\pi i\mu}\int_\gamma v_r(t)\,\frac{dt}{t} \quad ,$$

$$v_\theta'(t_o) = 0 \quad .$$

It is therefore merely necessary to solve the elementary problem in which the boundary of the hole has a uniform radial displacement

$$v_r = \epsilon = \frac{1}{2\pi i\mu}\int_\gamma v_r(t)\,\frac{dt}{t} \quad ,$$

and for this problem the complex potentials are

$$\varphi(z) = 0 \quad , \quad \psi(z) = - 2\mu\epsilon/z \quad .$$

Subtracting these from the potentials given by (5.15), (5.16) gives the solution of the original problem in the form

$$\varphi(z) = - \frac{1}{2\pi i\mu}\int_\gamma \frac{2\mu v_r\, t\,dt}{t-z} \quad ,$$

$$\psi(z) = \frac{1}{2\pi i}\int_\gamma \frac{2\mu v_r\, dt}{t(t-z)} + \frac{1}{2\pi i\mu}\int_\gamma \frac{2\mu v_r}{(t-z)}\,\frac{dt}{t} + \frac{1}{2\pi i\mu}\int_\gamma \frac{2\mu v_r\, dt}{(t-z)^2}$$

$$+ \frac{1}{2\pi i\mu}\,\frac{1}{z}\int_\gamma \frac{2\mu v_r\, dt}{t}$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{2\mu v_r \, dt}{t(t-z)} + \frac{1}{2\pi i_\kappa} \frac{1}{z} \int_\gamma \frac{2\mu v_r \, t dt}{(t-z)^2} \quad .$$

These agree with formulae (3.9), (3.11).

§ 6. Change in direction of the boundary

When there is a sudden change in the direction of the boundary at some point, Williams (1952) has shown that it is possible for the stresses to tend to infinity at that point even if the boundary stresses or displacements are finite and continuous.

Williams considers a sector of angle α with zero stresses or zero displacements along the radial edges $\theta = 0$ and $\theta = \alpha$. Solutions of the equations of elasticity in polar co-ordinates are then constructed using the Airy stress function U and the auxiliary harmonic function ψ of Chapter 1. The stresses are proportional to $r^{\lambda-1}$ and the edge conditions lead to transcendental equations for λ.

The required stresses and displacements are readily obtained from a simple choice of complex potentials:

Take $\varphi(z) = az^\lambda$, $\psi(z) = bz^\lambda$,

where $a = a_1 + ia_2$, $b = b_1 + ib_2$ and for the present, λ is real.

Then $\breve{\theta\theta} + \breve{rr} = 4\mathrm{Re}\ \bar{\Phi}(z) = 4\mathrm{Re}\ a\lambda r^{\lambda-1} e^{i(\lambda-1)\theta}$

$$= 4\lambda r^{\lambda-1} [a_1 \cos(\lambda-1)\theta - a_2 \sin(\lambda-1)\theta] , \qquad (5.20)$$

and $\breve{\theta\theta} - \breve{rr} + 2i\breve{r\theta} = 2e^{2i\theta} [\, \bar{z}\, \bar{\Phi}'(z) + \Psi(z) \,]$

$$= 2\lambda r^{\lambda-1} [a(\lambda-1)e^{i(\lambda-1)\theta} + be^{i(\lambda+1)\theta}] . \qquad (5.21)$$

Taking the real part of (5.21) gives

$$\breve{\theta\theta} - \breve{rr} = 2\lambda r^{\lambda-1} [\, (\lambda-1)\, \{a_1 \cos(\lambda-1)\theta - a_2 \sin(\lambda-1)\theta\} + b_1 \cos(\lambda+1)\theta - b_2 \sin(\lambda+1)\theta]$$

and so adding or subtracting (5.20),

$$\widetilde{rr} = \lambda r^{\lambda-1} [-(\lambda-3)\, a_1 \cos(\lambda-1)\theta + (\lambda-3)\, a_2 \sin(\lambda-1)\theta$$

$$- b_1 \cos(\lambda+1)\theta + b_2 \sin(\lambda+1)\theta] \quad ,$$

$$\widetilde{\theta\theta} = \lambda r^{\lambda-1} [(\lambda+1)\, a_1 \cos(\lambda-1)\theta - (\lambda+1)\, a_2 \sin(\lambda-1)\theta$$

$$+ b_1 \cos(\lambda+1)\theta - b_2 \sin(\lambda+1)\theta] \quad .$$

In addition, the imaginary part of (5.21) gives

$$\widetilde{r\theta} = \lambda r^{\lambda-1} [(\lambda-1)\, a_1 \sin(\lambda-1)\theta + (\lambda-1)\, a_2 \cos(\lambda-1)\theta$$

$$+ b_1 \sin(\lambda+1)\theta + b_2 \cos(\lambda+1)\theta] \quad .$$

For the displacements, the formula

$$2\mu(v_r + iv_\theta) = e^{-i\theta} [\kappa\, \varphi(z) - z\, \overline{\varphi'(z)} - \overline{\psi(z)}]$$

gives $2\mu(v_r + iv_\theta) = r^\lambda [\kappa a e^{i(\lambda-1)\theta} - \bar{a}\lambda e^{-i(\lambda-1)\theta} - \bar{b} e^{-i(\lambda+1)\theta}]$

and so $2\mu v_r = r^\lambda [(\kappa-\lambda)\, a_1 \cos(\lambda-1)\theta - (\kappa-\lambda)\, a_2 \sin(\lambda-1)\theta - b_1 \cos(\lambda+1)\theta$

$$+ b_2 \sin(\lambda+1)\theta] \, ,$$

$$2\mu v_\theta = r^\lambda [(\kappa+\lambda)\, a_1 \sin(\lambda-1)\theta + (\kappa+\lambda)\, a_2 \cos(\lambda-1)\theta + b_1 \sin(\lambda+1)\theta$$

$$+ b_2 \cos(\lambda+1)\theta] \, .$$

Although it has been assumed that λ is real, these stresses and displacements will satisfy the equations of elasticity for all values of λ. If λ is complex, the stresses and displacements are complex but their real and imaginary parts separately satisfy the equations and so either may be taken throughout.

Since the displacement components must be finite at the origin, it is always necessary to have $\mathrm{Re}(\lambda) \geq 0$. But then, if $0 < \mathrm{Re}(\lambda) < 1$, each of the three stress components becomes infinite at the origin. The possibility of solutions with λ in this range will now be considered for special edge conditions. The first three of these were discussed by Williams.

§ 7. "Free-free" edges

If both radial edges are prescribed to be free (of stress),

$$\breve{\theta\theta} = 0 \quad \text{and} \quad \breve{r\theta} = 0 \quad \text{when} \quad \theta = 0 ,$$

$$\breve{\theta\theta} = 0 \quad \text{and} \quad \breve{r\theta} = 0 \quad \text{when} \quad \theta = \alpha .$$

These conditions give

$$(\lambda+1)\, a_1 + b_1 = 0 , \quad (\lambda-1)\, a_2 + b_2 = 0 ,$$

$$(\lambda+1)\, a_1 \cos(\lambda-1)\alpha - (\lambda+1)\, a_2 \sin(\lambda-1)\alpha + b_1 \cos(\lambda+1)\alpha - b_2 \sin(\lambda+1)\alpha = 0 ,$$

$$(\lambda-1)\, a_1 \sin(\lambda-1)\alpha + (\lambda-1)\, a_2 \cos(\lambda-1)\alpha + b_1 \sin(\lambda+1)\alpha + b_2 \cos(\lambda+1)\alpha = 0 .$$

For these to be consistent, eliminating b_1, b_2 gives

$$(\lambda+1)\, a_1 [\cos(\lambda-1)\alpha - \cos(\lambda+1)\alpha] - a_2 [(\lambda+1) \sin(\lambda-1)\alpha - (\lambda-1) \sin(\lambda+1)\alpha] = 0 ,$$

$$a_1 [(\lambda-1) \sin(\lambda-1)\alpha - (\lambda+1) \sin(\lambda+1)\alpha] + (\lambda-1)\, a_2 [\cos(\lambda-1)\alpha - \cos(\lambda+1)\alpha] = 0 ,$$

or $a_1 (\lambda+1) \sin \lambda\alpha \sin \alpha + a_2 [\lambda \sin \alpha \cos \lambda\alpha - \sin \lambda\alpha \cos \alpha] = 0 ,$

$$a_1 [\lambda \sin \alpha \cos \lambda\alpha + \sin \lambda\alpha \cos \alpha] - a_2 (\lambda-1) \sin \lambda\alpha \sin \alpha = 0 ,$$

and eliminating a_1, a_2 gives

$$(\lambda^2-1) \sin^2 \lambda\alpha \sin^2 \alpha + \lambda^2 \sin^2 \alpha \cos^2 \lambda\alpha - \sin^2 \lambda\alpha \cos^2 \alpha = 0$$

or

$$\sin^2 \lambda\alpha = \lambda^2 \sin^2 \alpha . \tag{5.22}$$

This equation has no root λ such that $0 < \mathrm{Re}(\lambda) < 1$ unless $\alpha > 180^\circ$.

§ 8. "Clamped-clamped" edges

If both radial edges are fixed, then

$$v_r = 0 \quad \text{and} \quad v_\theta = 0 \quad \text{when} \quad \theta = 0 ,$$

$$v_r = 0 \quad \text{and} \quad v_\theta = 0 \quad \text{when} \quad \theta = \alpha .$$

These conditions give

$$(\varkappa-\lambda)\, a_1 - b_1 = 0 , \quad (\varkappa+\lambda)\, a_2 + b_2 = 0 ,$$

$$(\varkappa-\lambda)\, a_1 \cos(\lambda-1)\alpha - (\varkappa-\lambda)\, a_3 \sin(\lambda-1)\alpha - b_1 \cos(\lambda+1)\alpha + b_2 \sin(\lambda+1)\alpha = 0 ,$$

$$(\varkappa+\lambda)\, a_1 \sin(\lambda-1)\alpha + (\varkappa+\lambda)\, a_3 \cos(\lambda-1)\alpha + b_1 \sin(\lambda+1)\alpha + b_2 \cos(\lambda+1)\alpha = 0 .$$

As before, eliminating b_1, b_2 gives

$$(\varkappa-\lambda)\, a_1 [\cos(\lambda-1)\alpha - \cos(\lambda+1)\alpha] - a_2 [(\varkappa-\lambda) \sin(\lambda-1)\alpha + (\varkappa+\lambda) \sin(\lambda+1)\alpha] = 0 ,$$

$$a_1 [(\varkappa+\lambda) \sin(\lambda-1)\alpha + (\varkappa-\lambda) \sin(\lambda+1)\alpha] + (\varkappa+\lambda)\, a_2 [\cos(\lambda-1)\alpha - \cos(\lambda+1)\alpha] = 0 ,$$

or $$a_1 (\varkappa-\lambda) \sin \lambda\alpha \sin\alpha - a_2 [\varkappa \sin \lambda\alpha \cos\alpha + \lambda \sin\alpha \cos \lambda\alpha] = 0 ,$$

$$a_1 (\varkappa \sin \lambda\alpha \cos\alpha - \lambda \cos \lambda\alpha \sin\alpha] + a_3 (\varkappa+\lambda) \sin \lambda\alpha \sin\alpha = 0 ,$$

and eliminating a_1, a_3 this time gives

$$\varkappa^2 \sin^2 \lambda\alpha = \lambda^2 \sin^2 \alpha . \qquad (5.23)$$

For the range of possible values of the modified Poisson's ratio $(0 \le \eta \le 1/3)$, $\varkappa > 1$ and equation (5.23) has no root with $0 < \mathrm{Re}(\lambda) < 1$ unless $\alpha > 180^\circ$.

§ 9. "Clamped-free" edges

If the radial edge $\theta = 0$ is free and the edge $\theta = \alpha$ is clamped, then

$$\widetilde{\theta\theta} = 0 \quad \text{and} \quad \widetilde{r\theta} = 0 \quad \text{when} \quad \theta = 0 ,$$

$$v_r = 0 \quad \text{and} \quad v_\theta = 0 \quad \text{when} \quad \theta = \alpha .$$

These conditions give

$$(\lambda+1)\, a_1 + b_1 = 0 , \quad (\lambda-1)\, a_2 + b_2 = 0 ,$$

$$(\varkappa-\lambda)\, a_1 \cos(\lambda-1)\alpha - (\varkappa-\lambda)\, a_2 \sin(\lambda-1)\alpha - b_1 \cos(\lambda+1)\alpha + b_2 \sin(\lambda+1)\alpha = 0,$$

$$(\varkappa+\lambda)\, a_1 \sin(\lambda-1)\alpha + (\varkappa+\lambda)\, a_2 \cos(\lambda-1)\alpha + b_1 \sin(\lambda+1)\alpha + b_2 \cos(\lambda+1)\alpha = 0 .$$

Eliminating b_1, b_2 in this case gives

$$a_1 [(\varkappa-\lambda) \cos(\lambda-1)\alpha + (\lambda+1) \cos(\lambda+1)\alpha] - a_2 [(\varkappa-\lambda) \sin(\lambda-1)\alpha + (\lambda-1) \sin(\lambda+1)\alpha] = 0,$$

$$a_1 [(\varkappa+\lambda) \sin(\lambda-1)\alpha - (\lambda+1) \sin(\lambda+1)\alpha] + a_2 [(\varkappa+\lambda) \cos(\lambda-1)\alpha - (\lambda-1) \cos(\lambda+1)\alpha] = 0,$$

or $$a_1 [(\varkappa+1) \cos \lambda\alpha \cos\alpha + (\varkappa-1-2\lambda) \sin \lambda\alpha \sin\alpha] -$$

(continued)

$$- a_2[(\varkappa-1)\sin\lambda\alpha\cos\alpha - (\varkappa+1-2\lambda)\sin\alpha\cos\lambda\alpha] = 0 ,$$

$$a_1[(\varkappa-1)\sin\lambda\alpha\cos\alpha - (\varkappa+1+2\lambda)\sin\alpha\cos\lambda\alpha]$$

$$+ a_2[(\varkappa+1)\cos\lambda\alpha\cos\alpha + (\varkappa-1+2\lambda)\sin\lambda\alpha\sin\alpha] = 0 .$$

Finally, eliminating a_1 , a_2 and simplifying gives

$$(\varkappa+1)^2\cos^2\lambda\alpha + (\varkappa-1)^2\sin^2\lambda\alpha - 4\lambda^2\sin^2\alpha = 0$$

or
$$\varkappa\sin^2\lambda\alpha = \tfrac{1}{4}(\varkappa+1)^2 - \lambda^2\sin^2\alpha . \qquad (5.24)$$

Taking $\eta = 0{\cdot}3$, Williams finds that this equation has a root λ with $0 < \mathrm{Re}(\lambda) < 1$ whenever $\alpha > 63^\circ$.

For $\alpha = 180^\circ$, the equation becomes

$$\sin^2\lambda\pi = (\varkappa+1)^2/4\varkappa$$

or
$$\cos 2\lambda\pi = -(\varkappa+1/\varkappa) .$$

Since $\varkappa > 1$, the right hand side is less than -1 ; hence

$$2\lambda\pi = (2n+1)\pi + i\beta$$

where $n = 0, \pm 1, \pm 2, \ldots$ and $\cosh\beta = \frac{1}{2}(\varkappa+1/\varkappa)$. Thus for all values of η , the equation (5.24) has a root λ with $\mathrm{Re}(\lambda) = \frac{1}{2}$.

This result explains the lack of success with the inclusion problem of Chapter 4 when the included disc is <u>rough</u>. For if A is one of the four points of the plate at the end of an arc of contact with the disc, the edge conditions of this special case are satisfied at least along infinitesimal parts of the tangent line at A . Thus the stresses near A tend to infinity, varying as $r^{-\frac{1}{2}}$ where r is measured from A . The radial stress along the arc of contact, in particular, tends to infinity at A .

To examine the corresponding situation when the included disc is <u>smooth</u>, it is convenient to consider first the more general case:

§ 10. One edge free, one edge in contact with a smooth rigid edge

If $\theta = \alpha$ is taken to be the free edge, one has

$$\breve{r\theta} = 0 \quad \text{and} \quad v_\theta = 0 \quad \text{when} \quad \theta = 0 ,$$

$$\breve{r\theta} = 0 \quad \text{and} \quad \breve{\theta\theta} = 0 \quad \text{when} \quad \theta = \alpha .$$

The first two conditions give

$$(\lambda-1)\, a_2 + b_2 = 0 , \quad (\mu+\lambda)\, a_2 + b_2 = 0$$

and hence

$$a_2 = b_2 = 0 . \tag{5.25}$$

The remaining two conditions then give

$$\begin{aligned} a_1 (\lambda-1) \sin(\lambda-1)\alpha + b_1 \sin(\lambda+1)\alpha &= 0 , \\ a_1 (\lambda+1) \cos(\lambda-1)\alpha + b_1 \cos(\lambda+1)\alpha &= 0 , \end{aligned} \tag{5.26}$$

and on eliminating the ratio $a_1 : b_1$,

$$(\lambda-1) \sin(\lambda-1)\alpha \cos(\lambda+1)\alpha = (\lambda+1) \cos(\lambda-1)\alpha \sin(\lambda+1)\alpha$$

i.e. $(\lambda-1)\,[\sin 2\lambda\alpha - \sin 2\alpha] = (\lambda+1)\,[\sin 2\lambda\alpha + \sin 2\alpha]$

or

$$\sin 2\lambda\alpha = -\lambda \sin 2\alpha . \tag{5.27}$$

When $\alpha = 180^\circ$, this equation is satisfied by

$$\lambda = 0, \pm 1, \pm 2, \ldots \quad \text{and} \quad \lambda = \pm \tfrac{1}{2} , \pm \tfrac{3}{2}, \ldots .$$

The integer values are of no interest since they lead to stresses which are analytic at the origin. For the half integer values, the second of equations (5.26) is satisfied identically since $\cos \lambda\pi = 0$ while the first requires

$$a_1 (\lambda-1) + b_1 = 0 \tag{5.28}$$

since $\sin \lambda\pi \neq 0$.

The three stress components then become

$$\breve{r\theta} = a_1\, \lambda(\lambda-1) r^{\lambda-1} [\sin(\lambda-1)\theta - \sin(\lambda+1)\theta] ,$$

$$\breve{\theta\theta} = a_1\, \lambda r^{\lambda-1} [(\lambda+1) \cos(\lambda-1)\theta - (\lambda-1) \cos(\lambda+1)\theta] ,$$

$$\breve{rr} = a_1 \lambda r^{\lambda-1} [-(\lambda-3) \cos(\lambda-1)\theta + (\lambda-1) \cos(\lambda+1)\theta] .$$

To keep the displacements finite at the origin, λ has to be positive. But when $\lambda = \frac{1}{2}$,

$$\breve{r\theta} = \tfrac{1}{4} a_1 r^{-\frac{1}{2}} (\sin \theta/2 + \sin 3\theta/2) ,$$

so that although $\breve{r\theta} = 0$ for $\theta = 0$ and $\theta = \pi$ when $r \neq 0$, it is nevertheless indeterminate when $r = 0$. However when $r \to 0$ along any line θ = constant ($\neq 0$ or π) , $\breve{r\theta} \to \infty$. This solution therefore is inadmissible as a model for the problem of the smooth inclusion since $\breve{r\theta} = 0$ everywhere along the boundary of the hole.

This leaves $\lambda = 3/2$, for which

$$\breve{r\theta} = \tfrac{3}{4} a_1 r^{\frac{1}{2}} (\sin \theta/2 - \sin 5\theta/2) ,$$

$$\breve{\theta\theta} = \tfrac{3}{4} a_1 r^{\frac{1}{2}} (5 \cos \theta/2 - \cos 5\theta/2) ,$$

$$\breve{rr} = \tfrac{3}{4} a_1 r^{\frac{1}{2}} (3 \cos \theta/2 + \cos \theta/2) .$$

Although none of these becomes infinite, each has infinite derivatives at the origin. The corresponding displacements are given by

$$2\mu v_r = a_1 r^{\frac{3}{2}} [(\kappa-\tfrac{1}{2}) \cos \theta/2 - \tfrac{1}{2} \cos 5\theta/2] ,$$

$$2\mu v_\theta = a_1 r^{\frac{3}{2}} [(\kappa+\tfrac{1}{2}) \sin \theta/2 + \tfrac{1}{2} \sin 5\theta/2] .$$

In particular, when $\theta = 0$,

$$\breve{\theta\theta} = 3a_1 r^{\frac{1}{2}} , \quad 2\mu v_\theta = 0 , \tag{5.29}$$

and when $\theta = \pi$,

$$\breve{\theta\theta} = 0 , \quad 2\mu v_\theta = (\kappa+1) a_1 r^{\frac{3}{2}} . \tag{5.30}$$

In the case of the smooth inclusion, $\breve{\theta\theta}$ and v_θ correspond to the boundary <u>radial</u> stress and <u>radial</u> displacement respectively. Equations (5.29) and (5.30) show their behaviour in the neighbourhood of a point such as A , the end of one of the arcs of contact, with r measured from A .

CHAPTER 6

Mixed Boundary Conditions (continued)

In this chapter, the problem of an infinite plate with a smooth rigid circular inclusion is considered again to illustrate a treatment of singularities based on the analysis of Chapter 5.

§ 1. Résumé

It was shown in Chapter 4 that to make $\breve{r\theta} = 0$ at all points of the boundary, the complex potentials have the form

$$\left.\begin{aligned} \varphi(z) &= \tfrac{1}{4}Tz + \sum_{k=1}^{\infty} \alpha_k / z^{2k-1}, \\ \psi(z) &= -\tfrac{1}{2}Tz + \sum_{k=1}^{\infty} \beta_k / z^{2k-1}, \end{aligned}\right\} \tag{6.1}$$

where

$$\left.\begin{aligned} 3\beta_2 &= 2.1\,\alpha_1 + \tfrac{1}{2}T, \\ (2k+1)\beta_{k+1} &= 2k(2k-1)\alpha_k, \quad k \geq 2, \end{aligned}\right\} \tag{6.2}$$

and that the boundary radial stress and radial displacement are given by

$$\breve{rr} = T(\tfrac{1}{2}+U_2) + \beta_1 - 2\sum_{k=1}^{\infty} (2k-1)\,\alpha_k\, U_{2k},$$

$$2\mu v_r = T(\tfrac{1}{2}-\eta+\tfrac{1}{3}U_2) - \beta_1 + \sum_{k=1}^{\infty} (\kappa + \frac{2k-1}{2k-1})\,\alpha_k\, U_{2k},$$

where $U_{2k} \equiv \cos 2k\theta$.

If instead of (6.1) the complex potentials are written as

$$\Phi(z) = \tfrac{1}{4}T + \sum_{k=1}^{\infty} A_k / z^{2k},$$

$$\Psi(z) = -\tfrac{1}{2}T + \sum_{k=1}^{\infty} B_k / z^{2k},$$

so that $A_k = -(2k-1)\,\alpha_k$, $B_k = -(2k-1)\,\beta_k$, the relation between the coefficients becomes

$$B_2 = 2A_1 - \tfrac{1}{2}T \quad , \tag{6.3}$$

$$B_{k+1} = 2kA_k \quad , \; k \geq 2 \; , \tag{6.4}$$

and $$\widetilde{rr} = T(\tfrac{1}{2}+U_2) - B_1 + 2\sum_{k=1}^{\infty} A_k U_k \quad ,$$

$$2\mu v_r = T(\tfrac{1}{2}-\eta+\tfrac{1}{3}U_2) + B_1 - \sum_{k=1}^{\infty} \left(\frac{\kappa}{2k-1} + \frac{1}{2k+1}\right) A_k U_{2k} \; .$$

Some simplification occurs if A_1 is replaced by $A_1 - \frac{1}{2}T$ and B_1 replaced by $-A_0 + \frac{1}{2}T$ so that

$$\widetilde{rr} = A_0 + 2\sum_{k=1}^{\infty} A_k \; U_{2k} \; ; \tag{6.5}$$

it is then found that

$$2\mu v_r = T(1-\eta)\;(1+2U_2) - A_0 - \sum_{k=1}^{\infty} \left(\frac{\kappa}{2k-1} + \frac{1}{2k+1}\right) A_k \; U_{2k} \; . \tag{6.6}$$

The relation (6.3) now becomes

$$B_2 = 2A_1 - 3T/2$$

and can be brought into line with the others by replacing B_2 by $B_2 + 3T/2$. The complex potentials then become

$$\left.\begin{aligned} \Phi(z) &= \tfrac{1}{4}T - \tfrac{1}{2}T/z^2 + \sum_{k=1}^{\infty} A_k/z^{2k} \quad , \\ \Psi(z) &= -\tfrac{1}{2}T + \tfrac{1}{2}T/z^2 - 3T/2z^4 + \sum_{k=1}^{\infty} B_k/z^{2k} \quad , \end{aligned}\right\} \tag{6.7}$$

with $B_1 = -A_0$, $B_{k+1} = 2k\,A_k$ for $k \geq 1$.

The terms in T can be recognised as those which solve the extensional problem for an infinite plate with an <u>unstressed</u> circular hole.

§ 2. <u>Removal of singularities</u>

The analysis of Chapter 5 has shown that in the neighbourhood of each of the points A, B, C, D of Figure 1 of Chapter 4, the stresses and displacements vary as $r^{\frac{1}{2}}$ and $r^{\frac{3}{2}}$ respectively, where r is the distance from the point in question. Thus along the arc $0 \leq \theta \leq \pi/2$ of the boundary of the hole one has

$$\left.\begin{aligned} \widetilde{rr} &= 0 \quad \text{for} \quad \theta \leq \alpha, \\ \widetilde{rr} &< 0 \quad \text{for} \quad \theta > \alpha, \end{aligned}\right\} \tag{6.8}$$

with $\breve{rr} \propto (\theta-\alpha)^{\frac{1}{2}}$ for small positive values of $\theta - \alpha$,

and
$$\left.\begin{aligned} v_r &= 0 \quad \text{for} \quad \theta \geq \alpha \ , \\ v_r &> 0 \quad \text{for} \quad \theta < \alpha \ , \end{aligned}\right\} \tag{6.9}$$

with $v_r \propto (\alpha-\theta)^{\frac{3}{2}}$ for small negative values of $\theta - \alpha$. The two constants of proportionality are in fact related:

If $$\breve{rr} \sim - a(\theta-\alpha)^{\frac{1}{2}} \quad \text{when} \quad \theta \to \alpha + \ ,$$

then $$2\mu v_r \sim \tfrac{1}{3}(\kappa+1)\, a(\alpha-\theta)^{\frac{3}{2}} \quad \text{when} \quad \theta \to \alpha - \ , \tag{6.10}$$

but the analysis does <u>not</u> determine the value of a.

The discontinuity in the derivative of $\breve{rr}$ w.r.t. θ at the point $\theta = \alpha$ makes the series for $\breve{rr}$ (6.5) very slowly convergent. Using a theorem of Bromwich (which will be stated later), the coefficients A_k can be shown to be $O(k^{-\frac{1}{2}})$ as $k \to \infty$. In consequence, to represent $\breve{rr}$ at all accurately by a finite series will need an excessive number of terms; the same is also true of v_r but to a lesser extent. Only by removing analytically the dominant term or terms of the singularity and leaving "smooth" residual functions to be approximated can sufficient precision be achieved.

With this object, the method adopted was to take a function $f(\theta)$ possessing the essential features of the boundary radial stress listed at (6.8) together with the symmetry properties

$$f(-\theta) = f(\theta) \ , \ f(\pi-\theta) = f(\theta) \ .$$

Corresponding to the assumed loading

$$\breve{rr} = f(\theta) \ , \ \breve{r\theta} = 0$$

on the boundary of the hole and zero stresses at infinity, there are certain boundary displacements given (say) by

$$2\mu v_r = g(\theta) \ , \ 2\mu v_\theta = g_1(\theta) \ , \tag{6.11}$$

of which $g(\theta)$ has to be calculated.

Then, in the problem of the rigid inclusion, approximations to the boundary values $R \equiv \breve{rr}$ and $S \equiv 2\mu v_r - T(1-\eta)(1+2U_2)$ may be taken in the form

$$\tilde{R} = af(\theta) + a_0 + 2\sum_{k=1}^{n} a_k U_{2k} ,$$

$$\tilde{S} = ag(\theta) - a_0 - \sum_{k=1}^{n} a_k \left(\frac{\kappa}{2k-1} + \frac{1}{2k+1}\right) U_{2k} ,$$

respectively, and the coefficients $a, a_0, a_1, \ldots, a_n$ determined as before by the method of least squares. If the constant a is correctly chosen, it is clear that the remaining terms represent functions which at least have continuous derivatives.

The calculation of $g(\theta)$ is certainly possible if $f(\theta)$ possesses a Fourier series

$$f(\theta) = f_0 + 2\sum_{n=1}^{\infty} f_n \cos 2n\theta$$

with easily determined coefficients. For then, by (6.6) with $T = 0$,

$$g(\theta) = - f_0 - \sum_{n=1}^{\infty} f_n \left(\frac{\kappa}{2n-1} + \frac{1}{2n+1}\right) \cos 2n\theta . \qquad (6.12)$$

Guided by this, a simple choice for $f(\theta)$ possessing the necessary symmetries and having the required behaviour in the neighbourhood of the points $\theta = \pm\alpha, \pm(\pi-\alpha)$ was

$$\left.\begin{aligned} f(\theta) &= -(\cos 2\alpha - \cos 2\theta)^{\frac{1}{2}} \quad \text{for} \quad \alpha \le |\theta| \le \pi - \alpha , \\ &= 0 \quad \text{elsewhere} . \end{aligned}\right\} \qquad (6.13)$$

The coefficients f_n are given by

$$f_n = -\frac{2}{\pi}\int_{\alpha}^{\pi/2} (\cos 2\alpha - \cos 2\theta)^{\frac{1}{2}} \cos 2n\theta \, d\theta \qquad (6.14)$$

and a recurrence relation for them may be found by considering the function

$$U = (\cos 2\alpha - \cos 2\theta)^{\frac{3}{2}} \sin 2n\theta ,$$

for which $\frac{dU}{d\theta} = \frac{3}{2}(\cos 2\alpha - \cos 2\theta)^{\frac{1}{2}} 2 \sin 2\theta \sin 2n\theta$

$$+ (\cos 2\alpha - \cos 2\theta)^{\frac{3}{2}} 2n \cos 2n\theta$$

$$= \frac{3}{2}(\cos 2\alpha - \cos 2\theta)^{\frac{1}{2}} [\cos 2(n-1)\theta - \cos 2(n+1)\theta] +$$

(continued)

$$+ 2n \cos 2\alpha \, (\cos 2\alpha - \cos 2\theta)^{\frac{1}{2}} \cos 2n\theta$$

$$- n(\cos 2\alpha - \cos 2\theta)^{\frac{1}{2}} [\cos 2(n-1)\theta + \cos 2(n+1)\theta].$$

Integrating between the limits α and $\pi/2$ gives

$$(n+3/2)\, f_{n+1} = 2n \cos 2\alpha \, f_n - (n-3/2)\, f_{n-1} \, . \qquad (6.15)$$

This recurrence relation is stable. For on writing $u_n = n^{\frac{3}{2}} f_n$, so that when n is large,

$$u_{n+1} = (n+1)^{\frac{3}{2}} f_{n+1}$$

$$\simeq n^{\frac{3}{2}} (1+3/2n)\, f_{n+1} \, ,$$

$$u_{n-1} \simeq n^{\frac{3}{2}} (1-3/2n)\, f_{n-1} \, ,$$

it follows that $u_{n+1} - 2 \cos 2\alpha \, u_n + u_{n-1} = 0$. This difference equation has the solutions $\cos 2n\alpha$, $\sin 2n\alpha$ and so (6.15) has solutions which behave like $\cos 2n\alpha/n^{\frac{3}{2}}$, $\sin 2n\alpha/n^{\frac{3}{2}}$ for large values of n. Since both these solutions tend to zero, the successive calculation of f_n presents no difficulty.

To begin the process, one needs the values of f_0 and f_1 which may be expressed in terms of the complete elliptic integrals $E(k)$ and $K(k)$ of modulus $k = \cos\alpha$. Details are given in an Appendix where it is shown that

$$f_0 = - \frac{2\sqrt{2}}{\pi} [E(k) - k'^2 K(k)] \, ,$$

$$f_1 = \frac{2\sqrt{2}}{3\pi} [(k^2 - k'^2) E(k) + k'^2 K(k)] \, ,$$

k' being the complementary modulus, given by $k'^2 = 1 - k^2$.

By (6.12), the boundary radial displacement is given by $2\mu v_r = g(\theta)$ and

$$g(\theta) = \tfrac{1}{2} g_0 + \sum_{n=1}^{\infty} g_n \cos 2n\theta \qquad (6.16)$$

where $\tfrac{1}{2} g_0 = - f_0$, $g_n = - \left(\frac{\kappa}{2n-1} + \frac{1}{2n+1}\right) f_n$, $(n \geq 1)$.

It was shown above that $f^n = O(n^{-\frac{3}{2}})$ when $n \to \infty$ and so $g_n = O(n^{-\frac{5}{2}})$

when $n \to \infty$. It seemed probable that an excessive number of terms of the series (6.16) would need to be taken to calculate $g(\theta)$ with reasonable accuracy. Accordingly, some consideration was given to the introduction of a second auxiliary function, $h(\theta)$, which when subtracted from $g(\theta)$ would leave a function possessing a more rapidly convergent Fourier series.

Since $f(\theta)$ was chosen to have the essential features of the boundary radial stress given by (6.8), it was natural to assume that $h(\theta)$ should have the essential features of the boundary radial displacement given by (6.9). An obvious choice was therefore

$$h(\theta) = h(\cos 2\theta - \cos 2\alpha)^{\frac{3}{2}} \quad \text{for} \quad 0 \le \theta \le \alpha ,$$

$$= 0 \qquad \text{for} \qquad \alpha < \theta \le \pi/2 ,$$

h being a constant. The appropriate value of h can be inferred from (5.29), (5.30); for when $\theta \to \alpha +$,

$$f(\theta) \sim - (2 \sin 2\alpha)^{\frac{1}{2}} (\theta-\alpha)^{\frac{1}{2}}$$

whilst when $\theta \to \alpha -$,

$$h(\theta) \sim h(2 \sin 2\alpha)^{\frac{3}{2}} (\alpha-\theta)^{\frac{3}{2}} .$$

Consequently, $$h(2 \sin 2\alpha)^{\frac{3}{2}} = \frac{n+1}{3} (2 \sin 2\alpha)^{\frac{1}{2}}$$

or $$h = \frac{n+1}{3} \frac{1}{2 \sin 2\alpha} .$$

Confirmation that with this multiplier $h(\theta)$ has the appropriate form can be obtained by expanding $h(\theta)$ in the Fourier series

$$h(\theta) = \tfrac{1}{2}h_0 + \sum_{n=1}^{\infty} h_n \cos 2n\theta , \qquad (6.17)$$

and comparing the asymptotic forms of h_n and g_n when $n \to \infty$. These may be found from the following theorem of Bromwich (Bromwich (1926) p[494]):

Let $F(\psi)$ be a function of ψ which has bounded variation when $\psi \ge 0$; let γ be a function of ν such that $\nu \gamma \to \infty$ as $\nu \to \infty$.

Then, if $-1 < \mu < 1$,

$$\nu^{\mu} \int_0^{\gamma} \psi^{\mu-1} F(\psi) \sin \nu\psi \, d\psi \to F(0+) \Gamma(\mu) \sin(\mu\pi/2)$$

and if $0 < \mu < 1$,

$$\nu^{\mu} \int_0^{\gamma} \psi^{\mu-1} F(\psi) \cos \nu\psi \, d\psi \to F(0+) \Gamma(\mu) \cos(\mu\pi/2) .$$

The coefficient $g_n = -\left(\frac{\kappa}{2n-1} + \frac{1}{2n+1}\right) f_n$

$$\sim -\frac{\kappa+1}{2n} f_n$$

where $f_n = -\frac{2}{\pi} \int_{\alpha}^{\pi/2} (\cos 2\alpha - \cos 2\theta)^{\frac{1}{2}} \cos 2n\theta \, d\theta .$

Integration by parts gives

$$f_n = \frac{1}{n\pi} \int_{\alpha}^{\pi/2} \frac{\sin 2\theta}{(\cos 2\alpha - \cos 2\theta)^{\frac{1}{2}}} \sin 2n\theta \, d\theta$$

and writing $\theta = \alpha + \psi$ gives

$$f_n = \frac{1}{n\pi} \int_0^{\pi/2-\alpha} \psi^{-\frac{1}{2}} F(\psi) [\sin 2n\alpha \cos 2n\psi + \cos 2n\alpha \sin 2n\psi] d\psi ,$$

where $F(\psi) = \sin 2(\alpha+\psi) \left[\frac{\psi}{2\sin(2\alpha+\psi)\sin\psi} \right]^{\frac{1}{2}} .$

Hence, by Bromwich's theorem,

$$f_n \sim \frac{1}{n\pi} \sqrt{\frac{\sin 2\alpha}{2}} \, (\sin 2n\alpha . \sqrt{\pi} \cos \pi/4 + \cos 2n\alpha . \sqrt{\pi} \sin \pi/4)/(2n)^{\frac{1}{2}}$$

$$= \sqrt{\frac{2 \sin 2\alpha}{\pi}} \cos(2n\alpha - \pi/4)/(2n)^{\frac{3}{2}}$$

and so $$g_n \sim -(\kappa+1)\sqrt{\frac{2 \sin 2\alpha}{\pi}} \cos(2n\alpha - \pi/4)/(2n)^{\frac{5}{2}} . \qquad (6.18)$$

Similarly, for the coefficient

$$h_n = \frac{4h}{\pi} \int_0^{\alpha} (\cos 2\theta - \cos 2\alpha)^{\frac{3}{2}} \cos 2n\theta \, d\theta ,$$

two integrations by parts yield

$$h_n = \frac{4h}{\pi} \frac{3}{2n} \int_0^{\alpha} (\cos 2\theta - \cos 2\alpha)^{\frac{1}{2}} \sin 2\theta \sin 2n\theta \, d\theta$$

$$= \frac{4h}{\pi} \frac{3}{2n} \frac{1}{2n} \int_0^{\alpha} \frac{d}{d\theta} [\sin 2\theta (\cos 2\theta - \cos 2\alpha)^{\frac{1}{2}}] \cos 2n\theta \, d\theta$$

and the substitution $\theta = \alpha - \psi$ gives

$$h_n = \frac{4h}{\pi}\ \frac{3}{2n}\ \frac{1}{2n}\ \int_0^{\alpha} H(\psi)\ [\cos 2n\alpha \cos 2n\psi + \sin 2n\alpha \sin 2n\psi]d\psi\ ,$$

where $$H(\psi) \equiv -\frac{d}{d\psi}\ \{\sin 2(\alpha+\psi)\ [2 \sin(2\alpha-\psi) \sin \psi]^{\frac{1}{2}}\}\ .$$

Since, when $\psi \to 0$,

$$H(\psi) \sim -\frac{d}{d\psi}\ \{\sin 2\alpha (2 \sin 2\alpha)^{\frac{1}{2}}\ \psi^{\frac{1}{2}}\}$$

$$= -\ 2^{-\frac{1}{2}}\ (\sin 2\alpha)^{\frac{3}{2}}\ \psi^{-\frac{1}{2}}\ ,$$

Bromwich's theorem gives

$$h_n \sim -\frac{4h}{\pi}\ \frac{3}{2n}\ \frac{1}{2n}\ \frac{(\sin 2\alpha)^{\frac{3}{2}}}{2^{\frac{1}{2}}} \times$$

$$\times\ (\cos 2n\alpha .\ \sqrt{\pi} \cos \pi/4 + \sin 2n\alpha .\ \sqrt{\pi} \sin \pi/4)/(2n)^{\frac{1}{2}}$$

$$= -\ 2.3h \sin 2\alpha \sqrt{\frac{2 \sin 2\alpha}{\pi}}\ \cos\ (2n\alpha - \pi/4)/(2n)^{\frac{5}{2}}\ .$$

Comparing this with (6.18), it is seen that the dominant terms in the asymptotic expansions of g_n and h_n agree if

$$\mu+1 = 2.3h \sin 2\alpha$$

or $$h = \frac{\mu+1}{3}\ \frac{1}{2 \sin 2\alpha}\ ,\qquad \text{as before .}$$

Like the coefficients f_n , the coefficients h_n satisfy a recurrence relation which may be obtained by considering the function

$$U = (\cos 2\theta - \cos 2\alpha)^{\frac{5}{2}} \sin 2n\theta\ .$$

Differentiating and rearranging gives

$$\frac{dU}{d\theta} = \frac{5}{2}\ (\cos 2\theta - \cos 2\alpha\ )^{\frac{3}{2}}\ (-2 \sin 2\theta) \sin 2n\theta$$

$$+\ (\cos 2\theta - \cos 2\alpha)^{\frac{5}{2}}\ 2n \cos 2n\theta$$

$$= (5/2)\ (\cos 2\theta - \cos 2\alpha)^{\frac{3}{2}}\ [\cos 2(n+1)\theta - \cos 2(n-1)\theta]$$

$$+n\ (\cos 2\theta - \cos 2\alpha)^{\frac{3}{2}}\ [\cos 2(n+1)\theta + \cos 2(n-1)\theta]$$

$$-\ 2n \cos 2\alpha\ (\cos 2\theta - \cos 2\alpha)^{\frac{3}{2}}\ \cos 2n\theta$$

and integrating between the limits 0 and α gives

$$(n+5/2)\ h_{n+1} = 2n \cos 2\alpha .\ h_n - (n-5/2)\ h_{n-1} . \qquad (6.19)$$

Like the recurrence relation for f_n, this relation is stable and allows the values of h_n to be calculated from those of h_0 and h_1. In the Appendix, these are shown to be given by

$$h_0 = \frac{8\sqrt{2}}{3\pi} [2(k'^2-k^2) E(k') - k^2(k'^2-2k^2) K(k')] ,$$

$$h_1 = \frac{8\sqrt{2}}{5\pi} [2(k'^4+k^2) E(k') - k^2(1+k^2) K(k')] ,$$

where $k' = \sin\alpha$ and $k^2 = 1-k'^2$, as before.

Using the coefficients f_n and h_n, the function $g(\theta)$ is calculated by forming the differences

$$d_n = g_n - h_n ,$$

where $g_0 = -2f_0$, $g_n = -\left(\frac{n+1}{2n-1} + \frac{1}{2n+1}\right) f_n$, $(n \geq 1)$ and summing the series

$$g(\theta) = h(\theta) + \tfrac{1}{2}d_0 + \sum_{n=1} d_n \cos 2n\theta . \qquad (6.20)$$

In a trial with the nominal value $\alpha = 70^\circ$ and $\eta = 0{\cdot}25$, the coefficients f_n, g_n, h_n and d_n were calculated for $n = 0(1)60$. In this and all subsequent calculations, the simple approximations for the elliptic integrals $E(k)$ and $K(k)$ given by Hastings (1955) were used; they have a maximum error of $1{\cdot}5 \times 10^{-3}$.

The order of magnitude of the coefficients can be seen from the following list in which the third coefficient of each line is the numerically largest of the last few calculated:

$f_0 = -8{\cdot}40 \times 10^{-2}$, $f_1 = 7{\cdot}90 \times 10^{-2}$, $f_{57} = 5{\cdot}09 \times 10^{-4}$;

$g_0 = 1{\cdot}68 \times 10^{-1}$, $g_1 = -1{\cdot}84 \times 10^{-1}$, $g_{57} = -1{\cdot}34 \times 10^{-5}$;

$h_0 = 1{\cdot}42$, $h_1 = 9{\cdot}40 \times 10^{-1}$, $h_{57} = -1{\cdot}35 \times 10^{-5}$;

$d_0 = -1{\cdot}25$, $d_1 = -1{\cdot}12$, $d_{59} = -3{\cdot}63 \times 10^{-7}$.

For certain values of n, namely 28, 37, 46 and 55, each of the coefficients f_n, g_n and h_n was noticeably smaller than its neighbours. These results are in general accord with the asymptotic forms of the coefficients.

§ 3. Numerical results

Two versions of a program using the approximations

$$\tilde{R} = af(\theta) + a_0 + 2\sum_{k=1}^{n} a_k \cos 2k\theta ,$$

$$\tilde{S} = ag(\theta) - a_0 - \sum_{k=1}^{n} a_k \left(\frac{\pi}{2k-1} + \frac{1}{2k+1}\right) \cos 2k\theta$$

to the boundary values $R \equiv \breve{rr}$ and $S \equiv 2\mu v_r - T(1-\eta)(1+2\cos 2\theta)$, respectively, were run. In the first version, 90 boundary points P_i were taken, $i = 1(1)90$, with $\theta_i = \frac{1}{2}^\circ(1^\circ)89\frac{1}{2}^\circ$, each point carrying equal weight. 17 approximating functions were used, i.e. $n = 15$, and $g(\theta)$ was calculated from (6.20) using 60 terms of the summation.

Taking $\eta = 0{\cdot}25$, the coefficients a and the boundary radial stress and displacement were calculated with the assumption that $\alpha = 70^\circ(\frac{1}{4}^\circ)72^\circ$, these values being suggested by the earlier attempts.

The residuals were very sensitive to changes in α and whether judged by the smallness of the maximum absolute values:

$$R_R(\max) \equiv \max_{0 < \theta_i \leq \alpha} |R(P_i) - \tilde{R}(P_i)| ,$$

$$R_S(\max) \equiv \max_{\alpha < \theta_i < \pi/2} |S(P_i) - \tilde{S}(P_i)| ,$$

or the smallness of the root mean squares:

$$R_R(\text{rms}) \equiv \{\sum_{i=1}^{P} [R(P_i) - \tilde{R}(P_i)]^2/P\}^{\frac{1}{2}} ,$$

$$R_S(\text{rms}) \equiv \{\sum_{i=P+1}^{N} [S(P_i) - \tilde{S}(P_i)]^2/(N-P)\}^{\frac{1}{2}} ,$$

where P is the integer for which $\theta_P \leq \alpha < \theta_{P+1}$ and N the number of boundary points, there was a clear indication that the correct value of α was approximately $71^\circ 30'$. The relevant results are given in Table 1.

Further calculations with values of α at an interval of 1 minute showed that the residual displacements were smallest when $\alpha = 71^\circ 28'$ but the residual stresses were smallest for $\alpha = 71^\circ 30'$. However, the results for $71^\circ 30'$ were unacceptable because the radial displacement calculated at the point $\theta = 71\frac{1}{2}^\circ$ was negative $(2\mu v_r = -28{\cdot}4 \times 10^{-5})$. Since the residual stresses always tended to be smaller than the residual displacements, it was felt that α should be chosen to make $R_S(\max)$ and $R_S(\text{rms})$ as small as possible. A further calculation showed $\alpha = 71^\circ 28\frac{1}{2}'$ to be best in this respect; it also gave the least value to the root mean square of both sets of residuals taken together. Results are given in Table 2.

Table 1

(90 boundary points, 17 approximating functions)

α	$R_R(\max)$	$R_S(\max)$	$R_R(\text{rms})$	$R_S(\text{rms})$
$71^\circ 0'$	$69{\cdot}6 \times 10^{-5}$	366×10^{-5}	$16{\cdot}9 \times 10^{-5}$	116×10^{-5}
$71^\circ 15'$	$35{\cdot}2$	190	$8{\cdot}67$	$59{\cdot}8$
$71^\circ 30'$	$0{\cdot}88$	$16{\cdot}0$	$0{\cdot}21$	$4{\cdot}98$
$71^\circ 45'$	$44{\cdot}7$	187	$12{\cdot}1$	$58{\cdot}8$
$72^\circ 0'$	$93{\cdot}5 \times 10^{-5}$	392×10^{-5}	$25{\cdot}4 \times 10^{-5}$	123×10^{-5}

Table 2

(90 boundary points, 17 approximating functions)

α	$R_R(\max)$	$R_S(\max)$	$R_R(\text{rms})$	$R_S(\text{rms})$
$71^\circ 27'$	$5{\cdot}76 \times 10^{-5}$	$21{\cdot}3 \times 10^{-5}$	$1{\cdot}56 \times 10^{-5}$	$6{\cdot}78 \times 10^{-5}$
$71^\circ 28'$	$3{\cdot}25$	$5{\cdot}90$	$0{\cdot}96$	$2{\cdot}07$
$71^\circ 28\frac{1}{2}'$	$2{\cdot}27$	$1{\cdot}88$	$0{\cdot}66$	$0{\cdot}79$
$71^\circ 29'$	$1{\cdot}60$	$9{\cdot}7$	$0{\cdot}38$	$2{\cdot}96$
$71^\circ 30'$	$0{\cdot}88 \times 10^{-5}$	$16{\cdot}0 \times 10^{-5}$	$0{\cdot}21 \times 10^{-5}$	$4{\cdot}98 \times 10^{-5}$

Table 3

(91 boundary points, 17 approximating functions)

α	$R_R(\max)$	$R_S(\max)$	$R_R(\text{rms})$	$R_S(\text{rms})$
$71^\circ 27'$	$6{\cdot}20 \times 10^{-5}$	$17{\cdot}6 \times 10^{-5}$	$1{\cdot}81 \times 10^{-5}$	$5{\cdot}69 \times 10^{-5}$
$71^\circ 28'$	$3{\cdot}97$	$5{\cdot}23$	$1{\cdot}15$	$1{\cdot}91$
$71^\circ 28\frac{1}{2}'$	$2{\cdot}87$	$1{\cdot}76$	$0{\cdot}82$	$0{\cdot}74$
$71^\circ 29'$	$1{\cdot}76$	$7{\cdot}30$	$0{\cdot}50$	$2{\cdot}26$
$71^\circ 30'$	$1{\cdot}36 \times 10^{-5}$	$20{\cdot}0 \times 10^{-5}$	$0{\cdot}30 \times 10^{-5}$	$6{\cdot}15 \times 10^{-5}$

Table 4

(91 boundary points, 17 approximating functions, 120 terms in series for $g(\theta)$)

α	R_R(max)	R_S(max)	R_R(rms)	R_S(rms)
$71^\circ 28'$	$3{\cdot}97 \times 10^{-5}$	$4{\cdot}68 \times 10^{-5}$	$1{\cdot}15 \times 10^{-5}$	$1{\cdot}86 \times 10^{-5}$
$71^\circ 28\frac{1}{2}'$	2·87	1·58	0·82	0·78
$71^\circ 29'$	1·76	7·88	0·50	2·33

Since the value of α proved to be very close to $71\frac{1}{2}^\circ$, which corresponds to a boundary point, it seemed desirable to check the results by running a second version of the program using 91 boundary points $\theta = 0^\circ(1^\circ)90^\circ$. The corresponding results are given in Table 3 and confirm the value $\alpha = 71^\circ 28\frac{1}{2}'$. In a further check, some of these calculations were repeated but with 120 terms of the series $\sum d_n \cos 2n\theta$ taken to calculate $g(\theta)$, because some of the residual displacements were small enough to be comparable with the error introduced by truncating after 60 terms. The results (given in Table 4) showed no significant changes.

For $\alpha = 71^\circ 28\frac{1}{2}'$, the non-zero values of the radial stress and radial displacement calculated at the intermediate points $\theta = \frac{1}{2}^\circ(1^\circ)89\frac{1}{2}^\circ$ agreed to at least 4 decimal places with those found previously when these points were data points. Figure 1 shows the variation of the boundary radial stress, hoop stress and radial displacement. The boundary hoop stress $\breve{\theta\theta}$ can be calculated quite simply; for with

$$\Phi(z) = \tfrac{1}{4}T - \tfrac{1}{2}T/z^2 + \sum_{n=1}^{\infty} A_k/z^{2k} ,$$

$$\breve{\theta\theta} + \breve{rr} = 4\mathrm{Re}\,\Phi(z)$$

$$= T(1-2U_2) + 4\sum_{k=1}^{\infty} A_k\, U_{2k} .$$

However, by (6.5),

$$\breve{rr} = A_0 + 2\sum_{k=1}^{\infty} A_k\, U_{2k}$$

and so $\breve{\theta\theta} = T(1-2U_2) - 2A_0 + \breve{rr}$. (6.21)

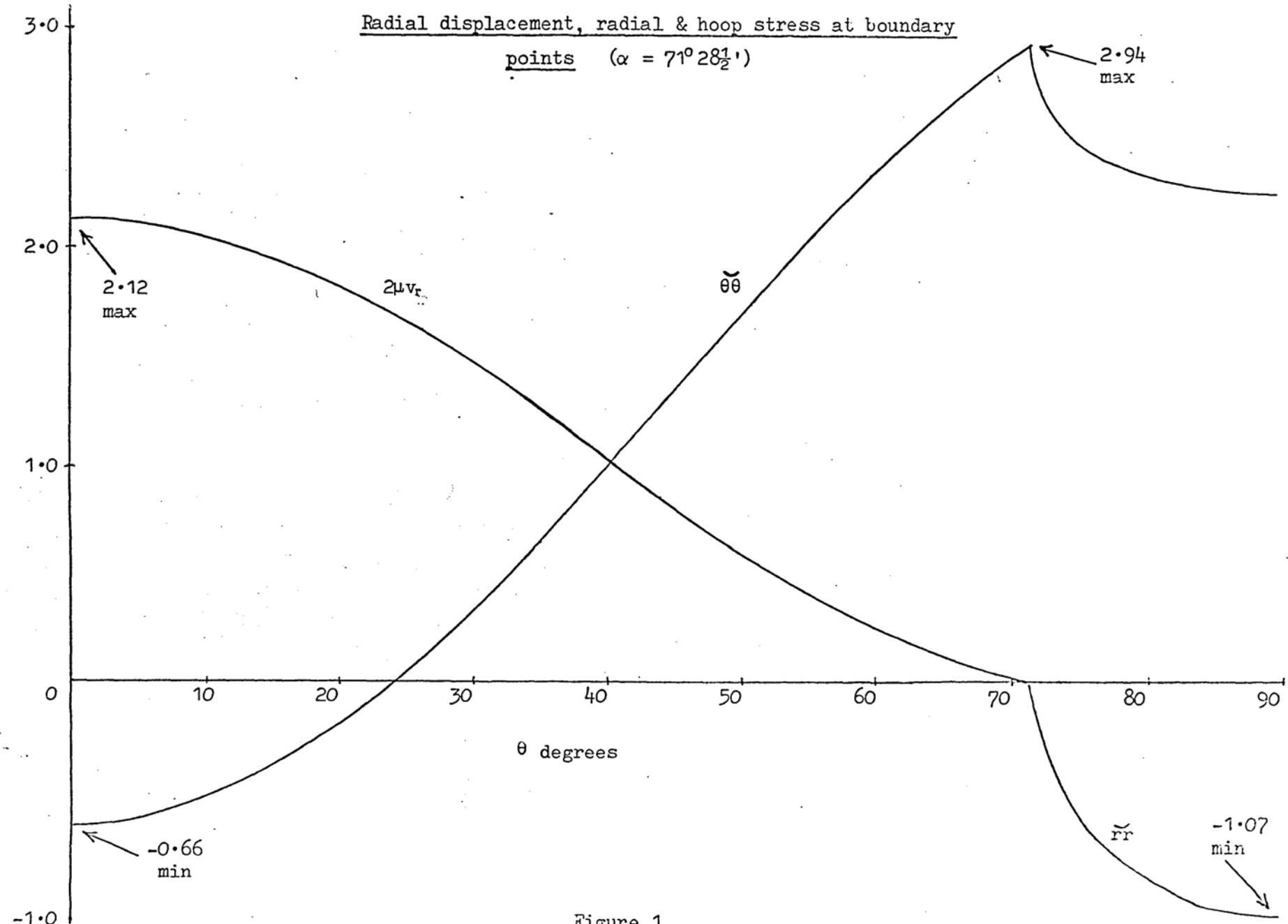

Radial displacement, radial & hoop stress at boundary
points (α = 71° 28½')
3·0
2·0
1·0
0
-1·0
2·12
max
2μv_r
θθ
2·94
max
10
20
30
40
50
60
70
80
90
θ degrees
-0·66
min
rr
-1·07
min

Figure 1

When the auxiliary function $f(\theta)$ is used, $A_0 = af_0 + a_0$.

Table 5

(Without the use of auxiliary functions)

k	R_R(max)	R_S(max)	R_R(rms)	R_S(rms)
0	136×10^{-3}	614×10^{-3}	136×10^{-3}	523×10^{-3}
1	328	326	163	257
2	408	200	140	152
4	286	33·2	67·7	28·2
8	72·4	44·9	10·9	33·3
16	34·3	19·8	5·52	11·6
32	6·57	7·99	1·25	3·75
36	$5·11 \times 10^{-3}$	$6·47 \times 10^{-3}$	$0·93 \times 10^{-3}$	$2·76 \times 10^{-3}$

Table 6

(Auxiliary functions calculated with $\alpha = 71^\circ 28\frac{1}{2}'$)

k	R_R(max)	R_S(max)	R_R(rms)	R_S(rms)
Aux	0	$66·2 \times 10^{-5}$	0	$49·5 \times 10^{-5}$
0	$1·61 \times 10^{-5}$	65·9	$1·61 \times 10^{-5}$	49·3
1	5·51	65·4	2·74	48·9
2	7·78	65·3	2·94	48·8
4	7·31	65·0	3·34	48·6
8	28·3	8·77	7·39	5·01
15	2·87	1·76	0·82	0·74
16	1·66	1·63	0·49	0·72
20	$1·15 \times 10^{-5}$	$1·34 \times 10^{-5}$	$0·28 \times 10^{-5}$	$0·59 \times 10^{-5}$

Table 7

(Auxiliary functions calculated with $\alpha = 71^\circ 32'$)

k	R_R(max)	R_S(max)	R_R(rms)	R_S(rms)
Aux	0	$49·7 \times 10^{-5}$	0	$21·5 \times 10^{-5}$
0	$0·04 \times 10^{-5}$	49·7	$0·04 \times 10^{-5}$	21·5
1	0·15	49·7	0·08	21·5
2	0·28	49·7	0·10	21·5
4	1·68	49·7	0·45	21·5
8	7·60	57·4	1·48	19·6
15	6·71	45·9	1·61	14·2
16	5·85	45·8	1·26	14·3
20	$9·28 \times 10^{-5}$	$37·8 \times 10^{-5}$	$2·22 \times 10^{-5}$	$10·9 \times 10^{-5}$

The much greater closeness with which the boundary conditions can be satisfied when the functions $f(\theta)$ and $g(\theta)$ are used is quite apparent from Tables 5 and 6. In Table 5, the results of approximating R and S by Fourier series alone are shown; the angle α is assumed to

lie between 71° and 72° and the index k refers to the suffix of the last coefficient included. In Table 6, the auxiliary functions $f(\theta)$ and $g(\theta)$ are taken first, calculated for $\alpha = 71^\circ 28\frac{1}{2}'$, followed by terms of the Fourier series. Even when terms up to $\cos 72\theta$ have been included, the results in the first case are inferior on all counts to those achieved with $f(\theta)$ and $g(\theta)$ alone.

Prompted by this, it seemed possible that the simple approximations

$$\tilde{R} = af(\theta) \quad , \quad \tilde{S} = ag(\theta) \quad ,$$

might enable the angle α to be determined quickly and reasonably accurately for other values of the modified Poisson's ratio η. Since $f(\theta) \equiv 0$ over $0 \leq \theta < \alpha$, one can choose the coefficient a to minimize

$$I = \sum_{i=P+1}^{N} [S(P_i) - ag(\theta_i)]^2 \quad ,$$

which requires $a = \sum_{P+1}^{N} S(P_i)\, g(\theta_i) / \sum_{P+1}^{N} [g(\theta_i)]^2$.

It is then a simple matter to find the value of α for which the minimum value of I is least.

Using this method, the angle α was located to the nearest tenth of a degree for various values of η. Since it is known that Poisson's ratio σ lies between 0 and $\frac{1}{2}$ and $\eta = \sigma/(1+\sigma)$, η was restricted to the range $0 \leq \eta \leq \frac{1}{3}$. It was soon apparent that α varies surprisingly little with η and this greatly reduced the amount of searching necessary. The following results were obtained:

η	0·0	0·1	0·2	0·25	0·3	0·33
α	$70^\circ 48'$	$71^\circ 06'$	$71^\circ 24'$	$71^\circ 30'$	$71^\circ 42'$	$71^\circ 54'$.

On searching at a smaller interval with $\eta = 0{\cdot}25$ it was found that the minimum value of I was least when $\alpha = 71^\circ 32'$, which differs slightly from the value $71^\circ 28\frac{1}{2}'$ obtained previously. The reason for the difference can be seen from Table 7 where for $\alpha = 71^\circ 32'$, it is clear that adding

trigonometric terms to the approximations produced hardly any alteration in the residuals.

For the accurate determination of α the original method was therefore used, again with 91 boundary points and 17 approximating functions. The values obtained above provided useful starting points and the results obtained are given below in Table 8. Occasionally, the minima of R_R and R_S corresponded to values of α differing by about one minute. In these cases, the value of α which gave the smaller root mean square of <u>all</u> the residuals was taken.

Table 8

(91 boundary points, 17 approximating functions)

η	α	R_R(max)	R_S(max)	R_R(rms)	R_S(rms)
0·0	$70^\circ 46'$	$0{\cdot}59 \times 10^{-5}$	$4{\cdot}57 \times 10^{-5}$	$0{\cdot}24 \times 10^{-5}$	$1{\cdot}78 \times 10^{-5}$
0·1	$71^\circ 00\frac{1}{2}'$	1·98	1·97	0·70	0·60
0·2	$71^\circ 18'$	3·01	1·77	0·85	0·83
0·25	$71^\circ 28\frac{1}{2}'$	2·87	1·76	0·82	0·74
0·3	$71^\circ 40'$	3·20	1·86	0·92	0·84
0·33	$71^\circ 48\frac{1}{2}'$	$3{\cdot}31 \times 10^{-5}$	$2{\cdot}21 \times 10^{-5}$	$0{\cdot}95 \times 10^{-5}$	$1{\cdot}02 \times 10^{-5}$

In a paper received after the computations in this Chapter had been carried out, Noble and Hussain (1967) consider the problem in which the included disc is both smooth and elastic with constants μ' and κ'. If one takes the complex potentials

$$\Phi(z) = \sum_{k=0}^{\infty} A_k' z^{2k} , \quad \Psi(z) = \sum_{k=0}^{\infty} B_k' z^{2k}$$

to represent the state of stress within the disc (denoted by primes), the boundary condition $\widetilde{r\theta}' = 0$ requires

$$B'_{k-1} = 2k A_k' \quad \text{for} \quad k \geq 1 ,$$

and the other relevant boundary values are found to be

$$\widetilde{rr}' = 2\sum_{k=0}^{\infty} A_k' U_{2k} , \qquad (6.22)$$

$$2\mu' v_r' = (\varkappa'-1)A_0' + \sum_{k=1}^{\infty} \left(\frac{\varkappa'}{2k+1} + \frac{1}{2k-1}\right) A_k' U_{2k} \quad . \tag{6.23}$$

The boundary conditions for the problem now become

$$\breve{rr} = \breve{rr}' = 0 \quad , \quad \text{for} \quad 0 \le \theta \le \alpha \quad ,$$

$$\breve{rr} = \breve{rr}', \; v_r = v_r', \; \text{for} \quad \alpha \le \theta \le \pi/2 \quad .$$

Using (6.5) and (6.6) with (6.22) and (6.23), the continuity of $\breve{rr}$ requires

$$A_0' = \tfrac{1}{2}A_0 \quad , \quad A_k' = A_k \quad \text{for} \quad k \ge 1 \; ;$$

one then has $$\breve{rr} = \breve{rr}' = A_0 + 2\sum_{k=1}^{\infty} A_k U_{2k} \quad , \tag{6.24}$$

$$2\mu(v_r - v_r') = T(1-\eta)(1+2U_2) - [1+\tfrac{1}{2}(\mu/\mu')(\varkappa'-1)]A_0$$

$$- \sum_{k=1}^{\infty} [\beta_k + (\mu/\mu')\gamma_k] A_k U_{2k} \quad , \tag{6.25}$$

where $$\beta_k = \frac{\varkappa}{2k-1} + \frac{1}{2k+1} \; , \; \gamma_k = \frac{\varkappa'}{2k+1} + \frac{1}{2k-1} \quad . \tag{6.26}$$

Equating the right hand sides of (6.24) and (6.25) to zero when $0 \le \theta \le \alpha$ and $\alpha \le \theta \le \pi/2$, respectively, gives "dual series" agreeing with those obtained by Noble and Hussain who use Airy stress functions.

By writing $\beta_k \simeq \frac{\varkappa+1}{2k}$ and $\gamma_k \simeq \frac{\varkappa'+1}{2k}$, Noble and Hussain show that to a first approximation,

$$\breve{rr} \propto \sin\theta \sqrt{\cos 2\alpha - \cos 2\theta} \quad \text{for} \quad \alpha \le \theta \le \pi/2 \quad . \tag{6.27}$$

The approximate value of α is obtained from a transcendental equation and then improved twice by a "perturbation scheme" in which β_1, γ_1 & β_2, γ_2 have their true values restored. Graphs are drawn for the special case $\varkappa = \varkappa'$.

When $\mu/\mu' = 0$, whatever the value of $\varkappa'$, equation (6.25) is identical with (6.6) so that a comparison can be made. From their graphs, Noble and Hussain's values are approximately:

η	0·0	0·2	0·3
α	71°9'	71°36'	71°51' .

Noble and Hussain judge the accuracy of their results by that achieved in the special case $\mu = \mu'$, $\varkappa = \varkappa'$. An exact solution for this case has been obtained by Stippes, Wilson and Krull (1962) who show that $\alpha = 70{\cdot}38^\circ$ $(70^\circ 23')$. Noble and Hussain obtain $70{\cdot}46^\circ$ $(70^\circ 28')$ and a straightforward adaptation of the previous program gave the following results:

Table 9

(91 points, 17 approximating functions)

α	$R_R(\max)$	$R_S(\max)$	$R_R(\text{rms})$	$R_S(\text{rms})$
$70^\circ 22'$	$5{\cdot}35 \times 10^{-5}$	$12{\cdot}0 \times 10^{-5}$	$1{\cdot}29 \times 10^{-5}$	$5{\cdot}16 \times 10^{-5}$
$70^\circ 23'$	$3{\cdot}87$	$4{\cdot}51$	$0{\cdot}98$	$2{\cdot}76$
$70^\circ 24'$	$2{\cdot}66$	$5{\cdot}74$	$0{\cdot}72$	$1{\cdot}96$
$70^\circ 25'$	$1{\cdot}49 \times 10^{-5}$	$14{\cdot}8 \times 10^{-5}$	$0{\cdot}55$	$3{\cdot}98 \times 10^{-5}$

where S now refers to the difference of radial displacements. The root mean square of all the residuals was least for $\alpha = 70^\circ 24'$.

It is of interest to note that if, knowing (6.27), one had taken

$$f(\theta) = -\sin\theta\sqrt{\cos 2\alpha - \cos 2\theta}\ ,$$

the need to calculate elliptic integrals would not have occurred.

§ 4. Stresses at internal points

If the stress components are required at internal points of the plate, the use of the auxiliary function $f(\theta)$ (which is essential for the determination of α and the calculation of the boundary stresses and displacements) introduces a difficulty because the stresses corresponding to the boundary loading $\breve{rr} = f(\theta)$, $\breve{r\theta} = 0$ do not have simple closed forms. It is therefore necessary to return to the original complex potentials (6.7):

$$\Phi(z) = \tfrac{1}{4}T - \tfrac{1}{2}T/z^2 + \sum_{k=1}^{\infty} A_k/z^{2k}\ ,$$

$$\Psi(z) = -\tfrac{1}{2}T + \tfrac{1}{2}T/z^2 - 3T/2z^4 + \sum_{k=1}^{\infty} B_k/z^{2k}\ ,$$

from which the required stresses can be found in the combinations

$$\tfrac{1}{2}(\breve{00} + \breve{rr}) = 2\,\mathrm{Re}\,\Phi(z)$$

$$= \mathrm{Re}\,[\tfrac{1}{2}T - T/z^2 + 2\sum_{k=1}^{\infty} A_k/z^{2k}]\ ,$$

and $$\tfrac{1}{2}(\breve{0\theta} - \breve{rr}) + i\breve{r\theta} = z\,\Phi'(z) + (z^2/r^2)\,\Psi(z)$$

$$= T/z^2 - \sum_{k=1}^{\infty} 2k\,A_k/z^{2k}$$

$$- \tfrac{1}{2}Tz^2/r^2 + \tfrac{1}{2}T/r^2 - \tfrac{3}{2}T/r^2z^2 + (z^2/r^2)\sum_{k=1}^{\infty} B_k/z^{2k}\ ,$$

$$= -\tfrac{1}{2}Tz^2/r^2 + (\tfrac{1}{2}T+B_1)/r^2 + (1-3/2r^2)T/z^2$$

$$- (1-1/r^2)\sum_{k=1}^{\infty} 2k\,A_k/z^{2k}\ ,$$

on using $B_{k+1} = 2k\,A_k$ for $k \geq 1$. Here r denotes $|z|$.

It is clear that summing the series $\sum_{k=1}^{\infty} 2k\,A_k/z^{2k}$ may set a limit to the accuracy with which the stresses can be calculated. But for large values of k,

$$A_k \sim af_k \sim a\sqrt{\frac{2\sin 2\alpha}{\pi}}\ \cos\,(2k\alpha - \pi/4)/(2k)^{\frac{3}{2}},$$

so that $2k\,A_k \sim a\sqrt{\dfrac{2\sin 2\alpha}{\pi}}\ \cos\,(2k\alpha - \pi/4)/(2k)^{\frac{1}{2}}$.

Hence, if the series $\sum_{k=1}^{\infty} 2k\,A_k/z^{2k}$ is truncated after n terms, an estimate of the magnitude of the truncation error is

$$|a|\sqrt{\frac{2\sin 2\alpha}{\pi}}\ \left|\sum_{n+1}^{\infty} \cos\,(2k\alpha - \pi/4)\ \frac{1}{(2k)^{\frac{1}{2}}}\ \frac{1}{z^{2k}}\right|$$

$$\leq |a|\sqrt{\frac{2\sin 2\alpha}{\pi}}\ \frac{1}{(2n)^{\frac{1}{2}}}\sum_{n+1}^{\infty}\ \frac{1}{r^{2k}}$$

$$= |a|\sqrt{\frac{2\sin 2\alpha}{\pi}}\ \frac{1}{(2n)^{\frac{1}{2}}}\ \frac{1}{r^{2n+2}}\ \frac{1}{1-1/r^2}\ .$$

When $\alpha = 71^\circ 28\tfrac{1}{2}$ and $\eta = 0{\cdot}25$ it is found that $a \simeq 2{\cdot}35$ giving $a\sqrt{(2\sin 2\alpha/\pi)} \simeq 1{\cdot}45$. Hence if 60 terms are taken, the truncation error is certainly negligible for $r \geq 1{\cdot}1$ because $1{\cdot}1^{-120} \simeq 1{\cdot}07 \times 10^{-5}$. In addition, when 14 approximating functions are used, the other coefficients range from $a_0 = -1{\cdot}41 \times 10^{-3}$ to $a_{12} = 1{\cdot}55 \times 10^{-6}$ with $a_{11} = 1{\cdot}20 \times 10^{-5}$, while the coefficients f range from $f_0 = -7{\cdot}23 \times 10^{-2}$ to $f_{57} = -1{\cdot}49 \times 10^{-4}$

with $f_{12} = -3{\cdot}54 \times 10^{-3}$. It was therefore reasonable to conclude that no appreciable error would be incurred by taking

$$A_k = af_k + a_k \quad \text{for} \quad k \leq 12 \quad \text{and} \quad A_k = af_k \quad \text{for} \quad k > 12 \ .$$

Values of the radial stress, hoop stress and shear stress calculated in this way for $r = 1{\cdot}2$, $1{\cdot}4$, $1{\cdot}6$, $2{\cdot}0$ and $2{\cdot}4$, $\theta = 0^\circ(10^\circ)90^\circ$ are given in Figures 2, 3 and 4 respectively. The results were the same whether 14 or 17 approximating functions were used.

§ 5. Calculation of the transverse displacement at points of the boundary

If the components of the displacement were required at internal points of the plate, there would be no difficulty in calculating them from the complex potentials

$$\varphi(z) = \tfrac{1}{4}Tz + \tfrac{1}{2}T/z - \sum_{k=1}^{\infty} A_k/(2k-1)z^{2k-1} \ ,$$

$$\psi(z) = -\tfrac{1}{2}Tz - \tfrac{1}{2}T/z + \tfrac{1}{2}T/z^3 - \sum_{k=1}^{\infty} B_k/(2k-1)z^{2k-1}$$

and the formula

$$2\mu(v_r + iv_\theta) = e^{-i\theta}\left[\kappa\,\varphi(z) - z\,\overline{\varphi'(z)} - \overline{\psi(z)}\right] \ .$$

For boundary points, however, this gives

$$\begin{aligned} 2\mu(v_r + iv_\theta) = \kappa\Big[&\tfrac{1}{4}T + \tfrac{1}{2}Te^{-2i\theta} - \sum_{k=1}^{\infty} \frac{A_k}{2k-1}\, e^{-2ik\theta}\Big] \\ &- \Big[\tfrac{1}{4}T - \tfrac{1}{2}Te^{2i\theta} + \sum_{k=1}^{\infty} A_k\, e^{2ik\theta}\Big] \\ &+ \Big[\tfrac{1}{2}Te^{-2i\theta} + \tfrac{1}{2}T + B_1 - \tfrac{1}{2}Te^{2i\theta} + \sum_{k=1}^{\infty} \frac{2k}{2k-1} A_k e^{2ik\theta}\Big] \ , \end{aligned}$$

so that

$$2\mu v_r = T(1-\eta)\,(1+2\cos 2\theta) + B_1 - \sum_{k=1}^{\infty} A_k \left[\frac{\kappa}{2k-1} + \frac{1}{2k+1}\right]\cos 2k\theta \ ,$$

$$2\mu v_\theta = -T(1-\eta)\,2\sin 2\theta + \sum_{k=1}^{\infty} A_k \left[\frac{\kappa}{2k-1} - \frac{1}{2k+1}\right]\sin 2k\theta \ , \qquad (6.28)$$

and it is clear that the second series, like the first (which has been considered previously), converges too slowly to be of use. Hence if the

Radial Stress $\breve{rr}$ ($\times 10^{-3}$ T)

Negative values denoted by bars.

($\eta = 0{\cdot}25$)

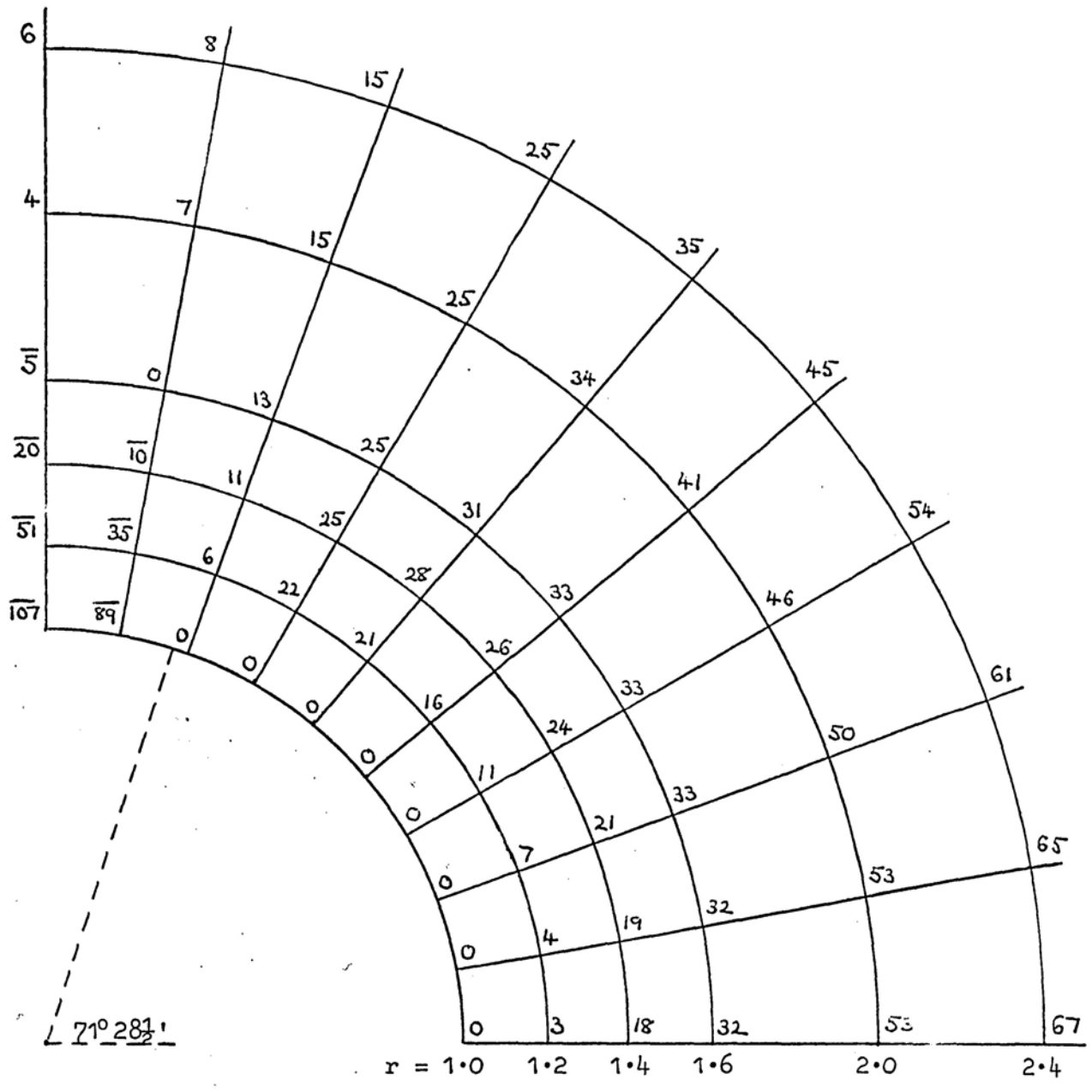


Figure 2

Hoop Stress $\breve{\theta\theta}$ (x 10^{-2}T)

Negative values denoted by bars.

(η = 0·25)

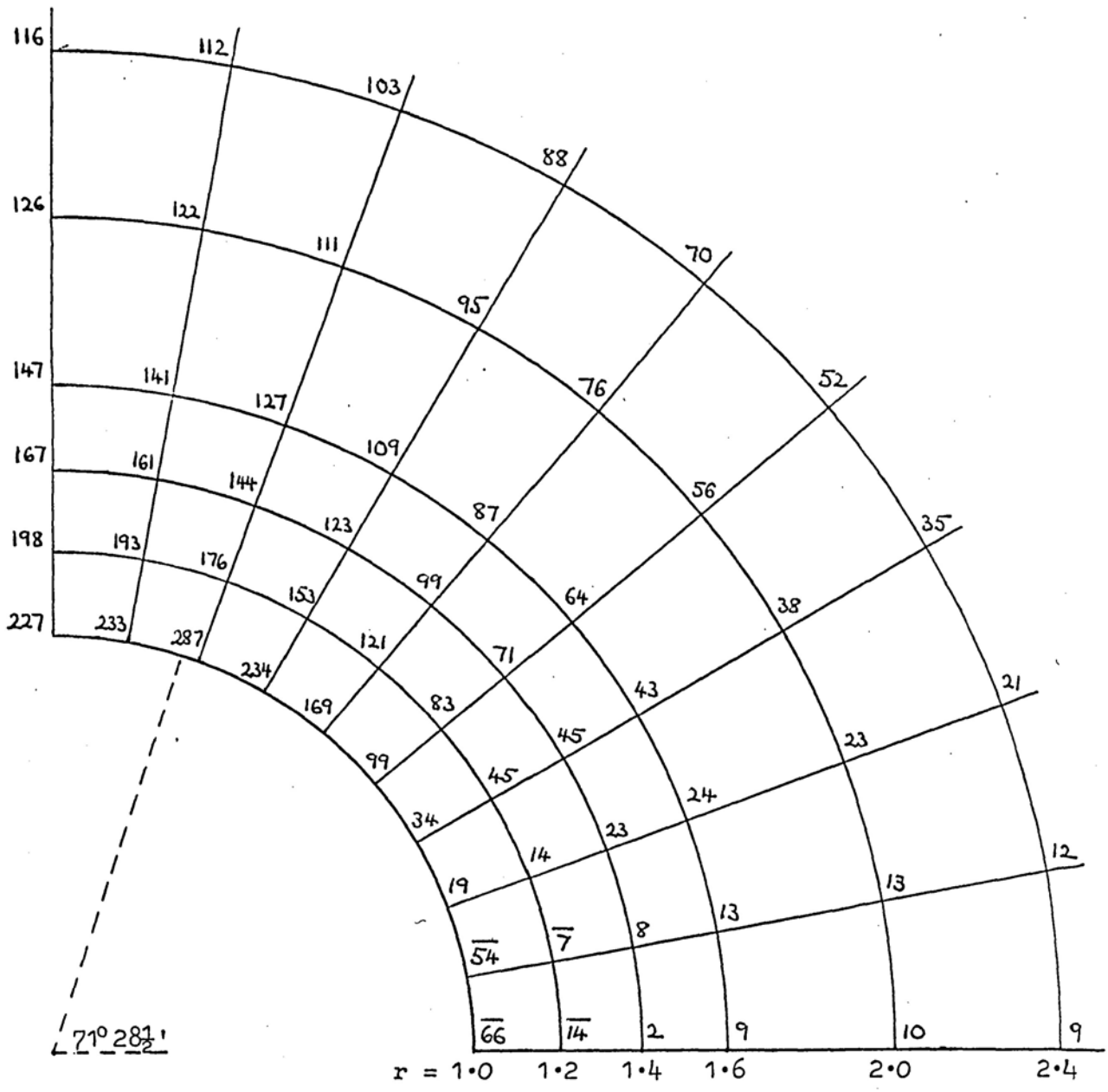


Figure 3

Shear Stress $\breve{r\theta}$ ($\times 10^{-2}T$)

Negative values denoted by bars.

($\eta = 0{\cdot}25$)

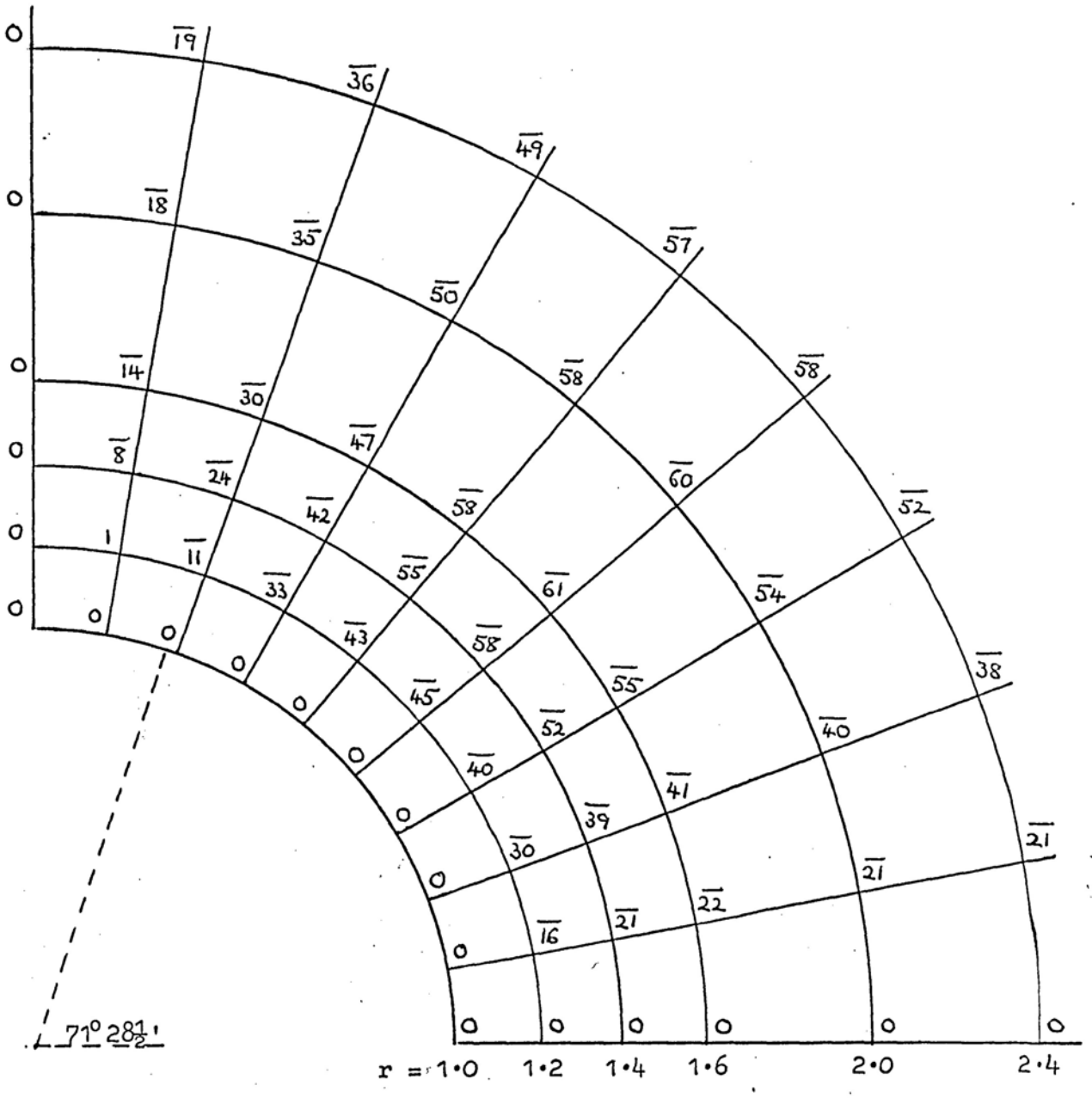


Figure 4

transverse displacement is required at boundary points it is necessary to calculate the auxiliary function $g_1(\theta)$ which is defined by (6.11) and which is the counterpart of $g(\theta)$.

The Fourier expansion of $g_1(\theta)$ follows immediately from (6.28). For if, with the previous notation,

$$g(\theta) = - f_0 - \sum_{n=1}^{\infty} f_n \left(\frac{\nu}{2n-1} + \frac{1}{2n+1}\right) \cos 2n\theta ,$$

then
$$g_1(\theta) = \sum_{n=1}^{\infty} f_n \left(\frac{\nu}{2n-1} - \frac{1}{2n+1}\right) \sin 2n\theta . \qquad (6.29)$$

It was not considered worthwhile to go to the trouble of constructing a function $h_1(\theta)$, analogous to $h(\theta)$, which would speed the convergence of (6.29). Instead, it was decided to dispense with $h(\theta)$ as well. If the previous values of the boundary radial displacement (and the boundary radial stress) could be reproduced when $g(\theta)$ is calculated from the Fourier series alone, one would be reasonably confident that the boundary transverse displacements derived from $g_1(\theta)$ would also be correct.

Preliminary calculations with $\eta = 0{\cdot}25$ showed that 150 terms of the Fourier series gave $g(\theta)$ almost as accurately as when the auxiliary $h(\theta)$ and 61 terms were used. The error did not exceed 1 unit in the 6^{th} decimal place except for a few values of θ close to $\alpha(=71^\circ 28\frac{1}{2}')$ where it reaches a maximum of 5 units.

Using 91 boundary points $\theta = 0^\circ(1^\circ)90^\circ$ and 17 approximating functions, the boundary radial displacements and radial stresses at the data points and the intermediate points $\theta = \frac{1}{2}^\circ(1^\circ)89\frac{1}{2}^\circ$ agreed to 4 decimal places with those obtained when the auxiliary function $h(\theta)$ was used.

The values of $2\mu v_\theta$ obtained are shown alongside the previously calculated values of $2\mu v_r$ in Figure 5.

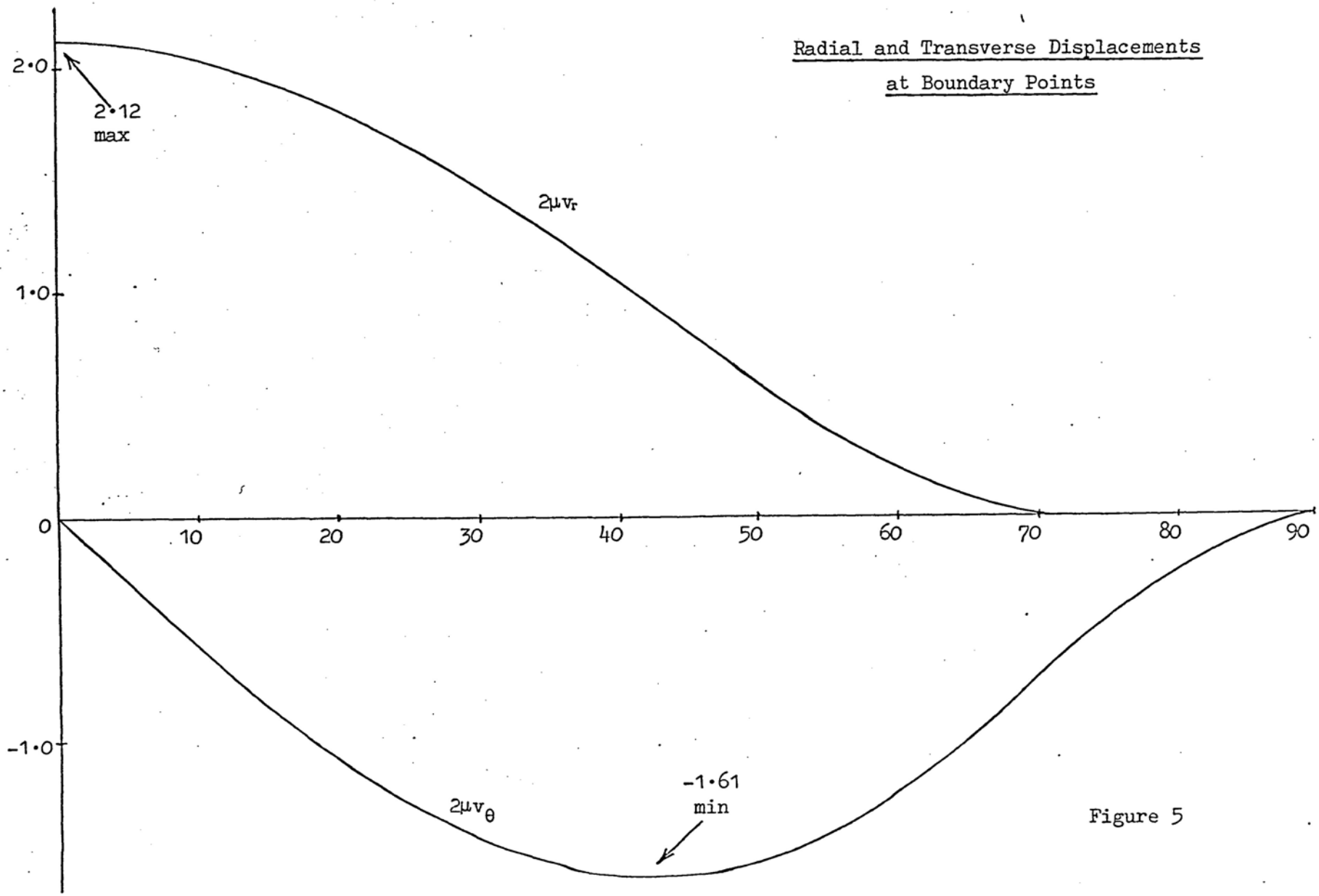

Radial and Transverse Displacements
at Boundary Points
2·0
1·0
0
-1·0
2·12
max
$2\mu v_r$
10
20
30
40
50
60
70
80
90
$2\mu v_\theta$
-1·61
min

Figure 5

§ 6. Appendix to Chapter 6

Evaluation of the coefficients f_0, f_1, h_0 and h_1.

The coefficient f_0 is given by

$$f_0 = -\frac{2}{\pi}\int_\alpha^{\pi/2} (\cos 2\alpha - \cos 2\theta)^{\frac{1}{2}}\, d\theta$$

$$= -\frac{2\sqrt{2}}{\pi}\int_0^{\beta} (\sin^2\beta - \sin^2\varphi)^{\frac{1}{2}}\, d\varphi \ ,$$

where $\beta = \pi/2 - \alpha$ and $\varphi = \pi/2 - \theta$.

The substitution

$$\sin\varphi = \sin\beta \sin\psi \ , \quad d\varphi = \frac{\sin\beta\cos\psi\, d\psi}{\sqrt{1-\sin^2\beta\sin^2\psi}} \ , \tag{6.30}$$

gives $$f_0 = -\frac{2\sqrt{2}}{\pi}\int_0^{\pi/2} \frac{k^2\cos^2\psi}{\Delta}\, d\psi \ ,$$

where $k = \sin\beta$ and $\Delta = \sqrt{1-k^2\sin^2\psi}$; hence

$$f_0 = -\frac{2\sqrt{2}}{\pi}\int_0^{\pi/2} \frac{1-k^2\sin^2\psi-(1-k^2)}{\Delta}\, d\psi$$

$$= -(2\sqrt{2}/\pi)\,[E(k) - k'^2\, K(k)] \ .$$

The coefficient f_1 is given by

$$f_1 = -\frac{2}{\pi}\int_\alpha^{\pi/2} (\cos 2\alpha - \cos 2\theta)^{\frac{1}{2}} \cos 2\theta\, d\theta$$

$$= \frac{2\sqrt{2}}{\pi}\int_0^{\beta} (\sin^2\beta - \sin^2\varphi)^{\frac{1}{2}} \cos 2\varphi\, d\varphi \ ,$$

and the substitution (6.30) in this case gives

$$f_1 = \frac{2\sqrt{2}}{\pi}\int_0^{\pi/2} \frac{k^2\cos^2\psi(1-2k^2\sin^2\psi)}{\Delta}\, d\psi \ ,$$

$$= \frac{2\sqrt{2}}{\pi}\left[\int_0^{\pi/2} 2k^2\cos^2\psi\ \Delta\, d\psi - \int_0^{\pi/2} \frac{k^2\cos^2\psi}{\Delta}\, d\psi\right] .$$

The second integral has been evaluated already; for the first, integration by parts gives

$$\int_0^{\pi/2} \cos^2\psi\ \Delta\, d\psi = [\sin\psi\cos\psi\ \Delta]_0^{\pi/2} - \int_0^{\pi/2} \sin\psi\, d(\cos\psi\ \Delta)$$

$$= \int_0^{\pi/2} \sin^2\psi\ (\Delta + k^2\cos^2\psi/\Delta)\, d\psi$$

$$= 2\int_0^{\pi/2} \sin^2\psi\ \Delta\ d\psi - (1-k^2)\int_0^{\pi/2} \sin^2\psi\ d\psi/\Delta\ .$$

Hence $$3\int_0^{\pi/2} \cos^2\psi\ \Delta\ d\psi = 2\int_0^{\pi/2} \Delta\ d\psi - k'^2\int_0^{\pi/2} \sin^2\psi\ d\psi/\Delta$$

and $$3k^2\int_0^{\pi/2} \cos^2\psi\ \Delta\ d\psi = 2k^2\int_0^{\pi/2} \Delta\ d\psi + k'^2\int_0^{\pi/2} \Delta\ d\psi - k'^2\int_0^{\pi/2} d\psi/\Delta$$

$$= (1+k^2)\int_0^{\pi/2} \Delta\ d\psi - k'^2\int_0^{\pi/2} d\psi/\Delta\ .$$

Thus $$f_1 = \frac{2\sqrt{2}}{\pi}\left[\frac{2}{3}(1+k^2)\,E(k) - \frac{2}{3}k'^2K(k) - E(k) + k'^2K(k)\right]$$

$$= \frac{2\sqrt{2}}{3\pi}\left[(k^2-k'^2)\,E(k) + k'^2K(k)\right]\ .$$

For the coefficient

$$h_0 = \frac{4h}{\pi}\int_0^{\alpha} (\cos 2\theta - \cos 2\alpha)^{\frac{3}{2}}\ d\theta$$

$$= \frac{8\sqrt{2}h}{\pi}\int_0^{\alpha} (\sin^2\alpha - \sin^2\theta)^{\frac{3}{2}}\ d\theta\ ,$$

the substitution (6.30) is replaced by the substitution

$$\sin\theta = \sin\alpha\ \sin\varphi\ ,\quad d\theta = \frac{\sin\alpha\ \cos\varphi\ d\varphi}{\sqrt{1-\sin^2\alpha\ \sin^2\varphi}}\ . \tag{6.31}$$

Since $\sin\alpha = \cos\beta = k'$, this gives

$$h_0 = \frac{8\sqrt{2}h}{\pi}\int_0^{\pi/2} \frac{k'^4\cos^4\varphi}{\Delta'}\ d\varphi\ ,$$

where $$\Delta' = \sqrt{1-k'^2\sin^2\varphi}\ .$$

Hence $$h_0 = \frac{8\sqrt{2}h}{\pi}\left[\int_0^{\pi/2} k'^2\cos^2\varphi\ \Delta'\ d\varphi - (1-k'^2)\int_0^{\pi/2} k'^2\cos^2\varphi\ d\varphi/\Delta'\right]$$

$$= \frac{8\sqrt{2}h}{\pi}\left\{\frac{1}{3}\left[(1+k'^2)\,E(k') - k^2K(k')\right] - k^2\left[E(k') - k^2K(k')\right]\right\}$$

$$= \frac{8\sqrt{2}h}{3\pi}\left[2(k'^2 - k^2)\,E(k') - k^2(k'^2 - 2k^2)\,K(k')\right]\ .$$

Lastly, for the coefficient

$$h_1 = \frac{4h}{\pi}\int_0^{\alpha} (\cos 2\theta - \cos 2\alpha)^{\frac{3}{2}}\cos 2\theta\ d\theta\ ,$$

a preliminary integration by parts leads to

$$h_1 = \frac{4h}{\pi}\int_0^{\alpha} \frac{3}{2}(\cos 2\theta - \cos 2\alpha)^{\frac{1}{2}} \sin^2 2\theta \, d\theta$$

$$= \frac{8\sqrt{2}h}{\pi}\int_0^{\alpha} 3(\sin^2\alpha - \sin^2\theta)^{\frac{1}{2}} \sin^2\theta \cos^2\theta \, d\theta$$

and the substitution (6.31) gives

$$h_1 = \frac{8\sqrt{2}h}{\pi}\int_0^{\pi/2} 3k'^4 \sin^2\varphi \cos^2\varphi \, \Delta' \, d\varphi \, .$$

Again, by integration by parts,

$$\int_0^{\pi/2} 3\sin^2\varphi \cos^2\varphi \, \Delta' d\varphi = [\sin^3\varphi \cos\varphi \, \Delta']_0^{\pi/2} - \int_0^{\pi/2} \sin^3\varphi \, d(\cos\varphi \, \Delta')$$

$$= \int_0^{\pi/2} \sin^4\varphi \, (\Delta' + k'^2\cos^2\varphi/\Delta') \, d\varphi$$

$$= \int_0^{\pi/2} \sin^2\varphi \, [(1-\cos^2\varphi)\Delta' + \cos^2\varphi(1-\Delta'^2)/\Delta'] d\varphi$$

so that

$$5\int_0^{\pi/2} \sin^2\varphi \cos^2\varphi \, \Delta' d\varphi = \int_0^{\pi/2} \sin^2\varphi \, \Delta' \, d\varphi + \int_0^{\pi/2} \sin^2\varphi \cos^2\varphi \, d\varphi/\Delta' \, .$$

After some further reduction, one finds that

$$5k'^2 \int_0^{\pi/2} \sin^2\varphi \cos^2\varphi \, \Delta' d\varphi = k'^2 \int_0^{\pi/2} \Delta' \, d\varphi - (1+k'^2) \int_0^{\pi/2} \cos^2\varphi \, \Delta' \, d\varphi$$

$$+ \int_0^{\pi/2} \cos^2\varphi \, d\varphi/\Delta' \, ,$$

from which, by using earlier results,

$$15k'^4 \int_0^{\pi/2} \sin^2\varphi \cos^2\varphi \, \Delta' \, d\varphi = 3k'^4 \, E(k') - (1+k'^2)\,[(1+k'^2)\,E(k') - k^2 K(k')]$$

$$+ 3[E(k') - k^2 K(k')]$$

$$= 2(k'^4 - k'^2 + 1)\, E(k') - k^2(2-k'^2)\, K(k') \, .$$

Hence, $$h_1 = \frac{8\sqrt{2}h}{5\pi}\,[2(k'^4+k^2)\, E(k') - k^2(1+k^2)\, K(k')] \, .$$

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