Capacity of Non-Malleable Codes

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***Abstract*—Non-malleable codes, introduced by Dziembowski, Pietrzak and Wichs (ICS 2010), encode messages s in a manner so that tampering the codeword causes the decoder to either output s or a message that is independent of s. While this is an impossible goal to achieve against unrestricted tampering functions, rather surprisingly non-malleable coding becomes possible against every fixed family $\mathcal{F}$ of tampering functions that is not too large (for instance, when $|\mathcal{F}| \leqslant 2^{2^{\alpha n}}$ for some $\alpha < 1$ where n is the number of bits in a codeword).**

In this work, we study the "capacity of non-malleable codes," and establish optimal bounds on the achievable rate as a function of the family size, answering an open problem from Dziembowski et al. (ICS 2010). Specifically,

- **We prove that for every family $\mathcal{F}$ with $|\mathcal{F}| \leqslant 2^{2^{\alpha n}}$, there exist non-malleable codes against $\mathcal{F}$ with rate arbitrarily close to $1 - \alpha$ (this is achieved w.h.p. by a randomized construction).**
- **We show the existence of families of size $\exp(n^{O(1)} 2^{\alpha n})$ against which there is no non-malleable code of rate $1 - \alpha$ (in fact this is the case w.h.p for a random family of this size).**
- **We also show that $1 - \alpha$ is the best achievable rate for the family of functions which are only allowed to tamper the first αn bits of the codeword, which is of special interest.**
  As a corollary, this implies that the capacity of non-malleable coding in the split-state model (where the tampering function acts independently but arbitrarily on the two halves of the codeword, a model which has received some attention recently) equals $1/2$.

We also give an efficient Monte Carlo construction of codes of rate close to 1 with polynomial time encoding and decoding that is non-malleable against any fixed $c > 0$ and family $\mathcal{F}$ of size 2^{n^c}, in particular tampering functions with, say, cubic size circuits.

I. Introduction

NON-MALLEABLE codes are a fascinating new concept put forth in [1], following the program on non-malleable cryptography which was introduced by the seminal work of Dolev, Dwork and Naor [2]. Non-malleable codes are aimed at protecting the integrity of data in situations where it might be corrupted in ways that precludes error-correction or even error-detection. Informally, a code is non-malleable if the corrupted codeword either encodes the original message, or a completely unrelated value. This is akin to the notion of non-malleable encryption in cryptography which requires the intractability of, given a ciphertext, producing a different ciphertext so that the corresponding plaintexts are related to each other.

A non-malleable code against a family $\mathcal{F}$ of tampering functions each mapping $\{0,1\}^n$ to $\{0,1\}^n$, consists of a randomized encoding function $\mathsf{Enc}\colon \{0,1\}^k \to \{0,1\}^n$ and a deterministic decoding function $\mathsf{Dec}\colon \{0,1\}^n \to \{0,1\}^k \cup \{\perp\}$ (where $\perp$ denotes error-detection) which satisfy $\mathsf{Dec}(\mathsf{Enc}(s)) = s$ always, and the following non-malleability property with error ϵ: For every message $s \in \{0,1\}^k$ and every function $f \in \mathcal{F}$, the distribution of $\mathsf{Dec}(f(\mathsf{Enc}(s))$ is ϵ-close to a distribution $\mathcal{D}_f$ that depends only on f and is independent[1] of s. In other words, if some adversary (who has full knowledge of the code and the message s, but not the internal randomness of the encoder) tampers with the codeword $\mathsf{Enc}(s)$ corrupting it to $f(\mathsf{Enc}(s))$, he cannot control the relationship between s and the message the corrupted codeword $f(\mathsf{Enc}(s))$ encodes.

In general, it is impossible to achieve non-malleability against arbitrary tampering functions. Indeed, the tampering function can decode the codeword to compute the original message s, flip the last bit of s to obtain a related message $\tilde{s}$, and then re-encode $\tilde{s}$. This clearly violates non-malleability as the tampered codeword encodes the message $\tilde{s}$ which is closely related to s. Therefore, in order to construct non-malleable codes, one focuses on a restricted class of tampering functions. For example, the body of work on error-correcting codes consists of functions which can flip an arbitrary subset of bits up to a prescribed limit on the total number of bit flips.

The notion of non-malleable coding becomes more interesting for families against which error-correction is not possible. A simple and natural such family is the set of functions causing arbitrary "additive errors," namely $\mathcal{F}_{\mathsf{add}} = \{f_\Delta \mid \Delta \in \{0,1\}^n\}$ where $f_\Delta(x) := x + \Delta$. Note that there is no restriction on the Hamming weight of Δ as in the case of channels causing bounded number of bit flips. While error-correction is impossible against $\mathcal{F}_{\mathsf{add}}$, *error-detection* is still possible — the work of Cramer et al. [3] constructed codes of rate approaching 1 (which they called "Algebraic Manipulation Detection" (AMD) codes) such that offset by an arbitrary $\Delta \neq 0$ will be detected with high probability. AMD codes give a construction of non-malleable codes against the family

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[1] The formal definition (see Definition4) has to accommodate the possibility that Dec error-corrects the tampered codeword to the original message s; and this is handled in a manner independent of s by including a special element $\underline{\mathsf{same}}$ in the support of $\mathcal{D}_f$.

$\mathcal{F}_{\mathsf{add}}$.

Even error-detection becomes impossible against many other natural families of tampering functions. A particularly simple such class consists of all constant functions $f_c(x) := c$ for $c \in \{0,1\}^n$. This family includes some function that maps all inputs to a valid codeword c^*, and hence one cannot detect tampering. Note, however, that non-malleability is trivial to achieve against this family — the rate 1 code with identity encoding function is itself non-malleable as the output distribution of a constant function is trivially independent of the message. A natural function family for which non-malleability is non-trivial to achieve consists of *bit-tampering functions* f in which the different bits of the codewords are tampered independently (i.e., either flipped, set to $0/1$, or left unchanged); formally $f(x) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$ for arbitrary 1-bit functions $f_1, f_2, \ldots, f_n$ [1].

The family $\mathcal{F}_{\mathsf{all}}$ of all functions $f : \{0,1\}^n \to \{0,1\}^n$ has size given by $\log\log|\mathcal{F}_{\mathsf{all}}| = n + \log n$. The authors of [1] show the existence of a non-malleable code against *any* small enough family $\mathcal{F}$ (for which $\log\log|\mathcal{F}| < n$). The rate of the code is constant if $\log\log|\mathcal{F}| \leqslant \alpha n$ for some constant $\alpha \in (0,1)$. The question of figuring out the optimal rates of non-malleable codes for various families of tampering functions was left as an open problem in [1]. In this work we give a satisfactory answer to this question, pinning down the rate for many natural function families. We describe our results next.

A. Our results

Our results include improvements to the rate achievable as a function of the size of the family of tampering functions, as well as limitations of non-malleable codes demonstrating that the achieved rate cannot be improved for natural families of the stipulated size. Specifically, we establish the following results concerning the possible rates for non-malleable coding as a function of the size of the family of tampering functions:

1) (Rate lower bound) We prove in Section III that if $|\mathcal{F}| \leqslant 2^{2^{\alpha n}}$, then there exists a (strong) non-malleable code of rate arbitrarily close to $1-\alpha$ which is non-malleable w.r.t $\mathcal{F}$ with error $\exp(-\Omega(n))$. This significantly improves the probabilistic construction of [1], which achieves a rate close to $(1-\alpha)/3$ using a delicate Martingale argument. In particular, for arbitrary small families, of size $2^{2^{o(n)}}$, our result shows that the rate can be made arbitrarily close to 1. This was not known to be possible even for the family of bit-tampering functions (which has size 4^n), for which $1/3$ was the best known rate[2] [1]. In fact, we note (in Section V-D) why the proof strategy of [1] is limited to a rate of $1/2$ even for a very simple tampering function such as the one that flips the first bit. As discussed in Section III-C, our probabilistic construction is equipped with an encoder and decoder that can be efficiently and exactly implemented with access to a uniformly random permutation oracle and its inverse (corresponding to the ideal-cipher model in cryptography). This is a slight additional advantage over [1], where only an approximation of the encoder and decoder is shown to be efficiently computable.
2) (Upper bound/limitations on rate) The above coding theorem shows that the "capacity" of a function family $|\mathcal{F}|$ for non-malleable coding is at least $1 - (\log\log|\mathcal{F}|)/n$. We also address the natural "converse coding question" of whether this rate bound is the best achievable (Section V). This turns out to be false in general due to the existence of uninteresting large families for which non-malleable coding with rate close to 1 is easy. But we do prove that the $1-\alpha$ rate is best achievable in "virtually all" situations:
    a) We prove that for *random* families of size $2^{2^{\alpha n}}$, with high probability it is not possible to exceed a rate of $1-\alpha$ for non-malleable coding with small error.
    b) For the family of tampering functions which leave the last $(1-\alpha)n$ bits intact and act arbitrarily on the first αn bits, we prove that $1-\alpha$ is the best achievable rate for non-malleable coding. (Note that a rate of $1-\alpha$ is trivial to achieve for this family, by placing the message bits in the last $(1-\alpha)n$ bits of the codeword, and setting the first αn bits of the codeword to all 0s.)

The result 2b, together with the existential result 1 above, pins down the optimal rate for non-malleable codes in the *split-state model* to $1/2$. In the split-state model, which was the focus of a couple of recent works [4], [5], the tampering function operates independently (but in otherwise arbitrary ways) on the two halves of the codeword, i.e., $f(x) = ((f_1(x_1), f_2(x_2))$ where x_1, x_2 are the two halves of x and f_1, f_2 are functions mapping $n/2$ bits to $n/2$ bits. The recent work [5] gave an explicit construction in this model with polynomially small rate. Our work shows that the capacity of the split-state model is $1/2$, but we do not offer any explicit construction. For the more restrictive class of bit-tampering functions (where each bit is tampered independently), in a follow-up work [6] we give an explicit construction with rate approaching 1 [6]. We also present in that work a reduction of non-malleable coding for the split-state model to a new notion of non-malleable two-source extraction.

Monte Carlo construction for small families. Our result 1 above is based on a random construction which takes exponential time (and space). Derandomizing this construction, in Section IV we are able to obtain an efficient Monte Carlo construction of non-malleable codes of rate close to 1 (with polynomial time encoding and decoding, and inverse polynomially small error) for an *arbitrary* family of size $\exp(n^c)$ for any fixed $c > 0$. Note that in particular this includes tampering functions that can be implemented by circuits of any fixed polynomial size, or simpler families such as bit-tampering adversaries. The construction does not rely on any computational hardness assumptions, at the cost of using a small amount of randomness.

[2] Assuming the existence of one-way functions, an explicit construction of non-malleable codes of rate close to 1 was proposed in [1]. This construction, however, only satisfies a weaker definition of non-malleability that considers computational indistinguishability rather than statistical security.

B. Proof ideas

Rate lower bound. Our construction of rate $\approx 1 - (\log\log|\mathcal{F}|)/n$ codes is obtained by picking for each message, a random blob of t codewords, such that blobs corresponding to distinct messages are disjoint. For each tampering function f, our proof analyzes the distribution of $\mathsf{Dec}(f(\mathsf{Enc}(s))$ for each message s separately, and shows that w.h.p. they are essentially close to the same distribution $\mathcal{D}_f$. In order to achieve sufficiently small error probability allowing for a union bound, the proof uses a number of additional ideas, including a randomized process that gradually reveals information about the code while examining the t codewords in each blob in sequence. The analysis ensures that as little information is revealed in each step as possible, so that enough independence remains in the conditional joint distribution of the codewords throughout the analysis. Finally, strong concentration bounds are used to derive the desired bound on the failure probability. The proof for the special case of bijective tampering functions turns out to be quite straightforward, and as a warm-up we present this special case first in Section III-A.

Monte Carlo construction. Since the analysis of the probabilistic code construction considers each message s separately, we observe that it only needs limited (t-wise) independence of the codewords. On the other hand, the code construction is designed to be sparse, namely taking $t = \mathrm{poly}(n, \log|\mathcal{F}|, 1/\epsilon)$ suffices for the analysis. This is the key idea behind our efficient Monte Carlo construction for small families with $\log|\mathcal{F}| \leqslant \mathrm{poly}(n)$.

The birthday paradox implies that picking the blob of codewords encoding each message independently of other messages, while maintaining disjointness of the various blobs, limits the rate to $1/2$. Therefore, we construct the code by means of a t-wise independent *decoding* function implemented via a random low-degree polynomial. After overcoming some complications to ensure an efficient encoding function, we get our efficient randomized construction for small families of tampering functions.

Rate upper bounds. Our main impossibility result for the family of adversaries that only tamper the first αn bits of the codeword uses an information theoretic argument. We argue that if the rate of the code is sufficiently large, one can always find messages s_0 and s_1 and a set $X_\eta \subseteq \{0,1\}^{\alpha n}$ such that the following holds: The first αn bits of the encoding of s_0 has a noticeable chance of being in X_η, whereas this chance for s_1 is quite small. Using this property, we design an adversary that maps the first αn bits of the encoding to a dummy string if they belong to X_η and leaves the codeword intact otherwise. This suffices to violate non-malleability of the code.

C. Subsequent work

After the original write-up of this work, numerous exciting developments have emerged, of which we recall a few. Faust et al. [7] describe a probabilistic construction of non-malleable codes that, for a tampering family of size bounded by $2^{2^{\alpha n}}$, attains a sub-optimal rate of $1 - O(\alpha)$ (with high probability and with a negligible error). However, this construction is the most interesting when the family of tampering functions is bounded in size by $2^{\mathsf{poly}(n)}$, in which case it attains a rate close to 1 and a polynomial dependence of the running time of the encoder and decoder functions on the security parameter $\log(1/\epsilon)$. In a different work, Faust et al. [8] define the generalized notion of *continuous* non-malleable codes which, intuitively, allow the adversary to tamper the same codeword multiple times (without the need to re-encode the message using fresh randomness for each tampering). Jafargholi and Wichs [9] obtain an improved analysis of the idea in [7] to show existence of non-malleable codes achieving the optimal rate of about $1 - \alpha$ and secure with respect to continuous tampering. Similar to [7], this result also enjoys a polynomial dependence of the encoder and decoder running times on the security parameter. This construction additionally comes with the ability of tamper detection for well behaved tampering functions, as shown to hold for our construction in Remark 11 (i.e., for tampering functions f with few fixed points such that $f(\mathcal{U}_n)$, where $\mathcal{U}_n$ is the uniform distribution on $\{0,1\}^n$, has sufficiently large min-entropy).

Further applications of non-malleable codes in non-malleable cryptography have been shown by Coretti et al. [10] (in the context of non-malleable public-key encryption) and by Agrawal et al. [11] (in the context of non-malleable string commitment schemes). Roughly speaking, these constructions use suitable non-malleable codes to extend, in a modular way, a non-malleable cryptographic protocol acting on single bits (such as a bit-commitment scheme) to one that can act on arbitrarily long strings while preserving non-malleability.

Organization. The rest of the article is organized as follows. Section II reviews preliminaries and notation including definitions of non-malleability used throughout the article. Probabilistic construction of general rate-optimal non-malleable codes is presented in Section III. Section IV provides a randomness efficient variation of this result, thereby obtaining Monte-Carlo constructions of non-malleable codes against any family $\mathcal{F}$ of adversaries such that $\log|\mathcal{F}| \leqslant \mathsf{poly}(n)$. Finally, Section V proves impossibility results complementing the achievability results of Sections III and IV.

II. Preliminaries

A. Notation

We use $\mathcal{U}_n$ for the uniform distribution on $\{0,1\}^n$ and U_n for the random variable sampled from $\mathcal{U}_n$ and independently of any existing randomness. For a random variable X, we denote by $\mathscr{D}(X)$ the probability distribution that X is sampled from. Moreover, for an event $\mathcal{E}$, we use $\mathscr{D}(X \mid \mathcal{E})$ to denote the conditional distribution of the random variable X on the event $\mathcal{E}$. Generally, we will use calligraphic symbols (such as $\mathcal{X}$) for probability distributions and the corresponding capital letters (such as X) for related random variables. For a discrete distribution $\mathcal{X}$, we denote by $\mathcal{X}(x)$ the probability mass assigned to x by $\mathcal{X}$. Two distributions $\mathcal{X}$ and $\mathcal{Y}$ being ϵ-close in statistical distance is denoted by $\mathcal{X} \approx_\epsilon \mathcal{Y}$. We will use $(\mathcal{X}, \mathcal{Y})$ for the product distribution with the two coordinates independently sampled from $\mathcal{X}$ and $\mathcal{Y}$. All unsubscripted logarithms are taken to the base 2. Support of a discrete random variable (or distribution) X is denoted by $\mathsf{supp}(X)$. With a slight abuse of notation, for various bounds we condition probabilities and expectations on random variables rather than events (e.g., $\mathbb{E}[X|Y]$, or $\Pr[\mathcal{E}|Y]$). In such instances, the notation means that the statement holds for *every* possible realization of the random variables that we condition on.

The *empirical distribution* of t objects $a_1, \ldots, a_t$ from a universe Ω is the probability distribution over Ω that sets, for each $i \in \Omega$, the probability mass on i to be $|\{j \in [t]\colon a_j = i\}|/t$.

B. Definitions

In this section, we review the formal definition of non-malleable codes as introduced in [1]. First, we recall the notion of *coding schemes*.

Definition 1 (Coding schemes)**.** A pair of functions $\mathsf{Enc}\colon \{0,1\}^k \to \{0,1\}^n$ and $\mathsf{Dec}\colon \{0,1\}^n \to \{0,1\}^k \cup \{\perp\}$ where $k \leqslant n$ is said to be a coding scheme with block length n and message length k if the following conditions hold.

1) The encoder Enc is a randomized function; i.e., at each call it receives a uniformly random sequence of coin flips that the output may depend on. This random input is usually omitted from the notation and taken to be implicit. Thus for any $s \in \{0,1\}^k$, $\mathsf{Enc}(s)$ is a random variable over $\{0,1\}^n$. The decoder Dec is, however, deterministic.
2) For every $s \in \{0,1\}^k$, we have $\mathsf{Dec}(\mathsf{Enc}(s)) = s$ with probability 1.

The *rate* of the coding scheme is the ratio k/n. A coding scheme is said to have relative distance δ, for some $\delta \in [0,1)$, if for every $s \in \{0,1\}^k$ the following holds. Let $X := \mathsf{Enc}(s)$. Then, for any $\Delta \in \{0,1\}^n$ of Hamming weight at most δn, $\mathsf{Dec}(X + \Delta) =\perp$ with probability 1. □

Remark 2. In this paper, we have followed the common coding-theoretic convention used in the original work of Dziembowski et al. [1]. Namely, a coding scheme is defined with respect to fixed choices of k and n. When there is no risk of ambiguity, this notion is implicitly extended to mean an infinite ensemble of coding schemes for various choices of block length n tending infinity (and message lengths at least Rn for a prescribed rate parameter R).

Before defining non-malleable coding schemes, we find it convenient to define the following notation.

Definition 3. For a finite set Γ, the function $\mathsf{copy}\colon (\Gamma \cup \{\underline{\mathsf{same}}\}) \times \Gamma \to \Gamma$ is defined as follows:

$$\mathsf{copy}(x,y) := \begin{cases} x & x \neq \underline{\mathsf{same}}, \\ y & x = \underline{\mathsf{same}}. \end{cases} \qquad \square$$

Naturally, the notation can be extended to random variables so that for random variables X and Y respectively supported on $\Gamma \cup \{\underline{\mathsf{same}}\}$ and Γ, the random variable $\mathsf{copy}(X,Y)$, supported on Γ, is defined according to the above rule.

The notion of non-malleable coding schemes from [1] can now be rephrased as follows.

Definition 4 (Non-malleability)**.** A coding scheme (Enc, Dec) with message length k and block length n is said to be non-malleable with error ϵ (also called *exact security*) with respect to a family $\mathcal{F}$ of tampering functions acting on $\{0,1\}^n$ (i.e., each $f \in \mathcal{F}$ maps $\{0,1\}^n$ to $\{0,1\}^n$) if for every $f \in \mathcal{F}$ there is a distribution $\mathcal{D}_f$ over $\{0,1\}^k \cup \{\perp, \underline{\mathsf{same}}\}$ such that the following holds for all $s \in \{0,1\}^k$. Define the random variable $S := \mathsf{Dec}(f(\mathsf{Enc}(s)))$, and let S' be independently sampled from $\mathcal{D}_f$. Then, $\mathscr{D}(S) \approx_\epsilon \mathscr{D}(\mathsf{copy}(S', s))$. □

Remark 5. The above definition allows the decoder to output a special symbol $\perp$ that corresponds to error detection. It is easy to note that any such code can be transformed to one where the decoder never outputs $\perp$ without affecting the parameters (e.g., the new decoder may simply output 0^k whenever the original decoder outputs $\perp$).

Dziembowski et al. [1] also consider the following stronger variation of non-malleable codes.

Definition 6 (Strong non-malleability)**.** A pair of functions as in Definition 4 is said to be a *strong* non-malleable coding scheme with error ϵ with respect to a family $\mathcal{F}$ of tampering functions acting on $\{0,1\}^n$ if the following holds. For any message $s \in \{0,1\}^k$, let $E_s := \mathsf{Enc}(s)$, consider the random variable

$$D_{f,s} := \begin{cases} \underline{\mathsf{same}} & \text{if } f(E_s) = E_s, \\ \mathsf{Dec}(f(E_s)) & \text{otherwise,} \end{cases}$$

and let $\mathcal{D}_{f,s} := \mathscr{D}(D_{f,s})$. It must be the case that for every pair of distinct messages $s_1, s_2 \in \{0,1\}^k$, $\mathcal{D}_{f,s_1} \approx_\epsilon \mathcal{D}_{f,s_2}$. □

Remark 7 (Computational security)**.** Dziembowski et al. also consider the case where statistical distance is replaced with computational indistinguishability with respect to a bounded computational model. As our goal is to understand information-theoretic limitations of non-malleable codes, we do not consider this variation in this work. It is clear, however, that our negative results in Section V apply to this model as well. A related (but incomparable) model that we consider in Section IV is when the distinguishability criterion is still statistical; however the adversary is computationally bounded (e.g., one may consider the family of polynomial sized Boolean circuits). For this case, we construct an efficient Monte Carlo coding scheme that achieves any rate arbitrarily close to 1.

Remark 8 (Efficiency of sampling $\mathcal{D}_f$)**.** The original definition of non-malleable codes in [1] also requires the distribution $\mathcal{D}_f$ to be efficiently samplable given oracle access to the tampering function f. We find it more natural to remove this requirement from the definition since even combinatorial non-malleable codes that are not necessarily equipped with efficient components (such as the encoder, decoder, and sampler for $\mathcal{D}_f$) are interesting and highly non-trivial to construct. It should be noted; however, that for any non-malleable coding scheme equipped with an efficient encoder and decoder, it can be shown that the following is a valid and efficiently samplable choice for the distribution $\mathcal{D}_f$ (possibly incurring a constant factor increase in the error parameter):

1) Let $S \sim \mathcal{U}_k$, and $X := f(\mathsf{Enc}(S))$.
2) If $\mathsf{Dec}(X) = S$, output $\underline{\mathsf{same}}$. Otherwise, output $\mathsf{Dec}(X)$.

Our Monte Carlo construction in Section IV is equipped with a polynomial-time encoder and decoder. So is the case for our probabilistic construction in Section III in the random oracle model.

III. Probabilistic construction of non-malleable codes

In this section, we introduce our probabilistic construction of non-malleable codes. Contrary to the original construction of Dziembowski et al. [1], where they pick a uniformly random truth table for the decoder and do not allow the $\perp$ symbol, our code is quite sparse. In fact, in our construction $\mathsf{Dec}(U_n) = \perp$ with high probability (which, as an added benefit, allows for almost sure tamper detection for certain well behaved tampering functions, see Remark 11). As we observe in Section V-D, this is the key to our improvement, since uniformly random decoders cannot achieve non-malleability even against extremely simple adversaries at rates better than $1/2$. Moreover, our sparse construction offers the added feature of having a large minimum distance in the standard coding sense; any tampering scheme that perturbs the codeword in a fraction of the positions bounded by a prescribed limit will be detected by the decoder with probability 1. Another advantage of sparsity is allowing a compact representation for the code. We exploit this feature in our Monte Carlo construction of Section IV. Our probabilistic coding scheme is described in Construction 1. We note that there is an extra parameter $\delta \in [0, 1/2)$ in this construction, which enforces the relative Hamming distance between every two possible codewords in the construction to be larger than δ. This added feature may be of interest for certain applications that require some degree of noise resilience in addition to non-malleability. Of course one can set $\delta = 0$ if the only goal is to achieve (strong) non-malleability.

We remark that Construction 1 can be efficiently implemented in the ideal-cipher model, which in turn implies an efficient approximate implementation in the random oracle model (see the discussion following the proof of Theorem 9 in Section III-C). In turn, this implies that the distribution $\mathcal{D}_f$ in Definition 4 for this construction can be efficiently sampled in both models (see Remark 8).

The main theorem of this section is the result below that proves non-malleability of the coding scheme in Construction 1.

- *Given:* Integer parameters $0 < k \leqslant n$ and integer $t > 0$ such that $t2^k < 2^n$, and a relative distance parameter δ, $0 \leqslant \delta < 1/2$.
- *Output:* A pair of functions $\mathsf{Enc}\colon \{0,1\}^k \to \{0,1\}^n$ and $\mathsf{Dec}\colon \{0,1\}^n \to \{0,1\}^k$, where Enc may also use a uniformly random seed (which is hidden from the notation), but Dec is deterministic.
- *Construction:*
    1) Let $\mathcal{N} := \{0,1\}^n$.
    2) For each $s \in \{0,1\}^k$, in an arbitrary order,
        - Let $E(s) := \emptyset$.
        - For $i \in \{1, \dots, t\}$:
            a) Pick a uniformly random vector $w \in \mathcal{N}$.
            b) Add w to $E(s)$.
            c) Let $\Gamma(w)$ be the Hamming ball of radius δn centered at w. Remove $\Gamma(w)$ from $\mathcal{N}$ (note that when $\delta = 0$, we have $\Gamma(w) = \{w\}$).
    3) Given $s \in \{0,1\}^k$, $\mathsf{Enc}(s)$ outputs an element of $E(s)$ uniformly at random.
    4) Given $w \in \{0,1\}^n$, $\mathsf{Dec}(s)$ outputs the unique s such that $w \in E(s)$, or $\perp$ if no such s exists.

Construction 1: Probabilistic construction of non-malleable codes.

Theorem 9. *Let $\mathcal{F}\colon \{0,1\}^n \to \{0,1\}^n$ be any family of tampering functions. For any $\epsilon, \eta > 0$, with probability at least $1 - \eta$, the coding scheme* $(\mathsf{Enc}, \mathsf{Dec})$ *of Construction 1 is a strong non-malleable code with respect to $\mathcal{F}$ and with error ϵ and relative distance δ, provided that both of the following conditions are satisfied.*

1) *$t \geqslant t_0$, for some*
$$t_0 = O\left(\frac{1}{\epsilon^6}\Big(n + \log\frac{|\mathcal{F}|}{\eta}\Big)\right). \tag{1}$$
2) *$k \leqslant k_0$, for some*
$$k_0 \geqslant n(1 - h(\delta)) - \log t - 3\log(1/\epsilon) - O(1), \tag{2}$$
where $h(\cdot)$ denotes the binary entropy function.

Thus by choosing $t = t_0$ and $k = k_0$, the construction satisfies
$$k \geqslant n(1-h(\delta)) - \log\log(|\mathcal{F}|/\eta) - \log n - 9\log(1/\epsilon) - O(1).$$
In particular, if $|\mathcal{F}| \leqslant 2^{2^{\alpha n}}$ for any constant $\alpha \in (0,1)$, the rate of the code can be made arbitrarily close to $1 - h(\delta) - \alpha$ while allowing $\epsilon = 2^{-\Omega(n)}$.

Remark 10. (Error detection) An added feature of our sparse coding scheme is the error-detection capability. However, observe that any probabilistic coding scheme that is non-malleable against all families of adversaries of bounded size over $\{0,1\}^n$ (such as Construction 1, Construction 2, and the probabilistic construction of [1]) can be turned into one having relative distance δ (and satisfying the same non-malleability guarantees) by composing the construction with a fixed code $\mathcal{C}$ of block length n and relative distance δ. That is, the new scheme would first encode the message using the non-malleable code and would then further encode the resulting vector using $\mathcal{C}$. Indeed, any class $\mathcal{F}$ of tampering functions for the composed code corresponds to a class $\mathcal{F}'$ of the same size or less for the original construction. Namely, each function $f' \in \mathcal{F}'$ equals $\mathsf{Dec}_{\mathcal{C}} \circ f$ ($\mathsf{Dec}_{\mathcal{C}}$ being the decoder[3] of $\mathcal{C}$) for

[3] If achieving strong non-malleability (Definition 6) is desired, extra care is necessary to ensure that the decoder function $\mathsf{Dec}_{\mathcal{C}}$ never corrects errors (since Definition 6 implicitly discourages error correction).

some $f \in \mathcal{F}$. We allow the possibility of $\delta > 0$ directly in our construction since doing so does not make the analysis any more complicated.

A. Proof of Theorem 9 for bijective adversaries

We first prove the theorem for adversaries that are bijective and have no fixed points. This case is still broad enough to contain interesting families of adversaries such as additive error adversaries $\mathcal{F}_{\mathsf{add}}$ mentioned in the introduction, for which case we reconstruct the existence proof of AMD codes (although optimal explicit constructions of AMD codes are already known [3], [12]).

As it turns out, the analysis for this case is quite straightforward, and significantly simpler than the general case that we will address in Section III-B.

Let $N := 2^n$, $K := 2^k$, and consider a fixed message $s \in \{0,1\}^k$ and a fixed bijective tampering function $f\colon \{0,1\}^n \to \{0,1\}^n$ such that for all $x \in \{0,1\}^n$, $f(x) \neq x$. We show that the non-malleability requirement of Definition 4 holds with respect to the distribution $\mathcal{D}_f$ that is entirely supported on $\{\perp\}$. That is, we wish to show that with high probability, the coding scheme $(\mathsf{Enc}, \mathsf{Dec})$ of Construction 1 is so that

$$\Pr[\mathsf{Dec}(f(\mathsf{Enc}(s))) \neq \perp] \leqslant \epsilon. \tag{3}$$

By taking a union bound over all choices of f and s, this would imply that with high probability, the code is non-malleable (in fact, strongly non-malleable) for the entire family $\mathcal{F}$.

Let $E(s) := \mathsf{supp}(\mathsf{Enc}(s))$ be the set of the t codewords that are mapped to s by the decoder. Let $E_1, \ldots, E_t$ be the codewords in this set in the order they are picked by the code construction. For any fixed $x \in \{0,1\}^n$, we know that $\Pr[\mathsf{Dec}(x) \neq \perp] \leqslant t(K-1)/(N-t) \leqslant \frac{\gamma}{1-\gamma}$, where $\gamma := tK/N$. This can be seen by observing that the code construction chooses the codewords uniformly at random and without replacement, combined with a union bound. Thus, in particular, $\Pr[\mathsf{Dec}(f(E_1)) \neq \perp] \leqslant \frac{\gamma}{1-\gamma}$ (since $f(E_1) \neq E_1$, the knowledge of $E_1 \in E(s)$ only decreases this probability). In fact, the same argument holds for $\mathsf{Dec}(f(E_2))$ conditioned on any realization of $f(E_1)$, and more generally, since $f(E_1), \ldots, f(E_t)$ are distinct, one can derive for each $i \in [t]$,

$$\Pr[\mathsf{Dec}(f(E_i)) \neq \perp \mid f(E_1), \ldots, f(E_{i-1})] \leqslant \frac{\gamma}{1-\gamma}. \tag{4}$$

Define indicator random variables $0 = X_0, X_1, \ldots, X_t \in \{0,1\}$, where $X_i = 1$ iff $\mathsf{Dec}(f(E_i)) \neq \perp$. From (4) and using Proposition 31, we can deduce that for all $i \in [t]$, $\Pr[X_i = 1 \mid X_0, \ldots, X_{i-1}] \leqslant \frac{\gamma}{1-\gamma}$. Now, using Proposition 35, letting $X := X_1 + \cdots + X_t$, it follows that $\Pr[X > \epsilon t] \leqslant \Big(\frac{e\gamma}{\epsilon(1-\gamma)}\Big)^{\epsilon t}$. Assuming $\gamma \leqslant \epsilon/4$, the above upper bound simplifies to $\exp(-\Omega(\epsilon t))$. By taking a union bound over all possible choices of s and f (that we trivially upper bound by $N|\mathcal{F}|$), it can be seen that, as long as $t \geqslant t_0$ for some choice of $t_0 = O\Big(\frac{1}{\epsilon}\log(\frac{N|\mathcal{F}|}{\eta})\Big)$, the probability that $(\mathsf{Enc}, \mathsf{Dec})$ fails to satisfy (3) for some choice of s and f is at most η.

Finally, observe that the assumption $\gamma \leqslant \epsilon/4$ can be satisfied provided that $K \leqslant K_0$ for some choice of $K_0 = \Omega(\epsilon N/t)$, or equivalently, when $k \leqslant k_0$ for some choice of $k \geqslant n - \log t - \log(1/\epsilon)$. Note that for this case the proof obtains a better dependence on ϵ compared to (1) and (2).

B. Proof of Theorem 9 for general adversaries

First, we present a proof sketch describing the ideas an intuitions behind the general proof, and then proceed with a full proof of the theorem.

1) Proof sketch: In the proof for bijective adversaries, we heavily used the fact that the tampering of each set $E(s)$ of codewords is a disjoint set of the same size. For general adversaries; however, this may not be true. Intuitively, since the codewords in $E(s)$ are chosen uniformly and almost independently at random (ignoring the distinctness dependencies), the tampered distribution $f(E(s))$ should look similar to $f(\mathcal{U}_n)$ for all s, if $|E(s)|$ is sufficiently large. Indeed, this is what is shown in the proof. The proof also adjusts the probability mass of $\underline{\mathsf{same}}$ according to the fraction of the fixed points of f, but we ignore this technicality for the proof sketch.

Note that the distribution $f(\mathcal{U}_n)$ may be arbitrary, and may assign a large probability mass to a small set of the probability space. For example, f may assign half of the probability mass to a single point. We call the points in $\{0,1\}^n$ receiving a noticeable share of the probability mass in $f(\mathcal{U}_n)$ the *heavy elements* of $\{0,1\}^n$, and fix the randomness of the code construction so that the decoder's values at heavy elements are revealed before analyzing each individual message s. Doing so allows us to analyze each message s separately and take a union bound on various choices of s as in the case of bijective adversaries. Contrary to the bijective case; however, the distribution $\mathcal{D}_f$ is no longer entirely supported on $\perp$; but we show that it still can be made to have a fairly small support; roughly $\mathsf{poly}(n, \log|\mathcal{F}|)$. More precisely, the proof shows non-malleability with respect to the choice of $\mathcal{D}_f$ which is explicitly defined to be the distribution of the following random variable:

$$D := \begin{cases} \underline{\mathsf{same}} & \text{if } f(U_n) = U_n, \\ \mathsf{Dec}(f(U_n)) & \text{if } f(U_n) \neq U_n \text{ and } f(U_n) \in H, \\ \perp & \text{otherwise,} \end{cases}$$

where $H \subseteq \{0,1\}^n$ is the set of heavy elements formally defined as

$$H := \{x \in \{0,1\}^n \colon \Pr[f(U_n) = x] > 1/r\},$$

for an appropriately chosen $r = \Theta(\epsilon^2 t)$.

Remark 11. By inspecting the above recipe for the distribution $\mathcal{D}_f$ in Definition 4, we see that the set of heavy elements H would be empty whenever the tampering function f is such that the min-entropy of the random variable $f(U_n)$ is at least $\log r$. By plugging in the final choices of r and t as derived later in (20) and (21), we see that the required min-entropy lower bound is

$$\log\log(|\mathcal{F}|2^n/\eta) + 4\log(1/\epsilon) + O(1).$$

In particular, when $\mathcal{F}| \leqslant 2^{2^{\alpha n}}$ for some $\alpha \in (0,1)$ and, say, $\eta = 2^{-n}$, there are no heavy elements provided that the min-entropy[4] of $f(U_n)$ is at least

$$\alpha n + \log n + 4\log(1/\epsilon) + O(1).$$

In this case, the distribution $\mathcal{D}_f$ becomes fully supported on $\{\underline{\mathsf{same}}, \perp\}$, which means whenever any tampering occurs, the coding scheme is able to detect an error. Moreover, when in addition to the above the function f has few fixed points; namely, when $\Pr[f(U) = U] = O(\epsilon)$ for $U \sim \mathcal{U}_n$, one can take $\mathcal{D}_f$ to be the singleton distribution having its whole mass on the error symbol $\perp$ (at the cost of a constant factor increase, from ϵ to $O(\epsilon)$, in the error parameter). This is for example the case when the family of tampering functions consists of additive noise functions (i.e., $f(x) = x + e$ for some $e \in \{0,1\}^n$), and in this special case we derive a construction of Algebraic Manipulation Detection codes (as defined in [3]) achieving rate $1 - o(1)$ that are always able to detect tampering whenever it happens. □

Although the above intuition is natural, turning it into a rigorous proof requires substantially more work than the bijective case, and the final proof turns out to be rather delicate even though it only uses elementary probability tools. The first subtlety is that revealing the decoder at the heavy elements creates dependencies between various random variables used in the analysis. In order to make the proof more intuitive, we introduce a random process, described as an algorithm Reveal, that gradually reveals information about the code as the proof considers the codewords $E_1, \ldots, E_t$ corresponding to the picked message s. The process outputs a list of elements in $\{0,1\}^k$, and we show that the empirical distribution of this list is close to the desired $\mathcal{D}_f$ for all messages s.

Roughly speaking, at each step $i \in [t]$ the analysis estimates the distribution of $\mathsf{Dec}(f(E_i))$ conditioned on the particular realizations of the previous codewords. There are three subtleties that we need to handle to make this work:

1) The randomness corresponding to some of the E_i is previously revealed by the analysis and thus such codewords cannot be assumed to be uniformly distributed any more. This issue may arise due to the revealing of the decoder's values at heavy elements in the beginning of analysis, or existence of cycles in the evaluation graph of the tampering function f. Fortunately, it is straightforward to show that the number of such codewords remain much smaller than t with high probability, and thus they may simply be ignored.
2) At each step of the analysis, the revealed information make the distribution of $\mathsf{Dec}(f(E_i))$ gradually farther from the desired $\mathcal{D}_f$. The proof ensures that the expected increase at each step is small, and using standard Martingale concentration bounds the total deviation from $\mathcal{D}_f$ remains sufficiently small with high probability at the end of the analysis.
3) Obtaining small upper bounds (e.g., $\exp(-cn)$ for some $c < 1$) on the probability of various bad events in the analysis (e.g., $\mathsf{Dec}(f(\mathsf{Enc}(s)))$ significantly deviating from $\mathcal{D}_f$) is not difficult to achieve. However, extra care is needed to ensure that the probabilities are much smaller than $1/(2^k|\mathcal{F}|)$ (to accommodate the final union bound), where the latter may easily be doubly-exponentially small in n. An exponential upper bound of $\exp(-cn)$ does not even suffice for moderately large families of adversaries such as bit-tampering adversaries, for which we have $|\mathcal{F}| = 4^n$.

2) Complete proof of Theorem 9: First, observe that by construction, the minimum distance of the final code is always greater than δn; that is, whenever $\mathsf{Dec}(w_1) \neq \perp$ and $\mathsf{Dec}(w_2) \neq \perp$ for any pair of vectors $w_1 \neq w_2$, we have

$$\mathsf{dist}_h(w_1, w_2) > \delta n,$$

where $\mathsf{dist}_h(\cdot)$ denotes the Hamming distance. This is because whenever a codeword is picked, its δn neighborhood is removed from the sample space for the future codewords. Let V denote the volume of a Hamming ball of radius δn. It is well known that $V \leqslant 2^{nh(\delta)}$, where $h(\cdot)$ is the binary entropy function.

Fix an adversary $f \in \mathcal{F}$. We wish to show that the coding scheme $(\mathsf{Enc}, \mathsf{Dec})$ defined by Construction 1 is non-malleable with high probability for the chosen f.

Let the random variable $U := U_n$ be a uniformly random element in $\{0,1\}^n$. Define $p_0 := \Pr[f(U) = U]$. In the sequel, assume that $p_0 < 1$ (otherwise, there is nothing to prove). For every $x \in \{0,1\}^n$, define $p(x) := \Pr[f(U) = x \wedge x \neq U]$. Observe that

$$\sum_x p(x) = 1 - p_0.$$

We say that a string $x \in \{0,1\}^n$ is *heavy* if

$$p(x) > 1/r,$$

for a parameter $r \leqslant t$ to be determined later. Note that the number of heavy strings must be less than r. Define

$$\begin{aligned} H &:= \{x \in \{0,1\}^n : p(x) > 1/r\}, \\ \gamma &:= t/N, \\ \gamma' &:= tK/N. \end{aligned}$$

Fix the randomness of the code construction so that $\mathsf{Dec}(x)$ is revealed for every heavy x. We will argue that no matter how the decoder's outcome on heavy elements is decided by the randomness of the code construction, the construction is non-malleable for every message s and the chosen function f with overwhelming probability. We will then finish the proof with a union bound over all choices of s and f.

Consider a random variable D defined over $\{0,1\}^k \cup \{\perp, \underline{\mathsf{same}}\}$ in the following way:

$$D := \begin{cases} \underline{\mathsf{same}} & \text{if } f(U_n) = U_n, \\ \mathsf{Dec}(f(U_n)) & \text{if } f(U_n) \neq U_n \text{ and } f(U_n) \in H, \\ \perp & \text{otherwise.} \end{cases} \quad (5)$$

[4] Recall that the min-entropy of a distribution $\mathcal{X}$ over a discrete set Ω is defined as $H_\infty(\mathcal{X}) := \min_{x \in \Omega} \log(1/\mathcal{X}(x))$.

For the chosen f, we explicitly define the distribution $\mathcal{D}_f$ as $\mathcal{D}_f := \mathscr{D}(D)$.

Now, consider a fixed message $s \in \{0,1\}^k$, and define the random variable $E_s := \mathsf{Enc}(s)$. That is, E_s is uniformly supported on the set $E(s)$ (this holds by the way that the encoder is defined). Observe that the marginal distribution of each individual set $E(s)$ (with respect to the randomness of the code construction) is the same for all choices of s, regardless of the ordering assumed by Construction 1 on the message space $\{0,1\}^k$.

Furthermore, define the random variable D_s as follows.

$$D_s := \begin{cases} \underline{\mathsf{same}} & \text{if } f(E_s) = E_s, \\ \mathsf{Dec}(f(E_s)) & \text{otherwise.} \end{cases} \tag{6}$$

Our goal is to show that the distribution of D_s (for the final realization of the code) is ϵ-close to $\mathcal{D}_f$ with high probability over the randomness of the code construction. Such assertion is quite intuitive by comparing the way the two distributions D_s and $\mathcal{D}_f$ are defined. In fact, it is not hard to show that the assertion holds with probability $1 - \exp(-\Omega(n))$. However, such a bound would be insufficient to accommodate a union bound of even moderate sizes such as 2^n, which is needed for relatively simple classes such as bit-tampering adversaries. More work needs to be done to ensure that it is possible to achieve a high probability statement with failure probability much smaller than $1/|\mathcal{F}|$, which may in general be doubly exponentially small in n.

The claim below shows that closeness of $\mathscr{D}(D_s)$ to $\mathcal{D}_f$ would imply non-malleability of the code.

Claim 12. *Suppose that for every $s \in \{0,1\}^k$, we have $\mathscr{D}(D_s) \approx_\epsilon \mathcal{D}_f$ for the choice of $\mathcal{D}_f$ defined in* (5)*. Then,* $(\mathsf{Enc}, \mathsf{Dec})$ *is a non-malleable coding scheme with error ϵ and a strong non-malleable coding scheme with error 2ϵ.*

Proof. In order to verify Definition 6, we need to verify that for every pair of distinct messages $s_1, s_2 \in \{0,1\}^k$, $\mathscr{D}(D_{s_1}) \approx_{2\epsilon} \mathscr{D}(D_{s_2})$. But from the assumption, we know that $\mathscr{D}(D_{s_1})$ and $\mathscr{D}(D_{s_2})$ are both ϵ-close to $\mathcal{D}_f$. Thus the result follows by the triangle inequality.

It is of course possible now to use [1, Theorem 3.1] to deduce that Definition 4 is also satisfied. However, for the clarity of presentation, here we give a direct argument that shows that non-malleability is satisfied with the precise choice of $\mathcal{D}_f$ defined in (5) and error ϵ. Let $s \in \{0,1\}^k$, and let $E_s := \mathsf{Enc}(s)$ and $S := \mathsf{Dec}(f(E_s))$. Let $S' \sim \mathcal{D}_f$ and $S'' \sim \mathscr{D}(D_s)$ be sampled independently. We need to show that

$$\mathscr{D}(S) \approx_\epsilon \mathscr{D}(\mathsf{copy}(S', s)). \tag{7}$$

From the definition of D_s in (6), since $\mathsf{Dec}(f(E_s)) = s$ when $f(E_s) = E_s$, we see that $\mathscr{D}(\mathsf{copy}(S'', s)) = \mathscr{D}(\mathsf{Dec}(f(E_s))) = \mathscr{D}(S)$. Now, since by assumption $\mathscr{D}(S') \approx_\epsilon \mathscr{D}(S'')$, it follows that $\mathscr{D}(\mathsf{copy}(S', s)) \approx_\epsilon \mathscr{D}(\mathsf{copy}(S'', s))$ which proves (7). □

Let the random variables $E_1, \ldots, E_t$ be the elements of $E(s)$, in the order they are sampled by Construction 1 (note the integer subscript, which is to be distinguished from the notation $E_s = \mathsf{Enc}(s)$ whose subscript is a k-bit string).

Define, for $i \in [t]$,

$$S_i := \begin{cases} \underline{\mathsf{same}} & \text{if } f(E_i) = E_i, \\ \mathsf{Dec}(f(E_i)) & \text{otherwise.} \end{cases}$$

We note that, no matter how the final code is realized by the randomness of the construction, the distribution D_s is precisely the empirical distribution[5] of $S_1, \ldots, S_t$ as determined by the code construction. To see this, note that the message s corresponds to t codewords $E_1, \ldots, E_t$, one of which randomly chosen by the encoder when the message s is given. Moreover, S_i determines the decoding of the ith codeword after being tampered by the adversary f (with the special symbol $\underline{\mathsf{same}}$ reserved for when f does not tamper the codeword). Therefore, the empirical distribution of the S_i (which is the distribution obtained by outputting S_i for uniformly random $i \in [t]$) is precisely the distribution of $\mathsf{Dec}(f(\mathsf{Enc}(s)))$ (again reserving $\underline{\mathsf{same}}$ for when f does not tamper the encoding), which is how D_s is defined.

Observe that the random variables $S_1, \ldots S_{i-1}$ are not independent (for example, the knowledge of $S_1 = s$ would skew the distribution of E_2 which in turn would affect the distribution of S_2). However, our goal is to set up the parameters so that the distribution of each S_i is not affected by much conditioned on $S_1, \ldots, S_{i-1}$. Thus in the sequel, for each $i \in [t]$, we analyze the distribution of the variable S_i conditioned on the values of $S_1, \ldots S_{i-1}$ and use this analysis to prove that the empirical distribution of the sequence $(S_1, \ldots, S_t)$ is close to $\mathcal{D}_f$.

In order to understand the empirical distribution of the S_i (conditioned on the previous ones $S_1, \ldots, S_{i-1}$), we consider the following process Reveal that considers the picked codewords $E_1, \ldots, E_t$ in order, gradually reveals information about the code construction, and outputs a subset of the S_i. We will ensure that

1) The process outputs a large subset of $\{S_1, \ldots, S_t\}$, and,
2) The empirical distribution of the sequence output by the process is close to $\mathcal{D}_f$ with high probability.

The above guarantees would in turn imply that the empirical distribution of the entire sequence S_i is also close to $\mathcal{D}_f$ with high probability. We define the process as follows.

Process Reveal*:*:

1. Initialize the set $\mathsf{Skip} \subseteq [t]$ with the empty set. Recall that the values of $\mathsf{Dec}(w)$ for all $w \in H$ are already revealed in the analysis, as well as $\mathsf{Dec}(\Gamma(w))$ for those for which $\mathsf{Dec}(w) \neq \perp$.
2. For each heavy element $w \in H$, if $\mathsf{Dec}(w) = s$, consider the unique $j \in [t]$ such that $E_j = w$.

[5] Recall the definition of empirical distribution from the preliminaries section.

```
       Reveal[6] $j$ and $E_j$, and add $j$ to Skip.
3.     For $i$ from 1 to $t$, define *the $i$th stage* as follows:
       3.1.   If $i \in$ Skip, declare a *skip* and continue the
              loop with the next $i$. Otherwise, follow the
              remaining steps.
       3.2.   Reveal $\Gamma(E_i)$. Note that revealing $E_i$ implies
              that $\mathsf{Dec}(E_i)$ is revealed as well, since
              $\mathsf{Dec}(E_i) = s$. Moreover, recall that for any
              $x \in \Gamma(E_i) \setminus E_i$, $\mathsf{Dec}(x) = \perp$ by the code
              construction.
       3.3.   If $\mathsf{Dec}(f(E_i))$ is not already revealed:
              3.3.1. Reveal $\mathsf{Dec}(f(E_i))$.
              3.2.2. If $\mathsf{Dec}(f(E_i)) = s$, consider the
                     unique $j \in [t]$ such that $E_j = f(E_i)$.
                     It must be that $j > i$, since $\mathsf{Dec}(E_j)$
                     has not been revealed before. Reveal
                     $j$ and add it to Skip.
              3.3.3. Declare that an *unveil* has happened
                     if $\mathsf{Dec}(f(E_i)) \neq \perp$. If so, reveal
                     $\mathsf{Dec}(f(x))$ for all $x \in \Gamma(f(E_i)) \setminus E_i$
                     to equal $\perp$.
       3.4.   Reveal and output $S_i$.
```

For $i \in [t]$, we use the notation Reveal_i to refer to all the information revealed from the beginning of the process up to the time the ith stage begins. We also denote by $\mathsf{Next}(i)$ the least $j > i$ such that a skip does not occur at stage j; define $\mathsf{Next}(i) := t+1$ if no such j exists, and define $\mathsf{Next}(0)$ to be the index of the first stage that is not skipped. Moreover, for $w \in \{0,1\}^n$, we use the notation $w \in \mathsf{Reveal}_i$ as a shorthand to denote the event that the process Reveal has revealed the value of $\mathsf{Dec}(w)$ at the time the ith stage begins.

By the way the code is constructed, the decoder's value at each given point is most likely $\perp$. We make this intuition more rigorous and show that the same holds even conditioned on the information revealed by the process Reveal.

Claim 13. *For all $i \in [t]$ and any $a \in \mathsf{supp}(\mathsf{Reveal}_i)$ and any $x \in \{0,1\}^n$,*

$$\begin{aligned}\Pr[\mathsf{Dec}(x) \neq \perp \mid (\mathsf{Reveal}_i = a) \wedge (x \notin \mathsf{Reveal}_i)] \\ \leqslant \gamma'/(1-3\gamma V),\end{aligned}$$

where the probability is over the randomness of the code construction.

Proof. Suppose $x \notin \mathsf{Reveal}_i$, and observe that Reveal_i at each step reveals at most the values of the decoder at $2V$ points; namely, $\Gamma(E_i)$ and $\Gamma(f(E_i))$. Moreover, before the first stage, decoder's value is revealed at up to r heavy points and its Hamming neighborhood at radius δn. In total, the total number of points at which decoder's value is revealed by the information in Reveal_i is at most

$$(|H| + 2(i-1))|V| \leqslant (2t+r)V \leqslant 3\gamma V N.$$

Let

$$\mathcal{C} := \bigcup_s E(s)$$

be the set of all codewords of the coding scheme. Some of the elements of $\mathcal{C}$ are already included in Reveal_i, and by assumption we know that none of these is equal to x.

The distribution of each unrevealed codeword, seen in isolation, is uniform over the $N(1-3\gamma V)$ remaining vectors in $\{0,1\}^n$. Thus by taking a union bound on the probability of each such codeword hitting the point x (which is the only way to make $\mathsf{Dec}(x) \neq \perp$, we deduce that

$$\begin{aligned}\Pr[\mathsf{Dec}(x) \neq \perp \mid \mathsf{Reveal}_i = a] &\leqslant \frac{tK}{N(1-3\gamma V)} \\ &= \gamma'/(1-3\gamma V).\end{aligned}$$

□

Ideally, for each $i \in [t]$ we desire to have E_i almost uniformly distributed, conditioned on the revealed information, so that the distribution of $\mathsf{Dec}(f(E_i))$ (which is described by S_i when E_i does not hit a fixed point of f) becomes close to $\mathsf{Dec}(f(U_n))$. However, this is not necessarily true; for example, when the process Reveal determines the decoder's value on the heavy elements, the value of, say, E_1 may be revealed, at which point there is no hope to ensure that E_1 is nearly uniform. This is exactly what the set Skip is designed for, to isolate the instances when the value of E_i is already determined by the prior information. More precisely, we have the following.

Claim 14. *Suppose that $i \notin$ Skip when the ith stage of Reveal begins. Then, for any $a \in \mathsf{supp}(\mathsf{Reveal}_i)$,*

$$\mathscr{D}(E_i \mid \mathsf{Reveal}_i = a) \approx_\nu \mathcal{U}_n,$$

where $\nu := (3\gamma V)/(1-3\gamma V)$.

Proof. Note that, without any conditioning, the distribution of E_i is exactly uniform on $\{0,1\}^n$. If at any point prior to reaching the ith stage it is revealed that $\mathsf{Dec}(E_i) = s$, either line 2 or line 3.2.2 of process Reveal ensures that i is added to the set Skip.

If, on the other hand, the fact that $\mathsf{Dec}(E_i) = s$ has not been revealed when the ith stage begins, the distribution of E_i becomes uniform on the points in $\{0,1\}^n$ that have not been revealed yet. As in Claim 13, the number of revealed points is at most $(2t+r)V \leqslant 3\gamma V N$. Thus, the conditional distribution E_i remains $((3\gamma V)/(1-3\gamma V))$-close to uniform by Proposition 32. □

[6] In a rigorous sense, by revealing a random variable we mean that we condition the probability space on the event that a particular value is assumed by the variable. For example, revealing E_i means that the analysis branches to a conditional world where the value of E_i is fixed to the revealed value. In an intuitive sense, one may think of a reveal as writing constraints on the realization of the code construction on a blackboard, which is subsequently consulted by the analysis (in form of the random variable Reveal_i that the analysis defines to denote the information revealed by the process before stage i).

For each $i \in [t]$, define a random variable $S'_i \in \{0,1\}^k \cup \{\underline{\mathsf{same}}, \perp\}$ as follows (where U_n is independently sampled from $\mathcal{U}_n$):

$$S'_i := \begin{cases} \underline{\mathsf{same}} & \text{if } f(U_n) = U_n, \\ \mathsf{Dec}(f(U_n)) & \text{if } f(U_n) \neq U_n \wedge f(U_n) \in \mathsf{Reveal}_i, \\ \perp & \text{otherwise.} \end{cases} \tag{8}$$

Note that $\mathscr{D}(S'_1) = \mathcal{D}_f$.

Intuitively, S'_i is the "cleaned up" version of the random variable S_i that we are interested in. As defined, S'_i is an independent random variable, and as such we are more interested in its *distribution* than value. Observe that the distribution of S'_i is randomly determined according to the randomness of the code construction (in particular, the knowledge of Reveal_i completely determines $\mathscr{D}(S'_i)$). The variable S'_i is defined so that its distribution approximates the distribution of the actual S_i conditioned on the revealed information before stage i. Formally, we can show that conditional distributions of these two variables are (typically) similar. Namely,

Claim 15. *Suppose that $i \notin \mathsf{Skip}$ when the ith stage of Reveal begins. Then, for any $a \in \mathsf{supp}(\mathsf{Reveal}_i)$,*

$$\mathscr{D}(S_i \mid \mathsf{Reveal}_i = a) \approx_\nu \mathscr{D}(S'_i \mid \mathsf{Reveal}_i = a),$$

where $\nu := (3\gamma V + \gamma')/(1 - 3\gamma V)$.

Proof. First, we apply Claim 14 to ensure that

$$\mathscr{D}(E_i \mid \mathsf{Reveal}_i = a) \approx_{\nu'} \mathcal{U}_n,$$

where $\nu' = (3\gamma V)/(1 - 3\gamma V)$. Thus we can assume that the conditional distribution of E_i is exactly uniform at cost of a ν' increase in the final estimate.

Now, observe that, conditioned on the revealed information, the way S_i is sampled at stage i of Reveal can be rewritten as follows:

1) Sample $E_i \sim \mathcal{U}_n$.
2) If $f(E_i) = E_i$, set $S_i \leftarrow \underline{\mathsf{same}}$.
3) Otherwise, if $f(E_i) \in \mathsf{Reveal}_i$, set S_i to $\mathsf{Dec}(f(E_i))$ as determined by the revealed information.
4) Otherwise, reveal $\mathsf{Dec}(f(E_i))$ (according to its conditional distribution on the knowledge of Reveal_i) and set S accordingly.

This procedure is exactly the same as how S'_i is sampled by (8); with the difference that at the third step, S'_i is set to $\perp$ whereas S_i is sampled according to the conditional distribution of $\mathsf{Dec}(f(E_i))$. However, we know by Claim 13 that in this case,

$$\Pr[\mathsf{Dec}(f(E_i)) \neq \perp \mid \mathsf{Reveal}_i = a] \leqslant \gamma'/(1 - 3\gamma V).$$

Thus we see that S_i changes the probability mass of $\perp$ in $\mathscr{D}(S'_i)$ by at most $\gamma'/(1 - 3\gamma V)$. The claim follows. □

Recall that the distribution of S'_1 is the same as $\mathcal{D}_f$. However, for subsequent stages this distribution may deviate from $\mathcal{D}_f$. We wish to ensure that by the end of process Reveal, the deviation remains sufficiently small.

For $i \in [t-1]$, define Δ_i as

$$\Delta_i := \mathsf{dist}(\mathscr{D}(S'_{i+1}), \mathscr{D}(S'_i)).$$

where $\mathsf{dist}(\cdot)$ denotes statistical distance. Note that Δ_i is a random variable that is determined by the knowledge of Reveal_{i+1} (recall that Reveal_i determines the exact distribution of S'_i). We show that the conditional values attained by this random variable are small in expectation.

Claim 16. *For each $i \in [t-1]$, and all $a \in \mathsf{supp}(\mathsf{Reveal}_i)$,*

$$\mathbb{E}[\Delta_i \mid \mathsf{Reveal}_i = a] \leqslant \frac{2\gamma'}{r(1 - 3\gamma V)}, \tag{9}$$

Moreover, $\Pr[\Delta_i \leqslant 2/r \mid \mathsf{Reveal}_i = a] = 1$ (the expectation and the probability are over the randomness of the code construction).

Proof. Recall that the distribution of S'_{i+1} is different from S'_i depending on the points at which the decoder's value is revealed during stage i of Reveal. If a skip is declared at stage i, we have $\mathsf{Reveal}_{i+1} = \mathsf{Reveal}_i$ and thus, $\Delta_i = 0$. Thus in the following we may assume that this is not the case.

However, observe that whenever for some $x \in \{0,1\}^n$, the decoder's value $\mathsf{Dec}(x)$ is revealed at stage i, the new information affects the probability distribution of S'_i only if $\mathsf{Dec}(x) \neq \perp$. This is because when $\mathsf{Dec}(x) = \perp$, some of the probability mass assigned by S_i to $\perp$ in (8) is removed and reassigned by S'_{i+1} to $\mathsf{Dec}(x)$, which is still equal to $\perp$. Thus, changes of this type can have no effect on the distribution of S'_i. We conclude that only revealing the value of $\mathbb{E}_i$ and an unveil (as defined in line 3.3.3 of process Reveal) can contribute to the statistical distance between S'_i and S'_{i+1}.

Whenever an unveil occurs at stage i, say at point $x \in \{0,1\}^n$, some of the probability mass assigned to $\perp$ by S'_i is moved to $\mathsf{Dec}(x)$ in the distribution of S'_{i+1}. Since we know that $x \notin H$, the resulting change in the distance between the two distributions is bounded by $1/r$, *no matter* what the realization of x and $\mathsf{Dec}(x)$ are. Overall, using Claim 13, the expected change between the two distributions contributed by the occurrence of an unveil is upper bounded by the probability of an unveil occurring times $1/r$, which is at most

$$\frac{\gamma'/r}{1 - 3\gamma V}. \tag{10}$$

The only remaining factor that may contribute to an increase in the distance between distribution of S'_i and S'_{i+1} is the revealing of E_i at stage i. The effect of this reveal in the statistical distance between the two distributions is $p(E_i)$, since according to (8) the value of S'_{i+1} is determined by the outcome of $f(U_n)$, and thus the probability mass assigned to $\mathsf{Dec}(E_i)$ by S'_{i+1} is indeed $\Pr[f(U_n) = E_i]$. Let $\mathcal{D}_E$ be the distribution of E_i conditioned on the knowledge of Reveal_i. Observe that, since the values $\{p(x) \colon x \in \{0,1\}^n\}$ defines a probability distribution on N points, we clearly have

$$\sum_{x \in \mathsf{supp}(\mathcal{D}_E)} p(x) \leqslant 1. \tag{11}$$

On the other hand, by the assumption that a skip has not occurred at stage i, we can deduce using the argument in

Claim 14 that $\mathcal{D}_E$ is uniformly supported on a support of size at least $N(1-3\gamma V)$. Therefore, using (11), the expected contribution to Δ_i by the revealing of E_i is (which is the expected value of $p(E_i)$) is at most

$$\frac{1}{N(1-3\gamma V)} \leqslant \frac{\gamma'/r}{(1-3\gamma V)}, \tag{12}$$

where the inequality uses $r \leqslant \gamma' N = tK$. The desired bound follows by adding up the two perturbations (10) and (12) considered.

Finally, observe that each of the perturbations considered above cannot be more than $1/r$, since stage i never reveals the decoder's value on a heavy element (recall that all heavy elements are revealed before the first stage begins and the choices of E_i that correspond to heavy elements are added to Skip when Reveal begins). Thus, the conditional value of Δ_i is never more than $2/r$. □

Using the above result, we can deduce a concentration bound on the summation of the differences Δ_i.

Claim 17. *Let $\Delta := \Delta_1 + \cdots + \Delta_{t-1}$, and suppose*

$$\frac{\gamma'}{1-3\gamma V} \leqslant \frac{\epsilon r}{32t}. \tag{13}$$

Then,

$$\Pr[\Delta \geqslant \epsilon/8] \leqslant \exp(-\epsilon^2 r^2/(2048t)) =: \eta_0, \tag{14}$$

where the probability is over the randomness of the code construction.

Proof. For $i \in [t-1]$, define $\Delta'_i := \Delta_i r/2$, $\Delta'_0 := 0$, and $\Delta' := \Delta'_1 + \cdots + \Delta'_{t-1}$. Since Reveal_i determines Δ'_{i-1}, by Claim 16 we know that

$$\mathbb{E}[\Delta'_i \mid \Delta'_0, \ldots, \Delta'_{i-1}] \leqslant \nu,$$

where $\nu := \frac{\gamma'}{1-3\gamma V} \leqslant \epsilon r/(32t)$ In the above, conditioning on $\Delta'_0, \ldots, \Delta'_{i-1}$ instead of Reveal_i (for which Claim 16 applies), is valid in light of Proposition 31, since the knowledge of Reveal_i determines $\Delta'_0, \ldots, \Delta'_{i-1}$.

Moreover, again by the Claim 16, we know that the Δ'_i are between 0 and 1. Using Proposition 33, it follows that

$$\Pr[\Delta \geqslant \epsilon/8] = \Pr[\Delta' \geqslant \frac{\epsilon r}{16t} \cdot t] \leqslant \eta_0. \qquad □$$

Next, we prove a concentration bound for the total number of unveils that can occur in line 3.3.3 of process Reveal.

Claim 18. *Let u be the total number of unveils that occur in process* Reveal*. Assuming $\gamma'/(1-3\gamma V) \leqslant \epsilon/8$ (which is implied by* (13)*), we have*

$$\Pr[u \geqslant \epsilon t/4] \leqslant \exp(-\epsilon^2 t/128) \leqslant \eta_0,$$

where the probability is over the randomness of the code construction.

Proof. Let $X_1, \ldots, X_t$ be indicator random variable such that $X_i = 1$ iff an unveil occurs at stage i, and let $X_0 := 0$. Recall that an unveil can only occur at a stage that is not skipped. Thus, if $i \in [t]$ when the ith stage begins, we can deduce that $X_i = 0$.

Consider $i \in [t]$ such that $i \notin \mathsf{Skip}$ when the ith stage begins. An unveil occurs when $\mathsf{Dec}(f(E_i)) \notin \mathsf{Reveal}_i$. In this case, by Claim 13, we get that

$$\Pr[\mathsf{Dec}(f(E_i)) \neq \perp \mid \mathsf{Reveal}_i] \leqslant \gamma'/(1-3\gamma V).$$

Since Reveal_i determines all the revealed information in each prior stage, and in particular the values of $X_0, \ldots, X_{i-1}$, we can use Proposition 31 to deduce that

$$\Pr[X_i = 1 \mid X_0, \ldots, X_{i-1}] \leqslant \gamma'/(1-3\gamma V).$$

Finally, Proposition 33 derives the desired concentration bound on the number of unveils, which is $X_1 + \cdots + X_t$. □

We are now ready to wrap up the proof and show that with overwhelming probability, the empirical distribution of $S_1, \ldots, S_t$ is ϵ-close to $\mathcal{D}_f$.

Suppose that process Reveal outputs a subset of the S_i. Let $T \subseteq [t]$ be the set of indices i such Reveal outputs S_i in the end of the ith stage. Note that $T = [t] \setminus \mathsf{Skip}$, where Skip denotes the skip set when Reveal terminates. Observe that $|\mathsf{Skip}|$ is at most the total number of unveils occurring at line 3.3.3 of Reveal plus r (which upper bounds the number of heavy elements in H). Thus, using Claim 18 we see that, assuming (13),

$$\Pr[t - |T| \geqslant r + \epsilon t/4] \leqslant \eta_0. \tag{15}$$

Let δ_i for $i \in [t]$ denote the statistical distance between S'_i and $\mathcal{D}_f$. We know that δ_i is a random variable depending on Reveal_i. Thus, the value of δ_i becomes known to a particular fixed value conditioned on the outcome of every Reveal_j, $j \geqslant i$. Define $\delta_0 := \max_i \delta_i$, which is a random variable that becomes revealed by the knowledge of Reveal_t in the end of the process.

Using Claim 15, we thus know that for any $a \in \mathsf{supp}(\mathsf{Reveal}_i)$ and $i \in T$,

$$\mathscr{D}(S_i \mid \mathsf{Reveal}_i = a) \approx_{\nu_0 + \delta_0} \mathcal{D}_f,$$

where

$$\nu_0 := (3\gamma V + \gamma')/(1-3\gamma V).$$

Let $\mathcal{S}$ denote the empirical distribution of $\{S_i : i \in T\}$, and define $S_0 := \perp$. From the above conclusion, using Proposition 31 we can now write, for $i \in T$,

$$\mathscr{D}(S_i \mid (S_j : j \in T \cap \{1, \ldots, i-1\}) \approx_{\nu_0 + \delta_0} \mathcal{D}_f.$$

Recall that $|\mathsf{supp}(\mathcal{D}_f)| \leqslant r + 2$. Assuming that

$$\nu_0 + \delta_0 \leqslant \epsilon/4, \tag{16}$$

Proposition 38 implies (after simple manipulations) that with probability $1 - \eta_1$, where

$$\eta_1 \leqslant 2^{r+4-\Omega(\epsilon^2 |T|)}, \tag{17}$$

$\mathcal{S}$ is $(\epsilon/2)$-close to $\mathcal{D}_f$.

Recall that $\mathscr{D}(S'_1) = \mathcal{D}_f$. Using the triangle inequality for statistical distance, for every $i \in [t]$ we can write

$$\mathsf{dist}(S'_i, \mathcal{D}_f) = \mathsf{dist}(S'_i, S'_1) \leqslant \Delta_1 + \cdots + \Delta_{i-1} \leqslant \Delta,$$

and thus deduce that $\delta_0 \leqslant \Delta$. Recall that by Claim 17, we can ensure that, assuming (13), $\Delta \leqslant \epsilon/8$ (and thus, $\delta_0 \leqslant \epsilon/8$) with probability at least $1 - \eta_0$. Thus under the assumption that

$$\nu_0 \leqslant \epsilon/8, \tag{18}$$

and (13), which we recall below

$$\frac{\gamma'}{1 - 3\gamma V} \leqslant \frac{\epsilon r}{32t},$$

we can ensure that $\nu_0 + \delta_0 \leqslant \epsilon/4$ with probability at least $1-\eta_0$. Moreover, conditioned on the event $\nu_0+\delta_0 \leqslant \epsilon/4$ (recall that δ_0 is a random variable), we have already demonstrated that with probability at least $1 - \eta_1$, $\mathcal{S}$ is $(\epsilon/2)$-close to $\mathcal{D}_f$. After removing conditioning on the bound on δ_0, we may deduce that overall (under the assumed inequalities (13) and (18)), with probability at least $1 - O(\eta_0 + \eta_1)$,

$$\mathcal{S} \approx_{\epsilon/2} \mathcal{D}_f,$$

which in turn, implies that the empirical distribution of $S_1, \ldots, S_t$ becomes ϵ'-close to uniform, where

$$\epsilon' := \epsilon/2 + (1 - |T|/t).$$

Finally, we can use (15) to ensure that (assuming (13)), $\epsilon' \leqslant \epsilon$ and $|T|/t \geqslant 1 - \epsilon/2$ with probability at least $1 - O(\eta_0 + \eta_1)$ as long as

$$r \leqslant \epsilon t/4. \tag{19}$$

By comparing (17) with (14), we also deduce that $\eta_1 = O(\eta_0)$ (and also that (19) holds) as long as $r \leqslant r_0$ for some

$$r_0 = \Omega(\epsilon^2 t). \tag{20}$$

Altogether, we arrive at the conclusion that under assumptions (13), (18), and by taking $r := r_0$, with probability at least $1 - O(\eta_0)$,

$$(\text{empirical distribution of } (S_1, \ldots, S_t)) \approx_\epsilon \mathcal{D}_f,$$

which ensures the required non-malleability condition for message s and tampering function f. By taking a union bound over all possible choices of s and f, the probability of failure becomes bounded by

$$O(\eta_0 K |\mathcal{F}|) =: \eta_2.$$

We can now ensure that $\eta_2 \leqslant \eta$ for the chosen value for r by taking $t \geqslant t_0$ for some

$$t_0 = O\left(\frac{1}{\epsilon^6}\Big(\log \frac{|\mathcal{F}|N}{\eta}\Big)\right). \tag{21}$$

Furthermore, in order to satisfy assumptions (13), (18), and the requirement $tKV \leqslant 1$ which is needed to make the construction possible, it suffices to have $K \leqslant K_0$ for some

$$K_0 = \Omega(\epsilon^3 N/(tV)).$$

Using the bound $V \leqslant 2^{nh(\delta)}$, where $h(\cdot)$ is the binary entropy function, and taking the logarithm of both sides, we see that it suffices to have $k \leqslant k_0$ for some

$$k_0 \geqslant n(1 - h(\delta)) - \log t - 3\log(1/\epsilon) - O(1).$$

This concludes the proof of Theorem 9.

C. *Efficiency in the random oracle model*

One of the main motivations of the notion of non-malleable codes proposed in [1] is the application for *tamper-resilient security*. In this application, a stateful system consists of a public functionality and a private state $s \in \{0,1\}^k$. The state is stored in form of its non-malleable encoding, which is prone to tampering by a family of adversaries. It is shown in [1] that the security of the system with encoded private state can be guaranteed (in a naturally defined sense) provided that the distribution $\mathcal{D}_f$ related to the non-malleable code is efficiently samplable. In light of Remark 8, efficient sampling of $\mathcal{D}_f$ can be assured if the non-malleable code is equipped with an efficient encoder and decoder.

Although the code described by Construction 1 may require exponential time to even describe, it makes sense to consider efficiency of the encoder and the decoder in the *random oracle model*, where all involved parties have oracle access to a shared, exponentially long, random string. The uniform decoder construction of [1] is shown to be efficiently implementable in the random oracle model in an *approximate* sense (as long as all involved parties query the random oracle a polynomial number of times), assuming existence of an efficient algorithm implementing a uniformly random permutation Π (over $\{0,1\}^n$) and its inverse Π^{-1}.

We observe that Construction 1, for the distance parameter $\delta = 0$ (which is what needed for strong non-malleability as originally defined in [1]) can be *exactly* implemented efficiently (without any further assumptions on boundedness of the access to the random oracle) assuming access to a uniformly random permutation and its inverse (i.e., the so-called ideal-cipher model). This is because our code is designed so that the codewords are picked uniformly at random and without replacement. More precisely, the encoder, given message $s \in \{0,1\}^k$, can sample a uniformly random $i \in [t]$, and output $\Pi(\phi(s,i))$, where $\phi\colon \{0,1\}^k \times [t] \to \{0,1\}^n$ is any fixed injective function that interprets (s,i) as an element of $\{0,1\}^n$ (recall the the code construction assumes $t2^k \leqslant 2^n$).

As noted in [1], efficient approximate implementations of uniformly random permutations exist in the random oracle model. In particular, [13] (following [14]) shows such an approximation with security $\mathsf{poly}(q)/2^n$, where q is the number of queries to the random oracle.

IV. A Monte Carlo construction for computationally bounded adversaries

An important feature of Construction 1 is that the proof of non-malleability, Theorem 9, only uses limited independence of the permutation defining the codewords $E(s)$ corresponding to each message. This is because the proof analyzes the distribution of $\mathsf{Dec}(f(\mathsf{Enc}(s)))$ for each individual message separately, and then takes a union bound on all choice of s.

More formally, below we show that Theorem 9 holds for a broader range of code constructions than the exact Construction 1.

Definition 19 (ℓ-wise independent schemes)**.** Let $(\mathsf{Enc}, \mathsf{Dec})$ be any randomized construction of a coding scheme with block length n and message length k. For each $s \in \{0,1\}^k$, define $E(s) := \mathsf{supp}(\mathsf{Enc}(s))$ and let $t_s := |\mathsf{supp}(\mathsf{Enc}(s))|$. We say that the construction is ℓ-wise independent if the following are satisfied.

1) For any realization of $(\mathsf{Enc}, \mathsf{Dec})$, the distribution of $\mathsf{Enc}(s)$ (with respect to the internal randomness of Enc) is uniform on $\mathsf{supp}(\mathsf{Enc}(s))$.
2) The distribution of the codewords defined by the construction is ℓ-wise independent. Formally, we require the following. Let $\mathcal{C} := \bigcup_{s\in\{0,1\}^k} \mathsf{supp}(\mathsf{Enc}(s))$. Suppose the construction can be described by a deterministic function[7] $E\colon \{0,1\}^k \times \mathbb{N} \times \mathbb{N} \to \{0,1\}^n$ such that for a bounded random oracle $\mathcal{O}$ over $\mathbb{N}$ (describing the random bits used by the construction), the sequence
$$(E(s,i,\mathcal{O}))_{s\in\{0,1\}^k, i\in[t_s]}$$
enumerates the set $\mathcal{C}$. Moreover, for any set of distinct t indices $S = \{(s_j, i_j)\colon j \in [\ell], s_j \in \{0,1\}^k, i_j \in [t_s]\}$, we have
$$\mathscr{D}(E(s_1,i_1,\mathcal{O}),\ldots,E(s_\ell,i_\ell,\mathcal{O})) = \mathscr{D}(\Pi(1),\ldots,\Pi(\ell))$$
for a uniformly random bijection $\Pi\colon [2^n] \to \{0,1\}^n$.

Lemma 20. *Let* $(\mathsf{Enc}, \mathsf{Dec})$ *be any randomized construction of a coding scheme with block length n and message length k. For each $s \in \{0,1\}^k$, define $E(s) := \mathsf{supp}(\mathsf{Enc}(s))$. Suppose that for any realization of* $(\mathsf{Enc}, \mathsf{Dec})$*, and for every $s_1, s_2 \in \{0,1\}^k$, we have*

1) *$|E(s_1)| \geqslant t_0$, where t_0 is the parameter defined in Theorem 9.*
2) $|E(s_2)| = O(|E(s_1)|)$.

Moreover, suppose that $k \leqslant k_0$, for k_0 as in Theorem 9. Let $t := \max_s |E(s)|$. Then, assuming that the construction is $(3t)$-wise independent, the conclusion of Theorem 9 for distance parameter $\delta = 0$ holds for the coding scheme $(\mathsf{Enc}, \mathsf{Dec})$.

Proof. We argue that the proof of Theorem 9 holds without any technical change if

1) The codewords in $\mathsf{supp}(\mathsf{Enc}(\mathcal{U}_k))$ are chosen not fully independently but $(3t)$-wise independently, and
2) Each set $E(s)$ is not necessarily of exact size t but of size at least t_0 and $\Theta(t)$.

The key observation to be made is that the proof analyzes each individual message $s \in \{0,1\}^k$ separately, and then applies a union bound on all choices of s. Thus we only need sufficient independence to ensure that the view of the analysis on each individual choice of the message is statistically the same as the case where the codewords are chosen fully independently.

Observe that the bulk of the information about the code looked up by the analysis for analyzing each individual message is contained in the random variable Reveal_{t+1} defined in the proof of Theorem 9, that is defined according to how the process Reveal evolves. Namely, Reveal_{t+1} summarizes all the information revealed about the code by the end of the process Reveal.

For a fixed message $s \in \{0,1\}^n$ the process Reveal iterates for $|E(s)| \leqslant t$ step. At each step, the location of at most two codewords in $\mathsf{supp}(\mathsf{Enc}(\mathcal{U}_k))$ is revealed. Moreover, before the process starts, the values of the decoder on the heavy elements in H, which can correspond to less than t codewords, are revealed by the process. The only other place in the proof where an independent codeword is required is the union bound in the proof of Claim 13, which needs another degree of independence. Altogether, we conclude that the proof of Theorem 9 only uses at most $3t$ degrees of independence in the distribution of the codewords picked by the construction.

Moreover, for each message s, the analysis uses the fact that $|E(s)| \geqslant t_0$ to ensure that the code does not satisfy non-malleability for the given choice of s and tampering function remains below the desired level. Since $|E(s)|$ for different values of s are assumed to be within a constant factor of each other, the requirement (20) may also be satisfied by an appropriate choice of the hidden constant. Finally, using the fact that $\max_s |E(s)| = O(\min_s |E(s)|)$, we can also ensure that assumptions (13), and (18) can be satisfied for appropriate choices of the hidden constants in asymptotic bounds. □

In order to implement an efficient ℓ-wise independent coding scheme, we use the bounded independence property of polynomial evaluations over finite fields. More precisely, we consider the coding scheme given in Construction 2.

The advantage of using the derandomized Monte Carlo construction is that the number of random bits required to describe the code is dramatically reduced from $O(tnK)$ bits (which can be exponentially large if the rate of the code is $\Omega(1)$) to only $O(tn)$ bits, which is only polynomially large if $t = \mathsf{poly}(n)$. In order to efficiently implement the derandomized construction, we use bounded independence properties of polynomial evaluation. Using known algorithms for finite field operations and root finding, the implementation can be done in polynomial time.

Lemma 21. *Consider the pair* $(\mathsf{EncMC}, \mathsf{DecMC})$ *defined in Construction 2. For every $\eta > 0$, there is a $t_0 = O(n + \log(1/\eta))$ such that for every $t \geqslant t_0$ (where t is a power of two), with probability at least $1 - \eta$ the following hold.*

1) $(\mathsf{EncMC}, \mathsf{DecMC})$ *is a $(9t)$-wise independent coding scheme.*
2) *For all $s \in \{0,1\}^k$,* $|\mathsf{supp}(\mathsf{EncMC}(s))| \in [t, 3t]$.

[7] As an example, in Construction 1, all the values t_s are equal to the chosen t, and moreover, one can take $E(s,i,\mathcal{O}) = \Pi(s,i)$, where $\Pi\colon \{0,1\}^k \times [2^{n-k}] \to \{0,1\}^n$ is a uniformly random bijection defined by the randomness of $\mathcal{O}$.

- *Given:* Integer parameters $0 < k \leqslant n$ and integer $t > 1$ which is a power of two. Let $b := \log(2t)$ and $m := n - k - b$.
- *Output:* A coding scheme $(\mathsf{EncMC}, \mathsf{DecMC})$ of block length n and message length k.
- *Randomness of the construction:* A uniformly random polynomial $P \in \mathbb{F}_{2^n}[9t-1]$.
- *Construction of* EncMC*:* Given $s \in \{0,1\}^k$,
  1) Initialize a set $E \subseteq \{0,1\}^n$ to the empty set.
  2) For every $z \in \{0,1\}^b$,
     a) Construct a vector $y := (s, 0^m, z) \in \{0,1\}^n$ and regard it as an element of $\mathbb{F}_{2^n}$.
     b) Solve $P(X) = y$, and add the set of solutions (which is of size at most $9t-1$) to E.
  3) Output a uniformly random element of E.
- *Construction of* DecMC*:* Given $x \in \{0,1\}^n$, interpret x as an element of $\mathbb{F}_{2^n}$, and let $y := P(x)$, interpreted as a vector $(y_1, \ldots, y_n) \in \{0,1\}^n$. If $(y_{k+1}, y_{k+2}, \ldots, y_{k+m}) = 0^m$, output $(y_1, \ldots, y_k)$. Otherwise, output $\perp$.

Construction 2: The Monte Carlo Construction.

Proof. Let $N := 2^n$ and $K := 2^k$. Consider the vector $X := (X_1, \ldots, X_N) \in \mathbb{F}_{2^n}^N$, where $X_i := P(i)$ and each i is interpreted as an element of $\mathbb{F}_{2^n}$. Since the polynomial P is of degree $9t-1$, the distribution of $X_1, \ldots, X_N$ over the randomness of the polynomial P is $(9t)$-wise independent with each individual X_i being uniformly distributed on $\mathbb{F}_{2^n}$. This standard linear-algebraic fact easily follows from invertibility of square Vandermonde matrices.

Note that the decoder function DecMC in Construction 2 is defined so that

$$\mathsf{DecMC}(U_n) = \begin{cases} \perp & \text{with probability } 1 - 2tK/N \\ s \in \{0,1\}^k & \text{with probability } 2t/N. \end{cases} \tag{22}$$

For $s \in \{0,1\}^k$, let $E(s) := \mathsf{supp}(\mathsf{EncMC}(s))$. Note that the encoder, given s, is designed to output a uniformly random element of $E(s)$. Since the definition of the $\mathsf{EncMC}(s)$ is so that it exhausts the list of all possible words in $\{0,1\}^n$ that can lie in $\mathsf{DecMC}^{-1}(s)$, it trivially follows that $(\mathsf{EncMC}, \mathsf{DecMC})$ is always a valid coding scheme; that is, for any realization of the code and for all $s \in \{0,1\}^n$, we have $\mathsf{DecMC}(\mathsf{EncMC}(s)) = s$ subject to the guarantee that $|E(s)| > 0$.

Fix some $s \in \{0,1\}^k$. Let $Z_1, \ldots, Z_N \in \{0,1\}$ be indicator random variable such that $Z_i = 1$ iff $\mathsf{DecMC}(i) = s$ (when i is interpreted as an n-bit string). Recall that $(Z_1, \ldots, Z_N)$ is a $(9t)$-wise independent random vector with respect to the randomness of the code construction. Let $Z := Z_1 + \cdots + Z_N$, and note that $Z = |E(s)|$. From (22), we see that

$$\mathbb{E}[Z] = \mathbb{E}[|E(s)|] = 2t\ .$$

Using Theorem 36 with $\ell := t/4$ and $A := \mathbb{E}[Z]/2 = t$, we see that

$$\Pr[|Z - 2t| \geqslant t] \leqslant 8(3/4)^{t/4}.$$

By taking a union bound over all choices of $s \in \{0,1\}^k$, we conclude that with probability at least $1 - \eta_0$, where we define $\eta_0 := 8N(3/4)^{t/4}$, the realization of $(\mathsf{EncMC}, \mathsf{DecMC})$ is so that

$$(\forall s \in \{0,1\}^k)\colon |E(s)| \in [t, 3t].$$

This bound suffices to show the desired conclusion. □

By combining the above tools with Theorem 9, we can derive the following result on the performance of Construction 2.

Theorem 22. *Let $\mathcal{F}\colon \{0,1\}^n \to \{0,1\}^n$ be any family of tampering functions. For any $\epsilon, \eta > 0$, with probability at least $1-\eta$, the pair* $(\mathsf{EncMC}, \mathsf{DecMC})$ *in Construction 2 can be set up to achieve a non-malleable coding scheme with respect to $\mathcal{F}$ and with error ϵ. Moreover, the scheme satisfies the following.*

1) *The code achieves $k \geqslant n - \log\log(|\mathcal{F}|/\eta) - \log n - 9\log(1/\epsilon) - O(1)$.*
2) *The number of random bits needed to specify the code is $O\Big((n + \log(|\mathcal{F}|/\eta))n/\epsilon^6\Big)$.*
3) *The encoder and the decoder run in worst case time* $\mathsf{poly}(\log(|\mathcal{F}|/\eta)n/\epsilon)$.

Proof. Let t_0 and k_0 be the parameters promised by Theorem 9. We instantiate Construction 2 with parameter $t := t_0$ and $k := k_0$. Observe that this choice of t is large enough to allow Lemma 21 to hold. Thus we can ensure that, with probability at least $1 - \eta$, $(\mathsf{EncMC}, \mathsf{DecMC})$ is a $(9t)$-wise independent coding scheme where, for every $s \in \{0,1\}^k$, $|E(s)| \in [t_0, 3t_0]$. Thus we can now apply Lemma 20 to conclude that with probability at least $1 - 2\eta$, $(\mathsf{EncMC}, \mathsf{DecMC})$ is a strong non-malleable code with the desired parameters.

The number of random bits required to represent the code is the bit length of the polynomial $P(X)$ in Construction 2, which is $9tn$. Plugging in the value of t from (21) gives the desired estimate.

The running time of the decoder is dominated by evaluation of the polynomial $P(X)$ at a given point. Since the underlying field is of characteristic two, a representation of the field as well as basic field operations can be computed in deterministic polynomial time in the degree n of the extension using Shoup's algorithm [15].

The encoder is, however, slightly more complicated as it needs to iterate through $O(t)$ steps, and at each iteration compute all roots of a given degree $9t-1$ polynomial. Again, since characteristic of the underlying field is small, this task can be performed in deterministic polynomial time in the degree $9t - 1$ of the polynomial and the degree n of the extension (e.g., using [16]). After plugging in the bound on t from (21), we obtain the desired bound on the running time. □

As a corollary, we observe that the rate of the Monte Carlo construction can be made arbitrarily close to 1 while keeping the bit-representation of the code as well as the running time of the encoder and decoder at $\mathsf{poly}(n)$ provided that $\epsilon = 1/\mathsf{poly}(n)$ and $|\mathcal{F}| = 2^{\mathsf{poly}(n)}$. In particular, we see that the Monte Carlo construction achieves strong non-malleability even with respect to such powerful classes of adversaries as polynomial-sized Boolean circuits (with n outputs bits) and virtually any interesting computationally bounded model.

Remark 23. Since in this construction the error ϵ is only polynomially small, for cryptographic applications such as tamper-resilient security it is important to set up the code so as to ensure that $1/\epsilon$ is significantly larger than the total number of tampering attempts made by the adversary.

Caveat. We point out that any explicit coding scheme for computationally bounded models (such as polynomial-sized Boolean circuits) necessarily implies an explicit lower bound for the respective computational model. This is because a function in the restricted model cannot be powerful enough to compute the decoder function, as otherwise, the following adversary would violate non-malleability:

> Consider fixed tuples $(s_1, x_1), (s_2, x_2) \in \{0,1\}^k \times \{0,1\}^n$, where $s_1 \neq s_2$, $\mathsf{Dec}(x_1) = s_1$ and $\mathsf{Dec}(x_2) = s_2$. Given a codeword $x \in \{0,1\}^n$, compute $s := \mathsf{Dec}(x)$. If $s = s_1$, output x_2. If $s = s_2$, output x_1. Otherwise, output x.

Remark 24. (Alternative Monte Carlo construction) In addition to Construction 2, it is possible to consider a related Monte Carlo construction when polynomial evaluation is performed at the encoder and root finding is done by the encoder. More precisely, the encoder, given $s \in \{0,1\}^k$, may sample $i \in [t]$ uniformly at random, and output $P(s,i)$ where (s,i) is interpreted as an element of $\mathbb{F}_{2^n}$ (possibly after padding). The drawback with this approach is that the rate of the code would be limited by $1/2$, since for larger rates there is a noticeable chance that the encoder maps different messages to the same codeword.

V. Impossibility bounds

In this section, we show that the bounds obtained by Theorem 9 are essentially optimal. In order to do so, we consider three families of adversaries. Throughout the section, we use k and n for the message length and block length of coding schemes and define $N := 2^n$ and $K := 2^k$.

A. General adversaries

The first hope is to demonstrate that Theorem 9 is the best possible for *every* family of the tampering functions of a prescribed size. We rule out this possibility and demonstrate a family $\mathcal{F}$ of tampering functions achieving $\log\log|\mathcal{F}| \approx n$ for which there is a non-malleable code achieving rate $1-\gamma$ for arbitrarily small $\gamma > 0$.

Let $S \subseteq \{0,1\}^n$ be any set of size at least $N^{1-\alpha}$ and at most $N/2$. Consider the family $\mathcal{F}$ of functions satisfying the property that

$$(\forall f \in \mathcal{F})(\forall x \in S)\colon f(x) = x.$$

We can take the union of such families over all choices of S; however, for our purposes it suffices to define $\mathcal{F}$ with respect to a single choice of S. Observe that

$$|\mathcal{F}| N^{N-|S|} \geqslant N^{N/2},$$

which implies

$$\log\log|F| \geqslant n-1.$$

However, there is a trivial coding scheme that is non-malleable with zero error for all functions in $\mathcal{F}$. Namely, the encoder Enc is a deterministic function that maps messages to distinct elements of S, whereas the decoder Dec inverts the encoder and furthermore, maps any string outside S to $\perp$. In this construction, we see that

$$(\forall f \in \mathcal{F})(\forall x \in \{0,1\}^k)\colon \mathsf{Dec}(f(\mathsf{Enc}(x))) = x,$$

since f necessarily fixes all the points in S (in particular, in Definition 4 one can take $\mathcal{D}_f := \mathscr{D}(\underline{\mathsf{same}})$). Finally, observe that the rate of this coding scheme is at least $1-\gamma$. In fact, this result holds for any $\gamma \geqslant 1/n$, implying that the rate of the code can be made $1-o(1)$.

B. Random adversaries

The observation in Section V-A rules out the hope for a general lower bound that only depends on the size of the adversarial family. However, in this section we show that for "virtually all" families of tampering functions of a certain size, Theorem 9 gives the best possible bound. More precisely, we construct a family $\mathcal{F}$ of a designed size M as follows: For each $i \in [M]$, sample a uniformly random function $f_i\colon \{0,1\}^n \to \{0,1\}^n$ and add f_i to the family. Since some of the f_i may turn out to be the same (albeit with negligible probability), $|\mathcal{F}|$ may in general be lower than M (which can only make a lower bound stronger).

We prove the following.

Theorem 25. *For any $\alpha > 0$, there is an M_0 satisfying*

$$\log\log M_0 \leqslant \alpha n + O(\log n)$$

such that with probability $1-\exp(-n)$, a random family $\mathcal{F}$ with designed size $M \geqslant M_0$ satisfies the following: There is no coding scheme achieving rate at least $1-\alpha$ and error $\epsilon < 1$ that is non-malleable with respect to the tampering family $\mathcal{F}$.

Proof. We begin with the following simple probabilistic argument:

Claim 26. *Let $\mathcal{C} \subseteq [q]^N$ be a multi-set of vectors each chosen uniformly and independently at random. For any integer $\ell \in [N]$ and parameter $\gamma > 0$, there is an $M_0 = O(\ell q^\ell \log(qN/\gamma))$ such that as long as $|\mathcal{C}| \geqslant M_0$, the following holds with probability at least $1-\gamma$: For every $S \subseteq [N]$ with $|S| \leqslant \ell$, the set of vectors in $\mathcal{C}$ restricted to the positions picked by S is equal to $[q]^{|S|}$.*

Proof. Fix any choice of the set S (where, without loss of generality, $|S| = \ell$) and let $\mathcal{C}_S$ be the set of vectors in $\mathcal{C}$ restricted to the positions in S. For any $w \in [q]^{|S}$, we have

$$\Pr[w \notin \mathcal{C}_S] = \Big(1 - \frac{1}{q^\ell}\Big)^{|\mathcal{C}|} \leqslant \exp(-\Omega(|\mathcal{C}|/q^\ell)).$$

By taking a union bound on all the choices of w and S, the probability that $\mathcal{C}$ does not satisfy the desired property can be seen to be at most

$$(qN)^{\ell} \exp(-\Omega(|\mathcal{C}|/q^{\ell})),$$

which can be made no more than γ for some

$$|\mathcal{C}| = O\Big(q^{\ell}(\ell \log(qN) + \log(1/\gamma))\Big) . \qquad \square$$

Let $\gamma > 0$ be a parameter to be determined later. By Claim 26, with probability at least $1-\gamma$ over the randomness of the family $\mathcal{F}$, we can ensure that for all sets $S \subseteq \{0,1\}^n$ of size at most $4N^{\alpha}$, and for all functions $f_S \colon S \to \{0,1\}^n$, there is a function $f \in \mathcal{F}$ that agrees with f_S on all points in S. This guarantee holds if we take $\mathcal{F} \geqslant M_0$ for some

$$M_0 = O\Big(N^{\alpha} N^{(4N^{\alpha})} \log(N/\gamma)\Big).$$

Overestimating the above bound yields

$$\log\log M_0 \leqslant \alpha n + \log\log(N/\gamma) + O(1)$$

which is at most $\alpha n + O(\log n)$ for $\gamma = \exp(-n)$. Assuming that the family $\mathcal{F}$ attains the above-mentioned property, we now proceed as follows.

Consider any coding scheme $(\mathsf{Enc}, \mathsf{Dec})$ with block length n and message length k which is non-malleable for the family $\mathcal{F}$ randomly constructed as above and achieving rate at least $1-\alpha$ for some $\alpha > 0$ and any non-trivial error $\epsilon < 1$. For any message $s \in \{0,1\}^k$, let

$$E(s) := \mathsf{supp}(\mathsf{Enc}(s)) \subseteq [N]$$

and observe that $E(s) \cap E(s') = \emptyset$ for all $s \neq s'$. Observe that

$$\mathbb{E}[|E(U_k)|] \leqslant N^{\alpha}$$

by the disjointness property of the $E(s)$ and the assumption on the rate of the code. By Markov's bound,

$$\Pr[|E(U_k)| \geqslant 2N^{\alpha}] < 1/2$$

implying that for at least half of the choices of $s \in \{0,1\}^k$, we can assume $|E(s)| \leqslant 2N^{\alpha}$. Take two distinct vectors $s_1, s_2 \in \{0,1\}^k$ satisfying this bound.

Now, let $S := E(s_1) \cup E(s_2)$, where $|S| \leqslant 4N^{\alpha}$ as above. Consider any $c_1 \in E(s_1)$ and $c_2 \in E(s_2)$ and define $f_S \colon S \to \{0,1\}^n$ such that

$$(\forall x \in E(s_1))\colon \; f_S(x) = c_2,$$

and,

$$(\forall x \in E(s_2))\colon \; f_S(x) = c_1.$$

By the choice of $\mathcal{F}$, we know that there is $f \in \mathcal{F}$ that agrees with f_S on all the points in S. This choice of the adversary ensures that

$$\Pr[\mathsf{Dec}(f(\mathsf{Enc}(s_1))) = s_2] = 1$$

and

$$\Pr[\mathsf{Dec}(f(\mathsf{Enc}(s_2))) = s_1] = 1$$

with respect to the randomness of the encoder. Since the two distributions $\mathsf{Dec}(f(\mathsf{Enc}(s_1)))$ and $\mathsf{Dec}(f(\mathsf{Enc}(s_2)))$ are maximally far from each other and moreover, the adversary f always tampers codewords in $E(s_1)$ and $E(s_2)$ to a codeword corresponding to a different message, we conclude that there is no choice of $\mathcal{D}_f$ in Definition 4 that ensures non-malleability with any error less than 1. $\square$

C. General adversaries acting on a subset of positions

An important family of adversaries is the one that is only restricted by the subset of bits it acts upon. More precisely, let $T \subseteq [n]$ be a fixed set of size αn, for a parameter $\alpha \in (0,1)$. For $x \in \{0,1\}^n$, we use the notation $x_T \in \{0,1\}^{|T|}$ for the restriction of x to the positions in T. Without loss of generality, assume that T contains the first $|T|$ coordinate positions so that $x = (x_T, x_{\bar{T}})$, where $\bar{T} := [n] \setminus T$. We consider the family $\mathcal{F}_T$ of all functions $f \colon \{0,1\}^n \to \{0,1\}^n$ such that

$$f(x) = (g(x_T), x_{\bar{T}})$$

for some $g \colon \{0,1\}^{|T|} \to \{0,1\}^{|T|}$. Observe that $|\mathcal{F}_T| \leqslant N^{(\alpha N^{\alpha})}$ which implies $\log\log|\mathcal{F}_T| \leqslant \alpha n$.

We prove the following lower bound, which is a variation of the classical Singleton bound for non-malleable codes. What makes this variation much more challenging to prove is the fact that 1) non-malleable codes allow a randomized encoder, and 2) non-malleability is a more relaxed requirement than error detection, and hence the proof must rule out the case where the decoder does not detect errors (i.e., outputs a wrong message) while still satisfies non-malleability.

Theorem 27. *Let $T \subseteq [n]$ be of size αn and consider the family $\mathcal{F}_T$ of the tampering functions that only act on the coordinate positions in T (as defined above). Then, there is a $\delta_0 = O((\log n)/n)$ such that the following holds. Let $(\mathsf{Enc}, \mathsf{Dec})$ be any coding scheme which is non-malleable for the family $\mathcal{F}_T$ and achieves rate $1-\alpha+\delta$, for any $\delta \in [\delta_0, \alpha]$ and error ϵ. Then, $\epsilon \geqslant \delta/(16\alpha)$. In particular, when α and δ are absolute constants, $\epsilon = \Omega(1)$.*

Before proving the theorem, we state the following immediate corollary.

Corollary 28. *Let $\mathcal{F}$ be the family of split-state adversaries acting on n bits. That is, each $f \in \mathcal{F}$ interprets the input as a pair (x_1, x_2) where $x_2 \in \{0,1\}^{\lfloor n/2 \rfloor}$ and $x_2 \in \{0,1\}^{\lceil n/2 \rceil}$, and outputs $(f_1(x_1), f_2(x_2))$ for arbitrary tampering functions f_1 and f_2 (acting on their respective input lengths).*

Moreover, for a fixed constant $\delta \in (0,1)$, let $\mathcal{F}_\delta$ be the class of tampering functions where $f \in \mathcal{F}_\delta$ iff every bit of $f(x)$ depends on at most $\lfloor \delta n \rfloor$ of the bits of x.

Let $(\mathsf{Enc}_1, \mathsf{Dec}_1)$ (resp., $(\mathsf{Enc}_\delta, \mathsf{Dec}_\delta)$) be any coding scheme which is non-malleable for the class $\mathcal{F}$ (resp., $\mathcal{F}_\delta$) achieving error at most ϵ and rate R (resp., R_δ). Then, for every fixed constant $\gamma > 0$, there is a fixed constant $\epsilon_0 > 0$ such that if $\epsilon \leqslant \epsilon_0$, the following bounds hold.

(i) *$R \leqslant 1/2 - \gamma$,*
(ii) *$R_\delta \leqslant 1 - \delta - \gamma$.*

The proof of Theorem 27 uses basic tools from information theory, and the core ideas can be described as follows. Assume that the codeword is (X_1, X_2) where the adversary acts on X_1, which is of length αn. We show that for any coding scheme with rate slightly larger than $(1-\alpha)n$, there is a set $X_\eta \subseteq \{0,1\}^{\alpha n}$ such that

1) For some message s_0, X_1 lies in X_η with noticeable probability.
2) For a "typical" message s_1, X_1 is unlikely to land in X_η.
3) There is a vector $w \in \{0,1\}^{\alpha n}$ that cannot be extended to a codeword (w, w') that maps to either s_0 or s_1 by the decoder.

We then use the above properties to design the following strategy that violates non-malleability of the code: Given (X_1, X_2), if $X_1 \in X_\eta$, the adversary tampers the codeword to (w, X_2), which decodes to a message outside $\{s_0, s_1\}$. This ensures that $\mathsf{Dec}(f(\mathsf{Enc}(s_0)))$ has a noticeable chance of being tampered to an incorrect message. Otherwise, the adversary leaves the codeword unchanged, ensuring that $\mathsf{Dec}(f(\mathsf{Enc}(s_1)))$ has little chance of being tampered at all. Thus there is no choice for a distribution $\mathcal{D}_f$ that sufficiently matches both $\mathsf{Dec}(f(\mathsf{Enc}(s_0)))$ and $\mathsf{Dec}(f(\mathsf{Enc}(s_1)))$.

Proof of Theorem 27: Throughout the proof, we use standard information theoretic tools, such as the notation $H(X)$ for the Shannon entropy of a discrete random variable X and $I(X;Y)$ for the mutual information between discrete random variables X and Y. We will need the following standard information-theoretic fact.

Claim 29. *Suppose $H(X) \leqslant r$ and let $p(x) := \Pr[X = x]$. For any $\eta > 0$, and define*

$$X_\eta := \{x \in \mathsf{supp}(X) \colon p(x) > \frac{1}{2^{r/(1-\eta)}}\}.$$

Then, $\Pr[X \in X_\eta] \geqslant \eta$ *and* $|X| < 2^{r/(1-\eta)}$.

Proof. The upper bound on $|X_\eta|$ is immediate from the definition of X_η. Let $\bar{X_\eta} := \mathsf{supp}(X) \setminus X_\eta$. We need to show that $\Pr[X \in \bar{X_\eta}] \leqslant 1-\eta$. If this is not the case, we can write

$$\begin{aligned} H(X) &\geqslant \sum_{x \in \bar{X}_\eta} p(x) \log(1/p(x)) \\ &\geqslant \sum_{x \in \bar{X}_\eta} \frac{r p(x)}{1-\eta} \\ &= \Pr[x \in \bar{X_\eta}] r/(1-\eta) > r, \end{aligned}$$

a contradiction. □

Suppose there is a coding scheme $(\mathsf{Enc}, \mathsf{Dec})$ that is non-malleable for the family $\mathcal{F}_T$ and achieving rate at least $1-\alpha+\delta$, for an arbitrarily small parameter $\delta \in (0, \alpha]$. Let $S \sim \mathcal{U}_k$, $X := \mathsf{Enc}(S)$ and suppose $X = (X_1, X_2)$ where $X_1 := X_T$ and $X_2 := X_{\bar{T}}$.

For any $s \in \{0,1\}^k$, define $E(s) := \mathsf{supp}(\mathsf{Enc}(s))$. Observe that

$$\mathbb{E}_S |E(S)| \leqslant N/N^{1-\alpha+\delta} = N^{\alpha-\delta}$$

By Markov's bound, for any $\gamma \in (0,1]$,

$$\Pr[|E(S)| > N^{\alpha-\delta}/\gamma] < \gamma. \tag{23}$$

By the assumption on rate, $H(S) \geqslant n(1-\alpha+\delta)$. Also, $H(X_2|S) \leqslant H(X_2) \leqslant n - |T| = n(1-\alpha)$. Thus,

$$I(X_2; S) = H(S) - H(S \mid X_2)$$

Using the chain rule for mutual information,

$$\begin{aligned} I(X_1; S) &= I(X_1, X_2; S) - I(X_2; S \mid X_1) \\ &= (H(S) - H(S \mid X_1, X_2)) \\ &\quad - (H(X_2 \mid X_1) - H(X_2 \mid S, X_1)) \\ &\geqslant H(S) - H(X_2 \mid X_1) & (24) \\ &\geqslant H(S) - H(X_2) & (25) \\ &\geqslant (1-\alpha+\delta)n - (1-\alpha)n = \delta n, & (26) \end{aligned}$$

where (24) holds because $S = \mathsf{Dec}(X_1, X_2)$ and thus $H(S \mid X_1, X_2) = 0$, in addition to non-negativity of entropy; (25) uses the fact that conditioning does not increase entropy; and (26) holds because of the assumption on the rate of the code and the length of X_2. From this, we can deduce that

$$H(X_1 \mid S) = H(X_1) - I(X_1; S) \leqslant H(X_1) - \delta n.$$

Note that the latter inequality in particular implies that $H(X_1) \geqslant \delta n$, and that $\mathsf{supp}(X_1) \geqslant 2^{\delta n}$. By Markov's bound,

$$\begin{aligned} &|\{s \in \{0,1\}^k \colon H(X_1 \mid S = s) > (H(X_1) - \delta n)(1+4\gamma)\}| \\ &\qquad < \frac{2^k}{1+4\gamma} \leqslant (1-2\gamma)2^k. \end{aligned} \tag{27}$$

By combining (23) and (27) using a union bound, there is a choice of $s_0 \in \{0,1\}^k$ such that

$$|E(s_0)| \leqslant N^{\alpha-\delta}/\gamma,$$

and,

$$H(X_1 \mid S = s_0) \leqslant (H(X_1) - \delta n)(1+4\gamma).$$

We can take $\gamma := \delta/(8\alpha)$ so that the above becomes

$$|E(s_0)| \leqslant 8\alpha N^{\alpha-\delta}/\delta, \tag{28}$$

and,

$$H(X_1 \mid S = s_0) \leqslant H(X_1) - \delta n/2.$$

For a parameter $\eta > 0$, to be determined later, we can now apply Claim 29 to the conditional distribution of X_1 subject to $S = s_0$ and construct a set $X_\eta \subseteq \{0,1\}^{\alpha n}$ such that

$$\begin{aligned} \Pr[X_1 \in X_\eta \mid S = s_0] &\geqslant \eta, \\ |X_\eta| &\leqslant 2^{(H(X_1)-\delta n/2)/(1-\eta)}. \end{aligned} \tag{29}$$

Let $\eta' := \Pr[X_1 \in X_\eta]$, and let $h(\cdot)$ denote the binary entropy function. Using a simple information-theoretic rule that follows from the definition of Shannon entropy, we can write

$$\begin{aligned}
H(X_1) &= h(\eta') + \eta' H(X_1 \mid X_1 \in X_\eta) \\
&\quad + (1-\eta')H(X_1 \mid X_1 \notin X_\eta) \\
&\leqslant h(\eta') + \eta' \cdot \frac{H(X_1) - (\delta/2)n}{1-\eta} \\
&\quad + (1-\eta')H(X_1 \mid X_1 \notin X_\eta) \qquad (30)\\
&\leqslant h(\eta') + \eta' \cdot \frac{H(X_1) - (\delta/2)n}{1-\eta} \\
&\quad + (1-\eta')H(X_1), \qquad (31)
\end{aligned}$$

where (30) is due to the upper bound on the support size of X_η and (31) holds since conditioning does not increase entropy. After simple manipulations, (31) simplifies to

$$\eta' \leqslant \frac{2h(\eta')(1-\eta)}{\delta n - 2\eta H(X_1)} \leqslant \frac{2h(\eta')}{n(\delta - 2\eta\alpha)}. \qquad (32)$$

Now, we take $\eta := \delta/(4\alpha)$, so that the above inequalities, combined with the estimate $h(\eta') = O(\eta' \log(1/\eta'))$ yields

$$\begin{aligned}
h(\eta')/\eta' \geqslant \delta n/4 &\Rightarrow \log(1/\eta') = \Omega(\delta n) \\
&\Rightarrow \eta' \leqslant \exp(-\Omega(\delta n)).
\end{aligned}$$

From the above inequality, straightforward calculations ensure that

$$\eta' \leqslant \eta/4 = \delta/(16\alpha), \qquad (33)$$

as long as $\delta \geqslant \delta_0 = O((\log n)/n)$.

From (33), recalling that $\eta' = \Pr[X_1 \in X_\eta]$ and using Markov's bound,

$$|\{s\colon \Pr[X_1 \in X_\eta \mid S = s] > \eta/2\}|/2^k < 1/2.$$

Combined with (23) and a union bound, there is a fixed $s_1 \in \{0,1\}^k$ such that

$$|E(s_1)| \leqslant 8\alpha N^{\alpha-\delta}/\delta, \text{ and, } \Pr[X_1 \in X_\eta \mid S = s_1] \leqslant \eta/2. \qquad (34)$$

Assuming the chosen lower bound for δ, we can also ensure that, using (28), that $|E(s_0) \cup E(s_1)| < N^\alpha$. Thus, there is a fixed string $w \in \{0,1\}^{\alpha n}$ that cannot be extended to any codeword in $E(s_0)$ or in $E(s_1)$; i.e.,

$$\Pr[X_1 = w \mid (S = s_0) \vee (S = s_1)] = 0,$$

which in turn implies

$$(\forall x_2 \in \{0,1\}^{n(1-\alpha)})\colon \mathsf{Dec}(w, x_2) \notin \{s_0, s_1\}. \qquad (35)$$

Now, we consider the following tampering strategy $f\colon \{0,1\}^{|T|} \times \{0,1\}^{n-|T|} \to \{0,1\}^{|T|} \times \{0,1\}^{n-|T|}$ acting on the coordinate positions in T:

- Given $(x_1, x_2) \in \{0,1\}^{|T|} \times \{0,1\}^{n-|T|}$, if $x_1 \in X_\eta$, output (w, x_2).
- Otherwise, output (x_1, x_2).

Suppose the coding scheme $(\mathsf{Enc}, \mathsf{Dec})$ satisfied Definition 4 for a particular distribution $\mathcal{D}_f$ over $\{0,1\}^n \cup \{\underline{\mathsf{same}}, \perp\}$ for the tampering function f.

Since f does not alter any string with the first component outside X_η, (34) implies that

$$\Pr[f(X_1, X_2) = (X_1, X_2) \mid S = s_1] \geqslant 1 - \eta/2. \qquad (36)$$

On the other hand, by (29) and (35),

$$\Pr[\mathsf{Dec}(f(X_1, X_2)) \notin \{s_0, s_1\} \mid S = s_0] \geqslant \eta. \qquad (37)$$

By (37) and Definition 4, $\mathcal{D}_f$ must be ϵ-close to a distribution D_0 that assigns at most $1-\eta$ of the probability mass to $\{\underline{\mathsf{same}}, s_0, s_1\}$. On the other hand, by (36), $\mathcal{D}_f$ must be ϵ-close to a distribution D_1 that assigns at least $1-\eta/2$ of the probability mass to $\{\underline{\mathsf{same}}, s_1\}$. Thus, the statistical distance between D_0 and D_1 is at least $\eta/2$ (from the distinguisher corresponding to the event $\{\underline{\mathsf{same}}, s_1\}$). By triangle inequality, however, D_0 and D_1 are (2ϵ)-close. Therefore, $\epsilon \geqslant \eta/4$ and the result follows.

D. The rate 1/2 barrier for the uniform coding scheme.

Dziembowski et al. [1] consider the uniformly random coding scheme $(\mathsf{Enc}, \mathsf{Dec})$ in which the decoder Dec maps any given input $x \in \{0,1\}^n$ to a uniform and independent random string in $\{0,1\}^k$. Moreover, the encoder, given $s \in \{0,1\}^k$, outputs a uniformly random element of $\mathsf{Dec}^{-1}(s)$. In this section, we show that the uniform coding scheme cannot achieve a rate better than $1/2$ even with respect to very simple tampering functions, namely, any tampering function $f\colon \{0,1\}^n \to \{0,1\}^n$ that is a bijection (for example, one may think of f as the function that flips the first bit of the input).

The high level intuition is quite simple: according to the definition of non-malleability, for a uniformly random message $S \in \{0,1\}^k$, the random variable $\mathsf{Dec}(f(\mathsf{Enc}(S)))$ is "essentially independent" of S (as made precise in Definition 4). On the other hand, this random variable should have almost maximal entropy (i.e., k) since the decoder is a uniformly random function. Thus the entropy of the tuple $(S, \mathsf{Dec}(f(\mathsf{Enc}(S))))$ should be close to $2k$. On the other hand, this entropy is also at most n since since $(S, \mathsf{Dec}(f(\mathsf{Enc}(S))))$ is a deterministic function of $\mathsf{Enc}(S)$ which is an n-bit string. Therefore the rate k/n cannot be much more than $1/2$. The following theorem makes this intuition rigorous.

Theorem 30. *Let* $(\mathsf{Enc}, \mathsf{Dec})$ *be a coding scheme with message length k and block length n that is randomly constructed as follows. The decoder* Dec *is taken to be a random function mapping each input $x \in \{0,1\}^n$ to a uniform and independent random string in $\{0,1\}^k$. Moreover, the encoder, given $s \in \{0,1\}^k$, outputs a uniformly random element of* $\mathsf{Dec}^{-1}(s)$. *Let ϵ denote the error of the $\mathcal{C}$ as a non-malleable coding scheme against any family of adversaries containing some bijective function. Then, with probability $1 - o_k(1)$ over the randomness of the code construction, the rate R of the code must satisfy*

$$R \leqslant 1/2 + O(\epsilon) + o_k(1).$$

Proof. Let $\mathcal{C}$ denote the randomized coding scheme defined by the pair $(\mathsf{Enc}, \mathsf{Dec})$. Let $Y \sim \mathcal{U}_n$ be a uniform string and

$S := \mathsf{Dec}(Y)$. Note that $S \in \{0,1\}^k$. Consider an independent random variable $Y' \sim \mathcal{U}_n$ and let $S' := \mathsf{Dec}(Y')$. Over the randomness of both $\mathcal{C}$ and (Y, Y'), the two random variables S and S' are independent and uniformly distributed on $\{0,1\}^k$. Therefore, we can write down the collision probability of S and S' as

$$\Pr_{\mathcal{C},Y,Y'}[S = S'] = \mathbb{E}_{\mathcal{C}} \Pr_{Y,Y'}[S = S'] = 1/K,$$

where $K := 2^k$. Using Markov's bound, for a small parameter $\delta > 0$ to be determined later,

$$\Pr_{\mathcal{C}}[\Pr_{Y,Y'}[S = S'] \geqslant (1/K)^{1-\delta}] \leqslant (1/K)^{\delta}.$$

Let c be a particular instantiation of $\mathcal{C}$ as determined by the random coin tosses of the code construction. We use $\mathscr{D}(S \mid \mathcal{C} = c)$ to denote the distribution of S as defined by the particular choice of c. So far we have shown that for at least $1 - (1/K)^{\delta}$ fraction of the choices of c, the code $\mathcal{C}$ is such that

$$\Pr_{Y,Y'}[S = S' \mid \mathcal{C} = c] < (1/K)^{1-\delta},$$

and thus, the collision entropy[8] of $\mathscr{D}(S \mid \mathcal{C} = c)$ is at least $(1-\delta)k$. Since the collision entropy lower bounds the Shannon entropy, we see that there is a good chance (at least $1 - (1/K)^{\delta}$) that after fixing the code $\mathcal{C}$, we have

$$H(S \mid \mathcal{C} = c) \geqslant (1-\delta)k. \tag{38}$$

Consider any bijective tampering function $f\colon \{0,1\}^n \to \{0,1\}^n$. In particular this implies that $f(Y) \neq Y$. Since $\mathsf{Dec}(Y)$ and $\mathsf{Dec}(f(Y))$ are independent and uniform over $\{0,1\}^k$ (with respect to the randomness of $\mathcal{C}$), we see that

$$\begin{aligned}\Pr_{\mathcal{C},Y}[\mathsf{Dec}(Y) = \mathsf{Dec}(f(Y))] = \mathbb{E}_{\mathcal{C}} \Pr_{Y}[&\mathsf{Dec}(Y) \\ &= \mathsf{Dec}(f(Y))] = 1/K.\end{aligned}$$

Similar to the above, using Markov's inequality we see that for at least $1 - (1/K)^{\delta}$ fraction of the choices of c, the code $\mathcal{C}$ is such that

$$\Pr_{Y}[\mathsf{Dec}(Y) = \mathsf{Dec}(f(Y)) \mid \mathcal{C} = c] < (1/K)^{1-\delta}. \tag{39}$$

In the sequel, we assume $\mathcal{C}$ to be such that both (38) and (39) are satisfied (by a union bound, this is the case with probability at least $1 - 2(1/K)^{\delta}$ over the randomness of the code construction). Since the following discussion focuses on this fixed outcome of the code $\mathcal{C}$, in the sequel we find it convenient to remove the conditioning $\mathcal{C} = c$ from the notation.

Observe that Y has the same distribution as $\mathsf{Enc}(S)$ (according to the way the encoder is defined), and in particular $\mathsf{Enc}(S)$ is uniformly distributed on $\{0,1\}^n$ (observe that although after fixing the code, the distribution of S may become non-uniform over $\{0,1\}^k$, the distribution of $\mathsf{Enc}(S)$ remains uniform over $\{0,1\}^n$ due to the way the encoder function is defined). Thus we can rewrite (39) as

$$\Pr[S = \mathsf{Dec}(f(\mathsf{Enc}(S)))] < (1/K)^{1-\delta},$$

where the probability is over S and the internal coin tosses of Enc. By averaging, there is some fixed $s \in \{0,1\}^k$ such that

$$\Pr[s = \mathsf{Dec}(f(\mathsf{Enc}(s)))] < (1/K)^{1-\delta}. \tag{40}$$

Assume now that the coding scheme is non-malleable with respect to any family containing f and error at most ϵ. Let $\mathcal{D}_f$ be the distribution over $\{0,1\}^k \cup \{\perp, \underline{\mathsf{same}}\}$ from Definition 4 and S'' be a fresh sample from $\mathcal{D}_f$. Thus, non-malleability implies that

$$\mathscr{D}(\mathsf{Dec}(f(\mathsf{Enc}(s)))) \approx_{\epsilon} \mathscr{D}(\mathsf{copy}(S'', s)). \tag{41}$$

In the above equation, the probability mass of s in the left hand side distribution is at most $(1/K)^{1-\delta}$ according to (40). The corresponding mass in the right hand side distribution is, however, $\Pr[S'' \in \{s, \underline{\mathsf{same}}\}]$. Therefore, the two distributions being ϵ-close implies that

$$\mathcal{D}_f(\underline{\mathsf{same}}) < (1/K)^{1-\delta} + \epsilon. \tag{42}$$

Let $\mathcal{D}'_f$ be any distribution over $\{0,1\}^k \cup \{\perp\}$ obtained from $\mathcal{D}_f$ by redistributing the probability mass of $\mathcal{D}_f$ on $\underline{\mathsf{same}}$ arbitrarily to other points in the sample space, and let S_0 be a fresh sample from $\mathcal{D}'_f$. Note that according to (42), the distributions $\mathcal{D}_f$ and $\mathcal{D}'_f$ are $((1/K)^{1-\delta} + \epsilon)$-close. Using this and (41) (which is valid for all s according to Definition 4), we get that

$$\begin{aligned}\mathscr{D}(S, \mathsf{Dec}(f(\mathsf{Enc}(S)))) &\approx_{\epsilon} \mathscr{D}(\mathsf{copy}(S'', S)) \\ &\approx_{\epsilon'} \mathscr{D}(S, \mathsf{copy}(S_0, S)) \\ &= \mathscr{D}(S, S_0),\end{aligned} \tag{43}$$

where $\epsilon' := (1/K)^{1-\delta} + 2\epsilon$ (note that since S_0 is never equal to $\underline{\mathsf{same}}$, we always have $\mathsf{copy}(S_0, S) = S_0$). Observe that (43) in particular implies that

$$\mathscr{D}(\mathsf{Dec}(f(\mathsf{Enc}(S)))) \approx_{\epsilon'} \mathscr{D}(S_0),$$

but since $\mathscr{D}(\mathsf{Enc}(S)) = \mathscr{D}(Y) = \mathcal{U}_n$, and f is a bijection, the left hand side distribution is the same as $\mathscr{D}(S) = \mathscr{D}(S')$ and thus, $\mathscr{D}(S') \approx_{\epsilon'} \mathscr{D}(S_0)$, and

$$\mathscr{D}(S, S_0) \approx_{\epsilon'} \mathscr{D}(S, S'),$$

where we recall that the random variable S' is independent from S with the same distribution. Plugging this back into (43), we see using the triangle inequality that

$$\mathscr{D}(S, \mathsf{Dec}(f(\mathsf{Enc}(S)))) \approx_{2\epsilon'} \mathscr{D}(S, S'). \tag{44}$$

Using this result and (38), and recalling that S and S' are independent random variables (even conditioned on $\mathcal{C}$) we have that

$$H(S, S') = 2H(S) \geqslant 2(1-\delta)k. \tag{45}$$

On the other hand, since the random variable $(S, \mathsf{Dec}(f(\mathsf{Enc}(S))))$ is a deterministic function of $\mathsf{Enc}(S)$, the latter being an n-bit string, we also have

$$H(S, \mathsf{Dec}(f(\mathsf{Enc}(S)))) \leqslant n. \tag{46}$$

[8] The collision entropy $H_2(X)$ (also known as Rényi entropy) is defined as $H_2(X) = -\log(\Pr[X = X'])$, where X' is an independent copy of X. It is a standard exercise to show that $H_2(X) \leqslant H(X)$.

We can now combine (44), (45) and (46) to conclude, using Proposition 37, that with probability at least $1-2(1/K)^{\delta}$ over the randomness of the code construction, the coding scheme $\mathcal{C}$ must satisfy

$$2(1-\delta)k \leqslant n(1+\epsilon')+1 = n(1+O((1/K)^{1-\delta}+\epsilon))+1.$$

Assuming that $\delta \leqslant 1/2$ and $\epsilon + 2^{-k/2} \leqslant 1/2$, we can slightly simplify the above to get that the rate R of the code must satisfy

$$R \leqslant (1-O(\delta)) \cdot (1/2 + O(\epsilon + 2^{-k/2})) + o(1).$$

As the message length k grows to infinity, we can pick δ to be a sub-constant term, say $\delta = 1/\sqrt{k}$ to conclude that with probability at least $1-2^{1-\delta k} = 1-o_k(1)$ over the randomness of the code construction, the coding scheme $\mathcal{C}$ must satisfy

$$R \leqslant 1/2 + O(\epsilon) + o_k(1)$$

as desired.

□

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Appendix

A. Useful tools concerning random variables

In many occasions in the paper, we deal with a chain of correlated random variables $0 = X_0, X_1, \ldots, X_n$ where we wish to understand an event depending on X_i conditioned on the knowledge of the previous variables. That is, we wish to understand

$$\mathbb{E}[f(X_i) \mid X_0, \ldots, X_{i-1}].$$

The following proposition shows that in order to understand the above quantity, it suffices to have an estimate with respect to a more restricted event than the knowledge of $X_0, \ldots, X_{i-1}$. Formally, we can state the following, where X stands for X_i in the above example and Y stands for $(X_0, \ldots, X_{i-1})$.

Proposition 31. *Let X and Y be possibly correlated random variables and let Z be a random variable such that the knowledge of Z determines Y; that is, $Y = f(Z)$ for some function f. Suppose that for every possible outcome of the random variable Z, namely, for every $z \in \mathsf{supp}(Z)$, and for some real-valued function g, we have*

$$\mathbb{E}[g(X) \mid Z = z] \in I. \tag{47}$$

for a particular interval I. Then, for every $y \in \mathsf{supp}(Y)$,

$$\mathbb{E}[g(X) \mid Y = y] \in I.$$

Similarly, suppose for some distribution $\mathcal{D}$, and all $z \in \mathsf{supp}(Z)$,

$$\mathscr{D}(X \mid Z = z) \approx_\epsilon \mathcal{D}.$$

Then, for all $y \in \mathsf{supp}(Y)$,

$$\mathscr{D}(X \mid Y = y) \approx_\epsilon \mathcal{D}.$$

Proof. Let $T = \{z \in \mathsf{supp}(Z) \colon f(z) = y\}$, and let $p(z) := \Pr[Z = z \mid Y = y]$. Then,

$$\mathbb{E}[g(X) \mid Y = y] = \sum_{z \in T} p(z) \mathbb{E}[g(X) \mid Z = z].$$

Since by (47), each $\mathbb{E}[g(X) \mid Z = z]$ lies in I and $\sum_{z \in T} p(z) = 1$, we deduce that

$$\mathbb{E}[g(X) \mid Y = y] \in I.$$

Proof of the second part is similar, by observing that if a collection of distributions is statistically close to a particular distribution $\mathcal{D}$, any convex combination of them is equally close to $\mathcal{D}$ as well. □

Proposition 32. *Let the random variable $X \in \{0,1\}^n$ be uniform on a set of size at least $(1-\epsilon)2^n$. Then, $\mathcal{D}(X)$ is $(\epsilon/(1-\epsilon))$-close to $\mathcal{U}_n$.*

We will use the following tail bounds on summation of possibly dependent random variables, which are direct consequences of Azuma's inequality.

Proposition 33. *Let $0 = X_0, X_1, \ldots, X_n$ be possibly correlated random variables in $[0,1]$ such that for every $i \in [n]$ and for some $\gamma \geqslant 0$,*

$$\mathbb{E}[X_i \mid X_0, \ldots, X_{i-1}] \leqslant \gamma.$$

Then, for every $c \geqslant 1$,

$$\Pr[\sum_{i=1}^{n} X_i \geqslant cn\gamma] \leqslant \exp(-n\gamma^2(c-1)^2/2),$$

or equivalently, for every $\delta > \gamma$,

$$\Pr[\sum_{i=1}^{n} X_i \geqslant n\delta] \leqslant \exp(-n(\delta-\gamma)^2/2).$$

Proof. The proof is a standard Martingale argument. For $i \in [n]$, define

$$X_i' := X_i - \gamma,$$

and

$$S_i := \sum_{j=1}^{i} X_i' = \sum_{j-1}^{i} X_i - i\gamma.$$

By assumption, S_i is a super-martingale, that is, assuming $S_0 := 0$,

$$\mathbb{E}[S_{i+1} \mid S_0, \ldots, S_i] \leqslant S_i.$$

Thus, by Azuma's inequality, for all $t \geqslant 0$,

$$\Pr[S_n \geqslant t] \leqslant \exp(-t^2/(2n)).$$

Substituting $t := (c-1)n\gamma$ proves the claim. □

In a similar fashion (using Azuma's inequality for submartingales rather than super-martingales in the proof), we may obtain a tail bound when we have a lower bound on conditional expectations.

Proposition 34. *Let $0 = X_0, X_1, \ldots, X_n$ be possibly correlated random variables in $[0,1]$ such that for every $i \in [n]$ and for some $\gamma \geqslant 0$,*

$$\mathbb{E}[X_i \mid X_0, \ldots, X_{i-1}] \geqslant \gamma.$$

Then, for every $\delta < \gamma$,

$$\Pr[\sum_{i=1}^{n} X_i \leqslant n\delta] \leqslant \exp(-n(\delta-\gamma)^2/2).$$

The following tail bound is similar in flavor to the one given by Proposition 33, but only applies to indicator random variables. However, it can be better when the individual expectations are low and the target deviation from mean is very large.

Proposition 35. *Let $0 = X_0, X_1, \ldots, X_n \in \{0,1\}$ be indicator, possibly dependent, random variables such that for every $i \in [n]$,*

$$\mathbb{E}[X_i \mid X_1, \ldots, X_{i-1}] \leqslant p,$$

for some $p \in [0,1]$. Let $X := X_1 + \cdots + X_n$. Then, for every $c \geqslant 1$,

$$\Pr[X > cnp] \leqslant (e/c)^{cnp}.$$

Proof. We closely follow the standard proof of Chernoff bounds for independent indicator random variables (see, e.g., [17]). Using Markov's bound on the exponential moment of X, we can write, for a parameter $t > 0$ to be determined later,

$$\begin{aligned}\Pr[X > cnp] &\leqslant \frac{\mathbb{E}[\exp(tX)]}{\exp(tcnp)} \\ &= \frac{\mathbb{E}[\exp(tX_1)\cdots\exp(tX_n)]}{\exp(tcnp)}. \end{aligned} \tag{48}$$

However, we can write down the expectation of product as the following chain of conditional expectations

$$\begin{aligned}\mathbb{E}_{(X_1,\ldots,X_n)}[\exp(tX)] = \mathbb{E}_{X_1}\Big[e^{tX_1}\mathbb{E}_{(X_2|X_1)}\big[e^{tX_2}\ldots \\ \mathbb{E}_{(X_n|X_1,\ldots,X_{n-1})}e^{tX_n}]\ldots\big]\Big] \\ \leqslant (p\exp(t)+1)^n.\end{aligned}$$

where the inequality uses the fact that the X_i are Bernoulli random variables and thus

$$\begin{aligned}\mathbb{E}[\exp(tX_i) \mid X_1, \ldots, X_{i-1}] &\leqslant p\exp(t) + (1-p)\exp(0) \\ &\leqslant p\exp(t) + 1.\end{aligned}$$

Using the inequality $(1+x)^n \leqslant \exp(nx)$ the above simplifies to

$$\mathbb{E}[\exp(tX)] \leqslant \exp(np\exp(t)),$$

and thus, plugging the above result into (48),

$$\Pr[X > cnp] \leqslant \frac{\exp(np\exp(t))}{\exp(tcnp)}.$$

Choosing $t := \ln c$ yields the desired conclusion. □

For summation of ℓ-wise independent random variables, we use the following tail bound from [18]:

Theorem 36. *Let $\ell > 1$ be an even integer, and let $X_1, \ldots, X_n \in [0,1]$ be ℓ-wise independent variables. Define $X := X_1 + \cdots + X_n$ and $\mu := \mathbb{E}[X]$. Then,*

$$\Pr[|X-\mu| \geqslant A] \leqslant 8\Big(\frac{\ell(\mu+\ell)}{A^2}\Big)^{\ell/2}.$$

We use the following standard result relating statistical distance between distributions to the difference in Shannon entropy.

Proposition 37. *Let X and Y be random variables over $\{0,1\}^n$ such that the statistical distance between the distributions of X and Y is at most ϵ. Then, $|H(X)-H(Y)| \leqslant \epsilon n + h(\epsilon)$, where $h(\cdot)$ is the binary entropy function.*

Proof. This is a standard result proved, for example, in [19] ((4) on page 3281) using coupling techniques. For completeness, here we include a description due to Radhakrishnan[9].

First, we suppose X and Y are coupled so that they maintain their individual marginal distributions but $\Pr[X \neq Y] \leqslant \epsilon$ (it is a standard fact that coupling in this fashion is always possible for any two distributions over the same sample space). Let $Y = X + e$ where $e \in \{0,1\}^n$ is the error vector and addition is coordinate-wise XOR. Let $Z \in \{0,1\}$ denote the error indicator; i.e., $Z = 1$ iff $e \neq 0$. Note that $\Pr[Z=1] \leqslant \epsilon$, and thus

$$\begin{aligned} H(e \mid Z) &= \Pr[Z=0] \cdot H(e \mid Z=0) \\ &\quad + \Pr[Z=1] \cdot H(e \mid Z=1) \\ &\leqslant \Pr[Z=1] \cdot H(e) \leqslant \epsilon n. \end{aligned}$$

We have

$$\begin{aligned} H(X) &= H(Y+e) \leqslant H(Y,e) \leqslant H(Y) + H(e) \\ &= H(Y) + H(e,Z) \leqslant H(Y) + H(Z) + H(e \mid Z) \\ &\leqslant H(Y) + h(\epsilon) + \epsilon n. \end{aligned}$$

Since the above argument is symmetric with respect to X and Y, we can also derive

$$H(Y) \leqslant H(X) + h(\epsilon) + \epsilon n$$

and the claim follows. □

B. Approximating distributions by fuzzy correlated sampling

In this appendix, we show that it is possible to sharply approximate a distribution $\mathcal{D}$ with finite support by sampling possibly correlated random variables $X_1, \ldots, X_n$ where the distribution of each X_i is close to $\mathcal{D}$ conditioned on the previous outcomes, and computing the empirical distribution of the drawn samples.

Lemma 38. *Let $\mathcal{D}$ be a distribution over a finite set Σ such that $|\mathsf{supp}(\mathcal{D})| \leqslant r$. For any $\eta, \epsilon, \gamma > 0$ such that $\gamma < \epsilon$, there is a choice of*

$$n = O((r + 2 + \log(1/\eta))/(\epsilon - \gamma)^2)$$

such that the following holds. Suppose $0 = X_0, X_1, \ldots, X_n \in \Sigma$ are possibly correlated random variables such that for all $i \in [n]$ and all values $0 = x_0, x_1 \ldots, x_n \in \mathsf{supp}(\mathcal{D})$,

$$\mathscr{D}(X_i \mid X_0 = x_0, \ldots, X_{i-1} = x_{i-1}) \approx_\gamma \mathcal{D}.$$

Then, with probability at least $1-\eta$, the empirical distribution of the outcomes $X_1, \ldots, X_n$ is ϵ-close to $\mathcal{D}$.

Proof. First, we argue that without loss of generality, we can assume that $|\Sigma| \leqslant r+1$. This is because if not, we can define a function $f\colon \Sigma \to \mathsf{supp}(\mathcal{D}) \cup \{\star\}$ as follows:

$$f(x) := \begin{cases} x & \text{if } x \in \mathsf{supp}(\mathcal{D}) \\ \star & \text{otherwise.} \end{cases}$$

Observe that for any distribution $\mathcal{D}'$ over Σ, $\mathsf{dist}(\mathcal{D}', \mathcal{D}) = \mathsf{dist}(f(\mathcal{D}'), \mathcal{D})$, since the elements outside $\mathsf{supp}(\mathcal{D})$ always contribute to the statistical distance and we aggregate all such mass on a single extra point $\star$, and by doing so do not affect the statistical distance. Thus the empirical distribution of $(X_1, \ldots, X_n)$ is ϵ-close to $\mathcal{D}$ iff the empirical distribution of $(f(X_1), \ldots, f(X_n))$ is.

Now suppose $|\Sigma| \leqslant r+1$. Let $A \subseteq \Sigma$ be any non-empty event, and denote by $\mathcal{D}'$ the empirical distribution of the outcomes $X_1, \ldots, X_n$. Let $p := \mathcal{D}(A)$, and define indicator random variables

$$Y_i := \begin{cases} 0 & X_i \notin A, \\ 1 & X_i \in A. \end{cases}$$

for $i \in [n]$ and $Y_0 := 0$. Observe that

$$\mathcal{D}'(A) = \frac{\sum_{i=1}^n Y_i}{n},$$

and, by the assumption on the closeness of conditional distributions of the X_i to $\mathcal{D}$,

$$\mathbb{E}[Y_i \mid Y_0, \ldots, Y_{i-1}] \in [p-\gamma, p+\gamma].$$

By Propositions 33 and 34, we can thus obtain a concentration bound

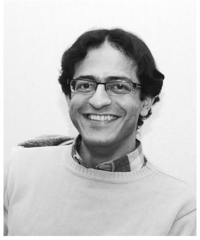
$$\Pr[|\mathcal{D}'(A) - p \,|> \epsilon] \leqslant 2\exp(-(\epsilon-\gamma)^2 n/2).$$

Now we can apply a union bound on all possible choices of A and conclude that

$$\Pr[\neg(\mathcal{D}' \approx_\epsilon \mathcal{D})] \leqslant 2^{r+2}\exp(-(\epsilon-\gamma)^2 n/2),$$

which can be ensured to be at most η for some choice of

$$n = O((r+2+\log(1/\eta))/(\epsilon-\gamma)^2). \qquad \square$$



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[9] Personal communication.

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