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Unsteady turbulent jets and plumes

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Declaration

I herewith certify that all material in this dissertation which is not my own work has been properly acknowledged.

John Craske
January 10, 2016

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Abstract

This thesis investigates the physics of statistically unsteady axisymmetric turbulent jets and plumes using theory and direct numerical simulation. The focus is on understanding and modelling the physics that govern the behaviour of radially integrated quantities, such as the integral scalar flux, momentum flux and buoyancy flux. To this end, a framework is developed that generalises previous approaches, making no assumption about the longitudinal velocity profile, turbulence transport or pressure. The framework is used to develop well-posed integral models that exhibit a good agreement with simulation data. In the case of passive scalar transport, shear-flow dispersion is observed to be dominant in comparison with longitudinal turbulent mixing. A dispersion closure for free-shear flows based on the classical work of Taylor (*Proc. R. Soc. Lond. A*, vol. 219 1954*b*, pp. 186-203) is therefore developed. In the analysis of jets whose source momentum flux undergoes an instantaneous step change, it is demonstrated that a momentum–energy framework, of the kind used by Priestley & Ball (*Q. J. R. Meteorol. Soc.*, vol. 81 1955, pp. 144-157), is the natural choice for unsteady free-shear flows. The framework is used to demonstrate why existing top-hat models of unsteady jets and plumes are ill-posed and that jets and plumes with Gaussian velocity profiles remain approximately straight-sided and are insensitive to source perturbations. Contrary to the view that the unsteady jet and plume equations are parabolic, it is shown that the generalised system of equations is hyperbolic. In unsteady plumes, the relative orientation of three independent families of characteristic curves determines whether propagating waves are lazy, forced or pure. To relate findings that are based on the momentum–energy framework to the classical mass–momentum framework, an unsteady entrainment coefficient is defined that generalises the decomposition proposed by Kaminski *et al.* (2005, *J. Fluid Mech.*, vol. 526, pp. 361-376).

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List of symbols

The following list consists of symbols that are used frequently throughout the thesis. Their description is followed by the page numbers on which their definitions can be found. Additional symbols are defined within individual chapters where necessary.

B_m	Mean integral buoyancy, 56, 98, 212
B_m^*	Characteristic integral buoyancy, 245
$\mathcal{B}$	Buoyancy normalised by a steady state, 236, 240
C_m	Mean scalar integral concentration, 56, 94, 110
$\mathcal{C}$	Scalar concentration normalised by a steady state, 127, 187
E_m	Mean energy flux, 101, 146, 214
E_f	Turbulent energy flux, 146
E_g	Gross energy flux, 146
E_p	Pressure transport integral, 146
γ_m	Dimensionless mean energy flux, 146, 181
γ_f	Dimensionless turbulent energy flux, 146
γ_p	Dimensionless pressure transport integral, 146
γ_g	Dimensionless gross energy flux, 145
γ_0	Leading-order dimensionless energy flux, 188
γ_1	Perturbed dimensionless energy flux, 188
F_m	Mean scalar/buoyancy flux, 51, 103, 106, 114, 186, 213
F_f	Turbulent scalar/buoyancy flux, 114
F_g	Gross scalar/buoyancy flux, 115
F_m^*	Characteristic scalar/buoyancy flux, 117, 222
F_s	Source scalar/buoyancy flux, 52, 98, 218
$\mathcal{F}$	Scalar/buoyancy flux normalised by a steady state, 239
θ_m	Dimensionless mean scalar/buoyancy flux, 115, 213
θ_f	Dimensionless turbulent scalar/buoyancy flux, 115, 213
θ_g	Dimensionless gross scalar/buoyancy flux, 213
θ_0	Leading-order dimensionless scalar/buoyancy flux, 127, 187
θ_1	Perturbed dimensionless scalar/buoyancy flux, 127, 187

M_m	Mean momentum flux, 50, 51, 101, 106, 114, 144
M_f	Turbulent momentum flux, 50, 145
M_g	Gross momentum flux, 145
M_m^*	Characteristic momentum flux, 164, 197
M_p	Pressure integral, 145
M_s	Source momentum flux, 29, 92, 179
$\mathcal{M}$	Momentum flux, normalised by a steady state, 197, 200, 236
β_f	Dimensionless turbulent momentum flux, 50, 146
β_p	Dimensionless pressure integral, 146
β_g	Dimensionless gross momentum flux, 145
β_u	Dimensionless radial velocity variance integral, 157
β_v	Dimensionless azimuthal velocity variance integral, 157
P_m	Leading-order turbulence production, 146
P_f	Longitudinal component of turbulence production, 146
P_g	Gross turbulence production/energy redistribution, 146
P_p	Pressure redistribution, 146
δ_m	Dimensionless leading-order turbulence production, 146
δ_f	Dimensionless longitudinal turbulence production, 146
δ_g	Dimensionless gross turbulence production/redistribution, 145
δ_p	Dimensionless pressure redistribution, 146
Q_m	Mean volume flux, 29, 51, 101, 114, 144
Q_s	Source volume flux, 92, 179
$\mathcal{Q}$	Volume flux normalised by a steady state, 200, 236
α	The entrainment coefficient 150, 216
α_0	The steady-state entrainment coefficient 52, 217
A_m	Characteristic area 144
b_m	Characteristic buoyancy 98
c_m	Characteristic concentration 94, 114
r_m	Characteristic radius 93, 114, 144, 192
t_m	Characteristic time scale 75
w_m	Characteristic velocity 93, 114, 144, 222
r_s	Characteristic source radius, 49, 92
τ_s	Source time scale, 92, 160, 218

r_d	Threshold radius/upper limit of integration 105
t_v	Asymptotic virtual source (in time) 118, 165, 189
z_v	Asymptotic virtual source (in space) 115, 189
χ^A	The value of χ after a step change, 164, 189, 203, 218, 231
χ^B	The value of χ before a step change, 164, 189, 203, 218, 231
$\bar{\chi}$	Ensemble average of a quantity χ , 44, 83
$\langle\chi\rangle$	Cross-sectional average of a quantity χ , 129, 134
χ'	Fluctuation of a quantity χ with respect to $\bar{\chi}$, 44, 83
χ_0	The steady-state value of χ , 127, 156, 200, 217, 236
η	Similarity variable in radial dimension 49, 123
λ	Dimensionless velocity/similarity variable 124, 131, 170, 197, 245
λ^*	Eigenvalue/dimensionless front position/velocity 168, 227, 245

Chapter 1

Introduction

1.1 Motivation

1.1.1 *Turbulence in the 21st century*

With computers of unlimited power and memory it would, in principle, be possible to solve the equations that govern the motion of fluids at an arbitrarily large scale and resolution. Seemingly, one would be able to obtain accurate weather forecasts, exact predictions for the drag exerted on structures and for the transport of contaminants by turbulence. However, with such predictions one would not necessarily understand the underlying physical processes or their relative importance. Full simulation of the equations of motion might not indicate the optimal way of controlling a particular flow, or provide the intuition that is needed in the solution of design problems. Moreover, direct simulation of energetically large problems would necessitate a correspondingly large expenditure in computational energy. Since the use of resource-intensive simulations is neither feasible nor sensible in many contexts, models of reduced complexity that capture the most significant physics of a flow continue to play a central role in fluid mechanics. Arguably, to understand a particular flow is to be able to simplify the full equations that govern its behaviour.

Whilst the work in this thesis exploits the most recent advances in computational techniques for observational purposes, its ultimate aim is to provide theoreticians and practitioners with a relatively simple model for unsteady turbulent jets and plumes. In obtaining this model we develop a framework that resolves several open questions and provides insight into a wide range of problems involving both steady and unsteady jets and plumes.

1.1.2 *Unsteady jets and plumes*

Turbulent jets and plumes are regions of intense fluid motion and mixing that are driven by buoyancy and/or momentum. An example of a turbulent plume can be seen in figure 1.1 which displays a photograph of Mount St. Helens following its eruption on May 18, 1980. Due to the presence of volcanic ash one can observe a large number of scales in the flow, ranging from large billows that span the width of the eruption column down to smaller structures that appear to be independent of the overall shape of the flow. The information captured in the photograph suggests



FIGURE 1.1. Mount St. Helens eruption, May 18, 1980 (U.S. Geological Survey 1980). Photograph by Robert Krimmel, courtesy of the U.S. Geological Survey. <http://earthshots.usgs.gov/earthshots/Mount-St-Helens>; accessed July 19, 2015.

that this flow is responsible for vigorous mixing with its environment and that its shape and intensity change over time.

The jets and plumes considered in this thesis are free-shear flows (i.e. flows that are not confined by solid boundaries), such as the eruption column depicted in figure 1.1. In contrast to their steady-state counterparts, unsteady jets and plumes are the result of source conditions that vary in time. In the presence of turbulence, we refer to such flows as being *statistically* unsteady, which means that functions such as the mean or expected velocity of the flow are time dependent. If the time scale associated with the statistical unsteadiness is large compared to that of the turbulence, unsteady plumes can be regarded as quasi-steady (see, e.g., Delichatsios, 1979). In comparison to unsteady plume theory, quasi-steady plume theory, which we discuss in the following section, is relatively well-developed. Yet, owing to time-dependent forcing from both natural and man-made sources, unsteady plumes are prevalent in engineering and the physical sciences. In addition to the transport of ash particles and buoyancy in volcanic eruptions (Woods, 2010), examples include the diurnal heating of the surface of the earth (see, e.g., Hunt *et al.*, 2003), the transport of smoke and heat from rapidly developing fires (Hunt, 1991) and the heat load in buildings (Linden, 1999).

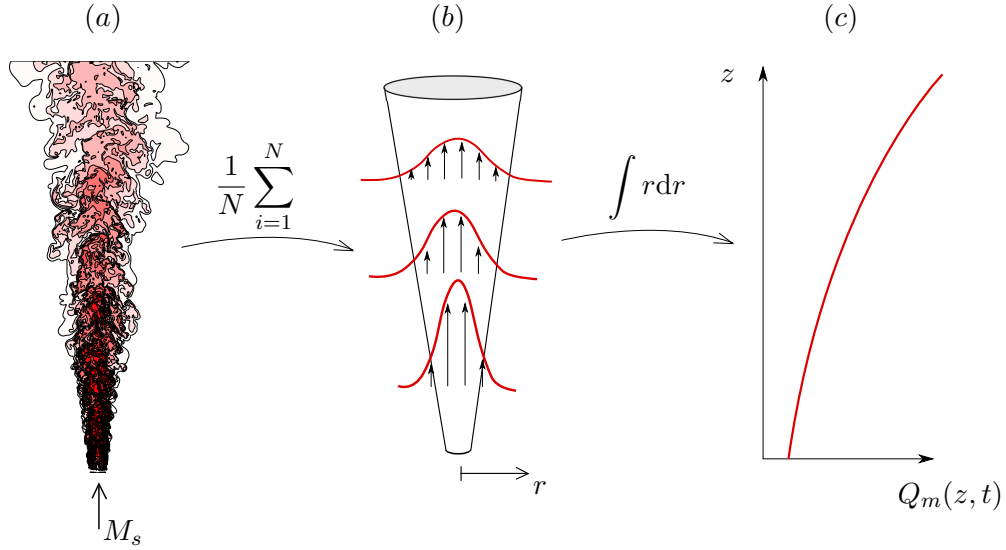


FIGURE 1.2. Analysis of the bulk properties of a turbulent jet using ensemble averaging and radial integration. (a) Isoregions of velocity in a turbulent jet driven by source momentum flux M_s . (b) Schematic representation of the ensemble averaged velocity. (c) Typical volume flux Q_m .

This thesis provides a framework and a series of models that allow one to understand and predict the behaviour of unsteady plumes. In the tradition of classical plume theory, our focus is on integrals of ensemble-averaged quantities over the radius of the plume, such as the volume flux, momentum flux and buoyancy flux, which will be defined precisely in chapter 2. Hence, we start with an ensemble of N three-dimensional direct simulations of a flow (figure 1.2a), before averaging over the ensemble and the azimuthal direction to obtain statistical quantities (figure 1.2b). Finally, we integrate the statistical quantities with respect to r to obtain integral quantities such as the volume flux Q_m , which depend on the longitudinal coordinate z and time t (figure 1.2c). We will also consider the behaviour of derived quantities, such as the characteristic radius and velocity of the plume, demonstrating precisely how they are affected by statistically unsteady forcing. The structure of propagating waves within unsteady plumes and the physics that determines the rate at which disturbances spread along the axis of a plume will also be discussed. Our overall description of unsteady plumes will be informed by a collection of smaller studies into unsteady transport phenomena. These studies will include unsteady passive scalar transport in an otherwise steady jet and momentum transport in a jet whose source momentum flux changes abruptly.

There are several existing models of unsteady plumes (see, e.g., Scase & Hewitt, 2012), which will be discussed in detail in the relevant chapters of this thesis.

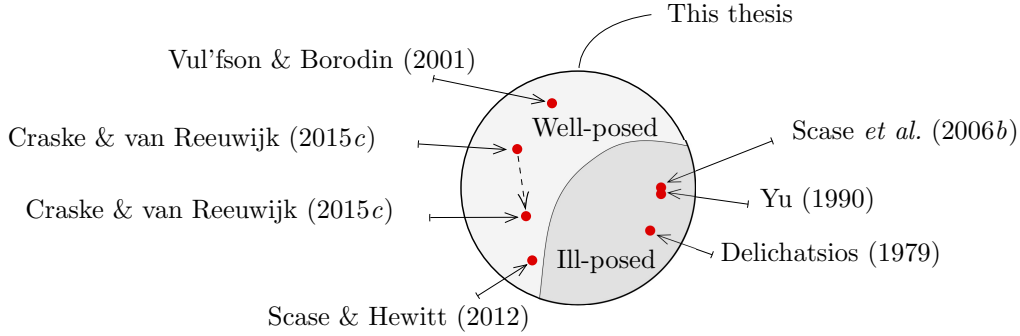


FIGURE 1.3. The space of all unsteady plume models, captured by the framework described in this thesis. Existing plume models are either well-posed or ill-posed (Scase & Hewitt, 2012), and form a subset of the generalised framework developed here.

Our contribution is distinct from those of our predecessors because we develop a generalised framework that relies on very few assumptions. Existing models form a relatively small subset of the models that can be obtained from the generalised system that we propose. Consequently, we can relate the properties of individual models to the underlying physics of the problem and, following work by Scase & Hewitt (2012), for example, explain why some models are ill-posed. The way in which this thesis encompasses a range of existing models and allows us to identify their properties is depicted in figure 1.3; the existing models to which the figure refers will be discussed in §1.2.

The scientific community can benefit from this research in a number of ways. First, from an enhanced understanding of the physics of jets and plumes, leading to a significant improvement in one's ability to make accurate predictions about their behaviour. Second, from high-resolution simulation data pertaining to several canonical unsteady flow problems. Third, from the way in which we combine the classically distinct fields of turbulence, plume theory and dispersion theory. The framework that we have developed unifies and finds application in a wide range of related steady and unsteady problems. For instance, one can infer or relate properties of quasi-steady plumes by making bulk observations of unsteady plumes. In particular, we will show that the propagation velocity of a front in an unsteady jet can be related in a precise way to the entrainment coefficient of a steady jet. In fact, we will see that properties that are unobservable in the steady state can play a non-trivial role in the dynamics of unsteady jets and plumes.

1.2 Historical summary

In plume theory one arguably finds one of the greatest success story of turbulence modelling in the form of the entrainment hypothesis (Taylor, 1945), which was communicated to a wide audience by Batchelor (1954). The entrainment hypothesis couples the ambient flow that is induced, and subsequently engulfed, by a jet or a plume to its momentum flux. Consequently, the entrainment hypothesis forms the backbone of all bulk models of jets and plumes and remains a subject of continual research and debate (see, e.g., Turner, 1986; Carazzo *et al.*, 2006; da Silva *et al.*, 2014). Although the seminal reference for plume theory is the model of Morton *et al.* (1956), describing conservation laws for volume, momentum and buoyancy in a plume, it was preceded by similarity solutions for steady plumes proposed by Schmidt (1941*a,b*) and, at a similar time in the Soviet Union, by Zeldovich (1937). Earlier still, Tollmien (1926) had developed similarity solutions for a turbulent jet. Of particular relevance to the present study is that Priestley & Ball (1955) developed a plume model based on energy conservation of the mean flow at around the same time as Morton *et al.* (1956) had decided to focus on volume conservation.

Following the foundational papers described above, many aspects of plume theory and its applications have been developed in greater detail. Morton (1959) introduced the now widely used flux-balance parameter Γ to quantify the relative balance between buoyancy and inertia in a plume. The flux-balance parameter is useful because it allows one to identify ‘forced’ (inertia-dominated) plume behaviour, for which $0 \leq \Gamma < 1$ (Morton, 1959), and ‘lazy’ (buoyancy-dominated) plume behaviour, for which $1 < \Gamma$ (Hunt & Linden, 2001). Jets, for which $\Gamma = 0$, and ‘pure’ plumes, for which $\Gamma = 1$, are limiting cases or fixed points. In particular, the ‘pure’ plume is an asymptotic limit, typically approached at distances that are large relative to the length scale associated with the source (the far-field), and is consequently associated with a virtual point source of zero volume flux (Hunt & Linden, 2001). These ideas were clarified in the scale diagrams of Morton & Middleton (1973) and later by van den Bremer & Hunt (2010).

Based on the experimental observations of Ricou & Spalding (1961), Rooney & Linden (1996) obtained a similarity model for non-Boussinesq plumes (i.e. plumes driven by relatively large density differences), which was subsequently generalised to stratified environments by Woods (1997). Solutions universally valid for both Boussinesq and non-Boussinesq axisymmetric plumes were given by van den Bremer & Hunt (2010), and later extended to two-dimensional plumes (van den Bremer & Hunt, 2014*a,b*). Plumes whose buoyancy varies non-monotonically with mixing have been considered by Caulfield & Woods (1995), for application in the analysis

of volcanic eruptions (see, e.g., Woods, 1988). More recently, the effect of chemical reactions within a plume have been investigated (Conroy *et al.*, 2005; Conroy & Llewellyn Smith, 2008; Campbell & Cardoso, 2010). Following the work of Linden *et al.* (1990), plume theory has been applied extensively in the field of natural ventilation (see, e.g., Linden, 1999, for further details) and has also been used to analyse fires (Morton, 1965; Heskestad, 1998). For further details about the history of plume theory the reader is referred to review articles by Turner (1966), List (1982), Linden (2000), Kaye (2008), Woods (2010) and Hunt & van den Bremer (2011).

A substantial number of experiments have been performed involving jets and plumes, including early measurements of gaseous plumes by Rouse *et al.* (1952, axisymmetric and planar), Ricou & Spalding (1961, axisymmetric) and Lee & Emmons (1961, planar). In the absence of precise ways of measuring the flow directly, some of these early experiments employed ingenious methods to infer properties of a flow via bulk measurements (e.g., the calculation of entrainment using observations of a ‘filling box’ in the classical experiments of Baines & Turner, 1969). More detailed observations of the velocity in jets and plumes have been obtained using hot-wire measurements (e.g. Panchapakesan & Lumley, 1993*a,b*; Hussein *et al.*, 1994; Shabbir & George, 1994), and laser-Doppler velocimetry (Kotsovinos & List, 1977; Kotsovinos, 1977; Hussein *et al.*, 1994; Papanicolaou & List, 1988). Recent experiments have employed particle image velocimetry (Wang & Law, 2002; Ezzaamel *et al.*, 2015). A notable feature of experimental studies of jets and plumes that is relevant to the present study is the observation that the longitudinal turbulent transport of momentum and buoyancy is significant (i.e. in excess of 10%), in spite of the fact that it is typically neglected from the theoretical work discussed in the previous paragraph.

Relative to the history of plume theory, advances in high-performance computing have occurred only recently (Alfonsi, 2011). Consequently, direct numerical simulations (DNSs) of jets and plumes, in which one resolves all the scales occurring in the flow, are limited in number. Stanley *et al.* (2002) obtained DNS of a compressible planar jet to study near-field mixing and flow evolution at a Reynolds number, Re , equal to 3000; DNS of passive scalar transport in jet subjected to a cross flow with $Re = 5000$ has been undertaken by Muppidi & Mahesh (2008); DNS of a subsonic (compressible) axisymmetric jet at $Re = 4700$ has been undertaken by Wang *et al.* (2010); an axisymmetric turbulent plume at $Re = 7700$ has been simulated by Plourde *et al.* (2008) and DNSs of forced and lazy plumes have been obtained by Taub *et al.* (2015). A crucial point to note about DNS of jets and

plumes is that the simulation of spatially large domains is done at the expense of a high Reynolds number. For example, the axisymmetric plume simulated by Plourde *et al.* (2008) is, to date, of relatively high Reynolds number ($Re = 7700$). However, consisting of only 8 source diameters in length, the plume in question only encompasses the ‘near-field’. To the best of our knowledge, nobody has conducted DNS of unsteady jets and plumes.

Investigations into unsteady plumes are sparse in comparison to those into steady-state plumes and will be discussed in greater detail in chapters 5, 6, 7 and 8. In our opinion, this is because they are difficult to investigate, not because there is a shortage of applications. To summarise briefly, Turner (1962) initiated studies of unsteady plumes with a model for a starting plume. Following refinements by Middleton (1975), the first use of a system of partial differential equations to describe unsteady plumes appears to be Delichatsios (1979). Indicative of the difficulties associated with unsteady plume modelling is the fact that subsequent unsteady plume models (e.g. Yu, 1990; Vul’fson & Borodin, 2001; Scase *et al.*, 2006*b*) adopted quite different assumptions and approaches. More recently, the basis of unsteady plume models has been called into question with the discovery that several are ill-posed (Scase & Hewitt, 2012). As noted by Scase *et al.* (2006*b*), the experimental investigation of unsteady plumes is challenging, owing partly to the need to run an ensemble of statistically equivalent experiments. Nevertheless, Scase *et al.* (2008) obtained experimental data pertaining to a plume whose source buoyancy flux undergoes a sudden reduction.

1.3 Preview

This thesis is about modelling unsteady plumes. The models are a representation of the ‘real’ jets and plumes that we will observe using direct numerical simulation. Being far simpler than real jets and plumes, the *integral* models with which we are concerned disregard many features of the flow. Our criterion for the models is that they should correctly predict a designated number of large-scale features of the flow. For example, in unsteady jets one would like to be able to predict the rate at which disturbances travel along the length of the jet. We also require that the models are well-posed, making it possible to obtain a unique solution using numerical methods, if necessary.

In spite of the success of steady plume theory, we will see that there are properties of unsteady jets and plumes that cannot be inferred from *integral* observations of a quasi-steady state. Interestingly, such properties *can* be inferred from a quasi-steady state when one looks at more detailed features of the flow. In the case of

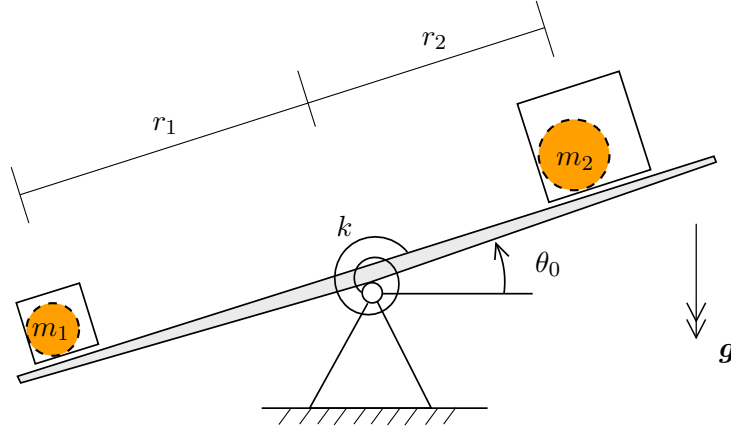


FIGURE 1.4. A system with unknown properties m_1 , m_2 and k in static equilibrium.

jets and plumes this will involve looking at radial distributions of mean-flow and turbulent quantities. Before complicating the story with specific details relating to jets and plumes, we consider a simple example.

1.3.1 Modelling

Consider the structure illustrated in figure 1.4. With a measuring tape, a protractor and a stop watch, we would like to deduce several properties of this system. In particular, we intend to establish the mass m_1 and m_2 of the contents of each box and the stiffness of the spring k . When the system is in static equilibrium the angle θ_0 can be expressed as a *known* function of dimensionless parameters:

$$\theta_0 = f_0 \left(\frac{r_1}{r_2}, \frac{m_1}{m_2}, \frac{k}{r_1 m_1 g} \right) = f_0 (\Pi_1, \Pi_2, \Pi_3). \quad (1.1)$$

By moving the position of each box, we can determine Π_2 and Π_3 by recording θ_0 for at least two different values of Π_1 . However, without opening the boxes, it is not possible to determine the particular values of m_1 , m_2 and k from static equilibrium alone. Instead, we perturb the system from static equilibrium and observe the frequency ω with which the structure oscillates. The equations of motion provide a known relationship between ω and the dimensionless parameters:

$$\theta_1 \stackrel{\text{def}}{=} \frac{\omega^2 r_1^2 m_1}{k} = f_1 (\Pi_1, \Pi_2, \Pi_3). \quad (1.2)$$

Knowledge of Π_1 , Π_2 , Π_3 and θ_1 allows us to determine m_1 , m_2 and k . Although we could have looked inside each box in the steady state, we determined something

about the contents non-invasively by perturbing the system from equilibrium. In contrast to f_0 , f_1 contains information about the moment of inertia, which is a functional of the system. To establish the distribution of mass in each box, which we have implicitly neglected, we would need to consider additional properties of the system.

1.3.2 The jet equations

Like the example in the previous section, some properties of plumes are unobservable in a steady state. The plume equations are a radical simplification of the full equations of fluid motion and therefore sacrifice considerable information pertaining to the ‘internal’ structure of a plume. The loss of information is both useful and unfortunate. It is useful because it means that steady-state predictions of a plume are not affected by the assumptions one makes about its internal structure and are therefore very robust. Many different pictures of a plume map onto the same steady-state plume model. It is unfortunate, however, that in situations in which a plume is perturbed from its steady-state the missing information plays a crucial role in determining the response. To see this, consider the system of ordinary differential equations

$$\left. \begin{aligned} \beta \frac{dY_0}{dz} &= 0, \\ \gamma \frac{d}{dz} \left(\frac{Y_0^2}{X_0} \right) &= \delta \frac{Y_0^{5/2}}{X_0^2}, \end{aligned} \right\} \quad (1.3)$$

in the dependent variables X_0 and Y_0 and the system parameters β , γ and δ . We assume that the parameters of the system are constant, but notice from (1.3) that they cannot be determined uniquely from the steady-state solution (X_0, Y_0) . Specifically, the system is invariant under the transformation $\beta \mapsto a\beta$, $\gamma \mapsto b\gamma$ and $\delta \mapsto b\delta$ for $a, b \neq 0$. The quantity Y_0 appears to be conserved regardless of the value of β , and the solution to the entire system is determined once we fix δ/γ . Conversely, if we observe the solution to (1.3) we can infer δ/γ , but neither γ nor β .

However, the full unsteady version of (1.3),

$$\left. \begin{aligned} \frac{\partial X}{\partial t} + \beta \frac{\partial Y}{\partial z} &= 0, \\ \frac{\partial Y}{\partial t} + \gamma \frac{\partial}{\partial z} \left(\frac{Y^2}{X} \right) &= \delta \frac{Y^{5/2}}{X^2}, \end{aligned} \right\} \quad (1.4)$$

depends on β , γ and δ independently, and we are forced to address the physics that

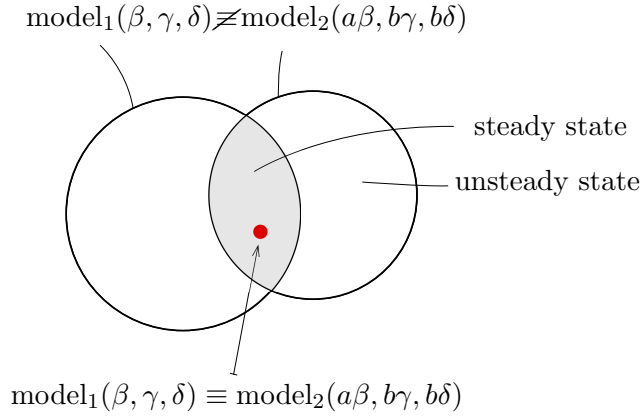


FIGURE 1.5. Equivalence and inequivalence of a model under the parameter transformation $(\beta, \gamma, \delta) \rightarrow (a\beta, b\gamma, b\delta)$. Models that are equivalent in a steady state are not necessarily equivalent in an unsteady state.

these parameters represent if we are to obtain accurate predictions using (1.4). The relationship between two models describing steady and unsteady states is illustrated in figure 1.5. Whilst the models in figure 1.5 are equivalent in a steady state, an unsteady state exposes fundamental differences between them. We will see that β , γ and δ correspond to properties of the velocity field in jets and plumes. In particular, we will find that from an integral perspective the jets depicted in figure 1.6 are equivalent in a steady state. Yet, in an unsteady state, they have very different dynamical properties.

1.4 Aim, objectives and scope

The aim of this thesis is to advance the understanding of unsteady turbulent jets and plumes using DNS and to develop a general framework and model that can be used for making accurate predictions.

The objectives of the work are

- (a) Upgrade and extend an existing DNS code to facilitate the accurate simulation of steady and unsteady free-shear flows, faithfully accounting for their behaviour on unbounded domains.
- (b) Simulate and investigate the unsteady release of a passive scalar into an otherwise steady axisymmetric turbulent jet.
- (c) Simulate and investigate axisymmetric turbulent jets whose source momentum flux is subjected to an instantaneous step change.
- (d) Simulate and investigate axisymmetric turbulent plumes whose source buoyancy flux is subjected to an instantaneous increase.

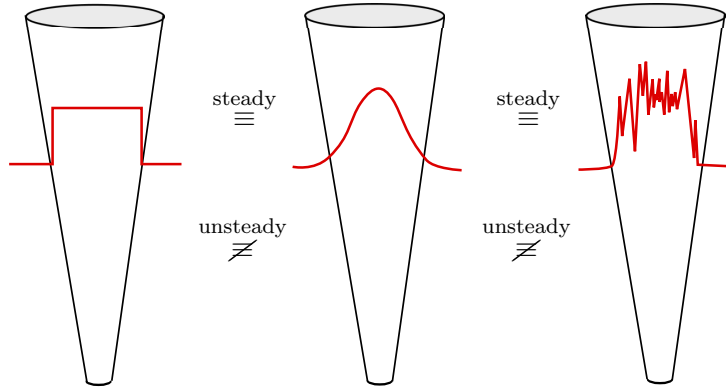


FIGURE 1.6. In a steady state the assumed radial distribution of velocity does not affect the integral model and the cases illustrated in the figure are equivalent. In an unsteady state, the cases are not equivalent because the internal features of the flow determine the structure of the governing integral equations.

In keeping with plume theory, the physics on which we focus occurs at a macroscopic scale and we will therefore not consider small-scale features of the flow explicitly. To establish a comprehensive picture in the most general case, we will study the transport of passive and active scalars in a series of separate investigations. In particular, we consider canonical cases involving a step-change in source conditions, from which a wide variety of other fundamental flows can be constructed. For generality we are primarily concerned with the behaviour of jets and plumes at relatively large distances from their sources. Though interesting extensions, we will not discuss planar plumes, non-Boussinesq plumes or plumes in stratified environments. We have devoted the chapters in this thesis to a systematic and thorough investigation of the basic ingredients of unsteady plumes rather than extensions to more exotic problems.

Based on an overarching theoretical framework we will propose particular models for unsteady jets and plumes and show that their predictions are in good agreement with DNS data. However, the purpose of this work is not to establish a set of parameter values that produce a best-fit to simulation data. We focus on the fundamental differences between various models and the physics that they represent rather than fine tuning model parameters.

1.5 Outline

In chapter 2 we discuss the governing equations and the various assumptions that are commonly used in their simplification. Specifically, we explain how the Navier-

Stokes equations, describing local balances in the flow, are related to the classical integral equations for jets and plumes, and to the generalised unsteady framework that we propose. Chapter 3 discusses methodology, focusing on the implementation and verification of the code that we have used for direct numerical simulation. Validation of the code is provided in chapter 4, with a discussion about how we obtain integral quantities from the data.

We start investigating the physics of unsteady jets and plumes in chapter 5, which considers the unsteady release of a passive scalar into an otherwise steady jet. This relatively simple linear problem allows us to understand how the radial distribution of a scalar affects the transport of the radially averaged (integral) scalar concentration. In particular, we will show that shear-flow dispersion plays a dominant role and will develop an unsteady integral model based on Taylor’s (1953) model for dispersion in pipes. In chapter 6 we analyse DNS results of a jet whose source momentum flux is subjected to an instantaneous step change, both up and down. In the analysis of this non-linear problem we extend the momentum–energy framework used by Priestley & Ball (1955) and Kaminski *et al.* (2005) to incorporate unsteadiness. We show that the momentum–energy framework has several advantages over the classical mass–momentum framework, and use it in a diagnostic capacity to interpret the DNS results. In contrast to the linear scalar transport considered in chapter 5, the unsteady transport of momentum and energy considered in chapter 6 involves possible changes in both the radius and the characteristic velocity of the jet. Chapter 7 focuses on the modelling of unsteady jets using the momentum–energy framework. We identify several special features of unsteady jets with a Gaussian velocity profile, including their tendency to remain straight-sided (conical) and their neutral response to source perturbations. These features are contrasted with the properties of top-hat unsteady jet models and used to explain why the latter are ill-posed. The model for unsteady jets that we propose focuses on a correct description of energy transport in the jet and incorporates higher-order energy transport in the form of shear-flow dispersion, based on the approach developed in chapter 5.

In chapter 8 we combine the findings reported in chapters 5, 6 and 7 to develop a generalised framework and model for unsteady turbulent plumes. In contrast to the top-hat unsteady plume equations, which are parabolic, we show that the generalised plume equations are typically hyperbolic. We relate the behaviour of the flux-balance parameter and radius of the unsteady plume to profile coefficients in the governing integral equations. In contrast to the top-hat unsteady plume equations (for details see Scase & Hewitt, 2012), realistic assumptions regarding

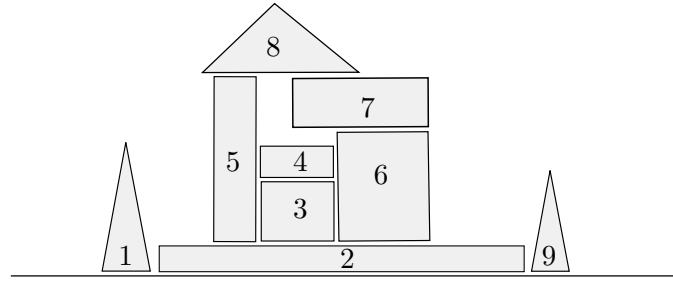


FIGURE 1.7. Schematic of the relationship between the chapters in this thesis. The foundations are covered by blocks lower down in the picture, whilst more developed analysis and theory is covered by blocks higher up in the picture. The dependence of chapter n on a preceeding chapter $n - m$ is indicated by block n resting on block $n - m$.

the velocity profile and turbulent buoyancy flux result in a system of equations that is well-posed. The model that we propose for unsteady plumes incorporates dispersion of both energy and buoyancy, exhibits a good agreement with the DNS data and, in certain circumstances, possesses an analytical solution. In chapter 9 we draw conclusions and make recommendations for future work. The structure of the thesis and the dependencies of each chapter are indicated schematically in figure 1.7.

Chapter 2

Preliminaries

The purpose of this chapter is to provide the fundamental equations of motion and a broad context for the work that will be described in detail in subsequent chapters. We will state the governing equations, discussing their properties and the challenges that one faces in seeking their solution. In many cases of practical relevance the governing equations can be simplified, typically by assuming that their solution possesses some form of symmetry in the limit of a large or small parameter. In fact, one can obtain a hierarchy of equations, each adopting a cruder perspective and/or increasingly onerous assumptions. This chapter will explain where the equations with which we are concerned sit within the hierarchy and the way in which they relate to the underlying physics.

2.1 The governing equations

2.1.1 *The compressible Navier-Stokes equations*

In this section we summarise the equations that govern the motion of a fluid, for further details the reader is referred to Batchelor (1967). Under the assumption of the continuum hypothesis one assumes the existence of continuous functions, such as density $\rho(\mathbf{x}, t)$, of the spatial coordinates $\mathbf{x}$ and time t , that describe the dynamical and thermodynamical state of a fluid. For mass conservation it is necessary that the density ρ and velocity $\mathbf{u}$ of the fluid are coupled:

$$\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \mathbf{u}, \quad \text{where} \quad \frac{D}{Dt} \stackrel{\text{def}}{=} \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \quad (2.1)$$

is the material derivative. Consideration of the forces acting on a parcel of fluid leads to an equation for the fluid acceleration:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p_t - \rho \mathbf{g} + \nabla \cdot \mathbf{S}. \quad (2.2)$$

Here p_t is the total pressure (inclusive of a hydrostatic component), $\mathbf{g} \stackrel{\text{def}}{=} (0, 0, g)$ (in Cartesian coordinates) is the gravitational acceleration and $\mathbf{S}$ is the deviatoric stress tensor:

$$\mathbf{S} \stackrel{\text{def}}{=} \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top \right) - \frac{2\mu}{3} (\nabla \cdot \mathbf{u}) \mathbf{I}, \quad (2.3)$$

for the viscosity μ and the identity $\mathbf{I}$. One of the assumptions of this model is that the shear stress in a fluid can be completely characterised by the velocity gradient $\nabla \mathbf{u}$.

To solve (2.1) and (2.2) for the five unknowns $(\mathbf{u}, p_t, \rho)$ it is necessary that they are supplemented with a thermodynamic equation for internal energy, for example, and an equation of state (see Batchelor, 1967). We will refer to the resulting system as L_1 . In the following sections we will describe a sequence of approximations $L_2, L_3, \dots$ that use assumptions to simplify the governing equations, such that $L_1 \supset L_2 \supset \dots$

2.1.2 The Boussinesq approximation

To simplify equations (2.1) and (2.2) we assume that density differences $\Delta\rho$ in the flow are small, relative to a reference density ρ_0 , and manifest themselves solely as a buoyancy force in equation 2.2. This assumption constitutes the basis of the Boussinesq approximation (see, e.g., Spiegel & Veronis, 1960) of the equations governing the motion of a fluid. In general ρ is a function of pressure, temperature and possibly additional state variables, such as salinity. Here we restrict our attention to pressure and temperature. The requirement that $\Delta\rho/\rho_0 \ll 1$, typically implies that pressure differences and temperature differences in the flow are small relative to the reference state. In this section we describe conservation equations for volume, momentum and temperature and outline the assumptions on which they are based. For a detailed treatment of the conditions for which the Boussinesq approximation is valid, the reader is referred to Spiegel & Veronis (1960), Batchelor (1967) and Tritton (1988).

It should be noted that density differences in the flow can arise from hydrostatic variations of the environment density $\rho_d(z)$, in addition to density differences induced by the flow. We can assume that the variation in ρ_d is small by considering flows whose vertical length scale is small relative to the height over which hydrostatic variations in pressure p and density ρ occur (e.g. the ‘scale-height’ of the atmosphere). Furthermore, we assume that both the characteristic velocity of the flow and velocity scale associated with unsteadiness are small relative to the speed of sound, the former being the assumption of a low *Mach number*. Moreover, we assume that density differences arising from thermal effects are relatively small. Under these conditions a fluid behaves as if it is incompressible (see Batchelor, 1967, for further details). If, in addition, one assumes that the internal heating of a fluid due to viscous dissipation and molecular heat conduction are relatively small, the law of mass conservation (2.1) becomes a volumetric constraint on the

flow field:

$$\nabla \cdot \mathbf{u} = 0. \quad (2.4)$$

Under the conditions described above, the leading-order contribution made by density differences in equation (2.2) is the gravitational forcing term $-\rho\mathbf{g}$. Thus, on dividing by the reference density ρ_0 and simplifying $\mathbf{S}$, the Boussinesq approximation of (2.2) is

$$\frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{b} + \nu \nabla^2 \mathbf{u}, \quad (2.5)$$

where, in a Cartesian frame of reference, the buoyancy $\mathbf{b}$ is defined by

$$\mathbf{b} \stackrel{\text{def}}{=} (0, 0, b), \quad \text{where } b \stackrel{\text{def}}{=} \frac{\rho_d - \rho}{\rho_0} g, \quad (2.6)$$

and p is the kinematic pressure relative to that of the hydrostatically balanced environment, such that for constant ρ_d , $p_t \stackrel{\text{def}}{=} \rho_0 p - \rho_d g z$.

A transport equation for temperature can be obtained from the first law of thermodynamics (Batchelor, 1967) as a form of energy conservation. In the energy equation, the Boussinesq approximation involves neglecting the effect of both pressure variations and viscous dissipation on the internal energy (see Tritton, 1988, for further details). Consequently, energy conservation can be expressed as a transport equation for temperature T according to

$$\frac{DT}{Dt} = \kappa \nabla^2 T. \quad (2.7)$$

For an ideal gas the coefficient of thermal expansion, $(-\partial \ln \rho / \partial T)_p$, is equal to $1/T$ and, under the assumption that $\Delta \rho / \rho_0 \ll 1$, the equation of state relating ρ and T can be linearised to give $\rho_0 - \rho \stackrel{\text{def}}{=} (T - T_0) \rho_0 / T_0$. If we further assume that the environment is unstratified, such that $\rho_d = \rho_0$, equation (2.7) implies that buoyancy is conserved*.

The incompressible Navier-Stokes equations, which will be denoted $L_2(\mathbf{u}, p, b)$, are therefore a system of five dependent variables and can be expressed in a con-

*In a stratified environment buoyancy is not conserved. More generally, since the equation for buoyancy is a derived relation, a nonlinear equation of state or mixing relation will result in buoyancy is not being conserved (see, e.g., Woods, 2010, and references therein for further details and examples).

servative form:

$$\left. \begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) &= -\nabla p + \mathbf{b} + \nu \nabla^2 \mathbf{u}, \\ \frac{\partial b}{\partial t} + \nabla \cdot (\mathbf{u} b) &= \kappa \nabla^2 b. \end{aligned} \right\} \quad (2.8)$$

We will regard L_2 as a subset of L_1 (the compressible Navier-Stokes equations) because by invoking the Boussinesq approximation we have removed various physical phenomena (e.g. sound waves) from the problem. In spite of the Boussinesq approximation, only for special cases have solutions to the governing equations (2.8) been obtained (see, e.g., Drazin & Riley, 2006). Thus, in subsequent sections we will look for further simplifications of (2.8). For details regarding the mathematical properties of the incompressible system, the reader is referred to Ladyzhenskaya (1963), Temam (2001) and Chorin & Marsden (2013).

2.1.3 The Reynolds-averaged Navier-Stokes equations

In view of their sensitive dependence on initial and boundary conditions, it proves convenient to describe turbulent flows using random variables. To this end, we define the ensemble average of a random variable χ as

$$\bar{\chi} \stackrel{\text{def}}{=} \int_{\Omega} \chi \, d\mu, \quad (2.9)$$

where Ω is a probability space and μ the measure or probability assigned to subsets of that space (see, e.g., Frisch, 1995). Physically, $\bar{\chi}$ represents the expected value of a variable resulting from known initial or boundary conditions[†]. For example, $\bar{\mathbf{u}}$ at a given spatial and temporal point in the flow is the velocity vector that one would expect to measure, or would be most likely to measure, in a single realisation of the flow. As noted by Monin & Yaglom (1973, p.215), the assumption made in pursuing this statistical approach to turbulence is the existence of probability distributions for each dependent variable. However, a measured velocity $\mathbf{u}$ from a single realisation, being a real vector, will never coincide with $\bar{\mathbf{u}}$ exactly; the extent to which $\mathbf{u}$ deviates from $\bar{\mathbf{u}}$ is given by the underlying probability distribution and the dynamics of the process. Since the process of ensemble averaging is a mathematical construction, it is natural to ask to what extent it corresponds to the physics that is observable in a single realisation.

[†]More precisely, initial and boundary conditions whose probability distributions are known or assumed.

The approach taken in this thesis is to analyse and model the ensemble-averaged flow. Our analysis and models therefore correspond to the expected behaviour of the flow, rather than the flow one might observe in a single realisation, such as a single experiment or field observation. Nevertheless, we will demonstrate in chapters 5 and 6 that the ensemble average captures several features that are common to each member of the ensemble. While we will not be in a position to examine or predict features of the flow that are particular to individual realisations, the ensemble-average approach we adopt is consistent with previous work in the field (see, e.g., Scase *et al.*, 2008; Landel *et al.*, 2012) and allows us to develop models for aspects of the flow that are broadly characteristic of any given realisation and useful to engineers seeking to make predictions *a priori*, with limited information of the system's boundary and initial conditions.

As noted by Monin & Yaglom (1973, p.215), the mean $\bar{\chi}$ is an average over all possible values of χ , i.e. over an infinitely large set of experiments or realisations of the flow. In practice, therefore, we approximate $\bar{\chi}$ using a finite set of observations from N statistically independent realisations of the flow, under the assumption that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \chi_i = \bar{\chi}. \quad (2.10)$$

Statistical estimators will be discussed in detail in §3.6.2. In the case of statistically unsteady flows, $\bar{\chi}$ depends on time, which is why we refrain from taking time averages. Fluctuations in χ will be denoted χ' and defined such that $\overline{\chi'} = 0$.

Application of the ensemble averaging operator to (2.8) yields

$$\nabla \cdot \bar{\mathbf{u}} = 0, \quad (2.11)$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) = -\nabla \bar{p} - \nabla \cdot (\overline{\mathbf{u}' \otimes \mathbf{u}'}) + \bar{\mathbf{b}} + \nu \nabla^2 \bar{\mathbf{u}}, \quad (2.12)$$

$$\frac{\partial \bar{b}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}} \bar{b}) = -\nabla \cdot (\overline{\mathbf{u}' b'}) + \kappa \nabla^2 \bar{b}. \quad (2.13)$$

Since it is not possible to isolate the behaviour of $(\bar{\mathbf{u}}, \bar{p}, \bar{b})$ from higher-order statistics (e.g. $\overline{\mathbf{u}' \otimes \mathbf{u}'}$), the system (2.11)-(2.12), which we will refer to as $L_3(\bar{\mathbf{u}}, \bar{p}, \bar{b} | \overline{\mathbf{u}' \otimes \mathbf{u}'}, \dots)$, is not closed. For closure one is forced to consider an infinite system of equations involving all moments of the flow (see, e.g., Monin & Yaglom, 1973; Fursikov, 2012). In the present work we will see that the situation is analogous to the spatially-dependent information that is lost when one integrates (2.11)-(2.12) to obtain equations for the bulk quantities in a plume. In the case of (2.11)-(2.12),

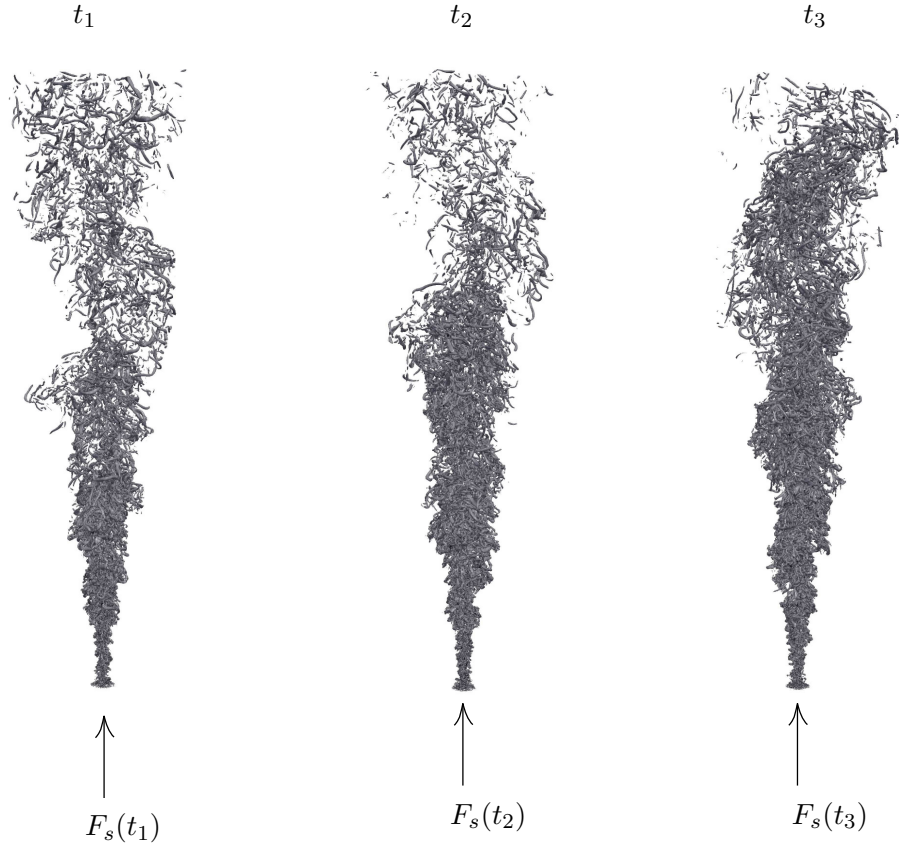


FIGURE 2.1. Isosurfaces of the second invariant of $\nabla \mathbf{u}$ in a statistically unsteady turbulent plume at several times.

one hopes, in analogy with the viscous stresses in (2.2), that the second-order moments of the flow can be related to $\nabla \mathbf{u}$ (see, e.g., Pope, 1975), so that the equations can be truncated at a finite order.

2.2 Turbulent jets and plumes

Jets and plumes are free-shear flows driven by isolated sources of momentum or buoyancy, respectively (see, for example, the turbulent plume in figure 2.1). In the absence of additional boundaries or forcing from the environment, their behaviour is determined entirely by their source conditions. Moreover, jets and plumes have a progressively weaker dependence on the precise details of their source conditions, with respect to the longitudinal coordinate z , because they mix with their surrounding environment (Fischer *et al.*, 1979). Consequently, it is the behaviour of conserved quantities such as the momentum flux (jets) and buoyancy flux (plumes) that are of primary interest (these quantities will be defined in §2.4.1).

2.2.1 High-Reynolds-number jets and plumes

As far as the mean flow is concerned, one typically assumes that, in the absence of solid boundaries, jets and plumes become independent of molecular effects such as viscous stresses and thermal diffusion (see, e.g., Tennekes & Lumley, 1972). Indeed, our concerns in this thesis relate to the dynamics of the mean flow. We do not consider evolution equations for higher order statistics such as the turbulence kinetic energy, in which viscous effects always play a crucial role and distinguish turbulence from the conservative systems found in statistical mechanics (Frisch, 1995).

More precisely, we say that molecular effects cease to play an *explicit* role in the governing equations for $\bar{\mathbf{u}}$ and $\bar{b}$ if their determination in the limit $\text{Re} \stackrel{\text{def}}{=} w_m r_m / \nu \rightarrow \infty$ whilst the Prandtl number $\text{Pr} \stackrel{\text{def}}{=} \nu / \kappa = O(1)$, is a regular perturbation problem (see, e.g., Van Dyke, 1968):

$$\nabla \cdot \bar{\mathbf{u}} = 0, \quad (2.14)$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) = -\nabla \bar{p} - \nabla \cdot (\overline{\mathbf{u}' \otimes \mathbf{u}'} + \bar{\mathbf{b}}, \quad (2.15)$$

$$\frac{\partial \bar{b}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}} \bar{b}) = -\nabla \cdot (\overline{\mathbf{u}' b'}), \quad (2.16)$$

is valid uniformly. Here w_m and r_m are the characteristic velocity and length scales of the flow. In contrast to wall-bounded flows, in which the passage to infinite Reynolds number results in a singular perturbation problem, the limiting behaviour of free-shear flows is valid uniformly over their radial extent.

2.2.2 Axisymmetric jets and plumes

Hereafter, unless stated otherwise, we employ a cylindrical coordinate system (r, ϕ, z) in the radial, azimuthal and longitudinal directions, respectively, for which the longitudinal axis Oz is coincident with the axis of the flow, as indicated in figure 2.2. Thus, we express the mean velocity $\bar{\mathbf{u}}$ in terms of radial, longitudinal and azimuthal components as $(\bar{u}, \bar{v}, \bar{w})$.

Deviations from axisymmetry in the near field (close to the source) become arbitrarily small further from the source and it is appropriate to regard the statistics of the flow as homogeneous in the azimuthal dimension ϕ , which means that $\partial_\phi \bar{\chi} = 0$ for all ensemble averages $\bar{\chi}$. In the remaining longitudinal direction and radial direction the flow is inhomogeneous. Our focus is on the behaviour of the longitudinal velocity $\bar{w}$ and buoyancy $\bar{b}$ or passive scalar concentration $\bar{c}$. Using (2.14)-(2.16), it is therefore appropriate to consider conservation equations for vol-

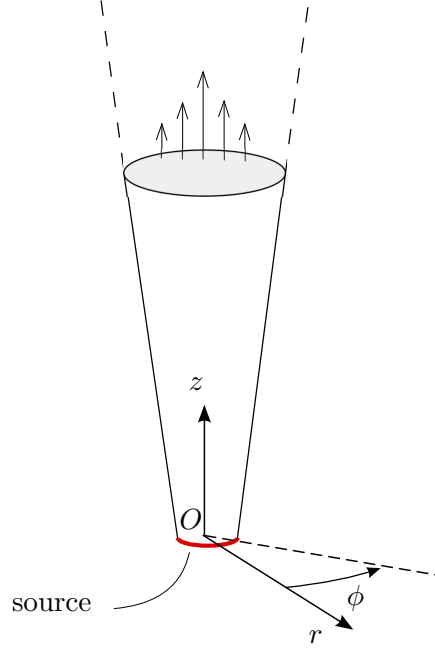


FIGURE 2.2. The cylindrical coordinate system used to describe axisymmetric free-shear flows.

ume, the longitudinal specific momentum (hereafter referred to as momentum for brevity) and buoyancy:

$$\frac{1}{r} \frac{\partial(r\bar{u})}{\partial r} + \frac{\partial\bar{w}}{\partial z} = 0, \quad (2.17)$$

$$\frac{\partial\bar{w}}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(r\bar{u}\bar{w} + r\overline{u'w'}) + \frac{\partial}{\partial z}(\bar{w}^2 + \overline{w'^2}) = -\frac{\partial\bar{p}}{\partial z} + \bar{b}, \quad (2.18)$$

$$\frac{\partial\bar{b}}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(r\bar{u}\bar{b} + r\overline{u'b'}) + \frac{\partial}{\partial z}(\bar{w}\bar{b} + \overline{w'b'}) = 0. \quad (2.19)$$

We denote the system (2.17)-(2.19) as $L_4(\bar{u}, \bar{w}, \bar{b} | \bar{p}, \overline{u'w'}, \dots)$ to highlight the fact that assumptions are required for $\bar{p}, \overline{u'w'}$ etc. for closure[‡]. We regard L_4 as a subset of L_3 , due to the fact that to obtain the former we neglect viscous effects in the mean flow and assume axisymmetry.

[‡]Whilst it is possible to eliminate $\bar{p}$ from (2.18) using equations for the conservation of lateral momentum (Hussein *et al.*, 1994), doing so introduces additional unknowns. For conciseness, we retain $\bar{p}$ as an unknown quantity.

2.2.3 The statistically steady state

In practice, temporal variation in source conditions and atmospheric conditions results in flows being statistically unsteady. However, at timescales much smaller than the timescale characterising an unsteady forcing, yet much larger than the timescale associated with turbulence, the equations can be regarded as statistically steady, or quasi-steady (Woods, 2010). In statistically-steady jets and plumes an ensemble-averaged quantity is independent of time, and equations (2.18) and (2.19) become

$$\frac{1}{r} \frac{\partial}{\partial r} (r \bar{u} \bar{w} + r \overline{u'w'}) + \frac{\partial}{\partial z} (\bar{w}^2 + \overline{w'^2}) = -\frac{\partial \bar{p}}{\partial z} + \bar{b}, \quad (2.20)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r \bar{u} \bar{b} + r \overline{u'b'}) + \frac{\partial}{\partial z} (\bar{w} \bar{b} + \overline{w'b'}) = 0. \quad (2.21)$$

With the diagnostic continuity relation (2.17), equations (2.20) and (2.21) constitute a system $L_5 \subset L_4$ governing the behaviour of statistically-steady axisymmetric jets and plumes.

2.2.4 Boundary layer theory

In analogy with Prandtl's (1904) treatment of laminar boundary layers, the equations governing *slender* free-turbulent boundary layers can be simplified significantly. One assumes that the flow is slender such that the ratio of radial and longitudinal length scales r_m/L is much less than unity. With this assumption, the continuity equation implies that the characteristic radial velocity is $O(w_m r_m/L)$. Noting that $\partial_z \sim 1/L$ and $\partial_r \sim 1/r_m$ the steady-state system L_5 becomes (Tennekes & Lumley, 1972)

$$\frac{1}{r} \frac{\partial(r\bar{u})}{\partial r} + \frac{\partial \bar{w}}{\partial z} = 0, \quad (2.22)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r \bar{u} \bar{w} + r \overline{u'w'}) + \frac{\partial(\bar{w}^2)}{\partial z} = \bar{b}, \quad (2.23)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r \bar{u} \bar{b} + r \overline{u'b'}) + \frac{\partial(\bar{w} \bar{b})}{\partial z} = 0, \quad (2.24)$$

which we will refer to as the turbulent boundary layer equations L_6 . Use of an eddy-viscosity closure of the form $\overline{u'w'} = -\nu_T \partial_r \bar{w}$ renders (2.23) equivalent to the laminar system that Prandtl (1904) considered (provided that ν_T is assumed to be constant). However, whilst laminar boundary layers become arbitrarily slender in the limit $\text{Re} \rightarrow \infty$, $\nu_T \sim w_m r_m$ is independent of ν for high Re , hence turbulent boundary layers approach a finite rate-of-spread (George, 1989).

2.3 Self-similarity

The concept of self-similarity is fundamental to free-shear flows (Townsend, 1956, provides many examples). Indeed, it was on self-similarity that the foundational models of jets (Tollmien, 1926) and plumes (Zeldovich, 1937) were based. Self-similarity is found in physical problems in which there exists a large separation of scales. A simple example is provided by the gravitational potential induced by a solid of typical length scale r_s . At distances $z \gg r_s$ the solid acts like a point mass and the potential $V(z) \sim 1/z$ is uniform over spherical surfaces around the solid. When $z/r_s = O(1)$, on the other hand, the symmetry is broken, and one must integrate the potential induced by each part of the solid.

An analogous situation is found in free-shear flows at distances $z \gg r_s$, where r_s is the source radius and the jet or plume becomes independent of z/r_s . Consequently, one finds that the radial dependence of all quantities in a jet or a plume at relatively large distances from its source are independent of z and have a ‘self-similar’ form. An example of self-similarity is the radial profiles of the mean longitudinal velocity at different longitudinal locations in a plume, displayed in figure 2.3. More precisely, for a given variable φ , one can form the dimensionless relation

$$\frac{\varphi}{Az^n} = f\left(\eta, \frac{z}{r_s}\right), \quad \text{where} \quad \eta \stackrel{\text{def}}{=} \frac{r}{r_s(z)}, \quad (2.25)$$

for a constant A , which typically corresponds to a conserved quantity of the system. As $z/r_s \rightarrow \infty$ one supposes that f tends to a finite limit:

$$\lim_{z/r_s \rightarrow \infty} \frac{\varphi}{Az^n} = f(\eta). \quad (2.26)$$

That a power-law dependence on z emerges in (2.26) follows from the absence of a distinguished length scale in the problem. For a given function g of z we require that $g(z_1)/g(z_0) = g(z_1/z_0)$ and find that g must be a power-law monomial $g(z) = z^n$ (see e.g. Barenblatt, 1996, for a discussion of the equivalent condition placed on the dimensions of any physical quantity). Whether the similarity function $f(r/r_m)$ is universal in the sense that it is identical for all jets or all plumes, is a matter that has been debated (George, 1989). The physical interpretation of (2.26) is that the flow has reached a state of equilibrium in which the balances in the governing equations are independent of z .

Under the assumption of self-similarity, radial integrals such as

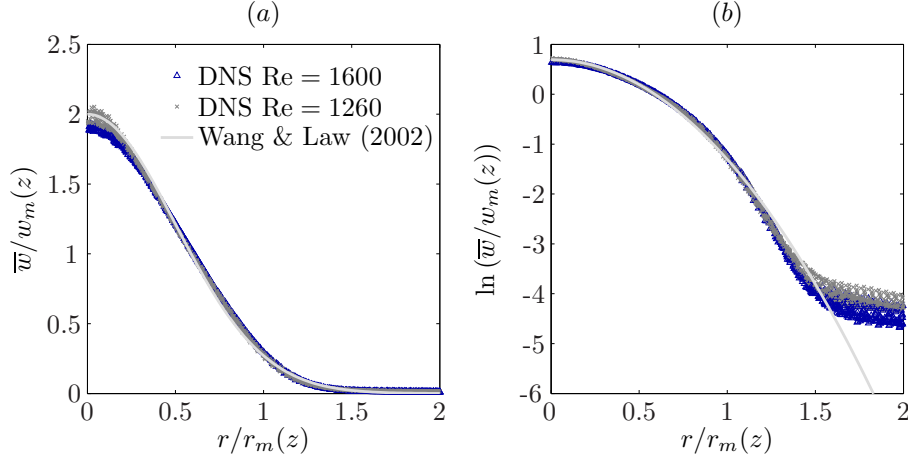


FIGURE 2.3. Dimensionless mean longitudinal velocity in an axisymmetric plume, where $w_m(z)$ and $r_m(z)$ are characteristic velocity and length scales at a distance z from the source. We compare DNS results to the experimental data of Wang & Law (2002). (a) Linear scale; (b) logarithmic scale, illustrating that the process is self-similar for $\bar{w}/w_m > \exp(-4)$.

$$\frac{1}{r_m^2} \int_0^\infty f(\eta) r dr, \quad (2.27)$$

which we assign to dimensionless variables, are constant. Hence, self-similarity implies that radial integrals of all quantities in the governing equations (2.22)-(2.24) are known, to within arbitrary constants (i.e. functionals of the similarity profiles) that are independent of z . In the case of a jet, the assumption of self-similarity implies that the jet radius $r_m \propto z$ and the characteristic velocity $w_m \propto M_m/z$, where

$$M_m \stackrel{\text{def}}{=} 2 \int_0^\infty \bar{w}^2 r dr, \quad (2.28)$$

is the conserved momentum flux of the jet. Moreover, due to self-similarity, it follows that the longitudinal turbulent transport of momentum

$$M_f \stackrel{\text{def}}{=} 2 \int_0^\infty \bar{w}'^2 r dr \stackrel{\text{def}}{=} \beta_f M_m. \quad (2.29)$$

Profile coefficients, such as β_f , represent a particular assumption about the radial dependence of quantities in a free-shear flow. The present study considers how the profile coefficients affect the structure of the governing equations. In

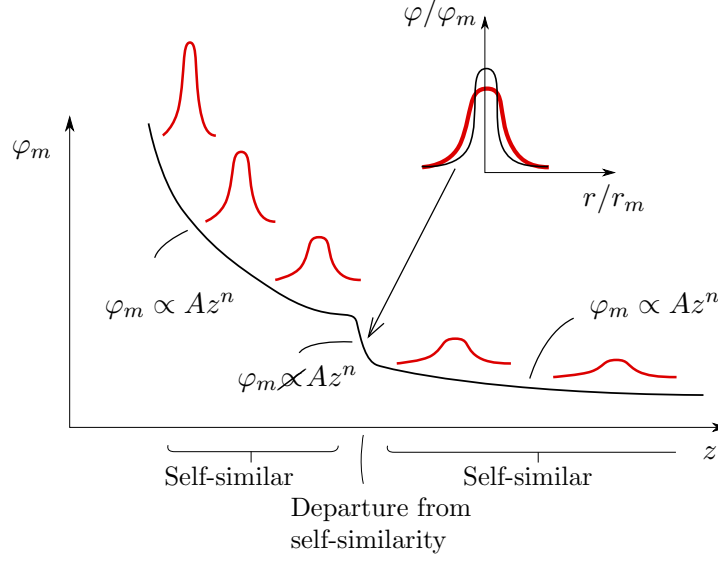


FIGURE 2.4. Departure from self-similarity in the vicinity of a deviation from a longitudinal power-law scaling.

the case of a steady state we will see that they play an insignificant role. In an unsteady state, on the other hand, the profile coefficients determine the structure of the governing equations in a fundamental way. In other words, the unsteady equations depend, non-trivially, on the assumptions one makes about information that is lost in the process of integration.

Higher-order effects result from a local departure from a power-law scaling in a jet or a plume due, for example, to unsteadiness (see figure 2.4). Such a departure, or perturbation, necessarily implies a corresponding departure from self-similarity. In the case of the transport of a passive scalar c , the perturbation might manifest itself in $\bar{c}$ or $\overline{w'c'}$ or both, in order to satisfy the governing equations. The consideration of departures from self-similarity is the subject of dispersion theory (Taylor, 1953), and will be incorporated into our model for unsteady jets in chapters 5 and 7.

2.4 Integral models

2.4.1 Classical quasi-steady plume theory

In jets and plumes one is often interested in radial integrals, as illustrated in figure 2.5. In this regard, integrals of the volume flux, momentum flux and buoyancy

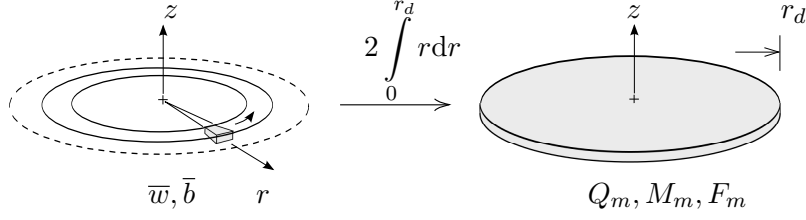


FIGURE 2.5. Integral quantities obtained from an *axisymmetric* free-shear flow. Here r_d denotes the upper limit of integration, which is assumed to be infinity in this chapter.

flux, defined according to

$$Q_m \stackrel{\text{def}}{=} 2 \int_0^\infty \bar{w} r dr, \quad M_m \stackrel{\text{def}}{=} 2 \int_0^\infty \bar{w}^2 r dr, \quad F_m \stackrel{\text{def}}{=} 2 \int_0^\infty \bar{w} \bar{b} r dr, \quad (2.30)$$

respectively, are particularly relevant. For a plume, self-similarity is consistent with the view that the plume is driven by a point source of buoyancy flux, which in an unstratified environment is a conserved quantity. Noting the dimensions $[Q_m] = L^3 T^{-1}$, $[M_m] = L^4 T^{-2}$ and $[F_m] = L^4 T^{-3}$, it follows that

$$Q_m \propto F_m^{1/3} z^{5/3}, \quad M_m \propto F_m^{2/3} z^{4/3}. \quad (2.31)$$

Jets, which are driven by a conserved flux of momentum M_m , have a volume flux $Q_m \propto M_m^{1/2} z$. Thus, without considering physics beyond dimensional analysis, the longitudinal increase in the volume flux of jets and plumes indicates that they entrain fluid from their environments.

More generally, the classical integral model for plumes was proposed by Morton *et al.* (1956) and can be obtained by integrating (2.22)-(2.24) to give I_7 , the simplest model of plumes in our hierarchy:

$$\frac{dQ_m}{dz} = 2\alpha_0 M_m^{1/2}, \quad (2.32)$$

$$\frac{dM_m}{dz} = \frac{F_m Q_m}{M_m}, \quad (2.33)$$

$$\frac{dF_m}{dz} = 0, \quad (2.34)$$

where (2.32) constitutes the entrainment hypothesis of Taylor (1945), relating the radial velocity of fluid entering the plume to local scales via an entrainment coefficient α_0 :

$$\lim_{r \rightarrow \infty} (ur) = -\alpha_0 M_m^{1/2}. \quad (2.35)$$

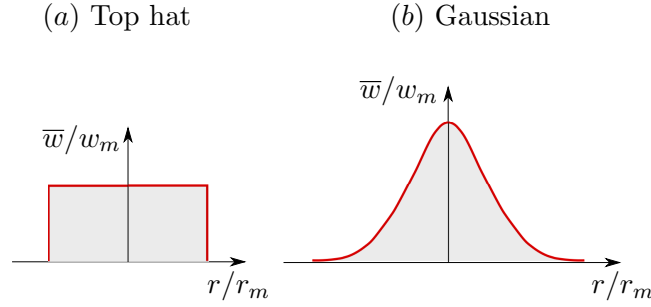


FIGURE 2.6. Top-hat and Gaussian descriptions of the mean longitudinal velocity $\bar{w}$ in a plume.

In spite of their simplicity, the formulation of (2.32)-(2.34) contains several details that are relevant to the present study. Based on the boundary layer equations (2.22)-(2.24), the classical plume equations neglect the longitudinal pressure gradient $\partial_z \bar{p}$, the turbulent momentum flux $\overline{w'^2}$ and the turbulent buoyancy flux $\overline{w'b'}$. Furthermore, by expressing the buoyancy integral as $F_m Q_m / M_m$ in (2.33), Morton *et al.* (1956) assumed that, in addition to being self-similar, profiles of $\bar{w}$ and $\bar{b}$ have the same relative width[§]. Since it was not necessarily assumed that profiles of higher-order statistics in the plume are self-similar, the plume model of Morton *et al.* (1956) is based on quasi-similarity (Morton, 1971) or, equivalently, partial self-preservation (George, 1989). Indeed, whilst (2.32)-(2.34) admit an exact (fully) self-similar solution corresponding to a point source of buoyancy flux F_s

$$\left. \begin{aligned} Q_m &= \frac{6\alpha_0}{5} \left(\frac{9\alpha_0}{10} \right)^{1/3} F_s^{1/3} z^{5/3}, \\ M_m &= \left(\frac{9\alpha_0}{10} \right)^{2/3} F_s^{2/3} z^{4/3}, \\ F_m &= F_s, \end{aligned} \right\} \quad (2.36)$$

the governing equations (2.32)-(2.34) can be integrated numerically or approximated in the vicinity of a finite, rather than infinitesimal, source (see, e.g. van den Bremer & Hunt, 2010, for a general description). In the vicinity of a finite source, where the flow might be regarded as partially self-similar, one sees a departure from the far-field power-law scaling that is required for full self-similarity at all orders of statistics (see George, 1989, for further details).

[§]Although Morton *et al.* (1956) worked with characteristic scales for velocity, buoyancy and width, in place of the integrals Q_m , M_m and F_m , their formulation is equivalent to (2.32)-(2.34)

2.4.2 Top-hat unsteady plume theory

A generalisation of the quasi-steady plume model of Morton *et al.* (1956) was developed by Scase *et al.* (2006b), who considered statistically unsteady plumes governed by the equations

$$\frac{\partial}{\partial t} \left(\frac{Q_m^2}{M_m} \right) + \frac{\partial Q_m}{\partial z} = 2\alpha_0 M_m^{1/2}, \quad (2.37)$$

$$\frac{\partial M_m}{\partial z} = \frac{F_m Q_m}{M_m}, \quad (2.38)$$

$$\frac{\partial}{\partial t} \left(\frac{F_m Q_m}{M_m} \right) + \frac{\partial F_m}{\partial z} = 0. \quad (2.39)$$

The rigorous derivation of (2.37)-(2.39) relies on the assumption that the longitudinal velocity in the plume has a top-hat radial distribution, as indicated in figure 2.6(a). As mentioned in §2.3 and §1.3.2, with knowledge limited to the volume flux Q_m and momentum flux M_m of the plume, one can conceive of an infinite number of plausible radial dependencies. Indeed, the *steady* plume equations (2.32)-(2.34) are consistent with the view that the velocity profile has a top-hat form (figure 2.6a) or a Gaussian form (figure 2.6b)[¶]. In contrast, the *unsteady* plume equations (2.37)-(2.39) are only valid for top-hat velocity profiles, and are therefore restricted by a stringent assumption about the underlying physics. Notice, in particular, that the entrainment coefficient is the only free parameter appearing in (2.37)-(2.39). These matters will be discussed at length in chapters 6 and 7.

In spite of their close correspondence with the steady plume equations and rigorous derivation, Scase & Hewitt (2012) have shown that the system (2.37)-(2.39) is ill-posed. We will demonstrate in this thesis that unlike the steady plume equations, the unsteady top-hat plume equations are missing an important piece of physics. This thesis will pin-point the missing physics and demonstrate that (2.37)-(2.39) is a degenerate case amongst a family of well-posed unsteady plume models, the latter being labelled as I_4 in figure 2.7.

[¶]Classically, one employs a different value of the entrainment coefficient when dealing with plumes, whose velocity is assumed to have Gaussian, rather than top-hat, profile (see, e.g. Carazzo *et al.*, 2006). However, in steady plumes the distinction is superficial because it relates to the choice of the radial scale associated with the flow, rather than the physical consequences of a particular velocity profile. Indeed, the distinction only manifests itself if one chooses to express the plume equations explicitly in terms of a radial, velocity, and buoyancy scale (see Morton *et al.*, 1956). In contrast, for a given α_0 , (2.32)-(2.34) are valid, regardless of the assumed form of the velocity profile.

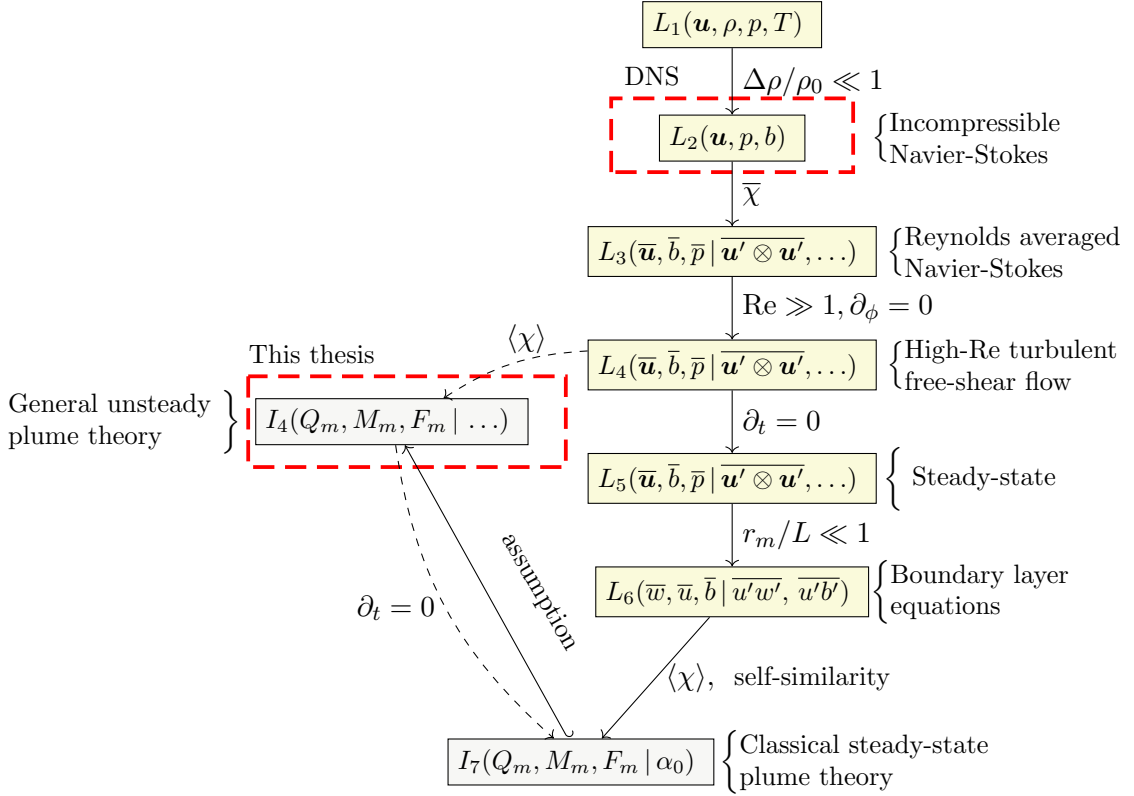


FIGURE 2.7. Hierarchy of governing equations on which classical plume theory (I_7) is based. The framework for unsteady jets and plumes proposed in this thesis (I_4) retains information that is normally lost on integration of the Reynolds averaged equations, and therefore generalises previous approaches.

2.5 A hierarchy of models for unsteady jets and plumes

The aim of this section was to describe the relationship of the bulk steady-state plume equations (2.32)-(2.34) (I_7) to the full Navier-Stokes equations (L_1). Each model in the hierarchy illustrated in figure 2.7 is a simplification of its predecessor. Consequently, whilst a model lower in the hierarchy is easier to solve, it necessarily admits fewer solutions or modes and/or makes assumptions about the missing information. One of the central findings of this thesis is that by invoking a particular set of assumptions, existing unsteady plume models (i.e. a subset of I_4) implicitly disregard a large number of feasible solutions. Worse still, as we demonstrate in chapters 7 and 8 (see also Scase & Hewitt, 2012), the formulation adopted in several existing models is ill-posed and therefore unfeasible.

Scaling algebraically with the Reynolds number, the difference in the com-

plexity or degrees of freedom between the descriptions depicted in figure 2.7 is arbitrarily large. At each stage in the process of simplification one relies on either asymptotic limits (e.g. $\text{Re} \rightarrow \infty$) and/or closure of unknown terms using the primary unknowns (e.g. $\overline{u'w'} = -\nu_T \partial_r \bar{w}$). In this regard the steady-state plume equations, which are a relation $I_7(Q_m, M_m, F_m | \alpha_0)$ between just three dependent variables and involve only a single free parameter, are relatively simple. The effectiveness of the steady plume equations stems from the fact that they rely on very few approximations further to the assumption of an asymptotically small dependence on the area of the source. Indeed, we will show in chapter 8 that even the incorporation of longitudinal turbulence transport makes no difference to the *structure* of the steady plume equations.

In our discussion of the governing equations, and indeed the development of plume theory, the unsteady plume equations (2.37)-(2.39) are slightly anomalous. Rather than simplifying and inheriting selected properties from a more general formulation, such as the Reynolds-averaged equations, they can be viewed as an extension of the steady plume equations^{||}. The situation is illustrated in figure 2.7, where the arrow from I_7 to the set of possible unsteady plume equations I_4 , represents a particular assumption. The route to the unsteady plume equations taken in this thesis is different. We integrate the Reynolds-averaged equations ($L_4 \rightarrow I_4$) before making additional assumptions. Consequently, we demonstrate that the assumptions made about radial distributions, turbulence and self-similarity have a profound effect on the structure of a generalised system of unsteady plume equations. Notably, the relatively large family of unsteady plume models that we will obtain map *onto* the classical steady-state plume equations I_7 . It is therefore not possible to understand fully the dynamics of unsteady plumes by extrapolating from the steady state plume equations.

The resulting framework can be used in a diagnostic capacity to inspect results from DNS of the full incompressible Navier-Stokes equations (see L_2 in figure 2.7). In a prognostic capacity we will use the framework to develop and understand models for passive scalar transport (chapter 5) and energy transport (chapter 7). Chapter 8 will combine energy transport and scalar transport to understand and model the behaviour of unsteady plumes, as illustrated in figure 2.8.

^{||}We note that the unsteady plume equations developed by Scase *et al.* (2006b) were derived rigorously from local conservation equations. However, the assumption of top-hat distributions means that they only incorporate information from the local (i.e. dependent on r and z) equations that is evident in the steady plume equations. Thus, the unsteady plume model of Scase *et al.* (2006b) can be regarded as the simplest or canonical extension of the steady plume equations.

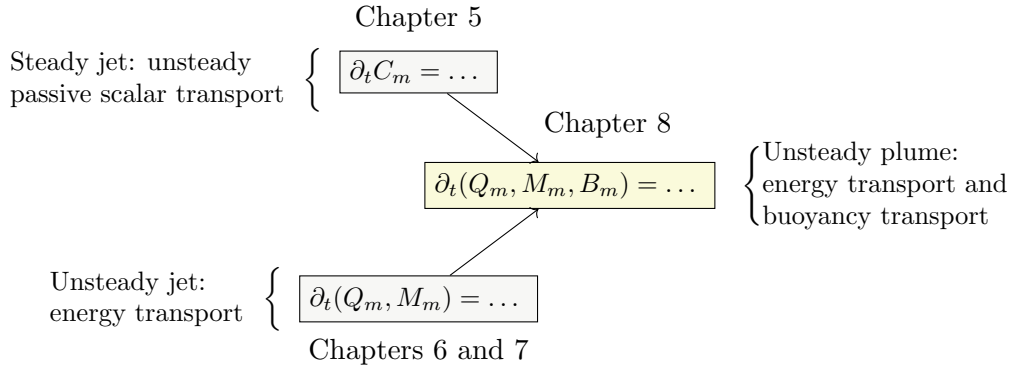


FIGURE 2.8. Unsteady plumes incorporate the physics of passive scalar transport in addition to the coupling between momentum and energy transport found in unsteady jets. The integral scalar concentration C_m and the integral buoyancy B_m will be defined precisely in chapters 5 and 8, respectively.

Chapter 3

Methodology

This chapter focuses on the numerical methods that were used to obtain the data that will be described in subsequent chapters. Some of the material contained in this chapter was published in appendix A of Craske & van Reeuwijk (2015*b*) and is reproduced here with permission.

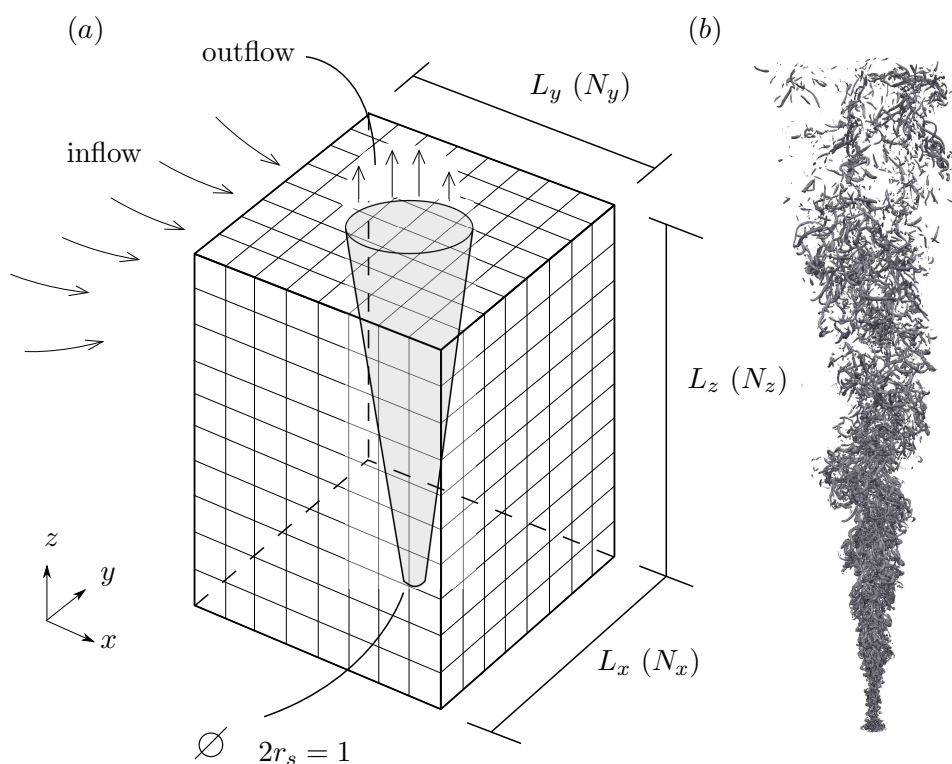


FIGURE 3.1. (a) Computational domain consisting of a rectangular cuboid of dimensions $L_x \times L_y \times L_z$, discretised using $N_x \times N_y \times N_z$ computational control volumes. (b) Isosurfaces of the second invariant of $\nabla \mathbf{u}$ in a turbulent plume.

3.1 Direct numerical simulation

Direct numerical simulation (DNS) of the incompressible Navier-Stokes equations involves a discrete approximation of the relations

$$\nabla \cdot \mathbf{u} = 0, \quad (3.1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = -\nabla p + \frac{\vartheta}{T_0} \mathbf{g} + \nu \nabla^2 \mathbf{u}, \quad (3.2)$$

$$\frac{\partial \vartheta}{\partial t} + \nabla \cdot (\mathbf{u} \vartheta) = \kappa \nabla^2 \vartheta, \quad (3.3)$$

describing the conservation of volume, momentum and relative temperature* $\vartheta \stackrel{\text{def}}{=} T - T_0$. Without loss of generality, we assume that the constant coefficient of thermal expansion is equal to T_0^{-1} . The distinguishing feature of DNS, in contrast to methods such as large-eddy simulation or Reynolds-averaged modelling, is that the dynamically relevant scales in the flow are fully resolved. In order to resolve the small scales responsible for viscous dissipation or thermal diffusion, one must employ a sufficiently small grid spacing $\Delta \mathbf{x}$. The challenge facing numericists is that the smallest ‘Kolmogorov’ scale $\eta \sim Re^{-3/4}$, and therefore the total number of computational cells N^3 that one requires in a three-dimensional simulation scales according to $N^3 \sim Re^{9/4}$, at the very least. As we will demonstrate in the following section, in practice $N^3 \sim Re^{9/4}$ turns out to be an optimistic estimation.

3.2 Description of the code

The name of the code we employ is SPARKLE and it is based on a finite volume discretisation of the governing equations. SPARKLE has been used extensively in the simulation of turbulent thermal convection (for details see van Reeuwijk, 2007, and references therein), shear-driven turbulence (Jonker *et al.*, 2013) and problems involving the small-scale aspects of turbulent entrainment (van Reeuwijk & Holzner, 2014). The original version of SPARKLE was second-order accurate and provided only limited means of simulating open boundary conditions for free-shear flows. As part of the present thesis, SPARKLE was upgraded to fourth-order accuracy and equipped with special inflow and outflow boundary conditions that are suitable for the simulation of jets and plumes in unconfined environments (see §3.5 and appendix A).

SPARKLE employs the symmetry-preserving method of Verstappen & Veldman (2003), which conserves mass, momentum and energy, and is accurate to fourth order. The challenge of obtaining fully conservative schemes at orders higher than

*For comparison with the incompressible Navier-Stokes equations in the previous chapter, note that the buoyancy $b \stackrel{\text{def}}{=} g\vartheta/T_0$ in an *unstratified* environment. In practice, we employ ϑ as a dependent variable in the simulations, rather than b .

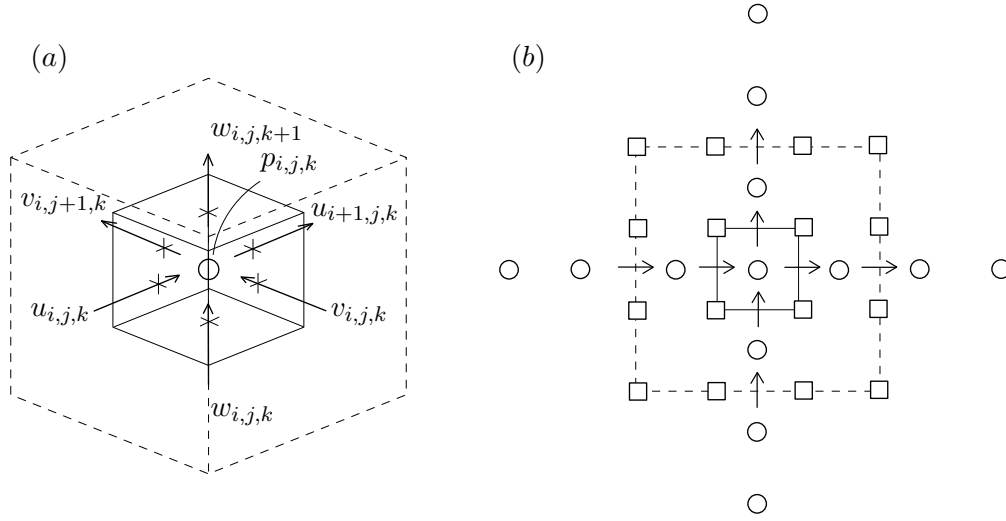


FIGURE 3.2. Staggered arrangement of variables relative to a single computational cell, where $\rightarrow$ denotes the location of a flux, $\circ$ the location of the transported quantity and, in the case for which $\circ$ represents a velocity, $\square$ represents the location of lateral transporting velocities. (a) Control volume; (b) typical two-dimensional section.

two is formidable, and as such has received much attention from numericists (see e.g., Morinishi *et al.*, 1998). To achieve conservation and fourth-order accuracy, the scheme of Verstappen & Veldman (2003) adopts a finite volume approach over two nested control volumes. In three dimensions the volume of the larger control volume is 3^3 times that of the smaller (see figure 3.2a). Standard central differencing is applied to the fluxes at the boundary of each control volume independently. These operations are then combined, using Richardson extrapolation, to eliminate their leading, second-order, error. Each control volume is centred on the location of the transported variable and fluxes are defined on the centre of each face (see figure 3.2b). The staggered arrangement of variables over the grid means that transporting velocities and their transported quantities need to be interpolated to the centre of each control volume face, before their product can be taken to obtain a flux. In the case of the transport of scalar terms, which are defined on the centre of each cell, the velocities coincide with the point at which the flux is defined and the only interpolation that is required is of the scalar variable. Figure 3.2(b) shows a two-dimensional section of a typical control volume. We use $\rightarrow$ to denote the location of a flux and $\circ$ to denote the location of the transported quantity. In the case for which $\circ$ represents a velocity, $\square$ represents the location of lateral transporting velocities.

3.2.1 Operators

Using the approach described above, the discrete approximation to the Navier-Stokes equations is

$$\left. \begin{aligned} D\mathbf{u} &= 0, \\ \frac{\delta \mathbf{u}}{\delta t} + N(\mathbf{u})\mathbf{u} &= -\mathbf{G}p + \frac{I_3 \vartheta}{T_0} \mathbf{g} + \nu L\mathbf{u}, \\ \frac{\delta \vartheta}{\delta t} + N(\mathbf{u})\vartheta &= \kappa L\vartheta. \end{aligned} \right\} \quad (3.4)$$

Here, capitalised letters (with the exception of T_0) denote differencing operators, $\delta/\delta t$ is a discrete approximation of a time derivative, p is pressure and $\mathbf{u}$ is a vector of velocities (u, v, w) with respect to a Cartesian coordinate system (x, y, z) . The differencing approach designed by Verstappen & Veldman (2003) ensures that on periodic domains the inner products

$$\langle D\mathbf{u}, \phi \rangle = -\langle \mathbf{u}, \mathbf{G}\phi \rangle, \quad \langle N(\mathbf{u})\mathbf{v}, \mathbf{w} \rangle = -\langle \mathbf{v}, N(\mathbf{u})\mathbf{w} \rangle, \quad (3.5a, b)$$

for a given scalar quantity ϕ , are satisfied, which is consistent with the skew-symmetry one finds in the advection and gradient operators on a continuum, and in turn means that the total energy on a periodic domain is changed only as a result of external forcing and/or viscous dissipation.

To describe the spatial differencing operators, we focus on the case of a grid of uniform spacing $\Delta x, \Delta y$ and Δz , which is consistent with the grids that were employed for the simulations reported in this thesis. In general, however, the code supports a non-uniform grid spacing in z . We will use Ω to denote the effective volume that appears when conservation equations for the large and small control volumes are combined:

$$\Omega \stackrel{\text{def}}{=} (3^5 - 3^3) \Delta x \Delta y \Delta z = 216 \Delta x \Delta y \Delta z. \quad (3.6)$$

With these definitions, to take an example, the third component G_3 of the gradient operator $\mathbf{G}$ results in

$$G_3 p_{i,j,k+1/2} = \frac{\Delta x \Delta y}{\Omega} \left[\underbrace{3^5 (p_{i,j,k+1} - p_{i,j,k})}_{\text{Inner}} - \underbrace{3^2 (p_{i,j,k+2} - p_{i,j,k-1})}_{\text{Outer}} \right], \quad (3.7)$$

in which differences corresponding to the inner (small) and outer (large) control volumes have been indicated. In (3.7) the subscript $k + 1/2$ refers to the fact that

whilst p is defined on the cell centres, G_3p is defined on the horizontal faces of control volumes, i.e. coincident with the location of w . The first two components of $\mathbf{G}$ are obtained in a similar manner.

The divergence operator D is defined according to

$$\begin{aligned} D\mathbf{u}_{(i,j,k)+1/2} = & \frac{\Delta y \Delta z}{\Omega} \left[3^5 (u_{i+1,j,k} - u_{i,j,k}) - 3^2 (u_{i+2,j,k} - u_{i-1,j,k}) \right] + \\ & \frac{\Delta z \Delta x}{\Omega} \left[3^5 (v_{i,j+1,k} - v_{i,j,k}) - 3^2 (v_{i,j+2,k} - v_{i,j-1,k}) \right] + \\ & \frac{\Delta x \Delta y}{\Omega} \left[3^5 (w_{i,j,k+1} - w_{i,j,k}) - 3^2 (w_{i,j,k+2} - w_{i,j,k-1}) \right]. \end{aligned} \quad (3.8)$$

The diffusion operator is found from D and $\mathbf{G}$, according to $L \stackrel{\text{def}}{=} D\mathbf{G}$, which, for example, results in

$$\begin{aligned} L\vartheta_{i,j,k} = & \frac{\Delta x^2 \Delta y^2}{\Omega} \left[3^{10} \left(\frac{\vartheta_{i,j,k+1} - \vartheta_{i,j,k}}{\Omega} - \frac{\vartheta_{i,j,k} - \vartheta_{i,j,k-1}}{\Omega} \right) \right. \\ & - 3^7 \left(\frac{\vartheta_{i,j,k+2} - \vartheta_{i,j,k-1}}{\Omega} - \frac{\vartheta_{i,j,k+1} - \vartheta_{i,j,k-2}}{\Omega} \right) \\ & - 3^7 \left(\frac{\vartheta_{i,j,k+2} - \vartheta_{i,j,k+1}}{\Omega} - \frac{\vartheta_{i,j,k-1} - \vartheta_{i,j,k-2}}{\Omega} \right) \\ & \left. + 3^4 \left(\frac{\vartheta_{i,j,k+3} - \vartheta_{i,j,k}}{\Omega} - \frac{\vartheta_{i,j,k} - \vartheta_{i,j,k-3}}{\Omega} \right) \right] + \frac{\Delta y^2 \Delta z^2}{\Omega} \left[\dots \right] + \dots \end{aligned} \quad (3.9)$$

The discrete Laplacian of the velocity: $L\mathbf{u}$, has the same form as (3.9).

The advection of the cell-centred scalar quantity ϑ is defined by interpolating ϑ to the location of the advecting velocities (see figure 3.2). For advection in the z -direction this results in

$$\begin{aligned} N(w) \vartheta_{i,j,k} = & \frac{\Delta x \Delta y}{\Omega} \left\{ \dots \right. \\ & 3^5 \left[w_{i,j,k+1} \left(\frac{\vartheta_{i,j,k} + \vartheta_{i,j,k+1}}{2} \right) - w_{i,j,k} \left(\frac{\vartheta_{i,j,k-1} + \vartheta_{i,j,k}}{2} \right) \right] \\ & \left. - 3^2 \left[w_{i,j,k+2} \left(\frac{\vartheta_{i,j,k} + \vartheta_{i,j,k+3}}{2} \right) - w_{i,j,k-1} \left(\frac{\vartheta_{i,j,k-3} + \vartheta_{i,j,k}}{2} \right) \right] \right\}. \end{aligned} \quad (3.10)$$

In contrast, when applying the advection operator to a velocity, it is necessary to interpolate both the advecting and advected components of velocity to the centre of each control volume face. For example, the longitudinal transport of horizontal

momentum is approximated according to

$$N(w) u_{i,j,k} = \frac{\Delta x \Delta y}{\Omega} \left\{ \dots \right. \\ \left. 3^5 \left[I_1 w_{i-1/2,j,k+1} \left(\frac{u_{i,j,k} + u_{i,j,k+1}}{2} \right) - I_1 w_{i-1/2,j,k} \left(\frac{u_{i,j,k-1} + u_{i,j,k}}{2} \right) \right] \right. \\ \left. - 3^2 \left[I_1 w_{i-1/2,j,k+2} \left(\frac{u_{i,j,k} + u_{i,j,k+3}}{2} \right) - I_1 w_{i-1/2,j,k-1} \left(\frac{u_{i,j,k-3} + u_{i,j,k}}{2} \right) \right] \right\}. \quad (3.11)$$

Here I_1 is a fourth-order interpolation operator over the first spatial dimension:

$$I_1 w_{i-1/2,j,k} \stackrel{\text{def}}{=} \frac{9}{8} \left(\frac{w_{i,j,k} + w_{i-1,j,k}}{2} \right) - \frac{1}{8} \left(\frac{w_{i+1,j,k} + w_{i-2,j,k}}{2} \right). \quad (3.12)$$

Similarly, it is necessary to interpolate buoyancy in equation 3.2 to the location of $w_{i,j,k}$:

$$I_3 \vartheta_{i,j,k-1/2} \stackrel{\text{def}}{=} \frac{9}{8} \left(\frac{\vartheta_{i,j,k} + \vartheta_{i,j,k-1}}{2} \right) - \frac{1}{8} \left(\frac{\vartheta_{i,j,k+1} + \vartheta_{i,j,k-2}}{2} \right). \quad (3.13)$$

For non-uniform grids the spatial discretisation has a similar form to the examples provided above. However, for conservation on non-uniform grids, care must be taken to ensure that it is the fluxes, e.g. $\Delta y_j \Delta z_k u_{i,j,k}$, that are interpolated, rather than the velocities, because for that case Δy_j and Δz_k vary over the domain. For further details see Verstappen & Veldman (2003).

To advance the solution for a variable ϕ in time, the code employs a third-order variable-time-step Adams-Bashforth scheme. To this end, the problem is regarded as an ordinary differential equation

$$\frac{d\phi}{dt} = f(t, \phi). \quad (3.14)$$

Third-order accuracy is obtained by utilising previous values of the right-hand side f :

$$\phi^{n+1} = \phi^n + \Delta t_n \sum_{i=0}^m \alpha_i f(t_{n-i}, \phi^{n-i}), \quad (3.15)$$

where ϕ^n denotes the values over the entire computational domain of variable ϕ at the time t_n . The coefficients α_i are determined by examining the Taylor series of each contribution $f(t_{n-i}, \phi^{n-i})$ and eliminating the leading-order error. To achieve

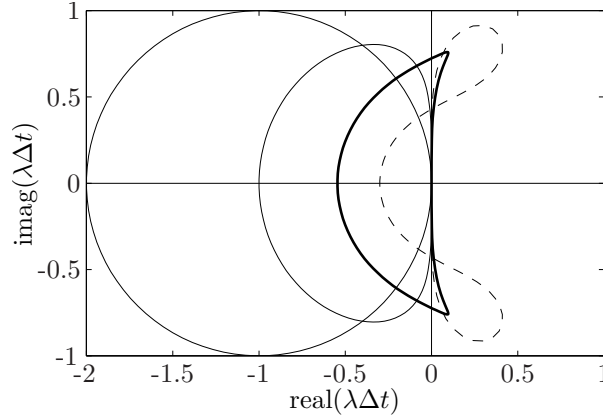


FIGURE 3.3. Stability domain for Adams-Bashforth time integration schemes. $O(\Delta t)$, $O(\Delta t^2)$ [—]; $O(\Delta t^3)$ [— · —]; $O(\Delta t^4)$ [---]. Here, λ is an eigenvalue that is (for advection-dominated problems) typically proportional to $W/\Delta z$, for a local maximum velocity W and corresponding grid spacing Δz .

third-order accuracy we use

$$\begin{aligned}\alpha_0 &= 1 + \frac{\Delta t_n(2\Delta t_{n-1} + \Delta t_{n-2})}{2\Delta t_{n-1}(\Delta t_{n-1} + \Delta t_{n-2})} + \frac{\Delta t_n^2}{3\Delta t_{n-1}(\Delta t_{n-1} + \Delta t_{n-2})}, \\ \alpha_1 &= -\frac{\Delta t_n(\Delta t_{n-1} + \Delta t_{n-2})}{2\Delta t_{n-1}\Delta t_{n-2}} - \frac{\Delta t_n^2}{3\Delta t_{n-1}\Delta t_{n-2}}, \\ \alpha_2 &= \frac{\Delta t_n\Delta t_{n-1}}{2\Delta t_{n-2}(\Delta t_{n-1} + \Delta t_{n-2})} + \frac{\Delta t_n^2}{3\Delta t_{n-2}(\Delta t_{n-1} + \Delta t_{n-2})},\end{aligned}\tag{3.16}$$

which, if the time increments are uniform, reduce to $\alpha_0 = 23/12$, $\alpha_1 = -4/3$ and $\alpha_2 = 5/12$. The stability properties of Adams-Bashforth schemes employing a uniform time step Δt are shown in figure 3.3. In contrast to first- $O(\Delta t)$ and second- $O(\Delta t^2)$ order schemes, the third-order scheme that we employ is stable for pure advection (the imaginary axis), although it results in a comparatively stringent constraint on the time step in general.

At every time step an intermediate velocity $\mathbf{u}^*$ is obtained by applying all forcing terms involving velocity, in addition to the pressure gradient of the previous time step:

$$\mathbf{u}^* - \mathbf{u}^n = \sum_{i=0}^2 \alpha_i \Delta t_n \mathbf{f}^{n-i} - \Delta t_n \mathbf{G} p^{n-1},\tag{3.17}$$

where $\mathbf{f}^{n-i}$ are explicit acceleration terms. Defining the current pressure to be $p^n = p^{n-1} + \pi^n/\Delta t_n$ results in the following correction to the velocity field

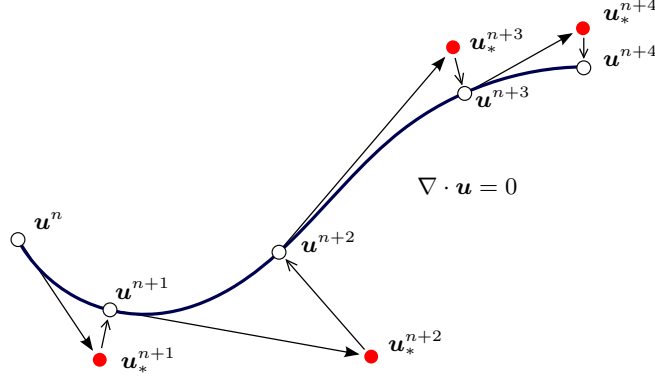


FIGURE 3.4. Correction of the intermediate velocity $\mathbf{u}_*$ using the pressure correction projection, such that $D\mathbf{u}^{n+1} = 0$.

$$\mathbf{u}^{n+1} - \mathbf{u}^* = -\mathbf{G}\pi^n. \quad (3.18)$$

Applying the divergence operator D to this equation and defining the projection to be such that $D\mathbf{u}^{n+1} = 0$, as illustrated in figure 3.4, results in the Poisson equation

$$L\pi^n = D\mathbf{u}^*. \quad (3.19)$$

The Poisson equation can be solved efficiently by decomposing the projection π into Fourier modes $\hat{\pi}_{mn}$, of wave numbers m and n , over the homogeneous directions x and y (see figure 3.5, which illustrates the $\hat{\pi}_{14}$ mode). For each Fourier mode the problem is reduced to a finite difference equation in the remaining z direction, which is solved by inverting a sparse matrix of the form

$$T = \begin{bmatrix} \times & \times & \times & \otimes & \times & \times & \times \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \times & \times & \times & \otimes & \times & \times & \times \end{bmatrix}. \quad (3.20)$$

Here, the six entries on each row denoted by $\times$ are grid-dependent numbers that are independent of the wave number mn , whereas the leading diagonal $\otimes$ is dependent on both the wave number and the grid, and must therefore be computed for each mode. The matrix T is augmented with an additional 6 rows, which determine the value of a set of ghost cells (see §3.2.3), and therefore the boundary conditions, at the top and bottom of the domain.

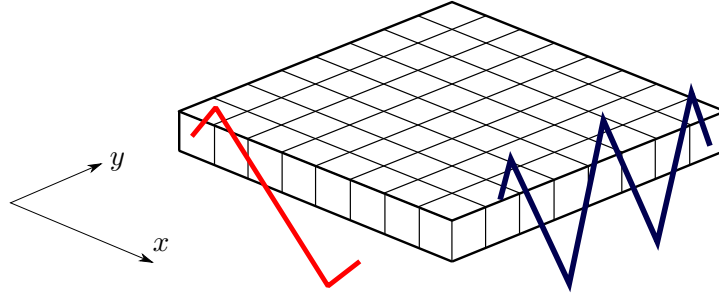


FIGURE 3.5. Two-dimensional pressure correction $\hat{\pi}_{14}$ using cosine modes. In the x -direction is the 1st mode and in the y -direction is the 4th mode.

3.2.2 Domain decomposition

The parallelisation we employ to solve (3.4) is a two-dimensional decomposition as illustrated in figure 3.6. Typically, each process handles a subset of the (y, z) -domain and the entire x -domain as indicated in figure 3.6(a). After performing Fourier transforms in the x -direction, Fourier transforms in the y -direction are performed by aligning the decomposition with Oy , as illustrated in figure 3.6(b). Finally, to solve the matrix equations for each of the two-dimensional lateral Fourier modes, the decomposition aligns itself with Oz .

Each process handles a set of interior cells surrounded by a set of ghost cells, as indicated in figure 3.7. Accounting for the number of ghost cells $N_g = 3$, the total number of cells handled by each process is equal to

$$N_{\text{ptotal}} = (N_x + 2N_g) (N'_y + 2N_g) (N'_z + 2N_g), \quad (3.21)$$

where N'_y and N'_z are the number of cells handled by each process in the y - and z -direction, respectively. During each time step the processes update their ghost cells using either information from specified boundary conditions (see left-hand-side of figure 3.7) or neighbouring processes (see right-hand-side of figure 3.7).

3.2.3 Boundary conditions

Like differential operators of a continuum, discrete differential operators are singular and require additional information in the form of boundary conditions to be invertible. A related issue is that the interior operators described in §3.2.1 involve relatively large stencils[†]. Consequently, in the vicinity of boundaries the stencils rely on information that lies outside the computational domain. Therefore, one can

[†]The Laplacian operation $L\phi_{i,j,k}$ has a stencil that spans seven points in each direction.

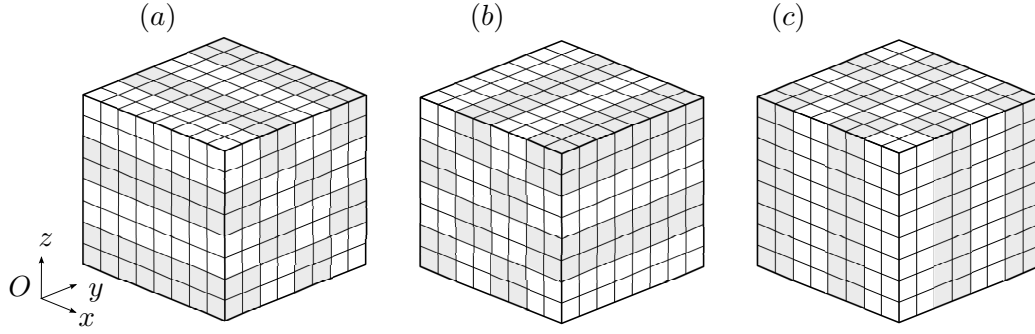


FIGURE 3.6. Two-dimensional domain decomposition. The decomposition aligns itself with Ox followed by Oy followed by Oz , in order to take Fourier transforms and solve the resulting matrix for the Poisson equation.

either provide ‘ghost cells’, in addition to boundary cells, as illustrated in figure 3.7 or modify the interior operators in the vicinity of a boundary. The two approaches are in fact equivalent, because in appealing to interior points to define the values of variables at the ghost cells, one is implicitly defining modified operators in the vicinity of a boundary. Nevertheless, we take the former approach and introduce ghost cells because it allows one to treat boundary closure schemes separately from the interior operators in a flexible manner.

The entire computational domain will be denoted $\hat{\Omega}$. The set of computational nodes can be partitioned into those nodes forming the interior of the domain Ω and those nodes forming the boundary of the domain $\partial\Omega$ (inclusive of ghost cells), thus $\hat{\Omega} = \Omega + \partial\Omega$. Looking in a wall-normal direction, the boundary closure problem is to find the values of m ghost and boundary values in terms of n interior values:

$$\begin{bmatrix} B_{\partial} & B \\ m \times m & m \times n \end{bmatrix} \begin{pmatrix} \phi_{\partial} \\ m \times 1 \\ \phi \\ n \times 1 \end{pmatrix} = \begin{pmatrix} r_{\partial} \\ m \times 1 \end{pmatrix}, \quad (3.22)$$

where B_{∂} operates on $\phi_{\partial} \in \partial\Omega$ and B operates on $\phi \in \Omega$. When different from zero, the right-hand-side values r_{∂} define inhomogeneous boundary conditions, and in general depend on space and time. For each boundary point (3.22) provides a system of m coupled equations to close the problem. Setting $\overline{B} \stackrel{\text{def}}{=} -B_{\partial}^{-1}B$, we find that

$$\phi_{\partial} = \overline{B}\phi + B_{\partial}^{-1}r_{\partial}. \quad (3.23)$$

The benefit of this approach is that it can be used for a low-order or high-order treatment of boundary values, in addition to inflow and outflow conditions (see

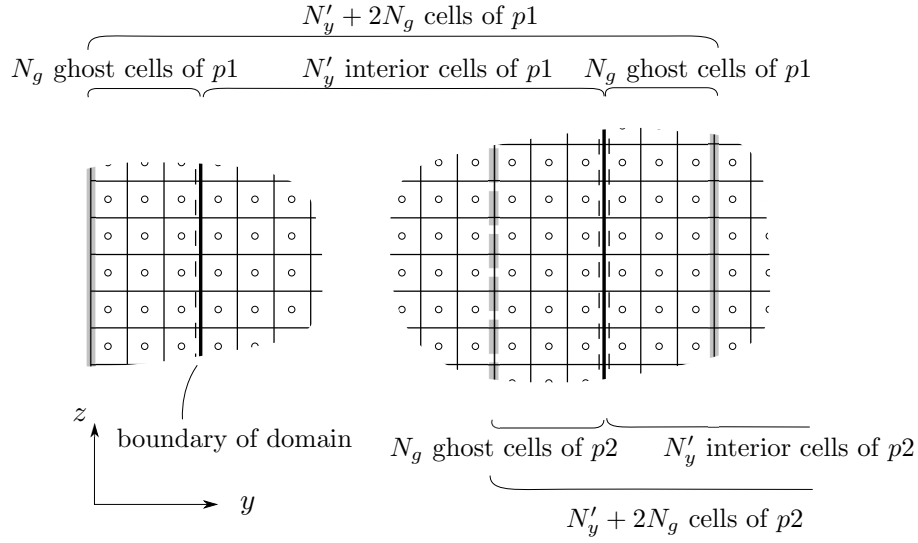


FIGURE 3.7. Distribution of cells within a process $p1$ and neighbouring process $p2$. Each process handles a set of interior cells bounded by $N_g = 3$ layers of ghost cells. At boundaries between processes the values at the ghost cells of one process are equal to those at the interior points of a neighbouring process.

§3.5), and is independent of the particular interior discretisation that is employed.

In practice, the entries of the boundary operator $[B_\partial B]$ are obtained from the inverse transpose of a matrix V , which is similar to a Vandermonde matrix:

$$V_{ij} \stackrel{\text{def}}{=} \frac{h_i^j}{j!}, \quad i, j \in \{0, \dots, m+n-1\}, \quad (3.24)$$

where h_i is the signed distance between a given point and the boundary. The matrix V is therefore comprised of Taylor series coefficients:

$$V \begin{pmatrix} f(0) \\ f^{(1)}(0) \\ f^{(2)}(0) \\ \vdots \end{pmatrix} = \begin{pmatrix} f(h_1) \\ f(h_2) \\ f(h_3) \\ \vdots \end{pmatrix}. \quad (3.25)$$

The boundary operator is obtained from V^{-1} by selecting relevant rows:

$$\begin{bmatrix} B_{\partial} & B \\ m \times m & m \times n \end{bmatrix} = \begin{matrix} R \\ m \times (m+n)(m+n)^2 \end{matrix} V^{-1}, \quad (3.26)$$

where the Boolean matrix R_{ij} indicates the i^{th} boundary condition of order $j - 1$. This formulation accounts for both the physical boundary condition that one wishes to impose (e.g. $f(0) = 0$) and the extrapolation required to determine the value of the remaining ghost cells[‡]. For example, using an overall stencil size of $m + n = 3 + 4 = 7$ one can specify a Neumann boundary condition and two high-order extrapolations with

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} \text{Physical} \\ \text{Computational} \\ \text{Computational} \end{matrix} \quad (3.27)$$

$f \quad f^{(1)} \quad f^{(2)} \quad f^{(3)} \quad f^{(4)} \quad f^{(5)} \quad f^{(6)}$

and

$$r_{\partial} = \begin{pmatrix} \varphi(0) \\ 0 \\ 0 \end{pmatrix}, \quad (3.28)$$

where $\varphi(0)$ is the boundary value. The right hand sides of the remaining rows are set to zero to give an extrapolation that, in this case, assumes that $f^{(5)} = f^{(6)} = 0$. As an example, the weights of such a boundary operator for a cell-centred variable such as ϑ , subject to a physical Neumann boundary condition are

$$\begin{bmatrix} B_{\partial} & B \\ m \times m & m \times n \end{bmatrix} = \begin{bmatrix} -\frac{1}{120} & \frac{1}{24} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{24} & \frac{1}{120} & 0 \\ \frac{1}{720} & -\frac{1}{120} & \frac{1}{48} & -\frac{1}{36} & \frac{1}{48} & -\frac{1}{120} & \frac{1}{720} \\ -\frac{3}{640} & \frac{25}{384} & -\frac{75}{64} & \frac{75}{64} & -\frac{25}{384} & \frac{3}{640} & 0 \end{bmatrix}, \quad (3.29)$$

where, for simplicity, we have assumed a uniform grid spacing such that $h = 1$.

Whilst the general approach described in the previous paragraphs is useful for its flexibility, in practice the use of high-order extrapolation to determine boundary values has an adverse effect on the stability of the resulting systems (Carpenter *et al.*, 1993). For the simulation of free-shear flows on open domains where (a) the precise boundary conditions corresponding to an unbounded domain are not known exactly and (b) in comparison to wall-bounded flow, the accuracy of the solution

[‡]Note that a polynomial-based extrapolation for a ghost value in terms of n known values is equivalent to the assumption that the n th order derivative $f^{(n)}(0) = 0$.

is expected to depend less sensitively on the order of accuracy of the boundary conditions, the use of high order boundary closures might be unwarranted. It is therefore useful to consider relatively low-order symmetric boundary closures, similar to those suggested by Verstappen & Veldman (2003). We impose a given boundary condition on pairs of points consisting of a ghost point and its image in the physical domain. For a Neumann or Dirichlet boundary condition applied to a cell-centred variable

$$\begin{bmatrix} B_{\partial} & B \end{bmatrix} = \begin{bmatrix} \times & \circ & \circ & \circ & \circ & \times & \circ & \dots \\ \circ & \times & \circ & \circ & \times & \circ & \circ & \dots \\ \circ & \circ & \times & \times & \circ & \circ & \circ & \dots \end{bmatrix}, \quad (3.30)$$

where $\times$ denotes a non-zero entry. For a Dirichlet boundary condition applied to a wall-centred variable

$$\begin{bmatrix} B_{\partial} & B \end{bmatrix} = \begin{bmatrix} \times & \circ & \circ & \circ & \times & \circ & \circ & \dots \\ \circ & \times & \circ & \times & \circ & \circ & \circ & \dots \\ \circ & \circ & \times & \circ & \circ & \circ & \circ & \dots \end{bmatrix}. \quad (3.31)$$

A Neumann boundary condition applied to a wall-centred variable incorporates an upwind discretisation:

$$\begin{bmatrix} B_{\partial} & B \end{bmatrix} = \begin{bmatrix} \times & \circ & \circ & \circ & \times & \circ & \circ & \dots \\ \circ & \times & \circ & \times & \circ & \circ & \circ & \dots \\ \circ & \circ & \times & \times & \circ & \circ & \circ & \dots \end{bmatrix}. \quad (3.32)$$

The entries $\times$ can be obtained from either the first (Dirichlet) or second (Neumann) entries of the corresponding 2×2 matrix V^{-1} .

3.3 Verification

In this section we *verify* that the operators implemented in the code correspond to those described in §3.2.1 and, therefore, that they provide an approximation with the expected order of accuracy.

Following Wesseling (2001), the local truncation error of a discrete operator L is defined:

$$\varepsilon_l \stackrel{\text{def}}{=} L(\varphi(\mathbf{x}) - \phi), \quad (3.33)$$

where $\varphi(\mathbf{x})$ is the exact solution and ϕ is its numerical approximation. The order of the local truncation error is equal to the order of the error made in approximating a continuous differential operator. Importantly, the error is independent of previous times and errors incurred at other points in the domain. However, at a single point the error resulting from integration will have a dependence on all local truncation errors according to the properties of the inverse discrete operator L^{-1} . We quantify this error by defining a *global truncation error*.

The global truncation error is defined as the difference between the numerical solution and the exact solution:

$$\varepsilon_g \stackrel{\text{def}}{=} \varphi(\mathbf{x}) - \phi. \quad (3.34)$$

In general, the relationship between the local and the global truncation error is not trivial because it depends on the properties of the inverse discrete operator:

$$\varepsilon_g = L^{-1}\varepsilon_l. \quad (3.35)$$

To quantify the global error over the entire domain it will prove useful to define a norm for the space to which ϕ belongs (Iserles, 1996), which on uniform grids we define as

$$\|\phi\|^2 \stackrel{\text{def}}{=} \sum_{\Omega} \phi^2 h^d, \quad (3.36)$$

where $d = 3$ for a three-dimensional domain. This definition is consistent with the L_2 -norm of the function space containing φ in the limit of the grid spacing $h \rightarrow 0$, and is therefore an appropriate choice. We note that for non-uniform grids the volume h^d , will vary spatially. For our purposes the norm (3.36) indicates the distance that lies between a numerical approximation and an exact solution.

3.3.1 Taylor-Green vortex

To verify the spatial discretisation we simulate a Taylor-Green vortex (Taylor & Green, 1937), which satisfies the Navier-Stokes equations and has the exact ana-

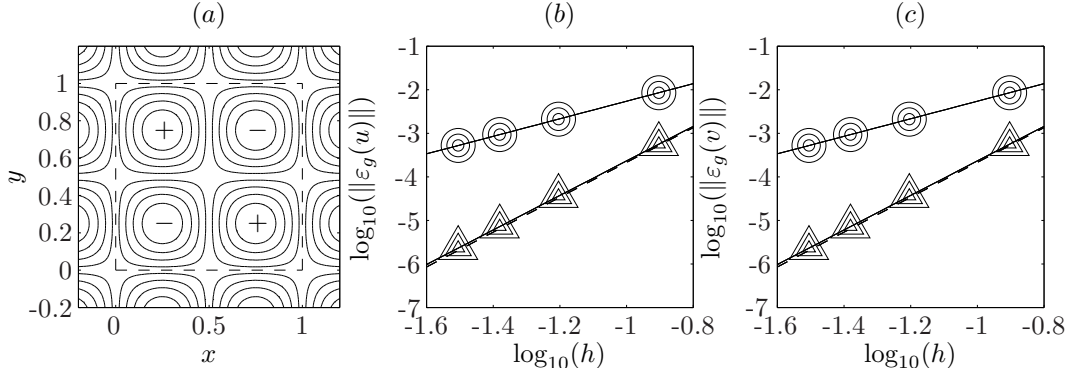


FIGURE 3.8. Norm of global truncation error in the simulation of the Taylor-Green vortex, whose stream function is shown in (a), on a uniform grid in three orientations: $O(h^2)$ scheme $\circ$; $O(h^4)$ scheme $\triangle$; $\|\varepsilon_g\| \propto h^4$ ——. Different sized symbols correspond to different orientations of the vortex.

lytical solution:

$$u(x, y, t) = \sin(2\pi x) \cos(2\pi y) \exp\left(-\frac{8\pi^2}{\text{Re}}t\right), \quad (3.37)$$

$$v(x, y, t) = -\cos(2\pi x) \sin(2\pi y) \exp\left(-\frac{8\pi^2}{\text{Re}}t\right), \quad (3.38)$$

$$p(x, y, t) = \frac{1}{4} \left[\cos(4\pi x) + \cos(4\pi y) \right] \exp\left(-\frac{16\pi^2}{\text{Re}}t\right), \quad (3.39)$$

on the domain $(x, y) \in [0, 1] \times [0, 1]$, with free-slip boundary conditions ($\mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \nabla(\mathbf{u} \times \mathbf{n}) = 0$ on $\partial\Omega$). We use the two-dimensional solution (3.37)-(3.39) to test all three dimensions by applying the permutation (xyz) , which aligns the axis of the vortex along each coordinate in turn. This approach allows one to check that on uniform grids the discretisation is indeed isotropic. In figure 3.8(b,c) we plot the norm, taken over all points in the domain, of the global truncation error ε_g against the grid size h . The global truncation error is defined as the difference, $\varepsilon_g \stackrel{\text{def}}{=} \phi_{ijk}^n - \varphi(\mathbf{x}, t_n)$, between the numerical solution and the exact solution at the end of the simulation, which corresponds here to $t_n = 1$. The simulations employ a uniform grid of spacing $h = \Delta x = \Delta y = \Delta z$. Figure 3.8 shows that the discretisation preserves the symmetry of the exact solution (the errors for u and v are identical) and for all orthogonal orientations of the vortex yields an global truncation error that scales according to $O(h^4)$. Shown for comparison are the results obtained for the same problem using the original second-order-accurate discretisation.

3.3.2 Buoyancy driven flow

To test the transport of active scalars we will employ an approach described by Coleman & Sandberg (2010). The approach involves the use of arbitrary test functions with known derivatives that can be substituted into the equations of motion to obtain a corresponding forcing function. The forcing is applied directly in the simulation to balance the equations and therefore yield an approximation to the original test function. The only restriction placed on the choice of test functions is that they satisfy a given boundary condition.

3.3.2.1 SYMMETRIC TEST. Here we use a simple function to test the convergence of a buoyancy driven flow. We would like to obtain a relative temperature field described by

$$\vartheta(x, y, z) = \sin(\pi x) \sin(\pi y) \sin(\pi z). \quad (3.40)$$

We substitute this field into the transport equation for temperature with unknown forcing:

$$\frac{\partial \vartheta}{\partial t} + \mathbf{u} \cdot \nabla \vartheta - \frac{1}{\text{Pe}} \nabla^2 \vartheta = r_N(x, y, z, t) + r_L(x, y, z, t), \quad (3.41)$$

where the velocity $\mathbf{u}$ is dimensionless and Pe is the Péclet number. Here we assume that the velocity field describes a Taylor-Green vortex with its axis directed in the positive z -direction. The terms on the right-hand-side of (3.41) are specified to balance the equation, which, for ϑ specified by (3.40), results in

$$\begin{aligned} r_N(x, y, z, t) = & \pi \sin(\pi z) \left(\sin(2\pi x) \cos(2\pi y) \cos(\pi x) \sin(\pi y) \right. \\ & \left. - \cos(\pi y) \sin(\pi x) \sin(2\pi y) \cos(2\pi x) \right) \exp\left(-\frac{8\pi^2}{\text{Re}} t\right), \end{aligned} \quad (3.42)$$

for advection and

$$r_L(x, y, z, t) = -3 \frac{\pi^2}{\text{Pe}} \sin(\pi x) \sin(\pi y) \sin(\pi z), \quad (3.43)$$

for diffusion. Temperature is an active scalar that provides a forcing in the momentum equations. In order to preserve the Taylor-Green solution it is therefore necessary to balance this buoyancy force with an artificial force. Because the scalar variables are cell-centred, the relative temperature appearing in the equation for w , is approximated as

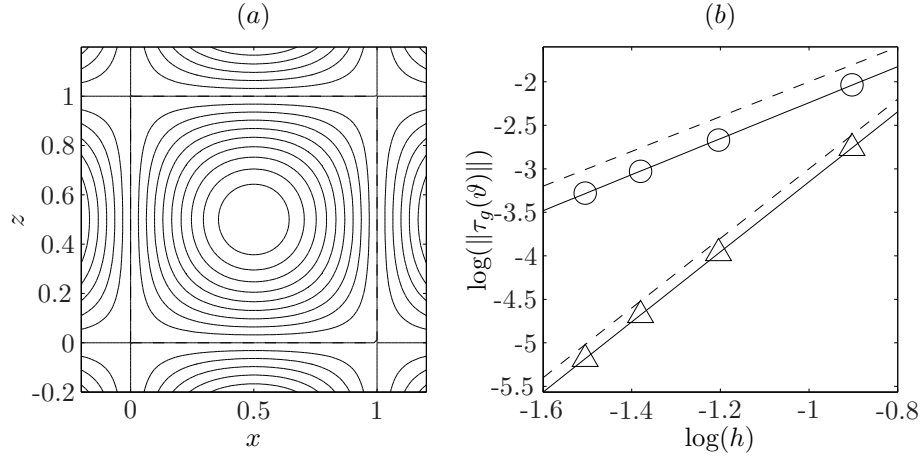


FIGURE 3.9. (a) Isolines of analytical temperature test function. (b) Norm of global truncation error of temperature. $O(h^2)$ scheme [O]; $O(h^4)$ scheme [Δ]; $\|\varepsilon_g\| \propto h^2, h^4$ [— — —].

$$\vartheta = I_3 \vartheta + O(h^4). \quad (3.44)$$

Thus the interpolation introduces an error in the vertical velocity which is distributed to the other transport equations. Although for this problem the vertical velocity field is everywhere zero, the presence of discretisation errors introduce a small $O(h^4)$ vertical circulation.

We test the numerical approximations to the problem by using the same grids that were employed in the Taylor-Green vortex described in §3.3.1. Figure 3.9 displays the convergence of the norm of the global truncation error of ϑ and confirms the $O(h^4)$ accuracy of the new scheme on uniform grids.

3.3.2.2 ASYMMETRIC TEST. In the previous test we used the two-point low-order symmetry boundary conditions described in §3.2.3, taking advantage of the inherent symmetry in the solution. To test how the boundary conditions respond to departures from symmetry, independently from the behaviour of the interior approximation, we employ a polynomial test function. We test a scalar field with the form

$$\vartheta(x, y, z) = x x_* y y_* z z_* x^l y^m z^n, \quad (3.45)$$

where $\chi_* \stackrel{\text{def}}{=} \chi - 1$. The equation for relative temperature is

$$\partial_t \vartheta + \mathbf{u} \cdot \nabla \vartheta - \frac{1}{\text{Pe}} \nabla^2 \vartheta = r_N + r_L. \quad (3.46)$$

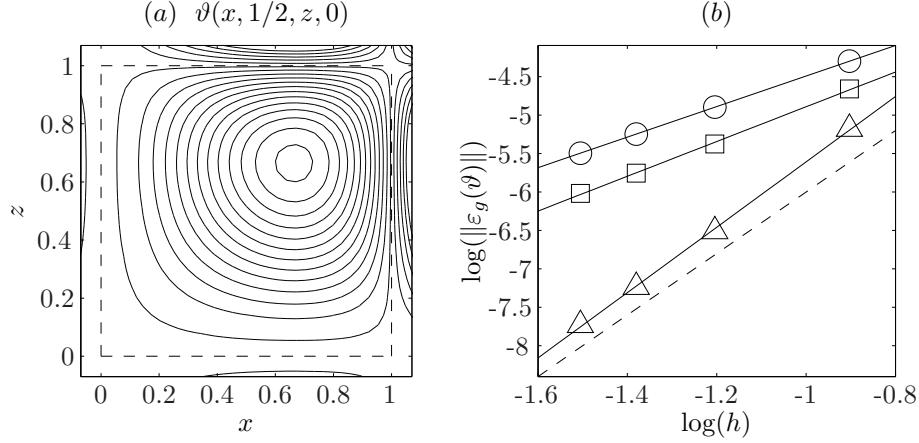


FIGURE 3.10. Asymmetric scalar function. (a) Analytical solution and initial conditions $\vartheta(x, y, z, 0)$ shown at $y = 1/2$. (b) Norm of global truncation error for $O(h^2)$ scheme [○]; $O(h^4)$ scheme with extrapolation boundary conditions [△]; $O(h^4)$ scheme with symmetry boundary conditions [□].

We will impose a Taylor-Green vortex with vertical vorticity for the velocity field, according to (3.37)-(3.39), which results in

$$\begin{aligned}
 x^{-l}y^{-m}z^{-n}(r_N + r_L) = & \sin(2\pi x) \cos(2\pi y) e^{-8\pi^2 t} \left(yzx_*y_*z_* + xyz_*y_*z_* + yzlx_*y_*z_* \right) \\
 & - \cos(2\pi x) \sin(2\pi y) e^{-8\pi^2 t} \left(xzx_*y_*z_* + xyz_*x_*z_* + xzm_*y_*z_* \right) \\
 & + \frac{1}{\text{Pe}} \left(yzy_*z_*(1+l) \left(\frac{lx_*}{x} + 2 \right) + xzx_*z_*(1+m) \left(\frac{my_*}{y} + 2 \right) + \right. \\
 & \left. xyx_*y_*(1+n) \left(\frac{nz_*}{z} + 2 \right) \right).
 \end{aligned} \tag{3.47}$$

Due to the asymmetric nature of the problem and the crucial role that boundaries are likely to play it is necessary to employ a high-order treatment of the boundary conditions in order to preserve the overall accuracy of the scheme. To this end, we use the extrapolation boundary conditions described in §3.2.3. Indeed figure 3.10 indicates that for this particular *boundary-layer*[§] problem, the use of the symmetric boundary conditions (see equation 3.30) degrades the order of the global truncation error. In contrast, the use of extrapolation boundary conditions (see equation 3.26) preserves the $O(h^4)$ accuracy implied by the interior operators.

[§]We note that this particular problem, whose boundary layer is a direct consequence of the specified boundary conditions is very different from the free-shear flows that form the focus of this study.

3.4 Scaling

3.4.1 *Scaling with respect to Re*

An unfortunate consequence of the central differencing scheme that we use to solve (3.1)-(3.3) is that it requires[¶] Δz to be small compared to ν/W , for a characteristic velocity W (representative of the largest velocity in the domain). Since $\text{Re} \propto W$, the restriction $\Delta z \leq \nu/W$ implies that $N \sim 1/\Delta z \sim \text{Re}^{-1}$, which is stronger than the restriction that $N \sim \text{Re}^{-3/4}$ for resolution of the smallest scales in the flow. Furthermore, owing to the fact that the code that we use is explicit, stability of the numerical simulation requires that $\Delta t \leq \Delta z/W$. Typically, in order to obtain converged statistics the total duration for which a simulation must be run scales in proportion to the characteristic source turnover time $\tau_s \stackrel{\text{def}}{=} r_s/W$, for a relevant source length scale r_s . Hence, one finds that $\tau_s/\Delta t \sim r_s/\Delta z \sim N$. If, in addition, one adopts the optimistic view that the number of operations n_{op} required in the simulation is linearly proportional to the number of cells $\times$ the number of time steps, then one concludes that $n_{\text{op}} \sim \text{Re}^4$.

For the particular case of free-shear flow simulation, the situation is worse. Studies of free-shear flow, including the present one, are typically concerned with the far-field behaviour of the flow, i.e. the flow at relatively large distances from the source, such that $L/r_s \gg 1$. If one assumes that the velocities in the free-shear flow scale according to a power-law such that $w_m \sim z^a$ and that on dimensional grounds the characteristic length scale of the flow $r_m \sim z$, then the characteristic timescale $t_m \sim z^{1-a}$. Consequently, the ratio between the source timescale τ_s and the largest timescale associated with the flow of the flow will be $t_m(L)/\tau_s \sim (L/r_s)^{1-a}$. To obtain converged statistics one must simulate in proportion to the larger timescale $t_m(L)$, with the consequence that the number of time steps $N_t \sim N(L/r_s)^{1-a}$ rather than $N_t \sim N$, and $n_{\text{op}} \sim \text{Re}^4(L/r_s)^{1-a}$. For plumes $a = -1/3$, but for jets $a = -1$ and $n_{\text{op}} \sim \text{Re}^4(L/r_s)^2$, which places a severe restriction on the combination of Reynolds number and domain size that one can simulate.

3.4.2 *Scaling with respect to processors*

In the simulation of a particular flow with a fixed Reynolds number Re , the time it takes for a parallelised code to execute reduces with the number of cores (processes) that are used in the computation. By using many cores it is therefore possible to

[¶]When the cell Reynolds number $\text{Re}_c \stackrel{\text{def}}{=} \Delta z W/\nu$ is $O(1)$, central differencing schemes admit non-physical oscillations (Gresho & Lee, 1981). Although similar conditions apply to Δx and Δy , here we focus on Δz .

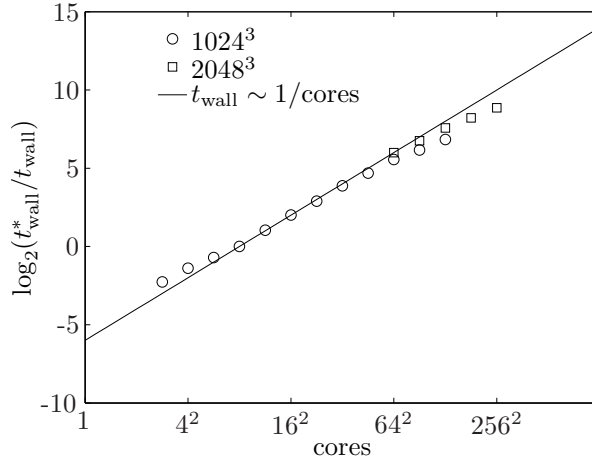


FIGURE 3.11. The observed scaling of the code (symbols, corresponding to grids of 1024^3 cells and 2048^3 cells) in comparison with the limit of perfect scaling (solid line). The results are presented relative to the wall-time $t_{\text{wall}}^* \stackrel{\text{def}}{=} t_{\text{wall}}(2^6)$.

simulate relatively high-Reynolds-number flows. One hopes that by using twice as many processes the program runs twice as fast. However, not every aspect of a program can be parallelised (see, e.g., Rodgers, 1985; Amdahl, 1967). For example, because a point in an incompressible flow is influenced by all other points in the domain, it is necessary for the separate processes to share information. Moreover, the use of more processes increases the number of ghost cells that must be employed (see equation 3.21). Nevertheless, local operators such as advection and diffusion can be parallelised very effectively.

To test the performance of the code, we run multiple simulations of a particular flow, each employing a different number of computing cores. We run the simulations for approximately 500 time steps, which ensures that the performance statistics (in this case the total wall time) are fully converged. Figure 3.11 displays the observed relationship between the total wall time t_{wall} and the number of cores used. A perfect scaling for a fully parallelised code implies that $t_{\text{wall}} \propto 1/\text{cores}$, corresponding to the solid line in figure 3.11. It is evident that over a large range of cores the observed scaling is good in comparison with the theoretical optimum. However, with a grid resolution of 1024^3 the use of 128^2 cores results in the number of internal cells ($N'_y = N'_z = 8$) being comparable to the number of layers of ghost cells on each face ($N_g = 3$) and the scaling deteriorates. The use of a larger number of internal cells relative to the number of ghost cells alleviates the problem, although we note that the use of multiple ghost cells as described in §3.2.3 is disadvantageous in this respect.

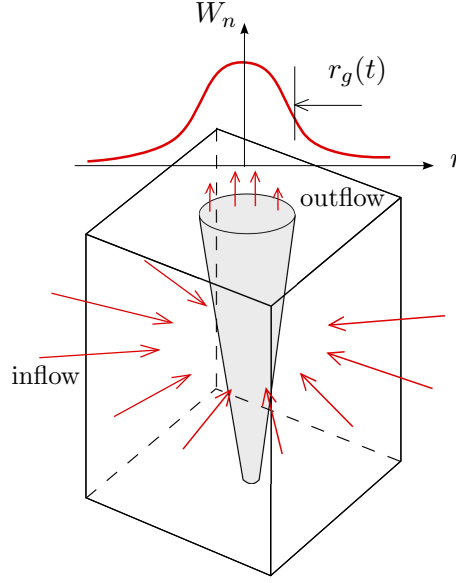


FIGURE 3.12. At the top of the domain a non-uniform convective outflow boundary condition is imposed and on the vertical faces of the domain an axisymmetric inflow boundary condition is imposed.

3.5 Inflow and outflow boundary conditions

This section is a summary of the work that was undertaken to implement inflow and outflow boundary conditions in the code. Full details were published in Craske & van Reeuwijk (2013) and are reproduced in appendix A with permission.

3.5.1 Background

Our aim is to simulate the behaviour of jets and plumes in semi-infinite *unconfined* domains. However, the computational domain that we employ describes a region of finite extent. It is therefore necessary to provide open boundary conditions that allow fluid to enter the computational domain through the vertical faces and leave the domain through the top face (see figure 3.12). There are many possible boundary conditions that allow fluid to enter and leave a domain, ranging from specification of a wall-normal velocity $\mathbf{u} \cdot \mathbf{n}$ to more elaborate conditions involving $\partial_t \mathbf{u}$. However, with each is associated the particular force, exerted by the boundary on the fluid, that is necessary to enforce the boundary condition (i.e. a Lagrangian multiplier). Since we are interested in unconfined domains, we would like this force to be relatively small, such that the fluid enters and leaves the domain in a manner that is consistent with the behaviour of an unconfined domain. For incompressible flows, in which the pressure field ensures that every part of the flow field is instant-

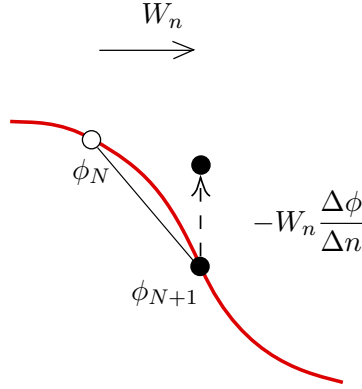


FIGURE 3.13. The convective boundary condition determines the time rate of change of ϕ_{N+1} on the boundary by accounting for the wall-normal gradient of ϕ and the characteristic wall-normal velocity W_n .

neously affected by every other part, the challenge of specifying innocuous outflow boundary conditions is formidable (see Sani & Gresho, 1994, for details). The approach that we have developed is based on the widely-used convective boundary condition (Orlanski, 1976). Rather than imposing a time-invariant condition on ϕ , the convective boundary condition relates the time dependence of ϕ to advection. In terms of continuous operators and the solution φ , the condition corresponds to

$$\frac{\partial \varphi}{\partial t} + W_n \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } n = 0, \quad (3.48)$$

where n is an (outward) wall-normal coordinate and W_n is a characteristic wall-normal velocity. If $\partial_n \varphi = 0$, then (3.48) implies that $\partial_t \varphi = 0$. In contrast to a condition such as $\varphi = 0$, which would imply either destruction or production of the quantity φ at the boundary, (3.48) allows the flow field to determine the behaviour of φ naturally. The basic approach in a discrete implementation of (3.48), taking the top face of the domain as an example, is illustrated in figure 3.13.

The approach can be extended to determine the time-dependent values of ghost cells ϕ_∂ . Using the boundary operator discussed in §3.2.3, equation (3.48) is implemented as

$$\frac{\delta \phi_\partial}{\delta t}_{m \times 1} = -W_n \begin{bmatrix} B_\partial & B \\ m \times m & m \times n \end{bmatrix} \begin{pmatrix} \phi_\partial \\ m \times 1 \\ \phi \\ n \times 1 \end{pmatrix}, \quad (3.49)$$

where each row of the operator $[B_\partial \ B]$ contains a discrete approximation to the

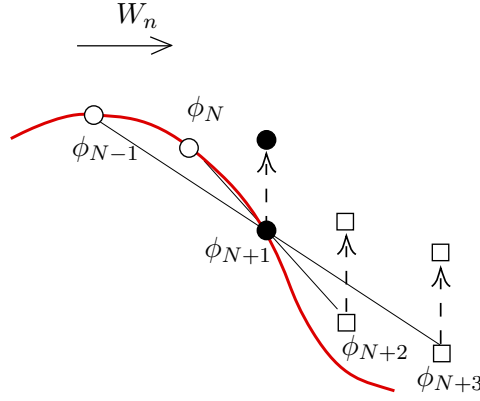


FIGURE 3.14. The convective boundary condition is used to determine the behaviour ghost cell values ϕ_{N+2} , ϕ_{N+3} , according to the wall-normal gradient of ϕ and the characteristic wall-normal velocity W_n .

gradient $\partial_n \phi$ on the boundary (see equation 3.32). The behaviour of ghost cells at a boundary is illustrated schematically in figure 3.14.

3.5.2 Implementation

Our contribution to the development of the convective boundary condition for free-shear flows lies in the specification of a spatially and temporally-variable advecting velocity W_n . Whilst W_n is typically assumed to be constant, the profiles of flow quantities in jets and plumes reveal a strong radial dependence (see chapter 4). In the case of velocity, equating W_n with the average outflow velocity results in a *domain dependent* artificial forcing on the outflow boundary. Instead, we assume that $W_n(r, t)$ is a Gaussian profile defined in terms of the total volume flux $Q_m(L_z)$ and a time-dependent radial length scale $r_g(t)$, as illustrated in figure 3.12. Since the length scale $r_g(t)$ is determined dynamically, based on an integral of the observed velocity profile at $z = L_z$, the method is self-calibrating, and

$$W_n \stackrel{\text{def}}{=} \frac{Q_m(L_z, t)}{\pi r_g(t)^2} \exp\left(-\frac{r(x, y)^2}{r_g(t)^2}\right). \quad (3.50)$$

On the vertical faces of the domain (see figure 3.12) we impose *inflow* boundary conditions using an inhomogeneous form of (3.48) for the horizontal components of velocity. In this way, despite the cuboidal shape of the domain, the inflow boundary conditions account for a radial pressure gradient in a manner that is consistent with an axisymmetric flow field. A detailed description of the open boundary conditions can be found in appendix A.

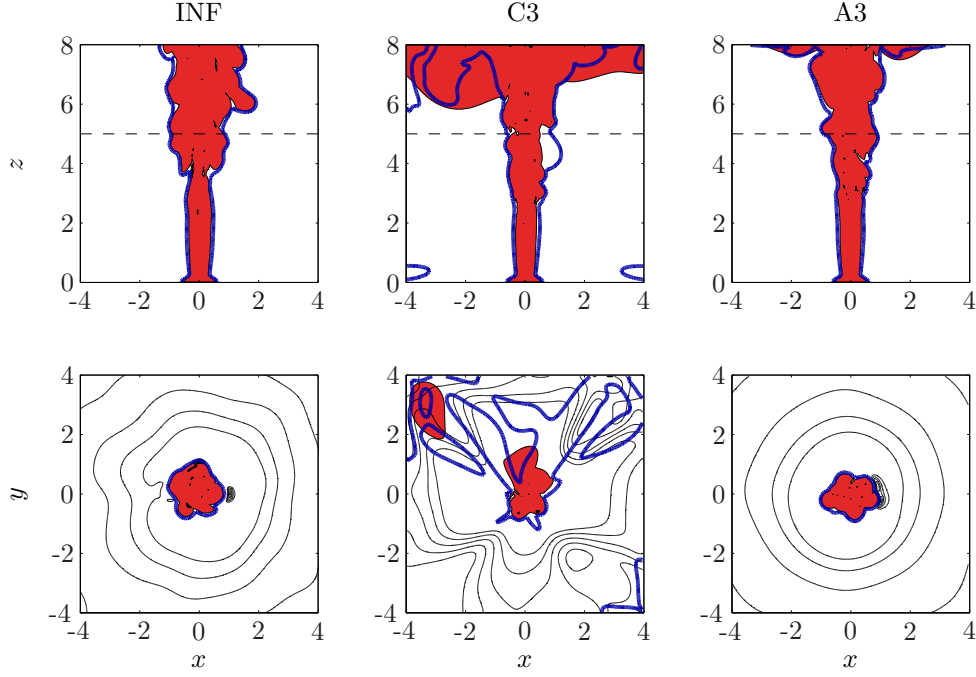


FIGURE 3.15. Vertical (top row) and horizontal (bottom row) slices through a large closed domain (INF), a small domain employing classical convective open boundary conditions (C3) and a small domain employing open boundary conditions that account for the statistical axisymmetry of the flow (A3). The slices indicate regions that exceeded an instantaneous normalised temperature threshold, $\vartheta > 0.005$ [red]; an instantaneous threshold of normalised enstrophy, $\frac{1}{2}\|\nabla \times \mathbf{u}\|^2 > 0.005$ [blue]; and, in the bottom figures only, isolines of absolute normalised horizontal velocity, $\sqrt{u^2 + v^2}$ [black]. The horizontal slice was taken at $z = 5$ [---]. All coordinates are normalised with respect to the source diameter.

3.5.3 Results

To investigate the suitability of the proposed boundary conditions we conducted a series of simulations of a turbulent plume employing either classical open boundary conditions (uniform W_n and homogeneous Neumann inflow conditions) or our proposed boundary conditions (non-uniform $W_m(r, t)$ and axisymmetric inflow conditions). We compared the results to those obtained from the simulation of a turbulent plume in a relatively large ‘filling box’ employing solid boundaries, of dimensions $L_x L_y L_z = 32^2 \times 16$ source diameters. Each of the domains employing open boundary conditions was of height 8 source diameters and had an aspect ratio $L_x/L_z = L_y/L_z$ of either 1/2, 3/4, 1 or 2.

Figure 3.15 displays temperature and enstrophy fields pertaining to the filling box (INF), the classical open boundary conditions (C3, aspect ratio 1) and the

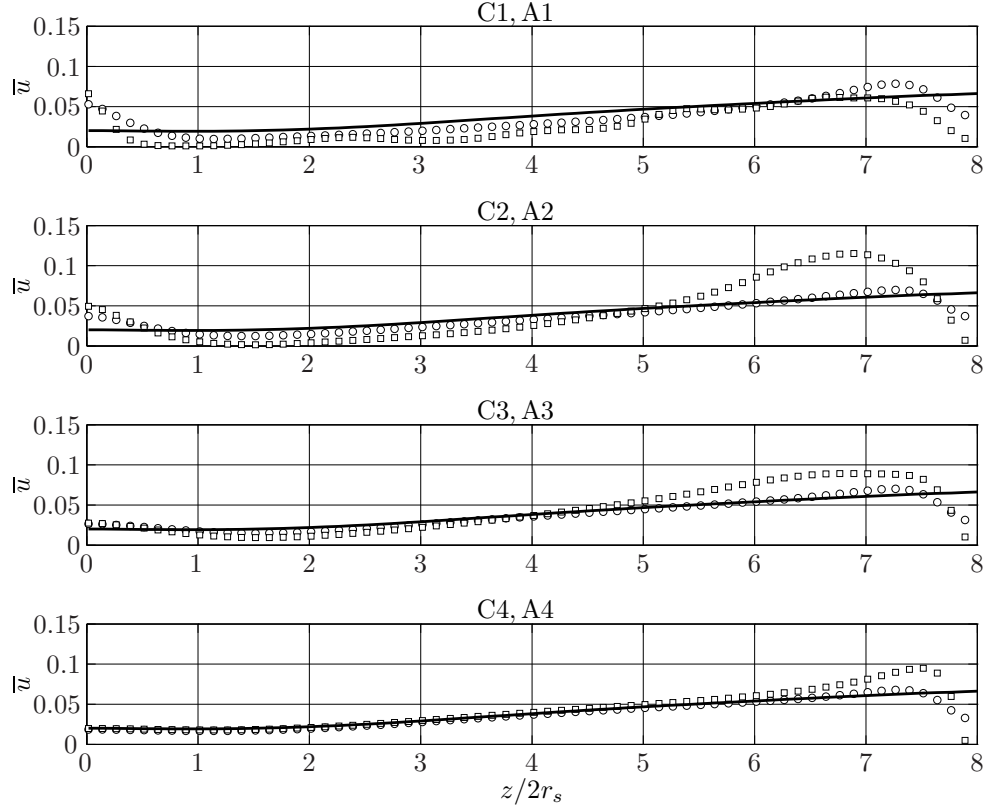


FIGURE 3.16. Time and azimuthally-averaged normalised radial velocity, $\bar{u}$ on a vertical shell on which $r/r_s = 4$. Data from the simulation INF [—] is compared to data from simulations employing uniform convective open boundary conditions, Cn [$\square$] and axisymmetric open boundary conditions, An [$\circ$], for $n = 1, 2, 3, 4$, on domains of different aspect ratio.

proposed boundary conditions (A3, aspect ratio 1). Note that the domain of INF is much larger than the window displayed in figure 3.15 and provides a good representation of the flow in an unconfined domain. It is clear from C3 that a failure to account for the localised nature and statistical axisymmetry of the flow results in severe disturbance and the spurious transport of temperature and enstrophy into the flow's ambient, which were observed to accumulate over time. These non-physical effects are particularly pronounced near the top of the domain and close to the corners, where they are liable to interfere with the inflow open boundary conditions. Mitigation of the effects evident in C3 requires relatively large domains, on which it is expensive to simulate. By comparison, the temperature and enstrophy fields in A3 exhibit a good agreement with those in INF, and therefore with those that one would expect to find in unbounded domains.

In addition to the problems evident in figure 3.15, the classical convective open

boundary conditions can modify the flow rates into and out of the domain, thereby altering important physical properties such as the entrainment rate. Perhaps the most compelling shortcoming of classical open boundary conditions for these flows is that they produce simulation results that depend on the domain size, which we demonstrate by looking at entrainment into the plume in figure 3.16. Specifically, figure 3.16 displays the variation, with respect to z , of the radial velocity in the ambient. The simulations using classical open boundary conditions C1, C2, C3 and C4 employ aspect ratios of 1/2, 3/4, 1 and 2, respectively. Similarly, the simulations of the proposed axisymmetric boundary conditions A1, A2, A3 and A4 employ aspect ratios of 1/2, 3/4, 1 and 2, respectively. It is evident from figure 3.16 that when classical open boundary conditions are used, the entrainment velocity is strongly affected by the aspect ratio of the domain; for a reasonable agreement with INF, one would be forced to select an aspect ratio of 2. The dependence of the results on aspect ratio is partly due to the specification of W_n as the average outflow velocity, which is quantity that depends on the area over which one is averaging. In contrast, since we specify W_n using information from the plume, the simulations A1, A2, A3 and A4 exhibit a weaker dependence on aspect ratio than C1, C2, C3 and C4. In particular, A2, with an aspect ratio of 3/4 is in reasonably good agreement with INF.

3.6 Statistics

3.6.1 Cylindrical coordinates

Although the simulations are conducted on a uniform Cartesian grid, as displayed in figure 3.17(a), owing to the axisymmetry of the flow it is convenient to process the data with respect to a cylindrical coordinate system. To do so, we partition the domain into cylindrical shells and assign each cell to a particular shell, according to its distance from $(x, y) = (0, 0)$. The domain and the cylindrical shells are depicted in figure 3.17(c). Each shell has thickness of $\Delta r \stackrel{\text{def}}{=} \Delta x = \Delta y$. We therefore define the radii associated with the boundary and the centre of each shell according to

$$\begin{aligned} r_b(i) &\stackrel{\text{def}}{=} i\Delta x, & i = 0, 1 \dots N_x/2 + 1, \\ r_c(i) &\stackrel{\text{def}}{=} \Delta x/2 + i\Delta x, \end{aligned} \quad (3.51)$$

as indicated in figure 3.18. The i th shell is defined as a set of cells:

$$\text{shell}_i \stackrel{\text{def}}{=} \{(x_c, y_c) \mid r_b(i)^2 \leq x_c^2 + y_c^2 < r_b(i+1)^2\}. \quad (3.52)$$

The number of cells belonging to i th shell will be denoted $N_a(i)$. A typical shell is

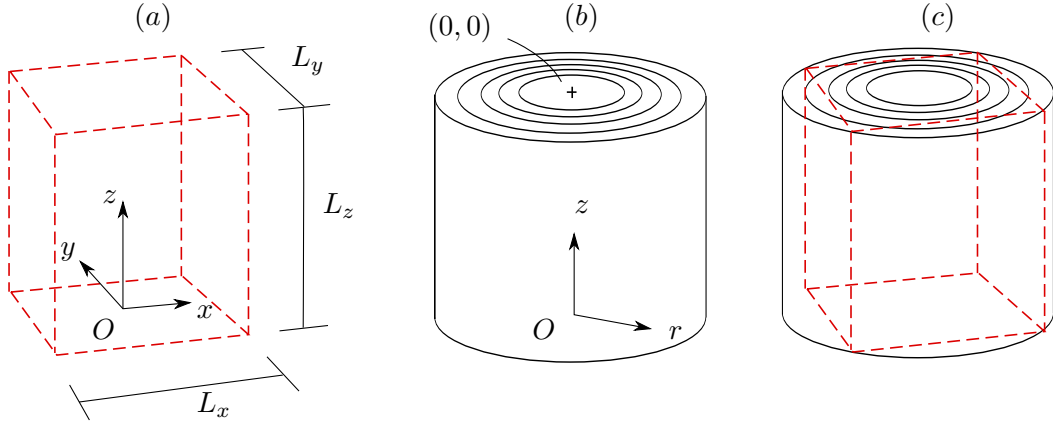


FIGURE 3.17. Discretisation of Cartesian domain into cylindrical shells. The shells are parallel to Oz and centred on $(x, y) = (0, 0)$, which is located at the centre of the square horizontal planes of the domain.

depicted in figure 3.19.

3.6.2 Estimators

Under the assumption of statistical homogeneity in the azimuthal direction, we regard the observed values of a flow variable at a fixed radial and longitudinal location as identically and independently distributed random variables. In this section we consider the ensemble average and the spatial average of these observations as estimators for their expected value. We will regard the azimuthal average of a sequence of N_a observations of a random variable X as a function

$$a(X) \stackrel{\text{def}}{=} \frac{1}{N_a} \sum_{\text{shell}} X_j, \quad (3.53)$$

where X_j is the j th observation of X belonging to a particular shell. Similarly, the ensemble average of a sequence of N_b observations from independent realisations of the flow is defined:

$$b(X) \stackrel{\text{def}}{=} \frac{1}{N_b} \sum_{\text{ensemble}} X_j, \quad (3.54)$$

where X_j is the j th member of the ensemble. The overall mean or average used in this work is $\bar{X} \stackrel{\text{def}}{=} b(a(X)) = a(b(X))$. Furthermore, both b and a are idempotent such that $a \circ a = a$ and $b \circ b = b$. It is convenient to identify fluctuations in X over *space* with the function $a' \stackrel{\text{def}}{=} X - a(X)$ and fluctuations of the spatial mean a over the *ensemble* with $b' \stackrel{\text{def}}{=} a(X) - b(a(X))$.

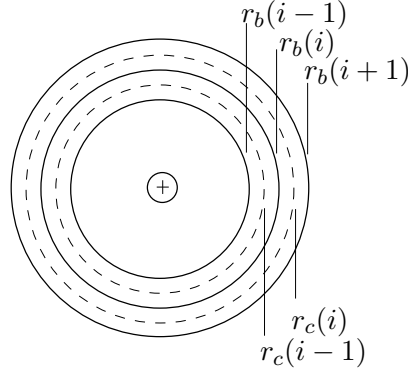


FIGURE 3.18. Discretisation of the domain into cylindrical shells, for which $\Delta r = \Delta x = \Delta y$. The variables r_b and r_c correspond to the radial location of the shell boundaries and shell centres, respectively.

The fluctuations over space and the ensemble are defined such that $a \circ a' \stackrel{\text{def}}{=} 0$ and $b \circ b' \stackrel{\text{def}}{=} 0$. These definitions allow us to decompose X according to

$$X \stackrel{\text{def}}{=} \bar{X} + X' \stackrel{\text{def}}{=} \bar{X} + a'(X) + b'(X). \quad (3.55)$$

In this work we will be interested in the covariance of X with another random variable Y :

$$\overline{X'Y'} \stackrel{\text{def}}{=} \overline{XY} - \bar{X}\bar{Y}. \quad (3.56)$$

Using (3.55) in (3.56) and assuming that a' and b' are uncorrelated, one finds

$$\overline{X'Y'} = \underbrace{a(a'(X)a'(Y))}_{\text{spatial covariance}} + \underbrace{b(b'(X)b'(Y))}_{\text{ensemble covariance}}. \quad (3.57)$$

Consequently, the covariance $\overline{X'Y'}$ is comprised of a spatial covariance and an ensemble covariance. With the exception of shells at relatively small r , the number of independent samples that we use to obtain the spatial average is greater than the number of samples used in an ensemble, and therefore the first term on the right hand side of (3.57) typically dominates. On the axis of the flow, however, where only a relatively small number of samples contribute to the azimuthal average, the second term on the right-hand-side of (3.57) dominates.

For simulations of a statistically steady state, we replace the ensemble average with a time average:

$$c(X) \stackrel{\text{def}}{=} \frac{1}{t_{\text{run}}} \sum_{\text{time}} X_j \Delta t_j, \quad (3.58)$$

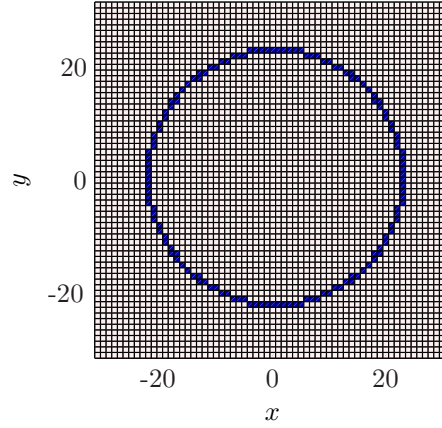


FIGURE 3.19. Shell number 22 in an axisymmetric coordinate system imposed on a Cartesian grid.

where X_j is the j th observation in time and Δt_j is the j th time step.

3.6.3 The azimuthal average

In this section we verify the azimuthal averaging routines by imposing a velocity field defined by the stream function

$$\psi = \left(x - \frac{1}{2}\right) \left(x + \frac{1}{2}\right) \left(y - \frac{1}{2}\right) \left(y + \frac{1}{2}\right), \quad (3.59)$$

uniformly in z , as shown in figure 3.20. We will utilise cylindrical coordinates $x = r \cos \phi$, $y = r \sin \phi$, where ϕ is the azimuthal angle, and examine the radial u and azimuthal v components of $\mathbf{u} \stackrel{\text{def}}{=} (u, v, 0)$, so that

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \begin{pmatrix} \partial_y \psi \\ -\partial_x \psi \end{pmatrix}. \quad (3.60)$$

Consistent with (3.53) we define the azimuthal average of the random function $\mathbf{u}$ according to

$$a(\mathbf{u}) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} \mathbf{u} d\phi. \quad (3.61)$$

Hence, we find

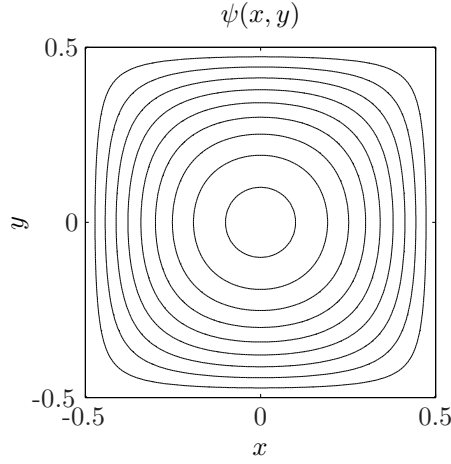


FIGURE 3.20. Stream function $\psi(x, y)$ used to test the azimuthal and ensemble averaging routines.

$$\left. \begin{aligned} a(u) &= 0, \\ a(v) &= \frac{1}{16} (r - 5r^3 + 6r^5 - 3r^7), \\ a(v^2) - a(v)^2 &= \frac{1}{512} (r^6 - 12r^8 + 44r^{10} - 48r^{12} + 17r^{14}), \\ a(u^2) - a(u)^2 &= \frac{1}{512} (r^6 - 8r^8 + 20r^{10} - 16r^{12} + 5r^{14}). \end{aligned} \right\} \quad (3.62)$$

To test the implementation of the discrete azimuthal average (3.53) we impose $\mathbf{u}$ on square computational domains consisting of 32^2 , 64^2 , 128^2 and 256^2 computational cells of uniform size over a horizontal plane $(x, y) \in [-1/2, 1/2]^2$. Each domain employs a nominal 4 cells in the z direction. In figure 3.21 we compare the calculation of the azimuthal averages using (3.53) with the analytical solutions (3.62). Since the first moment of the radial velocity $a(u)$ converges to zero, we normalise the residual by the grid spacing $h^2 = \Delta x^2$ to demonstrate that the error is $O(h^2)$. All the domains tested show a good agreement with the first moment of azimuthal velocity. Note that the analytical solutions (3.62) are only valid when the shell over which one is averaging lies entirely inside the domain (see, e.g., figure 3.19). When $r > 0.5$ the shell is truncated by the boundary of the domain ($L_x = L_y = 1$ in the present example) and the analytical solutions (3.62) in figure 3.21 do not apply. The second moments displayed in figure 3.21(c, d), discriminate among the different resolutions employed and demonstrate a good agreement with the analytical solution for the highest grid resolution $h = 1/256$.

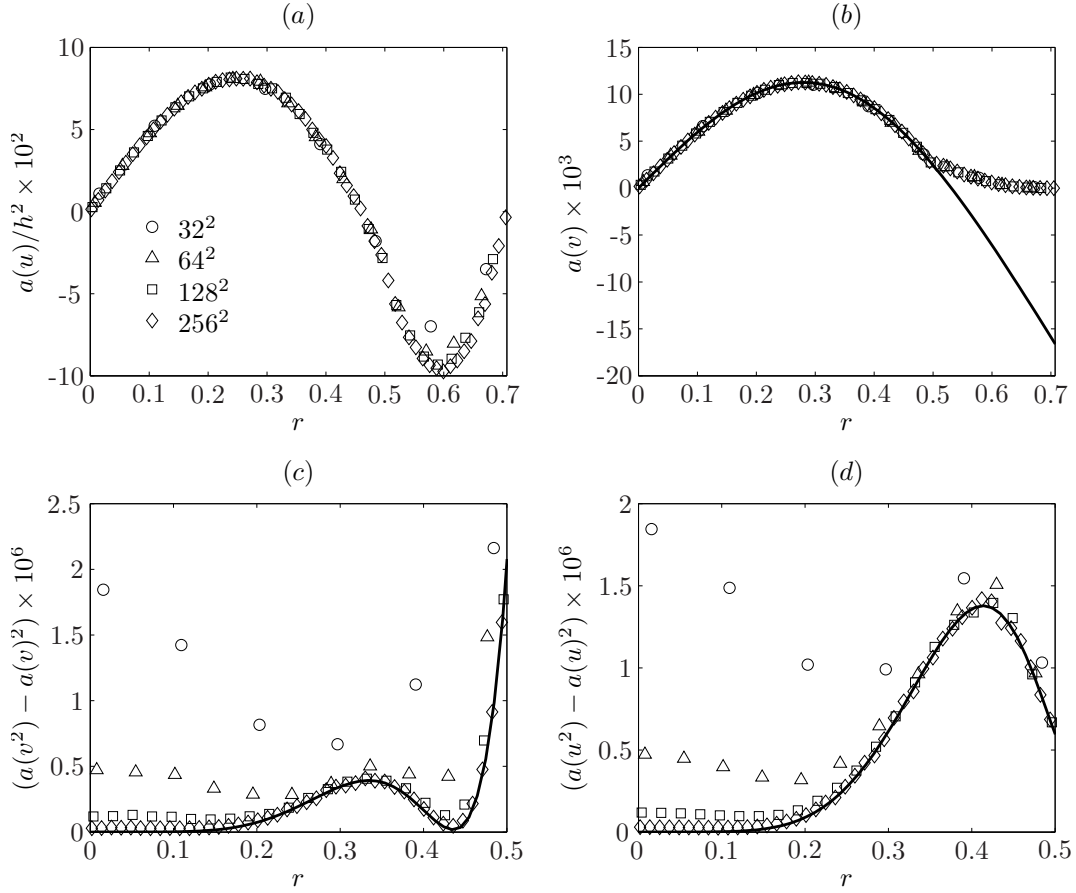


FIGURE 3.21. Azimuthal averages of velocity statistics obtained from the stream function (3.63). The solid line corresponds to the exact solution (3.62). The grid resolutions in (x, y) are $32^2, 64^2, 128^2$ and 256^2 . The azimuthal average of the radial velocity $a(u) \sim 0$, hence (a) demonstrates that the discrete approximation has an error of $O(h^2)$.

3.6.4 The ensemble average

To test the discrete ensemble average operation (3.54) we introduce a sample-dependent amplification, over three members of an ensemble:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \left(x - \frac{1}{2}\right) \left(x + \frac{1}{2}\right) \left(y - \frac{1}{2}\right) \left(y + \frac{1}{2}\right) \begin{pmatrix} 1 \\ 1/2 \\ 3/2 \end{pmatrix}. \quad (3.63)$$

Because the average value of the amplification has been chosen to be equal to unity, the first-order moments are unaffected by the modulation. The higher-order

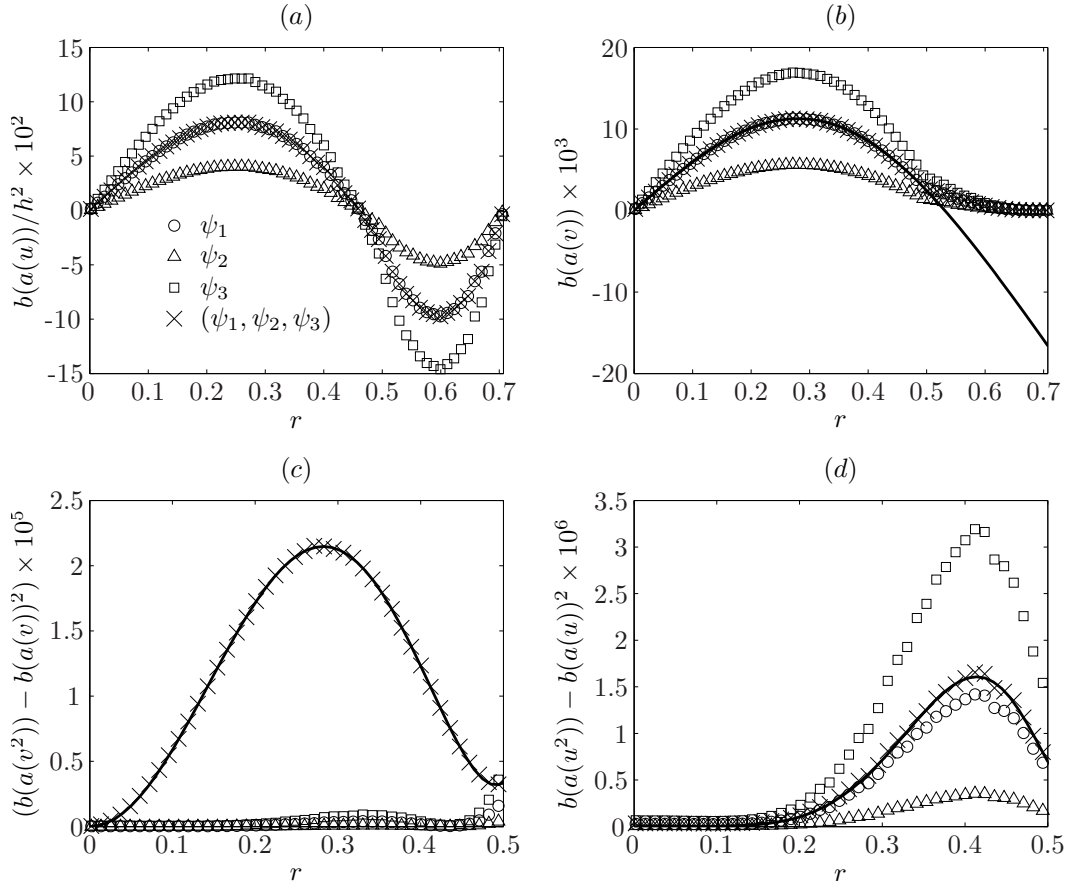


FIGURE 3.22. Azimuthal- and ensemble-averaged statistics. All members of the ensemble have the same radial dependence, with an amplitude of either $1/2$, 1 or $3/2$. Statistics pertaining to each member of the ensemble are displayed, in addition to their ensemble average. The exact solutions (3.64) are indicated with the solid lines. All data correspond to a grid spacing of $h = 1/256$.

moments, however, are affected by the modulation. The analytical solutions to this problem are given by

$$\left. \begin{aligned} b(a(u)) &= 0, \\ b(a(v)) &= \frac{1}{16} (r - 5r^3 + 6r^5 - 3r^7), \\ b(a(v^2)) - b(a(v))^2 &= \frac{1}{3072} (54r^6 - 216r^8 + 440r^{10} - 408r^{12} + 137r^{14}), \\ b(a(u^2)) - b(a(u))^2 &= \frac{1}{3072} (7r^6 - 56r^8 + 140r^{10} - 112r^{12} + 35r^{14}). \end{aligned} \right\} \quad (3.64)$$

Figure 3.22 displays the result of computing the left-hand-side of (3.64) using the

discrete data, compared to the analytical solution. Here, we use data that correspond to a grid resolution of 256^2 in the horizontal plane. It is evident that for the first-order statistics the ensemble average is equal to the azimuthal average of ψ_1 . For the higher-order statistics, fluctuations over the ensemble give rise to a non-vanishing contribution (i.e. the statistics obtained from the ensemble (ψ_1, ψ_2, ψ_3) are not equal to those obtained from ψ_1). In all cases the discrete approximation of the expected values is in good agreement with the analytical solution.

Chapter 4

Steady turbulent jets and plumes

In this chapter we describe direct numerical simulations of steady-state jets and plumes and provide validation against experimental data. Following a discussion of the results, we turn to the problem of defining radial integrals such as the volume flux and momentum flux, in preparation for the prominent role that these quantities will play in subsequent chapters.

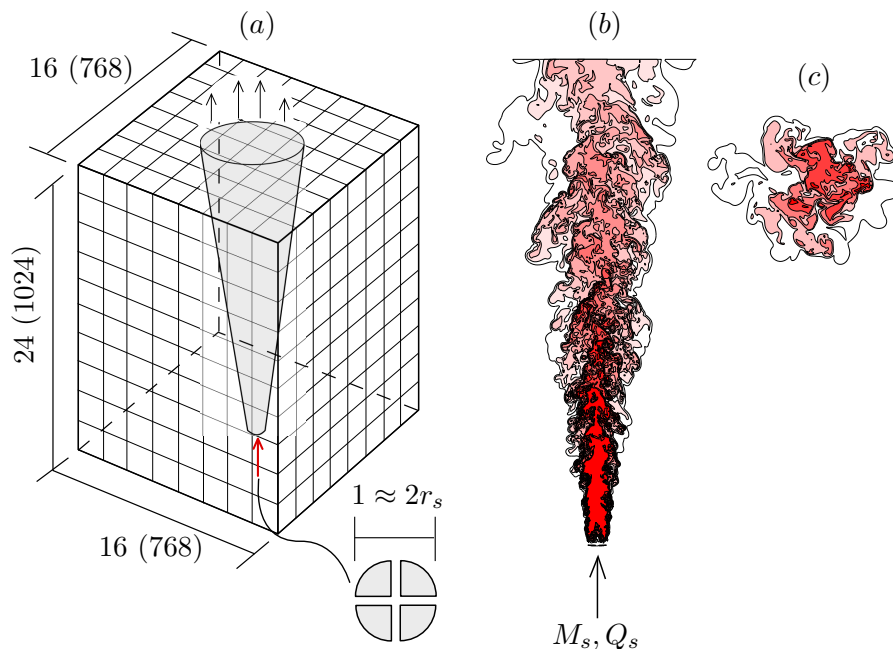


FIGURE 4.1. (a) Computational domain for a jet simulation, consisting of a rectangular cuboid of dimensions $L_x L_y L_z = 16^2 \times 24$ source diameters, discretised using $768^2 \times 1024$ computational control volumes. (b) Vertical slice through passive scalar isoregions. (c) Horizontal slice through passive scalar isoregions at $z/L_z = 5/6$.

4.1 Steady jets

4.1.1 Simulation details

The data presented herein were obtained from direct simulation of the full Navier-Stokes equations (3.1)-(3.3) in addition to a transport equation (identical to equa-

TABLE 4.1. Summary of steady-state turbulent jet simulation details.

	$(L_x \times L_y \times L_z)/r_s$	$N_x \times N_y \times N_z$	Re_s	t_{run}/τ_s
L_j	$44^2 \times 66$	$768^2 \times 1152$	4815	3323
H_j	$44^2 \times 66$	$768^2 \times 1152$	6810	3524
H_j^*	$44^2 \times 66$	$512^2 \times 768$	6810	3351

tion 3.3) for a passive scalar c , whose diffusivity is defined to give a Schmidt number Sc equal to 0.71. For details of the code the reader is referred to §3.2. We simulate an axisymmetric turbulent jet driven by an isolated source of steady momentum flux M_s and volume flux Q_s . The source is circular and located at the centre of the base of a cuboidal domain of size $44^2 \times 66$ characteristic source radii, r_s , where $r_s \stackrel{\text{def}}{=} Q_s/M_s^{1/2}$ and the Reynolds number is $\text{Re} \stackrel{\text{def}}{=} 2M_s^{1/2}/\nu$. The characteristic radius r_s is slightly less than the radial extent of the source owing to the source conditions, which are discussed below. The domain is discretised using a Cartesian grid consisting of $768^2 \times 1152$ computational cells, as indicated in figure 4.1(a), with a resolution of 48 cells over the physical source diameter, which corresponds to approximately 34 cells over $2r_s$.

We simulate two steady-state jets, L_j and H_j , of Reynolds numbers 4815 and 6810, respectively. Data for these cases are collected over approximately $t_{\text{run}}/\tau_s = 3000$ dimensionless time units, where the source turnover time $\tau_s \stackrel{\text{def}}{=} Q_s^2/M_s^{3/2}$. We eliminate initial transients from the problem by running the simulations for not less than 3000 dimensionless time units before collecting statistics. Although the sampling time and our allowance for transient behaviour appears to be excessively large, it should be noted that the turnover time in a jet scales in the longitudinal direction according to z^2 . Simulation H_j^* , which is used to check the convergence of the results, employs a grid of $512^2 \times 768$ and has a Reynolds number of 6810.

A constant mean momentum and scalar flux is maintained by imposing a Dirichlet boundary condition on both the longitudinal velocity w and the passive scalar c at the source. The resulting instantaneous concentration field is displayed in figure 4.1(b, c). To initiate the turbulence we apply perturbations of 1% to the velocities in the first cell above the source. A disadvantage with simply connected circular sources of momentum flux is that they result in a potential core region, of approximately $10r_s$ in longitudinal extent (Rajaratnam, 1976), preceding the region of fully developed flow, in which we are primarily interested. To reduce the longitudinal extent of the potential core, we employ the numerical equivalent

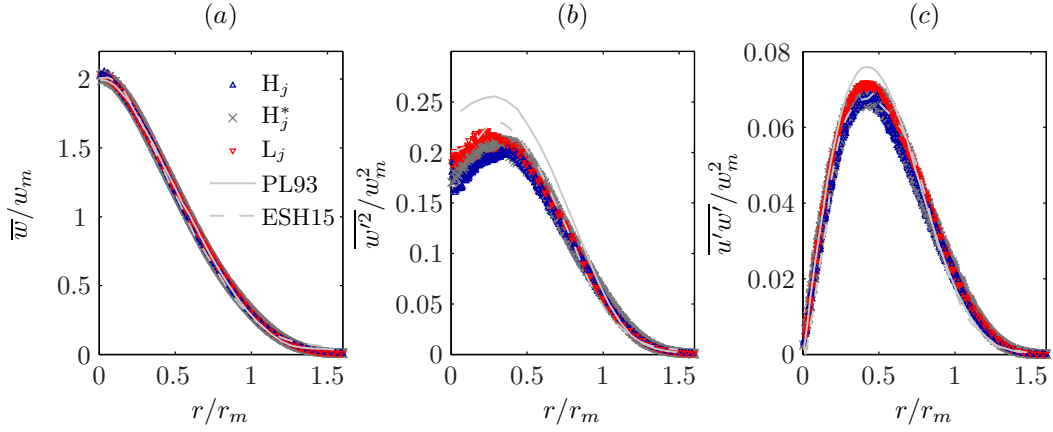


FIGURE 4.2. Dimensionless profiles of longitudinal velocity (a), longitudinal velocity variance (b) and Reynolds stress (c). The simulation data are compared to the experimental data of Panchapakesan & Lumley (1993a, PL93) and Ezzamel *et al.* (2015, ESH15). The displayed simulation data consists of approximately 80 different radial profiles taken from the interval $z/r_s \in [28, 55]$. The experimental data from ESH15 and PL93 were obtained from the single location $z/r_s = 30$ and an average over the interval $z/r_s \in [120, 240]$, respectively.

of a gauze used in experiments (see e.g. Hunt & Linden, 2001). In particular, we obstruct a purely circular source with two centralised orthogonal strips, thereby disconnecting the source into four identical discrete segments, as shown in figure 4.1(a). Consequently, the characteristic source radius r_s is slightly smaller than the physical extent of the source, whose overall diameter is spanned by 48 cells. It should be noted that simulations without the gauze yield similar results in the far field to those presented here, albeit with a smaller far-field region. On the vertical faces of the domain and the top, horizontal face we impose inflow and outflow boundary conditions, respectively, as described in §3.5. A summary of the simulation details is provided in table 4.1.

4.1.2 Validation

In figure 4.2 we compare radial profiles of mean longitudinal velocity $\bar{w}$, longitudinal velocity variance $\overline{w'^2}$, and the Reynolds stress $\overline{u'w'}$, to the experimental data of Ezzamel *et al.* (2015) and Panchapakesan & Lumley (1993a), hereafter referred to as ESH15 and PL93, respectively. Figure 4.2 displays approximately 80 radial profiles from the simulation data, each taken from the interval $z/r_s \in [28, 55]$ and normalised using the characteristic radius $r_m(z) \stackrel{\text{def}}{=} Q_m/M_m^{1/2}$ and characteristic velocity $w_m(z) \stackrel{\text{def}}{=} M_m/Q_m$. The same normalisation was applied to the exper-

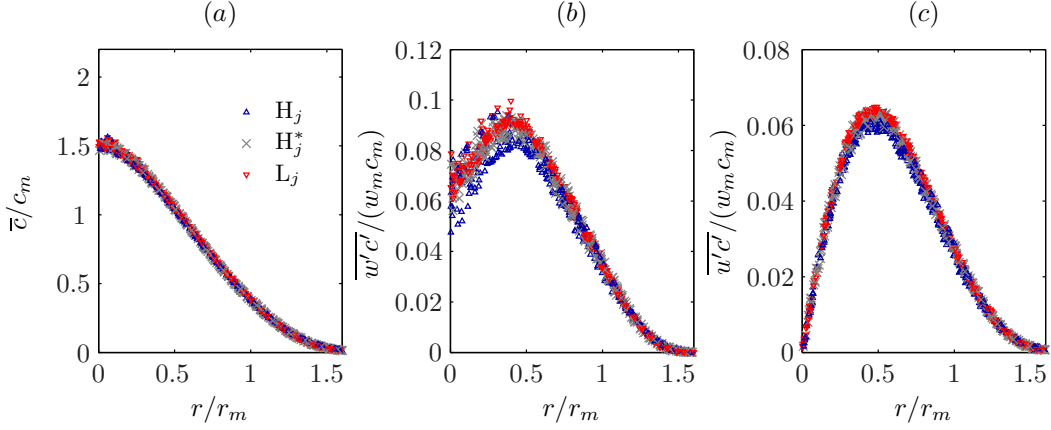


FIGURE 4.3. Steady-state self-similar concentration statistics: (a) mean scalar concentration; (b, c) dimensionless turbulent fluxes. Here $\overline{w'c'}$ corresponds to the longitudinal turbulent scalar flux and $\overline{u'c'}$ to the radial turbulent scalar flux.

imental data, although in PL93 the interval from which the data were obtained was $z/r_s \in [120, 240]$ and in ESH15 the data correspond to the single location $z/r_s = 30$. In the mean longitudinal velocity shown in figure 4.2(a) the simulation results exhibit a good collapse relative to each other, and to both ESH15 and PL93. In figure 4.2(b) it is evident that the simulation results approximately reproduce both the amplitude and the radial dependence of the dimensionless turbulent transport $\overline{w'^2}/w_m^2$ found in ESH15. With respect to PL93, the turbulent transport $\overline{w'^2}/w_m^2$ in both the simulation data and ESH15 is smaller in amplitude, although it has a similar radial dependence. The likely cause of this difference is the fact that the normalised longitudinal fluctuations w'/w_m adjust to equilibrium over extremely large distances. Indeed, PL93 point out that equilibrium is not attained until approximately $z/r_s = 140$. However, the effect that this has on the integral quantities in the jet is smaller than one might expect, because the values most strongly affected are concentrated on the centreline. In the Reynolds stress $\overline{u'w'}$, shown in figure 4.2(c), the numerical results exhibit a good agreement with those of ESH15, whose amplitude again lies slightly beneath the far-field data of PL93.

The dimensionless radial profiles of the steady scalar field $\bar{c}$ and the turbulence fluxes $\overline{u'c'}$ and $\overline{w'c'}$ are shown in figure 4.3. We define the characteristic concentration c_m , with which the quantities in figure 4.3 are normalised, using the *integral* concentration C_m :

$$C_m \stackrel{\text{def}}{=} 2 \int_0^{r_d} \bar{c} r dr, \quad (4.1)$$

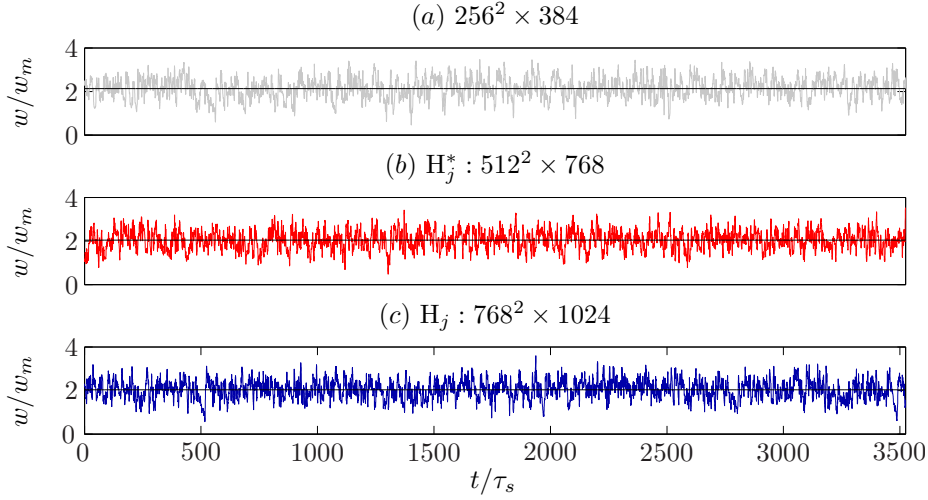


FIGURE 4.4. Time series of the longitudinal velocity w at $z/r_s = 33$, corresponding to different grid refinements: (a) low resolution; (b) medium resolution; (c) high resolution. The time scale $\tau_s \stackrel{\text{def}}{=} Q_s^2/M_s^{3/2}$ is the source turnover time. For the spectra corresponding to the time series see figure 4.5.

such that $c_m \stackrel{\text{def}}{=} C_m M_m / Q_m^2$. The upper limit of integration $r_d \gg r_m$ used in (4.1) will be discussed in §4.3.2. Consistent with the treatment of the velocity statistics, the profiles were taken from the range $z/r_s \in [28, 55]$ and are convincingly self-similar. Indeed, the comparison between case L_j and H_j in figures 4.2 and 4.3 supports the view that the large-scale, integral statistics are not significantly influenced by the Reynolds number and their comparison to case H_j^* demonstrates that the results are independent of the spatial grid resolution. Noteworthy in figure 4.3(b) is that the dimensionless longitudinal turbulent transport of c , namely $\overline{w'c'}$, is significantly less than that of w , namely $\overline{w'^2}$, in figure 4.2(b). In contrast, the radial transport terms in figures 4.2(c) and 4.3(c) are almost identical in form and amplitude.

Time series of the centreline velocity in the jets were obtained at several longitudinal locations; those corresponding to $z/r_s = 33$ are displayed in figure 4.4. Note that since we normalise the velocities using the characteristic scale w_m , it is to be expected that on the centreline the normalised velocities are centred around 2. Indeed, the normalised centreline velocity displayed in figure 4.2(a) is approximately equal to 2. To infer spatial properties of the turbulence from the temporal data we invoke Taylor's hypothesis. Taylor (1938) reasoned that for flows in which the turbulent fluctuations w' are small compared to the mean velocity $\bar{w}$ with which they are advected, the spatial structure of the turbulence can be regarded as frozen

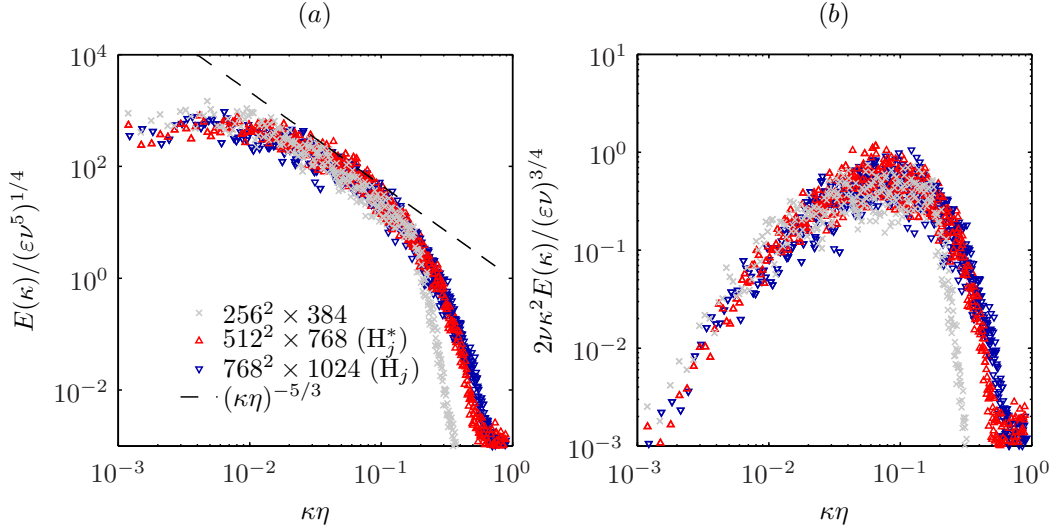


FIGURE 4.5. The effect of grid refinement in the convergence of the energy spectrum (a) and the dissipation spectrum (b). See §4.1.2 for definitions of the dissipation rate ε and the microscale η .

as it passes a fixed point. Consequently, one assumes that $\overline{w}\partial_z \sim \partial_t$ when $w' \ll \overline{w}$. To this end, the use of centreline data, where $\overline{w}$ is a maximum, is a sensible choice. Although, as figure 4.2(b) confirms, the turbulence intensity is not negligible (see also Mi *et al.*, 2013). Rather than employing a modified version of Taylor’s hypothesis (see, e.g., Mi & Antonia, 1994), we use the basic version and accept it as a first approximation. An estimation of the dissipation rate was therefore calculated according to $\varepsilon \approx 15\nu(\partial_t w')^2/\overline{w}^2$ and of the microscale according to $\eta \stackrel{\text{def}}{=} \nu^{3/4}/\varepsilon^{1/4}$. The Taylor Reynolds number for the flow, defined according to $\text{Re}_\lambda \stackrel{\text{def}}{=} \overline{w}'^2\sqrt{15/\varepsilon\nu}$ (Tennekes & Lumley, 1972), was found to be equal to 100 and 135 in simulations L_j and H_j , respectively.

Taking a Fourier transform of the longitudinal centreline velocity at $z/r_s = 33$, in addition to the lateral components of the velocity, and squaring the result yields the energy spectrum at the fixed location $\mathbf{x}/r_s = (0, 0, 33)$. More precisely, to obtain the spectrum we split the domain of each signal into three equal-sized windows. We then filter the signal over each window using a Hamming window, before averaging the result. Using Taylor’s hypothesis, we map the frequency domain f onto the wavenumber domain κ according to $\kappa = 2\pi f/\overline{w}$, where $\overline{w}$ is the mean velocity at the point in question. In figure 4.5 we display the energy spectrum and the dissipation spectrum at $z/r_s = 33$, which is relatively close to the lower bound ($z/r_s = 28$) of the sub-domain from which we sample to obtain longitudinal

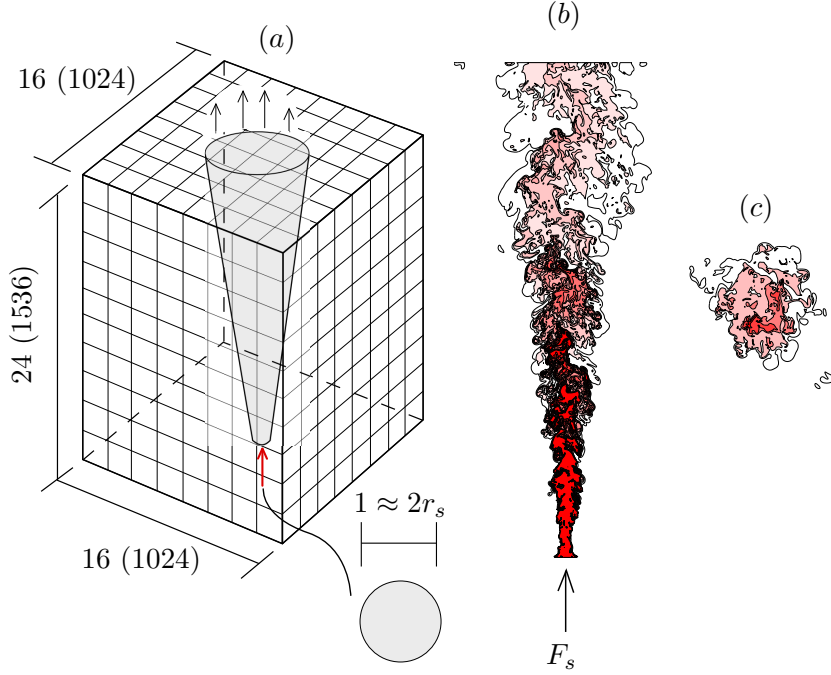


FIGURE 4.6. (a) Computational domain for a plume simulation, consisting of a rectangular cuboid of dimensions $L_x L_y L_z = 16^2 \times 24$ source diameters, discretised using $1024^2 \times 1536$ computational control volumes. (b) Vertical slice through passive scalar isoregions. (c) Horizontal slice through passive scalar isoregions at $z/L_z = 2/3$.

averages. The figure provides results from simulations employing a progressively higher resolution. Whilst the simulation employing $256^2 \times 384$ computational cells does not provide sufficient resolution at the high wave numbers on which dissipation occurs, the spectra obtained from $512^2 \times 768$ (H_j^*) and $768^2 \times 1152$ (H_j) are almost indistinguishable. Also noteworthy is the fact that when inferred from $\eta \stackrel{\text{def}}{=} \nu^{3/4}/\varepsilon^{1/4}$, the microscale that characterises the smallest length scales in the flow is an order of magnitude smaller than the length scale at which the dissipation appears to primarily occur.

4.2 Steady plumes

4.2.1 Simulation details

Statistically steady plumes were obtained from a direct numerical simulation of the Navier-Stokes equations on a domain of size $32 \times 32 \times 48$ source radii r_s , uniformly discretised using $1024 \times 1024 \times 1536$ computational control volumes. For plumes, which develop rapidly in the longitudinal direction, we employ a simply-connected circular source (see figure 4.6). In the simulation of plumes, we use open boundary

TABLE 4.2. Summary of steady-state turbulent plume simulation details.

	$(L_x \times L_y \times L_z)/r_s$	$N_x \times N_y \times N_z$	Re_s	t_{run}/τ_s
L_p	$44^2 \times 66$	$1024^2 \times 1536$	1260	182
H_p	$44^2 \times 66$	$1024^2 \times 1536$	1600	173
H_p^*	$44^2 \times 66$	$512^2 \times 768$	1660	357

conditions that allow fluid and, in particular, buoyancy, to enter and leave the finite computational domain in a manner that is consistent with a semi-infinite unbounded domain (see §3.5 and appendix A for further details).

The plumes we simulate are driven by an isolated circular source of buoyancy flux F_s located at the centre of the bottom face of the domain. Fluxes of volume and momentum at the source are equal to zero, hence these plumes are dominated by the effects of buoyancy in the vicinity of their source and are commonly referred to as being ‘lazy’ (Hunt & Kaye, 2005). The Prandtl number $\text{Pr} \stackrel{\text{def}}{=} \nu/\kappa$ used in the simulations is equal to 0.71, which corresponds to air. For plumes, as well as jets, we apply uncorrelated perturbations of amplitude 1% to the velocities in the first cell above the source to initiate turbulence. Through specification of the source buoyancy flux, we simulate plumes L_p and H_p with source Reynolds number $\text{Re} \stackrel{\text{def}}{=} 2F_s^{1/3}r_s^{2/3}/\nu$ equal to 1260 and 1600, respectively. In addition, we simulate plume H_p^* , employing half as many computational cells as H_p along any given dimension, to check convergence. Following initial transients, simulations L_p and H_p were run for a duration of approximately $170\tau_s$, where the source turnover time $\tau_s \stackrel{\text{def}}{=} r_s^{4/3}F_s^{-1/3}$. A summary of the simulation details is provided in table 4.2.

4.2.2 Validation

To validate the steady plume data we compare the simulation results to the experimental results of Wang & Law (2002) in figure 4.7. To normalise quantities involving buoyancy we define the characteristic buoyancy b_m using the *integral* buoyancy B_m :

$$B_m \stackrel{\text{def}}{=} 2 \int_0^{r_d} \bar{b} r dr, \quad (4.2)$$

such that $b_m \stackrel{\text{def}}{=} B_m M_m / Q_m^2$. In the leading-order quantities displayed in figure 4.7(a, d, c, f), i.e. dimensionless $\bar{w}$, $\bar{b}$, $\overline{u'w'}$ and $\overline{u'b'}$ (note that radial transport terms

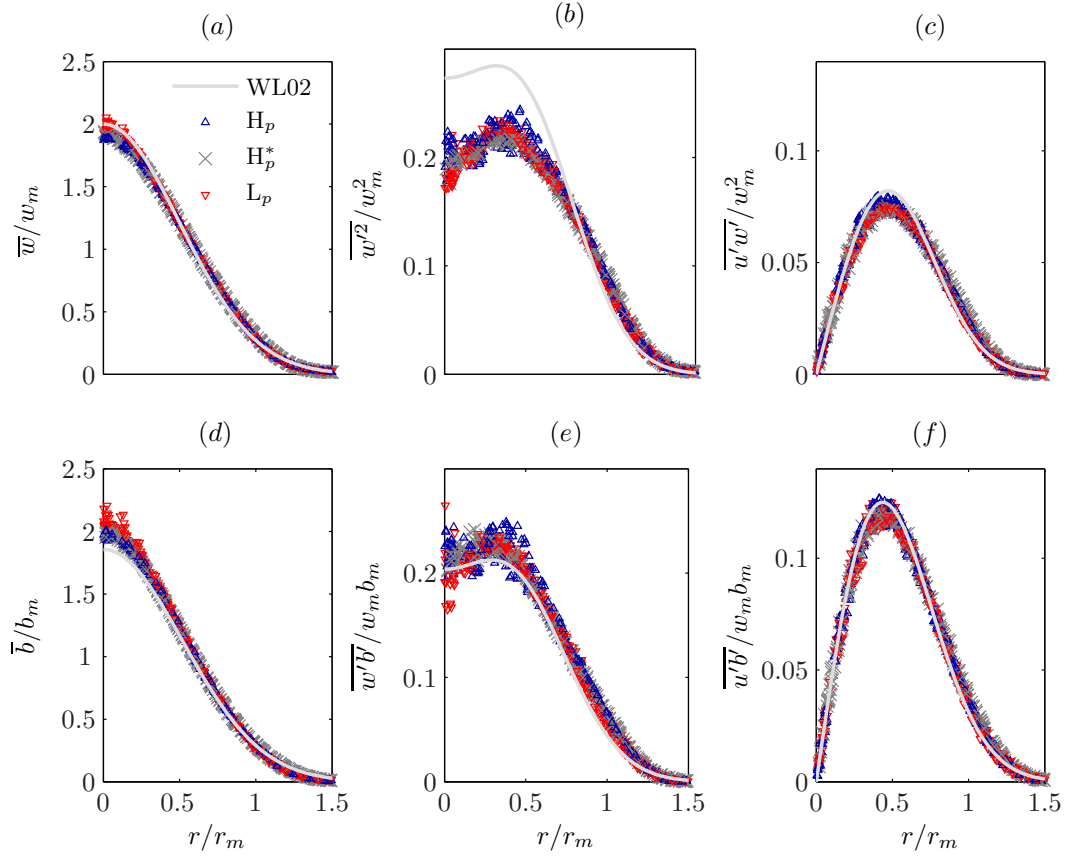


FIGURE 4.7. Dimensionless radial profiles of quantities in a steady-state plume. DNS data from simulations L_p , H_p^* and H_p are compared to the results of Wang & Law (2002), which consist of a fit to data obtained over the range $62 < z/r_s < 110$. The DNS data were obtained over the range $20 < z/r_s < 40$.

are an operand of ∂_r in the governing equations and therefore make a leading-order contribution) the simulations and experiments are in good agreement. In particular, the normalised simulation results comprise self-similar profiles which, consistent with the assumption of high Re , do not exhibit a dependence on the different source buoyancy fluxes used in L_p and H_p . Consistent with our observations of steady jets, there is, however, an observable discrepancy between the experimental data for w' and the simulation data for w' , although it should be noted that this discrepancy is made more pronounced by the fact that we have normalised the data using integral quantities rather than centreline values. Indeed, whilst there is a noticeable difference in the normalised centreline values of w' between the experiments and the simulations, owing to the apparently small difference in their widths, they share similar values of the normalised radial integral of $\overline{w'^2}r$. Furthermore, the profiles

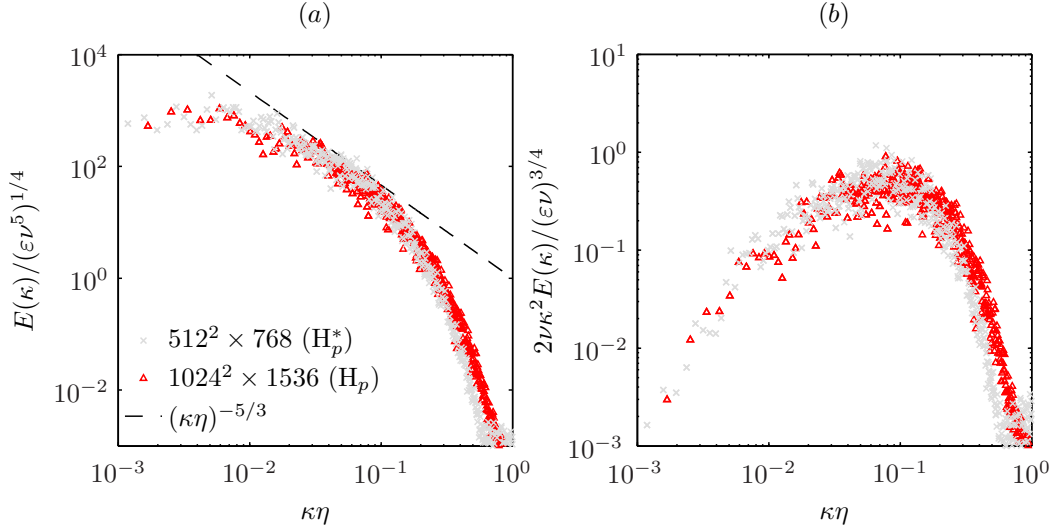


FIGURE 4.8. The effect of grid refinement in the convergence of the energy spectrum (a) and the dissipation spectrum (b) in a turbulent plume at $z/r_s = 24$. See §4.1.2 for definitions of the dissipation rate ε and the microscale η .

in Wang & Law (2002) were obtained over the range $62 < z/r_s < 110$, which is significantly further from the source than the range $20 < z/r_s < 40$ used in the present study. Since $\overline{w'^2}$ is a relatively high-order quantity it is reasonable to expect that it converges to a universal form at greater distances from the source than leading-order profiles such as $\overline{w}$.

In figure 4.8(a, b) we display the energy spectrum and dissipation spectrum, respectively, for simulations H_p and H_p^* , the latter employing half as many grid cells as the former along a given dimension. The spectra were obtained on the centreline of the plume at a longitudinal location $z/r_s = 24$, which is close to the lower end of the interval $z/r_s \in [20, 40]$ on which we will later focus. The spectra for H_p and H_p^* exhibit a reasonably good agreement for all but the highest wave numbers ($\kappa\eta \sim 1$). Consistent with the spectra obtained for the steady jet, we see an approximate correspondence of the energy spectrum with a $(\kappa\eta)^{-5/3}$ scaling in an inertial subrange ($0.01 < \kappa\eta < 0.1$) and note again that the estimated microscale η corresponds to scales that are smaller than those on which dissipation appears to primarily occur.

Owing to the presence of buoyancy, the Reynolds number in a plume increases with respect to the longitudinal coordinate z . At $z/r_s = 24$, which corresponds to the location at which the spectra displayed in figure 4.8 were obtained, the Taylor Reynolds number Re_λ is approximately equal to 100.

4.3 Integrals

4.3.1 The plume area

We are ultimately concerned with radial integrals of the governing equations. For example, instead of the local velocity $\bar{w}(r, z, t)$, we will be interested in the behaviour of the volume flux

$$Q_m \stackrel{\text{def}}{=} 2 \int_0^{r_d} \bar{w} r dr, \quad (4.3)$$

where r_d is a radial location that is typically chosen such that it lies outside the jet or the plume. Looking at higher-order integrals such as the momentum flux M_m and the energy flux E_m , defined according to

$$M_m \stackrel{\text{def}}{=} 2 \int_0^{r_d} \bar{w}^2 r dr, \quad E_m \stackrel{\text{def}}{=} 2 \int_0^{r_d} \bar{w}^3 r dr, \quad (4.4)$$

in addition to turbulent quantities, will allow us to characterise all aspects of the flow that are relevant to the governing integral equations.

Recall from §3.6.2 that the azimuthal average of a quantity φ at the location $r_c(k) = \Delta x/2 + k\Delta x$ is

$$a_k(\varphi) \stackrel{\text{def}}{=} \frac{1}{N_a(k)} \sum_{\text{shell}_k} \varphi_{ij}, \quad (4.5)$$

where N_a is the number of cells in the k^{th} shell. We would like to sum the azimuthal averages to obtain integrals over the radius of the flow:

$$\sum_{k=k_1}^{k_2} a_k(\varphi) \delta A_k \sim 2\pi \int_{r_c(k_1)}^{r_c(k_2)} \bar{\varphi} r dr. \quad (4.6)$$

Naively, one could define the area δA_k of a shell as $2\pi r_c(k) \Delta r$, which would indeed converge to the right-hand-side of (4.6) as $\Delta r \rightarrow 0$. However, the contribution $a_k(\varphi) \delta A_k$ should be exactly equal to the summation of $\varphi_{ij} \Delta x \Delta y$ over a shell. In other words we would like the following diagram to commute,

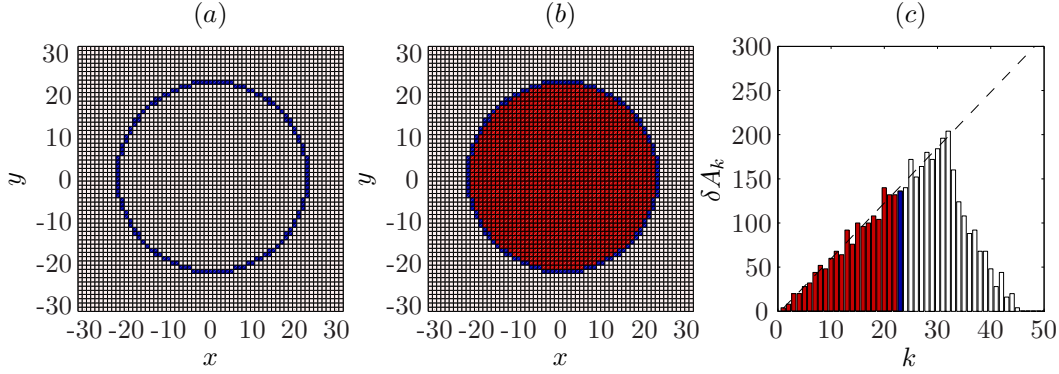


FIGURE 4.9. Determination of the exact area δA_k of a cylindrical shell of radius $r_c(k) = \Delta x/2 + k\Delta x$. (a) Shaded region denotes all cells belonging to shell $_{k_1}$. (b) Shaded region denotes all cells belonging to shell $_k$, where $k \leq k_1$, and therefore represents the discrete area A_{k_1} . (c) The exact area δA_k of each shell (bars) in comparison with $2\pi r_k \Delta r$ (dashed line).

$$\begin{array}{ccc}
 \mathbb{R}^{N_a(k)} & \xrightarrow{\frac{1}{N_a(k)} \sum_{\text{shell}_k}} & \mathbb{R} \\
 & \searrow \sum_{\text{shell}_k} \Delta x \Delta y & \downarrow \delta A_k \\
 & & \mathbb{R}
 \end{array}$$

which implies that $\delta A_k \stackrel{\text{def}}{=} N_a(k) \Delta x \Delta y$. This means, for example, that the volume flux in a shell is *exactly* equal to $\bar{w}_k \delta A_k$, for the azimuthally-averaged velocity $\bar{w}$.

The area allocated to each shell in the domain is illustrated in figure 4.9. Note that when $r_c > L_x/2$, δA_k starts to reduce towards zero, due to the fact that the shell is truncated by the four vertical faces of the domain. The area of the shell is equal to zero when $r_c > L_x/\sqrt{2}$. The dashed line in figure 4.9(c) indicates that for finite Δr , $\delta A_k \stackrel{\text{def}}{=} N_a(k) \Delta x \Delta y$ is close, but not equal, to $2\pi r_c(k) \Delta r$.

4.3.2 Boundary terms

Although it is conventional to let the upper limit of integration $r_d \rightarrow \infty$ in integrals such as (4.3), there are several practical problems associated with such an approach. First, when $r_d \rightarrow \infty$ Q_m in (4.3) no longer represents the volume flux in the turbulent part of the free-shear flow with which we are primarily concerned. Second, integrals such as (4.3) do not necessarily converge to a finite value when $r_d \rightarrow \infty$, and even when they do, are sensitive to the assumed nature of the induced flow in

the ambient (see Kotsovinos, 1977, for further details). Third, in most experimental or numerical setups the domain is of finite lateral extent, which therefore forces one to integrate to a finite radial limit. These problems led Kotsovinos & List (1977) to use a finite value of r_d in defining integrals such as (4.3). In this section we explain the implications of integrating to a finite point in the radial direction.

In a steady state, a term appearing in the integrated equations of motion might have the form

$$2 \int_0^{r_d} \frac{\partial \bar{w} \bar{c}}{\partial z} r dr, \quad (4.7)$$

where $\bar{c}$ and $\bar{w}$ are the local dependent variables. To incorporate integrals such as the mean flux

$$F_m \stackrel{\text{def}}{=} \int_0^{r_d} \bar{w} \bar{c} r dr, \quad (4.8)$$

it proves convenient to commute differentiation with integration. However, due to the fact that the upper limit of the integral might depend on z (and, more generally, time), integration and differentiation are not commutative, the ‘commutator’ being given by Leibniz’s rule according to

$$\frac{dF_m}{dz} - 2 \int_0^{r_d} \frac{\partial \bar{w} \bar{c}}{\partial z} r dr = \bar{w} \bar{c} \frac{dr_d^2}{dz} \Big|_{r=r_d}. \quad (4.9)$$

Evidently for the *boundary flux* on the right-hand-side of (4.9) to be significant there must be a non-zero longitudinal scalar flux $\bar{w} \bar{c}$ at $r = r_d$ and the normal direction of the bounding surface must have a finite longitudinal component. Equation (4.9) is useful because it allows one to understand the integral balances in a free-shear flow at any radial location. For example, if one selects r_d to be the point at which $\bar{u} = 0$ in a self-similar jet*, one finds a balance between longitudinal transport by the mean flow and radial transport by turbulence†:

$$\overline{u'c'} \Big|_{r=r_d} = \frac{dr_d}{dz} \bar{w} \bar{c} \Big|_{r=r_d}. \quad (4.10)$$

Figure 4.10 illustrates the balance described by (4.10).

More generally, we will often wish to integrate transport equations of the form

*Volume conservation $\nabla \cdot \mathbf{u} = 0$ ensures that at least one such point exists for a given z .

†This follows from integration of the local steady-state transport equations for the scalar $\bar{c}$ from $r = \infty$ to $r = r_d$.

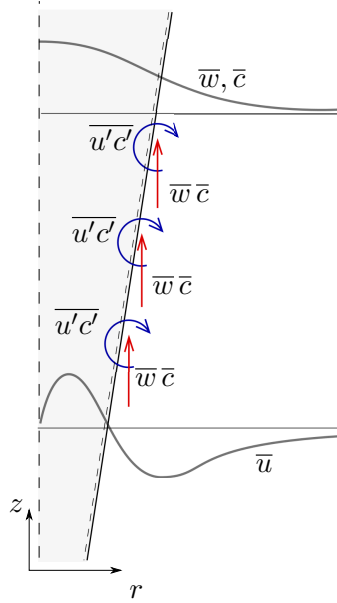


FIGURE 4.10. Example of boundary fluxes in a scalar transport equation. Here the boundary is defined by points at which $\bar{u} = 0$ and therefore $\overline{u'c'} = (dr_d/dz)\bar{w}\bar{c}$ on $r = r_d$.

$$\frac{\partial \bar{c}}{\partial t} + \frac{\partial \bar{w}\bar{c}}{\partial z} = \dots, \quad (4.11)$$

which will result in integral equations of the form

$$\frac{\partial C_m}{\partial t} + \frac{\partial F_m}{\partial z} = \frac{C_m M_m^{3/2}}{Q_m^2} \partial_m^C + \dots, \quad (4.12)$$

where C_m is the integral quantity of interest, F_m is the flux of that integral quantity and the dimensionless boundary contribution

$$\partial_m^C \stackrel{\text{def}}{=} \frac{Q_m^2}{C_m M_m^{3/2}} \left(\bar{c} \frac{Dr_d^2}{Dt} \right)_{r=r_d}, \quad \text{and} \quad \frac{D}{Dt} \stackrel{\text{def}}{=} \frac{\partial}{\partial t} + \bar{w}(r_d) \frac{\partial}{\partial z}. \quad (4.13)$$

When C_m is equal to the volume flux Q_m (i.e. when equation 4.11 is an integral momentum balance), the dimensionless boundary contribution is

$$\partial_m^Q \stackrel{\text{def}}{=} \frac{Q_m}{M_m^{3/2}} \left(\bar{w} \frac{Dr_d^2}{Dt} \right)_{r=r_d}. \quad (4.14)$$

The form of ∂_m^Q suggests that to minimise the effect of boundary fluxes, we should choose r_d such that $\bar{w}(r_d, z, t)$ is relatively small[‡]. Conversely, we do not wish for

[‡]This is further supported by the fact that the longitudinal velocity in free-shear flows typically decays exponentially with the similarity variable $r/r_m(z)$.

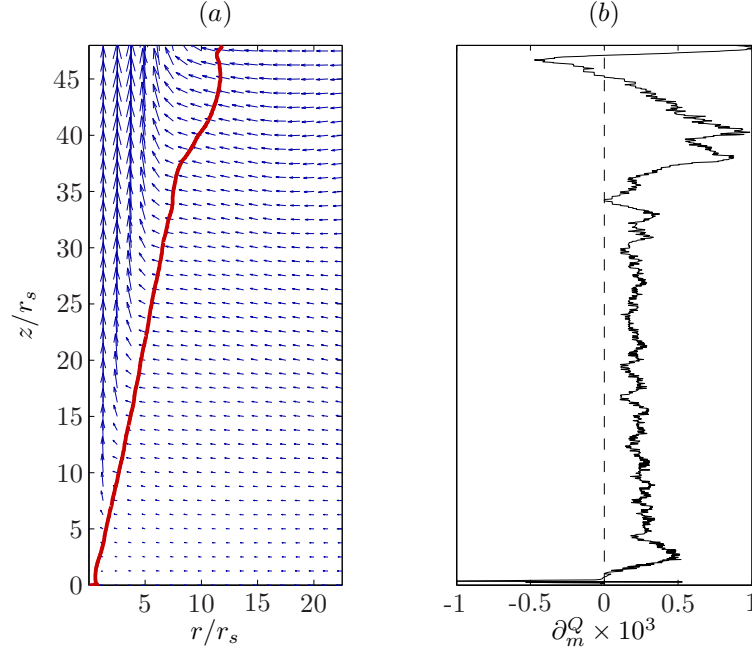


FIGURE 4.11. (a) Velocity field $r\mathbf{u} \stackrel{\text{def}}{=} (ru, rw)$ (blue arrows) and upper limit of integration $r_d(z)/r_s$, in a steady plume. (b) Dimensionless boundary contribution ∂_m^Q to the momentum equation.

$\bar{w}(r_d, z, t)$ to be so small that our integrals depend sensitively on the ambient (see, e.g., the remarks at the start of this section or Kotsovinos, 1977). Thus, we define r_d such that

$$\bar{w}(r_d, z, t) = \epsilon w_m(z, t), \quad (4.15)$$

where $w_m \stackrel{\text{def}}{=} M_m/Q_m$ is the characteristic velocity in the flow and, in practice, $\epsilon \stackrel{\text{def}}{=} 0.01$.

Figure 4.11(a) displays the threshold radius $r_d(z)$, defined according to (4.15), in a steady plume. Consistent with the self-similarity of the flow, $r_d(z)$ scales linearly with respect to z for $5 < z/r_s < 35$. Above $z/r_s = 35$ the outflow boundary condition appears to cause some distortion of the induced flow field, on which the precise location at which $\bar{w} = 0.01 w_m$ depends sensitively. A relatively small $O(0.01)$ variation in the longitudinal velocity in the ambient results in a significant change in r_d . Nevertheless, the effect of the boundary flux of momentum is relatively small, as evidenced by the $O(0.001)$ values of the dimensionless parameter ∂_m^Q in figure 4.11(b).

With the aim of capturing the turbulent jet rather than the ambient motion, there are other possible definitions of r_d , some of which circumvent the sensitivity

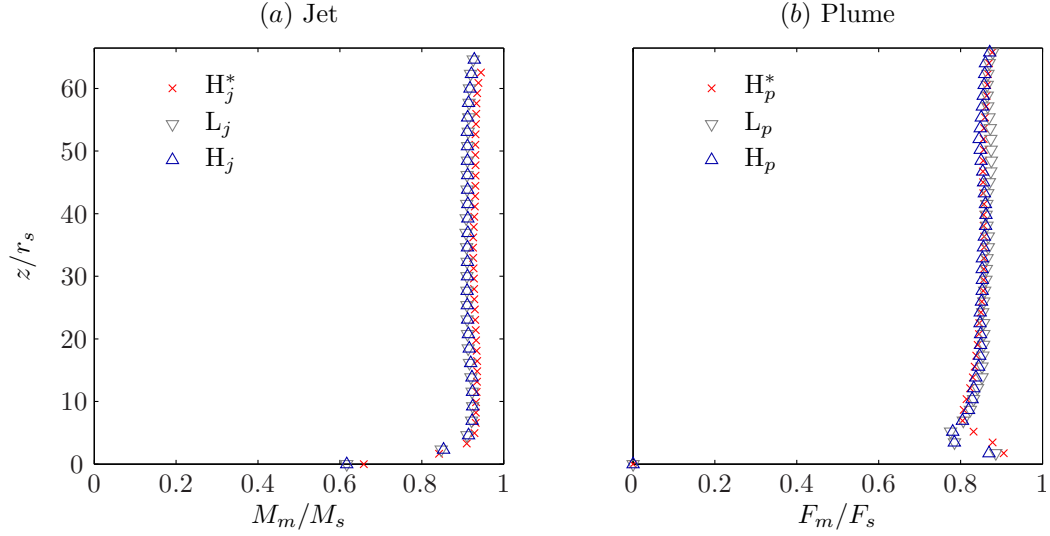


FIGURE 4.12. (a) Mean momentum flux M_m in a steady jet. (b) Mean buoyancy flux F_m in a steady plume.

that our definition (4.15) has on the ambient velocity. Examples include using $\overline{u'w'}$, or simply specifying a linear relation $r_d \propto z$, with a coefficient of proportionality that is substantially larger than the entrainment coefficient. However, we choose to use (4.15) because it is an intrinsic property of the mean flow field and means that the magnitude of the boundary contribution (4.14), for which $\overline{w}(r_d)/w_m = \epsilon$ is known *a priori*, depends solely on the behaviour of the threshold radius r_d .

4.3.3 Conserved quantities in jets and plumes

We will discuss the behaviour of integral quantities in both steady and unsteady jets and plumes extensively in chapters 5-8. Therefore, to conclude the present chapter we restrict our attention to the integral momentum flux M_m in a jet and the integral buoyancy flux F_m in a plume. Owing to the absence of external forces, in the case of the jet, and sources/sinks of buoyancy, in the case of the plume, M_m and F_m are conserved quantities in jets and plumes, respectively. Note that, due to the presence of buoyancy, the momentum flux M_m in a *plume* is not a conserved quantity. Figure 4.12(a) displays the momentum flux M_m that we observe in simulations of the steady jets (described in §4.1.1), whilst figure 4.12(b) displays the buoyancy flux F_m in the steady plumes (described in §4.2.1). As expected, both M_m and F_m are approximately invariant with respect to z , although one can discern variations in M_m and F_m in the near field (e.g. $z/r_s < 10$).

To understand the behaviour of M_m and F_m it is instructive to consider the

possible cause of their variation with respect to z . First, statistical estimators obtained over a finite sampling period only provide an approximation to the underlying random variables of a process. Here, estimators obtained over a finite period of time Δt are time invariant when $\Delta t \rightarrow \infty$. The relatively small variations in F_m at $z/r_s \approx 50$ are due to the fact that over a finite sampling period $\partial_t F_m \neq 0$ which, in general, implies that $\partial_z F_m \neq 0$. Second, as discussed in the previous section, the fluxes in jets and plumes can be affected by boundary contributions. However, as previously demonstrated, with a suitable selection of r_d such contributions are relatively small. Third, and most significantly, the expectation of conservation of a *mean* flux in a jet or a plume, such as M_m or F_m is predicated on the assumption that the corresponding *turbulent* fluxes evolve in constant proportion to the mean flow (see, e.g., figures 4.2 and 4.3). In their near-fields, jets and plumes do not exhibit the self-similarity that is evident in figures 4.2, 4.3 and 4.7. Consequently, turbulence transport (e.g., radial integrals of $\overline{w'b'}$ and $\overline{w'^2}$) cannot be characterised in terms of mean-flow quantities and one no longer sees conservation in the mean fluxes. This point will be elaborated in chapters 5-8, in which the integral conservation equations will be examined in detail.

Chapter 5

Shear-flow dispersion in turbulent jets

In this chapter we investigate the transport of a passive scalar in a fully-developed turbulent axisymmetric jet at $\text{Re} = 4815$. The material we present has been accepted for publication (Craske *et al.*, 2015) and is reproduced here with permission. The particular problem we consider is the response of the concentration field to an instantaneous change in the scalar flux at the source. To analyse the time evolution of this statistically unsteady process we take an ensemble average over 16 independent direct numerical simulations. We find that the evolution of the scalar integral $C_m(z, t)$ is a self-similar process, with front position and spread both scaling as $\sqrt{t}$. One of the central results is that the longitudinal mixing of C_m is observed to be primarily caused by shear-flow dispersion. A mathematical model of the problem is constructed by applying Taylor's (1953) classical theory for shear-flow dispersion to a turbulent jet to obtain a closure that couples the integral scalar flux to the integral concentration C_m . Model predictions using the dispersion closure are in good agreement with the simulation data, and application of the dispersion closure to a two-dimensional jet results in an integral transport equation that is fully consistent with Landel *et al.* (*J. Fluid Mech.*, vol. 711, 2012, pp. 212–258).

5.1 Introduction

Shear-flow dispersion (Taylor, 1953, 1954*b*; Aris, 1956) is one of the primary sources of mixing in integral models for passive scalar transport and as such has wide-ranging practical applications, including contaminant transport in the atmosphere and ocean, nutrient delivery, the spread of smoke from fires and the discharge of waste effluent in streams. Caused by lateral (cross-stream) gradients in a mean velocity, shear-flow dispersion was first identified by Taylor (1953), who examined the transport of a solute in both laminar (Taylor, 1953) and turbulent pipe flow (Taylor, 1954*b*). It was demonstrated that for sufficiently large times (Taylor, 1954*a*) the effective longitudinal (streamwise) mixing is determined by a balance between longitudinal advection and radial mixing.

The canonical example of shear-flow dispersion is the release of a passive scalar into a bounded, one-dimensional flow in a pipe or a plane channel (see figure 5.1). At $t = 0$, one assumes that the mean scalar concentration $\bar{c}$, where the overline denotes an ensemble average, is uniformly distributed over the pipe for $z \leq 0$ and

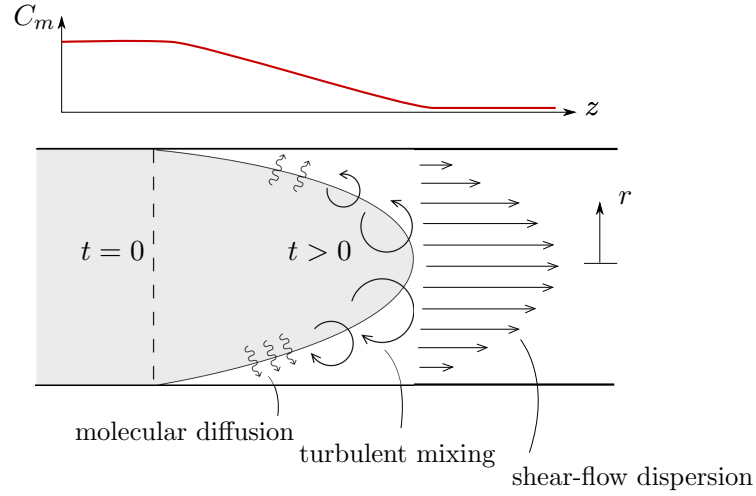


FIGURE 5.1. Longitudinal mixing processes in a pipe flow.

equal to zero for $z > 0$. One is typically interested in the integral concentration

$$C_m(z, t) \stackrel{\text{def}}{=} 2 \int_0^{r_d} \bar{c} r \, dr, \quad (5.1)$$

where the upper limit of integration r_d corresponds to the radial extent of the flow. Hence, for a pipe of radius r_m , $r_d = r_m$ and $C_m = r_m^2 \langle \bar{c} \rangle$, where $\langle \rangle$ denotes an average over the pipe's cross section. At $t = 0$ the dependence of C_m on z corresponds to a Heaviside step function. For time $t > 0$ the step change in C_m will become progressively smoother owing to three distinct, yet closely related, physical processes.

- D1 The diffusive flux $-\kappa \partial_z \bar{c}$ (molecular diffusion).
- D2 The ensemble covariance $\overline{w\bar{c}} - \bar{w}\bar{c}$ (turbulent mixing).
- D3 The spatial covariance $\langle \bar{w}\bar{c} \rangle - \langle \bar{w} \rangle \langle \bar{c} \rangle$ (shear-flow dispersion).

Molecular diffusion is a manifestation of an averaging operation over individual particles in the fluid and is expected to play a negligible role at large scales in high-Reynolds-number flows. Turbulent mixing is a result of the chaotic motion of the flow, the covariance $\overline{w\bar{c}} - \bar{w}\bar{c}$ corresponding to a turbulent scalar flux. Shear-flow dispersion, on the other hand, is the result of a correlation of the average velocity $\bar{w}$ with the average concentration $\bar{c}$ over *space*, and enters the problem

because the integral concentration C_m is the primary unknown. When regions of high mean concentration coincide spatially with regions of high longitudinal mean velocity (see figure 5.1), $\langle \bar{w} \bar{c} \rangle \neq \langle \bar{w} \rangle \langle \bar{c} \rangle$, and one finds a dispersive flux. Notice that the turbulent scalar flux and the dispersive flux in D2 and D3 are defined relative to the ensemble average scalar flux, $\bar{w} \bar{c}$, and the spatially averaged flux, $\langle \bar{w} \rangle \langle \bar{c} \rangle$, respectively. In many large-scale practical applications, such as open channel flow $D3 \gg D2 \gg D1$ (see e.g. Elder, 1959). Moreover, Taylor (1953, 1954*b*) demonstrated that longitudinal dispersion is inversely proportional to the turbulent diffusion coefficient associated with the flow, indicating that mixing by turbulence inhibits shear-flow dispersion. Further details can be found in §5.5.1, in which we summarise the dispersion theory of Taylor (1953) in the context of pipe flow. For a general introduction to dispersion theory the reader is referred to Chatwin & Allen (1985).

Taylor's original analysis of pipe-flow dispersion was subsequently generalised to pipes of arbitrary cross section by Aris (1956), who obtained solutions for the longitudinal moments of the solute concentration and demonstrated that the longitudinal dispersion coefficient and longitudinal turbulent diffusion coefficient are additive. Gill (1967) later considered a general series expansion, involving higher longitudinal derivatives of the mean concentration, to determine the radial dependence of the concentration. Subsequent work has addressed the way in which the concentration profile approaches a Gaussian form (Chatwin, 1970), the asymptotic behaviour for small times (Chatwin, 1977) and the concentration at large distances from its centre of mass (Haynes & Vanneste, 2014). Following the work of Brenner (1980*b*), a generalised Taylor dispersion theory emerged, which was not restricted to the unidirectional flows on which previous studies had focused. Generalised Taylor dispersion theory has been applied in a wide range of fields including sedimentation (Brenner, 1979), flows through porous media (Brenner, 1980*a*) and chemically reacting flows (Shapiro & Brenner, 1986). However, in spite of the many refinements that have been made to Taylor's theory of dispersion it has not, to the authors' knowledge, been applied explicitly in the analysis of scalar dispersion in jets.

More generally, passive scalar transport in turbulent jets has received significant attention in the literature (e.g. Paranthoen *et al.*, 1988; Tong & Warhaft, 1995; Warhaft, 2000). However, the primary focus has been on steady releases; unsteady releases, for which shear-flow dispersion is expected to be significant, have received relatively little attention. An exception is the work by Landel *et al.* (2012), who investigated unsteady scalar transport in a two-dimensional jet, created in the laboratory by confining the jet to a narrow gap between two parallel plates. They

observed that the core region of the jet, which is primarily associated with longitudinal advective transport, is surrounded by eddies that are responsible for mixing. Moreover, this structure was observed to be self-similar with height, which means that when normalised by suitable length and velocity scales the behaviour of the jet is independent of its streamwise coordinate. Longitudinal mixing was attributed to the stretching of fluid that occurs between the core and the eddies. Using a mixing-length assumption, justified by the self-similarity of the flow, Landel *et al.* (2012) formulated an advection-diffusion equation for the transport of the integral $C_m(z, t)$ of a scalar given by

$$\frac{\partial C_m}{\partial t} + K_a M_m^{1/2} \frac{\partial}{\partial z} \left(\frac{C_m}{z^{1/2}} \right) = K_d M_m^{1/2} \frac{\partial}{\partial z} \left(z^{1/2} \frac{\partial C_m}{\partial z} \right). \quad (5.2)$$

Here z is the longitudinal (streamwise) coordinate, M_m is the jet's integral momentum flux, and K_a and K_d are empirical dimensionless parameters accounting for advection and eddy diffusivity, respectively.

In spite of the fact that mixing length models can provide only a limited representation of turbulence (Pope, 2000), Landel *et al.* (2012) show that equation (5.2) provides an accurate means of predicting integral scalar concentrations in two-dimensional jets. However, it is neither clear what physical process determines the value of K_a and K_d , nor possible to state *a priori* which parts of the model can be applied to axisymmetric jets. Furthermore, it is unclear whether the mixing is caused by the longitudinal turbulent scalar flux or shear-flow dispersion (or both). In this chapter we show that for axisymmetric jets, the longitudinal turbulent scalar flux is not able to account for the observed mixing of the scalar integral and that it is shear-flow dispersion that is the predominant cause of the mixing.

The chapter is structured as follows. In §5.2 we define the problem, introducing the governing equations and the variables that are of primary interest. In §5.3 we describe the DNS used to obtain the data and explain how the ensemble average was constructed. In §5.4 we demonstrate that the scalar transport can be viewed as a self-similar process by analysing the propagation speed of the scalar and its longitudinal spreading rate. We explain how Taylor's model for shear-flow dispersion in pipes (Taylor, 1953) can be applied to jets in §5.5, and compare the resulting dispersion closure to DNS observations in §5.6. We apply our dispersion closure to planar jets in §5.7, which reveals that it is in one-to-one agreement with the model proposed by Landel *et al.* (2012). A summary of the findings and ideas for future work are provided in §5.8.

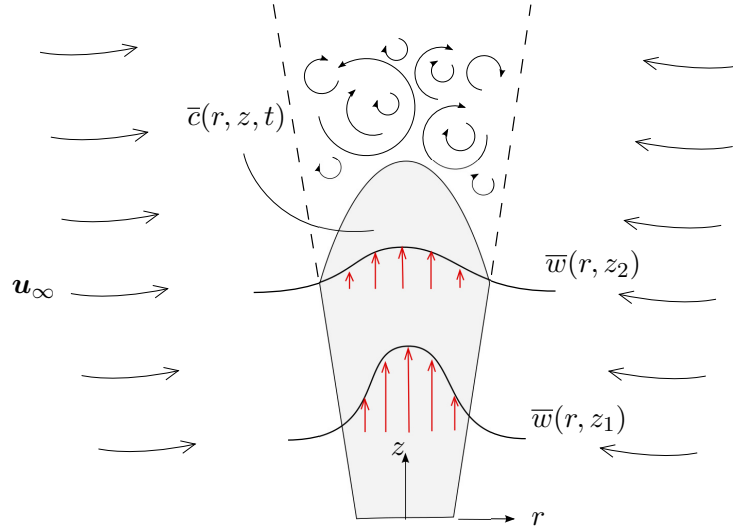


FIGURE 5.2. Definition sketch. The grey area represents regions to which the scalar has spread at a particular time t and $\bar{w}$ is the ensemble-average longitudinal velocity.

5.2 Problem definition

A schematic for the flow considered is provided in figure 5.2. The turbulent jet is statistically steady, axisymmetric and produces a mean velocity field $(\bar{u}, \bar{w})$ in coordinates (r, z) , where r denotes the radial, and z the longitudinal, coordinate. For $t < 0$, the scalar concentration is equal to zero everywhere. For $t \geq 0$, a continuous release of scalar is activated at the inlet (hereafter referred to as the source), which produces a front that propagates and spreads as it travels along the jet. Due to the non-uniform velocity distribution, the scalar will be advected at different rates depending on its position in the jet: fluid elements in the core regions will tend to travel faster than elements in the outer region, thereby modifying the radial distribution of the scalar downstream.

As discussed in chapter 2, the motion of the jet is governed by the incompressible Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}, \quad (5.3)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (5.4)$$

We use cylindrical coordinates (r, ϕ, z) , where r is a radial coordinate, ϕ an azimuthal coordinate and z a longitudinal coordinate, with corresponding velocity

field $\mathbf{u}(r, \phi, z, t) = (u, v, w)$. The fluid is of constant uniform density ρ , the pressure relative to a hydrostatic balance is denoted p and the kinematic viscosity of the fluid is ν . The passive scalar is governed by the transport equation

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = \kappa \nabla^2 c, \quad (5.5)$$

where κ is the molecular diffusion coefficient.

Following §2.2.2 the velocity field is assumed to be statistically stationary and, due to the fact that the mean azimuthal component of velocity is equal to zero, swirl free. It is therefore convenient to use the time and azimuthal average velocity $\bar{\mathbf{u}} = (\bar{u}(r, z), 0, \bar{w}(r, z))$. As explained in §3.6.2, for the scalar concentration c , which is statistically unsteady, we restrict ourselves to an ensemble and azimuthal average $\bar{c}(r, z, t)$.

Our focus is on the behaviour of integrals over lateral slices of the jet. Thus, the volume flux $Q_m(z)$ and the mean momentum flux $M_m(z)$,

$$Q_m(z) \stackrel{\text{def}}{=} 2 \int_0^{r_d} \bar{w} r dr, \quad M_m(z) \stackrel{\text{def}}{=} 2 \int_0^{r_d} \bar{w}^2 r dr, \quad (5.6a, b)$$

will play an important role, where the threshold radius r_d , delineating the extent of the jet, was discussed and defined precisely in §4.3.2. In addition to the integral scalar concentration C_m defined in (5.1), we define the integral mean scalar flux F_m , and the integral turbulent scalar flux F_f , according to

$$F_m(z, t) \stackrel{\text{def}}{=} 2 \int_0^{r_d} \bar{w} \bar{c} r dr, \quad F_f(z, t) \stackrel{\text{def}}{=} 2 \int_0^{r_d} \overline{w'c'} r dr. \quad (5.7a, b)$$

Consistent with our treatment in previous chapters, the volume flux Q_m and momentum flux M_m will be used to define characteristic length and velocity scales $r_m \equiv Q_m/M_m^{1/2}$ and $w_m \equiv M_m/Q_m$, respectively. We define a mean concentration c_m for the flow from the integral concentration C_m according to $c_m \equiv C_m/r_m^2$.

Taking a time and ensemble average of (5.3) and (5.4) for $\text{Re} \stackrel{\text{def}}{=} 2M_m^{1/2}/\nu \gg 1$ and integrating over lateral slices of the jet yields the integral jet equations (see e.g. Rajaratnam, 1976, or §2.4.1)

$$\frac{dQ_m}{dz} = 2\alpha_0 M_m^{1/2}, \quad \frac{dM_m}{dz} = 0, \quad (5.8a, b)$$

which express volume conservation and momentum conservation, respectively, and α_0 is the classical entrainment coefficient (see e.g. Morton *et al.*, 1956). It is

immediately apparent from (5.8*a, b*) that $Q_m = 2\alpha_0 M_m^{1/2}(z - z_v)$ for a conserved momentum flux M_m and a virtual origin z_v , the latter corresponding to the location at which the volume flux is equal to zero.

The ensemble average of (5.5) can be integrated over lateral slices of the jet to give

$$\frac{\partial C_m}{\partial t} + \frac{\partial}{\partial z}(F_m + F_f) = 0, \quad (5.9)$$

which is an integral conservation equation for the scalar and is the principal focus of this work. In particular, our discussion of the problem will focus on the mean and turbulent dimensionless scalar flux, defined as

$$\theta_m \stackrel{\text{def}}{=} \frac{F_m}{w_m C_m}, \quad \theta_f \stackrel{\text{def}}{=} \frac{F_f}{w_m C_m}, \quad (5.10)$$

respectively.

5.3 Simulation details

We simulate an axisymmetric turbulent jet driven by an isolated source of steady momentum flux M_s and volume flux Q_s . The reader is referred to §4.1.1 for details of the simulation and to §3.2 for details of the code. Here we focus on a description of the unsteady aspects of the simulations pertaining to the scalar transport.

To release a passive scalar into the flow at $t = 0$ we impose a step change in the scalar flux at the source:

$$F_g(z = 0, t) = F_s H(t), \quad (5.11)$$

where H is the Heaviside step function and $F_g = F_m + F_f$ is the gross scalar flux. The transport of the scalar is governed by (5.5), with the diffusivity $\kappa = \nu/\text{Sc}$ chosen such that the Schmidt number $\text{Sc} = 0.71$.

Azimuthally averaged data were obtained by partitioning the domain into concentric cylindrical cells as described in §3.6 and §4.3.1. During the course of the steady simulations we write a set of 16 complete three-dimensional field files to disk at time intervals of approximately $250\tau_s$. These field files are used to obtain independent initial velocity conditions for the simulation of 16 statistically unsteady scalar transport simulations, which we shall refer to as U_j1-16. For the scalar field we impose an initial condition of zero concentration uniformly throughout the domain. An unsteady scalar transport simulation is then created by imposing a

TABLE 5.1. Passive scalar simulation details. L_j , H_j and H_j^* are steady-state simulations, as described in §5; U_j1-16 are simulations pertaining to the unsteady release of a passive scalar.

	$L_x L_y L_z / r_s$	$N_x N_y N_z$	Re_s	$Sc \stackrel{\text{def}}{=} \nu / \kappa$	t_{run} / τ_s
L_j	$44^2 \times 66$	$768^2 \times 1152$	4815	0.707	3323
H_j	$44^2 \times 66$	$768^2 \times 1152$	6810	0.707	3524
H_j^*	$44^2 \times 66$	$512^2 \times 768$	6810	0.707	3524
U_j1-16	$44^2 \times 66$	$768^2 \times 1152$	4815	0.707	331

constant flux F_s of scalar at the source, as indicated by (5.11).

During each individual unsteady simulation, rather than outputting discretely sampled instantaneous data, a time average is taken over an interval in time that is much smaller than the characteristic timescale of the problem. Specifically, we employ a time interval not exceeding $2\tau_s$, which prevents artificial smoothing of the front that time averaging would otherwise cause. In the unsteady simulations the azimuthal direction remains statistically homogeneous, which enables us to take azimuthal averages. In addition, the reliability of the statistics was further improved by performing an ensemble average over the 16 unsteady simulations. Therefore, a mean quantity $\bar{\chi}$ obtained from the unsteady simulations is a function of r , z and t . A summary of the simulation details can be found in table 5.1.

From the steady-state simulation data we see that the entrainment rate $\alpha_0 = 0.069$, the virtual origin $z_v = -4.2r_s$ and the dimensionless fluxes $\theta_m = 0.89$ and $\theta_f = 0.08$. Having demonstrated in §4.1.2, figure 4.3, that the integral statistics such as $\overline{w'c'}$ are approximately independent of the Reynolds number Re , we choose to use case L_j at $Re = 4815$, rather than case H_j with $Re = 6810$, to obtain the unsteady data, which ensures that the scalar field has ample resolution and that the maximum cell Péclet number, $\max(\mathbf{u} \cdot \Delta \mathbf{x}) / \kappa$, for grid spacing $\Delta \mathbf{x}$, remains small.

5.4 Propagation and spreading of the scalar front

5.4.1 Scaling

Following a step-change in the source scalar flux, a disturbance in the integral concentration C_m propagates and spreads in the longitudinal direction in the form of a front. The objective of this section is to track the front position, which we

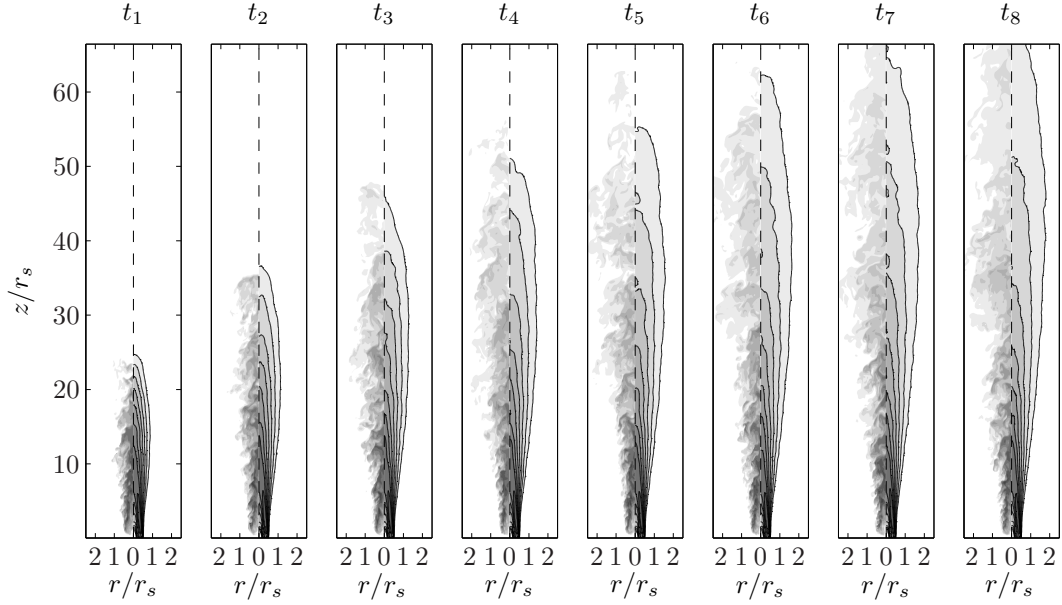


FIGURE 5.3. Isolines of scalar concentration $c(r, z, t = t_i)$, at times $t_i/\tau_s \approx 37i, i = 1 \dots 8$, where $\text{Re} = 4815$. Left side: instantaneous radial slice; right side: azimuthal and ensemble average.

denote $z^*(t)$, and determine the rate at which it spreads. Figure 5.3 displays the scalar field at several instants in time. On the left-hand-side of each window is a vertical slice through the scalar field of a single member of the ensemble and on the right-hand-side is the ensemble and azimuthally averaged scalar field. As the scalar is introduced into the domain it is mixed by the turbulent jet and transported in the positive z -direction. At each time the scalar field has a similar form, in which the widest part of a given isoregion lies approximately midway between the origin and its leading edge in z .

A robust criterion for identifying the location of the scalar front is to define z^* implicitly as

$$F_m[z^*(t), t] = F_m^* \quad (5.12)$$

where $F_m^* \equiv F_{m0}/2$ and is therefore equal to the arithmetic mean of the mean steady-state scalar flux F_{m0} before and after the step change. The value of F_{m0} was obtained from the steady-state data by averaging $F_m(z)$ over the interval $z/r_s \in [28, 55]$, and is slightly lower than F_s owing to the contribution from the turbulent scalar flux (i.e. $F_s = F_m + F_f$). In addition, we define the positions $z_{1/4}(t)$ and $z_{3/4}(t)$, which correspond to the longitudinal distance at which $F_m = F_{m0}/4$ and $F_m = 3F_{m0}/4$, respectively, and will allow us to quantify the longitudinal spread

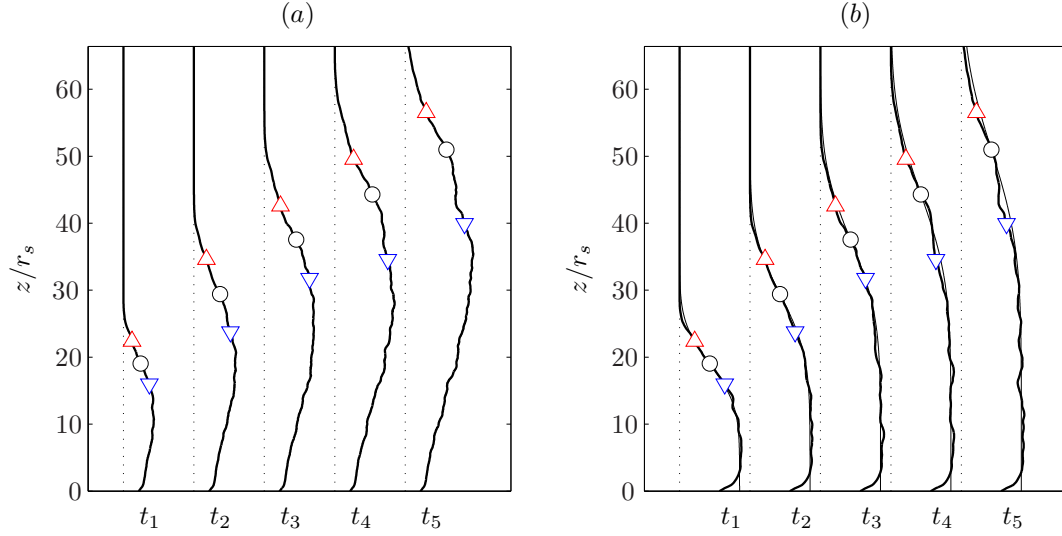


FIGURE 5.4. Profiles of (a) the integral concentration C_m and (b) the integral flux F_m against z/r_s , at times $t_i/\tau_s \approx 37i, i = 1 \dots 5$ with position of the front z^*/r_s [o] and the limits of the longitudinal spread of the front $z_{1/4}/r_s$ [$\triangle$] and $z_{3/4}/r_s$ [∇]. The thin line in (b) denotes the function $\text{erfc}[(z - z^*)/\sigma]$.

of the scalar.

Figure 5.4 shows typical profiles of C_m and of F_m obtained from the ensemble-averaged simulation data, with circles marking the location of the front z^* . In the region $r_s \ll z \ll z^*$ the concentration field is in a quasi-steady state and the integral concentration C_m increases linearly in z to a maximum behind the front (figure 5.4). Ahead of its maximum, C_m tends to zero smoothly with increasing z/z^* . It is observed that the propagation speed of the front decreases with increasing t . Evident in both the behaviour of C_m and F_m is that the longitudinal extent of the front increases in time, which indicates the presence of some form of longitudinal mixing.

As discussed in §2.1.3, our focus is on the behaviour of the ensemble-averaged flow or expected flow, rather than the behaviour of an individual realisation. It is nevertheless desirable that the ensemble average accurately captures notable features that might be regarded as typical amongst individual realisations. The relationship of the ensemble average to a single realisation of the flow is illustrated in figure 5.5. Although the single realisation deviates slightly from the ensemble average, the position of the travelling front and its longitudinal extent are faithfully captured by the ensemble average.

Owing to the fact that the velocity in the jet $w_m \sim 1/z$, one expects for dis-

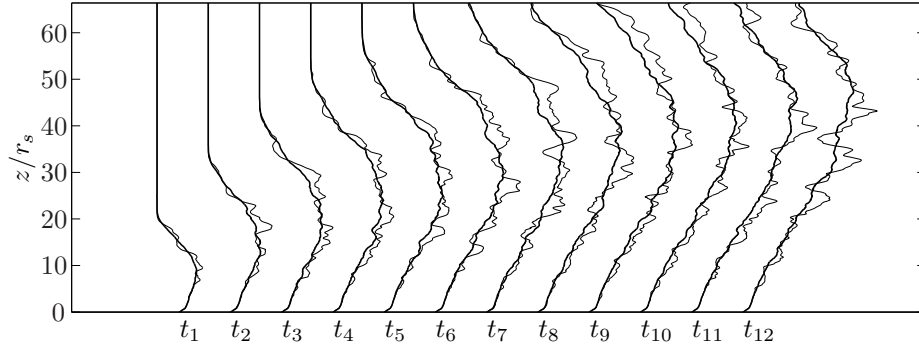


FIGURE 5.5. Evolution of a single realisation of the integral concentration C_m of the flow (thin line) compared with the ensemble average (thick line) at times $t_i/\tau_s \approx 26i$, $i = 1 \dots 12$.

turbances to propagate according to $z^* \sim \sqrt{t}$. To investigate this we plot $(z^* - z_v)^2$ against $t - t_v$ in figure 5.6, where t_v can be regarded as the location of the virtual source in time. Using the steady-state value of z_v , $t_v = -1.5\tau_s$ was obtained by fitting a straight line to $(z - z_v)^2$, whose agreement with the simulation data in figure 5.6 confirms the expected scaling relation. Both z_v/r_s and t_v/τ_s are small in comparison to the size of the domain and the total duration of the simulations, respectively, and consequently play an insignificant role in the far-field scaling observed in figure 5.6. Interestingly, $z^* \sim \sqrt{t}$ corresponds to the classical dispersion scaling, although for planar jets and plumes, for example, in which $w_m \propto 1/z$, one would expect to find a different power-law scaling. It is noteworthy that the propagation rate of the front appears to be slightly greater than w_m (figure 5.6, dashed line). Furthermore, from figure 5.6, $z_{1/4}$ and $z_{3/4}$ can be seen to scale in proportion to $\sqrt{t}$, which suggests that the scaling associated with the position of the front is identical to that associated with its longitudinal rate of spread, and that the process is therefore self-similar.

The scalar is released continuously at the source with a constant source flux F_s , therefore

$$\int_0^\infty C_m dz = F_s t. \quad (5.13)$$

Since $C_m \propto z$ in the steady state, we conclude that we can normalise the profiles of C_m by $F_m^* t/z^*$ in order to observe self-similarity. Profiles of C_m normalised in this way, plotted against the normalised streamwise coordinate z/z^* , are shown in figure 5.7(a). The approximate collapse indicates that, following a step change in

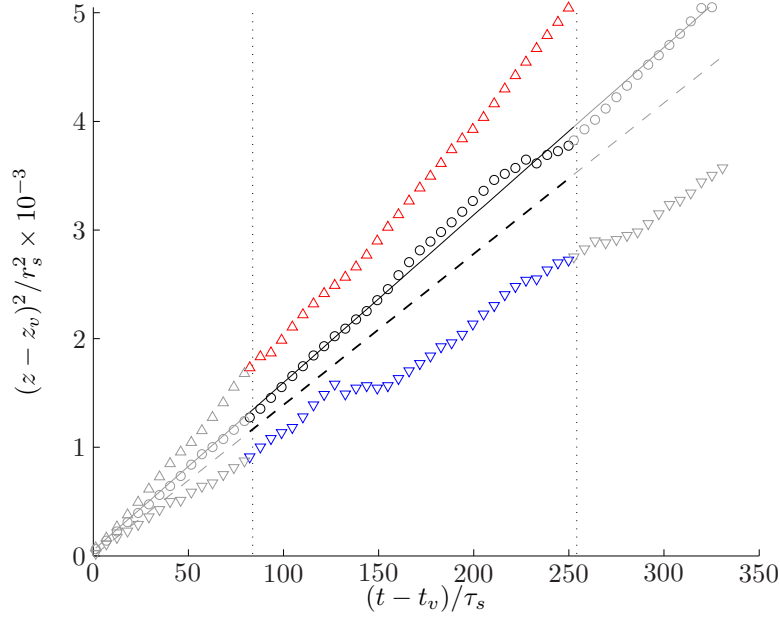


FIGURE 5.6. Position of the scalar front [o] and corresponding linear fit (solid line), compared to the ‘top-hat’ propagation (dashed line). The approximate extent of the front is indicated by $z_{1/4}(t)$ [$\triangle$] and $z_{3/4}(t)$ [∇]. The region between the two vertical dotted lines corresponds to the time domain that was used to obtain averaged properties of the front and the self-similar profiles that are shown in figures 5.7, 5.8 and 5.10.

the flux of concentration at the source, the transport of a passive scalar in a jet is indeed a self-similar process that can be described completely by the variable z/z^* .

5.4.2 Longitudinal mixing

To quantify the mixing at the front, we fit the data in figure 5.4(b) to a function of the form $F_m/F_m^* = \text{erfc}[(z - z^*(t))/\sigma]$. This choice is motivated by the fact that it provides a good approximation to the solution of an advection-diffusion equation if it is assumed that the characteristic velocity in the vicinity of the front is approximately constant. The longitudinal spread of the scalar flux $\sigma(t) = \sqrt{2D_e t}$ and D_e is an effective diffusivity. The error function is indicated with a thin line in figure 5.4(b) and exhibits an excellent agreement with the simulation data. From the definition of the longitudinal positions $z_{1/4}$ and $z_{3/4}$, which represent percentiles of F_m , we infer $\sigma = 1.048(z_{1/4} - z_{3/4})$. The fit allows us to calculate an average diffusion coefficient $D_e = 0.74M_m^{1/2}$. In an effort to account for the observed level of mixing, we assume that the turbulent Schmidt number is approximately unity and compare the value of D_e to the uniform turbulent viscosity ν_T obtained from a

gradient-diffusion hypothesis. We use a subscript m in ν_{Tm} to emphasise that ν_{Tm} is an integral representation of the turbulent viscosity. Based on the measurements of Hussein *et al.* (1994), Pope (2000) reports that in jets $\nu_{Tm} \approx 0.033 M_m^{1/2*}$. The significant difference between this estimation and D_e implies that turbulence alone cannot account for the mixing observed at the front. An intuitive way of understanding the difference between ν_{Tm} and D_e is that it implies the longitudinal extent of the front (determined by D_e) exceeds the radius of the jet (determined by ν_{Tm}). We will demonstrate that shear-flow dispersion accounts for the difference by examining the way in which the concentration profile is distorted in the vicinity of the front.

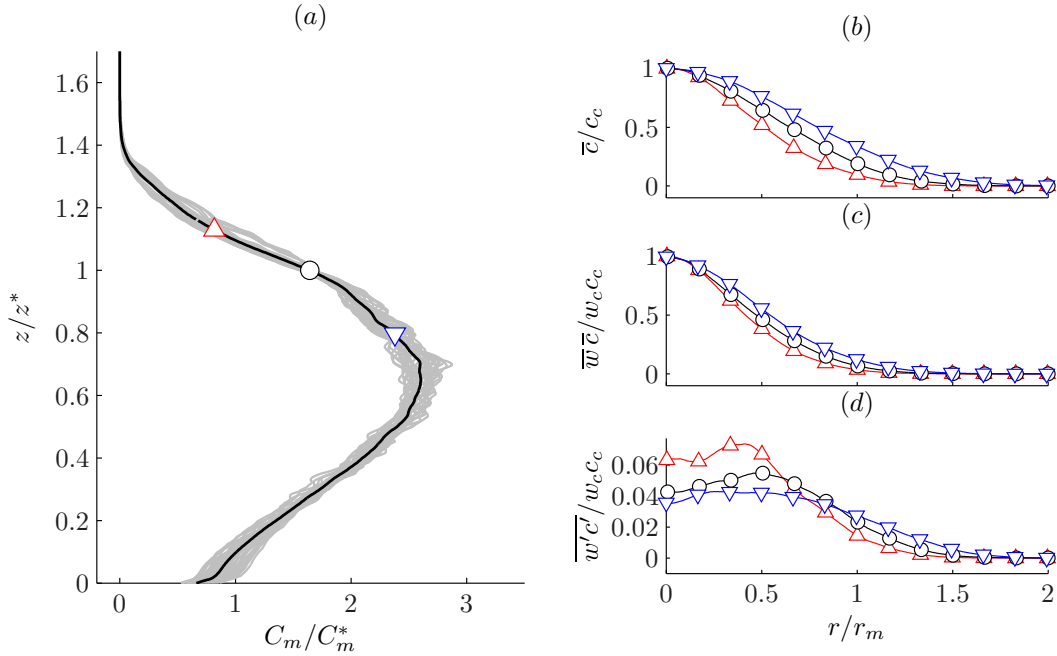


FIGURE 5.7. (a) Self-similar profiles of C_m for fixed times (grey) and their average (black); (b) normalised profiles of scalar $\bar{c}/c_c$; (c) normalised profiles of mean scalar flux $\bar{w}\bar{c}/w_c c_c$; (d) normalised profiles of turbulent scalar flux $\overline{w'c'}/w_c c_c$. In (a-d) the positions $z_{1/4}$, z^* and $z_{3/4}$ are marked with $\triangle$, $\circ$ and ∇ , respectively. The profiles displayed in (b-c) were obtained from a moving average at either $z_{1/4}$, z^* or $z_{3/4}$, over the time domain indicated in figure 5.6.

As described in §5.1 longitudinal mixing of an integral quantity such as C_m can occur due to a uniform increase in $\overline{w'c'}$ over the radius of the jet (turbulent mixing) or due to a distortion of the mean concentration profile $\bar{c}$ (shear-flow dispersion).

*In Pope (2000) ν_{Tm} is expressed in terms of the centreline velocity w_c and the radial location $r_{1/2}$ at which $\bar{w} = w_c/2$ (the jet half-width) as $\nu_{Tm} \approx 0.028 w_c r_{1/2}$. In a Gaussian jet $w_c = 2w_m$ and $r_{1/2} = \sqrt{\ln 2/2} r_m$, hence $\nu_{Tm} \approx 0.033 w_m r_m$.

Shear-flow dispersion results in a local increase in the correlation $\langle \bar{w} \bar{c} \rangle$, involving only mean-flow quantities, and therefore to a local increase in the mean scalar flux. It is therefore appropriate to examine radial profiles of $\bar{c}$ and $\bar{w}$ in the vicinity of the front, which are shown alongside the fluxes $\bar{w} \bar{c}$ and $\overline{w'c'}$ in figure 5.7. The profiles displayed in figures 5.7(b-d) are normalised using centreline values $c_c(z, t) \stackrel{\text{def}}{=} \bar{c}(0, z, t)$ and $w_c(z) \stackrel{\text{def}}{=} \bar{w}(0, z)$ of the mean concentration and the mean longitudinal velocity, respectively. The central observation that can be made from figures 5.7(b-d) is that as z/z^* increases the profiles become narrower. The scalar is transported by advection to regions of relatively large z/z^* when it is located close to the centreline of the jet, where the longitudinal velocities are highest. Consequently, the scalar profile becomes more peaked with downstream distance, which is evident in figure 5.7(b). Figures 5.7(c, d) show that the turbulent flux does not scale in exact proportion to the mean flux in the vicinity of the front. Rather, the turbulent flux appears to increase at large values of z/z^* , although it continues to make a relatively small contribution to the total flux.

To quantify precisely the relative contribution to the total scalar flux made by the mean and turbulent flow, we plot the dimensionless scalar fluxes θ_m and θ_f , respectively, with respect to the normalised downstream distance in figure 5.8. The grey curves displayed in figure 5.8 correspond to different times and exhibit an approximate collapse with respect to the variable z/z^* , which is consistent with the self-similarity that was described in §5.4.1. In the steady state the radial dependence of the velocity and concentration profiles is approximately independent of z , which implies that θ_m is constant. Consequently, shear-flow dispersion is evident as a departure from the steady-state value of θ_m and is seen in figure 5.8 to be significant over a large extent of the z/z^* domain. Figure 5.8 also indicates that in a quasi-steady state, for which $z/z^* \ll 1$, the contribution of the dimensionless turbulent flux (θ_f) amounts to the expected 10%, but rises to 20% at the front, where $z = z^*$.

Figure 5.8 confirms that in the vicinity of $z = z^*$ the local distortion of the concentration profile $\bar{c}$, evident in figure 5.7(b), acts in correlation with $\bar{w}$ to increase the dimensionless scalar flux θ_m . In particular, the dimensionless scalar flux ranges from approximately 1.0 (for $z/z^* \ll 1$) to 2.0. Consequently, if the turbulent scalar flux, characterised by θ_f , were to evolve in exactly the same way as θ_m , one would expect to see a corresponding two-fold increase in the value of θ_f . However, figure 5.8 indicates that the dimensionless turbulent scalar flux does not maintain the same relative balance with the dimensionless mean scalar flux over the front. At $z/z^* = 1.4$, for example, θ_f increases to approximately five times its steady-state

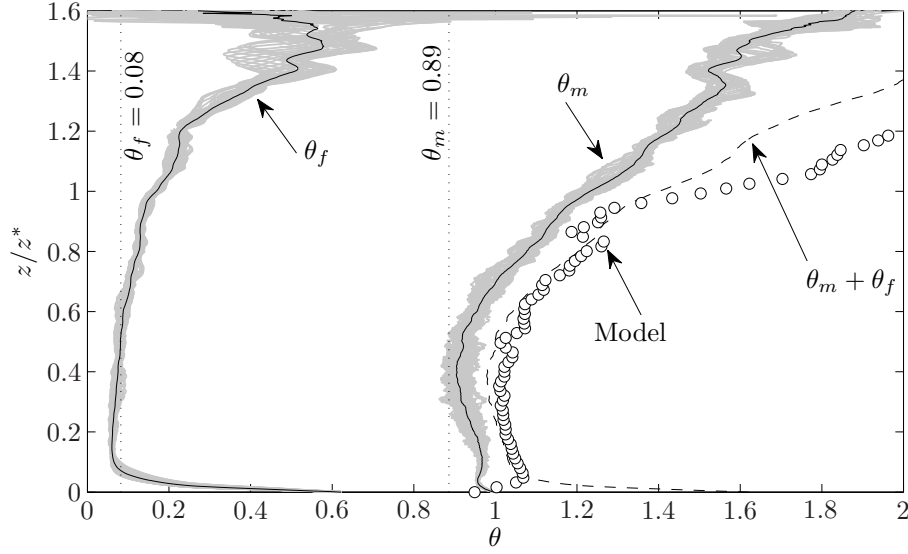


FIGURE 5.8. Evolution of the dimensionless mean scalar flux θ_m and the turbulent scalar flux θ_f . Grey lines correspond to the behaviour of θ at a fixed point in time, while the solid black line corresponds to their average, obtained over the time domain indicated in figure 5.6.

value, although θ_f continues to make a small contribution relative to that associated with the mean flow. Moreover, due to the fact that θ_f is an integral quantity, in addition to accounting for a local increase in the magnitude of the turbulent flux $\overline{w'c'}$ (see figure 5.7d), the increase in θ_f accounts for a narrowing of the *spatial* distribution of c' . It should also be noted that beyond z/z^* , the fluxes $\overline{w\bar{c}}$, $\overline{w'c'}$ and the concentration C_m used in their normalisation approach zero, which makes it difficult to obtain an accurate estimation of θ_f and θ_m from a finite dataset.

As pointed out previously, fluid located close to $r = 0$ is transported by relatively high longitudinal velocities, and can therefore propagate further than fluid located at the periphery of the jet. It is therefore natural that at large distances from the source one sees a narrowing of the scalar distribution (see figure 5.7b). At the leading edge of the scalar field, we can therefore expect the scalar profile to take the form of a spike at $r = 0$. For a Gaussian velocity profile, the largest velocity is found on the centreline and has a value of $2w_m$. This implies an upper bound of $\theta_m = 2$, which agrees reasonably well with figure 5.8. Hence, the mean scalar flux F_m cannot exceed $2C_m w_m$. Furthermore, if it is assumed that $dz^*/dt \approx w_m(z^*) \propto 1/z^*$, in accordance with figure 5.6, then fluid on the jet centreline will reach $z/z^* \approx \sqrt{2}$. At $z/z^* \approx \sqrt{2}$ in figure 5.8 we observe that the sum $\theta_m + \theta_f \approx 2$. For $z/z^* > \sqrt{2}$, we see a reduction in θ_f and that θ_m approaches a value of 2, as expected.

5.4.3 A naive model for scalar dispersion

It is possible to construct a naive model for scalar dispersion that explains several of the observations that were made in the previous section. To do so we follow Taylor (1953, §2) and consider dispersion by advection in a two-dimensional flow field. Whereas Taylor considered the dispersion of a solute in a circular pipe, the flow with which we are concerned is unconfined and requires several additional assumptions. In particular, we will assume that the dimensionless mean longitudinal velocity in a jet is a function of the similarity variable $\eta \stackrel{\text{def}}{=} r/r_m$. We will assume that at $t = 0$ a passive scalar of uniform concentration occupies a semi-infinite region $z < 0$, $\forall r^\dagger$. A jet with a point source of momentum at $z = 0$ transports the scalar in the longitudinal direction for $t > 0$.

We are interested in tracking the position of the interface of the scalar with the ambient fluid. To do so we will assume that the interface can be described by a set of points, each with a dimensionless radial position η_j that remains constant. In doing so we are implicitly accounting for mixing, because the radial position r_j of each point on the interface will necessarily increase such that $\eta_j \stackrel{\text{def}}{=} r_j/r_m$ remains constant. Thus, ambient fluid will cross the interface and dilute the scalar. In this respect, it is perhaps useful to regard the scalar as being comprised of many infinitesimal particles that spread out according to the spreading rate of the jet, and therefore do not follow the streamlines of the mean flow. Figure 5.9(a) displays the region occupied by the scalar, determined from the analysis that will be described below.

The longitudinal velocity of each point on the interface is

$$\frac{dz_j}{dt} = w_m f(\eta_j). \quad (5.14)$$

Because η_j does not depend on t , integration of (5.14), using the fact that $w_m \propto z^{-1}$, results in

$$\lambda_j = f(\eta_j). \quad (5.15)$$

where $\lambda_j \stackrel{\text{def}}{=} z_j^2 \alpha_0 / (M_s^{1/2} t)$. Hereafter we will omit the subscript j and therefore assume that the interface is a continuum. Note that if the front position z^* obeys $dz^*/dt = w_m$ then $z^* = M_s^{1/4} t^{1/2} / \alpha_0^{1/2}$ and therefore $z/z^* = \sqrt{\lambda}$.

The width of the scalar profile at a given λ is equal to

[†]Since we intend to assume that the jet has a Gaussian velocity profile, which does not possess compact support, the stated initial condition for the passive scalar simplifies the problem.

$$\eta = f^{-1}(\lambda). \quad (5.16)$$

Equation (5.40) allows us to determine the extent of the scalar at a given time, as illustrated in figure 5.9(a). However, in contrast to the simplified distribution of a scalar in a pipe flow given by Taylor (1953), the shaded region in figure 5.9(a) represents a non-uniform concentration.

We will express the scalar concentration in a form that is analogous to the mean longitudinal velocity:

$$\bar{c} = c_m(z, t)g(\eta), \quad (5.17)$$

and, like the dimensionless velocity profile $f(\eta)$, we specify c_m such that the integral of $g(\eta)$ over the jet is unity, which entails

$$g(\eta) = \begin{cases} f^{-1}(\lambda)^{-2}, & 0 \leq \eta \leq f^{-1}(\lambda), \\ 0, & f^{-1}(\lambda) < \eta. \end{cases} \quad (5.18)$$

Here we have assumed for simplicity that, unlike the longitudinal velocity function f , $g(\eta)$ has the form of a top-hat. Since $\bar{w} = w_m f(\eta)$, the dimensionless scalar flux θ_m is

$$\theta_m \stackrel{\text{def}}{=} 2 \int_0^\infty f g \eta d\eta = \frac{2}{f^{-1}(\lambda)^2} \int_0^{f^{-1}(\lambda)} f \eta d\eta. \quad (5.19)$$

If one assumes that the mean velocity $\bar{w}$ has a Gaussian form: $f(\eta) = 2 \exp(-2\eta^2)$, then $f^{-1}(\lambda) = \sqrt{\ln(2/\lambda)}/2$ and

$$\theta_m(\lambda) = \frac{2(1 - \lambda/2)}{\ln(2/\lambda)}, \quad 0 \leq \lambda \leq 2. \quad (5.20)$$

Figure 5.9(b) compares (5.20) to DNS observations and, perhaps surprisingly, reveals a good agreement for $\lambda > 1$. Notably, (5.20) shows a better agreement with the total dimensionless scalar flux $\theta_m + \theta_f$ than with the mean scalar flux θ_m . This is perhaps predictable because in a real jet the conservation relation for a passive scalar involves the total scalar flux (inclusive of turbulent transport) rather than the mean flux alone. It is also clear that the simple model described above gives poor predictions for $z/z^* \ll 1$, which is due to the fact that for simplicity we assumed that the scalar occupies the entire semi-infinite domain for $z < 0$. It is expected that the incorporation of a finite source for the scalar into the analysis

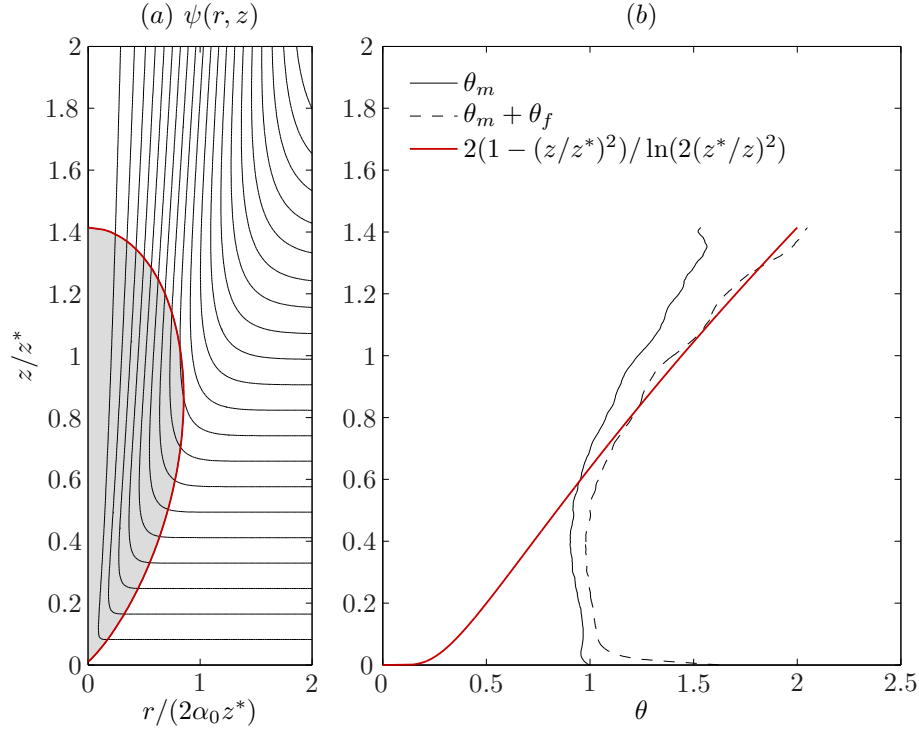


FIGURE 5.9. (a) Distribution of the passive scalar in a jet based on the simple model described in §5.4.3. The shaded region represents the areas to which the scalar has spread, although the concentration is not uniform within the shaded region. (b) Prediction of the dimensionless scalar flux compared to DNS observations of θ_m and θ_f .

would significantly improve the predictions for $z/z^* \ll 1$.

5.5 A model for shear-flow dispersion in jets

Although shear-flow dispersion was originally developed for pipe flow (Taylor, 1953, 1954b), it can be usefully extended to boundary-free shear flows, such as a jet. In this section we will summarise the key features of the approach and obtain a model for the dispersion of a passive scalar. In contrast to a jet, the steady-state distribution of a scalar in a pipe is uniform and bounded in the radial direction. Like jets however, both the velocity distribution and the scalar distribution are self-similar, which means that their radial dependence is invariant in the longitudinal direction. Fundamental to both dispersion in pipe flow and dispersion in jets is the idea that changes in the mean scalar concentration in the *longitudinal* direction can cause a local departure from self-similarity over the *lateral* (radial) dimension. Specifically, the departure from self-similarity in the scalar is caused by *lateral* gradients in the

longitudinal velocity and, in correlation with the longitudinal velocity, results in a local increase in the scalar flux.

Taylor demonstrated that to leading order the radial dependence of the modification to the uniform concentration is independent of the longitudinal coordinate. Consequently, one can define a dimensionless function $g_1(\eta)$ to describe the shape of the perturbed concentration profile, where $\eta \stackrel{\text{def}}{=} r/r_m$. Using this notation, the basic dimensionless steady-state concentration will be denoted $g_0(\eta)$. The functions g_0 and g_1 apply equally well to the analysis of pipe flow and jet flow, although for pipe flow the steady-state scalar concentration is constant, and therefore g_0 does not depend on η . The aim is to establish an expression for the dimensionless concentration

$$g(\eta) = g_0 + L_1 g_1, \quad (5.21)$$

where L_1 determines the degree to which the concentration profile departs from a steady state, and is therefore expected to depend in some way on the behaviour of the scalar in the longitudinal direction. Evidently, if the dimensionless velocity profile $\bar{w}/w_m = f_0(\eta)$ is known, (5.21) can be multiplied by f_0 and integrated with respect to η^2 (i.e. over a lateral slice of the pipe flow or jet). The resulting integral corresponds to the dimensionless flux $\theta_m \stackrel{\text{def}}{=} F_m/(w_m C_m)$:

$$\theta_m \stackrel{\text{def}}{=} \langle f_0 g_0 \rangle + L_1 \langle f_0 g_1 \rangle = \theta_0 + L_1 \theta_1. \quad (5.22)$$

We define $\theta_0 \stackrel{\text{def}}{=} \langle f_0 g_0 \rangle$ as the dimensionless steady-state scalar flux, and $\theta_1 \stackrel{\text{def}}{=} \langle f_0 g_1 \rangle$ as the additional flux arising from changes in the shape of $g(\eta)$. Here the operator $\langle \rangle$ is a suitable non-vanishing spatial average over η , which for jets we define in §5.5.2. The ultimate aim of a dispersion closure is to obtain an expression for θ_m in terms of the integral concentration C_m and its longitudinal derivatives.

Instead of working with the ensemble average scalar concentration $\bar{c}$, we work with the dimensionless concentration

$$\mathcal{C} \equiv \bar{c}/c_{m0} = \mathcal{C}_m (g_0 + L_1 g_1), \quad (5.23)$$

where $c_{m0}(z)$ is the mean steady-state concentration and $\mathcal{C}_m \stackrel{\text{def}}{=} c_m/c_{m0}$. Consequently, in the steady state $\mathcal{C}$ is independent of z in both jets and pipes, making it is possible to recast Taylor's approach to dispersion in pipe flow in a form that is applicable to jets. In the following sections we will substitute (5.23) into the longitudinal scalar transport equation to relate the unknown perturbed profile g_1

to $\partial_z \mathcal{C}_m$.

5.5.1 Shear-flow dispersion in a pipe

Following (Taylor, 1953, 1954b), we consider the evolution of a step change in the dimensionless concentration $\mathcal{C}$ in a pipe of radius r_m . Consistent with boundary layer theory we neglect second derivatives in the longitudinal direction. Unlike jets, in pipes the steady-state concentration c_{m0} is independent of z , and we can replace $\bar{c}$ with $\mathcal{C}$ in the scalar transport equation:

$$\frac{\partial \mathcal{C}}{\partial t} + \bar{w} \frac{\partial \mathcal{C}}{\partial z} = \frac{\kappa_{Tm}}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \mathcal{C}}{\partial r} \right), \quad (5.24)$$

where κ_{Tm} is the *integral* eddy diffusivity[‡] and $\bar{w}$ is the steady-state mean velocity field. For this problem Taylor applied a Galilean transform, defining a coordinate system that moves according to the longitudinal velocity or, equivalently, the velocity associated with the scalar flux:

$$\xi \stackrel{\text{def}}{=} z - \theta_0 w_m t, \quad \tau \stackrel{\text{def}}{=} t, \quad (5.25)$$

where θ_0 is constant and equal to unity for pipe flow. This coordinate transformation ensures that the equation governing the cross-sectional average concentration does not contain mean flux terms. We assume that the radial dependence of the velocity field does not vary in the longitudinal direction, such that $\bar{w} = w_m f_0(\eta)$, where $w_m = \langle \bar{w} \rangle$ is the cross-sectional-averaged longitudinal velocity, $\eta \equiv r/r_m$ is a normalised radius and $\langle f_0 \rangle = 1$. Equation (5.24) transforms according to

$$\frac{\partial \mathcal{C}}{\partial \tau} + w_m (f_0 - \theta_0) \frac{\partial \mathcal{C}}{\partial \xi} = \frac{\kappa_{Tm}}{r_m^2 \eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \mathcal{C}}{\partial \eta} \right). \quad (5.26)$$

The radial dependence of $\mathcal{C}$ can be decomposed into $g_0(\eta)$, which corresponds to the steady-state radial dependence of the concentration, and $g_1(\eta)$, which accounts for the leading order perturbation from $g_0(\eta)$:

$$\mathcal{C} = \mathcal{C}_m (g_0 + L_1 g_1). \quad (5.27)$$

Here, the variable $\mathcal{C}_m$ determines the amplitude of the concentration profile. For pipe flow, since the steady-state concentration is uniform, it can be assumed that $g_0 = 1$, without loss of generality, and we define $\mathcal{C}_m$ such that the cross sectional

[‡]In order for $\partial_r \kappa_T = 0$ we assume that the local eddy diffusivity κ_T is constant and therefore equal to κ_{Tm} .

average $\langle g_1 \rangle = 0$. In (5.27), L_1 determines the amplitude of the leading order perturbation.

Applying the averaging operation $\langle \rangle$ to (5.26) gives an equation for the mean concentration:

$$\frac{\partial \mathcal{C}_m}{\partial \tau} + w_m \langle f_0 g_1 \rangle \frac{\partial}{\partial \xi} (\mathcal{C}_m L_1) = 0. \quad (5.28)$$

Subtracting (5.28) from (5.26) yields

$$\begin{aligned} g_1 \frac{\partial (\mathcal{C}_m L_1)}{\partial \tau} + w_m (f_0 - \theta_0) \frac{\partial \mathcal{C}_m}{\partial \xi} + w_m \frac{\partial \mathcal{C}_m L_1}{\partial \xi} [(f_0 - 1)g_1 - \langle f_0 g_1 \rangle] \\ = \frac{\kappa_{Tm} \mathcal{C}_m L_1}{r_m^2 \eta} \frac{d}{d\eta} \left(\eta \frac{dg_1}{d\eta} \right). \end{aligned} \quad (5.29)$$

Taylor's key assumption was that for large time ($t \gg r_m^2 / \kappa_{Tm}$) there exists an asymptotic balance between the second and fourth terms in (5.29). Noting that $\partial_\xi = \partial_z$, one therefore assumes that

$$L_1 \frac{\kappa_{Tm}}{\eta} \frac{d}{d\eta} \left(\eta \frac{dg_1}{d\eta} \right) = - \frac{w_m r_m^2}{\mathcal{C}_m} (\theta_0 - f_0) \frac{\partial \mathcal{C}_m}{\partial z}, \quad (5.30)$$

which represents a balance between longitudinal advection and turbulent transport in the radial direction. Equation (5.30) has a separable form and implies that to within a constant multiplicative factor

$$L_1 = - \frac{w_m r_m^2}{\kappa_{Tm}} \frac{1}{\mathcal{C}_m} \frac{\partial \mathcal{C}_m}{\partial z}. \quad (5.31)$$

In principle (5.30) can be inverted to determine the form of the perturbation g_1 , and therefore the flux

$$F_m \stackrel{\text{def}}{=} w_m c_m r_m^2 (\langle f_0 g_0 \rangle + L_1 \langle f_0 g_1 \rangle). \quad (5.32)$$

Defining

$$\theta_m \equiv F_m / w_m c_m r_m^2, \quad \theta_0 \equiv \langle f_0 g_0 \rangle, \quad \theta_1 \equiv \langle f_0 g_1 \rangle, \quad (5.33a, b, c)$$

(5.32) can be expressed as $\theta_m = \theta_0 + L_1 \theta_1$. The parameter θ_0 is the basic steady-state dimensionless scalar flux arising from self-similar profiles of $\bar{c}$ and θ_1 is the dimensionless scalar flux arising from a local departure from self-similarity in $\bar{c}$.

5.5.2 Shear-flow dispersion in an axisymmetric jet

To apply Taylor's dispersion theory to jets we assume that the decomposition $\mathcal{C} = \mathcal{C}_m(g_0 + L_1 g_1)$ provides a leading order representation of the dimensionless concentration field. The use of the normalised radius $\eta \equiv r/r_m(z)$ allows us to map the conical geometry of the jet, for which $r_m = 2\alpha_0 z$, onto a cylindrical geometry. Moreover, the use of the dimensionless concentration $\mathcal{C}$ ensures that in the steady state the amplitude of the scalar perturbation $L_1 = 0$, in spite of the fact that $c_m \sim 1/z$, and therefore allows us to focus on departures from this state. Due to the geometry of the jet, it is unlikely that the ordinary differential equation satisfied by the perturbation $g_1(\eta)$ is the same as (5.30). However, since we are primarily interested in the integral quantity $\theta_1 \stackrel{\text{def}}{=} \langle f_0 g_1 \rangle$, rather than the particular form of $g_1(\eta)$, we will not attempt to obtain an exact equation for $g_1(\eta)$ in this work.

In the absence of a well-defined edge, we define the cross-sectional average $\langle \chi \rangle$ of a quantity χ in the jet according to

$$\langle \chi \rangle \equiv \frac{2}{r_m^2} \int_0^{r_d} \chi r dr, \quad (5.34)$$

such that $w_m = \langle \bar{w} \rangle$. For axisymmetric jets, a constant eddy viscosity ν_{Tm} can be expressed in terms of the entrainment coefficient according to $\nu_{Tm} = \alpha_0 w_m r_m / 3$ (see Craske & van Reeuwijk, 2015b). Assuming that the turbulent Schmidt number ν_{Tm} / κ_{Tm} is approximately unity, we therefore set $\kappa_{Tm} = \alpha_0 w_m r_m / 3$. Furthermore, the dimensionless mean scalar concentration $\mathcal{C}_m$ can be defined in terms of integral quantities, such that $\mathcal{C}_m \stackrel{\text{def}}{=} C_m / C_{m0}$, where C_{m0} is a steady-state integral concentration. Consequently, substitution of κ_{Tm} , $\mathcal{C}_m$ and the linear dependence of the jet width on z , $r_m = 2\alpha_0 z$, into (5.31) results in

$$L_1 = 6 \left(1 - \frac{z}{C_m} \frac{\partial C_m}{\partial z} \right). \quad (5.35)$$

In the steady state, for which $C_m \sim z$, it is clear from (5.35) that $L_1 = 0$. Using (5.22), the dispersion closure for the dimensionless scalar flux in an unsteady jet can be expressed as

$$\theta_m = \theta_0 + 6\theta_1 \left(1 - \frac{z}{C_m} \frac{\partial C_m}{\partial z} \right). \quad (5.36)$$

As noted previously, instead of calculating g_0 and g_1 exactly, we treat θ_0 and θ_1 as model parameters whose value can be determined from observation. Notable however is the fact that θ_0 can be determined from the steady-state data.

5.6 Model prediction

Substitution of (5.36) into (5.9) results in an advection-dispersion equation given by

$$\frac{\partial C_m}{\partial t} + (\theta_0 + 6\theta_1) \frac{M_m^{1/2}}{2\alpha_0} \frac{\partial}{\partial z} \left(\frac{C_m}{z} \right) = 6\theta_1 \frac{M_m^{1/2}}{2\alpha_0} \frac{\partial^2 C_m}{\partial z^2}. \quad (5.37)$$

Hereafter we will assume that z and t are coordinates relative to the location of a virtual source (z_v, t_v) . It is noteworthy that the advection parameter $\theta_0 + 6\theta_1$ (cf. K_a in Landel *et al.*, 2012) contains a contribution from both θ_0 and θ_1 , which are parameters that each have a particular physical significance and are precisely defined integrals. In particular, θ_0 depends on steady-state properties of the flow and can therefore be estimated *a priori*. Although $\theta_m = \theta_0$ in the steady state, recognising that the longitudinal turbulent scalar flux θ_f will affect the relevant advection velocity, we equate θ_0 with the observed steady-state value of $\theta_m + \theta_f \approx 1.0$. To estimate the value of the parameter θ_1 , we compare the coefficient of the mixing term on the right hand side of (5.37) with the mixing coefficient D_e that was observed in §5.4. Specifically, we set $\theta_1 = \alpha_0 D_e / (3M_m^{1/2}) = 0.017$. The prediction of θ_m that we obtain using (5.36) with $(\theta_0, \theta_1) = (1.0, 0.017)$ and the observed behaviour of C_m is shown in figure 5.8. Although the model appears to over-predict the magnitude of θ_m it shows a good agreement with the variation of θ_m with respect to z/z^* . Indeed, the model shows a better agreement with the magnitude of $\theta_f + \theta_m$ than θ_m , which is a consequence of the estimation of θ_1 being based on D_e , which is influenced by θ_m and θ_f .

Having established that the dispersion of a scalar in a jet is a self-similar process in §5.4, we seek a similarity solution to (5.37). We express (5.37) as

$$\frac{\partial C_m}{\partial t} + \frac{\lambda^* M_m^{1/2}}{2\alpha_0} \frac{\partial}{\partial z} \left(\frac{C_m}{z} \right) = D_e \frac{\partial^2 C_m}{\partial z^2}, \quad (5.38)$$

with C_m subject to the conditions

$$\lim_{z \rightarrow \infty} C_m = 0, \quad \int_0^\infty C_m dz = F_s t, \quad (5.39)$$

where F_s is the source flux and $\lambda^* \equiv \theta_0 + 6\theta_1$ is a constant that characterises the propagation speed of the front. Based on the observations reported above, $\lambda^* \approx 1.10$. Using the similarity variable (cf. §5.4.3)

$$\lambda \stackrel{\text{def}}{=} \frac{z^2 \alpha_0}{t M_m^{1/2}}, \quad (5.40)$$

we propose a solution for C_m of the form:

$$C_m(z, t) = \frac{2\alpha_0^{1/2} F_s t^{1/2}}{M_m^{1/4}} \Phi_m(\lambda). \quad (5.41)$$

Transformation of (5.38) according to (5.41) and (5.40) results in

$$\frac{d^2 \Phi_m}{d\lambda^2} - \frac{1}{2\lambda} \left[\text{Pe} \left(1 - \frac{\lambda}{\lambda^*} \right) - 1 \right] \frac{d\Phi_m}{d\lambda} - \frac{\text{Pe}}{4\lambda^2} \left(\frac{\lambda}{\lambda^*} - 1 \right) \Phi_m = 0, \quad (5.42)$$

where $\text{Pe} \stackrel{\text{def}}{=} \lambda^* M_m^{1/2} / (2\alpha_0 D_e) \approx 10.99$, based on our observations. The transformed boundary condition and integral constraint become

$$\lim_{\lambda \rightarrow \infty} \Phi_m = 0, \quad \int_0^\infty \frac{\Phi_m}{\lambda^{1/2}} d\lambda = 1. \quad (5.43)$$

Equation (5.42) has the general solution:

$$\Phi(\lambda) = c_1 \exp\left(-\frac{\text{Pe}\lambda}{2\lambda^*}\right) G\left(1, \frac{\text{Pe}}{2} + \frac{1}{2}, \frac{\text{Pe}\lambda}{2\lambda^*}\right) \lambda^{\text{Pe}/2} + c_2 \sqrt{\lambda}, \quad (5.44)$$

where G is the hypergeometric function. To determine exact expressions for the constants c_1 and c_2 , we note that for large λ (5.44) can be expressed as

$$\Phi(\lambda) \sim \left[c_1 2^{\text{Pe}/2-1/2} \left(\frac{\text{Pe}}{\lambda^*} \right)^{1/2-\text{Pe}/2} \Gamma\left(\frac{1}{2} + \frac{\text{Pe}}{2}\right) + c_2 \right] \lambda^{1/2}, \quad (5.45)$$

where Γ is the Gamma function. Hence

$$c_2 = \underbrace{-2^{\text{Pe}/2-1/2} \left(\frac{\text{Pe}}{\lambda^*} \right)^{1/2-\text{Pe}/2} \Gamma\left(\frac{1}{2} + \frac{\text{Pe}}{2}\right)}_{c_3} c_1, \quad (5.46)$$

to ensure that $\Phi \rightarrow 0$ as $\lambda \rightarrow \infty$. The value of the remaining degree of freedom c_1 is determined by

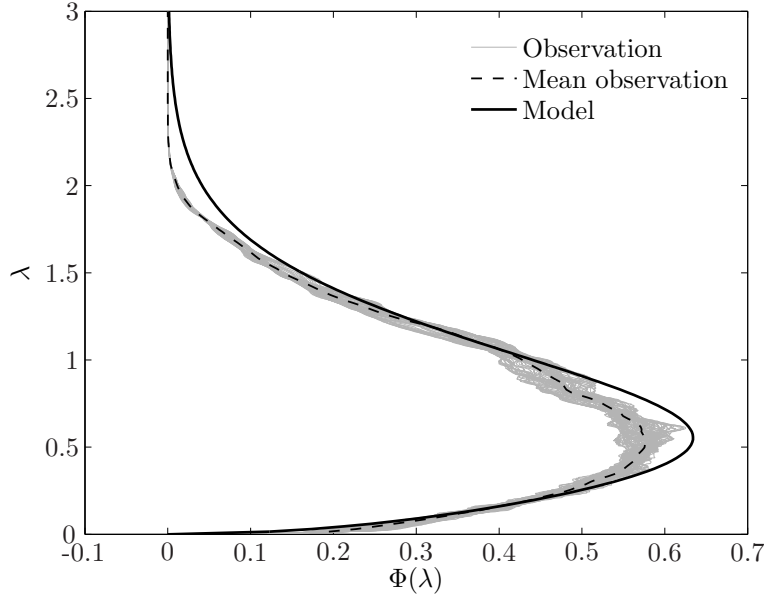


FIGURE 5.10. Comparison of observed normalised concentration with the analytical self-similar solution (5.44) with $c_1 = -25.79$ and $c_2 = 1.00$. The observed profiles were obtained over the time domain indicated in figure 5.6.

$$c_1 = \left\{ \int_0^\infty \left[\exp\left(-\frac{\text{Pe}\lambda}{2\lambda^*}\right) G\left(1, \frac{\text{Pe}}{2} + \frac{1}{2}, \frac{\text{Pe}\lambda}{2\lambda^*}\right) \lambda^{\text{Pe}/2-1/2} + c_3 \right] d\lambda \right\}^{-1}. \quad (5.47)$$

Using the observed values $\text{Pe} \approx 10.99$ and $\lambda^* \approx 1.10$, we find $c_1 = -25.79$ and $c_2 = 1.00$.

Figure 5.10 demonstrates that the similarity solution (5.44) agrees reasonably well with the observed data. In particular, the model reproduces the shape and the longitudinal extent of the scalar distribution. The model solution appears to slightly over-predict the peak concentration, and the concentration in the tail of the distribution at relatively large values of λ . We attribute the latter to higher-order contributions in the dimensionless scalar flux θ_m , for which (5.22) does not account (cf. the higher-order terms for dispersion in a pipe flow considered by Chatwin, 1970), which will be discussed in the following section. However, given that the model is based on a leading-order representation of the front involving two parameters, and that the specification of θ_0 was made *a priori*, the agreement is satisfactory.

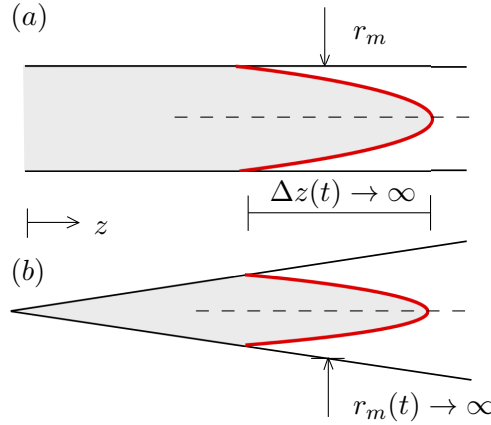


FIGURE 5.11. (a) Dispersion of a passive scalar in a pipe flow, where $\Delta z \gg r_m$ for large times. (b) Dispersion of a passive scalar in a jet, where $\Delta z = O(r_m)$.

5.6.1 Model limitations

A notable difference between dispersion in pipes and dispersion in jets is that when analysing the former one can wait for a sufficiently long time such that the longitudinal length scale Δz associated with a concentration gradient significantly exceeds the lateral (radial) length scale of the flow r_m (see Taylor, 1954*a*, for details). This situation is illustrated schematically in figure 5.11(a), and is what one typically associates with classical shear-flow or Taylor dispersion. In jets, as depicted in 5.11(b), the situation is different because r_m grows linearly with respect to z and one finds (cf. figure 5.7) that $\Delta z = O(r_m)$. The implication is that the dimensionless scalar flux modelled by equation (5.36) is an approximation, representing the leading terms of an infinite perturbation expansion in the ‘small’ parameter $r_m^2/(\kappa_T m t)$. For flow in pipes, the full perturbation expansion was used by Gill (1967) and Chatwin (1970) to examine transient dispersion problems.

The reasonably good agreement between the model’s predictions and observations in figure 5.10 suggests that the missing higher-order terms in the full series solution do not play a significant role in the jet when $\Delta z = O(r_m)$ and self-similarity with respect to z and t has been established. However, close to the source in the *near field*, where $\Delta z \ll r_m$, it is expected that they play a more prominent role, accounting for the precise way in which the scalar is released into the jet and for longitudinal length scales that might be small relative to the radius of the flow. The longitudinal extent of the near field is $O(r_s/\alpha_0)$, being determined by the length over which radial mixing occurs, known as the region of flow development (Rajaratnam, 1976). Hence, one would also expect that in problems requiring near-field prediction a more sophisticated analysis and model for turbulent trans-

port would be required.

5.7 Planar jets

Although the dispersion model described in this chapter has been developed for axisymmetric jets, it is built from general principles that are equally applicable in the analysis of planar jets or plumes. Central to the model are the geometric parameters $\theta_0 = \langle f_0 g_0 \rangle$ and $\theta_1 = \langle f_0 g_1 \rangle$, which correspond to the dimensionless scalar flux of the steady state and of the perturbed scalar profile, respectively. Both θ_0 and θ_1 are functionals of the dimensionless longitudinal velocity $f_0(\eta)$, the scalar concentration $g_0(\eta)$ and the perturbed concentration $g_1(\eta)$. Owing to the fact that $g_1(\eta)$ can be determined *a priori* by solving an ordinary differential equation involving g_0 , the dimensionless perturbation flux θ_1 can, in principle, be determined without consulting unsteady data.

In contrast, Landel *et al.* (2012)[§], whilst providing several analytical solutions and insights from experiments, relied on advection and dispersion parameters (K_a and K_d , respectively), whose relationship with the underlying turbulent scalar flux $\overline{w'c'}$ or mean flow scalar flux $\overline{w}\overline{c}$ was not made explicit. However, the approach taken in §5.5 to obtain a model for Taylor dispersion in axisymmetric jets is applicable to planar jets. Since planar jets are driven by line sources of momentum flux of infinite length, we will assume that the scalar is also released over a line source of infinite length coincident with the source of momentum flux. For planar jets we define the volume flux, momentum flux, and integral scalar concentration, per unit length, according to

$$Q_m(z) \stackrel{\text{def}}{=} 2 \int_0^{r_d} \overline{w} dr, \quad M_m(z) \stackrel{\text{def}}{=} 2 \int_0^{r_d} \overline{w}^2 dr, \quad C_m(z) \stackrel{\text{def}}{=} 2 \int_0^{r_d} \overline{c} dr, \quad (5.48a, b, c)$$

respectively. Consequently, it is convenient to define top hat variables r_m and w_m such that $Q_m \stackrel{\text{def}}{=} 2w_m r_m$ and $M_m \stackrel{\text{def}}{=} 2w_m^2 r_m$, hence $w_m \stackrel{\text{def}}{=} M_m/Q_m$ and $r_m = Q_m^2/(2M_m)$. For two-dimensional jets the averaging operator $\langle \rangle$ therefore becomes

$$\langle \chi \rangle \equiv \frac{1}{r_m} \int_0^{r_d} \chi dr, \quad (5.49)$$

which ensures that $w_m = \langle \overline{w} \rangle$. Consistent with the definitions above is a Gaussian

[§]Strictly, it should be noted that Landel *et al.* (2012) examined confined *quasi-two-dimensional* jets rather than the planar jets discussed here

velocity profile of the form

$$\bar{w} = w_m \sqrt{2} \exp\left(-\frac{\pi}{2} \eta^2\right) = w_m f_0(\eta), \quad (5.50)$$

where $\eta \stackrel{\text{def}}{=} r/r_m$. The planar steady jet equations are

$$\frac{dQ_m}{dz} = 2\alpha_0 \frac{M_m}{Q_m}, \quad \frac{dM_m}{dz} = 0, \quad (5.51a, b)$$

representing volume conservation and momentum conservation, respectively. Note that the α_0 defined by (5.51a, b) is a factor of $\sqrt{2}$ larger than the value of α_0 used by Landel *et al.* (2012), who scale the entrainment velocity on the maximum velocity, which is equal to $\sqrt{2}w_m$. The solution to (5.51a, b) is $Q_m = 2\sqrt{\alpha_0 z M_s}$, where $M_m = M_s$ is the momentum flux per unit length at the source. In a steady planar jet there is a balance between the longitudinal transport of mean kinetic energy and the production of turbulence (the energy balance in an axisymmetric jet will be derived and discussed in detail in the following chapter):

$$\frac{d}{dz} \left(\gamma_m \frac{M_m^2}{Q_m} \right) = \delta_m \frac{M_m^3}{Q_m^3}, \quad (5.52)$$

where γ_m and δ_m are the dimensionless energy flux and dimensionless turbulence production, respectively, defined as

$$\gamma_m \stackrel{\text{def}}{=} \frac{\langle \bar{w}^3 \rangle}{w_m^3}, \quad \delta_m \equiv \frac{4r_m}{w_m^3} \left\langle \overline{u'w'} \frac{\partial \bar{w}}{\partial r} \right\rangle. \quad (5.53)$$

Using (5.50), the dimensionless energy flux γ_m can be expressed as

$$\gamma_m \stackrel{\text{def}}{=} \langle f_0^3 \rangle = \frac{2}{\sqrt{3}}. \quad (5.54)$$

Moreover, invoking a gradient diffusion hypothesis:

$$\overline{u'w'} = -\nu_{Tm} \frac{\partial \bar{w}}{\partial r} = \pi \nu_{Tm} \frac{w_m}{r_m} \eta f_0(\eta), \quad (5.55)$$

the dimensionless turbulence production can be expressed as

$$\delta_m = -\frac{4\pi^2 \nu_{Tm}}{w_m r_m} \langle f_0^2 \eta^2 \rangle = -2\pi \frac{\nu_{Tm}}{w_m r_m}. \quad (5.56)$$

When (5.52) is combined with (5.51a, b) we find that the relationship between δ_m, γ_m and α_0 is $\alpha_0 = -\delta_m/2\gamma_m$, hence

$$\nu_{Tm} = \frac{2}{\sqrt{3}} \frac{\alpha_0}{\pi} w_m r_m. \quad (5.57)$$

In two dimensions, setting $\kappa_{Tm} = \nu_{Tm}$, equation (5.31) therefore becomes

$$L_1 = -\frac{\pi\sqrt{3}r_m}{2\alpha_0} \frac{1}{\mathcal{C}_m} \frac{\partial \mathcal{C}_m}{\partial z}. \quad (5.58)$$

The steady-state integral concentration C_{m0} in a planar jet scales according to $\sqrt{z}$. Therefore, noting that for a planar jet $r_m = 2\alpha_0 z$, substitution for $\mathcal{C}_m \stackrel{\text{def}}{=} C_m/C_{m0} \sim C_m/\sqrt{z}$ in (5.58), we find

$$L_1 = \pi\sqrt{3} \left(\frac{1}{2} - \frac{z}{C_m} \frac{\partial C_m}{\partial z} \right). \quad (5.59)$$

Substitution of (5.59) and $\theta_m = \theta_0 + L_1\theta_1$ into the transport equation for a scalar in a jet:

$$\frac{\partial C_m}{\partial t} + \frac{\partial}{\partial z} \left(\theta_m \frac{M_m^{1/2}}{\sqrt{4\alpha_0 z}} C_m \right) = 0, \quad (5.60)$$

gives

$$\frac{\partial C_m}{\partial t} + \left(\theta_0 + \pi\sqrt{\frac{3}{4}}\theta_1 \right) \frac{M_m^{1/2}}{\sqrt{4\alpha_0}} \frac{\partial}{\partial z} \left(\frac{C_m}{z^{1/2}} \right) = \theta_1 \pi M_m^{1/2} \sqrt{\frac{3}{4\alpha_0}} \frac{\partial}{\partial z} \left(z^{1/2} \frac{\partial C_m}{\partial z} \right). \quad (5.61)$$

Hence when our dispersion closure is applied to planar jets we recover the same model (to within constant prefactors) as that proposed by Landel *et al.* (2012). More precisely, (5.61) is in one to one correspondence with (5.2) when

$$K_a = \frac{1}{\sqrt{4\alpha_0}} \left(\theta_0 + \pi\sqrt{\frac{3}{4}}\theta_1 \right), \quad K_d = \theta_1 \pi \sqrt{\frac{3}{4\alpha_0}}. \quad (5.62)$$

It is important to note, however, the fundamental physical difference between a *statistically* two-dimensional turbulent planar jet and the quasi-two-dimensional jet studied by Landel *et al.* (2012) by confining the jet to the narrow gap between two parallel plates. Although the framework we use applies equally well to each of these cases, one would expect for dimensionless parameters such as the entrainment coefficient α_0 , to depend on the particular case considered.

Not straightforward to anticipate from (5.2) is that the advection parameter K_a depends on both θ_0 and θ_1 . Using (5.62), it is useful to compute the values of θ_0 and θ_1 that correspond to $K_a = 1.65$ and $K_d = 0.09$, as reported by Landel

et al. (2012):

$$\theta_0(K_a = 1.65, K_d = 0.09) = 0.995 \text{ (0.968)}, \quad (5.63)$$

$$\theta_1(K_d = 0.09) = 0.010 \text{ (0.017)}, \quad (5.64)$$

where the values of θ_0 and θ_1 that we report in the present chapter are shown in parenthesis, and we have taken $\alpha_0 = \sqrt{2} \cdot 0.068$, based on the measurements of Landel *et al.* (2012) (the factor $\sqrt{2}$ arising due to the characteristic scales we use to define α_0). It is noteworthy that K_a and K_d imply that $\theta_0 \approx 1.0$, which is in excellent agreement with our observations. Whereas K_a and K_d were obtained by fitting to experimental data, θ_0 can be obtained theoretically as the dimensionless scalar flux arising from Gaussian profiles of $\bar{w}$ and $\bar{c}$ of equal spread. On the other hand, θ_1 appears to take different values in axisymmetric jets compared to quasi-two-dimensional jets, which is perhaps not surprising when one considers that θ_1 depends on the lateral mixing provided by turbulence and the longitudinal scaling of the velocity field.

In planar jets, the dispersion closure used in (5.61) has the form

$$\theta_m = \theta_0 + \pi\sqrt{3} \left(\frac{1}{2} - \frac{z}{C_m} \frac{\partial C_m}{\partial z} \right) \theta_1. \quad (5.65)$$

While it may appear inconvenient that (5.65) contains pre-factors that differ to those in the equivalent expression for axisymmetric jets (5.36), the advantage is that the physical meaning of θ_0 and θ_1 is unchanged. In principle, one is therefore able to make predictions *a priori* about dispersion in planar/quasi-two-dimensional jets in addition to axisymmetric jets.

5.8 Conclusions

We have analysed the transport of a passive scalar in a statistically axisymmetric turbulent jet and found that the longitudinal mixing is primarily the result of shear-flow dispersion rather than turbulent transport (see, e.g., figure 5.8). Consequently, estimations of the amount of longitudinal mixing based exclusively on turbulence that do not account for the radial dependence of the mean flow are likely to significantly underestimate the total longitudinal mixing.

While shear-flow dispersion in turbulent jets is ostensibly different to shear-flow dispersion in pipe flows, for which the classical theory was first developed (Taylor, 1953), there are several similarities. Different from pipe flow is the fact that in a jet the radial steady-state scalar concentration is non-uniform, and will,

according to the lateral transport properties of the scalar, be concentrated in regions of relatively high longitudinal velocity. From an integral perspective, this property of the scalar field is characterised by a constant dimensionless flux parameter θ_0 , which can also account for the longitudinal turbulent scalar flux if necessary. Similar to pipe flow is the fact that longitudinal changes in an otherwise steady-state scalar field result in a local perturbation of the scalar's radial dependence. In jets the effect of a perturbation of amplitude L_1 on the local scalar flux can be accounted for with the parameter θ_1 . Together, θ_0 and $L_1\theta_1$ provide a dispersion closure in an integral model that shows good agreement with DNS data. When applied to planar jets the approach yields a differential equation that is in one-to-one correspondence with that proposed by Landel *et al.* (2012) and provides additional physical insight.

In principle, shear-flow dispersion finds application in any problem in which there are lateral gradients in a velocity field and is therefore relevant to a wide range of situations in industrial and environmental fluid mechanics. In particular, one should expect to find shear-flow dispersion in both laminar and turbulent free-shear flows, including turbulent plumes and wakes. The axisymmetric turbulent jet that we have analysed is relatively simple because it is statistically two-dimensional and self-similar sufficiently far from the source. Consequently, the behaviour of a propagating step-change in the integral concentration could be characterised completely in terms of a similarity variable $\lambda \propto z^2/t$. However, one expects for shear-flow dispersion to also play a role in more complicated three-dimensional problems such as plumes that are influenced by a cross wind and multiple, coalescing jets and plumes. The present study indicates that unless shear-flow dispersion arising from advection by the mean flow is correctly understood and parameterised, longitudinal mixing in such flows will not be predicted correctly.

Outstanding among issues deserving further attention is a formulation for the precise way in which the scalar profile is perturbed as a result of a step change in the longitudinal concentration profile. Such a formulation would allow one to estimate a dispersion coefficient (cf. θ_1 in the present study) for a variety of different free-shear flows such as planar jets and plumes *a priori*. A related issue for consideration is the role of higher-order perturbations to the scalar profile such as those considered in the context of pipe flow by Taylor (1954*a*), Gill (1967) and Chatwin (1970).

In the following chapter we consider the more complicated nonlinear problem of a jet whose source momentum undergoes a sudden change. There, we will find that dispersion manifests itself in the equation for the mean longitudinal kinetic energy. In particular, we will see that the dimensionless energy flux γ_m plays a role

that is analogous to θ_m in the present chapter. However, whilst $\theta_m \approx 1$, $\gamma_m > 1$, which has significant implications for the integral behaviour of both unsteady jets and plumes.

Unsteady turbulent jets: theory and observation

In this chapter we study the physics of unsteady turbulent jets. In contrast to the unsteady passive scalar transport discussed in chapter 5, here we consider statistical unsteadiness in the velocity field of the jet. We use direct numerical simulation to observe the effects of an instantaneous step change (both up and down) in the source momentum flux of the jet. We will demonstrate that the use of a momentum–energy framework of the kind used by Priestley & Ball (*Q.J.R. Meteorol. Soc.*, vol. 81, 1955, pp. 144–157) has several advantages over the classical mass–momentum formulation. Consequently, in place of the dimensionless scalar flux θ_m of chapter 5, our focus will turn to the dimensionless energy flux γ_m . We will show that the structure of the governing equations, the propagation speed of disturbances in the jet and the rate at which they spread depends crucially on the behaviour of the dimensionless energy flux. Most of the work reported in this chapter has been published in Craske & van Reeuwijk (2015*b*) and is reproduced here, in an adapted form, with permission.

6.1 Introduction

The plume theory developed by Morton *et al.* (1956, hereafter referred to as the MTT56 model), as introduced in §2.4.1, provides an elegant and simple means of describing the complex effects of turbulence in jets and plumes. The success of their approach can be attributed to the fact that such flows have a natural tendency to evolve spatially in a state of self-similarity (see §2.3 and, e.g., Tollmien, 1926; George, 1989), such that the evolution of a single characteristic length, velocity and buoyancy scale suffice for obtaining predictions of their integral quantities. In the event that such strict similarity is not attained in higher-order statistics of the flow, predictive models based on the similarity of lower-order quantities often remain useful, even in stratified environments (Morton, 1971).

The classical plume theory, describing the behaviour of statistically steady jets and plumes, has subsequently been extended to unsteady cases. Indeed, as indicated in §1.1.2 such processes prevail in many natural and man-made situations: the accidental release of contaminant, the melting of ice sheets, volcanic eruptions and natural ventilation are inherently ‘unsteady’ problems. Motivated by the need to predict the response of fire detectors, work on unsteady plumes, subsequent to

the starting plume model of Turner (1962), focused on the plumes above growing fires (see e.g. Delichatsios, 1979; Heskestad, 1998). As such, much attention was paid to time similarity solutions in which all variables are rescaled by the scales appearing at the leading edge of a front. Delichatsios (1979) showed that such similarity solutions require a power-law dependence of the source buoyancy flux on time and coincide with a quasi-steady approximation close to the source.

As noted in §1.1.2, it was recently discovered that the unsteady plume models of Delichatsios (1979), Yu (1990) and Scase *et al.* (2006*b*) are ill-posed (Scase & Hewitt, 2012), because they do not account for longitudinal mixing processes. Indeed, their governing integral equations are consistent with the view that lateral slices of the jet or plume do not interact longitudinally (Scase *et al.*, 2009). However, possible sources of longitudinal mixing such as turbulence and lateral gradients of mean velocity (causing dispersion) are evident in numerous experimental studies of steady jets and plumes (e.g. Kotsovinos & List, 1977; Panchapakesan & Lumley, 1993*a*). The findings of chapter 5 demonstrate that in jets passive scalar transport by shear-flow dispersion is significant and dominant in comparison to turbulence transport. By extension, one expects for there to be a manifestation of shear-flow dispersion in the velocity field of an unsteady jet.

A contribution to experimental data pertaining to unsteady plumes was made by Scase *et al.* (2008), who investigated the effects of a sudden reduction in buoyancy flux. In particular, Scase *et al.* (2008) found that source conditions that maintain turbulence in the plume cannot be changed so as to disconnect the plume into individual thermals, a finding which adds validity to the description of such flows using plume theory. Nevertheless, experimental data for unsteady plumes remain limited, which is partly due to the difficulty of obtaining many realisations of an ‘identical’ experiment (Scase *et al.*, 2006*b*). In the context of flow control, several experimental studies of unsteady jets have investigated the enhanced entrainment that is caused by periodic excitation of the flow (see e.g. Bremhorst & Hollis, 1990, and references therein). Similarly, Borée *et al.* (1997) observed a relatively large radial inflow in the vicinity of a rapid reduction in the velocity of an unsteady jet. For the ease with which statistics can be taken over homogeneous dimensions, numerical simulations offer a significant advantage over experiments. In addition, numerical simulations allow one to control the source conditions arbitrarily and therefore investigate a broad range of problems or, alternatively, a canonical case that is difficult to realise in a laboratory.

Building on chapter 5 our aim here is to understand how unsteadiness in the velocity field affects the integral behaviour of a turbulent jet. As in chapter 5, we

will be interested in establishing the physical processes responsible for longitudinal mixing so that their effects can be accurately represented in simple and robust integral models. We examine pure momentum jets, before considering plumes in chapter 8, in order to focus exclusively on the behaviour of the volume flux and the momentum flux. As discussed in §2.5, central to our approach is the fact that we derive the governing integral equations for unsteady jets from first principles, without invoking any assumptions about self-similarity, the form of radial dependence or the magnitude of turbulent transport terms. This allows us to examine the behaviour of both steady and unsteady jets from a integral perspective comprehensively, and therefore address the validity of some of the assumptions that are used in existing unsteady plume models.

In the present chapter we obtain a generalised system of integral equations by extending the formalism of Priestley & Ball (1955, hereafter referred to as PB55), who obtain a complete system of equations for steady plumes by integrating a mean kinetic energy equation, derived from mass conservation and momentum conservation, rather than integrating the mass conservation equation directly. This approach was used recently by Kaminski *et al.* (2005, hereafter referred to as KTC05) to investigate a decomposition of the entrainment coefficient for steady plumes. Here we will show that the energy equation can be used to obtain an unsteady area or mass conservation equation for unsteady jets at the integral level, which generalises the equations used in existing models to arbitrary profiles that are not restricted to remain self-similar.

Here we use the framework in a diagnostic capacity to analyse results from the direct numerical simulation (DNS) of steady and unsteady jets. In chapter 7, we will employ the framework as a prognostic tool and develop a dispersion closure for unsteady jets. In addition, in chapter 7 we demonstrate the role that dispersion plays in determining the behaviour of the area of the jet and the response of integral fluxes to source perturbations. The present chapter is organised as follows. In §6.2 we develop the framework by deriving a system of governing integral equations for unsteady jets that describe the conservation of mass, momentum and energy. Simulation details are provided in §6.3, and in §6.4 we present results from the simulation of steady and unsteady jets. In particular, in §6.4.2 we determine the propagation velocity of the front appearing in the unsteady jets, and show that its evolution is a self-similar process. In §6.5 we develop theory applicable to the front, relating the observations of §6.4.2 to two types of dispersion, before drawing conclusions in §6.6.

6.2 The governing equations

In this section we will derive a system of integral equations for an unsteady jet. The two main differences between our approach and the approach adopted by the plume theory of MTT56 is that i) we retain higher-order terms that are typically neglected, and ii) following PB55 and KTC05 we supplement the classical mass–momentum formulation with a conservation equation for mean kinetic energy. We then demonstrate that a mass conservation equation can be derived from the momentum–energy pair, which gives unknown parameters appearing in the former an alternative and more direct physical interpretation.

6.2.1 Reynolds equations

As discussed in chapter 2, we consider a round turbulent jet orientated in the longitudinal (z) direction whose flow is statistically axisymmetric and swirl-free. Since the mean motion in the jet is in the longitudinal direction we will focus on the ensemble-averaged continuity equation and longitudinal momentum equation, which in cylindrical coordinates are

$$\frac{1}{r} \frac{\partial(r\bar{u})}{\partial r} + \frac{\partial\bar{w}}{\partial z} = 0, \quad (6.1)$$

$$\frac{\partial\bar{w}}{\partial t} + \frac{1}{r} \frac{\partial(r\bar{u}\bar{w})}{\partial r} + \frac{\partial\bar{w}^2}{\partial z} + \frac{1}{r} \frac{\partial(r\overline{u'w'})}{\partial r} + \frac{\partial\overline{w'^2}}{\partial z} = -\frac{\partial\bar{p}}{\partial z} + \nu\nabla^2\bar{w}. \quad (6.2)$$

An equation for the mean longitudinal kinetic energy can be obtained by multiplying (6.2) by $2\bar{w}$ and utilising (6.1):

$$\begin{aligned} \frac{\partial\bar{w}^2}{\partial t} + \frac{1}{r} \frac{\partial(r\bar{u}\bar{w}^2)}{\partial r} + \frac{\partial\bar{w}^3}{\partial z} + 2\frac{\partial(\bar{p}\bar{w})}{\partial z} \\ + \frac{2}{r} \frac{\partial(r\overline{u'w'w})}{\partial r} + 2\frac{\partial(\overline{w'^2\bar{w}})}{\partial z} = 2\bar{p}\frac{\partial\bar{w}}{\partial z} + 2\overline{w'^2}\frac{\partial\bar{w}}{\partial z} + 2\overline{u'w'}\frac{\partial\bar{w}}{\partial r} \\ + \nu\nabla^2\bar{w}^2 - 2\nu\left(\frac{\partial\bar{w}}{\partial r}\right)^2 - 2\nu\left(\frac{\partial\bar{w}}{\partial z}\right)^2. \end{aligned} \quad (6.3)$$

To assess the magnitude of the viscous terms we consider a longitudinal velocity scale, w_m , a velocity scale for the turbulence, w_f , a radial length scale r_m and a longitudinal length scale L . A local Reynolds number can be defined using these integral scales: $\text{Re}_f \stackrel{\text{def}}{=} w_f r_m / \nu$. In classical plume theory it is assumed that $\text{Re}_f \rightarrow$

∞ and $r_m/L \rightarrow 0$, which implies that $w_f/w_m \sim (r_m/L)^{1/2}$ (see e.g. Tennekes & Lumley, 1972, for details). In this limit it is necessary that $\text{Re}_f \gg (L/r_m)^{1/2}$ in order that the viscous terms are negligible in both the mean momentum and mean kinetic energy equation. In contrast to steady problems, in unsteady problems there is no guarantee that $r_m/L \ll 1$, owing to the possibility of sudden changes in the longitudinal direction. Thus, we take the limit $\text{Re}_f \rightarrow \infty$ independently and retain terms that are $O(r_m/L)$ relative to the leading-order balance:

$$\frac{\partial \bar{w}}{\partial t} + \frac{1}{r} \frac{\partial(r\bar{u}\bar{w})}{\partial r} + \frac{\partial \bar{w}^2}{\partial z} + \frac{1}{r} \frac{\partial(r\bar{u}'\bar{w}')}{\partial r} + \underbrace{\frac{\partial \bar{w}'^2}{\partial z}}_{O(r_m/L)} = - \underbrace{\frac{\partial \bar{p}}{\partial z}}_{O(r_m/L)}, \quad (6.4)$$

describing the mean longitudinal momentum, and

$$\begin{aligned} \frac{\partial \bar{w}^2}{\partial t} + \frac{1}{r} \frac{\partial(r\bar{u}\bar{w}^2)}{\partial r} + \frac{\partial \bar{w}^3}{\partial z} + \overbrace{2 \frac{\partial(\bar{p}\bar{w})}{\partial z} + 2 \frac{\partial(\bar{w}'^2\bar{w})}{\partial z}}^{O(r_m/L)} \\ + \frac{2}{r} \frac{\partial(r\bar{u}'\bar{w}'\bar{w})}{\partial r} = \underbrace{2\bar{p} \frac{\partial \bar{w}}{\partial z} + 2\bar{w}'^2 \frac{\partial \bar{w}}{\partial z}}_{O(r_m/L)} + 2\bar{u}'\bar{w}' \frac{\partial \bar{w}}{\partial r}, \end{aligned} \quad (6.5)$$

describing the mean longitudinal kinetic energy. Here we note that for steady jets, their slenderness $r_m/L \sim 2\alpha \approx 0.2$, where α is the classical entrainment coefficient, so that the assumption $r_m/L \ll 1$ is questionable even for steady jets. Collectively, the terms on the right-hand side of (6.5) represent a sink for the kinetic energy carried by the mean longitudinal flow, acting either to redistribute the energy to the other mean flow components or to produce turbulence kinetic energy. In §§6.2.4–6.2.5 we show that entrainment can be viewed as a consequence of this energy sink. In general the jet does work on its surrounding fluid, redistributing its momentum flux over an area that increases, at the expense of a longitudinally diminishing flux of kinetic energy. It should be noted that in the system (6.1), (6.4) and (6.5), there exist only two independent equations, because the equation for kinetic energy is a mechanical equation that was obtained using the continuity and momentum conservation equations. The aspect of these equations that is most pertinent to the present study is the fact that $\bar{w}$, w' , etc. have an *a priori* unknown dependence on r , which is manifested as profile constants when the equations are integrated.

6.2.2 Integral equations

Where in chapter 5 our focus was on predicting the behaviour of the integral concentration C_m , here we focus on predicting the behaviour of the longitudinal volume flux and the longitudinal specific momentum flux (hereafter referred to as the momentum flux for brevity),

$$Q_m(z, t) \stackrel{\text{def}}{=} 2 \int_0^{r_d(z, t)} \bar{w} r dr, \quad M_m(z, t) \stackrel{\text{def}}{=} 2 \int_0^{r_d(z, t)} \bar{w}^2 r dr, \quad (6.6a, b)$$

respectively. Unlike the statistically steady jet considered in chapter 5, in which Q_m and M_m were known functions of z , in this chapter Q_m and M_m are unknown functions of z and t . We continue to use the subscript m to denote quantities that are based on the integrals of a mean value. We remind readers that the threshold radius r_d delineates the edge of the jet and is defined as the point in the flow at which the longitudinal velocity reduces to 1/100 of its mean centreline value at a given z and t (see §4.3.2 for a detailed discussion).

From the definitions of Q_m and M_m naturally follow characteristic length and velocity scales:

$$r_m \stackrel{\text{def}}{=} \frac{Q_m}{M_m^{1/2}}, \quad w_m \stackrel{\text{def}}{=} \frac{M_m}{Q_m}, \quad (6.7a, b)$$

which are consistent with a top-hat interpretation of the radial profiles such that $Q_m \stackrel{\text{def}}{=} w_m r_m^2$ and $M_m \stackrel{\text{def}}{=} w_m^2 r_m^2$, although here we make no assumptions regarding radial dependence. Additionally, we define a characteristic area for the jet, $A_m \stackrel{\text{def}}{=} r_m^2$. Although a transport equation for the area A_m can be obtained directly using mass conservation and applying an appropriate kinematic boundary condition, this procedure is not straightforward and requires that stringent assumptions are made about the velocity profile (see e.g. Scase *et al.*, 2006b). The difficulty in obtaining a canonical transport equation for A_m is reflected in the variety of slightly different equations that have been adopted in unsteady plume models (compare, for example, Yu, 1990; Delichatsios, 1979). Nevertheless, a generic equation for A_m that encompasses these different approaches is

$$\frac{1}{\gamma_g} \frac{\partial A_m}{\partial t} + \frac{\partial Q_m}{\partial z} = 2\alpha M_m^{1/2}, \quad (6.8)$$

where α is the classical entrainment coefficient used by MTT56 and $1/\gamma_g$ is a free parameter. By looking at integral transport equations for the volume flux and the momentum flux, we will reveal the physical meaning of γ_g and obtain an explicit

expression for α .

Integration of (6.4) and (6.5), over a horizontal disk of radius $r_d(z, t)$, and dividing by π results in integral conservation equations for momentum and energy:

$$\frac{\partial Q_m}{\partial t} + \frac{\partial(\beta_g M_m)}{\partial z} = 0, \quad (6.9)$$

$$\frac{\partial M_m}{\partial t} + \frac{\partial}{\partial z} \left(\gamma_g \frac{M_m^2}{Q_m} \right) = \delta_g \frac{M_m^{5/2}}{Q_m^2}. \quad (6.10)$$

The prognostic equations (6.9) and (6.10) suggest that in unsteady jets it is useful to regard Q_m and M_m as integrals of momentum and energy, respectively, rather than fluxes. A similar equivalence between volume flux and the integral of (specific) momentum is found in the shallow-water equations. To obtain (6.9)-(6.10), we neglect momentum and energy transport on the boundary of the jet, where $r = r_d$, as discussed in §4.3.2. This assumption is justified for a slender jet provided that $\epsilon \ll 1$ in (4.15). Indeed, defining the boundary of the jet in terms of ϵ means that the effects of non-lateral entrainment can be quantified as $O(\epsilon^2)$ and $O(\epsilon^3)$ terms in the momentum (6.9) and energy (6.10) equations, respectively. In addition, (6.9)-(6.10) do not include a contribution from temporal changes in r_d .

In the system of equations (6.9)-(6.10), β_g , γ_g and δ_g are dimensionless parameters, or profile constants, defined as

$$\beta_g \stackrel{\text{def}}{=} \frac{M_g}{M_m}, \quad \gamma_g \stackrel{\text{def}}{=} \frac{E_g Q_m}{M_m^2}, \quad \delta_g \stackrel{\text{def}}{=} \frac{P_g Q_m^2}{M_m^{5/2}}, \quad (6.11)$$

which represent a dimensionless momentum flux, a dimensionless energy flux and a dimensionless turbulence production, respectively. In §6.2.4 we will demonstrate that the γ_g defined above is indeed equal to the γ_g appearing in the area equation (6.8). The dimensionless parameters allow all unknown integrals in the system to be related to Q_m and M_m , which are the dependent variables. We will refer to the quantity

$$M_g \stackrel{\text{def}}{=} 2 \underbrace{\int_0^{r_d(z,t)} \bar{w}^2 r dr}_{M_m} + 2 \underbrace{\int_0^{r_d(z,t)} \overline{w'^2} r dr}_{M_f} + 2 \underbrace{\int_0^{r_d(z,t)} (\bar{p} - p_d) r dr}_{M_p}, \quad (6.12)$$

as the gross momentum flux, comprising integrals of the mean transport of longitudinal momentum M_m , the turbulent transport of longitudinal momentum M_f and pressure M_p . Here, p_d is the pressure in the ambient: $p_d = p(r_d, z)$. Similarly,

the gross energy flux

$$E_g \stackrel{\text{def}}{=} \underbrace{2 \int_0^{r_d(z,t)} r \bar{w}^3 r dr}_{E_m} + \underbrace{4 \int_0^{r_d(z,t)} r \bar{w} \overline{w'^2} r dr}_{E_f} + \underbrace{4 \int_0^{r_d(z,t)} r (\bar{p} - p_d) \bar{w} r dr}_{E_p}, \quad (6.13)$$

includes contributions from the mean flow, in addition to those from turbulence and pressure. The gross turbulence production is defined to include pressure redistribution:

$$P_g \stackrel{\text{def}}{=} \underbrace{4 \int_0^{r_d(z,t)} \overline{u'w'} \frac{\partial \bar{w}}{\partial r} r dr}_{P_m} + \underbrace{4 \int_0^{r_d(z,t)} \overline{w'^2} \frac{\partial \bar{w}}{\partial z} r dr}_{P_f} + \underbrace{4 \int_0^{r_d(z,t)} \bar{p} \frac{\partial \bar{w}}{\partial z} r dr}_{P_p}. \quad (6.14)$$

We will use subscripts in the dimensionless quantities that are consistent with the definitions above, e.g.

$$\gamma_m \stackrel{\text{def}}{=} \frac{E_m Q_m}{M_m^2}, \quad \gamma_f \stackrel{\text{def}}{=} \frac{E_f Q_m}{M_m^2}, \quad (6.15a, b)$$

represent the dimensionless mean energy flux and dimensionless turbulent energy flux, respectively. The system (6.8)-(6.10) is a generalisation of the approaches adopted by PB55 and MTT56 in the absence of buoyancy. Whilst MTT56 focus on volume and mean momentum conservation using (6.8) and (6.9), PB55 invoke mean momentum and mean energy conservation using (6.9) and (6.10). Central to the current work is the fact, noted by PB55, that only when a particular radial dependence of quantities such as $\bar{w}$ and $\overline{w'^2}$ is assumed (e.g. Gaussian), can parameters such as γ_m and γ_f be determined. In particular, $\gamma_m \geq 1$ in general, where equality holds only when the longitudinal velocity is assumed to be distributed uniformly. In a leading-order analysis of the steady state, in which the mean energy equation (6.10) reduces to a balance between two terms, the value of γ_m has received little attention. This is because, provided that it remains constant, γ_m does not affect the steady-state solution independently. Indeed, in such circumstances γ_m only enters the problem via the ratio δ_m/γ_m . However, with the addition of temporal derivatives, γ_m plays an independent role in the governing equations.

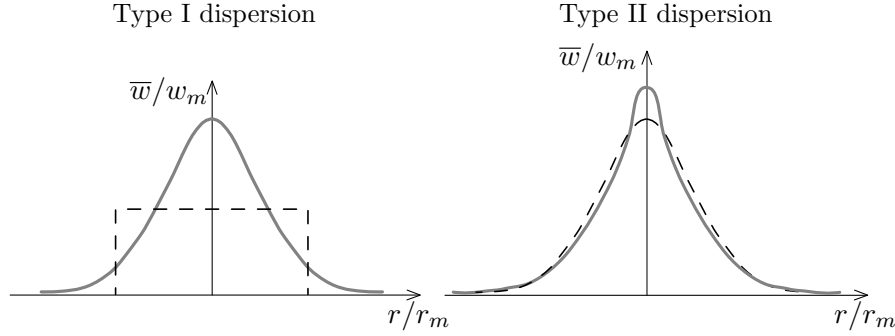


FIGURE 6.1. Schematic representation of type I dispersion, arising from non-uniform velocity profiles, and type II dispersion, arising from a departure from self-similarity.

6.2.3 Dispersion and non-uniform velocity profiles

The fact that the radial distribution of velocity in real jets is non-uniform implies that shear provides a means of spreading regions of the flow over the longitudinal dimension. Indeed, parcels of fluid located on the jet's centreline travel faster than those that are located on its periphery. In this work we will refer to all effects arising from non-uniform velocity profiles as dispersive. As noted in chapter 5, a rigorous treatment of dispersion in shear flows involves a series of perturbations to the flow. Radial mixing of the n^{th} perturbation is balanced by longitudinal advection of the $(n-1)^{\text{th}}$ perturbation. Here, we define type I dispersion as resulting from the zeroth, leading-order, description of the system using constant profile coefficients. Type I dispersion therefore deals with non-uniform, but *self-similar* profiles. This leading-order description necessarily includes radial mixing by turbulence. However, in order to preserve self-similarity in the vicinity of steep longitudinal gradients, one assumes that such mixing occurs on infinitesimal time scales, which is physically unrealistic. To compensate, type II dispersion corresponds to the first perturbation of the system and becomes relevant in the vicinity of steep longitudinal gradients. Type II dispersion therefore corresponds to a departure from self-similarity and provides a more realistic picture of the overall system in an unsteady context. Type II dispersion corresponds to the leading-order amount by which the otherwise constant profile coefficients can deviate from their steady-state values. Radial mixing of this first perturbation is balanced by advection of the zeroth-order profile. As described in chapter 5, in both pipes and jets the departure from self-similarity of an advected quantity results in a local increase or reduction in the integral flux of that quantity, owing to its correlation with a non-uniform

velocity profile. Velocity profiles corresponding to type I and type II dispersion are illustrated schematically in figure 6.1. We will see that the classification of type I and type II dispersion provides a useful means of understanding the processes that determine the effective rate with which quantities are transported in a jet and the rate at which they are mixed longitudinally respectively.

The effects of type I dispersion can be readily appreciated if one considers the transport of a passive scalar. If the radial distribution of the scalar is much narrower (wider) than that of the velocity field, then the scalar will be concentrated around regions of relatively high (low) velocity and therefore transported relatively rapidly (slowly). Indeed, such effects account for the theoretical and observed propagation speed of a front in an unsteady plume (see e.g. Turner, 1962; Delichatsios, 1979). Although such a process can be regarded as advection (Landel *et al.*, 2012), it is determined entirely by the radial velocity profile and therefore, in terms of its effect on the integral behaviour of the jet, fits comfortably with the notion of dispersion. In this work we will show that the dispersive perspective is made more compelling by the fact that type I dispersion causes a separation of the system's characteristic curves and influences the behaviour of the jet's area. Indeed, the notion of dispersion provides a unifying concept for all the processes that are dominant in unsteady jets.

In this regard it is noteworthy that when a smooth, normalised, velocity profile $f \stackrel{\text{def}}{=} \bar{w}/w_m$ decreases monotonically in r , its associated energy profile $f^2 \stackrel{\text{def}}{=} \bar{w}^2/w_m^2$, will necessarily be narrower than f . On these grounds alone, one can expect the dimensionless flux of energy to be greater than that of momentum. Only in the non-physical limit of top-hat profiles will one find that f^2 and f have the same width, and therefore that the dimensionless energy and momentum fluxes are equal. One must also bear in mind that, in a jet, momentum is a conserved quantity, whereas the energy of the mean flow is not. Some of the energy entering a horizontal plane is removed from the mean flow and converted into turbulence kinetic energy.

Using the observation that the longitudinal velocity profile in a jet is approximately Gaussian and self-similar, the value of the profile parameter γ_m can be determined exactly, which is how PB55 determined their so-called profile constants. In terms of w_m and r_m a Gaussian profile is expressed as

$$\bar{w} = 2w_m \exp\left(-2\frac{r^2}{r_m^2}\right), \quad (6.16)$$

which is consistent with the definitions $Q_m \stackrel{\text{def}}{=} w_m r_m^2$ and $M_m \stackrel{\text{def}}{=} w_m^2 r_m^2$. Using (6.16), the dimensionless mean energy flux is found exactly as

$$\frac{E_m Q_m}{M_m^2} \stackrel{\text{def}}{=} \gamma_m \stackrel{\text{def}}{=} \frac{2}{w_m^3 r_m^2} \int_0^\infty \bar{w}^3 r dr = \frac{4}{3}. \quad (6.17)$$

To focus on the effects of dispersion it is necessary to clearly distinguish between those effects that can be attributed to the shape of radial profiles, and those that are caused by the relative magnitude of turbulent transport terms. In fact, the momentum equation (6.9) contains no information about the shape of the underlying profiles and can be regarded as comprising zeroth moments, or mean values. In contrast, having been obtained by multiplying the momentum equation by $2\bar{w}$, the mean energy equation (6.10) is of higher-order, and therefore contains information about the profile shapes. With this in mind, the turbulent transport parameter β_f represents the magnitude of the turbulent transport, regardless of the shape of the profile of $\bar{w}'^2$. However, the energy transport term, γ_f , depends not only on β_f , but also on the correlation of $\bar{w}'^2$ with $\bar{w}$ over the radial dimension. Therefore, shape effects can be isolated by examining the central moment $\gamma_f - 2\beta_f$, which provides a decomposition that is analogous to the Reynolds decomposition $\overline{XY} - \bar{X}\bar{Y}$ for fluctuating quantities X and Y . Here, we must employ $2\beta_f$ rather than β_f , in the light of the fact that the mean energy equation was obtained by multiplying the momentum equation by $2\bar{w}$, rather than $\bar{w}$. In a similar way, shape effects pertaining to the transport of mean energy and pressure work are obtained as $\gamma_m - 1$ and $\gamma_p - 2\beta_p$, respectively. In sum, these contributions are equal to $\gamma_m + \gamma_f + \gamma_p - (1 + 2\beta_f + 2\beta_p) = \gamma_g + 1 - 2\beta_g$, which corresponds to the dimensionless energy flux arising from non-uniform radial dependences.

6.2.4 The entrainment coefficient from an energetics perspective

In a steady setting, MTT56 obtain ordinary differential equations in Q_m and M_m by considering volume conservation and momentum conservation, respectively. In an unsteady setting, volume conservation provides an evolution equation for the area, or storage, of the jet (Scase *et al.*, 2006b). This viewpoint is complicated by the fact that the area then assumes a precise, physical interpretation (see, e.g., Scase *et al.*, 2006b, who assume a top-hat distribution of longitudinal velocities). However, the fact that

$$\frac{\partial A_m}{\partial t} \stackrel{\text{def}}{=} \frac{\partial}{\partial t} \left(\frac{Q_m^2}{M_m} \right) \stackrel{\text{def}}{=} 2 \frac{Q_m}{M_m} \frac{\partial Q_m}{\partial t} - \frac{Q_m^2}{M_m^2} \frac{\partial M_m}{\partial t}, \quad (6.18)$$

suggests that to understand the physics that determine the behaviour of A_m , it is equations describing the temporal change of momentum, $\partial_t Q_m$, and the temporal

change of the mean kinetic energy, $\partial_t M_m$, that should be consulted. Conversely, particular assumptions made in obtaining a volume conservation equation manifest themselves in the kinetic energy balance. It is in this way that there exists an incompatibility between the plume models of PB55 and MTT56 (see Morton, 1971, for details).

An equation describing the evolution of the area of the jet can be obtained by substituting (6.9) and (6.10) into (6.18), which results in

$$\frac{1}{\gamma_g} \frac{\partial A_m}{\partial t} + \frac{\partial Q_m}{\partial z} = 2\alpha M_m^{1/2}, \quad (6.19)$$

where

$$\alpha \stackrel{\text{def}}{=} - \underbrace{\frac{\delta_g}{2\gamma_g}}_{\alpha_{\text{prod}}} + \underbrace{\frac{Q_m}{2\gamma_g M_m^{5/2}} \frac{\partial}{\partial z} \left[(\gamma_g + 1 - 2\beta_g) M_m^2 \right]}_{\alpha_{\text{disp}}} + \underbrace{\frac{Q_m}{\gamma_g M_m^{3/2}} (\beta_g - 1) \frac{\partial M_m}{\partial z}}_{\alpha_{\text{turb}}}. \quad (6.20)$$

Equation (6.19) is different to the volume conservation equation obtained by KTC05 because it describes an unsteady state, without buoyancy, and accounts for a possible dependence of the entrainment coefficient on turbulence transport and pressure. An additional difference is the choice made by KTC05 to define their top-hat radius (cf. r_m in the present formulation) in terms of profiles of velocity and buoyancy, rather than velocity alone. In both KTC05 and the present formulation the entrainment coefficient is affected by the shape of the longitudinal velocity profile. In an unsteady setting, the use of the mean energy equation has produced the factor γ_g^{-1} in the area equation, whose physical significance as a dimensionless energy flux γ_g can now be understood. In the area equation γ_g has the effect of modifying the timescale on which changes in area occur. Alternatively, if γ_g remains constant, the first term of (6.19) can be expressed as the temporal derivative of a modified area: $\partial_t(A_m/\gamma_g)$. In either case, these effects are determined by the profile shapes and turbulent transport terms. The derivation of the area equation using the equations for the mean momentum and energy generalises the separate approaches (e.g. top-hat, Gaussian) adopted in unsteady plume models and makes clear their physical implications. Motivation for the particular decomposition used in (6.20) comes from the fact that it separates the influence of the integral intensity of the turbulence and pressure $(\beta_g - 1)$ from dispersive effects $(\gamma_g + 1 - 2\beta_g)$.

The term α_{prod} is the leading-order term responsible for entrainment. It is proportional to the ratio of the dimensionless integral production and redistribu-

tion of turbulence kinetic energy δ_g , and the gross dimensionless energy flux γ_g . Conventionally, α_{prod} is the only contribution to entrainment that is considered and, in the steady state, gives rise to the classical entrainment coefficient:

$$\alpha_0 \stackrel{\text{def}}{=} -\frac{\delta_g}{2\gamma_g}, \quad (6.21)$$

where, unless stated otherwise, the value of a quantity χ in the steady state will be denoted hereafter by χ_0 . In the steady-state equations the independent behaviour of δ_g and γ_g is unimportant, as it is only their ratio (6.21) that modifies the governing equations. In an unsteady setting, on the other hand, one finds several additional terms (e.g. $\gamma_g^{-1}\partial_t A_m$) that depend on γ_g independently, which means that the behaviour of both γ_g and δ_g , and not just their ratio, becomes relevant.

Together, the terms α_{disp} and α_{turb} account for a difference in the dimensionless flux of momentum compared to the dimensionless flux of energy and, since $A_m = Q_m^2/M_m$, the consequent effect that this has on area. Individually α_{disp} and α_{turb} are decomposed components of this process, accounting for non-uniform radial profiles and the intensity of turbulent fluxes, respectively. In a steady state, the fact that the dimensionless energy flux is typically greater than the dimensionless momentum flux is inconsequential, because $\partial_z M_m = 0$. In an unsteady setting, on the other hand, $\partial_z M_m \neq 0$ and α_{disp} and α_{turb} play an active role.

The dispersion term α_{disp} can be expanded as

$$\alpha_{\text{disp}} = \underbrace{\frac{Q_m}{\gamma_g M_m^{3/2}} (\gamma_g + 1 - 2\beta_g) \frac{\partial M_m}{\partial z}}_{\alpha_{\text{disp1}}} + \underbrace{\frac{Q_m}{2\gamma_g M_m^{1/2}} \frac{\partial}{\partial z} (\gamma_g + 1 - 2\beta_g)}_{\alpha_{\text{disp2}}}, \quad (6.22)$$

which makes explicit the individual contributions from dispersion of type I and type II, respectively. When all profiles are self-similar, the gross momentum and energy fluxes can be expressed as a constant proportion of M_m and M_m^2/Q_m , respectively, therefore the dimensionless fluxes γ_g and β_g are constant. In that case, it is evident from (6.22) that $\alpha_{\text{disp2}} = 0$. In general however, $\alpha_{\text{disp2}} \neq 0$, which will be the case whenever the longitudinal scaling of the jet variables differs from the asymptotic power-law scaling predicted by similarity theory. If the velocity profiles deform and similarity is lost, the effect of α_{disp2} is to ‘mix’ area in the longitudinal direction. Notably, since β_g is not an operand of ∂_z in α_{turb} , type II dispersion provides the only source of such mixing. The similarity drift identified by KTC05 also includes the effects of deformation that we label as α_{disp2} . However, as the name implies,

the similarity drift in KTC05 is used primarily to quantify the effect that a change in the relative widths of the velocity and buoyancy profiles has on entrainment. To refer to α_{disp2} as accounting for similarity drift in the present context is therefore slightly misleading, because here we are concerned with localised deformations in the velocity profile alone, arising from an unsteady forcing. In the present context it is more useful to regard α_{disp2} as the effective entrainment, or detrainment that corresponds to a change in the shape of the velocity profile. For example, in the development of a top-hat velocity profile into a Gaussian velocity profile the flow on the axis of the jet effectively ‘entrains’ from the rest of the jet and hence $\alpha_{\text{disp2}} > 0$.

The contribution α_{disp1} is perhaps best regarded as a consequence of the spatial acceleration of the flow. Even when the flow evolves in a state of self-similarity, α_{disp1} will be non-zero in general. Assuming $\gamma_g + 1 - 2\beta_g > 1$, positive spatial acceleration ($\partial_z M_m > 0$) will result in a positive contribution to the entrainment coefficient, whilst negative spatial acceleration ($\partial_z M_m < 0$) results in a negative contribution to the entrainment coefficient. The role of α_{disp1} can therefore be visualised as resulting from a relative convergence or divergence of the flow on a horizontal plane depending on whether $\partial_z M_m$ is greater than or less than zero, respectively. In general, the contribution to entrainment from α_{disp1} is fundamentally distinct from the Richardson number dependence that emerges in the presence of buoyancy. However, in the special case of statistically steady plumes, in which $\partial_z M_m$ can be replaced with the integral of buoyancy, α_{disp1} contributes to the Richardson number dependence identified by PB55 and KTC05.

6.2.5 The relation between the classical view of entrainment and turbulence mixing

The classical view of turbulent entrainment in jets pertains to the engulfment of quiescent fluid at the turbulence boundary and is based on the continuity equation. It is therefore useful to consider the relation between the classical view and the energetics perspective advocated in this thesis. Below we will argue that the continuity equation demonstrates *that* fluid is entrained, whilst the mean energy equation provides insight into *why* the fluid is entrained.

For the purposes of the present discussion we will adopt the classical approach, integrating (6.1) over a disk of radius $r_d \rightarrow \infty$ and assuming that Q_m remains bounded:

$$\frac{\partial Q_m}{\partial z} = -2ru|_{\infty}, \quad (6.23)$$

which simply states that longitudinal changes in the volume flux are accompanied

by an induced radial flow in the ambient. Equation (6.23) is an integral continuity equation and is valid for statistically steady and unsteady flows.

In a steady state the mean energy equation is useful (see, e.g., Kaminski *et al.*, 2005) because it reveals that the strength of the induced flow is equal to the ratio of turbulence production and the energy flux:

$$2ru|_{\infty} = \frac{\delta_g}{\gamma_g} w_m r_m. \quad (6.24)$$

Hence, the mean energy equation provides insight into why fluid is being entrained into the jet. If turbulence production were equal to zero there would be no induced flow in the ambient and $\partial_z Q_m = 0$. In a steady state the induced flow $ru|_{\infty}$ is equal to the rate at which fluid is entrained into the jet. However, in a statistically unsteady situation equation (6.23) is of limited use because it does not necessarily describe the rate at which ambient fluid is entrained into the jet. Indeed, the radius of the jet $r_m \stackrel{\text{def}}{=} Q_m/M_m^{1/2}$ might vary with time, which would mean that the rate at which fluid enters the jet does not correspond to the induced flow in the ambient.

In a statistically unsteady situation the mean energy equation is useful because, since it pertains to *why* fluid is entrained, it is able to go further than (6.23) and predict the rate at which fluid is entrained into the jet. Using (6.23) and the area equation (6.19), which was obtained from integral equations for the mean momentum (6.9) and energy (6.10),

$$2ru|_{\infty} - \frac{2r_m}{\gamma_g} \frac{\partial r_m}{\partial t} = -2\alpha r_m w_m, \quad (6.25)$$

where the entrainment coefficient $\alpha \stackrel{\text{def}}{=} \alpha_{\text{prod}} + \alpha_{\text{disp}} + \alpha_{\text{turb}}$ has a known dependence on the physical properties of the jet. Equation (6.25) states that $-2\alpha r_m w_m$ is the rate of entrainment relative to the inward/outward propagation rate of a characteristic jet radius at a fixed longitudinal location (for a discussion of several alternative definitions of an entrainment velocity the reader is referred to Turner, 1986). However, in jets that do not have a top-hat profile the edge of the jet is not always clearly defined. Evident from the left-hand side of (6.25) is that the appropriate radius from which to view entrainment is the effective entrainment radius $r_m \gamma_g^{-1/2}$, which is determined by the radial dependence of the jet's longitudinal velocity profile. As one would expect, for a top-hat jet with $\gamma_g = 1$, the effective entrainment radius coincides with the radius r_m . For a temporal jet $\partial_z Q_m = 0$ and therefore, using (6.23), $ru|_{\infty} = 0$. In that case, (6.25) indicates that the entrainment flux $-2\alpha r_m w_m$ is equal in magnitude and opposite in sign to the time rate of change of the effective area $r_m^2 \gamma_g^{-1}$.

Note that even though the momentum–energy framework naturally includes turbulence, it remains an integral-scale description that is not concerned with the small-scale aspects of turbulence. Consequently, we do not make statements about the dynamics of the turbulent-non-turbulent interface (see e.g. da Silva *et al.*, 2014, for details), although large-scale and small-scale entrainment are intimately connected (see van Reeuwijk & Holzner, 2014).

6.3 Simulation details

The data presented herein were obtained from direct simulation of the full Navier-Stokes equations as described in §3.2. The domain and discretisation employed were described in §4.1.1 and are summarised in table §6.1. Indeed, we will again refer to the steady-state simulations L_j , H_j and H_j^* , which were discussed in chapter 4, although here our focus is on the unsteady velocity field. For validation of the steady-state jet data from L_j , H_j and H_j^* , the reader is referred to §4.1.2.

For the unsteady simulations we impose a step change in the source momentum flux, doubling and halving the source momentum flux M_s for cases that we refer to as LH_j and HL_j , respectively, while keeping the source radius constant. To ensure reliable statistics we carry out an ensemble of simulations. Initial conditions for LH_j and HL_j were obtained from the two steady-state base simulations, L_j and H_j , respectively. During the course of one of these base simulations we write 16 complete three-dimensional field files to disk at dimensionless time intervals not less than $200\tau_s$, where the source time scale $\tau_s \stackrel{\text{def}}{=} Q_s^2/M_s^{3/2}$. The field files from each of the two base cases provide the initial conditions for an ensemble of 16 independent statistically unsteady simulations. In the case of the steady simulations we obtain statistics by averaging over time and the homogeneous azimuthal dimension. In the case of the unsteady simulations however, the time dimension is no longer statistically homogeneous. We therefore reduce the time interval over which averages are taken to not more than $2\tau_s$, and average over the ensemble, in addition to the azimuthal direction, as described in chapter 4. From the unsteady simulations, which are each of approximate duration $200\tau_s$, we obtain a sequence of 120 files describing the ensemble-averaged behaviour of the variables over the inhomogeneous spatial dimensions r and z . For details pertaining to acquisition of azimuthally-averaged data, the reader is referred to §3.6.2 and §4.3.1.

TABLE 6.1. Simulation details, where M_s^B and M_s^A are the source momentum fluxes before and after the step change, respectively.

	$(L_x \times L_y \times L_z)/r_s$	$N_x \times N_y \times N_z$	Re_s	Re_λ	t_{run}/τ_s	M_s^A/M_s^B
L_j	$44^2 \times 66$	$768^2 \times 1152$	4815	100	3323	-
H_j	$44^2 \times 66$	$768^2 \times 1152$	6810	135	3524	-
$LH_j 1-16$	$44^2 \times 66$	$768^2 \times 1152$	-	-	235	2
$HL_j 1-16$	$44^2 \times 66$	$768^2 \times 1152$	-	-	166	1/2

6.4 Results

6.4.1 The steady jet

Figure 6.2(a) shows an instantaneous and time-averaged slice through the longitudinal velocity field. Notable is the fact that entrainment into the jet is established close to the source, which is evident in the positive, and almost uniform, gradient of Q_m for $z/r_s > 0$. In figure 6.2(b, c) we display the longitudinal variation of Q_m and M_m in simulations L_j , H_j and H_j^* . Since plume theory only pertains to mean quantities, it is necessary to define a steady-state momentum flux:

$$M_{m0} \stackrel{\text{def}}{=} \frac{1}{z_t - z_b} \int_{z_b}^{z_t} M_m dz \leq M_s, \quad (6.26)$$

where $z_b/r_s = 28$ and $z_t/r_s = 55$, which, as expected, results in a good collapse of the simulation data from L_j , H_j and H_j^* (cf. figure 4.12). From dimensional analysis it follows that M_m should be constant and Q_m proportional to z/r_s , which is confirmed in the simulation results. At the top of the domain the linear behaviour in Q_m is slightly affected by an inclination of the streamlines in the ambient, which influences the integration limit r_d . A steady-state entrainment coefficient α_0 and virtual origin z_v were determined for each jet by fitting a straight line to $Q_m(z)$ over the interval $z \in [z_b, z_t]$, to give $\alpha_0 = 0.067$ and $z_v/r_s = -5.0$, when averaged over L_j and H_j . Hereafter we will use the notation $Q_{m0} = 2\alpha_0 M_{m0}^{1/2} (z - z_v)$ to denote the theoretical steady volume flux in the jet.

Figure 6.3 displays longitudinal profiles of β_g , γ_g and δ_g , and their constituent parts, which represent the dimensionless momentum flux, energy flux and turbulence production, respectively. The almost identical behaviour of these variables in L_j and H_j confirms that the statistics examined are practically independent

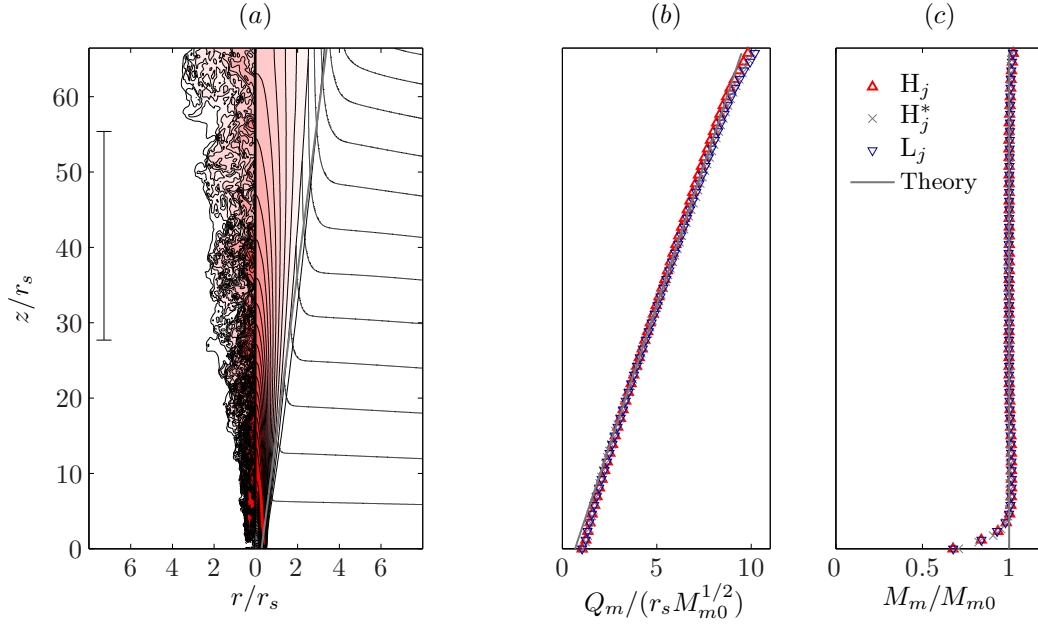


FIGURE 6.2. (a) Isoregions of instantaneous longitudinal velocity $w(r, \pi, z, t_0)$, where $t_0 \approx 1200\tau_s$ (left hand side) and average longitudinal velocity $\bar{w}(r, z)$ (right hand side), with darker shades corresponding to larger values. A linear fit to the theoretical plume width $r_m = 2\alpha_0(z - z_v)$ is indicated on the right hand side with a solid grey line, in addition to streamlines of the induced ambient flow; (b) normalised volume flux $Q_m / (r_s M_{m0}^{1/2})$ and (c) momentum flux M_m / M_{m0} . The interval $[z_b, z_t]$, used in equation (6.26) to obtain the steady state momentum flux M_{m0} , is indicated on the left hand side of (a).

of Reynolds number. Here, we are focusing on the dimensionless quantities, e.g. M_f / M_m , whose value should be independent of the dependent variables Q_m and M_m . The dimensionless quantities therefore indicate the relative contribution that is made by all integrals appearing in the governing integral equations (6.8)-(6.10). Of significance is the fact that the values of the dimensionless quantities differ, in some cases considerably, from those that are frequently assumed in models of steady and unsteady jets (see e.g. Morton *et al.*, 1956; Scase *et al.*, 2006b). As can be judged from the profile of $\beta_f \stackrel{\text{def}}{=} M_f / M_m$ in figure 6.3(d), despite the central role that turbulence plays in mixing momentum locally, turbulent transport terms at the integral level are an order of magnitude smaller than those associated with the mean flow, but not insignificant. This observation is consistent with previous findings from experiments, including Panchapakesan & Lumley (1993a, PL93) and Hussein *et al.* (1994). In fact, in β_p it is seen that a large portion of the turbulent transport of momentum flux is balanced by a reduction of pressure in the jet,

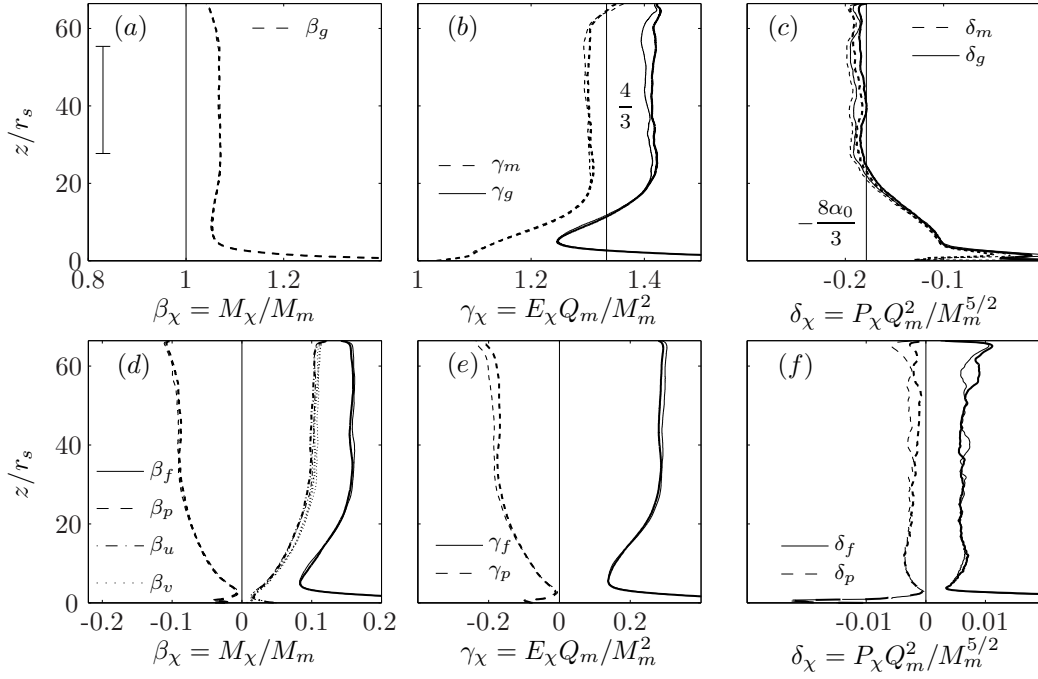


FIGURE 6.3. Dimensionless momentum flux β_g (a,d), dimensionless energy flux γ_g (b,e), dimensionless turbulence production and pressure redistribution δ_g (c,f), and their constituent parts, in a steady jet. (a-c) The leading order components; (d-f) higher-order components. Thin lines correspond to results from L_j , and thick lines correspond to results from H_j . Constant values corresponding to a Gaussian profile are displayed as vertical lines in (b,c). The interval over which averaging was performed to obtain the values reported in table 6.2 is indicated on the left-hand side of (a).

which can be estimated to a good approximation using the transverse components of momentum flux M_u and M_v (see e.g. Hussein *et al.*, 1994):

$$\beta_p = -\frac{1}{2}(\beta_u + \beta_v), \quad (6.27)$$

where

$$\beta_u \stackrel{\text{def}}{=} \frac{2}{M_m} \int_0^{r_d(z,t)} \overline{u'^2} r dr, \quad \beta_v \stackrel{\text{def}}{=} \frac{2}{M_m} \int_0^{r_d(z,t)} \overline{v'^2} r dr. \quad (6.28a, b)$$

In sum, the contribution to the gross momentum flux from pressure and turbulence accounts for approximately 7%.

Figure 6.3(b,e) displays dimensionless energy fluxes and confirms that $\gamma_m > 1$, which is consistent with the non-uniform profiles that one expects to find in real jets. At $z/r_s \approx 20$ the dimensionless energy flux γ_m appears to have converged

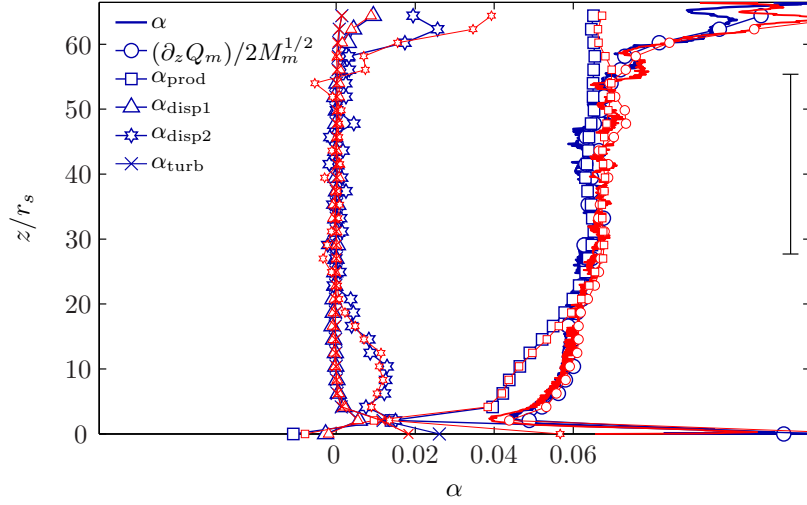


FIGURE 6.4. The entrainment coefficient $\alpha(z)$ and its constituent parts. Data for H_j and L_j shown in blue (dark grey, large symbols) and red (light grey, small symbols), respectively. The interval $z/r_s \in [28, 55]$, over which averaging was performed to obtain the value $\alpha_0 = 0.067$, is indicated on the right hand side of the figure.

to approximately 1.3, which is close to the exact value of $\gamma_m = 4/3$ that one would expect to see if the mean flow were perfectly Gaussian, and indicates the presence of dispersion. Below $z/r_s = 15$, the jet develops from patches of uniform velocity, owing to the source conditions (see §4.1.1), into a developed flow with significant radial dependence, resulting in the monotonic increase of γ_m from 1 to 1.3. The gross energy flux γ_g is slightly higher than the mean energy flux because the turbulent transport of mean energy γ_f exceeds the negative pressure-work term γ_p , as seen in figure 6.3(e). Very close to the source, where $z/r_s < 5$, the azimuthal inhomogeneity resulting from the source condition (see §6.3) is evidenced in large values of the variance $\overline{w'^2}$ and therefore in large values of β_f and γ_f . Further from the source, beyond the visible local minimum in β_f and γ_f , the flow becomes azimuthally homogeneous and β_f and γ_f are approximately constant.

In the dimensionless production term δ_m in figure 6.3(c), one sees that an approximately constant value is not established until $z/r_s > 20$. Below this height, although one might expect to see a reduction in the entrainment into the jet, because $\alpha_0 \stackrel{\text{def}}{=} -\delta_m/2\gamma_m$, the steady increase in $-\delta_m$ is approximately balanced by the steady increase in γ_m . This is an example of the way in which the independent behaviour of γ_m and δ_m is unimportant in the steady jet equations.

Figure 6.4 displays the constituent parts of the entrainment coefficient (6.20),

TABLE 6.2. The dimensionless parameters of a steady jet. Here TH = top-hat, G = Gaussian and PL93 = Panchapakesan & Lumley (1993*a*). The values displayed in the columns beneath H_j and L_j are given to within one standard deviation. The dimensionless parameters reported in this table are averages over the interval $z/r_s \in [28, 55]$, which is indicated in figure 6.3(*a*).

	TH	G	H_j	L_j	PL93
α	α_0	α_0	0.065 ± 0.001	0.069 ± 0.002	0.082
β_f	0.000	0.000	0.157 ± 0.001	0.160 ± 0.004	0.187
β_u	0.000	0.000	0.099 ± 0.002	0.102 ± 0.003	0.094
β_v	0.000	0.000	0.104 ± 0.001	0.108 ± 0.003	0.097
β_p	0.000	0.000	-0.088 ± 0.001	-0.091 ± 0.002	-
β_g	1.000	1.000	1.068 ± 0.001	1.069 ± 0.002	-
γ_m	1.000	1.333	1.303 ± 0.003	1.298 ± 0.005	1.305
γ_f	0.000	0.000	0.284 ± 0.003	0.292 ± 0.006	0.354
γ_p	0.000	0.000	-0.171 ± 0.001	-0.185 ± 0.006	-
γ_g	1.000	1.333	1.416 ± 0.006	1.404 ± 0.006	-
δ_m	$-2\alpha_0$	$-8\alpha_0/3$	-0.187 ± 0.003	-0.195 ± 0.006	-0.201
δ_f	0.000	0.000	0.006 ± 0.000	0.007 ± 0.001	-
δ_p	0.000	0.000	-0.002 ± 0.000	-0.003 ± 0.000	-

for both L_j and H_j . A moving-average filter of dimensionless length 1 was applied to raw data in order to obtain the data displayed in figure 6.4. The close agreement between α and $(\partial_z Q_m)/2M_m^{1/2}$ implies that the integral equations for momentum (6.9) and energy (6.10), and therefore area (6.8), are satisfied by the simulation data. As expected, the term α_{prod} dominates in the steady state, and provides an excellent means of inferring an entrainment coefficient in a way that is independent, yet consistent, with the direct calculation of $(\partial_z Q_m)/2M_m^{1/2}$. However, close to the source the role of α_{disp2} , whose effects are contained in the similarity drift term in KTC05, is also discernible. There, α_{disp2} has the effect of making a positive contribution to entrainment, which is the result of a gradual deformation of the source velocity profile over the region of flow development. Correspondingly, in this region α_{prod} increases monotonically to its far-field value, which ultimately dominates $\alpha(z)$. Beyond $z/r_s = 55$, both α_{disp2} and $(\partial_z Q_m)/2M_m^{1/2}$ appear to be affected by the boundary. Indeed, in figure 6.2 it is evident in the streamlines at the top of the domain that the boundary induces a small longitudinal velocity in the ambient, to which the longitudinal derivative appearing in α_{disp2} , and therefore $\partial_z Q_m/(2M_m^{1/2})$, is sensitive. Notable, however, is that the turbulence

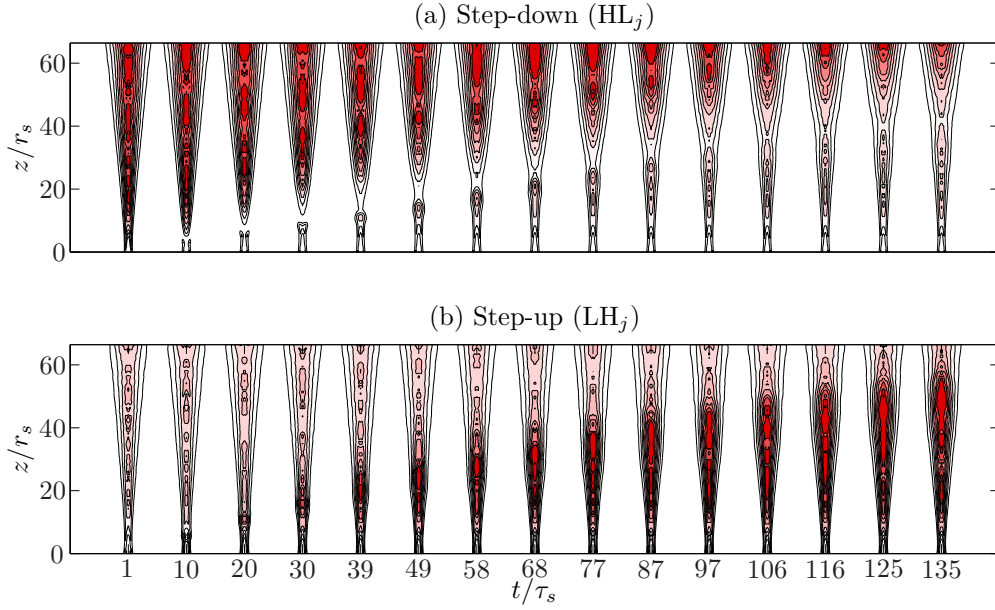


FIGURE 6.5. Isolines of the normalised ensemble-averaged longitudinal kinetic energy $\bar{w}^2/w_{m0}^L$, where $w_{m0}^L(z)$ is the characteristic longitudinal velocity in the steady state simulation L_j .

production term α_{prod} , which provides the dominant contribution to entrainment, is not affected by the boundary.

The values of the dimensionless integrals are summarised in table 6.2, along with the values corresponding to top-hat and Gaussian profiles. In particular, the dimensionless parameters that we have observed show a good agreement with those of PL93 and do not reveal anything unexpected. Notable, however, is the significant contribution from $O(r_m/L)$ transport terms such as $\beta_f \approx 0.15$ and $\gamma_f \approx 0.29$, which is partially disguised when they appear in sum with other terms (e.g. $\beta_p + \beta_f \approx 0.07$). In a steady state, other than creating a mis-match between the source fluxes and the far-field fluxes (e.g. $M_{m0} \neq M_s$), such terms have little effect on the perceived momentum and energy balances. Also noteworthy is the fact that the turbulence production and redistribution terms δ_f and δ_p are almost two orders of magnitude smaller than δ_m . With respect to the mean-flow transport of energy, we note that the observation that $\gamma_m \approx 1.3$ differs considerably from the top-hat assumption $\gamma_m = 1$, the implication of which will be discussed in the following sections.

6.4.2 The unsteady jet

Figure 6.5 displays isolines of the dimensionless longitudinal kinetic energy, $\bar{w}^2/w_{m0}^L$, where $w_{m0}^L(z)$ is the characteristic longitudinal velocity in the steady-state simulation L_j , on instantaneous longitudinal slices through the domain at several times. For the unsteady simulations the timescale τ_s is defined according to $\tau_s \stackrel{\text{def}}{=} r_s^2/M_m^{*1/2}$, where M_m^* is a constant characteristic momentum flux that will be defined precisely at the end of this section. In each case a front, negative in figure 6.5(a) and positive in figure 6.5(b), propagates and spreads in the longitudinal direction. When the source momentum flux is reduced the region with relatively high momentum flux appears to pinch away from the region of relatively low momentum flux beneath. This behaviour is not observed when the source momentum flux is increased. Clear from figure 6.5 is that the leading edge of the the region of high momentum flux in figure 6.5(b) travels at approximately the same speed as the trailing edge of the region of high momentum flux in figure 6.5(a). Ascertaining the rate at which propagation and spreading occurs forms the focus of this section, and in §6.5 we will develop theory relating to these processes.

The averaging over the ensemble is depicted in figure 6.6, which displays the longitudinal profile of momentum flux for each individual simulation at $t = 47\tau_s$, in addition to their ensemble average. In general the amplitude of the deviations from the ensemble average appears to scale in constant proportion to the average momentum flux at that height. In contrast to what one might expect, a significant increase in the relative magnitude of deviations from the ensemble is not discernible at the level of the front. In figure 6.6 it is possible to identify a primary front in each case, whose location corresponds approximately to the leading edge of the disturbance. In HL_j it is also possible to identify a secondary front, that develops below the primary front, and an accompanying trough in M_m . The secondary front in HL_j has an appearance that is similar to the primary front in LH_j .

In figure 6.7(a, b), which displays longitudinal profiles of M_m averaged over the ensemble, at the dimensionless times $t_1 = 31\tau_s, t_2 = 55\tau_s, t_3 = 79\tau_s, t_4 = 103\tau_s$, one can discern qualitatively the way in which the front evolves. The front propagates and spreads in the longitudinal direction. More subtle, is the fact that in figure 6.7(a) the front appears to spread more rapidly than that in figure 6.7(b). Indeed, on the basis that HL_j comprises relatively high velocities ahead of relatively low velocities, one might expect to see rarefaction-like behaviour. In LH_j , on the other hand, the relatively high velocities occur behind the front and one might expect to see shock-like behaviour. However, it is evident in figure 6.7(b) that a shock is prevented from occurring due to longitudinal mixing. Figure 6.7(c, d) displays the

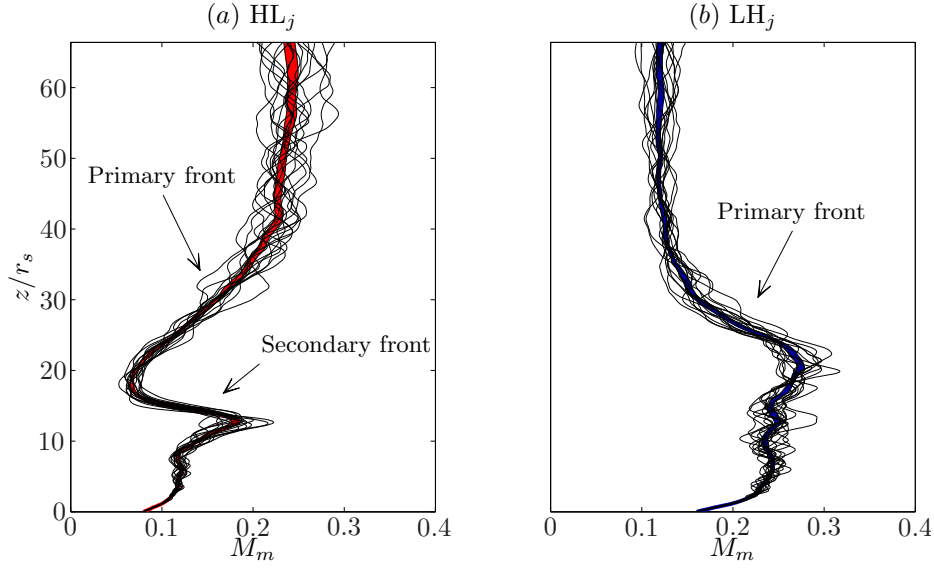


FIGURE 6.6. Individual simulations (thin lines) and their ensemble average (thick line) at $t = 47\tau_s$. The thickness of the line depicting the ensemble average is equal to twice the estimated standard deviation of the true mean at that location.

profiles of the jet radius r_m . The general behaviour of r_m is difficult to anticipate, because it depends on both Q_m and M_m . It is observed in figure 6.7(c,d) that the jet remains approximately straight-sided in the vicinity of the front, excepting the local increase in r_m in HL_j , which is coincident with the local minimum in M_m . We will return to the topic of straight-sidedness in detail in chapter 7.

The way in which the ambient flow field is affected by the unsteady jets is displayed in Figure 6.8. Most noticeable is the divergence in the streamlines in the vicinity of negative longitudinal gradients of the momentum flux (i.e. regions in which $\partial_z M_m < 0$). This behaviour is consistent with the view that, for non-uniform velocity profiles, α_{disp1} leads to a reduction in entrainment ($\alpha_{\text{disp1}} < 0$), or relative divergence, when the flow decelerates in the longitudinal direction. On the other hand, in the region in figure 6.8(a) in which $\partial_z M_m > 0$ the behaviour of $\alpha_{\text{disp1}} > 0$ indicates an enhanced entrainment resulting from the acceleration of the jet in the longitudinal direction. When $\partial_z M_m > 0$, both the spacing of the streamlines evident in figure 6.8(a) and the observation of an enhanced radial inflow reported by Borée *et al.* (1997), support the qualitative predictions that can be obtained from the framework that was outlined in §6.2.4. Indeed, we will demonstrate in chapter 7 that if entrainment were not affected by $\partial_z M_m$, the local area Q_m^2/M_m of the jet would necessarily increase or decrease in order to accommodate the local deceleration ($\partial_z M_m < 0$) or acceleration ($\partial_z M_m > 0$) of the flow, respectively.

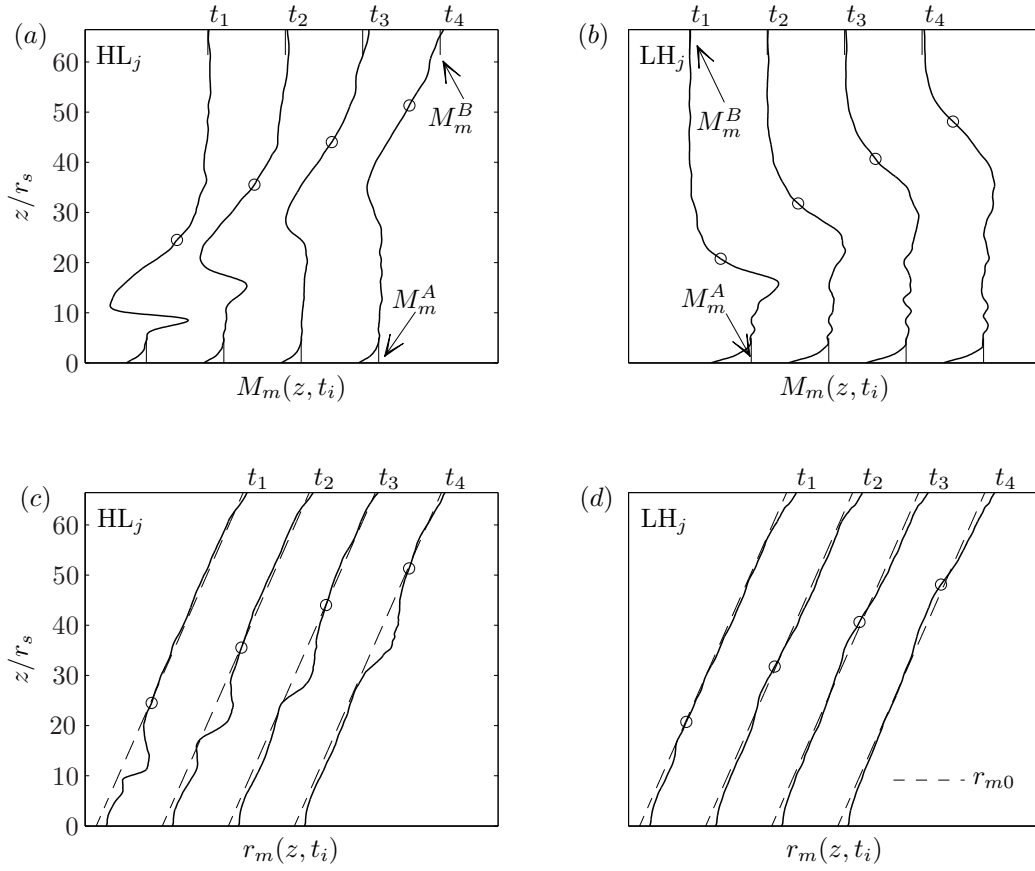


FIGURE 6.7. Ensemble-averaged, instantaneous profiles of mean momentum flux M_m (a, b) and radius r_m (c, d). For comparison, the steady state radius $r_{m0} = 2\alpha_0 z$ is also displayed and the location $z^*(t)$, which is defined according to $M_m(z^*) = (M_m^A + M_m^B)/2$, is marked $\circ$. The time to which each profile corresponds is $t_i = 31\tau_s + 24\tau_s i$, where $i = 0 \dots 3$.

An advantage of examining entrainment via balances of mean momentum and energy in an unsteady jet is that it is not necessary to develop a storage model for the volume of the jet independently. Indeed, the area Q_m^2/M_m is a statistical quantity that characterises the lateral spread of the jet rather than an observable physical threshold corresponding to a precise lateral extent. Consequently, as demonstrated in equation (6.20) and the previous paragraph, the entrainment properties of the unsteady jet can be inferred from the behaviour of Q_m and M_m . By definition, at the upper limit r_d of the integrals Q_m and M_m , the longitudinal velocity $\bar{w}(r_d, z, t)$ is relatively small. Therefore, assuming that the unsteady flow remains slender, there will be a relatively small flux of momentum and energy at $r = r_d$. It is consequently not necessary to account for the effects of non-lateral

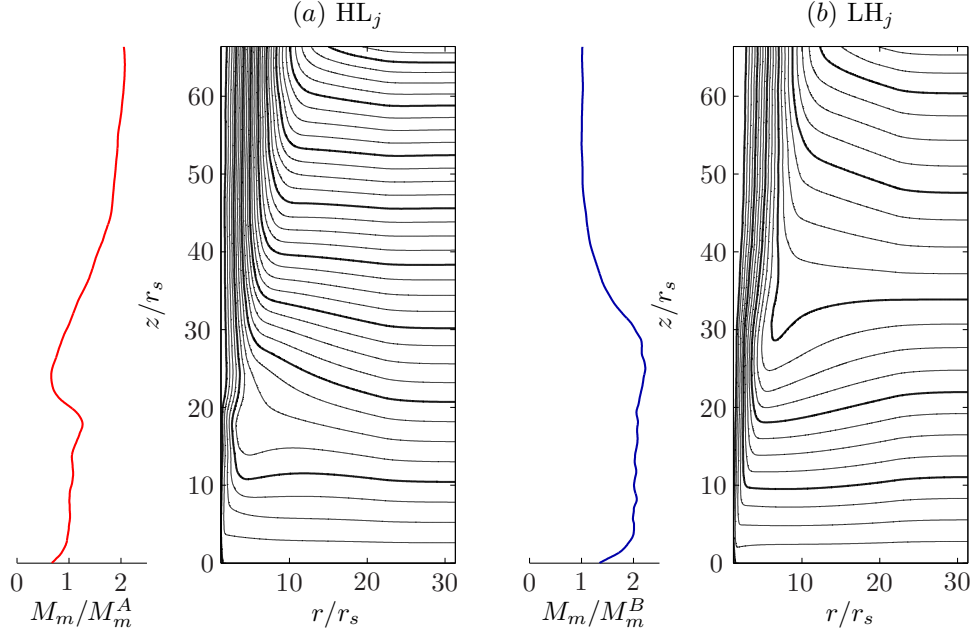


FIGURE 6.8. Isolines of the stream function $\psi(r, z, t)$ at $t/\tau_s = 64$, where $\bar{u} \stackrel{\text{def}}{=} -r^{-1}\partial_z\psi$ and $\bar{w} \stackrel{\text{def}}{=} r^{-1}\partial_r\psi$, displayed alongside the normalised momentum flux.

entrainment in addition to the terms that already appear in (6.20). In situations in which the flow is not slender, or $\bar{w}(r_d, z, t)$ is not relatively small, the entrainment relation (6.20) can be extended in a straightforward manner to account for longitudinal boundary fluxes.

To track the location of the leading front, $z^*(t)$, we use an intermediate value of the momentum flux to provide the implicit equation

$$M_m(z^*) = \frac{M_m^A + M_m^B}{2}, \quad (6.29)$$

where M_m^B and M_m^A are constants, corresponding to the steady mean momentum fluxes before and after the step change, respectively. The front position is depicted in figure 6.7 with a circle. Figure 6.9(a, b) displays the relation $z^*(t)$, determined numerically, and, with respect to the shaded isoregions of $\partial_z M_m$, demonstrates that it coincides approximately with the inflection points in HL_j and LH_j, in addition to giving a smooth representation of the front position. Given the normalisation employed, the linear behaviour of the data in figure 6.9(c) reveals that the front position evolves according to $z^* \sim \sqrt{t}$.

The scaling observed in figure 6.9 can be interpreted by assuming that the

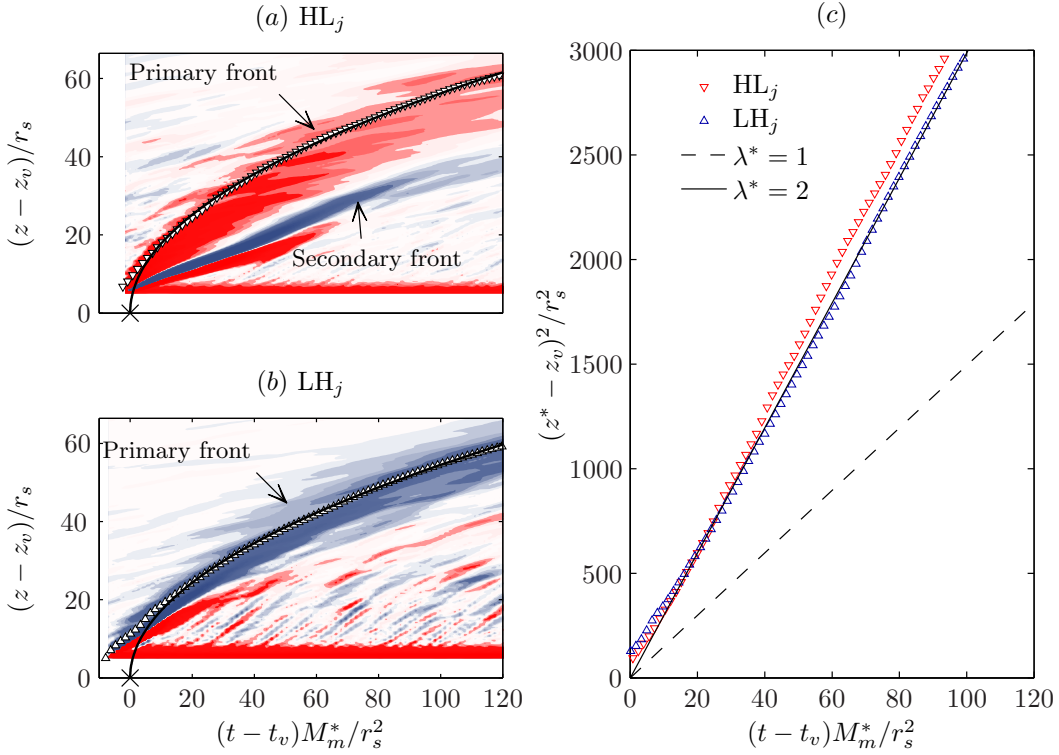


FIGURE 6.9. Location of front $z^*(t)$ determined from simulation data. Windows (a) and (b) display isoregions of $\partial_z M_m$ and a parabolic fit to the data with a bold line. The location of the asymptotic virtual source (z_v, t_v) is marked $\times$. Window (c) displays the front position with respect to a quadratically scaled longitudinal axis, in addition to the theoretical location of the front when the longitudinal velocity is assumed to have a top-hat form ($\lambda^* = 1$) and a Gaussian form ($\lambda^* = 2$).

front propagates at a rate given by

$$\frac{dz^*}{dt} = \lambda^* \frac{M_m^{*1/2}}{2\alpha_0(z^* - z_v)}, \quad (6.30)$$

where M_m^* is a suitable constant momentum flux, characteristic of the motion at the leading edge of the front, and λ^* is a constant, which in §6.5.1 we show can be interpreted as an eigenvalue of the system. In fact, under certain simplifying assumptions that will be discussed in chapter 7, the appropriate momentum flux M_m^* can be obtained exactly according to

$$M_m^* = \frac{1}{4} \left(\frac{M_m^A - M_m^B}{M_m^{A1/2} - M_m^{B1/2}} \right)^2, \quad (6.31)$$

when the front is regarded as a shock (see e.g. Toro, 1997). Integration of (6.30)

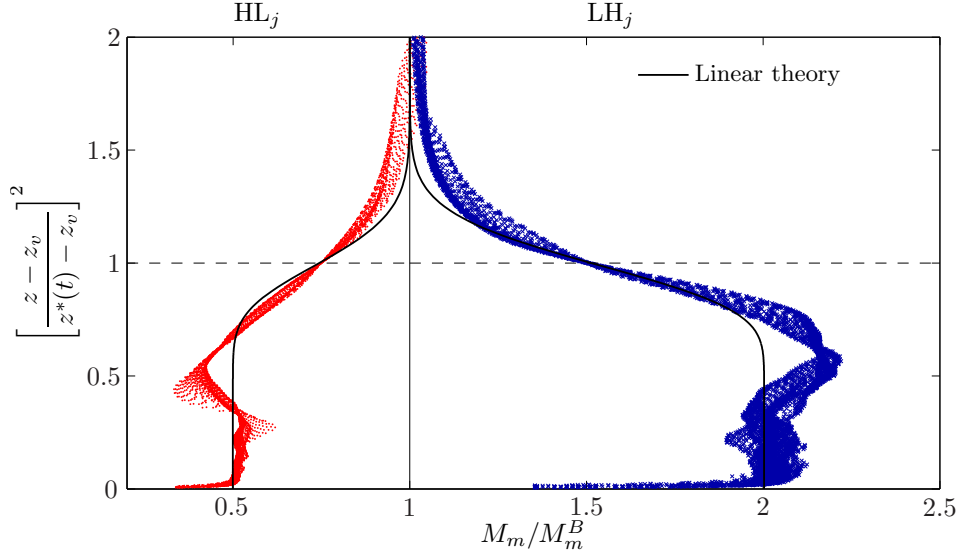


FIGURE 6.10. Self-similarity of the simulation results for the momentum flux M_m over the time interval $[63, 159]\tau_s$ compared to a similarity solution of a corresponding linear advection-dispersion equation.

reveals that

$$(z^* - z_v)^2 = \frac{\lambda^* M_m^{*1/2}}{\alpha_0} (t - t_v), \quad (6.32)$$

where t_v is a constant of integration that corresponds to the location of an asymptotic virtual source in time. Equation (6.32) describes the scaling observed in figure 6.9 and, with (6.31), provides a normalisation for fronts of different strengths. A similar argument was used by Abraham (1996) to deduce that the total mass entrained into an unsteady gas jet has a cubic dependence on its longitudinal penetration. Observable in figure 6.9 is that $\lambda^* \approx 2$ in both simulations, which suggests that the front propagates at $2w_m$ rather than w_m . For Gaussian profiles $2w_m$ corresponds to the maximum velocity whilst w_m corresponds to the mean velocity. In the next section we will look at the hyperbolic character of the governing equations in greater detail and show that the eigenvalue λ^* can be inferred with reasonable accuracy using the steady-state data. In doing so we will be able to demonstrate that it is the velocity profile of the jet that is chiefly responsible for determining the propagation speed of the front.

In figure 6.10 the momentum flux over the time interval $[63, 159]\tau_s$ is plotted with respect to a similarity variable obtained by rescaling the longitudinal coordinate by the observed front position $z^*(t)$. The observable collapse of the data confirms that the length scale associated with the spread of the front scales in pro-

portion to $z^*(t)$. In turn, one can infer that the processes responsible for the mixing of the front scale according to the local integral properties of the jet, rather than an independent scale. The picture that emerges in physical coordinates is similar to the cap of the starting plume envisaged by Turner (1962): for all times the longitudinal extent of the front remains in constant proportion to the local radius of the jet $r_m(z^*(t), t)$. The self-similarity evident in figure 6.10 suggests that unsteady jets attain an equilibrium state. In this state one expects the dynamics of the front to be invariant with time, provided that all quantities are non-dimensionalised using the local scales $w_m(z, t)$ and $r_m(z, t)$. This kind of self-similarity is weaker than the similarity that one finds in steady jets, because it does not imply that the radial dependence of $\bar{w}/w_m$ is everywhere the same. Furthermore, one would not expect to find similarity in the near field. Indeed, near-field (small-time) effects are evident in figure 6.10 as regions in which the data do not collapse. It is natural to expect that an unsteady jet produced by any change in source conditions that takes place over a finite time will eventually attain a state of similarity such that

$$\frac{\bar{w}}{w_m} = f\left(\frac{r}{r_m}, \frac{z^2 \alpha_0}{M_m^{*1/2} t}, \frac{M_m^A}{M_m^B}\right). \quad (6.33)$$

In converging to such a state, longitudinal mixing will ensure that the jet has a progressively weaker dependence on the precise way in which its source conditions were changed.

6.5 Theory of the front

In this section we analyse the propagation speed and spreading rate of the front in greater detail. We will show that the former is determined by the values of the dimensionless fluxes γ_g and β_g , and is significantly influenced by the steady-state longitudinal velocity profile. Mixing of the front, in contrast, requires that the dimensionless fluxes γ_g and/or β_g depart from their steady-state values, which implies a local departure from similarity in the jet. We distinguish these effects by identifying them as type I and type II dispersion, respectively.

6.5.1 Type I dispersion and characteristic curves

Here we will determine the characteristic curves of the system (6.9)-(6.10) in the general case, for which it will be assumed that the rates with which energy and momentum are transported are different, i.e. $\gamma_g \neq \beta_g$. Using the momentum-energy formulation we express the governing equation for unsteady jets as

$$\frac{\partial}{\partial t} \begin{pmatrix} Q_m \\ M_m \end{pmatrix} + \begin{bmatrix} 0 & \beta_g \\ -\gamma_g \frac{M_m^2}{Q_m^2} & 2\gamma_g \frac{M_m}{Q_m} \end{bmatrix} \frac{\partial}{\partial z} \begin{pmatrix} Q_m \\ M_m \end{pmatrix} = \begin{pmatrix} 0 \\ \delta_g \frac{M_m^{5/2}}{Q_m^2} \end{pmatrix}. \quad (6.34)$$

Looking locally, we assume that the dimensionless terms β_g , γ_g and δ_g are constants, thereby restricting analysis to type I dispersion. This allows us to understand, to leading order, the effect that they have on the classification of the system. Characteristic surfaces on which a function $\Phi(z, t)$ is constant, satisfy the following condition:

$$\det \begin{bmatrix} -\lambda^* & \beta_g \frac{Q_m}{M_m} \\ -\gamma_g \frac{M_m}{Q_m} & 2\gamma_g - \lambda^* \end{bmatrix} = 0, \quad (6.35)$$

where the eigenvalue λ^* represents a dimensionless velocity defined by

$$\lambda^* \stackrel{\text{def}}{=} \frac{dz}{dt} \frac{Q_m}{M_m} = -\frac{\partial \Phi}{\partial t} \left(\frac{\partial \Phi}{\partial z} \right)^{-1} \frac{Q_m}{M_m}. \quad (6.36)$$

The relation (6.35) implies that

$$\lambda^* = \gamma_g \left(1 \pm \sqrt{1 - \frac{\beta_g}{\gamma_g}} \right). \quad (6.37)$$

In the eigenvalues (6.37) we see that when the rates with which momentum and energy are transported are equal (i.e. when $\beta_g = \gamma_g$) the eigenvalues, and therefore the characteristic curves, overlap. In that case it can be shown that the eigenvectors of the system are not linearly independent and the system is parabolic (see e.g. Whitham, 1974). Indeed, it is a parabolic form of the equations that is implied by the top-hat model of Scase *et al.* (2006b). However, substitution of the values of γ_g and β_g obtained from the steady state (see table 6.2) suggests that

$$\lambda^* \approx 1.4 \pm 0.7, \quad (6.38)$$

and therefore that the system is hyperbolic. Unlike parabolic systems, in hyperbolic systems fast-propagating disturbances can overtake slower disturbances. Indeed, there is a close analogy between the unsteady jet equations and the shallow-water equations for supercritical flow. In both cases there are generally two families of characteristic curves and information is only able to travel downstream.

The existence of two independent families of characteristic curves in unsteady jets is the fundamental difference between the behaviour of the generalised system of integral equations (6.9)-(6.10) and existing top-hat models of unsteady jets. Physically, the separation of characteristic curves is the result of a non-uniform mean velocity profile and therefore of mean shear. While energy is transported by the mean flow $4/3$ times faster than momentum in Gaussian jets, (6.37) suggests that the leading characteristic propagates at *twice* the speed associated with the advection of momentum (assuming $\beta_g = 1$). However, this conclusion relies on the velocity $w_m \equiv M_m/Q_m$ having a unique value on the leading characteristic. For infinitesimal perturbations the leading characteristic indeed propagates at twice the local characteristic velocity w_m . For finite *nonlinear* perturbations, however, the correct value of w_m to use at the front follows from conservation laws and the use of the so-called Rankine-Hugoniot jump conditions (see e.g. Toro, 1997), which will be discussed further in chapter 7. For finite perturbations, similar to a tidal bore, energy, unlike momentum, is not conserved across step changes in the flow. Thus the front propagation speed that is likely to be observed in practice depends on the magnitude of the step change M_m^A/M_m^B , the extreme case being that of a starting jet ($M_m^A/M_m^B \rightarrow \infty$), which will be discussed in §7.5.1.

Only in the area between the characteristic curves can the jet deviate from its steady-state behaviour, following an instantaneous change in the source conditions. The precise way in which the jet deviates from its steady-state behaviour is determined by the position of the characteristic curves and therefore by the relative speeds with which momentum and energy are advected. This has important consequences for the area $A_m \equiv Q_m^2/M_m$ of the jet, which is defined by a particular balance between momentum and energy in the flow.

The characteristic curves are paths along which the total derivatives of several quasi-invariant quantities are decoupled. Using the eigenvectors of the system, the invariants are defined in terms of the total derivatives:

$$dY_m = -\frac{\gamma_g}{\lambda^*} \frac{M_m}{Q_m} Q_m^{-\gamma_g/\lambda^*} dQ_m + Q_m^{-\gamma_g/\lambda^*} dM_m, \quad (6.39)$$

where the integrating factor $Q_m^{-\gamma_g/\lambda^*}$ has been introduced. The quasi-invariants are therefore found to be

$$Y_m = M_m Q_m^{-\gamma_g/\lambda^*}, \quad (6.40)$$

to within a constant factor. We note that when the system is parabolic and $\lambda^* = 1$, the quasi-invariants are identical and equal to $M_m/Q_m = w_m$. Along the charac-

teristic curves the governing equations become

$$\frac{dY_m}{dt} = \delta_g \frac{M_m^{5/2}}{Q_m^{2+\gamma_g/\lambda^*}}. \quad (6.41)$$

The quasi-invariants prove particularly useful in understanding the behaviour of the area of the jet A_m . It can be shown that along fast and slow characteristic curves emanating from an instantaneous change in M_m , the area retains its steady state form, $A_m = 4\alpha_0^2 z^2$. In chapter 7 we show that the behaviour of the area between these curves depends on their separation, i.e. on the discriminant $1 - \beta_g/\gamma_g$, for which there exists a distinguished value that renders the area insensitive to changes in Q_m and M_m . Thus, there exists a value of $1 - \beta_g/\gamma_g$ that ensures straight-sidedness in the jet for all time. Remarkably, this value is obtained when the jet is assumed to have a Gaussian form, with $\beta_g = 1$ and $\gamma_g = 4/3$.

6.5.2 Type II dispersion and front mixing

In the previous section we established type I dispersion under the assumption that β_g and γ_g are constants, equal to their steady-state values. However, such an assumption is unrealistic in the vicinity of a steep front where self-similarity will, in general, not be maintained. This naturally raises the question of what effect a departure from self-similarity will have on the local fluxes and, therefore, on the longitudinal mixing at the front. Indeed, in §6.4.2 a non-negligible longitudinal spreading of the front was observed, which was found to scale according to $\sqrt{t}$. In this section we begin by estimating the mixing coefficient associated with the observed longitudinal spreading rate, before looking at the local variations in the dimensionless fluxes in the jet.

The spread of the front can be obtained by assuming that the process can be described by an advection-dispersion equation of the form

$$\frac{\partial M_m}{\partial t} + \lambda^* \frac{M_m^{*1/2}}{2\alpha_0(z - z_v)} \frac{\partial M_m}{\partial z} = D_e \frac{\partial^2 M_m}{\partial z^2}. \quad (6.42)$$

Equation (6.42) can be expressed as an ordinary differential equation using the similarity scaling that was observed in §6.4.2. In particular, it will be assumed that M_m has the self-similar form $M_m = f(z/z^*)$. The relationship $z^*(t)$, found in (6.32), implies that an appropriate similarity variable is

$$\lambda \stackrel{\text{def}}{=} \frac{z^2 \alpha_0}{t M_m^{*1/2}}, \quad (6.43)$$

so that $\lambda(z^*) = \lambda^*$. Partial derivatives with respect to t and z can then be expressed as

$$\frac{\partial}{\partial t} = -\frac{\lambda}{t} \frac{d}{d\lambda}, \quad \frac{\partial}{\partial z} = \frac{2\lambda}{z} \frac{d}{d\lambda}, \quad \frac{\partial^2}{\partial z^2} = \frac{2\lambda}{z^2} \left(\frac{d}{d\lambda} + 2\lambda \frac{d^2}{d\lambda^2} \right). \quad (6.44a, b, c)$$

Transformation of (6.42) according to (6.43) therefore yields

$$\frac{d^2 M_m}{d\lambda^2} = \frac{1}{2\lambda} \left[\text{Pe} \left(1 - \frac{\lambda}{\lambda^*} \right) - 1 \right] \frac{dM_m}{d\lambda}, \quad (6.45)$$

where

$$\text{Pe} \stackrel{\text{def}}{=} \frac{\lambda^* M_m^{*1/2}}{2\alpha_0 D_e}. \quad (6.46)$$

The dispersive Péclet number Pe is useful here, because it indicates the longitudinal extent of the front relative to the distance travelled by the front. Solutions of (6.45), subject to the boundary conditions $M_m(0) = M_m^A$ and $M_m(\infty) = M_m^B$, can be expressed in terms of Kummer's confluent hypergeometric function M (Abramowitz & Stegun, 1970, p.504). When $\text{Pe} \gg 1$ it is reasonable to suppose that the second term in the square brackets of (6.45) can be neglected. In such circumstances, the solution can be expressed as

$$M_m = M_m^A + \frac{M_m^B - M_m^A}{\Gamma(\text{Pe}/2 + 2)} \left(\frac{\text{Pe}\lambda}{2\lambda^*} \right)^{\text{Pe}/2+1} \exp\left(-\frac{\text{Pe}\lambda}{2\lambda^*}\right) M\left(1, \frac{\text{Pe}}{2} + 2, \frac{\text{Pe}\lambda}{2\lambda^*}\right). \quad (6.47)$$

where M is Kummer's function and Γ is the Gamma function. Also of interest is the dependence of $dM_m/d\lambda$ on Pe at the level of the front. Differentiation of (6.47) and setting $\lambda = \lambda^*$ results in

$$\left(\frac{\lambda^*}{M_m^A - M_m^B} \right) \frac{dM_m}{d\lambda} \bigg|_{\lambda=\lambda^*} = -\sqrt{\frac{\text{Pe}}{4\pi}} + O\left(\frac{1}{\sqrt{\text{Pe}}}\right). \quad (6.48)$$

Alternatively, a numerical solution to (6.45) can be readily obtained by prescribing arbitrary conditions at $\lambda = \lambda^*$, such as $M_m = 0$ and $dM_m/d\lambda = 1$, and then solving the initial value problem forwards and backwards over the intervals $[\lambda^*, L_z]$ and $[0, \lambda^*]$, respectively. A solution satisfying the boundary conditions $M_m(0) = M_m^A$ and $M_m(\infty) = M_m^B$ can then be obtained by using the invariance of equation (6.45) with respect to the transformation $M_m \mapsto aM_m + b$.

An estimation of the dispersion coefficient D_e can be obtained by matching the

similarity solution to the data displayed in figure 6.10. Specifically, the matching is performed by comparing the gradient of the similarity solution at the level of the front to that observed in the transformed simulation data. To do this we average over each of the instantaneous profiles comprising figure 6.10. To quantify mixing effects, rather than the spreading resulting from the rarefaction behaviour of HL_j , we use LH_j for our estimation. The gradient of the similarity solution of (6.42) was obtained to leading order by assuming that the effects of longitudinal mixing are relatively weak compared to the advection of the front. This procedure suggests that $D_e/M_m^{*1/2} \approx 0.22$, which defines the similarity solution that is shown in figure 6.10 with a solid line. Referring back to §6.3, it should be noted that in theory the relatively small, yet finite time interval $\delta t < 2\tau_s$ over which we average the data is responsible for a small amount of smoothing over the longitudinal direction. However, this contribution scales according to $\delta t^2/t^2$, whose value is no greater than 2×10^{-4} for the data presented in figure 6.10, from which the value of D_e was inferred. Accounting for the observed steepness of the front, we estimate that our inferred value of D_e overestimates the actual longitudinal mixing coefficient due to time averaging by not more than 1%. Figure 6.10 reveals that (6.42) provides a reasonably accurate first approximation to the front in each case. An interesting aspect of this result is that (6.42) ignores the coupling that exists between Q_m and M_m . In chapter 7 we will account for the surprising effectiveness of (6.42), and show that it implies that unsteady jets remain approximately straight-sided.

Having obtained an estimation for D_e , the logical next step is to ascertain whether the longitudinal mixing is due to turbulence or shear-flow dispersion of the kind identified by Taylor (1953). As described in chapter 5, shear-flow dispersion is caused by lateral gradients in the mean longitudinal velocity acting over longitudinal gradients of a transported variable. As discovered by Taylor (1953), dispersion produces a local deformation in the radial dependence of the transported quantity, which, owing to its correlation with a non-uniform velocity can enhance or diminish the total flux of the quantity.

We make an estimation of the average turbulence eddy viscosity ν_{Tm} using the gradient diffusion hypothesis:

$$\overline{u'w'} = -\nu_{Tm} \frac{\partial \bar{w}}{\partial r}, \quad (6.49)$$

and evaluate the dominant production term by assuming that $\bar{w}$ follows a Gaussian distribution (6.16):

$$P_m \stackrel{\text{def}}{=} 4 \int_0^\infty \overline{u'w'} \frac{\partial \bar{w}}{\partial r} r dr = -4\nu_{Tm} \int_0^\infty \left(\frac{\partial \bar{w}}{\partial r} \right)^2 r dr = -8\nu_{Tm} \bar{w}_m^2. \quad (6.50)$$

On using the definition of δ_m and the relation $\delta_m = -2\alpha_0\gamma_m$, implied by (6.21) to relate P_m to α_0 , we find

$$\nu_{Tm} = \frac{\alpha_0}{3} M_m^{*1/2}. \quad (6.51)$$

The observed dispersion coefficient D_e is several times larger than the eddy viscosity ν_{Tm} ; indeed $D_e/M_m^{*1/2} = 0.22$ compared to $\nu_{Tm}/M_m^{*1/2} = 0.023$. Although the estimation (6.51) does not account for the fact that ν_{Tm} might change locally at the front, its value relative to D_e suggests that the mixing of the front is primarily a dispersive phenomenon. In chapter 7 we apply the model for scalar dispersion developed in chapter 5 to energy dispersion.

We are now in a position to address the origin of the longitudinal mixing observed in figure 6.10. In §6.5.1 we assumed that γ_g remains constant in the vicinity of a front. This amounts to assuming that the self-similarity of the jet is unaffected by a shock or rarefaction, which, from a physical point of view, is unrealistic. In fact, unless the velocity profiles have a top-hat form, we expect that non-uniform profiles of velocity will redistribute energy in the longitudinal direction and therefore smooth steep fronts. This behaviour is confirmed in figure 6.11, which displays a moving average of the dimensionless mean (*a*) and dimensionless turbulence (*b*) longitudinal kinetic energy relative to steady-state profiles. Positive (negative) values indicate parts of the flow where the mean or turbulence energy is higher (lower) than the steady-state value at the same location. In LH_j the negative longitudinal gradient in M_m at the front means that fluid of relatively high energy, transported from behind the front, is concentrated on the axis of the jet and surrounded by fluid of relatively low energy. In HL_j, in which the longitudinal gradient in M_m is positive, the situation is reversed and fluid with relatively high energy is removed from the axis of the jet at the front. It is notable that the deformation of both the mean energy profile and the turbulence energy profile has a similar form.

The moving average of the dimensionless fluxes displayed in figure 6.12 confirms that both γ_m and γ_f increase in LH_j and decrease in HL_j in the vicinity of the front. Noteworthy is the fact that in spite of the departures from steady-state similarity observable in $\overline{w'^2}$ in figure 6.11, figure 6.12 shows that the dimensionless

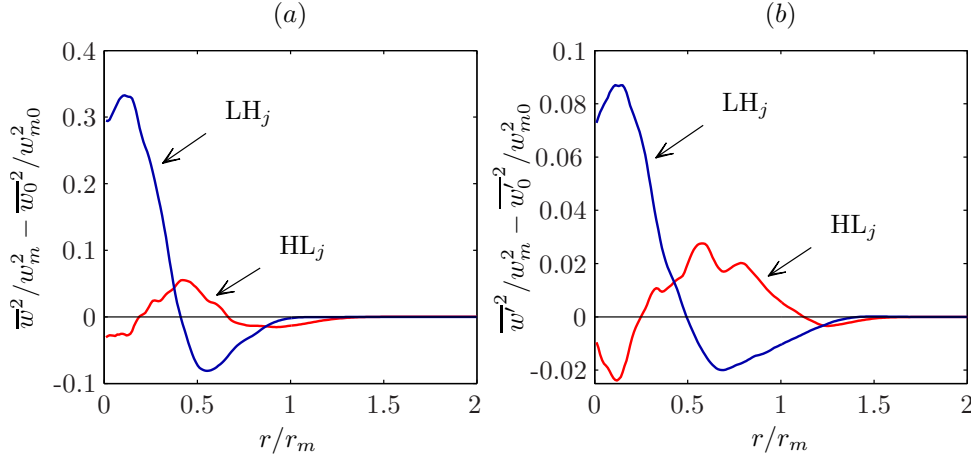


FIGURE 6.11. Moving averages (following the front), relative to steady state profiles. (a) Normalised longitudinal mean kinetic energy profile and (b) longitudinal turbulence kinetic energy. Steady state profiles were obtained by averaging profiles in both LH_j and HL_j over the range $z/r_s \in [28, 55]$.

turbulent transport β_f is not significantly affected by the front. To appreciate this, note that the deviation from self-similarity of $\overline{w'^2}$ shown in figure 6.11 appears in a product with r when integrated over the jet to give β_f . The dispersion described in the previous paragraph arises due to the redistribution of a quantity over the radius of the jet, rather than a change in the area integral of that quantity. Such effects belong exclusively to the mean energy equation, because there, unlike in the momentum equation, the quantities appear in a product with $\overline{w}$. In particular, the parameter describing dispersion in $\overline{w w'^2}$ is $\gamma_f - 2\beta_f$ (see §6.2), and has a behaviour that is similar to $\gamma_m - 1$ in the vicinity of the front in LH_j. We also note that the significant increase in γ_m in HL_j upstream of the front (see bottom of figure 6.12(a)) is coincident with the trough in M_m that develops slightly ahead of the secondary, slower front (see also figure 6.7), whose dynamics are beyond the scope of the present investigation.

6.6 Conclusions

This chapter has focused on observations of steady and unsteady jets. We developed an unsteady momentum–energy framework that generalises the approach employed by Priestley & Ball (1955) and Kaminski *et al.* (2005), in the absence of buoyancy. In the study of an unsteady plume presented in chapter 8, buoyancy effects are included. The framework described in §6.2.2 is useful because it allows one to understand the behaviour of unsteady jets from an integral perspective. Notably,

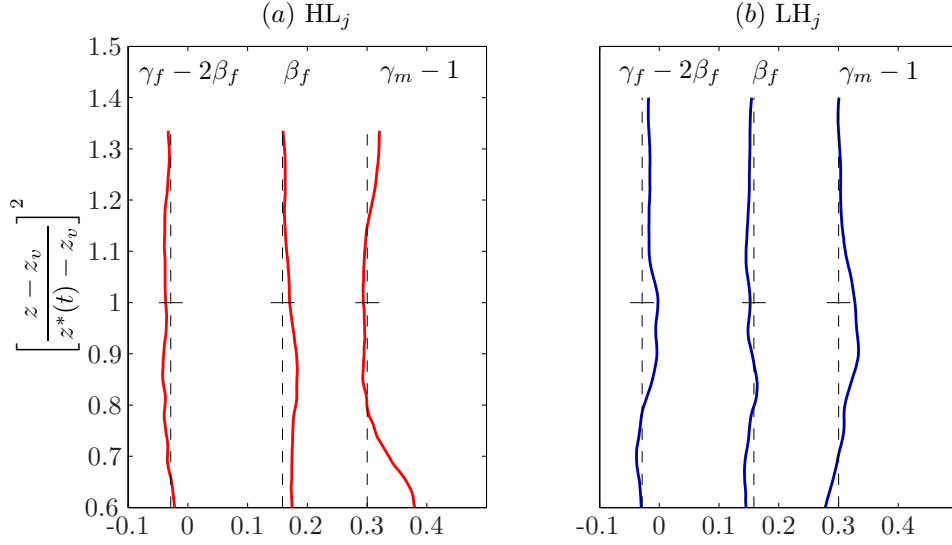


FIGURE 6.12. Moving average of energy flux parameters, at the level of the front. Dashed lines correspond to the average parameter values from the steady data, reported in §6.4.1.

making no assumptions about the internal features of the flow such as turbulence and the velocity profile, we did not *model* the integral behaviour of the jet. Instead, we used the framework in a diagnostic capacity. For example, in §6.5 we deduced that for constant profile coefficients, the governing integral equations are generally of a hyperbolic type. Moreover, the characteristic curves of the hyperbolic system are determined by dimensionless fluxes of momentum and energy.

From the DNS results, we observed that the propagation speed of a front in an unsteady jet is approximately equal to $\lambda^* w_m^*$, where $\lambda^* \approx 2$ and $w_m^* \stackrel{\text{def}}{=} M_m^*/2\alpha_0 z$ is a top-hat velocity that characterises the motion at the front (see figure 6.9). Following relatively small changes in M_m at the source, $M_m^* \approx M_m$, and the front's propagation speed will be $2w_m$, which, for Gaussian profiles, is equal to the velocity on the axis of the jet. Examination of the governing integral equations confirms that for Gaussian profiles the fastest characteristic indeed propagates at $2w_m$.

We observed that the behaviour of the front accords with a self-similar process (see figure 6.10) in z and t , whose position and longitudinal extent scales according to $z^* \sim \sqrt{t}$. This allowed us to determine the longitudinal spreading rate of the front using a similarity transformation (see §6.5.2) and, consequently, a mixing coefficient D_e . By comparing D_e to the typical magnitude of the eddy viscosity it was demonstrated that the mixing is primarily due to dispersion. Here, in analogy with the scalar dispersion studied in the previous chapter, we find dispersion of the

mean longitudinal kinetic energy. Indeed, just as the mean scalar flux F_m was found to depend on the shape of the concentration profile relative to the mean velocity, the mean energy flux E_m depends on the shape of the mean squared velocity profile relative to the mean velocity.

The notion of energy dispersion unifies several distinct concepts that are prevalent in the behaviour of unsteady jets. In particular, we identify type I dispersion with non-uniform profiles of velocity that remain self-similar with height. Under such conditions the profile of kinetic energy is narrower than the profile of velocity, which means that the dimensionless energy flux is greater than the dimensionless momentum flux. Consequently, the resulting integral equations have a hyperbolic character. In chapter 7, we will show that type I dispersion determines the behaviour of the area of the jet in the vicinity of a front and the rate at which perturbations at the source grow or decay in the longitudinal direction. Type II dispersion, in contrast, concerns departures from self-similarity that modify the local fluxes, and was shown to be chiefly responsible for the longitudinal mixing in the unsteady jets. In addition, type II dispersion provides the only way in which the area of the jet can be mixed in the longitudinal direction. Both type I and type II dispersion reside in the mean energy equation. In the following chapter we will produce a simple closure for unsteady jets based on Taylor dispersion and discuss the properties of unsteady jet models in general.

Chapter 7

Unsteady turbulent jets: model development and analysis

In this chapter we develop an integral model for the unsteady turbulent jet that was analysed in chapter 6. The model accounts for the difference in the rate at which momentum and energy are advected (type I dispersion) and for the local deformation of velocity profiles that occurs in the vicinity of a sudden change in the momentum flux (type II dispersion). To account for the latter, we use the dispersion model that was introduced in chapter 5. Using the framework that was developed in chapter 6, we show that unsteady jets with Gaussian velocity profiles remain approximately straight-sided when their source area is fixed. Straight-sidedness provides an algebraic means of reducing the order of the governing equations and leads to a simple advection-dispersion relation. In addition to ensuring straight-sidedness, we show that the Gaussian velocity profile has the special feature of being insensitive to source perturbations. Profiles that are more peaked than the Gaussian profile attenuate perturbations and, following an increase (decrease) in the source momentum flux, lead to a local decrease (increase) in the area of the jet. Conversely, profiles that are flatter than the Gaussian amplify perturbations and lead to a local increase (decrease) in the area of the jet. Most of the work reported in this chapter has been published in Craske & van Reeuwijk (2015*c*) and is reproduced here, in an adapted form, with permission.

7.1 Introduction

The objective of this chapter is to develop a robust model describing the evolution of the integral properties of an unsteady turbulent jet, making use of the observations and theory of chapter 6. Whilst there are several unsteady jet models based on extensions of classical plume theory (e.g. Delichatsios, 1979; Yu, 1990; Scase *et al.*, 2006*b*, which will be discussed in detail in §7.2.2), it has recently been shown, for the more general case of plumes, that most of these are ill-posed (Scase & Hewitt, 2012). Specifically, the models are ill-posed because they do not account for longitudinal interaction in the jet and therefore admit the development of unbounded short-wave modes. One exception to this case is the model developed by Scase & Hewitt (2012), who introduced a longitudinal mixing term based on an eddy diffusivity closure, which renders the system of equations well-posed.

In chapters 5 and 6 dispersion was identified as a core process governing the

dynamics of unsteady jets. We distinguished as dispersive all transport processes resulting from non-uniform velocity profiles rather than uniform top-hat velocity profiles. In the case of energy transport, realistic velocity profiles were found to give rise to two distinct types of dispersion:

1. Type I dispersion results from the fact that laterally non-uniform, albeit self-similar, profiles of longitudinal velocity cause energy to be advected faster than momentum.
2. Type II dispersion results from a departure from self-similarity. In particular, laterally non-uniform velocity profiles acting over longitudinal gradients can cause a departure from self-similarity.

It is notable that Scase & Hewitt (2012) incorporate longitudinal mixing into an unsteady jet/plume model using Prandtl's mixing length theory (see also Landel *et al.*, 2012, as discussed in chapter 5). Moreover, Scase & Hewitt (2012) show that the inclusion of turbulent mixing eliminates the growth of short-wave modes that render the original model of Scase *et al.* (2006*b*) ill-posed. However, for jets, the observations we reported in chapter 5 and 6 suggest that type II dispersion rather than turbulence transport *per se* provides the dominant source of longitudinal mixing.

Both type I and type II dispersion account for the longitudinal interaction of lateral slices of a jet, which is a feature that is absent from top-hat models of jets and plumes (Scase *et al.*, 2009; Scase & Hewitt, 2012). However, it is type I, rather than type II dispersion that represents the leading-order effect of laterally non-uniform velocity profiles. Consequently, type I dispersion has a fundamental influence on the way in which quantities such as momentum, energy and area are distributed over the longitudinal dimension of an unsteady jet. More precisely, type I dispersion determines the relative rates with which momentum and energy are advected, the separation of characteristic curves and the extent to which the area of an unsteady jet departs from its steady-state behaviour.

Here we capitalise on the momentum-energy framework presented in chapter 6 to develop a robust and accurate integral model that describes the dominant transport processes of unsteady jets. A notable difference between this approach and that of previous unsteady plume models is that the use of the momentum-energy framework ensures consistency between the governing equations for area, momentum and the energy of the mean flow, regardless of the profile of velocity that is assumed. In order to obtain a closed system of equations that can be used in a

prognostic capacity, a dispersion closure is developed that captures the propagation (type I dispersion) and the spreading rate (type II dispersion) of disturbances in the jet.

The work is organised as follows. Following a summary of the momentum-energy framework in §7.2.1, in §7.2.2 we review several existing unsteady plume models and show that, from an energetics perspective, none of the ill-posed unsteady plume models employ a realistic assumption regarding the radial profile of longitudinal velocity. We show in §7.2.3 that in the absence of buoyancy the mixing term suggested by Scase & Hewitt (2012) does not conserve momentum, because it introduces non-physical source terms in both the momentum and mean energy equation. In §7.3 we develop a dispersion closure for jets based on the observations reported in chapter 6 and the model developed for scalar dispersion in chapter 5. We compare predictions obtained using our model to both direct numerical simulation (DNS) results and the regularised unsteady plume model of Scase & Hewitt (2012) in §7.4. In §7.5 we show theoretically that under certain conditions Gaussian jets remain approximately straight-sided and use this result to propose several possible simplifications of the model developed in §7.3. Finally, in §7.6 we analyse the governing equations in the absence of mixing to demonstrate the effect that type I dispersion has on the growth of perturbations and the behaviour of a jet's velocity and radius in the vicinity of a step change in the momentum flux.

7.2 Unsteady jet models

7.2.1 Exact formulation

We consider the equations governing the motion of fluid induced by a source of volume and specific momentum flux (Q_s, M_s) located at $z = 0$. As discussed in detail in chapters 2 and 6, we focus on the evolution of integral quantities of the flow, whose governing equations are obtained by integrating pointwise three-dimensional conservation equations over a horizontal disk centred on the axis of the jet. The radius of the disk is $r_d(z, t)$, which is defined according to $\bar{w}(r_d, z, t) = \epsilon \bar{w}(0, z, t)$ (see §4.3.2 for details), where $\epsilon \ll 1$ and $\bar{w}$ is the ensemble-average longitudinal velocity. Hence r_d encompasses most of the longitudinal motion comprising the jet and we neglect longitudinal fluxes of momentum and energy entering the jet from the ambient. Following chapter 6, integration over the horizontal disk of pointwise conservation equations for volume, momentum and the energy associated with the mean flow results in the following system:

$$\frac{1}{\gamma_g} \frac{\partial A_m}{\partial t} + \frac{\partial Q_m}{\partial z} = 2\alpha M_m^{1/2}, \quad (7.1)$$

$$\frac{\partial Q_m}{\partial t} + \frac{\partial(\beta_g M_m)}{\partial z} = 0, \quad (7.2)$$

$$\frac{\partial M_m}{\partial t} + \frac{\partial}{\partial z} \left(\gamma_g \frac{M_m^2}{Q_m} \right) = \delta_g \frac{M_m^{5/2}}{Q_m^2}. \quad (7.3)$$

As noted in §6.2.2, given the form of (7.2)-(7.3), it will occasionally prove useful to regard Q_m and M_m not as fluxes, but as the integrals of (specific) momentum and (specific) energy of the mean flow at a given height. The parameter α is the classical entrainment coefficient (see e.g. Morton *et al.*, 1956; Turner, 1986, and §6.2.4). As in chapter 6, we define length and velocity scales for the flow according to $r_m \stackrel{\text{def}}{=} Q_m/M_m^{1/2}$ and $w_m \stackrel{\text{def}}{=} M_m/Q_m$, respectively. Equations (7.1)-(7.3) comprise an extension to unsteady problems of the integral equations posed by Priestley & Ball (1955) and later by Kaminski *et al.* (2005), though in the absence of buoyancy. In particular, the only assumptions made in our derivation are that the Reynolds number $\text{Re} \gg 1$ and that longitudinal fluxes entering the jet from the ambient at $r = r_d$ are negligible. In (7.1)-(7.3), we use Greek letters to denote dimensionless parameters, corresponding to unknown fluxes and turbulence production terms. For the definitions of these terms the reader is referred to chapter 6.

Although the momentum-energy approach of Priestley & Ball (1955) predates the mass-momentum formulation of Morton *et al.* (1956), the latter emerged as the standard model for plumes (see e.g. Hunt & van den Bremer, 2011). Only recently, when Kaminski *et al.* (2005) showed that a momentum-energy formulation allows one to establish an expression for the entrainment coefficient in terms of turbulence production and several profile constants has there been a revival of interest in the momentum-energy approach. In the intervening years, Fox (1970) also recognised that consideration of the mean flow energetics allows one to understand properties of turbulent entrainment that are not accessible at the level of volume or mass conservation alone. In the analysis of unsteady jets, and by implication unsteady plumes, employing a mean-flow energy equation (7.3) brings further advantages that will be discussed in this and the following chapter.

As discussed in chapter 6, pertaining to both type I and type II dispersion, the dimensionless energy flux

$$\gamma_m \stackrel{\text{def}}{=} 2 \frac{Q_m}{M_m^2} \int_0^{r_d(z,t)} \bar{w}^3 r dr, \quad (7.4)$$

has a dominant role in both steady and unsteady jets. Notable is the fact that when $\bar{w}$ is assumed to have a top-hat form, $\gamma_m = 1$, yet when $\bar{w}$ is assumed to have a Gaussian form $\gamma_m = 4/3$. The difference of $1/3$ between these values greatly exceeds the contributions β_f, β_p and γ_f , due to turbulent transport (see table 6.2). Perhaps more importantly, γ_m provides a means of accounting for both type I and type II dispersion in terms of the mean flow: type I dispersion is determined by the steady-state value of γ_m , whilst type II dispersion is determined by the response of γ_m to local changes in Q_m , M_m and/or their derivatives. Therefore, to develop a simple model for unsteady jets that accounts for dispersion, we focus on the role of γ_m and set $\gamma_g = \gamma_m, \beta_g = 1, \delta_g = \delta_m$ in (7.1)-(7.3):

$$\frac{1}{\gamma_m} \frac{\partial}{\partial t} \left(\frac{Q_m^2}{M_m} \right) + \frac{\partial Q_m}{\partial z} = 2\alpha M_m^{1/2}, \quad (7.5)$$

$$\frac{\partial Q_m}{\partial t} + \frac{\partial M_m}{\partial z} = 0, \quad (7.6)$$

$$\frac{\partial M_m}{\partial t} + \frac{\partial}{\partial z} \left(\gamma_m \frac{M_m^2}{Q_m} \right) = \delta_m \frac{M_m^{5/2}}{Q_m^2}. \quad (7.7)$$

Equations (7.5)-(7.7) focus on the transport of momentum and energy by the mean flow rather than turbulence. The implication of (7.6)-(7.7) is that there is no longitudinal mixing of momentum Q_m . In addition, the propagation speed of disturbances in the jet depends only on M_m, Q_m and the local value of γ_m , which in turn is determined by an assumed mean velocity profile. In reality, the presence of turbulent transport modifies the propagation speed and, if there is a local change in the relative intensity of turbulence β_f , can result in a discernible redistribution of momentum. In spite of these simplifications we will see that turbulence continues to play a notable role via δ_m in the final term of (7.7), which represents the conversion of energy in the mean flow to turbulence kinetic energy.

The system (7.5)-(7.7) appears to be over-determined, because in the original three-dimensional equations the mean-energy equation is obtained from a combination of the momentum equation and the continuity equation. Thus, in (7.5)-(7.7) there are two independent equations and a third, whose consistency with the others depends on the closure that is employed for α . For the expression for α that ensures consistency between (7.5)-(7.7) and an accompanying discussion, the reader

is referred to §6.2.4.

7.2.2 Existing unsteady jet models

In this section we compare the formulation (7.5)-(7.7), which makes no assumption regarding the radial profile of longitudinal velocity, to the unsteady plume models of Delichatsios (1979), Yu (1990) and Scase *et al.* (2006b). For the purposes of comparison, we will assume that γ_m is constant, which is equivalent to assuming that the velocity profiles remain self-similar and that type II dispersion is equal to zero. By adopting the traditional perspective of area and momentum conservation, in the absence of buoyancy the models are captured by

$$\frac{1}{\gamma_m} \frac{\partial}{\partial t} \frac{Q_m^2}{M_m} + \frac{\partial Q_m}{\partial z} = 2\alpha_0 M_m^{1/2} + c \frac{2Q_m}{M_m} \left(1 - \frac{1}{\gamma_m}\right) \frac{\partial M_m}{\partial z}, \quad (7.8)$$

$$\frac{\partial Q_m}{\partial t} + \frac{\partial M_m}{\partial z} = 0. \quad (7.9)$$

Here, $\alpha_0 \stackrel{\text{def}}{=} -\delta_m/(2\gamma_m)$ is the steady-state entrainment coefficient and we have introduced c to account for an inconsistency between a model's equations and the exact area equation (7.5). Thus, the values $c = 0$ or $c = 1$ indicate whether the model is inconsistent or consistent, respectively, when considering mean energy and momentum conservation.

The models of Delichatsios (1979) and Yu (1990) are given by $(\gamma_m, c) = (2, 0)$ and $(\gamma_m, c) = (1, 1)$, respectively. Both claim to assume Gaussian profiles for the longitudinal velocity. However, the comparison shows that in neither of these models does one find the exact theoretical value $\gamma_m = 4/3$, which can be obtained by assuming a Gaussian distribution in (7.4). Instead, in Delichatsios (1979), whose equations are ostensibly based on the Lagrangian formulation employed by Middleton (1975) but are not derived explicitly, one finds $\gamma_m = 2$. A testament to the ambiguity and consequent difficulty surrounding the unsteady area equation is that Yu (1990), also assuming a Gaussian profile, finds $\gamma_m = 1$. However, as pointed out by Scase & Hewitt (2012), the derivation of Yu (1990) is questionable in light of the divergent integral appearing in their mass conservation equation. With $(\gamma_m, c) = (1, 1)$, the model of Yu (1990) is consistent with (7.8); the model of Delichatsios (1979), on the other hand, is inconsistent in this respect. In conclusion, it appears that neither of these models can be referred to as genuine Gaussian models. In particular, the model of Yu (1990) has the characteristics of a top-hat model, which explains why Scase & Hewitt (2012) identify it as having a similar

TABLE 7.1. Comparison of unsteady plume models applied to a jet. For further details see equations (7.8)-(7.9).

	γ_m	c
Delichatsios (1979)	2	0
Yu (1990)	1	1
Scase <i>et al.</i> (2006 <i>b</i>)	1	1
Gaussian	4/3	1
Top-hat	1	1

dynamic response to the model of Scase *et al.* (2006*b*). Indeed, the model of Scase *et al.* (2006*b*), which is derived explicitly using top-hat profiles, is fully consistent with (7.8)-(7.9). Noteworthy, however, is the fact that whilst the equations of Scase *et al.* (2006*b*) are only valid for uniform (top-hat) profiles, the system (7.5)-(7.7) is valid for arbitrary velocity profiles.

Not included in table 7.1, which summarises the comparison of the unsteady plume models, is the model of Vul'fson & Borodin (2001), which is unique in stipulating that the plume remains conical, or straight-sided, independently of time-dependent changes that occur at its source; although Scase & Hewitt (2012) remark that the appropriateness of such an assumption is an open question. In this regard we have discovered that Gaussian jets, even if their momentum flux is subjected to a sudden change, have the remarkable property of remaining straight-sided, provided that their source area is fixed. This feature of the jet equations will be discussed in greater detail in §7.5, and, more generally, in §7.6 we illustrate the effect that any underlying velocity profile has on both the local area and the downstream growth-rate of perturbations. Therefore for jets, compared to the models listed in table 7.1, the model of Vul'fson & Borodin (2001) provides the only system that is fully consistent with the assumption of a Gaussian profile.

With the exception of the refinement suggested by Scase & Hewitt (2012), which will be discussed in the following section, none of the models above accounts for the possibility of longitudinal mixing, i.e. for type II dispersion or turbulent transport. However, clear from the results presented in chapter 6 is the fact that the longitudinal spreading of disturbances in unsteady jets is significant and should therefore be included.

7.2.3 The regularisation of ill-posed models

In Scase & Hewitt (2012) it was pointed out that the unsteady plume models of Delichatsios (1979), Yu (1990) and Scase *et al.* (2006*b*) are ill-posed in their admission of the unbounded growth of short-wave modes. It is not a coincidence, given the discussion of the previous section, that the only model identified as being well-posed was that of Vul'fson & Borodin (2001). To render the models well-posed it was recommended (Scase & Hewitt, 2012) that a diffusion term be introduced in the momentum equation. In this section we identify several problems with the diffusion term that was proposed, and suggest that a more appropriate closure involves the mixing of energy in the mean flow via shear-flow dispersion.

In the absence of buoyancy, the area and momentum equations proposed in Scase & Hewitt (2012) are

$$\frac{\partial}{\partial t} \left(\frac{Q_m^2}{M_m} \right) + \frac{\partial Q_m}{\partial z} = 2\alpha_0 M_m^{1/2}, \quad (7.10)$$

$$\frac{\partial Q_m}{\partial t} + \frac{\partial M_m}{\partial z} = \frac{\beta_1}{2\alpha_0} \frac{Q_m^2}{M_m} \frac{\partial}{\partial z} \left[M_m^{1/2} \frac{\partial}{\partial z} \left(\frac{M_m}{Q_m} \right) \right], \quad (7.11)$$

where β_1 is a dimensionless constant. Unlike the original system of equations, the system (7.10)-(7.11) does not admit steady-state solutions of the form $Q_m = 2\alpha_0 M_m^{1/2} z$, i.e. solutions in which the jet radius $r_m \stackrel{\text{def}}{=} Q_m/M_m^{1/2} = 2\alpha_0 z$. This behaviour is in contrast to that of plumes, for which the model was derived, in which the classical steady-state radius $r_m = 6/5\alpha_0 z$ is unaffected by the additional mixing term. To understand the cause of the problem, it is useful to rearrange (7.11) as

$$\frac{\partial Q_m}{\partial t} + \frac{\partial}{\partial z} \left[M_m - \frac{\beta_1}{2\alpha_0} \frac{Q_m^2}{M_m^{1/2}} \frac{\partial}{\partial z} \left(\frac{M_m}{Q_m} \right) \right] = -\frac{\beta_1}{2\alpha_0} M_m^{1/2} \frac{\partial}{\partial z} \left(\frac{Q_m^2}{M_m} \right) \frac{\partial}{\partial z} \left(\frac{M_m}{Q_m} \right). \quad (7.12)$$

The closure can now be understood as a combination of a modified flux of momentum on the left-hand side and a source term on the right-hand side. Thus, due to the source term, a steady jet does not conserve momentum using the closure proposed by Scase & Hewitt (2012). In fact, a steady-state power-law solution to the system (7.10)-(7.11) can be found:

$$M_m = M_s \left(1 - \frac{z}{z_v}\right)^\Phi, \quad (7.13)$$

$$Q_m = -\frac{2\alpha_0 M_s^{1/2} z_v}{\Phi/2 + 1} \left(1 - \frac{z}{z_v}\right)^{\Phi/2+1}, \quad (7.14)$$

where z_v is the location of a virtual source of zero momentum flux and zero volume flux, $z_v \stackrel{\text{def}}{=} -(\Phi/2 + 1)Q_s/(2\alpha_0 M_s^{1/2})$, and M_s and Q_s are positive quantities denoting the mean momentum flux and volume flux at $z = 0$, respectively. The exponent in (7.13) and (7.14) is found to be

$$\Phi \stackrel{\text{def}}{=} \frac{2\beta_1 - \sqrt{8\beta_1 + 1} + 1}{\beta_1 - 1}. \quad (7.15)$$

In contrast to plumes, in jets the diffusion term has the undesirable effect of modifying the classical power-law dependences, because jets do not contain a forcing term on the right-hand side of their momentum equation. This illustrates the fact that modification of the plume equations is best tackled at the simpler level of the jet equations, which place a greater restriction on the possible forms that can be introduced.

Of further interest is the fact that (7.10) and (7.11) imply a mean energy balance

$$\frac{\partial M_m}{\partial t} + \frac{\partial}{\partial z} \left[\frac{M_m^2}{Q_m} - \frac{\beta_1}{2\alpha_0} \frac{Q_m^2}{M_m^{1/2}} \frac{\partial}{\partial z} \left(\frac{M_m^2}{Q_m^2} \right) \right] = -2\alpha_0 \frac{M_m^{5/2}}{Q_m^2} - \frac{\beta_1}{\alpha_0} M_m^{1/2} \left(\frac{\partial Q_m}{\partial z} \right) \frac{\partial}{\partial z} \left(\frac{M_m}{Q_m} \right), \quad (7.16)$$

which illustrates that a modification of the momentum and/or area equation manifests itself in the mean energy equation. We argue that it is more beneficial to account for longitudinal mixing in the mean energy equation directly, in a manner that is physically realistic.

7.3 A dispersion closure for turbulent jets

In this section we develop a dispersion closure for unsteady jets by reconsidering the transport of a passive scalar quantity studied in chapter 5. In particular, recall the flux F_m of the scalar quantity $\bar{c} = c_m(z)g(\eta, z)$ in a known steady velocity field $\bar{w} = w_m(z)f_0(\eta)$, where $\eta = r/r_m$. We will assume that the upper limit of all radial integrals $r_d \rightarrow \infty$. In §7.3.1 we consider type I dispersion by examining a self-similar steady state. In §7.3.2 we consider departures from this steady state

and apply the approach taken in §5.5 to obtain an expression for type II dispersion.

7.3.1 Type I dispersion

One of the main advantages of employing the mean energy equation is that it contains nonlinear terms whose value depends on the correlation of mean flow properties over the radius. To appreciate the leading-order effect that this correlation has on the flux F_m , we will consider a self-similar steady state in which the function $g = g_0(\eta)$ and $f = f_0(\eta)$. Indeed, f_0 and g_0 can be defined such that their integrals are equal to unity:

$$2 \int_0^\infty f_0(\eta) \eta d\eta = 2 \int_0^\infty g_0(\eta) \eta d\eta = 1. \quad (7.17)$$

When f_0 and g_0 differ from uniform distributions of the same width, their correlation might prevent us from equating the top-hat product $w_m c_m r_m^2$ with F_m :

$$F_m \stackrel{\text{def}}{=} 2w_m c_m r_m^2 \int_0^\infty f_0(\eta) g_0(\eta) \eta d\eta \neq w_m c_m r_m^2, \quad (7.18)$$

in general. Thus, spatial averaging introduces additional unknown correlations in the same way that they are introduced at a fixed location by a temporal or ensemble average. If the scalar profile g_0 is narrower than a non-uniform monotonically decreasing distribution f_0 , then $F_m > w_m c_m r_m^2$, due to the concentration of the scalar in regions where $\bar{w}$ is relatively large. To see this, assume that $f_0(\eta) = 2 \exp(-2\eta^2)$ and

$$g_0(\eta) = \frac{2}{\mu^2} \exp\left(-2\frac{\eta^2}{\mu^2}\right). \quad (7.19)$$

Here μr_m is the lateral length scale associated with the radial dependence of $\bar{c}$. Using (7.19), the scalar flux can be determined exactly:

$$F_m \stackrel{\text{def}}{=} 2w_m c_m r_m^2 \int_0^\infty \frac{4}{\mu^2} \exp\left[-2\eta^2 \left(1 + \frac{1}{\mu^2}\right)\right] \eta d\eta = \theta_0 w_m c_m r_m^2, \quad (7.20)$$

where, for the particular case of Gaussian profiles f_0 and g_0 ,

$$\theta_0 = \frac{2}{1 + \mu^2}. \quad (7.21)$$

The parameter θ_0 appeared in the unsteady plume model of Delichatsios (1979) to account for the difference in lateral spread between radial profiles of buoyancy and momentum. In chapter 6 we identified those profile effects that are associated with the steady state as giving rise to type I dispersion, and showed that they were responsible for a difference in the rates at which energy and momentum are transported in a jet. A Gaussian approximation to the steady energy distribution in a jet shows that $f_0^2(\eta) = 4 \exp(-4\eta^2)$. Hence $\mu^2 = 1/2$ and θ_0 , which in the case of energy we denote as γ_0 , is equal to $4/3$.

7.3.2 Type II dispersion

A difficulty associated with type II dispersion, unlike type I, is that it cannot be investigated without perturbing a jet from a steady self-similar state. By definition, the mixing brought about by type II dispersion results from a departure from self-similarity. As chapter 5 pointed out, the classical dispersion theory of Taylor (1953), originally based on pipe flow, quantifies the mixing that is caused by departures from self-similarity. Fundamental to both dispersion in pipe flow and dispersion in jets is the idea that changes in the mean scalar concentration in the longitudinal direction cause the local departure from self-similarity over the lateral dimension. Here, we follow the heuristic approach developed in chapter 5 and recast the classical dispersion balance for pipes in a form that can be applied to jets.

Recall from chapter 5 that the leading-order model for the dimensionless scalar flux θ_m was

$$\theta_m = \theta_0 + L_1 \theta_1, \quad (7.22)$$

where

$$L_1 = -\frac{3r_m}{\alpha_0} \frac{1}{\mathcal{C}_m} \frac{\partial \mathcal{C}_m}{\partial z}, \quad (7.23)$$

and $\mathcal{C}_m$ is the characteristic concentration c_m normalised by the steady-state concentration $c_{m0} \sim 1/z$, such that $\mathcal{C}_m = c_m/c_{m0}$. The dimensionless flux θ_m consists of the sum of a steady component θ_0 and an unsteady component $L_1 \theta_1$. Evident from equation (7.23) is that the contribution from the latter depends on the extent to which longitudinal gradients $\partial_z c_m$ deviate from their steady-state value. The constant geometric parameter θ_1 , which depends on f_0 and g_1 , determines the sensitivity of the dimensionless flux θ_m to the departure from similarity that is quantified by L_1 .

We assume that, like the scalar flux θ_m , the dimensionless energy flux γ_m in a jet can be decomposed into a steady component γ_0 and an unsteady component $L_1\gamma_1$:

$$\gamma_m = \gamma_0 + L_1\gamma_1, \quad (7.24)$$

where, like θ_1 , the constant geometric parameter γ_1 can in principle be determined from the steady self-similar velocity profile f_0 and the deformation profile f_1 , which correspond to g_0 and g_1 in §5.5.

To determine L_1 we focus on the ratio w_m/w_{m0} in place of $\mathcal{C}_m \stackrel{\text{def}}{=} c_m/c_{m0}$ in (7.23). In addition, we define the local steady-state mean velocity w_{m0} such that $\partial_z w_{m0} = -2\alpha_0 w_{m0}/r_m$ and (7.23) becomes

$$L_1 = -\frac{3r_m}{\alpha_0} \left(\frac{2\alpha_0}{r_m} + \frac{1}{w_m} \frac{\partial w_m}{\partial z} \right). \quad (7.25)$$

In terms of integral quantities, the dimensionless energy flux (7.24) can therefore be expressed as

$$\gamma_m = \gamma_0 - 6 \left[1 + \frac{Q_m^2}{2\alpha_0 M_m^{3/2}} \frac{\partial}{\partial z} \left(\frac{M_m}{Q_m} \right) \right] \gamma_1. \quad (7.26)$$

The parameters γ_0 and γ_1 require calibration before the model can be used. Although it is possible to obtain values of γ_0 and γ_1 based on the geometry of the steady-state velocity profile alone, a more convenient and robust approach is to derive their values directly from the observations reported in chapter 6. In the steady state we found that $\gamma_m = 1.3$. For the unsteady jets, we estimated a dimensionless dispersion coefficient, $D_e/M_m^{*1/2} = 0.22$, by comparing self-similar profiles of $M_m(z, t)$ to a solution of an advection-dispersion equation for the mean energy. In §7.5.2 we will show that under certain assumptions $D_e/M_m^{*1/2} = 3\gamma_1/(2\alpha_0)$, which, when $\alpha_0 = 0.07$, provides a reasonable first estimation $\gamma_1 = 0.01$. For further detail regarding the relationship between γ_1 and D_e the reader is referred to §7.5.2. In addition, we note that γ_0 is equal to the steady-state value of γ_m and therefore set

$$\gamma_0 = 1.3, \quad \gamma_1 = 0.01. \quad (7.27a, b)$$

7.4 Model prediction

Substitution of the dispersion closure (7.26) into (7.6)-(7.7) results in

$$\frac{\partial Q_m}{\partial t} + \frac{\partial M_m}{\partial z} = 0, \quad (7.28)$$

$$\frac{\partial M_m}{\partial t} + \gamma_0 \frac{\partial}{\partial z} \left(\frac{M_m^2}{Q_m} \right) = -2\alpha_0 \gamma_0 \frac{M_m^{5/2}}{Q_m^2} + 6\gamma_1 \frac{\partial}{\partial z} \left[\frac{M_m^2}{Q_m} + \frac{Q_m^2}{4\alpha_0 M_m^{1/2}} \frac{\partial}{\partial z} \left(\frac{M_m^2}{Q_m^2} \right) \right], \quad (7.29)$$

where we have used the steady-state relationship $\delta_m = -2\alpha_0\gamma_0$. Notably, (7.28)-(7.29) admit well-established power-law solutions describing a steady state. The addition of the final term in (7.29) affects neither the exponent of the steady-state solutions nor the predicted spreading rate of the jet. The system (7.28) and (7.29) can be readily solved numerically using a central differencing scheme.

In this section we compare the predictions of our model (7.28)-(7.29) and the modified model of Scase & Hewitt (2012), described by equations (7.10)-(7.11), which will hereafter be referred to as the Gaussian jet model (GJM) and the top-hat jet model (TJM), respectively, to the DNS results discussed in chapter 6. To solve the system (7.28) and (7.29) for GJM we discretise spatial derivatives by taking second-order central differences and advance the solution in time using a fourth-order Runge-Kutta method. At the base of the domain we prescribe unsteady source conditions $Q_m(0, t) = Q_s(t)$ and $M_m(0, t) = M_s(t)$. In particular, we impose a rapid smooth increase or decrease in the source momentum flux according to

$$M_s(t) = M_m^B + \frac{1}{2} (M_m^A - M_m^B) \left[1 + \operatorname{erf} \left(\frac{t - t_v}{t_*} \right) \right], \quad (7.30)$$

where M_m^B and M_m^A are the mean momentum fluxes before and after the sudden change, respectively, and we ensure that t_* is approximately two orders of magnitude smaller than the timescale relevant to the source. During this change we hold the source radius, $r_s = Q_s/M_s^{1/2}$, constant and therefore adjust Q_s accordingly. The time t_v is the temporal location of an asymptotic virtual source and was determined from the DNS data, as described in §6.4.2. For relatively large values of z we assume that any dependent variable X_m behaves according to the steady state:

$$\lim_{z \rightarrow \infty} \frac{\partial X_m}{\partial z} = \frac{n}{z - z_v} X_m \quad (7.31)$$

where $X_m \sim z^n$ in the steady state and z_v is the location of a virtual origin. In practice, (7.31) is approximated numerically, and imposed on all dependent variables at a value of z sufficiently large to ensure that the condition does not

interfere with the unsteady solution. The equations are solved over the time domain $[t_v - \delta t, t_\infty]$, where $\delta t = O(t^*)$, which ensures that the transition (7.30) is smooth and approximately symmetric, and t_∞ is a time that is approximately equal to the duration of the DNS to which we compare. The DNS data were obtained from an ensemble-average of individual, statistically unsteady, simulations, as described in §6.3.

To obtain numerical solutions to the adjusted top-hat model proposed by Scase & Hewitt (2012), TJM, we set $\beta_1 = 3\gamma_1$, so that the diffusive flux in (7.16) is equal to the dispersive flux in our model. For TJM a central differencing scheme proved impractical, and the flux-limiting scheme described by Kurganov & Tadmor (2000) was employed instead. The results presented in Scase & Hewitt (2012) were also obtained via the use of a flux-limiting scheme.

The numerical approximation to both GJM and TJM was performed using a grid of 2000 points over the domain $z/r_s \in [0, 89]$. In addition, we checked convergence by halving the total number of points that were used. In figures 7.1 and 7.2 the full- and half-resolution approximations are shown with lines and symbols, respectively. We note that the use of 2000 points, which makes the calculation computationally expensive, was necessitated by the difficulty of achieving convergence with TJM. The timescale τ_s for the unsteady simulations was defined according to $\tau_s \stackrel{\text{def}}{=} r_s^2 / M_m^{*1/2}$, where $M_m^{*1/2}$ is a momentum flux that characterises the motion of the front (see §6.4.2 or §7.5 in the present chapter). The DNS data shown in figures 7.1 and 7.2 were obtained from integrals of ensemble-averaged quantities.

Figure 7.1 indicates that the unsteady model developed here, GJM, provides an excellent description of both the propagation speed and the spreading rate of both types of front in M_m . In contrast, TJM significantly underpredicts and overpredicts the value of M_m upstream of the front in the step-down case (figure 7.1a) and the step-up case (figure 7.1b), respectively. In particular, the spike in M_m predicted by TJM is extremely difficult to approximate numerically and appears to be non-physical. At the level of the front TJM overpredicts and underpredicts M_m , in the step-down and the step-up case, respectively, and displays a poor agreement with the slope that is observed in the DNS data. Furthermore, the inconsistency of TJM with the steady-state solution is evidenced in the increase in M_m that occurs upstream of the front. Notable from the DNS results are the local maxima in M_m in the near-field region, $z/r_s < 20$, which GJM does not reproduce. Indeed, close to the source the jet has a top-hat profile, which slowly decays into an approximately Gaussian profile as the flow develops (see §6.4.1). Accordingly, unsteady top-hat models can reproduce certain features of the flow in the near field but do not

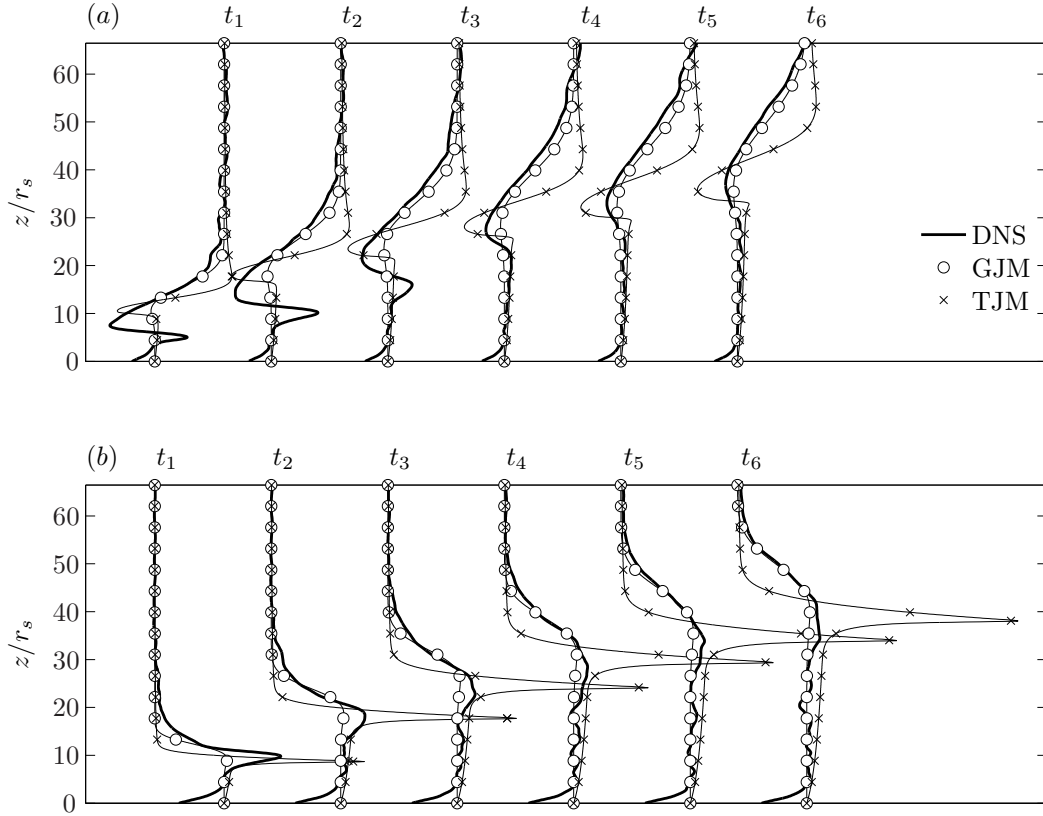


FIGURE 7.1. Mean momentum flux M_m following (a) a step-down and (b) a step-up in the momentum flux at the source. Direct numerical simulation compared with theoretical prediction at times approximately given by $t_n = 18\tau_s n$.

accurately capture its far-field behaviour.

From figure 7.1, one may be inclined to suppose that a larger value of the diffusion coefficient β_1 in TJM would give an improved agreement with the DNS data. In fact, increasing β_1 has an adverse effect on the predictive capabilities of TJM because it increases the strength of the spurious source of momentum and energy that was identified in §7.2.3. A substantial increase in β_1 results in an overestimation of the total mean flow energy in the domain and, consequently, an overestimation of the propagation speed of the front. Additionally, with a significant increase in its value, β_1 would exceed the bounds that are implied by the values of eddy viscosity on which it is based (Scase & Hewitt, 2012).

The local radius of the jet is a useful quantity to examine because, representing a balance between Q_m and $M_m^{1/2}$, its behaviour is not easy to anticipate *a priori*. The radius is also significant as the square root of the ‘conserved’ quantity in the mass conservation equation (7.10). Perhaps surprisingly, figure 7.2(b) shows that

the radius of the jet is approximately unaffected by the local increase in momentum flux, and is well-predicted by GJM. In contrast, TJM predicts a significant local increase in the radius in figure 7.2(b). In §7.6.2 we show that this property of TJM is due to the fact that it assumes a top-hat, rather than Gaussian, distribution of longitudinal velocity. In figure 7.2(a), which displays the step-down case, the DNS data reveal that the jet radius r_m increases locally behind the front, which is not captured by GJM and is in direct contradiction with the reduction in radius predicted by TJM. However, it is helpful to note that the local increase in area observed in the DNS occurs upstream of the leading edge of the front and corresponds to the local minimum in M_m in figure 7.1(a). By implication, since $r_m \stackrel{\text{def}}{=} Q_m/M_m^{1/2}$, Q_m appears to be relatively insensitive to the changes that occur in this region. Furthermore, relative to the steady-state jet radius $2\alpha_0 z$, the local increase in r_m that is evident in figure 7.2(a) diminishes in amplitude as z increases. The physics associated with the step-down case appear to be quite different to the step-up case and accurate prediction of the region upstream of the leading front, particularly in the near field, represents a challenge for a single, two-variable integral model. It should also be noted that the local increase in r_m seen in figure 7.2(a) corresponds to an increase in the threshold radius r_d at the same level.

In making an assessment of the models' ability to predict the radius of the unsteady jets it is necessary to bear in mind that $r_m \stackrel{\text{def}}{=} Q_m/M_m^{1/2}$ is not the same as the 'effective top-hat' radius employed by Scase *et al.* (2008) and Scase *et al.* (2009). The 'effective top-hat' radius, defined in Scase *et al.* (2008), is based on the total concentration of a passive tracer at a given height. Indeed, the use of a passive tracer is a valuable aid in experiments that do not permit direct access to the velocity field. However, the 'effective top-hat' radius will not, in general, coincide with r_m , which is the only length scale that can be formed from Q_m and M_m . With full access to the velocity field from the DNS data, we focus directly on a comparison of the models' prediction of Q_m and M_m , or any variable, such as r_m , that can be expressed in terms of Q_m and M_m . Consequently, we note that the behaviour of r_m shown in figure 7.2 is not inconsistent with the narrowing of an unsteady plume observed by Scase *et al.* (2008), since the latter was inferred from the 'effective top-hat' radius.

Collectively, figures 7.1 and 7.2 suggest that a top-hat description of unsteady jets does not reproduce observations satisfactorily and that the dispersion model of GJM is physically more realistic. We have also pointed out that top-hat models, such as TJM, are difficult to solve numerically, in spite of the fact that they are rendered well-posed with the inclusion of a longitudinal mixing term. The reason

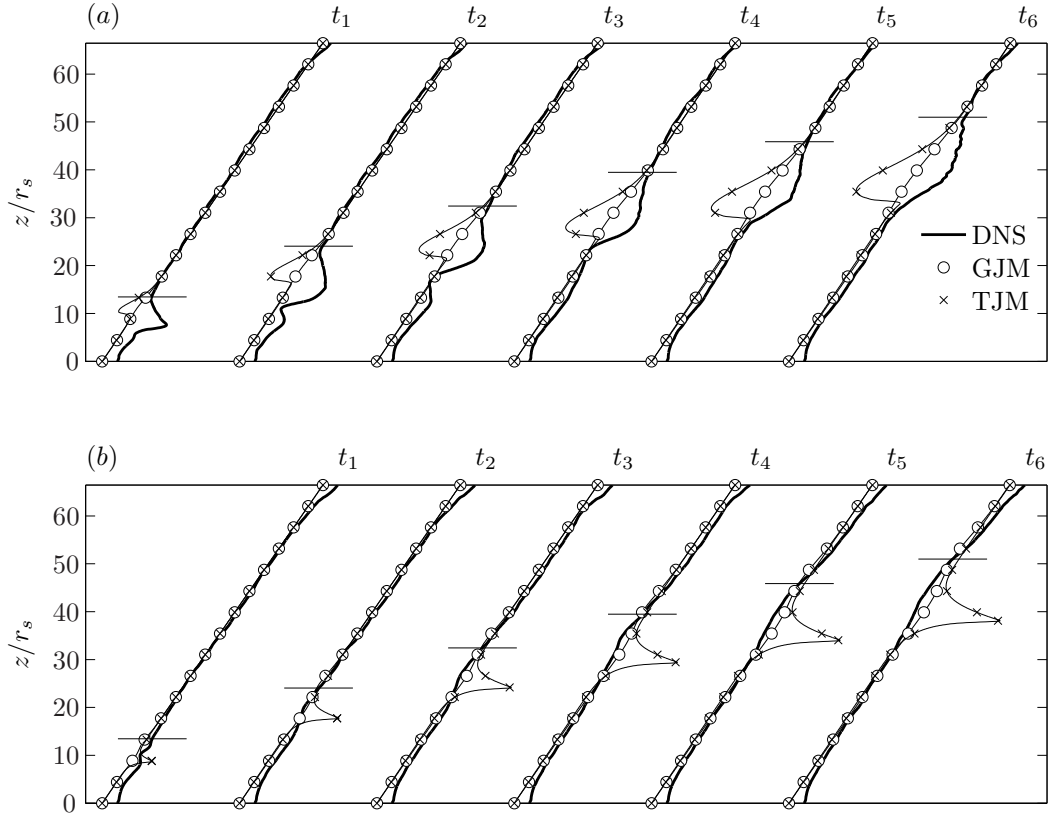


FIGURE 7.2. Jet radius $r_m \stackrel{\text{def}}{=} Q_m/M_m^{1/2}$ following (a) a step-down and (b) a step-up in the momentum flux at the source. Direct numerical simulation compared with theoretical prediction at times approximately given by $t_n = 18\tau_s n$. The location of the front is indicated with a horizontal line.

for this difficulty is that they do not contain type I dispersion. In other words, they invoke the non-physical assumption that velocity profiles are uniform. In §7.6 we explain the consequences of this issue in more detail, showing that if the velocity profiles are assumed to be non-uniform ($\gamma_m > 1$) perturbations applied at the source of the jet have an algebraic downstream growth, or even decay, in place of the exponential growth associated with top-hat profiles. Furthermore, we show that when $\gamma_m < 4/3$, the area upstream of a sudden increase in the momentum flux increases, allowing the jet to store momentum and energy locally.

7.5 A straight-sided approximation

For problems involving a step change in M_m at a source of constant area, A_s , it is natural to ask in what way the area in the vicinity of the front, where $z = z^*$, departs from its steady-state form. When $z/z^* \ll 1$ and $z/z^* \gg 1$, we expect the

area to remain unchanged, so that $A_m = Q_m^2/M_m = 4\alpha_0^2 z^2$. In their unsteady plume model, Vul'fson & Borodin (2001) go further and assume that the area is independent of time for all z , and therefore assumes its steady-state form. In this section we will show that for jets, such a 'straight-sided' assumption can be justified theoretically and has several fundamental implications for the governing system of equations. We will assume that the source area A_s remains fixed. In §7.5.1 we will neglect type II dispersion and therefore assume that γ_m is a constant equal to γ_0 . In §7.5.2 we extend the analysis to include type II dispersion and obtain a simple advection-dispersion equation that serves as a good approximation to the full model (7.28)-(7.29).

7.5.1 A straight-sided equation for momentum and energy (type I dispersion only)

The unsteady equation for area (7.5), neglecting turbulent transport and type II dispersion, so that $\beta_g = 1$ and $\gamma_g = \gamma_0$, can be expressed as

$$\frac{\partial A_m}{\partial t} = -\gamma_0 \frac{\partial Q_m}{\partial z} + 2\alpha_0 \gamma_0 M_m^{1/2} + \frac{2Q_m}{M_m} (\gamma_0 - 1) \frac{\partial M_m}{\partial z}. \quad (7.32)$$

Substitution of the straight-sided solution $Q_m = 2\alpha_0 M_m^{1/2} z$, into (7.32) reveals that

$$\frac{\partial A_m}{\partial t} = \frac{4\alpha_0 z}{M_m^{1/2}} \left(\frac{3}{4} \gamma_0 - 1 \right) \frac{\partial M_m}{\partial z}. \quad (7.33)$$

Equation (7.33) expresses the fact that when $\gamma_0 = 4/3$ the jet will remain straight-sided for all time. Further, (7.33) suggests that when $\gamma_0 < 4/3$ or $\gamma_0 > 4/3$ the area will increase or decrease, respectively, following a step increase in M_m . Indeed, the unsteady similarity solutions obtained by Scase *et al.* (2006b) can be rederived for arbitrary γ_0 (see §7.6.3) and reveal that when $\gamma_0 = 4/3$, the spreading angle of the jet is identical to the steady-state case: $r_m = 2\alpha_0 z$. In addition, the similarity solutions show that when $\gamma_0 < 4/3$ or $\gamma_0 > 4/3$ the spreading angle of the jet is less than or greater than the steady-state spreading angle, respectively. When $\gamma_0 = 1$, one recovers the similarity solution for top-hat profiles obtained by Scase *et al.* (2006b) and the behaviour $r_m = 2\alpha_0 z/3$. Contrary to the top-hat plume theory in which there is no means for faster fluid to overtake slower fluid (see e.g. Scase *et al.*, 2009), we find that specification of a realistic, albeit constant, value of γ_0 facilitates such overtaking in the form of type I dispersion.

In the light of equation (7.33), the Gaussian velocity profile can be usefully understood as providing momentum and energy fluxes whose ratio is such that the steady-state area of the jet can be preserved. To visualise this, one can consider a

control volume located at the front of a starting jet. Energy, M_m , is transported into the control volume with dimensionless flux γ_0 , and it is removed from the control volume owing to turbulence production at a dimensionless rate δ_m . If $\gamma_0 < 4/3$ then energy is added to the control volume at a reduced rate, which results in a relative surplus of momentum, Q_m , and hence the accumulation of area Q_m^2/M_m . Conversely, $\gamma_0 > 4/3$ results in the reduction or dispersion of area.

The use of the straight-sided approximation is highly desirable from a modelling perspective, as it provides an algebraic coupling between Q_m and M_m and therefore reduces the dimension of the system by one. For straight-sided jets, using the relations $Q_m = 2\alpha_0 M_m^{1/2} z$ and $\delta_m = -8\alpha_0/3$, which both follow from $\gamma_0 = 4/3$, the momentum equation and mean energy equation (7.6)-(7.7) become

$$\frac{\partial Q_m}{\partial t} + \frac{\partial}{\partial z} \left(\frac{Q_m^2}{4\alpha_0^2 z^2} \right) = 0, \quad (7.34)$$

$$\frac{\partial M_m}{\partial t} + \frac{2}{3} \frac{\partial}{\partial z} \left(\frac{M_m^{3/2}}{\alpha_0 z} \right) = -\frac{2}{3} \frac{M_m^{3/2}}{\alpha_0 z^2}. \quad (7.35)$$

Both (7.34) and (7.35) admit discontinuities at $z = z^*(t)$ that propagate according to

$$\frac{dz^*}{dt} = \frac{Q_m^*}{2\alpha_0^2 z^{*2}} = \frac{M_m^{*1/2}}{\alpha_0 z^*}. \quad (7.36)$$

The nonlinearity of (7.34) and (7.35) ensures that a positive step change ($\partial_z M_m < 0$) in M_m gives rise to a shock, whose propagation velocity must be determined from a Rankine-Hugoniot condition (see e.g. Toro, 1997). In order to determine a unique propagation velocity one must choose whether to impose momentum conservation or energy conservation over the shock. Likening the progression of a positive step change in a jet's momentum flux to the evolution of a tidal bore, we impose momentum conservation (7.34) over the discontinuity. Substituting $Q_m = 2\alpha_0 M_m^{1/2} z$ in (7.34), we find

$$M_m^{*1/2} = \frac{1}{2} \left(\frac{M_m^A - M_m^B}{M_m^{A1/2} - M_m^{B1/2}} \right). \quad (7.37)$$

Here, we use the superscripts B and A to denote the value of a variable or function before/ahead of and after/behind a front, respectively. The characteristic momentum flux (7.37) provided the motivation for the choice of scaling in figure 10 of chapter 6. It is also of interest that for a starting jet, for which $M_m^B = 0$, (7.37) is consistent with the result obtained by Ruban & Vonatsos (2008). There,

the Bernoulli equation was used along the axis of a planar jet to deduce that the propagation velocity is equal to half the maximum velocity in the jet immediately behind the shock. We note, however, that in other respects their laminar boundary layer solution is very different to the radially integrated turbulent flow considered here. In the present work, if $M_m^B = 0$ in (7.37) then $dz^*/dt = w_m^A$, where w_m^A is the top-hat velocity immediately behind the shock. Noting that the top-hat velocity w_m^A is equal to half the maximum velocity of a Gaussian profile confirms the consistency between (7.37) and the results of Ruban & Vonatsos (2008). In turn, these deductions each agree closely with the observations reported by Turner (1962) for a starting plume.

In (7.35) the inclusion of type I dispersion alone, and not the mixing provided by type II dispersion, has produced a simple first approximation of an unsteady jet. In spite of the absence of longitudinal mixing, equation (7.35) does not admit the non-physical growth of short-wave modes that were found in the predictions of several unsteady plume models (Scase & Hewitt, 2012). Indeed, we show in §7.6.1 that the evolution of short-wave modes is determined by type I dispersion, and that their growth can therefore be prevented by adopting a physically realistic assumption regarding the jet's velocity profile.

Although it is clear that the mixing in jets prevents the formation of discontinuities, like other hyperbolic systems, the value of (7.34) or (7.35) is that it provides a description of their limiting behaviour, without the further complications that are introduced when additional mixing terms are present.

7.5.2 A straight-sided equation for energy (type I and type II dispersion, approximate momentum conservation)

In equations (7.34)-(7.35) we established a description of a straight-sided unsteady jet that satisfies transport equations for both momentum and mean energy. In general, if the straight-sided relation $Q_m = 2\alpha_0 M_m^{1/2} z$ is imposed strictly then one cannot expect to satisfy equations for momentum and mean energy simultaneously unless $\gamma_m = 4/3$. Indeed, the solution for the two dependent variables of the problem is defined by two governing equations. The Gaussian profile, for which $\gamma_m = 4/3$, is the distinguished case for which the three equations (momentum, mean energy and straight-sidedness) are consistent.

As an approximate treatment it is useful to relax the momentum equation and consider a solution that satisfies both straight-sidedness and the mean energy equation exactly. The extent to which the momentum equation will be violated in this approximation depends on the value of γ_0 and γ_1 . In particular, if we let

$\gamma_0 = 4/3$, then the level of approximation in the momentum equation depends on the amount of longitudinal mixing, and therefore on γ_1 , whose value is small in comparison to $4/3$. Approximating the momentum balance in this way allows us to obtain a straight-sided version of the full unsteady jet model developed in §7.3, which reduces to a simple advection-dispersion equation. As such, it can be readily employed to aid a first estimation of a dispersion coefficient based on experimental or numerical observations.

With the straight-sided solution $Q_m = 2\alpha_0 M_m^{1/2} z$, our model for energy transport (7.29) simplifies to a nonlinear advection-dispersion equation:

$$\frac{\partial M_m}{\partial t} + \frac{3\gamma_0}{4\alpha_0} \frac{M_m^{1/2}}{z} \frac{\partial M_m}{\partial z} = \frac{3\gamma_1}{2\alpha_0} \frac{\partial}{\partial z} \left(M_m^{1/2} \frac{\partial M_m}{\partial z} \right). \quad (7.38)$$

To understand the leading-order processes described by equation (7.38) we consider dimensionless perturbations from $M_m = M_m^*$:

$$M_m = M_m^* (1 + \epsilon \mathcal{M}_1 + \epsilon^2 \mathcal{M}_2 + \dots). \quad (7.39)$$

Here, we have employed the characteristic momentum flux M_m^* , rather than M_m^B for example, because it is M_m^* that determines the propagation velocity of the front. In doing so, we find that the linearised system of equations obtained below can provide a good approximation to the fully nonlinear problem, in which the step-change forcing is relatively large. Substituting this expansion into (7.38) and retaining terms to $O(\epsilon)$ yields

$$\frac{\partial \mathcal{M}_1}{\partial t} + \frac{3\gamma_0}{4\alpha_0} \frac{M_m^{*1/2}}{z} \frac{\partial \mathcal{M}_1}{\partial z} = \frac{3\gamma_1}{2\alpha_0} M_m^{*1/2} \frac{\partial^2 \mathcal{M}_1}{\partial z^2}, \quad (7.40)$$

which is a linear advection diffusion equation for a spatially varying velocity field, and has exactly the same form as the equation that was used in §6.5.2 to estimate the mixing coefficient D_e . Indeed, comparison of (7.40) above with (6.42) shows that $D_e/M_m^{*1/2} = 3\gamma_1/(2\alpha_0)$. Assuming self-similarity of the process in terms of the variable λ :

$$\lambda \stackrel{\text{def}}{=} \frac{z^2 \alpha_0}{t M_m^{*1/2}}, \quad (7.41)$$

where the dimensionless front velocity $\lambda(z^*) = \lambda^* = 2$, equation (7.40) can be expressed as

$$\frac{d^2 \mathcal{M}_1}{d\lambda^2} = \frac{1}{2\lambda} \left[\text{Pe} \left(1 - \frac{\lambda}{\lambda^*} \right) - 1 \right] \frac{d\mathcal{M}_1}{d\lambda}, \quad (7.42)$$

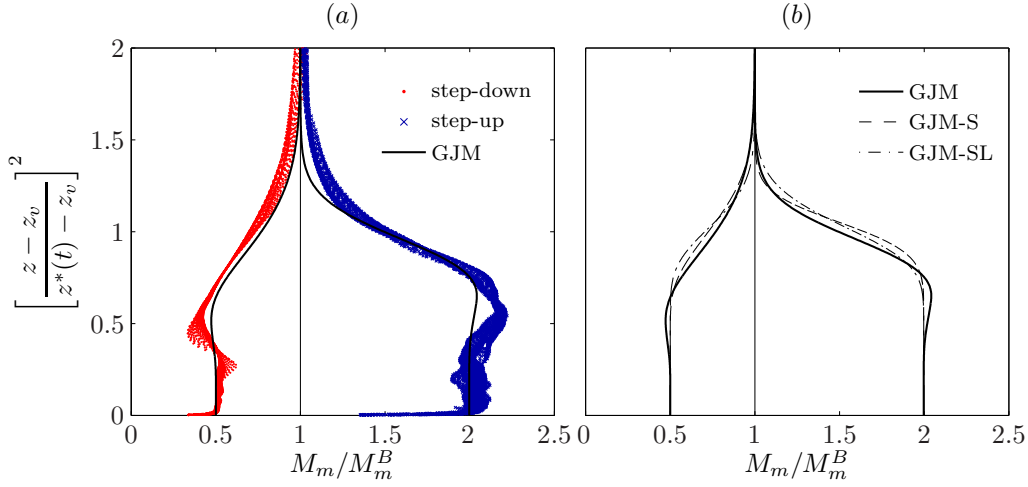


FIGURE 7.3. (a) Self-similarity of the dimensionless momentum flux M_m/M_m^B from DNS results compared to predictions using the full model GJM (7.28)–(7.29). (b) Predictions obtained using the nonlinear, straight-sided model GJM-S (7.38), and linear straight-sided model GJM-SL (7.40) are compared to GJM. The constant z_v is the value of z at an asymptotic virtual source. In each plot the left-hand side refers to the step-down case, and the right-hand side to the step-up case.

where the dispersive Péclet number,

$$\text{Pe} \stackrel{\text{def}}{=} \frac{\gamma_0}{2\gamma_1}, \quad (7.43)$$

is consistent with the definition provided in §6.5.2. Indeed, in chapter 6 the theoretical justification for using the linear advection dispersion equation was not provided, yet it was found to give a reasonable agreement with the data. As described in this section, it is the tendency of the jet to remain straight-sided that results in the surprising effectiveness of (7.40). As discussed in chapter 5, the Péclet number defined in (7.43) has a meaning that is equivalent to the ratio of advection and dispersion parameters (K_a/K_d) used in the recent study of a quasi-two-dimensional jet by Landel *et al.* (2012).

In figure 7.3 we display self-similar, far-field predictions from the general nonlinear model GJM, the straight-sided model GJM-S (7.38), and the linear straight-sided model GJM-SL (7.40), in addition to the collapsed DNS data. In GJM-S and GJM-SL we set $\gamma_0 = 4/3$, which is consistent with the value of γ_m required for straight-sidedness. We do not include predictions for TJM, as it proved infeasible to provide the number of points that were required for a far-field solution in that case. Notably, having been obtained from the constraint $r_m = 2\alpha_0 z$, both the

linear and nonlinear straight-sided solutions are monotonic, and are therefore not able to reproduce the extrema evident in both the DNS results and the full model. As one might expect, in the step-up (step-down) case the nonlinear model GJM-S predicts a steeper (less steep) front than the linear approximation GJM-SL. The linear, GJM-SL, and nonlinear, GJM-S, straight-sided predictions exhibit a good agreement, which demonstrates the effectiveness of the former when M_m^* is chosen correctly, i.e. as the solution of the idealised nonlinear shock problem. It is also evident in figure 7.3 that the full model GJM is not able to exactly reproduce the values $M_m/M_m^B > 1$ at relatively large values of z/z^* in the step-up case. An attempt to correct this by increasing γ_1 would mean that γ_1 no longer faithfully represents those features of the flow on which its definition in §7.3.2 was originally based.

7.5.3 Model limitations

As discussed in §5.6.1 in the context of passive scalar transport, the models developed in this chapter are unlikely to give accurate predictions in the near field of an unsteady jet, as evidenced for $t = 18\tau_s$ in figure 7.1. This is because the imposed step change at the source gives rise to a longitudinal length scale that is much smaller than the local radius of the flow, necessitating a more refined description of turbulent transport and dispersion. Beyond the near field, where $z > O(r_s/\alpha_0)$, the longitudinal scale of the flow becomes comparable to the local radial scale and the travelling front assumes a self-similar form in z and t , as shown in figure 7.3, that can be adequately described by the models developed in this chapter. Noting that $z^2 \sim M_s^{1/2}t/\alpha_0$ along the path of the front, the times associated with the ‘near-field’ are therefore $t < O(\tau_s/\alpha_0)$.

More generally, imposed changes in source conditions on time scales Δt less than $Q_s^2/M_s^{3/2}$ (the source turn-over time) will require a more refined description (e.g. additional terms in the perturbation expansion discussed in §5.6.1). Conversely, changes in source conditions for which $\Delta t \gg Q_s^2/M_s^{3/2}$ might be adequately described by a classical quasi-steady theory over longitudinal scales $\Delta z \ll M_s\Delta t/Q_s$.

In addition to issues associated with near-field effects, one suspects that highly nonlinear problems, such as starting jets, might present a challenge to the simplified models developed in this chapter. In particular, one would expect the dimensionless longitudinal turbulent transport terms β_f and γ_f , in addition to pressure, to play a significant role in such problems. An eddy-viscosity-based closure for β_f and γ_f that modifies neither the steady-state jet solutions nor the steady-state plume

solutions has the form

$$\beta_f \propto \frac{1}{M_m^{1/2}} \frac{\partial Q_m}{\partial z}, \quad \gamma_f = 2\beta_f, \quad (7.44a, b)$$

and can be incorporated into (7.28)-(7.29) without difficulty. Noteworthy in this regard, however, is the questionable use of an eddy-viscosity hypothesis to describe longitudinal turbulent fluxes such as $\overline{w'^2}$. On the other hand, it is interesting that the lateral flux $\overline{u'w'}$, which ultimately determines the contribution from type II dispersion, can be successfully described using an eddy viscosity hypothesis (see e.g. Pope, 2000).

7.6 The hyperbolic jet

In this section we move back to the original system (7.6) and (7.7) and investigate several properties of the hyperbolic system that one obtains by assuming that the parameter γ_m remains constant. With this assumption we therefore neglect type II dispersion and focus exclusively on type I dispersion. The justification for this restriction is that type I dispersion is of leading-order and has several interesting properties that are readily exposed in a simplified setting. In addition, the use of a constant value of γ_m is equivalent to the approach taken in previous unsteady jet and plume models. In contrast to the previous sections, whose focus was on modelling the various transport processes, the purpose of this section is to investigate the parameter space γ_m . Here we make no assumption regarding straight-sidedness and ascribe to the jet an arbitrary radial profile of longitudinal velocity.

In particular, we will demonstrate that type I dispersion determines the growth rate of perturbations in the jet, which pertains to the analysis performed by Scase & Hewitt (2012) for plumes. In addition, we will show that type I dispersion is responsible for the behaviour of r_m and w_m behind the leading front of the hyperbolic problem that one obtains by assuming that γ_m depends on neither z nor t . Although we have emphasised that in real jets $\gamma_m \sim 4/3$, understanding the way in which $\gamma_m < 4/3$ and $\gamma_m > 4/3$ influences the behaviour of an unsteady jet is valuable for several reasons. First, it explains why models that assume a top-hat distribution of velocities admit the unbounded growth of disturbances. Secondly, it provides a tentative indication as to why the velocities in a jet might naturally converge towards a Gaussian distribution.

7.6.1 The growth rate of source perturbations

In order to determine the cause of the difficulties associated with several unsteady plume models, Scase & Hewitt (2012) analysed their response to infinitesimal perturbations of the source conditions. Here we repeat the analysis for unsteady jets, focusing on a single harmonic of frequency σ . It is convenient to work in terms of the dimensionless variables

$$\zeta \stackrel{\text{def}}{=} \frac{\alpha_0 z^2 \sigma}{M_{m0}^{1/2}}, \quad \tau \stackrel{\text{def}}{=} \sigma t, \quad (7.45a, b)$$

where M_{m0} is a constant steady-state momentum flux. We will assume that the dimensionless energy flux γ_m is constant. In terms of the dimensionless variables (7.45a, b), linearised equations enforcing the conservation of momentum and energy, in terms of the dimensionless volume flux $\mathcal{Q}_1$ and momentum flux $\mathcal{M}_1$, can be expressed as

$$\left(\frac{\partial}{\partial \tau} + \begin{bmatrix} 0 & 1 \\ -\gamma_m & 2\gamma_m \end{bmatrix} \frac{\partial}{\partial \zeta} - \begin{bmatrix} 0 & 0 \\ \frac{\gamma_m}{2\zeta} & -\frac{\gamma_m}{4\zeta} \end{bmatrix} \right) \begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{M}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (7.46)$$

We decompose the problem into Fourier modes and, focusing on the harmonic of frequency σ , assume a solution of the form

$$\begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{M}_1 \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{Q}}_1(\zeta) \\ \hat{\mathcal{M}}_1(\zeta) \end{pmatrix} \exp(i\tau). \quad (7.47)$$

It follows from (7.46) that

$$\frac{d^2 \hat{\mathcal{M}}_1}{d\zeta^2} + \left(2i + \frac{1}{2\zeta} \right) \frac{d\hat{\mathcal{M}}_1}{d\zeta} + \left(\frac{i}{4\zeta} - \frac{1}{\gamma_m} \right) \hat{\mathcal{M}}_1 = 0, \quad (7.48)$$

which is a confluent hypergeometric equation, whose general solution is

$$\hat{\mathcal{M}}_1(\zeta) = \exp[-i\zeta(1+\phi)] \sqrt{\zeta} \left[c_1 M \left(\frac{6\phi+1}{8\phi}, \frac{3}{2}, 2\zeta\phi i \right) + c_2 U \left(\frac{6\phi+1}{8\phi}, \frac{3}{2}, 2\zeta\phi i \right) \right], \quad (7.49)$$

where M and U are independent solutions of Kummer's equation (Abramowitz & Stegun, 1970, p.504), c_1 and c_2 are arbitrary constants, and

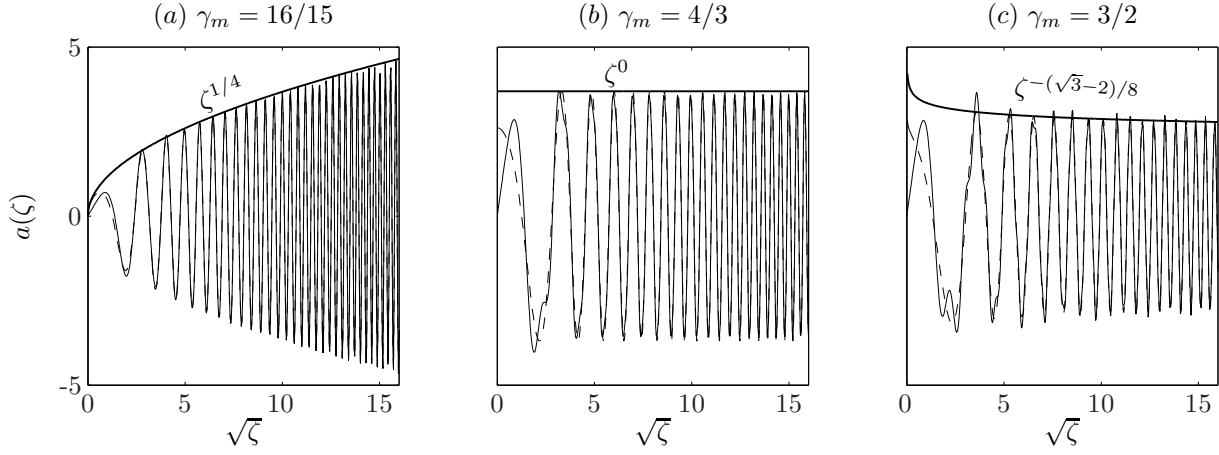


FIGURE 7.4. The response of jets with different dimensionless energy fluxes γ_m , to harmonic source perturbations. The continuous lines correspond to (7.49), with $c_1 = 1$ and $c_2 = 2$, the dashed lines correspond to the asymptotic solution (7.51) and the thick line to the modulus of (7.51).

$$\phi \stackrel{\text{def}}{=} \sqrt{1 - \frac{1}{\gamma_m}}. \quad (7.50)$$

The limiting form of (7.49) for large z is given by (Abramowitz & Stegun, 1970, p.508)

$$\hat{\mathcal{M}}_1(\zeta) \sim c_1 \exp[i\zeta(\phi - 1)] \frac{\Gamma(3/2)}{\Gamma\left(\frac{6\phi + 1}{8\phi}\right)} (2\phi i)^{a(\phi) - 1/2} \zeta^{a(\phi)}, \quad (7.51)$$

where Γ is the Gamma function. The solution (7.49) along with the asymptotic form (7.51) are plotted in figure 7.4 for several values of the dimensionless energy flux γ_m . It is evident that when $\gamma_m = 16/15$ the amplitude of the perturbations grow with respect to ζ , whereas when $\gamma_m \geq 1$ the perturbation amplitude remains constant or decays. When interpreting figure 7.4 it should be noted that $\sqrt{\zeta} \propto z$.

More precisely, the asymptotic solution (7.51) reveals the growth rate of the perturbations in ζ , with exponent

$$a(\phi) = \frac{1}{8\phi} - \frac{1}{4}, \quad (7.52)$$

which is displayed in figure 7.5. When $\gamma_m > 4/3$ the exponent is less than one and the perturbations decay. However, when $\gamma_m < 4/3$ the perturbations grow without bound. The Gaussian profile, or straight-sided jet, for which $\gamma_m = 4/3$, is the special case for which the amplitude of the perturbations remains constant. It is

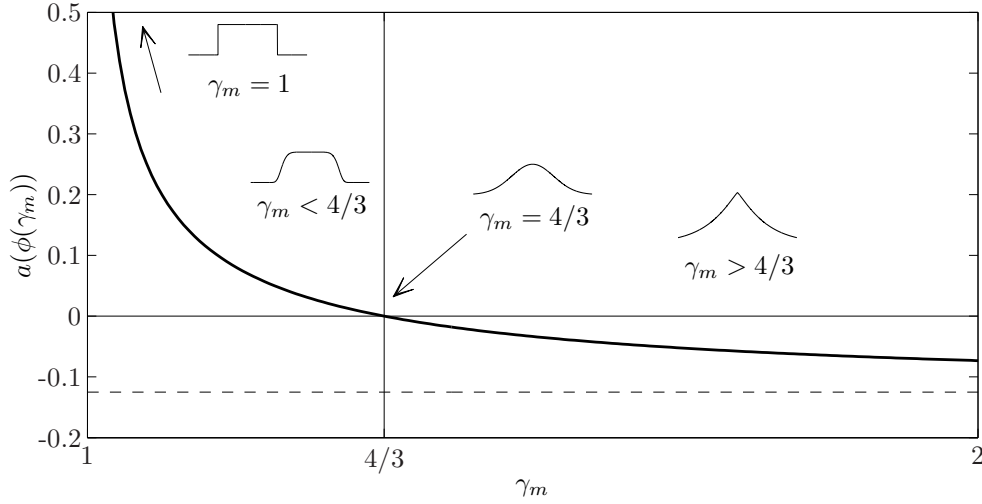


FIGURE 7.5. Dependence of the power-law growth of source perturbations applied to a jet on the dimensionless energy flux γ_m . The dashed line indicates the value $-1/8$ of $a(\phi)$ as $\phi \rightarrow \infty$.

tempting to speculate that it is in order to yield constant-amplitude perturbations that the flow chooses to be Gaussian in form. In the limit $\gamma_m \rightarrow 1$ the perturbations grow exponentially, and the limiting form of the solution is

$$\hat{\mathcal{M}}_1(\zeta) \sim c_3 \exp \left[\left(\frac{1}{2} + \frac{i}{2} \right) \sqrt{2\zeta} - i\zeta \right], \quad (7.53)$$

whilst in the straight-sided limit $\gamma_m \rightarrow 4/3$ the solution degenerates into a single harmonic of constant amplitude:

$$\hat{\mathcal{M}}_1(\zeta) \sim c_4 \exp \left(-i\frac{\zeta}{2} \right), \quad (7.54)$$

for constants c_3 and c_4 .

7.6.2 The response of the jet area

Under the assumption that γ_m remains constant, characteristic curves of the system (7.6)-(7.7) are paths along which the total derivatives of several quasi-invariant quantities are decoupled. Using the eigenvectors of the system the invariants are defined in terms of the total derivatives (§6.5.1):

$$dY_m^n = -\frac{\gamma_m}{\lambda_n^*} \frac{M_m}{Q_m} Q_m^{-\gamma_m/\lambda_n^*} dQ_m + Q_m^{-\gamma_m/\lambda_n^*} dM_m, \quad (7.55)$$

where the integrating factor $Q_m^{-\gamma_m/\lambda_n^*}$ has been introduced. Here λ_n^* is an eigenvalue

of the original system of equations (§6.5.1, equation (6.37)):

$$\lambda_n^* = \gamma_m \left[1 + (-1)^n \left(1 - \frac{1}{\gamma_m} \right)^{1/2} \right], \quad n = 1, 2, \quad (7.56)$$

where $n = 1$ and $n = 2$ correspond to the slower and faster of the two characteristic curves, respectively. Consequently, the quasi-invariants are

$$Y_m^n = \frac{M_m}{Q_m^{\gamma_m/\lambda_n^*}}. \quad (7.57)$$

If we consider a case in which the turbulence production is zero, then the original governing equations (7.6)-(7.7) become a homogeneous system and $dY_m^n/dt = 0$ along characteristic curves. Steady-state solutions to this idealised case correspond to a jet that does not entrain and therefore has a constant volume flux and a constant momentum flux. If a discontinuity is imposed at $(z, t) = (0, 0)$ then Q_m and M_m will be constant in a region S , bounded by fast and slow characteristic curves that emanate from $(0, 0)$. Therefore, in the homogeneous case, the value of the invariants, and hence the values Q_m^S and M_m^S , of Q_m and M_m in S , respectively, can be determined by tracing fast and slow characteristic curves to points outside S at which the solution is known:

$$\frac{M_m^S}{(Q_m^S)^{\gamma_m/\lambda_1^*}} = \frac{M_m^B}{(Q_m^B)^{\gamma_m/\lambda_1^*}}, \quad \frac{M_m^S}{(Q_m^S)^{\gamma_m/\lambda_2^*}} = \frac{M_m^A}{(Q_m^A)^{\gamma_m/\lambda_2^*}}. \quad (7.58a, b)$$

The solution of these equations is

$$M_m^S = X_m^{1/(2\phi)} M_m^A, \quad Q_m^S = X_m^{1/(2\phi)+1/2} Q_m^A, \quad (7.59a, b)$$

where ϕ was defined in (7.50) and

$$X_m \stackrel{\text{def}}{=} \left(\frac{M_m^A}{M_m^B} \right)^{1-\phi} \left(\frac{Q_m^B}{Q_m^A} \right). \quad (7.60)$$

Alternatively, (7.59a, b) can be manipulated to give the velocity w_m^S and radius r_m^S in the region S ,

$$w_m^S = X_m^{-1/2} w_m^A, \quad r_m^S = X_m^{1/(4\phi)+1/2} r_m^A, \quad (7.61a, b)$$

where

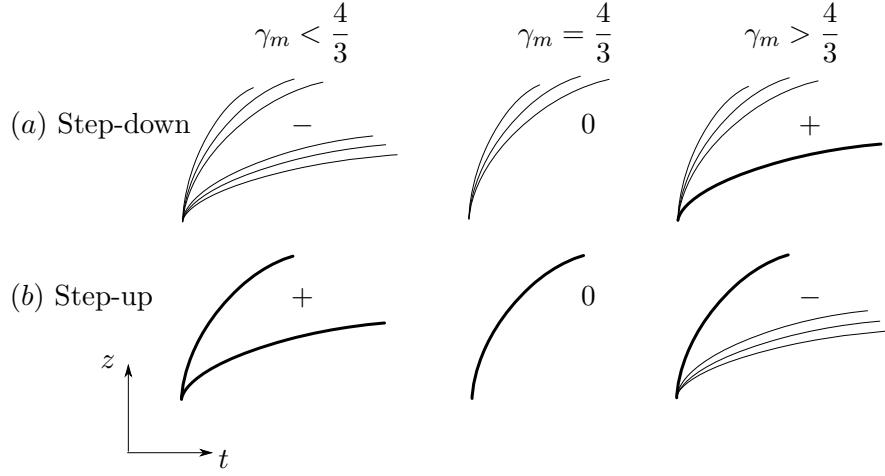


FIGURE 7.6. Response of system characteristic curves following a negative (step-down, *a*) and positive (step-up, *b*) change in the source momentum flux. A single bold line denotes a compression wave, three thin lines denote a rarefaction wave, and the symbols $+$, $-$, 0 denote a positive, negative and zero area change, respectively.

$$X_m \stackrel{\text{def}}{=} \left(\frac{w_m^A}{w_m^B} \right)^{1-2\phi} \left(\frac{r_m^B}{r_m^A} \right)^{2\phi}. \quad (7.62)$$

If it is assumed that the area is held constant at the source, then

$$w_m^S = \left(\frac{w_m^A}{w_m^B} \right)^{\phi-1/2} w_m^A, \quad r_m^S = \left(\frac{w_m^A}{w_m^B} \right)^{1/(4\phi)-\phi} r_m^A. \quad (7.63a, b)$$

If $\gamma_m = 4/3$, then $\phi = 1/2$ and $r_m^S = r_m^A$, indicating that the jet is straight-sided as expected. Equations (7.63*a, b*) are useful because they can provide the change in radius, r_m , and change in velocity, w_m^S between characteristic curves *a priori*. Although this analysis is based on the homogeneous system, knowledge of w_m^S allows one to infer whether each characteristic curve represents a compression or rarefaction wave. This information is summarised in figure 7.6, which shows that although the nature of the leading wave is determined by the sign of the step change in M_m , it is the value of γ_m that will determine the nature of the slower characteristic. Specifically, figures 7.7(*a*) and 7.7(*b*) provide contours of w_m^S/w_m^A and r_m^S/r_m^A , respectively. The value of each of these variables in the region S is determined by the relative change in source velocity w_m^A/w_m^B , and the profile parameter $\phi(\gamma_m)$ defined in (7.50). Whether the jet velocity or radius increases or decreases in the region S is therefore indicated by the quadrant of figure 7.7(*a, b*) to which the pair $(\phi, w_m^A/w_m^B)$ belongs. The analysis performed in this section supports

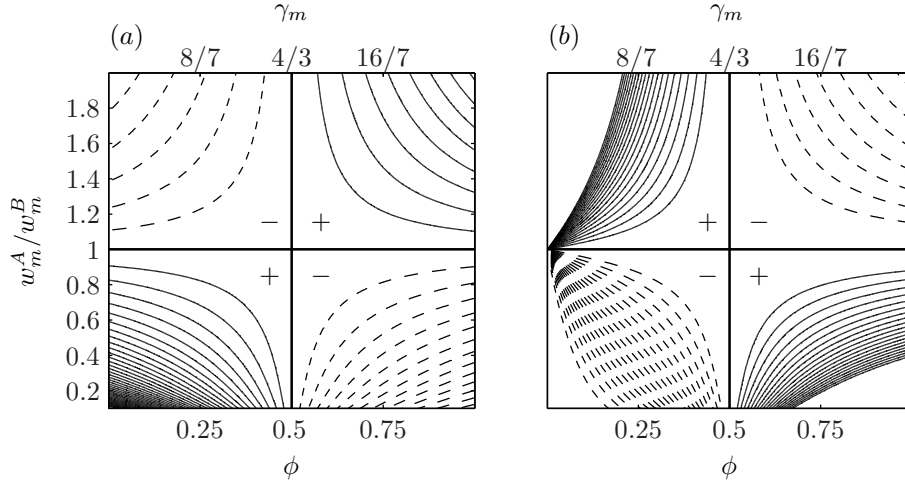


FIGURE 7.7. Response of relative jet velocity w_m^S/w_m^A in (a) and relative radius r_m^S/r_m^A in (b), in the region between characteristic curves for the homogeneous hyperbolic problem corresponding to equations (7.6)-(7.7) with constant γ_m and $\delta_m = 0$.

the inference that was made using the area equation (7.32) in §7.5. Namely, when $\gamma_m < 4/3$ the radius of the jet increases or decreases in response to an increase or decrease in the source momentum flux, respectively, whilst for $\gamma_m > 4/3$ the behaviour is reversed. Real jets, for which $\gamma_m \approx 4/3$, appear to occupy the special position between these states, in which the area is unaffected by changes in M_m .

7.6.3 Jet similarity solutions

In this section we re-derive the jet similarity solutions presented in Scase *et al.* (2006b), without making an assumption about the velocity profile of the jet. Specifically, we seek solutions to unsteady transport equations for momentum (7.6) and mean energy (7.7):

$$\frac{\partial Q_m}{\partial t} + \frac{\partial M_m}{\partial z} = 0, \quad (7.64)$$

$$\frac{\partial M_m}{\partial t} + \frac{\partial}{\partial z} \left(\gamma_m \frac{M_m^2}{Q_m} \right) = -2\alpha_0 \gamma_m \frac{M_m^{5/2}}{Q_m^2}, \quad (7.65)$$

of the form

$$Q_m(z, t) = c_1 z^{a_1} t^{b_1}, \quad M_m(z, t) = c_2 z^{a_2} t^{b_2}, \quad (7.66a, b)$$

for constants a_1, a_2, b_1, b_2, c_1 and c_2 . In (7.65) we have related the dimensionless

turbulence production δ_m to the steady-state entrainment coefficient according to $\delta_m = -2\alpha_0\gamma_m$. The solution to this problem is found to be

$$Q_m(z, t) = \left(\frac{\gamma_m \alpha_0}{8 - 5\gamma_m} \right)^2 \frac{z^3}{t}, \quad M_m(z, t) = \frac{1}{4} \left(\frac{\gamma_m \alpha_0}{8 - 5\gamma_m} \right)^2 \frac{z^4}{t^2}. \quad (7.67a, b)$$

Using the definitions $r_m \stackrel{\text{def}}{=} Q_m/M_m^{1/2}$ and $w_m \stackrel{\text{def}}{=} M_m/Q_m$, one finds that the characteristic radius and velocity scale according to

$$r_m(z) = \frac{2\gamma_m \alpha_0}{8 - 5\gamma_m} z, \quad w_m(z, t) = \frac{z}{4t}, \quad (7.68a, b)$$

respectively. Hence the behaviour of the velocity w_m depends on neither the entrainment coefficient nor the shape of the radial profile of longitudinal velocity. The radius r_m , on the other hand, in spite of its time independence, is affected by both the entrainment coefficient and the velocity profile. For top-hat and Gaussian profiles the radius is given by

$$r_m(z, t | \gamma_m = 1) = \frac{2\alpha_0 z}{3}, \quad (7.69)$$

$$r_m(z, t | \gamma_m = 4/3) = 2\alpha_0 z, \quad (7.70)$$

respectively. Both the behaviour of w_m in (7.68a, b) and that of r_m for the top-hat profile (7.69) agree with the findings presented in Scase *et al.* (2006b). Notably however, when the velocity profile is assumed to have a Gaussian form (7.70) shows that it retains its steady-state form, which is in agreement with the alternative derivation of straight-sidedness presented in §7.5.

The form of (7.68a, b) suggests that by redefining the entrainment coefficient, the similarity solutions can be represented in a form that is universally applicable to arbitrary velocity profiles. Recalling (6.19) and assuming that γ_m is constant results in

$$\alpha \stackrel{\text{def}}{=} \underbrace{-\frac{\delta_m}{2\gamma_m}}_{\alpha_{\text{prod}}} + \underbrace{\frac{Q_m}{M_m^{3/2}} \left(1 - \frac{1}{\gamma_m} \right) \frac{\partial M_m}{\partial z}}_{\alpha_{\text{disp1}}}. \quad (7.71)$$

Therefore, using $\delta_m = -2\gamma_m \alpha_0$ and the solutions (7.67a, b), we find that

$$\alpha = \frac{3\gamma_m \alpha_0}{8 - 5\gamma_m}, \quad (7.72)$$

and we can therefore state that for all velocity profiles

$$r_m(z, t) = \frac{2\alpha z}{3}. \quad (7.73)$$

Superficially, (7.73) has the same form as the top-hat solution (7.69), which was obtained by Scase *et al.* (2006*b*). However, in general, in (7.73) α is not the same as the steady-state entrainment coefficient α_0 . Indeed, when $\gamma_m = 4/3$, $\alpha = 3\alpha_0$, and the angle of spread of the jet is indistinguishable from its steady-state solution. More general solutions can be found that account for the dimensionless turbulent transport terms β_f and γ_f .

7.7 Conclusions

In this chapter we have used the momentum-energy framework described in chapter 6 in a prognostic capacity to develop a model for unsteady jets that includes energy dispersion ((7.28) and (7.29)). The use of the mean energy equation allowed us to account for profile-related transport processes that are not evident at the level of momentum conservation. By faithfully accounting for longitudinal transport processes the model that we have developed is easily solved numerically using a central differencing scheme. For jets, our proposed model for type II dispersion appears to be more appropriate than the turbulence mixing term suggested by Scase & Hewitt (2012) for plumes, the latter resulting in the non-physical increase in a jet's momentum flux with height (see e.g. (7.13) or figure 7.1). A comparison of the predictions obtained from the dispersion model with DNS results confirms that the closure provides a good representation of type II dispersion, and reproduces the location and longitudinal extent of fronts in the jets' momentum flux, over the parameter space investigated.

In addition, in §7.6 we determined the response of the jet to source perturbations, whose unbounded growth was found to be prevented by a sufficient amount of type I dispersion. The precise quantity of type I dispersion that causes the amplitude of perturbations to remain constant coincides with that obtained from a Gaussian velocity profile. Independently, in §7.5, we showed that Gaussian jets with a fixed source area remain straight-sided. We used this fact to develop a simplified version of our model equations. Both nonlinear and linear versions of the straight-sided model show a good agreement with the original model and the DNS data. Of interest in this respect is the connection with the model of steady plumes of Priestley & Ball (1955), which, in contrast to the model of Morton *et al.* (1956), also predicts straight-sidedness. Indeed, in §7.6.3 we rederived the similarity solu-

tions obtained by Scase *et al.* (2006*b*), without making an assumption about the velocity profile of the jet, to find that top-hat profiles lead to narrowing of the jet (as identified by Scase *et al.*, 2006*b*), Gaussian profiles result in no change in the radius of the jet, and profiles that are relatively peaked result in an increased jet radius.

Chapter 8

An energy-based framework for unsteady turbulent plumes

The penultimate chapter of this thesis combines the observations and theory of chapters 5, 6 and 7 to develop a generalised framework for unsteady plumes, which exposes several fundamental aspects of their physics. The framework allows one to understand how the structure of the governing integral equations depends on the assumptions one makes about the longitudinal velocity profile, turbulence and pressure. In general the governing integral equations are hyperbolic, becoming parabolic in the limiting case of a ‘top-hat’ model, and travelling waves in the plume can be classified as lazy, pure or forced according to the particular assumptions that are invoked to close the integral equations. Under the assumption that velocity and buoyancy profiles in the plume have a Gaussian form, we develop a well-posed model for unsteady plumes that exhibits a good agreement with DNS data. The material we present in this chapter has been submitted for publication (Craske & van Reeuwijk, 2015*a*).

8.1 Introduction

As discussed in chapters 1, 6 and 7, a number of models for statistically unsteady plumes have been developed (see e.g. Turner, 1962; Middleton, 1975; Delichatsios, 1979; Yu, 1990; Vul’fson & Borodin, 2001; Scase *et al.*, 2006*b*) as extensions of the popular steady-state plume model of Morton *et al.* (1956). In contrast to steady plumes, unsteady plumes have mean source fluxes of volume, momentum and buoyancy that vary in time.

Models of unsteady plumes can be traced back to Turner (1962), who conceived of a starting plume as a steady plume (Morton *et al.*, 1956) capped with a thermal (Turner, 1957). Middleton (1975) provided a more detailed description of the starting plume, subject to the condition that the flow is self-similar. Delichatsios (1979) subsequently used a system of partial differential equations to describe the behaviour of an unsteady plume, with independent variables describing time and the longitudinal coordinate. There, the equations were based ostensibly on the assumption of Gaussian velocity profiles and the starting plume models of Turner (1962) and Middleton (1975), although a derivation of the equations was not provided. As discussed in the context of unsteady jets in chapter 6, a variety of other

unsteady plume models have appeared subsequently, including Yu (1990, based on Gaussian velocity profiles) and Vul'fson & Borodin (2001, based on the assumption that the plume is 'straight-sided'). Perhaps the most rigorous and comprehensively investigated unsteady plume model is that of Scase *et al.* (2006*b*, referred to as TPM in the present chapter), which was based on a 'top-hat' description of the variables within the plume. TPM has subsequently been used to investigate the rise height and stall time of Boussinesq plumes subjected to a reduction in their source buoyancy flux (Scase *et al.*, 2006*a*) and for unstratified environments has been compared to laboratory observations (Scase *et al.*, 2008). In Scase *et al.* (2009) TPM was used to predict the behaviour of a plume whose source buoyancy flux undergoes a rapid increase, and a comparison to an implicit large eddy simulation showed that TPM correctly predicted the scaling associated with the longitudinal position of a self-similar pulse structure in the plume. For a comparison between some of the key features of existing unsteady plume models in the more restrictive context of unsteady jets, the reader is referred to §7.2.2 and §7.2.3.

As discussed in chapter 7 the physics and mathematical properties of unsteady plume models have recently been reappraised, owing to the discovery of Scase & Hewitt (2012) that the models of Delichatsios (1979), Yu (1990) and Scase *et al.* (2006*b*) are ill-posed. Each of these ill-posed models admits the arbitrarily-large unbounded growth of short-wave modes, which prevents one from obtaining convergence in numerical approximations. Whilst Scase & Hewitt (2012) cited the likely cause of this behaviour as the absence of longitudinal diffusion, in chapter 7 we demonstrated that, in the case of jets, it is the assumption of a top-hat velocity profile that is chiefly responsible for the problems. In this chapter we show that the same is true for unsteady plumes and that if one assumes a Gaussian distribution of longitudinal velocity the governing equations are, in general, well-posed.

The aim here is to extend the framework developed in chapter 6 to understand and model the bulk behaviour of unsteady plumes. In §8.2 we extend the integral equations for unsteady jets that were obtained in §6.2.2 to unsteady plumes. The direct numerical simulations that were undertaken to investigate unsteady plumes are described in §8.3 and the results are discussed in §8.4. Before making model-specific assumptions, in §8.5 we analyse the properties of the governing integral equations and their physical implications. In particular, we consider the characteristic curves of the system in §8.5.1 and describe how they determine the structure of travelling waves in §8.5.3. In §8.5.4 we revisit the power-law similarity solutions of Scase *et al.* (2006*b*) from a more general perspective and in §8.5.5 we investigate the response of an unsteady plume to harmonic source perturbations. In §8.6 we

focus on modelling the unsteady plume equations, incorporating a closure for shear-flow dispersion in the transport of buoyancy and energy in §8.6.1. Comparisons between the model's predictions and the DNS results are made in §8.6.2, before we propose a simplified form of the model in §8.6.3. To complement chapters 6 and 7, which focused on momentum and energy transport, our focus in this chapter will be on the effects of buoyancy.

8.2 Governing equations

8.2.1 Reynolds equations

The flow with which we are concerned is a round high-Reynolds-number turbulent plume orientated in the vertical (z) direction, whose flow is statistically axisymmetric and swirl-free. The plume is comprised of fluid that is of lower density than its surroundings and therefore has positive buoyancy

$$b \stackrel{\text{def}}{=} \frac{\rho_d - \rho}{\rho_0} g, \quad (8.1)$$

where ρ is the fluid density, $\rho_d(z)$ is the density of the environment at a height z and ρ_0 is a constant reference density. Using the Boussinesq approximation (see §2.1.2) and the assumption that the environment is unstratified, which implies that ρ_d is constant, the equations governing the ensemble-averaged quantities are (see chapter 2)

$$\frac{1}{r} \frac{\partial(r\bar{u})}{\partial r} + \frac{\partial\bar{w}}{\partial z} = 0, \quad (8.2)$$

$$\frac{\partial\bar{w}}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(r\bar{u}\bar{w} + r\overline{u'w'}) + \frac{\partial}{\partial z}(\bar{w}^2 + \overline{w'^2}) = -\frac{\partial\bar{p}}{\partial z} + \bar{b}, \quad (8.3)$$

$$\frac{\partial\bar{b}}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(r\bar{u}\bar{b} + r\overline{u'b'}) + \frac{\partial(\bar{w}\bar{b} + \overline{w'b'})}{\partial z} = 0. \quad (8.4)$$

Following chapter 6, we multiply equation (8.3) by $2\bar{w}$ to obtain an equation for the mean longitudinal kinetic energy of the plume:

$$\begin{aligned} \frac{\partial\bar{w}^2}{\partial t} + \frac{1}{r} \frac{\partial(r\bar{u}\bar{w}^2)}{\partial r} + \frac{\partial\bar{w}^3}{\partial z} + 2\frac{\partial(\bar{p}\bar{w})}{\partial z} \\ + \frac{2}{r} \frac{\partial(r\overline{u'w'w})}{\partial r} + 2\frac{\partial(\overline{w'^2w})}{\partial z} = 2\bar{p}\frac{\partial\bar{w}}{\partial z} + 2\overline{w'^2}\frac{\partial\bar{w}}{\partial z} + 2\overline{u'w'}\frac{\partial\bar{w}}{\partial r} + 2\bar{b}\bar{w}. \end{aligned} \quad (8.5)$$

Recall that having been obtained via invertible manipulations, the momentum–energy system (8.3),(8.5) and (8.4) is equivalent to the mass–momentum system (8.2),(8.3) and (8.4). In the following section we will show that the momentum–energy system constitutes the natural choice for developing and understanding integral models of unsteady plumes.

8.2.2 Integral equations

Integration of (8.3),(8.5) and (8.4) over the radial direction r yields the integral equations

$$\frac{\partial Q_m}{\partial t} + \frac{\partial(\beta_g M_m)}{\partial z} = B_m, \quad (8.6)$$

$$\frac{\partial M_m}{\partial t} + \frac{\partial}{\partial z} \left(\gamma_g \frac{M_m^2}{Q_m} \right) = \delta_g \frac{M_m^{5/2}}{Q_m^2} + 2\theta_m \frac{M_m B_m}{Q_m}, \quad (8.7)$$

$$\frac{\partial B_m}{\partial t} + \frac{\partial}{\partial z} \left(\theta_g \frac{M_m B_m}{Q_m} \right) = 0, \quad (8.8)$$

where the dimensionless profile coefficients θ_g , β_g , γ_g and δ_g were defined and discussed in chapters 5 and 6. In this chapter, θ refers to a dimensionless flux of buoyancy (cf. the dimensionless passive scalar flux introduced in chapter 5). The presence of buoyancy distinguishes unsteady plumes from unsteady jets and gives rise to an additional dependent variable. Specifically, we define the *integral* buoyancy B_m and the characteristic buoyancy b_m :

$$B_m \stackrel{\text{def}}{=} 2 \int_0^{r_d} \bar{b} r dr \stackrel{\text{def}}{=} b_m r_m^2. \quad (8.9)$$

We note that the integral buoyancy B_m is not typically used in *steady* plume theory, in which the effects of buoyancy are expressed in terms of the buoyancy flux F_m (see, e.g., Hunt & van den Bremer, 2011, and references therein). However, in the *unsteady* integral equations (8.6)–(8.8) the use of B_m is convenient from both a mathematical and a conceptual perspective. With B_m as a dependent variable, the temporal derivatives in (8.6)–(8.8) are decoupled and each obeys a typical conservation equation. Moreover, consistent with the problem of scalar transport considered in chapter 5, it is natural to view the buoyancy flux F_m as an unknown quantity requiring assumptions, because it depends on the correlation of $\bar{w}$ and $\bar{b}$. Indeed, with the use of B_m as the dependent variable rather than F_m , the problem has the same form as the classical dispersion problem (Taylor, 1953), discussed in chapter 5. Here, the integral buoyancy B_m plays the role of the

integral concentration C_m of chapter 5.

In obtaining (8.6)-(8.8) it was assumed that the mean velocity $\bar{w}$ and buoyancy $\bar{b}$ are equal to zero at $r = r_d$. For a discussion pertaining to the neglected boundary fluxes, the reader is referred to §4.3.2. Thus, we define the buoyancy flux according to

$$F_m \stackrel{\text{def}}{=} 2 \int_0^{r_d} \bar{w} \bar{b} r dr, \quad (8.10)$$

which is consistent with our definition of the (passive) scalar flux in chapter 5.

As described in chapter 6, information about turbulence, pressure and the shape of the underlying velocity profile in (8.6)-(8.8) is captured by dimensionless profile coefficients, which allow one to express the exact integral equations in terms of the dependent variables Q_m , M_m and B_m . To this end, we supplement the definitions of the dimensionless profile coefficients in chapter 6 with the gross dimensionless buoyancy flux:

$$\theta_g \stackrel{\text{def}}{=} \underbrace{\frac{2Q_m}{B_m M_m} \int_0^{r_d} \bar{w} \bar{b} r dr}_{\theta_m} + \underbrace{\frac{2Q_m}{B_m M_m} \int_0^{r_d} \overline{w' b'} r dr}_{\theta_f}, \quad (8.11)$$

which is analogous to the dimensionless (passive) scalar flux of chapter 5. Note that θ_g is used in (8.8) because it accounts for the total transport of buoyancy, whereas θ_m is used in (8.7) because the forcing of the mean-flow energy by buoyancy does not include turbulent transport θ_f .

The profile coefficients β_g , γ_g , θ_g and δ_g correspond to the gross fluxes of momentum, energy and buoyancy, and the gross production of turbulence kinetic energy (including the redistribution of energy via pressure). The values of the profile coefficients observed in the DNS simulations of steady plumes (see §4.2.1) are reported in table 8.1, and will be discussed further in §8.4.

By definition, the assumed shape of the mean velocity profile does not affect the volume flux or the momentum flux in the plume, i.e. the volume flux and the momentum flux are equal to Q_m and M_m , respectively, regardless of the behaviour of $\bar{w}$. Similarly, by definition, the shape of the buoyancy profile does not affect the mean integral buoyancy B_m . For this reason, in self-similar steady-state models of jets and plumes it is conventional (Turner, 1973), and entails no loss of generality, to regard $\bar{w}$ as having a uniform distribution of amplitude w_m over $r \leq r_m$, which

TABLE 8.1. The dimensionless parameters of a steady plume (the reader is referred to chapters 5 and 6 for their definitions). The steady-state entrainment coefficient in a plume is denoted α_0 . Here TH = top-hat, G = Gaussian, and L_p and H_p refer to the simulations of a steady-state plume described in chapter 4.

	TH	G	H_p	L_p
α	α_0	α_0	0.11	0.12
β_f	0.00	0.00	0.19	0.20
β_u	0.00	0.00	0.11	0.12
β_v	0.00	0.00	0.11	0.12
β_p	0.00	0.00	-0.09	-0.11
β_g	1.00	1.00	1.10	1.09
γ_m	1.00	1.33	1.28	1.29
γ_f	0.00	0.00	0.34	0.35
γ_p	0.00	0.00	-0.17	-0.19
γ_g	1.00	1.33	1.44	1.45
δ_m	$-2\alpha_0$	$-8\alpha_0/3$	-0.21	-0.23
δ_f	0.00	0.00	0.04	0.05
δ_p	0.00	0.00	-0.02	-0.03
δ_g	0.00	0.00	-0.19	-0.22
θ_m	0.00	0.00	1.01	1.02
θ_f	0.00	0.00	0.17	0.18
θ_g	0.00	0.00	1.18	1.20

is otherwise known as a top-hat distribution (see §2.4.1). However, as established in chapter 6, the assumed velocity profile does affect the mean energy flux E_m in the plume. Unlike steady jets and plumes, in unsteady jets and plumes the energy flux plays an independent role in the governing equations and therefore the assumption of a particular velocity profile *does* entail a loss of generality. Recall from chapter 6 that for a Gaussian mean velocity profile $\bar{w} = 2w_m \exp(-2r^2/r_m^2)$, the dimensionless energy flux γ_m can be determined exactly:

$$\gamma_m = \frac{8}{r_m^2} \lim_{r_d \rightarrow \infty} \int_0^{r_d} \exp\left(-6\frac{r^2}{r_m^2}\right) r dr = \frac{4}{3}. \quad (8.12)$$

In contrast, for top-hat velocity profiles the dimensionless energy flux is minimal (assuming that $\bar{w} \geq 0$), such that $\gamma_m = 1$. A variation in the radial dependence of the velocity profile will result in $\gamma_m > 1$. Although Hewitt & Bonnebaigt (2014) remark that integral models of turbulent plumes remove information pertaining to

radial dependence, we find that this is not necessarily the case. Use of the mean kinetic energy equation exposes the fact that γ_m depends on the radial profile of longitudinal velocity and determines the response of the plume to source perturbations (see §7.6.1 or §8.5.5 in the present chapter).

We remind readers that one motivation for working with the mean kinetic energy equation (8.5) rather than the continuity equation (see the classical treatment described in §2.4.1) is that the integral of $\nabla \cdot \mathbf{u} = 0$,

$$\frac{\partial Q_m}{\partial z} = -2ru|_{\infty}, \quad (8.13)$$

does not provide an explicit prognostic equation (for definiteness and consistency with the classical approach, we have assumed that $r_d \rightarrow \infty$). Rather, (8.13), which is ostensibly identical to the steady-state volume flux equation given by Morton *et al.* (1956), relates the volume flux in the plume to the induced radial velocity in the environment. Indeed, (8.13) can be used in place of the integral energy equation (8.7), but the modeller is then faced with the task of providing a closure for $ru|_{\infty}$. In the steady state, Morton *et al.* (1956) replace the right-hand-side of (8.13) with $2\alpha_0 M_m^{1/2}$, thereby defining the classical entrainment coefficient α_0 , to obtain a closed system of equations. In the unsteady case however, an induced radial flow in the environment does not necessarily correspond to entrainment into the plume, because the radius of the plume might change with respect to time (cf. the temporal jet studied by van Reeuwijk & Holzner, 2014, for example).

Efforts to obtain a prognostic equation for the area $r_m^2 \stackrel{\text{def}}{=} Q_m^2/M_m$ of the plume rely on particular assumptions regarding the velocity profile of the jet and, in all but the simplest cases (e.g. the rigorously derived top-hat model of Scase *et al.*, 2006b), have questionable foundations owing to the fact that plumes do not have a well-defined edge. The use of the momentum-energy framework circumvents these issues altogether and allows one to understand unsteady plume models in a broader context.

In contrast to the difficulties associated with the mass-momentum framework, the momentum-energy framework allows one to obtain a prognostic equation for the area of the plume without assumption. Following §6.2.4, we multiply (8.6) by $2Q_m/M_m$ and add the result to (8.7) multiplied by $-Q_m^2/M_m^2$:

$$\frac{1}{\gamma_g} \frac{\partial}{\partial t} \left(\frac{Q_m^2}{M_m} \right) + \frac{\partial Q_m}{\partial z} = 2\alpha M_m^{1/2}, \quad (8.14)$$

where

$$\alpha \stackrel{\text{def}}{=} -\frac{\delta_g}{2\gamma_g} + \frac{Q_m}{2\gamma_g M_m^{5/2}} \frac{\partial}{\partial z} [(\gamma_g - 2\beta_g + 1) M_m^2] \\ + \frac{Q_m}{\gamma_g M_m^{3/2}} (\beta_g - 1) \frac{\partial M_m}{\partial z} + \left(\frac{1}{\gamma_g} - \frac{\theta_m}{\gamma_g} \right) \frac{Q_m B_m}{M_m^{3/2}}. \quad (8.15)$$

The advantage of (8.14) and (8.15) in comparison with (8.13) is that not only does it account explicitly for the temporal change of the area of the jet, it accounts for the process of entrainment in terms of known properties of the velocity, buoyancy and turbulence profiles.

For details regarding the physical interpretation of the first three terms on the right-hand-side of (8.15), the reader is referred to chapter 6, as they are inherited from the equations describing an unsteady jet. The final contribution in (8.15) is unique to plumes and is proportional to a Richardson number, or the flux-balance parameter of Morton (1959). If $\theta_m > 1$, the buoyancy provides slightly more forcing in the energy equation (8.7) than one would expect from identically distributed $\bar{b}$ and $\bar{w}$. Noting that the area Q_m^2/M_m is inversely proportional to the integral energy M_m , the effect of $\theta_m > 1$ is to reduce the entrainment coefficient. Conversely, when $\theta_m < 1$ the buoyancy provides slightly less forcing in the energy equation than one might expect and the entrainment coefficient increases.

In §8.5.4 we derive generalised time-dependent similarity solutions of (8.6)-(8.8), which show that for Gaussian plumes the entrainment coefficient increases, relative to its steady-state value α_0 , in such a way as to ensure that the radius r_m retains its steady-state dependence on z .

8.2.3 The steady state

Using the momentum-energy formulation (8.6)-(8.8) it is interesting to see how the profile coefficients can be incorporated into the classical steady-state plume solutions. For constant source buoyancy flux F_s and constant profile coefficients, the latter assumption being consistent with far-field self-similarity, a steady-state

solution to (8.6)-(8.8) is

$$\left. \begin{aligned} Q_m &= Q_{m0} \stackrel{\text{def}}{=} \frac{6\alpha_0}{5} \left(\frac{9\alpha_0}{10} \right)^{1/3} \left(\frac{F_s}{\beta_g \theta_g} \right)^{1/3} z^{5/3}, \\ M_m &= M_{m0} \stackrel{\text{def}}{=} \left(\frac{9\alpha_0}{10} \right)^{2/3} \left(\frac{F_s}{\beta_g \theta_g} \right)^{2/3} z^{4/3}, \\ B_m &= B_{m0} \stackrel{\text{def}}{=} \frac{6\alpha_0}{5} \left(\frac{10}{9\alpha_0} \right)^{1/3} \left(\frac{F_s^2 \beta_g}{\theta_g^2} \right)^{1/3} z^{1/3}, \end{aligned} \right\} \quad (8.16)$$

where the constant steady-state entrainment coefficient, which follows from substitution of (8.16) into (8.7), is

$$\alpha_0 \stackrel{\text{def}}{=} -\frac{\delta_g}{2\gamma_g} \left(\frac{8\beta_g \theta_m}{5\gamma_g} - \frac{3}{5} \right)^{-1}. \quad (8.17)$$

The effects of θ_g and β_g are felt only via α_0 and an effective buoyancy flux $F_s/(\beta_g \theta_g)$, and are therefore typically unobservable from integrals of a steady plume. Noting that $F_s = \theta_g F_m / \theta_m$, (8.16) can also be expressed in terms of the mean buoyancy flux F_m . The solutions (8.16) might be useful to experimentalists who wish to compare data pertaining to a known *source* buoyancy flux F_s to the solution of the plume equations in a way that accounts for turbulent transport. Comparison of the system (8.16) to the classical plume solutions of Morton *et al.* (1956) shows that the flux-balance parameter of Morton (1959) is

$$\Gamma \stackrel{\text{def}}{=} \frac{5Q_m B_m}{8\beta_g \alpha_0 M_m^{3/2}} = \frac{5Q_m^2 F_m}{8\theta_m \beta_g \alpha_0 M_m^{5/2}}, \quad (8.18)$$

which characterises the relative importance of buoyancy compared to inertia in the flow*. As discussed in §1.2, when $0 < \Gamma < 1$, the plume is dominated by inertia and referred to as being ‘forced’; when $1 < \Gamma$ the plume is dominated by buoyancy effects and is referred to as being ‘lazy’. Jets and pure plumes correspond to the special cases for which $\Gamma = 0$ or $\Gamma = 1$, respectively. Thus, for the steady-state similarity solutions (8.16), describing a pure plume, $\Gamma = 1$. The appearance of β_g and θ_m in (8.18) is noteworthy because it shows that for a pure plume the classical definition of Γ , namely $(5Q_m^2 F_m)/(8\alpha_0 M_m^{5/2}) > 1$, since $\theta_m \approx 1$ and $\beta_g > 1$, in general.

*Indeed, it is occasionally useful recast (8.18) in terms of characteristic scales as $\Gamma = 5b_m r_m / 8\beta_g \alpha_0 w_m^2$.

TABLE 8.2. Simulation details. Here F_s^B and F_s^A are the source buoyancy fluxes before and after the step change, respectively. The source time scale τ_s is defined according to $\tau_s \stackrel{\text{def}}{=} r_s^{4/3} F_s^A^{-1/3}$.

	$(L_x \times L_y \times L_z)/r_s$	$N_x \times N_y \times N_z$	Re_s	t_{run}/τ_s	F_s^A/F_s^B
L_p	$32^2 \times 48$	$1024^2 \times 1536$	1260	360	-
H_p	$32^2 \times 48$	$1024^2 \times 1536$	1600	170	-
$\text{LH}_p 1\text{-}24$	$32^2 \times 48$	$1024^2 \times 1536$	1600	30	2

8.3 Simulation details

We conduct direct numerical simulations of unsteady plumes on a domain of size $32 \times 32 \times 48$ source radii r_s , uniformly discretised using $1024 \times 1024 \times 1536$ computational control volumes, as described in §4.2.1. For a detailed description of the finite volume method used to discretise the governing equations, the reader is referred to chapter 3.

Our ultimate aim was to obtain simulations of a plume whose source buoyancy flux undergoes a step change from F_s^B to F_s^A , where $F_s^B < F_s^A$. Therefore, we use steady-state simulations L_p and H_p , with source buoyancy fluxes of F_s^B and F_s^A , respectively, in order to validate the results and, in the case of L_p , to provide a set of initial conditions for the unsteady simulations. This approach is equivalent to the approach that was taken for jets, which was described in chapters 5 and 6. The Reynolds number $\text{Re}_s \stackrel{\text{def}}{=} 2F_s^{1/3} r_s^{2/3}/\nu$ was equal to 1260 and 1600 in L_p and H_p , respectively. During simulation L_p we write a complete three-dimensional description of the flow field to disk at time intervals much larger than the turnover time. This information provides the initial conditions for the unsteady simulations.

Using the three-dimensional field data obtained from L_p , unsteady plumes were created by imposing a step-change in the source buoyancy flux from F_s^B to F_s^A . Using 24 statistically independent initial conditions, we repeat this process to obtain an ensemble of 24 unsteady plumes. The unsteady plume data were subsequently averaged over the ensemble and over the statistically homogeneous azimuthal dimension of the flow. Details of the simulations are summarised in table 8.2 and for validation of the steady-state data the reader is referred to §4.2.2.

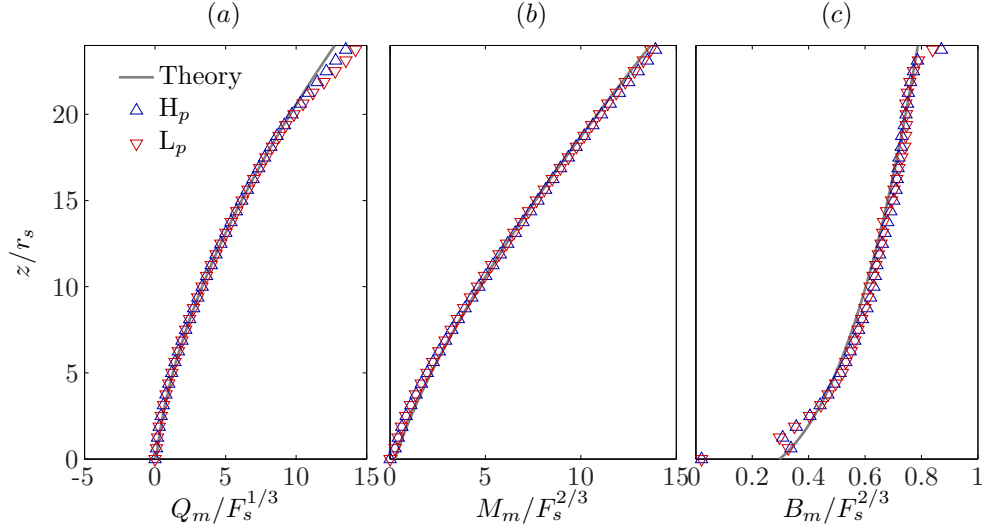


FIGURE 8.1. Comparison of the steady-state DNS results to classical plume theory (8.16).

8.4 Simulation results

Since the primary focus of this study are the integrals Q_m , M_m and B_m , we consider their steady-state behaviour in comparison to classical plume theory in figure 8.1. The DNS results for both L_p and H_p are in good agreement with the classical power-law solutions, with the exception of small deviations in the behaviour of the integral buoyancy B_m at the very bottom and top of the domain, and in Q_m at the top of the domain. It is worth noting that the theoretical predictions shown in figure 8.1 were based on the steady-state system (8.16), which accounts for θ_g and β_g via an effective buoyancy flux $F_s/(\beta_g\theta_g)$.

Profile coefficients indicate the relative importance of all terms in the governing integral equations (8.6)-(8.8). Whilst conventionally the plume equations are defined uniquely by choosing a suitable constant entrainment coefficient α_0 , the generalised framework employed here requires specification of information such as the dimensionless energy flux γ_m and turbulence transport β_f . The profile coefficients play a crucial role in determining the structure and, consequently, the behaviour of the unsteady plume equations. Table 8.1 displays the value of the profile coefficients evaluated from the steady-state data provided by simulation L_p and H_p . The values were obtained by averaging over the interval $[20, 40] \ni z/r_s$, in which the profile coefficients were observed to have reached an approximately constant value. The near field, in which the profile coefficients exhibit appreciable variation, will not be considered.

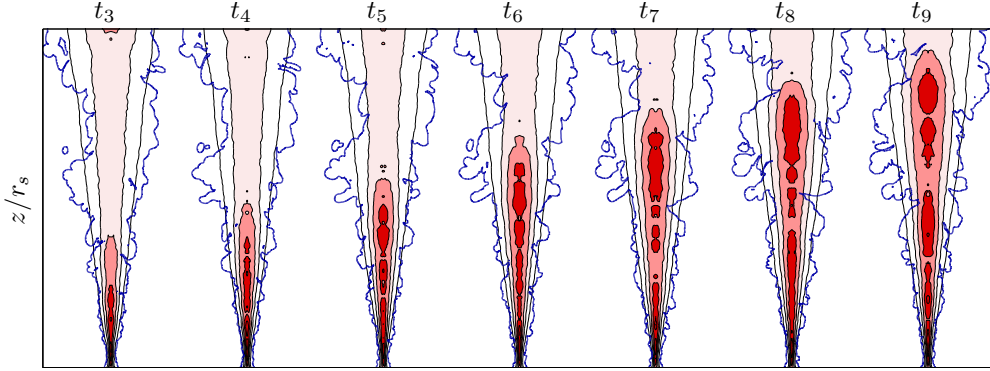


FIGURE 8.2. Isoregions of the ensemble and azimuthally-averaged buoyancy (red) and threshold of the instantaneous enstrophy (blue) in an unsteady turbulent plume at times $t_n \approx 1.8\tau_s n$, $n = 3, \dots, 9$, where $\tau_s \stackrel{\text{def}}{=} r_s^{4/3} F_s^A - 1/3$. The buoyancy displayed in the figure has been non-dimensionalised using the local characteristic buoyancy scale $b_{m0} \stackrel{\text{def}}{=} B_{m0} M_{m0}^{1/2} / Q_{m0}$ of a steady plume.

The observed steady-state entrainment coefficient $\alpha_0 \approx 0.115$, inferred from dQ_m/dz , is at the low end of the values that are reported from laboratory experiments, which typically range from 0.12 - 0.17 (Carazzo *et al.*, 2006). Also noteworthy is the fact that $\gamma_m \approx 1.28$ is close to the value $4/3$, associated with Gaussian profiles, rather than 1, which is the value associated with top-hat velocity profiles. Contributions to momentum, energy and buoyancy transport from turbulence (i.e. β_f , γ_f and θ_f) are of the order of 20% and are therefore not insignificant. In the overall energy balance, a leading-order contribution is made by the dimensionless turbulence production δ_m , in comparison to which δ_f and δ_p are relatively small.

Following a step change in the buoyancy flux at the source of an otherwise steady plume, a disturbance or wave propagates through the domain in the direction of the mean flow. In referring to this disturbance as a wave, we follow Whitham (1974) and regard the wave as a recognisable signal propagating through the domain with a certain velocity. Here the notion of a wave is perhaps more appropriate than the notion of a front, which was employed in chapter 6, as it encompasses a wide variety of possible disturbances that can be comprised of several fronts. More precisely, in §8.5.1 we will see that the wave is comprised of characteristic surfaces along which wavefronts travel (see e.g. Whitham, 1974).

The unsteady plume is illustrated in figure 8.2, which displays the azimuthal- and ensemble-averaged normalised buoyancy field at several time instances, in addition to the instantaneous boundary of the plume corresponding to a single member of the ensemble. The boundary of the plume is defined by an isoline associated with a relatively small value of enstrophy, and therefore separates turbulent and

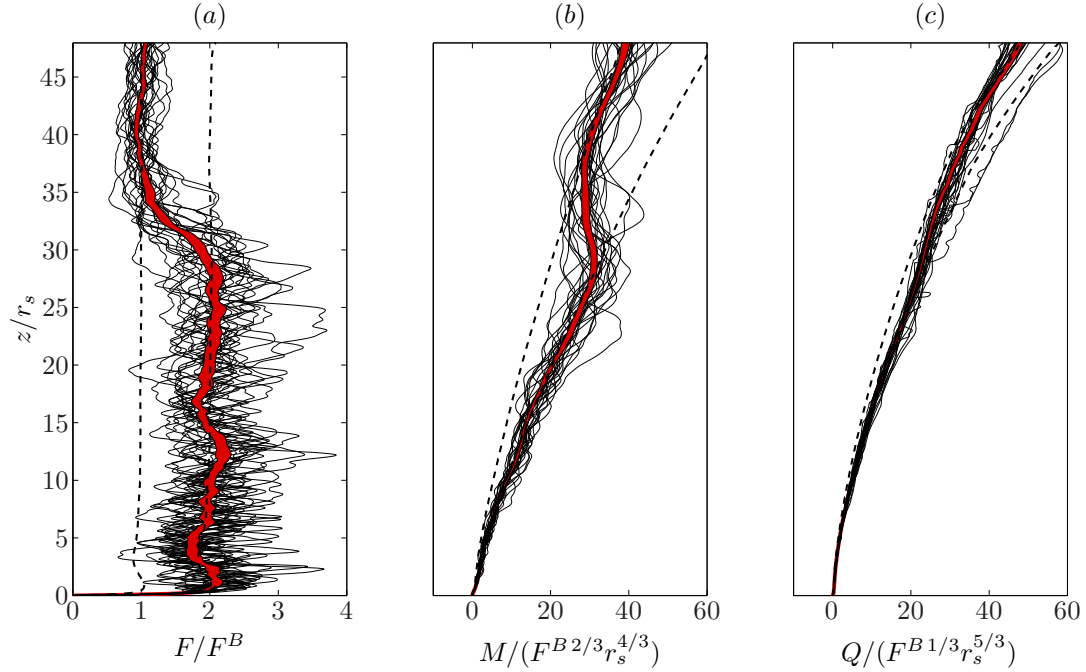


FIGURE 8.3. (a) Dimensionless buoyancy flux, (b) dimensionless momentum flux and (c) dimensionless volume flux corresponding to individual members of the ensemble (thin lines) and their ensemble average (thick line) at $t/\tau_s = 15$. The width of the line denoting the ensemble average is equal to twice the standard deviation of the sample, normalised by $\sqrt{n}$, where $n = 24$ is the number of members of the ensemble. The dashed lines correspond to steady-state data before and after the step change in the source buoyancy flux and F_m^B is the mean buoyancy flux before the step change.

non-turbulent parts of the flow. In spite of the irregular and meandering profile that one observes in a single simulation, in the averaged buoyancy field one observes a smooth region, whose shape is approximately invariant with respect to time, penetrating progressively further into the domain.

Our primary focus is on the behaviour of the integral quantities such as the buoyancy flux, momentum flux and the volume flux. Figure 8.3 displays the longitudinal dependence of these quantities some time after the step change in the buoyancy flux at the source. Integrals from each member of the ensemble are displayed and show significant variation in comparison to the relatively smooth profile that is obtained from their ensemble average. In the buoyancy flux F_m the step change has propagated to approximately $z/r_s = 30$. Upstream and downstream of the travelling wave the mean buoyancy flux is equal to F_m^A and F_m^B , respectively. Since the derivative of the volume flux Q_m depends on M_m and the derivative of M_m depends on F_m , the longitudinal profile of Q_m is smoother than M_m , which is

in turn smoother than F_m . Consequently, the effects of the step change on Q_m are comparatively difficult to discern.

According to classical plume theory (8.16), when $\theta_m = \beta_g = 1$, the characteristic velocity in a steady plume is

$$w_m = \frac{5}{6\alpha_0} \left(\frac{9\alpha_0}{10} \right)^{1/3} \left(\frac{F_m}{z - z_v} \right)^{1/3}, \quad (8.19)$$

where z_v is the location of an asymptotic virtual source. Hence, one presumes that, sufficiently far from the finite source the location z^* of the propagating wave obeys

$$\frac{dz^*}{dt} = \lambda^* w_m = \frac{5\lambda^*}{6\alpha_0} \left(\frac{9\alpha_0}{10} \right)^{1/3} \left(\frac{F_m^*}{z^* - z_v} \right)^{1/3}, \quad (8.20)$$

where λ^* is a constant of proportionality and F_m^* is a characteristic buoyancy flux such that $F_m^B \leq F_m^* \leq F_m^A$. One can integrate equation (8.20) to find that

$$z^* - z_v = \lambda^* \left(\frac{10}{9\alpha_0} \right)^{1/2} F_m^{*1/4} (t - t_v)^{3/4}, \quad (8.21)$$

where t_v is the time for which $z^*(t_v) = z_v$. To simplify matters we will employ a first-order estimation of F^* according to[†]

$$F_m^* = \frac{1}{2} (F_m^A + F_m^B). \quad (8.22)$$

We determine $z^*(t)$ in a simple and reliable manner from to

$$F_m(z^*(t), t) = F_m^*. \quad (8.23)$$

Consequently, if F_m is monotonic z^* is single valued. To ensure a unique definition of z^* in situations in which F_m is not monotonic, we take the maximum value of z^* satisfying (8.23). With the exception of a small region in the near field, we found (8.23) to be sufficient in defining a unique value of z^* . The position $z^*(t)$, determined according to (8.23) is displayed in figure 8.4(a) with respect to the normalised buoyancy flux at several time instances, and evidently provides a useful indication of the wave's position. The approximate self-similarity of F_m , when plotted with respect to z/z^* in figure 8.4(b) will be discussed at the end of this section.

[†]Note that for the case examined here it is possible to obtain an exact expression for the characteristic buoyancy flux associated with a shock according to $F_m^{*1/3} = \frac{2}{3} \left(\frac{F_m^A - F_m^B}{F_m^{A2/3} - F_m^{B2/3}} \right)$. For details see Whitham (1974) or compare to the characteristic momentum flux in unsteady jets §7.5.1, equation (7.37).

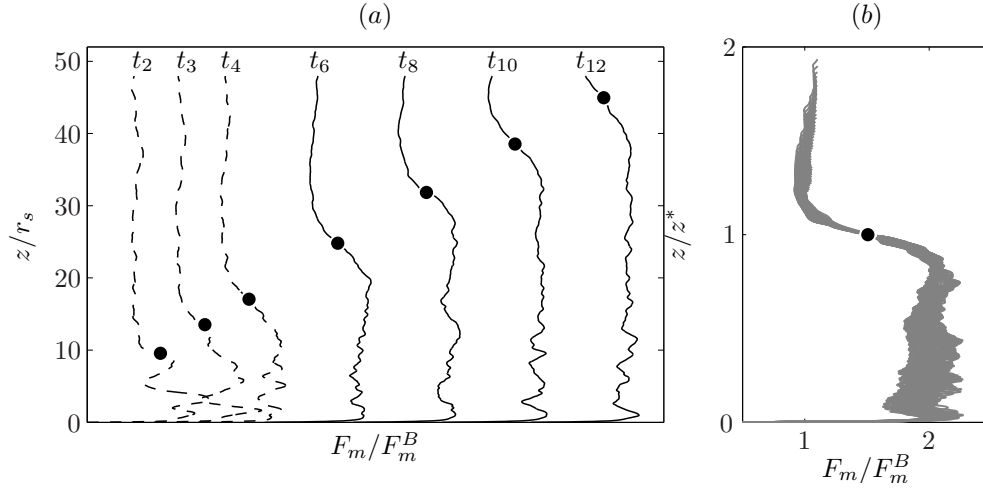


FIGURE 8.4. (a) Normalised buoyancy flux at the times $t_n \approx 1.8\tau_s n$. The black circle denotes $z^*(t)$, which approximately corresponds to the location of the wave. Profiles described with dashed lines are strongly influenced by mixing that is associated with near-field effects and are excluded from the self-similarity plot (b), which demonstrates the self-similarity of the normalised buoyancy flux.

In figure 8.5 we test the hypothesis that $z^* \sim t^{3/4}$ and determine the coefficient λ^* by plotting the observed location of the front $z^*(t)$. Evident from the isoregions of the normalised buoyancy displayed in figure 8.5 is that in the far field the wave's propagation adheres to the anticipated $z^* \sim t^{3/4}$ scaling. In addition, we find that the constant of proportionality $\lambda^* \approx 1.8$ in (8.21). In other words, with reference to (8.20), we observe that the front propagates at nearly twice the local top-hat velocity w_m , which is slightly less than the observed value of λ^* in an unsteady jet (see chapter 6).

The form of the propagating wave in the integrals Q_m, M_m and B_m is displayed as thick black lines in figure 8.6, which each correspond to a particular time instances. At any given time, upstream and downstream of the wave, the integrals Q_m, M_m and B_m satisfy a quasi-steady balance, and therefore the classical power-law solutions $B_m \sim z^{1/3}$, $M_m \sim z^{4/3}$ and $Q_m \sim z^{5/3}$. The relatively weak dependence of B_m on z means that in comparison with M_m and Q_m , the wave is more prominent in B_m . To within a constant rescaling factor, the longitudinal dependence of the integrals does not appear to alter significantly with respect to time. Although the main focus of this study is on the integrals Q_m, M_m and B_m , we note for future reference that the pressure integral $\beta_p M_m$ warrants further attention. As one might expect, the dimensionless pressure integral β_p increases at the leading edge of the travelling wave and is therefore expected to influence the dynamics of

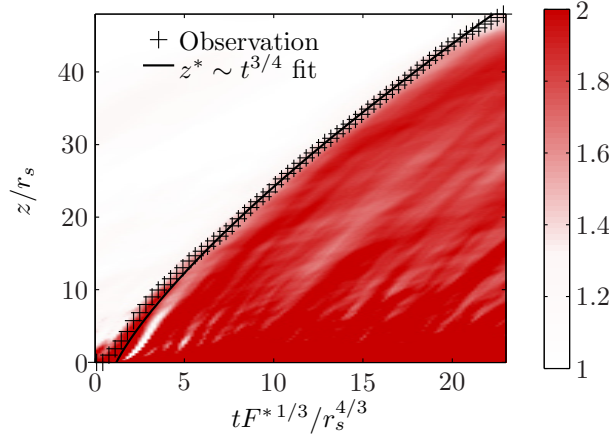


FIGURE 8.5. Normalised buoyancy $b_m(z, t)/b_{m0}(z)$ in the plume, where b_{m0} is the steady-state characteristic buoyancy, in addition to points denoting the location of the travelling wave. The crosses correspond to the observed position of the front, whilst the line denotes a best fit to the front position, of the form $z^* - z_v \propto (t - t_v)^{3/4}$.

the wave. Further understanding of this aspect of the flow would require a detailed analysis of the lateral components of the turbulence kinetic energy.

Rather than B_m, M_m and Q_m , it is perhaps the derived quantities $r_m \stackrel{\text{def}}{=} Q_m/M_m^{1/2}$ and $\Gamma \propto Q_m B_m/M_m^{3/2}$ that are more telling about the dynamics of the system. This is because in a steady state the characteristic radius r_m and the flux-balance parameter Γ , unlike Q_m, M_m and B_m , are independent of the buoyancy flux F_m and therefore exhibit the same behaviour upstream and downstream of the disturbance. In fact, without scrutinising the governing equations, it is difficult to predict whether r_m and Γ will increase or decrease in the vicinity of the propagating wave.

Figure 8.7 demonstrates that the behaviour of neither r_m nor Γ is strongly affected by the step-change in the buoyancy flux. One is inclined to conclude that r_m and Γ are slightly reduced in the vicinity of the wave, although the observed change, being not more than 15% of their steady-state values, is relatively small. It is apparent in figure 8.7 that both r_m and Γ increase slightly at the top of the domain. This is due to a small co-flow in the ambient at that level, which results in a relatively large value of r_d being used to define Q_m at the top of the domain (see §4.3.2). The corresponding increase in Q_m is evident in figure 8.1(a).

Based on the data displayed in figures 8.6 and 8.7, and the findings of chapters 5 and 6, one might suppose that the wave evolves in a self-similar manner such that its dependence on z is independent of t , to within a constant factor. The

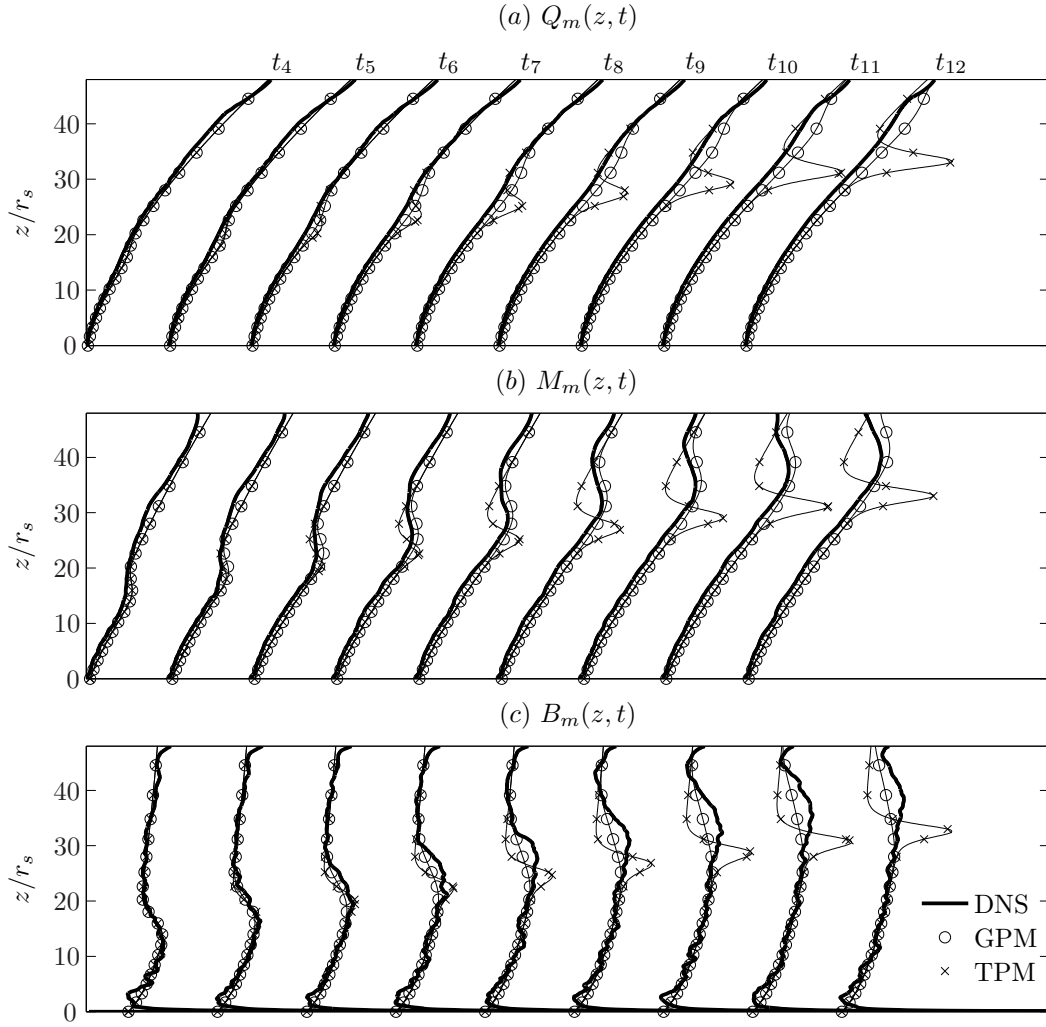


FIGURE 8.6. Model prediction of (a) volume flux Q_m , (b) momentum flux M_m , and (c) integral buoyancy B_m at times $t_n \approx 1.8\tau_s n$. GPM refers to the Gaussian plume model described in this thesis, whilst TPM refers to the top-hat plume model described by Scase & Hewitt (2012).

buoyancy flux profiles displayed in figure 8.4(a) support this view for sufficiently large times. For small times (see $t_2 \dots t_4$) we observe that the disturbance in the buoyancy flux is oscillatory and contains a local peak at approximately $z/r_s = 9$ that breaks down before t_2 and appears to result in an abrupt increase in the longitudinal extent of the wave. Consequently, between t_3 and t_6 the wave appears to become steeper, presumably because it is adjusting to a far-field equilibrium. The behaviour is evidently associated with the near field, because at larger times the disturbance is approximately monotonic in the vicinity of the leading edge of the wave, and appears to spread and converge towards a self-similar form. Having

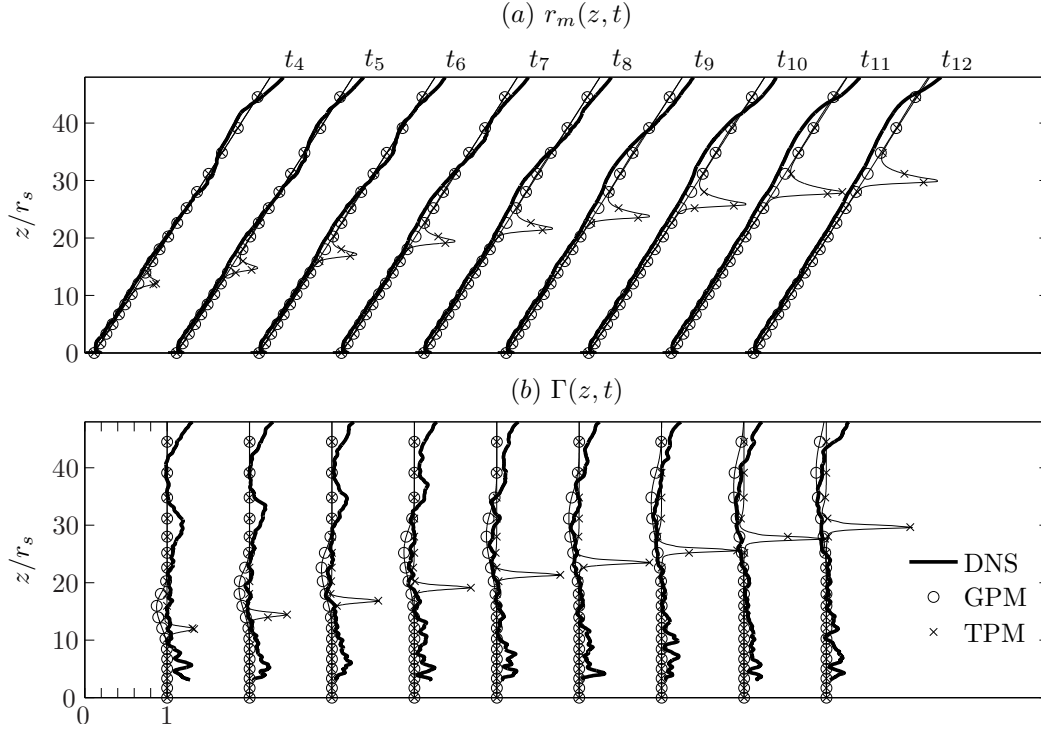


FIGURE 8.7. Model prediction of (a) the characteristic radius r_m (b) the flux-balance parameter Γ at times $t_n \approx 1.8\tau_s n$. GPM refers to the Gaussian plume model described in this thesis, whilst TPM refers to the top-hat plume model described by Scase & Hewitt (2012). Note that the profiles in (b) are separated by a distance of 1 unit, as indicated by the scale in the bottom-left corner of the figure.

established that $z^* \sim t^{3/4}$ for the process to be self-similar it is necessary that the longitudinal extend of the front also scales according to $t^{3/4}$. In figure 8.4(b) we rescale the longitudinal coordinate using the observed front position and observe an approximate collapse of the data, and therefore self-similarity, for $t \geq t_6$.

8.5 Unsteady plume properties

To understand the leading-order role played by the profile coefficients in (8.6)-(8.8) we will start by assuming that they are constants. Consideration of the possibility that, for example, the dimensionless buoyancy flux θ_m changes in the vicinity of the front is postponed until §8.6. In this section we refrain from attributing particular values to the profile coefficients. Instead, we aim to understand the way in which the properties of the unsteady plume equations (8.6)-(8.8) depend on the profile coefficients. In §8.5.1 we demonstrate that the generalised unsteady plume equa-

tions are hyperbolic and in §8.5.2 we discuss the behaviour of the characteristic radius r_m and the flux-balance parameter Γ . In §8.5.3 we examine the structure of travelling waves in an unsteady plume, before deriving similarity solutions in §8.5.4 and analysing the response of a plume to harmonic source perturbations in §8.5.5.

8.5.1 The hyperbolic system

A logical starting point for understanding the leading-order physics associated with the system (8.6)-(8.8) is to analyse its characteristic curves. Without making any assumption about the value of the unknown profile coefficients, the unsteady plume equations can be expressed as

$$\left\{ \frac{\partial}{\partial t} + \begin{bmatrix} 0 & \beta_g & 0 \\ -\gamma_g \frac{M_m^2}{Q_m^2} & \gamma_g \frac{2M_m}{Q_m} & 0 \\ -\theta_g \frac{B_m M_m}{Q_m^2} & \theta_g \frac{B_m}{Q_m} & \theta_g \frac{M_m}{Q_m} \end{bmatrix} \frac{\partial}{\partial z} \right\} \begin{pmatrix} Q_m \\ M_m \\ B_m \end{pmatrix} = \mathbf{R}. \quad (8.24)$$

where $\mathbf{R}$ consists of the right-hand-side sink or source terms appearing in (8.6)-(8.8). Characteristic curves of the system satisfy the following relation

$$\det \begin{bmatrix} -\lambda^* & \beta_g \frac{Q_m}{M_m} & 0 \\ -\gamma_g \frac{M_m}{Q_m} & 2\gamma_g - \lambda^* & 0 \\ -\theta_g \frac{B_m}{Q_m} & \theta_g \frac{B_m}{M_m} & \theta_g - \lambda^* \end{bmatrix} = 0, \quad (8.25)$$

where λ^* is a dimensionless velocity defined by

$$\lambda^* \stackrel{\text{def}}{=} -\frac{\partial_t \Phi}{\partial_z \Phi} \frac{Q_m}{M_m} = \frac{dz}{dt} \frac{Q_m}{M_m}, \quad (8.26)$$

and $\Phi(z, t)$ is an invariant along the characteristic curves. The relation (8.25) implies that

$$\lambda^* = \begin{cases} \lambda_1 = \gamma_g + \sqrt{\gamma_g^2 - \gamma_g \beta_g}, \\ \lambda_2 = \theta_g, \\ \lambda_3 = \gamma_g - \sqrt{\gamma_g^2 - \gamma_g \beta_g}. \end{cases} \quad (8.27)$$

To date, the assumption of top-hat velocity profiles and the omission of turbulence has resulted in the unsteady plume equations being regarded as a parabolic system (Scase *et al.*, 2009). However, use of the generalised framework described in §6.2.2 and §8.2 reveals that in general the unsteady plume equations comprise a hyperbolic system. Evident from (8.27) is that in the ‘top-hat’ limit $\gamma_g, \beta_g, \theta_g \rightarrow 1$, $\lambda^* = 1$, and the characteristic curves collapse onto a single family propagating at the local characteristic velocity. In that case the system cannot be decomposed using linearly independent eigenvectors and should therefore be regarded as parabolic (see Whitham, 1974). The top-hat unsteady plume formulation is therefore a degenerate case amongst a wide variety of possible alternatives, each possessing quite different dynamical properties. One should also note that the top-hat formulation represents a non-physical limit in the sense that a discontinuous mean velocity field is not realisable in a real turbulent plume.

Note that a leading-order representation of a Gaussian unsteady plume is obtained by neglecting turbulent transport ($\beta_g = \gamma_g = 0$) and using the fact that $\gamma_g = \gamma_m = 4/3$ for Gaussian profiles. Equation (8.27) shows that for Gaussian profiles $\lambda_1 = 2$, which implies that the fastest characteristic propagates at twice the local characteristic velocity, as discussed in §6.5.1. Under certain circumstances, which will be described in §8.5.3, one can assume that the propagation of a wave resulting from a step change in the buoyancy flux follows the fastest characteristic curve. Therefore, the propagation velocity of the wave deduced from purely theoretical arguments, is in close agreement with the observations reported in §8.4, for which $\lambda^* \approx 1.8$. We tentatively attribute the fact that the observed propagation velocity is slightly less than one would expect to find in a Gaussian plume ($\lambda^* \approx 2.0$) to pressure, which typically leads to a reduction in both β_g and γ_g .

8.5.2 The behaviour of the plume radius and flux-balance parameter

Following §7.5.1, it is useful to consider how the response of the area of the plume is affected by the profile coefficients. Substitution of the relation $Q_m = r_m M_m^{1/2}$ into (8.13), reveals that

$$\frac{1}{M_m^{1/2}} \frac{\partial r_m^2}{\partial t} = -\frac{\delta_g}{\gamma_g} + \frac{2r_m}{M_m} \left(\frac{3}{4} - \frac{\beta_g}{\gamma_g} \right) \frac{\partial M_m}{\partial z} + \frac{16\alpha_0}{5} \left(\frac{\beta_g}{\gamma_g} - \frac{\beta_g \theta_m}{\gamma_g} \right) \Gamma - \frac{\partial r_m}{\partial z}. \quad (8.28)$$

We will assume that at a particular time the plume is straight-sided (conical), such that $\partial_z r_m = 6\alpha_0/5$ in the right-hand-side of (8.28) and therefore, using (8.17),

$$\frac{1}{M_m^{1/2}} \frac{\partial r_m^2}{\partial t} = \frac{2r_m}{M_m} \left(\frac{3}{4} - \frac{\beta_g}{\gamma_g} \right) \frac{\partial M_m}{\partial z} - \frac{2\delta_g}{\gamma_g} \left(1 - \left(\frac{\theta_m + \Gamma(\theta_m - 1)}{2\theta_m - 3\gamma_g/(4\beta_g)} \right) \right). \quad (8.29)$$

Inspection of (8.29) implies that

$$\frac{1}{M_m^{1/2}} \frac{\partial r_m^2}{\partial t} = 0 \quad \forall \left(\Gamma, \frac{r_m}{M_m} \frac{\partial M_m}{\partial z} \right) \quad \text{iff} \quad \frac{3\gamma_g}{4\beta_g} = 1 \quad \text{and} \quad \theta_m = 1. \quad (8.30)$$

When the profile coefficients are such that $\gamma_g/\beta_g = 4/3$, the plume remains straight-sided for all time, provided that it has a constant source area. The difference between straight-sidedness in jets (see §7.5.1) and plumes is that in plumes it is necessary that an independent condition on θ_m is satisfied. Indeed, θ_m places an additional control on the behaviour of $\partial_t M_m$ relative to $\partial_t Q_m$ in (8.6)-(8.7)) and therefore affects the behaviour of Q_m^2/M_m . Equation (8.30) is useful because it allows one to relate the ‘internal’ properties of a plume, such as γ_g , to an ‘external’ observable such as r_m at the integral level. This is in contrast to steady-state plumes, for which it is not possible to distinguish between a Gaussian or a top-hat distribution of velocity by observing Q_m , M_m and F_m .

Having addressed how the area of an unsteady plume is affected by the value of its profile coefficients, it is natural to consider the behaviour of the flux-balance parameter Γ . If we assume that the conditions for straight-sidedness (namely $\gamma_g/\beta_g = 4/3$ and $\theta_m = 1$) are satisfied, then

$$B_m = \frac{4\beta_g M_m}{3z} \Gamma. \quad (8.31)$$

Substitution of (8.31) into (8.8) and using (8.7) one obtains,

$$\frac{\partial \Gamma}{\partial t} = 0 \quad \forall \frac{\partial M_m}{\partial z} \quad \text{iff} \quad \theta_g = \gamma_g, \quad (8.32)$$

provided that $\Gamma = 1$ at $t = 0$ and $\Gamma = 1$ at the source. Only when the dimensionless buoyancy flux is equal to the dimensionless energy flux will the plume remain pure ($\Gamma = 1$), regardless of the temporal dependence of $M_m(z, t)$. Noting that we have assumed $\theta_m = 1$ for straight-sidedness, it is only possible that $\theta_g = \gamma_g$, if a significant component of the total buoyancy flux is transported by turbulence. Table 8.1 suggests that $\theta_g \approx 1.2$, whilst $\gamma_g \approx 1.4$, and therefore we would not expect for the plume to remain exactly pure. However, assuming an approximate equality between γ_g and θ_g will prove to be a useful idealisation when we discuss modelling in §8.6.

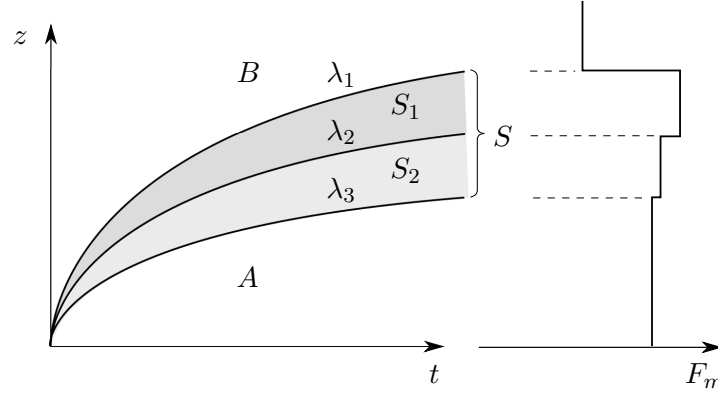


FIGURE 8.8. The three distinct characteristic curves in a generalised formulation of the unsteady plume equations. Regions A and B denote points in (z, t) space upstream and downstream of a wave, whilst regions S_1 and S_2 denote regions of the wave. The depicted behaviour of F_m in the wave is for schematic purposes only.

8.5.3 The structure of waves in a plume

The way in which the properties of the wave depend on the profile constants can be established by examining the behaviour of the invariants associated with (8.24). The left eigenvectors corresponding to the eigenvalues (8.27) are

$$L = \begin{bmatrix} -\frac{1}{(1+\phi)} \frac{M_m}{Q_m} & 1 & 0 \\ \frac{c_1 B_m}{Q_m} & -\frac{c_2 B_m}{M_m} & 1 \\ -\frac{1}{(1-\phi)} \frac{M_m}{Q_m} & 1 & 0 \end{bmatrix}, \quad (8.33)$$

where $\phi \stackrel{\text{def}}{=} \sqrt{1 - \beta_g/\gamma_g}$,

$$c_1 \stackrel{\text{def}}{=} \frac{\theta_g(\gamma_g - \theta_g)}{2(\beta_g\gamma_g - 2\gamma_g\theta_g + \theta_g^2)}, \quad \text{and} \quad c_2 \stackrel{\text{def}}{=} \frac{\theta_g(\beta_g - \theta_g)}{\beta_g\gamma_g - 2\gamma_g\theta_g + \theta_g^2}. \quad (8.34)$$

In general L is invertible, which means that the unsteady plume system (8.24) can be decomposed into a system of ordinary differential equations. Each differential equation corresponds to the derivative of a quasi-invariant quantity, Y_1, Y_2 or Y_3 , along a characteristic curve. In particular, introducing the integrating factor

$Q_m^{c_1}/M_m^{c_2}$,

$$dY_2 = c_1 \frac{B_m Q_m^{c_1-1}}{M_m^{c_2}} dQ_m - c_2 \frac{B_m Q_m^{c_1}}{M_m^{c_2+1}} dM_m + \frac{Q_m^{c_1}}{M_m^{c_2}} dB_m, \quad (8.35)$$

hence

$$Y_2 = \frac{B_m Q_m^{c_1}}{M_m^{c_2}}. \quad (8.36)$$

The remaining invariants are identical to those in an unsteady jet (see §7.6.2):

$$Y_1 = \frac{M_m}{Q_m^{1/(1+\phi)}}, \quad Y_3 = \frac{M_m}{Q_m^{1/(1-\phi)}}. \quad (8.37)$$

When the system is forced on the right-hand-side due to the production of turbulence kinetic energy and buoyancy, $\mathbf{R} \neq 0$ and the ‘invariants’ need not be constant along characteristic curves. It is nevertheless instructive to consider the homogeneous problem, for which $\mathbf{R} = 0$, such that Y_1 , Y_2 and Y_3 are constant along characteristics and therefore determine the behaviour of the original dependent variables Q_m , M_m and B_m .

the value of Y_1 , Y_2 and Y_3 at a given point in the domain can be determined by tracing their respective characteristic curves to the source at $z = 0$. We consider the case for which $\lambda_3 < \lambda_2 < \lambda_1$. We denote with S , S_1 and S_2 the regions $[\lambda_3, \lambda_1]$, $[\lambda_2, \lambda_1]$ and $[\lambda_3, \lambda_2]$, respectively, as indicated in figure 8.8. Hence

$$\frac{M_m^S}{M_m^A} = \left(\frac{M_m^A}{M_m^B} \right)^{(1-\phi)/(2\phi)} \left(\frac{Q_m^B}{Q_m^A} \right)^{1/(2\phi)}, \quad (8.38)$$

$$\frac{Q_m^S}{Q_m^A} = \left(\frac{M_m^A}{M_m^B} \right)^{(1/\phi-\phi)/2} \left(\frac{Q_m^B}{Q_m^A} \right)^{1/(2\phi)+1/2}, \quad (8.39)$$

and

$$\frac{B_m^{S_1}}{B_m^B} = \left(\frac{Q_m^B}{Q_m^S} \right)^{c_1} \left(\frac{M_m^S}{M_m^B} \right)^{c_2}, \quad \frac{B_m^{S_2}}{B_m^A} = \left(\frac{Q_m^A}{Q_m^S} \right)^{c_1} \left(\frac{M_m^S}{M_m^A} \right)^{c_2}. \quad (8.40)$$

The values of M_m and Q_m in the wave are not affected by the location of the second characteristic, which is associated with λ_2 . The integral buoyancy B_m , however, takes distinct values in S_1 and S_2 according to the position of the second characteristic. If it is assumed that both the source radius is fixed and the flux-balance parameter at the source is equal to unity, we may say that $Q_m^A \propto F_m^{A1/3}$, $Q_m^B \propto F_m^{B1/3}$, $M_m^A \propto F_m^{A2/3}$ and $M_m^B \propto F_m^{B2/3}$:

$$\frac{M_m^S}{M_m^A} = \left(\frac{F_m^A}{F_m^B} \right)^{1/(6\phi)-1/3}, \quad \frac{Q_m^S}{Q_m^A} = \left(\frac{F_m^A}{F_m^B} \right)^{1/(6\phi)-\phi/3-1/6}. \quad (8.41)$$

Therefore, following a step change in the source buoyancy flux $F_s^B \mapsto F_s^A$, the parameter $\phi \stackrel{\text{def}}{=} \sqrt{1 - \beta_g/\gamma_g}$, which depends on γ_g and β_g determines the behaviour of Q_m^S and M_m^S . In particular, for a given F_m^A and F_m^B , ϕ will determine the step-change in the plume velocity w_m either side of each characteristic curve and therefore whether each characteristic is associated with shock or rarefaction behaviour. For Gaussian plumes or, more generally, when $\gamma_g/\beta_g = 4/3$ (which implies that $\phi = 1/2$), the behaviour of the wave is particularly simple, because $Q_m^S = Q_m^A$ and $M_m^S = M_m^A$. In that case the leading characteristic associated with λ_1 is a rarefaction wave or a compression wave, according to whether $F_m^A < F_m^B$ or $F_m^A > F_m^B$, respectively. Furthermore, in agreement with the deductions made in the previous section, when $\phi = 1/2$ the plume radius r_m does not deviate from its steady-state value in the region S , although, as noted previously, for this to be the case it is necessary that $\theta_m = 1$. Since these properties are inherited from unsteady jets, the reader is referred to chapter 7 for further details. Here we will focus on the additional properties resulting from the third characteristic.

Using (8.40) one can determine how the flux-balance parameter Γ behaves in regions S_1 and S_2 :

$$\begin{aligned} \frac{\Gamma^{S_1}}{\Gamma^B} &= \left(\frac{Q_m^B}{Q_m^S} \right)^{c_1-1} \left(\frac{M_m^S}{M_m^B} \right)^{c_2-3/2} = \left(\frac{F_m^A}{F_m^B} \right)^{n_1}, \\ \frac{\Gamma^{S_2}}{\Gamma^A} &= \left(\frac{Q_m^A}{Q_m^S} \right)^{c_1-1} \left(\frac{M_m^S}{M_m^A} \right)^{c_2-3/2} = \left(\frac{F_m^A}{F_m^B} \right)^{n_2}, \end{aligned} \quad (8.42)$$

where

$$n_k \stackrel{\text{def}}{=} \left(c_2 - \frac{3}{2} \right) \left(\frac{1}{6\phi} + \frac{(-1)^{k-1}}{3} \right) - (c_1 - 1) \left(\frac{1}{6\phi} - \frac{\phi}{3} + \frac{(-1)^{k-1}}{6} \right), \quad k = 1, 2. \quad (8.43)$$

In the flux-balance parameter Γ , unlike Q_m and M_m , the characteristic associated with $\lambda_2 = \theta_g$, can result in a discontinuity. In general, distinct values of Γ are found in regions S_1 and S_2 and depend on γ_g , β_g and θ_g .

In a straight-sided plume ($\gamma_g/\beta_g = 4/3$, $\phi = 1/2$), equation (8.43) reduces to

$$n_1 = \frac{2c_2}{3} - \frac{c_1}{3} - \frac{4}{3}, \quad n_2 = 0. \quad (8.44)$$

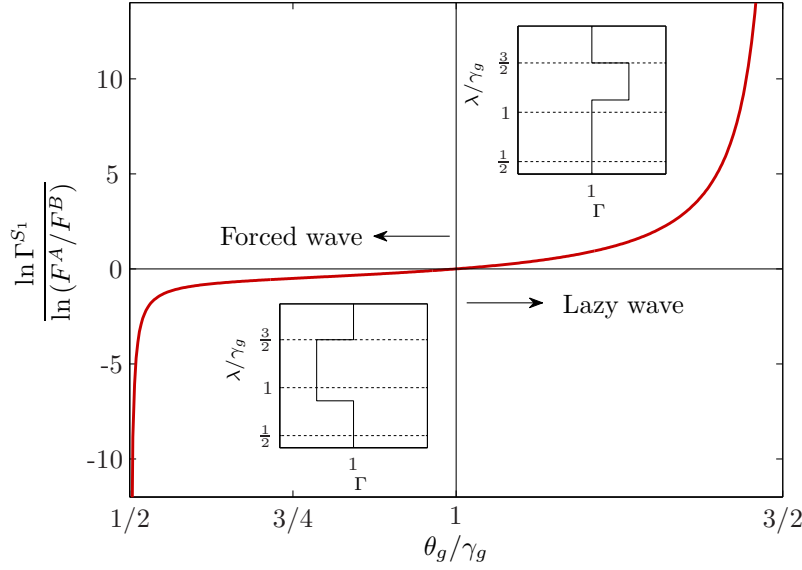


FIGURE 8.9. Wave structure in a straight-sided plume ($\gamma_g/\beta_g = 4/3$). The special case for which $\Gamma = 1$ in the wave corresponds to $\theta_g = \gamma_g$.

Therefore, substituting for c_1 and c_2 using (8.34) and assuming $\Gamma^A = \Gamma^B = 1$,

$$\Gamma^{S_1} \left(\frac{\theta_g}{\gamma_g} \right) = -\frac{2}{3} \left(\frac{5(\theta_g/\gamma_g)^2 - 9\theta_g/\gamma_g + 3}{4(\theta_g/\gamma_g)^2 - 8\theta_g/\gamma_g + 3} \right), \quad \Gamma^{S_2}(\theta_g) = 1, \quad (8.45a,b)$$

and the behaviour of Γ over the wave depends exclusively on the ratio of the dimensionless buoyancy flux θ_g to the dimensionless energy flux γ_g . Whilst the plume remains pure in S_2 (8.45b), the value of Γ in S_1 (8.45a) depends on the location of the second characteristic curve. If $\theta_g/\gamma_g < 1$, there is a deficit of buoyancy in S_1 and that part of the plume becomes forced ($\Gamma^{S_1} < 1$). Conversely, if $\theta_g/\gamma_g > 1$ additional buoyancy enters S_1 and that part of the plume becomes lazy. These results are consistent with the condition established in §8.5.2 that $\theta_g/\gamma_g = 1$ in order for the plume to remain pure and are illustrated in figure 8.9.

8.5.4 Time-dependent similarity solutions

Scase *et al.* (2006b) identified time-dependent similarity solutions of the unsteady plume equations under the assumption of a top-hat velocity profile. The work was motivated by the fact that, in the absence of a conserved quantity, such solutions provide the most natural scaling for unsteady plumes. Using the framework described in §8.2 it is useful to re-derive the similarity solutions, without making an assumption about the velocity or buoyancy profile of the plume. We will demon-

strate that the solutions support the findings reported in §§8.5.2-8.5.3 and reduce to the solutions identified by Scase *et al.* (2006*b*) as a special case.

For simplicity, we will neglect the turbulent transport of momentum and energy, setting $\beta_g = 1$ and $\gamma_g = \gamma_m$ and focus on the leading-order contribution to turbulence production, i.e. $\delta_g = \delta_m$. Therefore, we seek a similarity solution to the unsteady transport equations for momentum (8.6) and mean energy (8.7):

$$\left. \begin{aligned} \frac{\partial Q_m}{\partial t} + \frac{\partial M_m}{\partial z} &= B_m, \\ \frac{\partial M_m}{\partial t} + \frac{\partial}{\partial z} \left(\gamma_m \frac{M_m^2}{Q_m} \right) &= \alpha_0 \left(\frac{6\gamma_m}{5} - \frac{16\theta_m}{5} \right) \frac{M_m^{5/2}}{Q_m^2} + 2\theta_m \frac{M_m B_m}{Q_m}, \\ \frac{\partial B_m}{\partial t} + \frac{\partial}{\partial z} \left(\theta_g \frac{M_m B_m}{Q_m} \right) &= 0, \end{aligned} \right\} \quad (8.46)$$

in which we have expressed δ_m in terms of α_0 using (8.17), for comparison with the findings of Scase *et al.* (2006*b*). Power-law similarity solutions of (8.46) have the form

$$Q_m(z, t) = c_1 \frac{z^3}{t}, \quad M_m(z, t) = c_2 \frac{z^4}{t^2}, \quad B_m(z, t) = c_3 \frac{z^3}{t^2}. \quad (8.47a, b, c)$$

Solutions for which $c_3 = 0$ correspond to a jet, and were described in §7.6.3. Here we focus on solutions for which the buoyancy is non-zero $c_3 \neq 0$. In particular

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{\alpha_0^2 (8\theta_m - 3\gamma_m^2)}{\theta_g^2 (4\theta_g + 8\theta_m - 5\gamma_m - 4\theta_g \theta_m)^2} \begin{pmatrix} 2\theta_g \\ 1 \\ 4 - 2\theta_g \end{pmatrix}. \quad (8.48)$$

Noting that the radius $r_m \stackrel{\text{def}}{=} Q_m/M_m^{1/2}$ and velocity $w_m \stackrel{\text{def}}{=} M_m/Q_m$, one finds

$$r_m(z) = \left(\frac{8\theta_m - 3\gamma_m}{12\theta_g(1 - \theta_m) + 24\theta_m - 15\gamma_m} \right) \frac{6\alpha_0 z}{5}, \quad w_m(z, t) = \frac{z}{2\theta_g t}, \quad (8.49a, b)$$

respectively. The velocity w_m is independent of the entrainment coefficient, the assumed radial dependence of both the longitudinal velocity and the buoyancy, and depends only on the total dimensionless buoyancy flux θ_g , in addition to z and t . In contrast, the radius r_m is independent of t but is affected by the entrainment coefficient and the assumed radial dependences, which determine γ_m and θ_m . In

particular the radius is given by

$$r_m(z, t | \gamma_m, \theta_m = 1) = \frac{2\alpha_0 z}{3}, \quad (8.50)$$

$$r_m(z, t | \gamma_m = 4/3, \theta_m = 1) = \frac{6\alpha_0 z}{5}, \quad (8.51)$$

for top-hat and Gaussian profiles, respectively. Note that the radius of the plume is independent of the dimensionless gross buoyancy flux θ_g (inclusive of turbulent transport) when $\theta_m = 1$. Equation (8.49b), for $\theta_g = 1$ and (8.50) are identical to the solutions obtained by Scase *et al.* (2006b). Consistent with the analysis of §8.5.2, equation (8.51) indicates that the spreading rate of the unsteady Gaussian plume is identical to that of the steady-state. Moreover, with $\beta_g = 1$, the flux-balance parameter

$$\Gamma \stackrel{\text{def}}{=} \frac{5Q_m B_m}{8\alpha_0 M_m^{3/2}} = (2 - \theta_g) \frac{5r_m}{4\alpha_0 z} \begin{cases} = 5/6, & \text{(top-hat)} \\ > 1 & \text{if } 0 < \theta_g < 4/3, \quad \text{(Gaussian)} \\ = 1 & \text{if } \theta_g = 4/3, \quad \dots \\ < 1 & \text{if } 4/3 < \theta_g < 2, \quad \dots \end{cases} \quad (8.52)$$

As discovered by Scase *et al.* (2006b), time-dependent self-similar top-hat plumes are forced ($\Gamma = 5/6 < 1$). Gaussian plumes, on the other hand, admit forced, pure and lazy behaviour according to the value of θ_g .

As suggested in §7.6.3, given (8.49a), the difference between (8.50) and (8.51) can be accounted for with an entrainment coefficient of the form

$$\alpha \stackrel{\text{def}}{=} \left(\frac{8\theta_m - 3\gamma_m}{12\theta_g(1 - \theta_m) + 24\theta_m - 15\gamma_m} \right) \frac{9\alpha_0}{5}, \quad (8.53)$$

and therefore

$$r_m = \frac{2\alpha z}{3}. \quad (8.54)$$

In fact, (8.53) is identical to the generalised entrainment coefficient in (8.13) when $Q_m = c_1 z^3/t$, $M_m = c_2 z^4/t^2$ and $B_m = c_3 z^3/t^2$. In time-dependent similarity solutions for the top-hat plume $\alpha = \alpha_0$. For the Gaussian plume, however, the entrainment coefficient $\alpha = 9\alpha_0/5$ increases relative to the steady-state, in such a way that the radius of the plume $r_m = 6\alpha_0 z/5$ retains its steady-state form.

8.5.5 Response to source perturbations

The final property of the generalised unsteady plume equations that we consider is the response of a plume to harmonic perturbations applied to the source. Indeed, from a modelling perspective it is necessary to understand whether an unsteady plume model is stable with respect to source perturbations. Scase & Hewitt (2012) showed that top-hat unsteady plume models are ill-posed because they admit the unbounded (exponential) growth of high-frequency source perturbations. Using the framework described in this thesis, it is possible to adopt a more general approach and analyse the response of the unsteady plume equations without making any assumptions about the velocity profile, turbulent transport or pressure. It is consequently possible to establish whether a given plume model is well-posed. More generally, one can establish a relation between the dimensionless profile coefficients that ensures well-posedness and understand how this relates to the underlying physics of an unsteady plume.

Using the dimensionless variables

$$\zeta \stackrel{\text{def}}{=} \frac{3}{4} \frac{z\sigma}{w_{m0}}, \quad \tau \stackrel{\text{def}}{=} \sigma t, \quad (8.55)$$

where the steady-state velocity

$$w_{m0} \stackrel{\text{def}}{=} \frac{5}{6\alpha_0} \left(\frac{9\alpha_0}{10} \right)^{1/3} \left(\frac{F_s}{\beta_g \theta_m} \right)^{1/3} z^{-1/3}, \quad (8.56)$$

we consider a harmonic source perturbation of amplitude ϵ and seek solutions of the form

$$\left. \begin{aligned} Q_m &= Q_{m0}(1 + \epsilon \mathcal{Q}_1 \dots), \\ M_m &= M_{m0}(1 + \epsilon \mathcal{M}_1 \dots), \\ B_m &= B_{m0}(1 + \epsilon \mathcal{B}_1 \dots). \end{aligned} \right\} \quad (8.57)$$

At $O(\epsilon)$ the unsteady plume equations can be expressed as At $O(\epsilon)$ the unsteady plume equations can be expressed as

$$\left(\frac{\partial}{\partial \tau} + \begin{bmatrix} 0 & \beta_g & 0 \\ -\gamma_g & 2\gamma_g & 0 \\ -\theta_g & \theta_g & \theta_g \end{bmatrix} \frac{\partial}{\partial \zeta} - \frac{1}{\zeta} \begin{bmatrix} 0 & -\beta_g & R_{13} \\ R_{21} & R_{22} & R_{23} \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{pmatrix} \mathcal{Q}_1 \\ \mathcal{M}_1 \\ \mathcal{B}_1 \end{pmatrix} = 0, \quad (8.58)$$

where

$$\begin{aligned} R_{13} &\stackrel{\text{def}}{=} (\beta_g^{2/3}/\theta_m^{1/3}), & R_{21} &\stackrel{\text{def}}{=} -\frac{3}{4}\gamma_g - 2(\theta_m\beta_g)^{2/3} + 4\beta_g\theta_m, \\ R_{22} &\stackrel{\text{def}}{=} \frac{3}{8}\gamma_g + 2(\theta_m\beta_g)^{2/3} - 5\beta_g\theta_m, & R_{23} &\stackrel{\text{def}}{=} 2(\theta_m\beta_g)^{2/3}. \end{aligned} \quad (8.59)$$

to eliminate δ_g .

Here, for simplicity and definiteness, we restrict our attention to a simple and realistic set of parameter values for which the source perturbation problem has an analytical solution. In particular, we will assume Gaussian velocity profiles ($\gamma_m = 4/3$) and neglect the turbulent transport of momentum ($\beta_g = 1$ and $\gamma_g = 4/3$). We address the way in which θ_g affects the downstream development of source perturbations. The assumed values of β_g , θ_m and γ_g ensure that the plume remains straight-sided (see §8.5.2) such that $\mathcal{M}_1 = 2\mathcal{B}_1$ for the linearised perturbation, and thus (8.58) simplifies to

$$\left(\frac{\partial}{\partial \tau} + \begin{bmatrix} 2 & 0 \\ \theta_g/2 & \theta_g \end{bmatrix} \frac{\partial}{\partial \zeta} - \frac{1}{\zeta} \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \right) \begin{pmatrix} \mathcal{M}_1 \\ \mathcal{B}_1 \end{pmatrix} = 0. \quad (8.60)$$

We assume an oscillatory solution of the form

$$\begin{pmatrix} \mathcal{M}_1 \\ \mathcal{B}_1 \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{M}}_1(\zeta) \\ \hat{\mathcal{B}}_1(\zeta) \end{pmatrix} \exp(i\tau). \quad (8.61)$$

Substitution of (8.61) into (8.60) and elimination of $\hat{\mathcal{B}}_1$ results in

$$\frac{d^2 \hat{\mathcal{M}}_1}{d\zeta^2} + \left(\frac{5}{2\zeta} + i \left(\frac{1}{2} + \frac{1}{\theta_g} \right) \right) \frac{d\hat{\mathcal{M}}_1}{d\zeta} - \left(\frac{1}{2\theta_g} - i \left(\frac{1}{\theta_g\zeta} - \frac{1}{2\zeta} \right) \right) \hat{\mathcal{M}}_1 = 0, \quad (8.62)$$

whose solution is

$$\hat{\mathcal{M}}_1(\zeta) = \left(c_1 M \left(a, \frac{5}{2}, i \frac{\theta_g - 2}{2\theta_g} \zeta \right) + c_2 U \left(a, \frac{5}{2}, i \frac{\theta_g - 2}{2\theta_g} \zeta \right) \right) \exp \left(-i \frac{\zeta}{2} \right), \quad (8.63)$$

where

$$a \stackrel{\text{def}}{=} \frac{3\theta_g - 4}{2\theta_g - 4}, \quad (8.64)$$

and the hypergeometric functions M and U are independent solutions of (8.62) (Abramowitz & Stegun, 1970, p.504). At the source (8.63) implies that $\mathcal{B}_1(0) = \mathcal{M}_1(0) \forall c_1, c_2$, which means that to leading order $\Gamma(0) = 1$. To ensure finite $\hat{\mathcal{M}}_1(0)$, we set $c_2 = 0$ and assume that $c_1 = 1$ without loss of generality. The asymptotic

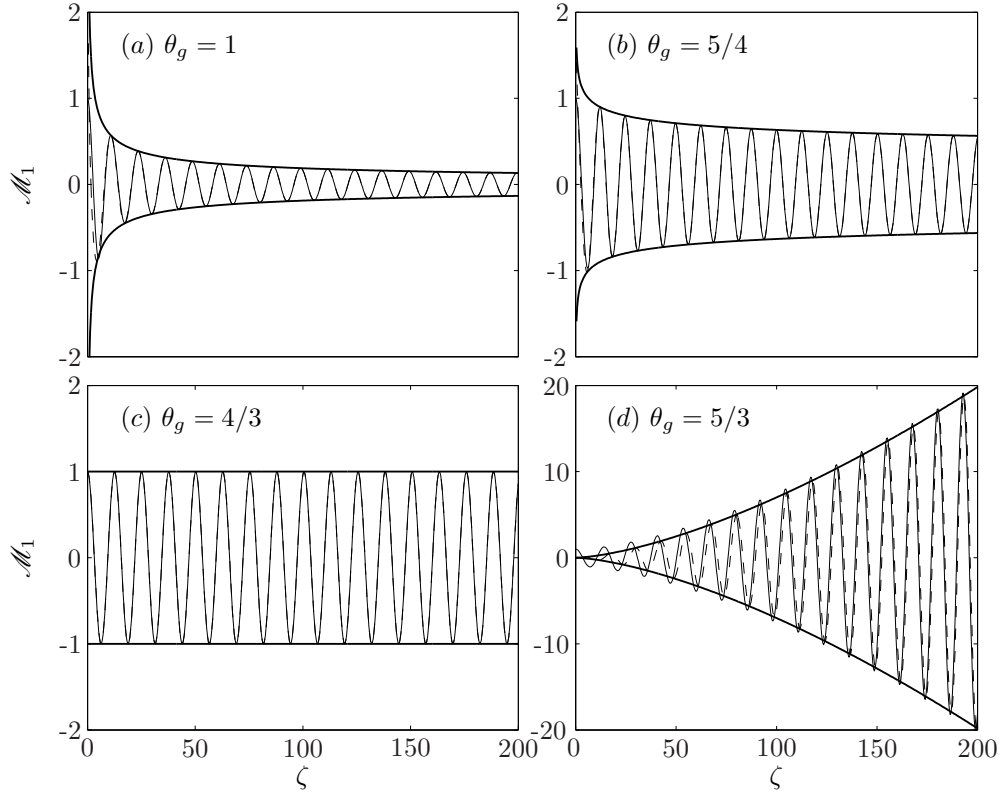


FIGURE 8.10. Stability of the system for $\gamma_m = 4/3$ in response to source perturbations with respect to the dimensionless longitudinal coordinate $\zeta \propto z^{4/3}\sigma$. The thin solid line corresponds to the exact solution of the linearised problem (8.63) and the thin dashed line to the asymptotic solution (8.65). The thick line denotes the envelope of the asymptotic solution. The dimensionless buoyancy flux $\theta_g = 4/3$ is the special case for which perturbations exhibit neutral growth. For models employing realistic values $\theta_g \leq 4/3$, the system is well-posed, in the sense that source perturbations are bounded and it is possible to obtain convergent numerical approximations.

expansion of (8.63) for $\zeta \rightarrow \infty$ is

$$\hat{\mathcal{M}}_1(\zeta) \sim \frac{3\sqrt{\pi}}{4\Gamma(5/2 - a)} \left(\frac{2\theta_g}{\theta_g - 2} \right)^a \exp\left(-i\frac{\zeta}{2}\right) \zeta^{-a}, \quad (8.65)$$

where Γ is the Gamma function. The exponent $a(\theta_g)$ therefore determines the growth rate of the perturbations as illustrated in figure 8.10. When $0 < \theta_g < 4/3$ (a forced wave, according to §8.5.3) the growth rate is negative, but when $4/3 < \theta_g < 2$ (a lazy wave) the growth rate is positive. Between growth and decay of source perturbations lies the special behaviour associated with $\theta_g = 4/3$, for which the plume is pure and exhibits a neutral response to source perturbations. Noting that

θ_g is likely to be less than $4/3$ in practice, this analysis demonstrates that the generalised unsteady plume equations are well-posed for physically realistic values of the dimensionless profile coefficients.

8.6 A Gaussian unsteady plume model

In this section we propose a model for unsteady plumes. Here, we assign values to the dimensionless profile coefficients that are appropriate to Gaussian plumes, thereby closing the system (8.6)-(8.8). In §8.6.1 we consider the possibility that the profile coefficients change in the vicinity of a step-change in the buoyancy flux, in analogy with the shear-flow dispersion that was investigated in chapter 5. In §8.6.2 we compare the model to the DNS results and in §8.6.3 we propose a simpler form of the model by invoking additional assumptions.

8.6.1 Higher-order effects: shear-flow dispersion

In the previous section we established the response of an unsteady *self-similar* plume by assuming the the plume's profile coefficients are constant. In practice, however, the longitudinal gradients that are produced by unsteady changes in the plume force a local departure from self-similarity (see §5.4.2 and §6.5.2). Taylor (1953) analysed an equivalent situation in pipe flow, showing that in the vicinity of a step increase in the radially-averaged scalar concentration there exists a deformation in the otherwise radially uniform concentration field. In correlation with a non-uniform velocity field, the deformation results in a local increase in the scalar flux, which to leading order satisfies a dispersive analogue of Fick's law. In chapters 5 and 6 a dispersion closure analogous to Taylor's closure was developed for scalar transport and energy transport in jets, respectively. Here we apply the dispersion closure to buoyancy transport and energy transport.

In order to focus on longitudinal changes that constitute a departure from the steady state, it is convenient to examine dimensionless quantities. For a given quantity $f_m(z, t)$, we define $\mathcal{F}_m \stackrel{\text{def}}{=} f_m/f_{m0}$, where f_{m0} is the steady-state value of f_m (cf. $\mathcal{C}_m$ in chapter 5). We can compute the dimensionless longitudinal gradient of $\mathcal{F}_m$ to quantify the departure from the steady state:

$$\frac{r_m}{\mathcal{F}_m} \frac{\partial \mathcal{F}_m}{\partial z} = \frac{r_m}{f_m} \frac{\partial f_m}{\partial z} - \frac{r_m}{f_{m0}} \frac{\partial f_{m0}}{\partial z}, \quad (8.66)$$

which is equal to zero in a steady state. In (8.66) it is convenient to *define* the

steady-state locally:

$$\frac{\partial f_{m0}}{\partial z} = \frac{6\alpha_0 n}{5} \frac{f_{m0}}{r_m}, \quad (8.67)$$

which means that the second term on the right hand side of (8.66) is equal to n , when $f_{m0} \sim z^n$. This is a sensible choice in general, because it means that departures from a steady state are defined relative to the local (in z) size of the plume rather than the size it should have with respect to a fixed virtual source. In particular, we define the dimensionless velocity and buoyancy:

$$\mathcal{U}_m \stackrel{\text{def}}{=} \frac{w_m}{w_{m0}}, \quad \mathcal{B}_m \stackrel{\text{def}}{=} \frac{b_m}{b_{m0}}. \quad (8.68)$$

Using (8.66), with $\mathcal{F}_m$ replaced with either $\mathcal{U}_m$ or $\mathcal{B}_m$, we follow the approach taken in §5.5 and propose a closure for the dimensionless buoyancy flux θ_m and the dimensionless energy flux γ_m :

$$\gamma_m = \gamma_0 - \frac{5r_m}{\alpha_0} \frac{1}{\mathcal{U}_m} \frac{\partial \mathcal{U}_m}{\partial z} \gamma_1 = \gamma_0 - 6 \left(\frac{1}{3} + \frac{5Q_m^2}{6\alpha_0 M_m^{3/2}} \frac{\partial}{\partial z} \left(\frac{M_m}{Q_m} \right) \right) \gamma_1, \quad (8.69)$$

$$\theta_m = \theta_0 - \frac{5r_m}{\alpha_0} \frac{1}{\mathcal{B}_m} \frac{\partial \mathcal{B}_m}{\partial z} \theta_1 = \theta_0 - 6 \left(\frac{5}{3} + \frac{5Q_m^3}{6\alpha_0 M_m^{3/2} B_m} \frac{\partial}{\partial z} \left(\frac{M_m B_m}{Q_m^2} \right) \right) \theta_1. \quad (8.70)$$

Note that the fractions $1/3$ and $5/3$ in (8.69) and (8.70), respectively, result from the fact that $w_{m0} \sim z^{-1/3}$ and $b_{m0} \sim z^{-5/3}$, respectively. A virtue of the dispersion closure is that the incorporation of (8.70) into (8.6)-(8.8) does not affect the steady-state solutions. In contrast, the diffusive mixing term for momentum proposed by Scase & Hewitt (2012) was not extended to buoyancy transport because it would have modified the exponents associated with the steady-state solutions. For the purposes of establishing a relative simple representation of an unsteady plume, we neglect turbulent transport and assume $\gamma_g = \gamma_m$, $\theta_g = \theta_m$ and $\beta_g = \beta_m$, which results in the following model

$$\frac{\partial Q_m}{\partial t} + \frac{\partial M_m}{\partial z} = B_m, \quad (8.71)$$

$$\frac{\partial M_m}{\partial t} + \gamma_0 \frac{\partial}{\partial z} \left(\frac{M_m^2}{Q_m} \right) = \delta_m \frac{M_m^{5/2}}{Q_m^2} + \frac{5\gamma_1}{\alpha_0} \frac{\partial}{\partial z} \left(\frac{Q_m}{M_m^{1/2} \mathcal{U}_m} \frac{\partial \mathcal{U}_m}{\partial z} \right) + 2\theta_m \frac{M_m B_m}{Q_m}, \quad (8.72)$$

$$\frac{\partial B_m}{\partial t} + \theta_0 \frac{\partial}{\partial z} \left(\frac{M_m B_m}{Q_m} \right) = \frac{5\theta_1}{\alpha_0} \frac{\partial}{\partial z} \left(\frac{Q_m}{M_m^{1/2} \mathcal{B}_m} \frac{\partial \mathcal{B}_m}{\partial z} \right). \quad (8.73)$$

Note that in the interests of brevity, θ_m on the right-hand-side of (8.72) has not been expanded and should be replaced with (8.70).

8.6.2 Comparison with DNS results

We solve the system (8.71)-(8.73) using fourth-order-accurate central differences and a fourth-order Runge-Kutta method for integration with respect to t . To adequately describe both near and far-field scales in the plume we employ a stretched computational grid, for which the ratio of the first and last computational cells $\Delta z_1/\Delta z_N$ is approximately equal to 0.1. The boundary condition imposed on the outflow face at $z = L_z$ is that the longitudinal gradient of all dependent variables is constant, and by setting $L_z = 72r_s$ we ensure that this choice has a negligible influence on the solution in those parts of the domain with which we are primarily concerned.

A significant difficulty in obtaining an accurate and informative comparison with the DNS results is to correctly account for near-field effects. As described in §8.4, at relatively small times a disproportionately large amount of mixing in the wave occurs (see t_2, t_3, t_4 in figure 8.4), which is caused by the flow development in the vicinity of the finite source. These near-field mixing effects are beyond the scope of the model developed in this work. Therefore, to obtain a fair comparison with the observed far-field behaviour of the unsteady plume, the initial conditions for the model are defined by fitting to the buoyancy flux at $t = t_4$ in figure 8.6(a), which is approximately the smallest time for which the buoyancy flux in the wave is monotonic. Initial conditions for Q_m, M_m and B_m are derived from the fit to the observed buoyancy flux $F_m(z, t_4)$ by substituting $F_m(z, t_4)$ for F_s in the steady-state solutions (8.16), with $\beta_g = 1$ and $\theta_g = 1$. For $t > t_4$, Q_m, M_m and B_m are held constant at the source.

We compare the predictions obtained from (8.71)-(8.73), which we refer to as the Gaussian plume model (GPM) to those of the adjusted top-hat plume model (TPM) proposed by Scase & Hewitt (2012). To obtain predictions from TPM it was necessary to employ a flux-limiting scheme, and to this end we again follow Kurganov & Tadmor (2000). For both TPM and GPM we use 2000 computational points, which is sufficient for full convergence, as was verified by halving the number of points to 1000. For the parameters of GPM we assume Gaussian profiles and neglect turbulence transport, setting $\gamma_0 = 4/3$ and $\theta_0 = 1$. For the higher order mixing terms we use $\gamma_1 = \theta_1 = 0.005$, and we will discuss the sensitivity of the results on this choice in §8.6.4. For the parameter ε (i.e. β_1 in chapter 7) that appears in the regularisation proposed by Scase & Hewitt (2012) we set $\varepsilon = 5\gamma_1$,

which ensures that the diffusive flux of energy in TPM is equal to the dispersive flux of energy in GPM.

Figures 8.6 and 8.7 compare the predictions of (8.71)-(8.73) to those of the regularised top-hat model described by Scase & Hewitt (2012). The predictions of Q_m and M_m using GPM displayed in figure 8.6 are in reasonably good agreement with the DNS data. In particular, GPM correctly represents the position and the spreading rate of the wave in M_m . In the volume flux Q_m it appears that GPM predicts that the wave travels slightly faster than the DNS observations suggest. Nevertheless, GPM provides a better overall agreement with the DNS observations than TPM, which predicts a large over-shoot in Q_m , M_m and B_m .

There is a noticeable difference between the GPM predictions and the DNS observations in B_m in the vicinity of the wave. The observations suggest that the front in B_m is much steeper and faster than that which is predicted by GPM. The primary reason for this difference is the use of $\theta_g = 1$ i.e. the assumption of zero turbulence transport. As we will demonstrate below, when the turbulent transport of buoyancy is accounted for ($\theta_g > 1$), the wave in B_m predicted by TPM becomes steeper and exhibits a better agreement with the observed B_m in figure 8.6. However, the purpose of this thesis is not to investigate fine-tuning of the various profile coefficients to give an optimised agreement with the DNS data. Rather, we have presented the basic mean-flow Gaussian case, for which $\theta_g = \theta_m = 1$, $\beta_g = 1$ and $\gamma_g = \gamma_m = 4/3$ because it is a simple, yet realistic starting point from which further refinements can easily be made.

In figure 8.7 the predictions from the GPM exhibit a satisfactory agreement with both the radius of the plume and the flux-balance parameter. TPM predicts a local increase in both r_m and Γ , which we do not observe in the DNS simulations. Consistent with the analysis of the hyperbolic problem in §8.5.1, for $\theta_g < 4/3$, GPM predicts a reduction, albeit relatively small, in Γ in the vicinity of the wave.

We now revisit the question of why the front in B_m observed in the DNS data in figure 8.6 appears to be steeper than that which is predicted by GPM. To answer this question it is useful to understand how the structure of the wave depends on θ_g . Figure 8.11 displays the quantities $Q_m(z, t)$, $M_m(z, t)$ and $B_m(z, t)$ normalised by the steady-state $Q_{m0}(z)$, $M_{m0}(z)$ and $B_{m0}(z)$, at a particular time. Each of the predictions displayed in figure 8.11 corresponds to a different choice of the dimensionless parameter θ_g . More precisely, we add a constant $\theta_f \in 0.5, \dots, 2.0$ to θ_0 in equation (8.73) to account for turbulence transport. We note that values of θ_g much less or greater than unity are physically unrealistic and have only been included in figure 8.11 for the purposes of providing a more complete representation

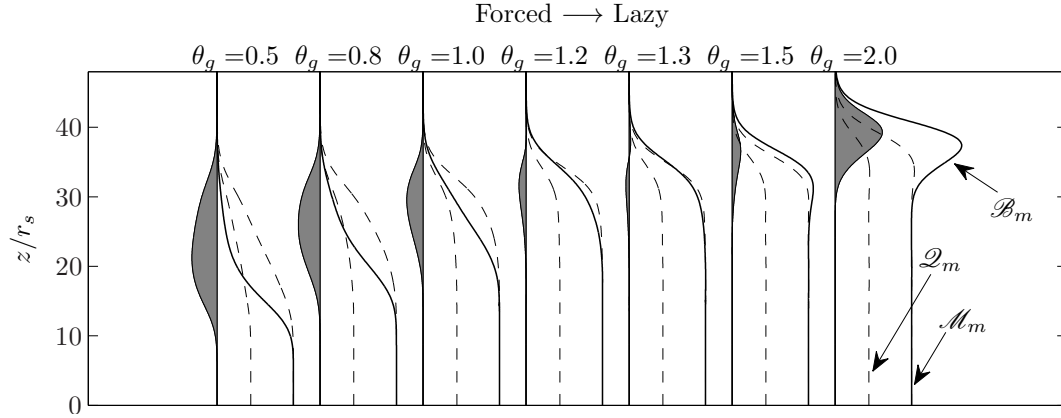


FIGURE 8.11. Wave structure for different values of θ_g . The solid lines are normalised integral quantities $\mathcal{Q}_m \stackrel{\text{def}}{=} Q_m/Q_{m0}$, $\mathcal{M}_m \stackrel{\text{def}}{=} M_m/M_{m0}$ and $\mathcal{B}_m \stackrel{\text{def}}{=} B_m/B_{m0}$, and the shaded region denotes the extent to which $\Gamma(z, t)$ is different from unity. For $\theta_g \approx 4/3$, $\Gamma = 1$ and the wave is pure.

of the system's response. For larger values of θ_g the wave appears to propagate faster than for smaller values of θ_g , due to the distribution of buoyancy within the wave. In agreement with the analysis of §8.5.3, it is also clear that the wave is lazy when $\theta_g > 4/3$ and forced when $\theta_g < 4/3$. Although it is tempting to attribute the comparatively smooth behaviour of B_m predicted by GPM in figure 8.6(c) to longitudinal mixing, this would be incorrect. The additional spreading of the wave in the integral buoyancy B_m relative to the DNS observations is primarily due to the relative rate at which buoyancy, momentum and energy are advected through the system (i.e. the basic structure of the characteristic curves discussed in §8.5), rather than higher-order mixing effects. This is evident in the prediction of B_m corresponding to $\theta_g = 1$ in figure 8.11, where one sees a much larger longitudinal spread than in the predictions corresponding to the canonical value of $\theta_g = 4/3$, in spite of the fact that the longitudinal mixing (dispersion) parameter θ_1 is identical.

8.6.3 Idealised similarity solution

Motivated by the observations reported in §8.4 and the theory developed in §8.5, the solution we derive in this section is based on three assumptions. 1. The plume remains straight-sided ($r_m = 6\alpha z/5 \quad \forall t$), hence $\gamma_g/\beta_g = 4/3$ and $\theta_m = 1$ (see §8.5.2). 2. The plume remains pure ($\Gamma = 1 \quad \forall t$), hence $\theta_g = \gamma_g$ (see §8.5.2). 3. The imposed change in the buoyancy flux is relatively small. For definiteness, we will neglect the turbulent transport of momentum and set $\beta_g = 1$, which implies that $\gamma_g = 4/3$. Under these conditions the structure of the wave described in §8.5.3

is particularly simple because it consists of a single front propagating along the fastest characteristic. Behind the front the plume behaves in accordance with a steady state. Noting from (8.27) that when $\gamma_g/\beta_g = 4/3$ the dimensionless velocity of the fastest characteristic is $\lambda^* = \lambda_1 = 3\gamma_g/2$, we find that $\theta_g = 2\lambda^*/3$ for consistency.

Consider the transport equation for the integral buoyancy:

$$\frac{\partial B_m}{\partial t} + \frac{\partial}{\partial z} \left(\theta_g \frac{M_m B_m}{Q_m} \right) = 0. \quad (8.74)$$

Since we are assuming that the plume remains pure and straight-sided with $\theta_m = 1$, we have

$$\frac{Q_m}{M_m^{1/2}} = \frac{6\alpha_0}{5} z, \quad \frac{M_m^{3/2}}{Q_m} = \frac{5B_m}{8\alpha_0}, \quad (8.75)$$

hence

$$\frac{M_m}{Q_m} = \left(\frac{25}{48\alpha_0^2} \frac{B_m}{z} \right)^{1/2}, \quad (8.76)$$

and thus

$$\frac{\partial B_m}{\partial t} + \frac{5\sqrt{3}}{12\alpha_0} \frac{\partial}{\partial z} \left(\frac{\theta_g B_m^{3/2}}{z^{1/2}} \right) = 0. \quad (8.77)$$

To include longitudinal spreading of the wave due to dispersion, we employ (8.70), which in this case reduces to

$$\theta_g = \underbrace{\theta_0}_{2\lambda^*/3} - 6 \left(\frac{5}{3} + \frac{z^3}{B_m} \frac{\partial}{\partial z} \left(\frac{B_m}{z^2} \right) \right) \theta_1. \quad (8.78)$$

Note that $\theta_1 \ll \theta_0$, implying that for smoothly varying B_m the effect of θ_1 on θ_g is relatively small and does not violate the condition that $\theta_g \approx 2\lambda^*/3$. The transport equation for the integral buoyancy becomes

$$\frac{\partial B_m}{\partial t} + \frac{5\sqrt{3}\lambda^*}{18\alpha_0} \frac{\partial}{\partial z} \left(\frac{B_m^{3/2}}{z^{1/2}} \right) = \frac{5\sqrt{3}\theta_1}{6\alpha_0} \frac{\partial}{\partial z} \left(\frac{3B_m^{3/2}}{z^{1/2}} + 2z^{5/2} \frac{\partial}{\partial z} \left(\frac{B_m^{3/2}}{z^2} \right) \right). \quad (8.79)$$

We consider a perturbation expansion for B_m , where ϵ depends on the magnitude of the imposed step change in the source buoyancy flux:

$$B_m = B_m^* (1 + \epsilon \mathcal{B}_1 + \epsilon^2 \mathcal{B}_2 + \dots), \quad (8.80)$$

where

$$B_m^* = \frac{6\alpha_0}{5} \left(\frac{10}{9\alpha_0} \right)^{1/3} z^{1/3} F_m^{*2/3}. \quad (8.81)$$

To first order, the condition of straight-sidedness and constant Γ can be expressed as

$$\mathcal{M}_1 = \mathcal{B}_1, \quad \mathcal{Q}_1 = \mathcal{B}_1/2. \quad (8.82)$$

At $O(\epsilon)$ we find

$$\frac{\partial \mathcal{B}_1}{\partial t} + \frac{5}{3\alpha_0} \left(\frac{\lambda^*}{2} + 4\theta_1 \right) \left(\frac{9\alpha_0 F_m^*}{10z} \right)^{1/3} \frac{\partial \mathcal{B}_1}{\partial z} = \frac{5\theta_1}{3\alpha_0} \left(\frac{9\alpha_0 F_m^*}{10} \right)^{1/3} z^{2/3} \frac{\partial^2 \mathcal{B}_1}{\partial z^2}. \quad (8.83)$$

Recalling that

$$\frac{dz^*}{dt} = \lambda^* \frac{5}{6\alpha_0} \left(\frac{9\alpha_0}{10} \right)^{1/3} \frac{F_m^{*1/3}}{z^{*1/3}}, \quad (8.84)$$

along the characteristic curves of the original hyperbolic system, it is natural to define a similarity variable

$$\lambda \stackrel{\text{def}}{=} \left(\frac{9\alpha_0}{10} \right)^{2/3} \frac{z^{4/3}}{F_m^{*1/3} t}, \quad (8.85)$$

such that $\lambda(t, z^*) = \lambda^*$. Assuming that the process is indeed self-similar, the governing equation (8.83) can be significantly simplified. Using (8.85),

$$\frac{d^2 \mathcal{B}_1}{d\lambda^2} = \frac{1}{2\lambda} \left[\text{Pe} \left(1 - \frac{\lambda}{\lambda^*} \right) - \frac{11}{2} \right] \frac{d\mathcal{B}_1}{d\lambda}, \quad (8.86)$$

where λ^* is the dimensionless velocity of the front and $\text{Pe} \stackrel{\text{def}}{=} 3\lambda^*/(4\theta_1) = 300$ for $\theta_1 = 0.005$ and $\lambda^* = 2$. Equation (8.86) has an analytical solution that can be expressed in terms of a hypergeometric function. In practice $\theta_1 \ll 1$, therefore $\text{Pe} \gg 1$, and it is appropriate to use an asymptotic form of the solution to (8.86):

$$\mathcal{B}_1 \sim \mathcal{B}_1^A + \frac{\mathcal{B}_1^B - \mathcal{B}_1^A}{\sqrt{\pi \text{Pe}}} \frac{\lambda}{\lambda^*} \exp \left(\frac{\text{Pe}}{2} \left(\ln \frac{\lambda}{\lambda^*} - \frac{\lambda}{\lambda^*} + 1 \right) \right) G \left(1, \frac{\text{Pe}}{2} + 2, \frac{\text{Pe}\lambda}{2\lambda^*} \right), \quad (8.87)$$

where G is the hypergeometric function and $\mathcal{B}_1^B$ and $\mathcal{B}_1^A$ are the values of $\mathcal{B}_1$ before and after the step change, respectively. Here, we have defined the steady-state functions Q_{m0} , M_{m0} and B_{m0} using $F_{m0} = F_m^{* \text{def}} = (F_m^A + F_m^B)/2$, hence $\mathcal{B}_1^B = -\mathcal{B}_1^A$. Figure 8.12 plots the solution to (8.86) in addition to the analytical asymptotic solution (8.87). Predictions corresponding to a Gaussian-based estimation using

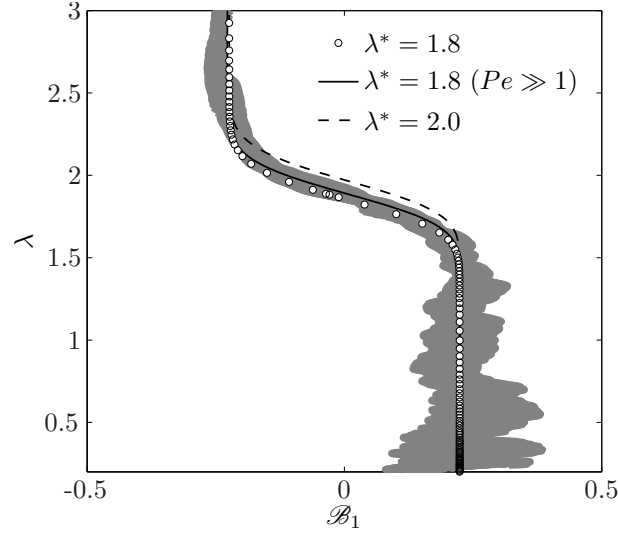


FIGURE 8.12. Predictions using a linearised, straight-sided, constant- Γ model. The circles denote the prediction that is based on the observed propagation speed ($\lambda^* = 1.8$) and the dashed line shows prediction based on the theoretical propagation speed ($\lambda^* = 2.0$). The solid line denotes the prediction that is based on the observed propagation speed ($\lambda^* = 1.8$), assuming that $Pe \gg 1$ such that the asymptotic solution (8.87) is valid. The grey lines correspond to observed values from DNS at several times. Note that the top-hat wave would be located at $\lambda = 1$.

$\gamma_g/\beta_g = 4/3$ and $\beta_g = 1$ (see §8.5), such that $\lambda^* = 2$ in addition to the observed value of $\lambda^* = 1.8$ (see §8.4) are shown. Evident from figure 8.12 is that the similarity solution (8.87) exhibits a good agreement with the both the observed behaviour of $\mathcal{B}_1$ and the full differential equation (8.86). Note that the observed integral buoyancy is approximately self-similar, albeit over the limited sample that we were able to provide in the far-field. The similarity scaling employed to derive (8.86) and the observation of self-similarity in figure 8.12 support the view that the length scales in the plume vary according to $t^{3/4}$. This scaling applies to both the position of a disturbance and its longitudinal extent, and therefore provides some resolution of the open question debated in Scase *et al.* (2009) as to the longitudinal scaling of a propagating pulse structure in a plume.

8.6.4 Sensitivity to changes in θ_1

Although the purpose of this study is not to calibrate model parameters, such as θ_1 , it is useful to understand the sensitivity of the proposed model to changes in the assumed values of the parameters. Further insight is provided by considering a small region of order $\tilde{\lambda} \stackrel{\text{def}}{=} \lambda - \lambda^*$ in the vicinity of the leading edge of the wave

(see, e.g. figure 8.12). For $\text{Pe} \gg 1$ and $\tilde{\lambda} \ll \lambda^*$, (8.86) becomes

$$\frac{d^2 \mathcal{B}_1}{d\tilde{\lambda}^2} + \frac{\text{Pe} \tilde{\lambda}}{2\lambda^{*2}} \frac{d\mathcal{B}_1}{d\tilde{\lambda}} = 0. \quad (8.88)$$

From (8.88) it is clear that for the highest derivative to play a role in this equation, $\tilde{\lambda} = O(\text{Pe}^{-1/2})$. More precisely, the solution of 8.88 is

$$\mathcal{B}_1 = \left(\frac{\mathcal{B}^B + \mathcal{B}^A}{2} \right) + \left(\frac{\mathcal{B}^B - \mathcal{B}^A}{2} \right) \text{erf} \left(\sqrt{\frac{\text{Pe}}{\lambda^*}} \frac{\tilde{\lambda}}{2} \right). \quad (8.89)$$

The fact that (8.86) can be reduced to (8.89) for $\text{Pe} \gg 1$ explains why the error function in §5.4.2 provided a good estimation of θ_1 . Indeed, advection-dispersion problems involving an underlying velocity that scales according to a power law all have a form that is similar to (8.86) and (6.45). However, in dimensional coordinates the longitudinal scaling of a wave, e.g. $\tilde{z}$, will depend on the particular problem considered ($\tilde{z} \sim t^{3/4}$ for plumes, whereas $\tilde{z} \sim t^{1/2}$ for jets). The fact that the solution to each of these problems can be reduced to an error function is because $\text{Pe} \gg 1$ implies that the variation in the advecting velocity over the longitudinal extent of the wave is insignificant.

Differentiation of (8.89) at $\tilde{\lambda} = 0$ provides a useful means of relating the steepness of the wave to the mixing parameter θ_1 :

$$\left(\frac{\lambda^*}{\mathcal{B}^B - \mathcal{B}^A} \right) \frac{d\mathcal{B}_1}{d\tilde{\lambda}} \Big|_{\lambda=\lambda^*} = \sqrt{\frac{\text{Pe}}{4\pi}}. \quad (8.90)$$

More generally, one can differentiate (8.86) to determine the sensitivity of a solution to changes in θ_1 , without making an assumption about Pe . Figure 8.13 displays the solution of (8.86), along with the DNS data and a region denoting the sensitivity of the solution with respect to changes in θ_1 . The extent of the shaded region corresponds to the leading-order amount by which the solution would change if θ_1 were to change by a factor of 2 in either direction. As one expects, the parts of the solution that are subject to the greatest uncertainty are those where its second derivative is largest.

8.7 Conclusions

This chapter has extended the framework developed in chapter 6 for unsteady jets to unsteady plumes. Existing unsteady plume models form a subset of the models that can be derived from the generalised framework that was discussed in §8.2 and, consequently, their individual properties can be understood and related to

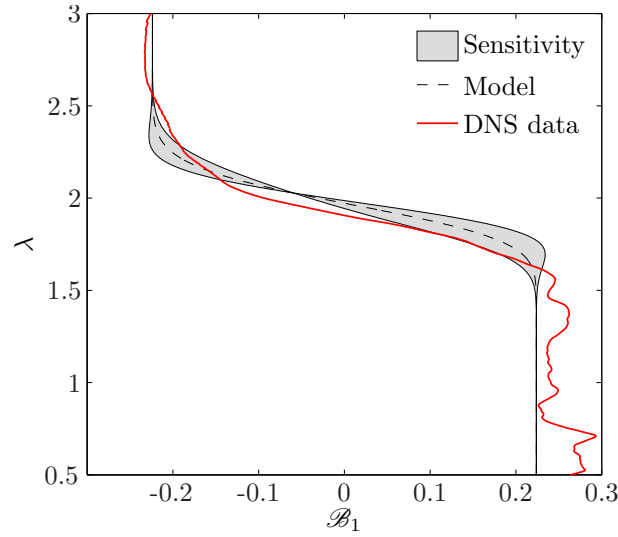


FIGURE 8.13. Sensitivity of idealised similarity solution to changes in θ_1 , where the sensitivity is defined as $d\mathcal{B}_1/d\log_2\theta_1$, evaluated at $\theta_1 = 0.005$. The extent of the shaded region in the figure therefore represents the leading-order amount by which the solution would change if θ_1 were to change by a factor of 2 in either direction. Here the DNS data consists of an average of the individual the profiles that are displayed in figure 8.12.

the physics of the flow. Notably, the structure of the governing integral equations depends on the assumptions one makes about features of the flow that are typically lost upon integration. The structure, for example, determines how the radius of the plume responds to changes in the buoyancy flux and whether the plume is stable to infinitesimal perturbations applied to the source.

Scase & Hewitt (2012) introduced an eddy-diffusion regularisation to obtain a well-posed top-hat model of an unsteady plume. However, the theory reported in this chapter shows that unsteady plume models do not require regularisation in general, and can be derived from first principles if a generalised framework is employed. Whilst the model of Scase *et al.* (2006*b*) was derived rigorously, since it is based on the mass-momentum framework it relies on the assumption of a top-hat velocity profile. The top-hat unsteady plume model is a degenerate case in the large set of possible unsteady plume models. Each of these models comprises, in general, a hyperbolic system of equations with three distinct characteristic curves. The growth rate of source perturbations, the behaviour of the radius of the plume and the flux balance parameter in an unsteady plume depend crucially on the particular assumptions that are made about the velocity profile, turbulence and pressure. The top-hat model is a singular case because it is a parabolic system and admits the exponential growth (Scase & Hewitt, 2012), rather than algebraic

growth or decay, of source perturbations.

Conclusions

9.1 Summary

In this thesis we sought to obtain a robust model for unsteady turbulent jets and plumes by investigating their physics. To this end, we developed a framework that generalises previous models of jets and plumes and reveals aspects of their behaviour that were previously hidden. Natural and man-made variability in source conditions, such as diurnal heating (Hunt *et al.*, 2003), transient fires (Heskestad, 1998), heating and cooling systems in buildings (Hunt, 1991; Linden, 1999), ensures that many of the jets and plumes one encounters in practice are statistically unsteady. In situations in which the variability occurs on time scales that are large, relative to those associated with the jet or plume, a quasi-steady approximation might suffice. However, for a thorough understanding of the underlying physics and predictive capabilities in more stringent conditions, the models and insights provided in this thesis will be useful, and encompass quasi-steady-state theories as a special case.

Whilst there are several existing unsteady plume theories, Scase & Hewitt (2012) demonstrated that most are ill-posed, which is problematic for two reasons. First, from a practical perspective, ill-posed models require regularisation in order to be useful. Secondly, if one supposes that the prediction of unsteady jets and plumes is essentially a well-posed problem, then models that are ill-posed are missing fundamental physical processes. The framework developed in this thesis allows one to see the missing physical processes.

Using direct numerical simulation, we developed an understanding of unsteady jets and plumes by looking at unsteady passive scalar transport in chapter 5 and unsteady mean-flow energy transport in chapters 6 and 7. Incorporating the theory and models developed in the previous chapters, an unsteady buoyancy flux was introduced in chapter 8. In chapter 5 it was discovered that shear-flow dispersion dominates turbulent transport in mixing the scalar along the axis of a jet. By looking at the mean kinetic energy budget in chapter 6, we established that the radial dependence of the longitudinal velocity profile plays a crucial role in determining the structure of the governing integral equations. In particular, *profile coefficients* appearing in the governing integral equations determine the arrangement of characteristic curves associated with the system and therefore the rate at

which disturbances in the jet travel. Chapter 7 demonstrated the precise way in which the profile coefficients determine the behaviour of the jet's radius and its response to source perturbations. Remarkably, unsteady jets with Gaussian mean velocity profiles are a distinguished case because they remain straight-sided, are insensitive to source perturbations and, consequently, result in well-posed models.

Many of the properties that were identified in chapter 7 are inherited by unsteady plumes. In addition to the two families of characteristic curves that one finds in unsteady jets, unsteady plumes contain a third family of curves determined by the dimensionless buoyancy flux. Thus, a particular property of unsteady plumes is that travelling waves can be lazy, pure or forced, depending on the relative position of the characteristic curve associated with the buoyancy flux. In unsteady plumes, like unsteady jets, the Gaussian velocity profile emerges as a distinguished case, resulting in straight-sided plumes that are typically insensitive to source perturbations. Utilising the model for shear-flow dispersion of a scalar, developed in chapter 5, and energy dispersion, developed in chapter 7, in chapter 8 we obtained a well-posed integral model for unsteady plumes.

9.2 Methodology and theoretical approach

Our observations of unsteady turbulent jets and plumes came from high-resolution DNS pertaining to several canonical problems. Acquisition of the data was challenging because the simulations were conducted on relatively large domains and, being statistically unsteady, had to be run several times to make up an ensemble. Indeed, obtaining the ensembles placed a high demand on the total computational time and the disk space required to store the independent initial conditions and output data. Nevertheless, DNS provided us with a means of controlling the source conditions and making detailed observations of the unsteady flow, which would have been prohibitively difficult in a laboratory.

The findings of this work suggest that restricting one's attention to integral quantities is a valuable approach, from both a practical and a theoretical perspective. From a practical perspective integral models allow one to obtain accurate predictions using a relatively small number of variables. From a theoretical perspective one is forced to understand, and therefore appropriately parameterise, information about the flow that is lost upon integration. The entrainment coefficient of Taylor (1945) is perhaps the canonical example of such a parameterisation, and one that has enjoyed much success (see, e.g., Woods, 2010). However, the present work has demonstrated that the assumption of a constant entrainment coefficient is inadequate in the modelling of unsteady jets and plumes (see §6.2.4 and

§8.2.2).

A robust integral model for unsteady jets and plumes requires an accurate understanding of their leading-order physics. Whilst it is likely that one could reproduce our integral models' predictions using a relatively simple closure for turbulence in a three-dimensional model, without further analysis such an approach would reveal very little about the leading-order physics of jets and plumes. An agreement between a comparatively crude integral model and data from direction simulation or experiment, on the other hand, makes an extremely strong statement about the physics of a problem. Perhaps nowhere is this viewpoint demonstrated more clearly than in the success of self-similar models for steady free-shear flows (see, e.g., Rodi, 1972).

It nevertheless appears that with the advent of high-performance computing there is less tendency to understand and develop integral models of a flow. In this respect, the techniques employed in this thesis are unusual. Despite having access to detailed information about the flow field from DNS, we focused on the behaviour of integrals, looking in more detail at the flow field only when the behaviour of the integrals suggested that it was necessary. An example of this approach is provided in chapter 5, in which longitudinal mixing due to turbulence transport was unable to account for the spreading rate that was observed in §5.4.2, hence we investigated the effect of local deformations to the mean scalar profile. Inspired by the elegant and resourceful use of the integral perspective in classical dispersion theory, we hope that future work involving direct simulation makes appropriate use of such methods.

9.3 The physics of unsteady turbulent jets and plumes

Several of the observations and conclusions drawn in this thesis relate to the particular properties of unsteady turbulent jets and plumes. For example, in chapter 7 we found that unsteady *axisymmetric* turbulent jets with Gaussian mean velocity profiles are straight-sided and exhibit a neutral response to source perturbations. Since this is a precise result, which is in reasonably good agreement with the DNS observations reported in chapter 6, it is useful to speculate on the inverse problem: do steady jets have Gaussian velocity profiles because integrals of Gaussian profiles are insensitive to unsteadiness? Indeed, it seems inconceivable that a steady jet could have a velocity profile that is flatter than a Gaussian profile, because the results of chapter 7 suggest that small perturbations would consequently grow without bound. In the near-field, however, the mean velocity profile in a typical jet is flatter (i.e. the dimensionless energy flux γ_m is less than $4/3$) than a Gaus-

sian profile (Rajaratnam, 1976). There, however, the longitudinal evolution of the profile is accompanied by a corresponding increase in the production of turbulence kinetic energy. Thus, we wish to focus our attention on why jets might ‘choose’ to have Gaussian velocity profiles in their *far fields*.

More generally, one must bear in mind that Gaussian profiles are not the only distributions that satisfy the criterion that the dimensionless energy flux γ_m be equal to $4/3$ for straight-sidedness and a neutral response. Thus, additional constraints would be required in order for the Gaussian profile to emerge as a unique solution to the problem. Nevertheless, one imagines that the Gaussian profile is in some sense optimal.

The role of turbulent transport terms in jets, and buoyancy transport in plumes, makes the problem more interesting. Are the buoyancy flux and turbulent transport in a plume balanced to ensure straight-sidedness and insensitivity to source perturbations? Whilst the velocity profile in laminar jets and plumes can be calculated exactly (see, e.g., Rajaratnam, 1976; Brand & Lahey, 1967), the same is not true for turbulent jets and plumes. Therefore, consideration of the optimal profiles and balances between various quantities in jets and plumes from an integral perspective might provide valuable insight and indicate fruitful avenues for future research.

9.4 Modelling

The notion of a top-hat velocity profile as an abstraction of the smooth mean velocity profiles that one observes in experiments is one that is employed widely in steady-state models of jets, plumes and fountains (see, e.g., Turner, 1973). Indeed, whether one assumes Gaussian, top-hat, or other velocity profiles has no influence on the form of the classical steady-state power-law solutions. However, in problems with more complicated dynamics, such as unsteady plumes, the assumed velocity profile plays a crucial role in determining the response of the system. In an unsteady plume the influence that the assumed velocity profile has on the system is due to the presence of a temporal derivative in the mean energy equation. The existence of a non-zero temporal derivative means that the energy flux, the production of turbulence kinetic energy and the buoyancy flux each make independent contributions to the overall balance. The role played by the assumed form of the velocity profile is almost impossible to see in the classical volume-momentum approach but becomes readily apparent when the system is formulated using equations for momentum and energy.

Since our models account for neither changes in the dimensionless turbulence

production nor changes in the dimensionless turbulent transport, we presume that relatively simple turbulence closure schemes would lead to reasonably accurate predictions of the flows considered in this thesis. Indeed, this viewpoint is consistent with the notion that shear-flow dispersion plays the leading-order role in such flows, and that the turbulent properties evolve sufficiently rapidly that they remain in local equilibrium (see, e.g., Pope, 1975). However, as we discuss in the following section, we suspect that for starting jets and plumes (e.g. Turner, 1962), which involve a turbulent-non-turbulent interface perpendicular to the axis of the flow, the situation might be quite different.

9.5 Recommendations for future research

In many practical situations it is appropriate to consider time-dependent sources of momentum and/or buoyancy. A pertinent example is the prediction of buoyancy-driven ventilation flows in buildings, in which one typically deals with diurnal heat sources or those that vary over the course of an event or lecture, for example (see, e.g., Shrinivas & Hunt, 2014). In these cases the notion of a quasi-steady plume is an approximation of the unsteady plumes that one encounters in practice. We therefore hope that several of the classical problems in natural ventilation might benefit from the incorporation of unsteady effects (see, e.g., Linden, 1999). In this respect, we recommend that full advantage is taken of the fact that the models we propose are well-posed, regardless of whether higher-order mixing terms are included. Therefore, in situations for which one's primary aim is to determine the *position* rather than longitudinal spreading rate of waves and fronts in an unsteady plume, we recommend the use of the hyperbolic system discussed in §8.5.1. Indeed, much insight can be gained from an analysis of a system's characteristic curves. In applications involving large systems, such as natural ventilation in buildings, one might expect for the additional information provided by higher-order derivatives to be overshadowed by uncertainties elsewhere in the system.

Recognising that, unlike DNS, experiments provide the opportunity for studying the far-field behaviour of jets and plumes it is useful to consider the ways in which our findings could be incorporated into laboratory work or field observations. The proposed framework might be useful to experimentalists wishing to investigate plumes using unobtrusive bulk techniques, such as measurements of volume fluxes or imaging using shadow-graph techniques, dye or perhaps smoke. Experimentalists might wish to relate bulk properties of unsteady plumes, such as the propagation velocity and spreading rate of disturbances, to 'internal' aspects such as the velocity profile and the production of turbulence kinetic energy. Indeed,

amongst other methods, one can infer entrainment by recording the spreading rate of a free-shear flow or observing the rate at which a scalar is transported by the flow.

Absent from the present work was a rigorous derivation of the perturbation (e.g. g_1 in chapter 5) that gives rise to longitudinal mixing in shear-flow dispersion. The reason that this was not included is that in free-shear flows, unlike the pipe flow considered by Taylor (1953), one cannot seek an asymptotic solution in which the radius of the flow is small with respect to the longitudinal length scale. In each of the flows we have considered the longitudinal length scale is necessarily proportional to the radial length scale because the flow is conical. Thus, all terms in an expansion of the solution remain finite and a more general treatment, analogous to the initial development of dispersion in a tube considered by Chatwin (1977), is required. To examine this issue further our recommendation is that laminar flows should be considered, for which the boundary-layer equations have a closed form solution.

A valuable contribution to the understanding of mixing in both planar and axisymmetric jets and plumes would be to explain the differences between dispersion in jets compared with dispersion in plumes. We found that the Péclet number Pe , which characterises the position of a travelling wave relative to its longitudinal extent, depends on the flow considered. For passive scalar transport in a jet we found that $Pe \approx 10$ (see §5.6). For energy transport in an unsteady jet, however, $Pe \approx 70$ (see §7.5.2), and for buoyancy transport in an unsteady plume, $Pe \approx 300$ (see §8.6.3). Since these differences appear to be systematic and suggest that shear-flow dispersion is suppressed in plumes, one would like to know more about the underlying physics. An analysis into the equivalent laminar flow problems, as suggested in the previous paragraph, might provide a tractable route to such understanding.

Further attention needs to be given to the role of turbulence in unsteady plumes, particularly starting plumes or plumes whose source buoyancy flux is reduced to zero. Unlike the case of scalar transport considered in chapter 5, whether shear-flow dispersion dominates longitudinal turbulence mixing in unsteady plumes is not clear. Based on our analysis, we speculate that for relatively small changes in the buoyancy flux in unsteady plumes, such that the problem can be linearised, shear-flow dispersion will provide the dominant source of longitudinal mixing. In starting plumes, however, we expect that pressure and turbulence play an active role. Unlike the approach that we have taken, in starting plumes one typically distinguishes two distinct parts of the flow: a rising vortex ring which is fed with

mass, momentum and buoyancy from a quasi-steady plume beneath (Turner, 1962). Since several studies have focused on modelling the rising vortex and its interaction with the flow beneath (e.g. Middleton, 1975; Ai *et al.*, 2006), it might be worthwhile to relate them to the system of partial differential equations developed in the present work.

Statistically unsteady planar jets and plumes have not been discussed, although preliminary work that we have undertaken indicates that unsteady planar free-shear flows behave quite differently to unsteady axisymmetric free-shear flows. It would therefore appear that paramount to obtaining a deeper understanding of complex confining geometries, the interaction of multiple free-shear flows or near-field behaviour, is the identification of the primary differences between axisymmetric and planar free-shear flows. Nevertheless, the framework that we propose can be readily extended to semi-bounded flows, such as wall jets and plumes, without significant modification, and might therefore provide a useful means of simplifying a wide range of confined flows.

Analogies between the unsteady problems that we have considered and steady-state problems can be usefully exploited. As pointed out by Scase *et al.* (2006*b*) a steady plume in an unstably-stratified ambient (studied by Batchelor, 1954) is analogous to a plume whose source buoyancy flux is reduced. Similarly, there are similarities between a plume in a stably-stratified environment and a plume whose source buoyancy flux is increased (in both cases $\partial_z F_m < 0$). In exploring such connections one would be interested to know whether shear-flow dispersion has a steady-state counterpart and in what respects the physics of the two types of problem are fundamentally different.

Turbulent entrainment continues to be a subject of debate and investigation (e.g. Kaminski *et al.*, 2005; Hunt & Kaye, 2005; Carazzo *et al.*, 2006; Gladstone & Woods, 2014). In this thesis we made progress in understanding the physics behind entrainment (see, e.g., §6.2.4) and developed a framework that will hopefully prove useful in guiding future work and interpreting the results of previous experiments. By examining unsteady jets and plumes one inevitably learns more about steady jets and plumes. The work on entrainment undertaken for this thesis has led to the development of a set of consistent entrainment relations for steady jets and plumes, which clarifies the assumptions behind existing models and their connection with the underlying physics (van Reeuwijk & Craske, 2015). It is our hope that the approach can be applied to entrainment in other contexts such as wall jets, wall plumes, gravity currents and turbulent fountains.

A conspicuous and challenging gap in current work on entrainment lies be-

tween a small-scale description (see, e.g., van Reeuwijk & Holzner, 2014; da Silva *et al.*, 2014) and the large-scale viewpoint adopted in this work (see also Carazzo *et al.*, 2006; Turner, 1986). Obtaining a connection between these two perspectives is crucial to obtaining a comprehensive understanding of turbulent entrainment and has significant implications for flow control and the atmospheric sciences. Our large-scale perspective shows that one can control entrainment via temporal forcing. However, the precise way in which such changes in entrainment are accommodated by the contortions of a turbulent-non-turbulent interface (TNTI) remain to be explored. The problem becomes particularly interesting when one considers the effect that changes in the shape of the mean velocity profile have on entrainment (see, e.g., the region of flow development in jets). Ideally, one aims to relate the decomposition of entrainment that we obtain (see chapter 6, equation (6.20) and chapter 8, equation (8.15)) to the area, topology and local small-scale entrainment velocity associated with the TNTI. It is worth emphasizing that our predictions regarding the radius of a free-shear flow relate to the characteristic scale $r_m \stackrel{\text{def}}{=} Q_m/M_m^{1/2}$, whose precise relationship with the location of the TNTI remains unclear.

9.6 Final remarks

In the development of plume theory it is interesting that early work focused on mean kinetic energy transport in place of the continuity equation (e.g. Rouse *et al.*, 1952; Priestley & Ball, 1955; Townsend, 1956). With Morton *et al.* (1956) however, the emphasis in the latter half of the twentieth century shifted towards the conservation of volume and, therefore, the entrainment coefficient (see, e.g., Turner, 1986). Given that the continuity equation is a balance that is of lower order than the mean kinetic energy equation, this sequence of events is perhaps surprising*. Nevertheless, the mass-momentum approach advocated by Morton *et al.* (1956) proved such a successful model of steady plumes that it formed the basis of subsequent unsteady plume models. Yet, we have demonstrated that the mass-momentum based approach is not the natural choice for unsteady jets and plumes. The use of a momentum-energy based approach frees one from having to make rigid assumptions about the precise nature of the flow and consequently makes apparent why some unsteady plume models are ill-posed whilst others are well-posed. The

*We acknowledge that whether the energy-based perspective indeed prevailed at the inception of plume theory is difficult to determine. Since both approaches ultimately refer to the same local governing equations it is somewhat artificial to draw a distinction between them. Nevertheless, based on recent review articles (Linden, 2000; Kaye, 2008; Woods, 2010), it is clear that volume conservation, along with the entrainment hypothesis, emerged as the favoured approach.

momentum–energy approach also provides the natural setting for incorporating the effects of shear-flow dispersion. With complicated free-shear flows, such as those involving chemical reactions, confining geometries and variable ambient conditions receiving progressively more attention, one is reminded to consider carefully the most appropriate models and methods of analysis. It is therefore crucial that the fundamental insights and methods of fluid dynamicists over fifty years ago are not forgotten, for they occasionally reveal aspects of a problem that, with growing literature on the subject, are becoming increasingly difficult to see.

List of publications

- CRASKE, J. & VAN REEUWIJK, M. 2013 Robust and accurate open boundary conditions for incompressible turbulent jets and plumes. *Computers & Fluids*, **86**, 284–297.
- CRASKE, J. & VAN REEUWIJK, M. 2015*b* Energy dispersion in turbulent jets. Part 1. Direct simulation of steady and unsteady jets. *Journal of Fluid Mechanics*, **763**, 500–537.
- CRASKE, J. & VAN REEUWIJK, M. 2015*c* Energy dispersion in turbulent jets. Part 2. A robust model for unsteady jets. *Journal of Fluid Mechanics*, **763**, 538–566.
- CRASKE, J., DEBUGNE, A. L. R. & VAN REEUWIJK, M. 2015 Shear flow dispersion in turbulent jets. *Journal of Fluid Mechanics* (accepted).
- CRASKE, J. & VAN REEUWIJK, M. 2015*a* An energy-based framework for modelling unsteady turbulent plumes (submitted to the *Journal of Fluid Mechanics*).

References

- ABRAHAM, J. 1996 Entrainment characteristics of transient gas jets. *Numerical Heat Transfer, Part A: Applications* **30** (4), 347–364.
- ABRAMOWITZ, M. & STEGUN, I. A. 1970 *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*. Dover (eighth printing).
- AI, J. J., LAW, A. W. K. & YU, S. C. M. 2006 On Boussinesq and non-Boussinesq starting forced plumes. *Journal of Fluid Mechanics* **558**, 357–386.
- AKSELVOLL, K. & MOIN, P. 1996 Large-eddy simulation of turbulent confined coannular jets. *Journal of Fluid Mechanics* **315**, 387–411.
- ALFONSI, G. 2011 On direct numerical simulation of turbulent flows. *Applied Mechanics Reviews* **64** (2).
- AMDAHL, G. M. 1967 Validity of the single-processor approach to achieving large scale computing capabilities. *AFIPS Conference Proceedings* .
- ARIS, R. 1956 On the dispersion of a solute in a fluid flowing through a tube. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **235** (1200), 67–77.
- BAINES, W. D. & TURNER, J. S. 1969 Turbulent buoyant convection from a source in a confined region. *Journal of Fluid Mechanics* **37** (1), 51–80.
- BARENBLATT, G. I. 1996 *Scaling, Self-similarity, and Intermediate Asymptotics: Dimensional Analysis and Intermediate Asymptotics*. Cambridge University Press.
- BATCHELOR, G. K. 1954 Heat convection and buoyancy effects in fluids. *Quarterly Journal of the Royal Meteorological Society* **80** (345), 339–358.
- BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.
- BOERSMA, B. J., BRETHOUWER, G. & NIEUWSTADT, F. T. M. 1998 A numerical investigation on the effect of the inflow conditions on the self-similar region of a round jet. *Physics of Fluids* **10** (4), 899–909.
- BORÉE, J., ATASSI, N., CHARNAY, G. & TAUBERT, L. 1997 Measurements and image analysis of the turbulent field in an axisymmetric jet subject to a sudden velocity decrease. *Experimental Thermal and Fluid Science* **14** (1), 45–51, Tenth Symposium on Turbulent Shear Flows.
- BRAND, R. S. & LAHEY, F. J. 1967 The heated laminar vertical jet. *Journal of Fluid Mechanics* **29**, 305–315.

- VAN DEN BREMER, T.S. & HUNT, G.R. 2014*a* Two-dimensional planar plumes and fountains. *Journal of Fluid Mechanics* **750**, 210–244.
- VAN DEN BREMER, T. S. & HUNT, G. R. 2010 Universal solutions for Boussinesq and non-Boussinesq plumes. *Journal of Fluid Mechanics* **644**, 165–192.
- VAN DEN BREMER, T. S. & HUNT, G. R. 2014*b* Two-dimensional planar plumes: non-Boussinesq effects. *Journal of Fluid Mechanics* **750**, 245–258.
- BREMHORST, K. & HOLLIS, P. G. 1990 Velocity field of an axisymmetric pulsed, subsonic air jet. *AIAA Journal* **28** (12), 2043–2049.
- BRENNER, H. 1979 Taylor dispersion in systems of sedimenting nonspherical Brownian particles. I. Homogeneous, centrosymmetric, axisymmetric particles. *Journal of Colloid and Interface Science* **71** (2), 189–208.
- BRENNER, H. 1980*a* Dispersion resulting from flow through spatially periodic porous media. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* **297** (1430), 81–133.
- BRENNER, H. 1980*b* A general theory of Taylor dispersion phenomena. *Physico Chemical Hydrodynamics* **1**, 91–123.
- CAMPBELL, A. N. & CARDOSO, S. S. S. 2010 Turbulent plumes with internal generation of buoyancy by chemical reaction. *Journal of Fluid Mechanics* **655**, 122–151.
- CARAZZO, G., KAMINSKI, E. & TAIT, S. 2006 The route to self-similarity in turbulent jets and plumes. *Journal of Fluid Mechanics* **547**, 137–148.
- CARPENTER, M. H., GOTTLIEB, D. & ABARBANEL, S. 1993 The stability of numerical boundary treatments for compact high-order finite-difference schemes. *Journal of Computational Physics* **108** (2), 272 – 295.
- CAULFIELD, C. P. & WOODS, A. W. 1995 Plumes with non-monotonic mixing behaviour. *Geophysical and Astrophysical Fluid Dynamics* **79** (1-4), 173–99.
- CHATWIN, P. C. 1970 The approach to normality of the concentration distribution of a solute in a solvent flowing along a straight pipe. *Journal of Fluid Mechanics* **43**, 321–352.
- CHATWIN, P. C. 1977 The initial development of longitudinal dispersion in straight tubes. *Journal of Fluid Mechanics* **80**, 33–48.
- CHATWIN, P. C. & ALLEN, C. M. 1985 Mathematical models of dispersion in rivers and estuaries. *Annual Review of Fluid Mechanics* **17** (1), 119–149.
- CHEN, C. J. & RODI, W. 1980 *Vertical turbulent buoyant jets: a review of experimental data*. Pergamon, Oxford.

- CHEN, J. H. 1973 Numerical boundary conditions and computational modes. *Journal of Computational Physics* **13** (4), 522–535.
- CHORIN, A. J. & MARSDEN, J. E. 2013 *A Mathematical Introduction to Fluid Mechanics*. Springer New York.
- COLEMAN, G. N. & SANDBERG, R. D. 2010 A primer on direct numerical simulation of turbulence -methods, procedures and guidelines. *Tech. Rep.*. Aerodynamics & Flight Mechanics Research Group School of Engineering Sciences, University of Southampton.
- CONROY, D. T. & LLEWELLYN SMITH, S. G. 2008 Endothermic and exothermic chemically reacting plumes. *Journal of Fluid Mechanics* **612**, 291–310.
- CONROY, D. T., LLEWELLYN SMITH, S. G. & CAULFIELD, C. P. 2005 Evolution of a chemically reacting plume in a ventilated room. *Journal of Fluid Mechanics* **537**, 221–253.
- CRASKE, J., DEBUGNE, A. L. R. & VAN REEUWIJK, M. 2015 Shear flow dispersion in turbulent jets. *Journal of Fluid Mechanics* (accepted).
- CRASKE, J. & VAN REEUWIJK, M. 2013 Robust and accurate open boundary conditions for incompressible turbulent jets and plumes. *Computers & Fluids* **86**, 284–297.
- CRASKE, J. & VAN REEUWIJK, M. 2015a An energy-based framework for modelling unsteady turbulent plumes (submitted to the *Journal of Fluid Mechanics*).
- CRASKE, J. & VAN REEUWIJK, M. 2015b Energy dispersion in turbulent jets. Part 1. Direct simulation of steady and unsteady jets. *Journal of Fluid Mechanics* **763**, 500–537.
- CRASKE, J. & VAN REEUWIJK, M. 2015c Energy dispersion in turbulent jets. Part 2. A robust model for unsteady jets. *Journal of Fluid Mechanics* **763**, 538–566.
- DELICHATSIOS, M. A. 1979 Time similarity analysis of unsteady buoyant plumes in neutral surroundings. *Journal of Fluid Mechanics* **93**, 241–250.
- DRAZIN, P. G. & RILEY, N. 2006 *The Navier-Stokes Equations: A Classification of Flows and Exact Solutions*. London Mathematical Society Lecture Note Series 334. Cambridge University Press.
- ELDER, J. W. 1959 The dispersion of marked fluid in turbulent shear flow. *Journal of Fluid Mechanics* **5**, 544–560.
- ENGQUIST, B. & MAJDA, A. 1977 Absorbing boundary conditions for the numerical simulation of waves. *Mathematics of Computation* **31**, 629–651.
- EZZAMEL, A., SALIZZONI, P. & HUNT, G.R. 2015 Dynamical variability of axisymmetric buoyant plumes. *Journal of Fluid Mechanics* **765**, 576–611.

- FERZIGER, J. H. 1987 Simulation of incompressible turbulent flows. *Journal of Computational Physics* **69** (1), 1–48.
- FERZIGER, J. H. & PERIĆ, M. 2002 *Computational Methods for Fluid Dynamics*. Springer Berlin.
- FISCHER, H. B., LIST, E. J., KOH, R. C. Y., IMBERGER, J. & BROOKS, N. 1979 *Mixing in Inland and Coastal Waters*. Academic Press.
- FOX, DOUGLAS G. 1970 Forced plume in a stratified fluid. *Journal of Geophysical Research* **75** (33), 6818–6835.
- FRISCH, U. 1995 *Turbulence: The Legacy of A. N. Kolmogorov*. Cambridge University Press.
- FURSIKOV, A. V. 2012 The closure problem for the Friedman-Keller infinite chain of moment equations, corresponding to the Navier-Stokes system. In *Fundamental Problematic Issues in Turbulence* (ed. A. Gyr, W. Kinzelbach & A. Tsinober). Birkhäuser Basel.
- GEORGE, W. K. 1989 The self-preservation of turbulent flows and its relation to initial conditions. In *Advances in turbulence* (ed. W.K. George & R.E.A. Arndt), pp. 39–72. Hemisphere.
- GILL, W. N. 1967 A note on the solution of transient dispersion problems. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **298** (1454), 335–339.
- GLADSTONE, C. & WOODS, A. W. 2014 Detrainment from a turbulent plume produced by a vertical line source of buoyancy in a confined, ventilated space. *Journal of Fluid Mechanics* **742**, 35–49.
- GRESHO, P. M. 1991*a* Incompressible fluid dynamics: some fundamental formulation issues. *Annual Review of Fluid Mechanics* **23** (1), 413–453.
- GRESHO, P. M. 1991*b* Some current CFD issues relevant to the incompressible Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering* **87** (2-3), 201–252.
- GRESHO, P. M. & LEE, R. L. 1981 Don't suppress the wiggles -they're telling you something! *Computers & Fluids* **9** (2), 223–253.
- HAN, T. Y., MENG, J. C. S. & INNIS, G. E. 1983 An open boundary condition for incompressible stratified flows. *Journal of Computational Physics* **49** (2), 276–297.
- HATTORI, T., NORRIS, S. E., KIRKPATRICK, M. P. & ARMFIELD, S. W. 2013 Comparison of non-reflective boundary conditions for a free-rising turbulent axisymmetric plume. *International Journal for Numerical Methods in Fluids* .

- HAYNES, P.H. & VANNESTE, J. 2014 Dispersion in the large-deviation regime. Part 1: shear flows and periodic flows. *Journal of Fluid Mechanics* **745**, 321–350.
- HEDSTROM, G. W. 1979 Nonreflecting boundary conditions for nonlinear hyperbolic systems. *Journal of Computational Physics* **30** (2), 222–237.
- HESKESTAD, G. 1998 Dynamics of the fire plume. *Philosophical Transactions: Mathematical, Physical and Engineering Sciences* **356** (1748), 2815–2833.
- HEWITT, R. E. & BONNEBAIGT, R. 2014 A note on unsteady laminar plumes/jets. *International Journal of Heat and Mass Transfer* **70** (0), 851–855.
- HUNT, G. R. & VAN DEN BREMER, T. S. 2011 Classical plume theory: 1937–2010 and beyond. *IMA Journal of Applied Mathematics* **76** (3), 424–448.
- HUNT, G. R. & KAYE, N. B. 2005 Lazy plumes. *Journal of Fluid Mechanics* **533**, 329–338.
- HUNT, G. R. & LINDEN, P. F. 2001 Steady-state flows in an enclosure ventilated by buoyancy forces assisted by wind. *Journal of Fluid Mechanics* **426**, 355–386.
- HUNT, J. C. R. 1991 Industrial and environmental fluid mechanics. *Annual Review of Fluid Mechanics* **23** (1), 1–42.
- HUNT, J. C. R., VRIELING, A. J., NIEUWSTADT, F. T. M. & FERNANDO, H. J. S. 2003 The influence of the thermal diffusivity of the lower boundary on eddy motion in convection. *Journal of Fluid Mechanics* **491**, 183–205.
- HUSSEIN, H. J., CAPP, S. P. & GEORGE, W. K. 1994 Velocity measurements in a high-Reynolds-number, momentum-conserving, axisymmetric, turbulent jet. *Journal of Fluid Mechanics* **258**, 31–75.
- ISERLES, A. 1996 *A First Course in the Numerical Analysis of Differential Equations*. Cambridge University Press.
- JONKER, H. J. J., VAN REEUWIJK, M., SULLIVAN, P. P. & PATTON, E. G. 2013 On the scaling of shear-driven entrainment: a DNS study. *Journal of Fluid Mechanics* **732**, 150–165.
- KAMINSKI, E., TAIT, S. & CARAZZO, G. 2005 Turbulent entrainment in jets with arbitrary buoyancy. *Journal of Fluid Mechanics* **526**, 361–376.
- KAYE, N. B. 2008 Turbulent plumes in stratified environments: A review of recent work. *Atmosphere-Ocean* **46** (4), 433–441.
- KLEMP, J. B. & LILLY, D. K. 1978 Numerical simulation of hydrostatic mountain waves. *J. Atmos. Sci.* **35** (1), 78–107.
- KOTSOVINOS, N. E. 1977 Plane turbulent buoyant jets. Part 2. Turbulence structure. *Journal of Fluid Mechanics* **81** (01), 45–62.

- KOTSOVINOS, N. E. & LIST, E. J. 1977 Plane turbulent buoyant jets. Part 1. Integral properties. *Journal of Fluid Mechanics* **81** (01), 25–44.
- KURGANOV, A. & TADMOR, E. 2000 New high-resolution central schemes for nonlinear conservation laws and convection-diffusion equations. *Journal of Computational Physics* **160** (1), 241–282.
- LADYZHENSKAYA, O. A. 1963 *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach Science Publishers.
- LANDEL, J. R., CAULFIELD, C. P. & WOODS, A. W. 2012 Streamwise dispersion and mixing in quasi-two-dimensional steady turbulent jets. *Journal of Fluid Mechanics* **711**, 1–47.
- LEE, S. & EMMONS, H. W. 1961 A study of natural convection above a line fire. *Journal of Fluid Mechanics* **11**, 353–368.
- LINDEN, P. F. 1999 The fluid mechanics of natural ventilation. *Annual Review of Fluid Mechanics* **31** (1), 201–238.
- LINDEN, P. F. 2000 Convection in the environment. In *Perspectives in Fluid Dynamics: A Collective Introduction to Current Research* (ed. G. K. Batchelor, H. K. Moffatt & M. G. Worster), pp. 289–345. Cambridge University Press.
- LINDEN, P. F., LANE-SERFF, G. F. & SMEED, D. A. 1990 Emptying filling boxes: the fluid mechanics of natural ventilation. *Journal of Fluid Mechanics* **212**, 309–335.
- LIST, E. J. 1982 Turbulent jets and plumes. *Annual Review of Fluid Mechanics* **14** (1), 189–212.
- MATHUR, S. R. & MURTHY, J. Y. 1997 Pressure boundary conditions for incompressible flow using unstructured meshes. *Numerical Heat Transfer, Part B: Fundamentals* **32** (3), 283–298.
- MI, J. & ANTONIA, R. A. 1994 Corrections to Taylors hypothesis in a turbulent circular jet. *Physics of Fluids* **6** (4), 1548–1552.
- MI, J., XU, M. & ZHOU, T. 2013 Reynolds number influence on statistical behaviors of turbulence in a circular free jet. *Physics of Fluids* **25** (7), 075101.
- MIDDLETON, J. H. 1975 The asymptotic behaviour of a starting plume. *Journal of Fluid Mechanics* **72**, 753–771.
- MILLER, M. J. & THORPE, A. J. 1981 Radiation conditions for the lateral boundaries of limited-area numerical models. *Quarterly Journal of the Royal Meteorological Society* **107** (453), 615–628.
- MIYAUCHI, T., TANAHASHI, M. & SUZUKI, M. 1996 Inflow and outflow boundary conditions for direct numerical simulations. *JSME International Journal. Series B, Fluids and Thermal Engineering* **39** (2), 305–314.

- MOIN, P. & MAHESH, K. 1998 Direct numerical simulation: a tool in turbulence research. *Annual Review of Fluid Mechanics* **30** (1), 539–578.
- MONIN, A. S. & YAGLOM, A. M. 1973 *Statistical Fluid Mechanics: Mechanics of Turbulence Vol. 1*. MIT Press.
- MORINISHI, Y., LUND, T. S., VASILYEV, O. V. & MOIN, P. 1998 Fully conservative higher order finite difference schemes for incompressible flow. *Journal of Computational Physics* **143** (1), 90–124.
- MORTON, B. R. 1959 Forced plumes. *Journal of Fluid Mechanics* **5** (01), 151–163.
- MORTON, B. R. 1965 Modeling fire plumes. *Symposium (International) on Combustion* **10** (1), 973–982, tenth Symposium (International) on Combustion.
- MORTON, B. R. 1971 The choice of conservation equations for plume models. *J. Geophys. Res.* **76** (30), 7409–7416.
- MORTON, B. R. & MIDDLETON, J. 1973 Scale diagrams for forced plumes. *Journal of Fluid Mechanics* **58** (01), 165–176.
- MORTON, B. R., TAYLOR, G. I. & TURNER, J. S. 1956 Turbulent gravitational convection from maintained and instantaneous sources. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **234** (1196), 1–23.
- MUPPIDI, S. & MAHESH, K. 2008 Direct numerical simulation of passive scalar transport in transverse jets. *Journal of Fluid Mechanics* **598**, 335–360.
- NITTA, T. 1962 The outflow boundary condition in numerical time integration of advection equations. *Journ. Met. Soc. Japan* **40**, 13–24.
- OL'SHANSKII, M. A. & STAROVEROV, V. M. 2000 On simulation of outflow boundary conditions in finite difference calculations for incompressible fluid. *International Journal for Numerical Methods in Fluids* **33** (4), 499–534.
- ORLANDSKI, I. 1976 A simple boundary condition for unbounded hyperbolic flows. *Journal of Computational Physics* **21** (3), 251–269.
- PANCHAPAKESAN, N. R. & LUMLEY, J. L. 1993a Turbulence measurements in axisymmetric jets of air and helium. Part 1. Air jet. *Journal of Fluid Mechanics* **246**, 197–223.
- PANCHAPAKESAN, N. R. & LUMLEY, J. L. 1993b Turbulence measurements in axisymmetric jets of air and helium. Part 2. Helium jet. *Journal of Fluid Mechanics* **246**, 225–247.
- PAPANICOLAOU, P. N. & LIST, E. J. 1988 Investigations of round vertical turbulent buoyant jets. *Journal of Fluid Mechanics* **195**, 341–91.

- PARANTHOEN, P., FOUARI, A., DUPONT, A. & LECORDIER, J. C. 1988 Dispersion measurements in turbulent flows (boundary layer and plane jet). *International Journal of Heat and Mass Transfer* **31** (1), 153–165.
- PLOURDE, F., PHAM, M. V., KIM, S. D. & BALACHANDAR, S. 2008 Direct numerical simulations of a rapidly expanding thermal plume: structure and entrainment interaction. *Journal of Fluid Mechanics* **604**, 99–123.
- POPE, S. B. 1975 A more general effective-viscosity hypothesis. *Journal of Fluid Mechanics* **72**, 331–340.
- POPE, S. B. 2000 *Turbulent Flows*. Cambridge University Press.
- PRANDTL, L. 1904 Über flüssigkeitsbewegung bei sehr kleiner reibung. In *Verh. III. Intern. Math. Kongr., Heidelberg*, pp. 484–491. Teubner, Leipzig, 1905.
- PRIESTLEY, C. H. B. & BALL, F. K. 1955 Continuous convection from an isolated source of heat. *Quarterly Journal of the Royal Meteorological Society* **81** (348), 144–157.
- RAJARATNAM, N. 1976 *Turbulent Jets*. Elsevier Scientific Publ.
- RAYMOND, W. H. & KUO, H. L. 1984 A radiation boundary condition for multi-dimensional flows. *Quarterly Journal of the Royal Meteorological Society* **110** (464), 535–551.
- VAN REEUWIJK, M. 2007 Direct simulation and regularization modeling of turbulent thermal convection. PhD thesis, Technische Universiteit Delft.
- VAN REEUWIJK, M. & CRASKE, J. 2015 Energy-consistent entrainment relations for jets and plumes (submitted to the *Journal of Fluid Mechanics*).
- VAN REEUWIJK, M. & HOLZNER, M. 2014 The turbulence boundary of a temporal jet. *Journal of Fluid Mechanics* **739**, 254–275.
- VAN REEUWIJK, M., JONKER, H. J. J. & HANJALIĆ, K. 2008 Wind and boundary layers in Rayleigh-Bénard convection. I. Analysis and modeling. *Phys. Rev. E* **77**, 036311.
- RICOU, F. P. & SPALDING, D. B. 1961 Measurements of entrainment by axisymmetrical turbulent jets. *Journal of Fluid Mechanics* **11** (01), 21–32.
- RODGERS, D. P. 1985 Improvements in multiprocessor system design. *SIGARCH Comput. Archit. News* **13** (3), 225–231.
- RODI, W. 1972 The prediction of free turbulent boundary layers by use of a two-equation model of turbulence. PhD thesis, Imperial College London.
- ROONEY, G. G. & LINDEN, P. F. 1996 Similarity considerations for non-Boussinesq plumes in an unstratified environment. *Journal of Fluid Mechanics* **318**, 237–250.

- ROUSE, H., YIH, C. S. & HUMPHREYS, H. W. 1952 Gravitational convection from a boundary source. *Tellus* **4** (3), 201–210.
- RUBAN, A. I. & VONATSOS, K. N. 2008 Discontinuous solutions of the boundary-layer equations. *Journal of Fluid Mechanics* **614**, 407–424.
- SALVETTI, M., ORLANDI, P. & VERZICCO, R. 1996 Numerical simulations of transisitional axisymmetric coaxial jets. *AIAA Journal* **34**, 736–743.
- SANI, R. L. & GRESHO, P. M. 1994 Résumé and remarks on the open boundary condition minisymposium. *International Journal for Numerical Methods in Fluids* **18** (10), 983–1008.
- SCASE, M. M., ASPDEN, A. J. & CAULFIELD, C. P. 2009 The effect of sudden source buoyancy flux increases on turbulent plumes. *Journal of Fluid Mechanics* **635**, 137–169.
- SCASE, M. M., CAULFIELD, C. P. & DALZIEL, S. B. 2006a Boussinesq plumes and jets with decreasing source strengths in stratified environments. *Journal of Fluid Mechanics* **563**, 463–472.
- SCASE, M. M., CAULFIELD, C. P. & DALZIEL, S. B. 2008 Temporal variation of non-ideal plumes with sudden reductions in buoyancy flux. *Journal of Fluid Mechanics* **600**, 181–199.
- SCASE, M. M., CAULFIELD, C. P., DALZIEL, S. B. & HUNT, J. C. R. 2006b Time-dependent plumes and jets with decreasing source strengths. *Journal of Fluid Mechanics* **563**, 443–461.
- SCASE, M. M. & HEWITT, R. E. 2012 Unsteady turbulent plume models. *Journal of Fluid Mechanics* **697**, 455–480.
- SCHMIDT, W. 1941a Turbulente ausbreitung wines stomes erhitzier luft. Teil I. *Z. angew Math. Mech.* pp. 265–278.
- SCHMIDT, W. 1941b Turbulente ausbreitung wines stomes erhitzier luft. Teil II. *Z. angew Math. Mech.* pp. 351–363.
- SHABBIR, A. & GEORGE, W. K. 1994 Experiments on a round turbulent buoyant plume. *Journal of Fluid Mechanics* **275**, 1–32.
- SHAPIRO, M. & BRENNER, H. 1986 Taylor dispersion of chemically reactive species: Irreversible first-order reactions in bulk and on boundaries. *Chemical Engineering Science* **41** (6), 1417–1433.
- SHRINIVAS, A. B. & HUNT, G. R. 2014 Transient ventilation dynamics induced by heat sources of unequal strength. *Journal of Fluid Mechanics* **738**, 34–64.
- DA SILVA, C. B., HUNT, J. C. R., EAMES, I. & WESTERWEEL, J. 2014 Interfacial layers between regions of different turbulence intensity. *Annual Review of Fluid Mechanics* **46** (1), 567–590.

- SOMMERFELD, A. 1949 *Partial Differential Equations in Physics*. Academic Press.
- SPIEGEL, E. A. & VERONIS, G. 1960 On the Boussinesq approximation for a compressible fluid. *The Astrophysical Journal* **131**, 442.
- STANLEY, S. A., SARKAR, S. & MELLADO, J. P. 2002 A study of the flow-field evolution and mixing in a planar turbulent jet using direct numerical simulation. *Journal of Fluid Mechanics* **450**, 377–407.
- STEVENS, D. P. 1990 On open boundary conditions for three dimensional primitive equation ocean circulation models. *Geophysical & Astrophysical Fluid Dynamics* **51** (1-4), 103–133.
- TAUB, G. N., LEE, H., BALACHANDAR, S. & SHERIF, S. A. 2015 An examination of the high-order statistics of developing jets, lazy and forced plumes at various axial distances from their source. *Journal of Turbulence* **16** (10), 950–978.
- TAYLOR, G. I. 1938 The spectrum of turbulence. *Proceedings of the Royal Society of London. Series A - Mathematical and Physical Sciences* **164** (919), 476–490.
- TAYLOR, G. I. 1945 Dynamics of a mass of hot gas rising in air. *Tech. Rep.*. U.S. Atomic Energy Commission. Los Alamos National Laboratory Research Library. Report 236.
- TAYLOR, G. I. 1953 Dispersion of soluble matter in solvent flowing slowly through a tube. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **219** (1137), 186–203.
- TAYLOR, G. I. 1954a Conditions under which dispersion of a solute in a stream of solvent can be used to measure molecular diffusion. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **225** (1163), 473–477.
- TAYLOR, G. I. 1954b The dispersion of matter in turbulent flow through a pipe. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **223** (1155), 446–468.
- TAYLOR, G. I. 1958 Flow induced by jets. *J. Aero./Space Sci.* **25**, 464–465.
- TAYLOR, G. I. & GREEN, A. E. 1937 Mechanism of the production of small eddies from large ones. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **158** (895), 499–521.
- TEMAM, R. 2001 *Navier-Stokes equations: theory and numerical analysis*. Amer Mathematical Society.
- TENNEKES, H. & LUMLEY, J. L. 1972 *A first course in turbulence*. The MIT press.
- THOMPSON, K. W. 1987 Time-dependent boundary conditions for hyperbolic systems. *Journal of Computational Physics* **68** (1), 1–24.

- THOMPSON, K. W. 1990 Time-dependent boundary conditions for hyperbolic systems, II. *Journal of Computational Physics* **89** (2), 439–461.
- TOLLMIEEN, W. 1926 Berechnung turbulenter ausbreitungsvorg ange. *Z.A.M.M.* **6** (6), 468–478.
- TONG, C. & WARHAFT, Z. 1995 Passive scalar dispersion and mixing in a turbulent jet. *Journal of Fluid Mechanics* **292**, 1–38.
- TORO, E. F. 1997 *Riemann solvers and numerical methods for fluid dynamics: a practical introduction*. Springer.
- TOWNSEND, A. A. 1956 *The Structure of Turbulent Shear Flow*. Cambridge University Press.
- TRITTON, D. J. 1988 *Physical Fluid Dynamics*. Clarendon Press.
- TURNER, J. S. 1957 Buoyant vortex rings. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **239** (1216), 61–75.
- TURNER, J. S. 1962 The ‘starting plume’ in neutral surroundings. *Journal of Fluid Mechanics* **13** (03), 356–368.
- TURNER, J. S. 1966 Jets and plumes with negative or reversing buoyancy. *Journal of Fluid Mechanics* **26** (04), 779–792.
- TURNER, J. S. 1973 *Buoyancy Effects in Fluids*. Cambridge University Press.
- TURNER, J. S. 1986 Turbulent entrainment: the development of the entrainment assumption, and its application to geophysical flows. *Journal of Fluid Mechanics* **173**, 431–471.
- VAN DYKE, M. 1968 *Perturbation Methods in Fluid Mechanics*. Acad. Press.
- VERSTAPPEN, R. W. C. P. & VELDMAN, A. E. P. 2003 Symmetry-preserving discretization of turbulent flow. *Journal of Computational Physics* **187** (1), 343–368.
- VUL’FSON, A. N. & BORODIN, O. O. 2001 Self-similar propagation regimes of a nonstationary high-temperature convective jet in the adiabatic atmosphere. *Journal of Applied Mechanics and Technical Physics* **42**, 255–261.
- WALCHSHOFER, C., STEINER, H. & BRENN, G. 2011 Robust outflow boundary conditions for strongly buoyant turbulent jet flames. *Flow, Turbulence and Combustion* **86**, 713–734.
- WANG, H. & LAW, W. K. 2002 Second-order integral model for a round turbulent buoyant jet. *Journal of Fluid Mechanics* **459**, 397–428.
- WANG, H. J. & DAVIDSON, M. J. 2003 Jet interaction in a still ambient fluid. *Journal of Hydraulic Engineering* **129** (5), 349–357.

- WANG, Z., HE, P., LV, Y., ZHOU, J., FAN, J. & CEN, K. 2010 Direct numerical simulation of subsonic round turbulent jet. *Flow, Turbulence and Combustion* **84** (4), 669–686.
- WARHAFT, Z. 2000 Passive scalars in turbulent flows. *Annual Review of Fluid Mechanics* **32** (1), 203–240.
- WESSELING, P. 2001 *Principles of Computational Fluid Dynamics*. Springer.
- WHITHAM, G. B. 1974 *Linear and Nonlinear Waves*. Wiley.
- WOODS, A. W. 1988 The fluid dynamics and thermodynamics of eruption columns. *Bulletin of Volcanology* **50**, 169–193.
- WOODS, A. W. 1997 A note on non-Boussinesq plumes in an incompressible stratified environment. *Journal of Fluid Mechanics* **345**, 347–356.
- WOODS, A. W. 2010 Turbulent plumes in nature. *Annual Review of Fluid Mechanics* **42** (1), 391–412.
- YU, H. Z. 1990 Transient plume influence in measurement of convective heat release rates of fast-growing fires using a large-scale fire products collector. *Journal of Heat Transfer-transactions of the ASME* **112** (1), 186–191.
- ZELDOVICH, Y. B. 1937 The asymptotic laws of freely-ascending convective flows. *Zh. Eksp. Teor. Fiz.* **7**, 1463–1465, (in Russian). English translation in Selected Works of Yakov Borisovich Zeldovich, **1**, 1992 (J. P. Ostriker ed.), 82–85, Princeton University Press.

Robust open boundary conditions

We show that a popular convective open boundary condition (OBC) is unsuitable in the direct simulation of incompressible turbulent jets and plumes, because 1) the boundary condition modifies their spreading rate; 2) the results are domain dependent; and 3) the boundary condition is liable to cause instability and therefore requires domains that are much larger than the area of interest. We demonstrate the accuracy of new axisymmetric OBCs compared to the standard OBC by conducting direct numerical simulation (DNS) of a turbulent plume and a turbulent jet. The new OBCs conform to the fundamental features of statistically axisymmetric turbulent flows, regardless of the computational domains on which they are imposed. They do not contain tunable parameters and are dynamical, accounting for the strength and extent of a flow at a given time, which eliminates the need for calibration to particular cases. The implementation presented herein is computationally efficient and robust in the vicinity of turbulent flows. This appendix has been published in Craske & van Reeuwijk (2013) and is reproduced here with permission.

A.1 Introduction

There is often a mismatch between the natural domain of a fluid flow and the computational domain imposed by a numerical code with which it is simulated. Natural domains of effectively infinite extent arise frequently in nature and are usually truncated when they are represented as a finite set of points by a computer. It is the role of open boundary conditions (OBCs) to faithfully account for the behaviour of the fluid outside the computational domain and its interaction with the fluid inside the computational domain.

The inflow OBC (IOBC) and the outflow OBC (OOBC) play different roles. The IOBC guides the flow into the domain, ensuring that it has a magnitude and direction that are consistent with the flow that would have occurred on a domain of infinite extent. The OOBC, on the other hand, must allow the fluid to leave the domain without being disturbed by the presence of the boundary. The IOBC is therefore a physical OBC that is defined by known features of the flow outside the computational domain, whereas the OOBC is a computational OBC that must appeal to interior points to establish how it should behave (Nitta, 1962). Such a specification exemplifies the different roles fulfilled by inflow and outflow OBCs.

Significant progress has been made in this field for situations in which the

flow is compressible (Thompson, 1987). The hyperbolic nature of such systems ensures that they have real-valued characteristics, which, with the knowledge of outwardly propagating information, can be used to obtain non-reflecting boundary conditions. Thompson (1987) extended the original one-dimensional solution by Hedstrom (1979) by accounting for multi-dimensions in non-rectangular coordinate systems, before generalising the work to all boundary conditions Thompson (1990). However, the incompressible case remains problematic due to its elliptic character and warrants further attention. The most popular approach in the case of time-dependent problems is to specify a convective boundary condition Wesseling (2001). The convective OBC and other incompressible OBCs will be given a detailed review in the next section.

We focus on incompressible, statistically axisymmetric flows with localised turbulence, such as turbulent jets, plumes and fountains. Although such flows are of considerable practical interest, they are particularly difficult to simulate with OBCs due to their unsteadiness, which can cause instability, and the significant effect that OBCs can have on the flow that is induced in the ambient. Specifically, we consider a high-Reynolds-number plume driven by a horizontal source of buoyancy that develops vertically by entraining fluid from its surrounding ambient. In this application the presence of temperature as an active scalar is responsible for further complexity, affecting the pressure field and giving rise to destabilising forces. However, we will also demonstrate that the proposed treatment of OBCs generalises to a problem not involving buoyancy with simulations of a turbulent jet. Above the source in vertically orientated jets and plumes there exists an approximately conical region of predominantly vertical motion that is highly turbulent. As the fluid inside this region rises its eddies entrain fluid from their surrounding ambient and decelerate. The flow in the surrounding ambient is approximately irrotational (Taylor, 1958).

Before discussing the relevant literature, a demonstration is given of the importance of correctly accounting for the natural, unconfined behaviour of the flows described above on the boundaries of a computational domain. Figure A.1 displays data from a direct numerical simulation (DNS) of a statistically axisymmetric turbulent plume on a closed domain Ω_{INF} , and on two smaller domains, $\Omega_{\text{C3}} = \Omega_{\text{A3}} \subset \Omega_{\text{INF}}$, with OBCs. The domain of Ω_{INF} is much larger than the window displayed in figure A.1, which therefore provides a good representation of the fluid in an unconfined domain. Simulation C3 employs a classical convective OBC (see, e.g., Ferziger & Perić, 2002; Wesseling, 2001) that will be described in detail in the next section. It is clear from figure A.1 that a failure to account for the lo-

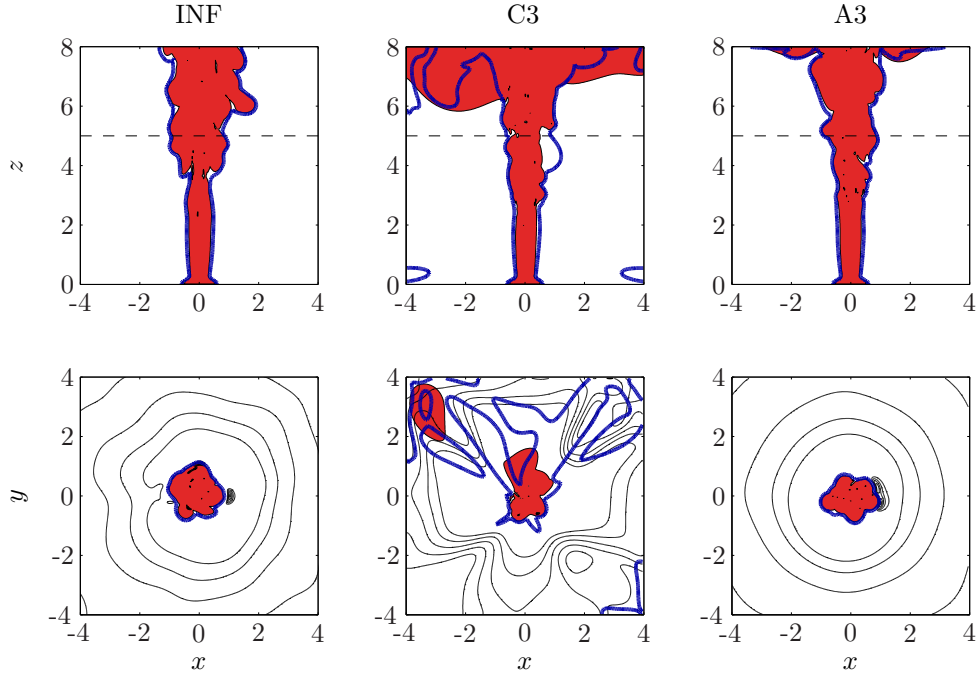


FIGURE A.1. Vertical (top row) and horizontal (bottom row) slices through a large closed domain, Ω_{INF} ; a small domain employing classical convective OBCs, Ω_{C3} ; and a small domain employing OBCs that account for the statistical axisymmetry of the flow, Ω_{A3} . The slices indicate regions that exceed an instantaneous normalised temperature threshold, $\vartheta > 0.005$ [■]; an instantaneous threshold of normalised enstrophy, $\frac{1}{2}\|\nabla \times \mathbf{u}\|^2 > 0.005$ [—]; and, in the bottom figures only, isolines of absolute normalised horizontal velocity, $\sqrt{u^2 + v^2}$ [—]. The horizontal slice was taken at $z = 5$ [— —].

calised nature and statistical axisymmetry of the flow results in severe disturbance and the spurious transport of temperature and enstrophy into the flow's ambient, which slowly accumulates as time progresses. These non-physical effects are particularly pronounced near the top of the domain and close to the corners, where they are liable to interfere with the IOBCs. The interaction of turbulent fluid with the IOBCs is undesirable because it tends to destabilise the simulation. Mitigation of the risk of this interaction requires wider domains, on which it is more expensive to simulate. In addition, the classical convective OBCs can modify the flow rates into and out of the domain, thereby altering important physical properties such as the entrainment rate. Perhaps the most compelling shortcoming of classical OBCs for these flows is that they produce simulation results that depend on the domain size, which we will demonstrate in the following sections.

In this paper we describe robust and accurate OBCs (simulation A3 in figure

A.1) that do not suffer from the aforementioned shortcomings. In order to retain the computational efficiency of Poisson solvers employing Fast Fourier Transforms (FFTs), an important design criterion was that the OBCs must not require sophisticated boundary conditions for pressure. In addition, we required that the computed flow satisfies the continuity equation at each point in the domain to within machine precision, which is a crucial requirement of all simulations of incompressible flows. A further criterion was that the OBCs should respond dynamically to the flow without relying on inflexible case-specific input parameters. After providing a summary of relevant previous work in §A.2, in §A.3 we provide details of the DNSs that were conducted for this investigation. In §A.4 we define the proposed OBCs and in §A.5 we compare their performance to that of classical convective OBCs on domains of different aspect ratio. Before drawing conclusions in §A.7, we demonstrate in §A.5.4 and §A.5.5 that the proposed OBCs are robust when they are applied in a simulation of a high-Reynolds-number plume and jet, respectively.

A.2 Background

There are many alternative combinations of velocities and gradients of velocities that can be prescribed at an OBC (Ol'shanskii & Staroverov, 2000), the most common of which are summarised in Table A.1. A popular choice is to specify the passive condition that wall normal gradients of a scalar term ϕ are equal to zero, $\partial\phi/\partial n = 0$ (Gresho, 1991*b*; Ferziger, 1987; Wesseling, 2001). The advantage of this approach over the more stringent Dirichlet condition, $\phi = f(x, y)$, is in the elimination, to first order, of an artificial boundary layer (Wesseling, 2001). However, when it is applied to the wall normal velocity in a time-dependent flow, the Neumann condition $\partial\mathbf{u}/\partial n = 0$ has been found to cause significant non-physical fluctuations in the pressure field upstream of an OOB (Miyauchi *et al.*, 1996) and is therefore liable to cause instability. In time-dependent flows this problem appears to be typical of OBCs that require a condition to be satisfied instantaneously, rather than providing a prognostic equation which tolerates and responds to deviations from a desired condition. It is therefore on *convective* OBCs, which provide this prognostic equation and are the appropriate choice for convection dominated flows (Wesseling, 2001), that the present work focuses.

Without providing a recipe for how the condition should be imposed in practice, Sommerfeld (1949) proposed a convective condition of this kind. He argued that waves emanating from a singularity on the interior of a domain should only propagate outwards, towards infinity. Orlanski (1976) subsequently presented an implementation of this condition to describe the propagation of a single wave, with

phase velocity C in the scalar field ϕ :

$$\frac{\partial \phi}{\partial t} + C \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma, \quad (\text{A.1})$$

in which n represents a coordinate that is an outward directed normal to a boundary surface Γ . Orlanski (1976) demonstrated that the solution defined by the use of (A.1) was free from reflection. Although this and much of the early work studying non-reflecting OBCs (Chen, 1973; Engquist & Majda, 1977, e.g.) examined equations which are hyperbolic in nature, Orlanski (1976) himself foresaw the utility of this approach to parabolic and elliptic problems. Indeed, Han *et al.* (1983) later demonstrated its effectiveness when applied to parabolic equations in examining a laminar wake.

In the spirit of Orlanski's original work several investigations have employed a local approximation to a one-dimensional phase velocity (Stevens, 1990; Miller & Thorpe, 1981) for use in (A.1), using the values of variables at a previous time step close to the boundary. An extension of this approach that is able to describe waves propagating obliquely to the boundary was proposed by Raymond & Kuo (1984) but has subsequently been reported as being unstable Hattori *et al.* (2013). In that case the practical problems resulting from a theoretically more refined OBC are typical of the difficulties that this area of work presents. Hattori *et al.* (2013) found that, among schemes utilising a local approximation of the phase velocity, the scheme by Stevens (1990) provides the most accurate description of a plume. In the work of Stevens (1990), C is defined using a local velocity that is corrected by an approximate phase velocity, allowing waves to leave the domain even when fluid is entering the domain. This idea was used previously in the work of Klemp & Lilly (1978) in which the correction was estimated as the speed of a typical gravity wave. Both approaches indicate the importance of correctly describing the speed with which disturbances propagate across an OBC. In all of these schemes C must be clipped to upper and lower bounds to ensure the CFL stability criterion is satisfied and the spatial differencing scheme remains *upwind*. The results of the comparative study by Hattori *et al.* (2013) also indicate that for jets and plumes the aspect ratio of a domain plays an important role in determining the robustness of a simulation and the duration for which it can be successfully run.

Application of (A.1) to turbulent flows that involve a wide range of velocities at an OBC necessitates a pragmatic choice of C . A popular choice (Akselvoll & Moin, 1996; Miyauchi *et al.*, 1996; Sani & Gresho, 1994) is to use the average wall-normal velocity, $U_n = \langle \mathbf{u} \cdot \mathbf{n} \rangle$ over the OBC:

$$\frac{\partial \phi}{\partial t} + U_n \frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma. \quad (\text{A.2})$$

Following the early work there have been many subsequent investigations resulting in an extensive literature on the subject (Sani & Gresho, 1994; Gresho, 1991*a*), but unfortunately no agreement on the most suitable approach; the OBC Minisymposium in 1991 is reported to have been an “exercise in frustration” (Sani & Gresho, 1994). OOBcs are considered to be among the most difficult types of BC (Gresho, 1991*b*), because they are not defined by any particular natural condition. For turbulent flows this is particularly problematic; the specification of correct OOBcs requires *a priori* knowledge of the solution that the simulation was intended to provide in the first place (Moin & Mahesh, 1998). This situation is indicative of the fact that there are no perfect OOBcs (Sani & Gresho, 1994), at least for fully turbulent flow, and suggests that effort should be concentrated on finding the most innocuous OBCs.

With $C = 0$, (A.1) specifies a Dirichlet BC for all components of velocity at an OOBc, which was found to cause a large error in the pressure field close to the OOBc (Ol’shanskii & Staroverov, 2000), in addition to oscillations which extended far upstream. Ol’shanskii & Staroverov (2000) and Miyauchi *et al.* (1996) both found that the use of instantaneous local velocities from a previous time step to define C yields poor results. The investigations of Ol’shanskii & Staroverov (2000) suggest that a sensible choice of the spatial variation of $C = C(x, y)$, if available, provides the best form of (A.1), suppressing the upstream influence of the BC. The fact that (A.2) is still a popular choice of OOBc (Ferziger & Perić, 2002) is testimony to the fact that the problem of specifying robust OOBcs is of a practical as much as a theoretical nature (Sani & Gresho, 1994); a simple, accurate and robust OOBc is more desirable than a theoretically perfect OOBc that is extremely sensitive and requires careful calibration.

IOBCs serve a different purpose from OOBcs and therefore require separate consideration. The most common IOBCs are summarised in Table A.1. Free-slip BCs have been applied to confining walls in the simulation of jets (Salveti *et al.*, 1996), although large domains such as Ω_{INF} are required to prevent overturning motions (Baines & Turner, 1969). Boersma *et al.* (1998) applied traction-free IOBCs to avoid this problem, allowing the passage of fluid through the walls of their domain. Other studies of jets have employed homogeneous Neumann IOBCs (Walchshofer *et al.*, 2011), which can be regarded as a subset of the more general traction OBCs. On domains that do not conform with the symmetry of a flow the application of

TABLE A.1. A summary of some of the possible choices for an OOBC and IOBC. Boundary conditions (BCs) marked “*” will be defined in §A.4. The symbols f and $\mathbf{F}$ are used to denote generic functions over a spatial domain.

Form (Name)	Comments
OOBCs	
(a) $\partial\phi/\partial z = 0$, (Neumann).	Produces erroneous disturbance in pressure field. Not suitable for turbulent flows (Miyauchi <i>et al.</i> , 1996; Ferziger & Perić, 2002).
(b) $\phi = f(\mathbf{x})$, (Dirichlet).	Not suitable for turbulent flows. Creates an artificial boundary layer (Wesseling, 2001) and produces erroneous disturbance in pressure field (Ol’shanskii & Staroverov, 2000). Correct selection of $f(\mathbf{x})$ not possible.
(c) $p = p_\infty$, (Pressure).	Physically unrealistic for localised turbulent outflows and unstable.
(d) $\partial\phi/\partial t = -U_n\partial\phi/\partial z$, (Classical convective).	Prognostic and generally stable. Not representative of local accelerations, causing a disturbance in the pressure field.
(e)* $\partial\phi/\partial t = -C(\mathbf{x})\partial\phi/\partial z$, (Axisymmetric convective).	Prognostic and stable. Improved representation of local accelerations compared to (d).
IOBCs	
(a) $\partial\phi/\partial n = f(\mathbf{x})$ ($= 0$), [Neumann (homogeneous)].	Not axisymmetric in general unless specified in $f(\mathbf{x})$. Can produce disturbances in the pressure field. Not robust.
(b) $\phi = f(\mathbf{x})$, (Dirichlet).	Stringent condition, enforced instantaneously, that may not be physically realistic.
(c) $p + \frac{1}{2}\ \mathbf{u}\ = 0$, (Total pressure).	Ideal for irrotational flows (Gresho, 1991b). Requires specification of flow direction to achieve axisymmetry. Unstable in the presence of vorticity. Problematic in stratified environments.
(d) $-p\mathbf{n} + \nu\partial\mathbf{u}/\partial n + \nu\nabla u_n = \mathbf{F}(\mathbf{x})$, (Traction).	Axisymmetry can be achieved if $\mathbf{F}(\mathbf{x})$ specified correctly. In general leads to an inhomogeneous pressure BC. Likely to be unstable in the presence of vorticity because of similarity with (a) and (c). See (Gresho, 1991b; Boersma <i>et al.</i> , 1998).
(e) $\partial\phi/\partial t = U_n\partial\phi/\partial n$, (Classical convective).	Prognostic and stable, but not axisymmetric in general. Is equivalent to homogeneous (a) in steady state.
(f)* $\partial\phi/\partial t = U_n[\partial\phi/\partial n - f(\mathbf{x})]$, (Axisymmetric convective).	Prognostic and stable, forces axisymmetry in a steady state through $f(\mathbf{x})$. Is equivalent to inhomogeneous (a) in a steady state.

traction IOBCs is not always convenient, in general requiring the specification of a spatially varying stress distribution. Even on domains that do conform with the symmetry of a flow, without a correctly formulated inhomogeneous term, the Neumann condition will not, in general, describe the unbounded domain correctly.

When the ambient can be assumed to be approximately irrotational, total pressure IOBCs [Table A.1, IOBCs (c)] are an attractive alternative (Mathur & Murthy, 1997), in which a domain's geometry can be accounted for by specifying the direction of the inflow. However, total pressure BCs are liable to instability in the vicinity of rotational flows with a high Reynolds number (see §A.5.4) and as a result require larger domains than the IOBCs proposed in the present work. The restriction of total pressure IOBCs to either large domains or flows of low Reynolds number makes them less attractive for the flows considered in this study. We have found that newly developed IOBCs [Table A.1, IOBCs (f)], discussed in detail in A.4, are much more tolerant to unsteadiness in the flow field compared to IOBCs that are enforced instantaneously. In A.6 we show that with the specification of a local wall-normal inflow velocity gradient a prognostic IOBC can be used to produce a velocity field that is identical to that obtained using total pressure IOBCs. In §A.5.4 we show that, unlike total pressure IOBCs, the prognostic IOBC is robust for flows with a high Reynolds number.

A.3 Simulation details

A.3.1 Governing equations

To investigate the behaviour of OBCs we used DNS to approximate the Navier-Stokes equations in the Boussinesq limit to describe the behaviour of a turbulent plume or jet. Here, a uniformly spaced orthogonal coordinate system is used, described by the variables x and y in the lateral directions and z in the vertical direction. The point (0,0,0) is central to the horizontal domain and is the location of a circular source of buoyancy with diameter D and integral buoyancy flux B , where $[D] = L$, and $[B] = L^4 T^{-3}$. In addition to the material properties of the fluid, D and B are the only dimensional parameters of the system. In dimensionless form the equations describing the conservation of momentum, volume and relative temperature are given by, respectively

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \text{Fr}^{-2} \vartheta \hat{\mathbf{e}}_z + \frac{1}{\text{Re}} \nabla^2 \mathbf{u}, \quad (\text{A.3})$$

$$\nabla \cdot \mathbf{u} = 0, \quad (\text{A.4})$$

$$\frac{\partial \vartheta}{\partial t} + \mathbf{u} \cdot \nabla \vartheta = \frac{1}{\text{Pr Re}} \nabla^2 \vartheta. \quad (\text{A.5})$$

The dimensionless dependent variables $\mathbf{u}$, p and ϑ are a vector of velocities, kinematic pressure and relative temperature, respectively. The parameter $\text{Re} = B^{1/3} D^{2/3} / \nu$ is a Reynolds number based on B , D and the kinematic viscosity, ν . The Prandtl number is defined as $\text{Pr} = \nu / \kappa$, where κ is the thermal diffusivity and the Froude number Fr is defined as $\text{Fr} = (\beta g)^{-1/2} B^{1/3} D^{-5/6}$, where g is gravitational acceleration and β is the coefficient of thermal expansion. The relative temperature is $\vartheta = T - T_0$, where T is absolute temperature and T_0 is a reference temperature. We assume that the pressure is independent of temperature and density and that the function relating these last two variables is a linear equation of state, $\rho(\vartheta) / \rho_0 = 1 - \beta \vartheta$, defining $\beta = -\rho_0^{-1} (\partial \rho / \partial T)_{T=T_0}$. The vector $\hat{\mathbf{e}}_z$ is a unit vector in the vertical direction and the integral buoyancy flux is defined as

$$B = -\beta g \kappa \int_{\Gamma_0} \nabla \vartheta \cdot \mathbf{n} dA, \quad (\text{A.6})$$

where Γ_0 is the bottom face of the domain, and $\mathbf{n}$ is an outward-facing normal. For jets $\vartheta = 0$, $\text{Fr}^{-2} \vartheta \hat{\mathbf{e}}_z = 0$, $B = 0$ and $\text{Re} = w_0 D / \nu$, where w_0 is the characteristic velocity at the source.

A.3.2 Numerical method

All spatial derivatives are approximated to second order accuracy over a staggered grid. Symmetry preserving central differencing was used throughout the computational domain, preventing the discrete advection operator from erroneously contributing to the production or dissipation of the variance of a dependent variable (Verstappen & Veldman, 2003). The prognostic equations were integrated in time using a second order adaptive Adams-Bashforth scheme. At each time step a pressure correction method is used to find the pressure field that will enforce a solenoidal velocity field. The Poisson equation for pressure is efficiently solved by employing an FFT over the two uniformly discretised horizontal directions to find eigenvectors in x and y , thereby reducing the problem to a tridiagonal system in z for each wavenumber. The use of the FFT in solving the Poisson equation

requires that either pressure is periodic in x and y , equal to zero on Γ_V or has a wall-normal gradient equal to zero on Γ_V . The later two cases reduce the space of eigenfunctions to either sines or cosines, respectively. For more details see (van Reeuwijk *et al.*, 2008; van Reeuwijk, 2007).

A.3.3 Computational domain, initial and boundary conditions

In each simulation the domain, Ω , is a square cuboid of height H , and base area $(\Phi H)^2$, where Φ is the domain's aspect ratio. The boundary surface of Ω is comprised of a bottom face Γ_0 , a top face Γ_1 and vertical faces Γ_V . The coordinate system has its origin in the centre of Γ_0 .

The sets of simulations INF, Cn and An , where $n = 1, 2, 3$ or 4 , were used to investigate both OOBcs and IOBCs on domains of different aspect ratios. We designed the height and aspect ratio of Ω_{INF} such that, over an interrogation window of height $8D$ and width $8D$, it was able to provide us with an accurate picture of how the fluid would behave in an unconfined domain. In the remaining sets we tested different aspect ratios but kept the domain height, $H = 8D$, constant. The discretised domains used in these simulations are each subsets of the domain used for the reference simulation: $\Omega_{C1} = \Omega_{A1} \subset \dots \subset \Omega_{C4} = \Omega_{A4} \subset \Omega_{\text{INF}}$, each employing $N = 32$ computational cells per source diameter, D , uniformly in x , y and z . The simulations $\{\text{CR}, \text{AR}\}$ and $\{\text{CRJ}, \text{ARJ}\}$ are used in a comparison to establish whether the proposed axisymmetric OBCs are robust enough to support a high-Reynolds-number flow in plumes and jets, respectively. Accordingly, they have a higher resolution than the other simulations. A summary of these details can be found in Table A.2.

For simulations of a plume a constant buoyancy flux B was prescribed over an approximately circular heat source of diameter D located in the centre of Γ_0 . For simulation of jets $\{\text{A3J}, \text{ARJ}, \text{CRJ}\}$ a velocity profile of the form

$$w = \frac{w_0}{2} \left[1 - \tanh \left(\frac{2r - D}{4 \Delta x} \right) \right] \quad (\text{A.7})$$

was specified on Γ_0 , where Δx is the constant grid spacing in the x -direction. The boundary Γ_0 was impermeable and employed free-slip conditions, $\partial(\cdot)/\partial n = 0$, for the tangential components of velocity. The closed domain of the reference simulation INF was defined by applying these conditions on velocity to its entire boundary. The classical OBCs used in Cn and the axisymmetric OBCs used in An are the central topic of this paper and will be described in detail in §A.4. In all simulations plume theory (Morton *et al.*, 1956) was used to obtain sensible initial

conditions for $\mathbf{u}$ and ϑ and the transients arising from this condition were discarded before statistics were obtained.

TABLE A.2. Index of simulations for investigating open boundary conditions.

Simulation	Φ	H/D	Cells	N/D	Re	Pr	Fr
INF	2	16	$1024^2 \times 512$	32	500	1	1
C1	1/2	8	$128^2 \times 256$	32	500	1	1
C2	3/4	8	$192^2 \times 256$	32	500	1	1
C3	1	8	$256^2 \times 256$	32	500	1	1
C4	2	8	$512^2 \times 256$	32	500	1	1
A1	1/2	8	$128^2 \times 256$	32	500	1	1
A2	3/4	8	$192^2 \times 256$	32	500	1	1
A3	1	8	$256^2 \times 256$	32	500	1	1
A4	2	8	$512^2 \times 256$	32	500	1	1
CR	1	8	$768^2 \times 768$	96	1700	1	1
AR	1	8	$768^2 \times 768$	96	1700	1	1
CRJ	1	8	$768^2 \times 768$	96	5000	1	∞
ARJ	1	8	$768^2 \times 768$	96	5000	1	∞
A3J	1	8	$256^2 \times 256$	32	2000	1	∞

A.4 Open boundary conditions (OBCs)

A.4.1 Outflow open boundary condition (OOBC)

In the set of simulations Cn we followed convention and applied (A.2) on Γ_1 to all components of velocity and all dependent scalar variables, which collectively will be denoted ϕ . To the right hand side (RHS) of (A.2) we added the viscous stresses in x and y , as recommended by Miyauchi *et al.* (1996). This classical convective OOBC is discretised as

$$\frac{d\phi_{ijk}}{dt} = -U_n \left(\frac{\delta\phi}{\delta n} \right)_{ijk} + f_{ij} \quad \text{on } \Gamma_1, \quad (\text{A.8})$$

where the subscripts $i, j, k \in \mathbb{N}$ are used to denote a point, (x_i, y_j, z_k) , in the numerical domain. Horizontal diffusion is approximated to second order in f_{ij}

and $(\delta\phi/\delta n)_{ijk}$ is a first order *upwind* approximation to the wall-normal (vertical) gradient of ϕ at (x_i, y_j, z_k) adjacent to, or on, Γ_1 . If the variable ϕ is cell-centred in z , (i.e. u, v, p , or ϑ) then the point (x_i, y_j, z_k) is located in the ghost cell, half a cell above Γ_1 . For the vertical velocity, w , the boundary condition is applied at a level coincident with that of Γ_1 . The use of the average outflow velocity U_n in (A.8) is problematic, not least because it depends on the area of the outflow. Indeed, a plume will produce a fixed volume flow rate at the level of the outflow such that $U_n \rightarrow 0$ as $\Phi \rightarrow \infty$, and the OBC would then not permit any subsequent change in ϕ at the boundary. It is more desirable to use a specification of C that does not depend on the size of the domain and provides a more accurate description of the spatial variation of outflow velocities.

Walchshofer *et al.* (2011) obtained good results from simulations of a buoyant jet by defining $C(x, y)$ using a Gaussian profile, which agrees well with experimental data for the mean vertical velocity in jets and plumes (Shabbir & George, 1994). In the simulations *An* we applied an extension of the approach of Walchshofer *et al.* (2011), allowing the radial spread of the Gaussian profile to vary in time:

$$C(x, y, t) = \frac{Q_\infty(t)}{\pi r_g(t)^2} \exp \left[-\frac{r(x, y)^2}{r_g(t)^2} \right], \quad (\text{A.9})$$

where $r(x, y) = (x^2 + y^2)^{1/2}$, $r_g(t)$ is a dimensionless length scale describing the width of the plume and $Q_\infty(t)$ is the dimensionless volume flux in the unbounded domain. The relation between $Q_\infty(t)$ and the actual dimensionless volume flux, $Q(t) = \int_{\Gamma_1} w dA$, is

$$Q_\infty = \pi r_g(t)^2 Q(t) \left\{ \int_{\Gamma_1} \exp \left[-\frac{r(x, y)^2}{r_g(t)^2} \right] dA \right\}^{-1}. \quad (\text{A.10})$$

Walchshofer *et al.* (2011) used a fixed $r_g = (6\alpha/5)(H/D + z_v)$, where α is the entrainment coefficient, which was estimated from experimental data (Chen & Rodi, 1980) and z_v defines the location of a virtual source. The disadvantage of using a fixed r_g is that it is not appropriate for flows whose discharge at the outflow is of a different horizontal extent to that on which r_g is based. A fixed r_g also neglects temporal variation in the spread of the vertical velocity profile at the outflow and variation of z_v . To partially alleviate the problem of uncertainty in the estimation of a fixed r_g an intentionally narrow (small α) profile can be imposed, which as Walchshofer *et al.* (2011) point out, is more stable because it reduces the non-physical spreading at the outflow. For the present work we generalise their approach by allowing the flow to determine $r_g(t)$ dynamically and therefore extend

the applicability of the method to a wider class of problems.

The ratio of the first to the zeroth moment of the Gaussian distribution provides a simple means of estimating $r_g(t)$ at each time step of a simulation:

$$r_g(t) = \frac{2}{\sqrt{\pi}} \frac{\sum_{\Gamma_1} r_{ij} \max[w_{ijk}(t), 0]}{\sum_{\Gamma_1} \max[w_{ijk}(t), 0]}, \quad (\text{A.11})$$

where $r_{ij} = (x_i^2 + y_j^2)^{1/2}$. We exclude negative values of w_{ijk} in (A.11) to prevent spurious inflow on Γ_1 from having an unduly large influence in the determination of $r_g(t)$. The modified OOBC is more general than (A.8); the uniform convective velocity U_n can be recovered from (A.9) in the limit $r_g(t) \rightarrow \infty$. It should be noted that there are several sources of error in the approximation in (A.11). Even if the velocities in the plume are exactly Gaussian in form, both the truncation error of the numerical integration and the assumption that $w \equiv 0$ outside the domain render (A.11) inexact. More importantly, the instantaneous velocities in a plume are not, in general, Gaussian in form and are neither axisymmetric nor distributed centrally over the domain. However, compared to $C = U_n$, the Gaussian form is a significant improvement, relative to which the aforementioned sources of approximation are small. Indeed, the outflow's most important attributes, namely its velocity and integral length scale, are captured dynamically by the OOBC. Although the OOBC uses a prescribed velocity profile, (A.9), the precise form of this profile is only likely to be responsible for effects that are of higher order than those considered here.

In addition to providing an improved representation of the convecting velocities at the OOBC, the dynamic specification of $r_g(t)$ has the advantage of not depending on parameters that must be specified *a priori*. This means, for example, that to within the approximation provided by (A.11), representative vertical velocities can be obtained for both jets and plumes, whose spreading rates are known to differ (Carazzo *et al.*, 2006).

The proposed modifications to (A.1) result in the following semi-discrete finite difference scheme:

$$\frac{d\phi_{ijk}}{dt} = -\frac{Q_\infty(t)}{\pi r_g(t)^2} \exp\left[-\frac{r_{ij}^2}{r_g(t)^2}\right] \left(\frac{\delta\phi}{\delta n}\right)_{ijk} + f_{ij} \quad \text{on } \Gamma_1. \quad (\text{A.12})$$

In order to avoid an infinitely stiff response of the fluid at the OOBC, both (A.8) and (A.12) require that non-zero vertical velocities on Γ_1 are specified at $t = 0$. If this requirement is not satisfied then $\mathbf{u}$ will remain zero throughout the simulation.

A.4.2 Inflow open boundary condition (IOBC)

The entrainment caused by a localised patch of turbulence developing in the vertical direction will drive an axisymmetric flow in its ambient. From the perspective of the ambient, the turbulence can be modelled as a vertical line sink. The fully irrotational flow induced by the line sink is a unique solution to Laplace's equation (see, e.g., Taylor, 1958). The gradual vertical variation of the strength of the line sink ensures that vertical velocities in the ambient are small. If these vertical velocities are neglected altogether then (A.4) implies that the horizontal velocities in the ambient are

$$(u, v) = -\frac{q(z, t)}{2\pi r}(\cos \theta, \sin \theta), \quad (\text{A.13})$$

where $\theta = \arctan(y/x)$ and $q(z, t)$ is the entrainment flux (the two-dimensional volume flux). At each height this solution is consistent with the unique two-dimensional axisymmetric incompressible flow induced by a point sink. In principle q could be coupled to the outflow, resulting in $q(t) = Q(t)/H$. However, this is not realistic for plumes in which the entrainment flux varies in z . In A.6 we show that both under- and over-estimation of q leads to significant deviations from axisymmetry in the induced flow field. To account for its vertical variation $q(z, t)$ can be found by integrating (A.4) over a horizontal slice, Ω_h , which results in

$$q(z, t) = \frac{\partial}{\partial z} \int_{\Omega_h} w(x, y, z, t) dA. \quad (\text{A.14})$$

We can define an inhomogeneous Neumann boundary condition on Γ_V that ensures that the horizontal velocities $\mathbf{u}_h = (u, v)$ will be consistent with (A.13):

$$\frac{\delta \mathbf{u}_h}{\delta n} = \mathbf{f}_{ijk} \quad \text{on } \Gamma_V. \quad (\text{A.15})$$

Here $\mathbf{f}_{ijk}$ is the wall-normal gradient of horizontal velocity obtained from the two-dimensional axisymmetric solution ((A.13)):

$$\mathbf{f}_{ijk} = -\frac{q_k(t)}{2\pi} \left[\frac{\partial}{\partial n} \left(\frac{\cos \theta}{r}, \frac{\sin \theta}{r} \right) \right]_{ijk}. \quad (\text{A.16})$$

Without any tolerance for deviations from axisymmetry the IOBC (A.15) is not robust. Therefore, whilst ensuring axisymmetry in the steady-state, we allow temporary departures from axisymmetry by setting

$$\left(\frac{d\mathbf{u}_h}{dt}\right)_{ijk} = C \left[\mathbf{f}_{ijk} - \left(\frac{\delta\mathbf{u}_h}{\delta n}\right)_{ijk} \right] \quad \text{on } \Gamma_V. \quad (\text{A.17})$$

The relaxation velocity $C \geq 0$ in (A.17) determines the rate at which the wall normal gradients converge to those specified in $\mathbf{f}_{ijk}$ and makes (A.17) an *upwind* discretisation. Although the role of C here is the same as its role on the OOBC, the nature of the flow on the IOBC suggest that its influence should be given a different interpretation. For OOBCs a sensible specification of C is essential due to the unsteady, inhomogeneous nature of the flow on Γ_1 . For IOBCs however, which admit an approximately steady flow into the domain, it is the correct specification of the inhomogeneous Neumann condition $\mathbf{f}_{ijk}$, to which the solution tends, that is most important. For this work we set $C = |U_n|$, where U_n is the average wall-normal velocity on Γ_V , which means that all points on Γ_V respond to a departure from the imposed gradient $\mathbf{f}_{ijk}$ at the same rate. It is possible to exploit the more artificial role of C on the IOBC by setting it to a higher value than $|U_n|$, which allows the IOBCs to modify the velocity field over much shorter timescales. Ultimately however, the value used to define C will be restricted by a criterion for numerical stability. Even beneath this limit, we observed that a high value of C can cause problems if the IOBCs must occasionally allow turbulence to leave the domain, because C then admits a more physical interpretation as the convecting velocity.

Under the assumption that vertical velocities in the ambient are small, the correct specification $q_k(t) = q(z_k, t)$ at every vertical level is crucial for attaining axisymmetry. If the vertical dependence of q is neglected and a spatially uniform distribution is used, the steady state implied by (A.17) will not, in general, be axisymmetric. The reason for this is that in two dimensions (A.13) is the unique solution to the incompressible, axisymmetric flow induced by a sink of strength $q(z, t)$. The incorrect specification of q on the boundary can only yield a solution if the condition of axisymmetry is violated. Indeed, an under- or over- specification of the magnitude of $q(z, t)$ at a given height will result in fluid being drawn into the domain predominantly through the middle of each face or through the corners of the domain, respectively. We demonstrate this effect in §A.6 using data from simulations of a sink in a quasi-two-dimensional domain. In three dimensions an incorrect specification will cause a departure from axisymmetry or a divergence/convergence in the horizontal velocity field.

In *Cn* and *CR* we did not supply any axisymmetric information to the IOBC, setting $\mathbf{f}_{ijk} = 0$ uniformly over Γ_V in (A.17). Therefore, in *Cn* and *CR* the OOBC

and the IOBC are identical, defined using the face average normal velocity U_n and its absolute value $|U_n|$, respectively.

A.4.3 Pressure

The relationship of pressure to the IOBCs and OOBBCs developed in the previous sections requires further explanation. The fact that pressure does not appear explicitly in the OBCs considerably simplifies its treatment because there is no need to deviate from the standard FFT-based pressure solver, which we will describe below for completeness. In order to solve the Poisson equation we impose a homogeneous Neumann condition to pressure over all boundaries, with the exception of the plane-average pressure over Γ_1 , which we set to zero to render the problem well-posed. This condition allows fluid to enter or leave the domain in the event of the global divergence of the flow being different from zero, which is a requirement for the existence of a solution to the Poisson equation. It is equivalent to applying a uniform correction to the velocities at the outflow to ensure zero global divergence.

Although the standard pressure solver enforces a solenoidal velocity field the dominant processes at the IOBC require a more direct coupling, for reasons explained below. Indeed, if a Neumann boundary condition is applied to pressure on the side walls at the pressure correction step, then, if required, extra fluid can be drawn into or out of the domain only via the outflow face. In that scenario the sides play a passive role. Such behaviour is not consistent with the flow that is induced in unconfined domains; in reality the induced flow is driven by regions of low pressure within the jet or plume and does not have an intrinsic preference for entering the domain through a particular face. In order that the induced flow field is correctly established, it is important that the treatment of pressure reflects this fact. Application of a Dirichlet boundary condition to pressure on the side walls at the pressure correction step allows the sides to play an active role. In that scenario the pressure field can draw fluid into the domain via both the sides and the top of the domain to eliminate a global divergence, which is consistent with the behaviour of unconfined domains. In this manner, active side boundary conditions allow turbulent entrainment into the jet or plume to determine the magnitude of velocities on the side walls in a natural way; pressure is dominant in determining the magnitude of the inflow whilst the IOBCs applied to velocity steer the flow in the correct direction.

For the active approach the Dirichlet condition is only applied to pressure on the side walls at the pressure correction step; a Neumann condition is applied to pressure prior to the pressure correction step. This means that in a steady state

the pressure field will have zero wall normal gradients. However, as explained in the paragraph above, the boundary condition applied at the pressure correction step is crucial in determining the route the flow takes to a steady state.

A.5 Results

A.5.1 General observations

In this and the following sub-sections we analyse data from DNS to quantify the effects that OBCs and domain aspect ratio have on a simulated flow field. In particular the data allows us to identify the origin of the disturbances seen in figure A.1. OBCs that behave correctly should be independent of their domain size and, in the interests of accuracy and computational efficiency, we would like to know which OBCs most closely satisfy this stipulation.

Qualitatively, figure A.1 indicates the form and location of the disturbances caused by classical OBCs. At the top of Ω_{C3} regions of relatively warm rotational flow have spread across the horizontal extents of the domain. On the horizontal slice it is clear that the classical OBCs have caused a significant departure from axisymmetry. In fact, the vertical slice taken at $y = 0$ does not capture the full extent of the three dimensional problems in C3; the horizontal slice shows that in the corner regions of Ω_{C3} warm fluid is transported into the ambient, far beneath the level of the OOBBC. Compared to INF and A3 the contours of horizontal velocity in C3 indicate that the fluid enters Ω_{C3} predominantly through the centre of each face, rather than through the corners. The contours of horizontal velocity in C3 are similar to those obtained from a simulation employing classical OBCs on a quasi-two-dimensional domain in §A.6, in which the departure from axisymmetry is clearly evident.

In contrast, the horizontal velocities in A3 show close agreement with the reference INF. They exhibit axisymmetry and a comparison with INF suggests that the correct volume of fluid is being entrained into the plume. Indeed, the horizontal velocities in A3 appear to exhibit greater axisymmetry than those of the reference simulation INF. The reason for this is the small influence that the square geometry of INF has on the sensitive ambient flow field, which highlights a further advantage of being able to specify appropriate OBCs.

The disturbances in figure A.1 indicate that at the top of the domain the plume in C3 is not entraining fluid correctly. If the OOBBC in C3 was behaving correctly then anomalies in the temperature field would be entrained into the plume and ejected from the domain via the OOBBC. Moreover, the IOBCs in C3 prevent

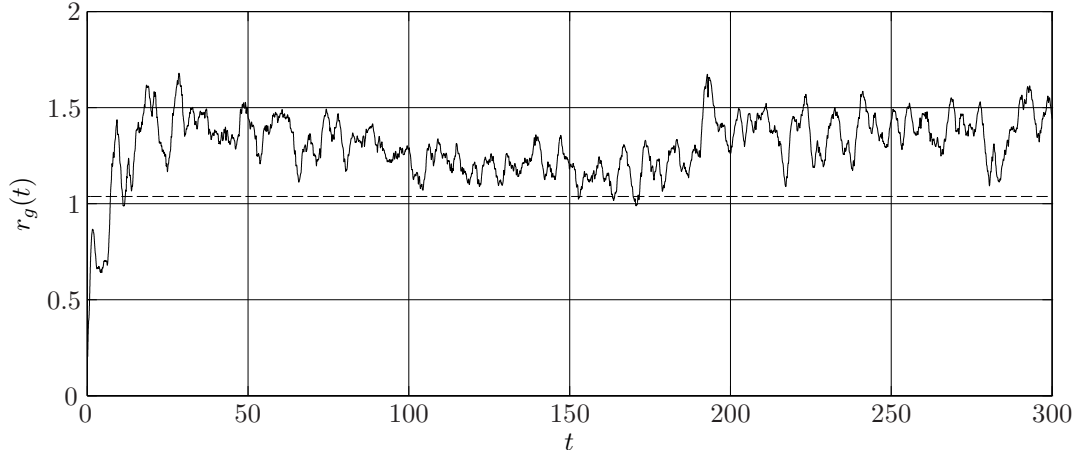


FIGURE A.2. Evolution of instantaneous outflow parameter $r_g(t)$ in plume A3 [O] and jet A3J [$\triangle$]. The variable $r_g(t)$ is determined dynamically at each time step.

the fluid from entering the domain axisymmetrically, although evidence of their influence in figure A.1 is combined with disturbances arising from the OOBBC. The cause of the underlying departure from axisymmetry is the homogenous Neumann condition to which the boundary conditions imposed on Cn tend in the steady state. It can be shown that a Neumann IOBC produces vertical vorticity when applied to a flat, vertical boundary in the vicinity of a singularity in the flow field. This is a highly undesirable feature of any flow field that is known to be approximately irrotational in its far-field ambient. The rotational force applied at the boundary in these situations has the effect of squeezing a given inflow into a smaller portion of the face through which it enters the domain. This effect was evident in the velocity field of quasi-two-dimensional simulations, which are discussed in §A.6, that were conducted to isolate the effects of the IOBCs from those of the OOBBCs (figure A.8).

A.5.2 Pressure gradient and entrainment

Figure A.2 depicts the evolution of the dynamically determined $r_g(t)$ in A3 and A3J. This clearly demonstrates the response of $r_g(t)$ to changes in the length scale of the flow at the OOBBC. In this case, the changes can arise due to the different spreading rates of jets and plumes (Carazzo *et al.*, 2006), differences in the length of the region over which the flow develops and fluctuations in time, which appear to be more significant in the plume than the jet. Whilst it appears that $r_g(t)$ for the jet steadily decreases, examination of $r_g(t)$ beyond $t = 300$ reveals that this is

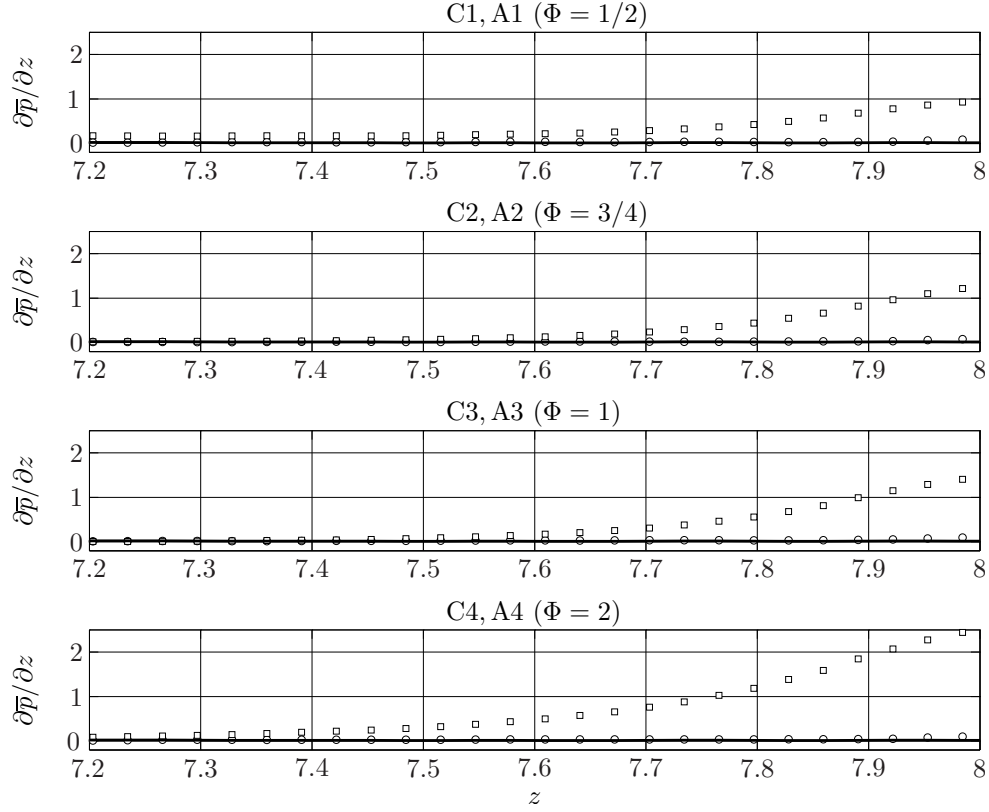


FIGURE A.3. Time averaged, normalised vertical pressure gradient, $\partial\bar{p}/\partial z$, on the centreline $r = 0$. Data from simulation INF [—] is compared to data from simulations employing uniform convective OBCs, Cn , $[\square]$ and axisymmetric OBCs, An , $[\circ]$, each employing domains of different aspect ratio Φ . The vertical extent of the domain has been truncated for clarity.

a transient effect, and that $r_g(t) \sim 0.7$.

Figure A.2 clearly shows that the new OOBBC responds dynamically to high intensity entrainment events (Plourde *et al.*, 2008), which account for a variation in $r_g(t)$ in A3 of approximately 40% on Γ_1 . In this section we quantify how the ability of the new OBCs to appropriately characterise the instantaneous convecting velocities (OOBCs) and their wall-normal gradients (IOBCs) alleviates a spurious pressure gradient on Γ_1 and allows the plume to entrain fluid correctly.

One can understand the poor performance of the standard OOBBCs, and in particular the cause of the large spreading of the plume at the top of Ω_{C3} , by comparing the centreline pressure gradients in Cn to those in An and INF. Figure A.3 shows that the classical OOBBCs produce an adverse pressure gradient on Γ_1 that is at least an order of magnitude larger than the pressure gradient at this location in An and INF. It is clear that both the magnitude of this anomaly and

the distance that it extends into the domain's interior are dependent on aspect ratio: the anomaly is over twice as large in C4 as it is in C2.

Following Gresho (1991*b*) we reveal the origin of this non-physical force by subtracting the governing equation for the wall-normal velocity, (A.3), from the wall-normal component of (A.8):

$$\frac{\partial \bar{p}}{\partial z} = (\bar{U}_n - \bar{w}) \frac{\partial \bar{w}}{\partial z} - f_z \quad \text{on } \Gamma_1, \quad (\text{A.18})$$

where the over-bar denotes a time average and f_z accounts for horizontal advection of vertical momentum, diffusion of momentum in the vertical direction and the turbulent transport of vertical momentum. Equation (A.18) demonstrates the influence on p implied by the classical OOBBC (A.8). The first term on the RHS indicates that if $f_z \approx 0$ and the local vertical velocity deviates from U_n a pressure gradient must exist, proportional to $\partial \bar{w} / \partial z$, to balance the forces in the vertical direction. It is in this term that there is a dependence on the domain of a simulation. Far from the axis of the plume, where $\partial \bar{w} / \partial z \approx 0$, these deviations will have little effect on the pressure field. Close to the axis of the plume however, where on average $\bar{w} \gg \bar{U}_n$ and $\partial \bar{w} / \partial z < 0$, we expect a large adverse pressure gradient. The consequence of this pressure gradient is a horizontal divergence in the flow just beneath Γ_1 , whose effect is evidenced in the accumulation of enstrophy and temperature at the top of the domain of C3 in figure A.1. It is important to acknowledge that the discrepancy arises from both spatial and temporal inhomogeneity in the convecting velocity.

The dependence of Cn on aspect ratio is highly problematic. In the limit $\Phi \rightarrow \infty$, $U_n \rightarrow 0$ and $\partial \bar{p} / \partial z \approx -\bar{w} \partial \bar{w} / \partial z$ on Γ_1 so that the OOBBC becomes infinitely stiff in response to convected perturbations in the flow field which are supposed to be transported across Γ_1 . This is particularly unfortunate because it means that an attempt to improve the accuracy of a simulation by increasing its aspect ratio in such circumstances could result in enhanced inaccuracy. Despite being limited to a relatively small vertical region, the elliptic nature of pressure is able to communicate disturbances throughout the entire domain instantaneously. In contrast, the simulations employing axisymmetric OOBBCs do not develop a significant adverse pressure gradient on Γ_1 and in this respect are independent of aspect ratio. Unlike the classical OOBBCs, the dynamically specified outflow profile only appeals to the flow for information and therefore accords with the profile that would be observed on an unconfined domain. As a result, $\bar{C} \approx \bar{w}$ everywhere on Γ_1 and the magnitude of non-physical effects is dramatically reduced.

One of the defining properties of a turbulent plume is that its vertical volume

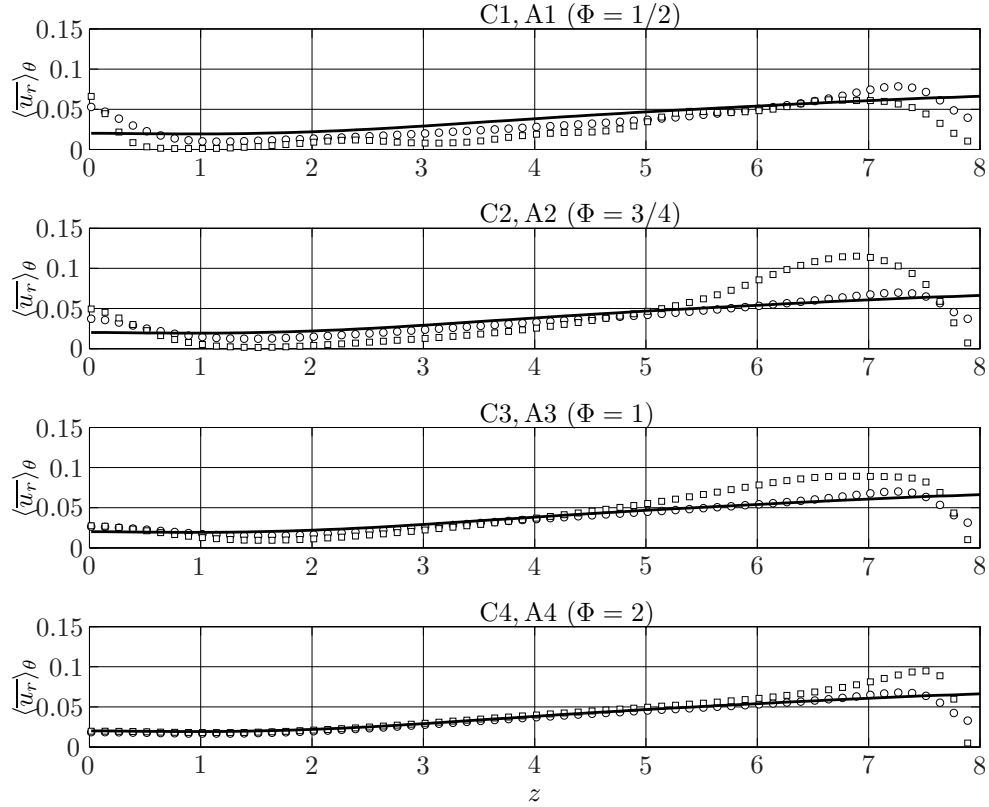


FIGURE A.4. Time and azimuthally averaged normalised velocity, $\langle \bar{u}_r \rangle_\theta$ on a vertical shell on which $r = 2$. Data from the simulation INF [—] is compared to data from simulations employing uniform convective OBCs, Cn [$\square$] and axisymmetric OBCs, An [$\circ$], on domains of different aspect ratio Φ .

flux increases with height owing to radial entrainment. It is therefore crucial that numerical simulations reproduce this process accurately. To assess this we look at how the plume's volume flux varies over the height of a domain and how this depends on the OBCs employed and on the aspect ratio of the domain.

Figure A.4 displays the vertical variation in the azimuthal and temporal mean radial inflow $\langle \bar{u}_r \rangle_\theta$, at $r = 2$. By continuity $4\pi \langle \bar{u}_r \rangle_\theta$ at $r = 2$ is equal to the vertical rate of change of volume flux over a disk of radius 2 units. The data indicates a dependence of $\langle \bar{u}_r \rangle_\theta$ on domain aspect ratio when classical OBCs are used. In particular, unless an aspect ratio of $\Phi = 2$ is used, there is a significant difference between Cn and INF when $z \gtrsim 5$. Although these inaccuracies arise from the combined influence of the IOBC and OOBC, it is believed that those arising from the OOBC are dominant. The adverse pressure gradient, evident in figure A.3, forces the plume to spread outwards at the OOBC which results in a significant reduction in the rate at which its volume flux increases. However, just

below this region inflow velocities increase significantly compared to those in INF. This correlation suggests that below the OOB the flow is dominated by large scale overturning motions. Although they are limited to the top of the domain, it is these rotational events that appear to cause instability when a total pressure IOBC is applied on Γ_V .

In contrast to Cn , for all but the smallest aspect ratio under consideration, the axisymmetric OBCs are in close agreement with INF and approximately independent of Φ . At a given height, if axisymmetric IOBCs are adopted instead of classical convective IOBCs, a comparatively smaller aspect ratio is sufficient to attain a given level of approximation to INF. Furthermore, compared to Cn , the agreement with INF in An is over a larger vertical extent, permitting more efficient use of the computational domain. This is extremely significant for the computational cost of simulations of plumes, in which additional vertical space is also required at the base of the domain to allow the flow to develop.

A.5.3 Axisymmetry

Figure A.5 quantifies the axisymmetry of the flow field over a cylindrical shell of dimensionless radius $r = 4$ in C3 and A3 over (t, z) space. We define an indicator for the axisymmetry of the flow on the shell as

$$S(z, t) = \left[\frac{\left(\int_0^{2\pi} u_r d\theta \right)^2}{\int_0^{2\pi} u_r^2 d\theta} \right]_{r=4}. \quad (\text{A.19})$$

If the flow is perfectly axisymmetric over the shell at a given height then $S(z, t) = 1$. In the limit in which all the flow enters the shell at the midpoint of each face $S(z, t) = 0$. The use of (t, z) space allows us to understand not only the temporal rate with which a flow field converges to a particular distribution of radial velocities, but also how this convergence depends on vertical location in a domain.

Isolines of $S(z, t)$ in figure A.5 show that in C3 the flow field is far from being axisymmetric. The departures from axisymmetry in this case are non-uniform in space and time. In space the most significant departures from axisymmetry occur at the base and at the top of the domain. In time the axisymmetry progressively deteriorates. Apart from small discrepancies at the base of the domain, the axisymmetry of simulation A3 is excellent in comparison. The normalised form of $S(z, t)$ and the very small velocities that occur in the ambient at the base of the domain, suggest that the discrepancies in A3 are insignificant. Unlike C3, the axisymmetry in A3 does not show obvious signs of deterioration over time and agrees

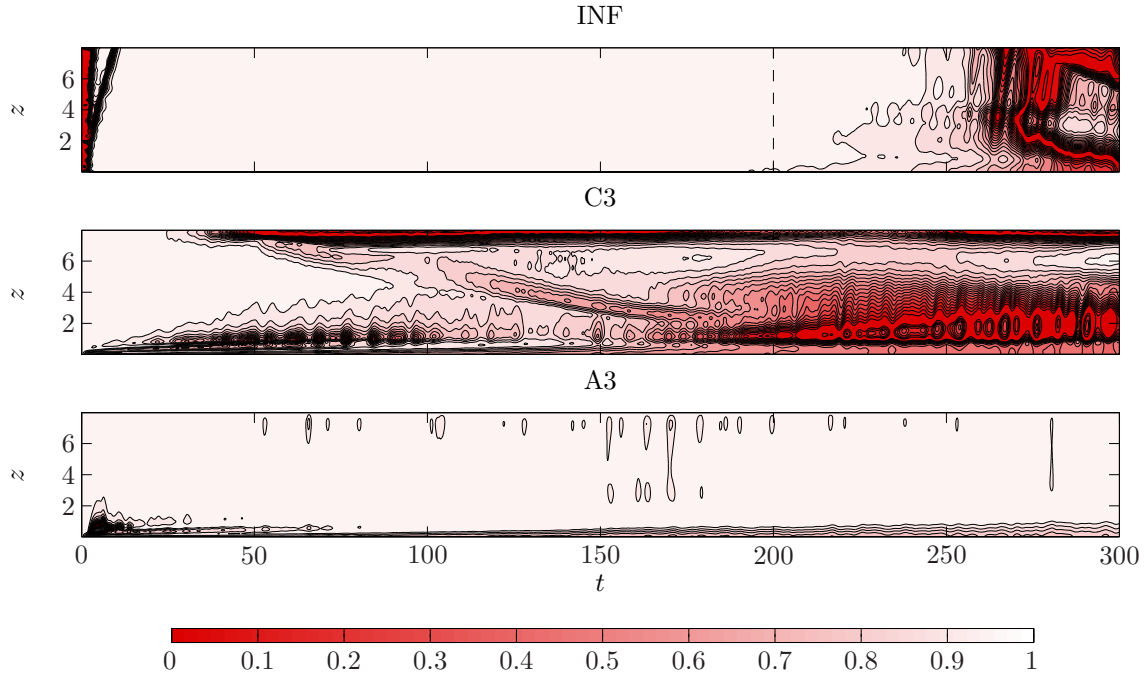


FIGURE A.5. Isolines of the axisymmetry parameter, $S(z, t)$, on a vertical shell with $r = 4$, in simulation INF (top), C3 (middle) and A3 (bottom). Isolines range from 0 (no axisymmetry, red) to 1 (full axisymmetry, white) in increments of 0.05. Upper limit beyond which the closed box geometry in simulation INF starts to significantly influence the velocities in the ambient [— —].

well with the reference simulation INF when $t \lesssim 200$. When $t \gtrsim 200$ it is evident that the geometry of the closed domain of INF starts to significantly influence the velocities in the ambient and INF no longer provides an adequate approximation of an unconfined domain.

Figure A.5 allows us to infer several things about the way in which deviations in axisymmetry engulf the domain. The first disturbances to axisymmetry appear to occur close to Γ_0 . These are not large disturbances, but they are significant relative to the small radial flux of radial momentum entering the domain at that level. After a time which is comparable to the time it takes for a fluid parcel to travel the entire height of the domain, $t \approx 16$, deviations in axisymmetry beneath Γ_1 appear. These are due to the horizontal divergence in the flow field beneath Γ_1 and its subsequent interaction with the sides of the domain. Following this interaction, significant departures from axisymmetry propagate downwards at a constant rate. From $t \approx 160$ the flow field is highly non-axisymmetric at the top of the domain and in the region $0 \lesssim z \lesssim 4$, which one may infer correspond to the regions in which the effects of the OOBBC and IOBC are likely to dominate, respectively. In the region

$5 \lesssim z \lesssim 7$ at large times C3 exhibits moderate axisymmetry in comparison to the rest of its domain. Therefore, the undesirable blocking effect of the classical OBC results in the re-entrainment of fluid into the plume some distance beneath Γ_1 that is apparently more axisymmetric than the fluid that is either entrained through the IOBC or disrupted immediately beneath Γ_1 .

Most importantly, the transition to an approximately steady state in C3 occurs over a time scale that is comparable to the time interval over which statistics for such flows are normally obtained (Plourde *et al.*, 2008), which places an onerous requirement on the total duration of such simulations. If classical convective OBCs are employed the simulation will either exhibit large scale transients over a moderately axisymmetric field ($t \lesssim 80$), a steady state over a predominantly non-axisymmetric field ($t \gtrsim 240$), or a combination of the two. The large time scales that are characteristic of the flow in the ambient make the selection of appropriate OBCs imperative.

A.5.4 High-Reynolds-number plume

In order to demonstrate the robustness of the new OBCs in this section we subject them to a more stringent test by simulating a high-Reynolds-number plume. For these simulations we chose a single aspect ratio of $\Phi = 1$, as it is typical of the aspect ratio that is likely to be employed in practice.

Figure A.6 displays results for a simulation with classical convective OBCs (left window) and the new axisymmetric OBCs (right window). The left hand side and right hand side of each window contains instantaneous and time averaged results, respectively. Also shown are isolines of the stream function ψ , defined by

$$ru_r = \frac{\partial \psi}{\partial z}, \quad rw = -\frac{\partial \psi}{\partial r}. \quad (\text{A.20})$$

The inset windows show contours of absolute horizontal velocity at $z = 7$.

Compared to AR, simulation CR exhibits significant non-physical effects that are likely to be dependent on aspect ratio. It is visible from figure A.6 that the plume in CR has a larger horizontal spread at a given height than in AR. This, once again, indicates that inappropriate OBCs cause a fundamental change in important plume properties, such as the rate with which they entrain fluid from their ambient. The streamlines in CR show that fluid is drawn into the domain near the edges of the outflow face, which results in a counter-flow throughout the height of the domain. Furthermore, the prescription of zero vertical velocities on Γ_V indicates that this disturbance is non-axisymmetric, taking place predominantly in

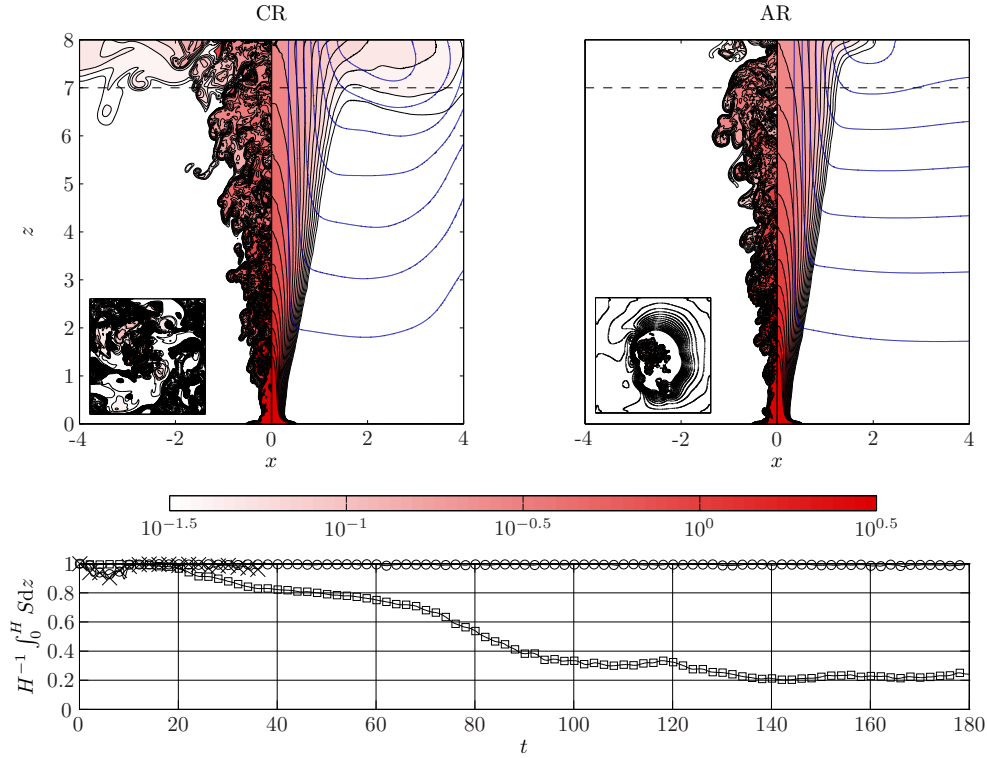


FIGURE A.6. Top: vertical slice through instantaneous (LHS) and time averaged (RHS) temperature field for simulations CR and AR, and isolines of normalised stream function ψ [—]. Each sub-window displays instantaneous isolines of normalised absolute horizontal velocity $\sqrt{u^2 + v^2}$ [—] and temperature over the horizontal slice on which $z = 7$ [---]. Bottom: evolution of global axisymmetry, $H^{-1} \int_0^H S(z, t) dz$ in simulation CR [$\square$] and AR [$\circ$]. Results from a simulation employing total pressure BCs on Γ_V until time $t = 36$, at which point instability occurred [$\times$].

the corner regions. Therefore, the approximately horizontal ambient streamlines in AR provide an indirect indication of axisymmetry. In contrast with CR, the vertical velocities in the corners of the outflow face in AR are small, which is consistent with the physics of the unconfined problem. This difference can be attributed to the profile of convecting velocity, $C(x, y)$, employed at the outflow. In AR the negligible convecting velocity applied to the corners of the outflow results in a stiff response, which prevents the non-physical development of vertical velocities that are seen in CR.

Although very mild distortion in the mean temperature field can be observed in AR it is limited to a region of small vertical extent and, unlike CR, does not appear to interact with the sides of the domain. In addition, the use of a non-uniform

convective outflow velocity was found to significantly reduce the occurrence of non-physical oscillations just beneath Γ_1 , which were regularly observed in equivalent simulations employing classical convective OBCs due to steep gradients in the vertical velocity near the OBC.

In the lower half of figure A.6 we show how the vertical integral of $S(z, t)$ evolves over time in simulations CR and AR. The results indicate that AR remains almost 100% axisymmetric throughout the duration of the simulation, whereas CR degenerates into a strongly non-axisymmetric field when $t \gtrsim 60$. It should be noted in the interpretation of figure A.6 that the duration over which statistics are normally obtained corresponds to approximately 60 units of dimensionless time. It is therefore impossible to obtain both statistically stationary and axisymmetric data from CR.

The lower half of figure A.6 includes incomplete data obtained from a simulation that was identical to AR with the exception of employing total pressure BCs on Γ_V [Table A.1, IOBCs (c)]. The total pressure BCs become unstable after a certain time ($t \approx 36$ in this case), which is typical behaviour when simulating turbulent plumes on domains with a small aspect ratio. The instability appears to originate at the top of the domain, where patches of high vorticity are most likely to interact with the sides of the domain. Because the total pressure BCs do not account for vorticity, the presence of vorticity close to the boundary results in very large pressure gradients. Although the simulation of lower Reynolds number flows or the use of larger domains can mitigate, or at least postpone, such instabilities, these approaches are at odds with the reasons for employing DNS and its viability, respectively.

A.5.5 *High-Reynolds-number jet*

Figure A.7 displays data obtained from the simulation of a high-Reynolds-number jet, with $\text{Re} = 5000$. Both the vertical and horizontal slice through CRJ suggest that the performance of the classical convective OBCs is improved by the absence of buoyancy from the problem. Although it could be argued that buoyancy is able to help the flow exit from the domain, our experience has indicated that the problems arising from its presence far outweigh any advantages. These problems include the need to correctly remove buoyancy from the domain; inflow through the OBC can result in the non-physical transport of either relatively dense or relatively buoyant fluid into the domain. The possibility of either of these events is greatly increased when classical convective boundary conditions are employed, in which the uniform convecting velocity ensures that the periphery of the OBC plays an active role. In

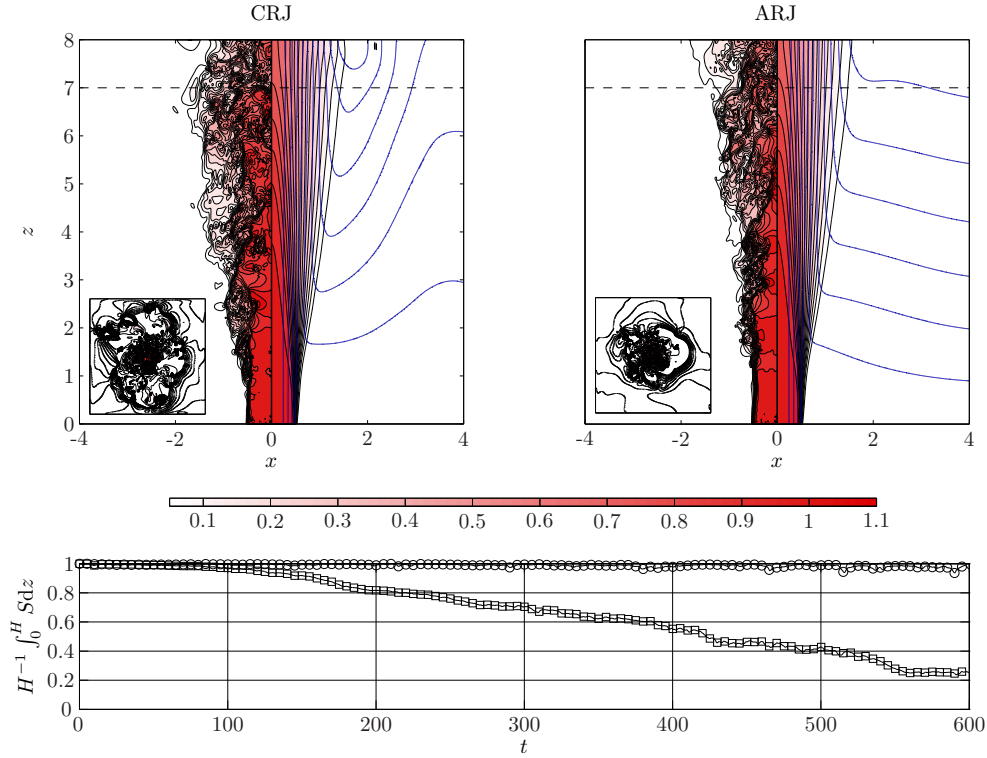


FIGURE A.7. Top: vertical slice through instantaneous (LHS) and time averaged (RHS) vertical velocity field for simulations CRJ and ARJ, and isolines of normalised stream function ψ [—]. Each sub-window displays instantaneous isolines of normalised absolute horizontal velocity $\sqrt{u^2 + v^2}$ [—] and vertical velocity over the horizontal slice on which $z = 7$ [— —]. Bottom: evolution of global axisymmetry, $H^{-1} \int_0^H S(z, t) dz$ in simulation CRJ [$\square$] and ARJ [$\circ$].

spite of the absence of these complications, the streamlines in the environment in CRJ indicate that approximately half of the vertical extent of the jet is entraining fluid via the OIBC rather than the IOBC, which is evidently non-physical. The lower half of figure A.7 shows that axisymmetry in CRJ gradually deteriorates, resulting in a poor representation of an unconfined domain in addition to large-time unsteadiness. Simulation ARJ is statistically steady and provides a more accurate representation of the induced flow in the environment. Also noteworthy is the apparently larger region of flow development in jets (figure A.7) compared to plumes (figure A.6). A useful feature of the proposed OBCs is that these differences are accommodated for automatically. Owing to the vertical variation of entrainment close to the source, it is likely that the region of flow development is responsible for the inclination of the streamlines in the ambient in simulation ARJ, compared to AR.

A.6 Two-dimensional simulations

In order to isolate the effects of the IOBC in a simplified context we conducted simulations of the quasi-two-dimensional flow induced by a sink with a dimensional volume flux $w_0\pi D^2/4$ in the centre of a shallow rectangular domain. The governing equations used in these simulations were those described in §A.2. The analytical solution of this problem (ANA), in terms of the dimensionless variables u_r and r , is given by

$$u_r = -\sqrt{u^2 + v^2} = -\frac{D}{8H} \frac{1}{r}. \quad (\text{A.21})$$

The normalised height of the rectangular domains was $H/D = 1/2$, where D is the sink diameter, and their aspect ratio was $\Phi = 64$. Each simulation employed a grid resolution of 4 cells per sink diameter.

On each cell over the volume sink, located in the centre of Γ_0 , we specified a constant negative vertical velocity, w_0 , to give $Re = w_0 D/\nu = 250$. All dependent variables are non-dimensionalised using w_0 and D . On the vertical inflow faces, Γ_V , we implemented five different IOBCs. In C2D we applied a classical convective IOBC by setting $\mathbf{f}_{ijk} = 0$ in (A.17). In P2D we specified total pressure IOBCs [Table A.1, IOBCs (c)]. The remaining simulations A2D, AL2D and AH2D all employ the axisymmetric IOBC ((A.17)) described in §A.4.2. However, for simulations A2D, AL2D and AH2D we defined the two-dimensional volume flux at a given height, $z = z_k$:

$$q_k(t) = \lambda \left(\frac{\delta}{\delta z} \sum_{ij} w_{ijk}(t) \delta A \right), \quad (\text{A.22})$$

where $\lambda = 1$ in A2D, such that $q_k(t)$ accords with the actual two-dimensional volume flux. To demonstrate the effects of incorrectly specifying $q_k(t)$, in AL2D and AH2D we intentionally under- and over-specified the two-dimensional volume flux, adopting $\lambda = 1/2$ and $\lambda = 2$, respectively.

The effects of applying a boundary condition that does not account for a problem's continuous axisymmetry are evident in figure A.8. Whilst in all cases the flow is seen to have a discrete axisymmetry, continuous axisymmetry is only attained by P2D and A2D. The results from P2D and A2D are almost indistinguishable, in spite of their different boundary conditions. Both total pressure IOBCs and the axisymmetric IOBCs admit the solution for the flow induced by the sink on an unconfined domain. However, this correspondence is only true in a steady state. Unlike the axisymmetric IOBCs, the total pressure IOBCs are not prognostic and

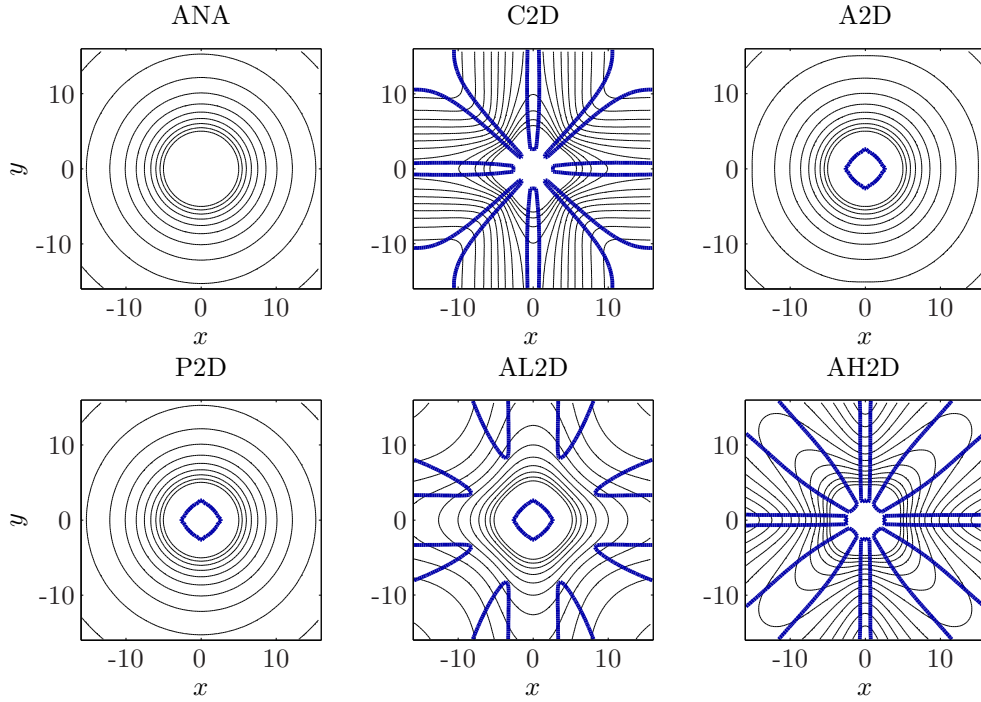


FIGURE A.8. Uniformly spaced isolines of normalised instantaneous absolute horizontal velocity, $\sqrt{u^2 + v^2}$, [—] and instantaneous threshold of normalised enstrophy $\frac{1}{2}\|\nabla \times \mathbf{u}\|^2 > 3.0 \times 10^{-7}$ [—]. Each window shows a horizontal slice at $z = 1/4$ through the domain of a simulation.

it is to this fact that small differences between either ANA and A2D or P2D and A2D can be attributed. In the absence of an unsteady forcing A2D attains full axisymmetry in the limit $t \rightarrow \infty$.

In simulation C2D most of the flow enters the domain through the middle of each vertical face. The contours showing an enstrophy threshold reveal that the IOBCs in C2D produce vorticity either side of the centre of each vertical face. The results from AL2D and AH2D help to explain the departure from axisymmetry seen in C2D. In AL2D and AH2D the flow predominantly enters the domain through the middle of each vertical face and the corner regions, respectively. In the steady state the boundary condition used in C2D is equivalent to the homogeneous Neumann condition $\partial \mathbf{u}_h / \partial n = 0$ and as such illustrates a limiting case: no account is made of the volume flux ($q_k = 0$) and the flow is constrained to enter through the middle of each vertical face.

The steady-state IOBC implied by (A.17) forces the flow to attain a specified wall-normal gradient, whose amplitude is controlled by $q_k(t)$. When $q_k(t)$ is specified to differ from the value that accords with the unconfined axisymmetric solution

it causes a departure from continuous axisymmetry. Such flows are a legitimate solution of the boundary value problem, although they represent a different balance of the terms in the governing equation. In such cases viscous stresses diffuse a non-zero vorticity that is produced at the boundary by the artificial torque that is required to force the flow through the middle or corner of each face.

A.7 Conclusions

We have presented OBCs whose behaviour conforms with the most important features of physically unconfined, statistically axisymmetric turbulent flows. The OBCs are dynamic and consequently do not require specification of the magnitude or extent of a statistically axisymmetric flow. The OBCs have been applied to a high-Reynolds-number jet and plume, and provide a significant improvement over classical OBCs for these flows.

The OOBC described in this paper allows fluid to leave a computational domain without causing significant disturbance by accounting for spatial and temporal variation in the velocity with which fluid is convected across a boundary. The IOBC forces fluid to enter the domain in the correct manner by accounting for the strength of the entraining axisymmetric flow at each vertical level in the domain. Most importantly, the new OBCs are independent of the computational domain. In contrast, classical convective boundary conditions are not invariant with aspect ratio for the flows under consideration, giving rise to serious disturbances when both large and small aspect ratios are employed. The prognostic form of the new OBCs allows the flow to adjust innocuously to departures from axisymmetry and transient turbulent events. Although the necessity of this feature is not immediately apparent at the IOBC, where the flow is approximately steady and irrotational, it provides a safeguard against transient events that would otherwise cause instability.

The new OBCs require very few additional computations at each time step and do not require complex boundary conditions for pressure. Of primary benefit is their ability to support computational domains that are not significantly bigger than the diameter or height of the jet or plume under consideration. In addition, compared to the classical OBCs, the new OBCs substantially reduce the vertical extent of contamination beneath the outflow and therefore use the computational domain more efficiently. They are able to faithfully reproduce flow induced in an unconfined ambient and do not contaminate the process of entrainment as much as classical OBCs. However, even when axisymmetry preserving OBCs are adopted the results indicate that it is important to use a sufficiently large aspect ratio,

which we estimate must be such that $\Phi > 3/4$.

The basic principle outlined in this paper can be readily extended to other problems. Any localised turbulence that is transported in a direction that is predominantly normal to an outflow boundary can be treated using this approach. It is not necessary for the flow to be statistically stationary; the dynamic OOBBC will naturally adjust to the scale of a flow at each time step. This means that for problems in which the fluxes of buoyancy or momentum at the source are time dependent (see, e.g., Scase *et al.*, 2006b) an OOBBC can be implemented using the method described in this paper without alteration. The OOBBC in simulations of planar jets and plumes is amenable to a simplified version of the treatment discussed here. Furthermore, we have successfully applied the technique in simulations of a turbulent fountain, for which an OBC at the level of the source must support both inflow and outflow. For that purpose we impose an outflow profile for the convecting velocity that is annular and scaled dynamically to suit the most important features of the outflow. The method presented applies equally well to non-rectangular domains because it does not depend on the geometry or size of a computational domain. In situations where the simulation of multiple sources of buoyancy or momentum is required (Wang & Davidson, 2003) the basic form of the OOBBC and IOBC described in this paper can still be used. To describe multiple sources both the convecting velocity profile and the inhomogeneous condition applied at the IOBC would need to contain a contribution from each source and a dependence on their location. The dynamic determination of such an offset would avoid the need to make *a priori* specifications and could also be used to account for the meandering of a plume. Whilst there is room to improve the methods described here, complicated refinements such as the use of higher order moments of the flow, are unlikely to bring substantial benefits.

A more refined determination of the convecting velocity C , used to determine the rate with which the IOBCs are enforced, warrants further attention. Here, a balance between the computational benefits of rapidly enforcing the correct distribution of inflow and robustness must be sought. In this regard it would be worth considering a definition of C whose relatively large initial value decays to $C = U_n$ before statistics are obtained. In theory, this would allow a correct inflow to be established quickly without exposing the simulation to the risk of instability at later times.

