Measures and amalgamation properties in ω-categorical structures

A thesis submitted for the degree of Doctor of Philosophy in Mathematics
by

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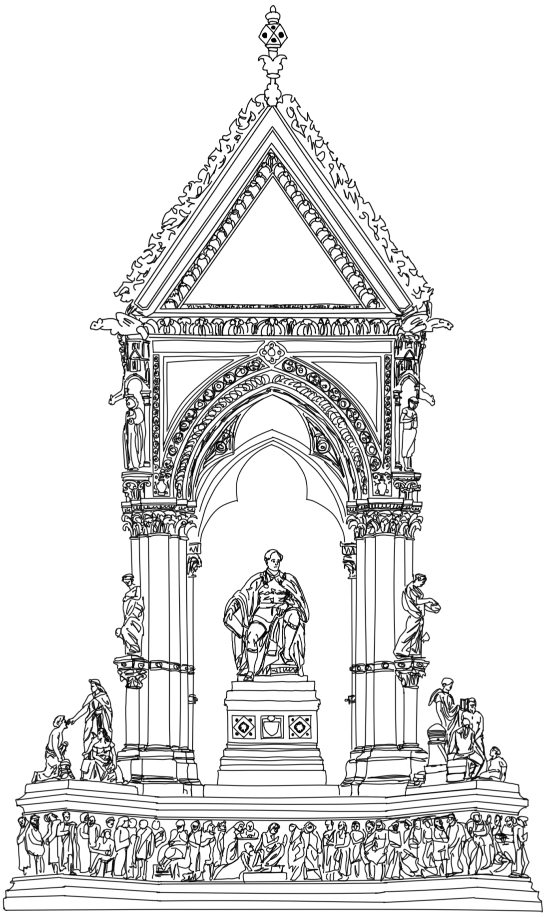

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To Lucas

Statement of Originality

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Abstract

This thesis explores the relationship between measures and amalgamation properties in simple ω-categorical structures. In particular, we prove that a stronger version of the independence theorem is satisfied by simple ω-categorical structures where a formula forks over $\emptyset$ if and only if it is assigned measure zero by every invariant Keisler measure. This will be used to show that certain classes of simple ω-categorical Hrushovski constructions have non-forking (over $\emptyset$) formulas which are assigned measure zero by every invariant Keisler measure. Until a recent article of Chernikov, Hrushovski, Kruckman, Krupinski, Moconja, Pillay and Ramsey, it was unknown whether there were any simple structures where forking and being universally measure zero disagreed. We give the first simple ω-categorical example of this phenomenon.

We also study ω-categorical MS-measurable structures in the sense of Macpherson and Steinhorn. Our main results show that various classes of ω-categorical Hrushovski constructions and the generic tetrahedron-free 3-hypergraph fail to be MS-measurable in spite of being supersimple of finite rank. These examples answer negatively a question of Elwes and Macpherson on whether being supersimple ω-categorical of finite SU-rank implies being MS-measurable. Our results on ω-categorical Hrushovski constructions complement previous work of Evans who had given a counterexample to this question. Unlike Evans' example, our Hrushovski constructions may satisfy independent n-amalgamation for all n over finite algebraically closed sets. We also develop tools to study the measures of higher amalgamations in MS-measurable structures. With these, we show that the generic tetrahedron-free 3-hypergraph is not MS-measurable giving the first one-based ω-categorical supersimple example of this phenomenon and showing that this structure cannot be elementarily equivalent to the ultraproduct of an asymptotic class of finite structures. Finally, we show that the universal homogeneous two-graph has a unique invariant Keisler measure not induced by any invariant type.

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I will always be grateful to my partner Katie for her support, love, and help during my PhD. I also thank her for volunteering to read some of my papers and making suggestions on better use of the English language. The paragraph below contains few of the many instances of "Hence" that have been removed from the body of this thesis due to her proof-reading, but which still wanted a home:

Hence etc.

David and Lotte also contributed to eradicating improper uses of "Substantially" and "Disanalogy" from this thesis. I am sure those words will find their home in future, less proof-read, writings of mine.

I would like to thank my parents and my sister for their support during my PhD. In particular, I thank my parents for the harmony with which we managed to live and work while being completely isolated during the Covid-19 pandemic. I thank my sister for helpful conversations in important moments of my PhD. This thesis is dedicated to her son who was born roughly at the same time as the first full draft of this thesis was completed. I would also like to thank my grandmother Rosanna, whom I miss very much.

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1 Introduction

There are various ways in which measures naturally arise in model theory. On one hand, they give us a good way of comparing sizes of definable sets in infinite structures. On the other, we are often interested in infinite structures which capture the asymptotic behaviour of classes of finite structures for which the size of definable sets satisfies certain uniformity assumptions [50]. Such structures usually have well-behaved measures.

Consider an infinite graph G as a first order structure in the language consisting of a binary edge relation $E(x, y)$. We would like to have a good notion of size on the definable subsets of G. For example, we would like to show, under adequate model theoretic and uniformity assumptions, that, for $a, b \in G$, if the sets defined by $E(x, a)$ and $E(x, b)$ are large, then so must be their intersection, i.e. the set of vertices sharing an edge with both a and b. Ideally, we would also like to have some information about the size of the latter when compared to the initial sets (cf. Corollary 3.2.16). A way to study such results is looking at finitely additive probability measures on the Boolean algebra of subsets of G defined by formulas with parameters from G. We are especially interested in measures which are invariant under the action of the automorphism group $\mathrm{Aut}(G)$. Invariant Keisler measures generalise this idea to arbitrary first-order structures. Formally, an invariant Keisler measure on a first-order structure $\mathcal{M}$ in the variable x is an $\mathrm{Aut}(M)$-invariant finitely additive probability measure on the Boolean algebra of definable subsets of M^x with parameters from M (i.e. $\mathrm{Def}_x(M)$). For brevity of language, we shall usually speak of the measure of formulas, rather than the measure of a set defined by a formula.

Let us look at the random graph [62]. Given n vertices, toss a coin for each pair of distinct vertices to decide whether to join them by an edge. We allow for the coin to be unfair with the probability of an edge being $p \in (0,1)$. In this way, we obtain a finite graph G_n. As n increases, the probability of certain events, such as G_n containing a triangle, tends to 1, while the probability of others, such as G_n having an isolated vertex, tends to 0. Indeed, one can prove that for any first order sentence in the language of graphs, the probability of its truth in G_n tends to either 0 or 1. Hence, the collection of all sentences whose probability tends to 1 forms a complete theory. This theory has a unique countable model up to isomorphism, i.e. it is ω-categorical, and we call this model the random graph R.

The theory of the random graph has quantifier elimination, which means that any definable subset of R is a Boolean combination of finitely many vertices and sets defined by formulas of the form $\phi(x, A, B)$, where for A and B finite and disjoint this formula asserts that x is joined by an edge to every vertex of A and none of B. Since the action of $\mathrm{Aut}(R)$ on R is transitive, any $\mathrm{Aut}(R)$-invariant finitely additive probability measure on the definable subsets of R assigns measure zero to the finite subsets of R (and measure 1 to its cofinite subsets). By quantifier elimination and finite additivity, the measure of any definable subset of R can be written as the sum of the measures of formulas of the form of $\phi(x, A, B)$.

We can also construct the random graph by building a sequence of finite graphs Ω_n as follows [2]: start with a single vertex for Ω_1. Then, obtain Ω_{n+1} by adding a vertex to Ω_n and tossing a coin (with probability $p \in (0,1)$ of heads) for each vertex of Ω_n to decide whether the new vertex forms an edge with it. The union of the various Ω_n is isomorphic to R with probability 1. This construction induces an $\mathrm{Aut}(R)$-invariant Keisler measure μ_p on the definable subsets of R, where for any finite and disjoint A and B we have that

$$\mu_p(\phi(x, A, B)) = p^{|A|}(1-p)^{|B|}.$$

Setting $p = 0$ or $p = 1$ in the definition above also yields $\mathrm{Aut}(R)$-invariant Keisler measures on the random graph. In particular, the latter two measures correspond to the two invariant types on R. A type over R assigns value 0 or 1 to every definable subset of R, and it can be viewed as a $\{0,1\}$-valued Keisler measure on R. Conversely, Keisler measures are a generalisation of types where definable sets are assigned values in $[0,1]$.

The invariant Keisler measures on R have been studied by Albert in [2], who provides a full classification of them:

Theorem 1.0.1. *Let μ be an* $\mathrm{Aut}(R)$*-invariant Keisler measure in the singleton variable x for the random graph R. Then, there is a unique probability measure ν on* $[0,1]$ *such that for A, B finite and disjoint,*

$$\mu(\phi(x, A, B)) = \int_0^1 p^{|A|}(1-p)^{|B|}\mathrm{d}\nu.$$

Essentially, Albert's theorem asserts that any invariant Keisler measure on R is an integral average of the measures μ_p. Indeed, the measures μ_p play a special role among the $\mathrm{Aut}(R)$-invariant Keisler measures: they form the space of $\mathrm{Aut}(R)$-ergodic Keisler measures for R. As we shall see in Chapter 3, for an ω-categorical structure, ergodic measures are particularly well-behaved. Most importantly, any invariant measure can be obtained as an integral average of them [52]. This observation of Albert will be essential to the work of this thesis.

With the exception of Albert's work, most early work on Keisler measures focused on a particular class of theories: NIP theories. These are a generalisation of stable theories which include dense linear orders, real closed fields, and real exponential fields. Indeed, the study of Keisler measures begins with the work of Keisler [42], who noticed that some of the aspects of the theory of forking for types in stable theories can be extended to NIP theories by looking at measures instead. Subsequently, a very sophisticated theory for Keisler measures was developed in the NIP context [37, 39, 38] with various techniques to reduce questions about measures to problems regarding types [38, Proposition 4.6, Lemma 4.8].

This thesis focuses on studying invariant Keisler measures in the context of simple theories. These are a class of theories characterised by a well-behaved notion of independence. They include the random graph, pseudofinite fields and the theory of algebraically closed fields with a generic automorphism. Their intersection with the class of NIP theories consists of stable theories, which includes vector spaces over $\mathbb{Q}$, algebraically closed fields and the theory of an infinite set with equality.

Invariant Keisler measures in a strictly simple context (i.e. simple but not stable) are not very well understood, and there is essentially no systematic study of them. Nevertheless, they naturally appear in the analysis of many simple theories. This is especially true in the context of simple structures which arise from classes of finite structures for which the asymptotic behaviour of definable sets is particularly good. A major motivating result on this front is the Chatzidakis, Van Den Dries and Macintyre theorem on definable sets in finite fields [9] extending the Lang-Weil estimates [47]:

Theorem 1.0.2. *Let $\phi(\overline{x},\overline{y})$ be a formula in the language of rings $\mathcal{L}_{ring}$ with $|\overline{x}| = n, |\overline{y}| = m$. Then, there is $C > 0$ and a finite set $D \subseteq \{0,\dots,n\} \times \mathbb{Q}^{>0}$ of pairs (d,μ) such that for each finite field $\mathbb{F}_q$ and $\overline{b} \in \mathbb{F}_q^m$,*

$$\left|\left|\phi(\mathbb{F}_q^n,\overline{b})\right| - \mu q^d\right| \leq Cq^{(d-\frac{1}{2})} \tag{$\triangle$}$$

for some $(d, \mu) \in D$.

Moreover, for each $(d, \mu) \in D$*, there is a formula* $\psi_{(d,\mu)}(\overline{y})$ *which in each* $\mathbb{F}_q$ *defines the set of tuples* $\overline{b}$ *for which* ($\triangle$) *holds.*

A pseudofinite field is a model of the common theory of finite fields. More generally, a complete theory is pseudofinite if for any sentence of the theory there is a finite structure satisfying it. From the result above, one can assign each definable subset of a pseudofinite field a dimension and a measure [8]: since D is finite, each m-tuple $\overline{b}$ from a pseudofinite field will satisfy exactly one of the $\psi_{(d,\mu)}(\overline{y})$. Then, the set defined by $\phi(\overline{x}, \overline{b})$ can be assigned dimension d and measure μ. The resulting measure and dimension are particularly well-behaved (e.g. they are definable and satisfy a version of Fubini's Theorem).

Macpherson and Steinhorn used Theorem 1.0.2 as a template in order to isolate a property of classes of finite structures with interesting asymptotic behaviour and a property of theories of infinite structures which in virtue of being ultraproducts of such classes end up having a good notion of dimension and measures on them [50]. Hence, they came with the definition of 1-dimensional asymptotic classes (later generalised to N-dimensional in [18]) and MS-measurable structures. In addition to finite fields, various classes of finite structures form asymptotic classes, such as the Payley graphs [50], whose ultraproduct is elementarily equivalent to the random graph, and finite simple groups of Lie rank at most d [59]. Any non-principal ultraproduct of an asymptotic class is MS-measurable, but an MS-measurable structure need not be pseudofinite or elementarily equivalent to an ultraproduct of an asymptotic class [19]. From a model theoretic point of view, MS-measurable structures are always supersimple of finite SU-rank. In the ω-categorical context, the generic k-hypergraphs (including the random graph), and the smoothly approximable structures of [11], which include ω-categorical ω-stable structures, are all MS-measurable. Indeed, the final pages of [11, p.184-186] already show an interest in the measures naturally arising in smoothly approximable structures.

In the supersimple finite rank ω-categorical context, MS-measurability appeared to be so pervasive that it was unclear whether there could be any such structure which was not MS-measurable. This was asked by Elwes and Macpherson in [19] and will be the main question of Chapter 5 and also vital to Chapter 6:

Question 5.0.1. *Does being ω-categorical supersimple of finite SU-rank imply MS-measurability?*

This question was recently answered in the negative by Evans [23]. The main idea behind Evans' proof is that MS-measurable structures must satisfy a weak version of independent n-amalgamation. Hence, building supersimple ω-categorical Hrushovski constructions of SU-rank 1 for which such amalgamation property fails, he gave the first counterexample to Question 5.0.1.

Something that remained unclear from Evans' result was whether satisfying strong enough independent amalgamation properties could imply being MS-measurable. In fact, it is possible to build ω-categorical Hrushovski constructions which are supersimple of finite SU-rank and satisfying independent n-amalgamation for all n (cf. Subsection 9.2.2, [35], and [68, Exercise 6.2.28]). The initial question of the PhD project which constitutes this thesis was whether one could prove that some supersimple ω-categorical Hrushovski constructions of finite SU-rank still failed in being MS-measurable in spite of satisfying independent n-amalgamation for all n (over finite algebraically closed sets). We managed to show this for various classes of these structures.

We also wanted to understand better MS-measurability in the context of ω-categorical Hrushovski constructions. Firstly, we wanted to understand whether any not one-based supersimple ω-categorical Hrushovski construction could be MS-measurable. In fact, in [19], Elwes and Macpherson also ask the following:

Question 5.0.2. *Is every ω-categorical MS-measurable structure one-based?*

In simple ω-categorical one-based structures, non-forking independence corresponds to a notion of independence that we call weak algebraic independence. The only known supersimple ω-categorical structures which are not one-based are Hrushovski constructions. Hence, if any ω-categorical Hrushovski construction is MS-measurable, Question 5.0.2 can be answered in the negative. Secondly, if we were not able to find any MS-measurable ω-categorical Hrushovski constructions, we wanted to give general reasons for which these structures tend to be not MS-measurable. We wanted to possibly relate these reasons to being not one-based. Finally, we wanted to understand whether, by investigating the MS-measurability of ω-categorical Hrushovski constructions we could obtain further insights on the question of whether they are pseudofinite or not.

The question of pseudofiniteness lurks in the background of this thesis. All of the structures that we show to be not MS-measurable are structures for which it is an open question whether they are pseudofinite or not. Regarding ω-categorical Hrushovski constructions, we know that these structures cannot have the finite submodel property [16], i.e. they satisfy sentences which cannot be satisfied by any of their finite substructures. It is currently an open question of Macpherson whether there are ω-categorical pseudofinite structures without the finite submodel property [49]. So, if any of these were pseudofinite, they would yield a negative answer to the question of Macpherson. Alternatively, proving that any of these are not pseudofinite would also be a substantial result since at the moment we have very few tools to show non-pseudofiniteness. In fact, from a conversation with Macpherson, it seems that there are no known examples of simple ω-categorical structures which have been proven to not be pseudofinite. The generic tetrahedron-free 3-hypergraph is another important example of a simple ω-categorical structure for which it is an open question of Cherlin whether it is pseudofinite or not [10]. In this thesis, we show that many classes of ω-categorical

Hrushovski constructions (Chapters 5 and 6) and the generic tetrahedron-free 3-hypergraph (Section 7.3) are not MS-measurable, thus showing that they cannot be elementarily equivalent to the ultraproduct of an asymptotic class [50, Lemma 5.4]. This is potentially a step forwards in understanding whether these structures are pseudofinite. A pseudofinite structure is elementarily equivalent to the ultraproduct of a class of finite structures. In a pseudofinite structure which is not MS-measurable, this class cannot be an asymptotic class. Our results somehow reflect the fact that the standard techniques for showing pseudofiniteness through algebraic or probabilistic constructions (cf. [46, Introduction]) are not very fruitful when investigating the pseudofiniteness of ω-categorical Hrushovski constructions or the generic tetrahedron-free 3-hypergraph.

Chronologically, the first major result of this thesis that we obtained is the theorem of Chapter 5 that a certain class of supersimple ω-categorical Hrushovski constructions of finite SU-rank is not MS-measurable. In contrast with Evans' result, some of these structures also satisfy independent n-amalgamation over finite algebraically closed sets for all $n \in \mathbb{N}$. An essential tool for this theorem was a form of transfer from non-forking independence of parameters to probabilistic independence of formulas defined over those parameters which is explored in unpublished work of Hrushovski [33]. I thank him for kindly sharing his work with me. In Section 3.1, we prove this transfer result in an MS-measurable context, which is simpler than the one considered by Hrushovski. Essentially, the transfer relies on stability of the formula $\mu(\phi(x,a) \wedge \psi(x,b)) = \alpha$, which was first proved in [31].

After obtaining this result, we learned about [12] in which the authors find a counterexample to a question which had been open for a while in the context of invariant Keisler measures for simple theories. Forking gives a natural notion of smallness for definable sets in a first order structure. Similarly, being universally measure zero, i.e. being assigned measure zero by every invariant Keisler measure, also gives a notion of smallness. Both sets form ideals in the Boolean algebra of definable subsets of a structure. It is a folklore result that if a formula forks over $\emptyset$, then it is universally measure zero [12, Fact 1.2]. In stable theories, the two ideals coincide. However, it was unknown whether this would also be the case for simple theories. In [12], the authors give the first known example of a simple structure with formulas which do not fork over $\emptyset$ but still are universally measure zero. They also study definably amenable groups and prove some positive results for these in simple and small theories.

A natural question stemming from the work of [12] is whether there are examples of ω-categorical simple theories where the forking and universally measure zero ideals differ. In fact, their example is very much not ω-categorical and their positive results regarding definable amenability in groups defined in simple and small theories made it unclear whether an ω-categorical counterexample could be built (this is covered in more detail in the introduction to Chapter 6). Fortunately, it transpires that exactly those Hrushovski constructions that we showed to not be MS-measurable have formulas which do not fork over $\emptyset$ but are universally measure zero.

Adapting the proof of non-MS-measurability to this question illuminated why the structures considered in Chapter 5 fail to be MS-measurable. Using the work of Jahel and Tsankov [41] and focusing on ergodic measures in ω-categorical structures, we proved a form of transfer from weak algebraic independence of parameters to probabilistic independence in an ergodic measure. From this, we obtained that a stronger version of the independence theorem holds in simple ω-categorical structures where the forking ideal and the universally measure zero ideal agree. The transfer result is proved in Section 3.2, while the stronger version of the independence theorem appears in Section 6.2. We understood that the reason for which the ω-categorical Hrushovski constructions we had been studying were not MS-measurable is that they failed in satisfying this strong independence theorem and we could build constructions for which the strong independence theorem fails because these structures are not one-based. In an MS-measurable structure, the forking ideal and the universally measure zero ideal agree. Hence, the ω-categorical Hrushovski constructions which are supersimple of finite rank, but for which the strong independence theorem fails, cannot be MS-measurable. However, we also understood that it was possible to build Hrushovski constructions satisfying this theorem.

This brings us again to the other topic of this thesis: amalgamation properties. In an n-amalgamation problem one is given a series of types p_I in the variables $x_I := (x_i | i \in I)$ for each subset I of $[n] = \{1, \dots, n\}$ of size $n-1$, and we are asking whether there is a type $p_{[n]}$ in the variable $x_{[n]}$ such that its restriction to the variables in I corresponds to p_I for each I and possibly satisfying some additional constraints. For example, suppose that we are working with a 3-hypergraph. A 4-amalgamation problem may consist of types p_I for each $I \subseteq [4]$ of size 3 asserting that the vertices in x_I form a 3-hyperedge. A solution to the amalgamation problem would consist of a type $p(x_1, x_2, x_3, x_4)$ asserting that each triplet of vertices forms a 3-hyperedge. In an independent n-amalgamation problem, we are asking that each p_I implies that the variables are independent and that $p_{[n]}$ does too. We say that a theory has independent n-amalgamation over a set of parameters A, when every independent n-amalgamation problem over A has a solution. Independent 3-amalgamation corresponds to the independence theorem in the context of simple theories. Hence, all simple theories satisfy independent 3-amalgamation over models. The topic of independent n-amalgamation is explored further in Section 7.1, where formal definitions are given.

Again and again, different amalgamation properties interact with the existence of measures. In the work of Evans [23], MS-measurability implies a weak version of independent n-amalgamation. In Chapter 6, we see that it also implies a stronger version of the independence theorem and that failure of the latter implies that there are non-forking formulas which are universally measure zero. In the same vein, in the work of Kruckman [46], disjoint n-amalgamation implies pseudofiniteness of an ω-categorical structure.

We conclude this thesis by studying the connection between measures and independent n-amalgamation for $n > 3$. Here, the connections with MS-measurability are more subtle. On one hand, in the generic tetrahedron-free 3-hypergraph, MS-measurability fails, and the proof of this exploits the failure of independent 4-amalgamation over $\emptyset$. This also shows that being supersimple ω-categorical of finite rank and one-based does not imply MS-measurability. On the other, there are MS-measurable structures for which independent 4-amalgamation over the empty set fails: the universal homogeneous two-graph. Moreover, as we have already seen, satisfying arbitrarily strong independent n-amalgamation properties fails in implying MS-measurability. Indeed, we give an example of some ω-categorical Hrushovski constructions satisfying independent n-amalgamation for all n and the strong independence theorem and still failing at being MS-measurable.

1.1 Summary of Chapters

Chapter 1: Introduction

We began with an introduction and the current summary of chapters. In the next section, Section 1.2, we give some of our basic notational conventions and provide some background knowledge that is required for this thesis. In particular, we give brief reminders of the basic properties of $\mathcal{M}^{eq}$, ω-categorical structures, Fraïssé limits and simple theories.

Chapter 2: Invariant Keisler measures and MS-measurability

We introduce invariant Keisler measures and MS-measurable structures. In Section 2.1, we define invariant Keisler measures and explain their correspondence with regular Borel probability measures on the space of types. In

Section 2.2, we discuss MS-measurable structures. Firstly, we list some basic definitions and properties in Subsection 2.2.1. Then, in Subsection 2.2.2, we prove a folklore theorem of Ben Yaacov showing that the notion of independence induced by the dimension in an MS-measurable structure corresponds to non-forking independence. We conclude with some basic results about ω-categorical MS-measurable structures and invariant Keisler measures in simple ω-categorical structures in Subsection 2.2.3. Our main original result in this chapter is Corollary 2.2.26, which is essentially proven in Theorem 2.2.25. In it, we prove that in an ω-categorical MS-measurable structure we may naturally take the dimension of the MS-dimension-measure to be SU-rank:

Corollary 2.2.26. *Suppose that the ω-categorical structure $\mathcal{M}$ is MS-measurable with dimension-measure $h = (d, \mu)$. Then, $\mathcal{M}$ is also MS measurable via the dimension-measure $h' = (SU, \mu')$, where μ' is as in Theorem 2.2.25. Hence, if an ω-categorical structure $\mathcal{M}$ is MS-measurable, there is a dimension-measure on $\mathcal{M}$ where the dimension is given by SU-rank.*

Chapter 3: Transfer of Independence

This chapter is dedicated to exploring the transfer from independence of parameters to probabilistic independence of sets defined over those parameters. In Section 3.1, we study this for MS-measurable ω-categorical structures. The main result of this section is a weaker version of an unpublished result of Hrushovski [33]:

Theorem 3.1.9. *Let $\mathcal{M}$ be MS-measurable, ω-categorical with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Suppose that a_1, a_2, a_3 from $\mathcal{M}^{eq}$ are such that $a_i \mathop{\smile\!\!\!\!\!\!\mid\;} a_j$ for $1 \leq i < j \leq 3$. Then, $\mathrm{d}(\mathrm{tp}(a_3/a_1) \cup \mathrm{tp}(a_3/a_2)) = \mathrm{d}(a_3)$ and*

$$\mu(\mathrm{tp}(a_3/a_1) \cup \mathrm{tp}(a_3/a_2)) = \frac{\mu(a_3/a_1)\mu(a_3/a_2)}{\mu(a_3)}.$$

Then, in Section 3.2, we study this transfer for invariant Keisler measures in ω-categorical structures. Our main result is showing transfer from weak algebraic independence to probabilistic independence for an ergodic measure:

Corollary 3.2.16. *Let $\mathcal{M}$ be ω-categorical. Let μ be an ergodic measure on M^{eq} in the variable x. Suppose that $\operatorname{acl}^{eq}(\emptyset) = \operatorname{dcl}^{eq}(\emptyset)$. Let a, b be tuples from $\mathcal{M}^{eq}$ such that $\operatorname{acl}^{eq}(a) \cap \operatorname{acl}^{eq}(b) = \operatorname{acl}^{eq}(\emptyset)$. Then, for any $\mathcal{L}^{eq}$-formulas $\phi(x, y), \psi(x, z)$, where y and z are in the same sort as a and b respectively,*

$$\mu(\phi(x, a) \wedge \psi(x, b)) = \mu(\phi(x, a))\mu(\psi(x, b)).$$

This result is based on recent work of Jahel and Tsankov [41]. Subsection 3.2.1 introduces ergodic measures. Subsection 3.2.2 looks at the Hilbert spaces induced by an invariant Keisler measure. In it we prove that we are in a context in which the results of Jahel and Tsankov can be used. Finally, our main result, Corollary 3.2.16 is proved in Subsection 3.2.3.

Chapter 4: ω-categorical Hrushovski constructions

This chapter is an introduction to ω-categorical Hrushovski constructions, following [24, 68]. We introduce these structures and their basic properties in Section 4.1. In Section 4.2, we find some equations that must be satisfied by any MS-measurable Hrushovski construction in Theorem 4.2.2.

Chapter 5: Non-MS-measurability of a Hrushovski construction

In this chapter, we prove that a certain class of ω-categorical Hrushovski constructions which are supersimple of SU-rank 2 are not MS-measurable. These structures are given in Constructions 5.1.1. We introduce them and their basic properties in Section 5.1. Our main result takes most of Section 5.2 and provides a counterexample to a question of Elwes and Macpherson [19]:

Theorem 5.2.3. *The structures $\mathcal{M}_f$ satisfying the conditions of Construction 5.1.1 are not MS-measurable. Hence, there are ω-categorical supersimple finite SU-rank structures which are not MS-measurable. Indeed, there are supersimple ω-categorical structures of finite SU-rank and with independent n-amalgamation over finite algebraically closed sets for all n which are not MS-measurable.*

This theorem is essentially proven in Proposition 5.2.2, where we give a set of equations that the measures of definable sets would have to satisfy if the structures given in Construction 5.1.1 were MS-measurable.

Chapter 6: Forking and Universally Measure Zero formulas

In this chapter, we study the relationship between forking and universally measure zero formulas. We call the set of formulas forking over $\emptyset$, $F(\emptyset)$ and the set of formulas which are assigned measure zero by every invariant Keisler measure $\mathcal{O}(\emptyset)$. We introduce and discuss these notions in Section 6.1. In Section 6.2, we prove that simple ω-categorical structures where these two sets coincide satisfy a stronger version of the independence theorem:

Theorem 6.2.3. *Let $\mathcal{M}$ be a simple ω-categorical structure with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Suppose that $F(\emptyset) = \mathcal{O}(\emptyset)$, i.e. a formula forks over the empty set if and only if it is universally measure zero. Then, $\mathcal{M}$ satisfies the strong independence theorem over the empty set.*

This will allow us, in Section 6.3, to show that the Hrushovski constructions of Construction 5.1.1 have formulas which do not fork over $\emptyset$ but are universally measure zero, giving an ω-categorical example to the question addressed in [12]:

Theorem 6.3.4. *There are ω-categorical supersimple theories T of finite SU-rank with a formula $\phi(x, a)$ which does not fork over the empty set, but which is universally measure zero.*

Chapter 7: Measures and higher amalgamation properties

We study the interactions between measures and higher amalgamation properties focusing on three 3-hypergraphs: the generic tetrahedron-free 3-hypergraph, the universal homogeneous two-graph, and another class of ω-categorical Hrushovski constructions.

We begin introducing higher amalgamation properties in Section 7.1. Then, in Section 7.2, we proceed by finding equations that the measures of independent

4-amalgamations must satisfy in MS-measurable structures in Theorem 7.2.1 and Corollary 7.2.2. These equations will be immediately helpful for showing the main result of Section 7.3:

Theorem 7.3.9. *The generic tetrahedron-free* 3*-hypergraph is not MS-measurable.*

In Section 7.4, we look at the universal homogeneous two-graph, an MS-measurable structure not satisfying independent 4-amalgamation over $\emptyset$. After proving some of its basic properties in Subsection 7.4.1, we prove in Subsection 7.4.2 that it has a unique invariant Keisler measure (while not having any invariant types):

Theorem 7.4.11. *There is a unique invariant Keisler measure μ on the universal homogeneous two-graph $\mathcal{G}$ in the singleton variable. For $d, a_1, \ldots, a_n \in G$, we have that*

$$\mu(d/a_1 \ldots a_n) = \left(\frac{1}{2}\right)^{n-1},$$

and $\mu(d) = 1$.

We also discuss connections of this result to the work of Jahel on unique ergodicity [40].

Finally, we conclude the thesis with Section 7.5, where we give an example of a supersimple ω-categorical Hrushovski construction which is not MS-measurable in spite of satisfying independent n-amalgamation over finite algebraically closed sets for all n and the strong independence theorem.

Conclusions 8

We conclude the main body of this thesis with some questions which either remain open at the end of this thesis or which naturally arise from our work. Sections 8.1 and 8.2 are dedicated to to further questions on MS-measurable structures and invariant Keisler measures in simple ω-categorical structures respectively.

Appendix 9: amalgamation properties in Hrushovski constructions

The purpose of the appendix is proving the main properties which we claim to be true of the Hrushovski constructions described in Constructions 5.1.1 and 7.5.1. In particular, we need to prove these structures are supersimple and that they can satisfy independent n-amalgamation for all n over finite algebraically closed sets. Moreover, we need to prove that the structures of Construction 7.5.1 satisfy the strong independence theorem over finite algebraically closed sets.

Section 9.1 is a general discussion on how to prove simplicity and the strong independence theorem in ω-categorical Hrushovski construction following existing literature [27, 35, 69, 22] and simplifying some conditions.

Section 9.2 generalises the tools used for proving simplicity to independent n-amalgamation over finite algebraically closed sets. Many ideas of this section are folklore and follow [35] and [69]. However, our discussion in Subsection 9.2.2 of how to obtain independent n-amalgamation for all n in an ω-categorical Hrushovski construction is original and follows different lines from the approach previously suggested in the literature. This culminates in Theorem 9.2.19, which is the main original result of the Appendix.

Finally, Section 9.3 is dedicated to proving the various properties claimed in the body of this thesis for Constructions 5.1.1 and 7.5.1.

1.2 Notation and Background

This is a thesis in model theory. Chapters 1-4 of Tent & Ziegler's book [64] should cover sufficient background for the reader. The main model theoretic concepts of this thesis are introduced in the subsections below.

Let $\mathcal{L}$ be a first-order language. We work with a complete countable first-order $\mathcal{L}$-theory T with infinite models. By $\mathcal{M}$ we denote a model. Usually, when working in an ω-categorical context, it will denote the countable model of an ω-categorical theory. We write M for when we consider $\mathcal{M}$ as a set. We write $\mathbb{M}$ to denote the monster model of a given theory. This is a highly saturated and homogeneous model, and we refer the reader to [64, Section 6.1] for details on its construction and basic properties. We use greek letters $\phi, \psi, \chi, \ldots$ to refer to formulas. The letters $x, y, z, \ldots$ indicate variables, and may also indicate a finite tuple of variables. Similarly, the lowercase letters $a, b, c, \ldots$ indicate parameters from M, and may also indicate a finite tuple. Sometimes, for both variables and parameters, we may overline the letters to make explicit that we are allowing tuples, thus writing $\overline{x}, \overline{y}, \overline{z}, \ldots$ for variables and $\overline{a}, \overline{b}, \overline{c}, \ldots$ for parameters. We indicate subsets of M (or small subsets of $\mathbb{M}$) by the uppercase $A, B, C, \ldots$ We use the letters, p, q, r, s to indicate types. For $A \subseteq \mathbb{M}$ small, we write $S_x(A)$ for the space of complete types over A in the variable x. We usually write "type" to denote a complete type and specify when we are working with a partial type. For a, a' tuples from $\mathcal{M}$ and $A \subseteq M$, we write $a \equiv_A a'$ to say that a and a' have the same type over A. We just write $a \equiv a'$ to indicate that the two tuples have the same type over the empty set.

1.2.1 A few reminders on imaginaries and $\mathcal{M}^{eq}$

Given an $\mathcal{L}$-structure $\mathcal{M}$, model theorists are often interested in working with a larger structure $\mathcal{M}^{eq}$ such that for each $\emptyset$-definable equivalence relation $E(\overline{x}, \overline{y})$ on $\mathcal{M}$, $\mathcal{M}^{eq}$ has a sort S_E whose elements correspond to the elements of the quotient of $\mathcal{M}$ with respect to E. In this thesis, we will often phrase results in terms of $\mathcal{M}^{eq}$. The reader might be interested in [56, Ch.16] or [54, Ch. 1.1] for details.

We begin with an $\mathcal{L}$-structure $\mathcal{M}$ with $\mathcal{L}$-theory T. For each $\emptyset$-definable equivalence relation $E(\overline{x}, \overline{y})$ on n-tuples of M, we add a sort S_E to the language and a function f_E from n-tuples of M to the sort S_E. With this, we obtain the language $\mathcal{L}^{eq}$. The structure $\mathcal{M}^{eq}$ is obtained as follows: we identify the sort $S_=$

associated with the equivalence relation of identity with $\mathcal{M}$ (with the various $\mathcal{L}$-formulas interpreted as usual). Meanwhile, for any other $\emptyset$-definable equivalence relation E on n-tuples, the sort S_E is identified with the set $\{\bar{a}/E : \bar{a} \in M^n\}$ and the function f_E is interpreted as the function sending each $\bar{a} \in M^n$ to $\bar{a}/E$. We shall often call variables or parameters in the sort $S_=$ **real**, while elements in other sorts are usually called **imaginaries**. The theory T^{eq} is the $\mathcal{L}^{eq}$-theory of $\mathcal{M}^{eq}$. We list the major properties of $\mathcal{M}^{eq}$ in the theorem below:

Theorem 1.2.1. *(i) Let $E_1, \ldots, E_k$ be $\emptyset$-definable equivalence relations in T and $\phi(x_1, \ldots, x_k)$ be an $\mathcal{L}^{eq}$-formula with the variable x_i in the sort S_{E_i}. Then, there is some $\mathcal{L}$-formula $\psi(y_1, \ldots, y_k)$ (where y_i is possibly an n-tuple) such that*

$$T^{eq} \vdash (\forall y_1, \ldots, y_k \in S_=)\psi(y_1, \ldots, y_k) \leftrightarrow \phi(f_{E_1}(y_1), \ldots, f_{E_k}(y_k));$$

(ii) every $\mathcal{L}^{eq}$-definable $X \subseteq M^n$ in $\mathcal{M}^{eq}$ (possibly with parameters from M^{eq}) is already $\mathcal{L}$-definable in M (with parameters from M);

(iii) any $\sigma \in \mathrm{Aut}(M)$ extends uniquely to an automorphism in $\mathrm{Aut}(M^{eq})$.

Remark 1.2.2. In this thesis, we study $\mathrm{Aut}(M)$-invariant regular Borel probability measures on $S_x(M)$. In Remark 2.1.3, we shall note that these can be identified with $\mathrm{Aut}(M^{eq})$-invariant Borel probability measures on $S_x(M^{eq})$.

Remark 1.2.3. The other important aspect of $\mathcal{M}^{eq}$ for our thesis is that it preserves the relevant model-theoretic properties of $\mathcal{M}$: λ-saturation, strong λ-homogeneity, stability, simplicity and ω-categoricity are preserved from $\mathcal{M}$ to $\mathcal{M}^{eq}$. We will recall this fact after the definition of each of these properties.

For $A \subseteq \mathcal{M}^{eq}$, we write $\mathrm{dcl}^{eq}(A)$ and $\mathrm{acl}^{eq}(A)$ for, respectively, the definable closure and the algebraic closure of A in $\mathcal{M}^{eq}$.

Remark 1.2.4. Any tuple from M^{eq} is interdefinable with a singleton from M^{eq} [56, Lemma 16.13]. In particular, there is no harm in identifying tuples from M^{eq} with elements of M^{eq} and we shall do so when phrasing results in terms of M^{eq}.

In this thesis, we frequently work with structures that have weak elimination of imaginaries.

Definition 1.2.5. We say that $\mathcal{M}$ has **weak elimination of imaginaries** if for every $e \in \mathcal{M}^{eq}$ there is a real tuple a such that $e \in \mathrm{dcl}^{eq}(a)$ and $a \in \mathrm{acl}^{eq}(e)$.

We shall make use of the following well known fact:

Proposition 1.2.6. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure with weak elimination of imaginaries. Then, for $A = \mathrm{acl}(A)$, $B = \mathrm{acl}(B)$ subsets of M,*

$$\mathrm{acl}^{eq}(A) \cap \mathrm{acl}^{eq}(B) = \mathrm{acl}^{eq}(A \cap B).$$

Proof. If $e \in \mathrm{acl}^{eq}(A)$, then

$$\mathrm{acl}^{eq}(e) \cap \mathcal{M} \subseteq \mathrm{acl}(A) \subseteq A,$$

where the second inclusion holds by A being algebraically closed. Hence, for $e \in \mathrm{acl}^{eq}(A) \cap \mathrm{acl}^{eq}(B)$,

$$\mathrm{acl}^{eq}(e) \cap \mathcal{M} \subseteq A \cap B.$$

But then, since by weak elimination of imaginaries, $e \in \mathrm{dcl}^{eq}(\mathrm{acl}^{eq}(e) \cap \mathcal{M})$,

$$e \in \mathrm{dcl}^{eq}(A \cap B) \subseteq \mathrm{acl}^{eq}(A \cap B).$$

□

The main consequence of weak elimination of imaginaries which we shall be using in this thesis is that it implies that imaginary algebraic closure and definable closure agree on sets which are algebraically closed with respect to M:

Fact 1.2.7 (Proposition 3.2 & Proposition 3.4 in [7]). *Let $\mathcal{M}$ have weak elimination of imaginaries. Then, for any $A \subseteq M$,*

$$\mathrm{acl}^{eq}(A) = \mathrm{dcl}^{eq}(\mathrm{acl}(A)).$$

The property above is equivalent to **elimination of strong types**. That is, given $A \subseteq M$, if $a \equiv_{\mathrm{acl}(A)} a'$, then $a \equiv_{\mathrm{acl}^{eq}(A)} a'$.

1.2.2 A few reminders on ω-categorical structures

We say that a theory is **ω-categorical** if it has a unique countable model up to isomorphism. We say that an $\mathcal{L}$-structure is ω-categorical if its theory is. Some of the basic theory of ω-categorical structures is covered in [64, Chapter 4], and we refer the reader to that for details. We will make constant implicit use of the Ryll-Nardzewski theorem [30, Theorem 7.3.1]:

Theorem 1.2.8 (Ryll-Nardzewski, Engeler and Svenonius)**.** *Let T be a complete theory with infinite models in a countable language $\mathcal{L}$. Then, the following are equivalent:*

1. *T is ω-categorical;*
2. *for $\mathcal{M} \models T$, and $A \subset M$ finite, there are finitely many complete n-types over A;*
3. *for $\mathcal{M} \models T$, and $A \subset M$ finite, any complete n-type over A is isolated by an $\mathcal{L}_A$-formula; and*
4. *some/every countable model $\mathcal{M} \models T$ is such that for every finite subset $A \subseteq M$, $\mathrm{Aut}(M/A)$, the group of automorphism of M fixing A pointwise, is **oligomorphic**. That is, for each $n \in \mathbb{N}$, the action of $\mathrm{Aut}(M/A)$ on n-tuples from M has finitely many orbits.*

When $\mathcal{M}$ is the countable model of an ω-categorical structure, for $A \subset M$ finite, the A-definable subsets of M are exactly its $\mathrm{Aut}(M/A)$-invariant subsets. We also have that any model $\mathcal{M}$ of an ω-categorical theory is ***ω-saturated***, i.e. all types over finite tuples from M are realised in $\mathcal{M}$. This is a consequence of types being isolated by formulas [64, Remark 4.3.4]. We shall frequently use this fact when speaking of choosing tuples realising certain given types. From ω-saturation, we have that the countable model of an ω-categorical structure is **strongly ω-homogeneous** in the sense of [64, Definition 6.1.5]: if $\overline{a}$ and $\overline{a}'$ are finite tuples such that $\overline{a} \equiv \overline{a}'$, then there is an automorphism $f \in \mathrm{Aut}(M)$ such that $f(\overline{a}) = \overline{a}'$ [30, Lemma 10.1.5].

If $\mathcal{M}$ is ω-categorical, then so is $\mathcal{M}^{eq}$ in the sense of T^{eq} having a unique countable model as a multi-sorted structure. In particular, any $\emptyset$-definable equivalence relation on n-tuples of M forms a definable subset of M^{2n}, and so by Ryll-Nardzewski there are finitely many of these. Taking $\mathcal{M}$ to be the countable model of an ω-categorical structure, $\mathcal{M}^{eq}$ has countably many countable sorts and is the unique countable model of T^{eq}. Moreover, both ω-saturation and strong ω-homogeneity are preserved when moving from $\mathcal{M}$ to $\mathcal{M}^{eq}$ [54, Lemma 1.5].

1.2.3 A few reminders on Fraïssé limits

In the final chapter of this thesis, Chapter 7, we work with some ω-categorical structures which are Fraïssé limits of classes of finite structures. Many interesting examples of ω-categorical structures can be obtained as Fraïssé limits. Moreover, Hrushovski constructions rely on some of the basic ideas behind them. Here we give just few basic definitions, but we refer the reader to [49, 46] for more information on these structures.

Definition 1.2.9. Let $\mathcal{C}$ be class of finite structures in a finite relational language $\mathcal{L}$ closed under isomorphisms. We say that $\mathcal{C}$ is a **Fraïssé class** if it satisfies the following properties:

- (Hereditary Property) $\mathcal{C}$ is closed under substructures;
- (Joint Embedding Property) for $A, B \in \mathcal{C}$ there is $C \in \mathcal{C}$ such that both A and B embed in C; and
- (Amalgamation Property) for $A, B_1, B_2 \in \mathcal{C}$ such that $f_i : A \to B_i$ are embeddings for $i \in \{1,2\}$, there is $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ for $i \in \{1,2\}$ such that $g_1 \circ f_1 = g_2 \circ f_2$;

Often, the above is called a strong Fraïssé class, though such distinctions go beyond the interests of this thesis.

We say that $\mathcal{C}$ has the **strong amalgamation property**, also known as disjoint amalgamation property, if in the amalgamation property condition C and the g_i can be chosen such that $g_1(B_1) \cap g_2(B_2) = f_1(A)$. If, in addition to this, every tuple of C for which a relation in $\mathcal{L}$ holds intersects trivially either $g_1(B_1) \setminus g_1(f_1(A))$ or $g_2(B_2) \setminus g_2(f_2(A))$, we say that $\mathcal{C}$ has the **free amalgamation property**.

If $\mathcal{C}$ forms a Fraïssé class, we can build an ω-categorical homogeneous structure from it, which we call its **Fraïssé limit**:

Theorem 1.2.10 ([49])**.** *Suppose that $\mathcal{C}$ is a Fraïssé class. Then, there is an ω-categorical structure $\mathcal{M}$ such that:*

- *(Age) $\mathcal{C}$ is set of finite substructures of $\mathcal{M}$ up to isomorphism;*
- *(Homogeneity) $\mathcal{M}$ is **homogeneous**. That is, any partial isomorphism between finite substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$; and*
- *(Extension Property) let $A \subset M$ be finite and $B \in \mathcal{C}$ be such that there is an embedding $f : A \to B$. Then, there is an embedding $g : B \to M$ such that $g(f(a)) = a$ for all $a \in A$.*

Moreover, $\mathcal{M}$ has $\operatorname{acl}(A) = A$ for all finite $A \subseteq M$ if and only if $\mathcal{C}$ has the strong amalgamation property.

In Chapter 7, we consider the generic tetrahedron-free 3-hypergraph and the universal homogeneous two-graph. Both of these are the Fraïssé limits of Fraïssé classes with the strong amalgamation property.

1.2.4 A few reminders on simple theories

Throughout this thesis we work with simple theories. These are a natural generalisation of stable theories which are characterised by a well-behaved notion of independence. A reader interested in the background may want to look at [43, Chapters 1-3] or [68, Chapters 1-2]. In this thesis we will not require knowledge of simple theories beyond their characterisation through the Kim-Pillay Theorem [64, Theorem 7.3.13]:

Theorem 1.2.11 (Kim-Pillay Theorem [44])**.** *Let $\mathbb{M}$ be a monster model for the complete theory T. Suppose that there is a ternary relation $a \mathop{\smile\!\!\!\!\!\mid}^{\star}_{C} B$ between a finite tuple and two small subsets of $\mathbb{M}$ which is invariant under automorphisms and satisfying the following properties:*

- *(Monotonicity and Transitivity) $a \mathop{\smile\!\!\!\!\!\mid}^{\star}_{C} BD$ if and only if $a \mathop{\smile\!\!\!\!\!\mid}^{\star}_{C} B$ and $a \mathop{\smile\!\!\!\!\!\mid}^{\star}_{BC} D$;*
- *(Symmetry) for finite tuples a, b, $a \mathop{\smile\!\!\!\!\!\mid}^{\star}_{C} b$ if and only if $b \mathop{\smile\!\!\!\!\!\mid}^{\star}_{C} a$;*

- *(Finite Character)* $a \mathop{\smile\!\!\!\!\!\!\mid}^{\star}_{C} B$ *if and only if* $a \mathop{\smile\!\!\!\!\!\!\mid}^{\star}_{C} b$ *for every finite tuple from* B*;*
- *(Local Character) there is a cardinal* κ *such that for any* a *and* B *there is* $B_0 \subsetneq B$ *such that* $|B| < \kappa$ *and* $a \mathop{\smile\!\!\!\!\!\!\mid}^{\star}_{B_0} B$*;*
- *(Existence) for any* a, B, C*, there is some* $a' \equiv_B a$ *such that* $a' \mathop{\smile\!\!\!\!\!\!\mid}^{\star}_{B} C$*; and*
- *(Independence Theorem over models) let* M *be a model and* a, b, c_0, c_1 *be tuples such that*

$$a \mathop{\smile\!\!\!\!\!\!\mid}^{\star}_{M} b, c_0 \mathop{\smile\!\!\!\!\!\!\mid}^{\star}_{M} a, c_1 \mathop{\smile\!\!\!\!\!\!\mid}^{\star}_{M} a, \text{ and } c_0 \equiv_M c_1.$$

Then, there is $c^\star$ *such that*

$$c^\star \equiv_{Ma} c_0, c^\star \equiv_{Mb} c_1, \text{ and } c \mathop{\smile\!\!\!\!\!\!\mid}^{\star}_{M} ab.$$

Then, T *is simple and* $\mathop{\smile\!\!\!\!\!\!\mid}^{\star}$ *corresponds to non-forking independence.*

In Subsection 2.2.2, we shall call a ternary relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\star}$ satisfying all of the conditions above except the independence theorem a notion of independence. Hence, having a notion of independence satisfying the independence theorem over models implies being simple. We never use explicitly any properties of forking or dividing, however, we give the definitions below for completeness:

Definition 1.2.12. Working in a monster model $\mathbb{M}$, for $A \subseteq \mathbb{M}$ small, we say that $\phi(x, b)$ **divides over** $\boldsymbol{A}$ if there is some $k \in \mathbb{N}$ and a sequence $(b_i)_{i<\omega}$ of realisations of $\mathrm{tp}(b/A)$ such that $\{\phi(x, b_i) | i < \omega\}$ is k-inconsistent (i.e. any k many formulas from the set are inconsistent). We say that $\psi(x, c)$ **forks over** $\boldsymbol{A}$ if it implies a finite disjunction of formulas each dividing over A. A partial type π forks over A if it implies a formula forking over A.

In simple theories, a formula forks over A if and only if it divides over A. Moreover, as anticipated from the Kim-Pillay Theorem, non-forking induces a notion of independence as follows:

Definition 1.2.13. We say that a is **non-forking independent** from B over C, written $a \mathop{\smile\!\!\!\!\!\!\mid}_{C} B$, or $a \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_{C} B$ when discussing multiple notions of independence, if $\mathrm{tp}(a/BC)$ does not fork over C.

Later in this thesis, we will talk of structures satisfying the independence theorem over different sets than models (such as algebraically closed sets):

Definition 1.2.14. We say that a structure $\mathbb{M}$ satisfies the independence theorem over $A \subseteq \mathbb{M}$ if for any tuples a, b, c_0, c_1 such that

$$a \mathop{\downfree}_A b, c_0 \mathop{\downfree}_A a, c_1 \mathop{\downfree}_A a, \text{ and } c_0 \equiv_A c_1,$$

there is $c^\star$ such that

$$c^\star \equiv_{Aa} c_0, c^\star \equiv_{Ab} c_1, \text{ and } c \mathop{\downfree}_A ab.$$

In Section 2.2.3, we will discuss SU-rank. This is defined as follows:

Definition 1.2.15. For an ordinal α and p a complete type over a small subset of $\mathbb{M}$, we define $SU(p) \geq \alpha$ as follows:

- $SU(p) \geq 0$ for any p;
- $SU(p) \geq \alpha + 1$ if there is some $B \supset A$ such that there is an extension $q \in S(B)$ of $p \in S(A)$ which forks over A and such that $SU(q) \geq \alpha$; and
- for a limit ordinal λ, $SU(p) \geq \lambda$ if $SU(p) \geq \beta$ for any $\beta < \lambda$.

If $SU(p) \geq \alpha$, but $SU(p) \not\geq \alpha + 1$, we write $SU(p) = \alpha$. Otherwise, if $SU(p) \geq \alpha$ for all ordinals, we write $SU(p) = \infty$. For a partial type $\pi(x)$ over A,

$$SU(\pi(x)) := \sup\{SU(p) | \pi \subset p, p \in S_x(A)\}.$$

This also defines SU-rank for formulas since these can be considered as partial types.

We say that T is **supersimple** if for any small A and p a complete type in a finite tuple over A, there is some finite $A_0 \subseteq A$ such that p does not fork over A_0. Equivalently, T is supersimple if $SU(p)$ has ordinal value (i.e. $SU(p) < \infty$) for all complete types in finite variables over small parameter sets [43, Proposition 2.5.6]. We say that the SU-rank of T is $SU(x = x)$, where x is the singleton variable. We say that T has finite SU-rank if $SU(x = x)$ is finite.

Remark 1.2.16. We have that for T simple and complete types $p(x) \subseteq q(x)$ of ordinal-valued SU-rank,

$$SU(p) = SU(q) \text{ if and only if } q \text{ is a non-forking extension of } p,$$

by which we mean that q does not fork over the parameter set over which p is defined. Moreover, for T supersimple of finite SU-rank, SU-rank is **additive** from the Lascar inequalities [43, Proposition 2.4.19]:

$$SU(ab/A) = SU(a/Ab) + SU(b/A).$$

This implies that when T has finite SU-rank, $SU(\overline{x} = \overline{x})$ is finite for any finite length variable $\overline{x}$.

2 Invariant Keisler Measures and MS-measurability

In this thesis, we investigate two natural ways in which measures appear in a model theoretic context: invariant Keisler measures and MS-measurable structures. In this chapter we introduce both notions. We also prove some basic and folklore facts about both invariant Keisler measures and MS-measurable structures. We conclude the chapter with an original result about ω-categorical MS-measurable structures, Corollary 2.2.26, which proves that we may naturally take the dimension on ω-categorical MS-measurable structures to be SU-rank.

Keisler measures are finitely additive probability measures on the Boolean algebra of definable sets in a fixed variable with parameters from a given set. In this thesis, we always take the parameter sets to be models. Furthermore, we are particularly interested in the context where the measures are invariant under the action of the automorphism group of $\mathrm{Aut}(M)$. It is easy to see that Keisler measures are a generalisation of types, and invariant Keisler measures are a generalisation of invariant types: a type is just a Keisler measure only taking values 0 and 1. However, the space of invariant Keisler measures for a structure can be much richer than the space of its invariant types. Indeed, we will see much later in Theorem 7.4.11 that there are theories with no invariant types but with invariant Keisler measures.

The focus of this thesis is invariant Keisler measures in simple ω-categorical structures. Most known examples in this context also have measures which interact particularly well with a natural notion of dimension. These are MS-measurable structures [50]. A structure is MS-measurable if it has a dimension-measure function assigning each definable set a dimension and a measure in a way that the function is finite, definable and satisfies an algebraicity condition and a version of Fubini's theorem (see Definition 2.2.7). Being MS-measurable is a property of a theory: if a structure is MS-measurable, then so is any other structure with the same theory. Furthermore, MS-measurable structures are supersimple and of finite SU-rank [19].

Many important examples of supersimple structures are MS-measurable. The most notable case and original motivation for the definition of MS-measurability are pseudofinite fields [9]. In an ω-categorical context, the universal homogeneous k-hypergraphs (including the random graph) and the smoothly approximable structures of [11] (in particular ω-stable ω-categorical structures) are MS-measurable [50]. The random graph and smoothly approximable structures are also elementarily equivalent to ultraproducts of well-behaved classes of finite structures: n-dimensional asymptotic classes [50]. Any such class naturally gives rise to an MS-measurable structure. While not all MS-measurable structures can be obtained in this way (cf. [19, Theorem 3.12, Example 3.13]), many natural examples of MS-measurable structures are pseudofinite and arise from such classes of finite structures.

The phenomenon of MS-measurability is very frequent in supersimple ω-categorical structures. Indeed, the original motivation for the work of this thesis was the question 5.0.1 of Elwes and Macpherson [19] of whether all supersimple ω-categorical structures with finite SU-rank are MS-measurable. A counterexample to Question 5.0.1 was given in [23]. In this thesis, we find further counterexamples to this question: some more ω-categorical Hrushovski constructions in Chapter 5 and Section 7.5 (Constructions 5.1.1 and 7.5.1) and the generic tetrahedron-free 3-hypergraph in Section 7.3.

Since one of our main objectives is proving that certain ω-categorical structures are not MS-measurable, we will need tools to simplify such proofs. An initial difficulty is that a structure may possibly have many different notions of dimension. So we might need to check that for none of the possible dimensions on a structure there is a system of measures inducing an MS-dimension-measure function. A way to circumvent this difficulty would be to prove that in a specific class of structures any dimension satisfying certain properties must correspond to a specific dimension. For example, this was the approach adopted by Evans in [23], where, in order to prove that certain ω-categorical Hrushovski constructions are not MS-measurable, he first proves that any dimension function on his class of Hrushovski constructions must be a scaled version of SU-rank. This can also be achieved for other classes of ω-categorical Hrushovski constructions, and indeed such generalisations were part of the early work of this PhD project [51].

Instead, we circumvent this issue by proving that if an ω-categorical structure is MS-measurable, then it has a dimension-measure function where the dimension is given by SU-rank:

Corollary 2.2.26. *Suppose that the ω-categorical structure $\mathcal{M}$ is MS-measurable with dimension-measure $h = (d, \mu)$. Then, $\mathcal{M}$ is also MS measurable via the dimension-measure $h' = (SU, \mu')$, where μ' is as in Theorem 2.2.25. Hence, if an ω-categorical structure $\mathcal{M}$ is MS-measurable, there is a dimension-measure on $\mathcal{M}$ where the dimension is given by SU-rank.*

This is the main result of this chapter, and it is discussed in its final Subsection 2.2.3. The proof is mainly contained in Theorem 2.2.25.

One of the main ingredients for the proof of Corollary 2.2.26 is a folklore result of Ben Yaacov that the notion of independence induced by the dimension of an MS-dimension measure corresponds to non-forking independence. In Subsection 2.2.2 we prove this result.

The structure of this chapter is as follows. Firstly, we introduce invariant Keisler measures in Section 2.1 partially following the presentation in [61, Chapter 7]. Then, Section 2.2 is dedicated to MS-measurable structures. It begins with some basic definitions and remarks in Subsection 2.2.1, all of which can be found in the original article [50]. Then, we move to prove Ben Yaacov's result that the dimension of an MS-dimension measure induces non-forking independence in Subsection 2.2.2. We conclude with a section on MS-measurable ω-categorical structures in Section 2.2.3. As mentioned above, the main result is Corollary 2.2.26. In this final section, we also explore some basic results for invariant Keisler measures in simple ω-categorical structures.

2.1 Invariant Keisler Measures

Keisler measures are finitely additive probability measures on the space of definable subsets of a structure. In this section, we introduce some of the key concepts around invariant Keisler measures. Chapter 7 of Simon's book [61] is a good introduction to the subject and is our main reference for this section.

Let T be a complete $\mathcal{L}$-theory, where $\mathcal{L}$ is a countable first-order language and $\mathcal{M} \models T$. Let $\mathrm{Def}_x(M)$ denote the Boolean algebra of $\mathcal{L}(M)$-formulas in the free variable x up to $\mathrm{Th}(\mathcal{M}_M)$-equivalence, where $\mathcal{M}_M$ is the expansion of $\mathcal{M}$ by constant symbols for each element of M and $\mathcal{L}(M)$ is the corresponding expansion of $\mathcal{L}$. We may also think of $\mathrm{Def}_x(M)$ as the set of definable subsets of

M^x by formulas in the variable x with parameters from M. In general, we will speak interchangeably of the two perspectives. Let $S_x(M)$ be the Stone space of types over M in the variable x with the usual Stone topology. We write $[\phi(x,a)]$ for the clopen set of types containing the formula $\phi(x,a)$. Recall that every clopen set in $S_x(M)$ is of this form and that the clopen sets yield a basis for the topology on $S_x(M)$.

This space is always compact and Hausdorff. Moreover, when M is countable, it is also metrizable. This can be seen by taking an enumeration of the $\mathcal{L}(M)$-formulas $\phi_1, \phi_2, \ldots$ and then for $p_1, p_2 \in S_x(M)$, letting $d(p_1, p_2) = \frac{1}{2^n}$, where n is the smallest natural number such that ϕ_n is not contained in both p_1 and p_2. The automorphism group $\mathrm{Aut}(M)$ naturally acts on $\mathrm{Def}_x(M)$ and on $S_x(M)$, where for $\tau \in \mathrm{Aut}(M)$, $\tau \cdot \phi(x,a) = \phi(x, \tau(a))$ and for $p \in S_x(M)$,

$$\tau \cdot p = \{\phi(x, \tau(a)) : \phi(x,a) \in p\}.$$

The action of an automorphism of M on $S_x(M)$ corresponds to a self-homeomorphism of $S_x(M)$. This follows from the fact that for proving continuity it is sufficient to check that the preimage of the elements of the basis of clopen sets is still clopen.

Definition 2.1.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We say that $\mu : \mathrm{Def}_x(M) \to [0,1]$ is a **Keisler measure** if it is a finitely additive measure such that $\mu(x = x) = 1$. We say that μ is $\mathrm{Aut}(M)$-**invariant** if for any $\tau \in \mathrm{Aut}(M)$,

$$\mu(\tau \cdot \phi(x,a)) = \mu(\phi(x,a)).$$

If $\mathcal{M}$ is strongly ω-homogeneous (e.g. when it is a monster model or the countable model of an ω-categorical theory), this is equivalent to saying that, if $a \equiv a'$, then $\mu(\phi(x,a)) = \mu(\phi(x,a'))$ (i.e. the measure is invariant in the sense of Definition 2.2.1).

From [61][§7.1], we know that there is a correspondence between Keisler measures on $\mathrm{Def}_x(M)$ and regular Borel probability measures on $S_x(M)$. We denote by $\mathcal{B}_x(M)$ the set of Borel subsets of $S_x(M)$. We say that $\mu : \mathcal{B}_x(M) \to [0,1]$ is **regular** if for any $B \in \mathcal{B}_x(M)$,

$$\sup\{\mu(F) : F \subseteq B, F \text{ is closed}\} = \inf\{\mu(O) : B \subseteq O, O \text{ is open}\}. \tag{Reg}$$

We have that a Keisler measure $\mu : \mathrm{Def}_x(M) \to [0,1]$ uniquely extends to a regular σ-additive probability measure $\mu' : \mathcal{B}_x(M) \to [0,1]$. Conversely, any regular Borel probability measure on $S_x(M)$ induces a Keisler measure by considering its restriction to clopen sets. When M is countable, and so $S_x(M)$ is metrizable, every Borel probability measure on $S_x(M)$ is regular [3, Theorem 7.1.7], and so there is a correspondence between Keisler measures on $\mathrm{Def}_x(M)$ and σ-additive Borel probability measures on $S_x(M)$.

This correspondence still holds between $\mathrm{Aut}(M)$-invariant Keisler measures in the variable x and $\mathrm{Aut}(M)$-invariant regular Borel probability measures on $S_x(M)$. We prove this known fact below. From here, we shall speak interchangeably of the two.

Lemma 2.1.2. *Let $\mu : \mathrm{Def}_x(M) \to [0,1]$ be an $\mathrm{Aut}(M)$-invariant Keisler measure. Then, its induced regular Borel probability measure μ' is invariant under $\mathrm{Aut}(M)$.*

Proof. We follow the construction of μ' in [61][§7.1]. We have that μ assigns a measure μ' to every clopen set $[\phi(x,a)]$. For $\mathcal{O}$ open, $\mu'(O)$ is defined as:

$$\mu'(O) := \sup\{\mu(A) | A \subseteq O \text{ clopen }\}.$$

Similarly, for F closed

$$\mu'(F) := \inf\{\mu(C) | F \subseteq C \text{ clopen }\}.$$

Since $\sigma \in \mathrm{Aut}(M)$ is a homeomorphism of $S_x(M)$, there is a correspondence between the clopen sets $A \subseteq O$ and the clopen sets $A' \subseteq \sigma O$, where every clopen set $A' \subseteq \sigma O$ is of the form σA for some $A \subseteq O$ and viceversa. Hence,

$$\mu'(O) = \sup\{\mu(A) | A \subseteq O \text{ clopen }\}$$

$$= \sup\{\mu(\sigma A) | A \subseteq O \text{ clopen }\}$$
$$= \sup\{\mu(A') | A' \subseteq \sigma O \text{ clopen }\} = \mu'(\sigma O).$$

Here the second equality follows from $\mathrm{Aut}(M)$-invariance of μ, while the third follows by the correspondence between clopen subsets of O and of σO. The same argument works for closed sets, and finally for Borel sets using the definition of μ' in (Reg). □

Remark 2.1.3. In this thesis, we will often want to work in $\mathcal{M}^{eq}$. For x a variable in the real sort, we can identify $\mathrm{Aut}(M)$-invariant Borel probability measures on $S_x(M)$ and $\mathrm{Aut}(M^{eq})$-invariant Borel probability measures on $S_x(M^{eq})$. We also identify $\mathrm{Aut}(M)$-invariant Keisler measures in the variable x on M and $\mathrm{Aut}(M^{eq})$-invariant Keisler measures in x on M^{eq}. For this, consider Theorem 1.2.1. From (i), we can identify $S_x(M)$ and $S_x(M^{eq})$ as sets since any type over M in x entirely determines a type over M^{eq} in x and vice-versa by restriction. Moreover, from (ii), $S_x(M)$ and $S_x(M^{eq})$ have exactly the same clopen sets and so we can identify their Stone topologies. Hence, $S_x(M)$ and $S_x(M^{eq})$ are the same topological space. From (iii), we can identify $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M^{eq})$. In particular, we can identify the action of $\mathrm{Aut}(M)$ on $S_x(M)$ and the action of $\mathrm{Aut}(M^{eq})$ on $S_x(M^{eq})$.

Given the various correspondences explained in this section, we shall safely use μ to denote both the Keisler measure and the corresponding regular Borel probability measure and, with measures in a real variable, we do not distinguish working with respect to M and with respect to M^{eq}.

2.2 MS-measurable structures

In this section, we introduce *MS*-measurable structures and some basic facts about them. We also prove some original results on ω-categorical *MS*-measurable structures. We begin with Subsection 2.2.1, where we introduce the notion of *MS*-measurability following [50] and [19]. Then, in Subsection 2.2.2, we prove the folklore result that dimension independence in MS-measurable structures corresponds to non-forking independence. Finally, in Subsection 2.2.3, we find a set of equations which hold in ω-categorical MS-measurable structures. We also show that if an ω-categorical structure is MS-measurable, then it is MS-measurable with dimension given by SU-rank.

2.2.1 Basic definitions

Let $\mathcal{M}$ be a structure. By $\mathrm{Def}(M)$ we mean the set of non-empty definable subsets of M defined by formulas with parameters from M. Hence, we may view $\mathrm{Def}(M)$ as

$$\bigcup_{n\in\mathbb{N}^{>0}} \mathrm{Def}_n(M),$$

where $\mathrm{Def}_n(M)$ is the set of non-empty definable subsets of M^n with parameters from M.

Definition 2.2.1. Let X be any set and consider a function $g : \mathrm{Def}(\mathrm{M}) \to X$. For $A \subseteq M$, we say that g is A-**invariant** if $g(\phi(M^{|\overline{x}|}, \overline{a})) = g(\phi(M^{|\overline{x}|}, \overline{a}'))$ whenever $\overline{a} \equiv_A \overline{a}'$. When $A = \emptyset$, we say g is **invariant**. The function g is **definable** if for any formula $\phi(\overline{x}, \overline{y})$ and $k \in X$, the set of $\overline{a} \in M^{|\overline{y}|}$ such that $g(\phi(M^{|\overline{x}|}, \overline{a})) = k$ is definable over the empty set. We say that g is **finite** if the set of values of $g(\phi(M^{|\overline{x}|}, \overline{a}))$ for $\overline{a} \in M^{|\overline{y}|}$ is finite.

To avoid cumbersome notation, when the model $\mathcal{M}$ is clear or irrelevant to our calculations, we sometimes write $g(\phi(\overline{x}, \overline{a}))$ instead of $g(\phi(M^{|\overline{x}|}, \overline{a}))$.

Note that definability always implies invariance. In an ω-categorical context, by Ryll-Nardzewski, invariance implies definability and finiteness.

Remark 2.2.2. The notion of A-invariance we just introduced is *a priori* stronger than that of $\mathrm{Aut}(M/A)$-invariance introduced in Definition 2.1.1. However, in this thesis, we will always work in structures where the two notions agree (i.e. we work with A being finite in strongly ω-homogeneous structures).

Before introducing the notion of an MS-measurable structure, we follow the notation of [67] to speak of a dimension function. The additivity condition is clarified in Remark 2.2.4.

Definition 2.2.3. We call the invariant function $\dim : \mathrm{Def}(\mathrm{M}) \to \mathbb{N}$ a **dimension** if it satisfies the following conditions:

- (Algebraicity) for X finite and non-empty, $\dim(X) = 0$;
- (Union) for $X, Y \in \mathrm{Def}(M)$, $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$; and
- (Additivity) for finite tuples $\bar{a}, \bar{b}, \bar{c}$ from M, $\dim(\bar{a}\bar{b}/\bar{c}) = \dim(\bar{a}/\bar{b}\bar{c}) + \dim(\bar{b}/\bar{c})$.

Remark 2.2.4. In the additivity condition, by $\dim(\bar{a}/B)$ we mean $\dim(\mathrm{tp}(\bar{a}/B))$. For a partial type $\pi(\bar{x})$ over $B \subseteq M$, we define

$$\dim(\pi(\bar{x})) = \min\{\dim(\phi(\bar{x}, \bar{b}) \mid \pi(\bar{x}) \vdash \phi(\bar{x}, \bar{b})\}.$$

Remark 2.2.5. In the course of this paper, we work with definable dimensions in an ω-categorical context. Hence, for $\bar{a}, \bar{b}$ finite tuples from M we have that

$$\dim(\bar{a}/\bar{b}) = \dim(\phi(\bar{x}, \bar{b})),$$

where $\phi(\bar{x}, \bar{b})$ is a formula isolating $\mathrm{tp}(\bar{a}/\bar{b})$.

Definition 2.2.6. We say that the function $\mu : \mathrm{Def}_{\bar{x}}(M) \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ is a **measure** if it is a finitely additive function, i.e. for X and Y definable and disjoint in the same variable,

$$\mu(X \cup Y) = \mu(X) + \mu(Y).$$

We can view a Keisler measure as a measure taking values in $[0, 1]$.

We give here a slightly simplified definition of MS-measurability than the original following [50, Proposition 5.7]:

Definition 2.2.7. The $\mathcal{L}$-structure $\mathcal{M}$ is ***MS*-measurable** if there is a dimension-measure function $h : \mathrm{Def}(\mathrm{M}) \to \mathbb{N} \times \mathbb{R}^{>0}$, with notation $h(X) = (\mathrm{d}(X), \mu(X))$, satisfying the following conditions:

1. The function h is finite and definable.

2. For $\overline{a} \in M^n$, $h(\{\overline{a}\}) = (0, 1)$.
3. (Additivity) For $X, Y \subseteq M^n$ definable and disjoint,

$$\mu(X \cup Y) = \begin{cases} \mu(X) + \mu(Y), & \text{for } \mathrm{d}(X) = \mathrm{d}(Y); \\ \mu(Y) & \text{for } \mathrm{d}(X) < \mathrm{d}(Y). \end{cases}$$

4. (Fubini) For $n \geq 2$, let $X \subseteq M^n$ be definable and for $1 \leq m < n$, let $\pi : M^n \to M^m$ be a projection of M^n to m-many coordinates. Suppose there is $(k, \nu) \in \mathbb{N} \times \mathbb{R}^{>0}$ such that for any $\overline{a} \in \pi(X)$, we have that $h(\pi^{-1}(\overline{a}) \cap X) = (k, \nu)$. Then, $d(X) = d(\pi(X)) + k$ and $\mu(X) = \mu(\pi(X))\nu$.

Remark 2.2.8. The original definition of the Fubini condition is more complex and indeed one of the main purposes of [50, Proposition 5.7] is obtaining an equivalent definition of an *MS*-measurable structure where the Fubini condition is phrased only in terms of projections rather than definable functions. Given a candidate dimension-measure on a structure $\mathcal{M}$, it is easier to check whether it satisfies the version of Fubini described above rather than the original version. However, in later chapters, when making computations with an *MS*-dimension-measure, it might be more convenient to use the original phrasing of the Fubini condition. We include it below:

5. (Fubini*) Suppose that $X, Y \in \mathrm{Def}(M)$ and that $f : X \to Y$ is a definable surjection. Since h is finite and definable, we know that Y is partitioned by non-empty definable sets $Y_1, \ldots, Y_r$, where, for $1 \leq i \leq r$,

$$Y_i := \{\overline{b} \in Y | h(f^{-1}(\overline{b})) = (d_i, \mu_i)\},$$

for $(d_1, \mu_1), \ldots, (d_r, \mu_r) \in \mathbb{N} \times \mathbb{R}^{>0}$ distinct. Let $h(Y_i) = (e_i, \nu_i)$ for each $1 \leq i \leq r$. Let

$$c := \max\{d_i + e_i | 1 \leq i \leq r\},$$

and suppose (possibly relabeling $Y_1, \ldots, Y_r$) that it is attained by $d_1 + e_1, \ldots, d_s + e_s$ for $s \leq r$. Then,

$$h(X) = (c, \mu_1\nu_1 + \cdots + \mu_s\nu_s).$$

Remark 2.2.9. From the definition it follows that being MS-measurable is a property of a theory, i.e. if $\mathcal{M}$ is MS-measurable, then so is any elementarily equivalent structure. We say that a many sorted structure $\mathcal{M}^*$ is MS-measurable if every restriction of $\mathcal{M}^*$ to finitely many sorts is MS-measurable. We have that if $\mathcal{M}$ is MS-measurable, then so is $\mathcal{M}^{eq}$ [50, Proposition 5.10]. We further discuss this in Remark 3.1.6.

Remark 2.2.10. An important, and somehow hidden, assumption of the definition of MS-measurability is that of **positivity**. That is, for $h(X) = (k, \nu)$, $\nu > 0$. This will be vital to our proof of non-measurability since we will show that a given definable set must be assigned measure zero.

Remark 2.2.11. As noted in [50, Proposition 5.3], for A-definable X, we have an induced A-invariant Keisler measure on the definable subsets of X.

Remark 2.2.12. The dimension part of an MS-dimension-measure is a definable dimension function, as defined above. Additivity follows from [67, p.919].

In an MS-measurable context, for $\pi(\bar{x})$ a partial type over $B \subseteq M$, we define

$$\mu(\pi(\bar{x})) = \inf\{\mu(\phi(\bar{x},\bar{b}))|\pi(\bar{x}) \vdash \phi(\bar{x},\bar{b}), \mathrm{d}(\phi(\bar{x},\bar{b})) = \mathrm{d}(\pi(\bar{x}))\},$$

and we write $\mu(\bar{a}/B)$ for $\mu(\mathrm{tp}(\bar{a}/B))$. For an ω-categorical MS-measurable structure, $\mu(\bar{a}/\bar{b}) = \mu(\phi(\bar{x},\bar{b}))$, for any formula $\phi(\bar{x},\bar{b})$ isolating $\mathrm{tp}(\bar{a}/\bar{b})$.

Remark 2.2.13. We shall make frequent use of the fact that, in an ω-categorical MS-measurable structure, Fubini implies that

$$h(\bar{a}\bar{b}) = (\dim(\bar{a}\bar{b}), \mu(\bar{a}\bar{b})) = (\dim(\bar{a}/\bar{b}) + \dim(\bar{b}), \mu(\bar{a}/\bar{b})\mu(\bar{b})).$$

2.2.2 Dimension-independence

It is easy to obtain a notion of independence from an invariant dimension. In particular, in an MS-measurable structure, dimension-independence corresponds to non-forking independence. In [19], this result is attributed to unpublished work of Ben-Yaacov. In this section, we prove it briefly. In the next subsection, this will help to show that in ω-categorical MS-measurable structures we may take the dimension to be SU-rank without harm.

Definition 2.2.14. Let $\mathrm{d} : \mathrm{Dim}(\mathbb{M}) \to \mathbb{N}$ be an invariant dimension, where $\mathbb{M}$ is a monster model. Let $\bar{a}$ be a tuple and B, C be small subsets of $\mathbb{M}$. We say that $\bar{a}$ is d-independent from B over C, writing $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_C B$ if

$$\mathrm{d}(\bar{a}/BC) = \mathrm{d}(\bar{a}/C).$$

Definition 2.2.15. Let $\mathbb{M}$ be a monster model for the complete theory T. Let $\mathop{\smile\!\!\!\!\!\mid}^{\star}$ be a ternary relation between a finite tuple and two small subsets of $\mathbb{M}$. Write

$$a \mathop{\smile\!\!\!\!\!\mid}^{\star}_{C} B,$$

where a is finite and $B, C \subseteq \mathbb{M}$ are small. We say that $\mathop{\smile\!\!\!\!\!\mid}^{\star}$ is a **notion of independence** (in the sense of [44, Definition 4.1]) if it is invariant under automorphism of $\mathbb{M}$ and it satisfies the following conditions:

- (Monotonicity and Transitivity) $a \mathop{\smile\!\!\!\!\!\mid}^{\star}_C BD$ if and only if $a \mathop{\smile\!\!\!\!\!\mid}^{\star}_C B$ and $a \mathop{\smile\!\!\!\!\!\mid}^{\star}_{BC} D$;
- (Symmetry) for finite tuples a, b, $a \mathop{\smile\!\!\!\!\!\mid}^{\star}_C b$ if and only if $b \mathop{\smile\!\!\!\!\!\mid}^{\star}_C a$;
- (Finite Character) $a \mathop{\smile\!\!\!\!\!\mid}^{\star}_C B$ if and only if $a \mathop{\smile\!\!\!\!\!\mid}^{\star}_C b$ for every finite tuple from B;
- (Local Character) there is a cardinal κ such that for any a and B there is $B_0 \subsetneq B$ such that $|B| < \kappa$ and $a \mathop{\smile\!\!\!\!\!\mid}^{\star}_{B_0} B$; and
- (Existence) for any a, B, C, there is some $a' \equiv_B a$ such that $a' \mathop{\smile\!\!\!\!\!\mid}^{\star}_B C$.

Notions of independence are particularly important in the context of simple theories. In fact, by the Kim-Pillay theorem [64, Theorem 7.3.13], if a theory has a notion of independence which also satisfies the independence theorem over models, the theory is simple and the notion of independence corresponds to non-forking independence. Additionally, we know from [44, Theorem 4.2, Claim I] that non-forking independence is the strongest notion of independence in any theory. Both of these results can be helpful in showing that a given notion of independence corresponds to non-forking independence (or that a theory is simple). We will use the latter to show that dimension-independence corresponds to non-forking independence in MS-measurable structures.

We begin by proving that dimension-independence yields a notion of independence. This is easy to prove from the basic properties of an invariant dimension:

Proposition 2.2.16. *Let* $\mathrm{d} : \mathrm{Dim}(\mathbb{M}) \to \mathbb{N}$ *be an invariant dimension. The relation of* d*-independence is a notion of independence.*

Proof. Invariance follows by invariance of the dimension d.
For monotonicity and transitivity first note that

$$\mathrm{d}(\bar{a}/ABC) \leq \mathrm{d}(\bar{a}/AB) \leq \mathrm{d}(\bar{a}/A)$$

Hence, we have that

$$\begin{aligned}\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{A} BC &\Leftrightarrow \mathrm{d}(\bar{a}/ABC) = \mathrm{d}(\bar{a}/A) \\ &\Leftrightarrow \mathrm{d}(\bar{a}/AB) = \mathrm{d}(\bar{a}/A) \text{ and } \mathrm{d}(\bar{a}/ABC) = \mathrm{d}(\bar{a}/AB) \\ &\Leftrightarrow \bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{A} B \text{ and } \bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{AB} C.\end{aligned}$$

For symmetry, note that our dimension is additive (by definability and fibration according to [67][p.919]). That is,

$$\begin{aligned}\mathrm{d}(\bar{a}\bar{b}/A) &= \mathrm{d}(\bar{a}/A\bar{b}) + \mathrm{d}(\bar{b}/A) \\ &= \mathrm{d}(\bar{b}/A\bar{a}) + \mathrm{d}(\bar{a}/A).\end{aligned}$$

Now, $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{A} \bar{b}$ means that $\mathrm{d}(\bar{a}/A\bar{b}) = \mathrm{d}(\bar{a}/A)$. Hence, comparing the two equations resulting from additivity, $\mathrm{d}(\bar{b}/A\bar{a}) = \mathrm{d}(\bar{b}/A)$, which yields $\bar{b} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_{A} \bar{a}$.

Finite character and local character follow from the definition of $\mathrm{d}(\bar{a}/B)$ (cf. Remark 2.2.4).

The proof of existence is the same as the standard proof in the case of non-forking independence (see for example [43][Fact 2.1.3]). Let $\bar{a} \in \mathbb{M}^{|\bar{x}|}$, $A \subset C$, and let $\mathrm{d}(\bar{a}/A) = k$. Now, consider the set of formulas:

$$\Gamma := \{\psi(\bar{x}', \bar{c}) | \bar{x}' \subseteq \bar{x}, \bar{c} \subset C, \mathrm{d}(\psi(\bar{x}', \bar{c})) < k\}.$$

We consider the partial type $q' = \mathrm{tp}(\bar{a}/A) \cup \{\neg\psi(\bar{x}', \bar{c}) | \psi(\bar{x}', \bar{c}) \in \Gamma\}$. Suppose by contradiction q' is inconsistent. Then, by compactness,

$$\mathrm{tp}(\bar{a}/A) \cup \{\neg\psi_1(\bar{x}'_1, \bar{c}_1), \ldots, \neg\psi_n(\bar{x}'_n, \bar{c}_n)\}$$

is inconsistent, and so $\mathrm{tp}(\bar{a}/A) \vdash \bigvee_{i=1}^{n} \psi_i(\bar{x}'_i, \bar{c}_i)$. But since $\mathrm{d}(\psi_i(\bar{x}'_i, \bar{c}_i)) < k$ for each $i \leq n$ by the union property of the dimension, $\mathrm{d}(\bar{a}/A) < k$, which contradicts our initial assumption.

Now, q' can be extended to a complete type q over C. Note that $\mathrm{d}(q) = k$, for if we had $\mathrm{d}(q) < k$, this would be witnessed by some formula $\psi(\bar{x}', \bar{c})$, and this formula would be in Γ by construction, contradicting the consistency of q'. $\square$

Hence, in an MS-measurable context, the dimension part of an MS-dimension-measure yields a notion of independence. We shall prove this coincides with non-forking independence. In order to prove this, first recall Lemma 3.5 from [19]:

Lemma 2.2.17. *Let $\mathcal{M}$ be MS-measurable. Let $X \subseteq M^{|\bar{x}|}$ be definable and $\phi(\bar{x}, \bar{y})$ be such that there is an indiscernible sequence $(\bar{b}_i)_{i<\omega}$ for which $\{\phi(\bar{x}, \bar{b}_i) | i < \omega\}$ is inconsistent and $\phi(M^{|\bar{x}|}, \bar{b}_i) \subseteq X$ for each $i < \omega$. Then, $\mathrm{d}(X) > \mathrm{d}(\phi(\bar{x}, \bar{b}))$.*

From this result, it is easy to prove that dimension independence implies non-forking independence.

Proposition 2.2.18. *Suppose that $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_{C} B$. Then, $\mathrm{tp}(\bar{a}/BC)$ does not divide over C. Hence, by simplicity, $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{d}}_{C} B$ implies $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_{C} B$, where $\mathop{\smile\!\!\!\!\!\!\mid}^{\mathrm{f}}_{C}$ is non-forking independence.*

Proof. Suppose by contrapositive that $\mathrm{tp}(\bar{a}/BC)$ divides over C. So there is $\phi(\bar{x}, \bar{d}) \in \mathrm{tp}(\bar{a}/BC)$ such that $\{\phi(\bar{x}, \bar{d}_i) | i < \omega\}$ is k-inconsistent and $(\bar{d}_i)_{i<\omega}$ is C-indiscernible with $\bar{d}_0 = \bar{d}$. Let $\chi(\bar{x})$ be a formula defined over C witnessing $\mathrm{d}(\bar{a}/C) = k$. Consider $\phi(\bar{x}, \bar{d}_i) \wedge \chi(\bar{x})$. Since $\phi(\bar{x}, \bar{d}) \in \mathrm{tp}(\bar{a}/BC)$, we have that $\models \exists \bar{x}(\phi(\bar{x}, \bar{d}) \wedge \chi(\bar{x}))$, and since $\bar{d}_i \equiv_C \bar{d}$, $\models \exists \bar{x}(\phi(\bar{x}, \bar{d}_i) \wedge \chi(\bar{x}))$. Thus, $\phi(\mathbb{M}^{|\bar{x}|}, \bar{d}_i) \wedge \chi(\mathbb{M}^{|\bar{x}|}) \subseteq \chi(\mathbb{M}^{|\bar{x}|})$, and $\{\phi(\bar{x}, \bar{d}_i) \wedge \chi(\bar{x}) | i < \omega\}$ is inconsistent. The conditions of Lemma 2.2.17 are met and

$$\mathrm{d}(\chi(\bar{x})) > \mathrm{d}(\phi(\bar{x}, \bar{d}) \wedge \chi(\bar{x})).$$

Since $\phi(\bar{x}, \bar{d}) \wedge \chi(\bar{x}) \in \mathrm{tp}(\bar{a}/BC)$, we have that $\mathrm{d}(\bar{a}/BC) < \mathrm{d}(\bar{a}/C)$, implying that $\bar{a}$ is not d-independent from B over C. Our claim then holds by contrapositive. $\square$

The other implication holds by the following result of Kim and Pillay [44][Theorem 4.2, Claim I]:

Lemma 2.2.19. *Let T be an arbitrary theory and $\mathop{\smile\!\!\!\!\!\!|}^{\star}$ be a notion of independence. Suppose that $a \mathop{\smile\!\!\!\!\!\!|}^{f}_{C} B$. Then, $a \mathop{\smile\!\!\!\!\!\!|}^{\star}_{C} B$.*

This yields the desired result:

Theorem 2.2.20. *Let $\mathcal{M}$ be an MS-measurable structure. Then, dimension independence is the same as non-forking independence.*

Remark 2.2.21. An alternative way to prove that dimension independence is the same of non-forking independence is by noting how the independence theorem over models is essentially proven in Theorem 2.18 of [31]. Then, Theorem 2.2.20 can be deduced from the Kim-Pillay Theorem 1.2.11.

2.2.3 MS-measurability in an ω-categorical context

In this subsection, we introduce some basic equations which must be satisfied in ω-categorical MS-measurable structures through Corollary 2.2.24. Observations regarding these equations allow us to prove that in an ω-categorical context we may always take the dimension part of an MS-dimension-measure to be SU-rank, as shown in Corollary 2.2.26. This is an helpful tool for simplifying proofs of non-MS-measurability of ω-categorical Hrushovski constructions. In parallel, we also note that we can prove similar results for an invariant Keisler measure in a simple ω-categorical theory. In particular, Corollary 2.2.27 gives sufficient conditions for a function on types over finitely many parameters to induce an invariant Keisler measure.

The following result is a standard fact about ω-categorical MS-measurable structures. As far as I can tell, a version of this is first proven in Elwes' PhD thesis [18, Lemma 5.2.1]:

Lemma 2.2.22. *Let $\mathcal{M}$ be ω-categorical and MS-measurable. Let $B \supseteq A$ be finite subsets of M. Suppose p is a partial type over A. By ω-categoricity, p has finitely many complete extensions to B. Let $p_1, \ldots, p_n$ be those which do not fork over A. Then,*

$$\mu(p) = \sum_{i=1}^{n} \mu(p_i). \tag{2.1}$$

Proof. Suppose the complete extensions of p to B are $p_1, \ldots, p_m$ with $m \geq n$ where $p_1, \ldots, p_n$ have maximal dimension. By Theorem 2.2.20, these are the non-forking extensions of p to B. Consider the sets

$$p_i(M^{|\bar{x}|}) := \left\{ \bar{c} \in M^{|\bar{x}|} \middle| \bar{c} \models p(\bar{x}) \right\}.$$

By ω-categoricity, these are disjoint definable sets such that $\bigcup_{i=1}^{m} p_i(M^{|\bar{x}|}) = p(M^{|\bar{x}|})$. Hence, by additivity:

$$\mu(p) = \mu\left(\bigcup_{i=1}^{n} p_i(M^{|\bar{x}|}) \cup \bigcup_{i>n}^{m} p_i(M^{|\bar{x}|})\right) = \mu\left(\bigcup_{i=1}^{n} p_i(M^{|\bar{x}|})\right) = \sum_{i=1}^{n} \mu(p_i).$$

Where the second last equality holds by additivity in the case $\mathrm{d}(X) < \mathrm{d}(Y)$, and the last equality due to additivity in the case of $\mathrm{d}(X) = \mathrm{d}(Y)$. $\square$

It is worth observing that the same idea yields an analogous result in the context of invariant Keisler measures for an ω-categorical simple structure.

Lemma 2.2.23. *Let $\mathcal{M}$ be the countable model of an ω-categorical and simple theory (or its monster model). Suppose that μ is an $\mathrm{Aut}(M)$-invariant Keisler measure on $\mathrm{Def}_x(M)$. Let $B \supseteq A$ be finite subsets of M. Suppose that p is a partial type over A. Then,*

$$\mu(p) = \sum_{i=1}^{n} \mu(c_i/B),$$

where $\mathrm{tp}(c_1/B), \ldots, \mathrm{tp}(c_n/B)$ are the complete non-forking extensions of p to B.

Proof. The essential ingredient of this result is that if a formula forks over $\emptyset$, then it must be assigned measure zero by any invariant Keisler measure [12, Fact 1.2]. By Ryll-Nardzewski, since types over finite sets are isolated by formulas, we have that the only completions of p which are possibly assigned positive measures are those which do not fork over $\emptyset$. By monotonicity we have that, since $A \subseteq B$,

$$\text{if } c \underset{}{\perp} B, \text{ then } c \underset{A}{\perp} B.$$

Hence, by additivity of the measure we have

$$\mu(p) = \sum_{i=1}^{n} \mu(c_i/B).$$

$\square$

By applying Fubini (see Remark 2.2.13) to Lemma 2.2.22, we get equations in terms of types over the empty set which must be satisfied by ω-categorical MS-measurable structures. Since being MS-measurable is a property of a theory, we assume throughout that we are working with an ω-saturated model.

Corollary 2.2.24. *Let $\mathcal{M}$ be ω-categorical and MS-measurable. Take $\bar{a}, \bar{b}, \bar{c}$ to be tuples from M. Let $\mathrm{tp}(\bar{c}_1/\bar{a}\bar{b}), \ldots, \mathrm{tp}(\bar{c}_n/\bar{a}\bar{b})$ be the finitely many complete non-forking extensions of $\mathrm{tp}(\bar{c}/\bar{a})$ to $\bar{a}\bar{b}$. Then, for $\bar{a} = \emptyset$,*

$$\frac{\mu(\overline{ac})\mu(\overline{ab})}{\mu(\bar{a})} = \sum_{i=1}^{n} \mu(\overline{abc_i}). \tag{2.2}$$

For $\bar{a} = \emptyset$, we simply have

$$\mu(\bar{c})\mu(\bar{b}) = \sum_{i=1}^{n} \mu(\overline{bc_i}).$$

We shall use the above corollary in our proof of non-measurability of some Hrushovski constructions in Chapter 5. There is a partial converse to this corollary. In fact, if an ω-categorical structure has a function $\mu : S(\emptyset) \to \mathbb{R}^{>0}$ satisfying equations of the form of (2.2) and an algebraicity condition, then it is MS-measurable. This can be extracted from the proof of Theorem 2.2.25. In this theorem, we show that in an ω-categorical MS-measurable structure we may choose our dimension to be any definable dimension yielding non-forking independence as its notion of independence.

Theorem 2.2.25. *Suppose that $\mathcal{M}$ is an ω-categorical MS-measurable structure with dimension-measure $h = (\mathrm{d}, \mu)$. Let $D : \mathrm{Def}(M) \to \mathbb{N}$ be any definable dimension for $\mathcal{M}$ whose induced notion of independence is non-forking independence. Then, $\mathcal{M}$ has a dimension-measure $h' = (D, \mu')$, where μ' agrees with μ on complete types over finite sets of parameters.*

Proof. By Ryll-Nardzewski, D is invariant and finite. The dimension-measure $h = (d, \mu)$ induces a function $\mu^* : S_{\mathrm{fin}}(M) \to \mathbb{R}^{>0}$ given by $\mu^*(\overline{a}/\overline{b}) = \mu(\overline{a}/\overline{b})$, where $S_{\mathrm{fin}}(M)$ denotes the space of types over finite sets of parameters from M in a finite tuple. We can write any set X definable over $\overline{a}$ as a finite disjoint union of the sets of realisations of complete types $p_1, \ldots, p_m$ over $\overline{a}$. Say, for $n \leq m$, that $p_1, \ldots, p_n$ have maximal D-dimension. Then, we define,

$$\mu'(X) = \sum_{i=1}^{n} \mu^*(p_i). \tag{2.3}$$

Claim: The function $\mu' : \mathrm{Def}(M) \to \mathbb{R}^{>0}$ is well-defined, i.e. it does not depend on the choice of parameters over which X is defined.

Proof of Claim. Suppose that X is defined both over $\overline{a}$ and $\overline{b}$. If X is definable over both $\overline{a}$ and $\overline{b}$, it is also definable over $\overline{ab}$. Hence, we may assume without loss of generality that $\overline{a} \subseteq \overline{b}$ to prove that the definition of μ' does not depend on the choice of parameters. Each p_i over $\overline{a}$ has finitely many complete extensions of maximal D-dimension to $\overline{b}$, say $p'_{i,1}, \ldots, p'_{i,m_i}$, which, by assumption and existence, are the non-forking extensions of p_i to $\overline{b}$. By the dimension theorem 2.2.20, these are the extensions of p_i of maximal d-dimension, and so, by MS-measurability, we know that

$$\mu(p_i) = \sum_{j=1}^{m_i} \mu(p'_{ij}).$$

But then,

$$\sum_{i=1}^{n} \mu(p_i) = \sum_{i=1}^{n} \sum_{j=1}^{m_j} \mu(p'_{ij}),$$

where the equation on the left is the definition of $\mu'(X)$ as defined over $\overline{a}$ and the equation on the right is the definition of μ' for X as defined over $\overline{b}$. This yields that μ' is well defined. □

Now, we need to verify that $h' = (D, \mu')$ satisfies the various conditions of Definition 2.2.7. Firstly, invariance of h' follows from invariance of D and invariance of μ', which is inherited from μ by definition, since non-forking independence is invariant. Furthermore, μ' inherits positivity from μ since in the context of 2.3 each of the p_i is assigned positive measure by μ^*. Similarly, h' satisfies algebraicity. The dimension D has the corresponding property by definition. For the measure μ', note that since $\overline{x} = \overline{a}$ isolates a complete type over $\overline{a}$ we have that

$$\mu'(\{\overline{a}\}) = \mu(\{\overline{a}\}),$$

by definition of μ'. Invariance is inherited from invariance of d and of μ. Since invariance implies finiteness and definability in an ω-categorical context, h is finite and definable.

For additivity, consider $X, Y \subseteq M^n$ definable and disjoint. Possibly adding unused parameters to the formulas defining them, we may assume that they are defined over the same set of parameters $\overline{a}$. Let $p_1, \ldots, p_n$ be the complete types over $\overline{a}$ of maximal D-dimension which contain the formula defining X and let $q_1, \ldots, q_l$ be the corresponding complete types of maximal D-dimension for Y. If $D(X) < D(Y)$, $p_1, \ldots, p_n$ have smaller dimension than $q_1, \ldots, q_l$, yielding that the complete types over $\overline{a}$ of maximal D-dimension in whose realisations lie in $X \cup Y$ are precisely $q_1, \ldots, q_l$. Hence, by definition of μ',

$$\mu'(X \cup Y) = \sum_{i=1}^{l} \mu^*(q_i) = \mu'(Y).$$

Instead, if $D(X) = D(Y)$, $p_1, \ldots, p_n$ and $q_1, \ldots, q_l$ have the same D-dimension, which from the definition of μ' implies that $\mu'(X \cup Y) = \mu(X) + \mu(Y)$.

Finally, we need to prove Fubini. Firstly, by construction, μ' and μ agree on complete types over finite sets of parameters, and so

$$\mu'(\overline{a}\overline{b}) = \mu(\overline{a}\overline{b}) = \mu(\overline{a}/\overline{b})\mu(\overline{b}) = \mu'(\overline{a}/\overline{b})\mu'(\overline{b}). \tag{2.4}$$

Now, suppose that X is defined by $\phi(\overline{x},\overline{y},\overline{b})$ and suppose that for every $\overline{a} \vDash \exists\overline{y}\phi(\overline{x},\overline{y},\overline{b})$, $h'(\phi(\overline{a},\overline{y},\overline{b})) = (k,\nu)$. Let $\text{tp}(\overline{a_1}/\overline{b}),\ldots,\text{tp}(\overline{a_m}/\overline{b})$ be the types of maximal D-dimension in the disjoint union of types equivalent to $\exists y\phi(\overline{x},\overline{y},\overline{b})$. Let $\text{tp}(a_i\overline{c_{j_i}}/\overline{b})$ $1 \le i \le m, 1 \le j_i \le n_i$ be the types of maximal D-dimension in the union of complete types equivalent to $\phi(\overline{x},\overline{y},\overline{b})$. Note that for each each $\overline{a_i}$ there must be at least one $\overline{c} \vDash \phi(a_i,\overline{y},\overline{b})$ such that $D(\overline{c}/a_i\overline{b}) = k$ by extension.

We have by definition of μ' that:

$$\mu'(X) = \sum_{i=1}^{m}\sum_{j_i=1}^{n_i}\mu(\overline{a_i c_{j_i}}/\overline{b})$$

By the union and additivity properties of D we have that for any $i \in \{1,\ldots,m\}, j_i \in \{1,\ldots,n_i\}$:

$$D(\phi(\overline{x},\overline{y},\overline{b})) = D(\overline{a_i c_{j_i}}/\overline{b}) = D(\overline{c_{j_i}}/\overline{a_i}\overline{b}) + D(\overline{a_i}/\overline{b}) = k + D(\exists\overline{y}\phi(\overline{x},\overline{y},\overline{b})). \quad (2.5)$$

The last equality follows by assumption and definition of the a_i and shows the dimension behaves according to the Fubini condition.

Note that by 2.5, $\text{tp}(\overline{c_{j_i}}/\overline{a_i}\overline{b})$ are the types of maximal dimension in the disjoint union of types equivalent to $\phi(\overline{a_i},\overline{y},\overline{b})$. To see this, note that if $\text{tp}(\overline{a_i}\overline{d}/\overline{b})$ is a type not of maximal dimension whose set of solutions is in the disjoint union giving X, we have

$$\begin{aligned} D(\overline{d}/\overline{a_i}\overline{b}) &= D(\overline{a_i}\overline{d}/\overline{b}) - D(\overline{a_i}/\overline{b})) \\ &< D(\overline{a_i c_{j_i}}/\overline{b}) - D(\overline{a_i}/\overline{b}) \\ &= D(\overline{c_{j_i}}/\overline{a_i}\overline{b}). \end{aligned}$$

So, by definition of μ' over definable sets, and by assumption:

$$\mu'(\phi(\overline{a_i},\overline{y},\overline{b})) = \sum_{j=1}^{n_i}\mu(\overline{c_{j_i}}/\overline{a_i}\overline{b}) = \nu. \quad (2.6)$$

We have that

$$\mu'(X) = \sum_{i=1}^{m}\sum_{j_i=1}^{n_i}\mu(\overline{a_i c_{j_i}}/\overline{b})$$

$$\begin{aligned}
&= \sum_{i=1}^{m} \sum_{j_i=1}^{n_i} \mu(\overline{c_{j_i}}/\overline{a_i}\overline{b})\mu(\overline{a_i}/\overline{b}) && \text{by Fubini on } \mu \\
&= \sum_{i=1}^{m} \mu(\overline{a_i}/\overline{b}) \sum_{j_i=1}^{n_i} \mu(\overline{c_{j_i}}/\overline{a_i}\overline{b}) \\
&= \sum_{i=1}^{m} \mu(\overline{a_i}/\overline{b})\nu && \text{by (2.6)} \\
&= \mu'(\exists y\phi(\overline{x},\overline{y},\overline{b}))\nu && \text{by definition of } \mu'.
\end{aligned}$$

Due to the last equations and (2.5), the Fubini condition is also satisfied. Hence, (D, μ') yields an MS-dimension-measure for $\mathcal{M}$. □

An important consequence of this theorem is that, for MS-measurable ω-categorical structures, we may take the dimension to be SU-rank. In a structure of finite SU-rank, SU-rank is a dimension function: it is additive by the Lascar inequalities [43, Prop. 2.5.19]. In an ω-categorical structure, it is definable being invariant. Since MS-measurable structures have finite SU-rank, in ω-categorical MS-measurable structures, SU-rank yields a definable dimension. By definition, it induces non-forking independence as its notion of independence. Hence, it satisfies all of the conditions of Theorem 2.2.25, and we get the following corollary.

Corollary 2.2.26. *Suppose that the ω-categorical structure $\mathcal{M}$ is MS-measurable with dimension-measure $h = (d, \mu)$. Then, $\mathcal{M}$ is also MS measurable via the dimension-measure $h' = (SU, \mu')$, where μ' is as in Theorem 2.2.25. Hence, if an ω-categorical structure $\mathcal{M}$ is MS-measurable, there is a dimension-measure on $\mathcal{M}$ where the dimension is given by SU-rank.*

This statement is known to be false outside of an ω-categorical context. In particular, in [19, Remark 3.8], Elwes and Macpherson show that there are MS-measurable structures where SU-rank is not definable. They also give an example of an ω-categorical structure for which we can artificially choose the dimension in the MS-measure and SU-rank to differ. However, our theorem does prove that if there is an MS-measure in the ω-categorical case, there is no harm in taking the dimension to be SU-rank.

Corollary 2.2.26 is a powerful tool in the context of MS-measurable ω-categorical structures. In particular, if we wish to prove that a certain ω-categorical structure is not MS-measurable, it can be helpful to use the assumption that the dimension coincides with SU-rank. For example, Evans' proof of non-measurability of some Hrushovski constructions [23] employs a highly non-trivial theorem showing that in the class of structures he is considering, any dimension function corresponds to a scaled SU-rank. Given our result, we do not need to prove such theorems in the course of disproving MS-measurability of a given structure. Indeed, for other ω-categorical structures such a theorem is false (e.g. the example in [19, Rem. 3.8]).

To conclude, observe that the initial part of the proof of Theorem 2.2.25 can be used to also prove a statement about invariant Keisler measures. By $\mathrm{loc}(a/B)$, i.e. the **locus** of a over B, we mean the set of realisations of $\mathrm{tp}(a/B)$ in the model we are working with.

Corollary 2.2.27. *Let $\mathcal{M}$ be the countable model of an ω-categorical and simple theory (alternatively, its monster model). Write $S^{\mathrm{nf}}_{\mathrm{fin}}(M)$ for the set of types over finitely many parameters from M in the variable x which do not fork over $\emptyset$.*
Suppose there is an invariant function $\mu^* : S^{\mathrm{nf}}_{\mathrm{fin}}(M) \to [0,1]$ such that $\mathrm{loc}(a/B) = \mathrm{loc}(c/D)$ implies $\mu^*(a/B) = \mu^*(c/D)$. Suppose further that μ^* satisfies the following conditions:

(a) Let $d_1, \ldots, d_n$ isolate the complete types in the variable x over $\emptyset$. Then,

$$\sum_{i=1}^{n} \mu^*(d_i) = 1.$$

(b) Let b be a tuple in the variable x and $\bar{a}$ another tuple from M. Let c be a single point in M. Let $\mathrm{tp}(b_1/\bar{a}c), \ldots, \mathrm{tp}(b_m/\bar{a}c)$ be the non-forking extensions of $\mathrm{tp}(b/\bar{a})$ to $\bar{a}c$. We have that

$$\sum_{i=1}^{m} \mu^*(b_i/\bar{a}c) = \mu^*(b/\bar{a}).$$

Then, μ^* determines uniquely an $\mathrm{Aut}(M)$*-invariant Keisler measure* $\mu : \mathrm{Def}_x(M) \to [0,1]$*, where for* $\phi(x,\bar{a})$*, letting* $\mathrm{tp}(b_1/\bar{a}), \ldots, \mathrm{tp}(b_l/\bar{a})$ *be the complete types over* $\bar{a}$ *in the variable* x *which do not fork over* $\emptyset$ *and contain the formula* $\phi(x,\bar{a})$*,*

$$\mu(\phi(x,\bar{a})) = \sum_{i=1}^{l} \mu^*(b_i/\bar{a}).$$

Conversely, each $\mathrm{Aut}(M)$*-invariant Keisler measure determines such a function.*

Proof. The most important aspect of the proof is that μ is well-defined, i.e. if two formulas, possibly over different parameters define the same subset of $M^{|x|}$, then they are assigned the same measure. Note that for $A \subseteq B$ finite and p a type over A non-forking over $\emptyset$, by monotonicity and transitivity of non-forking independence, p' is a non-forking extension of p to B if and only if p' is an extension of p to B which does not fork over $\emptyset$. Types over finite sets of parameters which do fork over $\emptyset$ are assigned measure zero by definition of μ. With these two observations, one can run exactly the same argument as that for the Claim at the beginning of the proof of Theorem 2.2.25 in order to show that μ is well-defined.

Invariance of μ is inherited from μ^*. Additivity follows from the definition of μ similarly to the proof in Theorem 2.2.25: consider the formulas $\phi(x,\bar{a})$ and $\psi(x,\bar{b})$ defining disjoint definable sets. Let $p_1, \ldots, p_n$ be the complete types (in the variable x) over $\overline{ab}$ non-forking over $\emptyset$ containing $\phi(x,\bar{a})$ and $q_1, \ldots, q_m$ be the complete types over $\overline{ab}$ non-forking over $\emptyset$ containing $\psi(x,\bar{b})$. Since $\phi(x,\bar{a})$ and $\psi(x,\bar{b})$ are disjoint, the p_i and the q_i are distinct types and any type over $\overline{ab}$ non-forking over $\emptyset$ containing $\phi(x,\bar{a}) \vee \psi(x,\bar{b})$ is amongst them. Hence, by definition of μ,

$$\mu(\phi(x,\bar{a}) \vee \psi(x,\bar{b})) = \sum_{i=1}^{n} \mu(p_i) + \sum_{i=1}^{m} \mu(q_i) = \mu(\phi(x,\bar{a})) + \mu(\psi(x,\bar{b})).$$

Here the last equality follows from definition of μ taking $\phi(x,\bar{a})$ and $\psi(x,\bar{b})$ as being defined over $\overline{ab}$. This yields additivity as desired. Finally, $\mu(x = x) = 1$ by condition (a).

The converse follows from Lemma 2.2.23 and the fact that formulas which fork over $\emptyset$ are assigned measure 0 by any invariant Keisler measure [12, Fact 1.2] $\square$

This corollary is a helpful tool in proving the existence of an invariant Keisler measure because one only needs to prove that a function satisfying the above conditions exists.

3
Transfer of Independence

A tool which has proven to be essential for the study of measures in ω-categorical structures is a form of transfer of independence from notions of independence in a structure to probabilistic independence between formulas defined over independent parameters with respect to an invariant measure. We first observed this phenomenon in ω-categorical MS-measurable structures and then in the context of ergodic invariant Keisler measures.

We begin with Section 3.1 where we study transfer of independence in an MS-measurable context. The main result of the section is Theorem 3.1.9 which relates non-forking independence to probabilistic independence in an MS-measure:

Theorem 3.1.9. *Let $\mathcal{M}$ be MS-measurable, ω-categorical with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Suppose that a_1, a_2, a_3 from $\mathcal{M}^{eq}$ are such that $a_i \mathop{\smile\hskip-0.9em\raise0.9ex\hbox{$|$}\hskip0.5em} a_j$ for $1 \leq i < j \leq 3$. Then, $\mathrm{d}(\mathrm{tp}(a_3/a_1) \cup \mathrm{tp}(a_3/a_2)) = \mathrm{d}(a_3)$ and*

$$\mu(\mathrm{tp}(a_3/a_1) \cup \mathrm{tp}(a_3/a_2)) = \frac{\mu(a_3/a_1)\mu(a_3/a_2)}{\mu(a_3)}. \tag{3.1}$$

This result is a weaker version of results obtained by Hrushovski in his unpublished notes [33]. In Chapter 5, this theorem will be essential to obtain the equations that show that a given class of ω-categorical Hrushovski constructions is not MS-measurable.

Then, we move to considering $\mathrm{Aut}(M)$-invariant measures for an ω-categorical structure in Section 3.2 . In this context, we prove a stronger result: a weaker notion of independence than non-forking independence, weak algebraic independence, also transfers to probabilistic independence for a specific class of invariant Keisler measures: ergodic Keisler measures. The main result of Section 3.2 is Corollary 3.2.16, which will be essential in Chapter 6 for the study of invariant Keisler measures in simple ω-categorical structures:

Corollary 3.2.16. *Let $\mathcal{M}$ be ω-categorical. Let μ be an ergodic measure on M^{eq} in the variable x. Suppose that $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Let a, b from M^{eq} be such that $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}} b$. Then, for any $\mathcal{L}^{eq}$-formulas $\phi(x,y), \psi(x,z)$, where y and z are in the same sort as a and b respectively,*

$$\mu(\phi(x,a) \wedge \psi(x,b)) = \mu(\phi(x,a))\mu(\psi(x,b)).$$

It is natural to focus on ergodic measures since every invariant Keisler measure can be written as an integral average of them [52]. This was already noticed in Albert's classification of the invariant Keisler measures on the random graph [2]. Recent results of Jahel and Tsankov [41] are crucial in order to obtain the transfer result of Corollary 3.2.16.

We begin Section 3.2 with a brief introduction to ergodic measures in Subsection 3.2.1 explaining their utility for the study of invariant Keisler measures. Then, we study the Hilbert spaces arising from invariant Keisler measures in Subsection 3.2.2 and prove some of their basic properties. In particular, we want to show the folklore result that the action of the automorphism group $\mathrm{Aut}(M)$ on the Hilbert space $L^2(\mu)$ associated to the invariant Keisler measure μ is a unitary representation. This result is Proposition 3.2.9. With this knowledge we will be able to directly obtain Corollary 3.2.16 as an application of results of Jahel and Tsankov [41]. This will be explored in Subsection 3.2.3.

3.1 Transfer for MS-measurable structures

In this section, we obtain another set of equations that hold for MS-measurable ω-categorical structures. The main idea is that in an MS-measurable structure, non-forking independence induces probabilistic independence in the measure. This is shown explicitly in Theorem 3.1.9 and Corollary 3.1.10, which yields equations that we will use later in our proof of non-MS-measurability of some Hrushovski constructions in Chapter 5. We thank Ehud Hrushovski for sharing his notes [33]. In there, he proves a more general version of Theorem 3.1.9, which implies the results in this section. Working in an ω-categorical MS-measurable context greatly simplifies the tools and the amount of theory required to obtain the probabilistic independence theorem. We give proofs for these results in this section.

Definition 3.1.1. Let $\mathbb{M}$ be the monster model of a complete $\mathcal{L}$-theory. Let $R(x, y)$ be an invariant relation on $\mathbb{M}$ in the sense that for $a, a' \in \mathbb{M}^{|x|}, b, b' \in \mathbb{M}^{|y|}, ab \equiv a'b'$ implies

$$R(a, b) \text{ if and only if } R(a', b').$$

We say that $R(x, y)$ is **stable** if there is no infinite indiscernible sequence $(a_i b_i)_{i<\omega}$ such that

$$R(a_i, b_j) \text{ if and only if } i \leq j.$$

We say that an $\mathcal{L}$-formula $\delta(x, y)$ is **stable** if there is no infinite sequence $(a_i b_i)_{i<\omega}$ such that

$$\models \delta(a_i, b_j) \text{ if and only if } i \leq j.$$

Note that every $\mathcal{L}$-formula $\delta(x, y)$ yields an invariant relation on $\mathbb{M}$. By [64, Lemma 7.1.1], stability of $\delta(x, y)$ is equivalent to the invariant relation defined by $\delta(x, y)$ on $\mathbb{M}$ being stable.

For more on stability, we direct the reader to [54, §1.2], though in this thesis we only make use of the finite equivalence relation theorem [54, Lemma 2.11].

Remark 3.1.2. Let $R(x,y)$ be a stable relation on $\mathbb{M}$ and consider the expansion of $\mathbb{M}$ to an $\mathcal{L}'$-structure $\mathbb{M}'$ where $\mathcal{L}' = \mathcal{L} \cup \{R'(x,y)\}$, where $R'(x,y)$ is interpreted as the relation $R(x,y)$ on $\mathbb{M}$. Then, $R'(x,y)$ is a stable formula in $\mathbb{M}'$. In fact, the same indicernible sequence that yields stability of $R(x,y)$ as a relation yields stability of $R'(x,y)$ as a formula.

Let $\mathcal{M}$ be a small ω-saturated $\mathcal{L}$-structure and $R(x,y)$ an invariant relation on it. By ω-saturation of $\mathcal{M}$, there is a unique invariant extension of $R(x,y)$ to $\mathbb{M}$. Hence, we say that $R(x,y)$ (as a relation on $\mathcal{M}$) is **stable** if its unique invariant extension to $\mathbb{M}$ is stable.

For an ω-saturated $\mathcal{L}$-structure $\mathcal{M}$, let $\mu : \mathrm{Def}_{\overline{y}}(M) \to [0,1]$ be an invariant measure (in the sense of Definition 2.2.1) and $\phi(\overline{x}_1,\overline{y}), \psi(\overline{x}_2,\overline{y})$ be $\mathcal{L}$-formulas. For $\alpha \in [0,1]$, we define the relation $R_\alpha(\overline{x}_1,\overline{x}_2)$ on $M^{|\overline{x}_1|} \times M^{|\overline{x}_2|}$ by

$$R_\alpha(\overline{a},\overline{b}) \text{ if and only if } \mu(\phi(\overline{a},\overline{y}) \wedge \psi(\overline{b},\overline{y})) = \alpha. \tag{3.2}$$

From [31, Prop.2.25], we know that $R_\alpha(\overline{x}_1,\overline{x}_2)$ is stable.

We shall begin by showing that whether $R_\alpha(\overline{a},\overline{b})$ holds of a pair of independent tuples only depends on the individual types of $\overline{a}$ and $\overline{b}$. This can be seen as a consequence of a commonly used corollary to the finite equivalence relation theorem [54, Lemma 2.11] (see [34] or [44] for similar uses). We need to introduce some notation to express the result.

Definition 3.1.3. Let $\delta(\overline{x};\overline{z})$ be an $\mathcal{L}$-formula and A a set of parameters. An instance of δ over A is a formula $\delta(\overline{x};\overline{a})$ in $\mathcal{L}_A$, i.e. the language obtained by adding to $\mathcal{L}$ constants for the elements of A. A complete δ-type over A is a maximally consistent set of instances of δ over A. By $S^{\overline{x}}_\delta(A)$ we mean the set of δ-types over A. By $FER_\delta(A)$ we denote the set of A-definable equivalence relations $E(\overline{x},\overline{y})$ with finitely many classes such that for any $\overline{b}$, $E(\overline{x},\overline{b})$ is elementary equivalent to a Boolean combination of instances of δ over A.

With the above notation, we can prove the following well-know application of the finite equivalence relation theorem:

Proposition 3.1.4. *Let $\mathcal{M}$ be an ω-saturated $\mathcal{L}$-structure with* $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$*. Let $\delta(\overline{x}, \overline{y})$ be a stable $\mathcal{L}$-formula. Suppose that $\overline{a}$ and $\overline{b}$ are tuples from $\mathcal{M}$ with $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\, \overline{b}$. Then, whether $\models \delta(\overline{a}, \overline{b})$ only depends on* $\mathrm{tp}(\overline{a})$ *and* $\mathrm{tp}(\overline{b})$*, but not on* $\mathrm{tp}(\overline{ab})$*.*

Proof. Let $\overline{a}$ and $\overline{a}'$ be such that $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\, \overline{b}, \overline{a}' \mathop{\smile\!\!\!\!\!\!|}\, \overline{b}$ and $\overline{a} \equiv \overline{a}'$. In particular, $\overline{a}$ and $\overline{a}'$ have the same δ-type over $\emptyset$.

Let $B \subset M$ and suppose $q_1, q_2 \in S_\delta^{\overline{x}}(B)$ are non-forking extensions of $\mathrm{tp}_\delta(\overline{a})$, the δ type of $\overline{a}$ over the empty set. By stability of $\delta(\overline{x}, \overline{y})$ and the finite equivalence relation theorem [54, Lemma 2.11], $q_1 \neq q_2$ if and only if there is a finite equivalence relation $E(\overline{x}, \overline{y}) \in \mathrm{FER}_\delta(\emptyset)$ such that $q_1(\overline{x}) \cup q_2(\overline{y}) \vdash E(\overline{x}, \overline{y})$. By $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$, $E(\overline{x}, \overline{y})$ is trivial, i.e. tuples with the same types over the emptyset will be in the same equivalence classes. In particular, since for $\overline{a}$, $E(\overline{x}, \overline{a})$ is a Boolean combination instances of δ over over $\emptyset$, the equivalence class of $\overline{a}$ is entirely determined by $\mathrm{tp}_\delta(\overline{a})$, and so $q_1 = q_2$. Hence, for any B there is a unique δ-type over B which is a non-forking extension of $\mathrm{tp}_\delta(\overline{a})$. In particular, $\mathrm{tp}_\delta(\overline{a}/\overline{b}) = \mathrm{tp}_\delta(\overline{a}'/\overline{b})$, which yields

$$\models \delta(\overline{a}, \overline{b}) \leftrightarrow \delta(\overline{a}', \overline{b}).$$

We can apply the same reasoning in the variable $\overline{y}$ to prove that for $\overline{a} \mathop{\smile\!\!\!\!\!\!|}\, \overline{b}$ and $\overline{a}' \mathop{\smile\!\!\!\!\!\!|}\, \overline{b}'$ with $\overline{a} \equiv \overline{a}'$ and $\overline{b} \equiv \overline{b}'$ we have

$$\models \delta(\overline{a}, \overline{b}) \leftrightarrow \delta(\overline{a}', \overline{b}').$$

□

From this and stability of $R_\alpha(\overline{x}, \overline{y})$ we immediately get the following result for an invariant Keisler measure:

Corollary 3.1.5. *Let μ be an invariant (in the sense of Definition 2.2.1) Keisler measure in the variable x on an ω-saturated $\mathcal{L}$-structure $\mathcal{M}$ with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Let $\phi_i(x, \bar{y}_i)$ for $i \in \{1,2\}$ be $\mathcal{L}$-formulas. Suppose that $\bar{a}_1, \bar{a}_2$ are tuples from M such that $\bar{a}_1 \mathop{\smile\!\!\!\!\!\!|}\; \bar{a}_2$. Then, the value of $\mu(\phi_1(x, \bar{a}_1) \wedge \phi_2(x, \bar{a}_2))$ only depends on $\mathrm{tp}(\bar{a}_1)$ and $\mathrm{tp}(\bar{a}_2)$, but not on $\mathrm{tp}(\bar{a}_1\bar{a}_2)$.*

Proof. Suppose that $\mu(\phi_1(x, \bar{a}_1) \wedge \phi_2(x, \bar{a}_2)) = \alpha$. Expand $\mathcal{M}$ to an $\mathcal{L}'$-structure where $\mathcal{L}' = \mathcal{L} \cup \{R_\alpha(\bar{x}, \bar{y})\}$ and $R_\alpha(\bar{x}, \bar{y})$ is interpreted as in 3.2. We know that this relation is stable from [31, Prop.2.25]. Hence, the result follows from Proposition 3.1.4 noting that by invariance of μ (in the sense of Definition 2.2.1), $\mathrm{tp}(\bar{a}_1)$ entirely determines the R_α-type of $\bar{a}_1$ over $\emptyset$. □

Let $\mathcal{M}$ be an $\mathcal{L}$ structure. Let $\bar{a}$ be a tuple from M and $B \subseteq M$. We write $\mathrm{loc}(\bar{a}/B)$ for the set of realisations of $\mathrm{tp}(\bar{a}/B)$ in $\mathcal{M}$.

Remark 3.1.6. As noted earlier in Remark 2.2.9, by Proposition 5.10 of [50], if $\mathcal{M}$ is MS-measurable, then so is $\mathcal{M}^{eq}$. Let $\chi(x)$ be an $\mathcal{L}^{eq}(M^{eq})$-formula and let $\mathcal{M}^\star$ and $\mathcal{M}^\bullet$ be restrictions of $\mathcal{M}^{eq}$ to finitely many sorts containing the sorts of the variables and parameters in $\chi(x)$. Inspecting the proof of Proposition 5.10, we can see that the dimension measures $h^\star$ on $\mathcal{M}^\star$ and $h^\bullet$ on $\mathcal{M}^\bullet$ induced by the dimension-measure h on $\mathcal{M}$ will agree on the value of the set defined by $\chi(x)$. Hence, we may write unambiguously

$$h(\chi(x))$$

for this value when working with MS-dimension-measures in $\mathcal{M}^{eq}$.

Lemma 3.1.7. *Let $\mathcal{M}$ be ω-categorical and a_1, a_2 be from M^{eq}. Then, there is an $\mathcal{L}^{eq}$ formula $\psi(x_1, x_2)$ such that*

$$\vDash \psi(a_1', a_2') \text{ if and only if } a_1' \equiv a_1 \text{ and } a_2' \equiv a_2 \text{ and } a_1' \mathop{\smile\!\!\!\!\!\!|}\; a_2'. \tag{3.3}$$

Moreover, if $\mathcal{M}$ is MS-measurable,

$$\mu\left(\psi(x_1, x_2)\right) = \mu(a_1)\mu(a_2).$$

Proof. The first part follows simply by Ryll-Nardzewski and invariance of non-forking independence. For the second part, we use Fubini. Firstly, since $\psi(x_1, x_2)$ isolates the types of its variables,

$$\models \exists x_1 \psi(x_1, b_2) \text{ if and only if } b_2 \in \mathrm{loc}(a_2).$$

So, $\mu(\exists x_1 \psi(x_1, x_2)) = \mu(a_2)$. Secondly, consider $\mu(\psi(x_1, b_2))$ for b_2 such that $\models \exists x_1 \psi(x_1, b_2)$. By invariance, $\mu(\psi(x_1, b_2)) = \mu(\psi(x_1, a_2))$. But now, for $b_1 \equiv a_1$, by the dimension theorem 2.2.20,

$$\models \psi(b_1, a_2) \text{ if and only if } \mathrm{d}(b_1/a_2) = \mathrm{d}(b_1).$$

By additivity, $\mu(a_1) = \mu(\psi(x_1, a_2))$. Considering Fubini applied to $\psi(x_1, x_2)$ yields the result. □

Fact 3.1.8. *If $\mathcal{M}$ is simple, ω-categorical with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$, the independence theorem holds over the empty set.*

Proof. By ω-categoricity, over finite sets, Lascar types and strong types coincide. By [43, Proposition 3.2.12], the independence theorem holds over $\mathrm{acl}^{eq}(\emptyset)$. Since $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$, the independence theorem holds over the empty set. □

At this point, we have the tools to show how, in an ω-categorical context, for a triplet of independent pairs, non-forking independence corresponds to probabilistic independence. We state the following result in terms of $\mathcal{M}^{eq}$.

Theorem 3.1.9. *Let $\mathcal{M}$ be MS-measurable, ω-categorical with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Suppose that a_1, a_2, a_3 from $\mathcal{M}^{eq}$ are such that $a_i \perp a_j$ for $1 \leq i < j \leq 3$. Then, $\mathrm{d}(\mathrm{tp}(a_3/a_1) \cup \mathrm{tp}(a_3/a_2)) = \mathrm{d}(a_3)$ and*

$$\mu(\mathrm{tp}(a_3/a_1) \cup \mathrm{tp}(a_3/a_2)) = \frac{\mu(a_3/a_1)\mu(a_3/a_2)}{\mu(a_3)}. \tag{3.4}$$

In the theorem, the set defined by $\mathrm{tp}(a_3/a_1) \cup \mathrm{tp}(a_3/a_2)$ is $\mathrm{loc}(a_3/a_1) \cap \mathrm{loc}(a_3/a_2)$. The statement about the dimension corresponds to the independence theorem over $\emptyset$. But we get a much stronger result: the events "$x \in \mathrm{loc}(a_3/a_1)$" and "$x \in \mathrm{loc}(a_3/a_1)$" are probabilistically independent in the measure μ' induced by μ on definable subsets of $\mathrm{loc}(a_3)$.

Proof. We write M instead of M^{eq} in the proof to simplify the notation. For $i \in \{1,2\}$, let $\phi_{i3}(x_i, x_3)$ isolate $\mathrm{tp}(a_i a_3)$. Let $\psi(x_1, x_2)$ be the formula satisfying the condition 3.3 shown to exist in Lemma 3.1.7. Our proof consists of calculating in different ways the measure of the set S defined by the formula

$$S(x_1, x_2, x_3) := \phi_{13}(x_1, x_3) \wedge \phi_{23}(x_2, x_3) \wedge \psi(x_1, x_2).$$

We apply Fubini, projecting onto the first two coordinates $\pi_{x_1 x_2} : M^3 \to M^2$. The formula $\exists x_3 S(x_1, x_2, x_3)$ is satisfied by any $b_1 b_2 \models \psi(x_1, x_2)$ by Fact 3.1.8. Hence,

$$\exists x_3 S(x_1, x_2, x_3) \equiv \psi(x_1, x_2),$$

and so, by Lemma 3.1.7,

$$\mu(\pi_{x_1 x_2}(S)) = \mu(a_1)\mu(a_2).$$

For $b_1 b_2 \in M$, we consider $\pi^{-1}_{x_1 x_2}(b_1 b_2) \cap S$, i.e. the set defined by $S(b_1, b_2, x_3)$. This definable set is non-empty if and only if $b_1 b_2 \models \psi(x_1, x_2)$. Also, by Fact 3.1.8, for any such pair, $S(b_1, b_2, x_3)$ will have maximal dimension. Finally,

$$\mu(S(b_1, b_2, M)) = \mu(S(a_1, a_2, M)) = \mu(\mathrm{tp}(a_3/a_1) \cup \mathrm{tp}(a_3/a_2)),$$

where the first equality follows from Corollary 3.1.5 and the second by definition of $S(x_1, x_2, x_3)$. Hence, by Fubini with respect to $\pi_{x_1 x_2}$ we obtain:

$$\mu(S) = \mu(\mathrm{tp}(a_3/a_1) \cup \mathrm{tp}(a_3/a_2))\mu(a_1)\mu(a_2). \tag{3.5}$$

Let us compute the measure with respect to $\pi_{x_3} : M^3 \to M$. The formula $\exists x_1 x_2 S(x_1, x_2, x_3)$ isolates $\mathrm{tp}(a_3)$ and so $\mu(\exists x_1 x_2 S(x_1, x_2, M)) = \mu(a_3)$. Meanwhile, for $b_3 \in M$, the formula $S(x_1, x_2, b_3)$ is consistent only if $b_3 \equiv a_3$. By invariance we may consider $S(x_1, x_2, a_3)$. To compute the measure of the set defined by this formula we apply Fubini again. Projecting onto x_1, $\exists x_1 S(x_1, x_2, a_3)$ isolates $\mathrm{tp}(a_2/a_3)$ and so, $\mu(\exists x_1 S(x_1, M, a_3)) = \mu(a_2/a_3)$. Meanwhile, for $b_2 \equiv_{a_3} a_2$ consider $S(x_1, b_2, a_3)$. Again, by invariance this defines a set with the same measure as $S(x_1, a_2, a_3)$. We claim that $\mu(S(M, a_2, a_3)) = \mu(a_1/a_3)$. By Lemma

3.1.7, there is a formula $\chi(x_1, x_2)$ over a_3 isolating the pairs $b_1 \equiv_{a_3} a_1$ and $b_2 \equiv_{a_3} a_2$ which are independent over a_3. So, we may consider $S(M, a_2, a_3)$ as the disjoint union

$$(S(M, a_2, a_3) \wedge \chi(M, a_2)) \sqcup (S(M, a_2, a_3) \wedge \neg\chi(M, a_2)).$$

But the definable set on the right of the disjoint union has lower dimension by the dimension theorem, and so, by additivity,

$$\mu(S(M, a_2, a_3)) = \mu(S(M, a_2, a_3) \wedge \chi(M, a_2)).$$

Consider the extension of $\mathrm{tp}(a_1/a_3)$ to a_2. By the same reasoning,

$$\mu(a_1/a_3) = \mu(\phi_{13}(M, a_3) \wedge \chi(M, a_2)).$$

Noting that $\chi(x_1, x_2)$ implies $\psi(x_1, x_2)$, we get that

$$\mu(S(M, a_2, a_3)) = \mu(a_1/a_3).$$

Hence, our calculations with respect to π_{x_3} yield:

$$\mu(S) = \mu(a_3)\mu(a_2/a_3)\mu(a_1/a_3). \tag{3.6}$$

Recall that by Fubini $\mu(a_i a_j) = \mu(a_i/a_j)\mu(a_j)$. Using this fact and comparing the equations 3.5 and 3.6, we obtain

$$\mu(\mathrm{tp}(a_3/a_1) \cup \mathrm{tp}(a_3/a_2)) = \frac{\mu(a_2a_3)\mu(a_1a_3)}{\mu(a_1)\mu(a_2)\mu(a_3)}. \tag{3.7}$$

Applying Fubini again, we get the desired result. □

Corollary 3.1.10. *Let $\mathcal{M}$ be MS-measurable, ω-categorical with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Suppose that a_1, a_2, a_3 from $\mathcal{M}^{eq}$ are such that $a_i \mathop{\smile\!\!\!\!\!\!\mid\,} a_j$ for $1 \leq i < j \leq 3$. Let $p_{ij}(x_i, x_j) = \mathrm{tp}(a_i, a_j)$. Then,*

$$\mathrm{d}(p_{12}(x_1, x_2) \cup p_{23}(x_2, x_3) \cup p_{13}(x_1, x_3)) = \sum_{i \leq 3} \mathrm{d}(a_i), \text{ and}$$

$$\mu(p_{12}(x_1, x_2) \cup p_{23}(x_2, x_3) \cup p_{13}(x_1, x_3)) = \frac{\mu(a_1a_2)\mu(a_1a_3)\mu(a_2a_3)}{\mu(a_1)\mu(a_2)\mu(a_3)}.$$

Proof. Considering the set defined by $p_{12}(x_1, x_2) \cup p_{23}(x_2, x_3) \cup p_{13}(x_1, x_3)$, we apply Fubini by projecting onto the first two coordinates. The projection isolates $\mathrm{tp}(a_1 a_2)$, and the fiber of $a_1 a_2$ isolates $\mathrm{tp}(a_3/a_1) \cup \mathrm{tp}(a_3/a_2)$. By Fubini

$$\mu(p_{12}(x_1, x_2) \cup p_{23}(x_2, x_3) \cup p_{13}(x_1, x_3)) = \mu(\mathrm{tp}(a_3/a_1) \cup \mathrm{tp}(a_3/a_2))\mu(a_1 a_2).$$

Substituting with equation 3.7, the result follows. □

3.2 Transfer for ergodic Keisler measures

In this section, we prove how weak algebraic independence of parameters transfers to probabilistic independence when looking at an ergodic Keisler measure. We begin with Subsection 3.2.1 introducing ergodic measures and the ergodic decomposition theorem 3.2.3. We prove our main transfer of independence result in Subsection 3.2.3 after studying some basic properties of the Hilbert spaces induced by invariant Keisler measures in Subsection 3.2.2. The main ingredient for our result is a recent theorem of Jahel and Tsankov [41, Theorem 3.2].

3.2.1 Ergodic measures

Ergodic measures will be an essential tool for the study of invariant Keisler measure in an ω-categorical context. They are better behaved than invariant measures and any invariant measure can be decomposed as an integral average of ergodic measures. Here we briefly introduce these measures in a general context and mention some basic results about them. Chapter 12 of Phelps' book [52] covers the theory we discuss at the adequate level of generality for our purposes.

We work in the context of a topological group G acting on a topological space X via a continuous action $\cdot : G \times X \to X$. When X is compact and Hausdorff, we call this action a ***G*-flow**. Let $\mathcal{B}(X)$ be the set of Borel subsets of X, and let $\mu : \mathcal{B}(X) \to [0, 1]$ be a Borel probability measure. We say that μ is ***G*-invariant** if for any $\tau \in G$ and any $A \in \mathcal{B}(X)$, we have that $\tau \cdot A \in \mathcal{B}(X)$ and $\mu(\tau \cdot A) = \mu(A)$.

Definition 3.2.1. We say that a G-invariant Borel probability measure μ is ***G*-ergodic** if for all $A \in \mathcal{B}(X)$, we have that if for all $\tau \in G$,

$$\mu(A \triangle \tau \cdot A) = 0,$$

then either $\mu(A) = 0$ or $\mu(A) = 1$.

An alternative definition tells us that any invariant function is constant [17, Prop.2.14]:

Proposition 3.2.2. *Let $(X, \mathcal{B}, \mu)$ be a probability space and G be a group acting on X such that μ is G-invariant. Then, the following are equivalent:*

1. *the measure μ is G-ergodic; and*
2. *any measurable function $f : X \to \mathbb{C}$, which is G-invariant almost everywhere (i.e. for any $\tau \in G$, $f \circ \tau = f$ a.e.) is constant almost everywhere.*

We are interested in studying G-ergodic measures since, when X is metrizable, they yield an ergodic decomposition of any G-invariant measure. Hence, their study is essential to the understanding of the G-invariant measures on X. Below, for $f \in C(X, \mathbb{C})$, we write $\mu(f) = \int_X f \mathrm{d}\mu$. From [52, p.77] we have:

Theorem 3.2.3 (Ergodic decomposition)**.** *Let X be a compact metrizable space with a group G acting continuously on it. Let μ be a G-invariant Borel probability measure on X. Then, the space $\mathfrak{M}(X)$ of G-invariant Borel probability measures on X is also metrizable and set of G-ergodic measures* $\mathrm{Erg}(X)$ *is a Borel subset of $\mathfrak{M}(X)$. Furthermore, there is a unique Borel probability measure $\mathfrak{m}$ on $\mathfrak{M}(X)$ such that for any $f \in C(X, \mathbb{C})$,*

$$\mu(f) = \int_{\mathrm{Erg}(X)} \nu(f) \mathrm{d}\mathfrak{m}(\nu).$$

Proof. The theorem is expressed in [52, p.77] and subsequent discussion. We provide an explanation of how to adapt that result to this context. Note that the G-flow on X implies that each function $g : X \to X$ given by $g(x) = g \cdot x$ is continuous. Recalling that in a metric space Baire sets and Borel sets coincide

[3][Cor.6.3.5], we get that there is a unique Borel probability measure $\mathfrak{m}$ on $\mathfrak{M}(X)$ such that for any $f \in C(X, \mathbb{R})$,

$$\mu(f) = \int_{\mathfrak{M}(X)} \nu(f) d\mathfrak{m}(\nu),$$

and $\mathfrak{m}(B) = 0$ for any Borel subset of $\mathfrak{M}(X)$ containing no G-ergodic measure. However, since $\mathrm{Erg}(X)$ is Borel (as mentioned in [52, p.77] for the metrizable case), and so is $\mathfrak{M}(X) \setminus \mathrm{Erg}(X)$, we get that

$$\begin{aligned}\int_{\mathfrak{M}(X)} \nu(f) d\mathfrak{m}(\nu) &= \int_{\mathrm{Erg}(X)} \nu(f) d\mathfrak{m}(\nu) + \int_{\mathfrak{M}(X)\setminus\mathrm{Erg}(X)} \nu(f) d\mathfrak{m}(\nu) \\ &= \int_{\mathrm{Erg}(X)} \nu(f) d\mathfrak{m}(\nu),\end{aligned}$$

as desired. □

3.2.2 The Hilbert space $L^2(\mu)$

Given the Borel probability space $(S_x(M), \mathcal{B}_x(M), \mu)$ induced by an invariant Keisler measure on M in the variable x, we wish to study the associated Hilbert space $L^2(\mu)$ defined below. Our main objective is proving a form of transfer from a weak form of algebraic independence to probabilistic independence using a result of Jahel and Tsankov [41] on unitary representations of oligomorphic permutation groups based on [66]. This will be done in the next section. We first need to show with few basic proofs that the context we are studying fits into that of the results of [41]. Our main result is Proposition 3.2.9 proving that the action of $\mathrm{Aut}(M)$ on $L^2(\mu)$ is a unitary representation. All of the proofs in this section are standard. We follow the discussion and notation of [41] and [66]. It is also worth noting that the study of the Hilbert spaces $L^2(\mu)$ is recently gaining interest given the work of Hrushovski and Chevalier [33, 13]. Indeed, Hrushovski's more general proof of Theorem 3.1.9 focuses on looking at the Hilbert space $L^2(\mu)$ from the point of view of continous logic, and his approach has definitely been illuminating in understanding the utility of the results of Jahel and Tsankov for my interests. Some basic knowledge of Hilbert spaces is required, such as

Chapter 1 of [15] and the Stone-Weierstrass Theorem [15, Theorem 8.1].

A Keisler measure μ on $\mathcal{M}$ in the variable x induces a Borel probability space $(S_x(M), \mathcal{B}_x(M), \mu)$. We are interested in studying the complex L^2-space $L^2(S_x(M), \mathcal{B}_x(M), \mu)$, which we abbreviate $L^2(\mu)$. An element $f \in L^2(\mu)$ stands for an equivalence class of complex measurable functions agreeing almost everywhere. Furthermore, since $L^2(\mu)$ is a Hilbert space, it is equipped with an inner product:

$$\text{for } f, g \in L^2(\mu), \langle f, g \rangle := \int_{S_x(M)} f \cdot \overline{g} \, \mathrm{d}\mu.$$

As usual, we write $||\cdot||_2$ for the norm associated with $L^2(\mu)$, where $||f||_2 = \langle f, f \rangle$.

We also denote $C(S_x(M))$ to be the set of continuous complex valued functions $f : S_x(M) \to \mathbb{C}$. We consider $C(S_x(M))$ as equipped with the supremum norm $||\cdot||_\infty$ and the $||\cdot||_2$ norm as standardly defined.

Let $\mathcal{D} \subseteq C(S_x(M))$ correspond to the set of indicator functions of clopen subsets of $S_x(M)$. We write $\mathcal{D}^*$ for the unital C^*-star algebra generated by $\mathcal{D}$. This is obtained by considering the closure of $\mathcal{D}$ under the operations of addition, multiplication, complex conjugation and multiplication by complex scalars. Since the product of the indicator function of $[\phi(x, a)]$ and $[\psi(x, b)]$ is the indicator function of $[\phi(x, a) \wedge \psi(x, b)]$, this space corresponds to all functions of the form

$$\sum_{i=0}^{n} \alpha_i \mathbb{1}_{A_i},$$

where $\alpha_i \in \mathbb{C}$ and $A_i \subseteq S_x(M)$ is clopen.

We can naturally consider $C(S_x(M))$ and $\mathcal{D}^*$ as subspaces of $L^2(\mu)$, keeping in mind that we are moving from considering these as spaces of functions to spaces of equivalence classes of functions. By construction of the measure on $S_x(M)$ from the associated Keisler measure, we can see that $\mathcal{D}^*$ is dense in $L^2(\mu)$:

Lemma 3.2.4. *$\mathcal{D}^*$ is dense in $L^2(\mu)$ with the $||\cdot||_2$-norm.*

Proof. It is sufficient to prove the corresponding statement for spaces of functions. That is, given, μ-measurable f such that

$$\int |f|^2 \mathrm{d}\mu < \infty,$$

for any $\epsilon > 0$, there is some $h \in \mathcal{D}^*$ such that

$$||f - g||_2 = \left(\int |f - g|^2 \mathrm{d}\mu \right)^{\frac{1}{2}} < \epsilon.$$

Note that $\mathcal{D}$ is a separating subset for $S_x(M)$. In fact, given $p, q \in S_x(M)$, we can always find a formula in p and not in q. Its indicator function separates p and q. Hence, we can apply the Stone-Weierstrass Theorem [15, Theorem 8.1] to see that $\mathcal{D}^*$ is dense in $C(S_x(M))$ in the supremum norm. Since μ is a probability measure, this implies also that $\mathcal{D}^*$ is dense in $C(S_x(M))$ in the $||\cdot||_2$-norm. In fact, for $g \in C(S_x(M))$, there is $h \in \mathcal{D}^*$ such that $||g - h||_\infty < \epsilon^{\frac{1}{2}}$, yielding

$$\begin{aligned} ||g - h||_2^2 &= \int |g - h|^2 \mathrm{d}\mu \\ &\leq \sup_{p \in S_x(M)} |g(p) - h(p)|^2 \\ &= ||g - h||_\infty^2 < \epsilon. \end{aligned}$$

The inequality follows from μ being a probability measure. By [58, Theorem 3.14] and compactness of $S_x(M)$, $C(S_x(M))$ is dense in $L^2(\mu)$ in the $||\cdot||_2$-norm. Hence, by transitivity of density, $\mathcal{D}^*$ is dense in $L^2(\mu)$ in the $||\cdot||_2$-norm. □

In the lemma above, we do not need to make any assumption about the invariance of μ. However, we are particularly interested in the context where μ is an invariant Keisler measure. For $\tau \in \mathrm{Aut}(M)$, $g : S_x(M) \mapsto \mathbb{C}$ a μ-measurable function, we write $g \circ \tau$ for the function $S_x(M) \mapsto \mathbb{C}$ given by

$$g \circ \tau(p) = g(\tau(p)).$$

We can similarly define this operation on equivalence classes of functions up to a.e. equivalence. We can observe that automorphisms of M induce bijections from $L^2(\mu)$ to itself.

Lemma 3.2.5. *Let $\tau \in \mathrm{Aut}(M)$, let $g \in L^2(\mu)$, then $g \circ \tau \in L^2(\mu)$ and*

$$\int_{S_x(M)} g(x)d\mu = \int_{S_x(M)} g \circ \tau(x)d\mu.$$

To prove this proposition we need to use the following classical theorem of measure theory about the pushforward of a measure [3][Theorem 3.6.1]:

Theorem 3.2.6. *Let X and Y be two sets with associated σ-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $\tau : X \to Y$ be an $(\mathcal{A}, \mathcal{B})$-measurable function (i.e. for each $B \in \mathcal{B}, \tau^{-1}(B) \in \mathcal{A}$). Let $\mu : \mathcal{A} \to [0, 1]$ be a measure. Then, the **pushforward measure** $\mu \circ \tau^{-1}$ is defined as*

$$\mu \circ \tau^{-1}(B) = \mu(\tau^{-1}(B)) \text{ for each } B \in \mathcal{B}.$$

Let g is a $\mathcal{B}$-measurable function on Y. We have that g is integrable with respect to the push-forward measure $\mu \circ \tau^{-1}$ if and only if $g \circ \tau$ is integrable with respect to μ. Moreover, in this case,

$$\int_Y g(y)\mathrm{d}(\mu \circ \tau^{-1}(y)) = \int_X g(\tau(x))\mathrm{d}\mu(x).$$

Proof of Lemma 3.2.5. The action of τ on $S_x(M)$ given by $\tau(x) = \tau \cdot x$ yields a $(\mathcal{B}_x(M), \mathcal{B}_x(M))$-measurable function by invariance of μ. Hence, the pushforward measure $\mu \circ \tau^{-1}$ is well defined. Additionally, note that since μ is an invariant measure, $\mu \circ \tau^{-1}(B) = \mu(B)$ for any $B \in \mathcal{B}_x(M)$. Suppose $g \in L^2(\mu)$. Then, by the theorem above we have that $g \circ \tau$ is integrable with respect to μ and

$$\int_{S_x(M)} g(x)\mathrm{d}\mu(x) = \int_{S_x(M)} g(y)\mathrm{d}(\mu \circ \tau^{-1}(y)) = \int_{S_x(M)} g(\tau(x))\mathrm{d}\mu(x).$$

For $g \in L^2(\mu)$ we deduce that $g \circ \tau \in L^2(\mu)$ since

$$\int_{S_x(M)} |g(x)|^2\mathrm{d}\mu = \int_{S_x(M)} |g(\tau \cdot x)|^2\mathrm{d}\mu.$$

□

Let $\mathcal{M}$ be a countable ω-categorical structure. We have that $\mathrm{Aut}(M)$ is a Polish group, that is, a separable completely metrizable topological group. In particular, in the topology of $G = \mathrm{Aut}(M)$, the pointwise stabilizers of finite sets $G_A = \mathrm{Aut}(M/A)$ for $A \subset M$ finite, are neighbourhoods of the identity and the set of cosets of these pointwise stabilizers form a basis of open sets for the topology [49][§4.1].

Remark 3.2.7. When $\mathcal{M}$ is the countable model of an ω-categorical structure, the restriction map from $\mathrm{Aut}(M^{eq})$ to $\mathrm{Aut}(M)$ yields a canonical topological group isomorphism (cf. [25, Remarks 1.10] and [41]), where $\mathrm{Aut}(M^{eq})$ is topologised as above. Hence, we can identify the actions of $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M^{eq})$ on $L^2(\mu)$ even when considering these as topological groups, and we shall do so in this thesis. The arguments in this and the following subsection also hold when considering the action of $\mathrm{Aut}(M)$ on a complex L^2-space associated with an invariant Keisler measure on M^{eq}, possibly in an imaginary sort, i.e. Hilbert spaces of the form $L^2(S_x(M^{eq}), \mathcal{B}_x(M^{eq}), \mu)$, where μ is an $\mathrm{Aut}(M)$-invariant regular Borel probability measure on $S_x(M^{eq})$ and x may be an imaginary variable. As noted in Remark 2.1.3, when x is a real variable, this is exactly the same space as $L^2(S_x(M), \mathcal{B}_x(M), \mu)$.

We can now introduce the concept of unitary representation [66, §3]:

Definition 3.2.8. Let $\mathcal{H}$ be a complex Hilbert space and $U(\mathcal{H})$ be its unitary group. A **unitary representation** of a topological group G is an homomorphism $\pi : G \to U(\mathcal{H})$ such that for each $f \in \mathcal{H}$ the map $\tau \mapsto \pi(\tau) \cdot f$ is continuous.

The action of $\mathrm{Aut}(M)$ on M naturally induces an action $\lambda : \mathrm{Aut}(M) \times L^2(\mu) \to L^2(\mu)$, where for $\sigma \in \mathrm{Aut}(M), f \in L^2(\mu)$,

$$\lambda(\sigma, f)(p) = f(\sigma^{-1}(p)) \text{ for all } p \in S_x(M).$$

This action is well defined and preserves integrals by Lemma 3.2.5. Furthermore, it induces a unitary representation of $L^2(\mu)$ (cf. §3 of [66]):

Proposition 3.2.9. *Consider the action $\lambda : \mathrm{Aut}(M) \times L^2(\mu) \to L^2(\mu)$ described above. Then, for each $\sigma \in \mathrm{Aut}(M)$, the map $\Lambda_\sigma : L^2(\mu) \to L^2(\mu)$ given by $f \mapsto \lambda(\sigma, f)$ is a **unitary operator**. The map $\pi : \mathrm{Aut}(M) \to U(L^2(\mu))$ from the group of automorphisms of M to the group of unitary operators of $L^2(\mu)$ given by $\sigma \mapsto \Lambda_\sigma$ is a homomorphism. Moreover, for each $f \in L^2(\mu)$ the map $\Psi_f : \mathrm{Aut}(M) \to L^2(\mu)$ given by $\tau \mapsto \pi(\tau) \cdot f$ is continuous. Hence, π is a **unitary representation** of* $\mathrm{Aut}(M)$.

Proof. We begin by proving that Λ_σ is a unitary operator. Firstly, Λ_σ maps functions in $L^2(\mu)$ to functions in $L^2(\mu)$ by Lemma 3.2.5. It is easy to see also that Λ_σ is linear and, by Lemma 3.2.5, we can see that it is bounded (with constant 1). Hence Λ_σ is a bounded linear operator. We need to check it is surjective and preserves the inner-product. To see surjectivity, take $f \in L^2(\mu)$. By Lemma 3.2.5, we know $f \circ \sigma \in L^2(\mu)$, and so $f = \Lambda_\sigma(f \circ \sigma)$. To see the inner-product is preserved, let $f, g \in L^2(\mu)$. Then, again by Lemma 3.2.5,

$$\langle \Lambda_\sigma(f), \Lambda_\sigma(g) \rangle = \int_{S_x(M)} f\bar{g}(\sigma^{-1}(x))\mathrm{d}\mu(x) = \int_{S_x(M)} f\bar{g}(x)\mathrm{d}\mu(x) = \langle f, g \rangle.$$

Hence, Λ_σ is a unitary operator.

Checking that π is a group homomorphism is easy. The trivial automorphism is clearly sent to the identity operator and we have that for $\sigma, \tau \in \mathrm{Aut}(M)$ and $a \in M$,

$$\Lambda_{\sigma\cdot\tau}(f)(a) = f(\tau^{-1}\sigma^{-1}(a)) = \Lambda_\sigma \circ \Lambda_\tau(f)(a).$$

Finally, we need to prove continuity of Ψ_f. Fix $\epsilon > 0, h \in L^2(\mu)$ and consider the open ball $B_\epsilon(h)$ in $L^2(\mu)$ centered at h of radius ϵ. We want to show that $\Psi_f^{-1}(B_\epsilon(h))$ is open. If this is empty, then we are done. Otherwise, there is some τ such that $\pi(\tau) \cdot f \in B_\epsilon(h)$. In particular, for some $\delta > 0$,

$$\|h - \pi(\tau) \cdot f\|_2 = \epsilon - \delta. \tag{3.8}$$

From Lemma 3.2.4 we may choose $g = \sum_{i=1}^m \alpha_i \mathbb{1}_{A_i}$ with A_i clopen and $\|f - g\|_2 < \delta/2$. Set $\bar{a} = \bigcup_{i=1}^m \bar{a}_i$, where $A_i = [\phi(x, \bar{a}_i)]$. This is a finite set, and so $\mathrm{Aut}(M/\bar{a})$ is a neighbourhood of the identity. Note that by construction of g, for $\sigma \in \mathrm{Aut}(M/\bar{a})$, $\pi(\sigma) \cdot g = g$ since for $p \in S_x(M)$,

$$\pi(\sigma) \cdot \mathbb{1}_{[\phi(x,\bar{a}_i)]}(p) = \mathbb{1}_{[\phi(x,\bar{a}_i)]}(\sigma^{-1} \circ p) = \mathbb{1}_{[\phi(x,\sigma(\bar{a}_i))]}(p) = \mathbb{1}_{[\phi(x,\bar{a}_i)]}(p),$$

where the final equality follows since σ fixes $\bar{a}_i$.

By the topology on $\mathrm{Aut}(M)$, $\tau\mathrm{Aut}(M/\bar{a})$ is open. Let $\tau\sigma \in \tau\mathrm{Aut}(M/\bar{a})$ where $\sigma \in \mathrm{Aut}(M/\bar{a})$. In order to prove continuity, we show that $\pi(\tau\sigma) \cdot f \in B_\epsilon(h)$, i.e. that $\tau \mathrm{Aut}(M/\bar{a}) \subseteq \Psi_f^{-1}(B_\epsilon(h))$. Firstly, note that $\|h - \pi(\tau\sigma) \cdot g\|_2 < \epsilon - \sigma/2$ since

$$\begin{aligned}
\|h - \pi(\tau\sigma) \cdot g\|_2 &\leq \|h - \pi(\tau) \cdot f\|_2 + \|\pi(\tau) \cdot f - \pi(\tau\sigma) \cdot g\|_2 \\
&= \epsilon - \delta + \|\pi(\tau) \cdot (f - \pi(\sigma) \cdot g)\|_2 \\
&= \epsilon - \delta + \|f - \pi(\sigma) \cdot g\|_2 \\
&= \epsilon - \delta + \|f - g\|_2 \\
&< \epsilon - \delta + \frac{\delta}{2} = \epsilon - \frac{\delta}{2}.
\end{aligned}$$

The first inequality is just the triangle inequality. The second equality follows from 3.8 and from the fact that π is a homomorphism. As we proved above, $\pi(\tau) = \Lambda_\tau$ is a unitary operator, which is used both for the second equality and for the third. The fourth equality follows from $\pi(\sigma) \cdot g = g$ since $\sigma \in \mathrm{Aut}(M/\bar{a})$. The final inequality follows by choice of g.

Hence, we have that

$$\begin{aligned}
\|h - \pi(\tau\sigma) \cdot f\|_2 &\leq \|h - \pi(\tau\sigma) \cdot g\|_2 + \|\pi(\tau\sigma) \cdot f - \pi(\tau\sigma) \cdot g\|_2 \\
&< \epsilon - \frac{\delta}{2} + \|f - g\|_2 < \epsilon.
\end{aligned}$$

Again, the first equality is the triangle inequality. The second inequality follows from the previous calculation on $\|h - \pi(\tau\sigma) \cdot g\|_2 < \epsilon - \delta/2$ and from $\pi(\tau\sigma)$ being a unitary operator. Since g was chosen so that $\|f - g\|_2 < \delta/2$, we get the final inequality. This yields that $\pi(\tau\sigma) \cdot f \in B_\epsilon(h)$ completing the proof of continuity. □

3.2.3 Transfer of independence for ergodic measures

We are now ready to prove the transfer of independence result of Corollary 3.2.16. The main ingredient for our proof will be Thereom 3.2 of [41], which we proceed to introduce below.

Let $\mathcal{H}$ be a complex Hilbert space and the action of $G = \mathrm{Aut}(M)$ on $\mathcal{H}$ be a unitary representation. For $A \subseteq M^{eq}$ we write G_A for the set of automorphisms of M fixing A in the standard action of $\mathrm{Aut}(M)$ on M^{eq}. We write, following [41],

$$\mathcal{H}_A = \overline{\{f \in \mathcal{H} | G_{A'} \cdot f = f \text{ for some finite } A' \subseteq A\}},$$

where for $S \subset \mathcal{H}$, $\overline{S}$ is its closure.

Definition 3.2.10. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K} \subseteq \mathcal{H}$ be a subspace. We denote the orthogonal complement of $\mathcal{K}$ in $\mathcal{H}$ (often denoted as $\mathcal{K}^{\perp}$) as $\mathcal{H} \ominus \mathcal{K}$. When $\mathcal{K}$ is a closed subspace, we have that $\mathcal{H} = \mathcal{K} \oplus (\mathcal{H} \ominus \mathcal{K})$. Given subspaces $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{H}$, and $\mathcal{H}_0 \subseteq \mathcal{H}_i$ for $i \in \{1, 2\}$ we say that $\mathcal{H}_1$ and $\mathcal{H}_2$ are **orthogonal** over $\mathcal{H}_0$ and write $\mathcal{H}_1 \perp_{\mathcal{H}_0} \mathcal{H}_2$ if $(\mathcal{H}_1 \ominus \mathcal{H}_0) \perp (\mathcal{H}_2 \ominus \mathcal{H}_0)$.

Note that when the $\mathcal{H}_i$ are all closed subspaces for $i \in \{1, 2, 3\}$ with $\mathcal{H}_1 \perp_{\mathcal{H}_0} \mathcal{H}_2$, we have that for $f \in \mathcal{H}_1$ we can decompose f into $f = f_1 + f_0$, where $f' \in \mathcal{H}_1 \ominus \mathcal{H}_0$ and $f_0 \in \mathcal{H}_0$. Similarly, for $g \in \mathcal{H}_2$ we can decompose it as $g = g_2 + g_0$. Now, by the orthogonalities $(\mathcal{H}_i \ominus \mathcal{H}_0) \perp \mathcal{H}_0$ for $i \in \{1, 2\}$ and $(\mathcal{H}_1 \ominus \mathcal{H}_0) \perp (\mathcal{H}_2 \ominus \mathcal{H}_0)$, we have that

$$\langle f, g \rangle = \langle f_0, g_0 \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{H}$.

In an ω-categorical structure, we have the following theorem of Jahel and Tsankov, which translates weak algebraic independence into orthogonality of the associated Hilbert spaces:

Theorem 3.2.11. *[41, Theorem 3.2] Let M be ω-categorical and $G = \mathrm{Aut}(M)$. Let $A, B \subseteq M^{eq}$ be algebraically closed with respect to acl^{eq}. Then, $\mathcal{H}_A \perp_{\mathcal{H}_{A \cap B}} \mathcal{H}_B$.*

Remark 3.2.12. Recently, [13] develops some results of this kind outside the context of ω-categorical structures.

Following Remark 3.2.7, below we allow $L^2(\mu)$ to be induced from an $\mathrm{Aut}(M)$-invariant Keisler measure on M^{eq} possibly in an imaginary variable x. We are especially interested in the case of μ being $\mathrm{Aut}(M)$-ergodic.

Note that the space of constant functions is closed in $L^2(\mu)$, hence, from Proposition 3.2.2 we get:

Corollary 3.2.13. *Let μ be an ergodic measure on M^{eq} in the variable x. Then, $L^2(\mu)_{\mathrm{dcl}^{eq}(\emptyset)}$ is generated by the constant indicator function $\mathbb{1}$.*

Proof. Suppose $f \in L^2(\mu)_{\mathrm{dcl}^{eq}(\emptyset)}$. Then, for some finite $C \subseteq \mathrm{dcl}^{eq}(\emptyset)$, f is G_C-invariant. However, since any element in $\mathrm{dcl}^{eq}(\emptyset)$ is fixed by $\mathrm{Aut}(M)$-automorphisms, $G_C = G_\emptyset$. This means that f is invariant almost everywhere, and therefore constant by ergodicity of the measure. □

For ergodic μ, in $L^2(\mu)$ the constant function $\mathbb{1}$ will be in the same equivalence class as $\mathbb{1}_\phi$, where ϕ isolates one of the finitely many types over the empty set, by Ryll-Nardzewski. In fact, when $\phi_1(x), \ldots, \phi_n(x)$ isolate the types over $\emptyset$, each $[\phi_i(x)]$ is $\mathrm{Aut}(M)$-invariant, and so an ergodic measure must concentrate on one of them: by definition each of them is assigned measure 0 or 1 and they cannot be all assigned measure 0.

Definition 3.2.14. Let $A, B, C \subseteq M^{eq}$. Then, A and C are **weakly algebraically independent** over B, written $A \mathop{\smile\!\!\!\!\!|}^{a}_{B} C$ if $\mathrm{acl}^{eq}(AB) \cap \mathrm{acl}^{eq}(BC) = \mathrm{acl}^{eq}(B)$.

Theorem 3.2.11 yields very powerful results when considering an ergodic measure and the inner product on $L^2(\mu)$.

Theorem 3.2.15. *Let $\mathcal{M}$ be an ω-categorical countable structure with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Consider the complex Hilbert space $\mathcal{H} = L^2(\mu)$, where μ is an ergodic measure. Suppose that $A, B \subseteq M^{eq}$ are algebraically closed and that $A \mathop{\smile\!\!\!\!\!|}^{a} B$. Let $f \in \mathcal{H}_A, g \in \mathcal{H}_B$. Then,*

$$\langle f, g\rangle = \langle f, \mathbb{1}\rangle \overline{\langle g, \mathbb{1}\rangle}.$$

Proof. Let f_0 and g_0 be the projections of f and g respectively on $\mathcal{H}_\emptyset$. Now, since $f \perp_{\mathcal{H}_\emptyset} g$, $\langle f, g\rangle = \langle f_0, g_0\rangle$. Since $f_0, g_0 \in \mathcal{H}_\emptyset$, by Lemma 3.2.13, $f_0 = \alpha\mathbb{1}$, $g = \beta\mathbb{1}$ for $\alpha, \beta \in \mathbb{C}$. This yields that

$$\langle f, g\rangle = \langle f_0, g_0\rangle = \langle \alpha\mathbb{1}, \beta\mathbb{1}\rangle = \alpha\overline{\beta}\langle \mathbb{1}, \mathbb{1}\rangle.$$

Being in a probability space, $\langle \mathbb{1}, \mathbb{1}\rangle = 1$. However, since $\langle f - f_0, f_0\rangle = 0$, we obtain that $\langle f, \mathbb{1}\rangle = \alpha$, and similarly, $\langle g, \mathbb{1}\rangle = \beta$. This yields the desired result. □

This is already essentially observed in Corollary 3.5 and Remark 3.6 of [41]. In particular, since we are interested in the measures of formulas, we have that

Corollary 3.2.16. *Let $\mathcal{M}$ be ω-categorical. Let μ be an ergodic measure on M^{eq} in the variable x. Suppose that $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Let a, b from M^{eq} be such that $a \mathop{\smile\!\!\!\!\!\!\,|}^{a} b$. Then, for any $\mathcal{L}^{eq}$-formulas $\phi(x, y), \psi(x, z)$, where y and z are in the same sort as a and b respectively,*

$$\mu(\phi(x, a) \wedge \psi(x, b)) = \mu(\phi(x, a))\mu(\psi(x, b)).$$

Proof. Note that the indicator functions $\mathbb{1}_{\phi(x,a)}$ and $\mathbb{1}_{\psi(x,b)}$ are in $\mathcal{H}_{\mathrm{acl}^{eq}(a)}$ and $\mathcal{H}_{\mathrm{acl}^{eq}(b)}$ respectively, and that

$$\begin{aligned}
\mu(\phi(x, a) \cap \psi(x, b)) &= \int_{S_x(M)} \mathbb{1}_{\phi(x,a)\cap\psi(x,b)} \mathrm{d}\mu \\
&= \int_{S_x(M)} \mathbb{1}_{\phi(x,a)} \cdot \mathbb{1}_{\psi(x,b)} \mathrm{d}\mu \\
&= \left(\int_{S_x(M)} \mathbb{1}_{\phi(x,A)} \mathrm{d}\mu\right) \overline{\left(\int_{S_x(M)} \mathbb{1}_{\phi(x,a)} \mathrm{d}\mu\right)} \\
&= \mu(\phi(x, a))\overline{\mu(\psi(x, b))} \\
&= \mu(\phi(x, a))\mu(\psi(x, b)).
\end{aligned}$$

Here, the third equality holds by Theorem 3.2.15 and the last one follows since our measure is real-valued. Hence, the result follows. □

Remark 3.2.17. This corollary has a partial converse which is well known in ergodic theory [41, Proposition 4.8]. Fix an invariant Keisler measure μ and suppose that for any formula $\phi(x, a)$ there is an automorphism $\sigma \in \mathrm{Aut}(M)$ such that

$$\mu(\phi(x, a) \wedge \phi(x, \sigma \cdot a)) = \mu(\phi(x, a))^2.$$

Then, μ is ergodic. For a proof of this, see [41, Proposition 4.8] recalling Lemma 3.2.4.

This result, existence for non-forking independence in simple theories and Theorem 3.1.9 imply that MS-measurable ω-categorical structures induce various ergodic Keisler measures. Let $A \subseteq M^{eq}$ be the algebraic closure of a finite set, where M is the countable model of an ω-categorical MS-measurable structures. Let $\chi(x)$ be a formula isolating a type over A (this can always be done as discussed in Lemma 3.2.18). Then, the induced $\mathrm{Aut}(M/A)$-invariant Keisler measure μ_χ is $\mathrm{Aut}(M/A)$-ergodic, where

$$\mu_\chi(\phi(x, a)) = \begin{cases} \dfrac{\mu(\phi(x, a) \wedge \chi(x))}{\mu(\chi(x))} & \text{if } d(\phi(x, a) \wedge \chi(x)) = d(\chi(x)), \\ 0 \text{ otherwise.} \end{cases}$$

We conclude this section with a brief discussion of the assumption of $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$ in Theorem 3.2.15 and Corollary 3.2.16. The assumption is needed in order to have the equality $L^2(\mu)_{\mathrm{acl}^{eq}(\emptyset)} = L^2(\mu)_{\mathrm{dcl}^{eq}(\emptyset)}$. The latter then yields the desired results since we know from Corollary 3.2.13 that $L^2(\mu)_{\mathrm{dcl}^{eq}(\emptyset)}$ is generated by the constant indicator function. However, we can also obtain the relevant independence results when $\mathrm{acl}^{eq}(\emptyset) \supsetneq \mathrm{dcl}^{eq}(\emptyset)$. The following lemma is common knowledge:

Lemma 3.2.18. *Let $\mathcal{M}$ be ω-categorical. Let $A \subseteq M$ be finite. Fix the variable x. Then, there is $A_0 \subseteq \mathrm{acl}^{eq}(A)$ finite such that for every b from M^{eq} in the variable x,*

$$\mathrm{tp}(b/A_0) \vdash \mathrm{tp}(b/\mathrm{acl}^{eq}(A)). \tag{3.9}$$

By this, we mean that if b and b' in the variable x have the same type over A_0, then they have the same type over $\mathrm{acl}^{eq}(A)$ in $\mathcal{M}^{eq}$.

Proof. This is substantially Lemma 2.4 in [25]. □

A useful consequences of the lemma is:

Corollary 3.2.19. *Let $\mathcal{M}$ be ω-categorical. Let μ be an invariant Keisler measure on $\mathcal{M}^{eq}$ in the variable x. Then, there is finite $A_0 \subseteq \mathrm{acl}^{eq}(\emptyset)$ such that*

$$L^2(\mu)_{A_0} = L^2(\mu)_{\mathrm{acl}^{eq}(\emptyset)}.$$

Proof. Let $A_0 \subseteq \mathrm{acl}^{eq}(\emptyset)$ be finite such that 3.9 from Lemma 3.2.18 holds. Suppose that $f \in L^2(\mu)$ is G_A-invariant for $G = \mathrm{Aut}(M)$ and A a finite subset of $\mathrm{acl}^{eq}(\emptyset)$. Then, f is also $G_{A'}$-invariant for $A' = A \cup A_0$. But then, by choice of A_0 (with respect to the variable x), the $G_{A'}$-orbit of any type in $S_x(M^{eq})$ is the same as its G_{A_0}-orbit. Hence, f is G_{A_0}-invariant. □

For an ω-categorical structure $\mathcal{M}$ and given the variable x and A_0 as above, there are finitely many types over A_0 in the variable x, isolated by formulas $\chi_1(x), \dots, \chi_m(x)$. By additivity, any invariant Keisler measure μ can be written as a weighted sum of measures μ_{χ_i} for $1 \leq i \leq m' \leq m$, where, for $\mu(\chi_i(x)) > 0$, μ_{χ_i} is the $\mathrm{Aut}(M/A_0)$-invariant Keisler measure induced from μ by

$$\mu_{\chi_i}(\phi(x,a)) = \frac{\mu(\phi(x,a) \wedge \chi_i(x))}{\mu(\chi_i(x))}.$$

From Corollary 3.2.19, we know that for any $\mathrm{Aut}(M/A_0)$-ergodic measure ν we have that for $a \mathop{\smile\!\!\!\!\!\mid}^{a} b$, and $\mathcal{L}^{eq}$-formulas $\phi(x,y), \psi(x,z)$,

$$\nu(\phi(x,a) \wedge \psi(x,b)) = \nu(\phi(x,a))\nu(\psi(x,b)).$$

Hence, in the context of $\mathrm{acl}^{eq}(\emptyset) \supsetneq \mathrm{dcl}^{eq}(\emptyset)$, in order to study invariant Keisler measures on an ω-categorical structure $\mathcal{M}$ we may naturally move to study the $\mathrm{Aut}(M/A_0)$-invariant Keisler measures.

4
ω-categorical Hrushovski constructions

In this chapter, we introduce ω-categorical Hrushovski constructions. We begin with Section 4.1, which gives a brief introduction to these structures following [24, 69] and [68]. Then, in Section 4.2, we find the form that the equations of Corollary 2.2.24 would take in the context of ω-categorical Hrushovski constructions if these were MS-measurable.

4.1 Building ω-categorical Hrushovski constructions

We focus on graphs and hypergraphs for simplicity of notation, but we can build Hrushovski constructions in relational languages in general [68]. By an n-hypergraph we always mean an n-uniform hypergraph, that is an n-hypergraph where each n-hyperedge consists of n distinct vertices. For fixed $n \geq 2$, let $\overline{\mathcal{K}}$ be the class of n-hypergraphs and $\mathcal{K}$ be the class of finite n-hypergraphs. Now, for $\alpha \in \mathbb{R}^{>0}$, and A a finite hypergraph, we define the **predimension** of A, $\delta(A)$ to be:

$$\delta(A) = \alpha|A| - |E(A)|,$$

where $E(A)$ denotes the set of n-hyperedges of A.

For $A \subseteq B, C \in \mathcal{K}$, suppose $g : A \to B$ and $f : A \to C$ are embeddings of A in B and C respectively. The **free amalgamation** of B and C over A, is the (unique) hypergraph $B \amalg_A C$ such that there are embeddings $h : B \to B \amalg_A C$, and $k : C \to B \amalg_A C$ such that $h(g(A)) = k(f(A)) = h(B) \cap k(C)$ and $E(B \amalg_A C) = h(E(B)) \cup k(E(C))$. We say that B and C are **freely amalgamated** over $A \subseteq B \cup C$ if $B \cup C = B \amalg_A C$, i.e. $B \cap C = A$ and $E(B \cup C) = E(B) \cup E(C)$. The idea is that if two hypergraphs B and C contain an induced copy of the same structure A, their free amalgamation is obtained by "gluing" together B and C on their copy of A without adding any additional hyperedges to the resulting hypergraph.

The predimension we introduced satisfies **submodularity**. That is, for $D \in \overline{\mathcal{K}}$, and $B, C \subset D$ finite, we have that

$$\delta(B \cup C) \leq \delta(B) + \delta(C) - \delta(B \cap C), \tag{4.1}$$

with equality holding if and only B and C are freely amalgamated over $B \cap C$.

Definition 4.1.1. Let $B \in \overline{\mathcal{K}}$, $A \subseteq B$ finite. We say that A is **d-closed** in B, and write $A \leq B$ if $\delta(A) < \delta(B')$ for any finite B' such that $A \subsetneq B' \subseteq B$. We say that A is **self sufficient** in B and write $A \leq^* B$ if $\delta(A) \leq \delta(B')$ for any finite B' such that $A \subseteq B' \subseteq B$.

Definition 4.1.2. Let $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be monotone increasing and unbounded with $f(0) = 0$. We let

$$\mathcal{K}_f := \{A \in \mathcal{K} : \delta(A') \geq f(|A'|) \text{ for any } A' \subseteq A\}.$$

Definition 4.1.3. We say that $\mathcal{K}_f$ has the **free amalgamation property** if given $A \leq B, C \in \mathcal{K}_f$ we have that $B \amalg_A C \in \mathcal{K}_f$.

Definition 4.1.4. We say that a structure $\mathcal{M}_f \in \overline{\mathcal{K}}$ is a generic model for $\mathcal{K}_f$ if every finite substructure of $\mathcal{M}_f$ is in $\mathcal{K}_f$ and $\mathcal{M}_f$ has the **extension property** for $\mathcal{K}_f$:

- **Extension Property:** given $A \leq M_f$ finite and $A \leq B \in \mathcal{K}_f$, there is an embedding of B over A in $\mathcal{M}_f$ such that $B \leq M_f$.

Remark 4.1.5. Suppose that $\mathcal{M}_f$ is a generic model for $\mathcal{K}_f$. By the extension property and f being strictly increasing with $f(0) = 0$, since $\emptyset \leq \mathcal{M}_f$, any $B \in \mathcal{K}_f$ is a finite substructure of $\mathcal{M}_f$.

When $\mathcal{K}_f$ has the free amalgamation property we can build a countable generic model for it whose theory is ω-categorical. We will call such structures ω-categorical Hrushovski constructions [69, Theorems 6.2.13 & 6.2.16]:

Theorem 4.1.6. *Suppose that $\{\emptyset\} \subsetneq \mathcal{K}_f$ has the free amalgamation property. Then, there is a countable generic model $\mathcal{M}_f$ for $\mathcal{K}_f$. Moreover, $\mathcal{M}_f$ is ω-categorical and* ***$\leq$-homogeneous****: let σ be an isomorphism between the finite substructures $A, A' \leq \mathcal{M}_f$. Then, σ extends to an automorphism of $\mathcal{M}_f$.*

From now on, by $\mathcal{M}_f$ we will denote the countable generic model for a non-trivial free amalgamation class $\mathcal{K}_f$.

There are various ways to ensure that f yields a free-amalgamation class. For example, [32] and later treatments, such as [24] and [69], note that it is sufficient for f to be continuous, piecewise smooth and with $f'(x)$ non-decreasing and such that $f'(x) \leq 1/x$. In the Appendix, we discuss another way to ensure closure under free amalgamations.

When $\alpha \in \mathbb{Q}$, choosing f with slow enough growth ensures that $\mathcal{M}_f$ is supersimple of finite SU-rank by [22, Theorem 3.6] (originally proved in [35]). This is explained further in the Appendix, in Theorem 9.1.5 and Section 9.1.

For $B \in \mathcal{K}_f$ or $B = \mathcal{M}_f$ and finite $A \subseteq B$ there is a unique smallest d-closed subset of B containing A which we call $\mathrm{cl}_B(A)$, i.e, the **closure** of A in B [68, Lemma 2.1]. In the context of $\mathcal{M}_f$, $\mathrm{cl}_{M_f}(A)$ corresponds to algebraic closure of A in $\mathcal{M}_f$ and its size is bounded above by $f^{-1}(\alpha|A|)$ [24, Theorem 3.19]. In general, we shall avoid the subscript when it is clear we are speaking of $\mathrm{cl}_{M_f}(A)$. By $\leq$-homogeneity, we have that for any finite tuple $\overline{a}$, $\mathrm{tp}(\overline{a})$ is entirely determined by the quantifier-free type of $\mathrm{acl}(\bar{a})$: suppose that the finite tuples $\bar{a}$ and $\bar{a}'$ are such that $\mathrm{qftp}(\bar{a}\bar{b}) = \mathrm{qftp}(\bar{a}'\bar{b}')$, where $\bar{a}\bar{b}$ and $\bar{a}'\bar{b}'$ are enumerations of $\mathrm{acl}(\bar{a})$ and $\mathrm{acl}(\bar{a}')$ respectively, then, $\bar{a} \equiv \bar{a}'$ [68, Corollary 2.4].

The predimension of a closed set induces a natural notion of **dimension**. For $D \in \mathcal{K}_f$ or $D = \mathcal{M}_f$, given $A \subseteq D$, we write $\mathrm{d}_D(A)$ for $\delta(\mathrm{cl}_D(A))$. If $\alpha \in \mathbb{N}$, then this dimension is also a natural number, and it may be re-scaled to be a natural number if $\alpha \in \mathbb{Q}$. As above, we omit the subscript and write $d(A)$ for $d_{\mathcal{M}_f}(A)$.

We write $d_D(A/B)$ for $d_D(AB) - d_D(B)$, where we write AB for $A \cup B$. For $A, B \in \mathcal{K}_f$ subsets of D, we say that A and B are **independent** over C in D if $d_D(A/BC) = d_D(A/B)$. This implies that $\mathrm{cl}_D(AB) \cap \mathrm{cl}_D(AC) = \mathrm{cl}_D(A)$ [22, Lemma 2.3]. When $\mathcal{M}_f$ is supersimple (of finite SU rank) we have that this notion of independence induced by the Hrushovski dimension in $\mathcal{M}_f$ is precisely non-forking independence and the Hrushovski dimension on $\mathcal{M}_f$ is SU-rank (see Theorems 9.1.5 and 9.1.15 in Appendix 9).

We have a clear picture of what independence looks like in $\mathcal{M}_f$ [43, Claim 6.2.9]:

Lemma 4.1.7. *Let $D \in \mathcal{K}_f$ or $D = \mathcal{M}_f$. Suppose $A = B \cap C \leq B, C \leq D$. Then, B and C are independent over A in D if and only if B and C are freely amalgamated over A and $\delta(B \cup C) = \delta(\mathrm{cl}_D(B \cup C))$.*

Remark 4.1.8. With the notation above, if B and C are independent over A in D, $BC \leq^* \mathrm{cl}_D(B \cup C)$. To see this, suppose by contradiction that there is some F such that $BC \subseteq F \subsetneq \mathrm{cl}_D(B \cup C)$ and $\delta(F) < \delta(BC)$. We may choose F to be maximal so that for $F \subsetneq H \subseteq \mathrm{cl}_D(B \cup C)$, $\delta(H) > \delta(F)$. Then,

$$BC \subseteq F \leq \mathrm{cl}_D(B \cup C) \leq D.$$

But this contradicts the definition of closure in D as the minimal d-closed subset of D containing BC.

Below we give two standard examples of ω-categorical Hrushovski constructions. Later, we will be working with similar structures built according to Constructions 5.1.1 and 7.5.1.

Example 4.1.9 (Example 3.21 of [24])**.** Work with graphs, $\alpha = 2$, and f piece-wise linear such that $f(1) = 2, f(2) = 3, f(5) = 5$ and for $t \geq 5$, $f(t) = \log(t) + 5 - \log(5)$. Note that since the predimension of a square is 4 and $f(4) > 4$, $\mathcal{K}_f$ does not contain squares. Meanhile, $\mathcal{K}_f$ contains pentagons since their predimension is 5. The class $\mathcal{K}_f$ forms a free amalgamation class. The associated countable structure $\mathcal{M}_f$ yields an ω-categorical graph of diameter 3, whose smallest k-cycle is a pentagon. Moreover, $\mathrm{Aut}(M_f)$ acts transitively on vertices and edges since vertices and edges are always closed in $\mathcal{M}_f$ (since $f(1) = 2$ and $f(2) = 3$) and by $\leq$-homogeneity.

Example 4.1.10 ([35] and Example 6.2.27 of [69])**.** Suppose that we are working with 3-hypergraphs, $\alpha = 1$ and $f(x) = \log_3(x) + 1$. Then, $\mathcal{K}_f$ is a free-amalgamation class and $\mathcal{M}_f$ is ω-categorical, supersimple of SU-rank 1 with weak elimination of imaginaries (see Theorem 9.1.5 and Lemma 9.1.13). Any finite set of unrelated vertices and three vertices forming a 3-hyperedges are in $\mathcal{K}_f$. Vertices are closed in $\mathcal{M}_f$ since $f(1) = 1$. Since $f(3) = 2$, two vertices in $\mathcal{M}_f$ have two possible closures: either they are already closed in $\mathcal{K}_f$ or there is a third vertex forming a 3-hyperedge with them. Hence, there are two possible closures for 2-types. This structure is standardly given as an example of a supersimple ω-categorical structure which is not one-based (i.e. non-forking independence is distinct from weak algebraic independence).

4.2 MS-measurable ω-categorical Hrushovski constructions?

In this section, we consider some equations which would hold in an ω-categorical Hrushovski construction $\mathcal{M}_f$ if it were MS-measurable. Suppose $A \subseteq M_f$, and $B = \mathrm{cl}_{\mathcal{M}_f}(A)$. Let $\bar{a}\bar{b}$ be an enumeration of B where $\bar{a}$ is an enumeration of A. As noted earlier, by $\leq$-homogeneity, the quantifier-free type of $\overline{a}\overline{b}$ determines the type of of $\overline{a}$ in $\mathcal{M}_f$. For $A \leq M_f$, we write $\theta_{\overline{a}}(\overline{x})$ for the formula isolating $\text{tp}(\overline{a})$.

Notation 4.2.1. By Fubini, any MS-dimension-measure h is invariant under permutation of variables. That is, for $\overline{x}_\sigma$ a permutation of the variables of $\overline{x}$, and $\phi_\sigma(\overline{x}) = \phi(\overline{x}_\sigma)$,

$$h(\phi_\sigma(x)) = h(\phi(x)).$$

Hence, if $\mathcal{M}_f$ is MS-measurable, for $A \leq M_f$, we may write $\mu(A) := \mu(\theta_{\overline{a}}(\overline{x}))$ without ambiguity. For $A \in \mathcal{K}_f$ we write $\mu(A)$ for $\mu(\theta_{\overline{a}}(\overline{x}))$, where $\overline{a}$ is an enumeration of a copy A' of A in M_f such that $A' \leq M_f$.

By the notation $\text{tp}(A/B)$ we mean $\text{tp}(\overline{a}/B)$ where $\overline{a}$ is an enumeration of A.

Theorem 4.2.2. *Let $\mathcal{M}_f$ be the ω-categorical Hrushovski construction for the class $\mathcal{K}_f$. Suppose that $\mathcal{M}_f$ is MS-measurable. Let $A \leq B, C \leq M_f$ be finite. Let $\text{tp}(C_1/B), \ldots, \text{tp}(C_n/B)$ be the finitely many non-forking extensions of $\text{tp}(C/A)$ to B. Let D_i be $\text{cl}(BC_i)$. Then, we have that:*

$$\frac{\mu(B)\mu(C)}{\mu(A)} = \sum_{i=1}^{n} \frac{\mu(D_i)}{|\text{Aut}(D_i/BC_i)|}, \tag{4.2}$$

where $|\text{Aut}(D_i/BC_i)|$ is the number of automorphisms of D_i fixing BC_i pointwise.

Proof. Let $\overline{a}\overline{b}$ be an enumeration of B where $\overline{a}$ is an enumeration of A. Similarly, let $\overline{a}\overline{c}$ be an enumeration of C. Let $\overline{a}\overline{b}\overline{c}_i\overline{d}_i$ be an enumeration of D_i where, $\overline{a}\overline{b}\overline{c}_i$ is the corresponding enumeration of ABC_i. By Corollary 2.2.24,

$$\frac{\mu(B)\mu(C)}{\mu(A)} = \sum_{i=1}^{n} \mu(\text{tp}(\overline{a}\overline{b}\overline{c}_i)) \tag{4.3}$$

We know that $\text{tp}(\overline{a}\overline{b}\overline{c}_i)$ is isolated by $\exists \overline{w}\theta_{\overline{a}\overline{b}\overline{c}_i\overline{d}_i}(\overline{xyzw})$. Note that

$$|\{e \in M_f | \theta_{\overline{a}\overline{b}\overline{c}_i\overline{d}_i}(\overline{a}\overline{b}\overline{c}_i\overline{e})\}|$$

is finite and counts the number of automorphisms of D_i fixing BC_i pointwise. Hence, by Fubini,

$$\mu(\text{tp}(\overline{a}\overline{b}\overline{c}_i))|\text{Aut}(D_i/BC_i)| = \mu(D_i).$$

This yields the desired equation. $\square$

Following Lemma 4.1.7, we can express the conditions for a hypergraph D_i to appear as $\mathrm{cl}(BC_i)$ in the theorem above in terms of $\mathcal{K}_f$.

Definition 4.2.3. Let $A_0 \le A_1, A_2$, $A_1 \amalg_{A_0} A_2 \subseteq D \in \mathcal{K}_f$. We say D is an **eventual closure** of $A_1 \amalg_{A_0} A_2$ if A_1 and A_2 are freely amalgamated over A_0 in D, $A_1 \cup A_2 \le^* D$, $\delta(A_1 \cup A_2) = \delta(D)$, and $A_i \le D$ for each $i \in \{0, 1, 2\}$.

Remark 4.2.4. Consider the closures D_i in Theorem 4.2.2. We have that there is $C_i \le M_f$ independent from B over A with $\mathrm{tp}(C_i/A) = \mathrm{tp}(C/A)$ and closure D_i if and only if there is $D_i' \in \mathcal{K}_f$, isomorphic to D_i as a hypergraph, which is an eventual closure of $B \amalg_A C$. The left-to-right implication follows by Lemma 4.1.7 and Remark 4.1.8. The right-to-left implication follows by the extension property, noting that $B \le D_i' \in \mathcal{K}_f$.

Hence, we may rephrase Theorem 4.2.2 as follows:

Corollary 4.2.5. *Let $\mathcal{M}_f$ be an ω-categorical Hrushovski construction with amalgamation class $\mathcal{K}_f$. Suppose $\mathcal{M}_f$ is MS-measurable. Let $A \le B, C \in \mathcal{K}_f$. Let $D_1, \ldots D_n$ be the eventual closures of $B \amalg_A C$. Then,*

$$\frac{\mu(B)\mu(C)}{\mu(A)} = \sum_{i=1}^{n} \frac{\mu(D_i)}{|\mathrm{Aut}(D_i/BC)|}.$$

5 Non-MS-measurability of a Hrushovski construction

In this chapter, we give a class of ω-categorical Hrushovski constructions. These are supersimple of finite SU-rank, as proven in the Appendix 9.3. However, we will prove in Section 5.2 that they are not MS-measurable. In Section 5.1, we introduce these structures and show how we have control on small enough eventual closures in them.

There are various reasons for wondering whether supersimple finite SU-rank ω-categorical Hrushovski constructions are MS-measurable. Firstly, we are interested in two questions mentioned by Elwes and Macpherson in their review article on MS-measurable structures:

Question 5.0.1. Does being ω-categorical supersimple of finite SU-rank imply MS-measurability?

Question 5.0.2. Is every ω-categorical MS-measurable structure one-based?

Being MS-measurable implies being supersimple of finite SU-rank [19]. Furthermore, in such structures D-rank, $S1$-rank and SU-rank are the same. Hence, the first question is simply asking whether supersimplicity and finite rank imply

MS-measurability in an ω-categorical context.

Supersimple ω-categorical Hrushovski constructions of finite SU-rank are essential structures for the study of these questions. These are the only examples of ω-categorical supersimple not one-based structures which we know of. Hence, finding out that any of these structures are not MS-measurable would answer the first question negatively. Finding out that any of these structures is MS-measurable would answer the second question negatively.

Another motivation for the study of MS-measurability in ω-categorical Hrushovski constructions is the fact that it would be very interesting to understand whether ω-categorical Hrushovski constructions are pseudofinite. We say that a complete theory is pseudofinite if any sentence in that theory has a finite model. Meanwhile, we say that a structure $\mathcal{M}$ has the finite submodel property if every sentence in its theory has a model which is a finite substructure of $\mathcal{M}$. A structure with the finite submodel property clearly has a pseudofinite theory. It is currently an open question of Macpherson [49] whether every ω-categorical pseudofinite structure has the finite submodel property:

Question 5.0.3. Does every ω-categorical pseudofinite structure have the finite submodel property?

It is unknown whether any non-trivial ω-categorical Hrushovski construction is pseudofinite, although certain non ω-categorical Hrushovski constructions are [60, 5]. We do know that the standard examples of ω-categorical Hrushovski constructions cannot have the finite submodel property [16]. Hence, if we could prove that any of these structures are pseudofinite, we should also be able to answer negatively Macpherson's question. Alternatively, proving that any of these structures is not pseudofinite would also be an interesting result since at the moment there are not many tools to show non-pseudofiniteness of a theory.

Many cases of MS-measurable structures are pseudofinite structures elementarily equivalent to a non-principal ultraproduct of some well-behaved class of finite structures, called an N-dimensional asymptotic class (e.g. pseudofinite fields, the random graph, the universal homogeneous two-graph) [50]. Any non-principal ultraproduct of such classes of structures is MS-measurable (and pseudofinite). Hence, proving that an ω-categorical Hrushovski construction is not MS-measurable shows that it cannot be elementarily equivalent to a non-principal ultraproduct of an asymptotic class. Meanwhile, if we could show that an ω-categorical Hrushovski construction is MS-measurable, this might give us some information on the sizes of definable sets in a potential class of finite structures with a non-principal ultraproduct elementarily equivalent to it.

We already know that some supersimple ω-categorical Hrushovski constructions are not MS-measurable from the work of Evans [23]. There, Evans first proves that any MS-measurable structure must satisfy a weak form of independent n-amalgamation using a version by Towsner of the Hypergraph Removal Lemma [65, 29, 57]. Then, he proves that, for a class of Hrushovski constructions (with a ternary relation and $\alpha = 1$), any dimension is a scaled version of the natural Hrushovski dimension. With these tools, Evans shows that some of these structures do not satisfy the weak n-amalgamation property.

Using Corollary 2.2.26, we may assume that if a Hrushovski construction is MS-measurable, then the dimension part of the MS-dimension-measure is the Hrushovski dimension (which corresponds to SU-rank as noted in Theorem 9.1.15). This allows us to circumvent the need to prove Evans' dimension theorem when working with different classes of ω-categorical Hrushovski constructions. Though it it worth noting that a similar dimension theorem to Evans' can be proved for other classes of ω-categorical Hrushovski constructions [51].

It is possible for Hrushovski constructions to satisfy the weak amalgamation condition (and even independent n amalgamation for all n over finite algebraically closed sets) as long as f is slow-growing enough (see [35] and Theorem 9.2.2). Indeed, a natural question which arises from Evans' paper is whether MS-measurability for an ω-categorical finite rank structure is implied by satisfying some strong enough form of independent n-amalgamation.

In this chapter, we show that there are ω-categorical supersimple Hrushovski constructions of finite rank which can satisfy independent n-amalgamation for all $n \in \mathbb{N}$ and which are not MS-measurable:

Theorem 5.2.3. *The structures $\mathcal{M}_f$ satisfying the conditions of Construction 5.1.1 are not MS-measurable. Hence, there are ω-categorical supersimple finite SU-rank structures which are not MS-measurable. Indeed, there are supersimple ω-categorical structures of finite SU-rank and with independent n-amalgamation over finite algebraically closed sets for all n which are not MS-measurable.*

As mentioned earlier, we begin with Section 5.1 where we introduce the ω-categorical Hrusvhovski constructions of Construction 5.1.1 and prove some basic facts about small algebraically closed sets in these structures. Then, Section 5.2 is dedicated to the proof of non MS-measurability, i.e. Theorem 5.2.3. Some basic model theoretic properties of these structures are proven in the Appendix 9.3.

5.1 The structure $\mathcal{M}_f$ and small eventual closures

We begin by giving the class of structures that we work with. The relevant notation and background on ω-categorical Hrushovski constructions is given in Section 4.1.

Construction 5.1.1. We work with graphs and set $\alpha = 2$. Let $f : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ be such that for $t \geq 6$, $f(3 \cdot t) \leq f(t) + 1$ and for $t \leq 6$, f is piece-wise linear with $f(1) = 2$, $f(4) = 5$, and $f(6) = 6$. An illustration of the initial values of f

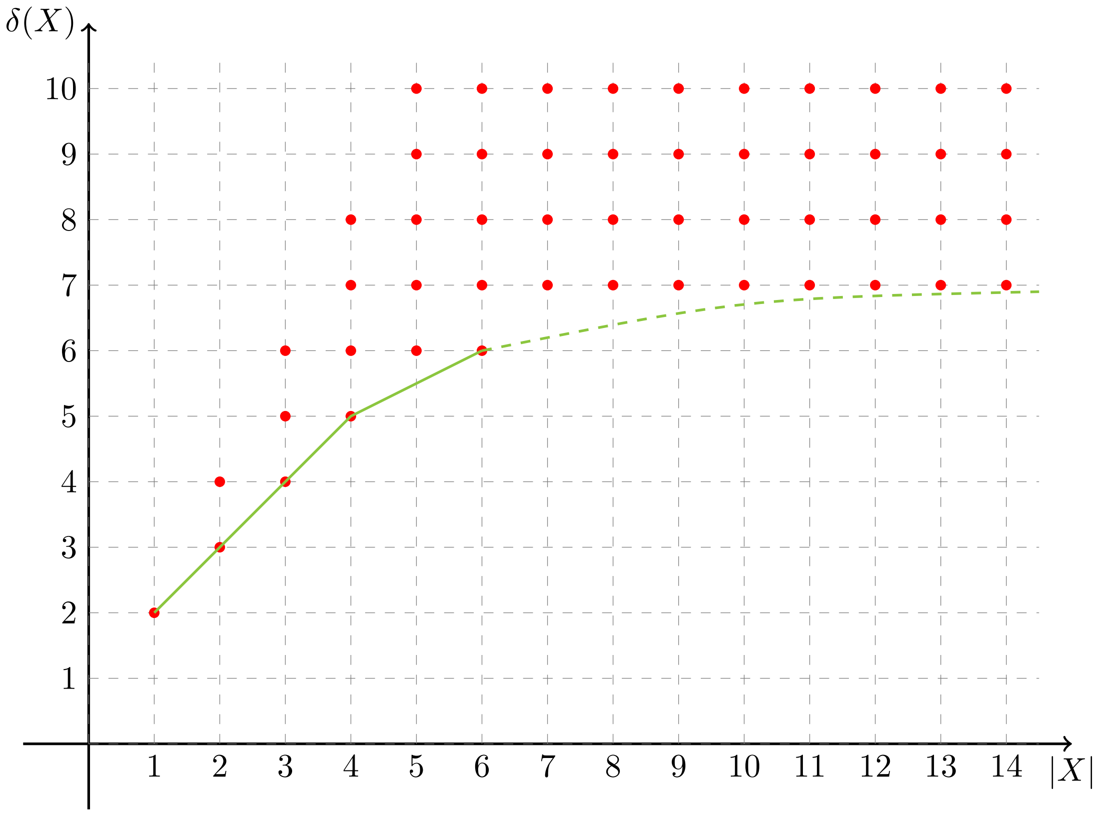


Figure 5.1: The initial values of f are represented by the line in the figure. The dots represent positions $(|X|, \delta(X))$ of the graphs X in $\mathcal{K}_f$ (some of the dots for $|X| \geq 5$ lie outside of the figure). Note that $f(1) = 2$ and $f(2) = 3$ are necessary to include vertices and edges in $\mathcal{K}_f$. $f(3) = 4, f(4) = 5, f(6) = 6$ imply that the smallest cycle in $\mathcal{K}_f$ is a 6-cycle. We also have that vertices and edges are always closed in $\mathcal{M}_f$.

is given by Figure 5.1. As proven in the Appendix 9.3, the class $\mathcal{K}_f$ has the free amalgamation property, and so, by Theorem 4.1.6, we can build an ω-categorical Hrushovski construction $\mathcal{M}_f$.

An example of a function satisfying the conditions of Construction 5.1.1 is given by taking the values specified for $t \leq 6$ and $\log_3(x) - \log_3(6) + 6$ for $t > 6$. Of course, there are uncountably many functions satisfying the conditions of our construction.

From now on, by $\mathcal{M}_f$ we mean an ω-categorical Hrushovski construction obtained following Construction 5.1.1.

In the Appendix 9.3, we prove various basic properties of the structures $\mathcal{M}_f$. The definition of independent n-amalgamation is given precisely in Section 7.1.

Lemma 5.1.2 (Basic properties of $\mathcal{M}_f$). *The ω-categorical Hrushovski construction $\mathcal{M}_f$ is supersimple of SU-rank 2. Furthermore, it has weak elimination of imaginaries. Finally, we may choose f with growth slow enough that $\mathcal{M}_f$ satisfies independent n amalgamation over finite algebraically closed sets for any $n \geq 3$ or even for all $n \in \mathbb{N}$.*

Proof. See Appendix 9.3. $\square$

As we shall see, the choice of values of $f(t)$ for $t \leq 6$ ensures that the eventual closures of graphs of small enough dimension are easy to enumerate.

Definition 5.1.3. Let $A \in \mathcal{K}_f$. Suppose $A \leq^* A' \in \mathcal{K}_f, \delta(A') = \delta(A)$, and $|A' \setminus A| = 1$. Then, we call A' a **one point extension** of A.

Since $\alpha = 2$, a one-point extension A' of A will be obtained by adding to A a single vertex v joined by two edges to A.

Lemma 5.1.4. *Let $A, B \in \mathcal{K}_f$, $A \leq^* B, \delta(A) = \delta(B)$, and $|B \setminus A| < 6$. Then, B is obtained from A by iterating one-point extensions.*

Proof. Note that with our choice of f for $X \in \mathcal{K}_f$ such that $|X| < 6$, $|E(X)| < |X|$. In fact, for $|X| < 6$, $f(X) > |X|$. Since $X \in \mathcal{K}_f$,

$$2|X| - |E(X)| = \delta(X) \geq f(|X|) > |X| \tag{5.1}$$

$$\Rightarrow |E(X)| < |X|. \tag{5.2}$$

Now, consider the graph X induced by the vertices in $B \setminus A$. Since $|X| < 6$, $|E(X)| < |X|$. Let $E(A; X) := \{\{u, v\} \in E(B) | u \in A, v \in X\}$. By $\delta(B) = \delta(A)$, we have that:

$$\begin{aligned}\delta(B) &= 2|B| - E(B) = 2|A| + 2|X| - |E(A)| - |E(X)| - |E(A;X)| \\ &= 2|A| - |E(A)| = \delta(A),\end{aligned}$$

and so since $|E(X)| < |X|$,

$$0 = 2|X| - |E(X)| - |E(A;X)| > |X| - |E(A;X)|,$$

which yields $|X| < |E(A;X)|$. By the pigeonhole principle, there must be a vertex v of X connected to two or more vertices in A. Note that if v were connected to three or more vertices, $\delta(A \cup \{v\}) < \delta(A)$, and so $A \not\leq^* B$. Hence, $A \cup \{v\}$ must be a one-point extension of A. The same argument can be iterated for $A \cup \{v\}$ so that we see B is constructed by iterating one-point extensions. $\square$

Remark 5.1.5. Let D be a proper eventual closure of $A_1 \amalg_{A_0} A_2$ with $|D \setminus A_1 \cup A_2| < 6$. Then, as noted in Lemma 5.1.4, D is obtained by iterating one-point extensions $D_1, \ldots, D_k$ for $k \leq 5$. One can see that each D_i is also an eventual closure.

We shall call an eventual closure of $A_1 \amalg_{A_0} A_2$ which is also a one-point extension a **one point closure**.

Given Lemma 5.1.4 and Corollary 4.2.5, we get:

Corollary 5.1.6. *Let $A \leq \mathcal{M}_f$, and $A \leq B, C \in \mathcal{K}_f$. Let $N = \lfloor f^{-1}(\delta(B \amalg_A C)) \rfloor$ so that N is the largest size of a graph of predimension $\delta(B \amalg_A C)$. Suppose that $N - |B \amalg_A C| < 6$ and that the maximal distance between a point in $B \setminus A$ and a point in $C \setminus A$ in $B \amalg_A C$ is ≤ 3. Then, there is no proper eventual closure for $B \amalg_A C$, and so,*

$$\mu(B \amalg_A C) = \frac{\mu(B)\mu(C)}{\mu(A)}.$$

Proof. Since no graph in $\mathcal{K}_f$ contains cycles of length < 6, $B \amalg_A C$ has no proper eventual closures. In fact, in a one point closure of $B \amalg_A C$ by a point v, v must be attached to a vertex in $B \setminus A$ and a vertex in $C \setminus A$ since B and C are closed in the eventual closure. But then, this one-point closure would contain a k-cycle for $k < 6$. $\square$

5.2 Proving non-measurability

We are ready to prove that the structures built in Construction 5.1.1 are not *MS*-measurable. We first find some equations that an *MS*-measure should satisfy and then note that these equations imply that some definable set has measure zero, contradicting the positivity condition of *MS*-measures.

Notation 5.2.1. We still follow the notation of 4.2.1. However, for clarity, we represent graphs pictorially. For example, for $A \in \mathcal{K}_f$ a path of length two, instead of writing $\mu(A)$, we may write $\mu\left(\ \right)$, so that it is clear from our notation which graph we are talking about. Note that when we speak of "the measure of A", we actually mean "the measure of the definable subset of $M_f^{|A|}$ consisting of copies of A which are such that $A \leq M_f$". By our choice of f, vertices and edges form complete types in $\mathcal{M}_f$.

Proposition 5.2.2. *Suppose that $\mathcal{M}_f$ is MS-measurable with dimension given by SU-rank. Assume without loss of generality that the measure of a single vertex is* 1*. Let λ be the measure of an edge. Then, μ must satisfy the following equations:*

$$\mu\left(\ \right) = \lambda^2 \tag{5.3}$$

$$\mu\left(\cdot \quad \cdot\right) + \mu\left(\ \right) = 1 \tag{5.4}$$

$$\mu\left(\ \right) = \lambda^3 \tag{5.5}$$

$$\mu\left(\ \right) + \mu\left(\ \right) = \lambda(1-\lambda^2) \tag{5.6}$$

$$\mu\left(\ \right) = \frac{\mu\left(\ \right)^2}{(1-\lambda^2)} \tag{5.7}$$

$$\mu\left(\ \right) = \lambda^4 \tag{5.8}$$

$$\mu\left(\ \right) + \mu\left(\ \right) = (1-\lambda^2)^2\lambda^2. \tag{5.9}$$

Proof. Equation 5.9 is obtained with a different technique from the rest. The proof for equations 5.3-5.8 consists in identifying various free amalgamations in the graphs involved and in the repeated use of Corollary 5.1.6 or the fact that all eventual closures we will consider are obtained by iterating one-point extensions and using Theorem 4.2.5. Again, to obtain these equations, we consider the eventual closures D_i of a free amalgamation $B \amalg_A C$. For clarity, we shall illustrate this through pictures colouring differently the components of $B \amalg_A C$ and the one-point extensions. We colour the copy of A in purple, the copy of $B \setminus A$ in blue and the copy of $C \setminus A$ in orange. When dealing with one-point extensions we shall colour the additional points in green. If you are reading the paper in black and white printing, the copy of $B \setminus A$ appears black, the copy of A appears grey, the copy of $C \setminus A$ appears light grey and any one-point extension will be very light grey. We also label the vertices for extra clarity.

Let us begin with equation 5.3. Consider a path of length two as a free amalgamation of two paths of length one over a point as in Figure 5.2. This free amalgamation has no further eventual closures. Since the measure of an edge is λ and the measure of a point is 1, equation 5.3 follows from Corollary 5.1.6.

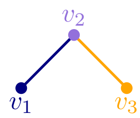


Figure 5.2: We consider a path of length two as the free amalgamation of the two edges $B := v_1v_2$ and $C := v_2v_3$ over the intermediate vertex $A := v_2$. Note that this path has already maximal size for its dimension as $f(3) = 4$. So, there are no further eventual closures.

Now, for equation 5.4, consider the free amalgamation of two points over the empty set. This does have a one point closure, namely a copy of a path of length two (by joining the vertex of the extension to the two distinct vertices in the amalgamation). There cannot be further extensions since $f(3) = 4$, and so the resulting graph has maximal size for its dimension. Since vertices have measure one in $\mathcal{M}_f$, equation 5.4 follows. Note that as a consequence of equations 5.4 and 5.3, we obtain that

$$\mu(\cdot \quad \cdot) = (1 - \lambda^2),$$

and we shall use this fact in future equations.

Equations (5.5)-(5.8) are explained in Figures 5.3-5.6.

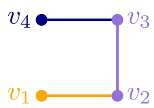
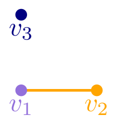


Figure 5.3: We consider a path of length three $v_1v_2v_3v_4$ as a free amalgamation of two paths of length two, $B := v_1v_2v_3$ and $C := v_2v_3v_4$, over the path of length one $A := v_2v_3$. There are no further eventual closures since $f(4) = 5$ and so we have maximal size for this dimension. This amalgamation yields equation 5.5 recalling that the two paths of length two in the amalgamation have measure λ^2, and that the path over which they are amalgamated has measure λ.



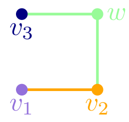


Figure 5.4: The disjoint union of an edge v_1v_2 and a vertex v_3 corresponds to the free amalgamation of the disjoint union of two vertices $B := v_1v_3$ and of the edge $C := v_1v_2$ over the vertex $A := v_1$. We can obtain a one point closure by joining a vertex w to the points v_2 and v_3. It is the unique one point closure since $|B \setminus A| = |C \setminus A| = 1$. There are no larger eventual closures since $f(4) = 5$. Thus, we get equation 5.6, recalling that the measure of two disjoint points is $(1 - \lambda^2)$.

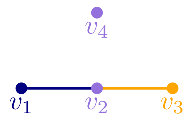


Figure 5.5: We think of the disjoint union of a vertex v_4 and a path of length two $v_1v_2v_3$ as a free amalgamations of two copies of the disjoint union of an edge and a vertex, $B := v_1v_2v_4$ and $C := v_2v_3v_4$, over the disjoint union of two vertices $A := v_2v_4$. The free amalgamation has dimension 6. Since $f(6) = 6$ and the vertices in $B \setminus A$ and $C \setminus A$ are at distance two from each other, from Corollary 5.1.6 we obtain the desired equation 5.7.

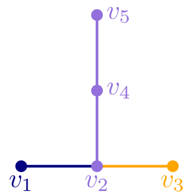


Figure 5.6: We consider the free amalgamation of two paths of length three, $B := v_1v_2v_4v_5$ and $C := v_3v_2v_4v_5$ over a path of length two $A := v_2v_4v_5$. Equation 5.8 follows similarly to the above by Corollary 5.1.6.

We are now ready to obtain equation 5.9. For it we can use Corollary 3.1.10 since the structures we are working with have weak elimination of imaginaries, as proven in Theorem 9.3.4 of the appendix, and $\mathrm{cl}(\emptyset) = \emptyset$. Since in $\mathcal{M}_f$ types of finite tuples are determined by the quantifier-free type of their closures, there are only two 2-types over the emptyset, $p_1(x, y)$ and $p_2(x, y)$, such that $p_i(a, b)$ implies $a \downarrow b$. These are isolated by the formulas $\phi(x, y)$ and $\psi(x, y)$, saying respectively that x and y are at distance two from each other and that they are at distance > 2. We consider the measure of $\psi(x_1, x_2) \wedge \psi(x_2, x_3) \wedge \phi(x_1, x_3)$. By Corollary 3.1.10, we have

$$\mu(\psi(x_1, x_2) \wedge \psi(x_2, x_3) \wedge \phi(x_1, x_3)) = \mu(\psi(x_1, x_2))^2 \mu(\phi(x_1, x_3)) = (1 - \lambda^2)^2 \lambda^2.$$

We wish to express $\psi(x_1, x_2) \wedge \psi(x_2, x_3) \wedge \phi(x_1, x_3)$ as a disjoint union of 3-types of maximal dimension in order to expand the left hand side of this equation. Let us look at the complete 3-types $\text{tp}(a_1a_2a_3)$ such that $a_1a_2a_3 \vDash \psi(x_1, x_2) \wedge \psi(x_2, x_3) \wedge \phi(x_1, x_3)$ and $a_i \mathop{\smile\!\!\!\!\!\!\;|}\; a_ja_k$ for $\{i, j, k\} = \{1, 2, 3\}$ distinct. Let A be the graph in $\mathcal{K}_f$ given by the following picture:

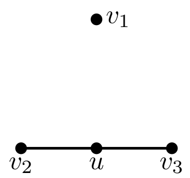


Figure 5.7: The graph A consists of the disjoint union of a path of length two v_2uv_3 and a vertex v_1. Note that if $A \leq M_f$, then $\mathcal{M}_f \vDash \psi(v_1, v_2) \wedge \psi(v_2, v_3) \wedge \phi(v_1, v_3)$ and v_i is independent from v_jv_k for $\{i, j, k\} = \{1, 2, 3\}$.

Again, since the types of these triples are determined by the quantifier-free types of their closures, we may focus on the graphs B in $\mathcal{K}_f$ such that, $A \leq^* B$, $\delta(B) = \delta(A) = 6$, and $v_1v_2, v_1v_2, v_2uv_3 \leq B$. Note that since $\delta(B) = 6$, B may have at most 6 points and must be obtained by iterating one-point extensions on the graph A.

Figure 5.8: Note that since $v_1v_2, v_1v_2, v_2uv_3 \leq B$, any one-point extension of A by a vertex w must be such that w has an edge with u. Since we avoid triangles, the only possible one-point extension is given by joining w to u and v_1. Call the graph so obtained B'. Note that B' doesn't have any one-point extension since any two points of B' have distance ≤ 3 from each other and $\mathcal{M}_f$ avoids cycles of size < 6.

Let $A(v_1, v_2, v_3, u)$ be the formula expressing $v_1v_2v_3u \cong A \leq \mathcal{M}_f$ and

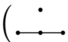
$$B'(v_1, v_2, v_3, u, w)$$

be the formula expressing $v_1v_2v_3uw \cong B' \leq \mathcal{M}_f$, where B' is described in Figure 5.8. From the argument in Figure 5.8, we can see that there are only two 3-types

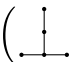
$p(x_1, x_2, x_3)$ which are completions of $\psi(x_1, x_2) \wedge \psi(x_2, x_3) \wedge \phi(x_1, x_3)$ and which satisfy our independence requirements, and they are isolated by $\exists u A(x_1, x_2, x_3, u)$ and $\exists u, w B'(x_1, x_2, x_3, u, w)$. Hence, by additivity,

$$\mu(\psi(x_1, x_2) \wedge \psi(x_2, x_3) \wedge \phi(x_1, x_3)) =$$
$$\mu(\exists u A(x_1, x_2, x_3, u)) + \mu(\exists u, w B'(x_1, x_2, x_3, u, w))$$

Consider the projection $\pi : M_f^4 \to M_f^3$ onto the first three coordinates. The restriction of this map to $A(M^4)$ is injective, and so by Fubini and algebraicity,

$\mu(\exists u A(x_1, x_2, x_3, u)) = \mu(A(x_1, x_2, x_3, u)) = \mu\,($  $)$.

Similarly, considering the projection $\pi' : M_f^5 \to M_f^3$ onto the first three coordinates and $B(M^5)$,

$\mu(\exists u, w B'(x_1, x_2, x_3, u, w)) = \mu(B'(x_1, x_2, x_3, u, w)) = \mu\,($  $)$.

This yields the desired equation. $\square$

Theorem 5.2.3. *The structures of the form $\mathcal{M}_f$ built following Construction 5.1.1 are not MS-measurable. Hence, there are ω-categorical supersimple finite SU-rank structures which are not MS-measurable. Indeed, there are supersimple ω-categorical structures of finite SU-rank and with independent n-amalgamation for all n which are not MS-measurable.*

Proof. By Corollary 2.2.26, we know that if $\mathcal{M}_f$ is MS-measurable, it also has an MS-dimension-measure where the dimension is given by SU-rank. However, the equations from Proposition 5.2.2 imply that $\lambda = 0$. This contradicts the positivity assumption for the measure in MS-measurable structures. To see this note that equations 5.5 and 5.6 imply that

$\mu\,($ $) = \lambda - 2\lambda^3.$

From equations 5.7, 5.8, and 5.9, we then get that

$$\frac{\lambda^2(1 - 2\lambda^2)^2}{(1 - \lambda^2)} + \lambda^4 = (1 - \lambda^2)^2 \lambda^2.$$

Simplifying, we obtain that $\lambda = 0$. $\square$

In the next chapter, we find more general reasons for which various classes of ω-categorical Hrushovski constructions are not MS-measurable by studying invariant Keisler measures in these structures. In particular, we prove that ω-categorical MS-measurable structures must satisfy a stronger version of the independence theorem. We also prove that in the structures $\mathcal{M}_f$ we introduced, the formula asserting that "x has distance two from a" does not fork over the empty-set but is universally measure zero. While our results make it implausible, it remains an open question whether some not one-based ω-categorical Hrushovski construction is MS-measurable.

6 Forking and Universally Measure Zero formulas

Keisler measures yield a natural notion of smallness for a definable subset of a structure: a subset X of M^x defined by $\phi(x, a)$ is **universally measure zero** if it is assigned measure zero by every invariant Keisler measure on M. A more classical model theoretic notion of smallness is forking: dividing captures the idea that a small subset of a model can be "moved enough" by automorphisms so as to not overlap with itself.

In stable theories, a definable set is universally measure zero if and only if it forks over $\emptyset$ [12]. This is also the case in ω-categorical NIP theories [4]. Until recently, it was unknown whether the two notions coincided in simple theories. The first counterexample, showing that in simple theories there can be non-forking formulas which are universally measure zero, is given in [12]. Using the same technique, the authors also give examples of groups with simple theory which are not definably amenable and prove some positive results in the context of small theories. Neither of the counterexamples given is ω-categorical. The theory of the first counterexample contains as a reduct the theory of the free action of the free group F_5 on an infinite set, which has infinitely many 1-types over a single parameter. This contradicts Ryll-Nardzewski. Meanwhile, on the definably amenable group case, all groups definable in an ω-categorical structure are definably amenable [12, Corollary 4.14]. Indeed, ω-categorical supersimple groups are actually amenable being finite-by-abelian-by-finite [26].

It is natural to ask whether by adding the assumption of ω-categoricity we can prove that being universally measure zero is the same as forking. Firstly, the known counterexample makes heavy use of a construction which is inherently not ω-categorical. Secondly, ω-categoricity implies that any invariant Keisler measure is also definable, i.e. the set of parameters a for which $\mu(\phi(x,a)) = \alpha$ is $\emptyset$-definable. In general, this is a substantially stronger assumption than invariance.

Another motivation for an ω-categorical counterexample comes from the study of ω-categorical MS-measurable structures [50]. In Section 2.2.3, we showed that in an ω-categorical MS-measurable structure we can always take the dimension part of the dimension-measure to be SU-rank. With this observation, we can see that if an ω-categorical structure is MS-measurable, then a formula forks over the $\emptyset$ if and only if it is universally measure zero. Hence, any simple ω-categorical structure with non-forking formulas which are universally measure zero cannot be MS-measurable, yielding further counterexamples to Question 5.0.1 of Elwes and Macpherson. Moreover, if we could prove that supersimple ω-categorical not one-based structures have such formulas, that would imply that these structures are not MS-measurable and answer positively Question 5.0.2.

It turns out that the ω-categorical Hrushovski constructions from Construction 5.1.1 we studied in Chapter 5 have formulas that do not fork over $\emptyset$ but are universally measure zero:

Theorem 6.3.4. *There are ω-categorical supersimple theories T of finite SU-rank with a formula $\phi(x, a)$ which does not fork over the empty set, but which is universally measure zero.*

More generally, in Theorem 6.2.3, we show that if forking and being universally measure zero agree in a simple ω-categorical structure, then it must satisfy a stronger version of the independence theorem. It is easy to build ω-categorical Hrushovski constructions for which this fails. These structures are extremely amenable in the sense of [36], which implies the existence of invariant types, and so invariant Keisler measures, in each variable.

We begin with Section 6.1, where we introduce the concept of universally measure zero formulas. We draw attention to how it is sufficient to study ergodic measures for the purposes of this question. In Section 6.2, we show how in simple ω-categorical structures, if forking and being universally measure zero agree, then they satisfy a stronger version of the independence theorem (Theorem 6.2.3). We also draw some implications for ω-categorical MS-measurable structures. Finally, in Section 6.3, we conclude giving the example of an ω-categorical structure, supersimple of finite SU-rank with a formula which does not fork over the empty set but which is universally measure zero.

6.1 Two notions of smallness

Keisler measures give us a natural notion of smallness for a definable set.

Definition 6.1.1. Let $\mathbb{M}$ be a monster model and $A \subset \mathbb{M}$ a small subset. We say that $\phi(x, a) \in \mathrm{Def}_x(\mathbb{M})$ is **universally measure zero** over A if $\mu(\phi(x, a)) = 0$ for any $\mathrm{Aut}(\mathbb{M}/A)$-invariant Keisler measure. We call $\mathcal{O}(A)$ the set of formulas which are universally measure zero over A. If $A = \emptyset$, we say that $\phi(x, a)$ is universally measure zero.

For $A \subseteq \mathbb{M}$ small, $\mathcal{O}_x(A) = \mathcal{O}(A) \cap \mathrm{Def}_x(\mathbb{M})$ forms an ideal in the Boolean algebra of $\mathrm{Def}_x(\mathbb{M})$ [12] in the following sense:

Definition 6.1.2. We say that a set $I \subseteq \mathrm{Def}_x(\mathbb{M})$ forms an **ideal** if

- $x \neq x \in I$;
- if $\models \forall x(\phi(x,a) \to \psi(x,b))$ and $\psi(x,b) \in I$, then $\phi(x,a) \in I$; and
- if $\phi(x,a), \psi(x,b) \in I$, then $\phi(x,a) \vee \psi(x,b) \in I$;

where the formulas $\phi(x,a), \psi(x,b)$ are in $\mathrm{Def}_x(\mathbb{M})$.

The definition of ideal is supposed to capture that of a set of "small" definable subsets of a structure. We recall the definitions of forking and dividing below [64].

Definition 6.1.3. Working in a monster model $\mathbb{M}$, for $A \subseteq \mathbb{M}$ small, we say that $\phi(x,b) \in \mathrm{Def}_x(\mathbb{M})$ **divides over** $\boldsymbol{A}$ if there is some $k \in \mathbb{N}$ and a sequence $(b_i)_{i<\omega}$ of realisations of $\mathrm{tp}(b/A)$ such that $\{\phi(x,b_i) | i < \omega\}$ is k-inconsistent. We say that $\psi(x,c) \in \mathrm{Def}_x(\mathbb{M})$ **forks over** $\boldsymbol{A}$ if it implies a finite disjunction of formulas each dividing over $\boldsymbol{A}$. We call $F_x(A)$, the set of formulas in the variable x with parameters from $\mathbb{M}$ which fork over A. We write $F(A)$ for the set of formulas with parameters from $\mathbb{M}$ forking over A.

Comparing the definition of forking with that of an ideal, we can see that $F_x(A)$ is the ideal in $\mathrm{Def}_x(\mathbb{M})$ generated by formulas dividing over A in the variable x [31]. When we work in a simple theory, a formula forks over A if and only if it divides over A [64, Proposition 7.2.15].

The purpose of this chapter is exploring further the relation between the notions of smallness given by forking and being universally measure zero. It is well known that if a formula forks over $\emptyset$, then it is universally measure zero [12, Fact 1.2]:

Fact 6.1.4. *For any theory T, we have that $F(\emptyset) \subseteq \mathcal{O}(\emptyset)$.*

Our main result is that there are ω-categorical supersimple theories for which $\mathcal{O}(\emptyset)$ strictly contains the set $F(\emptyset)$ of formulas forking over $\emptyset$.

Remark 6.1.5. In this thesis, we mainly study $\mathrm{Aut}(M)$-invariant measures on a countable ω-categorical structure $\mathcal{M}$. Any $\mathrm{Aut}(\mathbb{M})$-invariant measure on $\mathbb{M}$ induces an $\mathrm{Aut}(M)$-invariant measure on any small model $\mathcal{M} \prec \mathbb{M}$. Hence, if for $a \in M$, $\mu(\phi(x,a)) = 0$ for any $\mathrm{Aut}(M)$-invariant Keisler measure on M, then $\phi(x,a)$ is universally measure zero. In particular, for a small theory T, we have a countable ω-saturated model $\mathcal{N}$, and so any $\mathrm{Aut}(\mathbb{M})$-invariant Keisler measure μ on $\mathbb{M}$ is entirely determined by its restriction to formulas with parameters from N. By ω-saturation, working in an ω-categorical theory, no generality is lost by looking at the $\mathrm{Aut}(M)$-invariant Keisler measures for the countable model $\mathcal{M}$ instead of the $\mathrm{Aut}(\mathbb{M})$-invariant Keisler measures for the monster model $\mathbb{M}$.

Following the conventions set in Remarks 2.1.3 and 3.2.7, we shall use μ to denote both an invariant Keisler measure and the corresponding regular Borel probability measure. Moreover, when x is a real variable, we identify μ also with the corresponding invariant Keisler measure on M^{eq}. Again, following Remark 3.2.7, the various arguments in this chapter also hold when working in an $\mathrm{Aut}(M)$-invariant Keisler measure μ on M^{eq} in an imaginary variable.

If we are interested in studying the universally measure zero formulas for a structure, it is helpful to study $\mathrm{Aut}(M)$-ergodic measures on $S_x(M)$. From the ergodic decomposition 3.2.3, we get:

Corollary 6.1.6. *Let $\mathcal{M}$ be a countable structure and let μ be an* $\mathrm{Aut}(M)$*-invariant Borel probability measure on $S_x(M)$. Let $\mathfrak{M}_x(M)$ be the space of* $\mathrm{Aut}(M)$*-invariant Borel probability measures on $S_x(M)$. Then, there is a unique Borel probability measure $\mathfrak{m}$ on $\mathfrak{M}_x(M)$ such that for any $\mathcal{L}(M)$-formula $\phi(x,a)$,*

$$\mu([\phi(x,a)]) = \int_{\mathrm{Erg}_x(M)} \nu([\phi(x,a)]) \mathrm{d}\mathfrak{m}(\nu),$$

where $\mathrm{Erg}_x(M)$ *is the space of* $\mathrm{Aut}(M)$*-ergodic measures.*

Hence, if there is countable structure $\mathcal{M} \prec \mathbb{M}$ containing a such that $\phi(x,a)$ has measure zero for every ergodic measure on $S_x(M)$, then $\phi(x,a)$ is universally measure zero.

Remark 6.1.7. From [25, Corollary 1.2], we actually know that when $\mathcal{M}$ is the countable model of an ω-categorical theory, $\mathrm{Erg}_x(M)$ is closed in the space $\mathfrak{M}_x(M)$.

6.2 A strong independence theorem

In this section, we prove that, in a simple ω-categorical structure $\mathcal{M}$ with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$, if forking over the empty set is the same as being universally measure zero, then $\mathcal{M}$ satisfies a stronger version of the independence theorem over finite algebraically closed sets. We conclude with some consequences for ω-categorical MS-measurable structures.

Firstly, we note how the measure $\mu(\phi(x,a) \wedge \psi(x,b))$ only depends on the types of the individual parameters when a and b are weakly algebraically independent, as defined in Definition 3.2.14.

Corollary 6.2.1. *Let $\mathcal{M}$ be an ω-categorical structure with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Suppose that $a, b \in \mathcal{M}^{eq}$ are such that $a \mathop{\smile\!\!\!\!\!\!|}^{a} b$. Let $\phi(x,y), \psi(x,z)$ be $\mathcal{L}^{eq}$-formulas where y and z are in the sorts of a and b respectively. Then, for an arbitrary $\mathrm{Aut}(M)$-invariant Keisler measure μ,*

$$\mu(\phi(x,a) \wedge \psi(x,b))$$

only depends on $\mathrm{tp}(a)$ and $\mathrm{tp}(b)$.

Proof. Let $a' \mathop{\smile\!\!\!\!\!\!|}^{a} b'$ be such that $a' \equiv a$ and $b' \equiv b$. Then, by Corollary 3.2.16, we get that for any ergodic measure ν,

$$\begin{aligned}\nu(\phi(x,a) \wedge \psi(x,b)) &= \nu(\phi(x,a))\nu(\psi(x,b)) \\ &= \nu(\phi(x,a'))\nu(\psi(x,b')) = \nu(\phi(x,a') \wedge \psi(x,b')).\end{aligned}$$

But then, by the ergodic decomposition, Corollary 6.1.6, we have that for any $\mathrm{Aut}(M)$-invariant Keisler measure

$$\mu(\phi(x,a) \wedge \psi(x,b)) = \mu(\phi(x,a') \wedge \psi(x,b')).$$

□

As mentioned in the previous section, we know that $F(\emptyset) \subseteq \mathcal{O}(\emptyset)$. We are interested in studying what happens when $F(\emptyset) = \mathcal{O}(\emptyset)$ in order to find a structure where the two sets differs. In an ω-categorical context, by ω-saturation, nothing is lost by considering formulas with parameters from the countable model $\mathcal{M}$. In fact, if $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$, this will be witnessed by a formula with parameters from $\mathcal{M}$. To see this, recall the discussion in Remark 6.1.5.

Definition 6.2.2. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We say that $\mathcal{M}$ **satisfies the strong independence theorem** over $A \subseteq \mathcal{M}^{eq}$ if the following holds:

Let $a, b, c_0, c_1 \in M^{eq}$ be such that $a \mathop{\smile\!\!\!\!\!\!|}^{a}_{A} b$, $c_0 \equiv_A c_1$ and $c_0 \mathop{\smile\!\!\!\!\!\!|}_A a$, $c_1 \mathop{\smile\!\!\!\!\!\!|}_A b$. Then, there is $c^* \in \mathcal{M}^{eq}$ such that $c^* \equiv_{Aa} c_0, c^* \equiv_{Ab} c_1$, and $c^* \mathop{\smile\!\!\!\!\!\!|}_A ab$.

Here, the relation $\mathop{\smile\!\!\!\!\!\!|}$ denotes non-forking independence, while $\mathop{\smile\!\!\!\!\!\!|}^{a}$ denotes weak algebraic independence. From Corollary 6.2.1, we obtain that simple ω-categorical structures where forking coincides with being universally measure zero satisfy the strong independence theorem over the empty set.

Theorem 6.2.3. *Let $\mathcal{M}$ be a simple ω-categorical structure with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Suppose that $F(\emptyset) = \mathcal{O}(\emptyset)$, i.e. a formula forks over the empty set if and only if it is universally measure zero. Then, $\mathcal{M}$ satisfies the strong independence theorem over the empty set.*

Proof. Suppose that there are $a, b, c_0, c_1 \in \mathcal{M}^{eq}$ as in Definition 6.2.2. Let $\phi(x, a)$ and $\psi(x, b)$ isolate $\mathrm{tp}(c_0/a)$ and $\mathrm{tp}(c_1/b)$. By the existence property of non-forking independence, there is $b' \equiv b$ such that $b' \mathop{\smile\!\!\!\!\!\!|} a$. By Corollary 6.2.1, for any Keisler measure,

$$\mu(\phi(x, a) \wedge \psi(x, b)) = \mu(\phi(x, a) \wedge \psi(x, b')).$$

By simplicity, $\phi(x, a) \wedge \psi(x, b')$ does not fork over the empty set since the independence theorem holds over $\emptyset$. Hence, by $F(\emptyset) = \mathcal{O}(\emptyset)$, $\phi(x, a) \wedge \psi(x, b)$ is not universally measure zero, and so is non-forking over the empty set. This proves the strong independence theorem over the empty-set for $\mathcal{M}$. □

Remark 6.2.4. In general, a simple ω-categorical theory with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$ satisfies the independence theorem over the empty set (recall Fact 3.1.8). However, the condition of $F(\emptyset) = \mathcal{O}(\emptyset)$ yields a strengthening of the independence theorem, where the "base" of the amalgamation is weakly algebraically independent rather than non-forking independent. In fact, while non-forking independence implies weak algebraic independence, the converse does not hold in general.

Definition 6.2.5. We say that an ω-categorical simple structure $\mathcal{M}$ is **one-based** if, given $A, B \subseteq \mathcal{M}^{eq}$ algebraically closed, then,

$$A \underset{A\cap B}{\downarrow} B.$$

If $\mathcal{M}$ is one-based, for $A \subseteq M$ we have that

$$b \underset{A}{\downarrow^{a}} c \text{ if and only if } b \underset{A}{\downarrow} c.$$

In particular, in a one-based structure, satisfying the independence theorem over A is equivalent to satisfying the strong independence theorem over A.

However, there are not one-based simple ω-categorical structures. The only known examples of this are ω-categorical Hrushovski constructions. For these, to satisfy the strong independence theorem is a genuinely stronger requirement than satisfying the independence theorem.

We conclude this section with some consequences for MS-measurable structures in an ω-categorical context.

Lemma 6.2.6. *Let $\mathcal{M}$ be an MS-measurable structure. Then, $F(A) = \mathcal{O}(A)$ for any finite $A \subseteq M$.*

Proof. Suppose there is any formula $\phi(x, b)$ which does not fork over A, but is universally measure zero for any $\mathrm{Aut}(M/A)$-invariant Keisler measure. Let $c \models \phi(x, b)$ be such that $c \downarrow_A b$. By Theorem 2.2.20, $d(c/Ab) = d(c/A)$. Let

$\psi(x,A) \in \mathrm{tp}(c/A)$ be such that $d(\phi(x,A)) = d(c/A)$. By MS-measurability, we have an induced $\mathrm{Aut}(M/A)$-invariant Keisler measure μ_ψ on $\mathrm{Def}_x(M)$ given by

$$\mu_\psi(\chi(x,d)) = \begin{cases} \frac{\mu(\chi(x,d)\wedge\psi(x,A))}{\mu(\psi(x,A))} & \text{for } d(\chi(x,d)\wedge\psi(x,A))) = d(\psi(x,A)), \\ 0 & \text{otherwise.} \end{cases}$$

where μ is the measure in the dimension-measure. Being universally measure zero (over A), we have that $\mu_\psi(\phi(x,b)) = \mu(\phi(x,b)\wedge\psi(x,A)) = 0$, which contradicts positivity of the measure. $\square$

From Corollary 6.2.1, we obtain that ω-categorical MS-measurable structures satisfy the strong independence theorem over the algebraic closures of finite sets. Indeed, we also get the corresponding probabilistic independence statement. Recall from Lemma 3.2.18 that for $A \subseteq M^{eq}$ the algebraic closure of a finite set $A' \subseteq M$, and fixing the variable x, there is $A_0 \subset A$ finite such that types in the variable x over A are isolated by $\mathcal{L}^{eq}$-formulas with parameters from A_0.

Theorem 6.2.7. *Let $\mathcal{M}$ be ω-categorical and MS-measurable. Then, it satisfies the strong independence theorem over the algebraic closures of finite sets:*
Let $A \subseteq M^{eq}$ be the algebraic closure of a finite set, a_0, a_1, b, c tuples from M^{eq}. Suppose $a_0 \equiv_A a_1$, $b \mathop{\smile\!\!\!\!\!\!\!\;|}^{a}_A c$ and that $a_0 \mathop{\smile\!\!\!\!\!\!\!\;|}_A b$, $a_0 \mathop{\smile\!\!\!\!\!\!\!\;|}_A c$. Then, there is a^* such that $a^* \equiv_{Ab} a_0$, $a^* \equiv_{Ac} a_1$, and $a^* \mathop{\smile\!\!\!\!\!\!\!\;|}_A bc$. Moreover,

$$\mu(\mathrm{tp}(a_0/Ab) \cup \mathrm{tp}(a_1/Ac)) = \frac{\mu(a_0/Ab)\mu(a_1/Ac)}{\mu(a_0/A)}.$$

Proof. Let $\phi(x,b)$ and $\psi(x,c)$ isolate $\mathrm{tp}(a_0/Ab)$ and $\mathrm{tp}(a_1/Ac)$ respectively. Let $\chi(x)$ isolate $\mathrm{tp}(a_i/A)$, and let μ_χ be the $\mathrm{Aut}(M^{eq}/A)$ invariant Keisler measure induced by $\chi(x)$. We consider $\mu_\chi(\phi(x,b)\wedge\psi(x,c))$. By extension, we can find $c' \equiv_A c$ such that $c' \mathop{\smile\!\!\!\!\!\!\!\;|}_A b$. Since non-forking independence implies weak algebraic independence, by Corollary 6.2.1

$$\mu_\chi(\phi(x,b)\wedge\psi(x,c')) = \mu_\chi(\phi(x,b)\wedge\psi(x,c)).$$

However, the former has positive measure by the independence theorem over algebraically closed sets (and d-independence being the same as non-forking independence by Theorem 2.2.20). Hence, $\phi(x,b) \wedge \psi(x,c)$ must have a realisation which is independent from bc over A. Moreover, by Theorem 3.1.9, for $b \mathop{\smile\!\!\!\!\!\mid}_A c'$ we have that

$$\mu(\phi(x,b) \wedge \psi(x,c)) = \mu(\phi(x,b) \wedge \psi(x,c')) = \frac{\mu(\phi(x,b))\mu(\psi(x,c))}{\mu(\chi(x))}.$$

This yields the desired equation. □

6.3 The counterexample

We are now ready to introduce our example of a simple ω-categorical structure with $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$. It will be an ω-categorical Hrushovski construction. This is the same structure $\mathcal{M}_f$ as in Construction 5.1.1 from Chapter 5. The basic idea is that (non-trivial) simple ω-categorical Hrushovski constructions are not one-based. Hence, there are pairs which are weakly algebraic independent, but forking-dependent. This allows us to build Hrushovski constructions which are simple but which do not satisfy the strong independence theorem.

Theorem 6.3.1. *There is an ω-categorical graph $\mathcal{M}_f$ supersimple of SU-rank 2 with the following properties:*

- $\mathrm{Aut}(M_f)$ *acts transitively on the vertices of M_f;*
- *points are algebraically closed;*
- *edges are algebraically closed, but for $b \in M_f$ the formula $E(x,b)$ asserting that x has an edge with b forks over $\emptyset$;*
- *eor $a, b \in M_f$ with no edge between them, $\mathrm{acl}(ab) = ab$ or $\mathrm{acl}(ab) = abc$, where abc is a path of length two with endpoints a and b. In either case, $a \mathop{\smile\!\!\!\!\!\mid} b$;*
- *the smallest k-cycle in $\mathcal{M}_f$ is a 6-cycle; and*
- *the structure $\mathcal{M}_f$ has weak elimination of imaginaries.*

Furthermore, we can choose $\mathcal{M}_f$ to satisfy independent n-amalgamation over finite algebraically closed sets for any $n \in \mathbb{N}$, and also for all $n \in \mathbb{N}$.

Proof. The structure is an ω-categorical Hrushovski construction. It is the same construction as in Construction 5.1.1. In Appendix 9.3, we prove supersimplicity, weak elimination of imaginaries and note how higher independent amalgamation can be obtained. The other properties also follow by construction and basic calculations with the dimension, recalling that the Hrushovski dimension corresponds to SU-rank (Theorem 9.1.15). □

Remark 6.3.2. The graph $\mathcal{M}_f$ from Construction 5.1.1 is also **extremely amenable** in the sense of [36]. That is, every type in $S_x(\emptyset)$ extends to an $\mathrm{Aut}(M)$-invariant type over M. Since invariant types can be considered as invariant Keisler measures taking only the values 0 and 1, this guarantees that there are some invariant Keisler measures on $\mathcal{M}$ in each variable. To see that these structures are extremely amenable, for a finite tuple $\overline{a}$ from M, consider the type $p(\overline{x})$ given by

$$\bigcup_{B \subset M \text{ finite}} \{\mathrm{tp}(\overline{a}'/B) \mid \overline{a}' \equiv \overline{a}, \mathrm{acl}(\overline{a}B) = \mathrm{acl}(\overline{a}) \cup \mathrm{acl}(B),$$
$$\neg E(c,b) \text{ for } c \in \mathrm{acl}(\overline{a}), b \in \mathrm{acl}(B)\}.$$

Essentially, $p(\overline{x})$ asserts that $\overline{x}$ has no relation to $\mathcal{M}$ and is weakly algebraically independent from it. We can conclude that this type is consistent, complete and invariant by the extension property and because types of finite tuples are determined by the quantifier-free types of their algebraic closures.

From Proposition 1.2.6, when $\mathcal{M}$ has weak elimination of imaginaries, for $A, B \subseteq M$ algebraically closed and disjoint in $\mathcal{M}$, we have that $A \mathop{\smile\!\!\!\!\!\mid}^{a} B$. This allows us to see how the strong independence theorem fails in $\mathcal{M}_f$. For a and b sharing an edge, $a \mathop{\smile\!\!\!\!\!\mid}^{a} b$, but $\mathrm{tp}(a/b)$ forks over $\emptyset$. Meanwhile, for a and c at distance two from each other, $a \mathop{\smile\!\!\!\!\!\mid} c$. Hence, for $\mathcal{M}_f$ to satisfy the strong independence theorem over the empty set, $\mathcal{M}_f$ should include 5-cycles. But we have built $\mathcal{M}_f$ to exclude these. This implies the existence of a formula which does not fork over the empty-set, but is universally measure zero. We can give this formula explicitly:

Theorem 6.3.3. *Let $\mathcal{M}_f$ be the ω-categorical structure described in Theorem 6.3.1. Let $\phi(x,y)$ be the formula stating that the points x and y are exactly at distance two from* *each other. Then, for $a \in M_f$, the formula $\phi(x, a)$ is universally measure zero but does not fork over the emptyset.*

Proof. Let $a, b \in M_f$ share an edge. Now, $\phi(x, a) \wedge \phi(x, b)$ is inconsistent since $\mathcal{M}_f$ avoids 5-cycles. Meanwhile, $\phi(x, a)$ does not fork over the empty set since $c \mathop{\smile\!\!\!\!\!|}\, a$ for c at distance two from a. However, since $a \mathop{\smile\!\!\!\!\!|}^{a} b$, for an ergodic measure,

$$0 = \mu(\phi(x, a) \wedge \phi(x, b)) = \mu(\phi(x, a))\mu(\phi(x, b)) = \mu(\phi(x, a))^2,$$

where the second equality follows by Corollary 3.2.16 and the last equality by transitivity of the action of $\mathrm{Aut}(M)_f$. From the calculation above, we get that $\mu(\phi(x, a)) = 0$ for any ergodic measure. By Corollary 6.1.6, $\phi(x, a)$ is universally measure zero. □

Hence, we get the desired counterexample:

Theorem 6.3.4. *There are ω-categorical supersimple theories T of finite SU-rank with a formula $\phi(x, a)$ which does not fork over the empty set, but which is universally measure zero.*

Furthermore, by Lemma 6.2.6, we get another proof that this structure yields a counterexample to the question of Elwes and Macpherson [19].

Remark 6.3.5. The point at the heart of our proof is that it is possible to build simple ω-categorical Hrushovski constructions which do not satisfy the strong independence theorem. This can be done in different relational languages and with different ranks for the final structure. For example, using a 3-hypergraph, we can build a simple ω-categorical Hrushovski construction of SU-rank 1 with non-forking formulas which are universally measure zero.

Our result makes substantial use of the fact we work with not one-based ω-categorical structures. In fact, simple ω-categorical structures with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$ satisfy the independence theorem over the empty set. And in a one-based structure, satisfying this is equivalent to satisfying the strong independence theorem. Our counterexample relies on our simple structure not satisfying the strong independence theorem. For this to be possible we must work with a not one-based structure.

7
Measures and higher amalgamation properties

In this chapter, we explore the relation between measures and higher amalgamation properties. For this, we focus on three different 3-hypergraphs which exhibit different interesting features from the point of view of MS-measurability or their invariant Keisler measures: the generic tetrahedron-free 3-hypergraph, the universal homogeneous two-graph, and another class of ω-categorical Hrushovski constructions.

The initial motivation for this chapter was investigating whether the transfer properties from Chapter 3 generalise for the measures of higher amalgamations. In particular, we had proved that in an ω-categorical structure, if $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{a}} b$, then for μ an invariant Keisler measure,

$$\mu(\phi(x,a) \wedge \psi(x,b))$$

only depends on the individual types of a and b, but not on the type of the pair. Hence, we were interested in whether this generalised to a higher dimensional case as follows:

Question 7.0.1. Let $\mathcal{M}$ be ω-categorical and simple. Given independent parameters a, b, c, do we have that for formulas $\phi(w, ab), \psi(w, bc), \chi(w, ac)$ and an invariant Keisler measure μ,

$$\mu(\phi(w, ab) \wedge \psi(w, bc) \wedge \chi(w, ac)) \tag{7.1}$$

only depends on the types of the pairs ab, bc, ac and not on the type of the triplet?

When the three formulas do not fork over $\emptyset$ and their intersection has realisations which are independent from abc, we call the measure of a formula of the form of 7.1 a measure of an independent 4-amalgamation. More generally, in this chapter, we are interested in the measures of what we will call higher (independent) amalgamations.

A positive answer to Question 7.0.1 would have given us powerful tools to compute the measures of many conjunctions of formulas for which the tools of Chapter 3 are insufficient. Furthermore, in Chapter 6, using the transfer of independence results of Chapter 3, we managed to draw a clear connection between the strong independence theorem and the forking ideal coinciding with the universally measure zero ideal. It is reasonable to wonder whether we may be able to draw a similar connection between higher amalgamation properties and $\mathcal{O}(\emptyset) = F(\emptyset)$. Finally, another motivation for investigating Question 7.0.1 is the connection of stability of $\mu(\phi(x, a) \wedge \psi(x, b)) = \alpha$ with regularity lemmas. In particular, the fact that when a and b are independent the measure of the conjunction $\phi(x, a) \wedge \psi(x, b)$ only depends on $\mathrm{tp}(a)$ and $\mathrm{tp}(b)$ has already been fruitful in proving and generalising via model theoretic methods Tao's algebraic regularity lemma [63] in the work of [55] and [28, Theorem 6.4]. Recently, Chevalier and Levi [14] managed to generalise these results precisely by using the fact that a version of what we ask in Question 7.0.1 is true for a natural definable measure in ACFA (see the Stationarity Theorem of [14, Corollary 4.1.3], and the related discussion in the introduction to the article).

Unfortunately, the answer to Question 7.0.1 is negative, as we will see with the case of the universal homogeneous two-graph in Section 7.4.2. This points to an essential disanalogy from the binary to the ternary case.

In spite of Question 7.0.1 being answered in the negative, we still have powerful tools to compute the measures of higher amalgamations, especially in an MS-measurable context. We will see this in Section 7.2, where we give explicit equations that must be satisfied by the measure of an independent 4-amalgamation. This will allow us to explore some remaining questions about MS-measurable structures.

Our counterexample to the question of Elwes and Macpherson yielding an ω-categorical supersimple structure which is not MS-measurable relied heavily on ω-categorical Hrushovski constructions not being one-based and thus allowing for the strong independence theorem to fail. Hence, it is reasonable to ask whether every ω-categorical one-based supersimple structure is MS-measurable:

Question 7.0.2. Does being ω-categorical supersimple and one-based imply MS-measurability?

Note that every supersimple one-based ω-categorical structure has finite rank by [26].

The main result of this chapter is a negative answer to the revised question of Elwes and Macpherson. In fact we show in Section 7.3 that the generic tetrahedron-free 3-hypergraph is not MS-measurable:

Theorem 7.3.9. *The generic tetrahedron-free 3-hypergraph is not MS-measurable.*

This structure is supersimple, of SU-rank 1, one-based, homogeneous, with trivial algebraic closure and trivial non-forking independence, and still fails to be MS-measurable. With such a plethora of nice properties, it provides a negative answer to most reasonable refinements to the question of Elwes and Macpherson.

Furthermore, showing that the generic tetrahedron-free 3-hypergraph is not MS-measurable implies also that it cannot be elementarily equivalent to a non-principal ultraproduct of an asymptotic class of structures in the sense of [50, 19]. This gives us some interesting information regarding the question of Cherlin of whether this structure is pseudofinite [10].

The failure of MS-measurability in the generic tetrahedron-free 3-hypergraph points to a connection between MS-measurability and higher amalgamation properties. In particular, our proof relies on the failure of independent 4-amalgamation in this structure. However, as we shall see, this connection is much less clear-cut than that between independent 3-amalgamation (i.e. the independence theorem), or the strong independence theorem and MS-measurability. In particular, in Section 7.4, we study the case of the universal homogeneous two-graph. This structure is ω-categorical and MS-measurable [50]. However, independent 4-amalgamation over $\emptyset$ fails for it.

While studying the case of the universal homogeneous two-graph, we discovered that its space of invariant Keisler measures has some truly peculiar properties: it consists of a single invariant Keisler measure, which is not an invariant type in spite of $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$.[1]

Theorem 7.4.11. *There is a unique invariant Keisler measure μ on the universal homogeneous two-graph $\mathcal{G}$ in the singleton variable. For $d, a_1, \dots, a_n \in G$ distinct,*

[1]Without the requirement that $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$ there are many trivial examples of ω-categorical structures with no invariant types but with invariant Keisler measures. Consider the countable model $\mathcal{M}$ of the theory of an equivalence relation with two classes. This theory has no $\mathrm{Aut}(M)$-invariant types in $S_1(M)$: any type over the model is in one of the two equivalence classes and there is an automorphism switching them. However, there is an invariant Keisler measure assigning value $1/2$ to each equivalence class. The natural thing to do in this context would be to consider $\mathrm{Aut}(M/\mathrm{acl}^{eq}(\emptyset))$-invariant types and measures. Then, there are two $\mathrm{Aut}(M/\mathrm{acl}^{eq}(\emptyset))$-invariant types in $S_1(M)$, one for each equivalence class and we can view the $\mathrm{Aut}(M)$-invariant Keisler measure as their weighted sum where the two types are given equal weight $1/2$. Meanwhile, any weighted sum (with weights adding up to 1) of the two aforementioned types yields an $\mathrm{Aut}(M/\mathrm{acl}^{eq}(\emptyset))$-invariant Keisler measure. It is worth noting that in a NIP ω-categorical structure with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$ if there are no invariant types there are also no invariant measures [38, 4].

we have that

$$\mu(d/a_1 \dots a_n) = \left(\frac{1}{2}\right)^{n-1},$$

and $\mu(d) = 1$.

We also note that this result is directly related to Jahel's result that the universal homogeneous two-graph is uniquely ergodic [40]. This is also highly unusual since, in most other ω-categorical structures, being uniquely ergodic has no consequences for the size of the space of invariant Keisler measures.

While the connection between higher amalgamation properties and MS-measurability is still mysterious, one might hope that satisfying strong enough amalgamation properties (such as the strong independence theorem and independent n-amalgamation for all n) might be sufficient to imply MS-measurability. Indeed, in the case where algebraic closure and non-forking independence are trivial, we do know that satisfying independent n-amalgamation for all $n \in \mathbb{N}$ implies pseudofiniteness [46]. Whether MS-measurability can be implied by strong enough amalgamation properties is particularly relevant to the context of ω-categorical Hrushovski constructions. In fact, while we proved that some of these are not MS-measurable by failing to satisfy the strong independence theorem, we can also prove that the standard example of an ω-categorical supersimple Hrushovski construction does satisfy the strong independence theorem. As we will see in Appendix 9 (Remark 9.1.14), the standard choice of growth for f which implies simplicity also implies that the strong independence theorem holds. Hence, one may still hope that some ω-categorical Hrushovski constructions may be MS-measurable answering negatively the question of Elwes and Macpherson of whether an ω-categorical MS-measurable structure is one-based.

We conclude the chapter with another ω-categorical Hrushovski construction which satisfies the strong independence theorem and independent n-amalgamation for all n, but is not MS-measurable. Section 7.5 introduces this structure and proves non-MS-measurability.

The structure of this chapter is as follows. Firstly, we begin with Section 7.1 where we introduce higher amalgamation properties. Then, in Section 7.2, we obtain formulas for the measures of independent 4-amalgamations in an ω-categorical MS-measurable context. The main result is Theorem 7.2.1, which will be used repeatedly in later sections together with Corollary 7.2.2, which under some additional assumptions allows us to obtain formulas in a similar style to those we obtained for independent 3-amalgamations in Theorem 3.1.9. At this point, we are ready to study the three ternary structures which are the focus of this chapter. We begin with the generic tetrahedron-free 3-hypergraph in Section 7.3 and prove it is not MS-measurable. Then, we will move to the universal homogeneous two-graph in Section 7.4. In Subsection 7.4.1, we prove the folklore fact that this structure has trivial non-forking independence. In Subsection 7.4.2, we prove Theorem 7.4.11 and discuss the peculiar behaviour of the invariant Keisler measure in this structure. Finally, we conclude with Section 7.5 where we give an example of an ω-categorical Hrushovski construction which is not MS-measurable in spite of satisfying the strong independence theorem and independent n-amalgamation for all n over finite algebraically closed sets.

7.1 Higher independent amalgamation properties

In this section, we give a self-contained introduction to higher amalgamation properties. As we shall see, higher amalgamation properties correspond to satisfying a generalised version of the independence theorem, and so we will speak of the two interchangeably. Various important examples of simple theories satisfy independent n-amalgamation over models for all $n \in \mathbb{N}$, including $ACFA$ [34]. However, the first two simple structures which we will be studying in this chapter fail in satisfying independent 4-amalgamation over $\emptyset$. Hence, we will develop our study by focusing on whether a given set of complete types satisfies an independent amalgamation property, rather than focusing on whether the whole theory does. For our discussion, we adapt some of the exposition of [23] and [46].

Definition 7.1.1. Let T be simple and $a_1, \dots, a_n$ and A be parameters from $\mathbb{M}$ for $n \geq 2$. We write

$$\mathop{\downarrow}_{A} \{a_1, \dots, a_n\}$$

to indicate that $a_1, \dots, a_n$ form an independent set over A, that is, for any $i \in \{1, \dots, n\}$,

$$a_i \mathop{\downarrow}_{A} \{a_j | j \neq i, j \leq n\}.$$

We say that a complete type over A, $p(x_1, \dots, x_n)$ is **independent** over A if for some/any $a_1, \dots, a_n$ such that $\models p(a_1, \dots, a_n)$ we have that $\downarrow_A \{a_1, \dots, a_n\}$.

Remark 7.1.2. In Definition 7.1.1, note that if any tuple satisfying p is not independent over A, then every tuple satisfying p is not independent over A. Hence, being independent over a given set is a property of a complete type. This follows from $\mathrm{Aut}(\mathbb{M})$-invariance of independence.

Remark 7.1.3. Working in a simple theory, given a tuple $a_1, \dots, a_n$ independent over A, by monotonicity, transitivity and symmetry of non-forking independence we can obtain that for any two I, J disjoint subsets of $\{1, \dots, n\}$

$$(a_i | i \in I) \mathop{\downarrow}_{A} (a_j | j \in J).$$

Moreover, given the aforementioned properties of non-forking independence in simple theories, the independence condition in the definition can be replaced with saying for all $i \leq n$,

$$a_i \mathop{\downarrow}_{A} (a_j | j < i).$$

Below, we write $[n]$ for $\{1, \dots, n\}$, for $n \geq 3$ and given the variables $x_1, \dots, x_n$ and $I \subseteq [n]$, we write x_I for $\{x_i | i \in I\}$. Furthermore, given $J \subseteq I \subseteq [n]$ and given a type $p_I(x_I)$ we write its restriction to formulas in the variables from J as $p_I|_J$. Finally, for I a subset of $[n]$ of size k, we write $I \in [n]^k$.

Definition 7.1.4. For $I \in [n]^{n-1}$, let $p_I(x_I)$ be a family of independent types over A. Suppose further that for $I, J \in [n]^{n-1}$, we have that $p_I|_{I\cap J} = p_J|_{I\cap J}$. We say that $\{p_I(x_I)|I \in [n]^{n-1}\}$ is an ***n*-independent amalgamation problem** over A. A **solution** to an ***n*-independent amalgamation problem** over A is a type $p(x_{[n]})$ independent over A such that for each $I \in [n]^{n-1}$, $p|_I = p_I$. When a solution exists, we say that $\{p_I(x_I)|I \in [n]^{n-1}\}$ satisfy **independent *n*-amalgamation** over A. A simple theory T satisfies **independent *n*-amalgamation** over A if every n-independent amalgamation problem over A has a solution.

Definition 7.1.5. Let $\mathop{\smile\!\!\!\!\!\!|}_A \{a_1, \dots, a_{n-1}\}$. For each $I, J \in [n-1]^{n-2}$, let d_I be such that

$$d_I \mathop{\smile\!\!\!\!\!\!|}_A \{a_i|i \in I\} \text{ and } d_I \equiv_{A\{a_i|i\in I\cap J\}} d_J. \tag{7.2}$$

We say that $\{\mathrm{tp}(d_I/\{a_i|i \in I\}A)|I \in [n-1]^{n-2}\}$ satisfy the ***n*-generalised independence theorem** (n-GIT), over A if there is d such that $d \mathop{\smile\!\!\!\!\!\!|}_A a_1 \dots a_{n-1}$ and $d \equiv_{A\{a_i|i\in I\}} d_I$ for each $I \in [n-1]^{n-2}$. We say that T satisfies the ***n*-generalised independence theorem** over A if for any $a_1, \dots, a_{n-1}, \{d_I|I \in [n-1]^{n-2}\}$ satisfying the conditions in 7.2, $\{\mathrm{tp}(d_I/\{a_i|i \in I\}A)|I \in [n-1]^{n-2}\}$ satisfy the n-generalised independence theorem over A.

Remark 7.1.6. Note that for $\{\mathrm{tp}(d_I/\{a_i|i \in I\}A)|I \in [n-1]^{n-2}\}$ to satisfy the n-generalised independence theorem over A is equivalent to $\{\mathrm{tp}(d_I\{a_i|i \in I\}/A)|I \in [n-1]^{n-2}\} \cup \mathrm{tp}(a_1, \dots, a_{n-1}/A)$ satisfying independent n-amalgamation over A. Furthermore, the generalised 3-independence theorem over A is just the independence theorem over A in the sense of Definition 1.2.14. Hence, a simple theory always satisfies 3-GIT over models and independent 3-amalgamation over models.

Example 7.1.7. In this thesis, we are particularly interested in independent 4-amalgamation (or equivalently 4-GIT) over the emptyset. Consider an independent triplet $\mathop{\smile\!\!\!\!\!\!|} \{a_1, a_2, a_3\}$. Suppose that for each $\{i, j\} \in [3]^2$ there is d_{ij} such that $d_{ij} \mathop{\smile\!\!\!\!\!\!|} a_i a_j$ and for $\{i, j\}, \{i, k\} \in [3]^2$ with $j \neq k$, $d_{ij} \equiv_{a_i} d_{ik}$. From Definition

7.1.5, we say that the types $\mathrm{tp}(d_{ij}/a_ia_j)$ for $\{i,j\} \in [3]^2$ satisfy the **generalised 4-independence theorem** (4-GIT) over $\emptyset$ if there is d such that

$$d \mathop{\smile\!\!\!\!\!\!\;\mid}\, a_1a_2a_3, \quad d \equiv_{a_ia_j} d_{ij}.$$

There are various examples of simple (and supersimple) theories which do not satisfy independent 4-amalgamation (over the empty set or over models). Later in this chapter, we will study the case of the generic tetrahedron-free 3-hypergraph $\mathcal{H}$. This is the Fraïssé limit of all finite 3-hypergraphs omitting a tetrahedron, that is four vertices $abcd$ such that the ternary hypergraph relation $R(x,y,z)$ holds of each triplet. This structure is ω-categorical, supersimple of SU-rank 1 and with quantifier elimination. Given a_1, a_2, a_3 distinct vertices in $\mathcal{H}$ such that $\models R(a_1,a_2,a_3)$ and d_{12}, d_{13}, d_{23} such that $R(d_{12},a_1,a_2), R(d_{13},a_1,a_3), R(d_{23},a_2,a_3)$ we have that the types $\mathrm{tp}(d_{ij}/a_ia_j)$ for $1 \leq i < j \leq 3$ do not satisfy 4-GIT over $\emptyset$. However, choosing $a'_1a'_2a'_3$ such that $\models \neg R(a'_1,a'_2,a'_3)$, we have that given d_{12}, d_{13}, d_{23} such that $R(d_{12},a'_1,a'_2), R(d_{13},a'_1,a'_3), R(d_{23},a'_2,a'_3)$ the types $\mathrm{tp}(d_{ij}/a'_ia'_j)$ for $1 \leq i < j \leq 3$ do satisfy 4-GIT over $\emptyset$, since the resulting hypergraph contains no tetrahedron.

7.2 Measures of higher amalgamations

We are interested in studying the measures of higher amalgamations, especially independent 4-amalgamations. Consider an independent triplet $\mathop{\smile\!\!\!\!\!\!\;\mid}\, \{a_1,a_2,a_3\}$. Suppose that for $i < j \in \{1,2,3\}$ there are d_{ij} such that

$$d_{ij} \equiv_{a_i} d_{ik}, \quad d_{ij} \mathop{\smile\!\!\!\!\!\!\;\mid}\, a_ia_j.$$

We want to study the measure

$$\mu\left(\mathrm{tp}(d_{12}/a_1a_2) \cup \mathrm{tp}(d_{13}/a_1a_3) \cup \mathrm{tp}(d_{23}/a_2a_3)\right)$$

in an MS-measurable ω-categorical context. We may write $\phi(w,x,y), \psi(w,x,z)$, and $\chi(w,y,z)$ for the formulas isolating (respectively) $\mathrm{tp}(d_{12},a_1,a_2), \mathrm{tp}(d_{13},a_1,a_3)$, and $\mathrm{tp}(d_{23},a_2,a_3)$. Let

$$p_1(x,y,z), \ldots, p_n(x,y,z)$$

be the complete types such that for $i \leq n$, $a_1^i a_2^i a_3^i \vDash p_i(x,y,z)$ implies that $\downarrow \{a_1^i a_2^i a_3^i\}$ and that $\phi(w, a_1^i, a_2^i), \psi(w, a_1^i, a_3^i), \chi(w, a_2^i, a_3^i)$ satisfy 4-GIT over the empty set. As discussed in the previous section, this property only depends on the type of the triplet p_i.

Finally, we set the notation that

$$S(w,x,y,z) := \phi(w,x,y) \wedge \psi(w,x,z) \wedge \chi(w,y,z).$$

We would like to compute the measure of $S(w, a_1, a_2, a_3)$ where $a_1a_2a_3$ is an independent triplet for which $\phi(w, a_1, a_2), \psi(w, a_1, a_3), \chi(w, a_2, a_3)$ satisfy 4-GIT over $\emptyset$. Ideally, we would like to recover this measure from the measures of the types of different triplets, mimicking how in Theorem 3.1.9 we managed to express the measure of the type of a triplet in terms of the measures of the types of the pairs composing it. While this cannot be done in the general case, we can still recover some information.

Theorem 7.2.1. *Let $\mathcal{M}$ be MS-measurable ω-categorical with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Let $\downarrow \{a_1, a_2, a_3\}$ be an independent triplet such that $\phi(w, a_1, a_2), \psi(w, a_1, a_3), \chi(w, a_2, a_3)$ satisfy 4-GIT over the empty set. Let $d \downarrow a_1a_2a_3$ be such that $d \vDash S(w, a_1, a_2, a_3)$. Suppose further that $\mathrm{acl}^{eq}(d) = \mathrm{dcl}^{eq}(d)$. Let $p_1(x,y,z), \ldots, p_n(x,y,z)$ be as above. Then,*

$$\sum_{\substack{a_1^i a_2^i a_3^i \vDash p_i(x,y,z) \\ i \leq n}} \mu(S(w, a_1^i a_2^i a_3^i))\mu(a_1^i a_2^i a_3^i) = \frac{\mu(da_1a_2)\mu(da_1a_3)\mu(da_2a_3)\mu(d)}{\mu(da_1)\mu(da_2)\mu(da_3)}.$$

Proof. This proof relies again on considering the measure of $S = S(w,x,y,z)$ in two different ways. Firstly, we consider the projection onto the coordinates x, y, z, given by $\pi : S(M^4) \to R(M^3)$, where $R(x,y,z)$ is the formula isolating $\mathrm{tp}_{xy}(a_1a_2) \cup \mathrm{tp}_{xz}(a_1a_3) \cup \mathrm{tp}_{yz}(a_2a_3)$. The triplets $a_1'a_2'a_3'$ whose type is assigned maximal d-dimension are precisely the independent triplets since by additivity of the dimension,

$$d(a_1'a_2'a_3') = d(a_1'/a_2'a_3') + d(a_2a_3),$$

and $d(a_1'/a_2'a_3') = d(a_1'/a_2')$ if and only if $a_1' \mathop{\smile\!\!\!\!\!\!|}\, a_2'a_3'$ since d-independence agrees with non-forking independence by Theorem 2.2.20. By the independence theorem over the empty set, Fact 3.1.8, such triplets do exist. Meanwhile, the fibers of maximal dimension coming from independent triplets are given exactly by $S(w, a_1^i a_2^i a_3^i)$ since these are the triplets for which this formula does not fork over $\emptyset$. Hence, by Fubini we have that

$$\mu(S) = \sum_{\substack{a_1^i a_2^i a_3^i \models p_i(x,y,z) \\ i \le n}} \mu(S(w, a_1^i a_2^i a_3^i))\mu(a_1^i a_2^i a_3^i). \tag{7.3}$$

Now, we project onto w, x, y. Firstly, note that $\exists z S(w, x, y, z)$ isolates $\mathrm{tp}(da_1a_2)$. Hence, for $d'a_1'a_2' \models \exists z S(w, x, y, z)$, $\mu(S(d', a_1', a_2', z))$ is constant for any such triplet and equals $\mu(S(d, a_1, a_2, z))$. By Fubini,

$$\mu(S) = \mu(da_1a_2)\mu(S(d, a_1, a_2, z)). \tag{7.4}$$

Now, notice that

$$\mu(S(d, a_1, a_2, z)) = \mu(\psi(d, a_1, z) \wedge \chi(d, a_2, z)),$$

and using Theorem 3.1.9, and the fact that $\mathrm{acl}^{eq}(d) = \mathrm{dcl}^{eq}(d)$,

$$\mu(\psi(d, a_1, z) \wedge \chi(d, a_2, z)) = \frac{\mu(a_3/a_1d)\mu(a_3/a_2d)}{\mu(a_3/d)} = \frac{\mu(da_1a_3)\mu(da_2a_3)\mu(d)}{\mu(da_1)\mu(da_2)\mu(da_3)},$$

where the last equation is obtained by applying Fubini. Hence, comparing equations 7.3 and 7.4, we get the desired result.

□

Both in this and in the following result we only make use of the equivalence of dimension-independence and non-forking independence of Theorem 2.2.20. In particular, we do not need the full dimension theorem of Theorem 2.2.25.

Corollary 7.2.2. *Suppose that in the context of Theorem 7.2.1, $\mathrm{tp}_{xy}(a_1a_2) \cup \mathrm{tp}_{xz}(a_1a_3) \cup \mathrm{tp}_{yz}(a_2a_3)$ entirely determines a unique independent complete type for the triplet of parameters. Then, we have that*

$$\mu(S(w, a_1, a_2, a_3)) = \frac{\mu(d/a_1a_2)\mu(d/a_1a_3)\mu(d/a_2a_3)\mu(d)}{\mu(d/a_1)\mu(d/a_2)\mu(d/a_3)}. \tag{7.5}$$

Proof. In this case, for the first part of our calculations in Theorem 7.2.1, from equation 7.3, we have that

$$\mu(S) = \mu(S(w, a_1, a_2, a_3))\mu(\exists w S(w, x, y, z)).$$

However, by the lower dimensional case of Corollary 3.1.10,

$$\mu(\exists w S(w, x, y, z)) = \mu\left(\mathrm{tp}(a_1 a_2) \cup \mathrm{tp}(a_1 a_3) \cup \mathrm{tp}(a_2 a_3)\right) = \frac{\mu(a_1 a_2)\mu(a_1 a_3)\mu(a_2 a_3)}{\mu(a_1)\mu(a_2)\mu(a_3)}.$$

Comparing equations one gets the desired result. $\square$

Remark 7.2.3. Note, in order to obtain equation 7.5 with the assumptions of Theorem 7.2.1, the following (weaker) assumption is sufficient: for every independent completion $\mathrm{tp}(a'_1 a'_2 a'_3)$ of $\mathrm{tp}_{xy}(a_1 a_2) \cup \mathrm{tp}_{xz}(a_1 a_3) \cup \mathrm{tp}_{yz}(a_2 a_3)$ we have that:

- the types isolated by $\phi(w, a'_1 a'_2), \psi(w, a'_1 a'_3), \chi(x, a'_2 a'_3)$ satisfy 4-GIT; and
- the measure $\mu(S(w, a'_1 a'_2 a'_3))$ is constant (i.e. the measure only depends on the type of the pairs and not on the type of the independent triplet).

7.3 Non-MS-measurability of the generic tetrahedron-free 3-hypergraph

We work with the countable generic tetrehedron-free 3-hypergraph $\mathcal{H}$. This is the Fraïssé limit of all finite 3-hypergraphs which avoid a tetrahedron, i.e. a set of four vertices T such that for any three distinct $a, b, c \in T$, $R(a, b, c)$ holds, where $R(x, y, z)$ is the ternary hyperedge relation. This structure is homogeneous and ω-categorical, and so has quantifier elimination. It is supersimple [16], of SU-rank 1 with a trivial notion of non-forking independence in the sense that for $A, B, C \subseteq \mathcal{H}$,

$$A \underset{C}{\perp} B \text{ if and only if } A \cap C = A \cap (B \cup C).$$

Furthermore, $\mathcal{H}$ has degenerate algebraic closure, i.e. for any $A \subseteq \mathcal{H}, \mathrm{acl}(A) = A$, and is one-based [45]. Moreover, this structure is such that for any $A \subseteq \mathcal{H}, \mathrm{acl}^{eq}(A) = \mathrm{dcl}^{eq}(A)$ [45, Corollary 6.2].

The aim of this section is showing how the generic tetrahedron-free 3-hypergraph is not MS-measurable. By showing this, we are providing a counterexample to the conjecture of Elwes and MacPherson and to many reasonable refinements of it. Before this result, the only known supersimple ω-categorical structures of finite rank which are not MS-measurable were Hrushovski constructions, which are not one-based and have non-trivial algebraic closure. However, showing that the generic tetrahedron-free 3-hypergraph is not MS-measurable implies that even adding to the original conjecture assumptions as strong as being one-based, with trivial algebraic closure and of SU-rank 1, we still have examples of supersimple ω-categorical structures which are not MS-measurable.

Another reason for being interested in our result is that it is currently an open question of Cherlin [10] whether any generic $K^r_{(r+1)}$-free r-hypergraph (for $r \geq 2$) is pseudofinite, where by $K^r_{(r+1)}$ we mean the complete r-hypergraph on $r+1$-vertices. While our result does not settle the question in either direction, it does show that the generic tetrahedron-free 3-hypergraph is not elementary equivalent to the ultraproduct of an n-dimensional asymptotic class. We already knew this is the case for the generic triangle-free graph since it is not simple. We expect that similar techniques to the ones discussed here can show that no $K^r_{(r+1)}$-free r-hypergraph for $r \geq 2$ is MS-measurable.

Notation 7.3.1. We let a, b, c, d be distinct vertices in $\mathcal{H}$ such that

$$\models R(a, b, d) \wedge R(a, c, d) \wedge R(b, c, d) \wedge \neg R(a, b, c).$$

In particular, a, b, c form three vertices such that $\models \neg R(a, b, c)$, while abd are such that $\models R(a, b, d)$. The diagram is included in Figure 7.1 for clarity.

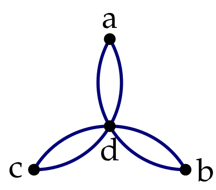


Figure 7.1: A visual representation of the 3-hypergraph formed by a, b, c, d. We depict the hyperedges by a navy blue line between three vertices. This would appear as black on black and white print. The lack of such a line between abc indicates that there is no hyperedge containing those three vertices.

Remark 7.3.2. Note that $\operatorname{tp}(abcd)$ is entirely determined by $\operatorname{tp}(abd), \operatorname{tp}(acd), \operatorname{tp}(bcd)$. In fact, since each of these triplets forms a hyperedge and since our structure avoids tetrahedrons

$$\models (R(a,b,d) \wedge R(a,c,d) \wedge R(b,c,d)) \to \neg R(a,b,c).$$

This will be relevant for applying Corollary 7.2.2. Suppose for a contradiction that $\mathcal{H}$ is MS-measurable. Let $e \downarrow abcd$. We want to compute the measure

$$\mu(\operatorname{tp}(e/abd) \cup \operatorname{tp}(e/acd) \cup \operatorname{tp}(e/bcd)).$$

Since there is only one complete 3-type over d satisfying

$$\operatorname{tp}(ab/d) \cup \operatorname{tp}(ac/d) \cup \operatorname{tp}(bc/d),$$

we can use Corollary 7.2.2 to get that

$$\mu(\operatorname{tp}(e/abd) \cup \operatorname{tp}(e/acd) \cup \operatorname{tp}(e/bcd)) = \frac{\mu(e/abd)\mu(e/acd)\mu(e/bcd)\mu(e/d)}{\mu(e/ad)\mu(e/bd)\mu(e/cd)}. \tag{7.6}$$

Note that we use Corollary 7.2.2 and Theorem 7.2.1 over the vertex d. This can be done safely since adding parameters preserves MS-measurability and since $\operatorname{acl}^{eq}(A) = \operatorname{dcl}^{eq}(A)$ for any $A \subseteq H$.

Notation 7.3.3. Let $N(w,x,y,z)$ be the formula asserting that x, y, z, w are distinct. We have the following notations:

$$\Phi_0(w,x,y,z) := N(w,x,y,z) \wedge \neg R(w,x,y) \wedge \neg R(w,x,z) \wedge \neg R(w,y,z)$$

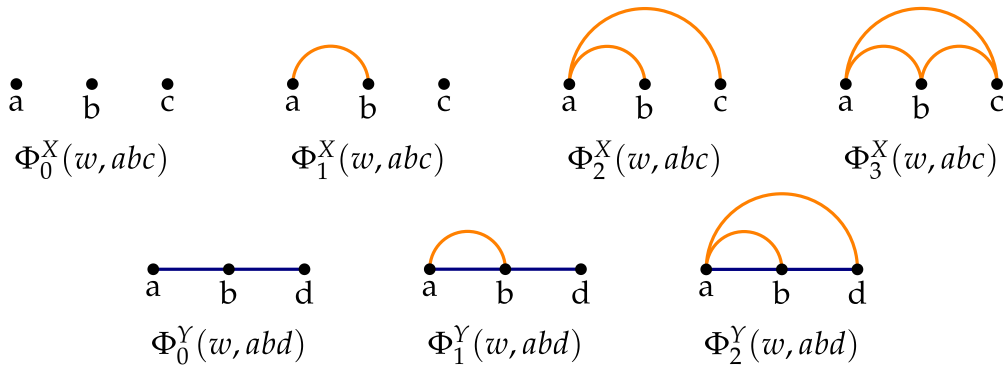


Figure 7.2: We use the following notation for formulas of the form $\Psi(w, A)$ isolating complete types (with all variables being distinct): the vertices of the parameter set A are drawn in black with hyperedges between them in navy blue. The latter would also look black in black & white print. Meanwhile, if $\Psi(w, A)$ asserts that $R(w, e, f)$ for parameters $e, f \in A$ we draw a line in orange between e and f. This would be light grey in black and white print. Conversely, if two verteces of the base have no orange line between them, we can deduce that Ψ asserts that the two vertices do not form a hyperedge with w. Clearly, while the depiction of the base of parameters represents visually the hypergraph, the same is not true for how we represent whether a pair of vertices in the parameter set forms a hyperedge with w. This is done to avoid the visual confusion that would result from depicting many hyperedges overlapping on the same vertex.

$$\Phi_1(w, x, y, z) := N(w, x, y, z) \wedge R(w, x, y) \wedge \neg R(w, x, z) \wedge \neg R(w, y, z)$$

$$\Phi_2(w, x, y, z) := N(w, x, y, z) \wedge R(w, x, y) \wedge R(w, x, z) \wedge \neg R(w, y, z)$$

$$\Phi_3(w, x, y, z) := N(w, x, y, z) \wedge R(w, x, y) \wedge R(w, x, z) \wedge R(w, y, z).$$

For $0 \leq i \leq 3$, we write $\Phi_i^X(w, x, y, z)$ for $\Phi_i(w, x, y, z) \wedge \neg R(x, y, z)$ and $\Phi_i^Y(w, x, y, z)$ for $\Phi_i(w, x, y, z) \wedge R(x, y, z)$. Note that $\Phi_3^Y(w, x, y, z)$ is inconsistent since it asserts that the various vertices form a tetrahedron. By quantifier elimination, all of the other formulas isolate complete 4-types since they specify the quantifier-free type of each triplet of variables in a ternary language. Finally, for $0 \leq i \leq 3$, we let

$$x_i = \mu(\Phi_i^X(w, a, b, c)) \text{ and } y_i = \mu(\Phi_i^Y(w, a, b, d)),$$

recalling that we chose a, b, c, d to be such that $\vDash \neg R(a, b, c)$ and $\vDash R(a, b, d)$. In Figure 7.2, we illustrate visually the formulas $\Phi_i^X(w, a, b, c)$ and $\Phi_i^Y(w, a, b, d)$ setting up some visual notation which will facilitate computations.

Lemma 7.3.4. *Suppose that $\mathcal{H}$ is MS-measurable. Assume that $\mu(x = x) = 1$ and that for distinct $a, b \in H$, $\mu(R(w, a, b)) = s$. The following equations hold:*

$$y_2 = s^2 \tag{7.7}$$

$$y_1 = s(1-2s) \tag{7.8}$$

$$y_0 = 1 - 3s + 3s^2 \tag{7.9}$$

Proof. Firstly, note that any two vertices and any two pairs of vertices have the same type over $\emptyset$ by homogeneity and the fact that the language is ternary. So, for $e, f \in \mathcal{H}$, we have $\mu(e) = \mu(ef) = 1$. We also recall that by quantifier elimination, $R(w, x, y)$ and $\neg R(w, x, y)$ isolate only one independent type of a triplet each.

We begin by considering equation 7.7. We have that $\mathrm{acl}^{eq}(a) = \mathrm{dcl}^{eq}(a)$ and $b \perp_a d$. Let $e \perp abd$ be such that $e \vDash \mu(R(e, a, b) \wedge R(e, a, d))$.

By quantifier elimination $\mathrm{tp}(e/ab)$ and $\mathrm{tp}(e/ad)$ are determined entirely by $\vDash R(e, a, b)$ and $\vDash R(e, a, d)$. Hence, by Theorem 3.1.9,

$$\begin{aligned} \mu(R(w, ab) \wedge R(w, a, d)) &= \mu(\mathrm{tp}(e/ab) \cup \mathrm{tp}(e/ad)) \\ &= \frac{\mu(e/ab)\mu(e/ac)}{\mu(e/a)} \\ &= s^2. \end{aligned}$$

For the last line of the equation note that $\mu(e/ab) = \mu(e/ad) = s$ and $\mu(e/a) = 1$. However, since $\mathcal{H}$ avoids tetrahedrons, we have that

$$\vDash \forall w R(w, ab) \wedge R(w, ad) \rightarrow \neg R(w, bd).$$

Hence, as desired,

$$y_2 = \mu(R(w, a, b) \wedge R(w, a, d) \wedge \neg R(w, b, d)) = \mu(R(w, a, b) \wedge R(w, a, d)) = s^2.$$

For 7.8, note that the extensions of $R(w, a, b) \wedge \neg R(w, a, d)$ to a complete type over abd are isolated by $\Phi_2(w, b, a, d)$ and $\Phi_1(w, a, b, d)$ respectively (i.e. either w has an hyperedge with bd or it does not). Hence, using Theorem 3.1.9, we have again that

$$s(1-s) = \mu(R(w, a, b) \wedge \neg R(w, a, d)) = \mu(\Phi_1(w, a, b, d)) + \mu(\Phi_2(w, b, a, d)).$$

In particular, this yields

$$y_1 + y_2 = s(1-s). \tag{7.10}$$

Since we know that $y_2 = s^2$, this yields $y_1 = s(1-2s)$ as desired.

Finally, 7.9 follows by using the same technique with respect to $\neg R(w,a,b) \wedge \neg R(w,a,d)$. The two completions are given by $\Phi_0(w,a,b,d)$ and $\Phi_1(w,b,d,a)$, yielding

$$y_0 + y_1 = (1-s)^2. \tag{7.11}$$

Manipulating the equation we get that $y_0 = 1 - 3s + 3s^2$. □

Lemma 7.3.5. *With the same assumptions as Lemma 7.3.4,*

$$x_3 = \frac{s^3}{(1-s)} \tag{7.12}$$

$$x_2 = \frac{s^2(1-2s)}{(1-s)}. \tag{7.13}$$

Proof. We begin with equation 7.12. We want to compute $x_3 = \mu(R(w,ab) \wedge R(w,a,c) \wedge R(w,b,c))$ using Theorem 7.2.1. Note that $\mathrm{tp}(ab) \cup \mathrm{tp}(ac) \cup \mathrm{tp}(bc)$ has two completions, i.e. $\mathrm{tp}(abc)$ and $\mathrm{tp}(abd)$. However, only for the first completion $R(w,a,b) \wedge R(w,ac) \wedge R(w,b,c)$ is consistent, since in the other case w forms a tetrahedron with abd. Hence, from Theorem 7.2.1, we have that

$$x_3(1-s) = s^3,$$

which yields the desired equation.

Finally, equation 7.13 is obtained by considering the extensions of $R(w,a,b) \wedge R(w,a,c)$ to a complete type over abc. The two extensions are $\Phi_3(w,a,b,c)$ and $\Phi_2(w,a,b,c)$. Hence, by Theorem 3.1.9 we have that $x_2 + x_3 = s^2$, which yields the desired result. □

Remark 7.3.6. At this point, we are ready to compute the measures for our main computation showing that the generic tetrahedron-free 3-hypergraph is not MS-measurable. For these, we want to consider the complete extensions of $\Phi_2(w,a,b,c) := N(w,a,b,c) \wedge R(w,a,b) \wedge R(w,a,c) \wedge \neg R(w,b,c)$ to $abcd$, where $abcd$ is defined as in Remark 7.3.1. Firstly, $\vDash \neg\Phi_2(d,a,b,c)$ since d forms a hyperedge with each of the pairs in a,b,c, so all of the extensions of $\Phi_2(w,a,b,c)$ to $abcd$ are non-forking. By quantifier-elimination, these extensions are entirely determined by the relations $R(w,d,f)$ for $f \in \{a,b,c\}$. Furthermore, we cannot have $R(w,a,d) \wedge R(w,d,b)$ nor $R(w,a,d) \wedge R(w,d,c)$ since in either cases w would form a tetrahedron (with adb and adc respectively). This implies that

$$\vDash \forall w(\Phi_2(w,a,b,c) \wedge R(w,a,d) \to \neg R(w,b,d) \wedge \neg R(w,c,d)),$$

and so that $\Phi_2(w,a,b,c) \wedge R(w,a,d)$ determines a complete type. We also have that $\vDash \forall w(\Phi_2(w,a,b,c) \to w \neq d)$ since $\vDash R(d,b,c)$ but $\vDash \forall w(\Phi_2(w,a,b,c) \to \neg R(w,b,c))$. Hence, we can deduce that there are a total of five completions of $\Phi_2(w,a,b,c)$ to a complete type over $abcd$, isolated by the following formulas:

$$\Psi_1(w,a,b,c,d) \equiv \Phi_2(w,a,b,c) \wedge \bigwedge_{f\in\{a,b,c\}} \neg R(w,d,f) \tag{7.14}$$

$$\Psi_2(w,a,b,c,d) \equiv \Phi_2(w,a,b,c) \wedge R(w,a,d) \tag{7.15}$$

$$\Psi_3(w,a,b,c,d) \equiv \Phi_2(w,a,b,c) \wedge R(w,b,d) \wedge \neg R(w,c,d) \tag{7.16}$$

$$\Psi_4(w,a,b,c,d) \equiv \Phi_2(w,a,b,c) \wedge R(w,c,d) \wedge \neg R(w,b,d) \tag{7.17}$$

$$\Psi_5(w,a,b,c,d) \equiv \Phi_2(w,a,b,c) \wedge R(w,c,d) \wedge R(w,b,d). \tag{7.18}$$

These complete types are illustrated in Figure 7.3. Note that each of these types is indeed consistent (and non-forking over $\emptyset$) since none of these formulas gives rise to a tetrahedron.

Remark 7.3.7. By observing Figure 7.3, we can also see that the $\Psi_i(w,a,b,c,d)$ can be decomposed as the conjunction of formulas isolating complete types over abd, acd, and bcd respectively. By inspecting the figures, we have that:

$$\Psi_1(w,a,b,c,d) \equiv \Phi_0^Y(w,b,c,d) \wedge \Phi_1^Y(w,a,c,d) \wedge \Phi_1^Y(w,a,b,d) \tag{7.19}$$

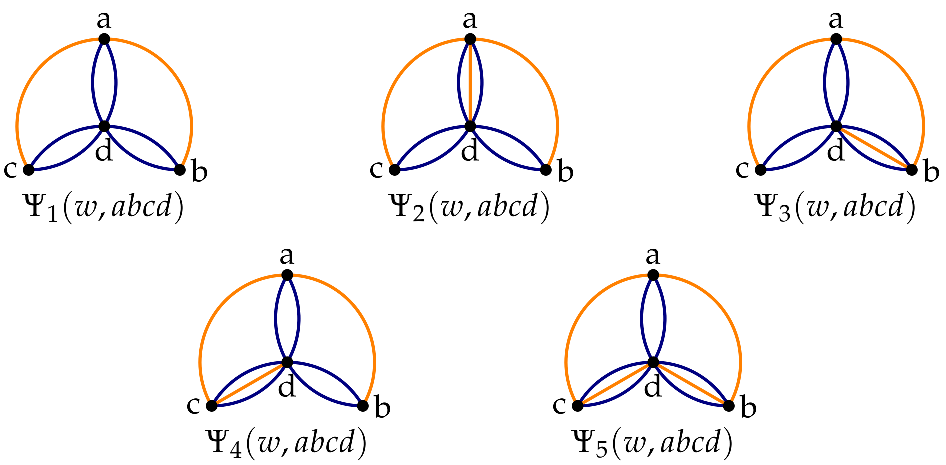


Figure 7.3: Since we are looking at extensions of $\Phi_2(w,a,b,c)$ to abcd, the various completions only depend on the relations $R(w,d,f)$ for $f \in \{a,b,c\}$. We begin with $\Psi_1(w,a,b,c,d)$ asserting that w forms no hyperedge with any of ad, bd, cd. Then, as noted in Remark 7.3.6, there is only one completion with $R(w,a,d)$, namely Ψ_2, for which we also have $\neg R(w,b,d) \wedge \neg R(w,c,d)$. Hence we are only left with considering the cases where we have $R(w,b,d) \wedge \neg R(w,c,d)$, $\neg R(w,b,d) \wedge R(w,c,d)$ and $R(w,b,d) \wedge R(w,c,d)$, which yield Ψ_3, Ψ_4 and Ψ_5 respectively.

$$\Psi_2(w,a,b,c,d) \equiv \Phi_0^Y(w,b,c,d) \wedge \Phi_2^Y(w,a,c,d) \wedge \Phi_2^Y(w,a,b,d) \tag{7.20}$$

$$\Psi_3(w,a,b,c,d) \equiv \Phi_1^Y(w,b,d,c) \wedge \Phi_1^Y(w,a,c,d) \wedge \Phi_2^Y(w,b,d,a) \tag{7.21}$$

$$\Psi_4(w,a,b,c,d) \equiv \Phi_1^Y(w,c,d,b) \wedge \Phi_2^Y(w,c,a,d) \wedge \Phi_1^Y(w,a,b,d) \tag{7.22}$$

$$\Psi_5(w,a,b,c,d) \equiv \Phi_2^Y(w,d,b,c) \wedge \Phi_2^Y(w,c,a,d) \wedge \Phi_2^Y(w,b,a,d) \tag{7.23}$$

Finally, we write b_i for $\mu(\Psi_i(w,a,b,c,d))$.

Lemma 7.3.8. *With the same assumption as Lemma 7.3.4, we have that*

$$b_1 = \frac{y_0 y_1^2}{(1-s)^3} \tag{7.24}$$

$$b_2 = \frac{y_0 y_2^2}{(1-s)^2 s} \tag{7.25}$$

$$b_3 = b_4 = \frac{y_1^2 y_2}{(1-s)^2 s} \tag{7.26}$$

$$b_5 = \frac{y_2^3}{(1-s)s^2} \tag{7.27}$$

Proof. Firstly, note that $b_3 = b_4$ by the automorphism sending $abcd$ to $acbd$. As noted in Remark 7.3.7, we can consider each $\Psi_i(w,a,b,c,d)$ as the conjunction of three complete types over abd, acd, and bcd respectively. As observed in Remark

7.3.2, $\mathrm{tp}(abcd)$ is entirely determined by $\mathrm{tp}(abd) \cup \mathrm{tp}(acd) \cup \mathrm{tp}(cbd)$. Hence, we can apply Corollary 7.2.2 to $\Psi_i(w,a,b,c,d)$. For example, recalling that the measure of every pair of distinct vertices is 1, when considering b_1 we have:

$$\begin{aligned} b_1 &= \frac{\mu(\Phi_0^Y(w,b,c,d))\mu(\Phi_1^Y(w,a,c,d))\mu(\Phi_1^Y(w,a,b,d))}{\mu(\neg R(w,a,d))\mu(\neg R(w,b,d))\mu(\neg R(w,c,d))} \\ &= \frac{y_0 y_1^2}{(1-s)^3}. \end{aligned}$$

Note that the numerator of the equation can be determined directly from the formulas in Remark 7.3.7. Meanwhile, for $0 \leq j \leq 3$, the denominator will be $(1-s)^{3-j}s^j$ where the j is the number of vertices $f \in \{a,b,c\}$ such that $\Psi_i(w,a,b,c,d)$ implies $R(w,f,d)$. These equation can be also read directly by inspection of the illustrations in Figure 7.3. □

Theorem 7.3.9. *The generic tetrahedron-free* 3*-hypergraph is not MS-measurable.*

Proof. We consider the equation resulting from additivity by considering the extensions of $\Phi_2(w,a,b,c)$ to $abcd$. These extensions are discussed in Remark 7.3.6 and illustrated in Figure 7.3. Since $b_3 = b_4$ (as noted in Lemma 7.3.8), we have

$$b_1 + b_2 + 2b_3 + b_5 = x_2.$$

The value of each of these variables is calculated in Lemma 7.3.8. Assuming $0 < s < 1$, the resulting equation simplifies to

$$\frac{s(1-2s+2s^2)}{(1-s)} = 0.$$

This equation has no real solutions for $0 < s < 1$, implying that $\mathcal{H}$ is not *MS*-measurable. □

Corollary 7.3.10. *There are ω-categorical supersimple theories with SU-rank* 1 *which are one-based, homogeneous, with degenerate algebraic closure, trivial non-forking independence and which are not MS-measurable. In particular, the generic tetrahedron-free* 3*-hypergraph is such an example.*

Our proof of Theorem 7.3.9 relied heavily on the assumption of the Fubini property of MS-measurable structures, which was needed in order to use Corollary 7.2.2. It is unclear whether the techniques that we used can be adapted to the context of an ergodic invariant Keisler measure, and it would be very interesting to see whether this could be done. We discuss this further in the conclusion.

7.4 Measures on the the universal homogeneous two-graph

In the previous section, we showed how failure of independent 4-amalgamation over the empty set can yield a failure of MS-measurability in the case of the generic tetrahedron-free 3-hypergraph. However, it is not true that such failure implies not being MS-measurable. Indeed, in this section, we give the example of the universal homogeneous two-graph as a structure which is MS-measurable but where independent 4-amalgamation over the empty set fails (and $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$). This structure shows some interesting differences between the measures of 3 and 4 amalgamations and provides a counterexample to Question 7.0.1. Furthermore, while studying it, we also found out it has various peculiar properties from the point of view of invariant Keisler measures. Indeed, in the main result of this section, Theorem 7.4.11, we show that this structure has a unique invariant Keisler measure, which is not given by an invariant type.

We begin with introducing the universal homogeneous two-graph $\mathcal{G}$ and some of its main properties in Subsection 7.4.1. We prove some basic model theoretic properties which are folklore knowledge. In particular, we prove it has trivial independence in Theorem 7.4.3. From that result, we can also deduce that for finite $A \subset G, \mathrm{acl}^{eq}(A) = \mathrm{dcl}^{eq}(A)$. We can also see that $\mathcal{G}$ does not satisfy independent 4-amalgamation over the empty set in spite of being MS-measurable. Then, in Subsection 7.4.2, we prove Theorem 7.4.11 that $\mathcal{G}$ has a unique invariant Keisler measure. From this, we can deduce various unusual properties of Keisler measures on $\mathcal{G}$. I thank David Evans and Dugald Macpherson for suggesting looking at measures in the universal homogeneous two-graph when discussing with them the connection between measures and amalgamation properties. I also thank Colin Jahel for a helpful discussion after I found out about the connections of my result with his proof that the universal homogeneous two-graph is uniquely ergodic [40].

7.4.1 Basic properties of the universal homogeneous two-graph

We say that a 3-hypergraph (V, T) is a **two-graph** if for every $U \subseteq V$ such that $|U| = 4$ there is an even number of triplets $a, b, c \in U$ such that $T(a, b, c)$, where T is the ternary hyperedge relation. Any graph (V, E) induces an associated two-graph (V, T), where for $a, b, c \in V$, $T(a, b, c)$ holds if and only if there is an odd number of E-edges in $\{a, b, c\}$. Finite two-graphs form a Fraïssé class and so there is a countable universal homogeneous two-graph $\mathcal{G}$. This structure is elementarily equivalent to the ultraproduct of a 1-dimensional asymptotic class, which consists of the two-graphs associated with the Payley graphs [50, Example 3.5-Example 3.6]. Hence, $\mathcal{G}$ is MS-measurable.

While we know that $\mathcal{G}$ is supersimple, we also want to show that it has trivial non-forking independence in order to study invariant Keisler measures in this structure. We begin by noting that trivial independence is a notion of independence on the universal homogeneous two-graph. We denote the monster model for the theory of the universal homogeneous two-graph $\mathbb{G}$, while we use $\mathcal{G}$ for the countable model.

Lemma 7.4.1. *For $A, B, C \subseteq \mathbb{G}$ we define the trivial independence relation by*

$$A \mathop{\smile\!\!\!\!\!\!\mid}^{0}_{C} B \text{ if and only if } A \cap C = A \cap (B \cup C).$$

We have that $\mathop{\smile\!\!\!\!\!\!\mid}^{0}$ is a notion of independence in the sense of Definition 2.2.15.

Proof. Invariance, monotonicity, transitivity and symmetry are all trivial consequences of Boolean algebra. Local character also holds since a finite tuple and a set will always have intersection of size $< \omega$. Existence follows from the fact that finite two-graphs form a Fraïssé class, the extension property and quantifier elimination. $\square$

In order to prove that non-forking independence is trivial, by the Kim-Pillay Theorem 1.2.11, we need to prove that trivial independence $\mathop{\smile\!\!\!\!\!\!\;\mid}^{0}$ satisfies the independence theorem over a model. In order to prove this, we will need the following lemma:

Lemma 7.4.2. *Let V be a 3-hypergraph consisting of five vertices. Hence, there are five distinct subsets of V consisting of four vertices $B_1, \dots, B_5$. Suppose that each of B_1, B_2, B_3, B_4 has an even number of 3-hyperedges. Then, B_5 has an even number of 3-hyperedges.*

Proof. Note that for $i < j$, $B_i \cap B_j$ consists of three vertices. For any subset $A \subseteq V$, we write $|T(A)|$ to indicate the number of 3-hyperedges in A. Since every 3-hyperedge is contained in one of $B_1, \dots, B_4$,

$$|T(V)| = \sum_{i=1}^{4} |T(B_i)| - \sum_{1 \le i < j < 4} |T(B_i \cap B_j)|.$$

Furthermore, since B_5 omits precisely the hyperedges which are in some $B_i \cap B_j$ for $1 \le i < j \le 4$,

$$|T(B_5)| = |T(V)| - \sum_{1 \le i < j < 4} |T(B_i \cap B_j)|.$$

In particular, this yields that

$$|T(B_5)| = \sum_{i=1}^{4} |T(B_i)| - 2 \left(\sum_{1 \le i < j < 4} |T(B_i \cap B_j)| \right),$$

which is even since each of $|T(B_i)|$ is even for $1 \le i \le 4$. $\square$

Theorem 7.4.3. *Non-forking independence is trivial for $\mathbb{G}$, i.e. $\mathop{\smile\!\!\!\!\!\!\;\mid}^{0} = \mathop{\smile\!\!\!\!\!\!\;\mid}$.*

Proof. Since $\mathop{\smile\!\!\!\!\!\mid}^0$ is a notion of independence, it is sufficient to prove that $\mathop{\smile\!\!\!\!\!\mid}^0$ satisfies the independence theorem over a model. Then, the theorem follows from the Kim-Pillay theorem 1.2.11.

Let M be a model of the theory of the generic homogeneous two-graph and let $\mathbb{G}$ be the monster model in which we work. Suppose that $\bar{a}_1$ and $\bar{a}_2$ are tuples disjoint from M and $\bar{d}_1$ and $\bar{d}_2$ are tuples respectively disjoint from $M\bar{a}_1$ and $M\bar{a}_2$ such that $\overline{d_1} \equiv_M \bar{d}_2$. Fix a vertex $c \in M$. Let a_1 and a_2 be vertices from $\bar{a}_1$ and $\bar{a}_2$ respectively and let d_1 and d_2 pick the same coordinate from $\bar{d}_1$ and $\bar{d}_2$, let

$$N(a_1, a_2) := |\{\bar{f} \in \{d_1a_1c, d_2a_2c, a_1a_2c\} | \models T(\bar{f})\}|.$$

Let $\bar{d}$ be a 3-hypergraph such that $\text{qftp}(\bar{d}) = \text{qftp}(\overline{d_i})$ for $i = 1, 2$. We define a 3-hypergraph V on $M\bar{a}_1\bar{a}_2\bar{d}$ where, for d_1, d_2, d picking the same coordinate from $\bar{d}_1, \bar{d}_2$, and $\bar{d}$ respectively,

$$\text{qftp}^V(M\bar{a}_1\bar{a}_2) = \text{qftp}^{\mathbb{G}}(M\bar{a}_1\bar{a}_2) \tag{7.28}$$

$$\text{qftp}^V(\bar{d}M\bar{a}_1) = \text{qftp}^{\mathbb{G}}(\bar{d}_1M\bar{a}_1) \tag{7.29}$$

$$\text{qftp}^V(\bar{d}M\bar{a}_2) = \text{qftp}^{\mathbb{G}}(\bar{d}_2M\bar{a}_2) \text{ and} \tag{7.30}$$

$$T^V(d, a_1, a_2) \text{ if and only if } N(a_1, a_2) \text{ is odd.} \tag{7.31}$$

Note that V is built so that every set of 4 vertices containing c has an even number of 3-hyperedges. We need to show that V is an actual two-graph. Any set of four vertices entirely contained within one of $M\bar{a}_1\bar{a}_2$, $\bar{d}M\bar{a}_1$, or $\bar{d}M\bar{a}_2$ will already have an even number of 3-hyperedges since each of these hypergraphs is a two-graph. So, given $a_1 \in \bar{a}_1, a_2 \in \bar{a}_2, d \in \bar{d}$, we need to consider the sets of four vertices $\{a_1, a_2, d, e\}$ for e in $M, \bar{a}_1$, and $\bar{d}$ respectively.

We begin with the case of $e \in M$. Note that each of

$$\{c, e, a_1, a_2\}, \{c, e, a_1, d\}, \{c, e, a_2, d\}, \{c, a_1, a_2, d\}$$

has an even number of 3-hyperedges due to conditions 7.28, 7.29, 7.30, and 7.31 respectively. Hence, by Lemma 7.4.2, $\{e, a_1, a_2, d\}$ has an even number of 3-hyperedges.

In the case of $e \in \bar{a}_1$, we have that each of

$$\{c, e, a_1, a_2\}, \{c, e, a_1, d\}, \{c, a_1, a_2, d\}, \{c, e, a_2, d\}$$

contain an even number of 3-hyperedges due to conditions 7.28 and 7.29 for the first two sets, and due to condition 7.31 for the last two. Hence, again, by Lemma 7.4.2, we must have that $\{a_1, a_2, d, e\}$ has an even number of vertices. The case for $e \in \bar{a}_2$ is analogous.

Finally, for $e \in \bar{d}$, we have that

$$\{c, e, a_1, a_2\}, \{c, a_1, a_2, d\}, \{c, e, a_1, d\}, \{c, e, a_2, d\}$$

all contain an even number of 3-hyperedges due to conditions 7.31 for the first two sets and conditions 7.29 and 7.30 for the last two sets. Again, we can use Lemma 7.4.2.

We can conclude that V is a two-graph. In particular, by compactness and the extension property for the universal homogeneous two-graph, we have that $\mathrm{qftp}^V(\bar{d}/M\bar{a}_1\bar{a}_2)$ is consistent in $\mathbb{G}$ and so realised by some $\bar{d}^*$ disjoint from $M\bar{a}_1\bar{a}_2$ with $\bar{d}^* \equiv_{M\bar{a}_1} \bar{d}_1$ and $\bar{d}^* \equiv_{M\bar{a}_2} \bar{d}_2$ by quantifier elimination. Hence, $\mathop{\smile\!\!\!\!\!\!|}^0$ satisfies the independence theorem over a model. $\square$

Remark 7.4.4. Note that the construction from the theorem above also implies that $\mathbb{G}$ satisfies the independence theorem over any non-empty set of vertices $C \subseteq \mathbb{G}$. We can also see that this implies that $\mathbb{G}$ satisfies the independence theorem over $\emptyset$ as we prove below.

Lemma 7.4.5. *The structure $\mathbb{G}$ satisfies the independence theorem over $\emptyset$.*

Proof. Suppose that the finite subsets of $\mathbb{G}$, A, B, D_1, D_2 are such that $A \downarrow B, D_1 \downarrow AB, D_2 \downarrow AB$ and $D_1 \equiv D_2$. Fix a vertex c_1 disjoint from the previous sets. Letting σ be an automorphism of $\mathbb{G}$ sending D_1 to D_2, let $c_2 = \sigma(c_1)$. By existence, we can assume that all of A, B, D_1, D_2, c_1, c_2 are disjoint. Now, fix another vertex e in $\mathbb{G}$ disjoint from everything. Since every two pairs have the same type in $\mathbb{G}$, $c_1 \equiv_e c_2$. For $i \in \{1,2\}$ $c_i \downarrow_e D_i$. Hence, we can apply the independence theorem over e to obtain c such that $D_1 \equiv_c D_2$, $c \downarrow_e D_1D_2$. Again, c can be chosen to be disjoint from AB too. Hence, we have $D_1 \downarrow_c AB, D_2 \downarrow_c AB$ and $D_1 \equiv_c D_2$. By the independence theorem over c, we can find D such that

$$D \equiv_{Ac} D_1, D \equiv_{Bc} D_2, D \underset{c}{\downarrow} AB.$$

In particular, we also have that

$$D \equiv_A D_1, D \equiv_B D_2, D \downarrow AB,$$

which yields the independence theorem over $\emptyset$. $\square$

Remark 7.4.6. From Remark 7.4.4 and Lemma 7.4.5, we can see for any finite $A \subseteq G, \mathrm{acl}^{eq}(A) = \mathrm{dcl}^{eq}(A)$. Firstly, note that $\mathcal{G}$ is such that $\mathrm{acl}(A) = A$ for any finite $A \subseteq G$ since $\mathrm{Age}(G)$ has the strong amalgamation property (Theorem 1.2.10). Hence, by [7, Proposition 3.2], it is sufficient to show that for any $A \subseteq \mathbb{G}$ and real tuple b from $\mathbb{G}$,

$$\mathrm{tp}(b/\mathrm{acl}(A)) \vdash \mathrm{tp}(b/\mathrm{acl}^{eq}(A)).$$

From [6, Remark 17.2] and [6, Theorem 18.14.4], it is sufficient to prove that for any finite $A \subseteq \mathbb{G}$ and finite real tuple b, $\mathrm{tp}(b/A)$ is an amalgamation base in the sense of [6, Definition 17.1]. Remark 7.4.4 implies that $\mathrm{tp}(b/A)$ is always an amalgamation base. Hence, we obtain that for any finite $A \subseteq G, \mathrm{acl}^{eq}(A) = \mathrm{dcl}^{eq}(A)$. Alternatively, we can use weak elimination of imaginaries [40, Proposition 4.5.2] and the fact that $\mathrm{acl}(A) = A$ for any finite set.

Now that we have seen that $\mathbb{G}$ has trivial non-forking independence, we can observe that it does not satisfy independent 4-amalgamation over the empty set:

Corollary 7.4.7. *The universal homogeneous two-graph* $\mathbb{G}$ *does not satisfy independent 4-amalgamation over the empty set.*

Proof. Suppose that $a, b, c \in \mathbb{G}$ are such that $\models \neg T(a, b, c)$. Then, the formulas $T(x, a, b), T(x, a, c), T(x, b, c)$ isolate complete types over pairs of vertices which do not satisfy 4-GIT over the empty set. In fact, if they did, we would have a set of four vertices with three hyperedges. However, since $\mathcal{G}$ has trivial independence, $\mathop{\smile\!\!\!\!\!|}\ \{a, b, c\}$. $\square$

Note that if we have a', b', c' such that $\models T(a', b', c')$, then $T(x, a', b'), T(x, a', c')$, and $T(x, b', c')$ do isolate types which satisfy 4-GIT over $\emptyset$.

From the corollary above, we can see that there are ω-categorical MS-measurable structures for which independent 4-amalgamation over $\emptyset$ fails. However, independent 4-amalgamation does hold over any non-empty set of vertices, as we prove below. Hence, it is still an open question whether an ω-categorical MS-measurable structure satisfies independent 4-amalgamation over models. Indeed, it is an open question whether an ω-categorical MS-measurable structure satisfies independent n-amalgamation for all $n \in \mathbb{N}$ over models.

The following fact is also folkore:

Theorem 7.4.8. *The universal homogeneous two-graph* $\mathbb{G}$ *satisfies independent 4-amalgamation over non-empty sets of vertices. It also satisfies independent n-amalgamation for all* $n > 4$ *over arbitrary sets of vertices.*

Proof. Let $\mathop{\smile\!\!\!\!\!|}_C \{A_1, A_2, A_3\}$ be subsets in $\mathbb{G}$ with C non-empty and let D_{ij} for $1 \leq i < j \leq 3$ be such that $D_{ij} \mathop{\smile\!\!\!\!\!|}_C A_iA_j$, and for $i, j, k \in \{1, 2, 3\}$ distinct, $D_{ij} \equiv_{A_i} D_{ik}$. We let V be a hypergraph on $DA_1A_2A_3C$, where D is a copy of the hypergraph on D_{ij} such that for $1 \leq i < j \leq 3$,

$$\mathrm{qftp}^V(D/A_iA_j) = \mathrm{qftp}^{\mathbb{G}} D_{ij}/A_iA_j).$$

Now, note that any four vertices contained entirely in DA_iA_jC or $A_1A_2A_3C$ will already have an even number of hyperedges. So in order to see that V is a two-graph, we only need to check that for $d \in D, a_i \in A_i$ for each $i \in \{1, 2, 3\}$, we have an even number of hyperedges in $\{d, a_1, a_2, a_3\}$. To see this, consider $c \in C$. Each subset of four elements of $\{d, a_1, a_2, a_3, c\}$ other than $\{d, a_1, a_2, a_3\}$ has an even number of hyperedges. Hence, by Lemma 7.4.2, there is an even number of hyperedges in $\{d, a_1, a_2, a_3\}$. By the extension property for the two-graph and triviality of independence, we can find such a D in $\mathbb{G}$, which will satisfy the 4-generalised independence theorem over C.

The proof of higher amalgamation properties for $n > 4$ is simpler to the extent that the quantifier-free type of any four vertices will already be determined by the amalgamation problem, and so we will know that a two-graph solving the amalgamation problem exists. $\square$

7.4.2 Keisler measures in the universal homogeneous two-graph

In this section, we prove that the universal homogeneous two-graph is such that it has a unique invariant Keisler measure. From this, we note how this structure has various peculiar properties regarding its Keisler measures.

The main ingredient for our proof will be the folklore fact that, fixing a vertex of $\mathcal{G}$, one can interpret the random graph.

Fact 7.4.9 (Lemma 3.2.10 in [1])**.** *Let $\mathcal{G}$ be the universal homogeneous two-graph and fix $c \in \mathcal{G}$. Define the graph $\Gamma(c)$ on $G \setminus \{c\}$ by setting for $a, b \in G \setminus \{c\}$,*

$$E(a, b) \text{ if and only if } \mathcal{G} \vDash T(a, b, c).$$

Then, $\Gamma(c)$ is isomorphic to the random graph and it is ***canonically embedded*** *over c in $\mathcal{G}$, i.e. X is an $\emptyset$-definable subset of $(\Gamma(c), E)$ if and only if it is a c-definable subset of $\mathcal{G}$.*

Remark 7.4.10. Note that from the construction of Fact 7.4.9 we can also see how, given $A, B \subseteq G \setminus \{c\}$, $A \equiv B$ in $(\Gamma(c), E)$ if and only if $A \equiv_c B$ in $\mathcal{G}$. In fact, since $A \equiv B$ in $(\Gamma(c), E)$, A and B agree on which pairs of vertices form a 3-hyperedge with c. Consider a triplet of vertices $a_1a_2a_3$ from A and the corresponding triplet

$b_1b_2b_3$ from B. Since $a_1a_2a_3$ and $b_1b_2b_3$ agree on which pairs form a hyperedge with c, they must agree on whether there is a hyperedge between them as every four vertices have an even number of hyperedges. From this, we can also see more generally that for $A \subseteq G \setminus \{c\}$, X is an A-definable subset of $(\Gamma(c), E)$ if and only if it is an Ac-definable subset of $\mathcal{G}$.

Now we are ready for our main result, showing that there is a unique invariant Keisler measure on the universal homogeneous two-graph. The main idea of the proof is that any invariant Keisler measure on $\mathcal{G}$ induces an invariant Keisler measure on the random graph canonically embedded in $G \setminus \{c\}$ where c is a fixed vertex of G. We have a classification of the invariant Keisler measures on the random graph from [2]. However, certain equations that the invariant Keisler measures on $\mathcal{G}$ must satisfy imply that only one induced measure on the random graph is possible. This, in turn, will imply that there is a unique invariant Keisler measure on $\mathcal{G}$ in the singleton variable. The result is related to Jahel's result that the universal homogeneous two-graph is uniquely ergodic [40, Ch.2] and can also be deduced from it. This will be discussed in later remarks.

Theorem 7.4.11. *There is a unique invariant Keisler measure μ on the universal homogeneous two-graph $\mathcal{G}$ in the singleton variable. For $d, a_1, \ldots, a_n \in G$ distinct, we have that*

$$\mu(d/a_1 \ldots a_n) = \left(\frac{1}{2}\right)^{n-1},$$

and $\mu(d) = 1$.

Proof. Let μ be an invariant Keisler measure on G. Fix $c \in G$. We have that μ induces the $\mathrm{Aut}(G/c)$-invariant Keisler measure μ' on definable subsets of $G \setminus \{c\}$ given by

$$\mu'(\phi(x, \overline{b})) = \mu(\phi(x, \overline{b})).$$

Note that this is still a Keisler measure since $\mu(x \neq c) = 1$ by transitivity of $\mathrm{Aut}(G)$ on the vertices of G. Now, μ' induces an $\mathrm{Aut}(\Gamma(c))$-invariant Keisler measure μ'' on $\Gamma(c)$ constructed as in Fact 7.4.9. From Remark 7.4.10, for any X definable over A in $(\Gamma(c), E)$, we know that X is Ac-definable in $\mathcal{G}$ and so we can set $\mu''(X) = \mu'(X)$. Since μ' is $\mathrm{Aut}(G/c)$-invariant, μ'' is $\mathrm{Aut}(\Gamma(c))$-invariant by Remark 7.4.10. Additivity and being a probability measure are inherited from μ'. From Albert's classification of invariant Keisler measures on the random graph [2], we know that there is a unique probability measure ν on $[0,1]$ such that

$$\mu''(\phi(x, A, B)) = \int_0^1 p^{|A|}(1-p)^{|B|}\mathrm{d}\nu, \tag{Alb}$$

where for A, B sets of distinct vertices in $\Gamma(c)$, $\phi(x, A, B)$ asserts that x is connected to every vertex in A and no vertex in B.

Claim: The measure ν concentrates on the point $\{\frac{1}{2}\}$, yielding

$$\mu''(\phi(x, A, B)) = \left(\frac{1}{2}\right)^{(|A|+|B|)}.$$

Proof. Let $a, b, a', b' \in G \setminus \{c\}$ be vertices such that $\vDash \neg T(c, a, b)$ and $\vDash T(c, a', b')$. By invariance, we have that

$$\mu(T(x, a, c)) = \mu(T(x, a, b)) \text{ and } \mu(\neg T(x, a, c)) = \mu(\neg T(x, a', b')). \tag{7.32}$$

Furtheremore, since every set of four vertices has an even number of hyperedges,

$$\mu(T(x, a, b)) = \mu(T(x, a, c) \wedge \neg T(x, b, c)) + \mu(\neg T(x, a, c) \wedge T(x, b, c))$$
$$\mu(\neg T(x, a', b')) = \mu(T(x, a', c) \wedge \neg T(x, b', c)) + \mu(\neg T(x, a', c) \wedge T(x, b', c))$$

Using the above equations, 7.32, and considering the measure μ'', we get that

$$\mu''(E(x, a)) = \mu''(E(x, a) \wedge \neg E(x, b)) + \mu''(\neg E(x, a) \wedge E(x, b)) \tag{7.33}$$
$$\mu''(\neg E(x, a)) = \mu''(E(x, a') \wedge \neg E(x, b')) + \mu''(\neg E(x, a') \wedge E(x, b')) \tag{7.34}$$

Hence, using Alb, we have that

$$\int_0^1 p \, \mathrm{d}\nu = 2\int_0^1 p(1-p)\mathrm{d}\nu = \int_0^1 (1-p)\mathrm{d}\nu. \tag{7.35}$$

Comparing the integrals at the opposite sides of the equation in 7.35, we get that

$$\int_0^1 p \, \mathrm{d}\nu = \frac{1}{2}.$$

Comparing this result from the equality between the first two integrals in 7.35, we get

$$\int_0^1 p^2 \,\mathrm{d}\nu = \frac{1}{4} = \left(\int_0^1 p \,\mathrm{d}\nu\right)^2.$$

In particular, this implies that

$$\int_0^1 \left(p - \frac{1}{2}\right)^2 \mathrm{d}\nu = 0.$$

Since $\left(p - \frac{1}{2}\right)^2$ is a non-negative function with value zero only for $p = 1/2$, this implies that ν concentrates at $\frac{1}{2}$. From Alb,

$$\mu''(\phi(x, A, B)) = \left(\frac{1}{2}\right)^{(|A|+|B|)},$$

as desired. □

Now, let $d, a_1, \ldots, a_n \in G$. Since every four vertices have an even number of hyperedges, for $1 < i < j \leq n$, whether $T(d, a_i a_j)$ is entirely determined by whether $T(d, a_1, a_i)$ and $T(d, a_1, a_j)$ hold. Hence, by quantifier elimination,

$$\mu(d/a_1, \ldots, a_n) = \mu\left(\bigwedge_{1<i\leq n} \epsilon_i T(x, a_1, a_i)\right),$$

where ϵ_i is either the empty symbol or $\neg$ and ϵ_i is $\neg$ if and only if $\neg T(x, a_1, a_i) \in \mathrm{tp}(d/a_1, \ldots, a_n)$. Consider $\Gamma(a_1)$ built as in Fact 7.4.9 and the $\mathrm{Aut}(\Gamma(a_1))$-invariant Keisler measure μ'' induced on $\Gamma(a_1)$ described above. By construction,

$$\mu\left(\bigwedge_{1<i\leq n} \epsilon_i T(x, a_1, a_i)\right) = \mu''\left(\bigwedge_{T(d,a_1,a_j)} E(x, a_j) \wedge \bigwedge_{\neg T(d,a_1,a_j)} \neg E(x, a_j)\right).$$

From Claim 1, we know that the latter measure is $(1/2)^{n-1}$, which yields the desired result. Note that by additivity and Ryll-Nardzewski, determining the measure of every complete type is sufficient to determine the whole measure.

What we obtained must be an invariant Keisler measure on $\mathcal{G}$ since the universal homogeneous two-graph is MS-measurable and so has at least one invariant Keisler measure, which is the one obtained in this theorem. For completeness, we prove this is actually an invariant Keisler measure following Corollary 2.2.27 since we have specified the behaviour of μ on all non-forking types in a single variable over finite sets of parameters.

Given trivial independence and simplicity, we cannot have two complete types over sets of parameters of distinct size with the same locus. To see this, suppose by contradiction that two types $\mathrm{tp}(c/\overline{a})$ and $\mathrm{tp}(d/\overline{b})$ have the same locus and $|\overline{b}| > |\overline{a}|$. We may also assume that $|\overline{a}| > 2$ since for $|\overline{a}| = 1$ there is a unique non-forking type which asserts that $x \neq a$ and any non-forking type over a larger set of parameters will assert that x is distinct from more parameters. Firstly, suppose that $\overline{a}$ and $\overline{b}$ are disjoint. Since $|\overline{b}| \geq 3$, there are more than two types over $\overline{b}$ non-forking over $\emptyset$, $\mathrm{tp}(d_1/\overline{b}), \ldots, \mathrm{tp}(d_n/\overline{b})$. By the independence theorem over $\emptyset$, we know that there are independent realisations for each of

$$\mathrm{tp}(c/\overline{a}) \cup \mathrm{tp}(d_i/\overline{b}).$$

But this contradicts $\mathrm{tp}(c/\overline{a})$ and $\mathrm{tp}(d/\overline{b})$ having the same locus since there are realisations of $\mathrm{tp}(c/\overline{a})$ which do not satisfy $\mathrm{tp}(d/\overline{b})$. If $\overline{a}$ and $\overline{b}$ have non-empty intersection, let a_1 be the first element of both tuples. As noted earlier, $\mathrm{tp}(d/\overline{b})$ is entirely determined by the formulas $T(d, a_1, b_j)$ for $j \leq |\overline{b}|$ and $\mathrm{tp}(c/\overline{a} \cap \overline{b})$ has multiple non-forking extensions to $\overline{b}$: one for each choice of whether $T(d, a_1, b_j)$ holds or not for $b_j \in \overline{b} \setminus \overline{a} \cap \overline{b}$. Hence, there are realisations of $\mathrm{tp}(c/\overline{a})$ which do not satisfy $\mathrm{tp}(d/\overline{b})$.

Now, we can proceed to show that μ induces an invariant Keisler measure following the additional conditions of Corollary 2.2.27. Since the value of $\mu(d/a_1 \ldots a_n)$ only depends on n, μ is invariant. Condition (a) is trivially met since there is a unique type over $\emptyset$ in the singleton variable. We only need to check condition (b), which will imply additivity. For this, begin with $\mathrm{tp}(d/a_1 \ldots a_n)$. By Fact 7.4.9, for $b \in M$ distinct from the a_i, there are precisely two non-forking extensions of $\mathrm{tp}(d/a_1 \ldots a_n)$ to $a_1 \ldots a_n b$, which are entirely determined by whether $T(x, a_1, b)$ or not, and both of which have measure $(1/2)^{n+1}$. Hence, condition (b) is met since

$$\left(\frac{1}{2}\right)^n = \mu(d/a_1 \dots a_n) = \mu(d_1/a_1 \dots a_n b) + \mu(d_1/a_1 \dots a_n b) = 2\left(\frac{1}{2}\right)^{n+1}.$$

□

Remark 7.4.12. After obtaining this result, I learned that the same proof idea lies behind Jahel's proof that the universal homogeneous two-graph is uniquely ergodic [40, Ch.4.5.3] (i.e. any minimal $\mathrm{Aut}(G)$-flow admits a unique $\mathrm{Aut}(G)$-invariant measure). One can see that our result implies that the flow induced from $\mathrm{Aut}(G)$ on $S_1^{\mathrm{nf}}(G)$ is minimal, where the latter is the set of non-forking types in one variable over G. By saying that the flow is minimal, we mean that there is no proper closed subset of $S_1^{\mathrm{nf}}(G)$ which is $\mathrm{Aut}(G)$-invariant. To see this, note that the invariant Keisler measure described in Theorem 7.4.11 has full support: every non-forking formula is assigned strictly positive measure, and so the smallest closed subset of measure 1 is the whole space [53, Definition 5.4]. In particular, suppose for a contradiction there was a proper subflow $X \subsetneq S_1^{\mathrm{nf}}(G)$. Since $\mathrm{Aut}(G)$ is amenable [40, Ch.2], there is an $\mathrm{Aut}(G)$-invariant probability measure μ^* on X. This induces another $\mathrm{Aut}(G)$-invariant Borel probability measure on $S_1^{\mathrm{nf}}(G)$, with support $X \subsetneq S_1^{\mathrm{nf}}(G)$. This contradicts the uniqueness result of Theorem 7.4.11.

Remark 7.4.13. Alternatively, we can also deduce Theorem 7.4.11 from Jahel's results by showing that $S_1^{\mathrm{nf}}(G)$ and $\mathrm{Gr}(\mathcal{G})$ are isomorphic as $\mathrm{Aut}(G)$-flows, where $\mathrm{Gr}(\mathcal{G})$ is the set of graphs on G which induce $\mathcal{G}$ as a two-graph. Then, one can use Jahel's proof that the $\mathrm{Aut}(G)$-flow on $\mathrm{Gr}(G)$ is uniquely ergodic [40, Lemma 4.5.4]. We prove this below

Lemma 7.4.14. *$S_1^{\mathrm{nf}}(G)$ and $\mathrm{Gr}(\mathcal{G})$ are isomorphic as $\mathrm{Aut}(G)$-flows.*

Proof. We need to build a continous bijection from $S_1^{\mathrm{nf}}(G)$ to $\mathrm{Gr}(\mathcal{G})$. To do this, given $p \in S_1^{\mathrm{nf}}(G)$, we define the graph $\Gamma(p)$ on G by having that for $a, b \in G$

$$\Gamma(p) \vDash E(a, b) \text{ if and only if } T(x, a, b) \in p.$$

We consider the map $\Gamma : S_1^{\mathrm{nf}}(G) \to \mathrm{Gr}(\mathcal{G})$ given by $p \mapsto \Gamma(p)$. The map is injective by quantifier elimination. Let V be a graph on G which induces $\mathcal{G}$ as a two-graph. For a fixed additional vertex d, consider the 3-hypergraph $\mathcal{V}d$ on Vd where for any $a, b, c \in G$,

$$T(d, a, b) \text{ if and only if } V \vDash E(a, b),$$
$$T(a, b, c) \text{ if and only if } \{a, b, c\} \text{ has an odd number of edges in } V.$$

The resulting hypergraph is a two-graph by construction. Let $p = \mathrm{tp}^{\mathcal{V}d}(d/G)$, we can see that $\Gamma(p) = V$, proving surjectivity. Furthermore, the map Γ preserves the action of $\mathrm{Aut}(G)$. Finally, to see this map is continous, we can see from [40, Ch. 4.5.3] that $\mathrm{Gr}(\mathcal{G})$ has a topology with a basis of clopen sets N_A, where for A a graph on a finite subset of G,

$$N_A := \{x \in \mathrm{Gr}(\mathcal{G}) | A \text{ is a subgraph of } x\}.$$

It is easy to verify that the map Γ is continous by checking that the preimage of clopen sets in this basis is a clopen set in the Stone topology on $S_1^{\mathrm{nf}}(G)$. □

Remark 7.4.15. The universal homogeneous two-graph has no $\mathrm{Aut}(G)$-invariant types over G. This is a trivial consequence of the fact that by homogeneity, every invariant type must have either that $T(x, a, b)$ or that $\neg T(x, a, b)$ for every pair of vertices, but neither of these conditions is consistent with every set of four vertices having an even number of hyperedges. Hence, it is possible for a structure to have no $\mathrm{Aut}(M)$-invariant types, but still have an invariant Keisler measure while $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$.

Remark 7.4.16. The fact $\mathcal{G}$ has no invariant types is a central ingredient to why we can use unique ergodicity of $\mathrm{Aut}(G)$ to deduce uniqueness of an invariant Keisler measure on $S_1^{\mathrm{nf}}(G)$. In general, knowing that the automorphism group of an ω-categorical structure is uniquely ergodic gives little information about the space of invariant Keisler measures on that structure. For example, the automorphism group of the random graph is uniquely ergodic [48], but we know that the random graph has a rich space of invariant Keisler measures [2]. Unique ergodicity of an automorphism group $\mathrm{Aut}(M)$ only tells us about the existence of a unique measure with respect to its action on its universal minimal flow. However, usually, the action of $\mathrm{Aut}(M)$ on $S_x^{\mathrm{nf}}(M)$ is not minimal: if the structure we are working with has any $\mathrm{Aut}(M)$-invariant types, those types will be minimal subflows of the action of $\mathrm{Aut}(M)$ on $S_x^{\mathrm{nf}}(M)$. Hence, the universal homogeneous two-graph is somehow peculiar in having the action of $\mathrm{Aut}(M)$ on $S_x^{\mathrm{nf}}(M)$ being minimal. It would be interesting to study invariant Keisler measures on other ω-categorical structures with no invariant types to see whether any of these exhibit similar phenomena.

Remark 7.4.17. Finally, we wanted to point out a difference between the measure of 3 and 4-amalgamations. In Corollary 3.1.5, we noted how, given an invariant Keisler measure on an ω-categorical structure with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$, for $a \mathop{\smile\!\!\!\!\!|} b$, we have that $\mu(\phi(x,a) \wedge \psi(x,b))$ only depends on the types of a and b and not on the type of the pair. Indeed, this is true even when $a \mathop{\smile\!\!\!\!\!|}^{\mathrm{a}} b$. For a long time, we were interested in whether one could prove an analogue for the measure of a 4-amalgamation roughly on the lines that if $\mathop{\smile\!\!\!\!\!|} \{a,b,c\}$, then

$$\mu(\phi(x,ab) \wedge \psi(a,ac) \wedge \chi(x,bc))$$

only depends on the types of the pairs ab, ac, bc and not on the type of the triplet. This, Question 7.0.1, was the initial motivation behind our study of invariant Keisler measures in ternary structures. However, the universal homogeneous two-graph is clearly a counterexample to this conjecture: consider abc and $a'b'c'$ such that the first triplet forms a hyperedge and the second does not. By quantifier elimination every pair has the same type, but for the unique invariant Keisler measure μ,

$$\mu(T(x,ab) \wedge T(x,ac) \wedge T(x,bc)) = \frac{1}{4} \quad \text{and}$$
$$\mu(T(x,a'b') \wedge T(x,a'c') \wedge T(x,b'c')) = 0,$$

since the formula in the second equation is inconsistent.

7.5 Non MS-measurability in spite of all the best amalgamation properties

In Chapters 5 and 6, we gave the example of the Hrushovski constructions of Construction 5.1.1 as structures which are not MS-measurable in spite of being supersimple, ω-categorical, of finite rank and satisfying independent n-amalgamation for all $n \in \mathbb{N}$ over finite algebraically closed sets. Our analysis of what induces the failure of MS-measurability is the failure of the strong independence theorem discussed in Chapter 6. It is natural to ask whether also satisfying this property and independent n amalgamation for all $n \in \mathbb{N}$ may imply MS-measurability for a supersimple ω-categorical structure. After all, the two supersimple ω-categorical (finite rank) examples we have found where MS-measurability fails, Construction 5.1.1 and the generic tetrahedron-free 3-hypergraph, seem to be connected to failure of the strong independence theorem and of the 4-generalised independence theorem respectively. However, the picture of the relation between amalgamation properties and MS-measurability is still more complicated, as we will show with the example in this section.

Here we give an ω-categorical Hrushovski construction $\mathcal{M}_f^R$ which is a supersimple 3-hypergraph of SU-rank 1 satisfying independent n-amalgamation for all $n \in \mathbb{N}$ and the strong independence theorem, but which is not MS-measurable. As part of the proof, we do take advantage of the fact that the class $\mathcal{K}_f^R$ for this Hrushovski construction avoids certain specific finite structures. However, these are not n-amalgamations. Similarly to the case of the generic tetrahedron-free 3-hypergraph, our hope is that our techniques may be adaptable to ergodic measures in order to show that in this structure the forking and universally measure zero ideal differ.

The relevant notation and background on ω-categorical Hrushovski constructions is given in Section 4.1. Below we introduce the class of ω-categorical

Hrushovski constructions which are the focus of this section.

Construction 7.5.1. We set $\alpha = 1$ and work with 3-hypergraphs with a ternary relation R. Let $f : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ be such that for $t \geq 6$, $f(3 \cdot t) \leq f(t) + 1$ and for $t \leq 6$, f is piece-wise linear with $f(1) = 1, f(3) = 2, f(6) = 3$. The associated class of hypergraphs, $\mathcal{K}_f^R$ has the free amalgamation property (Proposition 9.3.7) and so we may study the associated ω-categorical Hrushovski construction $\mathcal{M}_f^R$.

In Appendix 9.3.2, we prove the following properties:

Theorem 7.5.2. *The structure $\mathcal{M}_f^R$ is supersimple ω-categorical of SU-rank 1 and with weak elimination of imaginaries. Furthermore, $\mathcal{M}_f^R$ satisfies the strong independence theorem over finite algebraically closed sets. We can also choose f so that $\mathcal{M}_f^R$ satisfies independent n-amalgamation over finite algebraically closed sets for all $n \in \mathbb{N}$.*

Proof. The theorem consists of Corollary 9.3.9 and Theorem 9.3.11 of Appendix 9.3.2. □

It is worth also noting some basic features of the small algebraically closed substructures of $\mathcal{M}_f^R$. In particular, in order to apply Corollary 7.2.2, we show that there are triplets of vertices in $\mathcal{M}_f^R$ such that there is a unique independent 3-type isolated by the union of the types of each pair.

Lemma 7.5.3. *The structure $\mathcal{M}_f^R$ also satisfies the following properties:*

1. *vertices are closed and the action of $\mathrm{Aut}(M_f^R)$ on $\mathcal{M}_f^R$ is transitive;*
2. *between any two vertices there is at most one hyperedge containing both of them;*
3. *for $a, b \in \mathcal{M}_f^R$ either $ab \leq \mathcal{M}_f^R$ or there is $c \in \mathcal{M}_f^R$ such that $R(a, b, c)$ and $abc \leq \mathcal{M}_f^R$. In both cases, $a \mathop{\smile\!\!\!\!\!\!|}\; b$. Furthermore, $\exists y R(x, y, z)$ isolates a complete type $p(x, z)$; and*
4. *there is a unique independent 3-type containing $p(x_1, x_2) \cup p(x_1, x_3) \cup p(x_2, x_3)$. This type is also isolated by the formula*

$$\exists y_{12} y_{13}, y_{23} B(x_1, x_2, x_3, y_{12} y_{13}, y_{23}),$$

where $B(x_1, x_2, x_3, y_{12}y_{13}, y_{23})$ asserts that all of its variables are distinct and that for $1 \leq i < j \leq 3$, $R(x_i, y_{ij}, x_j)$. An illustration of the hypergraph with quantifier-free diagram isolated by $B(x_1, x_2, x_3, y_{12}y_{13}, y_{23})$ is given in Figure 7.4.

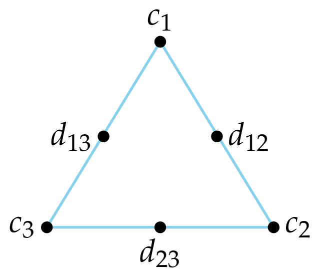


Figure 7.4: We depict the only 3-hypergraph in $\mathcal{K}_f^R$ such that $\models B(c_1c_2c_3d_{12}d_{13}d_{23})$. We represent each hyperedge by a straight line passing between three points. The lack of such a line between three vertices indicates that there is no hyperedge between them.

Proof. These various properties are consequences of our choice of the initial values of f, with which we bound the maximal size of hypergraphs of a given (small) predimension. In particular, (1) follows from $f(1) = 1$, which implies that the only possible hypergraphs of predimension 1 are vertices. Similarly, (2) and (3) follow from the choice of $f(3) = 2$, which implies that the only two hypergraphs of predimension 2 are two disjoint vertices or three vertices forming a hyperedge. In both cases, we also have that $d(a/b) = d(a) = 1$, which given the characterisation of non-forking independence in ω-categorical Hrushovski constructions implies that $a \mathop{\smile\!\!\!\!\!\mid\ } b$ (see Theorem 9.1.5). Finally, for (4), note that since any two vertices are contained in at most one hyperedge, the hyperedges between x_i and x_j and x_i and x_k cannot share their additional vertex for i, j, k distinct in $\{1, 2, 3\}$. Moreover, x_1, x_2, and x_3 cannot form a hyperedge if the type of their triplet is supposed to be independent. Hence, any hypergraph satisfying $p(x_1, x_2) \cup p(x_1, x_3) \cup p(x_2, x_3)$ such that x_1, x_2 and x_3 form an independent triplet must contain at least 6 vertices in its closure. The hypergraph illustrated in Figure 7.4 satisfies the type and is the unique 3-hypergraph with six vertices that does while having $d(c_1c_2c_3) = 3$, which is required in order for c_1, c_2, c_3 to form an independent triplet. However, since $f(6) = 3$, any hypergraph of predimension

3 must have at most six vertices. Hence, $p(x_1, x_2) \cup p(x_1, x_3) \cup p(x_2, x_3)$ isolates a complete type. □

Notation 7.5.4. From now on, we are interested in studying the measure of the formula $\exists y R(w, y, a)$ for $a \in \mathcal{M}_f^R$ and Boolean combinations of it. In particular, for $A \leq \mathcal{M}_f^R$ and $B \subseteq A$ we write

$$\Delta(w; B; A) := \bigwedge_{b \in B} \exists y R(w, y, b) \wedge \bigwedge_{a \in A \setminus B} \neg \exists y R(w, y, a).$$

We let $D \leq \mathcal{M}_f^R$ be consist of a copy of the hypergraph in Figure 7.4 and label its vertices accordingly as $c_1 c_2 c_3 d_{12} d_{13} d_{23}$.

Lemma 7.5.5. *Let μ be an ergodic Keisler measure on $\mathcal{M}_f^R$ in the variable w. Say that $\mu(\exists y R(w, y, a)) = p$. Then, we also have that:*

$$\mu(\Delta(w; c_1 c_2; c_1 c_2 d_{12})) = p^2 \tag{7.36}$$

$$\mu(\Delta(w; c_1 c_3 d_{12} d_{23}, D)) = p^4. \tag{7.37}$$

Proof. We begin with equation 7.36. From Lemma 7.5.3.4, we have that

$$\exists y R(w, y, c_1) \wedge \exists y R(w, y, c_2) \vdash \neg \exists y R(w, y, d_{12}). \tag{7.38}$$

In fact, if we had $\exists y R(w, y, d_{12})$ and $w \neq d_{12}$, $\mathcal{K}_f^R$ would have to contain a hypergraph of predimension ≤ 3 with more than 7 vertices, which is forbidden by our choice of f.[2] Hence,

$$\Delta(w; c_1 c_2; c_1 c_2 d_{12}) \equiv \exists y R(w, y, c_1) \wedge \exists y R(w, y, c_2).$$

But since $c_1 \mathop{\smile\!\!\!\!\!\!\,|}\, c_2$, we have by transfer of independence (Corollary 3.2.16) that

$$\mu(\exists y R(w, y, c_1) \wedge \exists y R(w, y, c_2)) = \mu(\exists y R(w, y, c_1)) \mu(\exists y R(w, y, c_2)) = p^2,$$

[2] For the purposes of studying invariant Keisler measures, the case of $w = d_{12}$ can be ignored since this definable set has measure zero. For this reason, we avoid discussing such cases in the main body of the proof. Nevertheless, the various statements about implications and equivalence of formulas that we make do indeed hold. For example, since we are working with a 3-uniform hypergraph, we actually have $\neg \exists y R(d_{12}, y, d_{12})$, making the implication 7.38 correct.

where the second equality follows by transitivity of the action of $\mathrm{Aut}(M_f^R)$ on the vertices of $\mathcal{M}_f^R$.

For equation 7.37, due to the same observation that yielded 7.38, we have that

$$\Delta(w; c_1c_3d_{12}d_{23}; D) \equiv \Delta(w; c_1c_3; c_1c_3d_{13}) \wedge \exists y R(w, y, d_{12}) \wedge \exists y R(w, y, d_{23}).$$

Furthermore, since $c_1c_3d_{13} \mathop{\smile\!\!\!\!\!\!\mid}^{a} d_{12}d_{13}$, by transfer of independence, we have

$$\mu(\Delta(w; c_1c_3d_{12}d_{23}; D)) = p^2\mu(\exists y R(w, y, d_{12}) \wedge \exists y R(w, y, d_{23})) = p^4.$$

For the first equality, note that

$$\mu(\Delta(w; c_1c_3; c_1c_3d_{13})) = p^2$$

from equation 7.36 since $c_1c_3d_{13} \equiv c_1c_2d_{12}$. Furthermore,

$$\mu(\exists y R(w, y, d_{12}) \wedge \exists y R(w, y, d_{23})) = p^2$$

since $d_{12} \mathop{\smile\!\!\!\!\!\!\mid} d_{13}$ and by transfer of independence. □

Remark 7.5.6. The results above also hold assuming (for a contradiction) that $\mathcal{M}_f^R$ is MS-measurable. This can be seen by substituting the uses of Corollary 3.2.16 with Theorem 3.1.9. Alternatively, one can use Remark 3.2.17. In fact, since $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$, if $\mathcal{M}_f^R$ is MS-measurable, the induced $\mathrm{Aut}(M)$-invariant Keisler measure in the singleton real variable w is $\mathrm{Aut}(M)$-ergodic.

Lemma 7.5.7. *Suppose that $\mathcal{M}_f^R$ is MS-measurable with $\mu(x = x) = 1$ and*

$$\mu(\exists y R(w, y, a)) = p.$$

Then,

$$\mu(\Delta(w; c_1c_3d_{12}d_{23}, D)) = \frac{p^4}{(1-p)}.$$

Proof. This proof will mainly rely on applying Corollary 7.2.2. Note that in D, we have that $\downarrow \{c_1, c_2, c_3\}$. Further note that $\Delta(w; c_1c_3d_{12}d_{23}; D)$ is equivalent to

$$\Delta(w; c_1c_3; c_1c_3d_{13}) \wedge \Delta(w; c_1d_{12}; c_1c_2d_{12}) \wedge \Delta(w, c_3d_{23}; c_2c_3d_{23}).$$

Furthermore, for $1 \leq i < j \leq 3, d_{ij} \in \mathrm{dcl}(c_ic_j)$, since any two points share at most one hyperedge. For $\Delta(w; B; A)$ as in Notation 7.5.4, write $\Delta'(w; B; A)$ for

$$\Delta(w; B; A) \wedge \theta_A(\bar{a}),$$

where $\theta_A(\bar{z})$ is the quantifier-free-diagram of the enumeration $\bar{a}$ of A.

From the observations above, we know that the formula $\Delta(w; c_1c_3d_{12}d_{23}; D)$ is further equivalent to the conjunction

$$\exists y_{13}\Delta'(w; c_1c_3; c_1c_3y_{13}) \wedge \exists y_{12}\Delta'(w; c_1y_{12}; c_1c_2y_{12}) \wedge \exists y_{23}\Delta'(w, c_3y_{23}; c_2c_3y_{23}).$$

As noted in Lemma 7.5.3.4, there is a unique independent 3-type isolated by $\mathrm{tp}(c_1c_2) \cup \mathrm{tp}(c_1c_3) \cup \mathrm{tp}(c_2c_3)$. Furthermore, for each of the three conjuncts in the formula above, there is a unique type over c_ic_j containing it and non-forking over $\emptyset$. Hence, from Corollary 7.2.2, we have that

$$\mu(\Delta(w; c_1c_3d_{12}d_{23}, D)) = \frac{\mu(\exists y_{13}\Delta'(w; c_1c_3; c_1c_3y_{13}))\mu(\exists y_{12}\Delta'(w; c_1y_{12}; c_1c_2y_{12}))\mu(\exists y_{23}\Delta'(w, c_3y_{23}; c_2c_3y_{23}))}{\mu(\exists yR(w, y, c_1))\mu(\exists yR(w, y, c_3))\mu(\neg\exists yR(w, y, c_2))}.$$

We can use again the fact that for $1 \leq i < j \leq 3, d_{ij} \in \mathrm{dcl}(c_ic_j)$ to get that

$$\exists y_{13}\Delta'(w; c_1c_3; c_1c_3y_{13}) \equiv \Delta(w; c_1c_3; c_1c_3d_{13}) \tag{7.39}$$

$$\exists y_{12}\Delta'(w; c_1y_{12}; c_1c_2y_{12}) \equiv \Delta(w; c_1d_{12}; c_1c_2d_{12}) \tag{7.40}$$

$$\exists y_{23}\Delta'(w, c_3y_{23}; c_2c_3y_{23}) \equiv \Delta(w, c_3d_{23}; c_2c_3d_{23}) \tag{7.41}$$

Note that, for any $1 \leq i < j \leq 3$, $c_ic_jd_{ij}$, and any permuation of these three vertices has the same type, isolated by the formula for a hyperedge. From Remark 7.5.6 and equation 7.36 from Lemma 7.5.5, each of these formulas has measure p^2. Since $\mu(\exists yR(w, y, a)) = p$ and $\mu(\neg\exists yR(w, y, a)) = 1 - p$, we can conclude that

$$\mu(\Delta(w; c_1c_3d_{12}d_{23}, D)) = \frac{p^6}{p^2(1-p)} = \frac{p^4}{(1-p)},$$

as required. $\square$

Theorem 7.5.8. *The structure $\mathcal{M}_f^R$ is not MS-measurable.*

Proof. Lemma 7.5.5 (with Remark 7.5.6) and Lemma 7.5.7 offer two distinct calculations for the measure of $\Delta(w; c_1c_3d_{12}d_{23}, D)$, yielding that

$$p^4 = \frac{p^4}{(1-p)}.$$

Solving this equation, we obtain that $p = 0$, contradicting positivity of the measure in MS-measurable structures. $\square$

Hence, from Theorem 7.5.2 and Theorem 7.5.8, we can conclude:

Corollary 7.5.9. *There are supersimple ω-categorical structures of SU-rank 1 satisfying the strong independence theorem and independent n-amalgamation for all $n \in \mathbb{N}$ over finite algebraically closed sets, which are not MS-measurable.*

8
Conclusions

In this final chapter we discuss some questions which are left at the end of this thesis and possible further work.

8.1 Remaining questions on MS-measurability in ω-categorical structures

Much of the work of this thesis was motivated by the question of Elwes and Macpherson [19] of whether ω-categorical MS-measurable structures are one-based (Question 5.0.2). In this thesis, we showed that MS-measurability fails in various ω-categorical Hrushovski constructions. Moreover, their non-MS-measurability is related to them being not one-based. The major result on this front was Theorem 6.2.7 showing that ω-categorical MS-measurable structures satisfy the strong independence theorem over the algebraic closures of finite sets. However, a supersimple ω-categorical structure can be not one-based and satisfy this property. An example of this are the structures in Construction 7.5.1, which are not one-based but satisfy the strong independence theorem over the algebraic closures of finite sets. This is proven in Corollary 9.3.9. Hence, while both our results and [23] show that many classes of supersimple ω-categorical Hrushovski constructions fail to be MS-measurable, we would still like to see whether one could directly prove that being not one-based implies non-MS-measurability [19]:

Question 5.0.2. *Is every ω-categorical MS-measurable structure one-based?*

Continuing the work of this thesis, we may study more amalgamation properties which can fail in supersimple ω-categorical structures and whose failure implies non-MS-measurability. As we noted, the strong independence theorem and the weak form of independent n-amalgamation from [23] are such properties, but the story clearly does not end there. For example, the failure of MS-measurability of Construction 7.5.1 seems to be related to the fact that certain configurations of hyperedges are forbidden in $\mathcal{M}_f^R$ (cf. Lemma 7.5.3.4).

For this reason, it is worth asking whether MS-measurability may imply other amalgamation properties in addition to simplicity and the strong independence theorem over the algebraic closures of finite sets. Chapter 7 yields some conflicting intuitions on this front. On one hand, the generic tetrahedron-free 3-hypergraph is not MS-measurable and the proof of this makes use of the fact this structure does not satisfy independent 4-amalgamation over $\emptyset$. However, the universal homogeneous two-graph is MS-measurable in spite of independent 4-amalgamation over $\emptyset$ failing. David Evans pointed out to me that [11, Example 5.1.6] also gives a supersimple ω-categorical structure not satisfying independent 4-amalgamation over $\mathrm{acl}(\emptyset)$. As we noted in Theorem 7.4.8, the universal homogeneous two-graph satisfies independent n-amalgamation over models for all $n \in \mathbb{N}$, while this property still fails for the generic tetrahedron-free 3-hypergraph. Hence, we may want to ask:

Question 8.1.1. Does being MS-measurable and ω-categorical imply independent n-amalgamation over models for all $n \in \mathbb{N}$?

This question motivates an investigation of which supersimple ω-categorical structures actually satisfy higher independent n-amalgamation properties over models. For example, it is unknown whether the smoothly approximable structures of [11] satisfy independent n-amalgamation over models for $n > 3$ and proving this (or disproving it) would be an interesting result in its own right. Regarding Question 8.1.1, it is worth noting that Constructions 5.1.1 and 7.5.1 should yield counterexamples to the converse. Both classes contain structures satisfying independent n-amalgamation over finite algebraically closed sets. While independent n-amalagation over models does not directly follow from this (cf. [22, p. 168]), we expect these structures to also satisfy this property via a proof similar to that of simplicity of ω-categorical Hrushovski constructions [22, 69]. Hence, they should show that independent n-amalgamation over models and ω-categoricity do not jointly imply MS-measurability. As we noted, we do not know any examples of one-based ω-categorical structures which satisfy independent n-amalgamation over models but which are not MS-measurable. Perhaps these conditions are sufficient to imply MS-measurability.

8.2 Further questions on invariant Keisler measures in simple ω-categorical structures

As we have seen in Lemma 6.2.6, MS-measurability in an ω-categorical context implies $F(\emptyset) = \mathcal{O}(\emptyset)$. It is then worth asking analogues of some of the questions that we have been asking on MS-measurability for $F(\emptyset) = \mathcal{O}(\emptyset)$:

Question 8.2.1. Does being supersimple and ω-categorical satisfying the strong indepence theorem imply $F(\emptyset) = \mathcal{O}(\emptyset)$?

Question 8.2.2. Does being supersimple, ω-categorical and one-based imply $F(\emptyset) = \mathcal{O}(\emptyset)$?

Question 8.2.3. Does being supersimple, ω-categorical with $F(\emptyset) = \mathcal{O}(\emptyset)$ imply being one-based?

Question 8.2.3 is an analogue of Question 5.0.2 of Elwes and Macpherson [19], which remains open in the MS-measurable context. Meanwhile, for questions

8.2.1 and 8.2.2, we do have a counterexample which is not MS-measurable given by the generic tetrahedron-free 3-hypergraph. If in Question 8.2.1 we add the condition of satisfying independent n-amalgamation for all n over finite algebraically closed sets, then Construction 7.5.1 is not MS-measurable but satisfies all of the hypotheses. We would expect these structures also to be such that $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$. It would be interesting to prove this, especially for the generic tetrahedron-free 3-hypergraph:

Conjecture 8.2.4. The generic tetrahedron-free 3-hypergraph is such that $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$.

Unfortunately, it is unclear how to adapt the proofs of non-MS-measurability of Chapter 7 to prove $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$. For example, the proof of non MS-measurability of the generic tetrahedron-free 3-hypergraph of Theorem 7.3.9 relied heavily on the Fubini property. In particular, both Theorem 7.2.1 and Remark 7.3.2, which are used in the proof, relied on the fact that in an MS-measurable structure the $\mathrm{Aut}(M/A)$-invariant Keisler measures induced by A-definable subsets are particularly well-behaved. For example, they will still be $\mathrm{Aut}(M/A)$-ergodic and satisfy Theorem 3.1.9. Meanwhile, in the context of arbitrary invariant Keisler measures it is looks implausible that given an $\mathrm{Aut}(M)$-ergodic Keisler measure, any $\mathrm{Aut}(M/A)$-invariant Keisler measure it induces on A-definable subsets is still going to be $\mathrm{Aut}(M/A)$-ergodic. Moreover, other results in Chapter 7, such as the negative answer to Question 7.0.1 with the universal homogeneous two-graph in Remark 7.4.17, show that at least when independent 4-amalgamation over $\emptyset$ fails, we cannot reduce questions about the measures of higher amalgamations to questions on the measures of the types being amalgamated in the same way as we did with the measures of 3-amalgamations in Chapter 3.

We are still lacking tools to deal with the measures of higher amalgamations when working with invariant Keisler measures even in a simple ω-categorical context and it would be worth developing some. Perhaps, unlike the direction taken in Chapter 7, it would be good to ask whether some positive results can be obtained when a structure satisfies independent 4-amalgamation. In particular, it would be interesting to study invariant Keisler measures in the universal homogeneous 3-hypergraph and see whether one can prove a similar result to Albert's classification of the invariant Keisler measures for the random graph [2]. This structure is the Fraïssé limit of all finite (3-uniform) 3-hypergraphs and, similarly to our explanation of the measures on the random graph in the introduction, any invariant Keisler measure on it is entirely determined by the measures of the formulas $\Phi(x, A, B)$, where for A a finite set of vertices of size n and $B \subseteq [A]^2$, $\Phi(x, A, B)$ expresses that x forms a hyperedge with each pair in B and does not form a hyperedge with each pair in $[A]^2 \setminus B$. We conjecture that the universal homogeneous 3-hypergraph has a similar classification of its invariant Keisler measures to the random graph:

Conjecture 8.2.5. Let $\mathcal{T}$ be the countable model of the universal homogeneous 3-hypergraph and μ an $\mathrm{Aut}(T)$-invariant Keisler measure on $\mathrm{Def}_x(T)$, where x is the singleton variable. Then, there is a unique probability measure ν on $[0, 1]$ such that

$$\mu(\Phi(x, A, B)) = \int_0^1 p^{|B|}(1-p)^{\binom{n}{2}-|B|}\mathrm{d}\nu.$$

We already know that the measures μ_p assigning $\Phi(x, A, B)$ the measure $p^{|B|}(1-p)^{\binom{n}{2}-|B|}$ are ergodic since they naturally induce an MS-dimension-measure [50, Proposition 5.12]. The main challenge is showing there are no more invariant Keisler measures than these. This problem is further discussed at the end of Ensley's PhD Thesis [21].

Going back to the structures that we have studied in this thesis, it would be interesting to see whether we could classify all invariant Keisler measures on them. In particular, our results on non-MS-measurability or on $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$ suggest that these structures have few invariant Keisler measures. For example, in the context of the generic tetrahedron-free 3-hypergraph, it is worth asking whether there is any invariant Keisler measure on $\mathcal{H}$ except for the unique invariant type on this structure. The following conjecture would imply Conjecture 8.2.4 and yield a negative answer to Question 8.2.2:

Conjecture 8.2.6. There is a unique $\mathrm{Aut}(H)$-invariant Keisler measure on the generic tetrahedron-free 3-hypergraph $\mathcal{H}$ in the singleton variable, which corresponds to the unique invariant type in $S_1(H)$.

The unique invariant type in $S_1(H)$ asserts for each $a, b \in \mathcal{H}$ that $\neg R(x, a, b)$. If this conjecture were true, measures on the generic tetrahedron-free 3-hypergraph would behave in same way as in the generic triangle-free graph: from [2] we know that the latter has a unique invariant Keisler measure, which is also its unique invariant type (asserting that x does not share an edge with every vertex in the model). Furthermore, it is reasonable to ask whether for $r \geq 2$ the generic $K^r_{(r+1)}$-free r-hypergraph behaves similarly. Can we prove it is not MS-measurable (we know this is the case for $r = 2, 3$)? Can we prove that in one variable it has a unique invariant Keisler measure corresponding to the unique invariant type over the countable model? We expect the generic tetrahedron-free 3-hypergraph to be the structure for which proving such things will be easier, but a general argument would be desirable.

For the Hrushovski constructions of Construction 5.1.1 and 7.5.1, we may ask similarly whether every $\mathrm{Aut}(M)$-ergodic Keisler measure is an $\mathrm{Aut}(M)$-invariant type. Proving this may be difficult since we do not know all of the invariant types of these structures, but it is worth asking whether there is some Hrushovski construction for which we can prove this:

Question 8.2.7. Is there some not one-based supersimple ω-categorical Hrushovski construction $\mathcal{M}$ for which all $\mathrm{Aut}(M)$-ergodic Keisler measures are $\mathrm{Aut}(M)$-invariant types?

If in an ω-categorical structure all $\mathrm{Aut}(M)$-ergodic measures are $\mathrm{Aut}(M)$-invariant types, we can prove that any invariant Keisler measure is a (possibly infinite) weighted sum of invariant types. This follows from the ergodic decomposition Theorem 3.2.3 since the space of ergodic measures would be countable by Ryll-Nardzewski. As we point out below, this is very similar to the behaviour of invariant Keisler measures in NIP ω-categorical structures.

In unpublished work with Braunfeld [4], we extend the results of Ensley [20] on measures in NIP ω-categorical structures in order to obtain a full classification of their invariant Keisler measures. In these structures, any $\mathrm{Aut}(M)$-invariant Keisler measure can be obtained as a weighted sum of $\mathrm{Aut}(M/\mathrm{acl}^{eq}(\emptyset))$-invariant types. This can also be deduced from the results of [38]. Moreover, we isolate the property of NIP ω-categorical structures which implies this classification of the measures: $\mathcal{M}$ has the **invariant extension property** (IEP) if any $\mathcal{L}(M)$-formula (in the variable x) which is not universally measure zero is contained in an $\mathrm{Aut}(M/\mathrm{acl}^{eq}(\emptyset))$-invariant type in $S_x(M)$. NIP ω-categorical structures have the IEP. Moreover, we already know that this property is true of some ω-categorical structures which have the independence property (i.e. are not NIP), such as the generic triangle-free graph. More generally, any structure for which every $\mathrm{Aut}(M)$-ergodic Keisler measure is an $\mathrm{Aut}(M)$-invariant type has the invariant extension property. It would be interesting to understand how common this phenomenon is in the realm of ω-categorical structures. As noted in Conjecture 8.2.6 and Question 8.2.7, the generic tetrahedron-free 3-hypergraph and ω-categorical Hrushovski constructions are good candidates for simple ω-categorical structures with the invariant extension property.

9 Appendix: amalgamation properties in Hrushovski constructions

In Chapter 5 and in Section 7.5, we introduced classes of ω-categorical supersimple Hrushovski constructions of finite SU-rank. These are given in Construction 5.1.1 and 7.5.1 respectively. In both cases, we specified the initial values and growth rate of functions f from which the Hrushovski constructions are built. We omitted from the body of the thesis the proofs that our choices of f imply the various properties we claim to hold for these structures. Firstly, proving simplicity for both structures and that the strong independence theorem holds for Construction 7.5.1 requires some effort. We include these technical results in this appendix. Secondly, we also claim that these structures can satisfy independent n-amalgamation for all $n \in \mathbb{N}$ over finite algebraically closed sets. Proving this requires generalising the methods used for proving simplicity and building new tools. In fact, it is unclear whether the growth rate for f which had been believed to yield independent n-amalgamation for all $n \in \mathbb{N}$ in [35] and [69] does actually work (cf. Remark 9.2.18).

We begin with a discussion of how to prove simplicity and the strong independence theorem in Section 9.1. The discussion on how to prove simplicity clarifies tools which are already discussed in the literature on Hrushovski constructions [27, 22, 68, 69]. The discussion on the strong independence theorem is original but follows the same structure. We also prove the folklore result that in these structures SU-rank agrees with the Hrushovski dimension.

Section 9.2 discusses sufficient conditions for a simple ω-categorical Hrushovski construction to satisfy independent n-amalgamation over finite algebraically closed sets for $n > 3$. These have been previously discussed in [35] and [69], where it is noted how independent n-amalgamation for arbitrarily large n can be obtained by slow enough growth-rates of f following a similar proof to that of simplicity. We give explicit proofs of why certain growth rates guarantee higher independent amalgamation properties. The initial part of our discussion generalises the tools for proving independent 3-amalgamation (and simplicity) to higher amalgamations. In particular, this is the case for Subsection 9.2.1. Subsection 9.2.2 discusses which growth rates guarantee different independent n-amalgamation properties. While proving independent n-amalgamation for all $n \leq N$ for a fixed N follows similar lines to what was expected in the literature, we think that a different choice of growth rate is needed to ensure independent n-amalgamation for all n. In Theorem 9.2.19, we give an explicit example of a growth rate which guarantees independent n-amalgamation for all n. This is the most original result in this appendix and it follows a different route from the standard approach.

Finally, Section 9.3 is dedicated to prove the properties which we attributed to the ω-categorical Hrushovski constructions of Constructions 5.1.1 and 7.5.1. Subsection 9.3.1 focuses on Construction 5.1.1 and Subsection 9.3.2 on Construction 7.5.1. For both classes of structures, we prove that they are supersimple of finite SU-rank with weak elimination of imaginaries in Theorems 9.3.4 and 9.3.9. For the constructions $\mathcal{M}_f^R$, we also prove that the strong independence theorem holds over finite algebraically closed sets in Theorem 9.3.9. Finally, we prove that these structures can satisfy independent n amalgamation for all n over algebraically closed sets with a similar choice of growth-rate for f to that described in Theorem 9.2.19.

9.1 Simplicity of ω-categorical Hrushovski constructions

In this section, we discuss under which conditions on f ω-categorical Hrushovski constructions are supersimple of finite SU-rank. This was first explored in [35]. We simplify some of the conditions for supersimplicity discussed in [27] and [22]. This will shorten the proofs of supersimplicity of the structures considered in the next section. Much of this material is implicit in the proof of Theorem 3.6 in [22]. Here we make the arguments explicit and correct a mistake in Remark 3.8 of the same article.

In parallel to these results, we investigate which conditions imply that an ω-categorical Hrushovski construction satisfies the strong independence theorem of Definition 6.2.2. In Chapter 6, we showed how failure of this amalgamation property in a simple ω-categorical structure with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$ implies that there are formulas which do not fork over $\emptyset$ but are universally measure zero. We prove that the growth rate of f which is standardly used for ensuring that $\mathcal{M}_f$ is supersimple and of finite SU rank is also sufficient for ensuring that $\mathcal{M}_f$ satisfies the strong independence theorem. In particular, this means that it is perfectly possible to build Hrushovski constructions satisfying this property (and indeed we do this in Construction 7.5.1). This is not in constrast with the fact that the Hrushovski construction from Construction 5.1.1 does not satisfy the strong independence theorem. In fact, for that construction, we are forcing a failure of this property by controlling some of the initial values of the function f rather than by controlling its growth rate.

We conclude with a general argument for why in these supersimple Hrushovski constructions SU-rank corresponds to the Hrushovski dimension. This is Theorem 9.1.15. This fact was well-known in the case of $\alpha = 1$ (cf. [69, Example 6.2.27], [23, Remarks 4.2]).

Throughout this section, $\mathcal{K}_f$ denotes a free amalgamation class built as in Definition 4.1.2. We begin by recalling some basic properties of the d-closed and self-sufficient relations which were introduced in Definition 4.1.1:

Lemma 9.1.1 (Lemma 3.10 in [24])**.** *Let $C \in \overline{\mathcal{K}}$ and let $\leq'$ stand for either $\leq$ or $\leq^*$. Then, the following hold:*

1. *let $A \leq' C$ be finite, and $B \subseteq C$. Then, $A \cap B \leq' B$;*
2. *let A and B be finite such that $A \leq' B \leq' C$. Then, $A \leq' C$; and*
3. *let A and B be finite with $A, B \leq' C$. Then, $A \cap B \leq' C$.*

Remark 9.1.2. In the definition below, following Lemma 4.1.7, when saying that B and C are **independent** over A in D we mean that B and C are freely amalgamated over A (which implies that $B \cap C = A$) and that $\delta(\mathrm{cl}_D(B \cup C)) = \delta(B \cup C)$.

Definition 9.1.3 (Independence Theorem Diagram)**.** Let D be a graph or a hypergraph. Suppose that D has substructures D_i and D_{ij} for $0 \leq i < j \leq 3$ all contained in $\mathcal{K}_f$. To simplify our notation, we allow inverting indices, e.g. $D_{12} = D_{21}$. We say that $(D_0; (D_i | i \in \{1, 2, 3\}); (D_{ij} | 1 \leq i < j \leq 3); D)$ is an **independence theorem diagram** with respect to $\mathcal{K}_f$, or simply that D is an **ITD**, if the following hold:

1. $D_{0j} = D_j$ for $1 \leq j \leq 3$;
2. $D_i \cap D_j = D_0$ for $1 \leq i < j \leq 3$;
3. $D_i, D_j \leq D_{ij}$ for $1 \leq i < j \leq 3$;
4. $D_{ij} \cap D_{jk} = D_j$ for $\{i, j, k\} = \{1, 2, 3\}$; and
5. any edge (or hyperedge) in D is entirely contained within some D_{ij} for $1 \leq i < j \leq 3$.
6. D_i and D_j are independent over D_0 in D_{ij} for $1 \leq i < j \leq 3$;

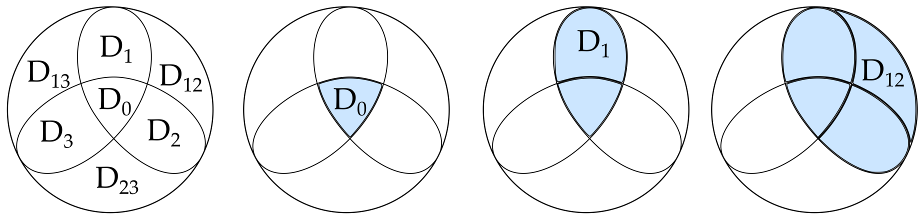


Figure 9.1: A visual representation of an independence theorem diagram. The first picture represents an ITD with its various parts labelled. To distinguish the parts more clearly, the last three pictures show as a shaded area D_0, D_1, and D_{12} respectively.

Figure 9.1 gives a visual representation of an independence theorem diagram.

We say that D is a **proper ITD** when it satisfies the following additional conditions:

7. $D_0 \subsetneq D_i$ for $1 \leq i \leq 3$; and
8. $D_i \cup D_j \subsetneq D_{ij} = \mathrm{cl}_D(D_i \cup D_j)$ for $1 \leq i < j \leq 3$.

We say that $\mathcal{K}_f$ is **closed under ITDs** if, whenever $(D; D_i; D_{ij})$ is an ITD, we have that $D \in \mathcal{K}_f$.

We note some basic properties of independence theorem diagrams and proper ITDs.

Lemma 9.1.4. *Suppose D is an ITD for $\mathcal{K}_f$. Then, for $0 \leq i < j \leq 3$ we have:*

1. $D_0 \leq D_{ij}$;
2. $D_{ij} \leq D$;
3. $D_{ij} \leq D_{ij} \cup D_{jk}$;
4. $D_{ij} \cup D_{jk} \leq^* D$.

Proof. Part 1 is a direct consequence of Lemma 9.1.1.1 and the definitions. Parts 2,3, and 4 are from [27, Lemma1.6]. □

From now on, for clarity of notation, when talking about D_i, D_j, or D_k we take $i, j, k \in \{1, 2, 3\}$ and we specify when we are talking about D_0.

It is well-known that closure under ITDs yields simplicity and various desirable properties [35, 69, 22]. We list these below so that we may refer back to them when needed. The theorem below is contained in [22, Lemma 2.3, Lemma 2.9, Corollary 2.24 & Theorem 3.6]:

Theorem 9.1.5. *Suppose that $\alpha \in \mathbb{N}$ and $\mathcal{K}_f$ is closed under ITDs. Then, $\mathcal{M}_f$ is supersimple of finite SU-rank. In particular, $\mathcal{M}_f$ satisfies the independence theorem over algebraically closed sets and non-forking independence is such that for a finite tuple a and finite $C \leq B \leq \mathcal{M}_f$,*

$$a \underset{C}{\downarrow} B \text{ if and only if } d(a/C) = d(a/B).$$

Also, $\mathcal{M}_f$ has weak elimination of imaginaries.

While closure under ITDs yields that $\mathcal{M}_f$ is supersimple of finite SU-rank, we also want to study an analogous notion which will imply that $\mathcal{M}_f$ satisfies the strong independence theorem. For this purpose, we introduce **Strong Independence Theorem Diagrams**:

Definition 9.1.6 (Strong Independence Theorem Diagram)**.** With the notation of Definition 9.1.3, we say that D is a **Strong Independence Theorem Diagram** (SITD) if it satisfies the conditions 1-5 of Definition 9.1.3 and the following modification of condition 6:

6*. D_1 and D_j are independent over D_0 in D_{1j} for $j \in \{2, 3\}$.

We say that D is a **proper SITD** if it further satisfies condition 7 of Definition 9.1.3 and the following modification of 8:

8*. $D_1 \cup D_j \subsetneq D_{1j} = \mathrm{cl}_D(D_1 \cup D_j)$ for $j \in \{2, 3\}$.

Finally, $\mathcal{K}_f$ is **closed under SITDs** if whenever D is a SITD, $D \in \mathcal{K}_f$.

In a strong independence theorem diagram we allow for D_{23} to be such that D_2 and D_3 are not independent over D_0, while still requiring that they are weakly algebraically independent over D_0, i.e. that they are still closed in D_{23} with closed intersection D_0.

Lemma 9.1.7. *Suppose that $(D; D_i; D_{ij})$ is a SITD for $\mathcal{K}_f$. Then,*

$$D_{ij} \leq D \text{ for each } 1 \leq i < j \leq 3.$$

Proof. For $1 \leq i < j \leq 3$, let $D_{ij} \subsetneq B \subseteq D$. Let $k \neq i, j$ and say without loss of generality that D_{ik} is such that D_i and D_k are independent over D_0 in D_{ik}.

Claim: $\delta(B \cap D_{ik}) \geq \delta(B \cap D_k) + \delta(D_i) - \delta(D_0)$.

Proof. Since D_i and D_k are independent over D_0 in D_{ik}, $(D_i \cup D_k) \leq^* D_{ik}$. Then, by Lemma 9.1.1.1,

$$B \cap (D_i \cup D_k) \leq^* B \cap D_{ik}.$$

Hence, since $B \cap D_i$ and D_k are freely amalgamated over D_0,

$$\delta(B \cap D_{ik}) \geq \delta(B \cap (D_i \cup D_k)) = \delta(B \cap D_k) + \delta(D_i) - \delta(D_0).$$

□

Recalling condition 5 of Definition 9.1.3, by inclusion-exclusion we get that:

$$\begin{aligned}\delta(B) &= \delta(B \cap D_{ik}) + \delta(B \cap D_{jk}) + \delta(D_{ij}) - \delta(B \cap D_k) - \delta(D_i) - \delta(D_j) + \delta(D_0)\\ &\geq \delta(B \cap D_{jk}) + \delta(D_{ij}) - \delta(D_j) + \delta(D_0)\\ &> \delta(D_{ij}).\end{aligned}$$

Here, the first equality follows by the claim, while the second by the fact that $D_j \leq B \cap D_{jk}$ from Lemma 9.1.1.1 □

Theorem 9.1.8. *Suppose that $\mathcal{K}_f$ is closed under strong independence theorem diagrams and that $\mathcal{M}_f$ has weak elimination of imaginaries. Then, $\mathcal{M}_f$ satisfies the strong independence theorem over finite algebraically closed sets.*

Proof. Note that since closure under SITDs implies closure under ITDs, $\mathcal{M}_f$ is supersimple of finite rank with non-forking independence behaving as in Theorem 9.1.5. Suppose that $D_0 \leq D_2, D_3 \leq \mathcal{M}_f$ are such that $D_2 \mathop{\smile\!\!\!\!\!\mid}^{a}_{D_0} D_3$. In particular, since D_2 and D_3 are algebraically closed, by weak elimination of imaginaries, $D_0 = D_2 \cap D_3 \leq \mathcal{M}_f$. Now, say that a_2 and a_3 are two tuples such that $a_2 \equiv_{D_0} a_3$ and $a_2 \mathop{\smile\!\!\!\!\!\mid}_{D_0} D_2$ and $a_3 \mathop{\smile\!\!\!\!\!\mid}_{D_0} D_3$. Set $D_{12} := \mathrm{acl}(a_2 D_2)$ and $D_{13} := \mathrm{acl}(a_3 D_3)$, and $D_{23} := \mathrm{acl}(D_2 D_3)$. Since types are entirely determined by the quantifier-free types of their closures, $\mathrm{qftp}(\mathrm{acl}(a_2D_0)) = \mathrm{qftp}(\mathrm{acl}(a_3D_0))$, and we can set $D_1 := \mathrm{acl}(a_2D_0)$. Since $D_0, D_1, D_2, D_3, D_{12}, D_{13}, D_{23}$ are all in $\mathcal{K}_f$, we can consider the associated strong independence theorem diagram D. Since $\mathcal{K}_f$ is closed under SITDs, $D \in \mathcal{K}_f$. Since $D_{23} \leq D$ by Lemma 9.1.7, by the extension property, there is an embedding of a copy D^* of D in $\mathcal{M}_f$ over D_{23} such that $D_{23} \leq D^* \leq \mathcal{M}_f$. Let a^* be the copy of a_2 (and a_3) in D_1^*, the copy of D_1 in D^*. By construction, since $D_1^* \leq D_{1j}^* \leq D^* \leq \mathcal{M}_f$ for $j \in \{2,3\}$, and the types of tuples being determined by the quantifier-free types of their closures, $a^* \equiv_{D_2} a_2$ and $a^* \equiv_{D_3} a_3$.

Finally, we require that $a^* \mathop{\smile\!\!\!\!\!\!|}\nolimits_{D_0} D_2D_3$, for which it is sufficient to show that $\delta(D/D_{23}) = \delta(D_1/D_0)$. This follows from the calculation below.

$$\begin{aligned}\delta(D/D_{23}) &= \delta(D) - \delta(D_{23}) && (9.1)\\ &= \delta(D_{12}) + \delta(D_{13}) - \delta(D_1) - \delta(D_2) - \delta(D_3) + \delta(D_0) && (9.2)\\ &= \delta(D_{13}) - \delta(D_3) = \delta(D_1) - \delta(D_0) = \delta(D_1/D_0). && (9.3)\end{aligned}$$

The second equality follows by inclusion-exclusion calculations, while the third and the fourth follow form the fact that D_1 and D_j are freely amalgamated over D_0, yielding $\delta(D_{ij}) = \delta(D_i) + \delta(D_j) - \delta(D_0)$. □

Hence, it is sufficient to include all ITDs in $\mathcal{K}_f$ in order to have supersimplicity and finite SU-rank for $\mathcal{M}_f$, and it is sufficient to include all SITDs to satisfy the strong independence theorem over algebraically closed sets. However, these properties are not easy to check because in order to see whether D is in $\mathcal{K}_f$ we need to verify that every subgraph $B \subset D$ is also in $\mathcal{K}_f$. The following lemmas help us simplifying this process. We prove the lemmas explicitly for independence theorem diagrams and include as corollaries the equivalent statement for strong independence theorem diagrams, which will have the same proof.

Lemma 9.1.9. *Let $C \leq D \in \mathcal{K}_f$, where D is an ITD. Then, C is an ITD.*

Proof. Call C_{ij} the intersection of C and D_{ij}. The only conditions that we need to check are that $C_i \leq C_{ij}$ and that C_i and C_j are independent over C_0 in C_{ij}.

Now, the first condition follows by Lemma 9.1.1.1 since $D_i \leq D_{ij}$ and $C_{ij} \subseteq D_{ij}$ implies $C_i = D_i \cap C_{ij} \leq C_{ij}$.

For the second condition, first note that since D_i and D_j are freely amalgamated over D_0 in D_{ij}, so must be C_i and C_j over C_0 in C_{ij}, being just intersections of these sets. We know that $D_i \cup D_j \leq^* \mathrm{cl}_{D_{ij}}(D_i \cup D_j), \mathrm{cl}_{C_{ij}}(C_i \cup C_j) \cap (D_j \cup D_j) = C_i \cup C_j$, and $\mathrm{cl}_{C_{ij}}(C_i \cup C_j) \cap \mathrm{cl}_{D_{ij}}(D_i \cup D_j) = \mathrm{cl}_{C_{ij}}(C_i \cup C_j)$. Hence, by Lemma 9.1.1.1, $C_i \cup C_j \leq^* \mathrm{cl}_{C_{ij}}(C_i \cup C_j)$. But then, by Lemma 4.1.7, we know that C_i and C_j are independent over C_0 in C_{ij}. □

The same proof works for SITDs noting that we just need to avoid checking that C_2 and C_3 are independent over C_0 in C_{23}:

Corollary 9.1.10. *Let $C \leq D \in \mathcal{K}_f$, where D is a SITD. Then, C is an SITD.*

We can use Lemma 9.1.9 to show that, when proving supersimplicity, we can reduce the number of hypergraphs for which we check $f(|D|) \leq \delta(D)$ by only focusing on proper ITDs. As above, the same proof works for SITDs.

Lemma 9.1.11. *The following are equivalent:*

1. *$\mathcal{K}_f$ is closed under ITDs;*
2. *$\mathcal{K}_f$ is closed under proper ITDs; and*
3. *for each proper ITD D we have that $f(|D|) \leq \delta(D)$.*

Proof. (1)⇒(2)⇒(3) follows from the definitions of proper ITD and $\mathcal{K}_f$. Let us focus then on (2)⇒(1). Let D be an ITD. For it to be in $\mathcal{K}_f$, we need to show that for any $C \subseteq D$, $f(|C|) \leq \delta(C)$. Note that if $f(|C|) > \delta(C)$, also $\mathrm{cl}_D(C)$ satisfies this condition since

$$f(|\mathrm{cl}_D(C)|) \geq f(|C|) > \delta(|C|) \geq \delta(\mathrm{cl}_D(C)),$$

where the first inequality holds since f is increasing and the last by definition of closure. Hence, we may assume that $C \leq D$. We shall show that if C is not a proper ITD, it must already be in $\mathcal{K}_f$, or that there is a proper ITD C', such that if C' is in $\mathcal{K}_f$, then so is C.

We label the intersections of C with D so that $C_i := D_i \cap C$ and $C_{ij} := D_{ij} \cap C$. By Lemma 9.1.9, C is an independence theorem diagram.

Suppose that $C_1 = C_0$, then C is in $\mathcal{K}_f$, being obtained by freely amalgamating C_{12} and C_{23} over C_2, and then freeely amalgamating this structure with C_{13} over C_3. Hence, if $C_i = C_0$ for any $1 \leq i \leq 3$, $C \in \mathcal{K}_f$.

Suppose that $C_{12} = C_1 \cup C_2$. Then, C is the free amalgamation of C_{23} and C_{13} over C_3. Thus, if $C_{ij} = C_i \cup C_j$, then $C \in \mathcal{K}_f$.

Finally, consider the substructure C' of C which constitutes the proper independence theorem diagram obtained by taking $C'_{ij} := \mathrm{cl}^C(C_i \cup C_j)$. Now, by construction and definition of closure, we must have $C'_{ij} \leq C_{ij}$. Furthermore, $C'_{ij} \leq C'$. Hence, C may be obtained as a free amalgamation of C' and C_{ij} over C'_{ij}, eventually repeating this operation for the different C_{ij}'s (see the Claim of Theorem 9.2.19).

So, we have seen that if all proper ITDs are in $\mathcal{K}_f$, then so are all ITDs, and so (2)$\Rightarrow$(1). We prove that (3)$\Rightarrow$(2) holds by considering a minimal counterexample. Suppose that D is a minimal proper ITD not in $\mathcal{K}_f$. Since $f(|D|) \leq \delta(D)$, there must be some $C \subset D$ such that $f(|C|) > \delta(|C|)$. Again, we may assume that $C \leq D$. By the previous lemma, C is an ITD. From the steps above, we know that it will be either a free amalgamation of hypergraphs in $\mathcal{K}_f$, or the free amalgamation of hypergraphs in $\mathcal{K}_f$ and some proper ITD, or it will be a proper ITD again. But, by minimality, any of these cases implies that $C \in \mathcal{K}_f$. So, it is sufficient to check the condition $f(|D|) \leq \delta(D)$ in proper ITDs. □

Again, the corresponding statement for SITDs follows:

Corollary 9.1.12. *The following are equivalent:*

1. *$\mathcal{K}_f$ is closed under SITDs;*
2. *$\mathcal{K}_f$ is closed under proper SITDs; and*
3. *for each proper SITD D we have that $f(|D|) \leq \delta(D)$.*

We have seen that we only need to prove that $f(|D|) \leq \delta(D)$ for the relevant ITDs in order to prove supersimplicity, or for the relevant SITDs in order to prove the strong independence theorem. Now, we give a sufficient condition on the growth rate of f in order to achieve these conditions. Note that it is the same condition for ITDs and SITDs:

Lemma 9.1.13. *Let $\alpha \in \mathbb{N}$ and $\mathcal{K}_f$ be a free amalgamation class. Let $n \in \mathbb{N}$. Suppose that for $n \leq t$, $f(3t) \leq f(t) + k$, for fixed $k \in \mathbb{N}$. Let D be an ITD (or a SITD). Say that D_{lm} has the biggest predimension d_{lm} among D_{12}, D_{13}, and D_{23}. Suppose that $f(n) \leq d_{lm}$. We have that if $\delta(D) \geq d_{lm} + k$, then $f(|D|) \leq \delta(D)$.*

Proof. This proof is essentially identical to the final part of the Theorem 3.6 in [22]. However, we repeat the argument for completeness and in order to avoid confusion with Remark 3.8 of [22], which follows the theorem and contains a mistake.

Let g be the inverse of f. Making the substitution $g(x) = t$ into $f(3t) \leq f(t) + k$, we get that $3g(x) \leq g(x+k)$ for $f(n) \leq x$. Then, for $f(n) \leq d_{lm}$:

$$|D| \leq \sum_{1 \leq i < j \leq 3} |D_{ij}| \leq \sum_{1 \leq i < j \leq 3} g(\delta(D_{ij})) \leq 3g(d_{lm}) \leq g(d_{lm} + k) \leq g(\delta(D))$$

Where the second inequality holds since $D_{ij} \in \mathcal{K}_f$ and the fourth holds since $\delta(D) \geq d_{lm} + k$ and g is increasing. Note that the resulting inequality is equivalent to $f(|D|) \leq \delta(D)$. □

Remark 9.1.14. Consider the case of $k = 1, n = 0$ in Lemma 9.1.13. As we have proven, in both an ITD and and SITD, $D_{ij} \leq D$. In particular, this implies that $\delta(D) > \delta(D_{ij})$ for each $1 \leq i < j \leq 3$. And since $\alpha \in \mathbb{N}$, we also have that $\delta(D) \geq \delta(D_{ij}) + 1$. Hence, under these circumstances, having that for all $t \in \mathbb{R}^{\geq 0}$, $f(3t) \leq f(t) + 1$ ensures that $\mathcal{M}_f$ is closed under SITDs (and so also under ITDs). Since functions of the form of f exist, e.g. $\log_3(x) + 1$, we can see that there are ω-categorical Hrushovski constructions which are supersimple of finite SU-rank and satisfying the strong independence theorem over finite algebraically closed sets. Example 4.1.10 is one such structure.

We have a method to obtain supersimple ω-categorical Hrushovski constructions of finite SU-rank, and a similar method to build ones that also satisfy the strong independence theorem over algebraically closed sets. With the tools developed in this section we will be able to verify these properties more easily.

To conclude, we note that under some very weak assumptions when $\mathcal{K}_f$ is closed under ITDs, SU-rank corresponds to the Hrushovski dimension. This fact is folklore knowledge and in [69, Example 6.2.17] and [23, Remarks 4.2] a more basic argument is given for the case of $\alpha = 1$. The basic facts about SU-rank being used are discussed in Subsection 1.2.4.

Theorem 9.1.15. *Suppose that $\alpha \in \mathbb{N}$ and that $\mathcal{K}_f$ is a free amalgamation class of m-hypergraphs closed under ITDs, so that $\mathcal{M}_f$ exists and satisfies the conditions of Theorem 9.1.5. Assume further that*

- *Vertices are closed: for $a \in M_f$, $a \leq \mathcal{M}_f$;*
- *The structure consisting of m-many vertices forming an m-hyperedge is in $\mathcal{K}_f$.*

Then, for $\bar{a}$ a finite tuple and finite $B \subseteq M_f$,

$$SU(\bar{a}/B) = d(\bar{a}/B). \tag{9.4}$$

In particular, $\mathcal{M}_f$ has SU-rank α.

Before proving the theorem, we make a couple of remarks to clarify its statement and prove a Lemma.

Remark 9.1.16. The additional conditions on $\mathcal{K}_f$ are very weak and clearly satisfied by the structures of Constructions 5.1.1 and 7.5.1. Any f such that $\mathcal{K}_f$ is a free amalgamation class closed under ITDs with $f(1) \leq \alpha, f(2) > \alpha$, and $f(m) \leq \alpha m - 1$ yields a class satisfying the required properties. In particular, since $\mathcal{K}_f$ is closed under substructures, if it does not contain an m-hyperedge, $\mathcal{M}_f$ simply consists of infinitely many vertices with no relations between them.

Remark 9.1.17. While SU-rank was defined on a monster model for a given simple theory, due to supersimplicity and ω-saturation of $\mathcal{M}_f$, all calculations regarding SU-rank of types over finite sets can be performed inside of $\mathcal{M}_f$.

Lemma 9.1.18. *Suppose that $\mathcal{K}_f$ satisfies the assumptions of Theorem 9.1.15. Let $a \in M_f$ be a singleton and $B \subseteq \mathcal{M}_f$ finite. Suppose further that $d(a/B) = k+1$ for $k \in \mathbb{N}$. Then, there is $C \supset B$ finite such that $d(a/C) = k$.*

Proof. Assume without loss of generality that $B \leq \mathcal{M}_f$. Let $B' = \mathrm{cl}(Ba)$ and let F be the free amalgamation over a of B' and an m-hyperedge E. This can be done since vertices are closed and m-hyperedges are in $\mathcal{K}_f$. We may assume by the extension property that $B' \leq F \leq \mathcal{M}_f$. Let $C' = E \setminus a$ and $C = B \cup C'$. Note that since $\delta(a) = \alpha, \delta(E) = \delta(C') + \alpha - 1$. If $\alpha = 1$, then, $a \in \mathrm{cl}(C')$, implying that $F = \mathrm{cl}(C)$. In this case, we must have that $d(a/B) = 1$ and since by algebraicity of the Hrushovski dimension $d(a/C) = 0$, C satisfies the conditions of the Lemma. Otherwise, $\alpha \geq 2$ and $C' \leq E$ since $\delta(E) = \delta(C') + \delta(a) - 1$. Furthermore, B and C' are disjoint and freely amalgamated over $\emptyset$ in $\mathcal{M}_f$.

Claim: Either $C \leq \mathcal{M}_f$ or $d(a/B) = 1$ and $F = \mathrm{cl}(C)$.

Proof of Claim. Consider $C \subsetneq T \leq F$. If $a \notin T$, since B' and E are freely amalgamated over a,

$$\delta(T) = \delta(B' \cap T) + \delta(C') > \delta(B) + \delta(C'),$$

where the inequality follows from $B \leq B'$. Otherwise, $a \in T$ and so $T = F$. We have,

$$\begin{aligned}\delta(F) &= \delta(B') + \delta(E) - \delta(a)\\ &= \delta(B') + \delta(C') - 1.\end{aligned}$$

If $d(a/B) = 1$, then $\delta(F) = \delta(B) + \delta(C')$ and so $\mathrm{cl}(C) = F$. Otherwise, $\delta(F) > \delta(B) + \delta(C') = \delta(C)$, and so $C \leq F \leq \mathcal{M}_f$. □

If $d(a/B) = 1$, then $a \in \mathrm{cl}(C)$ and so $d(a/C) = 0$ as desired. Otherwise, since $C \leq \mathcal{M}_f$,

$$\begin{aligned} d(a/C) &= \delta(F) - \delta(C) \\ &= [\delta(B') + \delta(E) - \delta(a)] - [\delta(B) + \delta(C')] \\ &= [\delta(B') - \delta(B)] + [\delta(E) - \delta(C') - \delta(a)] \\ &= d(a/B) - 1 = k. \end{aligned}$$

Here the first equality follows by $F = \mathrm{cl}(aC)$ and $C \leq \mathcal{M}_f$ and the second comes from considering both sets as free amalgamations. Since $d(a/C) = k$, we have obtained the desired result. □

Proof of Theorem 9.1.15. Since both SU-rank and the Hrushovski dimension are additive, it is sufficient to prove that the equality 9.4 holds for a singleton a. We can also assume that $B \leq \mathcal{M}_f$ since considering its closure in $\mathcal{M}_f$ preserves both the dimension and SU-rank. In fact, any extension of $\mathrm{tp}(a/B)$ to $\mathrm{tp}(a/\mathrm{cl}(B))$ is non-forking and so preserves SU-rank [64, Remark 7.1.3]. We prove by induction on n that for a a singleton and $B \leq \mathcal{M}_f$,

$$SU(a/B) \geq n \text{ if and only if } d(a/B) \geq n.$$

The base case follows from both SU-rank and the Hrushovski dimension being algebraic, i.e. they have value zero if and only if $a \in \mathrm{acl}(B)$ [69, Definition 5.1.1]. Now, suppose that our claim holds for $k \leq n$. Assume that $SU(a/B) \geq n+1$. Then, from Remark 9.1.17, there is some finite $B \leq C \leq \mathcal{M}_f$ such that $SU(a/C) \geq n$ and $\mathrm{tp}(a/C)$ is a forking extension of $\mathrm{tp}(a/B)$. By the characterisation of non-forking independence in Theorem 9.1.5, $d(a/C) < d(a/B)$. Since $d(a/C) \geq n$ by inductive hypothesis, $d(a/B) \geq n+1$.

Conversely, suppose that $d(a/B) \geq n+1$. Choose C as in Lemma 9.1.18 so that $d(a/C) \geq n$ and from the characterisation of non-forking independence, $\mathrm{tp}(a/C)$ is a forking extension of $\mathrm{tp}(a/B)$. By inductive hypothesis, $SU(a/C) \geq n$

and so by definition of SU-rank $SU(a/B) \geq n+1$ as desired.

From this, we can conclude that SU-rank agrees with the Hrushovski dimension. In particular, since any two vertices are closed and so have the same type, the SU-rank of the theory of $\mathcal{M}_f$ is the same as $SU(a)$ for $a \leq \mathcal{M}_f$. By the correspondence between SU-rank and Hrushovski dimension,

$$SU(x=x) = SU(a) = d(a) = \alpha.$$

□

9.2 Higher amalgamation in Hrushovski constructions

In this section, we study under which conditions ω-categorical Hrushovski constructions satisfy independent n-amalgamation over finite algebraically closed sets. We would like to treat higher amalgamation similarly to how we treated independent 3-amalgamation by independence theorem diagrams in the previous section. For this purpose, we introduce n-generalised independence theorem diagrams, n-GITDs, for $n \geq 2$ in Definition 9.2.1 and characterise independent n-amalgamation in ω-categorical Hrushovski constructions via being closed under n-GITDs. A similar idea is suggested in [69, Definition 6.2.18]. Previous literature [35, 69] omits explicit proofs of how to achieve independent n-amalgamation noting that one can follow similar lines to the standard proof of simplicity. We give the first explicit treatment of this context. While the tools used to prove simplicity do generalise to prove independent n-amalgamation for all $n \leq N$, it is unclear whether they actually yield independent n-amalgamation for all n. In particular, [35] and [69] suggest a choice of growth-rate of f which should yield independent n-amalgamation for all n. In Remark 9.2.18, we note there are some subtle reasons for which this choice does not yield the desired amalgamation properties via standard techniques. In order to circumvent this difficulty, we suggest a different choice of growth rate for f and prove in Theorem 9.2.19

that it does yield independent n-amalgamation over finite algebraically closed sets for all $n \in \mathbb{N}$. This is the main original result in this appendix. More generally, Subsection 9.2.2 discusses the relation between growth-rates for f and independent amalgamation properties.

We begin by introducing n-generalised independence theorem diagrams analogously to ITDs and proving some basic facts about them. We denote by $[n]^k$ the set of subsets of $[n] = \{1, \ldots, n\}$ containing k many elements. Similarly, for I a set of indices, we write $[I]^k$ for the set of subsets of I with k many elements. Throughout this section, we assume that $\alpha \in \mathbb{N}$, that $\mathcal{K}_f$ is as in Definition 4.1.2, and when $\mathcal{K}_f$ is a free amalgamation class, we let $\mathcal{M}_f$ be the associated ω-categorical Hrushovski construcion.

Definition 9.2.1. Let D be a finite l-hypergraph (possibly a graph). Let $\mathcal{K}' \subseteq \mathcal{K}$ be closed under substructures, where $\mathcal{K}$ is the class of finite l-hypergraphs. Fix $n \geq 2$. For $k < n$, for each $J \in [n]^k$ we let D_J be a substructure of D in $\mathcal{K}'$ such that, as a set of vertices,

$$D = \bigcup_{J \in [n]^{n-1}} D_J.$$

We define inductively what it means for

$$\left(D_0; (D_i)_{i \leq n}; (D_I)_{I \in [n]^2}; \ldots; (D_I)_{I \in [n]^{n-2}}; (D_I)_{I \in [n]^{n-1}}; D\right)$$

to be an n-GITD, i.e. an ***n*-Generalised Independence Theorem Diagram**. Firstly, we define an ***n*-Independence Diagram** $D \in \mathcal{K}'$ (i.e. an n-ID). For $n \geq 1$, we say that

$$\left(D_0; (D_i)_{i \leq n}; (D_I)_{I \in [n]^2}; \ldots; (D_I)_{I \in [n]^{n-2}}; (D_I)_{I \in [n]^{n-1}}; D\right)$$

is an n-ID if it satisfies the following conditions (taking $D_{[n]} := D$):

(i) $D \in \mathcal{K}'$;

(ii) for $I \subseteq J \subseteq [n], D_I \leq D_J$;

(iii) for $I, J \subseteq [n], D_I \cap D_J = D_{I \cap J}$ (where $D_\varnothing := D_0$);

(iv) for $I, J \subseteq [n]$, D_I and D_J are independent over $D_{I \cap J}$ in D; and

(v) for $I, J \subseteq [n]$, $\mathrm{cl}_D(D_I \cup D_J) = D_{I \cup J}$.

Now, we are ready to define an ***n*-Generalised Independence Theorem Diagram**. For $n \geq 2$, we say that

$$\left(D_0; (D_i)_{i \leq n}; (D_I)_{I \in [n]^2}; \ldots; (D_I)_{I \in [n]^{n-2}}; (D_I)_{I \in [n]^{n-1}}; D\right)$$

is an n-GITD if for $J \in [n]^{n-1}$,

$$\left(D_0; (D_i)_{i \in J}; (D_I)_{I \in [J]^2}; \ldots; (D_I)_{I \in [J]^{n-2}}; D_J\right)$$

is an $(n-1)$-ID (i.e. an $(n-1)$-Independence Diagram) and the following additional conditions are satisfied:

(iii) for $I, J \in [n]^{n-1}$, $D_I \cap D_J = D_{I \cap J}$; and

(vi) each hyperedge is entirely contained within some D_I for $I \in [n]^{n-1}$.

We say that $\mathcal{K}'$ is **closed under *n*-GITDs** if it contains all n-GITDs (where, by definition, all of the $(n-1)$-IDs forming the n-GITD are in $\mathcal{K}'$).

Often, for brevity we shall just say that D is an n-GITD or an n-ID leaving implicit that it has a decomposition into substructures of the form described above.

Remark 9.2.2. Since we work with ω-categorical Hrushovski constructions, in this section we are interested in classes $\mathcal{K}_f$ induced by an unbounded increasing function as in Definition 4.1.2. In order for $\mathcal{K}_f$ to induce a Hrushovski construction, we need it to be a free amalgamation class. Inspecting the above definition, we can see that $\mathcal{K}'$ is a free amalgamation class if and only if it is closed under 2-GITDs. Hence, we will be studying classes $\mathcal{K}_f$ which are closed under 2-GITDs. Moreover, when saying that a structure D is an n-GITD we will always be assuming implicitly that it is an n-GITD for the class $\mathcal{K}_f$.

Remark 9.2.3. For $\mathcal{K}_f$ a free amalgamation class, inspecting Definition 9.1.3 and 9.2.1 and recalling Lemma 9.1.4, we can see that any 3-GITD is an ITD and any proper ITD is a 3-GITD: our definition of a 3-GITD is slightly stronger than the definition of an ITD (due to condition (v) in Definition 9.2.1) but slightly weaker than the definition of a proper ITD (due to condition (7) in Definition 9.1.3). By Lemma 9.1.11, the free amalgamation class $\mathcal{K}_f$ is closed under 3-GITDs if and only it is closed under ITDs. Hence, if $\mathcal{K}_f$ is closed under k-GITDs for $k = 2, 3$, we can build an ω-categorical Hrushovski construction $\mathcal{M}_f$ which is supersimple of finite rank with weak elimination of imaginaries by Theorem 9.1.5.

Lemma 9.2.4. *Let $n \geq 2$ and suppose that D be is an n-ID. Then,*

$$\delta(D) = \delta(D_0) + \sum_{i=1}^{n} [\delta(D_i) - \delta(D_0)]. \tag{9.5}$$

Moreover,

$$\delta(D) = \sum_{k=0}^{n-1} (-1)^{n-1-k} \sum_{I \in [n]^k} \delta(D_I). \tag{9.6}$$

Proof. We prove equation 9.5 by induction on n. The base case for $n = 2$ follows from the characterisation of independence. From (iv) and (v), we know that

$$\delta(D) = \delta(D_1) + \delta(D_2) - \delta(D_0),$$

as desired. Assume as inductive hypothesis that equation 9.5 holds for $n - 1$. From (iv), we know that D_n and $D_{[n-1]}$ are independent over D_0. Hence, since from (v), $D = \mathrm{cl}_D(D_n \cup D_{[n-1]})$,

$$\delta(D) = \delta(D_n) + \delta(D_{[n-1]}) - \delta(D_0).$$

Note that $D_{[n-1]}$ is an $(n-1)$-ID by definition. Applying the inductive hypothesis to $\delta(D_{[n-1]})$ we get equation 9.5. For equation 9.6, since each D_I for $I \subsetneq [n]$ is an $|I|$-ID, we can use equation 9.5 to obtain

$$\begin{aligned}
&\sum_{k=0}^{n-1} (-1)^{n-1-k} \sum_{I \in [n]^k} \delta(D_I) \\
&= (-1)^{n-1}\delta(D_0) + \sum_{k=1}^{n-1} (-1)^{n-1-k} \sum_{I \in [n]^k} \delta(D_0) + \sum_{i \in I} [\delta(D_i) - \delta(D_0)] \\
&= \delta(D_0) \left(\sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n}{k} \right) + \sum_{k=0}^{n-2} (-1)^{n-k} \binom{n-1}{k} \sum_{i=1}^{n} [\delta(D_i) - \delta(D_0)] \\
&= \delta(D_0) + \sum_{i=1}^{n} [\delta(D_i) - \delta(D_0)] \\
&= \delta(D).
\end{aligned}$$

The first and last equalities are a consequence of equation 9.5. The second equality follows from counting the number of instances of each summand. The third equality follows from the alternating sum of the binomial coefficients being zero. This yields the desired result. □

Corollary 9.2.5. *Suppose that D is an n-ID. Then, any hyperedge in $\cup_{I\in[n]^{n-1}}D_I$ is entirely contained in one of the D_I. In particular, we have that*

$$\bigcup_{I\in[n]^{n-1}} D_I \leq^* D.$$

Proof. This statement is just a consequence of the fact that by counting vertices and edges (and since $D_I \cap D_J = D_{I\cap J}$), we have that

$$\delta(\cup_{I\in[n]^{n-1}}D_I) = \left[\sum_{k=0}^{n-1}(-1)^{n-1-k}\sum_{I\in[n]^k}\delta(D_I)\right] - \beta, \tag{9.7}$$

where β is the number of hyperedges in $\cup_{I\in[n]^{n-1}}D_I$ not contained in any of the D_I for $I \in [n]^{n-1}$ (cf. [68, Proposition 7.2]). From (v) in Definition 9.2.1, $D = \mathrm{cl}_D(\cup_{I\in[n]^{n-1}}D_I)$, and so

$$\delta(D) \leq \delta(\cup_{I\in[n]^{n-1}}D_I) = \sum_{k=0}^{n-1}(-1)^{n-1-k}\sum_{I\in[n]^k}\delta(D_I) - \beta.$$

But from equation 9.6 applied to $\delta(D)$, we can deduce that $\beta = 0$ and that

$$\delta(D) = \delta(\cup_{I\in[n]^{n-1}}D_I). \tag{9.8}$$

Hence, any hyperedge in $\cup_{I\in[n]^{n-1}}D_I$ is contained entirely in one of the D_I. Finally, from equation 9.8 and the fact that $D = \mathrm{cl}_D(\cup_{I\in[n]^{n-1}}D_I)$, we can deduce that $\cup_{I\in[n]^{n-1}}D_I \leq^* D$. Suppose for a contradiction that this is not the case. We can take A of maximal size such that $\cup_{I\in[n]^{n-1}}D_I \subsetneq A \subsetneq D$ and $\delta(A) < \delta(D)$. By maximality, $A \leq D$. But then, this contradicts that $D = \mathrm{cl}_D(\cup_{I\in[n]^{n-1}}D_I)$. □

Lemma 9.2.6. *Suppose that D is an n-GITD. Then,*

$$(D_n; (D_{in}|i \leq n); \ldots; (D_{I\cup\{n\}}|I \in [n-1]^{n-2}); \cup_{I\in[n-1]^{n-2}}D_{I\cup\{n\}})$$

is also an $n-1$-GITD. In particular, given an n-GITD D such that

$$D_{[n-1]} = \bigcup_{J \in [n-1]^{n-2}} D_J,$$

we have that D is also an $(n-1)$-GITD.

Proof. For $I \in [n-1]^{n-2}$, $D_{I\cup\{n\}}$ is an $(n-2)$-ID over D_n by definition. Hence, we only need to prove that any hyperedge in $\bigcup_{I\in[n-1]^{n-2}} D_{I\cup\{n\}}$ is entirely contained in one of the $D_{I\cup\{n\}}$. From the definition of an n-ID, we do already know that each hyperedge is contained in one of the $D_{I\cup\{n\}}$ or in $D_{[n-1]}$. Hence, we only need to worry about the possibility that there is a hyperedge which is entirely contained in $D_{[n-1]}$, but all of whose vertices are in $\bigcup_{I\in[n-1]^{n-2}} D_{I\cup\{n\}}$. In this case, we would also have that all of the vertices are contained in $\bigcup_{I\in[n-1]^{n-2}} D_I$, by considering the intersections of the $D_{I\cup\{n\}}$ with $D_{[n-1]}$. By inductive hypothesis, we know that $D_{[n-1]}$ is an $(n-1)$-ID and by Corollary 9.2.5, any hyperedge in $\bigcup_{I\in[n-1]^{n-2}} D_I$ is entirely contained in one of the D_I. But then, this hyperedge is contained in $D_{I\cup\{n\}}$ yielding the desired result. The additional statement follows from the fact that if $D_{[n-1]} = \cup_{J\in[n-1]^{n-2}} D_J$, then

$$D = \cup_{I\in[n-1]^{n-2}} D_{I\cup\{n\}}.$$

□

9.2.1 Closure under n-GITDs implies independent n-amalgamation

Here we prove how being closed under n-GITDs implies satisfying independent n-amalgamation over finite algebraically closed sets. This is proven in Theorem 9.2.12. Before that, a lot of set-up is needed. We roughly follow the lines of how simplicity of Hrushovski constructions is proven in [22].

Notation 9.2.7. In this subsection, we let $\mathcal{K}_f$ be closed under k-GITDs for $k = 2, 3$. In particular, we consider $\mathcal{M}_f$ simple and satisfying the consequences of Theorem 9.1.5.

We begin by characterising when a collection of finite closed sets D_i for $i \leq n$ forms an independent tuple over D_0. This will correspond to their closure forming an n-ID.

Lemma 9.2.8. *Let $D_0 \leq D_i \leq M_f$ be finite for $i \in \{1, \ldots, n\}$ and $n \geq 2$. For $I \subseteq [n]$, write $D_I := \mathrm{cl}(\{D_i | i \in I\})$ and let $D := D_{[n]}$. In $\mathcal{M}_f$ we have that*

$$\mathop{\downarrow}_{D_0} \{D_1, \ldots, D_n\} \text{ if and only if the following hold:}$$

$$\delta(D) = \sum_{k=0}^{n-1} (-1)^{n-1-k} \sum_{I \in [n]^k} \delta(D_I), \text{ and} \tag{a}$$

$$\mathop{\downarrow}_{D_0} \{D_i | i \in I\} \text{ for } I \in [n]^{n-1}. \tag{b}$$

Moreover, in this case, D forms an n-ID. Conversely, any n-ID $D \leq \mathcal{M}_f$ is such that $\downarrow_{D_0} \{D_1, \ldots, D_n\}$.

Proof. In the case of $n = 2$, we have that (a) is equivalent to $D_1 \downarrow_{D_0} D_2$ by the characterisation of independence in Theorem 9.1.5 (condition (b) only makes sense for $n > 2$). Write $S = [n]^{n-1}$ and for $R \in [S]^k$ with $k \leq n$, let $C_R = \cap_{I \in R} D_I$.
Claim: When, $\downarrow_{D_0} \{D_1, \ldots, D_n\}$, for $R \in [S]^k$,

$$C_R = D_{\cap_{I \in R} I}.$$

Proof of claim. We prove this by induction on k. For the base case of $k = 1$, there is nothing to prove. Hence, for $k > 1$, by inductive hypothesis, we may write

$$C_R := D_L \cap D_I,$$

where $L \subsetneq [n]$ and $I \in [n]^{n-1}$. Since we are assuming $\downarrow_{D_0} \{D_1, \ldots, D_n\}$, and independence is preserved under algebraic closures [6, Proposition 5.20], we have that

$$D_L \mathop{\downarrow}_{D_{L \cap I}} D_I. \tag{9.9}$$

Hence, from Lemma 4.1.7, $D_L \cap D_I = D_{L \cap I}$. □

From this claim, we get that, assuming $\mathop{\downarrow}_{D_0} \{D_1, \ldots, D_n\}$, D forms an n-ID. Looking at Definition 9.2.1, (i) follows from $D \subseteq \mathcal{M}_f$, (ii) and (v) follow from the definition of D_I, (iii) is the consequence of the claim and (iv) is observed in 9.9.

Now we are ready to prove the Lemma. Since D is an n-ID, we also have from equation 9.6 of Lemma 9.2.4 that (a) holds. By definition, $\mathop{\downarrow}_{D_0} \{D_1, \ldots, D_n\}$ implies (b). Hence, we already know that $\mathop{\downarrow}_{D_0} \{D_1, \ldots, D_n\}$ implies (a) and (b).

We want to show that (a) and (b) jointly imply $\mathop{\downarrow}_{D_0} \{D_1, \ldots, D_n\}$. Firstly, from (a),

$$\delta(D) = \sum_{k=0}^{n-1} (-1)^{n-1-k} \sum_{I \in [n]^k} \delta(D_I).$$

We also have that

$$\begin{aligned} d(D/D_{[n-1]}) &= \delta(D) - \delta(D_{[n-1]}) && (9.10) \\ &= \sum_{k=2}^{n-1} (-1)^k \sum_{K \in [n-1]^{n-k}} \delta(D_{K \cup \{n\}}) + \sum_{k=2}^{n-1} (-1)^{k-1} \sum_{K \in [n-1]^{n-k}} \delta(D_K). && (9.11) \end{aligned}$$

From (b) and $D_{[n-1]} \leq D \leq \mathcal{M}_f$, we have that for each of the K in the calculation above (since $|K| \leq n-2$), D_n is independent from D_K over D_0 in $D_{K \cup \{n\}}$. In particular,

$$\delta(D_{K \cup \{n\}}) = \delta(D_K \cup D_n) = \delta(D_K) + \delta(D_n) - \delta(D_0).$$

Using this in the context of equation 9.11, we have

$$d(D/D_{[n-1]}) = (\delta(D_n) - \delta(D_0)) \sum_{k=2}^{n-1} (-1)^k \binom{n-1}{n-k} = \delta(D_n) - \delta(D_0) \qquad (9.12)$$

Here the last equality follows from the alternated sum of the binomial coefficients being zero. Since $D_0 \leq D_n \leq \mathcal{M}_f$, we obtain $d(D/D_{n-1}) = d(D_n/D_0)$, which from Theorem 9.1.5 implies

$$D_n \mathop{\downarrow}_{D_0} D_{[n-1]}.$$

Since from (b) we already have that $\bigcup_{D_0} \{D_1, \ldots, D_{n-1}\}$, by simplicity of $\mathcal{M}_f$ we get that $\bigcup_{D_0} \{D_1, \ldots, D_n\}$.

Finally, note that given an n-ID $D \leq \mathcal{M}_f$, $\bigcup_{D_0} \{D_1, \ldots, D_n\}$. Condition (a) follows from Lemma 9.2.4. Condition (b) follows by induction on $n \geq 2$. □

We wish to show that in an n-GITD we have that $D_{[n-1]} \leq D$. In order to prove this, we first recall a basic fact about $\leq^*$:

Fact 9.2.9 (Lemma 3.4 of [22]). *Let $E \in \mathcal{K}_f$. Suppose that X and Y are freely amalgamated over $X \cap Y$ in E. Suppose further that $X \cap Y \leq^* X$. Then, $Y \leq^* X \cup Y$.*

Lemma 9.2.10. *Suppose that D is an n-GITD. Then, for each $J \in [n]^{n-1}$, $D_J \leq D$.*

Proof. By construction of D, we have that

$$\begin{aligned} \delta(D) &= \delta\left(\cup_{I \in [n]^{n-1}} D_I\right) && (9.13)\\ &= \delta\left(D_{[n-1]}\right) + \delta\left(\cup_{I \in [n-1]^{n-2}} D_{I \cup \{n\}}\right) - \delta\left(\cup_{I \in [n-1]^{n-2}} D_I\right). && (9.14) \end{aligned}$$

This can be seen by counting noting that for $J \in [n-1]^{n-2}$, each vertex in D_J is both in $D_{[n-1]}$ and in $\bigcup_{I \in [n-1]^{n-2}} D_{I \cup \{n\}}$. Write $X := D_{[n-1]}$ and $Y := \cup_{I \in [n-1]^{n-2}} D_{I \cup \{n\}}$. We have that $X \cap Y = \cup_{I \in [n-1]^{n-2}} D_I$. By definition of an n-GITD, we also have that X and Y are freely amalgamated over $X \cap Y$ in D since every hyperedge in D is contained in one of the D_J for $J \in [n]^{n-1}$. We can see that $X \cap Y \leq^* X$ by Corollary 9.2.5. Thus, we can apply Fact 9.2.9 to obtain that $Y \leq^* X \cup Y$. Now, we prove the main statement by induction on n, noting that the statement for $n = 2$ just says that for D_1 and D_2 freely amalgamated over D_0, $D_1 \leq D_1 \cup D_2$, which is a consequence of submodularity 4.1. By Lemma 9.2.6, we know that Y is an $(n-1)$-GITD. Hence, by inductive hypothesis, we have that for $I \in [n-1]^{n-2}$,

$$D_{I \cup \{n\}} \leq \bigcup_{I \in [n-1]^{n-2}} D_{I \cup \{n\}} = Y.$$

But, since $Y \leq^* X \cup Y$, we also have $D_{I \cup \{n\}} \leq X \cup Y = D$. This concludes the argument since we can show for any $J \in [n]^{n-1}$ that $D_J \leq D$. □

Corollary 9.2.11. *Suppose that $D \leq \mathcal{M}_f$ is an n-GITD. Then, in $\mathcal{M}_f$,*

$$\mathop{\downfree}_{D_0} \{D_1, \ldots, D_n\}.$$

Proof. We want to use Lemma 9.2.8. By definition of an n-GITD, for $I \subsetneq [n]$, $D_I = \mathrm{cl}_D(\{D_i | i \in I\})$. By Lemma 9.1.1.2, and Lemma 9.2.10 in the case of $|I| = n-1$, since $D \leq \mathcal{M}_f$, $D_I \leq \mathcal{M}_f$. Hence, $D_I = \mathrm{cl}_{\mathcal{M}_f}(\{D_i | i \in I\})$. Now, (a) holds for D from calculating the predimension of D by the same inclusion-exclusion formula mentioned in 9.7 and recalling that by (vi) in the definition of an n-GITD, any hyperedge is entirely contained in one of the D_I for $I \in [n]^{n-1}$. Meanwhile, (b) follows from the assumption that each of the D_I for $I \in [n]^{n-1}$ is an $(n-1)$-ID and so from Lemma 9.2.8. Hence, we can use again Lemma 9.2.8 to see that $\downfree_{D_0} \{D_1, \ldots, D_n\}$ as desired. $\square$

Theorem 9.2.12. *Let $N \geq 3$. Suppose that $\mathcal{K}_f$ is closed under n-GITDs for $2 \leq n \leq N$. Then, $\mathcal{M}_f$ satisfies independent n-amalgamation over the algebraic closures of finite sets for all $n \leq N$.*

Proof. This is a proof by induction. The base case of $N = 3$ is implied by the fact that being closed under proper ITDs yields independent 3-amalgamation over algebraic closures of finite sets [22]. Let $A \leq \mathcal{M}_f$ finite, $\downfree_A \{a_1, \ldots, a_{N-1}\}$ be in $\mathcal{M}_f$ and suppose that for each $I, J \in [N-1]^{N-2}$ there are a_I such that

$$a_I \mathop{\downfree}_{A} \{a_i | i \in I\} \text{ and } a_I \equiv_{A\{a_i | i \in I \cap J\}} a_J.$$

Write $D_0 := A$ and for $I \subseteq [N-1], D_I := \mathrm{acl}(D_0\{a_i | i \in I\})$. Furthermore, for $J \subseteq I \in [N-1]^{N-2}$ write $D^I_{J \cup \{N\}} := \mathrm{acl}(D_0 a_I \{a_j | j \in J\})$. Since types of finite sets are determined by the quantifier-free types of their algebraic closures, for $I, J \in [N-1]^{N-2}$ under appropriate enumerations, we have that

$$\mathrm{qftp}\left(D^I_{(I\cap J)\cup\{N\}}\right) = \mathrm{qftp}\left(D^J_{(I\cap J)\cup\{N\}}\right) \text{ and}$$
$$D^I_{\{N\}} \mathop{\downfree}_{D_0} D_I.$$

From Lemma 9.2.8, for $I \in [N-1]^{N-2}$, the structure $D^I_{\{N\}\cup I}$ is an $(N-1)$-ID. Similarly, $D_{[N-1]}$ is an $(N-1)$-ID. Hence, we can consider the associated N-GITD D. We label the copy of each of the $D^I_{\{N\}}$ in D as D_N, and a_n the associated copy of each of the a_I in D_N. Since $D_{[N-1]} \leq D$ by Lemma 9.2.10, by the extension axioms there is a copy D^* of D over $D_{[N-1]}$ such that $D_{[N-1]} \leq D^* \leq \mathcal{M}_f$. Let a_n^* be the copy of a_n in D^*. By Corollary 9.2.11, and the fact that types are determined by the quantifier-free types of closures, we have that $a_n^* \downarrow_A a_1 \dots a_{n-1}$, and $a_n^* \equiv_{A\{a_i | i \in I\}} a_I$ for $I \in [n-1]^{n-2}$, as desired, proving that the n-generalised independence theorem holds. □

9.2.2 Sufficient conditions for closure under n-GITDs

We now want to explain under which conditions we can get that $\mathcal{K}_f$ is closed under n-GITDs. In order for D to be in $\mathcal{K}_f$, we also need all of its proper substructures to belong to the class. In order to prove this, we need to introduce the notion of weak n-GITDs:

Definition 9.2.13. Let D be an n-GITD. Consider $B \subseteq D$ and for $I \subseteq [n]$ let $B_I = D_I \cap B$. Then, with this notation, we say that B is a **weak** n-GITD.

Remark 9.2.14. By definition of a weak n-GITD, in order to check that $\mathcal{K}_f$ is closed under n-GITDs it is sufficient to check that for all weak n-GITDs, $f(|B|) \leq \delta(B)$.

Lemma 9.2.15. *The following are still the case for a weak n-GITD B:*

(ii)* for $I \subseteq [n], B_I \leq B$;

(iii)* for $I, J \subseteq [n], B_I \cap B_J = B_{I\cap J}$ (where $B_\varnothing := D_0$);

(vi)* any hyperedge in B is entirely contained in one of the B_I for $I \in [n]^{n-1}$;

(vii)* suppose that (for $n \geq 3$) $B_{[n-1]} = \bigcup_{J\in[n-1]^{n-2}} B_J$. Then, B is a weak $(n-1)$-GITD; and

(viii)* for $I, J \subseteq [n]$, B_I and B_J are freely amalgamated over $B_{I\cap J}$.

Proof. Conditions (iii)* and (vi)* follow from the labelling of the B_I in B. Condition (ii)* follows from Lemma 9.1.1 and Lemma 9.2.10, while (viii)* follows from D_I and D_J being freely amalgamated over $D_{I\cap J}$. Finally, for condition (vii)*, suppose that $B_{[n-1]} = \bigcup_{J\in[n-1]^{n-2}} B_J$. Then,

$$B \subseteq \bigcup_{J\in[n-1]^{n-2}} D_{J\cup\{n\}},$$

where D is an n-GITD containing B as a subset. However, we know that $\cup_{J\in[n-1]^{n-2}} D_{J\cup\{n\}}$ is an $(n-1)$-GITD by Lemma 9.2.6. Thus, B is a weak $(n-1)$-GITD being a subset of an $(n-1)$-GITD. □

Lemma 9.2.16. *Let $\alpha, a \in \mathbb{N}$. Suppose that for $a \leq t$, $f(nt) \leq f(t) + k$, for fixed $k \in \mathbb{N}$. Let B be a weak n-GITD for $\mathcal{K}_f$. Say without loss of generality that $B_{[n-1]}$ has the biggest predimension among the B_I for $I \in [n]^{n-1}$ and call it $d_{[n-1]}$. Suppose that $f(a) \leq d_{[n-1]}$. We have that if $\delta(B) \geq d_{[n-1]} + k$, then $f(|B|) \leq \delta(B)$.*

Proof. We repeat the argument of 9.1.13. Let g be the inverse of f. Making the substitution $g(x) = t$ into $f(nt) \leq f(t) + k$, we get that $ng(x) \leq g(x+k)$ for $f(a) \leq x$. Then, for $f(a) \leq d_{[n-1]}$,

$$|B| \leq \sum_{I\in[n]^{n-1}} |D_I| \leq ng(d_{[n-1]}) \leq g(d_{[n-1]} + k) \leq g(\delta(B)), \qquad (\clubsuit)$$

as desired. The second inequality holds since $B_{[n-1]} \in \mathcal{K}_f$ and so $|B_I| \leq g(\delta(B_I))$ and the third holds since $\delta(B) \geq d_{[n-1]} + k$ and g is increasing. □

Remark 9.2.17. From Lemma 9.2.16, it is easy to see that for each $N \in \mathbb{N}$ there are various functions f_N such that $\mathcal{K}_{f_N}$ is closed under k-GITDs for $k \leq N$. One just needs a function which globally satisfies $f(Nt) \leq f(t) + 1$, such as $\log_N(x) + 1$. Suppose that B is a weak k-GITD for $k \leq N$ and that $d_{[k-1]}$ is the largest predimension of one of its $(k-1)$ components. Since $B_{[k-1]} \leq B$, if $d_{[k-1]} = \delta(B)$, $B_{[k-1]} = B$ and so $B \in \mathcal{K}_f$. Otherwise, since $\alpha \in \mathbb{N}$, $d_{[k-1]} + 1 \leq \delta(B)$, and so we can run the same argument as in ($\clubsuit$) to show that $|B| \leq g(\delta(B))$. Since this argument works for all weak k-GITDs (and so for all subsets of B) for $k \leq N$, we have that $B \in \mathcal{K}_f$.

Remark 9.2.18. It is natural to think that a similar choice of growth for f might yield that $\mathcal{K}_f$ is closed under n-GITDs for all $n \in \mathbb{N}$. In particular, in both [35] and [69, Exercise 6.2.28], it is suggested that (for $\alpha = 1$ and working with ternary hypergraphs) the inverse of $(x+1)!$ should yield this. The proof of this is never explicitly stated, but it is presumably supposed to use the same argument as Lemma 9.2.16. Unfortunately, there is a subtle reason for which the same style of argument is not sufficient to yield independent n-amalgamation for all n. In fact, it is not possible for any strictly increasing function tending to infinity to globally satisfy for all t that for each $n \in \mathbb{N}$, $f(nt) \leq f(t) + 1$. For example, choosing g to be $(x+1)!$ on the positive natural numbers, we have that

$$n(g(x)) \leq (g(x+1))$$

if and only if $x \geq n-2$. This will guarantee that for each n-GITD, $|D| \leq g(\delta(D))$ since $d_{[n-1]} \geq n-1$ due to $\perp_{D_0} \{D_1, \ldots, D_{n-1}\}$. However, the same is not true for weak n-GITDs. In particular, a substructure of an n-GITD may be such that $d_{[n-1]} < n-2$. In the case of ITDs this was not a problem since any proper substructure of an ITD was either a proper ITD or a free amalgamation of a proper ITD with elements in $\mathcal{K}_f$ as proven in Lemma 9.1.11. However, there is no clear analogue of this for higher amalgamations. It is *a priori* perfectly possible for a weak n-GITD to be such that $d_{[n-1]} < n-2$ and still not correspond to a lower amalgamation of elements of $\mathcal{K}_f$. For this reason we give a different choice of g (and consequently of f), which yields independent n-amalgamation for all n more clearly.

Theorem 9.2.19. *Let $\alpha \in \mathbb{N}$. Define $g : \mathbb{N}^{>0} \to \mathbb{N}^{>0}$ iteratively as follows:*

$$g(1) = 1$$
$$g(k+1) = 4^{g(k)} g(k).$$

Extend g to a piecewise linear function from the real numbers to the real numbers (by joining its entries in $\mathbb{N}$ by lines) and let f be the inverse of g. Then, $\mathcal{K}_f$ is a free amalgamation class and it is closed under n-GITDs for all $n \in \mathbb{N}$.

Proof. Note that for $\mathcal{K}_f$ to be a free amalgamation class, it is sufficient for it to be closed under 2-GITDs. A proper substructure of a 2-GITD is either a 1-ID or another 2-GITD. So, it is sufficient to show that for each 2-GITD, B, $|B| \leq g(\delta(B))$. To prove this, one can just repeat the same argument as (♣). Now, consider the case of n-GITDs. Let B be a weak n-GITD.

Claim: For each $I \in [n]^{n-1}$, let

$$B'_I = \mathrm{cl}_B\left(\bigcup_{J\in[I]^{n-2}} B_J\right),$$

and write $B' = \bigcup_{I\in[n]^{n-1}} B'_I$. If $B' \in \mathcal{K}_f$, then so is B.

Proof of Claim. From Lemma 9.2.15, we know that $B_I \leq B$. By definition of B'_I, $B'_I \leq B$. From Lemma 9.1.1, $B'_I \leq B_I$. Moreover, for $I \neq J \in [n]^{n-1}$, $B_I \cap B_J = B_{I\cap J}$ and any hyperedge in B is entirely contained in either B_I or B_J.

Suppose that $B' \in \mathcal{K}_f$. We define iteratively $B' \subseteq B^i \subseteq B$ for $i \leq n$ so that $B^n = B$ and each B^i is in $\mathcal{K}_f$. We begin with $B^0 := B'$. For $i \leq n$, suppose that we have built B^i such that $B'_I \leq B^i$ for each $I \in [n]^{n-1}$. Then, we build B^{i+1} by freely amalgamating $B_{[n]\setminus\{i+1\}}$ and B^i over $B'_{[n]\setminus\{i+1\}}$.

Firstly, we show by induction that for each $I \in [n]^{n-1}$, $B_I \leq B^i$. The base case is trivial. After that, assume by inductive hypothesis that $B_I \leq B^i$. Since the latter is freely amalgamated with $B_{[n]\setminus\{i+1\}}$ over $B'_{[n]\setminus\{i+1\}}$, we still have that $B_I \leq B^{i+1}$.

Now, we show that we can identify B^i as an induced substructure of B. We can see this by noting that in B, for $I \in [n]^{n-1}$, there are no hyperedges with vertices in both $B_I \setminus B'_I$ and $B \setminus B_I$. Consider a hyperedge of B intersecting B_I non-trivially but not entirely contained in B_I. Then, this hyperedge is entirely contained in some B_J for some different $J \in [n]^{n-1}$. The vertices in B_I which are in this hyperedge are also in B_J and so in $B_I \cap B_J = B_{I\cap J} \subseteq B'_I$. Hence, we can identify B^i as an induced substructure of B. Moreover, by the same argument, the free amalgamation of B^i and $B_{[n]\setminus\{i+1\}}$ over $B'_{[n]\setminus\{i+1\}}$ is still an induced substructure of B. When the process terminates, we get that $B^n = B$.

Hence, if $B' \in \mathcal{K}_f$, we also have that $B \in \mathcal{K}_f$ being obtained by iterating free amalgamations. $\square$

Given the claim, we can assume that $B = B'$. Let $d_{[n-1]}$ be the highest predimension of one of the $(n-1)$-components of B. We first note that any B_J for $J \in [n]^{n-2}$ is contained in

$$E := B_{[n-1]} \cup B_{[n-2]\cup\{n\}}.$$

For $I \in [n]^{n-1}$, let

$$E_I = \bigcup_{J\in[I]^{n-2}} B_J.$$

Each E_I is a subset of E. Moreover, if for $I, J \in [n]^{n-1}, E_I = E_J$, we have that $B_I = B_J = E_I$ (since we are taking $B = B'$). In fact,

$$E_I \subseteq B_I \cap B_J = B_{I\cap J} \subseteq E_I,$$

where the first inclusion holds since $E_I \subseteq B_I, B_J$ and the second since $I \cap J \in [n]^{n-2}$. Thus, $E_I = B_{I\cap J}$. Since $B_{I\cap J} \leq B$, we have that

$$B_I = \mathrm{cl}_B(E_I) = \mathrm{cl}_B(B_{I\cap J}) = B_{I\cap J},$$

and similarly $B_J = B_{I\cap J}$. Hence, any two distinct B_I and B_J determine distinct subsets of E.

The maximum size of any B_I for $I \in [n]^{n-1}$ is $g(d_{[n-1]})$. As we noted above, each B_I is associated to some $E_I \subseteq E$ and distinct B_I can only arise from distinct subsets of E. Therefore, regardless of n,

$$|B| \leq 2^{|E|} g(d_{[n-1]}) \leq 4^{g(d_{[n-1]})} g(d_{[n-1]}).$$

(Essentially, this inequality comes from the over-pessimistic scenario in which every single subset of E gives rise to a distinct B_I for $I \in [n]^{n-1}$ and each of these is of maximal size.) Hence, we are now ready to see that $\mathcal{K}_f$ is closed under

n-GITDs. If $d_{[n-1]} = \delta(B)$, then $B_{[n-1]} = B$ and we are done. Otherwise, since $B_{[n-1]} \leq B$, $d_{[n-1]} + 1 \leq \delta(B)$. But now,

$$|B| \leq 4^{g(d_{[n-1]})} g(d_{[n-1]}) = g(d_{[n-1]} + 1) \leq g(\delta(B)).$$

Hence, for each weak n-GITD, $B \leq g(\delta(B))$, which implies that $\mathcal{K}_f$ is closed under n-GITDs. □

9.3 The main properties of our Hrushovski constructions

We are finally ready to prove that the structures of Construction 5.1.1 and 7.5.1 can be built and satisfy the various properties claimed of them in the main body of the thesis. In Subsection 9.3.1, we prove that the structures of Construction 5.1.1 are supersimple of SU-rank 2, with weak elimination of imaginaries and f can be chosen so that they satisfy independent n-amalgamation for all n over finite algebraically closed sets. In Subsection 9.3.2, we prove that the structures of Construction 7.5.1 are supersimple of SU-rank 1 and such that they satisfy the strong independence theorem over finite algebraically closed sets and can also satisfy independent n-amalgamation for all n.

9.3.1 The structures of Construction 5.1.1 and their basic properties

In this section, we focus on our choice of structures $\mathcal{M}_f$ built as specified in Construction 5.1.1. We prove that these structures have the various properties mentioned in Chapters 5 and 6.

We begin by noting that $\mathcal{K}_f$ is a free amalgamation class and so the structures $\mathcal{M}_f$ may be built according to Theorem 4.1.6.

Proposition 9.3.1. *The class $\mathcal{K}_f$ is a free amalgamation class.*

Proof. We need to prove that $\mathcal{K}_f$ is closed under 2-GITDs. Note that a proper substructure of a 2-GITD (D_0, D_1, D_2, D) is either a substructure of one of D_1 or D_2 (and so already in $\mathcal{K}_f$) or another 2-GITD. Hence, we only need to check that for any 2-GITD D for which $D_1, D_2 \subsetneq D$, $f(|D|) \leq \delta(D)$. For $t \geq 6$, we have that $f(3t) \leq f(t) + 1$, and since f is increasing

$$f(2t) \leq f(3t) \leq f(t) + 1.$$

Thus, by Lemma 9.2.16, $D \in \mathcal{K}_f$ as long as $\max\{\delta(D_1), \delta(D_2)\} \geq 6$. Hence, we only need to focus on 2-GITDs where $\max\{\delta(D_1), \delta(D_2)\} < 6$. Note that

$$|D| = |D_1| + |D_2| - |D_0| \text{ and } \delta(D) = \delta(D_1) + \delta(D_2) - \delta(D_0).$$

So, in the context of Figure 5.1, it is sufficient to check that given any three dots p, q, and r (possibly with $q = r$ or $p = (0,0)$) lying above f, with $p = (p_1, p_2), q = (q_1, q_2), r = (r_1, r_2)$ such that $p_1 < q_1, r_1, p_2 < q_2, r_2$ and $q_2, r_2 < 6$, the fourth vertex of the parallelogram with edges $\overline{pq}$ and $\overline{pr}$ is still above the function f. Since $f(6) = 6$ and our function is increasing, this can be easily verified. □

Now we prove that $\mathcal{M}_f$ is supersimple of SU-rank 2. We proceed, step by step, by proving that for any proper ITD D for $\mathcal{K}_f$, $f(|D|) \leq \delta(D)$. Hence, the proof will follow by Lemma 9.1.11 and Theorem 9.1.5. We shall adopt the notation we already set for proper ITDs. To avoid confusion when speaking of the various element of an ITD, we always write D for the proper ITD, D_{ij} only for $i, j \in \{1, 2, 3\}$, and D_j for D_{0j}. Furthermore, given D, we shall assume without loss of generality that D_{12} has maximal dimension among the D_{ij}. We shall also write d_i for $\delta(D_i)$, d_{ij} for $\delta(D_{ij})$ and d for $\delta(D)$ to simplify our notation.

Lemma 9.3.2. *For all proper ITDs D for $\mathcal{K}_f$ with $d_{12} \leq 4$, $f(|D|) \leq \delta(D)$. For this proof, we only need to assume that $f(1) = 2, f(2) = 3, f(3) = 4$ and $f(6) \leq 6$.*

Proof. For $d_{12} \leq 4$, $|D_{12}| \leq 3$, and so $|D_{ij}| \leq 3$. Since we require $D_i, D_j \neq \emptyset$ and $D_i \cup D_j \subsetneq D_{ij}$ (for $i \neq j \in \{1, 2, 3\}$), we must have that $|D_{ij}| = 3$ for each

$i \neq j \in \{1,2,3\}$, $|D_i| = 1$ for each $i \in \{1,2,3\}$, and $|D_0| = 0$. There is only one graph satisfying these requirements, i.e. a 6-cycle as shown in Figure 9.2.

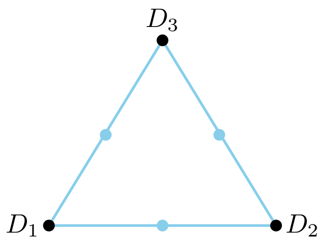


Figure 9.2: Each D_i for $1 \leq i \leq 3$ is a vertex. We can see that each D_{ij} must be a copy of a path of length 2 since $|D_{ij}| = 3$ and $d_{ij} \leq 4$. The resulting proper ITD is a 6-cycle.

□

Lemma 9.3.3. *For all proper ITDs D for $\mathcal{K}_f$ with $d_{12} \leq 5$, $f(|D|) \leq \delta(D)$. We only need to assume $f(1) = 2, f(2) = 3, f(3) = 4, f(4) = 5$, and $f(6) \leq 6$, and $f(12) \leq 7$.*

Proof. Given the previous lemma, we only need to prove the condition for $d_{12} = 5$. Note that since $f(3 \cdot 4) = f(12) \leq 7 = f(4) + 2$, by Lemma 9.1.13 we only need to check the case of $d = 6$.

We have that $d_0 > 0$ since otherwise, the condition that $d - d_{12} = 1$ forces $d_3 = 1$, which is impossible since no graph in $\mathcal{K}_f$ has predimension 1. Hence, we need to check the cases of $d_0 = 2$ and $d_0 = 3$ (note that for $d_0 > 3$ we cannot have $d = 6$ since $d > d_0 + 3$). By definition of an ITD, the following inclusion-exclusion formula holds:

$$|D| = \sum_{1 \leq i < j \leq 3} |D_{ij}| - \sum_{1 \leq i \leq 3} |D_i| + |D_0|. \tag{9.15}$$

Knowing the upper bounds for the sizes of the D_{ij} and the lower bounds for the sizes of the D_i, we can come to an upper bound for D:

$$|D| \leq \sum_{1 \leq i < j \leq 3} \lfloor f^{-1}(d_i + d_j - d_0) \rfloor - \sum_{1 \leq i \leq 3} \min\{|B| \text{ s.t. } \delta(B) = d_i\} + |D_0| := \beta \tag{9.16}$$

Furthermore, note that $(d_1 - d_0) + (d_2 - d_0) + d_0 = d_{12}$ and that $d_3 - d_0 = d - d_{12} = 1$.

For $d_0 = 2$, without loss of generality, we have $d_1 = 4, d_2 = 3, d_3 = 3$ (since we must have $d_1 - d_0 = 2, d_2 - d_0 = 1, d_3 - d_0 = 1$). Hence, we get that $\beta = 4 \cdot 2 + 3 - 2 \cdot 3 + 1 = 6$, and so $|D| \leq 6$. Since $f(6) \leq 6$, $f(|D|) \leq 6 = d$.

For $d_0 = 3$, we obtain that $d_1 - d_0 = 1, d_2 - d_0 = 1, d_3 - d_0 = 1$. Since $d_0 = 3$ implies that $|D_0| = 2$, $|D_i| \geq 3$ so $\beta = 3 \cdot 4 - 3 \cdot 3 + 2 = 5$, and so $f(|D|) \leq f(5) < 6 = d$, as desired. □

Theorem 9.3.4. *Let $\mathcal{K}_f$ and $\mathcal{M}_f$ be as in Construction 5.1.1. Then, $\mathcal{M}_f$ is supersimple of SU-rank 2. Moreover, $\mathcal{M}_f$ has weak elimination of imaginaries.*

Proof. From Theorem 9.1.5 and Lemma 9.1.11, we need to check that for any proper ITD D, $f(|D|) \leq \delta(D)$. We know from the lemmas above for any proper ITD D for $\mathcal{K}_f$ with $d_{12} \leq 5$, $f(|D|) \leq \delta(D)$. Since for $t \geq 6, f(3 \cdot t) \leq f(t) + 1$, by Lemma 9.1.13 we have that for $d_{12} \geq 6$, if $\delta(D) \geq d_{12} + 1$, then $f(|D|) \leq \delta(D)$. But $\delta(D) > d_{12}$ in any proper ITD. Hence, any proper ITD with $d_{12} \geq 6$ is such that $f(|D|) \leq \delta(D)$ and so the theorem follows. Finally, $\mathcal{M}_f$ has SU-rank 2 from Theorem 9.1.15. □

In the process of our proof of Theorem 9.3.4 we have proven that the smallest proper ITD in $\mathcal{K}_f$ has predimension 6. Our conditions in Construction 5.1.1 put no constraints on how slowly $f(t)$ grows for $t \geq 6$. Hence, we may make f slow growing enough to satisfy independent n-amalgamation over finite algebraically closed sets for all $n \leq N$ for fixed $N \in \mathbb{N}$ of our choice or for all n following the discussion of Lemma 9.2.16 and Theorem 9.2.19.

Lemma 9.3.5. *Let f be as in Construction 5.1.1. For $n \geq 3$, let D be a weak n-GITD with $d_{[n-1]} < 6$ being the maximal predimension of an $(n-1)$-component of D. Then, $D \in \mathcal{K}_f$.*

Proof. For our proof we consider a minimal counterexample D. That is, we suppose that D is a weak n-GITD for smallest possible n such that D is not in $\mathcal{K}_f$ in spite of $d_{[n-1]} < 6$, and having fixed n, we choose D of minimal size.

Since weak n-GITDs are closed under substructures by definition, any proper substructure of D is in $\mathcal{K}_f$. Note that $n > 3$ since we have already proved that $\mathcal{K}_f$ is closed under 3-GITDs for $d_{12} \leq 5$ in the previous lemmas. By Lemma 9.2.15, we may assume for each $I \in [n]^{n-1}$ that

$$E_I := \bigcup_{J \in [I]^{n-2}} D_J \subsetneq D_I$$

since otherwise, we could consider D as a weak $(n-1)$-GITD. From the Claim in the proof of Theorem 9.2.19, by minimality of D, we can also assume that

$$D_I = \mathrm{cl}_D(E_I).$$

There are exactly three graphs A_i for $i \in \{1, 2, 3\}$ in $\mathcal{K}_f$ of predimension ≤ 5 for which there is a $B_i \subsetneq A_i$ such that $A_i = \mathrm{cl}_{A_i}(B_i)$. These are illustrated in the following Figure 9.3. Moreover, in the same figure, we label as C^i_j for $j \leq 3$ the distinct sets of vertices in B_i which must be contained in distinct copies of D_J for $J \in [I]^{n-2}$ when considering A_i as a D_I for $I \in [n]^{n-1}$.

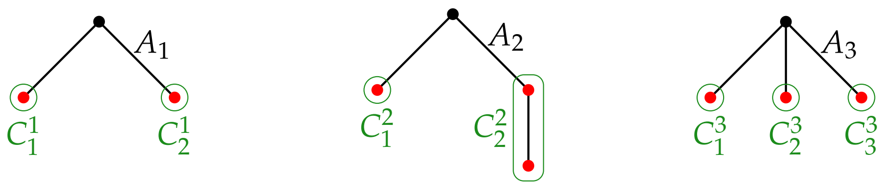


Figure 9.3: These are (up to isomorphism) the only three graphs in $\mathcal{K}_f$ of predimension ≤ 5 for which $B_i \subsetneq A_i$ and $\mathrm{cl}_{A_i}(B_i) = A_i$. This can be seen by simply enumerating the graphs in $\mathcal{K}_f$ of predimension ≤ 5 (there are only 7 of these). We represent the vertices in B_i in red. This appears as grey in black & white print. When considering A_i as a D_I for $I \in [n]^{n-1}$ we also need to consider B_i as E_I. If two vertices containing $A_i \setminus B_i$ in their closure were to belong to the same D_J for $J \in [I]^{n-2}$, since $D_J \leq D$, we would have that $D_I \subseteq E_I$, which contradicts our assumption. This tells us that certain vertices of B_i must be contained in distinct instances of D_J for $J \in [I]^{n-2}$. For A_i, we label these with C^i_j for $j \leq 3$ and in the figure we represent these sets of vertices by circling them in green (which also appears gray in black and white print). What we have is that if $D_I = A_i$, then no two of the C^i_j are contained in the same D_J for $J \in [I]^{n-2}$. Moreover, each C^i_j is entirely contained within one of the D_J for $J \in [I]^{n-2}$. This is trivial for all of them, but C^2_2, which does not consist of a single vertex. However, we have that C^2_2 consist of a path of length 1. If one of its vertices was in D_J and one in D_K for $J \neq K \in [I]^{n-2}$, that would contradict the fact that D_J and D_K are freely amalgamated over $D_{J \cap K}$. Hence, the two vertices must be contained in the same D_J.

We need to consider various different candidates for $D_{[n-1]}$.

Case 1: $D_{[n-1]} = A_1$.

In this context, since $\delta(A_1) = 4$, we must have that each D_I for $I \in [n]^{n-1}$ is of this form. Say without loss of generality that $C_1^1 = D_{[n-2]}$ and, following Theorem 9.2.19, let

$$E := D_{[n-1]} \cup D_{[n-2]\cup\{n\}}.$$

We have that E contains all of the D_J for $J \in [n]^{n-2}$. In particular, $\cup_{I\in[n]^{n-1}} E_I$ consists of only three vertices. Since two distict D_I can share at most one D_J for $J \in [n]^{n-2}$ we see that D must be a 3-GITD (an ITD) with D_1, D_2, D_3 consisting of distinct vertices. Hence, D is in $\mathcal{K}_f$ since hexagons are.

Case 2: $D_{[n-1]} = A_2$.

In this case, letting $D_{[n-2]} = C_2^2$, we can see that $D_{[n-2]\cup\{n\}}$ must also be of the form of A_2 (since no other candidate D_I contains an edge in its corresponding E_I). Taking

$$E := D_{[n-1]} \cup D_{[n-2]\cup\{n\}}$$

we can see that $\cup_{I\in[n]^{n-1}} E_I$ consists of three distinct D_J for $J \in [n]^{n-2}$ one for $D_{[n-2]}$ and two consisting of a single vertex. Again, we can deduce that D is a 3-GITD and so in $\mathcal{K}_f$.

Case 3: $D_{[n-1]} = A_3$ (and no D_I is of the form of A_2).

This is the most complicated case. Taking without loss of generality $D_{[n-2]}$ to be non-empty, we proceed as above by considering

$$E := D_{[n-1]} \cup D_{[n-2]\cup\{n\}}.$$

Suppose that both $D_{[n-1]}$ and $D_{[n-2]\cup\{n\}}$ are copies of A_3. No other D_I can be a copy of A_3 since, without loss of generality, such D_I would need to contain two vertices from $E_{[n-1]}$. But then, $D_I \cap D_{[n-1]} = D_{I\cap[n-1]}$ would contain two vertices which contain the vertex in $D_I \setminus E_I$ in their closure, contradicting $E_I \subsetneq D_I$. So any other D_I must be a copy of A_1 where $C_1^1 \in E_{[n-1]} \setminus D_{[n-2]}$ and

$C_2^1 \in E_{[n-2]\cup\{n\}} \setminus D_{[n-2]}$. Any possible resulting D must be a subgraph of the graph $D^\star$ illustrated in Figure 9.4.

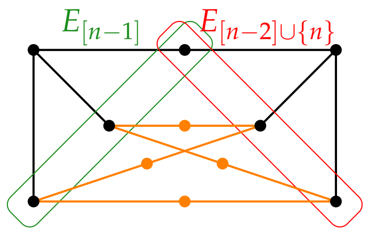


Figure 9.4: This is a depiction of the graph $D^\star$. We illustrate in black the vertices in E and we circled in green the copy of $E_{[n-1]}$ and in red the copy of $E_{[n-2]\cup\{n\}}$. The vertex in their intersection is $D_{[n-2]}$. As we noted in the main argument, any other D_I must be a copy of A_1, a path of length two with one end of the path in $E_{[n-1]} \setminus D_{[n-2]}$ and one in $E_{[n-2]\cup\{n\}} \setminus D_{[n-2]}$. Hence, we depicted in orange the vertices which can potentially be in D as part of some D_I. $D^\star$ consists of the graph containing E and all of these potential copies of D_I. We have that $|D^\star| = 11, \delta(D^\star) = 8$. Since $f(18) \leq 7$, we have that $f(|D^\star|) \leq \delta(D^\star)$. Moreover, removing any vertex but $D_{[n-2]}$ from $D^\star$ we obtain a graph which is a free amalgamation of smaller graphs in $\mathcal{K}_f$. Removing $D_{[n-2]}$ also yields a graph in $\mathcal{K}_f$. Hence, $D \in \mathcal{K}_f$.

The case where $D_{[n-1]}$ is a copy of A_3 and $D_{[n-2]\cup\{n\}}$ is a copy of A_1 is analogous. In particular, we again obtain that any D would be a subgraph of $D^\star$ as illustrated in Figure 9.4. □

Theorem 9.3.6. *In Construction 5.1.1, we can choose f so that $\mathcal{K}_f$ is closed under n-GITDs for all $n \in \mathbb{N}$. Hence, we can choose f so that $\mathcal{M}_f$ satisfies independent n-amalgamation for all $n \in \mathbb{N}$ over the algebraic closures of finite sets.*

Proof. We define the following function $g : \mathbb{N}^{>0} \to \mathbb{N}^{>0}$ similarly to Theorem 9.2.19. For $t \leq 6$, $g(t) = f^{-1}(t)$, where f for these values is defined as in Construction 5.1.1. Then, we iteratively define the values of g for $t > 6$ by setting

$$g(t+1) = 4^{g(t)} g(t).$$

Extending g to a piecewise linear function from $\mathbb{R}^{>0}$ to $\mathbb{R}^{>0}$ and taking f to be the inverse of the resulting function, we get that f satisfies all of the conditions of Construction 5.1.1. It is also easy to see that $\mathcal{K}_f$ is closed under n-GITDs for $n \in \mathbb{N}$. Consider a weak n-GITD D where $d_{[n-1]}$ is the maximal predimension of an $(n-1)$-component. If $d_{[n-1]} = \delta(D)$, we have that $D_{[n-1]} = D$ and we are done. So, we may assume that $d_{[n-1]} + 1 \leq \delta(D)$. If $d_{[n-1]} \leq 5$, we know that $D \in \mathcal{K}_f$ by Lemma 9.3.5. Otherise, $d_{[n-1]} \geq 6$. But then, as in Theorem 9.2.19,

$$|D| \leq 4^{g(d_{[n-1]})} g(d_{[n-1]}) = g(d_{[n-1]} + 1) \leq g(\delta(D)),$$

as desired. The claim follows from Theorem 9.2.12. □

9.3.2 The structures of Construction 7.5.1 and their basic properties

In this section, we focus on proving the various properties asserted of the structures $\mathcal{M}_f^R$ introduced in Construction 7.5.1. We follow the same structure as the previous section.

Proposition 9.3.7. *The class $\mathcal{K}_f^R$ is a free amalgamation class.*

Proof. Similarly to the proof of Lemma 9.3.1, we only need to check that $\mathcal{K}_f^R$ is closed under 2-GITDs where $D_1, D_2 \subsetneq D$ and $\max\{\delta(D_1), \delta(D_2)\} \leq 2$. Note that there are only two hypergraphs of predimension ≤ 2: a single vertex and a pair of distinct vertices. Any free amalgamation rising from these is in $\mathcal{K}_f^R$ since $f(18) \leq 4$. □

Lemma 9.3.8. *The class $\mathcal{K}_f^R$ is closed under SITDs.*

Proof. We consider the SITD D where d_{lm} is the maximal dimension of a D_{ij}. We need to show that $f(|D|) \leq \delta(D)$. For $d_{lm} \geq 3$, we have that $f(|D|) \leq \delta(D)$ by Lemma 9.1.13. Hence we can assume that $d_{lm} \leq 2$. In order for D to be a proper SITD, $d_1 = d_2 = d_3 = 1, d_0 = 0$. The only possible SITD with these properties is the graph given in Figure 7.4, which is included in $\mathcal{K}_f^R$. □

Corollary 9.3.9. *The structures $\mathcal{M}_f^R$ are supersimple of SU-rank 1 with weak elimination of imaginaries. Furthermore, they satisfy the strong independence theorem over finite algebraically closed sets.*

Proof. From Lemma 9.3.2, we know that $\mathcal{K}_f^R$ is closed under SITDs. Since any ITD is an SITD, $\mathcal{K}_f^R$ is closed under ITDs and Theorem 9.1.5 applies yielding supersimplicity, finite rank and weak elimination of imaginaries. Moreover, Theorem 9.1.8 implies that $\mathcal{M}_f^R$ satisfies the strong independence theorem over finite algebraically closed sets. Finally, $\mathcal{M}_f^R$ has SU-rank 1 from Theorem 9.1.15. $\square$

Lemma 9.3.10. *Let f be as in Construction 7.5.1. For $n \geq 3$ let D be a weak n-GITD with $d_{[n-1]} \leq 2$ being the maximal predimension of an $(n-1)$-component of D. Then, $D \in \mathcal{K}_f$.*

Proof. We follow the same strategy of Lemma 9.3.5 and consider a minimal counterexample. For $I \in [n]^{n-1}$, write $E_I = \cup_{J \in [I]^{n-2}} D_J$. For the same reasons as Lemma 9.3.5, we can take $E_I \subsetneq D_I$ and $D_I = \mathrm{cl}_D(E_I)$. Hence, we must have that each D_I consists of a hyperedge between three vertices with E_I consisting of two distinct vertices. Saying without loss of generality that $D_{[n-2]}$ is non-empty and considering

$$E := D_{[n-1]} \cup D_{[n-2]\cup\{n\}},$$

it is easy to see that $\cup_{I \in [n]^{n-1}} E_I$ consists of three vertices without any hyperedge between them since $D_{[n-1]}$ and $D_{[n-2]\cup\{n\}}$ are freely amalgamated over $D_{[n-2]}$. Hence, we can see that D must be a 3-GITD and in particular a subgraph of the 3-GITD represented in Figure 7.4. Thus, $D \in \mathcal{K}_f$. $\square$

Theorem 9.3.11. *The function f in Construction 7.5.1 can be chosen so that $\mathcal{M}_f^R$ satisfies independent n-amalgamation for all $n \in \mathbb{N}$ over finite algebraically closed sets.*

Proof. The proof is essentially the same as Theorem 9.3.6. Here we define $g : \mathbb{N}^{>0} \to \mathbb{N}^{>0}$ by $g(t) = f^{-1}(t)$ for $t \leq 3$ where these values are specified in Construction 7.5.1 and then setting again

$$g(t+1) = 4^{g(t)} g(t).$$

Then, the construction of f and the proof of closure under n-GITDs follow exactly the same argument as Theorem 9.3.6. $\square$

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