THE DISPERSION, IN FLUID FLOW THROUGH CIRCULAR TUBES, OF (RADIOACTIVE) LABELLED SOLUTES AND ERYTHROCYTES.

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ABSTRACT

This thesis is concerned with the interpretation of multiple indicator-dilution clearance curves obtained during physiological investigations of blood flow in the micro-circulation.

Sir Geoffrey Taylor obtained numerical solutions of the partial differential equation describing the dispersion of labelled material during flow through a circular tube by considering the combined effect of molecular motion and mass transport, i.e. diffusion and flow. This work has been extended, and numerical solutions of the equation obtained for the complete range of flow times, using a computer. Experimental verification of these analytical solutions for the change that occurs in a straight interface during flow is presented in this thesis, by studying the dispersion of a 'tracer' amount of iodide in an aqueous alkali chloride solution.

Numerical solutions of the equation

$$\frac{\partial c}{\partial t} = D\nabla^2 c - 2\bar{u}\left(1 - r^2/a^2\right)\frac{\partial c}{\partial x}$$

for c_m/c_o as functions of X and T, using the experimentally determined parameters, were shown to agree with the experimental results within the limit of error. The difficulties encountered in obtaining these results are described in chapter VI.

In chapter VII solutions of the above equation for different labelled components of the same flow regime are considered. It is shown that completely different clearance curves may be obtained depending on the value of D. These results are in excellent agreement with 'in vitro' determination of clearance curves for different labelled components of blood.

The physical significance of the partial differential equation in relation to these results, flow profiles, non-Newtonian viscosities and internal motion of the components is also discussed in this chapter. Physiological mechanisms introduced to explain 'in vivo' results using multiple indicator-dilution techniques are questioned with regard to this discussion. Support for the equation adopted is discussed briefly with reference to rheological investigations.

A brief account of the technique used in writing the computer program is contained as an appendix, written by Mr. C. Paige, as it has not been published.

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SYMBOLS

These are also defined when they first appear in the text.

a - radius of the tube

c - concentration

c_m - mean concentration over the cross-section

c_o - initial uniform concentration

$c(t)$, $c(r)$ - concentration as a function of time and radial position

D - diffusion coefficient

$f(T)$, $f(X)$ - functions of T and X

$k = \frac{a^2 - 2}{48\ D}$ - effective diffusion coefficient

k' - flow constant for the system

L - dispersion length

ℓ - length of tube

l_1 - length of mercury thread

$\dot{M} = \frac{M_2 - M_1}{\rho\ t}$ - mass flow rate

n - counting rate

P_{app} - applied pressure

ΔP - pressure difference between ends of the tube

Q - volume flow rate

r - radial position

Re - Reynold's number

t - time

$\bar{t}$ - mean circulation time

t_o - finite period of time

$\bar{u}$ - mean flow velocity

u_r - flow velocity at position r

u_o - flow velocity at the axis

$\underline{v}$ - convection velocity

V - volume

x - distance

x_o - finite distance

$\phi = \frac{a^2}{D}$ - constant for this series of experiments

τ - shear stress

τ_i - time constant

η - viscosity coefficient

$\dot{\gamma}$ - shear rate

Dimensionless parameters

$A_{to} = a^2/Dt_o$ $\qquad$ $A_{xo} = a^2\bar{u}/Dx_o$

$T = \bar{u}t/x_o$ $\qquad$ $X = x/\bar{u}t_o$

CHAPTER I

INTRODUCTION

In applying physical laws to the circulation in living animals many complicating factors must be considered. Blood is a complex heterogeneous fluid with anomalous properties of viscosity. The blood vessels are not rigid tubes but are distensible, so that their size depends on the blood pressure as well as upon the contraction of smooth muscle in the vessel walls. Also the flow is not steady but pulsatile in certain regions of the vascular bed. In order to obtain a more complete understanding of the dynamics of the circulation, analysis is made in terms of simple models. These may be mathematical equations[3d], or electrical circuits[62], or more literal models such as pipes carrying flowing fluids[3a]. Only when such simple models are thoroughly understood can a satiafactory analysis be given of the circulation.

A major problem in circulatory phenomena is the relation between the pressure required to drive the blood through the vessels and the volume flow rate produced. Investigations have been conducted in vivo and in vitro. The former has largely

been concerned with developing reliable techniques for measuring blood volume and flow rate, and the latter with the rheological properties of blood.

The in vivo investigations have utilised the indicator-dilution technique to measure the rate of flow and volume of blood in different circulatory beds[3c,3e]. Significant differences in the concentration of red blood cells in various parts of the circulation were shown to exist by employing separate labels for erythrocytes and plasma[3c,41,53]. Other experiments using this technique have revealed a difference in the mean circulation times between labelled erythrocytes and albumin[2p,17,20,23,29,38,39,48] in various regions of the circulation.

Rheological studies in vitro have shown that blood exhibits non-Newtonian properties[3a,70]. As there are frequent references to regions of Newtonian flow for whole blood in the literature[12] it is important to define what is meant by Newtonian and non-Newtonian. A Newtonian fluid[68] gives a linear shear stress to shear rate graph passing through the origin. In such cases the viscosity coefficient η is a constant defined by

$$\tau = \eta \dot{\gamma} \qquad (1)$$

where τ is the shear stress and $\dot{\gamma}$ the shear rate (velocity gradient). For fluids which exhibit non-Newtonian behaviour

the viscosity at a point is defined by

$$\eta^* = \tan^{-1}\left[\tau/\dot{\gamma}\right] \qquad (2)$$

This is sometimes referred to as the 'secant viscosity'(68). A'differential viscosity' η' may also be defined

$$\eta' = \tan^{-1}\left[\frac{d\tau}{d\dot{\gamma}}\right] \qquad (3)$$

if a sufficient portion of the curve is available. The viscosity of blood is less in small tubes (with a radius less than about 100 times the radius of the erythrocyte) than in larger tubes, for the same average shearing stress(3a,32). The effect appears to increase with increased wall stress and only occurs when red blood cells are present(66a). This is termed the 'sigma phenomenon' and has also been observed in many kinds of suspensions where the particles are large enough to be comparable in size with the tube radius(4). Decreasing values for the 'dynamic concentration' are also obtained as the tube bore is decreased(3c). The dynamic concentration is the ratio of red blood cell to plasma volume determined during flow using the indicator-dilution method.

A number of mechanisms have been cited in explanation of one or more of these phenomena. The presence of'slip' at the tube wall could contribute to an increase in apparent fluidity, but would not account for the reduced dynamic concentration, or the difference in circulation times of plasma and erythrocyte labels(28). Vand(67) has proposed that the natural exclusion

of particles from the wall surface is equivalent to having a layer at the wall with the viscosity of the suspending medium. This argument has been extended in a qualitative manner to account for certain blood flow observations. An extravascular plasma compartment has been proposed to explain the variation of red blood cell concentration within the circulatory bed[3c]. This could also account for a difference in circulation times. However, the total quantity of label was reported to be conserved in certain experiments[2p] so that the labelled protein must also return to the circulation in an interval of time comparable with the experimental observation for this theory to be valid. A difference in the diffusion coefficients of the labelled components might give a difference in mean circulation times[2p], although so far this has been discounted by other workers[2q]. A more detailed consideration of this aspect is given in the present thesis.

The concept of axial streaming of erythrocytes in the microcirculation is a much favoured explanation of these results. This was originally postulated by Fåhraeus to explain the reduction of haematocrit observed when blood flowed through narrow tubes[3a]. From continuity considerations the inward motion of blood corpuscles would result in both a difference in mean velocities of the corpuscles and the suspending phase, and in a reduction of the dynamic concentration[3a,66a]. A reduction in concentration of particles at the wall where the shear stress is greatest could also account for the increase

in apparent fluidity[3a,66a].

A theoretical explanation of axial streaming is more difficult and attempts to obtain experimental support by methods other than flow-dilution have also failed.

It has been shown theoretically[42] that mechanical exclusion of particle centres from the tube wall results in an effective increased concentration in the faster moving central stream of flow in a tube. These calculations were based on rigid spheres, and although similar considerations probably apply to a suspension of red blood cells the problem is complicated by their non-spherical shape and the fact that they are easily deformable[10,52]. Goldsmith and Mason[25] have shown experimentally that flexible fibres and oil drops migrate centrally in tube flow whereas rigid bodies do not, or at least to a much lesser extent. However, these observations referred to the motions of isolated particles and are not directly applicable to interactions which occur between particles brought into close proximity by the flow, as in dilute suspensions. Such cases are interpreted by a kinetic theory of particle interactions that has been developed from experimental observation[25]. In more concentrated suspensions Goldsmith and Mason used refractive index matching or visible tracers to study particle motions. 'Partial plug-flow' was observed corresponding to a velocity gradient at the wall which was greater than, and at the tube axis smaller than, the Poiseuille profile. These results indicate that axial migration is no

longer possible since the concentration profile was uniform (apart from natural exclusion at the wall), and many spheres were seen to move along the wall[25]. It would therefore seem that there is little justification in using the experimental observations of motions of isolated particles to postulate a similar hydrodynamic centripetal force in more concentrated suspensions, as has been done[2q].

High speed ciné photography[8] also contradicts the theory of axial migration of red blood cells in the microcirculation. At normal frame speeds the peripheral plasma layer appeared as a sharply defined clear zone, but at high speeds erythrocytes were observed to exhibit random motion, often in close proximity to the wall[8]. It was therefore suggested that the peripheral plasma layer observed in this and other experiments was an optical artefact. Bayliss[6] in a review of photometric evidence for the existence of a plasma layer was unable to draw any definite conclusions on the axial migration of red blood cells when a suspension is sheared. It has also been demonstrated by Wiederhielm and Billig (presented in a paper at the First International Conference on Hemorheology, Reykjavik, Iceland, 1966 and quoted by Wayland[68]) that experimental technique is important in determining the results obtained for light transmission studies in blood flowing through vessels of the microcirculation. They demonstrated that by varying the angle of illumination what appeared to be cell free layers could be obtained in all regions of the vessel. Quick freezing experiments

reported by Wiederhelm and Billig demonstrated a significant preferential orientation of erythrocytes parallel to the walls, in the vicinity of the walls. Such cell orientation could account for increased light transmittence[37,71] in the region of the wall. Possible artefacts due to the preparation of frozen sections make it difficult to substantiate the existence or non-existence of a cell-free layer.

The best indirect evidence for a plasmatic zone (axial streaming) in the microcirculation is the difference between the clearance curves for labelled erythrocytes and labelled albumin from the circulatory bed of the heart and lung[2p]. An analysis of the curves obtained in terms of a greater mean velocity for the red blood cells than the suspending phase has been given by Thomas[66b]. Reliable figures for the difference in mean circulation times in vivo cannot be obtained however, because of the uncertainty associated with recirculation[33]. Since the mean circulation times are more important in determining the relative contribution of molecular diffusion and axial streaming, efforts have been made to eliminate this uncertainty. This has been achieved in two ways : using an isolated muscle preparation in vivo[29] and using an artificial circulatory bed in vitro[2q]. Differences in mean circulation times have been found in both cases.

Since discussion of the in vivo results is complicated by physiological considerations the present discussion will be limited to consideration of the in vitro results. These

experiments by Thomas et al.[2q] and their interpretation in terms of axial streaming of erythrocytes initiated this present research. It will therefore be helpful to give a short summary of the experiments.

A series of 26, 200 micron bore nylon tubes constituted the artificial circulatory bed. Air pressure was used to drive a suspension of red blood cells through the tubes and flow measurement was made with a graduated pipette. The radioactively labelled suspension was initially contained in a short portion of the tubing specially adapted for the purpose. The activity in this section was monitored and the counters moved to a position 129cm. downstream. Flow was commenced and the clearance curves for P^{32} and I^{131} labelled components were monitored simultaneously by the two scintillation counters. The experiments were performed with suspensions containing :-

a) P^{32} and I^{131} labels in the form of inorganic ions

b) P^{32} in the form of inorganic phosphate and I^{131} labelled human serum albumin

c) P^{32} labelled erythrocytes and I^{131} labelled albumin.

The curves obtained in case c) were markedly different. The erythrocyte curve rose more steeply to a peak value which was appreciably ahead of the albumin curve. In case a) the iodide peak seemed to be slightly ahead, but there was no significant difference in mean circulation times. The phosphate curve was broader than the iodide curve, but this is consistent with a difference in diffusion coefficients of the two ions and

the analytical solution given by Taylor[64a] for the dispersion of diffusing solutes in a fluid flowing through a tube. Similarly in case b) the curves were fairly symmetric, with no significant difference in mean circulation times. The phosphate curve was narrower than the albumin curve, which was again consistent with the difference in diffusion coefficients and Taylor's theory.

The clearance curves were analysed using a mechanical integrator to determine the mean circulation times. The control measurement, when inorganic phosphate and albumin were used to label the suspension, gave a mean value for the ratio (mean circulation time I^{131}) / (mean circulation time P^{32}) of 1.01_6, slightly but significantly greater than unity. The exchange of an appreciable amount of phosphate by the red blood cells could give this result, but this was not considered possible. A mean value for the ratio (I^{131}) / (P^{32}) of 0.98_9, slightly but significantly below unity was obtained for the other type of control run. This was regarded as understandable, since it is known that appreciable concentrations of iodide can be taken up rapidly by red blood cells. The experimental curves show a large difference in mean circulation times of labelled cells and albumin. The source of a discrepancy between two experimental results at approximately 0.4 concentration was not traced, but the different properties of the erythrocytes in the various samples was regarded by the authors as a possible factor. No significant change in the ratio with variation of

shear stress at the tube wall was indicated for the range 12.5 - 100 dynes.cm.$^{-2}$, or with variation of concentration 0.2 - 0.4. This is of particular interest since Goldsmith and Mason[25] and workers in the field of blood rheology[3a] report concentration and shear dependence for their results. However, more experimental results were considered to be required before reliable quantitative conclusions could be drawn.

In summarising the results it was stated that differences in diffusion coefficients might be eliminated as a possible cause of this difference in circulation times, since the control runs obtained under similar hydrodynamic conditions to the experimental runs were in qualitative agreement with Taylor's theory[64a]. Complete clearance of labelled material was observed so that the true mean circulation time was obtained, which Taylor predicted was independent of diffusion coefficient. This was essentially confirmed by the control runs. It was presumed that electrical effects due to charge accumulating on the nylon wall could be ignored since the blood cells were suspended in a strongly ionised solution.

The authors concluded that a difference in mean circulation times of red blood cells and their suspending phase had been established, and that the effect could probably only be explained by radial migration of blood cells. An interesting possibility would be to deduce the true concentration profile of red blood cells from the shape of the clearance curves obtained. They

considered that the data was inadequate and that there were forbidding problems of analysis to be overcome.

Thomas and Rowlands discussed, in their papers[2q, 2p], the complex paths of blood corpuscles demonstrated by high speed photography[8] in connection with Taylor's mathematical analysis which assumes laminar flow[64a]. They consider that the volume vacated by a red cell is partially replaced by the suspending phase, so that small elements of the latter will also follow irregular paths. The net result is an additional effective diffusion of radioactive label. It was anticipated that this effective diffusion is a function of shear rate. It may also be a function of radial position, since the shear annuli become less convex as the distance from the axis is increased. The authors suggested 'shear-induced collision diffusion' as a possible descriptive term.

It was in order to obtain an assessment of the effects of molecular diffusion on the dispersion of labelled material in fluid flow through tubes that the present series of experiments was performed. A detailed description of the theory and experimentation involved will be given in subsequent chapters of this thesis.

A detailed examination of Taylor's analytical solutions for different D values shows that a significant difference in mean circulation times may be obtained even when diffusion is complete over the cross-section. These results, which apply to the dispersion curve monitored at a finite distance downstream,

would account for the small but significant differences observed in the mean circulation times for the control runs. A larger difference in mean circulation times for labelled components injected simultaneously may also be accounted for by a difference in diffusion coefficients. The particular circumstances under which such differences will be observed is considered after a more complete discussion of the Taylor theory.

It is maintained, by the author, that the complex paths observed for red blood cells in the microcirculation, using high speed cine photography, emerge as a result of this theory. The concept of 'shear-induced collision diffusion' would not be required in calculating the clearance curves. It is not clear to the author how it was proposed to determine the concentration profile for the red cells from the clearance curves, since the corpuscles have been shown to travel in such complex orbits.

Finally, the author would dispute that a difference in mean circulation times between erythrocytes and plasma has been determined. The results demonstrate a difference in mean circulation times for labelled red blood cells and labelled albumin injected simultaneously. The interpretation of such results requires careful consideration before being restated as a difference in circulation times between erythrocytes and plasma[2p], or as a difference in mean velocities of red blood

cells and their suspending phase[2q]. These comments might also be applied to all indicator dilution studies, i.e. to give careful consideration to exactly what measurement has been made.

Prior to a detailed analysis of these results in terms of the effects of molecular diffusion on the dispersion of labelled material during fluid flow through a tube it must be established that the equation[35a]

$$\frac{\partial c}{\partial t} = D\nabla^2 c - 2\bar{u}\left(1 - r^2/a^2\right)\frac{\partial c}{\partial x} \qquad (4)$$

gives an accurate description of the process. Taylor[64a] gives two analytical solutions of this equation for particular flow conditions. He also provided an experimental verification, but this was limited by an uncertainty in the value of D. It was hoped to obtain a more satisfactory verification of Taylor's solutions by employing a tracer diffusion coefficient. (The concept of tracer diffusion is discussed in chapter III.) Computer solutions of equation (4) have been obtained, thus enabling analysis of the effects of molecular diffusion to be extended to the complete range of flow conditions.

The major portion of this thesis is devoted to the significance of equation (4), its analytical solutions and their experimental verification. Finally certain inconsistencies existing in flow theory and measurement are discussed with regard to the behaviour of flowing materials.

CHAPTER II

THE PARTIAL DIFFERENTIAL EQUATION DESCRIBING THE DISPERSION OF LABELLED SOLUTES DURING FLUID FLOW THROUGH A TUBE, AND ITS ANALYTICAL SOLUTIONS.

Equation governing the dispersion of labelled material under the combined action of molecular diffusion and convection.

The effect of molecular diffusion on the dispersion of a labelled solute in a fluid flowing through a tube may be regarded as diffusion under the influence of an external force[35a]. It is analogous with a diffusion problem in a medium which is not at rest. Concentration changes take place due to convection as well as diffusion.

Considering the one dimensional case with flow in the x-direction only, where v is the convection velocity. The concentration change Δc due to convection in a volume element of unit cross-section bounded by two planes at x and $x+\Delta x$ during time Δt is

$$\Delta c = -\frac{1}{\Delta x}\left[(vc)_{x+\Delta x} - (vc)_x\right]\Delta t$$

$$= -\frac{\partial}{\partial x}(vc)\,\Delta t + \ldots\ldots$$

The differential equation for the rate of change of concentration, due to both diffusion and convection, becomes

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} - \frac{\partial}{\partial x}(\underline{v}c)$$

where D is the diffusion coefficient. Since fluids are regarded as isotropic[35a] this is

$$\frac{\partial c}{\partial t} = D\nabla^2 c - \nabla.(\underline{v}c)$$

in three dimensions. For $\underline{v}$ constant and flow in the x-direction only, this is

$$\frac{\partial c}{\partial t} = D\nabla^2 c - \underline{v}\frac{\partial c}{\partial x}$$

Taylor's analytical solutions[64a].

In a circular tube of radius 'a' the mean flow has a parabolic profile given by

$$u_r = u_o (1 - r^2/a^2) \qquad (5)$$

where u_r and u_o are the flow velocities at position r and at the axis respectively, i.e. the velocities in the flow direction determined from momentum transport. This is discussed further in chapter VII..

Taylor assumes that the concentration is symmetrical about the central line of the tube so that the concentration, c, is a function of r,x, and t only. The equation for diffusion becomes

$$\frac{\partial c}{\partial t} = D\left[\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial x^2}\right] - u_o\left(1 - \frac{r^2}{a^2}\right)\frac{\partial c}{\partial x} \qquad (6)$$

The diffusion coefficient D is assumed independent of c. In practice this is only applicable to 'self diffusion' coefficients, reference(35b).

In the cases considered by Taylor the term $\frac{\partial^2 c}{\partial x^2}$ is neglected, being much less than $\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r}$.

The boundary condition which expresses the fact that the wall of the tube is impermeable is

$$\frac{\partial c}{\partial r} = 0 \qquad \text{at } r = a$$

Taylor gives approximate solutions which are valid in the following limiting conditions :-

A) Changes in c due to convective transport along the tube take place in a time which is so short that the effect of molecular diffusion may be neglected, i.e. dispersion is by convection alone.

B) The time necessary for appreciable effects to appear owing to convective transport is long compared with the 'time of decay' during which radial variations of concentration are reduced to a fraction of their initial value by the action of molecular diffusion, i.e. molecular diffusion dominates in determining the dispersion.

Solution for condition A).

If at time t = 0 the tube contains unlabelled material c = 0 and the reservoir labelled material of uniform concentration, $c = c_o$, the solution for condition A) is[64a]

$$\left.\begin{aligned} c_m &= c_o && x < 0 \\ c_m &= c_o \left(1 - \frac{x}{u_o t}\right) && 0 < x < u_o t \\ c_m &= 0 && x > u_o t \end{aligned}\right\} \qquad (7)$$

where c_m is the mean concentration over the cross-section of the tube, defined by $c_m = \frac{2}{a^2} \int_0^a c\, r\, dr$. The solution is obtained by imagining that the constant initial concentration for $x < 0$ consists of thin laminae which are distorted in time into paraboloids.

Solution for condition B).

To find when condition B) is valid Taylor calculates how rapidly a concentration which varies with r degenerates into a uniform concentration. The expression obtained is

$$t_o \gg \frac{a^2}{(3.8)^2 D} \qquad (8)$$

where t_o is a finite period of time.

Taylor proceeds to consider the transfer of labelled material across a plane travelling with the mean speed of flow $\bar{u}$ ($= \frac{1}{2}u_o$) when condition B) is applicable. Longitudinal molecular diffusion is neglected. The dispersion of labelled

material relative to this plane is exactly as though it were being diffused by a process obeying Fick's second law, with an effective diffusion coefficient k.

i.e. $$\frac{\partial c}{\partial t} = k \frac{\partial^2 c}{\partial x^2} \qquad (9)$$

where $$k = \frac{a^2 \bar{u}^2}{48 D}$$

Equation (9) is the equation governing longitudinal dispersion when condition B) is applicable.

For labelled material of uniform concentration c_o entering a tube filled with unlabelled material (c = 0) at a uniform rate at x = 0, t = 0 the solution of equation (9) is

$$c_m/c_o = 1/2 \pm 1/2 \operatorname{erf}\left(\tfrac{1}{2}(x-\bar{u}t)k^{-\frac{1}{2}}t^{-\frac{1}{2}} \right) \qquad \begin{matrix} x \leq \bar{u}t \\ x > \bar{u}t \end{matrix} \qquad (10)$$

where $$\operatorname{erf} z = 2\pi^{-\frac{1}{2}} \int_0^z e^{-z^2} dz$$

<u>Validity range for condition B).</u>

In a subsequent paper[64b] Taylor restates the condition (equation 8) for $k = \frac{a^2 \bar{u}^2}{48 D}$ to be a valid representation of dispersion in a tube as

$$\frac{4L}{a} \gg \frac{\bar{u}a}{D} \gg 6.9 \qquad (11)$$

where L represents the distance in which the greater part of

the change in concentration takes place. This expression incorporates the condition for longitudinal molecular diffusion to be negligible compared with the dispersive effect represented by k.

Aris[5] removed the restriction represented by the second inequality by defining the effective diffusion coefficient as

$$k = D + \frac{k a^2 \bar{u}^2}{D}$$

Philip[50] calculated the minimum dispersal time at which $k = D + ka^2\bar{u}^2/D$ is valid, obtaining

$$\frac{Dt}{a^2} >> \frac{1}{15}$$

This is almost identical with Taylor's condition, equation(8).

Calculation of the analytical solutions of the type described by equation (10) are referred to as the Taylor curves in the present text. The expression $k = \frac{a^2\bar{u}^2}{48\ D}$ is used. The validity of the Taylor solution is considered with regard to equation (8).

Computer solution.

The analytical solutions of equation (6) given by Taylor are only applicable under two sets of flow conditions A) and B). To obtain solutions of the partial differential equation

$$\frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial r^2} + \frac{1}{r}\frac{\partial c}{\partial r} + \frac{\partial^2 c}{\partial x^2}\right) - 2\bar{u}\left(1 - \frac{r^2}{a^2}\right)\frac{\partial c}{\partial x} \qquad (6)$$

for a wide range of flow conditions a computer must be used.

A computer program of the above equation was written for us by Mr. C.Paige of the Institute of Computer Science. The form of the numerical solution required was determined by the experimental measurements, and the boundary conditions were specified by the experimental arrangement.

The most convenient experiments consist of labelled solution contained in a tube being displaced by flow of unlabelled material from a reservoir. The boundary conditions in this case are

$$\frac{\partial c}{\partial r} = 0 \qquad \text{at } r = a, \quad \text{all } t$$

$$\left.\begin{aligned} c &= 0 \qquad \text{for } x < 0 \\ c &= c_o \qquad \text{for } x > 0 \end{aligned}\right\} \quad \text{at } t = 0$$

The mean concentration as a function of time is measured at a finite distance x_o downstream. Alternatively, flow is stopped after a finite period of time t_o and the mean concentration as a function of distance along the tube determined. The print-out from the computer is therefore required to give the mean concentration with respect to the initial concentration (c_m/c_o) as a function of time or distance. A brief explanation of the method employed in writing the program is contained in the appendix written by Mr. Paige. The data for a particular experiment is fed into the computer in the manner described in the appendix (p.163) and shown in the sample of data (p.165).

Presentation of results.

Taylor's solutions must be modified to allow for unlabelled displacing labelled material rather than vice versa. The solution for condition A) is

$$\left.\begin{array}{ll} c_m = 0 & x < 0 \\ c_m = c_o \frac{x}{u_o t} = c_o \frac{x}{2\bar{u}t} & 0 < x < 2\bar{u}t \\ c_m = c_o & x > 2\bar{u}t \end{array}\right\}$$

To facilitate direct comparison of all the curves dimensionless parameters are employed. In this manner the relative importance of molecular diffusion in determining the dispersion may be more readily assessed.

For experiments determining c_m as f(x) after a finite time t_o, the dimensionless parameter $X = \frac{x}{\bar{u}t_o}$ is used.

The theoretical dispersion curve for purely convective transport is

$$c_m/c_o = X/2 \qquad (12)$$

When c_m as f(t) is determined at a finite distance x_o downstream the dimensionless parameter $T = \frac{\bar{u}t}{x_o}$ is used. The theoretical dispersion curve for purely convective transport under these flow conditions is

$$c_m/c_o = 1/2T \qquad (13)$$

The Taylor analytical solution for labelled displaced by unlabelled material under condition B) is

$$c_m/c_o = 1/2 \mp 1/2 \operatorname{erf}\left[\frac{x - \bar{u}t}{2\sqrt{kt}}\right]$$

The dimensionless parameters X and T are again used in calculating the results. In addition

$$A_{to} = \frac{\bar{u}a}{D}\,\frac{a}{\bar{u}t_o} = \frac{a^2}{Dt_o}$$

and

$$A_{xo} = \frac{\bar{u}a}{D}\,\frac{a}{x_o}$$

are used to describe the curves.

The theoretical dispersion curve for condition B) (molecular diffusion dominates in determining the dispersion) when flow is stopped after a time t_o is

$$c_m/c_o = 1/2 \mp 1/2 \operatorname{erf}\left[(X - 1) \sqrt{\frac{12}{A_{to}}} \right] \qquad (14)$$

When the curve is monitored at a finite distance x_o downstream the analytical solution is

$$c_m/c_o = 1/2 \pm 1/2 \operatorname{erf}\left[\frac{(1 - T)}{\sqrt{T}} \sqrt{\frac{12}{A_{xo}}} \right] \qquad (15)$$

The dispersion curves represented by equations (14) and (15)

are thus characterised by A_{to} and A_{xo} respectively. Identical curves for c_m/c_o as f(X) should be obtained for a particular value of A_{to}, i.e. $\bar{u}$, a, t_o, and D may be varied as long as A_{to} remains constant. Similarly, for measurements at a fixed distance, $\bar{u}$, a, D and x_o may be varied and as long as the value of A_{xo} is not altered the dispersion curve for c_m/c_o as f(T) should be constant. This demonstrates the advantage of plotting all the results on a dimensionless scale, and emphasizes the importance of the parameter A in determining the dispersion.

Value of the Taylor inequality.

The condition for Taylor's analytical solutions (14) and (15) to be valid may be expressed as

$$(3.8)^2 \gg A_{to} \qquad (16)$$

Direct comparison of experimental results with the Taylor solution for c_m/c_o as f(X) at a given A_{to} should therefore provide a numerical value for this inequality. The flow conditions for which the dispersion of labelled material is given by equations (14) and (15) will be determined. Similarly the flow conditions required for solutions of type A) to be valid may also be determined.

Comparison of results.

To obtain the experimental points for c_m/c_o as f(X) the following values must be known :-

a) the mean flow velocity, $\bar{u}$

b) the radius of the tube, a

c) the diffusion coefficient, D

d) the length of time t_o, during which flow takes place

e) the initial concentration, c_o

f) the final concentration as a function of distance along the tube, i.e. c as f(x)

The value of A_{to} may then be calculated and the corresponding c_m/c_o as f(X) curve plotted.

The Taylor curve given by equation (14) for the same A_{to} value must be calculated and c_m/c_o as f(X) plotted.

The computer must be programmed for the same A_{to} value. The form of the data input requires that $\bar{u}$, t_o, a, D and the total length be specified. However, as has been shown these need not be the same as the experimental conditions provided A_{to} is identical.

The theoretical curve corresponding to purely convective transport is represented by equation (12) and is independent of A_{to}.

In this manner all the theoretical curves may be compared with the experimental results to decide which best describes the dispersion obtained under the specified flow conditions.

Similarly the experimental points obtained for c_m/c_o as f(T) must be compared with the Taylor equation (15) and the computer solution for the same A_{xo} value. The theoretical

curve for purely convective transport, equation (13), is independent of A_{xo}.

Unfortunately computer solutions for a series of A_{xo} values were not available. It should be possible to obtain solutions of this type from the present program, although trial runs seemed to indicate instability. The computation time is much longer, and it is possible that larger errors are involved. A better method is probably to obtain cross-checks for the c_m/c_o as f(T) curves from the computed solutions for c_m/c_o as f(X). This may be achieved in the following manner.

A particular c_m/c_o as f(T) curve is characterised by A_{xo} where

$$A_{xo} = \frac{\bar{u}a}{D}\,\frac{a}{x_o}$$

If a and D are constant for all A_{xo} and A_{to}, as in this case, extrapolation is simplified.

Then $$A_{xo} = \frac{a^2}{D}\,\frac{\bar{u}}{x_o} = \phi\frac{\bar{u}}{x_o}$$

where ϕ is a constant equal to (a^2/D).

The computer print-out for a particular set of flow conditions $\bar{u}$, a, D, t_o gives c as f(x) at many time intervals, i.e. A_{to}, during computation of the final result. The precise number of results is controlled by the number of time steps, T, and the print indicator instruction, N, in the data input. Values of T, on the A_{xo} curves corresponding to these t_o values may be calculated from

$$T = \frac{\bar{u}t}{x_o} = \frac{\bar{u}}{x_o} t_o$$

but $$A_{to} = \frac{a^2}{D} \frac{1}{t_o} = \phi / t_o$$

therefore

$$T = A_{xo} / A_{to}$$

c_m/c_o at T is the same as c_m/c_o at $x = \bar{u}t_o/T$ on the corresponding A_{to} curve.

This may be expressed

$$x = \frac{A_{to}}{A_{xo}} \bar{u}t_o = \frac{a^2}{D} \frac{\bar{u}}{A_{xo}} = \frac{\phi}{A_{xo}} \bar{u}$$

The distance x is calculated for each A_{xo} value using the appropriate $\bar{u}$ from the computer data for the A_{to}-type results. The print-out tabulates values of c_m/c_o at sufficiently small intervals of x for the results (c_m/c_o) to be determined directly, without plotting each curve. Thus for each A_{xo} curve values of c_m/c_o at $T = A_{xo}/A_{to}$ may be determined. The number of points is governed by how many A_{to}-type results are available.

CHAPTER III

CRITICAL SURVEY OF EXPERIMENTAL AND THEORETICAL DETERMINATIONS OF THE DISPERSION OF LABELLED SOLUTES.

Concentration dependence of D.

The partial differential equation (6) governing the dispersion of material in fluid flowing through circular tubes assumes that D is independent of concentration. As stated in the last chapter, this is only valid for self-diffusion coefficients, i.e. where no chemical concentration gradient exists[35b]. Self-diffusion in the strict sense (diffusion of a select group of particles within a medium of exactly equal particles) is unobservable. However, results approaching self-diffusion may be obtained using radioactive isotopes as indicators for diffusion measurements, provided the mass ratio of the isotopes, or of the molecules containing the isotopes, is sufficiently close to unity. When it is possible to carry out measurements closely approaching self-diffusion it means that D is sufficiently independent of concentration to avoid difficulties arising from the dependence on concentration.

Experimental value of D.

For the purpose of this series of experiments it was

decided to use the 'tracer diffusion' of iodide ion in aqueous alkali chloride solution at 25°C. 'Tracer diffusion' refers to the measurement of diffusion coefficients of ions present at vanishingly small concentrations in an otherwise uniform electrolyte. This has the advantage of approximating to a system of uniform chemical composition, so that D is independent of concentration and in addition may be determined by two independent techniques. One method employs a radioactive isotope as tracer at very low concentrations compared with the supporting electrolyte. The other involves chemical determination of considerably higher concentrations of the diffusing species, followed by extrapolation to zero concentration.

These results were published in a paper by Stokes, Woolf and Mills[63]. Iodide ion was selected as the diffusing species because a convenient and accurate method of chemical analysis in the presence of alkali chlorides was available. They obtained agreement between the two techniques with an average deviation of 0.4%. This was consistent with the estimated precision of the measurements.

Tracer diffusion coefficients for iodide ion in sodium, potassium and lithium chlorides at different concentrations of the supporting electrolyte were presented. It was decided to adopt the iodide ion diffusion coefficient determined with a twice normal concentration of potassium chloride as supporting electrolyte. This value

$$D = 1.85\ 10^{-5}\ cm.^2\ sec.^{-1}$$

was used in calculation of all the theoretical curves given in chapter V.

Preparation of solutions.

The 2N potassium chloride solution was prepared by diluting twice the molecular weight of dry Analar grade salts with distilled water to give 1 litre of solution. The radioactive tracer used was I^{131}, carrier free in weak thiosulphate solution. 1 mg.cm.$^{-3}$ of potassium iodide was added as a carrier. Stokes[63] found that unless this carrier was added, low and obviously erroneous diffusion coefficients were obtained. They suggest that this is attributable to the presence of either radiocolloids or other oxidation states of iodine in the carrier-free solution. The effect would not be evident in self-diffusion because of exchange with the large amount of iodide ion already present.

Critical examination of previous experimental verifications of Taylor's analytical solutions.

Experimental verification of Taylor's analytical solution B) has been limited by the dependence of diffusion coefficients on concentration.

Taylor[64a] describes experiments involving the dispersion of a 1% potassium permanganate solution. This is a transparent dark purple solution when observed in a tube. Direct colorometric comparison was made under the same conditions so that determi

nation was limited to c as f(x) when flow was stopped after a finite time t_o.

A value of $0.7\ 10^{-5} cm.^2 sec.^{-1}$ was selected for D in determining the appropriate flow conditions, equation (8). The reason for selecting this value was not discussed. Reference was made in the paper to measurements of the diffusion coefficient for potassium permanganate by Furth and Ullman (1927) given in the tables by Landolt and Börnstein. D varies from $0.435 - 1.5\ 10^{-5} cm.^2 sec.^{-1}$ in the range $0 < c < 0.01$ gm.ml.$^{-1}$.

In determining the theoretical distribution of c, a value of D was selected which most corresponded to the observed distribution of concentration. This was assumed constant and Taylor's analytical solution calculated. Independent quantitative experimental verification was not therefore achieved by these experiments. The results for the experimental and theoretical distributions of concentration demonstrated a fair agreement when the value of D was assumed in this manner.

The effect of molecular diffusion, as opposed to purely convective transport (laminar flow), in determining the dispersion is manifest in the shape of the curves. In the former case a sigmoid curve for c_m/c_o as f(X) is predicted, whereas in the latter it is a straight line (equations 14 and 12 respectively). Taylor's experimental results demonstrate the effect of molecular diffusion, under certain flow conditions, in that a sigmoid curve was obtained. Provided Taylor's condition B) applies, a variation in D would be expected to be manifest as a change in the slope of the curve, but not in the

sigmoid configuration. An increase in D will result in a larger gradient, and a decrease in a smaller gradient, for the curve. A precise knowledge of D is therefore only important in providing an exact and independent proof of equations (14) and (15), i.e. when the value of D is in no way dependent upon the theory. It is obvious from equations (14) and (15) that a large variation in D causes a much smaller change in c_m/c_o.

Flow conditions corresponding to analytical solutions of type A) (laminar flow) were also used by Taylor. The distibution of concentration is independent of D in this case. The experiments demonstrated agreement with the theoretical values for c as f(x) when flow was stopped at time t_o.

Results for flow conditions between these two limits were also presented to show the effect of molecular diffusion on the distribution of concentration in the transition zone between clear water and dye solution. The tendency of molecular diffusion to prevent dispersion was demonstrated. From these measurements it was possible to deduce the variation of D with c, although certain inaccuracies make the results of doubtful value The results were not inconsistent with those of Ullman.

These experiments described by Taylor, which constitute the most comprehensive series known to the author, are therefore limited by the following factors :

i) the uncertainty in D associated with its concentration dependence

ii) analytical solutions were only available under two extreme conditions A) and B)

iii) results for c as f(t) measured at a fixed distance downstream were not obtainable by this method. This would correspond more closely to the experimental investigations using the indicator dilution technique. It would be possible to extrapolate to this situation, but this would not constitute an experimental verification.

Rich and Goodman [55] have also investigated the effects of molecular diffusion on the dispersion of an indicator in fluid flow. A 1% Na_2CrO_4 solution was used with a label of radioactive $Na_2Cr^{51}O_4$. This should correspond to self-diffusion since there is no chemical concentration gradient. The value of D used in calculating Taylor's analytical solution B) was not discussed. A reference to 'Diffusion of chromates and dichromates of sodium and potassium' by Ö holm quoted in Chem. Abstr. 35 2054 (1941) was given at the end of the paper. This reads :

"The values of the diffusion constant at 20°C for Na_2CrO_4 were 3N, 0.659; 1N, 0.731; 0.5N, 0.768; 0.25N, 0.812. For $Na_2Cr_2O_7$ the values were 3N, 0.787; 2N, 0.780; 1N, 0.802; 0.5N, 0.821; 0.25N, 0.844. For K_2CrO_4 the values were . . . "

This must refer to the more conventional diffusion coefficient and not self-diffusion using radioactively labelled ions. These two values can be appreciably different [18b], so that a satisfactory experimental verification must include discussion of the diffusion coefficient used and its determination.

It may be that D was assumed from the experimental curve as in Taylor's experiments.

The study of Rich and Goodman was restricted to flow conditions corresponding to Taylor's analytical solution B). Effluent sampling was used to determine the mean volume concentration as a function of time. A method of converting from concentration within the tube to effluent concentration was given. The results for c as f(t) were obtained, which are more useful for comparison with the indicator-dilution method. Satisfactory experimental verification of the theoretical expressions was claimed. Whether this claim referred to the sigmoid configuration of the curves when D was assumed in the same manner as by Taylor, or if an independent value of D was used was not discussed. It is therefore impossible to decide whether this represents an independent experimental verification.

Experiments corresponding to flow conditions for Taylor's solution A) are described by Nanninga, Swann and Schubert[46]. Effluent sampling was used to determine the concentration as a function of time, so that a much more complex theoretical expression for c as f(t) was calculated than given by Taylor[64a]. The theoretical treatment describes the observed data precisely until approximately 95% of the tube volume was delivered, when a deviation from linearity was obtained. This was postulated to be due to adsorption effects or diffusion effects in the wall region.

These two sets of experiments[55,46] may be regarded as demonstrating agreement with Taylor's analytical solutions A) and B). However no estimate of the range of validity is given. Also for condition B) the value of D used in the theoretical calculation was not discussed.

Experimental verification of Taylor's theory for the effects of molecular diffusion on the dispersion of labelled material is also implied by the results of Bourdillon[9]. Changes in dispersion curves for a particular material in different flow regimes are demonstrated, together with changes due to different diffusion coefficients in the same flow regime. The values of D were not known precisely, so that these results cannot be claimed to provide an independent quantitative experimental verification.

Aims of the present series of experiments.

The tracer diffusion coefficient of iodide ion in 2N potassium chloride solution, as determined by the two independent methods of Stokes, Woolf and Mills[63], provides a perfect model to obtain an independent quantitative experimental verification of Taylor's theory. The experiments were designed (chapter IV) to give c_m/c_o as a function of t and x. It also has the advantage that all measurements may be made during the experiment.

The computer program, which was developed for this partic-

ular series of experiments, presents the oppurtunity of comparing theoretical and experimental results for a complete range of flow conditions. This has not been attempted previously to the author's knowledge. A numerical value for the Taylor inequality (equation 8) may also be estimated from the present series of experiments.

Application of the results.

As stated in chapter I, the complete series of experiments should enable clearance curves to be predicted for all cases where tracers are used to measure flow rates in steady flow (low Re). The different type of dispersion curves obtained when multiple indicators are used in the same flow regime may be discussed, with particular reference to blood flow. The application of these results will be limited by lack of knowledge about diffusion coefficients in many cases. The dispersion predicted by solution of equation (6) for faster flows (high Re) and non-Newtonian solutions and suspensions may be considered. However, if these solutions are correct the justification for using the equation under such circumstances must be reconsidered. These aspects of the work are discussed more fully after the presentation of the results.

CHAPTER IV

EXPERIMENTAL DETERMINATION OF THE DISPERSION OF LABELLED SOLUTES IN FLUID FLOW THROUGH A CIRCULAR TUBE.

In order to present the results in the manner described in chapter II, two series of experiments were required. Labelled solution of uniform concentration c_o contained in a precision bore tube was displaced by steady flow of unlabelled solution, $c = 0$, from a reservoir in both cases. The labelled solution was 2N KCl with I^{131} as a radioactive tracer. The preparation of the solutions is described in chapter III, and the amount of I^{131} used was determined by the activity required. 2N KCl was used as the unlabelled solution.

To determine c_m/c_o as a function of X, f(X), flow was stopped after a time t_o and the activity was monitored at different positions in the tube. To determine c_m/c_o as a function of T, f(T), the concentration at a distance x_o downstream from the tube entrance was recorded as a function of time.

Experimental arrangement.

The experimental arrangement that was used may be divided into two sections : the electronic counting system and the flow apparatus.

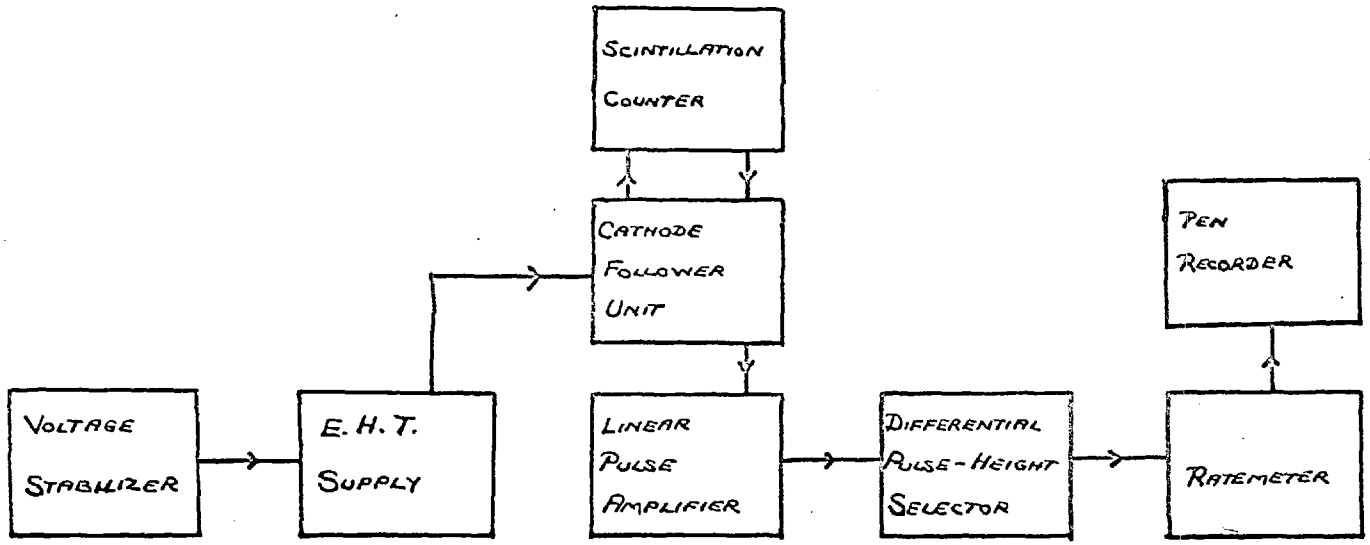


FIG.1. BLOCK DIAGRAM OF ELECTRONIC APPARATUS.

Counting arrangement for determination of c_m/c_o.

The radioactivity in the experimental tube was detected by a sodium-iodide scintillator attached to an IDL 663A scintillation counter. A modified Ekco cathode-follower head was used to match the detector to the main amplifier, a Nuclear Enterprises non-overloading linear pulse amplifier (NE 5202). The HV supply, from the stabilised Nuclear Enterprises 200 - 2000V EHT supply (NE 5307), was connected by the C.F. head to the scintillation counter as shown in the block diagram (fig.i.). The signal passed through a single channel differential pulse height selector (NE 5102) to the ratemeter (NE 5401). Rapid changes in the counting rate were monitored by a Lee^{d}s and Northrup pen-recorder (Speedomax G).

The I^{131} spectrum was monitored to obtain the optimum working conditions. The amplifier gain, discriminator bias and EHT were set accordingly for the whole series of experiments.

Linearity.

The linearity of the ratemeter was established using different strength I^{131} solutions in the ratio 4 : 3 : 2 : 1. A graph of strength against recorded activity was plotted (fig.ii.) and a straight line passing through the origin was obtained. All the results quoted used the 3K range of the ratemeter for which the linearity curve is shown.

Stability.

The variation in the counting rate was found to be less

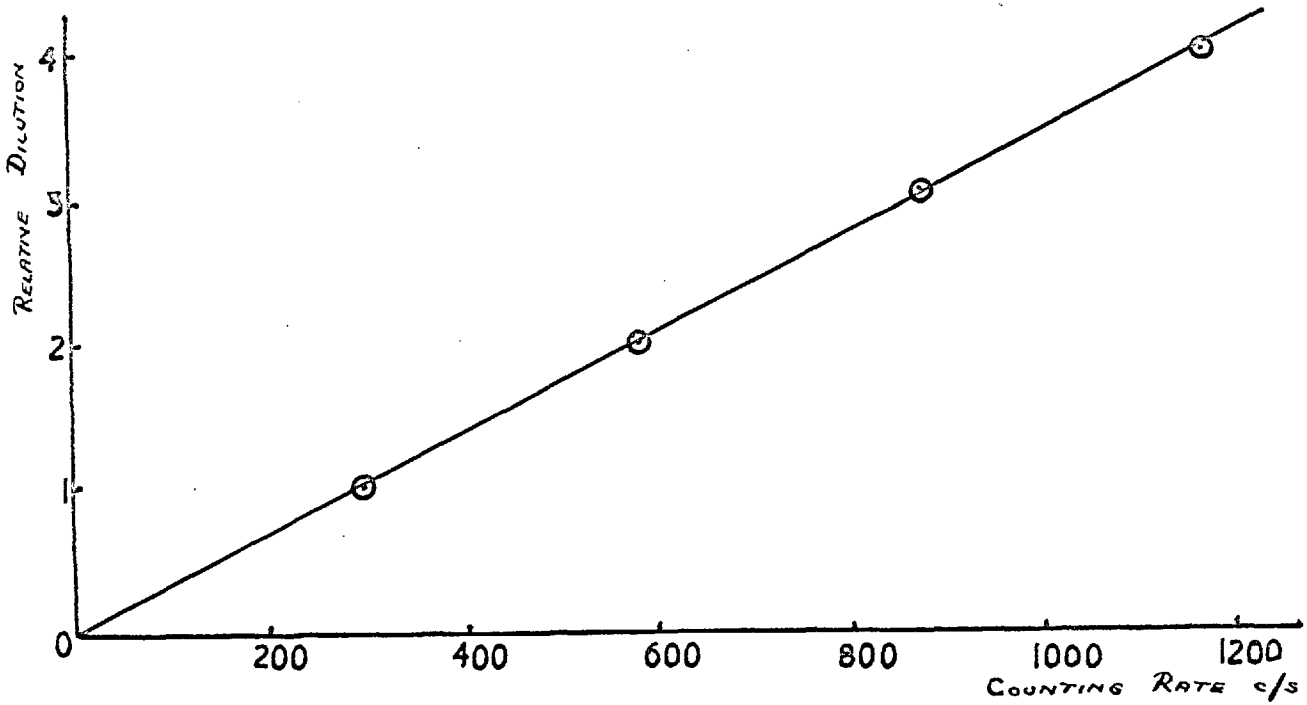


FIG.ii. LINEARITY OF THE RATEMETER.

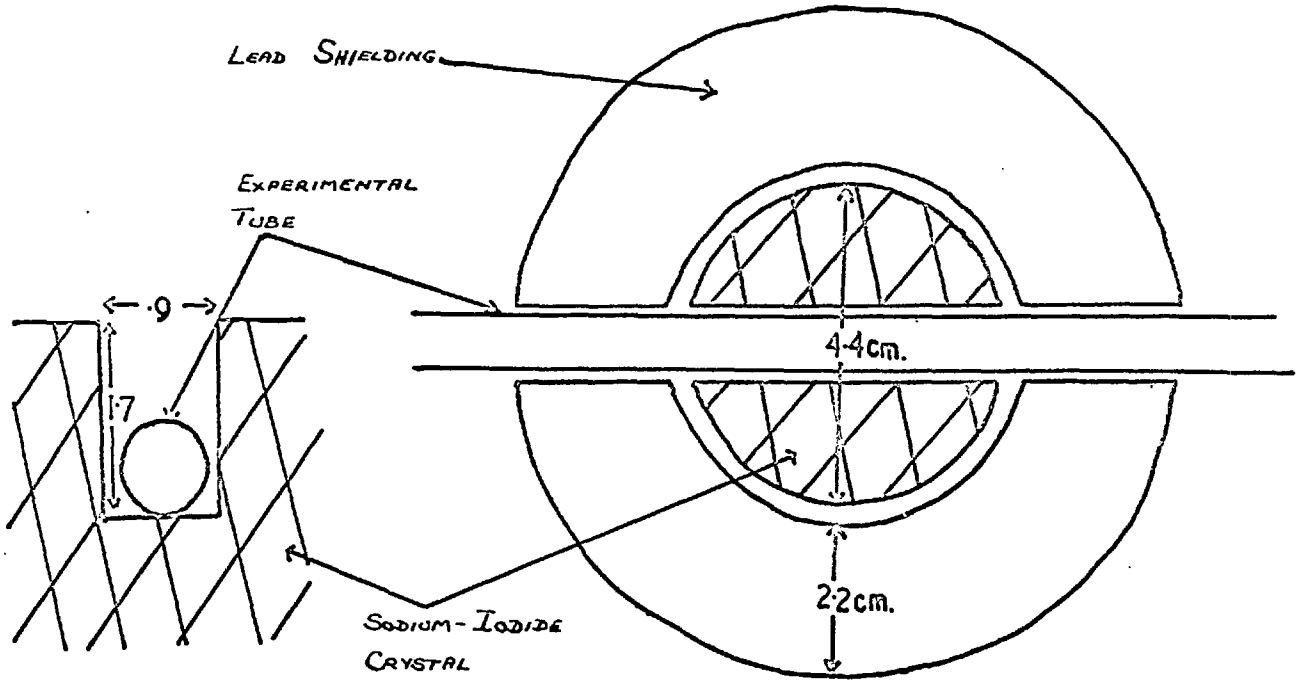


FIG.iii. COUNTING GEOMETRY.

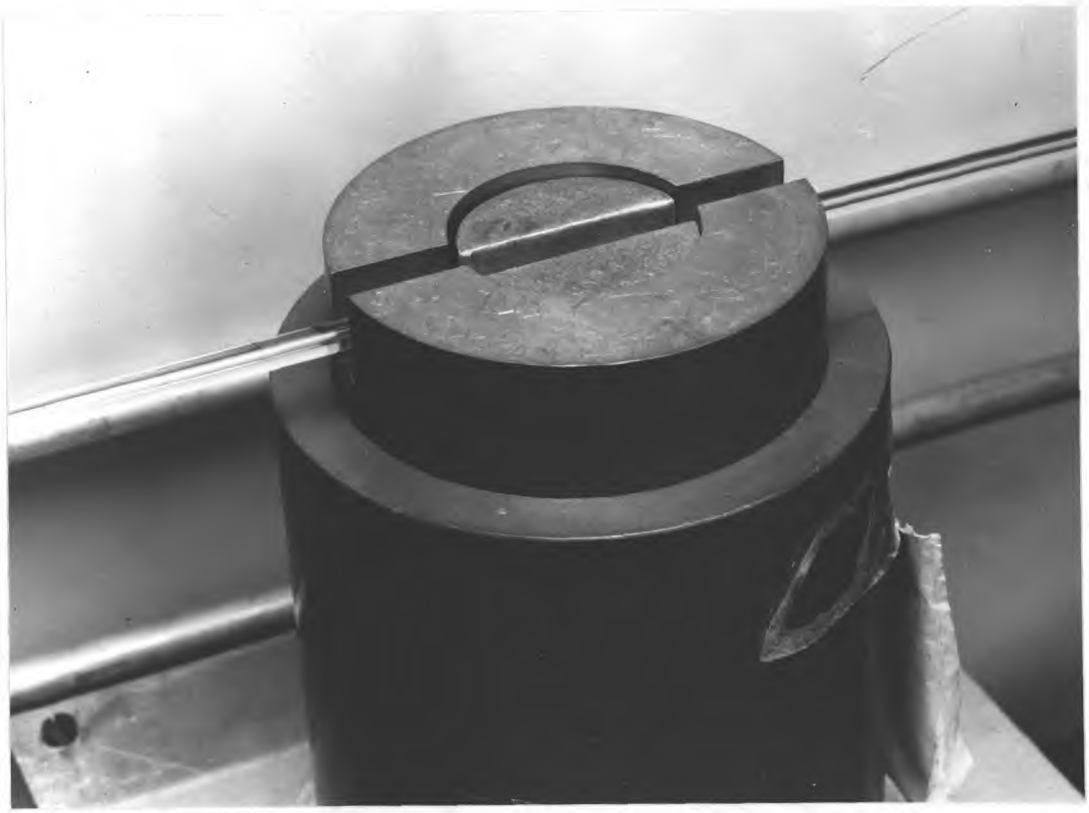


PLATE.1. SODIUM-IODIDE CRYSTAL AND POSITIONING OF THE EXPERIMENTAL TUBE.

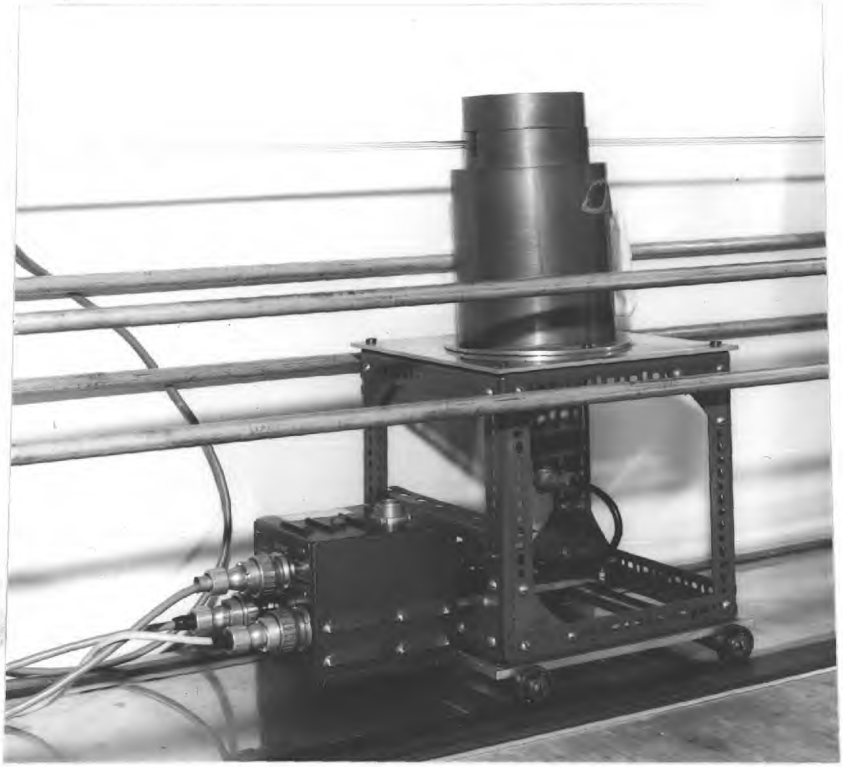


PLATE.2. SCINTILLATION COUNTER MOUNTED ON TROLLEY.

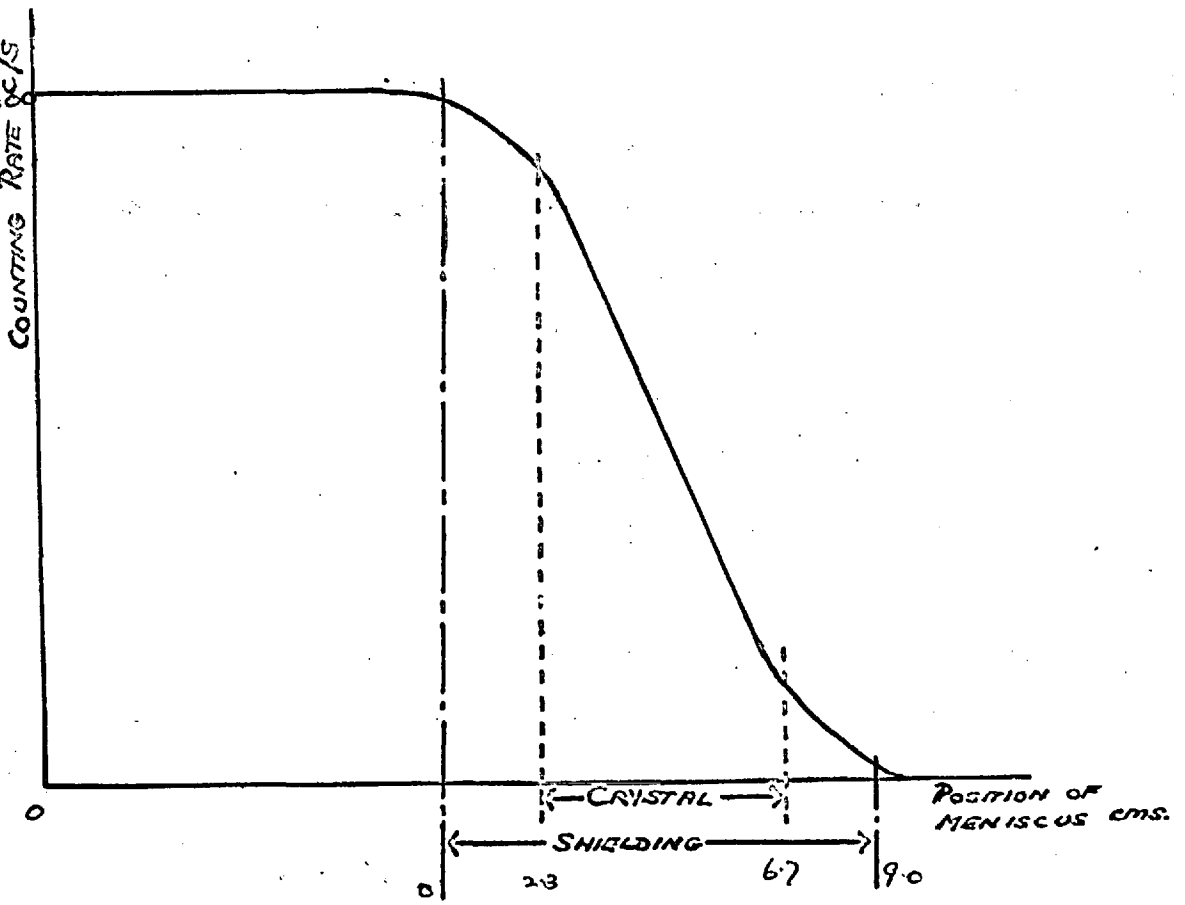


FIG. iv. DETERMINATION OF THE EFFECTIVE WINDOW OF THE SCINTILLATOR.

than the statistical fluctuation (i.e. $\pm \frac{100}{\sqrt{2n\tau}}$ % of mean) when observed for an hour. τ is the time constant for the ratemeter.

Counting geometry.

The sodium-iodide crystal was 4.4cm. in diameter, 2.4cm. high and had a rectangular channel 0.9cm. wide by 1.7cm. deep for positioning the experimental tube (fig.iii., plate 1). The scintillation counter was encased in lead (minimum 2.2cm. thick) and mounted on a trolley (plate 2).

The effective window of the crystal was found by part filling a precision bore tube with radioactive solution and plotting the position of the meniscus with respect to the outside casing against the recorded activity. The middle portion of the curve is a straight line giving activity per unit length. The effective length of the crystal was calculated by dividing the total activity (all the tube in the field of view containing radioactive solution) by the activity gradient (fig.iv.). The window length was found to be 4.74($\pm$0.04)cm. using 1.22, 2, 3 and 4mm. bore tubes.

When the apparatus was assembled it was found necessary to position the counter at least 20cm. from the end of the experimental tube to prevent an increased background count from the activity in the next section of the apparatus. This is discussed further in the chapter on development of the experimental technique.

For determination of c_m/c_o as f(X) it was important to

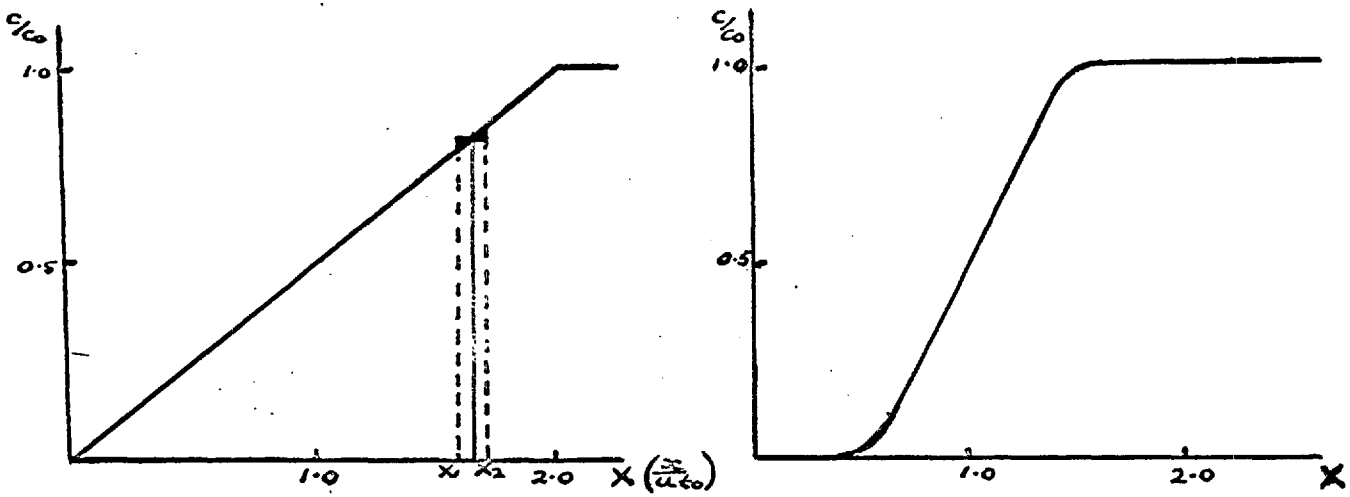


FIG.v. DISPERSION CURVES FOR c/c_o as f(X).

establish that the counting geometry was the same for the whole tube prior to starting flow, i.e. c_o was constant at all positions.

Time constant.

Determinations of c_m/c_o as f(T) require that $\tau \leq \frac{1}{5}$ of the decay time of c if the ratemeter is to reproduce it faithfully(61).

Measurement of c_m/c_o.

The ratio of the mean volume concentrations $(c_m/c_o)_{vol.}$ is determined by this technique, but the theoretical solutions involve the mean concentrations over the cross-section $(c_m/c_o)_{area}$. It must therefore be established that the two are equivalent. The counting geometry is symmetrical, so that if x is the position of the mid-point of the crystal $(c_m/c_o)_{vol.}$ at x will be approximately the same as $(c_m/c_o)_{area}$ provided the gradient of the $(c_m/c_o)_{area}$ curve is not rapidly changing or discontinuous in the region $x \pm \Delta x$. $2\Delta x$ is the effective length of the crystal. $(c_m/c_o)_{area}$ is measured at a distance x at time t so that the corresponding $(c_m/c_o)_{vol.}$ will be given by the area under the c_m/c_o as f(X) curve at time t between X_1 and X_2, where $X_1 = \frac{x-\Delta x}{ut}$, $X_2 = \frac{x+\Delta x}{ut}$. Fig.v. shows that for curves of type A) and B) the gradient is constant over the major region (this is also true for the computed results as

may be seen from subsequent graphs).

$$\therefore (c_m/c_o)_{area} \equiv (c_m/c_o)_{vol.} \quad \text{at x.}$$

There will be a small error introduced at the curved ends of the graph(of the order 1 - 2%)

Timing.

t_o was measured by starting a scaler counting at 50 c/s simultaneously with the commencement of flow. For c_m/c_o as f(T) an event marker was used at the start and finish of the experiment so that the time of flow was determined from the chart and the speed of the recorder.

Flow system.

Fig.vi. and plate 3 show the flow system.

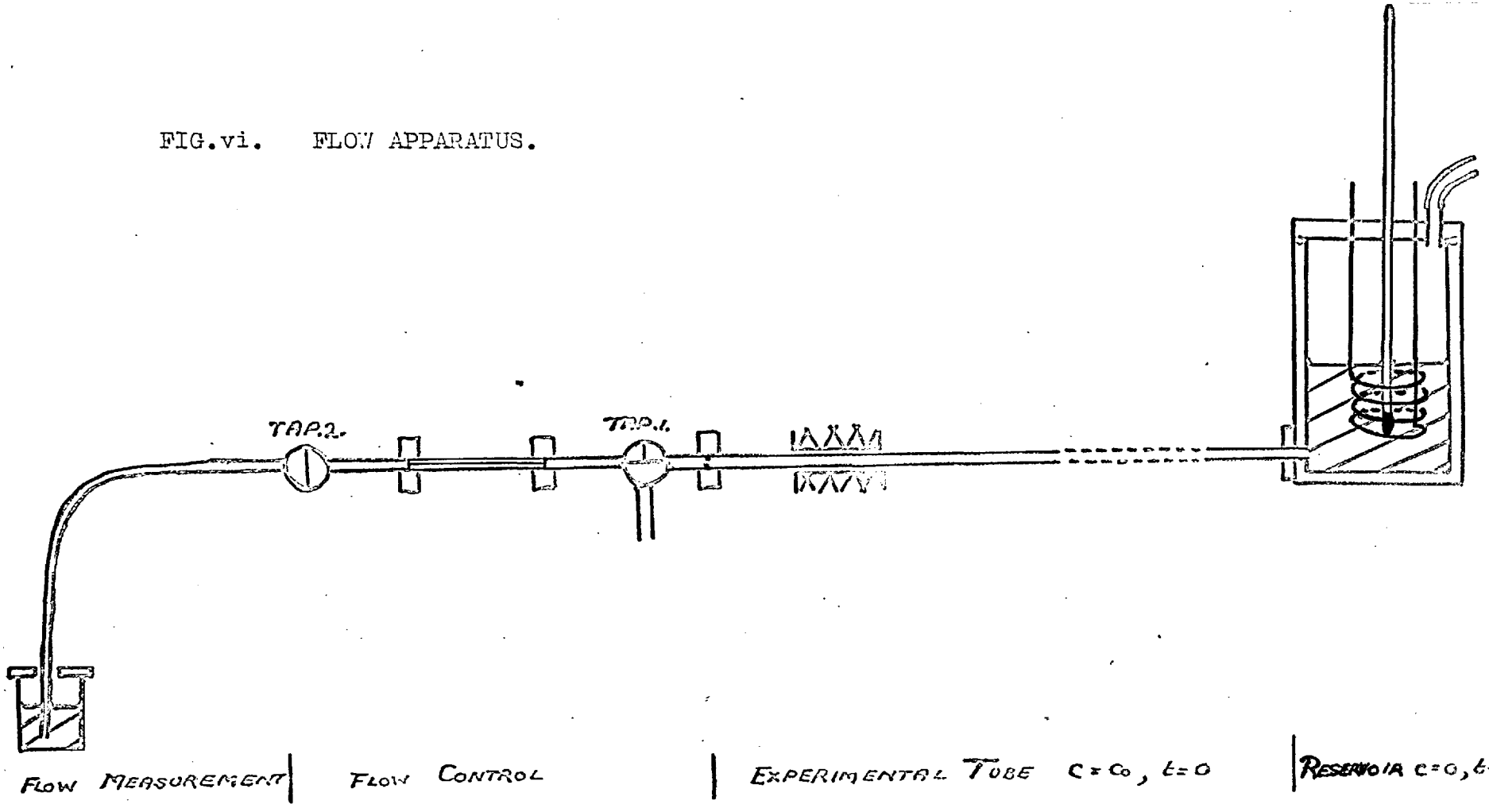
Perspex reservoir.

The reservoir has a circular cross-section. A stainless steel heating coil, thermometer (0 - 50°C in 0.1°) and pressure inlet are incorporated in the steel lid, which is fastened by four screws to the reservoir. Water from a constant temperature bath circulated through the heating coil. Fine control of the flow rate was obtained by applying a small pressure, which was recorded by a water or mercury manometer.

Experimental tube.

The term 'experimental tube' refers to the tube in which

FIG.vi. FLOW APPARATUS.



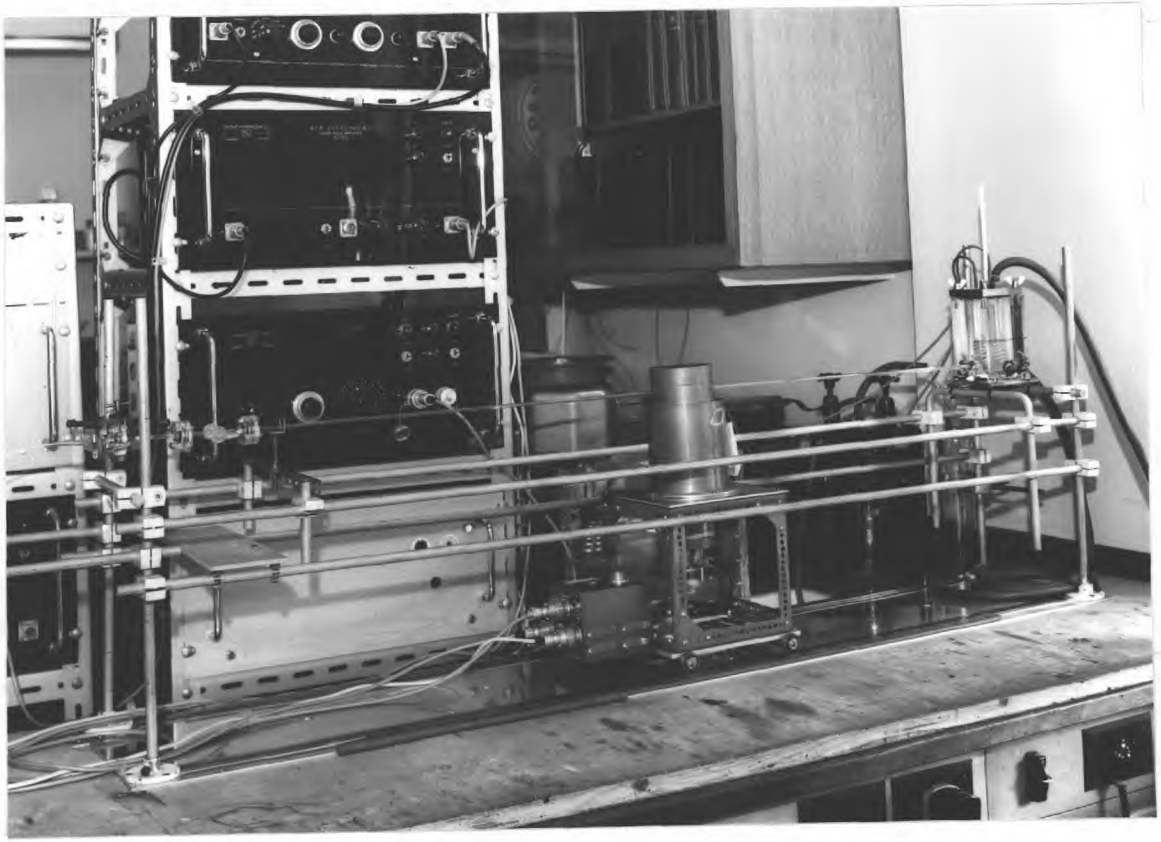


PLATE.3. EXPERIMENTAL ARRANGEMENT.

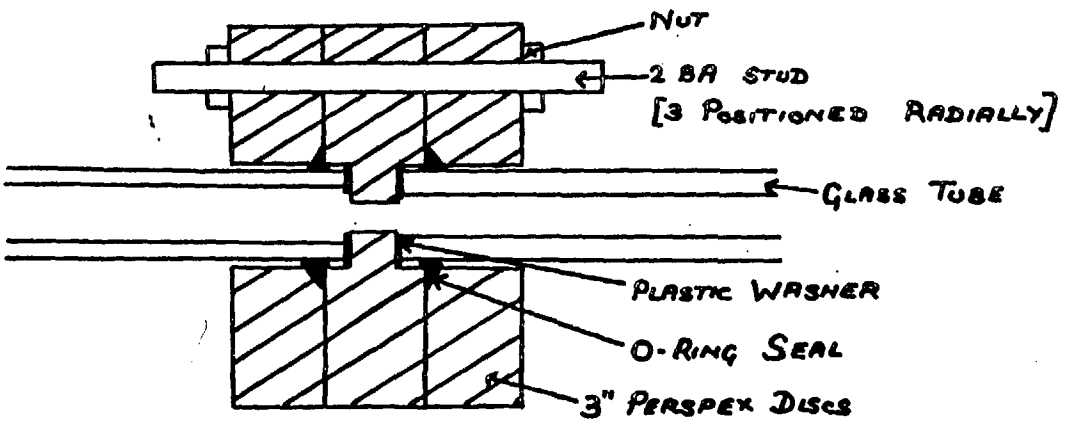


FIG.vii. TUBE JOINTS.

the measurements were made, this is shown in fig.vi. and plate 3. A precision bore glass tube, length 124cm. and diameter 0.122cm. was used. It was fitted flush with the inside wall of the resevoir by a perspex disc and O-ring seal. The manufacturers specify a possible 1% variation in the bore. The variation in r^2 was investigated using a cathetometer to measure the length of a mercury thread, l_1., at different positions in the tube. This was repeated for 3 threads of mercury and the variation in l_1 (hence r^2) was less than 1% in each case. Absolute determination of r^2 was also made using a travelling microscope to measure the length of a mercury thread and calculating its volume from the weight and density. The variation was less than 1%, and the mean value obtained was $3.715\ 10^{-3} cm.^2$ compared with $3.721\ 10^{-3} cm.^2$ It was decided to adopt the manufacturer's specification of 0.122cm. for the bore. A mean value of 0.122cm. was obtained for the tube diameter using a travelling microscope for direct measurement.

Taps.

Spring-loaded glass taps were used, lubricated with silicone stop-cock grease.

Joints.

The tubes were connected by the units shown in fig.vii. These were rigid components so that expansion could not occur.

Constrictions.

A length of capillary tube was used to control the flow rate through the experimental tube by providing a high downstream resistance. Tubes of different dimensions were used to ensure that only low pressures, $\lesssim$ 30cm. H_2O, were required to produce a particular flow rate. Anomalous results (steeper and/or displaced curves for c_w/c_o as f(T)) were obtained when high pressures were used to produce the required flow rate. Two possible explanations of this phenomena were considered, which are discussed further in chapter VI.

a) a deviation from a straight interface at the entrance to the experimental tube

b) a variation in diffusion coefficient with pressure.

Measurement of mean flow velocity.

In the final experimental arrangement a straight-through tap, positioned downstream from the constriction, was used to control the flow. The glass tubing of one arm of tap 2 was drawn out so that a polyethylene tube could be attached as shown in fig.vi. The whole system was filled with 2N KCl, and the outflow for the entire experiment was collected in a bottle which initially contained a small amount of 2N KCl solution. The bottle was weighed immediatly before and after the experiment to obtain the mass flow M_2-M_1, from which the volume flow $\frac{M_2 - M_1}{\rho}$ was calculated. ρ is the density of the 2N KCl solution. The experiment was timed by the recorder

for c_m/c_o as f(T) and a scaler for c_m/c_o as f(X) so that the volume flow rate $Q = \frac{M_2 - M_1}{\rho t}$ was obtained and hence the mean flow velocity $\bar{u}$ in the experimental tube, i.e. $Q = \pi a^2 \bar{u}$. This has the advantage of not relying on an indirect estimate of $\bar{u}$, minimising possible evaporation effects and facilitating a workable experimental procedure.

The methods employed previously are described in chapter VI.

Density of 2N KCl.

The density of 2N KCl solution was determined using a specific gravity bottle. A value of 1.09gm.cm.$^{-3}$ was used in calculation of all the results.

Determination of x.

The flow apparatus was supported by a scaffolding structure attached to the bench so that the position of the reservoir and tube were fixed. The track for the scintillation counter trolley was also fixed to the bench. 2.5cm. lengths were marked on the track corresponding to distances of the centre of the crystal from the beginning of the experimental tube. This facilitated easy measurement of c_m/c_o as f(X).

Cleaning procedure.

The experimental tube and reservoir were cleaned between experiments by washing out with water, hot detergent solution,

3 hot water rinses and finally dried by sucking air through the system. This procedure was adopted to minimise the likelihood of surface effects caused by other cleaning and drying agents. All beakers, syringes etc. used to transfer solutions were cleaned in a similar manner. The importance of this procedure is illustrated in chapter VI.

Safety procedure.

The normal precautions were taken in dispensing and handling the radioactive solution[72]. The total amount of activity in the tube during an experiment was $\sim 3.5\mu c$. This was washed out of the whole system after an experiment with unlabelled 2N KCl solution, the tube and reservoir were then cleaned through tap 1.

Temperature control.

The experiments must be conducted at 25°C, since that is the temperature at which D was determined[63]. The extent of the region to which this temperature control must be applied is obviously an experimental problem. Different regions of temperature control were used during the development stages (chapter VI), but for the series of experiments reported the entire surroundings of the apparatus were maintained at 25°C. The control was not very sensitive as electric fires were used, but since the experiments were all less than 10 minutes duration fairly effective control could be

achieved ($\pm 0.1^{\circ}$). The solutions were warmed prior to an experiment to $25^{\circ}C$, to reduce possible thermal diffusion at the interface[35c]. The laboratory was maintained at $25^{\circ}C$ for some time prior to an experiment to allow for thermal equilibrium of the flow tubes.

Preliminary experiments (n.b. chapter VI) demonstrated that pressure and temperature were important factors in obtaining agreement between theoretical and experimental results. This may be explicable by the considerable distortion of the interface between labelled and unlabelled solution caused by increased applied pressures, and local convection currents being established by temperature gradients in any part of the apparatus. A variation in D with these factors will also produce a small discrepency between theory and experiment.

It was found that experiments performed in the following manner, with the arrangement described gave consistent results which were in fairly good agreement with the analytical solutions of equation (6). Experiments using wider bore tubes and different flow measuring devices failed to give these results (chapter VI). It has not been possible to check the relative importance of the different factors mentioned in chapter VI (pressure, temperature, cleaning, history of solution etc.) since the present system was developed, due to lack of time. However, subsequent theoretical studies on

diffusion and viscosity have endorsed the author's opinion that any of these factors might be important under certain circumstances.

Experimental procedure for c_m/c_o as f(X).

The region between tap 1 and the collection bottle was filled with 2N KCl for all the experiments. Tap 2 controlled the start of the experiment. Tap 1 was a two-way tap allowing the experimental tube to be filled with radioactive solution and cleaned without interfering with the flow controlling region. This is more convenient from both the experimental and safety viewpoints.

1) The experimental tube was filled with radioactive solution. This was done by letting the solution flow down the tube from the reservoir under its own pressure head. A syringe was used to draw the solution slowly through the tap, tilting the tube slightly to avoid formation of air bubbles. The reservoir was cleaned using a fine tube attached to a syringe and finally dried with a tissue on a rod, avoiding the entrance to the tube. Tap 1 was turned to the straight through position.

2) Unlabelled 2N KCl solution was poured carefully down the side of the reservoir filling it to a mark 4.5cm. above the tube entrance.

3) The lid was screwed on to the reservoir and the temperature of the solution was checked - 25°C.

4) Initial activity in the tube (c_o) was recorded.

5) Collecting bottle was weighed.

6) A small pressure was applied for the required flow rate. The manometer was observed for a period of time to ensure that there were no leaks.

7) Tap 2 was turned to start flow simultaneously with the scaler counting at 50 c/s. Flow was stopped after a time t_o measured by the scaler.

8) Bottle reweighed.

9) Activity was monitored at different positions along the tube. A long time constant was used for greater stability. The activity was recorded for a few minutes in each position. The first position was subsequently checked to see that there was no significant decrease in c, indicative of a leak. (The effect of longitudinal molecular diffusion during the monitoring period was demonstrated to be negligible by checking every position twice over a two hour period.)

10) The remaining activity was washed out of the tube through tap 1. Background was recorded. Finally 2N KCl was washed through the constriction region to remove the activity, but it was not emptied.

11) Experimental tube and reservoir were cleaned through tap 1.

12) c_o and c as f(x) were determined from the recorder chart. The mean flow velocity $\bar{u}$ was calculated and a graph of c_m/c_o as f(X), i.e. $f(x/\bar{u}t_o)$, plotted. This was then compared with the theoretical curve for the same value of A_{to} (Taylor and/or computer).

Experimental procedure for c_m/c_o as f(T).

Stages 1 - 3 as for f(X).

4) Counter positioned at x_o and c_o was recorded.

5) Bottle weighed.

6) Pressure applied, observed for indication of a leak.

7) An appropriate value of time constant and chart speed were selected.

8) Tap 2 turned simultaneously with event marker.

9) Flow was stopped when c had almost reached the baseline - recorded by the event-marker.

10) Bottle reweighed.

11) and 12) as 10) and 11) previously.

13) The chart recorded c_o and c as f(t). The mean flow velocity was calculated and a graph of c_m/c_o as f(T), i.e. $f(\bar{u}t/x_o)$, plotted. This was then compared with the theoretical curve for the same value of A_{xo}.

CHAPTER V

R E S U L T S

T A B L E S

OF

D A T A

Table I. Experimental data for graphs 1 - 8.

Experiment	$P_{app.}$ cm.H_2O	Constriction lcm.	Constriction dcm.	t_o sec.	Qml.sec.$^{-1}$	$A_{to} = \frac{a^2}{Dt_o}$
Graph 1						
E2 9/1/67	~10.1	7.7	0.05	6.20	$8.599\ 10^{-2}$	32.4
E3 9/1/67	~ 8.0			6.28	$8.171\ 10^{-2}$	32.0
Graph 2						
E2 11/1/67	~22.8	10.1	0.04	14.38	$3.236\ 10^{-2}$	14.0
E1 12/1/67				14.26	$3.331\ 10^{-2}$	14.1
Graph 3						
E2 16/1/67	~22.8	15.8	0.03	28.0	$8.326\ 10^{-3}$	7.18
E2 12/1/67	~ 4.8	10.1	0.04	27.92	$2.328\ 10^{-2}$	7.20
Graph 4						
E2 13/1/67	~22.8	15.8	0.03	55.78	$7.807\ 10^{-3}$	3.61
E1 14/1/67				56.22	$7.629\ 10^{-3}$	3.58
Graph 5						
E2 14/1/67	~ 4.8	15.8	0.03	139.6	$5.783\ 10^{-3}$	1.44
E3 16/1/67				139.8	$5.803\ 10^{-3}$	1.44
Graph 6						
E2 6/2/67	~30.0	16.7	0.02	257.4	$1.552\ 10^{-3}$	0.781
E1 6/2/67				282.06	$1.661\ 10^{-3}$	0.713
Graph 7						
E1 31/1/67	~23.0	16.7	0.02	453.52	$1.481\ 10^{-3}$	0.443
E1 1/2/67				430.92	$1.524\ 10^{-3}$	0.467
Graph 8						
E2 26/1/67	~22.8	16.7	0.02	523.82	$1.494\ 10^{-3}$	0.384
E3 26/1/67				523.9	$1.524\ 10^{-3}$	0.384
E1 27/1/67				524.16	$1.468\ 10^{-3}$	0.384

Table II. Experimental data for graphs 9 - 16.

Experiment	Papp.cm.H_2O	Constriction lcm.	Constriction dcm.	x_o cm.	Qml.sec.$^{-1}$	$A_{xo}=\frac{\bar{u}a^2}{Dx_o}$
Graph 9						
E3 17/2/67	11	7.7	0.05	50	$6.895\ 10^{-2}$	23.7
E2 18/2/67					$8.097\ 10^{-2}$	27.9
Graph 10						
E1 17/2/67	45	10.1	0.04	40	$4.344\ 10^{-2}$	18.7
E3 16/2/67					$4.287\ 10^{-2}$	18.4
Graph 11						
E2 17/2/67	40	10.1	0.04	90	$3.870\ 10^{-2}$	7.40
E2 16/2/67					$3.691\ 10^{-2}$	7.06
Graph 12						
E1 7/9/66	64x13.6	16.7	0,02	99.2	$2.023\ 10^{-2}$	3.51
E1 14/9/66					$2.040\ 10^{-2}$	3.54
Graph 13						
E1 23/1/67					$8.031\ 10^{-3}$	1.38
E2 23/1/67	23	15.8	0.03	100	$7.758\ 10^{-3}$	1.34
E1 24/1/67					$8.303\ 10^{-3}$	1.43
Graph 14						
E3 30/8/66	18x13.6	16.7	0.02	99.2	$6.139\ 10^{-3}$	1.07
E2 13/9/66					$6.147\ 10^{-3}$	1.07
E2 6/2/67	30			25	$1.661\ 10^{-3}$	1.14
Graph 15						
E1 31/1/67	23	16.7	0.02	40	$1.481\ 10^{-3}$	0.637
E1 1/2/67					$1.524\ 10^{-3}$	0.656
Graph 16						
E1 31/8/66					$1.685\ 10^{-3}$	0.293
E2 31/8/66	55	16.7	0.02	99.2	$1.720\ 10^{-3}$	0.298
E1 1/9/66					$1.714\ 10^{-3}$	0.297

Table III. Calculation of c/c_o as f(X) for experiment E2 9/1/67 (graph 1).

$$X = \frac{x}{\bar{u}t_o} = \frac{\pi a^2}{Qt_o} x = 2.193\ 10^{-2}x$$

Baseline of recorder chart = 8.2divs.

c_o = 81.5 - 8.2 = 73.3divs.

x cm.	c chart divs.	c_m	c_m/c_o	$X = 2.193\ 10^{-2}x$
8.53	9.7	1.5	0.020	0.187
13.30	13.9	5.7	0.078	0.292
18.40	20.2	12.0	0.164	0.404
23.40	26.0	17.8	0.243	0.513
28.30	31.0	22.8	0.311	0.621
33.30	35.4	27.2	0.371	0.730
38.25	40.1	31.9	0.435	0.839
43.25	44.0	35.2	0.480	0.948
48.35	47.5	39.3	0.536	1.060
53.25	51.8	43.6	0.595	1.168
58.35	55.7	47.5	0.648	1.280
63.25	59.4	51.2	0.698	1.387
68.25	63.4	55.2	0.754	1.497
73.35	67.6	59.4	0.810	1.609
78.35	71.4	63.2	0.862	1.718
83.45	75.1	66.9	0.913	1.830
88.45	79.1	70.9	0.967	1.940
93.45	81.5	73.3	1.0	2.049

Table IV. Calculation of c/c_o as f(T) for experiment E1 17/2/67 (graph 10).

$$T = \frac{\bar{u}t}{x_o} = \frac{Q\,t}{\pi a^2 x_o} = 9.290\ 10^{-2}t$$

Baseline of recorder chart = 7.5divs.

c_o = 58.2 - 7.5 = 50.7divs.

t sec.	c chart divs.	c_m	c/c_o	$T = 9.29\ 10^{-2}t$
5.0	58.2	50.7	1.0	0.465
5.5	58.2	50.7	1.0	0.511
6.0	56.6	49.1	0.968	0.557
6.5	55.4	47.9	0.945	0.604
7.0	51.9	44.4	0.876	0.650
7.5	48.6	41.1	0.811	0.697
8.0	45.6	38.1	0.751	0.743
8.5	41.6	34.1	0.673	0.790
9.0	40.5	33.0	0.651	0.836
9.5	38.0	30.5	0.602	0.883
10.0	35.8	28.3	0.558	0.929
10.5	33.4	25.9	0.511	0.975
11.0	34.0	26.5	0.523	1.022
12.0	31.5	24.0	0.473	1.115
13.0	28.9	21.4	0.422	1.208
14.0	27.1	19.6	0.387	1.301
15.0	23.9	16.4	0.323	1.394
16.0	23.1	15.6	0.308	1.486
17.0	20.9	13.4	0.264	1.579
18.0	19.4	11.9	0.235	1.672
20.0	16.9	9.4	0.185	1.858
22.0	14.3	6.8	0.134	2.044
24.0	12.5	5.0	0.099	2.230
26.0	10.4	2.9	0.057	2.415
30.0	8.8	1.3	0.026	2.787
34.0	7.9	0.4	0.008	3.159

Table V. Computer data for graphs 1 - 8.

Graph	A_{to}	(M)	t_o sec.	$\bar{u}$ cm.sec.$^{-1}$	(Y)	$x_{tot.}$ cm.	(B,B')	(R)	(T)	(N)
1	33.5		6.0	4.534		50	0.25	20	270	30
2	14.4		14.0	3.965		120	0.20	16	640	80
3	7.18		28.0	1.983		110	0.20	12	640	80
4	3.59		56.0	0.940		100	0.20	12	700	70
5	1.44		140.0	0.427		100	0.125	12	1380	138
6	0.768		262.0 (419)	0.188		100	0.20	10	1680	210
7	0.480		419.0	0.188		100	0.20	10	1680	210
8	0.385		523.0 (1395)	0.0172		32	0.10	10	3280	410

N.B. The results for graphs 6 and 8 were taken from the printout for t_o=419 and 1395 respectively.

-st-

Table VI. Calculation of c_x/c_o as f(X) from the computer results for t_o=6.0sec. (graph 1).

$$X = \frac{x}{\bar{u}t_o} = \frac{1}{4.534}\,\frac{x}{6.0} = 3.676\ 10^{-2}x$$

x cm.	c_x/c_o	$X=3.676\ 10^{-2}x$
2.5	0.002	0.092
5.0	0.024	0.184
7.5	0.079	0.276
10.0	0.147	0.368
12.5	0.214	0.460
15.0	0.276	0.551
17.5	0.332	0.643
20.0	0.384	0.735
22.5	0.434	0.827
25.0	0.481	0.919
27.5	0.528	1.011
30.0	0.574	1.103
32.5	0.620	1.195
35.0	0.665	1.287
37.5	0.711	1.379
40.0	0.757	1.470
42.5	0.803	1.562
45.0	0.849	1.654
47.5	0.894	1.746
50.0	0.939	1.838
52.5	0.978	1.930

Table VII. Computer results for c/c_o as f(T).

1) Graph 9. $A_{xo} = 25.7$ $\quad A_{to} = \frac{a^2}{Dt_o} = \frac{2.011\ 10^2}{t_o}$

t_o sec.	A_{to}	$T = \frac{A_{xo}}{A_{to}}$	$\bar{u}$ cm.sec.$^{-1}$	$x = \frac{2.011\ 10^2\ \bar{u}}{A_{xo}}$	c/c_o
4.67	43.06	0.597	4.534	35.48	0.856
6.0	33.52	0.767	"	"	0.673
7.0	28.73	0.895	3.965	31.03	0.583
8.75	22.98	1.118	"	"	0.472
10.5	19.15	1.342	"	"	0.385
12.25	16.42	1.565	"	"	0.307
14.0	14.36	1.79	"	"	0.237
17.5	11.49	2.237	1.983	15.5	0.132
21.0	9.58	2.683	"	"	0.063

2) Graph 10. $A_{xo} = 18.6$

t_o sec.	A_{to}	T	$\bar{u}$ cm.sec.$^{-1}$	x cm.	c/c_o
6.0	33.52	0.555	4.534	49.02	0.921
7.0	28.73	0.647	3.965	42.87	0.797
8.75	22.98	0.809	"	"	0.648
10.5	19.15	0.971	"	"	0.548
12.25	16.42	1.133	"	"	0.468
14.0	14.36	1.295	"	"	0.397
17.5	11.49	1.619	1.983	21.44	0.270
21.0	9.58	1.942	"	"	0.168
24.5	8.21	2.266	"	"	0.095
28.0	7.18	2.590	"	"	0.048

Table VII. (contd.)

3) Graph 11. $A_{xo} = 7.2$

t_o sec.	A_{to}	T	$\bar{u}$ cm.sec.$^{-1}$	x cm.	c/c_o
14.0	14.36	0.500	1.983	55.4	0.998
17.5	11.49	0.627	"	"	0.857
21.0	9.58	0.752	"	"	0.733
24.5	8.21	0.877	"	"	0.635
28.0	7.18	1.003	"	"	0.543
33.6	5.99	1.202	0.940	26.25	0.405
39.2	5.13	1.404	"	"	0.281
44.8	4.49	1.604	"	"	0.181
50.4	3.99	1.805	"	"	0.107
56.0	3.59	2.006	"	"	0.057

4) Graph 12. $A_{xo} = 3.52$

28.0	7.18	0.490	0.940	53.7	1.0
33.6	5.99	0.588	"	"	0.945
39.2	5.13	0.686	"	"	0.850
44.8	4.49	0.784	"	"	0.757
50.4	3.99	0.882	"	"	0.659
56.0	3.59	0.981	"	"	0.558
70.0	2.87	1.226	0.427	24.4	0.327
84.0	2.39	1.473	"	"	0.155
98.0	2.05	1.717	"	"	0.058
112.0	1.80	1.956	"	"	0.017
126.0	1.60	2.200	"	"	0.003
140.0	1.44	2.444	"	"	0

6) Graph 14. $A_{xo} = 1.07$

140	1.44	0.743	0.427		0.920
157	1.28	0.836	0.188	35.3	0.804
174	1.156	0.926	0.0172	3.23	0.660
209.5	0.960	1.115	0.188	35.3	0.340

Table VII. (contd.)

6) Graph 13. $A_{xo} = 1.36$

t_o sec.	A_{to}	T	$\bar{u}$ cm.sec.$^{-1}$	x cm.	c/c_o
56.0	3.59	0.379	0.427	63.14	1.0
70.0	2.87	0.474	"	"	1.0
84.0	2.39	0.569	"	"	0.996
98.0	2.05	0.663	"	"	0.958
112.0	1.80	0.756	"	"	0.880
126.0	1.60	0.850	"	"	0.761
140.0	1.44	0.944	"	"	0.615
157.0	1.281	1.062	0.188	27.8	0.440
209.5	0.960	1.417	"	"	0.078
262.0	0.768	1.771	"	"	0.005

7) Graph 15. $A_{xo} = 0.645$

262	0.768	0.840	0.188	58.6	0.855
314	0.640	1.008	"	"	0.515
366.6	0.549	1.175	"	"	0.190
419	0.480	1.344	"	"	0.041

8) Graph 16. $A_{xo} = 0.297$

523	0.385	0.771	0.0172	11.65	0.965
628	0.320	0.928	0.094	63.65	0.769
697.5	0.288	1.031	0.0172	11.65	0.464
732	0.275	1.080	0.094	63.65	0.294
1046	0.1923	1.547	0.0172	11.65	0

Table VIII. Sample calculation for Taylor curve - graph 4.

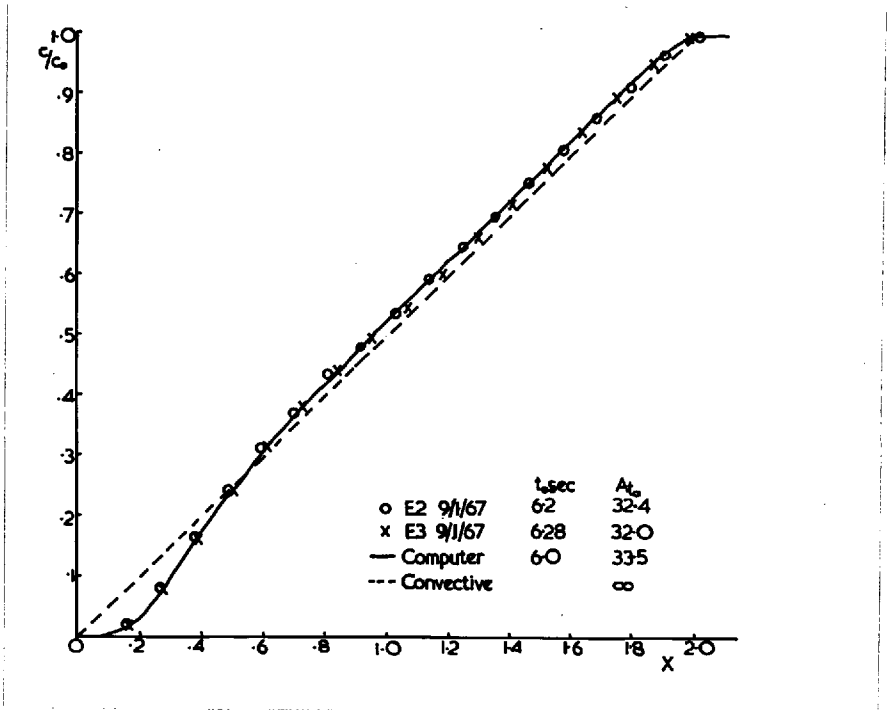
X	X-1	1.828(X-1)=Z	erf Z	½erf Z	c_x/c_o
0.40	0.60	1.0970	0.8792	0.4396	0.060
0.50	0.50	0.9140	0.8038	0.4019	0.098
0.60	0.40	0.7312	0.6989	0.3495	0.150
0.65	0.35	0.6398	0.6344	0.3172	0.183
0.70	0.30	0.5484	0.5620	0.2810	0.219
0.75	0.25	0.4570	0.4819	0.2410	0.259
0.80	0.20	0.3656	0.3949	0.1975	0.302
0.84	0.16	0.2925	0.3209	0.1605	0.339
0.88	0.12	0.2194	0.2437	0.1219	0.378
0.92	0.08	0.1462	0.1638	0.0819	0.418
0.96	0.04	0.0731	0.0823	0.0412	0.459
1.00	0	0	0	0	0.500
1.04	0.04	0.0731	0.0823	0.0412	0.541
1.08	0.08	0.1462	0.1638	0.0819	0.582
1.12	0.12	0.2194	0.2437	0.1219	0.622
1.16	0.16	0.2925	0.3209	0.1605	0.661
1.20	0.20	0.3656	0.3949	0.1975	0.698
1.25	0.25	0.4570	0.4819	0.2410	0.741
1.30	0.30	0.5484	0.5620	0.2810	0.781
1.35	0.35	0.6398	0.6344	0.3172	0.817
1.40	0.40	0.7312	0.6989	0.3495	0.850
1.50	0.50	0.9140	0.8038	0.4019	0.902
1.60	0.60	1.0970	0.8792	0.4396	0.940
1.70	0.70	1.280	0.9297	0.4649	0.965
1.80	0.80	1.462	0.9613	0.4807	0.981
1.90	0.90	1.645	0.9800	0.4900	0.990
2.00	1.00	1.828	0.9903	0.4952	0.995

Table IX. Sample calculation for Taylor curve - graph 15.

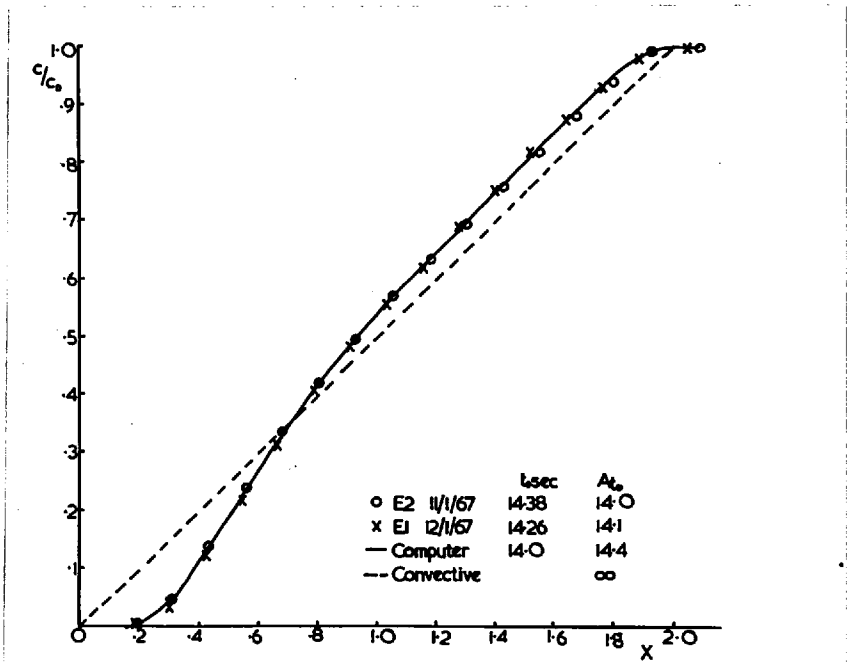
T	1-T	$T^{\frac{1}{2}}$	$\frac{1-T}{T^{\frac{1}{2}}}$	$4.313\frac{1-T}{T^{\frac{1}{2}}}=Z$	erf Z	$\frac{1}{2}$erf Z	c_m/c_o
0.50	.50	.7071	.7071	3.050	1.0	0.5	1.000
0.60	.40	.7746	.5164	2.227	0.9984	0.4992	0.9992
0.65	.35	.8062	.4341	1.872	0.9919	0.4960	0.9960
0.70	.30	.8367	.3586	1.547	0.9872	0.4936	0.9936
0.75	.25	.8660	.2887	1.245	0.9217	0.4609	0.9609
0.80	.20	.8944	.2236	0.9644	0.8274	0.4137	0.9137
0.84	.16	.9165	.1746	0.7530	0.7131	0.3566	0.8566
0.88	.12	.9381	.1279	0.5516	0.5647	0.2824	0.7824
0.92	.08	.9592	.0834	0.3597	0.3890	0.1945	0.6945
0.96	.04	.9798	.0408	0.1760	0.1966	0.0983	0.5983
1.00	0	1.0	0	0	0	0	0.5000
1.04	.04	1.020	.0392	0.1691	0.1890	0.0945	0.4055
1.08	.08	1.039	.0770	0.3321	0.3614	0.1807	0.3193
1.12	.12	1.058	.1134	0.4891	0.5109	0.2555	0.2445
1.16	.16	1.077	.1486	0.6409	0.6353	0.3177	0.1823
1.20	.20	1.095	.1826	0.7876	0.7346	0.3673	0.1327
1.25	.25	1.118	.2236	0.9644	0.8274	0.4137	0.0863
1.30	.30	1.140	.2632	1.135	0.8915	0.4458	0.0542
1.35	.35	1.162	.3012	1.299	0.9338	0.4669	0.0331
1.40	.40	1.183	.3381	1.458	0.9608	0.4804	0.0196
1.45	.45	1.204	.3738	1.612	0.9774	0.4887	0.0113
1.50	.50	1.265	.4082	1.761	0.9872	0.4936	0.0064

GRAPHS

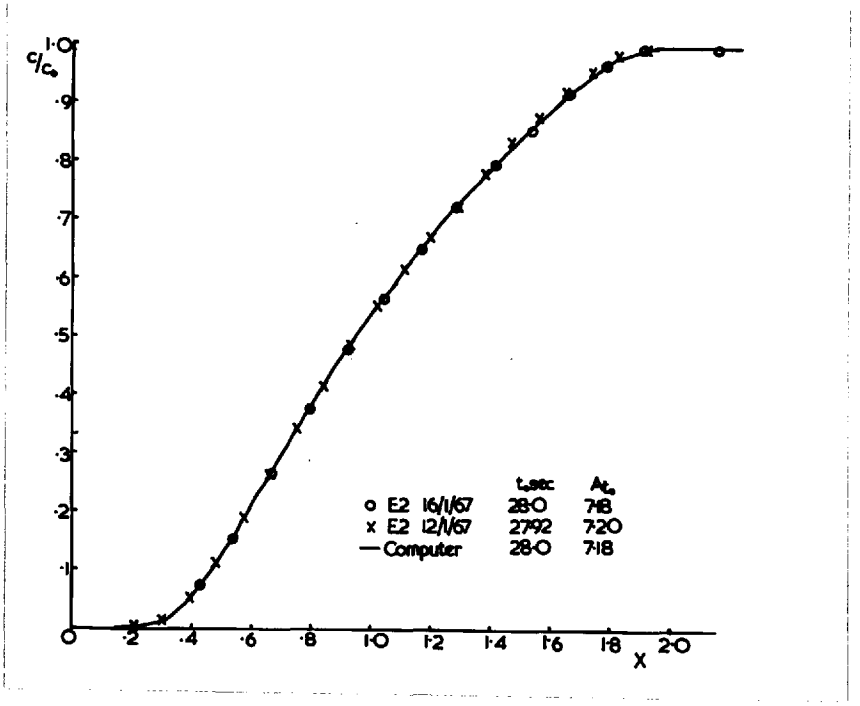
Fig. 1-8. A comparison of theoretical and experimental curves for c/c_0 against x $\left(\frac{x}{\bar{u} t_0}\right)$. The different curves were obtained by varying A_{t_0} $\left(\frac{a^2}{D}\frac{1}{t_0}\right)$. The theoretical calculations were based on Taylor's conditions A) and B), and the computer analyses. The experimental and Taylor results B) were displaced by the factors shown to agree with the computer result at $c_m/c_0 = 0.5$.



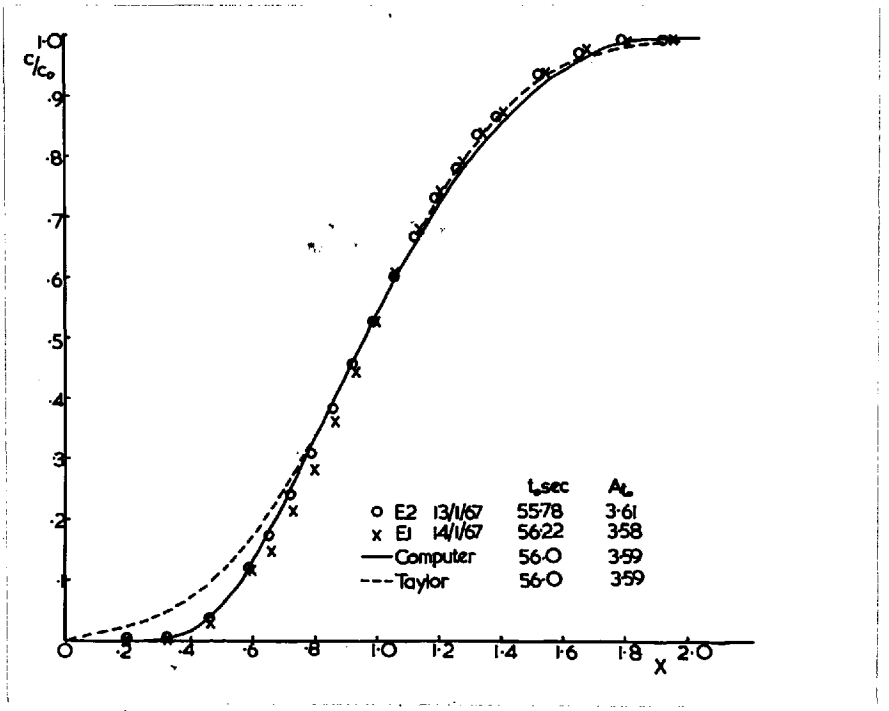
Graph.1. X decreased by 3% for experimental points.



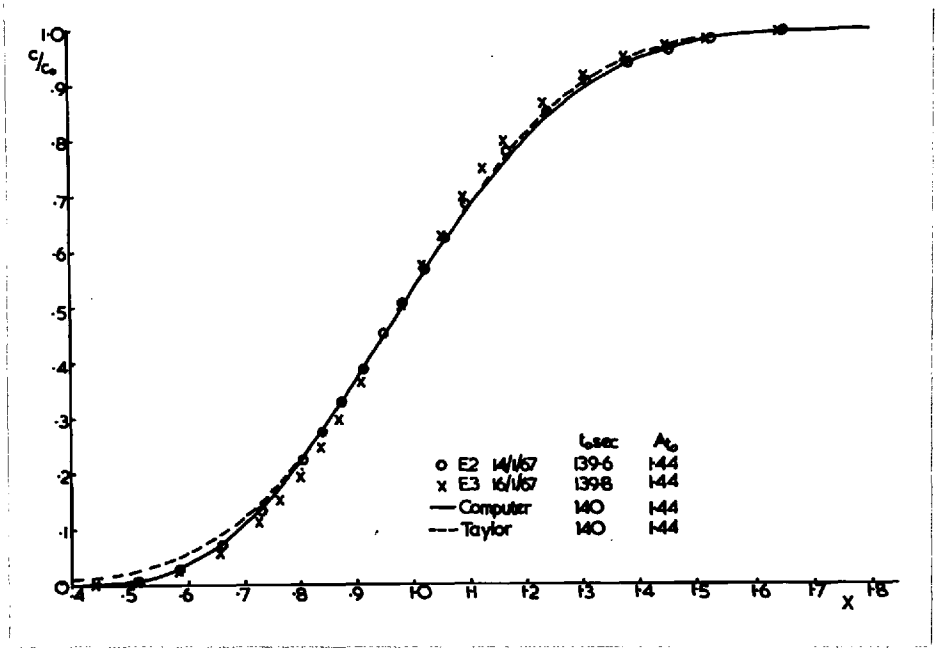
Graph.2. X decreased by 3% for experimental points.



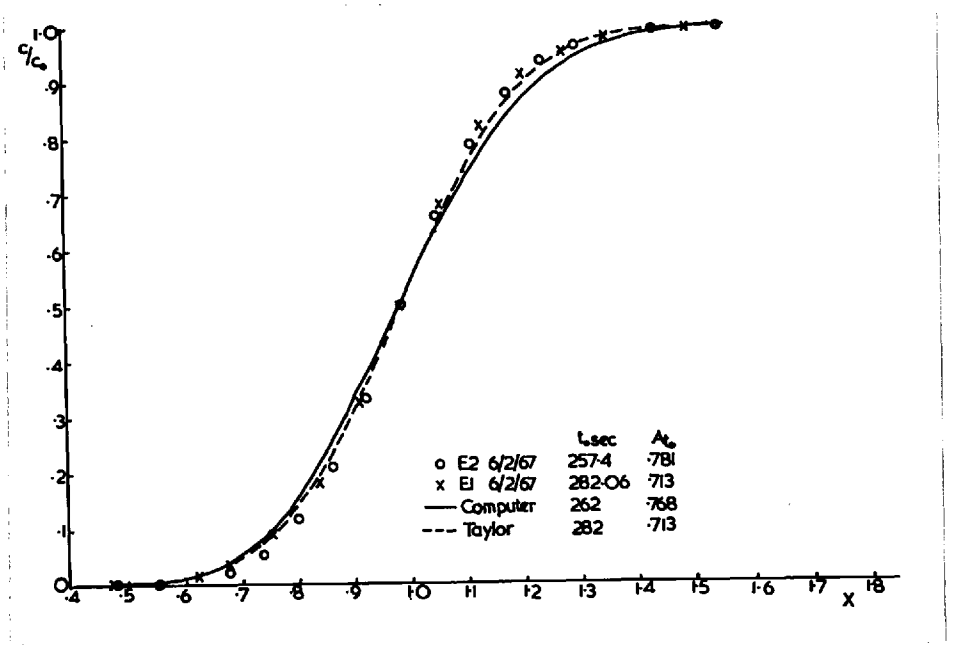
Graph.3. X decreased by 3% for experimental points.



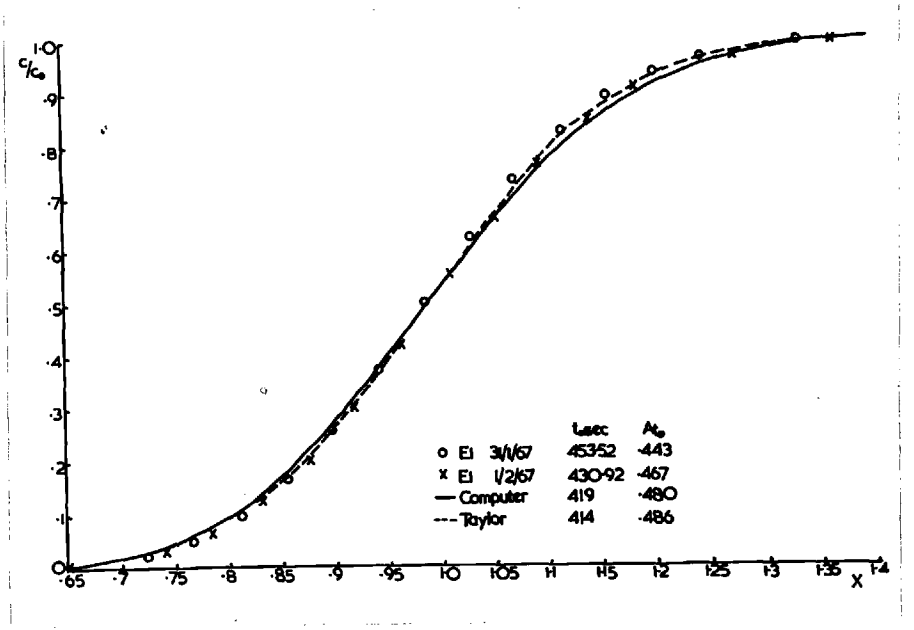
Graph.4. X decreased by 4% for experimental points.
X decreased by 4% for Taylor curve.



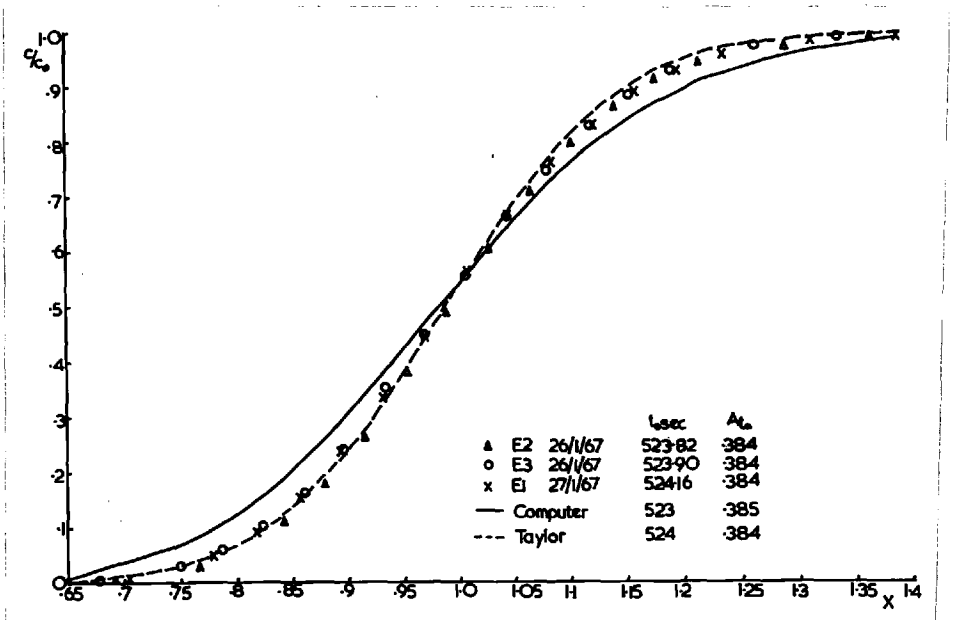
Graph.5. X decreased by 4% for experimental points.
X decreased by 2% for Taylor curve.



Graph.6. X decreased by 3% for experimental points.
X decreased by 1.5% for Taylor curve.

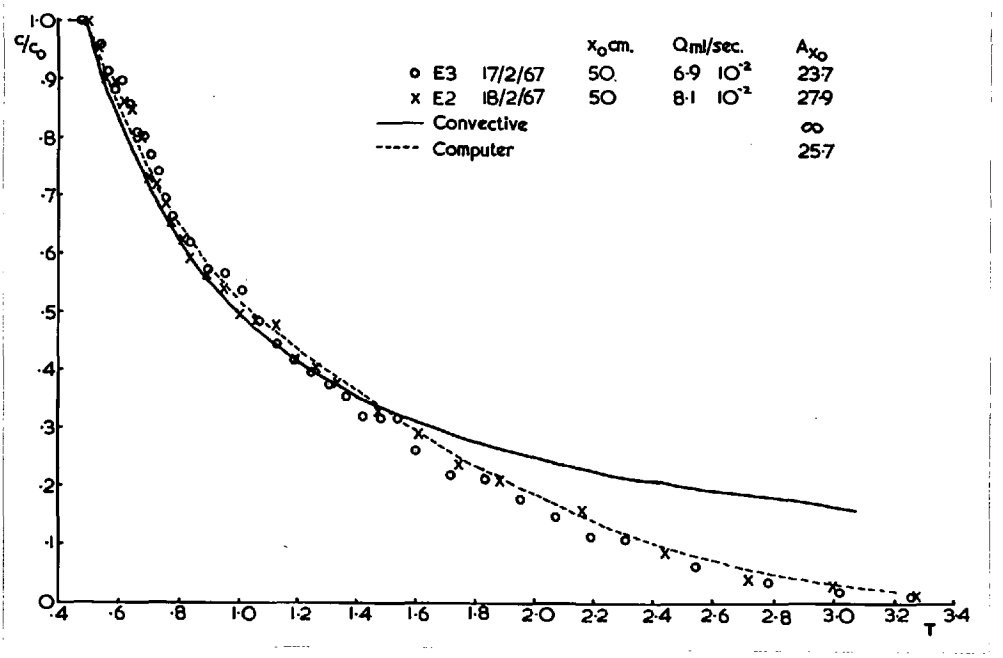


Graph.7. X decreased by 3% for experimental points.
X decreased by $1\frac{1}{4}$% for Taylor curve.

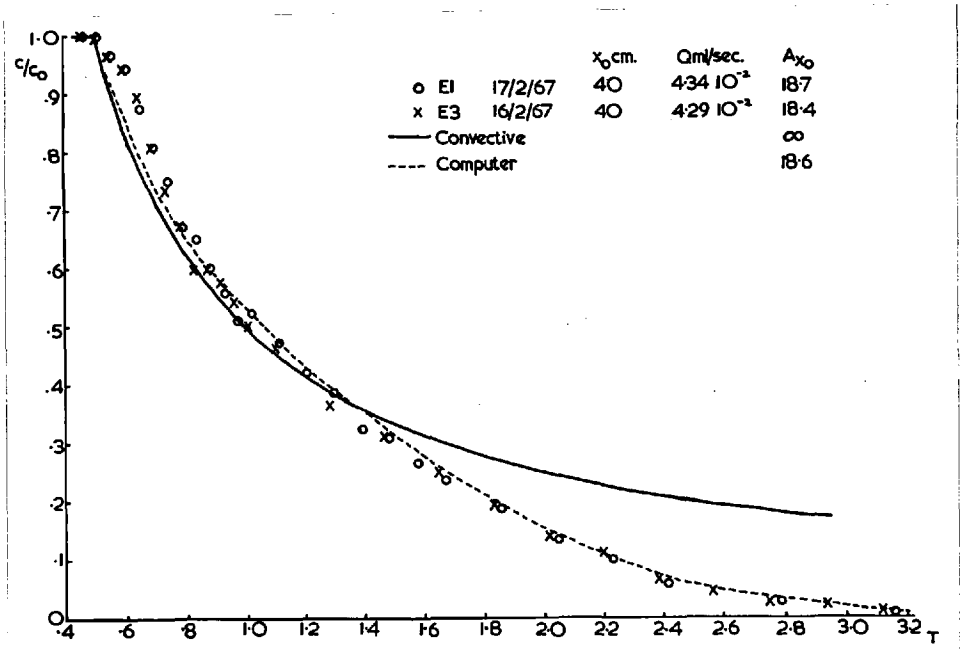


Graph.8. X decreased by 3% for experimental points.
X decreased by $1\frac{1}{4}$% for Taylor curve.

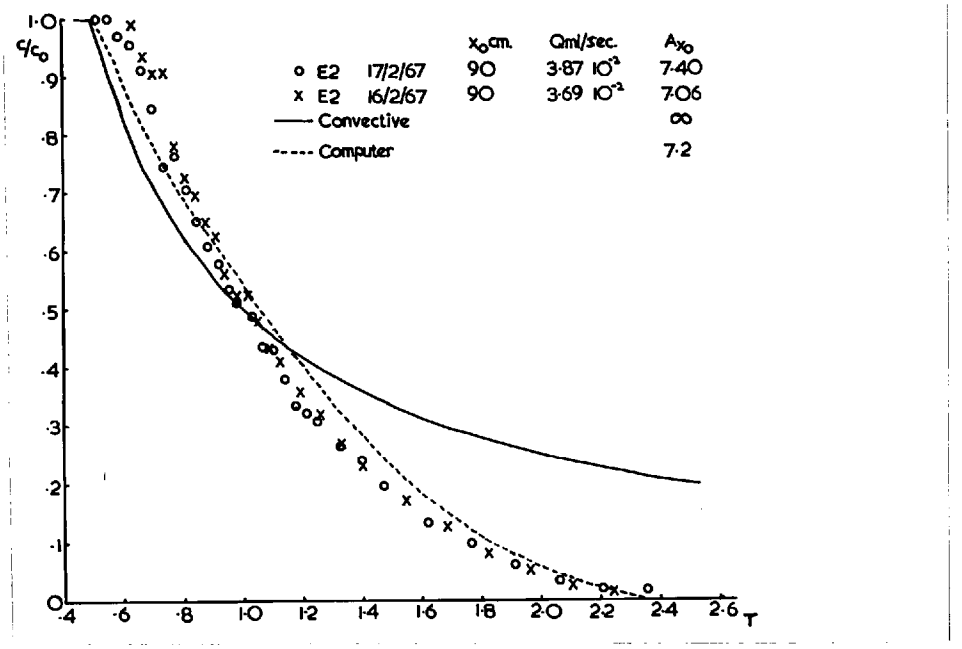
Fig. 9-16. A comparison of theoretical and experimental curves for c_m/c_0 against T ($\frac{\bar{a}t}{x_0}$). The different curves were obtained by varying P_{x_0} ($\frac{a^2}{D}\frac{\bar{v}}{x_0}$). Theoretical calculations were based on Taylor's conditions A) and B), and computer analyses. The experimental results were displaced by the factors shown to agree with the displaced Taylor curve B) (11½% a.b text) at $c_m/c_0 = 0.5$.



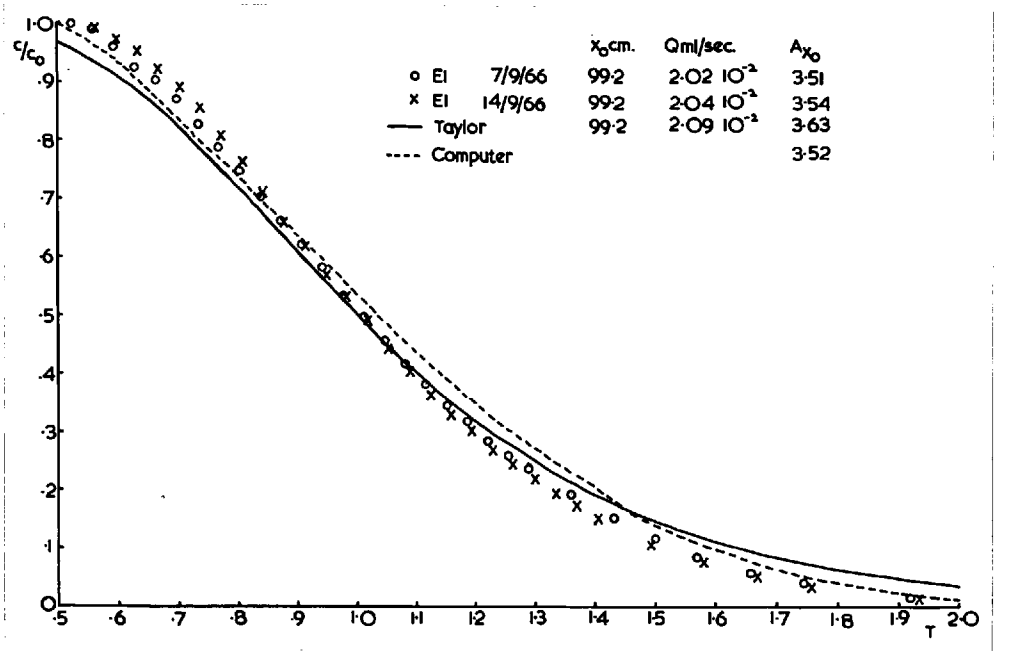
Graph.9. T decreased by 5% for experimental points, explicable by the uncertainty in time measurement at fast speeds.



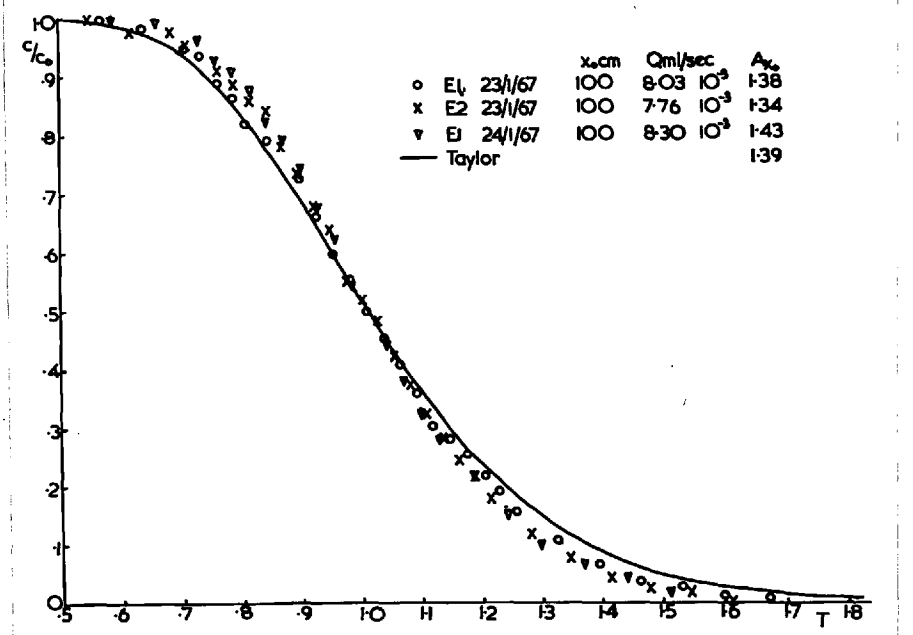
Graph.10. No displacement.



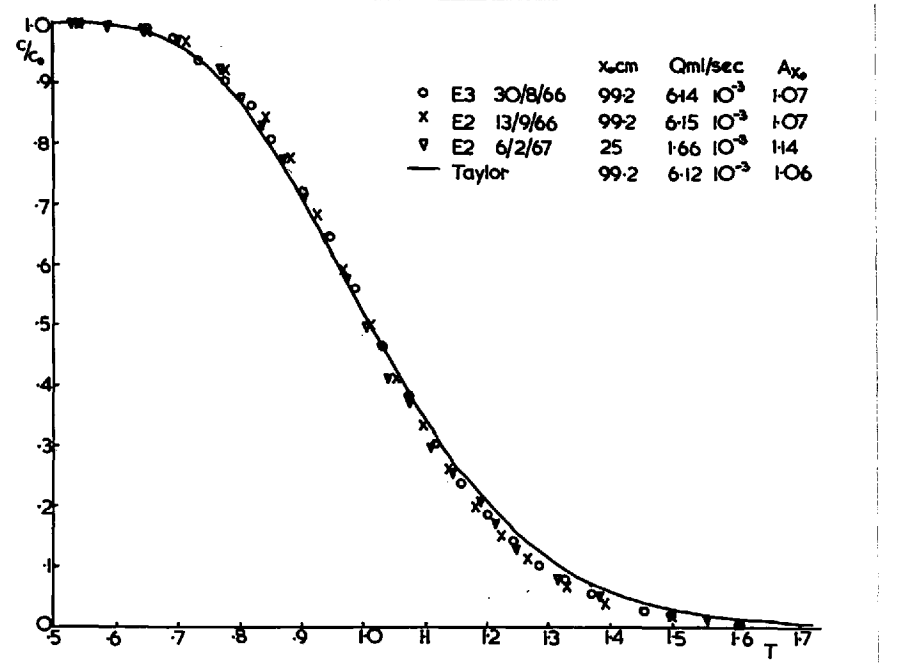
Graph.11. No displacement.



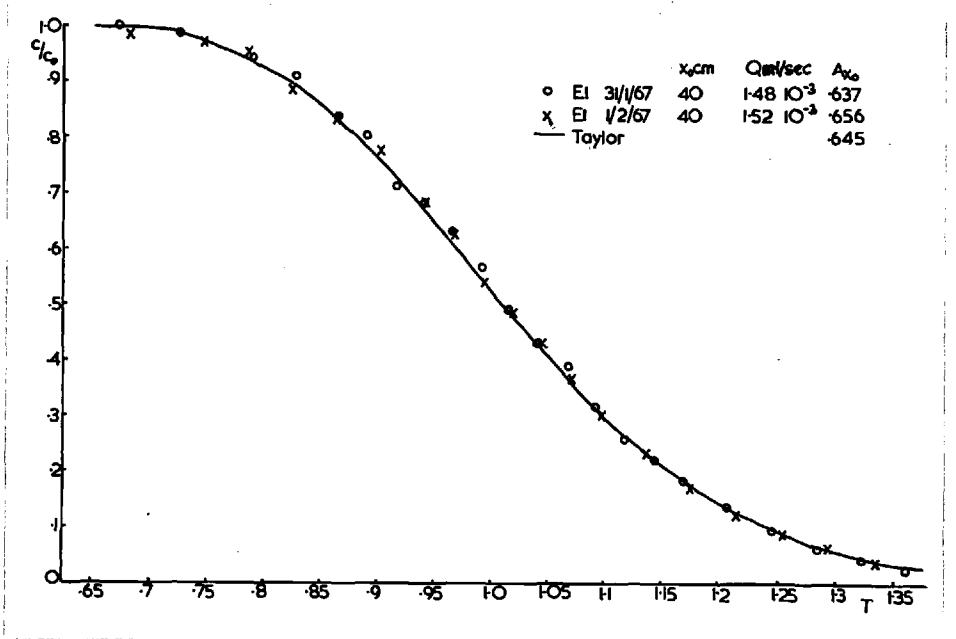
Graph.12. T increased by 1¼% for Taylor curve.



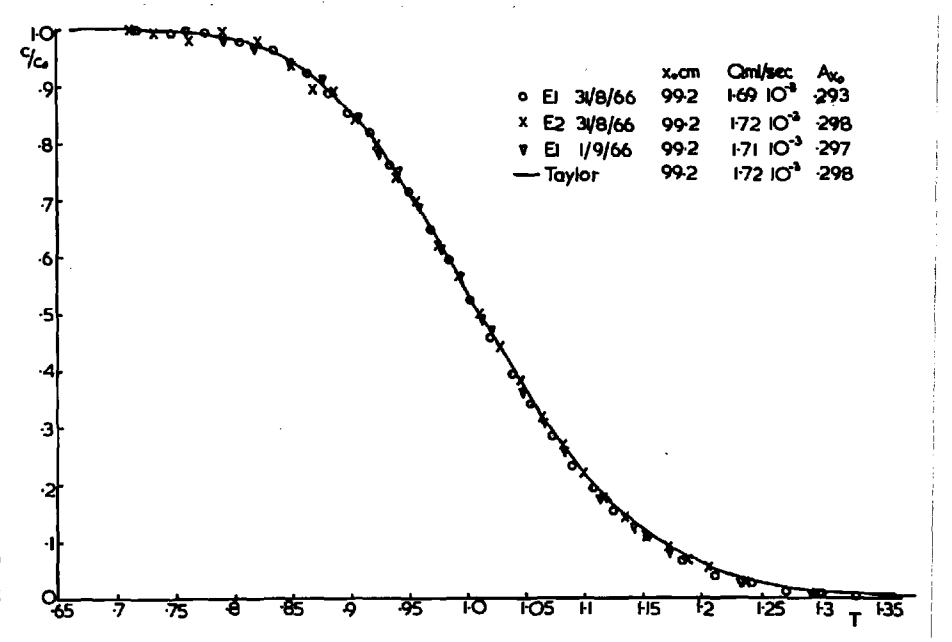
Graph.13. T increased by 1¼% for Taylor curve. T increased by 2% for experimental points.



Graph.14. T increased by 1¼% for Taylor curve. T increased by 1½% for experimental points.



Graph.15. T increased by 1¼% for Taylor curve. T increased by 3% for experimental points.



Graph.16. T increased by 1¼% for Taylor curve. T increased by 2% for experimental points.

CHAPTER V

RESULTS (contd. DISCUSSION and ANALYSIS)

The experimental points shown in graphs 1 - 16 were obtained using the arrangement and technique described in chapter IV with the exception of 12, 14 and 16. Apart from E2 6/2/67, graph 14, the arrangement for the earlier experiments was slightly different than described, but these results were consistent. The reasons for the modifications are explained in more detail in the following chapter. For the earlier experiments the room was not warmed, but a water jacket surrounded the section of the experimental tube between the reservoir and detector. This was maintained at 25°C. Room temperature was recorded and for the experiments reported, this ranged from 23.8 - 24.8°C, so that there were no large temperature gradients. The flow rate was determined from the mass flow collected from 4 - 6 times during the experiment. This was possible because flow was slow. A variation of less than 1% for the flow rates determined during any one experiment at the two slower flows was recorded (graphs 14 & 16) and less than 2% for the faster rate (graph 12). Pressures exceeding 30cm. H_2O were used (table II) to produce the required flow rate.

However, during the stage of development at which these results were obtained (September 1966) the pressure was not applied until just before the start of flow ($\leq$ 1min.). This may not be significant and is discussed more fully in chapter VI.

Using the experimental arrangement and technique described in chapter IV a complete series of self-consistent results was obtained. This method was developed experimentally as described in chapter VI. The deviation from the 'ideal' boundary conditions and the variation of the different parameters in the equation when the flow conditions were altered was investigated.

Calculation of results.

The following values were constant for all the experiments and are therefore not quoted in the tables of data.

Temperature = 25°C.

Diffusion coefficient, D = $1.85\ 10^{-5}$ cm.^{2}sec.$^{-1}$

Radius of experimental tube, a = 0.061 cm.

Density of 2N KCl, ρ = 1.09 gm.cm.$^{-3}$

The experimental data required to calculate X and T for the results given in graphs 1 - 8 and 9 - 16 is contained in tables I and II respectively.

The chart recorded c_o and c_∞ as f(x) when flow was stopped after a finite period of time t_o. Table III shows how these results

were converted to give c_m/c_o as f(X) for a particular experiment.

Similarly, table IV shows how c_m/c_o as f(T) was obtained from the experimental results.

The experimental points for each value of A_{to} and A_{xo} were plotted on separate graphs.

The computer was programmed for a corresponding series of A_{to} values. A description of the manner in which the computer must be programmed (the data input) is contained in the appendix (p.163). For this series of calculations S and S' of the data input were constant, equal to 0 and 4 respectively. The number of time steps was calculated from

$$\Delta t \leq \frac{1}{\frac{2u}{\Delta x} + \frac{4D}{(\Delta r)^2} + \frac{2D}{(\Delta x)^2}}$$

Table V gives the computer data for graphs 1 - 8. A sample of the data input for A_{to}=33.5 (t_o=6.0sec.) is shown in the appendix (p.165). The print-out gives c_m/c_o as f(x) at finite time intervals specified by the print indicator. Table VI shows how c_m/c_o as f(X) was obtained from the computed results.

The computer results for c_m/c_o as f(X) were plotted on transparent graph paper to enable simple direct comparison with the corresponding experimental and/or Taylor curves.

The computer solutions for c_m/c_o as f(T) were determined from the print-out for c_m/c_o as f(x) as described in chapter II. All the points which were available for each graph (A_{xo} value) were calculated as shown in table VII. The corresponding curves are shown in graphs 9 - 12 ; the computer points for c_m/c_o as f(T) are not shown in graphs 12 -16 since they are in good agreement with the Taylor curves and also very few points were available.

The Taylor analytical solution B) for fixed t_o is

$$c_m/c_o = 1/2 \pm 1/2 \text{ erf} \left[\sqrt{\frac{12}{A_{to}}} \left(X - 1 \right) \right] \qquad (14)$$

A sample calculation for c_m/c_o as f(X) at $A_{to}=3.59$ ($t_o=56$sec.) is given in table VIII. The values for the error function were taken from "Tables of the error function and its derivative - AMS41 " published by the U.S. Department of Commerce National Bureau of Standards.

The Taylor analytical solution B) for fixed x_o is

$$c_m/c_o = 1/2 \pm 1/2 \text{ erf} \left[\sqrt{\frac{12}{A_{xo}}} \left(\frac{1 - T}{\sqrt{T}} \right) \right] \qquad (15)$$

A sample calculation of c_m/c_o as f(X) for $A_{xo}=0.645$ ($x_o=40$, $Q=1.5\ 10^{-3}$) is given in table IX.

The curves for c_m/c_o as f(X) and f(T) were plotted on

transparent graph paper to facilitate easy comparison with the corresponding experimental and computed results.

The theoretical curves for purely convective transport (laminar flow) are

$$c_m/c_o = X/2 \qquad (12)$$

$$c_m/c_o = 1/2T \qquad (13)$$

These curves were also plotted on transparent graph paper.

Comparison of results.

A small variation in A_{to} or A_{xo} has a negligible effect on the shape of the curves. This may be demonstrated by theoretical calculation and is also noticeable in certain of the experimental results.

The graphs for corresponding A_{to} values were superimposed for comparison. It was found that the curves agreed in shape but not position. Similar results were obtained for corresponding A_{xo} curves.

In plotting graphs 1 - 8 it was decided to adjust the position of the experimental and Taylor curves to agree with the computer curve at $c_m/c_o=0.5$. The latter curve should give the most accurate description of the dispersion, since no terms in the partial differential equation (6) were ignored in obtaining this solution.

A constant displacement of $\sim 3\%$ was required for the experimental points. The shift in the Taylor curve was greatest for short times but decreased to a constant factor of $\sim 1\frac{1}{4}\%$ at long flow times, as might be expected. At these long times the computer solution is possibly inaccurate to 3 - 4% in c_m/c_o due to the size of the Δx step in the program. The finite difference method indicates that the curve tends to the Taylor solution when smaller steps are used (i.e. it becomes steeper and shifts to the right).

It was more difficult to decide on the position of the curves for c_m/c_o as f(T) , graphs 9 - 16, since an exact computer solution was not available. It was decided to displace the Taylor curve $1\frac{1}{4}\%$ to the right in accord with the previous result for $A_{to} \lesssim 0.7$. However, on reflection it would have been more consistent to have displaced the Taylor curve by the same amount for corresponding A_{to} and A_{xo} values. This would have resulted in a displacement of $\sim 3\%$ for all the experimental curves, except graph 9.

The shape of the curves is in good agreement and demonstrates how molecular diffusion becomes more important in determining the dispersion of labelled material as A_{to} and A_{xo} decrease. The displacement of the different curves is stated below each graph.

Discrepancy in the position of the curves.

Since all three curves (graphs 1 - 8) differ in position it is difficult to isolate and quantify the source

of error.

A systematic error of ~ 3% in the experimental results is suggested if the position of the computed results is regarded as correct. An error in the determination of X would give a displacement with respect to the ordinate.

$$X = \frac{x}{\bar{u}t_o} = \frac{\pi a^2 x}{Qt_o} = \frac{\pi a^2 \rho}{(M_2 - M_1)} x$$

A possible 1% error in the determination of 'a' and ρ could therefore account for this 3% discrepancy. The measurement of x could also contribute a systematic error in the calculation of X, due to the uncertain effect of the 'mixing region' at the tube entrance. It is impossible to quantify this effect without further extensive investigations. It is therefore conceivable that a maximum systematic error of 4 - 5% could be present in the determination of X.

Similarly an equivalent systematic error must be present in the calculation of T resulting in a displacement in the opposite direction for c_m/c_o as f(T).

$$T = \frac{\bar{u}t}{x_o} = \frac{Q}{\pi a^2 x_o} t = \frac{(M_2 - M_1)t}{\pi a^2 \rho x_o}$$

Graphs 9 - 16 suggest that this is the case.

The effect of the mixing region at the tube entrance

will be to create a diffuse, rather than sharp interface. Since both the Taylor and computed analytical solutions consider a sharp boundary this might explain the difference between the analytical and experimental results.

It is also possible that there might be an error in the computed solution due to the finite difference method employed. This may be as much as 3 - 4% for slow flows, long times, but is considered to be negligible for fast flows, short times.

Taylor's solution B) is also an approximate analytical solution which will be most accurate at slow flows, long times.

It is therefore difficult to assess the importance of the relative positions of the curves due to the possible errors involved. The author has chosen to plot the curves to demonstrate the excellent agreement in shape. Rich and Goodman[55] reported a similar discrepancy in the positions of the Taylor and experimental curves, although the displacement was not constant.

Shape of the curves.

The experimental points tend to be steeper than both the computed and Taylor curves for A_{to} = 3.6 - 0.71 and A_{xo} = 7.2 - 1.07 inclusive (graphs 4 - 6 & 11 - 14). This would be explained by an error in D : if D was slightly larger than $1.85\ 10^{-5}$ it would be more noticeable in this region

between the two analytical solutions of Taylor.

Conclusions.

The Taylor and computed analytical solutions of the differential equation

$$\frac{\partial c}{\partial t} = D\nabla^2 c - 2\bar{u}\left(1 - r^2/a^2\right)\frac{\partial c}{\partial x} \qquad (6)$$

agree with the experimental results within the estimated range of errors. This constitutes a completely independent experimental verification that the dispersion of labelled solutes during fluid flow through a tube will be determined by this partial differential equation, (6).

Value of the Taylor inequality.

Equation (16), chapter II expresses Taylor's condition B) as

$$A_{to} << (3.8)^2 \qquad (16)$$

i.e.

$$A_{to} << 14.4$$

Results for $A_{to} = 3.6$ show agreement between the Taylor, computed and experimental curves for the region $c_m/c_o = 0.32 - 1.0$. This corresponds to a 4x inequality for

equation (16). For a 10x inequality, A_{to}=1.44, divergence from the Taylor curve was obtained below c_m/c_o = 0.2. Thus condition B) is valid when

$$A_{to} \lesssim 1.44$$

i.e.
$$t_o \gtrsim 10 \frac{a^2}{(3.8)^2 D}$$

However, the experimental and computer results indicate that molecular diffusion will always have some effect on the dispersion of labelled solutes, i.e. it is impossible to obtain purely convective transport (laminar flow) for a finite value of D. These implications will be discussed in the following chapters.

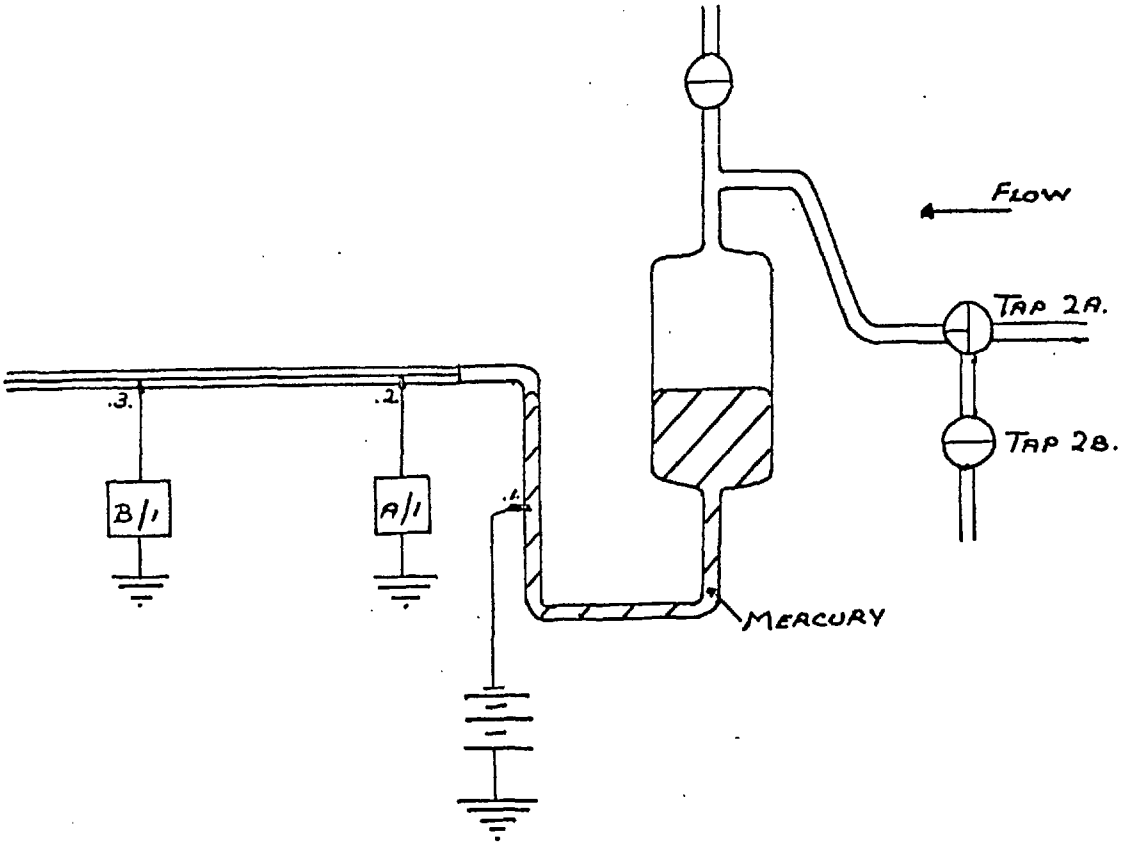
CHAPTER VI

DEVELOPMENT OF THE PRESENT EXPERIMENTAL SYSTEM AND TECHNIQUE.

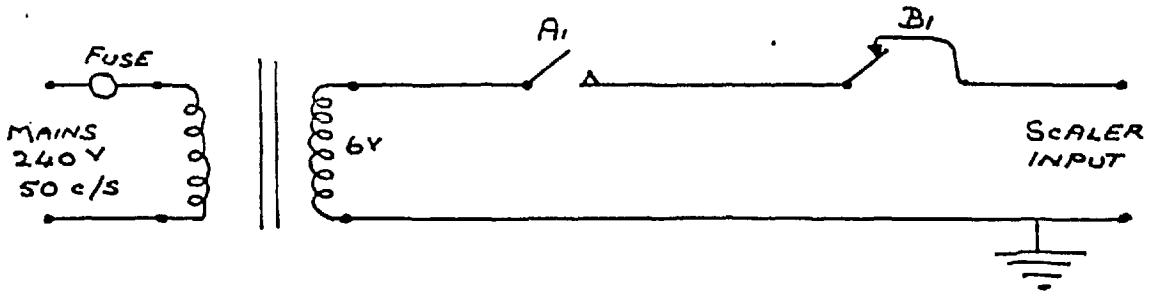
Once consistent results were obtained the majority of experimental results quoted in chapter V were acquired within 5 - 6 weeks. The following chronological account of the development of the experimental arrangement and technique described in chapter IV is given to help assess the accuracy of the method and to elucidate the difficulties involved. The basic method consisted of designing experiments corresponding to the bpundary conditions for the Taylor and computed numerical solutions. Flow conditions representing different values of the Taylor inequality were calculated, ensuring that Re was less than the critical value. All experimental results were repeated and compared for self-consistency. Comparison was also made with the Taylor theoretical curves. A deviation from the latter was considered in terms of an error in flow rate measurement, a variation in D, or in the boundary conditions, and the experimental arrangement was checked accordingly. The method will become more evident during the account of the development.

The experimentation was basically the same as described

FIG.viii. FLOW MEASURING DEVICE.



Circuit diagram.



in chapter IV : labelled material of uniform concentration $c = c_o$ was displaced by flow of unlabelled, $c = 0$, but otherwise identical material from a reservoir. In the original experimental design the electronic counting apparatus was as described in chapter IV, but the flow system was slightly different. The reservoir, experimental tube and flow controlling region were as shown in fig.vi., but the final section beyond tap 2 consisted of a flow measuring device. Initially the temperature of the solution in the reservoir was maintained at 25°C by water from a constant temperature bath circulating through the heating coil. The solution was stirred to facilitate more rapid equilibrium. A water jacket surrounded the experimental tube between the reservoir and the counter. This was maintained at 25°C by water circulating from the constant temperature bath. It was only possible to determine c_m/c_o as $f(T)$ with this arrangement.

The flow measuring device that was used is shown in fig.viii. The tap section occupied the same position as tap 2 in fig.vi. The transit time of the mercury meniscus between electrodes 2 and 3 was timed using the 12V relays A and B to start and stop a scaler counting at 50 c/s. The volume V between electrodes 2 and 3 was calculated by using an Harvard pump to produce a known flow rate Q and measuring the corresponding transit time, t_o

then $V = Qt_o$

An accurate value for the volume flow rate may then be obtained from the corresponding transit time.

To avoid contaminating the mercury this device was used prior to a series of experiments and a calibration curve plotted so that the volume flow rate might be determined directly from the applied pressure. A graph of applied pressure against reciprocal transit time gives a straight line cutting the pressure ordinate at δP, where the actual pressure difference between the ends of the tube is

$$\Delta P = P_{app} - \delta P$$

Employing Poiseuille's equation, the linear relationship between P_{app} and $1/t_o$ may be shown theoretically. The equation is

$$P_{app} = k'/t_o + \delta P$$

where k' is a constant for the flow system. For the system described δP was fairly large because of the back-pressure due to the surface tension of mercury. δP was assumed constant under the same external conditions and is analogous with yield stress. During an experiment using radioactive solution outflow occured through tap 2b, fig.viii. The dimensions of the constriction had been calculated so that the major pressure drop occured in that region, thus controlling the flow rate. It was therefore considered that flow would not vary appreciably for the two conditions.

Experiments to determine c_m as f(t), i.e. finite distance,

were performed corresponding to Taylor's condition B). The results for c_m/c_o as f(T) were calculated and compared with Taylor's analytical solutions. The curves did not agree with regard to position, which suggested a discrepancy in the flow rate determination. Flow rate measured by collection during the experiment was found to be greater than that determined from the calibration curve. In retrospect this is obvious, since δP must have been much less when the mercury timing device was not incorporated in the flow system.

The timing device was therefore abandoned in favour of collecting the outflow through tap 2. Preliminary experiments were at slow flow rates so that the mass collected during a time between identical events was recorded (drop falling to drop falling). A variation of less than 1% was obtained for the flow rate determined from successive collections at room temperature for a given applied pressure.

Experiments to determine c_m/c_o as f(T) were repeated. The outflow was collected for a six minute period during the experiment. The curves were inconsistent and deviations of 1 - 3% on both sides of the pivot point (c_m/c_o=0.5; T=1.0) were obtained. A water jacket was placed on the constriction, but this made little difference.

The flow rate determined at intervals during the experiment showed an initial increase before steady flow was established. This was considered to be due to the temperature gradient existing between the constriction and the termination of

the water jacket on the experimental tube, i.e. when the warmer solution reached the constriction the flow rate increased. The water jacket on the constriction was removed and the flow volume between the constriction and the end of the other water jacket was increased. This ensured that only solution at room temperature passed through the constriction during an experiment, so that the flow rate remained sensibly constant. To allow for the variation of η with room temperature the flow rate was determined during each experiment.

Consistent results for c_m/c_o as f(T) were then obtained. The position of the experimental curves corresponded closely with Taylor's solution, but the former were much steeper. D was calculated from the experimental points and the corresponding theoretical solutions calculated. The experimental points agreed fairly well with this curve, but D was approximately twice the adopted value. In the following series of experiments it was decided to switch the stirrer off about 5 minutes before flow was started to allow the solution to 'settle'. The resulting curves showed much better agreement with the slope of the Taylor curve for $D = 1.85\ 10^{-5} cm.^2 sec.^{-1}$.

(It is obvious from the kinetic theory for diffusion, discussed in chapter VII, that an increased value for D would be expected when mechanical energy is introduced to the system by stirring, but this had not been considered in

setting-up the experiment.)

The following series of experiments gave results which in some cases showed good agreement with the theoretical curves, but at other times were consistently steeper. This was puzzling since none of the parameters had changed noticeably. A change in the solution itself was regarded as a possible explanation, but results using different solutions were inconclusive. It was therefore decided to use freshly prepared solutions and proceed with experimentation at faster flow rates.

The end of the experimental tube was snapped off when being cleaned prior to this series of experiments. This meant that the detector was very near to the end of the tube and tap 1 when the apparatus was reassembled. Results obtained for c_m/c_o as f(T) exhibited an extended tail region with c_m failing to return to the baseline. A change in the counting geometry was considered responsible, so that the effect of radioactivity in the tap region was investigated. Firstly, the tap was filled with radioactive solution and a graph of activity against distance of the lead shielding from the tap (end of the tube) was plotted. The recorded activity was observed to decrease with distance from the tap and reach a normal background level at about 16cm. from the tube-end. Secondly, with the outside casing of the counter approximately 1.5cm. from the tap, the tube and tap region were gradually filled with radioactive solution. A graph of recorded

activity against position of the meniscus was plotted, as for the effective window of the crystal (fig.iv.). An initial plateau region was obtained with the tube full to ~2cm. beyond the casing, but this increased by approximately 5% when the tap region was full and a second plateau was obtained. When the counter was moved ~18cm. from the tap the recorded activity returned to the original plateau value, corresponding to a symmetrical counting geometry. For all the results quoted in chapter V the counter was at least 20cm. from the end of the tube.

Apart from the first curve obtained during the next series of experiments the results were still consistently steeper than the theoretical curves. A reference to cleaning and drying agents with regard to possible surface effects in a paper on flow through tubes made the author decide to use the cleaning procedure described in chapter IV. The method had not been standardised previously, but in the majority of cases a volatile drying agent had been used, e.g. alcohol.

The entire apparatus was dismantled and thoroughly cleaned before the next series of experiments. The syringes and beakers used to transfer solutions were also cleaned. On this occasion the experimental and theoretical results were in better agreement, but a variation in position with respect to the ordinate was obtained. Finally, after some trouble with leaks self-consistent results were obtained. These were in good agreement with the theory at very slow flows and

tended to be slightly steeper than the theoretical curves as the flow rate (A_{xo}) increased. Larger pressures than for the method described in chapter IV were used, but were not applied until just before flow was commenced ($\lesssim$ 1 min.). Examples of the results obtained at 3 different flow rates are shown in graphs 12, 14 and 16, chapter V. Room temperature was $\gtrsim$ 24°C for these experiments, so that the applied pressure was the only appreciably different parameter (table II). The late application of pressure for this series of experiments and the instability of the recording system do not permit any definite conclusions to be drawn about pressure effects from these results.

At this time computed solutions of equation (6) were not available so that it was difficult to judge whether this was a minor fault in the experimental system or whether Taylor's solution B) was no longer applicable. It was therefore decided to investigate the dispersion of labelled material under Taylor's condition A). In order to obtain condition A) and a low Reynold's number a wider bore tube was used.

The total flow volume for these experiments was large, so that the volume between the constriction and water jacket was increased to ensure steady flow. The flow rate was measured twice during an experiment by collection of the outflow. This method was not so accurate at fast flows, and variations of 2 - 10% were recorded between 9 pairs of these determinations. The variation was minimised by using an

event-marker in conjunction with the recorder to time the collection instead of a stop-watch. Variations of 2 - 3% were obtained using this method.

Using a 4mm. bore tube inconsistent results were obtained which were markedly different from Taylor's solution A). The curves obtained appeared to depend on how long before the start of flow and in what manner the pressure was applied. For one particular flow rate 3 types of curves were obtained under apparently identical conditions. The possible variables were considered to be room temperature and the initial boundary conditions. There appeared to be a slight correlation between room temperature and the type of result obtained. The possibility of fixing a shutter at the entrance to the experimental tube to establish the required boundary conditions was investigated. However, it was considered that the removal of the shutter would also cause a disturbence to the boundary. A better approximation to the ideal boundary conditions might be obtained where the removal of the shutter starts flow, but this was not tried. A few experiments were performed with a scintered glass plate over the tube entrance, since it was considered that this would give a better approximation to a straight interface. These results were also inconsistent. It appeared impossible to standardise the experimental technique using the wider bore tube. It was obvious that a considerable deviation from Taylor's boundary conditions existed, since the interface did not appear straight during the filling procedure.

The nature of the flow in the tube was questioned, but the author considered that the instability of the initial boundary conditions was more likely to be responsible. A simple theoretical investigation of the influence of the initial boundary conditions on the dispersion suggested that it might be easier to interpret results obtained when flow was stopped after a finite time t_o.

The same apparatus was used, with the water jacket removed from the experimental tube so that the counter could be moved along the length of the tube. The room temperature was maintained at 25°C to compensate for the removal of the water jacket and also eliminate the additional variable of room temperature. These experimental results were also inconsistent, although one experiment with $A_{to} \simeq 500$ gave good agreement with the solution for Taylor's condition A'), except in the region $c_m/c_o < 0.3$. In this region the experimental results were less than the theoretical results. During another experiment the counter was left 9cm. from the tube entrance and the pressure was removed after the flow had been stopped. An immediate increase in activity was observed indicating that back-flow had taken place along the tube. Pressure was reapplied immediately and the recorded activity decreased. This led the author to investigate the distortion effects on the primary interface when pressure was applied. Two methods were used to study this effect : a) the tube full and the reservoir empty b) using dye solutions. Changes in the boundary of labelled and unlabelled solution were also

investigated as a possible source of error in the measurement of x, since a displacement with respect to the pivot point was exhibited by the majority of experimental results. These effects were studied in different diameter tubes.

With the experimental tube completely filled with solution and the reservoir empty the meniscus was observed to travel down the tube on application of a pressure $\gtrsim$ 50cm. H_2. The distance of travel was dependent on the pressure applied. Motion was not due to a leak since it stopped after a short time and was reversible, i.e. the meniscus returned to its original position when the pressure was removed. No air bubbles were observed in the tube, although on certain occasions the solution did appear aerated. Prior to this the solution had been drawn through the tube from the reservoir using a syringe, but subsequently it was allowed to flow through the tube filling it under its own hydrostatic pressure. However, an entry effect was still observed on application of increased pressures. All joints in the system were rigid as shown in fig.vii. so that the reason for this behaviour could not be established.

An entry effect was also observed with the experimental tube filled with 2N KCl solution and the reservoir containing 2N KCl solution dyed with red ink. A region containing dye was observed extending several mm. down the tube. This was seen with no pressure applied and occured simultaneously when solution was poured into the reservoir. This meant that

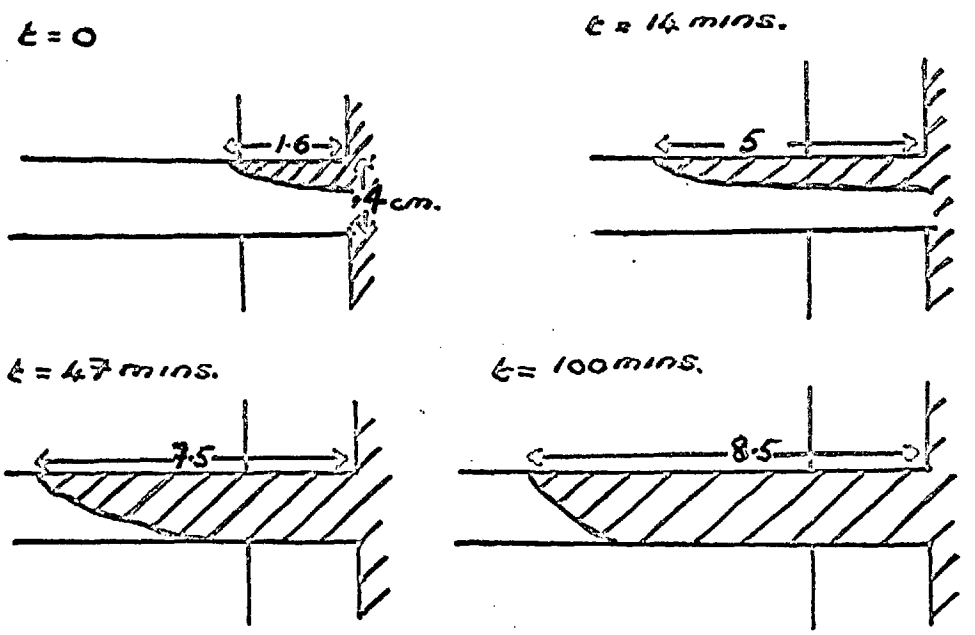


FIG. ix. . ENTRANCE EFFECT IN .4cm. BORE TUBE AS TIME INCREASED.

instead of a sharp interface there was a zone of mixing. In the 0.122cm. tube this entry was minimal and was impossible to perceive with dye-solution in the tube and clear solution in the reservoir. The extent of the mixing region increased with time and pressure. The latter effect was reversible to a limited extent. A disturbence to the solution in the reservoir, e.g. stirring, only produced a small change in the mixing region. The zone of mixing appeared uniform over the cross-section in the 0.122cm. tube. Entry effects were much larger in the 0.4cm. tube and were asymmetrical, not even approximating to a straight interface, fig.ix. Dye solution was observed to pass into the reservoir under resting conditions.

Entry effects appeared to depend on tube diameter, pressure and time. The manner in which the solution was introduced to the reservoir and any subsequent disturbance to the solution were secondary factors. The effect of a temperature gradient between the two solutions was not investigated experimentally. However, to eliminate an increased exchange due to thermal diffusion both solutions were warmed to 25°C prior to an experiment.

The theoretical implications of these entry effects are not discussed, but the experimental procedure was modified to minimise this possible source of error.

Attempts to use the 0.4cm. tube were abandoned because of the asymmetrical boundary conditions. It was considered that the mixing region in the 0.122cm. tube might be expected

to have no effect when c_m/c_o as f(X) is studied, or in slow flows where Taylor's solution for c_m/c_o as f(T) applies. However, where the total time for c_m/c_o as f(T) is short an extended curve displaced to the left might be produced as a result of this mixing zone. An uncertainty in the determination of x is also introduced by this effect.

A tube of 0.122cm. diameter was used to obtain all the results given in chapter V. This meant that the Reynold's numbers were large for the faster flows (condition A), but these were still $\lesssim 50$ so that this was considered to be unimportant compared with the variability of the boundary conditions in the larger tubes. Justification for this view was obtained from Taylor's experimental results, in which small tubes were used for all flow conditions and Reynold's numbers were $\gtrsim 110$ for fast flows, i.e. condition A) and laminar flow dispersion. Pressures of less than 30cm. H_2O were used to provide a fine control for the flow rate, the high downstream resistance of the constriction providing the overall control. Both solutions were warmed to 25°C prior to an experiment. The stirrer was no longer required. The radioactive solution was allowed to flow down the experimental tube under its own hydrostatic pressure.

Practical difficulties were encountered in determining the flow rate during an experiment at fast flows. Leaks also tended to occur through the two-way tap 2a, with this arrangement. The final system that was devised for determining the mean flow velocity eliminated both these difficulties

(n.b. chapter IV).

As stated previously the aim of this series of experiments was to verify Taylor's analytical solutions of equation (6), and to estimate the range for which they are valid. This was achieved for the particular experimental conditions described in chapter IV. It has also been demonstrated that the partial differential equation (6) may be solved by a computer to give the dispersion of labelled material for a range of flow conditions. Unfortunately, it was not possible to investigate the relative importance of the different experimental parameters in determining the dispersion, i.e. the effect on D of pressure, temperature and'history of the solution', due to lack of time. The phrase 'history of the solution' has been used to include such factors as method of setting-up the initial conditions, cleaning, stirring etc. Results shown in graphs 12, 14, and 16 suggest that at slow flow rates (condition B) pressure and temperature effects are not as important as discussed later in the text. However, the variation in these parameters was small and even if a 100% change in D was produced, which is unlikely, this would only cause a slight increase in the slope of the theoretical curve under conditions where Taylor's solution B) applies. When the implications of equation (6) are considered from a theoretical viewpoint it is possible to speculate on the results one might expect to obtain under different experimental conditions. This is

discussed further in the following chapters and the possible extensions of this work are summarised.

Parabolic flow profile.

In equation (6) a parabolic distribution is assumed for the velocities of planes at a position 'r' with respect to the tube axis, i.e. the mean momentum transfer in the flow direction. The validity of this assumption may be investigated by several experimental methods.

Firstly Poiseuille's equation, which applies to a Newtonian fluid flowing through a circular tube, is only explicable theoretically by a parabolic flow profile. Simple calculations based on Poiseuille's equation indicated a linear relationship between applied pressure and flow rate for the experimental arrangement used. These results were confirmed experimentally, showing that Poiseuille's equation was valid at these low Reynold's numbers. It was not possible to calculate the viscosity coefficient from Poiseuille's equation for the experimental arrangement described because of the varying pressure drops in the different tubes and the approximations involved.

The majority of experiments investigating flow profiles for materials possessing non-Newtonian as well as Newtonian viscosity coefficients (chapter I) study the dispersion of indicator substances[2q,11]. Unless the diffusion coefficient of the substance introduced is zero this dispersion is represented by equation (6), and is governed by diffusion

as well as the flow profile. The experiments described in this thesis, showing agreement between the dispersion curves obtained experimentally and the theoretical curves obtained from numerical analysis of equation (6), are therefore direct proof that this condition is met.

The investigation of the profile by means of probes was not possible in the present case because of the small diameter of the experimental tube.

CHAPTER VII

ANALYSIS OF RESULTS USING INDICATOR-DILUTION TECHNIQUES TO STUDY FLOW THROUGH TUBES.

The indicator-dilution technique.

The indicator-dilution technique has been widely used to obtain flow[3e] and volume[3c] measurements in blood circulation studies. The dispersion of labelled solutes during flow between two points is determined using rapid or constant injection at one site and rapid sampling at the other to plot the concentration-time curve. To eliminate any assumption regarding the type of flow involved the flow of indicator is obtained directly from the dispersion curve[3e]. However, it must be established that the flow of indicator represents the flow of total fluid before the two may be equated. This is also a necessary criterion in determining the volume of the system.

A suitable label for blood would have to be partitioned equally between the different components, but since no such label exists the normal procedure is to use either labelled cells, or albumin, or both[3c]. The labelled albumin is regarded as a plasma tracer, so that results for plasma volume

and flow are calculated from the dispersion curve. Cell volumes and flow are calculated from the dispersion curves for the labelled erythrocytes[3e].

Analysis of the curves obtained for the two separate labels in various parts of the circulation have demonstrated a difference in mean circulation times. Volume estimations using the theory of indicator-dilution[3e] have shown the dynamic haematocrit to be less than that of drawn blood. Results of indicator-dilution studies analysed in the manner described by Zierler[3e] and Robertson[3d] have been claimed to support such theories as axial streaming of red blood cells[2p], extravascular plasma compartment[26], difference in dynamic haematocrit[16,41,48,53], difference in circulation times of red cells and plasma in various parts of the circulation [2p,16,17,20,23,28,48], difference in the mean velocities between erythrocytes and plasma[66b], and cell separation theory in the kidney[38]. However, independent investigations of these physiological phenomena, using other techniques, have not proved positive. High speed ciné photography[8] and other photometric investigations[6] have failed to give complete support to theories of axial streaming. Quick freezing experiments on the kidney[40] have failed to reveal a tendency towards cell-free regions. It is difficult to account for the extra plasma space anatomically[3c]. Measurements of blood volume and flow by other techniques[3b,c,e] are not in perfect agreement with the indicator-dilution method.

An exponential extrapolation is normally made to eliminate the effects of recirculation. This has not been justified for all experimental measurements[33].

Theoretical analysis of the curves is normally considered in terms of multi-compartment theories and the downslope regarded as a series of exponential curves[3d]. The dispersion curves for different substances during blood flow through various parts of the circulation has been interpreted with regard to exchange rates between intra- and extra- vascular compartments. Computers have been used[3d] to determine the different exponentials and the corresponding rate constants for the more complex experimental curves.

Cardiovascular system.

Physiological investigations[60] have shown that the composition of whole blood, the anatomy of the circulation and the structure and material properties of the vessels may all vary. These different aspects of the cardiovascular system change with internal (chemical and neural) and external enviromental conditions in the same individual. Wide ranges of values for blood flow, pressure, composition etc. have also been obtained in normal healthy individuals under identical external conditions. It has also been shown that blood composition is determined by such factors as age, sex, race and heredity.

A very complex functional relationship of all these

factors is suggested by the homeostatic function of blood, i.e. a change in any factor could be compensated by changes in all, one, or a combination of the other factors. This is analogous to the response of any physical system to the introduction of energy. The law of conservation of energy specifies that some change must take place in the system. This could be achieved in many ways, but until more is known about the system and the manner in which the energy is introduced it is only possible to specify that a change must take place. Physical laws are used to describe one parameter as a function of other parameters of the system. The latter are specified by the boundary conditions, i.e. the experimental measurements. Laws formed in this manner only have a limited range of application.

In order to obtain a more complete understanding of the dynamics of the circulation the relative importance of the factors mentioned, under different flow conditions, must be determined. An example of this type of analysis is given in chapter I, with regard to the difference in clearance curves observed in vivo[2p,17,20,23,38,39,48]. Possible explanations of this effect, observed in vivo, were investigated in a similar in vitro arrangement[2q] so that certain variable parameters might be controlled, i.e. physiological mechanisms introduced to account for the observed differences might be correct but the possible mechanical factors must also be considered. However, the lack of a satisfactory law defining

the functional relationship for rheological properties of materials[2,2r,4,25] does not permit rigorous analysis even for this simple system.

Dispersion curves for labelled solutes during flow through circular tubes.

The experimental verification of the numerical solution for the partial differential equation (6) arising from the dispersion of labelled components in solutions flowing through tubes[49] has been presented in chapters IV and V. Theoretical assumptions made in establishing equation (6) are discussed later in this thesis to prevent possible misinterpretation of the results.

The author maintains that the solutions of equation (6), given in chapter III, for the longitudinal dispersion of solutes will apply to all solutes provided

i) $\bar{u}$ is the mean flow velocity determined from the mass flow rate during the entire experiment

i.e. $$\bar{u} = Q/\pi a^2 = \frac{M_2 - M_1}{\rho\, t\, \pi a^2}$$

ii) D is the diffusion coefficient (tracer diffusion) of the component in a suspension, or solution, of identical composition under the same pressure and temperature conditions. This means that the results are only directly applicable to radioactively- or dye- labelled solutes in an otherwise

identical solution. For the dispersion of dye particles or tracer spheres in an otherwise pure fluid a different approach must be used, since D may no longer be assumed isotropic. A completely rigorous analysis of this type must involve particle mechanics[1b,c,d].

It is now proposed to apply the analytical solutions of equation (6) to obtain a description of the dispersion curves expected for a 'slug' of blood containing labelled erythrocytes and albumin molecules. This is discussed for in vitro experiments : the mean flow velocity was determined from the mass transport, and the 'injected blood' was the same composition as the ' flowing' blood apart from the radioactive labels. It may be argued that solutions of equation (6) should not be applied, since D for the blood corpuscles must be anisotropic and the mean flow profile may not be parabolic. However, it is primarily the effects of using different indicators in the same flow regime that is being investigated, so that equation (6) gives an approximate description of this process.

Diffusion coefficients for albumin molecules and erythrocytes.

Since no figures are available for the diffusion coefficients of albumin molecules and erythrocytes in whole blood, it is necessary to estimate an order of magnitude for D. This may be achieved by comparison with diffusion data for similar macromolecules and particles in concentrated suspensions, and from theoretical considerations.

Diffusion coefficients are defined by Fick's law, which is[35a]

$$\frac{\partial c}{\partial t} = D\nabla^2 c$$

in 3-dimensions. In principle molecular diffusion must result from the thermal (kinetic) energy of the molecules, given by $\frac{1}{2}kT$. This forms the basis for Graham's law of diffusion in gases, and for an ideal gas the rate of displacement is related to the molecular mass. The diffusion coefficient decreases with increasing molecular weights. In a multi-component system the simple kinetic theory of collision and thermal motion is complicated by interatomic forces and intermolecular forces. The theoretical description of self-diffusion in solids is given in terms of holes in the crystal lattice[36b]. A similar theoretical description of diffusion in liquids has been considered by Frenkel[35b]. The latter process has also been considered in terms of a kinetic theory by Born[1a] and a reaction rate theory by Eyring[24]. X-ray diffraction studies have yielded experimental information on the mobility of components in solutions[7,21]. These theoretical considerations and comparison with diffusion data for proteins in acqueous solutions at similar concentrations suggest a value of $\leq 10^{-6}$ cm.^{2}sec.$^{-1}$ for the diffusion coefficient of serum albumin molecules.

It is implicit in the definition of diffusion coefficients that the motion is induced by thermal collisions. This may result in considerable Brownian motion when the concentration

of suspended particles is small. However, for a high concentration of suspended particles their displacement will be limited by inter-collisions. Thus, at relatively high haematocrits it may be assumed that the diffusion coefficient for the labelled erythrocytes is very small, tending to zero. Equation (6) also requires a homogeneous distribution of diffusing particles so that D is concentration independent. In applying the analytical solutions of equation (6) to the dispersion of labelled albumin molecules and erythrocytes during flow through a tube these values of D were used and it was assumed that diffusion was isotropic. The values of D selected represent orders of magnitude for the different components and it is to be expected that the exact values will vary for each blood sample.

Dispersion curves for labelled components initially confined to the same volume.

Fig.x. shows the theoretical curves for the longitudinal dispersion of labelled solutes during flow through a tube. The curves were calculated for different flow conditions, but constant values of a and D. However, theory predicts that the curves will be characterised by A_{to} (a^2/Dt_o). The form of the dispersion curves for two labelled components of a solution with different diffusion coefficients may therefore be predicted. In this case the same flow regime exists so that a, $\bar{u}$, and t_o are constant, but A_{to} for the two components will be different due to the value of D for each.

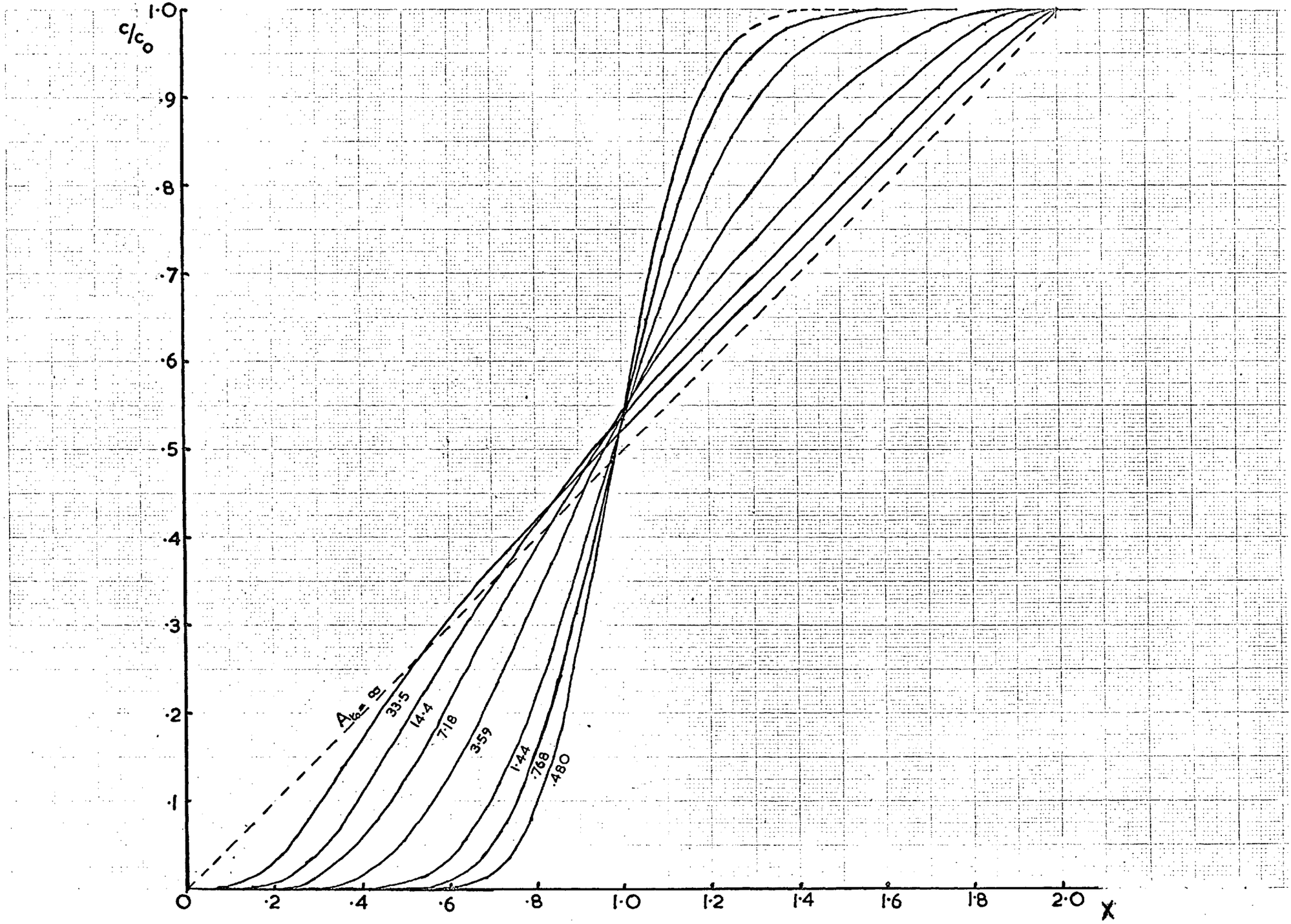


FIG.x. Numerical solution for the dispersion of labelled solutes during flow, c_m/c_o as f(X) for different A_{to} values.

Referring to fig.x. :-

$A_{to} = \infty$ corresponds to D = 0. The curve for $A_{to} = \infty$ represents Taylor's analytical solution A), i.e. laminar flow(lg).

For $A_{to} = 33.5$ let $D = D_1$

then $A_{to} = 14.4$ corresponds to $D = \frac{33.5}{14.4} D_1 = 2.32\ D_1$

$A_{to} = 7.18$ " " $D = 4.66\ D_1$

$A_{to} = 3.59$ " " $D = 9.32\ D_1$

$A_{to} = 1.44$ " " $D = 23.2\ D_1$

$A_{to} = .768$ " " $D = 51.8\ D_1$

$A_{to} = .480$ " " $D = 69.8\ D_1$

Curves with $A_{to} \leq 1.44$ correspond to Taylor's analytical solution B).

Dispersion curves for a rapid injection of blood containing labelled erythrocytes and albumin molecules.

It is assumed for rapid injection that the blood is initially distributed uniformly over the cross-section. The initial boundary conditions are :-

$$\left.\begin{array}{ll} c = c_o & 0 < x < x_i \\ c = 0 & 0 > x > x_i \end{array}\right\} \quad t = 0$$

where x_i is the injection length.

Initial conditions, t = 0.

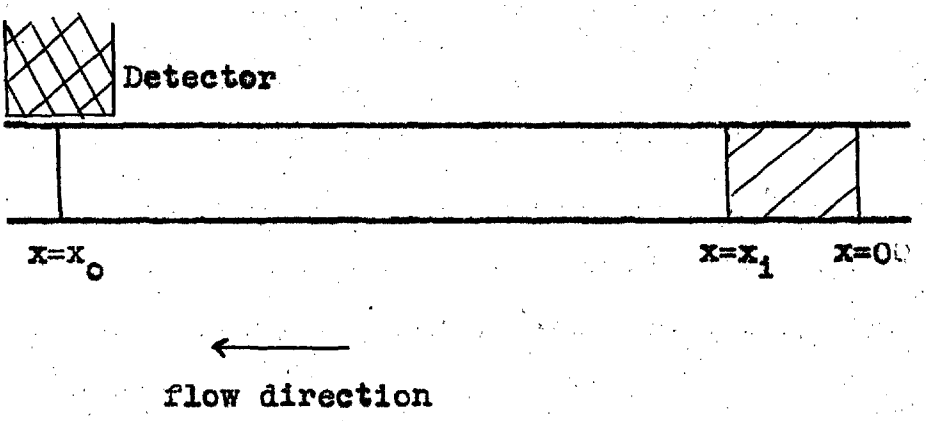
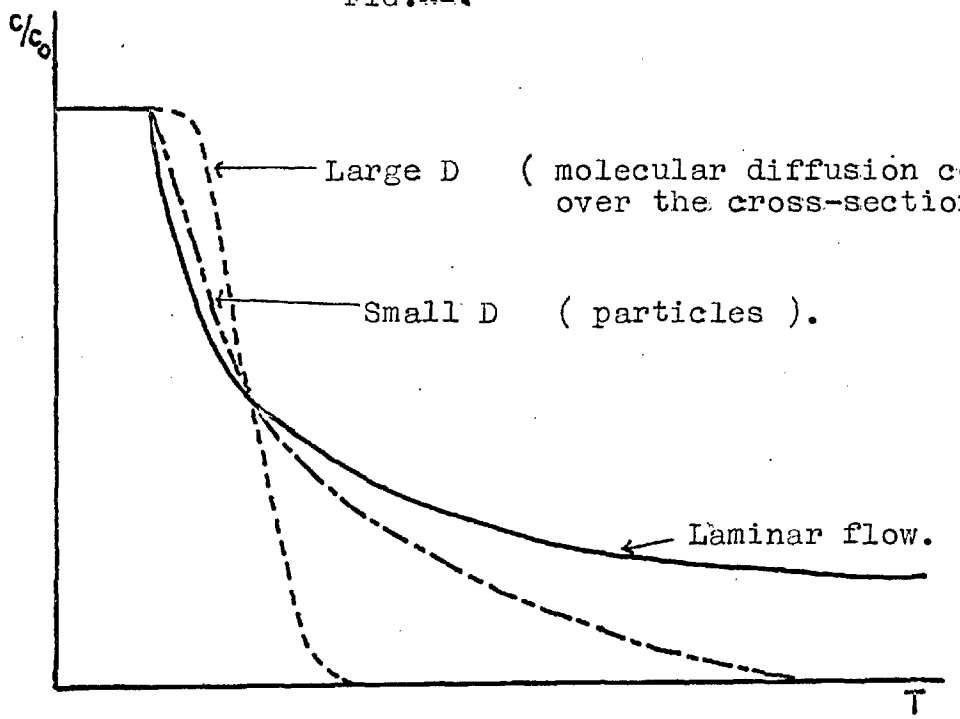
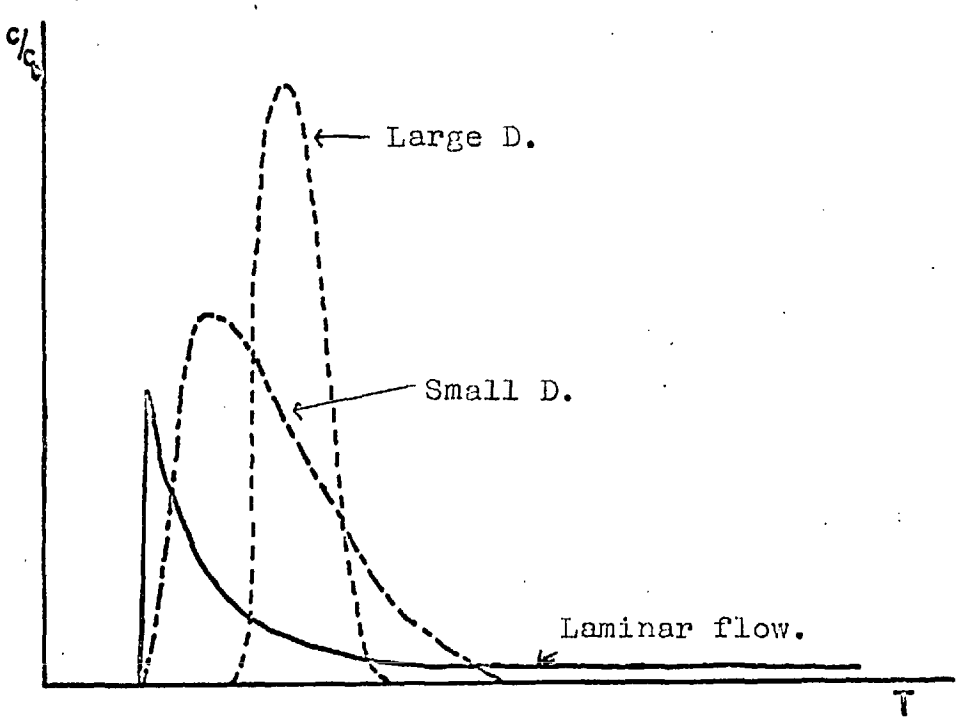


FIG.xi.



a) Multiple indicator dispersion curves, labelled replaced by unlabelled material.



b) Multiple indicator dispersion curves for rapid injection technique monitored at a finite distance downstream.

For the leading edge the solution of equation (6) for unlabelled replaced by labelled solution must be considered, i.e.

$$c_m/c_o = 1 - (c_m/c_o)_{x_o-x_i}$$

where $(c_m/c_o)_{x_o-x_i}$ is the result obtained for labelled material displaced by flow of unlabelled material, when the finite monitoring distance is x_o-x_i. For the trailing edge , the solution for labelled material displaced by unlabelled material is used and the fixed distance x_o.

To obtain the dispersion curve for a finite injection length these two effects must be added :-

$$c_m/c_o = 1 - (c_m/c_o)_{x_o-x_i} + (c_m/c_o)_{x_o} \qquad (18)$$

The form of the curves obtained for (c_m/c_o) depend on the value of the diffusion coefficients for the labelled components, as discussed for fig.x.. Curves for c_m/c_o as f(T) for three indicators with different diffusion coefficients are shown in diagram (a), fig.xi..

The form of the curves obtained from a combination of the theoretical 'wash-in' and 'wash-out' curves is shown in diagram (b), fig.xi.. It is immediately obvious that the shape of the curves for labelled components injected simultaneously may be completely different if the diffusion coefficients vary markedly. This also depends on the flow conditions.

Assuming that the diffusion coefficient for labelled erythrocytes tends to zero and D for labelled albumin molecules is much greater, of the order 10^{-6} cm.^{2}sec.$^{-1}$, curves of the type shown in fig.xi. would be obtained under the appropriate flow conditions. A_{xo} is the controlling parameter, i.e. $\frac{\bar{u}a^2}{Dx_o}$. When A_{xo} is large for both components, both curves will tend to laminar flow dispersion. For $A_{xo} \lesssim 1.44$ both curves will be described by Taylor's analytical solution B).

If D for one component tends to zero, A_{xo} for that component will be large for all values of a, $\bar{u}$, and x_o. When D is finite, say $\sim 10^{-6}$ cm.^{2}sec.$^{-1}$, $\frac{\bar{u}a^2}{x_o}$ must be $\sim 10^{-6}$ for Taylor's solution to be applicable. A considerable deviation from laminar flow dispersion was however observed for all experimental values of A_{xo}, i.e. $A_{xo} \lesssim 30$, n.b. chapter V.

The dispersion curves shown in diagram (b), fig.xi. are similar to the ones obtained using multiple indicator techniques to study blood flow in vivo and in vitro, i.e. in various parts of the microcirculation and in narrow tubes respectively[2p,16,17,23,26,29,38,39,48,53,2q,9]. A complete theoretical analysis of the in vivo measurements is not possible due to the many variables involved. However, this may be achieved for the in vitro experiments of Thomas[2q] assuming values for the diffusion coefficients of albumin and erythrocytes as described.

The following results were obtained during one of the experiments reported by Thomas (private communication). Flow rate in each tube, determined from the mass flow during the experiment, was $1.936\ 10^{-3}$ ml.sec.$^{-1}$. The distance from the leading edge of the plug to the monitoring point was 129cm. and from the trailing edge 144cm.

Therefore $\frac{\bar{u}a^2}{x_o} \simeq 5\ 10^{-6}$ cm.^{2}sec.$^{-1}$

Taylor's solution B) may therefore be applied to the dispersion of labelled albumin molecules for the assumed value of D. The dispersion curves obtained were very similar to those shown in diagram (b), fig.xi. The labelled erythrocyte curve was slightly ahead of the labelled albumin curve, the peak was ahead of the albumin peak and a cross-over was observed in the tail region.

Mean circulation times are defined by the relation

$$\bar{t} = \frac{\int_o^\infty t\ c(t)\ dt}{\int_o^\infty c(t)\ dt}$$

and may be determined from the dispersion curves using a mechanical integrator. A quantitive analysis has not been made, but it is obvious from fig.xi. that the mean circulation time for laminar flow dispersion would be infinite, and finite for the case where molecular diffusion dominates in

determining the dispersion. It would therefore be possible to have a difference in mean circulation times due to the different values of diffusion coefficients of the two labelled solutes.

Complete exponential clearance of labelled material is never observed[46], so that the clearance curve for a labelled component with a small diffusion coefficient will be closed and more weighted towards a shorter clearance time. It is thus more likely that the mean circulation time for this component ($D \to 0$) will be less than for a component with a larger diffusion coefficient, when Taylor's condition B) applies to the latter. The difference in mean circulation times between labelled erythrocytes and albumin molecules observed in vitro is therefore explicable by the effects of molecular diffusion. The effect will depend on $\frac{\bar{u}a^2}{x_o}$, and a value of $\sim 10^{-6}$ has been estimated as critical.

Since physiologists interpret these results in terms of the motions of erythrocytes and the suspending phase, it is important to consider the significance of equation (6) with regard to the motions of individual components. In order to do this the theoretical derivation of the equation must be reconsidered in the manner described in this thesis. This aspect of the work will be discussed further, but first additional evidence for the validity of equation (6) with regard to blood flow is presented by considering the control experiments of Thomas.

Difference in mean circulation times for two labelled components when Taylor's condition B) applies to both.

Thomas[2q] reported a small, but significant difference in mean circulation times for the control experiments, where

a) P^{32} and I^{131} labels in the form of inorganic ions were used

b) P^{32} in the form of inorganic phosphate and I^{131} labelled albumin were used.

The curve for phosphate ions in case a) was wider than for the iodide ions, but both were in qualitative agreement with curves predicted by Taylor's solution B). Diffusion coefficients of $\sim 10^{-5} - 10^{-6}$ are therefore indicated, with $D_{iodide} > D_{phosphate}$. For case b) the albumin curve was widest and more noticeably asymmetrical suggesting a smaller diffusion coefficient $\sim 10^{-6} - 10^{-7} cm.^2 sec.^{-1}$ for these components. The Taylor solution B) would not be completely valid in this range, but the shape of the curve may be estimated from the computed results, graphs 9 - 16, chapter V. All the curves are in qualitative agreement with the dispersion of labelled components predicted by equation (6), when the diffusion coefficients are assumed in the manner described. Alternatively assuming that equation ($_6$) predicts the dispersion of labelled components during flow through tubes, the diffusion coefficients calculated from the curves are of the same relative magnitude as predicted by the kinetic theory of diffusion.

Thomas[2q] states that the solution for Taylor's

condition B) predicts that the mean circulation time is independent of diffusion coefficient. This is not entirely correct, since a small but significant difference in mean circulation times is predicted for two labelled components when Taylor's condition B) applies. It arises from the finite injection length, which results in an asymmetrical dispersion curve when monitored at a finite distance downstream. The ascending portion of the curve is steeper than the descending portion. This is more noticeable for a long injection to monitoring length ratio, and for smaller values of D.

Assuming values of $2\ 10^{-5}$ and $1\ 10^{-5}$ $cm.^2 sec.^{-1}$ for the diffusion coefficients of iodide and phosphate ions respectively, the author calculated the dispersion curves for these two labelled components from equation (18)

$$c_m/c_o = 1 - (c_m/c_o)_{x_o - x_i} + (c_m/c_o)_{x_o}$$

where (c_m/c_o) are the Taylor solutions given in chapter II. The curves were analysed using a mechanical integrator and the mean circulation times calculated. A value of 0.991_o was obtained for the ratio (mean circulation time I^{131} label)/(mean circulation time P^{32} label) for the inorganic ions. This is in agreement with the slight, but significant differences determined by Thomas[2q] when the dispersion curves for the control experiments were analysed using a mechanical integrator.

For control experiments b), assuming that Taylor's condition B) applies for both components, a similar difference in mean circulation times is obtained. The ratio $\bar{t}\ I^{131}/\bar{t}\ P^{32}$ is slightly greater than unity in this case since the diffusion coefficient of I^{131} labelled albumin is assumed to be less than D for the phosphate ions. A ratio slightly, but significantly greater than unity was obtained by Thomas.

The results of this investigation were communicated to Dr. Thomas. He made an independent numerical calculation of the mean circulation times using Simpson's one third rule. These calculations also demonstrated a small difference in mean circulation times between the two labelled components, injected simultaneously, when Taylor's condition B) applied (private communication).

It has therefore been shown that when equation (6) applies and measurements are made at a fixed distance using multiple indicator techniques a difference in mean circulation times is likely to result. The equation has been applied successfully to predict the dispersion curves for labelled erythrocytes, albumin molecules and inorganic ions obtained during flow through narrow bore tubes in vitro. The latter results require further consideration for two reasons. Firstly, many critics question the application of equation (6) to blood flow since it includes a parabolic velocity

distribution. Secondly, if equation (6) applies, what support does this offer to the physiological explanations developed as a result of multiple indicator studies?

Parabolic velocity distribution.

The term $2\bar{u}\left(1 - r^2/a^2\right)$ in equation (6) represents the momentum transfer in the flow direction and is derived from Newton's definition of viscosity coefficients. If one returns to this original definition and considers subsequent developments concerning transport phenomena in gases, it is obvious that the parabolic distribution has no significance with respect to the motion of individual molecules, references 1a, 56a. In the present context it is also interesting to note that non-equilibrium statistical mechanics and reaction rate theory give a theoretical relationship between viscosity and self-diffusion coefficients[1a,24]. The velocity distribution refers to the planes within the fluid[45] and not molecules or particles at a position r with respect to the axis. For a plane parallel to the flow direction the net velocity of all the components is given by u_r at any instant in time.

Consequently, it is conceivable that macroscopic techniques, such as photography, used to study the motion of the planes may fail when molecular motion is considerable (D large), or when components have widely differing kinetic properties (as in a suspension), or when the velocity gradient is large. These comments do not apply to equation (6),

* this is not the sense in which the term fluid particle is used in fluid dynamics.

which should be valid when other techniques fail. Diffusion is observed as a result of the random internal motions of the components of a material, so that equation (6) represents the manner in which these motions are changed when flow occurs through a tube. The validity of equation (6) is entirely dependent on whether it describes the dispersion of labelled components during flow through a tube. It is hoped that the following discussion will clarify this argument.

Reconsidering the experimental results presented in chapter V, the iodide ions may be regarded as a tracer representing the behaviour of similar ionic components of a solution as D is of the same order of magnitude[35]. Laminar flow is normally defined as flow such that $v_x=v_z=0$ and $v(x,t)$[lg], and all fluid particles in a sheet parallel to the plane z = 0 are regarded as possessing the same velocity. Such flows are considered to occur in straight tubes, provided the Reynold's number does not exceed a certain critical value. For laminar flow, as defined, 'fluid particles'* would be distributed according to a laminar flow dispersion ($A_{t_o} = \infty$) for all flow conditions, since Re was much less than the critical value. But, when the dispersion was monitored after a time t_o where

$$t_o \gg \frac{a^2}{(3.8)^2 D}$$

it was found that the dispersion was completely different.

This condition was calculated by Taylor[64] and represents the condition "when the time necessary for appreciable effects to appear, owing to convective transport, is long compared with the 'time of decay' during which radial variations of concentration are reduced to a fraction of their initial value through the action of molecular diffusion". This means that when measurements are made after this time t_o it is impossible to predict the radial position of the labelled components with any certainty. During this time it is possible that the labelled component may have been transported to any radial position due to the combined effects of diffusion and convection. Solving equation (6) for the longitudinal dispersion, c_m/c_o as f(x), at this time, it is found to be much less than for the corresponding laminar flow dispersion, e.g. $A_{to} = 1.44$ and $A_{to} = \infty$ respectively, fig.x. The latter type of dispersion may only be obtained when "changes in c due to convective transport along the tube take place in a time which is so short that the effects of molecular diffusion may be neglected". For the same component these two distributions may only be obtained after different flow times. Both dispersions are represented by equation (6), in which the definition of a Newtonian fluid is applied to describe the bulk (momentum) transport, i.e. a parabolic velocity distribution for planes within the material. The diffusion term is introduced because these hypothetical planes are not intimately associated with internal motion of components

under all flow conditions. This approach is consistent with the theoretical development of transport phenomena in gases[1a,e].

An alternative explanation for the change in dispersion curves, which the author finds unacceptable, is that the molecular components remain in the same laminae during flow, but the velocity profile changes for different flow rates.

The inclusion of a parabolic flow term in equation (6) does not mean that a labelled molecular component (or any particulate component) of the flowing material must exhibit laminar flow, with the velocity at any point given by the parabolic distribution. The motion of any component of the material is governed by the bulk motion (characterised by $\bar{u}$ and a parabolic distribution, i.e. η) and its own intrinsic motion (characterised by D). A single molecular component does not therefore remain in the same radial position, or possess a constant velocity in the flow direction.

Dispersion of particles suspended in a fluid phase.

Equation (6) is based on a continuum theory. Initially, all components are assumed identical, moving only in the direction of flow, with identical velocities when in the same position with regard to the tube axis. It is also assumed that $u = 0$ at the wall and that the frictional resistance between each plane is constant, since their composition is identical. These assumptions give a parabolic velocity distribution for the planes[45]. A perturbation is then introduced to allow for the intrinsic motion and distinguishable

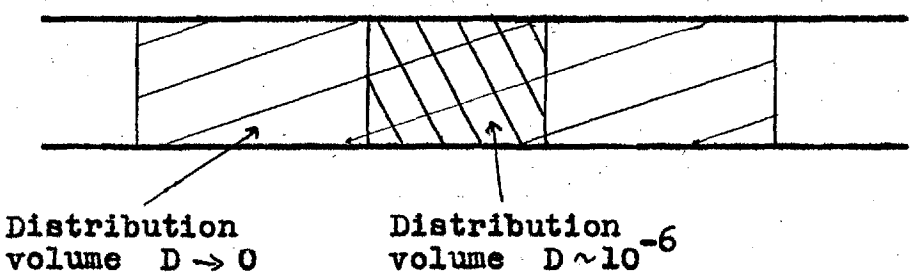
character of the components of the material, thus permitting different internal motions to be exhibited by different components. However, the equation does not enable individual elements of the components to be studied, merely the net effects. It would not be possible to consider the path of one labelled molecule (or macromolecule) during flow, but the probable position at any instant could be predicted for given boundary conditions. This is a familiar concept in modern physics, particularly in wave mechanics.

The dispersion for components of a concentrated suspension is considered in the same manner. It is regarded as a continuum, with the planes defining η made up of all the components. For any labelled component, the motion out of a plane is represented by D so that its dispersion is not determined entirely by the flow, but by the combined action of diffusion and flow. The motion of the particles is governed by the motion of the suspending phase and also their own interaction during flow.

The validity of equation (6), applied to dispersion of labelled components of a suspension, does not therefore depend on the viscosity coefficient remaining constant under all flow conditions, or on the velocity distribution being demonstrable by photographic techniques, but on whether it describes the dispersion of these components during flow (mass transport) through a tube.

If D for the labelled components tends to zero under stationary conditions, equation (6) does not necessarily mean that the particles will not move from a particular radial position during flow. A pressure differential may result from the motion of the other components and also collisions may occur between similar particles brought into contact by the flow. This is allowed for by the two terms in equation (6), i.e. the net effect of the particle motions and the flow.

The difference in the dispersion curves obtained for two labelled components of a suspension with different values of D, using solutions of equation (6), therefore represents an effective separation of the labelled components. For example, if flow is stopped after a time t_o, the volume in which the labelled component with $D \to 0$ is distributed is much greater than the volume in which $D \sim 10^{-6}$ is distributed.



Since c_m/c_o refers to concentration over the cross-section it is not possible to state where the particles are with regard to radial position. It is possible to solve equation (6) for c(r) at position x after a time t. However, this does not necessarily mean that a particle has remained in the same position r for the whole flow period since no distinction has been made between individual labelled particles. Also, it does not mean that a molecule (or group of molecules) in position r has velocity u_r at a time t, since there is a finite probability that it may just have come to that position from any other position in the tube.

The solution of equation (6) does not specify the path that a particle follows during flow. For example, laminar flow dispersion may be obtained for the labelled concentration as a f(x), f(t) and f(r), but this is no guarantee that the individual particles have remained in the same streamline, with a velocity given by $u_r = 2\bar{u}\ (1 - r^2/a^2)$ for the whole experiment. In order to establish that this had occured it would be necessary to study the path of an individual labelled particle, and this is not what equation (6) represents since it is based on a continuum theory and describes the net effects of motion.

It is therefore impossible to say with certainty, even when a 'laminar flow dispersion' is obtained, that the interface of labelled particles has assumed a parabolic shape. It might be a diffuse region with labelled particles distributed evenly over the tube section, but

with the concentration varying with distance in the same manner as for 'laminar flow' dispersion. Alternatively, for a finite injection length the labelled component might be concentrated in an axial region, but with the same longitudinal dispersion as one would expect for 'laminar flow'.

Assuming that equation (6) applies to the dispersion of labelled components of blood injected simultaneously, as suggested by the results of the experiments discussed earlier, it is only possible to draw one conclusion without further investigation. Namely, that the combined effect of molecular diffusion and flow produces an effective separation of labelled albumin and labelled erythrocytes when measurements are made after a time t_o, where $t_o \gtrsim a^2/D$ for the albumin molecules. For narrow tubes this means that the longitudinal distribution volume of labelled albumin is much less than that of labelled erythrocytes within a short time after the injection.

It is difficult to establish what this separation means in terms of the individual molecular motions of components without further investigations. However, it is possible to discuss the interpretations of previous experiments with regard to the net motion of the components.

Unfortunately the partial differential equation was not solved for radial distribution, but even if this showed that the labelled erythrocytes tended to be distributed in the axial region this does not necessarily imply axial streaming during flow. Exchange due to diffusion may have

taken place with unlabelled cells or with the fluid phase. In order to establish which interpretation is correct an independent investigation is required. Results using high speed ciné photography[8] suggest that there is not a completely cell free region adjacent to the wall.

When the dispersion curves are monitored after a finite period of time solution of equation (6) suggests that the longitudinal dispersion should be symmetrical for all components. The same value for $\bar{u}$ is used in equation (6) to give the different dispersions, but in this case $\bar{u}$ refers to the mean flow velocity determined from the mass transport of all components. It is impossible to separate the mean flow for each phase from this equation. The mean flow velocity calculated for each labelled component from the dispersion curves monitored after a finite time should correspond to the mean flow velocity of the material, when equation (6) applies. However, if $\bar{t}$ is calculated from the dispersion curves monitored at a fixed distance, as is the normal procedure in physiological investigations, a difference may be obtained for different labelled components depending on the flow parameters. The mean flow velocity must remain the same determined from both curves, according to equation (6), but the mean velocity for each component determined from $\bar{t}$ will vary. The difference in mean velocities of the labelled components is an artefact in this case due to the method of measurement and does not mean (as has been interpreted) that all the red

blood cells possess a mean velocity greater than the albumin molecules. It is merely that the slower moving labelled erythrocytes, remaining after the labelled albumin is cleared, will finally be cleared due to the effects of diffusion. During this time certain of the labelled erythrocytes cleared initially will be slowed by the effects of diffusion, but this is not recorded as these particles have passed the detector.

In order to establish the validity of equation (6) with regard to the dispersion of labelled components of blood, flow should be stopped after a time t_o and the corresponding dispersion curves monitored. It is also interesting that the present theory suggests that it is possible to obtain a similar difference in dispersion curves monitored at a distance x_o in wider tubes. In this case the monitoring distance must be greater than $\bar{u}a^2/D$, where D is the diffusion coefficient of the albumin molecules. These experiments have not been performed to the author's knowledge, but the possibility of axial streaming is not considered under these conditions. Unfortunately there was insufficient time to do these experiments in the present series.

It is impossible to apply equation (6) directly to in vivo studies, as the dimensions of the flow channels are continually varying. However, it is of interest that the results obtained for the clearance curves of labelled albumin and erythrocytes from an isolated muscle preparation[29] are

in excellent qualitative agreement with solutions of equation (6) when different values for D are used in the same flow regime, i.e. $2\bar{u}\left(1 - r^2/a^2\right)$ is the same for all components. These results even demonstrate the cross-over in the tail region predicted by solutions of equation (6). This is important in physiological investigations using indicator-dilution techniques, since it would appear to question the validity of exponential extrapolations made for recirculation. Such extrapolations are normally made in cardiac output studies[3b].

It is not disputed that an extravascular compartment would give a difference in mean circulation times, but it is considered that differences obtained in clearance curves using multiple indicator dilution techniques are insufficient evidence for the existence of such compartments. The results obtained by Goresky[26] for the dispersion of multiple indicators were dependent on the molecular weights of the indicator substance. This correlates very well with the results predicted by equation (6), which assumes that flow takes place through the same channel eliminating the possibiltiy of extravascular compartments. The equation is not directly applicable, since a multiplicity of flow channels is available in vivo. Similarly, the shunting mechanism discussed as a possible explanation of differences in $\bar{t}$ obtained when multiple indicator techniques are employed to study blood flow through the kidney does not have to be present to obtain such differences.

The successful application of equation (6) to blood flow studies using the multiple indicator-dilution technique suggests that the difference in clearance curves is due to different internal motions of the labelled components and the method of measurement. This is consistent with rheological[1f] investigation correlating changes in molecular structure with anomalous viscosities[2b,d,k,m,s,4,19,65]. Since the equation is based on a continuum theory, careful consideration is required before the results for the dispersion of a labelled component in a multicomponent system are interpreted in terms of a velocity distribution[2a,11]. The latter approach has developed from Einstein's theoretical treatment for η [22] in which the disturbance on solvent flow was considered. This was based on the success of the Poiseuille equation and the concept of laminar flow[25]. Extensions of this theory to moderate concentrations were attempted by Guth and Simha[30] and Vand[67]. Jeffrey[34] considered ellipsoidal particles in a similar manner. All these methods considered the disturbance to the streamlines when particles were added and the corresponding increase in viscosity. This is inconsistent with the justification for equation (6) discussed earlier, since the planes must contain suspended particles as well as the fluid phase.

The validity of equation (6) suggests that η for an incompressible fluid may vary with ΔP, D, a and l. For suspensions η has been shown to vary with shear stress[4,31],

tube radius[3a,4,32] and tube length[43]. Therefore non-Newtonian viscosity, as defined in chapter I, and a parabolic distribution for momentum transfer are not always incompatible.

The term "shear induced collision diffusion" introduced by Thomas[2q] is rendered redundant by equation (6). In this context, it is also difficult to envisage how the concentration profile may be obtained from the clearance curves, unless the laminae are given physical significance.

The lateral motion and increased kinetic energy of the 'molecular' components must result in cross-stresses even in 'laminar flow' in Newtonian fluids. This effect, which has been observed experimentally in gases[2n], becomes more complex in a multiphase system and has been observed during steady flow in polymer solutions[2f,j] and viscoelastic fluids[2e]. Similarly, the justification for equation (6) is consistent with viscous heating observed during capillary flow[2c] and with the dependance of turbulence on polymer concentration[2h]. Normal fluid flow will be interrupted by any object resulting in associated secondary flow[24,59]. A change in the internal flow pattern at abrupt expansions with an associated change in internal pressure[47] and turbulent effects[14,58] in pipe flow is also consistent with this interpretation of equation (6).

An increase in internal pressure for a multiphase system will result in different changes in the internal motion of all components. This interaction of components and the effect on viscosity has been investigated experimentally and

considered in terms of entanglements of polymers and their molecular weight distributions[2k,m,d,g]. Other experimentors prefer to consider the relaxation behaviour of high polymers in terms of an extension of the Eyring theory[2i].

Viscosity coefficients and studies of molecular motion have therefore become associated in rheology, but this is not the case in gas dynamics and equation (6) suggests that it should not be the case in fluid dynamics. In gas dynamics the internal motion prior to flow is an important factor in determining η, but the molecular trajectories are relatively unimportant[1a]. This is important in physiological investigations where it is the internal motion of the blood constituents that is of major interest when clotting mechanisms, gas and nutrient exchange in the microcirculation are considered.

On this basis the state of aggregation of red blood cells prior to flow must affect the flow properties of blood. This is consistent with in vitro investigations, which demonstrate that the flow properties of blood depend on whether aggregations of red blood cells are present or not. Aggregation in the stationary state has been correlated with fibrinogen concentration[15a,54,69]. Anomalous viscosity coefficients and the presence of a yield stress have also been correlated with fibrinogen concentration[15a,44,69]. The decrease in blood viscosity with increasing shear rate has been associated with the breakdown of these aggregates[13,15a,69,44,54]. Aggregation of red blood cells is often considered with regard to clot formation in the flowing

system, i.e. thrombus and emboli formation. Under the flow conditions discussed this would not occur, since aggregation is a reversible process[13,19]. It occurs when flow stops and the aggregates are dispersed as the shear stress is increased. It has been shown in vivo that decreased flow encourages aggregation, but no proof has been obtained that these aggregates plug vessels[51]. Infact it is only in the microcirculation that erythrocytes and plasma are observable as discrete entities. It is therefore the manner in which the blood consituents are broken down under different flow conditions to give separate phases for flow through the microcirculation that is of interest. Equation (6) suggests that this is determined by t ($\bar{u}$ and l), a^2 and D for the different components, i.e. the individual properties of the components and the mean flow velocity. The effect of the material properties of red blood cells on flow has been investigated experimentally. It was found that under the same flow conditions at the same cell percentage, hardened erythrocytes in a suspension possessed a higher viscosity than normal suspensions, references 15b,27. Since 'a' is also a factor in determining the relative internal motion of components, blood flow, internal and external enviroment act as feed-back mechanism to maintain the body's equilibrium.

The author believes that a similar mathematical analysis of blood flow, comparable with equation (6), is being attempted by Rodin and Isaacson[57].

Extensive study of the literature pertaining to multiple indicator techniques (reviewed in reference 3e) has led the author to conclude that the disperse and dispersive phases are regarded as separate continua, and yet interpenetrating. This is obviously inconsistent and it is hoped that the discussion associated with equation (6) has drawn attention to the type of relationship that exists between continuum and particulate theories of flow.

It is important to remember that the physical properties of the separate phases in a multiphase system are important in determining the internal motion during flow, not only the fluid phase. The experimental results presented in this thesis and the associated theoretical equation indicate that the internal motions of the component particles during flow are anisotropic, usually with motion perpendicular as well as in the direction of flow. The composition and physical properties of the material, the dimensions and material of the flow channel and the pressure and temperature conditions at which flow takes place, are all important parameters in determining the volume flow rate. ~~The significance of a viscosity coefficient in a closed system, where there is no net transport of momentum, is questionable. The internal motions of the components are the major concern in such systems, e.g. the circulation, and would be best described by anisotropic diffusion coefficients.~~

It is considered that the discussion associated with the experimental work contained in this thesis has demonstrated

that axial streaming does not have to occur for a difference in mean circulation times to be obtained in vitro, and also that there is no need for extravascular compartments and cell-shunting in vivo.

This thesis has endeavoured to illustrate certain inconsistencies which exist in flow theory and measurement, with particular reference to blood flow. The work originated from physiological investigations of blood flow and the interpretation that has been given to the results obtained. Tracer methods are used, which are a comparitively modern technological development from nuclear physics, but to interpret the results a continuum theory and concepts of laminar flow based on Newton's Laws are often retained. In the author's opinion these two approaches are not compatible.

A P P E N D I X

APPENDIX

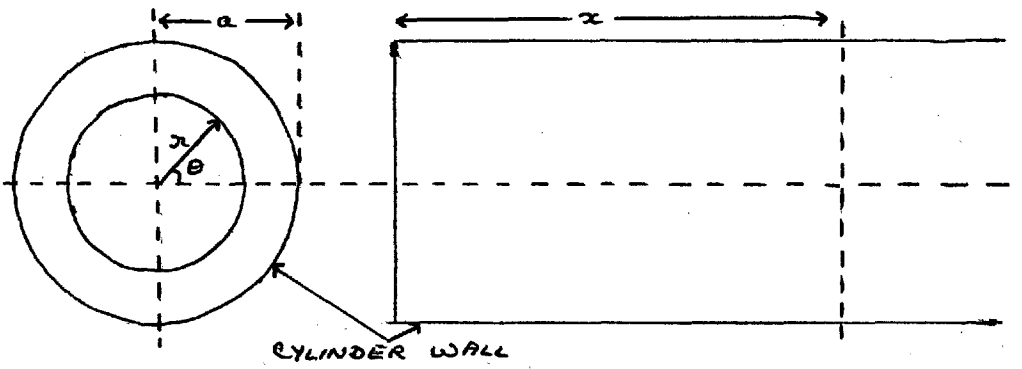
Numerical solution of the partial differential equation arising from the dispersion of material in solutions flowing through circular tubes.

C.C. Paige. 6/1/67

The independent variables are :-

t, time in seconds
x, distance along the tube in cms.
r, radial distance from the centre of the tube, in cms.

Because of cylindrical symmetry we need not consider the angle of rotation θ.



The dependent variable is the relative concentration $c = c(x,r,t)$. For zero concentration $c = 0$, for maximum concentration $c = 1$.

The partial differential equation is

$$\frac{\partial c}{\partial t} + 2U\left[1-\frac{r^2}{a^2}\right]\frac{\partial c}{\partial x} = \frac{D}{r}\frac{\partial}{\partial r}\left[r\frac{\partial c}{\partial r}\right] + D\frac{\partial^2 c}{\partial x^2} \qquad (1)$$

U is the mean velocity in the x direction in cm. sec.$^{-1}$ - it is constant for each run and is defined as flow rate divided by cross-sectional area. D is the diffusion coefficient, it is constant throughout this problem and is 1.85×10^{-5} cm^2.sec^{-1}. a is the inside radius of the tube in cms. (a = 0.061 cm. in this series of runs.) The boundary conditions are :-

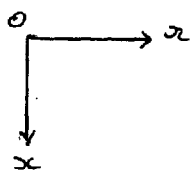
$$\frac{\partial c}{\partial r} = 0 \quad \text{at } r = a, \quad \text{all } t$$

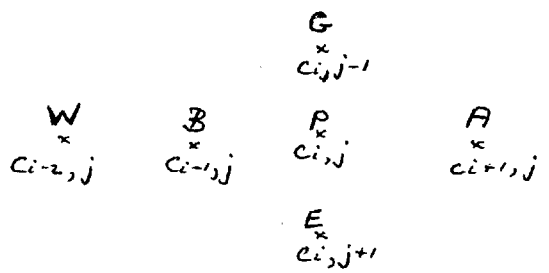
$$\left.\begin{aligned} c &= 0 \quad \text{for } x < 0 \\ c &= 1 \quad \text{for } x > 0 \end{aligned}\right\} \text{at } t = 0$$

The problem is to find $c(x,r,t)$ for $t > 0$, $x > 0$, $0 \leq r \leq a$. We can see from symmetry that we only need consider $0 \leq r \leq a$ and that $\frac{\partial c}{\partial r} = 0$ at $r = 0$.

Numerical Method.

After an examination of the many possible approaches it was decided that an explicit finite difference replacement scheme offered the most feasible solution in terms of computing time. A 6 point formula was used in the space dimensions





where if $c_{i,j} = c(x,r,t)$

then $c_{i+1,j+1} = c(x+\Delta x, r+\Delta r, t)$

This gives the approximation to the time derivative c at the point i,j of

$$c_t \doteq -2U\left[1-\frac{r^2}{a^2}\right]\left[\frac{c_{ij}-c_{ij-1}}{\Delta x}\right] + \frac{D}{(\Delta x)^2}\left[c_{ij+1} - 2c_{ij} + c_{ij-1}\right] + \frac{D}{(\Delta r)^2}\left[c_{i+1\,j} - 2c_{ij} + c_{i-1\,j}\right] + \frac{D}{r\Delta r}\left[\frac{1}{6}c_{i-2\,j} - c_{i-1\,j} + \frac{1}{2}c_{ij} + \frac{1}{3}c_{i+1\,j}\right] \tag{4}$$

(this assumes a uniform rectangular grid)

This particular finite difference scheme was chosen as it appears to be the only simple one that is stable for a reasonable time increment Δt.

To replace the time derivative we use the simple approximation

$$(c_{ij})_t \doteq \frac{c_{ij}(t+\Delta t) - c_{ij}(t)}{\Delta t} \tag{5}$$

From the Taylor expansion this leads to the approximation

$$\begin{aligned} c_t = {} & Dc_{rr} + \frac{D}{r}c_r + Dc_{xx} - 2U\left[1-\frac{r^2}{a^2}\right]c_x \\ & + 2U\left[1-\frac{r^2}{a^2}\right]\left[\frac{\Delta x}{2}c_{xx} - \frac{(\Delta x)^2}{6}c_{xxx} + \cdots\right] \\ & + D\left[\frac{(\Delta x)^2}{12}c_{4x} + 2\frac{(\Delta x)^4}{6!}c_{6x} + \cdots\right] \\ & + D\left[\frac{(\Delta r)^2}{12}c_{4r} - \frac{(\Delta r)^4}{6!}c_{6r} + \cdots\right] \\ & + \frac{D}{r}\left[\frac{(\Delta r)^3}{12}c_{4r} - \frac{(\Delta r)^4}{30}c_{6r} + \cdots\right] \\ & - \left[\frac{\Delta t}{2}c_{tt} + \frac{(\Delta t)^2}{6}c_{3t} + \cdots\right] \end{aligned} \tag{6}$$

We see that in the limit as Δx, Δr, $\Delta t \to 0$ in any order, the approximation approaches the true partial differential equation (1).

Now from(4) and (5) we can obtain the coefficients A, B etc. in the combined replacement scheme

$$c_{ij}(t+\Delta t) = Pc_{ij} + Ac_{i+1\,j} + Bc_{i-1\,j} + Wc_{i-2\,j} + Gc_{ij-1} + Ec_{ij+1} \tag{7}$$

(where everythig on the right hand side is at time t)

They are

$$P = 1 - \Delta t\left[\frac{2U}{\Delta x}\left(1-\frac{r^2}{a^2}\right) + \frac{2D}{(\Delta r)^2} - \frac{D}{2r.\Delta r} + \frac{2D}{(\Delta x)^2}\right]$$

$$A = \Delta t\left[\frac{D}{(\Delta r)^2} - \frac{D}{3r.\Delta r}\right]$$

$$B = \Delta t\left[\frac{D}{(\Delta r)^2} - \frac{D}{r.\Delta r}\right] \qquad (8)$$

$$W = \Delta t\left[\frac{D}{6r.\Delta r}\right]$$

$$G = \Delta t\left[\frac{2U}{\Delta x}\left(1-\frac{r^2}{a^2}\right) + \frac{D}{(\Delta x)^2}\right]$$

$$E = \Delta t\left[\frac{D}{(\Delta x)^2}\right]$$

from which we can see we will have stability if

$$\left[\frac{2U}{\Delta x}\left(1-\frac{r^2}{a^2}\right) + \frac{2D}{(\Delta r)^2} - \frac{D}{2r.\Delta r} + \frac{2D}{(\Delta x)^2}\right]\Delta t \leq 1 \qquad (9)$$

<u>Minimisation of Error</u>

Unfortunately this finite difference method has errors of the first order in Δt and Δx, and so it would seem these should be made very small to obtain a fairly accurate solution. However, an examination of the errors in (6) and the original equation show that the situation is not as bad as it first seems.

We obviously want to minimise the modulus of the main component of the error

$$2U\left[1-\frac{r^2}{a^2}\right]\frac{\Delta x}{2}C_{xx} - \frac{\Delta t}{2}C_{tt} \qquad (10)$$

now our original equation is

$$C_t = -2U\left[1-\frac{r^2}{a^2}\right]C_x + DC_{rr} + \frac{D}{r}C_r + DC_{xx}$$

and this gives us

$$C_{tx} = -2U\left[1-\frac{r^2}{a^2}\right]C_{xx} + D\left[C_{rrx} + \frac{1}{r}C_{rx} + C_{xxx}\right] \qquad (11)$$

$$C_{tt} = -2U\left[1-\frac{r^2}{a^2}\right]C_{xt} + D\left[C_{rrt} + \frac{1}{r}C_{rt} + C_{xxt}\right] \qquad (12)$$

and so our expression (10) that we want to minimise becomes

$$f\left[\frac{\Delta x}{2} - \frac{\Delta t}{2} f\right] c_{xx} - D\frac{\Delta t}{2}\left[c_{rrt} + f c_{rrx} + \frac{1}{r} c_{rt} - f/_{r} c_{rx} + c_{xxt} - f c_{xxx}\right] \qquad (13)$$

where

$$f = 2v\left[1 - \frac{r^2}{a^2}\right]$$

The term multiplied by D will usually be much smaller than the other (because of the value of D), and so if we could eliminate the first term the error would be greatly reduced. This obviously cannot be done completely as f is a function of the radius r - but this analysis does show that Δt could be as large as is possible compatible with stability with no deleterious effects - this conclusion is well supported by results.

Treatment of Boundary Conditions

We have to consider the boundary conditions

(i) x=0, $0 \leq r \leq a$, all $t > 0$ and $\frac{\partial c}{\partial r} = 0$ at $r = a$, all t

(ii) large x, $0 \leq r \leq a$, all $t > 0$

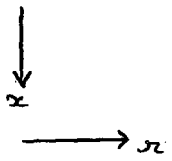
(iii) $r = 0$, $x > 0$, all $t > 0$ $\left.\begin{matrix} c = 0 \text{ for } x < 0 \\ c = 1 \text{ for } x > 0 \end{matrix}\right\}$ at $t = 0$.

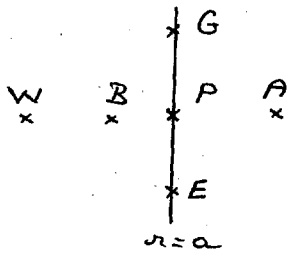
(iv) $r = a$, $x > 0$, all $t > 0$

The case of (i) is simple as this is a source of fluid with solute concentration zero, we just keep c = 0 at x =0 all t. For large enough x we know that c = 1, but as a large enough value of x (max) to be sure of this would involve too much computing time, it was considered sufficient to take a value x (max) beyond the region of interest and to extrapolate to find the corresponding value of c. This is reasonable as any diffusion will make the variation quite smooth, and indeed it was found that even extrapolation in the region of large variation was perfectly satisfactory.

Case (iv) is easily approximated to sufficient accuracy as here $\frac{\partial c}{\partial r} = 0$, and as our finite difference scheme treats

the points





We can use a central difference approximation at r = a and so say that $c_A = c_B$.

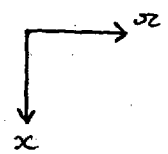
Boundary condition (iii) at r = 0 is more difficult, for although we know that $c_r = 0$ at r = 0, our equation involves the term $\frac{c_r}{r}$ at r = 0, which cannot be included in our finite difference scheme as it is. However, by de L'Hopitals theorem we have

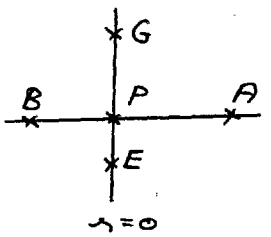
$$\lim_{r \to 0} \frac{c_r}{r} = \lim_{r \to 0} c_{rr}$$

and using this our equation becomes at r = 0

$$c_t = 2D c_{rr} + c_{xx} - 2U c_x$$

for which we can use a 5 point approximation to sufficient accuracy





and again we can say $c_B = c_A$.

Note that this gives the coefficient at P as

$$1 - \Delta t \left[\frac{2U}{\Delta x} + \frac{4D}{(\Delta r)^2} + \frac{2D}{(\Delta x)^2} \right]$$

and all other coefficients are ≥ 0, therefore this gives us an upper bound on Δt for stability

$$\Delta t \leq \frac{1}{\frac{2u}{\Delta x} + \frac{4D}{(\Delta r)^2} + \frac{2D}{(\Delta x)^2}}$$

N.B. Comparison with the Taylor Approximation

For slow flows, large times - the error in the computed solution due to the finite Δr is negligible compared with the Taylor error caused by neglecting the Dc_{rr} term. The effect of neglecting this may itself be negligible but cannot be tested with the present finite difference set up (due to the introduction of instability).

The more likely cause of the discrepency is the finite Δx in the computed solution. The term Dc_{xx} introduces an error negligible in comparison with this.

DESCRIPTION

The initial conditions are to be input in this order - they are denoted by the EXCHLF name given them in the program.

READ	DESCRIPTION
Υ	total length in cms.
S	number of length steps (if unequal)
if S ≠ 0 do {	J = 1(1)S
	READ (Y(J)) length divisions
}	REPEAT
if S = 0	READ (B) constant length interval
S'	number of extra length steps for extrapolation
B'	length of each extra length step
A	radius in cms.
R	number of radius steps if all equal (R must be even)
if R = 0 {	READ (R) number of unequal steps (again even)
	I = 2(1)R + 1
	READ (X(1)) radius divisions
}	REPEAT
Z	total time in seconds
T	number of time steps

READ	DESCRIPTION
N	print indicator, N must be even and also an integral fraction of T
D	diffusion coefficient cm.^{2} sec.$^{-1}$
U	mean velocity cm. sec.$^{-1}$

Any number of sets of data can be done on the same run.

Throughout we try to use the index I for cycling in the radius direction and K for time increasing. At any one time t the concentrations are stored in main variables C (or D) : at time t + Δt they are stored in D (or C).

The mean concentration is calculated by a Simpson integration across the tube.

An estimation of the computing instructions required is obtained from

(no. of x steps) x (no. of r steps) x (no. of t steps) x 0.0167

For Future Consideration

This metod of solution was found unsatisfactory for very slow flows and large times, as this requires too great a computing time to obtain sufficient accuracy.

The solution is too long to be used in a search program to find the distribution of diffusion with radius, but it could be used for testing different theories.

S A M P L E O F D A T A

JOB
LMK14EA1 BATE T7 19/12/66 R30
INPUT
0 LINEPRINTER 1000 LINES
COMPUTING 25000 INSTUCTIONS
STORE 60 BLOCKS

DATA

58
0
0.25
4
0.25
0.061
20
6
240
40
$1.85_{,}10^{-5}$
4.534

***Z

COMPUTER PROGRAM.

The program was written in EXCHLF autocode (November 1966) for execution on the London University Institute of Computer Science Atlas.

```
COMPILER EXCHLF
LMK14EA1,BATE PDE-TEST 7

TITLE
LARGE STEPS-CORRECT INITIAL FORMULA-6 POINT F.D. APPROX.
CALC.Crr,Cr/r AT r=0
IGNORE QUERIES

ROUTINE 3          »mean conc calcn,printout
NEWLINE 2
CAPTION
TIME=
PRINT(KC,1,4)
CAPTION
    TAU= UT/L=
PRINT(KUC/Y,1,4)
CAPTION
MEAN CONCENTRATION
NEWLINE

0'=0
J'=0
J=1(1)P
J=J'+M
C'=0
T'=-1
I=1(1)M
C'=C'+*(3+T')C(J'+I)X(I)
T'=-T'
REPEAT
C'=C'-C(J+M)X(M)
PRINT(2C'/*(3RA),1,5)
0'=0'+1
JUMP 30,10>0'
0'=0
NEWLINE
CAPTION
L=
PRINT(Y(J),3,4)
```

```
30) REPEAT
NEWLINE
CAPTION
CONCENTRATIONS WITH RADIUS AT TOTAL LENGTH
NEWLINE
J'=*(M)S
I=1(1)M
PRINT(C(J'+I),1,5)
REPEAT
NEWLINE
RETURN
CLOSE R

CHAPTER 0
C→15359
D→15359
Y→1023
H→30
U→40
A→40
B→40
G→40
X→40
W→40
CHANNEL 1 R
0=0
4)NEWLINE
CAPTION
TOTAL LENGTH-NO.STEPS-EXTRA STEPS-SIZE
NEWLINE

READ(Y)              »total length
PRINT(Y,4,2)
READ(S)              »length steps
Y(0)=0
JUMP 1,S=0           »equal length divisions
J=1(1)S
READ(Y(J))           »length divisions
REPEAT
JUMP 2
1)READ(B)            »constant length interval
S=*INTPT(Y/B+0.1)    »no.length steps
J=1(1)S
Y(J)=Y(J-1)+B
REPEAT

2)READ(S')           »no.extra length steps
READ(B')             »length each extra interval
J=1(1)S'
Y(J+S)=Y(J+S-1)+B'
REPEAT
Y(S+S'+1)=Y(S+S')+B'
```

```
PRINT(S,3,0)
PRINT(S',6,2)
PRINT(B',5,5)
NEWLINE
CAPTION
STEPS ARE
NEWLINE
P=S+S'+1
J=1(1)P
PRINT(Y(J),3,2)
REPEAT

READ(A)                    »radius
READ(R)                    »no.radius steps
JUMP 60, R=0
X(0)=A/R
M=R+1
X(1)=0
I=1(1)M
X(I+1)=X(I)+X(0)
REPEAT
X(0)=-X(2)
JUMP 61
60)READ(R)                 »no.radius steps
M=R+1
X(1)=0
I=2(1)M
READ(X(I))                 »radius divisions
REPEAT
X(0)=-X(2)
X(M+1)=2A-X(R)
61)NEWLINE
CAPTION
RADIUS
PRINT(A,3,3)
CAPTION
NO.STEPS
PRINT(R,3,0)
NEWLINE
CAPTION
R STEPS ARE
M=M+1
I=0(1)M
PRINT(X(I),3,4)
REPEAT
M=M-1
NEWLINE

READ(Z)                    »total time
READ(T)                    »no.of steps
READ(N)                    »print indicator,N an even integer, and
                             an integral fraction of T
```

```
C=Z/T                          »time step
NEWLINE
CAPTION
TOTAL TIME -NO. OF STEPS-TIME STEPS
NEWLINE
PRINT(Z,5,4)
PRINT(T,5,0)
PRINT(C,8,8)
READ(D)                        »diffusion coefficient
READ(U)                        »mean velocity
NEWLINE
CAPTION
DIFFUSION COEFFICIENT
PRINT(D,0,6)
CAPTION
MEAN VELOCITY
PRINT(U,9,4)
NEWLINE 2

C(1)=0
D(1)=0
W(1)=0
B(1)=0
X=X(2)-X(1)                    »r step
A(1)=4CD/*(XX)
»Y(1)                            is x step, C is t step
V=Y(1)Y (1)
H(1)=C*(2U/Y(1)+D/V)
E=CD/V
U(1)=1-E-H(1)-A(1)
D(2)=0
C(2)=0
H(2)=C*(2U*(1-X(2)X(2)/*(AA))/Y(1)+D/V)
W(2)=0
B(2)=0
A(2)=4CD/*(3XX)
U(2)=1-*(H(2)+A(2)+E)

I=3(1)M                        »radius cycle for coeffs.
D(I)=0
C(I)=0
H(I)=C*(2U*(1-X(I)X(I)/*(AA))/Y(1)+D/V)
W(I)=CD/*(6X(I)X)
B(I)=CD*(1/X-1/X(I))/X
A(I)=CD*(1/X+1/*(3X(I)))/X
U(I)=1-*(H(I)+E+A(I)+B(I)+W(I))
REPEAT
B(M)=B(M)+A(M)
A(M)=0
CAPTION
R    U    H    W    B    A    E    SUM
```

```
NEWLINE
I=1(1)M
PRINT(X(I),1,4)
PRINT(U(I),1,5)
PRINT(H(I),1,5)
PRINT(W(I),1,5)
PRINT(B(I),1,6)
PRINT(A(I),1,6)
PRINT(E,1,6)
PRINT(1-(U(I)+H(I)+W(I)+B(I)+A(I)+E),1,9)
NEWLINE
REPEAT
J'=0
J=1(1)P
J'=J'+M
I=1(1)M
C(I+J')=1
REPEAT
REPEAT
NEWLINE
CAPTION
FOR STABILITY ALL COEFFICIENTS ARE POSITIVE AND SUM IS 1

K=N(N)T
Q=2(2)N

J'=0
M'=M
J=1(1)P                    »length cycle
I'=J'                      »j-1 term for concentrations
J'=M'                      »j    "    "    "   & coeffs
M'=J'+M                    »j+1  "    "    "       "
D(J'+1)=H(1)C(I'+1)+U(1)C(J'+1)+EC(M'+1)+A(1)C(J'+2)     »r=0
          »could check here if 1 then out
D(J'+2)=H(2)C(I'+2)+U(2)C(J'+2)+A(2)C(J'+3)+EC(M'+2)     »r=dr
I=3(1)R
D(J'+I)=H(I)C(I'+I)+W(I)C(J'+I-2)+B(I)C(J'+I-1)+U(I)C(J'+I)
                    +A(I)C(J'+I+1)+EC(M'+I)
REPEAT
D(M')=H(M)C(J')+W(M)C(M'-2)+B(M)C(M'-1)+U(M)C(M')+EC(M'+M) »r=a
          »check here if 0 then mark
REPEAT
K'=I'-M
I=1(1)M
D(M'+I)=3D(J'+I)-3D(I'+I)+D(K'+I)  »end points in 1 direction
REPEAT

J'=0
M'=M
J=1(1)P                    »length cycle
```

```
I'=J'          >>j-1 term for concentrations
J'=M'          >> j   "    "      "     & coeffs
M'=J'+M        >>j+1  "    "      "       "
C(J'+1)=H(1)D(I'+1)+U(1)D(J'+1)+ED(M'+1)+A(1)D(J'+2)      >>r=0
          >> could check here if 1 then out
C(J'+2)=H(2)D(I'+2)+U(2)D(J'+2)+A(2)D(J'+3)+ED(M'+2)      >> r=dr
I=3(1)R

C(J'+I)=H(I)D(I'+I)+W(I)D(J'+I-2)+B(I)D(J'+I-1)+U(I)D(J'+I)
                    +A(I)D(J'+I+1)+ED(M'+I)
REPEAT
C(M')=H(M)D(J')+W(M)D(M'-2)+B(M)D(M'-1)+U(M)D(M')+ED(M'+M) >>r=a
          >>check here if 0 then mark
REPEAT
K'=I'-M
I=1(1)M
C(M'+I)==3C(J'+I)-3C(I'+I)+C(K'+I)   >>end points in 1 direction
REPEAT

REPEAT
JUMPDOWN(R3)
REPEAT
JUMP 4              >>fresh start
END
CLOSE

***Z
```

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