Proceedings of the 29th Annual International
Conference of the IEEE EMBS
Cité Internationale, Lyon, France
August 23-26, 2007.

An analogue bandpass filter realisation of the Continuous Wavelet Transform

Alexander J. Casson, David C. Yates, Shyam Patel and Esther Rodriguez-Villegas

Abstract— **The Continuous Wavelet Transform (CWT) is a highly useful signal processing technique for which low power implementations are desirable, for example to allow the transform to be used in battery powered, portable devices. This paper describes a Low Power CWT (LPCWT) implementation that is based around the mathematical approximation of a mother wavelet and uses application specific information to guide the approximation process. Simulations comparing the LPCWT and CWT are carried out with the two giving equivalent signal processing performance. The LPCWT is suitable for realisation in any desired circuit topology.**

I. INTRODUCTION

The Continuous Wavelet Transform (CWT) is a well known and highly useful signal processing technique. It is based around a *mother wavelet* function which acts as a bandpass filter. When applied to a signal this filter extracts the frequency content occurring at a particular time. The transform is usually carried out at a range of *scales*: altering the centre frequency and *shape* of the bandpass filter allows different time and frequency information to be determined. A full description of the transform is given in [1].

Classic uses of the CWT include data compression and feature extraction, for example in the detection of epileptic spikes. In [2] the authors have proposed a CWT based algorithm for the online detection of epileptic spikes from a wireless, mobile sensor (ambulatory EEG). Portable signal processing implementations are thus required.

Power consumption is a key requirement for battery operated, wireless, mobile systems with on-board signal processing. The CWT is thus ideally implemented in dedicated hardware circuits, removing the need for software. Digital circuit formulations, using a variety of methods for calculating the CWT, are possible. However, for low dynamic ranges analogue circuits can offer reduced power consumption compared to digital equivalents [3] and a direct analogue bandpass filter implantation offers an intrinsically real-time and low computational complexity CWT algorithm, minimizing the power consumption.

This paper analyses the design procedure to obtain an optimal CWT implementation which is suitable for this low power realisation (Section II). The design procedure is based around a wavelet approximation method that produces generalised transfer functions that are suitable for implementation in any desired technology. The trade-offs in the approximation procedure are highlighted and the epileptic spike detection algorithm [2] is used to illustrate how the trade-offs are best handled in an actual usage situation. This can then act as a guide to the optimal approximation method for other usage situations.

The performance of the proposed Low Power CWT (LPCWT) implementation is then compared to the MATLAB CWT function for the epileptic spike example (Section III). This gives a quantitative performance analysis of the LPCWT and shows that it is suitable as a substitute for the CWT function.

The authors are with the Electrical and Electronic Engineering Department, Imperial College London, SW7 2AZ {alexander.casson06, david.yates, e.rodriguez}@imperial.ac.uk.

II. WAVELET DESIGN PROCEDURE

A. Overview

The CWT implementation outlined below (termed the LPCWT as it is suitable for low power implementation) is based upon approximating the mother wavelet $\psi(t)$ and using this as a filter circuit. Other groups ([4]–[6]) have looked at using Padé and L_2 approximations to synthesise the required wavelet shape and give a transfer function suitable for implementation using dynamic translinear circuits. The method outlined below is similar to this, but the form of the mother wavelet required, the Mexican hat, is exploited to allow a simpler Maclaurin series approximation to be taken while still obtaining a rational transfer function. The resulting transfer functions are also noted to be implementation independent and can be realised using any desired circuit topology.

The design procedure is summarised as:

1) Based on the application determine the mother wavelet required. The Mexican hat wavelet is used as an example here.
2) Choose the scales at which analysis must be carried out. Reducing the number of scales used will reduce the circuity required and minimise the power consumption.
3) Mathematically approximate the wavelet shape using the procedure outlined below, exploiting the mathematical form if possible.
4) Determine a time delay T (see Section II-C) to ensure that the wavelet approximation is realisable.
5) Use the dynamic range and other application specific criteria to select the order of the approximation required.
6) Generate the required filter transfer functions and implement these in a suitable circuit.

1-4244-0788-5/07/$20.00 ©2007 IEEE

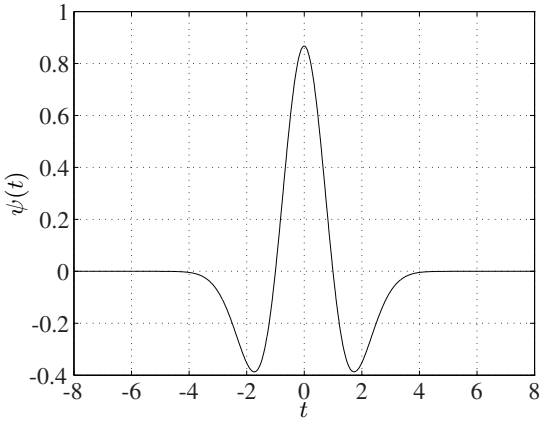


Fig. 1. The Mexican hat mother wavelet as defined in (3).

B. Wavelet approximation

The CWT of signal $x(t)$ at a scale a and at time b is defined as

$$W(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} x(t)\psi^* \left(\frac{t-b}{a} \right) dt. \tag{1}$$

This is equivalent to the convolution of the signal $x(t)$ with an impulse response

$$h(t) = \frac{1}{\sqrt{a}} \psi \left(\frac{-t}{a} \right). \tag{2}$$

It is well known that the output of a filter circuit is the convolution of the filter's impulse response and the input signal. Hence the CWT at a fixed scale can be carried out simply by using a filter with impulse response $h(t)$.

The Mexican hat mother wavelet is defined as

$$\psi(t) = \frac{2}{\pi^{1/4}\sqrt{3}}(1-t^2)e^{-t^2/2} \tag{3}$$

and is illustrated in Fig. 1. Given (3), (2) has the Fourier transform

$$H(j\omega) = \pi^{1/4}\sqrt{\frac{8}{3}a^5}\omega^2 e^{-a^2\omega^2/2}. \tag{4}$$

The Mexican hat CWT at scale a is thus carried out by a filter with transfer function $H(j\omega)$, (4).

However, two problems are present. Firstly, as can be seen in Fig. 1, $\psi(t)$ is symmetrical around $t = 0$ and is hence non-causal.[1] Any filter with this as its impulse response will necessarily have poles in the right hand complex plane and will be unstable. To correct for this a time delay T is introduced shifting $\psi(t)$ to the right. Note that this factor will also delay the filter output and this may need to be compensated for elsewhere. Substituting for $s = j\omega$, the required transfer function is modified to

$$H(s) = -\pi^{1/4}\sqrt{\frac{8}{3}a^5}s^2 e^{a^2s^2/2}e^{-sT} \tag{5}$$

$$= \frac{-\pi^{1/4}\sqrt{\frac{8}{3}a^5}s^2}{e^{sT-a^2s^2/2}}. \tag{6}$$

[1]This does not present an issue in computer implementations where arbitrary delays are easily introduced.

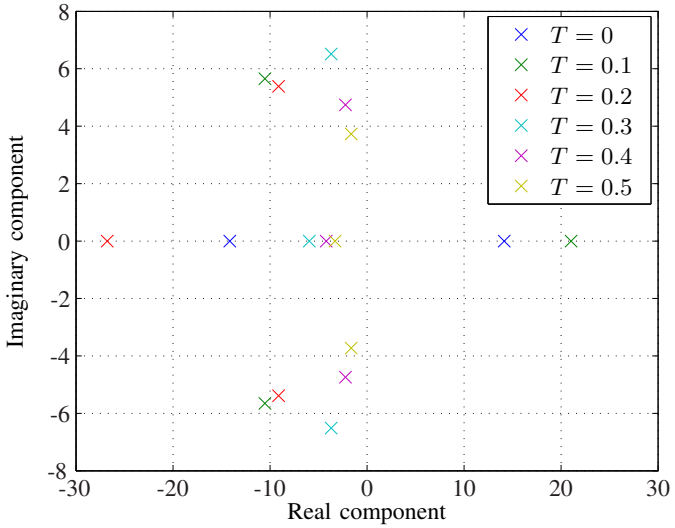


Fig. 2. Positions of poles of the third order transfer function (8) with $a = 0.1$ for varying values of T.

Here the form of (5) has been exploited to write (6) in rational form. However, the exponential term in the denominator means that the transfer function cannot be realised using standard filters which can only have rational, finite order transfer functions. As (6) is already rational only the denominator needs approximating and a Maclaurin series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{7}$$

can be used. This produces a stable and realisable approximation of the transfer function required for the LPCWT which is given by

$$H(s) = \frac{-\pi^{1/4}\sqrt{\frac{8}{3}a^5}s^2}{1 + sT + \left(\frac{T^2}{2} - \frac{a^2}{2}\right)s^2 + \left(\frac{T^3}{6} - \frac{Ta^2}{2}\right)s^3 + \cdots}. \tag{8}$$

C. Selecting the optimal approximation

To complete the wavelet design process it is necessary to choose the analysis scale a, time delay T and filter order to allow a stable, numerical, transfer function to be derived from the generalised function (8). These factors are partly inter-related, with suitable T values depending on the filter order. The positions of the transfer function poles at scale $a = 0.1$ for a third order filter as T varies are illustrated in Fig. 2. At scale $a = 0.1$ the value $T = 0.4$ is found to give stable transfer functions for up to a 7$^{\text{th}}$ order approximation allowing the filter order to be optimised within this range.

The filter order used represents a trade-off between a more accurate approximation of the wanted function and the end circuit complexity and power consumption. Fig. 3 shows the Bode magnitude plots for the ideal (time delayed) transfer function, (6), and different order LPCWT approximations of it, (8), for $a = 0.1$ and $T = 0.4$.

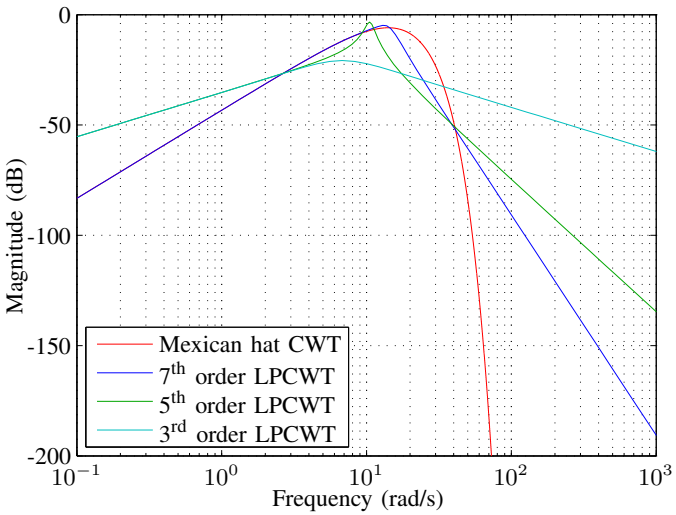


Fig. 3. Bode magnitude plots of the Mexican hat CWT transfer function (6) and different orders of the LPCWT (8) for $a = 0.1$ and $T = 0.4$.

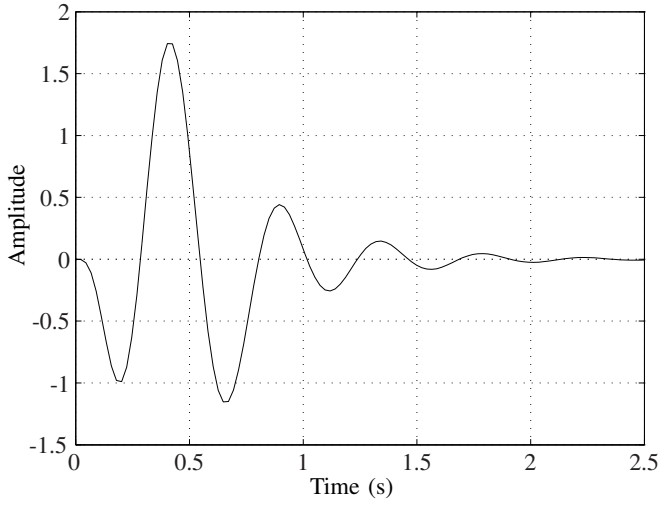


Fig. 4. The impulse response of (10) for comparison with the *shape* of Fig. 1. Note that numerical values differ due to the change in scale between the two figures.

At this point application specific information can be used to guide the approximation process. The application considered here is electroencephalogram (EEG) analysis. The international federation of clinical neurophysiology guidelines [7] recommend that EEGs be sampled to a resolution of 0.5μV and [8] notes that the maximum amplitude is typically 150μV. EEG recordings (analogue or digital) thus have a practical dynamic range of 50dB. From this, the frequency responses of Fig. 3 are only needed to match in the range 0–50dB as frequencies outside this range will be attenuated down to the noise floor.

The 3rd order Bode plot in Fig. 3 shows significant deviation from the ideal case at low frequencies. With a 7th order filter the low frequency approximation is perfect and error is only present for a limited range of higher frequencies which is likely to be acceptable. If wanted a low pass filter could be cascaded with the bandpass filter to improve this high frequency response.

The final approximation step is simply to substitute the chosen parameters into (9). For the cases $a = 0.025$, $T = 0.1$ and $a = 0.1$, $T = 0.4$, as required by the algorithm to be tested in Section III, the final transfer functions are given in (9) and (10). For comparison with the *shape* Fig. 1, Fig. 4 shows the impulse response of (10).

D. Notes

At this point two asides about the approximation method are suitable. Firstly, the Short Time Fourier Transform (STFT) of a signal $x(t)$ at time b is defined as

$$S(\omega, b) = \int_{-\infty}^{\infty} x(t)g(t-b)e^{j\omega t}dt \tag{11}$$

where $g(t-b)$ is the windowing function to be used. Comparing this to the approximation method it can be seen that although the CWT is being implemented, each proposed filter only operates at one fixed scale and so is essentially carrying out a STFT, albeit with a rather non-standard windowing function.

Secondly, in order for a function $\psi(t)$ to be a wavelet it must satisfy the *admissibility criteria* [1]. This ensures that the inverse CWT is possible and the practical consequences of it are that for $\psi(t)$ to be a wavelet

$$\int_{-\infty}^{\infty} \psi(t)dt = 0. \tag{12}$$

Numerical integration of the impulse response of (10) shows that this condition is satisfied to an accuracy of at least 10^{-16}. However, it should be noted that for applications, such as that considered in Section III, where the aim is only to extract signal information, there is no need for the signal to be reconstructed and so no need for the inverse transform. As a result the admissibility criteria need not apply to wavelets designed for these cases.

III. PERFORMANCE ANALYSIS

A. Methods

Having outlined the LPCWT wavelet design procedure it is now possible to asses the LPCWT's performance. As opposed to using arbitrary performance metrics, such as finding an error norm between the ideal and approximated wavelets, application specific performance measures are investigated. These reflect the most sensible design approach where for optimal design the approximation method must be tailored to the situation under consideration and allows a meaningful measure of how well the CWT is implemented in an actual application to be produced.

Testing in this case is carried out using the epileptic spike detection algorithm outlined in [2] using both the MATLAB CWT function and a MATLAB simulation of the LPCWT.

The algorithm to be tested is based upon the normalised wavelet power

$$P(a, b) = \frac{W^2(a, b)}{\sigma^2} \tag{13}$$

$$a\text{=}0.025\text{: } H(s) = \frac{-2.15\times10^{-4}s^2}{1.43\times10^{-12}s^7+3.23\times10^{-10}s^6+3.61\times10^{-8}s^5+2.65\times10^{-6}s^4+1.35\times10^{-4}s^3+0.005s^2+0.10s+1} \quad (9)$$

$$a\text{=}0.100\text{: } H(s) = \frac{-6.88\times10^{-3}s^2}{2.34\times10^{-8}s^7+1.34\times10^{-6}s^6+3.70\times10^{-5}s^5+6.79\times10^{-4}s^4+8.67\times10^{-3}s^3+0.075s^2+0.40s+1} \quad (10)$$

where σ^2 is the variance of the signal being analysed and is calculated in 4.5s blocks. The detection procedure is summarised as three rules:

1) The normalised wavelet power at scale $a = 0.025$ must be greater than a detection threshold.
2) The normalised wavelet power at the positions identified by rule 1 must decrease between scales $a = 0.025$ and $a = 0.1$.
3) Identified spikes less than 170ms apart are grouped into a single detection at the detection with the highest normalised power at scale $a = 0.025$.

The LPCWT is simulated in MATLAB by generating the impulse response functions corresponding to (10) and (11) and convolving these with the input signal in blocks of 4.5s. To make the MATLAB CWT function and the LPCWT directly comparable the delay T introduced in (5) is compensated for by delaying the samples that are chosen for comparison by a similar amount.[2]

To compare the two CWT methods, just one 22 minute data set is used, although in [2] 41 hours of data has been analysed using the approximation method and acceptable results found. The data set considered here contains 252 expert marked events that are deemed to be correctly detected if a detection is made within 85ms of the marker. The sensitivity of the procedure is then calculated as

$$\text{Sensitivity / \%} = \frac{\text{Number of correct detections}}{\text{Total number of marked events}}. \quad (14)$$

B. Results and analysis

Results comparing the algorithm performance with different order approximations of the LPCWT as the detection threshold is varied are shown in Fig. 5. The performance of the 5th order filter is comparatively poor, but this may be expected from the errors present in the Bode plot of Fig. 3. In contrast, the 7th order filter achieves sensitivities that are within a few percent, and so comparable to, the CWT function. Indeed these sensitivities are achieved with fewer false detections occurring.

It should be noted that in many applications the choice of mother wavelet to be used is essentially arbitrary. It is entirely possible for a non-ideal approximation of a mother wavelet to actually have a *shape* that allows it achieve better results than the actual mother wavelet. Future work may investigate optimising the wavelet design procedure to carry out wavelet design for ease of implementation and application performance instead of simply approximating an existing mother wavelet.

[2]For correct CWT operation in MATLAB it is necessary to alter the analysis scale with sampling frequency as: MATLAB scale = Wanted scale $(a)\times$ Sampling frequency. Note that from (1) the analysis scale will also affect the output coefficient amplitude and this must be compensated for. Data used in this analysis was sampled at 200Hz. Also, the impulse response function $h(t)$ ((9) and (10)) saved in MATLAB is intrinsically saved in discrete time as $h(t) * \delta(t - nM)$ where M is the sampling period. In the frequency domain the delta function introduces a factor of $1/M$ extra gain in the simulated transfer functions compared to (9) and (10).

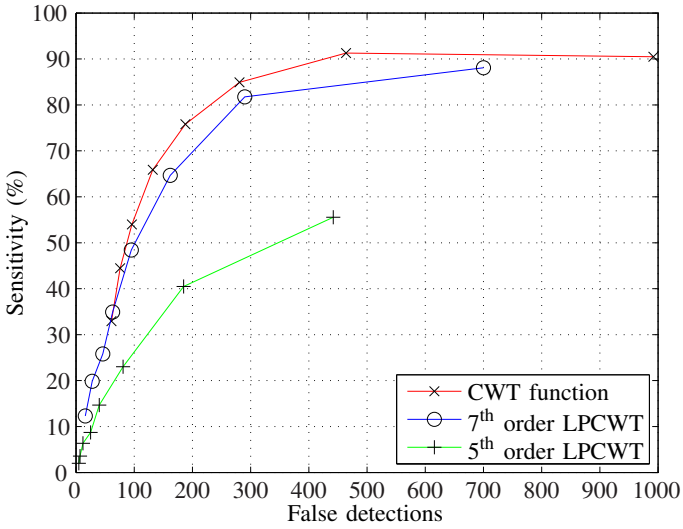


Fig. 5. Performance of the CWT function and LPCWT as the detection threshold is varied in the algorithm described in [2].

The detailed performance of the 7th order LPCWT compared to the MATLAB CWT function is shown in Fig. 6 by comparing the wavelet coefficients produced. Some differences are seen, but these would be expected from the approximation method used. Overall, the results all show that the 7th order LPCWT is a capable replacement for the MATLAB CWT function.

Preliminary work indicates that (9) and (10) are implementable as suitable electronic circuits. Future work will allow a three-way comparison between the MATLAB CWT function, a MATLAB simulation of the LPCWT (as carried out here), and the circuit level LPCWT implementation. This will also allow the power consumption to be quantified and optimal low power filter topologies will be investigated.

IV. CONCLUSION

A wavelet design method based around the mathematical approximation of a mother wavelet has been described. This approximation method illustrated how application specific information can be used to determine the optimum approximation, for example by considering the dynamic range required. The end result was the LPCWT giving generalised transfer functions that are suitable for implementation in any desired electronic topology. When applied to an actual usage situation simulations showed that the LPCWT is capable of giving results comparable to the MATLAB CWT function.

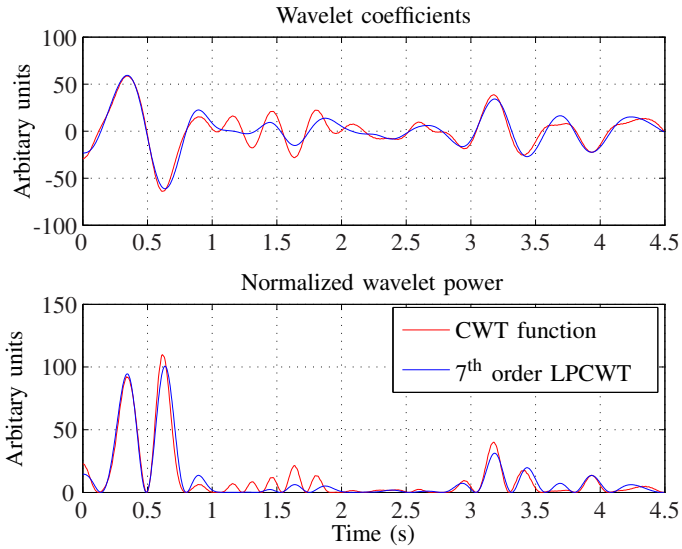


Fig. 6. Comparison of the wavelet coefficients produced by the CWT function and the 7^{th} order LPCWT.

REFERENCES

[1] S. Mallat, *A wavelet tour of signal processing*, 2nd ed. San Diego: Academic Press, 1999.

[2] A. J. Casson, D. C. Yates, S. Patel, and E. Rodriguez-Villegas, "Algorithm for AEEG data selection leading to wireless and long term epilepsy monitoring," in *IEEE Eng. Med. Biol. conference*, Lyon, August 2007.

[3] E. A. Vittoz, "Future of analog in the VLSI environment," in *IEEE symposium on circuits and systems*, New Orleans, May 1990.

[4] S. A. P. Haddad, S. Bagga, and W. A. Serdijn, "Log-domain wavelet bases," *IEEE Trans. Circuits Syst. I*, vol. 52, no. 10, pp. 2023–2032, 2005.

[5] S. A. P. Haddad, J. M. H. Karel, R. L. M. Peeters, R. L. Westra, and W. A. Serdijn, "Analog complex wavelet filters," in *IEEE international symposium on circuits and systems*, Kobe, May 2005.

[6] J. M. H. Karel, R. L. M. Peeters, R. L. Westra, and S. A. P. Haddad, "Wavelet approximation for implementation in dynamic translinear circuits," in *Proc. of the IFAC world congress*, Prague, July 2005.

[7] M. R. Nuwer, G. Comi, R. Emerson, A. Fuglsang-Frederiksen, J.-M. Guerit, H. Hinrichs, A. Ikeda, F. J. C. Luccas, and P. Rappelsberger, "IFCN standards for digital recording of clinical EEG," in *Recommendations for the practice of clinical neurophysiology: Guidelines of the international federation of clinical physiology*, G. Deuschl and A. Eisen, Eds. Amsterdam: Elsevier, 1999, pp. 11–14.

[8] C. D. Binnie, A. J. Rowan, and T. Gutter, *A manual of electroencephalographic technology*. Cambridge: Cambridge University Press, 1982.