

DIGITAL SIMULATION OF VISCOUS INCOMPRESSIBLE FLOW
AROUND A SOLID SPHERE

A thesis submitted for the
DIPLOMA OF IMPERIAL COLLEGE

by

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ABSTRACT

Numerical solutions of the time-dependent Navier-Stokes equations have been obtained for axi-symmetric incompressible Newtonian isothermal flow around a solid sphere at Reynolds numbers of 1, 10, 50, 200, 500, 1000 and 2000.

The Navier-Stokes equations were expressed in terms of a vorticity transport equation and a stream function equation. The two equations were solved simultaneously using finite-difference methods. The vorticity transport equation was solved using the Peaceman and Rachford alternating direction implicit method and the stream function equation was solved using the point iterative successive over-relaxation method.

The time-dependent problem solved was that of an impulsively accelerated viscous fluid stream which is initially at rest and which contains a stationary solid sphere. The 'late-time' solutions for Reynolds numbers of 1, 10, 50 and 200 approach solutions of the time-independent Navier-Stokes equations. The 'late-time' solutions for Reynolds numbers of 500, 1000 and 2000 show a behaviour which indicate the possibility of wake shedding at these Reynolds numbers.

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LIST OF SYMBOLS

<u>SYMBOL</u>	<u>EXPLANATION OR DEFINING EQUATION</u>
a	Coefficient defined by equation (4.3.41)
A	FORTTRAN symbol for a
$\bar{A}$	Arbitrary vector quantity
b_{1i}, b_{2i}	Coefficients defined by equations (4.3.37) and (4.3.38)
b_{3j}, b_{4j}	Coefficients defined by equations (4.3.39) and (4.3.40)
$\bar{B}$	Arbitrary vector quantity
c_i	Coefficient in equation (4.3.50)
C_{1i}, C_{2i}, C_{3i}	Coefficients defined by equations (4.3.22), (4.3.23) and (4.3.24)
$C'_{1i}, C'_{2i}, C'_{3i}$	Coefficients defined by equations (4.3.28), (4.3.29) and (4.3.30)
C_{1j}, C_{2j}, C_{3j}	Coefficients defined by equations (4.3.19), (4.3.20) and (4.3.21)
$C'_{1j}, C'_{2j}, C'_{3j}$	Coefficients defined by equations (4.3.25), (4.3.26) and (4.3.27)
C_{DF}	Viscous drag coefficient
C_{DP}	Pressure (form) drag coefficient
C_{DT}	Total drag coefficient
C_R	Coefficient defined in equation (4.3.43)
d	Diameter of the sphere
$D_i^{n+\frac{1}{2}}$	Coefficient defined by equation (4.3.17)
D_j^n	Coefficient defined by equation (4.3.16)
D_{ij}^n	Coefficient defined by equation (4.4.2)

<u>SYMBOL</u>	<u>EXPLANATION OR DEFINING EQUATION</u>
E^2	Differential operator defined by equation (3.1.9)
E_+^2	Differential operator defined by equation (3.1.21)
f	A general continuous function
$\bar{F}$	External body force
g	Vorticity function defined by equation (3.1.27)
$\bar{g}$	Gravity vector
g^*	Dimensionless vorticity function g
G	Vorticity function defined equation (4.4.1)
h	Mesh size in the Z-direction
h_1, h_2, h_3	Scale factor in the directions x_1, x_2, x_3
h_{3ij}	Scale factor in the x_3 -direction at node i, j
inl	Number of mesh lines in Z-direction
jnl	Number of mesh lines in θ -direction
k	Mesh size in the θ -direction
K	Dimensionless pressure coefficient
K_o	Dimensionless pressure coefficient at the front stagnation point
K_θ	Local dimensionless pressure coefficient
K_π	Dimensionless pressure coefficient at the rear stagnation point
l_1, l_2, l_w	Wake dimensionls defined in Table 1
L_1, L_2, L_w	Dimensionless wake dimensions defined in Table 1
P	Pressure
(r, θ, ϕ)	Spherical polar co-ordinates
r^*	Dimensionless radial co-ordinate (r/R)

<u>SYMBOL</u>	<u>EXPLANATION OR DEFINING EQUATION</u>
r_{∞}	The distance at which the outer boundary is located
R	Radius of sphere
Re	Reynolds number ($2RU_{\infty}/\nu$)
t	Time
t^*	Dimensionless time ($\frac{tU_{\infty}}{R}$)
u	Local velocity component in the x-direction
$u_r, u_{\theta}, u_{\phi}$	Local velocity components in the direction r, θ, ϕ
u_z	Local velocity component in the Z-direction
u_1, u_2, u_3	Local velocity components in the directions x_1, x_2, x_3
u_r^*, u_{θ}^*	Dimensionless local velocity components in the directions r, θ
$\bar{U}$	Velocity vector
$\bar{U}_o$	Initial velocity vector
U_{∞}	Velocity of the uniform stream
v	Local velocity component in the y-direction
W	Weighting factor defined in equation (4.2.13)
W_1, W_2, W_3	Weighting factors defined in equations (4.3.31) and (4.3.32)
x, y, z	Cartesian co-ordinates
x_1, x_2, x_3	Orthogonal curvilinear co-ordinates
$\bar{X}$	Position vector
X_i	Coefficient defined by equation (4.3.42)
Z	Co-ordinate direction defined as $Z = \ln r^*$
α	Weighting factor defined in equation (4.3.10)
ϵ	Error in the solution

<u>SYMBOL</u>	<u>EXPLANATION OR DEFINING EQUATION</u>
ϵ_{ψ}	Convergence criterion for the stream function defined by equation (4.6.2)
ϵ_g	Convergence criterion for the vorticity g function defined by equation (4.6.4)
ζ	Vorticity
ζ_s	Sphere surface vorticity
	Dimensionless vorticity $(R\zeta/U_{\infty})$
θ	Angular co-ordinate
θ_s	Angle of flow separation
μ	Fluid viscosity
ν	Fluid kinematic viscosity
ρ	Fluid density
τ	Stress term in a Newtonian fluid
ϕ	Co-ordinate representing the angle of rotation about the axis of symmetry
ϕ	Scalar quantity
ψ	Stream function
ψ^*	Dimensionless stream function $(\psi/U_{\infty}R^2)$
$\bar{\omega}$	Vorticity vector
ω_{ψ}	Displacement or relaxation factor for the stream function
$\omega_{\psi b}$	Displacement or relaxation factor for the stream function boundary used only for the parallel outflow boundary condition
ω_g	Displacement or relaxation factor for the surface vorticity $\hat{g}$ function
∇	Vector operator 'del'

<u>SUBSCRIPTS</u>	<u>EXPLANATION OR DEFINING EQUATION</u>
i	Mesh point index in the Z-direction (Fig. 4.2.1)
j	Mesh point index in the θ -direction (Fig. 4.2.1)
i, j	Indices of a mesh point in the flow region (Fig. 4.2.1)
ij	Abbreviation used in derivation of the finite-difference equation for i, j
∞	Outer boundary condition
o	Referring to $t^* = 0$
r	r-direction
Z	Z-direction
θ	θ -direction
ϕ	ϕ -direction
s	Surface of the sphere
$\mathcal{S}$	Flow separation

SUPERSCRIPTS

n	(n)th time step
*	Dimensionless variable

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CHAPTER 1
INTRODUCTION

INTRODUCTION

The prediction of the flow field around a solid sphere is of fundamental importance in chemical engineering. The study of fluid dynamics of a single sphere in uniform stream is used as a first step in the analysis of multi-particle systems that are encountered in chemical engineering. In most engineering problems overall heat or mass transfer rates are of importance and there are numerous experimental correlations available which fulfil this purpose to some extent. An optimum design of chemical process equipment requires understanding of the basic mechanisms of heat, mass and momentum transfer which these correlations do not provide. The solution of differential equations which describe transport phenomena not only provides fundamental insight into transport processes but also increases our ability to analyse complex engineering problems which so far have yielded crude answers after a large number of simplifying assumptions.

Due to their non-linearity, the partial differential equations which describe the mechanism of heat, mass and momentum transfer are of enormous complexity. The prediction of transfer rates from a solid sphere requires a knowledge of the flow pattern which can only be obtained from solutions of the equations of motion. For incompressible Newtonian fluid the equations of motion are the well known Navier-Stokes equations.

The numerical solution of the time-dependent Navier-Stokes equations is one of the most challenging problems in numerical analysis. In fact, at the present moment the mathematical analysis of the complete problem is not possible. However, this does not prevent us from attempting to develop numerical techniques to solve the Navier-Stokes equations for a particular flow, as in the present work for the flow around a solid sphere. It is hoped that the solutions obtained by these semi-intuitive methods may suggest the way for a complete analysis and at the same time provide answers to engineering problems.

The present work does not attempt to solve the heat and mass transfer equations because once a method is developed which can handle the momentum transfer equations, it can then be applied to the energy or mass transfer equations.

The numerical solutions of the time-independent Navier-Stokes equations have been reported (Al-Taha 1969; Hamielec et al. 1967; Le Clair et al. 1970). The simulation of the time-dependent flow is attempted because oscillation of the wake behind a sphere is reported at a Reynolds number of about 130 and a periodic shedding of the wake at a Reynolds number of about 500 (Torobin and Gauvin 1959). The difficulties encountered in the numerical solution of three-dimensional flows have yet to be overcome, also various experimental observations report different mechanisms

of vortex shedding, therefore axi-symmetric flow equations are solved to study whether there is wake instability under strict axial symmetry condition. Also to investigate whether the 'late-time' solutions of the time-dependent equations approach solutions of the time-independent Navier-Stokes equations.

The time-dependent problem that is considered is of an impulsively accelerated fluid stream, initially at rest and which contains a stationary solid sphere. This initial condition generates large time-dependent gradients of vorticity near the solid surface. A numerical method which shows stability under these conditions can be taken to provide accurate solutions as the time-dependent vorticity gradients relax. Also since no proof of convergence of the finite-difference equations is available a method can be taken to give consistent results with the differential equations if on starting from one known physical state it leads to another known physical state.

In the present work the Navier-Stokes equations are expressed in a time-dependent stream function-vorticity form. The two equations are then solved simultaneously using a finite-difference method. The stream function-vorticity formulation is advantageous because both the equations are second order partial differential equations, also the vorticity transport equation offers direct analogy with the diffusion equation. Therefore the method of solution for the vorticity transport

equation can be applied to solve the heat or mass transfer equations. The vorticity transport equation is solved using the Peaceman and Rachford alternating direction implicit method. Although, the method chosen complicates the updating of the vorticity field with time it has better numerical stability properties and is known to be accurate in comparison with other explicit or iterative implicit methods. The stream function equation is solved using a point iterative successive over-relaxation method.

The accuracy of the solution is judged by comparison of flow characteristics, such as the drag coefficients, the separation angle and the wake length with experimental values. Also the 'late-time' solutions are compared with the steady state solutions of the Navier-Stokes equations obtained by other workers. At the present moment, a complete error analysis of the solution is not possible. However, the form of errors introduced are known and some idea of the accuracy of solutions can be obtained. Truncation terms of the Taylor's series show what sort of errors are being introduced. For example, central difference approximation of the first vorticity partial derivatives in the convective term introduces 'noise' or dispersive errors in the solution. These errors can be damped down by the introduction of one-sided differences which lead to a solution with higher fluid viscosity than of the real fluid which is simulated.

The computer program and the numerical technique presented in this thesis was tested, first by solving the Navier-Stokes equations at low and intermediate Reynolds numbers ($Re=1, 10, 50$ and 200) when the expected steady state flow field is fairly well established and experimental measurements of the separation angle and the wake length are available. After the solutions at these Reynolds numbers were established solutions at higher Reynolds ($Re=500, 1000$ and 2000) were obtained. At these Reynolds numbers the expected flow pattern is complex and also the numerical difficulties are large due to increased non-linearity of the equations.

CHAPTER 2
COMPUTATIONAL FLUID DYNAMICS

2.1 INTRODUCTION

Theoretical fluid dynamics began with the theory of potential flows of an ideal incompressible fluid (Lamb 1932; Milne-Thomson 1968). Development of this theory extended over a long period and a large class of flows were successfully analysed. However, according to the Euler - D'Alembert paradox (Birkhoff 1968), the total force on a solid object placed in a potential flow is zero.

The inadequacies of the inviscid flow theory motivated the development of viscous flow theory by Navier (1822) and Stokes (1845). Theoretically, the Navier-Stokes equations explain why the potential flow theory was unable to predict drag force correctly but, the equations themselves were unsoluble due to their non-linear character. The few exact solutions that were obtained, contained no non-linear terms. The most important of these solutions was by Stokes (1851) for a steady 'creeping flow' around a solid sphere. The analytic solution completely neglected the inertial or non-linear terms in the Navier-Stokes equations. The predicted drag forces agreed well with experiments for Reynolds number less than unity.

Most practical flows are high Reynolds number flows. Prandtl (1904), observed that for such flows the viscous effect are confined to a thin layer near the solid boundary

and the flow outside this layer could be approximated by potential flow theory. On applying these assumptions, he found that the Navier-Stokes equations reduced from being elliptic to parabolic in space variables and the problem changed from being a boundary value problem to an initial value problem.

Boundary layer theory was a tremendous step forward towards bridging the gap between theoretical and experimental fluid dynamics. Excellent discussions of the theory and its applications are given by Schlichting (1968) and Rosenhead (1963).

The very nature of the boundary layer equations limited their applicability to flows with a predominant direction, which meant that it could not predict the flow pattern within the recirculatory wake behind a bluff body. Also, flows with intermediate Reynolds numbers could not be analysed by either boundary layer theory or by linearization of the Navier-Stokes equations.

The numerical solutions of the full set of Navier-Stokes equations was the only way to obtain insight into such a class of flows. It can be said that the classic paper by Thom (1928) for flow around a circular cylinder at a Reynolds number of 10 marked the beginning of computational fluid dynamics. Progress since then has been remarkable, both from the numerical technique point of view and the number of different problems analysed. This was illustrated by the variety of papers

presented at the International Symposium on High-Speed Computing in Fluid Dynamics at Monterey, California in August, 1968. (The Physics of Fluids, Supplement 11 (1969))

The success of computational fluid dynamics has mainly been in two-dimensional or axi-symmetric flows. However, methods are being developed for tackling three-dimensional problems (Ames 1969). The main difficulty in solving three-dimensional problems is excessive computer memory storage required and numerical stability.

So far, no general numerical method has emerged for the solution of the Navier-Stokes equations. This can be attributed partly to the lack of an effective error analysis of the solutions. The error analysis can form a basis for classification of methods.

The following sections briefly survey relevant existing numerical solutions of the Navier-Stokes equations.

2.2 TIME-INDEPENDENT SOLUTIONS

The vector form of Navier-Stokes equations for steady incompressible Newtonian isothermal flow can be written as:

$$(\bar{U} \cdot \nabla) \bar{U} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \bar{U} \quad \dots \quad (2.2.1)$$

and the continuity equation,

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Notes:

$$\nabla \cdot \bar{U} = 0 \quad (2.2.2)$$

The pressure P in equation (2.2.1) is the modified pressure, therefore the external body force is deriveable from a potential. The boundary condition on the solid surface is defined as:

$$\bar{U}|_s = 0 \quad (2.2.3)$$

Equations (2.2.1), (2.2.2), (2.2.3), the geometry of the solid object and the outer boundary conditions are sufficient to describe flow pattern around an obstacle.

In the case of two-dimensional axi-symmetric flows, the continuity equation can be satisfied by defining a stream function (ψ). On taking the curl of equation (2.2.1), the pressure term can be eliminated. The Navier-Stokes equations can then be written in the stream function-vorticity formulation, (See Chapter 3)

$$\psi_x \zeta_y - \psi_y \zeta_x = \nu \nabla_{xy}^2 \zeta \quad (2.2.4)$$

where the vorticity (ζ) is defined as:

$$\zeta = -\nabla_{xy}^2 \psi \quad (2.2.5)$$

Equations (2.2.4), and (2.2.5), which are the vorticity transport equation and the stream function equation, and must be solved simultaneously.

During the decade 1960-1970, the number of digital simulations reported for steady flows around solid spheres and other bluff bodies such as circular cylinders, have increased considerably. Almost all workers have adopted the stream function-vorticity formulation of the Navier-Stokes equations.

The earlier solutions by Thom (1928; 1932; 1933a; 1933b), Allen and Southwell (1955), and Apelt (1958) for flow around a circular cylinder and those of Kawaguti (1950) and Jenson (1959) for flow around a solid sphere were obtained using desk calculators. The calculational algorithms used were developed by Fox (1944; 1947; 1948), Fox and Southwell (1945), and Allen and Dennis (1951). The relaxation methods developed by these workers have proved to be extremely versatile. Their adaptation to automatic electronic computers to solve equations (2.2.4) and (2.2.5) was presented by Russell (1963).

Kawaguti (1950) was the first to use Thom's method to obtain the solution for steady flow around a sphere at a Reynolds number (Re) of 20, but no flow separation was observed. Jenson (1959) obtained solutions at $Re=5$, 10, 20, and 40, and successfully predicted flow separation behind a sphere at $Re=17$. Unlike Kawaguti, he employed spherical polar co-ordinates. Also by an exponential transformation of the radial co-ordinate he was able to concentrate more grid lines near the sphere surface, yet was able to work with a constant mesh size in the

transformed space. Thus he achieved greater accuracy.

Drag coefficients predicted by Jenson's solutions were generally greater than the experimental values, indicating that the boundary conditions used were not satisfactory. The sphere was assumed to be situated on the axis of a cylindrical pipe of diameter six times that of the sphere. The uniform parallel flow and zero vorticity was specified at the outer boundary condition.

Hamielec et al.(1967) used Jenson's method and extended the solutions upto $Re=100$ using IBM 7040 and 7094 digital computers. Their solutions with a finer mesh than Jenson showed a critical Reynolds number for flow separation of 22 as compared with 17 obtained by Jenson. The drag coefficients predicted were again higher than experimental values. This was clearly due to the location of the outer boundary not being far enough from the sphere surface.

Jenson's method has been thoroughly investigated by a number of workers. Al-Taha (1969) obtained solutions for spheres and oblate spheroids for Reynolds numbers between 0.0001 and 500 and investigated the effect of location of the outer boundary. Le Clair et al.(1970) studied the effect of the outer boundary more rigorously at low Reynolds numbers and showed that as the Reynolds number approaches zero, the drag on a sphere approaches zero via the Oseen drag rather than via the Stokes drag.

Comparison with other flow characteristics e.g. wake length, as predicted by numerical solutions with the available experimental measurements show similar dependence on choice of the location of the outer boundary condition. However, solutions reported for intermediate Reynolds numbers have all been physically realistic.

2.3 TIME-DEPENDENT SOLUTIONS

Time-dependent flows can be simulated by the solution of the following equation with appropriate initial and boundary conditions (c.f. equation 2.2.1).

$$\frac{\partial \bar{U}}{\partial t} + (\bar{U} \cdot \nabla) \bar{U} = - \frac{1}{\bar{\rho}} \nabla P + \nu \nabla^2 \bar{U} \quad (2.3.1)$$

In the case of two-dimensional or axi-symmetric flows, the time-dependent stream function-vorticity form of the Navier-Stokes equations can be written as:

$$\zeta_t + \psi_x \zeta_y - \psi_y \zeta_x = -\nu \nabla_{xy} \zeta \quad (2.3.2)$$

$$\zeta = -\nabla_{xy} \psi \quad (2.3.3)$$

The addition of the time-derivative in equations (2.3.1) and (2.3.2) brings to light some very interesting properties of the Navier-Stokes equations. Due to their non-linearity the 'late-time' solutions may not be the same as the steady state solutions of the equations.

Payne (1958) obtained solutions for an impulsively accelerated flow around a circular cylinder for $Re=40$ and 100. He used equation (2.3.2) in the following form,

$$\zeta_t + (u \zeta)_x + (v \zeta)_y = \nu \nabla_{xy} \zeta \quad (2.3.4)$$

where u and v are the velocity components in the x and y co-ordinate directions. The velocity components (u,v) are calculated by adding the velocity of classical inviscid steady flow to the velocity due to vorticity, given by:

$$u(x,y,t) = -\frac{1}{2\pi} \iint \zeta(X,Y,t) \frac{(y-Y)}{(x-X)(y-Y)} dX.dY \quad (2.3.5)$$

$$v(x,y,t) = -\frac{1}{2\pi} \iint \zeta(X,Y,t) \frac{(x-X)}{(x-X)(y-Y)} dX.dY \quad (2.3.6)$$

Payne used a simple explicit method to obtain the updated vorticity at a new time using equation (2.3.4). The velocity components (u,v) were then obtained from equations (2.3.5) and (2.3.6). This was repeated for each time step using updated values of u and v in equation (2.3.4). Although Payne's objective was to obtain steady state solutions he did not integrate for a sufficiently large times to achieve this, also no asymmetric effects were considered. However, Ingham (1968) repeated Payne's calculations using a finer mesh and integrated for a much longer time to obtain better steady state solutions. Payne's method unnecessarily complicates the solution for velocity components.

Fromm and Harlow (1963) solved equations (2.3.2) and (2.3.3) simultaneously and successfully simulated the development of a 'Vortex-street' behind a vertical flat plate. The vorticity equation was solved by the simple explicit method and the stream function equation was solved by Liebmann's method. Periodic boundary conditions were used and the problem corresponded to an infinite number of rectangular objects placed in series in a two-dimensional channel. The success of Fromm and Harlow initiated a number of studies (Hirota and Miyakoda 1965; Thoman and Szewczyk 1966; Rimon 1968) aimed at simulating time-dependent flows around other bodies, for example, the circular cylinder and the sphere.

The difficulty was and still is, the lack of a complete stability analysis of the finite-difference equations. The normal approach was to study the stability of the linear and non-linear terms in the Navier-Stokes equations independently and then to study the effect of their interaction, hoping that some insight would be gained into the complete stability analysis. In almost all solutions reported in the literature, the initial condition used was that of an impulsively accelerated flow. Under this initial condition very large vorticity gradients occur at early stages of calculation, hence any stability criterion derived by linear analysis is only a guide. For explicit methods, numerical experimentation is the only way to arrive at an empirical stability criterion.

Hirota and Miyakoda (1965) were the first to report the simulation of a Kármán vortex-street behind a circular cylinder at $Re=100$. Their solution is qualitative in nature since the mesh size used is about one-third the cylinder radius. Thoman and Szewczyk (1966) used an upstream directional difference scheme for the non-linear terms and managed to obtain solutions for the range of Reynolds numbers from 1 to 3×10^5 . For Reynolds numbers up to 30 solutions attained steady state. At larger Reynolds numbers asymmetric disturbances were introduced which caused alternate shedding of the vortices behind the cylinder. The accuracy of the solutions was improved by the choice of a cell structure which provided local cell dimensions consistent with the structure of solutions expected. The fluid left the domain of integration, about nine cylinder diameters downstream from the cylinder centre.

Rimon and Cheng (1969) reported time-dependent solutions for an axi-symmetric flow impulsively started from rest around a sphere for a range of Reynolds number from 1 to 1000. They used a time-dependent stream function-vorticity formulation of Navier-Stokes equations together with the Dufort - Frankel explicit method for updating vorticity values. The stream function equation was solved using a successive over-relaxation method and the parallel flow outer boundary condition was satisfied about twenty radii downstream from the centre of sphere.

Rimon and Cheng's solutions showed a number of inconsistencies as compared with the solutions of other workers (Jenson 1959; Al-Taha 1969; Le Clair et al. 1970). For example, the surface pressure at $Re=10$ was physically unrealistic. The reason for this and other inconsistencies was due to the outer boundary condition being improperly satisfied.

In view of the fact that vortex shedding occurs at about $Re=500$ (Torobin and Gauvin 1959). However, the time dependent behaviour of Rimon's solution at $Re=1000$ showed no marked difference from the solutions obtained at Reynolds numbers less than 500. Under experimental conditions, asymmetric disturbances may be responsible for the instability of the wake, as demonstrated by the solutions of Thoman and Szewczyk in case of the circular cylinder. However, Son and Hanratty (1969) using Peaceman and Rachford's alternating direction implicit method for updating the vorticity field and under strict symmetry conditions obtained quite different 'late-time' solutions for the circular cylinder at $Re=500$ as compared with the solutions of the time-dependent Navier-Stokes equations. The properties of the time-dependent Navier-Stokes equations suggest that even under strict symmetry conditions the solution behaviour with time should show a marked difference as the Reynolds number is increased.

CHAPTER 3
THEORETICAL ANALYSIS

3.1 DERIVATION OF EQUATIONS

The Navier-Stokes equations in vector form for an isothermal Newtonian flow are written as:

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\frac{1}{\rho} \nabla P_T + \mathbf{F} + \nu \nabla^2 \mathbf{U} \quad (3.1.1)$$

The Derivation of the above equation starting from the kinematics of fluid motion can be found in Schlichting (1968), Batchelor (1967) and Aris (1962). The main assumptions made in the derivation of equation (3.1.1) are that the fluid is isotropic, Newtonian, and that its properties are continuous functions of space and time.

The Principle of mass conservation leads to the equation of continuity which is expressed as:-

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0 \quad (3.1.2)$$

For an incompressible flow ($\rho = \text{constant}$), the continuity equation simply becomes:-

$$\nabla \cdot \mathbf{U} = 0 \quad (3.1.3)$$

Equations (3.1.1) and (3.1.3) together with initial and boundary conditions complete the description of laminar isothermal flow around an obstacle.

The external body force $\bar{F}$ in equation (3.1.1) can usually be derived from a potential e.g. if gravity is the only body force $\bar{F} = \nabla \cdot (\bar{g} \cdot \bar{X})$. This suggests that the total pressure P_T should be expressed as:

$$P_T = P_0 + \rho \bar{g} \cdot \bar{X} + P \quad (3.1.4)$$

where P_0 is a constant and $P_0 + \rho \bar{g} \cdot \bar{X}$ is the pressure when the fluid is at rest. P is the pressure due to motion and is referred to as the modified pressure. Hence equation (3.1.1) simplifies to:

$$\frac{\partial \bar{U}}{\partial t} + (\bar{U} \cdot \nabla) \bar{U} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \bar{U} \quad (3.1.5)$$

Taking the curl of (3.1.5) and using the vector identities (3.1.6),

$$\begin{aligned} \nabla \times (\nabla \times \bar{A}) &= 0 \\ \nabla \times (\nabla \phi) &= 0 \\ \nabla \times (\bar{A} \times \bar{B}) &= \bar{A} (\nabla \cdot \bar{B}) - \bar{B} (\nabla \cdot \bar{A}) + (\bar{B} \cdot \nabla) \bar{A} - (\bar{A} \cdot \nabla) \bar{B} \\ \bar{A} \times (\nabla \times \bar{A}) &= \frac{1}{2} \nabla (\bar{A} \cdot \bar{A}) - (\bar{A} \cdot \nabla) \bar{A} \end{aligned} \quad (3.1.6)$$

the pressure term in equation (3.1.5) was eliminated which becomes the vorticity transport equation,

$$\frac{\partial \bar{\omega}}{\partial t} + (\bar{U} \cdot \nabla) \bar{\omega} = (\bar{\omega} \cdot \nabla) \bar{U} + \nu \nabla^2 \bar{\omega} \quad (3.1.7)$$

where $\bar{\omega} = (\nabla \times \bar{U})$.

In the case of two-dimensional or axi-symmetric flows the vorticity vector has a component only in the direction of the normal to the plane of flow ($\bar{\omega}(0,0,\zeta)$) so that $(\bar{\omega} \cdot \nabla) \bar{u} = 0$. Hence equation (3.1.7) becomes:

$$\frac{\partial \zeta}{\partial t} + (\bar{u} \cdot \nabla) \zeta = \nu \nabla^2 \zeta \quad (3.1.8)$$

The vorticity transport equation (3.1.8) is analagous to the heat or mass transport equation. Thus the right hand side represents the diffusion of vorticity and the term containing the velocity vector on the left hand side of equation (3.1.8) represents the convection mode of vorticity transport.

If x_1, x_2, x_3 are the three orthogonal curvilinear co-ordinate directions and h_1, h_2, h_3 are the corresponding scale factors, then equation (3.1.8) can be expressed as:

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + \frac{1}{h_1 h_2} \frac{\partial}{\partial x_1} (h_2 u_1 \zeta) + \frac{1}{h_1 h_2} \frac{\partial}{\partial x_2} (h_1 u_2 \zeta) \\ = \nu E^2 (h_3 \zeta) \end{aligned} \quad (3.1.9)$$

$$\text{where } E^2 \equiv \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial x_1} \left(\frac{h_2}{h_1 h_3} \frac{\partial}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\frac{h_1}{h_2 h_3} \frac{\partial}{\partial x_2} \right) \right\}$$

and vorticity(ζ) is defined as:

$$\zeta = \frac{1}{h_1 h_2} \left(\frac{\partial (h_2 u_2)}{\partial x_1} - \frac{\partial (h_1 u_1)}{\partial x_2} \right) \quad (3.1.10)$$

The continuity equation (3.1.3) for two-dimensional or axi-symmetric flow can also be written in orthogonal curvilinear co-ordinates as:

$$\frac{\partial (h_2 h_3 u_1)}{\partial x_1} + \frac{\partial (h_1 h_3 u_2)}{\partial x_2} = 0 \quad (3.1.11)$$

If a stream function (ψ) is defined such that,

$$u_1 = - \frac{1}{h_2 h_3} \frac{\partial \psi}{\partial x_2} \quad (3.1.12)$$

and

$$u_2 = + \frac{1}{h_1 h_3} \frac{\partial \psi}{\partial x_1} \quad (3.1.13)$$

then equation (3.1.11) is eliminated by the substitution into it of the velocity components u_1 and u_2 into equation (3.1.11) i.e. the continuity equation is satisfied. Equations (3.1.9) and (3.1.10) can now be written in terms of the vorticity (ζ) and the stream function (ψ) as:

$$\frac{\partial \zeta}{\partial t} + \frac{1}{h_1 h_2} \left\{ \frac{\partial \psi}{\partial x_1} \frac{\partial}{\partial x_2} \left(\frac{\zeta}{h_3} \right) - \frac{\partial \psi}{\partial x_2} \frac{\partial}{\partial x_1} \left(\frac{\zeta}{h_3} \right) \right\} = \nu E^2 (h_3 \zeta) \quad (3.1.14)$$

$$\zeta = E^2 (\psi) \quad (3.1.15)$$

$$\text{where } E^2 \equiv \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial x_1} \left(\frac{h_2}{h_1 h_3} \frac{\partial}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\frac{h_1}{h_2 h_3} \frac{\partial}{\partial x_2} \right) \right\}$$

To simulate flow around a solid sphere the final equations described above are written in spherical polar co-ordinates, in which $x_1=r$, $x_2=\theta$, $x_3=\phi$, $h_1=1$, $h_2=r$ and $h_3=r \sin \theta$. Therefore velocity components, become:

$$u_r = - \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \quad (3.1.16)$$

$$u_\theta = + \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \quad (3.1.17)$$

the vorticity transport equation becomes:

$$\frac{\partial \zeta}{\partial t} + \frac{1}{r} \left\{ \frac{\partial \psi}{\partial r} \frac{\partial}{\partial \theta} \left(\frac{\zeta}{r \sin \theta} \right) - \frac{\partial \psi}{\partial \theta} \frac{\partial}{\partial r} \left(\frac{\zeta}{r \sin \theta} \right) \right\} = \nabla E^2(\zeta r \sin \theta) \quad (3.1.18)$$

and the stream function equation becomes:

$$\zeta = E^2(\psi) \quad (3.1.19)$$

$$\text{where } E^2 = \frac{1}{r} \left\{ \frac{\partial}{\partial r} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \right) \right\}$$

It is convenient to define a new dependent variable $g = \zeta r \sin \theta$ and solve the equations in terms of this variable rather than the prime vorticity function (ζ). The vorticity transport equation terms of the g function becomes:

$$\frac{\partial g}{\partial t} + \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial \psi}{\partial r} \left[\frac{\partial g}{\partial \theta} - 2 \cot \theta g \right] - \frac{\partial \psi}{\partial \theta} \left[\frac{\partial g}{\partial r} - \frac{2g}{r} \right] \right\} = \nabla E_+^2(g) \quad (3.1.20)$$

and the stream function equation becomes:

$$g = E_+^2(\psi)$$

$$\text{where } E_+^2 = \frac{\partial}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right)$$

It is further convenient, to render the variables dimensionless and to transform the equations from spherical space to rectangular space. It is generally known that a rectangular grid is always more convenient

when deriving finite-difference equations than any other type of grid, also in the case of the sphere the transformation has an added advantage since it allows more grid lines to be concentrated near the sphere surface where the gradients are largest. The dimensionless variables are defined below, (denoted by the superscript *)

$$r^* = \frac{r}{R} ; \quad t^* = \frac{U_\infty t}{R} ; \quad \zeta^* = \frac{R}{U} ; \quad g^* = \frac{g}{U_\infty} \quad (3.1.22)$$

$$\text{and } \psi^* = \frac{\psi}{U_\infty R}$$

For convenience the superscript will be omitted and the * dimensionless variables will be used exclusively.

On substitution of the dimensionless variables and by the transformation $r = e^Z$, the system of equations becomes the following. The velocity components becomes:

$$u_Z = - \frac{1}{e^{2Z} \sin \theta} \frac{\partial \psi}{\partial \theta} \quad (3.1.23)$$

$$u_\theta = \frac{1}{e^{2Z} \sin \theta} \frac{\partial \psi}{\partial Z} \quad (3.1.24)$$

the vorticity transport equation becomes:

$$e^{2Z} \frac{\partial g}{\partial t} + \frac{1}{e^Z \sin \theta} \left\{ \frac{\partial \psi}{\partial Z} \left[\frac{\partial g}{\partial \theta} - 2 \cot \theta g \right] - \frac{\partial \psi}{\partial \theta} \left[\frac{\partial g}{\partial Z} - 2g \right] \right\} = \frac{2}{Re} e^{2Z} E_+^2(g) \quad (3.1.25)$$

the stream function equation becomes:

$$e^{2Z} g = e^{2Z} E_+^2(\psi) \quad (3.1.26)$$

and, the function g becomes:

$$g = \zeta e^Z \sin \theta \quad (3.1.27)$$

where $e^{2Z} E_+^2 \equiv \frac{\partial}{\partial Z^2} - \frac{\partial}{\partial Z} + \frac{\partial}{\partial \theta} - \cot \theta \frac{\partial}{\partial \theta}$

Equations (3.1.23), (3.1.24), (3.1.25), (3.1.26) and (3.1.27) are the final form of the basic set of equations, which are solved with the appropriate initial and boundary conditions (See section 3.2).

The dimensionless parameter Re is the Reynolds number $\left(Re = \frac{2RU_\infty}{\nu} \right)$ based on the diameter of the sphere.

3.2 INITIAL AND BOUNDARY CONDITIONS

The most convenient and meaningful initial condition is that of an impulsively accelerated flow and it can be expressed as:

$$t = -0, \quad \bar{U} = 0 \quad \text{everywhere} \quad (3.2.1)$$

$$t = +0, \quad \bar{U} = \bar{U}_0$$

The initial velocity vector $\bar{U}_0$ satisfies the following conditions,

$$\begin{aligned} \nabla \cdot \bar{U}_0 &= 0 & \text{everywhere} \\ \bar{\omega}_0 = \nabla \times \bar{U}_0 &= 0 \end{aligned} \quad (3.2.2)$$

everywhere in the fluid except at the solid boundary

The above initial condition implies that $\zeta=0$ and $\psi=0$ (equation 3.1.26) initially everywhere, except at the sphere surface. The stream function equation (3.1.26) becomes:

$$e^{2Z} E_+^2(\psi) = 0 \quad (3.2.3)$$

The initial values of vorticity and of the function at the sphere surface are obtained by applying the no-slip condition to the potential flow solution of equation (3.2.3).

The boundary conditions for all times $t>0$ describing a sphere in streaming viscous flow are set as :

Sphere Surface: Applying the no-slip condition at $r = 1$ or $Z = 0$, for $0 \leq \theta \leq \pi$,

$$\bar{u}|_{Z=0} = 0 \Rightarrow \psi=0 ; \frac{\partial \psi}{\partial Z} = 0 ; \frac{\partial^n \psi}{\partial \theta^n} = 0 \quad (3.2.4)$$

therefore from equations (3.1.26) and (3.2.4) the function g at the sphere surface can be calculated from the following relationship,

$$(e^{2Z} g) \Big|_{Z=0} = \frac{\partial^2 \psi}{\partial Z^2} \Big|_{Z=0} \quad (3.2.5)$$

Symmetry axis ($\theta = 0, \pi$): The conditions of axial symmetry are defined as:

$$\psi = 0 ; \frac{\partial^n \psi}{\partial Z^n} = 0 ; \frac{\partial \psi}{\partial \theta} = 0 \quad (3.2.6)$$

for $\theta=0, \pi$ and $0 \leq Z \leq \infty$. The vorticity boundary condition can then be derived from equations (3.1.26) and (3.2.6) as:

$$g = 0 \Rightarrow \zeta = 0 \quad (3.2.7)$$

Inflow boundary: The uniform incoming fluid stream enter the domain of integration at $r = r_\infty$ or $Z = Z_\infty$ for the angle $0 \leq \theta \leq \pi/2$ defined from the upstream symmetry axis and has the stream function and vorticity distribution of:

$$\psi_\infty = \frac{1}{2} e^{2Z_\infty} \sin^2 \theta ; g = 0 \quad (3.2.8)$$

Outflow boundary: The fluid leaves the domain of integration at $r = r_\infty$ or $Z = Z_\infty$ from $\theta = \frac{1}{2} \pi$ to $\theta = \pi$. This outflow boundary condition is the most sensitive of all the boundary conditions. Mathematically the outflow condition is only well defined as $r \rightarrow \infty$, but due to limitations of computer storage and computation time a finite domain of integration has to be used, therefore any boundary condition that is used is necessarily an approximation. In the present work uniform parallel outflow boundary is used or that the outflow boundary which is the same as that used at the inflow boundary. The boundary-value problem thus formed is of the Dirichlet type, hence ease of numerical convergence and uniqueness of the solutions obtained is assured. The disadvantage is that the solutions will be dependent

on the choice of location of the outer boundary. The steady state solutions of Jenson (1959), Al-Taha (1969) and Le Clair et al. (1970) show that this dependency is more serious as far as drag calculations are concerned at low Reynolds numbers. At high Reynolds numbers, the wake length is seriously curtailed and computer solutions are difficult to assess since at present the only way to judge them is comparison with experimental results, which at high Reynolds numbers are ill-defined and difficult to measure precisely. Therefore use of the uniform parallel outflow boundary condition after some numerical experimentation is well justified.

Rimon and Cheng (1969) used the weaker parallel outflow boundary condition: $\frac{\partial \psi}{\partial x} = 0$, $\zeta = 0$ which is analysed by Cheng (1969) who used Burgers' equation as a model for the Navier-Stokes equations. Cheng (1969) concluded that the boundary errors introduced seem to be greater at low Reynolds numbers and this could be one of the reasons why Rimon and Cheng's solution at $Re=10$ was physically unrealistic. The parallel outflow boundary condition changes the boundary-value problem from the Dirichlet type to that of the Mixed type, hence difficulty in obtaining numerical convergence and loss of uniqueness of the solutions obtained can be expected.

In the present work the uniform parallel outflow

boundary condition was chosen to be appropriate. The parallel outflow boundary condition was also used when the solutions approached steady state in order to test the effect of the choice of the outflow boundary condition.

3.3 PROPERTIES OF THE DIFFERENTIAL EQUATIONS

Maximum possible consideration must be given to properties of the differential equations to be solved before any numerical solution is attempted. The present set of equations ((3.1.25), (3.1.26)) are both second order elliptic partial differential equations in the space variables (Z, θ) and equation (3.1.25) is parabolic in time which characterises the problem as being of the non-linear non-stationary kind. It is generally known that a non-linear non-stationary problem may not have a well-behaved solution for the entire time interval $t \geq 0$. Outside of a finite time interval the solution may either 'go to infinity' or else 'split up' , by losing its regularity, ceasing to satisfy the equations and beginning to form branches. Moreover, even if a solution exists for all $t \geq 0$, it may not approach the solution of the stationary problem as the boundary conditions and external forces are stabilized. In fact, depending on the values of the relevant parameters, a non-linear stationary boundary-value problem can have a unique solution, several solutions, or even no solutions at all.

In the general case, the questions of the existence and the uniqueness of the solutions of the Navier-Stokes equations for a given set of initial and boundary conditions cannot be satisfactorily answered. It is normally asserted that if a physical problem is well-posed, there exists a solution, which is unique and which depends continuously on the auxiliary data of the problem. In many cases even this condition seems to be too rigid since satisfactory solutions can be obtained for an ill-posed problem. The present formulation of the problem for time-dependent viscous flow around a solid sphere can be regarded as well-posed and it is reasonable to assume that a solution exists and is unique.

Ladyzhenskaya (1963) presented a number of theorems on the existence and uniqueness of solution of the general system of the Navier-Stokes equations. Some of her basic results concerning the non-linear non-stationary problem are stated below without discussion of the proofs. For the problem described as:

$$\begin{aligned} \frac{\partial \bar{U}}{\partial t} + (\bar{U} \cdot \nabla) \bar{U} &= - \frac{1}{\rho} \nabla p_T + \bar{F}(\bar{x}, t) + \nu \nabla^2 \bar{U} \\ \nabla \cdot \bar{U} &= 0 ; \bar{U}|_s = 0 ; \bar{U}|_{t=0} = \bar{a}(\bar{x}) \end{aligned} \quad (3.3.1)$$

in the space-time domain $Q_T = \Omega \times [0, T]$ and the external force $\bar{F}$ being derivable from a potential. Concerning the initial condition it was assumed that,

$$\nabla \cdot \bar{a}(\bar{x}) = 0 ; \bar{a}|_s = 0 \quad (3.3.2)$$

The basic results concerning the solution of the problem (3.3.1) are the following: If all the data of the problem are independent of one of the co-ordinates x_1, x_2, x_3 then the problem has a unique solution at all instants of time, with no restrictions whatsoever on the smallness of $\bar{F}$, $\bar{a}$ or the domain. The same is true in the three-dimensional problem if there is axial symmetry and if the axis of symmetry does not belong to the domain occupied by the fluid. In the general three-dimensional case, Ladyzhenskaya showed that the problem has a unique solution for all $t \geq 0$ under the condition that the forces $\bar{F}$ are derivable from a potential and that the 'generalized Reynolds number' is less than unity at the initial instant of time. However, if these conditions are not met, then it is proved that the problem has a unique solution for a certain time interval $t \in [0, T]$, whose size could be estimated by starting from the data of the problem.

Ladyzhenskaya defined a generalized Reynolds number as:

$$(\text{Re})_G = \frac{2\varphi C_n^*}{V} \quad (3.3.3)$$

where φ is defined by the integral,

$$\varphi^2(t) = \int_{\Omega} \sum_{k=1}^3 [\bar{u}_{x_k}(\bar{x}, t)]^2 dx \quad (3.3.4)$$

and C_{Ω}^* is defined as:

$$C_{\Omega}^* = 1/\sqrt{\lambda_1} \quad (3.3.5)$$

such that λ_1 , is the smallest eigenvalue of the problem
 $-\nabla^2 \bar{U} = \lambda \bar{U}$ in the domain Ω

In the description of axi-symmetric flow around a solid sphere given by section 3.1 and 3.2, the symmetry axis belongs to the domain occupied by the fluid. However, the strict mathematical symmetry imposed can change the behaviour of the solutions and one can hope for unique solutions within the interval $t \in [0, T]$ where T may possibly be larger in axis-symmetric case than in three-dimensional case.

3.4 CALCULATION OF DRAG COEFFICIENTS

The viscous and form drags were calculated using the following analysis:

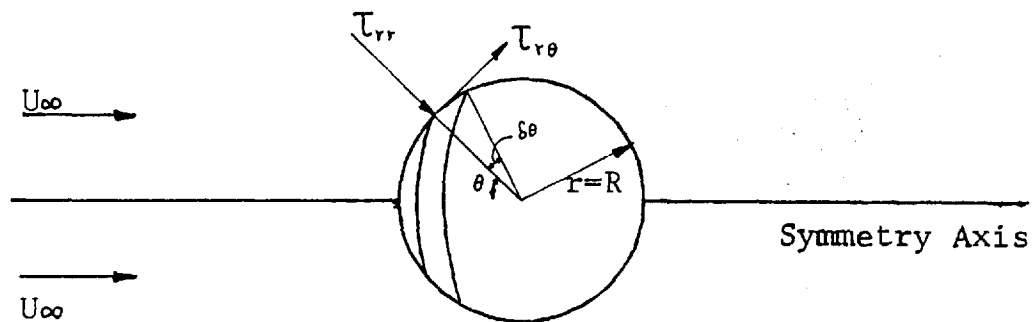


Fig.3.4.1 FLUID STRESSES AT SPHERE SURFACE.

In figure 3.4.1 τ_{rr} and $\tau_{r\theta}$ are the normal and tangential stresses acting on the sphere surface. By resolving these stresses into components parallel and perpendicular to the symmetry axis, the viscous and the form drags can be calculated. The drag coefficient is defined as:

$$C_D = \frac{\text{Drag force}}{\text{Cross-sectional area} \times \frac{1}{2} \rho U_\infty^2} \quad (3.4.1)$$

If the stresses are non-dimensionalised as, $\tau^* = \tau / \frac{1}{2} \rho U_\infty^2$, the viscous or friction drag coefficient C_{DF} can be expressed in terms of dimensionless variables as:

$$C_{DF} = 2 \int_0^\pi \tau_{r\theta}^* \Big|_{r=1} \sin^2 \theta \, d\theta \quad (3.4.2)$$

Starting from the definition of $\tau_{r\theta}$ (Appendix B), it can be shown that,

$$\tau_{r\theta}^* \Big|_{r=1} = \frac{4}{Re} \zeta_s^* \quad (3.4.3)$$

therefore,

$$C_{DF} = \frac{8}{Re} \int_0^\pi \zeta_s^* \sin^2 \theta \, d\theta \quad (3.4.4)$$

Similarly, the form or pressure drag coefficient C_{DP} can be written as:

$$C_{DP} = \int_0^\pi \tau_{rr}^* \Big|_{r=1} \sin 2\theta \, d\theta \quad (3.4.5)$$

or,

$$C_{DP} = \int_0^{\pi} K \sin 2\theta \, d\theta \quad (3.4.6)$$

where $K = (P_s - P_{\infty})/\frac{1}{2}\rho U_{\infty}^2$ in which P_s is the pressure at the sphere surface and P_{∞} is the pressure in the undisturbed fluid upstream at infinity.

The total drag coefficient C_{DT} is expressed as the sum of the two drag coefficients:

$$C_{DT} = C_{DF} + C_{DP} \quad (3.4.7)$$

The retrieval of surface pressure K is carried out using a procedure suggested by Jenson (1959). The procedure requires direct numerical integration of the Navier-Stokes equations, applied at the surface and the symmetry axis.

By application of the Navier-Stokes equations in the θ -directions at the sphere surface ($u_r = 0$, $u_{\theta} = 0$), the expression for the surface pressure can be written as; (Superscript * denoting dimensionless variables is omitted for convenience)

$$K = K_o + \frac{4}{Re} \int_0^{\theta} \left(\frac{\partial \zeta}{\partial r} + \zeta \right)_{r=1} d\theta \quad (3.4.8)$$

or in the transform space (Z, θ)

$$K = K_o + \frac{4}{Re} \int_0^{\theta} \left(\frac{\partial \zeta}{\partial Z} + \zeta \right)_{Z=0} d\theta \quad (3.4.9)$$

The stagnation pressure K_o in equation (3.4.9) is calculated by applying the Navier-Stokes equations in the r -direction at the symmetry axis and simplifying

$$\left(u_\theta = 0, \frac{\partial u_\theta}{\partial \theta} = 0, \frac{\partial u_r}{\partial \theta} = 0, \frac{\partial u_\theta}{\partial r} = 0 \right),$$

$$K_o = 1 + \frac{8}{\text{Re}} \int_0^\infty \left. \frac{1}{r} \frac{\partial \zeta}{\partial \theta} \right|_{\theta=0} dr + 2 \int_0^\infty \frac{\partial u_r}{\partial t} dr \quad (3.4.10)$$

In the transformed space (Z, θ) with u_r expressed in terms of stream function (ψ) the above equation becomes:

$$K_o = 1 + \frac{8}{\text{Re}} \int_0^\infty \left. \frac{\partial \zeta}{\partial \theta} \right|_{\theta=0} dZ - 2 \int_0^\infty \frac{1}{e^Z} \frac{\partial}{\partial t} \left[\left. \frac{\partial^2 \psi}{\partial \theta^2} \right|_{\theta=0} \right] dZ \quad (3.4.11)$$

The numerical technique used to solve the equations derived in this chapter is presented in Chapter 4.

CHAPTER 4
NUMERICAL TECHNIQUE

4.1 INTRODUCTION

The first step in numerical solution of partial differential equations by finite-difference methods is to formulate the original differential problem into a difference problem. The discretization process is normally carried out using Taylor series expansions of the functions about a general node in the mesh covering the domain of integration and relating the function values at nodes to those at neighbouring nodes. If on successively introducing finer meshes as the time step $\Delta t \rightarrow 0$, the difference problem approaches the differential problem the finite-difference formulation is said to be consistent.

The Navier-Stokes equations are based on the conservation of momentum, hence the discretization process must not permit undue accumulation of errors in the flux of momentum when summed over an arbitrary group of cells; in particular, over the entire field of computation. Ideally, the out-flux crossing a cell boundary should identically balance the corresponding influx, crossing the same boundary into the neighbouring cell. The integrals of motion (mass, momentum, and energy) over any part of the field of computation will then be conserved. A difference scheme is said to be conservative if it incorporates the above ideas.

In the case of linear partial differential equations

Lax's equivalence theorem (Richtmyer 1957) states that a stable computation of the difference problem will yield results which converge to the solution of the consistent differential problem. However, there is no analagous theorem for non-linear partial differential equations and since the Navier-Stokes equations are non-linear, then in the absence of a convergence theorem it can only be hoped that a stable, consistent and conservative difference scheme will yield a solution of the consistent differential problem. The conditions of consistency and conservation for a particular difference formulation of the Navier-Stokes equations cannot always be determined. If a difference formulation is conservative and consistent when applied to linear partial differential equations, it is assumed that conservation and consistency is preserved in the case of non-linear partial differential equations such as the Navier-Stokes equations. In the absence of mathematical proofs the above arguments can never be confirmed but can only be contradicted if the solutions obtained are physically unrealistic.

The complexity of the Navier-Stokes equations necessitates such heuristic arguments. Therefore, with this logic as a guide the numerical technique for simulation of viscous flow around a sphere is developed and is presented in the subsequent sections.

4.2 NOTATION

A five-star computational molecule with the following notation for approximation of derivatives is used throughout. (Fig. 4.2.1 and Fig. 4.2.2)

If $f(Z, \theta, t)$ is any continuous function to be approximated at the node i, j and has derivatives of all orders, then the function value and its derivatives can be represented by:

$$f_{i,j}^n = f((i-1)h, (j-1)k, n\Delta t) \quad (4.2.1)$$

$$\begin{aligned} \left(\frac{\partial f}{\partial Z} \right)_{i,j}^n &= \partial_i(f_{i,j}^n) + o[h^2] = \frac{1}{2h} \Delta_i(f_{i,j}^n) + o[h^2] \\ &= \frac{1}{2h} (f_{i+1,j}^n - f_{i-1,j}^n) + o[h^2] \end{aligned} \quad (4.2.2)$$

$$\begin{aligned} \left(\frac{\partial^2 f}{\partial Z^2} \right)_{i,j}^n &= \partial_i^2(f_{i,j}^n) + o[h^2] = \frac{1}{h^2} \Delta_i^2(f_{i,j}^n) + o[h^2] \\ &= \frac{1}{h^2} (f_{i+1,j}^n + f_{i-1,j}^n - 2f_{i,j}^n) + o[h^2] \end{aligned} \quad (4.2.3)$$

or in general,

$$\left(\frac{\partial^k f}{\partial Z^k} \right)_{i,j}^n = \partial_i^k(f_{i,j}^n) + o[h^2] \quad (4.2.4)$$

Similarly the partial derivatives with respect to θ can be represented as:

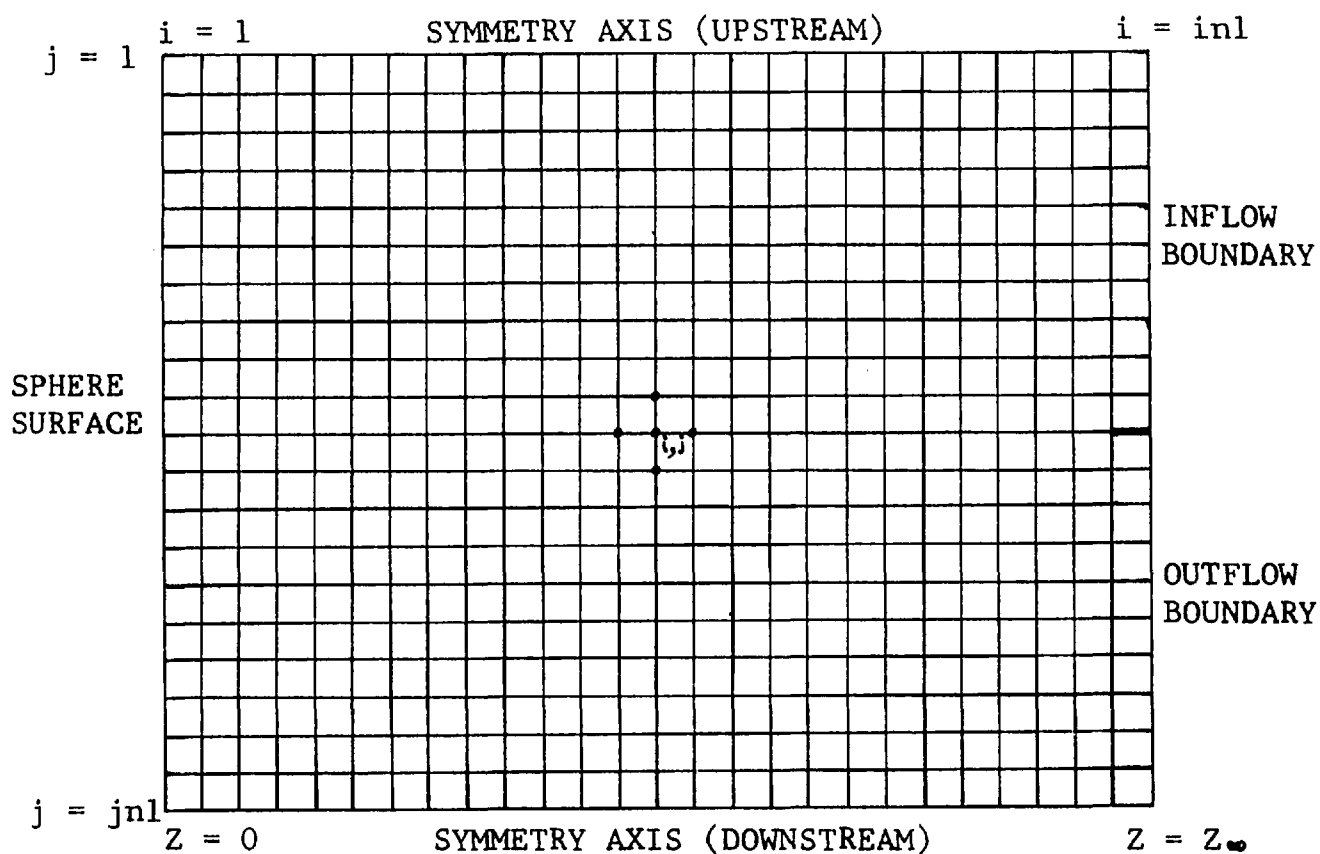


Fig. 4.2.1 RECTANGULAR GRID COVERING THE FLOW FIELD

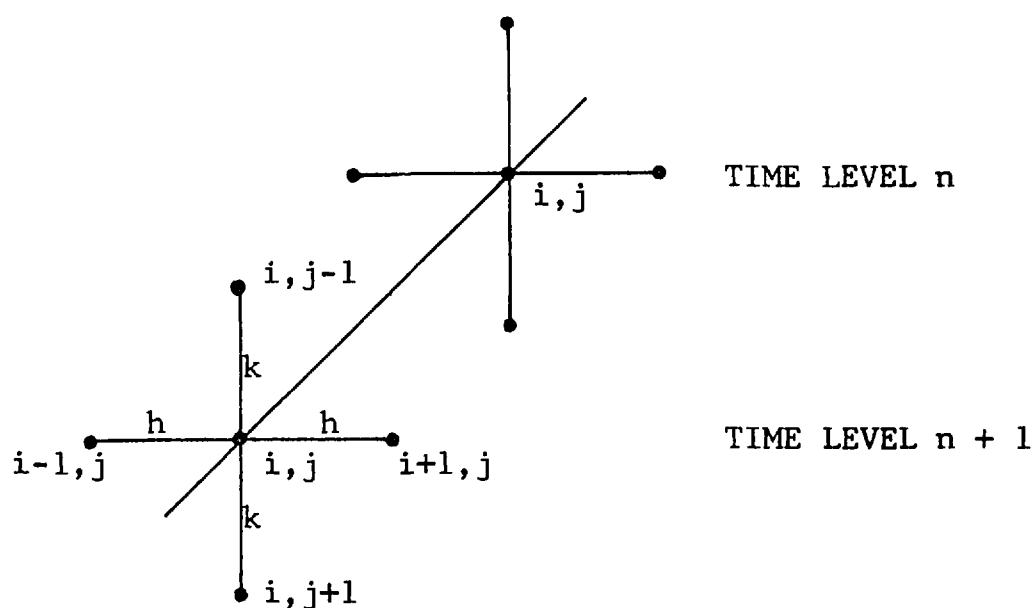


Fig. 4.2.2 A FIVE-POINT COMPUTATIONAL MOLECULE

$$\left(\frac{\partial f}{\partial \theta}\right)_{i,j}^n = \partial_j(f_{ij}^n) + o[k^2] \quad (4.2.5)$$

$$\left(\frac{\partial^2 f}{\partial \theta^2}\right)_{i,j}^n = \partial_j^2(f_{ij}^n) + o[k^2] \quad (4.2.6)$$

The time derivatives are expressed as:

$$\left(\frac{\partial^k f}{\partial t^k}\right)_{i,j}^n = \partial_n^k(f_{ij}^n) + o[\Delta t^2] \quad (4.2.7)$$

The derivatives are said to be approximated at the node i, j and at the n time-step. Each derivative approximation is of second order accuracy and is derived by using Taylor series expansion of the function about the node i, j i.e. central differencing is used. With the present notation the first order forward difference scheme for the time derivative can be written as:

$$\frac{1}{\Delta t} (f_{i,j}^{n+1} - f_{i,j}^n) = \partial_n(f_{ij}^{n+\frac{1}{2}}) \quad (4.2.8)$$

i.e. second order central difference about $n + \frac{1}{2}$ step.

The truncation error terms e.g. $o(h)$ are interpreted as:

$$\lim_{h \rightarrow 0} \left(\frac{\partial f}{\partial z}\right)_{i,j}^n = \partial_i(f_{ij}^n) \quad (4.2.9)$$

or

$$\lim_{h \rightarrow 0} o[h^2] = 0 \quad (4.2.10)$$

thus the consistency condition is satisfied by each derivative. Therefore the finite-difference equations derived with the above approximation are assumed to be consistent with the differential equation. The truncation error is formally defined at each node as:

$$\epsilon_{f_{ij}^n}^T = f_{i,j}^n - F_{i,j}^n \quad (4.2.11)$$

where $F_{i,j}^n$ is the numerical solution of the difference problem.

Biased differencing of first order derivatives

The geometrical interpretation of the first partial derivative is that it is the slope, in the direction of the derivative, of the tangent plane to the surface being approximated. Consider figure 4.2.3 as an illustration of this interpretation.

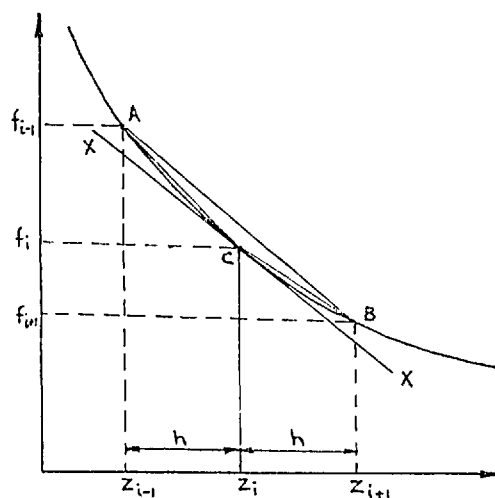


Fig. 4.2.3 GRADIENT APPROXIMATION

It is required to approximate the slope of the tangent XX at the point C or Z_i . If central difference Taylor series approximation is used, the slope of XX is same as that of line AB. The slope of AB is calculated by taking an arithmetic mean of the slopes of lines AC and CB i.e.

$$\left(\frac{df}{dz}\right)_i \approx \frac{1}{2h}(f_{i+1}-f_{i-1}) = \frac{1}{2h}(f_{i+1}-f_i) + \frac{1}{2h}(f_i-f_{i-1}) \quad (4.2.12)$$

$$= \text{slope AB} = \frac{1}{2} \text{ slope CB} + \frac{1}{2} \text{ slope AC}$$

The central difference approximation (4.2.12) will be good if the function is fairly smooth and h is sufficiently small. A better and more general approximation would be, if a more flexible averaging process is used. Therefore equation (4.2.12) is re-written as:

$$\left(\frac{df}{dz}\right)_i \approx \frac{W}{h}(f_{i+1}-f_i) + \frac{(1-W)}{h}(f_i-f_{i-1}) \quad (4.2.13)$$

where W can be considered to be a weighting factor. If $W=0.5$ equations (4.2.12) and (4.2.13) are identical. Any other value of W between zero and unity can be called a 'biased difference scheme'. The choice of W for a particular node can be made in a variety of ways, the simplest being the 'upstream biased' difference scheme. In this scheme if the flow direction at C is from A to C, then AC is taken to approximate XX and $W = 0$, but if the flow is from B to C, then BC is

taken to approximate $\Delta\Delta$ and $W = 1$. Clearly errors introduced by this scheme would be greater in most cases except when the gradients are very small in which both schemes are equally good. If there is some knowledge of the types of gradients to be approximated e.g. vorticity gradients are large near the sphere surface and very small near the outer boundary, a 'mixed scheme', comprising central and upstream differences for first order derivatives can be devised to yield a better flow simulation than with either of the two schemes used exclusively.

Using the weighting factor W , equations (4.2.2) and (4.2.5) can now be expressed as:

$$\left(\frac{\partial f}{\partial z}\right)_{i,j}^n = W \partial_i(f_{i+\frac{1}{2}j}^n) + (1-W) \partial_i(f_{i-\frac{1}{2}j}^n) + o[h] \quad (4.2.14)$$

$$\left(\frac{\partial f}{\partial \theta}\right)_{i,j}^n = W \partial_j(f_{ij+\frac{1}{2}}^n) + (1-W) \partial_j(f_{ij-\frac{1}{2}}^n) + o[k] \quad (4.2.15)$$

The notation system outlined so far is used to derive finite-difference equations for the Navier-Stokes equations as expressed in Chapter 3.

4.3 DERIVATION OF FINITE-DIFFERENCE EQUATIONS

In chapter 3 the form in which the vorticity transport equation is solved was derived. The equation (3.1.25) and the stream function equation (3.1.26)

are re-written below as:

the vorticity transport equation,

$$\frac{Re}{2} e^{2Z} \frac{\partial g}{\partial t} + \frac{Re}{2e^Z \sin \theta} \left\{ \frac{\partial \psi}{\partial Z} \left[\frac{\partial g}{\partial \theta} - 2 \cot \theta g \right] - \frac{\partial \psi}{\partial \theta} \left[\frac{\partial g}{\partial Z} - 2g \right] \right\}$$

$$= \frac{\partial^2 g}{\partial Z^2} - \frac{\partial g}{\partial Z} + \frac{\partial^2 g}{\partial \theta^2} - \cot \theta \frac{\partial g}{\partial \theta} \quad (4.3.1)$$

and the stream function equation,

$$\frac{\partial^2 \psi}{\partial Z^2} - \frac{\partial \psi}{\partial Z} + \frac{\partial^2 \psi}{\partial \theta^2} - \cot \theta \frac{\partial \psi}{\partial \theta} = g \quad (4.3.2)$$

where $g = \zeta e^Z \sin \theta$

The finite-difference approximation or discretization of equations (4.3.1) and (4.3.2) can formally be expressed as:

$$\begin{aligned} & \frac{Re}{2} e^{2Z_i} \frac{\partial g}{\partial t}_{i,j}^n + \frac{Re}{2} \frac{1}{e^{Z_i} \sin \theta_j} \left\{ \left(\frac{\partial \psi}{\partial Z} \right)_{i,j}^n \left[\left(\frac{\partial g}{\partial \theta} \right)_{i,j}^n - 2 \cot \theta_j g_{i,j}^n \right] \right. \\ & \quad \left. - \left(\frac{\partial \psi}{\partial \theta} \right)_{i,j}^n \left[\left(\frac{\partial g}{\partial Z} \right)_{i,j}^n - 2g_{i,j}^n \right] \right\} \\ & = \left(\frac{\partial^2 g}{\partial Z^2} \right)_{i,j}^n - \left(\frac{\partial g}{\partial Z} \right)_{i,j}^n + \left(\frac{\partial^2 g}{\partial \theta^2} \right)_{i,j}^n - \cot \theta_j \left(\frac{\partial g}{\partial \theta} \right)_{i,j}^n \end{aligned} \quad (4.3.3)$$

$$\left(\frac{\partial^2 \psi}{\partial Z^2} \right)_{i,j}^n - \left(\frac{\partial \psi}{\partial Z} \right)_{i,j}^n + \left(\frac{\partial^2 \psi}{\partial \theta^2} \right)_{i,j}^n - \cot \theta_j \left(\frac{\partial \psi}{\partial \theta} \right)_{i,j}^n = g_{i,j}^n \quad (4.3.4)$$

The discretization process is carried out at a typical node i, j in the mesh covering the domain

of integration and at the n time step. In equation (4.3.4) time does not appear explicitly, this means that any computational procedure must solve equation (4.3.3) first to obtain values of $g_{i,j}^{n+1}$ and equation (4.3.4) is then solved as a Poisson's equation to obtain values of $\psi_{i,j}^{n+1}$. If $\psi_{i,j}^{n+1}$ values are also required in equation (4.3.3) the entire step must be repeated until both the equations are satisfied simultaneously.

Assuming that an adequate mesh is chosen, i.e. h and k are sufficiently small, then the truncation error terms are negligible. Then a simple second order explicit finite-difference equation for the vorticity transport equation (4.3.3) can be written as:

$$\begin{aligned} \frac{Re}{2} e^{2Z_i} \partial_n(g_{ij}^n) + \frac{Re}{2} \frac{1}{e^{Z_i} \sin \theta_j} \left\{ \partial_i(\psi_{ij}^n) \left[\partial_j(g_{ij}^n) - 2 \cot \theta_j g_{ij}^n \right] \right. \\ \left. - \partial_j(\psi_{ij}^n) \left[\partial_i(g_{ij}^n) - 2 g_{ij}^n \right] \right\} = \partial_i^2(g_{ij}^n) - \partial_i(g_{ij}^n) + \partial_j^2(g_{ij}^n) \\ - \cot \theta_j \partial_j(g_{ij}^n) \end{aligned} \quad (4.3.5)$$

Richtmyer (1957) states that a second order time centered explicit scheme for the pure diffusion equation i.e. the linear terms, is unconditionally unstable. Hence the addition of the first partial derivatives in non-linear convective terms in equation (4.3.5) should render the above simple explicit scheme unconditionally unstable. However, the first order simple explicit scheme

is conditionally stable i.e.

$$\begin{aligned} \frac{\text{Re}}{2} e^{2Z_i} \partial_n(g_{ij}^{n+\frac{1}{2}}) + \frac{\text{Re}}{2} \frac{1}{e^{Z_i} \sin \theta_j} \left\{ \partial_i(\psi_{ij}^n) \left[\partial_j(g_{ij}^n) - 2 \cot \theta_j g_{ij}^n \right] \right. \\ \left. - \partial_j(\psi_{ij}^n) \left[\partial_i(g_{ij}^n) - 2 g_{ij}^n \right] \right\} = \partial_i^2(g_{ij}^n) - \partial_i(g_{ij}^n) + \partial_j^2(g_{ij}^n) \\ - \cot \theta_j \partial_j(g_{ij}^n) \end{aligned} \quad (4.3.6)$$

Linearizing equation (4.3.6) or assuming that the Reynolds number (Re) is small, it can be shown that the explicit scheme (4.3.6) is computationally stable if

$$\frac{a \Delta t}{e^{2Z_i} \text{Re}} \leq \frac{1}{2} \quad (\text{section 4.4})$$

Dufort and Frankel in 1953 suggested a stable second order explicit scheme. The second partial derivatives were approximated in the following manner, e.g.

$$\partial_i^2(g_{ij}^n)_{\text{DF}} = \partial_i^2(g_{ij}^n) + \frac{1}{h^2} [2g_{i,j}^n - (g_{i,j}^{n+1} + g_{i,j}^{n-1})] \quad (4.3.7)$$

and

$$\partial_j^2(g_{ij}^n)_{\text{DF}} = \partial_j^2(g_{ij}^n) + \frac{1}{k^2} [2g_{i,j}^n - (g_{i,j}^{n+1} + g_{i,j}^{n-1})] \quad (4.3.8)$$

Therefore the Dufort-Frankel scheme for the solution of equation (4.3.1) can be written as:

$$\frac{\text{Re}}{2} e^{2Z_i} \partial_n(g_{ij}^n) + \frac{\text{Re}}{2} \frac{1}{e^{Z_i} \sin \theta_j} \left\{ \partial_i(\psi_{ij}^n) \left[\partial_j(g_{ij}^n) - 2 \cot \theta_j g_{ij}^n \right] \right.$$

(4.3.9)

$$\begin{aligned}
& - \partial_j(\psi_{ij}^n) \left[\partial_i(g_{ij}^n) - 2g_{ij}^n \right] \Bigg\} = \partial_i^2(g_{ij}^n)_{DF} \quad (4.3.9) \\
& - \partial_i(g_{ij}^n) + \partial_j^2(g_{ij}^n)_{DF} - \cot \theta_j \partial_j(g_{ij}^n)
\end{aligned}$$

In the absence of non-linear convective terms, the Dufort-Frankel scheme is unconditionally stable. However, in practice in order to obtain sufficient accuracy a limitation on the time step Δt has to be imposed similar to that of the explicit scheme (4.3.6).

The Dufort-Frankel scheme requires larger computer storage than the conditionally stable simple explicit scheme. The choice between, the two schemes is usually made from an assessment of computer time and storage requirements.

An implicit scheme requiring the same storage as the Dufort-Frankel is better both for accuracy and numerical stability. A general iterative implicit scheme for solving the vorticity transport equation (4.3.1) can be written as:

$$\begin{aligned}
& \frac{\text{Re}}{2} e^{2Z_i} \partial_n(g_{ij}^{n+\frac{1}{2}}) + \frac{\text{Re}}{2} \frac{1}{e^{Z_i \sin \theta_j}} \left\{ \overline{\partial_i(\psi_{ij}^n)} \left[\overline{\partial_j(g_{ij}^n)} \right. \right. \\
& \left. \left. - 2 \cot \theta_j g_{ij}^n \right] - \overline{\partial_j(\psi_{ij}^n)} \left[\overline{\partial_i(g_{ij}^n)} - 2g_{ij}^n \right] \right\} = \overline{\partial_i^2(g_{ij}^n)} \\
& - \overline{\partial_i(g_{ij}^n)} + \overline{\partial_j^2(g_{ij}^n)} - \cot \theta_j \overline{\partial_j(g_{ij}^n)}
\end{aligned} \quad (4.3.10)$$

where $\partial_j(g_{ij}^n) = \alpha \partial_j(g_{ij}^n) + (1-\alpha) \partial_j(g_{ij}^n)$,

where α is weighting factor. All the other terms are expressed in a similar manner. Equation (4.3.10) can be solved by successive over or, under-relaxation technique and wherever required the most up to date values of the stream function can be used and these are obtained by successive over-relaxation sweeps with the stream function equation (4.3.4). The most common value of the weighting factor α used is one-half and the scheme is normally called the Crank-Nickolson scheme. The above implicit scheme converges very rapidly when the Reynolds number is small but for large Reynolds numbers, the number of iterations required become large as reported by Pearson (1965). For large Reynolds number flows an alternating direction implicit method with better stability and convergence properties is more suitable. Numerical experimentation with alternating direction methods shows them to be better than other methods, they require less computer storage and are more rapidly convergent.

The present work was initiated with the intention of solving the time-dependent Navier-Stokes equations for as large a Reynolds number as possible, but keeping the computer storage and time to minimum. For these reasons the original alternating direction implicit, (ADI) method proposed by Peaceman and Rachford (1955) was adopted. It is unconditionally stable for the linear diffusion equation

no stability analysis exists when the method is applied to the Navier-Stokes equations. Solutions obtained by Son and Hanratty (1969) for flow around a circular cylinder using the Peaceman and Rachford ADI method indicate its suitability for simulating flow around an obstacle. Son and Hanratty used the ADI method such that the surface vorticities were calculated at the $n + 1$ time step in a manner similar to an explicit method for the solution of the vorticity transport equation. In their technique the surface vorticities are not consistent at all time steps with the vorticity values in the adjacent fluid. The method described in this thesis overcomes this defect with a minimum number of additional iterations and since it approaches the steady state faster than Son and Hanratty's technique, it is believed that there is an overall saving of the computer time required to obtain the steady state solution for a particular Reynolds number. The method used will now be described.

Peaceman and Rachford Alternating Direction Implicit Method

In equation (4.3.10) all derivatives that are approximated require values of the function g at the unknown $n + 1$ time level, this means a large number of non-linear algebraic equations have to be solved iteratively. However, if derivatives in one of the directions are approximated at an unknown $n + \frac{1}{2}$ level, while the derivatives in the other direction are

approximated at the known n th. level, the sets of simultaneous equations thus formed can be solved easily without iteration. If the procedure is repeated for the next half step such that the derivatives previously approximated at the time step are now approximated at the $n + 1$ step and other derivatives are approximated at the $n + \frac{1}{2}$ time step level by the values obtained previously. The vorticity values can be obtained at the $n + 1$ level without iteration. The overall procedure is unconditionally stable if the non-linear convective terms are neglected. Thus Peaceman and Rachford ADI method has similar stability properties as those of the Dufort-Frankel method and those of the iterative implicit method.

In most cases it is unimportant, as to which derivatives must be approximated at $n + \frac{1}{2}$ time step, but in the present case the choice of the first direction can be used to save overall computational time. In section 3.2 it is stated that the sphere surface boundary condition for vorticity is a function of time and must be approximated. However, the symmetry axis boundary condition is constant and known; thus if the derivatives which involve the constant boundary condition are approximated at the $n + \frac{1}{2}$ level and then the iterations are only performed over the last half time step. This means that the derivatives with respect to angle (θ) must be approximated at the $n + \frac{1}{2}$ time step level.

The finite-difference equations can now be derived

for the vorticity transport equation (4.3.1) using Peaceman and Rachford ADI method as follows:

The first half time step in the θ -direction can be written as:

$$\begin{aligned} \frac{\text{Re}}{2} e^{2Z_i} \partial_n(g_{ij}^{n+\frac{1}{4}}) + \frac{\text{Re}}{2} \frac{1}{e^{Z_i \sin \theta_j}} \left\{ \overline{\partial_i(\psi_{ij}^n)} \left[\partial_j(g_{ij}^{n+\frac{1}{2}}) \right. \right. \\ \left. \left. - 2 \cot \theta_j g_{ij}^{n+\frac{1}{2}} \right] - \overline{\partial_j(\psi_{ij}^n)} \left[\partial_i(g_{ij}^n) - 2g_{ij}^n \right] \right\} = \partial_i^2(g_{ij}^n) \\ - \partial_i(g_{ij}^n) + \partial_j^2(g_{ij}^{n+\frac{1}{2}}) - \cot \theta_j \partial_j(g_{ij}^{n+\frac{1}{2}}) \end{aligned} \quad (4.3.11)$$

Similarly, the last half time step in the z -direction can be written as:

$$\begin{aligned} \frac{\text{Re}}{2} e^{2Z_i} \partial_n(g_{ij}^{n+\frac{3}{4}}) + \frac{\text{Re}}{2} \frac{1}{e^{Z_i \sin \theta_j}} \left\{ \overline{\partial_i(\psi_{ij}^n)} \left[\partial_j(g_{ij}^{n+\frac{1}{2}}) \right. \right. \\ \left. \left. - 2 \cot \theta_j g_{ij}^{n+\frac{1}{2}} \right] - \overline{\partial_j(\psi_{ij}^n)} \left[\partial_i(g_{ij}^{n+1}) - 2g_{ij}^{n+1} \right] \right\} = \partial_i^2(g_{ij}^{n+1}) \\ - \partial_i(g_{ij}^{n+1}) + \partial_j^2(g_{ij}^{n+\frac{1}{2}}) - \cot \theta_j \partial_j(g_{ij}^{n+\frac{1}{2}}) \end{aligned} \quad (4.3.12)$$

Equation (4.3.11) can be expressed more clearly in the following manner,

$$\begin{aligned}
& \frac{\operatorname{Re} e^{2Z_i}}{\Delta t} g_{ij}^{n+\frac{1}{2}} + \frac{\operatorname{Re}}{2} \frac{\overline{\partial_i(\psi_{ij}^n)}}{e^{Z_i \sin \theta_j}} \left[\partial_j(g_{ij}^{n+\frac{1}{2}}) - 2 \cot \theta_j g_{ij}^{n+\frac{1}{2}} \right] \\
& - \partial_j^2(g_{ij}^{n+\frac{1}{2}}) + \cot \theta_j \partial_j(g_{ij}^{n+\frac{1}{2}}) = \frac{\operatorname{Re} e^{2Z_i}}{\Delta t} g_{ij}^n + \frac{\operatorname{Re}}{2} \frac{\overline{\partial_j(\psi_{ij}^n)}}{e^{Z_i \sin \theta_j}} \\
& \left[\partial_i(g_{ij}^n) - 2g_{ij}^n \right] + \partial_i^2(g_{ij}^n) - \partial_i(g_{ij}^n)
\end{aligned}
\tag{4.3.13}$$

Similarly equation (4.3.12) becomes:

$$\begin{aligned}
& \frac{\operatorname{Re} e^{2Z_i}}{\Delta t} g_{ij}^{n+1} - \frac{\operatorname{Re}}{2} \frac{\overline{\partial_j(\psi_{ij}^n)}}{e^{Z_i \sin \theta_j}} \left[\partial_i^2(g_{ij}^{n+1}) - 2g_{ij}^{n+1} \right] - \partial_i(g_{ij}^{n+1}) \\
& + \partial_i(g_{ij}^{n+1}) = \frac{\operatorname{Re} e^{2Z_i}}{\Delta t} g_{ij}^{n+\frac{1}{2}} - \frac{\operatorname{Re}}{2} \frac{\overline{\partial_i(\psi_{ij}^n)}}{e^{Z_i \sin \theta_j}} \left[\partial_j(g_{ij}^{n+\frac{1}{2}}) \right. \\
& \left. - 2 \cot \theta_j g_{ij}^{n+\frac{1}{2}} \right] + \partial_j^2(g_{ij}^{n+\frac{1}{2}}) - \cot \theta_j \partial_j(g_{ij}^{n+\frac{1}{2}})
\end{aligned}
\tag{4.3.14}$$

Equation (4.3.13) can be expanded into sets of simultaneous equations of the following form.

$$C_{1j} p_{j-1}^{n+\frac{1}{2}} + C_{2j} p_j^{n+\frac{1}{2}} + C_{3j} p_{j+1}^{n+\frac{1}{2}} = D_j^n
\tag{4.3.15}$$

where,

$$D_j^n = C_{1i} P_{i-1}^n + C_{2i} P_i^n + C_{3i} P_{i+1}^n \quad (4.3.16)$$

for $i = 2, 3, \dots, i_n$ and $j = 2, 3, \dots, j_n$, where $i_n + 1$ and $j_n + 1$, are the total numbers of grid lines in the z and θ directions respectively. P is a general working variable substituted for the required g values.

Similarly, equation (4.3.14) can be expanded into sets of simultaneous equations of the form,

$$C'_{1i} P_{i-1}^{n+1} + C'_{2i} P_i^{n+1} + C'_{3i} P_{i+1}^{n+1} = D_i^{n+\frac{1}{2}} \quad (4.3.17)$$

where,

$$D_i^{n+\frac{1}{2}} = C'_{1j} P_{j-1}^{n+\frac{1}{2}} + C'_{2j} P_j^{n+\frac{1}{2}} + C'_{3j} P_{j+1}^{n+\frac{1}{2}} \quad (4.3.18)$$

The coefficients of equations (4.3.15), (4.3.16), (4.3.17) and (4.3.18) have been derived for both the central difference approximation of the first derivatives in the non-linear convective term and the upstream biased difference of the same derivatives. The coefficients for both approximations may be expressed as:

$$C_{1j} = -b_{4j} - KG_{ij} \left[w_1 \overline{\Delta_i(\psi_{ij})}_p + w_2 \overline{\Delta_i(\psi_{ij})}_n \right] \quad (4.3.19)$$

$$C_{2j} = \frac{1}{X_i} + \frac{2}{ak^2} - KG_{ij} \left[w_3 \overline{\Delta_i(\psi_{ij}^n)}_p - w_3 \overline{\Delta_i(\psi_{ij}^n)}_n \right] \quad (4.3.20)$$

$$+ 4k \cot \theta_j \overline{\Delta_i(\psi_{ij}^n)} \Big] \quad (4.3.20)$$

$$C_{3j} = -b_{3j} + KG_{ij} \left[W_1 \overline{\Delta_i(\psi_{ij}^n)_n} + W_2 \overline{\Delta_i(\psi_{ij}^n)_p} \right] \quad (4.3.21)$$

$$C_{1j} = b_{2i} - KG_{ij} \left[W_1 \overline{\Delta_j(\psi_{ij}^n)} + W_2 \overline{\Delta_j(\psi_{ij}^n)_p} \right] \quad (4.3.22)$$

$$C_{2i} = \frac{1}{X_i} - \frac{2}{ah} - KG_{ij} \left[W_3 \overline{\Delta_j(\psi_{ij}^n)_n} - W_3 \overline{\Delta_j(\psi_{ij}^n)_p} \right. \\ \left. + 4h \overline{\Delta_j(\psi_{ij}^n)} \right] \quad (4.3.23)$$

$$C_{3i} = b_{1i} + KG_{ij} \left[W_1 \overline{\Delta_j(\psi_{ij}^n)_p} + W_2 \overline{\Delta_j(\psi_{ij}^n)_n} \right] \quad (4.3.24)$$

$$C'_{1j} = -C_{1j} \quad (4.3.25)$$

$$C'_{2j} = -C_{2j} + \frac{2}{X_i} \quad (4.3.26)$$

$$C'_{3j} = -C_{3j} \quad (4.3.27)$$

$$C'_{1i} = -C_{1i} \quad (4.3.28)$$

$$C'_{2i} = -C_{2i} + \frac{2}{X_i} \quad (4.3.29)$$

$$C'_{3i} = -C_{3i} \quad (4.3.30)$$

The weighting factors W_1 , W_2 and W_3 in the coefficient equations satisfy the following relations,

$$W_1 + W_2 = 1 \quad (4.3.31)$$

$$W_3 = W_2 - W_1 \quad (4.3.32)$$

Therefore $W_1 = W_2 = 0.5$, the above coefficient become the central difference coefficients for both the convective and the diffusion term in the vorticity transport equation. If $W_1 = 1$ and $W_2 = 0$, the upstream biased differences are introduced in the convective term. Introduction of biased differences in this form is advantageous since it offers greater generality. The terms involving the stream function differences in equation (4.3.19) to (4.3.24) are defined as follows:

$$\overline{\Delta_i(\psi_{ij}^n)}_p = \overline{\Delta_i(\psi_{ij}^n)} + |\overline{\Delta_i(\psi_{ij}^n)}| \quad (4.3.33)$$

$$\overline{\Delta_i(\psi_{ij}^n)}_n = \overline{\Delta_i(\psi_{ij}^n)} - |\overline{\Delta_i(\psi_{ij}^n)}| \quad (4.3.34)$$

$$\overline{\Delta_j(\psi_{ij}^n)}_p = \overline{\Delta_j(\psi_{ij}^n)} + |\overline{\Delta_j(\psi_{ij}^n)}| \quad (4.3.35)$$

$$\overline{\Delta_j(\psi_{ij}^n)}_n = \overline{\Delta_j(\psi_{ij}^n)} - |\overline{\Delta_j(\psi_{ij}^n)}| \quad (4.3.36)$$

The above definitions of the stream function differences ensure that one of the parts in the coefficients is set to zero depending on the direction of velocity. In the calculation of the vorticity values the most up to date values of the stream function are used to evaluate the above differences.

The other coefficients used in the finite-difference equations derived so far are defined as:

$$b_{1i} = \frac{1}{a} \left(\frac{1}{h^2} - \frac{1}{2h} \right) \quad (4.3.37)$$

$$b_{2i} = \frac{1}{a} \left(\frac{1}{h^2} + \frac{1}{2h} \right) \quad (4.3.38)$$

$$b_{3j} = \frac{1}{a} \left(\frac{1}{k^2} - \frac{\cot \theta_j}{2k} \right) \quad (4.3.39)$$

$$b_{4j} = \frac{1}{a} \left(\frac{1}{k^2} + \frac{\cot \theta_j}{2k} \right) \quad (4.3.40)$$

$$a = \frac{2}{h^2} + \frac{2}{k^2} \quad (4.3.41)$$

$$X_i = \frac{a \Delta t}{e^{2Z_i} \text{Re}} \quad (4.3.42)$$

$$KG_{ij} = \frac{C_R}{h_{3ij}} \quad (4.3.43)$$

where $C_R = \frac{\text{Re}}{8hka}$ and $h_{3ij} = e^{Z_i} \sin \theta_j$

The sets of equations (4.3.15) and (4.3.17) are solved by Thomas's method as presented by Lapidus (1962) and is briefly described here. Consider a system of n simultaneous of the form,

$$\begin{aligned} C_{21} P_1 + C_{31} P_2 &= D_1 \\ C_{1r} P_{r-1} + C_{2r} P_r + C_{3r} P_{r+1} &= D_r \quad r=2, \dots, n-1 \\ C_{1n} P_{n-1} + C_{2n} P_n &= D_n \end{aligned} \quad (4.3.44)$$

or in matrix notation (4.3.44) is written as:

$$\underline{C} \cdot \underline{P} = \underline{D} \quad (4.3.45)$$

The matrix $\underline{C}$ is tridiagonal and the equations can be solved by Gaussian elimination from the first equation onwards by letting,

$$\begin{aligned} w_1 &= C_{21} \\ q_{r-1} &= \frac{C_{3r-1}}{w_{r-1}} \quad r=2, 3, \dots, n \\ w_r &= C_{2r} - C_{1r} q_{r-1} \end{aligned} \quad (4.3.46)$$

$$\begin{aligned} \text{and } x_1 &= \frac{D_1}{w_1} \\ x_r &= \frac{D_r - C_{1r} x_{r-1}}{w_r} \quad r=2, 3, \dots, n \end{aligned} \quad (4.3.47)$$

Equations (4.3.46) and (4.3.47) transform (4.3.44) such that,

$$\begin{aligned} P_n &= x_n \\ P_r &= x_r - q_r P_{r+1} \quad r=1, 2, \dots, n-1 \end{aligned} \quad (4.3.48)$$

If w , q and x are calculated in order of increasing r , it follows that (4.3.48) can be used to calculate the P_r in order of decreasing r , that is, $x_n, x_{n-1}, \dots, x_2, x_1$.

In the present work the stream function equation(4.3.2)

is solved by successive over-relaxation method and the finite-difference equation is derived, first by expressing equation (4.3.2) as:

$$\partial_i(\psi_{ij}^{n+1}) - \partial_i(\psi_{ij}^{n+1}) + \partial_j(\psi_{ij}^{n+1}) - \cot \theta_j \partial_j(\psi_{ij}^{n+1}) = g_{ij}^{n+1} \quad (4.3.49)$$

then, on expansion of (4.3.49), as:

$$\begin{aligned} \psi_{i,j}^{*n+1} = & b_{1i} \psi_{i+1,j}^{n+1} + b_{2i} \psi_{i-1,j}^{n+1} + b_{3j} \psi_{i,j+1}^{n+1} + b_{4j} \psi_{i,j-1}^{n+1} \\ & - C_i g_{i,j}^{n+1} \end{aligned} \quad (4.3.50)$$

where $C_i = \frac{e^{2Z_i}}{a}$. On introducing the stream function relaxation factor ω_ψ into (4.3.50) the finite-difference equation becomes:

$$\psi_{i,j}^{r+1} = (1 - \omega_\psi) \psi_{i,j}^r + \omega_\psi \psi_{i,j}^{*n+1} \quad (4.3.51)$$

where the counter r refers to the number of successive point iterations performed.

Equations (4.3.15), (4.3.17) and (4.3.51) together with the associated coefficient equations are the set of finite-difference equations that are solved to simulate the flow around a sphere.

4.4

STABILITY AND CONVERGENCE

If $g(z, \theta, t)$ is the exact solution of the initial value problem as defined in Chapter 3, and $G_{i,j}^n$ is the solution of the finite-difference equations derived in section 4.3. The error of approximation $\epsilon_{g_{i,j}}^n$ is given by:

$$\epsilon_{g_{i,j}}^n = g_{i,j}^n - G((i-1)h, (j-1)k, n\Delta t) \quad (4.4.1)$$

There are two questions which arise about the behaviour of the error of approximation. First, what happens to $\epsilon_{g_{i,j}}^n$ as $n \rightarrow \infty$? Secondly, what is its behaviour as $h, k, \Delta t \rightarrow 0$? The first question is one of numerical stability and the numerical method is considered to be stable if $\epsilon_{g_{i,j}}^n$ remains bounded over the entire time domain. The second question is of convergence and the method is said to be convergent if the error tends to zero. According to Lax's equivalence theorem for linear equations, the condition of numerical stability is both necessary and sufficient to establish convergence.

In the present complex case of solutions of the Navier-Stokes equations, numerically stable solutions are said to be convergent, unless the solutions themselves are physically unrealistic. Therefore a great deal of numerical experimentation is necessary to establish a convergent solution.

It is necessary to have some criterion to obtain stable solutions. In the present work the linear criterion

was found to be sufficient in the initial stages of calculation when the time-dependent vorticity gradients near the sphere surface are greatest. Rimon(1968) used the Dufort-Frankel method and found that the linear criterion $\frac{\nu \delta t}{h} \leq \frac{1}{4}$ suggested by Fromm (1964) was inadequate and did not guarantee that stable solutions could be obtained. After extensive experimentation, he suggested that the relative change in the vorticity in one time step $(\zeta_{i,j}^{n+1} - \zeta_{i,j}^n) / \zeta_{i,j}^n$ should not exceed a certain empirical value which depended on the size of the initial gradient. For flow around a sphere he suggested this empirical value to be about 1/20 of the maximum surface vorticity. In the present author's opinion this condition is unnecessarily strict. The linear criterion derived below is less stringent and has been found to be successful for the present ADI method and therefore shows the effectiveness of the present method over the Dufort-Frankel method in that larger time steps can be used.

Equation (4.3.6) for first order simple explicit method can be expanded as:

$$g_{i,j}^{n+1} = (1-2X_i)g_{i,j}^n + 2X_i \left\{ b_{1i}g_{i+1,j}^n + b_{2i}g_{i-1,j}^n + b_{3j}g_{i,j+1}^n + b_{4j}g_{i,j-1}^n \right\} - 2X_i D_{ij}^n \quad (4.4.2)$$

$$\text{where } D_{ij}^n = \frac{C_R}{h_{3ij}} \left\{ \Delta_i(\psi_{ij}^n) \left[\Delta_j(g_{ij}^n) - 4k \cot \theta_j g_{ij}^n \right] \right\} \quad (4.4.3)$$

$$- \Delta_j(\psi_{ij}^n) \left[\Delta_i(g_{ij}^n) - 4hg_{ij}^n \right] \Bigg\} \quad (4.4.3)$$

and,

$$X_i = \frac{a \Delta t}{e^{2Z_i} \text{Re}}$$

For 'creeping flow' the convective term is negligible and equation (4.4.2) becomes the pure diffusion equation in terms of the quantity g . Therefore it can be written as:

$$g_{i,j}^{n+1} = (1-2X_i)g_{i,j}^n + 2X_i \left\{ b_{1i}g_{i+1,j}^n + b_{2j}g_{i-1,j}^n + b_{3j}g_{i,j+1}^n + b_{4j}g_{i,j-1}^n \right\} \quad (4.4.4)$$

Comparison of equation (4.4.4) with the vorticity diffusion equation of Fromm (1964) the stability condition for the above equation is,

$$0 < X_i \leq \frac{1}{2} \quad \text{for all } i \quad (4.4.5)$$

The above condition can also be obtained from fundamental considerations outlined by Richtmyer (1957) and Ames (1969) which state that for stability all the coefficients in (4.4.4) should be positive.

This stability condition is only applied for the initial time steps after which the size of the time step is chosen to minimize computational time. This means that as the solution approaches steady state, much larger

time steps are used than indicated by (4.4.5). The strategy employed is that since most of the computing time is spent on the solution of the stream function equation therefore a time step is chosen such that a reasonable number of iterations are required to solve the stream function equation (between 50 and 100, depending on the Reynolds number and mesh size). As the solution approaches steady state the number of iterations decrease, at this stage the length of the time step is increased (usually doubled) to restore the original selected number of iterations.

4.5 APPROXIMATION OF BOUNDARY CONDITIONS

Sphere Surface Boundary Condition

In Chapter 3 Section 3.2, the sphere surface vorticity boundary condition was derived as:

$$(e^{2Z}g) \Big|_{Z=0} = \frac{\partial^2 \psi}{\partial Z^2} \Big|_{Z=0} \quad (4.5.1)$$

or,

$$g_{1,j}^{n+1} = \left(\frac{\partial^2 \psi}{\partial Z^2} \right)_{1,j}^{n+1}$$

where $g_{1,j}^{n+1} = \zeta_{1,j}^{n+1} \sin \theta_j$

The stream function derivative in (4.5.1) is evaluated by fitting a cubic polynomial to the stream function values at the nodes near the surface. Although both quartic and

and quintic polynomial formulae were tried, the variation in the values of the surface vorticity was not significant.

The equation used to evaluate (4.51) using a cubic polynomial can be written as:

$$g_{1,j}^{n+1} = \frac{1}{2h^2} (8\psi_{2,j}^{n+1} - \psi_{3,j}^{n+1}) \quad (4.5.2)$$

The quartic polynomial formula tried to evaluate (4.5.1) was the following.

$$g_{1,j}^{n+1} = \frac{1}{h^2} (6\psi_{2,j}^{n+1} - \frac{3}{2}\psi_{3,j}^{n+1} + \frac{2}{9}\psi_{4,j}^{n+1}) \quad (4.5.3)$$

and the quintic polynomial formula tried can be written as:

$$g_{1,j}^{n+1} = \frac{1}{72h^2} (576\psi_{2,j}^{n+1} - 216\psi_{3,j}^{n+1} + 64\psi_{4,j}^{n+1} - 9\psi_{5,j}^{n+1}) \quad (4.5.4)$$

Outflow Boundary Condition

The normal outflow boundary condition chosen in Chapter 3 was a uniform parallel flow, hence no difficulty arose in specifying this condition. To test the effect of variation of this boundary condition, parallel flow was specified when the solution had almost reached steady state.

The parallel outflow condition can be expressed as:

$$\frac{\partial \psi}{\partial x} = 0 \quad (4.5.5)$$

Where the x-direction is parallel to the symmetry axis.

Equation (4.5.5) in Z- θ co-ordinates becomes:

$$\frac{\partial \psi}{\partial Z} = \tan \theta \frac{\partial \psi}{\partial \theta} \quad (4.5.6)$$

The finite-difference formused of equation (4.5.6) was:

$$\psi_{inl,j}^{*n+1} = \psi_{in,j}^{n+1} = \frac{h}{2k \cot \theta} (\psi_{inl,j+1}^{n+1} - \psi_{inl,j-1}^{n+1}) \quad (4.5.7)$$

or,

$$\psi_{inl,j}^{r+1} = (1 - \omega_{\psi_b}) \psi_{inl,j}^r + \omega_{\psi_b} \psi_{inl,j}^{*n+1} \quad (4.5.8)$$

Where ω_{ψ_b} is the relaxation factor used at the outflow boundary.

Rimon (1968) used the parallel outflow boundary conditon for time-dependent viscous flow around a sphere and derived the following finite-difference equation on the assumption that the stream lines are parallel between the outer grid lines in and in+1,

$$\psi_{inl,j}^{n+1} = \psi_{in,j}^{n+1} - \left[\arcsin \left(\frac{e^{Z_{inl}} \sin \theta}{e^{Z_{in}}} \right) - \theta_j \right] \cdot \frac{1}{k} (\psi_{in,j}^{n+1} - \psi_{in,j-1}^{n+1}) \quad (4.5.9)$$

The vorticity $\zeta_{inl,j}^{n+1}$ at the outflow boundary condition was set to zero. Rimon's formula (4.5.9) was also tested, especially for $Re=10$ at which Rimon found his solution to be physically unrealistic. The vorticity at the boundary is non-zero when the parallel outflow boundary condition is used, but since the adjacent nodes on the boundary show vorticity values to be less than h^2 (the order of local truncation error) then vorticity values calculated by double numerical differentiation of the stream function are likely to introduce larger errors in the solution than those introduced by setting the boundary vorticity to zero.

4.6 COMPUTATIONAL PROCEDURE

The calculational algorithm is based on the assumption that the entire numerical field of the stream function and vorticity is known at some dimensionless time $n\Delta t$ or at the n th time step level, and it is required to determine the complete flow field at a new $n+1$ time step level.

The primary function of the main computational procedure is to solve equations (4.3.15), (4.3.17) and (4.3.51) and will now be described, starting from the initial condition i.e. $t=0$ or $n=0$.

(i) The initial values of the stream function ($\psi_{i,j}^0$) are obtained from the potential flow solution, given by:

$$\psi_{i,j} = \frac{1}{2} \left(1 - \frac{1}{e^{3Z_i}} \right) e^{2Z_i \sin^2 \theta_j} \quad (4.6.1)$$

The boundary conditions used are given in Chapter 3. The vorticity and function $g_{i,j}^o$ are set to zero everywhere in the computational flow field.

The point successive over-relaxation sweeps are then carried out using equation (4.3.51) until the following convergence criterion is satisfied,

$$\left| \psi_{i,j}^{r+1} - \psi_{i,j}^r \right| \leq \epsilon_\psi \quad (4.6.2)$$

The over-relaxation parameter ω_ψ is specified using the following equation given by Russell (1963) and found to be very near the optimum.

$$\omega_\psi = \frac{2}{1 + \pi \sqrt{[(inl^2 + jnl^2)/2]}} \quad (4.6.3)$$

For most solutions the stream function convergence criterion $\epsilon_\psi = 10^{-6}$ was used. However, for $Re=1000$, and $Re=2000$ solutions when the computer time became large, $\epsilon_\psi = 10^{-5}$ was used. Tests at various Reynolds numbers showed that both criteria were acceptable.

The surface vorticities or g values are then calculated using equation (4.5.2). This completes the initial solution and it can now be used to advance the velocity field in time.

(ii) The first step in updating the solution in time is to solve equation (4.3.13) as sets of simultaneous equations, given by (4.3.15) and (4.3.16). The solution provides the $g_{i,j}^{n+\frac{1}{2}}$ or vorticity values at the $n+\frac{1}{2}$ time level. The values of the stream function at the previous time step are used for this part of the procedure.

(iii) To calculate g values at the new $n+1$ time level, the surface vorticity values at the $n+1$ level are required. Since they are unknown, a first approximation is made, that is, the surface vorticities at the two time levels are assumed to be the same. Equation (4.3.14) is then solved as sets of simultaneous equations given by equations (4.3.17) and (4.3.18); again the stream function values available from the previous time step are used whenever required.

(iv) The stream function equation (4.3.49) is then solved using successive over-relaxation (SOR) sweeps with the finite-difference equation (4.3.51) until criterion (4.6.2) is satisfied. The stream function solution of the previous time step is used as an initial guess in the SOR technique. The surface g values are then calculated using equation (4.5.2).

(v) The newly calculated surface values of g from step (iv) are compared with the assumed values. If the difference is unacceptable the procedure is repeated from step (iii), until the following criterion is satisfied.

$$\left| g_{1,j}^{r+1} - g_{1,j}^r \right| \leq \epsilon_g \quad (4.6.4)$$

In repeating step (iii) the new available stream function values at the $n+1$ time level are used. The convergence criterion $\epsilon_g = 10^{-3}$ is used for most runs. However, numerical experiments showed that $\epsilon_g = 10^{-2}$ is also acceptable. In order to minimize the number of iterations required for convergence of the g values a surface relaxation factor (ω_g) is used such that,

$$g_{1,j}^{r+1} = (1 - \omega_g) g_{1,j}^r + \omega_g g_{1,j}^{*n+1} \quad (4.6.5)$$

where $g_{1,j}^{*n+1}$ is calculated by (4.5.2) in step (iv).

On successive iterations from step (iii) the number of SOR iterations required to solve the stream function equation decreased rapidly. With the proper combination of ϵ_g and ω_g , the overall convergence on the vorticity can be obtained in three to four iterations. The number of iterations required depends on Δt and Reynolds number, also when the time-dependent surface gradients are large, the total number of iterations is large. At low Reynolds numbers a relaxation factor of 0.5 on the surface vorticity gives good rates of convergence. This factor tends towards unity as the Reynolds number increases (See Table 1).

(vi) The new calculated values of the stream function $\psi_{i,j}^{n+1}$ and the function $g_{i,j}^{n+1}$ of the entire flow field now

replace the 'old' values i.e. the $n+1$ time level becomes the new n th time level and the entire procedure is repeated from step (ii), until the desired number of time steps are calculated or the set computing time is exhausted.

(vii) At the end of a desired number of time steps, drag coefficients and the surface pressure profiles are calculated using equations (4.4.4), (3.4.6), (3.4.7), (3.4.9) and (3.4.11). The numerical integrations in these equations are carried out using the trapezoidal rule. The vorticity gradients at the sphere surface and the upstream symmetry axis are normally evaluated by fitting a fourth order polynomial to the neighbouring nodes. Numerical tests at each Reynolds number are carried out to investigate the effect of the order of polynomials in the evaluation of the surface vorticity gradients.

(viii) At certain time intervals stream function and vorticity contours are plotted by the line-printer, the program for which was developed as part of the present simulation. The stream function and vorticity contour plots together with drag coefficients enable the program user to decide whether or not to proceed with the simulation.

(ix) As there is no simple way to establish from one time step to another whether or not the simulation is proceeding satisfactorarily, and also to gain greater insight into the numerical technique used, a high degree of user participation is necessary. This is achieved by

transferring at the end of each run the flow field in the computer-memory on to a magnetic tape. At the start of the next run the data are read-in and manipulated according to the decisions of the user which are based on the output of the previous run.

(x) A constant time step Δt is used over any one run. The whole algorithm provides complete freedom to change the step size whenever desirable i.e. either to minimize computing time for a complete solution over the entire time domain or to follow rapid changes in the flow field

The computer program based on the computational procedure described above is given in Appendix A and the numerical results obtained are presented in Chapter 5.

CHAPTER 5
NUMERICAL RESULTS

5.1 INTRODUCTION

Numerical solutions of the Navier-Stokes equations for flow around a solid sphere were obtained using the numerical technique developed in Chapter 4. In the present chapter the computer results for Reynolds numbers (Re) of 1, 10, 50, 200, 500, 1000 and 2000 are presented and the various difficulties encountered are discussed.

There are seven arbitrary constants which must be specified for each Reynolds number solution. In addition there is the time step Δt which is chosen using a numerical stability criterion. These constants are h , k , r_∞ , ϵ_ψ , ϵ_g , ω and ω_g . The mesh or grid constants h and k are chosen by numerical experimentation or from existing steady state solutions of the Navier-Stokes equations. The work by Al-Taha (1969) is very useful in this respect. As Al-Taha noted, the criterion $h, k \leq \frac{4}{Re}$ for stable steady state solutions given by Jenson (1959) is unnecessarily strict and can only be used at low Reynolds numbers. At high Reynolds numbers the computer storage and time becomes excessive if Jenson's criterion for choosing the mesh size is applied.

In the present work, the constants h , k and r_∞ were chosen from the results of Al-Taha (1969) up to Reynolds number (Re) of 500. For higher Reynolds numbers the results obtained up to $Re=500$ were extrapolated in order to assign these constants. For all Reynolds numbers

the choice of the mesh size was a compromise between solution accuracy and computing economy. The relaxation factors ω_ψ and ω_g were chosen to minimize computational time as discussed in chapter 3 section 6. The convergence criteria ϵ_ψ and ϵ_g were chosen by numerical experimentation and their values, together with the values of the other constants used, are given in Table 1.

Once a stable numerical solution has been obtained the most important question is; what is its accuracy? This depends on several factors, such as, the finite-difference scheme, the boundary condition approximations, the mesh size and the convergence criterion used. Assuming that a finite-difference scheme, and the mesh size have been chosen, the error in the solution is then the sum of the truncation error and the round-off error.

The truncation error depends on the difference scheme and the mesh size used. In the present work Taylor's series approximations for the derivatives are used, therefore the form of the truncation error introduced is known. The truncation error introduced by the first partial derivatives can be written as:

$$\left(\frac{\partial f}{\partial Z}\right)_{i,j}^n = \partial_i(f_{ij}^n) + o[h^2]_1 \quad (5.1.1)$$

where the truncation error term $o[h^2]_1$ can be expressed for Taylor's series as:

$$O[h^2]_1 = -\frac{h^2}{3!} \left(\frac{\partial^3 f}{\partial z^3} \right)_{i,j}^n - \frac{h^4}{5!} \left(\frac{\partial^5 f}{\partial z^5} \right)_{i+1,j}^n \quad 0 < l_i < 1 \quad (5.1.2)$$

Similarly, the truncation error term in the θ -direction is written as:

$$O[k^2]_1 = -\frac{k^2}{3!} \left(\frac{\partial^3 f}{\partial \theta^3} \right)_{i,j}^n - \frac{k^4}{5!} \left(\frac{\partial^5 f}{\partial \theta^5} \right)_{i,j+1}^n \quad 0 < l_i < 1 \quad (5.1.3)$$

The truncation error terms introduced by the second partial derivatives are expressed as:

$$\left(\frac{\partial^2 f}{\partial z^2} \right)_{i,j}^n = \partial_i (f_{ij}^n) + O[h^2]_2 \quad (5.1.4)$$

where,

$$O[h^2]_2 = -\frac{2h^2}{4!} \left(\frac{\partial^4 f}{\partial z^4} \right)_{i,j}^n - \frac{2h^4}{6!} \left(\frac{\partial^6 f}{\partial z^6} \right)_{i+1,j}^n \quad 0 < l_i < 1 \quad (5.1.5)$$

The truncation error term for the θ -direction is written as:

$$O[k^2]_2 = -\frac{2k^2}{4!} \left(\frac{\partial^4 f}{\partial \theta^4} \right)_{i,j}^n - \frac{2k^4}{6!} \left(\frac{\partial^6 f}{\partial \theta^6} \right)_{i,j+1}^n \quad 0 < l_i < 1 \quad (5.1.6)$$

The truncation terms (5.1.2) and (5.1.3) for the first order derivatives consist of odd order derivatives. The error introduced in the solution of the vorticity transport equation (4.3.1) by the approximation of the first derivatives is not diffusive and appears as waves of very small amplitude. These waves are noticeable near

the outer boundary where the vorticity tends towards zero. These dispersive errors are responsible for numerical instability. The solution remains stable if the amplitude of these waves remains small as the integration proceeds in time. However, by the artificial dampening of these dispersive errors, i.e. on introduction of 'pseudo viscosity', it is possible to obtain stable solutions. These solutions are meaningful only when the artificial viscosity introduced is less than the real viscosity of the fluid.

The truncation terms (5.1.5) and (5.1.6) for the second order derivatives consist of even order derivatives and the error introduced in the solution is diffusive and is generally small. The combined errors of all the truncation terms of the Navier-Stokes equations form a complex picture. It can be said, that there is some kind of an error distribution over the entire domain, which means that the flow simulation is more exact in some part of the flow field than in other parts. The accuracy of the simulation depends on the size of the local derivatives and the fineness of the grid.

The third generation computers are able to work with more than eight decimal places therefore, the round-off error introduced by the computer does not present any problem. The present work was done on a CDC 6600 machine which works with fourteen decimal places. The round-off errors introduced by specifying a particular convergence criterion is small, since the criterion is selected to

give a stable solution with a relative error of less than one percent.

In the absence of a rigorous error analysis the question of how the solutions should be assessed is crucial. Thom (1928) in his classic paper suggested to calculate the pressure in the flow field by numerical integration of the Navier-Stokes equations along two arbitrary paths to check the accuracy of numerical solutions. This has been tried unsuccessfully by various workers and by this author. The pressure evaluated at a point in the flow field is dependent on the path chosen. This is explained by the existence of an error distribution, as discussed earlier. The errors of numerical integration together with the accumulation of errors along different paths give different results. This test suggested by Thom is very strict and very few solutions can satisfy it.

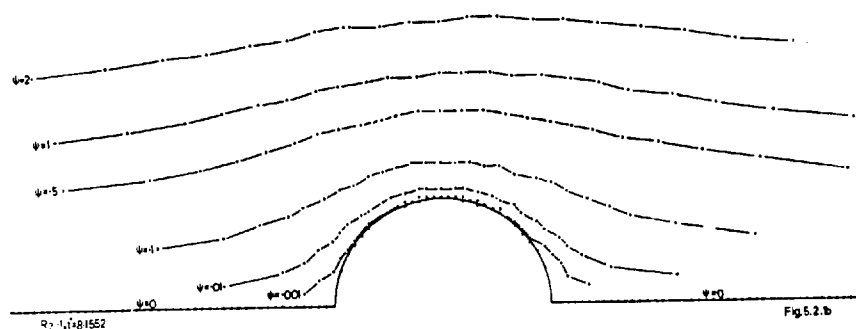
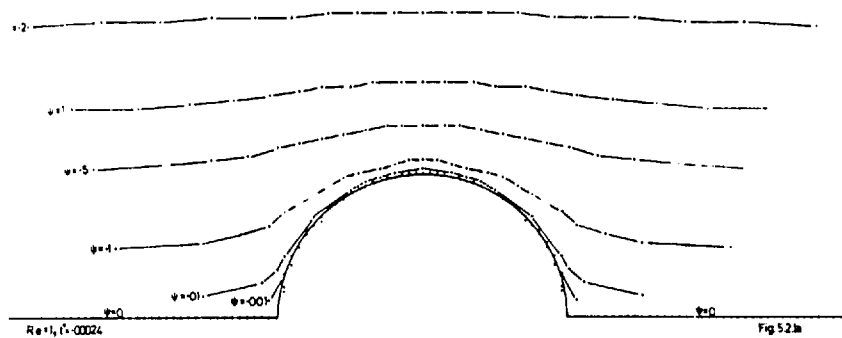
In the authors opinion most numerical solutions of the Navier-Stokes equations including the present work give a good approximation of the local velocities. Using this velocity field it is possible to obtain a good local approximation of the first velocity derivative. The second and higher velocity derivatives add significant errors of numerical differentiation and hence any integral of motion which involves the second derivative of the velocity field will yield grossly inadequate results.

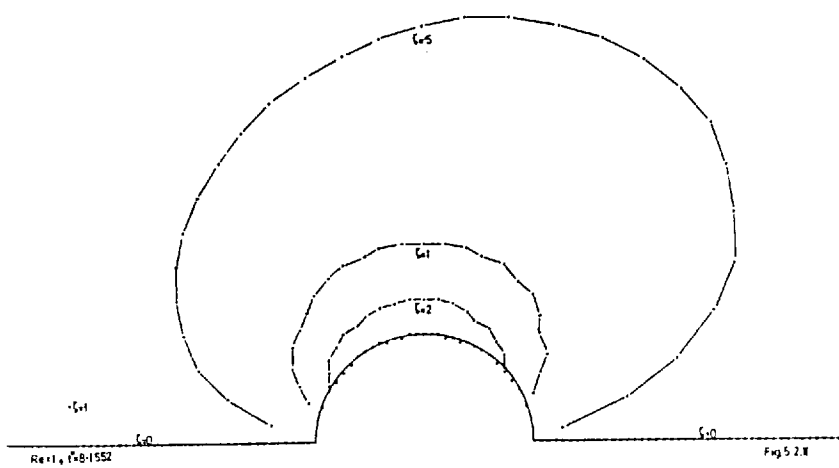
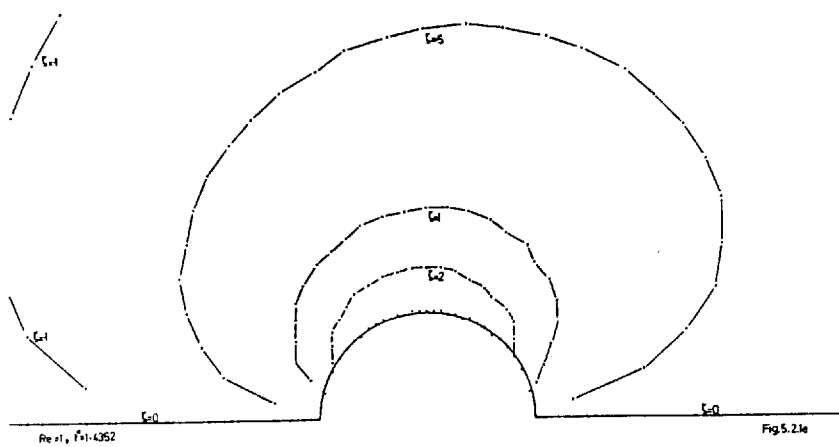
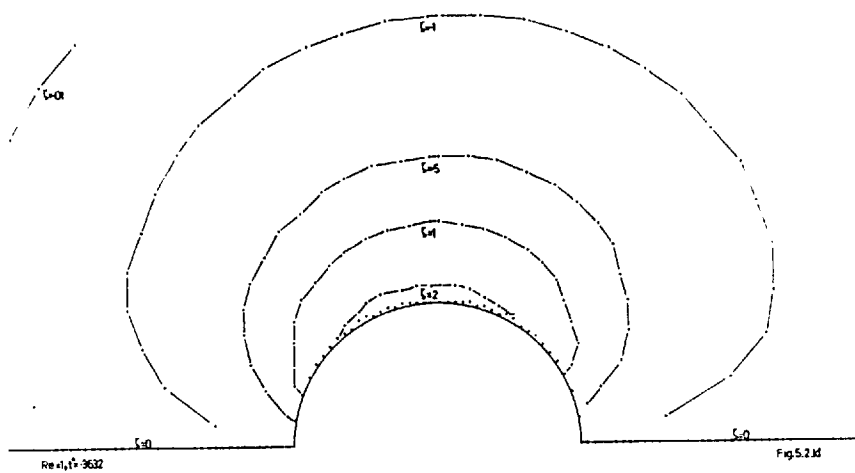
Solutions presented in subsequent sections are obtained starting with the potential flow initial condition. The uniform parallel outflow boundary condition is selected, however, parallel outflow boundary condition is also tested for $Re=1$, 10 and 500. A central difference scheme is adopted for both the convective and the diffusion terms in the vorticity transport equation, but in order to reduce the 'noise' level in the solution, the upstream biased differences are introduced in the convective term at $Re=50$ and 500. Also a 'mixed biased' difference scheme is tested at $Re=50$, which yields a better solution as compared to either the complete central difference solution or the upstream difference solution. The effect of variations in mesh size has been studied at $Re=500$.

The solutions at $Re=1000$ and 2000 provide an insight into the growth of numerical instabilities. The distinction between the physical instabilities i.e. those instabilities which show the properties of the Navier-Stokes equations and numerical instabilities is made by the rate at which each instability grew. The numerical instability is taken to grow at a much faster rate than the physical instability.

5.2 REYNOLDS NUMBER 1 SOLUTION

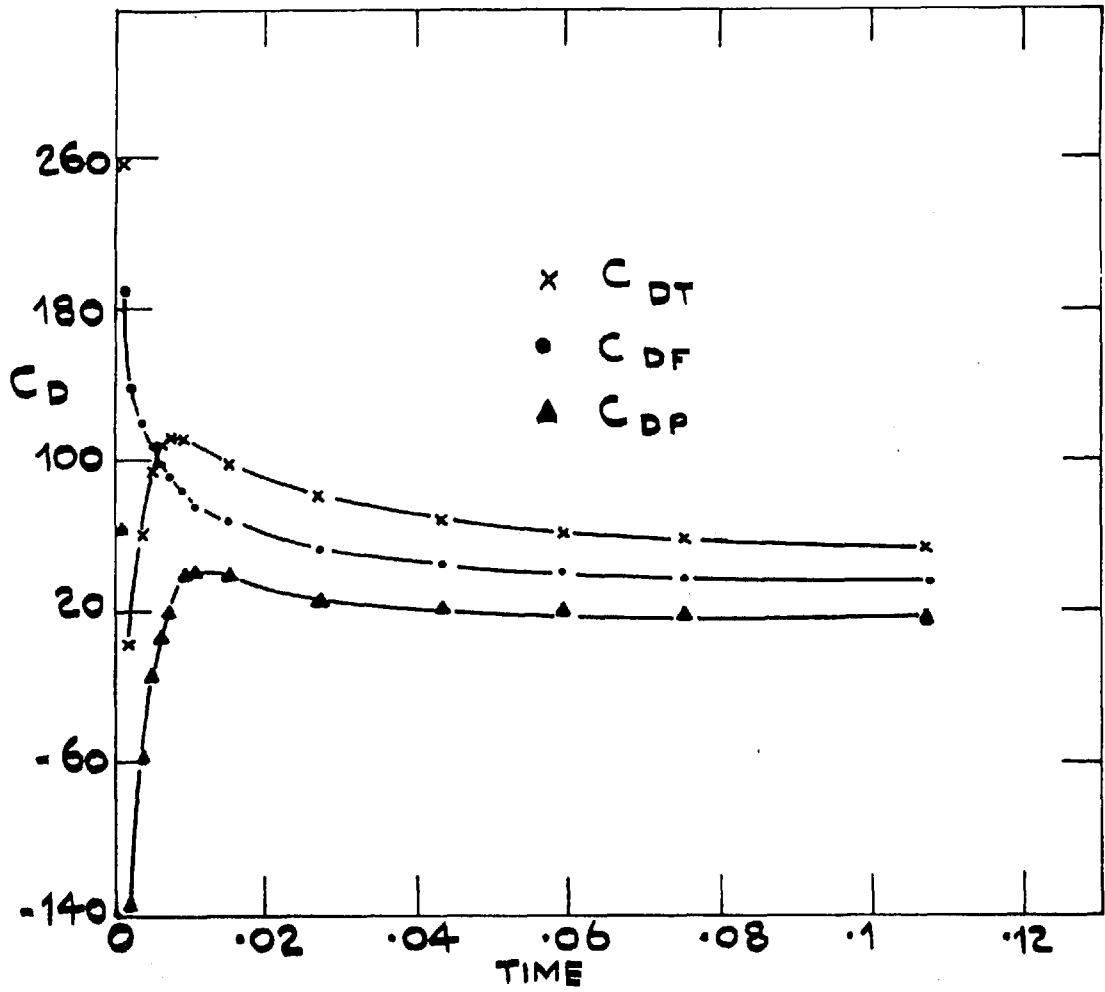
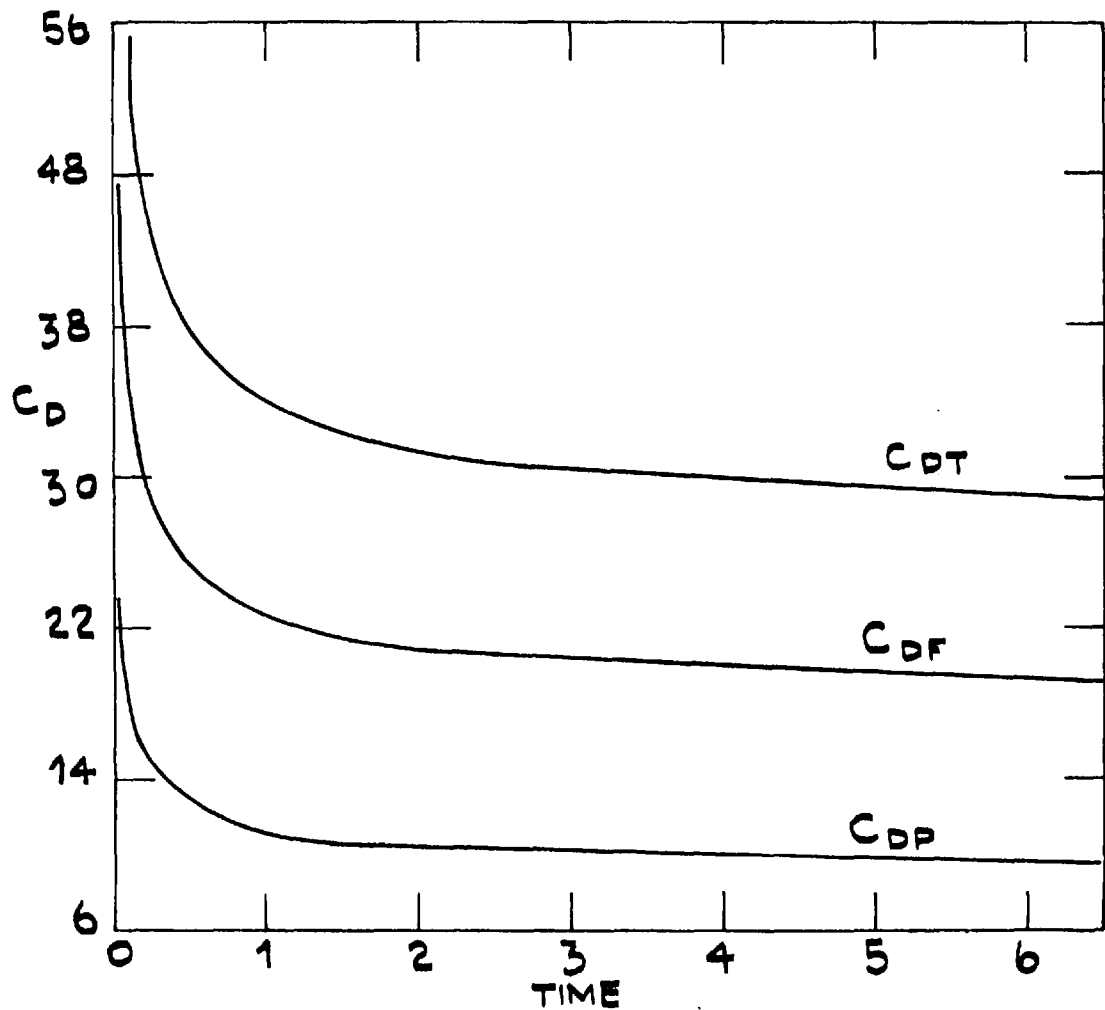
The development of the flow pattern, the generation of vorticity at the sphere surface and its diffusion into the fluid is shown by the computer line-printer plots from Fig. 5.2.1a to Fig. 5.2.1f. The straight lines between





the dots are added by hand for clarity and the kinks in the curves are due to the spacing limitation on line-printer. Fig. 5.2.1a shows stream lines at dimensionless time $t^*=0.00024$ ($t^* = \frac{tU_\infty}{R}$) which are very similar to the starting potential flow stream lines. Fig. 5.2.1b shows the steady state stream lines at $t^*=8.1552$ and they agree very well with the solutions of Al-Taha (1969) and Rimon (1968). Fig. 5.2.1c to Fig. 5.2.1f show the diffusion of vorticity and since the convective effects are small, the diffusion is almost purely in the radial direction. The steady state vorticity distribution at $t^*=8.1552$ shown by Fig. 5.2.1f. shows slight asymmetry between the flow upstream and downstream due to convective effects near the outer boundary. The vorticity contours agree with the solution of Al-Taha (1968).

The behaviour of the drag coefficients with time is shown in Fig. 5.2.2 and Fig. 5.2.3. Initially very large vorticity gradients occur at the sphere surface hence the calculated values of pressure or form drag (C_{DP}) are not reliable. At a Reynolds number of 1 the diffusion of vorticity is very rapid therefore the surface vorticity gradients quickly settle down to reasonable values, Fig. 5.2.2 shows the behaviour of the drag coefficients at small dimensionless times. The viscous drag (C_{DF}) always shows a smooth decrease whereas the pressure drag which initially is large and positive becomes large and negative and then increases to a positive value. After the pressure drag coefficient becomes positive it starts to decrease in

Fig. 5.2.2 Drag Coefficients vs. Time at $Re=1$ Fig. 5.2.3 Drag Coefficients vs. Time at $Re=1$

parallel with the viscous drag. This initial fluctuation of the pressure drag to a negative value appears at first sight to be unreasonable. However, it can be explained in terms of the diffusion of vorticity. The values plotted in Fig. 5.2.2 may be inaccurate due to numerical inaccuracies.

The computer results show that initially the vorticity diffuses equally in all radial directions. The upstream vorticity is then convected downstream and for a short time there is an accumulation of vorticity behind the sphere. This accumulation of vorticity causes the decrease in the pressure drag coefficient to a negative value. As this vorticity both diffuses and convects downstream, the pressure drag coefficient increases to a positive value. As the process of generation, diffusion and convection of vorticity from the sphere surface reaches a steady state, the drag coefficients gradually decrease to a steady state value as shown by Fig. 5.2.3.

The variation of surface vorticity with time is plotted in Fig. 5.2.4 and the variation of surface pressure with time is shown in Fig. 5.2.5. The radial diffusion of vorticity is confirmed by the curves in Fig. 5.2.4. The steady state curve at $t^* = 8.1552$ (superscript * in all the figures other than the computer plots is omitted for convenience) shows the expected slight asymmetry between conditions upstream and downstream and agrees with the curve of Al-Taha (1969) and Le Clair et al, (1970). Table 2 shows comparison of the present solution with the

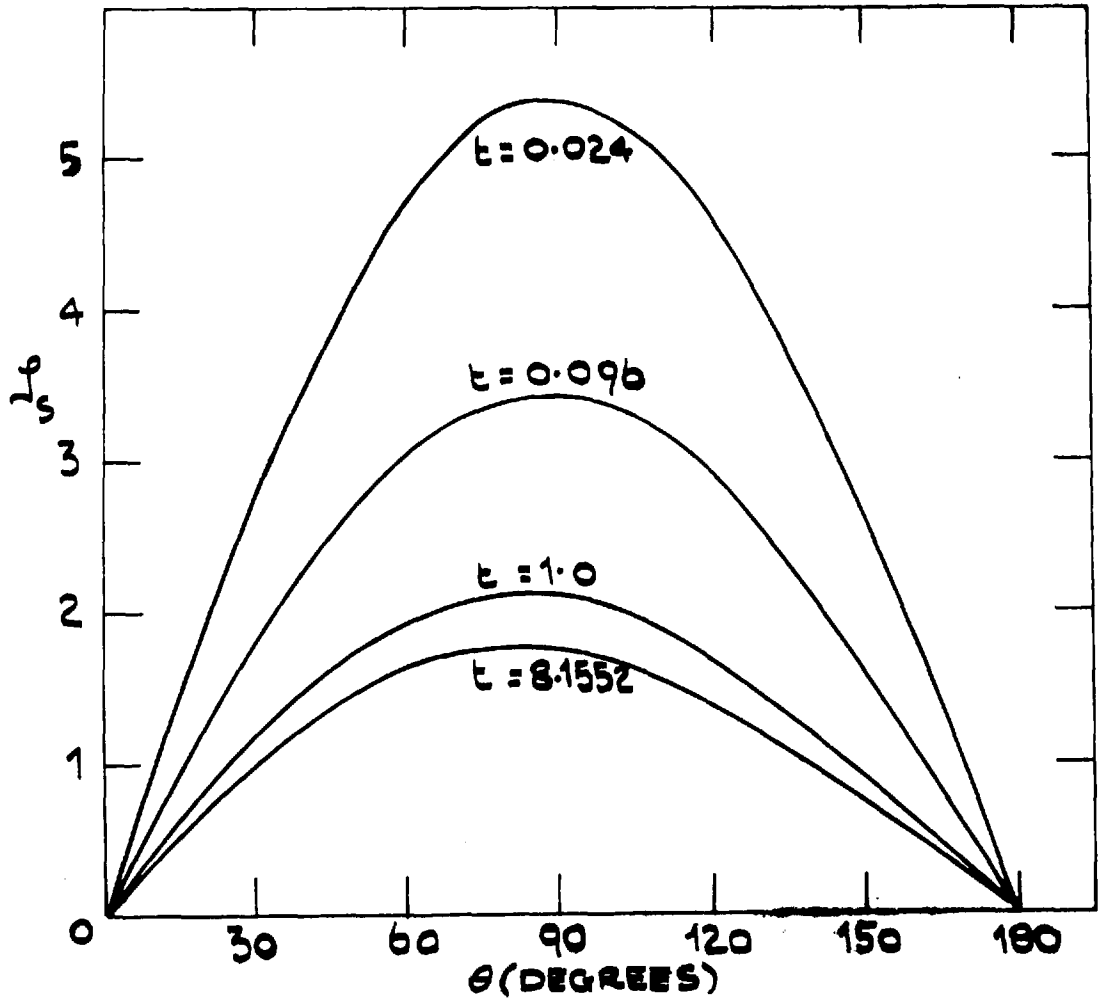


Fig. 5.2.4 Surface Vorticity vs. Time at $Re=1$

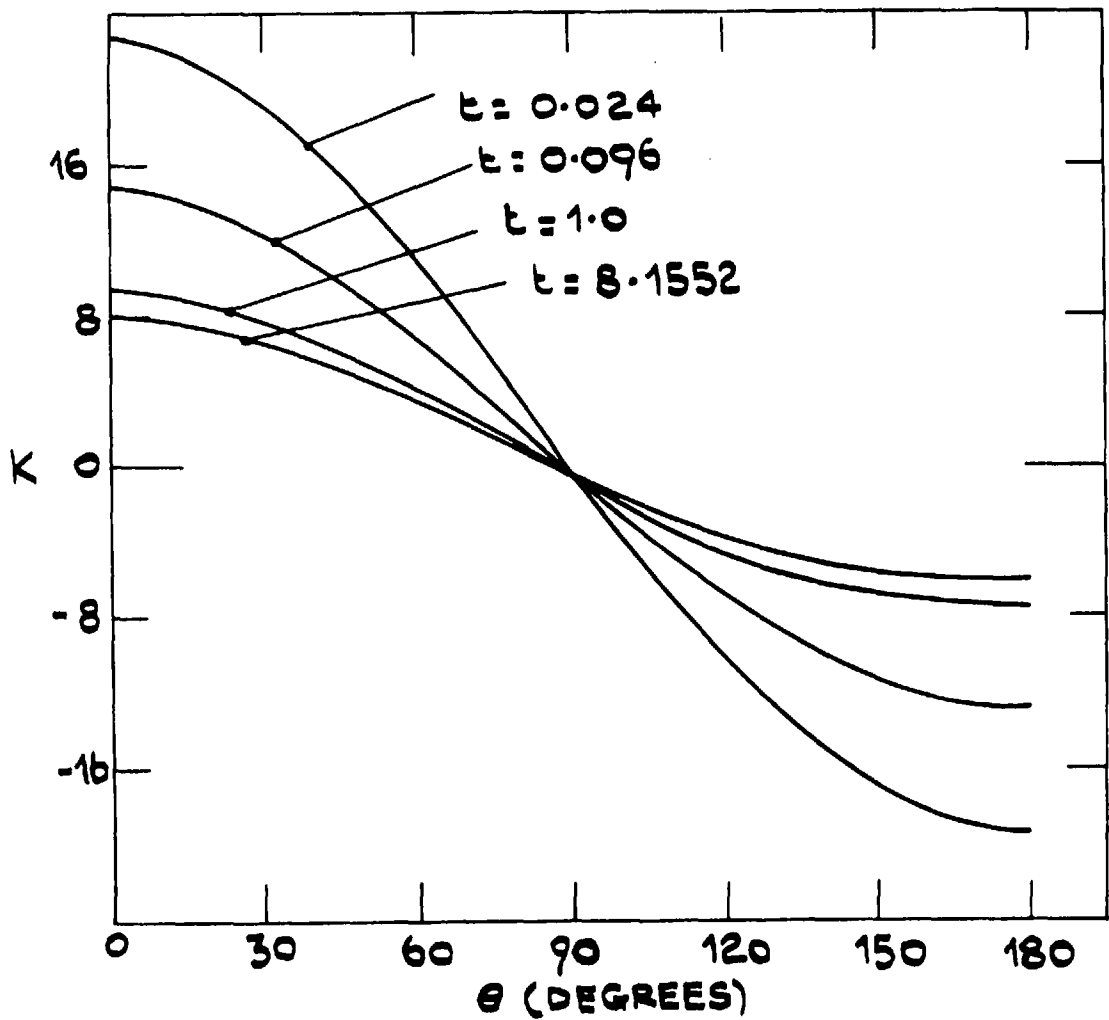


Fig. 5.2.5 Surface Pressure vs. Time at $Re=1$

solutions of Al-Taha and Le Clair et al.. The influence of the location of the outer boundary on the total drag can be judged from the figures in Table 2. The choice of the mesh size and the location of the outer boundary for the present solution is better both for computing economy and accuracy of the results. The ratio $C_{DF}:C_{DP}$ is approximately 2:1 as predicted by the Stokes solution. The steady state is very nearly reached as the solution approached the dimensionless time of unity, after which the changes in the flow pattern are extremely slow as shown by Fig. 5.2.1e and Fig. 5.2.1f.

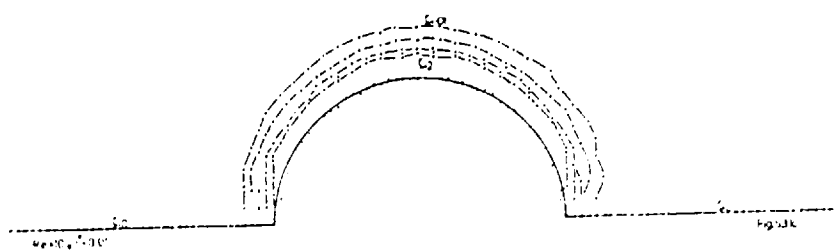
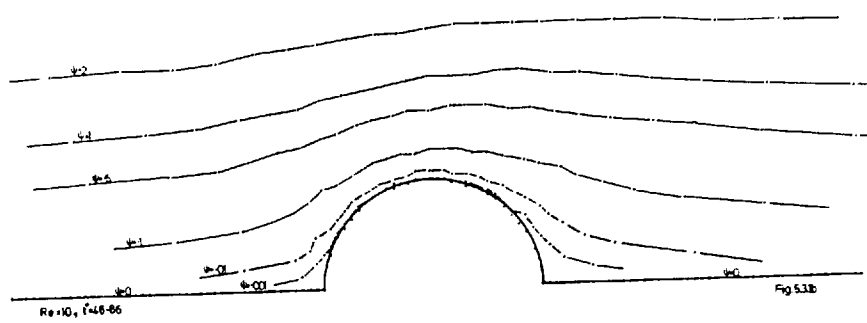
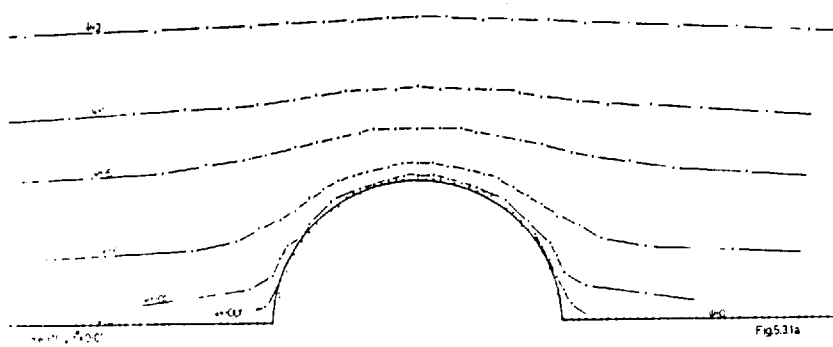
Using the solution at $t^*=8.1552$ as the steady state solution, the outflow boundary condition was changed from uniform parallel flow to parallel flow, similar to that suggested by Rimon (1968). The main results obtained with the parallel outflow boundary condition are presented in Table 3. The parallel outflow solution is not significantly different from the steady state solution. It appears that with the present mesh size and location of the outer boundary the parallel outflow boundary condition only adds more computing time because the rate of convergence of the stream function is slower. Also under-relaxation has to be used and the relaxation factor of 0.6 as given in Table 3 is the largest value that can be used in the present solution. Any value of the relaxation factor greater than 0.6 leads to divergence.

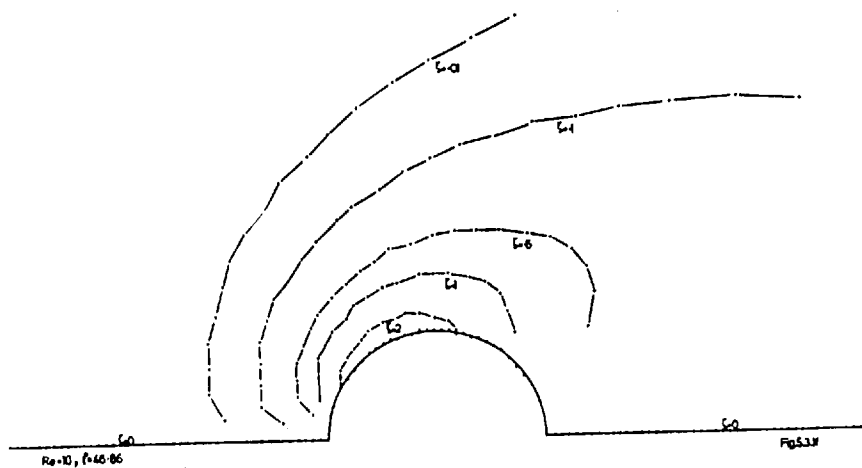
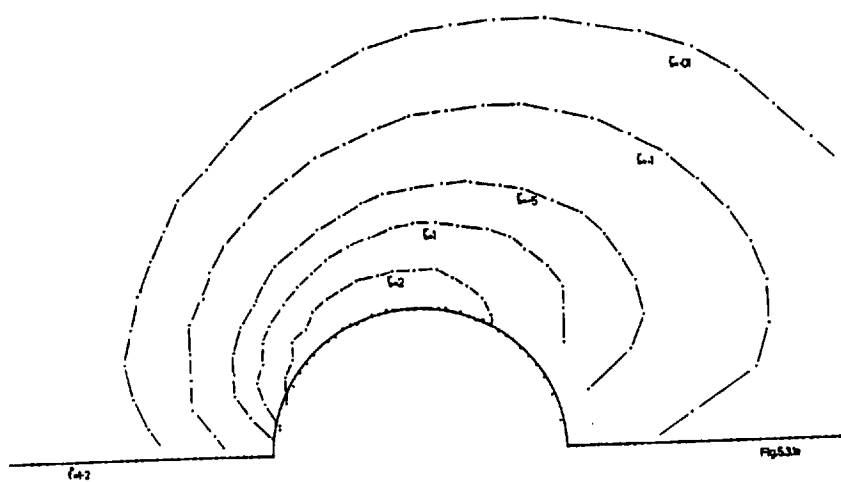
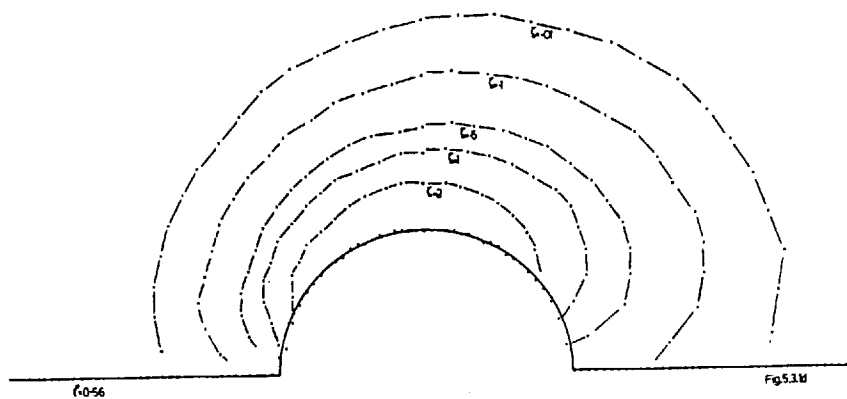
5.3 REYNOLDS NUMBER 10 SOLUTION

The development with time of the stream lines starting from the potential flow solution and the vorticity contours are shown by Fig. 5.3.1a to Fig. 5.3.1f. The stream line plots in Fig. 5.3.1a and Fig. 5.3.1b do not show a marked difference from the Reynolds number 1 solution shown in Fig. 5.2.1a and Fig. 5.2.1b. The steady state stream lines in Fig. 5.3.1b at $t^*=46.86$ agrees with the solution obtained by Al-Taha (1969) but not with that of Rimon (1968), whose solution shows flow separation at the rear of the sphere. Rimon's solution at $Re=10$ gives physically unrealistic surface pressure therefore the present steady state solution which agrees also with solution obtained by Le Clair et al. (1970) (Table 2) is considered to be more accurate.

The convection effects in the solution can be seen in the vorticity distribution plots Fig. 5.3.1c to Fig. 5.3.1f. As with the Reynolds number 1 solution, initially the diffusion near the sphere surface is the dominant mode of vorticity transport and after a dimensionless time of about unity the convection mode begins to distort the contours in the outer region. The steady state vorticity distribution shown in Fig. 5.3.1f at $t^*=46.86$ shows the expected vorticity contours with greater convection effects than at Reynolds number 1. The steady state vorticity contours agree with those obtained by Al-Taha (1969).

The behaviour of the drag coefficients with time is





shown by Fig. 5.3.2 and Fig. 5.3.3. The initial variation with time of the drag coefficients, shown by Fig. 5.3.2, follows a similar pattern to that shown by the Reynolds number 1 solution although the level of initial fluctuation of the pressure drag coefficient is different as is the starting value of the viscous drag coefficient. The numerical stability criterion (equation 4.4.5) allowed a time step ten times greater than that used in the $Re=1$ simulation, consequently the steady state solution at $t^*=46.86$ was obtained using less computer time than the $Re=1$ solution. The changes in the flow pattern occur faster and numerical technique was able to follow them without loss of accuracy. Fig. 5.3.3 shows that the solution reached steady state by the dimensionless time $t^* \approx 10$ (c.f $Re=1$, $t^* \approx 1$). The steady state drag coefficient agrees with the experimental value obtained by Lapple and Shepherd (1940) shown in Table 2.

The variation with time of the surface vorticity and surface pressure is shown in Fig. 5.3.4 and Fig. 5.3.5 respectively. Comparison of the steady state vorticities at $Re=1$ and $Re=10$ show (Fig. 5.2.4 and Fig. 5.3.4) greater asymmetry from upstream to downstream in the case of $Re=10$. The solution tends towards the formation of a recirculatory wake as obtained by Rimon (1968) but does not quite achieve it. The steady state surface pressure Fig. 5.3.5 shows a shallow minimum turning point as compared with Fig. 5.2.5 for $Re=1$ where the surface pressure continuously decreases from the upstream stagnation point

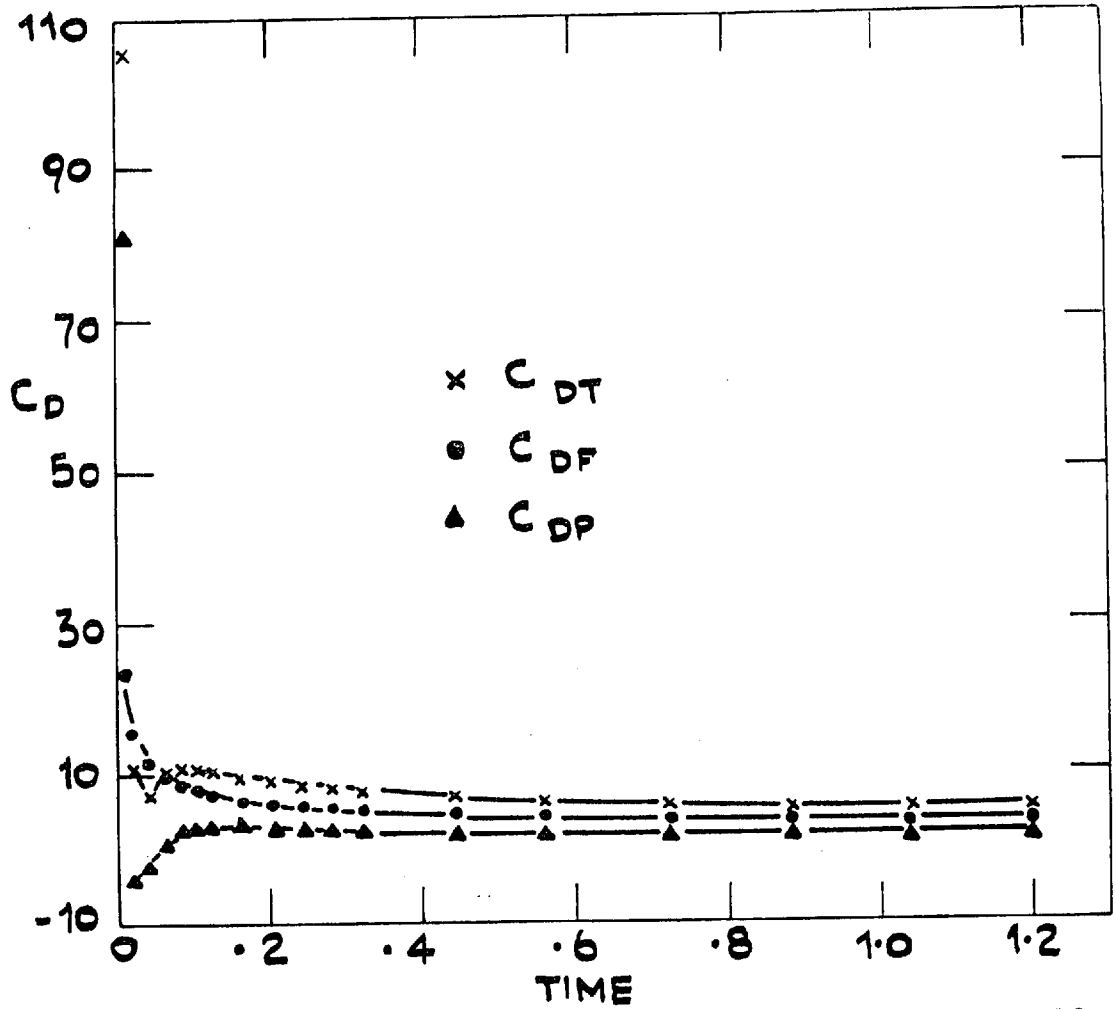


Fig. 5.3.2 Drag Coefficients vs. Time at $Re=10$

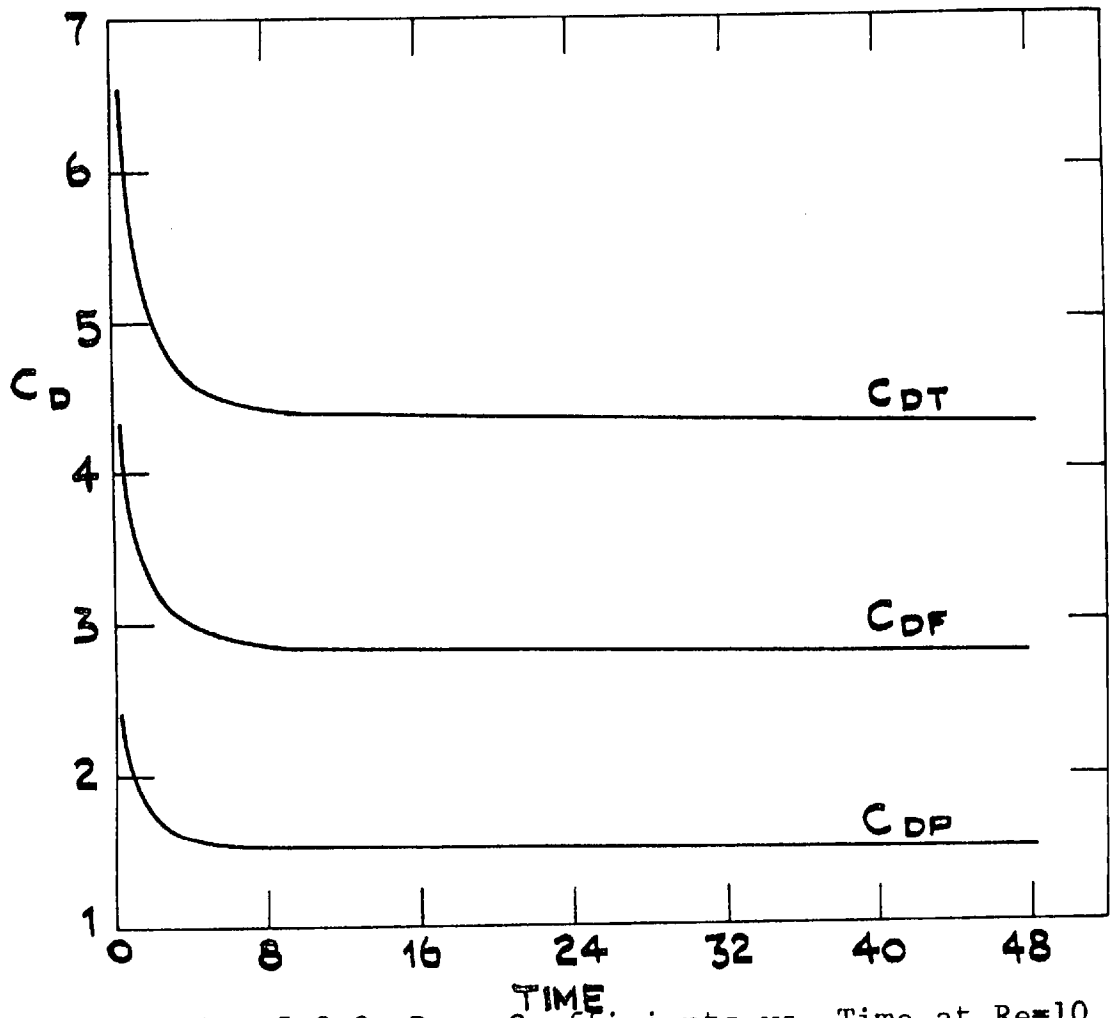
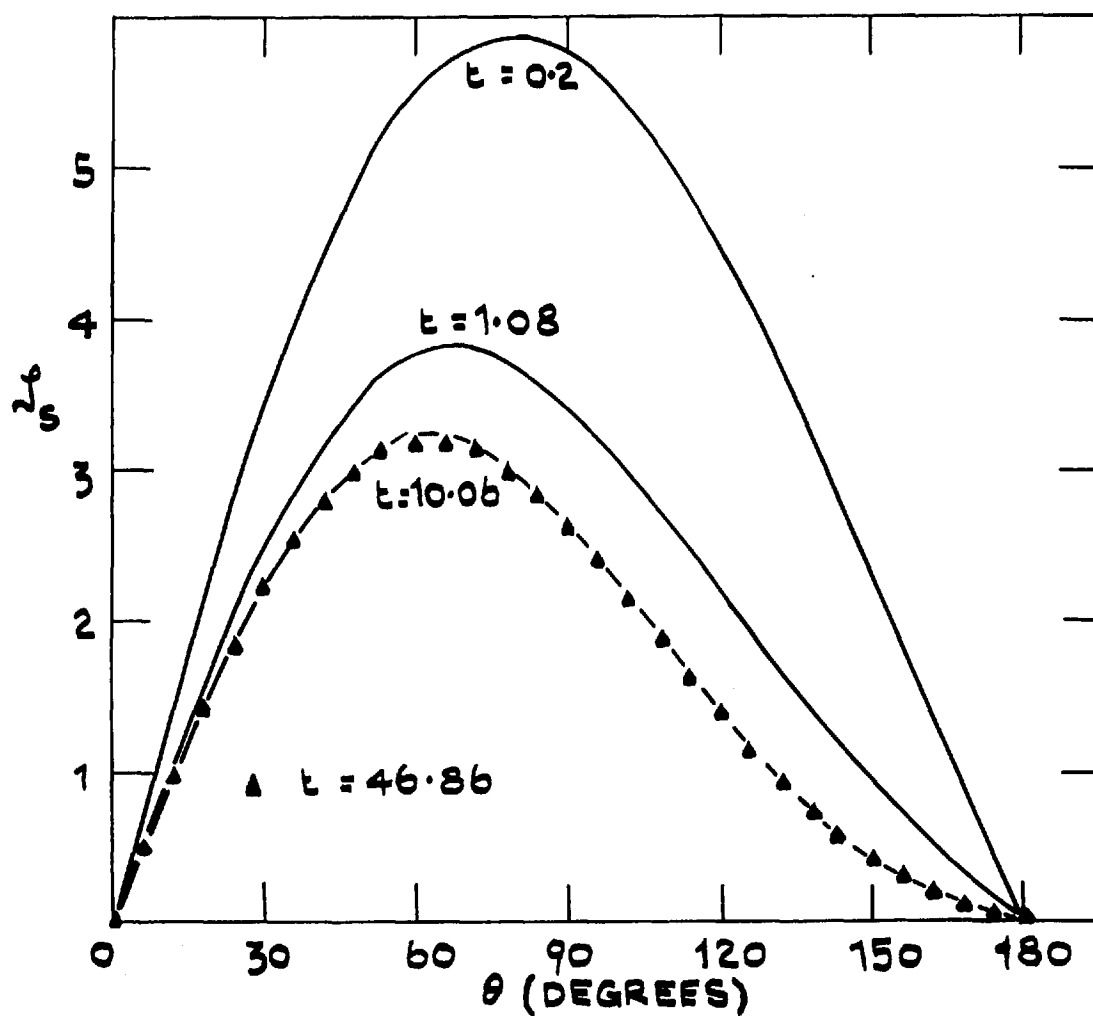
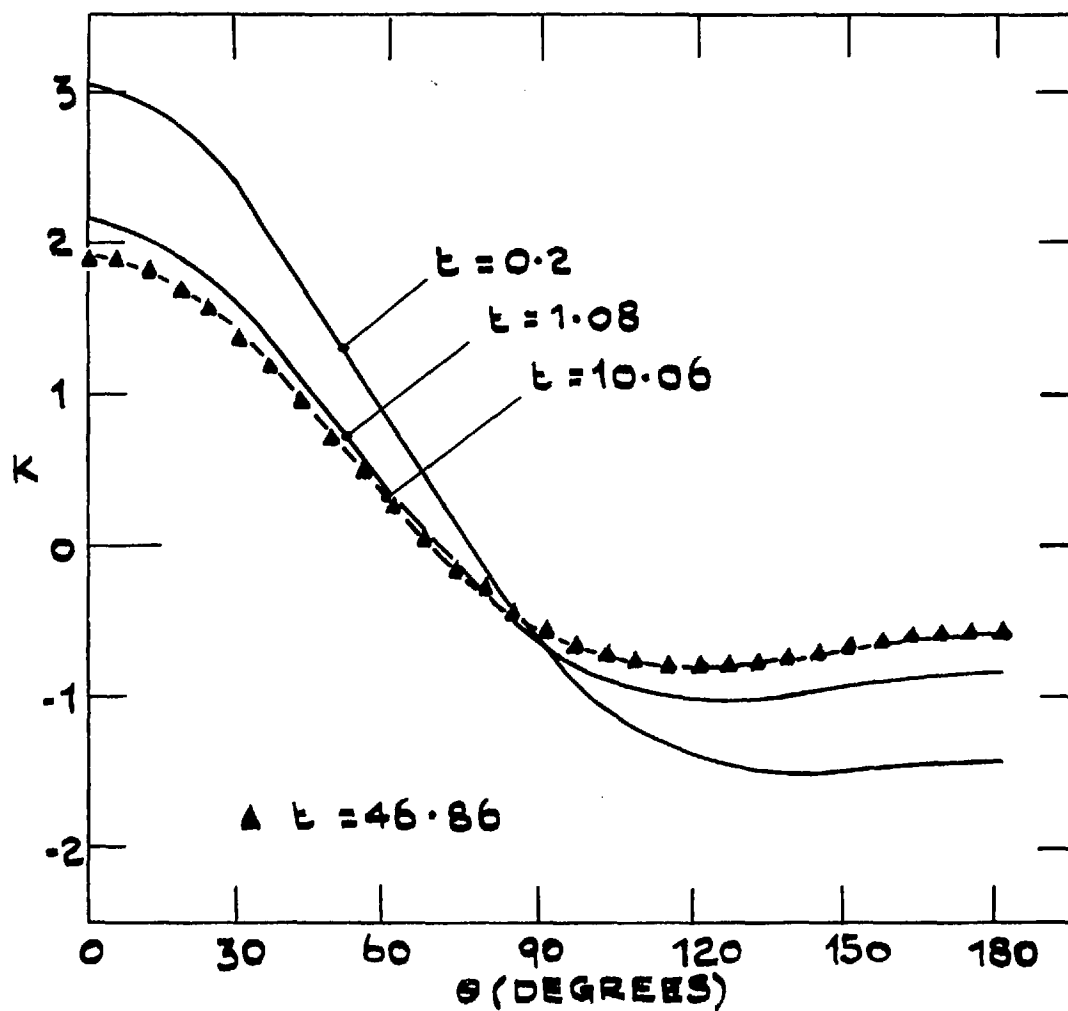


Fig. 5.3.3 Drag Coefficients vs. Time at $Re=10$

Fig. 5.3.4 Surface Vorticity vs. Time at $Re=10$ Fig. 5.3.5 Surface Pressure vs. Time at $Re=10$

to the rear of the sphere.

Table 2 shows the comparison of the present steady state solution with solutions by other workers. Rimon's (1968) solution is the only one with which the present $Re=10$ solution disagrees. Although the total drag coefficient calculated by Rimon is in reasonable agreement with the experimental value, the surface pressure profile used to calculate the drag coefficient does not. The stagnation pressure both upstream and downstream is in complete disagreement with expected values. In fact, Rimon's pressure profile showed a slight rise from the upstream stagnation pressure before decreasing thus making the solution physically unrealistic. Since Rimon used the parallel outflow boundary condition, it was decided to test whether this boundary condition was responsible for this oddity in the solution. Therefore, taking the steady state solution at $t^*=46.86$ as the initial solution, the parallel outflow boundary condition was imposed and the main results are shown in Table 3. As with the test at $Re=1$ no significant difference in the solution was detected. Once again the rate of convergence was affected and a under-relaxation factor of 0.6 had to be used at the outflow boundary. Values of the relaxation factor greater than 0.6 lead to divergence.

The remaining possible origin of the difference in the Rimon's solution was the position of the outer boundary therefore it is possible that in Rimon's solution it

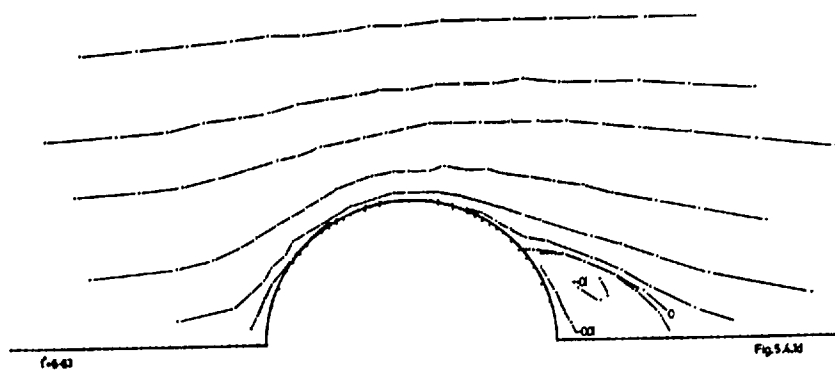
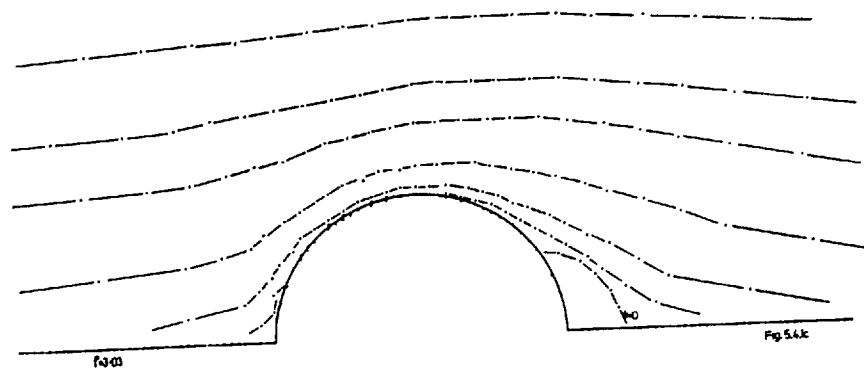
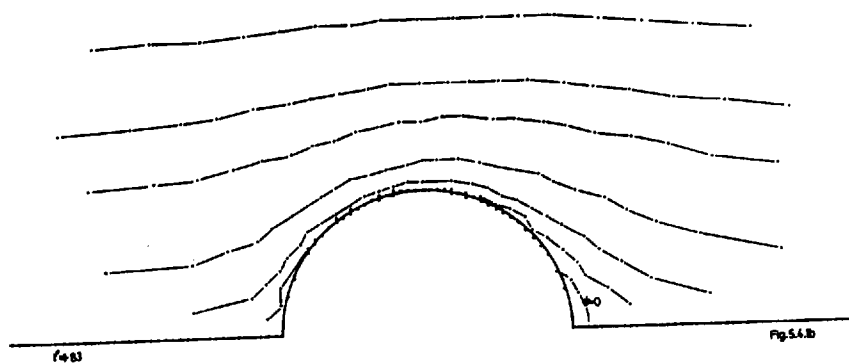
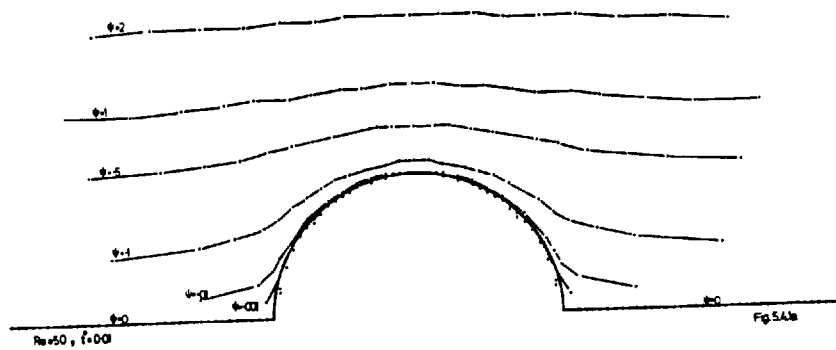
is too near to the sphere surface to properly satisfy the parallel boundary condition. The agreement of the present solution with the solution by Le Clair et al. (1970) with a finer mesh and larger domain of integration supports the validity of the present numerical technique.

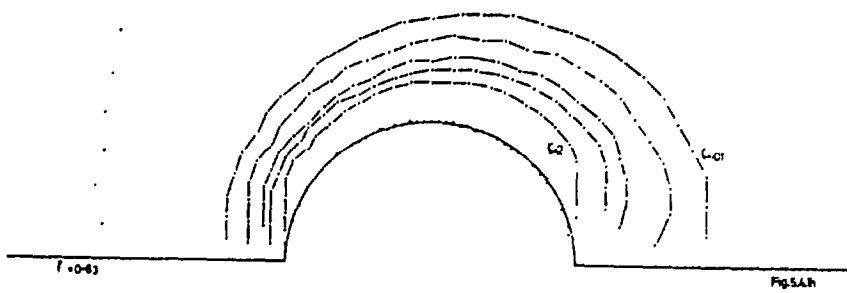
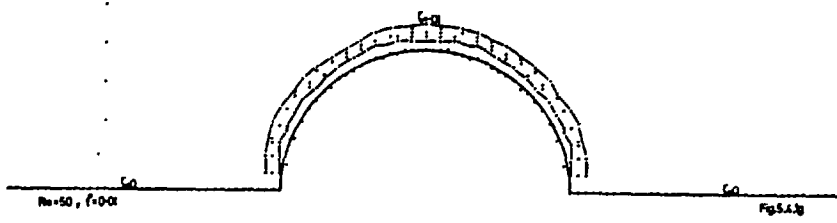
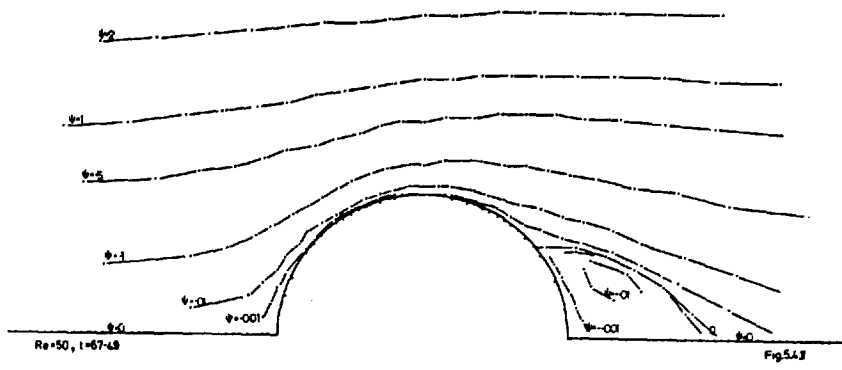
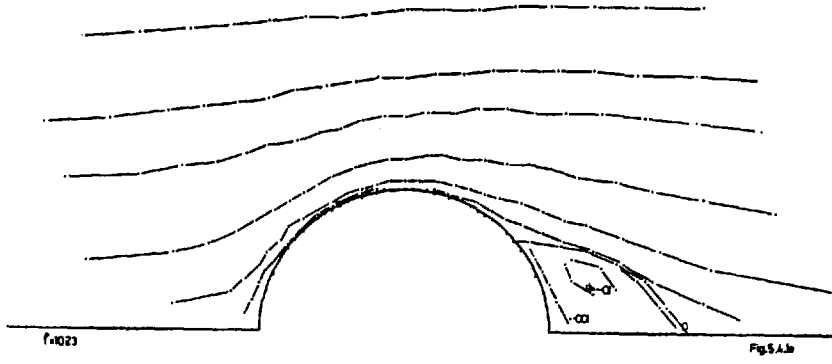
5.4 REYNOLDS NUMBER 50 SOLUTION

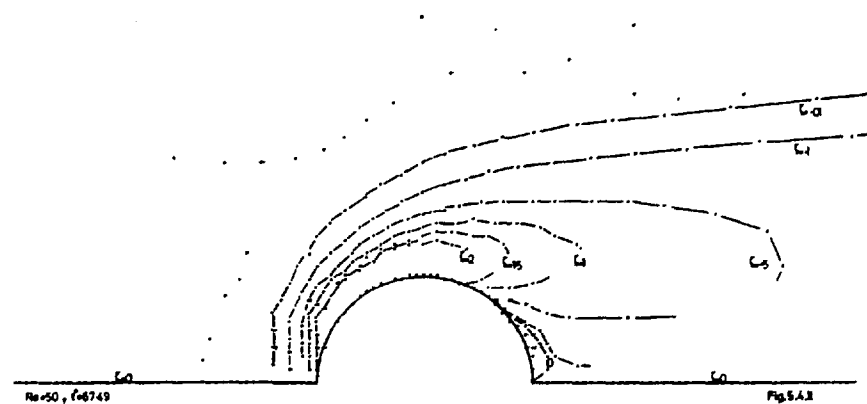
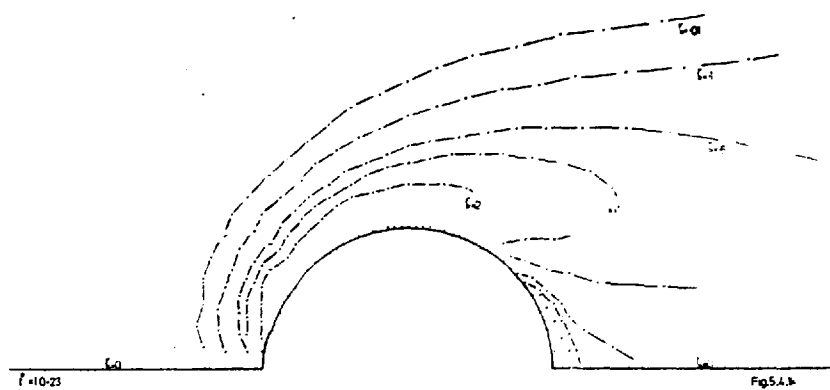
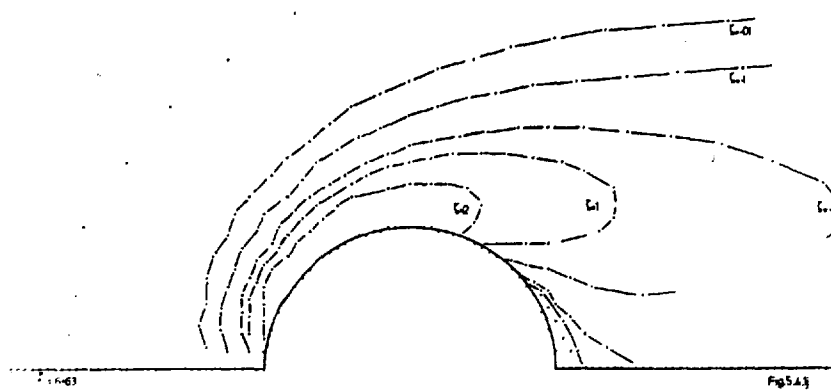
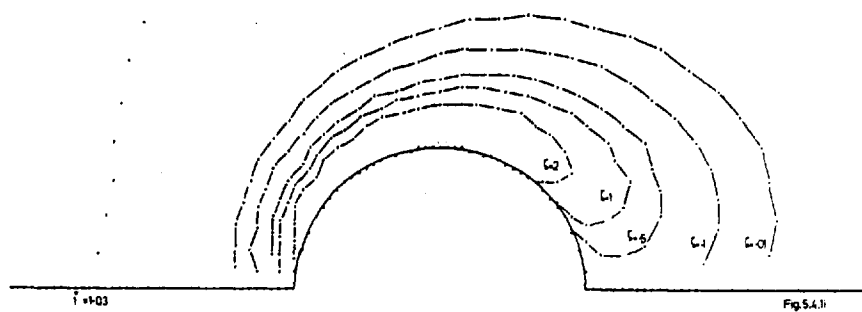
The development with time of a recirculatory wake at a Reynolds number of 50 starting from the potential flow solution is shown by the computer line-printer plots from Fig. 5.4.1a to Fig. 5.4.1f. The flow separation first appeared at about the dimensionless time $t^*=1.4$ and the wake is clearly visible in Fig. 5.4.1b $t^*=1.83$. The separation angle quite rapidly attains its steady state value while the wake length grows at a slower rate. Fig. 5.4.6 shows the rate of growth of both the separation angle (Θ_s) and the wake (L_w) in comparison with the development with time at other Reynolds numbers. The steady state stream line plot in Fig. 5.4.1f agrees with that of Al-Taha (1969).

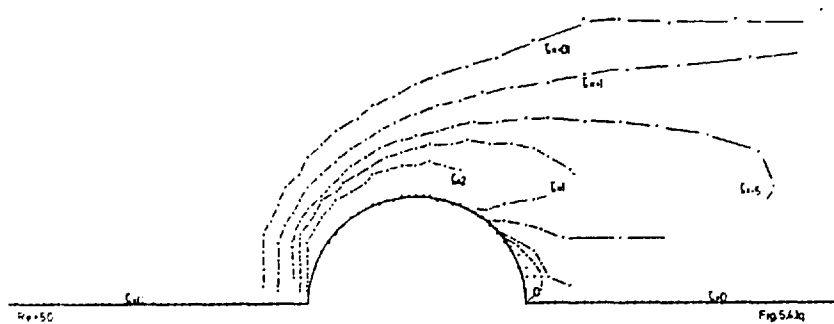
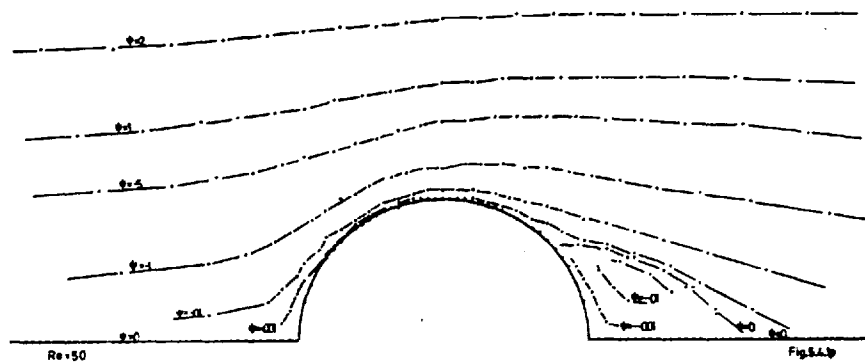
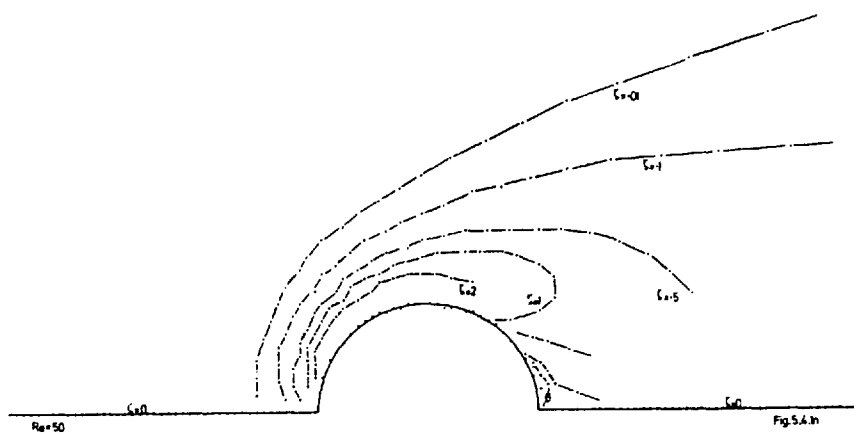
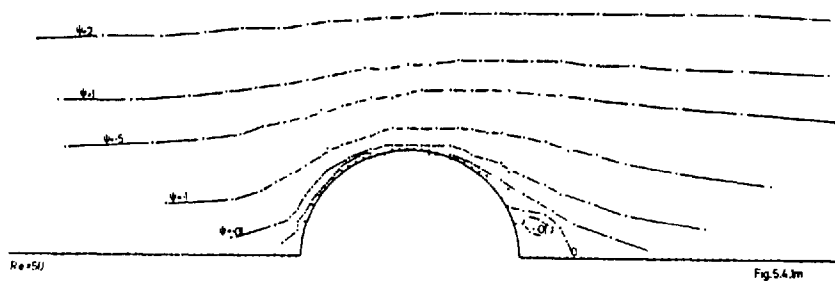
Also the flow characteristics, for example, the separation angle, the wake length and the drag coefficient agree with experimental values given in Table 2.

The generation and diffusion of vorticity with time from the sphere surface is shown by Fig. 5.4.1g to Fig. 5.4.1l. The diffusion of vorticity is dominant during





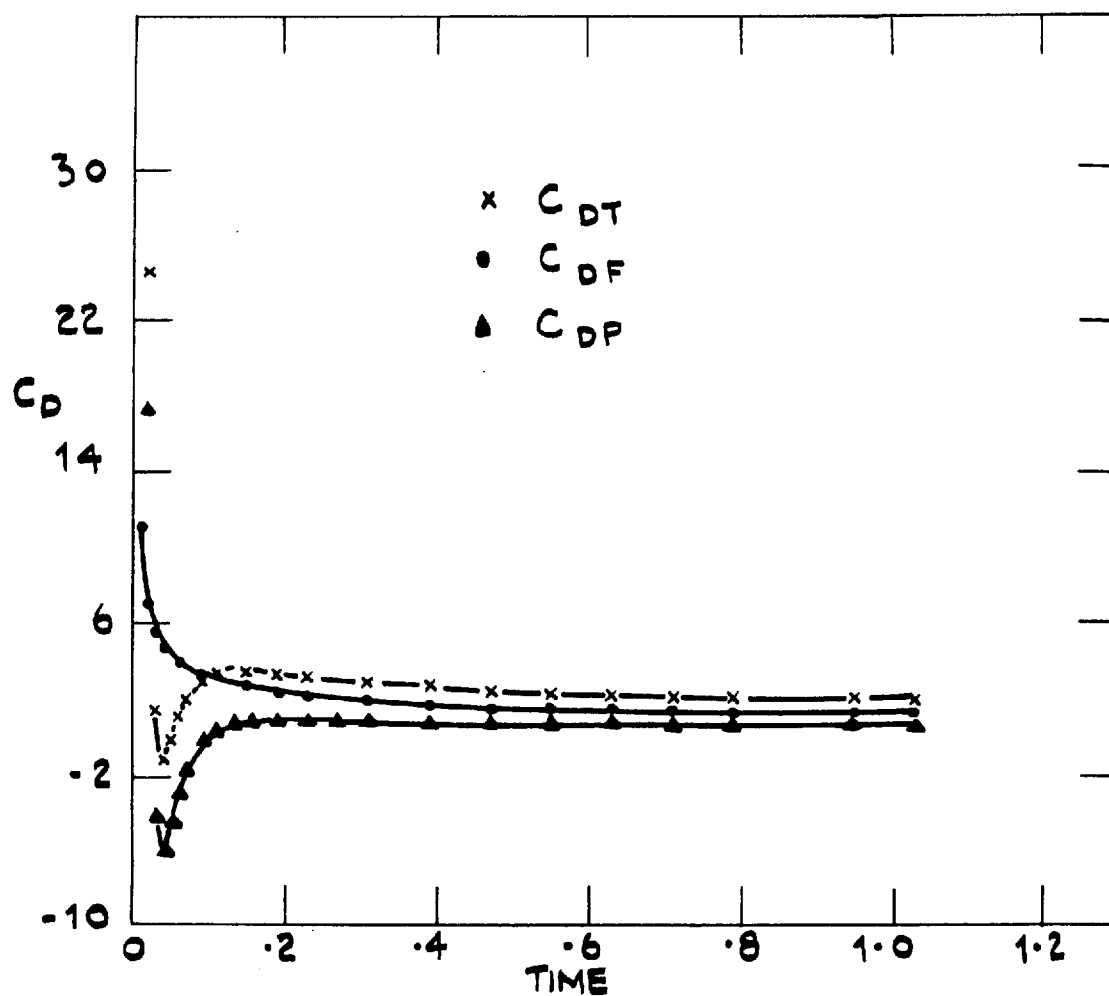
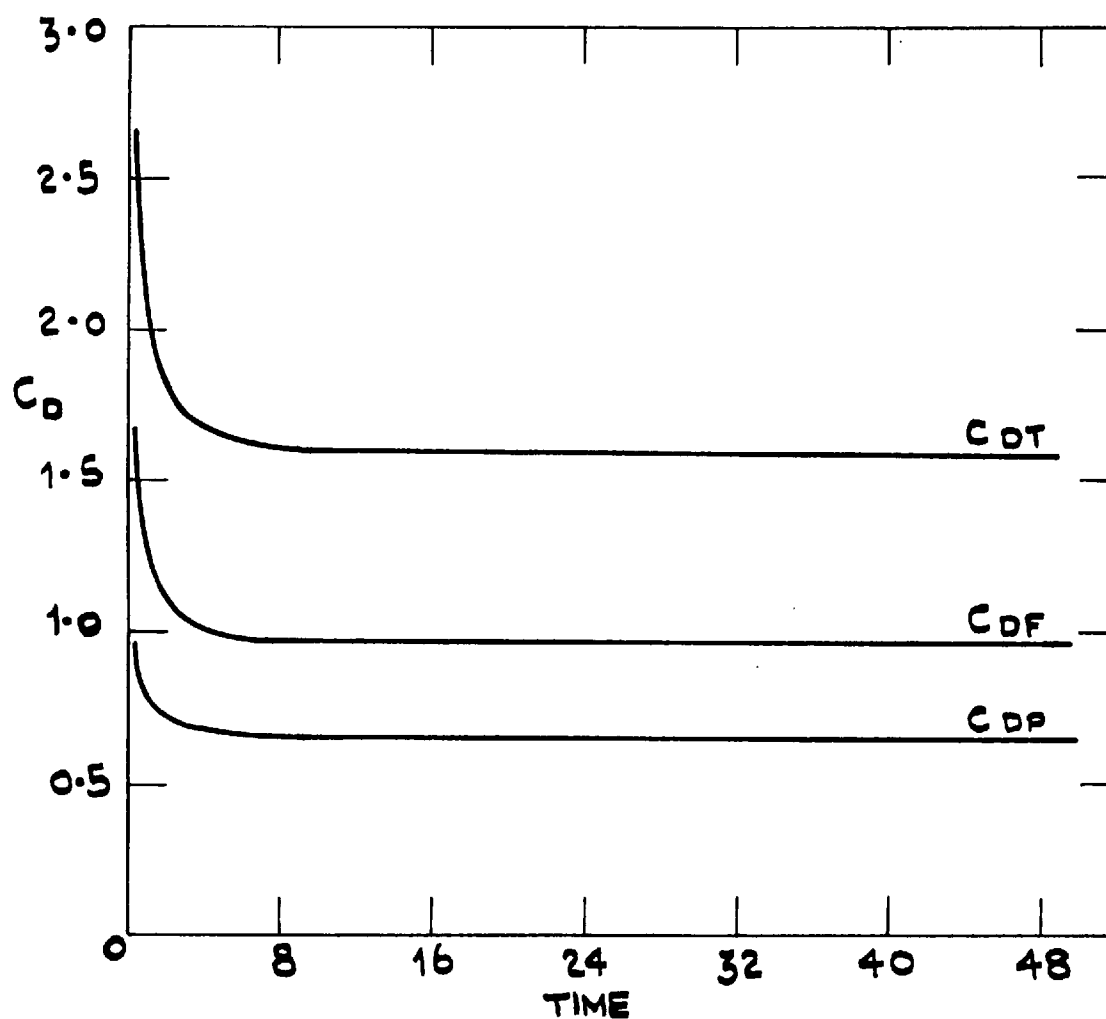




the initial stage of the flow development and is almost purely radial in direction. The convection effects in the vorticity transport become important earlier than in the case of the Reynolds number 10 solution. Fig. 5.4.1i at $t^* = 1.03$ shows the vorticity contours just before the flow separation occurs at the rear of the sphere. Once again the steady state vorticity contours in Fig. 5.4.1l at $t^* = 67.49$ are in good agreement with the solution of Al-Taha (1969). The linear numerical stability criterion derived in section 4.4 was successful and the full steady state solution was obtained using less overall computer time than the steady state solution of Al-Taha which used a point iterative successive over-relaxation technique.

The behaviour of the drag coefficients with time is shown in Fig. 5.4.2 and Fig. 5.4.3. Initially, the pressure drag coefficient shows the same behaviour (Fig. 5.4.2) as that discussed for $Re=1$ and 10. The drag coefficients reach a steady state value very quickly at $t^* \approx 8$ (c.f. $Re=1, t^* \approx 1$; $Re=10, t^* \approx 10$). However, the solution was integrated to a much longer time since small changes in the wake eddy occurred which settled down very slowly.

The variation with time of surface vorticity and pressure is given in Fig. 5.4.4 and Fig. 5.4.5. In comparison with $Re=1$ and 10 solutions the maximum surface vorticity occurs earlier in the front half of the sphere. The steady state surface vorticity and surface pressure at $t^* = 67.49$ are in good agreement with Al-Taha's (1969)

Fig. 5.4.2 Drag Coefficients vs. Time at $Re=50$ Fig. 5.4.3 Drag Coefficients vs. Time at $Re=50$

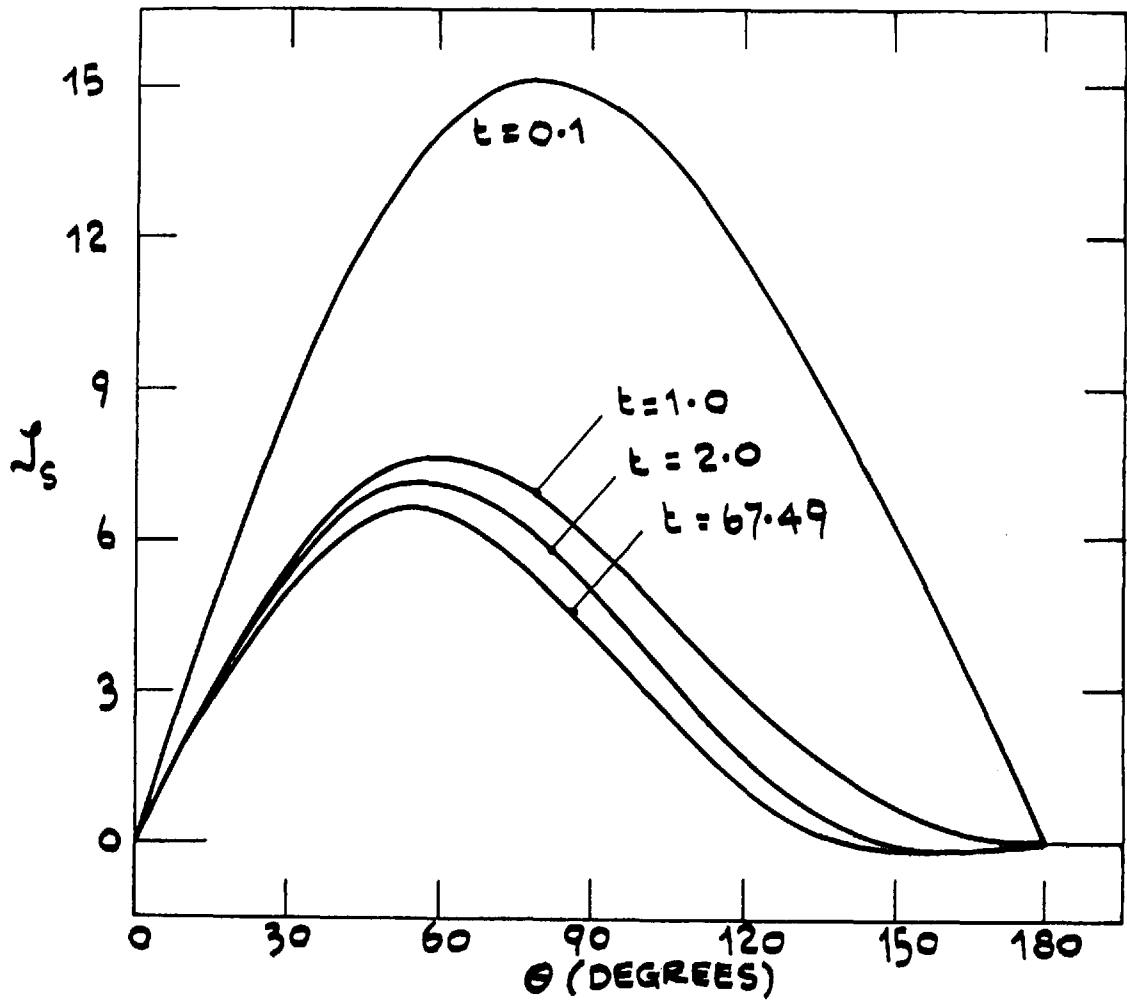
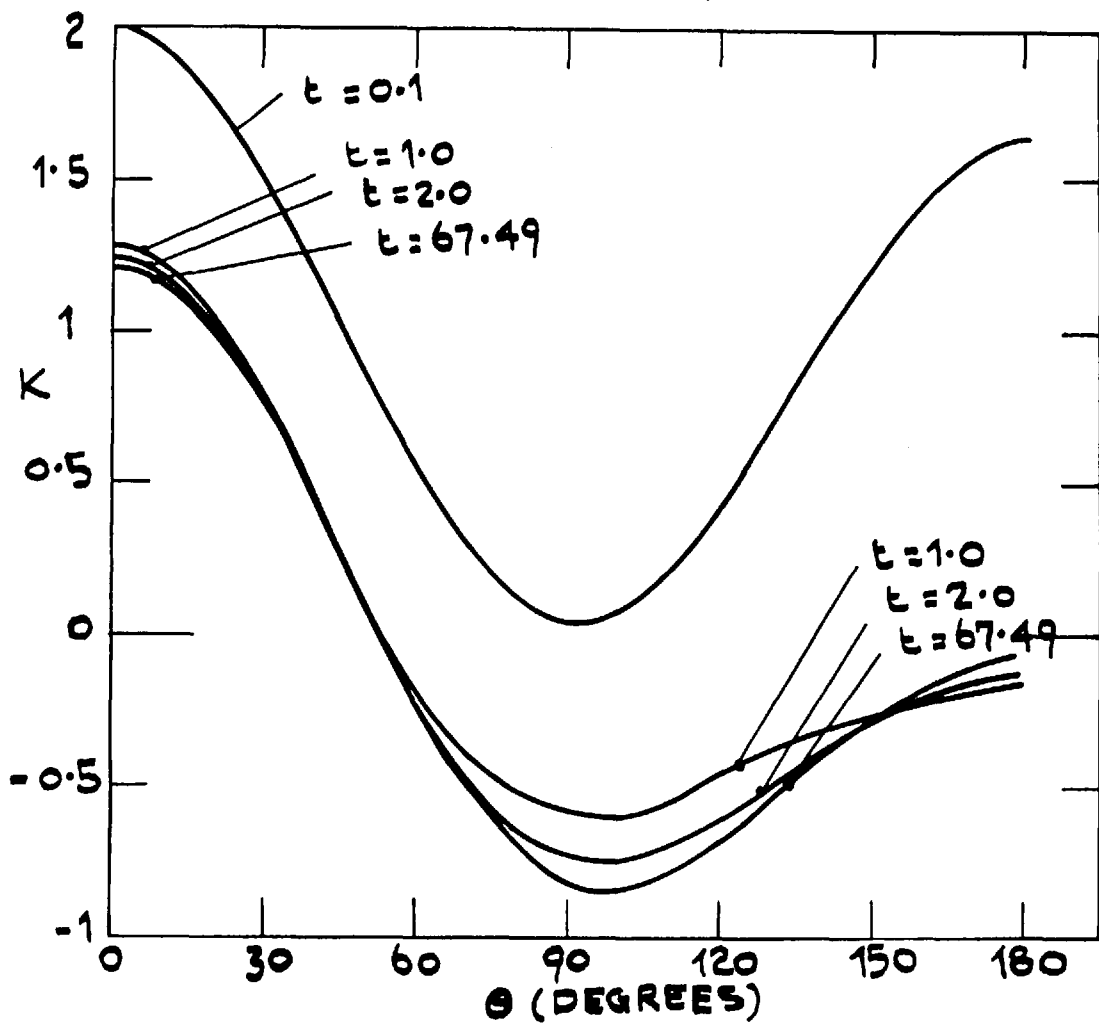
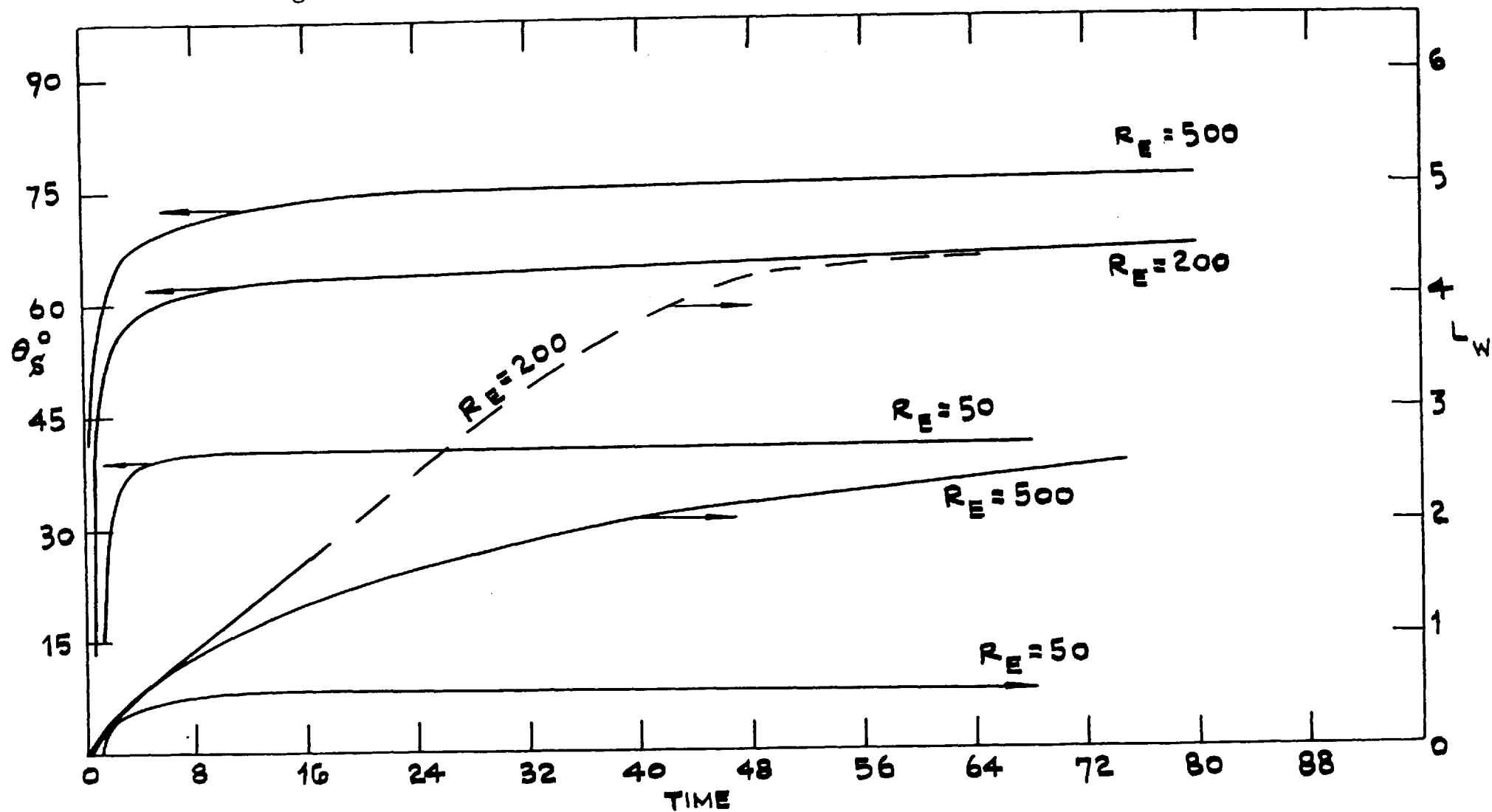
Fig. 5.4.4 Surface Vorticity vs. Time at $Re=50$ Fig. 5.4.5 Surface Pressure vs. Time at $Re=50$

Fig. 5.4.6 Separation Angle (θ_s) and Wake Length (L_w) vs. Time

solution. The present solution is considered to be more accurate since the choice of mesh size and location of the outer boundary is better as shown by Table 2. Also the separation angle and the wake length are in excellent agreement with the experimental measurements of Taneda (1956). The drag coefficient is slightly greater than the experimental value in accordance with Al-Taha's findings, which show the effect of the location of the outer boundary. Based upon the above comparison the steady state solution at Reynolds number of 50 is in every way satisfactory. It was decided to use it as a basis for comparison with solutions obtained using other finite-difference schemes. A small level of 'noise' is present in the solution of the vorticity transport equation near the outer boundary condition, therefore as discussed in Chapter 4 the first scheme that was tested was the introduction of upstream differences in the convective or non-linear term in the vorticity transport equation. The main results are presented in Table 4.

The upstream differences were introduced into the steady state solution in Fig. 5.4.1f and Fig. 5.4.1l. The steady state stream line and vorticity changed as a result of the introduction of upstream differences and the new steady state is shown by Fig. 5.4.1m and Fig. 5.4.1n. Comparison of the stream line in Fig. 5.4.1f with those of Fig. 5.4.1m shows that the wake eddy is much smaller when upstream differences are used in the convective term of the vorticity transport equation. Similarly, comparison of Fig. 5.4.1l and Fig. 5.4.1n shows

that the vorticity contours are more spaced out for the upstream difference scheme. This is due to the introduction of 'artificial viscosity' into the system. Also the dispersive errors introduced by the central difference approximation of the first derivatives in the convective terms are all damped out.

The drag coefficients given in Table 4 for the upstream difference solution agree better with the central difference solution than the separation angle or the wake length. Thus it appears that the drag coefficient by itself does not provide a sufficiently accurate yardstick for the assessment of solutions. If a central difference solution can be obtained with reasonable computing time than it should be attempted. However, in cases where the 'noise' level is such that the solution diverges an upstream biased difference solution should be attempted as a first approximation to the desired solution. The results also show that for the purpose of prediction of rates of heat and mass transfer, the upstream difference solutions may give values comparable with experimental transfer rates.

The truncation error introduced by the upstream difference solution is large. Therefore an attempt was made to obtain a better solution with the same mesh size by using both the central and the upstream differences at various points in the flow field. The 'mixed scheme' or scheme 3 (Table 4) was obtained after a great deal of numerical experimentation. The final scheme used is

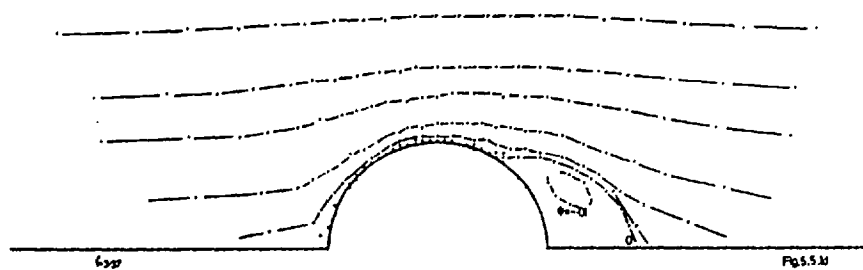
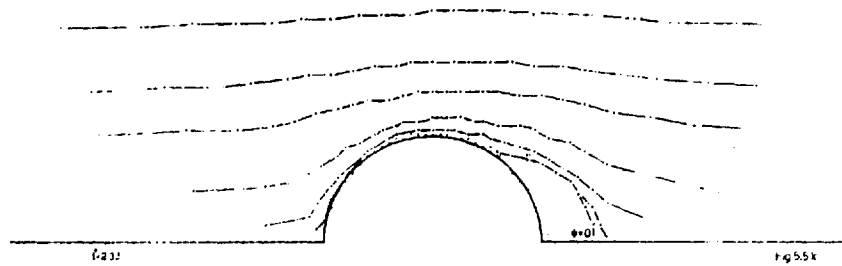
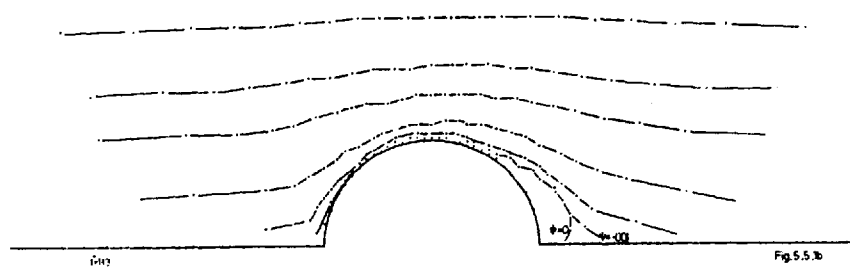
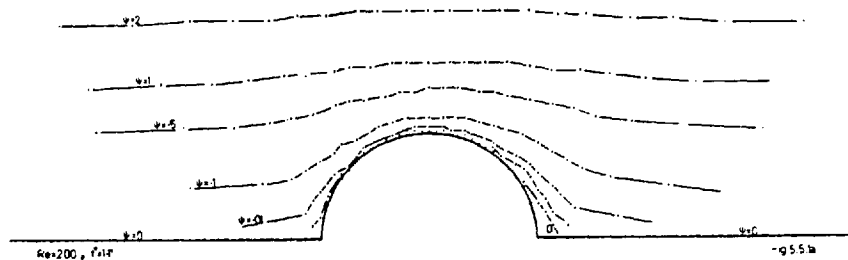
defined in Table 4. The essential feature of the scheme is to use central differences in regions of large vorticity and upstream differences in regions of small vorticity.

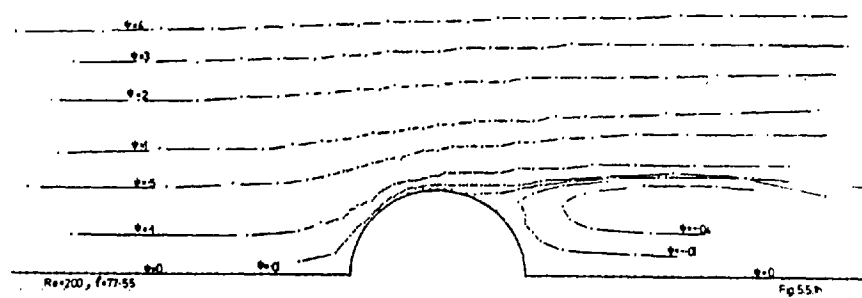
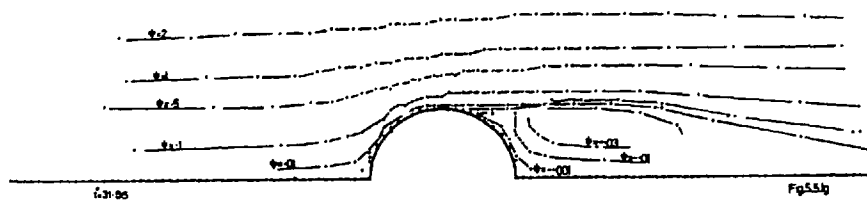
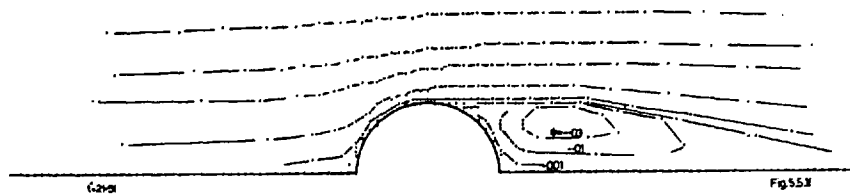
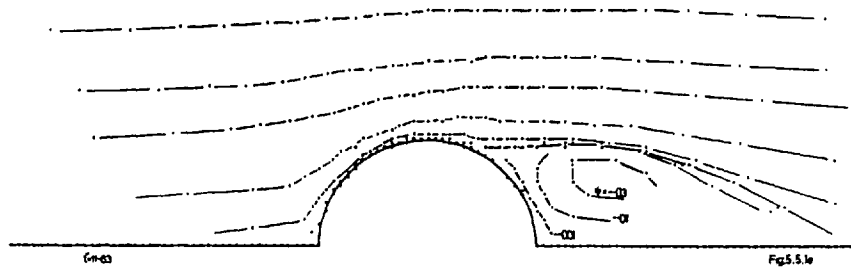
The stream line plot in Fig. 5.4.1p for the mixed scheme compares very well with Fig. 5.4.1f for the central difference scheme, also the vorticity distribution plot for the 'mixed scheme' shown in Fig. 5.4.1q again compares favourably with Fig. 5.4.1l and shows 'noise' only in the outer down stream region. Table 4 indicates that the 'mixed scheme' yields a longer wake length and a larger separation angle, both values are an improvement over the central difference scheme. The drag coefficient also shows a better agreement with experimental value. The drawback of the present 'mixed scheme' is the numerical experimentation required to locate the right regions to introduce upstream differences. It is hoped that the feasibility shown by the present scheme will serve as an indicator for the development of a more general scheme. One indication is that the choice between the upstream or the central difference scheme at any node should be made on the size of the local Reynolds number. If the local Reynolds number is larger than a particular value, than the upstream differences for the convective term should be used otherwise the central differences should be used. The flexibility introduced (Chapter 4) in the upstream difference scheme can be used to minimise truncation error i.e. the choice of 1, 2, and 3 can be made depending on the local Reynolds number and therefore a better solution is obtainable with a coarse mesh.

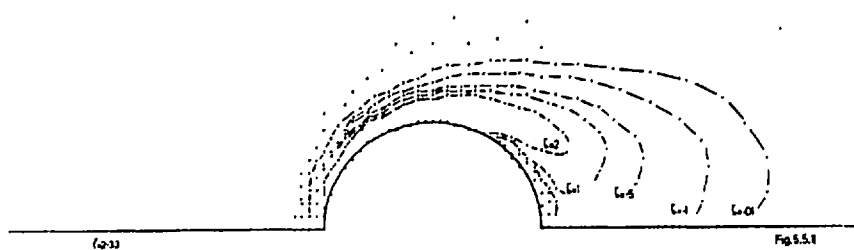
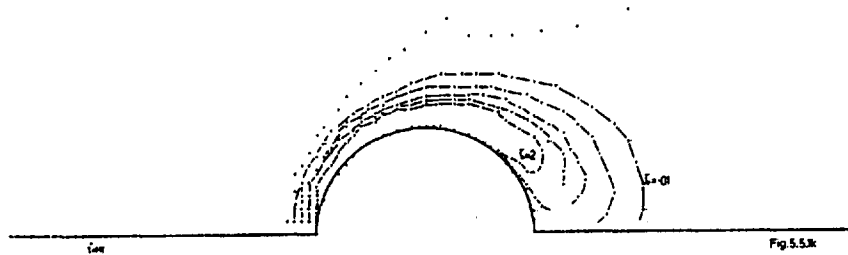
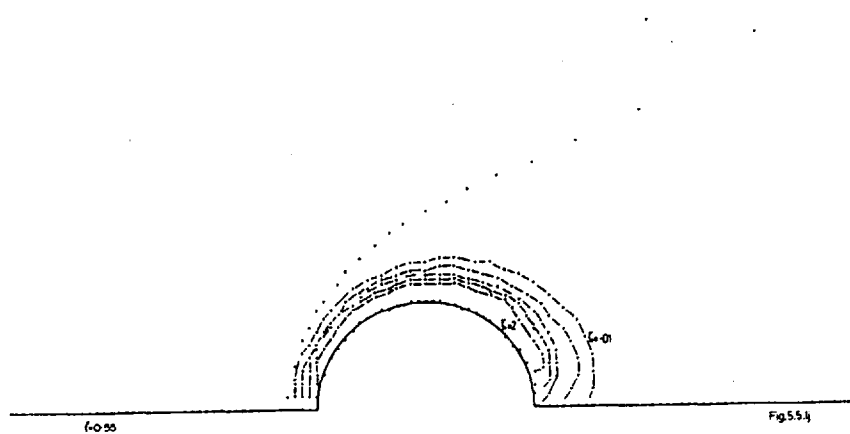
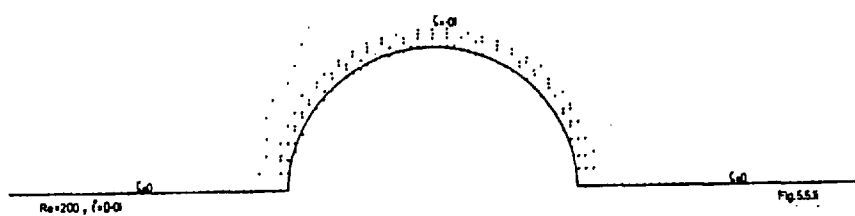
5.5 REYNOLDS NUMBER 200 SOLUTION

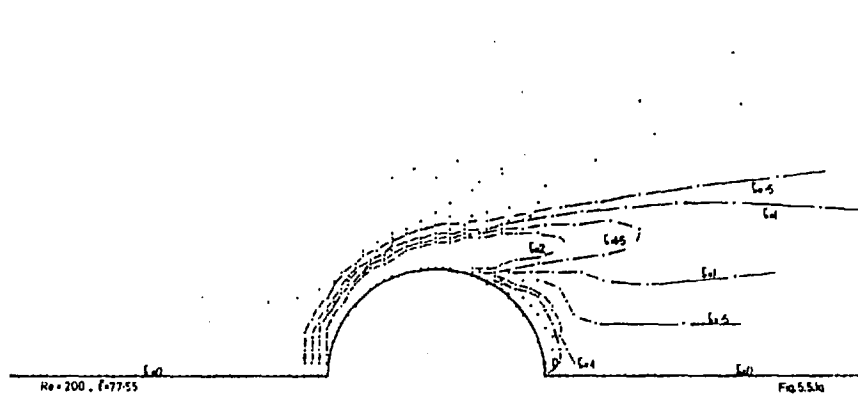
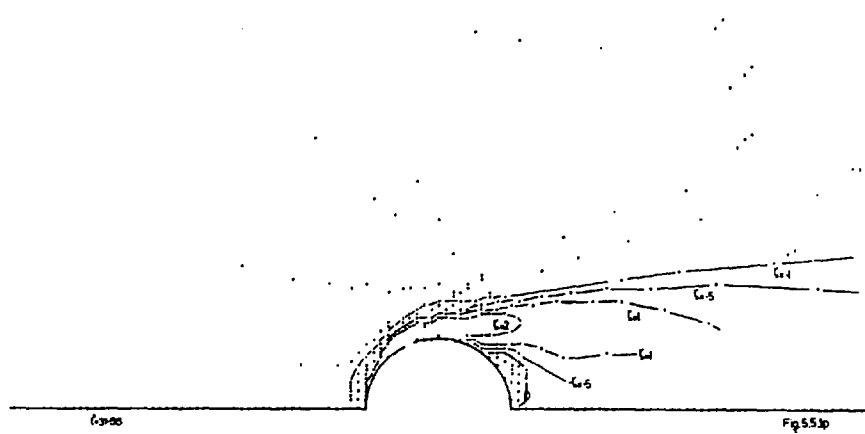
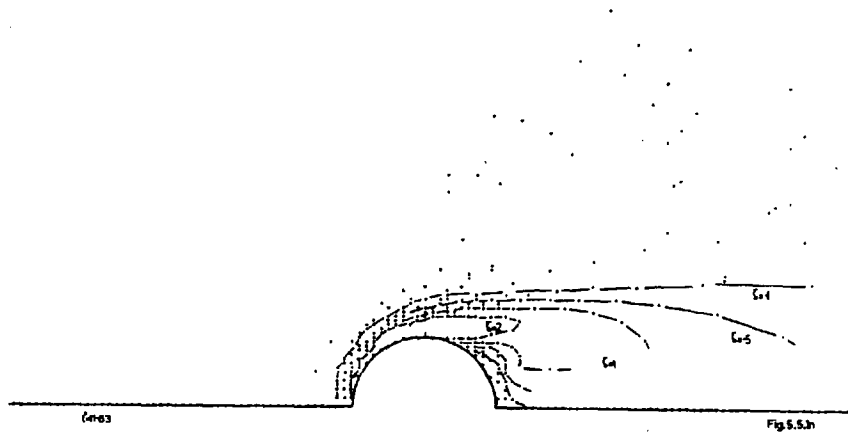
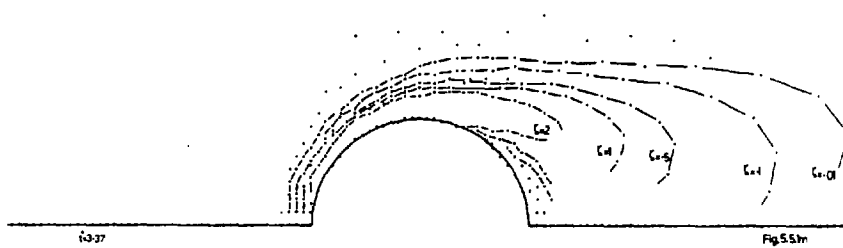
The simulation of the flow field at a Reynolds number of 200 starting from the potential flow field is shown by Fig. 5.5.1a to Fig. 5.5.1h. The flow separation first appeared at about dimensionless time $t^*=0.8$. The separation angle grows extremely fast with time as shown by Fig. 5.5.1a at $t^*=1.11$ and Fig. 5.4.6. The growth of the wake downstream is slower; it appears as if the wake begins to grow length-wise, after an almost steady state value of the separation angle has been attained. Once this growth begins, it continues almost linearly with time and reaches a high value in comparison with the reported experimental value. The broken-line part of the curve in Fig. 5.4.6 shows the development of a 'wake tail'. The wake length was defined as the distance between the sphere surface and the point where the zero stream line meets the downstream symmetry axis, and this point was calculated by linear extrapolation. Due to the narrow tailing effect the wake length obtained by linear extrapolation is not as accurate as one would like. This effect is shown by the broken-line curve in Fig. 5.4.6.

The final levelling of the growth of the wake length shown in Fig. 5.4.6 is due to the effects of the outflow boundary condition. The 'wake tail' seem to grow until the outflow boundary condition forces it to reach a steady state. Comparison of the curve for $Re=500$ in Fig. 5.4.6 with $Re=200$ curve shows that the final wake length for









$Re=500$ is smaller than at $Re=200$. This result is also due to the effect of the outflow boundary condition. The wake at $Re=200$ is actually smaller, but since the outflow boundary condition is the same for both Reynolds numbers, it automatically becomes a better approximation at $Re=200$ than at $Re=500$. Comparison with the steady state results of Al-Taha (1969) (Table 2) and with the time-dependent solution of the Navier-Stokes equations by Rimon (1968) confirm this. The wake length appears to be very large and is in disagreement with Taneda's (1956) experimental results. In fact, Taneda found that around this Reynolds number the end of the wake becomes ill-defined and oscillations occur. Experimental measurements are available only for the size of the wake bubble or eddy; the actual wake is thought to be much longer. The numerical results also show that the length of the wake bubble is less than that of wake length and Fig. 5.4.6 shows that after the dimensionless wake length has grown to value of approximately 2, the 'wake tail' begins to form. Therefore excluding the 'wake tail' the wake length ($L_w \approx 2$) when $Re=200$ is less than the final value of 2.55 for $Re=500$. The 'wake tail' did not appear for $Re=500$ due to the outflow boundary not being far enough from the sphere surface.

Rimon (1968) used a slightly coarser mesh and the outer boundary was 20.1 radii away compared with the present solution in which it is 11.023 radii away from the sphere centre. His solution shows a smaller wake length than

than the present solution, also it is smaller than the steady state solution of Al-Taha (1969) who used the same mesh size but located outer boundary at 6.686 radii away from the sphere centre. The difference between Rimon's and Al-Taha's solution can be explained either in terms of the parallel outflow boundary condition used by Rimon or that the solution was not integrated long enough in time to allow the full development of the wake to occur.

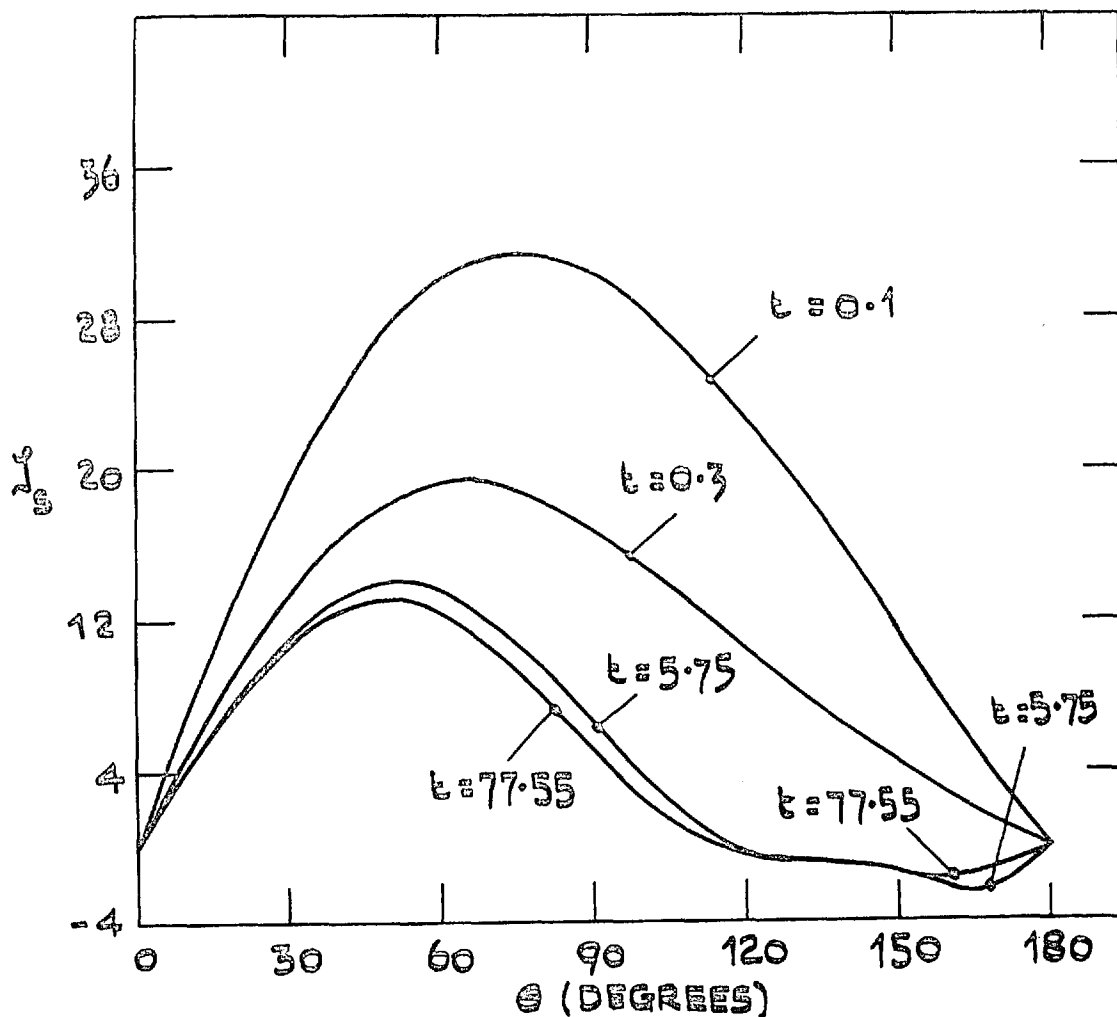
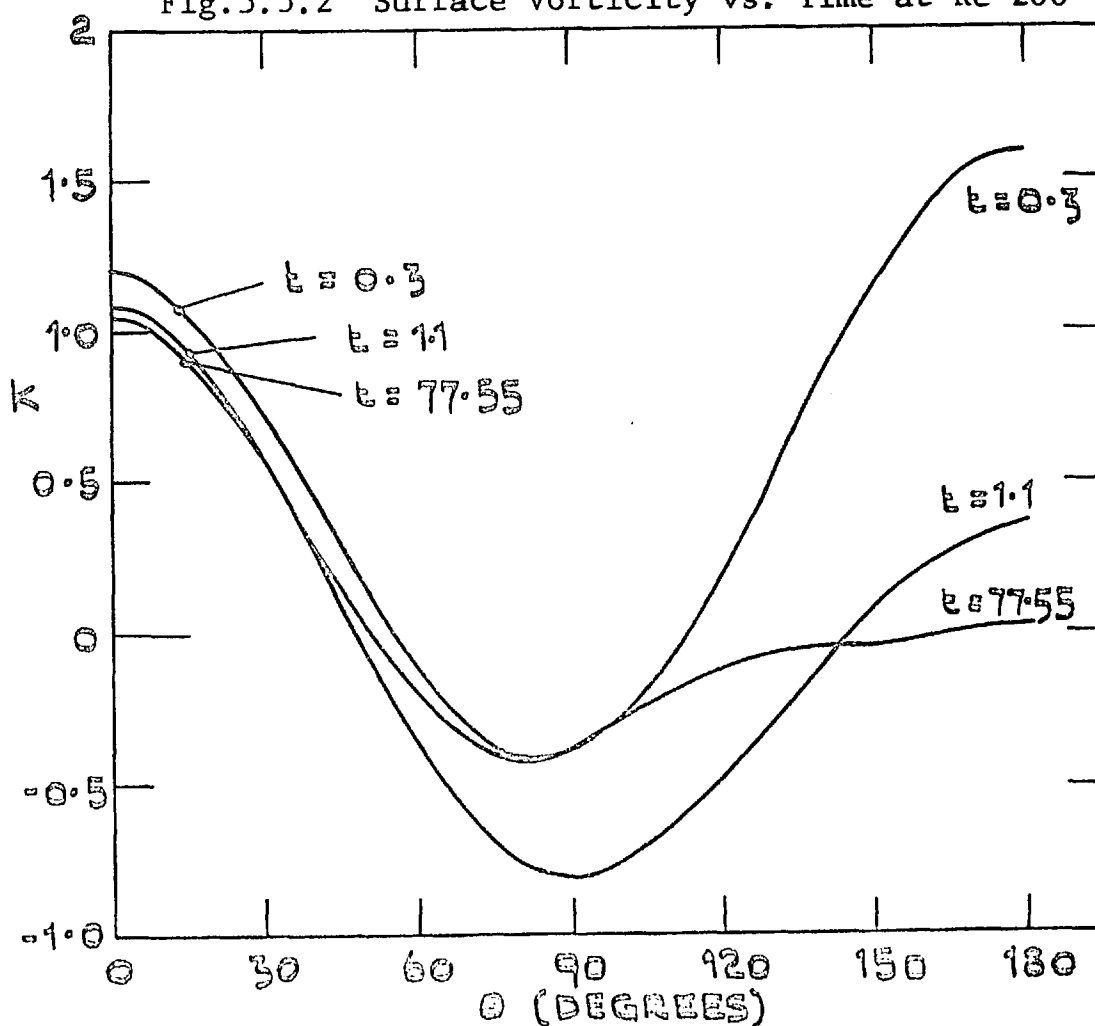
The 'wake tail' shown by present solution at $Re=200$ is a feature which was also detected by Son and Hanratty (1969) using a similar technique for the flow around a circular cylinder. Son and Hanratty located the outer boundary for the Reynolds number of 200 solution at 152.4 radii away from the cylinder centre. Their results show an ever increasing wake length, but as with the present work it is reasonable to suppose that if their solution had been integrated long enough in time a steady state would have reached. These results confirm the effect of the outflow boundary condition on the shape of the wake.

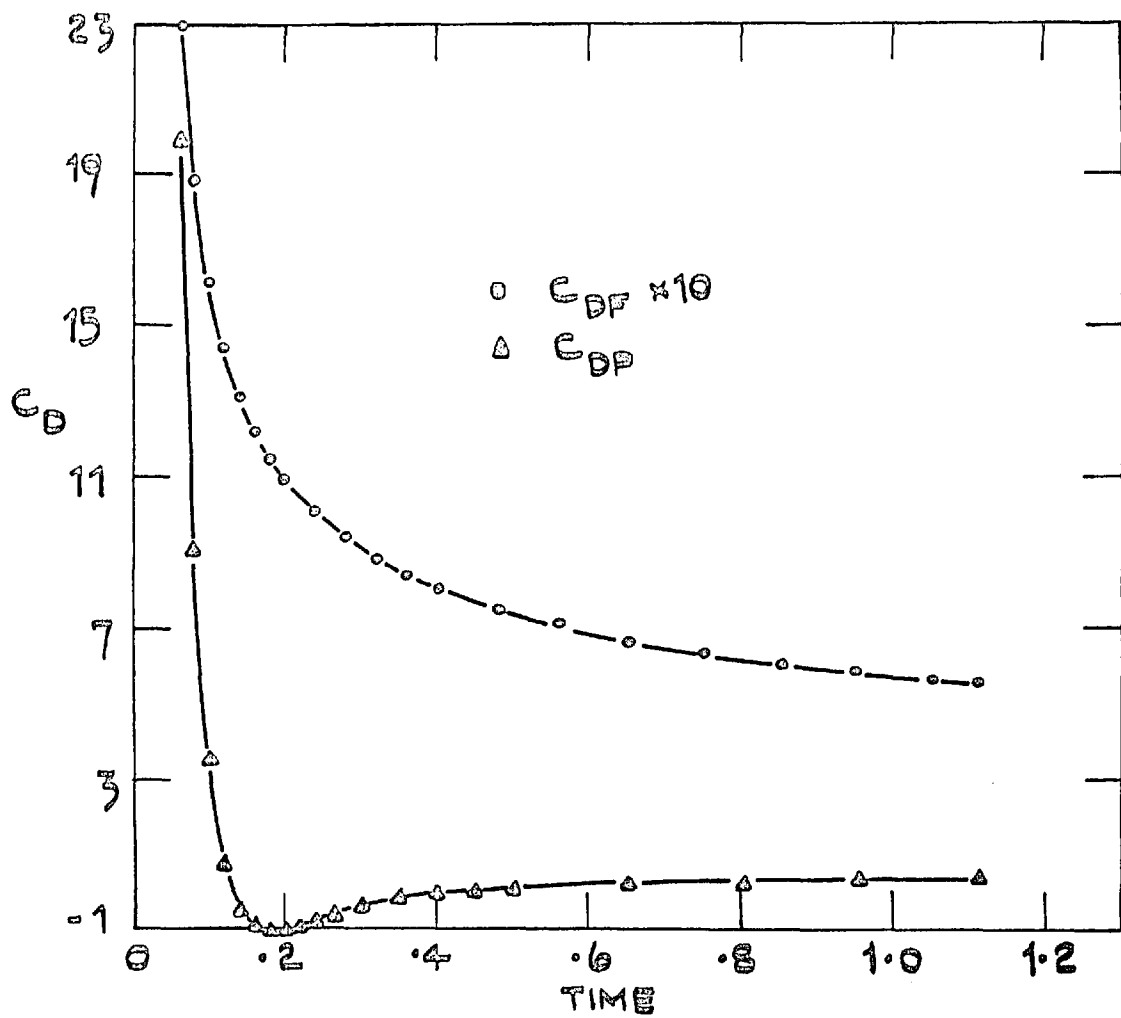
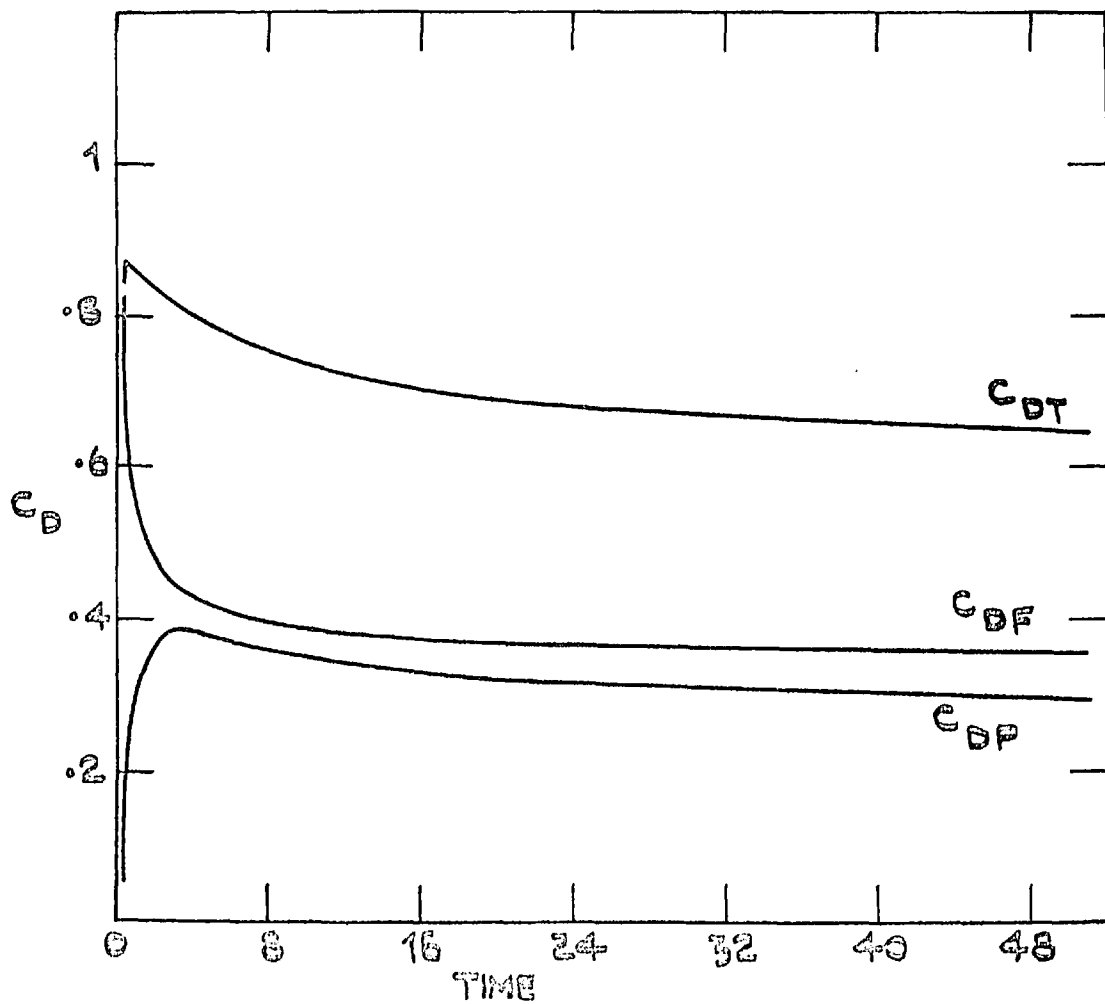
The generation, diffusion and convection of vorticity in time is shown by Fig. 5.5.1i to Fig. 5.5.1q. As expected the convection effects are much larger than for $Re=50$ and set in earlier in time. The 'noise' level in the solution is also higher than at $Re=50$ and hence there is a decrease in the overall accuracy of the solution.

The variation of surface vorticity and surface pressure with time is given by Fig. 5.5.2 and Fig. 5.5.3 respectively. The steady state surface vorticity shows good agreement with solutions by Al-Taha (1969) and Rimón (1968) but the pressure at the rear stagnation point (Table 2) is in disagreement with the value given by Al-Taha, Rimón and Le Clair et al. This is due to the different wake structure shown by the present solution as compared with solutions by these workers.

The behaviour of the drag coefficients with time is shown by Fig. 5.5.4 and Fig. 5.5.5. The initial variation of the pressure drag follows the same pattern as shown by other solutions at lower Reynolds number; the only difference is the period of this initial variation which for the present solution is greater than for previous solutions. Therefore greater reliability can be given to the actual values of the pressure drag coefficient. The variation with time of the total drag coefficient shown in Fig. 5.5.5 appears to be correct for moderate Reynolds numbers. This variation with time was also shown by solutions of Thoman and Szczczyk (1966) and Ingham (1968) for flow around a circular cylinder.

The steady state total drag coefficient shown in Table 2 is less than the experimental value. This difference is attributed to the smaller value of the pressure drag coefficient which depends on the structure of the wake. The wake structure itself is dependent on

Fig.5.5.2 Surface Vorticity vs. Time at $Re=200$ Fig.5.5.3 Surface Pressure vs. Time at $Re=200$

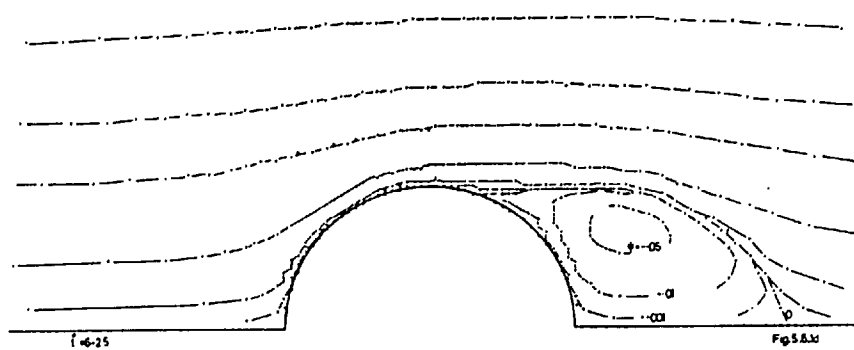
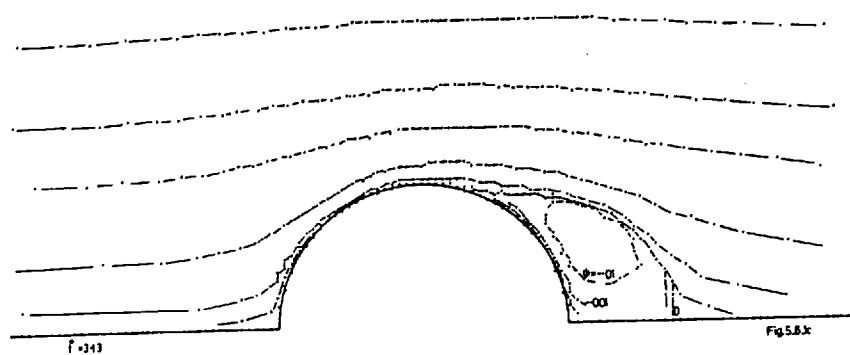
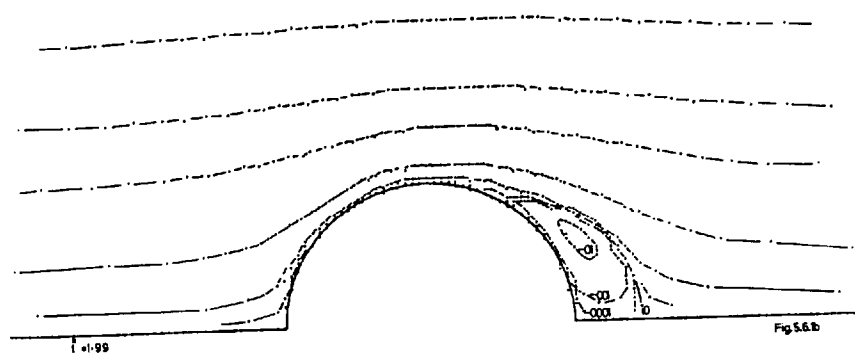
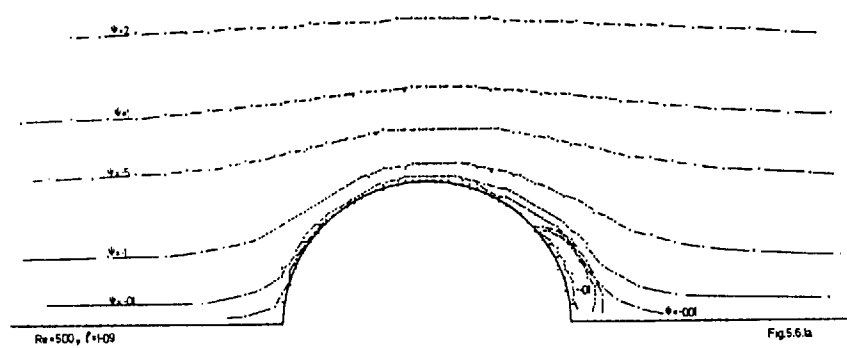
Fig.5.5.4 Drag Coefficients vs. Time at $Re=200$ Fig.5.5.5 Drag Coefficients vs. Time at $Re=200$

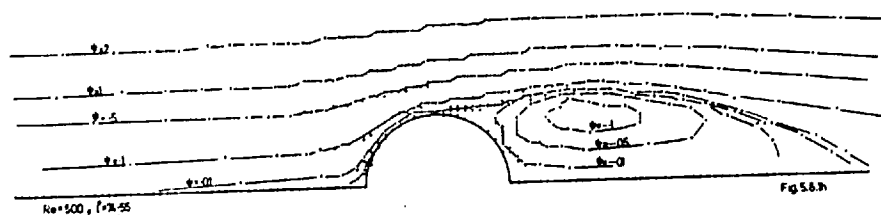
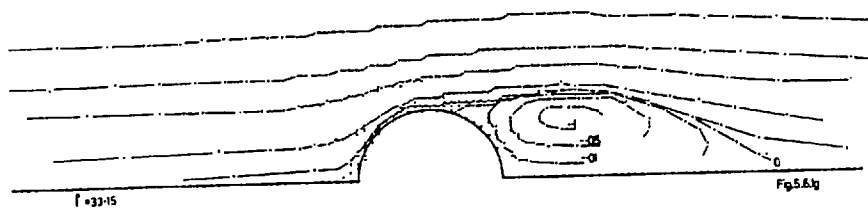
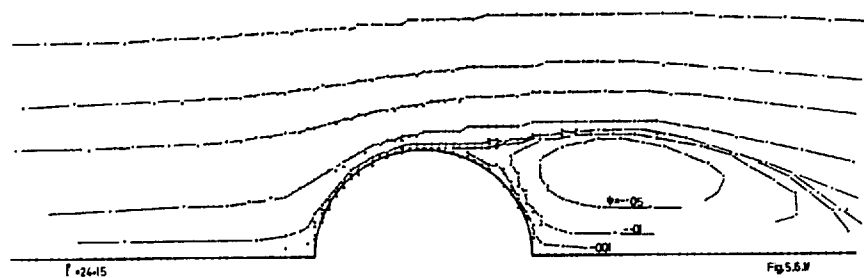
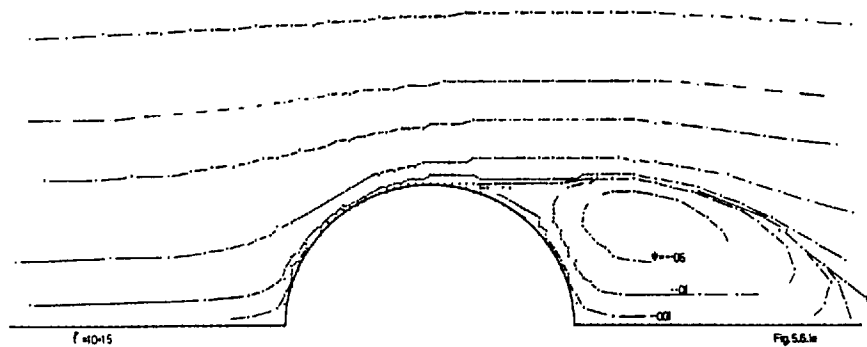
the outer boundary condition and the mesh size. Therefore the present value of the drag coefficient is consistent when compared with results obtained at lower Reynolds numbers.

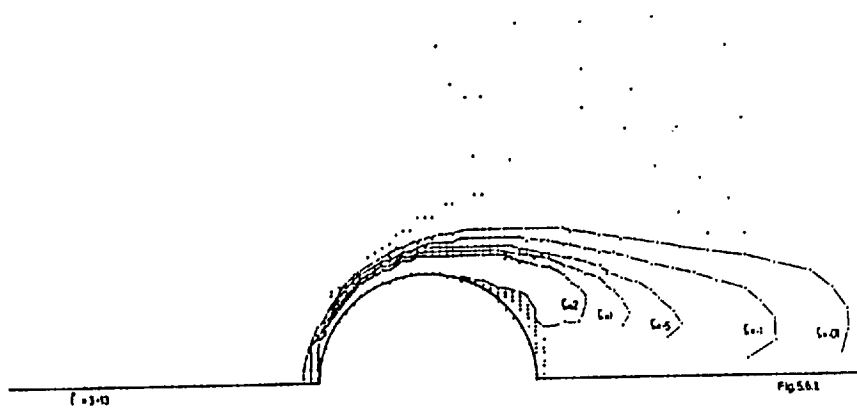
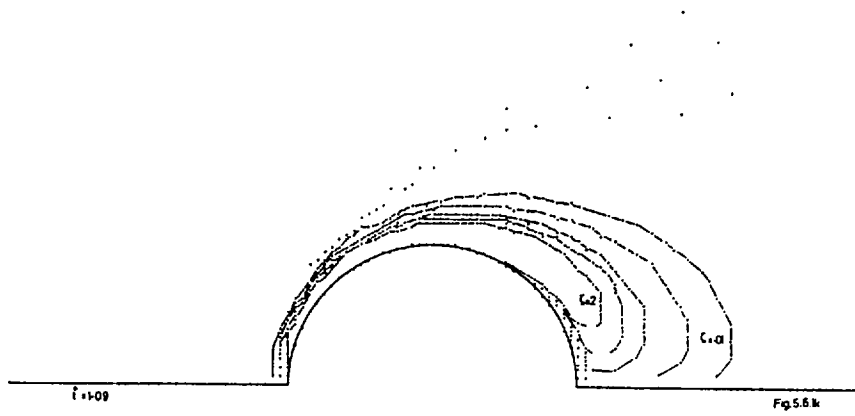
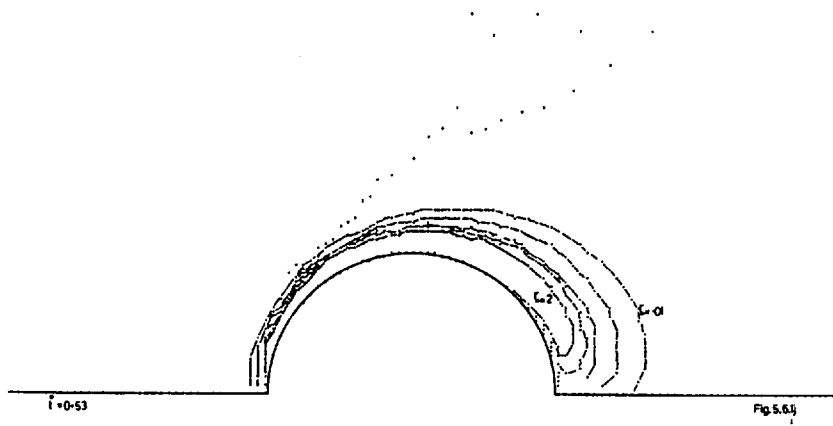
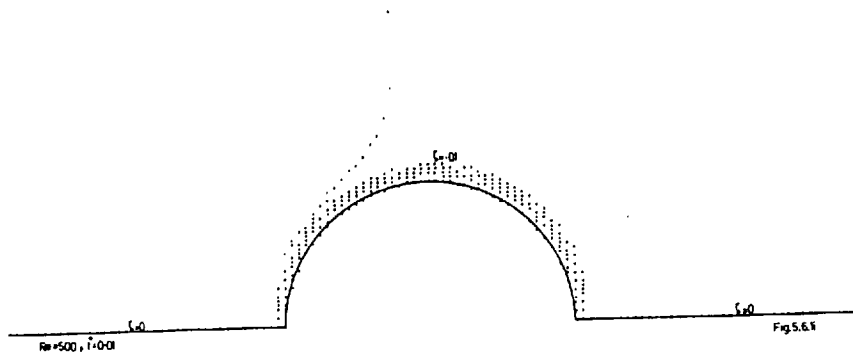
5.6 REYNOLDS NUMBER 500 SOLUTION

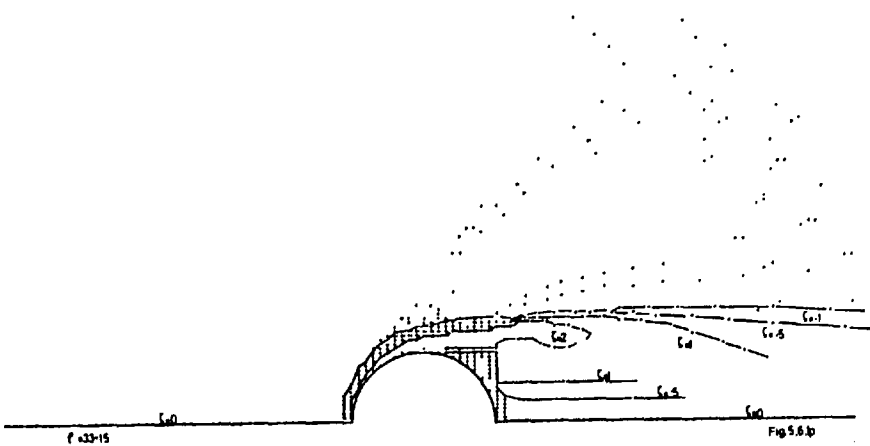
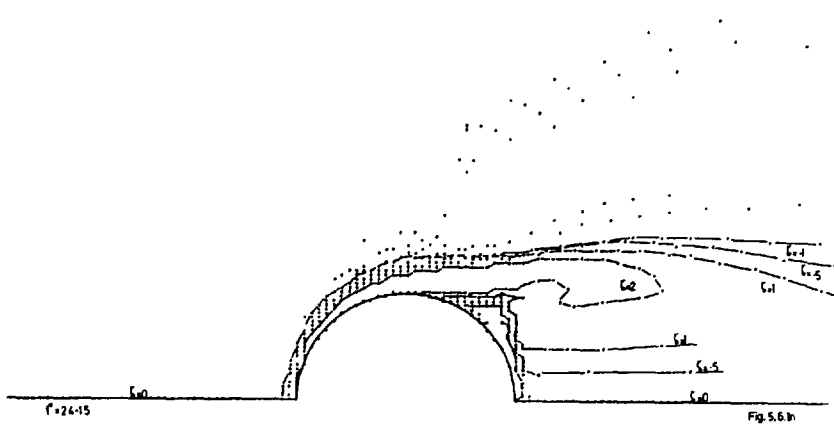
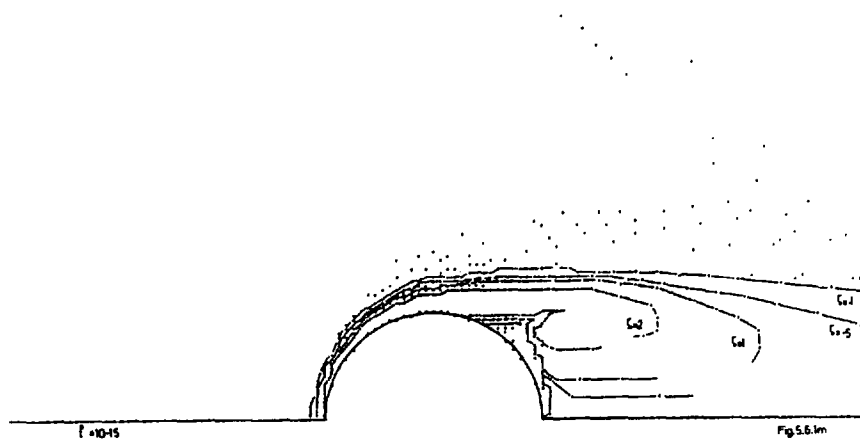
The development with time of the stream lines at $Re=500$ starting from the potential flow stream lines is shown by Fig. 5.6.1a to Fig. 5.6.1h. The generation, diffusion and convection of vorticity from the sphere surface is shown by Fig. 5.6.1i to Fig. 5.6.1q. The initial radial diffusion shown by Fig. 5.6.1i occurs only for a short time after which convection quickly distorts the vorticity contours and a wake develops. The 'noise' level in the solution is higher than at $Re=200$ but does not affect the numerical stability of the solution. The variation with time of the separation angle and the wake length is shown in Fig. 5.4.6. The recirculatory wake first appears at dimensionless time $t^* \approx 0.4$ and develops very rapidly along the sphere surface as indicated by the sharp increase in the separation angle and then it starts to grow length wise until a steady state is reached.

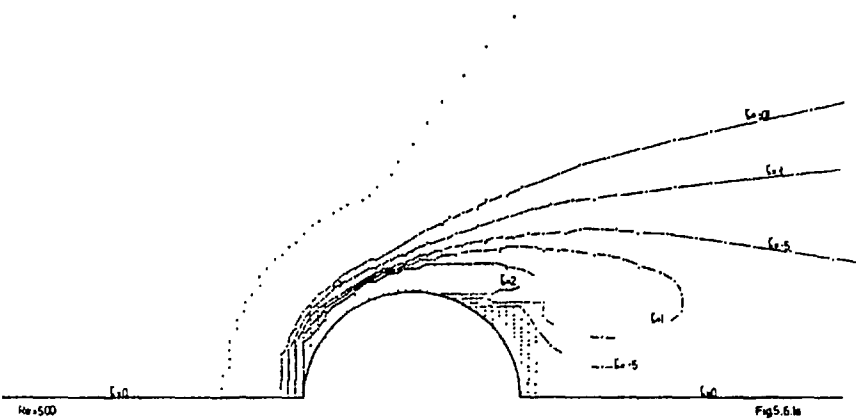
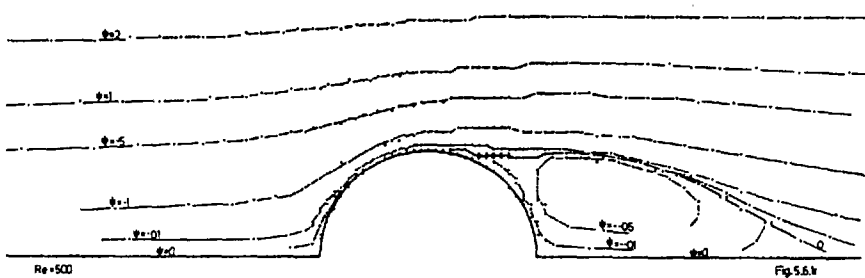
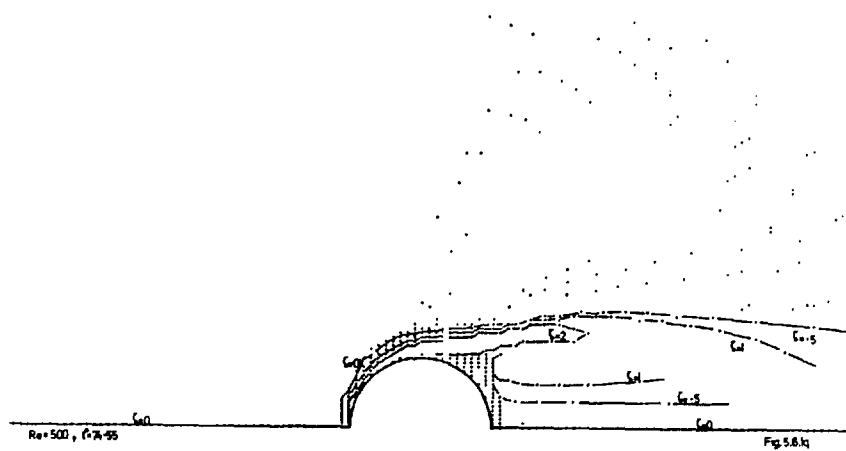
The variation of drag coefficients with time shows the same behaviour as at lower Reynolds numbers (Fig. 5.6.2 and Fig. 5.6.3. However, the steady state total drag coefficient does not appear to be in agreement with the







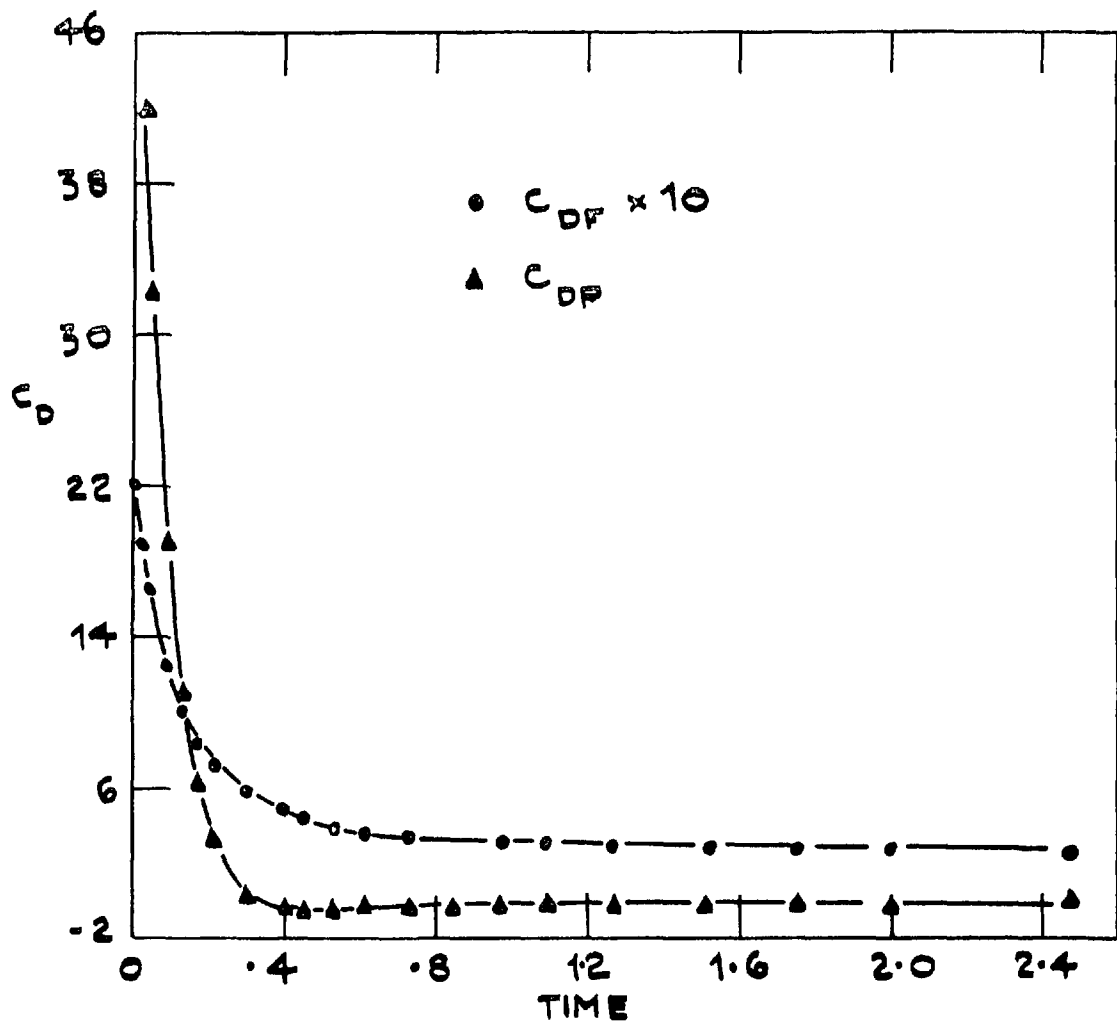
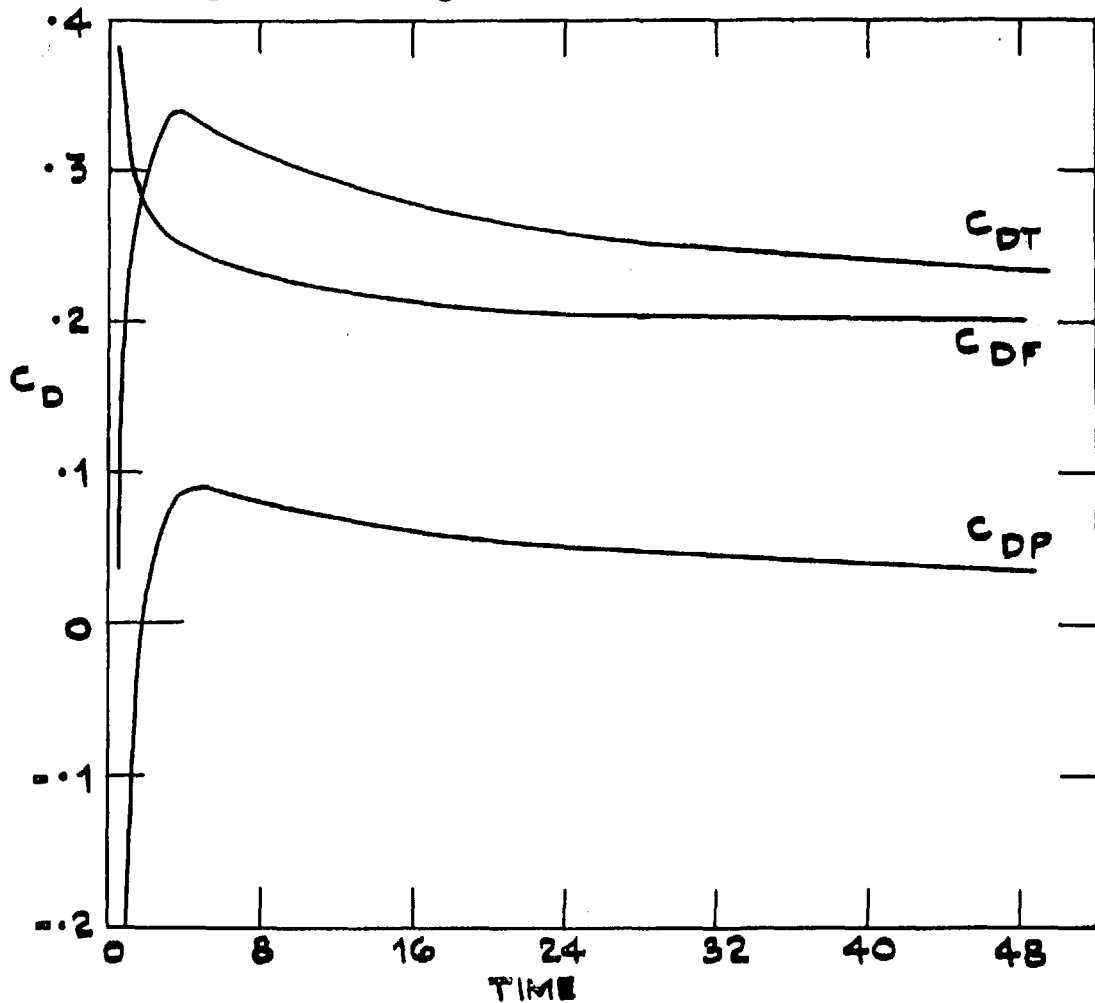


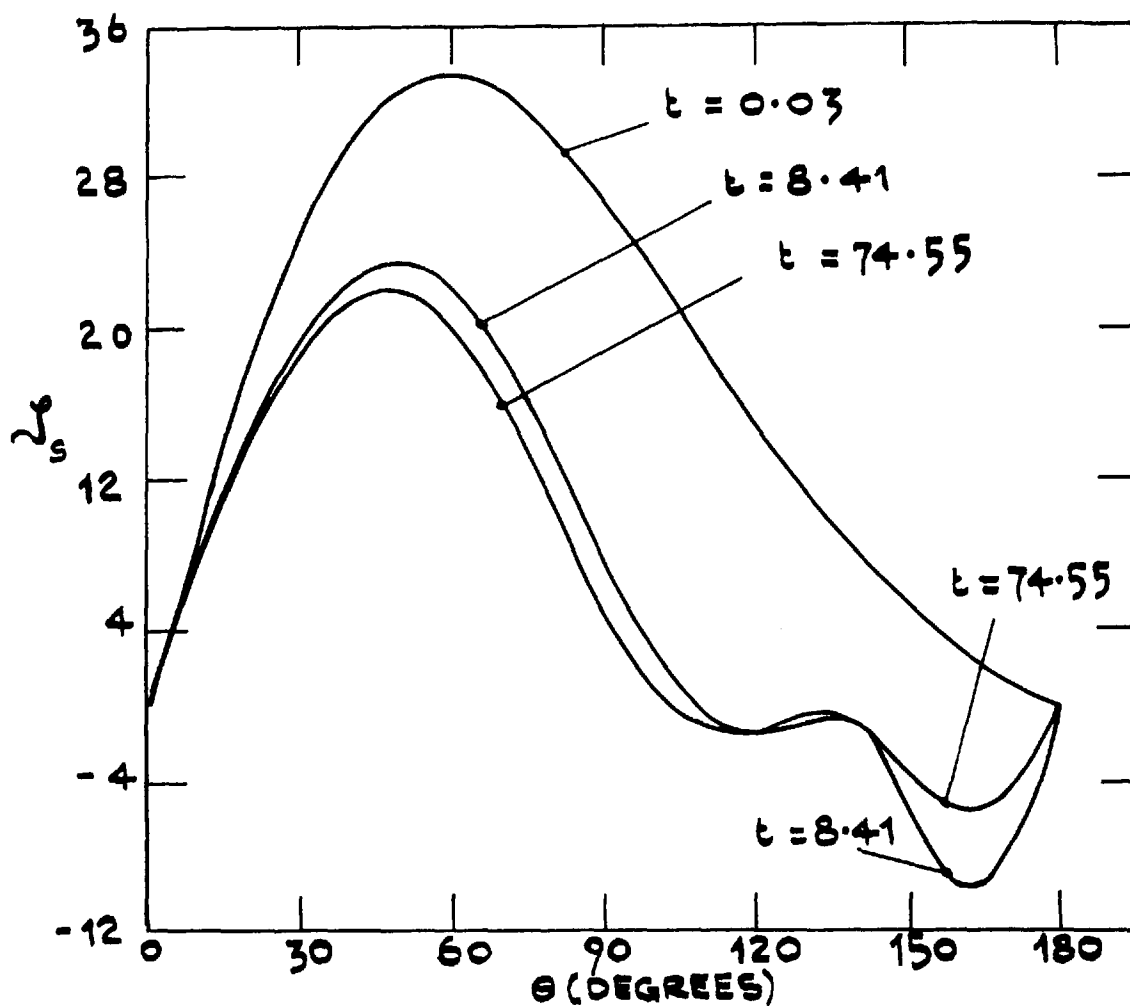
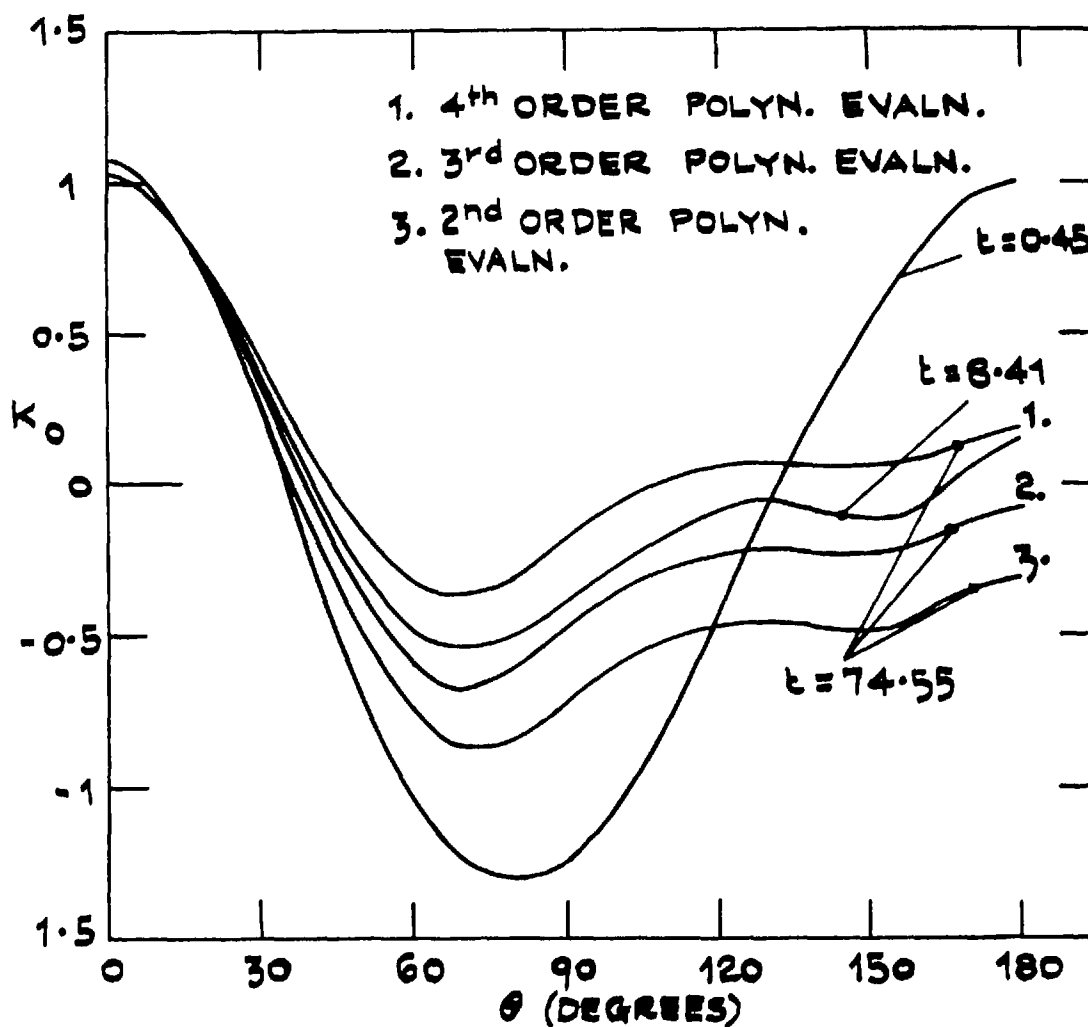


experimental value of Lapple and Shepherd(1940) and also the pressure drag coefficient is less than the viscous drag coefficient. The value of the pressure drag coefficient is an order of magnitude smaller than the expected value but it is in agreement with the calculated steady state value of Al-Taha (1969) who used the same mesh size. All the flow characteristics in the present solution have the expected values except for the pressure drag coefficient. Therefore at this stage of the simulation an investigation was carried out to explain this discrepancy.

The evaluation of the pressure drag coefficient requires surface gradients of vorticity. Numerical tests at lower Reynolds numbers by Al-Taha showed that evaluation of gradients by fitting fourth order polynomial gave accurate results and were not too affected by the order of the polynomial used. Table 5 shows the effect of the order of polynomial used at various Reynolds numbers and mesh sizes.

The Reynolds number 500 solution with $h=0.04$ and $k=3^\circ$ shows the greatest variation in the pressure drag coefficient with the order of polynomial used. It is clear that this mesh size is fairly coarse and the vorticity changes considerably in the region over which the polynomial is fitted. In fact, at this Reynolds number a boundary layer begins to form on the front half of the sphere (Fig. 5.6.1q) and a fourth order polynomial is fitted across the boundary layer. In this case the

Fig.5.6.2 Drag Coefficients vs. Time at $Re=500$ Fig.5.6.3 Drag Coefficients vs. Time at $Re=500$

Fig.5.6.4 Surface Vorticity vs. Time at $Re=500$ Fig.5.6.5 Surface Pressure vs. Time at $Re=500$

lowest order polynomial is most satisfactory and gives the value of the total drag coefficient which is in agreement with the experimental value. The effect of the order of polynomial on the calculation of pressure profiles is shown in Fig. 5.6.5 and it can be seen that the shapes of all the profiles are similar. The second order polynomial gives a deeper minimum in the surface pressure so that a better value of the pressure drag coefficient is obtained. The fourth order polynomial was found to give satisfactory results for all other Reynolds number solutions.

In order to reduce the 'noise' level in the solution which was high and also to determine the effect of the choice of difference scheme on drag calculations, a solution using upstream difference was obtained. Table 4 shows the main results in comparison with those of the previous solution. Fig. 5.6.1r and Fig. 5.6.1s show the stream line pattern and the vorticity distribution respectively. The upstream differences in the convective term of the vorticity transport equation introduced artificial viscosity as shown by Fig. 5.6.1s. The vorticity contours are more spaced out than in Fig. 5.6.1q. The 'noise' level is less but the overall solution is worse than the central difference solution. The pressure drag evaluation also shows no improvement. However, the solution attained steady state much more quickly than the central difference solution. If the upstream difference solution is used to predict heat or mass transfer rates it is likely to give results in

reasonable agreement with experimental values as shown by the total drag coefficient.

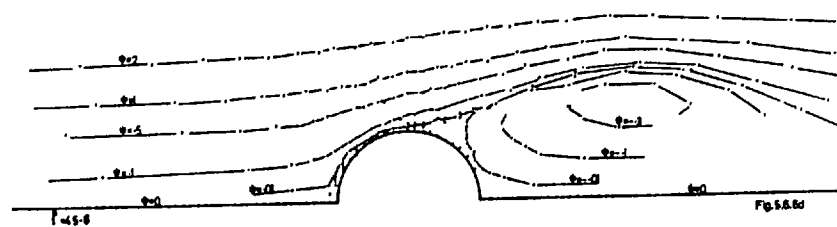
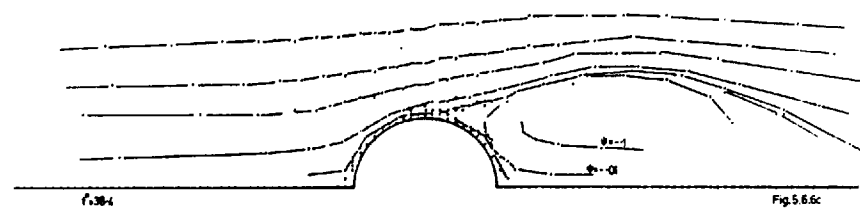
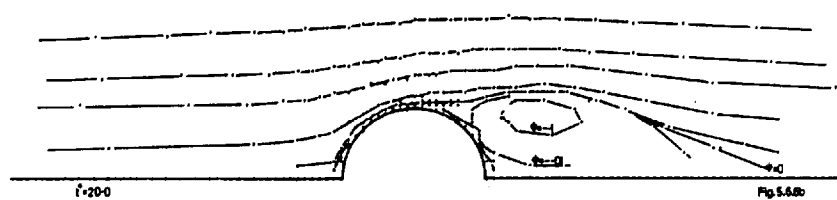
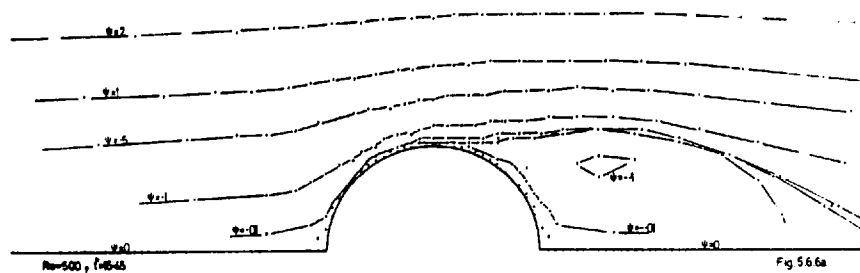
It was also decided to test the effect of the outflow boundary condition at this Reynolds number. Table 3 shows the main results using the parallel outflow boundary condition in comparison with the normal uniform parallel outflow boundary condition. No significant difference in the solutions was observed and as with the tests at $Re=1$ and 10 , the rate of convergence was slower when the parallel outflow boundary condition was used. An under-relaxation factor of 0.4 had to be used at the outflow boundary to obtain convergence and any value greater than 0.4 lead to divergence.

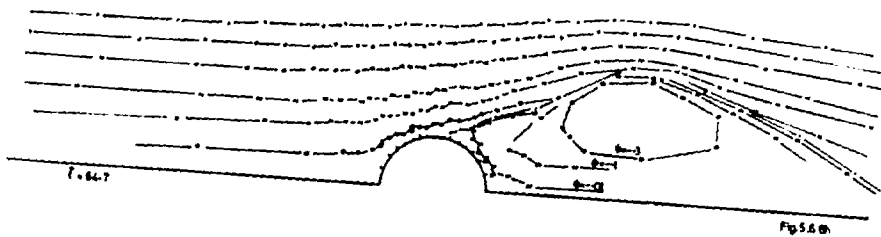
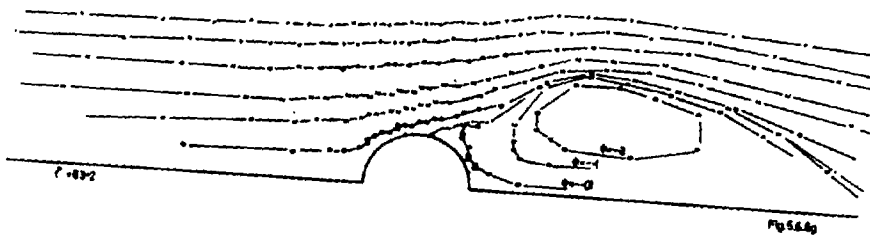
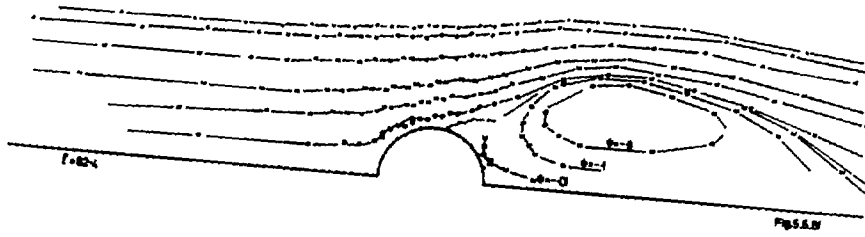
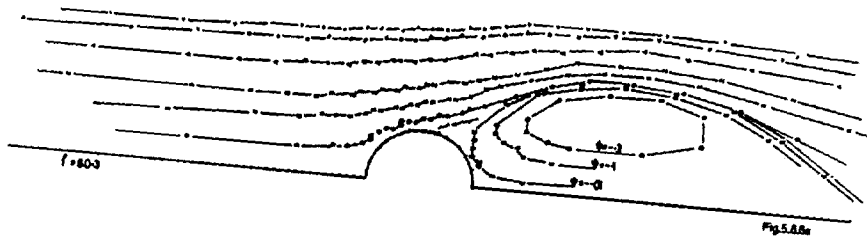
The results in Table 5 show that the original mesh size in the radial direction (Z -direction in the transformed space) is too large whereas the steps in the Θ -direction could be larger. Hence a solution was obtained such that the step size in the radial direction was halved and the step size in the Θ -direction was doubled so that the total number of nodes and the computing time for each time step was the same as for the original solution. To save overall computational time, the $Re=500$ solution at $t^*=15.05$ for the original mesh size ($h=0.04$, $k=3^\circ$) was used as an initial condition for the solution with the new mesh size ($h=0.02$, $k=6^\circ$). The pressure and total drag coefficients are plotted against time in Fig. 5.6.8 and Fig. 5.6.9 which show an instantaneous rise of the coefficients due to

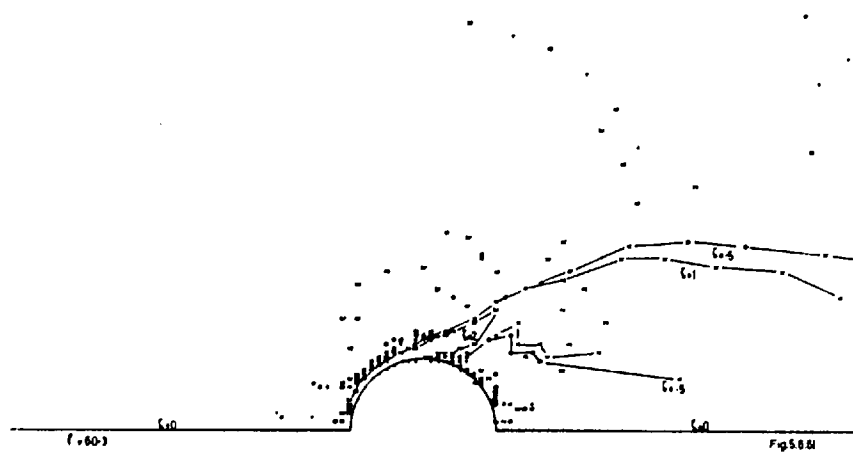
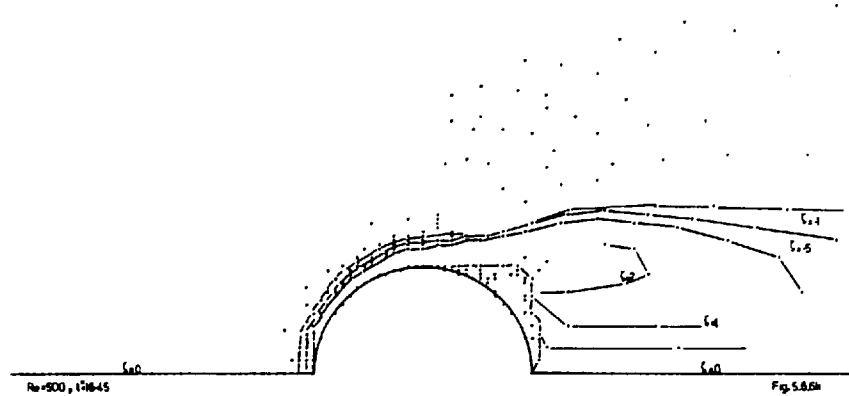
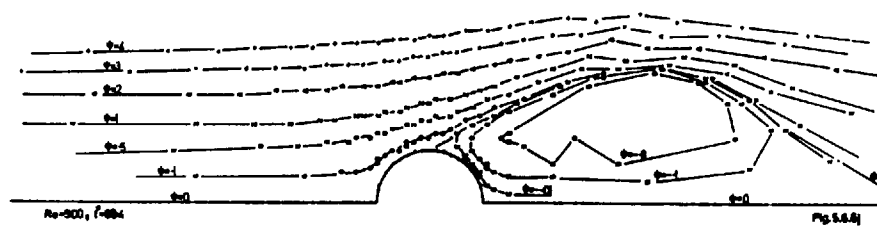
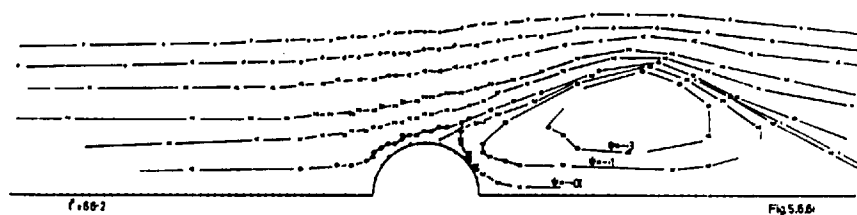
the effect of the changed mesh size on the evaluation procedure as discussed earlier. The coefficients then settle down and decrease in a similar manner to the original solution. However, the pressure and the total drag coefficient are always greater than those of the original solution so that the solution obtained with a mesh size of $h=0.02$ and $k=6^\circ$ is better than that obtained with $h=0.04$ and $k=3^\circ$.

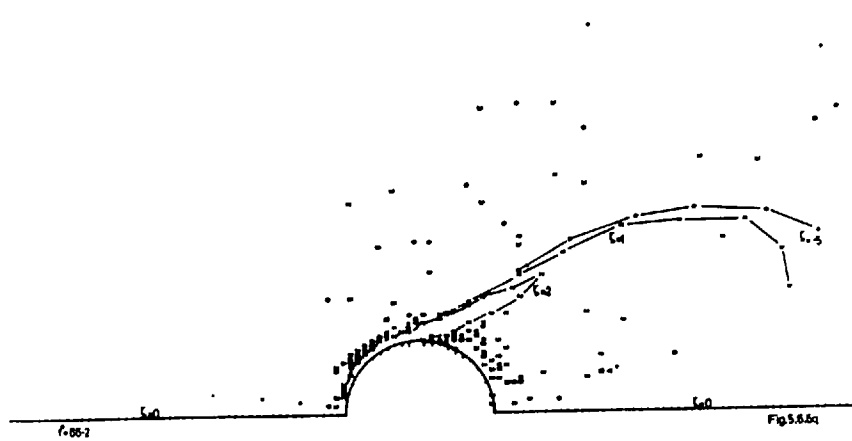
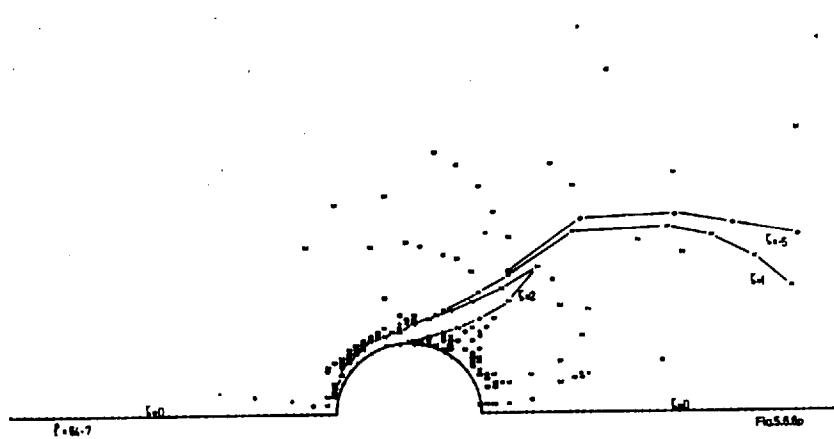
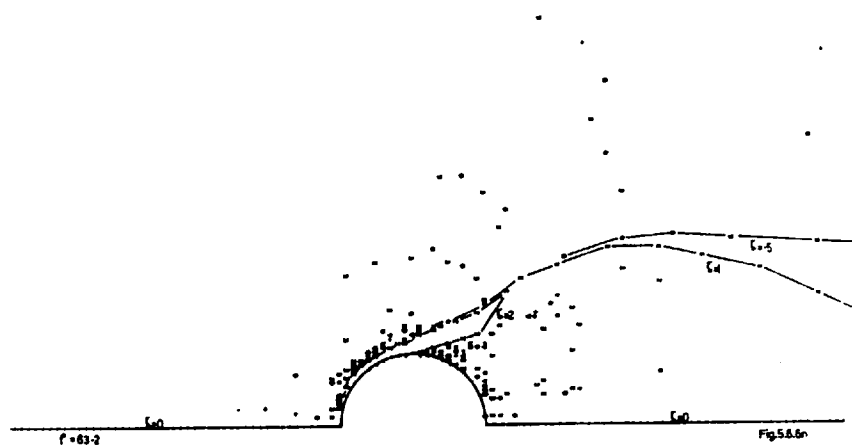
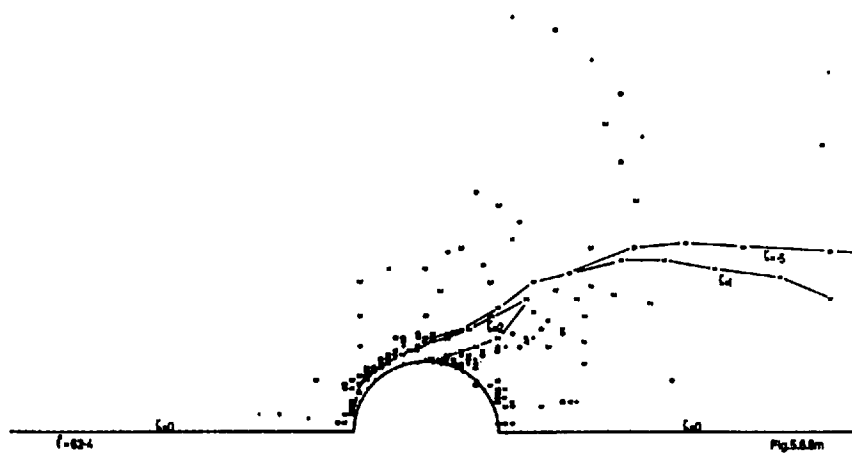
The development with time of the stream lines is shown by Fig. 5.6.6a to Fig. 5.6.6j. In comparison with the original solution, the main difference in the flow pattern in the early stages is the growth of the wake eddy, as shown by Fig. 5.6.10. The strength of the vortex increases with time and the drag coefficients decrease with time. The width of the eddy as measured from the downstream symmetry axis also increases and when it becomes larger than the sphere radius its rate of growth also becomes larger (Fig. 5.6.6d). At this point the eddy starts to gain energy from the main stream. This flow configuration is unstable and the vortex centre shifts downstream and a small secondary vortex appears (Fig. 5.6.6e and Fig. 5.6.6f). Before the rapid growth of the vortex becomes noticeable from the stream function plots the wake shows signs of instability as shown by the fluctuations in the minimum surface vorticity in Fig. 5.6.12. These fluctuations grow with time and eventually become too large for the numerical technique to handle.

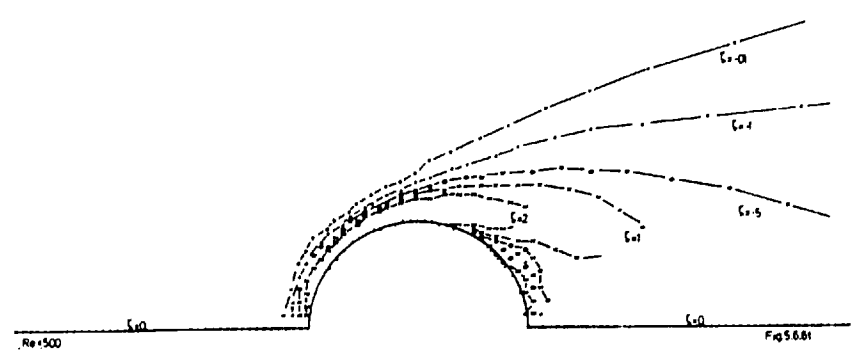
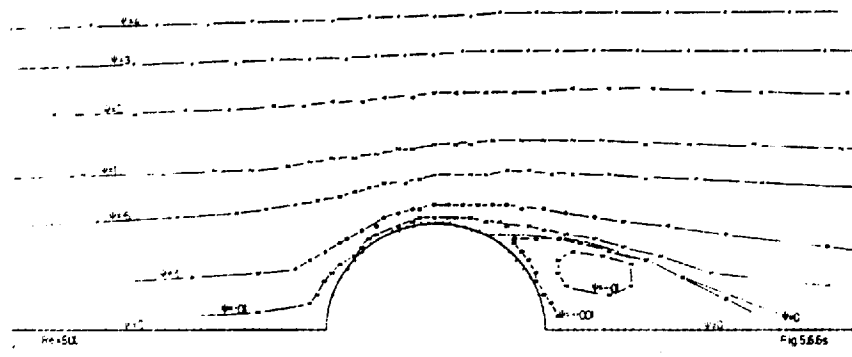
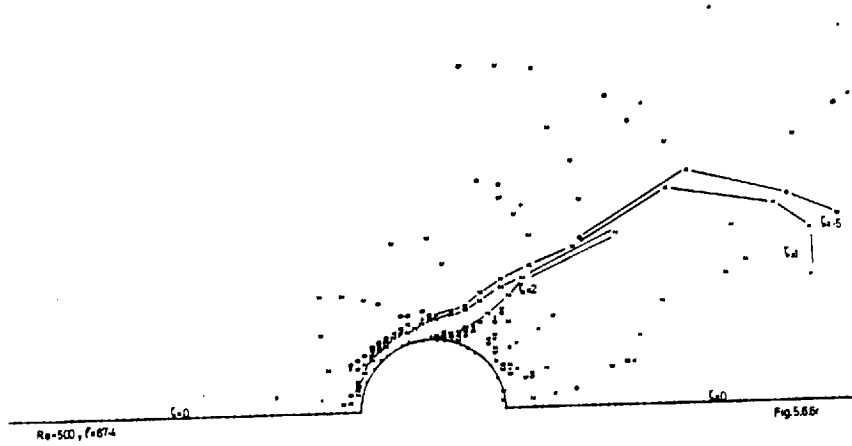
During the growth of the vortex, vorticity shedding occurs as shown by the lengthening and shortening of the





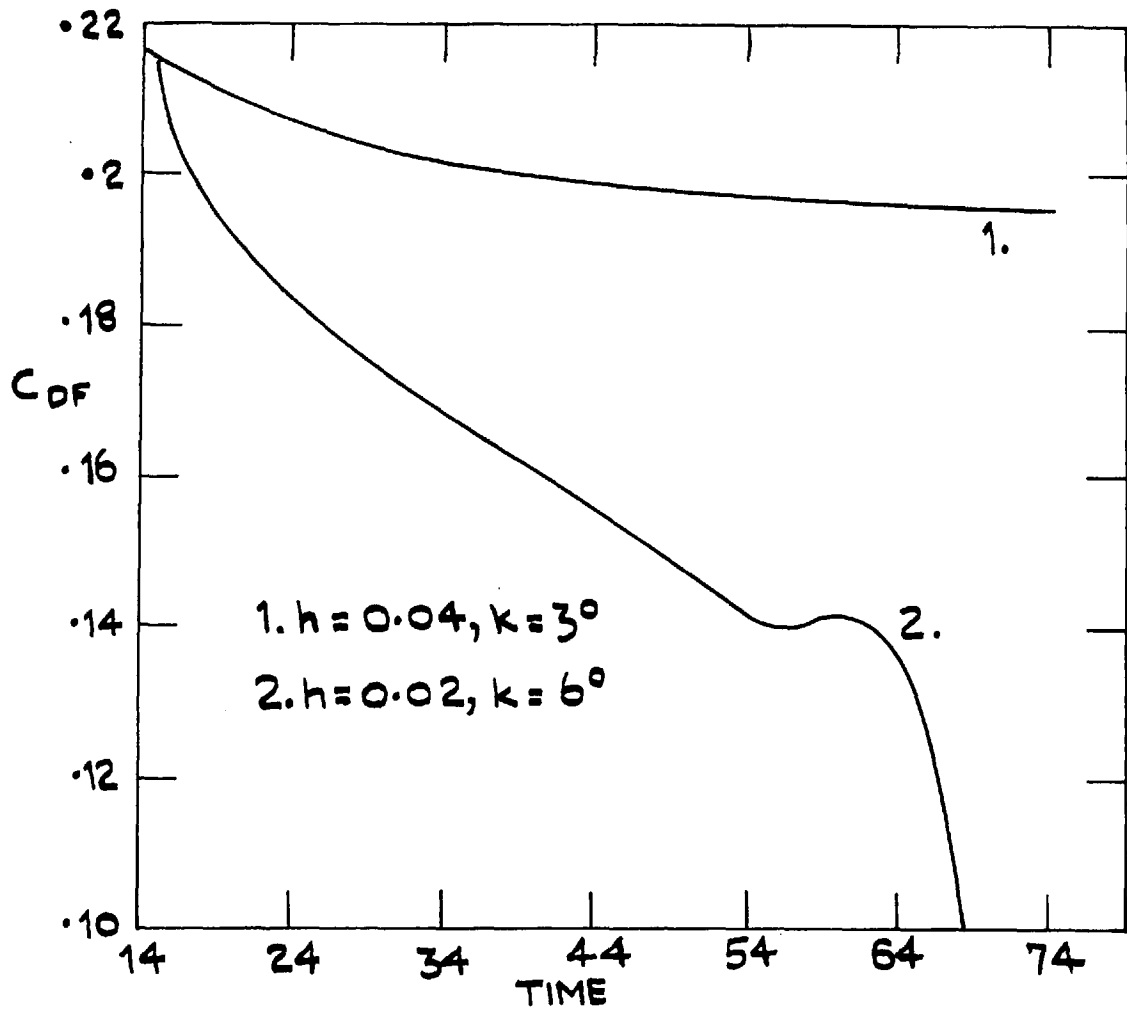
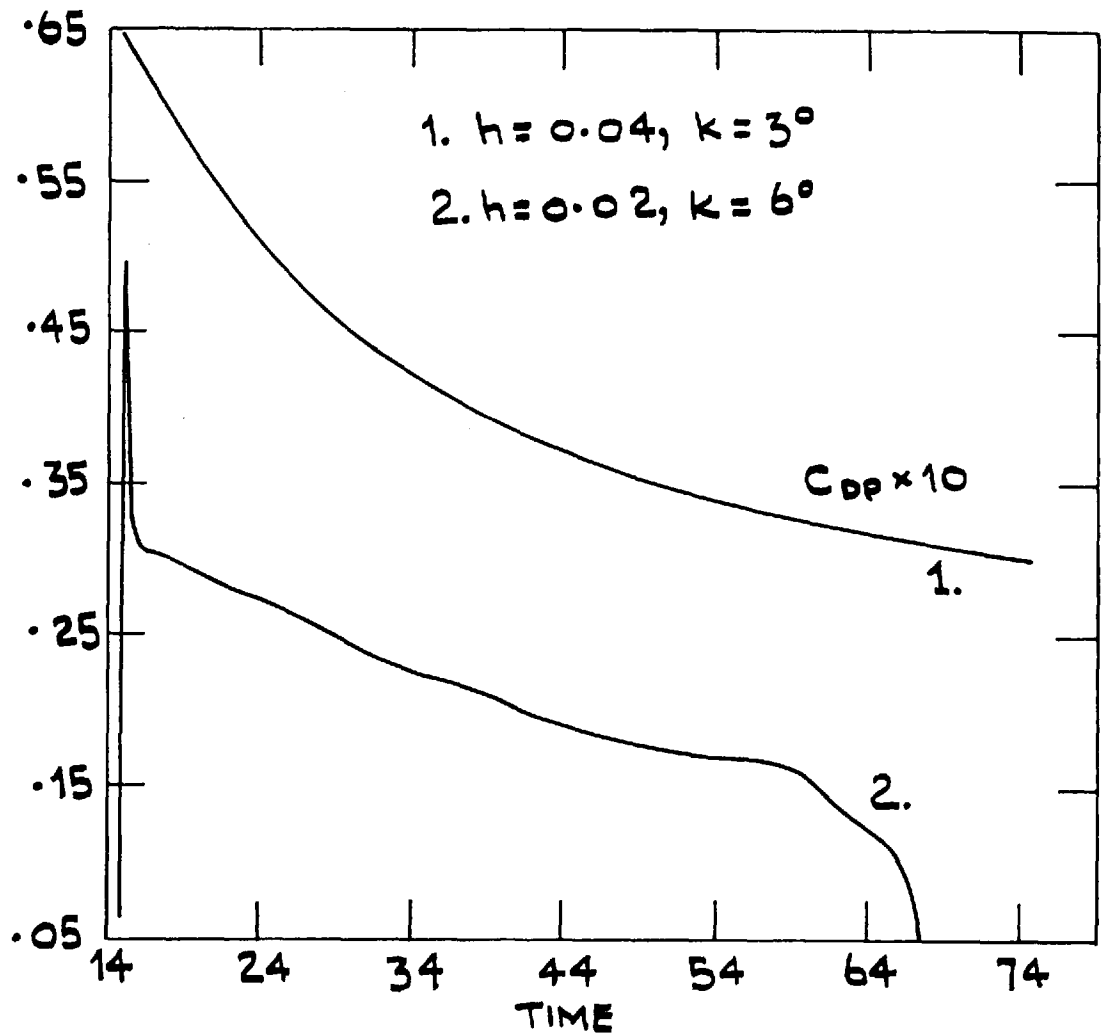






vorticity contours of value 2 in Fig. 5.6.6k to Fig. 5.6.6s. A study of the numerical output also recorded the presence of a vortex loop which appeared to move down-stream and eventually diffused. A vorticity loop is visible in Fig. 5.6.6r. As the solution proceeds in time distribution of vorticity becomes less regular and the 'noise' level increases.

At a dimensionless time of about 60 (Fig. 5.6.6e) the solution suddenly shows large changes in the flow pattern. The vortex centre moves both downstream and up into the main stream (Fig. 5.6.6f and Fig. 5.6.6g). By $t^* = 64.7$ the movement of the vortex is clearly visible in Fig. 5.6.6h. The increase in the wake length and the decrease in the separation angle are shown in Fig. 5.6.10. This movement of the vortex involves a large gain of energy by the vortex. Figures 5.6.6i and 5.6.6j show the flow trying to recover its original pattern. Meanwhile the irregularities in the vorticity increase and the vortex gains extra energy which it is unable to dissipate in the form of diffusion of vorticity. After $t^* \approx 64$, changes in the flow pattern occur extremely rapidly and the solution 'goes to infinity' in the wake region (vorticity becomes very large). The final breakdown of the solution is shown by figures 5.6.7 to 5.6.12. It was thought that the breakdown of the solution occurred because the vortex could not dissipate its energy and recover its original form. To test this hypothesis numerical tests were carried out with the very irregular solution at $t^* = 68.1$ by first using very

Fig.5.6.7 Drag Coefficients vs. Time at $Re=500$ Fig.5.6.8 Drag Coefficients vs. Time at $Re=500$

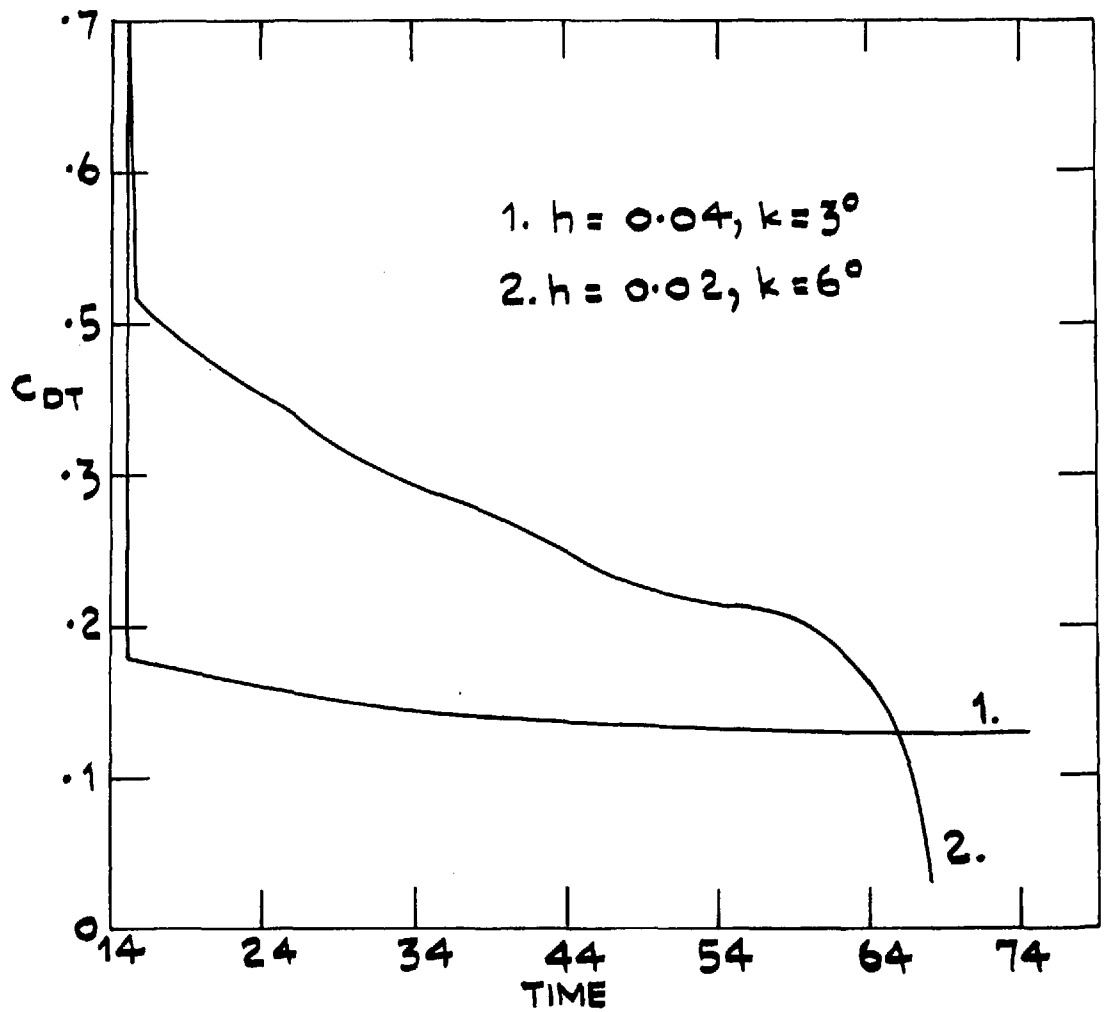


Fig.5.6.9 Drag Coefficient vs. Time at $Re=500$

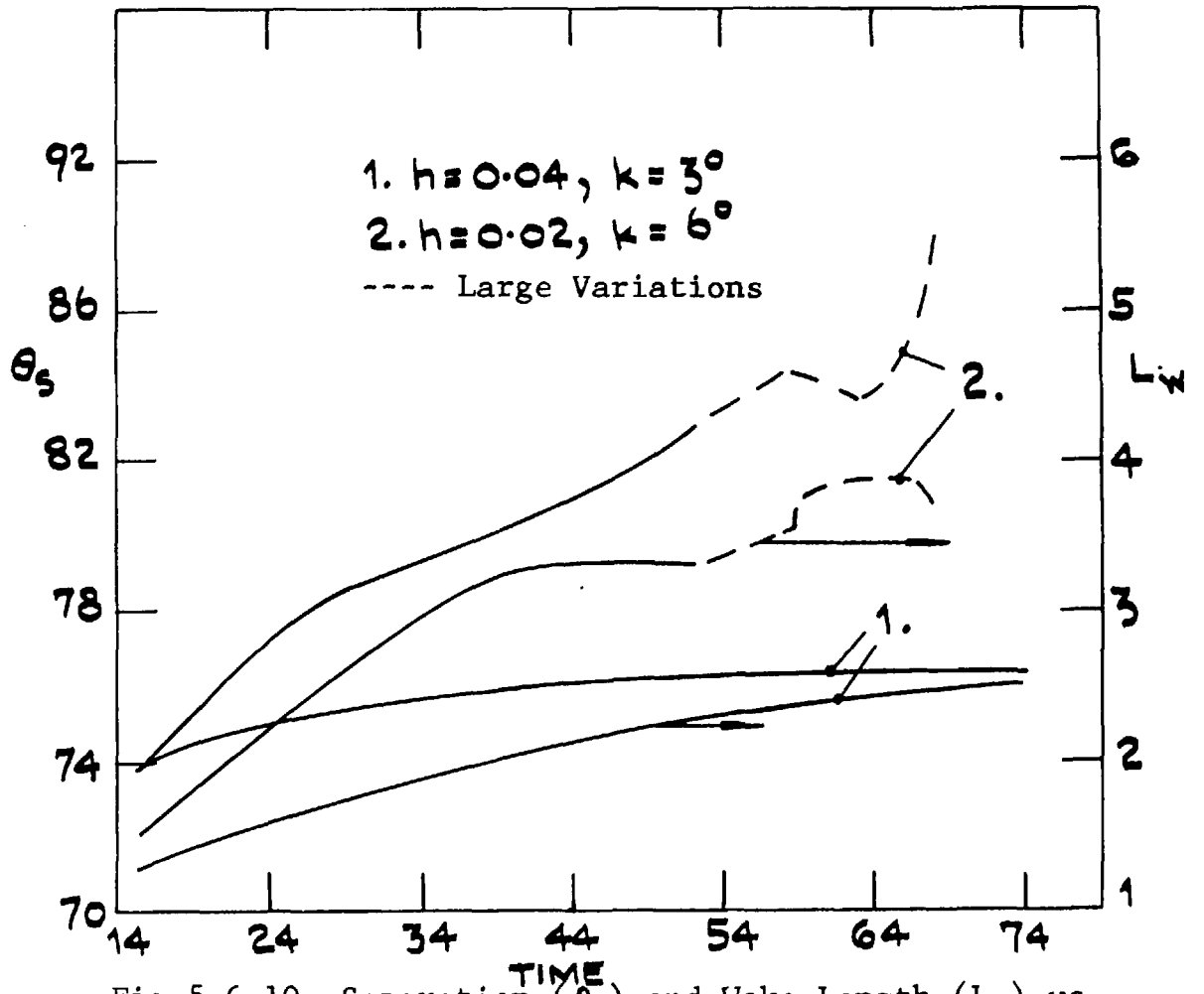
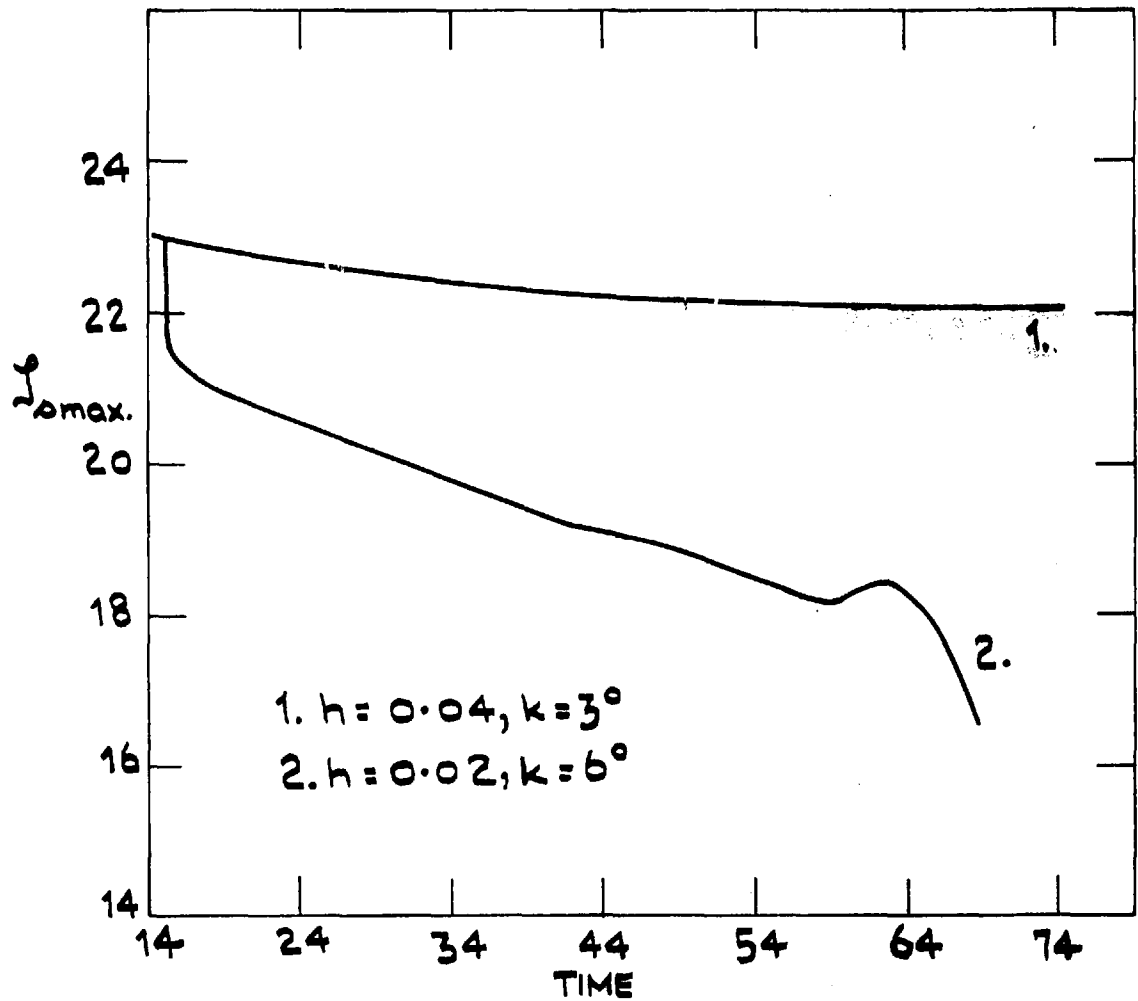
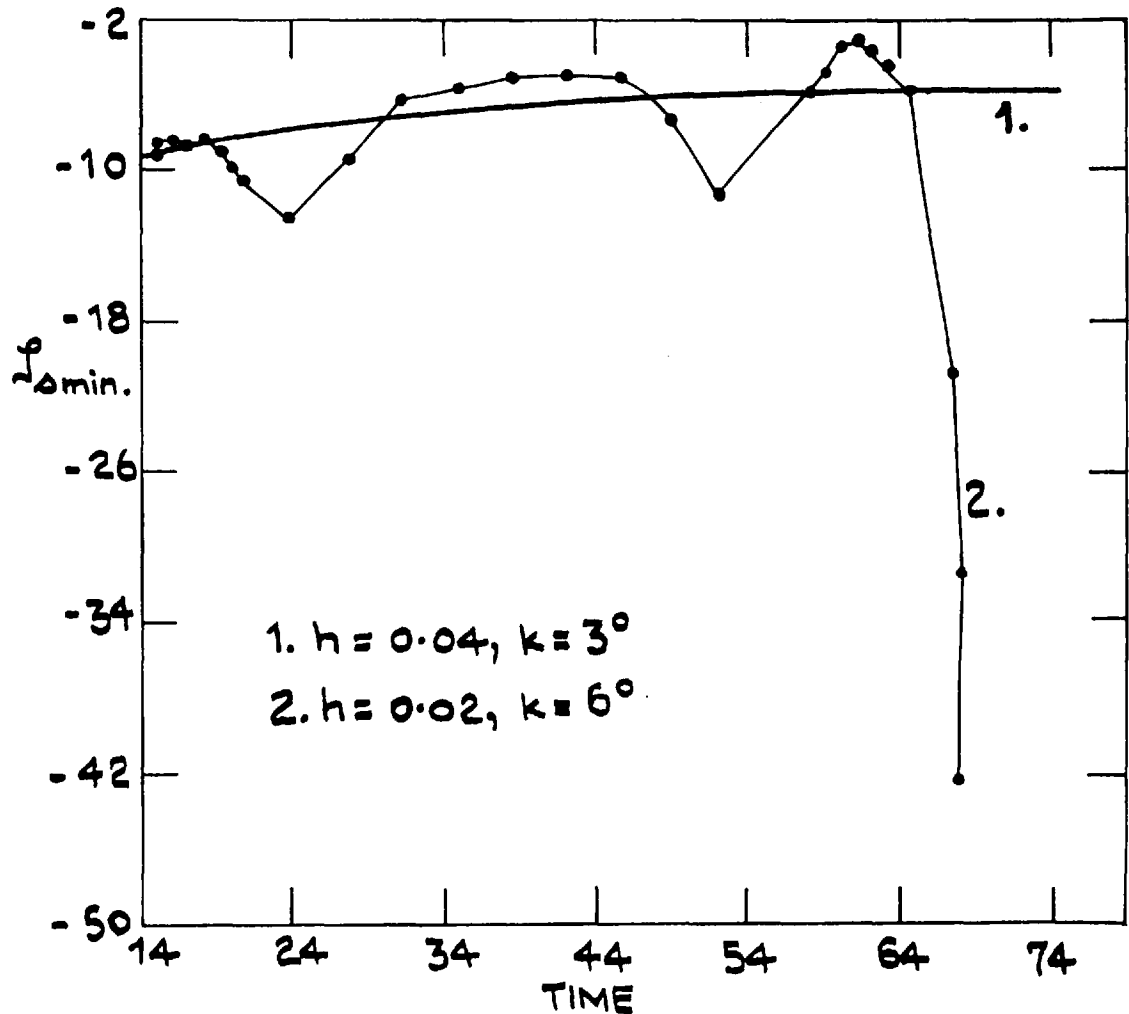


Fig.5.6.10 Separation (θ_s) and Wake Length (L) vs. Time at $Re=500$

Fig.5.6.11 Max. Surface Vorticity vs. Time at $Re=500$ Fig.5.6.12 Min. Surface Vorticity vs. Time at $Re=500$

small time steps to see if this had any affect on the behaviour of the solution and secondly by taking the outer boundary very far away from the sphere surface. These tests showed the same breakdown in the solution. The removal of the outer boundary had no effect and this was probably because the solution had already diverged too far from the true solution. However, the introduction of upstream differences and therefore 'pseudo viscosity' made the solution go back to a steady state form. The final steady state flow pattern and vorticity distribution is shown in figures 5.6.6s and 5.6.6t respectively. The main results obtained are also given in Table 4 and they compare favourably with the other steady state solutions obtained at Reynolds number of 500. Although the solution at $t^* = 68.1$ is very irregular, the introduction of 'pseudo viscosity' allows energy to be dissipated from the vortex and hence for it to be stabilised.

It is extremely difficult to conclude whether the central difference solution with $h=0.02$ and $k=6^\circ$ simulated the vortex shedding which has been observed experimentally at a Reynolds number of about 500. (Torobin and Gauvin 1959). The final breakdown of the solution is certainly a numerical feature. There are so many questions that need to be answered before the behaviour of the wake described above can be interpreted as one which would be observed experimentally.

The conclusion whcih can be drawn is that the 'late

time' solutions of the time-dependent Navier-Stokes equations at relatively high Reynolds numbers do not go to the solutions of the time-independent Navier-Stokes equations. However, the use of a coarse mesh and therefore by the introduction of large truncation errors leads to a steady state solution as shown by the original solution with $h=0.04$ and $k=3^\circ$. This conclusion is further supported by the solutions at $Re=1000$ and 2000 . With exactly the same mesh size the solution at Reynolds number of 2000 , soon started to 'go to infinity' after a short period of stable behaviour. Also the solution at $Re=1000$ shows signs of similar instabilities in the wake as the $Re=500$ solution. These instabilities show signs of growth and the eventual 'splitting up' of the solution. A comparison Figures 5.6.12, 5.7.7 and 5.8.7. shows that the perturbations introduced in the solutions by dispersive errors grow more rapidly with the increase in the Reynolds number and cause the flow to become unstable sooner.

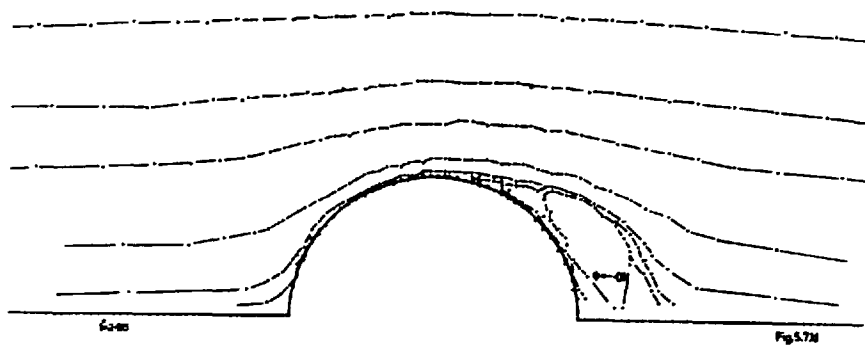
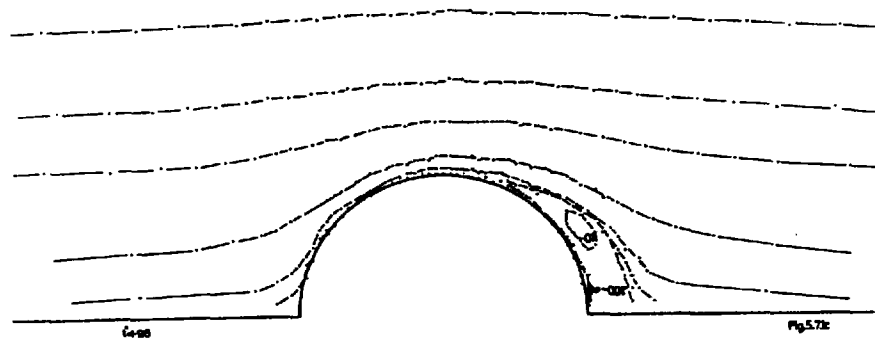
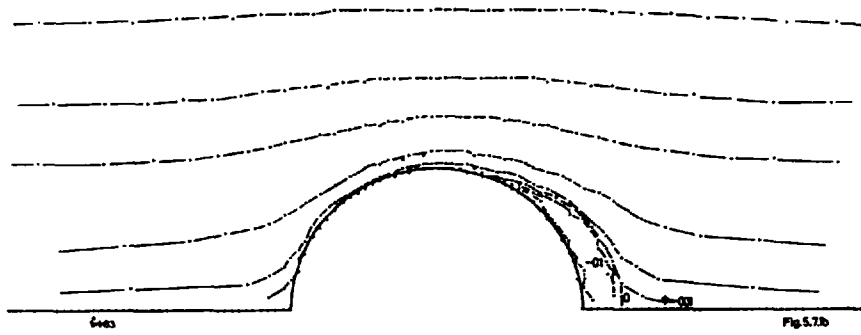
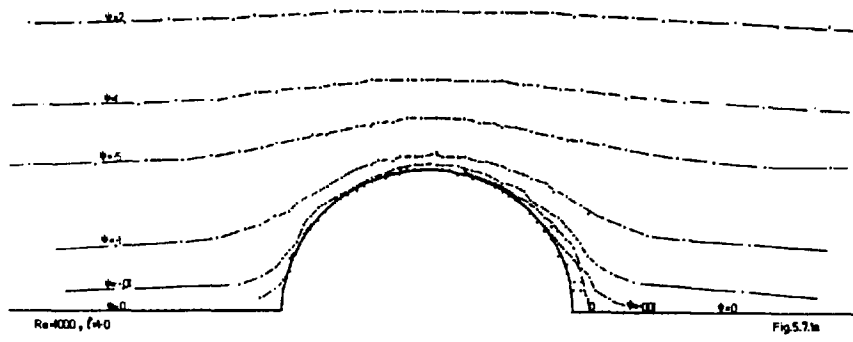
The truncation error analysis shows that the dispersive errors will be present even when the mesh is very fine. As the mesh is refined the solution approaches the true solution therefore these errors will act as perturbations in the flow which will either grow, decay or stabilise depending on the properties of the flow. The distinction between physical instability and pure numerical instability at the present time can only be made by the rate at which these instabilities grow. Physical instability is taken to grow at a much smaller rate than numerical instability.

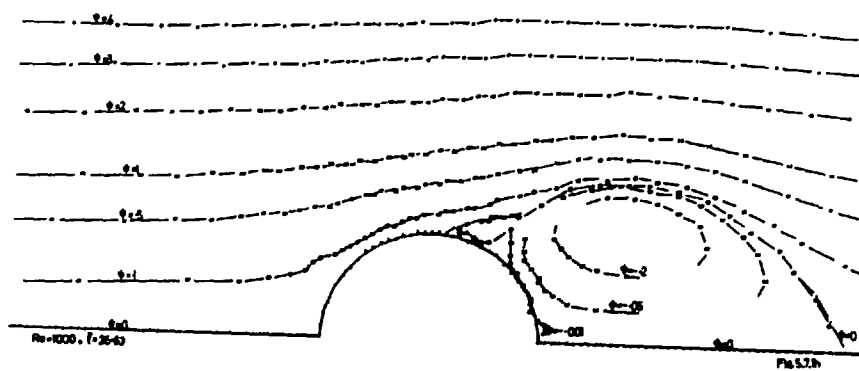
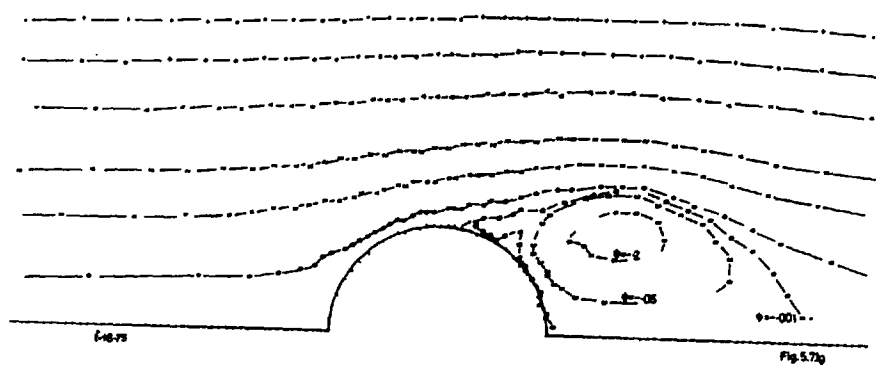
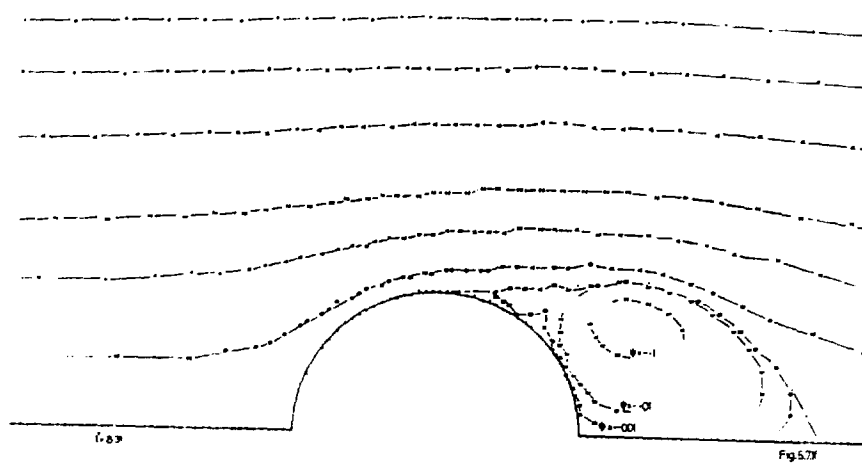
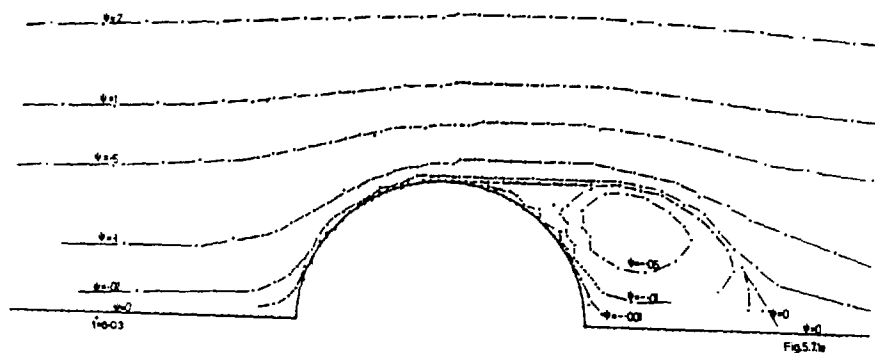
On this basis it can be suggested that an axi-symmetric wake behind a sphere at a Reynolds number of 500 is unstable if perturbations are present in the flow. To illustrate this figures 5.6.7 to 5.6.12 are presented. These figures all show a long period during which the physical instabilities grow before leading to a very short period of numerical instability and the final break-up of the solution.

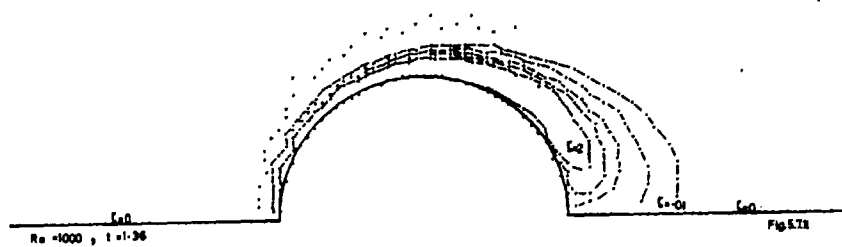
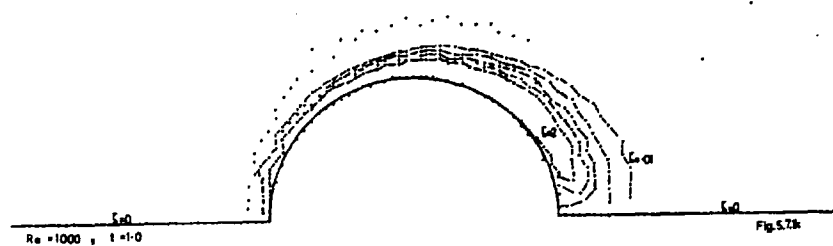
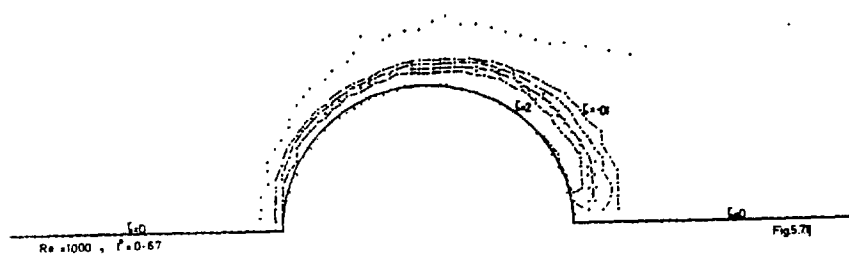
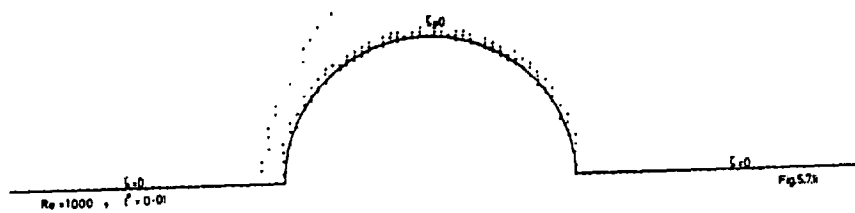
5.7 REYNOLDS NUMBER 1000 SOLUTION

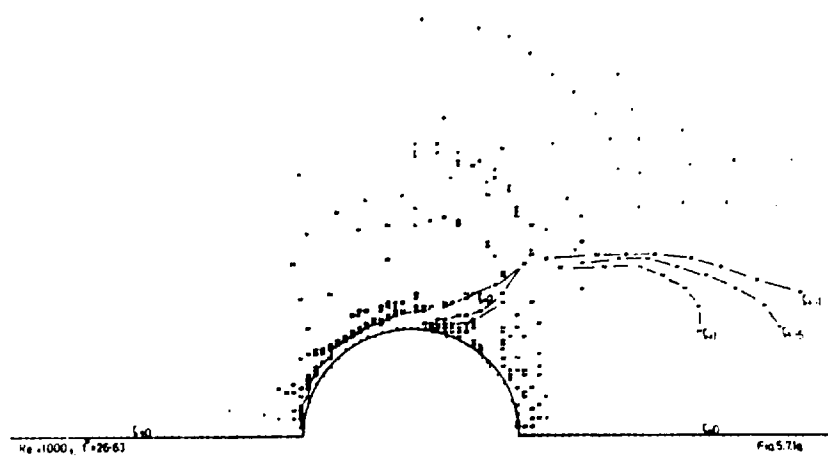
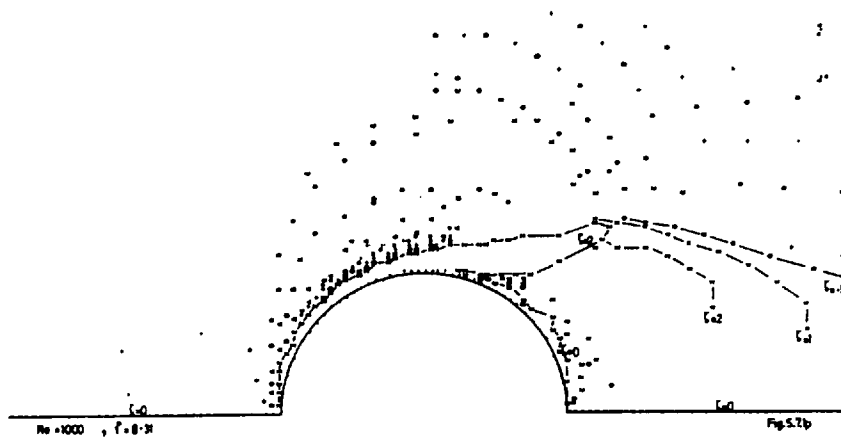
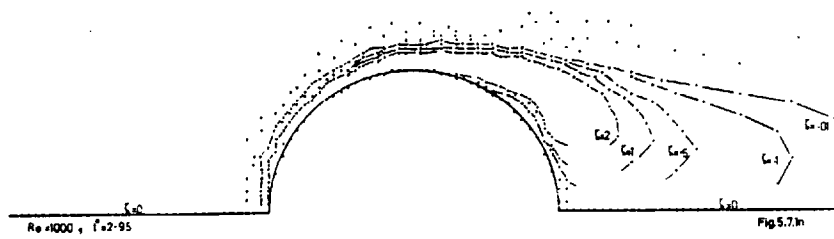
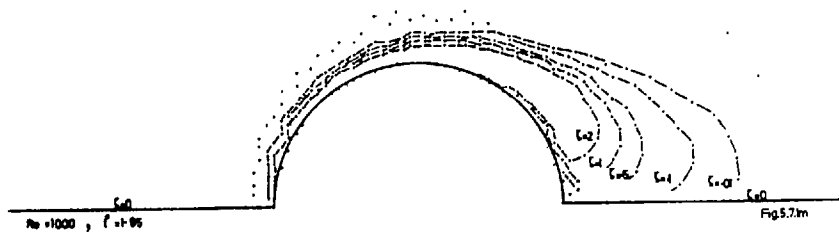
The development with time of the flow at a Reynolds number of 1000 starting from the potential flow solution is shown by figures 5.7.1a to 5.7.1h. The stream line plot at $t^* = 26.63$ in Fig. 5.7.1h shows a secondary vortex, the appearance of which at $Re=1000$ is also reported by Rimon (1968). The flow field also shows that the width of the wake eddy as measured from the downstream symmetry axis is greater than the sphere radius. This growth of the wake has not reached a steady state. However, the present simulation was stopped at this time due to the excessively large computing time required to continue the solution. The solution obtained upto $t^* = 26.63$ has all the expected features and with the information available from solutions at other Reynolds numbers it is possible to extrapolate the behaviour of the solution further in time.

The generation, diffusion and convection of vorticity with time from the sphere surface is shown by figures





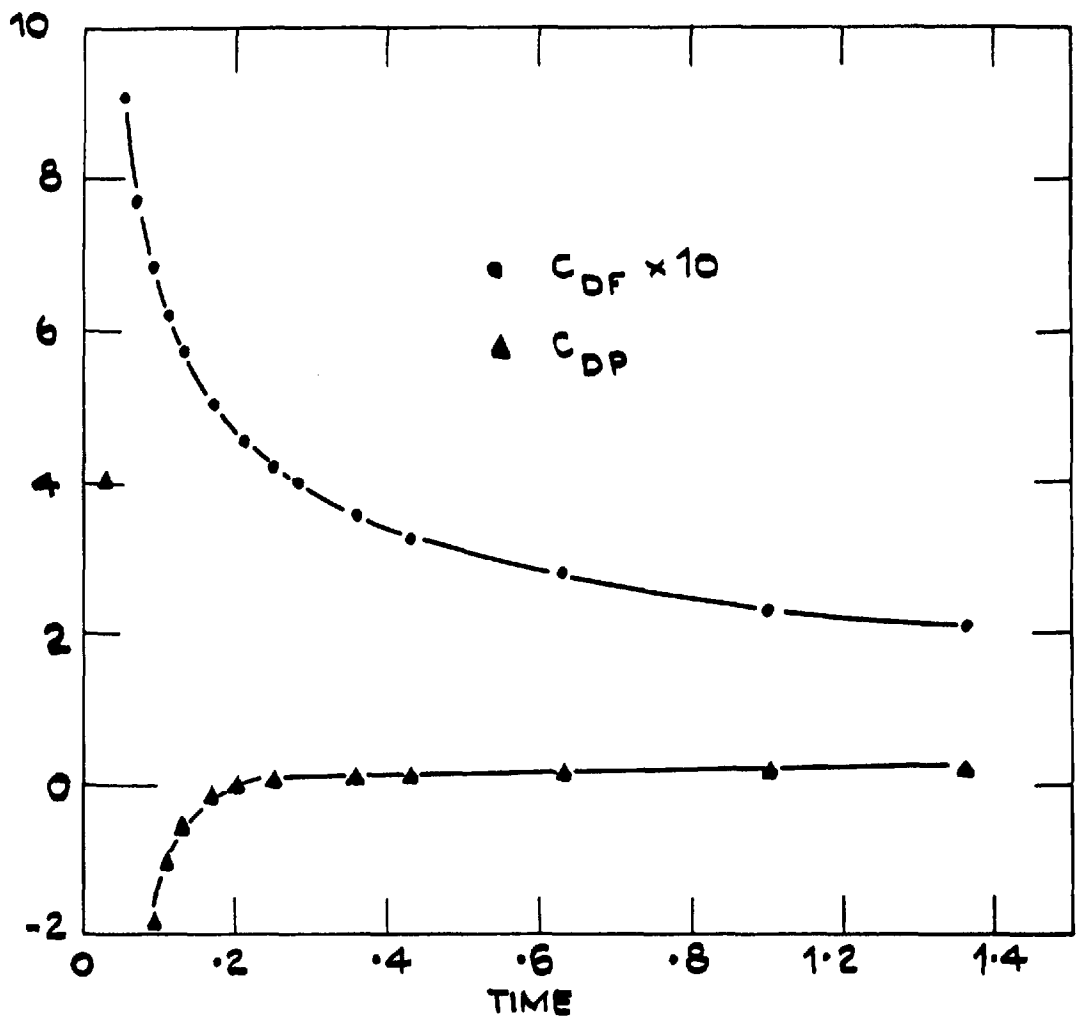
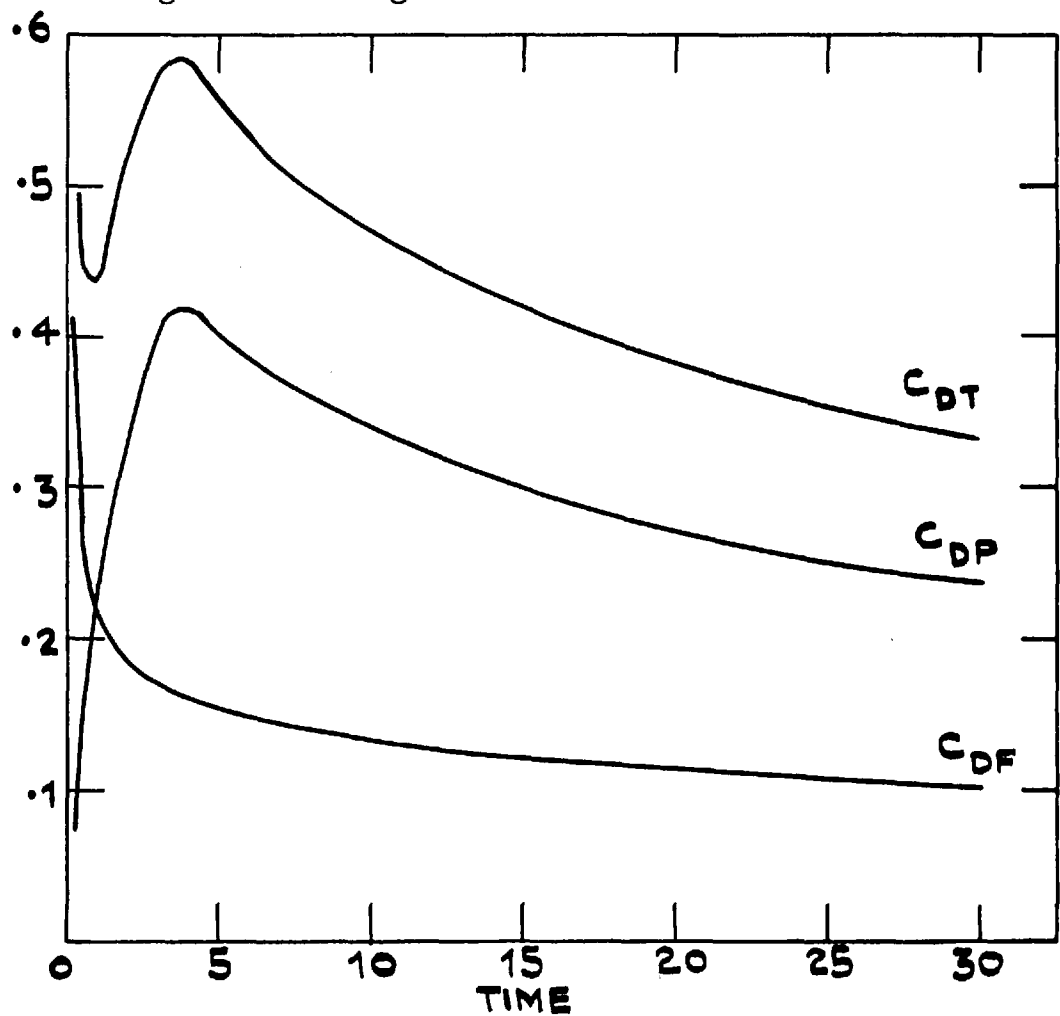


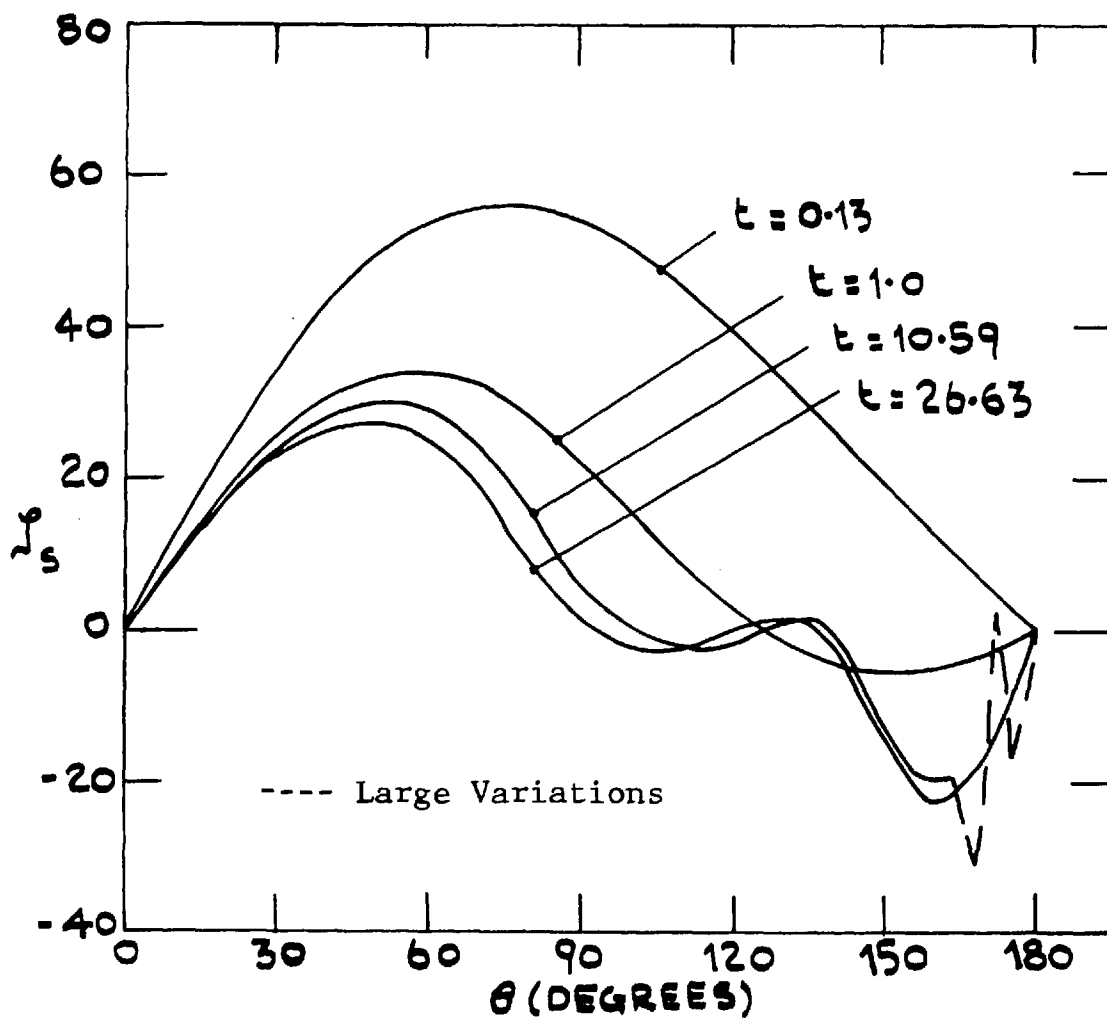
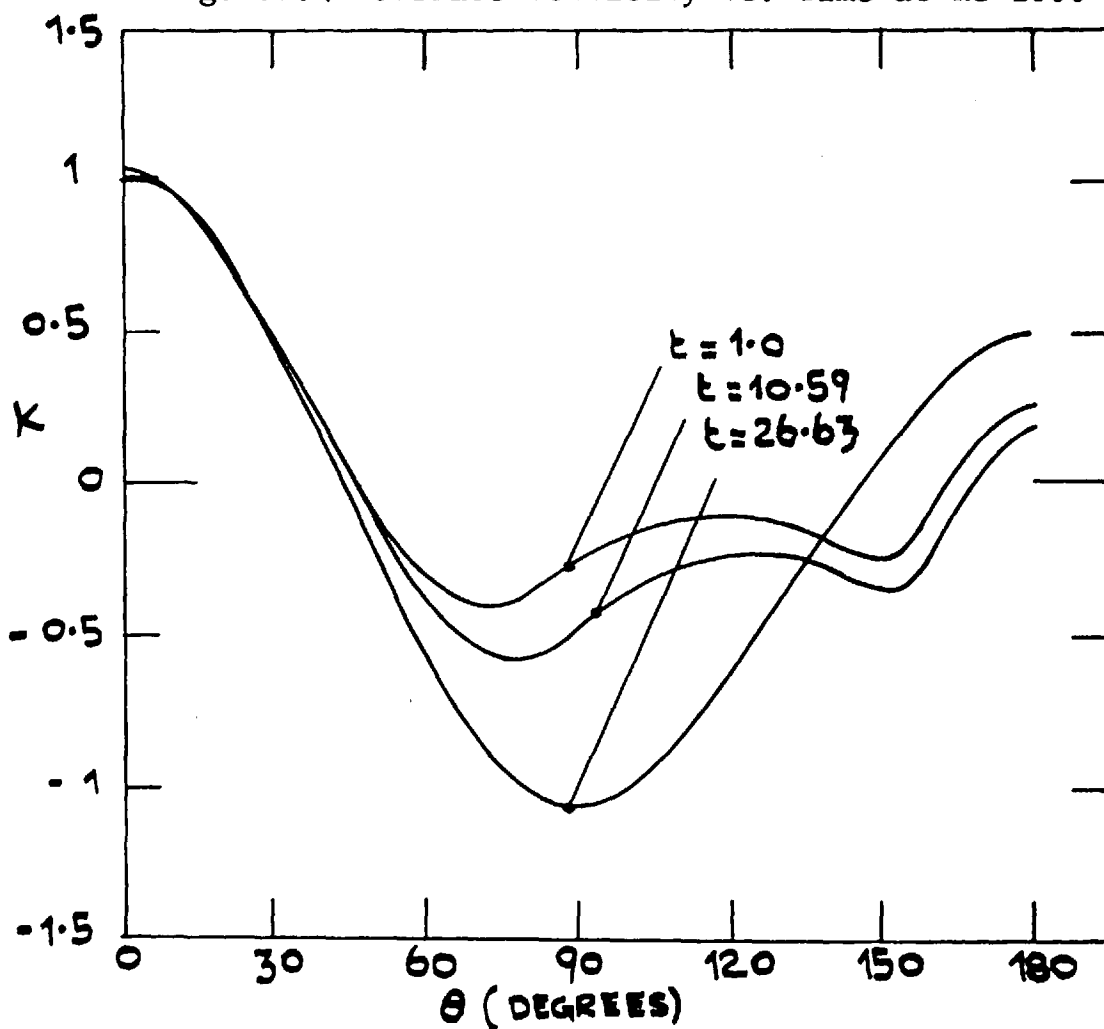


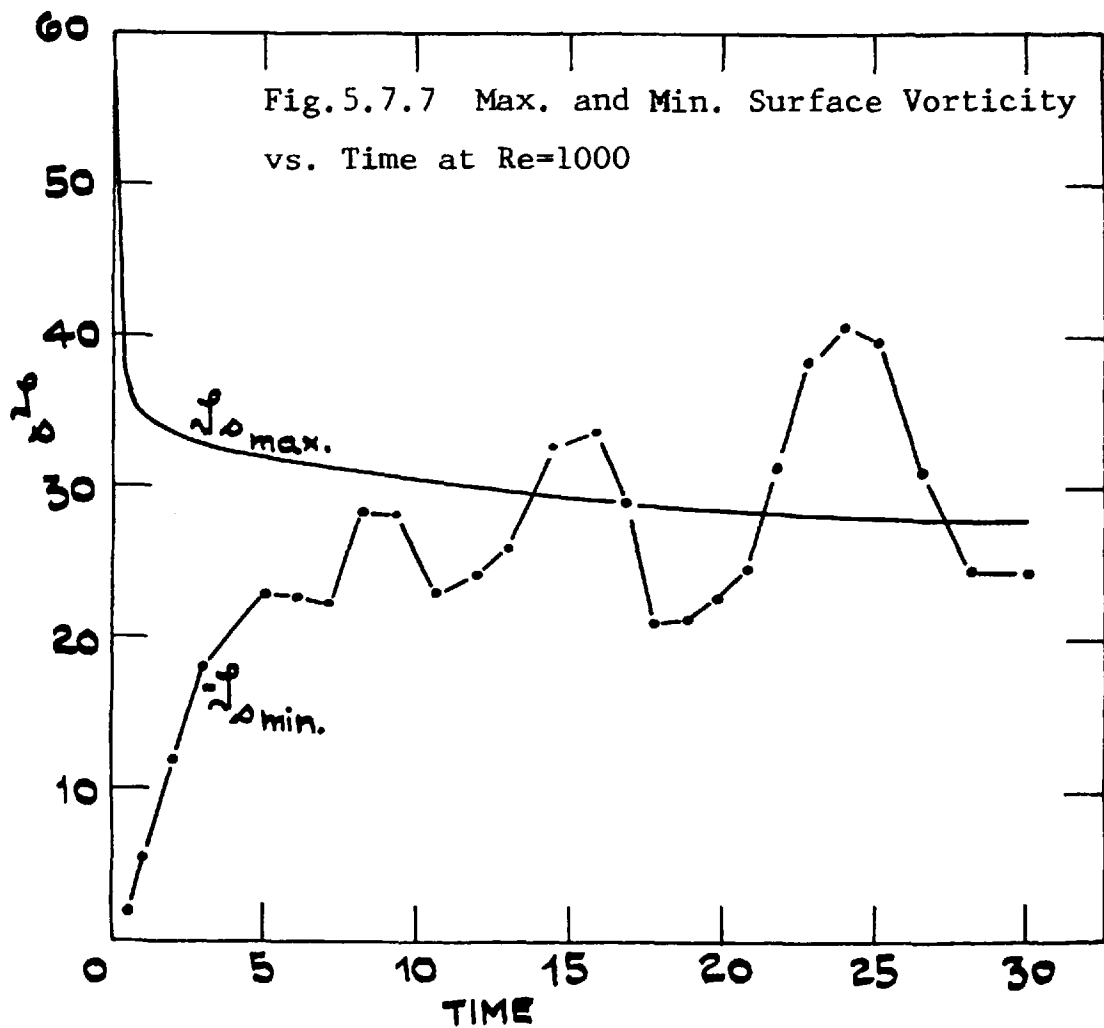
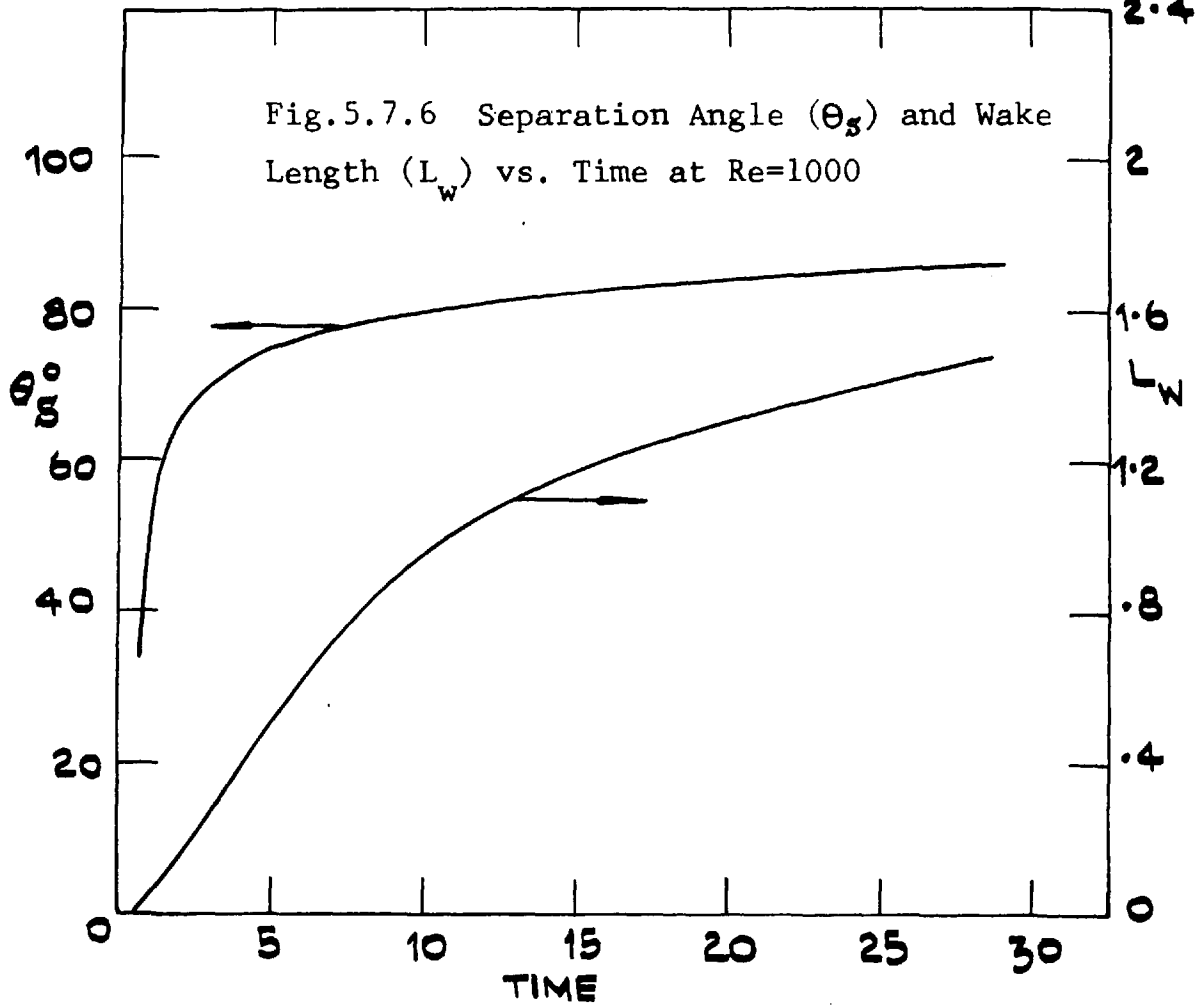
5.7.1i to 5.7.1q. The reduction in viscous or diffusion effects is compared with lower Reynolds numbers is shown by Fig. 5.7.1i and Fig. 5.7.1j. After an initial radial diffusion, convection carries the vorticity downstream and distortion of the vorticity contours occurs. As the solution proceeds in time the 'noise' level increases in the downstream section of the flow.

The variation of drag coefficients with time is shown in Fig. 5.7.2 and Fig. 5.7.3. The viscous drag coefficient shows a continuous fall with time in common with all the other solutions obtained. The pressure drag coefficient shows a sharp drop then a steep rise followed by a continuous decrease almost in parallel with the viscous drag coefficient. This behaviour by the pressure drag coefficient is displayed by all solutions at lower Reynolds numbers; the difference is the time interval over which this initial behaviour occurs. This time interval is very short at low Reynolds numbers and it increases as the Reynolds number increases. The solutions obtained for the flow around a solid sphere by Rimon (1968) using the Dufort-Frankel time difference scheme do not show this behaviour.

Comparison of the present solution at $t^* = 26.63$ with the solution of Rimon (1968) at $t^* = 19.0$ shows that the separation angle and the drag coefficient obtained by Rimon (1968) (See Table 2) are greater than the present values. However, Rimon used a coarser mesh and

Fig.5.7.2 Drag Coefficients vs. Time at $Re=1000$ Fig.5.7.3 Drag Coefficients vs. Time at $Re=1000$

Fig.5.7.4 Surface Vorticity vs. Time at $Re=1000$ Fig.5.7.5 Surface Pressure vs. Time at $Re=1000$



obtained a steady state solution. The present solution with a finer mesh is closer to the true solution. Fig. 5.7.7 shows fluctuations in the minimum surface vorticity which indicate instability in the wake. The fluctuations also show a steady growth in time similar to the behaviour of the minimum surface vorticity at $Re=500$ and at $Re=2000$ in Fig. 5.6.12 and Fig. 5.8.7 respectively. These fluctuations in vorticity are expected to grow causing the vortex to show similar behaviour to that observed for $Re=500$ and for the solution to 'break-up' as shown by the solutions at $Re=500$ and 2000 . The broken-line curve in Fig. 5.7.4 shows the region where these fluctuations in the minimum surface vorticity occur. The surface vorticity follows the expected pattern right round the sphere surface except near the downstream symmetry axis where the vorticity shows signs of irregularity. The behaviour of the entire solution suggests that these perturbations are not purely numerical in character. They display the property of the flow which allows the growth of perturbations to occur after a certain critical Reynolds number. Therefore it can be suggested that an axi-symmetric wake behind a solid sphere at a Reynolds number of 1000 is unstable if perturbations are present in the flow.

5.8 REYNOLDS NUMBER 2000 SOLUTION

The development with time of the flow pattern at a Reynolds number of 2000 starting from the potential flow solution is shown by figures 5.8.1a to 5.8.1g. The stream

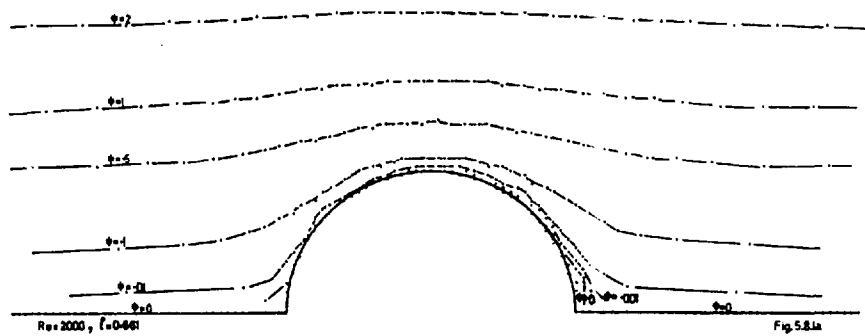


Fig. 5.8.1a

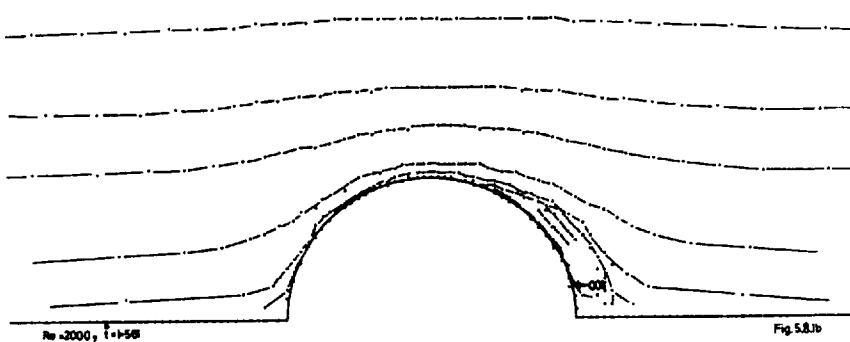


Fig. 5.8.1b

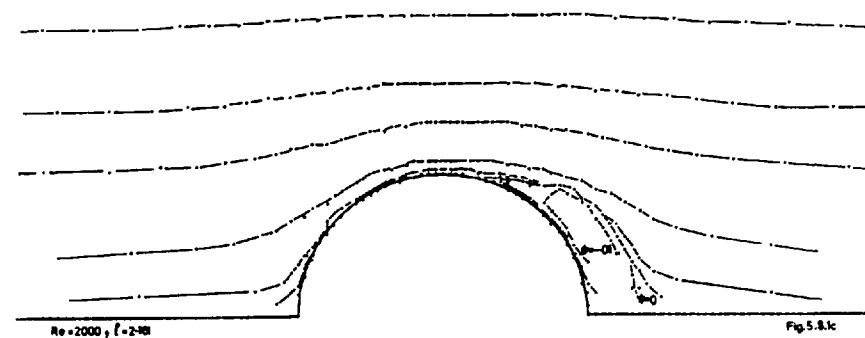


Fig. 5.8.1c

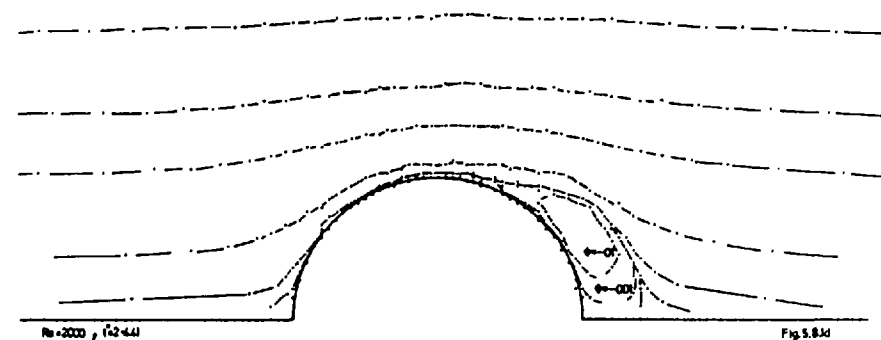
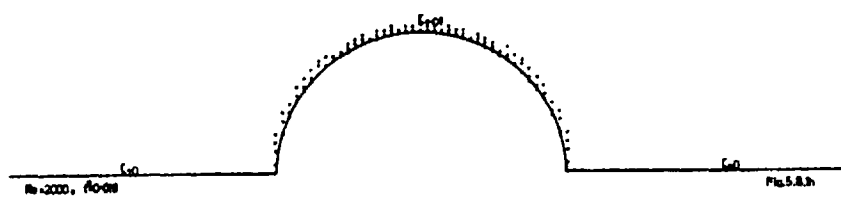
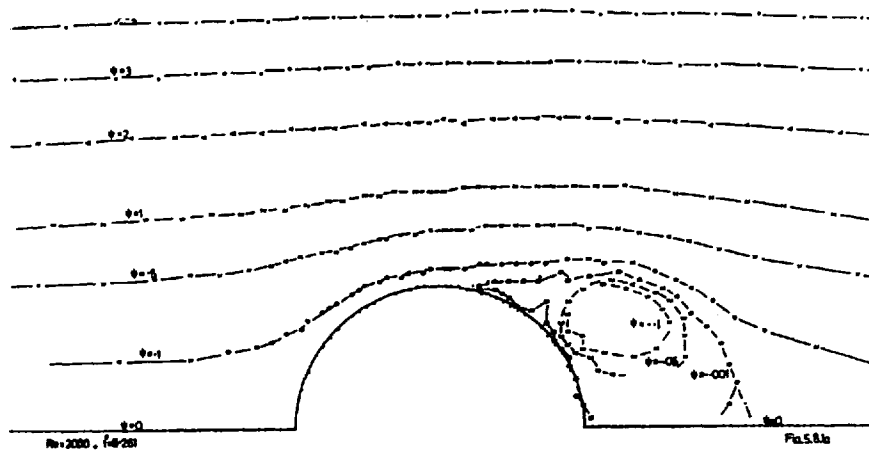
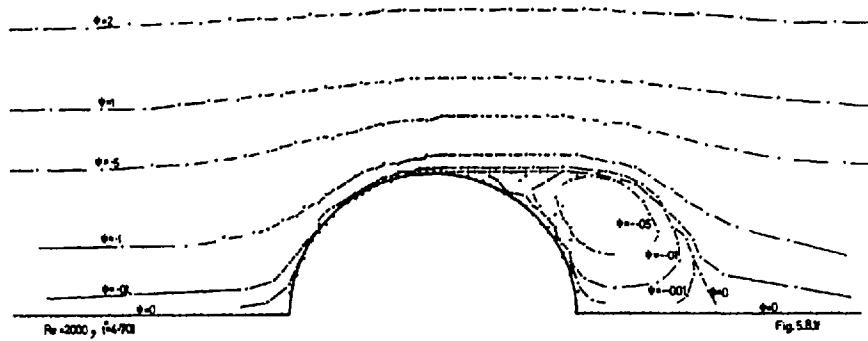
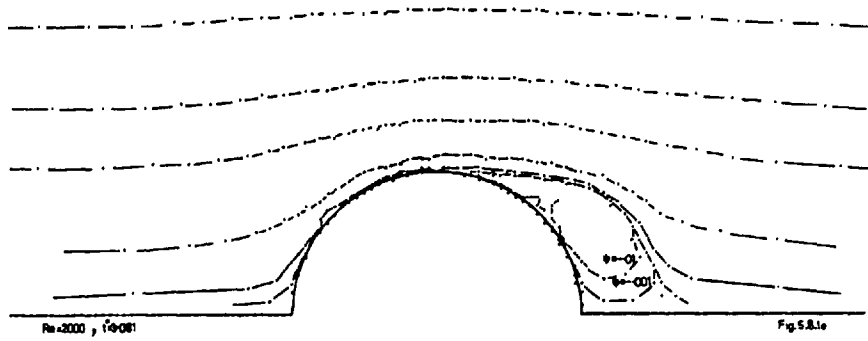
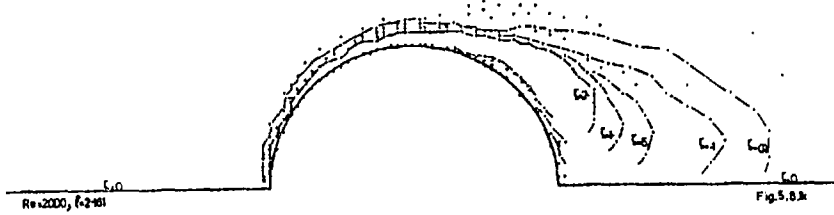
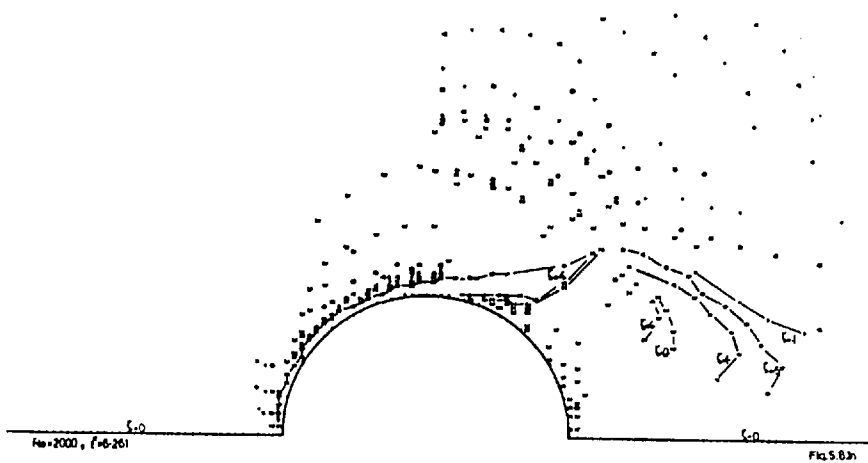
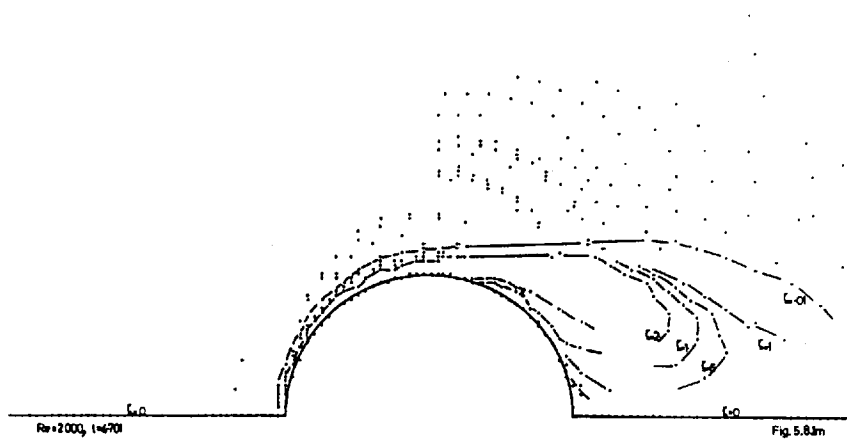
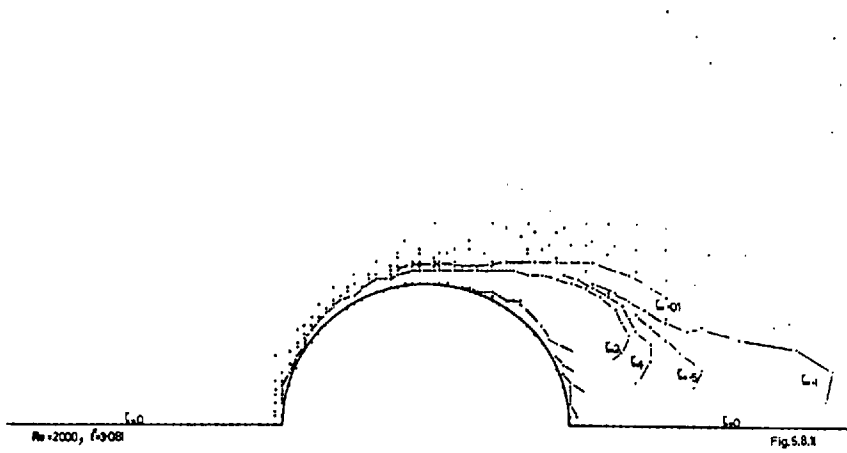


Fig. 5.8.1d

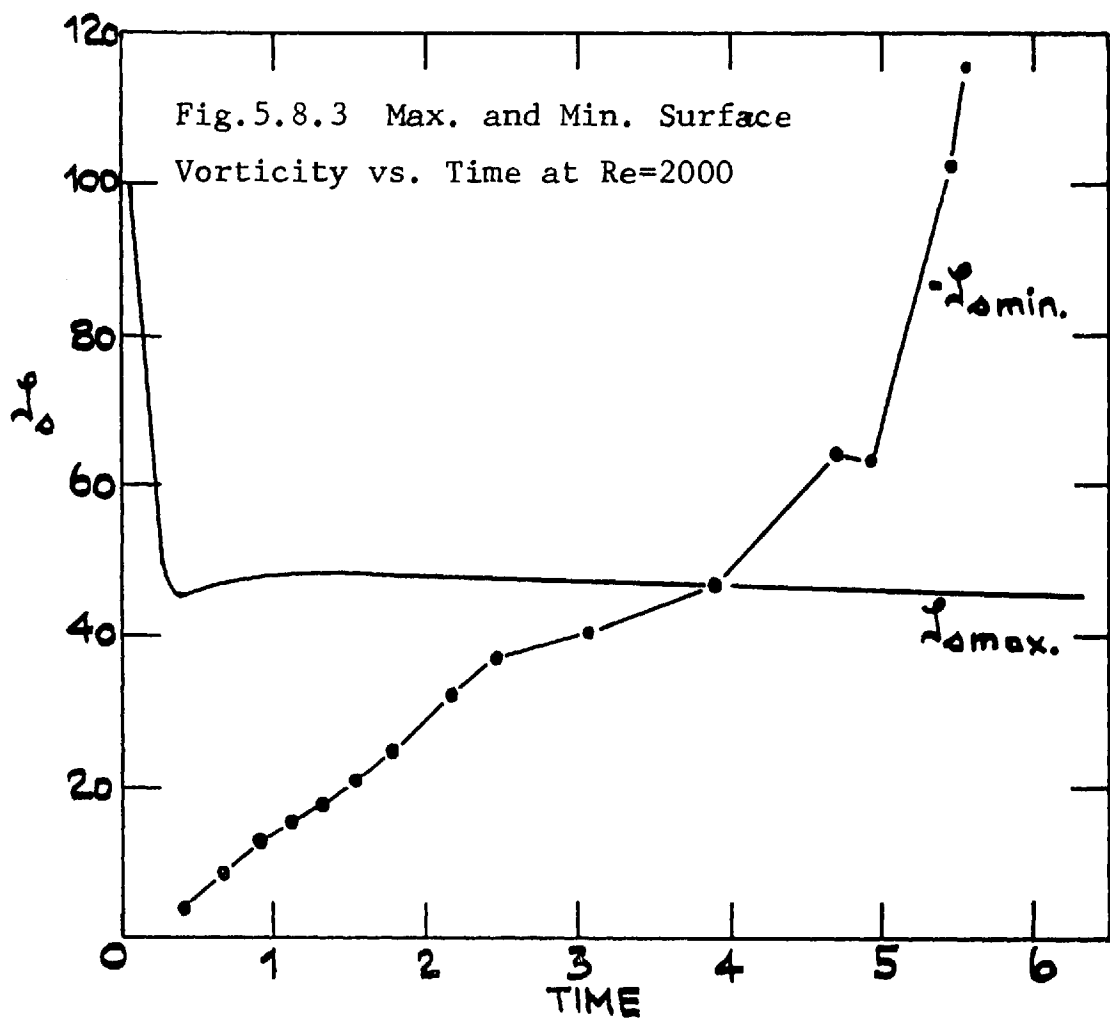
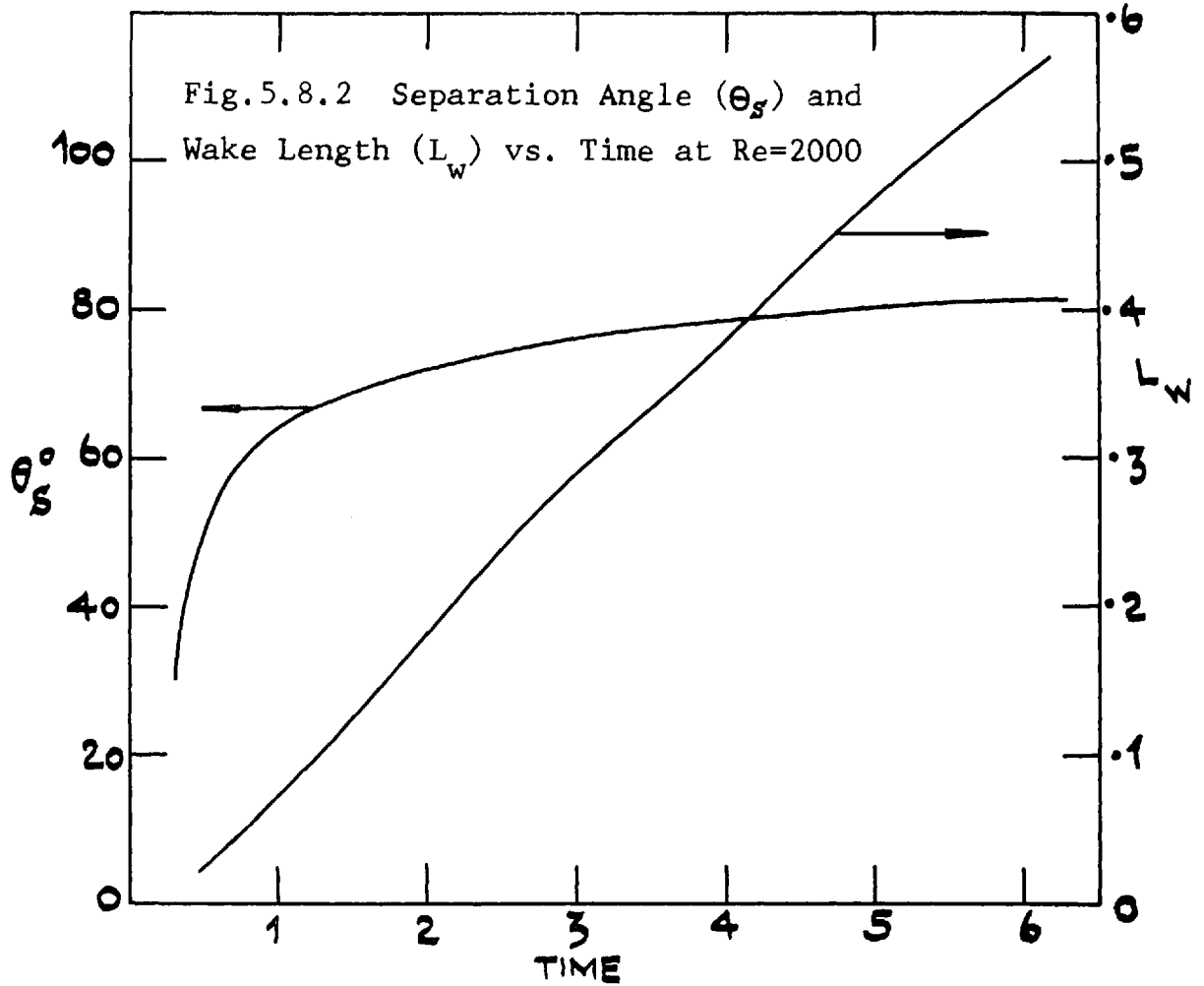


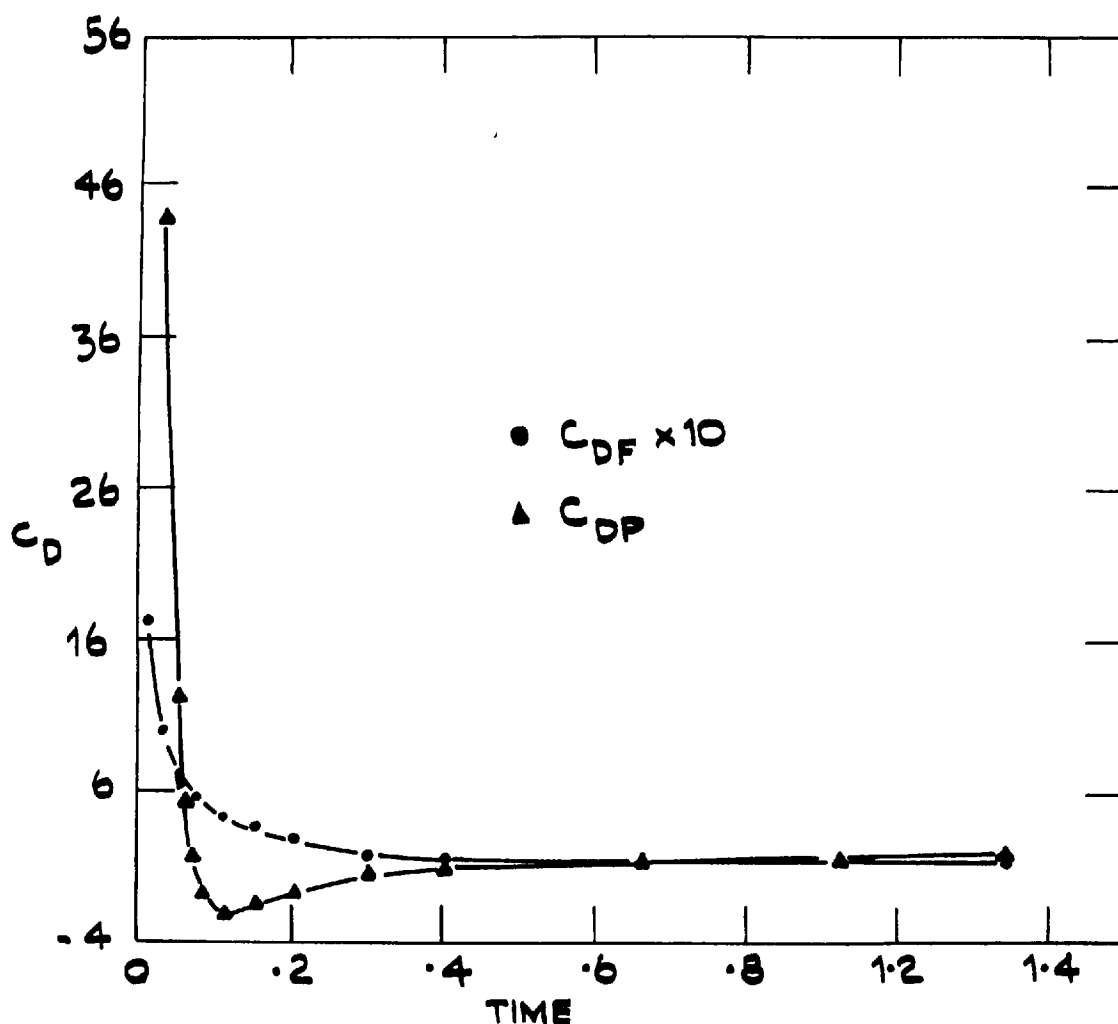
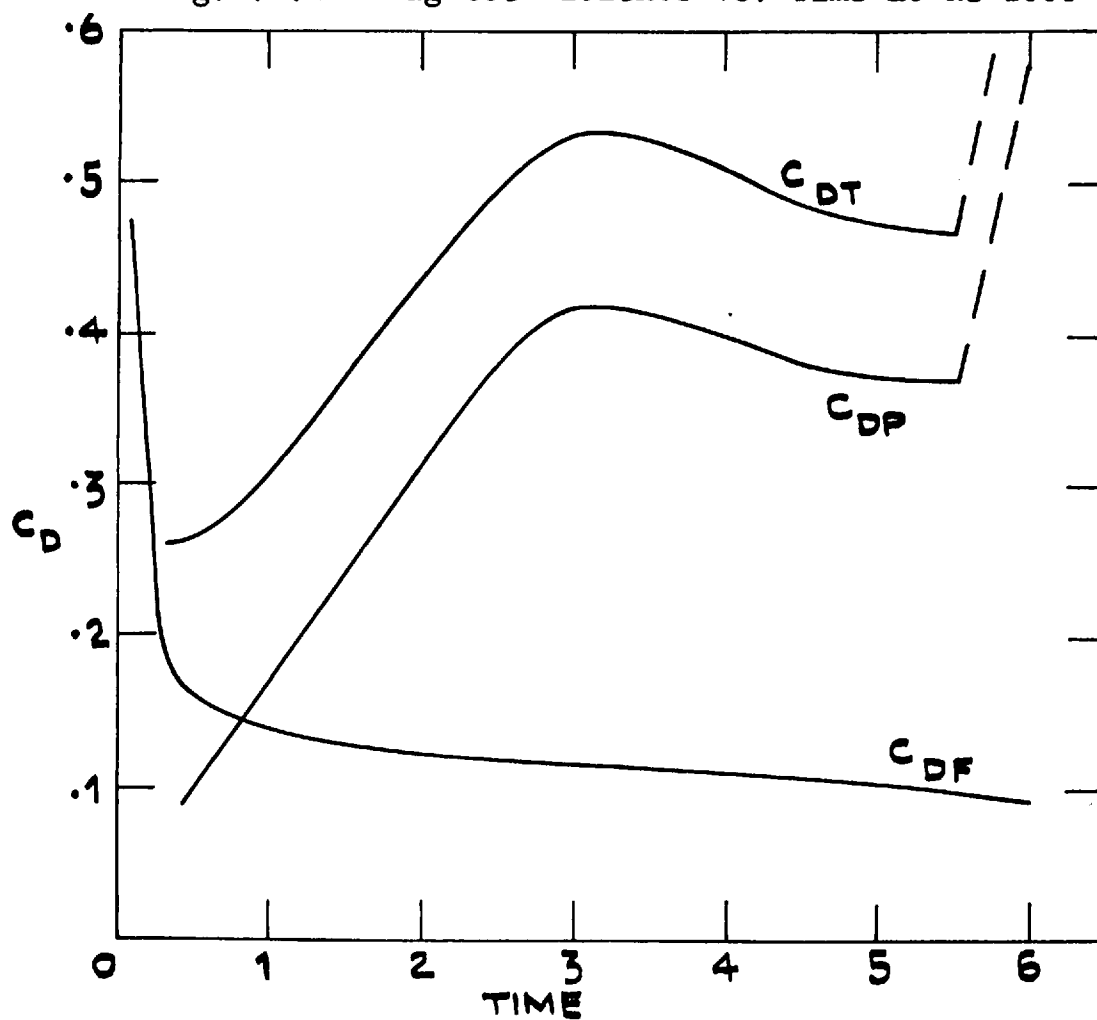


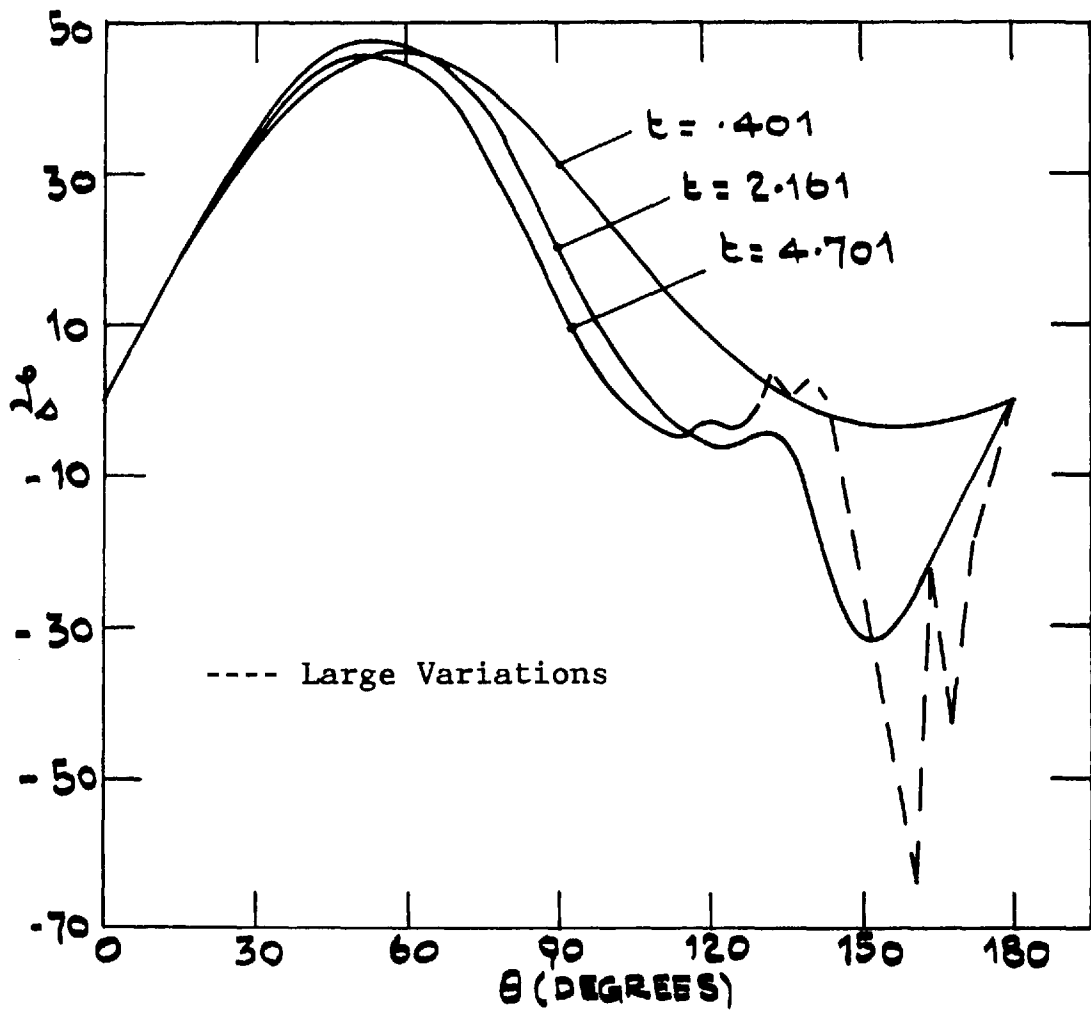
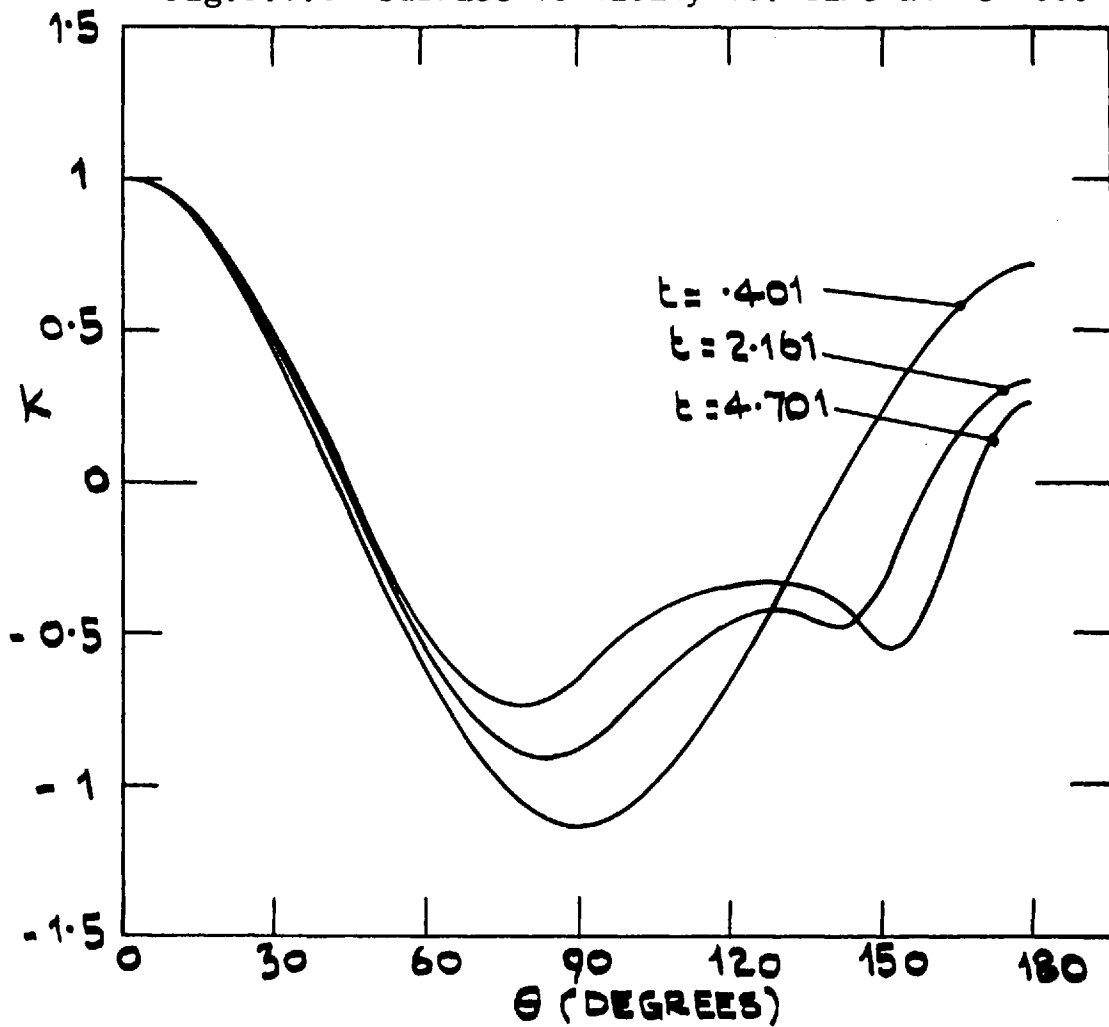


lines show the initial appearance of a recirculatory wake and its growth in time. A secondary vortex appears quite early on and is stronger and larger than at $Re=1000$. The separation angle in Fig. 5.8.2 shows signs of approaching a steady state but the wake length is growing linearly with time. The solution shows signs of numerical instability at $t^* \approx 5$ and a complete, 'break-down' occurs at $t^* \approx 6.5$. A comparison of Fig. 5.7.3 with 5.8.5 show that the change in the drag coefficients with time at $Re=2000$ is faster and appears to level-off before the solution becomes unstable. The irregularities present in the solution (Fig. 5.8.6) were also noted at $Re=1000$ but grew very much faster at $Re=2000$ which suggests that if perturbations are present in the flow the axi-symmetric wake at $Re=2000$ will become unstable sooner.

The generation, diffusion and convection of vorticity is shown in figures 5.8.1h to 5.8.1n. The vorticity contours are regular in the initial phase of the flow development but as the integration proceeds in time, the 'noise' level in the downstream section increases. The usefulness of the solution at $Re=2000$ becomes more apparent when it is compared with solutions at $Re=500$ and 1000 . A stricter numerical stability criterion was used for the present solution during the initial phase of the flow development than at $Re=1000$ and yet with exactly the same mesh the present simulation shows numerical instability earlier in time. Although the 'noise' level in the solution at $Re=2000$ is greater than at $Re=1000$, this alone cannot be the cause of faster growth of instabilities since no such



Fig.5.8.4 Drag Coefficients vs. Time at $Re=2000$ Fig.5.8.5 Drag Coefficients vs. Time at $Re=2000$

Fig.5.8.6 Surface Vorticity vs. Time at $Re=2000$ Fig.5.8.7 Surface Pressure vs. Time at $Re=2000$

behaviour was observed at lower Reynolds numbers. The solution shows the change in the properties of the Navier-Stokes equations at high Reynolds numbers. The initial stable solution gives the total drag coefficient which is in reasonable agreement with the experimental value of Lapple and Shepherd (1940).

CHAPTER 6

CONCLUSIONS

CONCLUSIONS

The results of the present work can be summarised as follows:

1. A computer program has been developed using a CDC 6600 digital electronic computer which solves the time-dependent Navier-Stokes equations for an axi-symmetric flow around a solid sphere. A time-dependent stream function-vorticity formulation of the Navier-Stokes equations has been derived and the vorticity transport equation has been solved using the Peaceman and Rachford alternating direction implicit method. The stream function equation has been solved using a point iterative successive over-relaxation method.
2. The computer program has been used to solve the problem of an impulsively accelerated viscous fluid stream which is initially at rest and which contains a stationary solid sphere. The solutions obtained are for Reynolds numbers of 1, 10, 50, 200, 500, 1000 and 2000, (based on the diameter of the sphere).
3. The 'late-time' solutions for Reynolds numbers of 1, 10, 50, and 200 approach solutions obtained by other workers for the time-independent Navier-Stokes equations. Also the steady state flow characteristics of the present solutions are in agreement with experimental values. A coarse mesh solution at a Reynolds number of 500 approaches steady state but as the mesh is refined the solution shows

velocity fluctuations in the wake and growth of the vortex occurs. As the vortex grows in width it becomes unstable and rapid 'break-up' of the solution occurs. The solution at a Reynolds number of 1000 shows similar velocity fluctuations in the wake and growth of the vortex. However, the $Re=1000$ solution was not integrated long enough in time because of the excessive computing time required. But the solution at a Reynolds number of 2000 with exactly the same mesh as for a Reynolds number of 1000 shows the rate of growth of velocity fluctuations in the wake to be greater than at $Re=1000$ and shows that an earlier 'break-up' of the solution occurs at $Re=2000$. An explanation of this behaviour has been presented.

4. The solutions at $Re=500$, 1000 and 2000 suggest that the wake at these Reynolds numbers is unstable if perturbations are present in the flow. However, it is not possible to say whether vortex shedding under strict axial symmetry is a possible phenomenon.

5. The non-linear properties of the Navier-Stokes equations suggest that the 'late-time' solutions of the time-dependent equation may not approach the solutions of the time-independent equations. This is shown by the solutions at Reynolds numbers of 500, 1000 and 2000. Since it is difficult to distinguish between numerical and physical instability, an explanation of the behaviour of solutions of the Navier-Stokes equations has been suggested, based on the assumption that the rate of

growth of physical instabilities is much slower than the rate of growth of numerical instabilities.

6. The Peaceman and Rachford alternating direction implicit method yields stable and quantitatively accurate numerical solutions of the Navier-Stokes equations in the intermediate range of Reynolds numbers.

7. It is possible to introduce upstream biased differences in the convective term of the vorticity transport equation and damp down all 'noise' in solutions. Therefore it is possible to obtain steady state solutions even at higher Reynolds numbers than reported in this thesis. However, the 'pseudo viscosity' introduced becomes comparable with the real viscosity of the fluid and thus solutions obtained do not show the true behaviour of the fluid. Since the steady state solutions with upstream differences give similar values of the drag coefficients as the central difference solutions, therefore upstream difference solutions can probably be used to predict steady state transfer rates with an accuracy that is similar to that of the drag coefficients.

8. In the present work, upstream differences are introduced into the convective term of the vorticity transport equation in a form which allows a 'mixed scheme' to be devised; that is, at certain points in the fluid a central difference scheme can be used and at other points in the fluid an upstream difference scheme can be

used. Using a 'mixed scheme' a steady state solution at a Reynolds number of 50 is obtained in which all 'noise' near the outer boundary condition is absent and the entire solution is better than the central difference solution. This technique can be used to obtain better solutions at other Reynolds numbers. However, due to considerable numerical experimentation involved to locate the nodes where the upstream differences should be introduced further work is needed to develop a method which automatically locates these regions.

9. The most sensitive boundary condition is the boundary through which the fluid leaves the domain of integration called the outflow boundary. The present work uses a uniform parallel outflow boundary condition. This boundary condition affects the calculated values of drag coefficients and other flow characteristics, such as, the wake length. The parallel but not uniform outflow boundary condition has been tested at Reynolds number of 1, 10 and 500. The results show no significant difference between the two outflow boundary conditions. However, convergence difficulties are encountered when the parallel outflow boundary condition is used.

10. The impulsive acceleration of the fluid stream generates very high vorticity gradients at the sphere surface during the initial stages of the flow development. The choice of the time step using the linear criterion of $\frac{\Delta t a}{e^{22i} Re} = \frac{1}{2}$ gives stable solutions. As the solution

proceeds in time increasingly larger time steps can be used and eventually time steps greater than those indicated by the linear criterion are employed without loss of stability.

APPENDICES

APPENDIX A

COMPUTER PROGRAM

A.1 The User's Guide to the Computer Program

The listing of the computer program in FORTRAN IV is presented in section A.3. The function of each subroutine in the program is described in the listing. Therefore the user should not find difficulty in understanding and using the program.

In the computer program, the input data and output of the results are handled by two subroutines called INPUT and OUTPUT. The main computations described in Chapter 4 Section 6 are performed by two subroutines called UPDATE and VARBC. The various options built in the program are controlled from a subroutined called CNTROL.

The main program (Fig. A.1.1) co-ordinates the function of the above mentioned subroutines. The program also consists of several auxillary subroutines which allow quick detection of errors.

To obtain results presented in this thesis the user should only specify appropriate counters and parameters in subroutin CNTROL which are defined in section A.2.

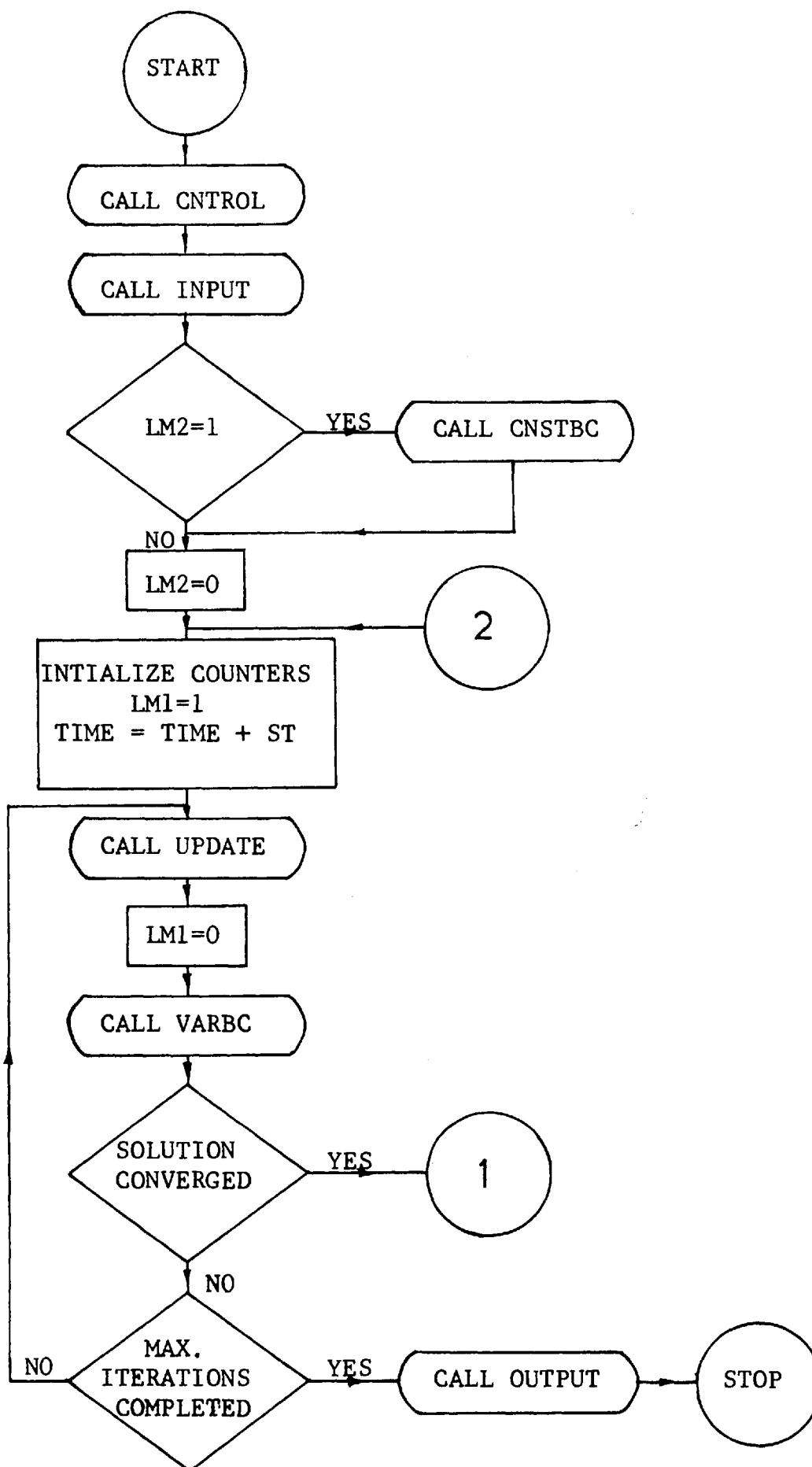


Fig. A.1.1 FLOW DIAGRAM OF THE MAIN COMPUTER PROGRAM

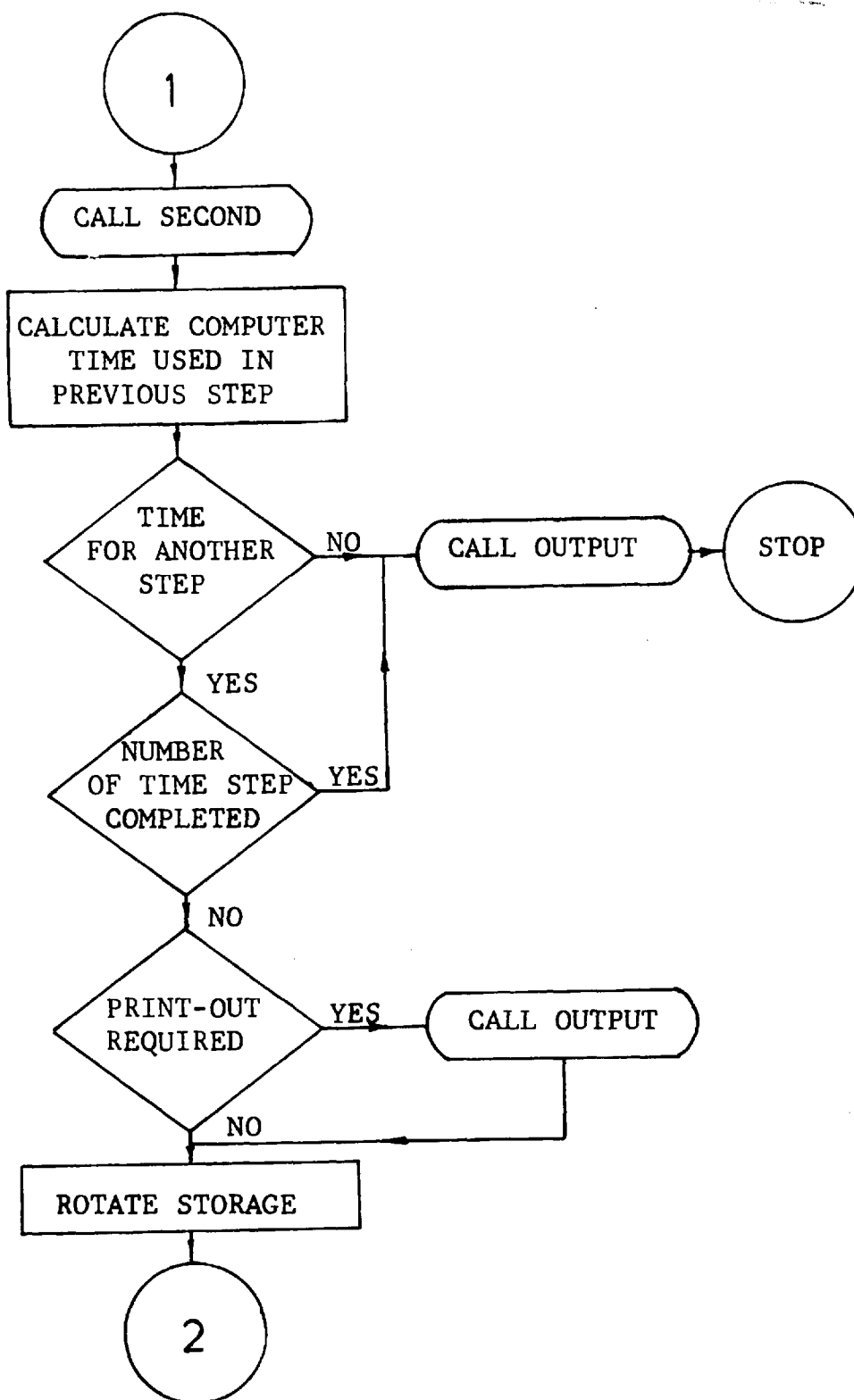


Fig. A.1.1 FLOW DIAGRAM OF THE MAIN COMPUTER PROGRAM

A.2 List of FORTRAN SYMBOLS used in the
Computer Program

Given below is a list of important FORTRAN symbols used in the program. All the other symbols used in the program are either defined in terms of these symbols or are self-explanatory.

<u>FORTTRAN SYMBOL</u>	<u>MEANING</u>
AKNOT	K_0
AK(J)	K_θ
B1(I)	b_{1i}
B2(I)	b_{2i}
B3(J)	b_{3j}
B4(J)	b_{4j}
C(I)	C_i
CCU	E_ψ
CCUOB	$E_{\psi b}$
CCG	E_g
CDF	C_{DF}
CDP	C_{DP}
CDT	C_{DT}
CPT	The Central Processor time set for a CDC 6600 machine
CR	C_R
CS(J)	$\cos \theta$
CT(J)	$\cot \theta$
DFU	ω_ψ
DFUB	$\omega_{\psi b}$

<u>FORTTRAN SYMBOL</u>	<u>MEANING</u>
DFG	ω_g
GO(I,J)	$g_{i,j}^n$
GN(I,J)	$g_{i,j}^{n+1}$
H3(I,J)	h_{3ij}
II	Input magnetic tape unit number
IN1	in1=total number of grid lines in the Z-direction
INPRV	Number of unconverged nodes in the initial condition solution
ISKIPN	Number of files to be skipped on the input magnetic tape
ISKIPO	Number of files to be skipped on the output magnetic tape
ITHETA(J)	θ in degrees
IXSTEP	Number of line-printer steps to be used for plotting quarter of the flow field. Normal setting as shown in the program listing
JJ	Output magnetic tape unit number
KNU	Number of stream lines to be plotted
KNV	Number of vorticity distributions contours to be plotted
KS	k in degrees
(All FORTRAN symbols begining with L are program control counters)	
LM1	If LM1=1 another step in time is taken otherwise iterations are performed on the existing solution

<u>FORTRAN SYMBOL</u>	<u>MEANING</u>
LM2	If LM2=1 time constant boundary conditions are reset
LM3	LM3=1 stream function equation is solved at $n + \frac{1}{2}$ time step
LM4 LM5	If LM4=1 and LM5=1 parallel outflow boundary condition solution is obtained for the stream function equation. If LM4=0 and LM5=0 normal uniform outflow boundary condition is set
LNP1	If LNP1=1 program starts from $t^*=0$ and uses the potential flow initial condition. Otherwise set as zero
LNP2	If LNP2=1 program reads in converged solution from a magnetic tape and updates it in time. Otherwise set as zero
LNP3	If LNP3=1 program reads in unconverged solution from a magnetic tape and performs the appropriate computations. Otherwise set as zero
LOP1	If LOP1=1 full output of results is produced and the current solution in the computer is transferred on a magnetic tape. Otherwise set as zero
LOP2	If LOP2=1 only drag coefficients are printed out and the current solution in the computer is transferred on a magnetic tape. Otherwise set as zero

<u>FORTRAN SYMBOL</u>	<u>MEANING</u>
LOP3	If LOP3=1 a table of stream function and vorticity values together the drag coefficients are printed out. Otherwise set as zero
M	The number of equation equal sections (1-10) that are to be searched when plotting contours
MNPR	The minimum number of unconverged nodes acceptable in a solution. Normally set as zero
MXITER	The maximum number of iteration allowed per time step for convergence of the surface vorticities
MXU	The maximum number of iteration allowed for convergence of the stream function values at $n + \frac{1}{2}$ time step
MXU	The maximum number of iterations allowed for convergence of the stream function values at $n + 1$ time step
NPRU1	The total number of unconverged nodes in the stream function field
NPRG1	The total number of unconverged surface vorticity nodes
NPR1NT	The number of time step after which print out of the stream function and vorticity field is produced
NTIMES	The total number of time steps by which the solution is advanced in time

<u>FORTRAN SYMBOL</u>	<u>MEANING</u>
OPTIME	The approximate CDC 6600 central processor time required for output of results
RE	Reynolds number
RPLOTU	The radial distance upto which the stream lines plot is produced
RPLOTV	The radial distance upto which the vorticity destribution plot is produced
SH	mesh size h
SK	mesh size k
SN(J)	Sin θ
ST	time step t
TIME	t^*
THETA(J)	θ
UO(I,J)	$\psi_{i,j}^n$
UN(I,J)	$\psi_{i,j}^{n+1}$
VCU(K)	Value of the stream function contour
VCV(K)	Value of the vorticity contour
W1	W1
W2	W2
W3	W3
Z(I)	Z_i (distance in Z-direction)

A.3 Listing of the Computer Program

```

C      PROGRAM MARK5 (INPUT,OUTPUT,TAPES=INPUT,TAPE6=OUTPUT,TAPE1,TAPE2)
C
C*****
C      PROGRAM SIMULATES VISCOUS INCOMPRESSIBLE FLOW AROUND A SOLID
C
C      SPHERE BY SOLVING TIME-DEPENDENT NAVIER-STOKES EQUATIONS
C
C      USING PEACEMAN AND RACHFORD ALTERNATING DIRECTION
C
C      IMPLICIT FINITE-DIFFERENCE METHOD
C*****
C
C      COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1      H3(61,61),THETA(61)
C      COMMON /UOGOC/ UO(61,61),GO(61,61)
C      COMMON /UNGNC/ UN(61,61),GN(61,61)
C      COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
C      COMMON /NPRC/ NPRU1,NPRG1,NPR1,NPRU,NPRG,NPR,MNPR,MXITER,INPRV
1      ,NPRINT,NTIMES,MXU,MXUP
C      COMMON /TSTEP/ RX(61),TIME
C      COMMON /LNPUT/ LNP1,LNP2,LNP3
C      COMMON /LOTPUT/ LOP1,LOP2,LOP3
C      COMMON /LMAIN/ LM1,LM2,LM3,LM4,LM5,LM6
C      COMMON /LRSTAB/ LRS1
C      COMMON /CPTIME/ CPT,OPTIME
C      SET PROGRAM CONTROL PARAMETERS TO START COMPUTATION
C      CALL CNTROL
C      READ OR EVALUATE INPUT DATA
C      CALL INPUT
C      SET TIME CONSTANT BOUNDARY CONDITIONS
C      IF(LM2.EQ.1) CALL CNSTBC
C      LM2=0
C      N=0$NPOLD=0$NPAGE=0$CPSEC1=0
1  CONTINUE
C      NITER=0
C      LM1=1
C      TIME=TIME+ST
C      IF(NPAGE.EQ.1) WRITE(6,100)
100  FORMAT(1H1)
C      2 CONTINUE
C      UPDATE VORTICITY FIELD IN TIME
C      CALL UPDATE
C      LM1=0
C      SET SURFACE VORTICITY BOUNDARY CONDITION
C      CALL VARBC
C      NITER=NITER+1
C      NPR=NPRU+NPRG
C      WRITE(6,101) NPR,NPRU,NPRG
101  FORMAT(10X,18,10X,18,10X,18)
C      TEST FOR CONVERGENCE
C      IF(NPR.LE.MNPR) GO TO 3
C      TEST FOR MAXIMUM ITERATIONS
C      IF(NITER.GE.MXITER) GO TO 7
C      GO TO 2
3  CONTINUE

```

```

C      CHECK COMPUTER TIME USED
      CALL SECOND(CPSEC)
      WRITE(6,102) CPSEC,NITER
102  FORMAT(1H0,1X,8HCPTIME =,F6.2,10X,7HNITER =,14)
      TLEFT=CPT-CPSEC
      DLT1=TLEFT-OPTIME
      DLT2=CPSEC-CPSEC1
      IF(DLT1.LE.DLT2) GO TO 6
      CPSEC1=CPSEC
      N=N+1
C      CHEECK PRINT OUT OF RESULT
      IF(NTIMES.EQ.N) GO TO 6
      NPNEW=N/NPRINT
      IF((NPNEW-NPOLD).EQ.1) GO TO 4
      NPAGE=0
5  CONTINUE
      NPOLD=NPNEW
      DO 4 I=1,IN1
      DO 4 J=1,JN1
      UO(I,J)=UN(I,J)
      GO(I,J)=GN(I,J)

      GO TO 1
4  CONTINUE
      NPAGE=1
      LOP1=0$LOP2=0$LOP3=1
      CALL OUTPUT
      GO TO 5
6  CONTINUE
      CALL OUTPUT
      STOP
7  CONTINUE
      LOP1=0$LOP2=1$LOP3=0
      CALL OUTPUT
      STOP
      END

```

SUBROUTINE CNTROL

```

C
C*****
C      THIS SUBROUTINE SUPPLIES ALL PROGRAM CONTROL PARAMETERS
C*****
C
COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1          H3(61,61),THETA(61)
COMMON /NPRC/ NPRU1,NPRG1,NPR1,NPRU,NPRG,NPR,MNPR,MXITER,INPRV
1          ,NPRINT,NTIMES,MXU,MXUP
COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
COMMON /TSTEP/ RX(61),TIME
COMMON /PLTPCN/ VCU(10),VCV(10),KNU,KNV,M
COMMON /PLOT/ XX1(59,10),YY1(59,10),XX2(30,10),YY2(30,10)
1          ,RPLLOT,IYS,IXSTEP,JNH1,JNH,KNP,LPLLOT,RPLLOTU,RPLLOTV
COMMON /LNP/ LNP1,LNP2,LNP3
COMMON /LOTPUT/ LOP1,LOP2,LOP3
COMMON /LNTAPE/ ISKIPN,II
COMMON /LOTAPE/ ISKIPO,JJ
COMMON /OUTBC/ DFUB,CCUOB,JNHH,JNHH2
COMMON /LMAIN/ LM1,LM2,LM3,LM4,LM5,LM6
COMMON /OMEGA/ W11,W21,W31,W12,W22,W32,W13,W23,W33
COMMON /VFIELD/ LF1,LF2,LF3,JCUT,INS,ISCAN,INSLOPE
COMMON /CPTIME/ CPT,OPTIME
C--- SET REYNOLDS NUMBER,MESH SIZE,INITIAL AND BOUNDARY CONDITIONS
C
RE=50.
TIME=0.
SH=0.04$KS=6
SK=3.14159*FLOAT(KS)/180.
ST=0.01
IN1=51$JN1=31
IN=IN1-1
JN=JN1-1
JNHH=JN/2
JNHH2=JNHH+2
C--- CENTRAL DIFFERENCE APPROXIMATION IN THE NON-LINEAR TERMS OF
C          VORTICITY TRANSPORT EQUATION
W11=0.5$W21=0.5$W31=0.0
C
C--- UNIFORM PARALLEL OUTER BOUNDARY CONDITION
LM2=0$LM3=0$LM4=0$LM5=0
C
C--- POTENTIAL FLOW INITIAL CONDITION
LNP1=1$LNP2=0$LNP3=0
II=1$JJ=2
ISKIPN=0$ISKIPO=0
C*****
C
C--- SET CONVERGENCE CRITERIA AND RELAXATION FACTORS
C
CCU=0.000001$CCG=0.001
DFU=1.905$DFG=0.65
DFUB=0.60$CCUOB=0.000001
MNPR=0$INPRV=0
MXITER=20
MXUP=150$MXU=150

```

```

C*****
C
C---  SET OUTPUT PARAMETERS
C
      LOP1=1$LOP2=0$LOP3=0
      CPT=600.$OPTIME=10.
      NTIMES=100$NPRINT=10
C*****
C
C---  SET TIME SAVING PARAMETERS FOR STREAM FUNCTION SOLUTION
C
      ISCAN=1$INSLOPE=0
      INS=16$JCUT=21
C*****
C
C---  SET PARAMETERS FOR STREAM FUNCTION AND VORTICITY CONTOUR PLOTS
C
      RPLOTU=4.$RPLOTV=4.$M=8$IXSTEP=60
      IYS=1.6666*FLOAT(IXSTEP-1)+1.5
      KNU=10$KNV=9
      VCU(1)=0.$VCU(2)=0.001$VCU(3)=0.01$VCU(4)=0.1$VCU(5)=0.5
      VCU(6)=1.$VCU(7)=2.$VCU(8)=-0.010$VCU(9)=-0.05$VCU(10)=-0.1
      VCV(1)=0.$VCV(2)=0.01$VCV(3)=0.1$VCV(4)=0.5$VCV(5)=1.
      VCV(6)=2.$VCV(7)=-0.01$VCV(8)=-0.1$VCV(9)=-0.5
      RETURN
      END

```

SUBROUTINE UPDATE

```

C
C*****
C      THIS SUBROUTINE UPDATES VALUES OF VORTICITY AND STREAM FUNCTION
C
C      USING PEACEMAN AND RACHFORD ALTERNATING DIRECTION IMPLICIT METHOD
C
C*****
C
      COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1          H3(61,61),THETA(61)
      COMMON /UOGOC/ UO(61,61),GO(61,61)
      COMMON /UNGNC/ UN(61,61),GN(61,61)
      COMMON /ICOEF/ B1(61),B2(61),C(61),SH4,ASH
      COMMON /JCOEF/ B3(61),B4(61),A,CR,SK4,ASK
      COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
      COMMON /NPRC/ NPRU1,NPRG1,NPR1,NPRU,NPRG,NPR,MNPR,MXITER,INPRV
1          ,NTIMES,NPRINT,MXU,MXUP
      COMMON /SLVCF/ C1(61),C2(61),C3(61),D(61),P(61),LSV1
      COMMON /TSTEP/ RX(61),TIME
      COMMON /LMAIN/ LM1,LM2,LM3,LM4,LM5,LM6
      COMMON /VFIELD/ LF1,LF2,LF3,JCUT,INS,ISCAN,INSLOPE
      COMMON /OUTBC/ DFUB,CCUOB,JNHH,JNHH2
      COMMON /OMEGA/ W11,W21,W31,W12,W22,W32,W13,W23,W33
      COMMON /LRSTAB/ LRS1
      W1=W11$W2=W21$W3=W31
      C1(1)=0.
      C2(1)=1.
      C3(1)=0.
C
C      UPDATE VALUES OF VORTICITY AT THE HALF TIME STEP --- COLUMN WISE
C
      IF(LM1.EQ.0) GO TO 500
      C1(JN1)=0.
      C2(JN1)=1.
      C3(JN1)=0.
      LSV1=1
      DO 200 I=2,IN
      D(1)=GN(1,1)
      D(JN1)=GN(1,JN1)
      DO 20 J=2,JN
      SKG=CR/H3(I,J)
      DIFUJ=UO(I,J+1)-UO(I,J-1)
      DIFUJP=DIFUJ+ABS(DIFUJ)
      DIFUJN=DIFUJ-ABS(DIFUJ)
      C1(J)=B2(1)-SKG*(W1*DIFUJN+W2*DIFUJP)
      C2(J)=RX(1)-ASH-SKG*(W3*DIFUJN-W3*DIFUJP+SH4*DIFUJ)
      C3(J)=B1(1)+SKG*(W1*DIFUJP+W2*DIFUJN)
      D(J)=C1(J)*GO(I-1,J)+C2(J)*GO(I,J)+C3(J)*GO(I+1,J)
      DIFUI=UO(I+1,J)-UO(I-1,J)
      DIFUIP=DIFUI+ABS(DIFUI)
      DIFUIN=DIFUI-ABS(DIFUI)
      C1(J)=-(B4(J)+SKG*(W1*DIFUIP+W2*DIFUIN))
      C2(J)=RX(1)+ASK-SKG*(W3*DIFUIP-W3*DIFUIN+SK4*CT(J)*DIFUI)
      C3(J)=-(B3(J)+SKG*(W1*DIFUIN+W2*DIFUIP)
20 CONTINUE
      CALL SOLVE

```

```

DO 21 J=1,JN1
GN(I,J)=P(J)
21 CONTINUE
200 CONTINUE
DO 22 J=1,JN1
DO 22 I=1,IN1
GO(I,J)=GN(I,J)
22 CONTINUE
IF(LM3.EQ.0) GO TO 500
C
C   SOLVE STREAM FUNCTION AT THE HALF TIME STEP
C
N=0
31 CONTINUE
NPRU=NPRU1
DO 310 J=2,JN
DO 310 I=2,IN
UGN=B1(I)*UN(I+1,J)+B2(I)*UN(I-1,J)+B3(J)*UN(I,J+1)+
1B4(J)*UN(I,J-1)-C(I)*GN(I,J)
UGN=UN(I,J)+DFU*(UGN-UN(I,J))
ERRORU=UGN-UN(I,J)
IF(ABS(ERRORU).LE.CCU) NPRU=NPRU+1
UN(I,J)=UGN
310 CONTINUE
N=N+1
IF(NPRU.EQ.0) GO TO 61
IF(N.EQ.MXU) STOP
GO TO 31
61 CONTINUE
WRITE(6,1000) N
CALL VARBC
C
C   UPDATE VALUES OF VORTICITY AT THE NEW TIME --- ROW BY ROW
C
500 CONTINUE
C1(IN1)=0.
C2(IN1)=1.
C3(IN1)=0.
LSV1=0
DO 100 J=2,JN
D(1)=GN(1,J)
D(IN1)=GN(IN1,J)
DO 10 I=2,IN
SKG=CR/H3(I,J)
DIFUI=UN(I+1,J)-UN(I-1,J)
DIFUIP=DIFUI+ABS(DIFUI)
DIFUIN=DIFUI-ABS(DIFUI)
C1(I)=B4(J)+SKG*(W1*DIFUIP+W2*DIFUIN)
C2(I)=RX(I)-ASK+SKG*(W3*DIFUIP-W3*DIFUIN+SK4*CT(J)*DIFUI)
C3(I)=B3(J)-SKG*(W1*DIFUIN+W2*DIFUIP)
D(I)=C1(I)*GO(I,J-1)+C2(I)*GO(I,J)+C3(I)*GO(I,J+1)
DIFUJ=UN(I,J+1)-UN(I,J-1)
DIFUJP=DIFUJ+ABS(DIFUJ)
DIFUJN=DIFUJ-ABS(DIFUJ)
C1(I)=-B2(I)+SKG*(W1*DIFUJN+W2*DIFUJP)
C2(I)=RX(I)+ASH+SKG*(W3*DIFUJN-W3*DIFUJP+SH4*DIFUJ)
C3(I)=-(B1(I)+SKG*(W1*DIFUJP+W2*DIFUJN)

```

```

10 CONTINUE
   CALL SOLVE
   DO 11 I=1,IN1
      GN(I,J)=P(I)
11 CONTINUE
100 CONTINUE
101 CONTINUE

C
C   SOLVE STREAM FUNCTION EQUATION AT THE NEW TIME
C
      N=0
      NSOLD=0
      LM4=1
30 CONTINUE
      N=N+1
      NPRU=NPRU1
C--- LM5=1 PARALLEL OUTFLOW BOUNDARY CONDITION OTHERWISE UNIFORM
C
C                                     OUTFLOW CONDITION
      IF(LM5.EQ.1) CALL VARBC
C--- ISCAN=1 NO COMPUTER TIME SAVING TECHNIQUE IS USED
      NSNEW=N/ISCAN
      ITEST=NSNEW-NSOLD
      NSOLD=NSNEW
      DO 300 J=2,JN
         INV=(1-ITEST)*(FLOAT(INSLOPE)*(J-2)+INS)+ITEST*IN
         IF(J.GE.JCUT) INV=IN
         NPRU=NPRU-(IN-INV)
         DO 300 I=2,INV
            UGN=B1(I)*UN(I+1,J)+B2(I)*UN(I-1,J)+B3(J)*UN(I,J+1)+
1B4(J)*UN(I,J-1)-C(I)*GN(I,J)
            UGN=UN(I,J)+DFU*(UGN-UN(I,J))
            ERRORU=UGN-UN(I,J)
            IF(ABS(ERRORU).LE.CCU) NPRU=NPRU-1
            UN(I,J)=UGN
300 CONTINUE
      IF(NPRU.EQ.0) GO TO 62
      IF(N.EQ.MXUP) GO TO 2000
      GO TO 30
62 CONTINUE
      LM4=0
      WRITE(6,1000) N
1000 FORMAT(10X,2HN=,I4)
      RETURN
2000 CONTINUE
      LRS1=0
      CALL RESTAB
      DFUB=DFUB-0.05
      GO TO 101
      END

```

SUBROUTINE VARBC

```

C
C*****
C      THIS SUBROUTINE SETS TIME DEPENDENT BOUNDARY CONDITONS
C*****
C
COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1          H3(61,61),THETA(61)
COMMON /UOGOC/ UO(61,61),GO(61,61)
COMMON /UNGNC/ UN(61,61),GN(61,61)
COMMON /ICOE/ B1(61),B2(61),C(61),SH4,ASH
COMMON /JCOEF/ B3(61),B4(61),A,CR,SK4,ASK
COMMON /TSTEP/ RX(61),TIME
COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
COMMON /NPRC/ NPRU1,NPRG1,NPR1,NPRU,NPRG,NPR,MNPR,MXITER,INPRV
1          ,NPRINT,NTIMES,MXU,MXUP
COMMON /LMAIN/ LM1,LM2,LM3,LM4,LM5,LM6
COMMON /OUT3C/ DFUB,CCUOB,JNHH,JNHH2
NPRG=NPRG1
RSH2=0.5/(SH*SH)
IF(LM4.EQ.1) GO TO 15
IF(LM1.EQ.1) GO TO 11
C--- EVALUATION OF SURFACE VORTICITY USING A CUBIC POLYNOMIAL FORMULA
DO 10 J=2,JN
UGN=(8.*UN(2,J)-UN(3,J))*RSH2
UGN=GN(1,J)+DFG*(UGN-GN(1,J))
ERRORG=(UGN-GN(1,J))/H3(1,J)
IF(ABS(ERRORG).LE.CCG) NPRG=NPRG-1
GN(1,J)=UGN
10 CONTINUE
RETURN
11 CONTINUE
DO 12 J=2,JN
GN(1,J)=(8.*UN(2,J)-UN(3,J))*RSH2
12 CONTINUE
RETURN
15 CONTINUE
C--- SET OUTER BOUNDARY CONDITION-PARALLEL FLOW
NPRUOB=JNHH-1
I=IN1
DO 300 J=JNHH2,JN
X=H3(1,J)/E(I-1)
IF(X.GE.1) GO TO 350
ANGLEJ=3.14159-THETA(J)
UGN=UN(I-1,J)-(ARSIN(X)-ANGLEJ)*(UN(I-1,J)-UN(I-1,J-1))/SK
GO TO 360
350 CONTINUE
UGN=0.5*(H3(1,J)**2)
360 CONTINUE
UGN=UN(I,J)+DFUB*(UGN-UN(I,J))
ERRORU=UGN-UN(I,J)
IF(ABS(ERRORU).LE.CCUOB) NPRUOB=NPRUOB-1
UN(I,J)=UGN
300 CONTINUE
NPRU=NPRU+NPRUOB
RETURN
END

```

SUBROUTINE SOLVE

```

C
C*****
C      THIS SUBROUTINE SOLVES FOR VORTICITY VALUES AT A NEW TIME BY
C      PARTIONING A TRIDIAGONAL MATRIX INTO AN UPPER AND A LOWER
C      TRIANGLE
C*****
C
COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1      H3(61,61),THETA(61)
DIMENSION W(61),G(61)
COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
COMMON /SLVCF/ C1(61),C2(61),C3(61),D(61),P(61),LSV1
C      LSV1=1 UPDATE COLUMN BY COLUMN OTHERWISE UPDATE ROW BY ROW
IF (LSV1.EQ.1) GO TO 100
W(1)=C2(1)
G(1)=D(1)/W(1)
DO 10 I=2,IN1
W(I)=C2(I)-C1(I)*C3(I-1)/W(I-1)
G(I)=(D(I)-C1(I)*G(I-1))/W(I)
10 CONTINUE
P(IN1)=G(IN1)
I=IN
DO 20 K=1,IN
P(I)=G(I)-C3(I)*P(I+1)/W(I)
I=I-1
20 CONTINUE
RETURN
100 CONTINUE
W(1)=C2(1)
G(1)=D(1)/W(1)
DO 30 J=2,JN1
W(J)=C2(J)-C1(J)*C3(J-1)/W(J-1)
G(J)=(D(J)-C1(J)*G(J-1))/W(J)
30 CONTINUE
P(JN1)=G(JN1)
J=JN
DO 40 K=1,JN
P(J)=G(J)-C3(J)*P(J+1)/W(J)
J=J-1
40 CONTINUE
RETURN
END

```

SUBROUTINE INPUT

```

C
C*****
C      THIS SUBROUTINE CONTROLS ALL INPUT DATA REQUIRED TO START OFF
C      SOLUTION PROCEDURE
C*****
C
COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1          H3(61,61),THETA(61)
COMMON /UOGOC/ UO(61,61),GO(61,61)
COMMON /UNGNC/ UN(61,61),GN(61,61)
COMMON /ICOE/ B1(61),B2(61),C(61),SH4,ASH
COMMON /JCOEF/ B3(61),B4(61),A,CR,SK4,ASK
COMMON /TSTEP/ RX(61),TIME
COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
COMMON /NPRC/ NPRU1,NPRG1,NPR1,NPRU,NPRG,NPR,MNPR,MXITER,INPRV
1          ,NPRINT,NTIMES,MXU,MXUP
COMMON /LNPUT/ LNP1,LNP2,LNP3
NPRU1=(IN-1)*(JN-1)
NPRG1=JN-1
NPR1=NPRU1+NPRG1
IF(LNP1.EQ.1) GO TO 10
IF(LNP2.EQ.1) GO TO 11
IF(LNP3.EQ.1) GO TO 12
WRITE(6,100)
100 FORMAT(1H1,55HINCORRECT CONTROL PARAMETERS ----- EXECUTION TERMINA
1TED)
STOP
10 CONTINUE
CALL GRID
CALL COEFF
CALL INTVAL
GO TO 13
11 CONTINUE
CALL INTAPE
DO 1 J=1,JN1
DO 1 I=1,IN1
UO(I,J)=UN(I,J)
GO(I,J)=GN(I,J)
1 CONTINUE
GO TO 13
12 CONTINUE
CALL INTAPE
13 CONTINUE
C--- EVALUATE CONSTANT COEFFICIENTS FOR VORTICITY TRANSPORT EQUATION
A=2.*((1./SH**2)+(1./SK**2))
SH4=4.*SH
SK4=4.*SK
ASH=2./(A*SH*SH)
ASK=2./(A*SK*SK)
CR=RE/(8.*SH*SK*A)
DO 14 I=1,IN1
RX(I)=(E(I)*E(I)*RE)/(A*ST)
14 CONTINUE
RETURN
END

```

SUBROUTINE OUTPUT

```

C
C*****
C      THIS SUBROUTINE CONTROLS OUTPUT OF RESULTS
C*****
C
      COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1          H3(61,61),THETA(61)
      COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
      COMMON /LOTPUT/ LOP1,LOP2,LOP3
      COMMON /LRSTAB/ LRS1
      LRS1=0
      IF(LOP1.EQ.1) GO TO 10
      IF(LOP2.EQ.1) GO TO 11
      IF(LOP3.EQ.1) GO TO 12
      WRITE(6,100)
100 FORMAT(1H1,49HINCORRECT GENERATION OF OUTPUT CONTROL PARAMETERS)
      RETURN
10 CONTINUE
      CALL RESTAB
      CALL CNPLTP
      CALL OUTAPE
      CALL PDCOEF
      RETURN
11 CONTINUE
      CALL OUTAPE
      CALL PDCOEF
      RETURN
12 CONTINUE
      CALL RESTAB
      CALL PDCOEF
      RETURN
      END

```

SUBROUTINE INTAPE

C

C*****

C THIS SUBROUTINE READS IN DATA FROM MAGNETIC TAPE

C*****

C

```

COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1             H3(61,61),THETA(61)
COMMON /UGOC/ UO(61,61),GO(61,61)
COMMON /UNGNC/ UN(61,61),GN(61,61)
COMMON /ICOE/ B1(61),B2(61),C(61),SH4,ASH
COMMON /JCOEF/ B3(61),B4(61),A,CR,SK4,ASK
COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
COMMON /LNTAPE/ ISKIPN,II
REWIND II

```

C

IF(ISKIPN.GT.0) CALL SKPFILE(ISKIPN,II)

C

```

READ(II)(Z(I),E(I),I=1,IN1)
READ(II)(ITHETA(J),THETA(J),SN(J),CS(J),CT(J),J=1,JN1)
READ(II)((H3(I,J),I=1,IN1),J=1,JN1)
READ(II)(B1(I),B2(I),C(I),I=1,IN1)
READ(II)(B3(J),B4(J),J=1,JN1)
READ(II)((UO(I,J),I=1,IN1),J=1,JN1)
READ(II)((GO(I,J),I=1,IN1),J=1,JN1)
READ(II)((UN(I,J),I=1,IN1),J=1,JN1)
READ(II)((GN(I,J),I=1,IN1),J=1,JN1)
RETURN
END

```

SUBROUTINE OUTAPE

```

C
C*****
C      THIS SUBROUTINE WRITES OUT DATA ON MAGNETIC TAPE
C*****
C
COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1          H3(61,61),THETA(61)
COMMON /UOGOC/ UO(61,61),GO(61,61)
COMMON /UNGN/ UN(61,61),GN(61,61)
COMMON /ICOE/ B1(61),B2(61),C(61),SH4,ASH
COMMON /JCOE/ B3(61),B4(61),A,CR,SK4,ASK
COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
COMMON /LOTAPE/ ISKIPO,JJ
REWIND JJ

C
IF (ISKIPO.GT.0) CALL SKPFILE (ISKIPO,JJ)

C
WRITE(JJ)(Z(I),E(I),I=1,IN1)
WRITE(JJ)(ITHETA(J),THETA(J),SN(J),CS(J),CT(J),J=1,JN1)
WRITE(JJ)((H3(I,J),I=1,IN1),J=1,JN1)
WRITE(JJ)(B1(I),B2(I),C(I),I=1,IN1)
WRITE(JJ)(B3(J),B4(J),J=1,JN1)
WRITE(JJ)((UO(I,J),I=1,IN1),J=1,JN1)
WRITE(JJ)((GO(I,J),I=1,IN1),J=1,JN1)
WRITE(JJ)((UN(I,J),I=1,IN1),J=1,JN1)
WRITE(JJ)((GN(I,J),I=1,IN1),J=1,JN1)
END FILE JJ
RETURN
END

```

SUBROUTINE SKPFILE (ISKIP,I)

```

C
C*****
C      THIS SUBROUTINE SKIPS FILES ON MAGNETIC TAPE ---CDC 6600
C*****
C
      N=0
1  CONTINUE
      READ(1) DUMMY
      IF (EOF,1) 10,11
10  N=N+1
      IF (N.EQ.ISKIP) RETURN
11  GO TO 1
END

```

SUBROUTINE CNPLTP

```

C
C*****
C      THIS SUBROUTINE EVALUATES PLOT POSITIONS OF CONTOURS AND CALLS
C
C      THE LINE-PRINTER PLOTTING ROUTINE
C*****
C
      DIMENSION VN(61,61)
      DIMENSION ZVC1(10),ZVC2(10),LVC(10)
      COMMON /PLTPCN/ VCU(10),VCV(10),KNU,KNV,M
      COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1      H3(61,61),THETA(61)
      COMMON /UNGNC/ UN(61,61),GN(61,61)
      COMMON /PLOTG/ XX1(59,10),YY1(59,10),XX2(30,10),YY2(30,10)
1      RPLLOT,IYS,IXSTEP,JNH1,JNH,KNP,LPLLOT,RPLLOTU,RPLLOTV
      COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
      EQUIVALENCE (VN(1,1),GN(1,1))
      WRITE(6,1000)
      WRITE(6,1001)
      WRITE(6,1002) (VCU(K),K=1,KNU)
      DO 11 K=1,KNU
      ZVC1(K)=0.
      ZVC2(K)=0.
      LVC(K)=1
11 CONTINUE
      IIN1=(IN*M/10)+1
      IIN=IIN1-1
      JNH=JN/2
      JNH1=JNH+1
      DO 100 J=2,JN
C--- SEARCH AND INTERPOLATE BETWEEN TWO I-LINES
C
C      --- STREAM FUNCTION CONTOURS
      DO 110 I=2,IIN1
      DO 110 K=1,KNU
      IF (UN(I-1,J).LE.VCU(K).AND.UN(I,J).GE.VCU(K)) GO TO 200
      IF (UN(I-1,J).GE.VCU(K).AND.UN(I,J).LE.VCU(K)) GO TO 200
      GO TO 110
200 IF (LVC(K).EQ.0) GO TO 201
      ZVC1(K)=SH*(FLOAT(I-2)+(VCU(K)-UN(I-1,J))/(UN(I,J)-UN(I-1,J)))
      ZVC2(K)=0.
      LVC(K)=0
      GO TO 110
201 ZVC2(K)=SH*(FLOAT(I-2)+(VCU(K)-UN(I-1,J))/(UN(I,J)-UN(I-1,J)))
110 CONTINUE
      DO 300 K=1,KNU
      ZVC1(K)=EXP(ZVC1(K))
      ZVC2(K)=EXP(ZVC2(K))
      IF (ZVC1(K).LE.1.) ZVC1(K)=0.
      IF (ZVC2(K).LE.1.) ZVC2(K)=0.
      YY1(J-1,K)=ZVC1(K)*SN(J)
      XX1(J-1,K)=ZVC1(K)*CS(J)
      IF (J.LT.JNH1) GO TO 300
      YY2(J-JNH,K)=ZVC2(K)*SN(J)
      XX2(J-JNH,K)=ZVC2(K)*CS(J)
300 CONTINUE

```

```

WRITE(6,1003) ITHETA(J),(ZVC1(K),K=1,KNU)
WRITE(6,1004) (ZVC2(K),K=1,KNU)
DO 15 K=1,KNU
ZVC1(K)=0.
ZVC2(K)=0.
LVC(K)=1
15 CONTINUE
100 CONTINUE
C---- PLOT STREAM FUNCTION CONTOURS
RPL0T=RPL0TU
KNP=KNU
CALL CNPLOT
DO 16 J=2,JN
DO 16 I=1,IIN1
VN(I,J)=GN(I,J)/H3(I,J)
16 CONTINUE
WRITE(6,1005)
WRITE(6,1001)
WRITE(6,1002) (VCV(K),K=1,KNV)
DO 17 K=1,KNV
ZVC1(K)=0.
ZVC2(K)=0.
LVC(K)=1
17 CONTINUE
DO 120 J=2,JN
DO 130 I=2,IIN1
DO 130 K=1,KNV
C--- SEARCH AND INTERPOLATE BETWEEN TWO I-LINES -VORTICITY CONTOURS
IF(VN(I-1,J).LE.VCV(K).AND.VN(I,J).GE.VCV(K)) GO TO 2000
IF(VN(I-1,J).GE.VCV(K).AND.VN(I,J).LE.VCV(K)) GO TO 2000
GO TO 130
2000 IF(LVC(K).EQ.0) GO TO 2001
ZVC1(K)=SH*(FLOAT(I-2)+(VCV(K)-VN(I-1,J))/(VN(I,J)-VN(I-1,J)))
ZVC2(K)=0.
LVC(K)=0
GO TO 130
2001 ZVC2(K)=SH*(FLOAT(I-2)+(VCV(K)-VN(I-1,J))/(VN(I,J)-VN(I-1,J)))
130 CONTINUE
DO 301 K=1,KNV
ZVC1(K)=EXP(ZVC1(K))
ZVC2(K)=EXP(ZVC2(K))
IF(ZVC1(K).LE.1.) ZVC1(K)=0.
IF(ZVC2(K).LE.1.) ZVC2(K)=0.
YY1(J-1,K)=ZVC1(K)*SN(J)
XX1(J-1,K)=ZVC1(K)*CS(J)
IF(J.LT.JNH1) GO TO 301
YY2(J-JNH,K)=ZVC2(K)*SN(J)
XX2(J-JNH,K)=ZVC2(K)*CS(J)
301 CONTINUE
WRITE(6,1003) ITHETA(J),(ZVC1(K),K=1,KNV)
WRITE(6,1004) (ZVC2(K),K=1,KNV)
DO 25 K=1,KNV
ZVC1(K)=0.
ZVC2(K)=0.
LVC(K)=1
25 CONTINUE
120 CONTINUE

```

```

C--- PLOT VORTICITY CONTOURS
      RPL0T=RPL0TV
      KNP=KNV
      CALL CNPLOT
      DO 26 J=2,JN
      DO 26 I=1,IIN1
      GN(I,J)=VN(I,J)*H3(I,J)
26 CONTINUE
1000 FORMAT(1H1,35X,51H***** COORDINATES OF STREAM FUNCTION CONTOURS **
1**)
1001 FORMAT(1H0,/,5X,5HANGLE,10X,1X,9HCONTOUR 1,1X,9HCONTOUR 2,1X,9HCON
1TOUR 3,1X,9HCONTOUR 4,1X,9HCONTOUR 5,1X,9HCONTOUR 6,1X,9HCONTOUR 7
2,1X,9HCONTOUR 8,1X,9HCONTOUR 9,1X,10HCONTOUR 10)
1002 FORMAT(20X,10(3X,1H=,F6.3))
1003 FORMAT(1H0,4X,15,10X,10F10.4)
1004 FORMAT(20X,10F10.4)
1005 FORMAT(2H1,38X,45H***** COORDINATES OF VORTICITY CONTOURS *****)
      RETURN
      END

```

SUBROUTINE CNPLOT

```

C
C*****
C   THIS SUBROUTINE PLOTS STREAM FUNCTION AND VORTICITY CONTOURS
C*****
C
  DIMENSION CHAR(120)
  COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1      H3(61,61),THETA(61)
  COMMON /PLOTG/ XX1(59,10),YY1(59,10),XX2(30,10),YY2(30,10)
1      ,RPLLOT,IYS,IXSTEP,JNH1,JNH,KNP,LPLLOT,RPLLOTU,RPLLOTV
  COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
  DATA BLANK,PLOT/1H,1H,/
  DX=RPLLOT/(FLOAT(IXSTEP-1))
  DXHLF=DX/2.
  DO 10 L=1,120
  CHAR(L)=PLOT
10 CONTINUE
  WRITE(6,100) CHAR
100 FORMAT(1H1,120A1)
  DO 11 L=1,119
  CHAR(L)=BLANK
11 CONTINUE
C--- PLOT CONTOURS IN THE UPSTREAM SECTION
  XPLOT=RPLLOT
1 CONTINUE
  DO 12 JJ=1,JNH
  DO 12 K=1,KNP
    IF (ABS(XPLOT-XX1(JJ,K)).LE.DXHLF) GO TO 13
  GO TO 12
13 LL=(FLOAT(IYS)/RPLLOT)*YY1(JJ,K)+0.5
  IF (LL.EQ.0.OR.LL.GT.IYS) GO TO 12
  CHAR(LL)=PLOT
12 CONTINUE
  IF (XPLOT.LT.1.) GO TO 2
  WRITE(6,101) CHAR
101 FORMAT(1X,1H.,120A1)
3 CONTINUE
  DO 14 L=1,IYS
  CHAR(L)=BLANK
14 CONTINUE
  IF (XPLOT.LE.(DX/1000.)) GO TO 4
  XPLOT=XPLOT-DX
  GO TO 1
2 CONTINUE
  CL=1.-((XPLOT)**2)
  LC=(FLOAT(IYS)/RPLLOT)*SQRT(CL)+0.5
  IF (LC.EQ.0.OR.LC.GT.IYS) GO TO 5
  CHAR(LC)=PLOT
5 CONTINUE
  WRITE(6,102) CHAR
102 FORMAT(2X,120A1)
  GO TO 3
4 CONTINUE
C--- PLOT CONTOURS IN THE DOWN STREAM SECTION
  XPLOT=DX
  NHJ=JNH-1

```

```

      NJ=JN-1
6  CONTINUE
      DO 15 JJ=JNH,NJ
      DO 15 K=1,KNP
        IF(ABS(XPLOT+XX1(JJ,K)).LE.DXHLEF) GO TO 16
        GO TO 17
16  LL=(FLOAT(IYS)/RPLET)*YY1(JJ,K)+0.5
      IF(LL.EQ.0.OR.LL.GT.IYS) GO TO 17
      CHAR(LL)=PLOT
17  IF(ABS(XPLOT+XX2(JJ-NHJ,K)).LE.DXHLEF) GO TO 18
      GO TO 15
18  LL=(FLOAT(IYS)/RPLET)*YY2(JJ-NHJ,K)+0.5
      IF(LL.EQ.0.OR.LL.GT.IYS) GO TO 15
      CHAR(LL)=PLOT
15  CONTINUE
      IF(XPLOT.LT.1.) GO TO 7
      WRITE(6,101) CHAR
8  CONTINUE
      IF(XPLOT.GE.(RPLET-DX/1000.)) RETURN
      DO 19 L=1,IYS
      CHAR(L)=BLANK
19  CONTINUE
      XPLOT=XPLOT+DX
      GO TO 6
7  CONTINUE
      CL=1.-((XPLOT)**2)
      LC=(FLOAT(IYS)/RPLET)*SQRT(CL)+0.5
      IF(LC.EQ.0.OR.LC.GT.IYS) GO TO 9
      CHAR(LC)=PLOT
9  CONTINUE
      WRITE(6,102) CHAR
      GO TO 8
      END

```

SUBROUTINE INTVAL

```

C
C*****
C      THIS SUBROUTINE SETS POTENTIAL FLOW INITIAL CONDITION
C*****
C
      COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1          H3(61,61),THETA(61)
      COMMON /UOGOC/ UO(61,61),GO(61,61)
      COMMON /UNGNC/ UN(61,61),GN(61,61)
      COMMON /ICOE/ B1(61),B2(61),C(61),SH4,ASH
      COMMON /JCOEF/ B3(61),B4(61),A,CR,SK4,ASK
      COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
      COMMON /NPRC/ NPRU1,NPRG1,NPR1,NPRU,NPRG,NPR,MNPR,MXITER,INPRV
1          ,NPRINT,NTIMES,MXU,MXUP
      COMMON /LRSTAB/ LRS1
      DO 10 J=1,JN1
      DO 10 I=1,IN1
      GO(I,J)=0.
10 CONTINUE
      DO 11 J=2,JN
      DO 11 I=2,IN
      UO(I,J)=0.5*(1.-1./E(I)**3)*H3(I,J)*H3(I,J)
11 CONTINUE
      DO 13 J=1,JN1
      UO(I,J)=0.
      UO(IN1,J)=0.5*H3(IN1,J)*H3(IN1,J)
13 CONTINUE
      DO 14 I=1,IN1
      UO(I,1)=0.
      UO(I,JN1)=0.
14 CONTINUE
500 CONTINUE
      NPRU=NPRU1
      DO 100 J=2,JN
      DO 100 I=2,IN
      UGN=B1(I)*UO(I+1,J)+B2(I)*UO(I-1,J)+B3(J)*UO(I,J+1)+B4(J)*UO(I,J-1)
1          -C(I)*GO(I,J)
      UGN=UO(I,J)+DFU*(UGN-UO(I,J))
      ERRORU=UGN-UO(I,J)
      IF (ABS(ERRORU).LE.CCU) NPRU=NPRU-1
      UO(I,J)=UGN
100 CONTINUE
      WRITE(6,1000) NPRU
1000 FORMAT(10X,I8)
      IF (NPRU.GT.INPRV) GO TO 500
      RSH2=0.5/(SH*SH)
      DO 110 J=2,JN
      GO(I,J)=(8.*UO(2,J)-UO(3,J))*RSH2
110 CONTINUE
      LRS1=1
      CALL RESTAB
      DO 120 J=1,JN1
      DO 120 I=1,IN1
      UN(I,J)=UO(I,J)
      GN(I,J)=GO(I,J)
120 CONTINUE
      RETURN
      END

```

SUBROUTINE GRID

```

C
C*****
C      THIS SUBROUTINE SUPPLIES GRID INFORMATION AT EACH NODE
C*****
C
      COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1          H3(61,61),THETA(61)
      COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
      DO 10 J=2,JN
        ITHETA(J)=KS*(J-1)
        THETA(J)=SK*FLOAT(J-1)
        SN(J)=SIN(THETA(J))
        CS(J)=COS(THETA(J))
        CT(J)=CS(J)/SN(J)
10  CONTINUE
        ITHETA(1)=0
        ITHETA(JN1)=180
        THETA(1)=0.
        THETA(JN1)=SK*FLOAT(JN)
        SN(1)=0.
        SN(JN1)=0.
        CS(1)=1.
        CS(JN1)=-1.
        DO 11 I=2,IN
          Z(I)=SH*FLOAT(I-1)
          E(I)=EXP(Z(I))
11  CONTINUE
          Z(1)=0.
          Z(IN1)=SH*FLOAT(IN)
          E(1)=1.
          E(IN1)=EXP(Z(IN1))
          DO 12 J=1,JN1
            DO 12 I=1,IN1
              H3(I,J)=E(I)*SN(J)
12  CONTINUE
          RETURN
        END

```

SUBROUTINE CNSTBC

```

C
C*****
C      THIS SUBROUTINE SETS UP TIME INDEPENDENT BOUNDARY CONDITIONS
C*****
C
      COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1          H3(61,61),THETA(61)
      COMMON /UNGNC/ UN(61,61),GN(61,61)
      COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
      DO 10 I=1,IN1
        UN(I,1)=0.
        UN(I,JN1)=0.
10    CONTINUE
      DO 11 J=1,JN1
        UN(1,J)=0.
        UN(IN1,J)=0.5*H3(IN1,J)*H3(IN1,J)
11    CONTINUE
      DO 12 I=1,IN1
        GN(I,1)=0.
        GN(I,JN1)=0.
12    CONTINUE
      DO 13 J=2,JN
        GN(IN1,J)=0.
13    CONTINUE
      RETURN
      END

```

SUBROUTINE COEFF

```

C
C*****
C      THIS SUBROUTINE EVALUATES CONSTANT COEFFICIENTS IN
C
C      FINITE-DIFFERENCE EQUATIONS
C*****
C
      COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1          H3(61,61),THETA(61)
      COMMON /ICOEF/ B1(61),B2(61),C(61),SH4,ASH
      COMMON /JCOEF/ B3(61),B4(61),A,CR,SK4,ASK
      COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
      A=2.*((1./SH**2)+(1./SK**2))
      BA=((1./SH**2)-0.5/SH)/A
      BB=((1./SH**2)+0.5/SH)/A
      DO 10 I=2,IN
        B1(I)=BA
        B2(I)=BB
        C(I)=(E(I)*E(I))/A
10    CONTINUE
      DO 11 J=2,JN
        B3(J)=((1./SK**2)-0.5*CT(J)/SK)/A
        B4(J)=((1./SK**2)+0.5*CT(J)/SK)/A
11    CONTINUE
      RETURN
      END

```

SUBROUTINE PDCOEF

```

C
C*****
C      THIS SUBROUTINE EVALUATES SURFACE PRESSURE AND DRAG COEFFICIENTS
C*****
C
      DIMENSION VN(61,61)
      DIMENSION VTH(61),VTH2(61),AK(61),AK2(61)
      COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1          H3(61,61),THETA(61)
      COMMON /UOGOC/ UO(61,61),GO(61,61)
      COMMON /UNGNC/ UN(61,61),GN(61,61)
      COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
      EQUIVALENCE (VN(1,1),GN(1,1))
      DO 10 J=2,JN
      DO 10 I=1,IN1
      VN(I,J)=GN(I,J)/H3(I,J)
10 CONTINUE
      S1=1.0/(12.0*SH)
      S2=1.0/(12.0*SK)
      S3=2.0/(SK*SK*ST)
C
C      EVALUATE STAGNATION PRESSURE COEFFICIENT
C
      YUZ=0.
      YVZ=0.5*S2*(48*VN(1,2)-36*VN(1,3)+16*VN(1,4)-3*VN(1,5))
      DO 1000 I=2,IN
      VZ=S2*(48*VN(I,2)-36*VN(I,3)+16*VN(I,4)-3*VN(I,5))
      UT=S3*(UN(I,2)-UO(I,2))/E(I)
      YVZ=YVZ+VZ
      YUZ=YUZ+UT
1000 CONTINUE
      A1=SH*YVZ
      A3=2.*SH*YUZ
      AKNOT=1.0+8.0*A1/RE
C
C      EVALUATE SURFACE PRESSURE
C
      RSH3=6.0/(SH*SH*SH)
      RSHP4=3.0/SH+4.
      VTH(1)=0.,$VTH(JN1)=0.
      VTH2(1)=0.,$VTH2(JN1)=0.
      DO 2000 J=2,JN
      Y=S1*(-25.0*VN(1,J)+48.0*VN(2,J)-36.0*VN(3,J)+16.0*VN(4,J)-
13.0*VN(5,J))
C      EVALUATION OF SURFACE PRESSURE USING STREAM FUNCTION VALUES
      Y2=RSH3*UN(2,J)/SN(J)-RSHP4*VN(1,J)
      VTH(J)=4.0*(Y+VN(1,J))/RE
      VTH2(J)=4.0*(Y2+VN(1,J))/RE
2000 CONTINUE
      AK(1)=AKNOT
      AK2(1)=AKNOT
      DO 2055 J=2,JN1
      AK(J)=AK(J-1)+0.5*SK*(VTH(J-1)+VTH(J))
      AK2(J)=AK2(J-1)+0.5*SK*(VTH2(J-1)+VTH2(J))
2055 CONTINUE

```

```

C
C      EVALUATE VISCOUS AND FORM DRAG COEFFICIENTS
C
      CF=0.
      CP=0.
      CP2=0.
      DO 2066 J=2,JN
      SNS=SN(J)*SN(J)
      SN2=2.0*SN(J)*CS(J)
      CF=CF+VN(1,J)*SNS
      CP=CP+AK(J)*SN2
      CP2=CP2+AK2(J)*SN2
2066  CONTINUE
      A2=SK*CF
      CDF=8.0*A2/RE
      CDP=SK*CP
      CDP2=SK*CP2
      CDT=CDF+CDP
      CDT2=CDF+CDP2
      WRITE(6,2010)
2010  FORMAT(1H1,10X,5HANGLE,11X,3HVTH,13X,4HVTH2,13X,2HK,15X,2HK2,
115X,2HKC,15X,3HVNS)
      DO 2040 J=1,JN1
      AKC=AK(J)-A3
      WRITE(6,2030) 1THETA(J),VTH(J),VTH2(J),AK(J),AK2(J),AKC,VN(1,J)
2040  CONTINUE
2030  FORMAT(10X,14,6(5X,F12.5))
      WRITE(6,2050)
2050  FORMAT(1X,///,20X,2HRE,10X,2HA1,10X,2HA2,10X,3HCDF,10X,3HCDP,
110X,3HCDT,8X,4HCDP2,7X,4HCDT2)
      WRITE(6,2070) RE,A1,A2,CDF,CDP,CDT,CDP2,CDT2
2070  FORMAT(16X,F8.3,2X,7(F12.4))
      DO 11 J=2,JN
      DO 11 I=1,IN1
      GN(I,J)=VN(I,J)*H3(I,J)
11  CONTINUE
      RETURN
      END

```

SUBROUTINE RESTAB

```

C
C*****
C      THIS SUBROUTINE PRINTS OUT STREAM FUNCTION AND VORTICITY FIELD
C
C                                  IN A TABULAR FORM
C*****
C
      DIMENSION VO(61,61),VN(61,61)
      COMMON /GRIDC/ Z(61),E(61),ITHETA(61),SN(61),CS(61),CT(61),
1      H3(61,61),THETA(61)
      COMMON /UOGOC/ UO(61,61),GO(61,61)
      COMMON /UNGNC/ UN(61,61),GN(61,61)
      COMMON /TSTEP/ RX(61),TIME
      COMMON /INJNC/ IN1,JN1,IN,JN,SH,SK,ST,KS,RE,CCU,CCG,DFU,DFG
      COMMON /NPRC/ NPRU1,NPRG1,NPR1,NPRU,NPRG,NPR,MNPR,MXITER,INPRV
1      ,NPRINT,NTIMES,MXU,MXUP
      COMMON /LRSTAB/ LRS1
      EQUIVALENCE (VO(1,1),GO(1,1)),(VN(1,1),GN(1,1))
      IF(LRS1.EQ.1) GO TO 10
C      LRS1=1 WRITE OLD VALUES OTHERWISE WRITE NEW VALUES
      WRITE(6,100) RE,TIME,NPR,NPRU,NPRG
      WRITE(6,200)
      INN=IN1-10
      DO 1000 K=1,INN,10
      K1=K+9
      WRITE(6,201) ((UN(I,J),I=K,K1),J=1,JN1)
      WRITE(6,202)
1000 CONTINUE
      WRITE(6,203) ((UN(I,J),I=IN1,IN1),J=1,JN1)
      WRITE(6,300)
      DO 500 J=2,JN
      DO 500 I=1,IN1
      VN(I,J)=GN(I,J)/H3(I,J)
500 CONTINUE
      DO 1010 K=1,INN,10
      K1=K+9
      WRITE(6,201) ((VN(I,J),I=K,K1),J=1,JN1)
      WRITE(6,202)
1010 CONTINUE
      WRITE(6,203) ((VN(I,J),I=IN1,IN1),J=1,JN1)
      DO 510 J=2,JN
      DO 510 I=1,IN1
      GN(I,J)=VN(I,J)*H3(I,J)
510 CONTINUE
      RETURN
10 CONTINUE
      TIME=TIME-ST
      IF(TIME.LT.0.0) TIME=0.
      WRITE(6,100) RE,TIME,NPR,NPRU,NPRG
      WRITE(6,200)
      INN=IN1-10
      DO 1030 K=1,INN,10
      K1=K+9
      WRITE(6,201) ((UO(I,J),I=K,K1),J=1,JN1)
      WRITE(6,202)
1030 CONTINUE

```

```

      WRITE(6,203) ((UO(I,J),I=IN1,IN1),J=1,JN1)
      WRITE(6,300)
      DO 501 J=2,JN
      DO 501 I=1,IN1
      VO(I,J)=GO(I,J)/H3(I,J)
501  CONTINUE
      DO 1040 K=1,INN,10
      K1=K+9
      WRITE(6,201) ((VO(I,J),I=K,K1),J=1,JN1)
      WRITE(6,202)
1040  CONTINUE
      WRITE(6,203) ((VO(I,J),I=IN1,IN1),J=1,JN1)
      DO 511 J=2,JN
      DO 511 I=1,IN1
      GO(I,J)=VO(I,J)*H3(I,J)
511  CONTINUE
      RETURN
100  FORMAT(1H1,/////,20X,14HREYNOLDS NO. =,F8.2,60X,12HTOTAL TIME =,F1
12.5,///,20X,5HNPR =,I8,20X,6HNPRU =,I8,20X,6HNPRG =,I8)
200  FORMAT(1H1,34X,68H*****  STREAM FUNCTION FOR VISCOUS LAMINAR FLOW
1PAST A SPHERE  *****)
201  FORMAT(5X,10G12.5)
202  FORMAT(1H ,56X,20H*****
203  FORMAT(5X,612.5)
300  FORMAT(1H1,30X,75H*****  VORTICITY DISTRIBUTION FOR VISCOUS LAMINA
1R FLOW PAST A SPHERE  *****)
      END

```

APPENDIX B

Components of the Stress Tensor for Newtonian Fluids
in Spherical Coordinates (r, θ, ϕ)

$$\tau_{rr} = -\mu \left[2 \frac{\partial u_r}{\partial r} - \frac{2}{3} (\nabla \cdot \mathbf{U}) \right]$$

$$\tau_{\theta\theta} = -\mu \left[2 \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) - \frac{2}{3} (\nabla \cdot \mathbf{U}) \right]$$

$$\tau_{\phi\phi} = -\mu \left[2 \left(\frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot \theta}{r} \right) - \frac{2}{3} (\nabla \cdot \mathbf{U}) \right]$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu \left[r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu \left[\frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{u_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} \right]$$

$$\tau_{\phi r} = \tau_{r\phi} = -\mu \left[\frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{u_\phi}{r} \right) \right]$$

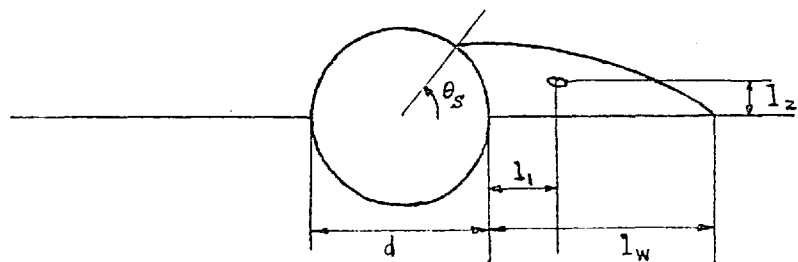
$$(\nabla \cdot \mathbf{U}) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi}$$

TABLES

TABLE 1

MAIN RESULTS AS SOLUTIONS APPROACH STEADY STATE

Re	1	10	50	200	500	1000 [†]	2000 [‡]
(h, k)	(0.07824, 6°)	(0.07824, 6°)	(.04, 6°)	(.04, 6°)	(.04, 3°)	(.01, 4°)	(.01, 4°)
r_{∞}	50	50	11.023	(11.023)	11.023	11.023	11.023
E_{ψ}	10^{-6}	10^{-6}	10^{-6}	10^{-6}	10^{-6}	10^{-5}	10^{-5}
E_g	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-2}	10^{-2}
ω_{ψ}	1.855	1.855	1.905	1.905	1.905	1.935	1.935
ω_g	0.5	0.5	0.65	0.75	0.8	0.8	0.95
C_{DF}	18.94	2.82	.9298	.3445	.1956	.1018	.1023
C_{DP}	9.46	1.53	.6547	.2852	.2981	.2457	.3762
C_{DT}	28.40	4.35	1.5845	.6297	.4937	.3475	.4785
θ_s	—	—	40.72°	66.32°	76.46°	86.38°	79.84°
L_1	—	—	.15	.80	.65	.60	.083
L_2	—	—	.42	.84	.86	1.09	.73
L_w	—	—	0.515	4.48	2.55	1.45	.4478
K_o	8.096	1.892	1.208	1.046	1.025	.956	1.016
K_{π}	-6.154	-.573	-.177	.025	-.442	.271	.256
$(\mathcal{J}_{s \max.}, \theta)$	(1.80, 72°)	(3.18, 60°)	(6.63, 54°)	(13.02, 54°)	(22.03, 48°)	(27.71, 48°)	(45.97, 52°)
$(\mathcal{J}_{s \min.}, \theta)$	—	—	(-.292, 156°)	(-1.5, 162°)	(-5.61, 162°)	(-31.3, 168°)	(-64.04, 160°)



$$L_1 = l_1/d ; L_2 = l_2/d ; L_w = l_w/d$$

† RESULTS QUOTATED AT $t^* = 26.63$

‡ RESULTS QUOTATED AT $t^* = 4.701$

TABLE 2

COMPARISON OF STEADY STATE RESULTS WITH SOLUTIONS OF RIMON (1968), AL-TAHA (1969)

Le CLAIR et al. (1970) AND WITH THE EXPERIMENTAL WORK OF TANEDA (1956)

REYNOLDS NO.=1

	(h,k)	r_{∞}	C_{DF}	C_{DP}	C_{DT}	K_o	K_{π}	θ_g	L_w
PRESENT WORK	(0.07824, 6°)	50	18.94	9.46	28.4	8.096	-6.154	—	—
EXPERIMENTAL	—	∞	—	—	26.5 [†]	—	—	—	—
AL-TAHA	(0.1, 12°)	6.05	22.89	11.33	34.22	9.416	-7.794	—	—
	(0.2, 12°)	20.86	22.78	11.14	33.92	9.538	-7.404	—	—
	(0.1, 12°)	54.598	18.92	9.19	28.12	8.505	-5.506	—	—
Le CLAIR et al.	(0.05, 3°)	150.	18.28	9.11	27.39	7.797	-5.898	—	—
	(0.1, 6°)	221.	18.46	8.97	27.43	7.995	-5.528	—	—
	(0.05, 3°)	1000.	18.29	9.09	27.37	7.810	-5.843	—	—

REYNOLDS NO=10

PRESENT WORK	(0.07824, 6°)	50.	2.82	1.53	4.35	1.892	-.573	—	—
EXPERIMENTAL	—	∞	—	—	4.10 [†]	—	—	—	—
RIMON	(0.05, 6°)	20.1	2.95	1.46	4.41	2.413	0.90	13.5°	.081
AL-TAHA	(0.2, 12°)	20.86	3.18	1.60	4.78	2.041	-.666	—	—
	(0.1, 12°)	6.05	3.10	1.68	4.78	1.985	-.730	—	—
	(0.05, 6°)	11.023	2.87	1.57	4.44	—	—	—	—
Le CLAIR et al.	(0.05, 3°)	90	2.80	1.54	4.34	1.869	-.606	—	—

Table 2 Continued.

REYNOLDS NO.=50

	(h, k)	r_∞	C_{DF}	C_{DP}	C_{DT}	K_O	K_π	θ_s	L_w
PRESENT WORK	(.04, 6°)	11.023	.9298	.6547	1.584	1.208	-.177	40.7°	.515
EXPERIMENTAL	—	~~∞~~	—	—	1.50 [†]	—	—	41°	.4
AL-TAHA	(0.05, 6°)	6.686	.9668	.7176	1.684	—	—	36°	.37
	(0.025, 3°)	6.686	.9693	.7223	1.692	1.206	-.266	36°	.31

REYNOLDS NO.=200

PRESENT WORK	(.04, 6°)	11.023	.3445	.2852	.6297	1.046	.025	66.3°	4.48
EXPERIMENTAL	—	~~∞~~	—	—	.77 [†]	—	—	65°	1.15
RIMON	(0.05, 6°)	20.1	.371	.367	.738	1.06	-.09	64°	1.05
AL-TAHA	(0.05, 6°)	6.686	.393	.323	.716	1.06	-.173	57°	1.26
	(0.025, 3°)	5.0	.364	.431	.795	1.05	-.096	56°	0.76
Le CLAIR et al.	(0.025, 3°)	12.	.372	.400	.772	1.05	-.143	—	—

REYNOLDS NO. =500

PRESENT WORK	(0.04, 3°)	11.023	.1956	.2981	.4937	1.025	-.442	76.5°	2.55
EXPERIMENTAL	—	~~∞~~	—	—	.55 [†]	—	—	—	—
AL-TAHA	(0.05, 6°)	6.686	.232	.058	.290	—	—	68°	2.01
	(0.025, 3°)	5.0	.205	.313	.518	1.022	.005	66.5°	1.05

Table 2 Continued.

REYNOLDS NO.=1000

	(h,k)	r_{∞}	C_{DF}	C_{DP}	C_{DT}	K_o	K_{π}	θ_s	L_w
PRESENT WORK	(0.01,4')	11.023	.1018	.2457	.3475	.956	.271	86.38°	1.45
EXPERIMENTAL	—	∞	—	—	.46 [†]	—	—	—	—
RIMON	(0.05,6')	20.1	.112	.355	.467	1.012	-.09	90°	1.45

REYNOLDS NO.=2000

PRESENT WORK	(0.01,4')	11.023	.102	.376	.478	1.016	.256	79.8°	.45
EXPERIMENTAL	—	~~∞~~	—	—	.42 [†]	—	—	—	—

† Experimental data by Lapple and Shepherd (1940)

EFFECT OF VARIATION OF OUTFLOW BOUNDARY CONDITON

	REYNOLDS NUMBER = 1	
	UNIFORM PARALLEL	PARALLEL ($\frac{\partial \psi}{\partial x} = 0$)
(h,k)	(0.07824, 6°)	(0.07824, 6°)
r_{∞}	50	50
I.C.	$t^* = 0$, P.F.	Uniform Parallel Soln.
t^*	8.1552	10.8352
C_{DF}	18.9374	18.8734
C_{DP*}	9.4580	9.3546
C_{DT}	28.395	28.2281
$C_{DT}(\text{Exp.})^{\dagger}$	26.5	26.5
$(\zeta_{s \text{ max.}}, \theta)$	(1.802, 72°)	(1.797, 76°)
K_o	8.096	8.115
K_{π}	-6.154	-5.988
ω_{ψ_b}	—	0.6

	REYNOLDS NUMBER = 10	
	UNIFORM PARALLEL	PARALLEL ($\frac{\partial \psi}{\partial x} = 0$)
(h,k)	(0.07828, 6°)	(0.07828, 6°)
r_{∞}	50	50
I.C.	$t^* = 0$, P.F.	Uniform Parallel Soln.
t^*	46.86	74.36
C_{DF}	28163	2.8382
C_{DP*}	1.5325	1.5190
C_{DT}	4.3488	4.3572
$C_{DT}(\text{Exp.})^{\dagger}$	4.1	4.1
$(\zeta_{s \text{ max.}}, \theta)$	(3.186, 60°)	(3.21, 60°)
K_o	1.892	1.897
K_{π}	-0.573	-0.556
ω_{ψ_b}	—	0.6

Table 3 Continued.

	REYNOLDS NUMBER = 500	
	UNIFORM PARALLEL	PARALLEL
(h,k)	$(0.04, 3^*)$	$(0.04, 3^*)$
r_∞	11.023	11.023
I.C.	$t^* = 0, \text{P.F.}$	Uniform Parallel Soln. at $t^* = 46.36$
t^*	74.55	50.84
C_{DF}	0.1956	0.1994
C_{DP^*}	0.0299	0.0355
C_{DT}	0.2255	0.2349
$C_{DT}(\text{Exp.})^\dagger$	0.566	0.566
θ_g	76.46°	76.05°
L_w	2.551	2.26
$(\zeta_{s \text{ max.}}, \theta)$	$(22.03, 48^\circ)$	$(22.27, 48^\circ)$
$(\zeta_{s \text{ min.}}, \theta)$	$(-5.61, 162^\circ)$	$(-6.12, 162^\circ)$
K_o	1.025	1.025
K_π	0.186	0.177
ω_{ψ_b}	—	0.4

* 4th order polynomial used to evaluate surface vorticity gradient

P.F. = Potential Flow Solution

† Experimental data by Lapple and Shepherd (1940)

TABLE 4

EFFECT OF VARIATION OF FINITE-DIFFERENCE SCHEME
IN THE NON-LINEAR CONVECTIVE TERM

REYNOLDS NO. = 50

	SCHEME 1	SCHEME 2	SCHEME 3
(h,k)	(0.04, 6°)	(0.04, 6°)	(0.04, 6°)
r_{∞}	11.023	11.023	11.023
I.C.	$t^*=0$, P.F.	Scheme 1 Solution	Scheme 1 Solution
t^*	67.49	84.19	107.89
C_{DF}	0.9298	0.9177	0.9234
C_{DP^*}	0.6547	0.6840	0.6505
C_{DT}	1.5845	1.6017	1.5740
$C_{DT}(\text{Exp.})^\dagger$	1.50	1.50	1.50
θ_g	40.72°	31.60°	41.19°
L_w	0.515	0.275	0.527
K_o	1.208	1.219	1.208
K_{π}	-.177	-.043	-.171
$(\zeta_{\text{max.}} \theta)$	(6.63, 54°)	(6.19, 60°)	(6.61, 54°)
$(\zeta_{\text{min.}} \theta)$	(-.292, 156°)	(-.147, 162°)	(-.299, 156°)

REYNOLDS NO. = 500

	SCHEME 1	SCHEME 2(a)	SCHEME 2(b)
(h,k)	(0.04, 3°)	(0.04, 3°)	(0.02, 6°)
r_{∞}	11.023	11.023	11.023
I.C.	$t^*=0$, P.F.	Scheme 1 Soln. at $t^*=11.023$	$\neq$
t^*	74.55	23.75	123.2
C_{DF}	74.55	23.75	123.2
C_{DP^*}	0.0299	.0566	.3006
C_{DT}	0.2255	.2297	.4983
$C_{DT}(\text{Exp.})^\dagger$	0.55	.55	.55
θ_g	76.46°	68.70°	63.90°

Table 4 Continued.

	SCHEME 1	SCHEME 2(a)	SCHEME 2(b)
L_w	2.551	1.305	1.163
K_o	1.025	1.026	1.024
K_π	.186	.187	.156
(ζ_{smax}, θ)	(22.03, 48°)	(17.27, 51°)	(18.56, 54°)
(ζ_{smin}, θ)	(-5.61, 162°)	(-3.08, 162°)	(-1.71, 162°)

‡ Scheme 1 solution with $h=0.02, k=6^\circ$ at $t=68.1$ and diverging
(See Section 5.6)

* 4th order polynomial used to evaluate the surface vorticity
gradient

P.F. = Potential flow solution

SCHEME 1 : Central differences (C.D.) everywhere $W1=0.5$;
 $W2=0.5$; $W3=0$

SCHEME 2 : Upstream biased differences (U.B.D.) everywhere
 $W1=1.0$; $W2=0$; $W3=1.0$

SCHEME 3 : 'Mixed' differences (M.D.) defined as:

C.D.	$1 < r < 2.2255$	$0^\circ < \theta < 60^\circ$
U.B.D.	$2.2255 < r < 11.023$	$0^\circ < \theta < 60^\circ$
C.D.	$1 < r < 2.6117$	$60^\circ < \theta < 120^\circ$
U.B.D.	$2.6117 < r < 11.023$	$60^\circ < \theta < 120^\circ$
C.D.	$1 < r < 11.023$	$120^\circ < \theta < 180^\circ$

† Experimental data by Lapple and Shepherd (1940)

TABLE 5

EFFECT OF ORDER OF POLYNOMIAL USED IN EVALUATION
OF THE SURFACE VORTICITY GRADIENT AND THE
PRESSURE DRAG (C_{DP})

<u>Re</u>	<u>ORDER</u>	<u>(h, k°)</u>	<u>I.C.</u>	<u>t</u>	<u>C_{DP}</u>	<u>C_{DT}</u>	<u>$C_{DT}(\text{Exp.})^\dagger$</u>
1	2	(.07824, 6°)	P.F.	8.1552	9.313	28.251	26.5
1	3	(.07824, 6°)	P.F.	8.1552	9.441	28.38	26.5
1	4	(.07824, 6°)	P.F.	8.1552	9.458	28.39	26.5
10	2	(.07824, 6°)	P.F.	46.86	1.499	4.315	4.10
10	3	(.07824, 6°)	P.F.	46.86	1.523	4.339	4.10
10	4	(.07824, 6°)	P.F.	46.86	1.532	4.349	4.10
50	2	(.04, 6°)	P.F.	67.49	.6376	1.567	1.50
50	3	(.04, 6°)	P.F.	67.49	.6453	1.575	1.50
50	4	(.04, 6°)	P.F.	67.49	.6547	1.584	1.50
200	2	(.04, 6°)	P.F.	77.55	.3053	.6498	.77
200	3	(.04, 6°)	P.F.	77.55	.2702	.6147	.77
200	4	(.04, 6°)	P.F.	77.55	.2852	.6297	.77
500	2	(.04, 3°)	P.F.	74.55	.2981	.4937	.55
500	3	(.04, 3°)	P.F.	74.55	.1428	.3383	.55
500	4	(.04, 3°)	P.F.	74.55	.0299	.2255	.55

Table 5 Continued.

Re	ORDER	(h, k')	I.C.	t	C _{DP}	C _{DT}	C _{DT} (Exp.) [†]
500	2	(.02, 6°)	t=15.05*	38.5	.2018	.3647	.55
500	3	(.02, 6°)	t=15.05*	38.5	.1951	.3581	.55
500	4	(.02, 6°)	t=15.05*	38.5	.2125	.3755	.55
1000	2	(.01, 4°)	P.F.	14.47	.2912	.4123	.46
1000	3	(.01, 4°)	P.F.	14.47	.2953	.4164	.46
1000	4	(.01, 4°)	P.F.	14.47	.3046	.4257	.46
2000	2	(.01, 4°)	P.F.	5.541	.3591	.4547	.42
2000	3	(.01, 4°)	P.F.	5.541	.3547	.4503	.42
2000	4	(.01, 4°)	P.F.	5.541	.3734	.4690	.42

* Refers to Re=500, h=0.04 and k=3 solution

† Experimental data by Lapple and Shepherd (1940)

P.F.=Potential flow solution

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