

Inductance and force calculations of circular coils with parallel axes shielded by a cuboid of high permeability

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ISSN 1751-8660

Received on 6th October 2017

Revised 29th January 2018

Accepted on 1st February 2018

E-First on 4th April 2018

doi: 10.1049/iet-epa.2017.0646

www.ietdl.org

Abstract: A detailed analysis is presented for a boundary value problem of circular coils with parallel axes shielded by a cuboid of high permeability. Field solutions are given by establishing a suitable ansatz of the magnetic scalar potential, which can satisfy the boundary conditions on six surfaces of the cuboid without difficulty. Analytic expressions are also given for the self and mutual inductance of shielded circular coils with rectangular cross section. By differentiating the self- and mutual magnetic energy with respect to the centre coordinates of the shielded coils, the total forces exerted on them are further obtained, which consist of self and mutual force components. Finally, the numerical results of the proposed method are compared with those of the finite-element method simulations, and the proposed method proves to be accurate and efficient enough for practical applications.

1 Introduction

In many cases of electric power applications, a static or slowly varying magnetic field should be confined to a certain region by magnetic covers [1–3]. Under such circumstances, the field and inductance of shielded coils will be changed owing to the existence of shielding material, which cannot be taken into account by the Neumann formula for inductance of coils in the free air [4–11]. Specific methods should be applied to this type of boundary value problem (BVP) to take the reflection effect of boundary into consideration. Normally, to evaluate the effects of shielding forms such as a pair of parallel plates, circular hollow cylinder or hollow sphere, we can refer to the relevant monographs [1–3]. However, for a shielding form of a cuboid, no results have been found as yet apart from some formulae published for the special case of this problem [12]. The absence of this shielding form in the literature is not due to its insignificance. In fact, the cuboidal shielding is commonly used, and it is more convenient to manufacture and install in practice. Moreover, it is not enough to simply employ the finite-element method (FEM) to calculate the effects of cuboidal shielding. For instance, using FEM to analyse an ‘inverse problem’ may lead to a rather cumbersome task [13] (the self and mutual inductance of a coupled coil system must satisfy some equations, so a direct solution with FEM will be infeasible). However, analytical solutions, by contrast, are competent at this complicated situation without the limitations of FEM.

In this work, analytical solutions will be given to the BVP mentioned in the title as a generalisation of the results for coaxial coils shielded by cuboid of high permeability [12]. Contrary to the intuition, this seemingly simple problem cannot be tackled easily with the routine method of magnetic vector potential (MVP) due to the presence of all components of the magnetic field. In addition, the solving process will become very tedious by starting with the Bessel-integral formulas of MVP of the circular rings [14, 15], since a complicated coefficient matrix of infinite dimensions will arise from the boundary conditions on the surfaces of the cuboid. Fortunately, these difficulties can be surmounted by using the magnetic scalar potential (MSP). In view of the physical meaning, a magnetic double layer with constant moment density will be created by a ring carrying the current I [16], as a consequence, the MSP will experience a jump quantity I while passing through the

area enclosed by the ring, and retain the continuity in the remaining area. Accordingly, the coefficients of the ansatz can be determined with the jump characteristics and orthogonality of the eigenfunctions [17]. With the MSP, the field, inductance, and forces between the shielded coils are easy to find. The validity of the results will be confirmed by the numerical calculations compared with those given by FEM simulations.

The configuration of this BVP is shown in Fig. 1 with the notations of geometric parameters. Two circular coils with parallel axes and centres (x_1, y_1, z_1) , (x_2, y_2, z_2) , are shielded by a cuboid with length, width, and height of $2a$, $2b$, $2l$, respectively. The axes of coils are parallel with the side $2l$ of the cuboid. The analysis will begin with the field of a single shielded circular ring, and the results are developed further to the circular coils with rectangular cross section.

2 Ansatz for MSP of a circular ring shielded by a cuboid of high permeability

We take central point of the cuboid as the origin of the Cartesian coordinate system (x, y, z) and let x, y, z -axes parallel to the sides of $2a, 2b$ and $2l$, respectively. The MSP $V(x, y, z)$ must satisfy the Laplace equation $\Delta V = 0$ and the magnetic field strength $\mathbf{H}$ will be given by negative gradient of V . Supposing that a circular ring (Ring 1) with radius r_1 and centre (x_1, y_1) is located in the plane $z = \zeta_1$ and carrying the current I_1 , then by separation of variables an ansatz can be established as

$$V(x, y, z) = I_1 \sum_{m,n=1}^{\infty} A_{mn} f_1(\lambda_m, x) \times f_2(\lambda_n, y) f_3^+(\lambda_{mn}, z, \zeta_1) \text{ for } z > \zeta_1 \quad (1a)$$

and

$$V(x, y, z) = I_1 \sum_{m,n=1}^{\infty} A_{mn} f_1(\lambda_m, x) \times f_2(\lambda_n, y) f_3^-(\lambda_{mn}, z, \zeta_1) \text{ for } z < \zeta_1, \quad (1b)$$

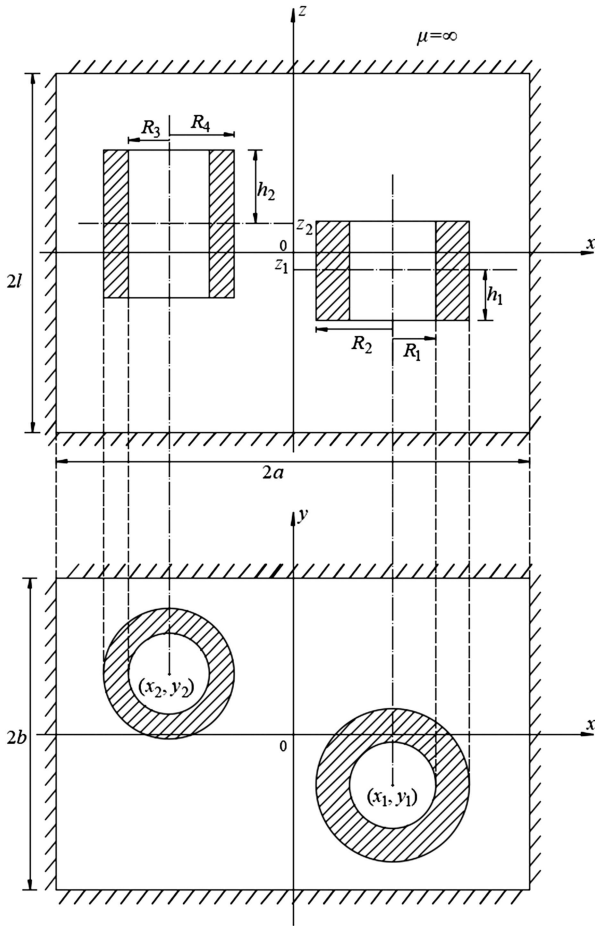


Fig. 1 Side and plan views of the circular coils with parallel axes shielded by a cuboid of high permeability

where A_{mn} are undetermined coefficients, λ_m , λ_n and λ_{mn} are eigenvalues such that

$$\lambda_{mn} = \sqrt{\lambda_m^2 + \lambda_n^2}. \quad (2)$$

In (1a) and (1b), the axial functions $f_3^\pm(\lambda_{mn}, z, \zeta_1)$ are prepared for later consideration of the jump feature. Furthermore, the eigenfunctions $f_1(\lambda_m, x)$ and $f_2(\lambda_n, y)$ should be the linear combination of trigonometric functions, namely

$$\begin{cases} f_1(\lambda_m, x) = A_1 \cos(\lambda_m x) + B_1 \sin(\lambda_m x), \\ f_2(\lambda_n, y) = A_2 \cos(\lambda_n y) + B_2 \sin(\lambda_n y). \end{cases} \quad (3)$$

The permeability of the cuboid can be regarded as infinity to simplify the analysis. It follows that the tangential components of $\mathbf{H}$ must vanish on the surfaces, namely

$$\begin{cases} H_y(\pm a, y, z) = 0, \\ H_x(x, \pm b, z) = 0, \end{cases} \quad (4a)$$

and

$$\begin{cases} H_z(\pm a, y, z) = 0, \\ H_z(x, \pm b, z) = 0. \end{cases} \quad (4b)$$

Since $\mathbf{H} = -\nabla V$, it follows that

$$\begin{cases} V(\pm a, y, z) = 0, \\ V(x, \pm b, z) = 0. \end{cases} \quad (4c)$$

Substituting (3) into (4c) yields the following linear systems for A_1 , B_1 and A_2 , B_2 , respectively

$$\begin{cases} \cos(\lambda_m a) A_1 + \sin(\lambda_m a) B_1 = 0, \\ \cos(\lambda_m a) A_1 - \sin(\lambda_m a) B_1 = 0, \end{cases} \quad (5a)$$

and

$$\begin{cases} \cos(\lambda_n b) A_2 + \sin(\lambda_n b) B_2 = 0, \\ \cos(\lambda_n b) A_2 - \sin(\lambda_n b) B_2 = 0. \end{cases} \quad (5b)$$

Hence, the existence of non-trivial solutions of (5a) demands

$$\begin{vmatrix} \cos(\lambda_m a) & \sin(\lambda_m a) \\ \cos(\lambda_m a) & -\sin(\lambda_m a) \end{vmatrix} = 0. \quad (6)$$

Thus the eigenvalues are given immediately

$$\lambda_m = \frac{m\pi}{2a}, \quad m = 1, 2, 3, \dots$$

Consequently, the corresponding eigenfunction f_1 can be found according to the Sturm–Liouville theory [18]

$$\begin{aligned} f_1(\lambda_m, x) &= \sin(\lambda_m a) \cdot \cos(\lambda_m x) \\ &\quad - \cos(\lambda_m a) \cdot \sin(\lambda_m x) = \sin[\lambda_m(a - x)]. \end{aligned} \quad (7a)$$

From (5b), we have the following result in a similar manner:

$$\lambda_n = \frac{n\pi}{2b}, \quad n = 1, 2, 3, \dots$$

and

$$\begin{aligned} f_2(\lambda_n, y) &= \sin(\lambda_n b) \cdot \cos(\lambda_n y) \\ &\quad - \cos(\lambda_n b) \cdot \sin(\lambda_n y) = \sin[\lambda_n(b - y)]. \end{aligned} \quad (7b)$$

The axial functions $f_3^\pm(\lambda_{mn}, z, \zeta_1)$ of the ansatz (1a) and (1b) should be treated in a slightly different manner. Considering that the total field should be separated into the primary and secondary parts, and the source can be regarded as a magnetic double layer, $f_3^\pm(\lambda_{mn}, z, \zeta_1)$ should be expressed as

$$f_3^+(\lambda_{mn}, z, \zeta_1) = A_3 e^{\lambda_{mn} z} + B_3 e^{-\lambda_{mn} z} + e^{-\lambda_{mn}(z - \zeta_1)} \quad \text{for } z > \zeta_1, \quad (8a)$$

$$f_3^-(\lambda_{mn}, z, \zeta_1) = A_3 e^{\lambda_{mn} z} + B_3 e^{-\lambda_{mn} z} - e^{-\lambda_{mn}(\zeta_1 - z)} \quad \text{for } z < \zeta_1. \quad (8b)$$

Moreover, for $z = \pm l$ the field components $H_x = -\partial V / \partial x$, $H_y = -\partial V / \partial y$ must vanish, that is

$$\begin{cases} A_3 e^{\lambda_{mn} l} + B_3 e^{-\lambda_{mn} l} + e^{-(l - \zeta_1)\lambda_{mn}} = 0, \\ A_3 e^{-\lambda_{mn} l} + B_3 e^{\lambda_{mn} l} - e^{-(l + \zeta_1)\lambda_{mn}} = 0. \end{cases} \quad (9)$$

[For simplicity, the square brackets $[]$ of $\cosh[\lambda_{mn}(l + \zeta_1)]$, $\sinh[\lambda_{mn}(l + z)]$, $\sin[\lambda_m(a - x)]$, $\cos[\lambda_n(b - y)]$, etc., will be omitted henceforth. Accordingly, the factor inside the round bracket always belongs to the argument of the preceding function. Unless otherwise stated, this rule will apply to all of the following formulae in this work.] Solving (9) gives

$$\begin{cases} A_3 = -\frac{\cosh \lambda_{mn}(l + \zeta_1) e^{-\lambda_{mn} l}}{\sinh(2l\lambda_{mn})}, \\ B_3 = \frac{\cosh \lambda_{mn}(l - \zeta_1) e^{-\lambda_{mn} l}}{\sinh(2l\lambda_{mn})}. \end{cases} \quad (10)$$

We, therefore, have the result of axial functions

$$f_3^\pm(\lambda_{mn}, z, \zeta_1) = \frac{2 \sinh \lambda_{mn}(l - z) \cosh \lambda_{mn}(l + \zeta_1)}{\sinh(2l\lambda_{mn})} \quad \text{for } z > \zeta_1, \quad (11a)$$

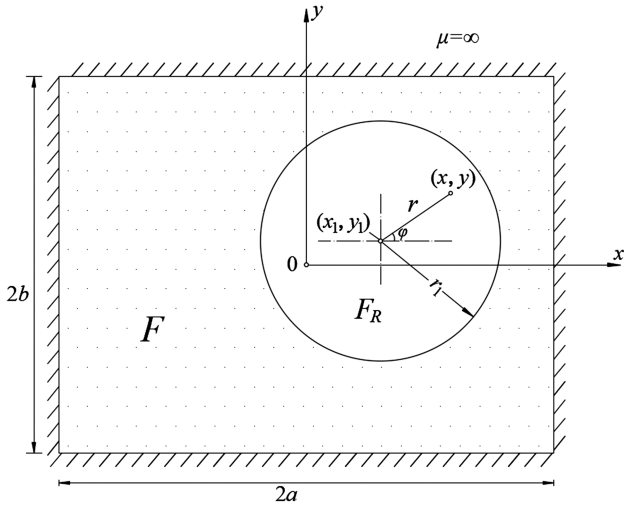


Fig. 2 Global and local coordinates of a circular ring shielded by a cuboid of high permeability

$$f_3^-(\lambda_{mn}, z, \zeta_1) = -\frac{2\sinh \lambda_{mn}(l+z)\cosh \lambda_{mn}(l-\zeta_1)}{\sinh(2l\lambda_{mn})} \text{ for } z < \zeta_1. \quad (11b)$$

Let F_R denotes the area encircled by the circular ring, it follows that $V(x, y, z)$ will obtain a spring quantity I_1 when passing through F_R and maintain the continuity on the remaining area F of plane $z = \zeta_1$ (see Fig. 2). This statement can be written in the following form:

$$V(x, y, \zeta_1+) - V(x, y, \zeta_1-) = \begin{cases} I_1 & \text{on } F_R, \\ 0 & \text{on } F, \end{cases} \quad (12)$$

where $V(x, y, \zeta_1+)$ and $V(x, y, \zeta_1-)$ are obtained by letting $z = \zeta_1$ in (1a) and (1b), respectively. With (1a) and (1b) and

$$f_3^+(\lambda_{mn}, \zeta_1, \zeta_1) - f_3^-(\lambda_{mn}, \zeta_1, \zeta_1) = 2, \quad (13)$$

it can be deduced from (12)

$$\sum_{m,n=1}^{\infty} A_{mn} \sin \lambda_m(a-x) \sin \lambda_n(b-y) = \begin{cases} 1/2 & \text{on } F_R, \\ 0 & \text{on } F. \end{cases} \quad (14)$$

Multiplying both sides of (14) by the eigenfunctions, we have

$$\sum_{p,q=1}^{\infty} A_{pq} \sin \lambda_p(a-x) \sin \lambda_q(b-y) \sin \lambda_m(a-x) \times \sin \lambda_n(b-y) = \begin{cases} \sin \lambda_m(a-x) \sin \lambda_n(b-y)/2 & \text{on } F_R, \\ 0 & \text{on } F, \end{cases} \quad (15)$$

where

$$\begin{cases} \lambda_p = \frac{p\pi}{2a}, & p = 1, 2, 3, \dots \\ \lambda_q = \frac{q\pi}{2b}, & q = 1, 2, 3, \dots \end{cases} \quad (16)$$

Next, we integrate the left-hand side of (15) with respect to (x, y) over the whole rectangle $2a \times 2b$, and the right-hand side (RHS) with respect to the local cylindrical coordinates (r, φ) over F_R , that is

$$\begin{aligned} & \sum_{p,q=1}^{\infty} A_{pq} \int_{-a}^a \sin \lambda_p(a-x) \sin \lambda_m(a-x) dx \\ & \times \int_{-b}^b \sin \lambda_q(b-y) \sin \lambda_n(b-y) dy \\ & = \frac{1}{2} \int_0^{2\pi} \int_0^{r_1} \sin \lambda_m(a-x) \sin \lambda_n(b-y) r dr d\varphi, \end{aligned} \quad (17)$$

where

$$\begin{cases} x = x_1 + r \cos \varphi, \\ y = y_1 + r \sin \varphi. \end{cases} \quad (18)$$

By means of the orthogonal relation

$$\begin{aligned} & \int_{-a}^a \sin \lambda_p(a-x) \sin \lambda_m(a-x) dx = a \delta_{pm}, \\ & \int_{-b}^b \sin \lambda_q(b-y) \sin \lambda_n(b-y) dy = b \delta_{qn}, \end{aligned} \quad (19)$$

where δ_{pm}, δ_{qn} are the Kronecker delta, it follows from (17)

$$\begin{aligned} A_{mn} &= \frac{1}{2ab} \int_0^{2\pi} \int_0^{r_1} \sin \lambda_m(a-x_1-r \cos \varphi) \\ & \times \sin \lambda_n(b-y_1-r \sin \varphi) r dr d\varphi. \end{aligned} \quad (20)$$

Applying the integrals

$$\int_0^{2\pi} \cos(\lambda_m r \cos \varphi) \cos(\lambda_n r \sin \varphi) d\varphi = 2\pi J_0(\lambda_{mn} r) \quad (21)$$

and

$$\begin{aligned} & \int_0^{2\pi} \cos(\lambda_m r \cos \varphi) \sin(\lambda_n r \sin \varphi) d\varphi \\ & = \int_0^{2\pi} \sin(\lambda_m r \cos \varphi) \cos(\lambda_n r \sin \varphi) d\varphi \\ & = \int_0^{2\pi} \sin(\lambda_m r \cos \varphi) \sin(\lambda_n r \sin \varphi) d\varphi = 0 \end{aligned} \quad (22)$$

to RHS of (20), where $J_n(x)$ is the Bessel function of the first kind of order n , we thus get the coefficient

$$A_{mn} = \frac{\pi r_1}{ab \lambda_{mn}} J_1(\lambda_{mn} r_1) \sin \lambda_m(a-x_1) \sin \lambda_n(b-y_1). \quad (23)$$

Eventually, combining (1a), (1b), (7a), (7b), (11a), (11b), and (23) gives the solution of MSP

$$\begin{aligned} V(x, y, z) &= \pm \frac{2\pi r_1 I_1}{ab} \sum_{m,n=1}^{\infty} J_1(\lambda_{mn} r_1) \sin \lambda_m(a-x_1) \\ & \times \sin \lambda_n(b-y_1) \sin \lambda_m(a-x) \sin \lambda_n(b-y) \\ & \times \frac{\sinh \lambda_{mn}(l \mp z) \cosh \lambda_{mn}(l \pm \zeta_1)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \quad z > \zeta_1, \\ & \quad z < \zeta_1. \end{aligned} \quad (24)$$

Moreover, by the definition

$$\mathbf{H} = -\nabla V, \quad (25)$$

we find the expressions of magnetic field immediately

$$\begin{cases}
H_x(x, y, z) = \pm \frac{2\pi r_1 I_1}{ab} \sum_{m,n=1}^{\infty} J_1(\lambda_{mn} r_1) \sin \lambda_m(a-x_1) \\
\quad \times \sin \lambda_n(b-y_1) \cos \lambda_m(a-x) \sin \lambda_n(b-y) \\
\quad \times \frac{\lambda_m \sinh \lambda_{mn}(l \mp z) \cosh \lambda_{mn}(l \pm \zeta_1)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \\
H_y(x, y, z) = \pm \frac{2\pi r_1 I_1}{ab} \sum_{m,n=1}^{\infty} J_1(\lambda_{mn} r_1) \sin \lambda_m(a-x_1) \\
\quad \times \sin \lambda_n(b-y_1) \sin \lambda_m(a-x) \cos \lambda_n(b-y) \\
\quad \times \frac{\lambda_n \sinh \lambda_{mn}(l \mp z) \cosh \lambda_{mn}(l \pm \zeta_1)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \\
H_z(x, y, z) = \frac{2\pi r_1 I_1}{ab} \sum_{m,n=1}^{\infty} J_1(\lambda_{mn} r_1) \sin \lambda_m(a-x_1) \\
\quad \times \sin \lambda_n(b-y_1) \sin \lambda_m(a-x) \sin \lambda_n(b-y) \\
\quad \times \frac{\cosh \lambda_{mn}(l \mp z) \cosh \lambda_{mn}(l \pm \zeta_1)}{\sinh(2l\lambda_{mn})}.
\end{cases} \quad (26)$$

$$\begin{pmatrix} z > \zeta_1 \\ z < \zeta_1 \end{pmatrix}.$$

Equations (26) lay the foundation for following derivations of inductance and forces of circular coils.

3 Self- and mutual inductance of circular coils with rectangular cross section shielded by a cuboid of high permeability

Expressions for the inductance of circular coils are deducible immediately from the magnetic field given by (26). Supposing that another circular ring (Ring 2) is placed in the aforesaid field of Ring 1, with the centre (x_2, y_2) and radius r_2 and lying in the plane $z = \zeta_2$, therefore the mutual inductance between the Rings 1 and 2 can be found by calculating the magnetic flux passing through the area encircled by the Ring 2, namely

$$\begin{aligned}
M &= \frac{\mu_0}{I_1} \int_0^{2\pi} \int_0^{r_2} H_z(x_2 + r \cos \varphi, y_2 + r \sin \varphi, \zeta_2) r dr d\varphi \\
&= \frac{4\mu_0 \pi^2 r_1 r_2}{ab} \sum_{m,n=1}^{\infty} J_1(\lambda_{mn} r_1) J_1(\lambda_{mn} r_2) \\
&\quad \times \sin \lambda_m(a-x_1) \sin \lambda_n(b-y_1) \sin \lambda_m(a-x_2) \\
&\quad \times \frac{\sin \lambda_n(b-y_2) \cosh \lambda_{mn}(l \pm \zeta_1) \cosh \lambda_{mn}(l \mp \zeta_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \\
&\quad \begin{pmatrix} \zeta_2 > \zeta_1 \\ \zeta_2 < \zeta_1 \end{pmatrix}.
\end{aligned} \quad (27)$$

When x_1, y_1, x_2, y_2 are all equal to zero, (27) will degenerate into (19a) and (19b) of [12]. For a circular ring of the infinitesimal cross section, the self-inductance is divergent [19].

To calculate the self- and mutual inductance for the shielded circular coils with rectangular cross section from (27), we consider a pair of circular coils, one has the centre (x_1, y_1) , inner and outer radii R_1, R_2 , axial length $2h_1$ and N_1 turns (Coil 1), another has the corresponding parameters of $(x_2, y_2), R_3, R_4, 2h_2$ and N_2 (Coil 2), and the middle planes of them are set in $z = z_1$ and $z = z_2$, respectively (see Fig. 1). By evaluating the axial integral with respect to the arguments ζ_1, ζ_2 of (27) and using the integral formula [20]

$$\begin{aligned}
w(\lambda_{mn}, R_1, R_2) &= \int_{R_1}^{R_2} r J_1(\lambda_{mn} r) dr \\
&= \frac{\pi}{2\lambda_{mn}} \{ R_2 [J_1(\lambda_{mn} R_2) H_0(\lambda_{mn} R_2) - J_0(\lambda_{mn} R_2) H_1(\lambda_{mn} R_2)] \\
&\quad - R_1 [J_1(\lambda_{mn} R_1) H_0(\lambda_{mn} R_1) - J_0(\lambda_{mn} R_1) H_1(\lambda_{mn} R_1)] \}, \\
\end{aligned} \quad (28)$$

where $H_n(x)$ is the Struve function of order n , we have the following results:

i. For $z_1 + h_1 \leq z_2 - h_2$

$$\begin{aligned}
M &= \kappa_{12} \sum_{m,n=1}^{\infty} w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \sin \lambda_m(a-x_1) \sin \lambda_n(b-y_1) \sin \lambda_m(a-x_2) \\
&\quad \times \frac{\sin \lambda_n(b-y_2) v_1(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})},
\end{aligned} \quad (29a)$$

where

$$\begin{aligned}
v_1(\lambda_{mn}, z_1, z_2) &= 4 \sinh(\lambda_{mn} h_1) \sinh(\lambda_{mn} h_2) \\
&\quad \times \cosh \lambda_{mn}(l + z_1) \cosh \lambda_{mn}(l - z_2) / \lambda_{mn}^2
\end{aligned}$$

and

$$\kappa_{12} = \frac{\mu_0 \pi^2 N_1 N_2}{ab h_1 h_2 (R_2 - R_1)(R_4 - R_3)}$$

is a coefficient relative to the number of turns and geometric parameters of both coils and cuboid.

ii. For $z_1 - h_1 \leq z_2 - h_2 < z_1 + h_1 \leq z_2 + h_2$

$$\begin{aligned}
M &= \kappa_{12} \sum_{m,n=1}^{\infty} w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \sin \lambda_m(a-x_1) \sin \lambda_n(b-y_1) \sin \lambda_m(a-x_2) \\
&\quad \times \frac{\sin \lambda_n(b-y_2) v_2(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})},
\end{aligned}$$

where

$$\begin{aligned}
v_2(\lambda_{mn}, z_1, z_2) &= \frac{1}{\lambda_{mn}^2} \left[\begin{aligned} &(h_1 + h_2 + z_1 - z_2) \lambda_{mn} \sinh(2l\lambda_{mn}) \\ &+ \sinh \lambda_{mn}(h_1 - l + z_1) \sinh \lambda_{mn}(h_2 - l - z_2) \\ &+ \sinh \lambda_{mn}(h_1 + l + z_1) \sinh \lambda_{mn}(h_2 - l + z_2) \\ &+ 2 \sinh(h_2 \lambda_{mn}) \sinh \lambda_{mn}(h_1 - l - z_1) \cosh \lambda_{mn}(l - z_2) \end{aligned} \right]
\end{aligned} \quad (29b)$$

iii. For $z_2 - h_2 \leq z_1 - h_1 < z_1 + h_1 \leq z_2 + h_2$ (see (29c))

In a similar manner, we deduce that the self-inductance, for a shielded circular coil with the centre (x_1, y_1, z_1) , inner and outer radii R_1, R_2 , axial length $2h_1$ and N_1 turns, is

$$\begin{aligned}
L &= \frac{\mu_0 \pi^2 N_1^2}{ab h_1^2 (R_2 - R_1)^2} \sum_{m,n=1}^{\infty} w^2(\lambda_{mn}, R_1, R_2) \\
&\quad \times \frac{\sin^2 \lambda_m(a-x_1) \sin^2 \lambda_n(b-y_1) v_4(\lambda_{mn}, z_1)}{\lambda_{mn} \sinh(2l\lambda_{mn})}
\end{aligned} \quad (30)$$

with the axial function given by

$$v_4(\lambda_{mn}, z_1) = \frac{1}{\lambda_{mn}^2} \left[\begin{aligned} &\cosh 2\lambda_{mn}(h_1 - l) - \cosh(2l\lambda_{mn}) \\ &+ \cosh(2h_1 \lambda_{mn}) \cosh(2z_1 \lambda_{mn}) \\ &- \cosh(2z_1 \lambda_{mn}) + 2h_1 \lambda_{mn} \sinh(2l\lambda_{mn}) \end{aligned} \right].$$

$$M = \kappa_{12} \sum_{m,n=1}^{\infty} w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \times \sin \lambda_m(a - x_1) \sin \lambda_n(b - y_1) \sin \lambda_m(a - x_2) \times \frac{\sin \lambda_n(b - y_2) v_3(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \quad (29c)$$

where

$$v_3(\lambda_{mn}, z_1, z_2) = \frac{2}{\lambda_{mn}^2} \left[h_1 \lambda_{mn} \sinh(2l\lambda_{mn}) + \sinh(\lambda_{mn} h_1) \cdot \cosh \lambda_{mn}(l - z_1) \sinh \lambda_{mn}(h_2 - l - z_2) + \sinh(\lambda_{mn} h_1) \cosh \lambda_{mn}(l + z_1) \sinh \lambda_{mn}(h_2 - l + z_2) \right].$$

4 Magnetic force exerted on circular coils shielded by a cuboid of high permeability

The magnetic forces between the coils are of practical importance. Before we commence the specific calculation, some preliminary discussions should be given for the force calculation of coils with the presence of magnetic materials. Unlike the force calculation of coils in the free air, in which we only need to consider the interaction between the coils, in this BVP the interaction between coils and that between coil and cuboid should be considered simultaneously. Hence, if we calculate the force from the magnetic energy, both mutual and self-magnetic energy (mutual and self-inductance) should be taken into account to obtain the total force. To clarify the process of force calculation, we will begin with the definition of magnetic force. With the aid of the obtained magnetic field (26) of Ring 1, the mutual magnetic force $F_2^{(m)}$ exerted on Ring 2 can be obtained immediately. According to the definition of magnetic force, it follows that [21, 22]

$$\begin{aligned} F_2^{(m)} &= \oint I_2 (d\mathbf{l}_2 \times \mathbf{B}_1) \\ &= I_2 r_2 \int_0^{2\pi} [B_z(x_2 + r_2 \cos \varphi, y_2 + r_2 \sin \varphi, z) \cos \varphi \mathbf{e}_x \\ &\quad + B_y(x_2 + r_2 \cos \varphi, y_2 + r_2 \sin \varphi, z) \sin \varphi \mathbf{e}_y \\ &\quad - \left(B_y(x_2 + r_2 \cos \varphi, y_2 + r_2 \sin \varphi, z) \sin \varphi \right. \\ &\quad \left. + B_x(x_2 + r_2 \cos \varphi, y_2 + r_2 \sin \varphi, z) \cos \varphi \right) \mathbf{e}_z] d\varphi, \end{aligned} \quad (31)$$

where ' $d\mathbf{l}_2$ ' is the infinitesimal element of Ring 2, and the local polar coordinate (r, φ)

$$\begin{cases} x = x_2 + r \cos \varphi \\ y = y_2 + r \sin \varphi \end{cases} \quad (32)$$

is introduced to assist the integration. Using the angular integrals

$$\int_0^{2\pi} \sin(\lambda_m r_2 \cos \varphi) \cos(\lambda_n r_2 \sin \varphi) \cos \varphi d\varphi = \frac{2\pi \lambda_m J_1(\lambda_{mn} r_2)}{\lambda_{mn}}, \quad (33a)$$

and

$$\begin{aligned} &\int_0^{2\pi} \cos(\lambda_m r_2 \cos \varphi) \cos(\lambda_n r_2 \sin \varphi) \cos \varphi d\varphi \\ &= \int_0^{2\pi} \cos(\lambda_m r_2 \cos \varphi) \sin(\lambda_n r_2 \sin \varphi) \cos \varphi d\varphi \\ &= \int_0^{2\pi} \sin(\lambda_m r_2 \cos \varphi) \sin(\lambda_n r_2 \sin \varphi) \cos \varphi d\varphi = 0 \end{aligned} \quad (33b)$$

it follows that

$$\begin{aligned} &\int_0^{2\pi} \sin \lambda_m(a - x_2 - r_2 \cos \varphi) \\ &\quad \times \sin \lambda_n(b - y_2 - r_2 \sin \varphi) \cos \varphi d\varphi \\ &= -2\pi \lambda_m J_1(\lambda_{mn} r_2) \cos \lambda_m(a - x_2) \sin \lambda_n(b - y_2) / \lambda_{mn}. \end{aligned} \quad (34)$$

We thus obtain the x-component of $F_2^{(m)}$

$$\begin{aligned} F_{2,x}^{(m)} &= -\frac{4\mu_0 \pi^2 r_1 r_2 I_1 I_2}{ab} \sum_{m,n=1}^{\infty} \lambda_m J_1(\lambda_{mn} r_1) J_1(\lambda_{mn} r_2) \\ &\quad \times \sin \lambda_m(a - x_1) \sin \lambda_n(b - y_1) \cos \lambda_m(a - x_2) \\ &\quad \times \frac{\sin \lambda_n(b - y_2) \cosh \lambda_{mn}(l \pm \zeta_1) \cosh \lambda_{mn}(l \mp \zeta_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \quad (35) \\ &\quad \left(\begin{array}{l} \zeta_2 \geq \zeta_1 \\ \zeta_2 < \zeta_1 \end{array} \right). \end{aligned}$$

Comparing (27) with (35), it is observed that

$$F_{2,x}^{(m)} = \frac{\partial W_m}{\partial x_2} = I_1 I_2 \frac{\partial M}{\partial x_2},$$

where W_m is the mutual magnetic energy. In fact, this is a general theorem which establishes the relationship between magnetic force and mutual inductance between two coils [23–27]. Consequently, the remaining force components $F_{2,y}^{(m)}$, $F_{2,z}^{(m)}$ are also deducible from differentiating (27) with respect to the corresponding coordinates y_2 , ζ_2 , and we will not repeat them here due to the space limitations.

Similarly, by differentiation with respect to the corresponding centre coordinates of coils in (29a)–(29c), the mutual force $F_2^{(m)}$ exerted on Coil 2 is obtained with no difficulty

i. For $z_1 + h_1 \leq z_2 - h_2$

$$\begin{aligned} F_{2,x}^{(m)} &= I_1 I_2 \frac{\partial M}{\partial x_2} \\ &= -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} \lambda_m w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\ &\quad \times \sin \lambda_m(a - x_1) \sin \lambda_n(b - y_1) \cos \lambda_m(a - x_2) \\ &\quad \times \frac{\sin \lambda_n(b - y_2) v_1(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \end{aligned} \quad (36a)$$

$$\begin{aligned}
F_{2,y}^{(m)} &= I_1 I_2 \frac{\partial M}{\partial y_2} \\
&= -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} \lambda_n w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \sin \lambda_m (a - x_1) \sin \lambda_n (b - y_1) \sin \lambda_m (a - x_2) \\
&\quad \times \frac{\cos \lambda_n (b - y_2) v_1(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})},
\end{aligned} \quad (36b)$$

$$\begin{aligned}
F_{2,z}^{(m)} &= I_1 I_2 \frac{\partial M}{\partial z_2} \\
&= -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \sin \lambda_m (a - x_1) \sin \lambda_n (b - y_1) \sin \lambda_m (a - x_2) \\
&\quad \times \frac{\sin \lambda_n (b - y_2) u_1(\lambda_{mn}, z_1, z_2)}{\sinh(2l\lambda_{mn})},
\end{aligned} \quad (36c)$$

where $v_1(\lambda_{mn}, z_1, z_2)$ is given in (29a), and $u_1(\lambda_{mn}, z_1, z_2)$ in (36c) is given by

$$\begin{aligned}
u_1(\lambda_{mn}, z_1, z_2) &= 4 \sinh(\lambda_{mn} h_1) \sinh(\lambda_{mn} h_2) \\
&\quad \times \cosh \lambda_{mn} (l + z_1) \sinh \lambda_{mn} (l - z_2) / \lambda_{mn}^2
\end{aligned}$$

ii. For $z_1 - h_1 \leq z_2 - h_2 < z_1 + h_1 \leq z_2 + h_2$

$$\begin{aligned}
F_{2,x}^{(m)} &= I_1 I_2 \frac{\partial M}{\partial x_2} \\
&= -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} \lambda_m w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \sin \lambda_m (a - x_1) \sin \lambda_n (b - y_1) \cos \lambda_m (a - x_2) \\
&\quad \times \frac{\sin \lambda_n (b - y_2) v_2(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})},
\end{aligned} \quad (37a)$$

$$\begin{aligned}
F_{2,y}^{(m)} &= I_1 I_2 \frac{\partial M}{\partial y_2} \\
&= -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} \lambda_n w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \sin \lambda_m (a - x_1) \sin \lambda_n (b - y_1) \sin \lambda_m (a - x_2) \\
&\quad \times \frac{\cos \lambda_n (b - y_2) v_2(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})},
\end{aligned} \quad (37b)$$

$$\begin{aligned}
F_{2,z}^{(m)} &= I_1 I_2 \frac{\partial M}{\partial z_2} \\
&= -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \sin \lambda_m (a - x_1) \sin \lambda_n (b - y_1) \sin \lambda_m (a - x_2) \\
&\quad \times \frac{\sin \lambda_n (b - y_2) u_2(\lambda_{mn}, z_1, z_2)}{\sinh(2l\lambda_{mn})},
\end{aligned} \quad (37c)$$

where $v_2(\lambda_{mn}, z_1, z_2)$ is given in (29b), and $u_2(\lambda_{mn}, z_1, z_2)$ in (37c) is given by

$$\begin{aligned}
u_2(\lambda_{mn}, z_1, z_2) &= \frac{1}{\lambda_{mn}^2} \left[\begin{aligned} &\sinh(2l\lambda_{mn}) + \sinh \lambda_{mn} (h_1 - l - z_1) \\ &\cdot \cosh \lambda_{mn} (h_2 + l - z_2) + \sinh \lambda_{mn} (h_1 - l + z_1) \\ &\cdot \cosh \lambda_{mn} (h_2 - l - z_2) - 2 \sinh(h_1 \lambda_{mn}) \\ &\cdot \cosh \lambda_{mn} (l + z_1) \cosh \lambda_{mn} (h_2 - l + z_2) \end{aligned} \right]
\end{aligned}$$

iii. For $z_2 - h_2 \leq z_1 - h_1 < z_1 + h_1 \leq z_2 + h_2$

$$\begin{aligned}
F_{2,x}^{(m)} &= I_1 I_2 \frac{\partial M}{\partial x_2} \\
&= -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} \lambda_n w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \sin \lambda_m (a - x_1) \sin \lambda_n (b - y_1) \cos \lambda_m (a - x_2) \\
&\quad \times \frac{\sin \lambda_n (b - y_2) v_3(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})},
\end{aligned} \quad (38a)$$

$$\begin{aligned}
F_{2,y}^{(m)} &= I_1 I_2 \frac{\partial M}{\partial y_2} \\
&= -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} \lambda_n w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \sin \lambda_m (a - x_1) \sin \lambda_n (b - y_1) \sin \lambda_m (a - x_2) \\
&\quad \times \frac{\cos \lambda_n (b - y_2) v_3(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})},
\end{aligned} \quad (38b)$$

$$\begin{aligned}
F_{2,z}^{(m)} &= I_1 I_2 \frac{\partial M}{\partial z_2} \\
&= \kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \sin \lambda_m (a - x_1) \sin \lambda_n (b - y_1) \sin \lambda_m (a - x_2) \\
&\quad \times \frac{\sin \lambda_n (b - y_2) u_3(\lambda_{mn}, z_1, z_2)}{\sinh(2l\lambda_{mn})},
\end{aligned} \quad (38c)$$

where $v_3(\lambda_{mn}, z_1, z_2)$ is given in (29c), and $u_3(\lambda_{mn}, z_1, z_2)$ in (38c) is given by

$$\begin{aligned}
u_3(\lambda_{mn}, z_1, z_2) &= \frac{2 \sinh(h_1 \lambda_{mn})}{\lambda_{mn}^2} \left[\begin{aligned} &\cosh \lambda_{mn} (l + z_1) \cosh \lambda_{mn} (h_2 - l + z_2) \\ &- \cosh \lambda_{mn} (l - z_1) \cosh \lambda_{mn} (h_2 - l - z_2) \end{aligned} \right].
\end{aligned}$$

In addition, the mutual force $F_1^{(m)}$ experienced by Coil 1 can be found in much the same manner.

i. For $z_1 + h_1 \leq z_2 - h_2$

$$\begin{aligned}
F_{1,x}^{(m)} &= I_1 I_2 \frac{\partial M}{\partial x_1} \\
&= -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} \lambda_n w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \cos \lambda_m (a - x_1) \sin \lambda_n (b - y_1) \sin \lambda_m (a - x_2) \\
&\quad \times \frac{\sin \lambda_n (b - y_2) v_1(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})},
\end{aligned} \quad (39a)$$

$$\begin{aligned}
F_{1,y}^{(m)} &= I_1 I_2 \frac{\partial M}{\partial y_1} \\
&= -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} \lambda_n w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \sin \lambda_m (a - x_1) \cos \lambda_n (b - y_1) \sin \lambda_m (a - x_2) \\
&\quad \times \frac{\sin \lambda_n (b - y_2) v_1(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})},
\end{aligned} \quad (39b)$$

$$\begin{aligned}
F_{1,z}^{(m)} &= I_1 I_2 \frac{\partial M}{\partial z_1} \\
&= \kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\
&\quad \times \sin \lambda_m (a - x_1) \sin \lambda_n (b - y_1) \sin \lambda_m (a - x_2) \\
&\quad \times \frac{\sin \lambda_n (b - y_2) s_1(\lambda_{mn}, z_1, z_2)}{\sinh(2l\lambda_{mn})},
\end{aligned} \quad (39c)$$

where $v_1(\lambda_{mn}, z_1, z_2)$ is given in (29a), and $s_1(\lambda_{mn}, z_1, z_2)$ in (39c) is given by

$$s_1(\lambda_{mn}, z_1, z_2) = 4\sinh(\lambda_{mn}h_1)\sinh(\lambda_{mn}h_2) \\ \times \sinh \lambda_{mn}(l + z_1)\cosh \lambda_{mn}(l - z_2)/\lambda_{mn}^2$$

ii. For $z_1 - h_1 \leq z_2 - h_2 < z_1 + h_1 \leq z_2 + h_2$

$$F_{1,x}^{(m)} = I_1 I_2 \frac{\partial M}{\partial x_1} \\ = -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} \lambda_m w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\ \times \cos \lambda_m(a - x_1) \sin \lambda_n(b - y_1) \sin \lambda_m(a - x_2) \\ \times \frac{\sin \lambda_n(b - y_2) v_2(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \quad (40a)$$

$$F_{1,y}^{(m)} = I_1 I_2 \frac{\partial M}{\partial y_1} \\ = -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} \lambda_n w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\ \times \sin \lambda_m(a - x_1) \cos \lambda_n(b - y_1) \sin \lambda_m(a - x_2) \\ \times \frac{\sin \lambda_n(b - y_2) v_2(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \quad (40b)$$

$$F_{1,z}^{(m)} = I_1 I_2 \frac{\partial M}{\partial z_1} \\ = \kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\ \times \sin \lambda_m(a - x_1) \sin \lambda_n(b - y_1) \sin \lambda_m(a - x_2) \\ \times \frac{\sin \lambda_n(b - y_2) s_2(\lambda_{mn}, z_1, z_2)}{\sinh(2l\lambda_{mn})}, \quad (40c)$$

where $v_2(\lambda_{mn}, z_1, z_2)$ is given in (29b), and $s_2(\lambda_{mn}, z_1, z_2)$ in (40c) is given by

$$s_2(\lambda_{mn}, z_1, z_2) \\ = \frac{1}{\lambda_{mn}^2} \left[\begin{aligned} &\sinh(2l\lambda_{mn}) - \sinh \lambda_{mn}(l - h_2 + z_2) \\ &\cdot \cosh \lambda_{mn}(h_1 + z_1 - l) + \sinh \lambda_{mn}(h_2 + z_2 - l) \\ &\cdot \cosh \lambda_{mn}(h_1 + z_1 + l) - 2\sinh(h_2\lambda_{mn}) \\ &\cdot \cosh \lambda_{mn}(l - z_2) \cosh \lambda_{mn}(l - h_1 + z_1) \end{aligned} \right]$$

iii. For $z_2 - h_2 \leq z_1 - h_1 < z_1 + h_1 \leq z_2 + h_2$

$$F_{1,x}^{(m)} = I_1 I_2 \frac{\partial M}{\partial x_1} \\ = -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} \lambda_m w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\ \times \cos \lambda_m(a - x_1) \sin \lambda_n(b - y_1) \sin \lambda_m(a - x_2) \\ \times \frac{\sin \lambda_n(b - y_2) v_3(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \quad (41a)$$

$$F_{1,y}^{(m)} = I_1 I_2 \frac{\partial M}{\partial y_1} \\ = -\kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} \lambda_n w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\ \times \sin \lambda_m(a - x_1) \cos \lambda_n(b - y_1) \sin \lambda_m(a - x_2) \\ \times \frac{\sin \lambda_n(b - y_2) v_3(\lambda_{mn}, z_1, z_2)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \quad (41b)$$

$$F_{1,z}^{(m)} = I_1 I_2 \frac{\partial M}{\partial z_1} \\ = \kappa_{12} I_1 I_2 \sum_{m,n=1}^{\infty} w(\lambda_{mn}, R_1, R_2) w(\lambda_{mn}, R_3, R_4) \\ \times \sin \lambda_m(a - x_1) \sin \lambda_n(b - y_1) \sin \lambda_m(a - x_2) \\ \times \frac{\sin \lambda_n(b - y_2) s_3(\lambda_{mn}, z_1, z_2)}{\sinh(2l\lambda_{mn})}, \quad (41c)$$

where $v_3(\lambda_{mn}, z_1, z_2)$ is given in (29c), and $s_3(\lambda_{mn}, z_1, z_2)$ in (41c) is given by

$$s_3(\lambda_{mn}, z_1, z_2) \\ = \frac{2\sinh(h_1\lambda_{mn})}{\lambda_{mn}^2} \left[\sinh \lambda_{mn}(l - z_1) \sinh \lambda_{mn}(l - h_2 + z_2) \right. \\ \left. + \sinh \lambda_{mn}(l + z_1) \sinh \lambda_{mn}(h_2 - l + z_2) \right]$$

As described above, the total force $\mathbf{F}$ experienced by a coil consists of the mutual and self-forces, which are induced by another coil and the magnetic cuboid, respectively. For example, when Coil 1 carrying the current I_1 is placed inside the cuboid alone, it will be attracted by the magnetic walls, that is, a self-magnetic force $\mathbf{F}_1^{(s)}$ will be induced. This force can be found by differentiating the self-magnetic energy with respect to the centre coordinates of Coil 1, *i.e.* (see (42a))

$$F_{1,y}^{(s)} = \frac{I_1^2}{2} \frac{\partial L}{\partial y_1} \\ = -\frac{\mu_0 \pi^2 I_1^2 N_1^2}{2ab h_1^2 (R_2 - R_1)^2} \sum_{m,n=1}^{\infty} \lambda_n w^2(\lambda_{mn}, R_1, R_2) \\ \times \frac{\sin^2 \lambda_m(a - x_1) \sin 2\lambda_n(b - y_1) v_4(\lambda_{mn}, z_1)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \quad (42b)$$

$$F_{1,z}^{(s)} = \frac{I_1^2}{2} \frac{\partial L}{\partial z_1} \\ = \frac{2\mu_0 \pi^2 I_1^2 N_1^2}{ab h_1^2 (R_2 - R_1)^2} \sum_{m,n=1}^{\infty} w^2(\lambda_{mn}, R_1, R_2) \\ \times \sin^2 \lambda_m(a - x_1) \sin^2 \lambda_n(b - y_1) \\ \times \frac{\sinh^2(\lambda_{mn}h_1) \sinh(2\lambda_{mn}z_1)}{\lambda_{mn}^2 \sinh(2l\lambda_{mn})}. \quad (42c)$$

Then the total force experienced by Coil 1 can be obtained as the sum of mutual and self-forces, that is

$$\mathbf{F}_1 = \mathbf{F}_1^{(m)} + \mathbf{F}_1^{(s)}. \quad (43)$$

The self-force $\mathbf{F}_2^{(s)}$ can be obtained by replacing $I_1, N_1, h_1, R_1, R_2, x_1, y_1, z_1$ in (42a)–(42c) with $I_2, N_2, h_2, R_3, R_4, x_2, y_2, z_2$, respectively.

$$F_{1,x}^{(s)} = \frac{I_1^2}{2} \frac{\partial L}{\partial x_1} = -\frac{\mu_0 \pi^2 I_1^2 N_1^2}{2ab h_1^2 (R_2 - R_1)^2} \sum_{m,n=1}^{\infty} \lambda_m w^2(\lambda_{mn}, R_1, R_2) \\ \times \frac{\sin 2\lambda_m(a - x_1) \sin^2 \lambda_n(b - y_1) v_4(\lambda_{mn}, z_1)}{\lambda_{mn} \sinh(2l\lambda_{mn})}, \quad (42a)$$

5 Numerical validations

5.1 Mutual inductance of shielded circular rings

The obtained results will be compared numerically with those given by FEM simulations. All of the following comparative calculations are performed with a personal computer of a 4-GHz 4-core CPU. For the mutual inductance of the shielded circular rings, the numerical results of (27) (using Wolfram Mathematica) are presented in Table 1, compared with those given by FEM software CST EM Studio. In the FEM simulations, two types of results are provided: (i) the results with an ideal magnetic cuboid, which are realised by imposing the boundary condition that the tangential component of $\mathbf{H}$ vanishes on all surfaces of the cuboid; (ii) the results with a more realistic cuboid of Mu-metal (a material dedicated to the magnetic shielding) [28] of thickness 5 mm and $\mu_r = 80,000$. The inner length, width, and height of the shielding cuboid are $2a = 2.8$, $2b = 3.6$, $2l = 2.4$ m, and the radii of the circular rings are $r_1 = 0.5$, $r_2 = 0.7$ m, respectively. The centres and vertical positions of both rings are variables in the comparisons, which are listed in the first six columns of Table 1. The values of the seventh column are correct to the last decimal place. In the FEM simulations, the infinitesimal cross section of the ring is modelled by a tiny square of 2 mm side length. The disparities between the results of (27) and those of FEM simulations with Mu-metal cuboid are $<0.8\%$, which is fairly small in the practical point of view.

We use the command 'ParallelSum' in Wolfram Mathematica to parallelise the evaluation of (27) (which will be computed in parallel by the four cores of CPU). The average elapsed time for one evaluation of (27) is only about 0.06 s, which is significantly shorter than that of FEM simulations (about 8 min). On average, in the FEM with ideal magnetic cuboid 65,000 4-node tetrahedral elements are required for one evaluation to achieve a result with the relative estimated error of ca. 0.037. Higher accuracy of FEM is hard to achieve within 30 min. The computation requirements for the FEM with Mu-metal cuboid are generally higher.

5.2 Forces and inductance of the shielded circular coils with rectangular cross section

For the mutual inductance of shielded circular coils with rectangular cross section, the numerical results of (29a)–(29c) will be compared again with those obtained by CST EM Studio. In the FEM simulations, two types of results are given again with the ideal and realistic cuboids of Mu-metal. Since Wolfram Mathematica appears inefficient for calculation of Struve function with large arguments, we replace $H_{0,1}(x)$ in (28) with the asymptotic expansion of Struve function for large x [29]

$$H_n(x) \simeq Y_n(x) + \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{\Gamma(k+1/2)}{\Gamma(n+1/2-k)(x/2)^{2k-n+1}}, \quad (44)$$

where $Y_n(x)$ is the Bessel function of the second kind of order n . Practically, we use (44) when $x > 15$, and use the command 'StruveH[n, x]' directly when $x \leq 15$. With this configuration, letting $m = 5$ in (44) is enough to give a result exact to the last decimal place in most cases, and all computations for the coils with rectangular cross section can be accelerated remarkably.

The inner length, width, and height of the shielding cuboid are $2a = 2.4$, $2b = 2.8$, $2l = 2.2$ m, and the parameters of the coils are $R_1 = 0.2$, $R_2 = 0.3$, $R_3 = 0.5$, $R_4 = 0.7$, $h_1 = 0.4$, $h_2 = 0.5$ m, $N_1 = 800$ and $N_2 = 1500$. The computations are implemented with different centre coordinates and vertical positions of both coils. The results are presented in Table 2, in which the disparities between the analytical methods and FEM with Mu-metal cuboid are $<1.5\%$. The average elapsed time of evaluation with (29a)–(29c) is about 2 s, which is also much faster than that of FEM simulation (ca. 5 min). About 1.2×10^5 4-node tetrahedral elements are required to obtain a result with the relative estimated error of ca. 6×10^{-5} in the FEM with an ideal magnetic cuboid.

For the numerical comparison of self-inductance of circular coils, the parameters of the cuboid are $2a = 2.4$, $2b = 2.6$, $2l = 1.8$ m. The self-inductance of two different coils is evaluated by (30) and CST EM Studio, respectively: $R_1 = 0.4$, $R_2 = 0.7$, $h_1 = 0.5$ m, $N_1 = 1200$ (Coil 1); $R_1 = 0.3$, $R_2 = 0.5$, $h_1 = 0.3$ m, $N_1 = 800$ (Coil 2). The results are presented in Table 3, in which the disparities between the analytical methods and FEM with Mu-metal cuboid are $<0.08\%$. In general, evaluations of (30) are slower than those of (29a)–(29c) (~ 15 s per evaluation to achieve a result correct to 7 decimal places). However, in practice, a value with more than 5 decimal places is seldom required, and consequently, the elapsed time of (30) can be reduced to <5 s. Hence, (30) is still fast and accurate enough for the practical applications. In FEM with an ideal magnetic cuboid, the average elapsed time is ca. 2.5 min and about 6.8×10^4 4-node tetrahedral elements are required to obtain a result with the relative estimated error of ca. 5.5×10^{-5} .

For the calculations of magnetic force experienced by shielded coils, the parameters of the cuboid are $2a = 2.7$, $2b = 2.5$, $2l = 2.8$ m, and those of the coils are $R_1 = 0.3$, $R_2 = 0.4$, $h_1 = 0.4$ m, and $N_1 = 1000$ (Coil 1); $R_3 = 0.5$, $R_4 = 0.6$, $h_2 = 0.6$ m, and $N_2 = 1400$ (Coil 2). To highlight the effect of geometric parameters on the final results, we set both currents $I_1 = I_2 = 1$ A in the force computations. The data of different centre coordinates and vertical positions of both coils are presented in Table 4. The results of x , y , z -components of total forces evaluated by (36a)–(43) (consisting of the self and mutual forces) are presented in Table 5, in which the force components of Coil 1 and Coil 2 are shown in the left and right half of the columns, respectively. The corresponding results obtained by CST EM Studio are shown separately in Table 6 (ideal magnetic cuboid) and Table 7 (Mu-metal cuboid of 5 mm thickness and $\mu_r = 80,000$) for comparison. Once again, the agreement between both methods is also quite satisfactory. Using the proposed method, the average elapsed time per evaluation is ca. 4 s

Table 1 Mutual inductance of circular rings calculated by proposed method compared with those by CST EM studio

x_1 , m	y_1 , m	x_2 , m	y_2 , m	ζ_1 , m	ζ_2 , m	M of (27), μH	M of CST EM Studio ¹ , μH	M of CST EM Studio ² , μH
0.2	0.6	-0.2	0.4	-0.1	0	0.620423149	0.6228356	0.6201419
0.3	0.7	-0.3	0.5	-0.3	0.2	0.194136043	0.1952462	0.1939384
0.4	0.8	-0.4	0.6	-0.5	0.4	0.080415074	0.08093547	0.0802417
0.5	0.9	-0.5	0.7	-0.7	0.6	0.040060664	0.04049235	0.0399218
0.6	1.0	-0.6	0.8	-0.9	0.8	0.021542040	0.02171746	0.0214290
0.7	1.1	-0.7	0.9	-1.1	1.0	0.012198136	0.01225323	0.0121106
0	-0.1	0.1	0	1.0	-1.1	0.051011290	0.05126982	0.0507826
0.1	-0.2	0.2	0.1	0.8	-0.9	0.060408568	0.06091200	0.0602028
0.2	-0.3	0.3	0.2	0.6	-0.7	0.077923063	0.07837036	0.0777289
0.3	-0.4	0.4	0.3	0.4	-0.5	0.098710926	0.09930903	0.0985254
0.4	-0.5	0.5	0.4	0.2	-0.3	0.088641448	0.08936629	0.0884586
0.5	-0.6	0.6	0.5	0	-0.1	-0.090960375	-0.09028645	-0.0910997

¹ Ideal magnetic cuboid.

² Mu-metal cuboid of 5 mm thickness and $\mu_r = 80,000$.

Table 2 Mutual inductance of shielded circular coils with rectangular cross section calculated by proposed method compared with those by CST EM studio

$x_1, \text{ m}$	$y_1, \text{ m}$	$x_2, \text{ m}$	$y_2, \text{ m}$	$z_1, \text{ m}$	$z_2, \text{ m}$	M of (29a)–(29c), mH	M of CST EM Studio ¹ , mH	M of CST EM Studio ² , mH
0.2	0.1	0.2	0.3	−0.5	0.4	67.8373815	67.83605	67.77952
0.3	0.3	−0.2	0.2	−0.6	0.35	42.9057236	42.90613	42.84882
0.4	0.4	−0.3	−0.3	−0.65	0.4	11.2904438	11.29011	11.24440
0.7	0.9	−0.4	−0.5	−0.7	0.6	1.7385515	1.738483	1.713719
0.1	0.3	0.2	0.35	−0.1	0.4	141.0263623	141.0255	140.9585
0.4	0.5	0.3	0.35	0.2	0.5	184.4761931	184.4753	184.3835
0.5	0.6	−0.2	−0.3	0.3	0.5	−6.2023781	−6.202204	−6.258865
0.4	0.6	−0.4	−0.6	0.2	0.3	−2.9399650	−2.939931	−2.980635
0.3	0.5	0.4	0.6	0.4	0.45	214.0339232	214.0327	213.9216
0.5	0.7	0.4	0.7	0.6	0.5	230.6797180	230.6786	230.5217
0.5	0.6	−0.4	−0.5	0.3	0.3	−3.2356475	−3.235596	−3.278134
0.6	0.8	−0.2	−0.3	0.5	0.6	−4.1076722	−4.107705	−4.160794

¹ Ideal magnetic cuboid.

² Mu-metal cuboid of 5 mm thickness and $\mu_r = 80,000$.

Table 3 Self-inductance of shielded circular coils with rectangular cross section calculated by proposed method compared with those by CST EM studio

	$x_1, \text{ m}$	$y_1, \text{ m}$	$z_1, \text{ m}$	L of (30), H	M of CST EM Studio ¹ , H	M of CST EM Studio ² , H
Coil 1	0.3	0.2	0.2	1.0398461	1.039825	1.039190
	0.5	0.3	0.3	1.0973400	1.097321	1.096544
	0.4	0.6	0.2	1.0551977	1.055179	1.054524
	0.2	0.4	0.35	1.1228045	1.122789	1.121966
	0.1	0.5	0.4	1.1762155	1.176203	1.175280
Coil 2	0.1	0.2	0.3	0.3531007	0.3530911	0.3529846
	0.2	0.3	0.4	0.3642378	0.3642290	0.3641024
	0.3	0.4	0.5	0.3865200	0.3865117	0.3863466
	0.4	0.6	0.55	0.4086214	0.4086139	0.4084119
	0.6	0.7	0.6	0.4452170	0.4452096	0.4449534

¹ ideal magnetic cuboid.

² Mu-metal of 5 mm thickness and $\mu_r = 80,000$.

Table 4 Parameters for force calculations of shielded circular coils with rectangular cross section

$x_1, \text{ m}$	$y_1, \text{ m}$	$x_2, \text{ m}$	$y_2, \text{ m}$	$z_1, \text{ m}$	$z_2, \text{ m}$
0	0	0.2	0.1	−0.6	0.6
0.3	0.4	0.4	0.5	−0.5	0.7
0.4	0.5	−0.3	−0.4	−0.7	0.5
0.6	0.7	−0.5	0.6	−0.9	0.3
0.5	0.6	0.45	0.55	−0.4	0.2
0.5	0.5	0.45	0.55	−0.1	0.4
0.7	0.6	0.65	0.55	0.2	0.7
0.75	0.4	0.7	0.45	0.1	0.5
0.3	0.6	0.35	0.55	0.8	0.7
0.65	0.55	0.7	0.6	0.9	0.75
0.5	0.7	−0.3	−0.5	0.4	0.7
0.4	0.2	−0.5	−0.3	0.6	0.5

(letting $m=7$ in (44)), while the elapsed time and relative estimated error for FEM are similar to those of mutual inductance of coils with rectangular cross section.

6 Conclusion

An analytic method is presented for the BVP of circular coils with parallel axes shielded by a cuboid of high permeability with the application of MSP. Considering that a ring carrying current is equivalent to a magnetic double layer with constant moment density, the MSP of a circular ring has been found by an ansatz of jump function. With the MSP, analytic solutions have been further obtained for the magnetic field, inductance, and forces between the circular rings and coils with rectangular cross section. Moreover, it

is found that the total force experienced by a shielded coil consists of the self and mutual forces, which originate from the self and mutual magnetic energy of the shielded coil, respectively. This feature of forces is different from those of coils placed in the free air, in which case the forces only depend on the mutual magnetic energy. Numerical validation shows good agreement between our method and FEM simulations, and the former is proved to be much faster than the latter in most cases.

7 Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant 51507115).

Table 5 Forces of shielded circular coils with rectangular cross section calculated by proposed analytic method

$F_{1,x}$, mN	$F_{1,y}$, mN	$F_{1,z}$, mN	$F_{2,x}$, mN	$F_{2,y}$, mN	$F_{2,z}$, mN
45.4219760	22.5828280	131.6971749	-41.2715710	-19.6938280	53.3527436
21.4806671	18.5471219	157.1731197	-12.8803249	7.4575470	190.9272291
-15.3648588	-17.5568965	-36.4361181	8.1048421	-4.5505443	128.0850695
-23.2018112	3.9743262	-123.7045906	2.7917440	89.7600076	37.8617696
5.5410113	16.5584194	429.6876075	19.8280526	75.4684398	-417.3160972
12.6363362	7.1014084	417.2749200	14.0333927	79.1781906	-338.5711806
21.1168303	21.1014339	453.1632333	45.0766079	44.9056825	-52.6241134
45.3065319	-3.9782873	392.9148601	86.2817696	43.3577624	-258.7447525
-11.9483549	41.5253550	79.2706822	27.7544900	53.3188540	582.6735034
9.0698642	9.0120695	157.3746683	80.4917495	80.3333411	851.5016859
8.5491367	29.8802238	-3.7290243	-14.2417949	-49.6497539	361.3481280
51.7322749	28.5786644	36.6931790	-75.9068965	-41.9093625	118.4849192

Table 6 Forces of shielded circular coils with rectangular cross section calculated by CST EM Studio (ideal magnetic cuboid), compared with results given in Table 5

$F_{1,x}$, mN	$F_{1,y}$, mN	$F_{1,z}$, mN	$F_{2,x}$, mN	$F_{2,y}$, mN	$F_{2,z}$, mN
45.421	22.584	131.72	-41.285	-19.697	53.336
21.470	18.541	157.20	-12.856	7.4407	190.94
-15.356	-17.569	-36.436	8.1360	-4.5351	128.10
-23.231	3.9322	-123.74	2.7403	89.840	37.841
5.5877	16.583	429.69	19.835	75.402	-417.32
12.692	7.0391	417.28	14.032	79.200	-338.58
21.151	21.135	453.17	44.980	44.827	-52.618
45.363	-4.0482	392.92	86.194	43.398	-258.75
-11.989	41.716	79.429	28.025	53.680	583.40
8.9803	8.9171	157.38	80.518	80.370	851.51
8.5581	29.861	-3.7230	-14.209	-49.627	361.39
51.771	28.598	36.692	-75.968	-41.920	118.49

Table 7 Forces of shielded circular coils with rectangular cross section calculated by CST EM Studio (Mu-metal cuboid of 5 mm thickness and $\mu_r = 80,000$), compared with results given in Table 5

$F_{1,x}$, mN	$F_{1,y}$, mN	$F_{1,z}$, mN	$F_{2,x}$, mN	$F_{2,y}$, mN	$F_{2,z}$, mN
45.424	22.583	131.75	-41.291	-19.703	52.805
21.473	18.554	157.18	-12.864	7.3917	190.14
-15.349	-17.556	-36.383	8.1463	-4.5002	127.75
-23.209	3.9654	-123.55	2.8039	89.563	37.780
5.5785	16.547	429.67	19.790	75.251	-417.33
12.678	7.0066	417.22	13.991	79.058	-338.77
21.119	21.110	453.02	44.891	44.743	-53.417
45.295	-4.0569	392.82	86.011	43.332	-259.05
-12.047	41.554	78.884	27.809	53.084	581.59
8.9308	8.8749	156.82	80.456	80.301	850.09
8.5750	29.869	-3.7739	-14.221	-49.599	360.56
51.813	28.620	36.613	-75.965	-41.928	118.07

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