

THE DEVELOPMENT OF CYCLONES

by

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ABSTRACT

The growth to maturity of a baroclinic disturbance on an initial zonal sheared flow is examined by integrating the quasi-geostrophic equations using semi-analytic energy conserving methods. Agreement with many observed features of large scale flow in the atmosphere is obtained using solutions for which the energy of the motion is artificially restricted to only a small number of eddy scales, while points of disagreement may be attributed principally to physical defects of the quasi-geostrophic approximations.

CONTENTS.

INTRODUCTION	6
CHAPTER 1. THE EQUATIONS OF MOTION AND BOUNDARY CONDITIONS.	
1.1 The Large Scale Equations.	10
1.2 Non-dimensional Coordinates.	13
1.3 Expansion in Terms of the Richardson Number.	16
1.4 Boundary Conditions.	19
1.5 Conservative Properties of the Quasi-geostrophic Equations.	24
CHAPTER 2. THE PERTURBATION PROBLEM.	
2.1 The Perturbation Equations.	28
2.2 The Initial State.	29
2.3 The Linear Solution.	31
2.4 Selection of the Dominant Wavelength.	33
2.5 The Structure of the Growing Disturbance.	36
2.6 The Adjustment of the Mean Zonal Flow (The Second Order Solution).	38
2.7 The Modified Growing Waves (The Third Order Solution).	43
2.8 The Stable and Neutral Eady Solutions.	47
CHAPTER 3. THE METHOD OF FUNCTIONAL REPRESENTATION.	
3.1 Introduction.	60
3.2 The Integrated Vorticity Equation.	61
3.3 Boundary Conditions for the Functional Representation Solutions.	63
3.4 A Three Parameter Solution.	65
3.5 An Analytic Solution without Energy Conservation	67
3.6 The Energy Conserving Solution E1.	71

3.7 The Energy Conserving Solution E2.	72
3.8 Comparison of the Different Three Parameter Solutions.	73
3.9 The Development of the Motion Patterns for the Three Parameter Solution.	76
3.10 Truncation Errors.	77
3.11 Improving the Accuracy of the Solutions.	81
CHAPTER 4. HIGHER WAVENUMBER COMPONENTS.	
4.1 The Eight Parameter Solution.	90
4.2 Comparison with the Atmosphere and Numerical Models of the Atmosphere.	93
4.3 Relative Flow on Horizontal and Isentropic Surfaces.	100
4.4 The Efficiency of Finite Amplitude Baroclinic Disturbances.	104
4.5 Physical Parameters Affecting the Intensity and Structure of the Disturbance.	108
CHAPTER 5. SURFACE FRICTION.	
5.1 The Ekman Boundary Layer.	134
5.2 Conservation Equations.	137
5.3 The Linear Perturbation Solution with Surface Friction.	138
5.4 The Finite Amplitude (Six Parameter) Solution with Surface Friction.	144
5.5 The Sixteen Parameter Solution.	152
CHAPTER 6. ADDITIONAL PHYSICAL EFFECTS.	
6.1 Introduction.	164
6.2 The Validity of the Quasi-geostrophic Approximations.	16
6.3 The Linear Perturbation Solution for Small Richardson Number.	166
6.4 Higher Order Richardson Number Equations.	170

6.5 Corrections to the Quasi-geostrophic Linear Perturbation Solution when the Richardson Number is not Large.	175
6.6 Methods of Modelling other Aspects of the Real Atmosphere.	178
APPENDIX I. GLOSSARY OF SYMBOLS USED.	185
APPENDIX II.	
(a) The Runge-Kutta Method.	191
(b) Newton's Method.	193
APPENDIX III. REFERENCES.	194
ACKNOWLEDGEMENTS.	197
LIST OF FIGURES.	
Figs 2.1 - 2.13	53-59
Figs 3.1 - 3.10	83-89
Figs 4.1 - 4.12	113-133
Figs 5.1 - 5.14	155-163
Figs 6.1 - 6.3	184

INTRODUCTION.

The purpose of the research described in this thesis is to investigate and explain, from a theoretical point of view, changes in growth and structure occurring during the lives of mid-latitude cyclone-anticyclone waves.

Previous investigators (Phillips (1956), Smagorinsky (1963), and several others) have integrated the equations of motion and thermodynamics as applied to the atmosphere using finite difference methods on a computer. The emphasis of these computer studies has been on simulating the motion of the atmosphere by solving equations representing 'model atmospheres' which resemble, as closely as possible the 'real atmosphere' (the ultimate limit on the complexity of the physical models examined being set by the speed and capacity of the computer used). Considerable success has been obtained, and subsequently 'experiments' have been conducted with the model atmospheres by altering various physical parameters present and analysing the resulting changes in the flow patterns. Such experiments are leading to a greater understanding of the physical processes at work in the atmosphere, and also to a recognition of absolute limits on the predictability of weather producing systems.

The approach of the research discussed below is somewhat different. We examine physical systems which, while broadly resembling the real atmosphere, possess the minimum of complexity. The physics of such simple systems can be more readily comprehended, and subsequently an understanding of the roles of the various atmospheric phenomena can be built up, step by step, by introducing additional physical effects.

Patterns of motion in the atmosphere are observed to be in a perpetual state of formation, evolution and dissipation. However the situation is not one of complete randomness, and it is possible to discern patterns having characteristic structures whose evolutions are, to a large extent, governed by particular physical processes. The existence of any recognisable structure is due to a subtle interplay between physical constraints on the distribution and movement of atmospheric properties, leading to less favoured conditions for patterns having slightly different structures. For example a particular pattern may be selected because it grows fastest, or alternatively because it has the most persistent structure.

It is convenient to regard atmospheric disturbances as being arranged in order of geometrical size. This thesis is concerned with large scale motion characteristically occupying the depth of the troposphere (≈ 10 km.), having a horizontal length scale of about 1000 km. and a time scale of the order of a day. The derivation (based on observed properties of large scale motion) of an approximate, but consistent, set of equations suitable for predicting the evolution of such disturbances (the quasi-geostrophic equations) is discussed in Chapter 1.

Unlike much previous work in the subject of large scale atmospheric dynamics we do not use finite difference methods to solve the equations. Instead we have developed two alternative methods which involve the expansion of the solution in a series of analytic functions of the space coordinates x , y , and z whose coefficients depend on time t . These methods have an advantage over finite differences in that they more readily permit an evaluation of the truncation error, an assessment of its physical effects, and modification of the solution to reduce its importance.

Before solving the large scale equations of motion it is necessary to define the state of the model atmosphere at an initial time. Actual patterns of motion observed in the atmosphere would seem to be the most appropriate initial state to use. However the analysis of the distribution of physical quantities in the atmosphere is based on a somewhat sparse set of measurements reinforced by subjective knowledge of the structure of atmospheric disturbances. Inevitably such an analysis filters out certain types of motion, possibly leading to inconsistencies whose physical consequences cannot be assessed without an extensive understanding of the mechanism of atmospheric processes (indeed it is the object of this research project to seek such an understanding). To avoid introducing any physical inconsistencies we adopt, instead, an initial state having a definite and well defined, but rather idealised, form. Patterns of motion occurring when this initial state is slightly disturbed are examined by means of the method of small perturbations. The resulting disturbance is resolved into a set of independent components, the one amplifying most rapidly being assumed to dominate the structure of the complete evolving disturbance. Subsequent modifications in the structure, due to non-linear interactions, are examined by calculating higher order perturbation solutions. This procedure is described in Chapter 2. The perturbation solution represents well the evolution of the disturbance during the early part of its life, but truncation errors, which dominate for large values of t , eventually lead to unrealistic results.

Chapters 3 and 4 contain a description of a second method of generating solutions - the method of functional representation. As mentioned above, the coarse network of meteorological observing stations imposes limitations on the accuracy of the analysis of the present weather.

One of the problems to be solved is whether lack of detail in the initial state results in fundamental errors in the numerical forecast. It is evident that if no significant errors arise when certain types of motion are neglected, it ought to be possible to devise numerical procedures which are more efficient by being selective about the features they predict. It is with these ideas in mind that the method of functional representation has been developed. Solutions are found for which the energy is distributed among a restricted number of eddy scales. The simplest model of a cyclone-anticyclone wave is one having only a single length scale, and this is discussed in Chapter 3. It has only three degrees of freedom but gives lifelike pressure patterns for all values of t , developing jets of realistic speed and a mean meridional circulation similar to that found in other numerical experiments and in the atmosphere. However examination of the truncation error reveals that higher wavenumber components are important when the disturbance approaches maximum intensity. It is found (Chapter 4) that increasing the number of degrees of freedom to eight allows the representation of additional features of atmospheric flow, such as frontal zones, and also a reduction of the truncation error.

The effect of introducing surface friction into the calculation is described in Chapter 5. Friction has little effect on the structure and growth rate of the disturbance during the early part of its life, but the continual dissipation of kinetic energy leads eventually to a final state having little kinetic energy or available potential energy. The computations suggest that baroclinic disturbances have a life-time of about a week. Methods of introducing additional physical effects into the model are suggested in Chapter 6.

CHAPTER 1. THE EQUATIONS OF MOTION AND BOUNDARY CONDITIONS.

1.1 The Large Scale Equations.

The derivation of equations of motion appropriate for large scale atmospheric flow has been discussed extensively by previous authors (see for example Charney (1948), Eady (1949)) and so it will not be considered in detail here. For simplicity the atmosphere will be assumed dry, frictionless and adiabatic. In addition we neglect (as is usually done) some effects of the earth's curvature, for example the horizontal component of the earth's rotation. We use the symbol * to denote dimensional variables (for the majority of the work we use a dimensionless system), and write the equations relative to the rotating earth in terms of the independent space and time coordinates (x^*, y^*, z^*, t^*) , where the x^* -axis is oriented in the eastward direction, y^* in the northward direction and z^* vertically upwards. $\underline{k}$ is the unit vector in the vertical direction.

The simplifications introduced are based on three observed features of large scale motion, (i) the gross variations of pressure and density can be described by standard distributions which are functions only of z^* , (ii) the winds are approximately horizontal and geostrophic, (iii) changes in the volume of a parcel of air are principally due to its displacement to regions where the density is different (in particular vertical displacements). The more rigorous discussions given by the authors cited above show the consistency of these approximations with typical values of physical variables measured on a large scale in the atmosphere.

$p_s^*(z^*)$ and $\rho_s^*(z^*)$ define the pressure and density of a

'standard static atmosphere', and from the equations of motion we deduce that they are related by the hydrostatic equation

$$\frac{dp_s^*}{dz^*} = -\rho_s^* g$$

The thermodynamics of the atmosphere are most readily expressible in terms of the variable Φ^* (= entropy/ c_p) which, since c_p is a constant, is just a multiple of the entropy. However for convenience of nomenclature we will refer to Φ^* as the entropy of the air. Thus

$$\Phi_s^* = \frac{c_v}{c_p} \log p_s^* - \log \rho_s^*$$

defines the entropy of the standard static atmosphere, and B (= $d\Phi_s^*/dz^*$) is the static stability.

Departures from the standard atmospheric values (represented by letters without subscripts, i.e. $p^*(x^*, y^*, z^*, t^*)$ and $\rho^*(x^*, y^*, z^*, t^*)$) are assumed to be small in comparison, and so correct to first order in the deviation quantities the entropy changes are given by

$$\Phi^*(x^*, y^*, z^*, t^*) = \frac{c_v}{c_p} \frac{p^*}{p_s^*} - \frac{\rho^*}{\rho_s^*}$$

and the geostrophic wind is

$$\underline{v}^* = \frac{1}{f_0} \underline{k} \times \nabla^* \psi^*$$

where

$$\psi^* = p^* / \rho_s^*$$

and f_0 is a constant. (We use the β -plane approximation, i.e. we write $f = f_0 + \beta_0 y^*$, where f_0 and β_0 are constants, and assume $\beta_0 y^* \ll f_0$.)

The flow pattern given by the geostrophic wind equation (together with $w^* = 0$) describes a balanced

non-divergent system where the air moves steadily along the isobars, and we may regard this as a basic 'dynamic state' for the atmospheric motion from which departures are small. Correct to first order deviations from both the static and dynamic states we find that the vertical equation of motion becomes

$$\frac{1}{\rho^*} \frac{\partial p^*}{\partial z^*} + g = 0$$

which may more conveniently be rewritten

$$\frac{\partial \psi^*}{\partial z^*} - B \psi^* - \phi^* g \approx \frac{\partial \psi^*}{\partial z^*} - \phi^* g = 0$$

Similarly from the horizontal equations of motion and the adiabatic equation we find the horizontal ageostrophic wind and the vertical velocity to the same order of approximation

$$\underline{v}_a^* = \frac{1}{f_0} \underline{k} \times \left[\frac{D}{Dt} \right]^* \underline{v}^* - \frac{\beta_0 y^*}{f_0} \underline{v}^* \quad (1)$$

$$w^* = -\frac{1}{B} \left[\frac{D}{Dt} \right]^* \phi^* \quad (2)$$

where

$$\left[\frac{D}{Dt} \right]^* = \frac{\partial}{\partial t^*} + u^* \frac{\partial}{\partial x^*} + v^* \frac{\partial}{\partial y^*}$$

Finally condition (iii) above implies that the continuity equation takes the form

$$\frac{\partial u_a^*}{\partial x^*} + \frac{\partial v_a^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} + w^* \frac{1}{\rho_s^*} \frac{d\rho_s^*}{dz^*} = 0 \quad (3)$$

Substitution for $\underline{v}_a^*$ from (1) into the continuity equation (3) provides the vorticity equation which has the form

$$\left[\frac{D}{Dt} \right]^* \left(\frac{\partial v^*}{\partial x^*} - \frac{\partial u^*}{\partial y^*} \right) + \beta_0 v^* - f_0 \frac{\partial w^*}{\partial z^*} - f_0 w^* \frac{1}{\rho_s^*} \frac{d\rho_s^*}{dz^*} = 0$$

After substituting for w^* from (2) and assuming a constant static stability B this becomes

$$\left[\frac{D}{Dt} \right]^* \left(\frac{\partial^2 \psi^*}{\partial x^{*2}} + \frac{\partial^2 \psi^*}{\partial y^{*2}} + \frac{\partial^2 \psi^*}{\partial z^{*2}} \frac{f_0^2}{gB} + \frac{\partial \psi^*}{\partial z^*} \frac{f_0^2}{gB} \frac{1}{\rho_s^*} \frac{d\rho_s^*}{dz^*} + f_0 \beta_0 y^* \right) = 0 \quad (4)$$

1.2 Non-dimensional Coordinates.

It is convenient to introduce a system of dimensionless coordinates

$$x = x^* \frac{f_0}{(gB)^{1/2} H}, \quad y = y^* \frac{f_0}{(gB)^{1/2} H},$$

$$z = z^* \frac{1}{H}, \quad t = t^* \frac{f_0}{h},$$

and also the non-dimensional quantities

$$\psi = \psi^* \frac{h}{gB H^2}, \quad \phi = \phi^* \frac{h}{BH},$$

$$K = -H \frac{1}{\rho_s^*} \frac{d\rho_s^*}{dz^*}, \quad \gamma = \beta_0 \frac{(gB)^{1/2} H h}{f_0^2},$$

H is a suitable scale height for the atmosphere and $h^2 = Ri = gB / (dU^*/dz^*)^2$ is the Richardson number, where dU^*/dz^* is a typical value of the vertical shear.

Taking the following to be representative of mid-latitude tropospheric large scale motion :- $g \sim 10^3 \text{ cm. sec.}^{-2}$, $f_0 \sim 10^{-4} \text{ sec.}^{-1}$, $\beta_0 \sim 10^{-13} \text{ cm.}^{-1} \text{ sec.}^{-1}$, $\rho_s^* \sim 10^{-3} \text{ gm. cm.}^{-3}$, $\frac{1}{\rho_s^*} \frac{d\rho_s^*}{dz^*} \sim 10^{-6} \text{ cm.}^{-1}$, $B \sim 10^{-7} \text{ cm.}^{-1}$, $Ri \sim 10^2$:- and choosing $H = 10 \text{ km.}$ we find that $K \sim 1$ and $\gamma \sim 1$.

Furthermore we can evaluate typical values for the horizontal length scale, the time scale and the pressure and temperature deviations (from the standard static values) for the large scale motion. These are

$$L^* = \frac{(gB)^{1/2} H}{f_0} \approx 1000 \text{ km.}, \quad \tau^* = \frac{h}{f_0} \approx 1 \text{ day}$$

$$\tau^* = \frac{\rho_s^* g B H^2}{h} \approx 10 \text{ mb.}, \quad \Theta^* = \frac{B H}{h} \Theta_s^* \approx 3^\circ \text{ K}$$

where Θ_s^* ($\sim 300^\circ \text{ K}$) is an average value for the potential temperature. It is clear that these are dependent on the physical parameters defining the basic state of the atmosphere, in particular on the Richardson number. The value $Ri = 100$ adopted above is good as a global average, but if we restrict our investigation to mid-latitudes the Richardson number can be somewhat smaller ($Ri = 10$, corresponding to a mean shear of about $30 \text{ m. sec.}^{-1} / 10 \text{ km.}$, or a temperature decrease of 30° C between latitudes 30° and 60° , is possible in the winter). We will adopt $Ri = 25$ in the subsequent work, since this is a typical mean value in the (dry) mid-latitude troposphere, and is therefore more representative for our purposes than $Ri = 100$. In this case $\tau^* \approx \frac{1}{2}$ day, $\tau^* \approx 20 \text{ mb.}$ and $\Theta^* \approx 6^\circ \text{ K}$, while γ is reduced to $\frac{1}{2}$. The length scale L^* is unaltered.

The vorticity equation (4) takes the non-dimensional form

$$\frac{D}{Dt} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - K \frac{\partial \psi}{\partial z} + \gamma y \right) = 0 \quad (5)$$

Eady (1949) and subsequently Green (1960) have examined the influence of the β and the mass divergence effects, represented non-dimensionally by γ and K respectively, on the growth of small amplitude disturbances superimposed on an initial zonal sheared flow (independent of latitude). They found that these terms are of definite importance when the depth scale of the motion is large, which applies particularly to very long waves ($\sim 10^4 \text{ km.}$). However the fastest growing waves occur in an intermediate band of moderately long wavelengths ($\sim 10^3 \text{ km.}$), and it was found that neither their structures

nor their growth rates are greatly influenced by either of these effects. This provides some justification for neglecting both χ and K altogether, an approximation which allows far greater symmetry (and ease of solution) of the equations. Consequently in the subsequent work we set $\chi = K = 0$, and regard f as a constant.

ψ is the stream function for the geostrophic wind defined by

$$\begin{aligned} u &= - \frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \psi}{\partial x} \end{aligned} \quad (6)$$

and all other dependent variables are expressed in terms of it. Thus we have the following relationships:-

$$\begin{aligned} u_a &= - \frac{1}{h} \frac{D}{Dt} \frac{\partial \psi}{\partial x} \\ v_a &= - \frac{1}{h} \frac{D}{Dt} \frac{\partial \psi}{\partial y} \\ w &= - \frac{1}{h} \frac{D}{Dt} \frac{\partial \psi}{\partial z} \end{aligned} \quad (7)$$

The non-dimensional horizontal momentum equation

$$\frac{1}{h} \frac{D}{Dt} \underline{v} + \underline{k} \times (\underline{v} + \underline{v}_a) + g_{rad} \psi = 0 \quad (8)$$

the adiabatic equation

$$\frac{1}{h} \left(\frac{D}{Dt} \varphi + w h \right) = 0 \quad (9)$$

the continuity equation

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (10)$$

and the hydrostatic equation

$$\frac{\partial \psi}{\partial z} - \varphi = 0 \quad (11)$$

provide a self consistent set known as the quasi-geostrophic equations (in our case for an atmosphere which is effectively incompressible).

A relation between the geostrophic wind and the entropy distribution is obtained by combining equations (6) and (11) to give the thermal wind equation

$$\frac{\partial \underline{v}}{\partial z} = \underline{k} \times \nabla \varphi \quad (12)$$

In all the above equations D/Dt represents the non-dimensional rate of change with time following a particle along a horizontal geostrophic trajectory, i.e.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}$$

1.3 Expansion in Terms of the Richardson Number.

The geostrophic (6) and ageostrophic (7) winds appear to be the first two terms of an expansion in a series of the form $\underline{q} = \sum_{n=0}^{\infty} \underline{q}^{(n)}$, where $\underline{q}$ is the three dimensional velocity vector and $\underline{q}^{(n)} \sim O(h^{-n})$ (i.e. $\underline{q}^{(0)} = (u, v, 0)$ and $\underline{q}^{(1)} = (u_a, v_a, w)$). This suggests that an alternative way of obtaining the quasi-geostrophic equations is to expand the physical variables in a series of this form and equate terms having the same order of magnitude (in powers of h^{-1}), whence we are led to the conclusion that they are correct to $O(Ri^{-1/2})$ (see also Eady (1949)) and evidently require a large value of the Richardson number for validity. In fact it is not possible to give a general derivation for the large scale equations in terms of this series alone, and it is necessary to expand in terms of more than one parameter (see for example Charney and Stern (1960) who consider several non-dimensional parameters). Nevertheless it

is still true to say that the momentum equation (8) contains no terms due to the rate of change with time of the ageostrophic wind because they are $O(Ri^{-1})$. In addition only horizontal geostrophic advection is included for the same reason.

In the adiabatic equation (9) there is a term representing the vertical advection of entropy, as well as the horizontal (geostrophic) advective terms. This arises because the product of the small vertical velocity ($\sim O(Ri^{-1/2})$) and the large vertical variation of entropy in the static atmosphere (formally of order $Ri^{+1/2}$) gives the non-dimensional advective term $\frac{1}{h} (w h)$ which is $O(Ri^{-1/2})$. However the deviation term $\frac{1}{h} w \frac{\partial \phi}{\partial z}$ is $O(Ri^{-1})$ and is excluded. Thus a necessary condition for the validity of the approximations is that the static stability should remain almost unchanged by any motion, i.e. $|\frac{\partial \phi}{\partial z}| \ll h$, or in dimensional coordinates $|\frac{\partial \phi^*}{\partial z^*}| \ll B$.

The non-dimensional vorticity equation is

$$\frac{1}{h} \frac{D \zeta}{Dt} - \frac{\partial w}{\partial z} = 0 \quad (13)$$

and so local changes of relative vorticity $\zeta (= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$ are, to this level of approximation, due only to geostrophic advection and stretching of the earth's vorticity. The stretching of relative vorticity is $O(Ri^{-1})$ and is neglected (the same applies to the twisting terms). Evidently a necessary condition for the validity of (13) is (in dimensional coordinates) $|\zeta^*| \ll f$.

The vorticity equation (13) can be written in the alternative form

$$\frac{D}{Dt} \nabla^2 \psi = 0 \quad (14)$$

and is the equivalent of Ertel's (1942) equation which

states that the potential vorticity Z_p^* (equal to the scalar product of the entropy gradient and the absolute vorticity vector divided by the density) is conserved along the trajectories in adiabatic frictionless flow. In non-dimensional coordinates the potential vorticity has the form

$$Z_p = Z_p^* \frac{\rho_s^*}{\rho_B} = 1 + R_i^{-1/2} \nabla^2 \psi + O(R_i^{-1}) \quad (14a)$$

and the equation (14) follows.

The quasi-geostrophic equations are generally regarded as being capable of describing much of the dynamics of large scale flow. Charney (1948) showed that the adoption of the hydrostatic and geostrophic approximations serves to exclude both sound and gravity waves. These are unlikely to have a very great effect on the large scale flow, and their exclusion is a positive advantage since their relatively high phase speeds necessitate calculation to an inordinate degree of accuracy. Nevertheless, it is becoming apparent that the inclusion of higher order terms is necessary for a correct prediction of the large scale flow, at least in parts of the flow field or at certain stages in the life history of the disturbance. Thus, as we suggest later in Chapter 6, it is frequently necessary to consider the effects of ageostrophic advection and changes of static stability in intense cyclones and anticyclones. Furthermore it may be shown that the inclusion of 'cross isobar advection' is necessary in order to predict correctly the differing structures of the cyclones and anticyclones (the latter usually being spread out over a larger area), and also the formation of sharp temperature gradients (fronts) in cyclones. However the inclusion of these terms considerably complicates the analysis, and the quasi-geostrophic solutions provide a

first estimate for the flow which can be improved later (see Chapter 6).

1.4 Boundary Conditions.

It would be convenient if the equations of motion could be solved throughout the whole atmosphere as then the only physical boundary present would be at the surface of the earth, however approximations that we have introduced restrict their validity to high latitudes. It is therefore necessary to 'invent' boundaries for the region of integration together with conditions to satisfy on them. Although these boundaries are necessarily artificial they are chosen to represent restrictions imposed on actual large scale motion in the atmosphere. It is necessary, however, to ensure that the boundary conditions are consistent in the light of the quasi-geostrophic approximations.

The most realistic condition to impose in the east-west direction is one of continuity around the earth, and this is ensured by restricting the solutions to those which are periodic with (or independent of) x .

For simplicity we ignore the topography and assume the earth's surface to be a rigid horizontal plane at which the vertical velocity vanishes. We next consider the conditions at upper levels. The influence of the stratosphere on growing tropospheric disturbances has been investigated by Eady (1949) and Green (1960) by considering a two layer atmosphere with the upper layer having a much larger static stability than the lower layer. For a linear (infinitesimal amplitude) solution it is possible to satisfy continuity of pressure

and vertical velocity at the interface (tropopause), with the solution decreasing exponentially with z above. However the discontinuities which appear in the entropy (indicating changes in the height of the tropopause (see Green)) result in computational difficulties when the growth to finite amplitude is examined. (A continuous distribution of static stability further complicates the analysis.) However Eady and Green found that the presence of the stratosphere affects principally the very long waves, while the structures of waves with wavelengths of order 10^3 km. (i.e. mid-latitude cyclone scale) did not differ greatly from those computed assuming a rigid horizontal tropopause (an infinitely stable stratosphere). Eady also considered more complicated systems having several layers with differing stabilities in order to simulate the effects of large cloud masses. Charney (1947) chose to examine baroclinic instability in an infinitely deep atmosphere. In his model ρ_s^* decreased exponentially with height, and he considered solutions having finite perturbation wind speeds at $z = \infty$ so that the kinetic energy of the growing wave $\frac{1}{2} \rho_s^* \bar{v}^2$ was vanishingly small at great heights. However the variation of f with latitude was found to be an essential parameter for the existence of Charney's solution. Furthermore Green showed that the mechanics of disturbances in a finite, but deep, atmosphere also depend strongly on the β effect. Evidently a model atmosphere extending to great heights (i.e. very much more than the density scale height) is not suited to our set of equations (with f constant). We therefore adopt the simplest case of a rigid horizontal upper lid (at a height H (≈ 10 km.) above the earth's surface) at which the vertical velocity vanishes.

It is more difficult to decide on suitable physically

realistic lateral boundaries for the region. Eady considered a system with three regimes side by side, the middle one (with a width of about 10^2 km.) having a low static stability B and representing a central mass of cloud, and the outer regimes (with a larger value of B) extending indefinitely. The solution decayed exponentially away from the central 'cloud mass' into the 'dry air'. An alternative model, with the three regimes having the same static stabilities but with the middle one having a larger meridional temperature gradient, is suggested by an examination of mean meridional cross sections of the atmosphere (see for example Heastie and Stephenson (1960)) which show a mid-latitude baroclinic zone of large shear (having a width of about 3000 km.) sandwiched between two relatively barotropic zones. However such a model is complicated (even if the discontinuous distribution of shear is replaced by a continuous one) because the mechanism of potential to kinetic energy conversion becomes obscured by the transfer of kinetic energy between the eddies and the zonal flow. Here we examine a simpler model obtained by ignoring the motion in the outer regimes altogether. Thus we consider the baroclinic zone to be bounded at its northern and southern extremities by rigid vertical walls. At these side walls we impose the condition that the normal geostrophic wind vanishes, and this has the effect of preventing any advection of potential vorticity (or entropy) through the sides.

It is necessary to find out if the above boundary conditions are sufficient to determine the motion uniquely, subject to the constraints of the quasi-geostrophic equations. It is evident that if we were solving the barotropic vorticity equation (i.e. equation (13) with $\partial w / \partial z \equiv 0$), then these boundary conditions would not determine the stream function uniquely from a given

vorticity distribution ζ ($= \partial^2 \psi / \partial x^2 + \partial^2 \psi / \partial y^2$), since there is an arbitrary additive (time dependent) term independent of x but varying linearly with y . This term, corresponding to an additional zonal wind independent of latitude, must be determined by imposing an extra condition to ensure that the mean momentum is constant (in the absence of friction): a property deduced from the zonally averaged momentum equation.

The baroclinic problem is more complicated because from a given distribution of potential vorticity Z_p ($= \zeta + h^{-1}(\partial^2 \psi / \partial x^2 + \partial^2 \psi / \partial y^2 + \partial^2 \psi / \partial \tilde{z}^2)$) the stream function ψ cannot be obtained uniquely since there is an arbitrary harmonic function of y and z (i.e. a function which satisfies $\partial^2 \psi / \partial y^2 + \partial^2 \psi / \partial \tilde{z}^2 = 0$). This, like the barotropic case, is not trivial since the stability characteristics of the flow are strongly dependent on the nature of the horizontal and vertical shears. We conclude that we do not have sufficient conditions on the boundaries to solve the problem, and a further constraint is required.

The extra boundary condition used here was adopted by Phillips (1956). Since the side walls are rigid we impose the additional condition that the total zonally averaged meridional wind vanishes there (i.e. $\overline{v} + \overline{v_a}^x = 0$), as well as our previous condition of zero normal geostrophic wind at the walls (i.e. $v = 0$), and we find on integrating the momentum equation (8) with respect to x and applying the result at the sides that

$$\frac{\partial \overline{u}^x}{\partial t} = 0$$

there. This is sufficient to resolve the non-uniqueness apart from an arbitrary constant (which is of no dynamical significance). This constraint is similar to, but less restrictive than, the extra condition necessary in the barotropic case (i.e. $\partial \overline{u}^{xy} / \partial t = 0$) since the latter is

equivalent, via the momentum equation, to the condition of zero mean meridional flow (i.e. $\overline{v + v_\alpha^x} = 0$) at all values of y .

Because there is no advection through the boundaries the evolution in time of the pattern of potential vorticity (and entropy) is uniquely determined from an initial distribution, and can be found by integrating along the trajectories.

To summarise the above discussion, we solve the quasi-geostrophic vorticity equation in a 'channel' which is infinite (with the stream function pattern repeated indefinitely) in the x direction but bounded at the sides, and above and below, by rigid plane surfaces. The determination of ψ then requires the solution of

$$\frac{D}{Dt} \nabla^2 \psi = 0 \quad (15)$$

subject to the boundary conditions :- ψ periodic with x ,

$$\frac{\partial \psi}{\partial x} = 0 \quad , \quad \frac{\partial^2 \overline{\psi}}{\partial t \partial y} = 0$$

at the sides, and

(16)

$$\frac{D}{Dt} \frac{\partial \psi}{\partial z} = 0$$

on the upper and lower lids. It is shown in the following section that the adoption of these boundary conditions is also sufficient to ensure that the total energy (and several other physical quantities) is conserved.

Neither the vorticity equation (15) nor the boundary conditions (16) involves the parameter h , and so the Richardson number does not have any 'controlling influence' on the solutions of this quasi-geostrophic system, but enters only as a 'scaling parameter' via the non-dimensional coordinate system (section 1.2). (This is not true for the β plane vorticity equation (5) since the

parameter γ is proportional to h .)

1.5 Conservative Properties of the Quasi-geostrophic Equations.

It is easily shown that in a hydrostatic atmosphere there is a constant ratio between the potential energy P^* and the internal energy I^* per unit volume, where

$$P^* = \frac{(\rho_s^* + \rho^*) g z^*}{x^*, y^*, z^*}$$

$$I^* = \frac{c_v (\rho_s^* + \rho^*) (T_s^* + T^*)}{x^*, y^*, z^*}$$

and ρ^* and T^* denote the departure of the density and temperature from the standard values ρ_s^* and T_s^* . Lorentz (1955) showed that only a small part of the 'total potential energy' $P^* + I^*$ can be converted into kinetic energy by an adiabatic rearrangement of mass, and he called this the available potential energy. More specifically, he defined the available potential energy as the difference between the total potential energy of the atmosphere and the total potential energy which would exist if the mass were redistributed (ignoring the influence of boundaries) under conservation of entropy to give a horizontal stable stratification. We now derive an equation, consistent with our previous approximations, which relates the rates of change of the kinetic and available potential energies per unit volume.

The energy equation can be derived from the quasi-geostrophic equations in two ways. We may either take the scalar product of the momentum equation (8) and $\underline{v}$ and make use of the continuity and adiabatic equations, or alternatively we may multiply the vorticity equation

(14) by ψ . In both cases averaging with respect to x, y and z gives after some manipulation

$$\frac{d}{dt} \left[\frac{1}{2} (u^2 + v^2) \right] - h(w\phi) = -\text{div} \left[\frac{1}{2} (u^2 + v^2) \underline{v} \right] - h \text{div} [\psi \underline{v}_a] - h \frac{\partial}{\partial z} (\psi w) \quad (17)$$

Here the operator 'div' signifies the divergence with respect to the horizontal coordinates (x, y) only, and d/dt is the total derivative with respect to time.

The two terms on the left hand side of (17) represent the rates of change of the kinetic energy and the available potential energy per unit volume within the region. A positive correlation between w and ϕ indicates a release of available potential energy. Only the rate of change of geostrophic kinetic energy is included since terms involving rates of change of the ageostrophic wind are excluded as being of higher order in h^{-1} (section 1.3).

The terms on the right hand side may be expressed as means over the boundaries of the region of integration. The first represents the flux of kinetic energy through the sides, and the remaining two terms (which may be combined and rewritten $-h[\underline{v}_a + w\underline{k}] \cdot \nabla \psi$) represent the work done by the pressure forces.

If we apply the boundary conditions proposed in the previous section, then the energy flux term vanishes because v is zero at the sides. In a similar way the last term on the right hand side disappears because there is no vertical motion at the top and bottom. Finally we note that ψ is independent of x at the sides and so

$$\overline{\psi v_a}^x = \psi \overline{v_a}^x = 0$$

there, and all the terms on the right hand side of (17) vanish.

After substituting for w from equation (7) the energy equation (17) may be written

$$\frac{1}{2} \frac{d}{dt} \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 + \left(\frac{\partial \psi}{\partial z} \right)^2 \right]^{x,y,z} = 0 \quad (18)$$

This states that the total energy E , where

$$E = \frac{1}{2} \left[\nabla \psi \cdot \nabla \psi \right]^{x,y,z}$$

is constant. Thus the kinetic energy and available potential energy per unit volume are defined by

$$(KE) = \frac{1}{2} \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right]^{x,y,z}$$

and

$$(PE) = \frac{1}{2} \left(\frac{\partial \psi}{\partial z} \right)^2^{x,y,z}$$

respectively. Since changes in (KE) can only arise from equal and opposite changes in (PE) , we will frequently (for the sake of brevity) refer to the latter as simply 'the potential energy'.

In dimensional coordinates we have

$$(KE)^* = \frac{1}{2} \int_s^* \underline{v}^{*2}$$

and

$$(PE)^* = \frac{1}{2} \theta_s^* \left(\frac{d\theta_s^*}{dz^*} \right)^{-1} \left(\frac{\theta^*}{\theta_s^*} \right)^2 \int_s^* g$$

and we see that $(PE)^*$ is consistent with the expression for the available potential energy derived from first principles by Lorentz (1955).

It is also possible to derive a number of other conservation equations, and we consider a few of them here. First we multiply the adiabatic equation (9) by Φ and apply the result at the top and bottom of the channel. Then averaging with respect to x and y gives the result that

$$\frac{d}{dt} \overline{\frac{1}{2} \phi^2}^{x,y} = 0 \quad (19)$$

on the lids.

Integrating the continuity equation (10) with respect to x and y we find that $\overline{w}^{x,y}$ is identically zero at all levels. We can also obtain the result $\overline{v_a}^{x,z} = 0$ from the continuity equation in a similar way. Hence from the momentum, vorticity and adiabatic equations we derive the following conservation properties of the quasi-geostrophic equations:-

$$\frac{d}{dt} \overline{u}^{x,y,z} = 0 \quad (20)$$

$$\frac{d}{dt} \overline{\zeta}^{x,y} = 0 \quad (21)$$

$$\frac{d}{dt} \overline{\phi}^{x,y} = 0 \quad (22)$$

Thus in a frictionless adiabatic atmosphere the mean momentum, vorticity and entropy are constant.

Finally the adiabatic equation and the boundary conditions imply that $\overline{\phi}^x$ is constant at the corners of the channel (i.e. the intersections of the walls and lids), and so

$$\frac{d}{dt} \overline{\left(\frac{\partial \phi}{\partial y} \right)}^{x,y} = 0 \quad (23)$$

on the upper and lower boundaries, i.e. the mean entropy gradient is constant on the lids.

If the quasi-geostrophic vorticity equation (15) is solved with the boundary conditions (16) by a method of successive approximations then each approximate solution can be chosen so as to satisfy some of these conservation equations exactly while others can be used as a check on the accuracy of the solution.

CHAPTER 2. THE PERTURBATION PROBLEM.

2.1 The Perturbation Equations.

We investigate the mechanics of large scale (potential energy converting) atmospheric disturbances in the mid-latitude troposphere by analysing the growth and structure of solutions of the quasi-geostrophic equations. We start from an exact solution of the vorticity equation (15) with the boundary conditions (16) which is chosen to represent the observed baroclinity of the atmosphere. The effect of superimposing an arbitrary small amplitude disturbance on this initial state is then examined by expanding the stream function ψ in a perturbation series

$$\psi = \sum_{n=0}^{\infty} \psi_n \quad (24)$$

where each ψ_n is $O(\epsilon^n)$, and ϵ is a small parameter. The zero order term ψ_0 is the stream function for the geostrophic wind in the initial state.

Because the side boundary conditions are linear each ψ_n can be made to satisfy them independently, but at the top and bottom we expand w as a perturbation series and equate the successive terms to zero. Substitution of (24) into the vorticity equation (15) and the boundary conditions (16) and equating terms of the same order gives a set of equations which may be solved successively to obtain each ψ_n .

Thus we derive the first order vorticity equation

$$\frac{\partial}{\partial t} \nabla^2 \psi_1 + J(\psi_0, \nabla^2 \psi_1) + J(\psi_1, \nabla^2 \psi_0) = 0 \quad (25a)$$

where ψ_1 is periodic with x and satisfies the conditions

$$\frac{\partial \psi_1}{\partial x} = 0, \quad \overline{\frac{\partial^2 \psi_1}{\partial t \partial y}} = 0$$

at the sides, and

$$\frac{\partial}{\partial t} \frac{\partial \psi_1}{\partial z} + J(\psi_0, \frac{\partial \psi_1}{\partial z}) + J(\psi_1, \frac{\partial \psi_0}{\partial z}) = 0 \quad (25b)$$

on the lids; the second order vorticity equation

$$\frac{\partial}{\partial t} \nabla^2 \psi_2 + J(\psi_0, \nabla^2 \psi_2) + J(\psi_1, \nabla^2 \psi_1) + J(\psi_2, \nabla^2 \psi_0) = 0 \quad (26a)$$

where ψ_2 is periodic with x and satisfies the conditions

$$\frac{\partial \psi_2}{\partial x} = 0, \quad \frac{\partial^2 \psi_2}{\partial t \partial y} = 0$$

at the sides, and

$$\frac{\partial}{\partial t} \frac{\partial \psi_2}{\partial z} + J(\psi_0, \frac{\partial \psi_2}{\partial z}) + J(\psi_1, \frac{\partial \psi_1}{\partial z}) + J(\psi_2, \frac{\partial \psi_0}{\partial z}) = 0 \quad (26b)$$

on the lids; etc. . J is the Jacobian with respect to the cartesian coordinates (x,y) .

2.2 The Initial State.

In searching for a suitable zero order state to start the series we note that if ψ_0 is any function of y and z only, then it is an exact solution. This is the stream function for a steady zonal flow with both horizontal and vertical shear and implies (from the thermal wind equation) associated horizontal temperature gradients.

A special case arises when there is no vertical shear (and hence no horizontal temperature gradient), that is when the stream function ψ_0 depends only on y . Growing disturbances can only obtain their kinetic energy from the kinetic energy of the zonal flow (there is no potential energy available). This problem has been studied by Kuo (1949) and others. In the more complicated situation having both horizontal and vertical

zonal wind shears present, the eddies can obtain their kinetic energy from both the kinetic energy of the zonal flow and from the release of potential energy. This problem has only been examined in some special cases (see for example Charney and Stern (1962), Green (1960)).

In this investigation we choose an initial state with no horizontal velocity gradients but with a constant westerly vertical shear, implying a constant meridional temperature gradient with cold air in the north and warm air in the south. This is the simplest situation having potential energy available for conversion into eddy kinetic energy, and represents approximately the mean state of the mid-latitude troposphere. It has been studied by Charney (1947), Eady (1949), Green (1960) and several others.

It is convenient to use a coordinate system moving with the average zonal wind. This is achieved by subtracting the mean zonal (geostrophic) wind together with its appropriate stream function (and geostrophic pressure distribution) from the solution. However in the atmosphere the winds at high levels are observed to be strong compared with those near the earth's surface. Thus our solution will be more readily comparable with atmospheric disturbances as shown on synoptic weather maps if they are displayed with a westerly wind added to them. When this is done (the westerly wind is chosen so that the wind speed at the surface in the initial state is zero) they are referred to as being drawn 'relative to the ground'.

Defining the Richardson number h^2 of section 1.2 in terms of the shear of the initial zonal flow, and choosing the origin of the coordinate system to be in the middle of the region (so that the lids are given by $z = \pm \frac{1}{2}$ and the sides by $y = \pm Y/2$, where Y is the

non-dimensional width of the channel) then the stream function for the initial flow is

$$\psi_0 = -y z$$

2.3 The Linear Solution.

The first order vorticity equation (25a) now reduces to

$$\left[\frac{\partial}{\partial t} + z \frac{\partial}{\partial x} \right] \nabla^2 \psi_1 = 0$$

On the boundaries ψ_1 satisfies

$$\frac{\partial \psi_1}{\partial x} = 0, \quad \frac{\partial^2 \psi_1}{\partial t \partial y} = 0$$

at $y = \pm Y/2$ and

$$\left[\frac{\partial}{\partial t} + z \frac{\partial}{\partial x} \right] \frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_1}{\partial x} = 0$$

at $z = \pm \frac{1}{2}$.

We consider the initial disturbance to be Fourier analysed into a sum of periodic components which, because of the linearity of the first order system, may be treated independently. We examine solutions of the form

$$\psi_1 \propto \text{Real Part} \left\{ \exp[i(\lambda x + \mu y - \omega t)] F(z) \right\} \quad (27)$$

where F and ω ($= \omega_r + i\omega_i$) are complex but λ and μ are real.

This represents a wave superimposed on the basic zonal flow propagating in the (x, y) plane at an angle $\tan^{-1}(\mu/\lambda)$ to the x -axis (the direction of shear) with a phase speed ω_r/v , where $2\pi v^{-1} = 2\pi(\lambda^2 + \mu^2)^{1/2}$ is the wavelength. The wave amplitude is proportional to $\exp[\omega_i t]$ and is dependent on z .

After substituting into the first order vorticity equation and the boundary conditions on the lids we obtain an eigen value problem, and Eady showed that non-trivial solutions for F exist provided that

$$\omega^2 = \frac{\lambda^2}{v^2} \left(1 + \frac{v^2}{4} - v \coth v \right) \quad (28)$$

The boundary conditions at the sides (zero normal velocity) are satisfied by adding the solution (27) to an identical expression with μ replaced by $-\mu$. This is equivalent to regarding the sides as perfect 'mirrors' which reflect the wave. Non-trivial solutions satisfying the boundary condition $\partial \psi_1 / \partial x = 0$ only exist if

$$\mu = n\pi / \gamma$$

where $n = 1, 2, 3, \dots$

The solution for one Fourier component may now be written (apart from an arbitrary multiplicative constant) in the following form which satisfies all the boundary conditions:-

$$\psi_1 = e^{\omega_i t} \left[b \sinh v z \sin(\lambda x - \omega_r t) - (a \sinh v z + C \cosh v z) \cos(\lambda x - \omega_r t) \sin \mu (y + \gamma/2) \right] \quad (29)$$

where

$$a = \omega_r \frac{v}{\lambda} / \left(\frac{v}{2} \coth \frac{v}{2} - 1 \right)$$

$$b = \omega_i \frac{v}{\lambda} / \left(\frac{v}{2} \coth \frac{v}{2} - 1 \right)$$

The solution represents a pattern of eddies and π/λ and π/μ are measures of their geometrical size (along the x and y axes). Their growth rate is $|\omega_i|$ ($= \sigma$) and $|\omega_r/\lambda|$ ($= c$) is their phase speed (the introduction of side walls parallel to the x -axis restricts the solution to zonal propagation only). We refer to v ($= (\lambda^2 + \mu^2)^{1/2}$) as the total wavenumber.

Solutions of the above form do not form a complete set. Apart from the constant horizontal translation of the wave, the time variable is separable from the space variables and so the structure of the solution is independent of time. For a complete solution this set of 'constant shape waves' must be supplemented by a set of non-constant shape solutions. In fact the general solution may be written in the form

$$\psi, \text{ \& Real Part } \left\{ \exp[i(\lambda x + \mu y)] G(z, t) \right\}$$

and by integrating the vorticity equation with respect to time we find that G satisfies the equation

$$\frac{\partial^2 G}{\partial z^2} - v^2 G = e^{-i\lambda z t} Z_0(z)$$

Z_0 , which is an arbitrary function of z arising from the integration, is chosen to describe the potential vorticity introduced by the initial disturbance. The Eady waves are solutions of the homogeneous equation (i.e. with the right hand side identically zero). The complete solution, which has been derived by Pedlosky (1964) is necessary to describe the evolution of an arbitrary initial disturbance. It is evident that the non-constant shape waves only exist if the initial disturbance introduces some potential vorticity into the flow, and that they govern its subsequent redistribution.

2.4 Selection of the Dominant Wavelength.

The constant shape (Eady) waves are of two types depending on the value of the total wavenumber v . If $v > 2.4$ (approx) then we find from the relation (28) that ω^2 is positive and ω is a real number, the growth rate σ is zero and the wave is neutral. If

$\nu < 2.4$ ω^2 is negative and ω is a pure imaginary number, the phase speed c is zero and the wave is stationary (relative to the mean zonal flow) and is non-neutral.

For each λ and μ satisfying $\lambda^2 + \mu^2 > 2.4^2$ there is a pair of (short wavelength) neutral solutions corresponding to positive and negative values of ω_r and representing waves moving eastwards and westwards relative to the mean zonal flow. The zonal phase speed c , which is a function of the total wavenumber ν (see Fig 2.1), is zero for $\nu \approx 2.4$ and increases with increasing ν to an asymptotic value of 0.5 (equal to the speed of the mean zonal wind on the lids).

The non-neutral (long) waves have amplitudes proportional to $\exp[\omega_i t]$ and either grow or decay according to whether ω_i is positive or negative. The growth rate depends on the geometrical size of the eddies (i.e. on both λ and μ) and it is found (see Fig 2.2) that for a given value of μ there is a value of λ ($= \lambda_{max}$) such that σ is a maximum ($= \sigma_{max}$). The largest value of σ_{max} occurs when μ is zero, i.e. when the width of the baroclinic zone Y is infinite, however for a finite width Y the largest σ_{max} is given by the smallest value of μ , namely $\mu = \pi/Y$ (when the eddy occupies the full width of the channel). For example if we take the width of the baroclinic zone to be 3000 km. , then assuming the value for the length scale L^* ($= 1000$ km.) adopted in section 1.2 we find that Y takes the value 3 and $\mu \approx 1$. Thus $\sigma_{max} \approx .25$ and $\lambda_{max} \approx 1.5$ (corresponding to a zonal wavelength of about 4000 km.). Taking $Ri = 25$ we find that $\tau_{\sigma_{max}}^* \approx 2$ days, and this is the time taken for the disturbance to about double its amplitude. Provided that μ is small (i.e. the baroclinic zone sufficiently wide) both λ_{max} and σ_{max} are slowly varying functions

of μ . For larger values of μ however $\lambda_{\max}$ and $\sigma_{\max}$ become more dependent on the width of the region and decrease with increasing μ to become zero for $\mu \approx 2.4$. If $\mu > 2.4$ (i.e. a zone width of less than about 1500 km. assuming the values of the physical parameters adopted previously) $\nu > 2.4$ for all values of λ (λ is necessarily real), and no unstable waves can exist.

We call the wave with the largest growth rate 'the dominant wave' since we expect it to outgrow all other initially infinitesimal components after sufficient time has elapsed. (This may not happen if the initial disturbance has an amplitude which is too large, for then non-linear interactions might become important enough for its structure to be altered before the selection has had time to occur.)

The non-constant shape (Pedlosky) solutions depend on the structure of the initial disturbance. At the beginning they grow only linearly with time (or even less rapidly), but Pedlosky has shown that if $\nu < 2.4$ (corresponding to the Eady non-neutral regime) then after a long time they decay at least as fast as t^{-1} , while if $\nu > 2.4$ they become oscillatory with the same frequencies as the Eady neutral solutions.

Consequently it is reasonable to assume that after sufficient time the structure of an infinitesimal initial disturbance approaches that of the dominant (exponentially growing) Eady wave, with structural changes occurring when the disturbance reaches finite amplitude. The time boundary condition

$$\psi \rightarrow \psi_0 \quad \text{as} \quad t \rightarrow -\infty$$

is consistent with this assumption since it eliminates the neutral, decaying and non-constant shape waves.

2.5 The Structure of the Growing Disturbance.

The unstable (growing) Eady solution has the form

$$\psi_1 = e^{\sigma t} [b Shvz \sin \lambda x - Chvz \cos \lambda x] \cos \mu y \quad (30)$$

We take $\mu = \pi/Y$, and choose λ and σ to be appropriate for the dominant wave.

The pattern of motion is illustrated in Fig 2.3. The solution represents a chain of intensifying regions of cyclonic and anticyclonic vorticity whose phase variation with height is such that the surface systems lie to the east of the upper level systems. For an infinite north-south extent of the baroclinic zone ($\mu = 0$) the phase lag is $\pi/2$, but it is less for finite widths. The surface lows are warm with the warmest air ahead of (to the east of) them, while the upper lows are cold with the coldest air behind. Vertical velocities are largest at mid-levels with ascent in the warm air ahead of the surface lows. Maximum changes in the horizontal wind speed and the horizontal temperature gradient occur at the top and bottom. For small μ they are largest between the systems but are largest at the side walls if μ is large (narrow baroclinic zone). As Eady noted, the general form of the solution agrees with that of cyclones and anticyclones in the very early stages of their life, though a more detailed comparison is best left until we have examined the effects of non-linear interactions.

Some insight into the working of the system can be obtained by examining zonally averaged correlations between physical quantities. In addition we can infer from flux gradients some properties of the non-linear solution. Such inferences are only tentative, since

if α and α' are two physical quantities, then expanding the product $\alpha\alpha'$ in a perturbation series and averaging with respect to x we obtain

$$\overline{\alpha\alpha'}^{\infty} = \overline{\alpha_0\alpha'_0}^{\infty} + \overline{(\alpha_0\alpha'_1 + \alpha_1\alpha'_0)}^{\infty} + \overline{(\alpha_0\alpha'_2 + \alpha_1\alpha'_1 + \alpha_2\alpha'_0)}^{\infty} + O(\epsilon^3)$$

and the contribution by terms of $O(\epsilon^2)$ can only be calculated in part from the first order solution.

The fluxes $\overline{w_1\phi_1}^{\infty}$, $\overline{v_1\phi_1}^{\infty}$ and $\overline{w_1v_1}^{\infty}$ are positive everywhere (see Fig 2.4). This indicates a net upward northward transport of heat, resulting in the release of potential energy (mainly in the middle of the region). Eady showed that the most efficient release of potential energy occurs when the particle paths have a slope of about half that of the isentropic surfaces, but that constraints on the motion do not allow the achievement of optimum trajectories everywhere.

The zonally averaged adiabatic equation is

$$\frac{\partial \overline{\phi}}{\partial t} + \frac{\partial (\overline{v\phi})}{\partial y} + k \overline{w} = 0$$

and the existence of flux gradients $\partial \overline{(v\phi)} / \partial y$, negative in the northern and positive in the southern latitudes, indicates a change in the zonally averaged temperature field. The linear solution does not alter the zonally averaged temperature distribution (it is periodic with x), and so we expect that the higher order perturbation solutions will describe an increase of temperature in the north and a decrease in the south (i.e. a decrease in the mean meridional temperature gradient). Flux gradients such as $\partial \overline{(w_1\phi_1)} / \partial z$ cannot be interpreted as implying features of higher order perturbation solutions since the terms they affect in the adiabatic equation are $O(Ri^{-1})$ (see section 1.3).

The momentum flux $\overline{u_1v_1}^{\infty}$ is zero everywhere and so

the linear Eady solution implies zero meridional transport of zonal momentum. Non zero correlations between u and v are associated with asymmetry in the horizontal flow which does not exist for the linear solution. Since $\partial(\overline{uv})/\partial y = \partial^2(\overline{uv})/\partial y^2$, the meridional flux gradient of relative vorticity is also zero. As in the previous paragraph, no momentum transfer is implied by flux gradients such as $\partial(\overline{wv})/\partial z$ which affect $O(Ri^{-1})$ terms in the momentum equations.

2.6 Adjustments of the Mean Zonal Flow (The Second Order Solution).

By substituting the linear solution into the Jacobian terms of the second order vorticity equation and boundary conditions (26) we obtain a system which may be solved for the second order solution ψ_2 . Assuming that only the dominant growing wave is of importance in deriving the finite amplitude solution, substitution for ψ_1 reveals that

$$J(\psi_1, \nabla^2 \psi_1) = 0$$

and

$$J\left(\psi_1, \frac{\partial \psi_1}{\partial t}\right) = -\frac{1}{2} e^{2\sigma t} \lambda \mu \approx b \sin 2\mu y$$

which suggests that the second order solution ψ_2 is independent of x . (If the second order solution contains a component dependent on x it must satisfy the same equations and boundary conditions as the first order solution and so is adequately represented by ψ_1 .)

Writing

$$\psi_2 = e^{2\sigma t} F_2(y, z)$$

we find from the second order vorticity equation that F_2 is determined from

$$\frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2} = 0$$

with the boundary conditions

$$\frac{\partial F_2}{\partial y} = 0$$

at $y = \pm \pi/2\mu$, and

$$2\sigma \frac{\partial F_2}{\partial z} - \frac{1}{2} \lambda \mu v b \sin 2\mu y = 0$$

at $z = \pm \frac{1}{2}$.

The solution is most easily derived as a Fourier series. Each term of the series is chosen so as to satisfy the conditions on one pair of boundaries while the Fourier coefficients are chosen so that the complete series satisfies the conditions on the other pair. Thus the solution may be written in two forms; if the side boundary conditions are satisfied identically the solution is

$$F_2(y, z) = F_S(y, z) = \frac{2}{\pi} \frac{\lambda v b}{\sigma} \sum_{n=0}^{\infty} A_n \operatorname{Sh}(2n+1)\mu z \operatorname{Sin}(2n+1)\mu y$$

(31)

where

$$A_n = \frac{(-1)^{n+1}}{\operatorname{Ch}\left[\frac{(2n+1)\mu}{2}\right]} \frac{1}{(2n-1)(2n+1)(2n+3)}$$

while satisfying the conditions on the lids identically gives

$$F_2(y, z) = F_L(y, z) = \frac{2}{\pi} \frac{\lambda v b}{\sigma} \left[\frac{\pi}{16} \frac{1}{\operatorname{ch}\mu} \operatorname{Sh} 2\mu z \operatorname{Sin} 2\mu y + \sum_{n=0}^{\infty} A'_n \operatorname{Sin}(2n+1)\pi z \operatorname{Sh}(2n+1)\pi y \right]$$

(32)

where

$$A'_n = \frac{(-1)^n}{\operatorname{Ch}\left[\frac{(2n+1)\pi^2}{2\mu}\right]} \frac{\mu^2}{(2n+1)\{(2n+1)^2\pi^2 + 4\mu^2\}}$$

The coefficients A_n and A'_n are functions of μ but are independent of λ and σ .

Both series of course represent the same function and converge well in the interior. However F_S converges only as fast as n^{-3} on the lids and it would be dangerous to use it for calculating second derivatives of ψ_2 there (and impossible for higher derivatives). F_L behaves in a similar manner at the sides. Both series are equally bad at the corners and it can be seen by differentiating along the boundaries that the wall and lid conditions for F_2 give different values of $\partial^2 F_2 / \partial y \partial z$ there. This inconsistency is confined to the corners and does not arise anywhere in the main body of the fluid. It probably occurs because a description of the flow near the corners requires a more accurate representation for the small scale eddying motion than is present here. Since $\mu < \pi^2/\mu$ (μ must be less than 2.4 for unstable Eady waves to exist) the convergence of F_S is more uniform than that of F_L , and for the computation of first derivatives three terms of F_S are found to be sufficient everywhere.

The second order solution represents the influence of the baroclinic disturbances on the mean zonal flow. The circulation in the zonally averaged cross section is shown in Fig 2.5, and has been calculated for three values of μ . Similar mean meridional circulation patterns have been obtained by other authors for models of the atmosphere (although they also include some physical effects that have been neglected here, see for example Phillips (1956) and Fig 2.7). The circulation is usually regarded as having a three cell structure, with direct cells near each of the side boundaries and an indirect cell in the middle. In some ways it is more convenient to regard it as consisting of only two cells

(see Fig 2.9), the vertical motion near the sides belonging to one large direct cell and the upward and downward motion in the interior being part of a central indirect cell.

The direct circulation, with upward motion in the warm southerly air and downward motion in the cold air in the north releases potential energy by altering the temperature field near the sides where the slope convection is inefficient (horizontal advection is small there). Note that no change in the mean meridional temperature gradient occurs at the walls because the side boundary condition $\partial \bar{u}^* / \partial t = 0$ implies that the vertical shear of the mean zonal wind is independent of time there.

The indirect cell, which does work against gravity, is more complicated to explain. The northward transport of heat by the slope convection in the cyclones and anticyclones reduces the mid-latitude zonally averaged meridional temperature gradient, and for thermal wind balance to be maintained there must be a corresponding reduction in the vertical shear of the mean zonal wind. Since the eddy momentum flux is zero (see section 2.5) this must be accomplished by a meridional (angular momentum conserving) ageostrophic flow, northwards at low levels and southwards at high levels. Continuity demands a region of ascent north of the central latitude and descent to the south. Thus the baroclinic convection has to drive the indirect circulation in order to satisfy the thermal wind equation. This imposes some limitations on the efficiency of the large scale potential energy conversion process.

When μ is small (i.e. a wide baroclinic zone) the reduction of the mid- latitude meridional temperature gradient is small and the indirect cell is relatively weak. The temperature changes are then mainly due to

eddy transfer of heat and are almost independent of height. For a narrow channel (μ large) the indirect cell is more important and the temperature changes, which at mid-levels are considerably reduced by the vertical motion, are concentrated at the top and bottom.

In addition to being reduced at mid-latitudes, the vertical shear of the zonal wind is increased at two intermediate latitudes near the northern and southern walls. By continuity there must be a horizontal ageostrophic flow connecting the up and down draughts of the direct and indirect circulations and, as at mid-latitudes, the zonal wind changes are ascribed to the effect of the earth's rotation and conservation of mean angular momentum.

The zonal vorticity produced at lower levels by the vertical motion of the indirect cell is positive in the north and negative in the south, and so the regions of cyclonic (anticyclonic) vorticity move northwards (southwards) at the bottom, while the movement is in the opposite direction at upper levels. There is some synoptic evidence for the northward migration of the surface cyclones (see later in section 4.2), and results from other numerical model of the atmosphere (for example Phillips (1956)) support this. An alternative suggestion for the observed behaviour of depressions is that small scale convection in the cold air north of the cyclone results in the formation of a 'heat low' and is responsible for the meridional movement, but it is apparent that more fundamental processes, like that described above, are more likely to be important.

It is tempting to compare the developing zonal flow with monthly mean data for the atmosphere (e.g. Heastie and Stephenson (1960)). However we must remember that our solution represents the changes in the zonal flow

due to one system of cyclones and anticyclones in the growing stage of its life, while those shown as monthly means for the atmosphere are averages of several different systems over their complete life cycles. Nevertheless some similarities are apparent when we compare the second order solution with the zonal mean flow for the southern hemisphere winter (see Fig 2.6), at least at upper levels where two regions of maximum westerly winds occur. However there is little correspondence at low levels, and in the equivalent cross section for the northern hemisphere only a single region of upper westerly flow occurs. Furthermore our computations do not agree with the results of Phillips's numerical model (1956) (see Fig 2.8) which resemble the atmospheric data for the northern hemisphere. It is likely that the β effect is important in determining the nature of the zonal wind distribution, but the difference between the two hemispheres suggests that friction and heating may also be important. However we defer further discussion until we have obtained a more accurate (energy conserving) solution in section 4.2.

2.7 The Modified Growing Waves (The Third Order Solution).

The perturbation method can be extended to obtain a third order solution. This is periodic with x , having the same zonal wavelength as ψ_1 , but is proportional to $\exp[3\sigma t]$, and represents the alterations in the structure of the disturbance (given by the first order solution) as a result of the changed zonal flow (by ψ_2). We can write ψ_3 in the form

$$\psi_3 = \sum_{n=0}^{\infty} e^{3\sigma t} \left[-B_n \operatorname{Sh} v_n z \operatorname{Sin} \lambda x + B_n' \operatorname{Ch} v_n z \operatorname{Cos} \lambda x \right] \operatorname{Cos} (2n+1) \mu y$$

where

$$v_n^2 = \lambda^2 + (2n+1)^2 \mu^2$$

The coefficients B_n and B'_n , which have a complicated algebraic form and are functions of both λ and μ , are calculated from the condition $w_3 = 0$ on the lids. (For convergence F_L must be used in the expression for ψ_2 when calculating the non-linear terms of w_3 .)

The series is rapidly convergent and only the first two terms are important, the first of which may be combined with ψ_1 (30) to give

$$\left[(B_n e^{\sigma t} - B'_n e^{3\sigma t}) S_{\lambda v z} S_{\mu \lambda c} + (B'_n e^{3\sigma t} - e^{\sigma t}) C_{\lambda v z} C_{\sigma \lambda c} \right] \cos \mu y \quad (33)$$

where B_n and B'_n are both positive.

This expression represents the modification of the first order solution and Fig 2.10 illustrates its time dependence resulting from the non-linear 'feedback' from the zonal flow. The coefficients of $S_{\mu \lambda c}$ and $C_{\sigma \lambda c}$ initially increase in magnitude like $e^{\sigma t}$ but eventually reach maxima and decrease to zero finally changing sign and becoming infinite like $e^{3\sigma t}$ as $t \rightarrow \infty$.

Although the solution eventually becomes infinite it appears that we have begun to produce a bounded solution for t less than about -1. Higher order solutions may be expected to improve this although, because of the nature of the expansion, they will always become exponential after sufficient time has elapsed.

It is interesting to note that the arbitrary constant multiplying the linear solution is of trivial importance in the non-linear solution, for if we multiply ψ_1 by the arbitrary constant $e^{\sigma t_0}$ the non-linear expression (33) has precisely the same form except that t is replaced by $t + t_0$, (i.e. the time origin is

changed).

We may write the non-linear component (33) in the form

$$D \cos(\lambda x - \alpha) \cos \mu y$$

where

$$D = \left[(b e^{\sigma t} - B_0 e^{3\sigma t})^2 S h^2 v z + (B_0' e^{3\sigma t} - e^{\sigma t})^2 C h^2 v z \right]^{1/2}$$

$$\alpha = -\arctan \left[\left(\frac{b - B_0 e^{2\sigma t}}{1 - B_0' e^{2\sigma t}} \right) \tanh v z \right]$$

Thus the amplitude D and the phase α are functions of both z and t . The amplitude at the surface and the phase difference between the top and bottom are shown in Fig 2.11. When the mean meridional entropy gradient becomes reduced (by ψ_2) the rate of growth of the baroclinic wave decreases, but at the same time the phase lag becomes smaller. This means that the upper trough catches up (and eventually overtakes) the surface cyclone as it approaches its maximum amplitude. The physical reasons for this behaviour are discussed later in section 3.8, with reference to energy conserving solutions, but it is interesting to note that, while growing 'open wave' cyclones are often observed to lie to the east of troughs at upper levels, occluded cyclones are usually in phase with the upper level troughs, i.e. a similar change in phase occurs in the atmosphere.

The maximum amplitude (at $t = -1$ approx.) corresponds to pressure deviations from the mean of about 15 mb. at the surface. This is somewhat less than is frequently observed in actual cyclones at their maximum development stage, but it has about the right order of magnitude and is comparable with values

obtained by Fourier analysis of synoptic patterns (see for example Gilchrist (1957)). Since ψ_2 grows exponentially the calculated zonally averaged temperature gradient eventually becomes reversed but the solution is probably not realistic at this stage.

The temperature changes implied by ψ_2 give increased meridional gradients near (but not at) the walls and decreases at central latitudes (see Fig 2.5). Thus we might expect the system to become more active near the sides and correspondingly less active in the middle of the region, hence the importance of the second term of ψ_3 proportional to $\cos 3\pi y$. One new feature resulting from the appearance of this term is that correlations such as $\overline{u_1 v_3}$ and $\overline{u_3 v_1}$ are non zero, and so meridional fluxes of zonal momentum are possible, evidently producing (in higher order terms) some readjustment of the zonal flow. This is discussed in more detail in section 4.2 during an examination of the eight parameter (energy conserving) functional representation solution.

In principle the perturbation method may be extended to give higher order solutions. For example ψ_4 has both zonal and periodic components. The zonal component includes temperature and momentum changes implied by the flux gradients $\partial(\overline{v_1 \phi_3})/\partial y$, $\partial(\overline{v_3 \phi_1})/\partial y$, $\partial(\overline{u_1 v_3})/\partial y$ and $\partial(\overline{u_3 v_1})/\partial y$, while the periodic term, which is proportional to $\{\sin\}_{\cos} 2\lambda x$, implies a departure from the simple symmetric structure of the highs and lows.

The perturbation series has the form of a power series

$$\psi = \sum_{n=0}^{\infty} F_n(x, y, z) E^n$$

where

$$\psi_n = F_n(x, y, z) E^n$$

and

$$E = \exp[\sigma t]$$

but its radius of convergence is not obvious (it may possibly be zero with the series converging only asymptotically as $t \rightarrow -\infty$). However since the exact solution must be bounded for all t the series cannot converge uniformly, and so further improvements are only obtained by employing an increasing degree of effort. Furthermore, the algebra, which involves multiple Fourier series, is extremely tedious and it would seem better to develop the solution in a different way. This is done in the next chapter.

2.8 The Stable and Neutral Eady Solutions.

In section 2.4 we adopted the hypothesis that the structure of an arbitrary initial disturbance approaches that of the dominant unstable wave, and subsequently deduced some aspects of its evolution. In this section we will examine briefly stable (decaying) and neutral waves in isolation.

The Eady decaying solution (corresponding to (29) with $\omega_r = 0$ and $\omega_i < 0$) has the form

$$\psi_i = e^{-\sigma t} \left[b_1 \sinh \lambda z \sin \lambda x + c_1 \cosh \lambda z \cos \lambda x \right] \sin \mu(y + y_0/2) \quad (34)$$

Comparing this with the unstable solution (30), it can be seen that besides decreasing exponentially with time it differs by having variations in the vertical of opposite sign. Thus the trough line is forward sloping

with the upper pressure systems lying ahead of those below (see Fig 2.12). In fact the pressure and entropy distributions along a line of latitude are mirror images of those for the unstable solution (cf. Fig 2.3)
 ((34) is identical to (30) with x replaced by $-x$), but upward (downward) motion occurs in the cold (warm) air. Thus the cold air moves northwards and ascends while the warm moves southwards and descends; hence the tendency for the system to return to the undisturbed state. Considering the non-linear interactions, we find that the higher order terms (of the perturbation series) for the diminishing solution are proportional to $\exp[-n\sigma t]$ ($n = 2, 3, 4, \dots$), and so decay more rapidly than the first order solution (34) and soon become negligible in comparison.

The Eady neutral solutions, as we have already noted in section 2.4, occur in pairs the members of which have phase speeds equal in magnitude but opposite in sign. The height at which the zonal wind is equal to the phase speed of a wave is known as its steering level. Thus the steering level of the westward moving wave is in the lower half of the atmosphere while the eastward wave has its steering level aloft. The dependence of the phase speed c on the total wavenumber ν (see Fig 2.1) implies that the shorter waves (λ large) or waves in narrower zones (μ large) have their steering levels closer to the lids.

The upper (eastward) wave may be written in the form

$$\psi_1 = (a \operatorname{Sh} \nu z + c \nu z) \cos \lambda(x - ct) \sin \mu(y + Y/2) \quad (35)$$

(where a is defined in (29)) and it is clear that the disturbed values of pressure, vorticity and entropy are all in phase, and that there is no variation of the phase

with height (apart from a change in sign of the entropy deviation ϕ_1). The vertical velocity is 90° out of phase and the correlations $\overline{\phi_1 w_1}^x$ and $\overline{\phi_1 v_1}^x$ are both zero. Thus no northward heat transfer occurs and no potential energy is released (the particle paths have, on average, the same inclination as that of the isentropic surfaces). The amplitudes of the pressure and temperature waves are greatest on the upper lid, while the vertical velocity maximum is at the steering level (since vorticity changes only occur as a result of zonal advection there). The structure of the lower westward wave is similar (see Fig 2.13) but its intensity is largest at the lower lid.

Because there are no meridional flux gradients of vorticity and entropy a single Eady stable wave is an exact solution of the full non-linear quasi-geostrophic equations. If however we imagine the atmosphere to be excited at a single short (neutral) wavelength, then (assuming that there are no non-constant shape waves) it is possible that both upper and lower waves might exist together. In this case there would be mutual interference between them producing some non-linear effects. This is a simple example of the more general problem of the interaction between a whole spectrum of waves. The evolution of such a complex disturbance depends not only on the wavelengths of the components present, but also on their relative amplitudes and phases at the initial time (say $t = t_0$).

As an example of this we consider a pair of neutral waves each of which has the same amplitude. If at $t = t_0$ the two waves are 180° out of phase then, due to their relative movement, their low (and high) pressure centres will move towards each other. The pressure extrema of the upper wave will then be less than half

a wavelength to the west of those of the lower wave. Clearly in this state the combined system has a structure rather like that of the Eady unstable solution (Fig 2.3), and indeed it is found that the interference between them is such as to produce a northward heat transfer with an increase in resultant amplitude as the two waves move together. This behaviour is made more obvious if we simply add together the stream functions for the two out-of-phase waves to give (assuming for simplicity that $\mu = \pi/\gamma$)

$$\psi_1 = \left[a \operatorname{Sh} \gamma z \operatorname{Sin} \lambda x \cos \lambda c(t-t_0) - C \operatorname{h} \gamma z \cos \lambda x \operatorname{Sin} \lambda x(t-t_0) \right] \cos \mu y \quad (36)$$

because this shows clearly its algebraic, and hence structural, similarity (when $t - t_0 < \pi/(\lambda c)$) with unstable solution (30). However its amplitude does not increase indefinitely, but varies sinusoidally in time with a period $\pi/(\lambda c)$.

An alternative case to consider is that occurring when the two waves are initially in phase with each other. The movement of the upper and lower waves then causes the combined system to adopt a forward sloping structure like that of the Eady decaying wave (Fig 2.12), and in a similar way to the previous case we find that the heat transfer is southwards and the resultant amplitude lessens as the two waves move out of phase.

It is evident that there is a possibility that certain combinations of neutral waves will interfere, release potential energy and grow in intensity (at least temporarily). This is considered in the remainder of this section.

The effect of the resulting heat transfer on the zonal flow can be found by calculating a second order perturbation solution, and for the out-of-phase (initially

growing wave it takes the form

$$\psi_2 = \sin^2 \lambda c (t - t_0) F_2(y, z)$$

where F_2 is algebraically the same as for the unstable second order solution (i.e. (31) or (32)) except that b and σ are replaced by a and λc . This is periodic in time, and so the heat transfer process is apparently reversible to this order of approximation, with the mean meridional entropy gradient at first decreasing and reaching a minimum when the upper and lower waves come into phase, and then increasing again back to its original undisturbed state in the second half of the cycle.

In order to find a third order solution we must solve the equation

$$\left[\frac{\partial}{\partial t} + z \frac{\partial}{\partial x} \right] \nabla^2 \psi_3 = 0$$

subject to the boundary conditions

$$\frac{\partial \psi_3}{\partial x} = 0, \quad \frac{\partial^2 \psi_3}{\partial t \partial y} = 0$$

at the sides, and

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + z \frac{\partial}{\partial x} \right] \frac{\partial \psi_3}{\partial z} - \frac{\partial \psi_3}{\partial x} = \\ & \frac{1}{4} \lambda \left[\left(v S h v z \frac{\partial F_2}{\partial y} - c h v z \frac{\partial^2 F_2}{\partial y \partial z} \right) \sin \lambda x \left(3 \sin \lambda c (t - t_0) - 4 \sin 3 \lambda c (t - t_0) \right) \right. \\ & \left. + \left(v c h v z \frac{\partial F_2}{\partial y} - S h v z \frac{\partial^2 F_2}{\partial y \partial z} \right) \cos \lambda x \left(3 \cos \lambda c (t - t_0) - 4 \cos 3 \lambda c (t - t_0) \right) \right] \cos \mu y \end{aligned} \quad (37)$$

on the lids.

As a first guess we might expect ψ_3 to have the form

$$\begin{aligned} \psi_3 = & \sum_{n=0}^{\infty} \left[A_n S h v_n z \sin \lambda x \cos \lambda c (t - t_0) + A'_n c h v_n z \cos \lambda x \sin \lambda c (t - t_0) \right] \cos(2n+1) \mu y \\ & + \sum_{n=0}^{\infty} \left[B_n S h v_n z \sin \lambda x \cos 3 \lambda c (t - t_0) + B'_n c h v_n z \cos \lambda x \sin 3 \lambda c (t - t_0) \right] \cos(2n+1) \mu y \end{aligned}$$

where $v_n^2 = \lambda^2 + (2n+1)^2 \mu^2$, since this satisfies the third order equation and the side conditions and it seems to have the right form to match the lid condition (37) thereby enabling the coefficients A_n , A'_n , B_n , B'_n to be found. However it must be noted that the first term of the first series in this assumed expression for ψ_3 has the same temporal and spatial variations as the first order solution (36), the latter satisfying an equation identical to the third order vorticity equation but with a homogeneous lid boundary condition (i.e. (37) with the right hand side identically zero). Furthermore the right hand side of the lid condition (37) also contains a term oscillating with the same frequency as the first order solution and so acts as a resonant forcing function. Thus the problem of solving these equations is similar to that of solving the resonant simple harmonic equation

$$\ddot{X} + k^2 X = \sin kt$$

Instead of the above form ψ_3 is given by

$$\begin{aligned} \psi_3 = & (t-t_0) [A_0 \sinh v_0 z \sin \lambda x \cos \lambda c(t-t_0) + A'_0 \cosh v_0 z \cos \lambda x \sin \lambda c(t-t_0)] \cos \mu y \\ & + \sum_{n=1}^{\infty} [A_n \sinh v_n z \sin \lambda x \cos \lambda c(t-t_0) + A'_n \cosh v_n z \cos \lambda x \sin \lambda c(t-t_0)] \cos(2n+1) \mu y \\ & + \sum_{n=0}^{\infty} [B_n \sinh v_n z \sin \lambda x \cos 3\lambda c(t-t_0) + B'_n \cosh v_n z \cos \lambda x \sin 3\lambda c(t-t_0)] \cos(2n+1) \mu y \end{aligned}$$

Combining the first term with ψ_1 (36) we obtain

$$\{[a + A_0(t-t_0)] \sinh v_0 z \sin \lambda x \cos \lambda c(t-t_0) + [A'_0(t-t_0) - 1] \cosh v_0 z \cos \lambda x \sin \lambda c(t-t_0)\} \cos \mu y$$

Evidently the short wave solution with non-linear modifications is not simply periodic with time because the coefficients are linear functions of t , indicating that the energy conversion process is irreversible. However, although a combined solution of this form may ultimately grow, it is still likely to be dominated by the exponentially growing waves, since its initial growth is (to this order of approximation) only linear.

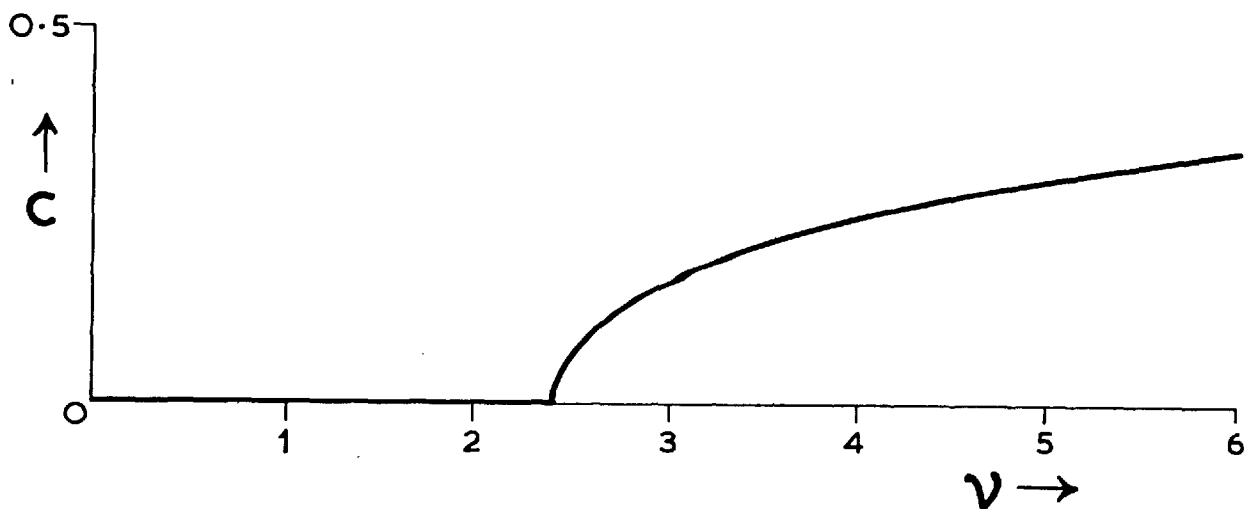


Fig 2.1 Phase speed c as a function of total wave number v .

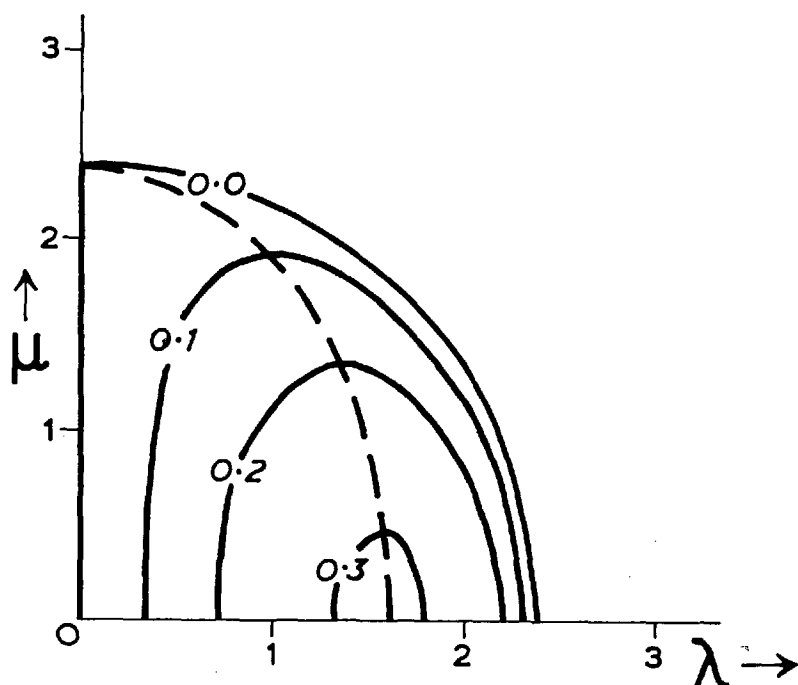


Fig 2.2 Growth rate σ as a function of the zonal and meridional wavenumbers λ and μ . The broken line is the line $\sigma = \sigma_{\max}$ (μ fixed with respect to λ)

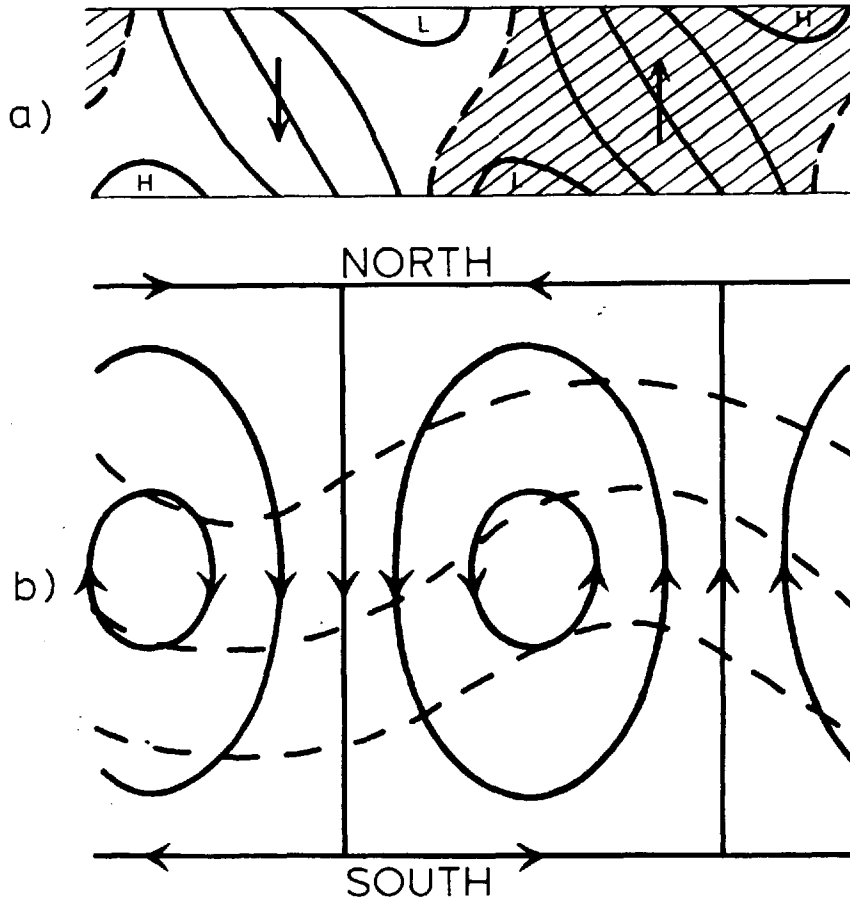


Fig 2.3 (a) Vertical section through the dominant wave along a line of latitude, with isopleths of ψ (—), warm air (shaded) and vertical velocities (arrows).

Fig 2.3 (b) Horizontal surface map of the above with stream lines of the geostrophic flow relative to the ground (—) and isotherms (— — —).

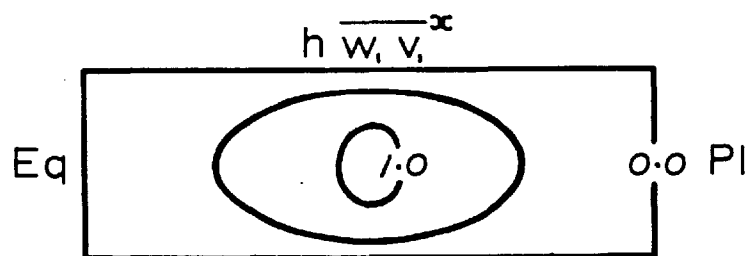
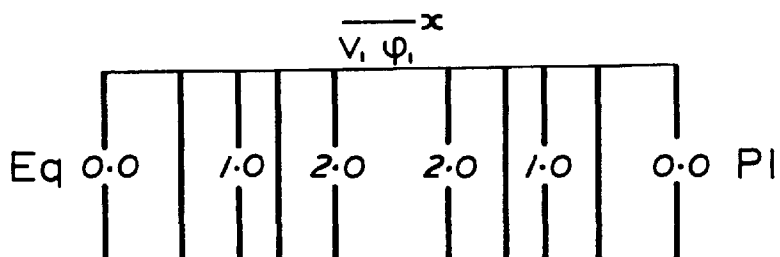
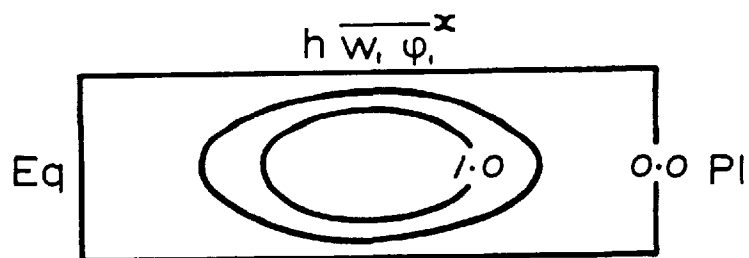


Fig 2.4 Zonally averaged fluxes as functions of y and z . Eq and Pl denote the equatorward and poleward walls.

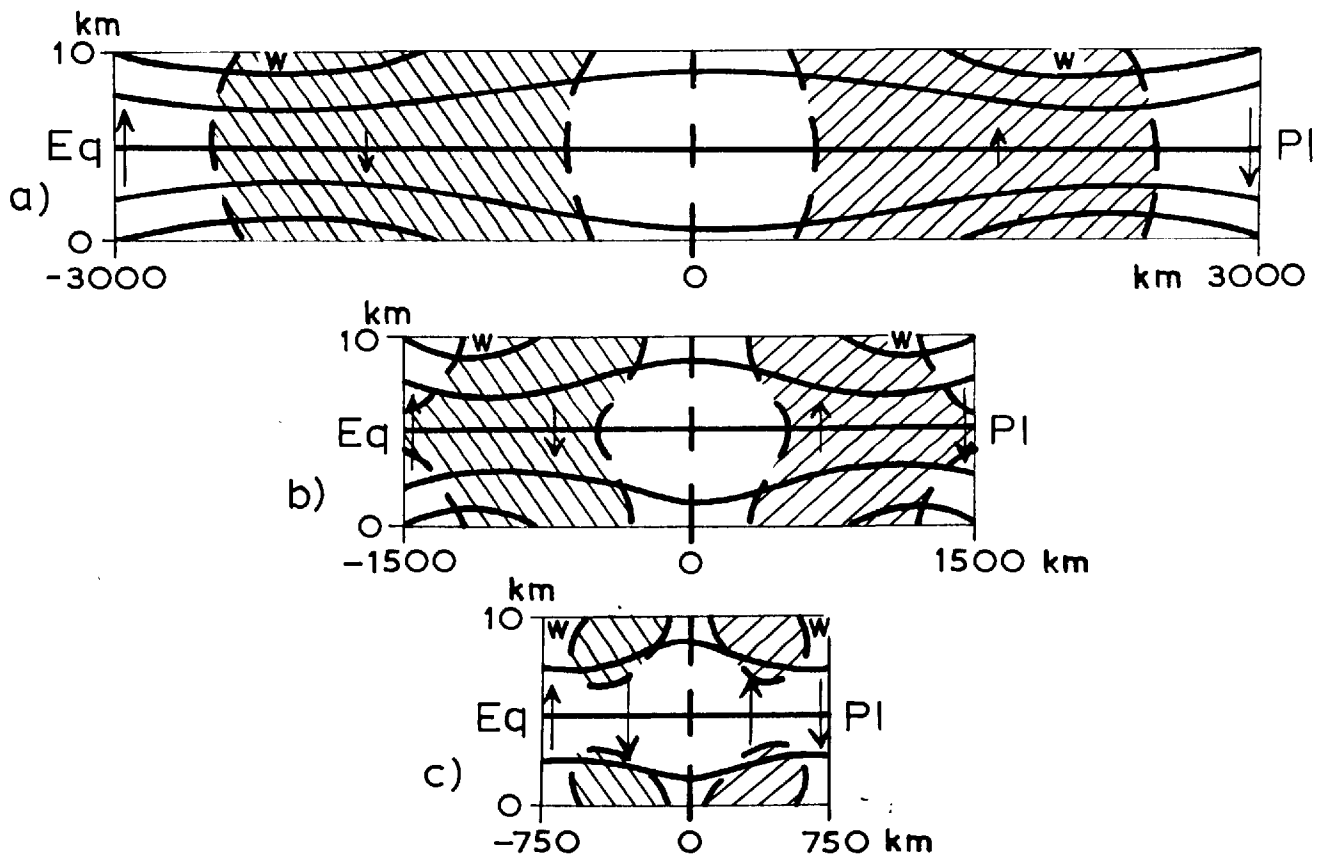


Fig 2.5 Zonally averaged meridional sections for baroclinic zone widths of (a) 6000 km ($\mu = 0.5$), (b) 3000 km. ($\mu = 1.0$) and (c) 1500 km. ($\mu = 2.0$), showing isothachs of $\bar{u}^x$ (—), isopleths of entropy change (---) with areas of maximum adiabatic heating (////) and maximum adiabatic cooling (\\\\) and vertical velocities (arrows). Amplitudes are arbitrary and are not comparable between diagrams. Pl and Eq denote the poleward and equatorward walls.

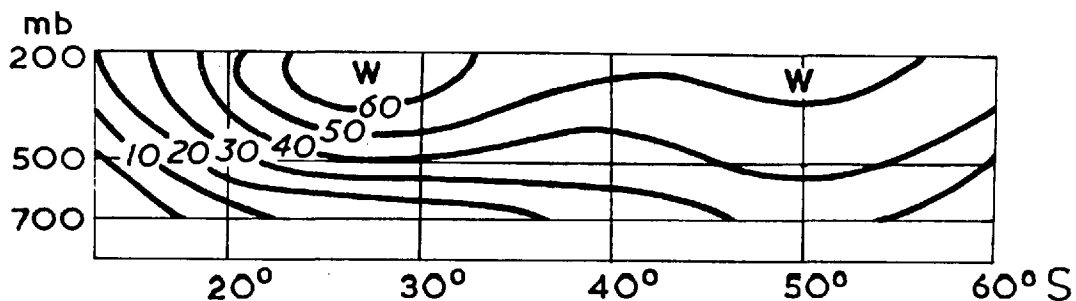


Fig 2.6 Zonally averaged meridional cross section of the southern hemisphere in July with isotachs of zonal wind (speeds in knots) after Heastie and Stephenson (1960).

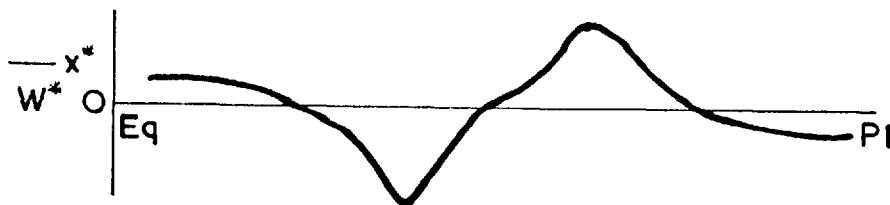


Fig 2.7 Zonally averaged mid level vertical velocity (arb. units) for Phillips's (1956) model, shown as a function of latitude and as a mean over the period 10 - 20 days. Upward motion shown positive.

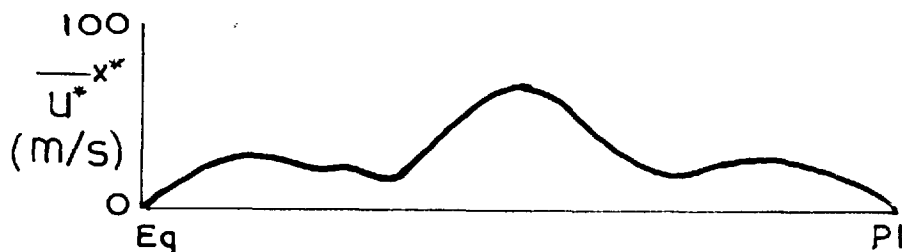


Fig 2.8 Mean zonal westerly wind at upper levels for Phillips's (1956) model shown as a function of latitude at day 20.

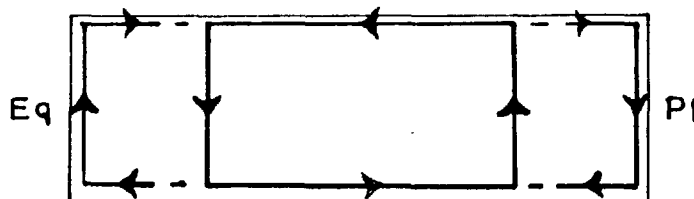


Fig 2.9 Schematic representation of the mean meridional circulation as an outer direct cell superimposed on an inner indirect cell.

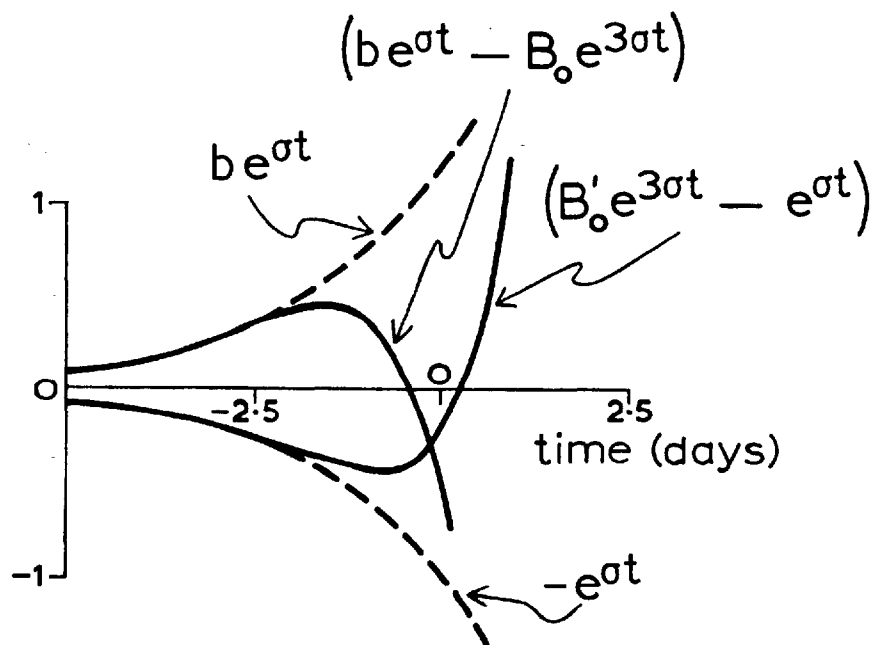


Fig 2.10 Variation of the coefficients of $\sin \lambda x$ and $\cos \lambda x$ with time for the linear solution (— — —) and the non-linear solution (——).

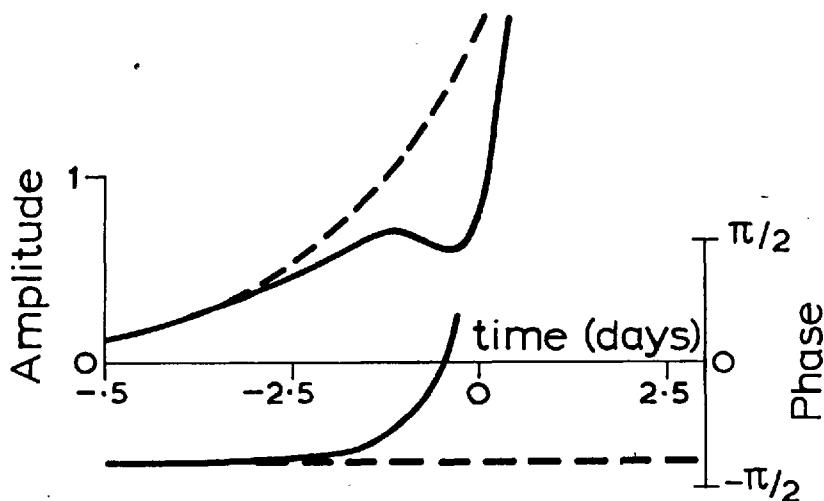


Fig 2.11 Variation with time of the amplitude (above) and phase difference between the pressure waves at the top and bottom (below) of the linear (— — —) and non-linear solutions (——).



Fig 2.12 Vertical section along a line of latitude of the Eady decaying solution with isopleths of ψ (—), warm air (shaded) and vertical velocities (arrows).



Fig 2.13 Vertical sections along a line of latitude of the (a) upper westerly and (b) lower easterly Eady (short) neutral waves, with isopleths of ψ (—), warm air (shaded) and vertical velocities. Horizontal arrows indicate the direction of travel relative to the mean zonal flow.

CHAPTER 3. THE METHOD OF FUNCTIONAL REPRESENTATION.

3.1 Introduction.

We have seen in the previous chapter that by expanding the stream function ψ in a perturbation series it is possible to predict some features of the growth of an initially infinitesimal disturbance superimposed on a sheared zonal flow. However the perturbation solutions represent the subsequent evolution well only in the early stages of the life of the disturbance. The balance of kinetic and potential energy is only correct to $O(\xi^n)$ and it breaks down when higher order terms become important, allowing unbounded solutions to develop.

In this chapter we outline an alternative method (called the method of functional representation) which provides solutions capable of satisfying the energy conservation equation (18) exactly and which consequently remain bounded. In addition the solutions satisfy the vorticity equation (14) identically, and the truncation errors (occurring because of the inability to represent non-linear terms correctly without including all scales of motion) arise only in conjunction with the boundary condition of zero vertical velocity at the upper and lower lids. Because of the relatively simple algebraic form of these solutions it is feasible to examine the effects of the truncation errors and introduce improvements which reduce their importance.

Each term of the perturbation series consists of a Fourier series (containing components representing the motion due to eddies of different scales) which has a time dependent amplitude proportional to $e^{n\sigma t}$. However the solutions developed below include only the most

important Fourier components each of which has its own time variation determined by a set of ordinary differential equations. The time dependent coefficients of the Fourier components are chosen so as to minimise the effects of the truncation errors, and also to ensure energy conservation. The spatial dependence of the solution is such as to satisfy the vorticity equation identically.

A similar method was described by Lorentz in (1960) for barotropic flow and in (1965) for baroclinic motion. In his baroclinic problem Lorentz used a two layer model in which vertical derivatives were expressed in terms of finite differences, whereas here we are able to express the derivatives exactly. Two layer models are known to yield a rather crude analysis of the linear perturbation problem, and so we expect our continuous model to show a comparable improvement over Lorentz's finite difference solution.

3.2 The Integrated Vorticity Equation.

For the perturbation method it was possible to derive a set of equations (and boundary conditions) which enable each term of the perturbation series to be determined from the previous terms. For the method of functional representation no such set can be found unless an order of precedence is devised, and so it is necessary to solve the full vorticity equation. In view of its non-linearity this is in general a very difficult problem. However the disturbance has the structure of the dominant growing Eady wave at the beginning of its life and this, as we now show, enables us to reduce the vorticity equation to a linear form without any approximation.

The initial zonal flow has a stream function $\psi_0 = -y^2$

and it is easily seen by substitution into (14a) that this corresponds to a state of constant potential vorticity (correct to $O(Ri^{-\frac{1}{2}})$). The superposition of an Eady wave does not alter this (only the non-constant shape waves depend on the potential vorticity introduced by the initial disturbance), and because potential vorticity is conserved along trajectories, it remains constant at all subsequent times. Thus if $\psi' = \psi - \psi_0$, then ψ' satisfies

$$\nabla^2 \psi' = 0 \quad (38)$$

The assumption of a simple initial flow has enabled us to integrate the vorticity equation (14) along the trajectories to give Laplace's equation. This is of course linear and can be solved by known methods. The only non-linearity now remaining in the problem arises from the boundary condition requiring vanishing vertical motion on the lids (the vertical velocity w is obtained from the adiabatic equation (9) which is non-linear).

Integration to a simple linear form is only possible for the simplified vorticity equation (13) for which local variations of relative vorticity are solely due to horizontal geostrophic advection and to stretching of the earth's vorticity. The full $O(Ri^{-\frac{1}{2}})$ vorticity equation (5), including both the β and the mass divergence effects, may be integrated along the trajectories to give

$$\nabla^2 \psi' - K \frac{\partial \psi'}{\partial z} = -(\gamma + K) \hat{y} \quad (39)$$

where $\hat{y}(x, y, z, t)$ is a function defining the northward displacement of a particle from its initial position (at $t = -\infty$). Since the system must be closed by an additional equation

$$\frac{D}{Dt} \hat{y} = \frac{\partial \psi'}{\partial x} \quad (40)$$

it is still non-linear, and the potential vorticity varies between trajectories. Thus the introduction of these more realistic approximations increases the complexity of the truncation problem by introducing a further non-linear equation.

3.3 Boundary Conditions for the Functional Representation Solutions.

The linear perturbation solution ψ_1 provides an accurate representation for the growing disturbance in the early stages of its life, and to ensure continuity with this we apply the time boundary condition

$$\psi' \rightarrow \psi_1 \quad \text{as} \quad t \rightarrow -\infty$$

ψ' is a harmonic function (i.e. it satisfies Laplace's equation (38)) which satisfies the linear side boundary conditions exactly. In terms of ψ' these are

$$\frac{\partial \psi'}{\partial x} = 0 \quad , \quad \overline{\frac{\partial \psi'}{\partial y}}^x = 0 \quad (41)$$

The solution is chosen so as to satisfy, as accurately as possible, the non-linear condition of vanishing vertical motion on the lids. To ensure this $\partial \phi / \partial t$ is chosen so that on the lids it accounts, as nearly as possible, for entropy changes implied by the horizontal advective term $\underline{v} \cdot \nabla \phi$. In general an exact balance is impossible because the quadratic advective term contains higher wavenumber components than the local time derivative, which is linear.

Writing the solution in the truncated form

$$\psi' = \sum_{\lambda=1}^n T_{\lambda}(t) F_{\lambda}(x, y, z)$$

where F_l is a harmonic function satisfying the side boundary conditions (41), we minimise the truncation error by applying the method of least squares in the form

$$\frac{\partial}{\partial \dot{T}_l} \left[\overline{w^{x,y}} \left(z = \pm \frac{1}{2} \right) \right] = 0 \quad (42)$$

($l = 1, 2, 3, \dots, n$) where the dot denotes a derivative with respect to time. Due to the symmetry of the problem $\overline{w^{x,y}}$ has the same algebraic form at both the upper and lower lids, and this condition gives n first order simultaneous ordinary non-linear differential equations which may be solved to obtain the T_l 's.

It is found that the solutions so obtained do not satisfy the energy conservation equation (18) exactly, but that only a slight modification of the procedure is required to ensure this. One way of doing this is to use one of the time dependent parameters, say $\dot{T}_j$, to ensure that the energy equation is satisfied, while the others are adjusted so as to minimise $\overline{w^{x,y}} \left(z = \pm \frac{1}{2} \right)$. This results in the following set of n equation:-

$$\frac{\partial}{\partial \dot{T}_l} \left[\overline{w^{x,y}} \left(z = \pm \frac{1}{2} \right) \right] = 0 \quad (43)$$

$$\dot{E} = 0$$

($l = 1, 2, \dots, j-1, j+1, \dots, n$) where E is the total energy.

An alternative (and more general) procedure is to use the method of Lagrange multipliers to minimise $\overline{w^{x,y}} \left(z = \pm \frac{1}{2} \right)$ subject to the restriction of the energy conservation equation. This gives $n+1$ equations

$$\frac{\partial}{\partial \dot{T}_l} \left[\overline{w^{x,y}} \left(z = \pm \frac{1}{2} \right) \right] + \Lambda \frac{\partial}{\partial \dot{T}_l} [\dot{E}] = 0$$

$$\dot{E} = 0 \quad (44)$$

($\lambda = 1, 2, 3, \dots, n$) from which we may eliminate the Lagrange multiplier Λ and solve for the n parameters T_λ .

It is clear that any of the conservation properties mentioned in section 1.5 may be imposed on the truncated solution (if they are not already satisfied) by using the method of Lagrange multipliers. However it is likely that too many constraints on the minimisation process will reduce its effectiveness. Nevertheless the energy equation represents a fundamental physical relation and it is convenient to have it exactly satisfied even for an approximate solution. Furthermore numerical experiments using finite difference approximations to the equations have indicated that it is essential to use energy conserving finite difference schemes for the solution to remain realistic over a long period of time.

3.4 A Three Parameter Solution.

We write the stream function for the dominant Eady unstable wave (30), with the exponential coefficients of $S_{im}\lambda x$ and $C_{os}\lambda x$ replaced by general functions of time, and add to it a further term. (Since the terms for each wavenumber have two time dependent parameters it is convenient to denote them by R and S instead of just T .) This additional term is chosen in order to represent, as nearly as possible, the temperature changes implied by (but not accounted for by) the linear solution. Substitution of (30) into the advective terms of the adiabatic equation indicates that the extra term should be a harmonic function independent of x (cf. section 2.5). It seems that it ought to have the form

$$\sum_{n=0}^{\infty} S_{o,n}(t) Sh(2n+1)\mu z S_{im}(2n+1)\mu y \quad (m = 2n+1)$$

since it would then satisfy both the side boundary conditions and Laplace's equation. Application of condition (42), which does not ensure energy conservation, gives the result that the $S_{0,m}$'s differ only by a multiplicative constant and are proportional to $\exp[2\sigma t]$, i.e. we reproduce the form of the perturbation solution which, as we have seen, is accurate for large negative values of t . Application of either of the energy conserving conditions (43) or (44), however, leads to an infinite set of non-linear simultaneous differential equations for the $S_{0,m}$'s, $R_{1,1}$ and $S_{1,1}$, and solving them is clearly a problem of some complexity. Instead we choose to modify the additional term by truncating the series so as to exclude the shorter wave components so that, retaining only the first term, we may write down a three parameter solution

$$\begin{aligned} \psi = & -y z + S_{0,1}(t) \sinh \mu z \sin \mu y \\ & + \left(R_{1,1}(t) \sinh \nu_{1,1} z \sin \lambda x + S_{1,1}(t) \cosh \nu_{1,1} z \cos \lambda x \right) \cos \mu y \end{aligned} \quad (45)$$

This expression describes the interaction of a disturbance, having only a single eddy scale, with the zonal flow. This solution (possessing three degrees of freedom) is the simplest representation for the evolving cyclone-anticyclone wave. Three time dependent parameters are necessary to describe (i) changes in the amplitude of the wave, (ii) changes in the structure of the wave (represented mainly by the phase difference between the waves at upper and lower levels) and (iii) changes in the zonal flow.

The functions $R_{\ell,m}$ and $S_{\ell,m}$ are the time dependent parameters associated with the (ℓ,m) component of the solution (i.e. the component having zonal and meridional wavenumbers $\ell\lambda$ and $m\mu$) while $\nu_{\ell,m}^2 = (\ell\lambda)^2 + (m\mu)^2$. For the $(0,m)$ components only one time dependent parameter is necessary since $\sin 0 \lambda x$ is identically zero.

Substitution for ψ into (7) gives an expression for the apparent vertical velocity, and this has the form

$$\begin{aligned}
 w = & \sin \lambda x \left[\cos \mu y \left\{ v \dot{R}_{1,1} \cosh v z + \lambda S_{1,1} (\cosh v z - v z \sinh v z) \right\} \right. \\
 & \left. + \cos^2 \mu y \lambda \mu S_{0,1} S_{0,1} (v \sinh \mu z \sinh v z - \mu \cosh \mu z \cosh v z) \right] \\
 & + \cos \lambda x \left[\cos \mu y \left\{ v \dot{S}_{1,1} \sinh v z + \lambda R_{1,1} (v z \cosh v z - \sinh v z) \right\} \right. \\
 & \left. + \cos^2 \mu y \lambda \mu R_{1,1} S_{0,1} (\mu \cosh \mu z \sinh v z - v \sinh \mu z \cosh v z) \right] \\
 & + \left[\sin \mu y \left\{ \mu \dot{S}_{0,1} \cosh \mu z \right\} + \frac{1}{2} \sin 2 \mu y \lambda \mu v R_{1,1} S_{1,1} \right]
 \end{aligned} \quad (46)$$

while the energy equation (18) becomes

$$v \sinh v [R_{1,1} \dot{R}_{1,1} + S_{1,1} \dot{S}_{1,1}] + 2 \mu \sinh \mu S_{0,1} \dot{S}_{0,1} - \frac{8}{\pi} \cosh \mu/2 \dot{S}_{0,1} = 0 \quad (47)$$

where v has been written instead of $v_{1,1}$ for brevity.

Application of any of the minimisation procedures (42), (43), (44) gives a set of equations which enable us to determine $S_{0,1}$, $R_{1,1}$ and $S_{1,1}$ as functions of time.

3.5 An Analytic Solution without Energy Conservation.

We define the coefficients

$$a_1 = \frac{4}{3\pi} \frac{\lambda v}{\cosh \mu/2}$$

$$a_2 = \frac{\lambda}{v} \left(1 - \frac{v}{2} \tanh \frac{v}{2} \right)$$

$$a_3 = \frac{8}{3\pi} \frac{\lambda \mu}{v} \left(\mu \cosh \mu/2 - v \sinh \mu/2 \tanh v/2 \right)$$

$$a_4 = \frac{\lambda}{v} \left(\frac{v}{2} \coth v/2 - 1 \right)$$

$$a_5 = \frac{8}{3\pi} \frac{\lambda \mu}{v} \left(v \sinh \mu/2 \coth v/2 - \mu \cosh \mu/2 \right)$$

where $a_2 q_4 = \sigma^2$ and $q_2/q_4 = b^2$. The a_i 's and also $a_3 q_4 - a_1 q_5$ are all positive for $u < 2.4$.

Minimising $\overline{w^{x,y}}$ on the lids with respect to $\dot{S}_{0,1}$, $\dot{R}_{1,1}$, and $\dot{S}_{1,1}$ using (42) gives the following equations** for a non energy conserving system (subsequently referred to as NE)

$$\dot{S}_{0,1} + a_1 R_{1,1} S_{1,1} = 0 \quad (48)$$

$$\dot{R}_{1,1} + a_2 S_{1,1} - a_3 S_{1,1} \dot{S}_{0,1} = 0 \quad (49)$$

$$\dot{S}_{1,1} + a_4 R_{1,1} - a_5 R_{1,1} \dot{S}_{0,1} = 0 \quad (50)$$

whose solutions are found as follows.

Multiplying (49) and (50) by $R_{1,1}$ and $S_{1,1}$ respectively and substituting for $(R_{1,1}, S_{1,1})$ from (48) we obtain the equations

$$R_{1,1} \dot{R}_{1,1} - \frac{a_2}{q_1} \dot{S}_{0,1} + \frac{a_3}{q_1} S_{0,1} \dot{S}_{0,1} = 0$$

$$S_{1,1} \dot{S}_{1,1} - \frac{a_4}{q_1} \dot{S}_{0,1} + \frac{a_5}{q_1} S_{0,1} \dot{S}_{0,1} = 0$$

which become on integration with respect to time

$$R_{1,1}^2 - \frac{2a_2}{q_1} S_{0,1} + \frac{a_3}{q_1} S_{0,1}^2 = C^2 \quad (51)$$

$$S_{1,1}^2 - \frac{2a_4}{q_1} S_{0,1} + \frac{a_5}{q_1} S_{0,1}^2 = D^2 \quad (52)$$

where C^2 and D^2 are constants.

** It is interesting to note the similarity of these equations with Euler's equations for a spinning top, probably one of the simplest non-linear systems in classical mechanics.

$R_{1,1}$ and $S_{1,1}$ are now eliminated from the three equations (48), (51) and (52) to give an equation involving only $S_{0,1}$

$$(\dot{S}_{0,1})^2 = (q_1 C^2 + 2q_2 S_{0,1} - q_3 S_{0,1}^2)(q_1 D^2 + 2q_4 S_{0,1} - q_5 S_{0,1}^2) \quad (53)$$

Applying the initial conditions $R_{1,1} \rightarrow 0$, $S_{1,1} \rightarrow 0$ and $S_{0,1} \rightarrow 0$ as $t \rightarrow -\infty$ we find that $C^2 = D^2 = 0$ and (53) may be written in the integral form

$$t = (q_3 q_5)^{-1/2} \text{Re} \int_{\alpha_1}^{S_{0,1}} \alpha^{-1} (\alpha_1 - \alpha)^{-1/2} (\alpha_2 - \alpha)^{-1/2} d\alpha \quad (53a)$$

where α is a dummy variable, and $\alpha_1 = 2a_2/a_3$ and $\alpha_2 = 2a_4/a_5$ are branch points of the integrand (the former being closer to the origin). Note that the ambiguity of sign is included in this expression since the integration may be performed along two paths, one of which encircles the branch point α_1 . Clearly $t \rightarrow \pm\infty$ as $S_{0,1} \rightarrow 0$ and so, regarded as a function of t , $S_{0,1}$ takes an infinite time to rise from zero (at $t = -\infty$) to its maximum value $2a_2/a_3$ (at $t = 0$) and then decreases, becoming zero again at $t = \infty$. (The expression (53a) is only valid for $\nu < 2.4$, however equations (48 - 50) have a short wave solution (i.e. $\nu > 2.4$) for which t is expressed in terms of $S_{0,1}$ as an elliptic integral.)

The integration of (53a) can be performed analytically using standard methods, whence we obtain on inversion

$$S_{0,1} = \frac{4q_2 q_4}{(q_3 q_4 - q_2 q_5) \text{Ch} 2\sigma t + (q_3 q_4 + q_2 q_5)}$$

$R_{1,1}$ and $S_{1,1}$ are given in terms of $S_{0,1}$ by (54)

$$R_{1,1} = \begin{cases} + \\ - \end{cases} \left(\frac{1}{q_1} S_{0,1} (2q_2 - q_3 S_{0,1}) \right)^{1/2} \begin{cases} t \leq 0 \\ t > 0 \end{cases}$$

$$S_{1,1} = - \left(\frac{1}{q_1} S_{0,1} (2q_4 - q_5 S_{0,1}) \right)^{1/2}$$

Graphs of these functions are shown in Fig 3.1 .

When t is large and negative we find that

$$S_{0,1} \sim K^2 \frac{2\lambda v b}{3\pi\sigma} \frac{1}{ch^{1/2}} e^{2\sigma t}$$

$$R_{1,1} \sim K b e^{\sigma t}$$

$$S_{1,1} \sim -K e^{\sigma t}$$

where

$$K = 4 a_4 \left[\frac{a_2}{a_1} \frac{1}{(a_3 a_4 - a_2 a_5)} \right]^{1/2}$$

showing that the solution starts with the structure of the growing perturbation solution. As $t \rightarrow \infty$ the solution takes a similar form except that $R_{1,1}$ has the opposite sign and the exponents of the exponentials are negative, i.e. at large values of t the solution decays and has the structure of Eady's diminishing wave (see section 2.8). This change in structure during the life of the disturbance is consistent with the results of the third order perturbation solution (section 2.7) which show that the phase difference between the upper and lower wave decreases and then changes sign. However for the NE solution the final state is identical with the initial undisturbed state leading to the rather suprising result that the dynamical process is reversible and achieves no permanent energy exchange.

The equations (43-50) satisfy two exact integrable relationships:-

(i)

$$\begin{aligned} & 2Shv (R_{1,1} \dot{R}_{1,1} + S_{1,1} \dot{S}_{1,1}) + 2\mu Sh\mu S_{0,1} \dot{S}_{0,1} \\ & - \frac{8}{\pi} ch^{1/2} \dot{S}_{0,1} + \frac{3\pi}{4} \left(\frac{32}{3\pi^2} - 1 \right) ch^{1/2} \dot{S}_{0,1} = 0 \end{aligned} \quad (55)$$

and (ii)

$$\begin{aligned} & v^2 (C_h^2 \frac{\mu}{2} R_{1,1} \dot{R}_{1,1} + S_h^2 \frac{\mu}{2} S_{1,1} \dot{S}_{1,1}) + 2 \mu^2 C_h^2 \frac{1}{2} S_{0,1} \dot{S}_{0,1} \\ & - \frac{8}{\pi} C_h \frac{\mu}{2} \dot{S}_{0,1} + \frac{3\pi}{4} \left(\frac{32}{3\pi^2} - 1 \right) C_h \frac{\mu}{2} \dot{S}_{0,1} = 0 \end{aligned} \quad (56)$$

The first of these is identical to the energy conservation equation (47) except for the last term. Thus, although the solution NE does not conserve energy exactly, it does satisfy a 'pseudo energy equation' of a similar form. The total energy is shown in Fig 3.2 as a function of time, and, as can be seen, the departure from its initial value is small (about 10% at the maximum). Evidently the last term of (55) does not have a great effect.

The second relationship (56) resembles the conservation equation

$$\frac{d}{dt} \frac{1}{2} \overline{\phi^2}^{x,y} (z = \pm 1/2) = 0$$

in much the same way (see section 1.5).

3.6 The Energy Conserving Solution E1.

We apply the energy conserving condition (43) using $\dot{S}_{0,1}$ to ensure that the energy equation is satisfied and $\dot{R}_{1,1}$ and $\dot{S}_{1,1}$ to minimise $\overline{w^2}^{x,y} (z = \pm 1/2)$. We then arrive at the following set of equations (which we call E1) :-

$$\dot{S}_{0,1} + a_1 R_{1,1} S_{1,1} \frac{(S_{0,1} S_h \frac{1}{2} - 3\pi/16\mu)}{(S_{0,1} S_h \frac{1}{2} - 2/\pi\mu)} \quad (57)$$

$$\dot{R}_{1,1} + a_2 S_{1,1} - a_3 S_{1,1} S_{0,1} = 0 \quad (58)$$

$$\dot{S}_{1,1} + a_4 R_{1,1} - a_5 R_{1,1} S_{0,1} = 0 \quad (59)$$

The last two equations are identical with equations (49) and (50) and the first, rewritten

$$\dot{S}_{0,1} + a_1 R_{1,1} S_{1,1} + R_{1,1} S_{1,1} \frac{(1 - \frac{3\pi^2}{32})}{(\frac{\pi}{2} \mu S_{0,1} S_{1,1}^{1/2} - 1)} = 0 \quad (60)$$

is clearly similar to the first NE equation (48) when the parameter $S_{0,1}$ is small. We conclude that the two solutions start in much the same way, but further examination is needed to establish the behaviour of E1 the amplitude becomes large.

Because $\frac{3\pi}{16\mu} < \frac{2}{\pi\mu}$, the numerator of the fraction in the second term of the first equation (57) is smaller than the denominator and so when $S_{0,1}$ approaches the value $3\pi/(16\mu S_{1,1}^{1/2})$, $S_{0,1}$ becomes small irrespective of the values of $R_{1,1}$ and $S_{1,1}$. Consequently this is an upper bound for $S_{0,1}$, the physical significance of which is discussed in section 3.8. (The vanishing of the denominator would correspond to the disappearance of the mean N-S temperature gradient, and this is never achieved.) The closeness of the zeros of the denominator and the numerator, i.e. $(\frac{2}{\pi\mu} \frac{1}{S_{1,1}^{1/2}} - \frac{3\pi}{16\mu} \frac{1}{S_{1,1}^{1/2}})$ is small compared with the maximum value of $S_{0,1}$, means that the solution 'notices' the upper bound relatively suddenly.

No analytic solutions could be found for E1 and the equations were solved numerically on a computer using the Runge-Kutta method (see Appendix II). The results are shown in Fig 3.1, and it is seen that, like NE, the solution is reversible in time.

3.7 The Energy Conserving Solution E2.

We now examine the application of the more general energy conserving method using Lagrange multipliers (44).

We obtain the following system of equations (referred to as E2):-

$$\begin{aligned}\dot{S}_{2,1} + a_1 R_{1,1} S_{1,1} + \Lambda \left[\frac{1}{r} + \tanh \frac{1}{2} S_{2,1} - \frac{2}{(\pi r^2 a_1 h)} \right] &= 0 \\ \dot{R}_{1,1} + a_2 S_{1,1} - a_3 S_{1,1} S_{2,1} + \Lambda \left[\frac{1}{v} + \tanh \frac{1}{2} R_{1,1} \right] &= 0 \\ \dot{S}_{1,1} + a_4 R_{1,1} - a_5 R_{1,1} S_{2,1} + \Lambda \left[\frac{1}{v} + \tanh \frac{1}{2} S_{1,1} \right] &= 0\end{aligned}$$

where

$$\Lambda = \frac{\lambda v}{2} \left(\frac{32}{3\pi^2} - 1 \right) R_{1,1} S_{1,1} / \left[S_{1,1}^2 + a_1^2 S_{2,1}^2 + 2 \left(S_{1,1} S_{2,1} - \frac{2}{\pi r^2} \right)^2 \right]$$

We see that, as with E1, the solution E2 is similar to NE for small values of the time dependent parameters.

These equations were also solved numerically using the Runge-Kutta method and the results are shown in Fig 3.1. As in the previous cases (NE and E1), E2 is reversible and returns, as $t \rightarrow +\infty$, to its initial state.

3.8 Comparison of the Different Three Parameter Solutions.

It is clear from Fig 3.1 that the solutions NE and E2 are very similar to each other, indicating that the variations in total energy implied by the pseudo energy equation (55) do not have any significant effect, and that we can regard NE as almost conserving energy. The solution E1 however behaves somewhat differently and this is discussed below.

The vertical velocity may be partitioned into a zonal mean component w_m ($= \bar{w}^x$) and an eddy component w' ($= w - w_m$). The minimisation constraints on NE and E2 are such that both $\overline{(w')^2}^{x,y}$ and $\overline{(w_m)^2}^{x,y}$ are made as small as possible at the upper and lower boundaries, but for E1 only the eddy component is minimised, whilst w_m is chosen so that the total energy is conserved. This latter condition may be shown to be equivalent to

$$\overline{[\psi_m w_m]}(z = \pm 1/4)^{x,y} = 0$$

i.e. there is no restriction on the magnitude of w_m on the lids provided that it is not correlated with ψ_m there. This additional freedom for the vertical velocity in E1 is responsible for its different behaviour.

It may be shown that

$$\overline{[(v\phi)^x \partial\phi_m/\partial y]}^{x,y,z} = \frac{1}{8} \lambda \nu R_{1,1} S_{1,1} \left\{ 1 - \frac{16 S_0^{1/2}}{3\pi} S_{0,1} \right\}$$

Early in the life of the disturbance $S_{0,1}$ is small, $R_{1,1} > 0$ and $S_{1,1} < 0$ so that $\overline{[(v\phi)^x \partial\phi_m/\partial y]}^{x,y,z}$ is negative, implying a down gradient heat flux (see Fig 3.3), consistent with the release of (zonal) potential energy. When $S_{0,1}$ reaches the value $3\pi/(16 S_0^{1/2}) \approx 1.2$ (i.e. its maximum value for the solution E1) the correlation between $(v\phi)^x$ and $\partial\phi_m/\partial y$ vanishes. Neither $(v\phi)^x$ nor $\partial\phi_m/\partial y$ is identically zero at this point (although the mean meridional entropy gradient $\partial\phi_m/\partial y$ is considerably reduced) and so we attribute the vanishing correlation to the uneven distribution of ϕ_m which is set up by the northward eddy transfer of heat (see Fig 3.5). After this stage ($t \approx -2$) the eddy is unable to release any further zonal potential energy (the correlation $\overline{[(v\phi)^x \partial\phi_m/\partial y]}^{x,y,z}$ changes sign giving an up-gradient heat transfer), perhaps because the restricted number of degrees of freedom prevents it adopting a more suitable structure, and its subsequent behaviour has a different character. This is illustrated by Fig 3.4 which shows that when t is less than about -2 the phase difference between the pressure wave at the top and bottom is almost constant in time but changes substantially afterwards (see also Fig 2.11). This is also the point at which E1 begins to behave differently from NE and E2.

Substituting the maximum value $3\pi/(16 S_0^{1/2})$ of $S_{0,1}$ for E1 into equations (57-59) we find that the solution

changes due to the vertical displacements exactly balance those due to the meridional eddy flux divergence (w_m has no direct limitation on its magnitude on the lids). On the other hand for both NE and E2 w_m is minimised on the lids, preventing a similar balance occurring, so that the northward heat flux leads to continued adiabatic heating in the north and cooling in the south. An up gradient heat transfer then occurs and zonal potential energy increases (Fig 3.8). Thus for the solutions NE and E2 to release potential energy they must drive an up gradient heat flux and so ultimately they release less potential energy than does E1 (see Fig 3.7). Furthermore NE and E2 reach their maxima earlier than E1.

3.9 The Development of the Motion Patterns for the Three Parameter Solution.

Horizontal maps of the geostrophic stream lines relative to the ground (i.e. isobars) and isotherms for E2 are shown in Fig 3.10 at intervals of 1 day. The westerly flow at upper levels becomes mainly concentrated in a belt meandering north-south, with the largest wind speeds on the trough and ridge lines near the side walls (Fig 3.10 (a)). As development proceeds closed circulations appear at upper levels and the westerly flow becomes weaker. At low levels (Fig 3.10 (b)) closed (warm) lows and (cold) highs (the former moving north and the latter south of the middle latitude as predicted by the second order perturbation solution in section 2.8) lie to the east of the upper (cold) troughs and (warm) ridges, and the whole pattern drifts eastwards with the speed of the mean zonal wind. After about $t = -2$ the phase difference decreases and the upper wave moves faster so as to catch up with the surface systems. Westerly flow

at low levels becomes considerably increased, since this is necessary in order to maintain the thermal wind balance as the temperature gradients become reduced (see section 2.6). We note that the mean meridional entropy gradient on the lids reverses near $t = -1$, thus violating the conservation equation (23) of section 1.5, and we must regard this as a sign that the solution is perhaps not realistic after this time.

3.10 Truncation Errors.

As mentioned earlier in this chapter, the only truncation errors which exist are those which arise from the inability to match entropy changes $\partial\phi/\partial t$ on the lids with those implied by $\underline{v} \cdot \nabla\phi$. In order to satisfy the adiabatic equation we interpret any components of the non-linear advection term not represented by the linear term $\partial\phi/\partial t$ as heating produced by a (spurious) vertical flow through the lids. Thus even though the vorticity equation (13) is satisfied identically, the vorticity distribution for a solution with a finite number of terms is not the same as that for an exact solution because the spurious vertical velocities through the upper and lower boundaries affect the stretching term.

After calculating the vertical flow through the lids we examine the error parameters e_1 and e_2 . We define e_1 by the following equation

$$e_1 = \frac{\int_{-1}^1 \int_{-1}^1 [h w(z = \pm \frac{1}{2})]^2 dx dy}{\int_{-1}^1 \int_{-1}^1 [\underline{v} \cdot \nabla\phi(z = \pm \frac{1}{2})]^2 dx dy} \quad (61)$$

This parameter can be regarded (roughly) as comparing entropy changes due to spurious flow through the lids with those which would occur without. If e_1 is large (i.e. close to unity or more) then the entropy changes

are poorly represented by $\partial\phi/\partial t$, but are well represented if e_1 is nearly zero.

The parameter e_2 assesses the effect of the truncation error on the stretching term. If $w(z=0)$ is the mid-level vertical velocity, then an estimate of the stretching is

$$[w(z=0) - w(z = \pm \frac{1}{2})] / \frac{1}{2}$$

while we may regard the expression

$$w(z=0) / \frac{1}{2}$$

as representing (very approximately) the stretching which would occur in the absence of truncation error.

Thus

$$e_2 = \frac{[w(z = \pm \frac{1}{2})]^2}{[w(z=0)]^2} \quad (62)$$

provides a measure of the change in the stretching term due to the truncation error compared with the stretching **calculated** ignoring the flow through the lids.

The parameters e_1 and e_2 for the solutions E1 and E2 are shown in Fig 3.9. It can be seen that both solutions become accurate for t greater than about -2, but that after the phase reversal is complete the truncation errors again become small as the disturbance decays (back to an exact solution of the full non-linear equations). However, this latter stage cannot be regarded as a realistic continuation of the behaviour of an initially growing solution, and indeed it is found (see later in Chapter 4) that the introduction of higher wavenumber components into the solution reduces the truncation error and results in a different (irreversible) behaviour at large values of t .

It is possible to consider the truncation error from a completely different point of view. Because of

the properties of Fourier series, minimisation of the mean square vertical flow through the lids has precisely the same effect as expanding w as a Fourier series and equating the leading terms to zero. In our case we equate to zero the coefficients of the terms of w which are also represented in $\partial\phi/\partial t$. Thus we may regard w as consisting of two parts, W and $\hat{W}$. W consists of all the Fourier components which vanish on the lids (usually the lower wavenumber components), while $\hat{W}$ consists of all the other components of w . We may now think of $W(x,y,z,t)$ as being the actual vertical velocity (vanishing identically on the lids), while $\hat{W}(x,y,z,t)$ is the truncation error, interpreted in the adiabatic equation as a spurious heating term Q .

Thus the adiabatic equation becomes

$$\frac{1}{h} \left[\frac{\partial \phi}{\partial t} + h W \right] = Q$$

where

$$Q = -\hat{W}$$

In a similar way the momentum equation becomes

$$\frac{1}{h} \frac{\partial \underline{v}}{\partial t} + \underline{k} \times [\underline{v} + \underline{V}_a] + \text{grad } \psi = \underline{F}$$

where

$$\frac{\partial U_a}{\partial x} + \frac{\partial V_a}{\partial y} + \frac{\partial W}{\partial z} = 0$$

and we interpret $\underline{F}(x,y,z,t)$ as a spurious 'friction force' produced by the truncation error and defined by

$$\underline{k} \cdot \nabla \times \underline{F} = \frac{\partial \hat{W}}{\partial z}$$

We note that if energy conservation is satisfied the potential energy and kinetic energy changes produced by $\underline{F}$ and Q exactly balance each other (a similar balance

holds for vorticity changes).

Fourier analysing w (given by equation (46)) we find that for the three parameter solution

$$\begin{aligned} W = & \sin \lambda \cos \mu y \left[v \dot{R}_{1,1} \cosh v z + \lambda S_{1,1} (\cosh v z - v z \sinh v z) \right. \\ & \left. + \frac{8}{3\pi} \lambda \mu S_{1,1} S_{0,1} (v \sinh \mu z \sinh v z - \mu \cosh \mu z \cosh v z) \right] \\ & + \cos \lambda \cos \mu y \left[v \dot{S}_{1,1} \sinh v z + \lambda R_{1,1} (v z \cosh v z - \sinh v z) \right. \\ & \left. + \frac{8}{3\pi} \lambda \mu R_{1,1} S_{0,1} (\mu \cosh \mu z \sinh v z - v \sinh \mu z \cosh v z) \right] \\ & + \sin \mu y \left[\mu \dot{S}_{0,1} \cosh \mu z + \frac{4}{3\pi} \lambda \mu v R_{1,1} S_{1,1} \right] \end{aligned}$$

and

$$\begin{aligned} \hat{W} = & \sin \lambda x \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos(2n+1)\mu y}{(2n-1)(2n+1)(2n+3)} \left[\frac{8}{\pi} \lambda \mu S_{1,1} S_{0,1} (v \sinh \mu z \sinh v z - \mu \cosh \mu z \cosh v z) \right] \\ & + \cos \lambda x \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos(2n+1)\mu y}{(2n-1)(2n+1)(2n+3)} \left[\frac{8}{\pi} \lambda \mu R_{1,1} S_{0,1} (\mu \cosh \mu z \sinh v z - v \sinh \mu z \cosh v z) \right] \\ & + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(2n+1)\mu y}{(2n-1)(2n+3)} \left[\frac{4}{\pi} \lambda \mu v R_{1,1} S_{1,1} \right] \end{aligned}$$

This treatment of the truncation error avoids the concept of flow through a 'rigid lid', and could be used to investigate more closely the physical consequences of the truncation. There is close similarity between the artificial friction and heating terms here and the notion of identifying residuals as constraints, introduced when solving an equation by relaxation methods (see Southwell (1946)). A closer study of this aspect is desirable, but we do not pursue it further in this thesis.

3.11 Improving the Accuracy of the Solutions.

The method of functional representation produces a solution by generating a sequence of approximate solutions each of which involves only a limited number of wavenumber components - the simplest of which is the three parameter solution. Since we are interested in the convergence of this sequence we examine the truncation error in order to find out which new wavenumber component to include so as to obtain the greatest improvement in accuracy. However the introduction of a new component alters the nature of the truncation error (which may perhaps be larger than before), and it is difficult to arrive at an a-priori decision as to which new component to include.

The method adopted here is to Fourier analyse the truncation error (i.e. $\hat{W}$) and find the largest term. A new component, having the same wavenumbers as this term, is then added to the expression for ψ - a procedure analogous to that of reducing the largest residual in a relaxation problem (Southwell (1946)). This method is somewhat subjective since since different components are large at different times. We have therefore concentrated our attention on the mature phase of the development of the disturbance and the choice is then fairly clear cut, at least until we reach an eight parameter solution.

After introducing one new component into the solution the appropriate equations were solved and a calculation performed to find out which further component should be included. The order in which the components were introduced is shown in Table 1. It is found that the terms appear in ascending order of $\mathcal{W}_{\ell,m}^2$ (where $\mathcal{W}_{\ell,m}^2 = (\ell\lambda)^2 + (m\mu)^2$). This seems to be a reasonable criterion to adopt for improvements after the eight parameter solution. (After the eight parameter solution it becomes difficult to pick

out the largest Fourier component because there are many to choose from, and also because no single wavenumber component dominates the truncation error for a long period of time.)

The squares of the total wavenumber for the components of the three parameter solution are distinctly smaller than those for the additional components which make up the four, six and eight parameter solutions. In a similar way a jump occurs in the value of $\nu_{l,m}^2$ between the eight and ten parameter solutions. This suggests that the three and eight parameter solutions might form relatively well balanced approximations to the exact solution.

Table 1.

(l,m) component	$\nu_{l,m}^2$	no. of parameters
(0,1)	1.0	} 3
(1,1)	3.3	
(0,3)	9.0	4
(1,3)	11.3	6
(2,2)	13.0	8
(3,1)	21.3	10

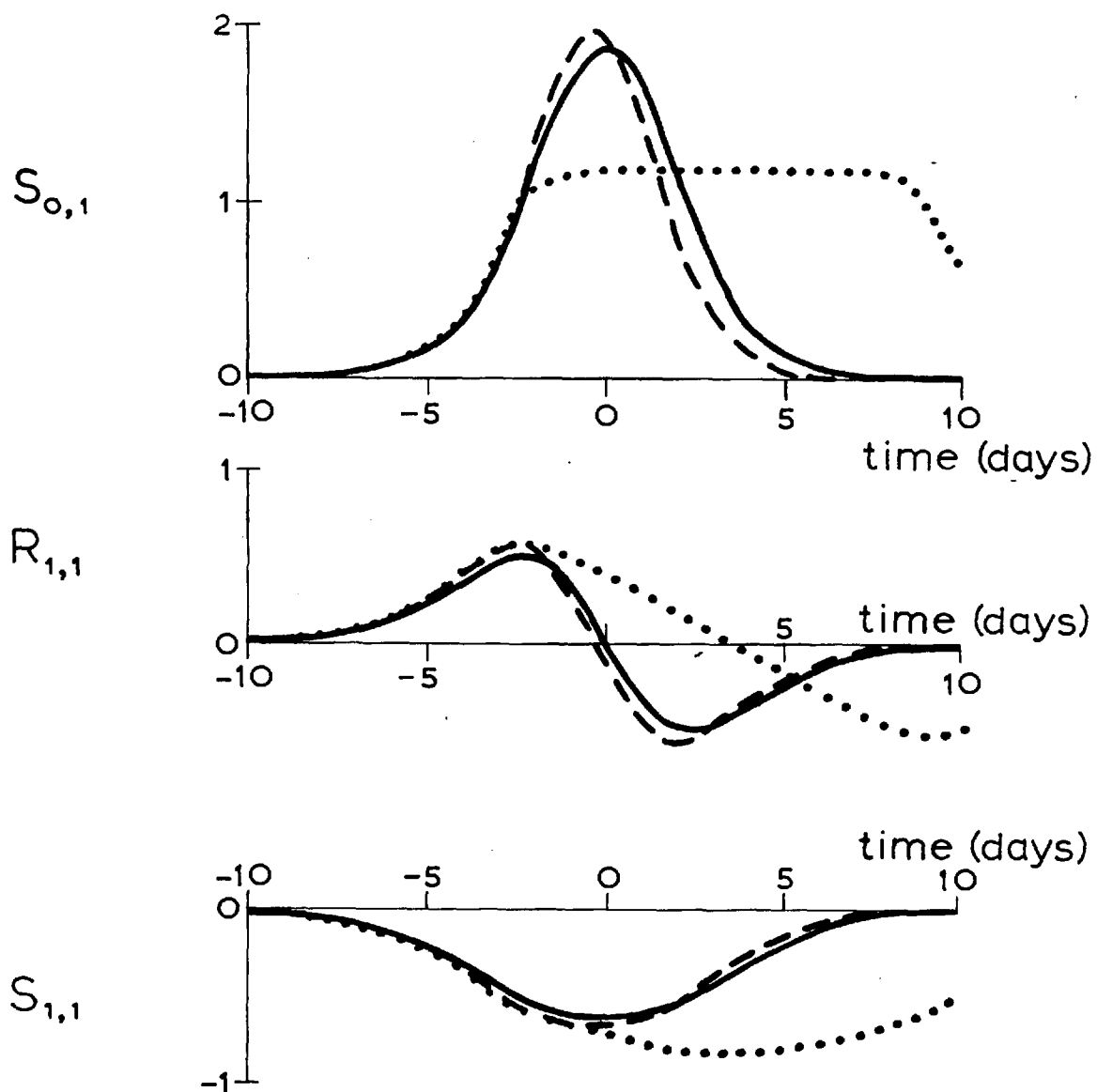


Fig 3.1 Variation of the functions $S_{0,1}$, $R_{1,1}$, $S_{1,1}$ with time for the three parameter solutions NE (— — —), E1 (.....) and E2 (- - -).

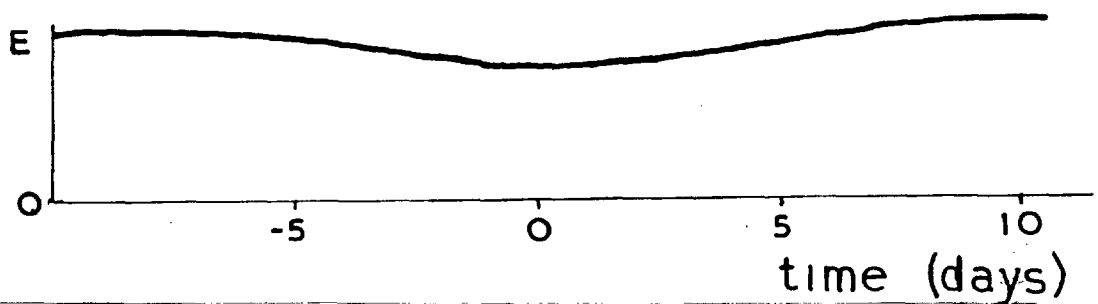


Fig 3.2 the total energy E as a function of time for the three parameter NE solution.

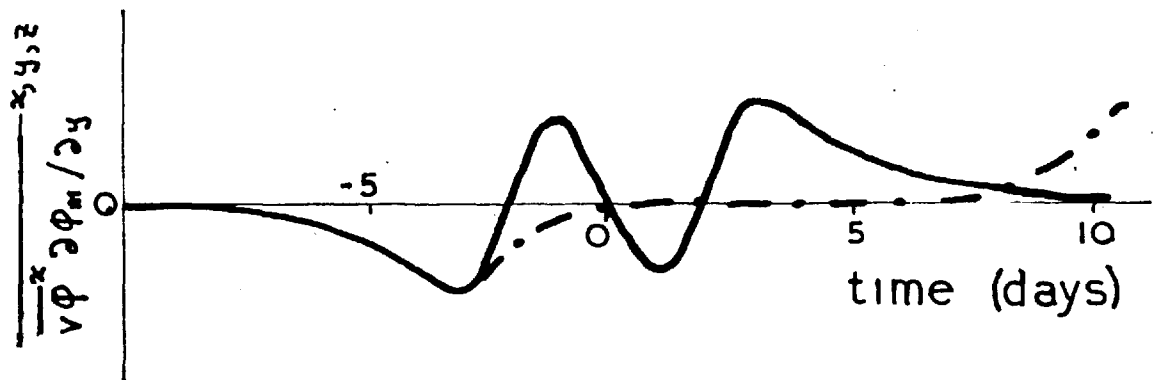


Fig 3.3 Variation with time of the correlation between the eddy heat flux $\overline{v\phi}^x$ and the zonally averaged meridional entropy gradient $\partial\phi_m / \partial y$ for the three parameter solutions E1 (---) and E2 (—).

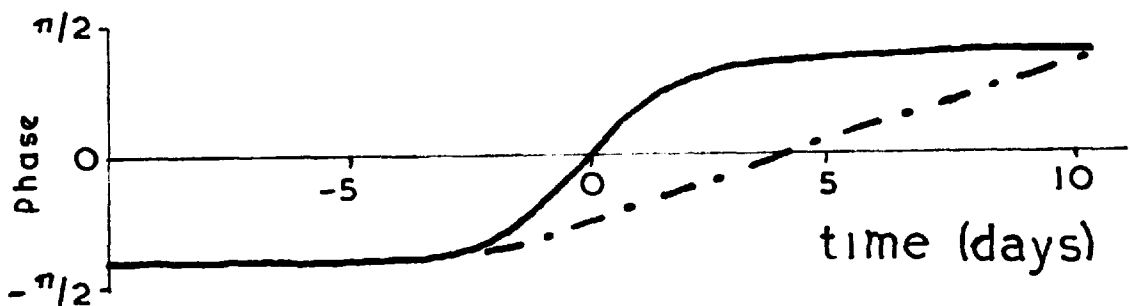


Fig 3.4 Phase difference between the pressure waves on the upper and lower lids shown as a function of time for the solutions E1 (---) and E2 (—).

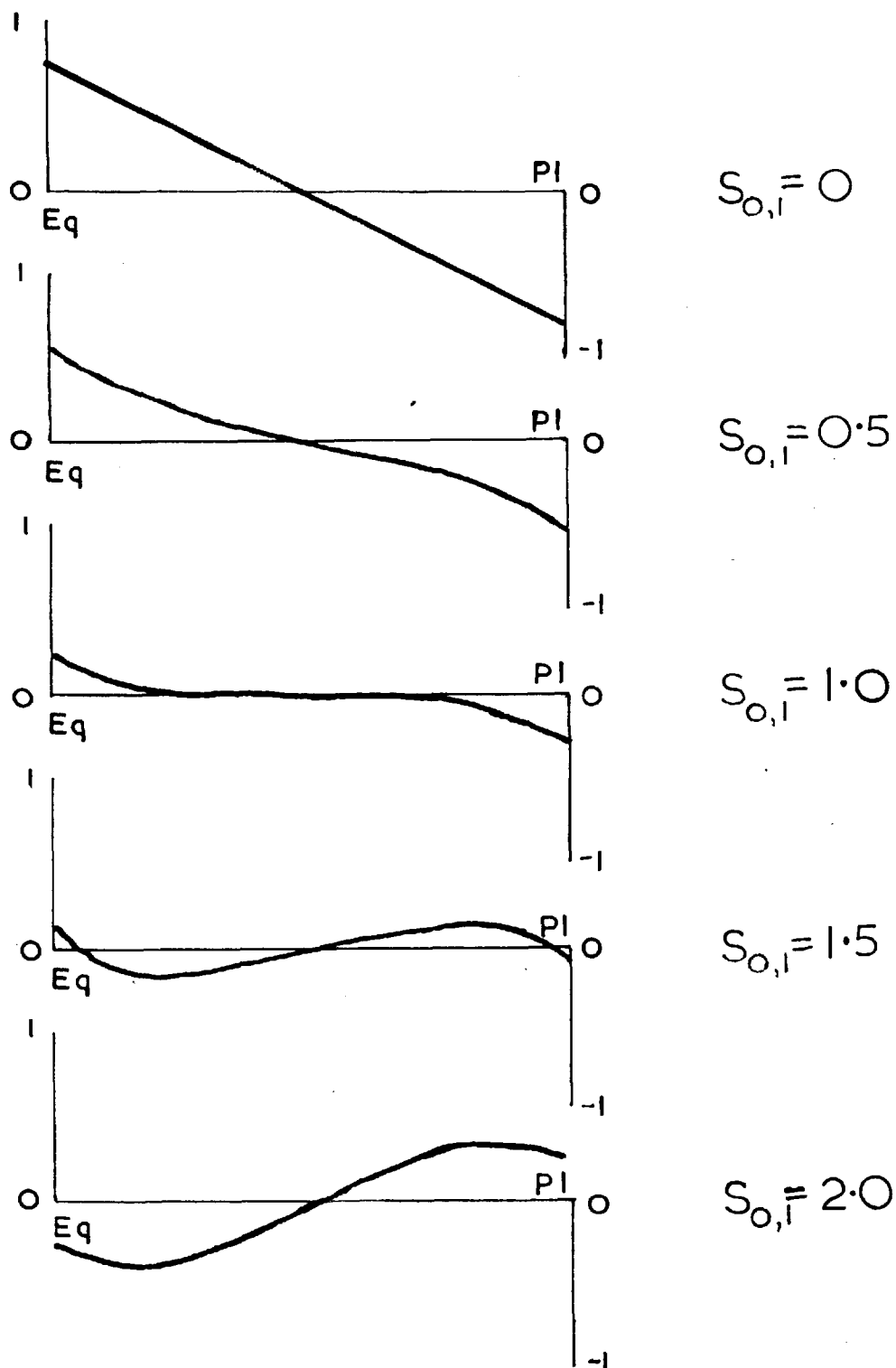


Fig 3.5 Variation of $\overline{\phi}^{x,z} = -y + 2 S_{0,1} S_{h/2} \sin y$ with latitude for various values of $S_{0,1}$.

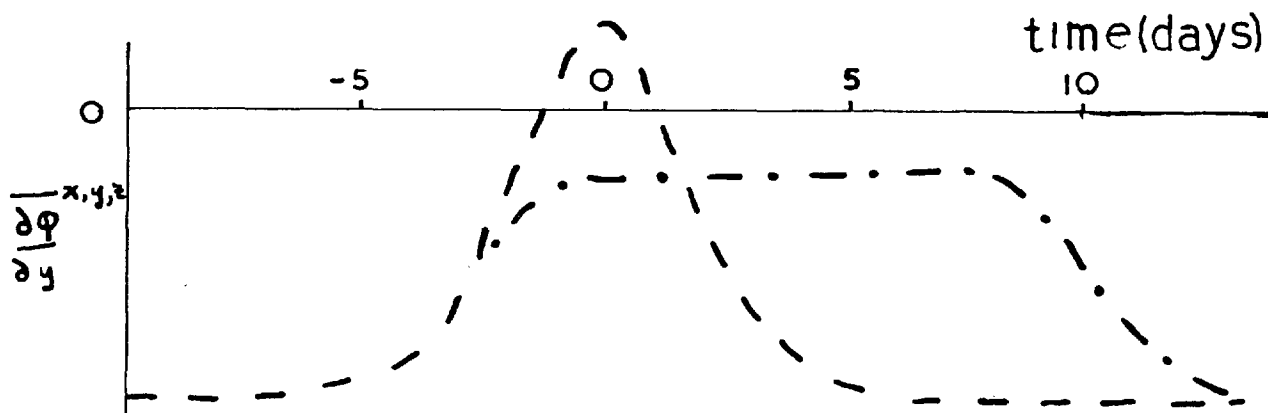


Fig 3.6 The mean meridional entropy gradient $\frac{\partial \phi}{\partial y}$ shown as a function of time for the three parameter solutions E1 (- · - ·) and E2 (- - -).

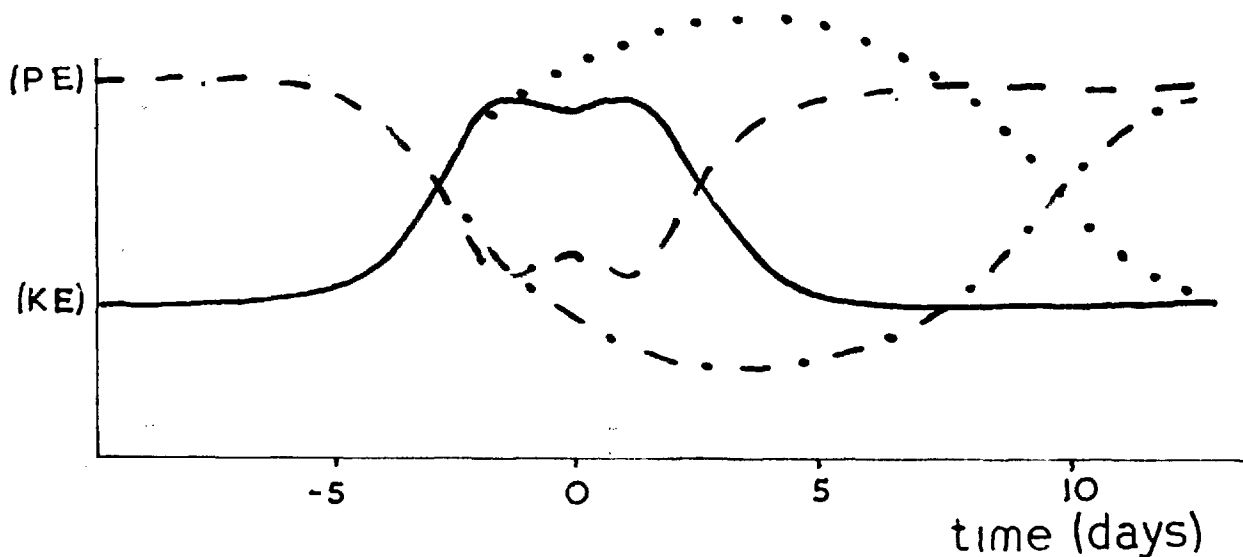


Fig 3.7 Kinetic and available potential energies per unit volume shown as a function of time for the three parameter solutions E1 and E2.

E1: (KE) - · - · ; (PE) - · - ·
 E2: (KE) - - - ; (PE) - - -

Fig 3.8 Variation of (a) zonal potential energy, and
(b) eddy potential energy per unit volume with time
for the three parameter solutions E1 (- · - ·) and
E2 (- - -).

Fig 3.9 Variation of the error parameters e_1 and e_2
with time for the three parameter solutions E1 (.....)
and E2 (——).

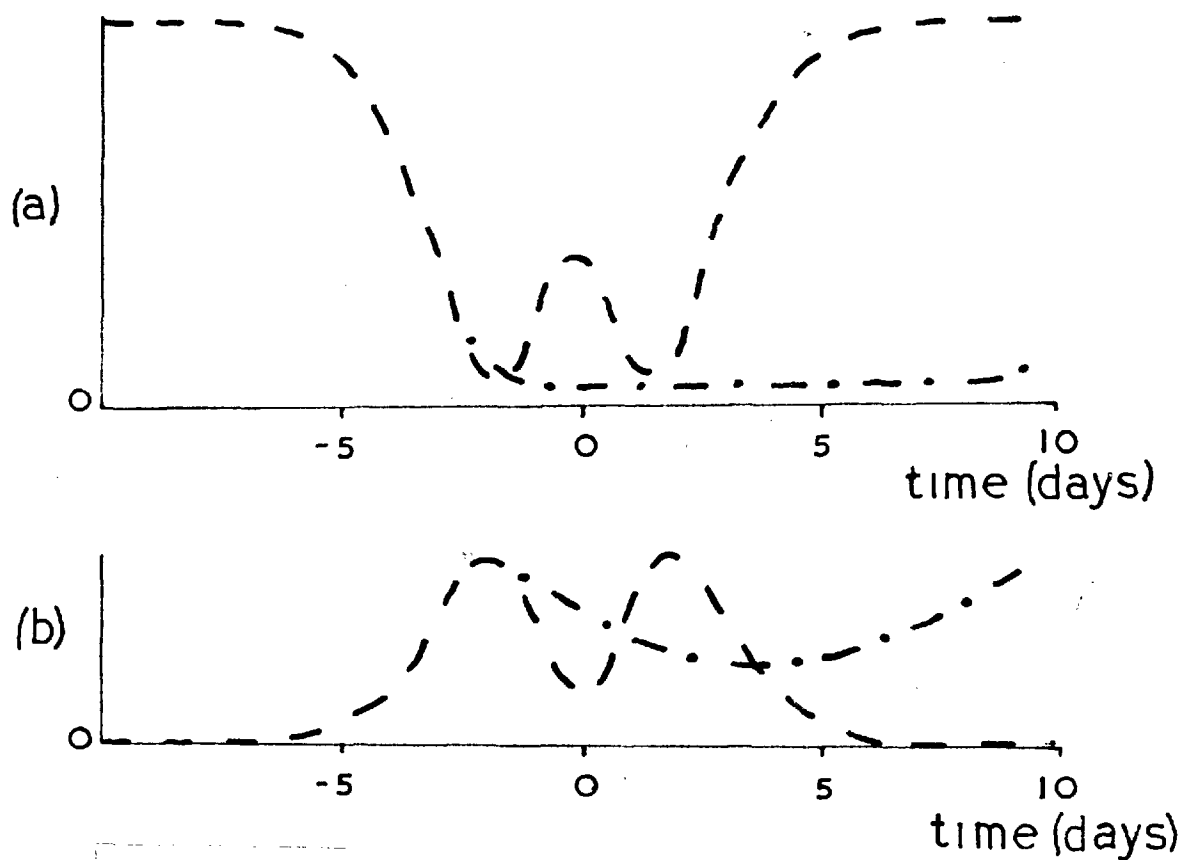


Fig 3.8

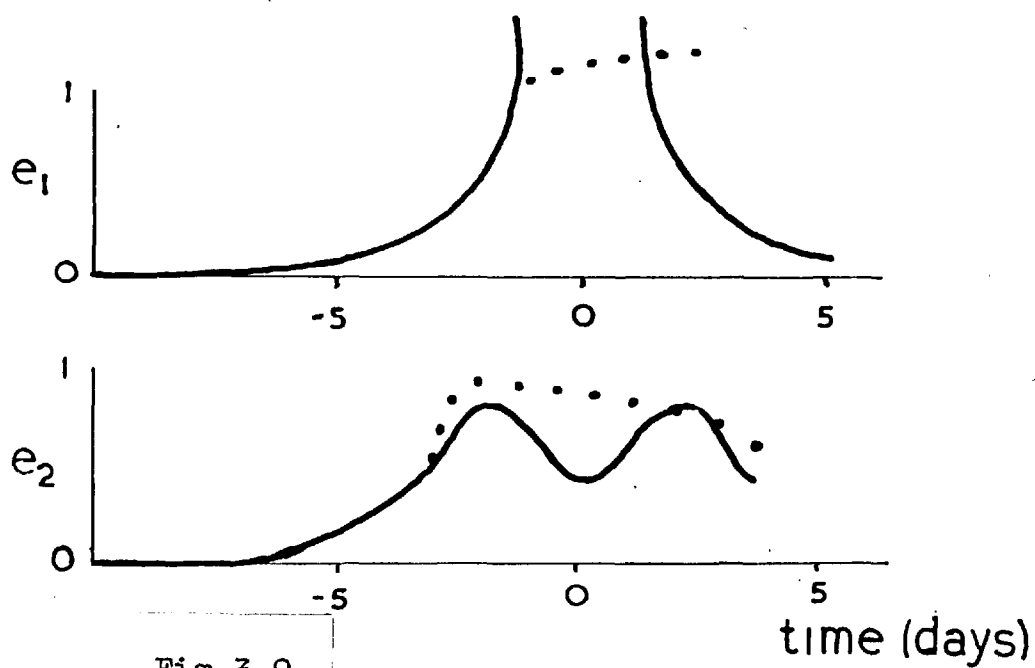
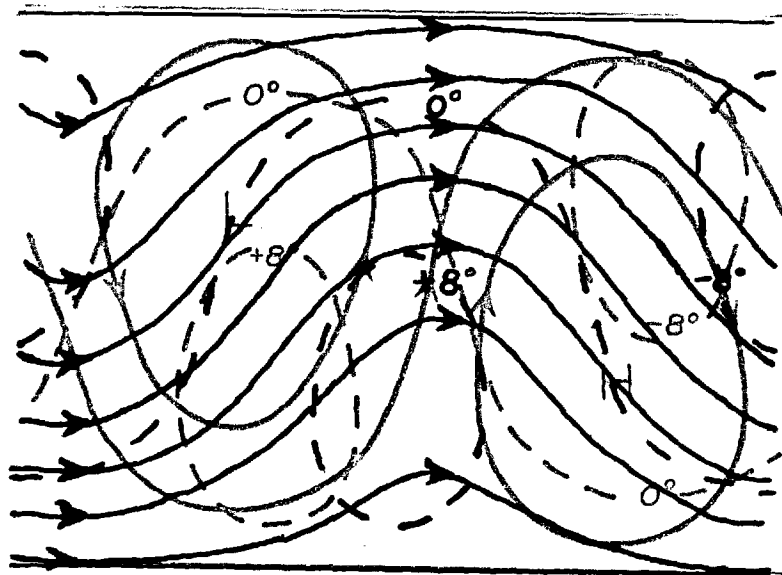


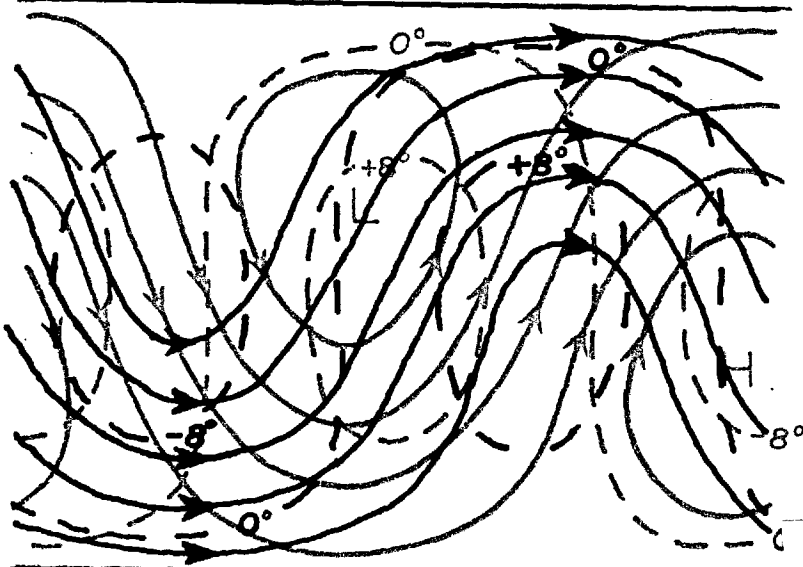
Fig 3.9

Fig 3.10 (a) Stream lines for the geostrophic flow relative to the ground (——) and isotherms (— — —) at the upper lid for the three parameter solution E2 shown every day.

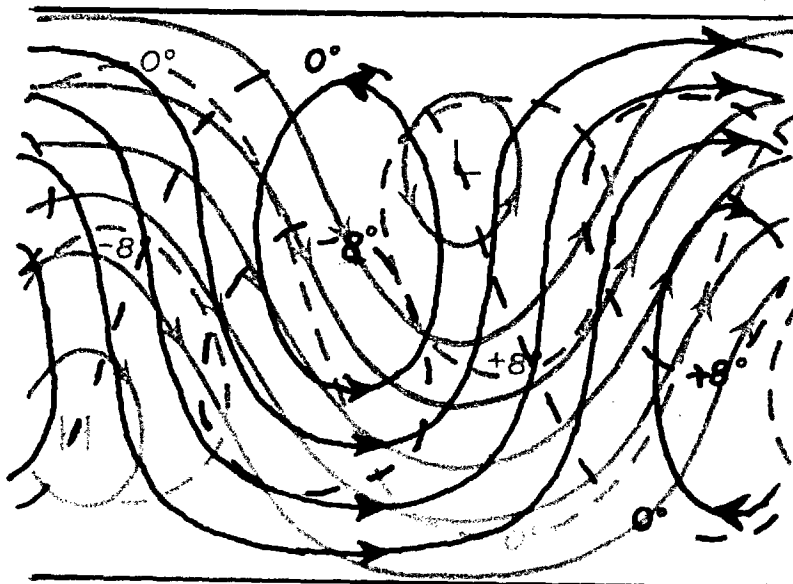
Fig 3.10 (b) Stream lines for the geostrophic flow relative to the ground (——) and isotherms (— — —) at the lower lid for the three parameter solution E2 shown every day.



$t=3$

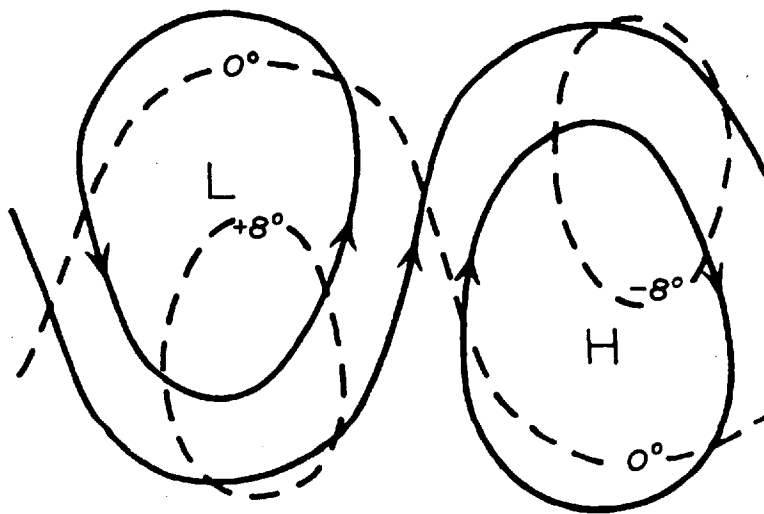


$t=2$

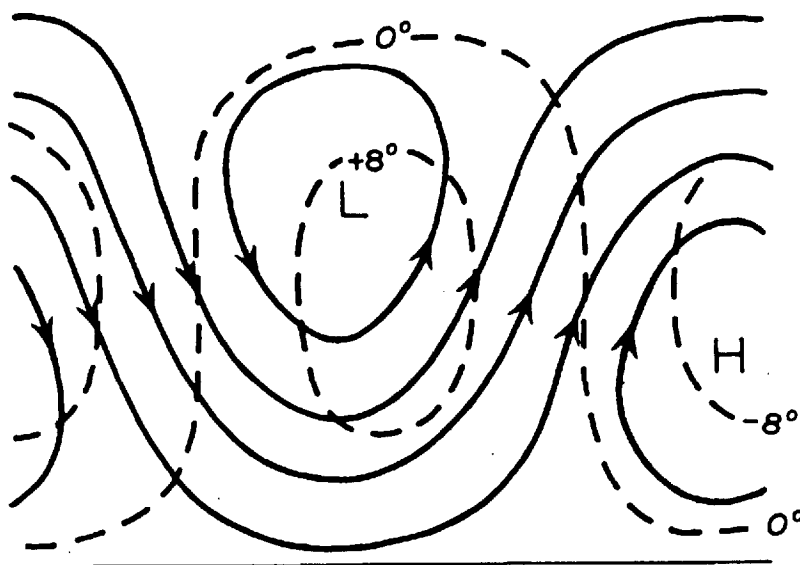


$t=1$

Fig 3.10 (a)
page 88.

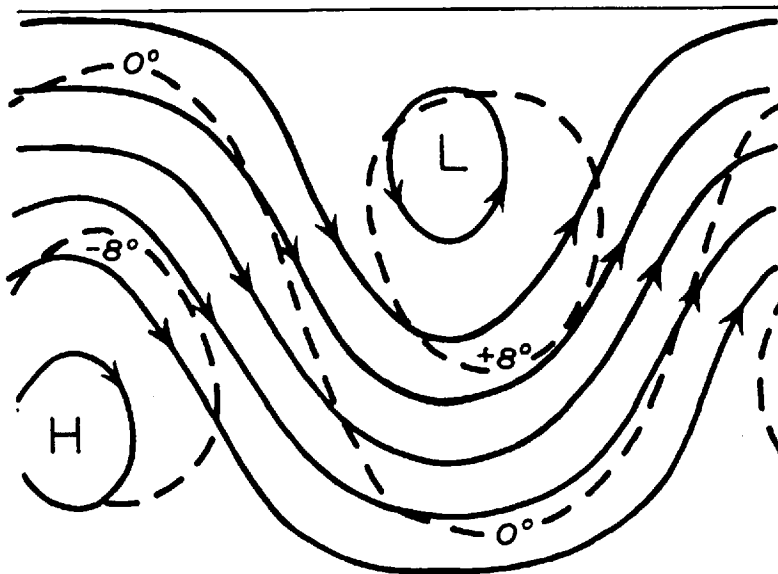


$t = -3$



$t = -2$

Fig 3.10 (b)
see opposite
page 88.



$t = -1$

CHAPTER 4. HIGHER WAVENUMBER COMPONENTS.

4.1 The Eight Parameter Solution.

The eight parameter solution has the form

$$\begin{aligned}\psi = & -yz + S_{0,1} Sh_{\mu} z \sin \mu y + S_{0,3} Sh_{3\mu} z \sin 3\mu y \\ & + (R_{1,1} Sh_{\nu,1} z \sin \lambda x + S_{1,1} Ch_{\nu,1} z \cos \lambda x) \cos \mu y \\ & + (R_{1,3} Sh_{\nu,3} z \sin \lambda x + S_{1,3} Ch_{\nu,3} z \cos \lambda x) \cos 3\mu y \\ & + (R_{2,2} Ch_{\nu,2} z \sin 2\lambda x + S_{2,2} Sh_{\nu,2} z \cos 2\lambda x) \sin 2\mu y\end{aligned}$$

and the time dependent parameters $S_{0,1}$, $S_{0,3}$, $R_{1,1}$, $S_{1,1}$, $R_{1,3}$, $S_{1,3}$, $R_{2,2}$, $S_{2,2}$ are found by solving the set of ordinary differential equations provided by the minimisation procedure (44) (imposing energy conservation by Lagrange multipliers).

The inclusion of higher wavenumber components in the solution does not change the broad features of the flow predicted by the three parameter solution during the growing stage of the life of the disturbance (see section 3.9), but it produces deformations in the flow field (see Figs 4.1 (a) and (b)) and gives an improved representation for the temperature changes. Thus the truncation error, as defined by e_1 (61), is considerably smaller (Fig 4.6) and the solution is more accurate. (Unfortunately the error parameter e_2 (62) was not calculated for the eight parameter solution.)

The variation of potential and kinetic energy is shown in Fig 4.5. We see that, unlike the three parameter solution (Fig 3.7), the eight parameter solution is

not reversible since the kinetic energy oscillates in time, successive maxima having slightly different amplitudes. Evidently the kinetic energy is shared between components with different wavenumbers and the disturbance never becomes sufficiently organised to transfer all the heat back to its original latitude (although the minimum kinetic energy attained is almost equal to the initial value). One may speculate that the inclusion of more wavenumber components in the calculation makes this eventual disorganisation more pronounced, and that a solution with a very great number of degrees of freedom has a final state which is 'turbulent', the kinetic energy being distributed among (and transferred between) eddies of many different scales. However in a more realistic model of the atmosphere friction prevents this happening and a radically different final state occurs (see Chapter 5).

The surface low (Fig 4.1 (b)) remains in the warm air during its growth while the cold air flows round it. At $t = -2$ and $t = -1$ days the cold air progresses eastwards south of the surface low, and ahead of this cold tongue the stream lines are more sharply curved forming a trough line which moves forwards around the low to adopt a NW-SE orientation. Its behaviour is similar to that of a cold front in the atmosphere, while the relatively straight stream lines north of the trough resemble isobars in the warm sector of a cyclone (see section 4.2). It is difficult to find a feature to identify with a warm front. At $t = -1$ the trough line begins to disappear and at the stage of maximum development ($t = 0$) there is little evidence of it at all, although there is an indication of a trough forming to the south west of the cyclone and to the west of the original trough line (a secondary cold front ?).

By this time, however, the mean meridional temperature gradient has changed sign and so a detailed interpretation is unjustified (according to equation (23) the mean meridional entropy gradient must remain constant on the lids).

The cyclones and anticyclones are mirror images of each other, unlike the atmosphere where the highs tend to be spread out over a larger area than the lows. This may be attributed to the neglect of certain physical mechanisms, such as cross isobar advection (i.e. advection with the ageostrophic wind), which enable the cyclonic vorticity to become more concentrated while the anticyclonic vorticity becomes more diffuse. An approximate method of introducing such effects is discussed in Chapter 6.

Like the trough line described above, a ridge oriented in a NE-SW direction becomes established north of the surface high. Fig 4.7 illustrates how the orientation of the troughs and ridges at low levels is such as to produce a net positive correlation between u and v in the north and a negative correlation in the south. As indicated in Fig 4.7, west of the low at northern latitudes the product of the small westerly wind component and the southward component gives a small negative contribution to $\overline{uv}$, but east of the low the combination of the relatively large westerly component of the flow and the northerly component gives a much larger positive contribution to $\overline{uv}$. A similar argument holds south of the middle latitude, and this results in a net flux of westerly momentum towards the sides.

At upper levels (Fig 4.1 (a)) the (cold) troughs and (warm) ridges, initially symmetric waves, become sharper on their leading edges. Warm air flows round north of the upper troughs in a manner which is reminiscent of spiral cirrus cloud patterns seen on satellite

photographs of cyclones, while the cyclonic vorticity (Fig 4.2 (a)) moves round south of the trough with the maximum concentrated in the southernmost tip of the wave. We can see (Fig 4.7) that the orientation of the upper troughs and ridges is such as to produce a net flux of westerly momentum towards the central latitude and away from the sides.

During the growing stage of the disturbance the upward motion at mid-levels (Fig 4.2 (b)) occurs in the warm air with the maximum lying a little ahead of the surface cyclone but extending southwards over the warm sector. The downward flow bears a similar relationship to the cold air, but in particular we note that in the vicinity of the surface cyclone descent occurs above the advancing cold air west of the surface trough line. At $t = -1$, however, the vertical velocity pattern breaks up and takes on quite a different appearance, with updraughts to the north, east and west of the surface cyclone and downward motion to the south.

The zonal average cross sections (Fig 4.3) for the eight parameter solution are very similar to that predicted by the second order perturbation solution (see section 2.6 and Fig 2.5), with two regions of high speed westerly flow aloft close to the side walls where the horizontal temperature gradients (and vertical shear) are large, and a region of lesser shear at mid-latitudes. After $t = -1$ the vertical shear and meridional temperature gradient change sign in the central region although the average shear remains westerly until $t = 0$.

4.2 Comparison with the Atmosphere and Numerical Models of the Atmosphere.

Although no particular cyclone in the atmosphere can

be regarded as being typical, it is usually possible to pick out certain features which are common to most cyclones. On November 24th. 1952 a severe cyclone developed over north-eastern Texas and its subsequent life was analysed by Petterssen and Bradbury (1954). We compare this developing depression (Fig 4.4 (a), (b) and (c)) with the results for the eight parameter solution described in the previous section.

The cyclogenesis occurred in a strong horizontal temperature gradient which existed over the United States on November 24th. 1952. As the development proceeded the cold upper trough (Fig 4.4 (a)), which was initially lying to the west of the intensifying frontal wave at the surface, moved quickly eastwards so as to catch up the evolving surface cyclone. It is particularly noticeable that the orientations of the upper trough and the distribution of cyclonic vorticity (with the maximum concentrated in the southern tip of the wave) on November 25th. and 26th. at 1500Z are very similar to those predicted by the eight parameter solution at $t = -2$ and $t = -1$ days (Fig 4.2 (a)).

Petterssen and Bradbury analysed the 1000mb. contours and the 1000-500mb. thickness (the latter of course being equivalent to the mean temperature field between the two pressure surfaces) and these are reproduced in Fig 4.4 (c). We see that the surface low, which moved towards the north, remained in the warm air while the cold air flowed round it producing a cold tongue advancing eastwards south of the surface pressure centre (November 26th. 1500Z) behind the occluding cold front (cf. Fig 4.1 (b) at $t = -2$ and $t = -1$). Petterssen and Bradbury did not publish any vorticity patterns at the ground, and in view of the complex frontal structure it is possible that they have a different appearance to those for the theoretical solution.

Petterssen and Bradbury calculated a vertical velocity distribution at 850 mbs. by integrating the vorticity equation (in a slightly more general form than ours) up from the ground. After comparing this with the distribution of rainfall they concluded that it bore some resemblance to the actual vertical motion. The updraughts (see Fig 4.4 (b)) occurred to the north of the developing cyclone and a little ahead of it but extended southwards in the warm air. However when occlusion began to take place at the surface (November 26 th. 1500Z) the pattern became somewhat different with widespread descending air south of the pressure centre and a general region of ascent to the north. It can be seen that this description is rather similar to that of the vertical velocities for the eight parameter solution (Fig 4.2 (b)).

It is evident that in spite of the quasi-geostrophic approximations, the idealised initial state and boundary conditions and the algebraic simplicity of the eight parameter solution it has been possible to predict features which bear considerable resemblance to those actually occurring in the atmosphere. The principal disagreement seems to be in the relative behaviour of the cyclones and anticyclones (the latter being relatively inactive in this case study) and in the layers close to the ground (where the theoretical solution does not depict a detailed frontal structure).

Results from the numerical (quasi-geostrophic) models of the atmosphere of Phillips (1956) and Huss and Mintz (1960) are illustrated in Figs 4.8 and 4.9. Like the eight parameter solution, they both predict the surface lows to move northwards and the highs southwards. Troughs and ridges form to the south of the surface lows and north of the highs. However the troughs are oriented

roughly in a SSW-NNE direction, unlike the SE-NW orientation given by the eight parameter solution. The ridge lines differ in a similar way, and this suggests an eddy flux of westerly zonal momentum at the surface towards the centre of the region (unlike the eight parameter solution). The vertical velocity patterns resemble those of the present model at $t = -2$ days, with upward motion slightly ahead of the surface lows and extending southwards in the warm air above the trough line.

A dissimilarity with Phillips's solution (and Huss and Mintz's) is that his predicts a region of maximum westerly wind in the centre of the region at upper levels (Fig 2.8) while the eight parameter solution has a minimum there (Fig 4.3). This was also referred to in section 2.6 during a discussion of the second order perturbation solution.

We now analyse the momentum changes in greater detail. For the eight parameter solution the rate of change of zonal momentum depends on two terms via the equation

$$\frac{\partial \bar{u}}{\partial t} = - \frac{\partial}{\partial y} \overline{(uv)}^x + h \bar{v}_a^x$$

The first of these is the eddy flux divergence while the second is the 'coriolis term', i.e. the change due to meridional angular momentum conserving flow. Phillips's system contains two additional terms, the loss of momentum due to surface friction and the momentum transfer by 'sub grid-scale eddies'.

The momentum budgets for the eight parameter solution (at $t = -2$ days) and Phillips's solution (the mean for the period 10 - 20 days) are shown in Tables 2 and 3. At upper levels the eddy flux gradient (b) for the eight parameter solution is positive in the centre and negative

$y = -\frac{\pi}{2\mu} \quad -\frac{3\pi}{8\mu} \quad -\frac{\pi}{4\mu} \quad -\frac{\pi}{8\mu} \quad 0 \quad \frac{\pi}{8\mu} \quad \frac{\pi}{4\mu} \quad \frac{3\pi}{8\mu} \quad \frac{\pi}{2\mu}$										(a)	
z											
+ .50	0	- 4	- 8	-12	-14	-12	- 8	- 4	0		
+ .25	0	- 2	- 4	- 7	- 7	- 7	- 4	- 2	0		
0	0	0	0	0	0	0	0	0	0		
- .25	0	+ 2	+ 4	+ 7	+ 7	+ 7	+ 4	+ 2	0		
- .50	0	+ 4	+ 8	+12	+14	+12	+ 8	+ 4	0		

z										(b)
+ .50	0	-11	-24	+11	+48	+11	-24	-11	0	
+ .25	0	- 4	- 6	+ 4	+12	+ 4	- 6	- 4	0	
0	0	0	0	0	0	0	0	0	0	
- .25	0	+ 4	+ 6	- 4	-12	- 4	+ 6	+ 4	0	
- .50	0	+11	+24	-11	-48	-11	+24	+11	0	

z										(c)
+ .50	0	+ 6	+16	-23	-62	-23	+16	+ 6	0	
+ .25	0	+ 2	+ 2	-11	-19	-11	+ 2	+ 2	0	
0	0	0	0	0	0	0	0	0	0	
- .25	0	- 2	- 2	+11	+19	+11	- 2	- 2	0	
- .50	0	- 6	-16	+23	+62	+23	-16	- 6	0	

Table 2. 8 parameter solution zonal momentum budget for $t = -2$ as a function of y and z . (a) $\partial \bar{u}^x / \partial t$, (b) $-\partial(uv)/\partial y$ and (c) $h \bar{v} a^x$. (Units arbitrary and not same scale as Table 3.)

j =	15	25	35	45	55	65	75	85	95	105	115	125	135	145	(a)
U	+3	+11	-8	-5	-33	+17	+11	+18	+13	-17	-9	-3	+5	+4	
L	+1	+3	-7	-6	-9	+11	+9	+9	+8	-4	-5	-5	+1	0	
U	-2	+4	-32	-15	-44	+45	+47	+54	+40	-21	-20	-21	-2	-2	(b)
L	0	+1	-10	-8	-6	-3	+5	+9	0	-5	-5	-7	0	-1	
U	+7	+13	+16	+14	+7	-25	-35	-34	-24	+2	+12	+14	+12	+6	(c)
L	-7	-13	-16	-14	-7	+25	+35	+34	+24	-2	-12	-14	-12	-6	
L	+8	+16	+17	+17	+3	-10	-31	-33	-16	+1	+13	+15	+14	+7	(d)
U	-1	-6	+7	-5	+5	-3	0	-2	-3	+3	-2	+4	-4	-1	
L	0	-2	+2	-1	+1	-1	-1	-1	-1	+1	0	+1	-1	0	(e)

Table 3. Phillips's (1956) zonal momentum budget for period 10-20 dys. U and L signify the upper and lower halves of the troposphere and j is the N-S coordinate with N and S walls at $j = 15$ and $j = 1$. (a) $\partial \bar{u}^{x*} / \partial t^*$, (b) $-\partial(u^*v^*) / \partial y^*$, (c) $f \bar{v} a^{x*}$, (d) $-k^* \bar{u}^{x*}$ and (e) $A \partial^2 \bar{u}^{x*} / \partial y^{*2}$. (Units arbitrary and not the same scale as Table 2.)

near the sides, and resembles the distribution of Phillips's values. This suggests that the orientation of the upper level troughs and ridges (not published for Phillips's solution) are similar in both models. At lower levels however the momentum flux gradient terms are not alike, that of the eight parameter solution having the same magnitude as at upper levels but with the opposite sign, while Phillips's has, generally speaking, the same sign as at upper levels but a very much smaller magnitude. As already mentioned this dissimilarity can be attributed to the different orientations of the surface trough and ridge lines for the two models.

The coriolis terms (c) compare very closely, evidently a result of the similar three cell meridional circulation patterns (see section 2.6 and Figs 2.5 and 2.7).

The effect of surface friction in Phillips's solution (d) is of considerable importance, but the small scale eddy flux gradient term (e) is small at all levels.

The increase of mid-latitude westerlies in Phillips's solution is due to the dominance at upper levels of the eddy flux gradient term, while the decrease at upper levels predicted by the eight parameter solution occurs because the coriolis term exceeds the eddy flux divergence in the middle of the region. One possible explanation for relatively greater importance of the eddy flux gradient term in Phillips's model is that different conditions existing at the side boundaries affect the internal flow. Both models impose the same condition $\partial \bar{u} / \partial t = 0$ at the side walls. Thus Phillips's model, starting from a state of rest, has zero wind speed on the walls at all subsequent times. The eight parameter solution, however, starts from an initial sheared zonal flow and so retains a large value of $\bar{u}^x$ at upper levels

on the side walls. Phillips's solution is therefore constrained to have small zonal wind speeds close to the side walls and may consequently have an increased eddy flux towards the interior.

Another possible explanation involves the variation with latitude of the coriolis parameter (an effect included in Phillips's model but not in the system of equations governing the eight parameter solution). Northward moving air conveys less absolute vorticity than southward flow. Thus an increased concentration of relative cyclonic vorticity occurs at the tip of the upper trough, while anticyclonic relative vorticity is similarly increased at the tip of the upper ridge. The curvature of the isobars becomes more pronounced and a larger eddy momentum flux occurs.

At low levels the most important terms in Phillips's solution are the coriolis and friction terms. To explain the different orientation of the troughs and ridges at low levels for the two models it is valid to invoke the argument involving the variation of the coriolis parameter f , but not that concerning the boundary conditions because both models have $\bar{u}^x = 0$ at the sides. The smaller magnitude of the eddy flux gradient in Phillips's solution is probably partly due to a reduction of surface wind speeds by friction.

In the atmosphere the zonal wind at mid-latitudes is westerly at all levels (see Fig 2.6), similar to Phillips's solution but not the eight parameter solution. Phillips's results suggest that it is maintained at upper levels by eddy transfer of momentum, but at low levels principally by the angular momentum conserving ageostrophic flow of the mean meridional inverse circulation (though eddy transfer also plays a part).

4.3 Relative Flow on Horizontal and Isentropic Surfaces.

Greater understanding of the physical processes responsible for the development of the system can be obtained from an examination of the flow relative to the moving cyclone. To do this would, technically, require a different coordinate system for each level, since the cyclone centre moves at different speeds at different levels (due to the phase change with time and the meridional movement of the centres of vorticity). However the mean zonal motion of the system (i.e. at the speed of the mean zonal wind) accounts for most of the movement at any level. Consequently we examine the flow patterns relative to the mean zonal wind.

At the surface the stream lines of the geostrophic flow relative to the mean zonal wind have a similar pattern to the surface isotherms (Fig 4.10 (c)). This is understandable since temperature changes are due to horizontal advection only, and so any differences between the stream lines are either due to the unsteadiness of the flow (i.e. the stream lines are not trajectories) or due to the truncation error (i.e. the spurious flow through the lids producing some of the temperature changes), both of which are evidently small. A similar analysis holds at upper levels.

At mid levels both vertical and horizontal advection produce temperature changes. According to Eady's (1949) criterion for the maximum efficiency for the release of potential energy the trajectories must have an inclination of about half that of the isentropic surfaces, implying that $\underline{v} \cdot \nabla \phi \approx -2h\omega$. Although this not achieved everywhere it is evident that when potential energy is released temperature changes at a particular level have the same sense as would be produced by horizontal advection but

with only about half the magnitude since upward (downward) motion occurs in regions of warm (cold) advection. Bearing these remarks in mind we can understand the relationship between the patterns of vertical motion at mid levels (Fig 4.10 (b)) and the mid level relative flow (Fig 4.10 (a)).

Initially (see $t = -3$) the largest horizontal relative winds at mid levels are in the N-S direction perpendicular to the isentropes, and regions of upward and downward flow coincide with northward and southward advection, with the maximum updraughts displaced to the north and maximum downdraughts displaced to the south where horizontal temperature gradients (and advection) are largest. As the disturbance intensifies the air at mid levels begins to circulate around the centres of vorticity, and the temperature field becomes distorted (principally by horizontal advection) resulting in the spreading out of the regions of upward and downward motion in the north and south ($t = -2$). Eventually the entropy field becomes twisted up by the cyclonic and anticyclonic motion and this coincides with a marked alteration in the vertical velocity pattern ($t = -1$). However in this new situation the vertical motion is still everywhere consistent with the scheme of the previous paragraph, namely a negative correlation between w and $\mathbf{v} \cdot \nabla \phi$.

We next consider the relative flow on isentropic surfaces. We define the total entropy $\Phi(x, y, z, t)$ (i.e. inclusive of the gross vertical variation of entropy in the standard static atmosphere) by

$$\Phi = h z + \phi \quad (62)$$

(It should be noted that this is consistent with the quasi-geostrophic adiabatic equation (9) if $\partial \phi / \partial z \ll h$).

The stream function χ for the geostrophic relative

flow on the isentropic surfaces is defined by

$$\underline{v} = \underline{k} \times [\nabla]_{\Phi}(\chi)$$

where $[\nabla]_{\Phi}$ signifies the gradient with respect to x and y with Φ held constant. χ is determined in terms of ψ in the following way.

We have, from the rules for partial derivatives,

$$\left[\frac{\partial \psi}{\partial x} \right]_{y, \Phi} = \left[\frac{\partial \psi}{\partial x} \right]_{y, z} + \left[\frac{\partial \psi}{\partial z} \right]_{x, y} \left[\frac{\partial z}{\partial x} \right]_{y, \Phi}$$

and so, using the hydrostatic equation

$$\left[\frac{\partial \psi}{\partial z} \right]_{x, y} = \phi$$

we find

$$\underline{v} = \underline{k} \times [\nabla]_z(\psi) = \underline{k} \times ([\nabla]_{\Phi}(\psi) - \phi [\nabla]_{\Phi}(z))$$

Substituting for ϕ from (62) we obtain

$$\underline{v} = \underline{k} \times [\nabla]_{\Phi}(\psi - \Phi z + \frac{1}{2} h z^2)$$

Thus

$$\chi = \psi - \Phi z + \frac{1}{2} h z^2 \quad (63)$$

is a stream function for the flow on the isentropic surface $\Phi = \text{const.}$, and it is unique apart from an arbitrary function of Φ .

In dimensional coordinates we have

$$\chi^* = \chi \frac{g B H^2}{h} = \frac{p^*}{p_s^*} - g \Phi^* z^* + \frac{1}{2} g B z^{*2}$$

which is equivalent to the Montgomery stream function

$$c_p T^* + g z^*$$

as can be verified directly by the use of approximations consistent with those of section 1.2.

The procedure adopted for finding the contours and stream lines on an isentropic surface was as follows:- the horizontal was divided up into a grid, and at each grid point (x_0, y_0) we found the root $z = z_0$ of the transcendental equation

$$hz + \phi(x_0, y_0, z, t_0) = \text{const}$$

using Newton's method (Appendix II), and then the stream function χ was calculated at each grid point from

$$\chi = \psi(x_0, y_0, z_0, t_0) - \frac{1}{2} h z_0^2 + \frac{1}{2} h z^2$$

The results are shown in Fig 4.11 (a) and (b).

We examined two isentropic surfaces; that which initially passed through the centre point of the atmosphere (say the 310°K isentropic surface), and a lower surface (300°K) which intersected the ground at the extreme south.

The flow on the 310°K isentropic surface (Fig 4.11 (a)) has a similar appearance to those drawn for actual cyclones (see for example Green, Ludlam and McIlveen (1966)). The air at low levels is conveyed northwards and upwards to higher levels, while there is a compensatory flow of high altitude air from the north down to lesser heights in the south. However since the system is not in a steady state the stream lines are not trajectories, and the apparent vertical velocity (i.e. $-\mathbf{v} \cdot \nabla \chi(z)$) is about twice the actual.

The up and down draughts 'lean backwards' but as the system intensifies they become oriented in a more direct N-S line (due to the decrease in the phase difference between the waves at the top and bottom). The slope of the isentropic surface becomes steeper in the vicinity of the cyclone but shallower in the anticyclone indicating changes in the vertical stability. The

accompanying larger values of Ri suggest that higher order Richardson number terms are important in the vicinity of the cyclone.

In the lower, $300^\circ K$, surface (Fig 4.11 (b)) there is a relative flow from east to west, but some N-S overturning is still apparent. For an exact solution the ground contours would not become closed, as happens here, since it is impossible for particles to 'get round' the corners of the region (i.e. the intersections of the walls and lids) where both v and w vanish. However this is a feature of the artificiality of the assumed boundaries. The steeper slopes of the surface, compared with those for the $310^\circ K$ surface, are probably representative and indicate that non quasi-geostrophic effects (violating the assumption $\partial\phi/\partial z \ll h$) are likely to be of more importance near the ground than elsewhere - consistent with frontogenesis principally near the surface of the earth.

4.4 The Efficiency of Finite Amplitude Baroclinic Disturbances.

By introducing the system of dimensionless coordinates in section 1.2 we combine all physical quantities into non-dimensional parameters. However with the assumption of a constant value of the coriolis parameter and the neglect of the mass divergence effect only one parameter remains - the ratio of the distance between the side walls Y^* and the length scale L^* , i.e. $Y^*/L^* = Y = \pi/\mu$. The non-dimensional zonal wavelength $2\pi/\lambda$ is obtained by assuming the Eady solution with the largest growth rate determines the structure of the disturbance and, as already noted in section 2.4, it is dependent on Y (as also is the growth rate).

Some of the effects of different zone widths on the initial growth of the mean meridional circulation were examined by means of the second order perturbation solution in section 2.6 (see Fig 2.5), and in order to discuss the influence of the walls on the finite amplitude solution we have found the eight parameter solutions for four values of μ : 0.5 , 1.0 , 1.5 and 2.0 .

The available potential energy $(PE)_0$ and the kinetic energy $(KE)_0$ present in the initial state is easily calculated to give in non-dimensional coordinates

$$(PE)_0 = \frac{1}{2} \overline{\left(\frac{\partial \psi_0}{\partial z}\right)^2}_{x,y,z} = \pi^2 / 24 \mu^2$$

$$(KE)_0 = \frac{1}{2} \left[\overline{\left(\frac{\partial \psi_0}{\partial x}\right)^2 + \left(\frac{\partial \psi_0}{\partial y}\right)^2} \right]_{x,y,z} = 1/6$$

Thus $(PE)_0$ is proportional to μ^{-2} while $(KE)_0$ is independent of μ . In dimensional coordinates we have

$$(PE)_0^* = \frac{1}{24} \frac{1}{B} (AY^*)^2 \rho_s^* g$$

$$(KE)_0^* = \frac{1}{6} \rho_s^* \left(\frac{dU_0^*}{dz^*} H \right)^2$$

(where A is the meridional entropy gradient $d\phi^*/dy^*$) showing that the potential energy is proportional to the square of the temperature difference between the northern and southern walls, while the kinetic energy is proportional to the square of the wind speed difference between the top and bottom. Therefore, for a given value of the Richardson number, the potential energy increases as the square of the N-S extent of the baroclinic zone but the kinetic energy is independent of it.

The available potential energy is measured from a zero state having horizontal isentropes (although this is not a state which can be attained by the motion) and

the fraction of (PE)₀ which is converted into kinetic energy is a measure of the efficiency ϵ of the energy conversion process. As we have seen in section 4.1 (and Fig 4.5) the kinetic energy oscillates in time, successive maxima having different intensities. Only the first maximum can properly be regarded as having grown from an initial state consisting of an infinitesimal disturbance superimposed on a sheared zonal flow, the others start from more complicated states. It is clear from Fig 4.5 that the efficiency depends to some extent on the condition of the atmosphere at the previous minimum of kinetic energy, and by performing a long term integration we could calculate an average efficiency for the motion. Here however we restrict our determination of ϵ to that of the first maximum only.

Eady (1949) defined an efficiency in a different way by comparing the calculated production of kinetic energy for an exponentially growing constant shape disturbance with the potential energy which would be released under ideal conditions, whereby air parcels are exchanged along paths having slopes half that of the isentropic surfaces. He then found that the efficiency $\hat{\epsilon}$ is equal to twice the growth rate of the disturbance.

The dependence of the efficiency ϵ on μ is shown in Table 4. We can see that the value of μ for which the motion is most efficient lies between 1.0 and 1.5, possibly the root of the equation

$$\mu = \lambda_{\max}(\mu)$$

that is $\mu = 1.35$. It is noticeable that the efficiency ϵ is relatively constant over a wide range of μ .

The dependence of $\hat{\epsilon}$ on μ differs from that of ϵ particularly at large values of μ . When the distance between the side walls is small Eady's measure

of efficiency $\hat{\epsilon}$ is also small. This happens because $\hat{\epsilon}$ measures the efficiency of constant shape disturbances capable of releasing potential energy by reducing the mean meridional entropy gradient, but only while building up 'eddy entropy gradients'. When the side walls are close together meridional heat transfer is impeded giving a small efficiency. ϵ , however, measures the efficiency of non-linear disturbances. As we have seen in section 3.8, these are not constant shape waves since the phase difference between the upper and lower systems changes with time in such a way that eddy entropy gradients are partially smoothed out leading to a further release of potential energy and an increased efficiency. Indeed it will be remembered that for the extreme case $\mu > 2.4$ the growth rate σ , and hence the efficiency $\hat{\epsilon}$, is identically zero but we showed in section 2.8 that a pair of neutral waves could release potential energy by non-linear interaction to give a non zero ϵ .

Table 4.

μ	λ_{max}	σ_{max}	$(PE)_0$	ϵ	$\hat{\epsilon}$
0.5	1.6	0.29	1.64	51%	58%
1.0	1.5	0.25	0.41	62%	50%
1.5	1.3	0.18	0.18	69%	36%
2.0	0.9	0.09	0.10	50%	18%

When μ is small $\hat{\epsilon}$ is larger than ϵ . It is difficult to see why this is so, but we presume that a finite amplitude wave is more able to detect the presence of the side walls, though distant from each other, than an infinitesimal constant shape wave.

The variation of available potential energy (PE) with time can be conveniently examined by plotting $(PE)/(PE)_0$ against $\hat{t}$ ($= 2 \sigma_{max} t$), since this effectively

separates the known dependence of σ_{max} and $(PE)_0$ on μ from other factors affecting the growth.

The results, shown in Fig 4.12, are basically similar, though two points of interest are evident. The first is that for large values of μ the energy conversion process occupies a relatively short time compared with the total. This is understandable when we realise that, if μ is close to 2.4, it requires only a small change in the zonally averaged temperature field for it to be unable to support a growing wave, whence the disturbance, behaving like a pair of neutral waves, changes its structure to that of a decaying wave.

The second item of immediate interest in Fig 4.12 is that for small μ the successive maxima of kinetic energy seem to bear less resemblance to each other than do the maxima of the more regular curves for large μ . However this observation is only based on the results of two maxima and cannot be asserted with any confidence until an integration over a long period of time has been performed.

4.5 Physical Parameters Affecting the Intensity and Structure of the Disturbance.

The initial state of the atmosphere is defined in terms of four quantities:- the width of the baroclinic zone Y^* , the height of the tropopause H , the mean meridional entropy gradient A , and the static stability B . Thus the Richardson number $Ri = h^2 = f_0 B / g A^2$ and we may write (see section 1.2)

$$L^* = \left(\frac{g^{1/2}}{f_0} \right) B^{1/2} H, \quad \tau^* = (g^{-1/2}) B^{1/2} A^{-1} \quad (63)$$

$$\pi^* = \left(\frac{\rho_s g^{3/2}}{f_0} \right) B^{1/2} A H^2, \quad \mu = \left(\frac{2\eta g^{1/2}}{f_0} \right) B^{1/2} H Y^{*-1} \quad (63)$$

One important problem to consider is how the structure and intensity of the disturbance depends on the parameters of the initial state Y^* , H , A , B . This is not just a question of examining the dimensionless coordinate system, because we cannot have dynamic similarity between different atmospheres if any of the physical quantities depend on the parameter μ which, as we have already seen, has some effect on the structure of the solution.

As an example we consider the effect of a fractional change Δ ($= \delta A / A$) in the value of the mean meridional entropy gradient A . From (63) it is seen that this results in fractional changes of Δ in the values of π^* and τ^* , but no changes in L^* or μ . Evidently an increased entropy gradient gives proportionately larger values of the pressure deviation, wind speed, vorticity and rate of growth, but no change in the zonal wavelength.

The situation is more complicated when we consider the effect of changing the other quantities since it is then necessary to examine solutions for different values of μ . A percentage increase Δ in B results in increases $\Delta/2$ % in L^* , τ^* , π^* and μ . The dependence of the dimensionless wavenumber λ_{max} and the growth rate σ_{max} on μ is non-linear but when μ is small they are only slightly dependent on μ , a larger value of μ implying slightly smaller values of λ_{max} and σ_{max} . The non-dimensional available potential energy in the initial state $(PE)_0$ is proportional to μ^{-2} and so decreases by Δ %. From the results of Table 4 we deduce that the efficiency ϵ is relatively independent of μ , implying that the dimensionless kinetic energy per unit volume at maximum development is proportional

to (PE)₀ and hence decreases by Δ %. It is reasonable to suppose that as a result the dimensionless wind speeds (pressure deviation and vorticity) are proportional to μ^{-1} and hence decrease by $\Delta/2$ %. Thus the change in dimensional wind δv^* is given by

$$\frac{\delta v^*}{v^*} = \frac{\delta v}{v} + \frac{\delta L^*}{L^*} - \frac{\delta \tau^*}{\tau^*} = -\frac{\Delta}{2} \%$$

Similarly the wavelength increases and the growth rate decreases by a little more than $\Delta/2$ %, while the pressure deviation remains unchanged.

Adopting the assumption, proposed in the previous paragraph, that the non-dimensional wind speed, vorticity and pressure deviation are proportional to μ^{-1} it is a simple matter to estimate the dependence of the dimensional solution on any combination of Y^* , H , A and B . The results are shown in Table 5.

Table 5.

Fractional changes in

	wind speed	vorticity	pressure deviation	zonal wavelength	growth rate
$\frac{\delta Y^*}{Y^*} = +\Delta$	$+\Delta$	$+\Delta$	$+\Delta$	$-(>0)$	$+(>0)$
$\frac{\delta H}{H} = +\Delta$	0	$-\Delta$	$+\Delta$	$+(>\Delta)$	$-(>0)$
$\frac{\delta A}{A} = +\Delta$	$+\Delta$	$+\Delta$	$+\Delta$	0	$+\Delta$
$\frac{\delta B}{B} = +\Delta$	$-\Delta/2$	$-\Delta$	0	$+(>\Delta/2)$	$-(>\Delta/2)$

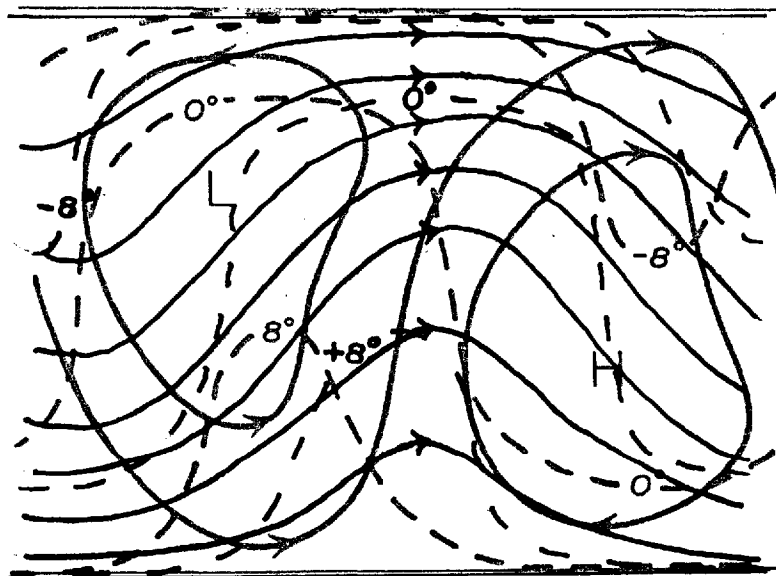
Fractional changes in dimensional physical quantities occurring as a result of fractional increases of $+\Delta$ in the parameters Y^* , H , A , B .

From Table 5 we deduce that high wind speeds are

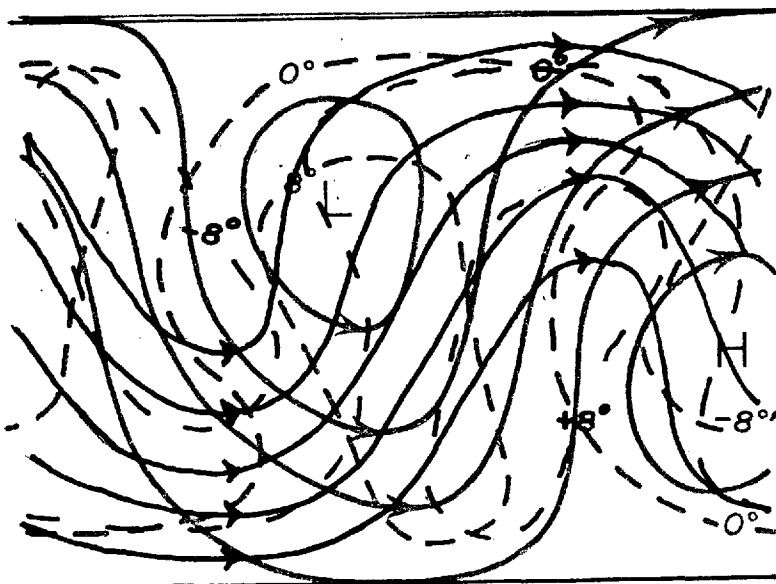
favoured by large values of Y^* and A , and small values of B . This is understandable since small values of the Richardson number lead to large accelerations and hence, for a wide baroclinic zone, to high wind speeds. Large vorticities arise for large values of Y^* and A , and small values of H and B - a shallow troposphere ensuring increased vortex stretching on the lids. Pressure deviations are greatest if Y^* , H and A are all large. This is understandable since, due to the hydrostatic relation, regions of high and low pressure at the ground are associated with columns of relatively cold and warm air. Evidently the pressure deviation p^* from its standard value p_s^* is directly related to the maximum adjustable temperature contrasts in the atmosphere, i.e. to AY^* , and to the depth of the troposphere. The disturbance has a long wavelength if Y^* is small and if H and B are large, but is unaffected by A . A large growth rate occurs if Y^* and A are large, and H and B are small, but principally if the Richardson number is small.

The disturbances examined here are assumed to be growing in a dry atmosphere. For an atmosphere containing water vapour conditions will be somewhat different due to the release of latent heat. One way of simulating a 'wet' atmosphere is to calculate the static stability B using the 'wet bulb potential temperature' instead of the potential temperature. Under these circumstances B typically takes a very much smaller value, and there is the implicit assumption that a parcel retains its water vapour content which condenses on ascent and evaporates on descent. However this does not represent the behaviour of the real atmosphere since precipitation of liquid water reduces the mixing ratio, so that descent is largely dry. Evidently a more realistic approximation

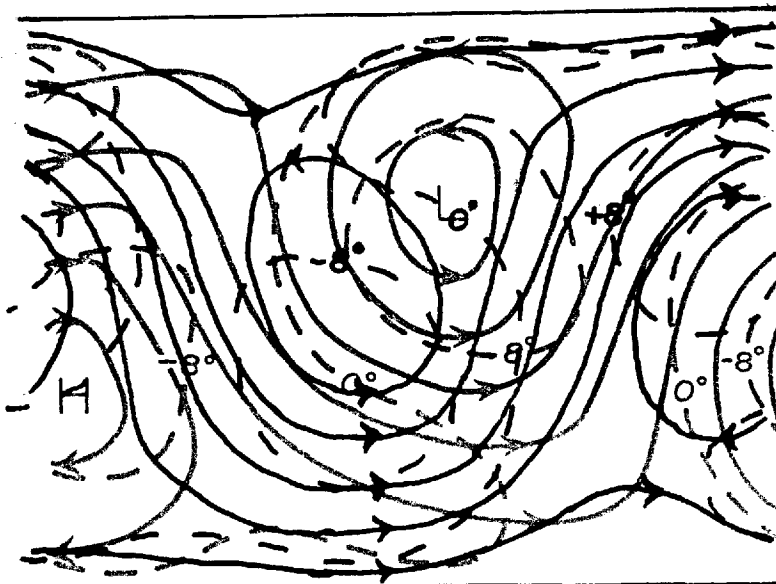
is to have different values of B for upward and downward motion, but this leads to computational difficulties. Nevertheless making the simplest assumption, namely that of a smaller B , we see from Table 5 that this results in higher wind speeds, larger vorticities, faster growth and a shorter wavelength. Central pressures are, however, unaffected. A further result is that the minimum width of the baroclinic zone for which amplifying Eady waves can exist is smaller than for a dry atmosphere (where the limiting width is about 1500 km. for the values of the parameters adopted in section 1.2 (see section 2.4)). Frequently one may observe in the atmosphere waves, rapidly amplifying in relatively narrow baroclinic zones, which develop into intense storms with wavelengths much less than 4000 km. (especially off the east coast of the North American continent in winter), and the above remarks suggest that water vapour is an important factor controlling the development. Eady (1949) performed a calculation with a wet convective regime sandwiched between two dry zones and obtained similar results for the wavelength and growth rate of a linear perturbation solution.



$t = -3$

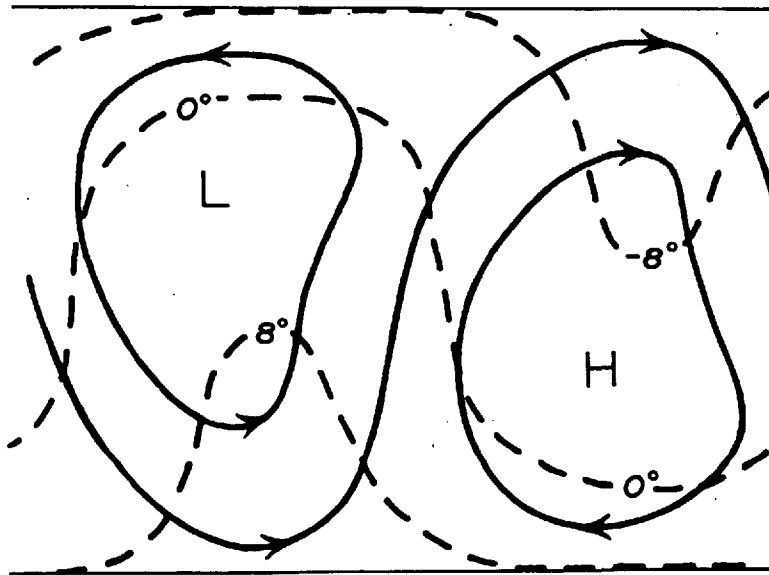


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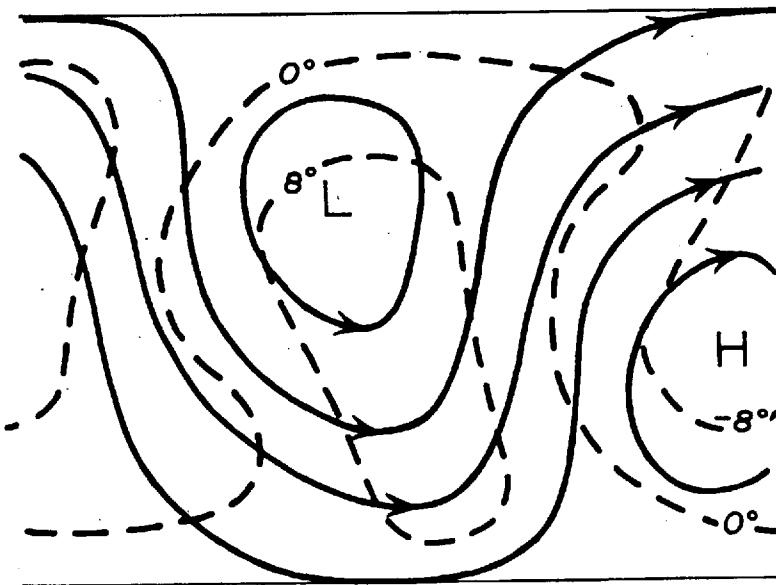


$t = -1$

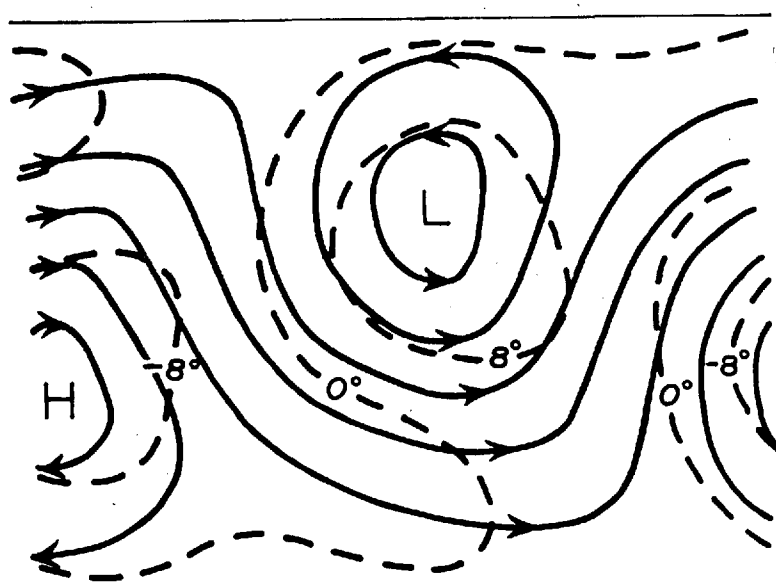
Fig 4.1 (a)
see page 115



$t = -3$

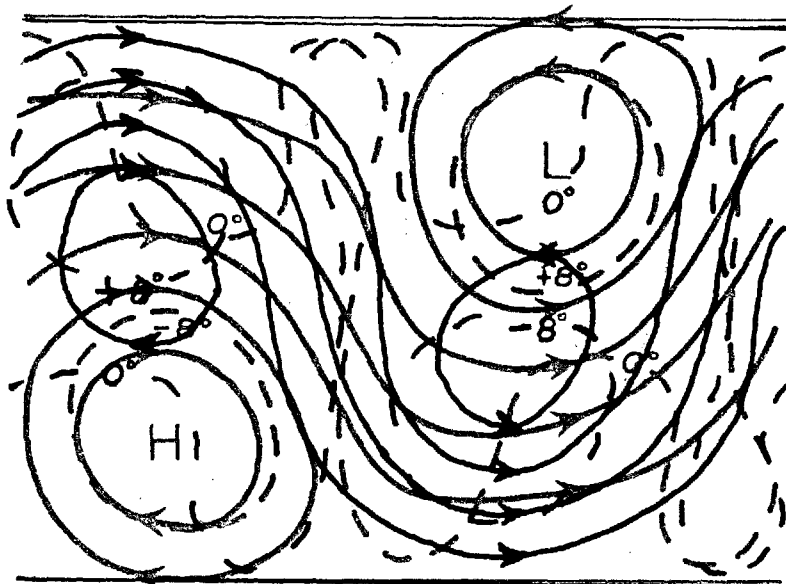


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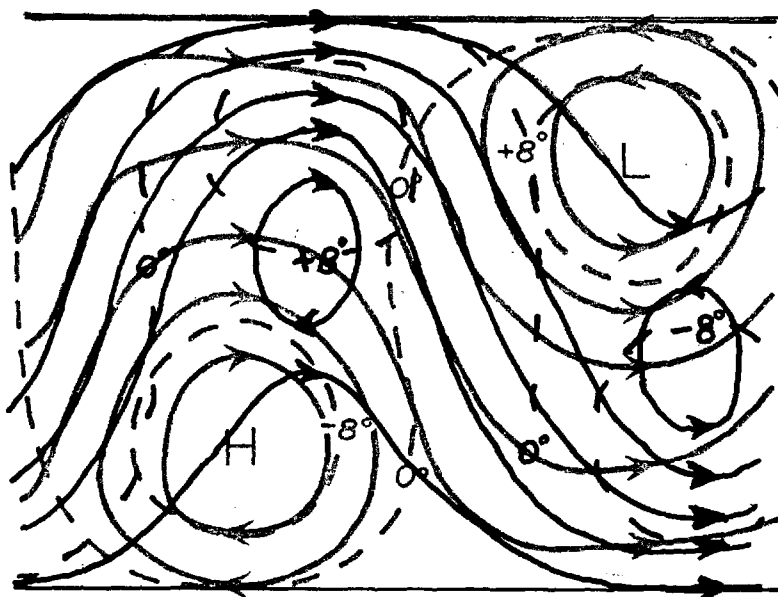


$t = -1$

Fig 4.1 (b)
see page 116



$t=0$



$t=1$

Fig 4.1 (a) (See also page 113). Stream lines for the geostrophic flow relative to the ground (—) and isotherms (---) at the upper lid for the 8 parameter solution shown every day ($t = 0$ is approximately the time of maximum development).

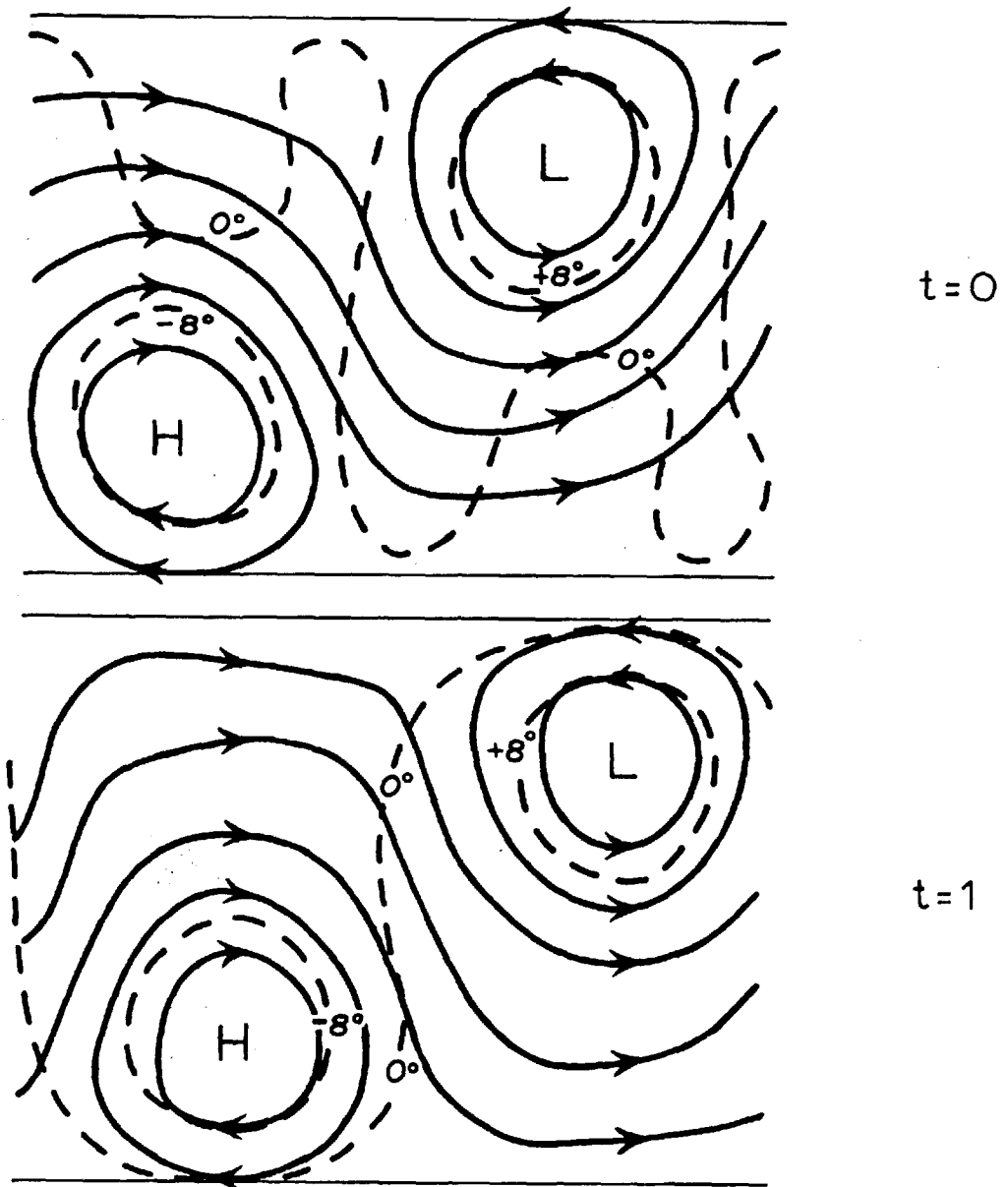
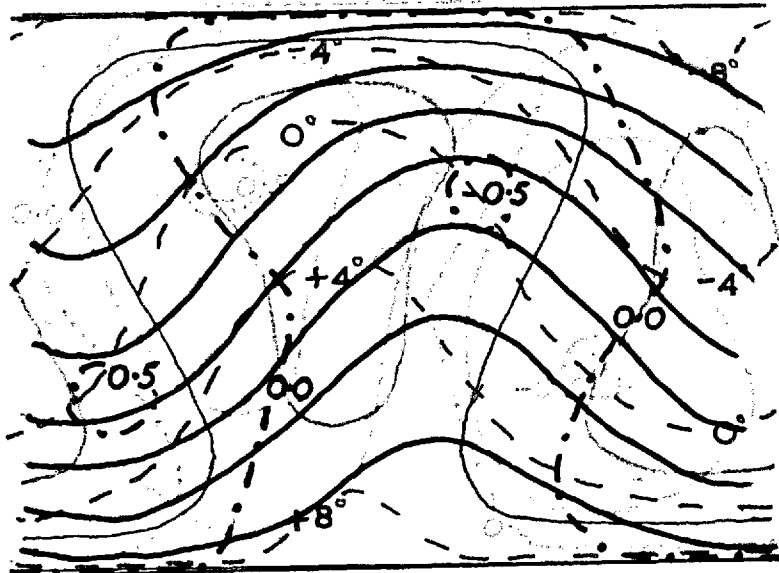


Fig 4.1 (b) (See also page 114). Stream lines for the geostrophic flow relative to the ground (—) and isotherms (---) at the lower lid for the 8 parameter solution shown every day ($t = 0$ is approximately the time of maximum development).

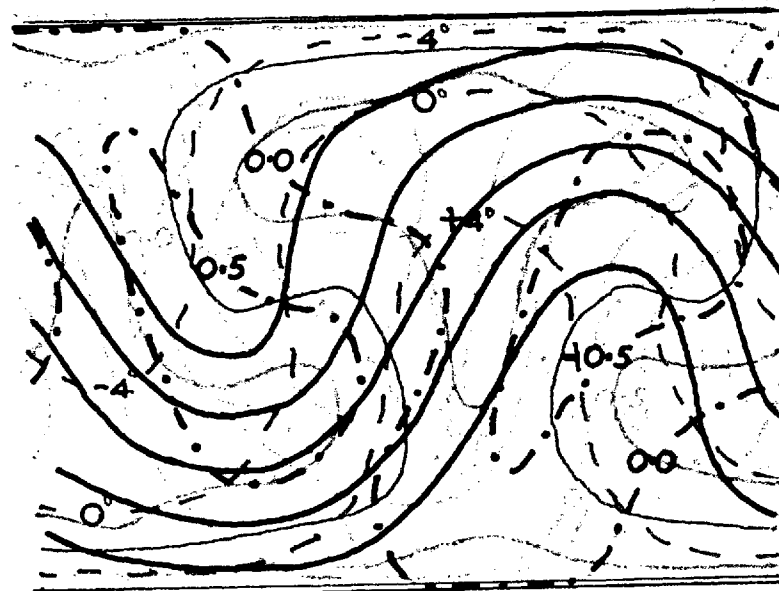
Fig 4.2 (a) Stream lines for the geostrophic flow relative to the ground (—) and isopleths of relative vorticity (-.-.-) at the upper lid for the 8 parameter solution shown every day. Unit of vorticity = f .

Fig 4.2 (b) Isotherms (- - -) and vertical velocity at mid levels for the 8 parameter solution shown every day. Upward motion shown red, zero vertical motion black and downward motion blue.

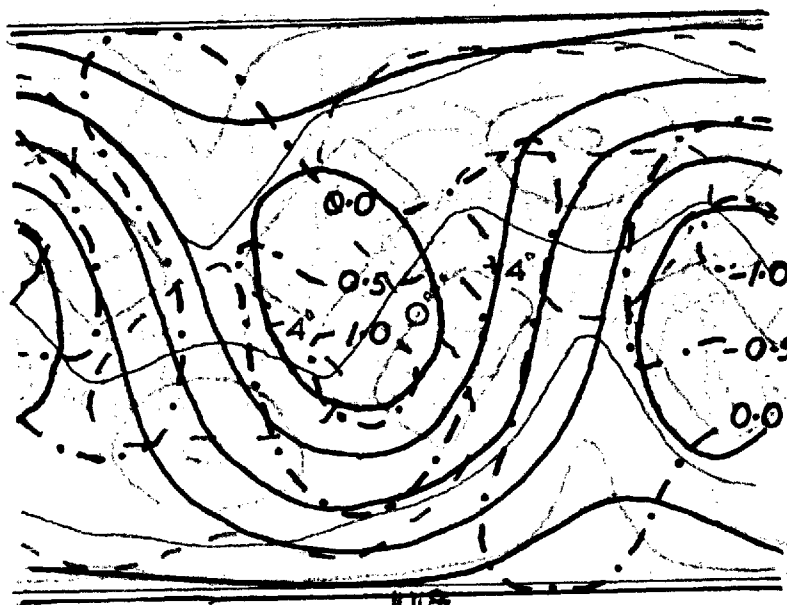
Fig 4.2 (c) Stream lines for the geostrophic flow relative to the ground (—) and isopleths of relative vorticity (-.-.-) at the lower lid for the 8 parameter solution shown every day. Unit of vorticity = f .



$t = -3$

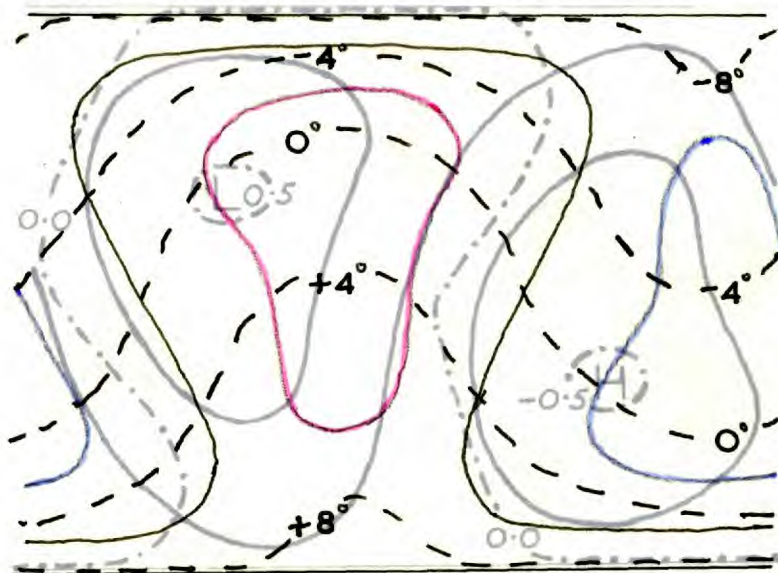


$t = -2$

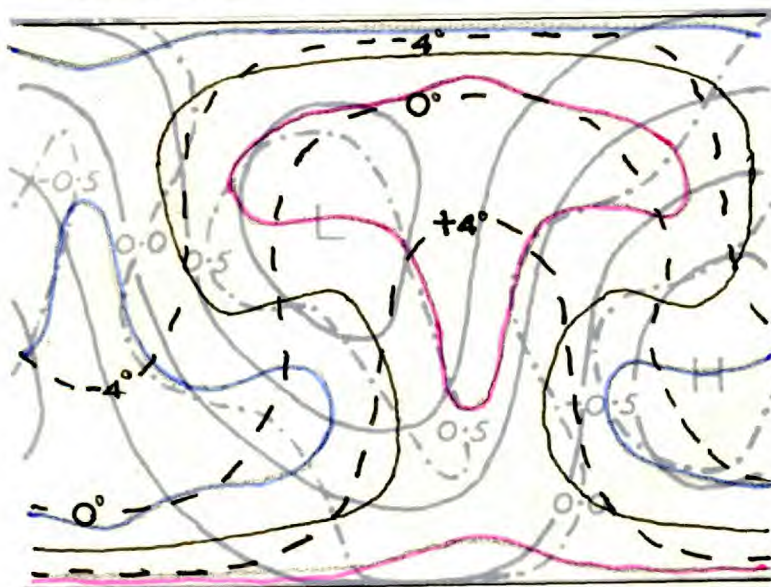


$t = -1$

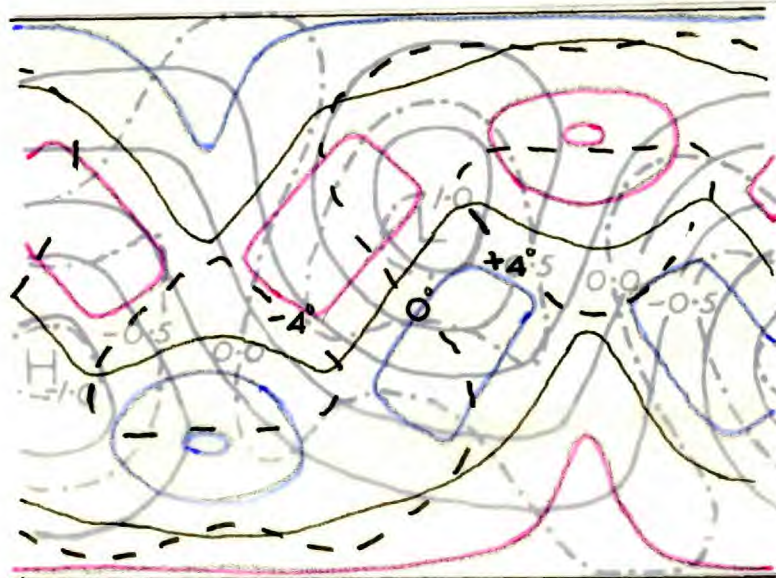
Fig 4.2 (a)
see opposite
page 117.



$t = -3$

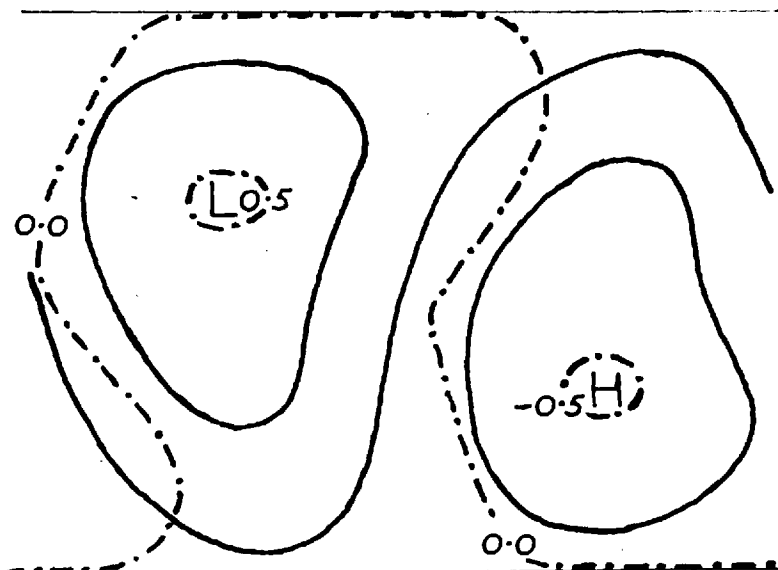


$t = -2$

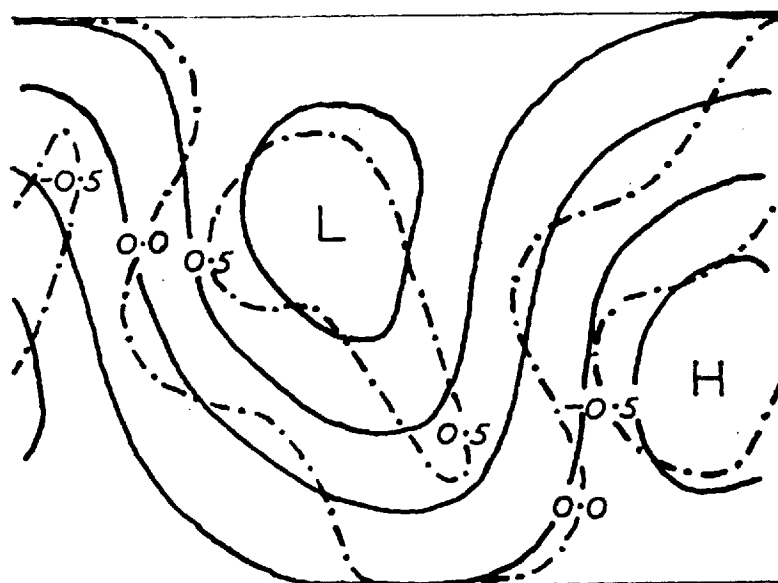


$t = -1$

Fig 4.2 (b)
see opposite
page 117.

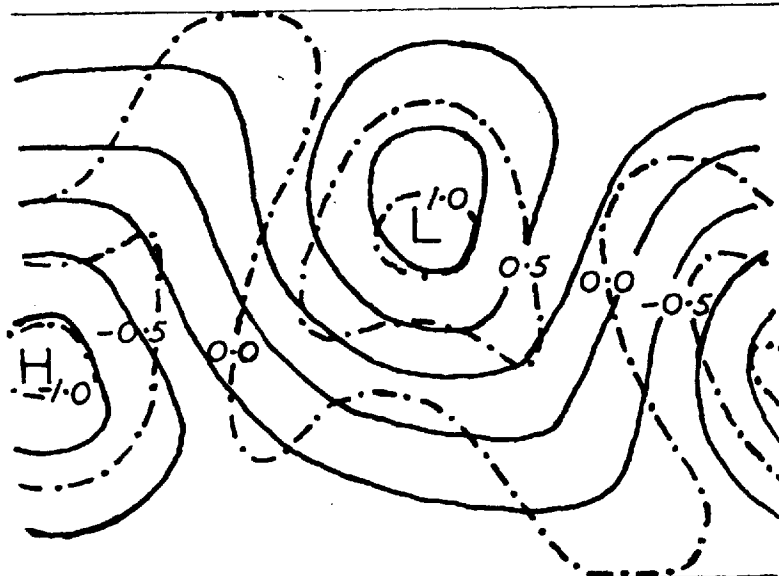


$t = -3$



$t = -2$

Fig 4.2 (c)
see opposite
page 117.



$t = -1$

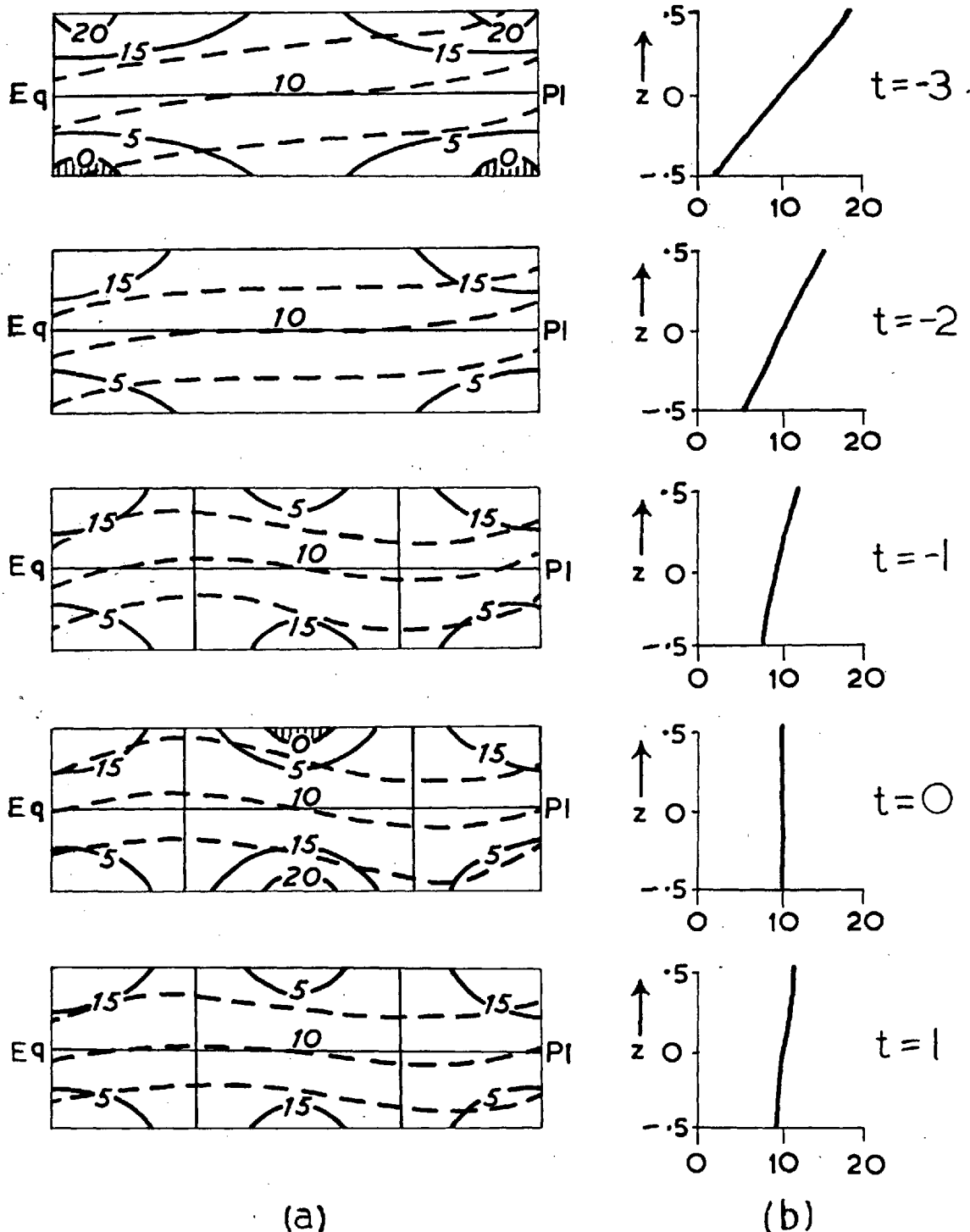


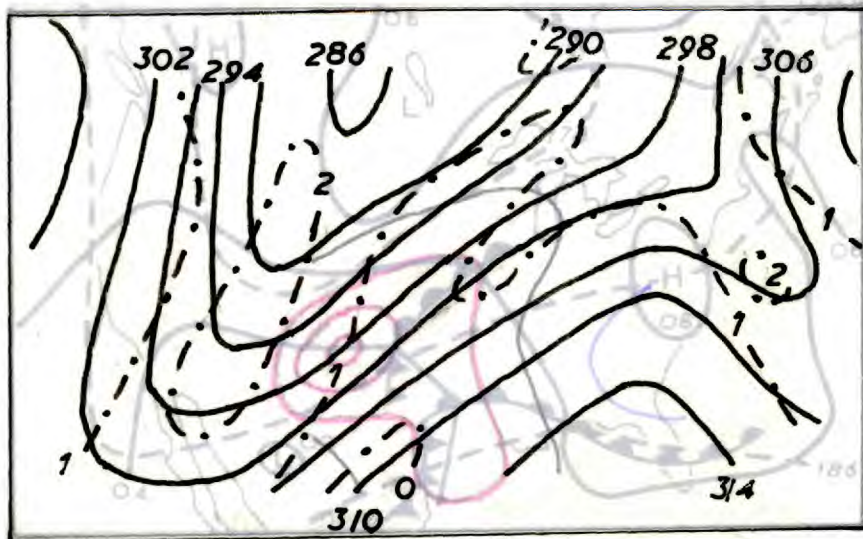
Fig 4.3 (a) Zonally averaged meridional cross sections for the 8 parameter solution with isotachs (—) and isentropes (---) shown every day. Wind speeds in m/s and easterlies shaded.

Fig 4.3 (b) Mean zonal wind $\bar{u}^{x,y}$ as a function of height shown every day.

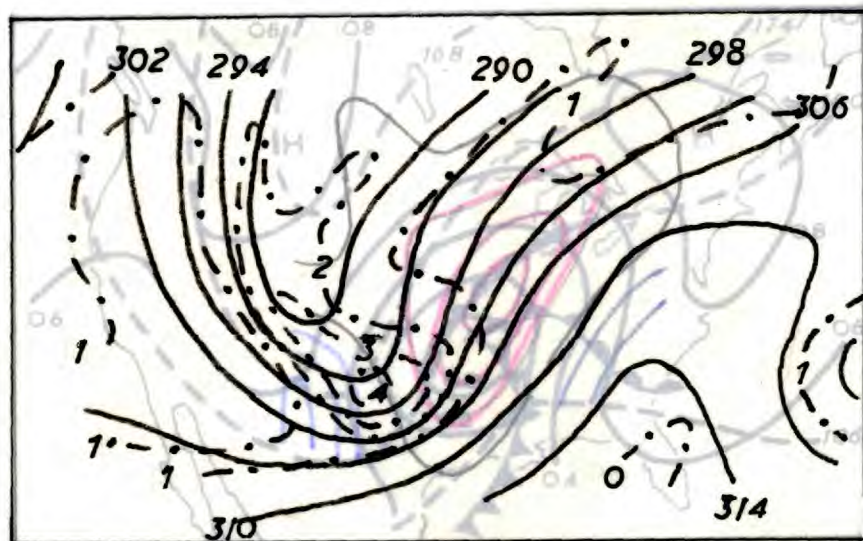
Fig 4.4 (a) Contours of the 300 mb. surface (——) and isopleths of absolute vorticity (-.-.-) (see Petterssen and Bradbury (1954)). Contours in 100's of ft. and vorticity in units of 10^{-4}sec^{-1} .

Fig 4.4 (b) Calculated vertical motion at 850 mb. (see Petterssen and Bradbury (1954)) . Upward motion shown red, zero vertical motion black and downward motion blue.

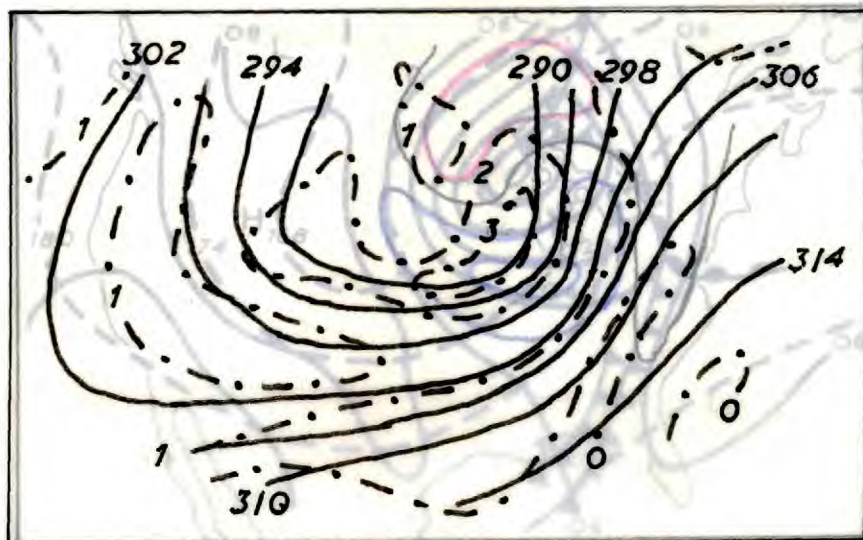
Fig 4.4 (c) Contours of 1000 mb. surface (——) and the 1000 - 500 mb. thickness (- - -) (see Petterssen and Bradbury (1954)). Contours and thickness lines in 100's of ft.



Nov 24 1952
1500Z

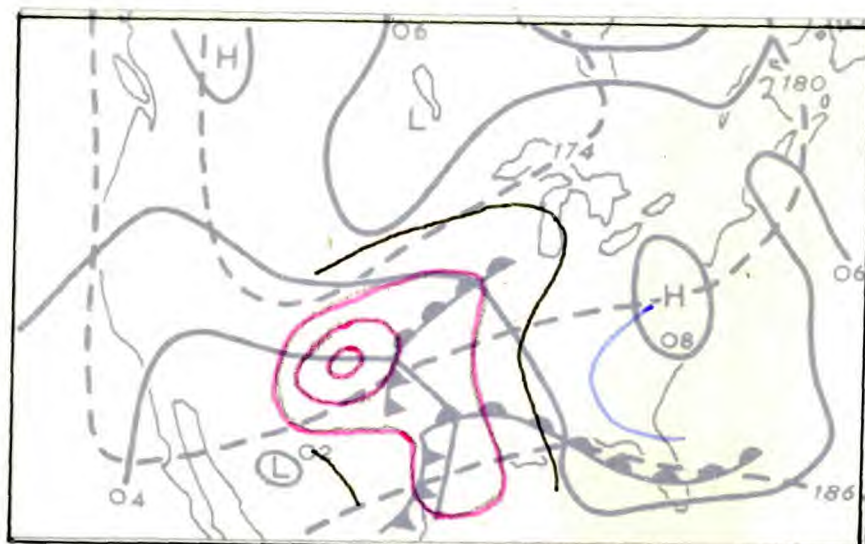


Nov 25 1952
1500Z

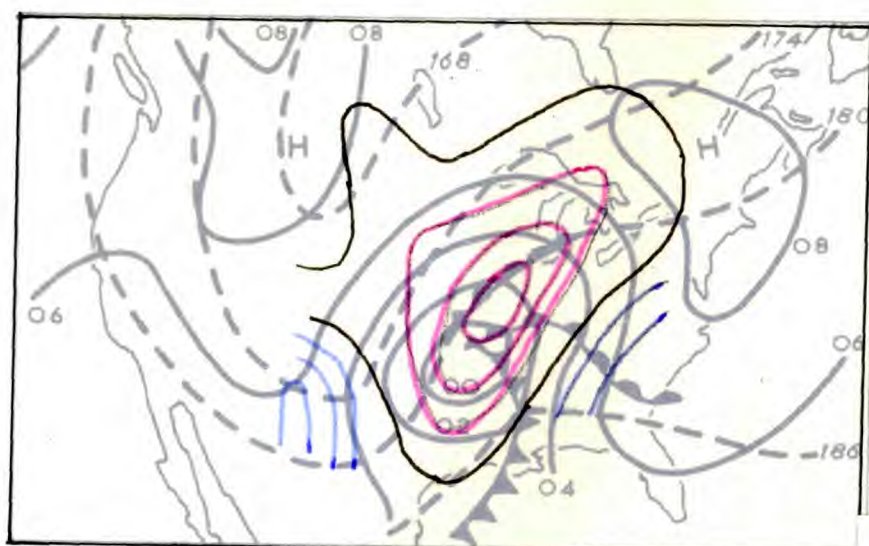


Nov 26 1952
1500Z

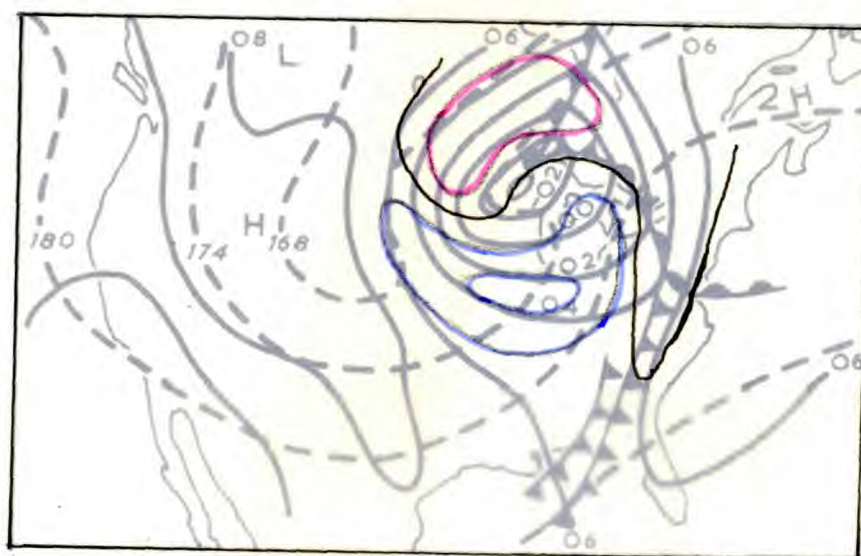
Fig 4.4 (a)
page 121.



Nov 24 1952
1500Z

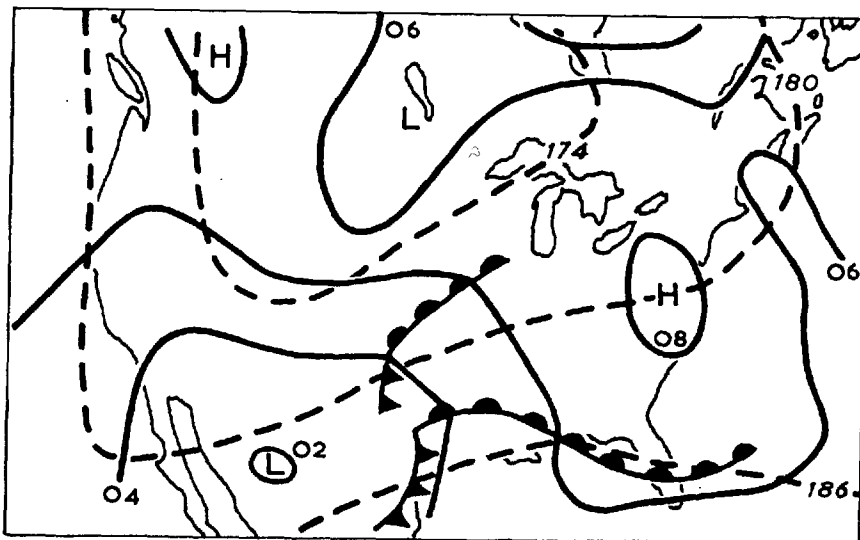


Nov 25 1952
1500Z

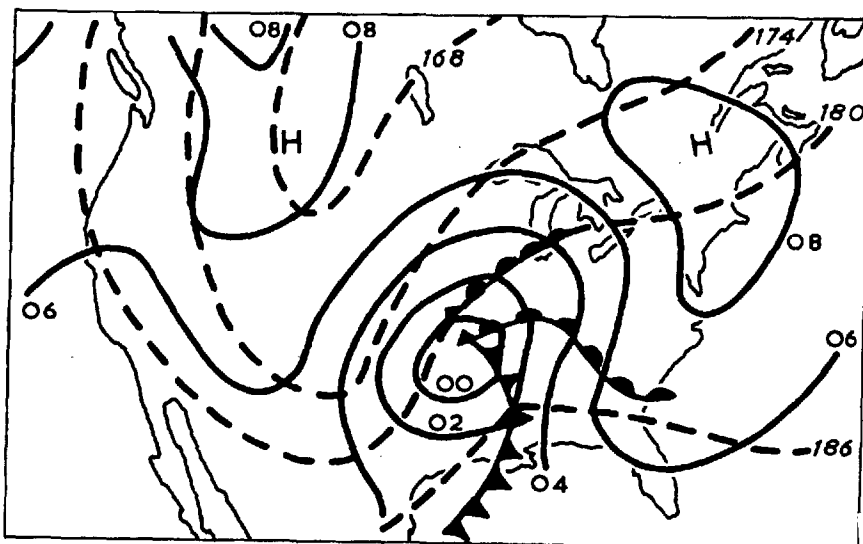


Nov 26 1952
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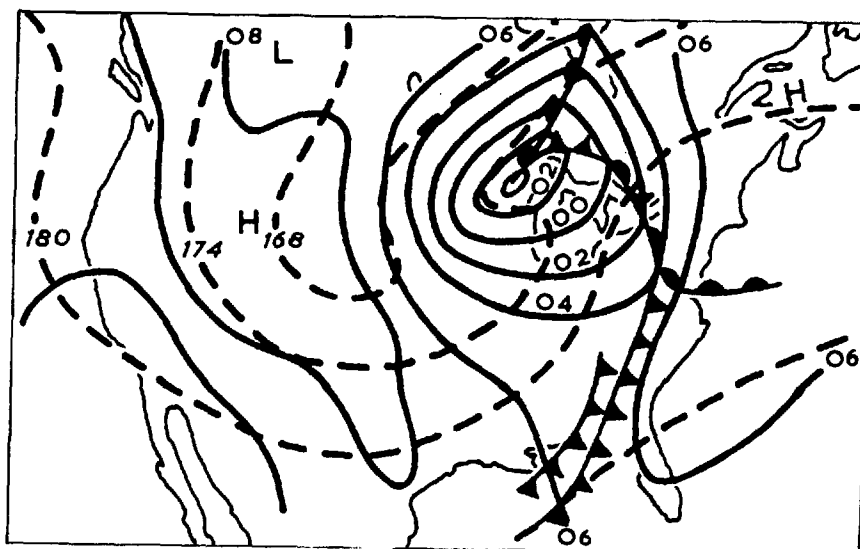
Fig 4.4 (b)
see opposite
page 121.



Nov 24 1952
1500Z



Nov 25 1952
1500Z



Nov 26 1952
1500Z

Fig 4.4 (c)
see opposite
page 121.

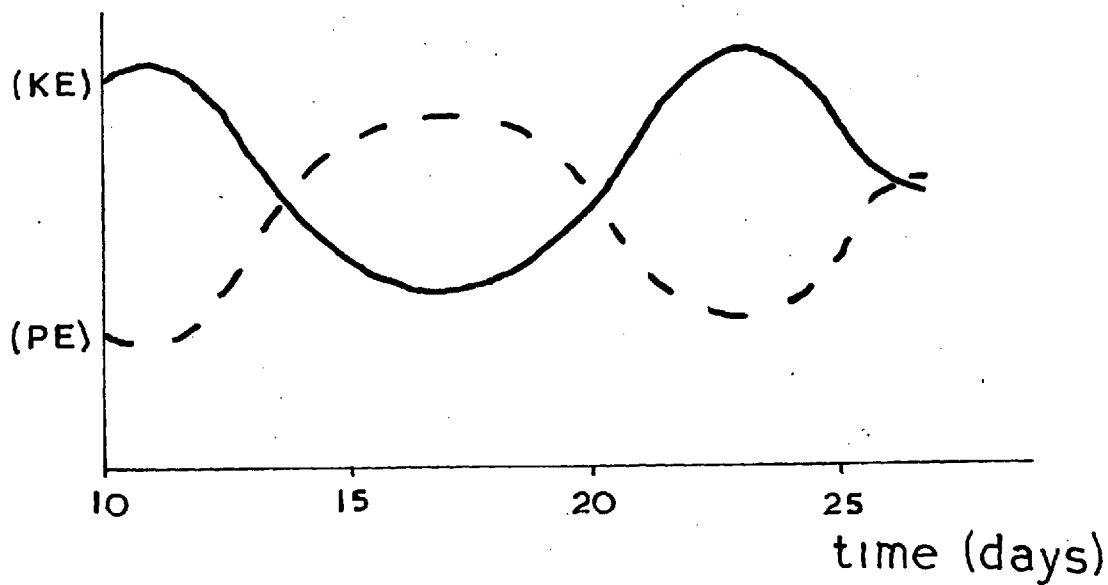
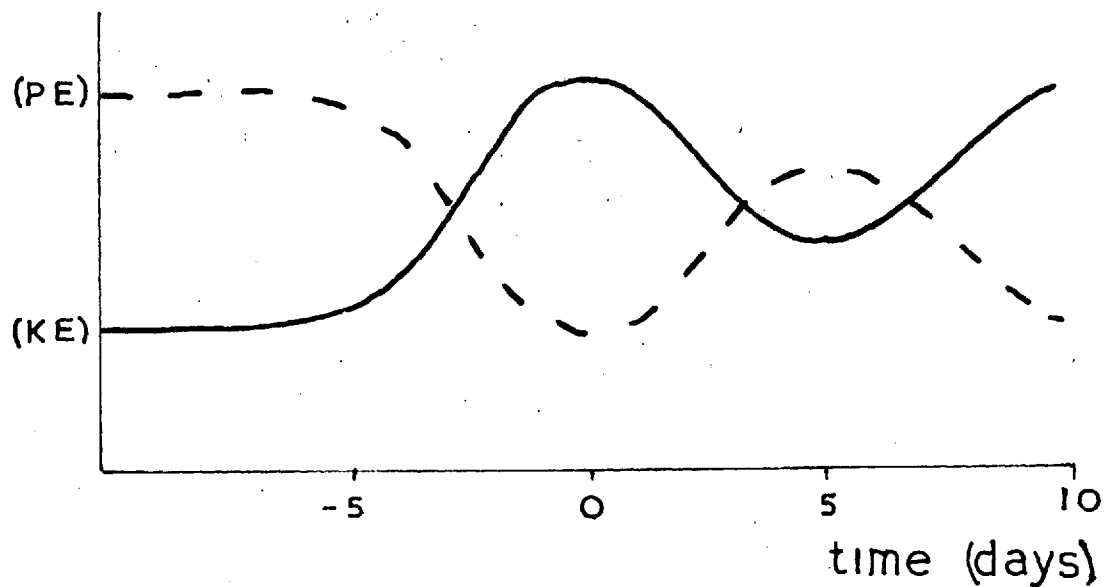


Fig 4.5 Variation of kinetic and available potential energy per unit volume with time for the eight parameter solution.

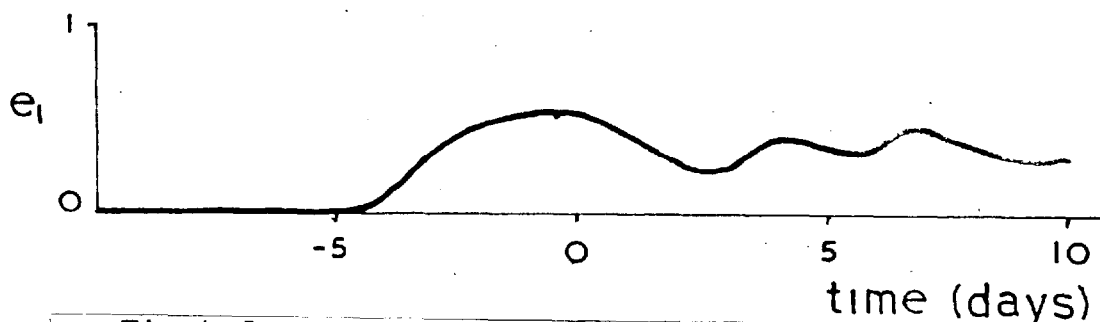


Fig 4.6 The error parameter e_1 for the eight parameter solution.

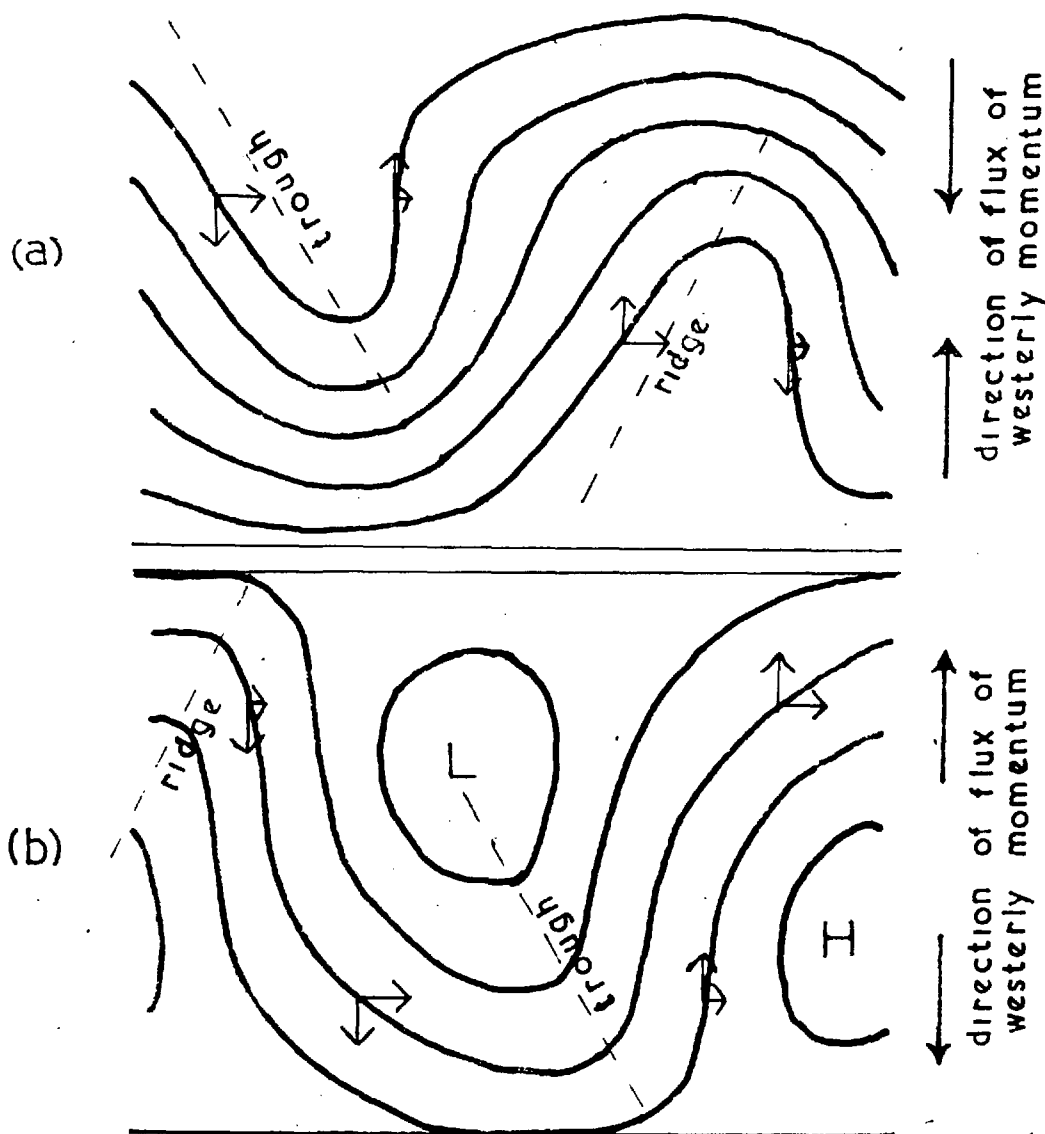


Fig 4.7 Isobars at (a) the upper lid, and (b) the lower lid at $t = -2$, showing how correlations between u and v are set up.

Fig 4.8 (a) 500 mb. isotherms (— — —) and vertical velocity at day 20 after Phillips (1956). Upward motion shown red, zero vertical motion black and downward motion blue.

Fig 4.8 (b) Contours of 1000 mb. surface at day 20 after Phillips (1956).

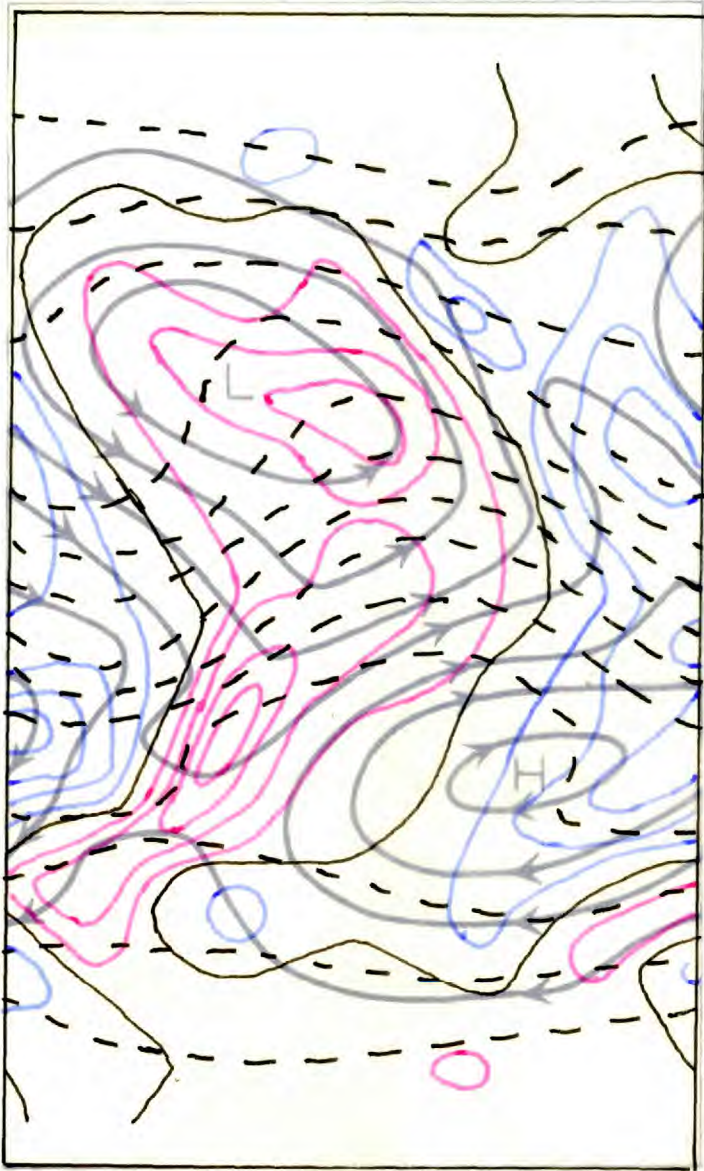


Fig 4.8 (a)
page 120.



Fig 4.9 (a) 500 mb. isotherms (---) and vertical velocity after Huss and Mintz (1960). Upward motion shown red, zero vertical motion black and downward motion blue.

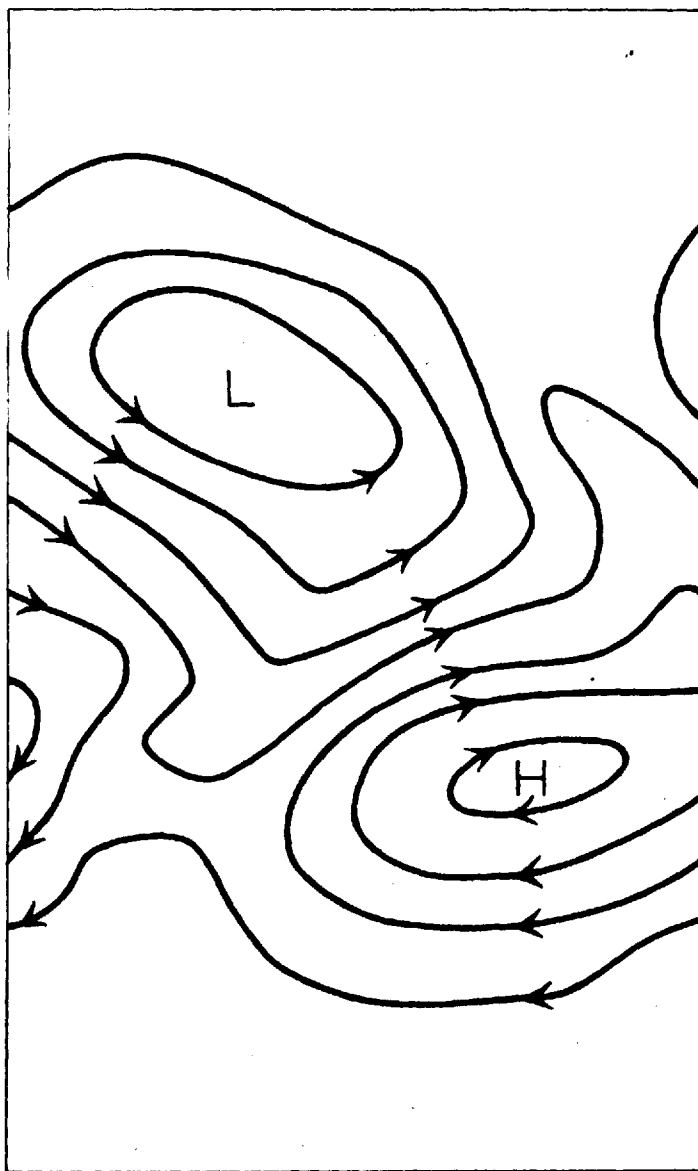


Fig 4.8 (b)
see opposite
page 126.

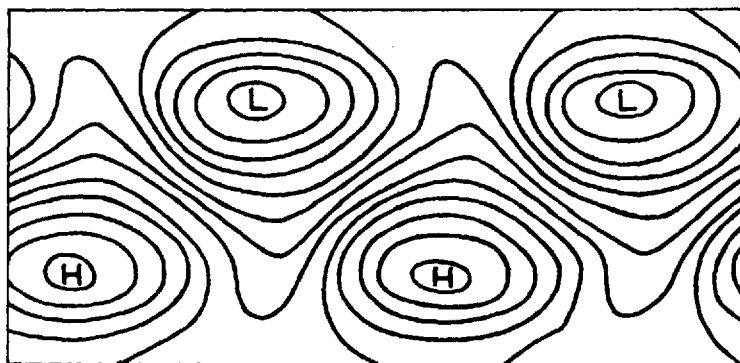
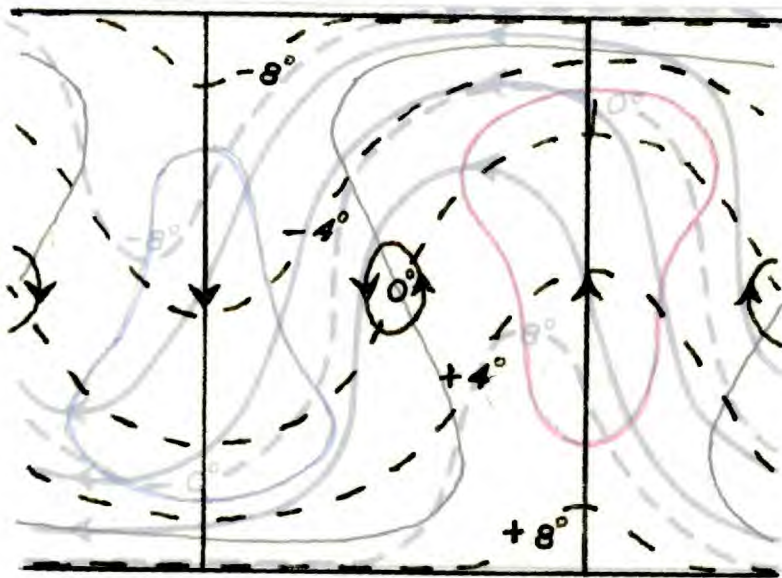


Fig 4.9 (b) Contours of 1000 mb. surface after
Huss and Mintz (1960).

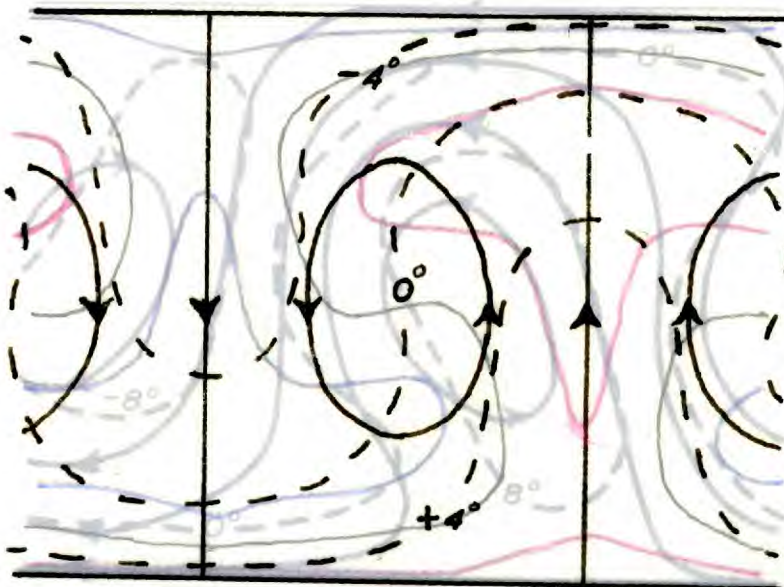
Fig 4.10 (a) Stream lines for the geostrophic flow relative to the initial mean zonal wind speed (—→) and isotherms (— — —) at mid levels for the eight parameter solution.

Fig 4.10 (b) Vertical velocities at mid levels. Upward motion shown red, zero vertical motion black and downward motion blue.

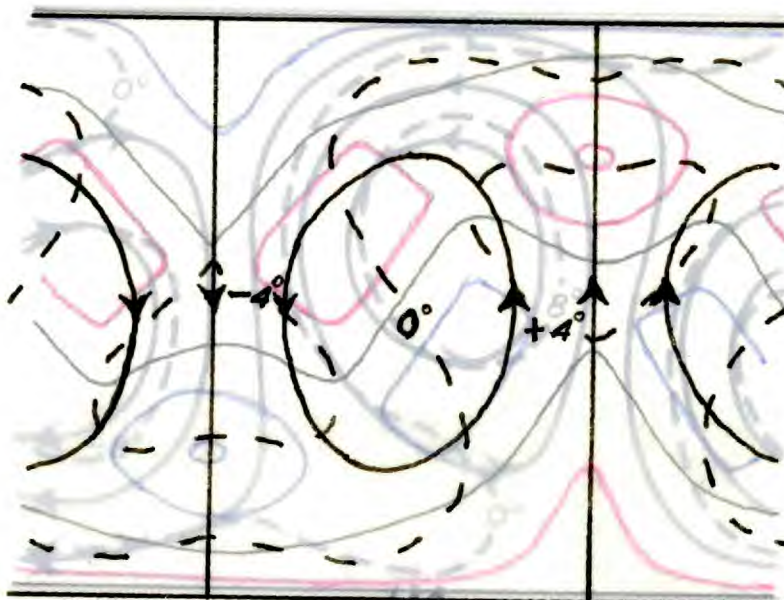
Fig 4.10 (c) Stream lines for the geostrophic flow relative to the initial mean zonal wind speed (—→) and isotherms (— — —) at the lower lid.



$t = -3$

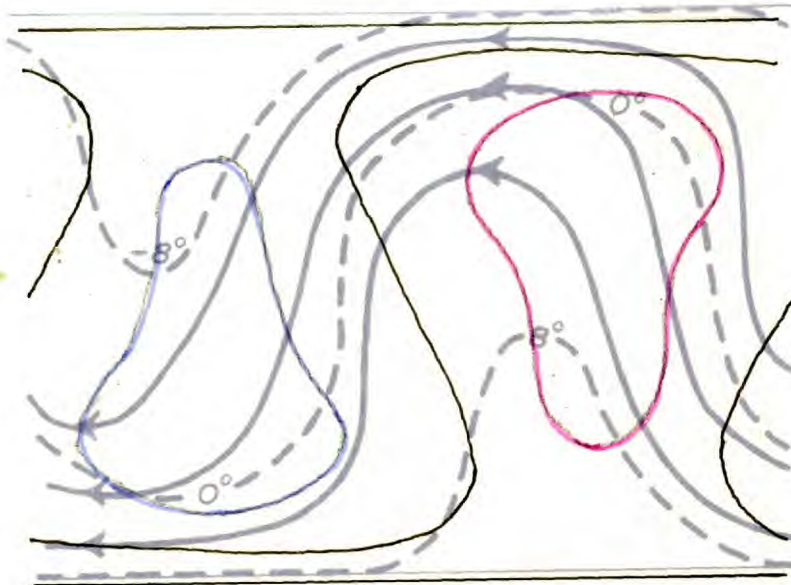


$t = -2$

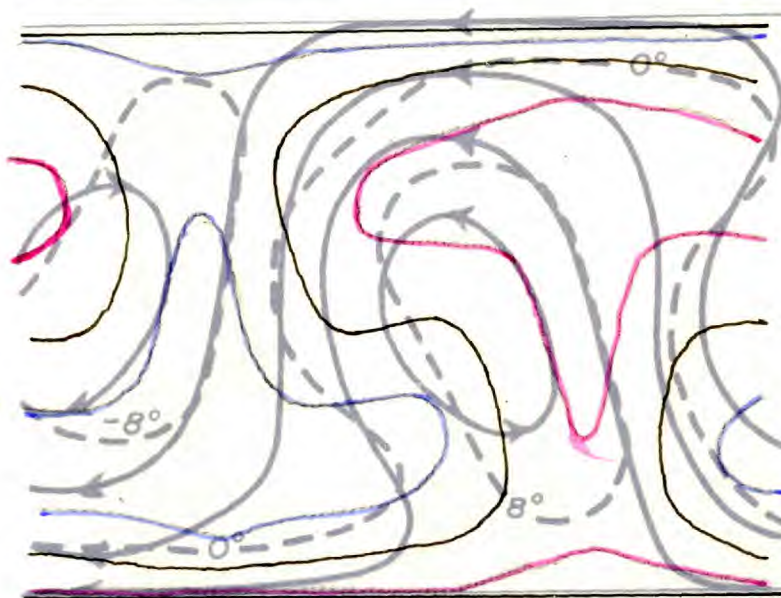


$t = -1$

Fig 4.10 (a)



$t = -3$

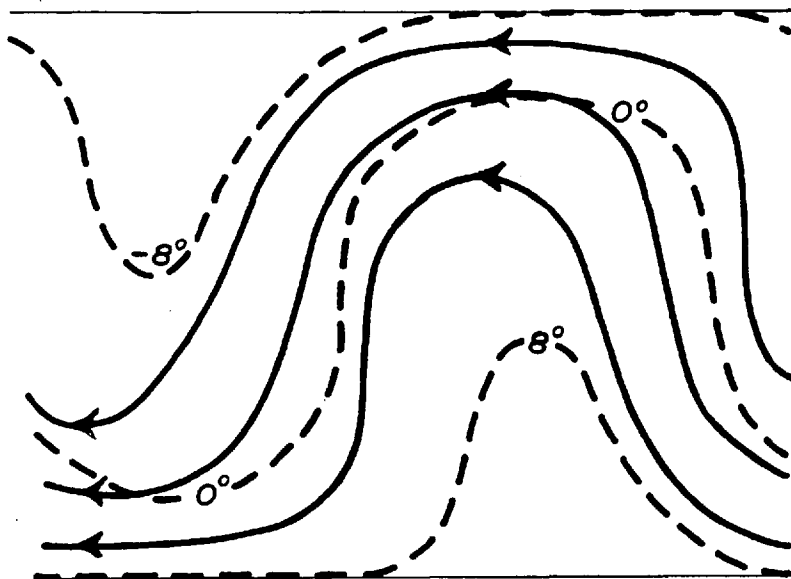


$t = -2$

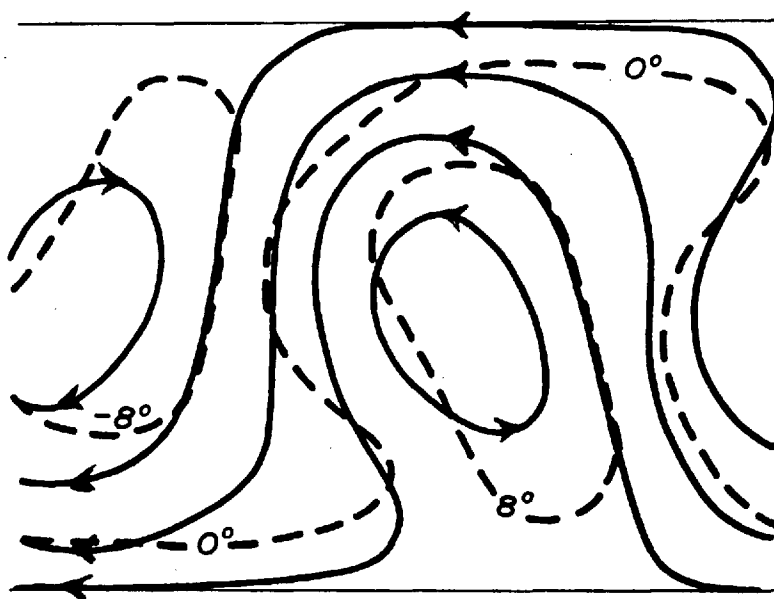


$t = -1$

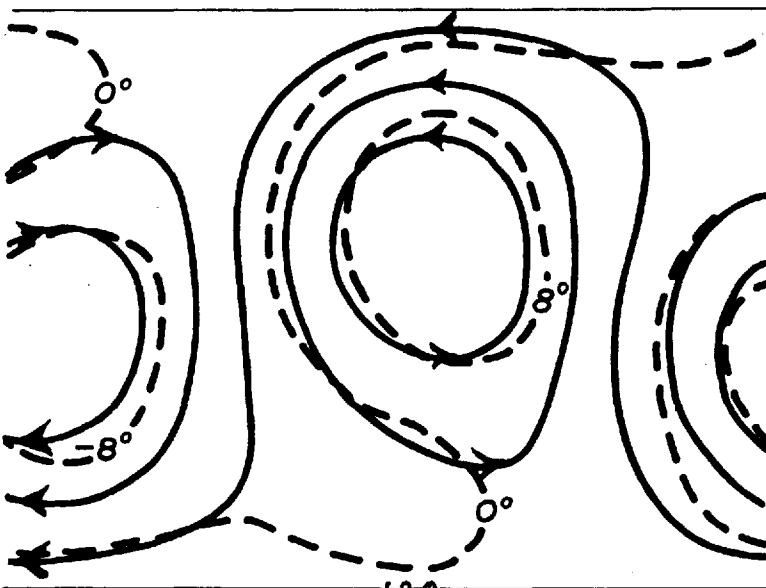
Fig 4.10 (b)
see opposite
page 128.



$t = -3$



$t = -2$

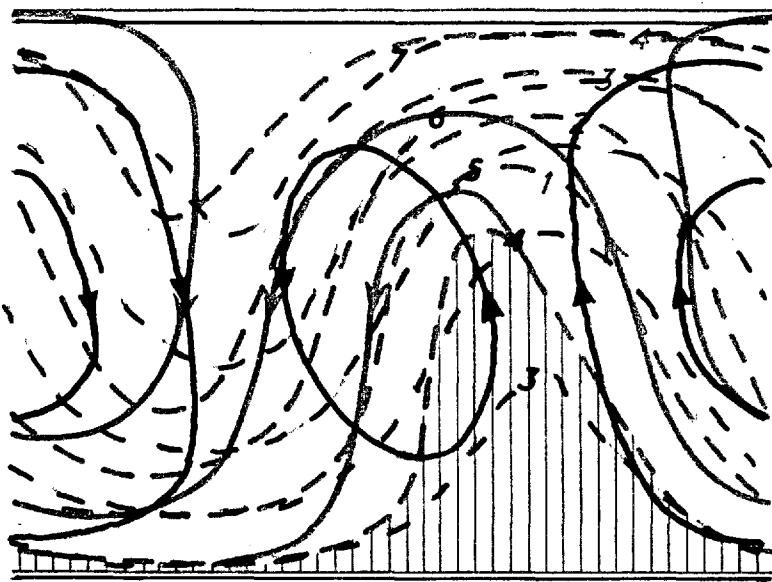


$t = -1$

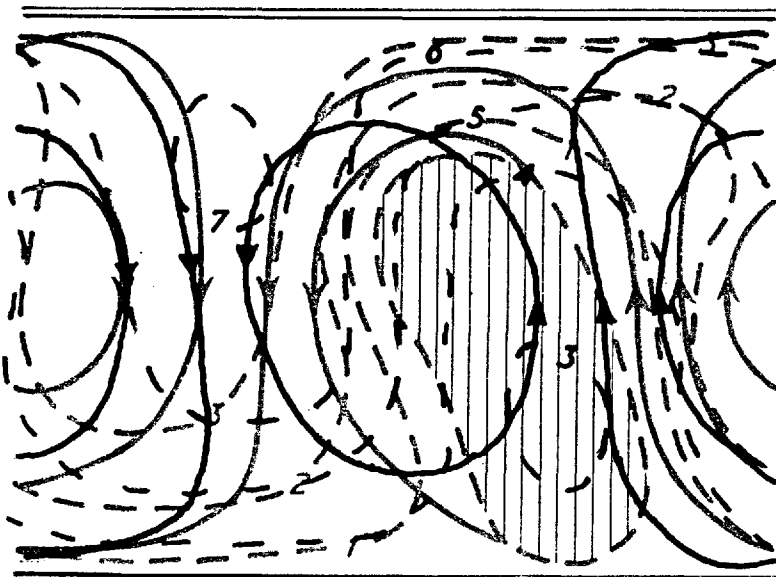
Fig 4.10 (c)
see opposite
page 128.

Fig 4.11 (a) Stream lines for the geostrophic flow relative to the initial mean zonal wind speed (——) and contours (— — —) on the 310°K isentropic surface for the eight parameter solution. Contour heights in km.

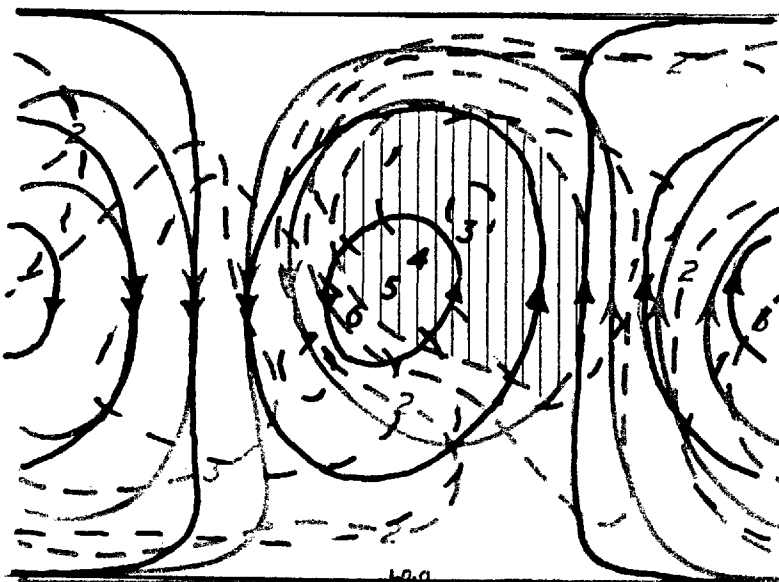
Fig 4.11 (b) Stream lines for the geostrophic flow relative to the initial mean zonal wind speed (——) and contours (— — —) on the 300°K isentropic surface for the eight parameter solution. Contour heights in km. Intersection with lower lid shaded.



$t = -3$

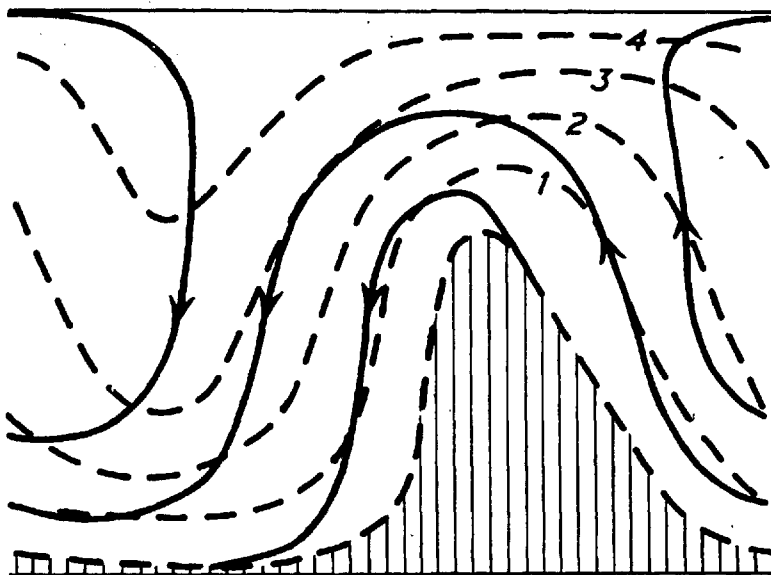


$t = -2$

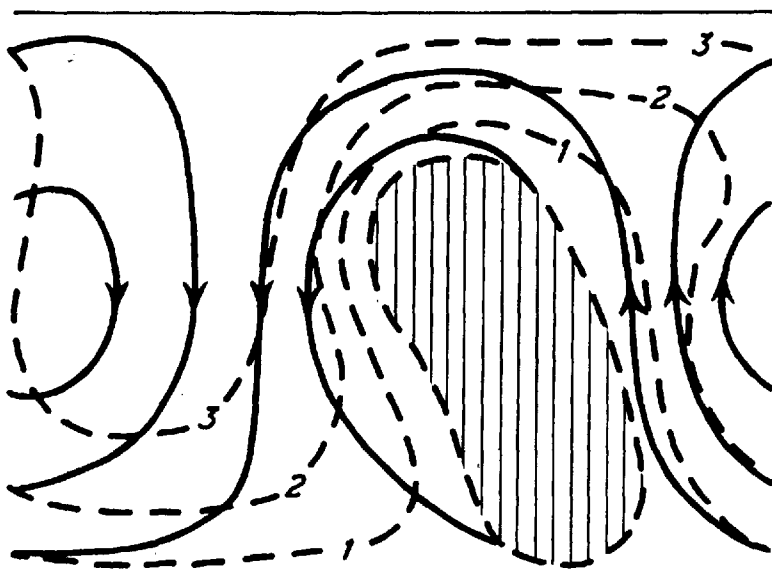


$t = -1$

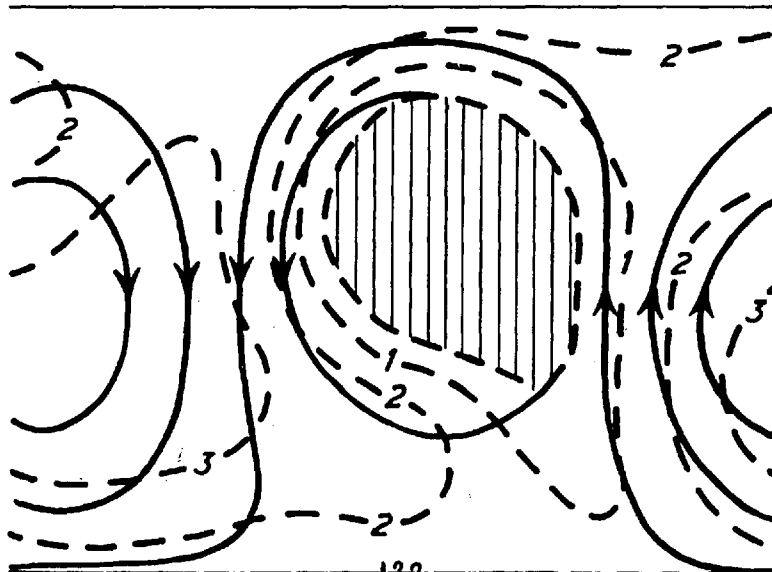
Fig 4.11 (a)
see opposite
page 131.



$t = -3$



$t = -2$



$t = -1$

Fig 4.11 (b)
see opposite
page 131.

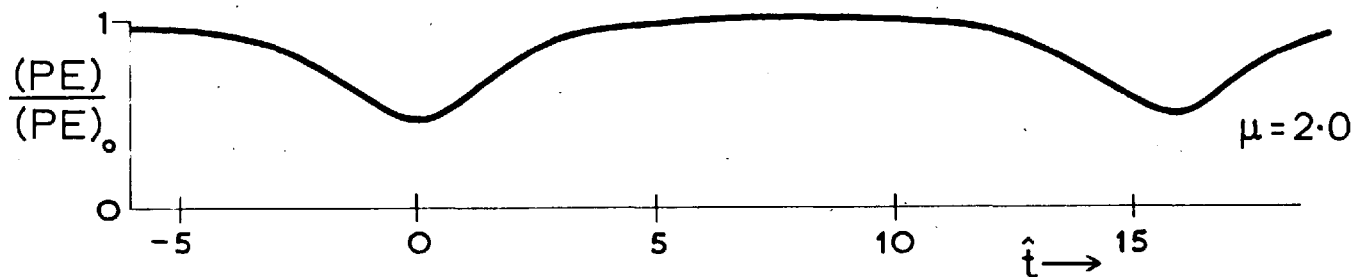
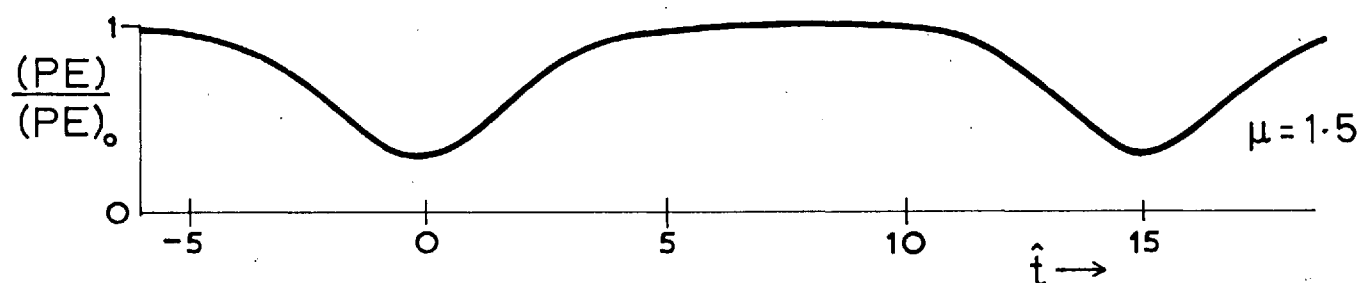
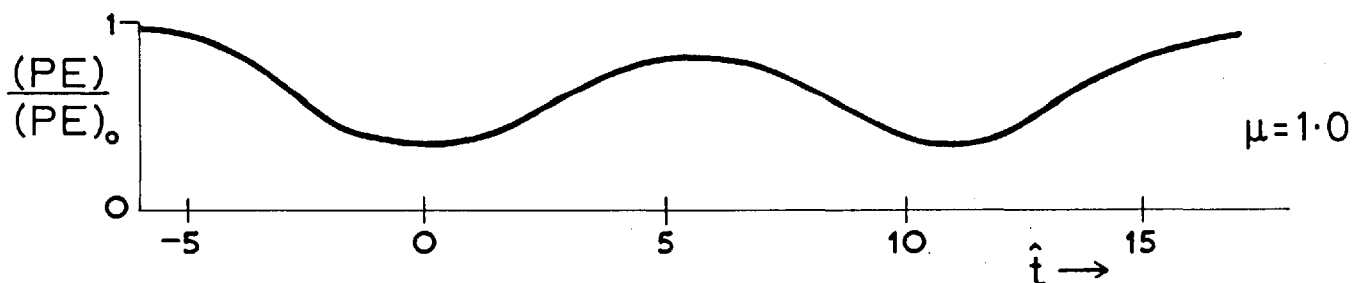
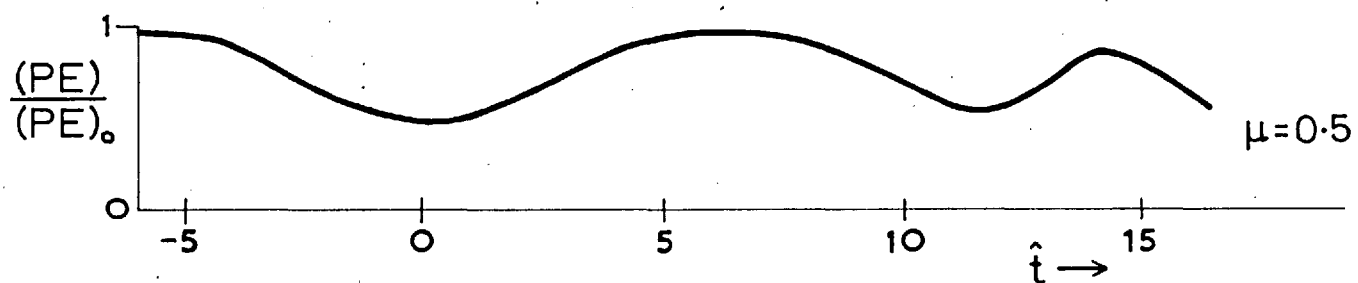


Fig 4.12 Variation of the scaled potential energy $\frac{(PE)}{(PE)_0}$ with the scaled time variable $\hat{t} = 2\sigma t$ for different values of μ .

CHAPTER 5. SURFACE FRICTION.

5.1 The Ekman Boundary Layer.

The absence of frictional dissipation in our calculations leads to the result that the total energy is conserved. Thus a state of minimum potential energy must correspond to a maximum of kinetic energy. The only ways in which the disturbance can then decay is either to convert its kinetic energy back into potential energy (as in the three parameter solution , see Chapter 3), or to transfer its kinetic energy to eddies of different scales.

In the atmosphere, however, kinetic energy is lost because the surface of the earth, which is not smooth, exerts a retarding force on air moving close to it. A further sink of kinetic energy is due to dissipation by eddy viscosity within the atmosphere. Thus a disturbance has a decay time determined by the roughness of the ground and the structure of the disturbance itself.

The influence of surface friction is communicated to the air above principally by eddying motion, which sets up and maintains a shear (and shearing stress) throughout a finite depth of the atmosphere. The nature of these eddies depends on the distribution of mean wind speed near the ground and also on its thermal stability, the surface roughness, and the height above the ground. Exact representation in a model atmosphere is impossible because the eddy scale is much smaller than that of cyclone waves and is difficult to resolve, but the effects of friction can be conveniently introduced using methods which rely on empirically determined constants.

Previous authors have included friction in their

models of the atmosphere in a variety of ways. For example Phillips (1956) takes the horizontal component of the vertical stress $\underline{\tau} = (\tau_{xz}, \tau_{yz})$ to be at a constant angle to the wind at the surface, proportional to its speed and decreasing rapidly with height. Mintz (1958) considers $\underline{\tau}$ to vary with height but only in its magnitude, whilst its direction is that of the surface wind. Smagorinsky (1965) assumes $\underline{\tau}$ to be proportional to $|\partial v / \partial z|^2$ and parallel to the wind shear in the main body of the atmosphere, while at the surface it is parallel to the wind and proportional to the square of its speed. These authors also take into account the transfer (horizontal and vertical) of momentum and heat by other 'sub-grid scale' eddies (e.g. small scale convection).

A simpler method of introducing friction into an atmospheric model such as ours was described by Charney and Eliassen (1949), and consists of bounding the atmosphere by an 'Ekman layer'. In this friction boundary layer (about half a kilometer deep) we assume that the geostrophic component of the wind is independent of height (justified because the shear due to the baroclinity is smaller than that due to the stress), and apply the theory of the 'Ekman spiral' for which the relevant equations have been given by Brunt (1939). The friction induces an ageostrophic flow across the isobars, and integration vertically through the layer shows that the mean cross isobar flow (towards the regions of low pressure) is proportional to the geostrophic wind in the layer. After applying the continuity equation (10) and the boundary condition $w = 0$ at the earth's surface we find that vertical velocities are produced at the top of the layer which are proportional to the geostrophic relative vorticity there.

In order to include surface friction in our model we solve the same vorticity equation as before but use the frictionally induced vertical velocity $w = k \zeta / h$ as a new lower lid boundary condition (k is a constant which depends principally on the roughness of the earth's surface). This condition is of course applied at the top of the friction layer (the bottom of the main body of the atmosphere) and it is convenient to choose the vertical coordinate in such a way that $z = -\frac{1}{2}$ refers to the top of Ekman layer while $z = +\frac{1}{2}$ refers (as before) to the upper lid. Thus we must satisfy

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = k \frac{\partial \phi}{\partial z}$$

at $z = -\frac{1}{2}$, where

$$k = \sin 2\alpha \left(\frac{K}{2f} \right)^{1/2} \frac{h}{H}$$

K is the eddy diffusivity (assumed to be independent of height) and α is the angle between the surface wind and the isobars. Taking the values given by Brunt, i.e. $K = 10 \text{ m}^2 \cdot \text{sec}^{-1}$ and $\alpha = 22.5^\circ$ together with those previously adopted in section 1.2, $f = 10^{-4} \text{ sec}^{-1}$, $H = 10 \text{ km.}$, $h^2 = 25$, we find that $k \approx 0.15$.

We consider the same initial zonal flow as in Chapter 2, namely a westerly wind independent of y but linearly increasing with height, and take the speed of the initial zonal wind at the bottom (i.e. at the top of the friction layer) to be zero so that initially there is no frictionally induced flow and it is a steady state exact solution. However, as in the frictionless calculation, it is mathematically convenient to use coordinates moving with the speed of the initial mean zonal wind. At first it was thought that a problem having friction layers at both the top and bottom of the atmosphere would be an instructive one, particularly

simple to solve because it would preserve symmetry about mid levels. An answer can be found by imposing frictionally induced vertical velocities at both the top and bottom but it is difficult to justify the presence of an initial cross isobar flow induced in one or both the Ekman layers with the requirement of a steady initial state (but see section 6.6).

5.2 Conservation equations.

For the present atmospheric model (with surface friction) the conservation equations of section 1.5 are modified. Thus it may be verified that the energy equation has the form

$$\frac{d}{dt} \overline{\frac{1}{2} (\nabla \psi_R)^2}^{x,y,z} = k \overline{[(\zeta \psi_R)(z=-\frac{1}{2})]}^{x,y} \quad (64)$$

where ψ_R is the stream function for the geostrophic wind relative to the ground. The term on the right hand side may be written

$$- k \left\{ \left(\frac{\partial \psi_R}{\partial x} \right)^2 + \left(\frac{\partial \psi_R}{\partial y} \right)^2 \right\}^{x,y} (z=-\frac{1}{2}) = - k \overline{v_R^2}^{x,y} (z=-\frac{1}{2})$$

and represents the energy dissipation due to friction.

It may also be verified that the following equations hold:-

$$\frac{d}{dt} \overline{\left(\frac{1}{2} \phi^2 \right)}^{x,y} = 0 \quad (65)$$

at the upper boundary ($z = +\frac{1}{2}$),

$$\frac{d}{dt} \overline{\left(\frac{1}{2} \phi^2 \right)}^{x,y} = k \frac{\partial}{\partial z} \overline{\left(\frac{1}{2} \phi^2 \right)}^{x,y} \quad (66)$$

at the top of the friction layer ($z = -\frac{1}{2}$),

$$\frac{d}{dt} \overline{u_R}^{x,y,z} = - k \overline{u_R}^{x,y} (z=-\frac{1}{2}) \quad (67)$$

$$\frac{d \overline{\zeta}^{x,y}}{dt} = 0 \quad (68)$$

$$\frac{d \overline{\phi}^{x,y}}{dt} = 0 \quad (69)$$

The last two conservation equations are identical with (21) and (22) for the frictionless atmosphere. It may further be shown that, while the mean meridional entropy gradient is conserved on the upper lid, at the top of the friction layer we have

$$\frac{d}{dt} \frac{\partial \overline{\phi}^{x,y}}{\partial y} = k \frac{\partial}{\partial z} \left(\frac{\partial \overline{\phi}}{\partial y} \right)^{x,y} \quad (70)$$

5.3 The Linear Perturbation Solution with Surface Friction.

As in Chapter 2 we examine constant shape solutions (27) of the linearised vorticity equation (25). Application of the linearised boundary conditions on the upper lid and at the top of the friction layer gives the following eigen-value condition for the existence of non-trivial solutions

$$\omega = - \frac{i k v \omega_H v}{2} \pm \sqrt{- \left(\frac{k v \omega_H v}{2} \right)^2 + \hat{\omega}^2 - i k \lambda (1 - \frac{v}{2} \omega_H v)} \quad (71)$$

where

$$\hat{\omega}^2 = \frac{\lambda^2}{v^2} \left(1 + \frac{v^2}{4} - v \omega_H v \right)$$

After some algebraic manipulation we find that the constant shape solution may be written in the form

$$\psi_i \sim e^{\omega_i t} \left[b^* S_h v z \sin(\lambda x - \omega_i t) - (a^* S_h v z + c_h v z) \cos(\lambda x - \omega_i t) \right] \sin_{\mu} (y + \frac{1}{2})$$

satisfying all the linearised boundary conditions. The parameters a^* and b^* are defined by

$$a^* = \left[\frac{\omega_r v}{\lambda} \left(\frac{v}{2} G H \frac{v}{2} - 1 \right) + \frac{k v^2}{2\lambda} \left(\frac{\omega_i v}{\lambda} + \frac{k v^2}{2\lambda} G H \frac{v}{2} \right) \right] / \left[\left(\frac{v}{2} G H \frac{v}{2} - 1 \right)^2 + \frac{k^2 v^4}{4\lambda^2} \right]$$

$$b^* = \left[\frac{\omega_i v}{\lambda} \left(\frac{v}{2} G H \frac{v}{2} - 1 \right) + \frac{k v^2}{2\lambda} \left(-\frac{\omega_r v}{\lambda} + G H \frac{v}{2} \left(\frac{v}{2} G H \frac{v}{2} - 1 \right) \right) \right] / \left[\left(\frac{v}{2} G H \frac{v}{2} - 1 \right)^2 + \frac{k^2 v^4}{4\lambda^2} \right]$$

The behaviour of these frictional waves is rather complicated. First we note that all solutions are either growing or decaying (except for the limiting cases $\lambda = 0$ and $\lambda = \infty$) i.e. $\omega_i \neq 0$ and there are no neutral waves. The dependence of the growth rate ω_i on the zonal wavenumber λ and the frictional constant k (when $\mu = 1$) is shown in Fig 5.2. It can be seen that the influence of friction is such as to encourage the diminishing waves ($\omega_i < 0$) and to reduce the growth rate of the unstable waves ($\omega_i > 0$). For large values of λ (in the regime where the Eady ($k = 0$) waves are neutral) we have, instead of a neutral pair, one strongly decaying and one weakly growing solution.

The Eady ($k = 0$) growing and decaying waves are stationary with respect to the mean zonal flow, but this is not so when we introduce friction into the calculation (see Fig 5.1). The phase speed ω_r/λ relative to the mean flow can only vanish when the expression under the square root in (71) is real and negative. This is impossible unless the imaginary term $-i k \lambda (1 - \frac{v}{2} G H \frac{v}{2})$ vanishes, which occurs when $\frac{v}{2} G H \frac{v}{2} = 2$ i.e. when $v \approx 1.9$. For $\mu = 1$ this corresponds to $\lambda \approx 1.6$. When λ is smaller than 1.6 the decaying wave has its steering level in the upper half of the atmosphere and hence moves eastwards relative to the mean flow, while the unstable solution is a lower (westward moving) wave. As $\lambda \rightarrow 0$ we find that

$$\omega_r/\lambda \rightarrow \mp \left(1 - \frac{\mu}{2} G H \mu \right) / \mu G H \mu$$

i.e. to a value independent of k . Evidently the $\lambda = 0$ Eady wave is a singular solution since it does not appear as the limit of the frictional solution as $k \rightarrow 0$. This singularity remains if we let $\mu \rightarrow 0$ as well, the limiting phase speeds as both λ and μ tend to zero being $\pm \frac{1}{2}$. The relevance of this limiting solution is discussed in the following paragraph.

Taking the limit as $\lambda \rightarrow 0$ of the unstable frictional solution we obtain

$$\lim_{\lambda \rightarrow 0} \psi_1 = \text{Const} \times (CH^{1/2} SH^{1/2} + SH^{1/2} CH^{1/2}) \cos \mu (y + \frac{y}{2}) \quad (72)$$

which satisfies the side boundary condition

$$\lim_{\lambda \rightarrow 0} \frac{\partial^2 \overline{\psi_1}}{\partial t \partial y} = 0 \quad (73)$$

if

$$\mu = n\pi / \gamma \quad (n = 0, 1, 2, \dots)$$

We note that its algebraic form is such that $\zeta(z = -\frac{1}{2}) \equiv 0$ i.e. no frictionally induced vertical velocities occur.

It may be verified that the stream function $\psi_0 + \lim_{\lambda \rightarrow 0} \psi_1$ is an exact solution of the full non-linear equations and boundary conditions. Evidently it represents a steady zonal flow which is different from the initial state $\psi_0 = -yz$ but, since it satisfies the side boundary condition (73), is 'dynamically accessible' from ψ_0 without violating the energy equation. This suggests the possibility of a baroclinic disturbance growing to finite amplitude and then decaying away under the action of friction leaving a final state having the form of the limiting $\lambda \rightarrow 0$ solution. This is verified later in section 5.4. It may be shown that the total energy $\frac{1}{2} \int \nabla (\psi_0 + \lim_{\lambda \rightarrow 0} \psi_1)^2 dx dy$ is greater than that of the initial state (i.e. $\psi_0 + \lim_{\lambda \rightarrow 0} \psi_1$ does not correspond to a possible final state) unless n is odd. It should be

noted that this solution is not an attainable final state for a frictionless atmosphere since (for n odd)

$\lim_{\lambda \rightarrow 0} \overline{u_1^{2n, 2n, 2}} \neq 0$, and this cannot be achieved without violating the conservation equation (20).

As has already been pointed out above the long wave differ from the Eady long waves in that they are not stationary relative to the mean flow. This may be understood by referring to the structure of the Eady solution. Eady's growing solution (Fig 2.3) has upward motion at mid levels associated with (i.e. slightly ahead of) the surface lows, while descent is similarly associated with the highs. Surface friction induces vertical motion at the bottom of the atmosphere, upward in cyclones and downward in anticyclones, having the same sense as at mid levels. The frictional growing solution is subject to the combined (additive) effect of both patterns of w and behaves as if it were in a deeper atmosphere than is the case, thereby having a steering level below the mid level. On the other hand Eady's decaying solution has descent associated with the cyclones and ascent with the anticyclones (see Fig 2.12), i.e. in the opposite sense to that of the frictionally induced motion. The frictional decaying solution therefore appears to be in a shallower atmosphere and has its steering level above the mid level. These ideas are illustrated in Fig 5.6 which shows the vertical motion profiles in the vicinity of the cyclone with and without friction.

For the short wave solutions ($\lambda > 1.6$) the situation is different, the upper (eastward moving) waves being unstable (weakly growing) and the lower waves being stable (strongly decaying). We explain the non-neutral character of the short waves by examining the effect of the frictionally induced vertical motion

on the short (neutral) Eady waves.

The frictionless neutral solutions (see Fig 2.13) have vertical velocities 90° out of phase with the vorticity maxima, and this enables us to consider (as a first approximation) that the baroclinically and frictionally induced vertical motion patterns act independently. The baroclinic vertical motion for the Eady ($k = 0$) solution is upward in the northward moving airstream to the east of the cyclones, and downward to the west. However the particle displacements are essentially along isentropic surfaces and no latitudinal heat transfer occurs. In addition to vertical and meridional motion the air is also subject to a zonal flow which is westward relative to the moving wave below the steering level, and eastward above. The effect of friction is to lift air passing through the centre of the cyclones and lower the air moving through the anticyclones. Thus below the steering level air flowing westwards through the cyclone towards the region of adiabatic (southward moving) descent has been lifted from a lower level, while air descended from a higher level in the anticyclone moves westwards into the region of northward moving ascent. Thus below the steering level the effective path of the particles has a slope less than that of the zonal mean isentropic surfaces, leading to a positive correlation between w and ϕ . Above the steering level the process is different since the lifted air moves eastwards towards the ascending northward flow and the air lowered in the anticyclones moves into the region of descent. This leads to an effective trajectory steeper than the meridional slope of the isentropic surfaces, and a negative value of $\overline{w\phi}^{y,z}$. These effects are illustrated in Fig 5.8. If the steering level is in the upper part of the atmosphere the transfer process occurring below dominates, i.e. $\overline{w\phi}^{y,z} > 0$ and

the wave is unstable. On the other hand for a lower wave the dominating process is that occurring above the steering level, leading to a negative correlation between w and Φ and a diminishing wave.

If μ is large ($\mu > 1.9$ approx.) (i.e. a narrow zone) then

$$1 - \frac{\mu}{2} \coth \mu < 0$$

and so (since $v > \mu$) the imaginary term under the square root in the expression (73) for ω never vanishes for any λ ($\neq 0$). Thus in a narrow baroclinic zone with surface friction all growing waves are upper waves progressing eastwards faster than the mean zonal wind. This can be seen clearly in Fig 5.3 which shows the phase speed ω_r/λ for the unstable wave (with $k = 0.15$) as a function of the wave numbers λ and μ .

The dependence of the growth rate ω_i for the unstable solution (also for $k = 0.15$) is shown in Fig 5.4. This diagram is similar to Fig 2.2 (for the frictionless Eady waves) apart from the absence of a short wave cut-off. Comparing these two figures we find that for the frictional solution the zonal wavenumber for the dominating wave (i.e. the wave with the largest growth rate for a given μ) is rather less dependent on μ , and that for small μ it has about the same value ($\lambda_{max} \approx 1.5$) as for the dominant Eady wave. Furthermore the dominant frictional wave for $\mu = 1$ is almost stationary relative to the mean zonal flow (and therefore similar to the frictionless solution).

The vertical cross section along a line of latitude through the dominant solution for $\mu = 1$ is shown in Fig 5.5 (cf. Fig 2.3). Its structure is similar to that of the Eady wave, the main differences being the

the larger variation of phase with height (of the pressure wave) and the lesser intensities of the vorticity centres at the bottom (compared with those aloft). It is evident that the introduction of friction has little influence on the structure of the dominant solution except to produce asymmetry about mid levels and a slightly smaller growth rate. When k is large the effect of friction is more pronounced but little, if any, change occurs in the dominant wavelength (see Fig 5.2).

5.4 The Finite Amplitude (Six Parameter) Solution with Surface Friction.

We assume, as before, that the structure of the infinitesimal disturbance approaches that of the dominant unstable solution. A finite amplitude solution may now be found by employing the method of functional representation in a similar way to that discussed in Chapter 3. However, because the frictional solution is not symmetric about mid levels, it is necessary to use twice the number of time dependent parameters for each Fourier component. Thus the six parameter solution is the simplest one can investigate, and this is the frictional equivalent of the frictionless three parameter solution (see section 3.4). We write it in the form (relative to coordinates moving with the initial mean zonal wind)

$$\begin{aligned} \psi = & -yz + (S_{0,1} Sh_{\mu} z + \hat{S}_{0,1} Ch_{\mu} z) \sin \mu y \\ & + \left[(R_{1,1} Sh_{\nu} z + \hat{R}_{1,1} Ch_{\nu} z) \sin \lambda x + (S_{1,1} Ch_{\nu} z + \hat{S}_{1,1} Sh_{\nu} z) \cos \lambda x \right] \cos \mu y \end{aligned} \quad (74)$$

where the parameters $S_{0,1}$, $\hat{S}_{0,1}$, $R_{1,1}$, $\hat{R}_{1,1}$, $S_{1,1}$, $\hat{S}_{1,1}$ are found by fitting this solution to the lid boundary conditions using the method of least squares.

With the introduction of friction we no longer have total energy conserved, but we can write down an energy equation including a frictional dissipation term. Thus inserting (74) into the energy equation (64) we find that it takes the form

$$\begin{aligned}
 & 2Shv \left(\dot{R}_{1,1} \dot{\hat{R}}_{1,1} + \dot{S}_{1,1} \dot{\hat{S}}_{1,1} + \dot{\hat{S}}_{1,1} \dot{\hat{S}}_{1,1} \right) \\
 & + 2\mu Sh\mu \left(\dot{S}_{0,1} \dot{\hat{S}}_{0,1} + \dot{\hat{S}}_{0,1} \dot{\hat{S}}_{0,1} \right) - \frac{8}{\pi} \left(a^{1/2} \dot{S}_{0,1} + Sh^{1/2} \dot{\hat{S}}_{0,1} \right) \quad (75) \\
 & = -k \left[v^2 \left\{ \left(\dot{R}_{1,1} Sh^{1/2} - \dot{\hat{R}}_{1,1} a^{1/2} \right)^2 + \left(\dot{S}_{1,1} a^{1/2} - \dot{\hat{S}}_{1,1} Sh^{1/2} \right)^2 \right\} \right. \\
 & \quad \left. + 2\mu^2 \left(\dot{S}_{0,1} Sh^{1/2} - \dot{\hat{S}}_{0,1} a^{1/2} \right)^2 \right]
 \end{aligned}$$

To be consistent we must use winds (and the geostrophic stream function) relative to the ground. In the frictionless case this was unnecessary because the kinetic energy of the winds relative to the moving coordinate system differed from that of winds relative to the ground only by an additive constant (the KE of the mean zonal flow), but the inclusion of friction causes a decay in the mean zonal wind, and hence a non zero rate of change of its KE. The left hand side of (75) represents the rate of change of total energy while the right hand side is the frictional dissipation.

Substitution into equations (65), (66) and (67) gives the following:-

$$\begin{aligned}
 & v^2 \left(\dot{R}_{1,1} a^{1/2} + \dot{\hat{R}}_{1,1} Sh^{1/2} \right) \left(\dot{R}_{1,1} a^{1/2} + \dot{\hat{R}}_{1,1} Sh^{1/2} \right) \\
 & + v^2 \left(\dot{S}_{1,1} Sh^{1/2} + \dot{\hat{S}}_{1,1} a^{1/2} \right) \left(\dot{S}_{1,1} Sh^{1/2} + \dot{\hat{S}}_{1,1} a^{1/2} \right) \quad (76) \\
 & + 2\mu^2 \left(\dot{S}_{0,1} a^{1/2} + \dot{\hat{S}}_{0,1} Sh^{1/2} \right) \left(\dot{S}_{0,1} a^{1/2} + \dot{\hat{S}}_{0,1} Sh^{1/2} \right) \\
 & - \frac{8}{\pi} \left(\dot{S}_{0,1} a^{1/2} + \dot{\hat{S}}_{0,1} Sh^{1/2} \right) = 0
 \end{aligned}$$

$$\begin{aligned}
& v^2 (\hat{R}_{1,1} c h^{1/2} - \hat{R}_{1,1} s h^{1/2}) (R_{1,1} c h^{1/2} - \hat{R}_{1,1} s h^{1/2}) \\
& + v^2 (\hat{S}_{1,1} s h^{1/2} - \hat{S}_{1,1} c h^{1/2}) (S_{1,1} s h^{1/2} - \hat{S}_{1,1} c h^{1/2}) \\
& + 2\mu^2 (\hat{S}_{0,1} c h^{1/2} - \hat{S}_{0,1} s h^{1/2}) (S_{0,1} c h^{1/2} - \hat{S}_{0,1} s h^{1/2}) \\
& - \frac{8}{\pi} (\hat{S}_{0,1} c h^{1/2} - \hat{S}_{0,1} s h^{1/2}) \\
& = k \left\{ \frac{v^3}{2} [(R_{1,1} s h^{1/2} - \hat{R}_{1,1} c h^{1/2})^2 + (S_{1,1} c h^{1/2} - \hat{S}_{1,1} s h^{1/2})^2] \right. \\
& \quad \left. + \mu^3 (S_{0,1} s h^{1/2} - \hat{S}_{0,1} c h^{1/2})^2 + \frac{8}{\pi} \mu (S_{0,1} s h^{1/2} - \hat{S}_{0,1} c h^{1/2}) \right\}
\end{aligned} \tag{77}$$

$$s h^{1/2} \hat{S}_{0,1} = k (S_{0,1} s h^{1/2} - \hat{S}_{0,1} c h^{1/2}) \tag{78}$$

As before we find a distribution for $\partial\phi/\partial t$ which accounts for, as nearly as possible, the temperature changes implied by the advection at the upper and lower boundaries (remembering that frictionally induced vertical advection now occurs at the bottom).

The entropy Φ has the form

$$\begin{aligned}
\Phi = & -y + \mu (S_{0,1} c h \mu z + \hat{S}_{0,1} s h \mu z) \sin \mu y \\
& + v [(R_{1,1} c h v z + \hat{R}_{1,1} s h v z) \sin \lambda x + (S_{1,1} s h v z + \hat{S}_{1,1} c h v z) \cos \lambda x] \cos \mu y
\end{aligned}$$

and we define new time dependent parameters

$$\begin{aligned}
B_{0,1} &= \mu (S_{0,1} c h^{1/2} + \hat{S}_{0,1} s h^{1/2}), & \hat{B}_{0,1} &= \mu (S_{0,1} c h^{1/2} - \hat{S}_{0,1} s h^{1/2}) \\
A_{1,1} &= v (R_{1,1} c h^{1/2} + \hat{R}_{1,1} s h^{1/2}), & \hat{A}_{1,1} &= v (R_{1,1} c h^{1/2} - \hat{R}_{1,1} s h^{1/2}) \\
B_{1,1} &= v (S_{1,1} s h^{1/2} + \hat{S}_{1,1} c h^{1/2}), & \hat{B}_{1,1} &= v (-S_{1,1} s h^{1/2} + \hat{S}_{1,1} c h^{1/2})
\end{aligned}$$

(representing the time dependence of the Fourier components of Φ at $z = \pm \frac{1}{2}$) and apply the minimisation conditions

$$\frac{\partial}{\partial \dot{B}_{0,1}} (\overline{w^2})^{x,y} = 0, \quad \frac{\partial}{\partial \dot{A}_{1,1}} (\overline{w^2})^{x,y} = 0, \quad \frac{\partial}{\partial \dot{B}_{1,1}} (\overline{w^2})^{x,y} = 0$$

at the top, and

$$\frac{\partial}{\partial \dot{B}_{0,1}} (\overline{w^2})^{x,y} = 0, \quad \frac{\partial}{\partial \dot{A}_{1,1}} (\overline{w^2})^{x,y} = 0, \quad \frac{\partial}{\partial \dot{B}_{1,1}} (\overline{w^2})^{x,y} = 0$$

at the bottom. This gives six equations from which we may calculate $S_{0,1}$, $\hat{S}_{0,1}$, $R_{1,1}$, $\hat{R}_{1,1}$, $S_{1,1}$, $\hat{S}_{1,1}$. The non-linear differential equations defining these functions are

$$\dot{S}_{0,1} + a_1 (R_{1,1} S_{1,1} - \hat{R}_{1,1} \hat{S}_{1,1}) + \frac{k\mu}{2} (S_{0,1} \tanh \frac{\nu}{2} - \hat{S}_{0,1}) = 0 \quad (79)$$

$$\dot{\hat{S}}_{0,1} - \frac{k\mu}{2} (S_{0,1} - \hat{S}_{0,1} \coth \frac{\nu}{2}) = 0 \quad (80)$$

$$\dot{R}_{1,1} + a_2 S_{1,1} - a_3 S_{1,1} S_{0,1} + b_3 \hat{S}_{1,1} \hat{S}_{0,1} + \frac{k\nu}{2} (R_{1,1} \tanh \frac{\nu}{2} - \hat{R}_{1,1}) = 0 \quad (81)$$

$$\dot{\hat{R}}_{1,1} - a_4 \hat{S}_{1,1} + a_5 \hat{S}_{1,1} S_{0,1} - b_5 S_{1,1} \hat{S}_{0,1} - \frac{k\nu}{2} (R_{1,1} - \hat{R}_{1,1} \coth \frac{\nu}{2}) = 0 \quad (82)$$

$$\dot{S}_{1,1} + a_4 R_{1,1} - a_5 R_{1,1} S_{0,1} + b_5 \hat{R}_{1,1} \hat{S}_{0,1} + \frac{k\nu}{2} (S_{1,1} \coth \frac{\nu}{2} - \hat{S}_{1,1}) = 0 \quad (83)$$

$$\dot{\hat{S}}_{1,1} - a_2 \hat{R}_{1,1} + a_3 \hat{R}_{1,1} S_{0,1} - b_3 R_{1,1} \hat{S}_{0,1} - \frac{k\nu}{2} (S_{1,1} - \hat{S}_{1,1} \tanh \frac{\nu}{2}) = 0 \quad (84)$$

where the coefficients $a_1 - a_5$ have been defined at the beginning of section 3.5 and

$$b_3 = \frac{2}{3\pi} \frac{\lambda\mu}{\nu} (\nu \coth \frac{\nu}{2} - \mu \sinh \frac{\nu}{2} \tanh \frac{\nu}{2})$$

$$b_5 = \frac{2}{3\pi} \frac{\lambda\mu}{\nu} (\mu \sinh \frac{\nu}{2} \coth \frac{\nu}{2} - \nu \coth \frac{\nu}{2})$$

In deriving these equations we have not imposed the energy equation (75) as a supplementary condition on the

minimisation process. However the solution does satisfy a pseudo energy equation

$$\begin{aligned}
 & v Sh v (R_{1,1} \dot{R}_{1,1} + \hat{R}_{1,1} \dot{\hat{R}}_{1,1} + S_{1,1} \dot{S}_{1,1} + \hat{S}_{1,1} \dot{\hat{S}}_{1,1}) \\
 & + 2\mu Sh \mu (S_{0,1} \dot{S}_{0,1} + \hat{S}_{0,1} \dot{\hat{S}}_{0,1}) - \frac{8}{\pi} (Ch \frac{\mu}{2} \dot{S}_{0,1} + Sh \frac{\mu}{2} \dot{\hat{S}}_{0,1}) \quad (85) \\
 & = -k [v \{ (R_{1,1} Sh \frac{v}{2} - \hat{R}_{1,1} Ch \frac{v}{2})^2 + (S_{1,1} Sh \frac{v}{2} - \hat{S}_{1,1} Ch \frac{v}{2})^2 \} \\
 & \quad + 2\mu^2 (S_{0,1} Sh \frac{\mu}{2} - \hat{S}_{0,1} Ch \frac{\mu}{2})^2] + \left(\frac{32}{3\pi^2} - 1\right) \lambda \mu (R_{1,1} S_{1,1} - \hat{R}_{1,1} \hat{S}_{1,1})
 \end{aligned}$$

which has a similar form to (75). Equations (79 - 84) also imply the following

$$\begin{aligned}
 & v^2 (\dot{R}_{1,1} Ch \frac{v}{2} + \hat{R}_{1,1} Sh \frac{v}{2}) (R_{1,1} Ch \frac{v}{2} + \hat{R}_{1,1} Sh \frac{v}{2}) \\
 & + v^2 (\dot{S}_{1,1} Sh \frac{v}{2} + \hat{S}_{1,1} Ch \frac{v}{2}) (S_{1,1} Sh \frac{v}{2} + \hat{S}_{1,1} Ch \frac{v}{2}) \\
 & + 2\mu^2 (\dot{S}_{0,1} Ch \frac{\mu}{2} + \hat{S}_{0,1} Sh \frac{\mu}{2}) (S_{0,1} Ch \frac{\mu}{2} + \hat{S}_{0,1} Sh \frac{\mu}{2}) \quad (86) \\
 & - \frac{8}{\pi} (\dot{S}_{0,1} Ch \frac{\mu}{2} + \hat{S}_{0,1} Sh \frac{\mu}{2})
 \end{aligned}$$

$$+ \lambda v \left(1 - \frac{32}{3\pi^2}\right) (R_{1,1} S_{1,1} - \hat{R}_{1,1} \hat{S}_{1,1}) = 0$$

$$\begin{aligned}
 & v^2 (\dot{R}_{1,1} Ch \frac{v}{2} - \hat{R}_{1,1} Sh \frac{v}{2}) (R_{1,1} Ch \frac{v}{2} - \hat{R}_{1,1} Sh \frac{v}{2}) \\
 & + v^2 (\dot{S}_{1,1} Sh \frac{v}{2} - \hat{S}_{1,1} Ch \frac{v}{2}) (S_{1,1} Sh \frac{v}{2} - \hat{S}_{1,1} Ch \frac{v}{2}) \\
 & + 2\mu^2 (\dot{S}_{0,1} Ch \frac{\mu}{2} - \hat{S}_{0,1} Sh \frac{\mu}{2}) (S_{0,1} Ch \frac{\mu}{2} - \hat{S}_{0,1} Sh \frac{\mu}{2}) \\
 & - \frac{8}{\pi} (\dot{S}_{0,1} Ch \frac{\mu}{2} - \hat{S}_{0,1} Sh \frac{\mu}{2}) \quad (87)
 \end{aligned}$$

$$+ \lambda v \left(1 - \frac{32}{3\pi^2}\right) (R_{1,1} S_{1,1} - \hat{R}_{1,1} \hat{S}_{1,1})$$

$$\begin{aligned}
 & = k \left\{ \frac{v^3}{2} [(R_{1,1} Sh \frac{v}{2} - \hat{R}_{1,1} Ch \frac{v}{2})^2 + (S_{1,1} Ch \frac{v}{2} - \hat{S}_{1,1} Sh \frac{v}{2})^2] \right. \\
 & \quad \left. + \mu^3 (S_{0,1} Sh \frac{\mu}{2} - \hat{S}_{0,1} Ch \frac{\mu}{2})^2 + \frac{8}{\pi} \mu (S_{0,1} Sh \frac{\mu}{2} - \hat{S}_{0,1} Ch \frac{\mu}{2}) \right\}
 \end{aligned}$$

These resemble (76) and (77) in much the same way as the pseudo energy equation resembles the actual energy equation. We note that equation (78) (the mean momentum equation) is satisfied identically by virtue of equation (80).

No analytic solutions could be found and so equations (79 - 84) were solved numerically using the Runge-Kutta procedure. They are shown in Fig 5.9. It is found that the last term in the pseudo energy equation (85) is negligible in comparison with the other terms (as in the frictionless solution NE described in section 3.5) and so the additional complication of satisfying the energy equation exactly by Lagrange multipliers was deemed unnecessary. Equations (86) and (87) are similarly sufficiently accurate approximations of equations (76) and (77).

We can see from Fig 5.9 that the six parameter solution with friction (6PF) behaves very differently from the three parameter frictionless solution (3P) (see Fig 3.1), particularly at large values of t . Unlike 3P the final state is not the same as the initial state (energy being irreversibly extracted by friction).

The parameters $S_{0,1}$ and $\hat{S}_{0,1}$ initially grow like $\exp\{2\omega_1 t\}$ (as for the frictionless solution) because of the dominance in equation (79) of the second term. However at large values of t they become nearly constant, while $R_{1,1}$, $\hat{R}_{1,1}$, $S_{1,1}$ and $\hat{S}_{1,1}$ oscillate with little change in amplitude or frequency. This suggests that as $t \rightarrow \infty$ the influence of friction becomes small, i.e. that the last terms in equations (79 - 8) are small, implying

$$\begin{aligned} S_{0,1} &\approx A_1 c h^{1/2} \\ \hat{S}_{0,1} &\approx A_1 s h^{1/2} \\ \hat{R}_{1,1} &\approx R_{1,1} f a h^{1/2} \\ \hat{S}_{1,1} &\approx S_{1,1} c a h^{1/2} \end{aligned}$$

Evidently $R_{1,1}$ and $\hat{R}_{1,1}$ are in phase, as are $S_{1,1}$ and $\hat{S}_{1,1}$. By substituting into equations (79 - 84) we find that the final solution suggested above is possible if

$$A_1 = \frac{3\pi}{8\mu \sinh \mu} \frac{(1 - v \cosh \mu v)}{(\mu \cosh \mu - v \cosh v)}$$

We then obtain

$$R_{1,1} = A_2 \cosh \frac{v}{2} \cos \frac{1}{2} \lambda t$$

$$\hat{R}_{1,1} = A_2 \sinh \frac{v}{2} \cos \frac{1}{2} \lambda t$$

$$S_{1,1} = A_2 \sinh \frac{v}{2} \sin \frac{1}{2} \lambda t$$

$$\hat{S}_{1,1} = A_2 \cosh \frac{v}{2} \sin \frac{1}{2} \lambda t$$

The constant A_2 is obtained by integrating equation (86) (the pseudo form of the conservation equation (65)) with respect to time to give

$$v^2 [(R_{1,1} \cosh \frac{v}{2} + \hat{R}_{1,1} \sinh \frac{v}{2})^2 + (S_{1,1} \sinh \frac{v}{2} + \hat{S}_{1,1} \cosh \frac{v}{2})^2] \\ + 2\mu^2 (S_{0,1} \cosh \frac{\mu}{2} + \hat{S}_{0,1} \sinh \frac{\mu}{2})^2 - \frac{3\pi}{2} (S_{0,1} \cosh \frac{\mu}{2} + \hat{S}_{0,1} \sinh \frac{\mu}{2}) = 0$$

whence substitution reveals that

$$A_2 = \frac{1}{v \cosh v} \left[\frac{3\pi}{4} \cosh \mu A_1 - \mu^2 \cosh^2 \mu A_1^2 \right]$$

The final state as $t \rightarrow \infty$ has the form

$$\psi = -y z + A_1 (\cosh \mu/2 \sinh \mu z + \sinh \mu/2 \cosh \mu z) \sin \mu y \quad (88) \\ + A_2 (\cosh \frac{v}{2} \sinh v z + \sinh \frac{v}{2} \cosh v z) \sin [\lambda(x + \frac{1}{2}t)] \cosh \mu y$$

It is readily verified that for this solution the geostrophic wind relative to the ground (and hence the relative vorticity) at the top of the friction layer is identically zero, i.e. no further frictionally induced flow occurs. Averaging with respect to x we find that the mean component of (88) has precisely the same form as the limiting ($\lambda \rightarrow 0$) perturbation solution (72). The eddy component of (88) consists of a wave having no phase variation with height, but its largest amplitude is at the top and moves with a phase speed $-\frac{1}{2}$ relative to the initial mean zonal flow, i.e. it is stationary relative to the ground. Thus the final state consists of an upper stationary wave but with no motion at the bottom. Closed eddies occur at the top giving rise to easterly winds at central latitudes, which since the zonal wind remains westerly at the sides (because of the side boundary condition $\partial \bar{u}' / \partial t = 0$) means that the average wind $\bar{u}^{x,y,z}$ is small.

It is of interest to know the rate at which the solution tends to the final state, i.e. the decay rate of the surface wave. This could be found by solving the set of linear differential equations obtained by setting $S_{0,1}$, $\hat{S}_{0,1}$, $R_{1,1}$, $\hat{R}_{1,1}$, $S_{1,1}$, $\hat{S}_{1,1}$ equal to their final values plus a small departure and substituting into equations (79 - 84) (neglecting second order terms). This has not been done here but instead the decay rate for the computed solution shown in Fig 5.9 has been calculated. From the decay rate σ_d we find the decay time $T_d^* = h/(f\sigma_d)$, the time taken for the surface vorticity to be reduced to half its value, and for the computed example (with $\mu = 1$, $h^2 = 25$ and $k = 0.15$) $T_d^* = 4\frac{1}{2}$ days. The growth rate of the disturbance from the initial state is, in this example, 0.15 (see Fig 5.2) giving a growth time $T_g^* = 3$ days. Thus a life time $T_g^* + T_d^*$ may be defined, in this case about a week.

5.5 The Sixteen Parameter Solution.

No horizontal maps of isobars and isotherms for the six parameter solution have been calculated. We have, however, computed flow patterns for the sixteen parameter solution with surface friction (16PF) which can be compared with the eight parameter frictionless solution (8P) described in chapter 4.

The variation of potential and kinetic energy is illustrated in Fig 5.12. Comparing this with Fig 4.5 we find that with surface friction the solution does not oscillate in time but tends to a final steady state as $t \rightarrow \infty$ (as does the six parameter solution of the previous section). The kinetic energy rises to a maximum (smaller than for the frictionless solution) and then decreases to a final small (non-zero) value. We note that it is impossible for the friction to extract all the kinetic energy because the side boundary condition $\partial \bar{u} / \partial t = 0$ implies that the wind speeds at the sides are unchanged. The efficiency of potential energy release is about 75%, higher than that of the frictionless solution. This occurs because the mean meridional entropy gradient at the top of the friction layer is not conserved (see equation (70)), which enables the isentropic surfaces to become almost horizontal near the bottom of the atmosphere.

The truncation error parameter is defined by

$$e_1 = \overline{(h\omega)^2}^{x,y} / \overline{(\underline{v} \cdot \nabla \phi)^2}^{x,y}$$

at the top, and

$$e_1 = \overline{(h\omega - k\zeta)^2}^{x,y} / \overline{(\underline{v} \cdot \nabla \phi + k\zeta)^2}^{x,y}$$

at the top of the friction layer, and is shown in Figs 5.10

and 5.11. e_1 is smaller at the bottom than at the top, where its maximum is comparable with that of the eight parameter solution (Fig 4.6). Unlike the 8P solution no short period oscillations occur (after $t = 0$).

Isobars and isotherms at the upper lid and at the bottom are shown in Fig 5.13 (a) and (b). Because the development is a little slower than in the frictionless solution maps are drawn every two days instead of every day. In the early stages the development of the patterns at both upper and lower levels is very similar to those for the frictionless case (Fig 4.1 (a) and (b)). At later stages in the life of the disturbance friction gives a marked reduction in the surface temperature gradients. This differs from the frictionless solution (Fig 4.1 (b)) where entropy is advected around the vorticity centres and the horizontal gradients increased. Closed isobars eventually disappear leaving a weak decaying westerly flow (N.B. isobars at the bottom are drawn at half the interval of those at upper levels).

The wave at upper levels increases in amplitude eventually producing closed isobars, and horizontal entropy gradients are increased, unlike those below. The upper wave catches up the lower wave but, unlike the frictionless solution, does not overtake it . As we showed in the previous section the final state has upper and lower patterns in phase and stationary relative to the ground. In the atmosphere 'open wave' warm sector depressions are observed to have a phase lag with height, but when occlusion occurs the upper wave comes into phase with the lower cyclone centre and the whole system slows up, stops, and then gradually decays. The 16PF solution corresponds more closely to this than does the 8P frictionless solution (which predicts a phase reversal and no change in the zonal drift of the system as a whole).

No momentum budget has been computed for the frictional solution, but it is evident from the zonal average cross sections (Fig 5.14) that the upper level mid-latitude easterlies, occurring later in the development, show even less agreement with Phillips's (1956) solution (which predicts upper mid-latitude westerlies) than does the frictionless solution (see section 4.2 and Fig 4.3). However the low wind speeds at the bottom agree more closely with those obtained by Phillips. Mid-latitude upper easterlies occur here because the indirect circulation (see section 2.6) produces vorticity at the surface which is cyclonic in the north and anti-cyclonic in the south. Thus an additional mean meridional circulation is induced by friction having the same sense as the indirect cell, and this leads to a larger decrease of westerly momentum at mid-latitudes aloft than for a frictionless atmosphere. The resulting mean wind $\overline{u}^{x,y}_i$ is small at all levels.

Fig 5.2 Growth rate ω_i as a function of zonal wavenumber λ for the linear perturbation solution with surface friction ($\mu = 1$). Shown for $k = 0.3, 0.15, 0.0$.

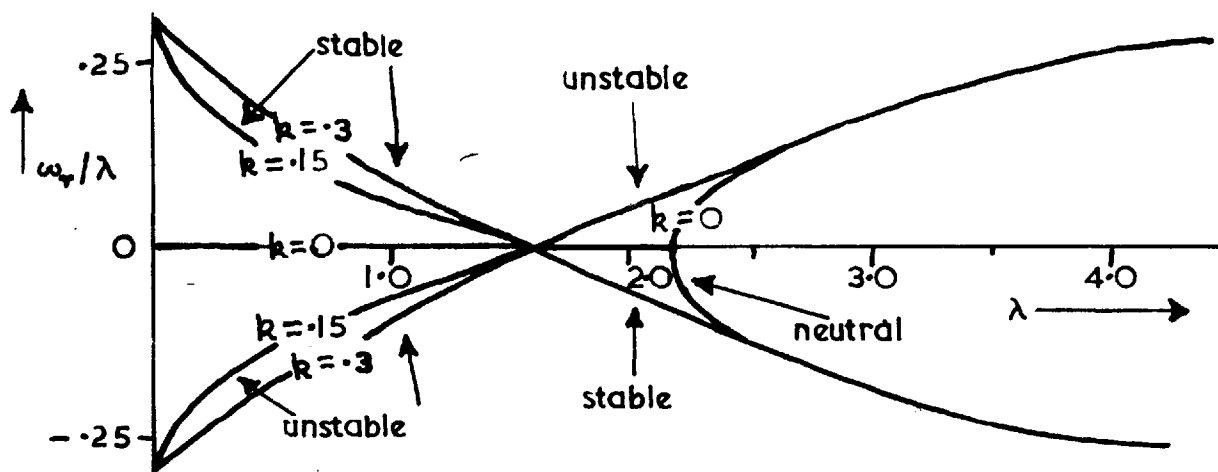


Fig 5.1 Phase speed ω_r/λ as a function of zonal wavenumber λ for the linear perturbation solution with surface friction ($\mu = 1$). Shown for $k = 0.3, 0.15, 0.0$.

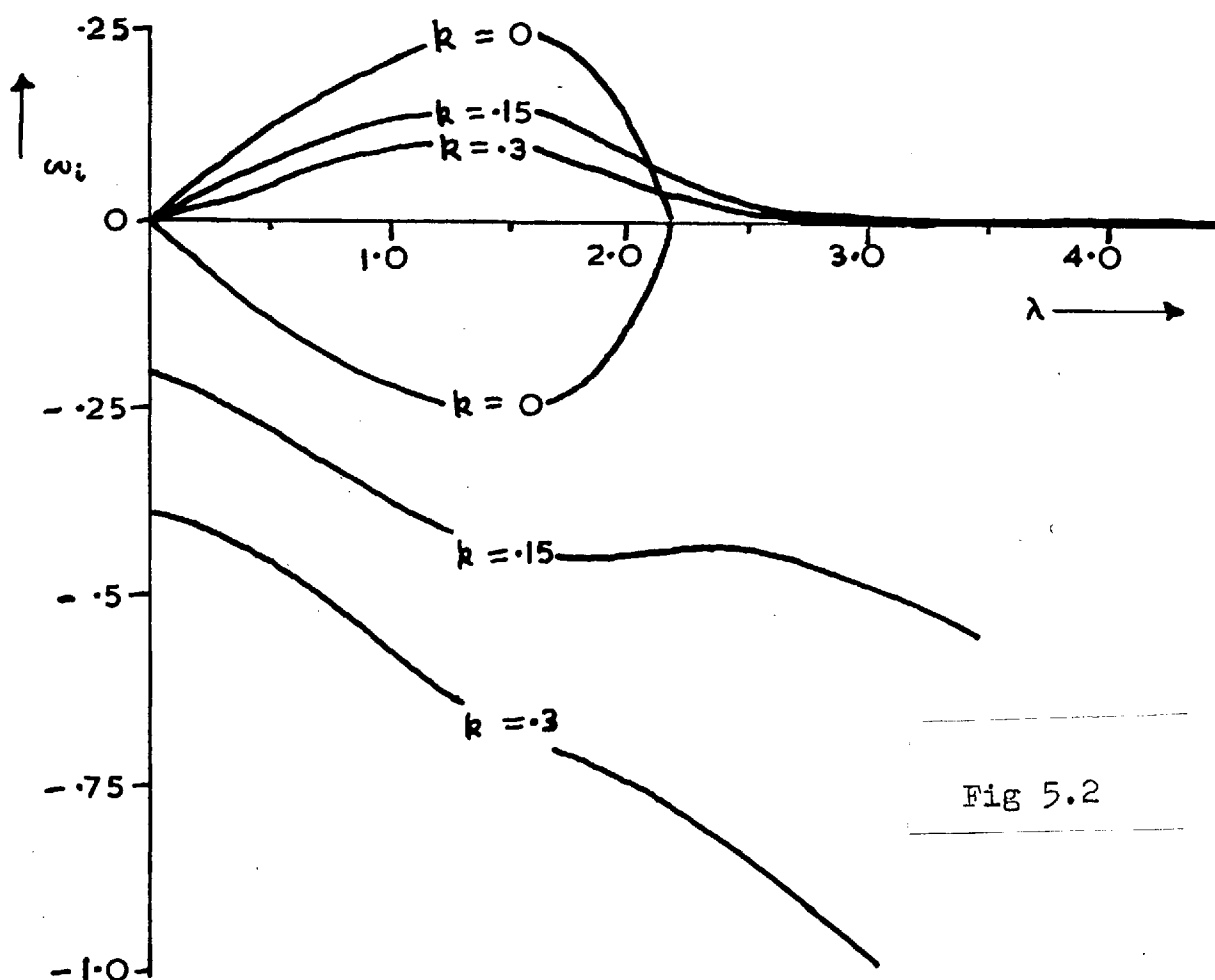


Fig 5.2

Fig 5.4 Growth rate ω_i as a function of zonal and meridional wavenumbers λ and μ for the unstable linear perturbation solution with surface friction for $R = 0.15$.

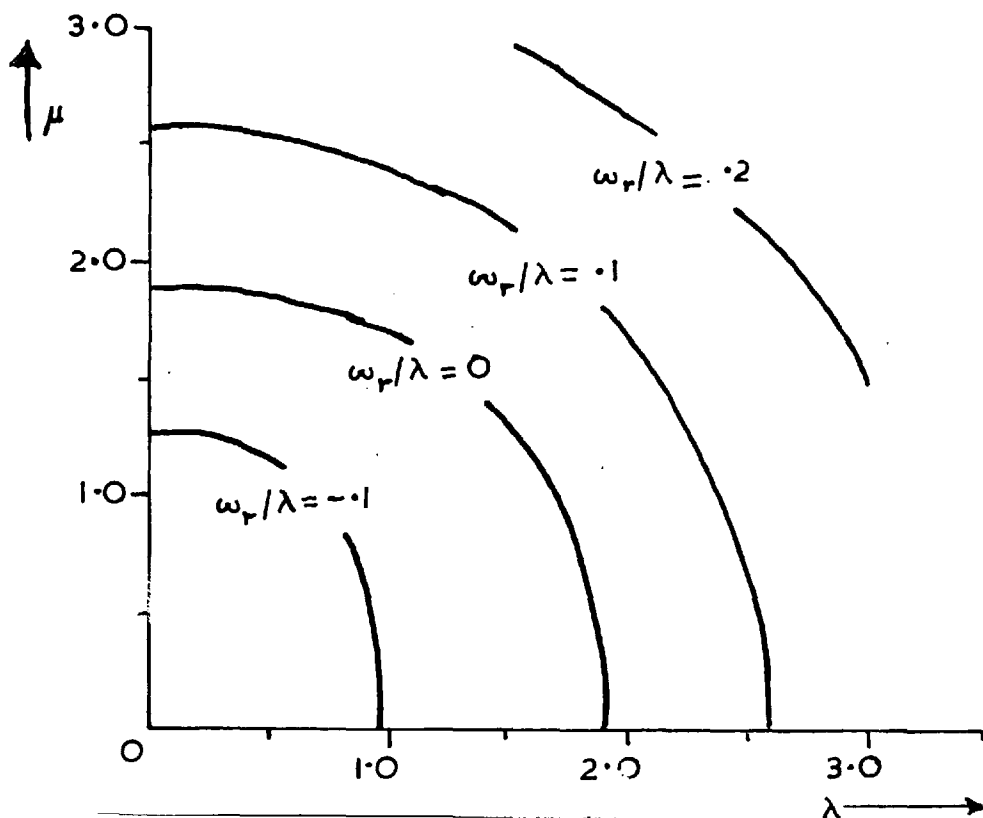


Fig 5.3 Phase speed ω_r/λ as a function of zonal and meridional wavenumbers λ and μ for the unstable linear perturbation solution with surface friction for $R = 0.15$.

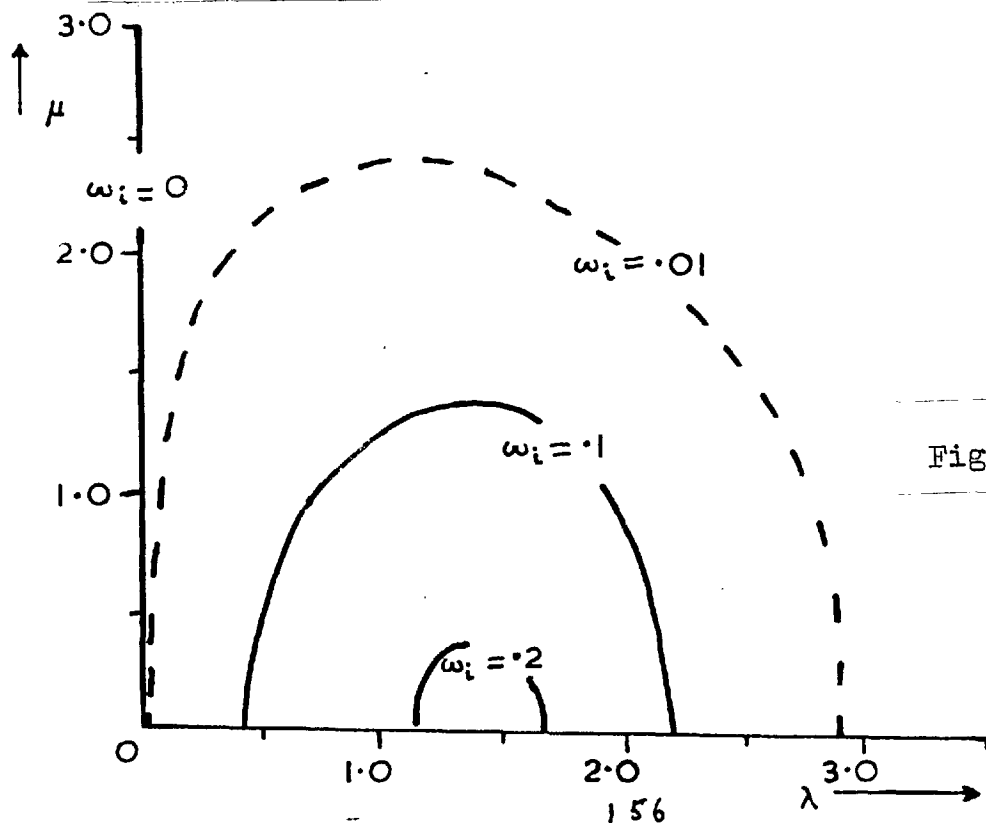


Fig 5.4

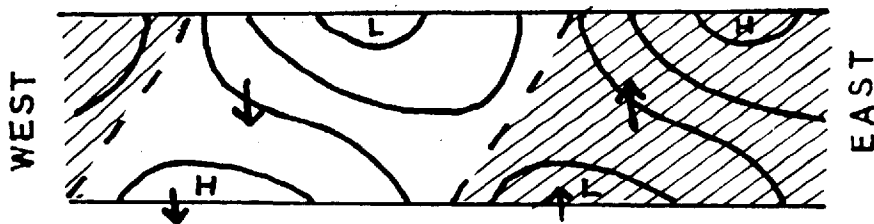


Fig 5.5 Vertical section along a line of latitude for the dominant unstable linear perturbation solution with surface friction ($\lambda = 1.5$, $\mu = 1$) for $R = 0.15$. Legend as in Fig 2.3 (a).

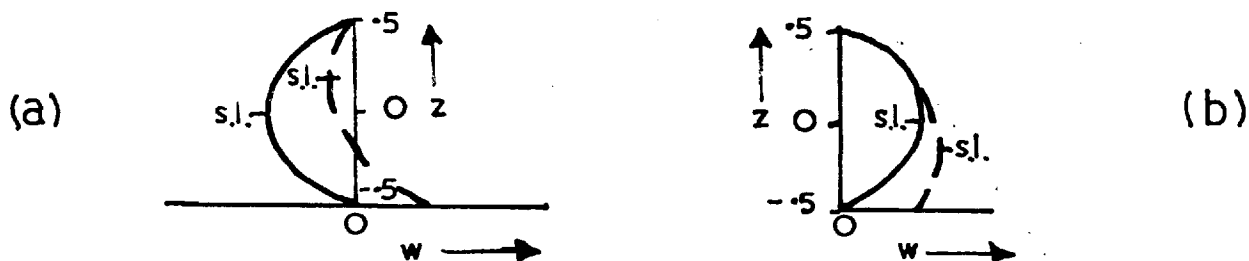


Fig 5.6 Variation of vertical velocity with height in the vicinity of the surface cyclone for (a) the decaying and (b) the growing long wave linear perturbation solutions. The full line represents the frictionless solution and the broken line represents the solution with surface friction.

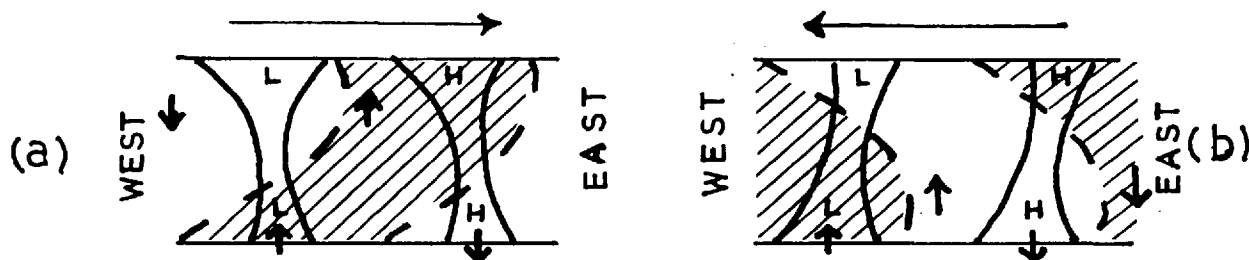
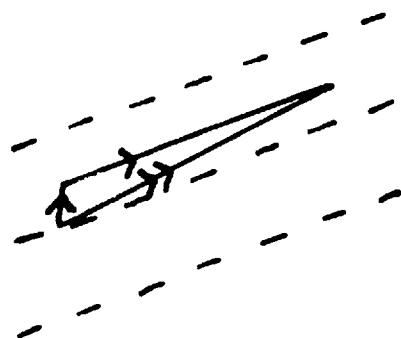
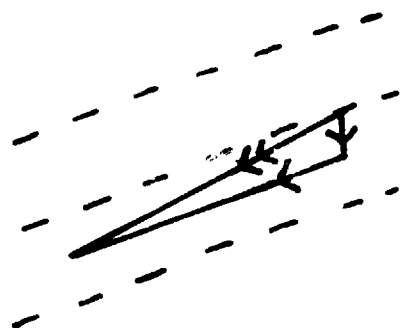


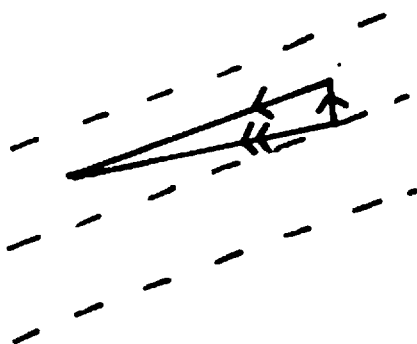
Fig 5.7 Vertical sections along a line of latitude for (a) the upper (growing) and (b) the lower (decaying) short wave linear perturbation solutions with surface friction. Legend as in Fig 2.13.



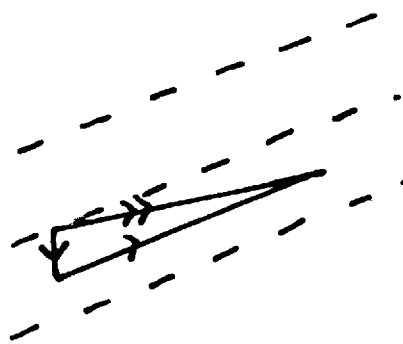
(a)



(b)



(c)



(d)

Fig 5.8 Schematic illustration of particle paths (in the $y - z$ plane) for short linear perturbation waves with surface friction. Broken lines are zonally averaged isentropic surfaces. Single arrows show the 'actual' frictionally induced vertical displacements and baroclinically induced isentropic displacements. Double arrows indicate the 'effective' displacements (a) above and (c) below the steering level in the vicinity of the cyclone, and (b) above and (d) below the steering level in the vicinity of the anticyclone.

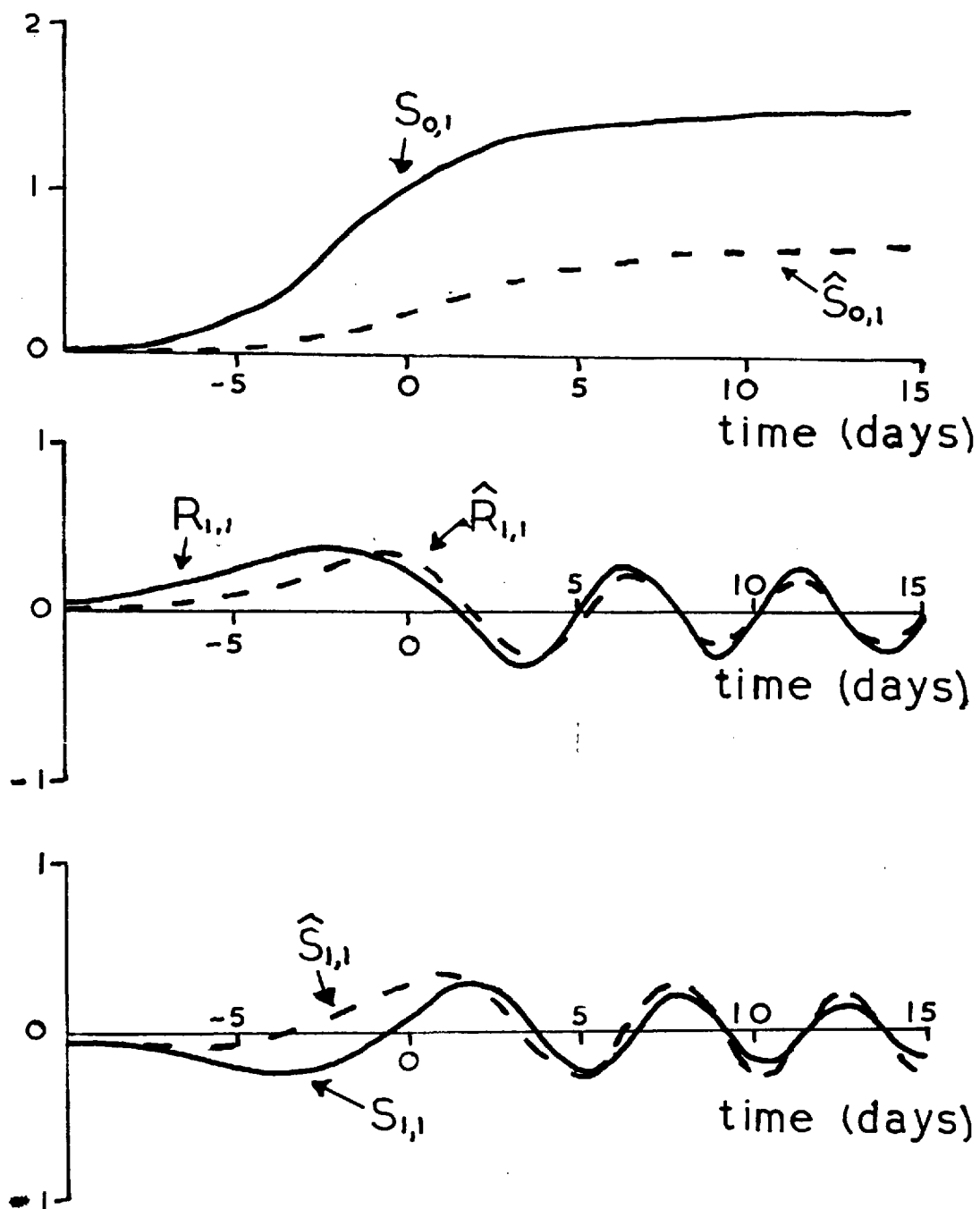


Fig 5.9 Variation of the time dependent parameters $S_{0,1}$, $\hat{S}_{0,1}$, $R_{1,1}$, $\hat{R}_{1,1}$, $S_{1,1}$, $\hat{S}_{1,1}$ for the six parameter solution with surface friction.

Fig 5.12 Variation of the available potential energy (PE) and the kinetic energy (KE) per unit volume with time for the 16 parameter solution with surface friction.

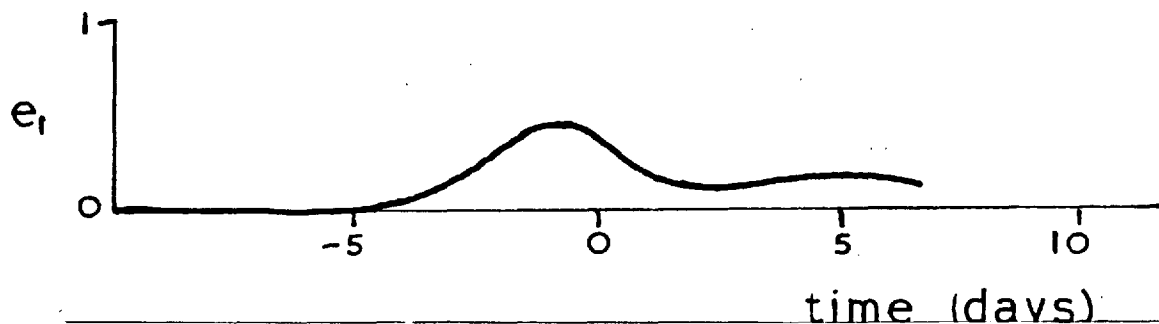


Fig 5.10 Variation of the error parameter e_1 with time at the upper boundary for the 16 parameter solution with surface friction.

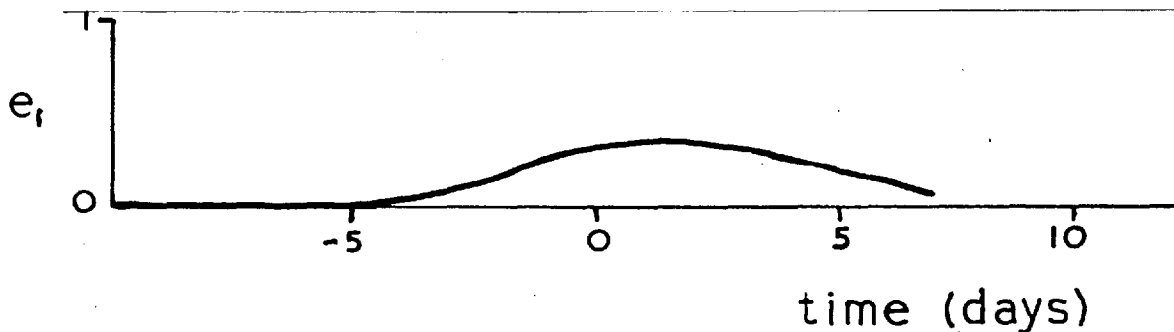


Fig 5.11 Variation of the error parameter e_1 with time at the top of the friction layer for the 16 parameter solution with surface friction.

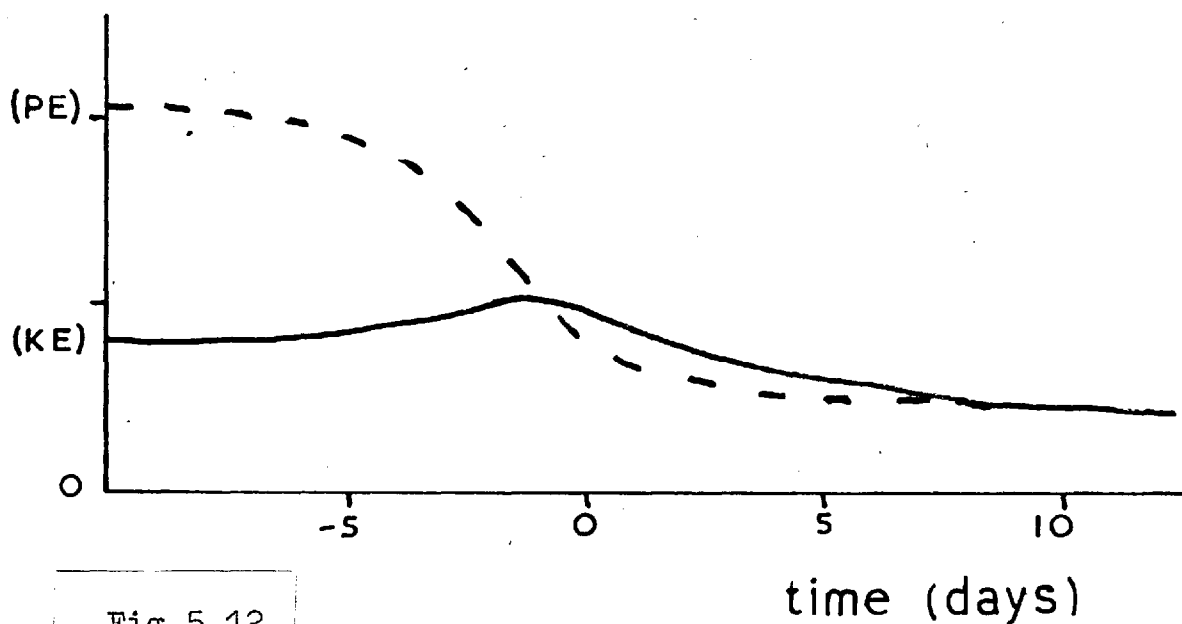
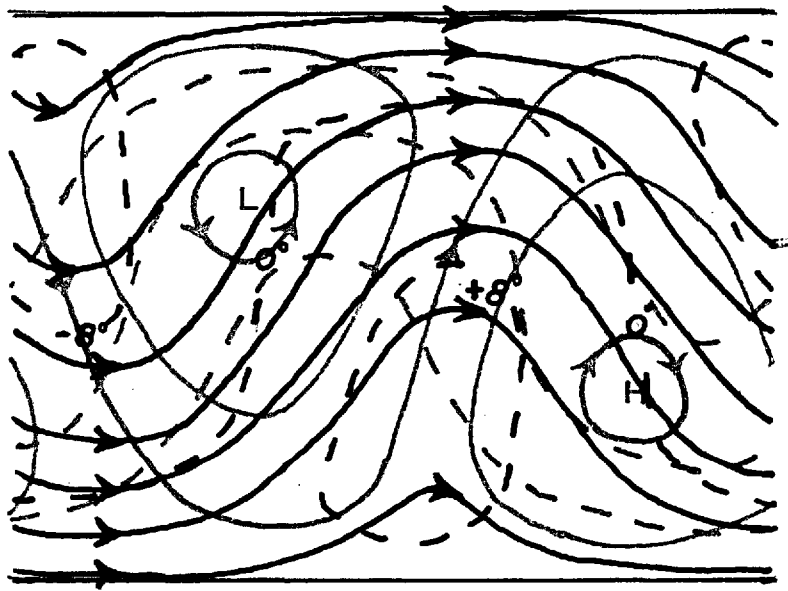


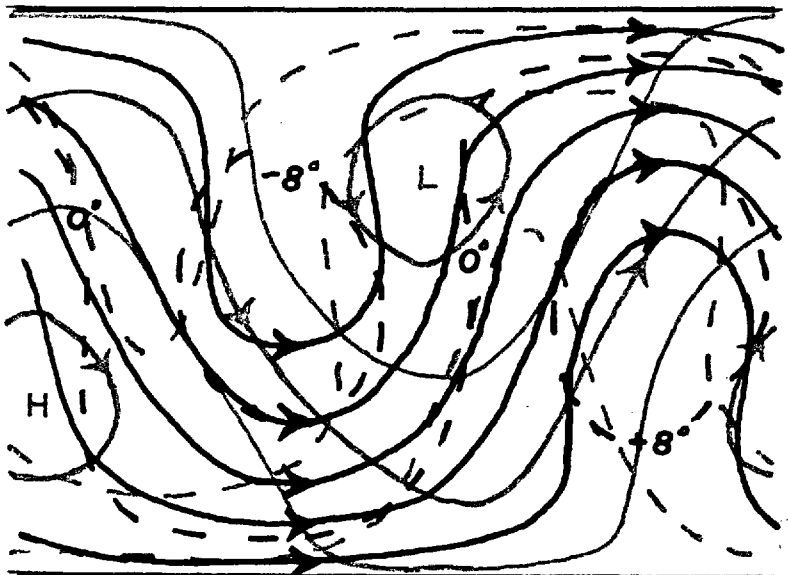
Fig 5.12

Fig 5.13 (a) Stream lines for the geostrophic flow relative to the ground (————) and isotherms (— — —) at the upper lid for the 16 parameter solution with surface friction shown every 2 days.

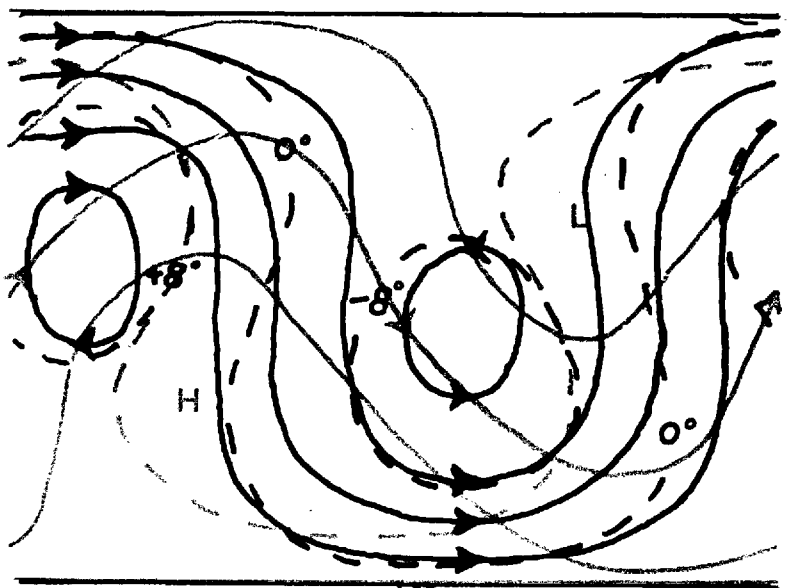
Fig 5.13 (b) Stream lines for the geostrophic flow relative to the ground (————) and isotherms (— — —) at the top of the friction layer for the 16 parameter solution with surface friction shown every 2 days.
N.B. Stream lines at half the interval of those in Fig 5.13 (a).



$t = -2$

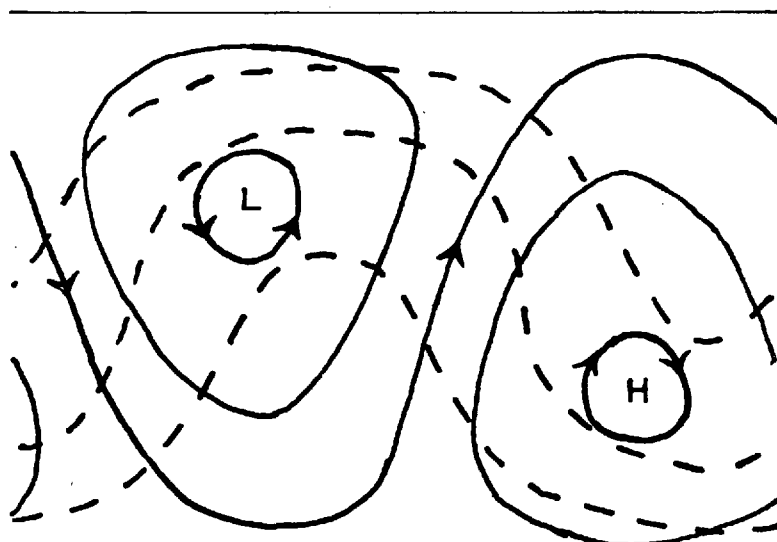


$t = 0$

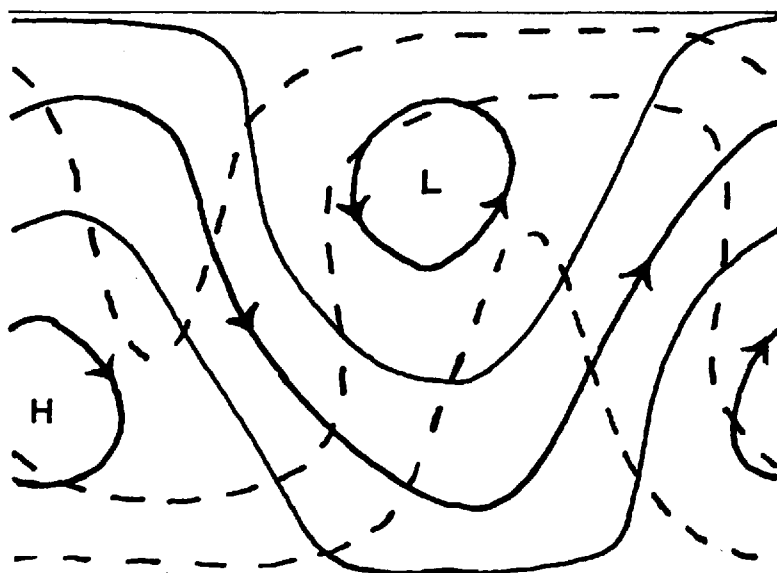


$t = 2$

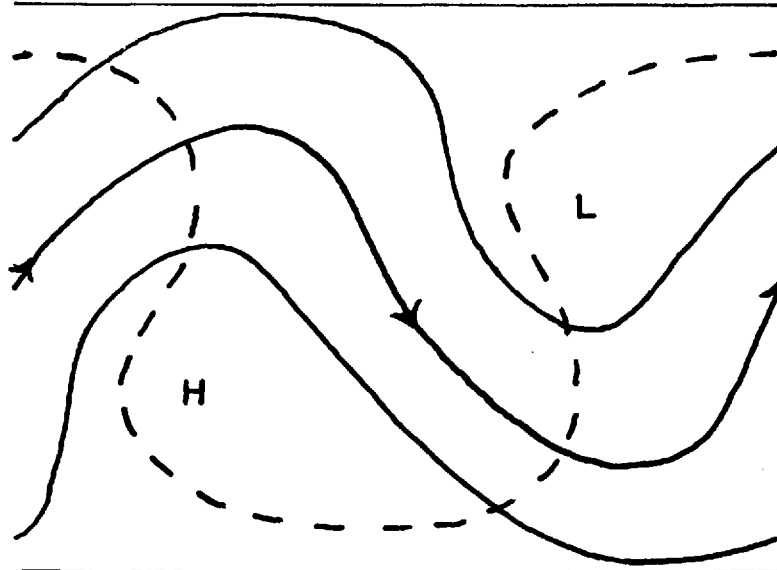
Fig 5.13 (a)
page 767.



$t = -2$



$t = 0$



$t = 2$

Fig 5.13 (b)
see opposite
page 161.

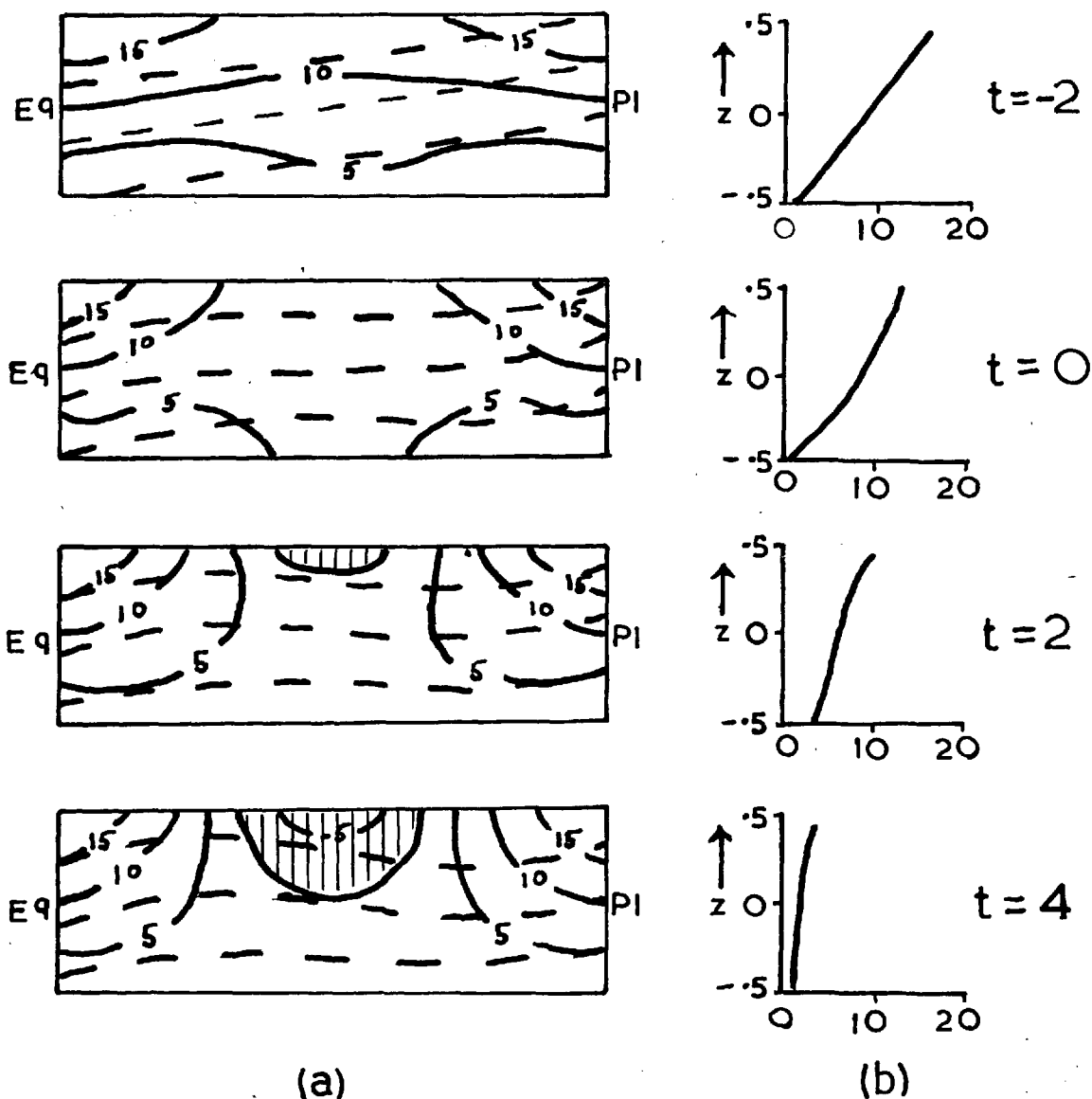


Fig 5.14 (a) Zonally averaged meridional cross sections for the 16 parameter solution with surface friction with isotachs (—) and isentropes (---) shown every 2 days. Wind speeds in m/s and easterlies shaded.

Fig 5.14 (b) Mean zonal wind $\bar{u}^{xy}$ as a function of height shown every 2 days.

CHAPTER 6. ADDITIONAL PHYSICAL EFFECTS.

6.1 Introduction.

As already mentioned in Chapter 4, some features of large scale atmospheric disturbances are absent from the solutions of the quasi-geostrophic equations. For instance - cyclones are observed to have a more compact structure than anticyclones, whereas in the quasi-geostrophic model they are mirror images of each other - frontal zones with sharp horizontal temperature gradients near the ground are found in the atmosphere associated with depressions, but in the model relatively weak zones are produced, having the same intensities in both cyclones and anticyclones.

One important assumption adopted in Chapter 1 is that cross isobar advection (by the ageostrophic wind) is negligible. However its inclusion in the momentum equation leads to the vorticity equation

$$\left[\frac{\partial}{\partial t^*} + \underline{v}^* \cdot \nabla^* \right] (\zeta^* + f) = (\zeta^* + f) \frac{\partial w^*}{\partial z^*}$$

where $\underline{v}^*$ refers to the total (geostrophic + ageostrophic) horizontal wind. This indicates that regions of large absolute vorticity $\zeta^* + f$ (i.e. cyclones) develop in a different way from regions where $\zeta^* + f$ is small, dynamic instability occurring when $\zeta^* + f < 0$.

Petterssen (1940) has shown that, when $w^* = 0$ (for example at the surface of the earth) and when the motion is adiabatic

$$\left[\frac{\partial}{\partial t^*} + \underline{v}^* \cdot \nabla^* \right] \frac{\partial \phi^*}{\partial n} = - \frac{1}{2} \frac{\partial \phi^*}{\partial n} (\hat{D} + \hat{B} G_s 2\alpha)$$

where $\partial \phi^* / \partial n$ is the magnitude of the horizontal gradient

of entropy ϕ^* , α is the angle between $\text{grad}^*\phi^*$ and the 'axis of dilatation', and $\hat{D}$ and $\hat{B}$ are the horizontal divergence and the 'total horizontal deformation' defined by

$$\hat{D} = \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}$$

$$\hat{B}^2 = \left(\frac{\partial u^*}{\partial x^*} - \frac{\partial v^*}{\partial y^*} \right)^2 + \left(\frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} \right)^2$$

For a model having geostrophic advection only $\hat{D} = 0$, and so baroclinic zones are produced by deformation alone. Evidently the inclusion of cross isobar flow in the adiabatic equation permits an additional mechanism for changing the horizontal gradients of ϕ^* - baroclinity becoming more pronounced when $\hat{D} < 0$ (e.g. in cyclones) and less intense when $\hat{D} > 0$.

In the following section it is shown that the criterion for the validity of the quasi-geostrophic approximations confirms the suggestion of section 1.3 that they are inappropriate when the Richardson number is not large. This indicates that ageostrophic advection (and other higher order Richardson number effects) is an important secondary process in the development of large scale atmospheric disturbances. Methods of calculating higher order Richardson number effects are discussed in sections 6.3 and 6.4. Corrections appropriate for the linear perturbation solution are found in section 6.5.

Section 6.6 contains a brief account of methods which might be used to find the influence of large scale heat sources and of the variation with latitude of the coriolis parameter f - the latter suggested in Chapter 4 as being important for the determination of meridional flux properties of the disturbances.

6.2 The Validity of the Quasi-geostrophic Approximations.

The results of the eight parameter solution with $\mu = 1$ (chapter 4) indicate a maximum value of about 6 for the non-dimensional relative vorticity ζ . In section 4.4 we showed that $\zeta_{\max} \propto \mu^{-1}$ (consistent with the proportionality of (PE)₀ to μ^{-2} and an efficiency independent of μ), and hence

$$|\zeta_{\max}^*| \approx 6 f / h \mu$$

(where $\zeta_{\max}^*$ is the dimensional maximum vorticity).

The quasi-geostrophic equations are valid if $\zeta^* \ll f$ (see section 1.3), and so the approximation is valid (as the disturbance approaches maturity) only if the Richardson number of the initial flow is somewhat greater than $36/\mu^2$. Typically, on a large scale $100 > Ri > 10$, and the mid latitude baroclinic zone has a width of about 3000 km. (i.e. $\mu \approx 1$) so that, for many practical cases, the Richardson number is not sufficiently large for the quasi-geostrophic equations to be applicable. We deduce that the influence of higher order Richardson number terms is often significant.

6.3 The Linear Perturbation Solution for small Richardson Number.

Neglecting, as before the compressibility and the variation of coriolis parameter with latitude, and introducing the dimensionless coordinates of section 1.2, we find that (excluding friction and heating) the (otherwise unapproximated) equations of motion, the adiabatic equation and the continuity equation have the non-dimensional form

$$\begin{aligned}
(Ri)^{-1/2} \left[\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right] u - v + \partial \psi / \partial x &= 0 \\
(Ri)^{-1/2} \left[\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right] v + u + \partial \psi / \partial y &= 0 \\
(Ri)^{-1/2} \frac{f^2}{gB} \left[\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right] w - \phi + BH \psi + \partial \psi / \partial z &= 0 \\
(Ri)^{-1/2} \left[\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right] \phi + w &= 0
\end{aligned} \tag{89}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

where u and v do not now refer to the components of the geostrophic wind. The dimensionless potential vorticity is

$$\begin{aligned}
\tau_p = \frac{\tau_p^* f_c^*}{fB} &= 1 + (Ri)^{-1/2} \left[\zeta + \frac{\partial \phi}{\partial z} \right] \\
&+ (Ri)^{-1} \left[\zeta \frac{\partial \phi}{\partial z} + \underline{k} \times \nabla \phi \cdot \frac{\partial \underline{v}}{\partial z} \right] \\
&+ (Ri)^{-1/2} \frac{f^2}{gB} \underline{k} \times \nabla \phi \cdot \nabla w
\end{aligned}$$

There are three dimensionless parameters and we consider their characteristic magnitudes. For large scale mid-latitude tropospheric disturbances $(Ri)^{-1/2}$ has a value within the range 0.3 - 0.1 while $f^2/gB \sim 10^{-4}$ and $BH \sim 10^{-1}$ (see section 1.2). Since f^2/gB is much smaller than $(Ri)^{-1/2}$ all terms including it will be neglected, and so the vertical equation of motion reduces to the hydrostatic equation. For simplicity we also neglect the term involving BH (it being principally associated with a distortion of the solution in the vertical) and write (as before)

$$\phi = \frac{\partial \psi}{\partial z}$$

Making the substitutions $u = z + u'$, $v = v'$, $w = w'$,
 $\phi = -y + \phi'$, $\psi = -yz + \psi'$, where u' , v' , w' , ϕ' ,
 ψ' , are small amplitude perturbations (of an initial
sheared zonal geostrophic flow) having the form

$$\begin{bmatrix} u' \\ v' \\ w' \\ \phi' \\ \psi' \end{bmatrix} = \begin{bmatrix} \hat{u}(z) \\ \hat{v}(z) \\ \hat{w}(z) \\ \hat{\phi}(z) \\ \hat{\psi}(z) \end{bmatrix} e^{i(\lambda x + \mu y - \omega t)}$$

we find, after linearization and elimination, that

$$X(1 - h^{-2}X^2) \frac{d^2 \hat{w}}{dz^2} - 2(\lambda - i\mu h^{-1}) \frac{d \hat{w}}{dz} - (v^2 X + i\lambda \mu h^{-1}) \hat{w} = 0 \quad (90)$$

where

$$X = \lambda z - \omega$$

This equation was derived by Eady (1949) (using
a slightly different notation). An eigen-value problem
arises when it is solved with the boundary condition
 $\hat{w} = 0$ at $z = \pm \frac{1}{2}$. For large values of the Richardson
number it reduces to

$$\frac{d^2 \hat{w}}{dz^2} - \frac{2\lambda}{X} \frac{d \hat{w}}{dz} - v^2 \hat{w} = 0$$

whose solutions may be shown to be those of the linearized
quasi-geostrophic equations (see Eady (1949)). These
are discussed in Chapter 2.

Arnason (1963) has examined solutions of (90) when
 h^2 is not large and has concluded that for $\mu = 0$ no

great differences (from the quasi-geostrophic solutions) arise provided that the Richardson number is greater than one (see Figs 6.1 and 6.2). Stone (1966) has discussed the structure of solutions of finite width ($\mu \neq 0$), and his results indicate that, for $Ri > 0.95$, baroclinic eddies with an intermediate zonal wavelength and a large meridional extent (small μ) have a maximum growth rate (as for the quasi-geostrophic solutions). He obtained expressions for the wavenumber and growth rate of the dominant wave (valid for small μ) which in our notation become

$$\lambda_{max} = \left(\frac{5}{2}\right)^{1/2} \frac{h}{(h^2+1)^{1/2}} - O(\mu^2)$$

$$\sigma_{max} = \left(\frac{5}{54}\right)^{1/2} \frac{h}{(h^2+1)^{1/2}} - O(\mu^2)$$

These agree closely with the calculations of Arnason (for $\mu = 0$), while the Eady quasi-geostrophic solution is given (very nearly) by the limit as $h \rightarrow \infty$. It should be noted that in dimensional coordinates the growth rate is given by

$$\sigma_{max}^* = \sigma_{max} / \gamma^* = \left(\frac{5}{54}\right)^{1/2} \frac{f}{(h^2+1)^{1/2}} - O(\mu^2)$$

and so increases with decreasing Richardson number . However the dimensional zonal wavelength of the disturbance increases with decreasing h if the Richardson number decrease is due to a larger shear, but decreases with decreasing h if it is due to a lesser static stability. In neither case is the dependence on the Richardson number very strong.

For $0.25 < Ri < 0.95$ Stone found that a different, essentially two dimensional (i.e. independent of x), kind of instability (called symmetric instability) is dominant, the eddies obtaining their kinetic energy from

the kinetic energy of the zonal flow. The largest growth rates then occur when μ is large and λ small (i.e. when the eddy axes are parallel to the wind shear). For $Ri < \frac{1}{4}$ 'Kelvin-Helmholtz' instability is the dominant process, with a maximum growth rate occurring for μ small and λ large (with the axes of the eddies perpendicular to the shear). Since on a large scale in the atmosphere h^2 is generally greater than unity (even if calculated with a wet adiabatic lapse rate), neither Kelvin-Helmholtz nor symmetric instability is relevant to our problem.

6.4 Higher Order Richardson Number Equations.

Because the solutions of equation (90) have a complicated algebraic form, we now discuss an approximate method of investigating the influence of a finite Richardson number.

We expand the variables in a Richardson number series (see section 1.3)

$$\begin{aligned} u &= u^{(0)} + u^{(1)} + u^{(2)} + \dots \\ v &= v^{(0)} + v^{(1)} + v^{(2)} + \dots \\ w &= w^{(0)} + w^{(1)} + w^{(2)} + \dots \\ \phi &= \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \dots \\ \psi &= \psi^{(0)} + \psi^{(1)} + \psi^{(2)} + \dots \end{aligned}$$

where the superscript $^{(n)}$ implies that a term is $O(h^{-n})$. After substituting into equations (89) (modified by the neglect of terms involving the parameters f^2/gB and BH)

we obtain a sequence of sets of equations by equating terms of the same order.

The zeroth order equations are

$$\begin{aligned} -v^{(0)} + \frac{\partial \psi^{(0)}}{\partial x} &= 0 \\ u^{(0)} + \frac{\partial \psi^{(0)}}{\partial y} &= 0 \\ -\phi^{(0)} + \frac{\partial \psi^{(0)}}{\partial z} &= 0 \\ w^{(0)} &= 0 \\ \frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} &= 0 \end{aligned}$$

i.e. the state of geostrophic balance.

The first order equations are

$$\begin{aligned} h^{-1} \left[\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y} \right] u^{(0)} - v^{(1)} + \frac{\partial \psi^{(1)}}{\partial x} &= 0 \\ h^{-1} \left[\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y} \right] v^{(0)} + u^{(1)} + \frac{\partial \psi^{(1)}}{\partial y} &= 0 \\ -\phi^{(1)} + \frac{\partial \psi^{(1)}}{\partial z} &= 0 \quad (91) \\ h^{-1} \left[\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y} \right] \phi^{(0)} + w^{(1)} &= 0 \\ \frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} + \frac{\partial w^{(1)}}{\partial z} &= 0 \end{aligned}$$

These are almost the same as the quasi-geostrophic equations (8 - 10) except that they contain a reference to entropy and stream function 'correction terms' $\phi^{(1)}$ and $\psi^{(1)}$.

However the first order potential vorticity equation is

$$\left[\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y} \right] \nabla^2 \psi^{(0)} = 0 \quad (92)$$

and is identical to the quasi-geostrophic potential vorticity equation (14).

The boundary conditions

$$v^{(0)} = 0, \quad \overline{v^{(0)}}^x = 0$$

at the sides, and

$$w^{(0)} = 0$$

on the lids are sufficient to solve (92) and to satisfy the energy conservation equation

$$\frac{1}{2} \frac{d}{dt} \left[(u^{(0)})^2 + (v^{(0)})^2 + (\phi^{(0)})^2 \right]^{x,y,z} = 0$$

These are identical to those for the quasi-geostrophic equations (see section 1.4) and it is evident that $\psi^{(0)}$ is the quasi-geostrophic stream function, whose evolution has been discussed in the earlier part of this thesis. However we are unable to find out any information about the correction term $\psi^{(1)}$ from the first order equations, and so we derive the second order equations

$$h^{-1} \left[\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y} \right] u^{(1)} + h^{-1} \left[u^{(1)} \frac{\partial}{\partial x} + v^{(1)} \frac{\partial}{\partial y} + w^{(1)} \frac{\partial}{\partial z} \right] u^{(0)} - v^{(2)} + \frac{\partial \psi^{(2)}}{\partial x} = 0$$

$$h^{-1} \left[\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y} \right] v^{(1)} + h^{-1} \left[u^{(1)} \frac{\partial}{\partial x} + v^{(1)} \frac{\partial}{\partial y} + w^{(1)} \frac{\partial}{\partial z} \right] v^{(0)} + u^{(2)} + \frac{\partial \psi^{(2)}}{\partial y} - \phi^{(2)} + \frac{\partial \psi^{(2)}}{\partial z} = 0$$

$$h^{-1} \left[\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y} \right] \phi^{(1)} + h^{-1} \left[u^{(1)} \frac{\partial}{\partial x} + v^{(1)} \frac{\partial}{\partial y} + w^{(1)} \frac{\partial}{\partial z} \right] \phi^{(0)} + w^{(2)} = 0$$

$$\frac{\partial u^{(2)}}{\partial x} + \frac{\partial v^{(2)}}{\partial y} + \frac{\partial w^{(2)}}{\partial z} = 0$$

The second order potential vorticity equation is

$$\left[\frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y} \right] \left\{ \nabla^2 \psi^{(1)} - h^{-1} \left[2 \left\{ \frac{\partial^2 \psi^{(0)}}{\partial x^2} \frac{\partial^2 \psi^{(0)}}{\partial y^2} - \left(\frac{\partial^2 \psi^{(0)}}{\partial x \partial y} \right)^2 \right\} + \left(\frac{\partial^2 \psi^{(0)}}{\partial x \partial z} \right)^2 + \left(\frac{\partial^2 \psi^{(0)}}{\partial y \partial z} \right)^2 - \frac{\partial^2 \psi^{(0)}}{\partial z^2} \left(\frac{\partial^2 \psi^{(0)}}{\partial x^2} + \frac{\partial^2 \psi^{(0)}}{\partial y^2} \right) \right] \right\} + \left[u^{(1)} \frac{\partial}{\partial x} + v^{(1)} \frac{\partial}{\partial y} + w^{(1)} \frac{\partial}{\partial z} \right] \nabla^2 \psi^{(0)} = 0 \quad (93)$$

and the boundary conditions

$$v^{(1)} = 0, \quad \overline{v^{(2)}}^x = 0$$

at the sides, and

$$w^{(2)} = 0$$

on the lids enable us to find $\psi^{(1)}$ once $\psi^{(0)}$ is known. These boundary conditions also ensure that the second order energy equation

$$\frac{1}{2} \frac{d}{dt} \int_{x,y,z} [u^{(0)} u^{(1)} + v^{(0)} v^{(1)} + \phi^{(0)} \phi^{(1)}] = 0$$

is satisfied.

We assume an initial state defined by

$$\psi_0^{(0)} = -yz$$

$$\psi_0^{(1)} = 0$$

which represents, as before, a steady sheared zonal flow independent of latitude, and it is easily found by substitution that it corresponds to a state of constant potential vorticity correct to $O(h^{-2})$. Thus the two vorticity equations (92) and (93) can be integrated along the trajectories to give (cf. section 3.2) to give

$$\nabla^2 \psi^{(0)} = 0 \tag{94}$$

and

$$\begin{aligned} \nabla^2 \psi^{(1)} = h^{-1} & \left[2 \left\{ \frac{\partial^2 \psi^{(0)}}{\partial x^2} \frac{\partial^2 \psi^{(0)}}{\partial y^2} - \left(\frac{\partial^2 \psi^{(0)}}{\partial x \partial y} \right)^2 \right\} \right. \\ & \left. + \left(\frac{\partial^2 \psi^{(0)}}{\partial x \partial z} \right)^2 + \left(\frac{\partial^2 \psi^{(0)}}{\partial y \partial z} \right)^2 + \left(\frac{\partial^2 \psi^{(0)}}{\partial z^2} \right)^2 - 1 \right] \end{aligned} \tag{95}$$

In principle the following method can be adopted

for finding $\psi^{(1)}$. First calculate the quasi-geostrophic stream function $\psi^{(0)}$ by solving (94) to as great an accuracy as possible (by the methods outlined in Chapter 3), next substitute $\psi^{(0)}$ into the non-linear terms on the right hand side of equation (95) and the corresponding boundary conditions, and finally solve the Poisson equation (95) for $\psi^{(1)}$. Higher order correction terms may be found by solving the higher order Richardson number equations in a similar way.

The results of Arnason (1960) and Stone (1966) (discussed in section 6.3) which imply that the structure of the dominant linear perturbation solution is little affected by the magnitude of h (> 1), provide some justification for believing that this method of successively obtaining the terms of the Richardson number series is convergent in the early stages of the life of the disturbance. Although convergence is not guaranteed at large values of t , the higher order correction terms found by the method are likely to indicate the nature of modifications to the disturbance at large amplitudes.

In view of the complicated form of the equations, the procedure outlined above is likely to be very tedious, however it is considerably simplified by introducing a new ~~time~~ dependent variable $\hat{\psi}^{(1)}$ defined by

$$\hat{\psi}^{(1)} = \psi^{(1)} + h^{-1} \left[\frac{1}{2} \{ (u^{(0)})^2 - z^2 + (v^{(0)})^2 \} - \{ (\phi^{(0)})^2 - y^2 \} \right]$$

for then the equation (95) reduces to Laplace's equation

$$\nabla^2 \hat{\psi}^{(1)} = 0$$

$\hat{\psi}^{(1)}$ is subject to the boundary conditions

$$\begin{aligned} \frac{\partial \hat{\psi}^{(1)}}{\partial x} &= - \frac{1}{h} \frac{\partial u^{(0)}}{\partial t} \\ \frac{\partial^2 \hat{\psi}^{(1)}}{\partial t \partial y} &= \frac{1}{h} \frac{\partial}{\partial t} \left[\phi^{(0)} \frac{\partial u^{(0)}}{\partial z} - u^{(0)} \frac{\partial \phi^{(0)}}{\partial z} \right] \end{aligned} \quad (96)$$

at the sides, and

$$\left[\frac{D}{Dt} \right]^{(0)} \frac{\partial \psi^{(1)}}{\partial z} + J(\hat{\psi}^{(1)}, \phi^{(0)}) = \quad (97)$$

$$2h^{-1} \left[\frac{\partial v^{(0)}}{\partial z} \cdot \left[\frac{D}{Dt} \right]^{(0)} v^{(0)} - \phi^{(0)} \left[\frac{D}{Dt} \right]^{(0)} \frac{\partial \phi^{(0)}}{\partial z} - y \frac{\partial \phi^{(0)}}{\partial z} \right]$$

on the lids, where

$$\left[\frac{D}{Dt} \right]^{(0)} = \frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y}$$

These boundary conditions are still somewhat complicated, but an approximate solution involving a limited number of wavenumber components can be found by employing a least squares fit technique similar to that used in finding the quasi-geostrophic functional representation solutions in Chapter 3. (Since the equations and boundary conditions are linear in $\hat{\psi}^{(1)}$ we can in principle find an exact solution corresponding to a given $\psi^{(0)}$, though it is probably not valid to include components with wavelengths more than an order of magnitude smaller than those included in the quasi-geostrophic solution $\psi^{(0)}$.)

6.5 Corrections to the Quasi-geostrophic Linear Perturbation Solution when the Richardson Number is not Large.

We assume a perturbation expansion for $\psi^{(0)}$ and $\psi^{(1)}$ having the form

$$\psi^{(0)} = \sum_{n=0}^{\infty} \psi_n^{(0)}$$

$$\psi^{(1)} = \sum_{n=0}^{\infty} \psi_n^{(1)}$$

where the subscript denotes the order in the perturbation

(ε) series and the super script the order in the Richardson number series. Substituting in the first and second order (Richardson number) equations and boundary conditions, we retain only terms of order $h^0 \varepsilon$ and $h^{-1} \varepsilon$ respectively. We write

$$\psi^{(0)} = \psi_0^{(0)} + \psi_1^{(0)} = -y z + e^{\sigma t} (b \operatorname{Sh} v z \operatorname{Sin} \lambda x - c \operatorname{Ch} v z \cos \lambda x) \cos \mu y$$

where λ and μ respond to the dominant Eady solution (see section 2.4) and introduce $\hat{\psi}_1^{(0)}$ which satisfies

$$\frac{\partial \hat{\psi}_1^{(0)}}{\partial x} = h^{-1} \frac{\partial^2 \psi_1^{(0)}}{\partial y \partial z} \quad (98)$$

at the sides, and

$$\left[\frac{\partial}{\partial t} + z \frac{\partial}{\partial x} \right] \frac{\partial \hat{\psi}_1^{(0)}}{\partial z} - \frac{\partial \hat{\psi}_1^{(0)}}{\partial x} = -2 h^{-1} \left[\left[\frac{\partial}{\partial t} + z \frac{\partial}{\partial x} \right] \left(\frac{\partial \psi_1^{(0)}}{\partial y} - y \frac{\partial^2 \psi_1^{(0)}}{\partial z^2} \right) + y \frac{\partial^2 \psi_1^{(0)}}{\partial x \partial z} \right] \quad (99)$$

on the lids.

$\hat{\psi}_1^{(0)}$ can then be expressed as the sum of two harmonic functions, the first satisfying the side boundary condition (98) but satisfying a homogeneous lid boundary condition (i.e. (99) with the right hand side set to zero), while the second satisfies the homogeneous side condition and the exact lid condition. The third part of the solution, namely a harmonic function satisfying homogeneous conditions on both side walls and lids, need not be included since it satisfies the same conditions as the quasi-geostrophic solution $\psi_1^{(0)}$, and is consequently already represented in the solution. $\psi_1^{(0)}$ is found to have the following form

$$\psi_1^{(0)} = h^{-1} e^{\sigma t} \frac{\sigma \mu}{\lambda} (b \operatorname{Sh} v z \cos \lambda x + c \operatorname{Ch} v z \operatorname{Sin} \lambda x) \operatorname{Sin} \mu y + h^{-1} e^{\sigma t} \sum_{n=1}^{\infty} (A_n \operatorname{Sh} v_n z \cos \lambda x + B_n \operatorname{Ch} v_n z \operatorname{Sin} \lambda x) \operatorname{Sin} 2n \mu y \quad (100)$$

where $v_n^2 = \lambda^2 + 4n^2 \mu^2$, and A_n and B_n are determined from

the simultaneous equations

$$\begin{aligned}
 & A_n \left[\sigma v_n c h^{v_n/2} \right] + B_n \left[\lambda \left(v_{n/2} S h^{v_{n/2}} - c h^{v_{n/2}} \right) \right] = \\
 & \frac{4}{\pi} v \left[\sigma v c h^{v/2} + b \lambda \left(c h^{v/2} - v_{n/2} S h^{v_{n/2}} \right) \right] (-1)^n \frac{n}{4n^2 - 1} \\
 & - \frac{8}{\pi} \mu \left[\sigma c h^{v/2} - \frac{\lambda b}{2} S h^{v/2} \right] (-1)^n \frac{n}{(4n^2 - 1)^2} \\
 & A_n \left[\lambda \left(S h^{v_n/2} - v_{n/2} c h^{v_{n/2}} \right) \right] + B_n \left[\sigma v_n S h^{v_{n/2}} \right] = \\
 & - \frac{4}{\pi} v \left[\sigma v b S h^{v/2} + \lambda \left(\frac{v}{2} c h^{v/2} - S h^{v/2} \right) \right] (-1)^n \frac{n}{4n^2 - 1} \\
 & + \frac{8}{\pi} \mu \left[\sigma b S h^{v/2} + \frac{\lambda}{2} c h^{v/2} \right] (-1)^n \frac{n}{(4n^2 - 1)^2}
 \end{aligned}$$

The series converges quickly in the interior but only as fast as n^{-2} on the lids. It will be remembered that a similar convergence difficulty arises when finding the second order quasi-geostrophic perturbation solution (section 2.6). Although it is only a local effect near the lids and probably does not affect the flow in the main body of the region, it must be remembered that the most important departures from the quasi-geostrophic solutions are likely to occur near the upper and lower boundaries (where the largest values of ζ arise). Thus the solution (100) may not be expressed in the best form.

The flow pattern at the surface is sketched in Fig 6.3 and it is clear that it differs little from the quasi-geostrophic solution (Fig 2.3). One noticeable point is that pressure centres are displaced northwards. The effect of this is that the cyclones move into regions where the horizontal temperature gradients are large while anticyclones move to places where the gradient is

small. This is not surprising since we expect larger gradients to be associated with regions of convergence (see section 6.1), and furthermore it agrees with the observed association of cyclones and fronts in the atmosphere.

6.6 Methods of Modelling Other Aspects of the Real Atmosphere.

In this last section we suggest methods of incorporating other physical effects into the model atmosphere. One feature which has not been considered so far in this thesis is that of radiative heating. We have seen that friction causes the eventual decay of the disturbance, leaving a final state having little available potential energy or kinetic energy (see Chapter 5). In the atmosphere an excess of incoming over outgoing radiation in the tropics causes an increase of temperature, while in polar regions a radiation deficit causes cooling. The baroclinic zone is thereby re-established, allowing the growth of a new disturbance.

Eliassen and Kleinschmitt () show that potential vorticity is not conserved in a non-adiabatic atmosphere. In terms of the quasi-geostrophic approximations the equation relating its change is

$$\frac{\partial}{\partial t} (\nabla^2 \psi) = \frac{\partial}{\partial z} \left[\frac{\partial \phi}{\partial t} + h w \right] \quad (101)$$

Evidently if entropy changes due to radiative heating are independent of height then potential vorticity is conserved. Thus the assumption of a heating distribution which is a function of x, y and t only, greatly simplifies the problem by enabling (101) to be integrated along the trajectories to give Laplace's equation. A simple heating

function of this sort is one which is independent of x and t but varies with y , changing sign between equator and pole. This represents approximately the distribution of heating in the atmosphere. We might examine a heat source having a linear variation with y , but an algebraically more convenient one is

$$\frac{D\phi}{Dt} + hw = -Q \sin \mu y$$

where Q is a constant.

Without friction in the calculation the lid boundary condition $w = 0$ and the side boundary condition $\partial \bar{u}^x / \partial t = 0$ allow an exact solution

$$\psi_0 = -t Q \frac{Sh \mu^2}{Ch \mu^2} \sin \mu y \quad (102)$$

This is the stream function for a zonal westerly sheared geostrophic flow increasing linearly with time. It is feasible to examine the growth of disturbances superimposed on this (unsteady) initial state, but no such calculations have been performed.

One difficulty which we anticipate concerns the intersections of the lids and side walls, where both v and w vanish. According to (102) radiation causes entropy at the corners to increase (or decrease) without limit. This is physically unreasonable and is likely to produce mathematical singularities. However they may be avoided by having friction layers (where mixing occurs) at both the top and bottom. We then impose the lid boundary conditions

$$\frac{\partial \phi}{\partial t} + \underline{v} \cdot \nabla \phi = -Q \sin \mu y - k_1 \frac{\partial \phi}{\partial z}$$

at the top, and

$$\frac{\partial \phi}{\partial t} + \underline{v} \cdot \nabla \phi = -Q \sin \mu y + k_2 \frac{\partial \phi}{\partial z}$$

at the bottom, where k_1 and k_2 are friction coefficients for the upper and lower friction layers defined as in section 5.1. We now find an exact solution of the form

$$\psi_0 = (S_{0,1} \sinh \mu z + \hat{S}_{0,1} \cosh \mu z) \sin \mu y \quad (103)$$

where

$$S_{0,1} = A_1 e^{\sigma t} - \frac{k_2 + k_1}{k_1 k_2} \frac{Q}{2 \mu^2 \sinh \mu/2}$$

and

$$\hat{S}_{0,1} = A_2 e^{\sigma t} - \frac{k_2 - k_1}{k_1 k_2} \frac{Q}{2 \mu^2 \cosh \mu/2}$$

and σ is a root of the equation

$$\sigma^2 + (k_1 + k_2) \mu \coth \mu \sigma + k_1 k_2 \mu^2 = 0 \quad (104)$$

It is easily seen that the roots of equation (104) are real and negative, and so as t becomes large

$$S_{0,1} \longrightarrow - \frac{k_2 + k_1}{k_1 k_2} \frac{Q}{2 \mu^2 \sinh \mu/2}$$

$$\hat{S}_{0,1} \longrightarrow - \frac{k_2 - k_1}{k_1 k_2} \frac{Q}{2 \mu^2 \cosh \mu/2}$$

The expression (103) then becomes a steady state solution where the radiative input of energy exactly equals the kinetic energy extracted by friction. This is a more suitable initial state than (102), and the method of functional representation can be used to investigate the behaviour of wave disturbances.

No account has been taken in our calculations of the curvature of the earth. The principal effect of this is the variation with latitude of the coriolis parameter f . As we mentioned in section 3.2 difficulties arise because the potential vorticity Z_p is not constant and the vorticity equation cannot be reduced to a linear

equation. It is of course possible to 'invent' artificial initial states for which Z_p is constant everywhere, but it is difficult to deduce general results from them. One such state is defined by

$$\psi_0 = -\frac{1}{2} \gamma y (z^2 - 2z_0 z) \quad (105)$$

where γ is given in section 1.2. It is easily verified that in this case the potential vorticity on a β -plane

$$Z_p = 1 + (Ri)^{-1/2} [\nabla^2 \psi + \gamma y]$$

is constant, so that integration of the vorticity equation along trajectories gives Laplace's equation

$$\nabla^2 [\psi - \psi_0] = 0$$

The expression (105) is the stream function for a zonal flow having westerly shear for $z > z_0$ and easterly shear for $z < z_0$. This initial state might be useful for examining the evolution of disturbances in the trade winds. The methods of solving the equations described in Chapter 3 are readily applicable to such a problem.

For the case of disturbances on a westerly flow with constant shear approximate methods may give reasonable results. It is shown in section 3.2 that, on a β -plane, integration of the quasi-geostrophic potential vorticity equation leads to the equations

$$\nabla^2 \psi + \gamma \hat{y} = 0 \quad (106)$$

$$\frac{D\hat{y}}{Dt} = \frac{\partial \psi}{\partial x}$$

where $\hat{y}$ is the northward displacement of a particle from its initial position (at $t = -\infty$). The results of Eady (1949) and Green (1960) (mentioned in section 1.2) imply that the variation of the coriolis parameter with

latitude does not greatly affect the structure and growth rate of the dominant linear perturbation solution. This suggests that the expansion

$$\psi = \sum_{n=0}^{\infty} \psi^{(n)}$$

$$\hat{y} = \sum_{n=0}^{\infty} \hat{y}^{(n)}$$

where the superscript $^{(n)}$ implies that the term is $O(\gamma^n)$, may be convergent in the early stages of the life of the disturbance, and may indicate the nature of the influence of a variable f on a finite amplitude solution (such as the modifications to the meridional flux of momentum discussed in section 4.2).

Substituting into (106) and equating terms of the same order gives

$$\nabla^2 \psi^{(0)} = 0 \quad (107)$$

$$\frac{\partial \hat{y}^{(0)}}{\partial t} + J(\psi^{(0)}, \hat{y}^{(0)}) = \frac{\partial \psi^{(0)}}{\partial x} \quad (108)$$

$$\nabla^2 \psi^{(1)} = -\gamma \hat{y}^{(0)} \quad (109)$$

As an example of the method we examine the linear perturbation solution. Writing

$$\psi^{(0)} = -y z + \psi_1^{(0)}$$

where $\psi_1^{(0)}$ is the first order perturbation solution (30) found by assuming a constant value for the coriolis parameter f , we neglect higher order perturbation terms and find the displacement $\hat{y}_1^{(0)}$ from

$$\left(\frac{\partial}{\partial t} + z \frac{\partial}{\partial x} \right) \hat{y}_1^{(0)} = \frac{\partial \psi_1^{(0)}}{\partial x}$$

$\hat{y}_1^{(0)}$ then has the form

$$\hat{y}_1^{(0)} = \frac{\lambda e^{\sigma t}}{\sigma^2 + \lambda^2 z^2} \left\{ (\sigma \cosh vz + \lambda z b \sinh vz) \sin \lambda x + (b \sigma \sinh vz - \lambda z \cosh vz) \cos \lambda x \right\} \cos \mu y$$

The correction term $\psi_1^{(1)}$ is now found by solving the Poisson equation (109), a calculation made much easier by expressing $\hat{y}_1^{(0)}$ as a Fourier series thus:-

$$\hat{y}_1^{(0)} = e^{\sigma t} \sum_{n=0}^{\infty} \left\{ A_n \cos 2n\pi z \sin \lambda x + B_n \sin 2n\pi z \cos \lambda x \right\} \cos \mu y$$

where

$$A_n = \lambda \int_{-1/2}^{1/2} \frac{\sigma \cosh vz + \lambda z b \sinh vz}{\sigma^2 + \lambda^2 z^2} \cos 2n\pi z \, dz$$

$$B_n = \lambda \int_{-1/2}^{1/2} \frac{b \sigma \sinh vz - \lambda z \cosh vz}{\sigma^2 + \lambda^2 z^2} \sin 2n\pi z \, dz$$

We then find

$$\psi_1^{(1)} = -\gamma e^{\sigma t} \sum_n \left\{ \frac{A_n}{4n^2\pi^2 - v^2} \cos 2n\pi z \sin \lambda x + \frac{B_n}{4n^2\pi^2 - v^2} \sin 2n\pi z \cos \lambda x \right\} \cos \mu y$$

The following procedure is suggested for finding the correction term $\psi^{(1)}$ for a finite amplitude solution. First find the solution $\psi^{(0)}$ assuming f to be constant (using the method of functional representation to obtain as great an accuracy as desired), next express $\hat{y}^{(0)}$ as a truncated triple Fourier series with general time dependent coefficients, and then derive and solve the set of ordinary non-linear differential equations for the coefficients by minimising the expression

$$\frac{\left(\frac{\partial \hat{y}^{(0)}}{\partial t} - \frac{\partial \psi^{(0)}}{\partial x} \right)^2}{x, y, z}$$

Finally solve the Poisson equation (109) for the correction term $\psi^{(1)}$.

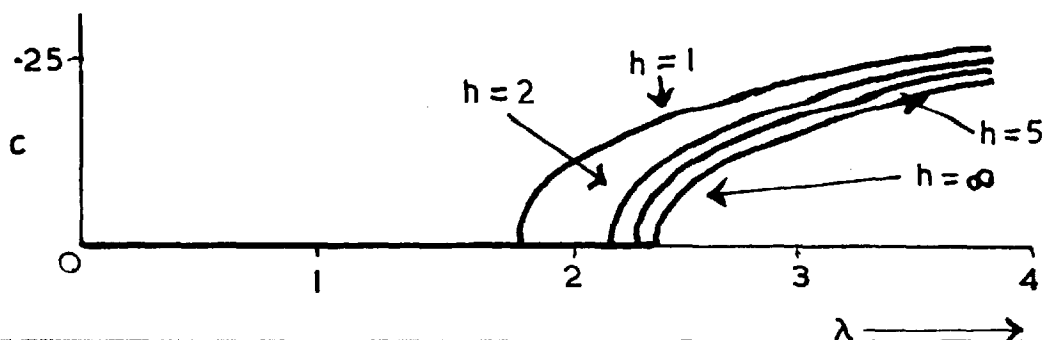


Fig 6.1. Variation of phase speed c with zonal wave-number λ ($\mu = 0$) for different values of h ($= (Ri)^{1/2}$) after Arnason (1963).

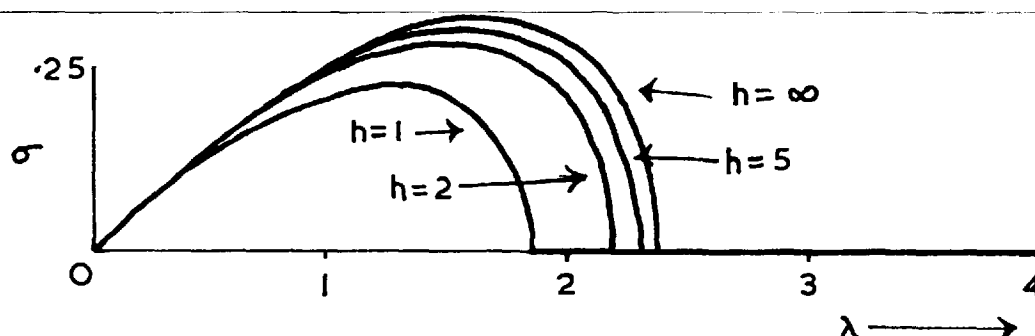


Fig 6.2 Variation of growth rate σ with zonal wave-number λ ($\mu = 0$) for different values of h ($= (Ri)^{1/2}$) after Arnason.

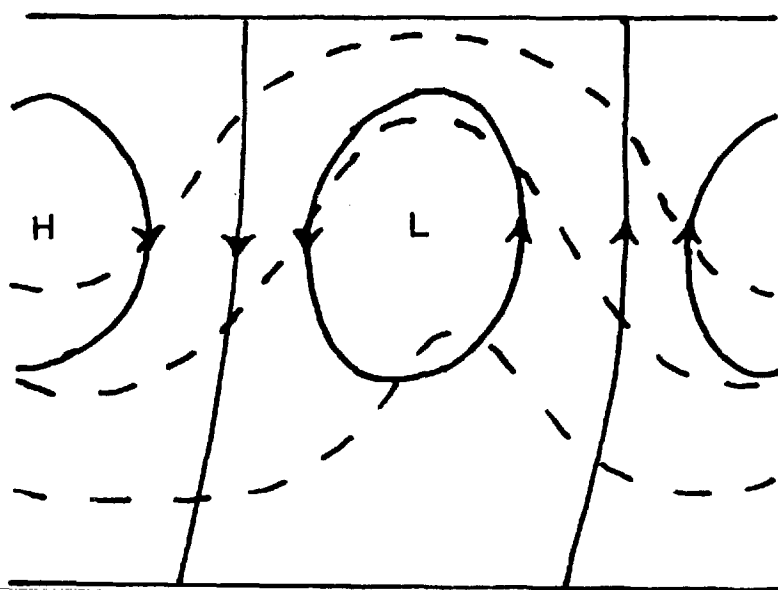


Fig 6.3 Isobars (—) and isotherms (---) at the surface for the perturbation solution with second order (Richardson number) effects.

APPENDIX I. GLOSSARY OF SYMBOLS USED.

$A = \partial \phi_0^* / \partial z^*$	initial meridional 'entropy' gradient
$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$	constants defined in section 3.5
$B = \partial \phi_s^* / \partial z^*$	static stability of the standard atmosphere
$c = \omega_r / \lambda $	dimensionless phase speed relative to the initial mean zonal flow
c_p	specific heat of air at constant pressure
c_v	specific heat of air at constant volume
E	dimensionless total energy per unit volume
ϵ_1, ϵ_2	parameters measuring the truncation error defined in section 3.10
f, f_0	coriolis parameter
g	acceleration of gravity
H	depth of the troposphere
$h = (Ri)^{1/2}$	square root of the Richardson number
I^*	internal energy per unit volume
$i = \sqrt{-1}$	
K	eddy diffusivity
$(KE)^*$	kinetic energy per unit volume
$(KE) = (KE)^* h^2 / [\rho_s^* g B H^2]$	dimensionless form of $(KE)^*$
$(KE)_0$	initial value of (KE)
k	frictional constant defined in section 5.1
$L^* = (g B)^{1/2} H / f_0$	horizontal length scale of disturbance

l, m, n	integers
(l, m) component	Fourier component with zonal wave number $l\lambda$ and meridional wavenumber $m\mu$
P^*	potential energy per unit volume
$(PE)^*$	available potential energy per unit volume
$(PE) = (PE)^* h^2 / [\rho_s^* g \beta H^2]$	dimensionless value of $(PE)^*$
$(PE)_0$	initial value of (PE)
p_s^*	pressure of standard atmosphere
p^*	departure of pressure from p_s^*
Q	heat source
$\underline{z}$	dimensionless velocity vector (u, v, w)
$Ri = h^2 = g \beta / [dU_0/dz^*]^2$	Richardson number
$R_{l,m}(t) ; \hat{R}_{l,m}(t) \}$ $S_{l,m}(t) ; \hat{S}_{l,m}(t) \}$	time dependent coefficients
T_s^*	temperature of standard atmosphere
T^*	departure of temperature from T_s^*
t^*	time
$t = t^* f_0 / h$	dimensionless value of t^*
$\hat{t} = 2\sigma - t$	scaled time variable
dU_0/dz^*	initial zonal wind shear
u^*	wind speed in x^* (zonal) direction relative to the initial mean zonal flow (geostrophic except in sections 6.1 - 6.5)
$u = u^* / [\frac{dU_0}{dz^*} H]$	dimensionless value of u^*
u_a^*	ageostrophic wind in the x^* (zonal) direction

$$u_a = u_a^* / \left[\frac{dU_0}{dz^*} H \right]$$

dimensional value of u_a^*

$$v^*$$

wind speed in y^* (meridional)
direction (geostrophic except in
sections 6.1 - 6.5)

$$v = v^* / \left[\frac{dU_0}{dz^*} H \right]$$

dimensionless value of v^*

$$v_a^*$$

ageostrophic wind in y^* (meridional)
direction

$$v_a = v_a^* / \left[\frac{dU_0}{dz^*} H \right]$$

dimensionless value of v_a^*

$$\underline{v}^*$$

horizontal wind vector ($u^*, v^*, 0$)
(geostrophic except in sections
6.1 - 6.5)

$$\underline{v} = \underline{v}^* / \left[\frac{dU_0}{dz^*} H \right]$$

dimensionless value of $\underline{v}^*$

$$w^*$$

vertical velocity

$$w = w^* h / [f_0 H]$$

dimensionless value of w^*

$$w_m = \overline{w}^*$$

dimensionless zonal mean vertical
velocity

$$x^*$$

coordinate in zonal direction

$$x = x^* f_0 / [(gB)^{1/2} H]$$

dimensionless value of x^*

$$Y^*$$

width of baroclinic zone

$$Y = Y^* f_0 / [(gB)^{1/2} H]$$

dimensionless value of Y^*

$$y^*$$

coordinate in meridional direction

$$y = y^* f_0 / [(gB)^{1/2} H]$$

dimensionless value of y^*

$$\hat{y}$$

dimensionless meridional displacement
of particle from its initial position

$$Z_{1P}^* = \frac{\nabla^* \Phi^* \cdot (\nabla^* x^* + f_k)}{\rho_s^* + \rho^*}$$

potential vorticity

$$Z_{1P} = Z_{1P}^* \rho_s^* / [fB]$$

dimensionless value of Z^*

$$z^*$$

vertical coordinate

$$z = z^* / H$$

dimensionless value of z^*

$\beta = \partial f / \partial y$	Rossby parameter
$\gamma = (\beta B)^{1/2} H h / f_0^2$	dimensionless value of β
ϵ	perturbation series parameter
ϵ	efficiency
$\hat{\epsilon} = 2\sigma$	Eady's definition of efficiency
ζ^*	vertical component of vorticity
$\zeta = \zeta^* f_0 / h$	dimensionless value of ζ^*
ζ_{max}	maximum value of ζ attained during the life of the disturbance
$\Theta^* = B H \Theta_s^* / h$	temperature scale for the disturbance
Θ_s^*	potential temperature for the standard atmosphere
Θ^*	departure of potential temperature from Θ_s^*
$H = -H \frac{1}{\rho_s^*} \frac{d\rho_s^*}{dz^*}$	dimensionless compressibility parameter
λ	dimensionless zonal wavenumber
λ_{max}	value of λ for the dominant wave for a given μ
μ	dimensionless meridional wavenumber
$\nu = (\lambda^2 + \mu^2)^{1/2}$	dimensionless total wavenumber
$\pi^* = \rho_s^* g B H^2 / h$	pressure scale for the disturbance
ρ_s^*	density of the standard atmosphere
ρ^*	departure of density from ρ_s^*
$\sigma = \omega_i $	dimensionless growth rate
σ_{max}	value of σ for the dominant wave for given μ
$\tau^* = h / f_0$	time scale for the disturbance
$\underline{\tau}$	horizontal component of the vertical stress

Φ^*	entropy/ c_p (referred to in the thesis as the 'entropy')
Φ_s^*	'entropy' of the standard atmosphere
φ^*	departure of 'entropy' from Φ_s^*
$\varphi = \varphi^* h / B H$	dimensionless value of φ^*
$\chi = \psi - \bar{\Phi} z + \frac{1}{2} h z^2$	dimensionless geostrophic stream function for flow relative to the initial mean zonal wind on the isentropic surface $\bar{\Phi} = \text{const}$
$\psi^* = p^* / f_s^*$	
$\psi = \psi^* h / [g B H^2]$	dimensionless geostrophic stream function for flow relative to the mean zonal wind
ψ_R	dimensionless geostrophic stream function for flow relative to the ground
ψ_n	n'th term of the perturbation (ε) expansion of ψ
$\psi^{(n)}$	n'th term of the Richardson number (h^{-1}) expansion of ψ
$\hat{\psi}^{(n)} = \psi^{(n)} + h^{-1} \left[\frac{1}{2} \{ (u^{(n)})^2 - \bar{u}^2 + (v^{(n)})^2 \} - \frac{1}{2} \{ (\varphi^{(n)})^2 - \bar{\varphi}^2 \} \right]$ $\omega = \omega_r + i \omega_i$	transformed value of $\psi^{(n)}$
$ \omega_r / \lambda = c$	dimensionless phase speed relative to the initial mean zonal wind
$ \omega_i = \sigma$	dimensionless growth
$[D/Dt]^* = \frac{\partial}{\partial t} + \underline{g}^* \cdot \nabla^*$	differentiation with respect to time following a particle ($\underline{g}^*$ horizontal and geostrophic except in sections 6.1 - 6.5)
$D/Dt = [D/Dt]^* h / f_0$	dimensionless form of $[D/Dt]^*$
d/dt	total derivative with respect to time

$\nabla^* = (\partial/\partial x^*, \partial/\partial y^*, \partial/\partial z^*)$ three dimensional gradient operator

∇

dimensionless form of ∇^*

div

dimensionless divergence with respect to the horizontal coordinates (x,y)

$$J(\alpha_1, \alpha_2) = \frac{\partial \alpha_1}{\partial x} \frac{\partial \alpha_2}{\partial y} - \frac{\partial \alpha_1}{\partial y} \frac{\partial \alpha_2}{\partial x}$$

dimensionless Jacobian operator

————— x

averaging operator over one zonal wavelength

————— x, y

averaging operator over one zonal wavelength and the width of the baroclinic zone

————— x, y, z

averaging operator over one zonal wavelength, the width of the baroclinic zone and the depth of the troposphere

APPENDIX II.

(a) The Runge-Kutta Method.

The Runge-Kutta method is a numerical procedure for solving a set of n first order simultaneous differential equations of the form

$$\frac{dT_l}{dt} = F_l(t, T_m)$$

(where $m = 1, 2, \dots, n$, and $l = 1, 2, \dots, n$,) from given initial values of the T_l 's.

The general formulation of the method involves a number of arbitrary constants which may be fixed so as to provide some optimising facility, such as - a minimum truncation error - least computation time - best use of the computer storage (for a discussion of different procedures see Martin (1958)). An " r 'th order" method involves successive computations at r intermediate sub-intervals of the basic time step δt , the truncation error being in general $O((\delta t)^{r+1})$.

The equations were solved on the University of London Atlas computer using the EXCHLF autocode language. The facility for solving differential equations by the fourth order Runge-Kutta method is included in the EXCHLF compiler as a standard routine which uses arbitrary constants as given by Gill (1951) (these are chosen so as to minimise the storage space in the computer). The sub-routine calculates the $4n$ quantities

$$k_{0,l} = \delta t F_l(t_0, T_m(t_0))$$

$$k_{1,l} = \delta t F_l(t_0 + \frac{1}{2}\delta t, T_m(t_0) + \frac{1}{2} k_{0,m})$$

$$k_{2,l} = \delta t F_l(t_0 + \frac{1}{2}\delta t, T_m(t_0) + (-\frac{1}{2} + \frac{1}{\sqrt{2}})k_{0,m} + (1 - \frac{1}{\sqrt{2}})k_{1,m})$$

$$k_{3,l} = \delta t F_l(t_0 + \delta t, T_m(t_0) + (-\frac{1}{\sqrt{2}})k_{1,m} + (1 + \frac{1}{\sqrt{2}})k_{2,m})$$

and then

$$T_l(t_0 + \delta t) = T_l(t_0) + \frac{1}{6} k_{0,l} + \frac{1}{3} (1 - \frac{1}{\sqrt{2}}) k_{1,l} + \frac{1}{3} (1 + \frac{1}{\sqrt{2}}) k_{2,l} + \frac{1}{6} k_{3,l}$$

The standard routine performs the integration for one fixed time step δt , but in order to reduce the truncation error the computation was repeated for two time steps each with an interval of $\frac{1}{2}\delta t$ and the results compared. The computer was programmed to accept the latter as a sufficiently accurate solution if, for each T_l ,

$$\frac{|T_l(t_0 + \delta t) - T_l(t_0 + 2(\frac{1}{2}\delta t))|}{|T_l(t_0 + \delta t) + T_l(t_0 + 2(\frac{1}{2}\delta t))|} \times 200 < 0.1 \quad (110)$$

If this criterion was not satisfied a further calculation was performed for four steps each with an interval of $\frac{1}{4}\delta t$, and this was compared with $T_l(t_0 + 2(\frac{1}{2}\delta t))$ using (110), and so on until an acceptable solution was obtained. The process was then repeated for the next time step to give $T_l(t_0 + 2\delta t)$.

For the differential equations given by the method of functional representation the calculations were started at $t = -20$ with the initial conditions

$$\begin{aligned}
R_{1,1} &= b e^{-20\sigma} \\
S_{1,1} &= -e^{-20\sigma} \\
\left. \begin{aligned} R_{\ell,m} \\ S_{\ell,m} \end{aligned} \right\} &= 0 \quad \ell, m \neq 1, 1
\end{aligned} \tag{111}$$

The above criterion (110) was chosen after integrating equations (48-50) numerically and comparing with the analytic solution (54). It was found that, unlike the analytic solution, the numerical solution amplified again at some large value of t , but that increasing the stringency of the condition (110) delayed this effect, a 0.1% allowable error between computations with different step intervals being chosen as a compromise between accuracy and computing time (test integrations were made for 5%, 1%, 0.5%, 0.1%, 0.01% allowable errors). It should be noted that the amplifying solution at large values of t is only partly due to truncation errors in the numerical integration procedure since the initial conditions (111) are used as an approximation to $R_{\ell,m} \rightarrow 0$ and $S_{\ell,m} \rightarrow 0$ as $t \rightarrow -\infty$.

(b) Newton's Method.

Newton's method was used to find the zero of the function $\Phi(x_0, y_0, z, t_0) + hz - G_{\infty}t$ (see section 4.3) by means of the iterative procedure

$$z_0^{(r+1)} = z_0^{(r)} - \left[\Phi(x_0, y_0, z_0^{(r)}, t_0) + h z_0^{(r)} - G_{\infty}t \right] / \left[\frac{\partial \Phi}{\partial z}(x_0, y_0, z_0^{(r)}, t_0) + h \right]$$

where $z_0^{(r)}$ is the r 'th approximation to the zero z_0 . The criterion for convergence was $|z_0^{(r)} - z_0^{(r+1)}| < .001$ (an accuracy of one decametre) and the first guess was taken as the height of the isentropic surface in the initial state $\frac{1}{h} [y + G_{\infty}t]$. The convergence was rapid.

APPENDIX III. REFERENCES.

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