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Finite Element Approximation of a Nonlinear Cross-Diffusion Population Model

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Summary We consider a fully discrete finite element approximation of the nonlinear cross-diffusion population model: Find u_i, the population of the $i^{\rm th}$ species, $i = 1$ and 2, such that

$$\frac{\partial u_i}{\partial t} - \Delta\,[\,c_i\,u_i + a_i\,u_i^2 + u_i\,u_j\,] - b_i\,\nabla\,.\,(u_i\,\nabla v) = g_i(u_1,u_2),$$

where $j \neq i$ and $g_i(u_1,u_2) := (\mu_i - \gamma_{ii}\,u_i - \gamma_{ij}\,u_j)\,u_i$. In the above, the given data is as follows: v is an environmental potential, $c_i \in \mathbb{R}_{\geq 0}$, $a_i \in \mathbb{R}_{>0}$ are diffusion coefficients, $b_i \in \mathbb{R}$ are transport coefficients, $\mu_i \in \mathbb{R}_{\geq 0}$ are the intrinsic growth rates, and $\gamma_{ii} \in \mathbb{R}_{\geq 0}$ are intra-specific, whereas γ_{ij}, $i \neq j$, $\in \mathbb{R}_{\geq 0}$ are interspecific competition coefficients. In addition to showing well-posedness of our approximation, we prove convergence in space dimensions $d \leq 3$. Finally some numerical experiments in one space dimension are presented.

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, with a Lipschitz boundary $\partial\Omega$ having normal ν. We consider a fully discrete finite element approximation of the following nonlinear cross-diffusion population model:
(P) Find $u_i :\ \Omega \times [0,T] \to \mathbb{R}$, the population of the $i^{\rm th}$ species, $i = 1$ and

2, such that

$$\frac{\partial u_i}{\partial t} - \nabla \,.\, \beta_i(u_1, u_2) = g_i(u_1, u_2) \qquad \text{in } \Omega_T := \Omega \times (0,T], \tag{1.1a}$$

$$\beta_i(u_1, u_2) \,.\, \nu = 0 \qquad \text{on } \partial\Omega \times (0,T], \tag{1.1b}$$

$$u_i(\cdot, 0) = u_i^0(\cdot) \geq 0 \qquad \text{in } \Omega; \tag{1.1c}$$

where, for $j \neq i$, the flux terms

$$\begin{aligned}\beta_i(w_1, w_2) &:= \nabla(c_i\, w_i + a_i\, w_i^2 + w_i\, w_j) + b_i\, w_i\, \nabla v \\ &\equiv (c_i + 2\, a_i\, w_i + w_j)\, \nabla w_i + w_i\, (\nabla w_j + b_i\, \nabla v)\,,\end{aligned} \tag{1.2a}$$

and the reaction terms

$$g_i(w_1, w_2) := (\mu_i - \gamma_{ii}\, w_i - \gamma_{ij}\, w_j)\, w_i \tag{1.2b}$$

are of Lotka-Volterra type. In the above, the given data is as follows: $v \in H^1(\Omega) \cap W^{1,s}(\Omega)$, $s > d$, is an environmental potential, $c_i \in \mathbb{R}_{\geq 0}$, $a_i \in \mathbb{R}_{>0}$ are diffusion coefficients, $b_i \in \mathbb{R}$ are transport coefficients, $\mu_i \in \mathbb{R}_{\geq 0}$ are the intrinsic growth rates, and $\gamma_{ii} \in \mathbb{R}_{\geq 0}$ are intra-specific, whereas γ_{ij}, $i \neq j, \in \mathbb{R}_{\geq 0}$ are interspecific competition coefficients.

We review briefly what is known about the system (P). Firstly, without loss of generality, one can take the coefficient of the cross-diffusion term $\Delta(u_1\, u_2)$ in both equations in (P) to be unity, by rescaling the unknowns $\{u_1, u_2\}$; see [7] for details. Secondly, the system (P) is strongly coupled with diffusion matrix

$$A(u_1, u_2) := \begin{pmatrix} c_1 + 2\, a_1\, u_1 + u_2 & u_1 \\ u_2 & c_2 + 2\, a_2\, u_2 + u_1 \end{pmatrix}. \tag{1.3}$$

Unfortunately, there is no maximum or comparison principle for such coupled systems. We note that

$$\underline{\xi}^T A(u_1, u_2)\, \underline{\xi} \geq \sum_{i=1}^{2} (c_i + (2\, a_i - \tfrac{1}{4})\, u_i)\, \xi_i^2 \qquad \forall\, \underline{\xi} \in \mathbb{R}^2\,. \tag{1.4}$$

If $8\, a_i \geq 1$ and $c_i > 0$, $i = 1, 2$, then $A(u_1, u_2)$ is positive definite for $u_1,\, u_2 \geq 0$. In this case of weak cross-diffusion, the existence of a global weak solution to (P) in any space dimension is easily proved in [6]. Obviously for general data, including strong cross-diffusion; that is, $c_i \in \mathbb{R}_{\geq 0}$ and $a_i \in \mathbb{R}_{>0}$, $i = 1, 2$, then $A(u_1, u_2)$ is not positive definite. Existence of a global weak solution to (P) for such general data has only been established recently. Using an exponential transformation of the unknown variables, $\{u_1,\, u_2\}$, existence of a global weak solution to (P) in one space dimension was established in [7]. Very recently existence of a global weak solution to (P) in up to three space dimensions has been established in [5] without using an exponential transformation, which restricted the proof in [7] to one space dimension. For other existence results for (P) under restricted choices of the coefficients, see the references in [7] and [5].

A key step of the multi-dimension existence proof in [5] is to establish and exploit an entropy inequality. As this will play a central role in our finite element approximation of (P), we review briefly this inequality here. Firstly, we introduce $F \in C^\infty(\mathbb{R}_{>0})$ such that for all $s \in \mathbb{R}_{>0}$

$$F(s) := s\,(\ln s - 1) + 1 \geq 0 \Rightarrow F'(s) = \ln s \Rightarrow F''(s) = s^{-1}\,. \quad (1.5)$$

Multiplying the i^{th} equation of (P) by $F'(u_i)$, and integrating over Ω yields for $i = 1,\,2$, with $j \neq i$, that

$$\frac{d}{dt}\int_\Omega F(u_i)\,\mathrm{d}x + \int_\Omega \left[\,(c_i\,u_i^{-1} + 2\,a_i + u_i^{-1}\,u_j)\,|\nabla u_i|^2 + \nabla u_j\,.\,\nabla u_i\right]\,\mathrm{d}x$$
$$= \int_\Omega [\,-b_i \nabla v\,.\,\nabla u_i + g_i(u_1,u_2)\,F'(u_i)\,]\,\mathrm{d}x\,. \quad (1.6)$$

Summing (1.6) over i yields that

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^{2} F(u_i)\,\mathrm{d}x + \int_\Omega \sum_{i=1}^{2}(c_i\,u_i^{-1} + 2\,a_i)\,|\nabla u_i|^2\,\mathrm{d}x$$
$$+ \int_\Omega \left|\left(\frac{u_2}{u_1}\right)^{\frac{1}{2}} \nabla u_1 + \left(\frac{u_1}{u_2}\right)^{\frac{1}{2}} \nabla u_2\right|^2 \mathrm{d}x$$
$$= \int_\Omega \sum_{i=1}^{2}[\,-b_i\,\nabla v\,.\,\nabla u_i + g_i(u_1,u_2)\,F'(u_i)\,]\,\mathrm{d}x\,. \quad (1.7)$$

Obviously, the bound (1.7) is only formal since e.g. a priori we do not know that $u_i(x,t) \in \mathbb{R}_{>0}$ for F to be well defined. To make this bound rigorous, and in constructing our numerical approximation of (P), one has to go through a regularization procedure. We introduce an alternative regularization procedure, which we believe to be more transparent, to that employed in [5]. We replace $F \in C^\infty(\mathbb{R}_{>0})$ for any $\varepsilon \in (0,1)$ by the regularized function $F_\varepsilon : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

$$F_\varepsilon(s) := \begin{cases} \frac{s^2-\varepsilon^2}{2\,\varepsilon} + (\ln\varepsilon - 1)\,s + 1 & s \leq \varepsilon\,, \\ (\ln s - 1)\,s + 1 & \varepsilon \leq s \leq \varepsilon^{-1}\,, \\ \frac{\varepsilon\,(s^2-\varepsilon^{-2})}{2} + (\ln\varepsilon^{-1} - 1)\,s + 1 & \varepsilon^{-1} \leq s\,. \end{cases} \quad (1.8)$$

Hence $F_\varepsilon \in C^{2,1}(\mathbb{R})$ with the first two derivatives of F_ε given by

$$F_\varepsilon'(s) := \begin{cases} \varepsilon^{-1}\, s + \ln \varepsilon - 1 & s \le \varepsilon\,, \\ \ln s & \varepsilon \le s \le \varepsilon^{-1}\,, \\ \varepsilon\, s + \ln \varepsilon^{-1} - 1 & \varepsilon^{-1} \le s \end{cases} \tag{1.9a}$$

$$\text{and} \qquad F_\varepsilon''(s) := \begin{cases} \varepsilon^{-1} & s \le \varepsilon\,, \\ s^{-1} & \varepsilon \le s \le \varepsilon^{-1}\,, \\ \varepsilon & \varepsilon^{-1} \le s\,; \end{cases} \tag{1.9b}$$

respectively. We introduce also

$$\lambda_\varepsilon(s) := [F_\varepsilon''(s)]^{-1} \qquad \text{and} \qquad \widetilde{\lambda}_\varepsilon(s) := \begin{cases} s & s \le \varepsilon^{-1}\,, \\ \varepsilon^{-1} & \varepsilon^{-1} \le s\,. \end{cases} \tag{1.10}$$

The corresponding regularised version of (P) is then
(P$_\varepsilon$) Find $u_{\varepsilon,i} :\ \Omega \times [0,T] \to \mathbb{R}$, $i = 1$ and 2, such that

$$\frac{\partial u_{\varepsilon,i}}{\partial t} - \nabla\,.\,\beta_{\varepsilon,i}(u_{\varepsilon,1}, u_{\varepsilon,2}) = g_{\varepsilon,i}(u_{\varepsilon,1}, u_{\varepsilon,2}) \qquad \text{in } \Omega_T\,, \tag{1.11a}$$

$$\beta_{\varepsilon,i}(u_{\varepsilon,1}, u_{\varepsilon,2})\,.\,\nu = 0 \qquad \text{on } \partial\Omega \times (0,T], \tag{1.11b}$$

$$u_{\varepsilon,i}(\cdot,0) = u_i^0(\cdot) \ge 0 \qquad \text{in } \Omega; \tag{1.11c}$$

where for $j \neq i$

$$\beta_{\varepsilon,i}(w_1,w_2) := (c_i + 2\,a_i\,\lambda_\varepsilon(w_i) + \lambda_\varepsilon(w_j)\,)\,\nabla w_i + \lambda_\varepsilon(w_i)\,(\nabla w_j + b_i\,\nabla v)\,, \tag{1.12a}$$

$$g_{\varepsilon,i}(w_1,w_2) := \mu_i\, w_i - (\gamma_{ii}\,\lambda_\varepsilon(w_i) + \gamma_{ij}\,\lambda_\varepsilon(w_j)\,)\,\lambda_\varepsilon(w_i). \tag{1.12b}$$

Multiplying the i^{th} equation of (P$_\varepsilon$) by $F_\varepsilon'(u_{\varepsilon,i})$, integrating over Ω and summing over i yields, on noting (1.10), the analogue of (1.7)

$$\begin{aligned} \frac{d}{dt}\int_\Omega \sum_{i=1}^2 F_\varepsilon(u_{\varepsilon,i})\,\mathrm{d}x &+ \int_\Omega \sum_{i=1}^2 (c_i\,[\lambda_\varepsilon(u_{\varepsilon,i})]^{-1} + 2\,a_i)\,|\nabla u_{\varepsilon,i}|^2\,\mathrm{d}x \\ &+ \int_\Omega \left| \left(\frac{\lambda_\varepsilon(u_{\varepsilon,2})}{\lambda_\varepsilon(u_{\varepsilon,1})}\right)^{\frac{1}{2}} \nabla u_{\varepsilon,1} + \left(\frac{\lambda_\varepsilon(u_{\varepsilon,1})}{\lambda_\varepsilon(u_{\varepsilon,2})}\right)^{\frac{1}{2}} \nabla u_{\varepsilon,2} \right|^2 \mathrm{d}x \\ &= \int_\Omega \sum_{i=1}^2 [\,-b_i\,\nabla v\,.\,\nabla u_{\varepsilon,i} + g_{\varepsilon,i}(u_{\varepsilon,1},u_{\varepsilon,2})\,F_\varepsilon'(u_{\varepsilon,i})\,]\,\mathrm{d}x\,. \end{aligned} \tag{1.13}$$

It is easily established from (1.8), (1.9a) and (1.10) that for $\varepsilon \in (0, e^{-2})$

$$F_\varepsilon(s) \ge \tfrac{\varepsilon}{2}\,s^2 - 2 \quad \forall\, s \ge 0 \qquad \text{and} \qquad F_\varepsilon(s) \ge \tfrac{s^2}{2\,\varepsilon} \quad \forall\, s \le 0\,; \tag{1.14a}$$

$$\max\{\lambda_\varepsilon(s),\, s\,F'_\varepsilon(s)\} \leq 2\,F_\varepsilon(s) + 1 \qquad \forall\, s \in \mathbb{R}\,, \tag{1.14b}$$

$$\lambda_\varepsilon(s)\,F'_\varepsilon(s) \geq s - 1 \qquad \forall\, s \in \mathbb{R}\,. \tag{1.14c}$$

From the inequalities (1.14a–c), and noting that $[1-s]_+ \leq 1 - [s]_-$ for all $s \in \mathbb{R}$, we obtain for $i = 1,\, 2$, with $j \neq i$, that

$$\begin{aligned} & g_{\varepsilon,i}(u_{\varepsilon,1}, u_{\varepsilon,2})\,F'_\varepsilon(u_{\varepsilon,i}) \\ &\qquad \leq C + 2\,\mu_i\,F_\varepsilon(u_{\varepsilon,i}) + [\,\gamma_{ii}\,\lambda_\varepsilon(u_{\varepsilon,i}) + \gamma_{ij}\,\lambda_\varepsilon(u_{\varepsilon,j})\,]\,[1-u_{\varepsilon,i}]_+ \\ &\qquad \leq C + 2\,(\mu_i + \gamma_{ii})\,F_\varepsilon(u_{\varepsilon,i}) + 2\,\gamma_{ij}\,F_\varepsilon(u_{\varepsilon,j}) \\ &\qquad\qquad + \tfrac{\varepsilon^{-1}}{2}\,(\gamma_{ii}+\gamma_{ij})\,[u_{\varepsilon,i}]_-^2 + \tfrac{\varepsilon}{2}\,(\gamma_{ii}\,[\lambda_\varepsilon(u_{\varepsilon,i})]^2 + \gamma_{ij}\,[\lambda_\varepsilon(u_{\varepsilon,j})]^2) \\ &\qquad \leq C + (2\,\mu_i + 4\,\gamma_{ii} + \gamma_{ij})\,F_\varepsilon(u_{\varepsilon,i}) + 3\,\gamma_{ij}\,F_\varepsilon(u_{\varepsilon,j})\,. \end{aligned} \tag{1.15}$$

It is crucial in bounding the above that the coefficients $\gamma_{ii},\, \gamma_{ij} \in \mathbb{R}_{\geq 0}$. Combining (1.13) and (1.15), and applying a Gronwall inequality yields the following uniform bounds

$$\sup_{t\in(0,T)} \int_\Omega \left[\sum_{i=1}^2 F_\varepsilon(u_{\varepsilon,i})\right] \mathrm{d}x + \int_{\Omega_T} \sum_{i=1}^2 a_i\,|\nabla u_{\varepsilon,i}|^2\,\mathrm{d}x\,\mathrm{d}t \leq C\,. \tag{1.16}$$

We see immediately from the above that the assumption $a_i \in \mathbb{R}_{>0}$ is crucial to obtain a uniform $L^2(0,T;H^1(\Omega))$ bound on $u_{\varepsilon,i}$. Although $u_{\varepsilon,i}$ can go negative, it follows from (1.16) and (1.14a) that

$$\sup_{t\in(0,T)} \int_\Omega \left[\sum_{i=1}^2 |\,[u_{\varepsilon,i}]_-\,|^2\right] \mathrm{d}x \leq C\,\varepsilon. \tag{1.17}$$

One can then use (1.16) and (1.17) to pass to the limit $\varepsilon \to 0$ in (P$_\varepsilon$) in order to prove existence of a non-negative solution to (P). As we have stated previously, we believe this procedure to be simpler and more transparent to the alternative regularization procedure adopted in [5].

It is the goal of this paper to introduce a fully discrete finite element approximation of (P) that is consistent with the entropy inequality (1.13). In order to derive a discrete analogue of (1.13), we adapt a technique introduced in [12,9] for deriving a discrete entropy bound for the thin film equation, a degenerate nonlinear fourth order parabolic equation. This technique has also been adapted to the thin film equation in the presence of surfactant, [2,4], and to a degenerate nonlinear second order parabolic system modeling bacterial pattern formation, [3].

We are not aware of any numerical analysis on the problem (P), except for the convergence of a semi-discretization in time (continuous in space) scheme in one space dimension, based on the exponential transformation of the unknown variables, $\{u_1,\, u_2\}$, see [7]. The layout of this paper is as follows. In §2 we formulate our fully discrete finite element approximation to (P) and derive a discrete analogue of the entropy bound (1.13). In §3 we establish convergence of our approximation in one, two and three space dimensions; and hence existence of a solution to (P) under basically the same assumptions as in [5]. In §4 we present some numerical computations in one space dimension. Finally, we note that the techniques in this paper can be easily adapted to other cross-diffusion systems; e.g. [8].

Notation and Auxiliary Results

We have adopted the standard notation for Sobolev spaces, denoting the norm of $W^{m,q}(G)$ ($m \in \mathbb{N}$, $q \in [1,\infty]$ and G a bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary) by $\|\cdot\|_{m,q,G}$ and the semi-norm by $|\cdot|_{m,q,G}$. For $q=2$, $W^{m,2}(G)$ will be denoted by $H^m(G)$ with the associated norm and semi-norm written, as respectively, $\|\cdot\|_{m,G}$ and $|\cdot|_{m,G}$. For ease of notation, in the common case when $G \equiv \Omega$ the subscript "Ω" will be dropped on the above norms and semi-norms. Throughout $(\cdot,\cdot)$ denotes the standard L^2 inner product over Ω.

For later purposes, we recall the following well-known Sobolev interpolation results, e.g. see [1]: Let $z \in H^1(\Omega)$ then the inequality

$$|z|_{0,r} \leq C|z|_{0,1}^{1-\sigma}\|z\|_1^{\sigma} \quad \text{holds for } r \in \begin{cases} [1,\infty] & \text{if } d=1, \\ [1,\infty) & \text{if } d=2, \\ [1,6] & \text{if } d=3; \end{cases} \tag{1.18}$$

where $\sigma = \frac{2\,(r-1)\,d}{r\,(d+2)}$ and C is a constant depending only on Ω and r. We recall also the following compactness result. Let X_0, X and X_1 be Banach spaces, X_k, $k=0,1$, reflexive, with a compact embedding $X_0 \hookrightarrow X$ and a continuous embedding $X \hookrightarrow X_1$. Then, for $\alpha_k > 1$, $k=0,1$, the embedding

$$\{\, \eta \in L^{\alpha_0}(0,T;X_0) : \tfrac{\partial \eta}{\partial t} \in L^{\alpha_1}(0,T;X_1) \,\} \hookrightarrow L^{\alpha_0}(0,T;X) \tag{1.19}$$

is compact.

For $q \in (1,\infty)$, let $\big(W^{1,q}(\Omega)\big)'$ denote the dual of $W^{1,q}(\Omega)$. It is convenient to introduce the "inverse Laplacian" operator $\mathcal{G} : \big(W^{1,q}(\Omega)\big)' \to W^{1,q'}(\Omega)$, $q' = \frac{q}{q-1}$, such that

$$(\nabla \mathcal{G} z, \nabla \eta) + (\mathcal{G} z, \eta) = \langle z, \eta \rangle_q \qquad \forall\, \eta \in W^{1,q}(\Omega), \tag{1.20}$$

and $\langle\cdot,\cdot\rangle_q$ denotes the duality pairing between $\big(W^{1,q}(\Omega)\big)'$ and $W^{1,q}(\Omega)$. It follows that $\|\mathcal{G}\cdot\|_{1,q'}$ is a norm on $\big(W^{1,q}(\Omega)\big)'$.

Throughout C denotes a generic constant independent of h, τ and ε, the mesh and temporal discretisation parameters and the regularization parameter. In addition $C(a_1, \cdots, a_I)$ denotes a constant depending on the arguments $\{a_i\}_{i=1}^I$.

2 Finite Element Approximation

We consider the finite element approximation of (P) under the following assumptions on the mesh:

(A) Let Ω be a polygonal or polyhedral domain if $d = 2$ or $d = 3$. Let $\{\mathcal{T}^h\}_{h>0}$ be a quasi-uniform family of partitionings of Ω into disjoint open simplices κ with $h_\kappa := \text{diam}(\kappa)$ and $h := \max_{\kappa \in \mathcal{T}^h} h_\kappa$, so that $\overline{\Omega} = \cup_{\kappa \in \mathcal{T}^h} \overline{\kappa}$. In addition, it is assumed for $d = 2$ or 3 that all simplices $\kappa \in \mathcal{T}^h$ are generically right-angled (for $d = 3$ this means that all tetrahedra have two vertices at which two edges intersect at right angles, see below for more details).

We note that a cube is easily partitioned into such tetrahedra.

Associated with $\mathcal{T}^h$ is the finite element space

$$S^h := \{\chi \in C(\overline{\Omega}) : \chi\,|_\kappa \text{ is linear} \,\forall\, \kappa \in \mathcal{T}^h\} \subset H^1(\Omega). \tag{2.1}$$

We introduce also

$$S^h_{\geq 0} := \{\chi \in S^h : \chi \geq 0 \text{ in } \Omega\} \subset \\ H^1_{\geq 0}(\Omega) := \{\eta \in H^1(\Omega) : \eta \geq 0 \; a.e. \text{ in } \Omega\}. \tag{2.2}$$

Let J be the set of nodes of $\mathcal{T}^h$ and $\{p_j\}_{j \in J}$ the coordinates of these nodes. Let $\{\chi_j\}_{j \in J}$ be the standard basis functions for S^h; that is $\chi_j \in S^h_{\geq 0}$ and $\chi_j(p_i) = \delta_{ij}$ for all $i, j \in J$. We introduce $\pi^h : C(\overline{\Omega}) \to S^h$, the interpolation operator, such that $(\pi^h \eta)(p_j) = \eta(p_j)$ for all $j \in J$. A discrete semi-inner product on $C(\overline{\Omega})$ is then defined by

$$(\eta_1, \eta_2)^h := \int_\Omega \pi^h(\eta_1(x)\,\eta_2(x))\,\mathrm{d}x = \sum_{j \in J} m_j\,\eta_1(p_j)\,\eta_2(p_j), \tag{2.3}$$

where $m_j := (1, \chi_j) > 0$. The induced discrete semi-norm is then $|\eta|_h := [\,(\eta, \eta)^h\,]^{\frac{1}{2}}$, where $\eta \in C(\overline{\Omega})$. We introduce also the L^2 projection $Q^h : L^2(\Omega) \to S^h$ defined by

$$(Q^h \eta, \chi)^h = (\eta, \chi) \quad \forall\, \chi \in S^h. \tag{2.4}$$

Similarly to the approach in [12] and [9], we introduce, for any $\varepsilon \in (0,1)$, $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ such that for all $z^h \in S^h$ and $a.e.$ in Ω

$$\Lambda_\varepsilon(z^h) \text{ is symmetric and positive definite,} \tag{2.5a}$$

$$\Lambda_\varepsilon(z^h)\,\nabla\pi^h[F'_\varepsilon(z^h)] = \nabla z^h. \tag{2.5b}$$

Firstly, we give the construction of Λ_ε in the simple case when $d = 1$. Given $z^h \in S^h$ and $\kappa \in \mathcal{T}^h$ having vertices p_j and p_k, we set

$$\Lambda_\varepsilon(z^h)\,|_\kappa := \begin{cases} \frac{z^h(p_k)-z^h(p_j)}{F'_\varepsilon(z^h(p_k))-F'_\varepsilon(z^h(p_j))} = \frac{1}{F''_\varepsilon(z^h(\xi))} & \\ \qquad\qquad \text{for some } \xi \in \kappa & \text{if } z^h(p_k) \neq z^h(p_j), \\ \frac{1}{F''_\varepsilon(z^h(p_k))} & \text{if } z^h(p_k) = z^h(p_j). \end{cases} \tag{2.6}$$

Clearly the piecewise constant construction in (2.6) satisfies the conditions (2.5a,b).

Following [9] we extend the above construction to $d = 2$ or 3. Let $\{e_i\}_{i=1}^d$ be the orthonormal vectors in $\mathbb{R}^d$, such that the j^{th} component of e_i is δ_{ij}, $i, j = 1 \to d$. Given non-zero constants ρ_i, $i = 1 \to d$; let $\widehat{\kappa}(\{\rho_i\}_{i=1}^d)$ be a reference simplex in $\mathbb{R}^d$ with vertices $\{\widehat{p}_i\}_{i=0}^d$, where $\widehat{p}_0$ is the origin and $\widehat{p}_i = \widehat{p}_{i-1} + \rho_i e_i$, $i = 1 \to d$. Given a $\kappa \in \mathcal{T}^h$ with vertices $\{p_{j_i}\}_{i=0}^d$, such that p_{j_0} is not a right-angled vertex, then there exists a rotation/reflection matrix R_κ and non-zero constants $\{\rho_i\}_{i=1}^d$ such that the mapping $\mathcal{R}_\kappa : \widehat{x} \in \mathbb{R}^d \to p_{j_0} + R_\kappa \widehat{x} \in \mathbb{R}^d$ maps the vertex $\widehat{p}_i$ to p_{j_i}, $i = 0 \to d$, and hence $\widehat{\kappa} \equiv \widehat{\kappa}(\{\rho_i\}_{i=1}^d)$ to κ. For all $\kappa \in \mathcal{T}^h$ and $z^h \in S^h$, we set

$$\widehat{z}^h(\widehat{x}) \equiv z^h(\mathcal{R}_\kappa \widehat{x}) \qquad \forall\, \widehat{x} \in \overline{\widehat{\kappa}}. \tag{2.7}$$

As $R_\kappa^T \equiv R_\kappa^{-1}$, we have that

$$\nabla z^h \equiv R_\kappa \widehat{\nabla} \widehat{z}^h, \tag{2.8}$$

where $x \equiv (x_1, \cdots, x_d)^T$, $\nabla \equiv (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d})^T$, $\widehat{x} \equiv (\widehat{x}_1, \cdots, \widehat{x}_d)^T$ and $\widehat{\nabla} \equiv (\frac{\partial}{\partial \widehat{x}_1}, \cdots, \frac{\partial}{\partial \widehat{x}_d})^T$. We then set

$$\Lambda_\varepsilon(z^h)\,|_\kappa := R_\kappa\, \widehat{\Lambda}_\varepsilon(\widehat{z}^h)\,|_{\widehat{\kappa}}\, R_\kappa^T, \tag{2.9}$$

where $\widehat{\Lambda}_\varepsilon(\widehat{z}^h)\,|_{\widehat{\kappa}}$ is the $d \times d$ diagonal matrix with diagonal entries, $k = 1 \to d$,

$$[\widehat{\Lambda}_\varepsilon(\widehat{z}^h)\,|_{\widehat{\kappa}}]_{kk} := \begin{cases} \frac{\widehat{z}^h(\widehat{p}_k)-\widehat{z}^h(\widehat{p}_0)}{F'_\varepsilon(\widehat{z}^h(\widehat{p}_k))-F'_\varepsilon(\widehat{z}^h(\widehat{p}_0))} \equiv \frac{z^h(p_{j_k})-z^h(p_{j_0})}{F'_\varepsilon(z^h(p_{j_k}))-F'_\varepsilon(z^h(p_{j_0}))} & \\ \qquad = \frac{1}{F''_\varepsilon(z^h(\xi))} \quad \text{for some } \xi \text{ between } p_{j_k} \text{ and } p_{j_0} & \\ & \text{if } z^h(p_{j_k}) \neq z^h(p_{j_0}), \\ \frac{1}{F''_\varepsilon(\widehat{z}^h(\widehat{p}_0))} \equiv \frac{1}{F''_\varepsilon(z^h(p_{j_0}))} & \text{if } z^h(p_{j_k}) = z^h(p_{j_0}). \end{cases} \tag{2.10}$$

It easily follows from (2.7) and (2.8) that $\Lambda_\varepsilon(z^h)$ constructed in (2.9) and (2.10) satisfies (2.5a,b). Throughout we make use of the fact that powers of the matrices $\Lambda_\varepsilon(z_1^h)$ and $\Lambda_\varepsilon(z_2^h)$ commute for any $z_i^h \in S^h$, see (2.9). It is the construction (2.9) and (2.10) that requires the right angle constraint on the partitioning $\mathcal{T}^h$. We note that this is not such a severe constraint, as there exist adaptive finite element codes that satisfy this requirement, see e.g. [11]. Another consequence of the right angle constraint on $\mathcal{T}^h$ is that

$$|\nabla \pi^h[\lambda_\varepsilon(\chi)]|_1^2 \leq (\nabla \chi, \nabla \pi^h[\lambda_\varepsilon(\chi)]) \qquad \forall\, \chi \in S^h\,. \tag{2.11}$$

In addition to $\mathcal{T}^h$, let $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$ be a partitioning of $[0,T]$ into possibly variable time steps $\tau_n := t_n - t_{n-1}$, $n = 1 \to N$. We set $\tau := \max_{n=1\to N} \tau_n$. For any given $\varepsilon \in (0,1)$, we then consider the following fully discrete finite element approximation of (P):
($\mathbf{P}_\varepsilon^{h,\tau}$) For $n \geq 1$ find $\{U_{\varepsilon,1}^n,\, U_{\varepsilon,2}^n\} \in [S^h]^2$ such that for $i = 1$ and 2, with $j \neq i$, and for all $\chi \in S^h$

$$\begin{aligned}\left(\tfrac{U_{\varepsilon,i}^n - U_{\varepsilon,i}^{n-1}}{\tau_n}, \chi\right)^h &+ (\,[c_i + 2\,a_i\,\Lambda_\varepsilon(U_{\varepsilon,i}^n) + \Lambda_\varepsilon(U_{\varepsilon,j}^n)]\,\nabla U_{\varepsilon,i}^n, \nabla\chi) \\ &+ (\Lambda_\varepsilon(U_{\varepsilon,i}^n)\,[\nabla U_{\varepsilon,j}^n + b_i\,\nabla(\pi^h v)\,], \nabla\chi) \\ = (\mu_i\, U_{\varepsilon,i}^n &- [\gamma_{ii}\,\lambda_\varepsilon(U_{\varepsilon,i}^{n-1}) + \gamma_{ij}\,\lambda_\varepsilon(U_{\varepsilon,j}^{n-1})]\,\lambda_\varepsilon(U_{\varepsilon,i}^n), \chi)^h\,,\end{aligned} \tag{2.12}$$

where $U_{\varepsilon,i}^0 \in S^h$ is an approximation of u_i^0 e.g. $U_{\varepsilon,i}^0 \equiv \pi^h u_i^0$ or $Q^h u_i^0$.

Below we recall some well-known results concerning S^h for any $\kappa \in \mathcal{T}^h$, $\chi, z^h \in S^h$, $m \in \{0,1\}$, $p \in [1,\infty]$, $s \in [2,\infty]$ if $d = 1$ and $s \in (d,\infty]$ if $d = 2$ or 3:

$$|\chi|_{1,p,\kappa} \leq C\, h_\kappa^{-1}\, |\chi|_{0,p,\kappa}\,; \tag{2.13}$$

$$|\chi|_{m,r,\kappa} \leq C\, h_\kappa^{-d\,(\frac{1}{p}-\frac{1}{r})}\, |\chi|_{m,p,\kappa} \qquad \forall\, r \in [p,\infty]\,; \tag{2.14}$$

$$\lim_{h\to 0} \|(I-\pi^h)\eta\|_{1,s} = 0 \qquad \forall\, \eta \in W^{1,s}(\Omega)\,; \tag{2.15}$$

$$|(I-\pi^h)\eta|_{m,s} \leq C\, h^{1-m}\, |\eta|_{1,s} \qquad \forall\, \eta \in W^{1,s}(\Omega)\,; \tag{2.16}$$

$$\textstyle\int_\kappa \chi^2\,\mathrm{d}x \leq \int_\kappa \pi^h[\chi^2]\,\mathrm{d}x \leq (d+2)\,\int_\kappa \chi^2\,\mathrm{d}x\,; \tag{2.17}$$

$$|(\chi, z^h) - (\chi, z^h)^h| \leq |(I-\pi^h)(\chi\, z^h)|_{0,1} \;\leq\; C\, h^{1+m}\, |\chi|_{m,p}\, |z^h|_{1,p'}\,; \tag{2.18}$$

where $p' = \frac{p}{p-1}$. It follows from (2.4) for all $\eta \in L^\infty(\Omega)$ that

$$(Q^h\eta)(p_j) = \frac{(\eta,\chi_j)}{(1,\chi_j)} \quad \forall\, j \in J \quad \Longrightarrow \quad |Q^h\eta|_{0,\infty} \leq |\eta|_{0,\infty}\,. \tag{2.19}$$

In addition, it holds for $m \in \{0,1\}$ that

$$|(I - Q^h)\eta|_{m,r} \le C\, h^{1-m}\, |\eta|_{1,r} \quad \forall\, \eta \in W^{1,r}(\Omega) \qquad \text{for any } r \in [2,\infty]. \tag{2.20}$$

It is also easily established that

$$|z^h|_{0,q} \le C\, h^{-1}\, \|\mathcal{G} z^h\|_{1,q} \quad \forall\, z^h \in S^h \qquad \text{for any } q \in (1,2]. \tag{2.21}$$

Finally, we note that (2.20) and (2.21) exploit the fact that we have a quasi-uniform family of partitionings $\{\mathcal{T}^h\}_{h>0}$.

We now recall two lemmas concerning $\Lambda_\varepsilon(\cdot)$.

Lemma 2.1 *Let the assumptions (A) hold and let $\|\cdot\|$ denote the spectral norm on $\mathbb{R}^{d\times d}$. Then for any given $\varepsilon \in (0,1)$ the function $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ satisfies*

$$\varepsilon\, \xi^T \xi \le \xi^T \Lambda_\varepsilon(z^h)\xi \le \varepsilon^{-1}\, \xi^T \xi \qquad \forall\, \xi \in \mathbb{R}^d,\ z^h \in S^h \tag{2.22}$$

and is continuous. In particular it holds for all z_1^h, $z_2^h \in S^h$, $\kappa \in \mathcal{T}^h$ that

$$\begin{aligned} \|(\Lambda_\varepsilon(z_1^h) - \Lambda_\varepsilon(z_2^h))\,|_\kappa\, \| &= \|(\widehat{\Lambda}_\varepsilon(\widehat{z}_1^h) - \widehat{\Lambda}_\varepsilon(\widehat{z}_2^h))\,|_{\widehat{\kappa}}\, \| \\ &\le \varepsilon^{-2} \max_{k=1\to d} \left[|z_1^h(p_{j_k}) - z_2^h(p_{j_k})| + |z_1^h(p_{j_0}) - z_2^h(p_{j_0})| \right], \end{aligned} \tag{2.23}$$

where we have adopted the notation (2.9) and (2.10).

Proof The proof is a simple modification of the proof of Lemma 2.1 in [3]. □

It follows from (1.10) that for all $\kappa \in \mathcal{T}^h$ and for all $z^h \in S^h$

$$|(I - \pi^h)\lambda_\varepsilon(z^h)|_{0,\infty,\kappa} \le h_\kappa\, |\nabla \lambda_\varepsilon(z^h)|_{0,\infty,\kappa} \le h_\kappa\, |\nabla z^h|_{0,\infty,\kappa}\,. \tag{2.24}$$

The following Lemma is an extension of (2.24) to $\Lambda_\varepsilon(\cdot)$.

Lemma 2.2 *Let the assumptions (A) hold. Then for any given $\varepsilon \in (0,1)$ the function $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ is such that for all $\kappa \in \mathcal{T}^h$*

$$\max_{x\in\kappa} \|\{\Lambda_\varepsilon(z^h) - \lambda_\varepsilon(z^h)\,\mathcal{I}\}(x)\| \le h_\kappa\, |\nabla z^h|_\kappa\, | \qquad \forall\, z^h \in S^h, \tag{2.25}$$

where $\mathcal{I}$ is the $d \times d$ identity matrix.

Proof The proof is a simple modification of the proof of Lemma 2.3 in [4]. □

Theorem 2.1 *Let the assumptions (A) hold and $\{U_{\varepsilon,1}^{n-1}, U_{\varepsilon,2}^{n-1}\} \in [S^h]^2$, $n \ge 1$. Then for all $\varepsilon \in (0, e^{-2})$, for all $h > 0$ and for all τ_n such that $\omega\, \tau_n \le 1$, where $\omega := \max\{2\,\mu_1 + \gamma_{11} + \gamma_{12}, 2\,\mu_2 + \gamma_{21} + \gamma_{22}\}$, there exists a solution $\{U_{\varepsilon,1}^n, U_{\varepsilon,2}^n\} \in [S^h]^2$ to the n-th step of $(\mathrm{P}_\varepsilon^{h,\tau})$.*

Proof For $i = 1$ and 2, with $j \neq i$, let $A^n_i : [S^h]^2 \to S^h$ be such that for all $\chi \in S^h$

$$\begin{aligned}((A^n_i(U_1,U_2),\chi)^h &= \big(U_i - U_i^{n-1},\chi\big)^h \\ &\quad + \tau_n \left[([c_i + 2\,a_i\,\Lambda_\varepsilon(U_i) + \Lambda_\varepsilon(U_j)\,]\,\nabla U_i, \nabla \chi) \right. \\ &\quad + (\Lambda_\varepsilon(U_i)\,[\nabla U_j + b_i\,\nabla(\pi^h v)\,], \nabla\chi) \\ &\quad \left. - (\mu_i\,U_i - [\gamma_{ii}\,\lambda_\varepsilon(U^{n-1}_{\varepsilon,i}) + \gamma_{ij}\,\lambda_\varepsilon(U^{n-1}_{\varepsilon,j})\,]\,\lambda_\varepsilon(U_i),\chi)^h \right]. \quad (2.26)\end{aligned}$$

Therefore, on noting (2.26), we have that (2.12) is equivalent to: Find $\{U^n_{\varepsilon,1}, U^n_{\varepsilon,2}\} \in [S^h]^2$ such that

$$A^n_i(U^n_{\varepsilon,1}, U^n_{\varepsilon,2}) = 0 \qquad i = 1,\,2\,. \tag{2.27}$$

Assume that for a given $R \in \mathbb{R}_{>0}$, there does not exist $\{U_1,U_2\} \in [S^h]^2_R := \{\,(z_1^h, z_2^h) \in [S^h]^2 : |z^h_1|^2_h + |z^h_2|^2_h \leq R^2\,\}$ with $A^n_i(U_1,U_2) = 0$, $i = 1,\,2$. It follows immediately from (2.26), (2.23), (1.10) and (1.9b) that A^n_i is continuous on $[S^h]^2_R$. Hence we can define the continuous function $B^n \equiv (B^n_1, B^n_2) : [S^h]^2_R \to [S^h]^2_R$, where $B^n_i(U_1,U_2) := -R\,A^n_i(U_1,U_2) /(\sum_{i=1}^2 |A^n_i(U_1,U_2)|^2_h)^{\frac{1}{2}}$. As $[S^h]^2_R$ is a convex and compact subset of the finite dimensional space $[S^h]^2$; the *Brouwer fixed point theorem*, see e.g. [10, Theorem 9.36], asserts that there exists $\{U_1,U_2\} \in [S^h]^2_R$ such that $B^n_i(U_1,U_2) = U_i$, $i = 1,\,2$. Moreover, we have that $|U_1|^2_h + |U_2|^2_h = R^2$. We will now prove a contradiction for R sufficiently large.

Choosing $\chi \equiv \pi^h[F'_\varepsilon(U_i)]$ in (2.26), and noting (2.5b) and (2.22), yields for $i = 1,\,2$, with $j \neq i$, that

$$\begin{aligned}((A^n_i(U_1,U_2), F'_\varepsilon(U_i))^h &= \big(U_i - U_i^{n-1}, F'_\varepsilon(U_i)\big)^h \\ &\quad + \tau_n \left[([c_i\,[\Lambda_\varepsilon(U_i)]^{-1} + 2\,a_i\,\mathcal{I} + [\Lambda_\varepsilon(U_i)]^{-1}\,\Lambda_\varepsilon(U_j)\,]\,\nabla U_i, \nabla U_i) \right. \\ &\quad - (\mu_i\,U_i - [\gamma_{ii}\,\lambda_\varepsilon(U^{n-1}_{\varepsilon,i}) + \gamma_{ij}\,\lambda_\varepsilon(U^{n-1}_{\varepsilon,j})\,]\,\lambda_\varepsilon(U_i), F'_\varepsilon(U_i)\,)^h \\ &\quad \left. + (\nabla U_j + b_i\,\nabla(\pi^h v), \nabla U_i)\,\right]. \quad (2.28)\end{aligned}$$

It follows from (1.9b) and (1.8) that for $i = 1,\,2$

$$\begin{aligned}(U_i - U^{n-1}_{\varepsilon,i}, F'_\varepsilon(U_i))^h &\geq (F_\varepsilon(U_i) - F_\varepsilon(U^{n-1}_{\varepsilon,i}),1)^h + \tfrac{\varepsilon}{2}\,|U_i - U^{n-1}_{\varepsilon,i}|^2_h \\ &\geq (F_\varepsilon(U_i) - F_\varepsilon(U^{n-1}_{\varepsilon,i}),1)^h + \tfrac{\varepsilon}{4}\,|U_i|^2_h - \tfrac{\varepsilon}{2}\,|U^{n-1}_{\varepsilon,i}|^2_h\,. \quad (2.29)\end{aligned}$$

Combining (2.28) and (2.29), and noting that $\sum_{i=1}^2 |U_i|^2_h = R^2$, (2.5a), $\lambda_\varepsilon(s) \geq 0$ for all s, $F'_\varepsilon(s) \geq 0$ if $s \geq 1$, $\mu_i, \gamma_{ii}, \gamma_{ij} \in \mathbb{R}_{\geq 0}$, (1.14a–c) and

$a_i \in \mathbb{R}_{>0}$ yields, similarly to (1.15), that

$$\begin{aligned}
\sum_{i=1}^{2}\big(\,(A_i^n(U_1,U_2),F_\varepsilon'(U_i))^h &\geq \tfrac{\varepsilon}{4}\,R^2+\sum_{i=1}^{2}[1-2\,\mu_i\tau_n]\,(F_\varepsilon(U_i),1)^h \\
&\quad+\tau_n\,\Big[\sum_{i=1}^{2}\big(\,[c_i\,[\Lambda_\varepsilon(U_i)]^{-1}+2\,a_i\,\mathcal{I}\,]\,\nabla U_i,\nabla U_i\big)+b_i\,(\nabla(\pi^h v),\nabla U_i)\,\Big] \\
&\quad+\tau_n\,|\,([\Lambda_\varepsilon(U_1)]^{-1}\Lambda_\varepsilon(U_2))^{\frac{1}{2}}\,\nabla U_1+([\Lambda_\varepsilon(U_2)]^{-1}\Lambda_\varepsilon(U_1))^{\frac{1}{2}}\,\nabla U_2\,|_0^2 \\
&\quad+\tau_n\sum_{i=1,j\neq i}^{2}(\gamma_{ii}\,\lambda_\varepsilon(U_{\varepsilon,i}^{n-1})+\gamma_{ij}\,\lambda_\varepsilon(U_{\varepsilon,j}^{n-1}),[U_i]_-)^h\;-C(U_{\varepsilon,1}^{n-1},U_{\varepsilon,2}^{n-1}) \\
&\geq \tfrac{\varepsilon}{4}\,R^2+\sum_{i=1}^{2}[1-2\,\mu_i\tau_n]\,(F_\varepsilon(U_i),1)^h-\tfrac{\tau_n\,\varepsilon^{-1}}{2}\sum_{i=1,j\neq i}^{2}(\gamma_{ii}+\gamma_{ij})\,|\,[U_i]_-|_h^2 \\
&\qquad-\tfrac{\tau_n\,\varepsilon}{2}\sum_{i=1,j\neq i}^{2}(\gamma_{ii}+\gamma_{ji})\,|\lambda_\varepsilon(U_{\varepsilon,i}^{n-1})|_h^2-C(U_{\varepsilon,1}^{n-1},U_{\varepsilon,2}^{n-1},\pi^h v) \\
&\geq \tfrac{\varepsilon}{4}\,R^2+\sum_{i=1,j\neq i}^{2}\,[1-\tau_n\,(2\,\mu_i+\gamma_{ii}+\gamma_{ij})\,]\,(F_\varepsilon(U_i),1)^h \\
&\qquad-C(U_{\varepsilon,1}^{n-1},U_{\varepsilon,2}^{n-1},\pi^h v)\,. \qquad (2.30)
\end{aligned}$$

Hence on noting our assumption on τ_n, and on choosing R sufficiently large we have that

$$\sum_{i=1}^{2}\big(\,(A_i^n(U_1,U_2),F_\varepsilon'(U_i))^h \geq \tfrac{\varepsilon}{4}\,R^2-C(U_{\varepsilon,1}^{n-1},U_{\varepsilon,2}^{n-1},\pi^h v)>0\,. \qquad (2.31)$$

Similarly to (2.29), on noting (1.8) and that $\sum_{i=1}^{2}|U_i|_h^2=R^2$, we have for R sufficiently large that

$$\begin{aligned}
\sum_{i=1}^{2}(U_i,F_\varepsilon'(U_i))^h &\geq \sum_{i=1}^{2}[\,(F_\varepsilon(U_i)-F_\varepsilon(0),1)^h+\tfrac{\varepsilon}{2}\,|U_i|_h^2\,] \\
&\geq \tfrac{\varepsilon}{2}\,R^2-2\,|\Omega|>0\,. \qquad (2.32)
\end{aligned}$$

Clearly (2.31) and (2.32) for R sufficiently large contradicts that $\{U_1, U_2\}$ is a fixed point of B^n

$$\sum_{i=1}^{2}(U_i, F'_\varepsilon(U_i))^h = \sum_{i=1}^{2}(B_i^n(U_1,U_2), F'_\varepsilon(U_i))^h$$
$$= -\frac{R\,\sum_{i=1}^{2}(A_i^n(U_1,U_2), F'_\varepsilon(U_i))^h}{(\sum_{i=1}^{2}|A_i^n(U_1,U_2)|_h^2)^{\frac{1}{2}}} < 0. \qquad (2.33)$$

Therefore, under the given assumptions on ε and τ_n, we have existence of a solution to (2.27) and hence (2.12), the n-th step of $(\mathrm{P}_\varepsilon^{h,\tau})$. □

Lemma 2.3 *Let the assumptions of Theorem 2.1 hold. Then for all $\varepsilon \in (0, e^{-2})$, for all $h > 0$, and for all $\tau_n > 0$ such that $\omega\,\tau_n \leq 1$ a solution $\{U^n_{\varepsilon,1}, U^n_{\varepsilon,2}\}$ to the n-th step of $(P_\varepsilon^{h,\tau})$ is such that*

$$(1-\omega\,\tau_n)\,\sum_{i=1}^{2}(F_\varepsilon(U^n_{\varepsilon,i}),1)^h + \tau_n\,\sum_{i=1}^{2}a_i\,|U^n_{\varepsilon,i}|_1^2$$
$$\leq (1+3\,\omega\,\tau_n)\,\sum_{i=1}^{2}(F_\varepsilon(U^{n-1}_{\varepsilon,i}),1)^h + C\,\tau_n\,[\,1+|\pi^h v|_1^2\,]\,. \quad (2.34)$$

Proof Similarly to (2.28) and (2.30), on choosing $\chi \equiv \pi^h[F'_\varepsilon(U^n_{\varepsilon,i})]$ in (2.12), and noting (2.5b), (1.9a) and (1.14b,c), we obtain for $i = 1, 2$, $j \neq i$, that

$$\begin{aligned}
&(U^n_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}, F'_\varepsilon(U^n_{\varepsilon,i}))^h + \tau_n\,(\nabla U^n_{\varepsilon,j} + b_i\,\nabla(\pi^h v),\ \nabla U^n_{\varepsilon,i})\\
&\quad + \tau_n\,(\,[\,c_i\,[\Lambda_\varepsilon(U^n_{\varepsilon,i})]^{-1} + 2\,a_i\,\mathcal{I} + [\Lambda_\varepsilon(U^n_{\varepsilon,i})]^{-1}\Lambda_\varepsilon(U^n_{\varepsilon,j})\,]\,\nabla U^n_{\varepsilon,i}, \nabla U^n_{\varepsilon,i})\\
&\quad = \tau_n\,(\mu_i\,U^n_{\varepsilon,i} - [\gamma_{ii}\,\lambda_\varepsilon(U^{n-1}_{\varepsilon,i}) + \gamma_{ij}\,\lambda_\varepsilon(U^{n-1}_{\varepsilon,j})\,]\,\lambda_\varepsilon(U^n_{\varepsilon,i}), F'_\varepsilon(U^n_{\varepsilon,i})\,)^h\\
&\quad \leq \tau_n\,\big[\,\mu_i\,[\,2\,(F_\varepsilon(U^n_{\varepsilon,i}),1)^h + |\Omega|\,]\\
&\qquad\qquad + (\gamma_{ii}\,\lambda_\varepsilon(U^{n-1}_{\varepsilon,i}) + \gamma_{ij}\,\lambda_\varepsilon(U^{n-1}_{\varepsilon,j}), [1-U^n_{\varepsilon,i}]_+)^h\big]\,. \qquad (2.35)
\end{aligned}$$

Similarly to (1.15), on noting that $[1-s]_+ \leq 1-[s]_-$ for all $s \in \mathbb{R}$, (1.14a,b) and (2.3), we have for $i = 1, 2$, $j \neq i$, that

$$\begin{aligned}
&(\gamma_{ii}\,\lambda_\varepsilon(U^{n-1}_{\varepsilon,i}) + \gamma_{ij}\,\lambda_\varepsilon(U^{n-1}_{\varepsilon,j}), [1-U^n_{\varepsilon,i}]_+)^h\\
&\quad \leq 2\,(\gamma_{ii}\,F_\varepsilon(U^{n-1}_{\varepsilon,i}) + \gamma_{ij}\,F_\varepsilon(U^{n-1}_{\varepsilon,j}),1)^h + \tfrac{\varepsilon^{-1}}{2}\,(\gamma_{ii}+\gamma_{ij})\,|[U^n_{\varepsilon,i}]_-|_h^2\\
&\qquad\qquad + \tfrac{\varepsilon}{2}\,[\,\gamma_{ii}\,|\lambda_\varepsilon(U^{n-1}_{\varepsilon,i})|_h^2 + \gamma_{ij}\,|\lambda_\varepsilon(U^{n-1}_{\varepsilon,j})|_h^2\,] + C\\
&\quad \leq 3\,(\gamma_{ii}\,F_\varepsilon(U^{n-1}_{\varepsilon,i}) + \gamma_{ij}\,F_\varepsilon(U^{n-1}_{\varepsilon,j}),1)^h\\
&\qquad\qquad + (\gamma_{ii}+\gamma_{ij})\,(F_\varepsilon(U^n_{\varepsilon,i}),1)^h + C\,. \qquad (2.36)
\end{aligned}$$

Similarly to (2.30), summing (2.35) over i and noting (2.29), (2.36) and (2.5a) yields the desired result (2.34). □

Remark 2.1 We note that (2.34) is a discrete analogue of the formal energy estimates (1.7) and (1.13). Furthermore with no reaction terms, $\mu_i = \gamma_{ii} = \gamma_{ij} = 0$, $i = 1$ and 2, with $j \neq i$, then $\sum_{i=1}^{2}, (F_\varepsilon(\cdot), 1)^h$ is a discrete Lyapunov functional for (P$_\varepsilon^{h,\tau}$). In addition, for such data $(U^n_{\varepsilon,i}, 1) = (U^0_{\varepsilon,i}, 1)$, for $n \geq 1$ and $i = 1$, 2.

Theorem 2.2 *Let the assumptions of Lemma 2.3 hold. Let $u_i^0 \in L^\infty(\Omega)$ with $u_i^0(x) \geq 0$ for a.e. $x \in \Omega$, $i = 1$, 2, and $v \in H^1(\Omega) \cap W^{1,\beta}(\Omega)$ with $\beta > d$. For $i = 1$, 2, let $U^0_{\varepsilon,i} \equiv Q^h u_i^0 \in S^h_{\geq 0}$, or $U^0_{\varepsilon,i} \equiv \pi^h u_i^0 \in S^h_{\geq 0}$ if $u_i^0 \in H^1(\Omega) \cap W^{1,\beta}(\Omega)$ with $\beta > d$. Then for all $\varepsilon \in (0, e^{-2})$, for all τ such that $\omega\,\tau \leq 1 - \delta < 1$ and for all $h > 0$ a solution $\{U^n_{\varepsilon,1}, U^n_{\varepsilon,2}\}_{n=1}^N$ to (P$_\varepsilon^{h,\tau}$) is such that*

$$\max_{n=1\to N} \sum_{i=1}^{2} \left[(F_\varepsilon(U^n_{\varepsilon,i}), 1)^h + \varepsilon^{-1} |\pi^h[U^n_{\varepsilon,i}]_-|_0^2 + |U^n_{\varepsilon,i}|_{0,1} \right] + \sum_{n=1}^{N} \tau_n \sum_{i=1}^{2} a_i\, \|U^n_{\varepsilon,i}\|_1^2$$
$$\leq C\, e^{\frac{4\,\omega\,T}{\delta}} \left[1 + |\pi^h v|_1^2 + \sum_{i=1}^{2} (F_\varepsilon(U^0_{\varepsilon,i}), 1)^h \right] \leq C\,. \qquad (2.37)$$

In addition

$$\sum_{n=1}^{N} \tau_n \sum_{i=1}^{2} \Bigg[|\Lambda_\varepsilon(U^n_{\varepsilon,i})|^r_{0,r} + |\pi^h[\lambda_\varepsilon(U^n_{\varepsilon,i})]\,|^r_{0,r} + |U^n_{\varepsilon,i}|^r_{0,r} + |\lambda_\varepsilon(U^n_{\varepsilon,i})|^r_{0,r} + \left\| \mathcal{G} \left[\frac{U^n_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}}{\tau_n} \right] \right\|^q_{1,q} \Bigg] \leq C\,, \qquad (2.38)$$

where $r = \frac{2\,(d+1)}{d}$ and $q = \frac{2\,(d+1)}{2\,d+1}$.

Proof It follows from (2.34) that for $n = 1 \to N$

$$\sum_{i=1}^{2} (F_\varepsilon(U^n_{\varepsilon,i}), 1)^h \leq C\,\tau_n\,[\,1 + |\pi^h v|_1\,] + e^{\frac{4\,\omega\,\tau_n}{\delta}} \sum_{i=1}^{2} (F_\varepsilon(U^{n-1}_{\varepsilon,i}), 1)^h\,. \qquad (2.39)$$

Hence it follows from (2.39), (2.19), (2.16) and our assumptions on u_i^0 and v that

$$\max_{n=1\to N} \sum_{i=1}^{2} (F_\varepsilon(U^n_{\varepsilon,i}), 1)^h \leq C\, e^{\frac{4\,\omega\,T}{\delta}} \left[1 + |\pi^h v|_1^2 + \sum_{i=1}^{2} (F_\varepsilon(U^0_{\varepsilon,i}), 1)^h \right] \leq C\,. \qquad (2.40)$$

Choosing $\chi \equiv 1$ in (2.12), and noting (2.3) and (1.10), yields that

$$(1 - \mu_i \, \tau_n) \, (U^n_{\varepsilon,i}, 1) \leq (U^{n-1}_{\varepsilon,i}, 1) \qquad n = 1 \to N, \quad i = 1, 2\,. \tag{2.41}$$

Hence it follows from (2.41) and (2.19) that

$$\max_{n=1\to N} (U^n_{\varepsilon,i}, 1) \leq e^{\frac{\omega T}{\delta}} \, (U^0_{\varepsilon,i}, 1) \leq C \, e^{\frac{\omega T}{\delta}} \qquad i = 1, 2\,. \tag{2.42}$$

In addition, it follows from (2.42), (1.14a) and (2.40) that for $n = 1 \to N$

$$\begin{aligned} |U^n_{\varepsilon,i}|_{0,1} &\leq (\pi^h |U^n_{\varepsilon,i}|, 1) \leq (U^n_{\varepsilon,i}, 1) - 2 \, (\pi^h [U^n_{\varepsilon,i}]_-, 1) \\ &\leq C \, [\, 1 + |\pi^h [U^n_{\varepsilon,i}]_-|^2_0 \,] \leq C \qquad i = 1, 2\,. \end{aligned} \tag{2.43}$$

The $\max$ over n bound in (2.37) then follows from (2.40), (2.3), (1.14a) and (2.43). Summing (2.34) for $n = 1 \to N$ and noting (2.40) yields the summation over n bound in (2.37) with $\|\cdot\|_1$ replaced by $|\cdot|_1$. Noting (2.43) and a Poincarè inequality yields the desired $\|\cdot\|_1$ bound in (2.37).

It follows from (2.9), (2.10), (1.10), (2.14), (1.18), (2.37) and (2.11) for the stated choice of r

$$\begin{aligned} |\Lambda_\varepsilon(U^n_{\varepsilon,i})|^r_{0,r} &\leq C \sum_{\kappa \in \mathcal{T}^h} h^d_\kappa \, |\pi^h [\lambda_\varepsilon(U^n_{\varepsilon,i})]|^r_{0,\infty,\kappa} \leq C \, |\pi^h [\lambda_\varepsilon(U^n_{\varepsilon,i})] \,|^r_{0,r} \\ &\leq C \, |\pi^h [\lambda_\varepsilon(U^n_{\varepsilon,i})] \,|^{r-2}_{0,1} \, \|\pi^h [\lambda_\varepsilon(U^n_{\varepsilon,i})] \,\|^2_1 \\ &\leq C \, [\, 1 + \|U^n_{\varepsilon,i}\|^2_1 \,] \qquad n = 1 \to N, \quad i = 1, 2\,. \end{aligned} \tag{2.44}$$

Hence the first two bounds in (2.38) follow from (2.44) and (2.37). The third and fourth bounds in (2.38) follow similarly.

It follows from (1.10), (2.19), (2.17), Sobolev embedding and (2.11) for any $\eta \in W^{1,q'}(\Omega)$, $q' = 2\,(d+1)$, and for $n = 1 \to N$ and $i, \, j = 1, \, 2$ that

$$\begin{aligned} |(\lambda_\varepsilon(U^{n-1}_{\varepsilon,i}) \, \lambda_\varepsilon(U^n_{\varepsilon,j}), Q^h \eta)^h| &\leq (\lambda_\varepsilon(U^{n-1}_{\varepsilon,i}), \lambda_\varepsilon(U^n_{\varepsilon,j})^h \, |\eta|_{0,\infty} \\ &\leq C \, |\pi^h [\lambda_\varepsilon(U^{n-1}_{\varepsilon,i})]|_{0,r} \, \|\pi^h [\lambda_\varepsilon(U^n_{\varepsilon,j})]\|_1 \, \|\eta\|_{1,q'} \\ &\leq |\pi^h [\lambda_\varepsilon(U^{n-1}_{\varepsilon,i})]|_{0,r} \, \|U^n_{\varepsilon,j}\|_1 \, \|\eta\|_{1,q'}\,. \end{aligned} \tag{2.45}$$

From (1.20), (2.4), (2.12), (2.16), (2.20) and (2.45) we obtain for any $\eta \in W^{1,q'}(\Omega)$ and for $n = 1 \to N$ and $i = 1, 2$ that

$$\begin{aligned} (\nabla \mathcal{G}[\tfrac{U^n_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}}{\tau_n}], \nabla \eta) &+ (\mathcal{G}[\tfrac{U^n_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}}{\tau_n}], \eta) \\ &= (\tfrac{U^n_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}}{\tau_n}, \eta) = (\tfrac{U^n_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}}{\tau_n}, Q^h \eta)^h \\ &\leq C \, M_n \left[1 + \sum_{k=1}^{2} \|U^n_{\varepsilon,k}\|_1 \right] \|\eta\|_{1,q'}\,, \end{aligned} \tag{2.46a}$$

$$\text{where} \quad M_n := 1 + \sum_{k=1}^{2} \left[\, |\Lambda_\varepsilon(U^n_{\varepsilon,k})|_{0,r} + |\pi^h [\lambda_\varepsilon(U^{n-1}_{\varepsilon,k})]|_{0,r} \right]. \tag{2.46b}$$

It follows from (2.46a,b), (1.10), (2.37), the first two bounds in (2.38) and (2.19) for the stated choice of q that for $i = 1,\, 2$

$$\sum_{n=1}^{N} \tau_n \left\| \mathcal{G}[\tfrac{U^n_{\varepsilon,i} - U^{n-1}_{\varepsilon,i}}{\tau_n}] \right\|_{1,q}^{q} \leq C \sum_{n=1}^{N} \tau_n \, M_n^q \left[1 + \sum_{k=1}^{2} \|U^n_{\varepsilon,k}\|_1 \right]^q$$
$$\leq C \left[\sum_{n=1}^{N} \tau_n \, M_n^r \right]^{\frac{2-q}{2}} \left[\sum_{n=1}^{N} \tau_n \left[1 + \sum_{k=1}^{2} \|U^n_{\varepsilon,k}\|_1 \right]^2 \right]^{\frac{q}{2}} \leq C \,. \quad (2.47)$$

Hence we obtain the desired fifth bound in (2.38). □

3 Convergence

Let

$$U_{\varepsilon,i}(t) := \tfrac{t - t_{n-1}}{\tau_n} \, U^n_{\varepsilon,i} + \tfrac{t_n - t}{\tau_n} \, U^{n-1}_{\varepsilon,i} \qquad t \in [t_{n-1}, t_n] \quad n \geq 1 \quad (3.1a)$$

and

$$U^+_{\varepsilon,i}(t) := U^n_{\varepsilon,i}, \qquad U^-_{\varepsilon,i}(t) := U^{n-1}_{\varepsilon,i} \qquad t \in (t_{n-1}, t_n] \quad n \geq 1. \quad (3.1b)$$

We note for future reference that

$$U_{\varepsilon,i} - U^{\pm}_{\varepsilon,i} = (t - t^{\pm}_n) \, \tfrac{\partial U_{\varepsilon,i}}{\partial t} \qquad t \in (t_{n-1}, t_n) \quad n \geq 1, \qquad (3.2)$$

where $t^+_n := t_n$ and $t^-_n := t_{n-1}$. We introduce also

$$\bar{\tau}(t) := \tau_n \qquad t \in (t_{n-1}, t_n] \quad n \geq 1. \qquad (3.3)$$

Using the above notation, ($\mathrm{P}^{h,\tau}_{\varepsilon}$) can be restated as:
Find $\{U_{\varepsilon,1},\, U_{\varepsilon,2}\} \in [\, C([0,T]; S^h) \,]^2$ such that for $i = 1$ and 2, with $j \neq i$, and for all $\chi \in L^2(0,T; S^h)$

$$\int_0^T \Big[\Big(\tfrac{\partial U_{\varepsilon,i}}{\partial t}, \chi \Big)^h + (\, [c_i + 2\, a_i \, \Lambda_\varepsilon(U^+_{\varepsilon,i}) + \Lambda_\varepsilon(U^+_{\varepsilon,j})] \, \nabla U^+_{\varepsilon,i}, \nabla \chi)$$
$$+ \, (\Lambda_\varepsilon(U^+_{\varepsilon,i}) \, [\nabla U^+_{\varepsilon,j} + b_i \, \nabla(\pi^h v) \,], \nabla \chi) \Big] \, \mathrm{d}t$$
$$= \int_0^T (\, \mu_i \, U^+_{\varepsilon,i} - [\, \gamma_{ii} \, \lambda_\varepsilon(U^-_{\varepsilon,i}) + \gamma_{ij} \, \lambda_\varepsilon(U^-_{\varepsilon,j})] \, \lambda_\varepsilon(U^+_{\varepsilon,i}), \chi)^h \, \mathrm{d}t \,. \quad (3.4)$$

Lemma 3.1 *Let the assumptions of Theorem 2.2 hold. In addition, let* $\{\mathcal{T}^h, \{\tau_n\}_{n=1}^N, \varepsilon\}_{h>0}$ *be such that* $\tau,\ \varepsilon\, h^{-\frac{d}{d+1}} \to 0$, *as* $h \to 0$, *with either (i)* $\tau_1 \leq C\, h^2$ *or (ii)* $u_i^0 \in H^1(\Omega)$ *if* $U^0_{\varepsilon,i} \equiv Q^h\, u_i^0$, $i = 1, 2$ *or (iii)* $u_i^0 \in H^1(\Omega) \cap W^{1,\beta}(\Omega)$, $\beta > d$, *if* $U^0_{\varepsilon,i} \equiv \pi^h\, u_i^0$, $i = 1, 2$. *Then there exists a subsequence of* $\{U_{\varepsilon,1}, U_{\varepsilon,2}\}_h$, *where* $\{U_{\varepsilon,1}, U_{\varepsilon,2}\}$ *solve* ($P_\varepsilon^{h,\tau}$) *and functions*

$$u_i \in L^2(0,T; H^1_{\geq 0}(\Omega)) \cap L^r(\Omega_T) \cap W^{1,q}(0,T; (W^{1,q'}(\Omega))') \tag{3.5}$$

with $u_i(\cdot, 0) = u_i^0(\cdot)$ *in* $(W^{1,q}(\Omega)\,)'$, $i = 1, 2$, *where* $r = \frac{2(d+1)}{d}$, $q = \frac{2\,(d+1)}{2\,d+1}$ *and* $q' = 2\,(d+1)$, *such that as* $h \to 0$

$$U_{\varepsilon,i},\ U^{\pm}_{\varepsilon,i} \to u_i \qquad \textit{weakly in } L^2(0,T; H^1(\Omega)) \cap L^r(\Omega_T), \tag{3.6a}$$

$$\mathcal{G}\frac{\partial U_{\varepsilon,i}}{\partial t} \to \mathcal{G}\frac{\partial u_i}{\partial t} \qquad \textit{weakly in } L^q(0,T; W^{1,q}(\Omega)), \tag{3.6b}$$

$$U_{\varepsilon,i},\ U^{\pm}_{\varepsilon,i} \to u_i \qquad \textit{strongly in } L^2(0,T; L^s(\Omega)), \tag{3.7a}$$

$$\lambda_\varepsilon(U^{\pm}_{\varepsilon,i}),\ \pi^h[\lambda_\varepsilon(U^{\pm}_{\varepsilon,i})] \to u_i \qquad \textit{strongly in } L^2(0,T; L^r(\Omega)), \tag{3.7b}$$

$$\Lambda_\varepsilon(U^{\pm}_{\varepsilon,i}) \to u_i\,\mathcal{I} \qquad \textit{strongly in } L^2(0,T; L^r(\Omega)); \tag{3.7c}$$

where $s \in [2, \infty]$ *if* $d = 1$, $s \in [2, \infty)$ *if* $d = 2$ *and* $s \in [2, 6)$ *if* $d = 3$.

Proof If (i) holds, that is $\tau_1 \leq C\, h^2$, then it follows from (2.13) and (2.19) that

$$\tau_1\, \|U^0_{\varepsilon,i}\|_1^2 \leq C\, |U^0_{\varepsilon,i}|_0^2 \leq C \qquad i = 1,\ 2\,. \tag{3.8}$$

Alternatively, (3.8) can be achieved with more regularity on u_i^0 via (2.20) and (2.16) if (ii) or (iii) hold, respectively. Noting (3.1a,b), (2.37), (2.38), (3.8), (2.19), (2.9) and (2.10) we have for $i = 1,\ 2$ that

$$\begin{aligned}
&\|U^{(\pm)}_{\varepsilon,i}\|^2_{L^2(0,T;H^1(\Omega))} + \varepsilon^{-1}\|\pi^h[U^{(\pm)}_{\varepsilon,i}]_-\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\quad + \|U^{(\pm)}_{\varepsilon,i}\|_{L^\infty(0,T;L^1(\Omega))} + \|\Lambda_\varepsilon(U^{(\pm)}_{\varepsilon,i})\|^r_{L^r(\Omega_T)} \\
&\quad + \|\lambda_\varepsilon(U^{(\pm)}_{\varepsilon,i})\|^r_{L^r(\Omega_T)} + \|\pi^h[\lambda_\varepsilon(U^{(\pm)}_{\varepsilon,i})\,]\,\|^r_{L^r(\Omega_T)} \\
&\quad + \|U^{(\pm)}_{\varepsilon,i}\|^r_{L^r(\Omega_T)} + \|\mathcal{G}\frac{\partial U_{\varepsilon,i}}{\partial t}\|^q_{L^q(0,T;W^{1,q}(\Omega))} \leq C\,.
\end{aligned} \tag{3.9}$$

In the above, and throughout, the notation $U^{(\pm)}_{\varepsilon,i}$ means with and without the superscript $\pm$. Although $U_{\varepsilon,i}$ can go negative, the amount it can is controlled by the regularization parameter ε through the second term in (3.9), the analogue of (1.17). Furthermore, we deduce from (3.2) and (3.9) for $i = 1,\ 2$ that

$$\|\mathcal{G}\,[U_{\varepsilon,i} - U^{\pm}_{\varepsilon,i}]\,\|_{L^q(0,T;W^{1,q}(\Omega))} \le C\tau\,\|\mathcal{G}\frac{\partial U_{\varepsilon,i}}{\partial t}\|_{L^q(0,T;W^{1,q}(\Omega))} \le C\,\tau\,. \tag{3.10}$$

Hence on noting (3.9), (3.10), (1.19) and the compact embedding $H^1(\Omega) \hookrightarrow L^s(\Omega)$, we can choose a subsequence $\{U_{\varepsilon,1}, U_{\varepsilon,2}\}_h$ such that the convergence results (3.5) with $H^1_{\geq 0}(\Omega)$ replaced by $H^1(\Omega)$, (3.6a,b) and (3.7a) hold.

We now consider (3.7c). Firstly, we note for $i = 1,\ 2$ that

$$\begin{aligned}\|u_i - \lambda_\varepsilon(U^{\pm}_{\varepsilon,i})\|_{L^2(0,T;L^r(\Omega))} &\\ \le \|u_i - \widetilde{\lambda}_\varepsilon(u_i)\|_{L^2(0,T;L^r(\Omega))} &+ \|\widetilde{\lambda}_\varepsilon(u_i) - \widetilde{\lambda}_\varepsilon(U^{\pm}_{\varepsilon,i})\|_{L^2(0,T;L^r(\Omega))} \\ &+ \|\widetilde{\lambda}_\varepsilon(U^{\pm}_{\varepsilon,i}) - \lambda_\varepsilon(U^{\pm}_{\varepsilon,i})\|_{L^2(0,T;L^r(\Omega))}.\end{aligned} \tag{3.11}$$

As $u_i \in L^r(\Omega_T)$, recall (3.5), it follows from (1.10) for $i = 1,\ 2$ that

$$\|u_i - \widetilde{\lambda}_\varepsilon(u_i)\|_{L^2(0,T;L^r(\Omega))} \to 0 \quad \text{as} \quad \varepsilon \to 0\,. \tag{3.12}$$

Noting (1.10), we have for $i = 1,\ 2$ that

$$\|\widetilde{\lambda}_\varepsilon(u_i) - \widetilde{\lambda}_\varepsilon(U^{\pm}_{\varepsilon,i})\|_{L^2(0,T;L^r(\Omega))} \le \|u_i - U^{\pm}_{\varepsilon,i}\|_{L^2(0,T;L^r(\Omega))}\,. \tag{3.13}$$

It follows from (1.10), (2.14) and (3.9) for $i = 1,\ 2$ that

$$\begin{aligned}&\|\widetilde{\lambda}_\varepsilon(U^{\pm}_{\varepsilon,i}) - \lambda_\varepsilon(U^{\pm}_{\varepsilon,i})\|_{L^2(0,T;L^r(\Omega))} \\ &\quad\le \|\varepsilon - [U^{\pm}_{\varepsilon,i}]_-\|_{L^2(0,T;L^r(\Omega))} \le \|\varepsilon - \pi^h\,[U^{+}_{\varepsilon,i}]_-\|_{L^2(0,T;L^r(\Omega))} \\ &\quad\le C\,h^{-d\,(\frac{1}{2}-\frac{1}{r})}\left[\varepsilon + \|\pi^h\,[U^{+}_{\varepsilon}]_-\|_{L^2(\Omega_T)}\right] \le C\,\varepsilon^{\frac{1}{2}}\,h^{-\frac{1}{r}}\,.\end{aligned} \tag{3.14}$$

Noting (2.24), (2.25), (2.14) and (3.9), we have for $i = 1,\ 2$ that

$$\begin{aligned}\|(I-\pi^h)[\lambda_\varepsilon(U^{\pm}_{\varepsilon,i})\,]\|_{L^2(0,T;L^r(\Omega))} &+ \|\lambda_\varepsilon(U^{\pm}_{\varepsilon,i})\,\mathcal{I} - \Lambda_\varepsilon(U^{\pm}_{\varepsilon,i})\|_{L^2(0,T;L^r(\Omega))} \\ &\le C\,h^{1-d\,(\frac{1}{2}-\frac{1}{r})}\,\|\nabla U^{\pm}_{\varepsilon}\|_{L^2(\Omega_T)} \le C\,h^{1-\frac{1}{r}}\,.\end{aligned} \tag{3.15}$$

Combining (3.11), (3.12), (3.13), (3.14) (3.15) and noting (3.7a) and our assumption on ε yields the desired result (3.7b,c). Finally, we note that (3.7a) and (3.9) $\Rightarrow u_i \ge 0$ *a.e.* $\Rightarrow H^1_{\geq 0}(\Omega)$ in (3.5). $\square$

Theorem 3.1 *Let all the assumptions of Lemma 3.1 hold. Then there exists a subsequence of* $\{U_{\varepsilon,1}, U_{\varepsilon,2}\}_h$*, where* $\{U_{\varepsilon,1},\, U_{\varepsilon,2}\}$ *solve (*$P_\varepsilon^{h,\tau}$*), and functions* $\{u_1, u_2\}$ *satisfying (3.5), (3.6a,b) and (3.7a–c). Furthermore, we have that* u_i*,* $i = 1,\ 2$*, fulfil* $u_i(\cdot,0) = u_i^0(\cdot)$ *in* $(W^{1,q}(\Omega)\,)'$*,* $q = \frac{2\,(d+1)}{2\,d+1}$*,* *and satisfy for* $i = 1$ *and* 2, $j \neq i$, *and for all* $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$, $q' = 2\,(d+1)$,

$$\int_0^T \big[\langle \tfrac{\partial u_i}{\partial t}, \eta\rangle_{q'}\,\mathrm{d}t + (\,[\,c_i + 2\,a_i\,u_i + u_j\,]\,\nabla u_i + u_i\,\nabla u_j + b_i\,\nabla v, \nabla\eta)\big]\,\mathrm{d}t = \int_0^T (\,[\,\mu_i - \gamma_{ii}\,u_i - \gamma_{ij}\,u_j\,]\,u_i, \eta)\,\mathrm{d}t\,. \quad (3.16)$$

Proof For any $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$, we choose $\chi \equiv \pi^h\eta$ in (3.4) and now analyse the subsequent terms. Firstly (2.18), the embedding $H^1(0,T;X) \hookrightarrow C([0,T];X)$, (2.21), (3.9) and (2.16) yield for $i = 1$, 2 and for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ and $\widetilde{\eta} \in H^1(0,T;W^{1,\infty}(\Omega))$ that

$$\begin{aligned}
&\left| \int_0^T \left[\left(\tfrac{\partial U_{\varepsilon,i}}{\partial t}, \pi^h\eta\right)^h - \left(\tfrac{\partial U_{\varepsilon,i}}{\partial t}, \pi^h\eta\right) \right] \mathrm{d}t \right| \\
&\quad\leq \left| \int_0^T \left[\left(\tfrac{\partial U_{\varepsilon,i}}{\partial t}, \pi^h[\eta - \widetilde{\eta}]\right)^h - \left(\tfrac{\partial U_{\varepsilon,i}}{\partial t}, \pi^h[\eta - \widetilde{\eta}]\right) \right] \mathrm{d}t \right| \\
&\qquad + \left| \int_0^T \left[\left(U_{\varepsilon,i}, \tfrac{\partial(\pi^h\widetilde{\eta})}{\partial t}\right) - \left(U_{\varepsilon,i}, \tfrac{\partial(\pi^h\widetilde{\eta})}{\partial t}\right)^h \right] \mathrm{d}t \right| \\
&\qquad + \left| (U_{\varepsilon,i}(\cdot,T), \pi^h\widetilde{\eta}(\cdot,T))^h - (U_{\varepsilon,i}(\cdot,T), \pi^h\widetilde{\eta}(\cdot,T)) \right| \\
&\qquad + \left| (U_{\varepsilon,i}(\cdot,0), \pi^h\widetilde{\eta}(\cdot,0)) - (U_{\varepsilon,i}(\cdot,0), \pi^h\widetilde{\eta}(\cdot,0))^h \right| \\
&\quad\leq C\,\|\mathcal{G}\tfrac{\partial U_{\varepsilon,i}}{\partial t}\|_{L^q(0,T;W^{1,q}(\Omega))}\,\|\pi^h[\eta - \widetilde{\eta}]\,\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \\
&\qquad + C\,h\,\|U_{\varepsilon,i}\|_{L^\infty(0,T;L^1(\Omega))}\,\|\pi^h\widetilde{\eta}\|_{H^1(0,T;W^{1,\infty}(\Omega))} \\
&\quad\leq C\,\|\eta - \widetilde{\eta}\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} + C\,h\,\|\widetilde{\eta}\|_{H^1(0,T;W^{1,\infty}(\Omega))}\,. \qquad (3.17)
\end{aligned}$$

Furthermore, it follows from (1.20) and (3.9) for $i = 1$, 2 and for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\begin{aligned}
&\left| \int_0^T \left(\tfrac{\partial U_{\varepsilon,i}}{\partial t}, (I - \pi^h)\eta\right) \mathrm{d}t \right| \\
&\qquad\leq C\,\|\mathcal{G}\tfrac{\partial U_{\varepsilon,i}}{\partial t}\|_{L^q(0,T;W^{1,q}(\Omega))}\,\|(I - \pi^h)\eta\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \\
&\qquad\leq C\,\|(I - \pi^h)\eta\|_{L^{q'}(0,T;W^{1,q'}(\Omega))}\,. \qquad (3.18)
\end{aligned}$$

Combining (3.17), the denseness of $H^1(0,T;W^{1,\infty}(\Omega))$ in $L^{q'}(0,T;W^{1,q'}(\Omega))$, (3.18), (2.15), (1.20) and (3.6b) yields for $i = 1$, 2 and for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\int_0^T \left(\tfrac{\partial U_{\varepsilon,i}}{\partial t}, \pi^h\eta\right)^h \mathrm{d}t \to \int_0^T \langle \tfrac{\partial u_i}{\partial t}, \eta\rangle_{q'}\,\mathrm{d}t \quad \text{as } h \to 0. \qquad (3.19)$$

It follows from (3.9) for $i,\ j = 1,\ 2$ and for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\begin{aligned}\left| \int_0^T (\Lambda_\varepsilon(U^+_{\varepsilon,j})\, \nabla U^+_{\varepsilon,i}, \nabla(I-\pi^h)\eta)\, \mathrm{d}t \right| \\ &\leq C\, \|\Lambda_\varepsilon(U^+_{\varepsilon,j})\|_{L^r(\Omega_T)}\, \|\nabla U^+_{\varepsilon,i}\|_{L^2(\Omega_T)}\, \|\nabla(I-\pi^h)\eta\|_{L^{q'}(\Omega_T)} \\ &\leq C\, \|(I-\pi^h)\eta\|_{L^{q'}(0,T;W^{1,q'}(\Omega))}\,. \end{aligned} \tag{3.20}$$

Similarly to (3.20), it follows from (3.9) and (3.5) for $i,\ j = 1,\ 2$ and for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ and $\widetilde{\eta} \in H^1(0,T;W^{1,\infty}(\Omega))$ that

$$\begin{aligned}&\left| \int_0^T ([\, \Lambda_\varepsilon(U^+_{\varepsilon,j}) - u_j\, \mathcal{I}\,)\, \nabla U^+_{\varepsilon,i}, \nabla\eta)\, \mathrm{d}t \right| \\ &\quad\leq \left| \int_0^T ([\, \Lambda_\varepsilon(U^+_{\varepsilon,j}) - u_j\, \mathcal{I}\,)\, \nabla U^+_{\varepsilon,i}, \nabla\widetilde{\eta})\, \mathrm{d}t \right| + C\, \|\nabla(\widetilde{\eta}-\eta)\|_{L^{q'}(\Omega_T)} \\ &\quad\leq C\, \|\Lambda_\varepsilon(U^+_{\varepsilon,j}) - u_j\, \mathcal{I}\|_{L^2(0,T;L^r(\Omega))}\, \|\nabla\widetilde{\eta}\|_{L^\infty(\Omega_T)} \\ &\qquad\qquad + C\, \|\nabla(\widetilde{\eta}-\eta)\|_{L^{q'}(\Omega_T)}\,. \end{aligned} \tag{3.21}$$

Combining (3.20), (2.15), (3.21), (3.7c), $H^1(0,T;W^{1,\infty}(\Omega))$ is dense in $L^{q'}(0,T;W^{1,q'}(\Omega))$ and (3.6a) yields for $i,\ j = 1,\ 2$ and for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\begin{aligned}&\int_0^T (\Lambda_\varepsilon(U^+_{\varepsilon,j})\, \nabla U^+_{\varepsilon,i}, \nabla(\pi^h\eta)\,)\, \mathrm{d}t \to \int_0^T (\, u_j\, \nabla u_i, \nabla\eta)\, \mathrm{d}t\,, \\ &\int_0^T (\nabla U^+_{\varepsilon,i}, \nabla(\pi^h\eta)\,)\, \mathrm{d}t \to \int_0^T (\nabla u_i, \nabla\eta)\, \mathrm{d}t \qquad \text{as } h \to 0\,. \end{aligned} \tag{3.22}$$

Similarly to the derivation of (3.22), one can show for $i = 1,\ 2$ and for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\int_0^T (\Lambda_\varepsilon(U^+_{\varepsilon,i})\, \nabla(\pi^h v), \nabla(\pi^h\eta)\,)\, \mathrm{d}t \to \int_0^T (\, u_i\, \nabla v, \nabla\eta)\, \mathrm{d}t \quad \text{as } h \to 0\,. \tag{3.23}$$

Similarly to the derivation of (3.22), it follows from (2.18), (3.9), (2.15) and (3.6a) for $i = 1,\ 2$ and for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\int_0^T \left(U_{\varepsilon,i}, \pi^h\eta \right)^h\, \mathrm{d}t \to \int_0^T (u_i, \eta)\, \mathrm{d}t \quad \text{as } h \to 0. \tag{3.24}$$

Noting a generalisation of (2.18), (2.13), (2.11), (2.16) and (3.9) yields for $i,\, j = 1,\, 2$ and for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\begin{aligned}
&\left|\int_0^T \left[(\lambda_\varepsilon(U^-_{\varepsilon,i})\,\lambda_\varepsilon(U^+_{\varepsilon,j}),\pi^h\eta)^h - (\pi^h[\lambda_\varepsilon(U^-_{\varepsilon,i})]\,\pi^h[\lambda_\varepsilon(U^+_{\varepsilon,j})],\pi^h\eta)\right]\,\mathrm{d}t\right| \\
&\quad \le \int_0^T |\,(I-\pi^h)\,(\,\pi^h[\lambda_\varepsilon(U^-_{\varepsilon,i})]\,\pi^h[\lambda_\varepsilon(U^+_{\varepsilon,j})]\,\pi^h\eta\,)\,|_{0,1}\,\mathrm{d}t \\
&\quad \le C\,h^2\,\|\pi^h[\lambda_\varepsilon(U^-_{\varepsilon,i})]\|_{L^2(0,T;H^1(\Omega))}\,\|\pi^h[\lambda_\varepsilon(U^+_{\varepsilon,j})]\|_{L^r(0,T;W^{1,r}(\Omega))} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\|\pi^h\eta\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \\
&\quad \le C\,h\,\|U^-_{\varepsilon,i}\|_{L^2(0,T;H^1(\Omega))}\,\|\pi^h[\lambda_\varepsilon(U^+_{\varepsilon,j})]\|_{L^r(\Omega_T)}\,\|\eta\|_{L^{q'}(0,T;W^{1,q'}(\Omega))} \\
&\quad \le C\,h\,\|\eta\|_{L^{q'}(0,T;W^{1,q'}(\Omega))}\,.
\end{aligned} \tag{3.25}$$

Similarly to (3.25), it follows from (3.9), (3.5), and (2.16) for $i,\, j = 1,\, 2$ and for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\begin{aligned}
&\left|\int_0^T \left[(\pi^h[\lambda_\varepsilon(U^-_{\varepsilon,i})]\,\pi^h[\lambda_\varepsilon(U^+_{\varepsilon,j})],\pi^h\eta) - (u_i\,u_j,\eta)\right]\,\mathrm{d}t\right| \\
&\quad \le \|\pi^h[\lambda_\varepsilon(U^+_{\varepsilon,j})]\|_{L^r(\Omega_T)}\,\|\pi^h\eta\|_{L^{q'}(\Omega_T)}\,\|\pi^h[\lambda_\varepsilon(U^-_{\varepsilon,i})]-u_i\|_{L^2(\Omega_T)} \\
&\qquad + \|u_i\|_{L^r(\Omega_T)}\,\|\pi^h\eta\|_{L^{q'}(\Omega_T)}\,\|\pi^h[\lambda_\varepsilon(U^-_{\varepsilon,j})]-u_j\|_{L^2(\Omega_T)} \\
&\qquad + \|u_i\|_{L^2(\Omega_T)}\,\|u_j\|_{L^r(\Omega_T)}\,\|(I-\pi^h)\eta\|_{L^{q'}(\Omega_T)} \\
&\quad \le C\,\left[h + \sum_{k=1}^{2}\|\pi^h[\lambda_\varepsilon(U^-_{\varepsilon,k})]-u_k\|_{L^2(\Omega_T)}\right]\,\|\eta\|_{L^{q'}(0,T;W^{1,q'}(\Omega))}\,.
\end{aligned} \tag{3.26}$$

Combining (3.25) and (3.26), and noting (3.7b), yields for $i,\, j = 1,\, 2$ and for all $\eta \in L^{q'}(0,T;W^{1,q'}(\Omega))$ that

$$\int_0^T (\lambda_\varepsilon(U^-_{\varepsilon,i})\,\lambda_\varepsilon(U^+_{\varepsilon,j}),\pi^h\eta)^h\,\mathrm{d}t \to \int_0^T (u_i\,u_j,\eta)\,\mathrm{d}t \quad \text{as } h\to 0. \tag{3.27}$$

Finally, combining (3.4), (3.19), (3.22), (3.23), (3.24) and (3.27) yields that $\{u_1,\, u_2\}$ satisfy (3.16). $\square$

4 Numerical Results

Before presenting some numerical results in one space dimension, we state briefly our algorithm for solving the resulting system of nonlinear algebraic equations for $\{U^n_{\varepsilon,1}, U^n_{\varepsilon,2}\}$ arising at each time level from the approximation (P$^{h,\tau}_\varepsilon$). We used the following iterative approach to solve (2.12) for

$\{U_{\varepsilon,1}^n, U_{\varepsilon,2}^n\}$: Given $U_{\varepsilon,i}^{n,0} \in S^h$, $i = 1,\, 2$, for $k \geq 1$ find $\{U_{\varepsilon,1}^{n,k}, U_{\varepsilon,2}^{n,k}\} \in [S^h]^2$ such that for $i = 1,\, 2$, with $j \neq i$, and for all $\chi \in S^h$

$$\left(\tfrac{U_{\varepsilon,i}^{n,k} - U_{\varepsilon,i}^{n-1}}{\tau_n}, \chi\right)^h + \left([c_i + 2\, a_i\, \Lambda_\varepsilon(U_{\varepsilon,i}^{n,k-1}) + \Lambda_\varepsilon(U_{\varepsilon,j}^{n,k-1})]\, \nabla U_{\varepsilon,i}^{n,k}, \nabla\chi\right)$$
$$+ \left(\Lambda_\varepsilon(U_{\varepsilon,i}^{n,k-1})\, [\nabla U_{\varepsilon,j}^{n,k} + b_i\, \nabla(\pi^h v)\,], \nabla\chi\right)$$
$$= (\mu_i\, U_{\varepsilon,i}^{n,k} - [\gamma_{ii}\, \lambda_\varepsilon(U_{\varepsilon,i}^{n-1}) + \gamma_{ij}\, \lambda_\varepsilon(U_{\varepsilon,j}^{n-1})\,]\, \lambda_\varepsilon(U_{\varepsilon,i}^{n,k-1}), \chi)^h\,. \tag{4.1}$$

(4.1) requires a linear solve at each iteration and is the natural extension of the iterative procedure proposed in [9] for solving a related finite element approximation of the thin film equation. We set, for $n \geq 1$, $U_{\varepsilon,i}^{n,0} \equiv U_{\varepsilon,i}^{n-1}$ and adopted the stopping criteria

$$\max_{i=1,\,2} |U_{\varepsilon,i}^{n,k} - U_{\varepsilon,i}^{n,k-1}|_{0,\infty} < tol \tag{4.2}$$

with $tol = 10^{-7}$, and set $U_{\varepsilon,i}^n \equiv U_{\varepsilon,i}^{n,k}$. Although we are unable to show convergence of the iteration (4.1), we observed good convergence properties in practice (at most 10 iterations and this maximum only being required in the very early stages of the evolution) with the exception of the experiments with strong transport ($b_1 = 20$ and 40).

Unless otherwise stated, in all experiments we chose a uniform partitioning of $\Omega = (0, 3)$ with mesh points $p_j = (j-1)h$, $j = 1 \to 301$, i.e. $h = 10^{-2}$; a uniform time step $\tau_n = \tau = 10^{-3}$ and set $\varepsilon = 5 \times 10^{-7}$.

No Reaction Terms

We repeated the experiments in [6] which show the behaviour of the two interacting species for different choices of the parameters and initial data. In each experiment we set $\mu_i = \gamma_{ii} = \gamma_{ij} = 0$, $v(x) = -1.5(x-0.5)^2$, and $U_{\varepsilon,i}^0 = \pi^h u_i^0$. We note that these discretisation parameters h and τ are exactly the same as those chosen in [6] for their finite difference approximation of (P). In contrast to their approximation, our approximation ($\mathrm{P}_\varepsilon^{h,\tau}$) conserves mass exactly, recall Remark 2.1. For these experiments, we integrated in time until a numerical stationary solution, $U_{\varepsilon,i}^S$, was achieved. This was determined by

$$\max_{i=1,\,2} |U_{\varepsilon,i}^{n,1} - U_{\varepsilon,i}^{n,0}|_{0,\infty} < 5 \times 10^{-11},$$

which is far more severe than the stopping criteria (4.2). In all of these experiments we found that

$$\max_{i=1,\,2,\; j\neq i} |\, [c_i + 2\, a_i\, \Lambda_\varepsilon(U_{\varepsilon,i}^S) + \Lambda_\varepsilon(U_{\varepsilon,j}^S)]\, \nabla U_{\varepsilon,i}^S$$
$$+ \Lambda_\varepsilon(U_{\varepsilon,i}^S)\, [\nabla U_{\varepsilon,j}^S + b_i\, \nabla(\pi^h v)\,]\, |_{0,\infty} < 10^{-7}, \tag{4.3}$$

which should be zero for an "exact" numerical stationary solution of $(\mathrm{P}_\varepsilon^{h,\tau})$.

In all the figures below $U^S_{\varepsilon,1}(\cdot)$ and $U^S_{\varepsilon,2}(\cdot)$ are plotted as a solid and dashed line, respectively. In addition, all numerical stationary solution times are correct to one decimal place.

A: Large and small cross-diffusion terms

We took $b_i = c_i = 1$, $u^0_1(x) = 10$, $u^0_2(x) = 20$, and $a_i = 0,\ 0.1,\ 10$; see Figure 4.1A. The numerical stationary solutions for $a_i = 0,\ 0.1,\ 10$ were achieved at times 5.0, 3.0 and 0.9, respectively.

B: Large diffusion coefficients c_i compared to a_i, i.e. $c_i \gg a_i$

All parameters were the same as in A, except $a_i = 0.01$, $c_i = 1, 10, 100$; see Figure 4.1B. The numerical stationary solutions for $c_i = 1,\ 10,\ 100$ were achieved at times 4.7, 1.6 and 0.2.

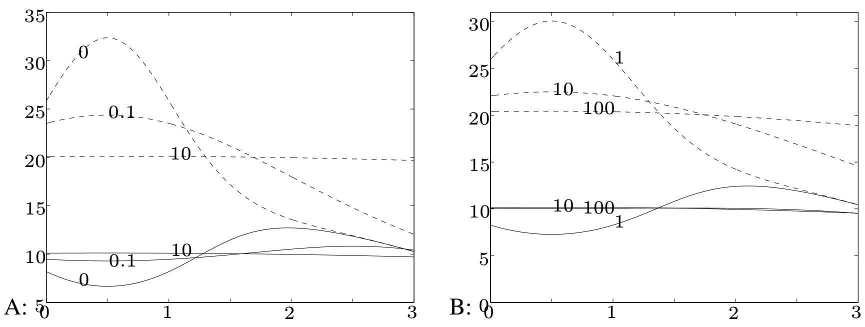

Fig. 4.1. A: Curves labelled with a_i values, B: with c_i values.

C: Segregation effects due to a large ratio of transport coefficients

All parameters were the same as in A, except $a_i = 1$ and $b_1 = 4,\ 8,\ 20,\ 40$; see Figure 4.2A. In addition, for $b_1 = 40$ we took $\tau = 5 \times 10^{-4}$ in order to achieve convergence of the iterative solver. The numerical stationary solutions for $b_1 = 4,\ 8,\ 20,\ 40$ were achieved at times 0.8, 0.7, 0.7 and 0.6. Only in the case of $b_1 = 40$, did $U^n_{\varepsilon,1}$ become negative and throughout the experiment always satisfied $\min\limits_{n=1,\cdots,N} \min\limits_{j\in J} U^n_{\varepsilon,1}(p_j) > -5 \times 10^{-5}$.

We repeated this experiment with $a_i = 0.1$; see Figure 4.2B. When $b_1 = 40$ we took $\tau = 2 \times 10^{-4}$ in order to achieve convergence of the iterative solver. The numerical stationary solutions for $b_1 = 4,\ 8,\ 20,\ 40$ were achieved at times 1.7, 1.4, 1.6 and 1.8. Only in the cases of $b_1 = 20$ and 40 did $U^n_{\varepsilon,1}$ become negative and throughout each experiment always satisfied $\min\limits_{n=1,\cdots,N} \min\limits_{j\in J} U^n_{\varepsilon,1}(p_j) > -3 \times 10^{-5}$.

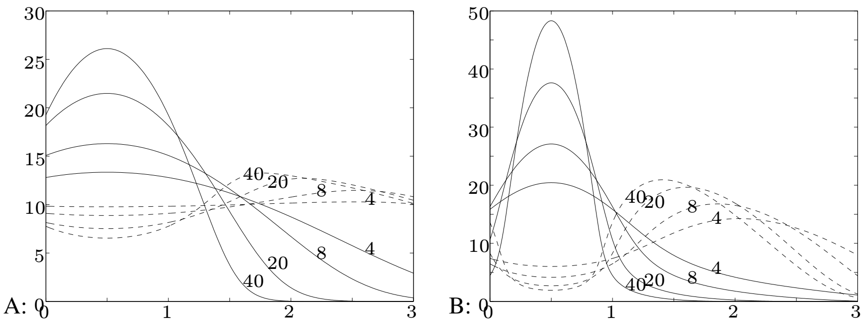

Fig. 4.2. A: Curves labelled with b_1 values with $a_i = 1$, B: with $a_i = 0.01$.

D: Discontinuous initial data

All parameters were the same as in A, except $a_i = 1$, $b_1 = 8$, $u_2^0 = 10$ and

$$u_1^0 = \begin{cases} 12 & 0 \leq x \leq 1.5 \\ 8 & 1.5 < x \leq 3 \end{cases};$$

see Figure 4.3A where u_i^0 is also plotted. The numerical stationary solution was achieved at time 0.7.

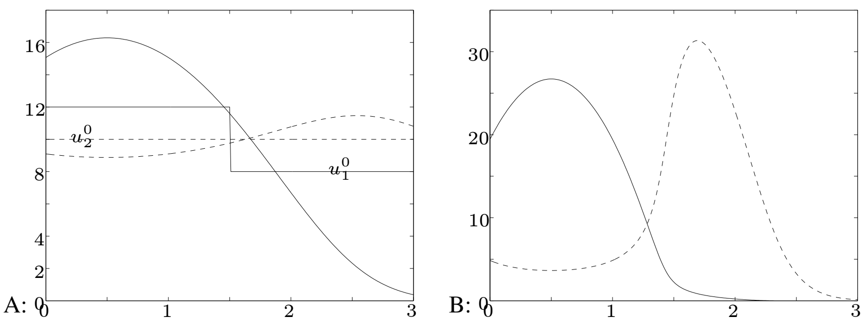

Fig. 4.3. A: Discontinuous initial data, B: Segregation of species.

E: Segregation of the two species

All parameters were the same as in A, except $a_1 = 1$, $a_2 = 0.01$, $b_1 = 40$ and $u_2^0 = 10$; see Figure 4.3B. The numerical stationary solution was achieved at time 1.5. Throughout the experiment $\min_{n=1,\cdots,N} \min_{j \in J} U_{\varepsilon,1}^n(p_j) > -4 \times 10^{-5}$.

There is very good agreement with the above figures and the corresponding figures in [6], except for Figure 4.1B in the case of $c_i = 1$. This is probably due to a typographical error in [6], rather than a significant difference in these different approximations of (P).

Reaction Terms

We now include reaction terms. All parameters were the same as in A above, except $a_1 = 0.1$, $a_2 = 0.05$, $\mu_i = 1$,

$$\gamma = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}, \quad \begin{pmatrix} 0.5 & 0.1 \\ 0.5 & 0.1 \end{pmatrix}, \quad \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0.5 & 0.5 \\ 0.1 & 0.1 \end{pmatrix},$$

which are labelled γ_1, γ_2, γ_3 and γ_4 in Figure 4.4A; where we plot the numerical stationary solutions. These were achieved at times 110, 108, 100 and 100. We note that the corresponding diffusion matrix, (1.3), is not positive definite on recalling (1.4); and $\omega = 2.2,\ 2.6,\ 3.0$ and 3.0, respectively, on recalling the definition in Theorem 2.1. Throughout the experiments $\min_{n=1,\cdots,N} \min_{j \in J} U^n_{\varepsilon,i}(p_j) > -5 \times 10^{-5}$. We repeated this experiment with

$$\gamma = \begin{pmatrix} 0.1 & 0.1 \\ 0.5 & 0.5 \end{pmatrix},$$

and even with this asymmetric initial data obtained the same numerical stationary solution as that labelled γ_4 in Figure 4.4A with $U^S_{\varepsilon,1}$ and $U^S_{\varepsilon,2}$ interchanged. When we repeated the experiment with $a_1 = a_2$, then the numerical stationary solutions satisfied $U^S_{\varepsilon,1} = U^S_{\varepsilon,2}$ for γ_1, γ_2 and γ_3; as to be expected.

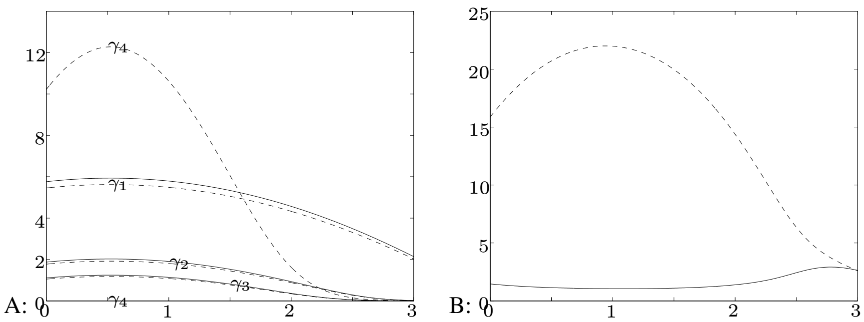

Fig. 4.4. A: Curves labelled with reaction matrices, B: $u_i(\cdot, 0.2)$.

Convergence Experiment

We take the initial data and all other parameters as in the previous experiment with γ_4, except $c_i = 0$ and $T = 0.2$. However, the uniform mesh parameters h, τ and ε were all varied. As we do not know the exact solution to (P), a comparison was made between the solutions of $(\mathrm{P}_{\varepsilon}^{h,\tau})$ on a coarse mesh, $U_{\varepsilon,i}$, with that on a fine mesh, u_i. The discretization parameters on the coarse meshes were $\tau = 256\,h^2/90$, $\varepsilon = 10^{-4}h$ and $h = 3/(\#J - 1)$ where $\#J = 2^k + 1$ with $k = 5, 6, 7$ and 8; while those for the fine mesh were the same except $\#J = 2^{11} + 1$. We repeated this experiment, but took $\tau = h/30$. We note that in both cases all the assumptions of Theorem 3.1 hold. In Figure 4.4B, we plot $u_i(\cdot, 0.2)$ the "true solution" of (P) and note that there has been a large change from the initial data. In Figure 4.5, we plot $|u_i(\cdot,t) - U_{\varepsilon,i}(\cdot,t)|_{0,\infty}$ versus time with the graphs labelled by $\#J$ and in Table 4.1 we give the values of $\mathcal{E}_i = |u_i(\cdot, 0.2) - U_{\varepsilon,i}(\cdot, 0.2)|_{0,\infty}$ correct to 3 s.f. for the different meshes. With $\tau \propto h^2$ the ratios of successive $\mathcal{E}_1$ are 3.38, 3.18 and 2.94 while those of $\mathcal{E}_2$ are 3.70, 3.72 and 3.41. With $\tau \propto h$ the ratios of successive $\mathcal{E}_1$ are 2.04, 2.05 and 2.10 while those of $\mathcal{E}_2$ are 1.93, 1.99 and 2.04.

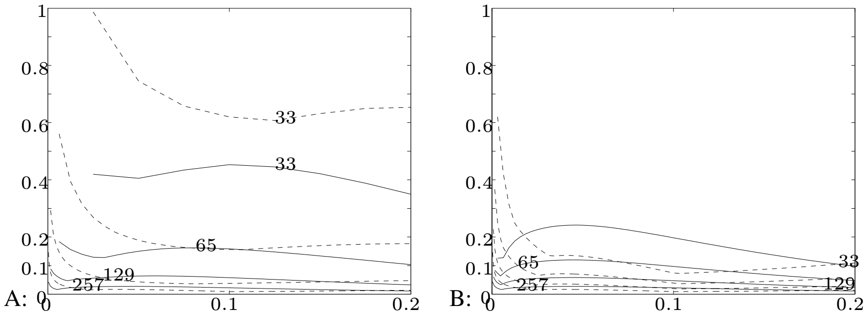

Fig. 4.5. A: $|u_i(\cdot,t) - U_{\varepsilon,i}(\cdot,t)|_{0,\infty}$ versus time with $\tau \propto h^2$, B: with $\tau =\propto h$.

A:

$\#J$	$\mathcal{E}_1$	$\mathcal{E}_2$
33	0.350	0.653
65	0.104	0.177
129	0.0326	0.0476
257	0.0111	0.0139

B:

$\#J$	$\mathcal{E}_1$	$\mathcal{E}_2$
33	0.0976	0.109
65	0.0477	0.0566
129	0.0232	0.0285
257	0.0111	0.0139

Table 4.1. $\mathcal{E}_i$ with A: $\tau \propto h^2$ and B: $\tau \propto h$.

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