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Management Engineering Section

Applications of Analogue Computers
in Operational Research

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ABSTRACT

The use of an analogue computer for solving several problems in operational research is described. The first concerns a production process where the output mean is subject to drift and items produced outside specification limits incur an additional cost. The problem is to determine how often the process should be stopped and re-set in order to minimise the average cost of production. Various cost parameters and trend variations are considered.

The use of an analogue for locating distribution centres is described, where it is required to minimise a linear weighted function of the distances between a number of linked points. An extension of the method to the problem of locating emergency centres is also described.

Various discrete production-inventory control models are considered and methods of simulating these on an analogue computer are given. Results obtained are used to formulate optimal re-order rule policies for prescribed demand patterns and performance criteria.

The use of an analogue for generating probability density functions is described and several examples are given.

Methods are described for simulating queueing systems using a sampled data analogue, and programmes for various sub-programme operations are presented. An example is given for a simple queue with geometrically distributed inter-arrival and service times.

The use of an analogue computer for simulating Industrial Dynamics systems is demonstrated, and an example is given for determining the optimal parameters when a number of production-inventory control systems are interconnected.

The use of an analogue computer ~~for~~ forecast rule simulation and investment analysis is also demonstrated.

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CHAPTER 1

INTRODUCTION

CHAPTER 1 : INTRODUCTION

This thesis describes the use of an analogue computer for solving problems in operational research. Operational research comprises a loosely defined amalgam of disciplines and techniques and this heterogeneous property is reflected in many O.R. problems. In addition, many problems are characterised by low accuracy input data and the need for human intervention during problem solution. When this is the case the application of conventional techniques may lead to analytic or computational requirements of questionable practicality and the use of analogue computation has been suggested as an alternative. This thesis describes an investigation that was carried out to examine this possibility.

Several problems were investigated and these are listed below:

- (1) Controlling a production process.
- (2) Locating a distribution centre.
- (3) Simulation of Production-Inventory Control Systems.
- (4) Generation of probability density functions.
- (5) Simulation of Queuing systems.
- (6) Industrial Dynamics.

- (7) Forecast Rule Simulation.
- (8) Investment Analysis.

1.1 ANALOGUE COMPUTATION

An analogue computer is essentially an electronic circuit, in which the components are interconnected so that the circuit variables (voltage or current, but usually voltage) will obey certain pre-specified mathematical relationships. The analogue components are grouped into functional blocks, according to the operations they perform on voltages fed into them, and a prescribed interconnection of these blocks constitutes the analogue programme. Some of the basic block operations are given below where (1) to (4) are accompanied by a sign change on the computer.

- (1) Multiplication by a constant.
- (2) Addition/subtraction.
- (3) Integration/differentiation
- (4) Multiplication/division.
- (5) Function generation.

Thus, given a flow graph representation of a problem or system to be simulated, the computer can be programmed by substituting for each branch of the graph a computing element that performs the branch operation. Connections between various computing

elements are made by flexible wire leads, and problem variables can be read from the computer as voltages which are displayed on a cathode ray tube or plotted on graph paper via a pen recorder.

To simplify programming, symbols are used to represent the various computing elements, and a flow graph drawn with these symbols constitutes the analogue programme. A list of analogue circuits and symbols is given in Table 1.1.

Various techniques are available for translating a given problem into a form suitable for implementation on the computer, and some of the most important methods are described in Appendix I.

The computer is controlled from a central control panel where various modes of operation can be selected.

Pot-set (PS) for setting potentiometers

Re-set (RS) for applying initial conditions to
integrators

Operate (OP) for generating the problem solution

Hold (H) for arresting the computer integrators
when it is required to examine the static
properties of a problem solution

Table 1.1(a) COMPUTER SYMBOLS

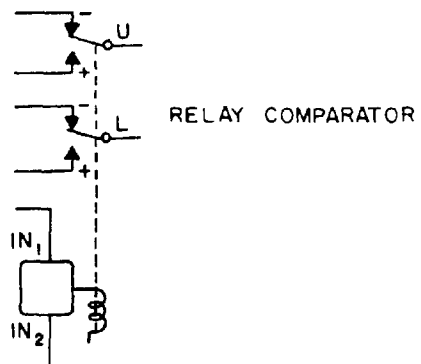
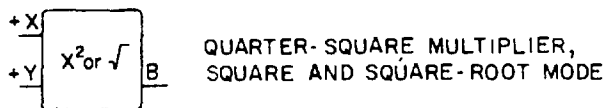
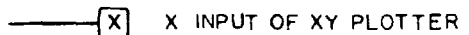
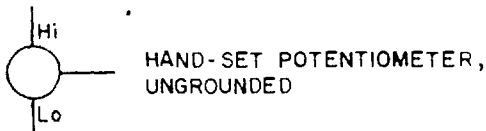
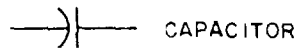
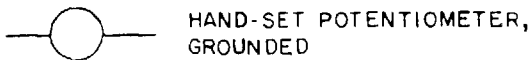
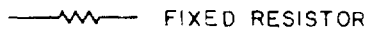
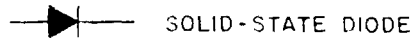
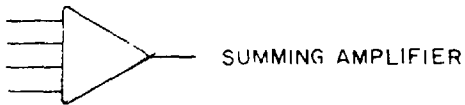
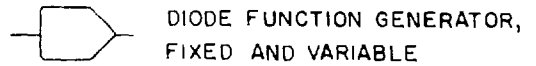
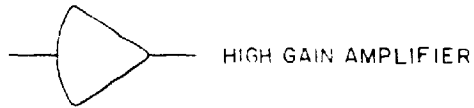
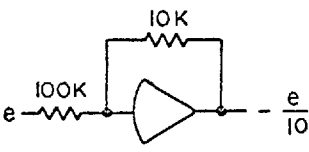
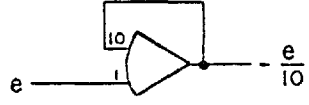
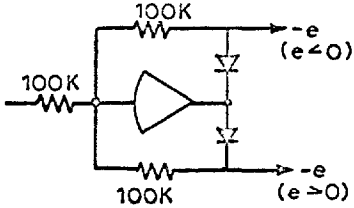
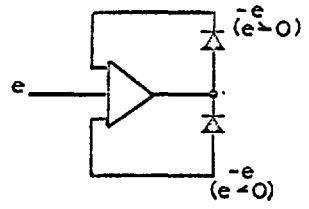
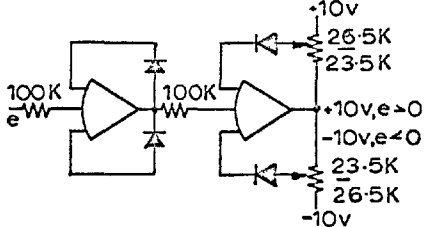
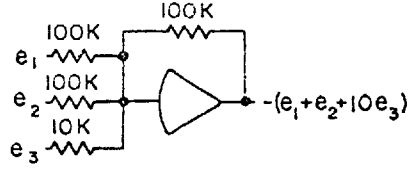
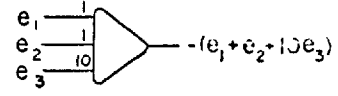
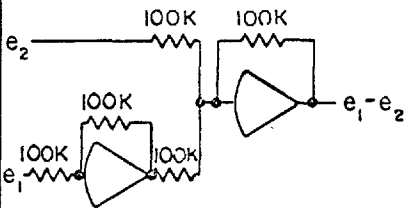
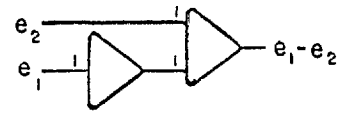
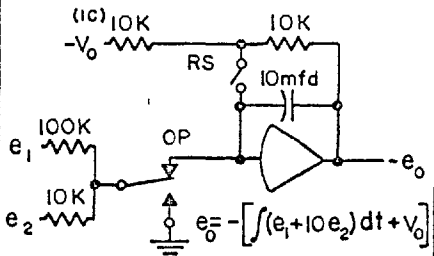
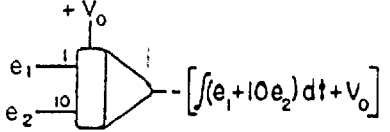


Table 1.1(b) SIMPLE CIRCUITS USING AMPLIFIERS AND POTENTIOMETERS

CIRCUIT DESCRIPTION	CIRCUIT	COMPUTER SYMBOL
1 GROUNDED POTENTIOMETER	$k = \text{POTENTIOMETER SETTING}$	
2 UNGROUNDED POTENTIOMETER	$e_2 + k(e_1 - e_2)$	
3 INVERTER		
4 MULTIPLICATION BY -10		
5 MULTIPLICATION BY -k for $1 \leq k \leq 10$ (for $k < 1$ use circuit 1 feeding circuit 3)	SETTING = $\frac{k}{10}$	
6. MULTIPLICATION BY 2		OR
7 MULTIPLICATION BY $\frac{1}{2}$	$\frac{e_1 + e_2}{2}$	

Table 1.1 (Continued)

CIRCUIT DESCRIPTION	CIRCUIT	COMPUTER SYMBOL
8 MULTIPLICATION BY $\frac{1}{10}$		
9(a) RECTIFICATION		
9(b) PRECISION LIMITING		AS FOR CIRCUIT
10. ADDITION		
11. SUBTRACTION		
12. INTEGRATION		

Repetitive Operation (RO) in some cases* it may be desirable to operate the computer repetitively. In this mode, the problem solution is generated repeatedly at high speed and the resulting waveforms are displayed continuously on an oscilloscope. This allows the investigator to vary system parameters and observe the effects of these changes instantaneously. The computer is switched to repetitive operation by depressing the appropriate mode selection switch on the computer control panel to alternatively operate and re-set the system integrators, while at the same time reducing the computer time scale.

Special purpose analogue techniques were used for simulation of discrete systems. These include analogue data storage, generation of random variables, distribution analysis, and conversion of a standard analogue computer to operate with discrete data. Details of these are given in Appendices IV, V and VI along with programming instructions for discrete systems.

* Chapters 3, 6 and 8.

1.2 COMPARISON OF ANALOGUE AND DIGITAL COMPUTERS

Analogue computers differ from digital computers in that they employ distinct computing elements for each mathematical operation. This method of parallel operation accounts for the high computing speeds that are possible and eliminates the need for data storage since data can be acted upon the instant it is available in the machine. This does not mean, however, that information storage is never used in analogue computation; in some cases it may be required and this has lead to the mating of analogue and digital machines to yield what is called a 'hybrid' computer.

Because of practical limitations on accuracy in the manufacture of analogue components the output accuracy of the computer is limited, and for computation requiring a high degree of precision, digital machines are superior. It should be borne in mind however that in many problems a high degree of accuracy is unnecessary and that results obtained by analogue computation may be acceptable when the average error can be estimated.

Assessing the relative merits of digital and analogue computers from a cost effectiveness point of view is difficult without referring to particular

problems where the availability of programmes and routines may be a significant factor. In general, where a problem can be handled on an analogue computer, it may be cheaper to do it that way, provided the degree of accuracy available is acceptable. Figure 1.1 shows qualitatively the relationship between cost and accuracy for various sizes of analogue and digital installations.

One important advantage of the analogue computer is that it allows for simple and easy access. This is particularly useful when the investigator wishes to obtain an insight into the behaviour of a system, where the inter-relationships are little understood. In such cases it is difficult to prescribe a complete course of investigation in advance and the process of feeding information to the system and analysing results becomes a continuous one in which the lack of interruptions caused by programming procedures is very welcome.

1.3 REVIEW OF ANALOGUE COMPUTER APPLICATIONS TO OPERATIONAL RESEARCH PROBLEMS

The application of control engineering concepts to problems in economics led to the use of analogue computers in this field. Subsequent application of

control theory to problems in production and inventory control similarly led to their formulation as analogue programmes. Thus, Morehouse et al. [1] described an analogue computer for simulating the behaviour of an industry inventory model, and Calvert et al. [2] described an analogue computer application to the study of a micro-economic inventory system. Philips [3] discussed an inventory system simulation and Townsend [4] described the use of an analogue for investigating a stock control system where the supply is discontinuous. Longley [5] studied analogue methods for a sampled data inventory control system, where the system variables were simulated with pulse and step waveforms.

The first successful treatment of a linear programming problem was described by Ablow and Brigham [6], who showed that linear and non-linear programming problems can be solved on the analogue by considering the dynamics of a massless particle moving in an n-dimensional space. Pyne [7] described in detail an analogue circuit to solve linear programming problems, and Neustadt [8] showed that the method can be extended to cover equality as well as inequality constraints and pointed out that the general method is useful for studying problems where the coefficients

are time dependent.

Fel'dbaum [9] discussed an analogue-digital device that combines steepest ascent and gradient techniques to optimise a function of several variables subject to constraints, and Favreau and Franks [10] have used an analogue to determine extrema of functions by random search.

A number of special purpose devices have been constructed that utilise similarities which exist between electrical networks and programming problems. Anderson [11] has described a current flow device that solves a transportation problem with a 50 x 20 matrix and the Scott Paper Co. list patents based on a similar principle for solving the transportation problem with multiple sources, destinations and intermediate points [12] and also for solving the caterer's problem [13]. Levi [14] has patented a device that utilises transformers to solve the transportation problem, while a device patented by Electrical and Musical Industrial Ltd. [15] utilises magnetic fluxes to solve a linear programming problem.

Vichnevetsky and Waha [16] have used a digital analogue method for optimising multistage decision processes using dynamic programming and Bang [17] has

solved certain problems in dynamic programming by means of an analogue computer.

Some problems associated with statistics have been solved using analogue devices. Jacobi [18] used an analogue technique for matrix rotation and Ambruzy [19] has developed an analogue circuit for quality control that utilises alternating currents to compute the average, range and variance of a set of sampled measurements. An analogue technique for computing co-variance matrices was developed by Connolly and Miller [20], and Alfson [21] has developed a simple analogue resistance network for the calculation of χ^2 in testing hypotheses. Analogue methods for determining distribution functions have been described by Brubaker and Korn [22], and Van Deusen [23] has described analogue techniques for determining the average value, mean square, variance and power spectrum of an input waveform.

Problems that involve solutions of integral or differential equations are conveniently handled on the analogue computer and Clarke et al. [24] used an analogue method for solving Lanchester's equations. Fetter and Goodman [25] considered an analogue for equipment investment, and Huggins [26] discussed an

analogue solution for the machine repair man problem.

A Goodyear Aircraft Corp. report [27] deals with analogue programmes for two simulations, one involving the breakdown of industrial machinery, the other relating to movement of vehicles through a traffic intersection.

Some other applications include an analogue method for locating depots by Brink and de Cani [28]; an analogue device for critical path analysis by Tottenham [29]; solutions of game problems on analogue computers by Rybashov and Dudnikov [30]; and an analogue method for controlling processes that are subject to trend by Eilon and Deziel [31], and more recently Clyne [32] has described an analogue computer for determining the optimal billet cut at the output of a steel mill, while Eilon and Deziel [33, 34] have described applications of analogue computers to forecasting, inventory control, process control and depot location. Lewis [35, 36] has also applied an analogue to forecasting and inventory control and Gillings [37] has developed a special purpose analogue for investment analysis.

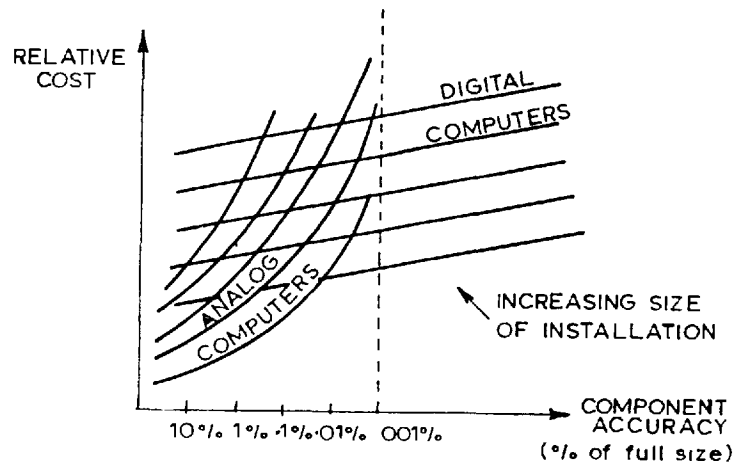


Figure 1.1 RELATIVE COST vs ACCURACY OF ANALOGUE AND DIGITAL COMPUTERS FOR DIFFERENT SIZES OF INSTALLATION

Figure reproduced from Korn [31]

CHAPTER 2

CONTROLLING PRODUCTION PROCESSES
SUBJECT TO TREND

Notation

x_0 = distance from the output mean to the specification
limit

b = value of x_0 at $t = 0$

a = rate at which the process mean drifts (standard
deviations per unit time)

t = time

r = rate of production

r_t = rate of acceptable production

$\phi(x)$ = normal distribution function

$\Phi(x)$ = cumulative normal distribution function

Q_t = total number of acceptable items produced

τ = time required to reset the process

c_r = cost of resetting the process

$\bar{r}$ = average rate of production

c_1 = cost per unit of acceptable output

r_s = rate of production of scrap

C = total cost

c_2 = cost per unit of defective that is oversized

c_3 = cost per unit of defective that is undersized

δ = distance between the specification limits (standard
deviations)

A = final value of the exponential shift

α = time constant

τ_1 = computer independant variable

Q = total number of items produced

σ = standard deviation

CHAPTER 2 : CONTROLLING PRODUCTION PROCESSES
SUBJECT TO TREND

2.1 INTRODUCTION

The object of this chapter is to describe the use of an analogue computer for determining the optimal reset policy for a production process. The process is assumed to be a normally distributed random variable with known time varying mean and constant standard deviation. In general, the output must remain within certain pre-specified limits otherwise the product has to be scrapped or reworked.

If the mean is subject to a trend variation it will be necessary to reset the process from time to time. By resetting the process, the output mean is adjusted to some predetermined value. The act of resetting, however, incurs a cost: this may be due to the time lost by an operator and the machine, or an actual cost associated with the resetting operation, or both. In a given situation, where details of all cost parameters are known, there are four cases that deserve special attention:

(1) The product characteristic is defined by a single specification limit beyond which the product is regarded as defective. Two cases may be defined here:

(a) Defectives cannot be repaired and become

scrap (the value of which is assumed to be zero).

(b) Defectives can be reworked (the cost of reworking is assumed to be known).

(2) The product characteristic is defined by upper and lower limits. Products above the upper limit (termed 'oversized') or below the lower limit ('undersized') are defectives. Here again two cases may be distinguished as in (a) and (b) above. In (b) the cost of reworking need not be the same for oversized as for undersized items.

This problem was treated by Hall and Eilon [38] who considered the situation for the case of a linear trend and suggested analytical methods for determining optimal resetting policies.

This chapter shows how the analogue computer can be programmed to solve the problem in the linear case (Section 2.2), and in addition describes the use of the analogue for solving the problem when the trend variation is non-linear (Section 2.3).

The analogue computer is particularly suited to solving problems of this kind where the expressions to be evaluated can be generated as solutions of implicit differential equations.

2.2 LINEAR TREND

1(a) Single Specification Limit when Defectives are Scrapped

The simplest case is that of the single specification limit. Figure 2.1 illustrates the case for a process with its mean initially at a distance - b units below the specification limit. (All distances are measured in units of the process standard deviation.) Products lying below the limit are termed acceptable, and products above the limit are defectives.

The distance between the process mean and the specification limit is shown on the horizontal plane as a function of time,

$$x_0 = -b + at \quad (2.1)$$

where "a" denotes the trend rate of the mean and is expressed in standard deviations per unit time. The distribution of output is shown vertically, where the probability that an item of output will have a dimension that is x standard deviations removed from the specification limit is $\phi(x,t)$. Thus, where r is the rate of production, the rate at which acceptable items are being produced is given by:-

$$\begin{aligned}
 r_t &= r \text{ (probability that } x < 0) \\
 &= r \int_{-\infty}^0 \phi(x, t) dx \\
 &= r \int_{-\infty}^{x_0} \phi(x - x_0) dx \\
 &= r\Phi(x_0)
 \end{aligned} \tag{2.2}$$

The total number of acceptable items Q_t that have been produced up to some time t is equal to the volume swept out between 0 and t by the unshaded portion of the distribution. This can be written:-

$$Q_t = \int_0^t r\Phi(x_0) dt \tag{2.3}$$

If the time required to reset the process is τ , the average rate of production ($\bar{r}$) is given by:-

$$\begin{aligned}
 \bar{r} &= \frac{Q_t}{t + \tau} = \frac{\int_0^t r \Phi(x_0) dt}{t + \tau} \\
 \frac{\bar{r}}{r} &= \frac{\int_0^{at} \Phi(x_0) d(at)}{a(t + \tau)}
 \end{aligned} \tag{2.4}$$

and it is required to determine 'at' for which $\frac{\bar{r}}{r}$ is a maximum.

The analogue computer was used to generate the expression in (2.4). This was accomplished by first

generating $\phi(x_0)$ integrating, and adding $\delta(-b)$ to the result, giving $1-\delta(x_0)$. This latter was then subtracted from 1 to give $\delta(x_0)$ which when integrated and divided by $a(t+\tau)$ yields (2.4).

The analogue circuits for performing these calculations are shown in Figures 2.2(a) and 2.2(b), and details of the circuits are given in Appendix II.1.

Results obtained from the computer for the linear case are shown in Figures 2.3, 2.4 and 2.5. Figure 2.3 shows the ratio $\frac{\bar{r}}{r}$ for b equal to 2, 4 and 6 and $a\tau$ equal to 1 and 10, plotted against x_0 . These results agree with those obtained by Hall and Eilon [38].

Figure 2.4 was obtained by evaluating the quotient

$$\frac{Q_t}{r_t} - at \quad (2.5)$$

to obtain a graph of the optimal value of 'at' as a function of $a\tau$ for various values of b . The circuit that was used to generate this function is shown in Figure 2.2(c). It is also possible to use the analogue computer to draw a curve for the optimal resetting policy, as shown in Figure 2.5. The procedure for using this nomograph is described in Appendix II.2.

Where the cost per item produced is c_1 and the cost of resetting the process is c_r , these results may be used by replacing τ by c_r/c_1 , and Q_t by $\frac{C}{c_1}$.

1(b) Single Specification Limit Case when Defectives are Reworked

When defectives are not scrapped but can be reworked and the cost of reworking is known the analogue can be programmed to generate the total process output as a function of the total cost. Taking the ratio of these quantities (output/cost) gives a measure of the cost efficiency of the process. The value of output for which the cost efficiency ratio is a maximum determines the optimal time between resets.

The analogue programme for generating these functions is similar to that described in Figure 2.2(a), except that for the cost case, both the rates of production outside the specification limits, r_s , and inside the specification limits, r_t , are generated. These functions are integrated, multiplied by their respective costs and added to give the total variable cost of production for a given level of output.

Mathematically the problem can be expressed as follows:-

If C = total cost, c_1 = cost of production within the limits, and c_2 = cost due to oversize, then,

$$C = c_2 \int_0^t r_s dt + c_1 \int_0^t r_t dt \quad (2.6)$$

or

$$\begin{aligned} \frac{C}{c_1} &= \frac{c_2}{ac_1} \int_0^{at} r [1 - \Phi(x_0)] d(at) + \dots \\ &\dots \cdot \frac{r}{a} \int_0^{at} \Phi(x_0) d(at) \end{aligned} \quad (2.7)$$

The production rates r_s and r_t are generated by the analogue as indicated in Figure 2.2(a). The circuit that was used to evaluate (2.7) is given in Figure 2.6.

Results obtained from the computer are shown in Figure 2.7 which is a nomograph that can be used to determine the optimal resetting policy for any combination of the reset cost and b . Directions for using this nomograph are included in Appendix II.3.

2(a) Double Specification Limits when Defectives are Scrapped

The process output may be required to lie between two specification limits, as shown in Figure 2.8. Here the output is represented by a normal curve and

the two specification limits form a slit through which some part of the curve can be seen at any time, the shaded parts represent the defective portion of the output. The distance between the mean of the distribution and the limit towards which it is drifting is denoted by x_0 .

The time integral of that portion of the curve 'visible' through the slit gives the total process output within the specification limits. Similarly the time integral of the shaded portions of the curve gives the total process output above and below the specification limits. The symmetry of the normal curve implies that scanning at a constant rate (which is implied in the case of a linear drift) in a direction either side of the mean gives the same result.

Thus, when the output above and below the specification limits is scrapped or when the penalty costs are the same in each case, for purposes of the calculation, the process may be initiated midway between the specification limits. Then, using one half the setup costs, the number of standard deviations the process should be allowed to drift from the starting point can be determined. The optimal policy is then to reset the process this distance below the starting point and allow the mean

to drift the same distance above it.

When the costs for over- and undersized items are not equal, or when defectives above the upper limit are reworked, but those below the lower limit are scrapped, the computer can be used to solve the problem by assuming that the process starts midway between the specification limits. The optimal production run is then calculated for drift in both directions using one half the setup costs for the calculation in each case. The optimal time between resets is then given by the sum of these quantities, and the optimal reset point is located at the optimal value of 'at' below the mid point.

The problem is to evaluate the expression

$$Q_t = \int_0^{at} \frac{r_t}{a} d(at) \quad (2.8)$$

where

$$r_t = r [\Phi(x_0) - \Phi(x_0 - \delta)]$$

δ = distance between the specification limits measured in units of the process standard deviation.

The analogue programme for evaluating the function in (2.8) is an extension of the basic circuit for the single limit case. This circuit is shown in Figure 2.9. Here two normal curves are generated; one corresponds

to the height of the curve intersected by the upper specification limit, the other corresponds to the height of the curve intersected by the lower specification limit. Each of these functions is integrated to determine the rates of production (i) below the upper, and (ii) below the lower specification limits. These are subtracted to give the rate of production r_t within the limits. r_t is then operated on to determine the cost output characteristic in the usual manner.

Computer results for the double limit case are shown in Figure 2.10 which shows the optimal value of α plotted against σ , where output not within the specification limits is scrap. The figure is plotted for two values of $\delta < 6$; when $\delta > 6$ the problem may be solved as an extension of the single limit case.

2(b) Double Specification Limits when Defectives are Reworked

The procedure for dealing with this case is similar to 2(a) except that now defectives are reworked at a cost of c_2 per unit if it is oversized and c_3 per unit if it is undersized (Circuit Figure 2.9(b)).

Results of the computer calculations are shown in Figures 2.11 and 2.12. Figure 2.11 relates to a

specific example, where $\frac{c_2}{c_1} = 3.0$, $\frac{c_3}{c_1} = 2.0$, the specification limits are separated by $\delta = 2.0$ and $c_r = 1.0$. Figure 2.12 is a nomograph that can be used to determine the optimal reset point for any set of cost function ratios, reset cost and δ . Directions for using the nomograph are contained in Appendix II.4.

2.3 NON-LINEAR TREND

The rate at which the process mean shifts may not be linear. Essentially the problem is similar to one that has already been described in that the total output is proportional to the volume swept out by the unshaded portion of the normal curve, i.e.

$$Q_t = r \int_0^t \Phi(x_0) dt \quad (2.9)$$

but now $x_0 = f(t)$ where $f(t)$ is non-linear.

In the analogue, the first step is to generate $\phi(x_0)$ (a method for doing this is described in Appendix II.1). $\phi(x_0)$ is then integrated with respect to x_0 to give the rate of output of good items. This rate is integrated with respect to t to give an output proportional to the total acceptable output from the process.

Exponential trend

Exponential trends, for example of the type $x_0 = -b + A(1 - e^{-\alpha t})$, may be of special interest in certain processing operations. As in the linear case, the problem of optimal resetting may be divided into two parts,

- (i) What should be the optimal length of the production run?
- (ii) Where should the process be set to initially?

For the purpose of discussion it may be convenient to distinguish the following categories:

- 1. The single limit case:
 - 1.1 If the starting point is specified, attention is focussed on question (i).
 - 1.2 A minimum starting point may be specified and questions (i) and (ii) need to be answered.
 - 1.3 If the duration of the production run is fixed, for administrative or technical reasons, question (ii) has to be answered.
- 2. The double limit case:

Here the same categories as in the

single limit case may be defined, and labelled as 2.1, 2.2, 2.3 respectively.

Case 1.1

The circuit diagram for this programme is shown in Figure 2.13, and the results are shown in Figure 2.14. In Figure 2.14 the single limit case is plotted for $b = 3$ and various values of A .

Cases 1.2 and 1.3

If, in the single limit case, a minimum starting point is specified, it will in fact be the point at which the process should be initially set. This reduces case 1.2 to case 1.1 and the question of the optimal production run is dealt with accordingly.

Similarly, in case 1.2 the initial setting should obviously be such as to minimise defectives, which implies as low a starting point as possible.

Case 2.1

In the double limit case the solution will evidently depend on the starting point and on the cost parameters for reclaiming defectives. This problem was solved for a particular case where $\frac{c_2}{c_1} = 1.8$, $\frac{c_3}{c_1} = 1.1$, $\delta = 2.0$, $A = 2.0$ and $at = 0.4$. The analogue circuit is similar to the one given in Figure 2.9 for the double limit case except that $\mathbb{I}(x_0)$ and $\mathbb{I}(x_0 - \delta)$ are generated using the method

indicated in Figure 2.13 for the non-linear case. Figure 2.15 shows the result that was obtained with the starting point midway between the specification limits.

Case 2.2

When a minimum starting point is specified, it should be selected when the desirable starting point (which can be determined for a given set of parameters) is found to be below it. If the desirable starting point is above the minimum point, the latter may be ignored, and the starting point would be such as to allow for a theoretically indefinite production run.

Case 2.3

If the production run is fixed in duration, the starting point will depend on the characteristics of the exponential curve in relation to the run. Broadly the three situations described in Figure 2.16 deserve consideration. Where (a) applies, the trend may be assumed for practical purposes to be linear and the previous analysis may be used. Where (c) applies the starting point is adjusted so that the curve 'settles down' at some point (depending on the relevant cost parameters) between the specification limits. When (b) occurs, the average rate of production may be computed for a number of possible starting points

(by using the method described for the non-linear trend in Figure 2.13) and the optimal starting point can thus be found. It should be pointed out that in situations (a) and (b) an optimal production run may be found to be of shorter duration than the specified fixed run, in which case a decision about an interim resetting of the process during this run will have to be made.

2.4 SUMMARY

The work that has been described shows that the analogue computer can be effectively used for determining an optimal resetting policy for processes where the output is normally distributed and the mean is subject to drift.

In the case of a linear drift generalized results are given for (1) the single, and (2) the double specification limit case. These results cover both the situation when defective items are (a) scrapped and are valueless; (b) they can be reworked (the cost of reworking being known).

When the drift is non-linear, generalized results depend on the nature of the drift. This chapter described how the computer can be used when the parameters

governing the model are known, and proceeds to discuss
a special example where the drift has an exponential
form.

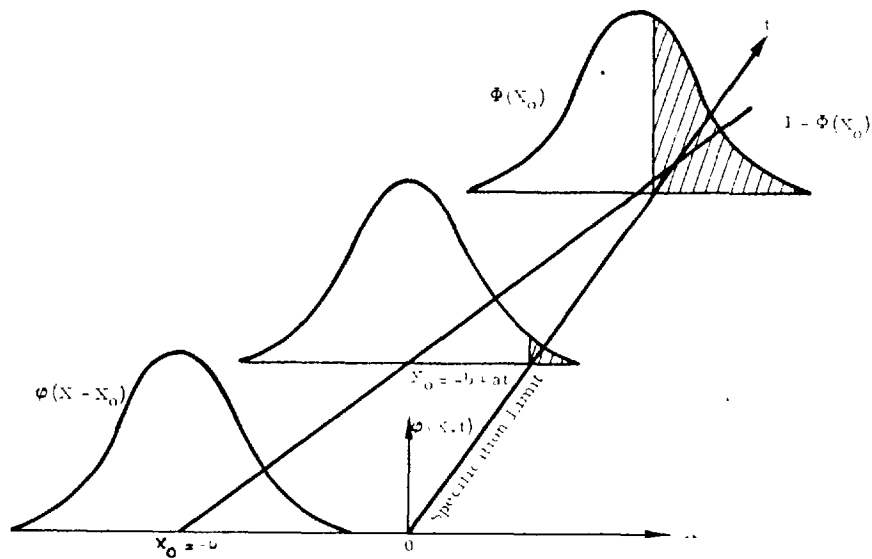


Figure 2.1 PROCESS SUBJECT TO A LINEAR TREND WITH A SINGLE SPECIFICATION LIMIT

$\Phi(x_0)$ Cumulative normal distribution up to the standardised deviate x_0 (this is proportional to the expected rate of production of good items r_t)

$1 - \Phi(x_0)$ is proportional to the expected rate of production of defectives r_s

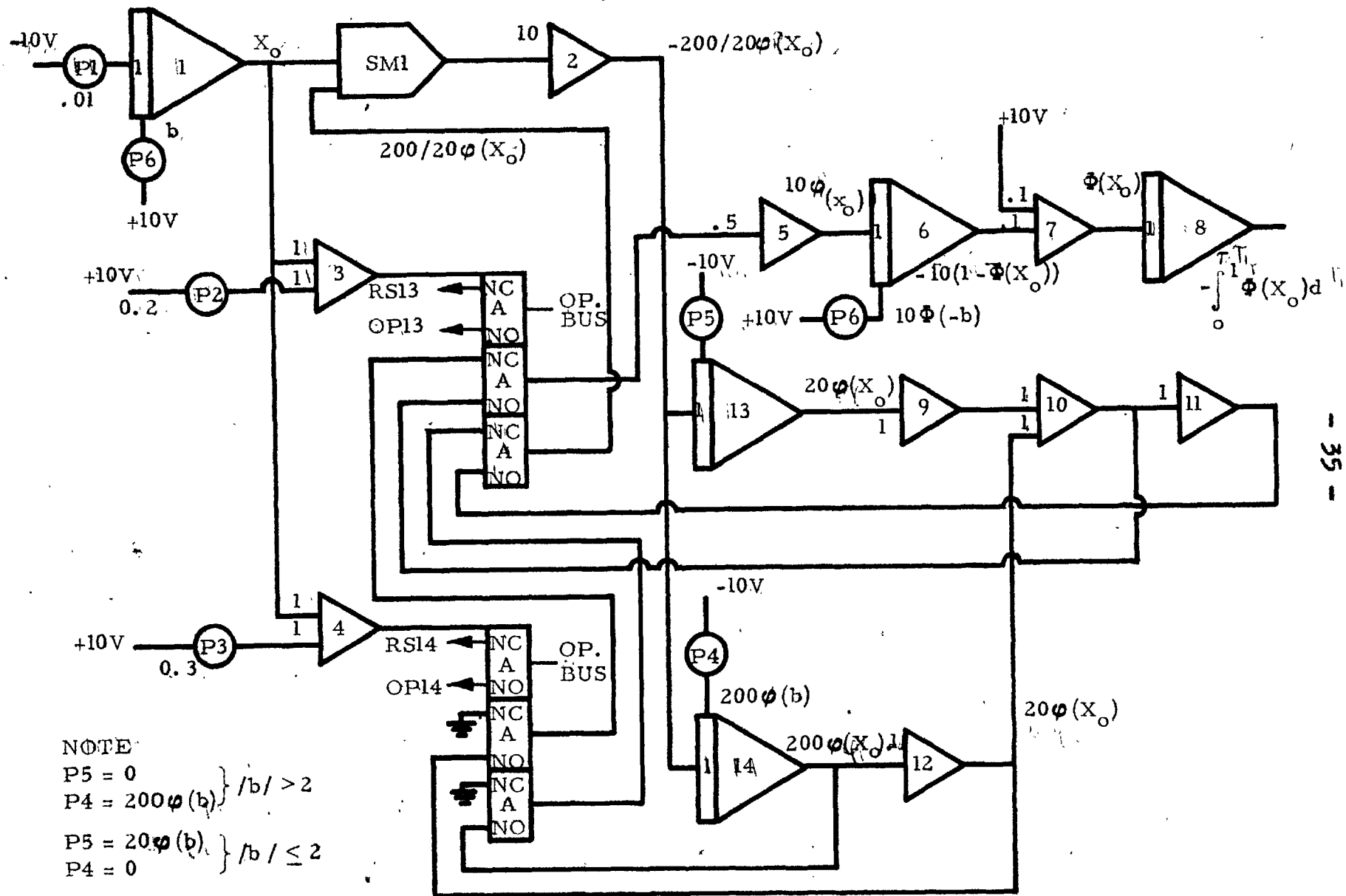


Figure 2.2(a) ANALOGUE CIRCUIT TO GENERATE

$$\int_0^{\tau_1} \Phi(X_0) d\tau_1$$

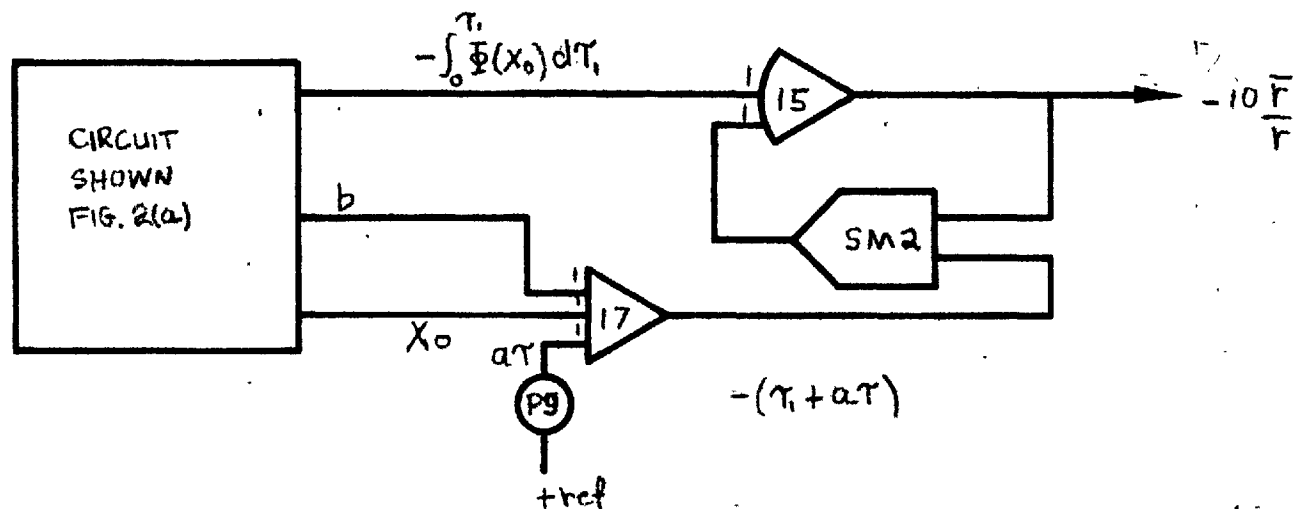


Figure 2.2(b) CIRCUIT TO GENERATE $\frac{x_t}{r_t}$ FOR THE SINGLE LIMIT CASE

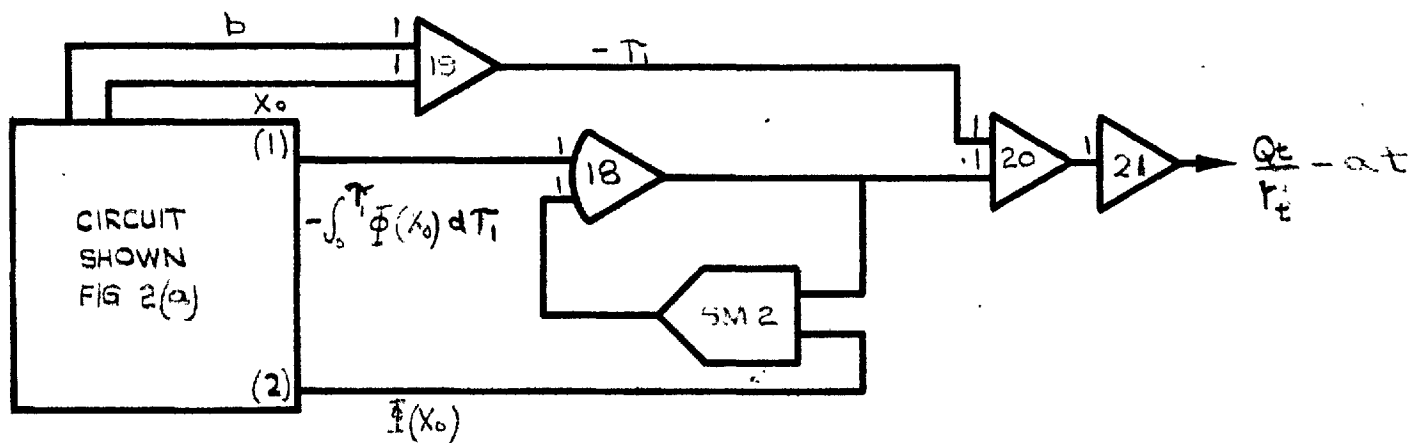


Figure 2.2(c) CIRCUIT TO GENERATE $\frac{Q_t}{r_t} - at$

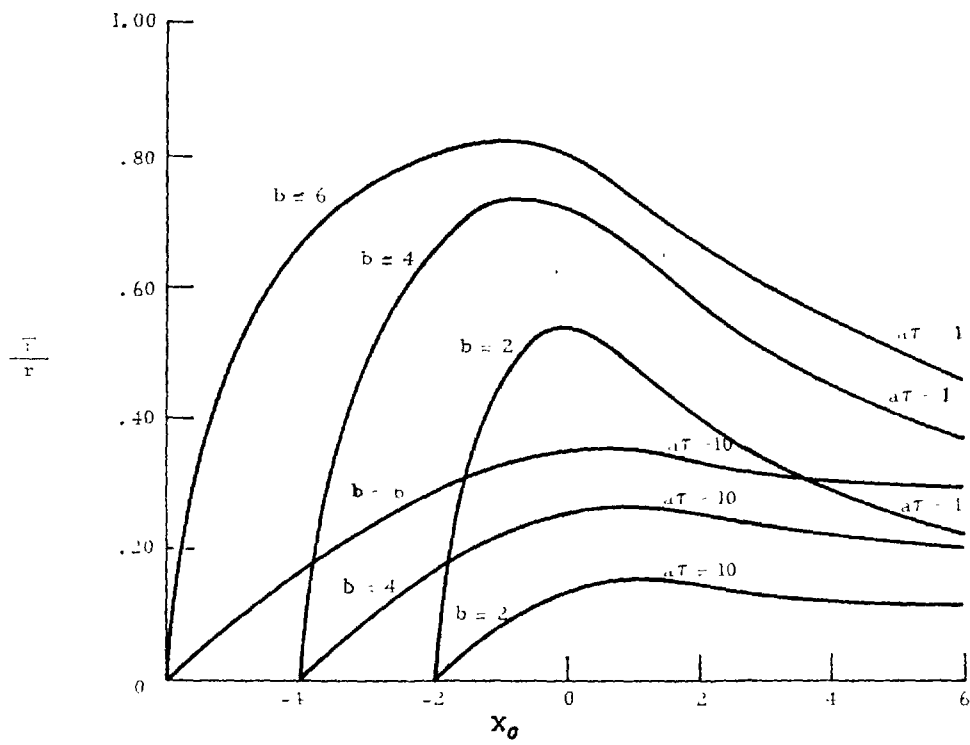


Figure 2.3 AVERAGE PRODUCTION RATE AS A FUNCTION OF x_0

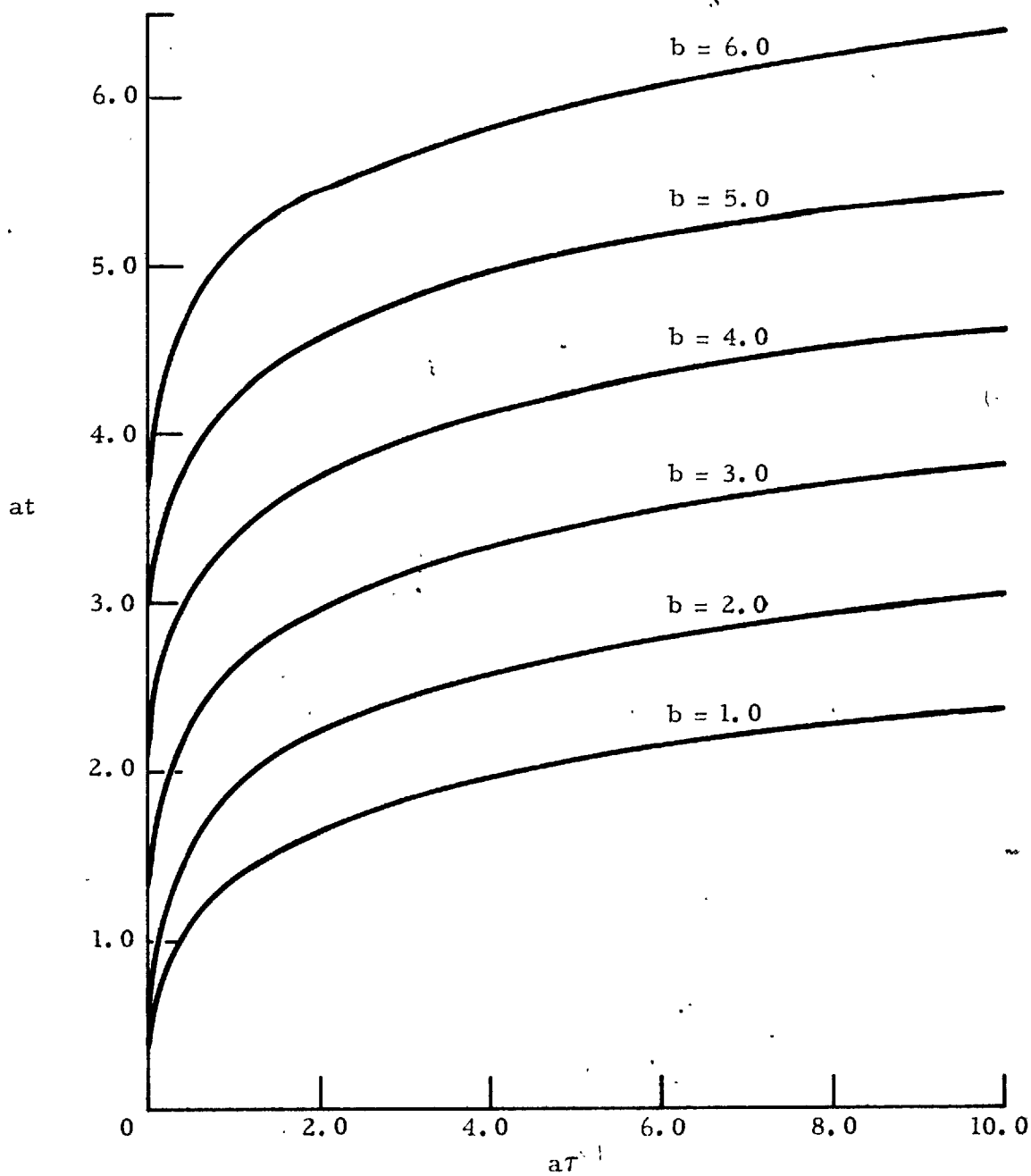


FIG.2.4 OPTIMAL PRODUCTION RUNS

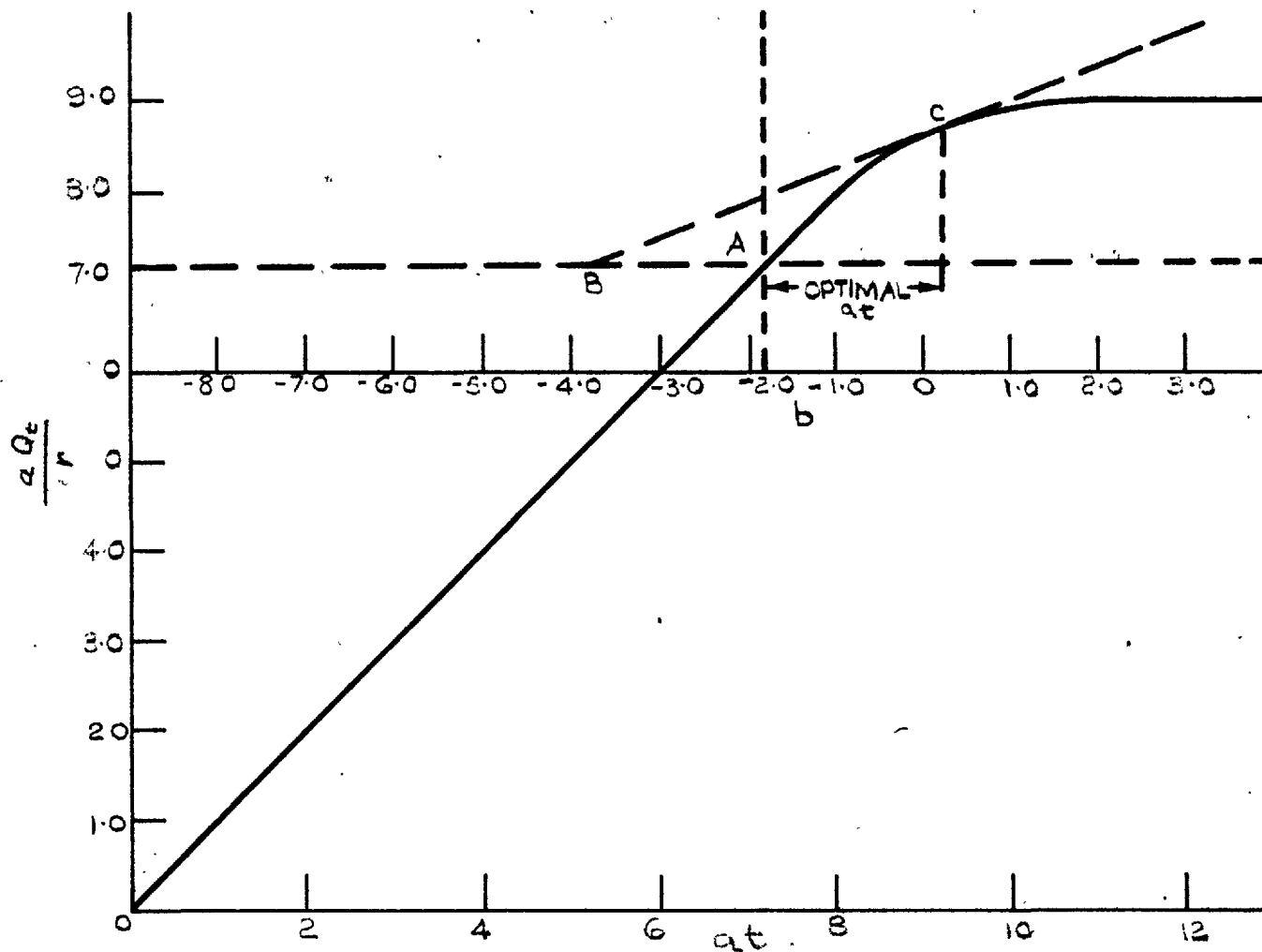


Figure 2.5 CURVE FOR DETERMINING THE OPTIMAL PRODUCTION RUN
(SINGLE LIMIT CASE; DEFECTIVES ARE SCRAPPED)

AN EXAMPLE IS SHOWN FOR $b=4.3$ and $a_T=2.0$

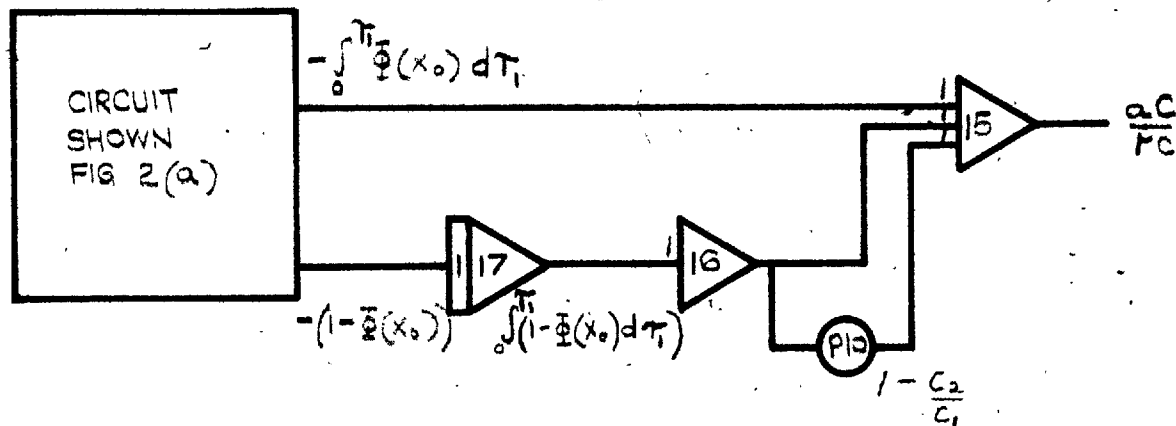
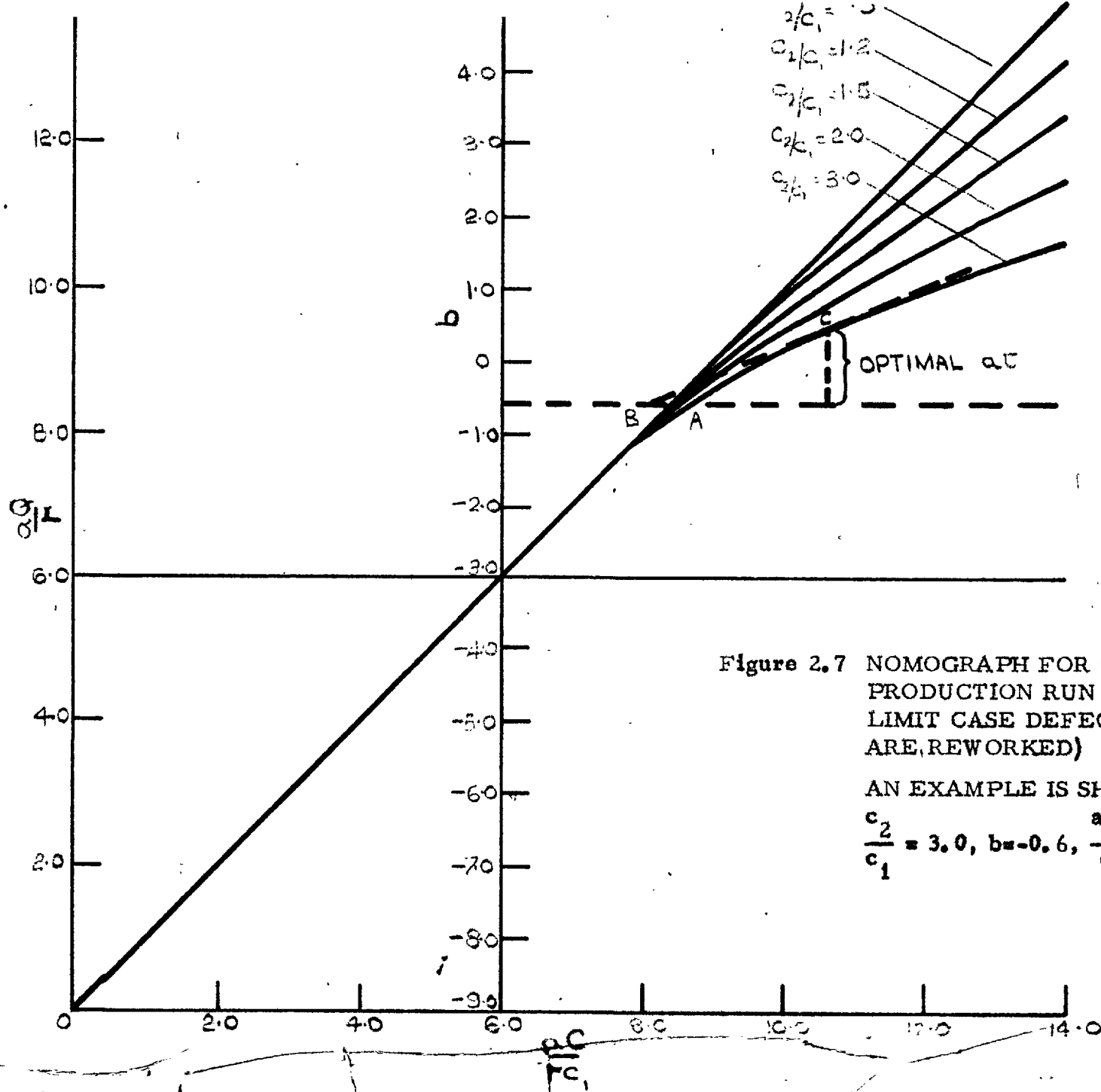


Figure 2.6 CIRCUIT TO GENERATE $\frac{aC}{rc_1}$ FOR THE SINGLE LIMIT CASE
WHEN DEFECTIVES ARE REWORKED



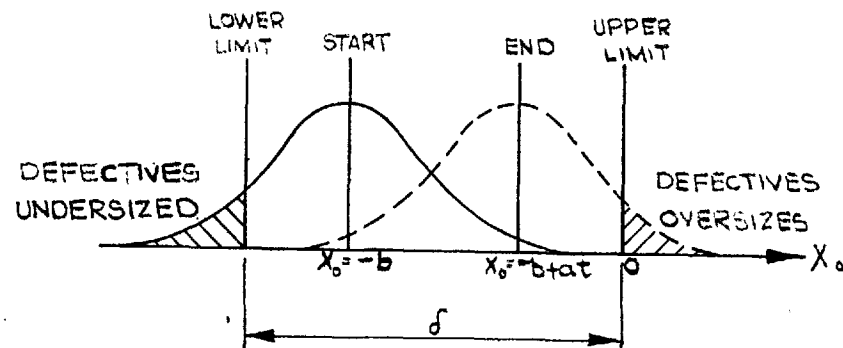


Figure 2.8 THE DOUBLE LIMIT CASE

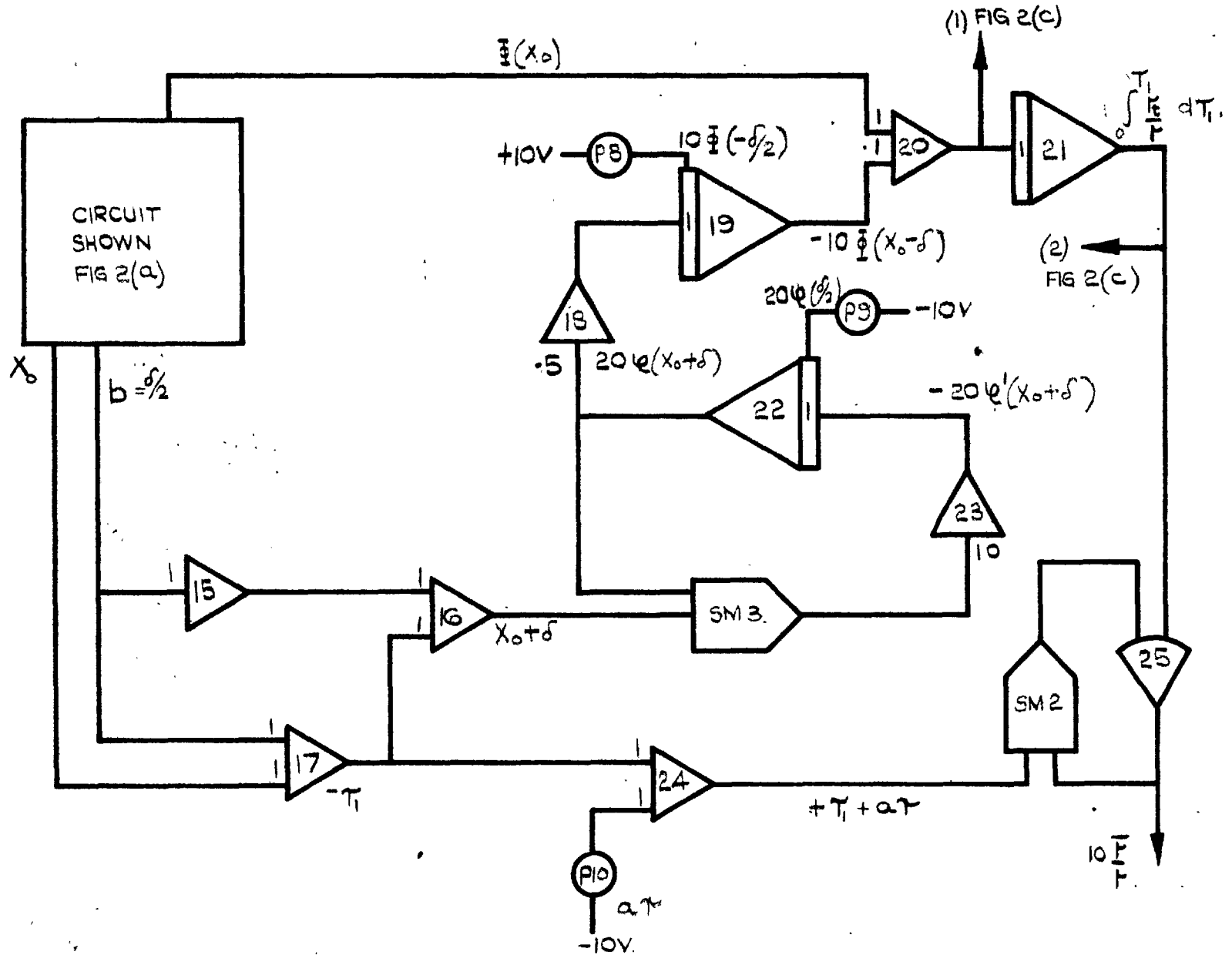


Figure 2.9(a) CIRCUIT FOR THE DOUBLE LIMIT CASE WHEN DEFECTIVES ARE SCRAPPED

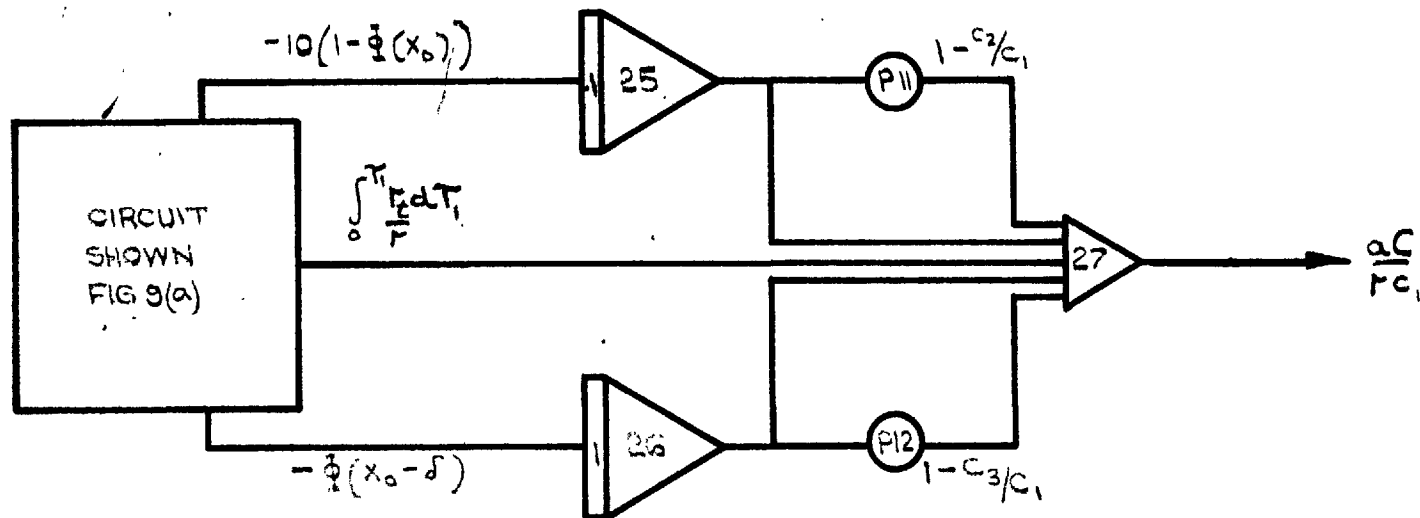


Figure 2.9(b) CIRCUIT FOR THE DOUBLE LIMIT CASE WHEN DEFECTIVES ARE REWORKED

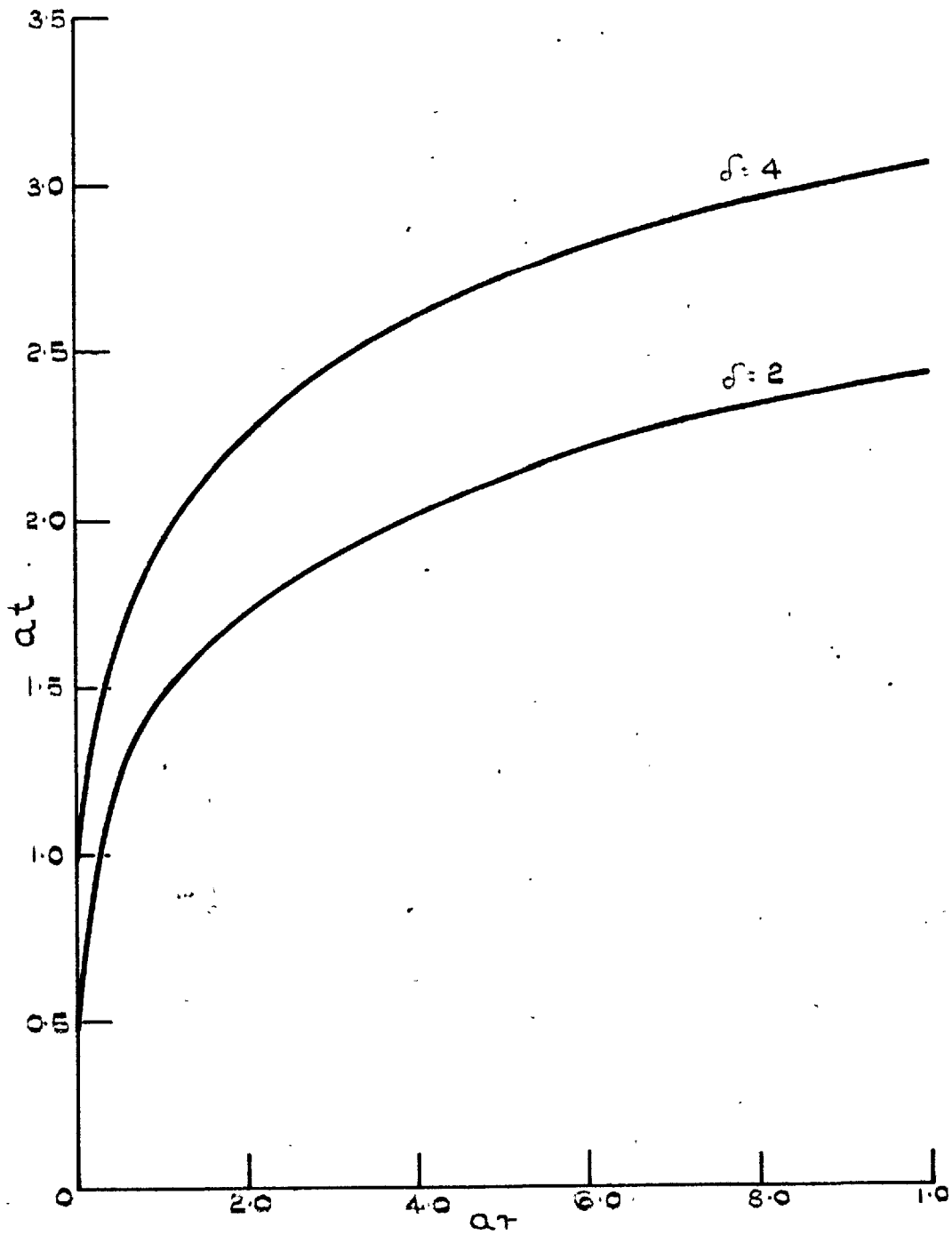


Figure 2.10 OPTIMAL PRODUCTION RUN FOR TWO VALUES OF δ

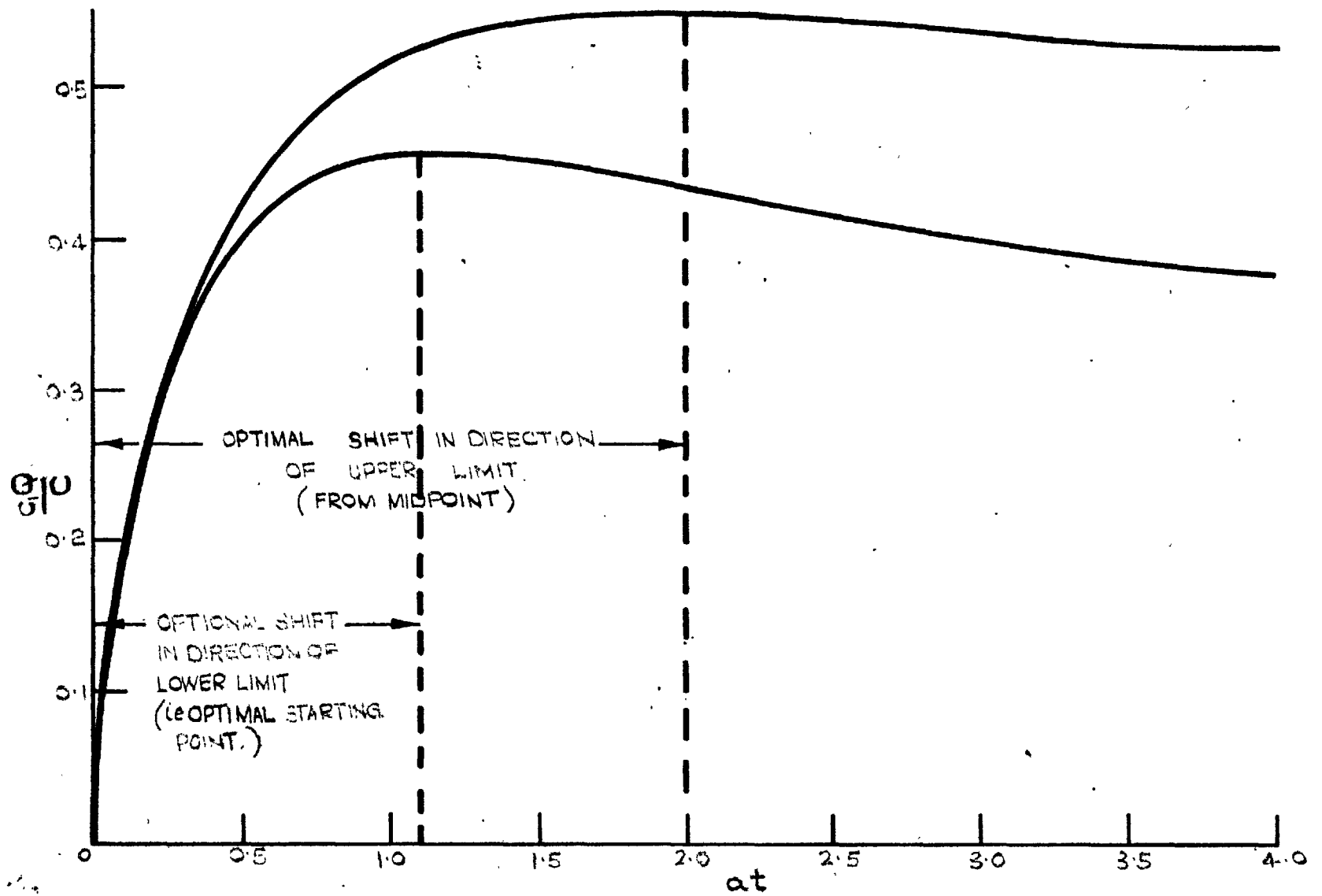


Figure 2.11 DOUBLE LIMIT CASE WITH $\frac{c_2}{c_1} = 3.0$, $\frac{c_3}{c_1} = 2.0$, $c_r = 1.0$ AND $\delta = 2.0$

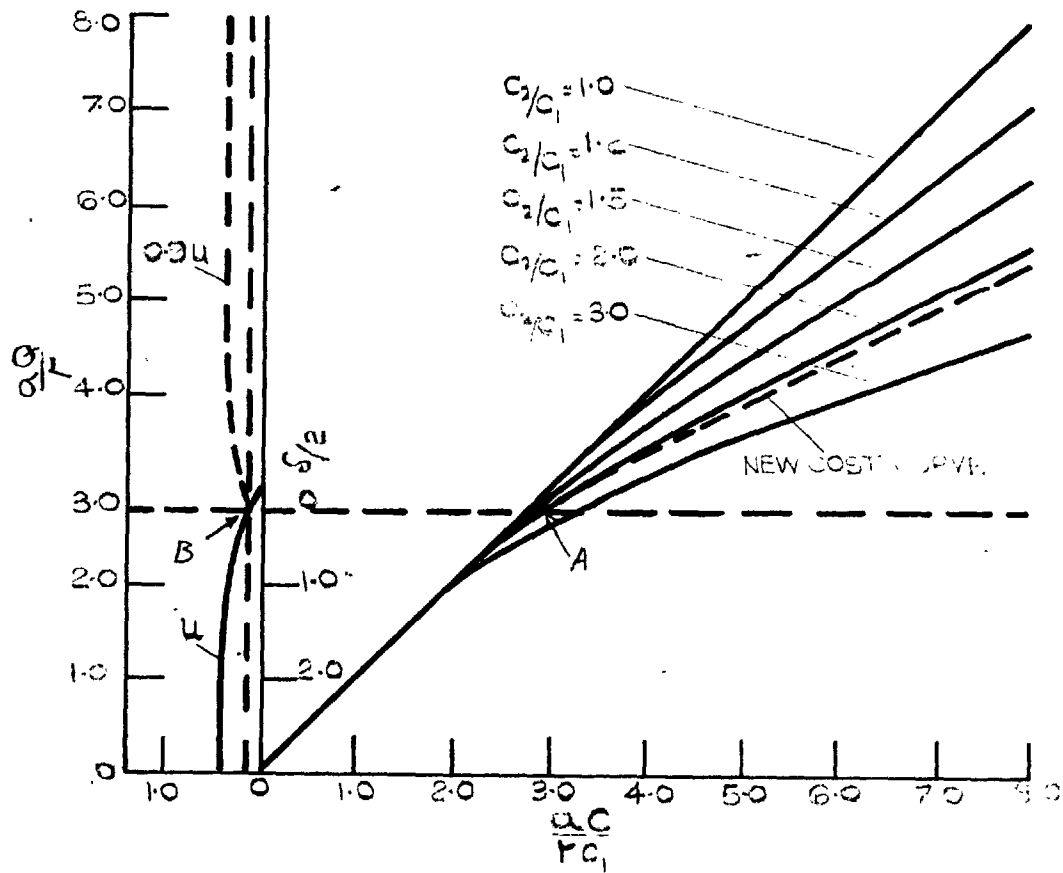


Figure 2.12 NOMOGRAPH FOR OPTIMAL RUN (DOUBLE LIMIT CASE WITH COSTS)

AN EXAMPLE IS SHOWN FOR $\delta=0.4$, $\frac{c_2}{c_1}=2.0$,

$$\frac{c_3}{c_1}=1.9 \text{ AND } \frac{ac}{c_1}=0.2$$

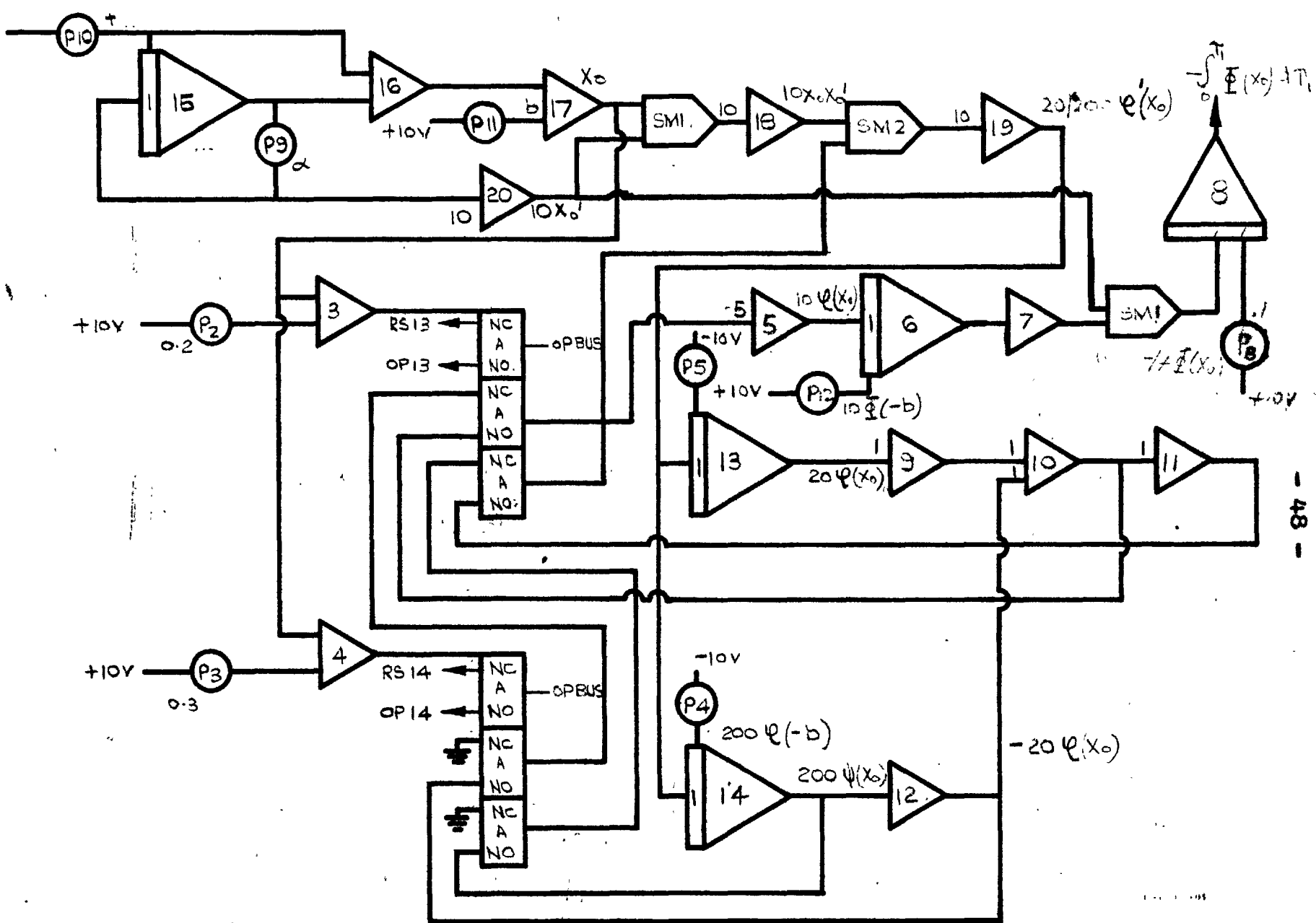


Figure 2.13 CIRCUIT FOR A NON-LINEAR CASE

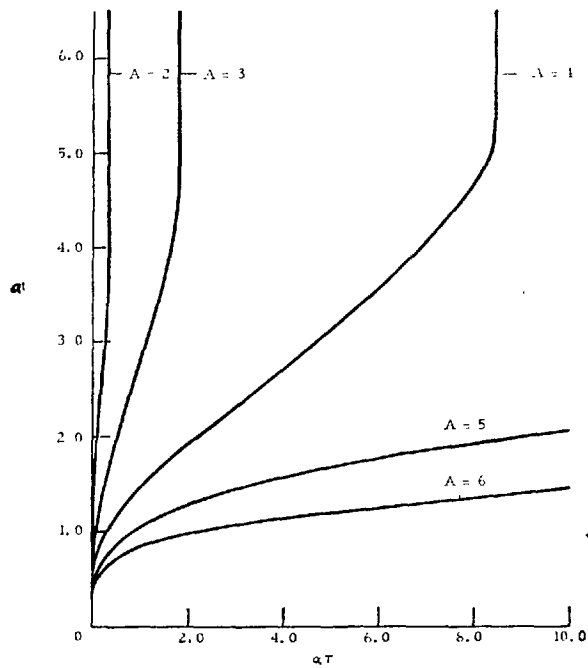


Figure 2.14 OPTIMAL PRODUCTION RUN FOR THE SINGLE LIMIT CASE, EXPONENTIAL TREND AND $b=-3$

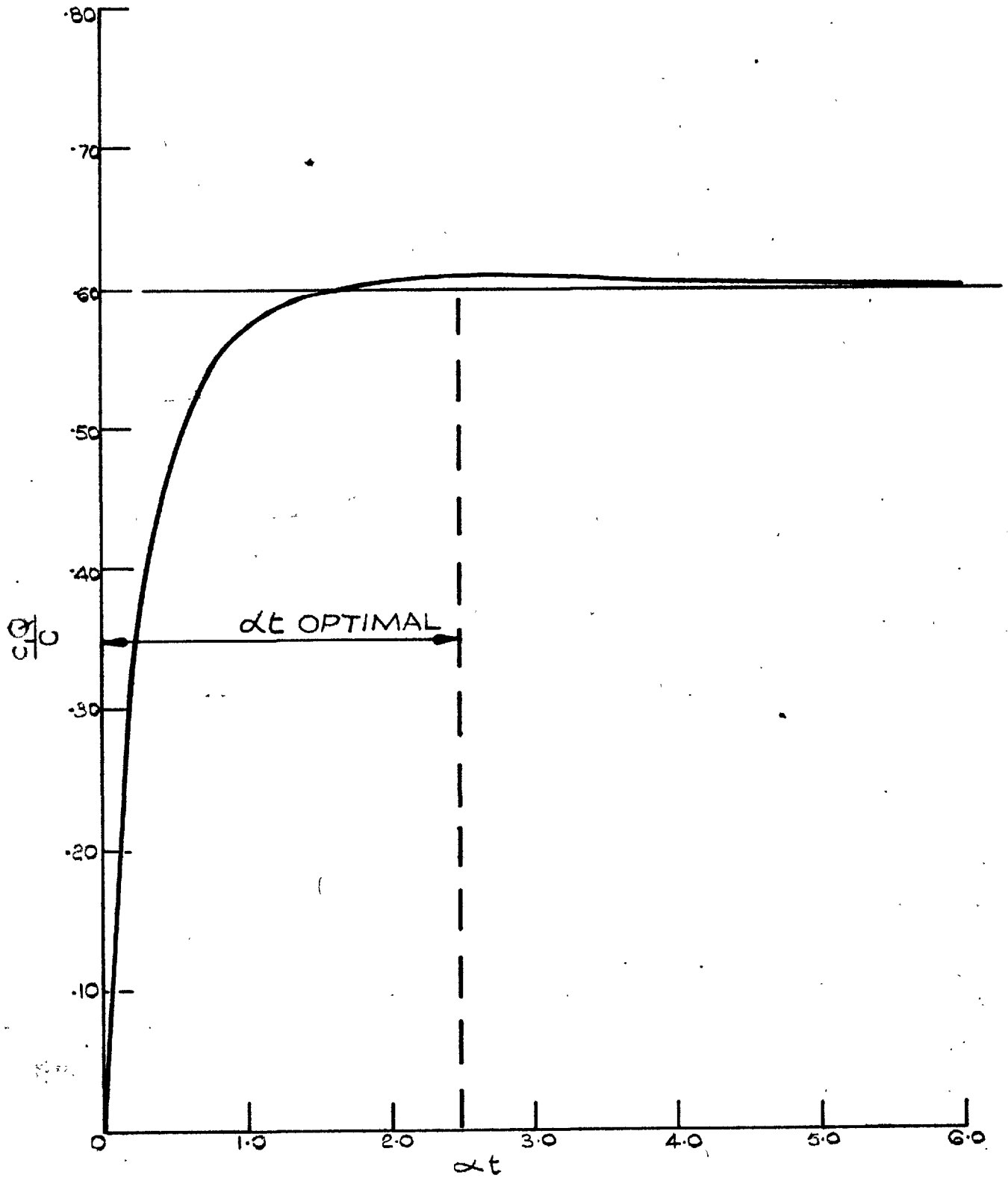


Figure 2.15 COST EFFICIENCY PLOTTED FOR AN EXPONENTIAL TREND

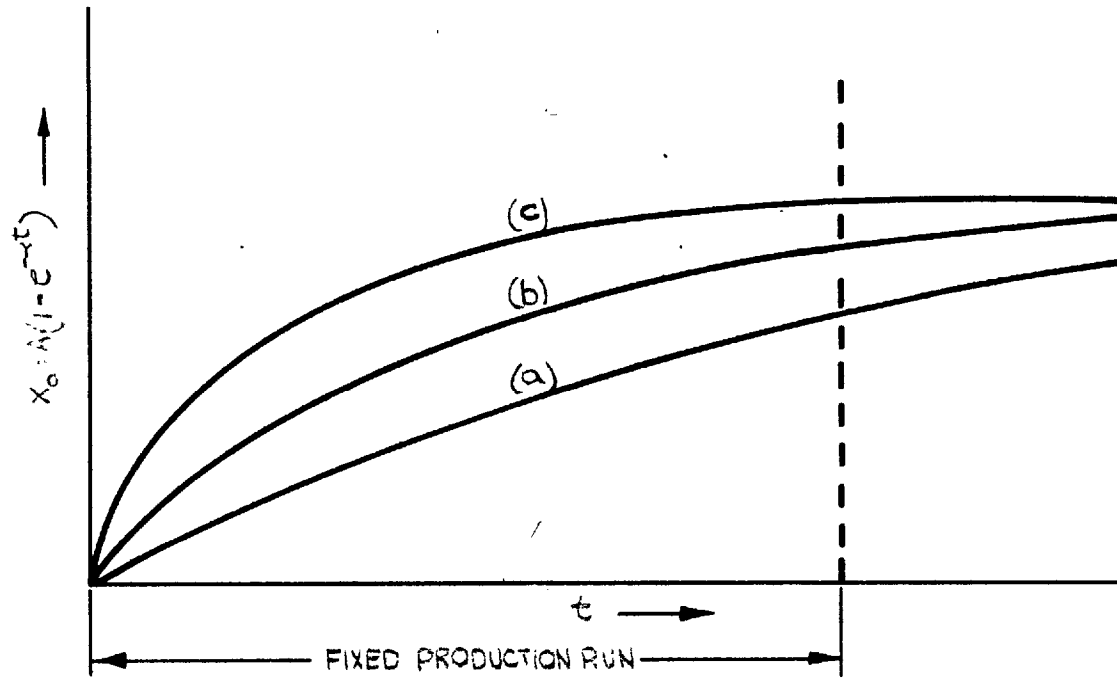


Figure 2.16 THREE CASES OF AN EXPONENTIAL TREND AND
FIXED PRODUCTION RUN

CHAPTER 3

SITING A DISTRIBUTION
CENTRE

Notation

x_i = co-ordinate of point i

y_i = ordinate of point j

d_{ij} = distance between points i and j

u_{ij} = weighting factor between i and j

D = total cost

D_0 = minimum total cost

N = number of points in network

$a_{ij} = y_i - y_j$

$b_{ij} = x_i - x_j$

φ = phase angle

w = radian frequency

t = time

x_A = co-ordinate of depot location

y_A = ordinate of depot location

CHAPTER 3 : SITING A DISTRIBUTION CENTRE

3.1 INTRODUCTION

This chapter describes an analogue computer application for finding the optimal location of a depot or a distribution centre which supplies goods to a number of customers. The cost of transportation is assumed to be a linear function of the distance involved and of the amount transported, and the problem is to find, for a given network of customers, the location of depot or depots that will result in minimum transportation costs.

Formulated in more general terms, the problem may be described as that of link-length minimization in networks, where the objective function to be minimized is related to the lengths of the links involved in the network. In a communication system, for example, a switching centre controls all the traffic flow in the network and if the location of the switching centre is suitably selected, the total length of cable required for the network can be minimized. A similar problem occurs in the location of an electric power distribution centre.

Several factors affect the way in which this problem may be treated. In the case where the

feasible distribution centre locations are confined to a predetermined set of points, a number of algorithms have been developed to solve the problem [41, 42, 43] . Where the set of feasible centre locations is a two dimensional continuum, various analogues have been developed [44, 45, 46] . In particular, Brink and de Cani [44] described a special purpose electronic analogue to determine the optimal location for a single distribution centre, and in addition to plot iso-cost lines, which may also be used to find the cost of locating the centre at any point in the plane.

The present ~~part~~^{chapter} pursues the analogue approach and describes how a general purpose electronic analogue computer can be used to solve the problem and to obtain iso-cost curves.

3.2 STATEMENT OF THE PROBLEM

If we have a network consisting of N points, each having the co-ordinates (x,y) , and if the distance between two points i and j is denoted by d_{ij} , then the objective function which involves all the links in this network is

$$D = \sum_{i=1}^N \sum_{j=1}^N u_{ij} d_{ij} \quad (3.1)$$

where u_{ij} is the weighting factor that should be applied to the link d_{ij} . Notice that with this double summation, each link appears twice, once as d_{ij} and then as d_{ji} , and this general formulation will allow for the fact that transportation takes place from point i to point j and vice versa, also that the weighting factor u_{ij} need not be the same as u_{ji} .

In most practical applications with which we are concerned the set of N points is divided into two subsets: n customers, who are not directly connected with each other, and m depots which are interconnected and which also have links with customers. We make the assumption that flow in any link is unidirectional (i.e. if there is flow from i to j , there is none in the reverse direction) so that the objective function is reduced to

$$D = \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m u_{jk} d_{jk} + \sum_{i=1}^n \sum_{j=1}^m u_{ij} d_{ij} \quad (3.2)$$

The first term in (3.2) corresponds to the cost of linking the distribution centres (the coefficient $\frac{1}{2}$ is applied to account for each link appearing twice in this double summation) and the second term corresponds to the cost of linking the customers to the centres.

In a communication system and in many commodity distribution systems the centres are directly linked and equation (3.2) applies. If the centres are not linked, the first term in (3.2) is deleted. The problem in general is to minimise the function (3.2) subject to any prevailing constraints.

Where the number of depots is greater than one, it will usually be necessary to allocate the customers among the centres before the optimal centre locations can be determined. The suggested procedure to be used is as follows:

1. An arbitrary initial set of centre locations is selected and the optimal allocation of customers to the centres is determined using the transportation technique of linear programming [47] or alternatively, the Ford-Fulkerson algorithm [48].
2. Having found the optimal allocation, the optimal location of each depot in relation to the set of its customers is determined.
3. It will then be necessary to re-determine the optimal customer allocation, using this new set of centre locations. If the optimal allocation changes, the optimal centre locations will have to be re-determined and so on, until successive iterations produce

no change in the optimal centre locations or the optimal allocation of customers to depots.

This chapter is concerned with the second phase of this optimisation procedure, namely the determination of optimal depot locations, after the allocation of customers to depots has been made.

3.3 THE ANALOGUE METHOD

If we use polar co-ordinates to indicate the position of point i in relation to point j , where d (the distance between the two points) and φ (the angle of the vector d) are our co-ordinates, then

$$a = y_i - y_j = d \sin \varphi \quad (3.3)$$

$$b = x_i - x_j = d \cos \varphi \quad (3.4)$$

and

$$d^2 = a^2 + b^2$$

We can now introduce a variable ωt , multiply equation (3.3) by $\cos \omega t$ and equation (3.4) by $\sin \omega t$, then add to obtain

$$a \cos \omega t + b \sin \omega t = \sqrt{a^2 + b^2} \sin (\omega t + \varphi) \quad (3.5)$$

Expression (3.5) is a sine wave, having its maximum amplitude equal to the distance between the two points i and j . If several such sine waves are generated, one for each pair of points in equation (3.2), and

multiplied by the appropriate weighting factors u , the sum of the peak values of the resultant set of waveforms is equal to the objective function D in equation (3.2).

The analogue computer can be programmed to generate sine and cosine waves as function of time; details of this programme are shown in Figure 3.1, and Table 3.1 gives a list of computer symbols. Multiplication by u_{ij} is accomplished by a potentiometer which can be varied manually between zero and one (for implementation on the analogue values of u must therefore be scaled, so that $u_{ij\max} = 1$). For practical considerations it is convenient to obtain $\max [\sqrt{a^2 + b^2} \sin (\omega t + \varphi)]$ as $\beta \frac{\omega}{2} \int_0^T \sqrt{a^2 + b^2} \sin (\omega t + \varphi) dt$ where $T = \frac{2\pi}{\omega}$

$$\beta = 1 \text{ for } \sin (\omega t + \varphi) \geq 0$$

$$\beta = 0 \text{ for } \sin (\omega t + \varphi) < 0$$

(Multiplying by β is equivalent to half wave rectification of the sine wave.) A block diagram of the system is shown in Figure 3.2 (in this and in subsequent diagrams $\omega = 1$ was taken), where x_A and y_A denote the co-ordinates of a single depot and x_i, y_i the co-ordinates of customer locations.

Integration over one period can be accomplished automatically on the computer by switching the machine "on" for one period (T seconds), then switching it "off"

and resetting the output to zero. When the computer is operated in repetitive mode and T made small, the operation can be performed several times per second. The values of x_A , y_A , x_i , and y_i are entered into the machine as dial settings and the objective function D is read out as the peak value of a waveform displayed on an oscilloscope. D is minimized by manually varying x_A and y_A while the machine is in operation. A circuit diagram for this system is shown in Figure 3.3(a) and Figure 3.3(b) shows typical circuit waveforms.

3.4 EXPERIMENTAL RESULTS

1. A single depot serving ten customers

As an example, a system having a single depot for ten customers was analysed. Table 3.2 gives the customer co-ordinates and weighting factors and the results are shown in Figure 3.4, where iso-cost lines are plotted for cost values equal to 1.05, 1.15, 1.30 and 1.60 times the minimum (denoted by D).

TABLE 3.2 Co-ordinates and the weighting factor u for ten customers

i	1	2	3	4	5	6	7	8	9	10
x	9.10	1.04	8.01	6.09	5.02	4.01	2.27	4.49	0.48	2.29
y	2.88	2.12	6.03	8.18	4.45	8.77	0.82	2.48	8.97	5.62
u	0.468	0.787	0.245	0.886	0.907	0.744	0.054	0.110	0.372	0.384

This iso-cost map provides a sensitivity analysis of the system and shows what deviations from the optimum are likely to occur for various possible locations of the depot.

2. A system with two depots

Another example was investigated for a case where two depots supply four customers, and where the depots are linked. The results are shown in Figure 3.4 and the computer diagram is shown in Figure 3.5. Here it was assumed that each customer is only served by one depot and that all the weighting factors $u=1$. The initial depot locations are shown at A and B in Figure 3.4 and the customers are at 1, 2, 3, 4. The allocation of customers to the centres was made on a distance basis, each customer being served by the nearest depot. The co-ordinates of A and B were then repeatedly varied, until no further improvement in the value of the objective function D could be obtained. In this case it was not found necessary to reallocate the customers between iterations. The optimal depot locations that were finally obtained are shown at A_0 and B_0 .

This result is very close to that obtained by Miehle [46], who used an iterative analytical technique

to solve the same problem, and whose result is also shown in Figure 3.5.

It is noteworthy however that this optimum is not a global one, since by choosing $x_A=5.0$ $y_A=2.0$ $x_B=2.0$ $y_B=3.0$ as starting points, a new optimum is reached at $x_A=4.90$ $y_A=2.42$ $x_B=2.12$ $y_B=2.78$ with $D_o=16.51$.*

3.5 DISCUSSION

The results for the two examples are summarized in Table 3.5, from which it appears that solution by the analogue method agree with the analytical computations to within 1%.

TABLE 3.3 Summary of results

	Customers	Depots	Objective function D_o	
			Analogue method	Analytical computations
Example 1	10	1	16.75	16.86
Example 2	4	2	16.88	16.76

In the case of the single depot system, it has been shown [45] that the objective function is convex

* This result was obtained by Mr. C.D.T. Watson-Gandy (Imperial College) using a digital analytic technique.

and that the optimal location of the depot involves a unique solution. Once a set of customers is allocated to a single depot, the method described here will therefore lead to the optimal solution. The program employed in Figure 3.3 was found to be flexible and easy to apply and the advantage of the method discussed here is that it employs a general purpose machine and does not require any equipment specially constructed for this problem.

It should be pointed out, however, that when there is more than one depot, the final solution may well depend on the initial selection of depot locations, since these initial conditions determine the allocation of customers to depots. In such cases it is necessary to repeat the procedure for a number of initial locations and compare the final results obtained in each case.

The number of computing amplifiers required to solve a distribution system with n customers and m centres is approximately $4 + 2m + 3n$ (see Figure 3.6), so that a machine with 100 amplifiers can handle a network involving about 30 customers.

If it is required to determine both the optimal number of distribution centres, as well as their locations, we can perform the analysis for $m = 1, 2, 3, \dots$ distribution centres and find the optimal solution for each case. The optimal results for all these cases are then compared and the lowest value of the objective function among these minima corresponds to the overall optimum.

The analogue method described here can be extended to handle the problem where the cost vs. distance relationship is non-linear. A computer diagram for such a case is shown in Figure 3.7, but notice that the number of amplifiers required to solve this problem with m centres and n customers is approximately $4 + 2m + 5n$.

A special case of the link-length minimization problem occurs where it is required to locate a point among a given set of points in such a way that the maximum distance from this point to any specified points in the network is minimized. This criterion might be adopted in locating emergency services, such as police or fire stations. The analogue technique in this case is similar to that which has been discussed in this paper, except that here the maximum of the distances between linked points must be identified. An analogue

computer diagram for performing this computation is given in Figure 3.8, and an example is given in Figure 3.9 for the case where it is required to locate an emergency centre among five customers.

3.6 CONCLUSION

The method described here involves a general purpose analogue computer for solving the problem of link-length minimization in a network. Although the method is restricted to a distribution system with n customers supplied from a single depot, it can be successfully employed for a multi-depot system, after an initial allocation of customers to depots is made, and the optimum location of the depots is found after a number of iterations. Similarly, this method can be used to determine the optimal number of depots for any given network.

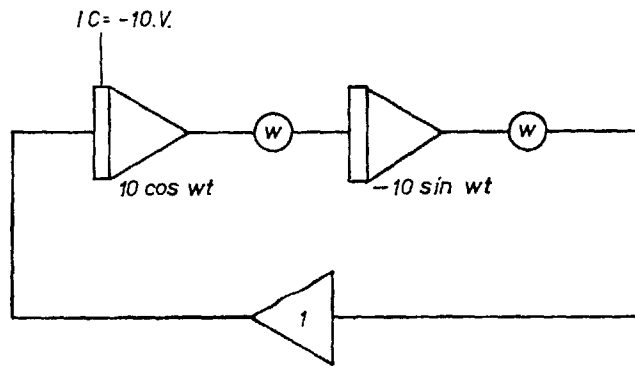


FIG 41. Harmonic oscillator loop

TABLE 31

SYMBOL	COMPONENT	OPERATION
	Inverting Amplifier	$Y = -X$
	Summing Amplifier	$Y = -X_1 - X_2$
	Integrator	$Y = -\int_0^t (X_1 + X_2) dt$
	Potentiometer	$Y = aX$ $0 \leq a \leq 1$
	Rectifier	$Y = -X \quad X < 0$ $Y = 0 \quad X \geq 0$
	Function Generator	$Y = f(X)$

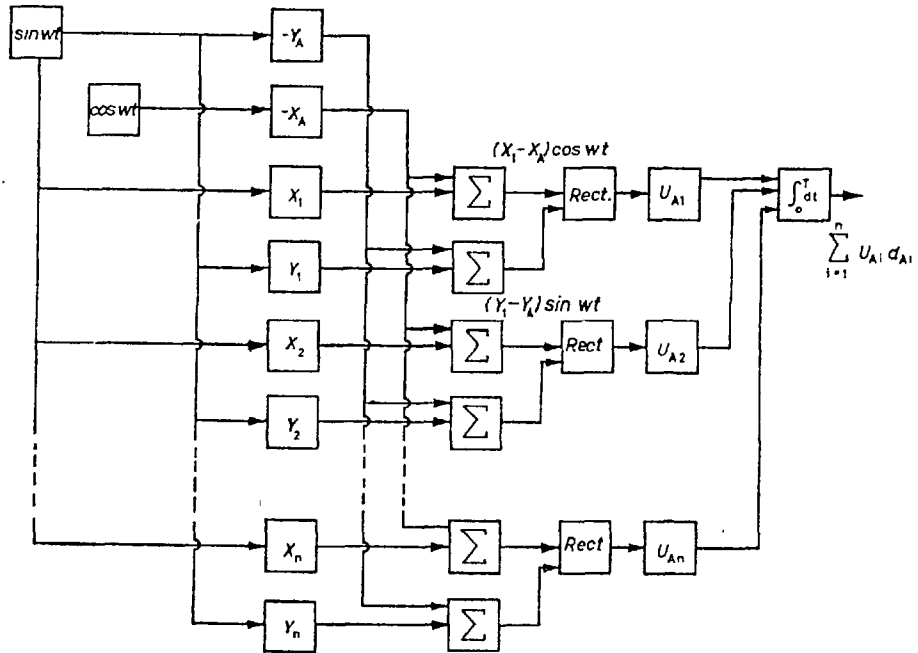


FIG.32 Block diagram for one depot and n customers

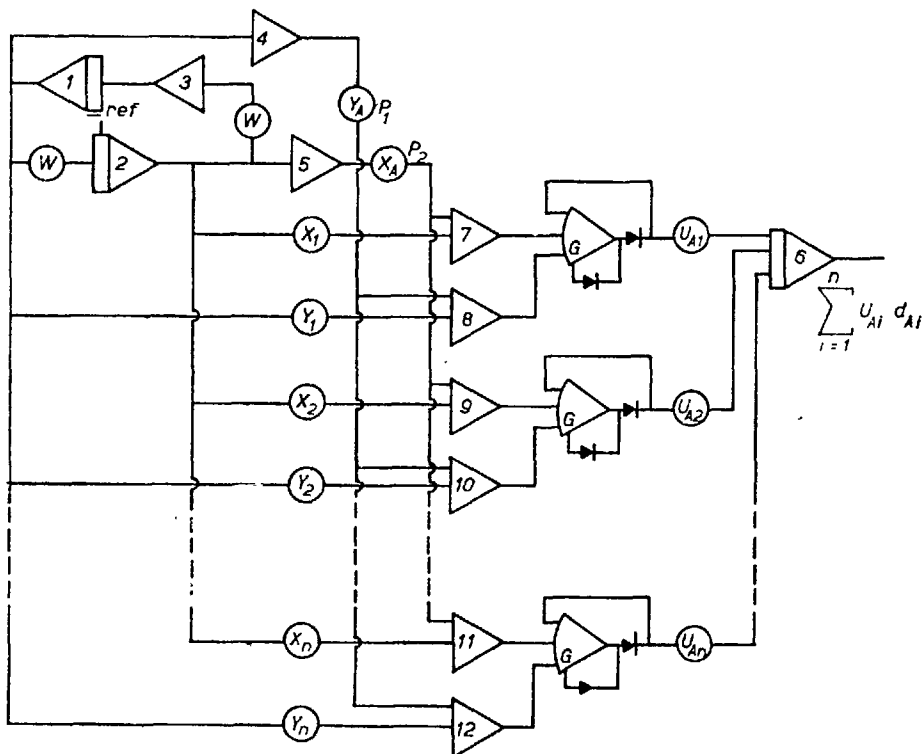


Figure 3.3(a) Computer diagram for one depot and n customers

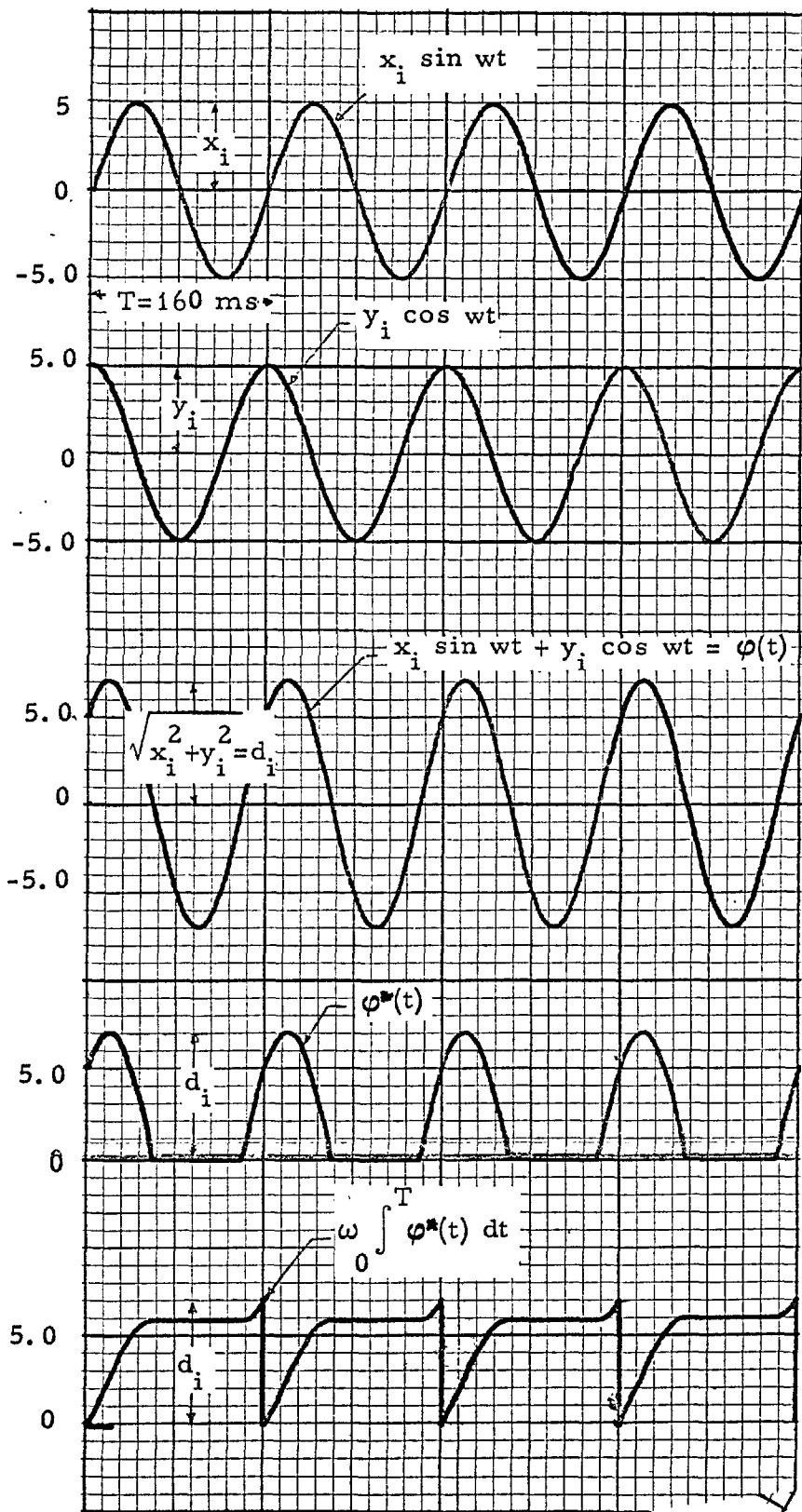


Figure 3.3(b) TYPICAL CIRCUIT WAVEFORMS

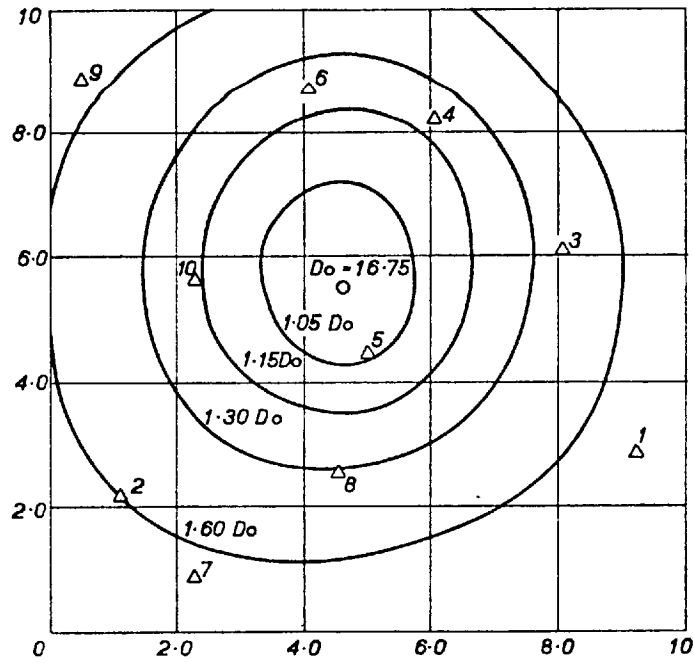


FIG.34 Optimal location and iso-cost contours for a single depot and ten customers. (Δ = customer location).

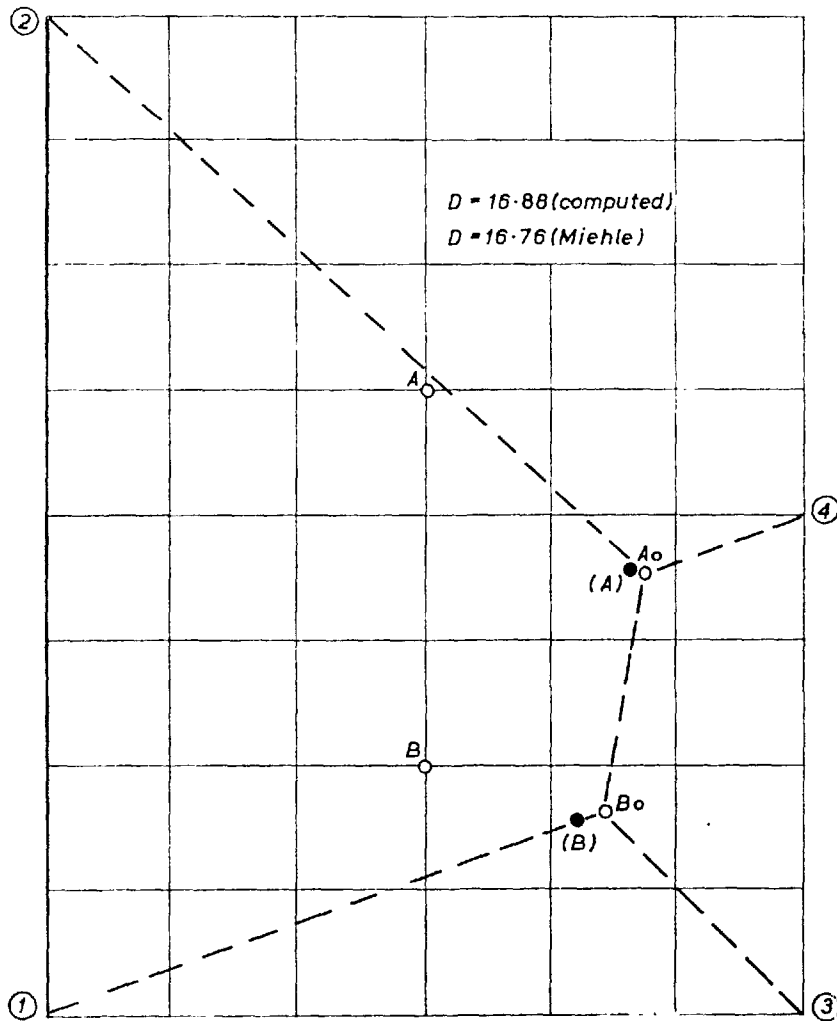


Figure 3.5 RESULTS OBTAINED FOR A SYSTEM WITH TWO DEPOTS, A AND B REPRESENT INITIAL DEPOT LOCATIONS, A_o AND B_o REPRESENT LOCATIONS DETERMINED FROM THE COMPUTER, (A) AND (B) REPRESENT LOCATIONS DETERMINED BY MIEHLE USING A NUMERIC ANALYTIC TECHNIQUE

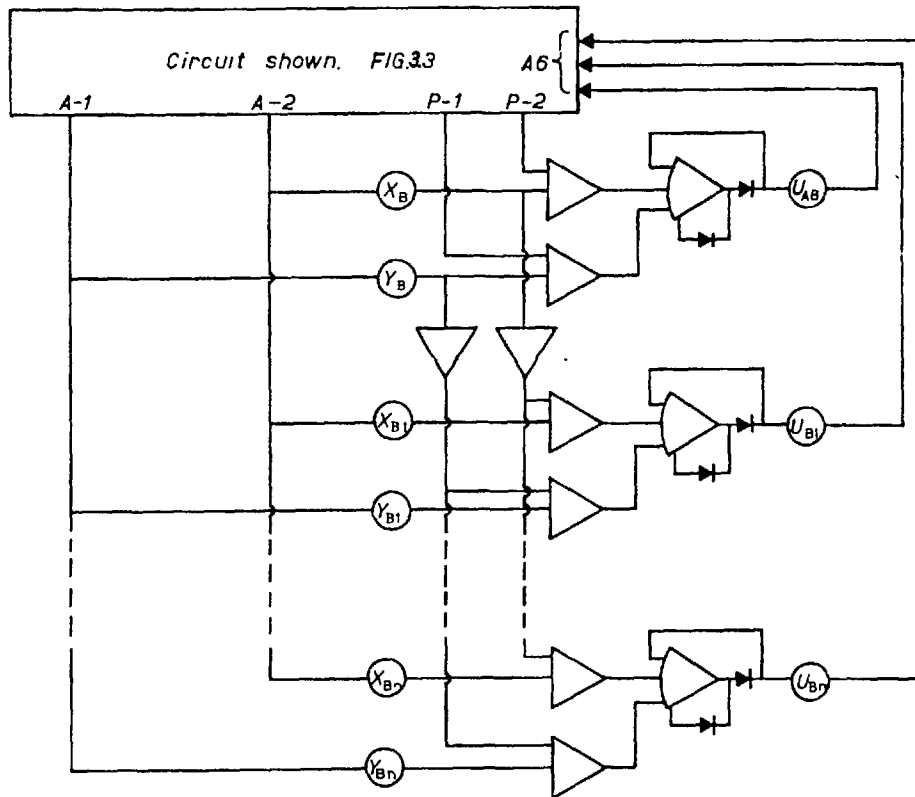


FIG.36. Computer diagram for a two depot system

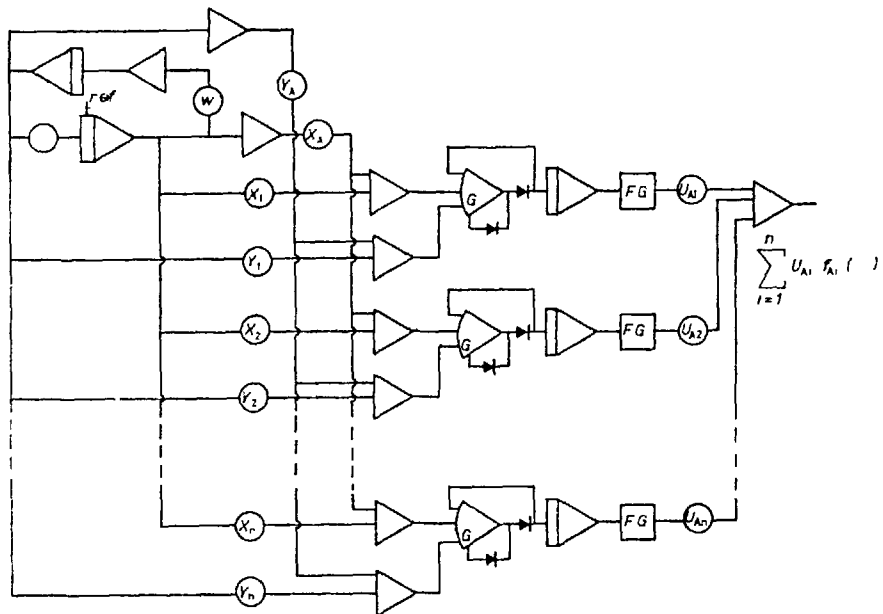


FIG.37 Analogue programme for minimizing D when cost is a non-linear function of distance

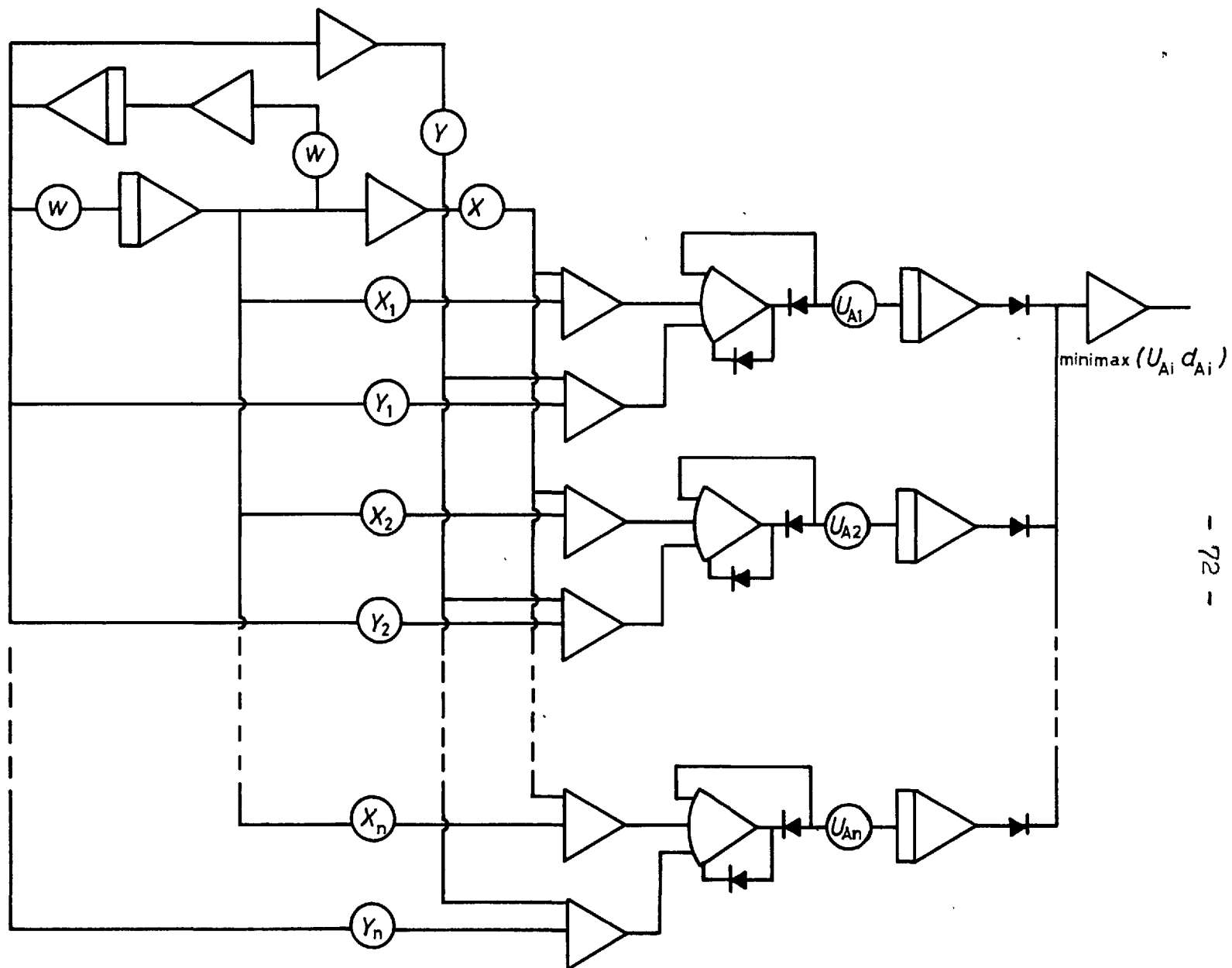


FIG. 3.8 ANALOGUE PROGRAMME FOR MINIMIZING THE MAXIMUM WEIGHTED DISTANCE BETWEEN LINKED POINTS.

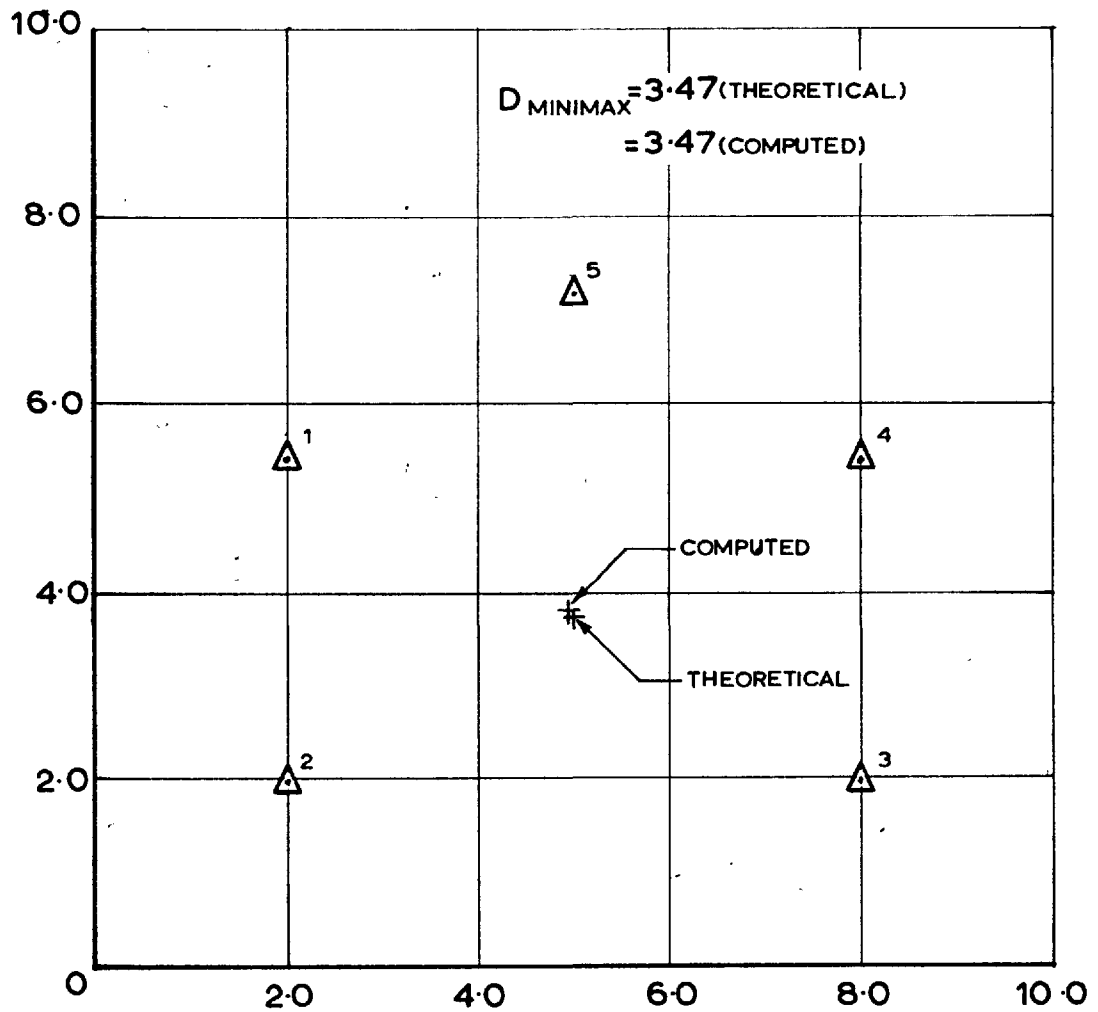


Figure 3.9 MINIMAX CENTRE LOCATION FOR A SINGLE DEPOT AND 5 CUSTOMERS

CUSTOMER No.	1	2	3	4	5
X	2.0	2.0	8.0	8.0	3.0
Y	5.19	2.0	2.0	5.19	7.19

CHAPTER 4

PRODUCTION-INVENTORY
CONTROL SYSTEMS

Notation

t = time

$D(t)$ = demand in period t

$i_o(t)$ = opening stock in period t

$i_c(t)$ = closing stock in period t

$\overline{i_c(t)}$ = average closing stock

$Q(t)$ = quantity re-ordered in period t

$F(t)$ = forecast made in period t of expected demand
in period $t+1$

L = production lag

$i_o(0)$ = initial opening stock level

k = re-order constant

R = **safety stock**

α = exponential smoothing constant

P_o = stockout probability

S_i = variance of variable i

e = exponential function

$\overline{D}$ = average demand

P_n = maximum stockout probability given that a step
has occurred

T_r = rise time

D = demand step magnitude

P_s = probability that a step occurs

S_Q = re-order quantity variance/demand variance

$n(t)$ = normal random variable

S_{ic} = closing stock variance/demand variance

k' = constant (cu-sum tracking system)

h = constant (cu-sum tracking system)

L_a = average run length at acceptable quality level

L_r = average run length at rejectable quality level

m = mean of a random variable

$E(x)$ = expected value of $x(t)$

c = cost

L'_r = run length at rejectable quality level

R_o = rounding interval

ϕ = half dead zone width (non-linear system)

M = constant (non-linear system)

F_∞ = exponentially smoothed forecast with constant α

$X(t)$ = input voltage

$Y(t)$ = output voltage

R_1, R_2 = resistance

C = capacitance

τ_1 = time constant

S = switch

ω_o = sample hold circuit cutoff radian frequency

τ' = time between samples

η = noise power

$R(\tau)$ = auto-correlation function

S_o = variance of sampled waveform

f_o = cutoff frequency of sample hold circuit

A = amplification factor of sample hold circuit

σ_u = standard deviation of the variable u

V = volts

F.M. = frequency modulation

A.M. = amplitude modulation

$S_{uu}(z)$ = power spectrum of the variable u

$h(t)$ = system response at time to an impulse applied
at $t=0$

R_{uu} = auto correlation function of the variable u

$f(t)$ = system time response

Φ = sum of squares of system impulse response

$\delta(t)$ = impulse function

E = forecast error

Ω = ohm

c_1 = cost per unit change in production level

c_2 = stockholding cost

CHAPTER 4 : PRODUCTION-INVENTORY
CONTROL SYSTEMS

4.1 INTRODUCTION

This chapter describes an investigation into the use of an analogue computer for simulating production-inventory control systems. The aim of the investigation was primarily to assess the feasibility of analogue computation as a tool for studying such systems, however much useful information concerning the models tested was obtained in the course of the investigation.

The use of an analogue computer for simulating production-inventory control systems was first suggested by Philips [3] who described a continuous inventory system simulation, while Townsend [4] extended the analogue method to the case where the demand is a continuous variable but supply is discontinuous. Most practical systems however, are strictly discontinuous i.e. all system variables are discrete or only known at discrete intervals. This was noted by control theorists in the 1950's who applied sampled data control theory to the analysis of such systems [51, 52]; However the extension of analogue techniques to cope with discrete systems is a comparatively recent development. Thus Longley [5] and Lewis [53]

have described the use of an analogue for simulating discrete production-inventory control systems while Eilon and Deziel [33, 54] have described the use of an analogue for simulating discrete production-inventory systems with both stochastic and deterministic demands. Much of the material presented here is drawn from the main body of work from which [33] and [54] were synthesised.

In all, four systems were investigated. Each of these is described below along with analogue diagrams and results. In some cases, specialised analogue techniques were used, e.g. data storage, statistical analysis of output data, etc., and details of these are given in Appendices IV, V and VI.

4.2 THE LINEAR SYSTEM

This section describes the use of an analogue computer for simulating a linear production-inventory control system. The system is a particular case of a general model described in [55] where the following assumptions are made:

- (1) The stock level is reviewed at fixed intervals and re-orders are placed on the factory based on; the physical stock level, stock on order

but not yet delivered, forecast demand.

- (2) Orders that cannot be filled at the end of a period are held over until the following period.
- (3) Orders in the pipeline cannot be altered.

It has been shown [51] that where the demand consists of a **stationary** random variable with zero auto-correlation, the linear re-order rule that minimises inventory fluctuations is;

$$Q(t) = R - i_c(t-1) - \sum_{t=L}^{t-1} [Q(t) - E(t)] + E(t) \quad (4.1)$$

where $Q(t)$ = re-order quantity

R = safety stock

$E(t)$ = expected demand in period t

L = lead time

And since for a prescribed stockout probability, the inventory variance is directly related to the amount of safety stock that must be carried, this rule, in effect minimises stock holding costs.

More generally however it may be required to minimise some combination of stockholding cost and re-order quantity fluctuations, subject as before to a prescribed stockout probability. This is

especially true where the production level cannot be adjusted easily or where a cost is incurred in doing so. When this is the case, re-order quantity fluctuations can be reduced at the expense of greater inventory variance by inserting a feedback constant k in the re-order rule. Thus (4.1) becomes

$$Q(t) = k \left[R - i_c(t-1) - \sum_{t=L}^{t-1} [Q(t) - E(t)] \right] + E(t) \quad (4.2)$$
$$0 \leq k \leq 1$$

and by suitable choice of k any cost function involving inventory and re-order quantity variance can be minimised (by expressing these variances as functions of k^*).

It frequently happens in practice that demand patterns are non-stationary. A particular case occurs where the demand consists of a normal random variable with its mean subject to abrupt change (step). Under these conditions, minimisation of inventory and re-order quantities is not enough, since the rate at which the system accommodates itself to changes in the demand average should be reflected in the performance

* A method for doing this is given in [54].

criterion (i.e. in the cost function to be minimised). As a measure of this, the system rise time T_r is defined:

T_r = time required for closing stock to
return to within 0.1 D of its initial
level

where D = step magnitude.

We are now required to minimise some function of the inventory variance, the re-order quantity variance, and the rise time.

Also, since the average demand $\bar{D}$ is no longer constant, we must specify some method of estimating this. Clearly, the choice of best forecasting method ought to be made in the light of the effect this choice will have on system performance and this implies that the forecasting 'method' should be determined by optimising the system objective function over a set of forecasting rules. There is a considerable body of theory which deals with this type of problem,^{*} however the inclusion of 'rise time' in the system objective function substantially complicates the

^{*} optimal control theory [59, 60, 61].

problem and it is doubtful whether application of these techniques would justify the effort involved. Therefore, a simple, and convenient choice of forecast rule has been made, this being simple exponential smoothing.

This section deals with a variant of the basic system where:

- (1) The demand is forecast using simple exponential smoothing with smoothing constant α .
- (2) Re-order quantities are smoothed using a smoothing constant k .
- (3) Demand consists of a step plus noise, where the noise is normally distributed with zero mean and negligible serial correlation.

It is required to determine how α and k should be chosen to optimise some objective function involving re-order quantity variance, closing stock variance, and rise time.

The system is shown schematically in Figure 4.1 where;

$D(t)$ = demand in period t

$i_o(t)$ = physical stock level at the start
of period t .

$Q(t)$ = quantity ordered from plant in period t .

$F(t)$ = forecast in period t of demand
expected in period $t + 1$

L = production lag

t = time : integer

and operation is as follows:

The demand is subtracted from $i_o(t)$ and at fixed intervals (periods) a new re-order quantity $Q(t)$ is computed. This is based on the stock level $i_o(t)$, the forecast $F(t)$, and the stock ordered previously but not yet delivered. $Q(t)$ is then fed to the factory which, after a time lag of L periods, adds $Q(t)$ units of stock to the storage. Thus, $i_o(t)$ is given by:

$$i_o(t+1) = \sum_0^t [Q(t-L) - D(t)] + i_o(0) \quad (4.3)$$

where $i_o(0)$ is the initial opening stock level.

The relationship between $Q(t)$ and the remaining system variables is termed the re-order rule.

In the linear system this is:

$$Q(t) = k \left[R - i_c(t-1) - \sum_{v=L}^{t-1} Q(v) \right] + (1+kL) F(t) \quad (4.4)$$

where,

k = re-order constant

$$\begin{aligned} i_c(t) &= \text{closing stock} \\ &= i_o(t) - D(t) \end{aligned}$$

$$\sum_{t-L}^{t-1} Q(t) = \text{stock on order but not yet delivered}$$

and $F(t)$ is a simple exponentially smoothed forecast given by,

$$F(t) = \alpha D(t) + (1-\alpha) F(t-1) \quad (4.5)$$

where α = constant

It was required to study the effect of parameters α , k , L , R on the system performance when the demand input to the system is made up of two components: one stochastic (a normal random variable) and the other deterministic (a step).

Ideally the system should exhibit the following characteristics:

- (a) provide suitably fast response to an abrupt change in the average level of demand (e.g. short rise time in response to a step.)
- (b) prevent spurious fluctuations in the demand from being transmitted to the re-order quantity (e.g. low re-order quantity variance).

- (c) maintain an average stock level high enough to insure a given stockout probability but not so high as to incur unnecessary stock holding costs.

For L fixed and given demand parameters the rise time and re-order quantity variance are each determined by α and k while the average stock level depends on α , k and R . However if the stockout probability is specified, then R depends on the closing stock variance (see Appendix III). Since this latter depends on α and k alone, R is similarly dependant, and the same applies to the average stock level. Thus the problem narrows to studying the effects of α , k and L on system performance and the remainder of this section describes the result of the study, which was carried out by analogue computer.

A flow graph of the system is shown in Figure 4.2 and the corresponding analogue programme is given in Figure 4.3.[‡] The system response to a step change in demand was of particular interest and this was simulated on the analogue by applying a step waveform to point X in Figure 4.3.

‡ A description of the 'sample hold circuit' used as basic element in this programme is given in Appendix IV.

Figure 4.4 shows the system response to a step change of 100 units, for $\alpha=1.0$, $k=0.5$ and $L=3$. It is apparent from the figure that the step causes $i_o(t)$ to fall $L \times 100$ units below its initial level and finally to rise 100 units above this level, while $i_c(t)$ falls $(L+1)100$ units and finally regains its initial level. On the other hand, $q(t)$ rises abruptly falling back to 100 units above its initial level. Figure 4.5 shows the system step response for various α , k combinations, where deviations are measured relative to the initial values. It is seen from the figure that reducing either α or k increases the time taken for the system variables to reach their equilibrium levels. As a measure of this, we define the rise time (T_r) as the time taken for the closing stock to rise to within 0.1D of its original level, in response to a step, where D is the magnitude of the step.

Also from Figure 4.5 it is apparent that the step response is symmetrical with respect to α and k . (For example, $\alpha=0.25$, $k=0.75$ gives the same result as $\alpha=0.75$, $k=0.25$). This follows from the symmetry of the system transfer function which is demonstrated in [44].

The minimum closing stock deflection is $(L+1)D$ units. It can be seen from equation (4.2) that the minimum is obtained for $\alpha + \alpha k L + k \geq 1$ and this is shown in Figure 4.6 where α, k combinations outside the region enclosed by the axes and the appropriate curve result in minimum deflection.

Figure 4.7 shows the system step response for $\alpha=0$. Here the rise time is seen to be infinite, and the closing stock falls to $(L+1+\frac{1}{k})D$ units below the initial level. When $k=1$ this system is equivalent to the familiar S - T stock control system [55].

System response to stochastic demand can be studied on the analogue in two ways, either by straightforward simulation, or in the case of non serially correlated demand, by the impulse response method.

Statistical simulation is carried out by feeding into the system a voltage waveform with statistical properties identical to the demand pattern being investigated. Methods for generating discrete voltage waveforms with arbitrary distribution functions, auto correlation and variance are described in Appendix V, and Appendix VI describes methods of analysing output data.

The system response to a step plus noise (where the noise is normally distributed with zero mean and zero auto correlation) is shown in Figures 4.8 - 4.10, for various α , k combinations. From the figures it is seen that increasing α and k from 0.25 causes the variance of $Q(t)$ and $i(t)$ to increase, while the rise time is reduced.

The impulse response method yields the input-output variance ratio for a random uncorrelated demand. The method is based on a fundamental theorem of linear systems which states that;

$$S_{yy}(z) = S_{xx}(z) \left| H(z) \right|^2 \quad (4.6)$$

where

$S_{yy}(z)$ = output power spectrum

$S_{xx}(z)$ = input power spectrum

$H(z)$ = system transfer function

.. It can be shown [56] that

$$S_{xx}(z) = \sum_{n=-\infty}^{+\infty} R_{xx}(n) z^{-n} \quad (4.7)$$

where $R_{xx}(n)$ is the input auto correlation function.

If the input is non serially correlated then

$$\begin{aligned} R_{xx}(n) &= K & n &= 0 \\ &= 0 & n &\neq 0 \end{aligned}$$

Where K = input variance

and replacing this in (4.7) gives,

$$S_{xx}(z) = K$$

which substituted into (4.6) yields

$$S_{yy}(z) = K |H(z)|^2 \quad (4.8)$$

Taking the inverse transform of (4.8),

$$R_{yy}(m) = K \sum_{n=m}^{\infty} h^2(n)$$

and for $m=0$ this becomes

$$\frac{R_{yy}(0)}{R_{xx}(0)} = K \sum_{n=0}^{\infty} h^2(n) \quad (4.9)$$

Thus the analogue method involves feeding an impulse to the system being simulated and summing the squares of the resulting system response. An analogue circuit for this computation is given in Figure 4.11. (In practice the sum was found to converge satisfactorily after 50 to 100 iterations.)

Figures 4.12 - 4.17 show some results obtained using this technique. Figure 4.12 shows S_Q (re-order quantity variance/demand variance) for $L=1$ and α , k combinations between 0 and 1. The shaded region in the figure indicates those values of α and k for which there is noise reduction. Figure 4.13 shows a similar

result for the closing stock variance. Here the function is seen to be saddle shaped, with saddle point at $\alpha=k=0.4$. As α and k tend to zero the output variance tends to infinity. Figures 4.14-4.17 show similar results for $L=3$ and $L=6$.

It may be required to determine the effect of α and k on the system rise time (T_r). Where this is the case, T_r can be measured by feeding a step to the system and plotting the response curve, for various α , k combinations. Alternatively the computer can be operated in repetitive mode and the response displayed continuously on an oscilloscope. Figures 4.18 - 4.20 show some results obtained using this technique for $L=1$, 3, and 6.

It is apparent from Figures 4.12 - 4.20 that as L is increased the variance ratios and the rise time increase for given α , k . Thus, in general, the lead time has a pronounced effect on system performance and this is accentuated when α and k are large. Figures 4.21 and 4.22(a) show how S_{ic} , S_Q and T_r vary as functions of L for $\alpha=k=0.25$ and $\alpha=k=1.0$. (In the case of $\alpha=k=1.0$ empirical curves were fitted to the data.)

From the foregoing results it is evident that no single α , k combination results in a system that is optimal in all respects. For example minimum S_Q is obtained for α and $k = 0$, which makes S_{ic} and T_r infinite. Similarly S_{ic} and T_r are minimised at the expense of S_Q . Thus, clearly the 'best' parameter combination will depend on the definition of 'best' system.

Although this definition can only be formulated with reference to a particular system, in some cases it may be possible to specify an index of performance which utilises the variables S_{ic} , S_Q and T_r . When this can be done, the optimal system parameters can be obtained as a straightforward optimisation using the data given in Figures 4.12 - 4.20, and in [44], which contains results for $L=2, 4$ and 5 .

For example, suppose the cost function C has been found to give an adequate indication of system performance where,

$$C = S_Q + S_{ic} + 0.1T_r$$

and it is required to determine the optimal choice of α and k for a system with $L=1$.

Using Figures 4.12, 4.15 and 4.18 we obtain Figure 4.22(b) where cost contours are plotted against α and k . From the figures it is apparent that $C_{\min}=4.65$ and the optimal parameter combination is $\alpha=k=0.38$.

The analogue computer has been used to study a linear production-inventory control system and a method for determining the optimal system parameters has been demonstrated.

4.3 NON-LINEAR SYSTEMS

Non-linear systems are those for which the output cannot be expressed as a simple linear combination of the inputs. Non-linearities may be introduced in a number of ways, notably, through the use of higher order terms in the system re-order rule, the use of tracking systems, and the presence of non-linearities in the basic system elements (for example where there is a limit to the amount a factory can produce). Two non-linear systems were investigated.

- (1) Linear system with cu-sum tracking to monitor the forecast; termed "System with Cu-sum tracking".
- (2) Non-linear system that utilises cu-sum tracking and a dead space element in the re-order rule; termed "Non-Linear System".

System (1) was suggested prior to the investigation while system (2) was developed by the author.

(1) System with Cu-sum Tracking

Figure 4.23 shows a block diagram of the system, and the cu-sum tracking element is shown in detail Figure 4.24.

Essentially, the tracking element compares the demand for a given period with the forecast made one period earlier, and determines the cumulative sum of these errors. When this cu-sum exceeds a specified level, an out of control signal is generated. The cu-sum of the errors is obtained as follows:

- (a) For the positive cu-sum, a constant k' is **subtracted** from the error and the result is added to the sum of values computed previously. Where this sum yields a negative number, it is made equal to zero.
- (b) For the negative cu-sum, k' is added to the error and this result is added to the previous cu-sum as above, with the exception that here, if the cu-sum is positive it is made equal to zero.

Thus we have;

$$S^+(m) = \max \left[0, \sum_{n=0}^m (x(n)-k') \right] \quad (4.10)$$

and

$$S^-(m) = \min \left[0, \sum_{n=0}^m (x(n)+k') \right] \quad (4.11)$$

where

$S^+(m)$ = positive cu-sum

$S^-(m)$ = negative cu-sum

$x(n)$ = input variable

k' = constant

When either cu-sum reaches a specified level, $\pm h$, an out of control signal is generated which resets both cu-sums to zero and initiates some form of corrective action within the larger system.

The choice of parameters k' and h will determine the behaviour of the tracking system; this can be expressed in terms of:

- (a) L_a - the number of periods the cu-sum is expected to run between out of control signals, when the error fed to the system is a random variable with zero mean.
- (b) L_r - the number of periods the system is expected to run without indicating out of control, when the input is a random

variable with mean m .

The relationship between L_a , L_r , k' , h and m has been determined [57, 58] for the case where the forecast error is normally distributed with zero auto correlation. Figure 4.25 shows a nomograph taken from Kemp [57] which gives k' and h for a particular choice of L_a and L_r .

When the cu-sum indicates that the forecasting system is out of control, a number of alternative courses of action may be open to the production controller and some of these are listed below:

- (1) If the forecast is normally held constant and only reversed when the cu-sum triggers,
 - (a) A new constant forecast may be fed to the system based on:
 - (i) The result of some extrinsic marketing information.
 - (ii) The current result of some routine forecast made from period to period (exponential smoothing).
 - (b) A new variable forecast can be fed to the system based on either (i) or (ii) above, and this is replaced by a constant forecast after some elapsed

interval T .

(2) If the forecast is updated from period to period, a new variable forecast may be fed to the system with:

- (a) new initial conditions
- (b) new forecast parameter
- (c) both (a) and (b) together,

where if either (a) or (c) are used, the forecasting system reverts to the original parameter after some elapsed interval T .

Figure 4.26(a) shows the system step response when corrective policy 1(a) is used. The solid line on the figure corresponds to the case where the revised forecast exactly equals the demand while the dotted line shows the effect of an error in the revised forecast.

When the demand consists of a step plus noise, the situation is best described in terms of confidence limits. Figure 4.26(b) shows 99% confidence intervals for $Q(t)$ and $i_c(t)$ for policy 1(a), given that the revised forecast equals the demand.

The confidence limits are obtained by adding $\pm 3\sigma_i$ to the step response up to point A in Figure 4.26 where the demand variance is assumed

to be 1.0, and the variances of $i_c(t)$ and $Q(t)$ are obtained from Figures 4.13 and 4.16. Beyond point A, two situations are possible; (a) that the cu-sum has triggered or, (b) that it has not.

The upper and lower confidence units are now determined from the step response and the 3σ value, on the assumption that either (a) or (b) may have occurred. Beyond point B in the figure, the limits are obtained on the assumption that the systems will certainly have triggered at B.

Figures 4.27(a) and 4.27(b) show analogue programmes for single and double sided cu-sum systems and Figures 4.28 - 4.32 illustrate their use for implementing policies 1(a), 1(b), 2(a), 2(b) and 2(c) respectively.

Figures 4.33 and 4.34 show some results obtained from the analogue for policies 1(a) and 1(b). From the figures it is apparent that the high variability of the process limits the usefulness of the result, nevertheless, it is clear that the effect of using a cu-sum is to reduce the variance of $i_c(t)$ and $Q(t)$ at the expense of a longer rise time (depending on how long it takes the cu-sum to detect a step).

Method 1(a) assumes that an accurate estimate of the step magnitude is available. Where this estimate is in error by E units, the closing stock will fall to $E(L+1)/k$ units below the initial level. Policies 1(b), 2(a), 2(b) and 2(c) remedy this shortcoming, but do so at the expense of increased variance.

These results are summarised qualitatively in Figures 4.35 and 4.36 which show the effect of policies 1(b) and 2(a), on the 99% confidence intervals for $i_c(t)$ and $q(t)$. The result for policy 2(b) is similar to 1(b) except that outside the $l+T$ interval the confidence interval depends on K and the initial α . The result for policy 2(c) is similar to 2(a) with the difference that beyond point A in Figure 4.36 the confidence interval is determined by K and the new value of α .

(2) Non-Linear System

The problem of determining an optimal re-order rule can be formulated in terms of the well known prediction problem [49] which has been extensively treated in the literature for the case of a linear predictor where both the input and the output function to be optimised are suitably defined, [46, 50, 51].

It has been shown [52] that where the aim of the system is to minimise the inventory variance, and the demand input consists of random noise ~~alone~~, then the optimal linear rule is obtained by setting $\alpha=0$ and $k=1$ in equation (4.4). If, however, it is desired to minimise some linear function of the inventory and re-order quantity variances with a random noise input, then the optimal linear rule is obtained for $\alpha=0$ and k between 0 and 1.

On the other hand, when the demand input is assumed to consist of a step plus noise, the situation becomes more complex. Clearly, in order for the inventory level to respond satisfactorily to a step, we must have $\alpha > 0$. However, it is noted that increasing α greatly increases the variance of both $i_c(t)$ and $Q(t)$ when the demand includes random noise.

This section deals with a non-linear re-order rule, where the input to the system is assumed to consist of random noise superimposed on a step (or series of steps) and the output is assumed to be some (unspecified) function of;

- (i) the inventory variance
- (ii) the re-order quantity variance
- (iii) the inventory rise time T_r .

Ideally the non-linear system should exhibit a favourable step response, while at the same time reacting minimally to random noise.

Figure 4.37 shows the basic non-linear system, where the block marked P is the linear system re-order rule (equation 4.4). Operation of the system is as follows; when the demand consists of noise alone, $Q(t)$ is computed as in the linear system and this is fed to the non-linear element shown. The output of this element is zero if $|Q(t)| < \varphi$ and $Q(t) - \varphi$ if $|Q(t)| > \varphi$ (or $Q(t) + \varphi$ if $|Q(t)| < -\varphi$).

This output is multiplied by M and fed to the factory as the amount to be manufactured. During this time $F'_{\alpha} = 0$ and the cu-sum remains inactive (though it may trigger occasionally according to the value of L_a). Figure 4.38 shows an analogue programme for the non-linear system.

When the demand consists of noise alone the system response is as shown in Figure 4.39 where demand fluctuations are seen to be largely self-compensating and changes in $Q(t)$ are relatively infrequent. By contrast, Figure 4.40 shows a result obtained for the linear system with the same demand and $\alpha = 0.15$, $k = 1.0$, $L = 3$.

When the demand consists of noise plus a step, the demand step is ultimately reflected as a change in the average level of $q(t)$. The effect of this is to shift the operating point of $q(t)$ away from the dead zone of the non-linear element, and thus inhibit its masking effect on the noise. A cu-sum tracking system is used to identify and correct this situation.

The following describes how this is accomplished:

- (a) The step is detected by the cu-sum method.
- (b) Its magnitude is estimated (by exponential smoothing).
- (c) The step component is subtracted from $q(t)$ and added to the output of the non-linear element.

The block marked T in Figure 4.37 contains the elements that perform these operations.

Thus, when a step occurs its presence is detected by the cu-sum system which caused Gate 1 to close feeding the output F'_{∞} to the 'long hold' element and from there to the non-linear element and the factory. The cu-sum is then reset and the output of the 'long hold' remains constant at F'_{∞} until the cu-sum again triggers.

Feeding $-F'\infty$ to the non-linear element removes the constant component of input to that element and ensures that the operating point lies somewhere on the flat portion of the characteristic. At the same time $+F'\infty$ is fed to the factory establishing a new average re-order level about which future orders will fluctuate.

An analogue programme for the non-linear system is given in Figure 4.38.

The system step response for $M=1.0$ and $M=1.25$ is shown in Figures 4.41 and 4.42. From these results it is seen that $i_o(t)$ does not return to its initial level, but rather to $i_o(0)-\varphi$. This is consistent with the requirement that changes in $Q(t) \angle \varphi$ should not be transmitted to the production facility. (Note that in the case of a negative step, $i_o(t)$ would return to $i_o(0)+\varphi$) Also from these figures, it is apparent that increasing M reduces the rise time, where rise time is defined here as the time required for $i_o(t)$ to return to within $0.1D$ of $i_o(0) \pm \varphi$.

The system response to a step plus noise is shown in Figure 4.43, and Figure 4.44 shows the linear system response to the same input. It is apparent from these results that the non-linear system reduces the overall fluctuation in $Q(t)$ while at the same time not

appreciably raising the variance of $i_c(t)$.

These results were checked by digital simulation and Figures 4.45 and 4.46 show some results that were obtained.

4.4 LINEAR SYSTEM WITH LOST SALES DUE TO STOCK SHORTAGE

This section deals with the behaviour of the linear system when demand that cannot be satisfied at the end of a period is lost. Hansmann [55] describes an approximate method of solution and derives a re-order rule for the case where a linear function of holding and stockout costs is to be minimised. In some cases, however, it may be necessary to take account of the re-order quantity variance as well as holding and stockout costs. A particular aspect of this problem was investigated by discrete data analogue simulation.

The following assumptions were made concerning system operation:

- (1) Orders that cannot be satisfied at the end of a period are lost.
- (2) Stockout probability is defined as the

proportion of periods in which a stockout occurs.*

- (3) Demand is a normal random variable with known mean and standard deviation σ and zero serial correlation.
- (4) The lead time equals one period.
- (5) The re-order rule is given by

$$Q(t) = k \left[R - i_c(t) + \sum_{t-L}^{t-1} Q(t) \right] + E(t)$$

where

$Q(t)$ = re-order quantity

k = smoothing constant

R = safety stock

$i_c(t)$ = closing stock

$E(t)$ = expected demand = constant

and system variables are expressed in units of σ_D .

It is required to determine values of the system parameters that optimise some objective criterion of

* This definition is an elementary one and may not be generally applicable. However, in the case where production is sold to a single customer, the definition reflects the proportion of time the customer is required to go elsewhere to fill his order requirements. On the assumption that this involves an additional fixed cost of ordering, the penalty cost will be directly related to the stockout probability as defined.

system performance and at the same time provide a 5% stockout probability.

An analogue programme for this system is given in Figure 4.47 and an amplitude distribution analyser (Appendix VI) was programmed along with the system to enable output distributions to be plotted with the simulation in progress. To simplify the analysis computer variables represent deviations of system variables from their initial levels, the latter being made equal to zero on the computer.

Figure 4.48 shows a computer output trace^{*} where stockout probability is plotted against R for k=0.5. From this figure the safety stock required to provide a given stockout probability can be determined (an example is shown for 5% probability) and by repeating this for different values of k, Table 4.1 was obtained. These values of k and R are plotted in Figure 4.49.

* These plots were obtained by using the distribution analyser to determine the proportion of time $i_c(t) < 0.002$ volts. A slight error is incurred here since

$$P\{i_c(t)=0\} = P\{i_c(t) < 0.002\} - P\{0 < i_c(t) < 0.002\}$$

However distribution analysis of $i_c(t)$ shows that the second term in this equation is small and can be justifiably neglected.

TABLE 4.1 VALUES OF k AND R TO PROVIDE 5% STOCKOUT
PROBABILITY

k	1.0	0.75	0.50	0.35	0.25	0.15
R	2.34	2.63	2.39	2.60	2.96	3.28

The system was simulated using values of k and R given in the Table, and Figure 4.50 shows some histograms of $Q(t)$ that were obtained. From these the relationship between R and σ_Q was determined and this is shown in Figure 4.51. Using Figures 4.49 and 4.51 the optimal system parameters can be obtained when the cost function depends on R and σ_Q .

In general the problem is to maximise some cost function of R and σ_Q subject to the constraining relationship given in Figure 4.51. This is illustrated with a simple example where the cost function is linear. Suppose the production facility is geared to operate at a fixed level $\bar{Q}(\bar{t})$ and that production rates either above or below this level incur an additional cost c_1 per unit of change. The average additional cost is then assumed to equal $c_1\sigma_Q$ and if the stockholding cost is c_2 per unit per period the objective function to minimise is;

$$C = \left[c_1\sigma_Q + c_2R \right] \sigma_D + c_2 \frac{\bar{D}}{2} \quad (4.13)$$

where the first two terms are multiplied by σ_D because R and $Q(t)$ have been expressed in units of σ_D .

Since $\bar{D}$ is assumed constant, minimising

$$C' = c_1 \sigma_Q + c_2 R \quad (4.14)$$

also minimises (4.13).

Referring to Figure 4.51 a line with slope $-\frac{c_2}{c_1}$ is tangent to the curve at a point which has as its co-ordinates the optimal combination of R and σ_Q , and from this value of R , the required smoothing constant k can be determined using Figure 4.49. Thus, referring to Figure 4.51, for $\frac{c_2}{c_1} = 1.0$ the optimum parameter values are $R=2.725$ $k=0.48$ which results in a value of $C'=3.325$.

Figure 4.52 shows optimal values of R and k' plotted against $\frac{c_p}{c_1}$. From this result and using Figure 4.51 cost function C' can be plotted against $\frac{c_p}{c_1}$. This is shown in Figure 4.53.

The analogue computer has been used to analyse a production-inventory control system with lost sales due to shortage, and the optimal system parameters for a given re-order rule and cost function have been determined.

4.5 SUMMARY

The analogue computer has been used to analyse four production-inventory control systems. In the first case, a linear system has been studied and results obtained from the analogue have been used to specify optimal system parameters for a Gaussian demand with steps and cost function involving S_{ic} , S_Q and T_r .

A variant of the basic system which employs cu-sum tracking has been studied and analysed. Judged on the results of this qualitative analysis the cu-sum policy 1(a) appears to be a reasonable one. Basically, the method trades off a longer rise time (depending on how long it takes the cu-sum to detect a step) against a reduction of variance in $i_c(t)$ and $Q(t)$. The method however assumes that some reasonably accurate estimate of the step magnitude is available at the time the cu-sum triggers. Where this estimate is in error by E units, the closing stock may fall $E(L+1)$ units below the 'safe' level. Policies 1(b) and 2(a) and (b) attempt to remedy this defect but do so at the expense of increased variance in $i_c(t)$ and $Q(t)$.

The non-linear system attempts to combine the advantages of both the S-T and s-S-T systems but does so at the cost of increased complexity. The inter-relationships between the system variables and parameters are as yet incompletely understood, and the conditions for stability of the system have not yet been explored. Nevertheless, for certain parameter values, the system response is superior to the linear system response in respect of all three variables, S_{ic} , S_Q and T_r .

In the case where sales are lost due to shortage a method for using the analogue to determine optimal system parameters has been demonstrated for a particular case where 5% stockout probability is required.

Experience obtained in this investigation indicates that the analogue computer provides a reliable method for determining production-inventory system response to deterministic and stochastic inputs. Where a number of controlling policies are to be investigated the analogue appears to be particularly suitable in as much as it provides an immediate indication of any prominent features or inadequacies. Within the framework of this requirement the speed and accuracy of the equipment appear to be sufficient.

Difficulties were encountered with the analogue approach when the operations were required that involve detailed logic, (for example where the lead time is a random variable). Although it is possible to incorporate logic elements into the analogue programme, on a general purpose machine these are available only in limited numbers. The simplest solution would be to use a 'hybrid' (digital-analogue) device.

A further difficulty was encountered in attempting to evaluate the statistical properties of the system output. The technique of overlaying traces is useful in this regard but suffers from being slow (the computer must operate at a speed compatible with the pen recorder, about 2 c.p.s.) and cannot approach the accuracy of other techniques.

The use of the analogue itself for analysing output distributions was not completely satisfactory since programming the analyser reduced the machine's capacity for handling an actual problem. As an alternative, a special purpose device might be useful.

The investigation has demonstrated the applicability of analogue computer techniques to the study of production-inventory control systems and elucidated the properties of some systems.

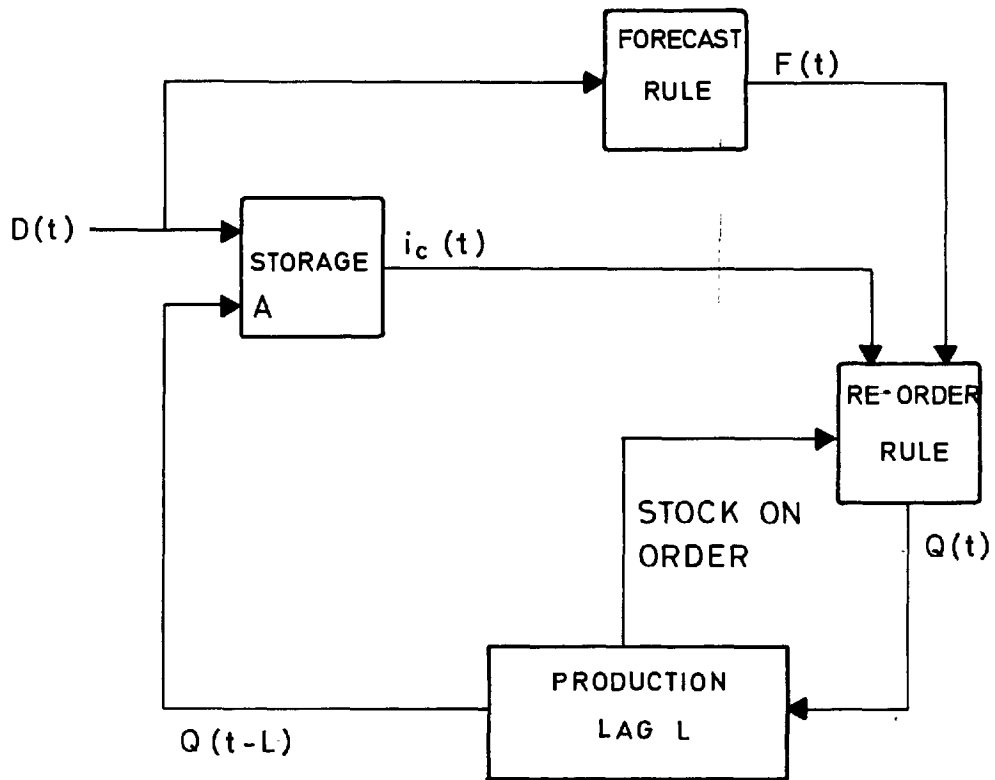
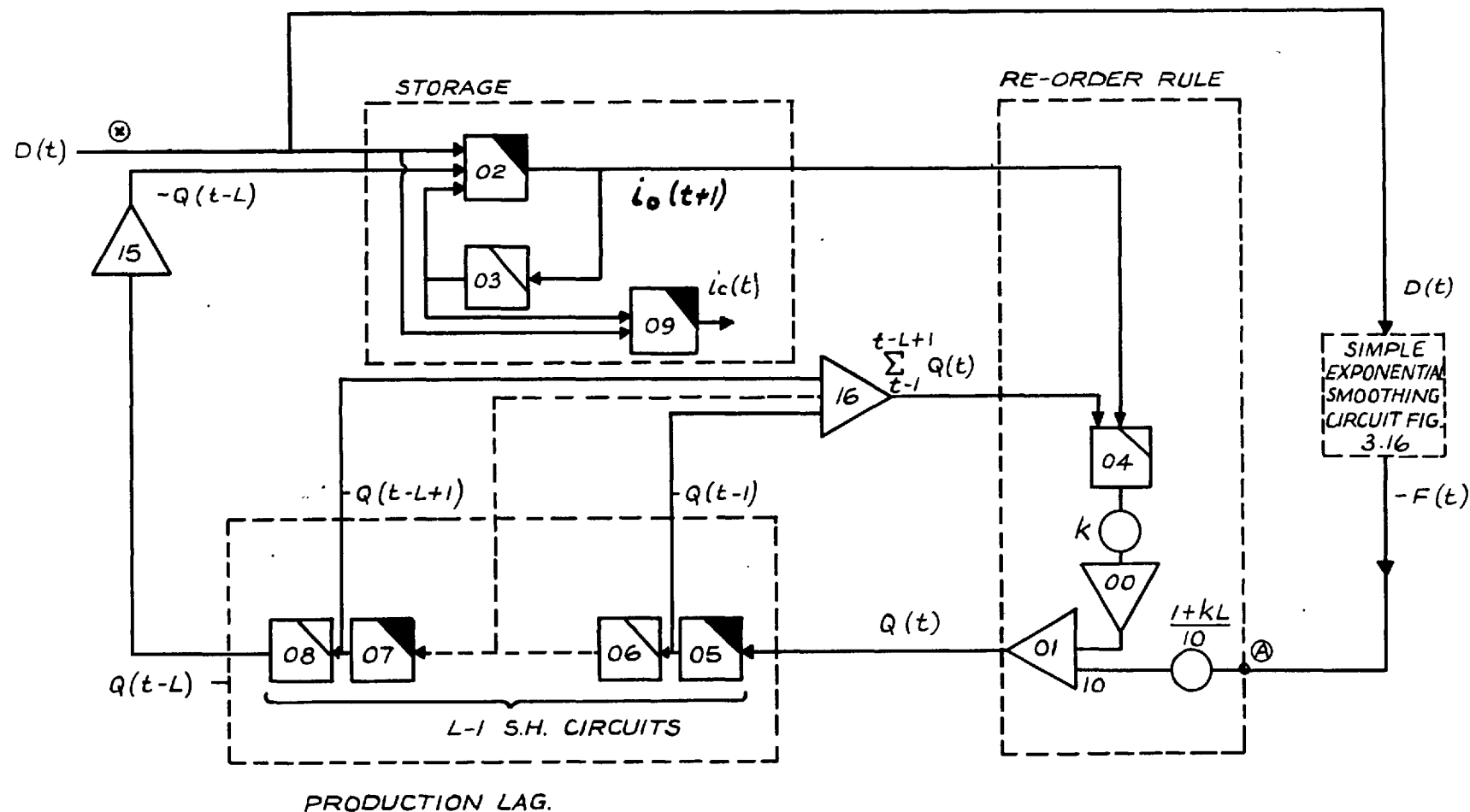


FIG 4.1 PRODUCTION-INVENTORY SYSTEM SCHEMATIC



FIG. 4.2 LINEAR SYSTEM FLOW GRAPH



NOTE: FOR PROGRAMMING CONVENIENCE $R=0$
VOLTAGES REPRESENT FLUCTUATIONS
ABOUT INITIAL SYSTEM VALUES.

FIG. 4,3 ANALOGUE PROGRAMME FOR LINEAR SYSTEM

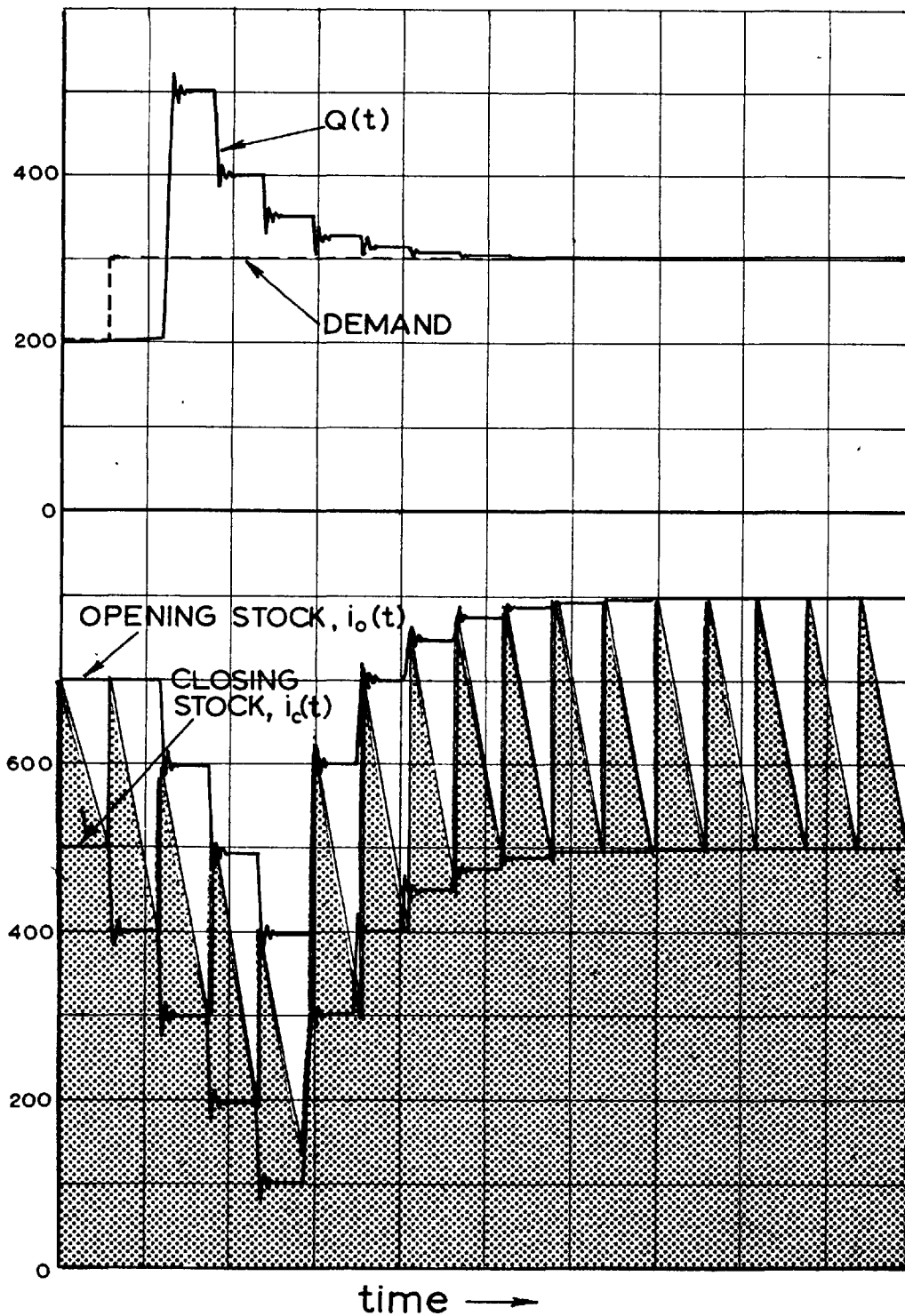


FIG. 4.4 LINEAR SYSTEM RESPONSE TO A DEMAND STEP
FOR $\alpha=0.5$, $k=1.0$, $L=3$

$\alpha = 1.0$

$\alpha = 0.75$

$\alpha = 0.5$

$\alpha = 0.25$

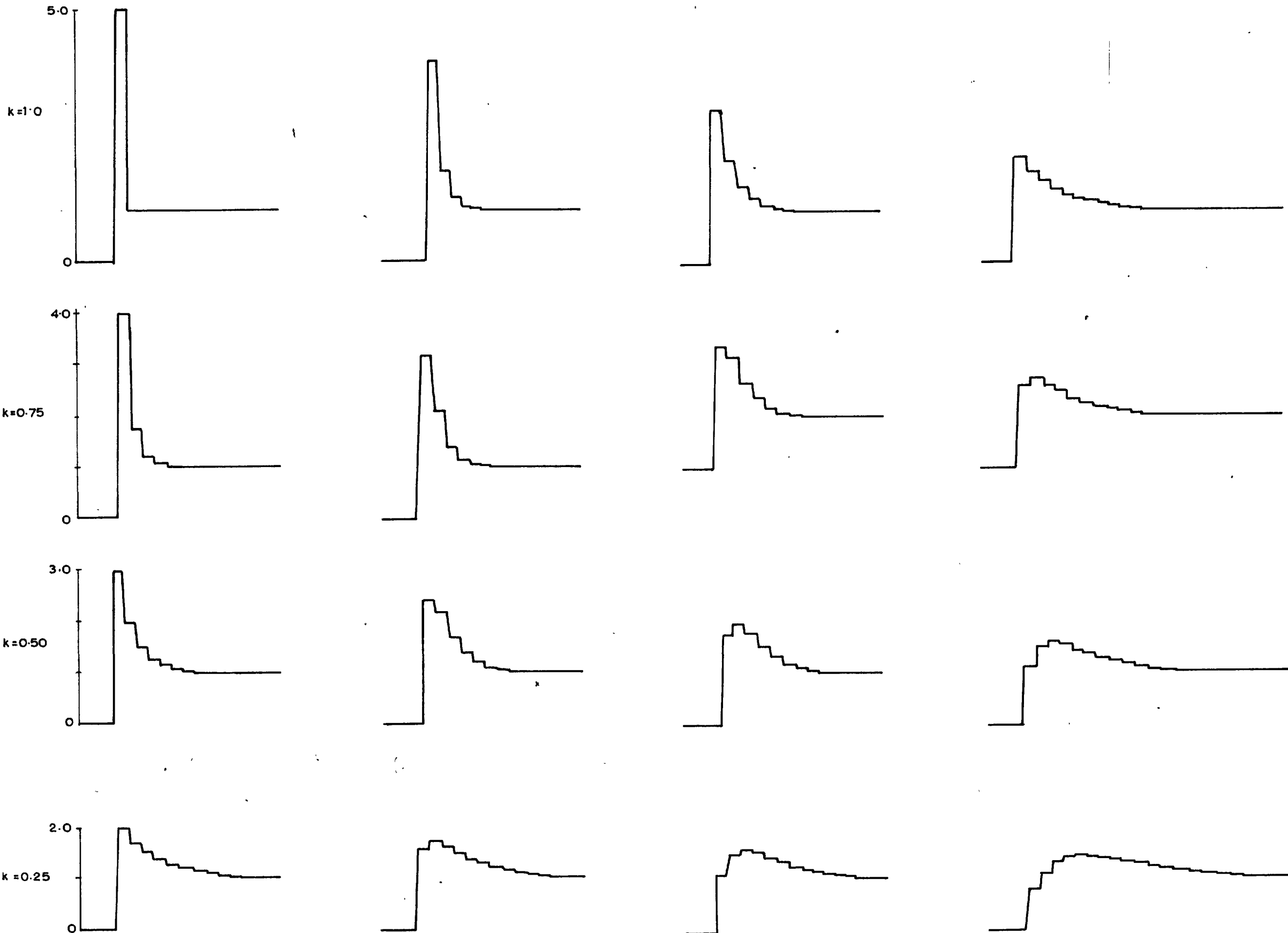


FIG. 4.5a) STEP RESPONSE $Q(t)$ LINEAR SYSTEM $L=3$ FOR SOME VALUES OF α AND k

$\alpha = 1.0$ $\alpha = 0.75$

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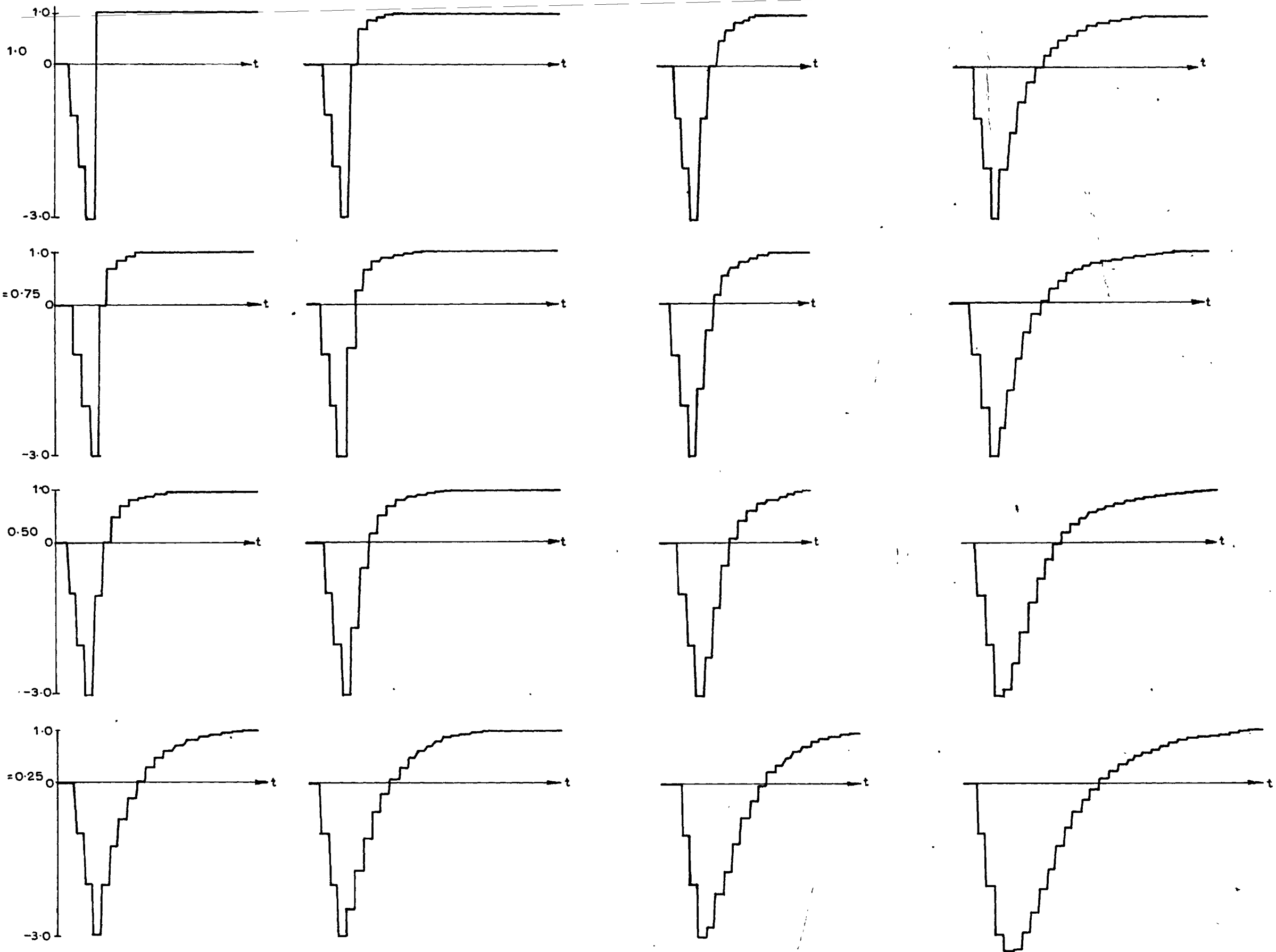
 $\alpha = 0.5$ $\alpha = 0.25$ 

FIG. 4.50 STEP RESPONSE $i(t)$ LINEAR SYSTEM $L=3$ FOR DIFFERENT VALUES OF α AND k

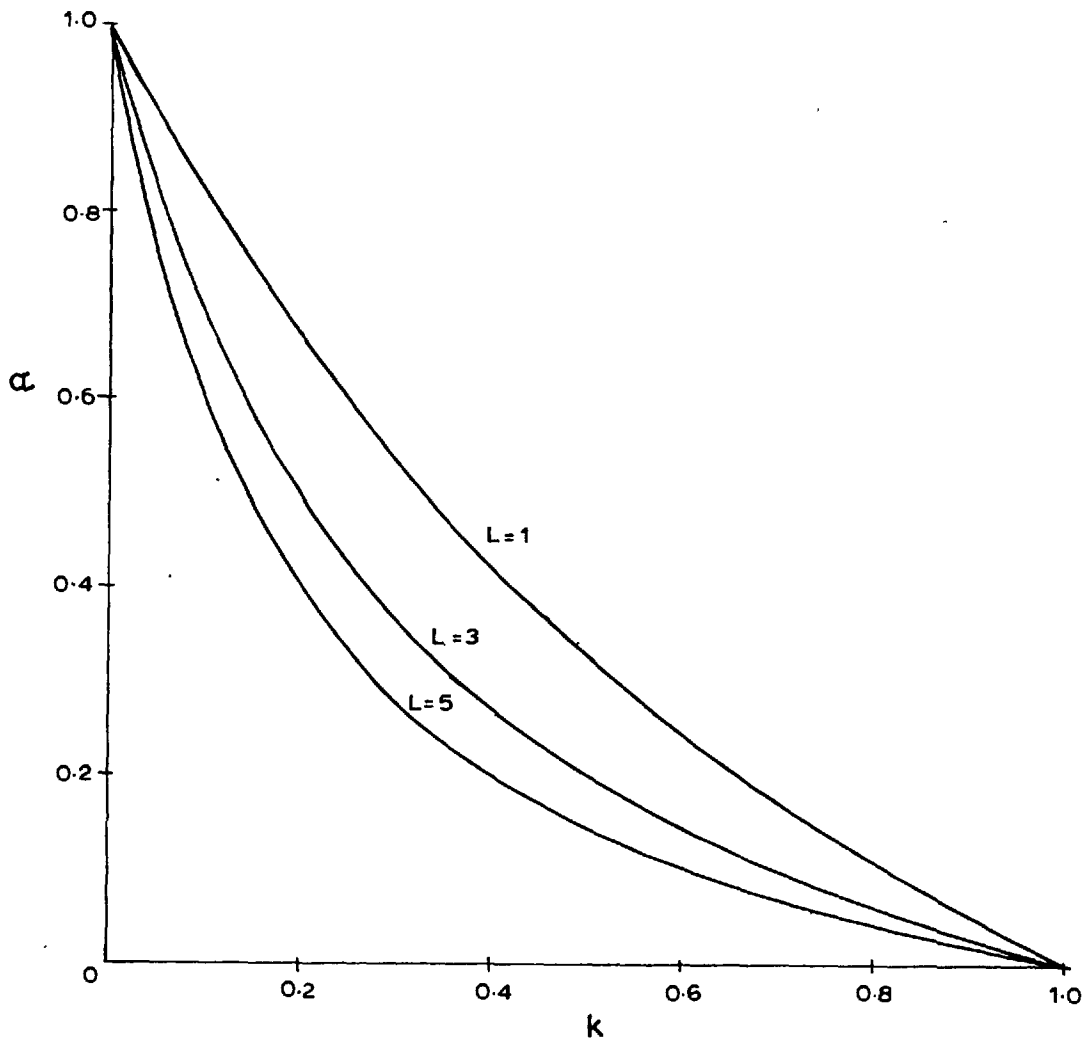
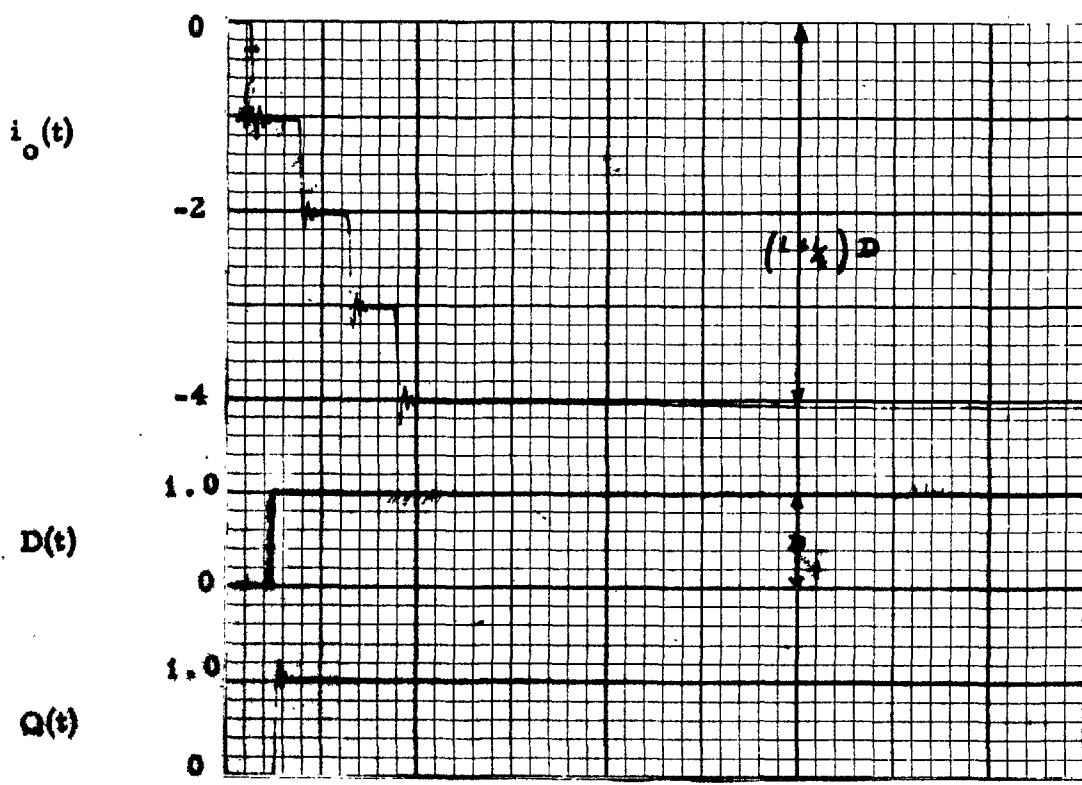
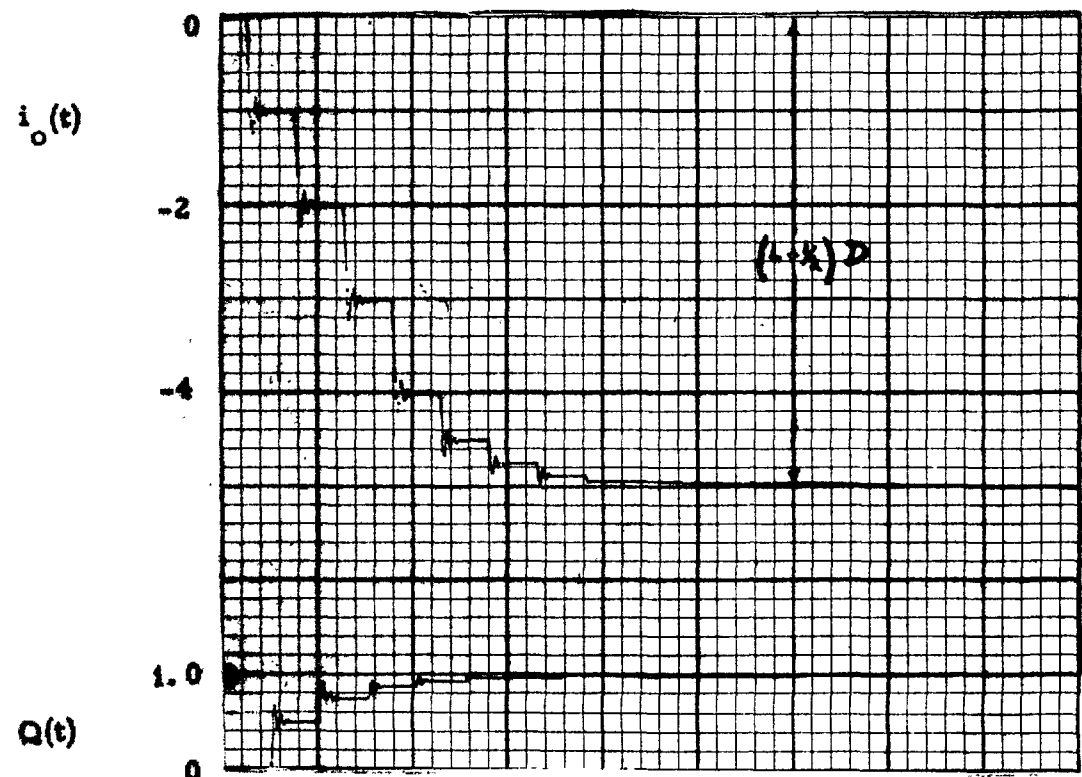


FIG. 4.6 α , k combinations outside the region enclosed by the axes and the appropriate curve result in minimum deflection of $i_c(t)$



(a)



(b)

time

FIG 4.7 LINEAR SYSTEM STEP RESPONSE
 $\alpha = 0, L = 4, 'a'k = 1.0, 'b'k = 0.5$

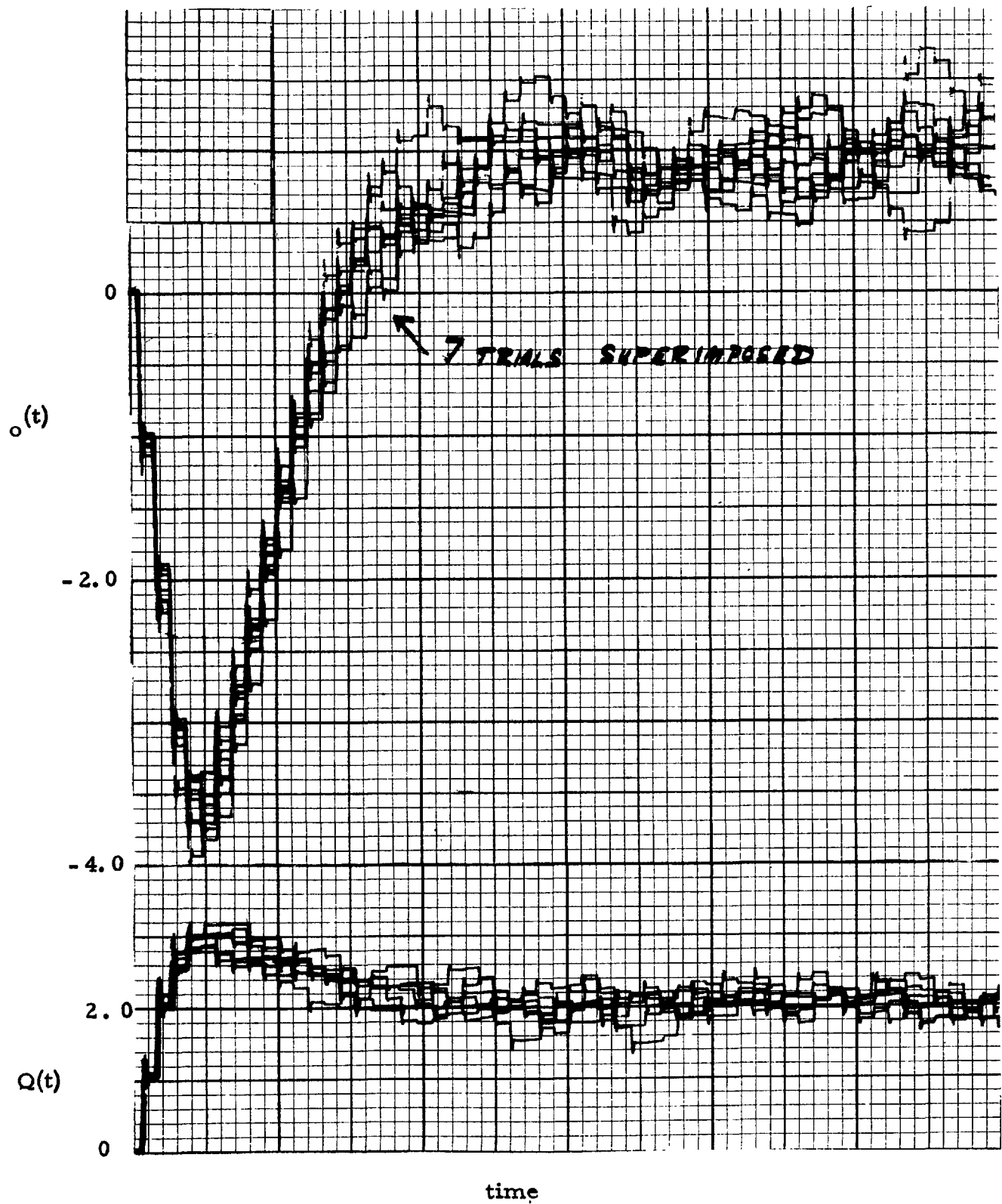


FIG 4.8 LINEAR SYSTEM RESPONSE TO A UNIT STEP PLUS NOISE
 $\sigma_D = 0.1$, $\alpha = 0.25$, $k = 0.25$, $L = 4$

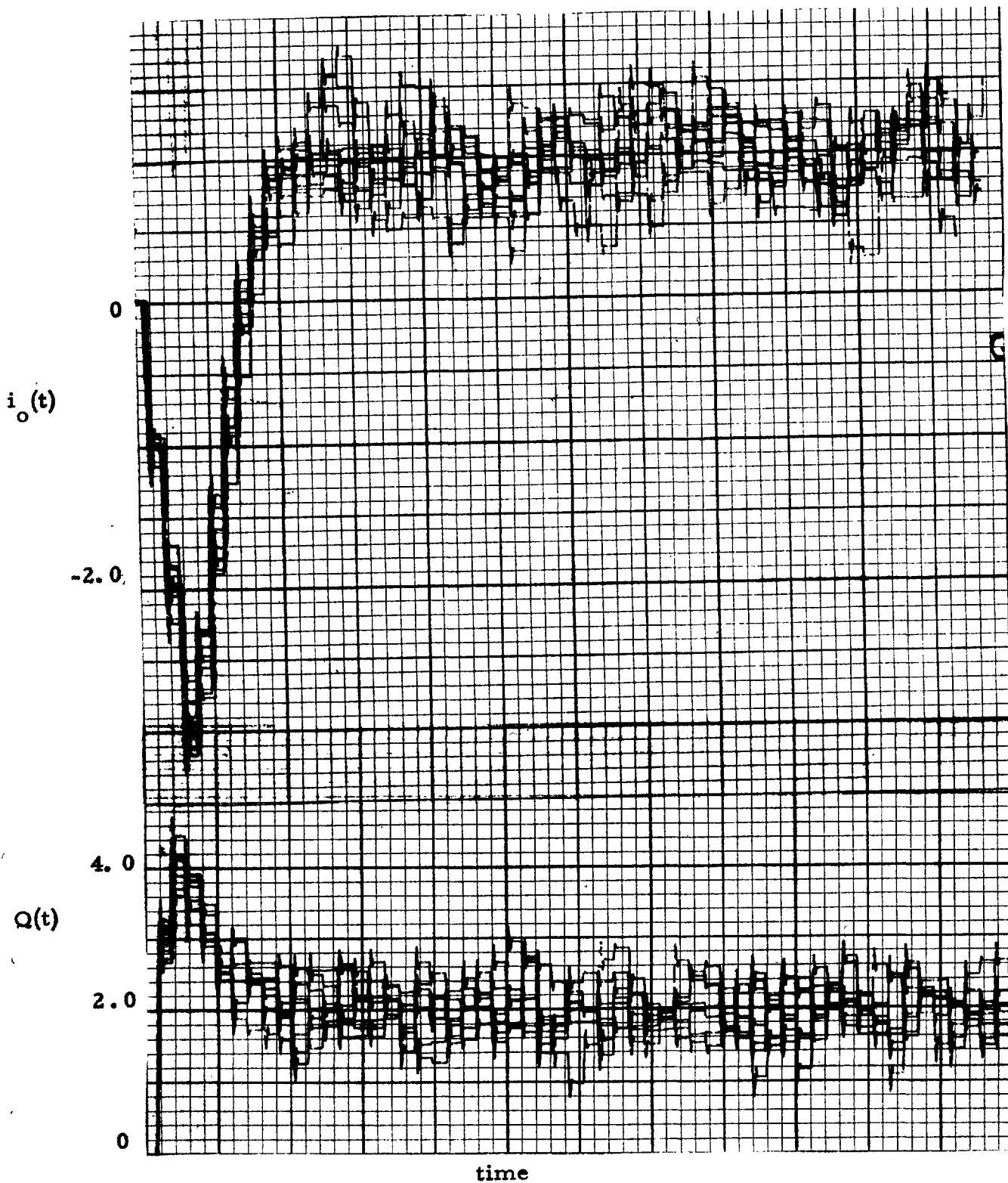


FIG 4.9 $\alpha = 0.50$, $k = 0.50$ (see Fig 1.5)

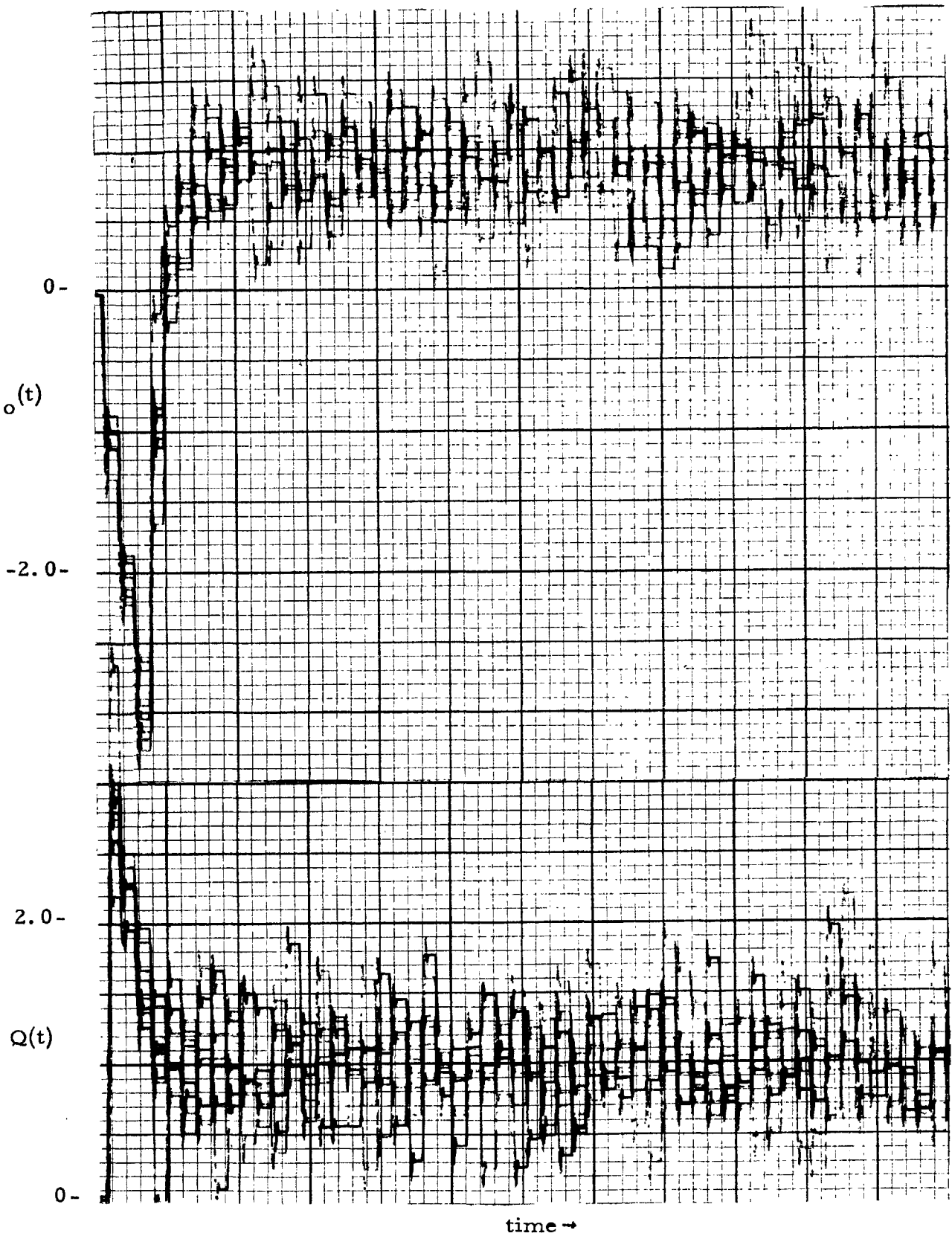


FIG 4.10 $\alpha=0.75$, $k=0.75$ (see Fig 1.5)

IMPULSE FUNCTION GENERATOR.

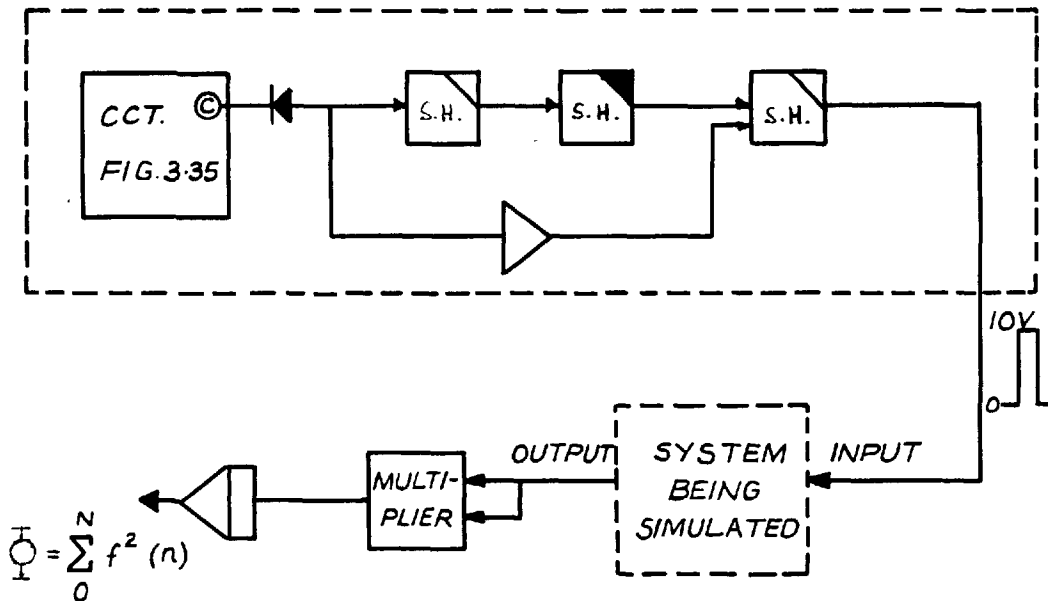


FIG. 4.11. CIRCUIT FOR OBTAINING THE SUM OF THE SQUARES OF THE IMPULSE RESPONSE

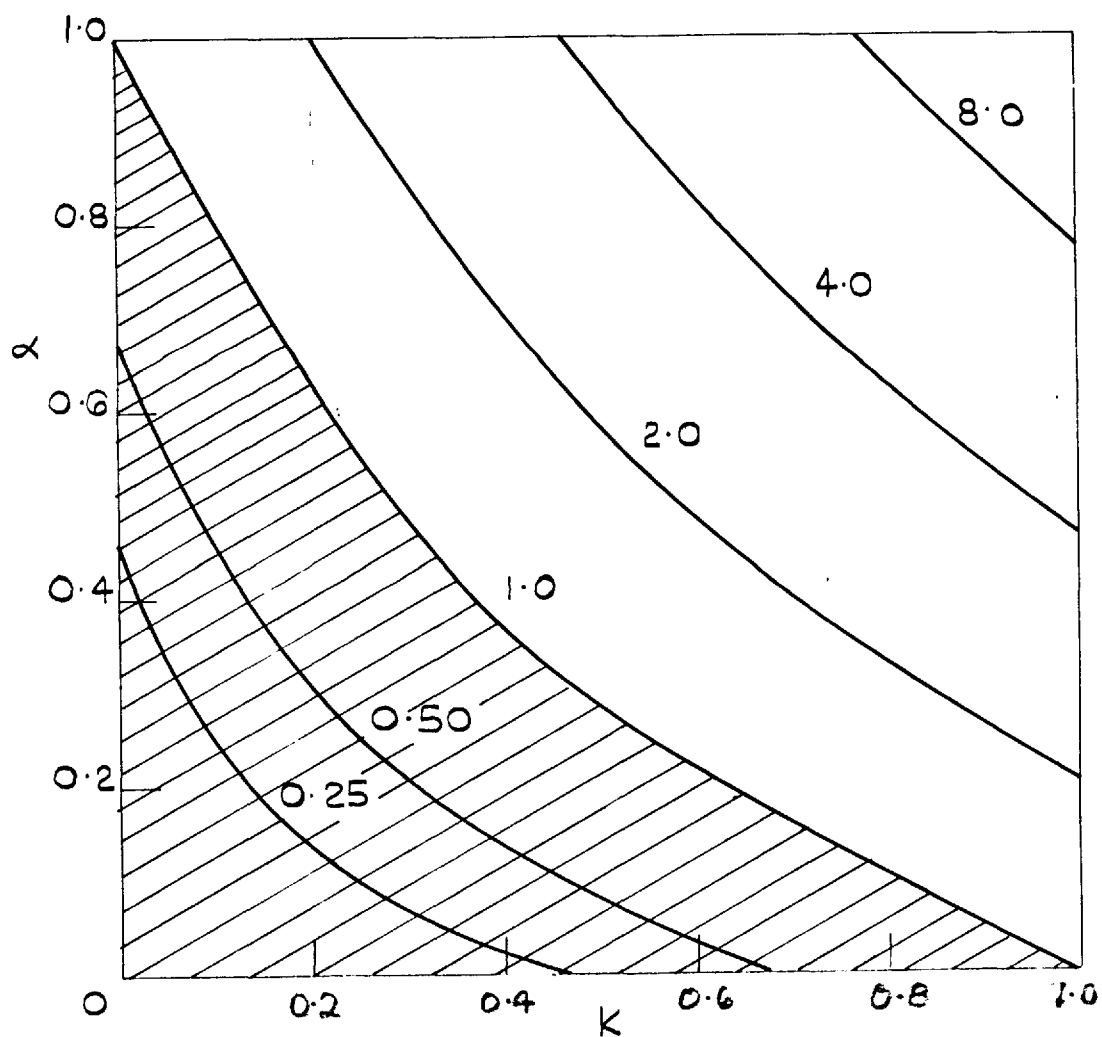


FIG. 4.12 ISOMERS OF S_Q , LINEAR SYSTEM $L = 1$

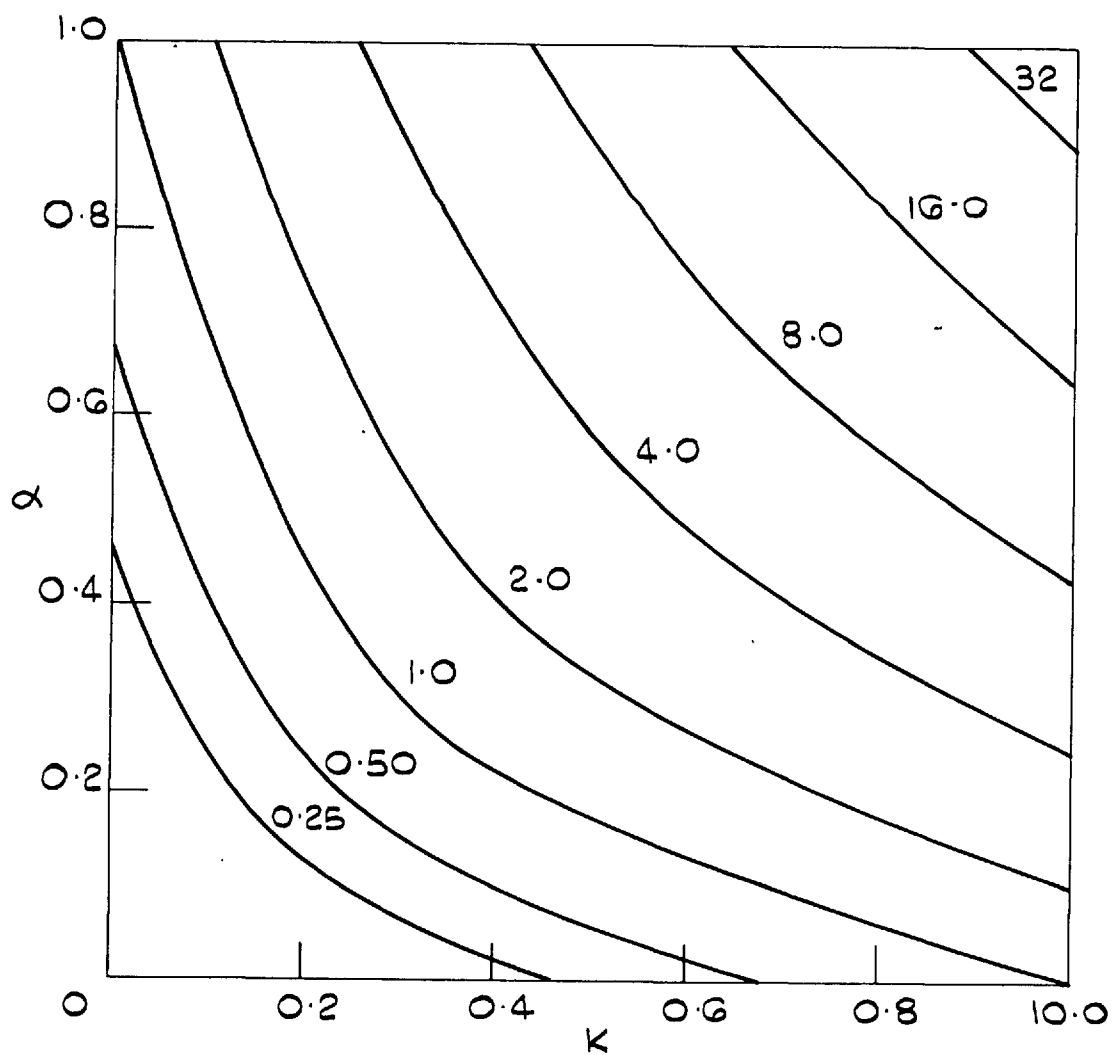


FIG. 4.13' ISOMERS OF S_Q LINEAR SYSTEM $L=3$

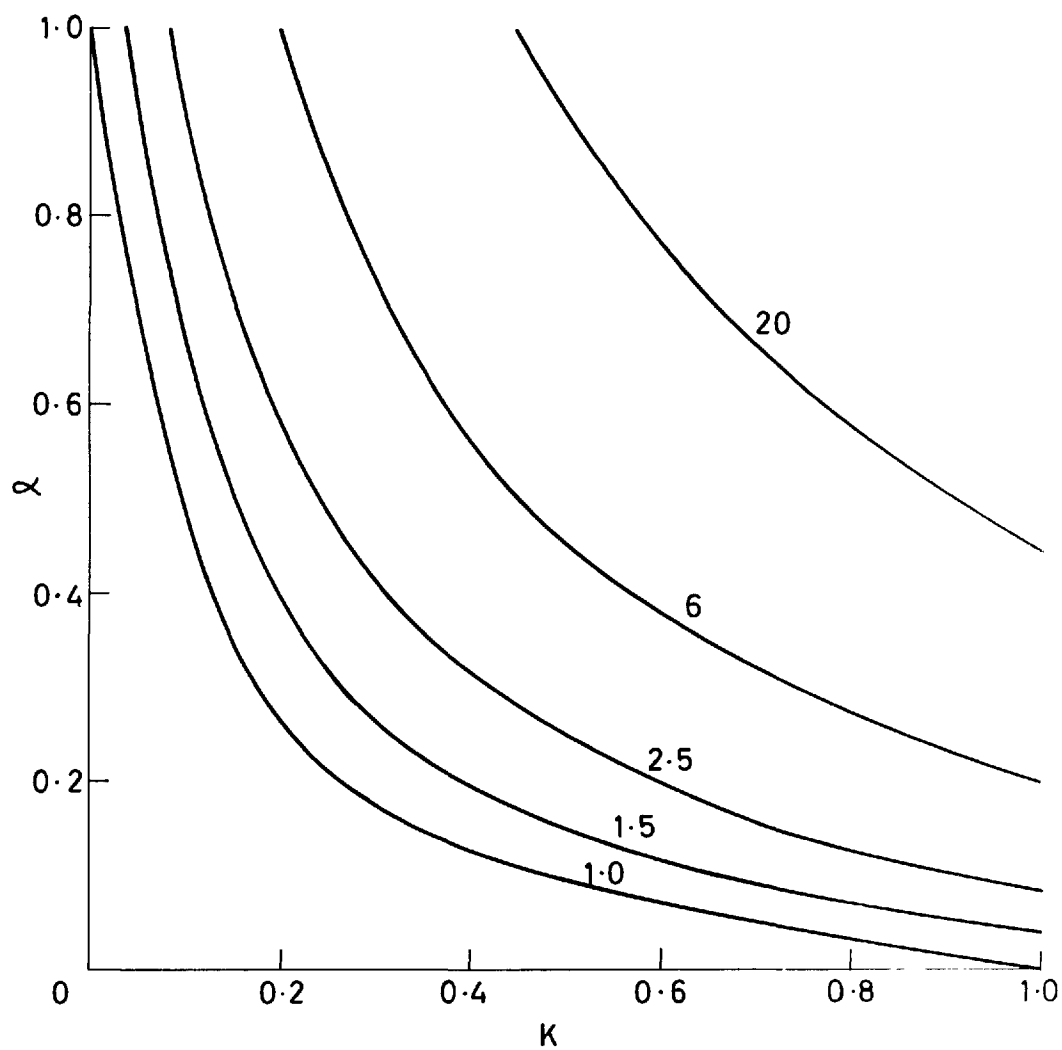


FIG. 4.14 ISOMERS OF S_Q LINEAR SYSTEM $L=6$

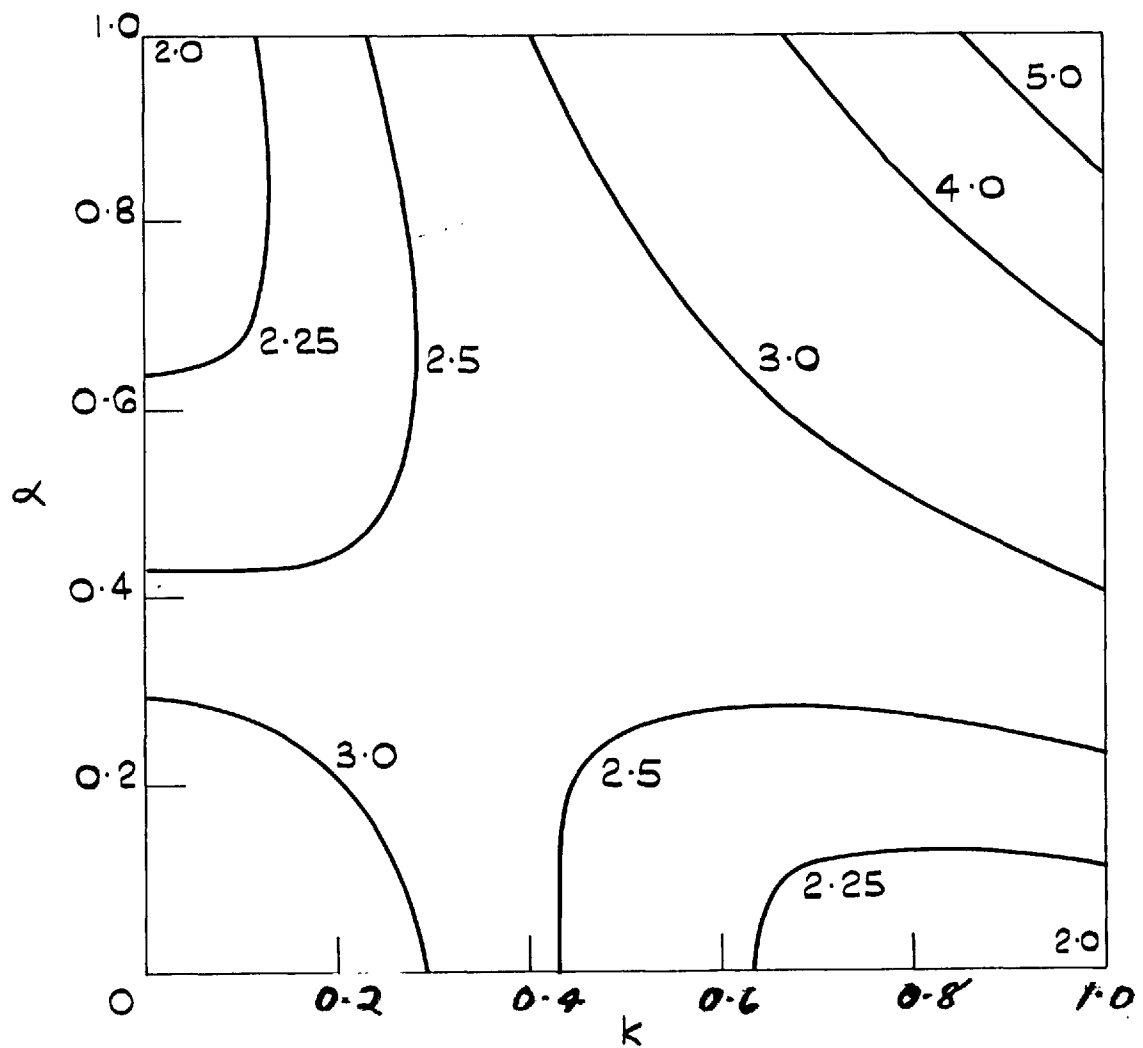


FIG. 4.15 ISOMERS OF S_{ic} LINEAR SYSTEM $L = 1$

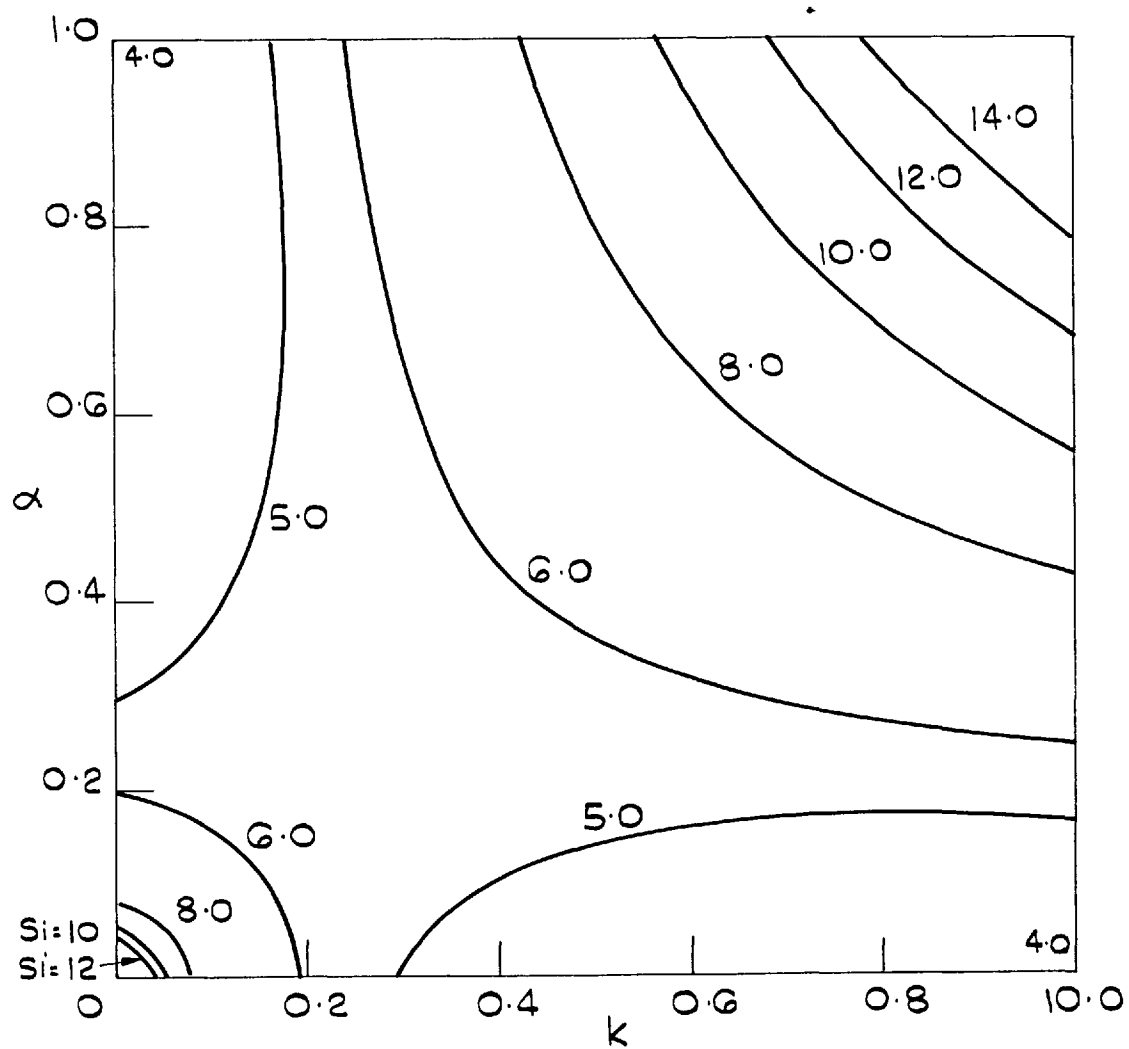


FIG. 4.16 ISOMERS OF S_{ic} LINEAR SYSTEM $L = 3$

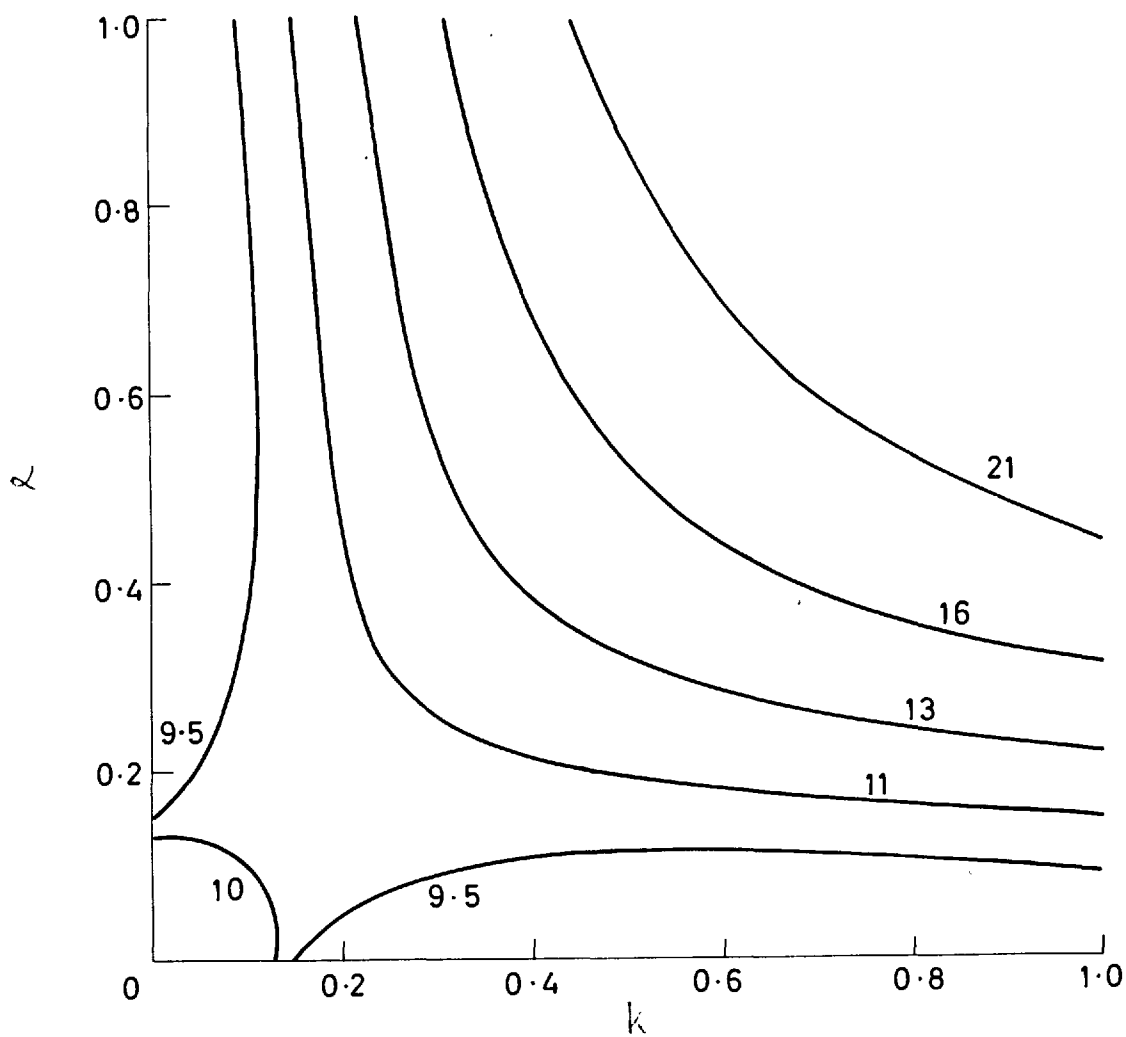


FIG. 4.17 ISOMERS OF S_{ic} LINEAR SYSTEM $L=6$

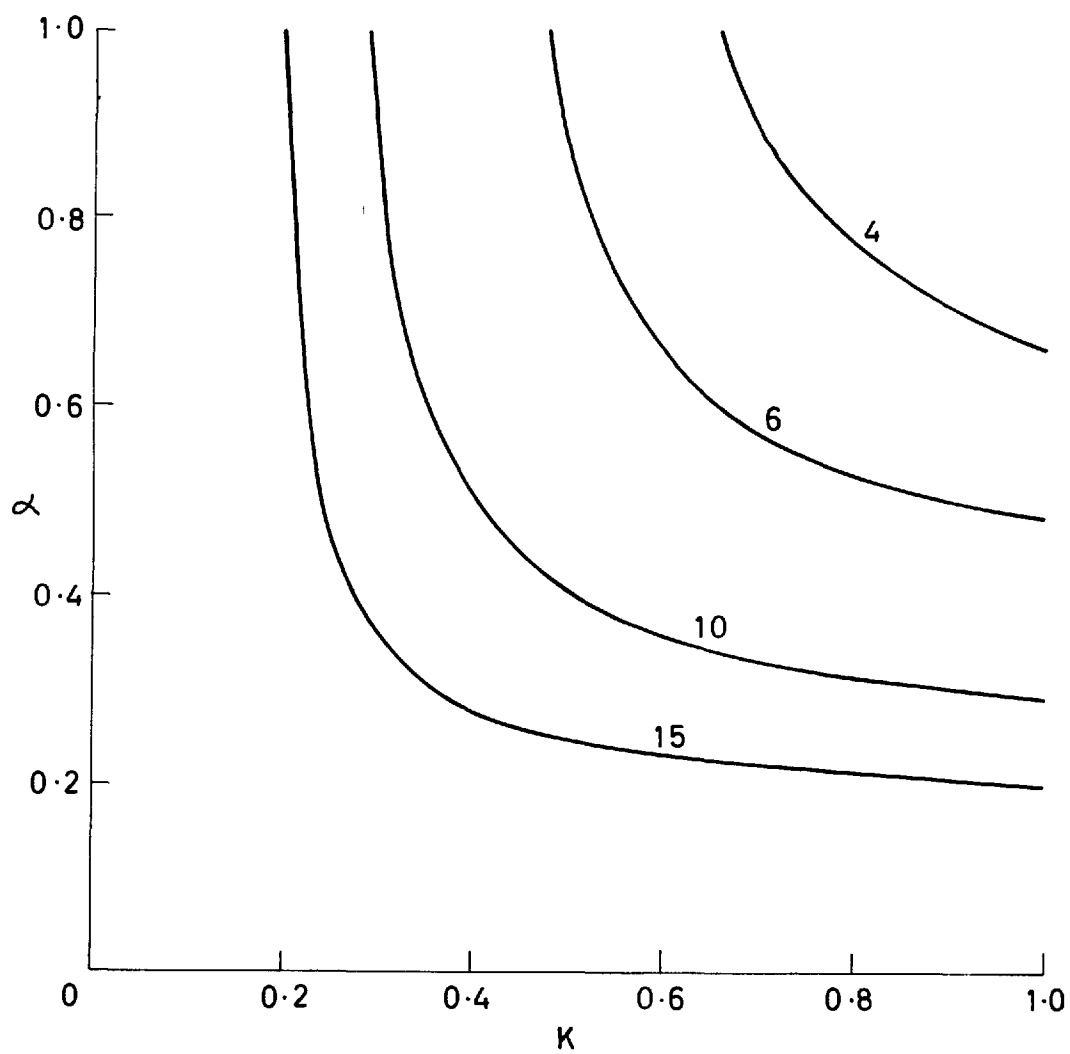


FIG. 4.18 ISOMERS OF T_r LINEAR SYSTEM $L=1$

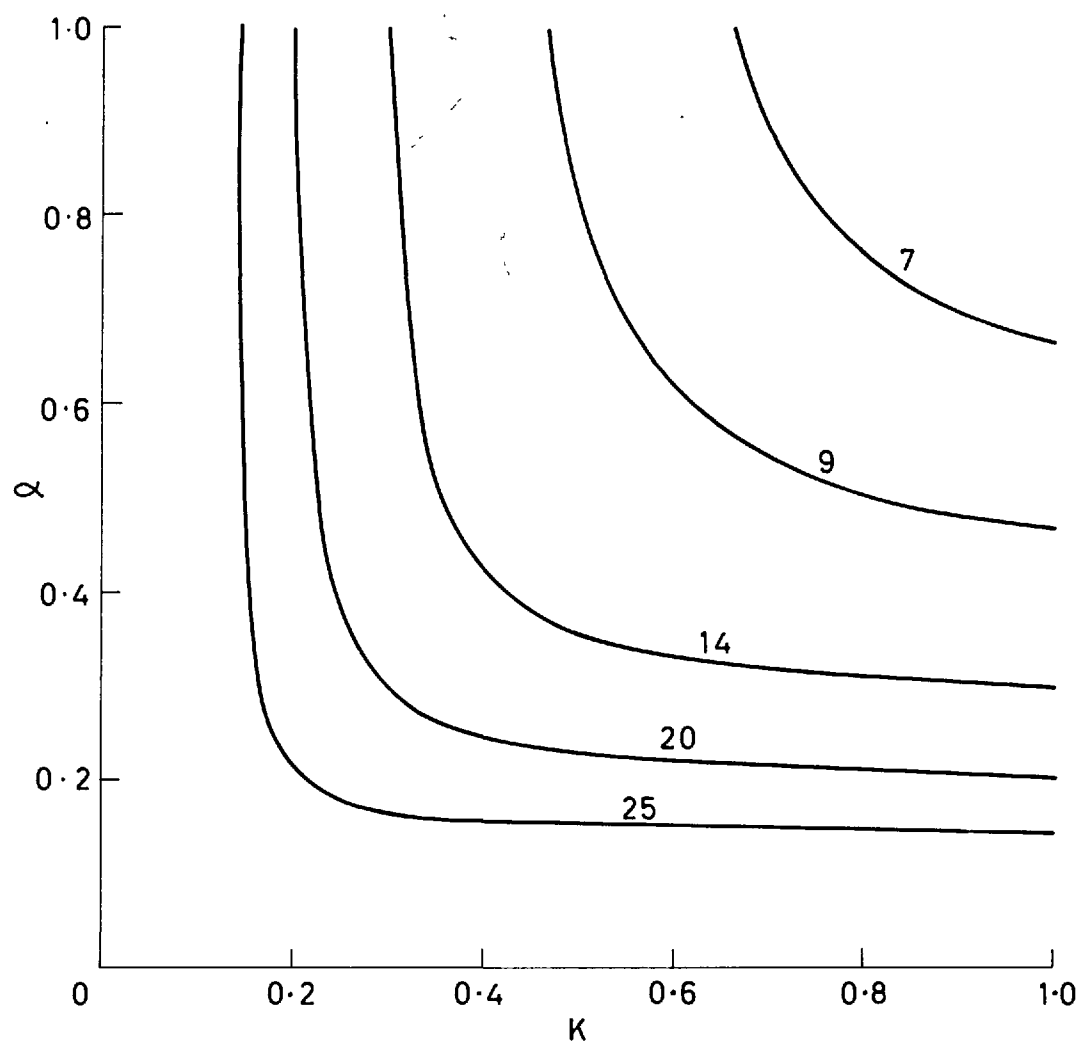


FIG. 4.19 ISOMERS OF T_r LINEAR SYSTEM $L=3$

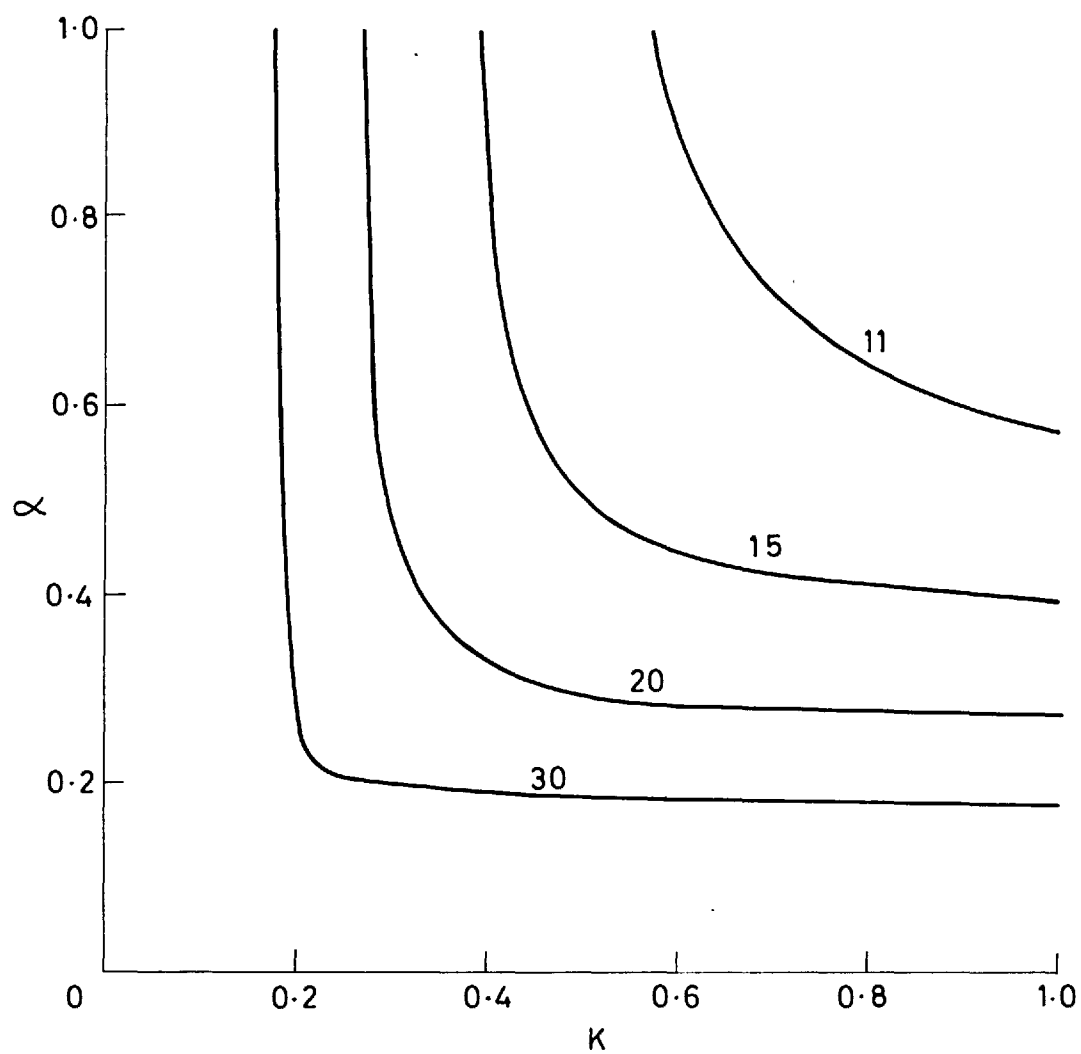


FIG. 4.20 ISOMERS OF T_r LINEAR SYSTEM $L=5$

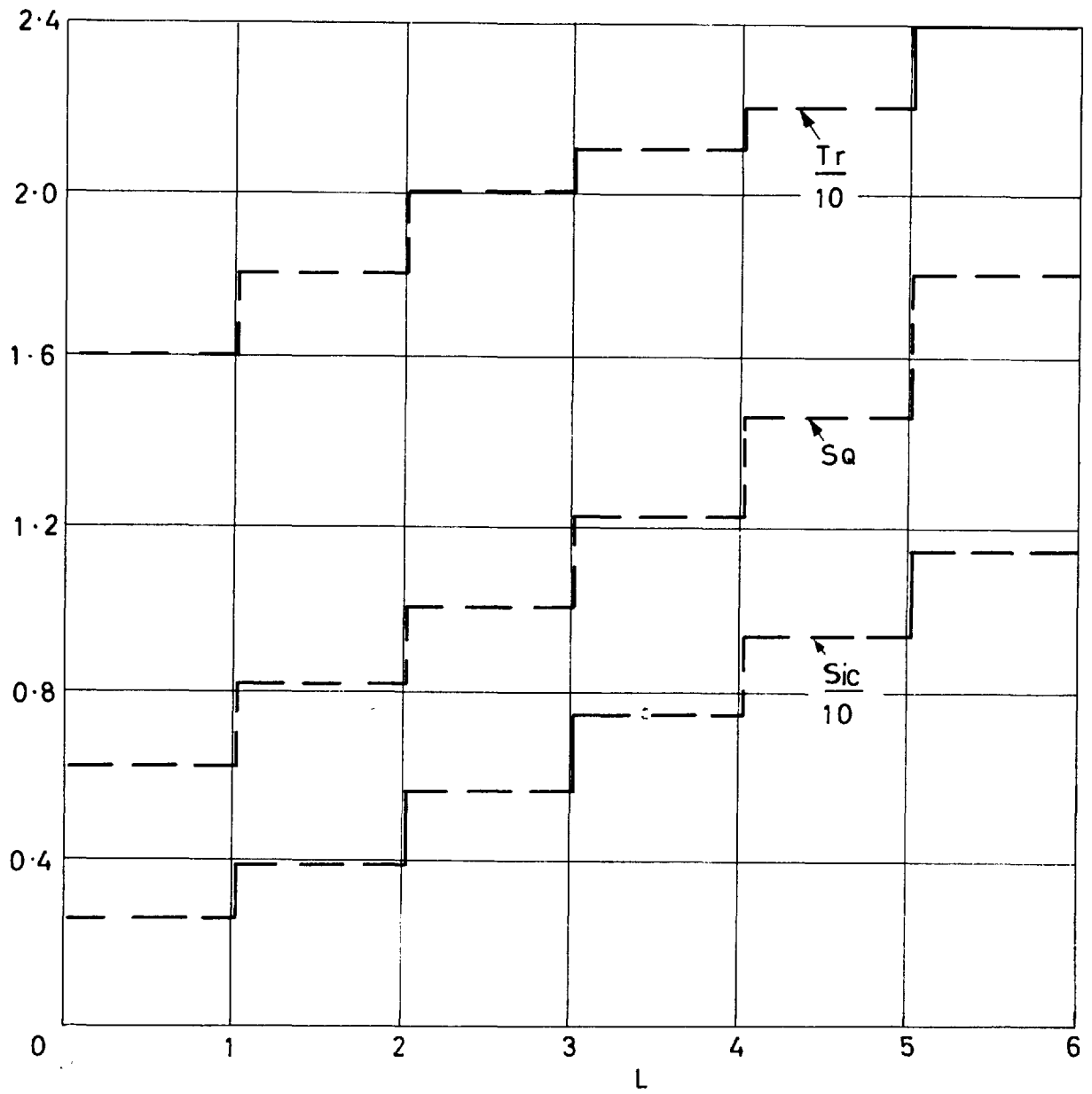


FIG. 4.21 VARIATION OF S_{ic} , S_Q , AND T_r WITH L FOR $\alpha=k=0.25$

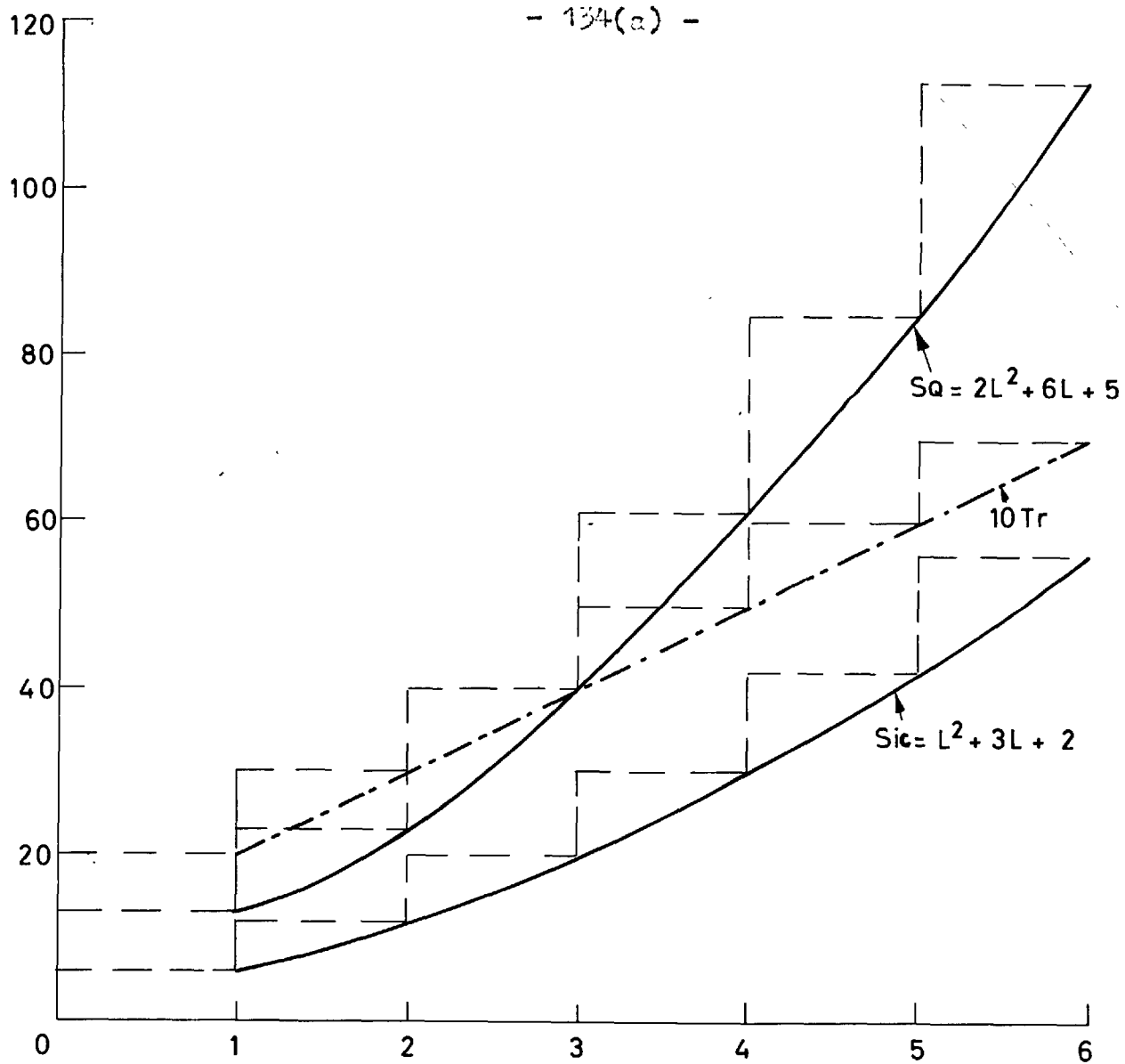


FIG. 4.22(a) VARIATION OF S_{ic} , S_Q AND T_r for $\alpha=k=1.0$

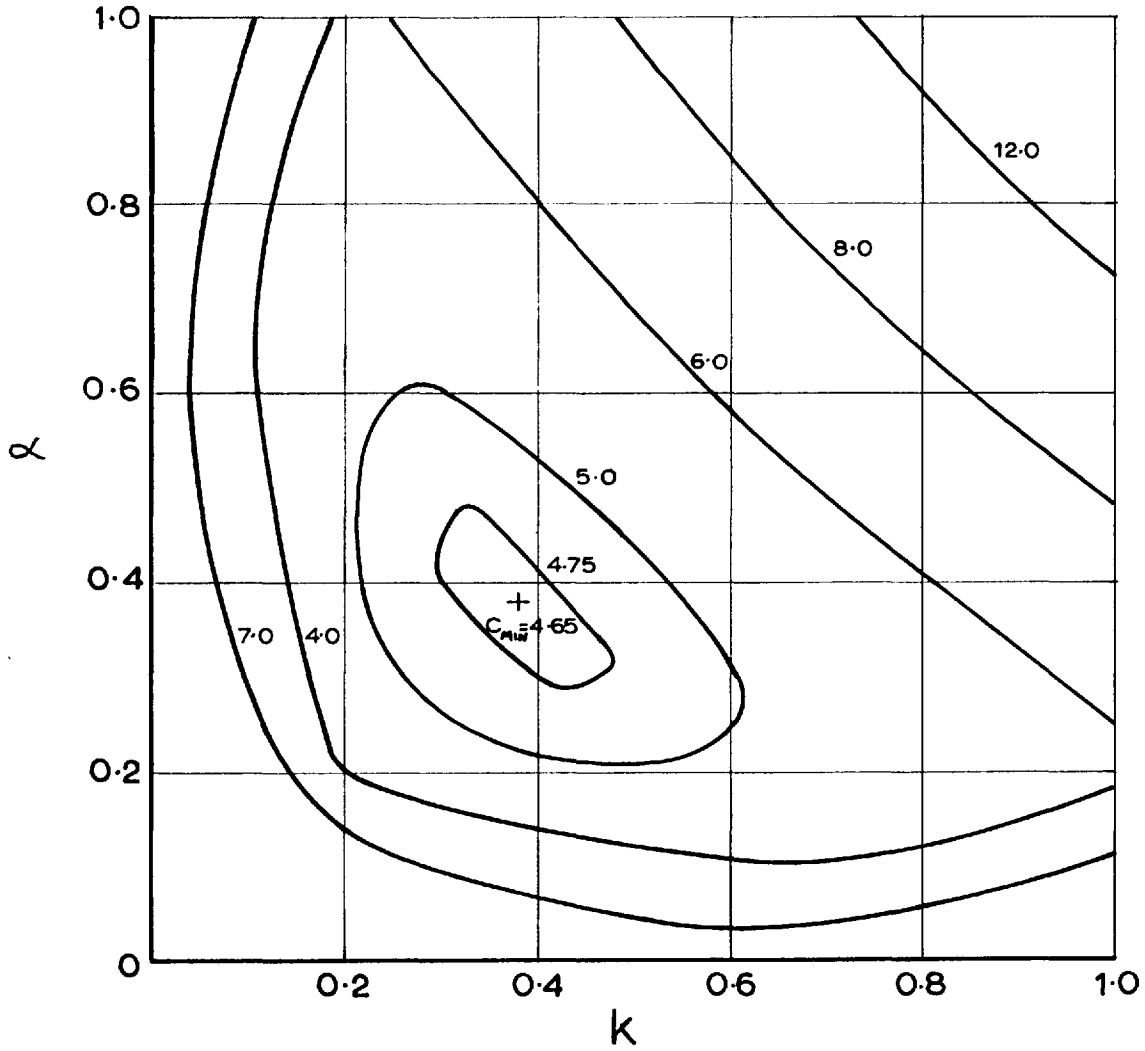


Figure 4.22(b) CONTOURS OF C vs α and k FOR
 $L=1.0$ AND $C=S_{ic}+S_Q+0.1T_r$

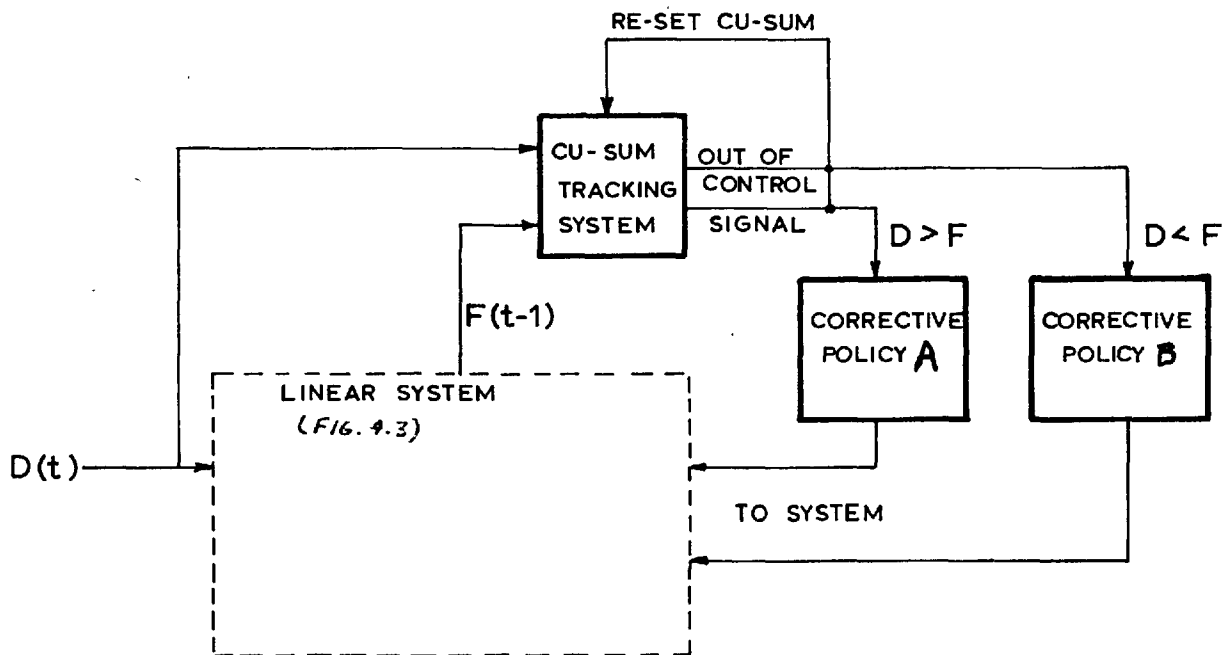


FIG. 4. 23 SYSTEM WITH CU-SUM TRACKING

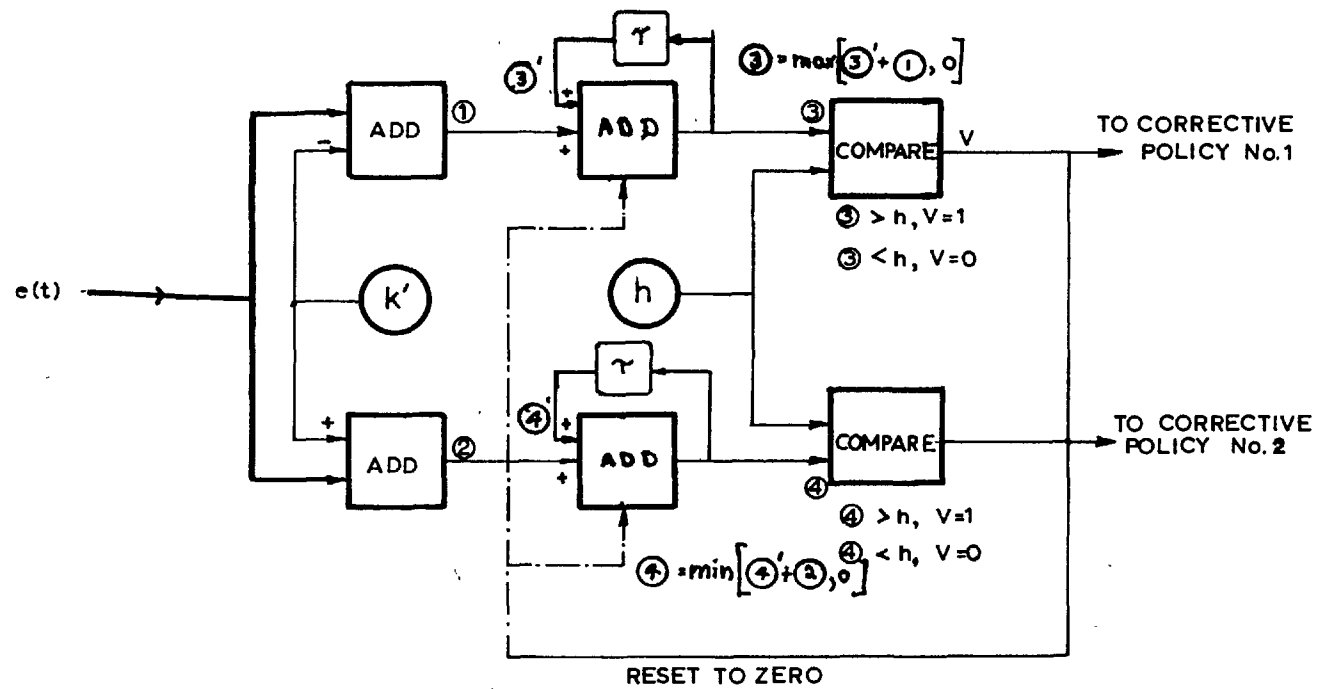
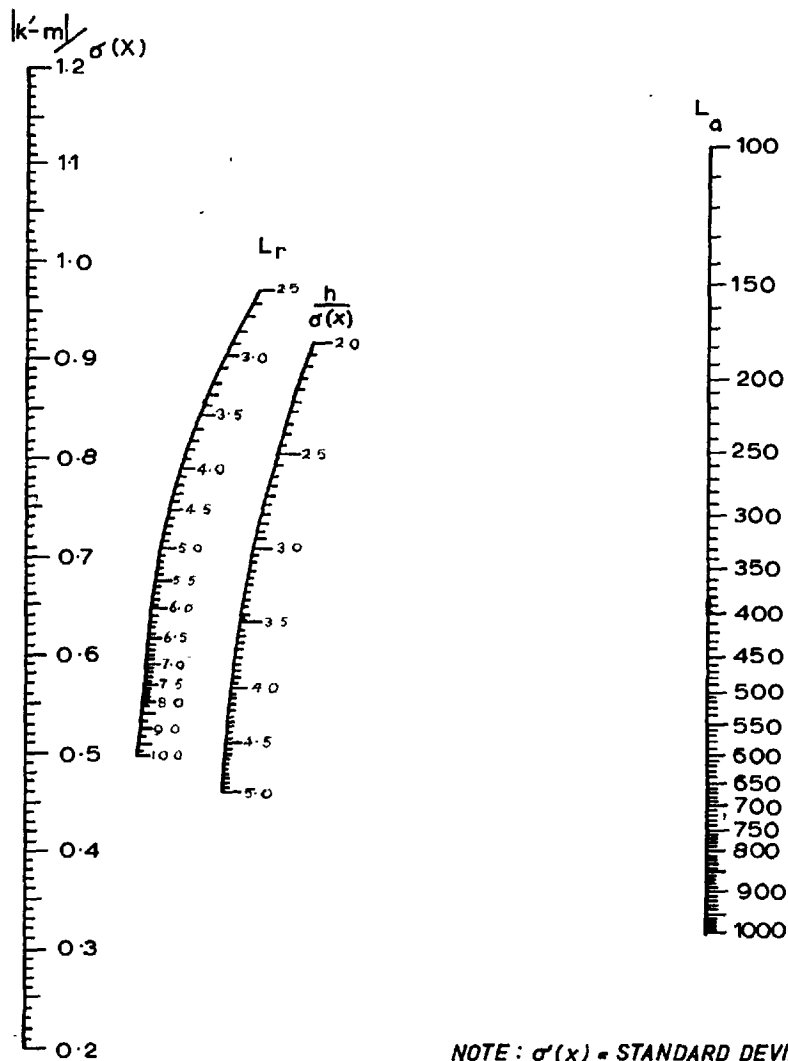


FIG. 4.24 CU-SUM TRACKING SYSTEM



NOTE: $\sigma(x)$ = STANDARD DEVIATION
OF THE TRACKING SYSTEM INPUT

FIG. 4.25 NOMOGRAPH RELATING k' AND h TO L_a , L_r AND m

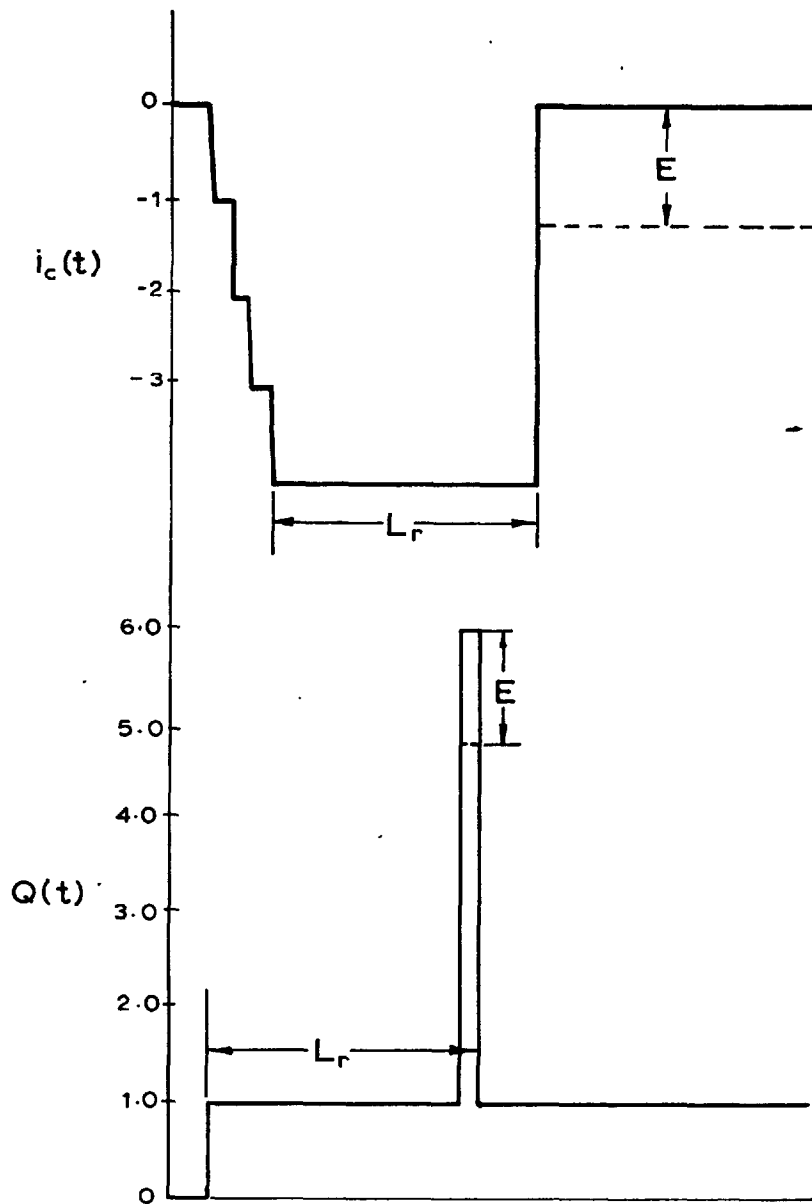


FIG.4.26(a) STEP RESPONSE SYSTEM WITH CU-SUM AND
AND CORRECTIVE POLICY
1a), $\alpha=k=1.0$, $L=3$, SOLID LINE SHOWS RESPONSE
WHEN $F=D$, DOTTED LINE SHOWS RESPONSE WHEN
 $F-D=E$

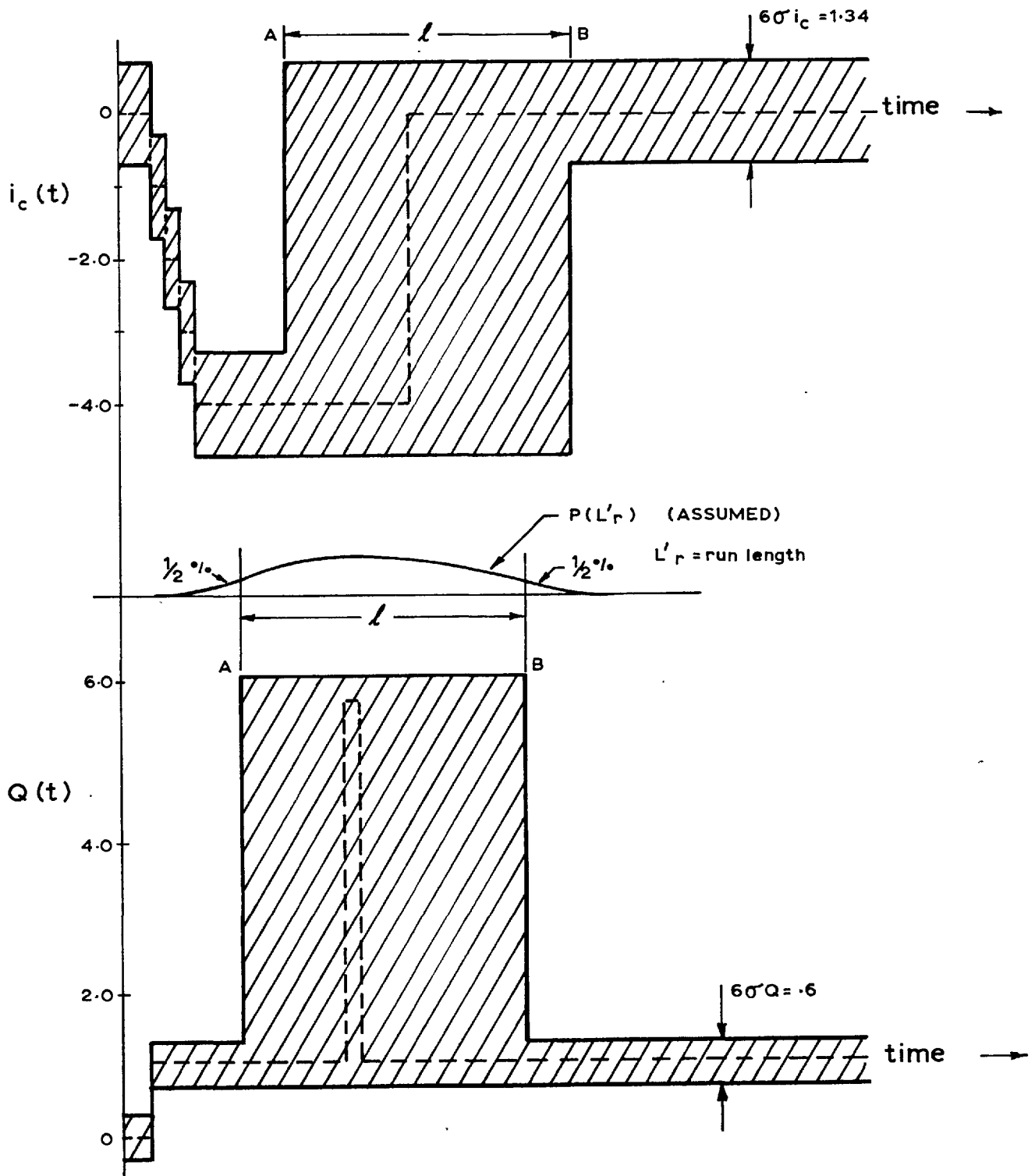


FIG. 4.2(b) CU-SUM SYSTEM POLICY 1(a), SHADED REGIONS DENOTE 99% CONFIDENCE INTERVAL ($k=1.0$, $L=3$, $\bar{D}=1.0$, $\sigma_D=0.1$, $F=\bar{D}$)

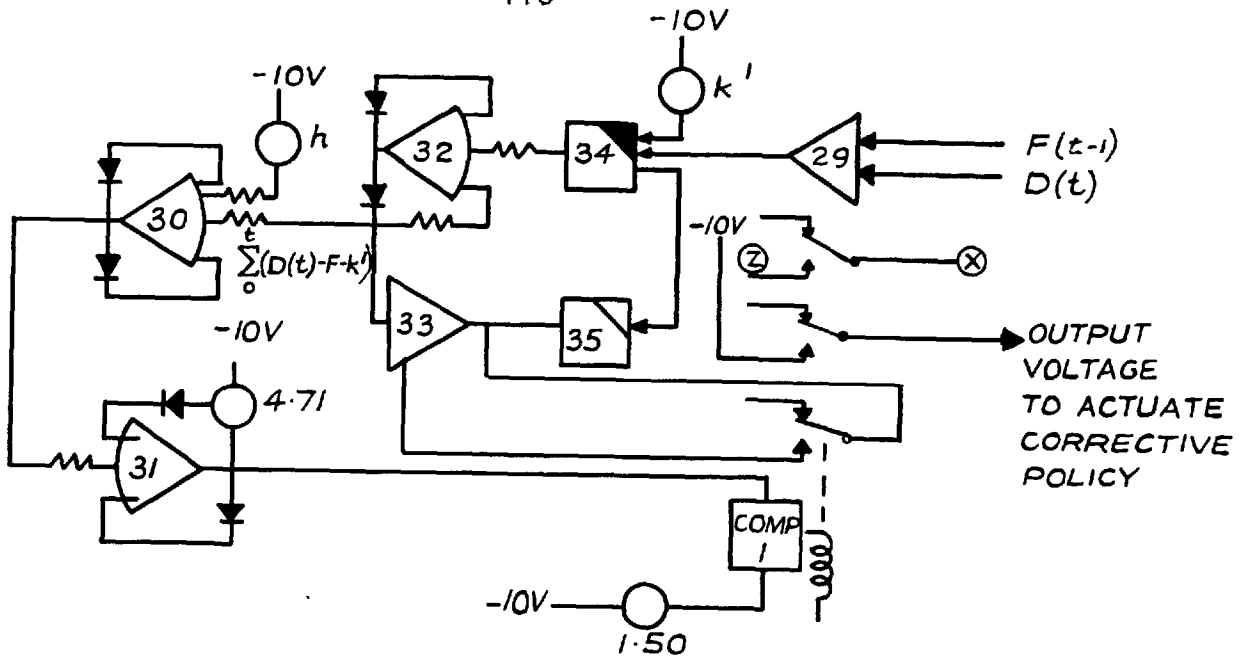


FIG. 4.27 (a) ANALOGUE PROGRAMME FOR SINGLE SIDED CU-SUM

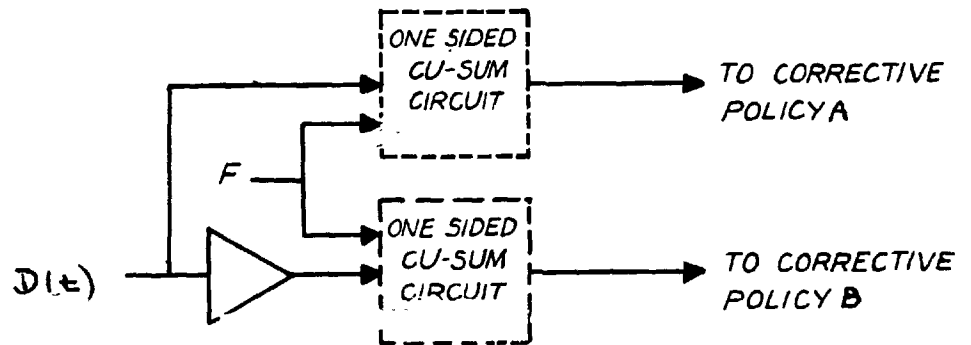


FIG. 4.27 (b) ANALOGUE PROGRAMME FOR TWO SIDED CU-SUM SYSTEM

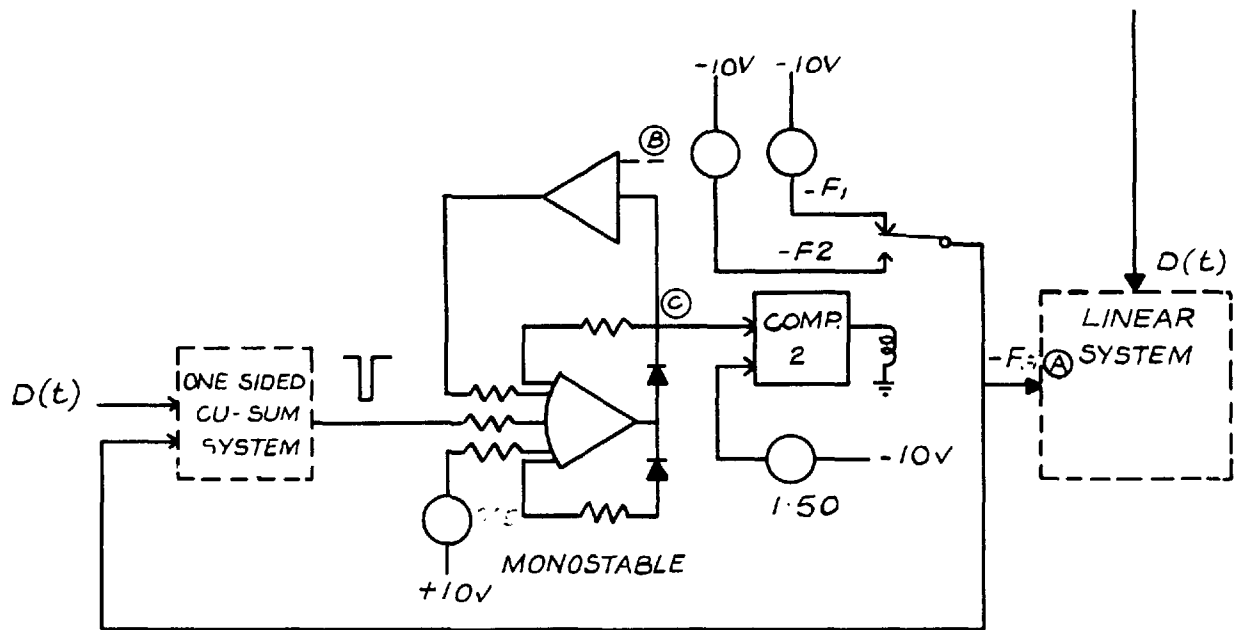


FIG. 4.28 ANALOGUE PROGRAMME FOR CU-SUM SYSTEM POLICY 1(a)
 F_1 = INITIAL FORECAST, F_2 = NEW FORECAST WHEN CU-SUM TRIGGERS

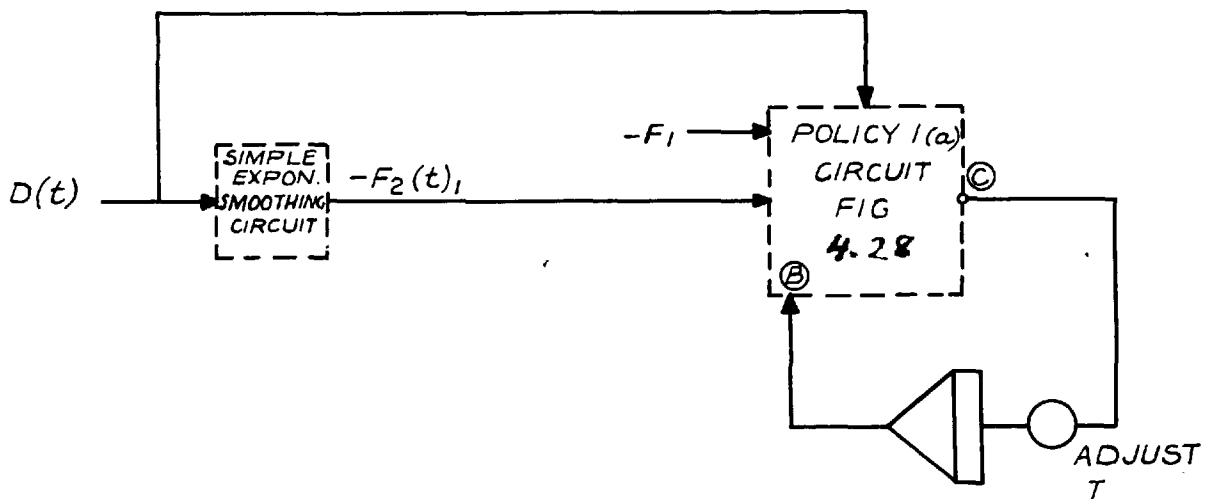


FIG. 4.29 ANALOGUE PROGRAMME FOR CU-SUM SYSTEM POLICY 1(b)

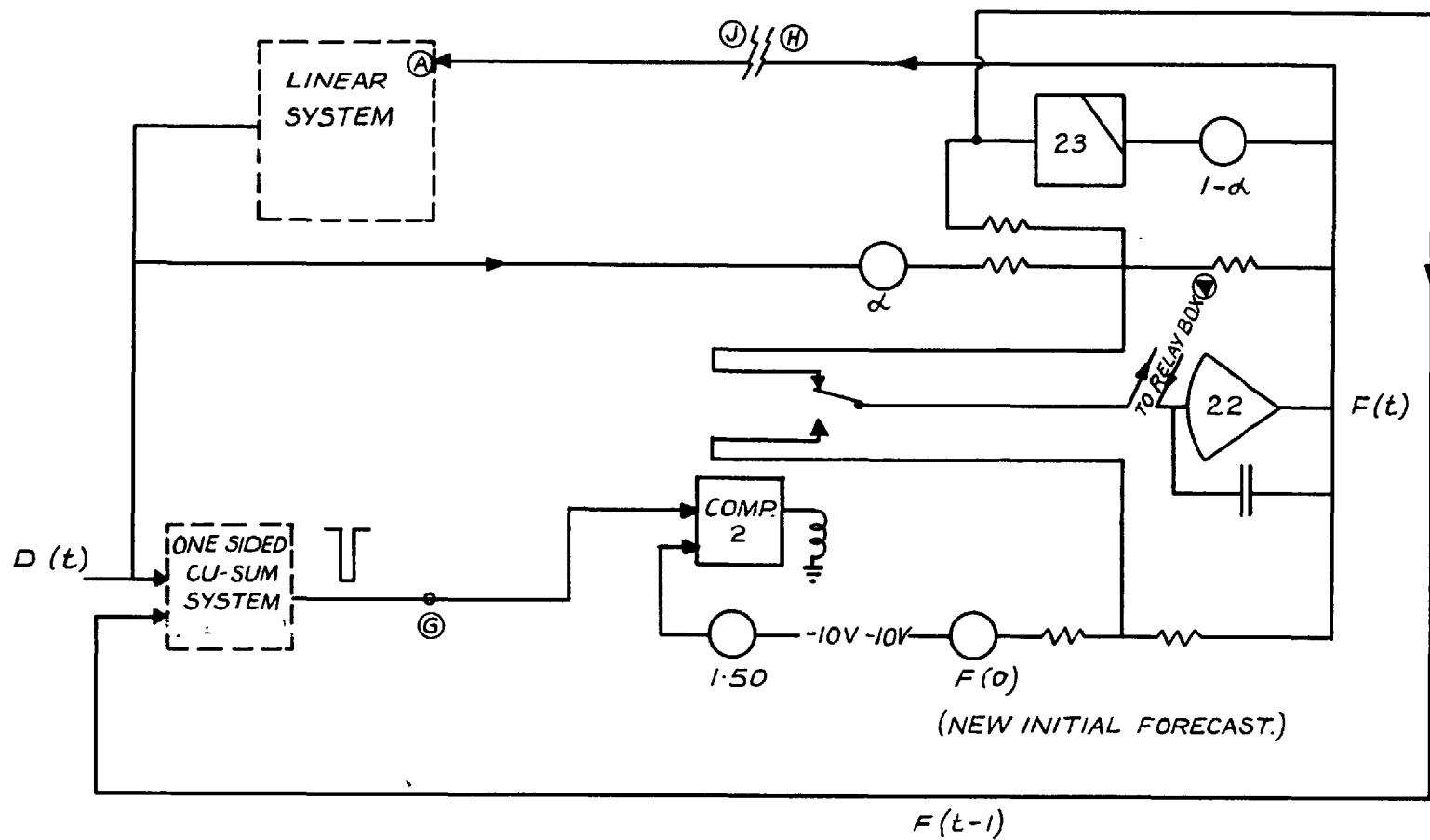


FIG. 4. 30 CU-SUM SYSTEM POLICY 2(a)

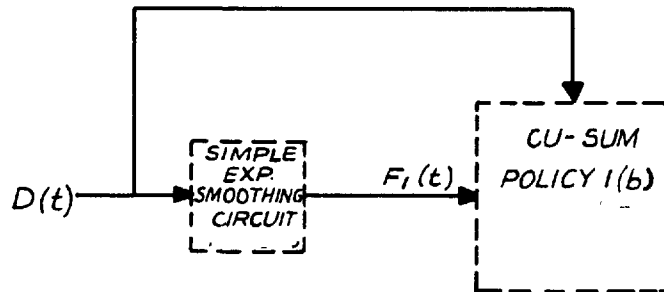


FIG. 4.31 ANALOGUE PROGRAMME.
CU-SUM SYSTEM POLICY 2(b)

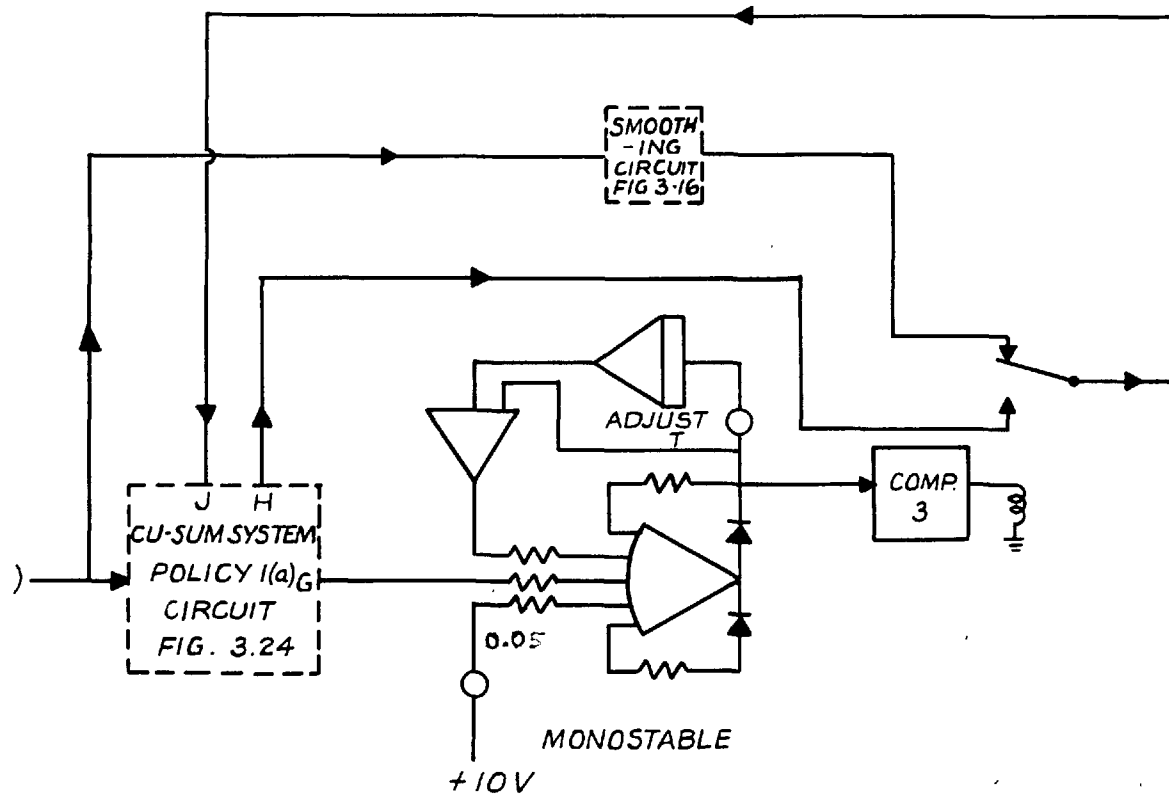


FIG. 4.32 ANALOGUE PROGRAMME CU-SUM SYSTEM POLICY 2(c)

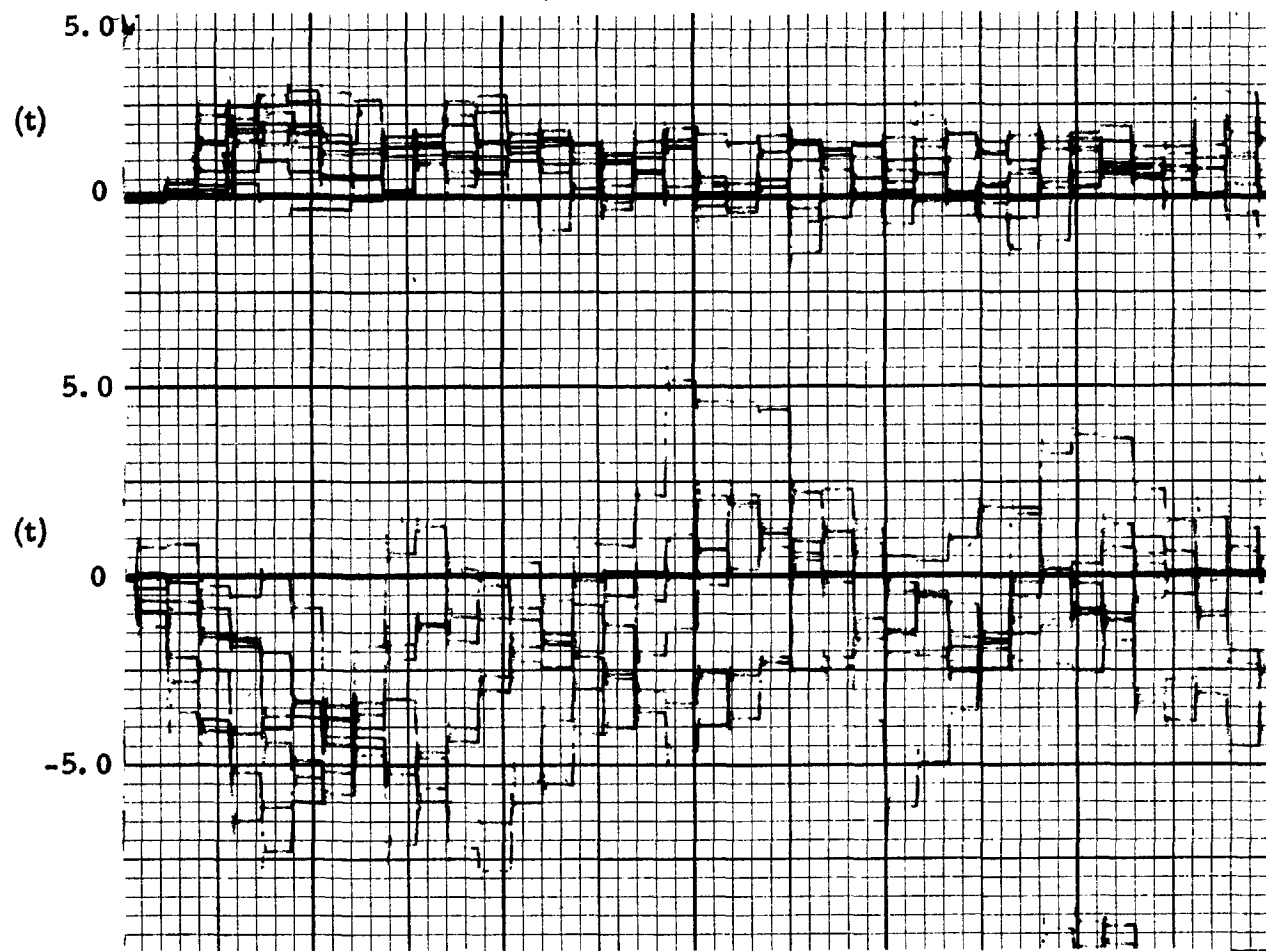


FIG 4.33 CU-SUM SYSTEM POLICY 1(a)

$\alpha=k=0.25$, $L=3$, $h=1.6$, $k'=0.32$, $D=\bar{D}=n(t)$, $\bar{D}=0.8$, $\sigma_D=0.8$

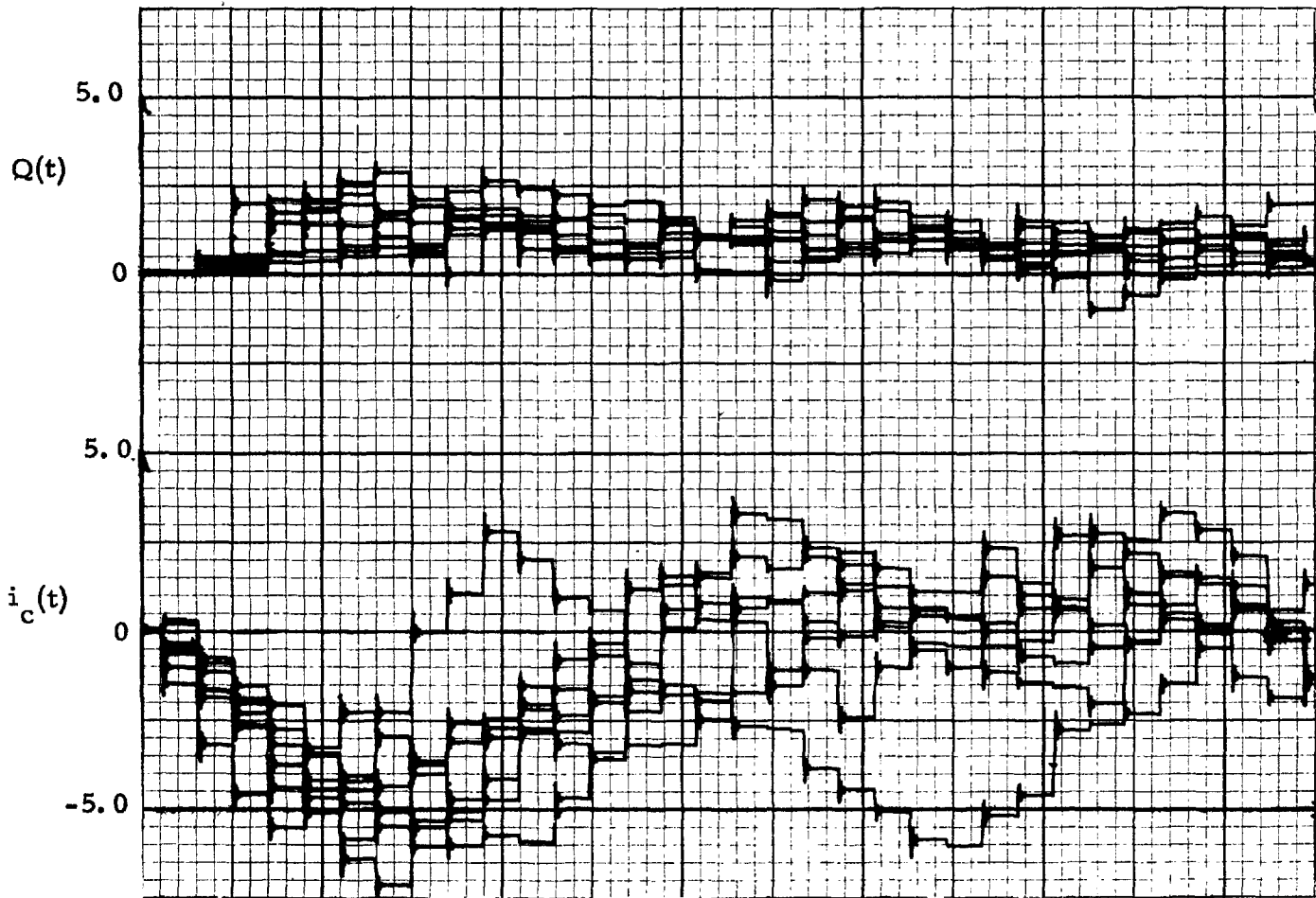


FIG 4. 34(a) CU-SUM SYSTEM POLICY 1(b)
— $\alpha=0.25, k=0.25, \bar{D}=\sigma_D=0.8$

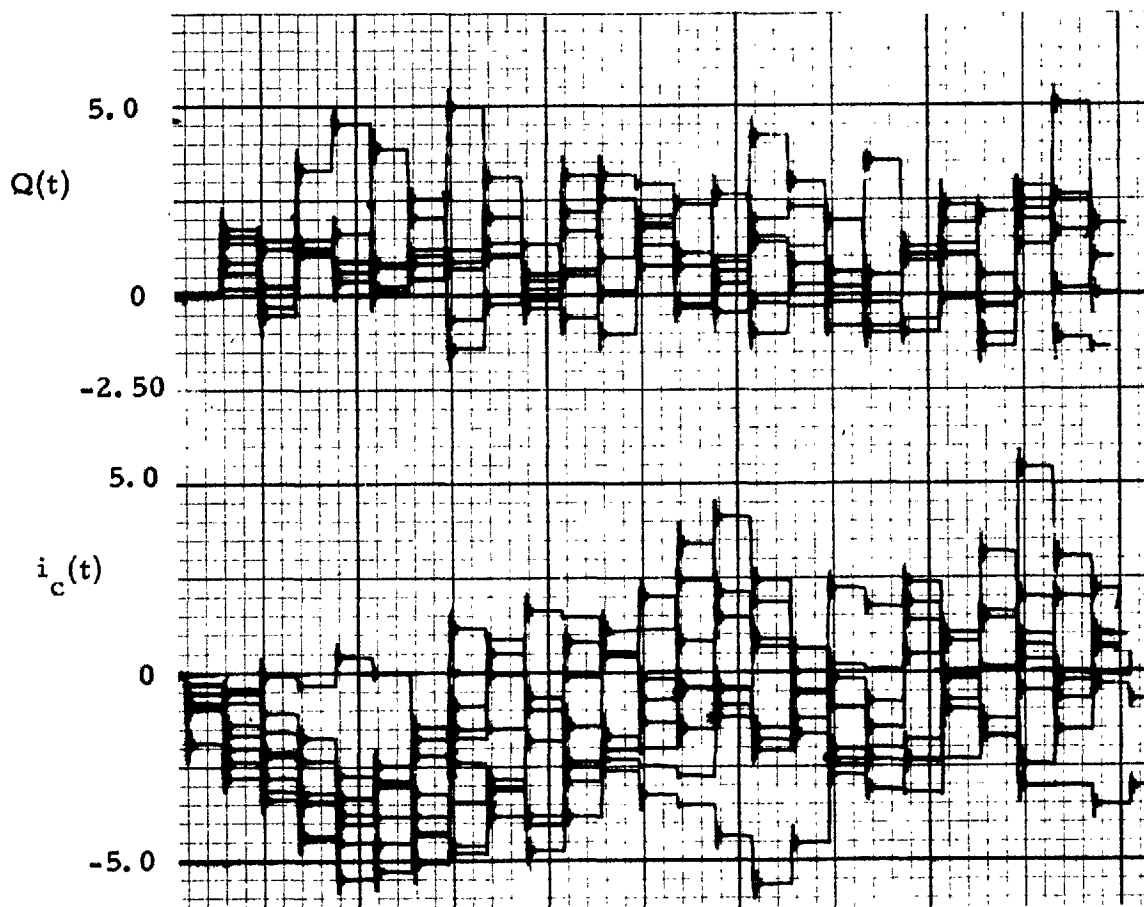


FIG 4.34(b) CU-SUM SYSTEM POLICY 1(b)
 $\alpha=k=1.0$

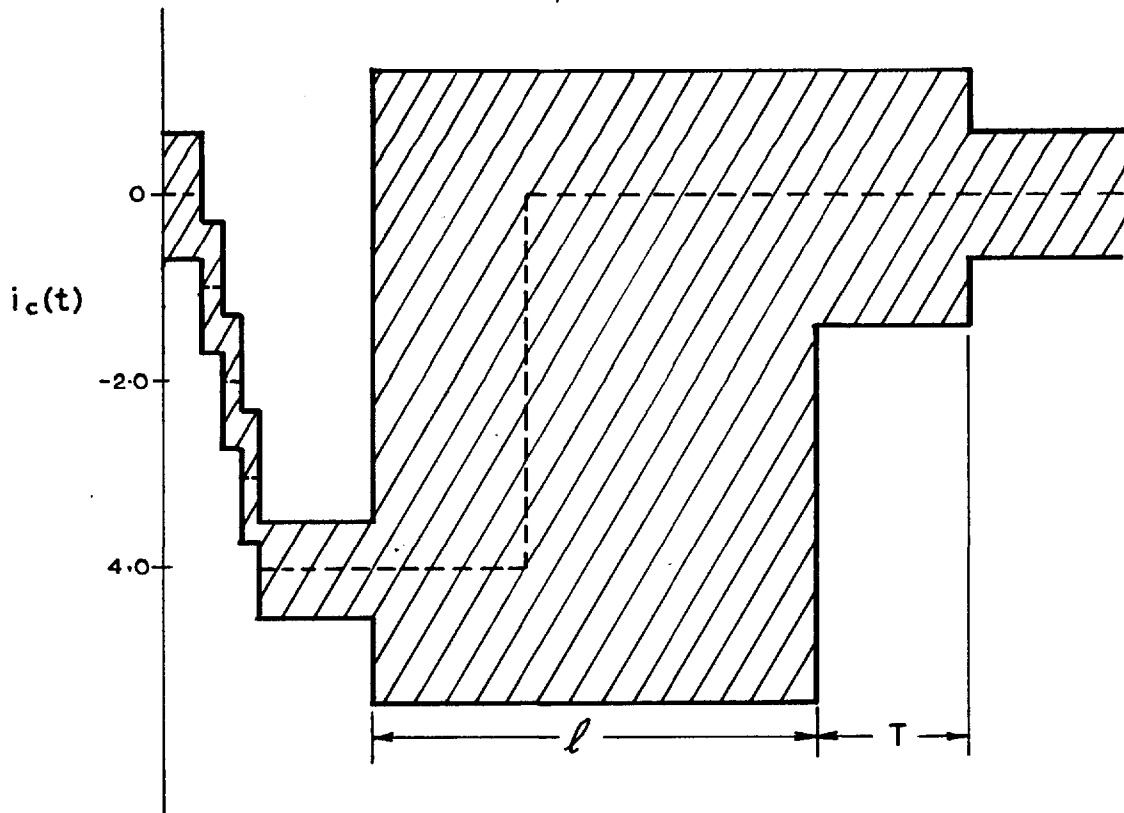


FIG. 4.35(a) CU-SUM POLICY 1b) SHADED REGION
DENOTES 99% CONFIDENCE INTERVAL
FOR $i_c(t)$ ($\alpha=k=1.0$, $\bar{D}=1.0$, $\sigma_D=0.1$)

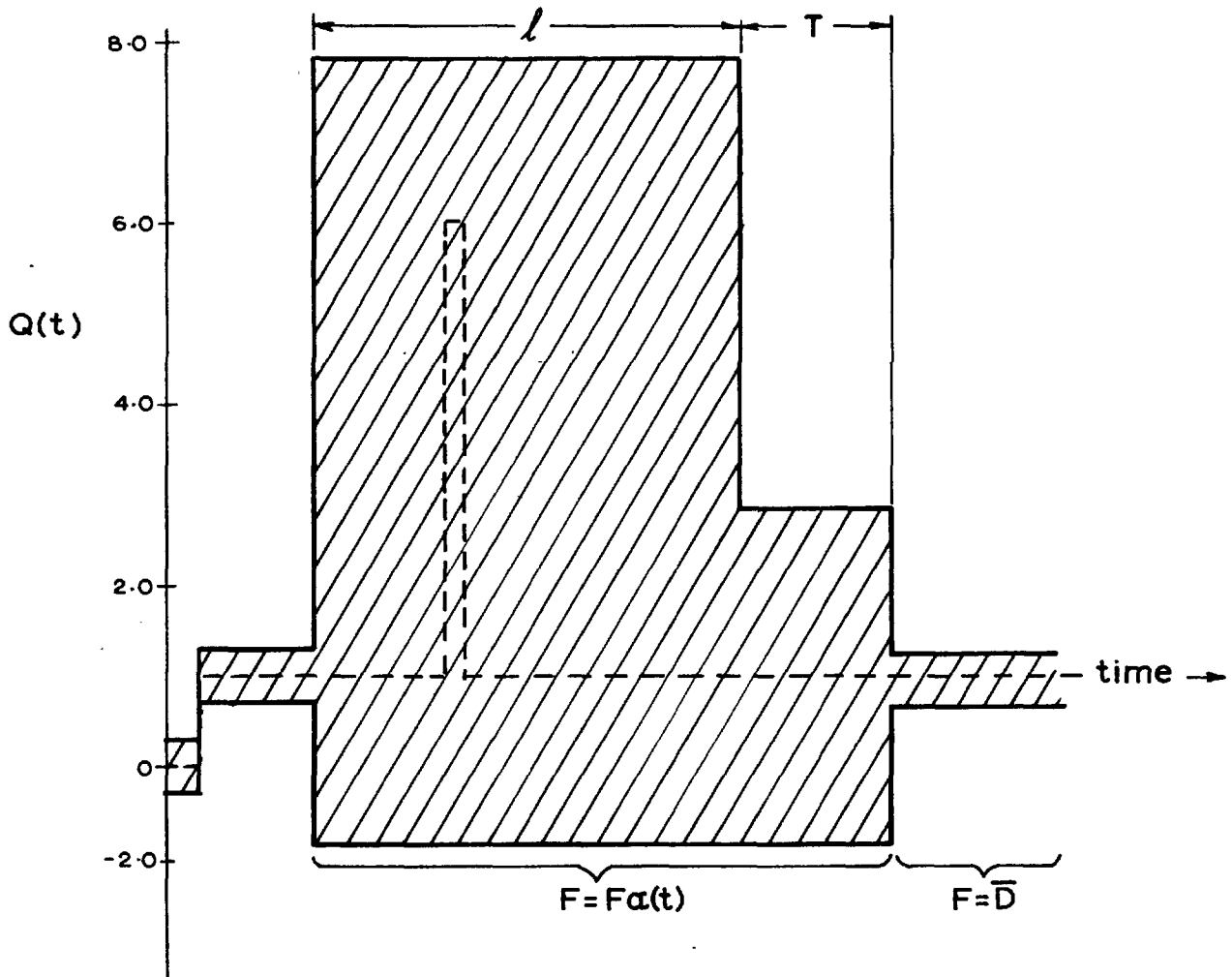


FIG. 4.35(b) CU-SUM POLICY 1(b) 99% CONFIDENCE INTERVALS FOR $Q(t)$

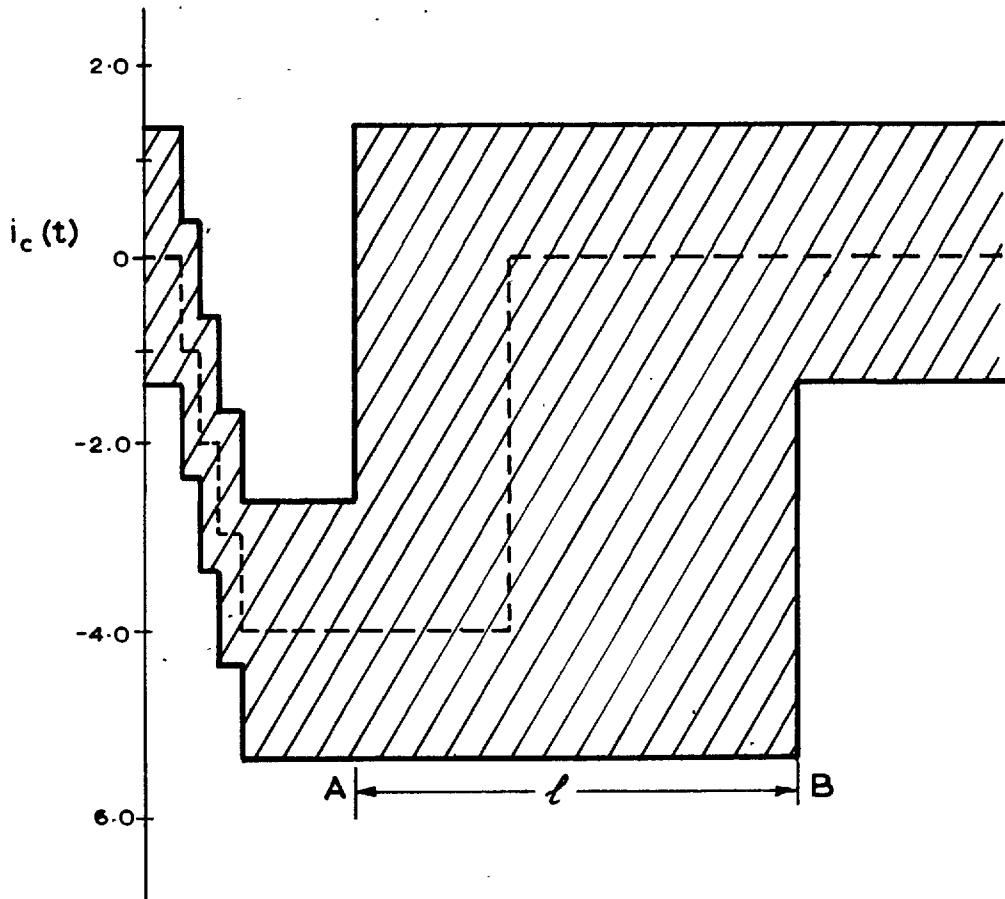


FIG. 4.36(a) CUSSUM POLICY 2(a) SHADED REGION DENOTES
99% CONFIDENCE INTERVAL FOR $i_c(t)$
($\alpha=k=1.0$, $\bar{D}=1.0$, $\sigma_D=0.1$)

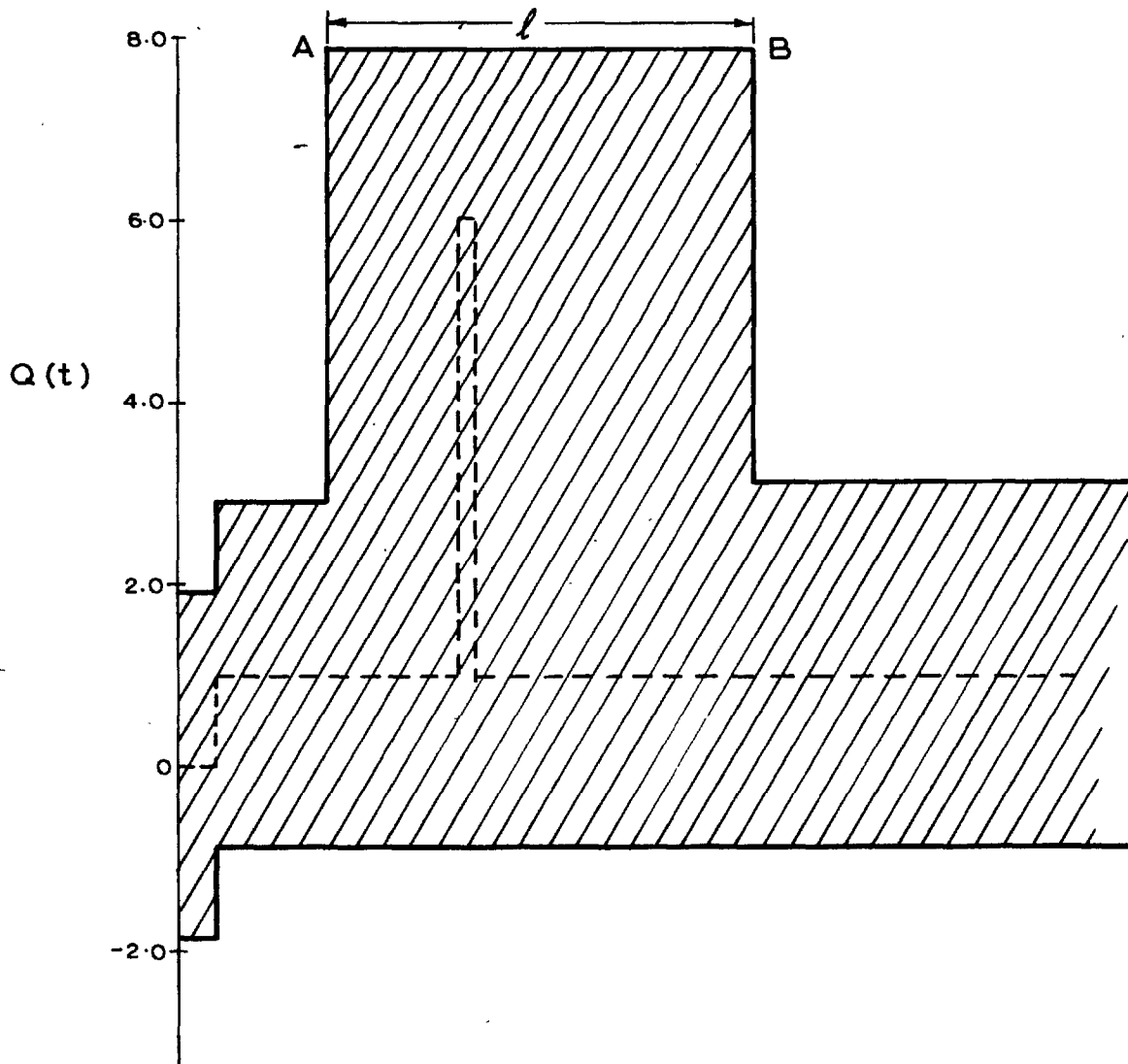


FIG. 4.36(b) CU-SUM POLICY 2(a) SHADED REGION DENOTES 99% CONFIDENCE INTERVAL

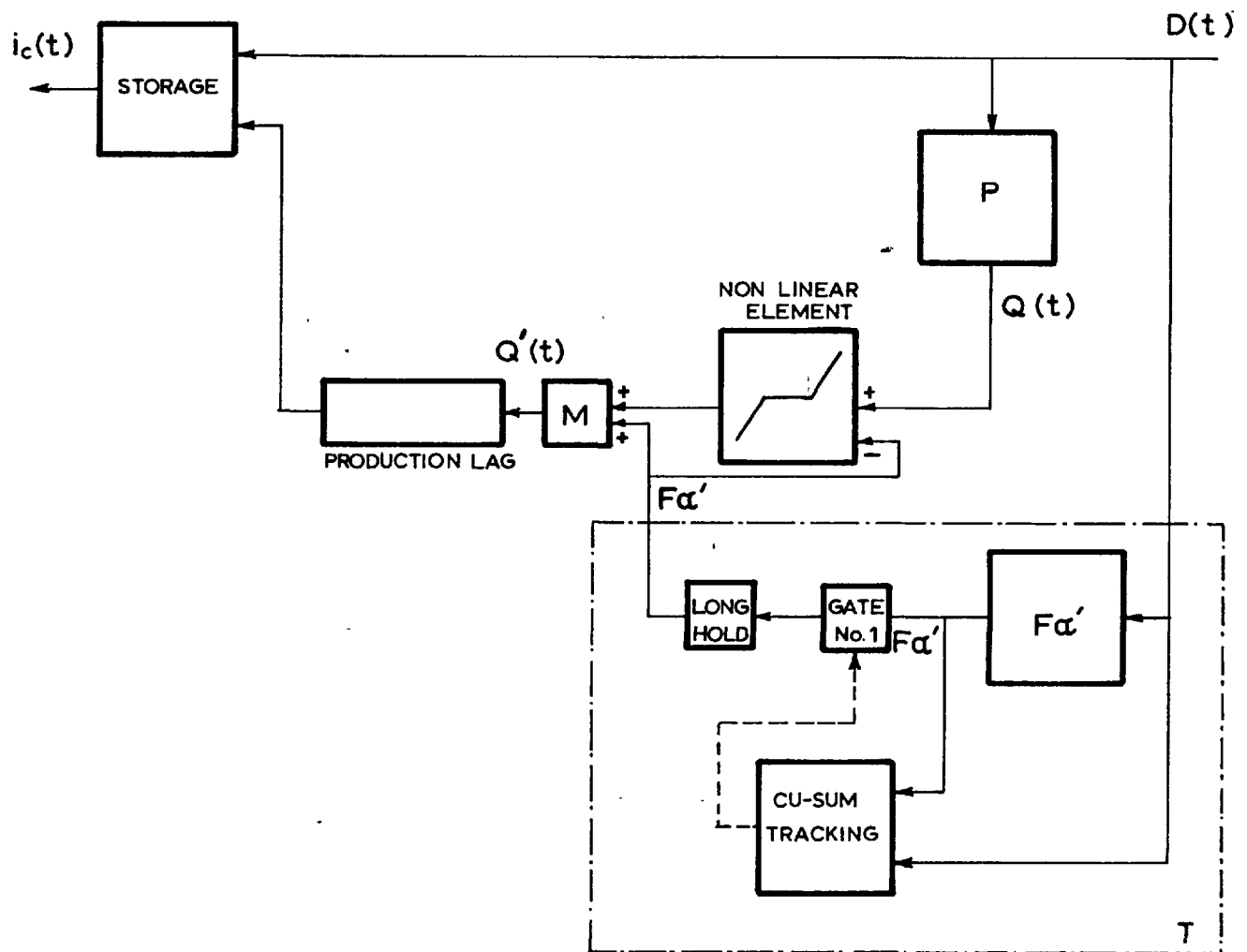


FIG. 4, 37 NON LINEAR SYSTEM SCHEMATIC

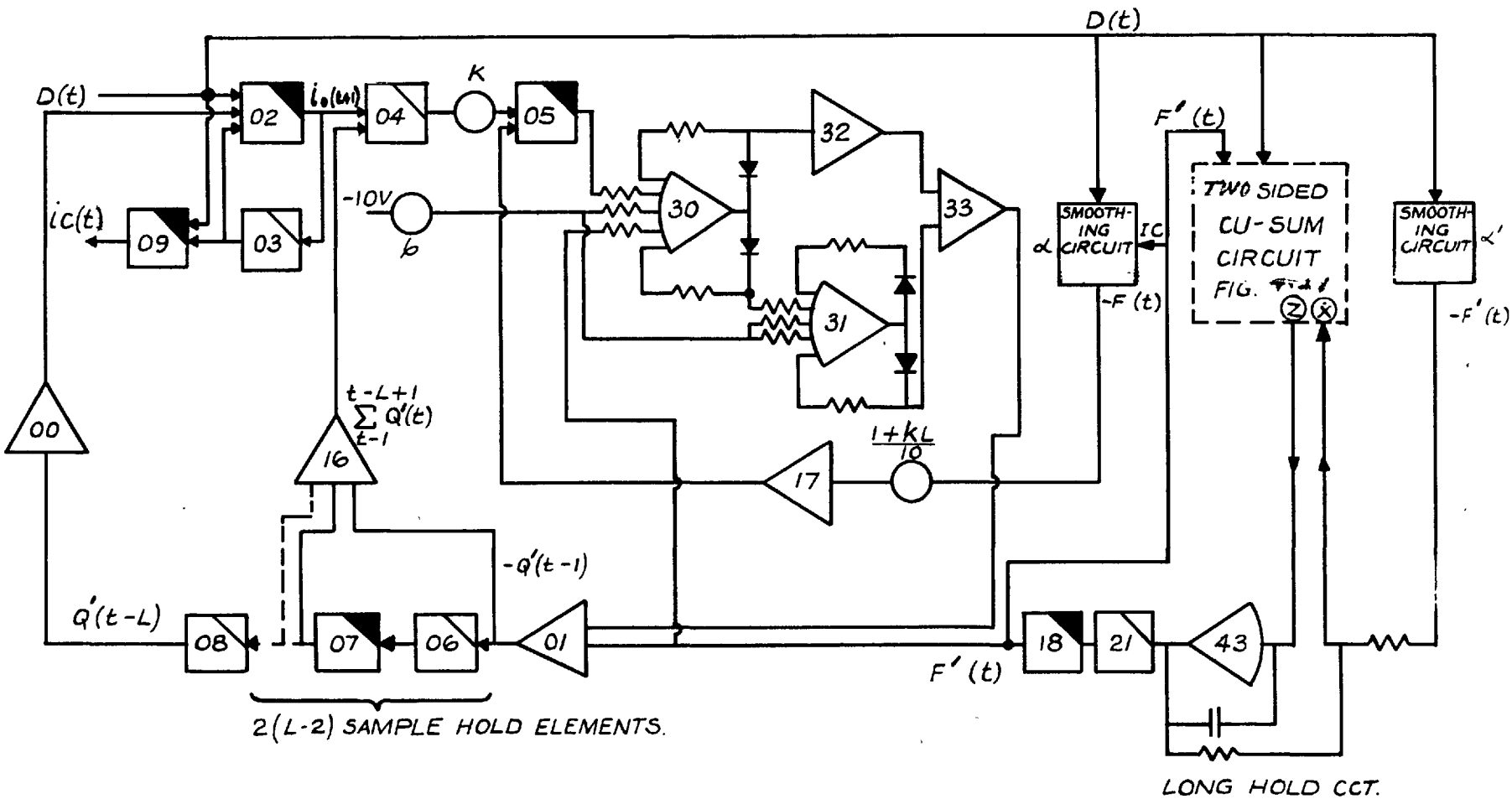


FIG. 4.38 ANALOGUE PROGRAMME FOR NON-LINEAR SYSTEM

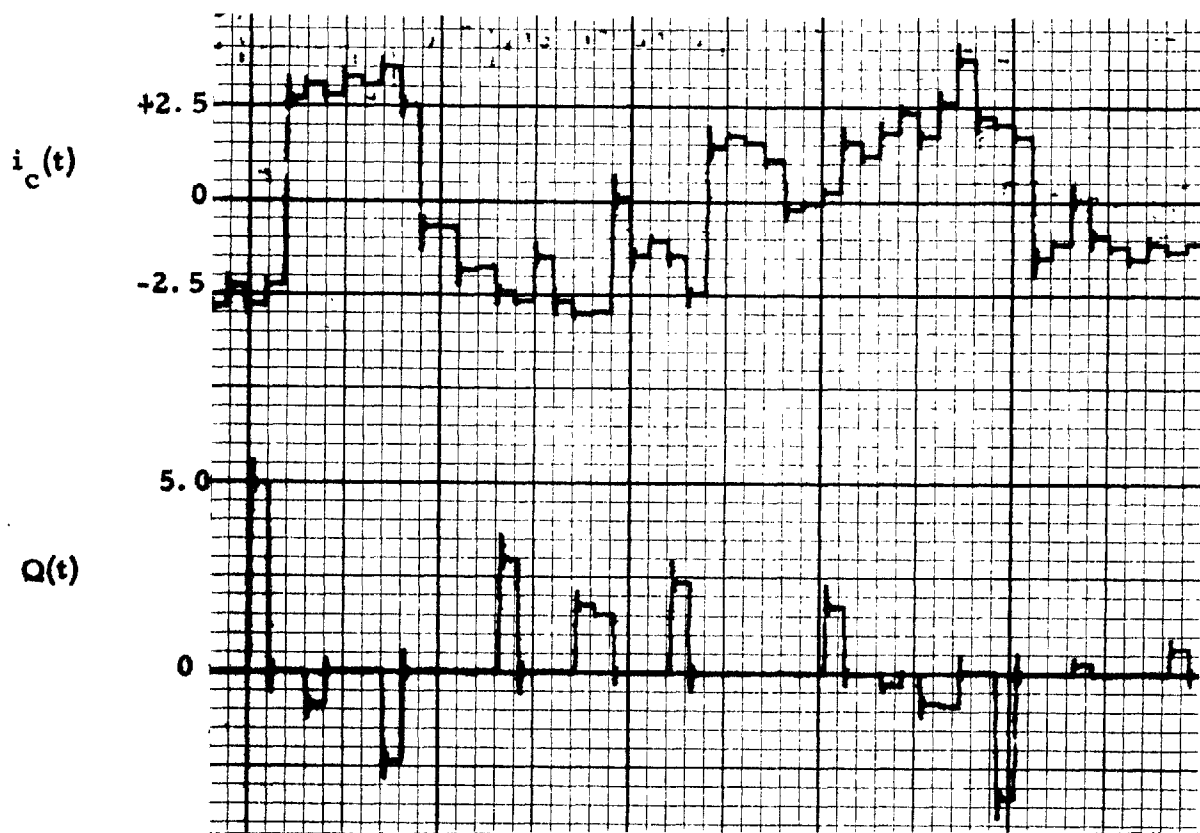


FIG 4.39 NON LINEAR SYSTEM NOISE RESPONSE
 $\alpha=0.15$, $k=1.0$, $L=3$, $\phi=2.0$, $\alpha'=0.4$, $M=2.25$, $k'=0.50$,
 $h=3.00$, $\sigma_n=1.0$

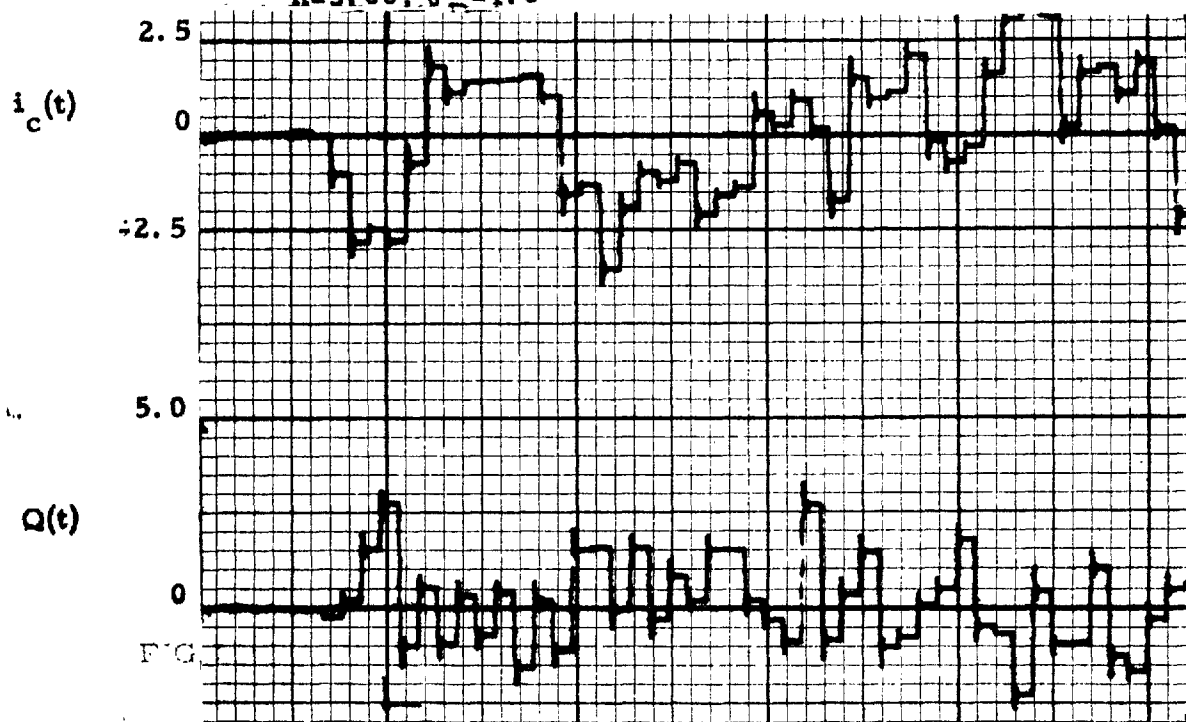


FIG 4.40 LINEAR SYSTEM, $\alpha=0.15$, $k=1.0$, $L=3.0$, $D(t)=n(t)$, $\sigma_D=1.0$

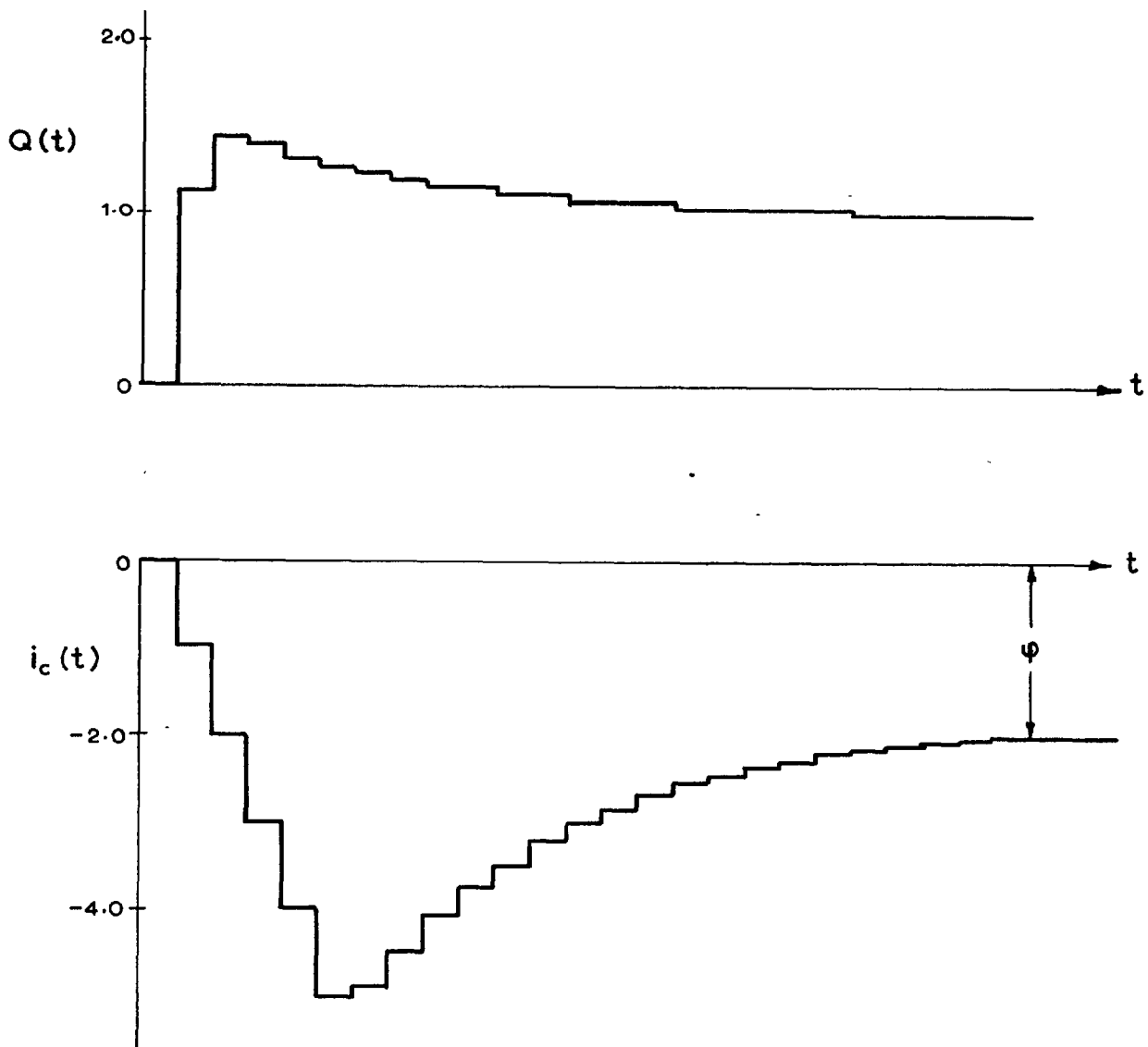


FIG. 4.41 NON LINEAR SYSTEM STEP RESPONSE
 $\alpha=0.15$, $k=1.0$, $k'=50$, $\lambda=300$, $\alpha'=0.15$, $\phi=200$,
 $M=1.0$, $L=3$

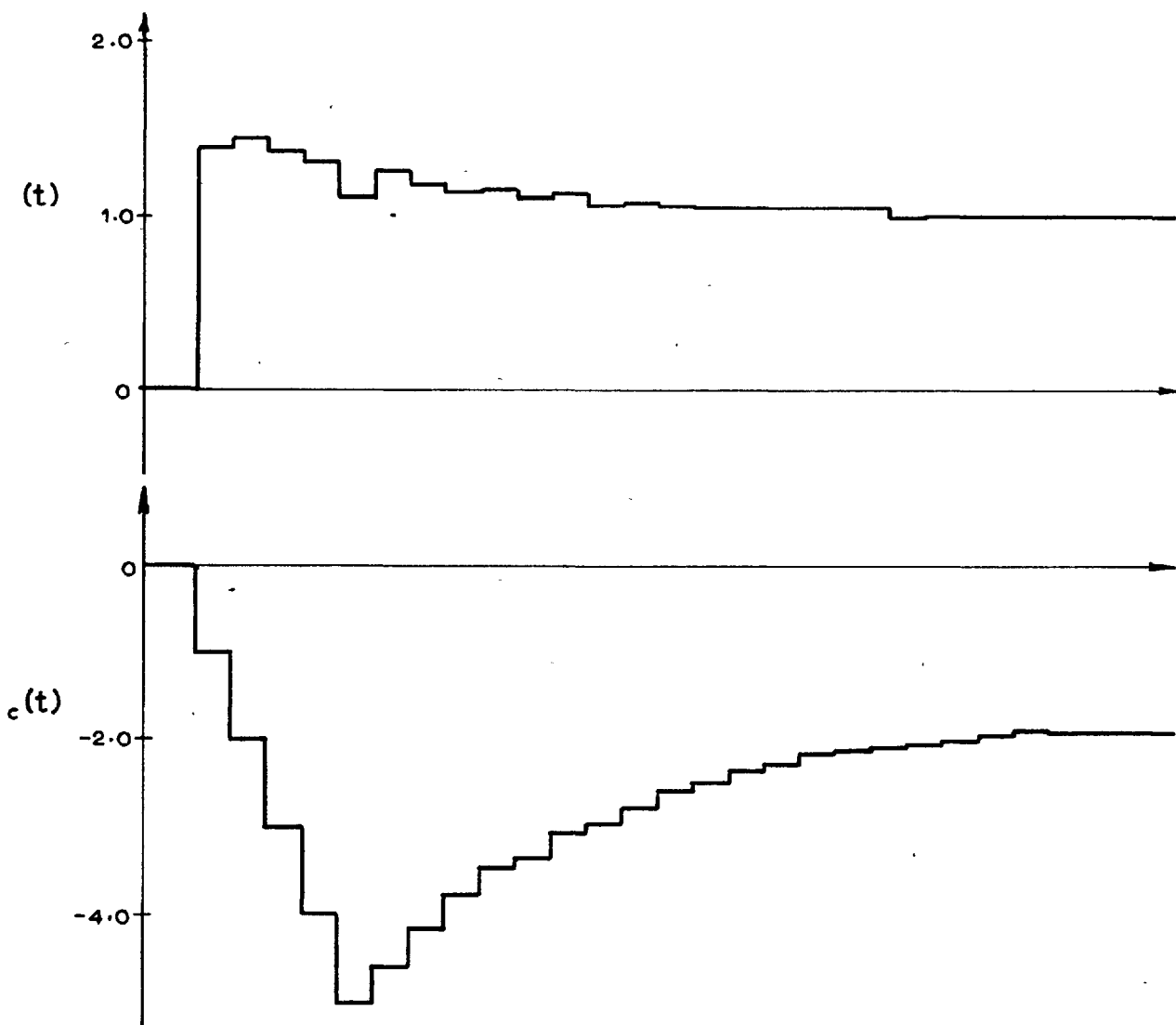


FIG. 4.42 NON LINEAR SYSTEM STEP RESPONSE
 $\alpha=0.15$, $k=1.0$, $k'=50$, $h=300$, $\alpha'=0.15$, $\varphi=200$,
 $M=1.25$, $L=3$



FIG 4.43 NON LINEAR SYSTEM RESPONSE TO A STEP PLUS NOISE

$\alpha=0.15$, $k=1.0$, $L=3$, $\phi=2.0$, $\alpha'=0.4$, $M=2.25$, $k'=0.05$, $h=3.00$, $\bar{D}=1.0$
 $\sigma_D=1.0$



FIG 4.44 LINEAR SYSTEM RESPONSE TO A STEP PLUS NOISE

$\alpha=0.15$, $k=1.0$, $L=3$, $D(t)=D+n(t)$, $D=\sigma=1.0$

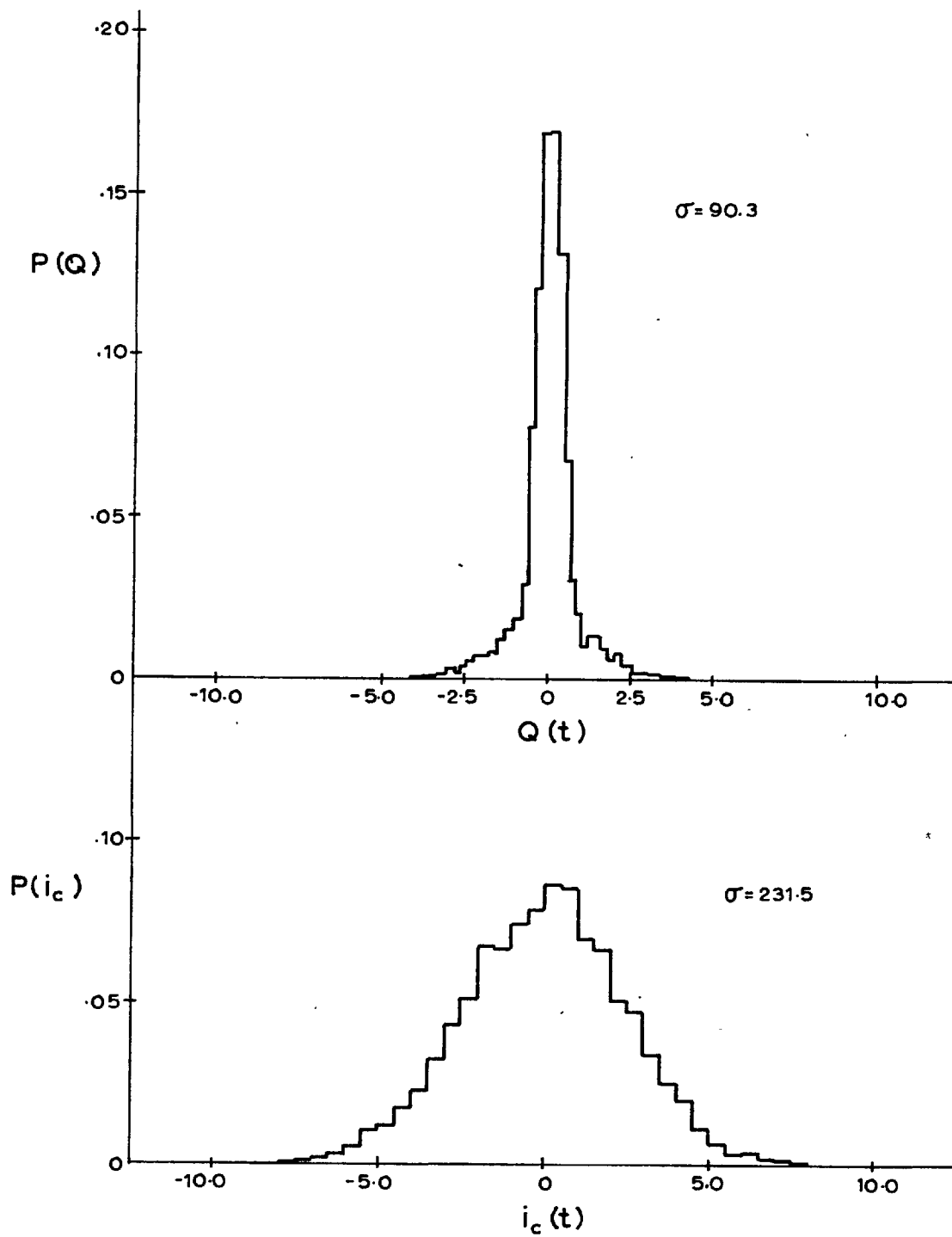


FIG. 4.45 NON LINEAR SYSTEM $L=3$, $k=1.0$, $\alpha=0.15$, $h=3.0$
 $\alpha=0.15$, $\varphi=2.0$, $M=1.0$, $k'=0.5$

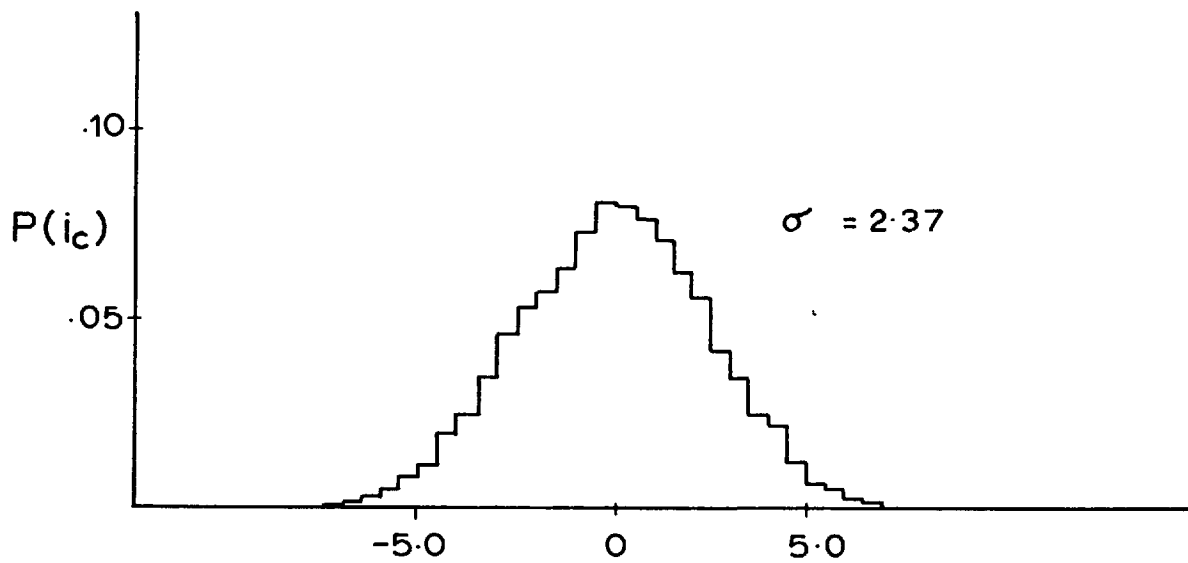
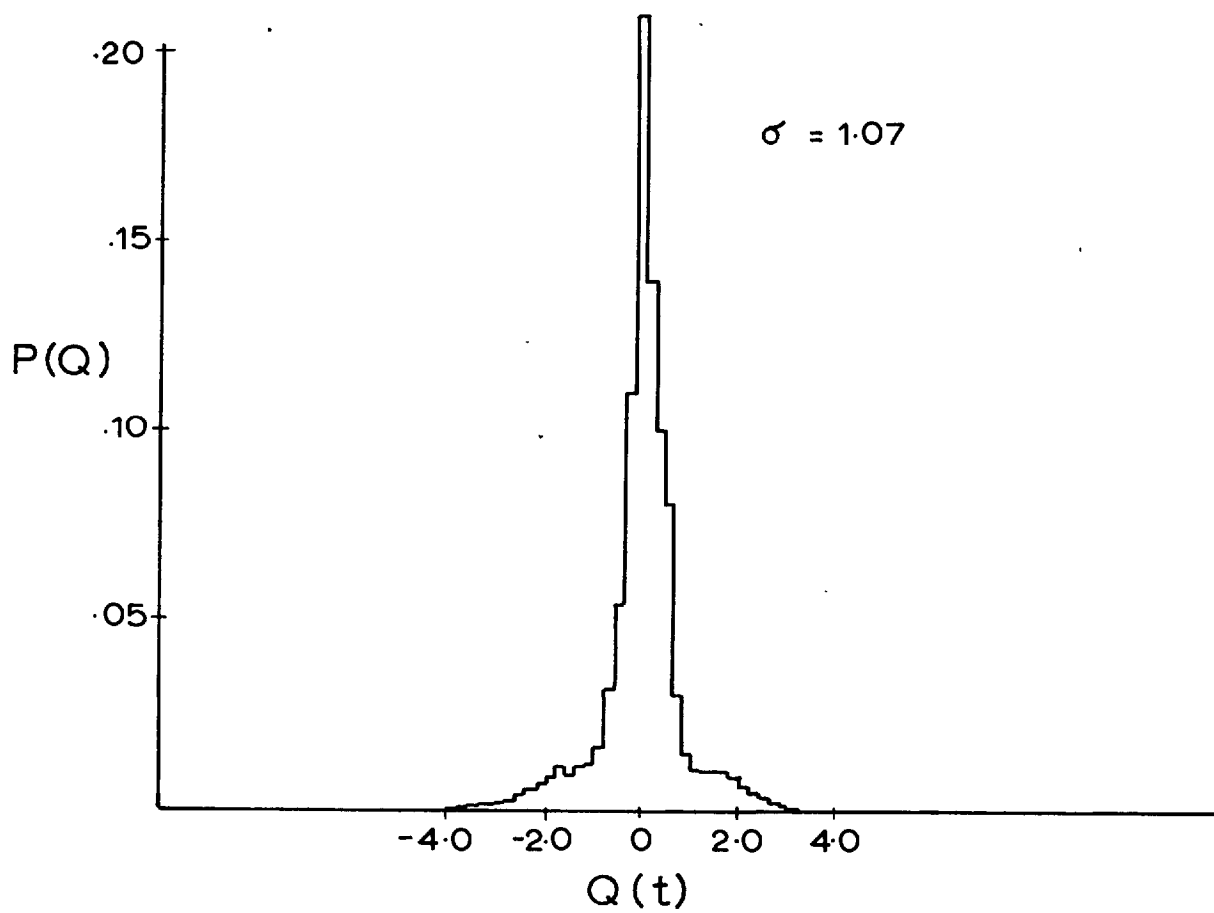


Figure 4.46 NON-LINEAR SYSTEM $L=3$, $k=1.0$, $\alpha=0.15$, $k'=0.5$,
 $h=3.0$, $\alpha'=0.15$, $\varphi=2.0$, $M=1.25$

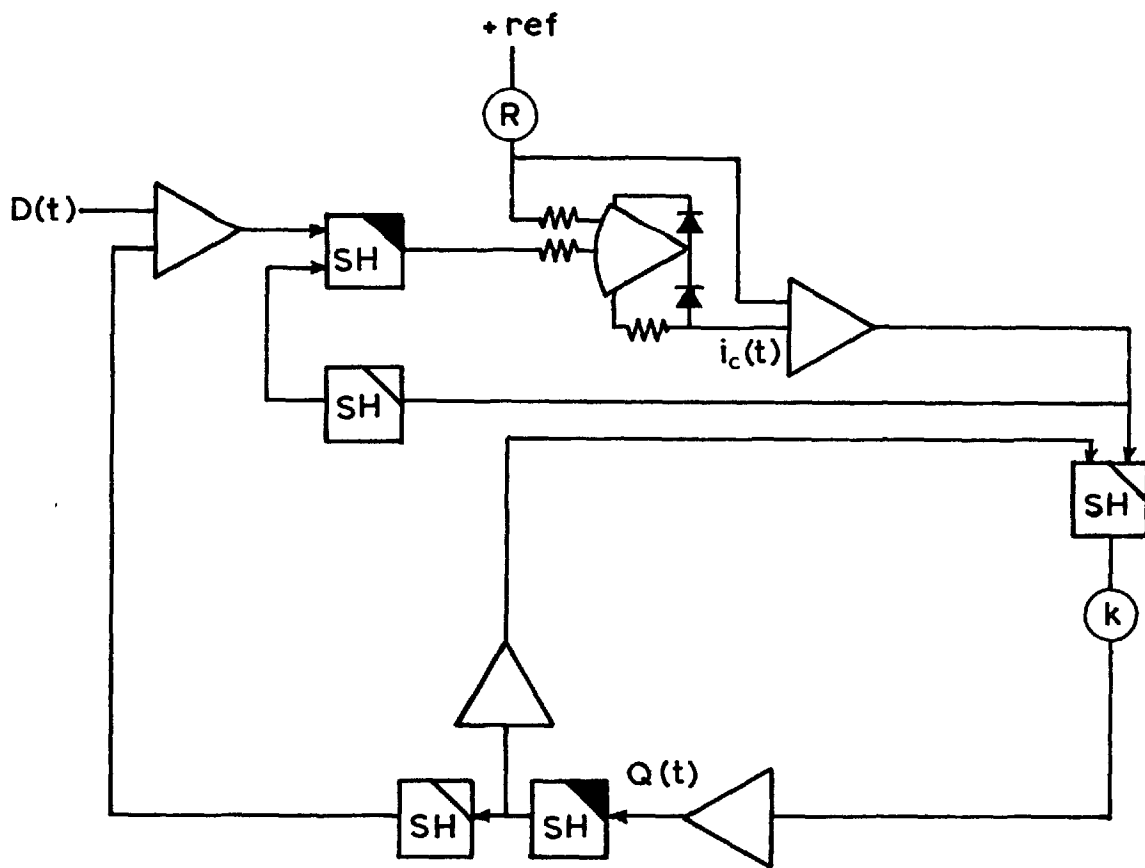


Figure 4.47 ANALOGUE PROGRAMME FOR SYSTEM WITH
LOST SALES DUE TO STOCK SHORTAGE;
L=1

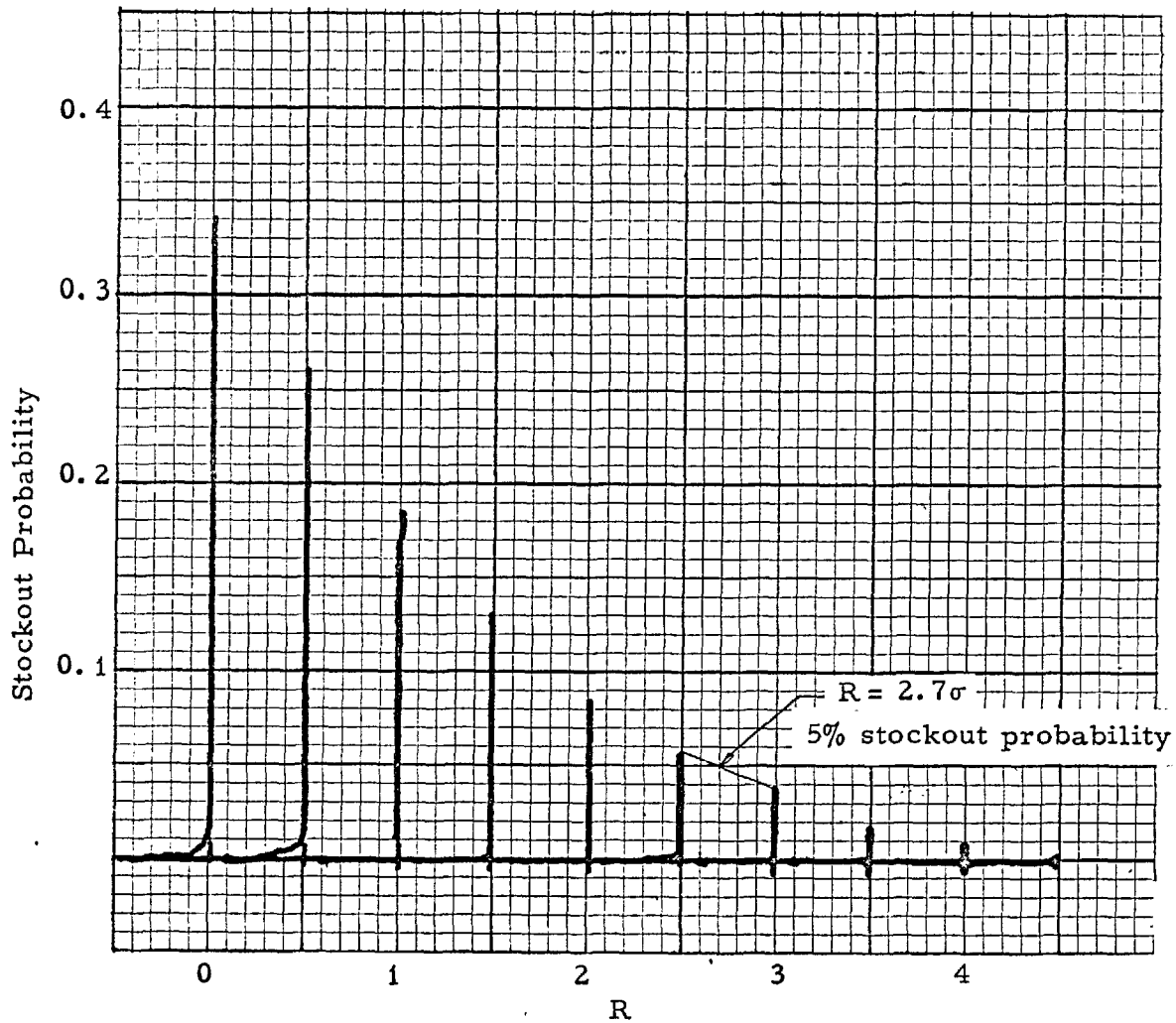


Figure 4.48 STOCKOUT PROBABILITY vs R FOR $k=0.5$

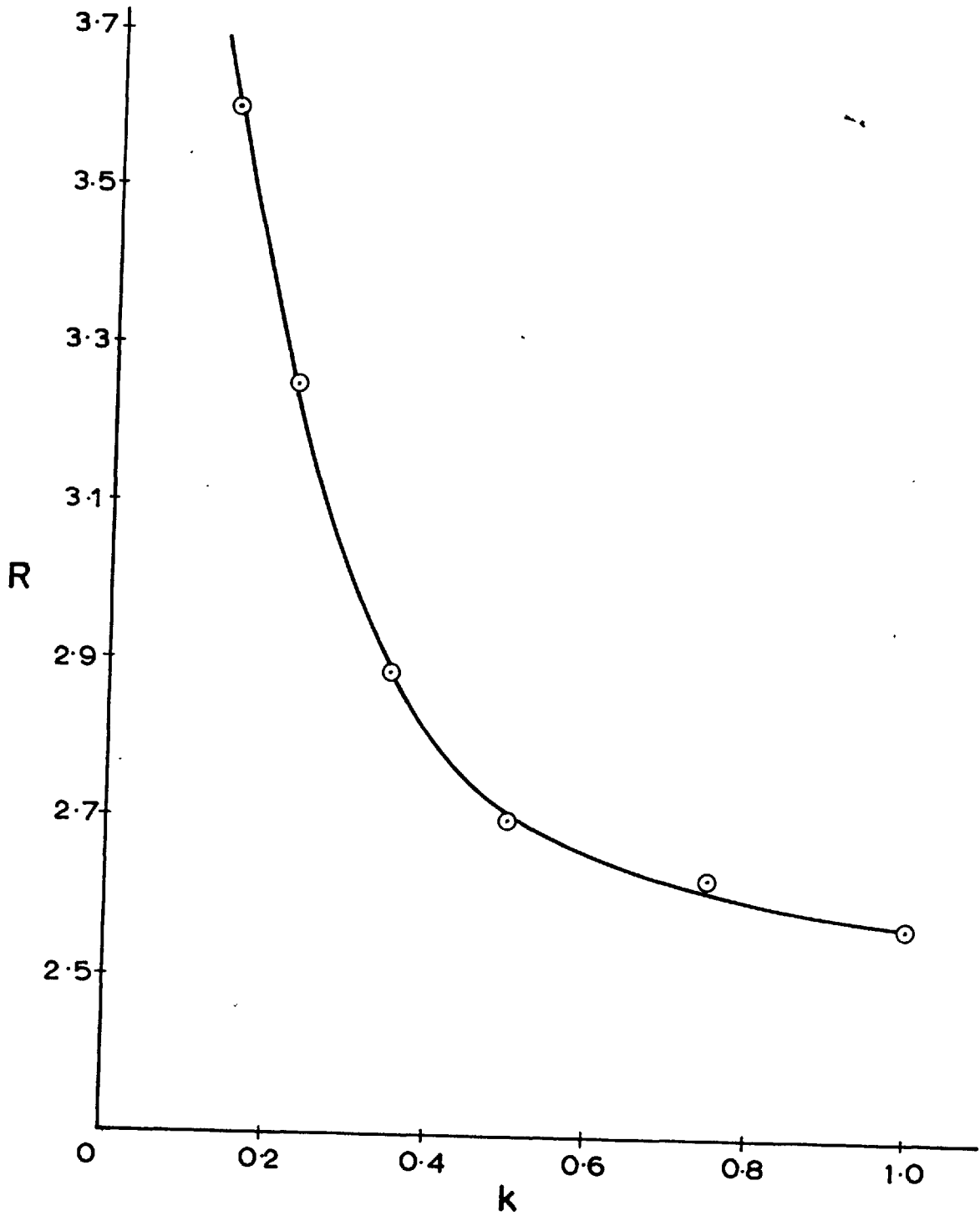


Figure 4.49 R vs k FOR 5% STOCKOUT PROBABILITY

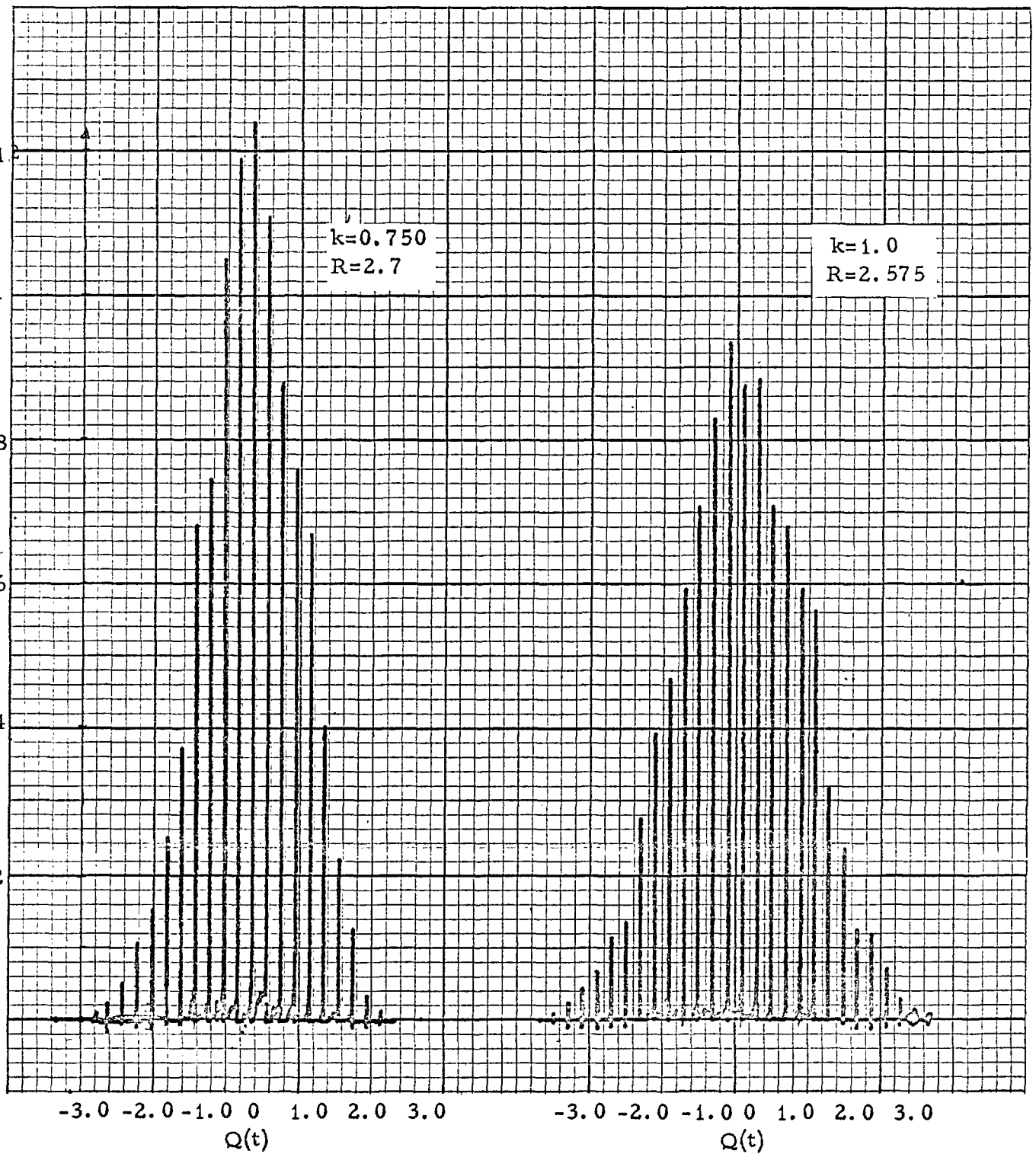


Figure 4.50 SAMPLE HISTOGRAMS OF $Q(t)$ FOR k AND R TO PROVIDE 5% STOCKOUT PROBABILITY

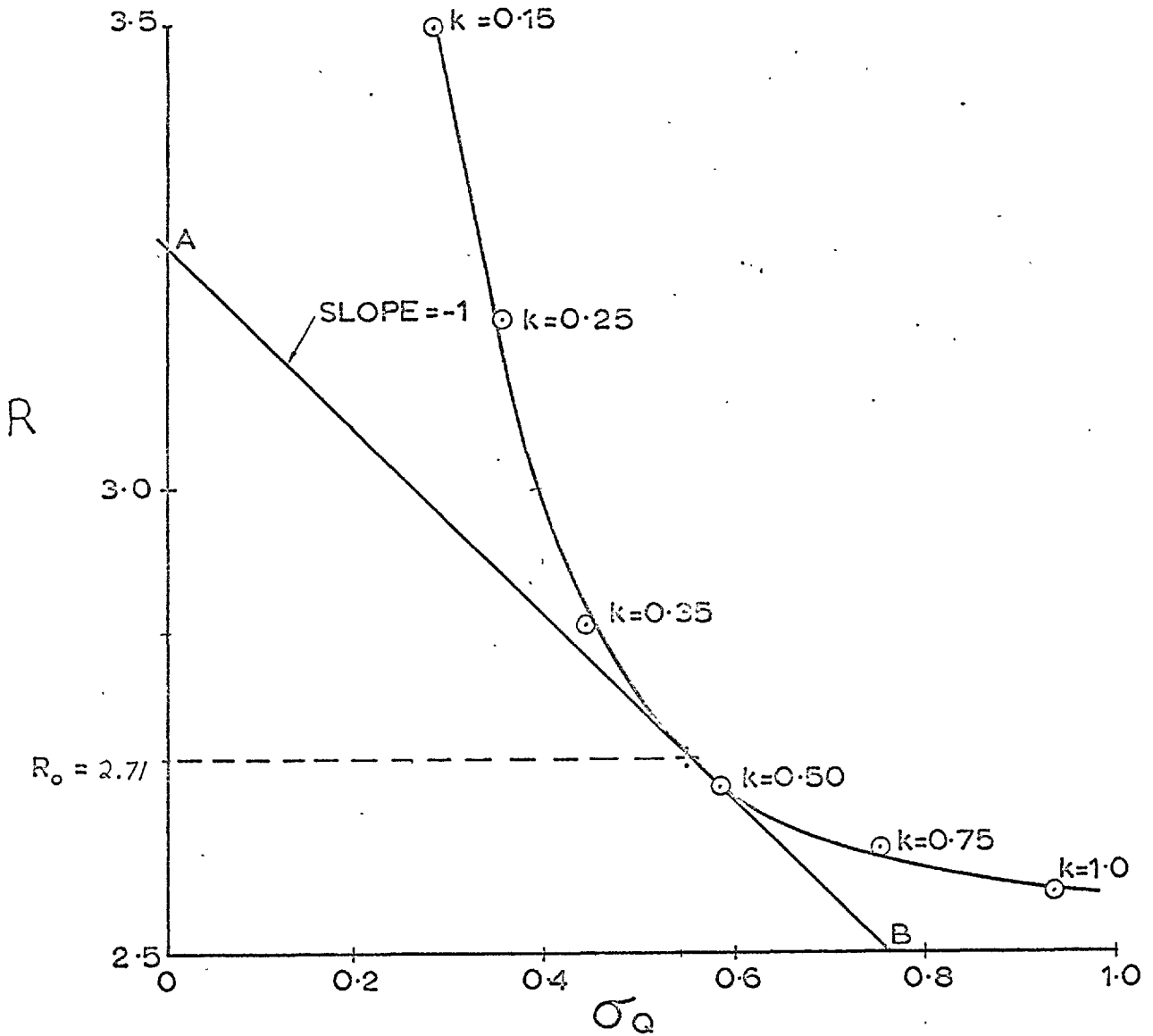


Figure 4.51 R vs σ_Q FOR 5% STOCKOUT PROBABILITY (LINE AB SHOWS CONSTRUCTION FOR DETERMINING THE OPTIMAL VALUE OF R WHEN $c_2/c_1=1.0$)

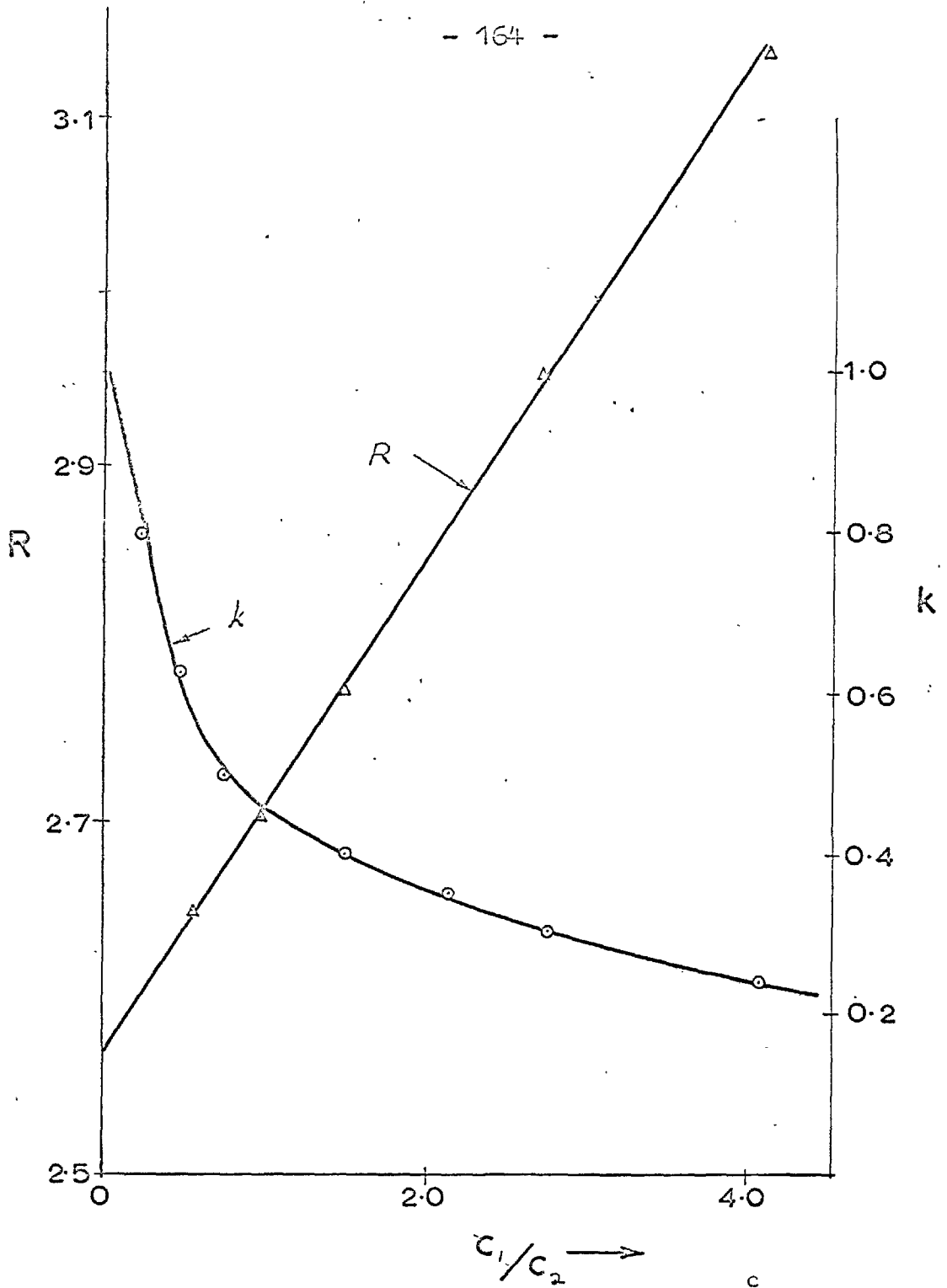


Figure 4.52 OPTIMUM PARAMETER VALUES vs $\frac{c_1}{c_2}$ FOR A
COST FUNCTION OF THE FORM $C = c_1 \sigma_Q + c_2 R + c_2 \frac{D}{2}$

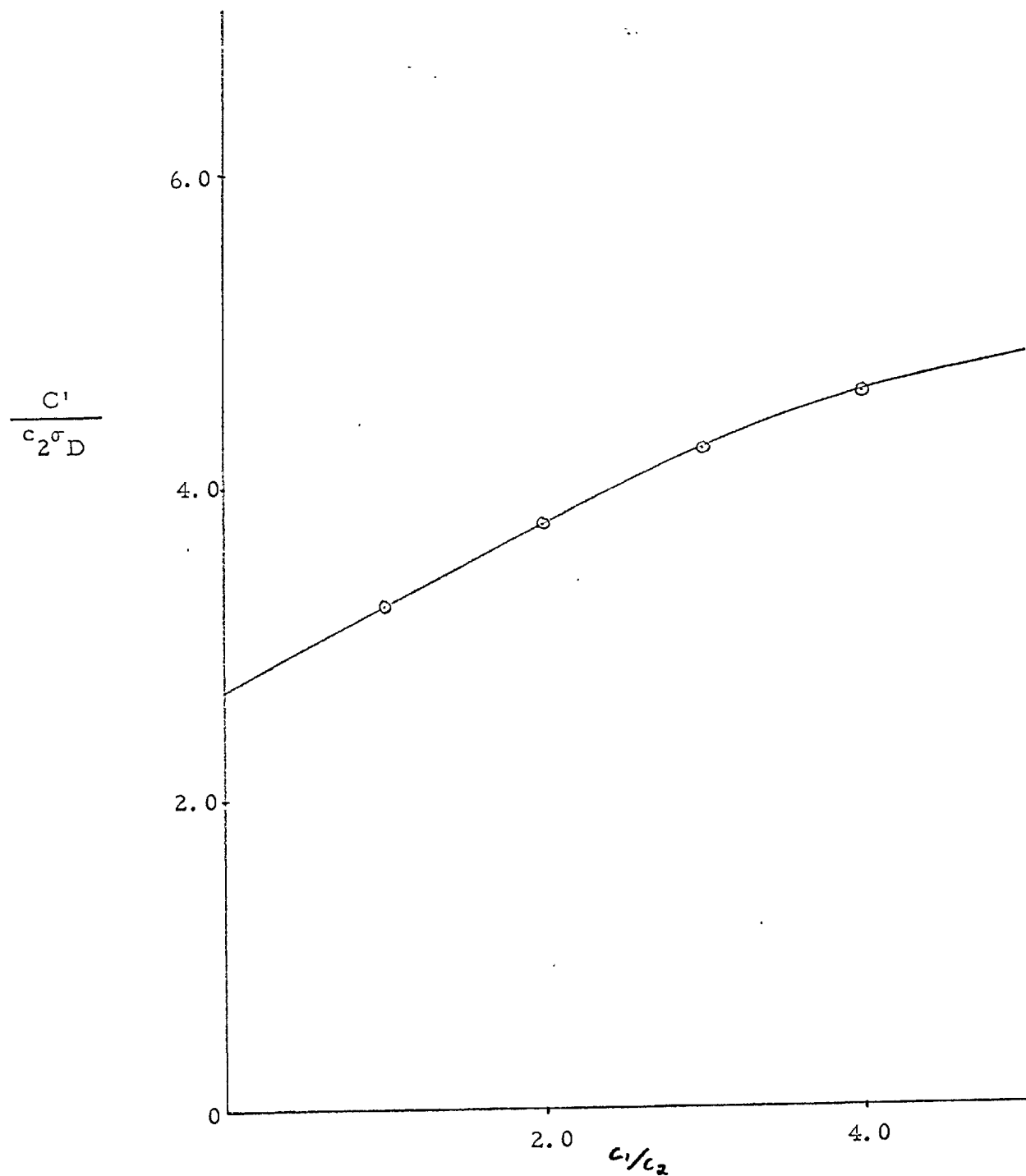


Figure 4.53 $\frac{C'}{c_2^{\sigma_D}}$ vs $\frac{c_1}{c_2}$

CHAPTER 5
GENERATION OF PROBABILITY
DENSITY FUNCTIONS

Notation

$P(x)$ = probability density function

$G(x)$ = probability distribution function

$\bar{x}$ = average

σ = standard deviation

$\mu, \alpha, \lambda, k, b, c$ = constants

$\text{sgn } \bar{}$ = sign of

$\Gamma(k)$ = gamma function

CHAPTER 5 : GENERATION OF PROBABILITY DENSITY FUNCTIONS

5.1 INTRODUCTION

Many operational research problems are concerned with the manipulation of probability density functions. Examples occur frequently in the literature and various techniques have been applied to their solution.

In at least one case [31] the analogue computer has been employed, and it appears that this technique might be useful for solving many such problems given that the appropriate probability density functions can be generated. This chapter deals with this requirement describing how the analogue computer was used to generate various probability density functions.

Three methods of function generation are commonly employed with analogue computers:

- (1) Diode function generation; where the function is approximated by a series of straight line segments using a diode-resistance network.
- (2) The 'direct' programming method (Appendix I).
- (3) The 'indirect' programming method (Appendix I).

Of these the third is in general the most satisfactory, although it applies only when the

density function can be obtained as the solution to some differential equation. Fortunately this is often the case, and the following describes how various well known probability density functions can be written implicitly as solutions to differential equations which can be solved on an analogue computer.

Preliminary Definitions

A probability density function is defined as $P(x)$ where

$$\int_u^{u+\Delta} P(x) dx = \text{probability that } u \leq x \leq u + \Delta u$$

and $P(x)$ is subject to the conditions:

- (i) $P(x) \geq 0$, $-\infty < x < +\infty$
- (ii) $\int_{-\infty}^{\infty} P(x) = 1$

A probability distribution function is defined as $G(u)$ where

$$G(u) = \int_{-\infty}^u p(x) dx \quad (5.1)$$

5.2 THE NORMAL DISTRIBUTION

The expression for the normal density curve is

$$P(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}} \quad (5.2)$$

where:

x = independent variable

$\bar{x}$ = average

σ = standard deviation

Setting,

$$z = \frac{x-\bar{x}}{\sigma}$$

(5.2) becomes,

$$P(z) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{z^2}{2}} \quad (5.3)$$

Differentiating both sides of (5.3) with respect to z yields,

$$P'(z) = -z P(z) \quad (5.4)$$

An analogue circuit for solving this differential equation is given in Figure 5.1. To obtain reasonable accuracy an amplitude scale factor of 200 was adopted over the range $-3\sqrt{2} \leq z \leq 2\sqrt{2}$. This factor was unsuitable, however, over the range $-2\sqrt{2} \leq z \leq 2\sqrt{2}$ and a comparator switch was used to change the scale factor from 200 to 20 for this range. The comparator was actuated at

$z = -3\sigma$ to initiate the function generation, and at $z = -2\sigma$ to switch the scale factor. This was accomplished by substituting a second integrator with appropriately scaled initial condition to operate over the range $z \geq -2\sigma$. It was found unnecessary to return to the large scale factor in the range $2\sigma \leq z \leq 3\sigma$ as errors in the function over this range had negligible effect on the final result.

Figures 5.2 and 5.3 show plots of the normal density function and distribution function for values of $\sigma = 1.0, 2.0$ and 3.0 .

5.3 THE RAYLEIGH DISTRIBUTION

The expression for the Rayleigh density curve is

$$P(x) = \frac{2x}{\mu^2} e^{-\left(\frac{x}{\mu}\right)^2} \quad x \geq 0 \quad (5.5)$$

where $\mu = \text{constant}$

$$\bar{x} = \frac{\mu\sqrt{\pi}}{2}$$

$$\sigma^2 = \mu^2 \left[2 - \frac{\pi}{2} \right]$$

setting $z = \frac{x}{\mu}$, (5.5) becomes,

$$P(z) = \frac{2z}{\mu} e^{-z^2} \quad (5.6)$$

and differentiating twice we obtain,

$$P''(z) = -4P(z) - 2z P'(z) \quad (5.7)$$

The analogue was programmed to solve (5.7) with initial conditions

$$P(0) = P''(0) = 0$$

$$P'(0) = \frac{2}{\mu}$$

The circuit is shown in Figure 5.4, and Figures 5.5 and 5.6 show some results that were obtained for $\mu=1.0$ to 4.0.

5.4 THE LAPLACE DISTRIBUTION

The expression for the Laplace density function is

$$P(x) = \frac{\alpha}{2} e^{-\alpha |x-\bar{x}|} \quad (5.8)$$

where, $\bar{x}$ = average

$$\text{and } \sigma^2 = \frac{2}{\alpha^2}$$

Setting $z = x-\bar{x}$, (5.8) becomes

$$P(z) = \frac{\alpha}{2} e^{-\alpha |z|} \quad (5.9)$$

and differentiating yields,

$$P'(z) = \frac{\alpha}{2} \left(\frac{1}{-\alpha \operatorname{sgn} z} \right) e^{-\alpha |z|} \quad (5.10)$$

where $\operatorname{sgn} \equiv$ sign of

An analogue circuit for solving (5.10) is shown in Figure 5.7, and the circuit operates as follows; comparator no. 1 switches to change the sign of $P'(z)$ as z goes from negative to positive. It was found

necessary to use a scaling factor of 200 over the range $z < -3.0$, and comparator no. 2 switches to reduce the scaling factor to 20 as z increases above -3.0 . It was not necessary to return to the large scale factor for $z \geq 3.0$, as errors over this range have negligible effect.

Some results are shown in Figures 5.8 and 5.9 for $\alpha = 0.5, 0.75, 1.0$.

5.5 THE GAMMA DISTRIBUTION

The expression for the Gamma density function is:

$$P(x) = \frac{\lambda (\lambda x)^{k-1} e^{-\lambda x}}{\Gamma(k)} \quad x \geq 0 \quad (5.11)$$

where $\Gamma(k)$ is the gamma function (i.e. $\Gamma(k) = (k-1)!$) and λ, k are constants, with,

$$\bar{x} = \frac{k}{\lambda}$$

$$\sigma^2 = \frac{k}{\lambda^2}$$

If we set $k = \lambda$ (so that $\bar{x}=1$), (5.11) becomes,

$$P(x) = \frac{k^k x^{k-1} e^{-kx}}{\Gamma(k)} \quad (5.12)$$

Now consider the differential equation,

$$y^{(k)} + \binom{k}{1} k y^{(k-1)} + \binom{k}{2} k^2 y^{(k-2)} + \dots + k^k = 0 \quad (5.13)$$

This has as its solution,

$$y = \lambda_1 e^{-kx} + \lambda_2 x e^{-kx} + \lambda_3 x^2 e^{-kx} + \dots + \lambda_k x^{k-1} e^{-kx} \quad (5.14)$$

with,

$$y(0) = \lambda_1$$

$$\dot{y}(0) = -k\lambda_1 + \lambda_2$$

$$\ddot{y}(0) = k^2\lambda_1 - 2k\lambda_2 + 2\lambda_3$$

$$\vdots$$

$$y^{(k-1)}(0) = (-1)^{k-1} k^{k-1} \lambda_1 + \dots + (k-1)! \lambda_k$$

If we choose initial conditions such that

$$\lambda_1 = \lambda_2 = \dots \lambda_{k-1} = 0$$

$$\lambda_k = \frac{k^k}{\Gamma(k)} = \frac{k^k}{(k-1)!}$$

then (5.14) is the same as (5.12) which is the required gamma distribution. Thus we set,

$$y(0) = 0$$

$$\dot{y}(0) = 0$$

$$\vdots$$

$$y^{(k-2)}(0) = 0$$

$$\text{and } y^{(k-1)}(0) = k^k$$

A generalized circuit for generating the gamma density function when λ and k are integers is given in Figure 5.10 and Table 5.1 gives scaled coefficient

settings for values $k=\lambda$ from 2 to 10.

Figures 5.11 and 5.12 show some results that were obtained for these values.

5.6 THE BETA DISTRIBUTION

The expression for the Beta density function is

$$P(x) = \begin{cases} A x^b (1-x)^c & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases} \quad (5.15)$$

where

$$A = \frac{\Gamma(b+c+2)}{\Gamma(b+1)\Gamma(c+1)}$$

$$\bar{x} = \frac{b+1}{b+c+2}$$

$$\sigma^2 = \frac{(b+1)(c+1)}{(b+c+2)(b+c+3)}$$

A circuit for generating the Beta density curve by the 'direct' method is given in Figure 5.13 for b and c integers, and some results are given in Figures 5.14 and 5.15 for b and $c \leq 2$.

5.7 THE MAXWELL DISTRIBUTION

The Maxwell density function is given by,

$$P(x) = \frac{\sqrt{2}}{\alpha^3 \sqrt{\pi}} x^2 e^{-\frac{x^2}{2\alpha^2}} \quad (5.16)$$

where

$$\bar{x} = 2\alpha \sqrt{\frac{2}{\pi}}$$

and

$$\sigma^2 = 3\alpha^2 - \frac{4\alpha^2}{\pi}$$

Substituting $z = \frac{x}{\alpha}$ in (5.16) we obtain,

$$P(z) = \frac{1}{\alpha} \sqrt{\frac{2}{\pi}} z^2 e^{-\frac{z^2}{2}}$$

which upon repeated differentiation yields,

$$P'(z) = \frac{2}{\alpha} \sqrt{\frac{2}{\pi}} e^{-z^2} - 2z P(z) \quad (5.17)$$

$$P''(z) = \frac{2}{\alpha} \sqrt{\frac{2}{\pi}} z e^{-z^2} - 6P(z) - 2zP'(z) \quad (5.18)$$

$$P'''(z) = -\frac{4}{\alpha} \sqrt{\frac{2}{\pi}} z e^{-z^2} - 8P'(z) - 2zP''(z) \quad (5.19)$$

From (5.17) we can write

$$-\frac{4}{\alpha} \sqrt{\frac{2}{\pi}} z e^{-z^2} = -2P'(z) - 4zP(z)$$

and substituting this into (5.19) we obtain,

$$P'''(z) = -4zP(z) - 10P'(z) - 2zP''(z) \quad (5.20)$$

The analogue was programmed to solve (5.20),
with initial conditions:

$$P(0) = P'(0) = P'''(0) = 0$$

$$P''(0) = \frac{2}{\alpha} \sqrt{\frac{2}{\pi}}$$

The analogue programme is shown in Figure 5.16 and some results are given in Figures 5.17 and 5.18 for $\alpha=1.0$ and 2.0 .

5.8 CONCLUSIONS

The analogue computer has been used to generate various probability density functions. Although in some cases the choice of distribution parameters is restricted to integral values, the analogue appears to be well suited to this task. Results obtained are accurate to within $\pm 2\%$ of peak values in the case of density functions and to within $\pm .04$ in the case of distribution functions.

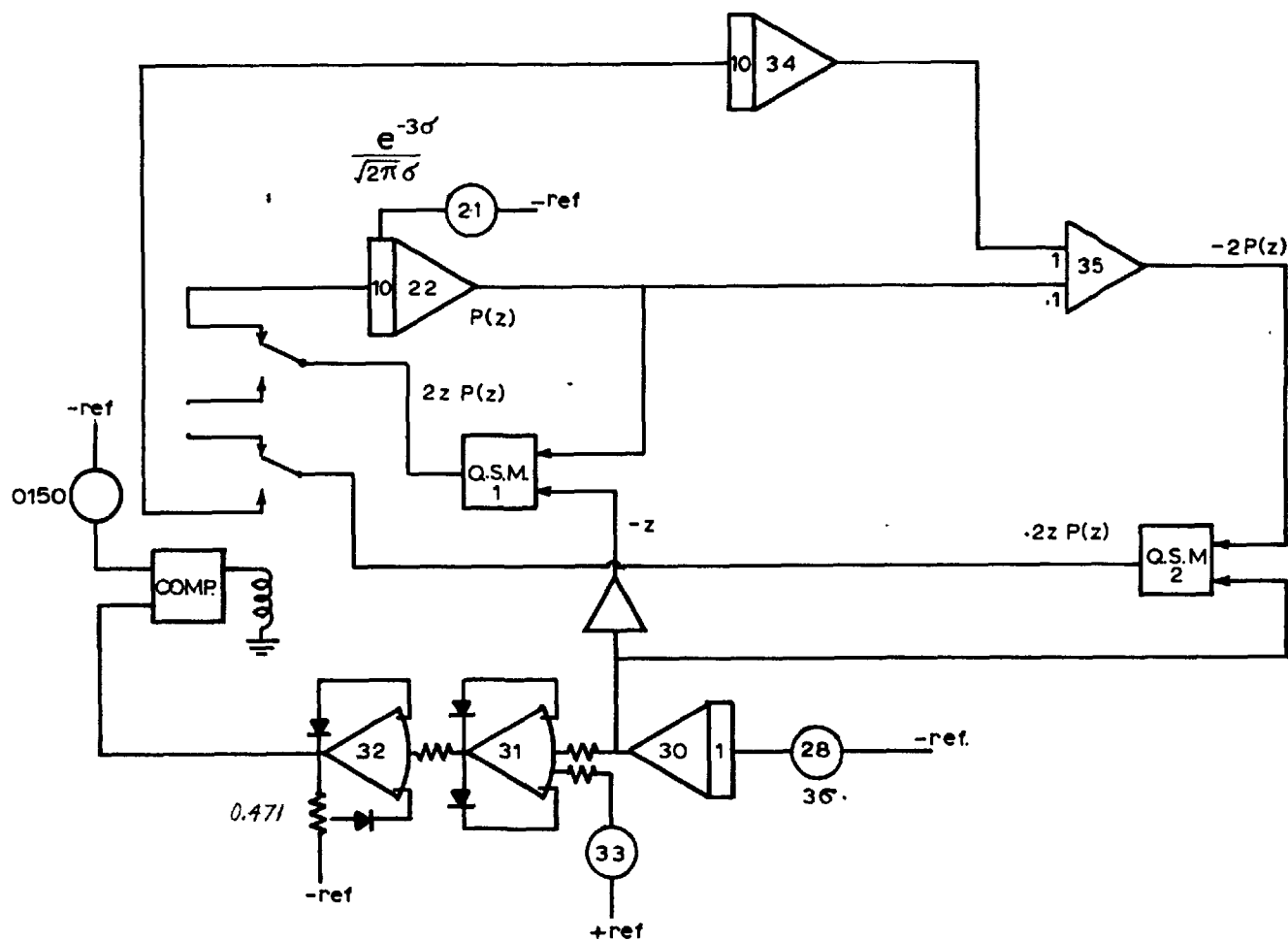


Figure 5.1 CIRCUIT TO GENERATE A NORMAL DENSITY FUNCTION

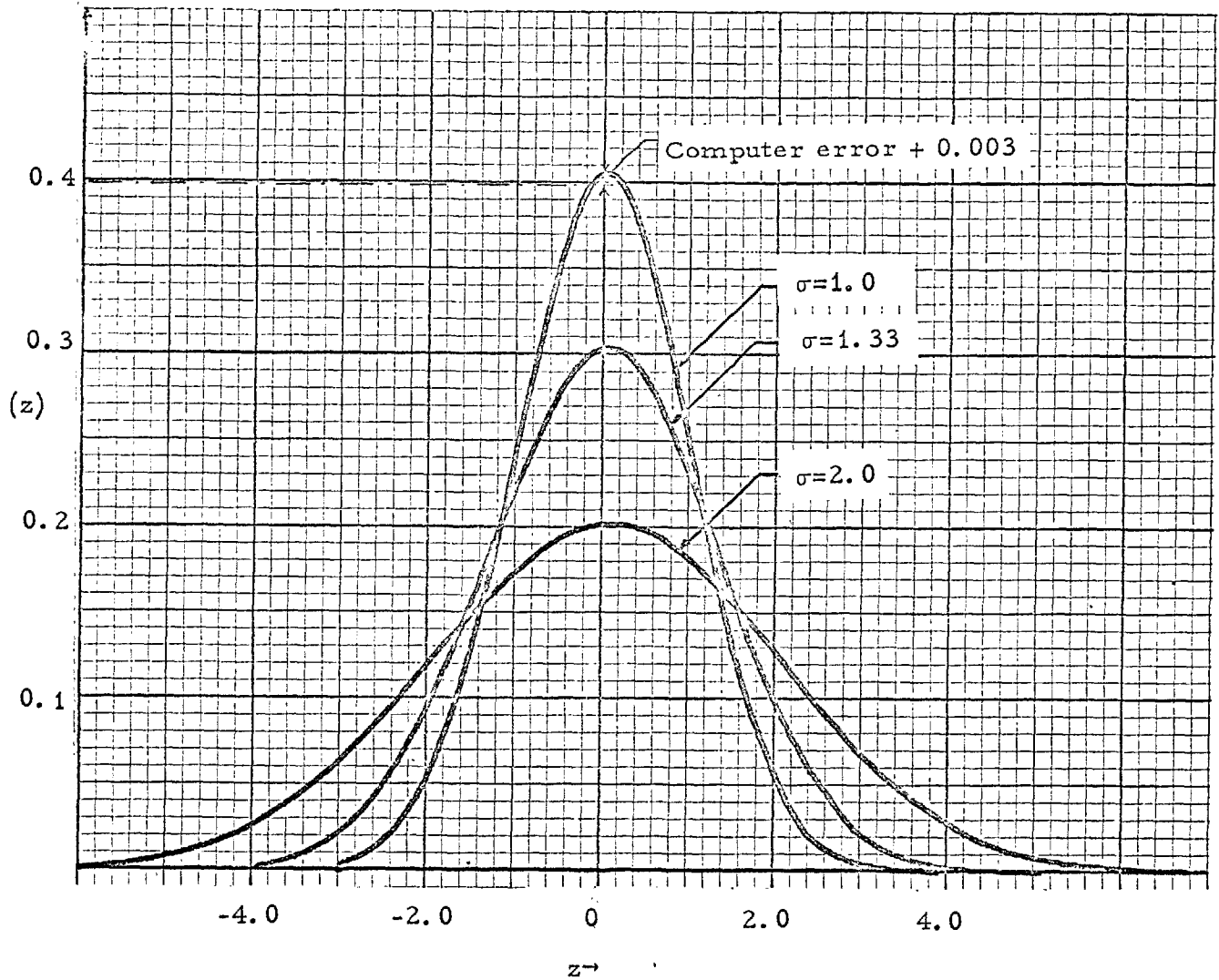


Figure 5.2 NORMAL DENSITY CURVES FOR $\sigma=1.0, 1.33, 2.0$

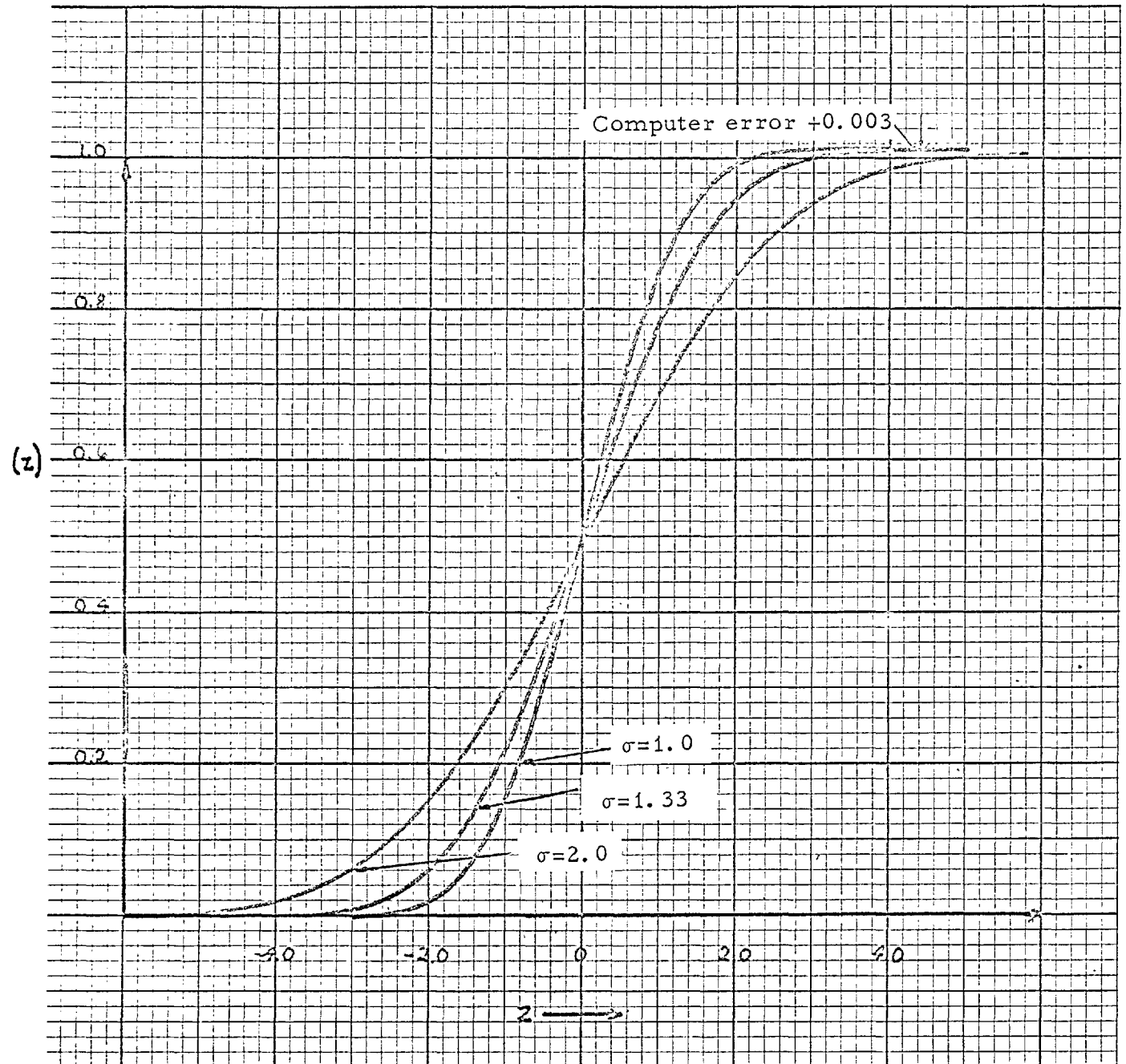


Figure 5.3 CURVES OF THE NORMAL DISTRIBUTION FUNCTION FOR
 $\sigma = 1.0, 1.33, 2.0$

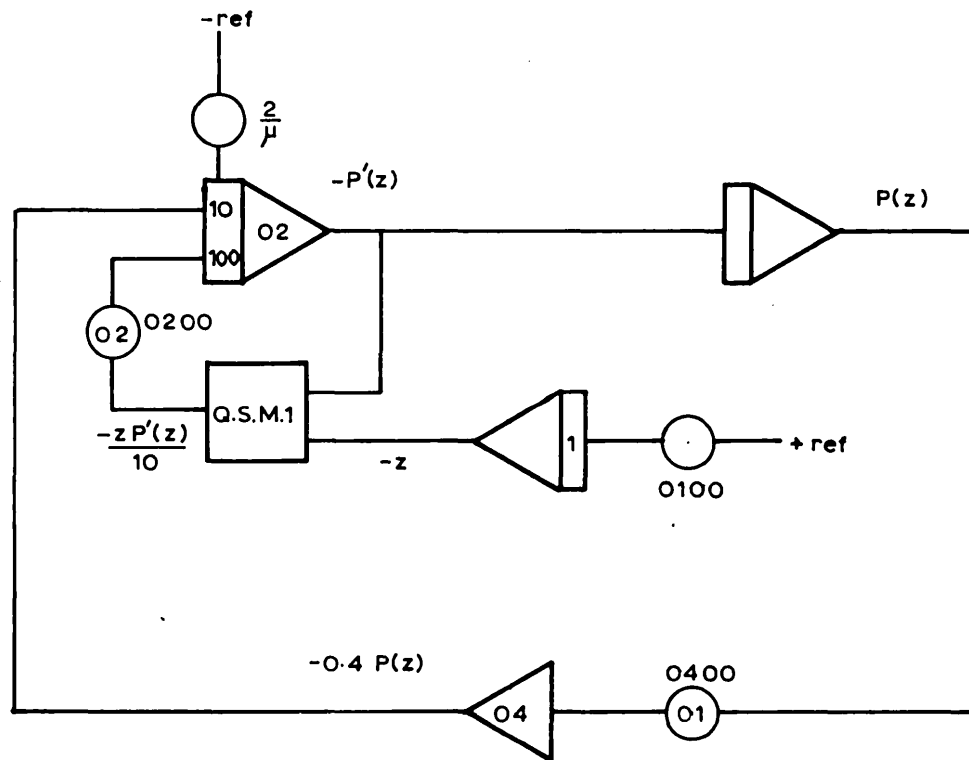


Figure 5.4 CIRCUIT FOR GENERATING A RAYLEIGH DENSITY FUNCTION

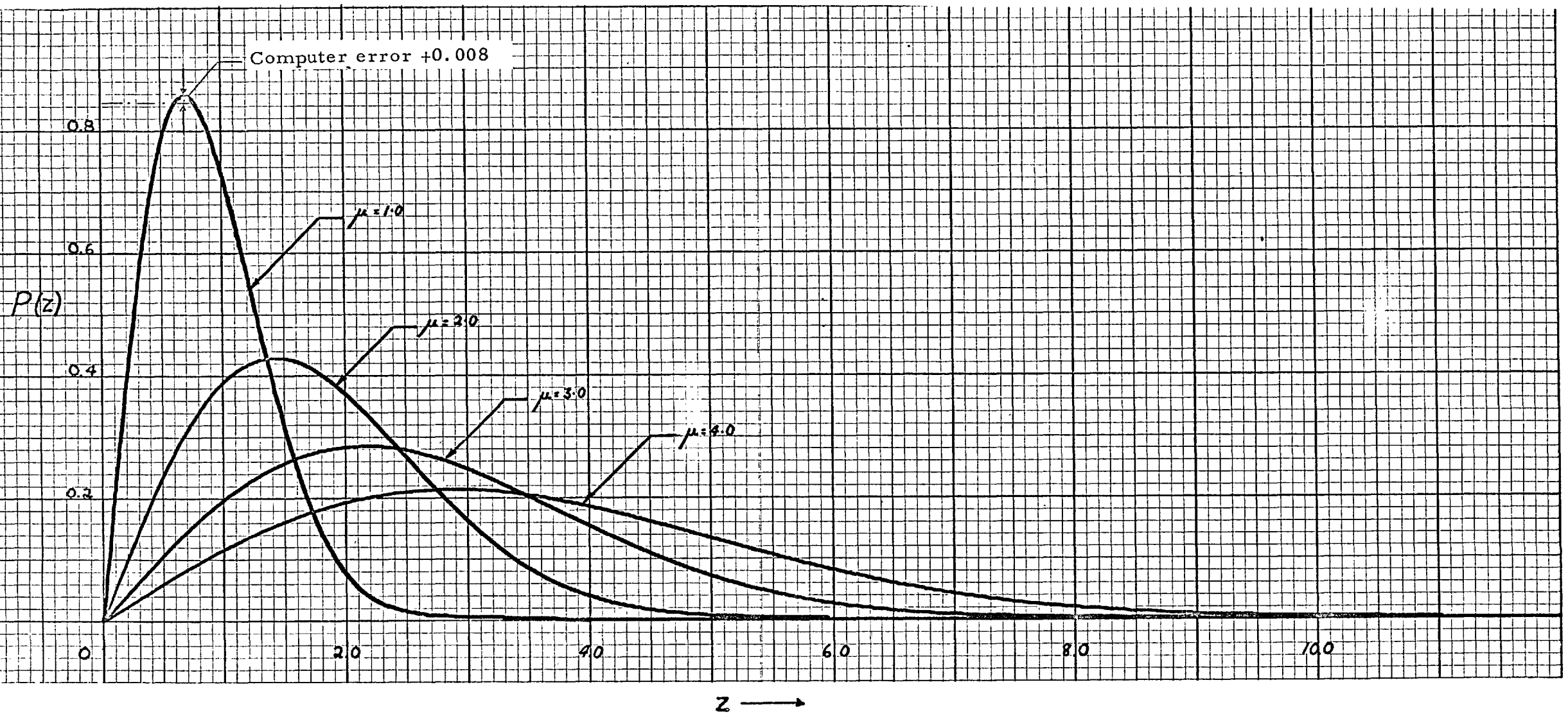


Figure 5.5 RAYLEIGH DENSITY FUNCTION FOR $\mu = 1.0, 2.0, 3.0, 4.0$

$G(z)$

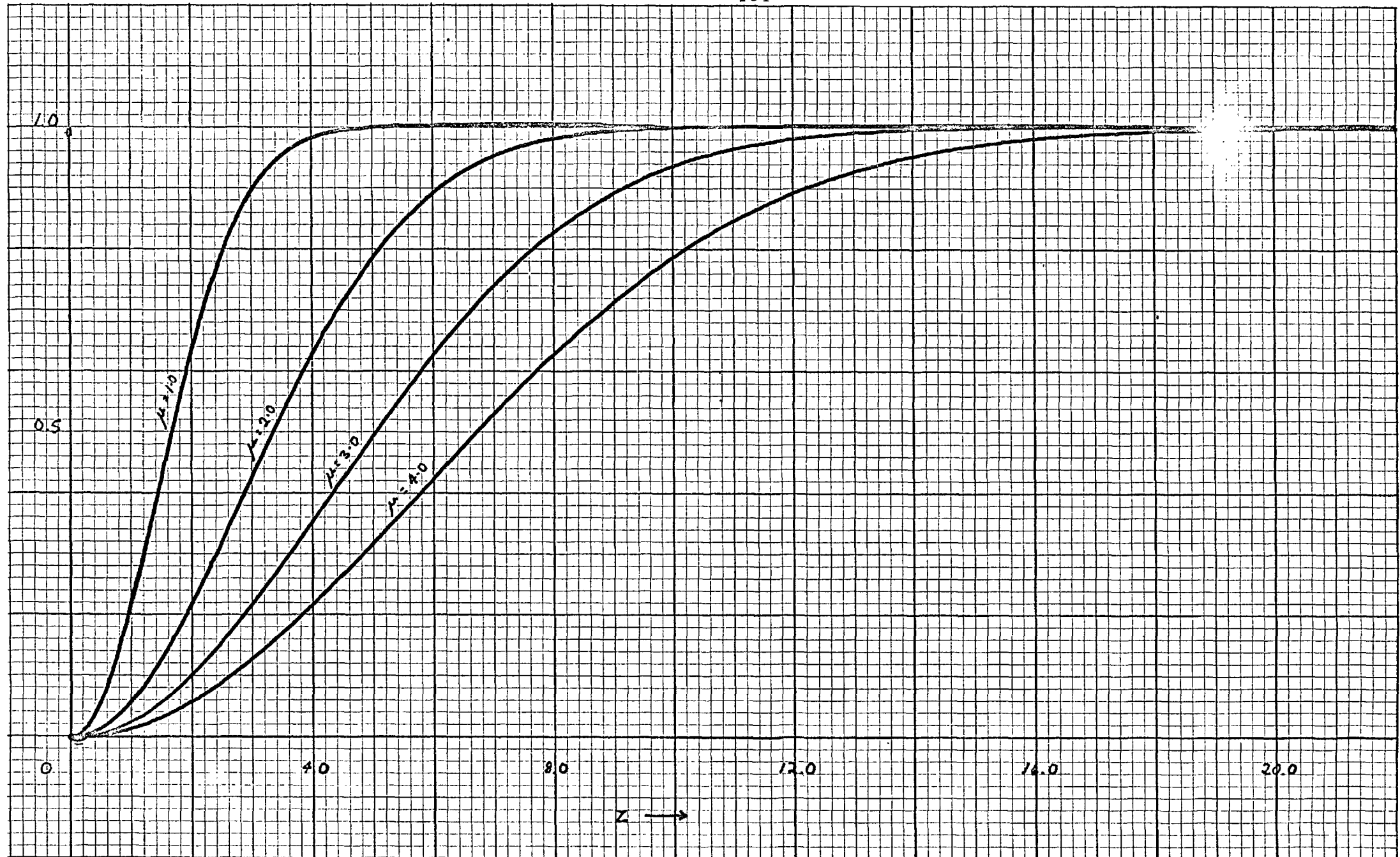


Figure 5.6 RAYLEIGH DISTRIBUTION FUNCTION FOR $\mu=1.0, 2.0, 3.0, 4.0$

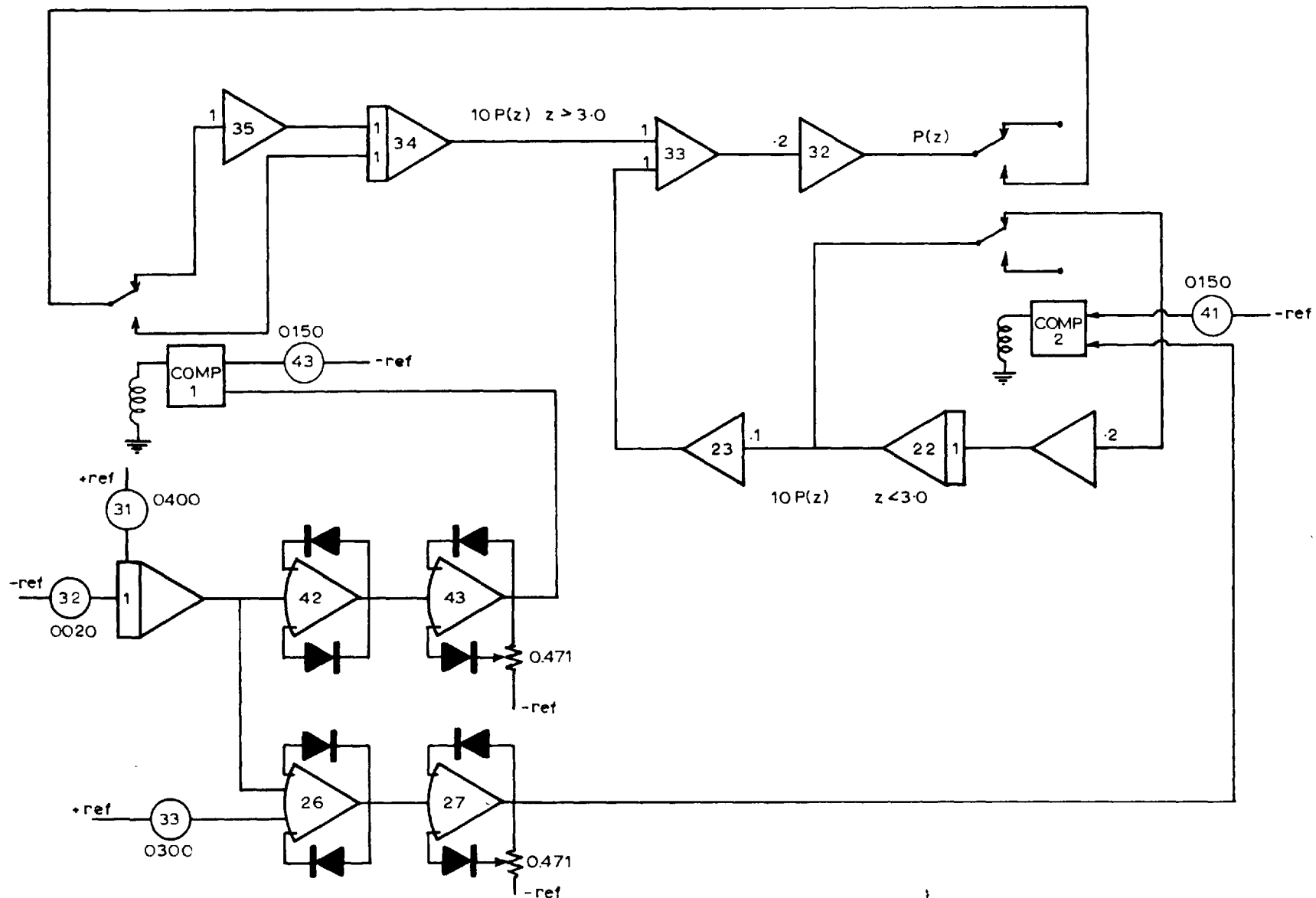


Figure 5.7 ANALOGUE CIRCUIT FOR GENERATING THE LAPLACE DENSITY FUNCTION

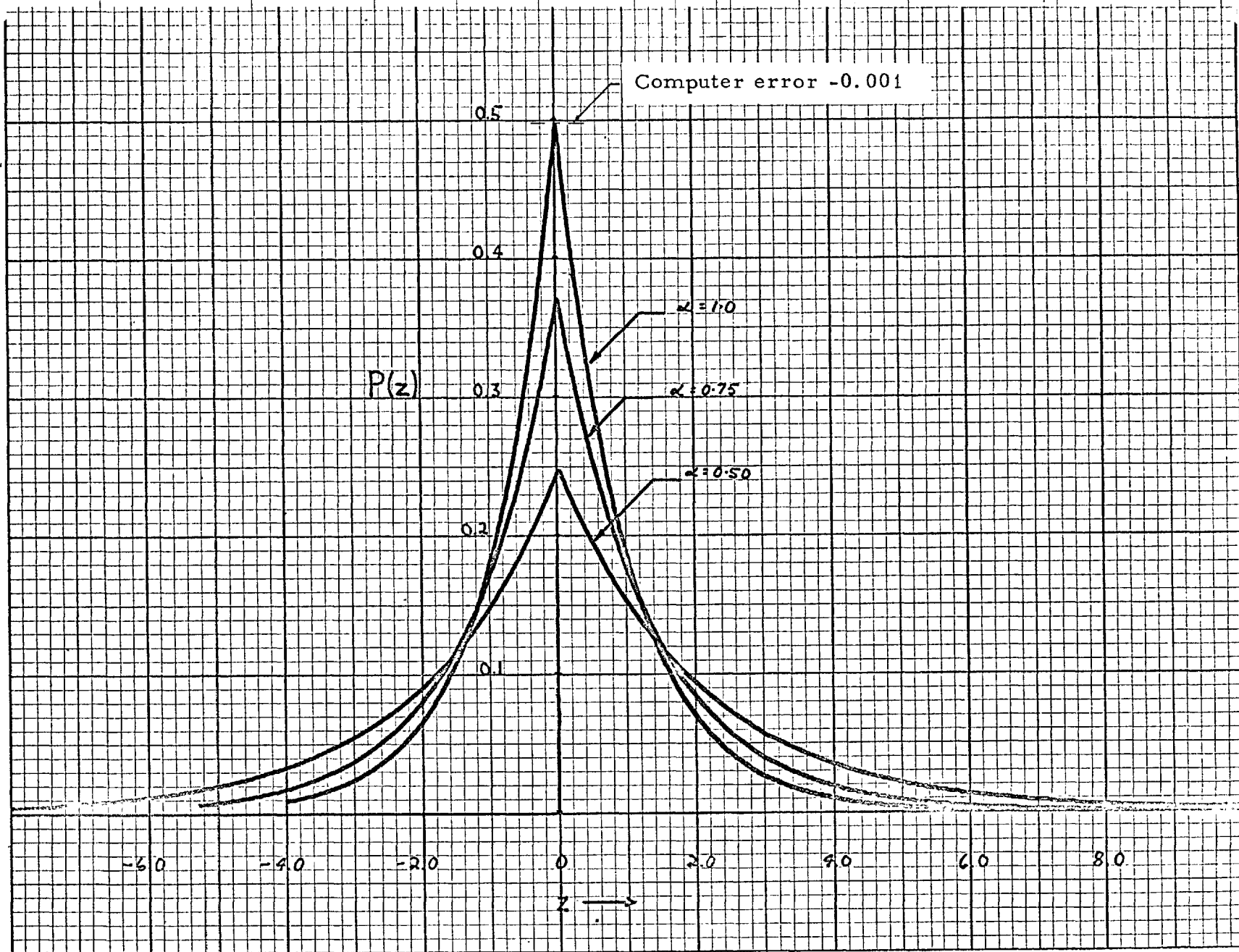


Figure 5.8 LAPLACE DENSITY FUNCTION $\alpha=0.5, 0.75, 1.0$

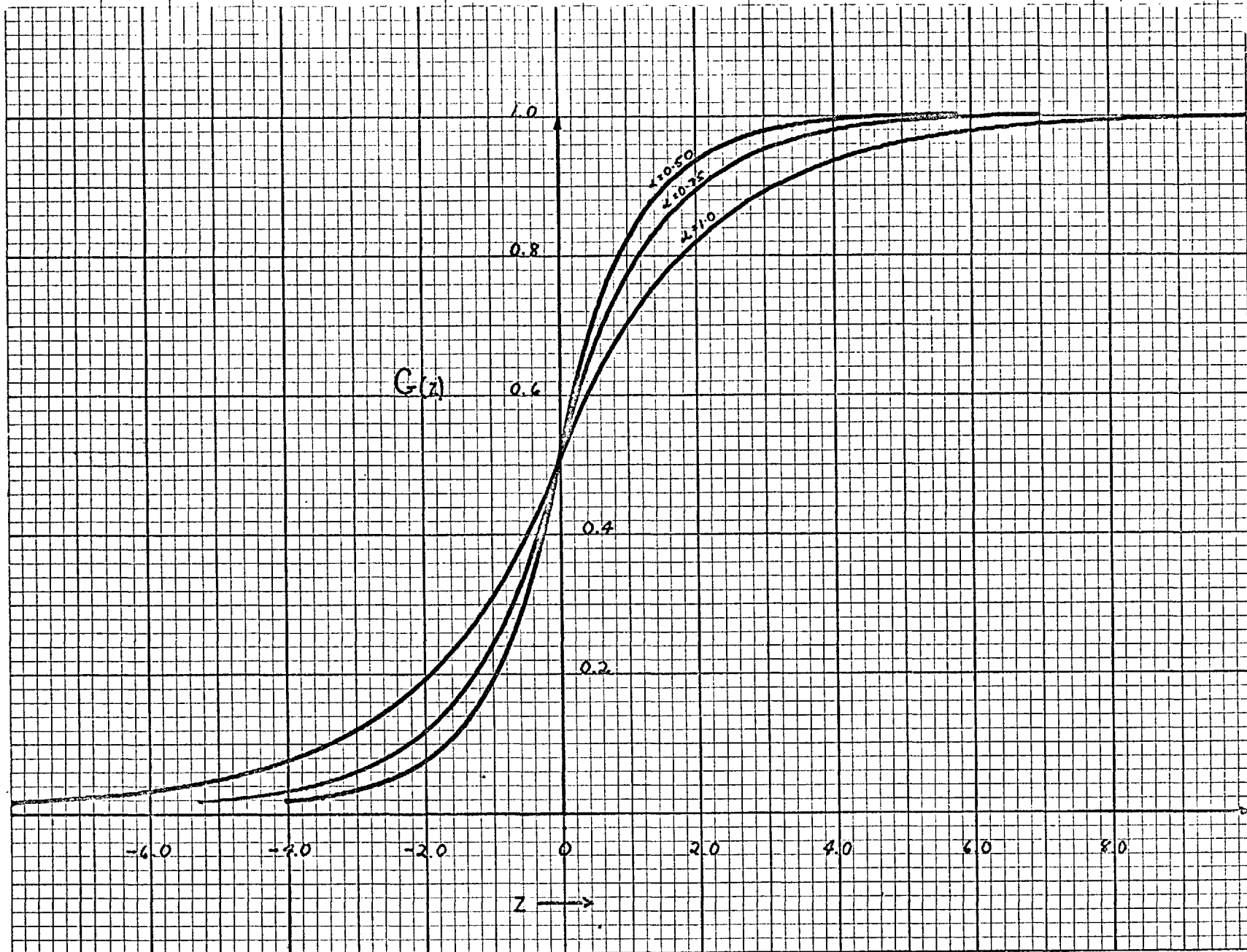


Figure 5.9 LAPLACE DISTRIBUTION FUNCTION FOR $\alpha=0.5, 0.75, 1.0$

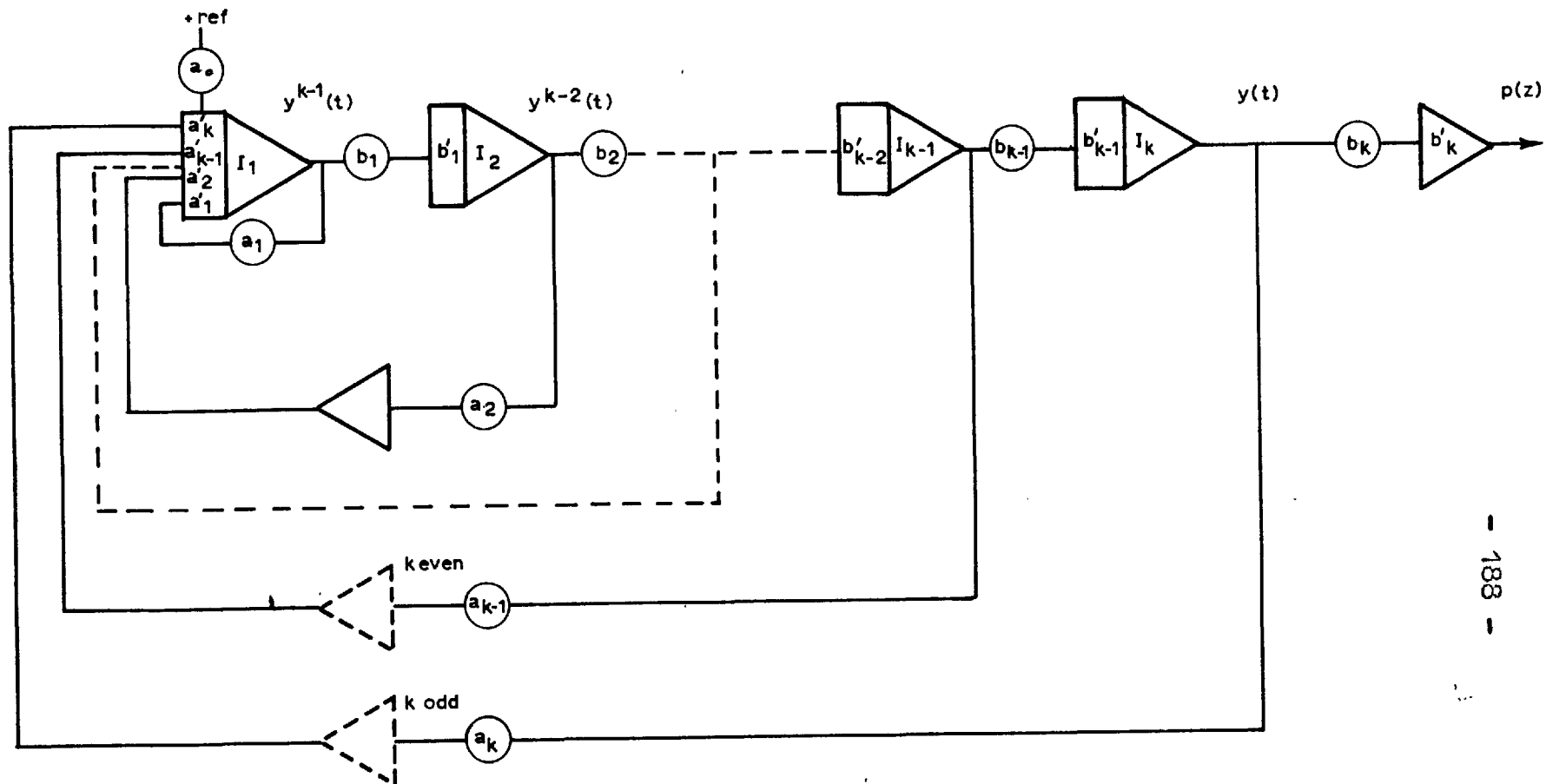


Figure 5.10 GENERALISED CIRCUIT FOR GENERATING THE GAMMA DENSITY FUNCTION

Table 5.1 Coefficient Settings for Gamma Density Function Generation
(See Figure 5.8)

k	k	2					3					4				
		A _k	a _k	a' _k	b _k	b' _k	A _k	a _k	a' _k	b _k	b' _k	A _k	a _k	a' _k	b _k	b' _k
0	1	0.400	-	-	-	-	1	0.270	-	-	-	1	0.512	-	-	-
1	4	0.400	10	0.400	10	9	0.900	10	1.000	10	16	0.800	1000	1.000	10	
2	4	0.100	10	1.000	1	27	0.270	10	0.100	10	96	0.480	1000	0.500	10	
3							27	0.270	10	1.000	1	256	0.256	1000	1.000	10
4												256	0.256	100	0.100	1
5																
6																
7																
8																
9																
10																

Table 5.1 (Continued)

k	k	5					6					7				
		A _k	a _k	a' _k	b _k	b' _k	A _k	a _k	a' _k	b _k	b' _k	A _k	a _k	a' _k	b _k	b' _k
0		1	0.625	-	-	-	1.0	0.933	-	-	-	1	0.823	-	-	-
1		25	0.250	100	1.000	10		0.360	100	1.000	10	49	0.490	100	1.000	10
2		250	0.250	100	1.000	10		0.540	100	1.000	10	1,029	0.102	1000	1.000	10
3		1250	0.125	100	0.500	10		0.432	100	1.000	10	12,005	0.120	1000	1.000	10
4		3125	0.625	10	1.000	10		0.194	100	1.000	10	84,035	0.840	100	1.000	10
5		3125	0.625	1	0.100	1		0.933	10	0.500	10	352,947	0.352	100	1.000	10
6								0.933	1	0.100	1	823,543	0.823	10	1.000	10
7												823,543	0.823	1	0.100	1
8																
9																
10																

Table 5.1 (Continued)

k	k	8					9					10				
		A _k	a _k	a' _k	b _k	b' _k	A _k	a _k	b _k	b' _k	A _k	a _k	b _k	b' _k	A _k	b' _k
0			1	0.838	-	-			1	0.774	-			1	1.000	-
1			64	0.640	100	1.000			81	0.810	1.000			100	1.000	1.000
2			1,792	0.179	100	1.000			2,916	0.292	1.000			45x10 ²	0.450	1.000
3			28,672	0.286	100	1.000			6,561	0.612	1.000			12x10 ⁴	0.120	1.000
4			286,726	0.286	100	1.000			59,049	0.827	1.000			21x10 ⁵	0.210	1.000
5			1,835,008	0.183	100	1.000			531,441	0.744	1.000			252x10 ⁵	0.252	1.000
6			7,340,032	0.734	10	1.000			4,782,969	0.446	1.000			21x10 ⁷	0.210	1.000
7			16,777,216	0.167	10	1.000			43,046,721	0.172	1.000			12x10 ⁸	0.120	1.000
8			16,777,216	0.167	1	0.200			387,420,489	0.387	1.000			45x10 ⁸	0.450	1.000
9									387,420,489	0.193	0.500			10 ¹⁰	10.00	0.500
10														10 ¹⁰	02.00	0.200

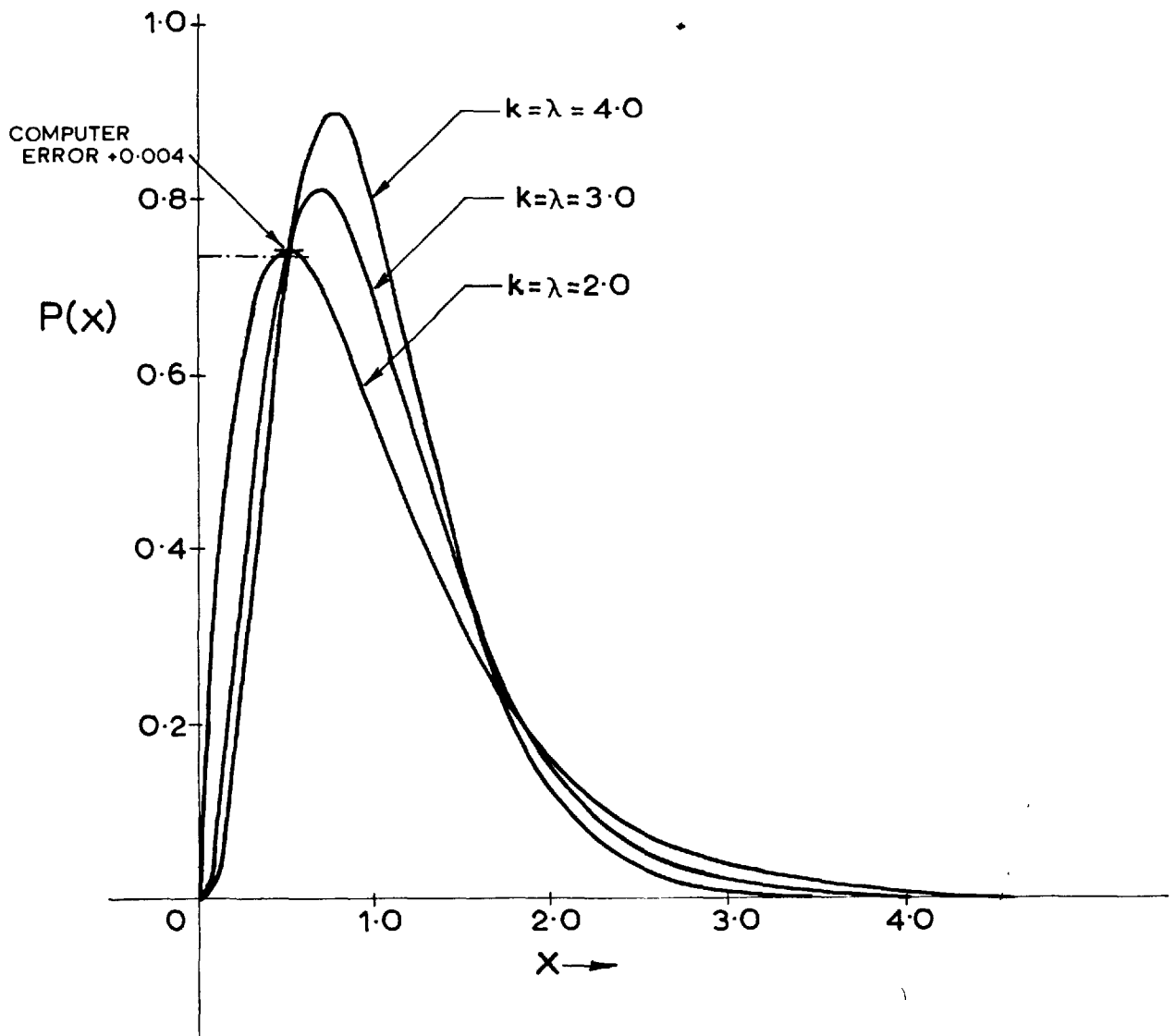


Figure 5.11(a) GAMMA DENSITY FUNCTION FOR $k=\lambda=2.0, 3.0, 4.0$

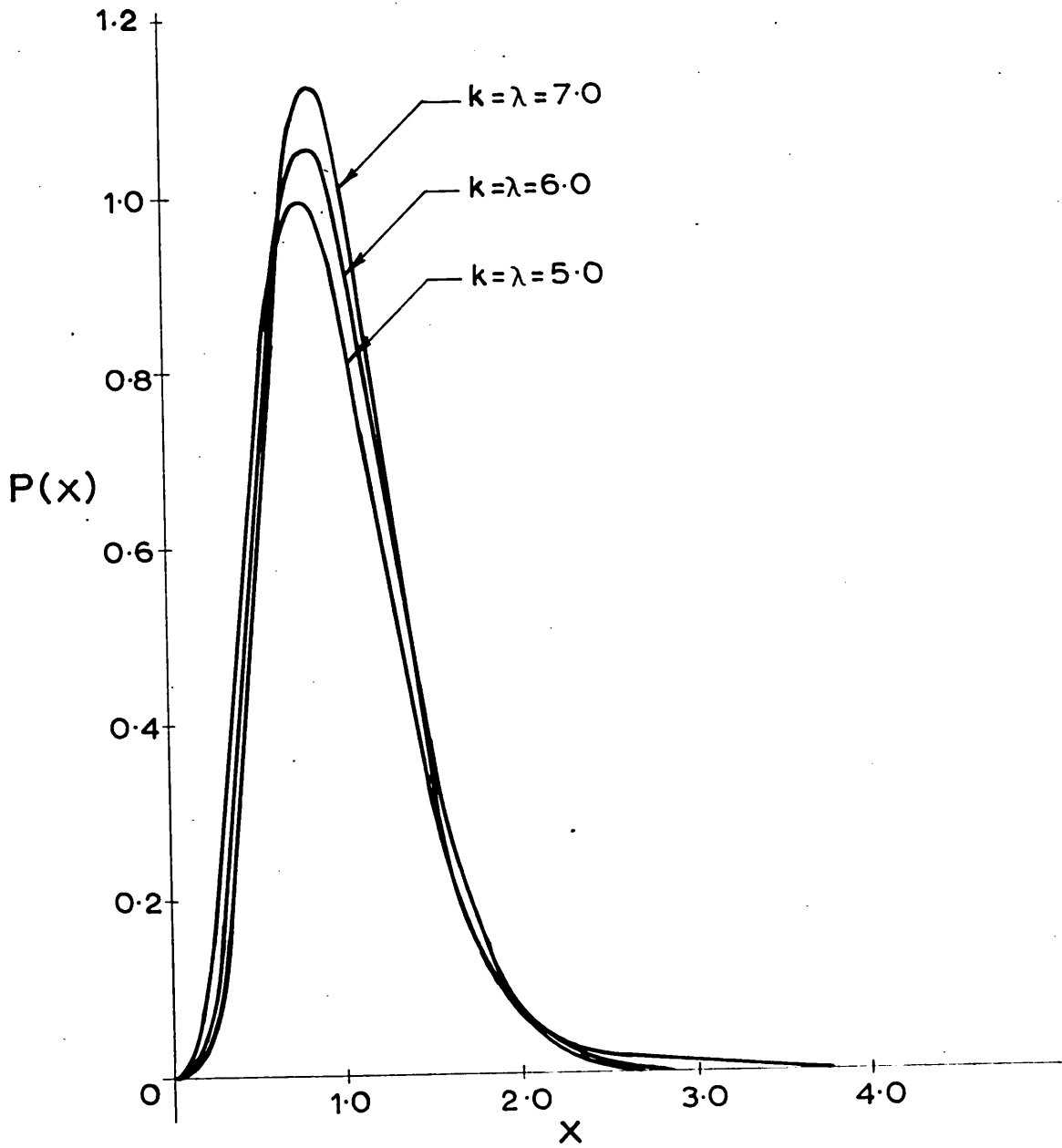


Figure 5.11(b) GAMMA DENSITY FUNCTION FOR $\lambda=k=5.0, 6.0, 7.0$

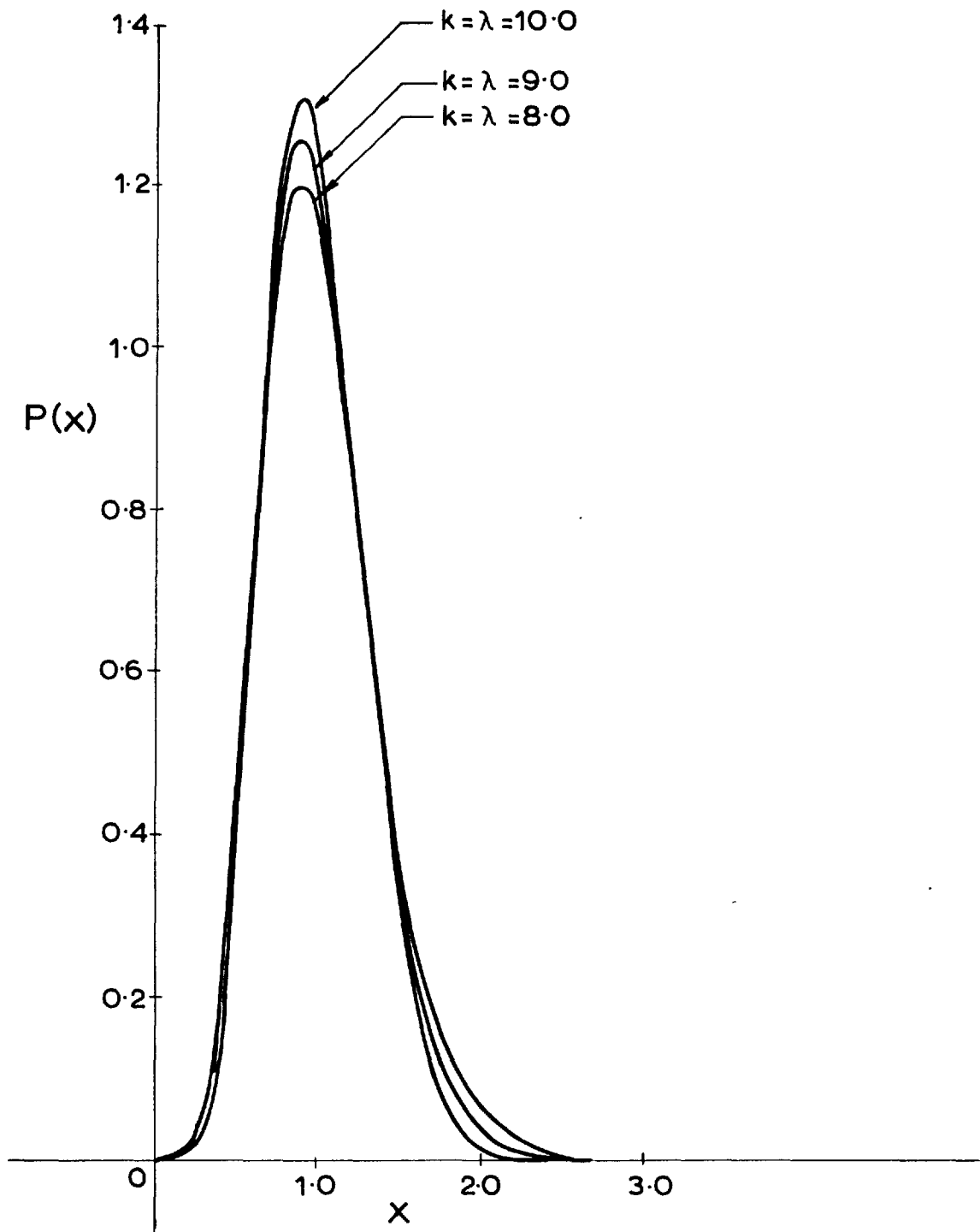


Figure 5.11(c) GAMMA DENSITY FUNCTION FOR $\lambda=k=8.0, 9.0, 10.0$

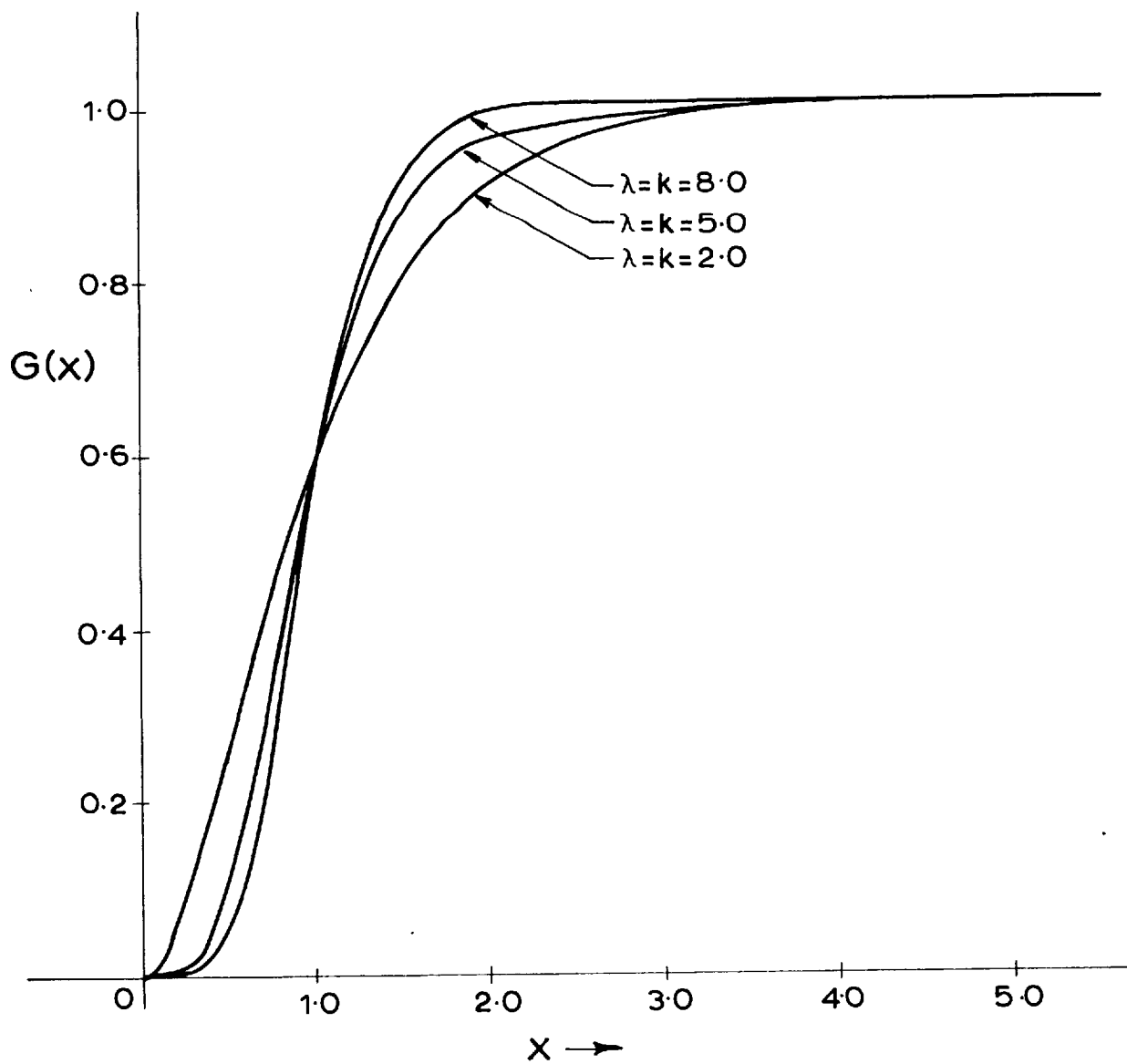


Figure 5.12 GAMMA DISTRIBUTION FUNCTION FOR $\lambda=k=2.0, 5.0, 8.0$

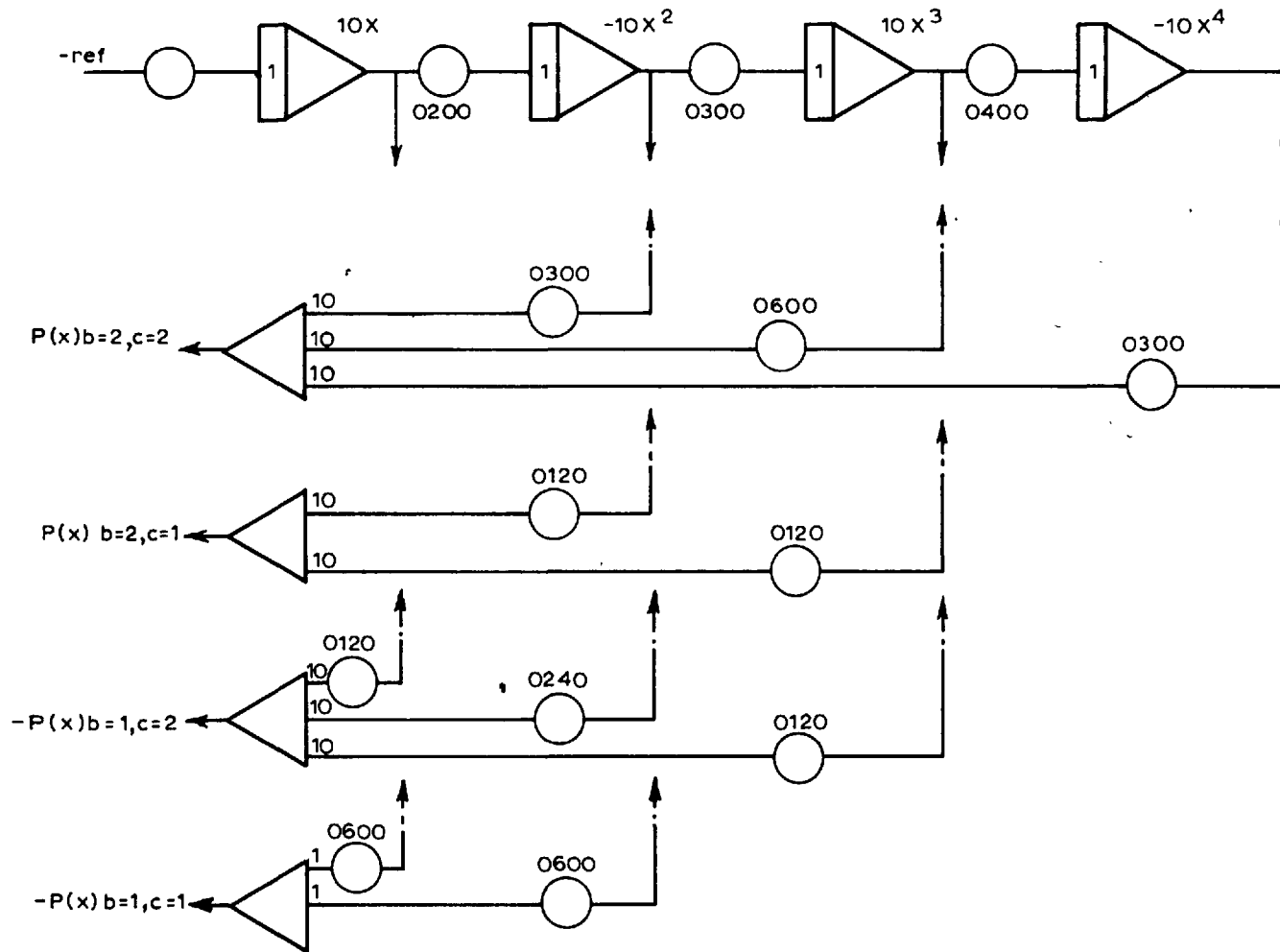


Figure 5.13 CIRCUIT FOR GENERATING BETA DISTRIBUTION WITH b AND c INTEGERS

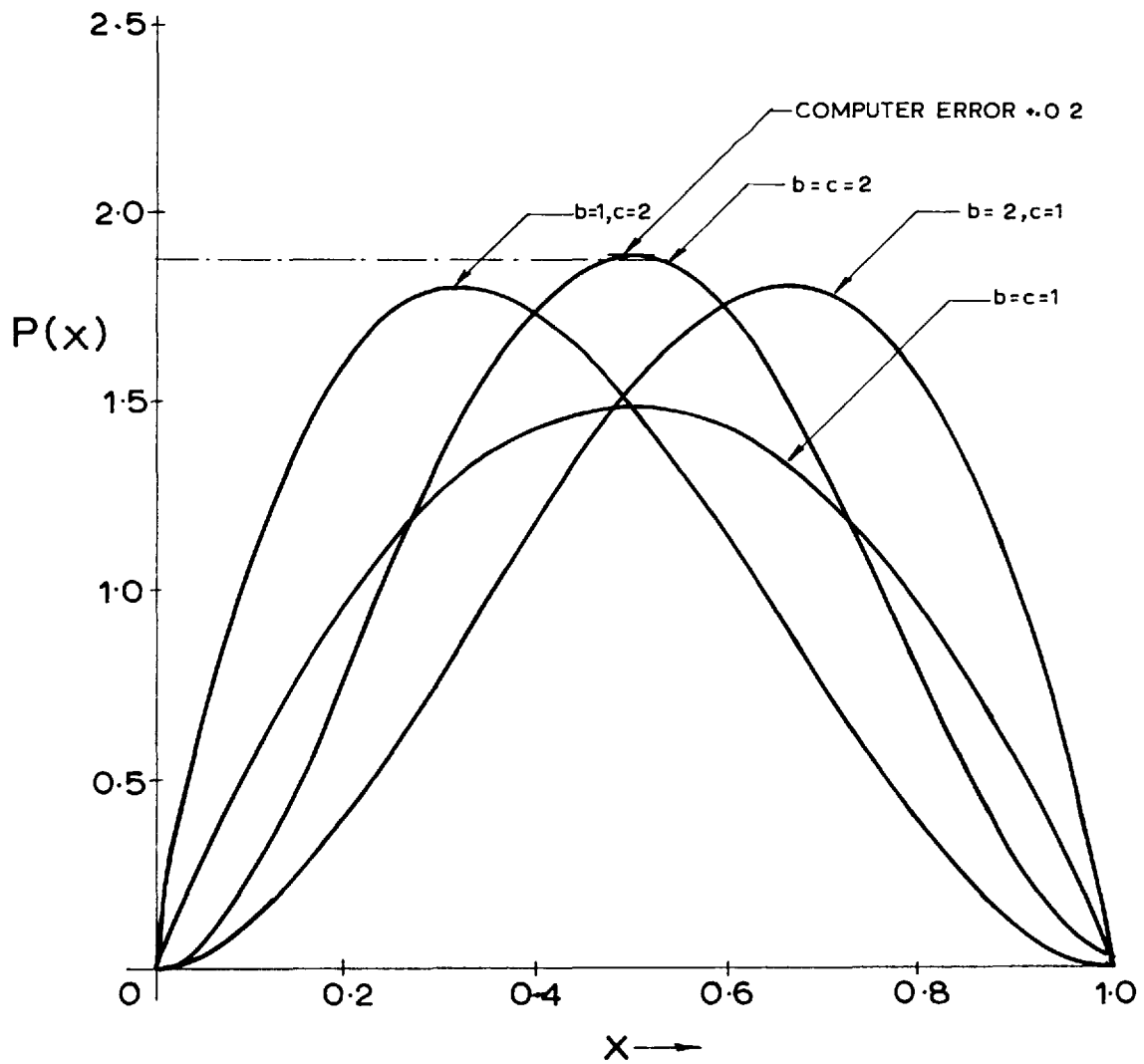


Figure 5.14 BETA DENSITY CURVE

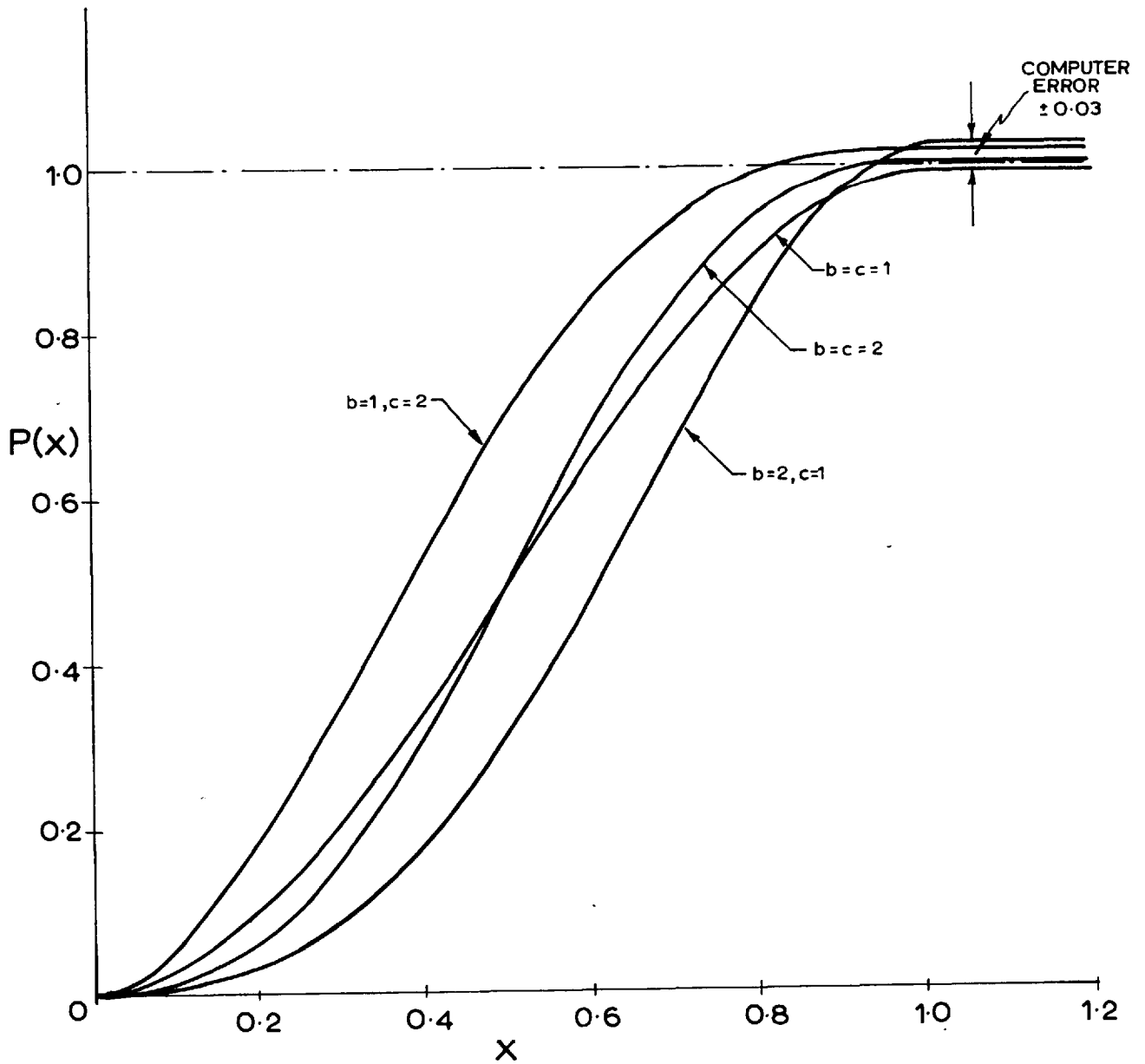


Figure 5.15 BETA DISTRIBUTION FUNCTION FOR $a=b=2$;
 $a=b=1$; $a=2, b=1$; $a=1, b=2$

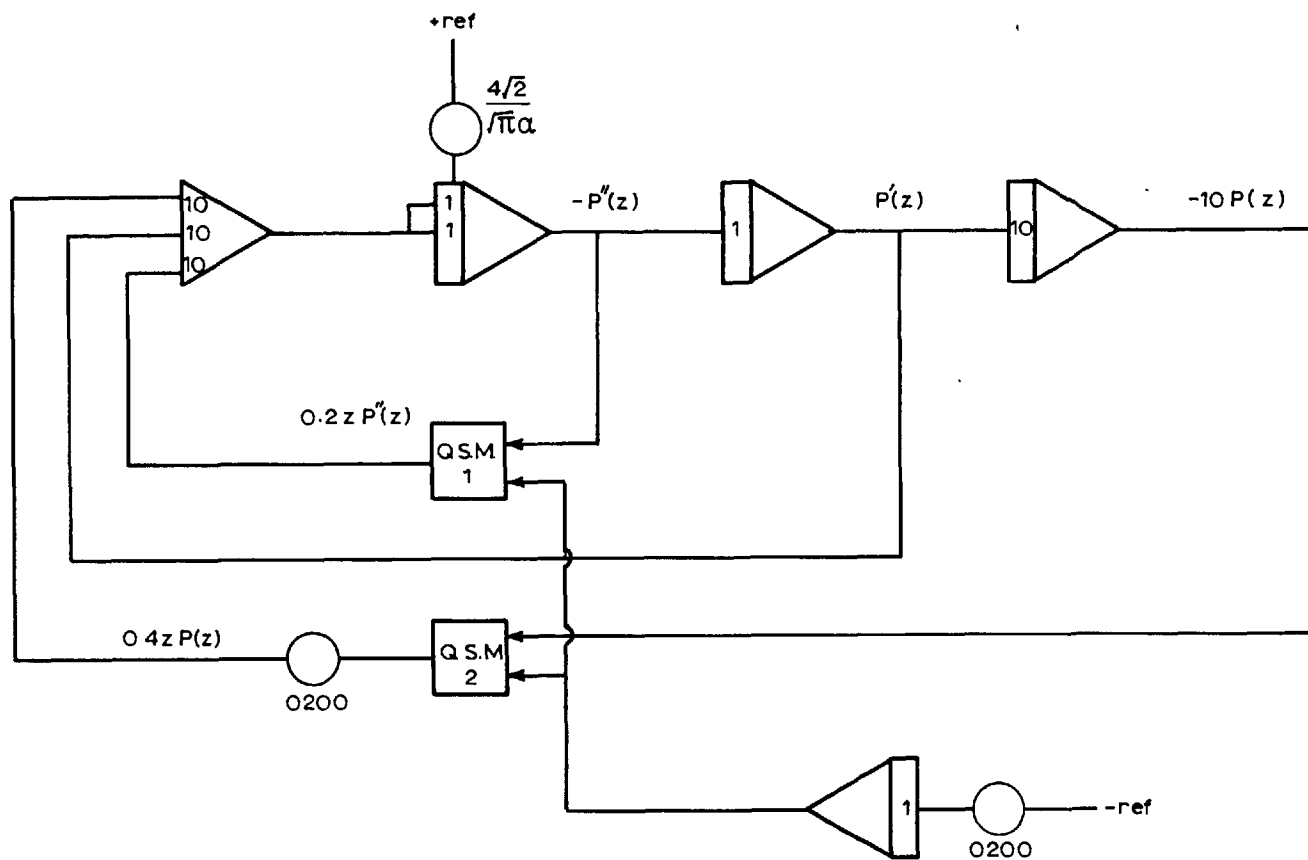


Figure 5.16 CIRCUIT FOR GENERATING A MAXWELL DENSITY FUNCTION

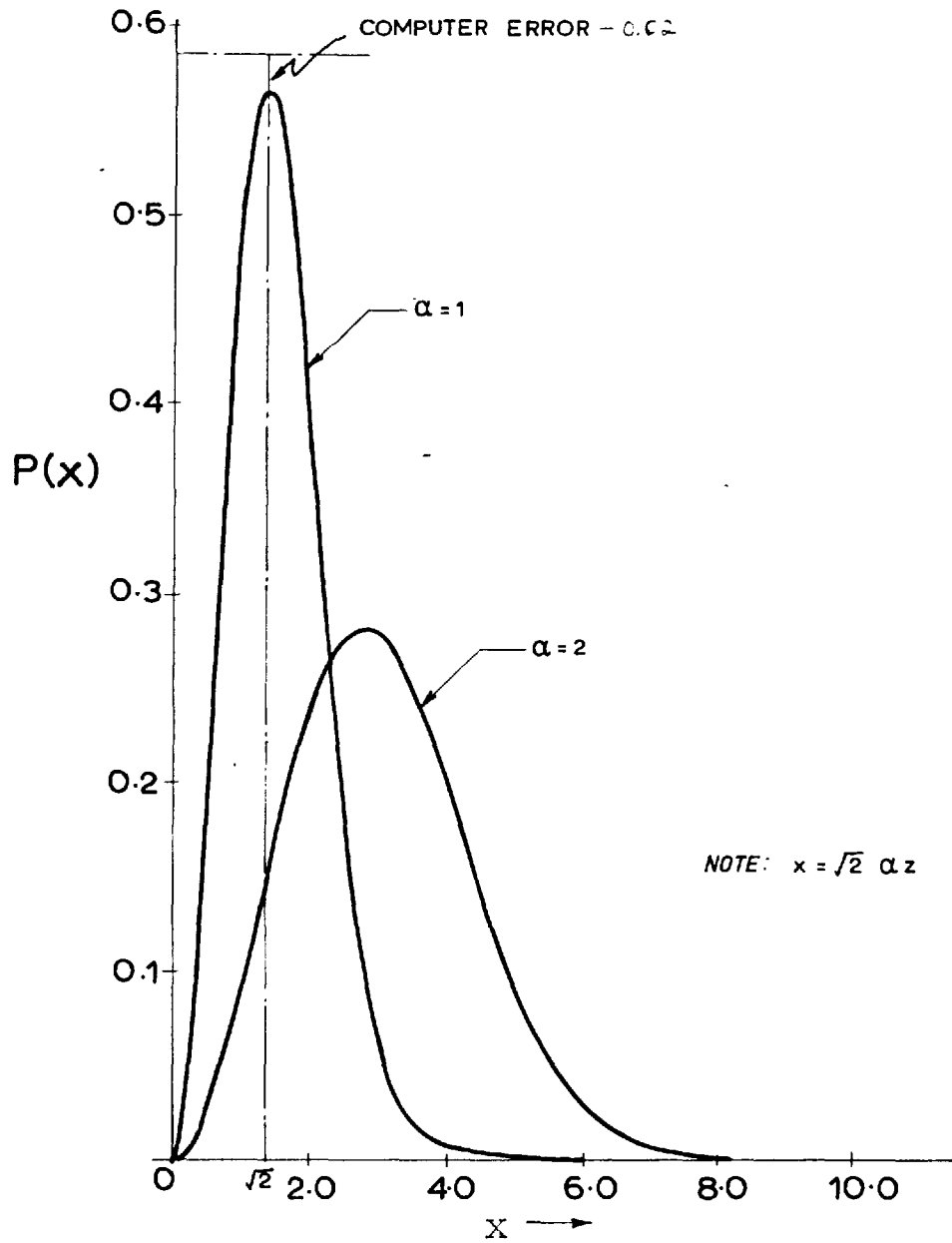


Figure 5.17 MAXWELL DENSITY CURVE

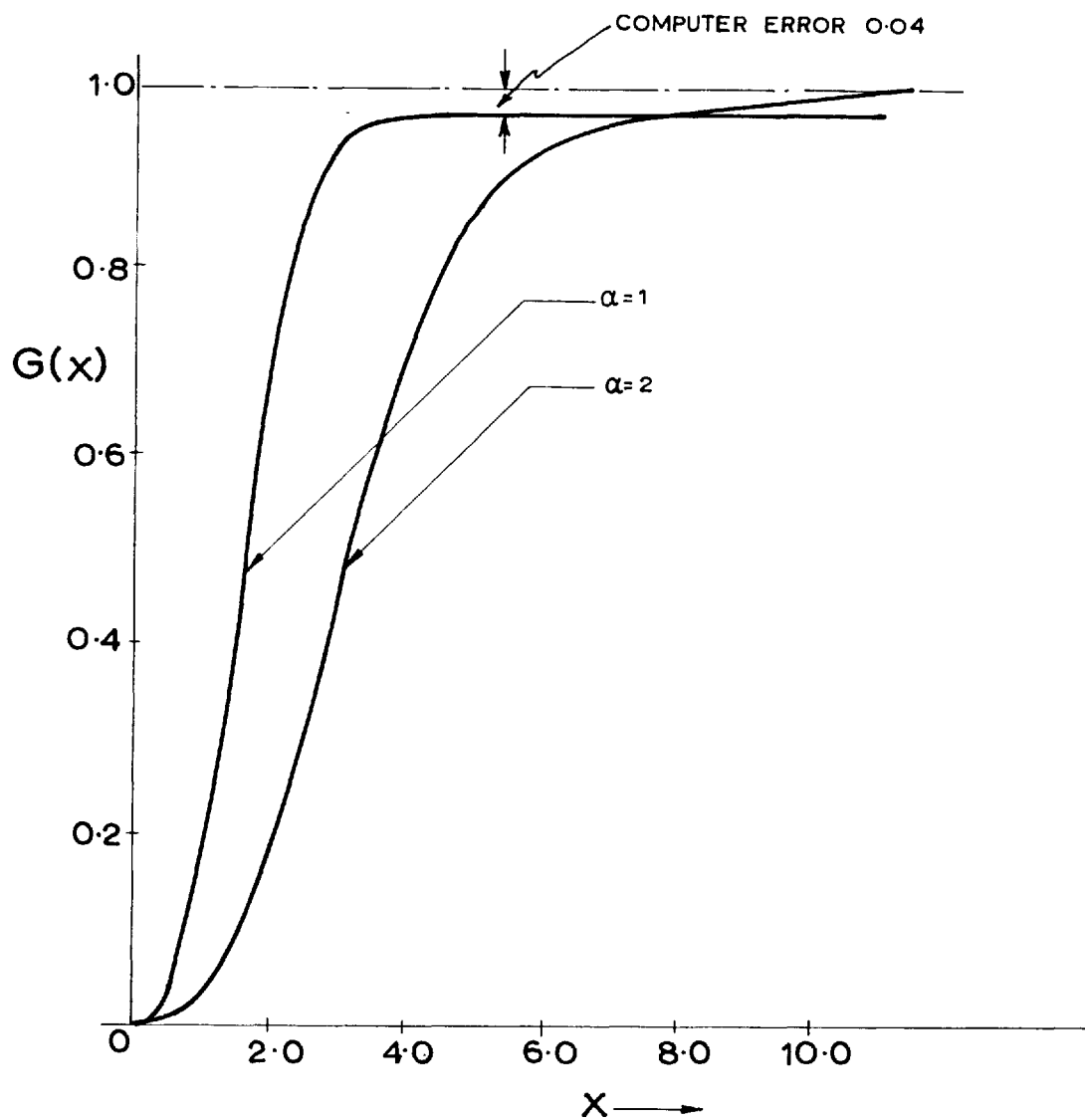


Figure 5.18 MAXWELL DISTRIBUTION FUNCTION FOR
 $\alpha=1, 2$

CHAPTER 6

QUEUEING SIMULATION

Notation

t_0 = ramp voltage

t = time

n = number of intervals between events

$L(t)=P(t)$ = probability that inter-arrival time equals t

p = probability that event occurs in a given time interval

FIFO = first in first out

ϕ = probability of an arrival

ϕ' = probability of a departure

λ = probability that the number of customers in the system increases by 1.

μ = probability that the number of customers in the system decreases by 1

ρ = service factor

N = number of customers in system

CHAPTER 6 : QUEUEING SIMULATION

6.1 INTRODUCTION

Queues occur when items arriving at a service facility can only be serviced if the facility is free. This results in some items being obliged to wait, and the group waiting is called a queue.

In general, a queue is characterised by an arrival rate for customers requiring service, a service rate and a queue discipline. This latter determines the order in which customers in the queue are served, the queue configuration (number of channels) and the effect of queue length on the arrival rate.

Though many queueing problems have been solved analytically, the complex nature of many actual systems often leads to analytical requirements of questionable practicality. Simulation is therefore widely used and this chapter describes an investigation into the use of an analogue computer to simulate queueing systems.

The investigation was in two parts; one deals with general methods for simulating queueing systems and the other describes a simple example where results that were obtained are compared with theory.

It was found in the course of the study that large sample Monte Carlo type simulation is difficult to achieve with standard analogue equipment. This difficulty stems from a limitation on the maximum operating speed of standard analogue computing amplifiers which prevents output histograms from being generated continuously. Nevertheless, new developments in the design of ultra high speed analogue computing equipment [69] promise to overcome this difficulty.

6.2 GENERAL ANALOGUE METHODS FOR QUEUEING SYSTEM SIMULATION

Queueing systems involve three distinct sub-programme operations; (1) generation of arrivals according to some probability distribution, (2) generation of service times, and (3) simulation of the queue discipline. For the purpose of analogue programming (1) and (2) can be treated as similar, where the problem is essentially to generate a sequence of pulses with time between pulses forming a prescribed distribution. Various methods for doing this are described in (a) below while (b) presents methods for simulating some aspects of queue discipline.

- (a) Generation of events with prescribed inter-arrival time distribution.

(i) General method: This method can be used to generate a pulse train with inter-arrival times distributed according to $L(x)$, where x is the time between arrivals.

The method is shown in Figure 6.1 where a Gaussian noise source feeds a timing circuit which determines the time between events. Each time an event occurs, a voltage pulse is fed from the output of the timer to the system being simulated.

Operation of the timer is as follows; a white noise generator feeds sample hold circuit no. 1 which operates automatically when the computer is switched to operate. Voltage x_1 at the output of the sample hold is normally distributed with known standard deviation. This voltage is fed to a diode function generator and transformed to x_2 . The function generator is set up so that x_2 will be distributed according to $L(x)$, (A method for doing this is described in Appendix V.) x_2 is then fed to a discriminator circuit which operates when $t_0 = x_2$ where t_0 is a ramp waveform initiated when the computer is

switched to operate. Thus the discriminator operates after x_2 seconds have elapsed causing a negative voltage to be fed to the reset relay of integrator 1, and to the sampling relay of sample hold 1. When the output of the integrator falls to zero, the discriminator de-activates, and the timing cycle is repeated, but this time with a different noise sample.

In practice, the time delay introduced by the relay comparator (about 10 m. sec.) insures that the discriminator will not de-activate until after t_0 has fallen to zero.

The method that has been described was used to generate a pattern of arrivals with inter-arrival time normally distributed. Figure 6.2 shows an output trace where each pulse corresponds to a single arrival and the mean arrival rate is 3.33 per second with standard deviation equal to 0.10. (Note that where the mean inter-arrival time is 3σ , the distribution will be truncated.)

The maximum arrival rate using this method depends on the switching speed of the discriminator, since this must be negligible compared with the average time between arrivals. It was found in

practice that the reset period at the end of each timing cycle requires 10 m. sec. Thus for a 50:1 ratio between mean inter-arrival time and switching interval, the maximum rate would be 2 arrivals/second. This implies a running time of 8.3 minutes to simulate 1,000 arrivals, and the method was judged to be too slow for ordinary ~~Monte Carlo~~ simulation when only standard analogue equipment is available.

(ii) Method for generating exponential inter-arrival times.

It can be shown [70] that where the probability of occurrence of an event in some small time interval Δt is a constant, λ , independent of any previous interval, then the probability distribution of time between events is exponential e.g.

$$P(t) = \lambda e^{-\lambda t}$$

where t = inter-arrival time.

A process with exponential inter-arrival times can be generated on the analogue by feeding the output of a Gaussian source to a threshold circuit as shown in Figure 6.3. Each time the noise voltage exceeds a pre-set level a single pulse is generated which

corresponds to an arrival. The arrival rate is determined by the discriminator level and the noise standard deviation, so that where p gives the instantaneous probability of the level being exceeded the mean inter-arrival time is $\frac{1}{p}$.

The speed of this method is limited by the maximum switching rate of the discriminator about 10 m. sec. for an on-off cycle.

In order for the assumption regarding higher order probabilities to hold true, the probability of a single event over a 10 m. sec. interval, must itself be small, say 0.01. This makes the average inter-arrival time 1 second, and the problem of slowness occurs as previously.

(iii) Method for generating geometric inter-arrivals.

The time between events is geometrically distributed when

$$P(n) = p(1-p)^n$$

p = probability that event occurs in one

time interval.

n = number of intervals between events.

The geometric distribution is a discrete time analogue of the exponential distribution and approximates it for small p .

Geometrically distributed inter-arrivals can be obtained on the analogue using the circuit^{in Fig. 6.4}. Here, a noise waveform is sampled at discrete intervals and fed to a discriminator. If the sampled voltage exceeds a given threshold level the discriminator output is +1.0 otherwise it is zero. As before, the arrival rate is determined by the noise variance and the threshold level, and the average number of intervals between arrivals is $\frac{1}{p}$.

(b) Queue Discipline

The queue discipline may be dictated by the controller or imposed by the actual queueing problem. A wide variety of situations may occur and these are too numerous to mention individually, nevertheless it may be useful to consider some general aspects of

queue discipline which can be simulated.

(i) The queue: the queue itself is simulated on the analogue by an integrator, or in the case of a geometric inter-arrival distribution, by a summation circuit. The number of queueing channels is limited only by the number of integrators/summation circuits available for this purpose.

(ii) Customer behaviour can be represented where:

- (A) Customers refuse to join a queue which contains more than n items.
- (B) Customers may shift from one queue to another when the difference in length exceeds a specified number.

Figures 6.5 and 6.6. show analogue circuits for each of the above cases.

(iii) Order of service must always be FIFO in analogue simulation. This assertion rests on the fact that for any other policy it is necessary to record the status of each customer in the queue. This in turn implies one integrator per customer, an equipment requirement that cannot in general be satisfied.

6.3 ANALOGUE SIMULATION OF A SIMPLE QUEUE

The analogue was used to simulate a single server queue where inter-arrival times and service times are geometrically distributed with mean values less than 10.

It should be pointed out that the assumption of geometric inter-arrivals and restrictions on arrival rates, leads to conditions which are seldom met in practice. However the example is intended as an illustration and more general problems could be tackled without any substantial alteration to the method, given the availability of ultra high speed analogue equipment.

The analogue circuit is shown in Figure 6.7 where inter-arrival times and service times are generated as previously described for the geometric distribution.

The probability of an arrival ϕ is determined by adjusting P1 in the circuit, and the probability of a customer service being completed ϕ' is determined by P2. Since the probability of both events occurring in one interval cannot in general be neglected, the probability that the number of customers in the system increases by one is given by;

$$\lambda = \phi (1 - \phi')$$

and the probability of a net decrease is;

$$\mu = \rho'(1-\rho)$$

Figure 6.8 is a computer output trace showing N (the number of customers in the system) vs time, for different values of the service factor $\rho = \frac{\lambda}{\mu}$.

Figures 6.9 and 6.10 show histograms of $P(N)$ for different values of ρ_1^* where the height of each line gives the measured probability, and dots correspond to values predicted by theory [71].

A modification of the basic system was introduced whereby an additional server is made available whenever the number of customers in the system exceeds x_1 and is withdrawn as soon as N falls below x_2 ($x_2 \leq x_1$). The circuit for this system is shown in Figure 6.11 where service times for the added server are determined from noise samples taken at intervals time shifted from the multivibrator sampling instants. This is done to provide independant service times for the two servers. Results are shown in Figures 6.12 and 6.13 for $\rho=0.56$, $x_2=1$ and $x_3=4$ and 5.

* These were obtained using the distribution analyser described in Appendix VI.

6.4 SUMMARY

Standard analogue equipment appears to be too slow for ~~Monte Carlo type~~ ^{normal} queueing simulation. The problem is essentially a technological one in that standard equipment does not match the speed requirements imposed by the simulation problem. It is envisioned that a new generation of analogue equipment presently being developed will possess the required capability and the techniques described in this chapter will be applicable without alteration.

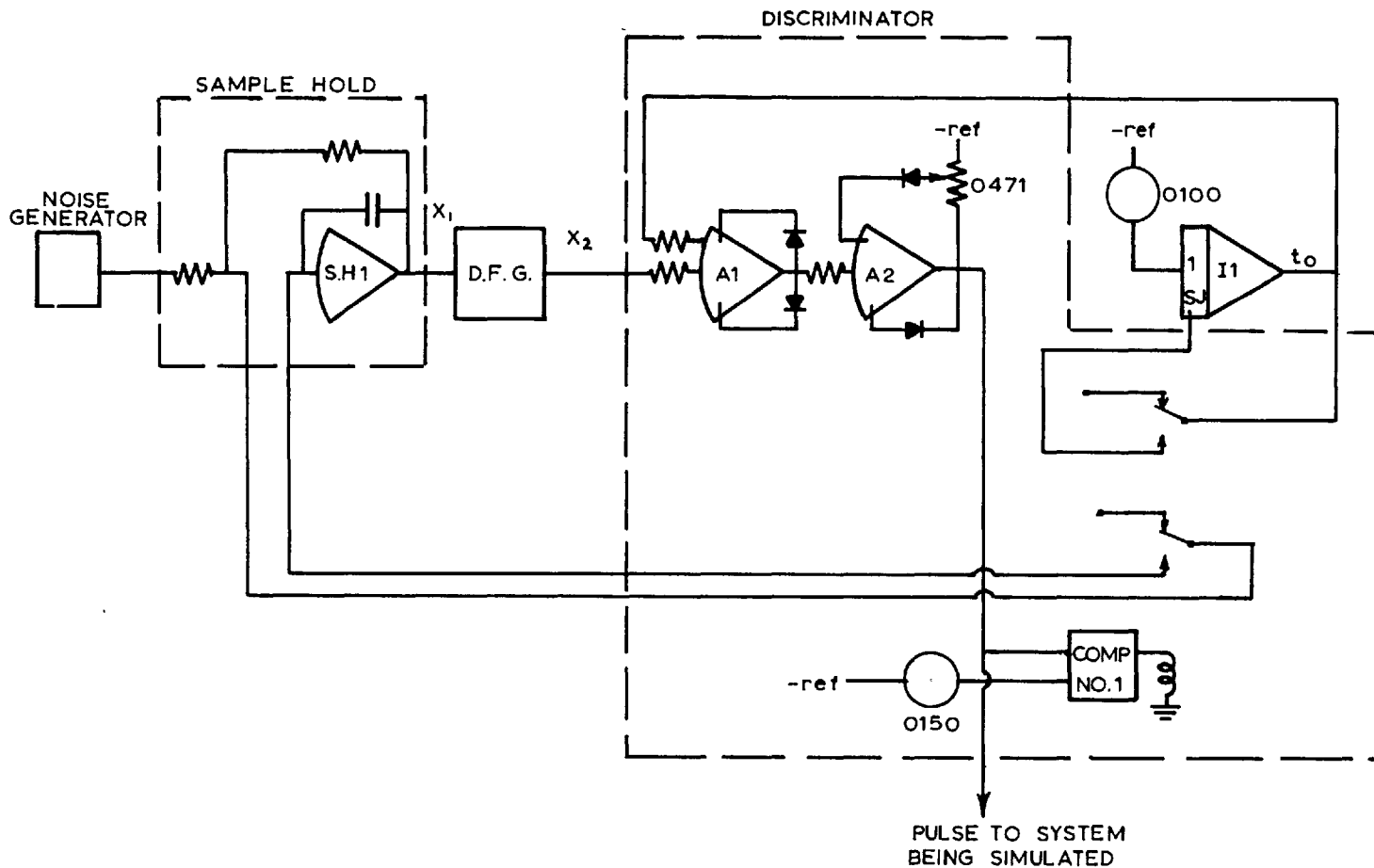


Figure 6.1 TIMING CIRCUIT METHOD FOR GENERATING PULSES WITH GENERAL INTER-ARRIVAL TIME DISTRIBUTION

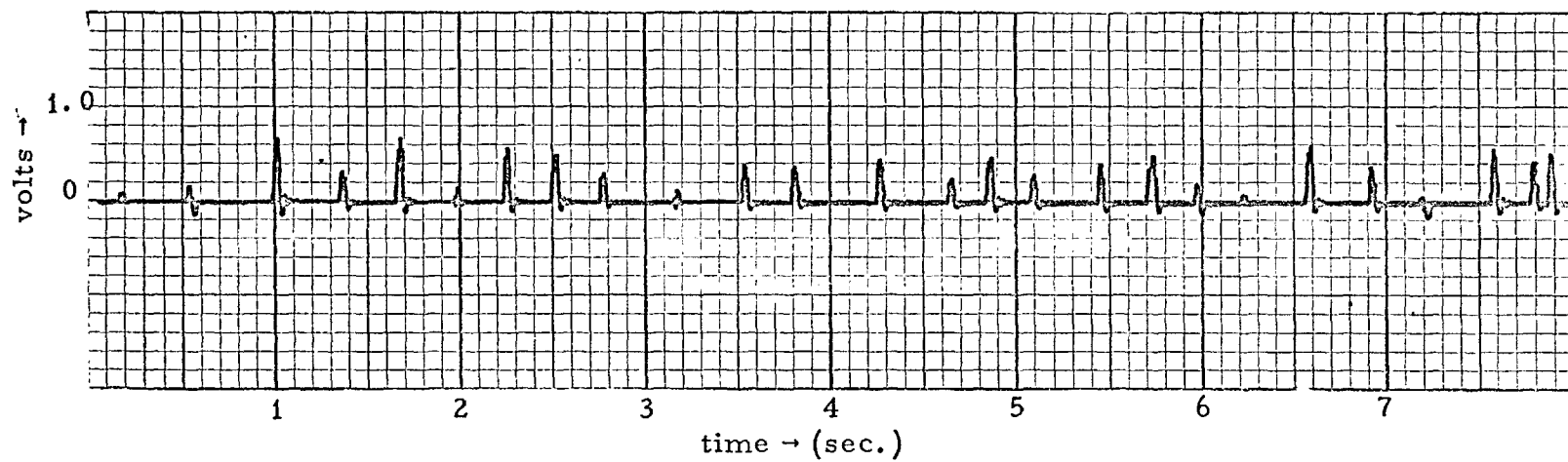


Figure 6.2 NORMALLY DISTRIBUTED ARRIVALS WITH MEAN
INTER-ARRIVAL TIME 0.30 sec.

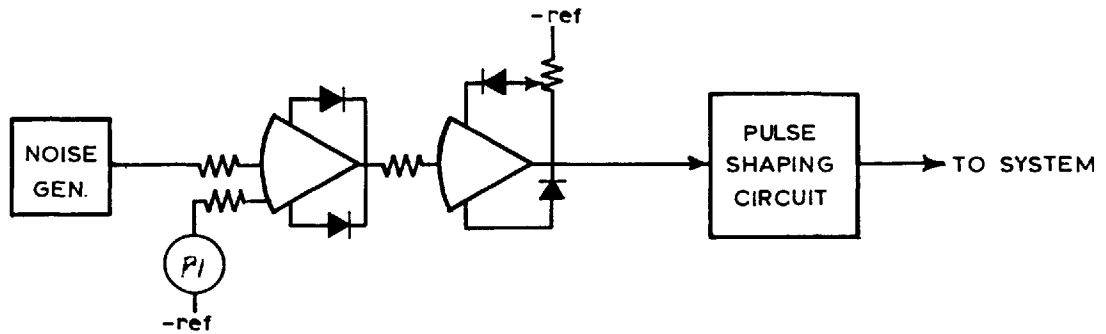


Figure 6.3 ANALOGUE METHOD FOR GENERATING A PULSE SEQUENCE WITH EXPONENTIAL INTER ARRIVAL TIMES

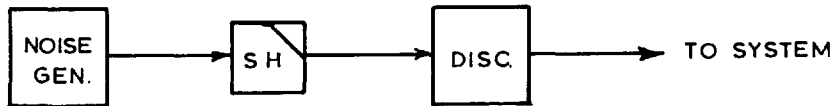


Figure 6.4 ANALOGUE METHOD FOR GENERATING A PULSE SEQUENCE WITH GEOMETRIC INTER ARRIVAL TIMES

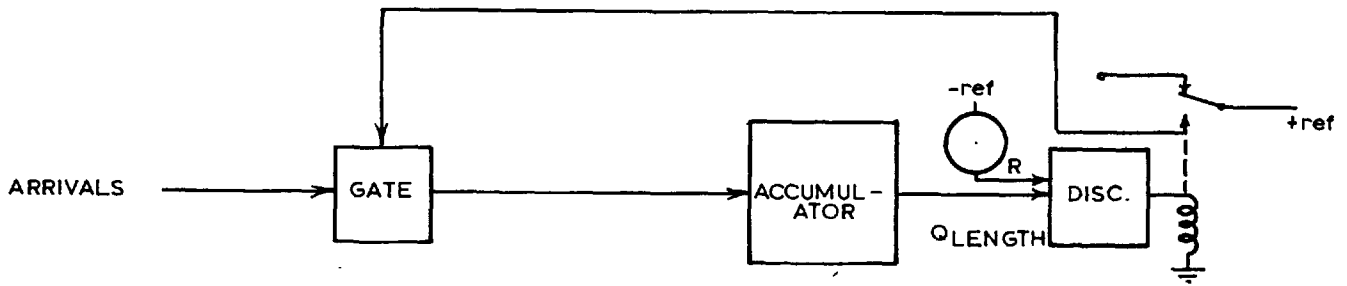


Figure 6.5 ANALOGUE CIRCUIT TO SIMULATE CUSTOMER DISCOURAGEMENT

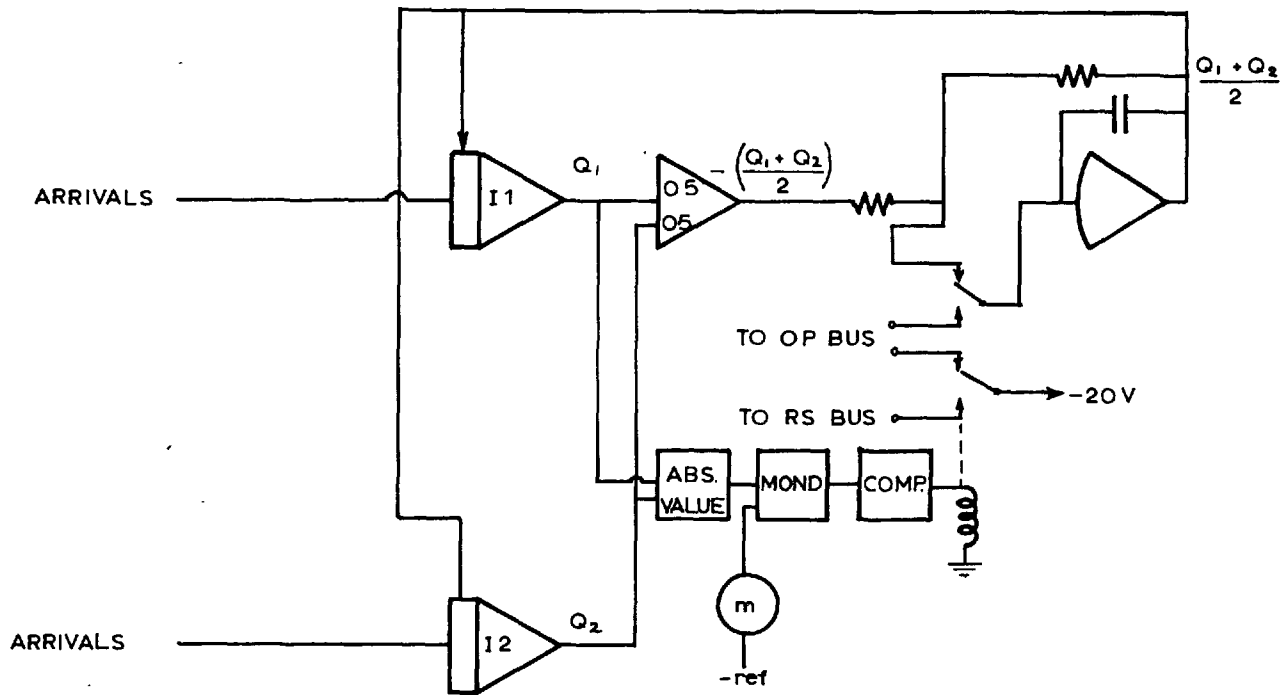


Figure 6.6 ANALOGUE CIRCUIT FOR CASE WHERE CUSTOMERS SHIFT QUEUE WHEN $Q_1 - Q_2 > m$

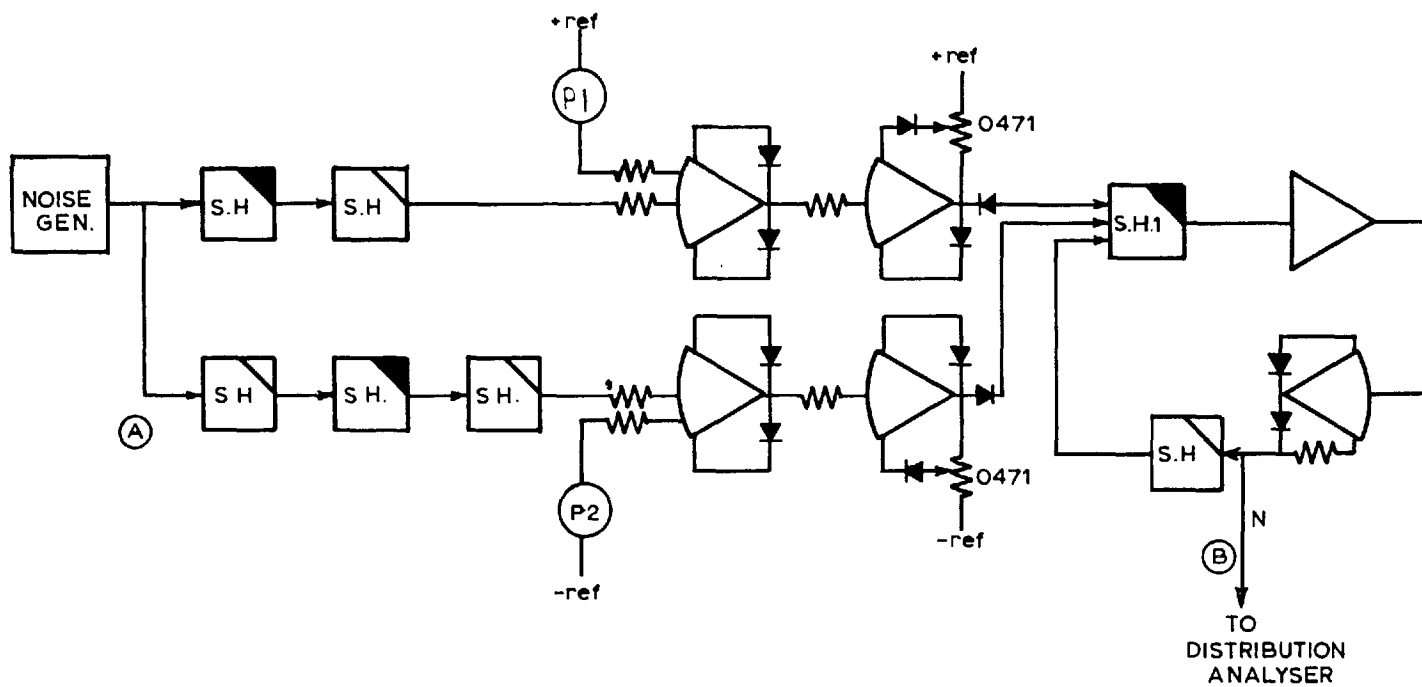


Figure 6.7 ANALOGUE PROGRAMME FOR SINGLE SERVER QUEUEING SIMULATION

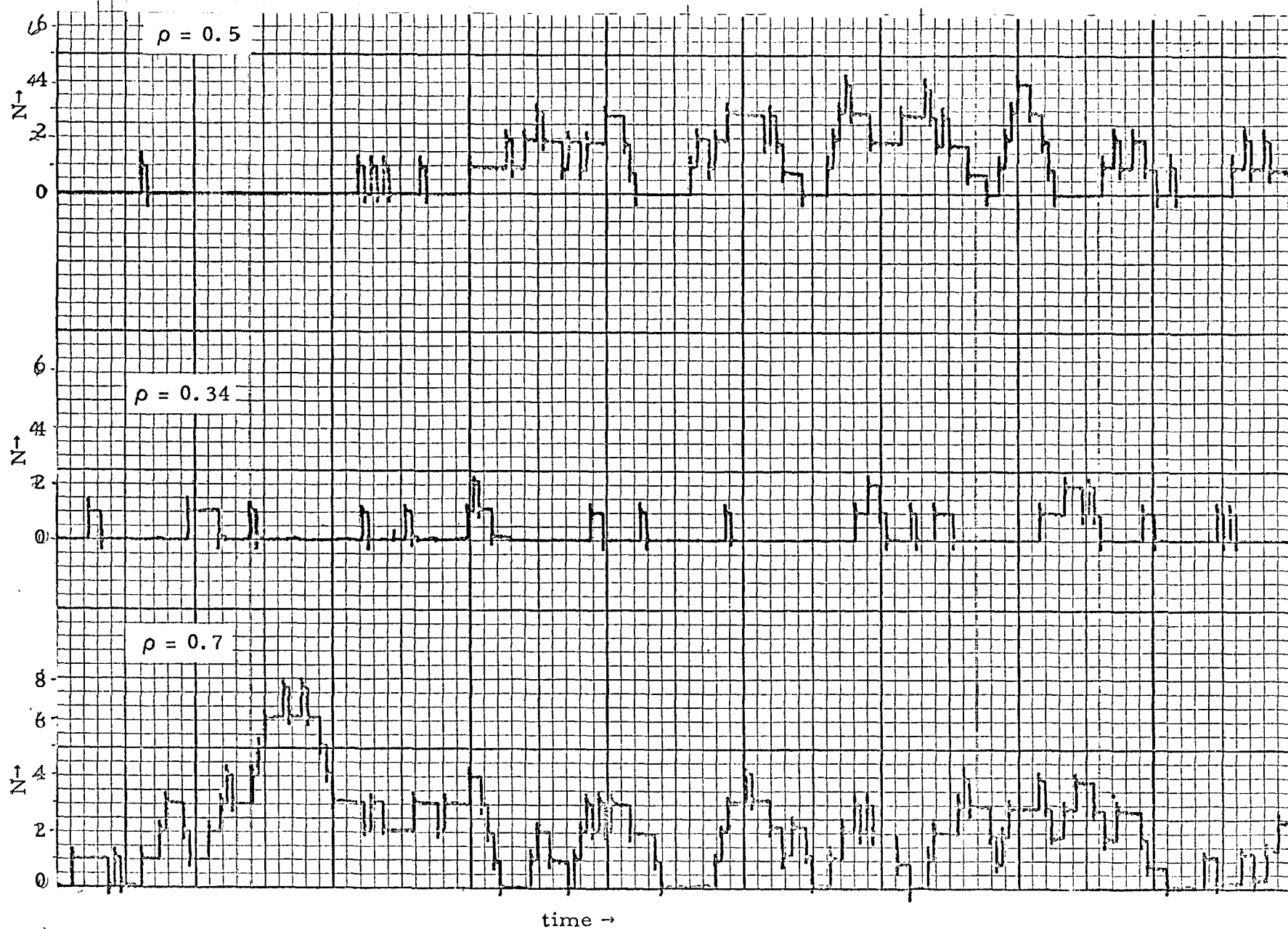


Figure 6.8 N VS TIME FOR $\rho=0.5, 0.34, 0.7$

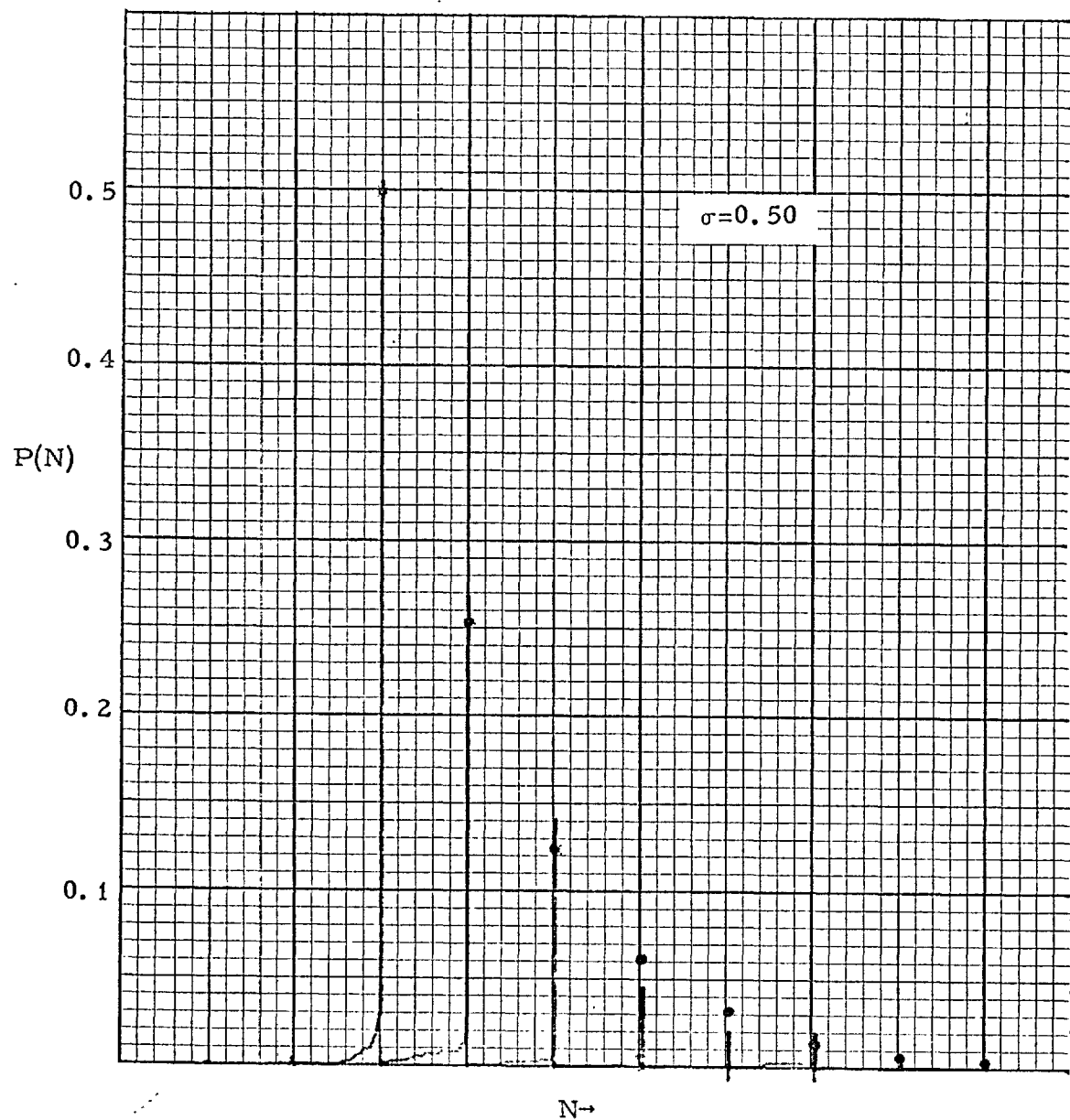


Figure 6.9 HISTOGRAM OF $P(N)$, $\rho=0.5$

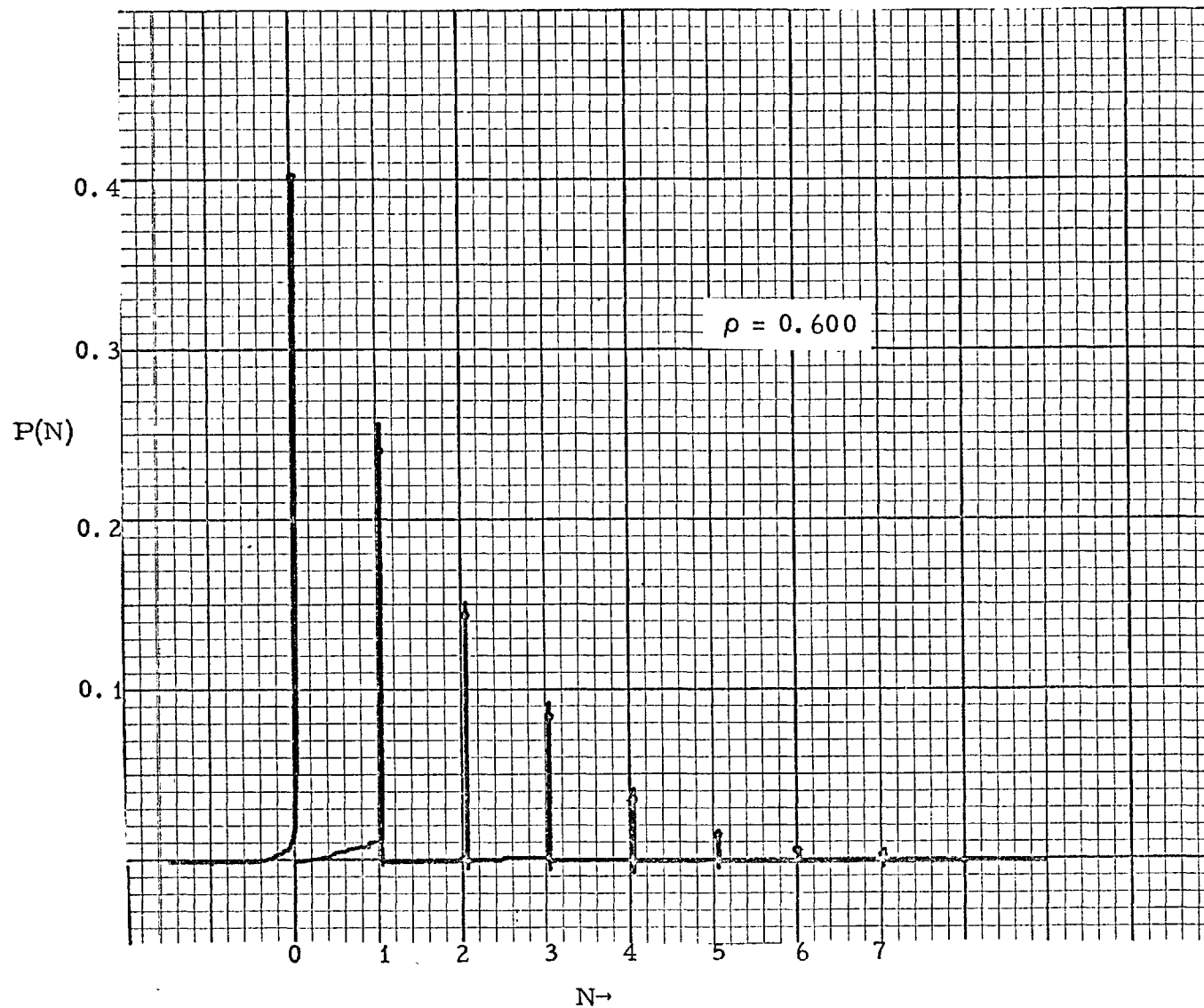


Figure 6.10 HISTOGRAM OF $P(N)$, $\rho=0.6$

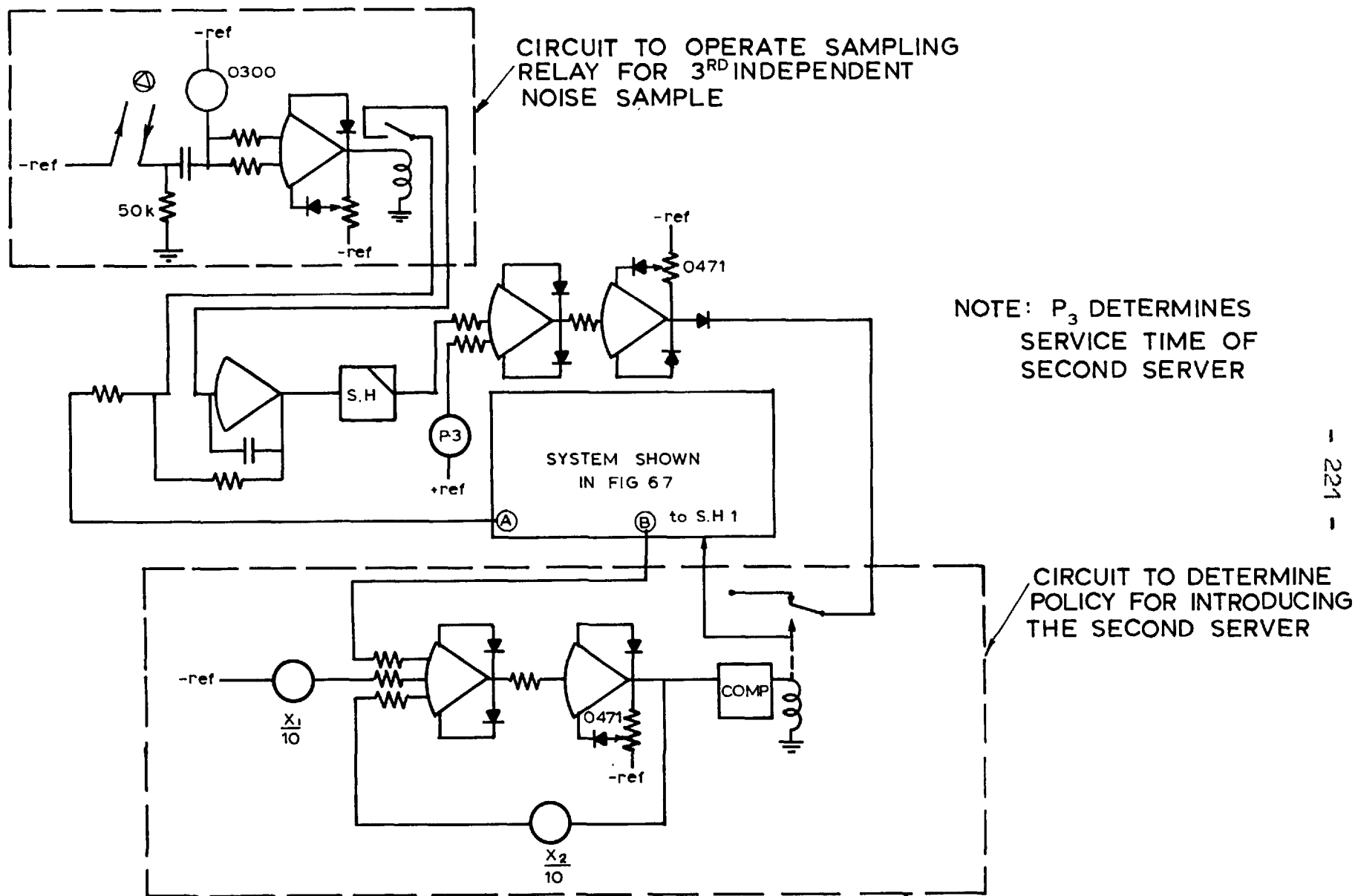


Figure 6.11 ANALOGUE PROGRAMME FOR TWO SERVER SYSTEM

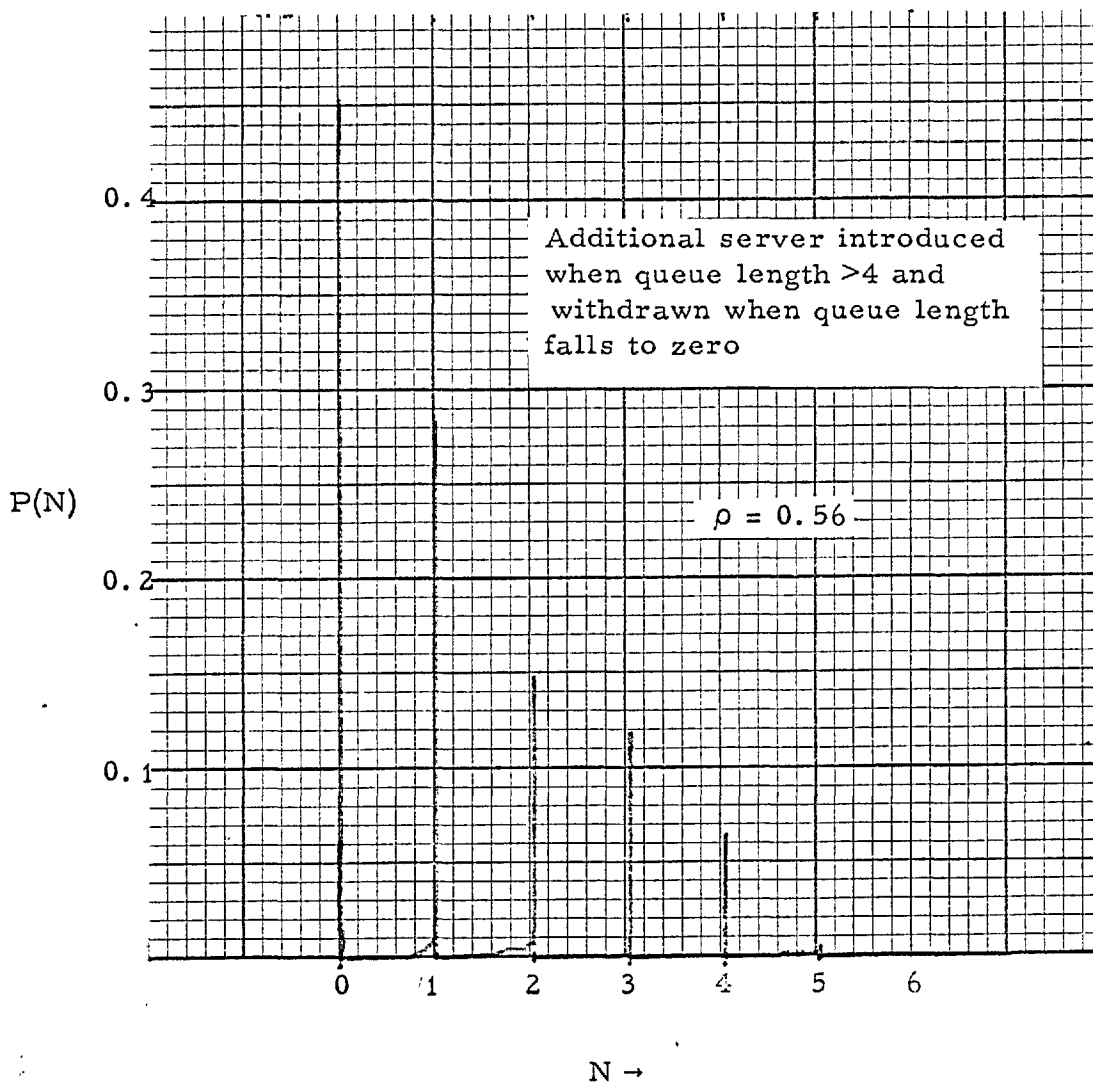


Figure 6.12 HISTOGRAM OF $P(N)$ FOR SYSTEM WITH
TWO SERVERS, $\rho=0.50$, $x_1=4$, $x_2=1$

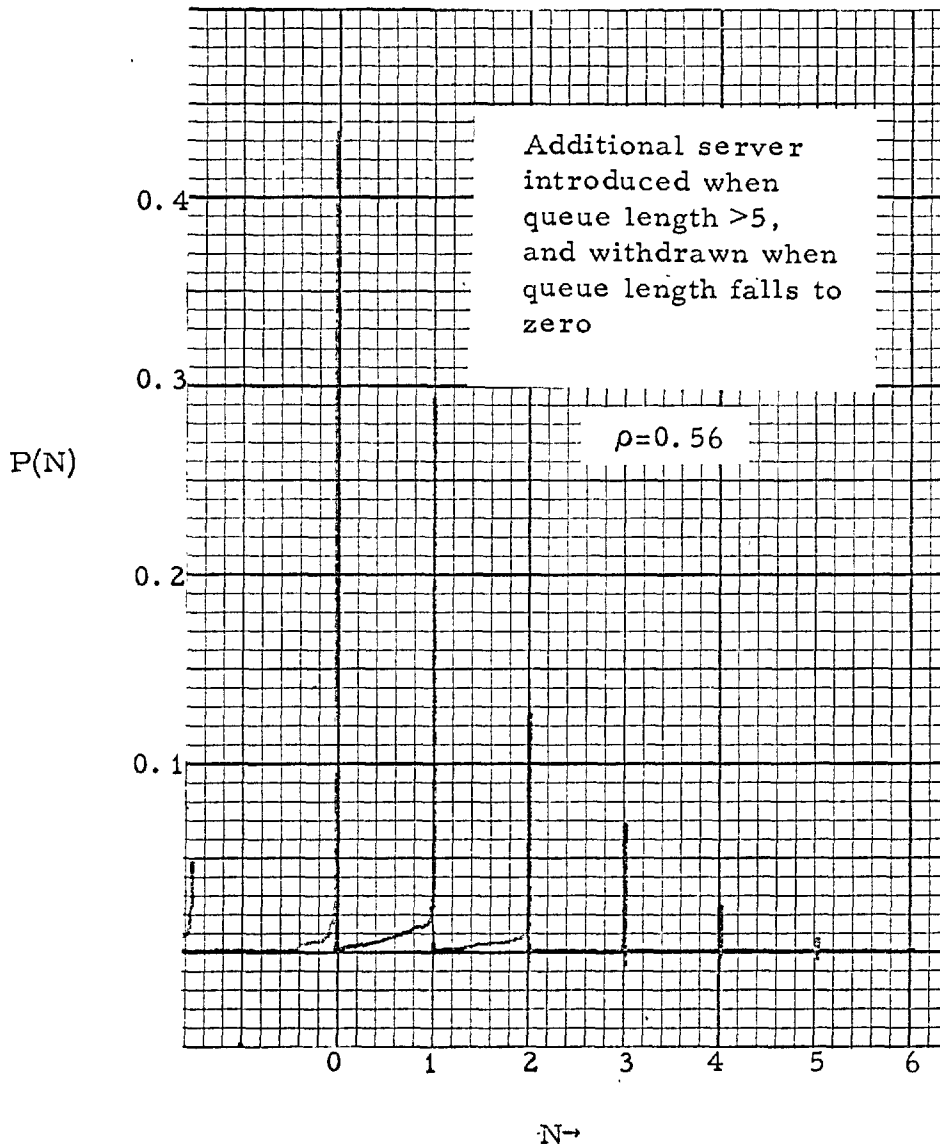


Figure 6.13 HISTOGRAM OF $P(N)$ FOR SYSTEM WITH TWO SERVERS FOR $\rho=0.56$, $x_1=5$, $x_2=1$

CHAPTER 7

INDUSTRIAL DYNAMICS

Notation

$D(t)$ = demand rate

$G(t)$ = delivery rate to customer

$U(t)$ = unsatisfied orders

$a(t)$ = delay in filling orders

a_m = minimum delay in filling orders

Δ_a = average additional delay in filling orders when
 $i(t)=R(t)$

$i(t)$ = inventory level

$R(t)$ = required inventory level

$F(t)$ = forecast

$Q(t)$ = re-order rate

$Q^*(t-L)$ = delivery rate from distributor

L = time lag (distributor)

$U_o(t)$ = required unsatisfied order level

c_s = cost per unit maximum absolute stock level
deviation

c_Q = cost per unit maximum absolute re-order rate
deviation

σ = abscissa of absolute convergence of the Laplace
transform $\frac{Q(s)}{(1+s)^L}$

c_n = cost incurred by system n when optimised
separately

C = cost incurred by n systems when optimised
together

$|x(t)|_m$ = maximum absolute value of $x(t)$

$F(s)$ = Laplace transform of $f(t)$

CHAPTER 7 : INDUSTRIAL DYNAMICS

7.1 INTRODUCTION

This chapter describes the use of an analogue computer for simulating Industrial Dynamics models. Industrial Dynamics has been defined [72] as the study of the information feedback character of industrial systems, and more generally can be described as the study of the dynamic behaviour of industrial systems on a macro-economic level. This latter definition has important implications for the use of analogue computation, since system variables considered on an aggregate (macro-economic) basis, can usually be regarded as continuous.

Current interest in Industrial Dynamics stems from work by Forrester [72] who postulates a series of increasingly elaborate models to describe the entire customer-producer-employment system. These models were simulated using a digital computer and to the extent that the assumptions made can be justified some useful results were obtained.

This chapter describes how an analogue computer can be used to simulate Industrial Dynamics models on a modest scale, and in particular deals with

Forrester's model of the retail sector. A simplified version of this model was analysed to determine the system parameters that minimise prescribed objective functions when two models operate in series.

7.2 FORRESTER'S MODEL OF THE RETAIL SECTOR

A block diagram of this model is shown in Figure 7.1 where demand orders $D(t)$, are fed to the receiving department which in turn supplies the customer at a rate $G(t)$ and adjusts the warehouse stock level $i(t)$ accordingly. The re-ordering department determines the required production rate $Q(t)$ and transmits this information to the factory whose output $Q^*(t-L)$ is fed to the warehouse. A forecasting system is used to predict future demand by exponential smoothing and auxiliary blocks determine the required inventory level $R(t)$ and the delay in filling orders $a(t)$.

The system defining equations are given in Table 7.1 and a corresponding analogue computer programme is given in Figure 7.2.

The system response to a unit step is shown in curve no. 1 Figure 7.3 for a particular set of parameter values while curve no. 2 shows the system response for the same parameters, but with the last

Table 7.1 System Equations

System Variable	Defining Equation	Equation No.
$U(t)$	$U'(t) = D(t) - G(t)$	(7.1)
$G(t)$	$G(t) = a(t)U(t)$	(7.2)
$F(t)$	$F'(t) = \infty D(t) - F(t)$	(7.3)
$i(t)$	$i(t) = i(0) - \int_0^t [G(t) - Q^{\times}(t-L)] dt$	(7.4)
$Q(t)$	$Q(t) = k \left\{ R(t) - i(t) + L F(t) - \int_0^t [Q(t) - Q^{\times}(t-L)] dt + [U(t) - U_0(t)] \right\}$	(7.5)
$Q^{\times}(t-L)$	$Q^{\times}(t-L) = \frac{1}{2\pi j} \int_{\sigma-jR}^{\sigma+jR} \frac{Q(s)}{(1+s)^L} e^{st} ds$	(7.6)
$U_0(t)$	$U_0(t) = [a_m + \Delta a] F(t)$	(7.7)
$a(t)$	$a(t) = a_m + \Delta a \frac{R(t)}{i(t)}$	(7.8)

Note: $Q^{\times}(t-L)$ results from passing $Q(t)$ through L first order lags connected in series.
Time scale, 1 sec. = 1 month throughout

term in the re-order rule deleted (equation (7.5)). It is apparent from the figure that this term introduces a time lag into the system response, however since a similar lag can be introduced in other ways (for example by reducing α or k), this term was judged to be superfluous and is omitted in subsequent simulations. The effect of $i(0)$ on the system step response is demonstrated in Figure 7.4 from which it is clear that reducing $i(0)$ causes the step response to be accentuated. as was to be expected from equation (7.8).

Figure 7.5 shows the effect of a_m and Δ_a on the step response. It is apparent from the figure that increasing $\frac{\Delta_a}{a_m}$ accentuates the effect of a step.

The effects of α and k on the system step response are illustrated in Figure 7.6 from which it is apparent that decreasing either constant smoothes $Q(t)$ while at the same time causing $i(t)$ to fluctuate more noticeably.

Figure 7.7 shows the effect of replacing $a(t)$ in equation (7.8) by a constant. From this result it is apparent that $a(t)$ improves the overall system performance by simultaneously reducing the amplitudes of fluctuations in both $i(t)$ and $Q(t)$ while introducing only a slight additional lag in $G(t)$.

An Industrial Dynamics model for the retail sector has been simulated and the use of the analogue for studying the effects of system parameters has been demonstrated.

7.3 SIMPLE PRODUCTION-INVENTORY CONTROL SYSTEMS IN SERIES

This section describes the use of an analogue computer to study the behaviour of a simplified industrial production-inventory model when two such systems are inter-connected. The system is shown schematically in Figure 7.8(a) where the order processing delay is neglected and demand orders are fed directly to the inventory. Re-order quantities are computed according to,

$$Q(t) = k \left[R - i(t) - \int_0^t [Q(t) - Q^*(t-L)] dt \right] + (1+kL) F(t)$$

where $F(t)$ is obtained as in (7.3) by exponential smoothing, and the system parameters are α , k and L with $R=0$ so that output waveforms are regarded as excursions about quiescent levels.

When a number of systems are connected in series Figure 7.8(b) the re-order quantities computed by the first are also fed as demand to the second and so on. This might occur in practice, for example where an assembly producer (system 1) feeds stock requirements

to a component manufacturer (system 2), who in turn places orders on a raw material supplier (system 3).

An analogue programme for systems in series is given in Figure 7.9 where $L=1$ for each system. The system response to a step is shown in Figure 7.10 for two values of α and k where α and k are made equal. It is apparent from this figure that increasing α and k for all three systems increases the magnitudes of the excursions of $q(t)$ from normal while reducing those of $i(t)$.

Figure 7.11 shows the effect of using different α , k parameters for each system in the chain, where in the first case α and L are increasing and in the second decreasing for subsequent systems.

Figure 7.12 shows a result obtained where α and k are taken as 0.5 for system no. 1 and the parameters of subsequent systems are manipulated to yield (a) parity among the peak absolute values of $i(t)$ and (b) parity among the peak values of $q(t)$. It is apparent from the figure that (a) is achieved with less overall disruptive effect than (b), since (b) is achieved only at the expense of vastly greater excursions of $i(t)$ from normal.

The following describes how this aspect of system behaviour was analysed. It was required to determine how α and k should be chosen in order to minimise a linear cost function involving the maximum absolute deviations of $i(t)$ and $Q(t)$. Two distinct situations may occur;

- (a) When the parameters of each ^{sub}system are chosen to minimise the costs c_n incurred by the ~~the~~ respective sub-systems,

$$\text{where } c_n = \min_{\alpha_n, k_n} \left[c_s |i_n(t)|_m + c_Q |Q_n(t)|_m \right] \quad (7.9)$$

and c_s = cost per unit maximum absolute deviation of $i(t)$

c_Q = cost per unit maximum absolute deviation of $Q(t)$

- (b) When the parameters of each system are chosen to minimise the overall cost function C .

$$C = \min_{\alpha_1, k_1, \alpha_2, k_2} \left[c_s |i_1(t)|_m + |i_2(t)|_m + c_Q |Q_1(t)|_m + |Q_2(t)|_m \right] \quad (7.10)$$

As a preliminary step the system transfer function was derived using standard control techniques [73] giving

$$\frac{Q(s)}{D(s)} = \frac{(k+\alpha k(1+L)+\alpha)s}{(s+\alpha)(s+k)} + \frac{Q(0)}{s} \quad (7.11)$$

$$\frac{i(s)}{D(s)} = \frac{1}{s} \left[\frac{(k+\alpha k(1+L)+\alpha)s}{(s+\alpha)(s+k)} - 1 \right] + \frac{i(0)}{s} \quad (7.12)$$

From equations (7.11) and (7.12) it is clear that the system transfer function is symmetrical with respect to α and k . From the uniqueness of the Laplace transform [74], it follows that the system response for any input (in particular for a step) will also be symmetrical, and use was made of this in the analysis which follows.

The system which was simulated is shown in Figure 7.10 for two models in series with a track hold circuit^{*} connected to each circuit point of interest (e.g. $i_1(t)$, $Q_1(t)$, $i_2(t)$, $Q_2(t)$).

The system was studied by multiplying the outputs from the track hold circuits by c_s and c_p , and summing as appropriate to yield c_1 , c_2 and C . By operating the computer in repetitive mode the effect of varying

* This circuit [75] holds the maximum value of an input waveform at its output over one reset cycle. The circuit is shown in Figure 7.13, and Figure 7.14 shows typical input-output waveforms. Re-setting is accomplished by feeding a 10v pulse to the summing junction through a 100 k Ω resistor.

parameters α and k can be observed continuously on an oscilloscope, and the system cost functions minimised for various $\frac{c_Q}{c_S}$ ratios by manually varying the α , k potentiometers. Figures 7.15 and 7.16 show results obtained in this manner for the case where the two systems are optimised (a) separately and (b) together.

From Figure 7.15 it is apparent that the optimal policies for (a) and (b) differ considerably over the range $0.25 \leq \frac{c_Q}{c_S} \leq 4.5$. Moreover, this difference accounts for a net increase in system operating costs when the systems are optimised separately instead of globally. Because the system transfer function is symmetrical in α and k , values of these parameters in Figure 7.15 can be interchanged.

It was noticed that when either α or k are zero, the inventory fails to regain its original level, in responding to a step. This may be unacceptable in practice, and where this is so the objective function must be re-defined to preclude such an occurrence.*

* This could be achieved by inserting a term of the

form $\int_0^t [i(t) - R]^2 dt$ in the system cost function.

7.4 SUMMARY

The use of an analogue computer for simulating models in Industrial Dynamics has been demonstrated, and the system parameters that minimise a linear cost function involving the maximum absolute deviations of $i(t)$ and $q(t)$ have been obtained for the case where two systems operate in series.

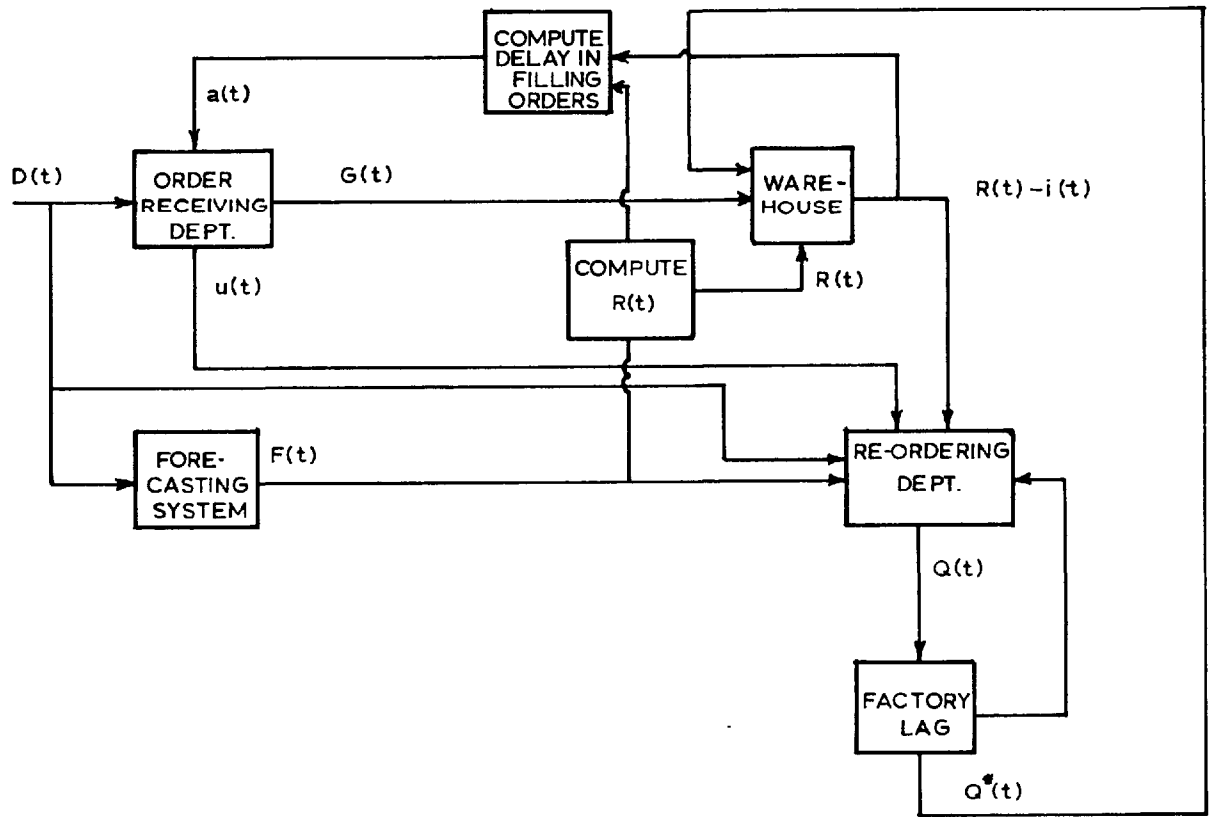


Figure 7.1 MODEL FOR THE RETAIL SECTOR

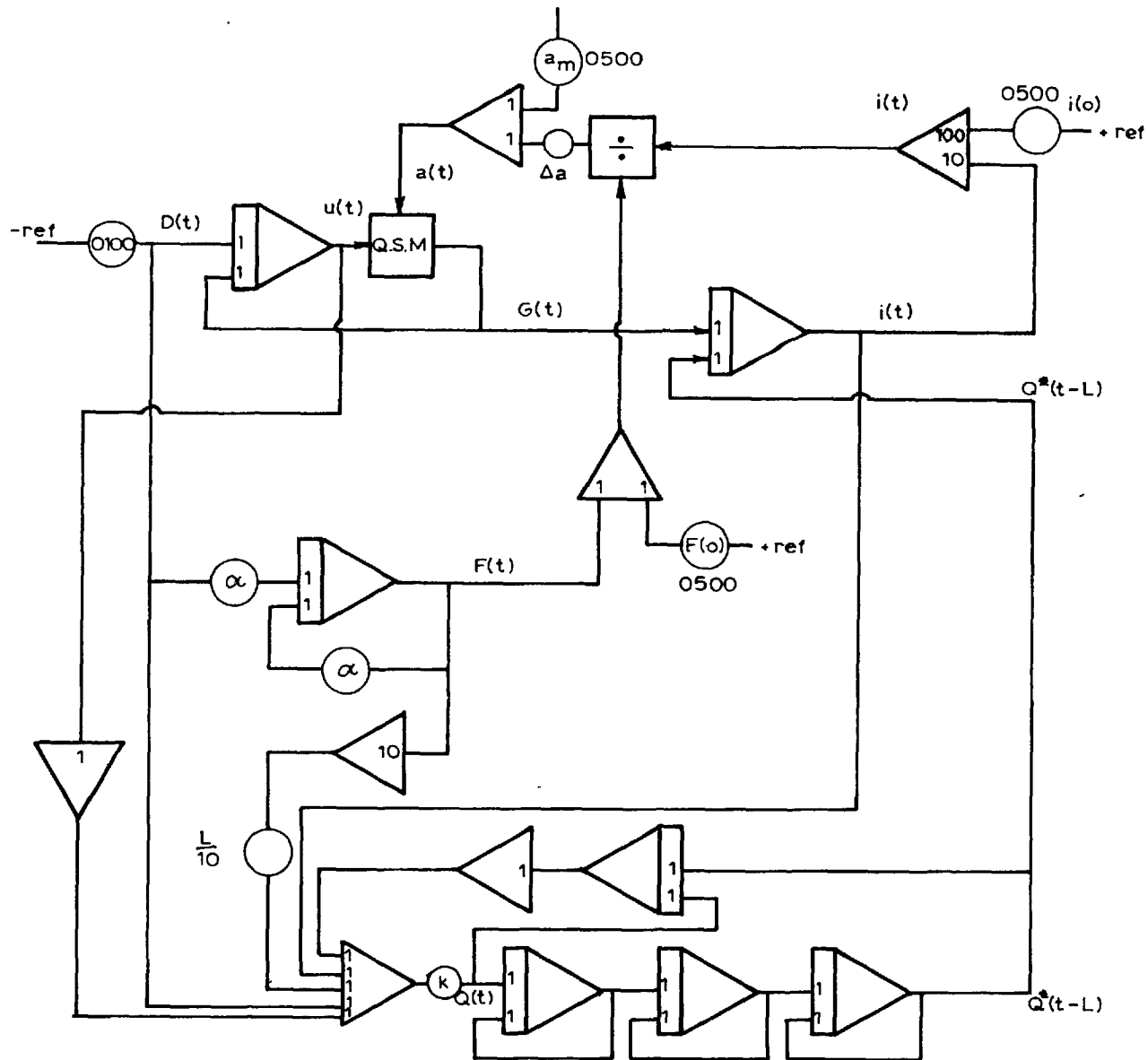


Figure 7.2 ANALOGUE CIRCUIT FOR RETAIL SECTOR SIMULATION

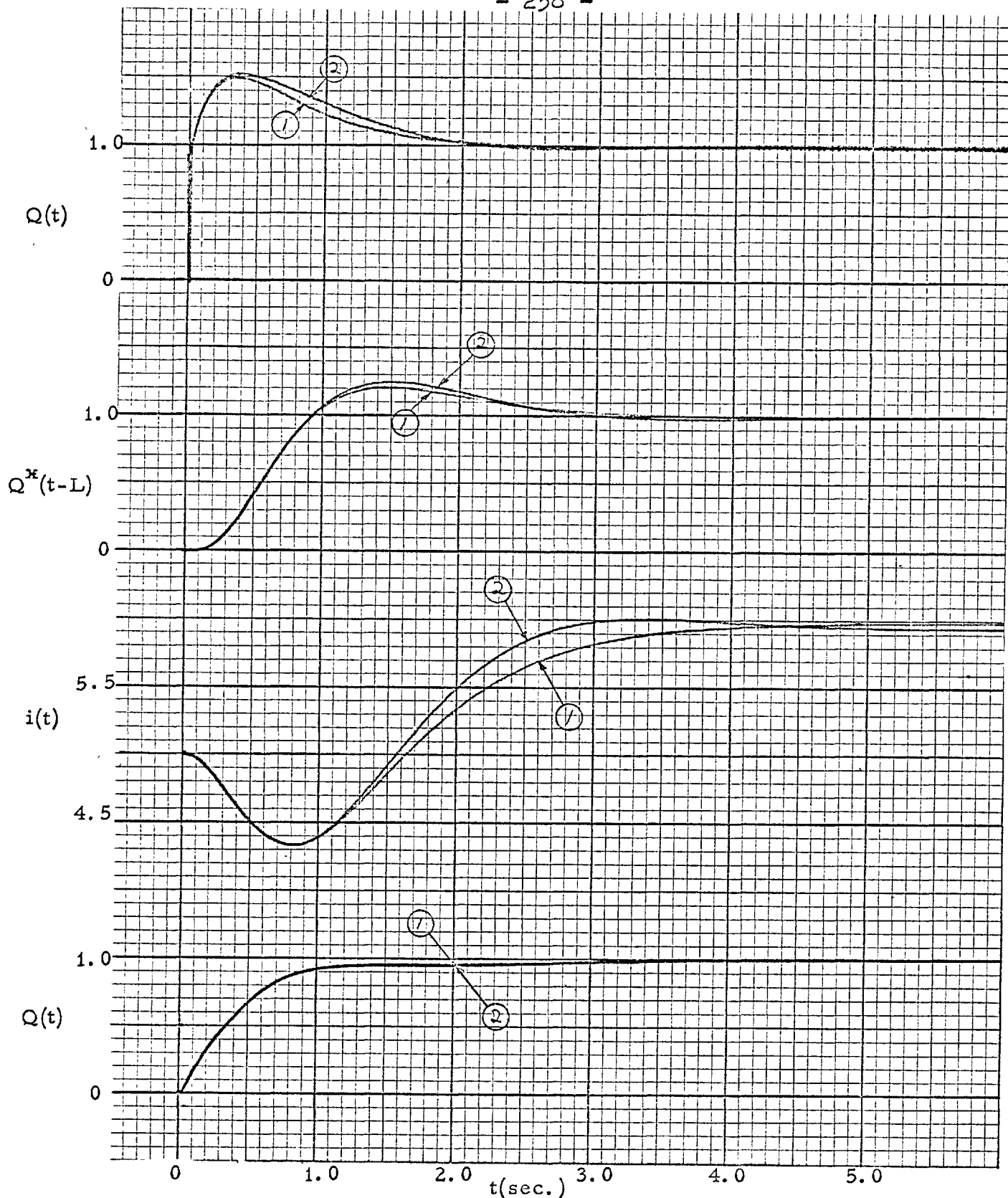


Figure 7.3 FORRESTOR'S MODEL OF THE RETAIL SECTOR STEP RESPONSE
 $\alpha=0.5$, $k=1.0$, $a_m=0.25$, $\Delta_a=0.25$, $R(0)=5.0$, $i(0)=5.0$, CURVE NO. 2
 SHOWS THE EFFECT OF NEGLECTING THE LAST TERM IN (7.5)

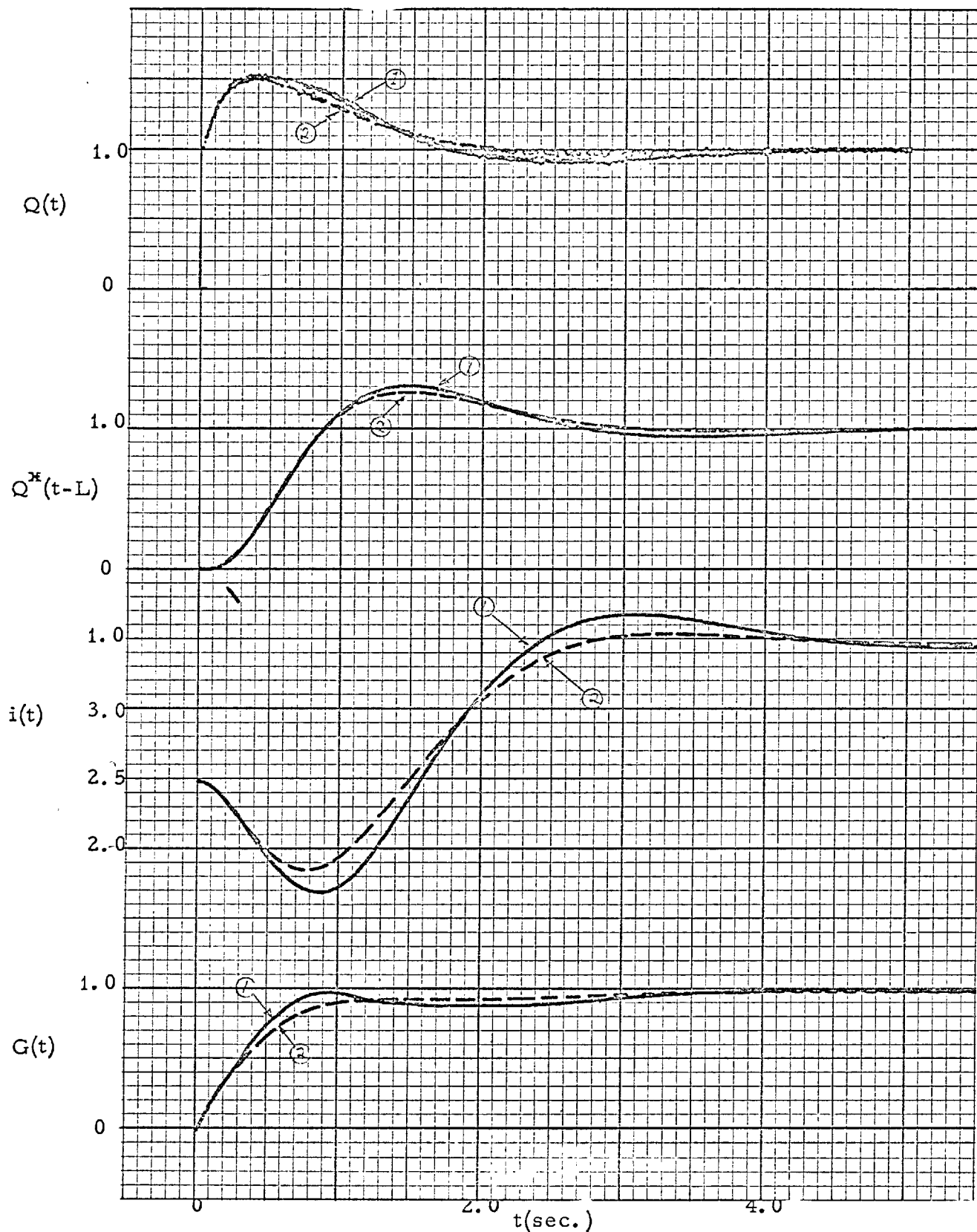


Figure 7.4 SYSTEM STEP RESPONSE $\alpha=0.50$, $k=1.0$, $\Delta t=0.50$, $a_{mn}=0.25$
 (1) $R(0)=i(0)=2.50$ (2) $R(0)=i(0)=5.0$

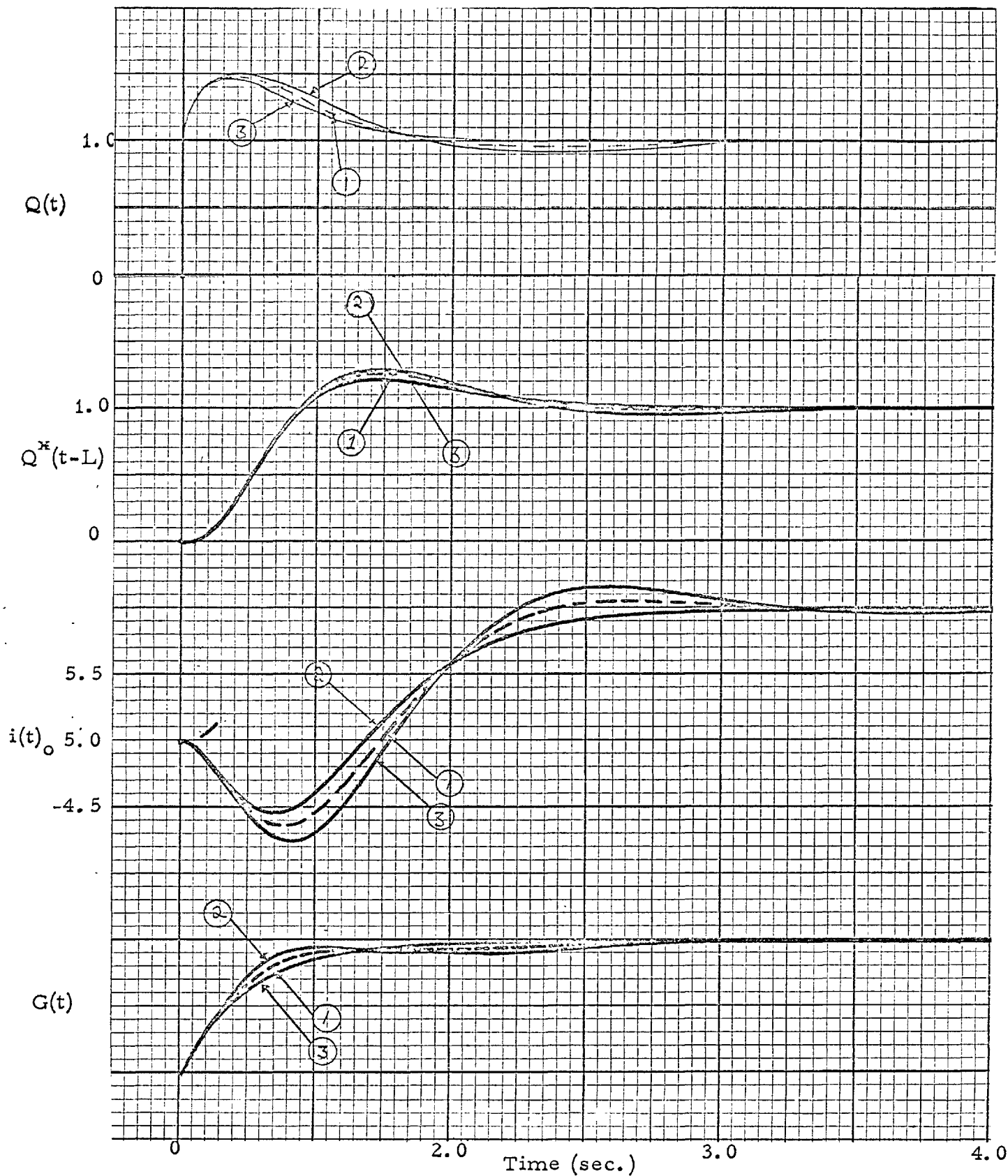


Figure 7.5 SYSTEM STEP RESPONSE $R_o = 5.00$, $i_o = 5.00$, $\alpha = 0.5$, $k = 1.0$

(1) $\Delta_a = 0.25$ $a_m = 0.25$ (2) $\Delta_a = 0.5$ $a_m = 0$ (3) $\Delta_a = 0.0$ $a_m = 0.5$

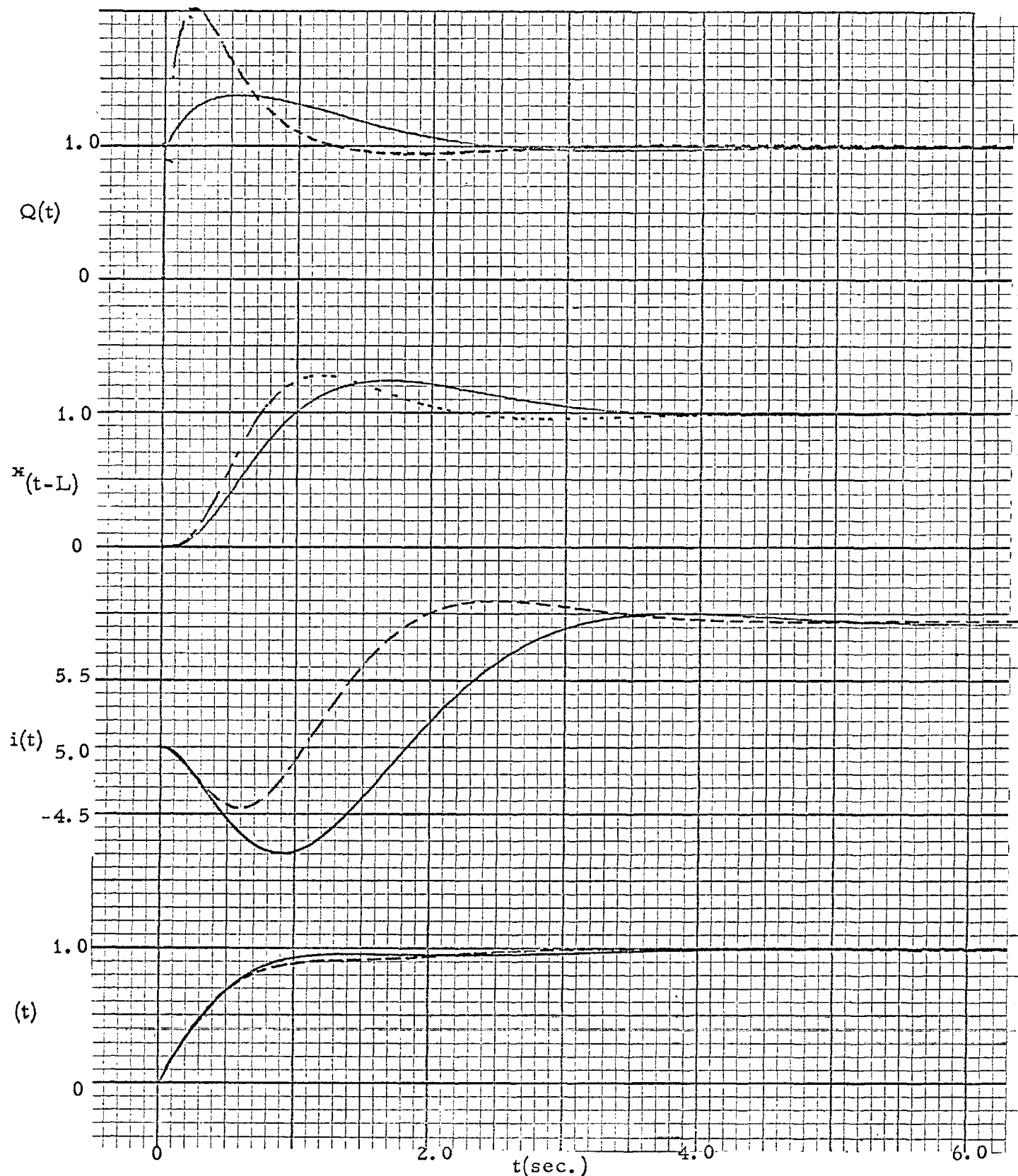


Figure 7.6 SYSTEM STEP RESPONSE $R(0)=i(0)=5.0$, $\Delta_a=0.25$, $a_m=0.25$, SOLID LINE SHOWS RESULT FOR $\alpha=k=0.5$, DOTTED LINE FOR $\alpha=k=1.0$

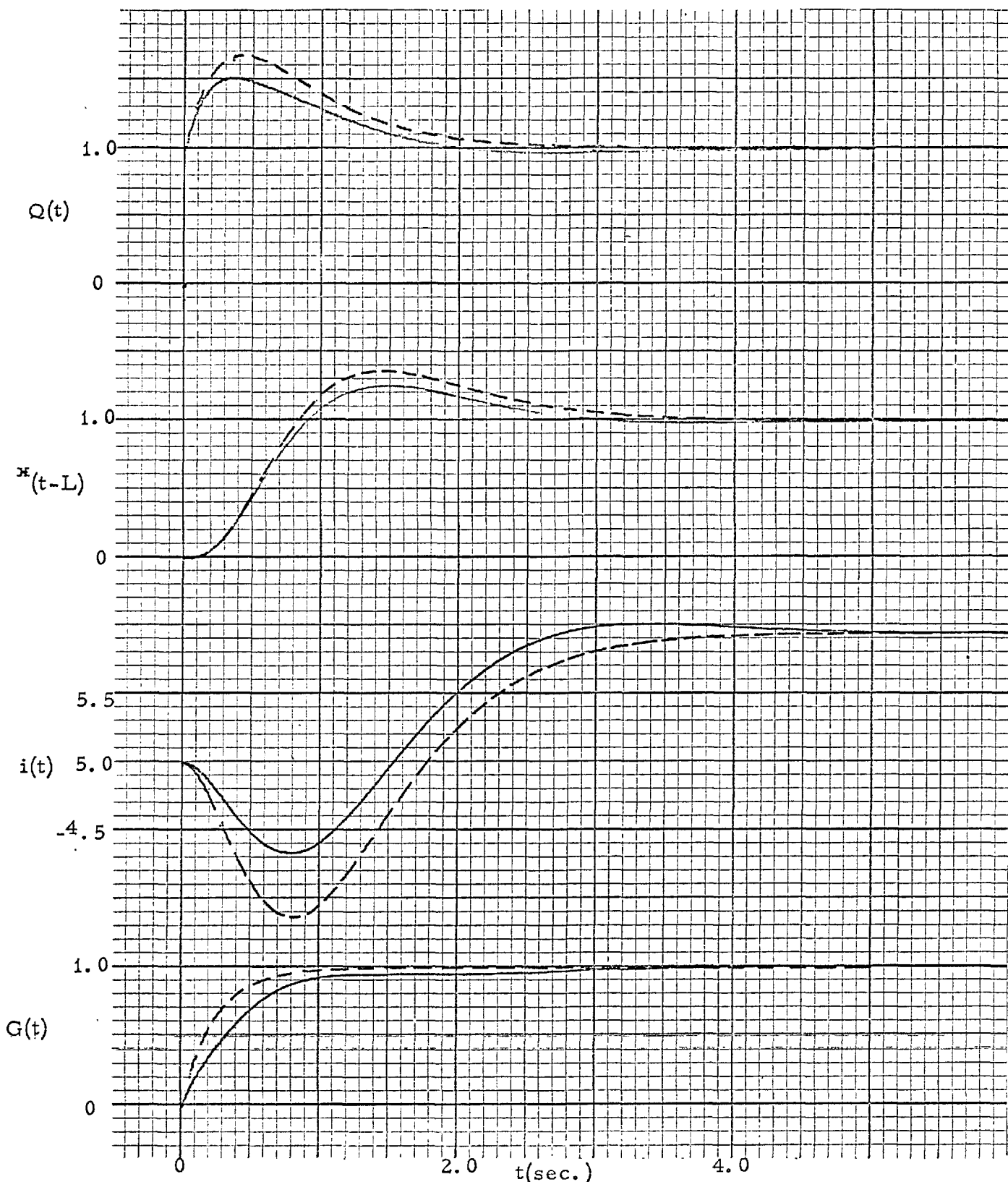


Figure 7.7 SYSTEM STEP RESPONSE FOR $R(0)=i(0)=5.0$, $\alpha=0.5$, $k=1.0$, SOLID LINE SHOWS RESPONSE FOR $a(t)$ OBTAINED FROM EQUATION (7.8) WITH $\Delta_a=0.25$, $a_m=0.25$, DOTTED LINE SHOWS RESPONSE FOR $a(t)=\text{const.}=0.5$

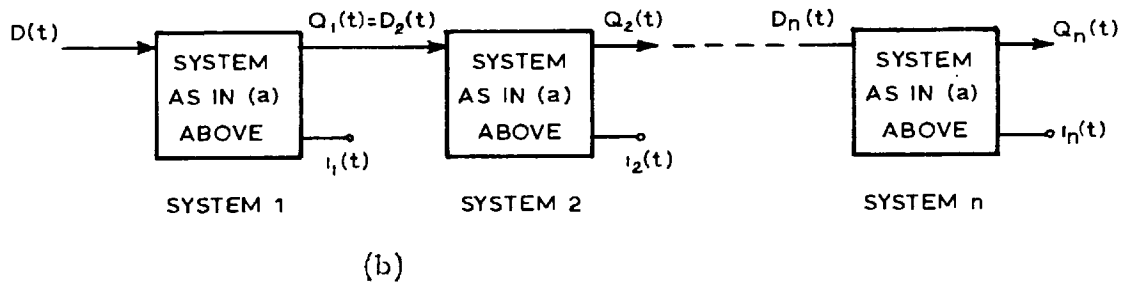
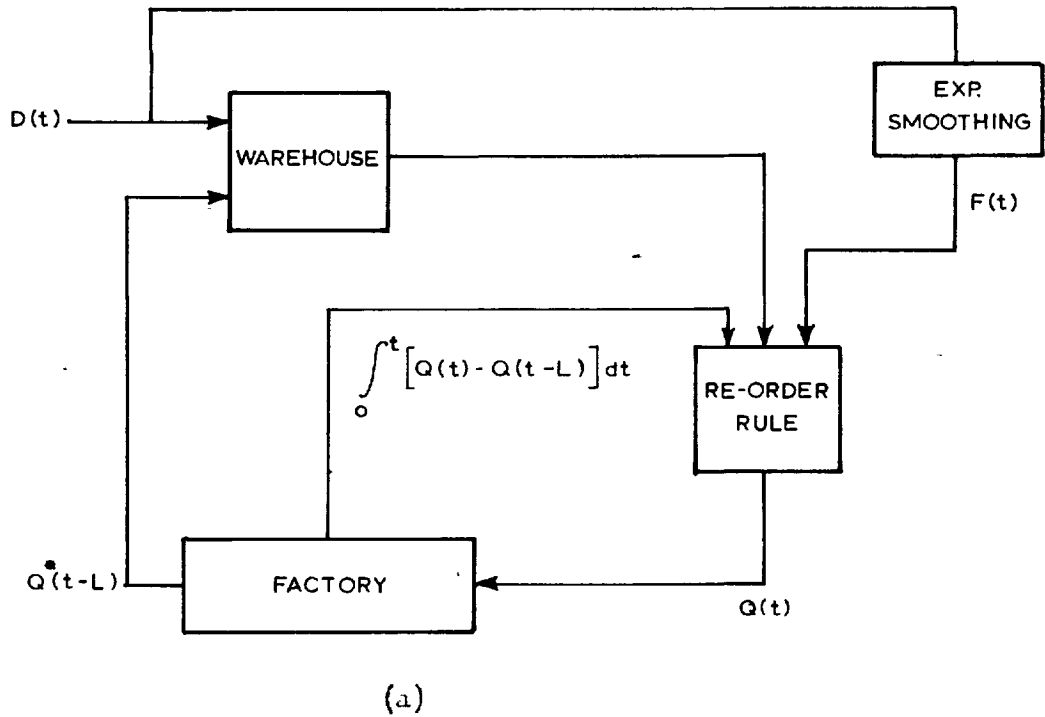


Figure 7.3 (a) SCHEMATIC DIAGRAM OF A SIMPLIFIED INDUSTRIAL PRODUCTION-INVENTORY CONTROL MODEL
(b) n SYSTEMS IN SERIES

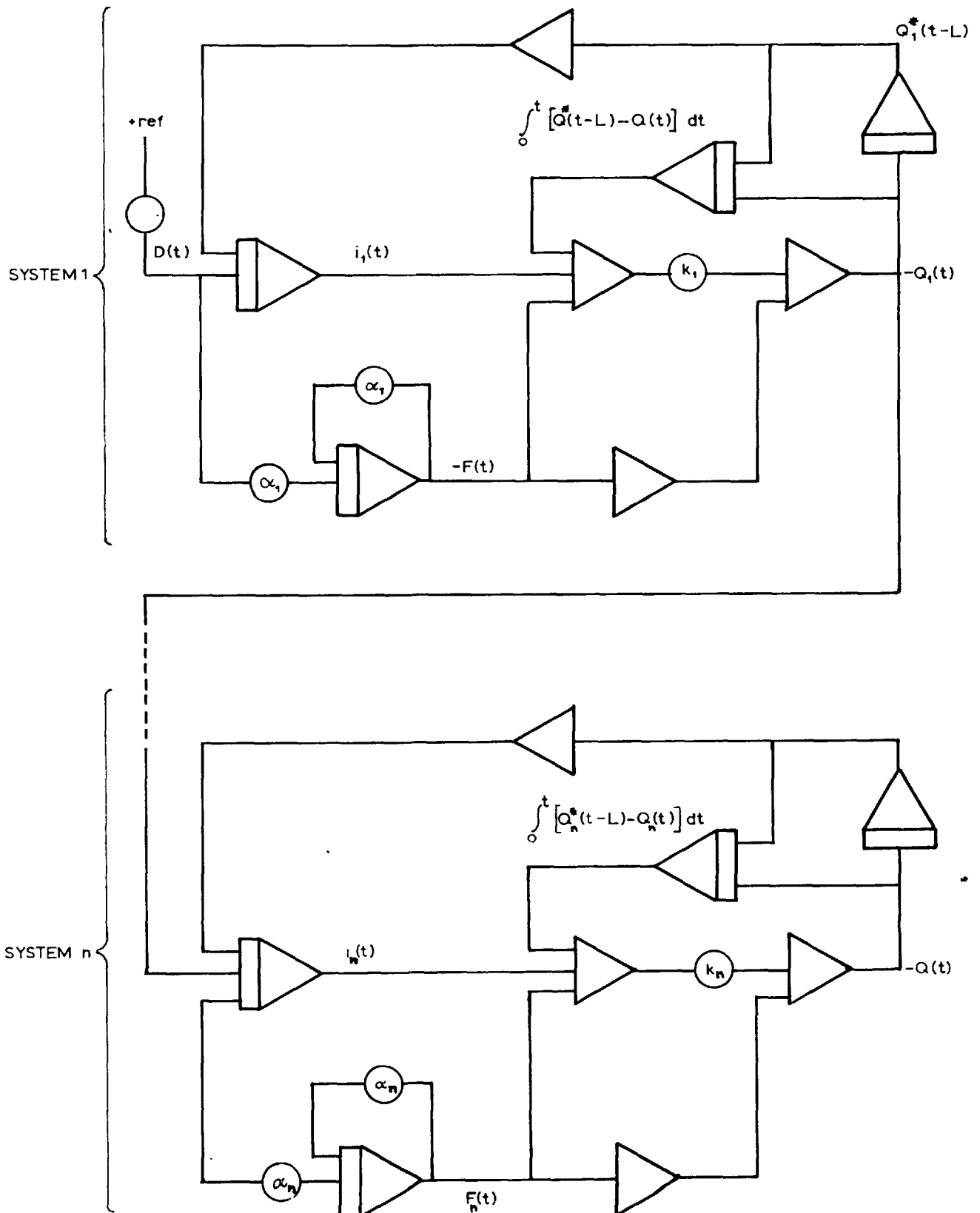


Figure 7.2 ANALOGUE PROGRAMME FOR n SIMPLIFIED PRODUCTION-
INVENTORY SYSTEMS IN SERIES

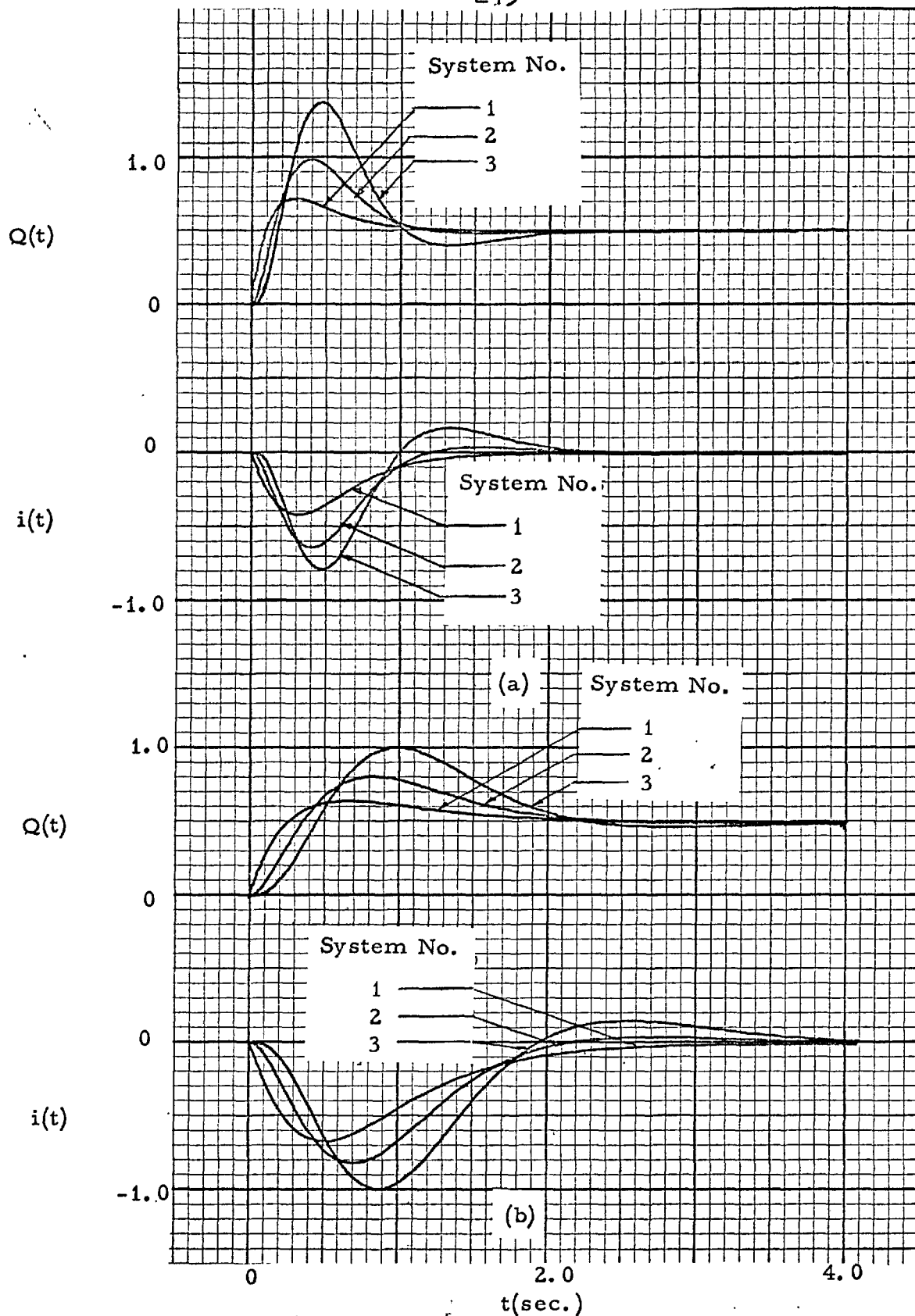


Figure 7.10 THREE SIMPLIFIED PRODUCTION-INVENTORY MODELS
IN SERIES RESPONSE TO A UNIT STEP (a) $\alpha=k=1.0$
(b) $\alpha=k=0.5$

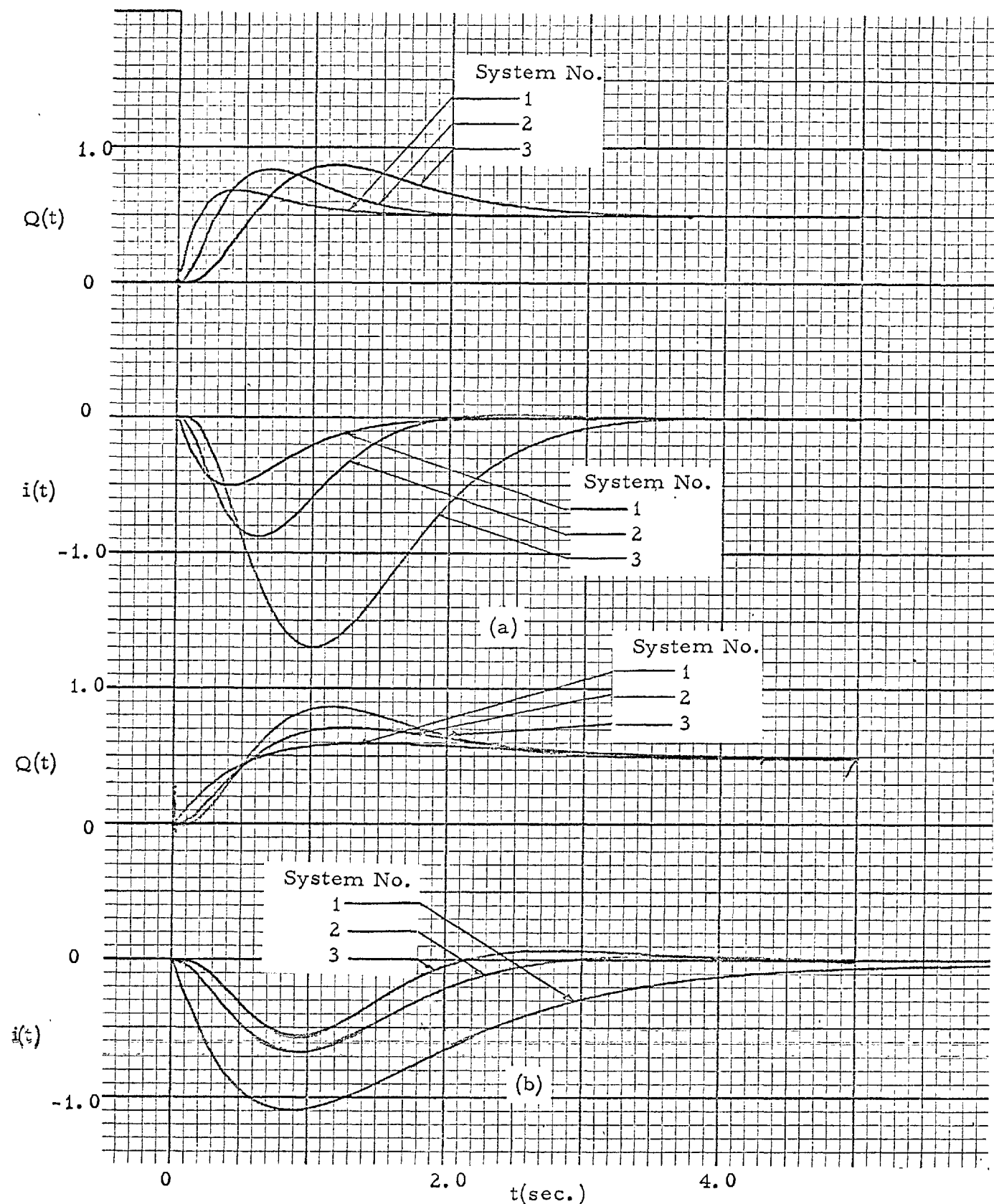


Figure 7.11 EFFECT OF VARYING PARAMETER VALUES FOR DIFFERENT MEMBERS OF CHAIN (a) $\alpha_1=k_1=0.75$, $\alpha_2=k_2=0.50$, $\alpha_3=k_3=0.25$, (b) $\alpha_1=k_1=0.25$, $\alpha_2=k_2=0.50$, $\alpha_3=k_3=1.0$

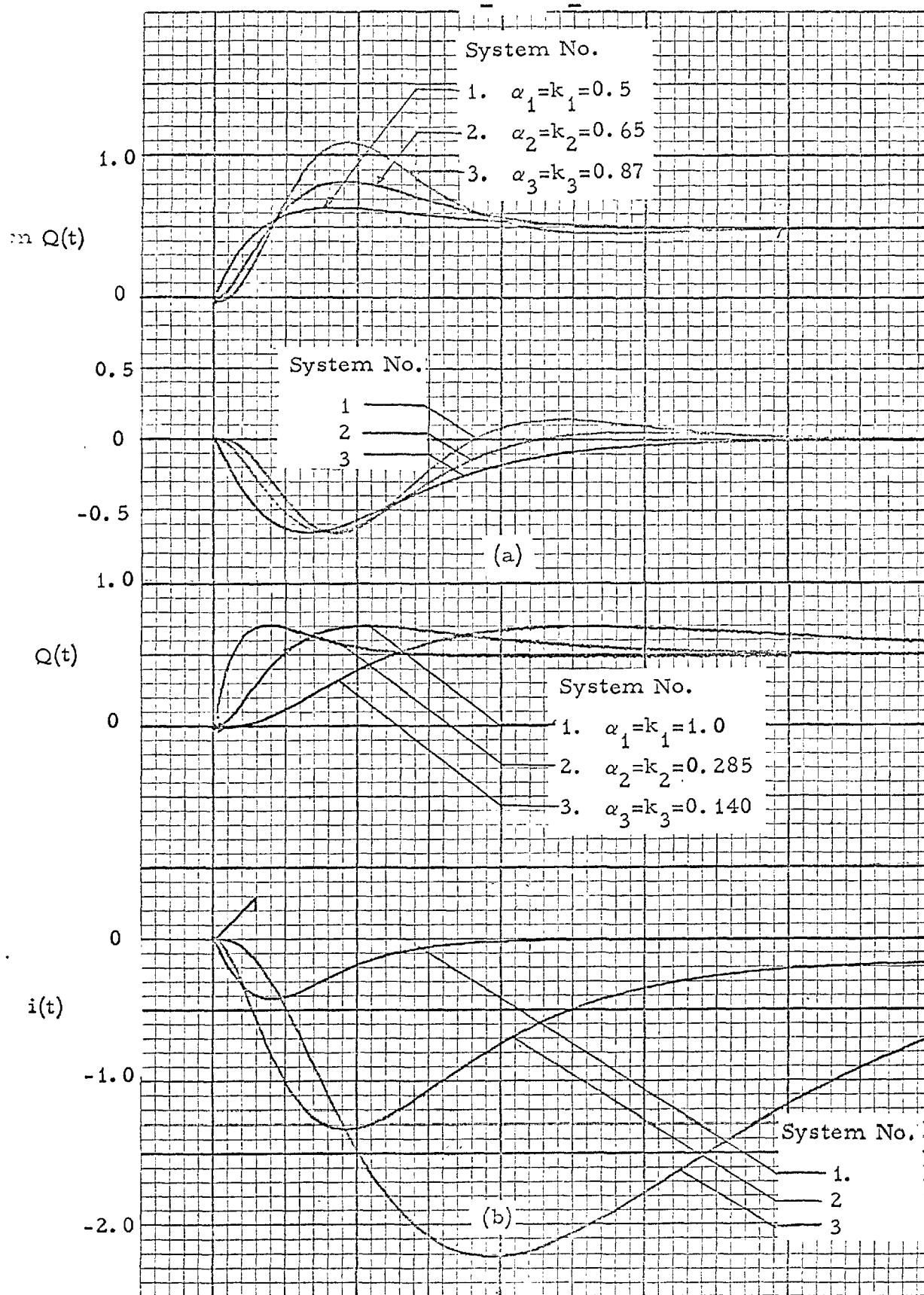


Figure 7.12 THREE SYSTEMS IN SERIES WITH α AND k ADJUSTED TO YIELD PARITY AMONG PEAK ABSOLUTE VALUES OF (a) $i(t)$ AND (b) $Q(t)$

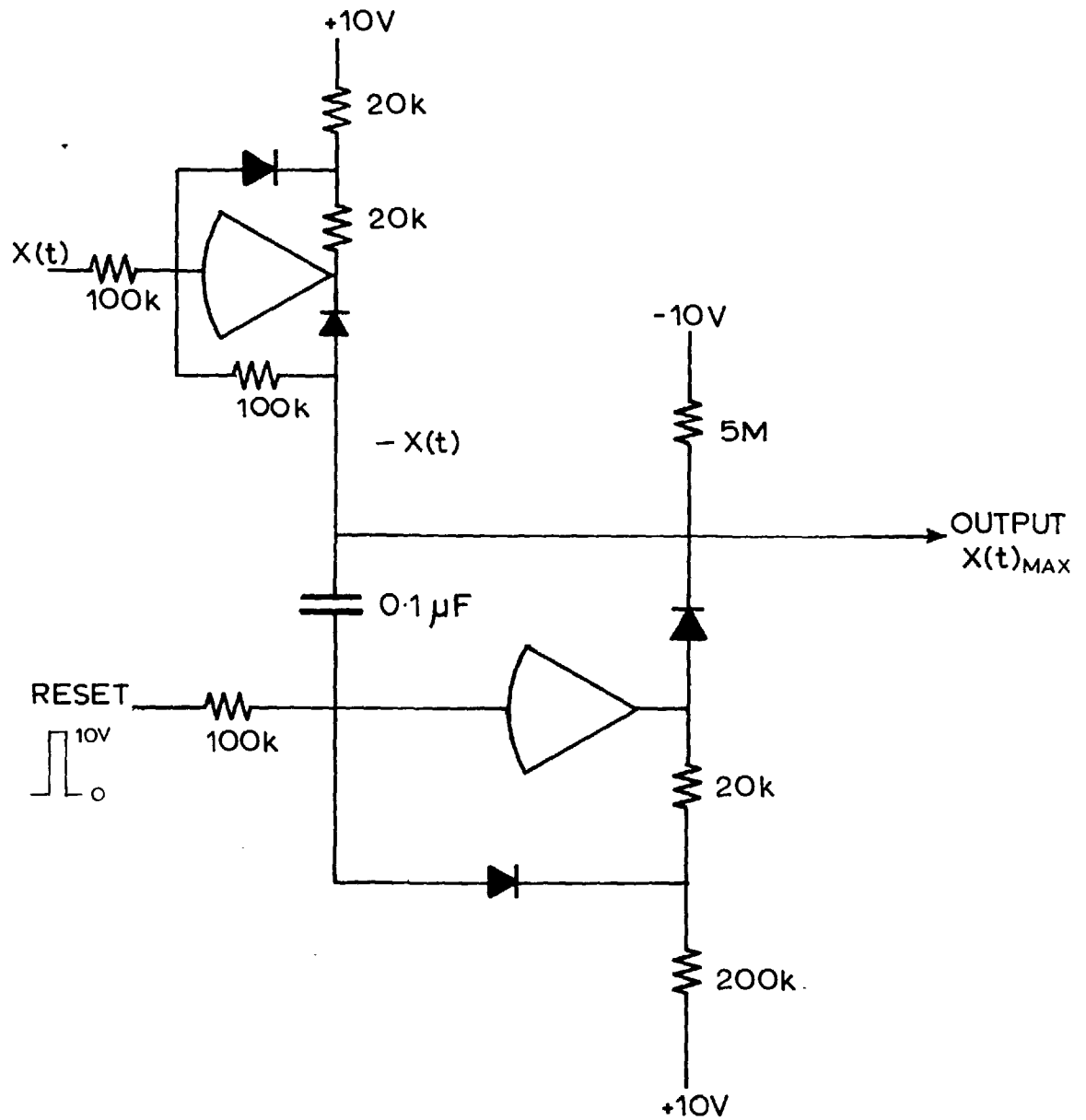


Figure 7.13 TRACK HOLD CIRCUIT FOR OBTAINING $x(t)_{max}$

- 249(a)

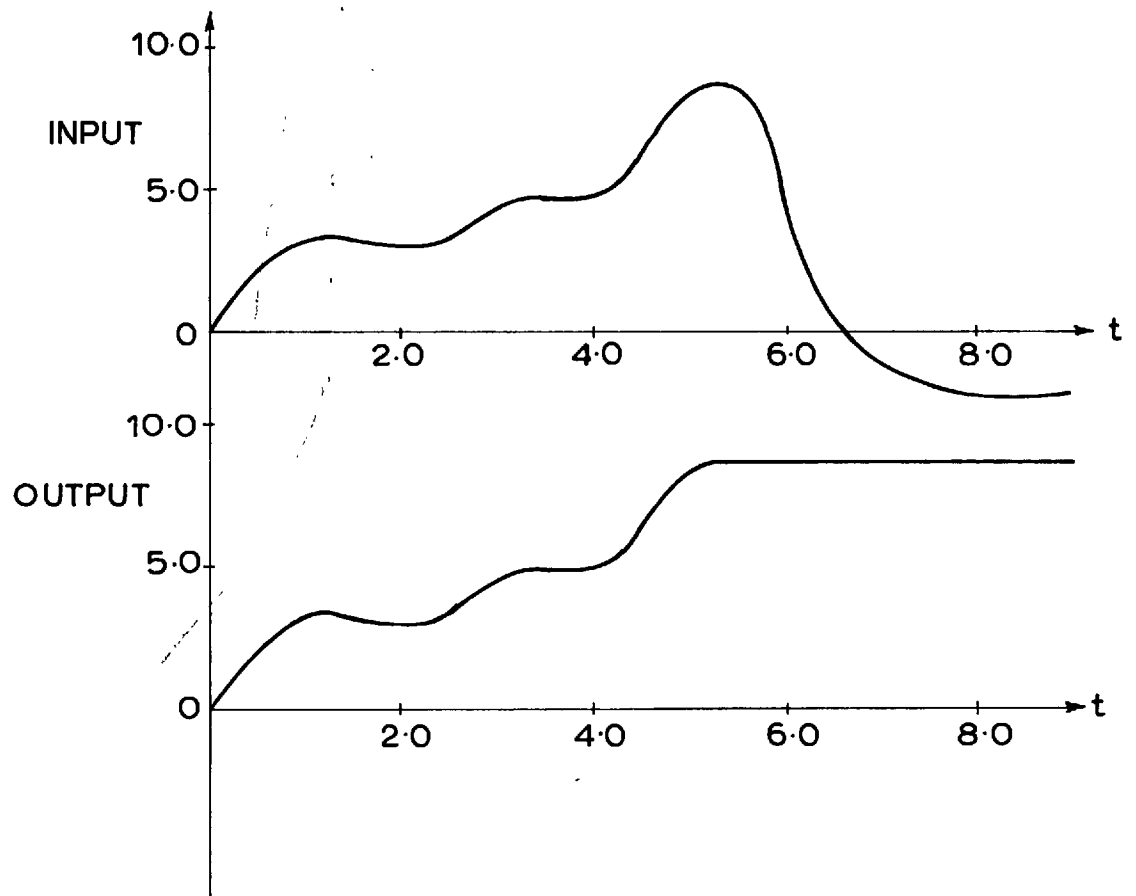


Figure 7.14 OUTPUT FROM TRACK HOLD CIRCUIT

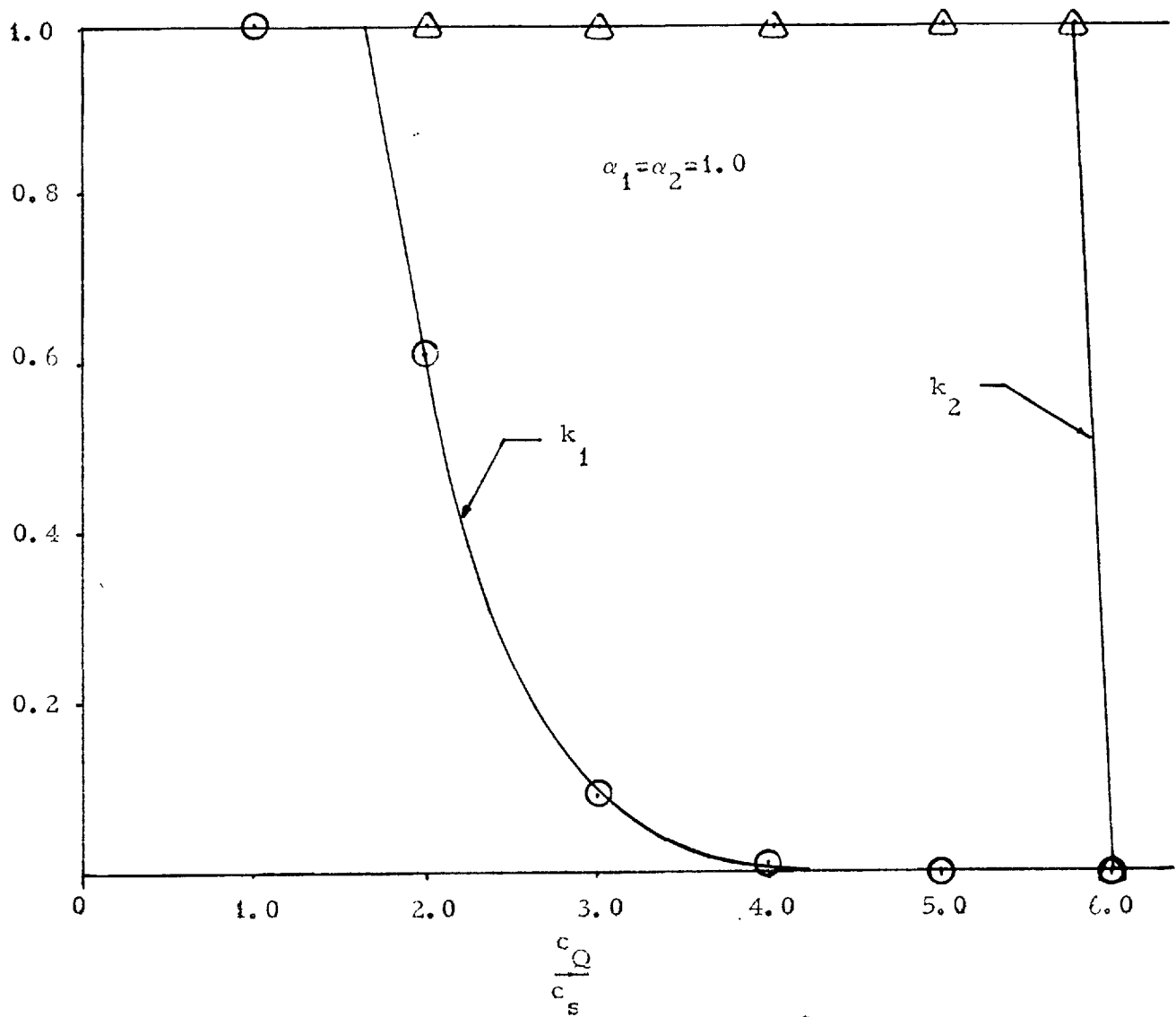


Figure 7.15(a) OPTIMAL PARAMETER VALUES vs $\frac{c_Q}{c_S}$ WHEN
SYSTEMS ARE OPTIMISED SEPARATELY

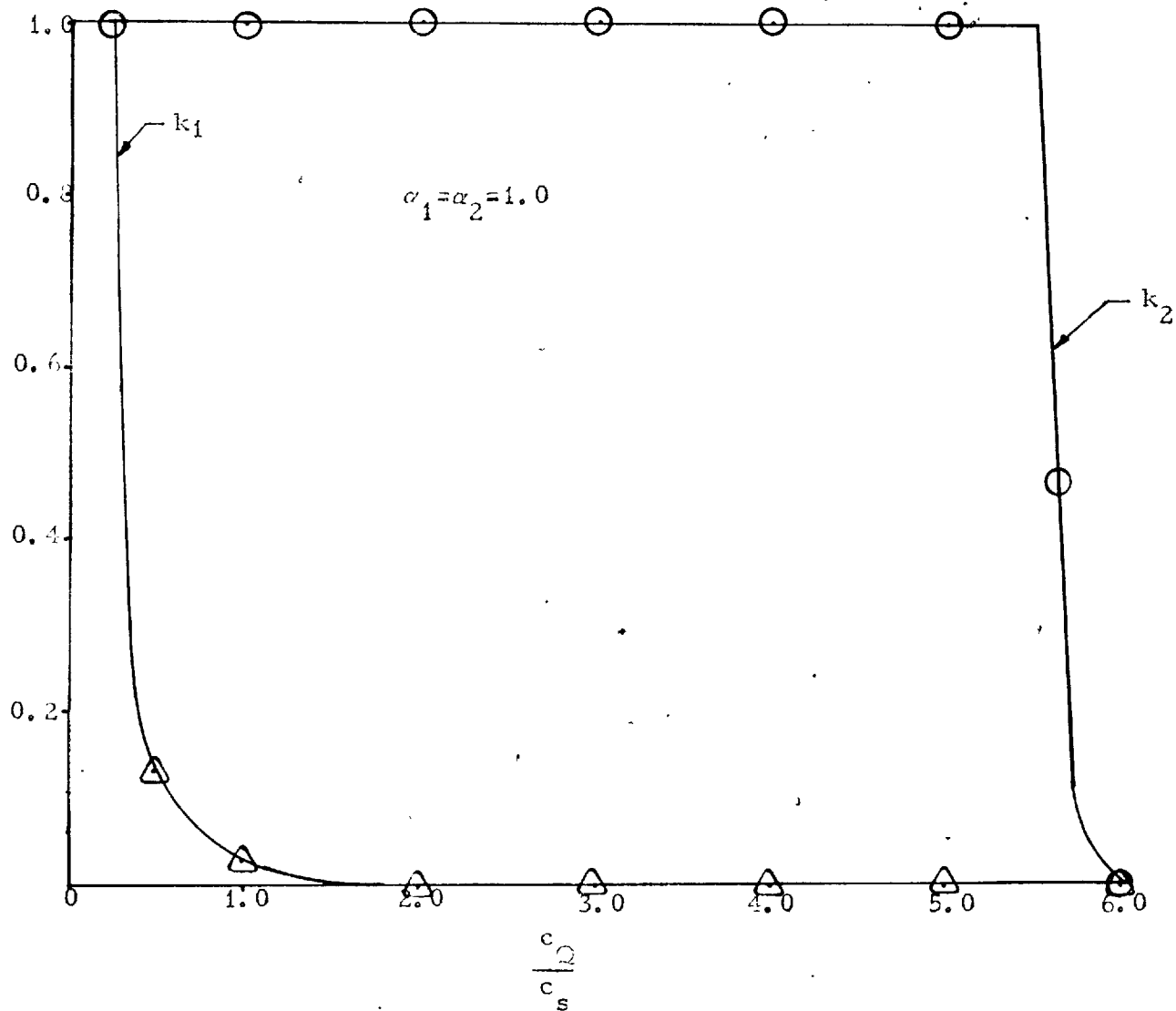


Figure 7.15(b) OPTIMAL PARAMETER VALUES vs $\frac{c_1}{c_2}$ WHEN
SYSTEMS ARE OPTIMISED TOGETHER

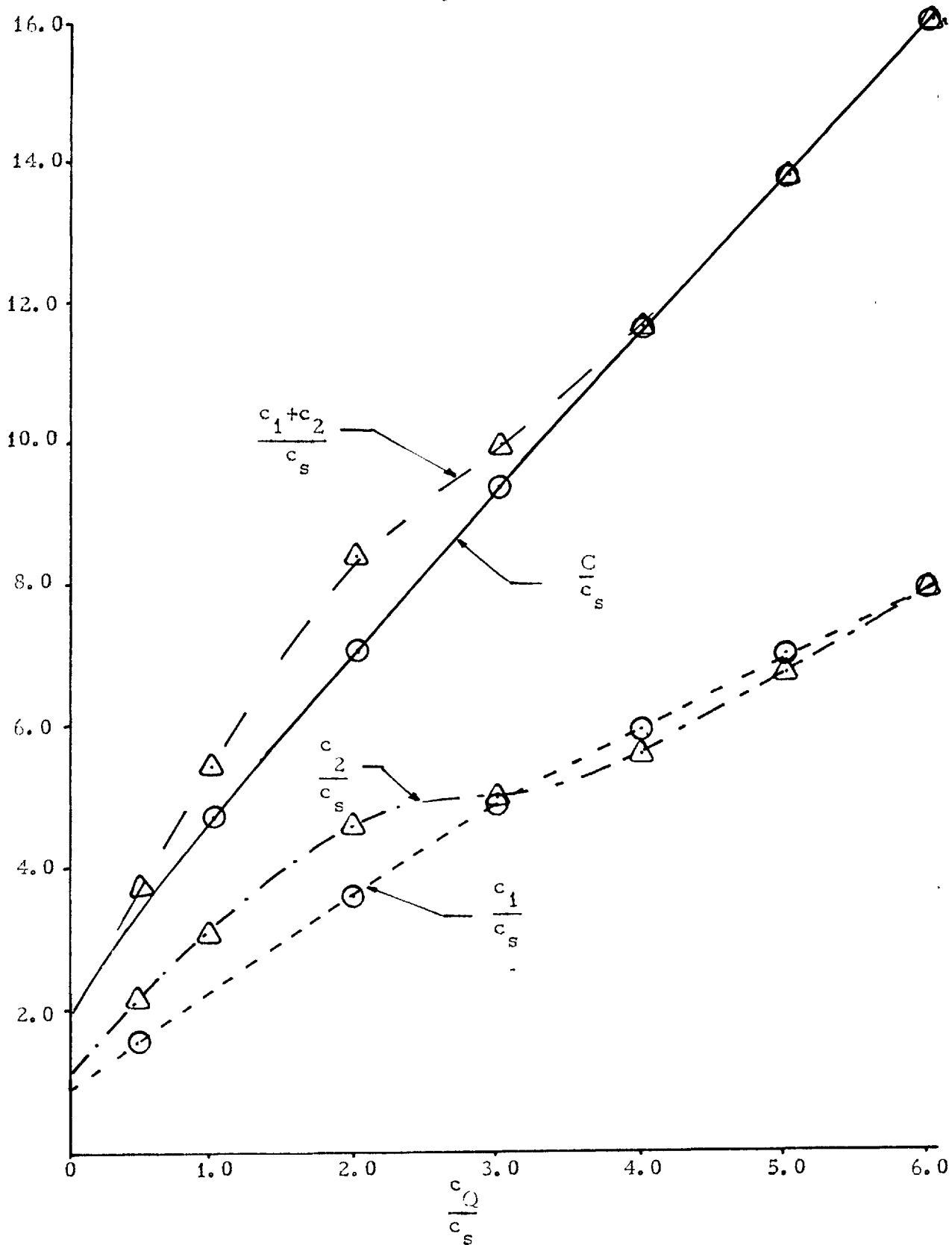


Figure 7.16 NORMALIZED COST VALUES vs $\frac{c_1}{c_s}$ WHEN SYSTEMS ARE OPTIMISED SEPARATELY AND TOGETHER

CHAPTER 8

FORECASTING AND
INVESTMENT ANALYSIS

Notation

$x(t)$ = input variable

$y(t)$ = output variable

i = equivalent continuous interest rate ($i = \ln(1+r)$)

r = interest rate

$A(t)$ = rate of capital flow

$P'(t)$ = present value of an element of flow at
time t

$P(t)$ = net present value of a project

$\bar{x}(t)$ = average of $x(t)$

σ = standard deviation

CHAPTER 8 : FORECASTING AND INVESTMENT ANALYSIS

8.1 FORECASTING

The use of an analogue computer for simulating forecast rules was first suggested by Otterman [64] who described an analogue circuit for computing the exponentially weighted average of a continuous variable. In this study we consider the use of a sampled data analogue computer for simulating forecast rules, where the variables are discrete. The extension of the analogue approach to include discrete variables was indicated by the fact that in practice data are generally available as a series of observations made at different points in time, rather than as continuous functions.

Two techniques can be used to arrive at an analogue circuit to handle forecasting rules, namely the flow graph technique and the transform method. These are briefly described below:

Flow graph technique

Consider for example, simple exponential smoothing, where:

$$\bar{x}(t) = \alpha x(t) + (1-\alpha)\bar{x}(t-1) \quad (8.1)$$

$\bar{x}(t)$ = current estimate of mean

$x(t)$ = current demand

α = constant between 0 and 1.

The corresponding flow graph is given in Figure 8.1 where τ indicates a delay of one time period.

The analogue programme for simulating this system is obtained by substituting a computing element for each branch operation and a summer for each node having more than one input. Delays are implemented on the computer by operating two sample hold circuits sequentially, resulting in an analogue programme as shown in Figure 8.2(a).

z Transform technique

The z transform can be obtained for the forecast rule using well-known mathematical techniques or by referring to tables [64]. Suppose, for example, we have obtained the z transform $H(z)$ of some forecast rule:

$$H(z) = \frac{a_0 + a_1 z + a_2 z^2 \dots a_n z^n}{b_0 + b_1 z + b_2 z^2 \dots b_m z^m} \quad (8.2)$$

The expression in (8.2) can be split up into partial fractions and written as:

$$H(z) = \frac{c_1}{1-d_1 z} + \frac{c_2}{1-d_2 z} \dots = D_1(z) + D_2(z) + D_3(z) \dots \quad (8.3)$$

Taking the inverse transform of any one of the partial fractions in (8.3) we have:

$$e_{OUT}(t) = C_k e_{IN}(t) - d_k e_{OUT}(t-1) \quad (8.4)$$

which has the same form as equation (8.1) and hence can be simulated using the analogue configuration of Figure 8.2(a). Since all transfer functions $D_k(z)$ can be realised in a similar manner, it is evident that $D(z)$ can be obtained by computing each $D_k(z)$ and summing the outputs.

Analogue Programmes

Using the techniques which have been described, analogue circuits to simulate the five forecasting rules given in Table 8.1 have been synthesised and these are given in Figure 8.2.

Figures 8.3 and 8.4 illustrate some of the results that can be obtained by this method. Figure 8.3 shows the simple exponential smoothing system response to noise plus a unit step where $\sigma \text{ noise} = 0.1$ Figure 8.4 shows similar results for double exponential smoothing.

...

The use of an analogue computer for simulating discrete data forecast rules has been demonstrated.

Further work in this area has since been carried out by Iacovou [76] who extends the approach to non-linear as well as linear systems.

8.2 INVESTMENT ANALYSIS

This section describes the use of an analogue computer for performing calculations in the field of investment analysis. Investment analysis in its broadest sense is concerned with assessing the financial attractiveness of a capital project where for the sake of objective clarity financial attractiveness is defined mathematically.

It is necessary to distinguish between two broad classes of investment problem, namely, investment under certainty and under uncertainty. This section deals with the former, where all future cash flows and interest rates are known.

When an investment decision is to be made under certainty there are two principal methods of project evaluation which may apply. These will be referred to as the Net Present Value method and the Yield method.

In the Net Present Value method we assume that capital inflows and outflows from a project are known

and can be expressed as a function of time $A(t)$, where $A(t)$ may be either continuous or discrete.

The present value of an element of flow at time t is given by

$$P'(t) = A(t) e^{-it} \quad (8.5)$$

where

i = rate of interest on capital

$P'(t)$ = present value of an element of flow

It will be clear from the foregoing that $A(t)$ may assume both positive and negative values depending on whether the net flow is into or out of the project at time t .

The net present value of the project is then given by

$$P(t) = \int_0^t P'(t) dt = \int_0^t A(t) e^{-it} dt \quad (8.6)$$

and where $A(t)$ is known only at discrete intervals (as is usually the case [77]),

$$P(n) = \sum_{m=0}^n A(m) \left[\frac{1}{1+r} \right]^m \quad (8.7)$$

where

$A(m)$ = net cash flow in year m

r = interest rate expressed in per cent

and r is related to i according to

$$\frac{1}{1+r} = e^{-i}$$

The net present value $P(n)$ or $P(t)$ is then used as an indication of the financial attractiveness of the project, being positive for profitable investments and negative otherwise.

The yield method on the other hand seeks to elicit the equivalent interest rate (measured in per cent per annum) on capital tied up in the project. This is accomplished by determining the value (or values) of r that solve the equation

$$\sum_{n=0}^m A(n) \left[\frac{1}{1+r} \right]^n = 0 \quad (8.8)$$

Thus, the equivalent interest rate is that value of r for which the sum of discounted cash flows is zero.

The use of an analogue for investment analysis was first suggested by Fetter and Goodman [25] who described a continuous investment analogue for studying various equipment replacement policies. Gillings [37] developed a special purpose analogue for performing net present value and equivalent discount rate calculations for discrete models (equations (8.7) and (8.8)) and the following describes how a general purpose analogue machine can be used to obtain similar results for the discrete case.

An analogue programme for computing $P(n)$ is given in Figure 8.5 where cash flows and the interest rate are entered via potentiometers to a string of sample hold circuits arranged in series to provide an n -period

delay. The discount factor $\left(\frac{1}{1+r}\right)^n$ is computed using two sample hold circuits arranged to form an infinite loop and each cash flow is multiplied by the appropriate discount factor using a quarter square multiplier and summed as shown to yield $P(n)$.

To determine the net present value of a project, the machine is switched to operate and $P(n)$ is plotted against time. This is shown in Figure 8.6 for different values of r , and a series of cash flows $A(n)$ as shown. To obtain the equivalent interest rate, the machine is switched to rep-op mode and the $P(n)$ waveform observed on an oscilloscope; r is then varied so that the $P(n)$ waveform returns to zero, the value of r read from the potentiometer is the equivalent interest rate. This is shown in Figure 8.6, for $r = 0.21$.

The use of an analogue computer for performing calculations in the field of investment analysis for two methods of project evaluation under certainty has been demonstrated. It should be mentioned that the usefulness of an analogue in this area appears to be primarily for illustrating the principles involved in investment appraisal calculations rather than practical analysis.

Table 8.1 Five Forecasting Rules

Rule	Expression	z Transform
(a) Simple exponen- tial smoothing	$\bar{x}(t) = \alpha x(t) + (1-\alpha)\bar{x}(t-1)$	$H(z) = \frac{\alpha}{1-(1-\alpha)z}$
(b) Double exponen- tial smoothing	$y(t) = (2 + \frac{\alpha}{1-\alpha})\bar{x}(t) + (1 + \frac{\alpha}{1-\alpha})b(t)$ where $b(t) = \alpha \bar{x}(t) + (1-\alpha)b(t-1)$	$H(z) = \frac{\alpha(2 + \frac{\alpha}{1-\alpha})}{1-(1-\alpha)z}$ $- \frac{(1+\alpha)(\frac{\alpha}{1-\alpha})}{[1-(1-\alpha)z]^2}$
(c) Moving average	$y_1(t) = \frac{x(t) + x(t-1) + \dots + x(t-N+1)}{N}$	$H(z) = \frac{1-z^N}{1-z}$
(d) Moving average corrected for trend	$y(t) = y_1(t) + \frac{6S(t)}{N(N+1)}$ where $S(t) = S(t-1) - Ny_1(t-1)$ $+ \frac{N-1}{2}x(t)$ $+ \frac{N+1}{2}x(t-N)$	$H(z) = \frac{4N-2}{N(N+1)} \left\{ \frac{1}{1-z} \right\}$ $+ \frac{2}{N} \left\{ \frac{z^N}{1-z} \right\} - \frac{6}{N(N+1)}$ $\left\{ \frac{z(1-z^N)}{(1-z)^2} \right\}$
(e) Second- order system	$y(t) = \alpha x(t) + 2(1-\alpha)y(t-1)$ $- (1-\alpha)y(t-2)$	$H(z) = \frac{\alpha}{1+2(1-\alpha)z-(1-\alpha)z^2}$

Analogue circuits for these rules are given in Figure 8.2

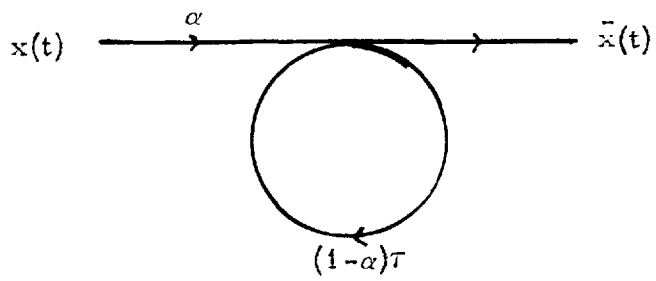
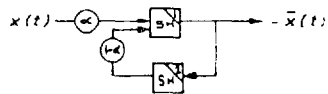
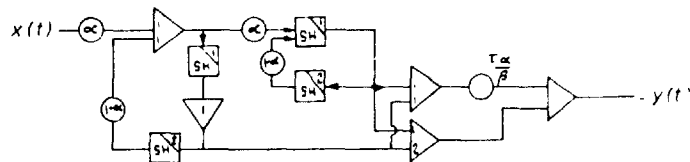


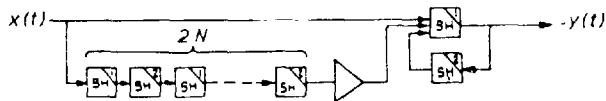
Figure 8.1 FLOW GRAPH FOR EXPONENTIAL SMOOTHING



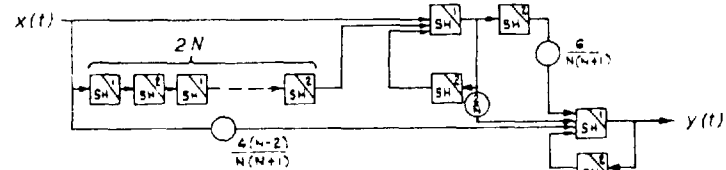
(a) SIMPLE EXPONENTIAL SMOOTHING



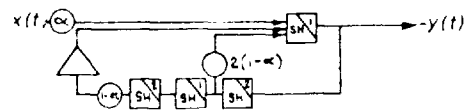
(b) DOUBLE EXPONENTIAL SMOOTHING



(c) MOVING AVERAGE



(d) MOVING AVERAGE CORRECTED FOR TREND



(e) SECOND ORDER SYSTEM

Figure 8.2 Computer diagrams for five forecast rules

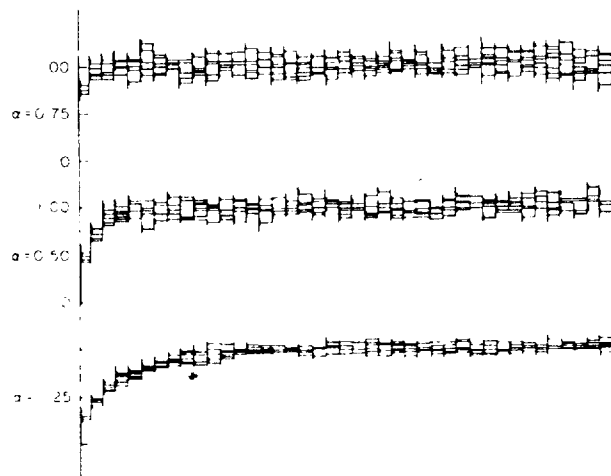


Fig. 8.3 Response of simple exponential smoothing to a step with noise
 $\sigma_D = 0.1$, $\bar{D}(t) = 1.0$

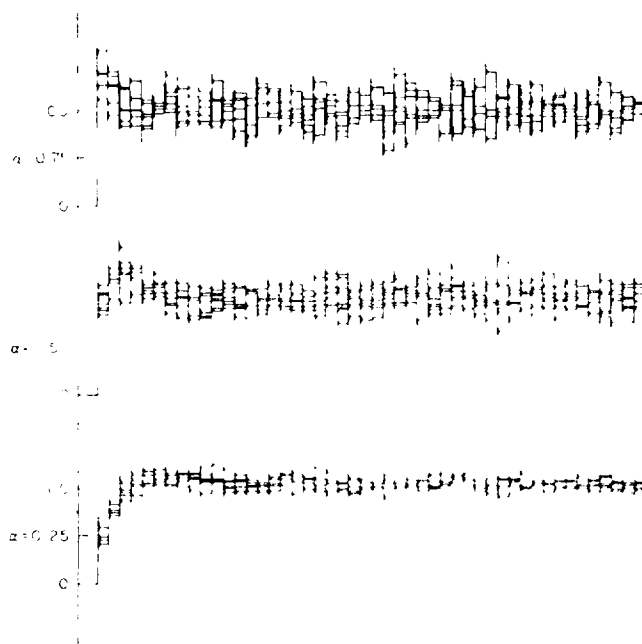


Fig. 8.4 Double exponential smoothing, response to a
 step with noise superimposed $\sigma_D = 0.1$, $\bar{D}(t) = 1.0$

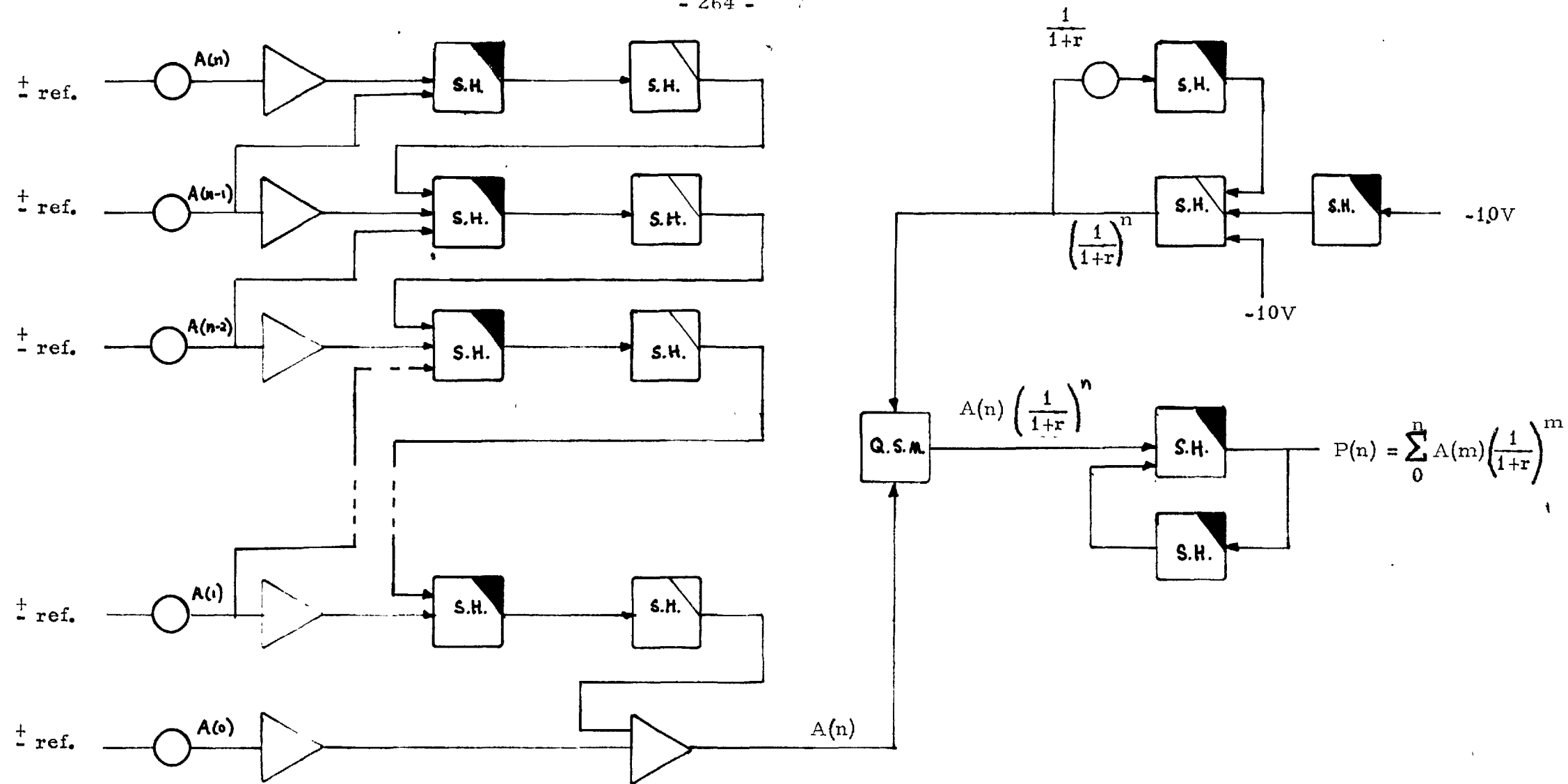


Figure 8.5 ANALOGUE PROGRAMME FOR D.C.F. METHODS OF INVESTMENT APPRAISAL, A_n =CASH FLOW IN YEAR n , r =INTEREST RATE, P =NET PRESENT VALUE

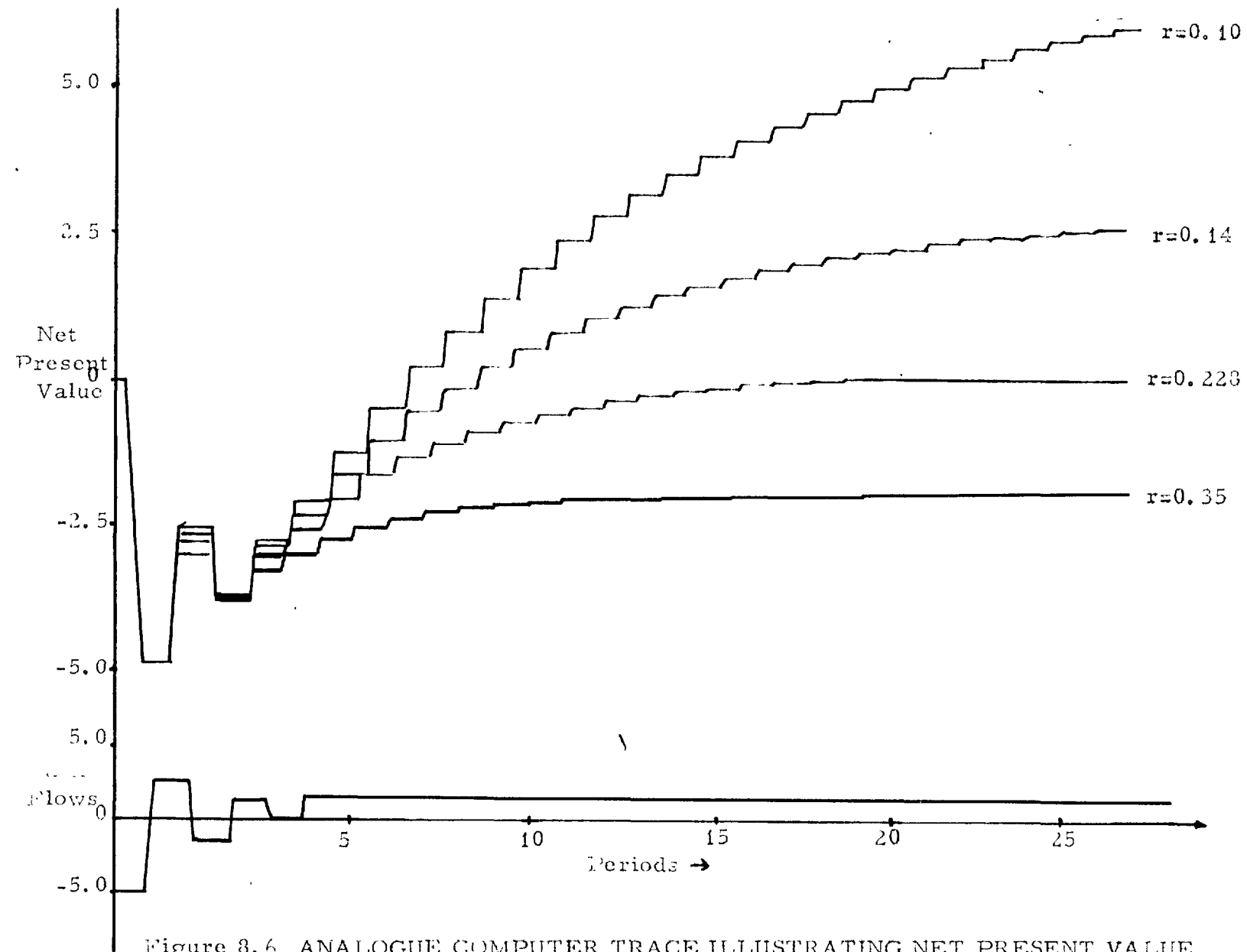


Figure 8.6 ANALOGUE COMPUTER TRACE ILLUSTRATING NET PRESENT VALUE AND EQUIVALENT DISCOUNT RATE CALCULATIONS

CHAPTER 9

CONCLUSIONS

CHAPTER 9 : CONCLUSIONS

9.1 SUMMARY

The use of analogue computers for solving various operational research problems has been demonstrated. It appears that the analogue provides advantages over other methods in case where rapid man-machine communication is a necessary feature of problem solution, and where accuracy is of secondary importance. Both of these are commonly encountered in Operational Research.

The use of the analogue for determining optimal reset policies in a process control problem illustrates its advantages for a class of problem where functions are to be manipulated which, although continuous, are awkward to handle analytically. Chapter 5 seeks to extend the analogue approach by describing how various probability density functions can be generated on the analogue and is intended to serve as basis for solving other problems in this area.

Chapter 2 illustrates the use of the analogue for a class of problem involving networks, where the problem that has been solved can be formulated as that of finding the absolute centres and medians of a graph.

Chapter 4 illustrates the use of the analogue for analysing and synthesising production-inventory control systems, and here the advantages of a direct man-machine interface are apparent.

The suitability of the analogue for simulating Industrial Dynamics models has been demonstrated and information has been obtained about the behaviour of industrial production-inventory systems when operating in series.

The use of the analogue for simulating queueing systems has been studied, and limitations imposed by standard analogue equipment have been pinpointed.

The use of the analogue for forecast rule simulation and investment analysis has also been dealt with.

9.2 FUTURE DEVELOPMENTS IN THE FIELD

The use of analogue computers for solving several O.R. problems has been demonstrated, however a serious drawback is apparent in some applications (notably discrete ~~Monte Carlo~~ simulation) in that the solution rate is not fast enough to allow large statistical samples to be obtained in a short period of time. The maximum sampling rate obtainable with available

commercial equipment is in the range 50 to 1,000 c.p.s.[⌘] This means that for simulation runs of 10,000 iterations the time required to obtain an output histogram with 10 intervals, is of the order of minutes. This in turn precludes the use of a continuous oscilloscope display of the output distribution and of course eliminates the quick man machine feedback loop normally such an attractive feature in analogue computation.

The technological problems that must be overcome in order to increase the sampling rate are:

- (i) Development of wide bandwidth d.c. amplifiers which permit use of very low computing resistances to reduce stray capacitance effects.
- (ii) Design of fast electronic switching devices to operate the sample hold circuits.

Considerable progress has been made along these lines. by Korn [69] who describes a fast digital analogue computing system with sampling rates in the range 10,000-100,000 c.p.s. and digital control of analogue units. This allows output histograms to

⌘ Using the TR 48 computer with reed relay switching unit the maximum rate was 50 c.p.s. Digital analogue equipment currently available utilises electronic switching and permits sampling rates of up to 10,000 c.p.s.

be generated continuously (at about $\frac{1}{2}$ second intervals for the example mentioned earlier) and enables the operator to vary system parameters and observe the effect on system performance immediately.

Using hybrid methods Korn has developed various devices for statistical averaging, distribution analysis, and packaged parameter optimisation. This latter employs simple digital analogue converters which allow the operator to set parameters manually for digitally controlled optimisation strategies.

These devices and techniques are described in detail in [78, 79, 80].

APPENDIX I

ANALOGUE PROGRAMMING TECHNIQUES

APPENDIX I ANALOGUE PROGRAMMING TECHNIQUES

Various techniques are available for translating a given problem into a form suitable for implementation on the computer. Some of the most important methods and a few examples are given below.

1. Direct Solution

This method is used when it is required to evaluate some function of an independent variable. For example suppose we wish to evaluate the expression

$$y(t) = \int_0^t \int_0^t A t f(t) dt dt \quad (I.1)$$

The computer programme for solving this equation using the direct method is shown in Figure I.1. (Note that multiplication, summation and integration operations are accompanied by a sign change on the computer.)

2. Indirect Solution

This method is used when the expression to be evaluated is a known function of both independent and dependant variables. For example, consider the equation

$$-y''(t) = a y'(t) + by(t) - f(t) \quad (I.2)$$

The computer programme to solve this equation is shown in Figure I.2.

3. Implicit Solution

This method is used when the function to be evaluated can be obtained as the solution of an implied equation. Consider the example where it is required to evaluate;

$$y = e^{-\frac{t^2}{2}}$$

Differentiating, we have

$$y' = -t e^{-\frac{t^2}{2}} = -ty \quad (I.3)$$

and this can be solved using the programme shown in Figure I.3.

4. Flow Graph Method

This method is useful in building a physical model of the system which is being investigated. For example consider the simple production and re-ordering system shown in Figure I.4 where I is the level of inventory, R is the quantity re-ordered and P is the quantity produced by the factory.

In this system let $R = -ki$, where i is taken to be

the inventory level above or below a given datum, and assume $P=R(1-e^{-\tau_1 t})$ where τ_1 is some constant (determined by the rate at which the factory is able to alter its production rate). A flow graph representation of this system, is shown in Figure I.5, and the corresponding computer programme is shown in Figure I.6. Amplifier 3 in this circuit is added to alter the polarity of P and the integrator 2 simulates the transfer function $\frac{1}{1+\tau_1 s}$.

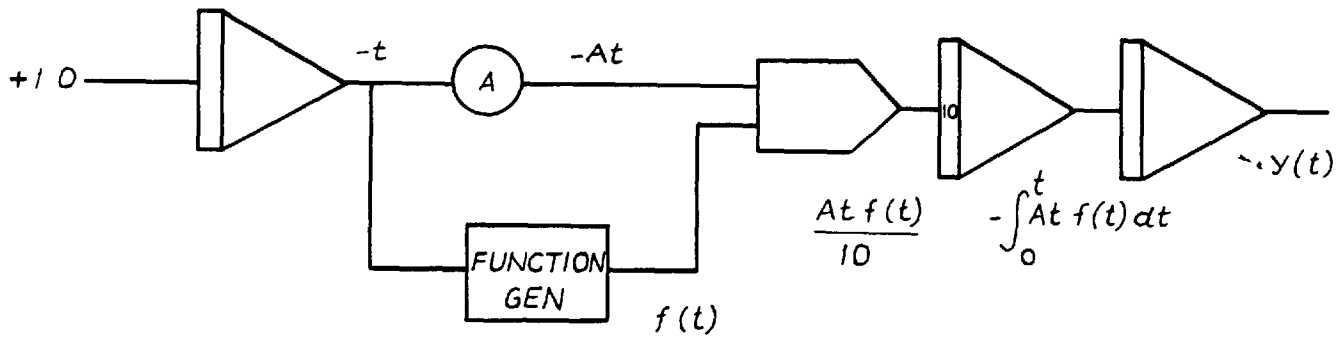


FIG. 1.1 EXAMPLE FOR COMPUTING THE FUNCTION 1.1 BY THE DIRECT METHOD

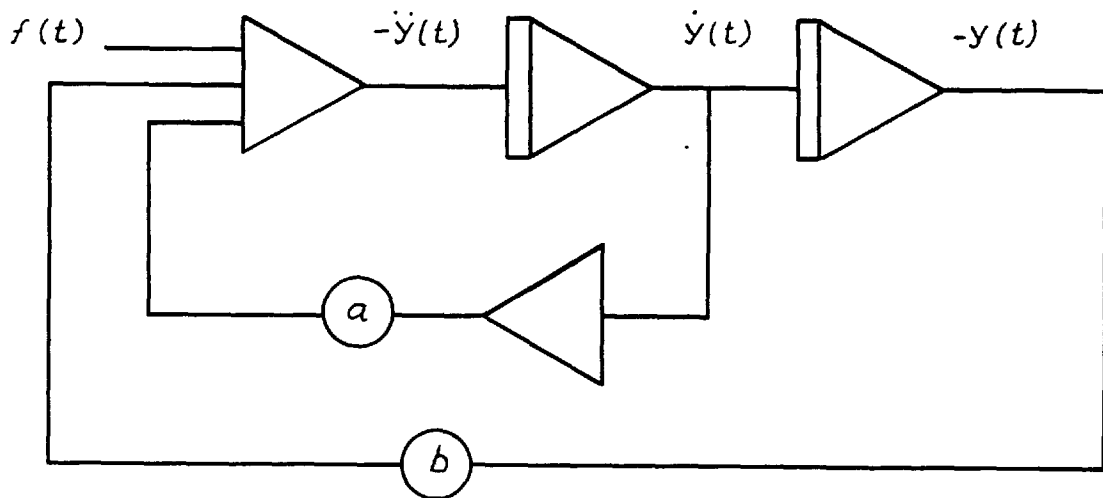


FIG. 1.2 EXAMPLE USING THE INDIRECT METHOD OF SOLUTION

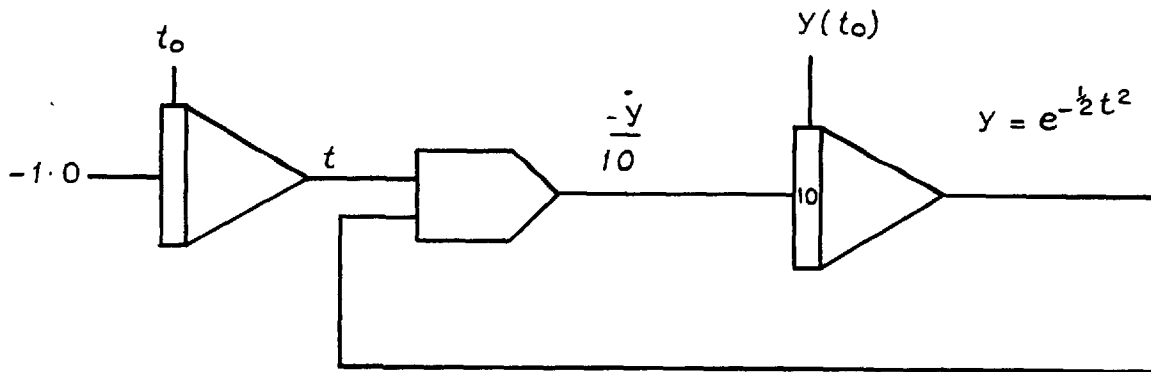


FIG. 1.3 CIRCUIT FOR SOLVING EQUATION 1.3

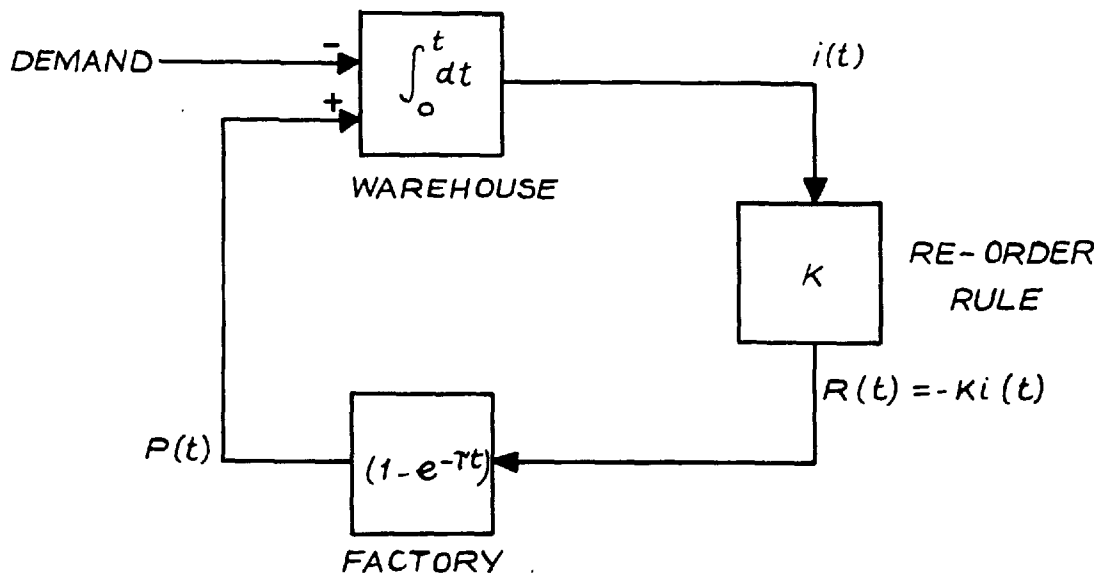


FIG. 1.4 PRODUCTION AND INVENTORY SYSTEM

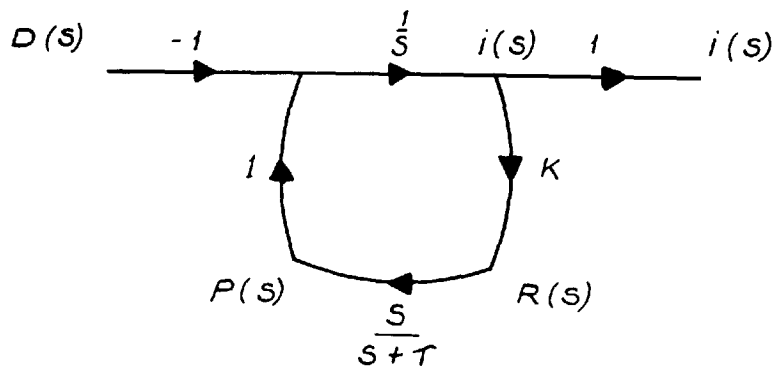


FIG. 1.5 FLOW GRAPH FOR THE SYSTEM SHOWN IN FIG. 1.4

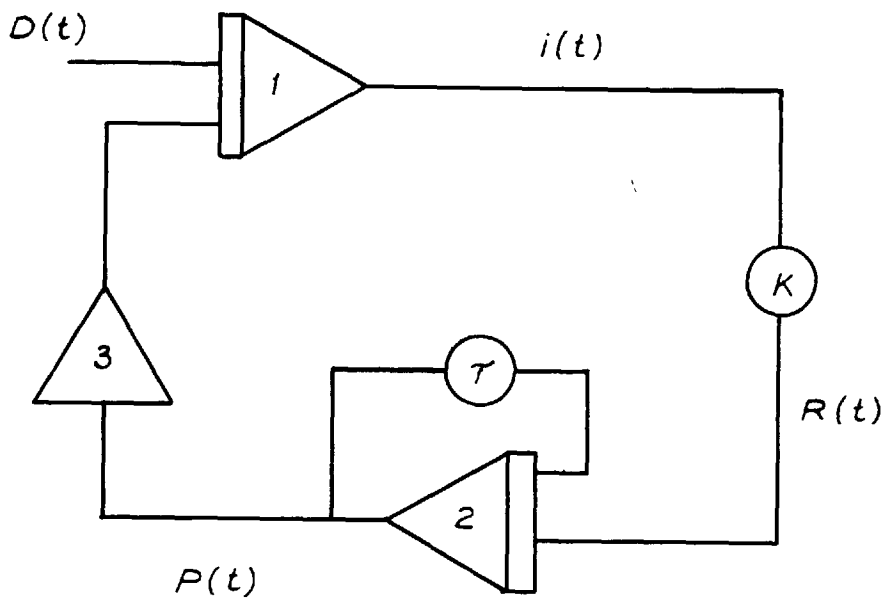


FIG. 1.6 ANALOGUE PROGRAMME FOR FLOW GRAPH
FIG. 1.5

APPENDIX II

ANALOGUE METHOD FOR GENERATING A NORMAL
DENSITY FUNCTION AND INSTRUCTIONS FOR
USING PROCESS CONTROL NOMOGRAPHS

APPENDIX II ANALOGUE METHOD FOR GENERATING
A NORMAL DENSITY FUNCTION AND INSTRUCTIONS
FOR USING PROCESS CONTROL NOMOGRAPHS

1. Analogue Method for Generating a Normal Density
Function

The expression for the normal distribution

$$\varphi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \quad (\text{II.1})$$

can be generated on the analogue computer by differentiating and solving the resultant non-linear differential equation. Thus,

$$\begin{aligned} \varphi'(x) &= \frac{(-x)}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \\ &= (-x) \varphi(x) \end{aligned}$$

A circuit for generating $\varphi(x)$ is shown in Figure 2.2(a) where the function is generated for values of x between the limits $\pm 3\sigma$.

To obtain reasonable accuracy an amplitude scale factor of 200 was adopted over the range $-3\sigma \leq x \leq -2\sigma$. This factor was unsuitable, however, over the range $-2\sigma \leq x \leq 2\sigma$ and a comparator switch was used to change

the scale factor from 200 to 20 for this range. The comparator was actuated at $x = -3\sigma$ to initiate the function generation, and at $x = -2\sigma$ to switch the scale factor. This was accomplished by substituting a second integrator with appropriately scaled initial condition to operate over the range $x \geq -2\sigma$. It was found unnecessary to return to the large scale factor in the range $2\sigma \leq x \leq 3\sigma$ as errors in the function over this range had negligible effect on the final result, (accurate to $\pm 4\%$ in the linear case).

Ignoring the function outside the limits $\pm 3\sigma$ does not appreciably alter the results since:

$$\int_{-\infty}^{-3\sigma} \psi(x) dx < 0.0014$$

In the non-linear case $x_0 = f(t)$ where $f(t)$ is not linear. Differentiating the expression in (II.1) with respect to t gives

$$\psi'(x) = -\psi(x) \cdot x \cdot \left(\frac{dx}{dt}\right)$$

The circuit that was used to compute $\Phi(x_0)$ is shown in Figure 2.13. Again, a comparator switch was used to regulate the amplitude scale as in the linear case and the function $\Phi(x_0)$ is obtained on the computer as:

$$1 - \Phi(x_0) = \int_0^{\tau_1} \psi(x_0) d\tau_1 \cdot \frac{dx_0}{d\tau_1} + \Phi(-b) \quad (\text{II.2})$$

where the computer independant variable τ_1 is set equal to αt in the case where $x_0 = f(t) = -b + A(1 - e^{-\alpha t})$.

2. Procedure for Employing the Nomograph of Figure II.5

A vertical line is drawn through the appropriate value of b cutting the curve at some point A. From A the distance αt is marked off in a negative horizontal direction forming a segment AB. From B a line is drawn tangent to the curve at C. The slope of this line gives the maximum rate of production times $\frac{1}{r}$. The horizontal distance between point A and the co-ordinate of C gives the optimal value of αt .

3. Procedure for Employing the Nomograph of Figure II.7

A horizontal line is drawn through the appropriate value of b , cutting the relevant cost curve at some point A. From A, $\frac{ac_r}{c_1}$ is marked off horizontally in the negative sense, forming a segment AB. From B a line is drawn tangent to the cost curve at C, and the slope of this line times $\frac{1}{r}$ gives the maximum ratio of output/cost. The vertical distance between the point of tangency and A gives the optimal value of αt .

4. Procedure for Employing the Nomograph of Figure II.12

A horizontal line is drawn through the appropriate value of $\frac{\delta}{2}$ intersecting the relevant cost curve at A, and intersecting the 'undersize curve' that lies to the left of the $\frac{\delta}{2}$ axis at B. The portion of the 'undersize curve' that lies below B, (i) has its ordinates multiplied by $1 - \frac{c_3}{c_1}$, (ii) is rotated through 180° about point B, (iii) is added horizontally to the appropriate cost curve. The new cost curve is treated in the manner described in section 3 of this appendix to determine the optimal value of a .

APPENDIX III

RELATIONSHIP BETWEEN

α , k AND R

APPENDIX III RELATIONSHIP BETWEEN
 α , k AND R FOR A GIVEN REQUIRED STOCKOUT
PROBABILITY

Stockout probability (P_o) may be defined in a number of ways, the simplest being:

$$P_o = \frac{n}{N} \quad (\text{III.1})$$

where n = number of periods in which a stockout occurs.

N = total number of periods

P_o = stockout probability

If the system is on the steady state, P_o is given by:

$$P_o = \int_{-\infty}^0 P(x) dx \quad (\text{III.2})$$

where $P(x)$ is the probability that the closing stock occupies level x , and for Gaussian demand, $P(x)$ is characterised by:

$$P(x) = \frac{1}{\sqrt{2\pi S_{ic}}} e^{-\frac{(x-\bar{i}_c)^2}{2S_{ic}}} \quad (\text{III.3})$$

where S_{ic} = closing stock variance

$\bar{i}_c$ = average closing stock

Thus, where P_o is given, $\bar{i}_c$ can be determined from (III.2) and (III.3), and this in turn determines

R according to:

$$R = \bar{i}_c + \overline{D(t)} \quad (\text{III.4})$$

where $\overline{D(t)}$ = average demand

However, if the system is not in the steady state (for example if a step occurs), the simple definition in (III.1) is seen to be inadequate since n , the number of stockouts in N periods will clearly be related to the number and magnitude of steps, if any, that occur.

A simple answer to this problem might be to define a stockout probability, P_m as the maximum probability of a stockout given that a step has occurred. P_m is then given by:

$$P_m = \int_{-\infty}^0 P(x) dx$$

where

$$P(x) = \frac{1}{\sqrt{2\pi S_{ic}}} e^{-\frac{(x-\bar{i}_c-g)^2}{2S_{ic}}} \quad (\text{III.5})$$

and g is the maximum deflection if $i_c(t)$ caused by a unit step. Where the step magnitude is different from unity, this can be accounted for in (III.5) by multiplying g by this magnitude. Thus, for a given maximum stockout probability $\bar{i}_c$ can be determined from (III.2) and (III.5) and this determined R according to (III.4).

This definition however, fails to take account of the rise time $(T_r)^*$ as a factor in determining probability of a stockout. The rise time may affect stockout probability in two ways:

- (i) by reducing the stock level so that the longer time over which the stock level is reduced the greater is the overall stockout probability.
- (ii) a long rise time increases the probability that a second step may occur before the system has fully recovered from the first one.

Where $S_{ic} \ll g$, the effect of (i) is negligible, since the probability of a stockout will be effectively zero for all $i_c(t)$ deflections except the maximum. However, (ii) will be important, since the expected number of stockouts in N periods will be $T_r P_s$ where P_s is the probability that a step occurs in any given period.

On the other hand when $S_{ic} \gg g$ a step has a negligible effect on the stockout probability and the

* T_r = time required for the closing stock to rise to within 0.1D of its initial level, in response to a step of magnitude D.

effect of T_r can likewise be ignored. However, the precise nature of the relationship between stockout probability and rise time where S_{ic} and g are of similar orders of magnitude, is not immediately apparent, and though the problem is outside the scope of this thesis its solution might have strong implications for the design of an optimal production-inventory control system.

APPENDIX IV

DISCRETE ANALOGUE
COMPUTATION AND
DATA STORAGE

APPENDIX IV : DISCRETE ANALOGUE COMPUTATION
AND DATA STORAGE

Standard analogue computing equipment is intended for continuous variable operation. Although modern 'hybrid' computers are capable of discrete as well as continuous operation, this equipment was not available for the study, and it was necessary to modify a standard analogue computer for discrete variable operation. The following describes how this was accomplished, and how a tape recorder was introduced to the system to provide analogue data storage.

Discrete variable operation is accomplished on the analogue using sample-hold elements. These sample an input voltage and hold it at the output for a subsequent time period usually much larger than the sampling interval. The result of sample-holding a sine wave is shown in Figure IV.1. (Note that the sample-hold operation is accompanied by a sign change on the computer.) By operating pairs of sample-hold elements at time shifted intervals (e.g. one element holds while the other samples and so on) various elementary operations can be performed. Figure IV.2 shows the circuit configurations for a one period delay and for summation.

The triangular spaces shown in the upper hand corner of each element indicate the time at which the element samples, thus elements with shaded triangles sample at one instant, and elements with unshaded triangles sample one half period later.

Sample-hold elements are synthesised from existing computer components, and details of the circuit are shown in Figure IV.3 where switch S closes to 'sample' and opens to 'hold'.

The circuit transfer function in 'sampling' mode is

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{R_1}{R_2}}{1 + \frac{1}{R_1 C} S}$$

where,

$E_o(s)$ = Laplace transform of output voltage

$E_i(s)$ = Laplace transform of input voltage

R_2 = Amplifier input resistance

R_1 = Amplifier feedback resistance

C = Feedback capacitance

$\frac{1}{R_1 C}$ = Circuit time constant (i.e. time required for output to reach 0.64v in the case of a unit step input)

Thus, if the input to the circuit is a step voltage (as will be the case if it is fed from a previous sample-and-hold) the output is given by,

$$y(t) = x_0 \frac{R_1}{R_2} \left(1 - e^{-\frac{t}{R_1 C}} \right) \quad (\text{IV.1})$$

where x_0 = step magnitude

t = sampling interval

From (IV.1) we see that for t sufficiently large $y(t)$ tends to x_0 . The time taken for $y(t)$ to approximate x_0 sufficiently is seen to be inversely related to the circuit time constant $\frac{1}{R_1 C}$.

By reducing this time constant the circuit speed can be increased. A limit was encountered however since for low values of R_1 and C the circuit does not function in the hold condition (e.g. the output voltage becomes noisy and C discharges noticeably over the 'hold' cycle). (The best RC combination was found to be $R_1=R_2=10k\Omega$, $C=0.02\mu F$.)

Reed relays were used for switching because of their high speed and favourable bounce characteristics. A number of these were obtained in a single unit comprising 3 coils and 36 n.o. contacts (Elliott Type ERS-3). The unit was mounted in an enclosed cabinet, and the relay contacts joined to external

patching terminals so that connection A-B in Figure IV.3 is made by wire leads from the computer patch board to terminals on the relay box. The relay box is shown in Figure IV.4 where the relay contact patching terminals are indicated.

Relays are actuated from a relay control circuit^{*} housed in the relay box. This circuit generates a sequence of time shifted pulses on two channels (one of each relay coil used) with frequency variable from 2 to 17 milliseconds and pulse width adjustable from 1 to 100 c.p.s. (see Figure IV.5). Control settings for the circuit are shown in Figure IV.4 and constructional details of the relay housing along with a diagram of the relay control circuit are given in Figures IV.6 and IV.7.

In practice this method of 'discretising' computer operation was found to be simple and reliable.

Data storage is not generally required in analogue computation since machine elements operate in parallel and can act on information the instant it is available in the machine. However, in this investigation two

* This circuit was designed by Dr. D.M. Elson, Mullard Research Laboratories, Redhill, Surrey.

factors distated the need for a storage facility.

- (1) To compare the response of different systems to a stochastic demand, it was required that the input data should be repeatable.
- (2) To analyse probability distributions where the process is non-stationary.

A tape recorder was used for data storage and since normal tape recording equipment (Amplitude Modulated) is unsuitable for accurate d.c. level recording^x it was necessary to use a frequency modulation system (Epsilon Type MR 800 Labcorder). This instrument is equipped with four recording channels; two F.M. for data logging, one A.M. for voice recording, and a second A.M. channel to record timing pulses generated internally and automatically recorded by the machine.

The machine must be modified for pulse operation, and details of this circuit alteration along with operating instructions are given in the Labcorder instruction manual.

^x A.M. equipment is unsuitable for d.c. level recording because residual magnetism cannot be erased from the tape.

When the recorder is used to feed a discrete waveform to the computer, it is important that the recorded waveform should be in time synchronism with the sample-hold circuits; this is to insure that sampling occurs at the sample-hold frequency and over a flat portion of the recorded waveform. To effect proper synchronisation a train of clock pulses are recorded on the timing channel along with each data waveform. (It was necessary to modify the internal circuitry for this purpose; details of this modification are shown in Figure IV.8.) These clock pulses are triggered by a waveform from the relay box and are naturally in time synchronisation with the recorded waveform.

When the machine is in replay, the recorded pulses are used to control the relay box, thus ensuring that the sample hold circuits are in synchronism with the recorded waveform. The scheme is shown in block diagram form Figure IV.9 where circled characters correspond to circuit points in Figures IV.7 and IV.8.

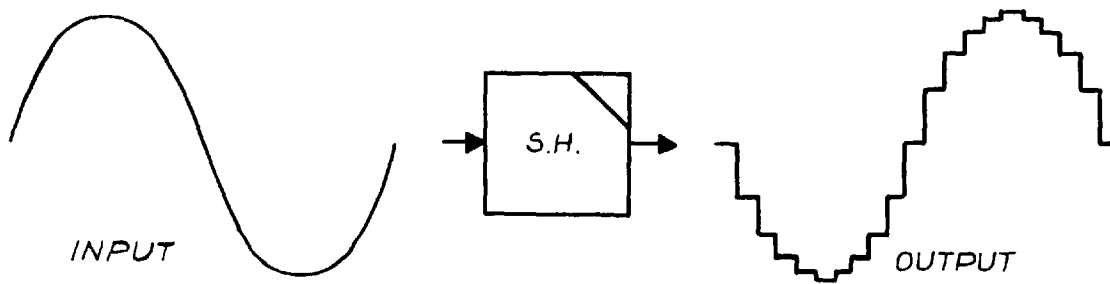


FIG. IV. 1 SINE WAVE FED TO A SAMPLE HOLD CIRCUIT

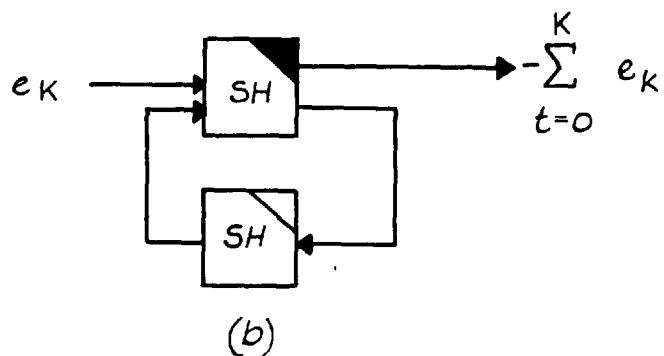
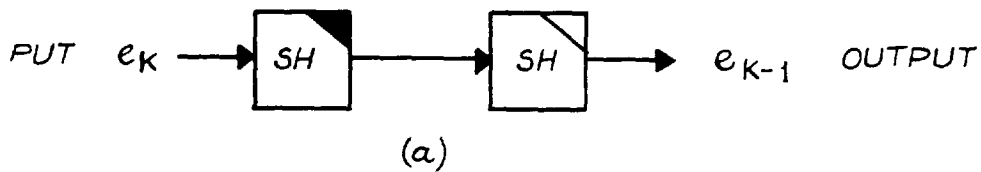


FIG. IV. 2 CIRCUIT CONNECTIONS FOR (a) ONE PERIOD TIME DELAY (b) SUMMATION

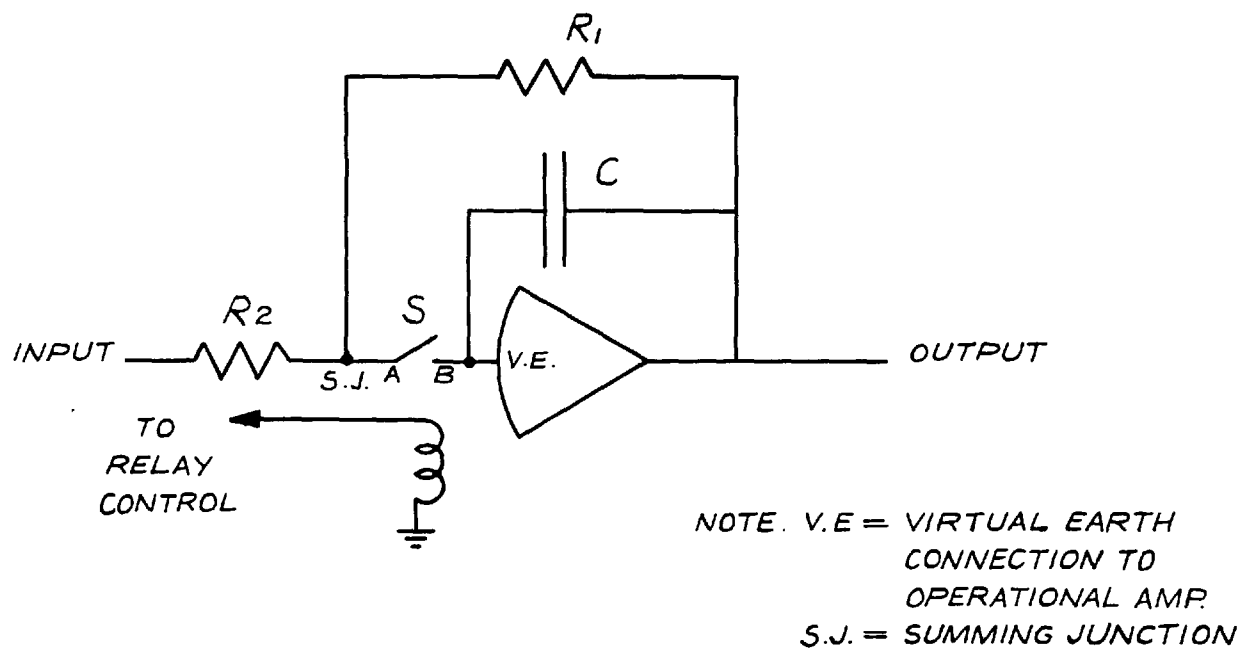


FIG. IV. 3 SAMPLE HOLD CIRCUIT

PATCHING TERMINALS
FOR RELAY CONTACTS

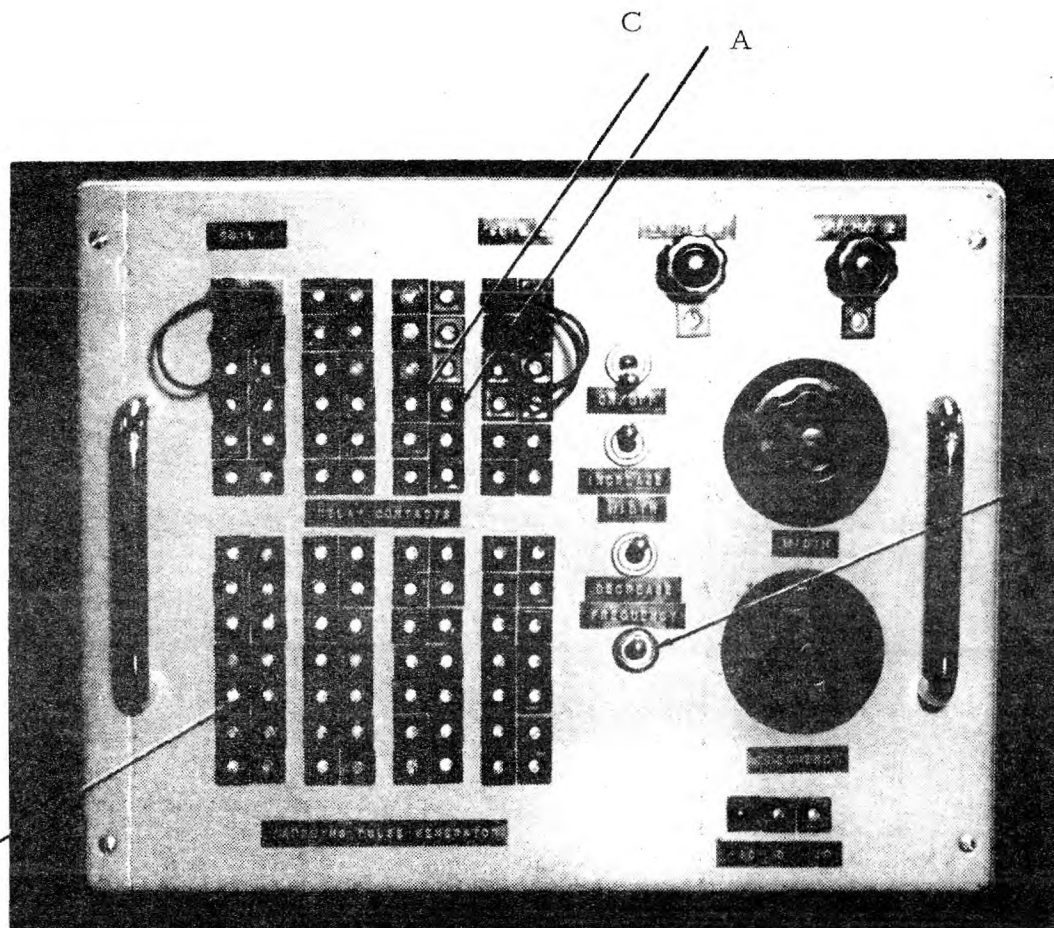


FIG. IV.4 CLOSE UP OF RELAY CONTROL UNIT SHOWING
PATCHING TERMINALS AND CONTROLS

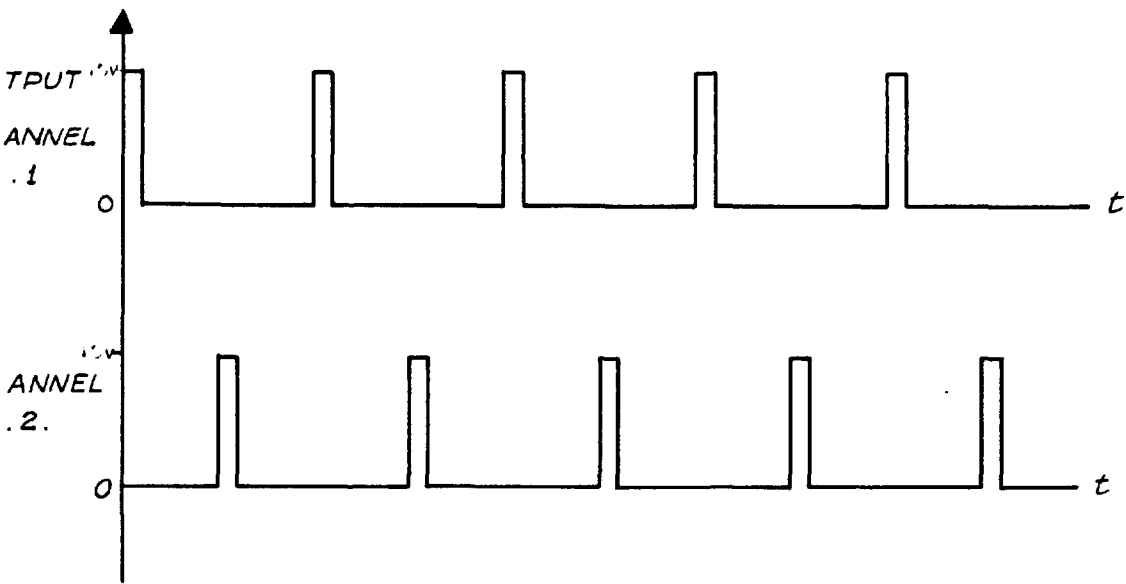


FIG. IV.5 OUTPUT FROM RELAY CONTROL CIRCUIT

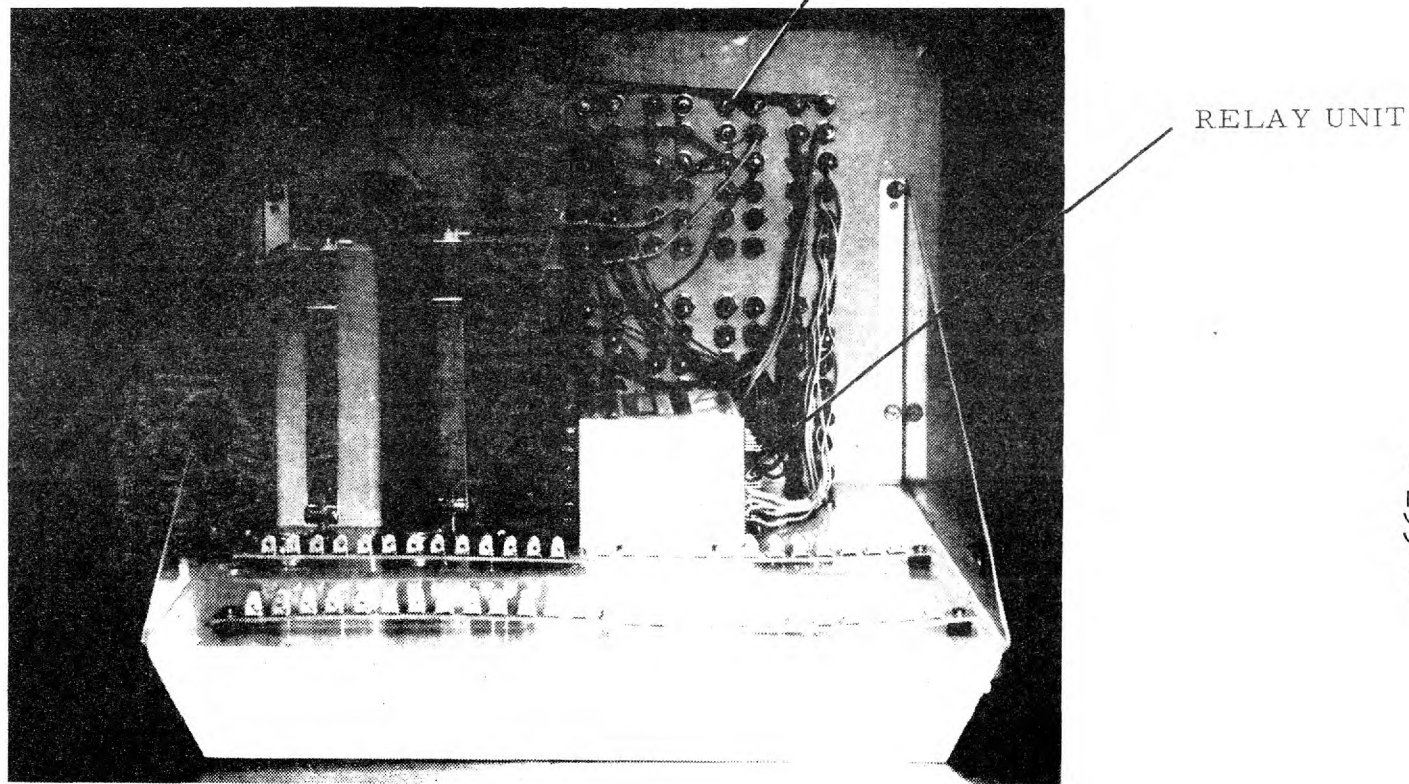
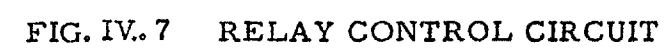


FIG. IV.6 RELAY HOUSING



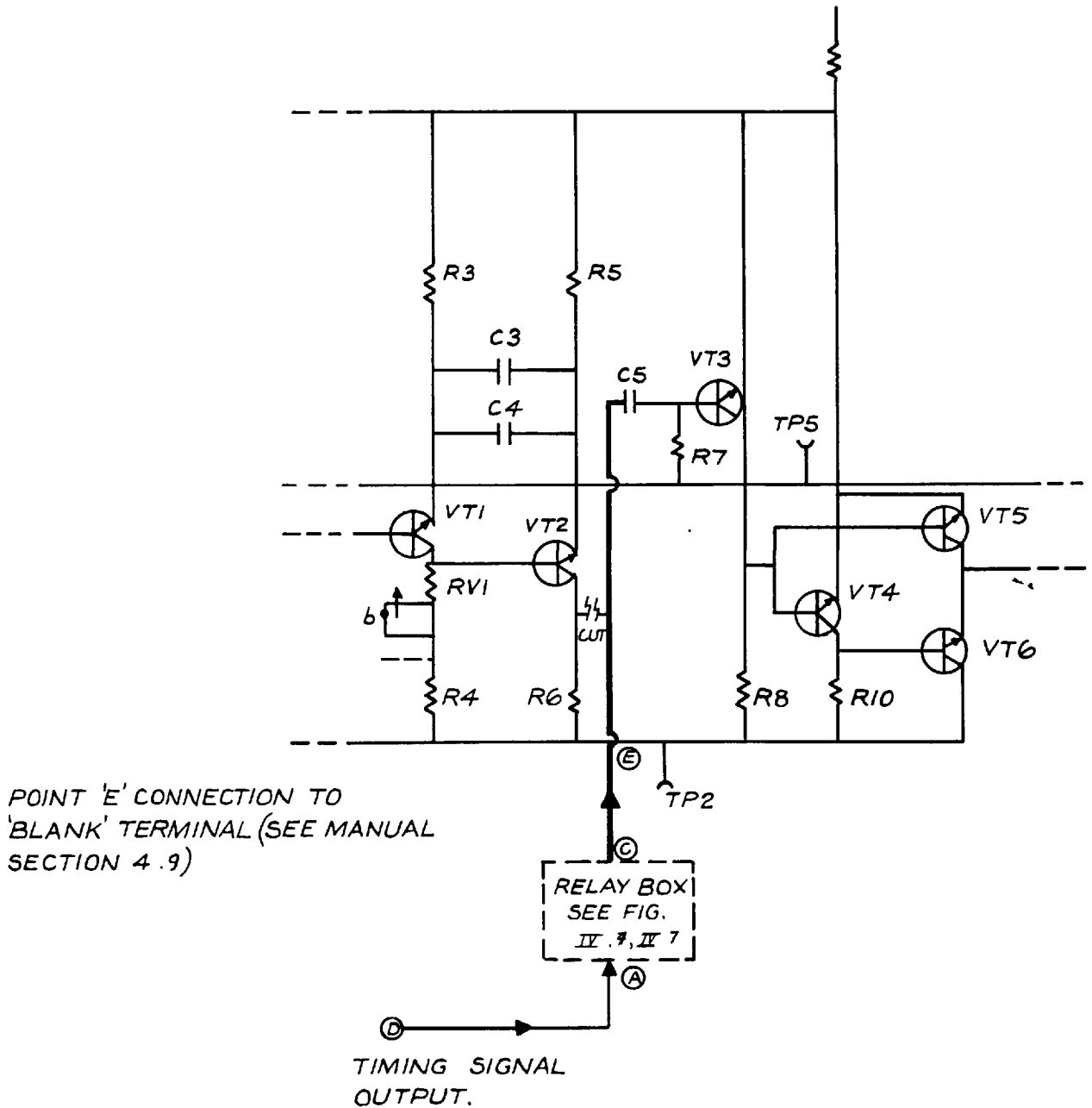
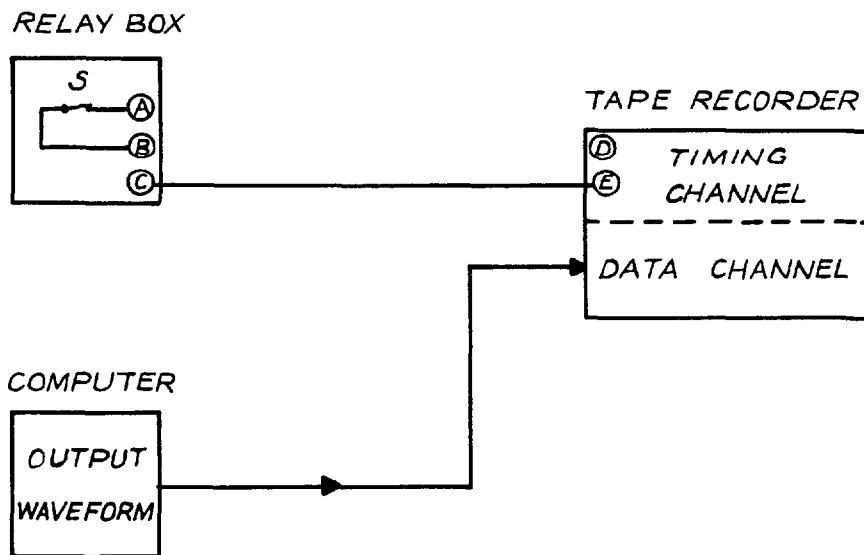
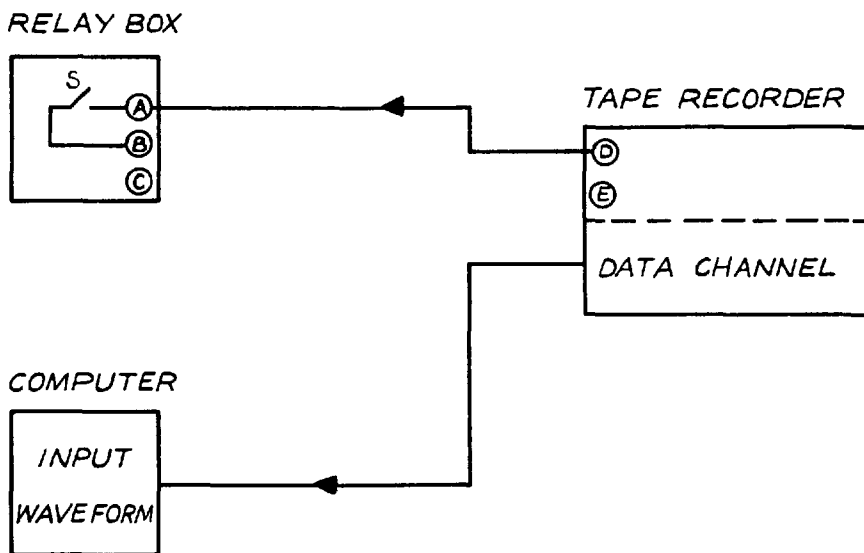


FIG. IV.8 HEAVY LINE SHOWS MODIFICATION TO RECORDER
TIMING CHANNEL CIRCUIT FIG. 11/2
INSTRUCTION MANUAL TYPE MR800 LABRECORDER



(a)



(b)

FIG. IV.9 SCHEME USED TO SYNCHRONISE COMPUTER CIRCUITS WITH RECORDED WAVEFORM
(a) RECORD (b) REPLAY

APPENDIX V

GENERATING DISCRETE NOISE WAVEFORMS
WITH ARBITRARY DISTRIBUTION FUNCTION
AND AUTO CORRELATION

APPENDIX V : GENERATING DISCRETE NOISE
WAVEFORMS WITH ARBITRARY DISTRIBUTION FUNCTION
AND AUTO CORRELATION

Stochastic demand patterns can be obtained by sampling the output of a white noise generator. The noise generator output is derived from random components of current emitted by an incandescent cathode and it can be shown from theory that this process is normal [62]. Sampling the noise waveform is statistically equivalent to low pass filtering and it can be shown [56, 63] that the sampled waveform is also normal. Where some other form of amplitude distribution is required. This can be obtained by passing the noise through a function generator before sampling, and a procedure for determining the characteristics of the function generator given the a priori and posterior distribution functions is outlined below.

Figure V.1 shows the two distribution functions, where $F_1(x)$ is the amplitude distribution function of the input noise and $P_2(x)$ denotes the required distribution function. The co-ordinates of the function generator characteristic are obtained graphically by construction.

Referring to Figure V.1, from any point A_i on $P_1(x)$ a horizontal line is drawn intersecting $P_2(x)$ at B_i . Vertical lines drawn from A_i and B_i meet the horizontal axis at u_i and v_i respectively. It is required to set up the function generator so that for an input less than or equal to u_i the output is less than or equal to v_i (since the probabilities of both events are required to be equal). This is assured if the function generator yields output v_i for input u_i . Thus, plotting v against u for various A_i gives the required function generator characteristic in Figure A.III.1(b).

The auto correlation function of white noise is related to the noise band width according to,

$$R(\tau) = \eta \frac{w_0}{4} e^{-w_0 |\tau|} \quad (V.1)$$

where $R(\tau)$ = auto correlation function

η = noise power (watts/c.p.s.)

w_0 = noise band width (radians/sec.)

since for most noise generators w_0 is very large, we have

$$R(\tau) \simeq \delta(t) \quad (V.2)$$

The auto correlation function of the sampled waveform is given by (V.1) where τ is the time between

samples, and w_o is the circuit band width (radians/sec.). Thus, the auto correlation between samples can be altered either by varying the sampling rate or the sample-hold circuit cutoff frequency. In practice it is simpler to adjust the former and where negligible auto correlation is required this is assured by making $\tau \leq 0.0023$ sec.* i.e. maximum sampling rate ≤ 4340 c.p.s.

The mean of the sampled waveform equals the mean of the input noise, and the sample variance is related to the noise variance according to;

$$\frac{S_S}{S_N} = \frac{R(O)_S}{R(O)_N} = \frac{\eta \frac{w_o(s)}{4}}{\eta \frac{w_o(N)}{4}} = \frac{w_o(s)}{w_o(N)}$$

where N $\Rightarrow$ noise generator output

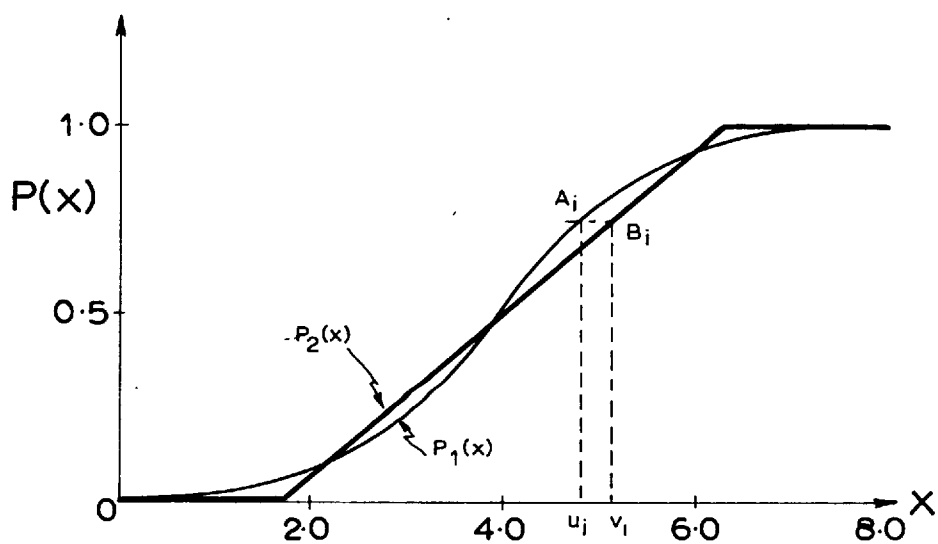
s $\Rightarrow$ sampled waveform

This calculation was carried out for the sample-hold circuit described in Appendix IV and the Bruel and Kjaer type 1402 noise generator for which $w_o = 3.18 \times 10^3$, thus,

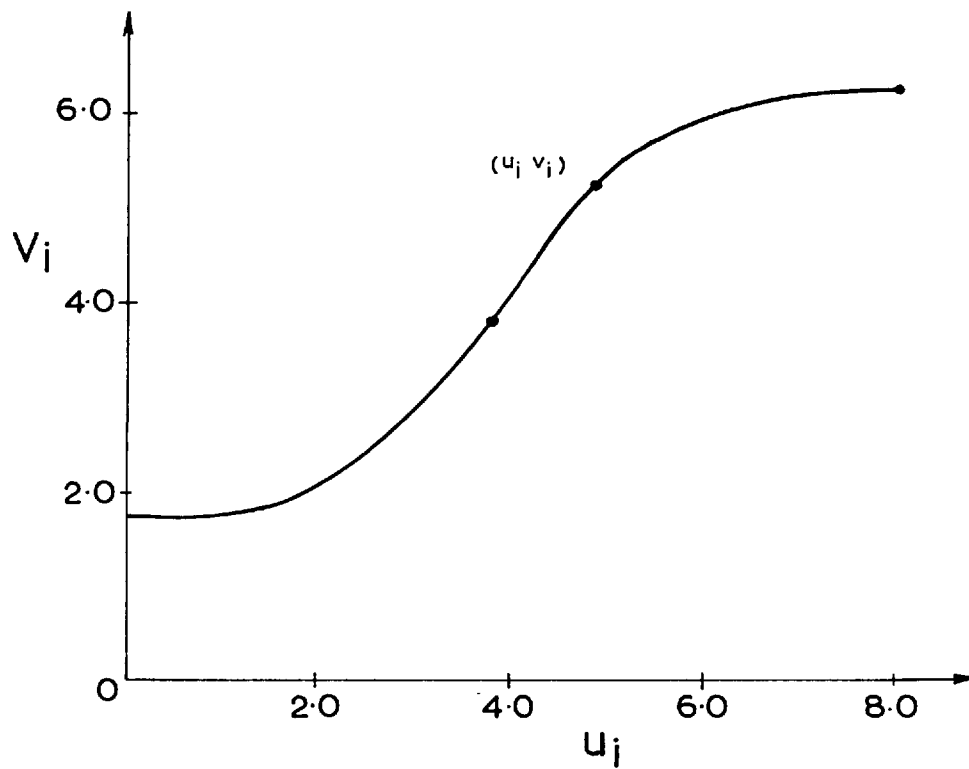
$$\frac{S_S}{S_N} = .039$$

* For the case where $R_1=R_2=10K\Omega$, $C=0.02\mu F$ (Appendix IM)

which agrees with measurements made on the sampled waveform (see Appendix VI).



(a)



(b)

Figure V.1 (a) PRIORI AND POSTERIORI DISTRIBUTION FUNCTIONS
(b) REQUIRED FUNCTION GENERATOR CHARACTERISTIC

APPENDIX VI

ANALYSIS OF STATISTICAL
DATA BY ANALOGUE COMPUTER

APPENDIX VI : ANALYSIS OF STATISTICAL
DATA BY ANALOGUE COMPUTER

When the analogue computer is used for statistical simulation it may be required to determine the statistical properties of output waveforms. Properties that may need to be determined are:

- (1) mean
- (2) variance
- (3) auto/cross correlation functions
- (4) amplitude probability distribution

Circuits for determining the exponentially mapped past mean, variance and auto/cross correlation of statistical variables are given in Figure VI.1 and properties of these estimates are described in [64, 65].

The amplitude probability distribution of a voltage waveform can be obtained using special purpose auxiliary devices, however these tend to be expensive to buy and rather elaborate to build. Alternatively, the analogue can be programmed to perform the analysis and a method for doing this is described here.

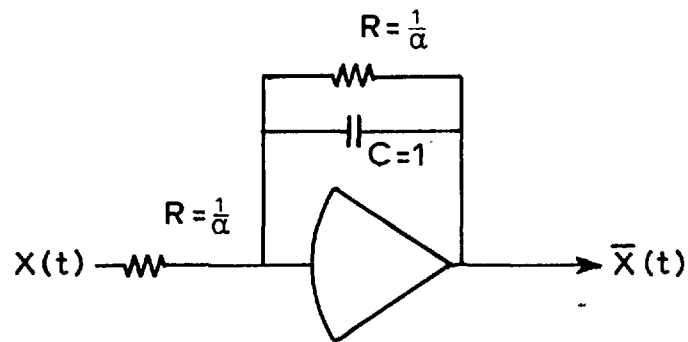
The waveform to be analysed may be either continuous or discrete; where it is continuous, the operating speed of the circuits employed may affect

the accuracy of the device and a number of authors have dealt with this [66, 67, 68]. Where the waveform is discrete, the principles involved are similar although the operating speed of the circuits is critical. Figure VI.2 shows a schematic diagram of the analyser and Figure VI.3 shows the corresponding analogue programme. The analyser measures the proportion of time the waveform spends between two voltage levels and by varying these levels over a series of repeated runs the amplitude probability distribution is obtained. Amplifiers 3 and 5 in Figure VI.2 are included to increase the switching speed of the discriminator, and matched diodes are used to ensure that the switching is symmetrical. Potentiometer 2 determines the lower limit and potentiometers 1 and 4 determine the interval width. The frequency of the square wave generator determines the run length though the number of samples per run will depend on the clock pulse rate which is varied by adjusting the frequency control on the relay box. If the x axis terminal on the pen recorder is connected to point R in the circuit the amplitude distribution can be plotted automatically.

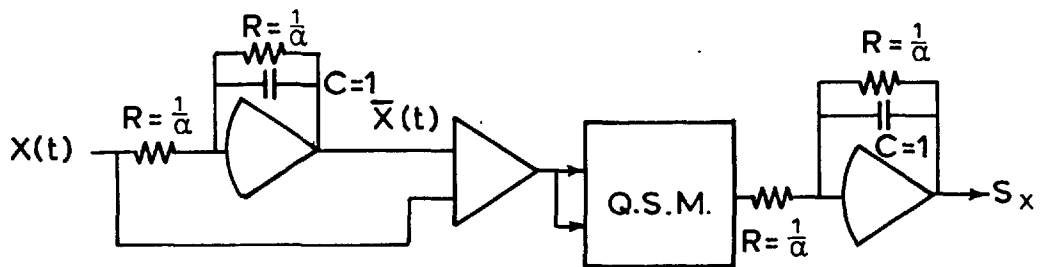
Probability estimates obtained from the analyser exhibit statistical variation caused by finite sample size. Where the input waveform is not serially

correlated, the estimate is binomially distributed with mean p and variance $\frac{p(1-p)}{n}$, for sample size n and desired probability p . Figure VI.4 shows the relationship between the sample size and confidence limits of the estimate for various values of p .

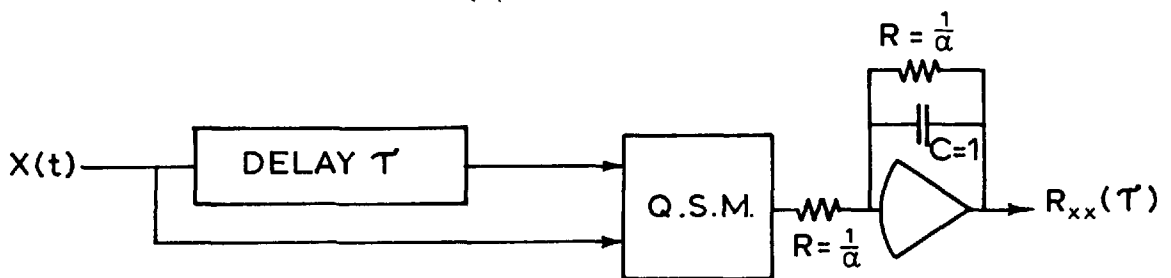
Figure VI.5 shows a result obtained using the analyser to determine the amplitude distribution of the noise generator output with an interval width of 0.8 volts and taking 1500 samples per interval.



(a)



(b)



(c)

Figure VI.1 CIRCUITS FOR DETERMINING THE (a) MEAN
(b) VARIANCE (c) AUTO CORRELATION

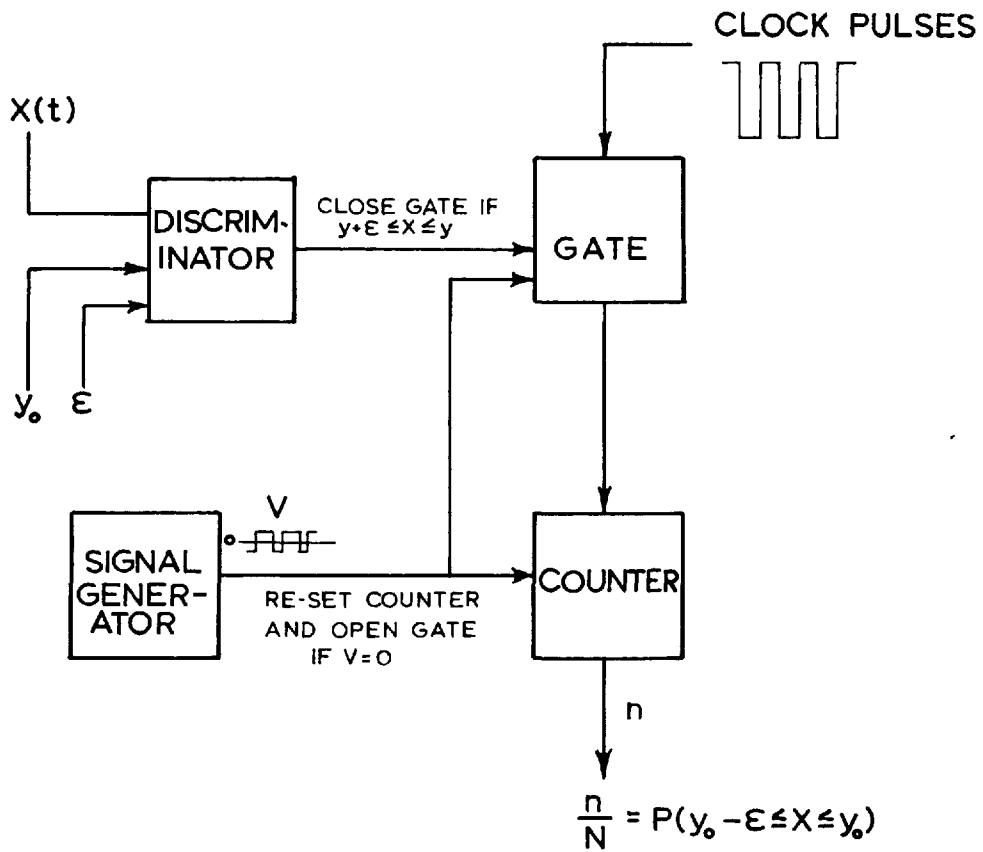


Figure VI.2 DISTRIBUTION ANALYSER SCHEMATIC

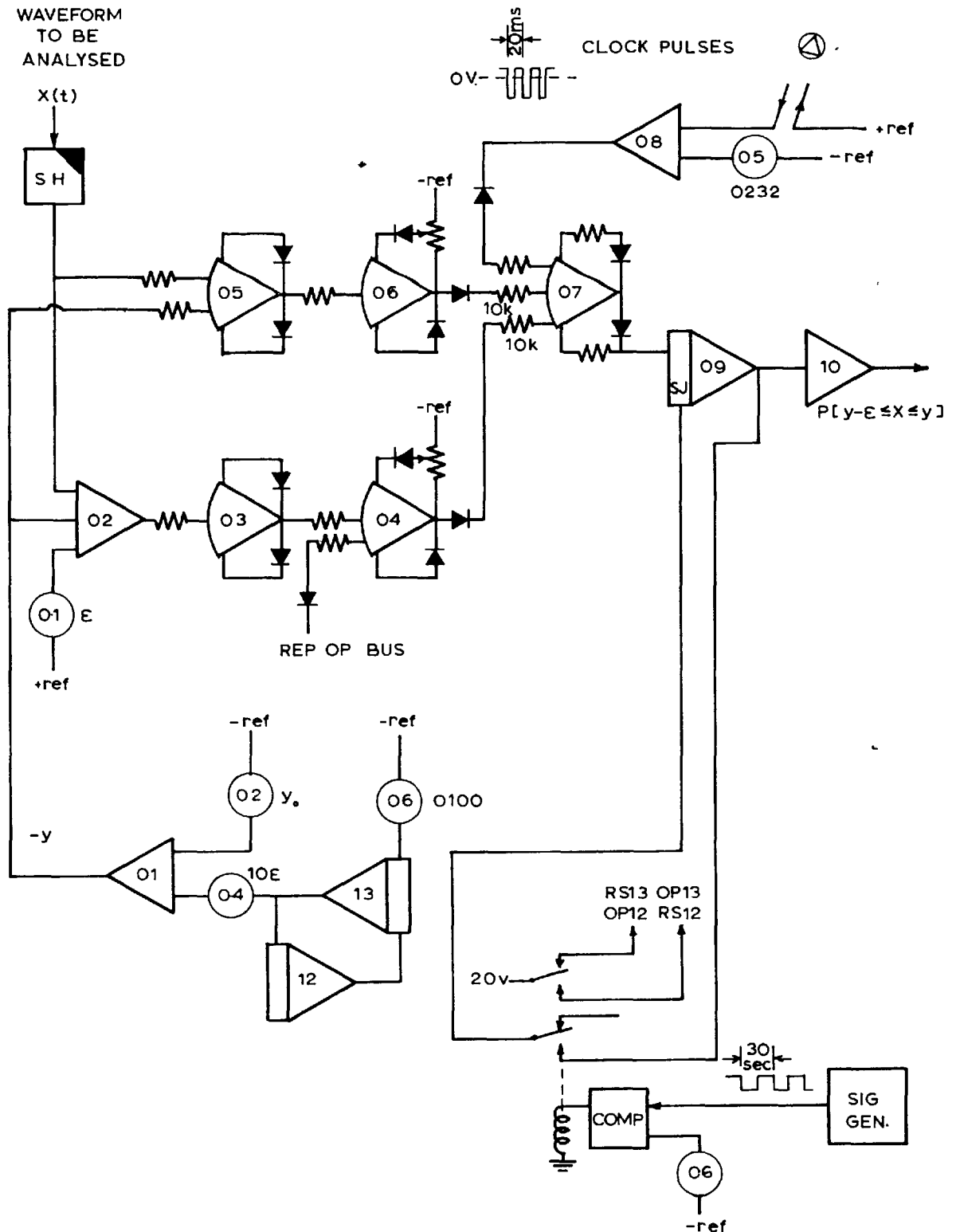


Figure VI.3 AUTOMATIC DISTRIBUTION ANALYSER

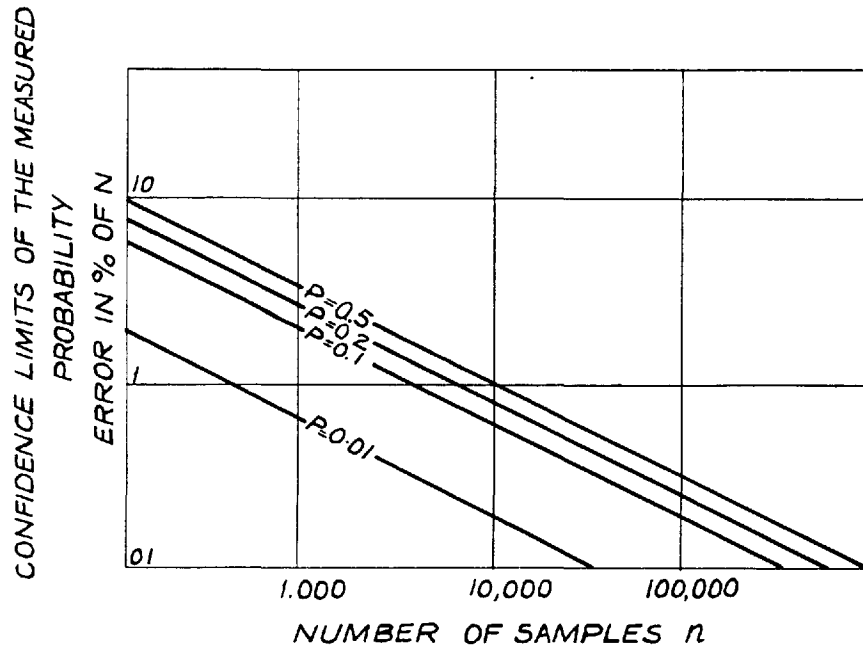


FIG. VI.4 CONFIDENCE LIMITS VS SAMPLE SIZE FOR SOME VALUES OF THE ESTIMATE P

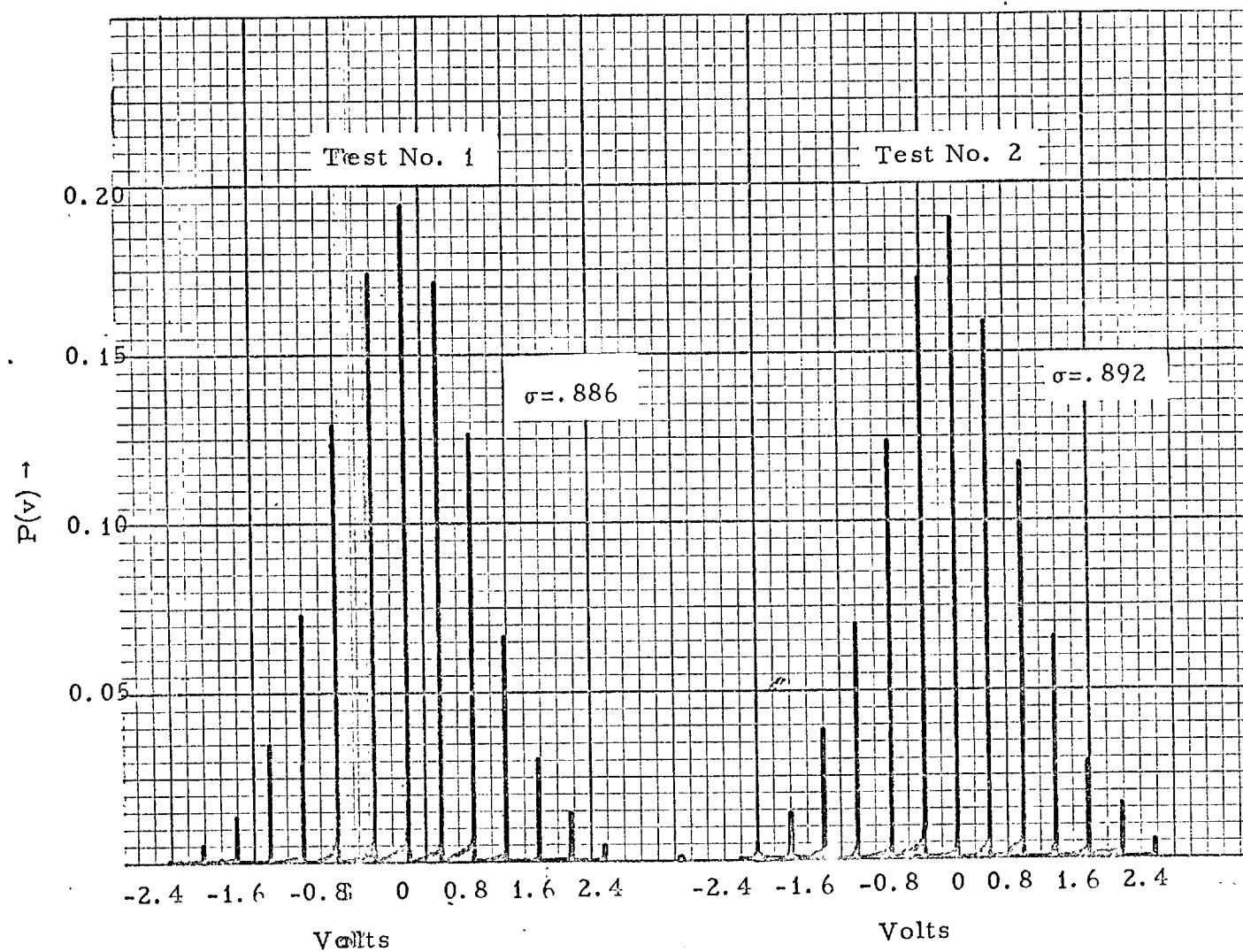


Figure VI.5 TESTS ON NOISE GENERATOR

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