

PROPERTY (τ) IN POSITIVE CHARACTERISTIC

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ABSTRACT

We prove a quantitative equidistribution statement for certain adelic homogeneous subsets in positive characteristic. As an application, we describe a proof of property (τ) for arithmetic groups in this context.

1. Introduction

Let $X = L/\Lambda$ be the quotient of a locally compact group L by a lattice $\Lambda \subset L$. Any subgroup $M \subset L$ acts on X by left multiplication. A **homogeneous measure** on X is, by definition, a probability measure μ that is supported on a single closed orbit $Y = M_Y g \Lambda$ of its stabilizer $M_Y = \text{Stab}(\mu)$. A **homogeneous set** is the support of a homogeneous measure.

Motivated by problems in number theory, the limiting behavior of a sequence $\{\mu_i\}$ of homogeneous measures has been extensively studied. When L is a Lie group and $\text{Stab}(\mu_i)$ is generated by unipotent subgroups, Mozes and Shah [24] used Ratner's celebrated measure classification theorems [30] and the Linearization techniques of Dani and Margulis [6] to obtain a very satisfactory classification theorem for the limiting measure; see also [11, 15, 8].

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More recently, the quantitative aspects of this problem have attracted considerable attention. Indeed, in a landmark paper, Einsiedler, Margulis, and Venkatesh [10] proved an effective equidistribution theorem for homogeneous measures when M and L are semisimple Lie groups and Λ is a congruence lattice, under some additional conditions; and in a followup work Einsiedler, Margulis, Venkatesh and the first named author [9] extended this result to certain adelic periods in the number field setting.

In this paper, we consider homogeneous measures in positive characteristic setting. Let us fix some notation in order to state our main results. Let F be a global function field, and let $\mathbf{G}$ be an absolutely almost simple, simply connected F -group. Throughout the paper, we assume $\mathbf{G}$ does not admit non-standard isogenies. This condition is always satisfied if $\text{char}(F) > 3$. More precisely, $\mathbf{G}$ admits non-standard isogenies only in the following cases: if $\text{char}(F) = 2$ and $\mathbf{G}$ is of type $B_n, C_n, (n \geq 2)$ or of type F_4 , another case is if $\text{char}(F) = 3$ and $\mathbf{G}$ is of type G_2 ; see [26, §1] and references therein.

Put $\mathbf{G}' = \mathbf{G} \times \mathbf{G}$, and let

$$\mathbf{H} = \{(h, h) : h \in \mathbf{G}\} \subset \mathbf{G}'$$

be the diagonal embedding of $\mathbf{G}$ in $\mathbf{G}'$.

Throughout the paper, Σ denotes the set of all places of F . For each place $v \in \Sigma$, let F_v be the completion of F at v , and let $\mathbb{A}_F$ (or simply $\mathbb{A}$ if there is no confusion) denote the ring of adèles over F .

Let $G' = \mathbf{G}'(\mathbb{A})$ and $X = G'/\mathbf{G}'(F)$. We let m_X denote the G' -invariant probability measure on X . Let $i: \mathbf{H} \rightarrow \mathbf{G}'$ be the above inclusion. For every $g \in G'$,

$$Y_g := gi(\mathbf{H}(\mathbb{A})/\mathbf{H}(F))$$

is a closed orbit in X equipped with a probability measure μ_g which is invariant under

$$H_g := g\mathbf{H}(\mathbb{A})g^{-1}.$$

Throughout the paper, we drop i from the notation and simply write

$$Y_g = g\mathbf{H}(\mathbb{A})/\mathbf{H}(F).$$

We will use the following notion of complexity which was introduced and studied in [9]. Let $\Omega_0 \subset G'$ be a compact open subgroup. Given (Y_g, μ_g) as above, define

$$(1) \quad \text{vol}(Y_g) = \mathfrak{m}_g(\Omega_0 \cap H_g)^{-1}$$

where m_g denotes a Haar measure on H_g which projects to μ_g . Note that a different choice of Ω_0 only changes the above function by a multiplicative constant; see [9, §2.3] for a proof.

The following is the main theorem of this paper.

THEOREM 1.1: *There exists $\kappa > 0$ so that the following holds. Let $g \in G'$, and let (Y_g, μ_g) be as above. Then*

$$\left| \int_{Y_g} f \, d\mu_g - \int f \, dm_X \right| \ll \text{vol}(Y_g)^{-\kappa} \mathcal{S}(f), \quad f \in C_c^\infty(X),$$

where $\mathcal{S}(f)$ denotes a certain adelic Sobolev norm, and the implied multiplicative constant depends on X . The exponent κ depends only on the type of $\mathbf{G}$ and F . If X is compact, then κ depends only on the type of $\mathbf{G}$.

Similar to the proofs in [10, 9], our proof relies on uniform spectral gap. However, our treatment deviates from loc. cit. in that instead of using effective ergodic theorems for the action of unipotent subgroups, we use averages over expanding pieces of root subgroups, see §7.2. This idea (which is due to Margulis, see [23]) makes the analysis in the case at hand more transparent. We also use the work of Prasad [29]; albeit our application here is less intricate than in [9], in particular, our treatment does not rely on [1], as $\mathbf{H} \subset \mathbf{G}'$ is simply connected and embedded.

As it was mentioned our proof relies on uniform spectral gap. However, it also allows us to give an independent proof of property (τ) in all cases except for groups of type A_1 . That is, if we assume property (τ) for groups of type A_1 , Drinfeld [7], and use the uniform estimates on decay of matrix coefficients for groups with Kazhdan's property (T) , Oh [25], we can deduce property (τ) in all other cases as well as our theorem.

In the number field setting, the proof of property (τ) has a long history and was completed thanks to several deep contributions; see [17, 25, 31, 16, 3, 4, 14]. In the positive characteristic setting, the existing literature is far less satisfactory beyond groups with property (T) , the aforementioned work of Drinfeld [7], and the work of Lafforgue [21, 22]. However, the statement itself is agreed upon by the experts. We hope the following theorem will contribute to filling this lacuna in the literature.

THEOREM 1.2 (Property τ): *Let the notation and assumptions be as above, in particular, we assume that $\mathbf{G}$ does not admit non-standard isogenies.*

Let $v \in \Sigma$ be such that $\mathbf{G}$ is isotropic over F_v . Then $L_0^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F))$, the orthogonal complement of $\mathbf{G}(\mathbb{A})$ -invariant functions, is isolated from the trivial representation as a representation of $\mathbf{G}(F_v)$. Moreover, this isolation (spectral gap) is independent of the F -form of $\mathbf{G}$.

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2. Notation and preliminaries

Let F be a global function field. Let Σ be the set of places on F , and let $\mathbb{A}$ be the ring of adèles over F . For each place $v \in \Sigma$, let F_v be the completion of F at v . Let $\mathfrak{o}_v$ be the ring of v -integers in F_v ; k_v denotes the residue field of $\mathfrak{o}_v$ and ϖ_v is a uniformizer of $\mathfrak{o}_v$. We will denote by $|x|_v$ the absolute value on F_v . Note that $\mathfrak{o}_v$ is the maximal compact subring of F_v . Let

$$B_v(a, r) = \{b \in F_v : |a - b|_v \leq r\}.$$

As in the introduction, let $\mathbf{G}$ be an absolutely almost simple, simply connected F -group which does not admit non-standard isogenies. This condition is always satisfied if $\text{char}(F) > 3$. More precisely, $\mathbf{G}$ admits non-standard isogenies only if $\text{char}(F) = 3$ and $\mathbf{G}$ is of type G_2 or $\text{char}(F) = 2$ and $\mathbf{G}$ is of type $B_n, C_n, (n \geq 2)$ or of type F_4 ; see [26, §1] and references therein.

Put $\mathbf{G}' = \mathbf{G} \times \mathbf{G}$. Let

$$\mathbf{H} = \{(h, h) : h \in \mathbf{G}\} \subset \mathbf{G}'$$

be the diagonal embedding of $\mathbf{G}$ in $\mathbf{G}'$. We also write $\mathbf{G}^1 = \mathbf{G} \times \{1\}$ and $\mathbf{G}^2 = \{1\} \times \mathbf{G}$. For every group $\mathbf{L}$, we put $L = \mathbf{L}(\mathbb{A})$. We will often simplify the notation and denote G^1 simply by G , but G^2 will always be denoted as G^2 .

Abusing the notation slightly, for every $g \in G'$ and a subset $B \subset G$, we let

$$\Delta_g(B) = g\{(\mathbf{g}, \mathbf{g}) : (\mathbf{g}, 1) \in B\}g^{-1}.$$

We adapt a similar notation for subsets of $\mathbf{G}(F_v)$ for all $v \in \Sigma$.

For $i = 1, 2$, let π^i denote the projection from $\mathbf{G}'$ onto the i -th component. Denote by $\iota^1 : \mathbf{G} \rightarrow \mathbf{G}'$ the map $g \mapsto (g, 1)$ and ι^2 is the map $g \mapsto (1, g)$. For every $v \in \Sigma$, we let $\pi_v : G' \rightarrow \mathbf{G}'(F_v)$ denote the natural projection, and $\iota_v : \mathbf{G}'(F_v) \rightarrow G'$ denotes the natural embedding in the v -th place. Put

$$\pi_v^i := \pi^i \circ \pi_v \quad \text{and} \quad \iota_v^i = \iota_v \circ \iota^i.$$

For $L = G^i, G', H$ and every $v \in \Sigma$, let $L_v = \iota_v(\mathbf{L}(F_v))$. Given $g = (g_v) \in G'$ and $w \in \Sigma$, we will let

$$g^w = \iota_w \circ \pi_w(g).$$

That is, $g_w^w = g_w$ and $g_v^w = 1$ for all $v \neq w$.

Let $\rho : \mathbf{G} \rightarrow \mathrm{SL}_N$ be an embedding defined over F . For any $v \in \Sigma$, let

$$K_v = \iota_v^1(\rho^{-1}(\mathrm{SL}_N(\mathfrak{o}_v))) \subset G_v$$

and let $K \subset G$ denote the product of K_v for all v . For $m \geq 1$, let

$$(2) \quad K_v[m] := \ker(K_v \rightarrow \mathrm{SL}_N(\mathfrak{o}_v/\varpi_v^m \mathfrak{o}_v));$$

note that $K_v = K_v[0]$. Let

$$K'_v := \{(g^1, g^2) : (g^i, 1) \in K_v\},$$

and define K' and $K'_v[m]$ accordingly. We choose Ω_0 in (1) to be K' . For any $L \subset K_v$, $v \in \Sigma$, and $m \geq 0$ we defined $L[m] := L \cap K[m]$.

For any subspace $\mathfrak{h} \subset \mathfrak{g}_w$ we write $\mathfrak{h}[0]$ for the preimage of the $\mathfrak{o}_w$ -integral $N \times N$ matrices under the restriction of the differential $D\rho : \mathfrak{g} \rightarrow \mathfrak{sl}_N$ to $\mathfrak{h}$. More generally, we write $\mathfrak{h}[m]$ for the preimage of the matrices all of whose entries have valuation at least m .

Recall that $X = G'/\mathbf{G}'(F)$ and $Y_g = gH/\mathbf{H}(F)$ where $g \in G'$. The homogeneous set Y_g is equipped with the $H_g = gHg^{-1}$ -invariant probability measure μ_g .

LEMMA 2.1: We have $\mathrm{Stab}(\mu_g) = H_g \cdot \mathbf{C}(F)$, where $\mathbf{C}$ denotes the center of $\mathbf{G}'$.

Proof. By virtue of [9, Lemma 2.2], we have

$$\mathrm{Stab}(\mu_g) = g\mathbf{H}\mathbf{N}(F)g^{-1}$$

where $\mathbf{N}$ denotes the normalizer of $\mathbf{H}$ in $\mathbf{G}'$. Since $\mathbf{H} = \{(h, h) : h \in \mathbf{G}\}$ and $\mathbf{G}$ is absolutely, almost simple, we conclude that $\mathbf{N} = \mathbf{H} \cdot \mathbf{C}$ which implies the claim. ■

LEMMA 2.2: For every $g \in G'$, there exists some $\hat{g} \in G^2$ so that $Y_g = Y_{\hat{g}}$.

Proof. Let $g = (g^{(1)}, g^{(2)})$, then

$$\begin{aligned} Y_g &= \{(g^{(1)}, g^{(2)})(h, h) : h \in \mathbf{G}(\mathbb{A})\} \\ &= \{(g^{(1)}h, g^{(2)}(g^{(1)})^{-1}g^{(1)}h) : h \in \mathbf{G}(\mathbb{A})\} \\ &= \{(h, g^{(2)}(g^{(1)})^{-1}h) : h \in \mathbf{G}(\mathbb{A})\}. \end{aligned}$$

The claim thus holds with $\hat{g} = (1, (g^{(2)}(g^{(1)})^{-1})_v)$. $\blacksquare$

In view of this lemma, from this point until the end of the paper, we will always assume $g \in G^2$.

2.1. PROPERTIES OF A SPLIT LIE ALGEBRA. Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$ and for any $v \in \Sigma$, let $\mathfrak{g}_v := \text{Lie}(\mathbf{G}) \otimes F_v$. We fix w such that $\mathbf{G}$ is F_w -split. In particular, G_w is a Chevalley group (see [33, §3.4.2]). Let Φ be the set of roots for G_w . Let $p_w := \text{char}(k_w)$.

Let $\mathfrak{a}$ be a maximal split torus of $\mathfrak{g}_w$. Then, $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}_\lambda$, where for any $\lambda \in \mathfrak{a}^*$

$$(3) \quad \mathfrak{g}_\lambda := \{v \in \mathfrak{g} : \forall a \in \mathfrak{a}, \text{ad}(a)v = \lambda(a)v\},$$

and $\Phi \subset \mathfrak{a}^*$ is the set of roots for $\mathfrak{g}_w$, i.e., the set of non-zero characters for which (3) is non-empty (see [32, §2.6]).

For any $\lambda \in \Phi$, there exists a one-dimensional Lie algebra $\mathfrak{g}_\lambda \subset \mathfrak{g}$, and a one-dimensional unipotent subgroup $U_\lambda = \{u_\lambda(r)\} \subset G_w$ such that $\mathfrak{g}_\lambda$ is the Lie algebra of U_λ (see [2, §2.3]). Set $L_\lambda = \langle U_\lambda, U_{-\lambda} \rangle$. Then, for any t we can also denote by $a_\lambda(t)$ the unique diagonal element in L_λ such that

$$a_\lambda(t)u_\lambda(1)a_\lambda(t^{-1}) = u(t^2).$$

Fix two opposite Borel subgroups B and B^- in G_w . Denote by U and U^- the unipotent radicals of B and B^- , respectively, and by $A = B \cap B^-$ a maximal split torus of G_w . For every nonnegative integer m , let

$$U_m = U \cap K_w[m], \quad U_m^- = U^- \cap K_w[m], \quad \text{and} \quad A_m = A \cap K_w[m].$$

Then, according to [33, §3.1.1] for any $m \in \mathbb{N}$, we have

$$(4) \quad K_w[m] = U_m^- A_m U_m.$$

Fix an ordering $\lambda_1, \dots, \lambda_d$ on Φ^+ once and for all; we use the ordering $-\lambda_1, \dots, -\lambda_d$ on Φ^- . In view of [33, §3.1.1] again, the product map

from $\prod_{i=1}^d U_{\pm\lambda_i}[m]$ to $U^\pm[m]$ is a bijection. For every $\lambda \in \Phi$ and every $g \in U^\pm[1]$, we define g_λ to be the λ -coordinate of g in this identification. Using (4), we can define g_λ for any $\lambda \in \Phi$ and $g = u^-au^+ \in K_w[1]$ by taking $g_\lambda = u_\lambda^\pm$.

For every non-negative integer m , we also define

$$A_{-m} = \{a \in A : aU_{2m}a^{-1} \subset U_0 \text{ and } a^{-1}U_{2m}^-a \subset U_0^-\}.$$

LEMMA 2.3: *There exists some $m_0 \in \mathbb{N}$ (depending only on the dimension of $\mathbf{G}$) such that the following holds. For every $g \in K_w[2m_0 + 1]$, there exists $\lambda \in \Phi$, so that*

$$|\{g' \in K_w A_{-m_0} K_w : \|(g'gg'^{-1})_\lambda - I\| \geq q_w^{-m_0} \|g'gg'^{-1} - I\|\}| \geq q_w^{-3m_0 \dim \mathbf{G}}.$$

Proof. We first note that for all $m \in \mathbb{N}$, all $g' \in K_w A_{-m} K_w$ and all $g \in K_w[2m+1]$, we have

$$g'gg'^{-1} \in K_w[1].$$

Therefore $(g'gg'^{-1})_\lambda$ is defined.

Let us begin with the following claim: one can assume

$$(5) \quad \|g_\lambda - I\| \geq q_w^{-c} \|g - I\| \quad \text{for some } \lambda \in \Phi$$

where $c \in \mathbb{N}$ depends only on $\dim \mathbf{G}$.

To see (5), note that since $g \in K_w[1]$, we can write $g = g^-g^0g^+$ where $g^0 \in A_1$, $g \in U_1$ and $g^- \in U_1^-$. Suppose now that (5) fails with $c = 1$, then $\|g^0 - I\| > \|g^\pm - I\|$. Let $\lambda \in \Phi$ be a simple root so that $|\lambda(g^0)|$ is maximal. The map

$$r \mapsto (u_\lambda(r)gu_\lambda(-r))_\lambda$$

is a rational function $f_1(r)/f_2(r)$ so that coefficients of f_i are bounded by $\ll \|g - I\|$ and $|f_2(r) - 1| \leq 1/2$ for all $|r|_w \leq 1$. Therefore, there exists some $r \in F_w$ with $|r|_w = 1$ so that if we put

$$\hat{g} = u_\lambda(r)gu_\lambda(-r),$$

then

$$\|\hat{g}_\lambda - I\| \gg \|\hat{g} - I\|.$$

Now suppose the claim is proved for $\hat{g}$. For $\beta > 0$ and $\star \in G$ put

$$E_\beta(\star) = \{g' \in K_w A_{-m_0} K_w : \|(g'\star g'^{-1})_\lambda - I\| \geq \beta \|g'\star g'^{-1} - I\|\}.$$

Then,

$$E_\beta(\hat{g})u_\lambda(r) \subset E_\beta(g) \quad \text{and} \quad |E_\beta(\hat{g})u_\lambda(r)| = |E_\beta(\hat{g})| \geq \beta^{3 \dim \mathbf{G}}.$$

We thus assume that (5) is satisfied, and will show the lemma holds with λ and $m_0 = c + 1$. For simplicity in the notation, let us assume $\lambda \in \Phi^+$, the argument when $\lambda \in \Phi^-$ is similar. Put $\tilde{g} = a_\lambda(\varpi_w^{-c-1})ga_\lambda(\varpi_w^{c+1})$. Then $\tilde{g} \in K_w[1]$. Using (4) and the decomposition of U_1 into one-dimensional unipotent subgroups, we can write

$$g = g^- g^0 h_1 u_\lambda(s') h_2, \quad h_1, h_2 \in U_1.$$

We now want to look at the decomposition of $\tilde{g}$ into a diagonal element and elements of the one-dimensional unipotent groups. Conjugation by $a_\lambda(\varpi_w^{-c-1})$ expands $u_\lambda(s')$ by q_w^{2c+2} , and for each other root $\sigma \neq \lambda$, it expands g_σ by at most q_w^{c+1} . We conclude from (5) that $s = q_w^{2c+2}s'$ satisfies $u_\lambda(s) = \tilde{g}_\lambda$. Moreover, for k such that $|s|_w = q^{-k}$, we have

$$q_w^{-c-1} \leq |s|_w = q_w^{-k} \leq q_w^{2c+2} q_w^{-2m-1} = q_w^{-1},$$

and $\tilde{g}^0 \in K_w[k+1]$, $\tilde{g}_\sigma \in K_w[k+1]$ for all $\sigma \neq \lambda$.

Since $K_w[k+1]$ is a normal subgroup of K_w , we have

$$\tilde{g} = h \tilde{u}_\lambda(s)$$

where $h \in K_w[k+1]$. Moreover, for every $g_2 \in K_w[k+1]$, we have

$$g_2 \tilde{g} g_2^{-1} = h'' u_\lambda(s)$$

where $h'' \in K_w[k+1]$.

Altogether, there is some

$$g_1 \in K_w A_{-c-1} K_w = K_w A_{-m_0} K_w$$

so that for every $g_2 \in K_w[m_0+1]$ we have $g_2 g_1 g g_1^{-1} g_2^{-1} \in K_w[1]$ and

$$\|(g_2 g_1 g g_1^{-1} g_2^{-1})_\lambda - I\| \geq q_w^{-m_0} \|g - I\|.$$

The proof is complete. ■

3. Construction of a good place

Recall that $Y_g = g\mathbf{H}(\mathbb{A})/\mathbf{H}(F)$, where $g = (1, (g_v))$. In this section we show the existence of a place w where the group

$$H_{g,w} = \Delta_g(G_w) = gH_w g^{-1}$$

has **controlled geometry**.

Recall the definition of K_v and K'_v from §2. For every $v \in \Sigma$, let

$$(6) \quad K_{v,Y_g} := H_{g,v} \cap K'_v = gH_v g^{-1} \cap K'_v = \Delta_g(K_v) \cap K'_v.$$

In this section, we will simply write Y for Y_g , and denote K_{v,Y_g} by $K_{v,Y}$.

The following is the main result of this section.

PROPOSITION 3.1 (Existence of a good place): *There exists a place w of F such that*

- (1) $\mathbf{G}$ is split over F_w ,
- (2) $\pi_w(K'_w)$ and $\pi_w(K_{w,Y})$ are hyperspecial subgroups of $\mathbf{G}'(F_w)$ and $\pi_w(H_g)$, respectively,
- (3) $g^w \in K'_w$, see §2 for the notation,
- (4) $q_w \ll \log(\text{vol } Y)^2$ where the implied constant depends on $\mathbf{G}$.

Let us begin by recalling some standard facts from Bruhat–Tits theory.

3.1. BRUHAT–TITS THEORY. Let $\mathbf{G}, F$ be as in our setting (see §2), and let $v \in \Sigma$. Then:

- (1) For any point x in the Bruhat–Tits building of $\mathbf{G}(F_v)$, there exists a smooth affine group scheme $\mathfrak{G}_v^{(x)}$ over $\mathfrak{o}_v$, unique up to isomorphism, such that: its generic fiber is $\mathbf{G}(F_v)$, and the compact open subgroup $\mathfrak{G}_v^{(x)}(\mathfrak{o}_v)$ is the stabilizer of x in $\mathbf{G}(F_v)$; see [33, 3.4.1].
- (2) If $\mathbf{G}$ is split over F_v and x is a special point, then the group scheme $\mathfrak{G}_v^{(x)}$ is a Chevalley group scheme with generic fiber $\mathbf{G}$; see [33, 3.4.2].
- (3) $\text{red}_v : \mathfrak{G}_v^{(x)}(\mathfrak{o}_v) \rightarrow \underline{\mathfrak{G}}_v^{(x)}(k_v)$, the reduction mod ϖ_v map, is surjective, which follows from the smoothness above; see [33, 3.4.4].
- (4) $\underline{\mathfrak{G}}_v^{(x)}$ is connected and semisimple if and only if x is a hyperspecial point. Stabilizers of hyperspecial points in $\mathbf{G}(F_v)$ will be called hyperspecial subgroups; see [33, 3.8.1] and [29, 2.5].

If $\mathbf{G}$ is quasi-split over F_v , and splits over $\widehat{F}_v$ (the maximal unramified extension of F_v), then hyperspecial vertices exist; and they are compact open subgroups with maximal volume. Moreover a theorem of Steinberg implies that $\mathbf{G}$ is quasi-split over $\widehat{F}_v$ for all v ; see [33, 1.10.4].

It is known that for almost all v , the groups K_v are hyperspecial; see [33, 3.9.1] (and §2 for the definition of K_v). We also recall that: for almost all v the group $\mathbf{G}$ is quasi-split over F_v ; see [27, Thm. 6.7].

3.2. ADELIC VOLUMES AND TAMAGAWA NUMBER. Fix an algebraic volume form ω on $\mathbf{G}$ defined over F . The form ω determines a Haar measure on each vector space $\mathfrak{g}_v := \text{Lie}(\mathbf{G}) \otimes F_v$ which also gives rise to a normalization of the Haar measure on $\mathbf{G}(F_v)$. We denote both these measures by $|\omega_v|$, and denote by $|\omega_{\mathbb{A}}|$ the product measure on $\mathbf{G}(\mathbb{A}_F)$. Then

$$(7) \quad |\omega_{\mathbb{A}}|(\mathbf{G}(\mathbb{A}_F)/\mathbf{G}(F)) = D_F^{\frac{1}{2} \dim \mathbf{G}} \tau(\mathbf{G}),$$

where $\tau(\mathbf{G})$ is the **Tamagawa number** of $\mathbf{G}$, and D_F is the discriminant of F . In the case at hand $\mathbf{G}$ is simply connected; it was recently proved that $\tau(\mathbf{G}) = 1$, see [12, §1.3].

3.3. THE QUASI-SPLIT FORM. The volume formula (7) relates the Haar measure on Y to the algebraic volume form ω (and the field F). However, the volume of our homogeneous set Y as a subset of X depends heavily on the amount of distortion coming from g .

Following [29, Sect. 0.4], we let $\mathcal{G}$ denote a simply connected algebraic group defined and quasi-split over F which is an inner form of $\mathbf{G}$. Let L be the field associated to $\mathcal{G}$ as in [29, Sect. 0.2];¹ it has degree $[L : F] \leq 3$. We note that $\mathcal{G}$ should be thought of as the least distorted version of $\mathbf{G}$.

Let ω^0 be a differential form on $\mathcal{G}$ corresponding to ω . This can be described as follows: Let $\varphi : \mathbf{G} \rightarrow \mathcal{G}$ be an isomorphism defined over some Galois extension of F . We choose ω^0 so that $\omega' = \varphi^*(\omega^0)$; it is defined over F . It is shown in [29, Sect. 2.0–2.1] that, up to a root of unity of order at most 3, this is independent of the choice of φ .

As was done in [29], we now introduce local normalizing parameters c_v which scale the volume form ω^0 to a more canonical volume form on $\mathcal{G}(F_v)$.

For every place $v \in \Sigma$, choose an $\mathfrak{o}_v$ -structure on $\mathcal{G}$, i.e., a smooth affine group scheme over $\mathfrak{o}_v$ with generic fiber $\mathcal{G}$ as in [29, Sect. 1.2].

Let $\{\mathcal{P}_v \subset \mathcal{G}(F_v)\}$ denote a coherent collection of parahoric subgroups with “maximal volume”; see [29, Sect. 1.2] for an explicit description. Let us recall that by a coherent collection we mean that $\prod_{v \in \Sigma} \mathcal{P}_v$ is a compact open subgroup of $\mathcal{G}(\mathbb{A}_F)$. Note that any F -embedding of $\mathcal{G}$ into GL_N gives rise to a coherent family of compact open subgroups of $\mathcal{G}(\mathbb{A}_F)$ which at almost all places satisfies

¹ In most cases L is the splitting field of $\mathcal{G}$ except in the case where $\mathcal{G}$ is a triality form of ${}^6\text{D}_4$ where it is a degree 3 subfield of the degree 6 Galois splitting field with Galois group S_3 . Note that there are three such subfields which are all Galois-conjugate.

the above requirements on $\mathcal{P}_v$; see §3.1. At the other places we may choose $\mathcal{P}_v$ as above and then use (1) in §3.1 to define the $\mathfrak{o}_v$ -structure on $\mathcal{G}(F_v)$. Let us also remark that maximality of the volume implies that the corresponding parahoric is either hyperspecial (if $\mathcal{G}$ splits over an unramified extension) or special with maximum volume (otherwise).

This allows us, in particular, to speak of “reduction modulo ϖ_v ”. For every place v of F , we let $\overline{\mathcal{M}}_v$ denote the reductive quotient of $\text{red}_v(\mathcal{P}_v)$; this is a reductive group over the residue field.

Let $r_v \in F_v$ be so that $r_v \omega_v^0$ is a form of maximal degree, defined over $\mathfrak{o}_v$, whose reduction mod ϖ_v is non-zero, and let $c_v = |r_v|_v$.

3.4. PRODUCT FORMULA. Let us use the abbreviation $D_{L/F} = D_L D_F^{-[L:F]}$ for the norm of the relative discriminant of L/F ; see [29, Thm. A]. It is shown in [29, Thm. 1.6] that

$$(8) \quad \prod_{v \in \Sigma} c_v = D_{L/F'}^{\frac{1}{2}\mathfrak{s}(\mathcal{G})} \cdot A,$$

where $A > 0$ depends only on $\mathbf{G}$ over $\bar{F}$ and $[F : \mathbb{Q}]$, $\mathfrak{s}(\mathcal{G}) = 0$ when $\mathcal{G}$ splits over F in which case $L = F$, and $\mathfrak{s}(\mathcal{G}) \geq 5$ otherwise; these constants depend only on the root system of $\mathcal{G}$.

It should be noted that the parameters c_v were defined using $\mathcal{G}$ and ω^0 but will be used to renormalize ω_v on $\mathbf{G}$, and hence on $\mathbf{H}$.

Abusing the notation, we will speak of hyperspecial subgroups of G_w , G'_w , and $H_{g,w}$ in the remaining parts of this section.

Let $\Sigma' \subset \Sigma$ denote the set of places where $\mathbf{G}$ is isomorphic to $\mathcal{G}$ over F_v and that K_v is hyperspecial. Then $\Sigma \setminus \Sigma'$ is finite.

LEMMA 3.2: *Let $v \in \Sigma'$. Then:*

- (1) $c_v |\omega_v| (K_{v,Y}) \leq 1$.
- (2) *Assume $g^v \notin K'_v$. Then*

$$c_v |\omega_v| (K_{v,Y}) \leq 1/2.$$

Proof. In view of the definition of $K_{v,Y}$, see (6), we have

$$(9) \quad K_{v,Y} = \Delta_g(K_v) \cap K'_v.$$

Since $v \in \Sigma'$, $\mathbf{G}$ is isomorphic to $\mathcal{G}$ over F_v and we have $c_v |\omega_v| (K_v) = 1$ which implies the first claim.

To see the second claim, first note that if g^v does not normalize K'_v , then g_v does not normalize $\pi_v^2(K_v)$, where $\pi_v^2 : G_v \rightarrow \mathbf{G}(F_v)$ is the natural projection; see §2. In particular,

$$[\pi_v^2(K_v) : \pi_v^2(K_v) \cap g_v^{-1} \pi_v^2(K_v) g_v] \geq 2.$$

Now since $v \in \Sigma'$, we have $\pi_v^2(K_v)$ is a hyperspecial vertex in $\mathbf{G}(F_v)$, $\pi_v^2(K_v)$ equals its normalizer in $\mathbf{G}(F_v)$; this and the above imply part (2) in view of (9).

We recall the argument that $\pi_v^2(K_v)$ equals its own normalizer for the convenience of the reader; $\pi_v^2(K_v)$ is the stabilizer of a hyperspecial point, x_v say. Since g_v normalizes $\pi_v^2(K_v)$, $\pi_v^2(K_v)$ also stabilizes $g_v x_v$. If $g_v x_v \neq x_v$, then $\pi_v^2(K_v)$ stabilizes the geodesic between these two points which is a contradiction. ■

3.5. THE VOLUME OF A HOMOGENEOUS SET. In view of our definition of volume and taking into account the choice of Ω_0 , the equation (7) implies that

$$(10) \quad \text{vol}(Y) = D_F^{\frac{1}{2} \dim \mathbf{G}} \prod_{v \in \Sigma} (|\omega_v|(K_{v,Y}))^{-1},$$

where $|\omega_v|$ is as before. This and (8) imply that

$$(11) \quad \text{vol}(Y) = AD_{L/F}^{\frac{1}{2} s(\mathcal{G})} D_F^{\frac{1}{2} \dim \mathbf{G}} \prod_{v \in \Sigma} (c_v |\omega_v|(K_{v,Y}))^{-1}.$$

Let us note the rather trivial consequence $\text{vol}(Y) \gg 1$ of (11).² Below we will assume implicitly $\text{vol}(Y) \geq 2$ (which we may achieve by replacing Ω_v by a smaller neighborhood at one place in a way that depends only on $\mathbf{G}$).

Proof Proposition 3.1. Recall that $\Sigma' \subset \Sigma$ denotes the set of places where $\mathbf{G}$ is isomorphic to $\mathcal{G}$ over F_v and that K_v and hence $K'_v = \{(\mathbf{g}_1, \mathbf{g}_2) : (\mathbf{g}_i, 1) \in K_v\}$ is hyperspecial. Furthermore, if $g^v \in K'_v$, then

$$K_{v,Y} = \Delta_g(K_v) \cap K'_v$$

is hyperspecial in $H_{g,v}$.

Now recall from Lemma 3.2 that for all places $v \in \Sigma'$, we have $c_v |\omega_v|(K_{v,Y}) \leq 1$. Moreover, if $v \in \Sigma'$ but $g^v \notin K'_v$, then

$$(12) \quad c_v |\omega_v|(K_{v,Y}) \leq 1/2.$$

² This would also follow trivially from the definition if only we would know that the orbit intersects a fixed compact subset.

By the Chebotarev density theorem, the set

$$\Sigma'' = \{v \in \Sigma' : \mathbf{G} \text{ is split over } F_v\}$$

has positive density, $c > 0$ say, which depends only on $\mathbf{G}$.

Let B be a constant depending on $\mathbf{G}$ which will be explicated later. Assume now (for a contradiction) that for every $v \in \Sigma''$ with $q_v \leq Bc^{-1} \log(\text{vol}(Y))^2$, we have $c_v |\omega_v| (K_{v,Y})^{-1} \geq 2$.

Recall also that we are assuming $\text{vol}(Y) \geq 2$. The above discussion, (11), and the prime number theorem imply that

$$\text{vol}(Y) \geq B' 2^{B \log(\text{vol}(Y))^2 / (\log B - \log c + 2 \log \log(\text{vol}(Y)))}$$

where B' depends only on $\mathbf{G}$. Now if $B \gg_{\mathbf{G}} 1$, then

$$B' 2^{B \log(\text{vol}(Y))^2 / (\log B - \log c + 2 \log \log(\text{vol}(Y)))} > \text{vol}(Y).$$

This contradiction finishes the proof. $\blacksquare$

4. Deduction of Theorem 1.2 from Theorem 1.1

In this section, we complete the proof of Theorem 1.2. The proof relies on Theorem 4.1 below, which in turn is a consequence of Theorem 1.1. It should be noted, however, that the proof of Theorem 1.1 requires a certain uniform decay of matrix coefficients for $H_{g,w}$ where w is a good place; see §7.1. In the setting of Theorem 4.1, the group $H_{g,w}$ has Kazhdan's property (T) , thus the required uniform rate follows from the work of Oh [25].

As before, we let π^1 denote the projection onto the first factor. Throughout this section, the space $\mathbf{G}(\mathbb{A})/\mathbf{G}(F)$ is equipped with the $\mathbf{G}(\mathbb{A})$ -invariant probability measure, we thus drop this measure from the notation.

THEOREM 4.1: *Assume the absolute rank of $\mathbf{G}$ is at least 2. There exists $\kappa_1 > 0$ which satisfies the following. Let $f_1, f_2 \in L_0^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F))$ —the orthogonal complement of $\mathbf{G}(\mathbb{A})$ -invariant functions—be $\pi^1(K_v)$ -finite functions. Then*

$$|\langle g.f_1, f_2 \rangle| \leq \dim \langle \pi^1(K_v).f_1 \rangle^{1/2} \dim \langle \pi^1(K_v).f_2 \rangle^{1/2} \Xi_{\mathbf{G}(F_v)}(\mathbf{g})^{-\kappa_1} \|f_1\|_2 \|f_2\|_2,$$

where $\mathbf{g} \in \mathbf{G}(F_v)$, $\langle \pi^1(K_v).f \rangle$ is the linear span of $\pi^1(K_v).f$, and $\Xi_{\mathbf{G}(F_v)}$ is the Harish-Chandra spherical function of $\mathbf{G}(F_v)$.

Let us first assume Theorem 4.1 and finish the proof of Theorem 1.2.

Proof of Theorem 1.2. First note that the theorem in the case where $\mathbf{G}$ is of type A_1 follows from [7]. We may, therefore, assume that the absolute rank of $\mathbf{G}$ is at least 2.

By [5], see also [14, Lemma 3.22], we may restrict ourselves to smooth functions in $L_0^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F))$. Now by [5, Thm. 2 and Cor.], Theorem 1.2 is equivalent to the existence of a uniform decay rate for matrix coefficients for $\pi^1(K_v)$ -finite functions in $L_0^2(\mathbf{G}(\mathbb{A})/\mathbf{G}(F))$ as claimed in Theorem 4.1. ■

Proof of Theorem 4.1. Let $g = (1, (g_u)) \in G'$ where $g_v = \mathbf{g}$ and $g_u = 1$ for all $u \neq v$.

In view of our definition of the volume of a homogeneous set, there exist positive constants κ_2 and κ_3 (depending only on the root system of $\mathbf{G}(F_v)$) such that

$$\|g\|^{\kappa_2} \ll \text{vol}(Y_g) \ll \|g\|^{\kappa_3}.$$

Thus, we conclude from Theorem 1.1 that if

$$f = f_1 \otimes \bar{f}_2 \quad \text{with } f_i \in C_c^\infty(\mathbf{G}(\mathbb{A})/\mathbf{G}(F)) \text{ for } i = 1, 2,$$

then

$$\begin{aligned} & \left| \int_X f \, d\mu_{Y_g} - \int_X f_1 \otimes \bar{f}_2 \, dm_X \right| \\ &= \left| \langle g \cdot f_1, f_2 \rangle_{\mathbf{G}(\mathbb{A})/\mathbf{G}(F)} - \int_{\mathbf{G}(\mathbb{A})/\mathbf{G}(F)} f_1 \int_{\mathbf{G}(\mathbb{A})/\mathbf{G}(F)} \bar{f}_2 \right| \ll \|g\|^{-\kappa} \mathcal{S}(f). \end{aligned}$$

The implied multiplicative constant depends on $\mathbf{G}(\mathbb{A})/\mathbf{G}(F)$ and so also on the F -structure of $\mathbf{G}$. We note, however, that thanks to [5], the above upgrades to a uniform effective bound on the decay of the matrix coefficients as in Theorem 4.1 with κ_1 independent of the F -form of $\mathbf{G}$. This implies Theorem 4.1. ■

Remark: We now discuss the independence of the above proof from other proofs of property (τ) (which, as mentioned earlier, we failed to locate in the literature beyond groups with property (T) , and the works of Drinfeld [7] and Laforgue [21, 22]).

Let w be a place obtained by applying Proposition 3.1 to Y_g . In particular,

- (1) $\mathbf{G}$ is split over F_w ,
- (2) K'_w and $K_{w,Y}$ are hyperspecial subgroup of G_w and H_w , respectively,
- (3) $q_w \ll (\log(\text{vol } Y_g))^2$ where the implied constant depends on $\mathbf{G}$.

Then by [25, Thm. 1.1], and our assumption that the absolute rank of $\mathbf{G}$ is at least two, the group G_w has property (T) . Therefore, there exists a constant κ_4 (which depends only on the type of $\mathbf{G}$) so that for any unitary representation $(\pi, \mathcal{H})$ of G_w without a non-zero G_w -invariant vector, all K_w -finite vectors $f_1, f_2 \in \mathcal{H}$, and all $h_w \in G_w$, we have

$$(13) \quad |\langle \pi(h_w).f_1, f_2 \rangle| \ll \dim \langle \pi_1(K_w).f_1 \rangle^{1/2} \dim \langle \pi_1(K_w).f_2 \rangle^{1/2} \|f_1\| \|f_2\| \|h_w\|^{-\kappa_4}$$

where the implicit constant depends on G_w . If $\text{char}(F) > 2$ this is stated in [25, Thm. 1.1]. A careful examination of the proof shows that (13) holds when $\text{char}(F) = 2$ as well; see [25, §4.7].

As we discussed above, our proof of Theorem 4.1 relied on Theorem 1.1. Moreover, the latter relies on a spectral gap input for certain representations of G_w ; see §7.1, in particular properties (S5) and (S6). In the case at hand, this input is provided thanks to (13). Thus the above proof is independent of assuming property (τ) in all cases considered here, except for groups of type A_1 .

5. Effective generation of a group at a good place

In this section, we fix a good place w and show some useful properties of G_w ; see §3 for the definition and the proof of the existence of a good place. In particular, $\mathbf{G}$ is split over F_w and we can apply the notation and results from §2.1.

Recall that $\mathfrak{o}_w$ is the maximal compact subring of F_w , ϖ_w is a uniformizer of $\mathfrak{o}_w$, and $\pi^1(K_w) = \mathbf{G}(\mathfrak{o}_w)$. For any $L > 0$ and $R \subset G$, denote by $R[L]$ the set of elements in R whose entries vanish in $\mathfrak{o}_w/\varpi_w^L \mathfrak{o}_w$. For any $L > 0$ and $\mathfrak{r} \subset \mathfrak{g}$ denote by $\mathfrak{r}[L]$ the set of elements in $\mathfrak{r}$ whose entries have valuation at least L .

For every $\lambda \in \Phi$, let A_λ be the scheme theoretic split torus in

$$\langle U_\lambda, U_{-\lambda} \rangle \cong \text{SL}_2$$

which we realize as a closed $\mathbb{Z}$ -subscheme of $\mathbf{G}$. Put

$$(14) \quad A'_\lambda = \Delta_g(A_\lambda) \subset H_g;$$

the elements in A'_λ will be denoted by $a'_\lambda(t)$. Similarly, put $U'_{\pm\lambda} = \Delta_g(U_{\pm\lambda}) \subset H_g$ and their elements by $u'_{\pm\lambda}(s)$.

Note that for every $\mathfrak{g} \in G_w$, $t \in F_w^\times$, and $s \in F_w$, we have

$$(15) \quad a'_\lambda(t) \mathfrak{g} a'_\lambda(t)^{-1} = a_\lambda(t) \mathfrak{g} a_\lambda(t)^{-1} \quad \text{and} \quad u'_\lambda(s) \mathfrak{g} u'_\lambda(-s) = u_\lambda(s) \mathfrak{g} u_\lambda(-s).$$

For $b \in F_w$ and $r > 0$, let

$$B(b, r) := \{s \in F_w : |b - s|_w < r\}.$$

LEMMA 5.1: *There exists an absolute constant κ_5 such that the following holds.*

(1) *Assume that $\text{char}(F) > 2$. For any $\lambda \in \Phi$ and every $|s|_w = 1$, we have*

$$U_\lambda[\kappa_5] \subset \{u_\lambda(r^2 - s^2) : |r|_w = 1\}.$$

(2) *Assume that $\text{char}(F) = 2$. Then*

$$\{u_\lambda(c^2) : |c|_w \leq 1\} \subset \{u_\lambda(r^2 - s^2) : |r|_w = 1\}.$$

Proof. Let us first assume that $\text{char}(F) > 2$. Let $\psi_s : F_w^2 \rightarrow F_w$ be defined by

$$\psi_s(t, r) = t - (r^2 - s^2).$$

Then,

$$\frac{\partial \psi_s}{\partial r}(0, 1) = -2 \neq 0.$$

Hence, by the implicit function theorem (see [20, §4]), there exists $0 < \alpha < 1$, such that for all $t \in B(0, \alpha^2)$ there exists $r \in B(s, \alpha)$ so that $\psi_s(t, r) = 0$, i.e., $t = r^2 - s^2$. This implies the first claim with $\kappa_5 = \alpha^2$.

If $\text{char}(F) = 2$, then $r^2 - s^2 = (r - s)^2$ which implies the claim. ■

LEMMA 5.2: *Recall that $\mathbf{G}$ does not admit any non-standard isogenies. There exist κ_6 and $\ell \geq 1$ with the following property. Let $\lambda \in \Phi$. Then, there exist $h_1, \dots, h_\ell \in H_{g,w}$ with $\|h_i\| \leq 1$ so that every $\mathbf{g} \in K_w[\kappa_6]$ can be written as*

$$\mathbf{g} = \mathbf{g}_1 \mathbf{g}_2 \cdots \mathbf{g}_\ell,$$

where for any $1 \leq i \leq \ell$, $\mathbf{g}_i \in h_i U_\lambda[0] h_i^{-1}$.

Proof. Clearly it suffices to prove this for a fixed $\lambda \in \Phi$. First note that since $G = G^1$ (see §2) for every $h = (h^1, h^2) \in H_{g,w}$, we have

$$h U_\lambda h^{-1} = (h^1, 1) U_\lambda (h^1, 1)^{-1} \subset G_w \quad \text{and} \quad \text{Ad}(h) \mathbf{u}_\lambda = \text{Ad}((h^1, 1)) \mathbf{u}_\lambda \in \mathfrak{g}_w.$$

Since $\mathbf{G}$ is simply connected and does not admit any non-standard isogenies, there exist $h'_1, \dots, h'_{\dim \mathbf{G}} \in H_{g,w}$ so that the F_w -span of $\{\text{Ad}(h'_i) \mathbf{u}_\lambda\}$ equals $\mathfrak{g}_w$, see [26, Prop. 1.11] and references therein. Moreover, since K_w is Zariski dense in G_w , we can choose h'_i so that $\|h'_i\| \leq 1$ for all i .

In consequence, by [2, Prop. 1.16] applied with $\{h'_i U_\lambda h_i'^{-1} : 1 \leq i \leq \dim \mathbf{G}\}$ and G_w , we conclude that the product map

$$\varphi : \prod_{j=1}^{\ell} U_j \rightarrow G_w$$

is a separable surjective morphism, where ℓ is absolute and

$$U_j \in \{h'_i U_\lambda h_i'^{-1} : 1 \leq i \leq \dim \mathbf{G}\} \quad \text{for every } 1 \leq j \leq \ell.$$

Thus, by the implicit function theorem (see [20, §3]), there exists κ_6 so that

$$G_w[\kappa_6] \subset \varphi \left(\prod_{j=1}^{\ell} U_j[0] \right)$$

where $U_j[0] := h'_j U_\lambda[0] h_j'^{-1}$. The proof is complete. $\blacksquare$

Remark 5.3: Lemma 5.2 is the only place in the paper where the assumption that $\mathbf{G}$ does not admit non-standard isogenies is used.

We also need the following lemma, which deals with the case that the characteristic of F equals 2, see Lemma 5.1.

LEMMA 5.4: Assume $\text{char}(F) = 2$ and let $\tau_0 \in \mathfrak{o}_w$ be nonzero. Let $\lambda \in \Phi$. There exists an absolute constant ℓ' with the following property. For every $r \in B(0, 1)$,

$$u_\lambda(r) = h_1 \cdots h_{\ell'}$$

where for all $1 \leq i \leq \ell'$, either $h_i \in \{u_\lambda(\tau_0 \alpha^2) : |\tau_0 \alpha^2| \leq |\tau_0|^{-1}\}$ or $h_i \in H_{g,w}$ with $\|h_i\| \leq 1$.

Proof. We give a hands on proof based on direct computations. Let us begin with the following observation: Let $\varphi_\lambda : \text{SL}_2 \rightarrow \langle U_\lambda, U_{-\lambda} \rangle$ be the natural isomorphism which maps the upper (resp. lower) triangular unipotent subgroup of SL_2 to U_λ (resp. $U_{-\lambda}$). There exist $0 < \kappa_7 < 1$ and $h_1, h_2, h_3 \in H_{g,w}$ with $\|h_i\| \leq 1$ so that every element

$$h \in \varphi_\lambda \left(\begin{pmatrix} \tau_0 & 0 \\ 0 & 1 \end{pmatrix} \text{SL}_2(\mathfrak{o}_w^2[\kappa_7]) \begin{pmatrix} \tau_0^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) =: L'_{\lambda, \tau_0}[\kappa_7]$$

can be written as a product

$$(16) \quad h = h_1 \cdots h_6$$

where for every $1 \leq j \leq 6$, $h_j \in h_i \{u_\lambda(\tau_0 \alpha^2) : |\tau_0 \alpha^2| \leq 1\} h_i^{-1}$ for some $1 \leq i \leq 3$.

This is a special case of Lemma 5.2, applied with SL_2 over F_w^2 and the root subgroup $\{u_\lambda(r) : r \in F_w^2\}$.

We now turn to the proof of the lemma. Let $|a|, |c| \leq \kappa_7$ and put $r' = \frac{r}{1+a^4}$, then

$$\begin{pmatrix} r' & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^2 & 0 \\ \tau_0^{-1}c^2 & a^{-2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & r' \end{pmatrix} = \begin{pmatrix} a^{-2} & r'a^2 + r'a^{-2} + \tau_0^{-1}c^2 \\ 0 & a^2 \end{pmatrix}.$$

This implies that

$$\begin{aligned} \Delta_g \left(\varphi_\lambda \left(\begin{pmatrix} r' & 1 \\ 1 & 0 \end{pmatrix} \right) \right) \varphi_\lambda \left(\begin{pmatrix} a^2 & 0 \\ \tau_0^{-1}c^2 & a^{-2} \end{pmatrix} \right) \Delta_g \left(\varphi_\lambda \left(\begin{pmatrix} 0 & 1 \\ 1 & r' \end{pmatrix} \right) \right) \\ = \varphi_\lambda \left(\begin{pmatrix} a^{-2} & r'a^2 + r'a^{-2} + \tau_0^{-1}c^2 \\ 0 & a^2 \end{pmatrix} \right). \end{aligned}$$

Now, we have

$$\begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix} \begin{pmatrix} a^{-2} & r'a^2 + r'a^{-2} + \tau_0^{-1}c^2 \\ 0 & a^2 \end{pmatrix} \begin{pmatrix} 1 & \tau_0(\tau_0^{-1}ac)^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & r'(1+a^4) \\ 0 & 1 \end{pmatrix};$$

applying φ_λ , we conclude that

$$\varphi_\lambda \left(\begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix} \right) \varphi_\lambda \left(\begin{pmatrix} a^{-2} & r'a^2 + r'a^{-2} + \tau_0^{-1}c^2 \\ 0 & a^2 \end{pmatrix} \right) u_\lambda(\tau_0(\tau_0^{-1}ac)^2) = u_\lambda(r),$$

where we also used $r = (1+a^4)r'$.

The claim follows from these computations and (16). $\blacksquare$

LEMMA 5.5: *There exist C_1 so that the following holds. Let $m_1 \in \mathbb{N}$, $m \geq 3m_1$, and $\lambda \in \Phi$. Let $g \in K_w[m]$ satisfy $\|g_\lambda - I\| \geq q_w^{-m_1}\|g - I\|$. Let $m_g \geq m$ be so that $g \in K_w[m_g] \setminus K_w[m_g + 1]$. For every $q_w^{m_1} \leq |t|_w \leq q_w^{(m_g - m_1)/2}$, and all $|s|_w \leq 1$, we have*

$$\|a'_\lambda(t)u'_\lambda(s)gu'_\lambda(-s)a'_\lambda(t^{-1}) - a_\lambda(t)g_\lambda a_\lambda(t^{-1})\| \leq C_1 q_w^{m_1} \max\{|s|_w, |t|_w^{-1}\}.$$

If we further assume that $|t|_w = q_w^{(m_g - m_1)/2} \geq C_1 q_w^{2m_1+3}$ and $|s|_w \leq C_1^{-1} q_w^{-2m_1-3}$,

$$\|a'_\lambda(t)u'_\lambda(s)gu'_\lambda(-s)a'_\lambda(t^{-1}) - I\| \geq q_w^{-m_1-2},$$

where the implied constant is absolute.

Proof. First note that in view of (15), we may replace a'_λ and u'_λ with a_λ and u_λ , respectively.

The proof is based on the following observation: let $g \in K_w[m']$ for some $m' \geq 1$, then for all $s \in B_F(0, 1)$ we have

$$u_\lambda(s)gu_\lambda(-s) = gh(s, g)$$

where $\|h(s, g)\| \ll \|g - I\|s\|_w$ and the implied constant is absolute.

Let g and t be as in the statement, and let $s \in B_w(0, 1)$. Conjugating the above with $a_\lambda(t)$, we conclude that

$$(17) \quad a_\lambda(t)u_\lambda(s)gu_\lambda(-s)a_\lambda(t^{-1}) = a_\lambda(t)ga_\lambda(t^{-1})h_t(s, g)$$

where $h_t(s, g) = a_\lambda(t)h(s, g)a_\lambda(t^{-1})$, hence,

$$\|h_t(s, g) - I\| \ll q_w^{m_g} \|g - I\|s\|_w \ll |s|_w$$

for an absolute implied constant.

Recall the condition $\|g_\lambda - I\| \geq q_w^{-m_1} \|g - I\|$. Moreover, note that the conjugation by $a_\lambda(t)$ has weight 2 for U_λ and weight ≤ 1 otherwise. We thus conclude that

$$\|a_\lambda(t)ga_\lambda(t^{-1}) - a_\lambda(t)g_\lambda a_\lambda(t^{-1})\| \leq q_w^{m_1} |t|_w^{-1}.$$

This and (17) finish the proof of the first claim.

To see the second claim first note that

$$a_\lambda(t)g_\lambda a_\lambda(t^{-1}) = u_\lambda(\tau) \quad \text{where } |\tau| \geq q_w^{-m_1-2};$$

moreover, if $|t|_w = q_w^{(m_g - m_1)/2} \geq C_1 q_w^{2m_1+3}$ and $|s|_w \leq C_1^{-1} q_w^{-2m_1-3}$, then

$$C_1 q_w^{m_1} \max\{|s|_w, |t|_w^{-1}\} \leq q_w^{-m_1-3}.$$

The second claim thus follows from the first claim and the above. $\blacksquare$

Recall from §3 that we fixed a Haar measure $\prod c_v |\omega_v|$ on H_g , and in particular, $c_w |\omega_w|$ on $H_{g,w}$. Unless otherwise stated explicitly, $c_w |\omega_w|$ is the measure of reference on $H_{g,w}$ throughout the paper.

LEMMA 5.6: *Let m_0 be as in Lemma 2.3. Let $m \geq 6m_0 + 1$. For $i = 1, 2$, let*

$$E_i \subset K_w A_{-m_0} K_w$$

be two subsets so that $|K_w A_{-m_0} K_w \setminus E_i| \leq q_w^{-4m_0 \dim \mathbf{G}}$.

Then for every $g' \in K'_w[m]$, there exist $\alpha_i \in \Delta_g(E_i)$ and $\lambda \in \Phi$ so that

$$\alpha_2 g' \alpha_1^{-1} = g \in K_w[m - 2m_0] \subset G_w \times \{1\}, \quad \|g_\lambda - I\| \gg \|g - I\|$$

where the implied constant depends only on $\dim \mathbf{G}$.

Proof. Let us write

$$g' = h\hat{g}$$

for some $h \in H_{g,w} \cap K'_w[m]$ and $\hat{g} \in K_w[m] = G_w \cap K'_w[m]$. If $\alpha \in \Delta_g(K_w A_{-m_0} K_w)$, then

$$g' \alpha^{-1} = (h \alpha^{-1}) \alpha \hat{g} \alpha^{-1}.$$

Note that since $\hat{g} \in K_w[m]$, for all α as above we have $\alpha \hat{g} \alpha^{-1} \in K_w[m - 2m_0]$. We will find $\alpha_1, \alpha_2 \in \Delta_g(E_i)$ so that $\alpha_2 g' \alpha_1^{-1} = \alpha_1 \hat{g} \alpha_1^{-1}$.

Since $\hat{g} \in K_w[m] \subset K_w[2m_0 + 1]$, in view of Lemma 2.3, there exists $\lambda \in \Phi$ so that

$$|E_{\hat{g}}| \geq q_w^{-3m_0 \dim \mathbf{G}}$$

where $E_{\hat{g}} = \{y \in K_w A_{-m_0} K_w : \|(y \hat{g} y^{-1})_\lambda - I\| \geq q_w^{-m_0} \|y \hat{g} y^{-1} - I\|\}$.

Since $h \in K'_w \cap H_{g,w}$, we have

$$h \cdot \Delta_g(K_w A_{-m_0} K_w) = \Delta_g(K_w A_{-m_0} K_w).$$

In particular, the map $f : \Delta_g(E_1) \rightarrow \Delta_g(K_w A_{-m_0} K_w)$ defined by $\alpha \mapsto h \alpha^{-1}$ is measure-preserving.

Altogether, and in view of our assumption on the relative measures of E_1 and E_2 , we may choose $\alpha_1 \in \Delta_g(E_1) \cap \Delta_g(E_{\hat{g}})$ with $\alpha_2 = f(\alpha_1)^{-1} \in \Delta_g(E_2)$. The lemma follows. ■

6. Non-divergence of unipotent orbits

In this section we show that when X is not compact, the cusps have small measure with respect to μ_g . We follow the proof of [10, Lemma 3.6.1], which relies on the non-divergence of unipotent flows.

Let $\mathbb{F}$ be a finite field such that F is a finite separable extension of $\tilde{F} := \mathbb{F}(T)$, the field of rational functions in one variable over $\mathbb{F}$; see, e.g., [34, Ch. III]. Let $\tilde{\mathbf{G}} = \text{Res}_{F/\tilde{F}}(\mathbf{G}')$. Then $\tilde{\mathbf{G}}$ is a semisimple $\tilde{F}$ -group and

$$\tilde{\mathbf{G}}(\mathbb{A}_{\tilde{F}})/\tilde{\mathbf{G}}(\tilde{F}) = \mathbf{G}'(\mathbb{A}_F)/\mathbf{G}'(F).$$

Fix a good place $w \in \Sigma$, and let $\tilde{w}$ be a place of $\tilde{F}$ so that $w|\tilde{w}$. We note that since w is a good place for $\mathbf{G}'$, the group $\tilde{\mathbf{G}}$ is isotropic over $\tilde{F}_{\tilde{w}}$ (however, it need not be split).

Let $K'(\tilde{w}) := \prod_{v|\tilde{w}} K'_v$. Since $\mathbf{G}$ is simply connected, it follows from strong approximation (see [28]) that one can write

$$\mathbf{G}'(\mathbb{A}) = K'(\tilde{w}) \left(\prod_{v|\tilde{w}} G_v \right) \mathbf{G}'(F) = K'(\tilde{w}) \tilde{\mathbf{G}}(\tilde{F}_{\tilde{w}}) \tilde{\mathbf{G}}(\tilde{F}).$$

Let $\Gamma = \tilde{\mathbf{G}}(\tilde{F}) \cap K'(\tilde{w})$ and $\tilde{G}_{\tilde{w}} = \tilde{\mathbf{G}}(\tilde{F}_{\tilde{w}})$; note that Γ is a congruence subgroup of $\tilde{G}_{\tilde{w}}$. The above implies that

$$K'(\tilde{w}) \backslash \mathbf{G}'(\mathbb{A}) / \mathbf{G}'(F) \cong \tilde{G}_{\tilde{w}} / \Gamma.$$

Let $\mathcal{O}$ be the ring of $\tilde{w}$ -integers in $\tilde{F}$; note that $\mathcal{O}$ is a PID. Let $\tilde{\mathfrak{g}} = \text{Lie}(\tilde{G}_{\tilde{w}})$, and let $\tilde{\mathfrak{g}}_{\mathcal{O}}$ be an $\mathcal{O}$ -lattice in the $\tilde{\mathfrak{g}}$ with the property that $[\tilde{\mathfrak{g}}_{\mathcal{O}}, \tilde{\mathfrak{g}}_{\mathcal{O}}] \subset \tilde{\mathfrak{g}}_{\mathcal{O}}$. As it is done in [10], for any $x \in X$, we set

$$\text{ht}(x) = \sup\{\|\text{Ad}(g)\mathbf{u}\|^{-1} : x = g\Gamma, g \in \tilde{G}_{\tilde{w}}, \mathbf{u} \in \tilde{\mathfrak{g}}_{\mathcal{O}} \setminus \{0\}\},$$

and

$$\mathfrak{S}(R) := \{x \in X : \text{ht}(x) \leq R\}.$$

LEMMA 6.1: For every $m \in \mathbb{N}$, let

$$Q'_m := \{((g_v^1), (g_v^2)) \in K' : (g_w^1, g_w^2) \in \pi_w(K'_w[m])\}.$$

There exists κ_8 so that for all $x \in X$ with $m \leq \text{ht}(x)^{-\kappa_8}$, the map $\mathbf{g} \mapsto \mathbf{g}x$ is injective on Q'_m .

Proof. Fix some $m \in \mathbb{N}$ and assume that $\mathbf{g}_1 x = \mathbf{g}_2 x$ for $\mathbf{g}_1, \mathbf{g}_2 \in Q'_m$. Fix $\tilde{g} \in \tilde{G}_{\tilde{w}} = \tilde{\mathbf{G}}(\tilde{F}_{\tilde{w}})$ such that the projection to the $\tilde{w}$ place of x is $\tilde{g}\Gamma$. Then,

$$\mathbf{g}_{1,\tilde{w}} \tilde{g}\Gamma = \mathbf{g}_{2,\tilde{w}} \tilde{g}\Gamma,$$

and so $\mathbf{g}_{2,\tilde{w}}^{-1} \mathbf{g}_{1,\tilde{w}}$ fixes $L_x := \text{Ad}(\mathbf{g})\tilde{\mathfrak{g}}_{\mathcal{O}}$. Note that L_x is an $\mathcal{O}$ -lattice in $\tilde{\mathfrak{g}}$. By the definition of the height, for all $\mathbf{u} \in L_x$ we have $\|\mathbf{u}\| \geq \text{ht}(x)^{-1}$.

Since the covolume of L_x is independent of x (and equals the covolume of $\tilde{\mathfrak{g}}_{\mathcal{O}}$), by lattice reduction theory, one can find a basis $\mathbf{u}_1, \dots, \mathbf{u}_d$ (where $d \leq 2N^2$) of L_x such that $\|\mathbf{u}_i\| \ll \text{ht}(x)^d$; see [18, Thm. 1.2] and references therein.

Thus, by choosing the constants m, κ_8 sufficiently large, we get

$$\|\mathbf{g}_{2,\tilde{w}}^{-1} \mathbf{g}_{1,\tilde{w}} \mathbf{u}_i - \mathbf{u}_i\| < \text{ht}(x)^{-1} \quad \text{for all } 1 \leq i \leq d.$$

In that case, the vector $\mathbf{g}_{2,\tilde{w}}^{-1}\mathbf{g}_{1,\tilde{w}}$ fixing the lattice L_x setwise implies that it is, in fact, fixing L_x pointwise. Thus, $\mathbf{g}_{2,\tilde{w}}^{-1}\mathbf{g}_{1,\tilde{w}}$ belongs to the center of $\tilde{G}_{\tilde{w}}$. Let $g' \in G'$ be so that $x = g'\mathbf{G}'(F)$. Then, we have $\mathbf{g}_2^{-1}\mathbf{g}_1 = g'\gamma g'^{-1}$ where $\gamma = (\gamma^1, \gamma^2) \in \mathbf{G}'(F)$ and γ^i embeds diagonally in $\mathbf{G}(\mathbb{A})$. Now since $\mathbf{g}_{2,w}^{-1}\mathbf{g}_{1,w}$ is central, we conclude that (γ_w^1, γ_w^2) is central, which is impossible if m is big enough. ■

PROPOSITION 6.2 (Non-divergence estimate): *There are positive constants κ_9 and κ_{10} , depending on N and $[F : \tilde{F}]$, so that for any $g \in G'$ we have*

$$\mu_g(X \setminus \mathfrak{S}(R)) \ll p_w^{\kappa_9} R^{-\kappa_{10}};$$

recall that $p_w = \text{char}(k_w)$.

Proof. The proof is similar to that of [10, Lemma 3.6.1] (see [10, App. B]) where one replaces the use of the non-divergence result by Kleinbock and Margulis [19] with Ghosh [13]. We will go over the main steps of the proof presented in [10, App. B] to explicate the differences.

Since $\mathcal{O}$ is a PID, discrete $\mathcal{O}$ -submodules of $\tilde{\mathfrak{g}}$ are free; see [13, Lemma 4.1]. Following [13, §4], for a discrete $\mathcal{O}$ -module $\Delta = \bigoplus_{i=1}^{\ell} \mathcal{O}\mathbf{u}_i$, we denote $\|\Delta\| = \|\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{\ell}\|_v$. Then, $\|\Delta\|$ is a “norm like” function in the language of [13] that plays the role of the covolume of Δ in the proof (see the proof of Lemma 4.3 and properties N1–N3 in [13]). We refer to Δ as an $\mathcal{O}$ -arithmetic lattice in

$$V = \bigoplus_{i=1}^{\ell} \tilde{F}_{\tilde{w}} \mathbf{u}_i \subset \tilde{\mathfrak{g}}.$$

Let $h \in \tilde{G}_{\tilde{w}}$. A subspace $V \subset \tilde{\mathfrak{g}}$ is called **h -rational** if $V \cap \text{Ad}_h \tilde{\mathfrak{g}}_{\mathcal{O}}$ is an $\mathcal{O}$ -arithmetic lattice in V ; we let $\|V\|_h = \|V \cap \text{Ad}_h \tilde{\mathfrak{g}}_{\mathcal{O}}\|$. The first step is to show that for any h there are no h -rational subspaces which are H_w invariant of low norm.

Let $U = \{u(t)\}$ be a one-parameter $\tilde{F}_{\tilde{w}}$ -unipotent subgroup of H_w , and use $\text{Ad}_{\tilde{\rho}(u(t))}$ in the definition of $h(t)$ in the Proof of [10, Lemma 3.6.1], i.e., $h(t) = \text{Ad}_{\tilde{\rho}(u(t))}|_{V \in \text{SL}(V)}$, where V is an h -rational space for some $h \in \tilde{G}_{\tilde{w}}$.

Now, one can follow the arguments in [10, Proof of Lemma 3.6.1] using the above notations, Lemma 6.1, [13, Thm. 5.2], and [13, Property N2] to show that there exist positive constants c, κ_9 such that for any $x \in \pi_v(Hg\Gamma)$

(18) there is no x -rational, H_w -invariant proper subspace of norm $\leq cp_w^{-\kappa_9}$,

where c is an absolute constant and κ_9 depends only on $\dim \mathbf{G}$.

Put $\xi = cp_{\bar{w}}^{-\kappa_9}$. Since the number of x -rational proper subspaces of norm at most ξ is finite and $U \subset H_w$, a.e. $h \in H_w$ has the property that hUh^{-1} does not leave invariant any proper x -rational subspace of norm $\leq \xi$. Alternatively, we may also conclude for a.e. $h \in H_w \cap K'_v$ that U does not leave any proper hx -rational subspace of norm $\leq \xi$ invariant.

Since μ_g is H -invariant and H -ergodic, by Mautner phenomenon μ_g is also U -ergodic. This also implies that the U -orbit of hx equidistributes with respect to μ_g for a.e. h . We choose $h \in H_w \cap K_v$ so that both of the above properties hold for $x' = hx$.

Let h' be such that $x' = h'\Gamma_{\bar{w}}$. Then, for any $h'\Gamma_{\bar{w}}$ -rational subspace V , if we let

$$\psi_V(t) := \|\mathrm{Ad}_{u(t)}^{-1} V\|_{u(t)^{-1}h'} = \|\mathrm{Ad}_{u(t)}^{-1}(V \cap \mathrm{Ad}_{h'}\mathfrak{g}\mathcal{O})\|,$$

then either ψ_V is unbounded or equals a constant $\geq \xi$. Thus, by [13, Thm 4.4] there exists a positive constant κ_{11} so that

$$\mu_g(\{t : |t|_{\bar{w}} \leq r, x'u(t) \notin \mathfrak{S}(\epsilon^{-1})\}) \ll p_{\bar{w}}^{\kappa_{11}}(\frac{\epsilon}{\xi})^\alpha \mu_g(\{t : |t|_{\bar{w}} \leq r\}),$$

for all large enough r and $\epsilon > 0$, where $\alpha = \kappa_{10}$ only depends on the degree of the polynomials appearing in the matrix entries for the elements of the one-parameter unipotent subgroup U . The lemma now follows as the U -orbit equidistributes with respect to μ_g . ■

7. Invariant subgroups for the measure μ

Recall that μ is an H_g -invariant measure on $Y_g = g\mathbf{H}(\mathbb{A})/\mathbf{H}(F)$. In this section, we will use tools from homogeneous dynamics combined with algebraic properties of good places which have been established in previous sections to prove Theorem 1.1. When X is not compact, the proof will also use the non-divergence result proved in Lemma 6.2.

7.1. ADELIC SOBOLEV NORMS. Let $C^\infty(X)$ denote the space of functions which are invariant by a compact open subgroup of $\mathbf{G}'(\mathbb{A})$. We follow [9, App. A] to define Sobolev norms on $C^\infty(X)$, and present some of its properties.

First, we define a system of projections $\mathrm{pr}_v[m]$ for any unitary $\mathbf{G}'(F_v)$ -representation for each place v such that $\sum_{m \geq 0} \mathrm{pr}_v[m] = 1$. For any $v \in \Sigma$, let $\mathrm{Av}_v[m]$ be the projection on $K_v[m]$ -invariant vectors—this is simply given

by averaging over the compact open subgroup $K_v[m]$ equipped with the probability Haar measure. Put $\text{pr}_v[0] := \text{Av}_v[0]$ and $\text{pr}_v[m] := \text{Av}_v[m] - \text{Av}_v[m-1]$ for $m \geq 1$.

Let $\mathcal{M}$ be the set of functions mapping Σ to the non-negative integers, such that almost every v is mapped to zero. For any $\underline{m} \in \mathcal{M}$ let

$$\|\underline{m}\| := \prod_v q_v^{m_v}, \quad \text{and} \quad \text{pr}[\underline{m}] := \prod_v \text{pr}_v[m_v].$$

Then $\text{pr}[\underline{m}]$ acts on any unitary $\mathbf{G}'(\mathbb{A})$ -representation, and $\sum_{\underline{m}} \text{pr}[\underline{m}]f = f$ for any $f \in C^\infty(X)$. Moreover, for any $f \in C^\infty(X)$ and $\underline{m} \in \mathcal{M}$, the function $\text{pr}[\underline{m}]f$ is invariant under $K_{\underline{m}} := \prod_v K_v[m_v]$.

Given an integer $d \geq 0$ we define a degree d Sobolev norm by

$$(19) \quad \mathcal{S}_d(f)^2 := \sum_{\underline{m}} \|\underline{m}\|^d \|\text{pr}[\underline{m}](1 + \text{ht}(x))^d f(x)\|_2^2,$$

where $\text{ht}(\cdot)$ is as defined in §6. For a compactly supported smooth function on X any of these Sobolev norms is finite. It is also easy to see that

$$(20) \quad \mathcal{S}_d(f) \leq \mathcal{S}_{d'}(f) \quad \text{if } d < d'.$$

We claim that in a similar way to [9, App. A], we have the following:

(S0) For any d , $\mathcal{S}_d$ is a pre-Hilbert norm on $C_c^\infty(X)$.

(S1) There exists d_0 , depending on N , and $[F : \tilde{F}]$ (see §6 for the definition of $\tilde{F}$) such that for all $d \geq d_0$, we have $\|f\|_{L^\infty} \ll_d \mathcal{S}_d(f)$.

(S2) Given d_0 , there are $d > d' > d_0$ and an orthonormal basis $\{e_k\}$ of the completion of $C_c^\infty(X)$ with respect to $\mathcal{S}_d$ which is orthogonal with respect to $\mathcal{S}_{d'}$ such that

$$\sum_k \mathcal{S}_{d'}(e_k)^2 < \infty \quad \text{and} \quad \sum_k \frac{\mathcal{S}_{d_0}(e_k)^2}{\mathcal{S}_{d'}(e_k)^2} < \infty.$$

(S3) Let $g \in \mathbf{G}(\mathbb{A})$ and $f \in C_c^\infty(X)$. We write $g.f$ for the action of g on f , and $\|g\| = \prod_{v \in \Sigma} \|g_v\|$. Then, for every $d \geq 0$

$$\mathcal{S}_d(g.f) \ll \|g\|^{4Nd} \mathcal{S}_d(f),$$

where the implied constant is absolute. If in addition $g \in K$, then we have $\mathcal{S}_d(g.f) = \mathcal{S}_d(f)$.

Now let us fix a good place w . Note that for every $\lambda \in \Phi$, we have

$$\|u'_\lambda(s)\| \leq (1 + |s|_w)^N \quad \text{for all } s \in F_w.$$

(S4) For any $d \geq d_0$, any $r \geq 0$, any $g \in K_w[r]$, and any $f \in C_c^\infty(X)$, we have

$$\|g \cdot f - f\|_\infty \leq q_w^{-r} \mathcal{S}_d(f).$$

(S5) There exist d_0, M , which depend only on N , so that the following holds. For every L , let Av_L be the operation of averaging over $K_w[L]$. Let $\lambda \in \Phi$ and $s \in F_w$. Put $\mathbb{T}_s = \text{Av}_L \star \delta_{u'_\lambda(s)} \star \text{Av}_L$ where $\star$ denotes convolution of operators. For all $x \in X$, all $f \in C_c^\infty(X)$, and $d \geq d_0$ we have

$$|\mathbb{T}_s f(x) - \int f \, dm_X| \ll q_w^{(d+2)L} \text{ht}(x)^d \|\mathbb{T}_s\|_{2,0} \mathcal{S}_d(f),$$

where $\|\mathbb{T}_s\|_{2,0}$ denotes the operator norm of $\mathbb{T}_s$ on $L_0^2(X, m_X)$. Moreover, we have

$$(21) \quad \|\mathbb{T}_s\|_{2,0} \ll (1 + |s|_w)^{-1/2M} q_w^{2dL}.$$

(S6) There exist d_0, M , which depend only on N , so that for all $d \geq d_0$ and all $\lambda \in \Phi$, we have

$$(22) \quad \left| \langle u_\lambda(s) f_1, f_2 \rangle_{L^2(\mu)} - \int f_1 \, d\mu \int \bar{f}_2 \, d\mu \right| \ll (1 + |s|_w)^{-1/2M} \mathcal{S}_d(f_1) \mathcal{S}_d(f_2).$$

Indeed, the above properties can be shown arguing in a similar way to [9, App. A4]. We only present a brief guided tour of the proof, and the implied changes in the statements. In the proof of property (S1), Lemma 6.1 replaces [9, Lemma 7.2], and so the implied constant only depends on N and $[F : \tilde{F}]$. In the proof of property (S3), for any place v and $g \in G_v$, the inequality

$$\text{ht}(gx) \ll \|g\|^{2N} \text{ht}(x)$$

follows from the definition of $\text{ht}(x)$ (see §6), and so (except possibly a different power in the result) the proof is the same.

Properties (S5) and (S6) follow from $1/M$ -temperedness (see [5, Thm. 2]) of the natural representation of $\mathbf{G}(F_w)$ on $L_0^2(X, m_X)$ and $L_0^2(X, \mu)$, respectively. In the case where $\mathbf{G}$ is type A_1 , this follows from property (τ) , [7]. In all other cases, $\mathbf{G}$ has absolute rank at least 2, and w is so that $\mathbf{G}$ is F_w -split (see §3). Therefore, $\mathbf{G}(F_w)$ has property (T) and the desired $1/M$ -temperedness follows from [25].

7.2. GENERIC POINTS. Let $\lambda \in \Phi$. Let $n \in \mathbb{N}$. For any $x \in X$ and $t = \varpi_w^{-n}$, define

$$(23) \quad D_n f(x) := q_w^n \int_{B(0, q_w^{-n})} f(a'_\lambda(t) u'_\lambda(s) x) \, ds - \int_X f \, d\mu;$$

see the beginning of §5, more specifically (14), for the definition of $a'_\lambda(t)$ and $u'_\lambda(s)$. In particular, we have $a'_\lambda(t), u'_\lambda(s) \in H_{g,w} \subset H_g$ for all t, s, λ . Therefore, they preserve the measure μ .

A point $x \in X$ is called n_0 -**generic** for λ with respect to the Sobolev norm $\mathcal{S}_d$ if, for any integer $\ell \geq n_0$, we have

$$(24) \quad |D_\ell f(x)| \leq q_w^{-\ell/8M} \mathcal{S}_d(f)$$

where M is as in (22).

PROPOSITION 7.1: *If d_0 is chosen large enough, depending on N , then for all $d \geq d_0$ the μ -measure of points which are not n_0 -generic for some $\lambda \in \Phi$ with respect to $\mathcal{S}_d$ is $\ll q_w^{-n_0/5M}$.*

Proof. Let $\lambda \in \Phi$ be arbitrary. For simplicity in the notation, we will write $a'(t)$ and $u'(s)$ for $a'_\lambda(t)$ and $u'_\lambda(s)$, respectively.

Let $\ell \geq n_0$ be an integer and d_0 satisfy (22). Let f be a fixed function in $C_c^\infty(X)$. Let us write $B = B(0, q_w^{-\ell})$. Put

$$A_1 = \{(s_1, s_2) \in B \times B : |s_1 - s_2|_w < q_w^{-11\ell/10}\}$$

and $A_2 = B \times B \setminus A_1$.

For $(s_1, s_2) \in B \times B$, put

$$F(s_1, s_2) = \langle a'(t) u'(s_1 - s_2) a'(t^{-1}) f, f \rangle_\mu.$$

If $(s_1, s_2) \in A_2$ and $|t|_w \geq q_w^\ell$, then

$$a'(t) u'(s_1 - s_2) a'(t^{-1}) = u'(\tau) \quad \text{with } |\tau| \geq q_w^{0.9\ell}.$$

Hence, for any $(s_1, s_2) \in A_2$ and $|t|_w \geq q_w^\ell$, by (22) we have

$$\left| F(s_1, s_2) - \left(\int_X f \, d\mu \right)^2 \right| \ll q_w^{-9\ell/20M} \mathcal{S}_{d_0}(f)^2.$$

Since the Lebesgue measure of A_1 is $q_w^{-21\ell/10}$, using the definition of $D_\ell f(x)$ and the Fubini's theorem, now we have

$$\begin{aligned} & \int_X |D_\ell f(x)|^2 d\mu \\ &= q_w^{2\ell} \int_B \int_B F(s_1, s_2) ds_1 ds_2 - \left(\int_X f d\mu \right)^2 \\ &= q_w^{2\ell} \left(\int_{A_1} F(s_1, s_2) ds_1 ds_2 + \int_{A_2} F(s_1, s_2) ds_1 ds_2 \right) - \left(\int_X f d\mu \right)^2 \\ &\ll 3q_w^{-9\ell/20M} \mathcal{S}_{d_0}(f)^2. \end{aligned}$$

Using Markov's inequality we arrive at

$$(25) \quad \mu(\{x \in X : |D_\ell f(x)| > \varepsilon\}) \ll 3\varepsilon^{-2} q_w^{-9\ell/20M} \mathcal{S}_{d_0}(f)^2.$$

To conclude, we use property (S2) of the Sobolev norms. That is, there are $d > d' > d_0$ and an orthonormal basis $\{e_k\}$ of the completion of $C_c^\infty(X)$ with respect to $\mathcal{S}_d$ which is orthogonal with respect to $\mathcal{S}_{d'}$ so that

$$(26) \quad \sum_k \mathcal{S}_{d'}(e_k)^2 < \infty \quad \text{and} \quad \sum_k \frac{\mathcal{S}_{d_0}(e_k)^2}{\mathcal{S}_{d'}(e_k)^2} < \infty$$

Put $c = (\sum_k \mathcal{S}_{d'}(e_k)^2)^{-1/2}$ and define

$$(27) \quad X' = \bigcup_{\ell \geq n_0, i \geq 1} \{x \in X : |D_\ell e_i(x)| \geq cq_w^{-\ell/8M} \mathcal{S}_{d'}(e_i)\}.$$

Using (25) and (26), we have

$$(28) \quad \mu(X') \ll q_w^{-n_0/5M}.$$

Let $f \in C_c^\infty(X)$, and write $f = \sum f_k e_k$ and suppose $x \notin X'$. Let $\ell \geq n_0$, then using the triangle inequality for D_ℓ we obtain

$$\begin{aligned} |D_\ell f(x)| &\leq \sum_k |f_k| |D_\ell e_k(x)| \leq cq_w^{-\ell/8M} \sum_k |f_k| \mathcal{S}_{d'}(e_k) \\ &\leq cq_w^{-\ell/8M} \left(\sum_k |f_k|^2 \right)^{1/2} \left(\sum_k \mathcal{S}_{d'}(e_k)^2 \right)^{1/2} \\ &= q_w^{-\ell/8M} \mathcal{S}_d(f), \end{aligned}$$

where the last inequality follows from the definition of c and since $\{e_i\}$ is an orthonormal basis with respect to $\mathcal{S}_d$.

This completes the proof in view of (28) as λ was arbitrary. $\blacksquare$

For the rest of this section, we fix some $d \geq d_0$ so that Proposition 7.1 holds, and write $\mathcal{S} = \mathcal{S}_d$.

For every m , let $Q_m = \{((g_v), 1) \in K : g_w \in \pi_w^1(K_w[m])\}$.

PROPOSITION 7.2: *There exists κ_{12} such that for all $c \in \mathbb{N}$ and all large enough $m \geq 3c$, the following holds. Let $\mathbf{g} \in Q_m$ satisfy $\mathbf{g}_w \neq 1$ and*

$$\|\mathbf{g}_{w,\lambda} - I\| \geq q_w^{-c} \|\mathbf{g}_w - I\|.$$

Let $\lambda \in \Phi$. Suppose $y_1, y_2 \in X$ are two $\lfloor m/10 \rfloor$ -generic points for the measure μ with respect to $\mathcal{S}$ satisfying that $y_2 = \mathbf{g}y_1$. Then,

$$|u_\lambda(\tau) \cdot \mu(f) - \mu(f)| \ll q_w^{-m\kappa_{12}} \mathcal{S}(f)$$

for all $\tau \in B(0, 1)$ and all $f \in C_c^\infty(X)$.

Proof. As in the proof of Proposition 7.1, we write $a'(t), u'(s)$ for $a'_\lambda(t), u'_\lambda(s)$; we also write $a(t), u(s)$ for $a_\lambda(t), u_\lambda(s)$. Recall from (15) that

$$a(t)\mathbf{g}a(t^{-1}) = a'(t)\mathbf{g}a'(t^{-1}) \quad \text{and} \quad u(s)\mathbf{g}u(-s) = u'(s)\mathbf{g}u'(-s).$$

Let us also recall that $a'(t)$ and $u'(s)$ belong to H_g , hence, they leave μ invariant, while $a(t), u(s) \in G_w$ and we do not have any a priori information regarding their action on μ .

For all s and t , put

$$\mathbf{g}_{s,t} = a'(t)u'(s)\mathbf{g}u'(-s)a'(t^{-1});$$

note that $\mathbf{g}_{s,t,v} = \mathbf{g}_v$ for all $v \neq w$.

Apply Lemma 5.5 with $m_1 = c$, and let $m \geq 3c$. Let us write

$$\ell = \lfloor (m_{\mathbf{g}_w} - c)/2 \rfloor$$

where $\mathbf{g}_w \in K_w[m_{\mathbf{g}_w}] \setminus K_w[m_{\mathbf{g}_w} + 1]$. Let $|t|_w = q_w^\ell$; we will always assume $|s|_w \leq |t|_w^{-1}$. By Lemma 5.5, we have

$$(29) \quad \|\mathbf{g}_{s,t,w} - a(t)\mathbf{g}_{\lambda,w}a(t^{-1})\| \leq C_1 q_w^{-\ell+c};$$

moreover, if $|t|_w = q_w^\ell \geq C_1 q_w^{2c+3}$, then

$$(30) \quad \|\mathbf{g}_{s,t,w} - I\| \geq q_w^{-c-2}.$$

Recall now that y_i , $i = 1, 2$, is $\lfloor m/10 \rfloor$ -generic for μ w.r.t. $\mathcal{S}$. This, together with our choice of the integer $\ell \geq \lfloor m/10 \rfloor$, implies that

$$(31) \quad |q_w^\ell \int_{B_w(0, q_w^{-\ell})} f(a'(t)u'(s)y_i) \, ds - \int_X f \, d\mu| \leq q_w^{-\ell/8M} \mathcal{S}(f)$$

for $i = 1, 2$.

Since $y_2 = \mathbf{g}y_1$, we get that $a'(t)u'(s)y_2 = \mathbf{g}_{s,t}a'(t)u'(s)y_1$. In particular,

$$f(a'(t)u'(s)y_2) = f(\mathbf{g}_{s,t}a'(t)u'(s)y_1).$$

Applying (31) with y_2 , we thus conclude that

$$(32) \quad \left| q_w^\ell \int_{B_w(0, q_w^{-\ell})} f(\mathbf{g}_{s,t}a'(t)u'(s)y_1) \, ds - \int_X f \, d\mu \right| \leq q_w^{-\ell/8M} \mathcal{S}(f).$$

Put $\hat{\mathbf{g}} = a(t)\mathbf{g}'a(t^{-1})$ where $\mathbf{g}'_v = \mathbf{g}_v$ for all $v \neq w$, and $\mathbf{g}'_w = \mathbf{g}_{w,\lambda}$. Then (32) and (29) imply that if $m \geq 5c$, then

$$\left| q_w^\ell \int_{B_w(0, q_w^{-\ell})} f(\hat{\mathbf{g}}a'(t)u'(s)y_1) \, ds - \int_X f \, d\mu \right| \ll q_w^{-\ell/9M} \mathcal{S}(f).$$

The above, (31), applied with y_1 and $\hat{\mathbf{g}}^{-1}.f$, and property (S3) yield

$$(33) \quad |\hat{\mathbf{g}}^{-1}.\mu(f) - \mu(f)| \ll q_w^{-\ell/9M} \mathcal{S}(f).$$

Recall that $\|\mathbf{g}_{w,\lambda} - I\| \geq q_w^{-c} \|\mathbf{g}_w - 1\|$. Therefore, if we write

$$u(\tau_0) = a(t)\mathbf{g}_{w,\lambda}a(t^{-1}),$$

then $q_w^{-c-2} \leq |\tau_0|_w \leq 1$.

For every ρ , put $\hat{\mathbf{g}}_\rho := a'(\rho)\hat{\mathbf{g}}a'(\rho^{-1})$. Then $\hat{\mathbf{g}}_{v,\rho} = \hat{\mathbf{g}}_v$ and $\hat{\mathbf{g}}_{w,\rho} = u(\rho^2\tau_0)$. Moreover, we have

$$(34) \quad \begin{aligned} |\hat{\mathbf{g}}_\rho.\mu(f) - \mu(f)| &= |\hat{\mathbf{g}}.\mu(a'(\rho^{-1}).f) - \mu(a'(\rho^{-1}).f)| \\ &\ll q_w^{-\ell/9M} \mathcal{S}(a'(\rho^{-1}).f), \end{aligned}$$

where we use the fact that μ is H_g -invariant in the first equality and (33) in the second line. In view of property (S3), we have

$$\mathcal{S}(a'(\rho^{-1}).f) \ll \max\{|\rho|_w^{4Nd}, |\rho|_w^{-4Nd}\} \mathcal{S}(f).$$

This and (34) imply

$$|\hat{\mathbf{g}}_\rho.\mu(f) - \mu(f)| \ll q_w^{-\ell/9M} \max\{|\rho|_w^{4Nd}, |\rho|_w^{-4Nd}\} \mathcal{S}(f).$$

Recall now that $\hat{g}^{-1}\hat{g}_\rho = u((\rho^2 - 1)\tau_0)$. Using the triangle inequality, the above, and (33) applied to $\hat{g}_\rho \cdot f$, we may conclude that

$$\begin{aligned} |u((\rho^2 - 1)\tau_0) \cdot \mu(f) - \mu(f)| &= |\hat{g}^{-1}\hat{g}_\rho \cdot \mu(f) - \mu(f)| \\ &\leq |\hat{g}^{-1}\hat{g}_\rho \cdot \mu(f) - \hat{g}_\rho \cdot \mu(f)| + |\hat{g}_\rho \cdot \mu(f) - \mu(f)| \\ &\ll q_w^{-\ell/9M} \mathcal{S}(f) + q_w^{-\ell/9M} \max\{|\rho|_w^{4Nd}, |\rho|_w^{-4Nd}\} \mathcal{S}(f) \\ &\ll q_w^{-\ell/9M} \mathcal{S}(f), \end{aligned}$$

for all $|\rho|_w = 1$. We now consider two cases:

CASE 1: $\text{char}(F) > 2$. Then by Lemma 5.1, $B(0, \kappa_5) \subset \{\rho^2 - 1 : |\rho|_w = 1\}$. This and the fact that $|\tau_0|_w \geq q_w^{c-2}$ imply that

$$|u(\tau) \cdot \mu(f) - \mu(f)| \ll q_w^{-\ell/9M} \mathcal{S}(f)$$

for all $\tau \in B(0, q_w^{c-2}\kappa_5)$. Using $u(\rho^2\tau) = a'(\rho)u(\tau)a'(\rho^{-1})$ and arguing as in (34) again, we conclude that so long as m (and hence ℓ) is large enough,

$$|u(\tau) \cdot \mu(f) - \mu(f)| \ll q_w^{-\ell/10M} \mathcal{S}(f)$$

for all $\tau \in B(0, 1)$; the proof is complete in this case.

CASE 2: $\text{char}(F) = 2$. Then by Lemma 5.1, we have

$$|u(\tau) \cdot \mu(f) - \mu(f)| \ll q_w^{-\ell/9M} \mathcal{S}(f)$$

for all $\tau = \tau_0\alpha^2$ and $\alpha \in B(0, 1)$. Using $u(\rho^2\tau) = a'(\rho)u(\tau)a'(\rho^{-1})$ and arguing as in (34) again, we conclude that so long as m is large enough,

$$(35) \quad |u(\tau) \cdot \mu(f) - \mu(f)| \ll q_w^{-\ell/10M} \mathcal{S}(f)$$

for all $\tau = \tau_0\alpha^2$ with $|\tau| \leq q_w^{c+2}$ (recall that $q_w^{c-2} \leq |\tau_0| \leq 1$).

Now by Lemma 5.4, for every $r \in B(0, 1)$

$$u_\lambda(r) = h_1 \cdots h_{\ell'}$$

where ℓ' is absolute and $h_i \in H_{g,w}$ with

$$\|h_i\| \leq 1 \quad \text{or} \quad h_i \in \{u_\lambda(\tau_0\alpha^2) : |\tau_0\alpha^2| \leq |\tau_0|^{-1}\}.$$

This and (35) complete the proof in this case as well. $\blacksquare$

Recall that $\text{vol}_{G'}$ denotes our fixed Haar measure on G' . Also recall that for every m , we let

$$\mathcal{Q}'_m = \{((g_v^1), (g_v^2)) \in K' : (g_w^1, g_w^2) \in \pi_w(K'_w[m])\}.$$

LEMMA 7.3: Assume n is so that

$$\mathrm{vol}_{G'}(Q'_n)^{-1} \leq \frac{\mathrm{vol}(Y)}{4(\#\mathbf{C}(F))} \mathrm{vol}(Y),$$

where $\mathbf{C}$ denotes the center of $\mathbf{G}'$. There exist points y_1, y_2 and $\lambda \in \Phi$ so that:

- (1) y_1 and y_2 are $\lfloor n/20 \rfloor$ -generic for λ and the measure μ w.r.t. $\mathcal{S}$,
- (2) $y_2 = \mathbf{g}y_1$ where $\mathbf{g} \in Q_{n-m_0} \setminus \mathrm{Stab}(\mu)$. Moreover, $\mathbf{g}_w \neq 1$ and we have

$$\|\mathbf{g}_{w,\lambda} - I\| \geq q_w^{-m_0} \|\mathbf{g}_w - I\|,$$

where m_0 is as in Lemma 2.3.

Proof. We may assume $\mathrm{vol}(Y)$ is large enough so that n maybe chosen with $\lfloor n/20 \rfloor \geq 2m_0 + 1$. Let

$$Y'_{\mathrm{good}}(R) = \mathfrak{S}(R) \cap \{y \in Y : y \text{ is } \lfloor n/20 \rfloor\text{-generic}\}.$$

Then by Propositions 7.1 and 6.2, if $R \gg_{q_w} 1$ is large enough, we have

$$(36) \quad \mu(Y'_{\mathrm{good}}(R)) \geq 1 - q_w^{-11m_0 \dim \mathbf{G}}.$$

Fix one such R for the rest of the argument, and put $Y'_{\mathrm{good}} = Y'_{\mathrm{good}}(R)$.

Let $\mathbf{H} = \Delta_g(K_w A_{-m_0} K_w)$. Put

$$Y_{\mathbf{H}} = \{(h, y) \in \mathbf{H} \times Y'_{\mathrm{good}} : hy \in Y'_{\mathrm{good}}\}.$$

Note that for every $h \in \mathbf{H}$ and all $y \in Y'_{\mathrm{good}} \cap h^{-1}Y'_{\mathrm{good}}$, we have $(h, y) \in Y_{\mathbf{H}}$. Since $\mathbf{H} \subset H_{g,w}$ leaves μ invariant, it follows from this observation and (36) that for every $h \in \mathbf{H}$, the fiber $\{y : (h, y) \in Y_{\mathbf{H}}\}$ has measure $\geq 1 - 2q_w^{-11m_0 \dim \mathbf{G}}$. Therefore, by Fubini's theorem, there exists a subset $Y_{\mathrm{good}} \subset Y'_{\mathrm{good}}$ with

$$\mu(Y_{\mathrm{good}}) \geq 1 - q_w^{-5m_0 \dim \mathbf{G}}$$

so that if for every $y \in Y_{\mathrm{good}}$ we put $\mathbf{H}_y = \{h \in \mathbf{H} : hy \in Y'_{\mathrm{good}}\}$, then

$$|\mathbf{H} \setminus \mathbf{H}_y| < q_w^{-5m_0 \dim \mathbf{G}}.$$

Let $\{Q'_n.x_i : i \in I\}$ be a disjoint covering of $\mathfrak{S}(R)$, then $\#I \leq \mathrm{vol}_{G'}(Q'_n)^{-1}$, moreover, since $\mu(Y_{\mathrm{good}}) \geq 0.9$, there exists some i_0 so that

$$\mu(Q'_n.x_{i_0} \cap Y_{\mathrm{good}}) \geq \frac{1}{2 \cdot (\#I)}.$$

Recall now from Lemma 2.1 that $\mathrm{Stab}(\mu) = H \cdot \mathbf{C}(F)$. We now claim that there exist $y'_1, y'_2 \in Q'_n.x_{i_0} \cap Y_{\mathrm{good}}$ so that $y'_2 = \mathbf{g}'y'_1$ for some $\mathbf{g}' \notin \mathrm{Stab}(\mu)$. Assume to the contrary that for every $y, y' \in Q'_n.x_{i_0} \cap Y_{\mathrm{good}}$, we have $y' = hy$

where $h \in \text{Stab}(\mu)$. Then, since $b \mapsto bx$ is injective for all $x \in \mathfrak{S}(R)$ and all $b \in Q'_n$, see Lemma 6.1, we have

$$\mu(Q'_n \cdot x_{i_0} \cap Y_{\text{good}}) \leq (\#\mathbf{C}(F)) \cdot \mathfrak{m}(Q'_n \cap H).$$

From this we conclude that

$$\begin{aligned} \text{vol}_{G'}(Q'_n) &\leq (\#I)^{-1} \leq 2\mu(Q'_n x_{i_0} \cap Y_{\text{good}}) \\ &\leq 2 \cdot (\#\mathbf{C}(F)) \cdot \mathfrak{m}(Q'_n \cap H) \leq 2 \cdot (\#\mathbf{C}(F)) \cdot \text{vol}(Y)^{-1} \end{aligned}$$

which contradicts our assumption and proves the claim.

We now use Lemma 5.6 to move $\mathbf{g}'$ to $\mathbf{g} \in Q_{m-2m_0}$ while keeping points generic; moreover, we need $\mathbf{g}_w \neq 1$ and that $\|\mathbf{g}_{w,\lambda} - I\| \gg \|\mathbf{g}_w - I\|$ for some $\lambda \in \Phi$.

Let $E_1, E_2 \subset K_w A_{-m_0} K_w$ be so that

$$\Delta_g(E_i) = H_{y'_i} \quad \text{and} \quad |K_w A_{-m_0} K_w \setminus E_i| \leq q_w^{-5m_0 \dim \mathbf{G}}.$$

For $i = 1, 2$, let $\alpha_i \in H_{y'_i}$ and $\lambda \in \Phi$ satisfy the conclusion of Lemma 5.6, and put $\mathbf{g} := \alpha_2 \mathbf{g}' \alpha_1^{-1}$. Note that $\mathbf{g} \in Q_{n-2m_0}$, since $\mathbf{g}_v = \mathbf{g}'_v$ for any $v \neq w$, and by Lemma 5.6 we have

$$\alpha_2 \mathbf{g}' \alpha_1^{-1} = \mathbf{g}_w \in K_w[n - 2m_0] \subset G_w \times \{1\}$$

satisfies $\|\mathbf{g}_{w,\lambda} - I\| \gg \|\mathbf{g}_w - I\|$. Moreover, since $\alpha_i \in \text{Stab}(\mu)$ and $\mathbf{g}' \notin \text{Stab}(\mu)$, we have $\mathbf{g} \notin \text{Stab}(\mu)$.

Let $y_i = \alpha_i y'_i$. Since $\alpha_i \in H_{y'_i}$, we have $y_i \in Y'_{\text{good}}$. Furthermore, since $y'_2 = \mathbf{g}' y'_1$,

$$y_2 = \alpha_2 y'_2 = \alpha_2 \mathbf{g}' \alpha_1^{-1} \alpha_1 y'_1 = \mathbf{g} y_1.$$

It only remains to show that $\mathbf{g}_w \neq 1$. Assume to the contrary that $\mathbf{g}_w = 1$. Then for every $\ell \geq \lfloor n/20 \rfloor$ and $i = 1, 2$, we have

$$\left| q_w^\ell \int_{B_w(0, q_w^{-\ell})} f(a'_\lambda(t) u'_\lambda(s) y_i) \, ds - \int_X f \, d\mu \right| \leq q_w^{-\ell/8M} \mathcal{S}(f)$$

for all $f \in C_c^\infty(X)$. Now since $\mathbf{g}_w = 1$ and $a'_\lambda(t), u'_\lambda(t) \in H_{g,w} \subset G'_w$, we have

$$a'_\lambda(t) u'_\lambda(s) y_2 = a'_\lambda(t) u'_\lambda(s) \mathbf{g} y_1 = \mathbf{g} a'_\lambda(t) u'_\lambda(s) y_1$$

for all t, s . This and the above imply that $\mu(\mathbf{g}^{-1} \cdot f) = \mu(f)$. That is, $\mathbf{g} \in \text{Stab}(\mu)$ which is a contradiction. The proof is complete. $\blacksquare$

COROLLARY 7.4: *There exist some κ_{13} and L so that*

$$|\mu(b.f) - \mu(f)| \ll \text{vol}(Y)^{-\kappa_{13}} \mathcal{S}(f),$$

for every $b \in K'_w[L]$ and every $f \in C_c^\infty(X)$.

Proof. Since $K'_w[L] = K_w[L] \cdot (H_g \cap K'_w[L])$ for every $L \geq 0$ and H_g leaves μ invariant, it suffices to find κ_{13} and L so that the claim holds for all $b \in K_w[L]$.

Let $n \in \mathbb{N}$ be so that

$$\text{vol}_{G'}(Q'_n)^{-1} \leq \frac{\text{vol}(Y)}{4(\#\mathbf{C}(F))} \text{vol}(Y);$$

see Lemma 7.3. Then by that lemma, there exist $y_1, y_2 = \mathbf{g}y_1$, and $\lambda \in \Phi$ which are both $\lfloor n/20 \rfloor$ -generic for λ and the measure μ , so that $\mathbf{g} \in Q_{n-2m_0}$, $\mathbf{g}_w \neq 1$, and

$$\|\mathbf{g}_{w,\lambda} - I\| \geq q_w^{-m_0} \|\mathbf{g}_w - I\|.$$

Let $m = n - 2m_0$. Assuming n is large enough, we have $\lfloor m/10 \rfloor \geq \lfloor n/20 \rfloor$. Apply Proposition 7.2 with $c = m_0$ and these $y_1, y_2 = \mathbf{g}y_1$, and m . We thus conclude that

$$(37) \quad |u_\lambda(\tau) \cdot \mu(f) - \mu(f)| \ll q_w^{-m\kappa_{12}} \mathcal{S}(f)$$

for all $\tau \in B(0, 1)$ and all $f \in C_c^\infty(X)$.

In view of Lemma 5.2, there exist $h_1, \dots, h_\ell \in H_g$ so that every $b \in K_w[\kappa_6]$ may be written as $b = b_1 \cdots b_\ell$, where $b_i \in h_i U_\lambda[0] h_i^{-1}$ for every $1 \leq i \leq \ell$.

Since $h_i \in H_g$ leave the measure μ invariant, and we may choose h_i so that $\|h_i\| \leq 1$, the above and (37) imply the claim with $L = \kappa_6$ and $\kappa_{13} = \star \kappa_{12}$. ■

Proof of Theorem 1.1. The proof goes along the same lines as the proof of [9, Thm. 1.5]; we recall the details for the convenience of the reader.

Let $f \in C_c^\infty(X)$. Let L be as in Corollary 7.4, and let Av be the operation of averaging over $K_w[L]$. Let us also fix some $\lambda \in \Phi$ and let $\delta_{u'_\lambda(s)}$ be the delta mass at $u'_\lambda(s)$ for some $s \in F_w$. Let $\mathbb{T}_s = \text{Av} \star \delta_{u'_\lambda(s)} \star \text{Av}$, where $\star$ denotes the convolution operation. In view of Corollary 7.4 and the fact that $u'_\lambda(s) \in H_g$ leaves μ invariant, we conclude that

$$(38) \quad |\mu(f) - \mu(\mathbb{T}_s f)| \leq \text{vol}(Y)^{-\star} (\mathcal{S}(\delta_{u'_\lambda(s)} \star \text{Av} \star f) + \mathcal{S}(f)).$$

Therefore,

$$(39) \quad \begin{aligned} |\mu(f) - m_X(f)| &\leq |\mu(\mathbb{T}_s f) - m_X(f)| + \text{vol}(Y)^{-\star} (\mathcal{S}(\delta_{u'_\lambda(s)} \star \text{Av} \star f) + \mathcal{S}(f)) \\ &\leq |\mu(\mathbb{T}_s f) - m_X(f)| + \text{vol}(Y)^{-\star} (1 + |s|_w)^{4d} \mathcal{S}(f) \end{aligned}$$

where we used $\mathcal{S}(\delta_{u'_\lambda(s)} \star \mathbf{A} v * f) \leq (1 + |s|_w)^{4Nd} \mathcal{S}(f)$. We now estimate the first term on the right side of the above. First recall from (S5) that

$$\left| \mathbb{T}_s f(x) - \int f \, d m_X \right| \ll q_w^{(d+2)L} \text{ht}(x)^d \|\mathbb{T}_s\|_{2,0} \mathcal{S}(f),$$

where $\|\mathbb{T}_s\|_{2,0}$ denotes the operator norm of $\mathbb{T}_s$ on $L_0^2(X, m_X)$. Therefore,

$$\begin{aligned} |\mu(\mathbb{T}_s f) - m_X(f)| &= \left| \int \mathbb{T}_s f - m_X(f) \, d\mu \right| \\ &\leq \left| \int_{\mathfrak{S}(R)} \mathbb{T}_s f - m_X(f) \, d\mu \right| + \left| \int_{X \setminus \mathfrak{S}(R)} \mathbb{T}_s f - m_X(f) \, d\mu \right| \\ &\leq q_w^{(d+2)L} R^d \|\mathbb{T}_s\|_{2,0} \mathcal{S}(f) + q_w^* R^{-*} \|f\|_\infty \\ &\leq ((1 + |s|_w)^{-1/2M} q_w^{(3d+2)L} R^d + q_w^* R^{-*}) \mathcal{S}(f); \end{aligned}$$

in the last inequality we used $\|\mathbb{T}_s\|_{2,0} \ll (1 + |s|_w)^{-1/2M} q_w^{2dL}$, see (21).

Optimizing $|s|_w$ and R , and using the fact that $q_w \ll (\log \text{vol}(Y))^2$, we get the theorem from the above and (39). ■

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References

- [1] A. Borel and G. Prasad, *Finiteness theorems for discrete subgroups of bounded co-volume in semi-simple groups*, Institut des Hautes Études Scientifiques. Publications Mathématiques **69** (1989), 119–171.
- [2] A. Borel and T. A. Springer, *Rationality properties of linear algebraic groups. II*, Tôhoku Mathematical Journal **20** (1968), 443–497.
- [3] M. Burger and P. Sarnak, *Ramanujan duals. II*, Inventiones Mathematicae **106** (1991), 1–11.
- [4] L. Clozel, *Démonstration de la conjecture τ* , Inventiones Mathematicae **151** (2003), 297–328.
- [5] M. Cowling, U. Haagerup and R. Howe, *Almost L^2 matrix coefficients*, Journal für die reine und angewandte Mathematik **387** (1988), 97–110.
- [6] S. G. Dani and G. A. Margulis, *Limit distributions of orbits of unipotent flows and values of quadratic forms*, in *I. M. Gelfand Seminar*, Advances in Soviet Mathematics, Vol. 16, American Mathematical Society, Providence, RI, 1993, pp. 91–137.

- [7] V. G. Drinfeld, *Proof of the Petersson conjecture for $GL(2)$ over a global field of characteristic p* , Funktsional'nyi Analiz i ego Prilozheniya **22** (1988), 34–54, 96.
- [8] M. Einsiedler, E. Lindenstrauss, P. Michel and A. Venkatesh, *Distribution of periodic torus orbits on homogeneous spaces*, Duke Mathematical Journal **148** (2009), 119–174.
- [9] M. Einsiedler, G. Margulis, A. Mohammadi and A. Venkatesh, *Effective equidistribution and property (τ)*, Journal of the American Mathematical Society **33** (2020), 223–289.
- [10] M. Einsiedler, G. Margulis and A. Venkatesh, *Effective equidistribution for closed orbits of semisimple groups on homogeneous spaces*, Inventiones Mathematicae **177** (2009), 137–212.
- [11] A. Eskin, S. Mozes and N. Shah, *Unipotent flows and counting lattice points on homogeneous varieties*, Annals of Mathematics **143** (1996), 253–299.
- [12] D. Gaitsgory and J. Lurie, *Weil's Conjecture for Function Fields. Vol. 1*, Annals of Mathematics Studies, Vol. 199, Princeton University Press, NJ, 2019.
- [13] A. Ghosh, *Metric diophantine approximation over a local field of positive characteristic*, Journal of Number Theory **124** (2005), 454–469.
- [14] A. Gorodnik, F. Maucourant and H. Oh, *Manin's and Peyre's conjectures on rational points and adelic mixing*, Annales Scientifiques de l'École Normale Supérieure. Quatrième Série **41** (2008), 383–435.
- [15] A. Gorodnik and H. Oh, *Rational points on homogeneous varieties and equidistribution of adelic periods*, Geometric and Functional Analysis **21** (2011), 319–392.
- [16] H. Jacquet and R. P. Langlands, *Automorphic Forms on $GL(2)$* , Lecture Notes in Mathematics, Vol. 114, Springer, Berlin–New York, 1970.
- [17] D. A. Kazhdan, *On the connection of the dual space of a group with the structure of its closed subgroups*, Funktsional'nyi Analiz i ego Prilozheniya **1** (1967), 71–74.
- [18] D. Kleinbock, R. Shi and G. Tomanov, *S -adic version of Minkowski's geometry of numbers and Mahler's compactness criterion*, Journal of Number Theory **174** (2017), 150–163.
- [19] D. Y. Kleinbock and G. A. Margulis, *Flows on homogeneous spaces and Diophantine approximation on manifolds*, Annals of Mathematics **148** (1998), 339–360.
- [20] F. V. Kuhlmann, *Valuation theoretic and model theoretic aspects of local uniformization*, in *Resolution of Singularities (Obergrugl, 1997)*, Progress in Mathematics, Vol. 181, Birkhäuser, Basel, 2000, pp. 381–456.
- [21] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Inventiones Mathematicae **147** (2002), 1–241.
- [22] V. Lafforgue, *Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale*, Journal of the American Mathematical Society **31** (2018), 719–891.
- [23] A. Mohammadi, *A special case of effective equidistribution with explicit constants*, Ergodic Theory and Dynamical Systems **32** (2012), 237–247.
- [24] S. Mozes and N. Shah, *On the space of ergodic invariant measures of unipotent flows*, Ergodic Theory and Dynamical Systems **15** (1995), 149–159.
- [25] H. Oh, *Uniform pointwise bounds for matrix coefficients of unitary representations and applications to Kazhdan constants*, Duke Mathematical Journal **113** (2002), 133–192.
- [26] R. Pink, *Compact subgroups of linear algebraic groups*, Journal of Algebra **206** (1998), 438–504.

- [27] V. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, Pure and Applied Mathematics, Vol. 139, Academic Press, 1994.
- [28] G. Prasad, *Strong approximation for semi-simple groups over function fields*, Annals of Mathematics **105** (1977), 553–572.
- [29] G. Prasad, *Volumes of S -arithmetic quotients of semi-simple groups*, Institut des Hautes Études Scientifiques. Publications Mathématiques **69** (1989), 91–117.
- [30] M. Ratner, *On Raghunathan’s measure conjecture*, Annals of Mathematics **134** (1991), 545–607.
- [31] A. Selberg, *On the estimation of Fourier coefficients of modular forms*, in *Theory of Numbers*, Proceedings of Symposia in Pure Mathematics, Vol. 8, American Mathematical Society, Providence, RI, 1965, pp. 1–15.
- [32] R. Steinberg, *Automorphisms of classical Lie algebras*, Pacific Journal of Mathematics **11** (1961), 1119–1129.
- [33] J. Tits, *Reductive groups over local fields*, in *Automorphic Forms, Representations and L -Functions. Part 1*, Proceedings of Symposia in Pure Mathematics, Vol. 33, American Mathematical Society Providence, RI, 1979, pp. 29–69.
- [34] A. Weil, *Basic Number Theory*, Die Grundlehren der mathematischen Wissenschaften, Vol. 144, Springer, New York, 1967.