

THE OPTIMAL CONTROL OF STOCHASTIC JUMP PROCESSES :
A MARTINGALE REPRESENTATIONAL APPROACH

BY

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ABSTRACT.

Using a martingale theoretic framework, dynamic programming methods and the techniques of absolutely continuous transformation of probability measures, the optimal control problem of a certain class of stochastic jump processes is formulated and analysed. The approach followed here is in many respects similar to that used in the control of systems governed by stochastic differential equations developed over the last decade.

The sample path of the jump process x_t is characterized by a sequence of random variables $(Z_0, T_1, Z_1, T_2, Z_2, \dots)$, described by a family of non-anticipative conditional probability distributions $(\mu_k, k \in Z^+)$ which in turn determines a basic probability measure P . It turns out that the family $(\mu_k, k \in Z^+)$ is in one-one correspondence with a family of so called "local descriptions" $(\Lambda^k, \lambda^k, k \in Z^+)$ of the process. Essentially the pair determines when and where the k th jump goes respectively.

The method of control is through the absolutely continuous transformations of the family of local descriptions, achieved through a family of controlled Radon Nidodym derivatives $(\alpha_u^k, \beta_u^k, k \in Z^+)$, where:

$$\alpha_u^k = \frac{d\Lambda_u^k}{d\Lambda^k}, \quad \beta_u^k = \frac{d\lambda_u^k}{d\lambda^k}.$$

The controls $u(t, \omega)$ are assumed predictable with respect to the increasing partial observations sigma field F_t^y .

The cost function to minimise is of the form:

$$J(u) = E_u \left\{ \int_{(0, T_F]} r_0^S c(s, x, u_s) d\tilde{p}^u(s, x) + G_F(\omega) \right\}$$

where $\tilde{p}^u(t,A)$ is the predictable increasing process under measure P_u associated with $p(t,A)$, the counting process of x_t .

Using dynamic programming a very general principle of optimality is first obtained for the problem. Further necessary and sufficient conditions of optimality for both complete and incomplete observations are then derived using martingale theory and Doob-Meyer decompositions of supermartingales.

Sufficient conditions to ensure the existence of an optimal control are given for the complete observations situation..

The special Markovian situation is analysed in the above light. Finally practical examples are provided to illustrate the theory.

TO MY FAMILY

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CHAPTER 1: INTRODUCTION.

1.1. Introductory and Historical remarks.

Over the last decade considerable progress has been made in the theory of optimal control of systems governed by stochastic differential equations. Using a combination of mathematical tools old and new, various fundamental results concerning optimality conditions and existence questions were established on a rigorous basis. The old tools include the familiar concepts in dynamic programming and Ito's calculus while the new ones which precipitated results that could only be speculated upon before are like martingale representation theorems, Doob-Meyer decompositions, Girsanov's technique of transforming measures and also an updated theory of stochastic integration.

The theme of the present thesis is to extend the above success story to a different kind of control system, namely a system modeled by stochastic jump processes. Whilst the principal tools we shall use are essentially those already mentioned above, in lesser or greater forms, our approach could on the other hand be concisely stated as dynamic programming involving martingale representational methods.

This work is a continuation and extension of that started in (20). The jump process framework and the martingale representation theory involved are based on that of Davis in (15). By adding on control parameters to a pair of Radon Nikodym derivatives which determine the stochastic rate and jump size distribution, we achieve control of the process in terms of absolutely continuous transformations of probability measures. This method has

the merit of not altering the sample path structure of the process during control. The objective is to choose a control policy based on some past observations of the process, complete or incomplete, such as to minimise a prespecified control dependent cost functional. Necessary and sufficient conditions characterizing an optimal control are then derived using dynamic programming and martingale theorems.

In the rest of this section we trace briefly the development of the part of stochastic control theory relevant to our studies here. A survey of some related literature on the optimal control of jump processes is then given in the next section. In the light of all this, we give motivations for the present work and a more detailed description of the contributions and contents of this thesis in section 1.3.

Early research on stochastic control theory were mainly devoted to discrete Markovian decision processes, inventory control, statistical decision theory and suchlikes during the 50's. See for example the works of Bellman(2), Howard(36), Arrow, Karlin and Scarf (1) for expositions. One could view these as the initial reactions to the discovery of the dynamic programming principle by Bellman.

Subsequent more mature works along similar lines were carried out during the early 60's on the control of discrete parameter processes by authors like Blackwell (5), Derman(22), Dubins and Savage (24) and Jewell(38). The typical set up is to control the transition probabilities of a given discrete state Markov chain x_t .

Associated with every control action and each type of jump is a specified cost function. Optimality conditions with nice properties could then be derived using discrete dynamic programming. This state of affairs was later extended to continuous time discrete state space type of control problems. Equally successful and perhaps more elegant results were obtained. See for example Miller (46), Kakamanu(39).

Meanwhile, led on by this initial success, and also by promising results in the field of deterministic control theory, attention was being drawn to the control of more general forms of continuous time stochastic processes, mostly Markovian diffusion types at first. Some of the pioneers engaged in such work were Fleming, Nisio(30), Kushner(42) and Florentin(32). The culmination of these efforts could be found in the excellent review article of Fleming (29). The system model used here is typically expressed as a stochastic differential equation of the form:

$$\begin{aligned} dx_t &= f(t, x_t, u_t)dt + \sigma(t, x_t)dw_t. \\ x_0 &= x. \end{aligned} \tag{1.1}$$

Here $x \in R^n$ is the initial state, $u_t \in R^m$ is the control, assumed Markov, w_t is a separable n vector Brownian motion process defined on some probability space $(\Omega, \mathcal{F}, P)$, σ is an $n \times n$ matrix valued function on $[0, 1] \times R^n$ and $g: [0, 1] \times R^n \times R^m \rightarrow R^n$ is a n -vector valued function. Various smoothness and uniform elliptic type of conditions were needed on the coefficients σ, g to guarantee a solution to the equation (1.1). The objective is to choose a specified kind of Markov control u so as to

minimise a cost function of the form:

$$J(u) = E \left\{ \int_0^\tau c(t, x_t, u_t) dt \right\}$$

where τ is the exit time from some cylinder $Q \subset R^{n+1}$ and c is some measurable cost rate. Note that due to the control dependence of the sample path x_t , the partial observation σ -field $F_t^Y = \sigma\{x_t\}$ also depends on the control u . This makes variational analysis difficult as the future admissibility of controls would depend on what controls were used in the past. Another disadvantage of the above type of model is the requirement of some kind of smoothness assumptions in the dependence of the admissible controls on the observations. This is clearly undesirable for certain kinds of optimal controls, like bang-bang type.

To avoid the abovementioned shortcomings, a new approach to the optimal control of systems described by stochastic differential equations was initiated by Benes in(4) at the beginning of the 70's. Although the kind of systems equation was still the same as that in(1.1) a radically different notion of what is a solution was now introduced. Instead of using directly the transformation $w_t \rightarrow x_t$ to characterize a solution, a transformation of probability measures $P \rightarrow P_u$ was defined such that some original random variable x_t defined by:

$$dx_t = \sigma(t, x_t) dB_t. \quad (1.2)$$

generate under P_u the measure in their sample space appropriate for the solution (1.1). Here B_t is some fixed n -vector Brownian motion process on some given probability space $(\Omega, \mathcal{F}, \mu)$. The formula for this transformation of measures was originally given in Girsanov (34) and is expressed by the Radon Nikodym derivative (for $n = 1$)

$$\frac{dP_u}{dP} = \exp\left\{ \int_0^1 \sigma^{-1} g(s, x_s, u_s) dw_s - \frac{1}{2} \int_0^1 (\sigma^{-1} g(s, x_s, u_s))^2 ds \right\} \quad (1.3)$$

Using this approach, the objectionable lipschitz type of condition on the admissible control laws can be avoided. And since the sample path of x_t is now given by (1.2), it is unchanged under the transformation of measures $P \rightarrow P_u$. Hence the observations sigma field, now generalized to $\mathcal{F}_t^Y = \sigma\{f(x_t), 0 \leq s \leq t, \text{ some measurable function } f\}$, is also independent of the control law used. Benes (4) and Duncan and Varaiya (25) gave sufficient conditions to ensure the existence of a solution to (1.1). Then by imposing certain convexity conditions on $g(t, x, U)$, they separately proved the existence of an optimal control law to a fairly standard type of cost function:

$$J(u) = E \int_0^1 c(t, x, u_t) dt.$$

Davis and Varaiya (19) then continued in this direction to establish some significant Hamilton-Jacobi type of dynamic programming optimality conditions for the partially observable control system. A so called principle of optimality was first derived and using Doob-Meyer decompositions for supermartingales and martingale representation theorems, more explicit optimality conditions similar to the Hamilton-Jacobi equation of deterministic control were obtained. Some of the techniques used here originated from Rishel (49) who considered the optimal control of a very general type of continuous time stochastic system. A concept introduced in this paper known as the relative completeness of the value function was needed in (19) to establish the principle of optimality.

The Doob-Meyer decompositions and martingale representation theorems used here were developed a few years earlier by authors like Meyer (45), Kunita and Watanabe (40), Clark (14) and Wong (60).

A development parallel to that of (19) and (49) was given by Striebel in (56). Very similar types of dynamic programming optimality conditions were obtained, though instead of the relative completeness concept, she used an alternative weaker condition known as the ϵ -lattice property. We shall have occasions to use this latter concept in this thesis.

The above is then a brief outline of the development of a fair part of stochastic control theory for the last ten or fifteen years. The present work on the optimal control of jump processes is a continuation of the above trend of research in that many of the techniques used here could also be found in (19). The main difference is that while the sample path there is given implicitly by a stochastic differential equation like (1.2), the sample path of our jump process is explicitly stated, and is essentially a piecewise constant step process with arbitrary jump sizes. This more explicit structure enables us to obtain better optimality conditions. Before we give more motivations for this work and also an outline of the contents of this thesis, it is desirable to look at some previous literature on the theory of jump processes and their control. This is done in the next section.

1.2. Review of some literature on jump processes.

Early works on the control of jump processes deal with the continuous time Markovian type of situation we have mentioned in the last section. See for example Miller (46), Kakumanu (39). Most of these were concerned with finite or countable state and action spaces and use conventional Markovian tools and techniques like the policy improvement scheme of Howard (36). The results obtained therefore do not easily extend to more general kind of situations, e.g. non-Markovian with continuous state and action spaces and non-anticipative type of information structure.

This state of affairs has improved over the past few years with the appearance of several papers concerning jump process theory. The first group of these could be represented by the works of Stone(54), Pliska(48) and Rishel (50). Although the techniques used here were essentially conventional, the first two authors managed to derive better results on Markovian type of systems with continuous state space, while Rishel gave a useful necessary condition of optimality for a fairly general type of jump processes with partial observations. We shall review these works in greater detail latter on.

The second group of these papers was initiated by the pioneering work of Bremaud (11) who presented a martingale theoretic approach to the analysis of point processes. By exploiting the analogies between point processes and the Brownian motion process, he obtained elegant results on likelihood ratios and estimation

theory very similar to those of Brownian motion already in vogue. This initiative was immediately taken up by several groups of authors: Boel, Varaiya and Wong (9), Chou and Meyer (13), Jacod (37) and Davis and Elliot (20). They separately derived the important martingale representation type of results for general jump processes. So as in the Wiener process situation, every martingale of a jump process can be represented by a stochastic integral of the jump process. The link between abstract theory and concrete applications was thus established and the stage set for problems like the optimal control of a jump process.

This task was first pursued by Boel (7) who in his thesis used the martingale framework of (9) to formulate and solve the optimal control problem of a general jump process. His approach and the tools used were essentially those in (19). However, due to the imprudent use of such tools which were more suited to continuous path type of processes, this initial work published in (8) was unnecessarily abstract and heavy in flavour of the L^2 - theory of (19). We shall also review this work in greater detail later on.

Meanwhile a more direct approach to the control problem was adopted by Davis and Elliot in their paper(20). Though many of the techniques used there still originated from (19), more emphasis was being given to the piecewise constant path structure of the jump process. The optimality conditions obtained thus have greater resemblance to previous literature on the control of jump processes.

The present work is a continuation and extension

of that in (20). The spirit is to exploit the powerful martingale theory developed over the last decade while retaining the distinctive characteristics of the jump process under consideration.

Let us now review in greater detail the works of Pliska (48), Stone (54), Rishel (50) and Boel and Varaiya (8). These will be needed in the future for comparison with our work.

(i) Pliska.

The part of Pliska's paper relevant to our study here considers the optimal control of a Markov jump process with general state and action spaces over a finite planning horizon. The results obtained there were thus related and extend those of others like Miller, Blackwell and Kakumanu on continuous time Markov decision theory.

Pliska's approach to the formulation of the controlled jump Markov process was through indexing a certain linear operator $A(f)$ with the decision rule or control action f . $A(f): B \rightarrow B$ corresponds to the infinitesimal generator of the Markov process under action f where B is the Banach space of all bounded borel measurable real valued functions on S , the state space. $A(f)$ is defined by:

$$A(f)v(s) = -\lambda(s, f(s))\{v(s) - \int_S v(z)Q(dz/s, f(s))\}. \quad (1.4)$$

where $s \in S$, $v \in B$, $f(s)$ is the decision rule mapping the state space S into a certain set of admissible control action r_s for each $s \in S$, Q is a sub-Markov kernel that determines the conditional state distribution and is also indexed by f , $\lambda(s, f(s))$ is the exponential holding rate

of the jump process under action f while in the state s .

Defining M to be the set of measurable control policies $\pi: R^+ \rightarrow F$, where F is the set of all the decision rules $\{r_s, s \in S\}$, Pliska showed that a Markov jump process with infinitesimal generator $A(\pi_t)$ exists for each $\pi \in M$.

The expected cost criteria used over the time horizon T is defined to be $v_T(\pi)$, where

$$v_t(s) = E_s \int_{T-t}^T r(s_\tau, \pi_t(s_\tau)) d\tau + \int_S P(T-t, T, s, dz) v(z) \quad (1.5)$$

and $\pi \in M$, r is the bounded, real valued reward rate and $P(.,.,.,.)$ is the transition function of the Markov process corresponding to the infinitesimal generator $A(\pi_t)$.

The control problem is to maximise the expected reward, i.e. to seek a $\pi^* \in M$ such that:

$$\hat{v}_T = v_T(\pi^*) = \sup_{\pi \in M} v_T(\pi) \quad (1.6)$$

One of the main results of the paper was in showing that $\hat{v}_T$ is the solution to the differential equation:

$$\dot{v}_t = \max_{f \in F} \{A(f)v_t + r(f)\} \quad (1.7)$$

$$v_0 = v \in B', \quad B, \quad 0 \leq t \leq T.$$

where F is an admissible class of decision rules.

The other main result is the statement of a necessary and sufficient condition for optimality. Thus $\pi \in M$ is optimal if and only if:

$$A(\pi_{T-t})\hat{v}_t + r(\pi_{T-t}) = \max_{f \in F} \{A(f)\hat{v}_t + r(f)\} \quad (1.8)$$

a.e. on $[0, T]$. The same holds if $\hat{v}_t$ is replaced by $v_t(\pi)$

Finally it is also worth noting that Pliska

gave a sufficient condition for the existence of an optimal policy in the sense of (1.6)

Remarks 1.1.

Though concise and neat, the above approach is not easily extended to more general kinds of situations with perhaps non-stationary rates and kernels and a more encompassing information structure. However it should be noted for its success in improvement on the Markovian decision theory of the last decade.

(ii) Stone.

A different kind of stochastic process is considered in Stone's formulation. It is the so called semi-Markov jump process and is roughly determined as follows:

Let $\{x_t, t \geq 0\}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{A}, P)$ and a state space Σ .

Let

$$Y_t = \begin{cases} t & \text{if } x_s = x_t, \text{ all } 0 \leq s \leq t. \\ t - \sup\{s: 0 \leq s \leq t, x_s \neq x_t\}, & \text{otherwise.} \end{cases}$$

Then $\{x_t, t \geq 0\}$ is a semi-Markov jump process if the two dimensional process $\{(X_t, Y_t), t \geq 0\}$ is a strong Markov process with stationary transition probabilities. See (55) for a more precise definition.

Thus the future evolution of the semi-Markov jump process at time t depends on the complete information of its sample path from the last jump epoch up to the present time t .

His method of control was through choosing a pair of measurable functions which determined the rate and condi-

-tional state jump distribution of the jump-process. The analysis was carried out in terms of the infinitesimal generator of the two dimensional stationary Markov process. This approach is hence in some ways similar to Pliska's though the underlying processes were not exactly identical.

The cost functional used in Stone's paper was of the form:

$$\phi_u(z) = E_z \left\{ \int_0^\tau e^{-\lambda t} c(Z_t^u, u(Z_t^u)) dt + e^{-\lambda \tau} c(Z_\tau^u) \right\}$$

where $z \in \Sigma \times R^+$, $\lambda > 0$ is some exponential discounting factor, τ is a random stopping time of $\{x_t^u\}$. c and C are Borel measurable functions and u is the control defined on $\Sigma \times R^+$ to some specified control space. The control problem is to seek a control $u^* \in U'$ such that

$$\phi_{u^*}(x, s) = \min_{u \in U'} \phi_u(x, s), \quad \text{all } (x, s) \in \Sigma \times R^+.$$

where u' is a specified set of controls.

By using dynamic programming and the infinitesimal generator characterization of the semi-Markov process, Stone derived some necessary and sufficient conditions of optimality very similar to those of Pliska's, though the technical details of the former seemed to be more involved.

Remarks 1.2.

The techniques and approach being basically the same as Pliska's, Stone's work therefore also contains the same disadvantages, i.e. not being easily extended to more general problems. On the other hand both models are sufficient for most practical applications.

(iii) Rishel.

Rishel's approach to the problem is an original one in that the theory used is essentially self contained and involved no Markovian or martingale methods. The jump process characterization is in terms of the conditional rate $q(a/X_j)$ and the conditional state jump distribution $\pi(dx_{j+1}/\tilde{X}_j)$, where $X_j = (x_0, s_1, x_1, \dots, s_j, x_j)$, $\tilde{X}_j = (x_0, s_1, x_1, \dots, s_j, x_j, s_{j+1})$ and $\{x_0, s_1, x_1, \dots, s_j, x_j, \dots\}$ is the discrete sequence of states and interarrival times which generate the jump process.

The first part of the paper used this characterization of the jump process to derive certain representation formulas of functionals of the process and of conditional expectations of such functionals. These are useful for controlling purposes, since, for example, the conditional expected remaining cost functional would have such a useful and explicit representation.

Part II of the paper constructed a controlled system of jump process by parametrizing the conditional rate and state jump distribution with the control variable u . Thus they are now denoted as $q(a, u/X_j)$ and $\pi(dx_{j+1}, u/\tilde{X}_j)$ respectively. The remaining cost functional to minimise is defined as:

$$J(u)(t) = E\left\{\int_t^T f(s, x_s, u_s) ds + g(x_T)/\sigma_t\right\}$$

where σ_t is the observation sigma field which is in general incomplete.

The main result, known as the minimum principle, was a set of necessary conditions for optimality and was derived by differentiating the cost performance under a control that slightly departed from the original one.

Hence the methods used were essentially perturbational analysis. An infinite system of differential equation was also derived, analogous to the adjoint equation of the Pontryagin's maximum principle in deterministic control. The main statement of the minimum principle ran as follows:

For $u(t)$ to be optimal, it is necessary that for each $t \in [0, T]$, $v \in U$, we have:

$$\min_{v \in U} E\{H(t, v)/\sigma_t\} = E\{H(t, u_t)/\sigma_t\} \quad (1.9)$$

where U is the control value space and $H(t, u_t)$ is a certain "Hamiltonian" function. The adjoint equation associated with (1.9) is:

$$\frac{dJ(u)(t)}{dt} = -H(t, u_t). \quad (1.10)$$

Remarks 1.3.

The results of Rishel are in some respects generalizations of the Markovian decision theory of Pliska and Stone, though the methods and approach are entirely different. One important difference is that while the optimality conditions of Pliska and Stone are necessary and sufficient, Rishel's condition is only a necessary one. This also accounts for the unusual explicitness of the minimum principle even when the observations are incomplete.

(iv) Boel and Varaiya.

The approach Boel and Varaiya used is closely related to ours. Both were inspired by the martingale theoretic type of results obtained in (19).

They first derived some very general necessary and sufficient dynamic programming conditions for the optimal control of a wide class of stochastic processes which also included the class of jump processes. These rather abstract conditions were then applied to the special case

of a jump process to obtain more significant results.

The method of control they adopted was through the change of probability measures. A stochastic jump process $(x_t, \mathcal{F}_t, P_0)$ was first defined over some given interval I on a given probability space $(\Omega, \mathcal{F}_0, P_0)$. The control law u has the effect of changing this basic process to $(x_t, \mathcal{F}_t, P_u)$ without altering the sample path of the process. P_u would in normal circumstances be mutually absolutely continuous with respect to the basic measure P_0 , though this was not required in the paper, achieved through considering all relevant processes and quantities up to null sets of P_u . The sigma field of observations was $\mathcal{F}_t^Y \subset \mathcal{F}_t$ and assumed to be increasing with time. The optimal control problem was to choose a control policy out of the class of $\mathcal{F}_t^Y$ predictable controls $\mathcal{U}$, such as to minimise the cost $J(u)$ given by:

$$J(u) = E_u \left\{ \int_0^{T_f} r_0^s c(s, x, u_s) d\tilde{p}^u(s, x) + r_0^{T_f} J_f \right\} \\ (0, T_f] \times X$$

Here c is the bounded cost rate, r the discount factor, $\tilde{p}^u(t, A)$ is the predictable increasing process associated with the jump process under measure P_u . J_f is the terminal cost, assumed $\mathcal{F}_{T_f}$ measurable.

The necessary and sufficient conditions for optimality obtained were similar in structure to those of (19), one form of which was a kind of stochastic analogue of the Hamilton-Jacobi equation, derived via martingale representation theorems. A special situation with Markov structure and observations was also considered, but due to the apparent failure of the relative completeness property, an extra assumption concerning discrete appro-

-ximation of a Markov control had to be introduced. Hence the Markov results obtained were not entirely satisfactory.

Remarks 1.4.

The work of Boel and Varaiya is probably the first attempt using the martingale theoretic approach in the control of a jump process. However, as we had remarked earlier, the L^2 flavour is a bit too strong, resulting in abstract conditions not easily related to practical circumstances. Another major shortcoming is the requirement that all jump times of the process should be totally inaccessible stopping times of $\mathcal{F}_t$. The reason is that the martingale representation theorems of (9) would otherwise be invalid. This means that certain jump process phenomena with discontinuities in the jump time distributions could not be modeled by the above approach.

1.3. Motivations, contributions and outline of thesis.

Systems that could be modeled by jump processes occur widely in the physical world, ranging from natural phenomena like gamma ray detection and sun spot activities to more mundane human affairs like inventory control and traffic regulation. It is hence desirable to formulate a sound mathematical model flexible enough to incorporate the bulk of these systems and to develop theory related to their control. This forms the basic motivation for the present work. As to the reasons for adopting the martingale approach, we believe they are:

(i). The martingale framework allows the optimal control problem to be formulated in its greatest generality.

The increasing information structure also fits nicely with the concept of $\mathcal{F}_t$ adaptability of the basic martingales of the jump process. We thus expect elegant results like those obtained in systems described by stochastic differential equations.

(ii) Powerful theorems on Martingale representation and supermartingale decompositions developed over the last decade were especially successful in the analysis of Brownian motion type of processes. It is hence logical to apply them to jump processes. Due to its inherently more explicit sample path structure, one should expect even more straightforward results.

(iii). Boel and Varaiya had used a similar approach, but due to the overuse of L^2 type of techniques and conventions more suited to Brownian motions, the results they obtained were unnecessarily abstract. This work plans to avoid such pitfalls and to present an overall picture more akin to the traditional results of Markov decision problems. Previous results could then be more easily related to ours enabling future research on such matters to have a better foundation.

Contributions of thesis.

The contribution of this work towards the optimal control of stochastic jump processes could be viewed as two fold.

(i). A significant improvement and extension of some of the previous results of Davis and Elliot (20). A much more integrated and concise outlook is presented. The details are as follows: Whilst the proofs of most of the results in (20) are based on the elementary one jump

situation, the main results here are derived directly for the general situation. This eliminates the need for extrapolation of results and hence reduces the possibilities of unforeseen negligence. This more direct approach also throws more light on the structure of the general jump process and on such matters as the absolutely continuous transformation of rates. It turns out that the mathematical description of the multijump process can be made to resemble closely its one jump counterpart. This is advantageous towards the simplification of concepts and notations.

Another improvement over the results in (20) is that in the proof of the principle of optimality we discarded the rather cumbersome relative completeness property and instead used a simpler concept called the ϵ -lattice property. Considerable effort is saved in the verification of the simpler property.

Finally in the analysis of the optimal control problem we made use of a stochastic dynamic programming lemma to reduce the problem of seeking optimal controls on a fixed interval to one of seeking optimal controls between two jump epochs. Again this is helpful in conceptual visualization and notational simplicity.

(ii). New results are obtained in several areas, mainly those concerning partial observations, the existence question and the Markov situation. In the partial observation situation, we derived formulas for the local descriptions of the observed process and reformulated the control problem in terms of these local descriptions. The problem and results obtained thus resemble those in the complete observations situation, except that one is now also confronted with complicated projections and with the nuisance of past control dependence of such projections.

For the rest of the new results, see outline of thesis.

Outline of Thesis.

In Chapter 2 we pave the foundation of the control problem by constructing the general jump process in the manner of (15). Related theorems on martingale representations and formulas for the absolutely continuous changes of rates and measures are stated. The procedure is normally to do it for the simple one jump process and then extend it to the general case. The jump process will be characterized by the so called 'local descriptions' (Λ, λ) which determine when and where a jump goes. Λ_t , the integrated jump rate of the basic jump process will be allowed to have discontinuities, meaning that the jump times are not required to be totally inaccessible stopping times of $F_t = \sigma\{x_s, 0 \leq s \leq t\}$. This is hence an improvement of the situation in (8). The results of this chapter are mainly extensions of that in (20).

Chapter 3 begins with the formulation of the general optimal control problem. The information pattern used F_t^Y is assumed to be incomplete and increasing. i.e. $F_t^Y \subset F_t$ and $F_s^Y \subset F_t^Y$ for $s < t$. The controls used are assumed to be F_t^Y predictable. Using a concept known as the ϵ -lattice properly, a fundamental theorem known as the principle of optimality is then derived using dynamic programming. This roughly states that the minimal expected cost after time t is smaller than the expected value, given the information at t , of the cost of using any control law on $[t, t+h)$ and thereafter returning to an optimal control. Further significant necessary and

sufficient conditions of optimality are then derived using martingale theory for both the case of complete and incomplete informations. Only in the complete information situation do we find a true Hamilton-Jacobi type of equation.

In the chapter 4 we give sufficient conditions to ensure the existence of an optimal control law in the case of complete observations. The approach used here is essentially that of Davis (16), the advantage being not requiring any sort of convexity assumptions in the controlled rates. It is hence different from the approaches of Benes (4) and Duncan and Varaiya (25).

Chapter 5 looks at the special situation when the jump process and its cost structure are Markovian. A version of the principle of optimality is proved and optimality conditions similar to that of Chapter 3 derived. The main result is to show that a Markovian optimal control is also optimal in the class of controls based on complete observations. Existence results similar to that of Chapter 4 are then obtained. The relation of these results to some of the past literature on Markov jump process control is exemplified by comparing our work with Pliska's and Stone's. Finally an interpretation of the Markov results in terms of infinitesimal generators is given.

Chapter 6 provides several examples in real life to which the above theory can be applied. It also contains the conclusion of the thesis.

CHAPTER 2: THE GENERAL JUMP PROCESS AND RELATED THEOREMS.

A class of jump process is formulated in this chapter and related theorems derived in preparation for the control problem analysed in future chapters. In our approach, we shall exploit the piecewise constant path property of the jump process. This property ensures that the $\text{jump}_{\text{process}}$ is well defined once we specify a countable set of random variables $(T_i, Z_i, i=1,2,\dots)$ and their joint distributions, provided the T_i 's do not have multiple accumulation points. Being more constructional and attentive to details, we feel that our approach has greater practical appeal over the more abstract approach of others like (8) who started off by naming a complicated probability measure on some abstract measurable space. On the other hand, like (8), we shall be concerned with tools like the martingale representation theorem and formulas for absolutely continuous changes of measures. These, as we shall see, are useful in the analysis of the control problem.

2.1. Definitions.

The basic jump process $(x_t, t \geq 0)$ takes values in a measurable space (X, S) which is also assumed to be a Blackwell space. This is not restrictive since most spaces have this property. (e.g. complete separable metric spaces) See (Meyer, 45 IIID15). Let z_0, z_∞ be fixed elements of X and for $i=1,2,\dots$, let $(Y^i, \mathcal{Y}^i)$ denote a copy of the measurable space

$$(Y, \mathcal{Y}) = ((R^+ \times X) \cup \{(\infty, z_\infty)\}, \sigma\{B(R^+) * S, \{(\infty, z_\infty)\}\}).$$

$$\text{Define } \Omega = \prod_{i=1}^{\infty} Y^i, \quad \mathcal{F}^0 = \sigma\left\{ \prod_{i=1}^{\infty} \mathcal{Y}^i \right\}.$$

Let $(T_i, Z_i): \Omega \rightarrow Y^i$ be the coordinate mapping and $\omega_k: \Omega \rightarrow \Omega_k$

where $\Omega_k = \prod_{i=1}^k Y^i$, is the projection operator onto Ω_k .

i.e.

$$\omega_k(\omega) = (T_1(\omega), Z_1(\omega), \dots, T_k(\omega), Z_k(\omega)).$$

Now set

$$T_\infty(\omega) = \lim_{k \rightarrow \infty} T_k(\omega).$$

and define the path of x_t by:

$$x_t(\omega) = \begin{cases} z_0 & \text{if } t < T_1(\omega), \\ Z_i(\omega) & \text{if } t \in [T_i(\omega), T_{i+1}(\omega)), \\ z_\infty & \text{if } t \geq T_\infty(\omega). \end{cases} \quad (2.1)$$

Thus the process starts off at an initial position z_0

and at the random time T_1 jumps to the random state Z_1 .

At the random time T_2 the process jumps to Z_2 and so on.

$T_\infty(\omega)$ is the accumulation time of the random times $(T_k, k=1, 2, \dots)$, and may be finite or infinite. z_∞ is the fixed

terminal state of the jump process. x_t is then right

continuous and have piecewise constant sample paths.

The random process x_t generates its own increasing family of sigma fields (F_t^0) given by:

$$F_t^0 = \sigma\{x_s, s \leq t\}.$$

We now construct a probability measure P on (Ω, F^0) by defining the following family of conditional distribution functions: μ_1 is a probability measure on $(Y^1, \mathcal{Y}^1)$ such that:

$$\mu_1((\{0\} \times X) \cup (R^+ \times \{z_0\})) = 0.$$

For $i=2, 3, \dots$, $\mu_i: \Omega_{i-1} \times \mathcal{Y} \rightarrow [0, 1]$ is a function such that:

(i) $\mu_i(\cdot, \Gamma)$ is measurable for each $\Gamma \in \mathcal{Y}^i$.

(ii) $\mu_i(\omega_{i-1}(\omega); \cdot)$ is a probability measure on $(Y, \mathcal{Y})$ for each $\omega \in \Omega$.

- (iii) $\mu_i(\omega_{i-1}(\omega); R^+ \times \{Z_{i-1}(\omega)\}) = 0$, for each $\omega \in \Omega$.
- (iv) $\mu_i(\omega_{i-1}(\omega); \{(\infty, z_\infty)\}) = 1$, if $T_{i-1}(\omega) = \infty$, each $\omega \in \Omega$.
- (v) $\mu_i(\omega_{i-1}(\omega); (0, t] \times X) = 0$, for $t \leq T_{i-1}(\omega)$, each $\omega \in \Omega$.

The conditions (iii) and (v) ensure that two successive jump times do not occur simultaneously and that the process does have an effective jump at each jump time.

The probability measure P on (Ω, F^0) is then defined as follows: for $\Gamma \in \mathcal{F}$ and $\eta \in \Omega_{i-1}$, let

$$\begin{aligned} P\{(T_1, Z_1) \in \Gamma\} &= \mu_1(\Gamma) \\ P\{(T_i, Z_i) \in \Gamma \mid \omega_{i-1} = \eta\} &= \mu_i(\eta; \Gamma), \quad i=2,3,\dots \end{aligned} \quad (2.1a)$$

Denote by F_t (F resp.) the sigma field obtained by augmenting F_t^0 (F^0 resp.) with all null subsets of P null sets of F^0 .

With the above set up it can now be shown that:-

- (i) $(T_i, i=1,2,\dots)$ are stopping times of the sigma field F_t .
- (ii) T_∞ is a predictable stopping time of F_t .
- (iii) $F_\infty = F$, where $F_\infty = \bigvee_{t \geq 0} F_t$.
- (iv) $\bigvee_n F_{T_n} = F_{T_\infty-} = F_{T_\infty} = F$.
- (v) $F_{(T_{k-1}+t) \wedge T_k} = F_{T_{k-1}} \vee \sigma\{x(T_{k-1}+s) \wedge T_k, s \in [0, t]\}$, for all $t \geq 0$.

Remarks 2.1

These properties of the stopping times and stopped sigma fields are proved in (15). Note also that $(T_i, i=1,2,\dots)$ are F_t stopping times because of assumptions (iii) and (iv) of the conditional distributions $(\mu_i, i=1,2,\dots)$ ensuring that we have 'genuine' jumps.

2.2. Martingale representational results: the single jump case.

It turns out that most of the significant results on general jump processes can be obtained from the special situation in which we have only one jump. Since this affords us with the simplification of notations, we begin by considering the on jump case and then deducing the corresponding results for the general case by means of concatenation.

In this section we shall describe the "local descriptions" of the jump process and state the martingale representation theorem associated with it.

The process x_t described in section 2.1 has only one jump if:

$$\mu_2(n; \{(\infty, z_\infty)\}) = 1, \text{ for all } n \in Y^1.$$

i.e.

$$T_2(\omega) = \infty. \quad \text{w.p.1.}$$

It can also be constructed directly by: $(\Omega, F^0) = (Y, \mathcal{F})$ and $P = \mu_1$, where as before, we require that:

$$\mu_1(R^+ \times \{z_0\}) = 0.$$

$$\mu_1(\{0\} \times X) = 0.$$

to ensure a genuine jump at each epoch.

The sample path of x_t is then:

$$x_t = \begin{cases} z_0 & \text{if } t < T_1(\omega), \\ z_1(\omega) & \text{if } t \geq T_1(\omega). \end{cases}$$

F_t is defined as before. For the remainder of this section, we shall drop the subscripts of T_1, Z_1, μ_1 and denote them respectively as T, Z, μ .

For $A \in \mathcal{S}$, define

$$F_t^A = \mu((t, \infty] \times A), \quad F_t = F_t^X, \quad c = \inf\{t: F_t = 0\}. \quad (2.2)$$

Then in view of its definition F_t is right continuous and monotonic decreasing. Hence, by standard analysis, there are only countably many points of discontinuity of F_t over the interval $[0, c)$. At each point of discontinuity $s \in [0, c)$, denote

$$\Delta F_s = F_s - F_{s-}.$$

F_t specifies the marginal distribution of T , i.e.

$$F_t = P(T > t).$$

The probability measure μ specifies the stochastic evolution of the one jump process x_t . However, from an application's point of view, it may be more convenient to specify this by the so called "local descriptions" of the jump process. This essentially is a pair of entities that determine conditionally when and where the jump occurs, given past information. It is also closely related to the "Levy systems" for Hunt processes. See (62). For the one jump process, the local descriptions is derived as follows:

For each $A \in \mathcal{S}$, the measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ defined by $F_\cdot^A$ is absolutely continuous w.r.t. that defined by $F_\cdot$. Hence by the Radon Nikodym theorem, there exists a non-negative measurable function $\lambda(A, \cdot)$ such that:

$$F_t^A - F_0^A = \int_{(0, t]} \lambda(A, s) dF_s. \quad (2.3)$$

So $\lambda(A, s) = \frac{dF_s^A}{dF_s}$, and it is easy to see that $\lambda(A, s)$ can be interpreted as $\text{prob}(Z \in A / T = s)$.

A regular version of this conditional probability can be

shown to exist by the Blackwell property of (X, s) . Thus $\lambda(., s)$ is a probability measure for each s . Now define:

$$\Lambda(t) = - \int_{(0, t]} dF_s \quad (2.4)$$

$$\tilde{\Lambda}(t) = \Lambda(t \wedge T)$$

The pair (Λ, λ) is what we called the local description of the jump process, and is due to the interpretation:

$$d\Lambda_t = P\{ T \in (t, t+dt] \mid T > t \}$$

$$\lambda(A, t) = P\{ Z \in A \mid T = t \}$$

One can verify directly from their definitions.

that the pair of functions (Λ, λ) have the following properties:

- (i) $\Lambda(t)$ is defined on $[0, c)$.
- (ii) $\Lambda(0) = 0$, and $\Lambda(t)$ is increasing and right continuous.
- (iii) $\Delta\Lambda_s = \Lambda_s - \Lambda_{s-} \leq 1$ for all points of discontinuities.
- (iv) If $\Delta\Lambda_s = 1$, then $\Lambda_t = \Lambda_s$ for $t \geq s$.

(2.5)

- (v) $\lambda(A, s) \geq 0$ for $A \in \mathcal{S}$, $s > 0$.
- (vi) $\lambda(A, .)$ is measurable for each $A \in \mathcal{S}$.
- (vii) For all $s \in (0, c)$, except on a set of $d\Lambda$ -measure 0, $\lambda(., s)$ is a probability measure on (X, s) and $\lambda(., c)$ is a probability measure if $c < \infty$ and $\Lambda(c-) < \infty$.

Now by (ii), we can decompose $\Lambda(t)$ into:

$$\Lambda(t) = \Lambda^c(t) + \Lambda^d(t)$$

where $\Lambda^c(t)$ is increasing and continuous and $\Lambda^d(t) =$

$$\sum_{s \leq t} \Delta\Lambda(s) .$$

The following theorem then shows how Λ_t is related to F_t .

Theorem 2.1.

There is a bijective correspondence between the set of probability measures μ on $(Y, \mathcal{Y})$ and the set of local descriptions (Λ, λ) satisfying the properties (2.5) (i)-(vii). Further, F_t is related to Λ_t by:

$$\begin{aligned} F_t &= \exp(-\Lambda_t^c) \prod_{s \leq t} (1 - \Delta \Lambda_s), & t < c \\ F_t &= 0, & t \geq c. \end{aligned} \quad (2.6)$$

Proof:

It is already clear how the pair (Λ, λ) is obtained from μ . The converse is obtained by defining F_t by (2.6) and for each $A \in \mathcal{S}$, define

$$\mu((0, t] \times A) = - \int_{(0, t]} \lambda(A, s) dF_s.$$

For a derivation of (2.6) see for example (37).

Remarks 2.2.

When extended to a general jump process, theorem 2.1 enables one to model such processes in terms of their local descriptions. This is a great advantage since in most practical situations, jump processes are specified by dynamical rates and conditional jump distributions rather than the abstract measure P .

For the single jump process, the basic family of martingales is obtained as follows:

For $A \in \mathcal{S}$, $t \in \mathbb{R}^+$, define:

$$\begin{aligned} p(t, A) &= I_{(t \geq T)} I(Z \in A) \\ \tilde{p}(t, A) &= - \int_{(0, t \wedge T]} \frac{dF_s^A}{F_{s-}} \end{aligned} \quad (2.7)$$

$$q(t, A) = p(t, A) - \tilde{p}(t, A).$$

Lemma 2.2.

$(q(t,A))_{t \geq 0}$ is a (F_t, P) martingale.

Proof: See (15) for details.

To consider the question of martingale representation, we need first to form stochastic integrals with respect to the family of martingales $q(t,A)$. First let us define a class of suitable integrands. Let $\mathcal{I}$ denote the set of measurable functions $g: Y \rightarrow \mathbb{R}$ such that $g(\infty, z) = 0$, for all $z \in X$. Now since for fixed (t, ω) the functions $p(t,A)$ and $\tilde{p}(t,A)$ are countably additive in A , we can define Stieltjes integrals of the form:

$$\int_{(0,t] \times X} g(s,z) p(ds,dz) \quad , \quad \int_{(0,t] \times X} g(s,z) \tilde{p}(ds,dz)$$

for suitable $g \in \mathcal{I}$.

Explicitly, one has

$$\begin{aligned} \int_{(0,\infty] \times X} g(s,x) p(ds,dx) &= g(T,Z) \\ \int_{(0,\infty] \times X} g(s,x) \tilde{p}(ds,dx) &= \int_{(0,\infty] \times X} I_{(s \leq T)} g(s,x) \frac{d\mu(s,x)}{F_{s-}} \end{aligned}$$

The stochastic integral $\int g dq$ is now defined as:

$$\int_{(0,t] \times X} g(s,x) dq(s,x) = \int_{(0,t] \times X} g(s,x) \{p(ds,dx) - \tilde{p}(ds,dx)\} \quad , \quad (2.8)$$

the difference of two Stieltjes integrals.

The following classes of integrands are required:

$$L^1(p) = \{g \in \mathcal{I} : E \int_{R^+ \times X} |g(s,x)| p(ds,dx) < \infty \}$$

$$L^1(\tilde{p}) = \{g \in \mathcal{I} : E \int_{R^+ \times X} |g(s,x)| \tilde{p}(ds,dx) < \infty \}$$

$$L^1_{loc}(p) = \{g \in \mathcal{I} : g I_{(t < \sigma_k)} \in L^1(p), k=1,2,\dots, \text{ for some sequence of } F_t \text{ stopping times } \sigma_k < T_\infty, \sigma_k \uparrow T_\infty \text{ a.s. } P\}$$

$L_{loc}^1(\tilde{p})$ is defined analogously. It turns out that:

$$(i) \quad L^1(p) = L^1(\tilde{p}) = \{g \in \mathcal{I} : \int_{R^+ \times X} |g(s,x)| d\mu(s,x) < \infty\}$$

$$(ii) \quad L_{loc}^1(p) = L_{loc}^1(\tilde{p}) = \{g \in \mathcal{I} : \int_{R^+ \times X} 1_{(s \leq t)} |g(s,x)| d\mu(s,x) < \infty, \forall t < c\}$$

See (15) Prop. 4 for proofs.

Suppose M_t , $t \geq 0$, is a uniformly integrable martingale of $\mathcal{F}_t$ such that $M_0 = 0$ a.s. Then there exists some measurable $h \in \mathcal{I}$ such that $M_t = E\{h(T,Z) / \mathcal{F}_t\}$ a.s. By direct evaluation, we have:

$$M_t = I_{(t \geq T)} h(T,Z) - I_{(t < T)} \frac{1}{\mathcal{F}_t} \int_{(0,t] \times X} h(s,x) d\mu(s,x) \quad (2.9)$$

Now for $g \in L_{loc}^1(p)$ define

$$M_t^g = \int_{(0,t] \times X} g(s,x) q(ds,dx) \quad (2.10)$$

as the difference of Stieltjes integrals. The M_t^g is a $(\mathcal{F}_t, P)$ local martingale. One of the main results in (15) states that all $(\mathcal{F}_t, P)$ local martingales have the form (2.10), as the following theorem shows.

Theorem 2.3.

M_t is a $(\mathcal{F}_t, P)$ local martingale if and only if $M_t = M_t^g$, for some $g \in L_{loc}^1(p)$.

Whereas in most martingale representation theorems dealing with continuous path processes, integrands analogous to g have no explicit representations, such a representation exists here. If we take M_t to be as defined by (2.9), then

$$\begin{aligned} g(t,x) &= h(t,x) + I_{(t < c)} \frac{1}{\mathcal{F}_t} \int_{(0,t] \times X} h(s,x) d\mu(s,x) \\ g(\infty, x) &= 0. \end{aligned} \quad (2.11)$$

See (15), remarks after Prop. 5.

Remarks 2.3.

The representation (2.11) is possible because the sigma field $\mathcal{F}$ is generated by the random variables (T, Z) , allowing all u.i. martingales to be expressed as (2.9). This is certainly not the case for Brownian motions, since one needs an infinite number (countable) of r.v.'s to generate the sigma field $\mathcal{F}$. This is perhaps an indication that one should not solely rely on methods used with stochastic differential equations when dealing with jump processes. Formula (2.11) will be used in latter chapters.

2.3. Absolutely continuous changes of rates: the single jump case.

In this section we summarise some results of (20) concerning the single jump process when we perform absolutely continuous changes of rates. The main result is that if we change the local descriptions of the process from (Λ, λ) to $(\bar{\Lambda}, \bar{\lambda})$ such that $\bar{\Lambda} < \Lambda$, $\bar{\lambda} < \lambda$, then the measure $\bar{\mu}$ corresponding to $(\bar{\Lambda}, \bar{\lambda})$ is also absolutely continuous with respect to μ . The converse situation is also true, i.e. if $\bar{\mu} < \mu$, then $\bar{\Lambda} < \Lambda$, $\bar{\lambda} < \lambda$. The former result will be used latter to formulate 'controlled rates' for the jump process.

Recall that the 'base rate' Λ_t has the decomposition:

$$\Lambda_t = \Lambda_t^c + \Lambda_t^d$$

Lemma 2.4.

Suppose we are given another integrated rate $\bar{\Lambda}_t$ satisfying the (i)-(iv) of (2.5) and such that $\bar{\Lambda}_t < \Lambda_t$, then the $\bar{F}_t$ associated with $\bar{\Lambda}_t$ has the form:

$$\bar{F}_t = F_t \exp\left(-\int_0^t (\alpha(s)-1) d\Lambda_s^c\right) \Pi_t \quad (2.12)$$

where

$$\alpha(s) = \frac{d\bar{\Lambda}(s)}{d\Lambda(s)} \quad (2.12)$$

$$\pi_t = \prod_{s \leq t} \frac{(1 - \alpha(s)\Delta\Lambda_s)}{(1 - \Delta\Lambda_s)} \quad (2.13)$$

Proof: See (20).

Remarks 2.4.

Clearly $\bar{F}_t \ll F_t$. If we require the converse, i.e. $F_t \ll \bar{F}_t$, a sufficient condition is to restrict $\alpha(s)$ so that

$$\frac{1}{n} \leq \alpha(s) \leq \min \left(n, \frac{-(1-\frac{1}{n})F_{s-}}{\Delta F_s} \right)$$

for any positive integer n . This condition will also ensure that Λ_t and $\bar{\Lambda}_t$ are mutually absolutely continuous. We shall require this property in deriving optimality criteria for the control problem and also in showing the existence of optimal controls.

Theorem 2.5.

Suppose the probability measures $\bar{\mu}, \mu$ on $(\Omega, \mathcal{F})$ have local descriptions (Λ, λ) and $(\bar{\Lambda}, \bar{\lambda})$ respectively with $\bar{\Lambda}_t \ll \Lambda_t$ on $(0, \bar{c})$, $\bar{\lambda}(\cdot, t) \ll \lambda(\cdot, t)$ a.e. $d\bar{\Lambda}_t$, where $\bar{c} = \inf\{t: \bar{F}_t = 0\}$; then $\bar{\mu} \ll \mu$ with Radon Nikodym derivative

$$L(t, x) = \alpha(t) \beta(t, x) \exp\left(-\int_0^t (\alpha(s) - 1) d\Lambda_s^c\right) \pi_t \cdot I_{(t \leq \bar{c})}. \quad (2.14)$$

where

$$\alpha(t) = \frac{d\bar{\Lambda}_t}{d\Lambda_t}$$

$$\beta(t, x) = \frac{d\bar{\lambda}}{d\lambda}(t, x)$$

and π_t is as in (2.13).

Proof: See (20) theorem 3.7.

Remarks 2.5.

The above theorem allows one to describe a class of jump processes parametrized by u with $\mu_u \ll \mu$ by their respective Radon Nikodym derivatives, (α_u, β_u) .

It is often necessary to work with the conditional expectation $L_t = E(L/F_t)$ rather than L itself. L_t is obtained through a direct calculation, as follows:

$$L_t = E(L/F_t) = L(T, Z) I_{(t \geq T)} + I_{(t < T)} \frac{1}{F_t} \int_{(t, \bar{c}] \times X} L(s, x) d\mu(s, x)$$

Substituting $L(s, x)$ from (2.14) gives:

$$\begin{aligned} L_t &= \alpha(T) \beta(T, Z) \exp(-\int_0^t (\alpha(s) - 1) d\Lambda_s^C) \Pi_{T-} I_{(t \geq T)} \\ &\quad + I_{(t < T)} \exp(-\int_0^t (\alpha(s) - 1) d\Lambda_s^C) \Pi_t. \end{aligned} \quad (2.15)$$

L_t is by definition a right continuous martingale.

2.4. The general jump process and corresponding results.

In this section we extend the results obtained in the previous sections for the one jump case to the general jump process. We shall, in doing so, use notations that preserve the 'one jump' appearance of the various formulas and equations.

Using the formulation of sec.2.1, write for each $k=1, 2, \dots, t \geq 0, A \in \mathcal{S}, \omega \in \Omega$

$$F_t^{kA} = F_t^{kA}(\omega) = \mu_k(\omega_{k-1}(\omega); (t, \infty] \times A)$$

$$F_t^k = F_t^{kX}(\omega)$$

$$p^k(t, A) = I_{(t \geq T_k)} I_{(Z_k \in A)} \quad (2.16)$$

$$\tilde{p}^k(t, A) = \tilde{p}^k(\omega_{k-1}(\omega); t, A) = - \int_{(0, t \wedge T_k]} \frac{dF_s^{kA}}{F_s^k}$$

$$q^k(t, A) = p^k(t, A) - \tilde{p}^k(t, A)$$

$$p(t, A) = \sum_{k=1}^{\infty} p^k(t, A)$$

$$\tilde{p}(t, A) = \sum_{k=1}^{\infty} \tilde{p}^k(t, A)$$

(2.16 contd.)

$$q(t, A) = p(t, A) - \tilde{p}(t, A) .$$

The following lemma is then a simple extension of lemma 2.2.

Lemma 2.6.

$q^k(t, A)$ and $q(t, A)$ are (F_t, P) martingales, all $t \geq 0$, $A \in S$, $k=1, 2, \dots$

As in sec.2.2, for each $k \in Z^+$, $\omega \in \Omega$, $A \in S$, $F_t^{kA} \ll F_t^k$, so there exists a Radon Nikodym derivative $\lambda_k(\omega_{k-1}(\omega); A, t)$ such that:

$$F_t^{kA} - F_0^{kA} = \int_{(0, t]} \lambda_k(A, s) dF_s^k$$

And as before we can choose λ_k to be a regular family of conditional distributions so that $\lambda_k(., s)$ is a probability measure, each $k \in Z^+$, $s \geq 0$, $\omega \in \Omega$. Now write:

$$\Lambda_k(t) = \Lambda_k(\omega_{k-1}(\omega); t) = - \int_{(0, t]} \frac{dF_s^k}{F_{s-}^k}$$

Then the family $(\Lambda_k, \lambda_k, k \in Z^+)$ constitutes the local descriptions of the general jump process. For each $k \in Z^+$, $\omega \in \Omega$, the pair (Λ_k, λ_k) satisfy the properties (i)-(vii) of (2.4).

As in theorem 2.1, there is a bijective correspondence between the measure μ_k and the pair of local descriptions (Λ_k, λ_k) satisfying (2.4) (i)-(vii).

To state the martingale representation theorem, we first define the class of integrands x to be all measurable functions $g: \Omega \times Y \rightarrow R$ such that there exists $g^1: Y \rightarrow R$ and for

$k=2,3,\dots$, measurable functions $g^k: \Omega^{k-1} \times Y \rightarrow R$ such that:

$$g(t, x, \omega) = \begin{cases} g^1(t, x) & \text{if } t \leq T_1(\omega) \\ g^k(\omega_{k-1}(\omega); t, x) & \text{if } t \in (T_{k-1}(\omega), T_k(\omega)] \\ 0 & \text{if } t \geq T_\infty(\omega). \end{cases} \quad (2.17)$$

$$g^1(\infty, x) = g^k(\omega_{k-1}; \infty, x) = 0.$$

Then $L^1(p)$ and $L^1_{loc}(p)$ are defined as in sec.2.2.

The martingale representation theorem for general jump processes states:

Theorem 2.7.

If M_t is a (F_t, P) local martingale, then there exists a $g \in L^1_{loc}(p)$ such that

$$M_t - M_0 = \int_{(0,t] \times X} g(s, x) q(ds, dx). \quad (2.18)$$

Proof: See (15), theorem 2.

Remarks 2.6.

Suppose $T_\infty = \infty$, then the above theorem could be restated as: M_t is a (F_t, P) local martingale if and only if there exists a $g \in L^1_{loc}(p)$ such that

$$M_t - M_0 = \int_{(0,t] \times X} g(s, x) q(ds, dx).$$

See remark 1 of (15) after theorem 2.

Suppose M_t is a uniformly integrable martingale of F_t with $M_0 = 0$ a.s. One would now naturally like to evaluate the integrand g for this martingale as in the one jump case. This is done as follows: From (15), the following formula holds for all $t > 0$.

$$M_t = M_{t \wedge T_1} + \sum_{k=2}^{\infty} (M_{t \wedge T_k} - M_{T_{k-1}}) I_{(t \geq T_{k-1})}. \quad (2.19)$$

Define

$$X_t^1 = M_{t \wedge T_1} \quad (2.20)$$

$$X_t^k = M_{(t+T_{k-1}) \wedge T_k} - M_{T_{k-1}} \quad ; \quad k=2,3,\dots$$

Then from (2.19)

$$M_t = \sum_{k=1}^{\infty} X_{(t-T_{k-1}) \vee 0}^k \quad (2.21)$$

For fixed k , define

$$H_t^k = F_{(t+T_{k-1}) \wedge T_k}$$

Using the optional sampling theorem (45, VT9) we see that

X_t^k is a martingale of H_t^k with $X_0^k=0$. And since H_t^k is generated by $F_{T_{k-1}}$ and the sample path of x_s for $s \in (T_{k-1}, (t+T_{k-1}) \wedge T_k]$, there exists a measurable function h^k such that:

$$X_t^k = E(h^k(\omega_{k-1}; T_k, Z_k) / H_t^k) \quad (2.22)$$

But by (2.20) and (2.18)

$$X_t^k = \int_{(0, (t+T_{k-1}) \wedge T_k] \times X} g^k(s, x) dq^k(s, x) \quad (2.23)$$

so that as in the one jump situation, one has:

$$g^k(t, x) = h^k(t, x) - I_{(t < c^k)} \frac{1}{F_t^k} \int_{(t, \infty] \times X} h^k(s, x) d\mu_k(s, x). \quad (2.24)$$

where

$$c^k(\omega_{k-1}) = \inf (t: F^k(\omega_{k-1}; t) = 0).$$

Equation (2.24) is an extension of (2.11).

We now consider the question of absolutely continuous changes of rates in the general case. Suppose for each $k \in \mathbb{Z}^+$, $\omega \in \Omega$, (Λ^k, λ^k) is changed to $(\bar{\Lambda}^k, \bar{\lambda}^k)$ such that:

$$\bar{\lambda}^k(\omega_{k-1}(\omega); \cdot) \ll \lambda^k(\omega_{k-1}(\omega); \cdot)$$

$$\bar{\lambda}^k(\omega_{k-1}(\omega); \cdot, t) \ll \lambda^k(\omega_{k-1}(\omega); \cdot, t) \quad \text{a.e. } d\bar{\lambda}_t^k.$$

The question one would naturally ask now is whether the probability measure $\bar{P}$ derived from the local descriptions $\{(\bar{\lambda}^k, \bar{\lambda}^k), k \in \mathbb{Z}^+\}$ is mutually absolutely continuous w.r.t. P . It turns out that it is a very severe restriction to require that $\bar{P} \sim P$ on an infinite interval. (Since two Poisson processes with different rates are strictly singular w.r.t. each other on the infinite interval). But since we shall be dealing with finite intervals in the control problem, such restrictions do not concern us. The following arguments show that if we consider the restrictions of $\bar{P}, P$ to $\{F_{T_k}, k \in \mathbb{Z}^+\}$ or to $\{F_{T_f}, T_f < \infty\}$, then the two restricted probability measures are indeed mutually absolutely continuous w.r.t. each other.

First, we note that for each $k \in \mathbb{Z}^+$, $n \in \Omega_{k-1}$, we can treat the k th jump, characterized by the pair of r.v.'s (T_k, Z_k) , as a "single jump" by itself, with zero probability of occurrence before the time T_{k-1} . The single jump analysis thus applies. Using Lemma 2.4 and theorem 2.5 we conclude that $\bar{\mu}_k(n; \cdot) \ll \mu_k(n; \cdot)$, with Radon Nikodym derivative

$$\begin{aligned} L_k(t, x) &= L_k(n; t, x) = \frac{d\bar{\mu}_k}{d\mu_k}(n; t, x) \\ &= \alpha_k(t) \beta_k(t, x) \exp\left\{-\int_0^t (\alpha_k(s) - 1) d\Lambda_s^{k, k} \right\} I_{(t \leq \bar{c}^k)}. \end{aligned} \quad (2.25)$$

where,

$$\alpha_k(t) = \alpha_k(n; t) = \frac{d\bar{\lambda}^k}{d\lambda^k}(n; t) \quad (2.26)$$

$$\beta_k(t, x) = \beta_k(n; t, x) = \frac{d\bar{\lambda}^k}{d\lambda^k}(n; t, x)$$

$$\pi_t^k = s_{\leq t} \frac{(1 - \alpha_k(s) \Delta \Lambda_s^k)}{(1 - \Delta \Lambda_s^k)} \quad (2.26 \text{ contd.})$$

$$\Lambda_t^{kc} = \Lambda_t^k - \sum_{s \leq t} \Delta \Lambda_s^k$$

$$\bar{c}^k = \bar{c}^k(n) = \inf(t: \bar{F}^k(n; t) = 0).$$

Now for $A_1 \in \mathcal{Y}^1$, $A_2 \in \mathcal{Y}^2$, we have

$$\bar{P}\{(T_1, Z_1) \in A_1, (T_2, Z_2) \in A_2\} = \int_{A_1} \bar{\mu}_2(a_1; A_2) \bar{\mu}_1(da_1)$$

And since $L_k = \frac{d\bar{\mu}_k}{d\mu_k}$, $k \in \mathbb{Z}^+$, we have

$$\bar{P}\{(T_1, Z_1) \in A_1, (T_2, Z_2) \in A_2\} = \int_{A_1 \times A_2} L_1(a_1) L_2(a_1; a_2) P(da_1, da_2)$$

so that, using induction, we conclude that for each $k \in \mathbb{Z}^+$, $A_i \in \mathcal{Y}^i$, $i=1, \dots, k$.

$$\begin{aligned} \bar{P}\{(T_i, Z_i) \in A_i, i=1, \dots, k\} &= \int_{A_1 \times \dots \times A_k} \prod_{i=1}^k L_i(a_1, \dots, a_i) P(da_1, \dots, da_k) \\ &= \int_{A_1 \times \dots \times A_k} \prod_{i=1}^k L_i dP_k \end{aligned}$$

This implies that $\bar{P}_k \ll P_k$, each $k \in \mathbb{Z}^+$ where $\bar{P}_k, P_k$ are the restrictions of $\bar{P}, P$ respectively to F_{T_k} .

Further:

$$\frac{d\bar{P}_k}{dP_k} = \prod_{i=1}^k L_i \quad (2.27)$$

As $k \rightarrow \infty$, $\frac{d\bar{P}_k}{dP_k}$ may not converge to a measurable function. So

although $\bar{P}_k \ll P_k$ for each $k \in \mathbb{Z}^+$, it is not necessarily

that $\bar{P} \ll P$. A counter-example to demonstrate this anomaly is given in (17).

However if $T_\infty = \infty$ a.s. P and $\alpha_k(\cdot) \leq N < \infty$ for each $k \in \mathbb{Z}^+$, then we also have $\bar{P}_{T_f} \ll P_{T_f}$, where $\bar{P}_{T_f}, P_{T_f}$ are the restrictions of $\bar{P}, P$ to $\{F_{T_f}, T_f < \infty\}$. This is intuitively obvious since the above conditions ensure that $T_\infty = \infty$ a.s. $\bar{P}$, so that the jumps do not 'pile up' at some finite time. The following arguments make this more precise.

Consider a set $A \in F_{T_f}$ such that $P(A) = 0$. To show that $\bar{P}_{T_f} \ll P_{T_f}$, it is sufficient to prove that $\bar{P}(A) = 0$. Now since $T_k > T_f$ a.s. $\bar{P}$ eventually as $k \rightarrow \infty$, we have

$$\bar{P}(A) = \bar{P}\left\{\bigcup_{k \in \mathbb{Z}^+} A_k\right\} = \sum_{k \in \mathbb{Z}^+} \bar{P}(A_k)$$

where $A_k = A \cap \{\omega : T_{k-1} \leq T_f < T_k\}$.

Clearly $A_k \in F_{T_k}$, so that since $\bar{P}_{T_k} \ll P_{T_k}$, we have $\bar{P}(A_k) = 0$, each $k \in \mathbb{Z}^+$. This implies $\bar{P}(A) = 0$. Hence $\bar{P}_{T_f} \ll P_{T_f}$.

The converse $P_{T_f} \ll \bar{P}_{T_f}$ is similarly shown.

We now give an expression for $L_{T_f} = \frac{d\bar{P}_{T_f}}{dP_{T_f}}$. Observe that

for each $k \in \mathbb{Z}^+$, we have

$$\begin{aligned} \frac{d\bar{P}_{T_f}}{dP_{T_f}} I_{(T_{k-1} \leq T_f < T_k)} &= E\left\{\frac{d\bar{P}_{T_f}}{dP_{T_f}} / F_{T_f}\right\} I_{(T_{k-1} \leq T_f < T_k)} \\ &= E\left\{\frac{d\bar{P}_{T_k}}{dP_{T_k}} / F_{T_f}\right\} I_{(T_{k-1} \leq T_f < T_k)} \end{aligned}$$

so that on the set $\{T_{k-1} \leq T_f < T_k\}$,

$$L_{T_f} = \prod_{i=1}^{k-1} L_i \exp\left\{-\int_0^{T_f} (\alpha_k(s) - 1) d\Lambda_s^{k_G}\right\} \prod_t^k \quad (2.28)$$

For any $t \in \mathbb{R}^+$ we have: L_t is a right continuous uniformly integrable martingale, with $\lim_{t \rightarrow \infty} L_t = L$ a.s. See (45), remark after theorem 6, chap. VI.

If as above, $T_\infty = \infty$ a.s., we obtain a more elegant system of notation as follows:

First note that all $\mathcal{F}_t$ measurable functions $h(t)$, right continuous in t , have the representation

$$h(t) = \sum_{k=1}^{\infty} I_{\{t \in [T_{k-1}, T_k)\}} h^k(\omega_{k-1}(\omega); t)$$

Hence we can define, for $t \in \mathbb{R}^+$, $A \in \mathcal{S}$, $\omega \in \Omega$:

$$\Lambda_t = \Lambda(t, \omega) = \sum_{k=1}^{\infty} I_{\{t \in [T_{k-1}, T_k)\}} \Lambda^k(\omega_{k-1}(\omega); t)$$

$$\lambda(A, t) = \lambda(A, t, \omega) = \sum_{k=1}^{\infty} I_{\{t \in [T_{k-1}, T_k)\}} \lambda^k(\omega_{k-1}(\omega); A, t)$$

$$\alpha(t) = \alpha(t, \omega) = \sum_{k=1}^{\infty} I_{\{t \in (T_{k-1}, T_k]\}} \alpha_k(\omega_{k-1}(\omega); t)$$

$$\beta(t, x) = \beta(t, x, \omega) = \sum_{k=1}^{\infty} I_{\{t \in (T_{k-1}, T_k]\}} \beta_k(\omega_{k-1}(\omega); t, x)$$

$$\Pi_t = \prod_{k=1}^{\infty} \left\{ \prod_{i=1}^{k-1} \Pi_{T_i}^i \right\} \prod_{t \in [T_{k-1}, T_k)} I_{\{t \in [T_{k-1}, T_k)\}} \quad (2.29)$$

Using these definitions, (2.28) becomes, for $t < \infty$,

$$L_t = \prod_{\substack{s \leq t \\ x_s \neq x_{s-}}} \alpha(s) \beta(s, x_s) \exp\left\{-\int_0^t (\alpha(r) - 1) d\Lambda_r^C\right\} \Pi_t \quad (2.30)$$

which looks nearly the same as (2.15) for the elementary one jump situation.

One interesting property of the likelihood ratio is given in the next lemma.

Lemma 2.8.

For any $t_1, t_2 \in \mathbb{R}^+$, with $t_1 \geq t_2$, we have

$$E(L_{t_1}^{t_2} / \mathcal{F}_{t_1}) = 1 \text{ a.s.},$$

where

$$L_{t_1}^{t_2} = \frac{L_{t_2}}{L_{t_1}}.$$

Proof:

We know that $t_1 \in [T_{k-1}, T_k)$ for some $k \in \mathbb{Z}^+$ w.p. 1

Hence using (2.28) and the definition of $L_{t_1}^{t_2}$, we get

$$L_{t_1}^{t_2} = \begin{cases} \exp\{-\int_{t_1}^{t_2} (\alpha_k(r) - 1) d\Lambda_r^{kc}\} \Pi_{t_1}^{kt_2}, & \text{if } t_2 < T_k \\ \alpha_k(T_k) \beta_k(T_k, Z_k) \exp\{-\int_{t_1}^{T_k} (\alpha_k(r) - 1) d\Lambda_r^{kc}\} \Pi_{t_1}^{kT_k} \cdot \\ \cdot \prod_{i=k+1}^{j-1} L_i(T_i, Z_i) \exp\{-\int_{T_{j-1}}^{t_2} (\alpha_j(r) - 1) d\Lambda_r^{jc}\} \Pi_{T_{j-1}}^{jt_2}, & \\ & \text{if } t_2 \in [T_{j-1}, T_j) \\ & \text{some } j > k. \end{cases}$$

where

$$\Pi_{t_1}^{it_2} = \prod_{t_1 < s \leq t_2} \frac{(1 - \alpha_i(s) \Delta \Lambda_s^i)}{(1 - \Delta \Lambda_s^i)}.$$

Now on the set $\{T_{k-1} \leq t_1 < T_k\}$,

$$\begin{aligned} E(L_{t_1}^{t_2} / \mathcal{F}_{t_1}) &= E(L_{t_1}^{t_2} / T_k > t_2, \mathcal{F}_{t_1}) P(T_k > t_2 / \mathcal{F}_{t_1}) \\ &\quad + \sum_{j>k} E(L_{t_1}^{t_2} / t_2 \in [T_{j-1}, T_j), \mathcal{F}_{t_1}) P(t_2 \in [T_{j-1}, T_j) / \mathcal{F}_{t_1}) \end{aligned}$$

(2.31)

Using (2.6), the first term on the right is

$$\exp\{-\int_{t_1}^{t_2} (\alpha_k(r) - 1) d\Lambda_r^{kc}\} \Pi_{t_1}^{kt_2} \cdot \exp\{-\int_{t_1}^{t_2} d\Lambda_r^{kc}\} \prod_{t_1 < s \leq t_2} (1 - \Delta \Lambda_s^k)$$

$$\begin{aligned}
 &= \exp\{-(\bar{\Lambda}_{t_2}^k - \bar{\Lambda}_{t_1}^k) \cdot \prod_{t_1 < s \leq t_2} (1 - \Delta \bar{\Lambda}_s^k)\} \\
 &= \frac{\bar{F}_{t_2}}{\bar{F}_{t_1}^k} = \bar{P}(T_k > t_2 / F_{t_1}) .
 \end{aligned}$$

A similar calculation shows that the second term on the right of (2.31) is

$$\bar{P}(t_2 \in [T_{j-1}, T_j) / F_{t_1}) .$$

Hence

$$\begin{aligned}
 E(L_{t_1}^{t_2} / F_{t_1}) &= \bar{P}(T_k > t_2 / F_{t_1}) + \sum_{j > k} \bar{P}(t_2 \in [T_{j-1}, T_j) / F_{t_1}) \\
 &= 1 \quad \text{a.s.}
 \end{aligned}$$

This completes the proof.

CHAPTER 3 : OPTIMALITY CONDITIONS FOR GENERAL JUMP PROCESSES.

In this chapter we formulate and solve the optimal control problem for general jump processes. Necessary and sufficient conditions which an optimal control must satisfy are derived using dynamic programming and martingale methods. These methods were first used by Davis and Varaiya (9) to obtain optimality criteria for controlled systems described by stochastic differential equations. In applying them, we shall take advantage of the piecewise constant path property of the processes we are investigating, a feature not shared by the Wiener process framework of (9). This extra structure will yield us with optimality criteria more explicit than those of (9).

The control action consists of changing the probability measure governing the stochastic evolution of the process, achieved through changing the local descriptions of the jump process. Only measures that are mutually absolutely continuous are considered, for this permits the comparisons of processes under different measures. The control themselves are dependent on the observations on the past history of the process, which are allowed to be partial or complete. The concepts of increasing sigma fields and adaptability are hence relevant here.

Our approach, though proceeding largely along similar lines as that of (8) and (56), differs from theirs in that our system model is practically oriented instead of being completely abstract. Thus we speak of controlled rates rather than abstract measures. People like engineers will find our approach and results easier to understand and use. One other point is that we do not require the rather

complicated relative completeness concept in our framework. This concept was introduced by Rishel in (49) and used by Davis and Varaiya in (9) and Boel and Varaiya in (8) to prove their versions of the principle of optimality. Huge efforts based on using Zorn's lemma were needed to demonstrate that the admissible class of controls are relatively complete. By using an alternative concept known as the ϵ -lattice property, first introduced by Striebel in (56), we show how these unjustified efforts can be avoided.

3.1. Mathematical formulation.

Let the measurable space $(X, \mathcal{S})$ and $(\Omega, \mathcal{F}^0)$ be as defined in sec.2.1. We assume that a base probability measure P is given on $(\Omega, \mathcal{F}^0)$ such that the process x_t given by (2.1) is a jump process. In practice P would be specified either through the set of conditional distributions $(\mu_k, k=1,2,\dots)$ or through the set of local descriptions $(\Lambda^k, \lambda^k, k=1,2,\dots)$. The accumulation time T_∞ is assumed to be ∞ a.s. P . As before, let $\mathcal{F}(\mathcal{F}_t \text{ resp.})$ be the completion of $\mathcal{F}^0(\mathcal{F}_t^0 \text{ resp.})$ with respect to P , where $\mathcal{F}_t = \sigma\{x_s, s \leq t\}$. The control law u has the effect of changing the probability measure from P to P_u with the property $P_u \sim P$. Before describing how this is achieved, the class of admissible control laws is defined. Let $[0, T_f]$ be the interval of interest, where $T_f < \infty$ is the terminal time. For $t \in [0, T_f]$, let $\mathcal{F}_t^Y$ be a given increasing family of sub-sigma fields of $\mathcal{F}_t$, i.e. $\mathcal{F}_t^Y \subset \mathcal{F}_t$, all $t \in [0, T_f]$. $\mathcal{F}_t^Y$ is to be interpreted as the information available to the

controller at time t . The control law $u(.,.)$ is defined to be a function $u(t, \omega): [0, T_f] \times \Omega \rightarrow U$, where U is the space of control values, assumed to be a metric space. The class of admissible control laws $\mathcal{U}$ is then defined as the set of controls u satisfying the following conditions:

- (i) $u(t, \omega)$ is F_t^Y predictable, each $t \in [0, T_f]$.
- (ii) (u_t, F_t^Y, P_u) is a measurable stochastic process.
- (iii) If $u, v \in \mathcal{U}$, then so does

$$(u, v, t) = \begin{cases} u(s) & \text{if } s \leq t, \\ v(s) & \text{if } s > t. \end{cases}$$

i.e. $\mathcal{U}$ is closed under concatenation.

- (iv) For each $u \in \mathcal{U}$, $A \in F_t$, $P_u(A)$ depends only on u_s , $s \leq t$.

Now, under the basic measure P , the general jump process has local descriptions (Λ, λ) , given by (2.29), where each of the components (Λ^k, λ^k) of (Λ, λ) satisfy a set of conditions similar to (2.5) (i)-(vii). The controlled measures P_u , mutually absolutely continuous with respect to P , is constructed as follows:

Definition 3.1.

For the measurable space (U, B_U) let

$$\alpha(t, u, \omega) : R^+ \times U \times \Omega \rightarrow R^+, \quad \beta(t, x, u, \omega) : R^+ \times X \times U \times \Omega \rightarrow R^+$$

be jointly measurable functions satisfying the following conditions: For all $(s, x, u, \omega) \in R^+ \times X \times U \times \Omega$.

- (i) $\alpha(t, u, \omega)$, $\beta(t, x, u, \omega)$ are F_t predictable.

- (ii) $0 < c_1 \leq \alpha(s, u, \omega) \leq \min(c_2, \frac{1}{\Delta \Lambda_s})$.

- (iii) $0 < c_1 \leq \beta(s, x, u, \omega) \leq c_3$.

- (iv) $\int_X \beta(s, x, u, \omega) \lambda(dx, s, \omega) = 1$.

(3.1)

where c_1, c_2, c_3 are positive constants.

For each $u \in U$, denote the functions $\alpha(., u, .)$, $\beta(., ., u, .)$ by α_u, β_u respectively.

The pair (α_u, β_u) plays the role of controlled Radon Nikodym derivatives and is analogous to the drift coefficients $f(t, x, u)$ in diffusion control. α_u and β_u controls respectively where and when the next jump of the process is going to occur. The F_t predictability of (α_u, β_u) means that the process is self exciting.

Since (α_u, β_u) are F_t predictable, there exists some measurable functions $\alpha_k: \Omega_{k-1} \times R^+ \times U \rightarrow R^+$, $\beta_k: \Omega_{k-1} \times R^+ \times X \times U \rightarrow R^+$ such that:

$$\begin{aligned} \alpha(t, u, \omega) &= \sum_{k=1}^{\infty} I_{\{t \in (T_{k-1}, T_k]\}} \alpha_k(\omega_{k-1}(\omega); t, u) \\ \beta(t, x, u, \omega) &= \sum_{k=1}^{\infty} I_{\{t \in (T_{k-1}, T_k]\}} \beta_k(\omega_{k-1}(\omega); t, x, u) \end{aligned} \quad (3.2)$$

$(\alpha_{k_u}, \beta_{k_u})$ are the component Radon Nikodym derivatives of (α_u, β_u) , and obviously satisfy conditions (3.1) (i)-(iv). Now for each $k \in Z^+$, $u \in U$, $\eta \in \Omega_{k-1}$, $A \in S$, $t \in [0, T_f]$, define:

$$\begin{aligned} \Lambda_k^u(\eta; t) &= \int_{(0, t]} \alpha_k(\eta; s, u_s) d\Lambda_k(\eta; s) \\ \lambda_k^u(\eta; \Lambda, t) &= \int_A \beta_k(\eta; t, x, u_t) \lambda_k(\eta; dx, t) \end{aligned} \quad (3.3)$$

It can then be verified, using (3.1)(i)-(iv) and (2.5) (i)-(vii) that the set $(\Lambda_k^u, \lambda_k^u, k=1, 2, \dots)$ qualify as local descriptions of the general jump process.

So by virtue of a one-one bijective theorem similar to theorem 2.1, a set of controlled conditional distributions $(\nu_k^u, k=1, 2, \dots)$ can be constructed. Since $\Lambda_k^u < \Lambda_k$, $\lambda_k^u < \lambda_k$ and vice versa (due the bounds away from zero of (α, β)),

the measures $\{\mu_k^u, k=1,2,\dots\}$ are also mutually absolutely continuous w.r.t. $\{\mu_k, k=1,2,\dots\}$, by theorem 2.5.

Finally, the measure P_u is obtained from $\{\mu_k^u, k=1,2,\dots\}$ using the same procedure as that for P in sec.2.1. Since $T_\infty = \infty$ a.s. P and α_u is uniformly bounded above, $T_\infty = \infty$ a.s. P_u , and P_u is mutually absolutely continuous w.r.t. P . See also sec.2.4. This completes the description of P_u .

Remarks 3.1.

The above approach to the control of jump processes is practically inspired since such processes are invariably described by 'rates' resembling our (α_u, β_u) in most applications. One other method of control is to start off by defining an abstract measure P_u absolutely continuous w.r.t. P without bothering about what processes we are considering. See for example (8) and (56).

The likelihood ratio $L_t(u) = E\left(\frac{dP_u}{dP} / \mathcal{F}_t\right)$ is evaluated as in sec.2.4, thus, for each $u \in U$, $t \in [0, T_f]$, we have

$$L_t(u) = \left\{ \prod_{\substack{s \leq t \\ x_s \neq x_{s-}}} \alpha_u(s) \beta_u(s, x_s) \exp\left\{-\int_0^t (\alpha_u(r) - 1) d\Lambda_r^C\right\} \pi_t(u) \right\} \quad (3.4)$$

where

$$\pi_t(u) = \sum_{k=1}^{\infty} \prod_{i=1}^{k-1} \pi_{T_i-}^i(u) \cdot I_{\{t \in [T_{k-1}, T_k)\}} \pi_t^k(u)$$

with

$$\pi_t^k(u) = \prod_{s \leq t} \frac{(1 - \alpha_k(\omega_{k-1}(\omega); s, u_s) \Delta \Lambda_s^k)}{(1 - \Delta \Lambda_s^k)}$$

$L_t(u)$ is a $(\mathcal{F}_t, P)$ martingale for each u .

To define the class of $(\mathcal{F}_t, P_u)$ predictable processes $\tilde{p}^u(t, A)$ associated with $p(t, A)$, first write for $u \in U$, $t \in [0, T_f]$, $A \in \mathcal{S}$, $\omega \in \Omega$,

$$\Lambda_t^u = \Lambda^u(t, \omega) = \sum_{k=1}^{\infty} I_{\{t \in [T_{k-1}, T_k)\}} \Lambda_k^u(\omega_{k-1}; t)$$

$$\lambda^u(A, t) = \lambda^u(A, t, \omega) = \sum_{k=1}^{\infty} I_{\{t \in [T_{k-1}, T_k)\}} \lambda_k^u(\omega_{k-1}; A, t)$$

Then

$$\begin{aligned} \tilde{p}^u(t, A) &= \int_{(0, t]} \lambda^u(A, s, \omega) \Lambda^u(ds, \omega) \\ &= \sum_{k=1}^{\infty} \int_{(0, t \wedge T_k]} \lambda_k^u(\omega_{k-1}; A, s) \Lambda_k^u(\omega_{k-1}; ds) \\ &= \sum_{k=1}^{\infty} \tilde{p}_k^u(t \wedge T_k, A) \end{aligned} \quad (3.5)$$

where

$$\tilde{p}_k^u(t, A) = \int_{(0, t]} \lambda_k^u(\omega_{k-1}; A, s) \Lambda_k^u(\omega_{k-1}; ds)$$

As in lemma 2.6, one can now show that $q^u(t, A) = \{p(t, A) - \tilde{p}^u(t, A)\}$ is a (F_t, P_u) martingale, all $t \geq 0$, $A \in \mathcal{S}$.

Hence, using the same integrands $g \in L_{loc}^1(p)$ defined in sec.2.4, the stochastic integrals $\int g dq^u$ are similarly defined. The martingale representation theorem under measure P_u is then also the same as that in theorem 2.7.

Having described the controlled system of jump process, we now come to the cost structure of the control problem. First, some definitions.

Definitions 3.2.

(a) Let $c(t, x, u, \omega): R^+ \times X \times U \times \Omega \rightarrow R^+$ be a non-negative function, jointly measurable w.r.t. $B_R^* \times S^* \times B_U^* \times F$ and such that

(i) $c(t, x, u, \omega)$ is F_t adapted, each (x, u, ω)

(ii) $0 \leq c(t, x, u, \omega) \leq c_4 < \infty$, some $c_4 \in R^+$, all (t, x, u, ω) .

(b) Let $r_s^t(\omega)$ be a non-negative function defined for

$\omega \in \Omega, s, t \in [0, T_F]$ such that $s \leq t$, and $r_s^t(\omega)$ is F_t^Y adapted and $B_R^* \times F$ measurable. $r_s^t(\omega)$ shall also be uniformly integrable

and has continuous sample paths for fixed ω . Also

$$r_{t_1}^{t_3} = r_{t_1}^{t_2} r_{t_2}^{t_3} \quad \text{a.s. } P \text{ for } t_1 \leq t_2 \leq t_3.$$

$$r_t^t = 1 \text{ a.s. } P$$

(c) Set $G_f(\omega): \Omega \rightarrow R^+$ be a non-negative F_{T_f} measurable function such that $G_f(\omega) < c_4 < \infty$, some $c_4 \in R^+$, all $\omega \in \Omega$.

For each control policy $u \in U$, a cost $J(u)$ is incurred, where

$$J(u) = E_u \left\{ \int_{(0, T_f] \times X} r_0^s c(s, x, u_s, \omega) d\tilde{p}^u(s, x) + r_0^{T_f} G_f(\omega) \right\}. \quad (3.6)$$

and $E_u(\cdot)$ denotes taking expectation w.r.t. the P_u measure.

Remarks 3.2.

By the boundedness assumptions on $c, \alpha_u, \beta_u, G_f$, $J(u) < \infty$ for each $u \in U$. $r_s^t(\omega)$ is the discounting factor and reflects the fact that future costs are weighted differently from present costs, usually less heavily. c is the instantaneous cost rate and G_f is the terminal cost incurred at time T_f . The predictable process $\tilde{p}^u(t, A)$ is chosen as the integrator as it is more general than $\Lambda(t, \omega)$, the basic integrated rate of the process (normally Lebesgue's measure). There is no difficulty involved in replacing $\tilde{p}^u(t, A)$ by some other increasing process $A(t)$ or Lebesgue's measure.

Remarks 3.3.

Since $q^u(t, A) = p(t, A) - \tilde{p}^u(t, A)$ is a (F_t^y, P_u) martingale, therefore the stochastic integral

$$\int_{(0, T_f] \times X} r_0^s c(s, x, u_s, \omega) (dp - d\tilde{p}^u)$$

is a zero mean (F_t^Y, P_u) martingale, so

$$E_u \left\{ \int_{(0, T_f] \times X} r_0^S c(s, x, u_s, \omega) (dp - d\tilde{p}^u) \right\} = 0$$

We can therefore rewrite (3.6) as

$$J(u) = E_u \left\{ \int_{(0, T_f] \times X} r_0^S c(s, x, u_s, \omega) dp(s, x) + r_0^{T_f} G_f \right\}$$

The cost thus increases only at the epochs on $[0, T_f]$.

Statement of the Optimal control problem:

The optimal control problem is to find a control $u^* \in U$ such that

$$J(u^*) = J^* = \inf_{u \in U} J(u) \quad (3.7)$$

Such a u^* is called an optimal control and is characterized by the property:

$$J(u^*) \leq J(u), \quad \text{all } u \in U.$$

Remarks 3.4.

The above fixed interval problem can be transformed into a random time interval problem by replacing the integrand of $J(u)$ by $I_{\{s < \tau(\omega)\}} c(s, x, u_s, \omega)$, where $\tau(\omega)$ is any bounded F_t stopping time.

Our control problem is formulated here without the assumption of total inaccessibility of the stopping times $\{T_i, i=1, 2, \dots\}$. What this amounts to is that the predictable process $\tilde{p}(t, A)$ can have discontinuities.

This clearly is a desirable feature, for in certain problems like inventory control, one would like to have a model in which there can be a positive probability of

occurrence of deliveries or demands at fixed instants of time. These instants would correspond to the discontinuities of the predictable process $\tilde{p}(t,A)$. Such problems are therefore included in our set up. One restriction of the model used by (8) is that they required total inaccessibility of the stopping times $\{T_i\}$. This is because their version of the martingale representation theorem in (9) is otherwise invalid. The assumption of total inaccessibility is also inherent in (50) and (48). This is because they were both dealing with conditional probability density (or rates) of occurrences of jump times, and therefore require the base integrated rate to be Lebesgue's measure which does not permit discontinuities.

3.2. The general principle of optimality.

In this section, the first set of necessary and sufficient conditions an optimal control must satisfy are derived. These conditions are known as the principle of optimality because dynamic programming techniques are involved. By using a concept known as the ϵ -lattice property, we are able to prove the principle using fairly straightforward techniques.

For $u,v \in U$, define

$$(u,v,t) = \begin{cases} u(s) & \text{if } s \leq t \\ v(s) & \text{if } s > t \end{cases}$$

Then since U is closed under concatenation, $(u,v,t) \in U$.

For $t \in [0, T_f]$, the following process is well defined.

$$\psi(u,v,t) = E_{u,v,t} \left\{ \int_{(t,T_f] \times X} r_t^S c(s,x,v_s) d\tilde{p}^v + r_t^f G_f / F_t^Y \right\} \quad (3.8)$$

$\psi(u,v,t)$ represents the conditional expected cost incurred after time t , evaluated at time t , given the information sigma field F_t^Y and that the control u is used up to time t , after which v is adopted. It is called the remaining cost function.

Now $\psi(u,v,t)$ is by definition integrable and

$$\psi(u,v,t) \in L^1(\Omega, F_t^Y, P_u)$$

(because of $E_{u,v,t}\{\psi(u,v,t)\} = E_u\{\psi(u,v,t)\}$, see (iv) in the definition of U in sec.3.1.)

Now L^1 is a complete lattice under the natural partial ordering for real valued functions. See (26) IV8-22. Hence the following P_u essential infimum exists.

$$\begin{aligned} W(u,t) &= \inf_{v \in U} \psi(u,v,t) \in L^1(\Omega, F_t^Y, P_u) \\ W(0) &= J^*. \end{aligned} \tag{3.9}$$

$W(u,t)$ represents the lowest expected cost achievable after time t , evaluated at time t , and given the information F_t^Y and that control $u \in U$ had been used up to time t . It is known as the value function of the control problem. Notice also that $(W(u,t), F_t^Y, P_u)$ is a stochastic process adapted to F_t^Y , as is the remaining cost $\psi(u,v,t)$.

Remarks 3.5.

It is usual to evaluate the costs at time 0. To do this one merely multiply $W(u,t)$ and $\psi(u,v,t)$ by r_0^t . This is allright since r_0^t is by definition F_t^Y adapted.

One can, if one wishes to, define an alternative remaining cost function as follows: let

$$\phi(u,v,t) = E\{L(u,v,t) \cdot \int_{(t, T_f]} r_t^S c(s,x,v) d\tilde{p}^v + r_t^T G_f / F_t^Y\}$$

Where

$$L(u,v,t) = \frac{dP}{dP}(u,v,t)$$

Then for similar reasons as $\psi(u,v,t)$

$$\phi(u,v,t) \in L^1(\Omega, F_t^Y, P_0).$$

$\phi(u,v,t)$ is also a sort of remaining cost function and is related to $\psi(u,v,t)$ by

$$\psi(u,v,t) = \frac{\phi(u,v,t)}{E\{L(u,v,t) / F_t^Y\}} = \frac{\phi(u,v,t)}{E\{L_t(u) / F_t^Y\}}$$

The value function corresponding to $\phi(u,v,t)$ can then be defined to be:

$$V(u,t) = \inf_{v \in U} \phi(u,v,t) \in L^1(\Omega, F_t^Y, P_u)$$

We shall be chiefly concerned with using $\psi(u,v,t)$ and $W(u,t)$ in this thesis.

Definition 3.3.

The class of controls U is said to possess the ϵ -lattice property, w.r.t. the remaining cost function $\psi(u,v,t)$ provided that for each $\epsilon > 0$, $u, v_1, v_2 \in U$, $t \in [0, T_f]$, there exists a $v \in U$ such that :

$$\psi(u,v,t) \leq \psi(u,v_i,t) + \epsilon, \quad i=1,2. \quad \text{a.s.} \quad (3.10)$$

Remarks 3.6.

This definition is adapted from a version used in (56). It replaces the stronger relative completeness property used by the authors of (19) and (8). This latter property is defined as: For each $\epsilon > 0$, $u \in U$, $t \in [0, T_f]$, there exists a $v \in U$ such that $\psi(u,v,t) \leq W(u,t) + \epsilon$. Clearly this condition implies (3.10) since $W(u,t) \leq \psi(u,v_i,t)$, a.s., $i=1,2$. In addition to its simplicity, the ϵ -lattice property proves to be easily verifiable in the Markov situation, when the relative completeness is difficult to show.

Lemma 3.1.

The class U of F_t^Y predictable controls has the ϵ -lattice property.

Proof:

In the definition (3.3), choose the policy $v \in U$ to be :

$$v(s) = v_1(s) I_{\{\psi(u, v_1, t) \leq \psi(u, v_2, t)\}} + v_2(s) I_{\{\psi(u, v_1, t) > \psi(u, v_2, t)\}} \\ \text{for } s \in [t, T_f].$$

The reason that $v \in U$ is that $v_1, v_2 \in U$ and that each of the indicator functions on the right is F_t^Y measurable, and hence F_s^Y measurable since $s \geq t$ and $F_s^Y \supset F_t^Y$. Clearly then

$$\psi(u, v, t) \leq \psi(u, v_i, t) + \epsilon \quad \text{a.s.} \quad i=1,2.$$

The reasoning above would fail if, for example, F_t^Y is not increasing.

Lemma 3.2.

For each $t \in [0, T_f]$, $h > 0$, such that $t+h \in [0, T_f]$, $u \in U$, we have

$$\inf_{v \in U} E_u \{ r_t^{t+h} \psi(u, v, t+h) / F_t^Y \} = E_u \{ \inf_{v \in U} r_t^{t+h} \psi(u, v, t+h) / F_t^Y \} \text{ a.s.}$$

Proof:

From Lemma 3.1, U has the ϵ -lattice property. The proof then follows as in theorem A2 of (56). See also the Appendix for a version of this proof.

Equipped with the above two lemmas, we are now in a position to state and prove the main theorem of this section.

Theorem 3.3. (principle of optimality.)

Given any $t \in [0, T_f]$, $h > 0$ such that $t+h \in [0, T_f]$, we have for all $u \in U$

$$W(u, t) \leq E_u \left\{ \int_{(t, t+h] \times X} r_t^s c(s, x, u) d\tilde{p}^u(s, x) / F_t^Y \right\} + E_u \{ r_t^{t+h} W(u, t+h) / F_t^Y \} \text{ a.s.} \quad (3.11)$$

$$W(u, T_f) = E_u \{ G_f(\omega) / F_{T_f}^Y \} \text{ a.s.} \quad (3.12)$$

Furthermore, u is optimal if and only if equality holds in (3.11).

Proof :

$$\text{Since } \inf_{v \in U} \psi(u, v, T_f) = \inf_{v \in U} E_u \{ G_f(\omega) / F_{T_f}^Y \} = E_u \{ G_f(\omega) / F_{T_f}^Y \} ,$$

equation (3.12) follows. To prove (3.11), note that for all $\bar{v} \in U$,

$$\begin{aligned} W(u, t) \leq \psi(u, \bar{v}, t) &= E_{u, \bar{v}, t} \left\{ \int_{(t, t+h] \times X} r_t^S c(s, x, \bar{v}_s) d\tilde{p}^{\bar{v}} / F_t^Y \right\} \\ &\quad + E_{u, \bar{v}, t} \left\{ \int_{(t+h, T_f] \times X} r_t^S c(s, x, \bar{v}_s) d\tilde{p}^{\bar{v}} + r_t^{T_f} G_f(\omega) / F_t^Y \right\} \end{aligned}$$

Now let $\bar{v} = (u, v, t+h)$, some $v \in U$, then it is easily seen that

$$\begin{aligned} W(u, t) &\leq E_u \left\{ \int_{(t, t+h] \times X} r_t^S c(s, x, u) d\tilde{p}^u / F_t^Y \right\} \\ &\quad + \inf_{v \in U} E_{u, v, t+h} \left\{ \int_{(t+h, T_f] \times X} r_t^S c(s, x, v) d\tilde{p}^v + r_t^{T_f} G_f / F_t^Y \right\} \\ &= E_u \left\{ \int_{(t, t+h] \times X} r_t^S c(s, x, u) d\tilde{p}^u / F_t^Y \right\} \\ &\quad + \inf_{v \in U} E_{u, v, t+h} \{ r_t^{t+h} \psi(u, v, t+h) / F_t^Y \} . \end{aligned}$$

But from lemma 3.2 and 3.1, we know that the infimum and the conditional expectation can be interchanged. And since

$$E_{u, v, t+h} \{ r_t^{t+h} \inf_{v \in U} \psi(u, v, t+h) / F_t^Y \} = E_u \{ r_t^{t+h} W(u, t+h) / F_t^Y \} .$$

the inequality (3.11) follows.

It remains to prove the final assertion. Suppose u is optimal, then from the definition of the value function,

$$\begin{aligned} W(u, 0) &= J^* = E_u \left\{ \int_{(0, t] \times X} r_0^S c(s, x, u) d\tilde{p}^u + \int_{(t, T_f] \times X} r_0^S c(s, x, u) d\tilde{p}^u + r_0^{T_f} G_f \right\} \\ &= E_u \left\{ \int_{(0, t] \times X} r_0^S c(s, x, u) d\tilde{p}^u \right\} + E_u \{ r_0^t \psi(u, u, t) \} \end{aligned} \quad (3.13a)$$

Now apply (3.11) with $t=0$, $h=t$ to get :

$$W(u, 0) \leq E_u \left\{ \int_{(0, t] \times X} r_0^S c(s, x, u) d\tilde{p}^u / F_t^Y \right\} + E_u \{ r_0^t W(u, t) \} \quad (3.13b)$$

Comparing (3.13a) and (3.13b) shows that

$$E_u\{\psi(u,u,t) - W(u,t)\} \leq 0.$$

But

$$\psi(u,u,t) \geq W(u,t) \quad \text{a.s.}$$

Hence we get

$$\psi(u,u,t) = W(u,t) \quad \text{a.s.} \quad (3.14)$$

which implies equality of (3.11). Conversely, if equality holds in (3.11), let $t=0$, $h=T_f$, then

$$J^* = W(u,0) = E_u\left\{\int_{(0,T_f] \times X} r_0^S(s,x,u) d\tilde{p}^u + r_0^{T_f} G_f\right\} = J(u)$$

and hence u is optimal.

This completes the proof.

Corollary 3.4.

For all $u \in U$, $t \in [0, T_f]$, the process M_t^u defined by:

$$M_t^u = r_0^t W(u,t) + E_u\left\{\int_{(0,t] \times X} r_0^S(s,x,u) d\tilde{p}^u / F_t^Y\right\} \quad (3.15)$$

is a (F_t^Y, P_u) submartingale. It is a martingale if and only if u is optimal.

Proof : This follows immediately from (3.11).

Corollary 3.5.

If $u^* \in U$ is an optimal control, then for each $t \in [0, T_f]$

$$W(u^*,t) = \psi(u^*,u^*,t) = E_{u^*}\left\{\int_{(t,T_f] \times X} r_t^S(s,x,u^*) d\tilde{p}^{u^*} + G_f / F_t^Y\right\}$$

$$W(u^*,T_f) = E_{u^*}\{G_f(\omega) / F_{T_f}^Y\}$$

Proof : Follows from (3.14) in the proof of theorem 3.3.

Remarks 3.7.

Theorem 3.3 is one of the main result in (19), being cast in a different framework from ours. They had to use the stronger relative completeness concept to prove the theorem, an unnecessary procedure, as we have shown. The advantage of using a simpler concept like the ϵ -lattice property becomes even clearer when we investigate the Markovian case, where we no longer have increasing information.

Corollary 3.4 is the version of the optimality principle proved in (56). The interpretation of M_t^u is the expected total cost evaluated at time t with the available information F_t^Y , given that control u was used up to time t , and an optimal control law used afterwards. The submartingale property means that if a control u is used for a longer time, the expected cost should increase. The martingale property says that this increase is zero if and only if u is optimal.

Since all uniformly integrable (F_t^Y, P_u) martingales have the form $E_u(\xi(\omega)/F_t^Y)$, some measurable r.v. $\xi(\omega)$; the optional sampling theorem of (45) is applicable. Hence for any two F_t^Y stopping times S, T such that $S \leq T$, a.s., we have:

$$E_u\{M_{T \wedge T_f}^u / F_{S \wedge T_f}^Y\} \geq M_{S \wedge T_f}^u \quad \text{a.s.}$$

for all $u \in U$, with equality holding if and only if u is optimal.

3.3. Complete information situation ($F_t^Y = F_t$).

Optimality conditions for the situation in which we have complete observations on the past is investigated in this section. We are treating this problem first because only in this special case do we obtain some significant results that are likely to be applicable in practical situations. Anyone dealing with such control problems who has access to the entire past observations of the process can just read this section and ignore the next one dealing with optimality conditions

with partial information.

A property that considerably simplifies the analysis in the complete information situation is that the remaining cost function at time t is independent of the past control. This is made clear by the following lemma.

Lemma 3.6.

If $F_t^Y = F_t$, then the remaining cost function $\psi(u, v, t)$ defined by (3.8) does not depend on the control u .

Proof :

From (3.8)

$$\psi(u, v, t) = E_{u, v, t} \left\{ \int_{(t, T_f]} r_t^s c(s, x, v) d\tilde{p}^v + r_t^{T_f} G_f / F_t \right\}$$

Using a result of Loeve in (44) sec.24.2, we get

$$\begin{aligned} \psi(u, v, t) &= \frac{E\{L(u, v, t) \int_{(t, T_f]} r_t^s c(s, x, v) d\tilde{p}^v + r_t^{T_f} G_f / F_t\}}{E\{L(u, v, t) / F_t\}} \\ &= \frac{E\{L_{T_f}(u, v, t) \int_{(t, T_f]} r_t^s c(s, x, v) d\tilde{p}^v + r_t^{T_f} G_f / F_t\}}{E\{L_{T_f}(u, v, t) / F_t\}} \\ &\quad \text{(by an application of} \\ &\quad \text{the iterated conditional expectation rule.)} \end{aligned}$$

$$= \frac{E\{L_t(u) L_t^{T_f}(u, v, t) \int_{(t, T_f]} r_t^s c(s, x, v) d\tilde{p}^v + r_t^{T_f} G_f / F_t\}}{E\{L_{T_f}(u, v, t) / F_t\}}$$

where

$$L_t^{T_f}(u, v, t) = \frac{L_{T_f}(u, v, t)}{L_t(u)}$$

From equation (3.4), we see that :

$$L_t^{T_f}(u, v, t) = \prod_{\substack{t < s \leq T_f \\ x_s \neq x_{s-}}} \alpha(s, v_s) \beta(s, x_s, v_s) \exp \left\{ - \int_t^{T_f} (\alpha_v(r) - 1) d\Lambda_r^c \right\} \Pi_t^{T_f}(v) .$$

where

$$\Pi_t^{T_f}(v) = \frac{\Pi_t^{T_f}(u, v, t)}{L_t(u, v, t)}$$

and is independent of u .

From lemma 2.8,

$$E\{L_t^{T_f}(v) / F_t\} = \frac{E\{L_t^{T_f}(u, v, t) / F_t\}}{L_t(u)} = 1 \quad a.s.$$

so that

$$\begin{aligned} \psi(u, v, t) &= E\{L_t^{T_f}(v) \{ \int_{(t, T_f] \times X} r_t^s c(s, x, v) dp^{\tilde{v}} + r_t^{T_f} G_f \} / F_t\} \\ &= \psi(v, t). \end{aligned}$$

and is independent of u . This completes the proof.

Corollary 3.7.

When $F_t^y = F_t$, the value function $W(u, t)$ defined in (3.9) is independent of the control u .

Proof:

$$W(u, t) = \inf_{v \in U} \psi(u, v, t) = \inf_{v \in U} \psi(v, t) = W(t)$$

independent of u .

Remarks 3.8.

The fact that there is only one value function at each time t is the main reason for treating the complete information case With such importance.

M_t^u defined in (3.15) is now given by :

$$M_t^u = \{ \int_{(0, t] \times X} r_0^s c(s, x, u) dp^{\tilde{u}} + r_0^t W(t) \} \quad (3.16)$$

and is a (F_t, P_u) submartingale for all $u \in U$, and a martingale if and only if u is optimal.

If $u^* \in U$ is optimal, then $r_0^t W(t)$ is given by

$$r_0^t W(t) = E_{u^*} \left\{ \int_{(t, T_f] \times X} r_0^S(s, x, u^*) d\tilde{p}^{u^*} + r_0^{T_f} G_f / F_t \right\} \quad (3.17)$$

For $t \in [0, T_f]$, define $\tau_t = \inf\{s \geq t : x_s \neq x_{t-}\}$. τ_t is clearly a F_t stopping time. It is the first jump time of the process after time t .

Let $\sigma_t = (\tau_t, T_f)$. Then σ_t is also a F_t stopping time.

Lemma 3.8.

For each $t \in [0, T_f]$,

$$r_0^t W(t) = \inf_{v \in U} E_v \left\{ \int_{(t, \sigma_t] \times X} r_0^S(s, x, v) d\tilde{p}^v + r_0^{\sigma_t} W(\sigma_t) / F_t \right\} \quad \text{a.s.}$$

Proof :

It is equivalent to proving that for each $u \in U$,

$$\begin{aligned} M_t^u &= \int_{(0, t] \times X} r_0^S(s, x, u) d\tilde{p}^u + r_0^t W(t) \\ &= \int_{(0, t] \times X} r_0^S(s, x, u) d\tilde{p}^u + \inf_{v \in U} E_v \left\{ \int_{(t, \sigma_t] \times X} r_0^S(s, x, v) d\tilde{p}^v + r_0^{\sigma_t} W(\sigma_t) / F_t \right\} \end{aligned}$$

Write $(u, v, t) = w$ for each $u, v \in U$. Then from the definition of the value function and optional sampling theorem,

$$\begin{aligned} M_{\sigma_t}^w &= \int_{(0, \sigma_t] \times X} r_0^S(s, x, w) d\tilde{p}^w + r_t^{\sigma_t} W(\sigma_t) \\ &\leq E_w \left\{ \int_{(0, T_f] \times X} r_0^S(s, x, w) d\tilde{p}^w + r_0^{T_f} G_f / F_{\sigma_t} \right\} \end{aligned}$$

And since $\sigma_t \geq t$ a.s., we can take conditional expectations given F_t on both sides and apply the iterated conditional expectations rule to give :

$$E_w \left\{ \int_{(0, \sigma_t] \times X} r_0^S(s, x, w) d\tilde{p}^w + r_t^{\sigma_t} W(\sigma_t) / F_t \right\}$$

$$\begin{aligned} &\leq E_W\{E_W\{\int_0^S r_0^S c(s,x,w) d\tilde{p}^w + r_0^T G_f / F_{\sigma_t}\} / F_t\} \\ &\quad (0, T_f] \times X \\ &= E_W\{\int_0^S r_0^S c(s,x,w) d\tilde{p}^w + r_0^T G_f / F_t\} \\ &\quad (0, T_f] \times X \end{aligned}$$

which can be rewritten as :

$$\begin{aligned} &\int_0^S r_0^S c(s,x,u) d\tilde{p}^u + E_V\{\int_0^S r_0^S c(s,x,v) d\tilde{p}^v + r_0^{\sigma_t} t_W(\sigma_t) / F_t\} \\ &\quad (0,t] \times X \quad (t, \sigma_t] \times X \\ &\leq \int_0^S r_0^S c(s,x,u) d\tilde{p}^u + E_V\{\int_0^S r_0^S c(s,x,v) d\tilde{p}^v + r_0^T G_f / F_t\} \\ &\quad (0,t] \times X \quad (t, T_f] \times X \end{aligned}$$

Taking $\inf_{v \in U}$ on both sides yields :

$$\begin{aligned} &\int_0^S r_0^S c(s,x,u) d\tilde{p}^u + \inf_{v \in U} E_V\{\int_0^S r_0^S c(s,x,v) d\tilde{p}^v + r_0^{\sigma_t} t_W(\sigma_t) / F_t\} \\ &\quad (0,t] \times X \quad (t, \sigma_t] \times X \\ &\leq \int_0^S r_0^S c(s,x,u) d\tilde{p}^u + r_0^T t_W(t) \\ &\quad (0,t] \times X \\ &= M_t^u \quad \text{a.s.} \end{aligned}$$

It remains to prove the inequality in the opposite direction.

Since the controls w and u agree on $[0, t]$, we use the optional sampling theorem and the submartingale property of the principle of optimality to yield :

$$\begin{aligned} M_t^u &= M_t^w \leq E_W\{M_{\sigma_t}^w / F_t\} = E_W\{\int_0^S r_0^S c(s,x,w) d\tilde{p}^w + r_0^{\sigma_t} t_W(\sigma_t) / F_t\} \\ &\quad (0, \sigma_t] \times X \\ &= \int_0^S r_0^S c(s,x,u) d\tilde{p}^u + E_V\{\int_0^S r_0^S c(s,x,v) d\tilde{p}^v + r_0^{\sigma_t} t_W(\sigma_t) / F_t\} \\ &\quad (0,t] \times X \quad (t, \sigma_t] \times X \end{aligned}$$

Taking $\inf_{v \in U}$ gives

$$M_t^u \leq \int_0^S r_0^S c(s,x,u) d\tilde{p}^u + \inf_{v \in U} E_V\{\int_0^S r_0^S c(s,x,v) d\tilde{p}^v + r_0^{\sigma_t} t_W(\sigma_t) / F_t\} \quad (3.19)$$

(3.18) and (3.19) establish the result.

Remarks 3.9.

The previous lemma states the stochastic dynamic programming equation which the value function satisfies. Using it, we have effectively reduced the problem of finding optimal controls on $[t, T_f]$ to one of finding optimal controls on $[t, \sigma_t]$. Thus if the next jump time after time t occurs before the terminal time T_f , then, conditioned on F_t , one can restrict one's attention to seeking optimal controls from the present time instant t till the next jump epoch. And due to the piecewise constant path property, an optimal control on $[t, \sigma_t]$, if it exists, and $t \in (T_{k-1}, T_k]$, will be only a function of $\omega_{k-1}(\omega)$ and t , i.e. of the form $u^*(\omega_{k-1}(\omega); t)$. This means that the optimality criteria has the same form on the interval $[t, \sigma_t]$.

This approach was also used by Stone in (54) to obtain necessary and sufficient optimality conditions for controlled semi-Markov jump processes.

Note that lemma 3.8 is even more obvious if we assume the existence of an optimal control u^* . For it is clear that optimizing over controls $v \in U$ is the same as optimizing over controls of the form (v, u^*, σ_t) , where

$$(v, u^*, \sigma_t)(s) = \begin{cases} v(s) & \text{if } s \leq \sigma_t \\ u^*(s) & \text{if } s > \sigma_t \end{cases}.$$

The following theorem is the main result of this section, it gives an optimality criteria in terms of a pointwise minimization. For this reason it is also known as the minimum principle.

Theorem 3.9. (Minimum principle.)

$u^* \in U$ is an optimal control if and only if there is a measurable function $g: \Omega \times Y \rightarrow R$ with $g \in L^1_{loc}(p)$ such that for each $t \in [0, T_f]$

$$M_t^* = J^* + \int_{(0, t] \times X} g(s, x) dq^*(s, x) \quad \text{a.s.} \quad (3.20)$$

and at almost $(d\lambda_t)$ every point, $t \in [0, T_f]$, u^* minimizes the Hamiltonian

$$\alpha(t, u_t) \int_X \{g(t, x) + r_0^t c(t, x, u_t)\} \beta(t, x, u_t) \lambda(dx, t) \quad (3.21)$$

(N.B. the superscript $*$ denotes the control u^*)

Proof:

From the principle of optimality the control u^* is optimal if and only if M_t^* is a (F_t, P_*) martingale. Hence if there exists a $g \in L_{loc}^1(p)$ such that (3.20) holds, then u^* is optimal. This proves the sufficiency part. To prove the necessity part, we need to show that a u^* satisfying (3.20) also minimizes the Hamiltonian (3.21).

Now for any control $u \in U$,

$$\begin{aligned} M_t^u &= \int_{(0,t] \times X} r_0^s c(s, x, u) dp^u + r_0^t w(t) \\ &= M_t^* + \int_{(0,t] \times X} r_0^s c(s, x, u) dp^u - \int_{(0,t] \times X} r_0^s c(s, x, u^*) dp^* \end{aligned}$$

Substituting equation (3.20) gives

$$\begin{aligned} M_t^u &= J^* + \int_{(0,t] \times X} g(s, x) (dp - dp^u + dp^u - dp^*) + \int_{(0,t] \times X} r_0^s c(s, x, u) dp^u - \int_{(0,t] \times X} r_0^s c(s, x, u^*) dp^* \\ &= J^* + \int_{(0,t] \times X} g(s, x) dq^u + \int_{(0,t] \times X} \{g(s, x) + r_0^s c(s, x, u)\} dp^u \\ &\quad - \int_{(0,t] \times X} \{g(s, x) + r_0^s c(s, x, u^*)\} dp^* \end{aligned} \quad (3.22a)$$

Writing

$$\begin{aligned} m_t^u &= \int_{(0,t] \times X} g(s, x) dq^u \\ a_t^u &= \int_{(0,t] \times X} \{ (g(s, x) + r_0^s c(s, x, u)) dp^u - (g(s, x) + r_0^s c(s, x, u^*)) dp^* \} \end{aligned}$$

so that

$$M_t^u = J^* + m_t^u + a_t^u \quad (3.22b)$$

where m_t^u is a (F_t, P_u) martingale and a_t^u is a predictable process of integrable variation. Moreover by (63), Theorem IV 32, M_t^u is a "semimartingale spéciale", and the decomposition (3.21) is unique.

Now from the Principle of optimality, M_t^u is a (F_t, P_u) submartingale. And from (45), VI T 16, M_t^u has a right continuous modification denoted also by M_t^u such that it admits the following unique decomposition.

$$M_t^u = J^* + \bar{m}_t^u + \bar{a}_t^u \quad (3.23)$$

with $\bar{m}_t^u$ being a (F_t, P_u) martingale and $\bar{a}_t^u$ a predictable increasing process. Comparing (3.22b) and (3.23), one has by uniqueness,

$$\bar{m}_t^u = m_t^u, \quad \text{and} \quad \bar{a}_t^u = a_t^u.$$

Hence we see that a_t^u is an (a.s.) increasing process. But since a_t^u can be written as

$$a_t^u = \int_{(0,t] \times X} \{ (g(s,x) + r_0^S c(s,x,u)) \alpha_u(s) \beta_u(s,x) - (g(s,x) + r_0^S c(s,x,u^*)) \alpha_*(s) \beta_*(s,x) \} \lambda(dx,s) d\Lambda_s.$$

We infer that the integrand in the above expression is non-negative, i.e.

$$\begin{aligned} & \int_X (g(s,x) + r_0^S c(s,x,u)) \alpha_u(s) \beta_u(s,x) \lambda(dx,s) \\ & - \int_X (g(s,x) + r_0^S c(s,x,u^*)) \alpha_*(s) \beta_*(s,x) \lambda(dx,s) \geq 0 \\ & \text{a.s. } d\Lambda_s. \end{aligned}$$

so that

$$\begin{aligned} & \alpha_*(s) \int_X (g(s,x) + r_0^S c(s,x,u^*)) \beta_*(s,x) \lambda(dx,s) \\ & = \min_{u \in U} \alpha(s,u) \int_X (g(s,x) + r_0^S c(s,x,u)) \beta(s,x,u) \lambda(dx,s) \quad \text{a.s. } d\Lambda_s \\ & \quad (3.24) \end{aligned}$$

Thus an optimal control u^* minimizes the Hamiltonian given by (3.21). This completes the proof.

Remarks 3.10

Equation (3.24) is a "true" minimum principle as one has a pointwise minimization over the space of control values U . This is analogous to results obtained in applying dynamic programming theory to deterministic controlled systems modelled by differential equations. Whereas in deterministic systems open and closed loop controls are equivalent, the feedback aspect here manifests itself through requiring all the terms inside the Hamiltonian (3.21) to be F_S adapted. The F_S adaptability also means that these terms may not be completely determined if the information available is partial. One thus expects less satisfactory results for this more general situation. This is dealt with in a latter section.

As we observed in remarks 2.3, integrands associated with a martingale representation have explicit representations due to the structure of the jump process. Thus one would expect $g(s,x)$ of the equation (3.20) to have an explicit form. This is obtained as follows:

Let u^* be the optimal control of theorem 3.9. From (3.20), for each $t \in [0, T_f]$, we have

$$M_t^* = J^* + \int_{(0,t] \times X} g(s,x) dq^* = \int_{(0,t] \times X} r_0^S c(s,x,u^*) d\tilde{p}^* + r_0^t W(t)$$

To enable us to use stopping time arguments like those in sec.2.4, let us extend the definition of M_t^* in (3.16) on to R^+ by defining $M_t^* = M_{T_f}^*$, for all $t \geq T_f$. Then M_t^* is a (F_t, P_*) martingale on R^+ .

From lemma 3.8, since u^* is optimal, we have

$$r_0^t W(t) = E_* \left\{ \int_{(t, \sigma_t] \times X} r_0^S c(s,x,u^*) d\tilde{p}^* + r_0^{\sigma_t} W(\sigma_t) / F_t \right\}$$

Hence M_t^* can be rewritten as

$$\begin{aligned} M_t^* &= J^* + \int_{(0,t] \times X} g(s,x) dq^* \\ &= E_* \left\{ \int_{(0,\sigma_t]} r_0^S c(s,x,u^*) d\tilde{p}^* + r_0^{\sigma_t} W(\sigma_t) / F_t \right\} \end{aligned}$$

Using the shorthand $T_{if} = T_i \wedge T_f$, we follow the conventions of sec.2.4 (2.19)-(2.21) and define for $r \geq 0$,

$$\begin{aligned} H_r^k &= F_{(r+T_{k-1}) \wedge T_k} \\ X_r^k &= M_{(r+T_{k-1}) \wedge T_k}^* - M_{T_{k-1}}^* \\ &= E_* \left\{ \int_{(0,T_{kf}]} r_0^S c(s,x,u^*) d\tilde{p}^* + r_0^{T_{kf}} W(T_{kf}) \right. \\ &\quad \left. - \int_{(0,T_{k-1f}]} r_0^S c(s,x,u^*) d\tilde{p}^* - r_0^{T_{k-1f}} W(T_{k-1f}) / H_r^k \right\} \end{aligned}$$

Then as in sec. 2.4, X_r^k is a zero mean (H_r^k, P_*) martingale.

Rearranging and noting that $\tilde{p}^{k*}(t,A)$ and $q^{k*}(t,A)$ are zero on $[0, T_{k-1}]$, we obtain

$$\begin{aligned} X_r^k &= \int_{(0, (r+T_{k-1}) \wedge T_k]} g^k(s,x) dq^{k*} \\ &= E_* \left\{ \int_{(0,T_{kf}]} r_0^S c(s,x,u^*) d\tilde{p}^{k*} + r_0^{T_{kf}} W(T_{kf}) - r_0^{T_{k-1f}} W(T_{k-1f}) / H_r^k \right\} \end{aligned}$$

So as in equation (2.22), we identify a measurable function h^k :

$$\begin{aligned} h^k(T_k, Z_k) &= h^k(\omega_{k-1}(\omega); T_k, Z_k) \\ &= \int_{(0,T_{kf}]} r_0^S c(s,x,u^*) d\tilde{p}^{k*} + r_0^{T_{kf}} W(T_{kf}) \end{aligned}$$

and using the shorthand $I_k(t) = I_{\{t \in (T_{k-1}, T_k]\}}$, $W(T_k \wedge T_f) \Big|_{T_k=t, Z_k=x} = W_k(t,x)$,

the integrand g^k is given by (2.24) for each $t \in [0, T_f]$ as:

$$\begin{aligned}
 I_k(t)g^k(t,x) &= I_k(t)\{ h^k(t,x) - I_{(t < c^{k*})} \frac{1}{F_t^{k*}} \int_{(t, \infty] \times X} h^k(s,x) d\mu_k^* \\
 &= I_k(t)\{ \int_{(0,t] \times X} r_{0X}^S c(s,x,u^*) d\tilde{p}^{k*} + r_{0X}^t W_k(t,x) \\
 &\quad - I_{(t < c^{k*})} \frac{1}{F_t^{k*}} \int_{(t, \infty] \times X} \{ \int_{(0,s \wedge T_f] \times X} r_{0X}^T c(\tau, \xi, u^*) d\tilde{p}^{k*} \\
 &\quad + r_{0X}^{s \wedge T_f} W_k(s,x) \} d\mu_k^*(s,x) \}
 \end{aligned}$$

Bringing the first integral into the second yields :

$$\begin{aligned}
 I_k(t)g^k(t,x) &= I_k(t)\{ r_{0X}^t W_k(t,x) \\
 &\quad - I_{(t < c^{k*})} \frac{1}{F_t^{k*}} \int_{(t, \infty] \times X} \{ \int_{(t, s \wedge T_f] \times X} r_{0X}^T c(\tau, \xi, u^*) d\tilde{p}^{k*} + r_{0X}^{s \wedge T_f} W_k(s,x) \} d\mu_k^*(s,x) \}
 \end{aligned}$$

where

$$c^{k*}(\eta) = \inf\{ t: F^{k*}(\eta; t) = 0 \}$$

The second term on the right is merely

$$I_k(t)E_*\{ \int_{(t, T_{kf}] \times X} r_{0X}^T c(\tau, \xi, u^*) d\tilde{p}^{k*} + r_{0X}^{T_{kf}} W(T_{kf}) / F_{T_{k-1}}, T_k > t \}$$

which upon the application of lemma 3.8 , becomes

$$\begin{aligned}
 &I_k(t)E_*\{ r_{0X}^t W(t) / F_{T_{k-1}}, T_k > t \} \\
 &= I_k(t) E_*\{ r_{0X}^t W(t) / F_{t-} \} \\
 &= I_k(t) \eta_t
 \end{aligned}$$

where

$$\eta_t = E_*\{ r_{0X}^t W(t) / F_{t-} \}$$

We thus conclude that

$$I_k(t)g^k(t,x) = I_k(t)\{ r_{0X}^t W_k(t,x) - \eta_t \}$$

The above is summarised in :

Corollary 3.10.

The integrand g in theorem 3.9 is given by

$$g(t,x) = g(t,x,\omega) = \begin{cases} g^1(t,x), & \text{if } t \leq T_1 \\ g^k(\omega_{k-1}(\omega); t,x) & \text{if } t \in (T_{k-1}, T_k] \end{cases}$$

where

$$I_k(t) g^k(\omega_{k-1}(\omega); t,x) = I_k(t) W_k(t,x) - \eta_t$$

$$\eta_t = E_* \{ r_0^t W(t) / F_{t-} \}$$

(3.25)

$$W_k(t,x) = W(T_k \wedge T_f) \Big|_{\substack{T_k=t \\ Z_k=x}}$$

$$I_k(t) = I_{\{t \in (T_{k-1}, T_k]\}}$$

Further the Hamiltonian that an optimal control minimizes is now given by :

$$H(t, u_t) = \alpha(t, u_t) \left\{ \int_X \{ r_0^t W(t,x) + r_0^t c(t,x, u_t) \} \beta(t,x, u_t) \lambda(dx, t) - \eta_t \right\}$$

(3.26)

where

$$W(t,x) = W(\sigma_t) \Big|_{\substack{\sigma_t=t \\ Z_{\sigma_t}=x}}$$

Remarks 3.11.

In the above notations, we thus have :

$$H(t, u_t^*) = \min_{u \in U} H(t, u) \quad (3.27)$$

for any optimal control $u^* \in U$.

Equation (3.27) is the main result of this section. All the terms that constitute the Hamiltonian $H(t, u_t)$ are F_t adapted. Hence one can theoretically perform the minimization (3.27) and obtain the value of the optimal control u^* at time t .

A slightly different version of the above problem is to define the cost function $\bar{J}(u)$ as :

$$\bar{J}(u) = E_u \left\{ \int_{(0, T_f]} r_0^s c(s, x_s, u_s) d\Lambda_s + r_0^{T_f} G_f(\omega) \right\} \quad (3.28)$$

The arguments of lemma 3.6, corollary 3.7, lemma 3.8 still hold true. Following a proof as in theorem 3.9, the Hamiltonian to minimise is now given by :

$$\bar{H}(t, u_t) = r_0^t c(t, x_t, u_t) + \alpha(t, u_t) \left\{ \int_X r_0^t W(t, x) \beta(t, x, u_t) \lambda(dx, t) - \eta_t \right\} \quad (3.29)$$

The cost function $\bar{J}(u)$ has the same form as Rishel's in (50), where Λ_s is taken to be s . The Hamiltonian $\bar{H}$ is indeed also in close agreement to Rishel's version given in equation (39) of his paper. On the other hand the results stated here are not restricted only to the special case $\Lambda_s = s$ but to more general forms of underlying probability measures.

The next theorem derives a differential equation analogous to Hamilton-Jacobi-Bellman equation obtained in Dynamic programming theory applied to dynamical systems. Since it is closely coupled to the minimum principle, we can also look at it as an "adjoint equation" analogous to that of the Pontryagin's maximum principle.

Theorem 3.11.

If $u^* \in U$ is an optimal control and $\Lambda(t)$ is continuous in t , then the following differentiation formula holds :

$$\frac{d\eta_t}{d\Lambda_t} = -H(t, u_t^*) \quad , \quad \text{each } t \in [0, T_f] \quad (3.30)$$

where η_t is defined in (3.25).

Proof :

From lemma 3.8 and the definition of η_t in (3.25), we have

$$\eta_t = E_* \left\{ \int_{(t, \sigma_t] \times X} r_0^S c(s, x, u^*) d\tilde{p}^* + r_0^{\sigma_t} W(\sigma_t) / F_{t-} \right\}$$

so that we get

$$\begin{aligned} I_k(t) \eta_t &= I_k(t) E_* \left\{ \int_{(t, T_{kf}] \times X} r_0^S c(s, x, u^*) d\tilde{p}^* + r_0^{T_{kf}} W(T_{kf}) / F_{T_{k-1}, T_k > t} \right\} \\ &= I_k(t) \frac{1}{F_t^{k*}} \int_{(t, \infty] \times X} \left\{ \int_{(t, s] \times X} r_0 c(\tau, \xi, u^*) d\tilde{p}^* + r_0^S W_k(s, x) \right\} I_{(s \leq T_f)} \\ &\quad + \left\{ \int_{(t, T_f] \times X} r_0 c(\tau, \xi, u^*) d\tilde{p}^* + r_0^{T_f} G_f \right\} I_{(s > T_f)} \} d\mu_k^*(s, x) \end{aligned}$$

Now since $\Lambda(t)$ is continuous in t , we have

$$\begin{aligned} F_t^{k*} &= \exp \left\{ - \int_0^t \alpha_k^*(r) d\Lambda_r^k \right\} \\ d\mu_k^*(s, x) &= \beta_k^*(s, x) \lambda^k(dx, s) dF_s^{k*} \\ &= F_s^{k*} \alpha_k^*(s) \beta_k^*(s, x) \lambda^k(dx, s) d\Lambda_s^k \end{aligned}$$

which means η_t is differentiable w.r.t. Λ_t , so that for each $t \in [0, T_f]$

$$\begin{aligned} I_k(t) \frac{d\eta_t}{d\Lambda_t^k} &= I_k(t) \left\{ \alpha_k^*(t) \eta_t \right. \\ &\quad + \frac{1}{F_t^{k*}} \frac{d}{d\Lambda_t^k} \int_{(t, \infty] \times X} \left\{ \int_{(t, s] \times X} r_0 c(\tau, \xi, u^*) d\tilde{p}^* I_{(s \leq T_f)} + \int_{(t, T_f] \times X} r_0 c(\tau, \xi, u^*) d\tilde{p}^* I_{(s > T_f)} \right\} \\ &\quad \cdot d\mu_k^*(s, x) \\ &\quad \left. + \frac{1}{F_t^{k*}} \frac{d}{d\Lambda_t^k} \int_{(t, \infty] \times X} \left\{ r_0^S W_k(s, x) I_{(s \leq T_f)} + r_0^{T_f} G_f I_{(s > T_f)} \right\} d\mu_k^*(s, x) \right\} \end{aligned}$$

The last term on the right is

$$\begin{aligned} &-I_k(t) \frac{1}{F_t^{k*}} \int_X r_0^S W_k(t, x) F_t^{k*} \alpha_k^*(t) \beta_k^*(t, x) \lambda^k(dx, t) \\ &= -\alpha_k^*(t) \int_X r_0^S W_k(t, x) \beta_k^*(t, x) \lambda^k(dx, t) \end{aligned}$$

The middle term on the right is, using the Leibniz's formula for differentiation,

$$-I_k(t) \int_X r_0^t c(\tau, \xi, u^*) \alpha_k^*(t) \beta_k^*(t, x) \lambda^k(dx, t)$$

so that we finally have

$$I_k(t) \frac{d\eta_t^k}{d\Lambda_t^k} = I_k(t) \alpha_k^*(t) \left\{ \eta_t - \int_X \{ r_0^t W_k(t, x) + r_0^t c(t, x, u_t^*) \} \beta_k^*(t, x) \lambda^k(dx, t) \right\}$$

and since this is valid for all $t \in [0, T_f]$, $k \in Z^+$, we conclude that

$$\begin{aligned} \frac{d\eta_t}{d\Lambda_t} &= \alpha^*(t) \left\{ \eta_t - \int_X \{ r_0^t W(t, x) + c(t, x, u_t^*) \} \beta^*(t, x) \lambda(dx, t) \right\} \\ &= -H(t, u_t^*) \quad (\text{using (3.26)}) \end{aligned}$$

This completes the proof.

Corollary 3.12.

If $u^* \in U$ is an optimal control and $\Lambda(t)$ is continuous then,

$$\frac{d\eta_t}{d\Lambda_t} = - \min_{u \in U} H(t, u) = -H(t, u_t^*) \quad (3.31)$$

for all $t \in [0, T_f]$.

Proof :

Follows from the minimum principle and the previous theorem.

Remarks 3.12.

(3.31) is in a form closely similar to Pliska's version for controlled Markov jump processes. Rishel too had a similar "adjoint" equation. Observe that for $r_0^t W(t)$ to be differentiable, the base rate Λ_t must be continuous (e.g. Lebesgue's measure). When Λ_t is not necessarily continuous, one can, in place of (3.30), derived an

integral equation. This is

$$\begin{aligned} \eta_t = J^* + \int_{(0,t]} \left\{ r_0^S W(s) - \frac{\gamma(s)}{1 - \alpha^*(s) \Delta \Lambda_s} \right\} \alpha^*(s) d\Lambda_s \\ + \sum_{s \leq t} \left\{ \frac{(\alpha^*(s) \Delta \Lambda_s)^2}{1 - \alpha^*(s) \Delta \Lambda_s} \right\} \gamma(s) \end{aligned} \quad (3.32)$$

where

$$\gamma(s) = \int_X \{ r_0^S W(s, x) + r_0^S c(s, x, u_s^*) \beta(s, x, u_s^*) \lambda(dx, s)$$

The one jump version of the above formula is derived in (20). Note that (3.32) reduces to (3.30) when $\Delta \Lambda_t = 0$, all t , as it should.

3.4. Meyer's method.

From Meyer's book (45), (VII T 28, T29), we know that for each right continuous potential of class (D), there exists an integrable, natural, increasing unique process (A_t) , which generates x_t and such that for every stopping time T (w.r.t. F_t in our case) we have

$$A_T = w \lim_{h \rightarrow 0} A_T^h$$

where the weak limit is taken in $\sigma(L_1, L_\infty)$, and where

$$A_t^h = \frac{1}{h} \int_0^t \{ x_s - E(x_{s+h} / F_t) \} ds, \quad t \in \mathbb{R}^+$$

We show in this section how the above result is related to our minimum principle. The methods used here will also be required in the next section dealing with partial information. Since Lebesgue's measure is used by Meyer in the above formula, we assume $\Lambda_s = s$ in the rest of this section. However, it is almost certain that one can replace s by more general forms of increasing functions without

invalidating Meyer's proofs of the theorems VII T28, T29 in (45). This means that the results in the sequel can be readily generalized to include more general forms of Λ_S .

Define for each $u \in U$, $t \in [0, T_f]$

$$\begin{aligned} w(u, t) &= r_0^t \psi(u, t) - r_0^t W(t) \\ &= E_u \left\{ \int_{(0, T_f] \times X} r_0^s c(s, x, u) d\tilde{p}^u + r_0^{T_f} G_f / F_t \right\} - M_t^u \end{aligned} \quad (3.33)$$

The first term on the right is a right continuous uniformly integrable martingale while the second is a (F_t, P_u) right continuous supermartingale (due to the negative sign) by the principle of optimality. And since $\lim_{t \rightarrow \infty} w(u, t) = 0$, $w(u, t)$ is a right continuous potential. It is of class (D) due to the uniform boundedness of the cost function. Hence $w(u, t)$ admits a Doob-Meyer decomposition, and there exists an integrable, increasing process A_t^u which generates $w(u, t)$, further this process is unique, and

$$A_t^u = w \lim_{h \rightarrow 0} \frac{1}{h} \int_{(0, t]} \{ w(u, s) - E_u(w(u, s+h) / F_s) \} ds \quad (3.34)$$

Now since M_t^u is a (F_t, P_u) submartingale, it also admits a Doob-Meyer decomposition. The increasing process associated with it must also be A_t^u as M_t^u and $w(u, t)$ differs by a (F_t, P_u) martingale. Hence we have

$$M_t^u = J^* + N_t^u + A_t^u$$

some (F_t, P_u) martingale N_t^u . So we can rewrite A_t^u of (3.34) as :

$$A_t^u = w \lim_{h \rightarrow 0} \frac{1}{h} \int_{(0, t]} \{ -M_s^u + E_u(M_{s+h}^u / F_s) \} ds \quad (3.35)$$

Now define for $u \in U$, $t \in [0, T_f]$

$$\begin{aligned} B_t^u &= w \lim_{h \rightarrow 0} \frac{1}{h} E_u \{ w(u, t) - w(u, t+h) / F_t \} \\ &= w \lim_{h \rightarrow 0} \frac{1}{h} E_u \{ M_{t+h}^u - M_t^u / F_t \} \end{aligned} \quad (3.36)$$

We know the above limit exists by following an argument similar to that in lemma 4.1 of (19).

The following lemma then proves that :

$$A_t^u = \int_{(0,t]} B_s^u ds \quad \text{a.s.} \quad (3.37)$$

Lemma 3.13.

Let $(\Omega, \mathcal{F}, P)$ be a given probability space and $f, f_n : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a sequence of bounded jointly measurable functions such that for each $s \in \mathbb{R}^+$

$$f_n(\cdot, s) \xrightarrow{n \rightarrow \infty} f(\cdot, s) \quad \text{weakly in } \sigma(L_1, L_\infty)$$

Then

$$\int_0^t f_n(\cdot, s) ds \xrightarrow{n \rightarrow \infty} \int_0^t f(\cdot, s) ds, \quad \text{for each } t \in \mathbb{R}^+ \\ \text{weakly in } \sigma(L_1, L_\infty)$$

Proof :

For $\theta \in L_\infty(\Omega)$ we have

$$\begin{aligned} & \int_{\Omega} \theta(\omega) \int_0^t \{ f_n(\omega, s) - f(\omega, s) \} ds \, dP(\omega) \\ &= \int_0^t \int_{\Omega} \theta(\omega) \{ f_n(\omega, s) - f(\omega, s) \} \, dP(\omega) \, ds \\ &= \int_0^t a_n(s) \, ds \end{aligned} \quad \begin{aligned} & \text{(having used Fubini's} \\ & \text{theorem)} \\ & (3.38) \end{aligned}$$

where by hypothesis, for each $s \in \mathbb{R}^+$

$$a_n(s) = \int_{\Omega} \theta(\omega) \{ f_n(\omega, s) - f(\omega, s) \} \, dP(\omega) \xrightarrow{n \rightarrow \infty} 0$$

But since the functions are all bounded, we can apply Lebesgue's bounded convergence theorem to give :

$$\int_0^t a_n(s) \, ds \xrightarrow{n \rightarrow \infty} 0 \quad \text{each } t \in \mathbb{R}^+$$

From (3.38), we now see that this implies the required result.

This completes the proof.

Remarks 3.13.

Compare the above lemma with lemma 4.1 of (19). A_t^u always has the form (3.37) by theorem VII T29 of (45). The above lemma shows that in this case B_t^u is given by (3.36). This is helpful in obtaining an explicit expression for B_t^u .

Lemma 3.14.

Suppose $u^* \in U$ is an optimal control, then for each $t \in [0, T_f]$

$$B_t^u = H(t, u_t) - H(t, u_t^*) \quad \text{a.s.} \quad (3.39)$$

where $H(t, u_t)$ is the Hamiltonian defined in (3.26).

Proof :

In the following, we shall use the strong limit, as it implies the weak limit if the former exists.

From (3.36) and the definition of M_t^u in (3.16), we have

$$B_t^u = \lim_{h \rightarrow 0} \frac{1}{h} \{ E_u \{ r_0^{t+h} W(t+h) - r_0^t W(t) / F_t \} + E_u \{ \int_{(t, t+h] \times X} r_0^S c(s, x, u) d\tilde{p}^u / F_t \} \}$$

if the limit exists.

Again using the shorthand $I_k(t) = I_{\{t \in (T_{k-1}, T_k]\}}$ we have

$$\begin{aligned} I_k(t) E_u \{ r_0^{t+h} W(t+h) / F_t \} &= I_k(t) \{ \frac{1}{F_t^{ku}} \int_{(t, t+h] \times X} r_0^S W_k(s, x) d\mu_k^u \\ &\quad + \frac{F_{t+h}^{ku}}{F_t^{ku}} r_0^{t+h} W(t+h) \} \end{aligned}$$

and

$$\begin{aligned} I_k(t) E_u \{ \int_{(t, t+h] \times X} r_0^S c(s, x, u) d\tilde{p}^u / F_t \} \\ = I_k(t) \{ \frac{F_{t+h}^{ku}}{F_t^{ku}} \int_{(t, t+h] \times X} r_0^S c(s, x, u) \alpha_k^u(s) \beta_k^u(s, x) \lambda^k(dx, s) ds + o(h) \} \end{aligned}$$

Hence provided the limit exists,

$$\begin{aligned} I_k(t) B_t^u &= I_k(t) \left\{ \lim_{h \rightarrow 0} \frac{1}{h} \left\{ \frac{1}{F_t^{ku}} \int_{(t, t+h] \times X} r_{0k}^{S W}(s, x) d\mu_k^u + \frac{F_{t+h}^{ku}}{F_t^{ku}} r_0^{t+h} W(t+h) \right. \right. \\ &\quad \left. \left. - r_0^t W(t) + \frac{F_{t+h}^{ku}}{F_t^{ku}} \int_{(t, t+h] \times X} r_{0k}^{S C}(s, x, u) \alpha_k^u(s) \beta_k^u(s, x) \lambda^k(dx, s) ds \right\} \right\} \end{aligned} \quad (3.40)$$

The first term on the right of (3.40) is

$$\begin{aligned} I_k(t) \lim_{h \rightarrow 0} \frac{1}{h} \frac{1}{F_t^{ku}} \int_{(t, t+h] \times X} r_{0k}^{S W}(s, x) \alpha_k^u(s) F_s^{ku} \beta_k^u(s, x) \lambda^k(dx, s) ds \\ = I_k(t) \alpha_k^u(t) \int_X r_{0k}^t W(t, x) \beta_k^u(t, x) \lambda^k(dx, t) \end{aligned}$$

The second and third term of (3.40) are

$$\begin{aligned} I_k(t) \lim_{h \rightarrow 0} \frac{1}{h} \left\{ \frac{F_{t+h}^{ku}}{F_t^{ku}} r_0^{t+h} W(t+h) - r_0^t W(t) \right\} \\ = I_k(t) \lim_{h \rightarrow 0} \left\{ \frac{F_{t+h}^{ku} - F_t^{ku}}{h F_t^{ku}} r_0^{t+h} W(t+h) + \frac{r_0^{t+h} W(t+h) - r_0^t W(t)}{h} \right\} \end{aligned}$$

Now since $\Lambda_t = t$, F_t^{ku} is absolutely continuous. And since for each $t \in \mathbb{R}^+$ $P_*(T_k = t / F_{T_{k-1}}) = 0$, we have

$$\begin{aligned} I_k(t) \eta_t &= I_k(t) E_* \{ r_0^t W(t) / F_{T_{k-1}}, T_k > t \} \\ &= I_k(t) E_* \{ r_0^t W(t) / F_{T_{k-1}}, T_k \geq t \} \\ &= I_k(t) E_* \{ r_0^t W(t) / F_t \} \\ &= I_k(t) r_0^t W(t) \end{aligned}$$

so that we can use theorem 3.11 to evaluate the last limit as

$$-I_k(t) \{ \alpha_k^u(t) \eta_t + H(t, u_t^*) \}$$

The last term on the right of (3.40) is easily evaluated as

$$I_k(t) \alpha_k^u(t) \int_X r_0^t c(t, x, u) \beta_k^u(t, x) \lambda^k(dx, t)$$

Combining the above limits, we obtain

$$\begin{aligned} I_k(t) B_t^u &= I_k(t) \alpha_k^u(t) \{ -\eta_t + \int_X \{ r_0^t W_k(t, x) + r_0^t c(t, x, u_t) \} \beta_k^u(t, x) \lambda^k(dx, t) \} \\ &\quad - H(t, u_t^*) \\ &= H(t, u_t) - H(t, u_t^*) \end{aligned}$$

This establishes the result.

From the above result, one immediately obtains the following corollary, a restatement of (3.27).

Corollary 3.15.

If $u^* \in U$ is an optimal control, then for all $u \in U$, $t \in [0, T_f]$,

$$B_t^u = H(t, u_t) - H(t, u_t^*) \geq 0 \quad \text{a.s.}$$

Proof :

This is easily inferred from (3.37), since

$$A_t^u = \int_0^t B_s^u ds$$

is (a.s.) an increasing process for all $u \in U$, implying the result

Remarks 3.14.

The above methods show how Meyer's constructional formula can be used effectively in jump processes theory. Such explicit calculations are probably not possible for continuous path processes.

The results in this section is ^{an} alternative interpretation of those derived in the last. They will be required in the next section where we consider the control problem for partial observations.

3.5. Optimal control with partial observations.

Often in practice, one does not have complete access to the entire past history of the jump process. Instead, only part of it may be available to the controller. This information is depicted by the observations sigma field F_t^Y of sec.3.1 and sec.3.2 and has the property $F_t^Y \subset F_t$, $F_s^Y \subset F_t^Y$ if $s \leq t$. In this section we shall use the framework of sec.3.1 and sec.3.2 to obtain optimality conditions for the problem (3.7) when the information available is partial. We have treated the complete information case before this because the results there really stand on their own and are needed later to show the existence of optimal controls. Moreover, these previous results are the more significant ones, as far as implementation is concerned, and hence should predominate the present apparently more general situation.

We assume throughout this section that $\Lambda_t = t$. The partial observation sigma field F_t^Y is defined as follows.

Let $(Z, \mathcal{Z})$ be a Blackwell space and $f : X \rightarrow Z$ be a measurable function. Let $y_t = f(x_t)$ and $F_t^Y = \sigma\{y_s, s \leq t\}$, where F_t^Y is assumed completed with respect to P . Since $f(\cdot)$ is a deterministic function, (y_t, F_t^Y, P) will also be a jump process, and represents the observed process derived from x_t . For example, one may have $X = \mathbb{R}^n$, $Z = \mathbb{R}^m$ with $m < n$, so that only certain of the co'ordinates of x_t are observable. The jump times of y_t must by definition coincide with some or all the jump times of x_t but not necessarily conversely. This means that, in contrast to the complete information situation, one may not be certain of the number of jumps that have occurred, or the actual jump sizes of those that are observed. Hence, the controller is additionally burdened by the task of having to predict the present rate and conditional jump distributions. This deficiency will manifest itself in the lack of a genuine minimum principle in the sense of giving a pointwise minimization. Note also that the measurable function f

cannot include any random component since the sigma field F_t^Y must be fixed for each t . Nonetheless, one can include these additional random elements by suitably extending the state space X .

For the derived jump process y_t , define for $t \in \mathbb{R}^+$, $B \in \mathcal{Z}$, the counting process

$$\begin{aligned} p_y(t, B) &= \sum_{s \leq t} I_{\{y_s(\omega) \neq y_{s-}(\omega), y_s(\omega) \in B\}} \\ &= \sum_{T_i \leq t} I_{\{f(X_{T_i}) \neq f(X_{T_i-}), f(X_{T_i}) \in B\}} \\ &= \int_{(0, t] \times X} I_{\{f(x) \neq f(x_{s-}), f(x) \in B\}} P(ds, dx) \end{aligned} \quad (3.41)$$

As in the x_t process, $p_y(t, B)$ counts the number of jumps of y_t before time t which end up in $B \in \mathcal{Z}$. Note that only 'genuine' jumps are counted to ensure that the jump times of y_t are actually F_t^Y stopping times.

Lemma 3.16.

For all $B \in \mathcal{Z}$, the predictable increasing process $\tilde{p}_y(t, B)$ associated with $p_y(t, B)$ under the measure P is given by :

$$\tilde{p}_y(t, B) = \int_{(0, t]} E \left\{ \int_X I_B(s, x) \lambda(dx, s) / F_s^Y \right\} ds \quad \text{a.s. } P \quad (3.42)$$

where

$$I_B(s, x)(\omega) = I_B(s, x) = I_{\{f(x) \neq f(x_{s-}), f(x) \in B\}}$$

Proof :

For $B \in \mathcal{Z}$, $t \in \mathbb{R}^+$, let

$$q_y(t, B) = p_y(t, B) - \int_{(0, t]} E \left\{ \int_X I_B(s, x) \lambda(dx, s) / F_s^Y \right\} ds$$

Using (3.41), this is :

$$q_y(t, B) = \int_{(0, t] \times X} I_B(s, x) \{ p(ds, dx) - \lambda(dx, s) ds \} \\ + \int_{(0, t] \times X} \{ \int I_B(s, x) \lambda(dx, s) - E \{ \int I_B(s, x) \lambda(dx, s) / F_s^y \} \} ds$$

Now for $h > 0$, we have

$$E \{ q_y(t+h, B) - q_y(t, B) / F_t^y \} = E \{ \int_{(t, t+h] \times X} I_B(s, x) dq(s, x) / F_t^y \} \\ + E \{ \int_{(t, t+h] \times X} \{ \int I_B(s, x) \lambda(dx, s) - E \{ \int I_B(s, x) \lambda(dx, s) / F_s^y \} \} ds / F_t^y \}$$

Since $F_t^y \subset F_t$, and $q(t, A)$ is a (F_t, P) martingale, each $A \in \mathcal{S}$, the first term on the right disappears. Similarly, since $s > t$ inside the integral of the second term, we apply VT 25 of (21) and infer that it is zero as well. Hence $q_y(t, B)$ is a (F_t^y, P) martingale and by the uniqueness of Doob-Meyer decomposition, the compensator term of $p_y(t, B)$ is

$$\tilde{p}_y(t, B) = \int_{(0, t] \times X} E \{ \int I_B(s, x) \lambda(dx, s) / F_s^y \} ds$$

This completes the proof.

The above lemma is adapted from an argument of (8). Note that $\tilde{p}_y(t, B)$ is necessarily an increasing process since $\lambda(A, s)$ is always positive.

Corollary 3.17.

With respect to the probability measure P_u , $u \in U$, the predictable increasing process $\tilde{p}_y^u(t, B)$ associated with $p_y(t, B)$ is :

$$\tilde{p}_y^u(t, B) = \int_{(0, t] \times X} E_u \{ \alpha_u(s) \int I_B(s, x) \lambda(dx, s) / F_s^y \} ds \quad \text{a.s.} \\ \text{each } t \in \mathbb{R}^+, B \in \mathcal{Z} \quad (3.43)$$

Furthermore the process defined by

$$q_y^u(t, B) = p_y(t, B) - \tilde{p}_y^u(t, B) \quad (3.44)$$

is a (F_t^y, P_u) martingale.

The interpretations of $p_y(t, B)$, $\tilde{p}_y(t, B)$, $q_y(t, B)$ ($p_y(t, B)$, $\tilde{p}_y^u(t, B)$, $q_y^u(t, B)$, resp.) are analogous to $p(t, A)$, $\tilde{p}(t, A)$, $q(t, A)$ ($p(t, A)$, $\tilde{p}^u(t, A)$, $q^u(t, A)$ resp.) of the basic process x_t under measure P (P_u resp.).

One would now naturally wish to obtain a local descriptions of the observed process y_t if one exists. It seems that the following assumption is necessary to ensure a meaningful interpretation.

Assume for each $t \in R^+$, $x' \in X$

$$\lambda(D(x'), t) > 0 \quad (3.45)$$

where

$$D(x') = \{x \in X : f(x) \neq f(x')\}$$

This merely says that if a jump occurs at time t , then whatever the present state is, the jump is observable with positive probability. As a result of this, for each $s \in R^+$, we have

$$\int_X I_Z(s, x) \lambda(dx, s) = \lambda(D(x_{s-}), s) > 0$$

and

$$\int_X h(s, x) I_Z(s, x) \lambda(dx, s) > 0$$

for any strictly positive measurable function $h(s, x)$.

We now formulate the local descriptions of y_t under measure P . For $t \in R^+$, let

$$\Lambda_y(t) = \tilde{p}_y(t, Z) = \int_{(0, t]} E \left\{ \int_X I_Z(s, x) \lambda(dx, s) / F_s^y \right\} ds \quad (3.46)$$

so that by analogy with the process x_t , $\Lambda_y(t)$ is the integrated rate of y_t . The conditional jump rate is then given by :

$$\begin{aligned}\lambda_y(B,t) &= \frac{\tilde{dp}_y(t,B)}{d\Lambda_y} = \frac{\tilde{dp}_y(t,B)}{dt} \cdot \frac{1}{\frac{d\Lambda_y}{dt}} \\ &= \frac{E\left\{ \int_X I_B(t,x) \lambda(dx,t) / F_t^Y \right\}}{E\left\{ \int_X I_Z(t,x) \lambda(dx,t) / F_t^Y \right\}} \quad \text{each } B \in \mathcal{Z}, t \in \mathbb{R}^+.\end{aligned}\tag{3.47}$$

(3.47) is well defined since the denominator of the right hand side is strictly positive by assumption (3.45). Further, $\lambda_y(\cdot, t)$ is probability measure as it should be. Hence the local description of y_t is the pair (Λ_y, λ_y) given by (3.46) and (3.47).

Similarly, since $\alpha_u(t)$, $\beta_u(t,x)$ are strictly positive for each $u \in U$, $(t,x) \in \mathbb{R}^+ \times X$, the local descriptions of y_t under measure P_u is given by :

$$\begin{aligned}\Lambda_y^u(t) &= \tilde{p}_y^u(t, Z) = \int_{(0,t]} E_u\left\{ \alpha_u(s) \int_X \beta_u(s,x) I_Z(s,x) \lambda(dx,s) / F_s^Y \right\} ds \\ \lambda_y^u(B,t) &= \frac{\tilde{dp}_y^u(t,B)}{dt} \cdot \frac{1}{\frac{d\Lambda_y^u}{dt}} \\ &= \frac{E_u\left\{ \alpha_u(t) \int_X \beta_u(t,x) I_B(t,x) \lambda(dx,t) / F_t^Y \right\}}{E_u\left\{ \alpha_u(t) \int_X \beta_u(t,x) I_Z(t,x) \lambda(dx,t) / F_t^Y \right\}}, \quad \text{each } t \in \mathbb{R}^+, B \in \mathcal{Z}\end{aligned}\tag{3.48}$$

Since $\Lambda_y^u \sim ds$, $\Lambda_y \sim ds$, we also have $\Lambda_y^u \sim \Lambda_y$. So the Radon Nikodym derivative corresponding to the change of measure $P \rightarrow P_u$ is

$$\begin{aligned}\alpha_y^u(t) &= \frac{d\Lambda_y^u}{d\Lambda_y} = \frac{d\Lambda_y^u}{dt} \frac{dt}{d\Lambda_y} \\ &= \frac{E_u\left\{ \alpha_u(t) \int_X \beta_u(t,x) I_Z(t,x) \lambda(dx,t) / F_t^Y \right\}}{E\left\{ \int_X I_Z(t,x) \lambda(dx,t) / F_t^Y \right\}}\end{aligned}\tag{3.49}$$

λ_y^u may not be absolutely continuous with respect to λ_y , but as we

shall see, this is not an essential point.

Remarks 3.15.

All the predictable projections of the various terms with respect to (F_t^Y, P_u) may not be independent of the past control. This is a feature of most partial information stochastic control problems. The reasons are as follows:

Let $h(\omega)$ be any integrable F_{T_f} measurable function (e.g. the remaining cost function). Then using the measure $P_{(u,v,t)}$, $u, v \in U$, $t \in [0, T_f]$ corresponding to the control :

$$(u, v, t) = \begin{cases} u(s) & \text{if } s \leq t \\ v(s) & \text{if } s > t. \end{cases}$$

we have from lemma 2.8 and sec.24.2 of (44)

$$E_{u,v,t}\{h(\omega)/F_t^Y\} = \frac{E\{L_0^t(u)L_t^{T_f}(v)h(\omega)/F_t^Y\}}{E\{L_0^t(u)L_t^{T_f}(v)h(\omega)/F_t^Y\}}$$

As $E\{L_t^{T_f}(v)/F_t\} = 1$, the denominator is $E\{L_0^t(u)/F_t^Y\}$. But although $L_0^t(u)$ is F_t measurable, it is not necessarily F_t^Y measurable, so that

$$E_{u,v,t}\{h(\omega)/F_t^Y\} = \frac{E\{L_0^t(u)L_t^{T_f}(v)h(\omega)/F_t^Y\}}{E\{L_0^t(u)/F_t^Y\}}$$

and is in general dependent on u as well as v . This explains why the partial information value function $r_0^t W(u, t)$ defined in (3.8) must also be indexed by u , in contrast to the complete observation situation.

Having described the process y_t and related processes, we now come to the control problem itself. First, let us state a couple lemmas.

Lemma 3.18.

For each $u \in U$, $t \in [0, T_f]$, the following weak limit in $\sigma(L_1, L_\infty)$ exists.

$$\bar{A}^u(t) = w \lim_{h \rightarrow 0} \frac{1}{h} E_u \{ r_0^{t+h} W(u, t+h) - r_0^t W(u, t) / F_t^Y \} \quad (3.50)$$

Proof :

From the principle of optimality we know that

$$M_t^u = E_u \left\{ \int_{(0,t] \times X} r_0^s c(s, x, u) d\tilde{p}^u / F_t^Y \right\} + r_0^t W(u, t)$$

is a (F_t^Y, P_u) submartingale, so as in sec.3.4, we can follow an argument of (19) lemma 4.1 to infer that the following weak limit in $\sigma(L_1, L_\infty)$ exists.

$$\begin{aligned} \zeta^u(t) &= w \lim_{h \rightarrow 0} \frac{1}{h} E_u \{ M_{t+h}^u - M_t^u / F_t^Y \} \\ &= w \lim_{h \rightarrow 0} \frac{1}{h} \{ E_u \{ r_0^{t+h} W(u, t+h) - r_0^t W(u, t) / F_t^Y \} \\ &\quad + E_u \{ \int_{(t, t+h] \times X} r_0^s c(s, x, u) d\tilde{p}^u / F_t^Y \} \} \end{aligned} \quad (3.51)$$

But the last term on the right is by VT25 of (21) equal to

$$\theta^u(t) = w \lim_{h \rightarrow 0} \frac{1}{h} \int_{(t, t+h] \times X} E_u \{ \int r_0^s c(s, x, u) \alpha_u(s) \beta_u(s, x) \lambda(dx, s) / F_s^Y \} ds$$

which clearly exists. Thus looking at (3.51), we infer that $\bar{A}^u(t)$ must exist. This completes the proof.

Lemma 3.19.

For $u \in U, t \in [0, T_f]$, the following weak limit in $\sigma(L_1, L_\infty)$ exists.

$$A^u(t) = w \lim_{h \rightarrow 0} \frac{1}{h} \int_{(0,t]} E_u \{ r_0^{s+h} W(u, s+h) - r_0^s W(u, s) / F_s^Y \} ds \quad (3.52)$$

and $r_0^t W(u, t)$ admits the decomposition

$$r_0^t W(u, t) = J^* + A^u(t) + n^u(t) \quad (3.53)$$

where $n^u(t)$ is some (F_t^Y, P_u) martingale.

Proof: One can follow an argument similar to that of (8) sec.4.1.

Remarks 3.16.

If $u \in U$ in Lemma 3.19 is 'value decreasing', i.e. if $r_0^t W(u, t)$ is a (F_t^Y, P_u) supermartingale, then the limit in (3.52) would exist straightaway as a result of Meyer's decomposition. The arguments of (8) sec.4.1 get rid of this unnecessary assumption so that the decomposition (3.53) is valid for all $u \in U$, and not just value decreasing ones. $A^u(t)$ of (3.52) is not necessarily an increasing process, though it is clearly F_t^Y predictable. It is increasing if u is value decreasing.

The principle of optimality, theorem 3.3, states that M_t^u is a (F_t^Y, P_u) submartingale for all $u \in U$ and a martingale if and only if u is optimal. Hence for $h > 0$, such that $t+h \in [t, T_f]$, we have

$$E_u \{ r_0^{t+h} W(u, t+h) - r_0^t W(u, t) + \int_{(t, t+h] \times X} r_0^s c(s, x, u) d\tilde{p}^u / F_t^Y \} \geq 0$$

with equality if and only if u is optimal.

But, by VT25 of (21), we have

$$\begin{aligned} & E_u \{ \int_{(t, t+h] \times X} r_0^s c(s, x, u) d\tilde{p}^u / F_t^Y \} \\ &= E_u \{ \int_{(t, t+h] \times X} E_u \{ \int_0^s r_0^s c(s, x, u) \alpha_u(s) \beta_u(s, x) \lambda(dx, s) / F_s^Y \} ds / F_t^Y \} \\ &= E_u \{ \int_{(0, t+h] \times X} E_u \{ \int_0^s r_0^s c(s, x, u) \alpha_u(s) \beta_u(s, x) \lambda(dx, s) / F_s^Y \} ds \\ &= \int_{(0, t] \times X} E_u \{ \int_0^s r_0^s c(s, x, u) \alpha_u(s) \beta_u(s, x) \lambda(dx, s) / F_s^Y \} ds / F_t^Y \} \end{aligned}$$

Therefore, if we now define

$$\bar{M}^u(t) = \int_{(0,t]} E_u \left\{ \int_X r_0^S c(s,x,u) \alpha_u(s) \beta_u(s,x) \lambda(dx,s) / F_s^Y \right\} ds + r_0^t W(u,t) \quad (3.54)$$

then $\bar{M}^u(t)$ is also a (F_t^Y, P_u) submartingale for all $u \in U$ and a martingale if and only if u is optimal. Using (3.53) and (3.54) we get

$$\bar{M}^u(t) = J^* + n^u(t) + \int_{(0,t]} E_u \left\{ \int_X r_0^S c(s,x,u) \alpha_u(s) \beta_u(s,x) \lambda(dx,s) / F_s^Y \right\} ds + A^u(t) \quad (3.55)$$

Since the weak limit $\bar{A}^u(t)$ of lemma 3.18 is known to exist, we conclude from lemma 3.13 that

$$A^u(t) = \int_0^t \bar{A}^u(s) ds$$

Using this, (3.55) becomes

$$\bar{M}^u(t) = J^* + n^u(t) + \int_{(0,t]} \left\{ E_u \left\{ \int_X r_0^S c(s,x,u) \alpha_u(s) \beta_u(s,x) \lambda(dx,s) / F_s^Y \right\} + \bar{A}^u(s) \right\} ds. \quad (3.56)$$

Now the last term on the right of (3.56) is certainly F_t^Y predictable, which means that (3.56) is also the Doob-Meyer decomposition of $\bar{M}^u(t)$. And as $\bar{M}^u(t)$ is a (F_t^Y, P_u) submartingale, the process

$$\int_{(0,t]} \left\{ E_u \left\{ \int_X r_0^S c(s,x,u) \alpha_u(s) \beta_u(s,x) \lambda(dx,s) / F_s^Y \right\} + \bar{A}^u(s) \right\} ds$$

is a.s. increasing. We thus conclude that

$$E_u \left\{ \int_X r_0^S c(s,x,u) \alpha_u(s) \beta_u(s,x) \lambda(dx,s) / F_s^Y \right\} + \bar{A}^u(s) \geq 0 \quad \text{a.s.}$$

with equality if and only if u is optimal.

The above results are summarised in the following theorem.

Theorem 3.20.

For all controls $u \in U$, $t \in [0, T_f]$, we have

$$E_u \left\{ \int_X r_0^t c(t,x,u_t) \alpha_u(t) \beta_u(t,x) \lambda(dx,t) / F_t^Y \right\} + \bar{A}^u(t) \geq 0 \quad \text{a.s.} \quad (3.57)$$

u is optimal if and only if equality holds in (3.57), where $\bar{A}^u(t)$ is given by (3.50).

The above theorem means that if $u^* \in U$ is an optimal control, then

$$\begin{aligned} E_{u^*} \{ \int_0^S c(s, x, u^*) \alpha_*(s) \beta_*(s, x) \lambda(dx, s) / F_s^Y \} + \bar{A}^*(s) \\ = \min_{u \in U} \{ E_u \{ \int_0^S c(s, x, u) \alpha_u(s) \beta_u(s, x) \lambda(dx, s) / F_s^Y \} + \bar{A}^u(s) \} \end{aligned} \quad (3.58)$$

Remarks 3.17.

(3.58) is not a true minimum principle since the predictable projection is dependent on the past control used up to time s , as made clear in remarks 3.15. However it is already more explicit than the corresponding results of Brownian motions type of processes encountered in (19).

The rest of this section is devoted to obtaining a more explicit expression for $\bar{A}^u(t)$.

Lemma 3.21.

Suppose $W(u, t)$ is differentiable w.r.t. t , each $u \in U$, then $\bar{A}^u(t)$ defined in (3.50) is given by

$$\bar{A}^u(t) = \alpha_y^u(t) \left\{ -r_0^t W(u, t) + \int_Z r_0^t W(u, t, z) \lambda_y^u(dz, t) + \frac{dW(u, t)}{dt} \right\} \text{ a.s.} \quad (3.59)$$

where

$$W(u, t, z) = W(u, \tau_y(t), y_{\tau_y(t)}) \Big|_{\substack{\tau_y(t)=t \\ y_{\tau_y(t)}=z}}$$

and $\tau_y(t)$, $y_{\tau_y(t)}$ are the first jump time and state of the process y_t after time t .

Proof :

Since the local description of the process y_t is well defined, and $W(u, t)$ is F_t^Y adapted, we can emulate a derivation of the formula (3.59) just as in lemma 3.14 of sec.3.4. The only notable difference is that whereas $\frac{dW(t)}{dt}$ is explicitly represented by $-H(t, u_t^*)$ there,

this may not be possible here, hence the differentiability assumption.

Theorem 3.22.

If $W(u, t)$ is differentiable with respect to t , each $u \in U$, then for all $u \in U$, $t \in [0, T_f]$,

$$\begin{aligned} \frac{dW(u, t)}{dt} + \alpha_y^u(t) \left\{ \int_Z r_0^t W(u, t, z) \lambda_y^u(dz, t) - r_0^t W(u, t) \right\} \\ + E_u \left\{ \int_X r_0^t c(t, x, u_t) \alpha_u(t) \beta_u(t, x) \lambda(dx, t) / F_t^Y \right\} \geq 0 \quad \text{a.s.} \end{aligned} \quad (3.60)$$

and u is optimal if and only if equality holds in (3.60).

Proof : Immediate from Theorem 3.20 and lemma 3.21.

Corollary 3.23.

If $u^* \in U$ is an optimal control, then for each $t \in [0, T_f]$,

$$\begin{aligned} \frac{dW(u^*, t)}{dt} = - \alpha_y^*(t) \left\{ \int_Z r_0^t W(u^*, t, z) \lambda_y^*(dz, t) - r_0^t W(u^*, t) \right\} \\ + E_{u^*} \left\{ \int_X r_0^t c(t, x, u_t^*) \alpha_*(t) \beta_*(t, x) \lambda(dx, t) / F_t^Y \right\} \quad \text{a.s.} \end{aligned} \quad (3.61)$$

Remarks 3.18.

All the equations in the theorems and lemmas of this section reduce to their corresponding counterparts in the complete information situation when we put $F_t^Y = F_t$. The optimal criteria in this section suffer from the drawback of a functional minimization, which is virtually intractable in all but the very special cases.

CHAPTER 5 : THE EXISTENCE OF OPTIMAL CONTROLS IN THE COMPLETE INFORMATION SITUATION.

4.1. Introduction.

In the last chapter optimality conditions were derived for which an optimal control, if it exists, must satisfy. These conditions are useful in providing insights to the evaluation of such controls. However, before one undertakes on such a task, a natural and more fundamental question to ask is whether an optimal control actually exists. In this chapter we provide the answer to the above question by stating some sufficient conditions which ensure the existence of an optimal control. As in most other literature on the subject, (e.g. (4), (25) and (16)) we shall be concerned only with the situation in which complete information is available.

The approach we follow here is basically that of Davis's in (16). We first show that the set of attainable densities (i.e. likelihood ratios $L(u)$, $u \in U$) is weakly sequentially compact and then use it to prove that a certain control constructed from the Hamilton Jacobi theory developed in the last chapter is optimal in the class U . This can be done as there is no mention of the existence of optimal controls in the derivation of the principle of optimality, thanks to the use of the ϵ -lattice property. Our approach, being basically constructional in spirit, is therefore quite different from that of (4) or (25), both of whom depended on the usual compactness-continuity arguments of most existence problems. The main difficulty of this approach lies in showing that the limits of sequences of admissible densities are also admissible. To do this, they needed the stipulation that the drift term $f(t, x, U)$ (which corresponds to our α_u, β_u) is convex on U . No such assumptions are needed in our present framework.

One other notable feature of our methods is that the framework is cast in $L_1(P)$ rather than $L_2(P)$, as in the case of Brownian motions

types of processes. This we feel is only natural, since jump processes have discontinuous sample paths and are therefore not square integrable. Again this represents a departure from the usual $L_2(P)$ treatment of Davis and Varaiya in (19) for continuous path processes.

4.2. On the Weak compactness of the class of attainable densities.

It proves useful to consider the following set of L_1 functions.

Definition 4.1.

Using the framework and notations of sec.3.1, let

$$\begin{aligned}\bar{\alpha}(t, \omega) &: R^+ \times \Omega \rightarrow R^+ \\ \bar{\beta}(t, x, \omega) &: R^+ \times X \times \Omega \rightarrow R^+\end{aligned}$$

be jointly measurable functions satisfying the following conditions :

- (i) $\bar{\alpha}(t, \omega)$ and $\bar{\beta}(t, x, \omega)$ are F_t predictable.
 - (ii) $0 < c_1 \leq \bar{\alpha}(t, \omega) \leq \min(c_2, \frac{1-\delta'}{\Delta\lambda_t})$ a.s. $d\lambda_t$, all $\omega \in \Omega$, some $\delta' > 0$.
 - (iii) $0 < c_1 \leq \bar{\beta}(t, x, \omega) \leq c_3$, a.s. $d\lambda_t$, all $(x, \omega) \in X \times \Omega$
 - (iv) $\int_X \bar{\beta}(t, x, \omega) \lambda(dx, t, \omega) = 1$ a.s. $d\lambda_t$, all $\omega \in \Omega$
- (4.1)

where c_1 , c_2 and c_3 are the same positive constants of Definition 3.1, and $\{\lambda(t, \omega), \lambda(A, t, \omega)\}$ are the local descriptions corresponding to the basic measure P .

Denote by G_1 all $\bar{\alpha}(\cdot)$ and G_2 all $\bar{\beta}(\cdot)$ functions satisfying the above conditions. The set of ordered pair of measurable functions $G = \{(\bar{\alpha}, \bar{\beta}) : \bar{\alpha} \in G_1, \bar{\beta} \in G_2\}$ shall be called the set of admissible rates of the jump process. The reason is that for each $u \in U$, the class of F_t predictable controls defined in sec.3.1, there exists measurable functions $\bar{\alpha} \in G_1$, $\bar{\beta} \in G_2$ such that

$$\bar{\alpha}(t, \omega) = \alpha(t, u(t, \omega), \omega)$$

$$\bar{\beta}(t, x, \omega) = \beta(t, x, u(t, \omega), \omega)$$

where (α, β) are the Radon Nikodym derivatives defined in definition 3.1 and which corresponds to the change of measure $P \rightarrow P_u$.

Now since the pair $(\bar{\alpha}, \bar{\beta}) \in G$ are F_t predictable, then as in sec. 3.1 there exists measurable functions

$$\bar{\alpha}_k : \Omega_{k-1} \times R^+ \rightarrow R^+, \quad \bar{\beta}_k : \Omega_{k-1} \times R^+ \times X \rightarrow R^+$$

such that $(\bar{\alpha}, \bar{\beta})$ coincide with $(\bar{\alpha}_k, \bar{\beta}_k)$ on the stochastic interval $(T_{k-1}(\omega), T_k(\omega)]$, each $k \in Z^+$, $\omega \in \Omega$. Hence $(\bar{\alpha}_k, \bar{\beta}_k)$ necessarily satisfy the conditions of definition 4.1 on this interval. Denote for each $k \in Z^+$, $\omega_{k-1}(\omega) \in \Omega_{k-1}$, by G_1^k, G_2^k all such $\bar{\alpha}_k, \bar{\beta}_k$ respectively. Set $G^k = \{(\bar{\alpha}_k, \bar{\beta}_k), \bar{\alpha}_k \in G_1^k, \bar{\beta}_k \in G_2^k\}$.

For each $k \in Z^+$, $\omega_{k-1} \in \Omega_{k-1}$, $(\bar{\alpha}_k, \bar{\beta}_k) \in G^k$ we can therefore define a second pair of local descriptions $(\bar{\Lambda}^k, \bar{\lambda}^k)$ mutually absolutely continuous with respect to the basic pair (Λ^k, λ^k) , with

$$\bar{\alpha}_k = \frac{d\bar{\Lambda}^k}{d\Lambda^k}, \quad \bar{\beta}_k = \frac{d\bar{\lambda}^k}{d\lambda^k}$$

Following the same arguments of sec. 2.4, we then infer that $\bar{\mu}_k(\omega_{k-1}; \cdot)$, the probability measure corresponding to the pair $(\bar{\Lambda}^k, \bar{\lambda}^k)$ is mutually absolutely w.r.t. $\mu_k(\omega_{k-1}; \cdot)$, each $k \in Z^+$, $\omega_{k-1} \in \Omega_{k-1}$, with Radon Nikodym derivative

$$\begin{aligned} L_k(t, x) &= L_k(\omega_{k-1}; t, x) = \frac{d\bar{\mu}_k}{d\mu_k}(\omega_{k-1}; t, x) \\ &= \bar{\alpha}_k(t) \bar{\beta}_k(t, x) \exp\left\{-\int_0^t (\bar{\alpha}_k(s) - 1) d\Lambda_s^{kc}\right\} \Pi_{t-}^k I_{\{t \leq \bar{c}^k\}} \end{aligned} \quad (4.2)$$

where $\Pi_t^k, \Lambda_t^{kc}, \bar{c}^k$ are defined analogously to that of (2.26).

Denote by $D(G^k)$ all $L_k(\cdot)$ of the form (4.2), where $(\bar{\alpha}_k, \bar{\beta}_k) \in G^k$.

Now for each $k \in \mathbb{Z}^+$, $\omega_{k-1} \in \Omega_{k-1}$, $\bar{\mu}_k(\omega_{k-1}; \cdot)$ is mutually absolutely continuous w.r.t. $\mu_k(\omega_{k-1}; \cdot)$. It therefore follows from the arguments of sec.2.4 that $\bar{P}_N$, $N \in \mathbb{Z}^+$, the probability measure on $(\Omega_N, \mathcal{F}_{T_N})$ corresponding to the family of local descriptions $\{\bar{\Lambda}^k, \bar{\lambda}^k, k=1, 2, \dots, N\}$, is also mutually absolutely continuous w.r.t. P_N , the restriction of P to $\mathcal{F}_{T_N}$. Moreover one has

$$\bar{L}^N(\omega_N) = \frac{d\bar{P}_N(\omega_N)}{dP_N(\omega_N)} = \prod_{k=1}^N L_k(\omega_k), \quad \text{each } \omega_N \in \Omega_N. \quad (4.3)$$

where $L_k(\cdot)$ is given by (4.2).

Now define for each $N \in \mathbb{Z}^+$,

$$D^N(G) = \{ \bar{L}^N(\cdot) \text{ of the form (4.3), where } L_k(\cdot) \in D(G^k), k=1, \dots, N \} \quad (4.4)$$

Clearly $D^N(G)$ corresponds to the set of attainable densities restricted to the stochastic interval $[0, T_N(\omega)]$. This means that for each $(\bar{\alpha}, \bar{\beta}) \in G$, one can define a corresponding $\bar{L}^N(\cdot) \in D^N(G)$, $N \in \mathbb{Z}^+$. Note that

$$D^N(G) \subset L_1(\Omega_N, \mathcal{F}_{T_N}, P_N)$$

and that for each $\bar{L}^N \in D^N(G)$, we have

$$\int_{\Omega} \bar{L}^N dP_N = 1, \quad \int_{\Gamma} \bar{L}^N dP_N = \bar{P}_N(\Gamma), \quad \Gamma \in \mathcal{F}_{T_N}.$$

In the above notations, the likelihood ratio $L_0^{T_N}(u)$, $u \in \mathcal{U}$, corresponding to the probability measure P_u , is an element of $D^N(G)$. This accounts for the name attainable densities.

The main result of this section is to prove a certain weak sequential compactness property of $D^N(G)$, namely that for each sequence $\{\bar{L}_n^N\} \in D^N(G)$, there exists a subsequence which converges in $\sigma(L_1, L_\infty)$ to an element $\bar{L}^N$ in $L_1(\Omega_N, \mathcal{F}_{T_N}^-, P_N)$. Moreover the element $\bar{L}^N$ has the property $\bar{L}^N > 0$ a.s. P_N .

For the rest of this chapter we shall assume that for each $k \in \mathbb{Z}^+$, the basic distribution function F^k given in (2.16) have the following form:

$$F^k(\omega_{k-1}; t) = F^k(T_{k-1}; t) = F_d^k(t - T_{k-1}) \quad , \quad \text{each } t \in T_{k-1}, \omega_{k-1} \in \Omega_{k-1}. \quad (4.4)$$

where $F_d^k(\cdot)$ is a deterministic distribution function. (i.e. $F_d^k(0)=1$, $F_d^k(s)$ monotonic decreasing, right continuous, s.t. $F_d^k(\infty)=0$ etc.)

Lemma 4.1.

For each $N \in \mathbb{Z}^+$, $\epsilon > 0$, there exists a $\delta > 0, \rho < \infty$ and $D \in F_{T_N}$ such that $0 < \delta \leq L^N(\omega_N) \leq \rho < \infty$ for $\omega_N \in D$, each $L^N \in D^N(G)$. Furthermore $\bar{P}_N(D) \geq 1 - \epsilon$, where $\bar{P}_N$ is the probability measure corresponding to L^N .

Proof:

First note that since L^N is an attainable density, there exists $L_k \in D(G^k)$, $k=1, \dots, N$ such that

$$L^N(\omega_N) = \prod_{k=1}^N L_k(\omega_{k-1}; T_k, Z_k) \quad (4.5)$$

$$L_k(\omega_{k-1}; t, x) = \bar{\alpha}_k(t) \bar{\beta}_k(t, x) \exp\left\{-\int_0^t (\bar{\alpha}_k(s) - 1) d\Lambda_s^{kc}\right\} \prod_{s < t} \frac{(1 - \bar{\alpha}_k(s) \Delta \Lambda_s^k)}{(1 - \Delta \Lambda_s^k)} I_{\{t \leq \bar{c}^k\}}$$

where $(\bar{\alpha}_k, \bar{\beta}_k) \in G^k$ is the pair of Radon Nikodym derivatives corresponding to L_k .

For notational convenience, let $\{s: \Delta \Lambda_s^k > 0\} = \{t_1, t_2, \dots\}$ and $a_i = \Delta \Lambda_{t_i}^k$. Observe that $0 \leq a_i < 1$, so that from standard analysis

$$\prod_{i=1}^{\infty} (1 - a_i) > 0 \iff \sum_{i=1}^{\infty} a_i < \infty$$

Hence for any constant c , $0 \leq c < 1$, we have

$$\prod_{i=1}^{\infty} (1 - a_i) > 0 \iff \prod_{i=1}^{\infty} (1 - ca_i) > 0 \iff \sum_{i=1}^{\infty} a_i < \infty$$

Denote

$$\gamma^k(s) = \min(c_2 \Delta \Lambda_s^k, 1 - \delta') < 1$$

Now from (2.12)

$$\begin{aligned} \bar{F}^k(\omega_{k-1}; t) &= \bar{P}_N(T_k > t/\omega_{k-1}) = \exp\left\{-\int_0^t \bar{\alpha}_k(s) d\Lambda_s^{kc}\right\} \prod_{s \leq t} (1 - \bar{\alpha}_k(s) \Delta \Lambda_s^k) \\ &\quad \exp\{-c_1 \Lambda_t^{kc}\} \prod_{t_i \leq t} (1 - c_1 a_i) \leq \prod_{t_i \leq t} (1 - c_1 a_i) \end{aligned} \quad (4.6)$$

And from (4.4)

$$\begin{aligned} c_1^2 \exp\{-(c_2-1) \Lambda_t^{kc}\} \prod_{t_i < t} \frac{(1 - \gamma^k(t_i))}{(1 - a_i)} &\leq L_k(\omega_{k-1}; t, x) \\ &\leq c_2 c_3 \exp\{(1 - c_1) \Lambda_t^{kc}\} \prod_{t_i < t} \frac{(1 - c_1 a_i)}{(1 - a_i)} \end{aligned} \quad (4.7)$$

There are now two separate cases to consider.

case 1 : $c^{k=\infty}$ or $c^{k<\infty}$, $F_{c_k}^k = 0$.

Clearly in this case $\Lambda_t^{k \uparrow \infty}$ as $t \uparrow c^k$. Suppose $\sum_{t_i \leq c^k} a_i = \infty$ so that

$$\prod_{t_i \leq c^k} (1 - \gamma^k(t_i)) = 0 = \prod_{t_i \leq c^k} (1 - a_i) = \prod_{t_i \leq c^k} (1 - c_1 a_i)$$

Then for any $\varepsilon' > 0$ there exists a $\tau^{k < c^k}$ such that

$$\begin{aligned} \prod_{t_i \leq \tau} (1 - c_1 a_i) &< \varepsilon' \\ \prod_{t_i \leq \tau} (1 - \gamma^k(t_i)) &> 0, \quad \prod_{t_i \leq \tau} (1 - a_i) > 0 \end{aligned} \quad (4.8)$$

Now define

$$\begin{aligned} D_k &= \Omega^{k-1}_X [0, \tau^k] \times X \\ \delta_k &= c_1^2 \exp\{-(c_2-1) \Lambda_{\tau^k}^{kc}\} \prod_{t_i < \tau^k} \frac{(1 - \gamma^k(t_i))}{(1 - a_i)} \\ \rho_k &= c_2 c_3 \exp\{(1 - c_1) \Lambda_{\tau^k}^{kc}\} \prod_{t_i < \tau^k} \frac{(1 - c_1 a_i)}{(1 - a_i)} \end{aligned} \quad (4.9)$$

Due to assumption (4.4), it is easily verified that $\delta_k, \rho_k, (\tau^k - T_{k-1})$ are all not dependent on ω_{k-1} , and depends only on ε' and k .

Using (4.6), (4.7) and (4.8) we now conclude that

$$0 < \delta_k \leq L_k(\omega_k) \leq \rho_k < \infty ,$$

$$\bar{P}_N(\omega_k(\omega) \in D_k) \geq 1 - \varepsilon' \quad \text{all } \omega_k \in D_k , \quad L_k \in D(G^k) . \quad (4.10)$$

On the other hand if $\sum_{t_i \leq c^k} a_i < \infty$, so that

$$\prod_{t_i \leq c^k}^{1-\gamma^k(t_i)} , \quad \prod_{t_i < c^k}^{1-a_i} , \quad \prod_{t_i < c^k}^{1-c_1 a_i} \quad \text{are all}$$

strictly positive, we then must have $\Lambda_t^{kc} \uparrow \infty$ as $t \uparrow c^k$.

Hence for given $\varepsilon' > 0$, there must exist a $\tau^{k < c^k}$ such that

$$\exp\{-c_1 \Lambda_{\tau}^{kc}\} < \varepsilon' ,$$

Again defining D_k , δ_k , ρ_k as in (4.9), equation (4.10) applies.

case 2: $c^{k < \infty}$, $F_{c^k}^k > 0$.

In this case since $\sum_{t_i \leq c^k} a_i < \infty$, we have $\prod_{t_i \leq c^k}^{1-\gamma^k(t_i)}$,

$\prod_{t_i < c^k}^{1-a_i}$, $\prod_{t_i < c^k}^{1-c_1 a_i}$ are again all strictly positive , and
that $\Lambda_{c^k}^{kc} < \infty$.

Thus choosing $\tau^k = c^k$, equation (4.10) applies with $\bar{P}_N(\omega_k(\omega)) = 1$.
(since $\bar{F}_{c^k}^k = 0$)

We have thus shown that (4.10) is true in general, for each $k=1, \dots, N$. Thus we can now apply the above argument for each $k=1, \dots, N$ and for $\varepsilon' > 0$ pick times $\tau^1, \tau^2, \tau^3, \dots, \tau^N$ such that

$$\bar{P}_N(T_k > \tau^k) < \varepsilon' , \quad \text{all } L \in D^N(G) .$$

Hence for any $\varepsilon > 0$, we can choose ε' small enough so that

$$\bar{P}_N\left(\bigcap_{k=1}^N \{T_k \leq \tau^k\}\right) \geq 1 - \varepsilon , \quad \text{all } L \in D^N(G) .$$

And since on D_k , L_k has bounds δ_k, ρ_k , we conclude that

$$0 < \delta \leq L^N(\omega_N) \leq \rho < \infty, \quad \text{for } \omega_N \in D,$$

$$\bar{P}_N(D) \geq 1 - \epsilon, \quad \text{all } L^N \in D^N(G).$$

where

$$\delta = \prod_{k=1}^N \delta_k$$

$$\rho = \prod_{k=1}^N \rho_k$$

$$D = \bigcap_{k=1}^N \{T_k \leq \tau^k\}.$$

This proves the lemma.

Lemma 4.2.

For each $N \in \mathbb{Z}^+$, the set of attainable densities $D^N(G) \subset L_1(\Omega_N, F_{T_N}, P_N)$ is weakly sequentially compact.

Proof:

It is equivalent to showing that $D^N(G)$ is uniformly integrable, i.e. for each $\epsilon > 0$, there exists a $\delta > 0$ such that $P_N(A) < \delta$ implies $\bar{P}_N(A) < \epsilon$, where $A \in F_{T_N}$.

Let $D \in F_{T_N}$ be as in lemma 4.1, then

$$\bar{P}_N(A) = \int_{A \cap D} L^N dP_N + \int_{A \cap D^c} L^N dP_N$$

From lemma 4.1 we then have for $\epsilon' > 0$

$$\bar{P}_N(A) \leq \rho' P_N(A) + \epsilon', \quad \text{all } L^N \in D^N(G), \quad \text{some } \rho' < \infty.$$

Hence for $\epsilon > 0$, choosing $\delta = \frac{\epsilon}{2\rho'}$, $\epsilon' = \frac{\epsilon}{2}$, we get $P_N(A) < \delta$

implies

$$\bar{P}_N(A) < \epsilon, \quad \text{all } L^N \in D^N(G), \quad A \in F_{T_N}.$$

This proves the lemma.

Lemma 4.3.

Suppose $\{L_n^N\}$ is a weakly convergent sequence in $D^N(G)$ with limit $L^N \in L_1(\Omega_N, F_{T_N}, P_N)$, then $L^N > 0$ a.s. P_N .

Proof:

By definition

$$\int_A L_n^N dP_N \xrightarrow{n \rightarrow \infty} \int_A L^N dP_N, \quad \text{each } A \in F_{T_N}.$$

Let $A = \{L^N = 0\}$, then from lemma 4.1 we have

$$0 = \lim_{n \rightarrow \infty} \int_A L_n^N dP_N = \lim_{n \rightarrow \infty} \int_{A \cap D} L_n^N dP_N \geq \delta P_N(A \cap D)$$

where

$$P_N(D) \geq 1 - \epsilon$$

Hence

$$P_N(A \cap D) = 0$$

and

$$P_N(A) = P_N(A \cap D^c) < \epsilon.$$

Since ϵ is arbitrary, we conclude that $P_N(A) = 0$.

i.e.

$$L^N > 0 \text{ a.s. } P_N.$$

This completes the proof.

Remarks 4.2.

The above proof of the weak sequential compactness of $D^N(G)$ is novel and differs from the usual L_2 techniques of Benes (4) or Duncan and Varaiya (25). The entire proof relies on the fact that the 'tail' of the distribution functions F^k is 'uniformly insignificant'. Note also that we do not require weak closure of the set $D^N(G)$ but only that the weak limit is strictly positive, a.s. P_N . This will be needed in the next section.

4.3. The Existence theorem.

The mathematical formulation of sec.3.1 with $F_t^Y = F_t$ shall be used in this section to show the existence of an optimal control in the class U of F_t predictable controls. In addition to the assumption stated in sec.3.1, we also need the following conditions.

(i) For each $(s, x, \omega) \in R^+ \times X \times \Omega$, $\alpha(s, \cdot, \omega)$, $\beta(s, x, \cdot, \omega)$ are continuous on U .

(ii) The cost rate $c(s, x, u, \omega)$ is continuous on U for fixed $(s, x, \omega) \in R^+ \times X \times \Omega$.

(iii) The interval of control $[0, T_f]$ is finite, i.e. $T_f < \infty$.

(iv) For each $(t, u, \omega) \in R^+ \times U \times \Omega$, define (4.12)

$$H(t, u, p, \omega) = \int_X \{p(t, x, \omega) + r_0^t c(t, x, u, \omega)\} \alpha(t, u, \omega) \beta(t, x, u, \omega) \lambda(dx, t, \omega)$$

where $p(t, x, \omega) : R^+ \times X \times \Omega \rightarrow R^+$ is in $L_1(p)$.

Assume that for each (t, p, ω) there is a $u_0 \in U$ such that

$$H(t, p, \omega) = H(t, u_0, p, \omega) = \inf_{u \in U} H(t, u, p, \omega)$$

This assumption is satisfied if for example U is compact.

Theorem 4.4. (The existence theorem.)

With the formulation of sec.3.1 and the assumptions (4.12) (i)-(iv), an optimal control u^* exists in the class U of F_t predictable controls.

Proof :

From the assumptions (4.12) (i)-(iv), $H(t, u, p, \omega)$ is clearly continuous on U for fixed (t, p, ω) . It is also evident that H is measurable with respect to $M = B(R^+) * B(R^+) * F$ for fixed u .

Let S be a countable dense subset of U , then we have

$$H(t, p, \omega) = \inf_{u \in S} H(t, u, p, \omega)$$

Thus

$$\{(t,p,\omega) : H(t,p,\omega) < a\} = \bigcup_{u \in S} \{(t,p,\omega) : H(t,u,p,\omega) < a\}$$

and hence H is M measurable.

Now in view of assumptions (4.12) (iv), we also have

$$H(t,p,\omega) \in H(t,U,p,\omega)$$

And according to a lemma of Benes in (3), these facts guarantee the existence of a jointly measurable F_t predictable mapping $y(t,p,\omega)$ such that

$$H(t,p,\omega) = H(t,y(t,p,\omega),p,\omega) \quad (4.13)$$

From the principle of optimality we know that for each $u \in U$, $t \in [0, T_f]$

$$M_t^u = \int_{(0,t] \times X} r_0^S c(s,x,u) d\tilde{p}^u + r_0^t W(t)$$

is a (F_t, P_u) submartingale, and that M_t^u admits a Doob-Meyer decomposition

$$M_t^u = J^* + m_t^u + a_t^u \quad (4.14)$$

where m_t^u is a zero mean (F_t, P_u) martingale and a_t^u is a predictable increasing process.

The martingale representation theorem can thus be used on m_t^u to give

$$m_t^u = \int_{(0,t] \times X} g(s,x) dq^u \quad (4.15)$$

where $g \in L_1(p)$ and is independent of u due to the form of the decomposition (4.14).

Now let $p(t,x,\omega) = g(t,x,\omega)$ in (4.13) and construct the control

$$u^*(t,\omega) = y(t,g(t,x,\omega),\omega) \text{ on } [0, T_f] \quad (4.16)$$

Clearly u^* is a well defined F_t predictable process taking values in U , i.e. $u^* \in U$.

To show the existence result , it is sufficient to prove that u^* is optimal in U .

From (4.14) and (4.15)

$$M_t^u = J^* + \int_{(0,t] \times X} g(s,x) dq^u + a_t^u \quad (4.17)$$

Using the control u^* constructed in (4.16), we get, using obvious shorthands

$$\begin{aligned} M_t^* &= \int_{(0,t] \times X} r_0^S c(s,x,u^*) d\tilde{p}^* + r_0^t W(t) \\ &= M_t^u + \int_{(0,t] \times X} r_0^S c^* d\tilde{p}^* - \int_{(0,t] \times X} r_0^S c^u d\tilde{p}^u \\ &= J^* + \int_{(0,t] \times X} g dq^u + \int_{(0,t] \times X} r_0^S c^* d\tilde{p}^* - \int_{(0,t] \times X} r_0^S c^u d\tilde{p}^u + a_t^u \\ &= J^* + \int_{(0,t] \times X} g dq^* + \int_{(0,t] \times X} g(\alpha_{*}\beta_{*} - \alpha_u\beta_u) d\lambda_s d\Lambda_s \\ &\quad + \int_{(0,t] \times X} r_0^S c^* d\tilde{p}^* - \int_{(0,t] \times X} r_0^S c^u d\tilde{p}^u + a_t^u \\ &= J^* + \int_{(0,t] \times X} g dq^* + a_t^* \end{aligned} \quad (4.18)$$

where

$$\begin{aligned} a_t^* &= a_t^u - \bar{a}_t^u \\ \bar{a}_t^u &= \int_{(0,t] \times X} \{g + r_0^S c^u\} \alpha_u \beta_u d\lambda_s d\Lambda_s - \int_{(0,t] \times X} \{g + r_0^S c^*\} \alpha_{*}\beta_{*} d\lambda_s d\Lambda_s \end{aligned} \quad (4.19)$$

Clearly $\bar{a}_t^u$ is a predictable increasing process by virtue of the construction of u^* .

To prove that u^* is optimal , it suffices to show that $a_t^* = 0$ a.s.

From (4.17)

$$M_t^u = J^* + a_t^u + \int_{(0,t] \times X} g dq^u$$

$$= J^* + a_t^u - \bar{a}_t^u + \int_{(0,t] \times X} g dq^* + \int_{(0,t] \times X} r_{0S}^{S u \tilde{u}} dp^u - \int_{(0,t] \times X} r_{0S}^{S * \tilde{u}} dp^*$$

Taking expectations w.r.t. P_u gives

$$E_u a_t^u = E_u a_t^u - E_u \bar{a}_t^u + E_u \left\{ \int_{(0,t] \times X} g dq^* + \int_{(0,t] \times X} r_{0S}^{S u \tilde{u}} dp^u - \int_{(0,t] \times X} r_{0S}^{S * \tilde{u}} dp^* \right\}$$

i.e.

$$E_u \left\{ \int_{(0,t] \times X} g dq^* + \int_{(0,t] \times X} r_{0S}^{S u \tilde{u}} dp^u - \int_{(0,t] \times X} r_{0S}^{S * \tilde{u}} dp^* \right\} = E_u \bar{a}_t^u \geq 0, \text{ all } u \in U$$

$$= 0 \text{ if } u = u^*.$$

(4.20)

Now for each $u \in U$, certainly

$$\int_{(0,t] \times X} r_{0S}^{S u \tilde{u}} dp^u + r_{0\psi}^t(u, t) \geq \int_{(0,t] \times X} r_{0S}^{S u \tilde{u}} dp^u + r_{0W}^t(t) = M_t^u$$

Hence

$$J(u) = E_u \left\{ \int_{(0,t] \times X} r_{0S}^{S u \tilde{u}} dp^u + r_{0\psi}^t(u, t) \right\} \geq E_u M_t^u \quad (4.21)$$

But for any $\varepsilon > 0$, we know from the infimum property that there exists a $u \in U$ such that

$$J(u) < J^* + \varepsilon$$

so from (4.19), (4.21) we obtain

$$J^* + \varepsilon > J(u) \geq E_u M_t^u = J^* + E_u \left\{ a_t^u + \int_{(0,t] \times X} g dq^u \right\}$$

$$= J^* + E_u a_t^* + E_u \bar{a}_t^u + E_u \int_{(0,t] \times X} g dq^* + E_u \int_{(0,t] \times X} g(\alpha_* \beta_* - \alpha_u \beta_u) d\lambda_S d\lambda_S$$

$$= J^* + E_u a_t^* + E_u \left\{ \int_{(0,t] \times X} g dq^* + \int_{(0,t] \times X} r_{0S}^{S u \tilde{u}} dp^u - \int_{(0,t] \times X} r_{0S}^{S * \tilde{u}} dp^* \right\} \quad (4.22)$$

From (4.20), (4.22), we infer that

$$E_u a_t^* < \varepsilon \quad \text{a fortiori.}$$

By using the Optional Sampling theorem, the above arguments can also be repeated for a F_t stopping time like $t \wedge T_N$, $N \in \mathbb{Z}^+$. We then have

$$E_{u_N} a_{t \wedge T_N}^* < \epsilon \quad \text{all } t \in [0, T_f] , N \in \mathbb{Z}^+.$$

Thus there exists a sequence $\{u_n\}_{n \in \mathbb{N}}$ such that

$$E_{u_n} a_{t \wedge T_N}^* \xrightarrow{n \rightarrow \infty} 0$$

Define $a_{t \wedge T_N}^*(K) = a_{t \wedge T_N}^* \wedge K$, some $K < \infty$

then clearly

$$E_{u_n} a_{t \wedge T_N}^*(K) \xrightarrow{n \rightarrow \infty} 0 \quad (4.23)$$

Now corresponding to the sequence of controls $\{u_n\}$ is a sequence of densities $\{L_0^{T_N}(u_n)\}_{n \in \mathbb{N}} \in D^N(G)$. So we can rewrite (4.23) as

$$E_{L_0^{T_N}(u_n)} a_{t \wedge T_N}^*(K) \xrightarrow{n \rightarrow \infty} 0$$

But from lemma 4.3, $\{L_0^{T_N}(u_n)\}_{n \in \mathbb{N}} \in D^N(G)$ is weakly sequentially compact with limit $L_0^{T_N} \in L_1(\Omega_N, F_{T_N}, P_N)$. Since $a_{t \wedge T_N}^*(K) \in L_\infty$, we have

$$E_{L_0^{T_N}(u_n)} a_{t \wedge T_N}^*(K) \xrightarrow{n \rightarrow \infty} E_{L_0^{T_N}} a_{t \wedge T_N}^*(K) = 0$$

for some subsequence n .

And since lemma 4.3 also says that $L_0^{T_N} > 0$ a.s. P_N , we have

$$a_{t \wedge T_N}^*(K) = 0 \quad \text{a.s. } P_N \quad \text{all } K < \infty, \text{ all } N \in \mathbb{Z}^+.$$

As this is true for all K and N , and as $T_N > t$ as $N \rightarrow \infty$ w.p.1, we have

$$a_t^* = 0 \quad \text{a.s.}$$

This proves that u^* is optimal.

Remarks 4.3.

The above proof of the existence of an optimal control is in the same spirit as that Davis's in (16). It is to be noted that the

Hamiltonian $H(t,u,p,\omega)$ defined in (4.12) (iv) is the same as that obtained in the minimum principle, theorem 3.9 if $p(t,x,\omega)$ is taken to be $g(t,x,\omega)$, the integrand of (4.15). In this case (with $p=g$) the condition (4.12) (iv) must also be a necessary one in order to guarantee the existence of an optimal control.

CHAPTER 5 : MARKOVIAN JUMP PROCESSES.

5.1. Introduction.

In this chapter a more restricted class of controlled jump processes is considered, namely those where the control u , the Radon Nikodym derivatives (α, β) , the basic local descriptions and the cost parameter $c(\cdot)$ at time t all depend only on the state at that time instead of the entire past history of the process up to t . To be completely general, these functions are all allowed to depend on the time t , measured from the origin. The result of the above specialization is a very wide class of controlled Markov jump process inclusive of most other models in the literature on the subject.

An analogous approach to ours was attempted in (8), but we believe their work contains a major shortcoming, namely they did not verify the relative completeness property needed to prove the principle of optimality. Instead an extra assumption dealing with the discrete approximation of a Markov control was introduced to circumvent this difficulty. By using the simpler ϵ -lattice property we give a direct proof of the Markovian principle of optimality in this chapter. It will be noticed that the main difficulty here is that we no longer have increasing information pattern, so that a lemma similar to lemma 3.1 is harder to prove.

After deriving a version of the general principle of optimality, we go on to produce results analogous to that in the complete information situation, i.e. the Markov minimum principle, a Markovian Hamilton Jacobi type of equation and also a Markovian existence result. The intuitively obvious fact that a Markovian optimal control is also optimal in the class of completely observable controls will also be proved. Finally we interpret the results in terms of the infinitesimal generator of the jump process and relate them to the work of others.

Recent literature on controlled Markovian jump processes include

(48), (54), (39), (46) and (51).

5.2. Mathematical description of the Markov controlled jump process.

A class of temporally non-homogeneous Markov jump process is formulated in this section to provide the basis for the control problem in the next few sections. The model we use will be of sufficient generality to include those of (48), (54), (8) and (51).

As in the general situation, we begin by a description of the basic Markov jump process. The notations and conventions used here are the same as before.

Let $\bar{\lambda}^m(x;t) : X \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a jointly measurable function with the following properties.

- (i) $\bar{\lambda}^m(x;0) = 0$ and $\bar{\lambda}^m(x;t)$ is increasing, continuous in t for all $x \in X$.
- (ii) $\bar{\lambda}^m(x;t) \leq Kt$, some $K < \infty$ and $\bar{\lambda}^m(x;t) \uparrow \infty$ as $t \uparrow \infty$, all $x \in X$.

(5.1)

Also let $\lambda^m(x;A,t) : X \times S \times \mathbb{R}^+ \rightarrow [0,1]$ be a measurable function such that

- (i) $\lambda^m(x;.,t)$ is probability measure on (X,S) for each $(x,t) \in X \times \mathbb{R}^+$.
- (ii) $\lambda^m(.,A,.)$ is jointly measurable w.r.t. $S^*B(\mathbb{R}^+)$, each $A \in S$.

(5.2)

Recall in the formulation of the general jump process in sec.2.1 that the stochastic evolution of the basic process x_t is determined upon the specification of the family of conditional probability distribution $\{\mu_i, i=1,2,\dots\}$. So we now emulate this procedure by defining for each $i \in \mathbb{Z}^+$, $T_{i-1} \in \mathbb{R}^+$, $Z_{i-1} \in X$ the conditional probability distribution of the pair (T_i, Z_i) according to :

$$\begin{aligned} \text{(i)} \quad \mu_i(\omega_{i-1};(t,\infty] \times X) &= \mu^m(T_{i-1}, Z_{i-1}; t) & (5.3) \\ &= \begin{cases} \exp\{-[\bar{\lambda}^m(Z_{i-1};t) - \bar{\lambda}^m(Z_{i-1};T_{i-1})]\} & \text{if } t > T_{i-1} \\ 1 & \text{if } t \leq T_{i-1} \end{cases} \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \mu_i(\omega_{i-1}; (0, t] \times A) &= \mu^m(T_{i-1}, Z_{i-1}; t, A) \\
 &= \begin{cases} - \int \lambda^m(Z_{i-1}; A, s) \mu^m(T_{i-1}, Z_{i-1}; ds) & \text{if } t > T_{i-1} \\ (0, t) & \\ 0 & \text{if } t \leq T_{i-1} \end{cases} \\
 &\quad A \in \mathcal{S}.
 \end{aligned}$$

(5.3contd.)

The μ_i 's defined as such is a genuine conditional probability distribution corresponding to the pair of local descriptions (Λ^m, λ^m) where

$$\Lambda^m(T_{i-1}, Z_{i-1}; t) = \begin{cases} \bar{\Lambda}^m(Z_{i-1}; t) - \bar{\Lambda}^m(Z_{i-1}; T_{i-1}) & \text{if } t > T_{i-1} \\ 0 & \text{if } t \leq T_{i-1} \end{cases}$$

This fact is verified easily by checking (Λ^m, λ^m) against the conditions laid down in (2.5) (i)-(vii) and also the one-one bijective theorem 2.1. Note that $\Lambda^m(x; t)$ is continuous in t , and as a result of condition (5.1) (ii), $T_i < \infty$ w.p.1, all $i \in \mathbb{Z}^+$.

With the family $\{\mu_i, i=1, 2, \dots\}$ defined as above, a probability measure P^m can now be constructed on $(\Omega, \mathcal{F})$ as in (2.1a) of sec.2.1. Observe that given T_{i-1}, Z_{i-1} , μ_i has the same form for all $i \in \mathbb{Z}^+$. This invariance property accounts for the following lemma.

Lemma 5.1.

Under measure P^m , the jump process x_t is also a Markov process.

Proof :

Define for $t \in \mathbb{R}^+$

$$\gamma(t) = \inf\{s > t, x_s \neq x_t\}$$

Then $(\gamma(t), x_{\gamma(t)})$ represents the co'ordinates of the first jump strictly after time t . Also define

$$\xi(t) = \sup\{s: 0 \leq s \leq t, x_s \neq x_t\}.$$

to be the jump time immediately prior to time t .

To prove that x_t is a Markov jump process it is sufficient to show that the conditional probability distribution of $(\gamma(t), x_{\gamma(t)})$ given the entire past F_t , depends only on the present state x_t .

Now for $t' > t \geq 0$,

$$\begin{aligned} P^m(\gamma(t) > t' / F_t) &= \frac{\mu^m(\xi(t), x(t); t')}{\mu^m(\xi(t), x(t); t)} \\ &= \frac{\exp\{-\{\bar{\Lambda}^m(x(t); t') - \bar{\Lambda}^m(x(t); (t))\}\}}{\exp\{-\{\bar{\Lambda}^m(x(t); t) - \bar{\Lambda}^m(x(t); (t))\}\}} \\ &= \exp\{-\{\bar{\Lambda}^m(x(t); t') - \bar{\Lambda}^m(x(t); t)\}\} \end{aligned}$$

and is independent of the past at time t .

Similarly for $A \in S$

$$\begin{aligned} P^m(x_{\gamma(t)} \in A / F_t) &= \frac{\mu^m(\xi(t), x(t); \infty, A) - \mu^m(\xi(t), x(t); t, A)}{\mu^m(\xi(t), x(t); t)} \\ &= \frac{\int_t^\infty \lambda^m(x(t); A, s) \exp\{-\{\bar{\Lambda}^m(x_t; s) - \bar{\Lambda}^m(x_t; \xi_t)\}\} \bar{\Lambda}^m(x_t; ds)}{\exp\{-\{\bar{\Lambda}^m(x_t; t) - \bar{\Lambda}^m(x_t; \xi_t)\}\}} \\ &= \int_t^\infty \lambda^m(x_t; A, s) \exp\{-\{\bar{\Lambda}^m(x_t; s) - \bar{\Lambda}^m(x_t; t)\}\} \bar{\Lambda}^m(x_t; ds) \end{aligned}$$

and is also independent of the past at time t .

This shows that the probability distribution of $(\gamma(t), x_{\gamma(t)})$ is dependent only on the present occupied state x_t , given the entire past F_t . Hence x_t is a Markov process.

Remarks 5.1.

Whereas an entire family of local descriptions $\{\Lambda^k, \lambda^k, k=1, 2, \dots\}$ is required to define the general jump process, one such pair $(\bar{\Lambda}^m, \lambda^m)$ suffices here in the Markov situation. This is intuitively clear since a self exciting Markov jump process does not distinguish the number or nature of jumps that had occurred in the past. The present

occupied state alone affects its future evolution.

In most applications $\bar{\lambda}^m(x;t)$ will be of the form $\int_0^t a(x,s)ds$ for some measurable function $a(.,.)$. Then $a(x,t)$ is interpreted as the rate of the jump process at t . Often in practice $a(x,t)$ is taken to be 1, so that $\bar{\lambda}^m(x,t) = t$. Then the counting process N_t derived from x_t is just a Poisson process with unit rate. The above calculation shows that the basic process can be much more general than this.

We now come to the formulation of a controlled system of Markov jump process. The space of control values U is assumed to be the same as that defined in sec.3.1. The class of Markovian controls u_M is taken to be controls of the form $u(t,\omega) = u(t, x_{t-}(\omega))$. As such, $u \in u_M$ is clearly F_t predictable and moreover $u_M \subset u$, where u is the class of F_t predictable controls previously defined.

The following class of markovian controlled Radon Nikodym derivatives is defined analogously to that in sec.3.1.

Let $\alpha^m(z;t,u) : X \times R^+ \times U \rightarrow R^+$, $\beta^m(z;t,x,u) : X \times R^+ \times X \times U \rightarrow R^+$

be jointly measurable functions such that :

- (i) $0 < c_1 \leq \alpha^m(z;t,u) \leq c_2$, all $(z,t,u) \in X \times R^+ \times U$
- (ii) $0 < c_1 \leq \beta^m(z;t,x,u) \leq c_3$, all $(z,t,x,u) \in X \times R^+ \times X \times U$
- (iii) $\int_X \beta^m(z;t,x,u) \lambda^m(z;dx,t) = 1$, all $(z,t,u) \in X \times R^+ \times U$

(5.5)

For $u \in u_M$, the pair of controlled local descriptions, $(\Lambda_u^m, \lambda_u^m)$, for the i th jump is defined as :

$$\begin{aligned}
 (i) \quad \Lambda_u^m(T_{i-1}, Z_{i-1}; t) &= \int_{(T_{i-1}, t]} \alpha^m(Z_{i-1}; s, u_s) d\Lambda^m(T_{i-1}, Z_{i-1}; ds) \\
 &= \begin{cases} \int_{(T_{i-1}, t]} \alpha^m(Z_{i-1}; s, u_s) \bar{\lambda}^m(Z_{i-1}; ds) & \text{if } t > T_{i-1} \\ 0 & \text{if } t \leq T_{i-1} \end{cases}
 \end{aligned}
 \tag{5.6}$$

$$(ii) \quad \lambda_u^m(Z_{i-1}; A, t) = \begin{cases} \int_A \beta^m(Z_{i-1}; t, x, u_t) \lambda^m(Z_{i-1}; dx, t) & \text{if } t > T_{i-1} \\ 0 & \text{if } t \leq T_{i-1}. \end{cases}$$

each $A \in \mathcal{S}$

(5.6contd.)

The conditional probability distribution μ_u^m corresponding to $(\lambda_u^m, \lambda_u^m)$ for the i th jump is then given by :

$$\mu_u^m(T_{i-1}, Z_{i-1}; t) = \begin{cases} \exp\{-\lambda_u^m(T_{i-1}, Z_{i-1}; t)\} & \text{if } t > T_{i-1} \\ 1 & \text{if } t \leq T_{i-1} \end{cases}$$

$$\mu_u^m(T_{i-1}, Z_{i-1}; t, A) = \begin{cases} \int_0^t \lambda_u^m(Z_{i-1}; A, s) \mu_u^m(T_{i-1}, Z_{i-1}; ds) & \text{if } t > T_{i-1} \\ 0 & \text{if } t \leq T_{i-1} \end{cases}$$

each $A \in \mathcal{S}$.

(5.7)

As in the basic process, it can readily be checked that $(\lambda_u^m, \lambda_u^m)$ defined by (5.6) is a genuine pair of local descriptions and that μ_u^m is the one-one bijective probability measure corresponding to it. Hence a probability measure P_u^m can be constructed from the family $\{\mu_u^m(T_{i-1}, Z_{i-1}; \cdot), i=1, 2, \dots\}$ and it can be shown using a similar argument as that in lemma 5.1 that x_t is a Markov process under P_u^m . Moreover due to the bounds on (α_u^m, β_u^m) and also that on $\bar{\lambda}^m(x, t)$, we have P_u^m is also mutually absolutely continuous w.r.t. P^m .

Note that the F_t predictability of u_t is compatible with the definitions (5.6) and (5.7). The reason is that the non-trivial parts of these definitions clearly concern only the values of (α_u^m, β_u^m) and u on $\xi(t) < t \leq \gamma(t)$, i.e. where $x_{t-} = x_{\xi(t)} = Z_{i-1}$. If $t = \xi(t) = T_{i-1}$ then the definitions become trivial.

5.3. The Markovian principle of optimality.

Before we define the Markovian optimal control problem, some redefinitions of the cost structure is necessary.

Assume in the sequel that the quantities $c(\cdot)$, $r(\cdot)$, $G_f(\cdot)$ of definition 3.2 have the following forms.

$$c(t, x, u, \omega) = c(t, x, u, x_{t-}(\omega))$$

$$r_s^t(\omega) = 1$$

$$G_f(\omega) = G_f(x_{T_f-}(\omega)).$$

This will ensure that the above quantities are 'observable' with Markovian information. For each $u \in U_M$, the Markovian cost function is now defined as :

$$J_m(u) = E_u^m \left\{ \int_{(0, T_f]} c(s, x, u_s, x_{s-}(\omega)) d\tilde{p}_m^u(s, x) + G_f(x_{T_f-}(\omega)) \right\} \quad (5.8)$$

Here $E_u^m(\cdot)$ refers to taking expectation w.r.t. the Markovian probability measure P_u^m , each $u \in U_M$. $\tilde{p}_m^u(t, A)$ denotes the predictable increasing process associated with the Markovian jump process x_t under measure P_u^m . Obviously, since the bounds on c and G_f are unchanged, $J_m(u) < \infty$, all $u \in U_M$.

The Markovian optimal control problem is then to seek a control $u^* \in U_M$ with the property :

$$J_m^* = J_m(u^*) = \inf_{u \in U_M} J_m(u) \quad (5.9)$$

As in the complete information situation in sec.3.3, for $u, v, \in U_M$, we define the remaining cost function by :

$$\begin{aligned}
 \psi_m(u, v, t) &= E_{u, v, t}^m \left\{ \int_{(t, T_f] \times X} c(s, x, v) d\tilde{p}_m^v + G_f / F_t \right\} \\
 &= E_v^m \left\{ \int_{(t, T_f] \times X} c(s, x, v) d\tilde{p}_m^v + G_f / F_t \right\} \quad (\text{by lemma 3.6}) \\
 &= E_v^m \left\{ \int_{(t, T_f] \times X} c(s, x, v) d\tilde{p}_m^v + G_f / F_t \right\} \quad (\text{by the Markov property}) \\
 &= \eta(v, t, x_t) \in L_1(\Omega, F_t, P_u^m) \quad (5.10)
 \end{aligned}$$

so that as in sec.3.3 we know that a Markovian value function $V(t, x_t)$ exists by virtue of the complete lattice property of $L_1(P_u^m)$. Thus

$$\begin{aligned}
 V(t, x_t) &= \inf_{v \in U_M} \eta(v, t, x_t) \\
 V(0) &= J_m^* \quad (5.11)
 \end{aligned}$$

To seek optimality criteria for the Markovian control problem, we naturally would like to establish a Markovian principle of optimality analogous to that in sec.3.2. At first sight this may seem trivial, but on closer examination, a difficulty arises. In contrast to the the general situation of sec.3.2, we now no longer have increasing observations pattern. This presents a problem in verifying the ϵ -lattice property made precise by the following definition.

Definition 5.1.

The class of controls u_M is said to possess the ϵ -lattice property with respect to the Markovian remaining cost $\eta(v, t, x_t)$ provided that for each $\epsilon > 0$, $u_1, u_2 \in u_M$, $t \in [0, T_f]$, $z \in X$, there exists a control $u \in u_M$ such that

$$\eta(u, t, z) \leq \eta(u_i, t, z) + \epsilon, \quad i=1,2.$$

This is an adaptation of the definition in sec.3.2 where we had increasing information.

Whereas in sec.3.2, the ϵ -lattice property is easily verified for the class of controls u of F_t^Y predictable controls (see lemma 3.1), the situation is not as simple here. Since the information available is no longer increasing, one cannot construct a control based on past observations, as we did in lemma 3.1. And since the principle of optimality relies critically on this property, we have to seek an alternative proof for it. This^{is} done through several lemmas.

First define a class of 'switching' controls u_t^n , each $t \in [0, T_f]$, $n \in \mathbb{Z}^+$ as follows:

Consider a partition on $[t, T_f]$ of the form :

$$\left[t + \frac{k}{2^n} (T_f - t), t + \frac{(k+1)}{2^n} (T_f - t) \right), \quad k=0,1,2,\dots,2^n-1.$$

Using the shorthand $d_n = \frac{(T_f - t)}{2^n}$, the class of controls u_t^n consists of all controls u^n defined on $[t, T_f]$ of the form :

$$u_s^n = u(s, x_{(t+kd_n)^-}(\omega)), \quad \text{for } s \in [t+kd_n, t+(k+1)d_n) \\ \text{and some } u \in U_M.$$

Observe that u^n is not a Markov control but is 'nearly' one if the partition is fine enough. On each of the partitions it 'remembers' the information at the left end point. Hence we can call u^n a 'pseudo-Markov' control. A pseudo Markov control is nearly as good as its Markov counterpart in the following sense.

Lemma 5.2.

For each $t \in [0, T_f]$, $n \in \mathbb{Z}^+$, $z \in X$, $u \in U_M$,

$$|n(u^n, t, z) - n(u, t, z)| \leq \frac{C}{2^n} \quad \text{a.s.} \quad (5.12)$$

where $u^n \in U_t^n$ is the pseudo Markov control corresponding to u , and $C < \infty$ is some positive constant.

Proof :

This is a consequence of the boundedness of the rates α_u^m , and of the costs c and G_f .

Define

$$\zeta_k^n(u) = \int_{(t+kd_n, t+(k+1)d_n] \times X} c(s, x, u) dp_m^u$$

Then clearly $\zeta_k^n(u) = \zeta_k^n(u^n)$ on those intervals of the partitions that contain no jumps of the process x_t , as u and u^n coincide on such intervals.

Suppose there are m jumps on the interval $[t, T_f]$, and let J denotes the indices of those intervals of the partitions containing at least one of the m jumps. Then

$$\begin{aligned} \Delta \eta^n &= |\eta(u^n, t, z) - \eta(u, t, z)| \\ &= |E_{u^n} \{ \sum_{k \in J} \zeta_k^n(u^n) + \sum_{k \notin J} \zeta_k^n(u^n) + G_f / x_t = z \} \\ &\quad - E_u \{ \sum_{k \in J} \zeta_k^n(u) + \sum_{k \notin J} \zeta_k^n(u) + G_f / x_t = z \} | \end{aligned}$$

Since $\zeta_k^n(u) \leq C^1 d_n$, some $C^1 < \infty$, due to the boundedness of the cost rate $c(\cdot)$, we note that

$$|E_{u^n} \{ \sum_{k \in J} \zeta_k^n(u^n) / x_t = z \} - E_u \{ \sum_{k \in J} \zeta_k^n(u) / x_t = z \}| \leq m C^1 d_n$$

And since $\zeta_k^n(u^n) = \zeta_k^n(u)$ for the indices $\{k \notin J\}$, we have

$$\theta = \sum_{k \notin J} \zeta_k^n(u^n) + G_f = \sum_{k \notin J} \zeta_k^n(u) + G_f$$

is uniformly bounded r.v.. But since u^n and u are different on at most m intervals of the partitions, we infer that

$$|E_{u^n} \{ \theta / x_t = z \} - E_u \{ \theta / x_t = z \}| \leq m C^2 d_n, \text{ some } C^2 < \infty$$

Now as the jump process x_t has a bounded rate, there exists a Poisson process with rate α^0 that dominates the point process N_t derived from x_t . We thus conclude that

$$\Delta \eta^n \leq \sum_{m=1}^{\infty} (C^1 + C^2) d_n p_m^0 = (C^1 + C^2) d_n \alpha^0 (T_f - t) = \frac{C}{2^n}, \text{ where } C < \infty.$$

where p_m^0 is the probability of m jumps on $[t, T_f]$ corresponding to the dominant Poisson process.

This completes the proof.

The next lemma shows that for any two pseudo Markov controls $u_1^n, u_2^n \in \mathcal{U}_t^n$, one can derive a third one $\bar{u}^n \in \mathcal{U}_t^n$ with the smallest remaining cost of the three. This seems intuitively obvious if we imagine $\bar{u}^n$ as a control that 'switches' between u_1^n and u_2^n at the left end points of the partitions with the intention of incurring the least remaining cost. We use dynamic programming to prove it.

Lemma 5.3.

For each $t \in [0, T_f]$, $n \in \mathbb{Z}^+$, $u_1^n, u_2^n \in \mathcal{U}_t^n$, there exists a $\bar{u}^n \in \mathcal{U}_t^n$ such that

$$\eta(\bar{u}^n, t, z) \leq \eta(u_i^n, t, z), \quad \text{all } z \in X, \quad i=1,2.$$

Proof :

Suppose for some $k \in \{0, 1, 2, \dots, 2^n - 1\}$ there exists a $\bar{u}^n \in \mathcal{U}_t^n$ on the interval $[t_k, T_f]$, such that

$$\eta(\bar{u}^n, t_k, z) \leq \eta(u_i^n, t_k, z), \quad \text{all } z \in X, \quad i=1,2.$$

where $t_k = t + k d_n$, $k = 0, 1, 2, \dots, 2^n - 1$.

We now show that the above hypothesis is also true for the number $k-1$.

Define

$$A_{k-1} = \{z \in X : \eta(u_1^n(k-1), t_{k-1}, z) \leq \eta(u_2^n(k-1), t_{k-1}, z)\}$$

where

$$u_i^{n(k-1)}(s) = \begin{cases} u_i^n(s) & \text{if } s \in [t_{k-1}, t_k) \\ \bar{u}^n(s) & \text{if } s \in [t_k, T_f] \end{cases}, \quad i=1,2.$$

Now extend the definition of $\bar{u}^n \in U_t^n$ on to $[t_{k-1}, T_f]$ by defining $\bar{u}^n$ on the interval $[t_{k-1}, t_k)$ as

$$\begin{aligned} \bar{u}^n(s) &= \bar{u}^n(s, x_{t_{k-1}}^-(\omega)) \\ &= \bar{u}_1^n(s, x_{t_{k-1}}^-(\omega)) I_{\{x_{t_{k-1}}^-(\omega) \in A_{k-1}\}} \\ &\quad + \bar{u}_2^n(s, x_{t_{k-1}}^-(\omega)) I_{\{x_{t_{k-1}}^-(\omega) \in A_{k-1}^c\}} \end{aligned}$$

all $s \in [t_{k-1}, t_k)$.

Clearly by construction we now have

$$\eta(\bar{u}^n, t_{k-1}, z) \leq \eta(\bar{u}_i^n, t_{k-1}, z), \quad \text{all } z \in X, \quad i=1,2.$$

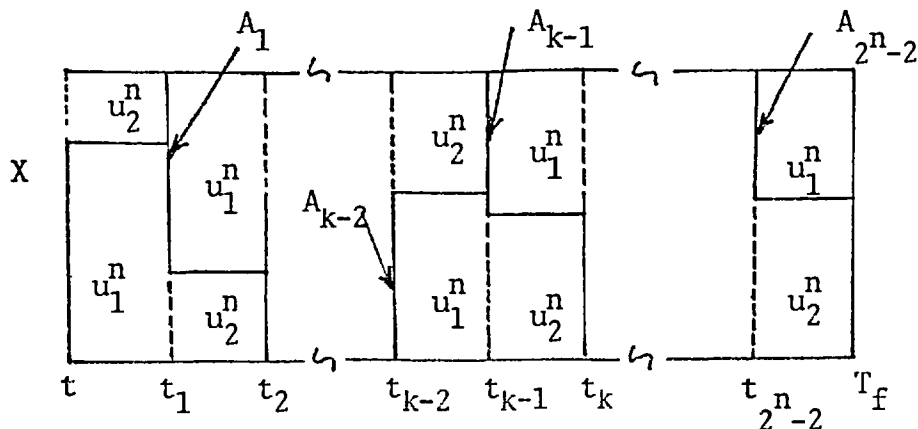
Hence the hypothesis is true for $k-1$.

But it is obvious that the hypothesis is true for $k=2^n-1$, i.e. on the last interval of the partitions, so that we conclude that it is true for all k . In particular we have

$$\eta(\bar{u}^n, t, z) \leq \eta(\bar{u}_i^n, t, z), \quad \text{all } z \in X, \quad i=1,2.$$

This completes the proof.

To illustrate the above construction, a diagram is given below.



The next lemma is the converse of lemma 5.2. It shows that given a pseudo Markov $\bar{u}^n$ one can derive a Markov control u that approximates $\bar{u}^n$.

Lemma 5.4.

Let $\bar{u}^n$ be the control constructed in lemma 5.3. Then there exists a Markov control $u \in U_M$ such that

$$|n(u, t, z) - n(\bar{u}^n, t, z)| \leq \frac{C}{2^n}, \quad \text{all } z \in X$$

where $C < \infty$ is the same positive constant as that in lemma 5.2.

Proof :

$u \in U_M$ is constructed by freeing the 'pseudoness' of $\bar{u}^n$ to convert it back into a Markov control. Hence let

$$u(s) = u(s, x_{s-}(\omega)) = \bar{u}^n(s, x_{s-}(\omega)) \quad , \quad \text{each } s \in [t, T_f]$$

Now since $\bar{u}^n$ is the one-one correspondent of u , an argument exactly similar to that in the proof of lemma 5.2 shows that

$$|n(u, t, z) - n(\bar{u}^n, t, z)| \leq \frac{C}{2^n} \quad \text{all } z \in X.$$

This completes the proof.

Lemma 5.5.

For each $t \in [0, T_f]$, $\epsilon > 0$, $u_1, u_2 \in U_M$, there exists a $u \in U_M$ such that

$$n(u, t, z) \leq n(u_1, t, z) + \epsilon, \quad \text{all } z \in X$$

Proof :

Fix ϵ , and take $n = \log_2(\frac{2C}{\epsilon})$, where C is the positive constant in the previous lemma. Using lemma 5.2 we can construct two pseudo Markov controls $u_1^n, u_2^n \in U_t^n$, such that

$$\eta(u_i^n, t, z) \leq \eta(u_i, t, z) + \frac{\epsilon}{2} \quad i=1,2. \quad (5.13)$$

From lemma 5.3, there exists another pseudo Markov control $\bar{u}^n$ such that

$$\eta(\bar{u}^n, t, z) \leq \eta(u_i^n, t, z) \quad , \quad i=1,2. \quad (5.14)$$

Finally from lemma 5.4 we can obtain a Markov control $u \in U_M$ from $\bar{u}^n$ such that

$$\eta(u, t, z) \leq \eta(\bar{u}^n, t, z) + \frac{\epsilon}{2} \quad (5.15)$$

Combining (5.13), (5.14) and (5.15), we get

$$\eta(u, t, z) \leq \eta(\bar{u}^n, t, z) + \frac{\epsilon}{2} \leq \eta(u_i^n, t, z) + \frac{\epsilon}{2} \leq \eta(u_i, t, z) + \epsilon \quad i=1,2.$$

i.e.

$$\eta(u, t, z) \leq \eta(u_i, t, z) + \epsilon \quad , \quad i=1,2.$$

This completes the proof.

Remarks 5.2.

The previous lemma is the ϵ -lattice property of the class of Markov controls U_M . The use of 'pseudo Markov' controls here does away with the need for convergence and continuity techniques used both in (18) and (7). This is due to the realization that a pseudo Markov control agrees with an actual Markov control on the intervals between jumps. Once again we find that it is advantageous to depart from methods more suited to Brownian motions type of processes and instead rely on the structure of the process for answers.

Theorem 5.6. (Markovian principle of optimality.)

For each $t \in [0, T_f]$, $h > 0$ such that $t+h \in [t, T_f]$, we have for all $u \in U_M$:

$$V(t, x_t) \leq E_u^m \left\{ \int_{(t, t+h] \times X} c(s, x, u_s) d\tilde{p}_m^u(s, x) / x_t \right\} + E_u^m \{ V(t+h, x_{t+h}) / x_t \}$$

(5.16)

$$V(T_f, x_{T_f}) = G_f(x_{T_f})$$

Furthermore equality holds if and only if u is optimal in u_M .

Proof :

Since the ϵ -lattice property holds for the class u_M with respect to $\psi_m(\cdot)$, we can follow a proof similar to the general situation in theorem 3.3.

Remarks 5.3.

Once again we have obtained a useful optimality condition without the need of assuming the existence of an optimal control. This means that theorem 5.6 can be used in the future to prove existence results in the Markovian situation. This approach is however not used by others like Pliska and Stone. Instead they both provided optimality criteria in their final forms without bothering about some intermediate but fundamental result like the above theorem. This is perhaps due to the fact that they were using more conventional tools of analysis like semigroup theory of Markov processes instead of the relatively novel approach we had followed.

5.4. The Markovian minimum principle.

In this section we prove the Markovian version of the minimum principle and the intuitively obvious fact that a Markovian optimal control is also optimal in the class of controls based on complete information of the past. This is because in the Markovian situation, only the present state affects future events. Hence information concerning the past history would be useless to the controller and can be discarded without regrets. Finally we shall derive a Hamilton

Jacobi type of equation analogous to that in theorem 3.11.

Although the whole of F_t is not available to the controller in choosing a control, it can still be used fictitiously during the intermediate stages to obtain a verifiable optimality criterion. In fact, by virtue of its formulation in sec.5.1 and sec.5.2 the Markovian situation is merely a subcase of the complete information control problem, only that certain of the quantities like $u, c(t, x, u)$ now acquire special forms. With such a perspective in mind, the analysis is considerably simplified.

First note that theorem 5.6 can be restated as :

$$M_m^u(t) = \int_{(0,t] \times X} c(s, x, u) d\tilde{p}_m^u(s, x) + V(t, x_t) \quad (5.17)$$

is a (F_t, P_u^m) submartingale for all $u \in U_M$, each $t \in [0, T_f]$, and is a martingale if and only if u is optimal in U_M .

This is a simple application of the Markov property :

$$E_u^m \{f(x_s, t \leq s \leq T_f / x_t)\} = E_u^m \{f(x_s, t \leq s \leq T_f / x_t)\}$$

for any integrable functional $f(\cdot)$ of x_s .

Theorem 5.7. (Markovian minimum principle.)

$u^* \in U_M$ is a Markovian optimal control if and only if there exists a measurable function $g_m: \Omega \times Y \rightarrow R^+$ with $g_m \in L_{loc}^1(p)$ such that

$$M_m^*(t) = J_m^* + \int_{(0,t] \times X} g_m(s, x) dq_m^*(s, x) \quad \text{a.s.} \quad (5.18)$$

and at almost $(d\Lambda_t^m)$ every point $t \in [0, T_f]$, u^* minimizes the Hamiltonian

$$\alpha^m(t, u_t) \int_X \{g_m(t, x) + c(t, x, u_t)\} \beta^m(t, x, u_t) \lambda^m(dx, t) \quad (5.19)$$

Proof :

Since the principle of optimality and the martingale representa-

theorem still applies, (5.18) is obvious from (5.17) and theorem 5.6. The rest of the proof is exactly as in theorem 3.9.

As in theorem 3.9, the integrand $g_m(t, x)$ must also be F_t adapted. However due to the special form of the value function $V(t, x_t)$ and corollary 3.10, $g_m(t, x)$ in fact depends only on x_{t-} . Thus we have from corollary 3.10,

$$g_m(t, x) = V(t, x) - V(t-, x_{t-}(\omega)) \quad \text{a.s.} \quad (5.20)$$

$$\text{where } V(t, x) = V(\sigma_t, x_{\sigma_t}(\omega)) \left| \begin{array}{l} \sigma_t = t \\ x_{\sigma_t}(\omega) = x. \end{array} \right.$$

We now have the following corollary.

Corollary 5.8.

A necessary condition for a Markov control $u^* \in U_M$ to be optimal is that it minimizes (a.s. $d\Lambda_t^m$) the Hamiltonian

$$H_m(t, u_t) = \alpha^m(t, u_t) \left\{ \int_X \{V(t, x) + c(t, x, u_t)\} \beta^m(t, x, u_t) \lambda^m(dx, t) - V(t-, x_{t-}) \right\} \quad (5.21)$$

The next theorem is the main result of this section.

Theorem 5.9.

A Markovian optimal control $u^* \in U_M$ is also optimal in the class of F_t predictable controls. i.e.

$$\inf_{u \in U_M} J_m(u) = \inf_{u \in U} J_m(u)$$

Proof :

Since a U optimal control is one that minimizes the Hamiltonian (5.21) as well (by theorem 3.9 and corollary 3.10), $u^* \in U_M$ therefore qualifies

as an optimal control in the class U .

Remarks 5.4.

There is an alternative explanation why the integrand $g_m(s, x)$ is dependent only on x_{t-} . If u^* is an optimal control in U_M then for $T_f \geq t_2 > t_1 \geq 0$, we have from (5.17)

$$M_m^*(t_2) - M_m^*(t_1) = \int_{(t_1, t_2] \times X} c(s, x, u_s) d\tilde{p}_m^u + V(t_2, x_{t_2}) - V(t_1, x_{t_1})$$

and is hence $F_{t_1}^{t_2} = \sigma\{x_s, t_1 \leq s \leq t_2\}$ measurable.

i.e. $M_m^*(t)$ is an additive functional of the Markov process (x_t, F_t, P_*^m) . By a result in (40) then one has the representation

$$M_m^*(t) = J_m^* + \int_{(0, t] \times X} g_m(s, x) dq_m^* \quad \text{a.s.}$$

where $g_m(s, x, \omega)$ has the form $g_m(s, x, x_{s-}(\omega))$.

Note also that $H_m(t, u_t)$ is observable to the controller since all its terms depend only on $x_{t-}(\omega)$ instead of ω as in the complete information situation.

Since $\Lambda^m(t)$ is continuous in t by assumption, theorem 3.11 is applicable and one gets

Theorem 5.10.

Suppose $u^* \in U_M$ is a Markovian optimal control, then for each $t \in [0, T_f]$, the following 'adjoint equation' holds :

$$\frac{dV(t-, x_{t-})}{d\Lambda_t^m} = -H_m(t, u_t^*) = -\min_{u \in U} H_m(t, u) \quad (5.22)$$

Proof : As in theorem 3.11.

Remarks 5.5.

The above result is analogous to theorem 12 of (48). The framework is however more general here, since we are dealing with a non-homogeneous process.

5.5. Markovian existence results.

It is perhaps more useful to obtain some results on the existence of Markovian optimal controls. Afterall, Markov jump process models are the most widely used in practical situations. In this section we present the Markovian version of the results derived in Chapter 4

As we had remarked in the previous section, the Markov control problem, by virtue of its formulation in sec.5.2 and sec.5.3, is merely a special case of the complete information situation. So with some appropriate modifications the results of Chapter 4 should apply without much difficulty. First let us re-examine the proof of the compactness property of the class of attainable densities. The obvious changes necessary are :

(i) The basic probability measure in the formulation of Chapter 4 should now be taken as P^m , corresponding to the pair of local descriptions (Λ^m, λ^m) .

(ii) The pair of 'attainable rates' $(\bar{\alpha}, \bar{\beta})$ of definition 4.1, apart from satisfying the conditions (4.1) (i)-(iv) under the probability measure P^m , should now have the form :

(5.23)

$$\begin{aligned}\bar{\alpha}(t, \omega) &= \bar{\alpha}(t, x_{t-}(\omega)) \\ \bar{\beta}(t, x, \omega) &= \bar{\beta}(t, x, x_{t-}(\omega)), \quad \text{each } t \in \mathbb{R}^+, x \in X, \omega \in \Omega.\end{aligned}$$

The assumption (ii) ensures that the $(\bar{\alpha}, \bar{\beta})$ would also serve as attainable rates for the Markovian Radon derivatives (α_u^m, β_u^m) .

Proceeding as in sec.4.2, a Markovian probability measure $\bar{P}^m$, mutually absolutely continuous with respect to P^m on any random interval

$[0, \tau]$, where $\tau < \infty$ a.s. P^m , can now be constructed corresponding to each pair $(\bar{\alpha}, \bar{\beta}) \in G$. Hence by restricting to F_{T_n} , each $N \in \mathbb{Z}^+$, the set of Markovian attainable densities $D_m^N(G)$ is defined analogously to that in (4.4). Proceeding the same way we did in sec.4.2, we can show that the lemma 4.3 also applies to the class $D_m^N(G) \subset L_1(\Omega_N, F_{T_N}, P_N^m)$. i.e. $D_m^N(G)$ is weakly sequentially compact in the topology of $\sigma(L_1, L_\infty)$, such that any weak sequentially limit of $D_m^N(G)$ is strictly positively, a.s. P_N^m .

The assumptions corresponding to those laid down in (4.18) are :

(i) For each $(s, x, \omega) \in \mathbb{R}^+ \times X \times \Omega$, $\alpha^m(s, \cdot, x_{s-}(\omega))$ and $\beta^m(s, x, \cdot, x_{s-}(\omega))$ are continuous on U .

(ii) The cost rate $c(s, x, u, x_{s-}(\omega))$ is continuous on U for fixed $(s, x, \omega) \in \mathbb{R}^+ \times X \times \Omega$.

(iii) For each $(t, u, \omega) \in \mathbb{R}^+ \times U \times \Omega$, define

$$H_m(t, u, p_m, x_{t-}(\omega)) = \alpha^m(t, u) \int_X \{p_m(t, x, x_{t-}(\omega)) + c(t, x, u)\} \beta^m(t, x, u) \lambda^m(dx, t)$$

where $p_m(t, x, x_{t-}(\omega)) : \mathbb{R}^+ \times X \times X \rightarrow \mathbb{R}^+$ is in $L_1(p)$.

Assume that for each (t, p, ξ) there is a $u_0 \in U$ such that

$$H_m(t, p_m, \xi) = H_m(t, u_0, p_m, \xi) = \inf_{u \in U} H_m(t, u, p_m, \xi) \quad (5.24)$$

We are now in a position to state the Markovian version of the existence theorem.

Theorem 5.11.

With the formulations of sec.5.2 and sec.5.3 and the assumptions (5.23) and (5.24), an optimal control $u^* \in u_M$ exists in the class of Markov controls.

Proof :

Since $D_m^N(G)$ is weakly sequentially compact as in lemma 4.3, the

proof proceeds in the same manner as in theorem 4.4.

Remarks 5.6.

As in the more general situation, if we take p_m to be the integrand g_m of (5.20) (which always exists by the principle of optimality and the martingale representation theorem) then the condition (5.24) (iii) is also a necessary one for the existence of a Markovian optimal control.

For a class of jump process with homogeneous rate Pliska gave a sufficient condition to ensure the existence of a measurable optimal policy in (48).

5.6. Infinitesimal generators and related topics.

Most authors in the literature on Markovian control problems have used Infinitesimal generators and the associated semigroup operators as their tools of analysis. In this section we provide an interpretation of the optimality principle in terms of the infinitesimal generator of the jump process and to see the relation of our work to others like Pliska and Stone.

For simplicity we take $\bar{\Lambda}^m(t) = t$ in the rest of this section.

For each $u \in U_M$, $t \in \mathbb{R}^+$, $h > 0$, consider the following operator :

$$T_h^t(u)f(x) = E_u^m(f(x_{t+h})/x_t=x) = E_{ux}^m(f(x_{t+h}))$$

where $f(\cdot)$ is a real valued bounded, measurable function on X .

Since for $s, h > 0$,

$$\begin{aligned} T_s^t(u)T_h^t(u)f(x) &= T_s^t(u)E_{ux}^m(f(x_{t+h})) \\ &= E_{ux}^m E_{ux_{t+s}}^m (f(x_{t+s+h})) = E_{ux}^m (f(x_{t+s+h})) \\ &= T_{h+s}^t(u)f(x) \quad \text{(having used the Markov} \\ &\quad \text{property and the iterated} \\ &\quad \text{conditional expectation.)} \end{aligned}$$

we conclude that

$$T_S^t(u)T_h^t(u) = T_{h+S}^t(u) , \text{ for each } t \in \mathbb{R}^+, u \in U_M.$$

Hence $T_S^t(u)$ is a semigroup operator. And as in Dynkin's book, (27) Chap. I, sec.6, we can now define the weak infinitesimal operator corresponding to $T_h^t(u)$ as :

$$\begin{aligned} A_t(u)f(x) &= w \lim_{h \rightarrow 0} \frac{1}{h} \{ T_h^t(u)f(x) - f(x) \} \\ &= w \lim_{h \rightarrow 0} \frac{1}{h} \{ E_{ux}^m(f(x_{t+h})) - f(x) \} \end{aligned} \quad (5.25)$$

provided the weak limit on L_∞ exists.

A calculation along similar lines as that in lemma 3.14 shows that the above limit does exist and is in fact

$$\begin{aligned} A_t(u)f(x) &= \alpha^m(t, u_t) \left\{ \int_X f(\xi) \beta^m(t, \xi, u_t) \lambda^m(d\xi, t) - f(x) \right\} \\ &= \alpha_u^m(t) \left\{ \int_X f(\xi) \lambda_u^m(d\xi, t) - f(x) \right\} \end{aligned} \quad (5.26)$$

This is clearly in agreement with the generator of a jump process. See for example (Breiman, 10, Chap. 15, sec.4), (Pliska, 47) or (54).

In the context of our control problem it is however necessary to consider operating on time varying functions like the Markovian value function $V(t, x_t)$ rather than time invariant functions like $f(\cdot)$. This presents no problem if we assume the existence of an optimal control $u^* \in U_M$ as in corollary 3.12. $V(t-, x_{t-})$ is then differentiable w.r.t. t . And as in lemma 3.14, we can show that for each $t \in [0, T_f]$ $u \in U_M$, we have

$$\begin{aligned} A_t(u)V(t-, x_{t-}) &= \frac{dV(t-, x_{t-})}{dt} \\ &+ \alpha_u^m(t) \left\{ \int_X \{ V(t, x) + c(t, x, u_t) \} \beta_u^m(t, x) \lambda^m(dx, t) - V(t-, x_{t-}) \right\} \end{aligned} \quad (5.27)$$

which in the notation of theorem 5.10 is just

$$A_t(u)V(t-,x_{t-}) = \frac{dV(t-,x_{t-})}{dt} + H_m(t,u_t) . \quad (5.28)$$

In terms of this infinitesimal generator, the Markovian minimum principle can be restated as :

Theorem 5.12.

If $u^* \in u_M$ is a Markovian optimal control , then

$$A_t(u)V(t-,x_{t-}) \geq A_t(u^*)V(t-,x_{t-}) = 0 \quad \text{a.s.} \quad d\Lambda_t^m \quad (5.29)$$

for all $u \in u_M$, $t \in [0, T_f]$.

Pliska's model of the Markov jump process is obtainable from ours if we make the following changes.

- (i) Further restrict u_M to all controls of the form $u(t,\omega) = u(x_{t-}(\omega))$, i.e. time invariant controls.
- (ii) With the basic rates $\bar{\lambda}^m(x,t) = t$, $\lambda^m(x;A,t) = \lambda^m(x;A)$, each $x \in X$, $A \in S$, $t \in \mathbb{R}^+$, let the controlled Radon Nikodym derivatives (α_u^m, β_u^m) be of the form

$$\alpha^m(x_{t-}; t, u(x_{t-})) = \alpha^m(x_{t-}; u(x_{t-}))$$

$$\beta^m(x_{t-}; t, x, u(x_{t-})) = \beta^m(x_{t-}, x, u(x_{t-}))$$
 i.e. the rates are also time invariant.
- (iii) Let the cost rates $c(t, x, u_t, x_{t-})$ be of the form $c(u(x_{t-}), x_{t-})$ and define the cost function to be

$$J_m(u) = E_u^m \left\{ \int_{(0, T_f]} c(u(x_{s-}), x_{s-}) ds + G(x_{T_f}) \right\}$$

To obtain Stone's model from our framework, it is necessary to slightly alter the structure of the information pattern available.

The reason is that Stone's process is not a Markov but rather a semi-Markov process. The information available to the controller therefore dates back to the last jump time from the present instant, i.e. at time t both the jump time and the state at the last epoch $\xi(t)$ are observable. The control one uses now is thus of the form

$$u(t, \omega) = u(\xi_t, x_{t-}; t)$$

Stone however further restricted this to

$$u(\xi_t, x_{t-}; t) = u(x_{t-}; t - \xi_t)$$

i.e. dependent only on the present state and the time elapsed since the last jump epoch.

The other changes necessary are :

$$(i) \quad \bar{\lambda}^m(x; t) = t, \quad \lambda^m(\xi_t, x_{t-}; A, t) = \lambda^m(x_{t-}; A, t - \xi_t)$$

$$(ii) \quad \alpha^m(\xi_t, x_{t-}; t, u(x_{t-}; t - \xi_t)) = \alpha^m(x_{t-}; t - \xi_t, u(x_{t-}; t - \xi_t))$$

$$\beta^m(\xi_t, x_{t-}; t, x, u(x_{t-}; t - \xi_t)) = \beta^m(x_{t-}; t - \xi_t, x, u(x_{t-}; t - \xi_t))$$

The process so obtained is no longer Markov and the superscript m should indicate the semi-Markov property of the process. Since the sigma field of observations $F_{\xi_t}^t = \sigma(x_s, \xi_{t-} \leq s \leq t)$ is locally increasing, the results of the section on complete information is still applicable. (fairly obvious if we view the region of control as $[t, \sigma_t]$). The changes required in the minimum principle are then trivial.

CHAPTER 6 : APPLICATIONS AND CONCLUSIONS.

6.1. Examples of practical applications.

In this section we give a few examples of practical situations in which the control theory developed so far can be applied. Due to the apparently discrete nature of physical objects and human endeavours, these situations certainly can be found in abundance in the world. It could therefore be remarked that efforts towards controlling such discrete phenomena are well justified.

(i) Optimizing the rate of machine operation.

The following situation is perhaps one that occur in most production lines in the modern factory.

Imagine a certain machine making some commodity on a production line. Suppose the machine can be operated in k modes, each mode being characterized by a parameter u_i , $i=1,2,\dots,k$; where $u_i < u_j$ if $j > i$. The higher the mode of operation would mean (say) higher speed or greater efficiency. Further suppose that corresponding to each mode of operation u_i , the machine manufacture imperfect products according to a Poisson process with rate $\alpha(u_i)$, where $0 < \alpha(u_i) < \alpha(u_j) < \infty$, all $i, j \in \{1, \dots, k\}$, $j > i$. Thus in terms of faulty products it is certainly unwise to operate the machine at a high mode.

Denote the jump process (in this case Poisson) formed by the occurrences of imperfect articles by N_t and let u_t be the mode used at time t , chosen according to the number of imperfections already occurred. i.e. u_t is the control based on Markov observations. We wish to find a control policy based on such observations to minimise the cost function

$$J(u) = E_u \left\{ \int_{(0, T_f]} c(s, u_s) ds + G(N_{T_f}) \right\}$$

where $T_f < \infty$, $c(.,.)$ is a non-negative, bounded measurable function which

is also decreasing in the parameter u . $G(\cdot)$ is a non-negative, bounded increasing function. In practical terms $c(t, u_i)$ would be the cost rate of operation at time t while engaging the mode u_i . Clearly it would be best to operate at the highest mode u_k if not for the increased rate of producing faulty articles. The cost of this is being summarised by $G(N_{T_f})$, a function of the total number of imperfect products at the end of the control period.

Since the process is Poisson, the Markovian analysis of Chapter 5 thus applies. Hence using theorem 5.10, the optimal mode of operation at time $t \in [0, T_f]$ is chosen according to

$$\frac{dV(t, N_{t-})}{dt} + \min_{u_i} \{ \alpha(u_i) \{ V(t, N_{t-}+1) - V(t, N_{t-}) \} + c(t, u_i) \} = 0.$$

$$V(T_f, N_{T_f}) = G(N_{T_f}) \quad (6.1)$$

If for example $\alpha(u_i)$ and $c(u_i)$ are linear and of the forms $\alpha_0 u_i$, $c_1 - c_0 u_i$ respectively, some positive constants α_0, c_0, c_1 , then the optimal control u^* is one that minimises

$$u_t \{ \alpha_0 \{ V(t, N_{t-}+1) - V(t, N_{t-}) \} - c_0 \}$$

i.e.

$$u_t^* = \begin{cases} u_k & \text{if } \alpha_0 \{ V(t, N_{t-}+1) - V(t, N_{t-}) \} - c_0 < 0 \\ u_1 & \text{if } \alpha_0 \{ V(t, N_{t-}+1) - V(t, N_{t-}) \} - c_0 \geq 0 \end{cases}$$

As required u_t^* is Markovian and F_t predictable. The solution of (6.1) could now be obtained by solving recursively the equation

$$\begin{aligned} \frac{dV(t, N_{t-})}{dt} + u_k \min \{ \alpha_0 \{ V(t, N_{t-}+1) - V(t, N_{t-}) \} - c_0, 0 \} \\ + u_1 \max \{ \alpha_0 \{ V(t, N_{t-}+1) - V(t, N_{t-}) \} - c_0, 0 \} = 0 \\ V(T_f, N_{T_f}) = G(N_{T_f}) \end{aligned} \quad (6.2)$$

(ii) Ferry control.

Another interesting situation is the operation of a ferry crossing. Assume that the rate $\alpha(u)$ of arrival of passenger ferries at a crossing is Poisson and is a function of the total number u of ferries in service, where $u=1, \dots, k$. Also suppose that the people arriving at the crossing forms another Poisson process with a fixed rate α_0 . Denote the former process by N_t^1 and the latter by N_t^2 . We assume for simplicity that a ferry arriving will always be able to accomodate the accumulation of potential passengers at the crossing and that the boarding time is negligible so that it can be considered instantaneous.

At any time t , the number of people x_t waiting at the crossing is then given by

$$x_t = N_t^2 - N_{\tau(t)}^1$$

where

$$\tau(t) = \sup_s \{ s \leq t : N_s^1 \neq N_t^1 \}$$

x_t is thus a jump process with jump sizes of $+1$ with rate α_0 and jump sizes of $-x_{t-}$ with rate u .

By controlling the number of ferries u in service, we wish to minimise the following cost function based on the Markov observation x_{t-} .

$$J(u) = E_u \left\{ \int_{(0, T_f]} c(s, u, x_{s-}) ds + G(x_{T_f}) \right\}$$

Since the system is obviously Markov, we can therefore apply the Markovian minimum principle and infer that an optimal control is one that satisfy

$$\begin{aligned} \frac{dV(t, x_{t-})}{dt} + \min_u \{ \alpha_0 \{ V(t, x_{t-}+1) - V(t, x_{t-}) \} + u \{ V(t, 0) - V(t, x_{t-}) \} \\ + c(t, u, x_{t-}) \} = 0 \end{aligned} \quad (6.3)$$

$$V(T_f, x_{T_f}) = G(x_{T_f})$$

To obtain an explicit form for the optimal control we could take

$G(x_{T_f}) = 0$ and $c(t, u, x_{t-})$ to be of the form

$$c(t, u, x_{t-}) = c_0 u + x_{t-}$$

some $c_0 > 0$. An optimal control u^* is then one that minimizes :

$$u_t \{ c_0 + V(t, 0) - V(t, x_{t-}) \}$$

i.e.

$$u_t^* = \begin{cases} k & \text{if } \{c_0 + V(t, 0) - V(t, x_{t-})\} < 0 \\ 1 & \text{if } \{c_0 + V(t, 0) - V(t, x_{t-})\} \geq 0 \end{cases}$$

Again u_t^* is Markovian as required and is F_t predictable. From (6.3) we then obtain

$$\begin{aligned} \frac{dV(t, x_{t-})}{dt} &+ \alpha_0 \{V(t, x_{t-}+1) - V(t, x_{t-})\} + x_{t-} \\ &+ \min k \{c_0 + V(t, 0) - V(t, x_{t-}), 0\} \\ &+ \max \{c_0 + V(t, 0) - V(t, x_{t-}), 0\} = 0 \end{aligned} \quad (6.4)$$

$$V(T_f, x_{T_f}) = 0$$

which can be solved recursively.

Remarks 6.1.

The above example could be pictured as a queueing system in which the queue length at time t is x_t . The service offered is however not the usual continually occurring type where the queue length decreases by one each time a service is completed. Here the service occurs at random according to the Poisson rate $\alpha(u)$ and clears the entire queue instantaneously each time it occurs. Another example in real life of the above situation is the collection of accumulating piles of letters by post office vans at a depot. The two contradicting costs are again the waiting cost of each letter and the service cost of the vans in operation.

(iii) Inventory control.

The problem of inventory control is a relatively old one when compared to topics like martingale representation and stochastic integration. We show here how an inventory system can be modelled and analysed using the control theory of jump processes we have developed.

Consider the situation where one operates an N period inventory system in which the demands (or purchase) for stocks arrive according to a Poisson process at a fixed rate α , say. Assume that the magnitude of the demands are random and identically distributed with a probability density of $\phi(\xi)$, where $\phi(\xi)$ is strictly positive and bounded above. To keep up the stock level, goods are ordered at the beginning of each period, i.e. immediately after the arrival of a demand. We assume that the orders are always fulfilled before the next arrival of demand.

Suppose $\{u_1, u_2, \dots, u_N\}$ is an ordering sequence, where u_j denotes the amount of goods ordered for the j th period $(T_{j-1}, T_j]$ at time T_{j-1} . Denote by $\{\xi_j, j=1, \dots, N\}$ corresponding to the demands at time $\{T_j, j=1, \dots, N\}$. One can then write down the stock level sequence $\{Z_j, j=0, 1, \dots, N\}$ corresponding to the two sequences $\{u_j\}, \{\xi_j\}$ as follows

$$\begin{aligned} Z_0 &= z_0 \text{ (fixed)} \\ Z_j &= u_j + Z_{j-1} - \xi_j, \quad j=1, \dots, N \end{aligned} \quad (6.5)$$

Z_j denotes the stock level immediately after time T_j , when the j th demand ξ_j has been fulfilled.

Given Z_{j-1} , the expected j th period cost under policy $\{u_j\}$ is defined as

$$\begin{aligned} E_{u_j} \{c(Z_{j-1}, \xi_j, u_j)\} &= E_{u_j} \{\bar{c}(u_j) + h(u_j + Z_{j-1}) + p(\xi_j - Z_{j-1} - u_j) I_{\{Z_{j-1} + u_j \leq \xi_j\}}\} \\ &= \bar{c}(u_j) + h(u_j + Z_{j-1}) + \int_{(Z_{j-1} + u_j, \infty]} p(\xi - Z_{j-1} - u_j) \phi(\xi) d\xi \\ &= E_{u_j, Z_{j-1}} \{c(Z_j)\} \end{aligned} \quad (6.6)$$

where $\bar{c}(\cdot)$, $h(\cdot)$, $p(\cdot)$ are all bounded non-negative functions. The practical interpretation of the various costs are :

$\bar{c}(u_j)$ the ordering cost for u_j items

$h(u_j + Z_{j-1})$ the storage cost for the j th period.

$p(\xi_j - Z_{j-1} - u_j)$ the shortage cost for the j th period.

The inventory problem is then defined as choosing an ordering policy $u = \{u_j, j=1, \dots, N\}$ based on the observations of the current stock level so as to minimise the following expected N period cost.

$$E_u \left\{ \sum_{j=1}^N c(Z_{j-1}, \xi_j, u_j) \right\} \quad (6.7)$$

The above set up can be viewed as a problem of controlled jump process if we identify the jump times with $\{T_j, j=0, 1, \dots, N\}$ and the states with $\{Z_j, j=0, 1, \dots, N\}$. The rate of jumps of this jump process is hence fixed at α and is not under control. The conditional state jump distribution of the j th jump under control u is given by :

$$\lambda_u^j(Z_{j-1}; (a, b), u_j) = \int_{(a, b]} \phi(u_j + Z_{j-1} - \xi) d\xi$$

Thus ϕ correspond to the controlled Radon Nikodym derivative $\beta_u(\cdot)$ and $\lambda_u^j(Z_{j-1}; \cdot, u_j)$ is mutually absolutely continuous w.r.t. Lebesgue's measure (since the state space is the real line here.)

A control $u \in U$ is of the form

$$u(t, \omega) = \sum_{j=1}^N u_j(Z_{j-1}) I_{\{t \in (T_{j-1}, T_j]\}}$$

and is hence Markov and time invariant between jumps.

The cost function (6.7) can then be written using our system of notations as

$$J(u) = E_u \left\{ \int_{(0, T_N]} c(x, u_t, x_{t-}(\omega)) dp(t, x) \right\}$$

$$= E_u \{ \int_{(0, T_N]} c(x, u_t, x_{t-}) d\tilde{p}^u(t, x) \} \quad (6.8)$$

Note that we have used a random terminal time T_N instead of the usual fixed time T_f , but this is no restriction by remark 3.4.

Since the cost rate and controls are only state dependent, the value function $W(t, \omega)$ will have the following form

$$W(t, \omega) = \sum_{j=1}^N W_{j-1}(Z_{j-1}) I_{\{t \in [T_{j-1}, T_j)\}}$$

for some measurable functions W_{j-1} , $j=1, \dots, N$.

We can now apply corollary 5,8 to characterize an optimal policy u^* as one which minimises

$$\alpha \int_{(0, \infty)} \{W(t, \xi) + c(\xi, u_t, x_{t-})\} \phi(u_t + x_{t-} - \xi) d\xi - W(t-)$$

Due to the time invariance property, this can be restricted as :

$$\begin{aligned} & \alpha \int_{(0, \infty)} \{W_j(\xi) + c(\xi, u_j^*, Z_{j-1})\} \phi(u_j^* + Z_{j-1} - \xi) d\xi - W_{j-1}(Z_{j-1}) \\ &= \min_u \{ \alpha \int_{(0, \infty)} \{W_j(\xi) + c(\xi, u, Z_{j-1})\} \phi(u + Z_{j-1} - \xi) d\xi - W_{j-1}(Z_{j-1}) \} \\ & \quad \text{each } Z_{j-1}, j \in \{1, \dots, N\} \end{aligned} \quad (6.9)$$

If we take $\alpha=1$ (no restriction) then clearly the L.H.S. of (6.9) is clearly zero by the definition of u^* and the value function. Hence (6.9) would reduce to the following recurrence equation.

$$W_{j-1}(Z_{j-1}) = \min_u \{ \int_{(0, \infty)} \{W_j(\xi) + c(\xi, u, Z_{j-1})\} \phi(u + Z_{j-1} - \xi) d\xi \}$$

This equation was also obtained by Scarf in (52) who showed that if $(h(.) + p(.))$ is convex and if $\bar{c}(.)$ is of the form

$$\bar{c}(x) = \begin{cases} 0 & \text{if } x = 0 \\ K + cx, & \text{if } x > 0, \text{ some } K, c, < \infty \end{cases}$$

then the optimal policy for the N period problem is characterized by a pair of critical numbers (s, S) , where $s < S$. The optimal policy is to order the amount $(S-x)$ if the present stock level is $x \leq s$ and not to order if $x > s$.

Remarks 6.2.

It is interesting to realise that one can apply the theory of jump process to inventory control, since at first glance, the demands which arrive at random are not subjected to control. Other references on inventory problems include (52) and (1).

(iv) Regulation of the rate of a Poisson process.

This example was first considered by Rishel in (50). We give a modified form of it here.

Consider a system which has an input Poisson process m_t with a fixed rate of arrivals α . We wish to control this process so that the resulting output process K_t is as near to a fixed rate r as possible in the sense of minimising

$$E_u \left\{ \int_0^{T_f} (K_t - rt)^2 dt \right\}$$

where u is the control based on observations of the input process m_t and $0 \leq u_t \leq 1$ is the probability of allowing the input process to join the output process.

To obtain optimality criteria for the problem, we let the jump process x_t be that corresponding to the jump times $\{T_i, i \in \mathbb{Z}^+\}$ of the input process m_t and the jump states $\{Z_i, i \in \mathbb{Z}^+, \text{ s.t. } Z_i = (m_i, k_i)\}$ where $m_i = m_{T_i}$, $k_i = K_{T_i}$.

Thus the controls are based on partial information in the sense that while all the jump times $\{T_i, i \in \mathbb{Z}^+\}$ are observable, only the first co'ordinate of the jump state is observable.

Hence in the notations of sec.3.5, the observed process $y(t)$ is

also the counting process $p(t)$ of the jump process x_t , so that

$$\alpha_y^u(t) = \alpha_t$$

Moreover

$$\lambda^u(\{x_t = (m_{t-}+1, K_{t-}+1)\}, t) = u$$

$$\lambda^u(\{x_t = (m_{t-}+1, K_{t-})\}, t) = 1-u$$

$$\lambda^u(\{x_t \neq (m_{t-}+1, K_{t-}+1) \text{ and } x_t \neq (m_{t-}+1, K_{t-})\}, t) = 0$$

The optimality criteria is hence obtained according to theorem

3.22 as :

$$\frac{dW(u, t-)}{dt} + (K_t - rt)^2 + \alpha_t \{ W(u, t, Z^1)u + W(u, t, Z^0)(1-u) - W(u, t-) \} \geq 0 \quad (6.10)$$

for all $u \in U$, with equality if and only if u is optimal, where

$$Z^1 = (m_{t-}+1, K_{t-}+1)$$

$$Z^0 = (m_{t-}+1, K_{t-})$$

Owing to the past control dependence of $W(u, t)$, the optimal control is not explicitly evaluable. But (6.10) could perhaps be used as a starting point for approximations.

(v) Controlled Poisson disorder process.

To illustrate the partial information situation, we give another interesting example which can be applied to practical problems connected with (say) the emission of electrons of a cathode ray tube.

Consider a Poisson process N_t whose initial exponential rate is $\alpha_1 > 0$. It is possible to alter this rate to $\alpha_2 > \alpha_1$ by exerting a control device (voltage change perhaps) characterised by the parameter u , $0 < u < 1$. Assume this change of rate can only occur at the jump times

of the Poisson process and that u is the probability of achieving this change. Apart from knowing that the initial rate is α_1 , the controller does not observe the change of the rate α_1 to α_2 . The problem is to seek a control based on the observations of the jump times over a finite interval $[0, T_f]$ so as to minimise

$$E_u \int_0^{T_f} (k_1 u_s^2 - k_2 N_s) ds$$

Here $k_1 u_s^2$ could represent the energy expended while $k_2 N_s$ is proportional to the number of occurrences of events which we like to maximise.

Using the convention of sec.3.5 we reformulate the problem as follows

Define a controlled jump process x_t with jump times coinciding with that of the Poisson process N_t , but whose state space X is in R^2 and each state Z_i is of the form

$$Z_i = (i, \alpha_1) \quad \text{or} \quad (i, \alpha_2)$$

The initial state of x_t is $(0, \alpha_1)$, and observations can only be made on the first co'ordinate of each state.

The state $\alpha(t, \omega)$ of the jump process x_t is determined by the set of measurable functions $\{\alpha_k(\omega_{k-1}; t), k \in Z^+\}$ defined by :

$$\alpha_k(\omega_{k-1}; t) = \alpha_k(Z_{k-1}; t) = \begin{cases} \alpha_1 & \text{if } Z_{k-1} = (k-1, \alpha_1) \\ \alpha_2 & \text{if } Z_{k-1} = (k-1, \alpha_2) \end{cases}$$

The conditional state jump distribution $\lambda_u(\cdot, t, \omega)$ corresponding to a control u is time invariant and of the form

$$\begin{aligned} \lambda_u(\omega_{k-1}; (k, \alpha_2)) &= \alpha_u(Z_{k-1}; (k, \alpha_2)) \\ &= \begin{cases} u & \text{if } Z_{k-1} = (k-1, \alpha_1) \\ 1 & \text{if } Z_{k-1} = (k-1, \alpha_2) \end{cases} \end{aligned}$$

$$\begin{aligned}\lambda_u(\omega_{k-1}; (k, \alpha_1)) &= \lambda_u(Z_{k-1}; (k, \alpha_1)) \\ &= \begin{cases} 1-u & \text{if } Z_{k-1} = (k-1, \alpha_1) \\ 0 & \text{if } Z_{k-1} = (k-1, \alpha_2) \end{cases}\end{aligned}$$

Since the observed counting process is the same as that corresponding to x_t , by (3.49) we have

$$\alpha_y^u(t) = E_u\{\alpha(t)/F_t^y\} = \alpha_1 \gamma_t^u + \alpha_2 (1-\gamma_t^u)$$

where $\gamma_t^u = P_u(\alpha(t) = \alpha_1 / F_t^y)$

Using theorem 3.22 we obtain the optimality criteria as

$$\begin{aligned}\frac{dW(u, t-)}{dt} &+ \{\alpha_1 \gamma_t^u + \alpha_2 (1-\gamma_t^u)\} \{W(u, t, Z^1) u_t + W(u, t, Z^0) (1-u_t)\} \gamma_t^u \\ &+ W(u, t, Z^1) (1-\gamma_t^u) - W(u, t-) \} + (k_1 u_t^2 - k_2 N_t) \geq 0\end{aligned}\tag{6.11}$$

for all $u \in U$, where

$$Z^1 = (N_{t-} + 1, \alpha_2), \quad Z^0 = (N_{t-} + 1, \alpha_1)$$

Further equality holds if and only if u is optimal.

Remarks 6.3.

Note that due to the complete observations of all jump times, a kind of separation principle is obtained in that the rate of the process could be estimated by itself. The equation (6.11) is however still too complicated to permit an explicit solution.

For a more general formulation of disorder problems, see (59)

6.2. Conclusions.

The thesis has successfully formulated a mathematical model for a very general class of controlled jump processes, flexible enough to be inclusive of most physical and economic systems dealing with such processes. Significant necessary and sufficient Hamilton -Jacobi type of optimality conditions were obtained for the optimal control problem, using relatively new techniques in martingale theory. This is especially true of the case when complete information of the past is available to the controller. It is also here that we managed to prove the existence of an optimal control law, subjected to certain conditions on the Radon Nikodym derivatives and cost rate of the process. By specializing to the Markovian situation, we managed to show that our results are generalizations of some of the previous literature on Markovian decision theory. It can thus be concluded that the martingale theoretic approach to such problems, though esoteric in some sense, certainly provides elegant and comprehensive results when applied pertinently.

APPENDIX.

The ϵ -lattice property and the interchange of the infimum and the conditional expectation operator.

The following is an adaptation of the arguments of Striebel in (56).

Definition.

Let $\{f_\gamma(\omega), \gamma \in C\}$ be a family of non-negative integrable functions on a probability space (Ω, F, P) , then the infimum in $L_1(\Omega, F, P)$ is defined as

$$f(\omega) = \inf_{\gamma \in C} f_\gamma(\omega) \quad \text{a.s. } P$$

where :

(i) $f(\omega)$ is F measurable .

(ii) for all $\gamma \in C$, $f(\omega) \leq f_\gamma(\omega)$ a.s. P (A1)

(iii) If $g(\omega)$ satisfies (i) and (ii) then

$$g(\omega) \leq f(\omega) \quad \text{a.s. } P$$

Lemma A.1

The infimum of $\{f_\gamma, \gamma \in C\}$ exists and is uniquely (a.s.P) given by :

$$\inf_{\gamma \in C} f_\gamma(\omega) = \frac{d\lambda}{dP}(\omega) \quad \text{a.s. } P \quad (A2)$$

where

$$\lambda_\gamma(A) = \int_A f_\gamma(\omega) dP, \quad \text{each } A \in F. \quad (A3)$$

and

$$\lambda(A) = \inf_{\substack{\{\gamma_i, A_i\} \\ i=1, \dots, n}} \sum_{i=1}^n \lambda_{\gamma_i}(A_i), \quad \text{each } A \in F. \quad (A4)$$

where the infimum is taken over all finite sets of indices $\{\gamma_i, i=1, \dots, n\}$ and disjoint sets $\{A_i \in F, i=1, \dots, n\}$ with $\bigcup_{i=1}^n A_i = A$.

Proof :

Since

$$\lambda_{\gamma}(A) = \int_A f_{\gamma}(\omega) dP, \quad \text{each } A \in \mathcal{F},$$

we have

$$\lambda(A) = \inf_{\substack{\{\gamma_i, A_i\} \\ i=1, \dots, n}} \sum_{i=1}^n \lambda_{\gamma_i}(A_i) = \inf_{\substack{\{\gamma_i, A_i\} \\ i=1, \dots, n}} \sum_{i=1}^n \int_{A_i} f_{\gamma_i}(\omega) dP \quad (A5)$$

By (26) III 7.5, 7.6, $\lambda(\cdot)$ exists and is a countably additive measure.

Since the infimum in (A5) is over some finite sets and indices, $\lambda(\cdot)$ must be absolutely continuous w.r.t. P , hence a Radon Nikodym derivative $\frac{d\lambda}{dP}$ is defined a.s. P uniquely by

$$\lambda(A) = \int_A \frac{d\lambda}{dP}(\omega) dP, \quad \text{each } A \in \mathcal{F}.$$

To prove (A2), we need to check that $\frac{d\lambda}{dP}$ satisfies (A1).

Clearly $\frac{d\lambda}{dP}$ satisfies (A1)(i). Suppose now that $\frac{d\lambda}{dP}(\omega) > f_{\gamma}(\omega)$ for $\omega \in A$ such that $P(A) > 0$, then

$$\begin{aligned} \lambda(A) &= \int_A \frac{d\lambda}{dP}(\omega) dP > \int_A f_{\gamma}(\omega) dP > \sum_{i=1}^n \int_{A_i} f_{\gamma}(\omega) dP \\ &\geq \inf_{\substack{\{\gamma_i, A_i\} \\ i=1, \dots, n}} \sum_{i=1}^n \int_{A_i} f_{\gamma_i}(\omega) dP = \lambda(A) \end{aligned}$$

which is a contradiction, hence we infer that

$$\frac{d\lambda}{dP}(\omega) \leq f_{\gamma}(\omega) \quad \text{a.s. } P$$

and (A1)(ii) is satisfied.

Suppose now there is a measurable $g(\omega)$ satisfying

$$g(\omega) \leq f_{\gamma}(\omega) \quad \text{a.s. } P, \text{ all } \gamma \in C.$$

we have

$$\begin{aligned} \int_A g(\omega) dP &\leq \int_A f_{\gamma}(\omega) dP = \sum_{i=1}^n \int_{A_i} f_{\gamma_i}(\omega) dP \\ &\leq \sum_{i=1}^n \int_{A_i} f_{\gamma_i}(\omega) dP, \quad \text{some } \{\gamma_i, i=1, \dots, n\} \end{aligned}$$

Taking infimum over the set $\{\gamma_i, A_i, i=1, \dots, n; A_i \in F, \bigcup_{i=1}^n A_i = A\}$ on both sides yields

$$\int_A g(\omega) dP \leq \lambda(A) = \int_A \frac{d\lambda}{dP}(\omega) dP$$

implying

$$g(\omega) \leq \frac{d\lambda}{dP}(\omega) \quad \text{a.s. } P.$$

Hence $\frac{d\lambda}{dP}$ satisfies (A1)(iii) as well, so that

$$\frac{d\lambda}{dP} = \inf_{\gamma \in C} f_{\gamma}(\omega) \quad \text{a.s. } P$$

This completes proof.

Definition

The family $\{f_{\gamma}(\omega), \gamma \in C\}$, has the ϵ -lattice property provided for all $\epsilon > 0$, $\gamma_1, \gamma_2 \in C$, there exists a $\gamma_0 \in C$ such that

$$f_{\gamma_0}(\omega) \leq f_{\gamma_i}(\omega) + \epsilon, \quad \text{a.s. } P \quad i=1, 2.$$

Lemma A2.

If the family $\{f_{\gamma}(\omega), \gamma \in C\}$ has the ϵ -lattice property, then $\lambda(A)$ defined by (A3) and (A4) satisfies

$$\lambda(A) = \inf_{\gamma \in C} \lambda_{\gamma}(A), \quad \text{each } A \in F. \quad (A6)$$

Proof :

For $A \in F$, $\epsilon > 0$, take $\gamma_1 \in C$, disjoint sets $\{A_i \in F, i=1, \dots, n\}$ with

$$\bigcup_{i=1}^n A_i = \sum_{i=1}^n A_i = A$$

and

$$\sum_{i=1}^n \lambda_{\gamma_i}(A_i) \leq \lambda(A) + \frac{\varepsilon}{2} \quad (A7)$$

From the ε -lattice property, there exists a $\gamma_0 \in C$ such that

$$f_{\gamma_0}(\omega) \leq f_{\gamma_i}(\omega) + \frac{\varepsilon}{2}, \quad i=1,2,\dots,n \quad \text{a.s. } P$$

From (A3) this means that

$$\lambda_{\gamma_0}(A_i) \leq \lambda_{\gamma_i}(A_i) + \frac{\varepsilon}{2} P(A_i), \quad i=1,2,\dots,n$$

Hence

$$\begin{aligned} \lambda_{\gamma_0}(A) &= \sum_{i=1}^n \lambda_{\gamma_0}(A_i) \leq \sum_{i=1}^n \lambda_{\gamma_i}(A_i) + \frac{\varepsilon}{2} P(A) \\ &\leq \lambda(A) + \varepsilon. \quad (\text{using (A7)}) \end{aligned}$$

Thus

$$\inf_{\gamma \in C} \lambda_{\gamma}(A) \leq \lambda_{\gamma_0}(A) \leq \lambda(A) + \varepsilon$$

From (A4) we also have

$$\inf_{\gamma \in C} \lambda_{\gamma}(A) \geq \lambda(A)$$

implying

$$\inf_{\gamma \in C} \lambda_{\gamma}(A) = \lambda(A)$$

This completes the proof.

Theorem A 3.

Let $\{f_{\gamma}(\omega), \gamma \in C\}$ be a family of non-negative integrable functions with the ε -lattice property, let $F^Y \subset F$ and let P^Y be the restriction of the measure P to F^Y . Then

$$E \left\{ \inf_{\gamma \in C} f_{\gamma}(\omega) / F^Y \right\} = \inf_{\gamma \in C} E \{ f_{\gamma}(\omega) / F^Y \} \quad \text{a.s. } P^Y$$

Proof :

Due to the linearity of the conditional expectation operation, $E\{f_Y(\omega)/F^Y\}$ on (Ω, F^Y, P^Y) is non-negative, integrable and satisfies the ε -lattice property. Let

$$\lambda_Y(A) = \int_A f_Y(\omega) dP, \quad A \in F$$

$$\lambda'_Y(B) = \int_B E\{f_Y/F^Y\} dP, \quad B \in F^Y$$

$$\lambda(A) = \inf_Y \lambda_Y(A), \quad A \in F$$

$$\lambda'(B) = \inf_Y \lambda'_Y(B), \quad B \in F^Y$$

From lemma A1 and A2, we have

$$\inf_{Y \in C} f_Y = \frac{d\lambda}{dP} \quad \text{a.s. } P \quad (A8)$$

$$\inf_{Y \in C} E\{f_Y/F^Y\} = \frac{d\lambda'}{dP} \quad \text{a.s. } P^Y \quad (A9)$$

But from the property of conditional expectation, for $B \in F^Y$

$$\lambda_Y(B) = \int_B f_Y(\omega) dP = \int_B E\{f_Y/F^Y\} dP = \lambda'_Y(B).$$

Hence

$$\lambda(B) = \lambda'(B) \quad (A10)$$

Thus from (A8), (A9), and (A10), we have for all $B \in F^Y$

$$\begin{aligned} \int_B \inf_Y E\{f_Y/F^Y\} dP &= \lambda'(B) = \lambda(B) = \int_B \inf_Y f_Y dP \\ &= \int_B E\{\inf_Y f_Y / F^Y\} dP \end{aligned}$$

And since $\inf_Y E\{f_Y/F^Y\}$ is F^Y measurable, we obtain

$$\inf_{Y \in C} E\{f_Y/F^Y\} = E\{\inf_{Y \in C} f_Y / F^Y\} \quad \text{a.s. } P^Y$$

This completes the proof.

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