U N I V E R S I T Y O F L O N D O N

F A C U L T Y O F E N G I N E E R I N G

Networks of Non-linear Resistors and Their Application as Analogues in the Solution of Physical Problems

A Thesis Presented for the Degree of Doctor of Philosophy

by

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ABSTRACT

This thesis is concerned with developing new methods for the solution of non-linear resistance networks, and the application of these new methods to the solution of resistance-network analogues of physical problems which cannot easily be obtained by existing analogue network techniques. The extension of existing non-linear theories involving resistance networks is also included.

A numerical method for the solution of small non-linear networks suitable for use with desk calculating machines is described. This method is derived from the existing theory of content and co-content. For large networks the method can be used with a digital computer. The method of solution by graphs, based on the theory of content and co-content is also investigated. New theorems of the content and co-content are presented together with suitable analogue devices to assist and speed up the computing process.

A new graphical method based on the voltage and current diagram of the resistance-network is used to

obtain the solution of the resistance network analogue of physical problems, and is successful in cases where the existing analogue network techniques failed to produce the solution. This method is not only superior to the existing analogue resistance network techniques in this respect, but it is also quicker, simpler to understand and more economical than using an analogue resistance network computer. The method is also used successfully in the design of complex valve amplifier circuits.

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INTRODUCTION

A physical system is said to be non-linear if the behaviour of each of its elements cannot be represented by an algebraic, differential or integral equation of the first degree. Strictly speaking all physical systems are non-linear, for linear systems never occur in nature. Most physical problems are however treated as linear systems because of the difficulty in analysis of the non-linear type, and in most cases, approximate solutions are essential in day to day practice.

The electrical system is not an exceptional case. Lack of knowledge of non-linear theories and lack of methods of solving non-linear problems however, compel it to be treated as a linear system, but this unfortunate factor may obscure interesting properties possessed by the non-linear system itself. Thus some parts of this thesis are devoted to the development and extension of existing non-linear theories even though they may seem to have no immediate application.

The solving of non-linear resistance networks will not only benefit the field of electrical

engineering, but the method could be applied to other engineering problems, such as those concerning the flow of liquid or gas in a pipe line, ie. the distribution of water supply, mine ventilation, the problem of the distributed load of non-linear springs, and building structures. Production engineers are looking for a solution to the problem of smooth flow on production lines, which is in many senses similar to the flow of currents through non-linear resistances.

Obtaining the solution of the non-linear electrical resistance network, by direct application of both of Kirchhoff's laws, brings in the problem of solving non-linear simultaneous algebraic equations. This is not only difficult but in some cases impossible. There are many other numerical methods which have been used from time to time for the solution of non-linear resistance networks, but most of these methods were developed for a particular type of circuit and limited in ≡ general application.

The first part of this thesis deals with a new numerical method which is developed from a suggestion made by Professor Cherry, and stated in the paper published by Miller[2] concerning the properties of"content" and "co-content" of a resistance network.

The method suggested by Professor Cherry is a form of the relaxation method, and has been proved to give excellent results. The time taken to reach the solutions depends however, on personal experience of the problem, and has proved to be rapid only if the approximate solutions are known beforehand. This thesis describes its development into a systematic process and experience of the problem or the knowledge of approximate solutions is no longer so necessary.

Following the numerical method, Miller's theory of content and co-content is investigated graphically. It has been extended to cover the cases where negative resistances are included in the circuit.

The application of this graphical method to the solution of resistance network analogues (Appendix 1 resistance network being used as an analogue of the finite-difference form of a non-linear differential equation) was successful, and at the time it was superior to the existing method when applied to some cases. This was because the existing iterative processes (9) (13) (due to Liebmann and Bailey, Karplus, etc.) failed to converge when certain values of negative resistances occurred in the circuit. Unfortunately

the solution by this graphical method is slow and tedious, mainly because many graphs have to be plotted and thus further development was concentrated on finding a means of accelerating the graph-plotting process. This is done mainly by the analogue device. (2.5 page 69).

Further study of the solution of differential equations by the resistance-network analogue has led to the following questions:-

a) Is it necessary to employ expensive equipment such as the equipment used by Hutcheon and Spalding[7], or by J. Ph. Korthals Altes[6] for the solution of a one dimensional problem ?

b) Is it possible to use cheaper equipment which is also applicable to the cases where the analogue techniques described by Liebmann and Bailey[9], Karplus[13], Hutcheon[11] and Altes[6] fail to converge ?

c) Is there a less complicated system which can be employed ?

These questions have led to the development of the new graphical system, described in the latter part of the thesis, which is based on Cherry's v-i rectangle diagrams[1]&[32].

The technique of the resistance-network analogue for solving boundary value problems involving differential equations has been known for a long time and because of

its ability to give good approximate solutions to the Laplace and Poisson equations when the exact mathematical analysis fails, it has been extensively used. The first known form of this analogue technique is the electrolytic tank which was first used by Adams[31] 85 years ago and has been developed to use resistors by Beuken in Holland for the solution of transient heat-flow problems in 1937, known as "electrical models". It has then been developed by many authors in many countries. The best known authors in this field in the last few years are the late G. Liebmann, and S.C Redshaw. This technique has been used extensively in the aerodynamic field, but it is most unfortunate that the iterative method is the basis for obtaining the solution of equations other than Laplace's. In 1957 M.E. Fisher[4] pointed out that the iterative method fails to converge when the solution becomes wave-like even for the linear equation. This obstacle is inherent in the iterative process of computing and cannot be overcome in a direct manner. To solve this problem then became part of the object of this thesis and led to the development of the new graphical method.

The iterative method by visual means has been known and practised by most of us in our childhood

in the form of jig-saw puzzles, and has remained in our subconscious minds and been used extensively in many forms. It has been known to be superior to the mathematical method because the eyes can interpret many problems quicker than the other senses even though they may not give very accurate results. This may be demonstrated by a problem of arranging furniture in a room to be built, where the size of the room is represented on a piece of paper, and all the furniture is represented by small pieces of paper of proportional size. They can then be arranged to fit the room much quicker than if they were represented by numerical dimensions. This visual iterative method together with the planar electrical diagram described by Cherry[1] forms the main part of the new graphical method. Cherry describes a representation of an electrical circuit by a set of continuous rectangles originated from the theory of squared-squares, a classical problem in geometry. Each of the rectangle sizes is determined by the V-I characteristic of the resistance element (Section 3.1). I have found that the solution of the ladder network containing non-linear resistances can be obtained very rapidly by drawing the V-I characteristic of the constituent

resistances on rectangular sheets of transparent glass and arranging them on top of one another to form Cherry's diagram. From this impressive result, the method is then further developed by using templates instead of glass sheets. This method is extended to cover a ladder network containing non-linear negative resistances, and it can be used to solve many one-dimensional Poisson's equations, and second-order differential equations including the eigenvalue problem. The fourth-order differential equation is also not beyond the capability of this template method, and I have shown that it can give a more accurate result than using two resistance networks coupled together by high value resistors, which latter was Liebmann's method.(3) This is because these high value resistors can be omitted altogether and therefore the approximation made for the losses in these resistors is no longer necessary.

This template method is found to be equally useful in the design of complex valve (or transistor) circuits. and it is no longer necessary to assume that the valve characteristics are linear. This method will give a precise solution of valves circuits for d.c. coupling. In the case of the cathode-coupled amplifier circuit, the "longtail pair", this method can give an accurate

solution even when both valves have different characteristics. Use of the existing analytical method in such a case only leads to great confusion.

Finally this template method can no doubt be applied to two-dimensional problems involving the resistance network analogue by further development, and with the assistance of the automatic means outlined in section. 3.7. It will have a wide application in all fields because of its flexibility, simplicity, and economy.

CHAPTER I

The Solution of the Non-linear Resistive Electrical Circuit by the theory of the Content of the Resistance Network.

In this section the theory of the "content" of the network, defined by W.Miller[2], is employed to obtain the solution of non-linear resistive circuit.

The iterative method derived from the theory of content, together with a desk calculating machine or slide rule, has been found to be a very powerful tool in obtaining the solution of a small non-linear resistive circuit. In a larger circuit the replacement of the desk calculating machine by the digital computer would be essential.

The non-linear equations associated with the non-linear circuit are transformed into approximate linear simultaneous equations. The solution obtained from the linear simultaneous equations is then the first approximate solution of the non-linear circuit. This solution is then improved by repeating the same process.

1.1 The Theory of the Content

The content G of a resistance element is defined as the area under the voltage-current characteristic of the element, from zero current to the operating

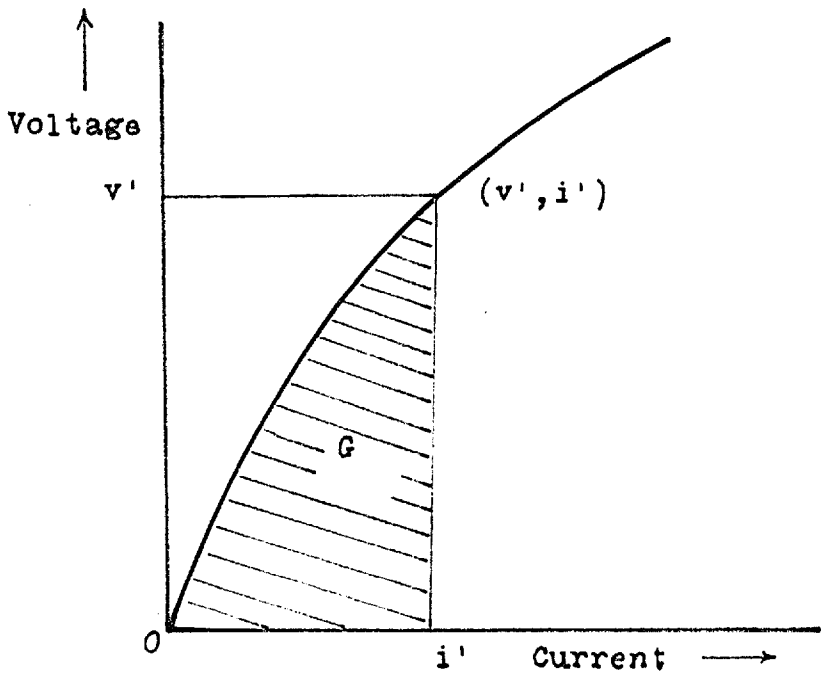


Fig. 1

current i' of the element. (i.e. The shaded area in Fig.1.)

$$\text{or} \quad G = \int_0^{i'} v\,di$$

Content has the dimensions of power and indeed for an ordinary (linear) resistance, the content is equal to half the power dissipation.

$$\int_0^{i'} v\,di = \int_0^{i'} Ri\,di = \frac{1}{2}R(i')^2$$

The content of a constant-voltage source E is equal to $-\int_0^{i'} E\,di = -Ei'$. This has a negative value if the current i' flows from the high potential to the low potential terminal through the external circuit; (it is equal to minus the power supplied by the generator.) The content of a constant-current generator is zero.

It was shown by Miller[2] that the total content of a two-terminal resistance network is equal to the sum of the contents of the constituent elements of the network. In addition, the content of the network has a stationary value at the real values of the voltages and currents of the network, with respect to fictitious variations of mesh currents (or of some other set of currents satisfying $\Sigma i = 0$ at all nodes). The voltage law $\Sigma v = 0$ round meshes is violated during the fictitious current

variation, but the V-I relationship of each resistance is maintained.

1.2 The application of the content to the non-linear circuit

For the solution of a non-linear resistive circuit with N unknown currents, each unknown current will first be given an arbitrary value. The arbitrary value of the currents is denoted by a letter"0" such as $I_{10}, I_{20}, \text{------} I_{(N-1)0}$ and I_{N0}. The amount of current which is added to, or subtracted from each arbitrary current to bring it to the correct value is denoted by a letter i_n. G_0 is the sum of the content of the network corresponding to the arbitrary valuc of the currents $I_{10}, I_{20} \text{-----} I_{N0}$, and G is the content of the network after the values of the arbitrary currents are corrected. $\{(I_{10}+i_1), (I_{20}+i_2) \text{-------} (I_{N0}+i_N)\}$ Therefore G is the function of the currents $(I_{10}+i_1,\ I_{20}+i_2, \text{----} I_{N0}+i_N)$.

By Taylor's theorem, G is expanded in terms of G_0 as follows:-

$$G = G_0 + \left(i_1\frac{\partial}{\partial I_{10}} + i_2\frac{\partial}{\partial I_{20}} + \text{----} i_N\frac{\partial}{\partial I_{N0}}\right)G_0$$
$$+\frac{1}{2}\left(i_1\frac{\partial}{\partial I_{10}} + i_2\frac{\partial}{\partial I_{20}} + \text{-------} i_N\frac{\partial}{\partial I_{N0}}\right)^2 G_0$$
$$+\frac{1}{3}\left(i_1\frac{\partial}{\partial I_{10}} + i_2\frac{\partial}{\partial I_{20}} + \text{------} i_N\frac{\partial}{\partial I_{N0}}\right)^3 G_0 \text{------} (1)$$

To obtain approximate values of i_1, i_2, i_3 the derivatives from the 2nd order upwards are neglected. The equation (1) becomes.

$$G = G_0 + (i_1\frac{\partial}{\partial I_{10}} + i_2\frac{\partial}{\partial I_{20}} + \text{-------} i_N\frac{\partial}{\partial I_{NO}})G_0$$

$$+\frac{1}{2}(i_1\frac{\partial}{\partial I_{10}} + i_2\frac{\partial}{\partial I_{20}} + \text{-------} i_N\frac{\partial}{\partial I_{NO}})^2 G_0 \text{ -------(2)}$$

Assume that N = 2, for simplicity in presenting the theory, then equation (2) becomes

$$G = G_0 + i_1\frac{\partial G_0}{\partial I_{10}} + i_2\frac{\partial G_0}{\partial I_{20}} + \frac{1}{2}i_1^2\frac{\partial^2 G_0}{\partial I_{10}^2} + \frac{1}{2}i_2^2\frac{\partial^2 G_0}{\partial I_{20}^2} +$$

$$+i_1 i_2\frac{\partial^2 G_0}{\partial I_{10}\partial I_{20}} \text{ ----------(3)}$$

If G is to have a minimum value, the $\frac{\partial G}{\partial i_1}$ and $\frac{\partial G}{\partial i_2}$ must be equal to zero.

Differentiating G with respect to i_1 and i_2, the equation (3) becomes:-

$$\frac{\partial G}{\partial i_1} = 0 + \frac{\partial G_0}{\partial I_{10}} + 0 + i_1\frac{\partial^2 G_0}{\partial I_{10}^2} + 0 + i_2\frac{\partial^2 G_0}{\partial I_{10}\partial I_{20}} \text{ -------(4)}$$

$$= \frac{\partial G_0}{\partial I_{10}} + i_1\frac{\partial^2 G_0}{\partial I_{10}^2} + i_2\frac{\partial^2 G_0}{\partial I_{10}\partial I_{20}} = 0 \text{ --------(5)}$$

Similarly

$$\frac{\partial G}{\partial i_2} = \frac{\partial G_0}{\partial I_{20}} + i_2\frac{\partial^2 G_0}{\partial I_{20}^2} + i_1\frac{\partial^2 G_0}{\partial I_{10}\partial I_{20}} = 0 \text{ -----(6)}$$

Since I_{10} and I_{20} have been given arbitrary values then the

corresponding values of $\frac{\partial G_0}{\partial I_{10}}$, $\frac{\partial G_0}{\partial I_{20}}$, $\frac{\partial^2 G_0}{\partial I_{10}^2}$, $\frac{\partial^2 G_0}{\partial I_{20}^2}$, $\frac{\partial^2 G_0}{\partial I_{10}\partial I_{20}}$ can be calculated.

The equations (5) and (6) then become linear simultaneous equations, whose solutions will be i_1 and i_2.

For N unknown currents in the circuit there will be N simultaneous equations.

In equation (2), the derivatives which are greater than the 2nd order are neglected, therefore the values of i_1 and i_2 for non-linear cases are only approximate. Thus it follows that the solutions of the unknown currents I_1 and I_2 are only the approximate solutions. (ie. $I_1 \doteq I_{10} + i_1$ and $I_2 \doteq I_{20} + i_2$). However, these approximate values of the currents can be improved by repeating the calculations and using these approximate values of the currents as the next arbitrary values.

In the linear case, the derivatives which are greater than the 2nd order are zero, and therefore the values of the currents i_1 and i_2 are now exact. Then after the arbitrary values of the currents are corrected, the values of the currents are the correct solutions.

1.3 Examples.

Examples are given for both linear and non-linear cases. In the linear case, the examples are to demonstrate

that the method gives the correct solution in one operation. This is because the value of i_n is an exact value. In the non-linear case i_n is merely an approximate value and the degree of approximation depends on the values of the terms omitted from equation (1). The values of the omitted terms also governs the rate of convergence. ie. If they are small the convergence will be rapid.

Eg. 1. Linear Case

The currents I_1 and I_2 of the circuit shown in Fig. 1.1 are to be evaluated. $R = 5\Omega$, $R_1 = 1\Omega$ $R_2 = 2\Omega$ and $E = 10$ volts.

The arbitrary values of the currents I_{10} and I_{20} are chosen to be 1 amp. each.

The value of the arbitrary content G_0 is calculated as follows:-

$$G_0 = -E(I_{10}+I_{20}) + \int_0^{I_{10}+I_{20}} Ridi + \int_0^{I_{10}} R_1 idi + \int_0^{I_{20}} R_2 idi$$

$$= -E(I_{10}+I_{20}) + \frac{1}{2}R(I_{10}+I_{20})^2 + \frac{1}{2}R_1 I_{10}^2 + \frac{1}{2}R_2 I_{20}^2 \quad \text{---(7)}$$

$$\frac{\partial G_0}{\partial I_{10}} = -E + R(I_{10}+I_{20}) + R_1 I_{10} = 1 \quad \text{----------(8)}$$

$$\frac{\partial G_0}{\partial I_{20}} = -E + R(I_{10}+I_{20}) + R_2 I_{20} = 2 \quad \text{----------(9)}$$

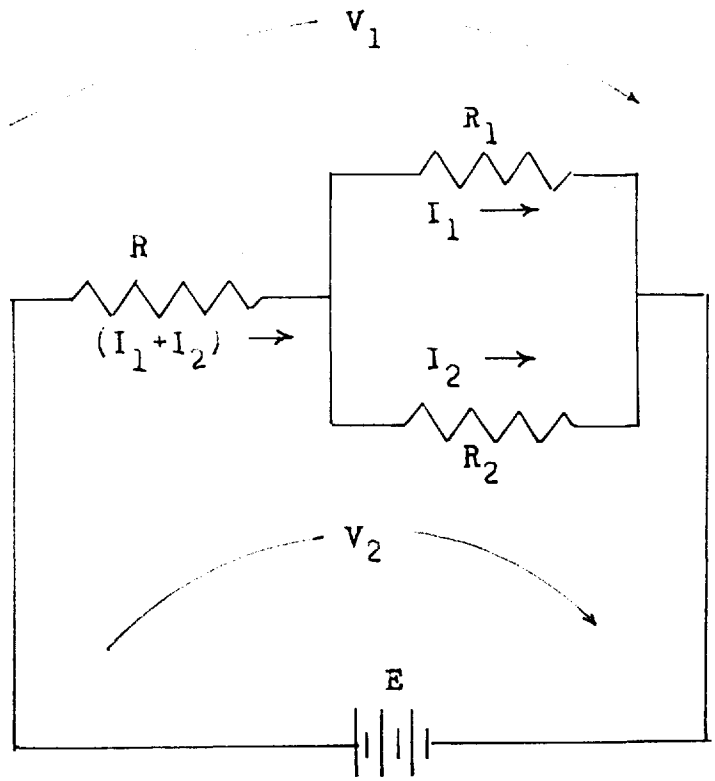


Fig.1.1

$$\frac{\partial^2 G_0}{\partial I_{10}^2} = R_1 + R = 6 \quad \text{----------(10)}$$

$$\frac{\partial^2 G_0}{\partial I_{20}^2} = R + R_2 = 7 \quad \text{----------(11)}$$

$$\frac{\partial^2 G_0}{\partial I_{10} \partial I_{20}} = R = 5 \quad \text{----------(12)}$$

Substitute (8) to (12) in equations (5) and (6)

(5) becomes $1 + 6i_1 + 5i_2 = 0$---- ---(13)

(6) becomes $2 + 5i_1 + 7i_2 = 0$--------(14)

evaluate i_1 and i_2

$$i_1 = \frac{3}{17} \text{ amps} \quad \text{----------(15)}$$

$$i_2 = -\frac{7}{17} \text{ amps} \quad \text{----------(16)}$$

The value of the current I_1 is then equal to $I_{10} + i_1$

$$I_1 = 1\frac{3}{17} \text{ amps.}$$

Similarly

$$I_2 = \frac{10}{17} \text{ amps.}$$

They both are the correct values of I_1 and I_2

Eg.2 <u>The Non-linear case.</u>

The currents I_1 and I_2 of a similar circuit to that shown in Fig.1.1 are to be evaluated. R and R_2 are linear resistors of the value 5 ohms and 2 ohms respectively, but the resistor R_1 is non-linear and has its V-I characteristic $V = kI^2$ with $k = 0.85$.

The arbitrary values of currents I_{10} and I_{20} are chosen to be 1 amp., and the arbitrary content G_0

is calculated from these values.

$$G_0 = -E(I_{10}+I_{20}) + \int_0^{I_{10}+I_{20}} Ridi + \int_0^{I_{10}} ki^2 di + \int_0^{I_{20}} R_2 i di$$

$$= -E(I_{10}+I_{20}) + \tfrac{1}{2}R(I_{10}+I_{20})^2 + \tfrac{k}{3}I_{10}^3 + \tfrac{1}{2}R_2 I_{20}^2 \quad --(17)$$

$$\frac{\partial G_0}{\partial I_{10}} = -E + R(I_{10}+I_{20}) + kI_{10}^2 \quad ----------(18)$$

$$\frac{\partial^2 G_0}{\partial I_{10}^2} = R + 2kI_{10} \quad ----------(19)$$

$$\frac{\partial^2 G_0}{\partial I_{10}\partial I_{20}} = R \quad ----------(20)$$

$$\frac{\partial G_0}{\partial I_{20}} = -E + R(I_{10}+I_{20}) + R_2 I_{20} \quad ----------(21)$$

$$\frac{\partial^2 G_0}{\partial I_{20}^2} = R + R_2 \quad ----------(22)$$

Substitute (17) to (22) into equation (5) and (6), and evaluate i_1 and i_2.

Since this is a non-linear case, i_1 and i_2 are merely approximate values, therefore the process has to be repeated until the correct values of the currents are obtained.

The table below shows the improved value of the currents I_1 and I_2 after each approximation together with the sum of the voltages around outer and inner meshes, V_1 and V_2.

	I_1 amps.	I_2 amps.	$V_1 \rightarrow 0$	$V_2 \rightarrow 0$ volts
1st approx.	1.1849	0.5722	0.1894	-0.0701
2nd approx.	1.1790	0.5864	0.0085	-0.0002
3rd approx.	1.1767	0.5880	0.0005	+0.0004

Any further approximation only improves the 3rd and 4th decimal places.

Eg.3

This example illustrates the calculation when there are three unknown currents. A linear case is first considered, followed by a non-linear case in the next example.

For N = 3 equation (3) becomes:-

$$G = G_0 + i_1\frac{\partial G_0}{\partial I_{10}} + i_2\frac{\partial G_0}{\partial I_{20}} + i_3\frac{\partial G_0}{\partial I_{30}} + \frac{i_1^2}{2}\frac{\partial^2 G_0}{\partial I_{10}^2} + \frac{i_2^2}{2}\frac{\partial^2 G_0}{\partial I_{20}^2}$$

$$\frac{i_3^2}{2}\frac{\partial^2 G_0}{\partial I_{30}^2} + i_1 i_2\frac{\partial^2 G_0}{\partial I_{10}\partial I_{20}} + i_1 i_3\frac{\partial^2 G_0}{\partial I_{10}\partial I_{30}} + i_2 i_3\frac{\partial^2 G_0}{\partial I_{20}\partial I_{30}}$$

$$\frac{\partial G}{\partial i_1} = \frac{\partial G_0}{\partial I_{10}} + i_1\frac{\partial^2 G_0}{\partial I_{10}^2} + i_2\frac{\partial^2 G_0}{I_{10}\ I_{20}} + i_3\frac{\partial^2 G_0}{\partial I_{30}\partial I_{10}} = 0 \quad ---(24)$$

$$\frac{\partial G}{\partial i_2} = \frac{\partial G_0}{\partial I_{20}} + i_2\frac{\partial^2 G_0}{\partial I_{20}^2} + i_1\frac{\partial^2 G_0}{I_{10}\ I_{20}} + i_3\frac{\partial^2 G_0}{\partial I_{30}\partial I_{20}} = 0 \quad ---(25)$$

$$\frac{\partial G}{\partial i_3} = \frac{\partial G_0}{\partial I_{30}} + i_3\frac{\partial^2 G_0}{\partial I_{30}^2} + i_1\frac{\partial^2 G_0}{\partial I_{10}\partial I_{30}} + i_2\frac{\partial^2 G_0}{\partial I_{20}\partial I_{30}} = 0 \quad ---(26)$$

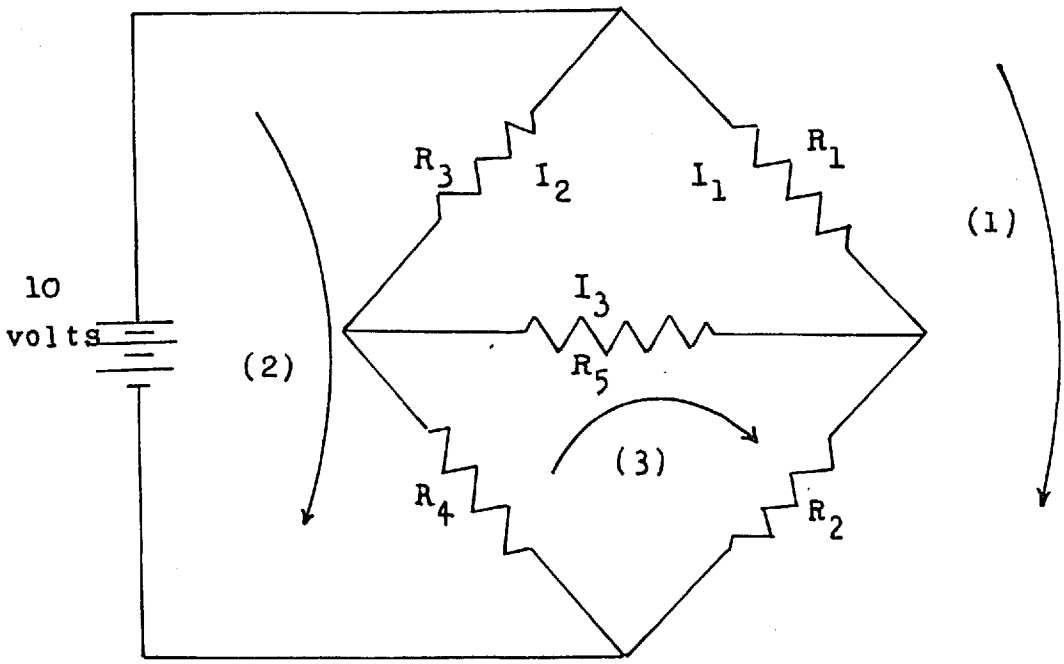


Fig.1.2

The currents I_1, I_2, I_3, of the bridge circuit shown in Fig.1.2 are to be evaluated.

$R_1 = 2\Omega$, $R_2 = 1\Omega$, $R_3 = 1\Omega$, $R_4 = 2\Omega$, $R_5 = 2\Omega$, $E = 10$volts

The arbitrary currents I_{10}, I_{20} and I_{30} are chosen to be 1 amp., and the arbitrary value of G_o is calculated.

$$G_o = -E(I_{10}+I_{20}) + \tfrac{1}{2}R_1 I_{10}^2 + \tfrac{1}{2}R_3 I_{20}^2 + \tfrac{1}{2}R_5 I_{30}^2 + \tfrac{1}{2}R_2(I_{10}+I_{30})^2 + \tfrac{1}{2}R_4(I_{20}-I_{30})^2 \quad \text{----------(27)}$$

$$\frac{\partial G_o}{\partial I_{10}} = -E + R_1 I_{10} + R_2(I_{10}+I_{30}) \quad \text{----------(28)}$$

$$\frac{\partial G_o}{\partial I_{20}} = -E + R_3 I_{20} + R_4(I_{20}- I_{30}) \quad \text{----------(29)}$$

$$\frac{\partial G_o}{\partial I_{30}} = R_5 I_{30} + R_2(I_{10}+I_{30}) - R_4(I_{20}-I_{30}) \quad \text{---------(30)}$$

$$\frac{\partial^2 G_o}{\partial I_{10}^2} = R_1 + R_2 \quad \text{----------(31)}$$

$$\frac{\partial^2 G_o}{\partial I_{20}^2} = R_3 + R_4 \quad \text{----------(32)}$$

$$\frac{\partial^2 G_o}{\partial I_{30}^2} = R_5 + R_2 + R_4 \quad \text{----------(33)}$$

$$\frac{\partial^2 G_o}{\partial I_{10}\partial I_{20}} = 0 \quad \text{----------(34)}$$

$$\frac{\partial^2 G_o}{\partial I_{10}\partial I_{30}} = R_2 \quad \text{----------(35)}$$

$$\frac{\partial^2 G_o}{\partial I_{20}\partial I_{30}} = -R_4 \quad \text{----------(36)}$$

Substitute(28) to (36) in the equations(24),(25) and (26) and obtain the values of i_1 i_2 and i_3.

This gives

$i_1 = 2$ amps.

$i_2 = 3$ amps.

$i_3 = 0$ amps.

The values of the currents I_1, I_2 ,I_3 are then 3,4, and 1 amps. respectively. These are the exactly correct results which are to be expected in a linear case.

Eg.4 Non-linear Case.

The circuit similar to that shown in Fig.1.2 is employed. The resistors R_1,R_2,R_3,R_4 and R_5 are non-linear and have the V-I characteristics as follows:-

R_1 $V = 3I^2$

R_2 $V = 2I + \frac{1}{2}I^2$

R_3 $V = I + I^2$

R_4 $V = 20I^2$

R_5 $V = 3I + 6I^2$

The arbitrary currents I_{10}, I_{20} and I_{30} are chosen to be 1.5 amps., 3.4 amps., and 0.4 amp., respectively.

The table below shows the value of the currents I_1 I_2 and I_3(amps.) after each approximation, together with the sum of the voltages(volts) around meshes 1,2 and 3.

	I_1	I_2	I_3	$V_1 \rightarrow 0$	$V_2 \rightarrow 0$	$V_3 \rightarrow 0$
Starting	3.4	1.5	0.4	181.56	4.35	-170.24
1st approx.	2	1.2	0.4	47.20	-1.2	-44,56
2nd approx.	1.34	1.3	0.4	10.063	-0.01	-9.691
3rd approx.	1.10	1.3	0.4336	1.18	0.026	-2.511
4th approx.	1.153	1.265	0.5336	0.16	0.0124	0.808

In four approximations the table shows the rapid rate of convergence. The values of the mesh voltages V_1, V_2, V_3 are reducing rapidly towards zero.

CHAPTER II

The Graphical Application of the Content to the Solution of the Non-linear Circuit.

The purpose of this section is to illustrate a new graphical method, incorporating the theory of content (or co-content), and its application to the solutions of a non-linear resistance network. The theory of content (or co-content) is extended to cover the non-linear circuit in which negative resistances are included. In addition the analogue computer is employed to speed up the process.

This graphical method can only be used for a simple network because of the lengthy process entailed in plotting each graph, despite the help of the analogue computer. Nevertheless if the process of graph plotting could be done more quickly by automatic means, the method can be extended to cope with a more complex network. This method is found to be very useful however when it is applied to a non-linear network which has a repetitive pattern such as a ladder network.

It was mentioned in the previous section that the content of a resistance network is the sum of the content of the constituent elements in the network.

In addition, the content of the network has a stationary value at the real values of the voltages and currents of the network, with respect to fictitious variations of mesh currents, (or of some other set of currents satisfying $\Sigma i = 0$ at all nodes).

It is therefore possible that the values of the content of the network can be plotted against fictitious values of mesh currents, and this graph will then display the stationary point of the content of the network. The value of the current corresponding to the position of the stationary points of the content of the network is then the real value of the current of the network.

In many cases there can be more than one stationary point. The values of the currents corresponding to these points are the real currents, and the circuit may exist in any one of these states.

2.1 Graphical theory of Content (or Co-content).

Before the theory of content can be applied graphically to the solution of the resistance network, some new theorems have to be provided. These theorems will help to indicate the position of the solutions by establishing in advance whether the total content of the network will have maximum value or minimum value or

both. Furthermore they will indicate whether the stationary value of the content is positive or negative.

a) Series Connected Network.

For the series connected network the total content of the network (including the content of a voltage generator) can have either maximum or minimum points depending mainly on the value of the resistances. If the total impedance of the network has a positive value, its V-I characteristic has a positive slope, and then the content of the network including the content of the constant voltage generator will have a minimum value. Correspondingly if the total impedance of the network has a negative value its V-I characteristic has negative slope, and then the content of the network including the content of the constant voltage generator will have a maximum value.

It may be proved as follows:- Consider the network shown in Fig. 2, and assume that R_1 R_2 and R_3 have linear characteristics, and that the constant voltage generator has a value E volts.

The total content of the network G, can be expressed in terms of

$$G = -Ei + \frac{1}{2}R_1 i^2 + \frac{1}{2}R_2 i^2 + \frac{1}{2}R_3 i^2$$

For G to have either a minimum or a maximum value $\frac{dG}{di}$ must be equal to zero. If G has a minimum value, $\frac{d^2G}{di^2}$ will have

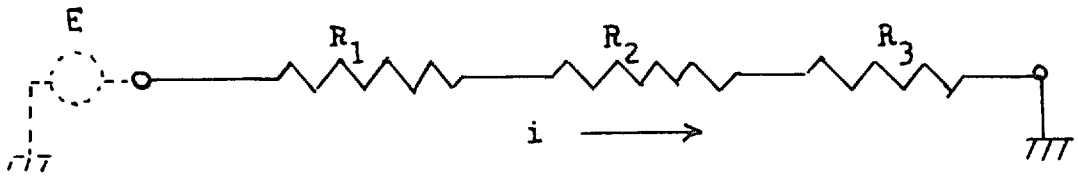


Fig.2

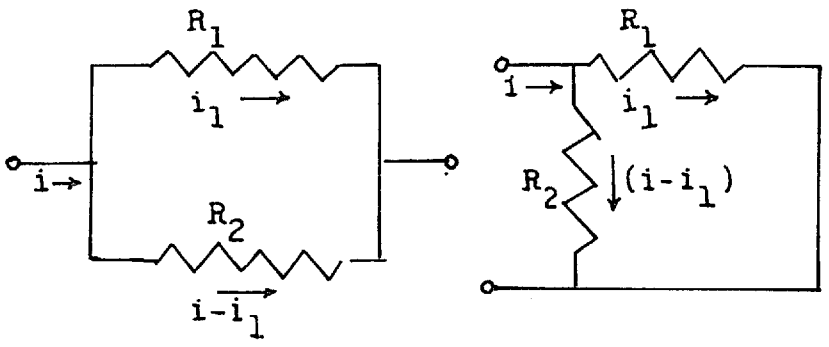


Fig2.1

a positive value and vice versa. Differentiating equation (1) with respect to i twice, gives:-

$$\frac{d^2G}{di^2} = R_1 + R_2 + R_3$$

which is the impedance of the network. Therefore for a positive impedance of the network, $\frac{d^2G}{di^2}$ will have a positive value and vice versa.

b) Parallel Connected Network.

The parallel connected network is shown in Fig. 2.1. The total content of the network including the content of a constant voltage generator will have maximum or minimum value for a condition similar to that above, ie. minimum for positive impedance and vice versa with respect to the variation of the current i. The interest in this circuit is concerned mainly with how i divides between the two internal paths. This solution can be obtained from the position of the stationary value of the sum of the contents of R_1 and R_2 plotted against i_1 or i_2 for each constant value of i. To indicate whether the stationary value is the maximum or minimum value of the content, the following theorems are provided:-

1) When R_1 and R_2 both have positive value, the sum of the content will have minimum value.

2) When R_1 and R_2 both have negative value, the sum

of the content will have maximum value.

3) When one of the resistors has a negative value and the other remains positive, then the total content may either have a maximum or a minimum value. This is determined by the numerical value of R_1 and R_2. If R_2 has a negative value and R_1 has a positive value, and if the numerical value of R_2 is greater than that of R_1 then the total content will have a maximum value. It will have a minimum value if R_2 is numerically smaller than R_1. This may be proved as follows:-

The content of the network is given as

$$G = \frac{1}{2}R_1 i_1^2 + \frac{1}{2}R_2(i-i_1)^2$$

for the linear case. The content of a constant current generator is equal to zero.

It then follows that $\frac{d^2G}{di_1^2} = R_1+R_2 = \frac{d^2G}{di_2^2}$

If R has a negative value, then for the term $\frac{d^2G}{di_2^2}$ or $\frac{d^2G}{di_1^2}$ to remain positive so that the total content has a minimum value, then R_2 must remain smaller than R_1 and vice versa

c). Positive and Negative Value.

The advance knowledge of the positive or negative value of the stationary point sometimes assists in

obtaining the solution of the network, but it is not as important as knowing whether the total content will have a maximum or minimum value. To give some idea of whether the stationary value of content is going to be positive or negative, the following theorems have been produced:-

1) When the network has a positive input impedance, and when its V-I characteristic can be drawn past the origin of the V-I axis, the total content of the network including the content of a constant voltage generator will have a minimum negative value with respect to the variation of the input current.

2) When the network has a negative input impedance and when its V-I characteristic can be drawn past the origin of the V and I axis, the total content of the network including the content of a constant voltage generator will have a maximum positive value.

3) In the case of the parallel network shown in Fig. 2.1 for each value of injected current the total content will have a maximum positive value or minimum positive value with respect to the variation of i_1 or i_2 if the input impedance of the network is positive and its characteristic line is in the 1st or 3rd quadrant of the V-I axis, and vice versa.

2.2 The solution of the Resistance Network by the Graphical Method.

To obtain the values of the currents flowing in the resistance network the following procedure may be adopted:-

1) Obtain the content of the input impedance as a function of the input current.

2) Add the content of the constant voltage generator to the content of the impedance of the network for a range of input currents which may give a graph similar to that shown in Fig. 2.2.

The minimum position then indicates the actual value of the input current.

To obtain the content of the input impedance of a series connected resistance, the values of the contents of each of these resistances which correspond to a value of the current i input current are added together algebraicly. The calculation is repeated for a range of values of i, thus giving the content of the impedance of the network as a function of the input current.

In the case of the parallel network shown in Fig. 2.1, the content of the input impedance for each value of the input current i is obtained by plotting the sum of the contents of R_1 and R_2 against the variation of the current i_1 or i_2 ($i_1 + i_2 = i$) i being kept constant.

The total content of the network including the content of a constant voltage generator.

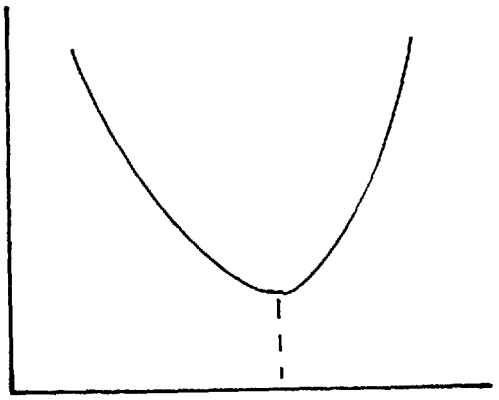


Fig. 2.2

The sum of the contents of R_1 and R_2

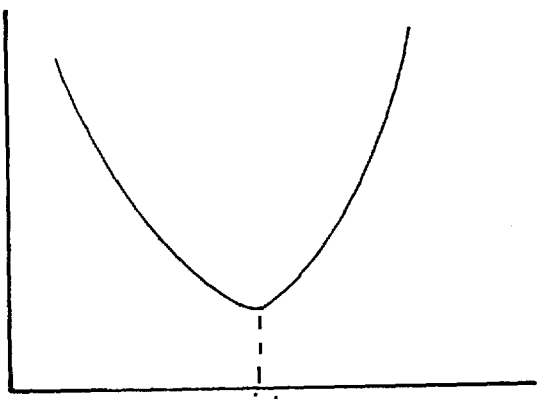


Fig. 2.3

Then the stationary value of the total content is the content of the impedance of the whole network at current i. The minimum position also gives the actual value of current i_1 or i_2 shown in Fig. 2.3. The calculation necessary to obtain the graph of Fig. 2.3 can be repeated for a range of values of i, and the various stationary values of the content give the content of the whole network as a function of the current i.

In some cases there can be more than one stationary point, and the values of the currents corresponding to the positions of the stationary point may not be the actual current which flows in the circuit, and thus further investigations have to be made to locate the actual values of the current. This only occurs where the V-I characteristic of the impedance of the network is a non-linear non-monotonic function as shown in Fig. 2.4

To help to determine the actual current in the network, the following method can be adopted:-

1) If the constant voltage generator has a value less than V_1 volts (see Fig 2.4 for the symbols) then the solution will lie between the values of the current $0\text{-}i_1$, and similarly if the voltage generator is greater than V_2 volts then the solution will lie beyond the value of the current i_2.

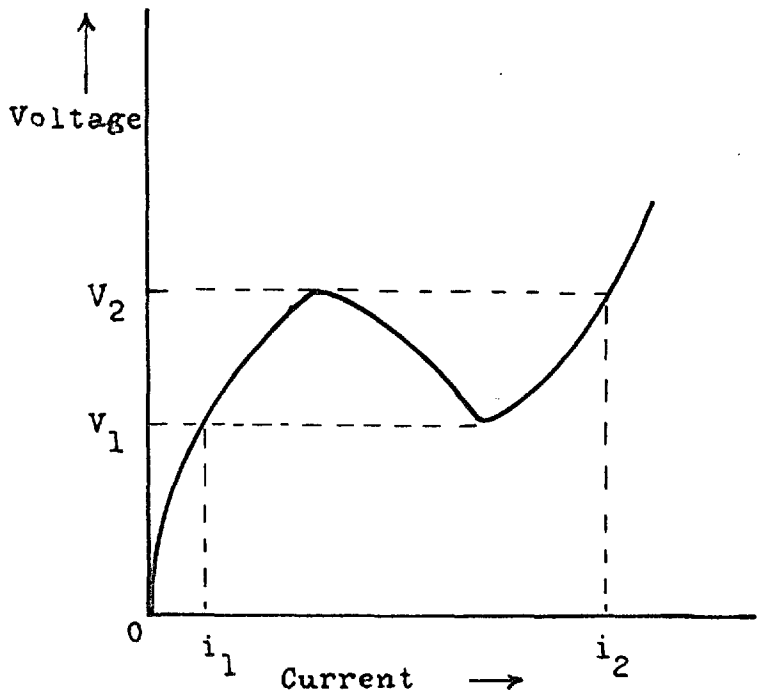


Fig. 2.4

2) If the constant voltage generator has a value in between V_1 and V_2 volts, then all the values of the current corresponding to the positions of all the stationary points of the total content are the real solutions, and the circuit may exist in any one of these states.

2.3 The solution of Ladder Networks containing Non-linear Resistances.

The solution of the ladder network shown in Fig. 2.5 may be obtained by a successive reduction method, ie. the number of sections is reduced step by step. The content of the impedance of the extreme right-hand section as a function of i_1 is first obtained by the method described in section 2.1b and 2.2. The next step is to add the content of the resistance r_1 to the content of the impedance of this section. This will give the content of the impedance looking in at aa' as a function of i_1.

The section on the right of aa' has now been reduced to an equivalent single resistance R' whose content is known as a function of i_1, and the method can be used again to find the content of the resistance looking in at bb' from the left. The network for the second step is shown in Fig. 2.6 and the content looking

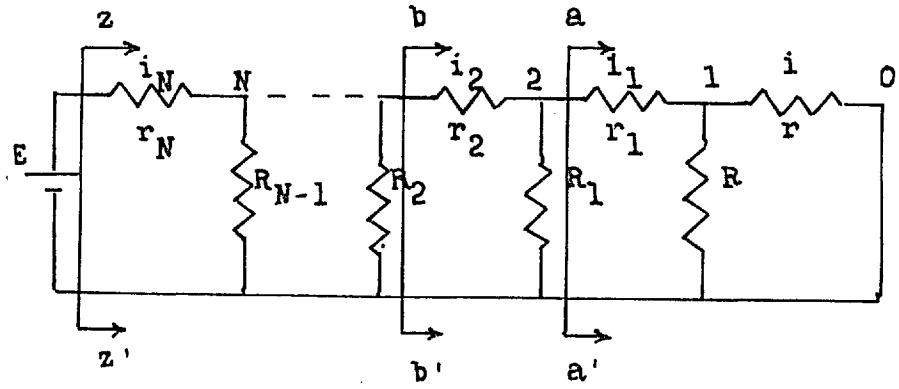


Fig. 2.5

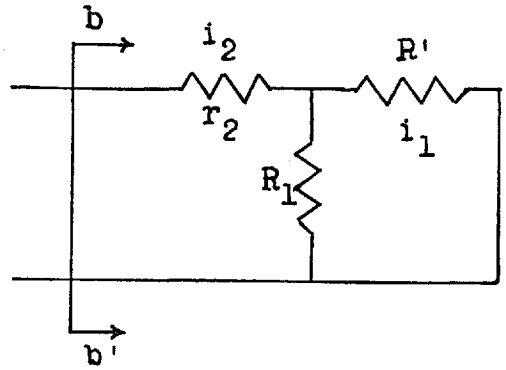


Fig. 2.6

in at bb' to be found as a function of i_2. Proceeding in this way, the input terminals zz' on the extreme left will eventually be reached after much laborious graphical work, and the total content will be known as a function i_N. To this is added the content, $-Ei_N$, of the generator, and if this grand-total-content is plotted against i_N, the minimum value will give the actual input current i_N'. It is then a simple matter to work forwards, from left to right on Fig. 2.5 to find the node voltages.

The solution obtained by this method is much more accurate than might be supposed, because although based on graphs showing a minimum, it is not the position of the minimum (ie. the value of the current), which is used in the next stage of the calculation, but the actual minimum value of the content, and this is well-defined.

When the left hand end is reached and the value of the input current is obtained, the value of this current may not be within the required limit of accuracy, but it can be improved by numerical calculation. This is done very rapidly because of the good start given by the value already obtained. To obtain the node voltages from left to right, the value of the voltage of node N (see

fig. 2.5) is obtained by subtracting the potential drop in the resistor r_N from the value of the voltage generator. The value of this potential drop may be obtained from the V-I characteristic of the resistor r_N or by calculation. When the value of the voltage of node N is found, the value of the current flowing in the resistor R_{N-1} is obtained similarly, from either V-I characteristic of R_{N-1} or by calculation. The value of this current is then subtracted from i'_N which then gives the value of the current i'_{N-1}. By repeating the same process along the ladder, the value of the voltage of node O will eventually be obtained. If this value does not correspond to that of the value given, ie. in this case zero, the value of i'_N is improved by slightly increasing or decreasing it, and then the calculation is repeated. This will eventually give a more exact value of i'_N.

In some cases one can estimate the value of i'_N and refine it as above quite quickly without any reference to the 'content' theory. However in the case where there are many negative resistances in the circuit, or where the resistances in the circuit are highly non-linear, estimation of the value of the input current i'_N may lead to a great amount of calculation.

Example In order to illustrate the consequences of some of the theorems discussed in section 2.1 to 2.3, one simple case will be worked out. For ease of comparison with the analytical results, the linear case is chosen. When the calculation is made on a non-linear problem the process will remain the same in principle, and an example of a non-linear case will be given in a further section in which the application of the non-linear ladder network is discussed.

The ladder network containing negative resistances as shown in fig.2.7, is considered. The series resistors r, r_1, r_2, r_3, r_4 and r_5 have a positive linear value of 1Ω. The shunt resistors R, R_1, R_2, R_3 and R_4 have a negative linear value of -1.621 Ω.

Graph 1 shows the content of the 1Ω resistor and the -1.621Ω resistor. Note that when the resistor has a negative value, its content will also have a negative value. The current flowing in the series resistors r to r_5 is given positive sign when it flows from the left-hand terminal to the right-hand terminal and vice versa. Similarly the current flowing in the shunt resistors, will have a positive value when it is flowing downwards.

Graph 2 shows the process of obtaining the value of the content of the extreme right-hand section, looking at a_1a_1' as a function of i_1. i_1 is first

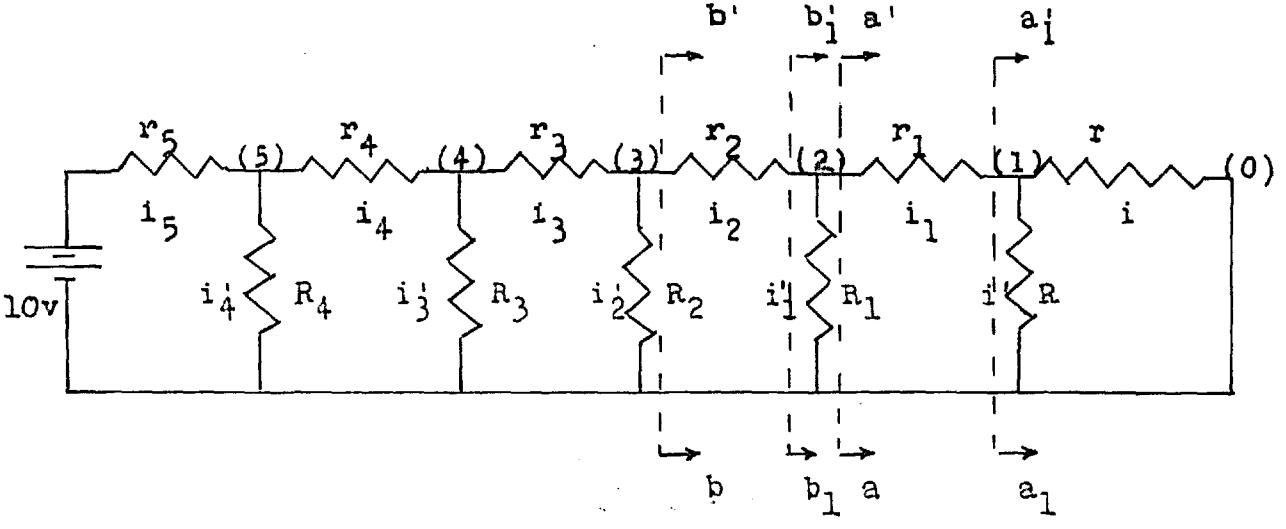


Fig.2.7

The value of the shunt resistors R to R_4 is -1.621Ω.

given a constant value of 1 amp., and the sum of the content of the resistors r and R is plotted against the variation of the current i', flowing through the resistor R.

Since R is numerically greater than r then the sum of the content has a maximum value with respect to the variation of the value of i'. The impedance looking at $a_1a'_1$ however, has a positive value, and therefore the value of its content as a function of i_1 has a positive value.

The next step is to repeat the process for the value of i_1 equal to 2,3,4,-1,-2,-3,-4 amps.

Graph 3a shows the content of the section $a_1a'_1$ as a function of i_1. 3b shows the total content including the content of the resistor r_1. By inspection of graph 3b it becomes obvious that the value of the impedance of the last section looking at aa' has a positive value and is numerically greater than the impedance of R_1. The impedance looking at b_1b_1' will then have a negative value with respect to the value of current i_2.

Graph 4 shows the process of obtaining the content of the impedance looking in at b_1b_1' as a function of i_2. The sum of the content of the resistor R_1 and the impedance looking in at aa', for each constant value of i_2 has a minimum negative value

with respect to the variation of the current i_1' as expected.

Graph 5 shows the content of the impedance looking at $b_1 b_1'$ as a function of i_2.

Graph 6 shows the total content of the whole network as a function of i_5.

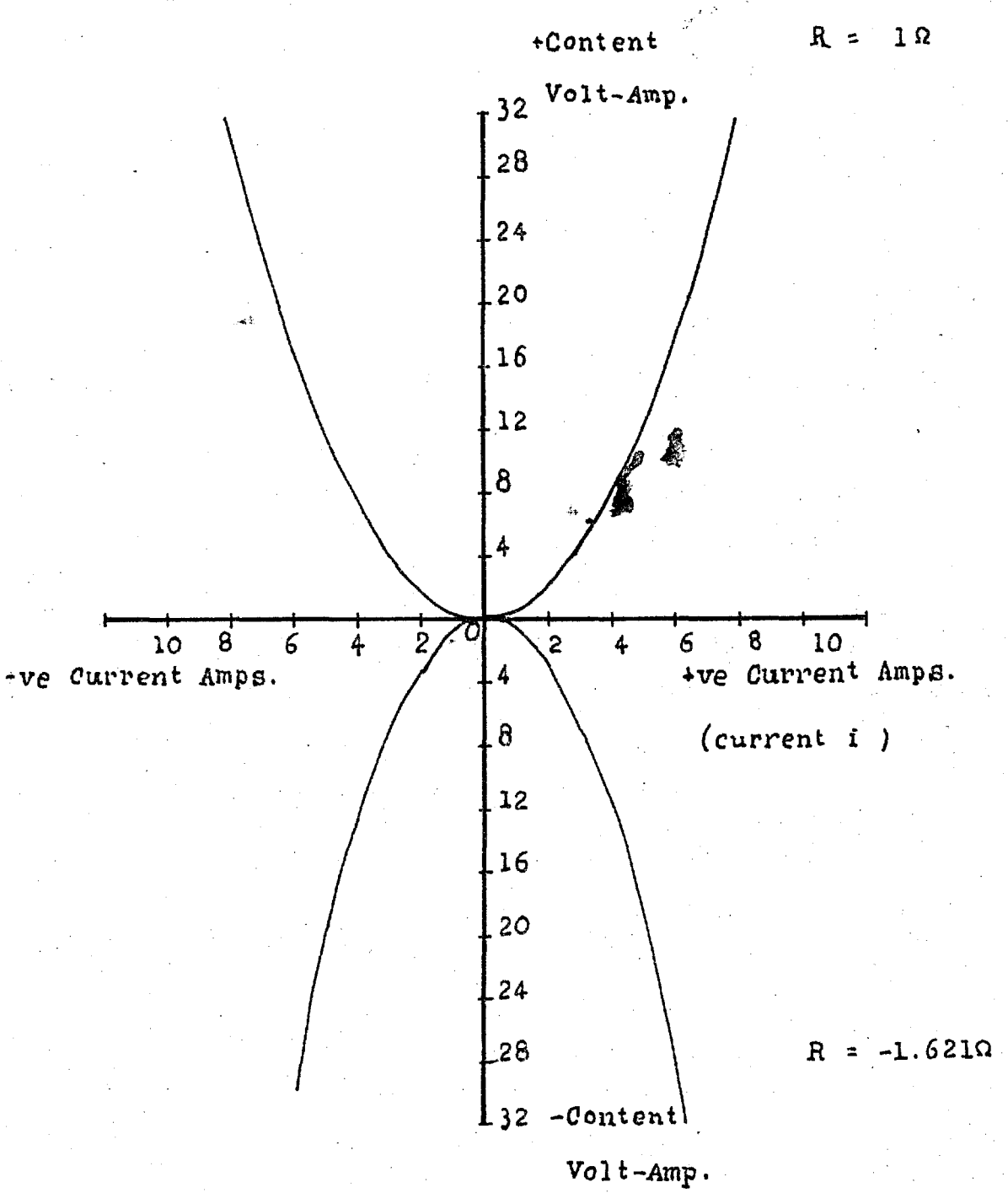
Graph 7 shows the content of the whole network including the content of E_1. The actual value of the current flowing into the network is determined by the position of the minimum value of the total content of the circuit.

The value of the current obtained by this method is 2.1 amps, which is indeed very close to the correct value of 2.135 amps.

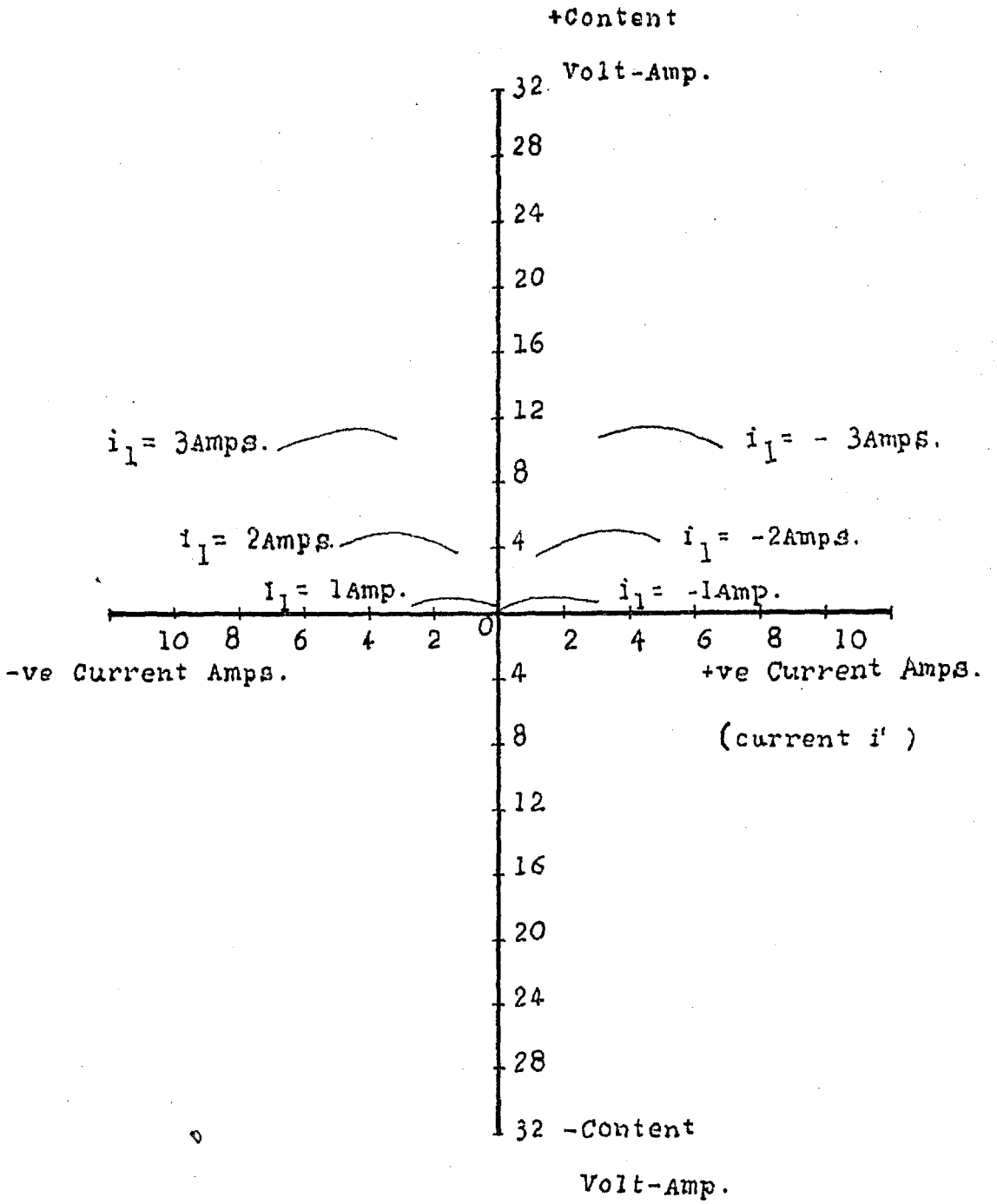
The values tabulated below show the voltage at each node computed from the value $i_5 = 2.1$ amps, and the exact value of the node voltages.

Node	0	1	2	3	4	5	6
Computed values	10	7.9	0.925	-6.621	-10.087	-7.333	-0.054
Exact values	10	7.865	0.878	-6.6674	-10.099	-7.302	-0.00

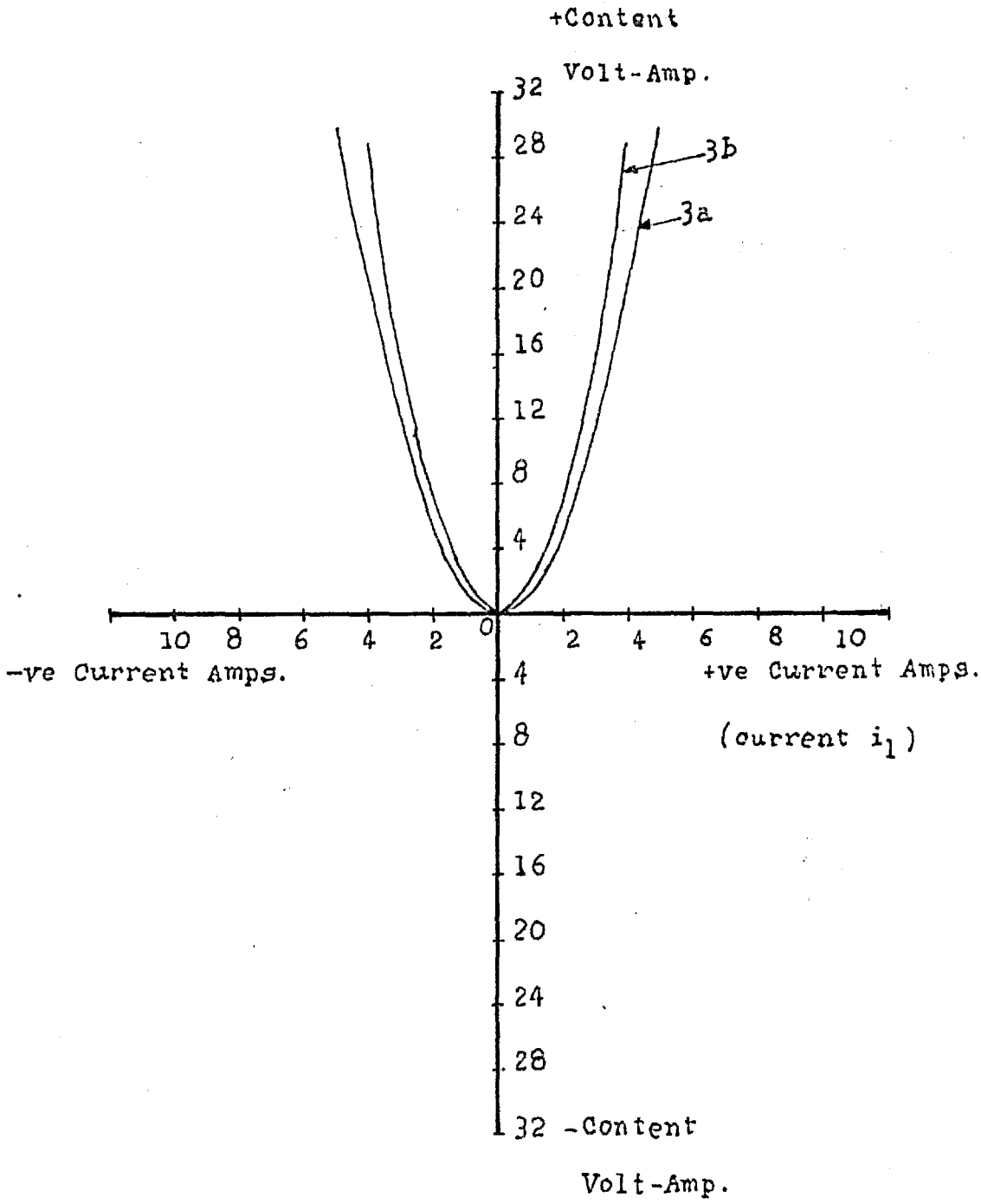
Graph 1


+Content
Volt-Amp.
R = 1Ω
32
28
24
20
16
12
8
4
10 8 6 4 2 0 2 4 6 8 10
-ve Current Amps.
+ve Current Amps.
(current i)
4
8
12
16
20
24
28
32 -Content
Volt-Amp.
R = -1.621Ω

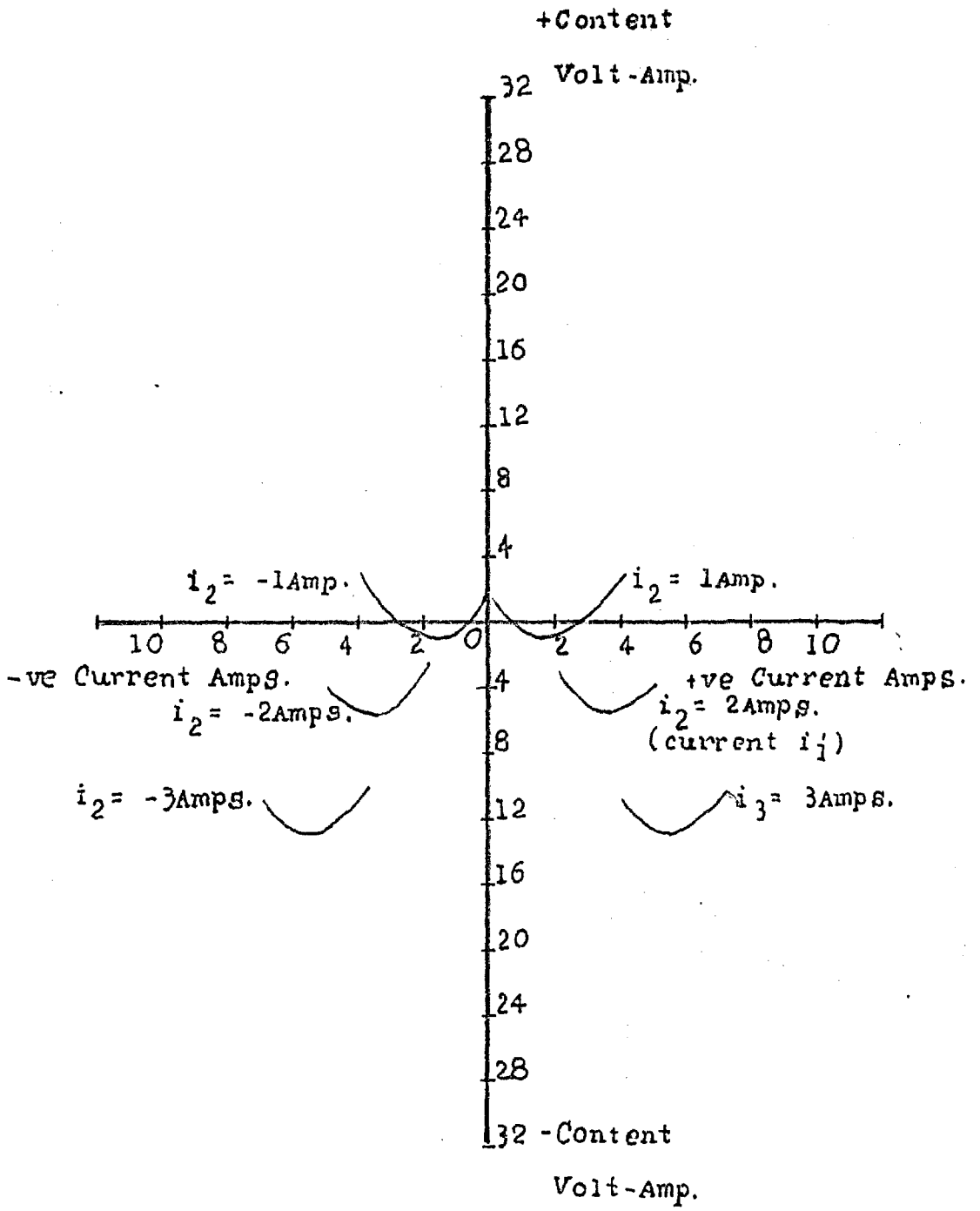
Graph 2


+Content
Volt-Amp.
32
28
24
20
16
12
8
4
i_1 = 3Amps.
i_1 = - 3Amps.
i_1 = 2Amps.
i_1 = -2Amps.
i_1 = 1Amp.
i_1 = -1Amp.
10 8 6 4 2 0 2 4 6 8 10
-ve Current Amps.
+ve Current Amps.
4
8
(current i')
12
16
20
24
28
32 -Content
Volt-Amp.

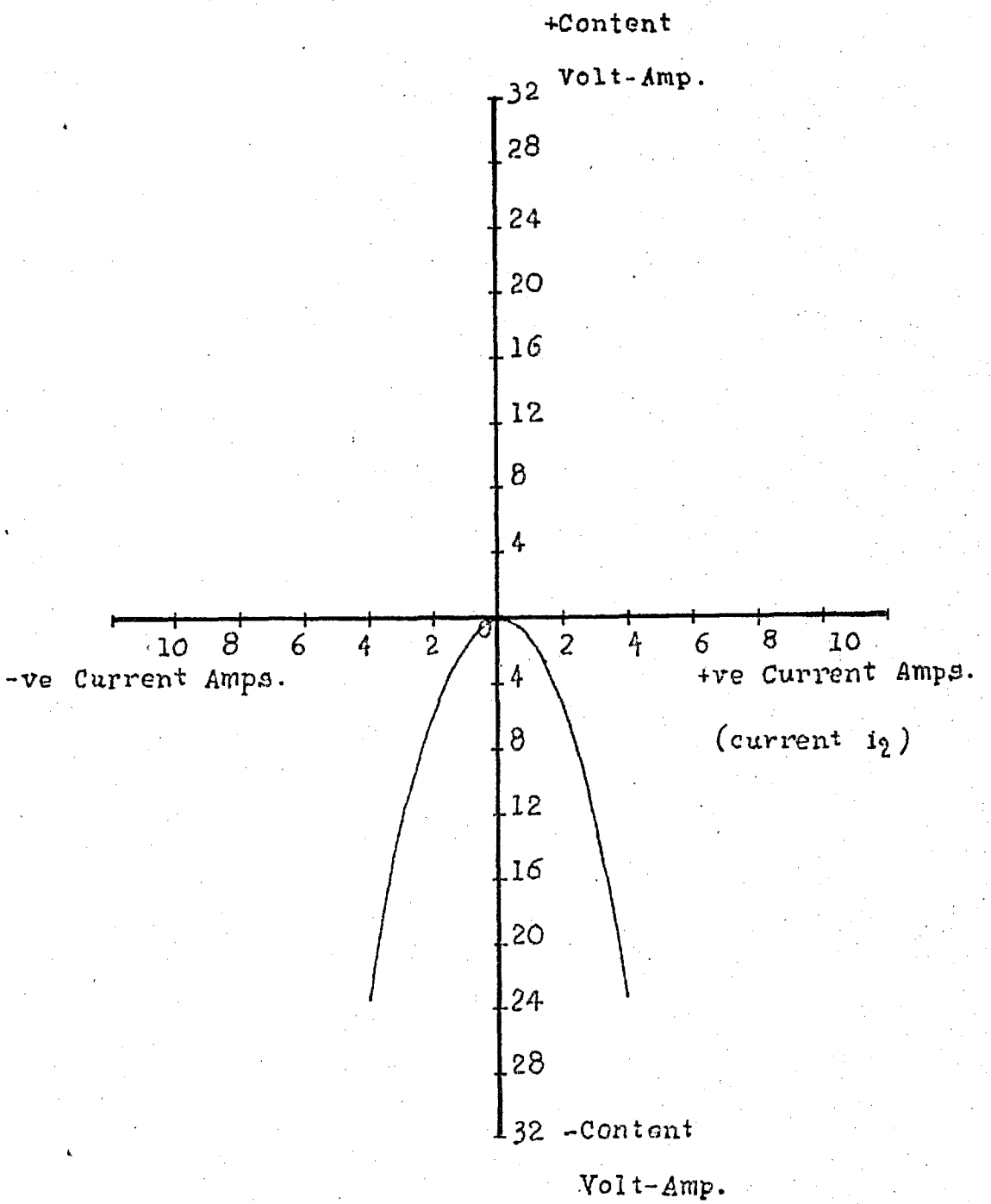
Graph 3a and 3b



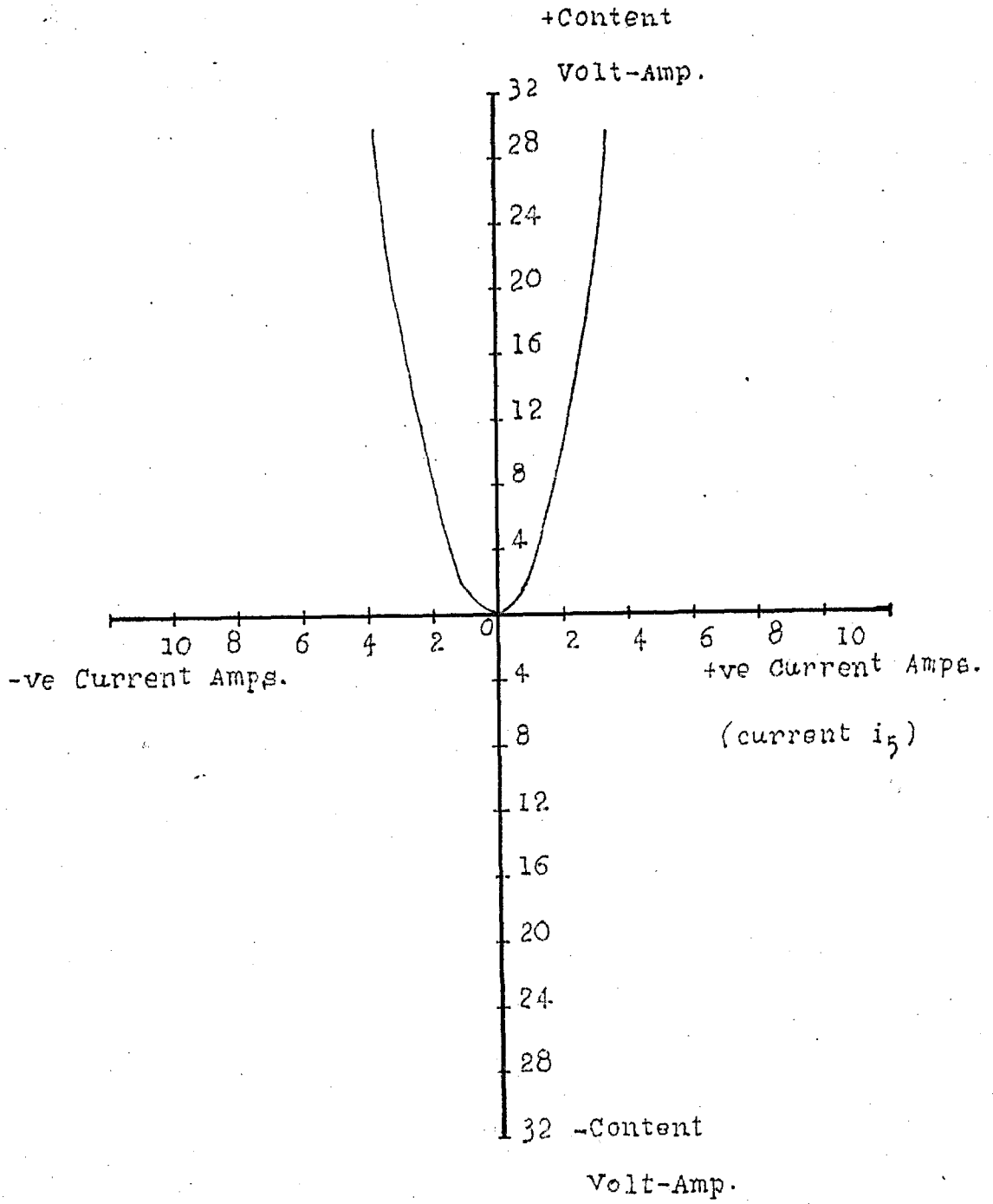
Graph 4


+Content
Volt-Amp.
32
28
24
20
16
12
8
4
i_2 = -1Amp.
i_2 = 1Amp.
10 8 6 4 2 0 2 4 6 8 10
-ve Current Amps.
+ve Current Amps.
4
i_2 = -2Amps.
i_2 = 2Amps.
8
(current i_1)
i_2 = -3Amps.
i_3 = 3Amps.
12
16
20
24
28
32 -Content
Volt-Amp.

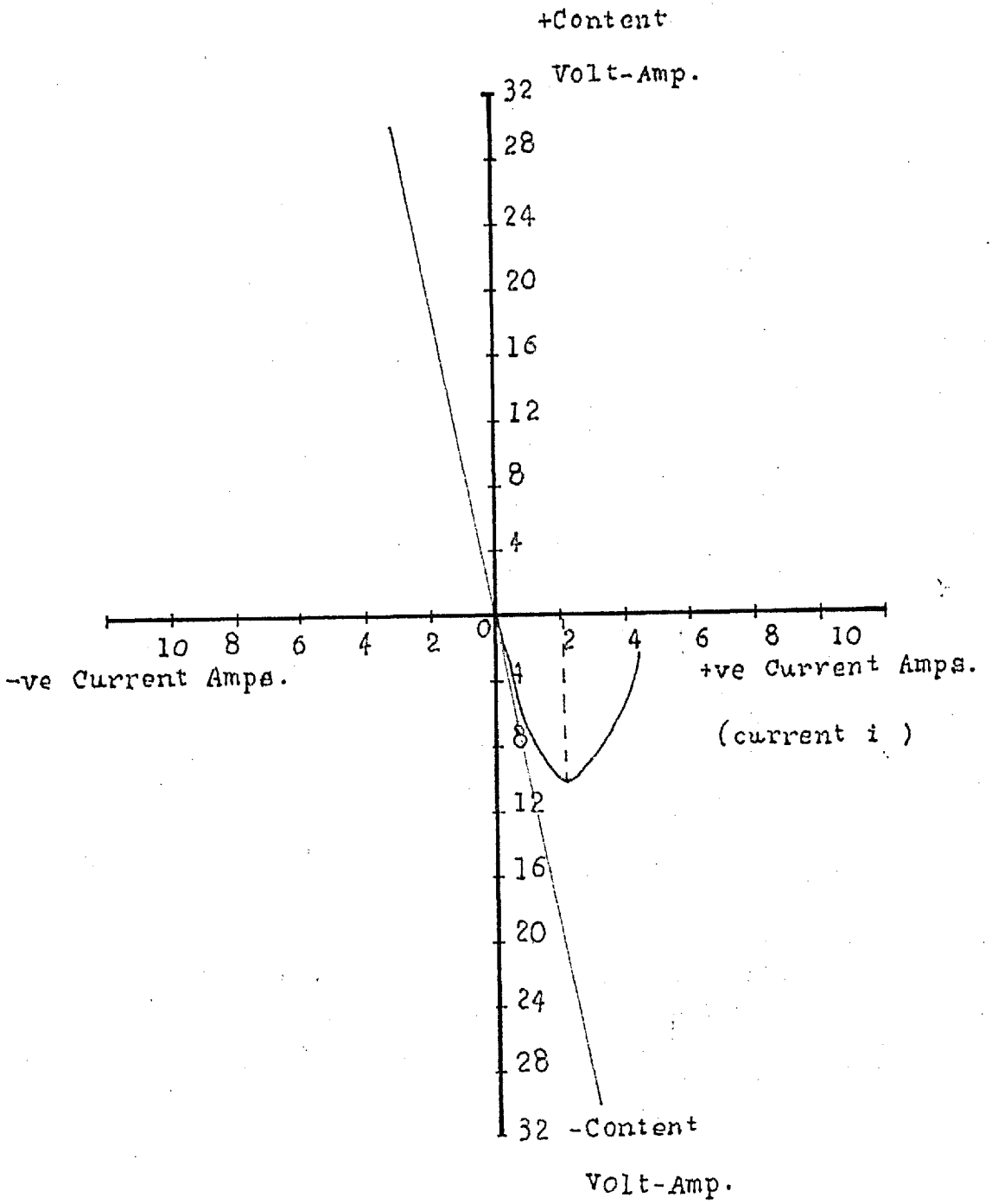
Graph 5


+Content
Volt-Amp.
32
28
24
20
16
12
8
4
10 8 6 4 2 0 2 4 6 8 10
-ve Current Amps.
+ve Current Amps.
(current i_2)
4
8
12
16
20
24
28
32 -Content
Volt-Amp.

Graph 6


+Content
Volt-Amp.
32
28
24
20
16
12
8
4
10 8 6 4 2 0 2 4 6 8 10
-ve Current Amps.
+ve Current Amps.
(current i_5)
4
8
12
16
20
24
28
32 -Content
Volt-Amp.

Graph 7


+Content
Volt-Amp.
32
28
24
20
16
12
8
4
10 8 6 4 2 0 2 4 6 8 10
-ve Current Amps.
+ve Current Amps.
(current i)
4
8
12
16
20
24
28
32 -Content
Volt-Amp.

2.4 The Co-content and its Graphical Application to the Solution of Non-linear Circuits

In the cases shown in sections 2.1 to 2.3, the solutions are given in terms of current components, but to obtain the voltage components some further calculation is necessary. If only the values of the voltages at each node are required, then the co-content of the network may be used in a similar way to the content. (The co-content of the resistive element is a dual quantity to the content of the resistive element, in the same sense as voltages to currents, and meshes to nodes.) In this case the co-content is plotted against the fictitious value of node voltages ie. $\Sigma v=0$ at all meshes. The current law $\Sigma i=0$ at all nodes is violated during the fictitious voltage variation.

The co-content of a resistance element, represented by J, is defined as the area under the V-I characteristic of the element from zero potential to the operating potential v', (Miller[2]), the unshaded area in Fig.2.8.

$$J = \int_0^{v'} i\,dv$$

The co-content of the network, (ie. it is the sum of the co-contents of the constituent elements of the network) has stationary values at the actual values of the voltages and currents of the network.

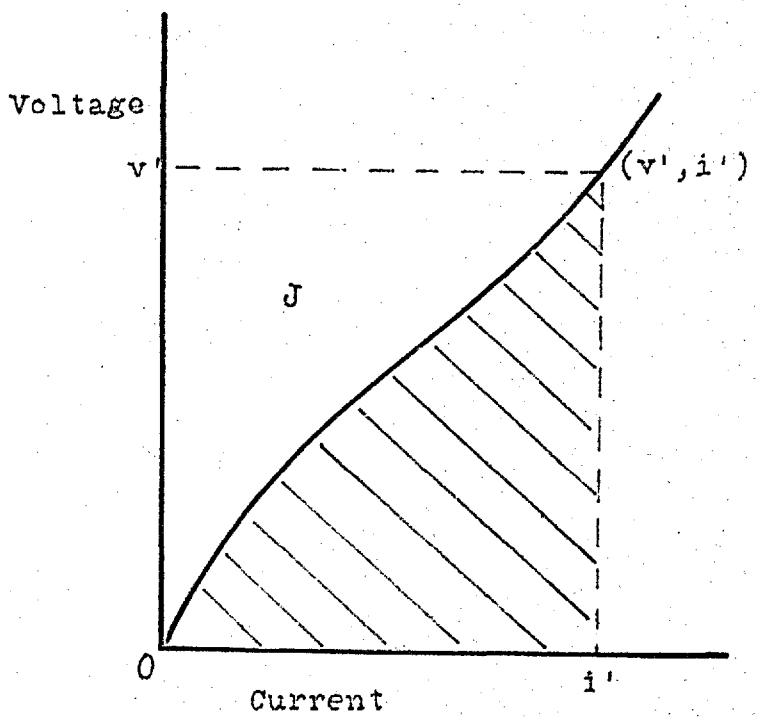


Fig.2.8

To illustrate the co-content method, the bridge circuit shown in Fig.2.9 is employed.

A constant voltage connected between the terminals has a value of 10 volts. R_1, R_5 and R_3 are linear resistances. R_1 and R_5 = 1Ω, and R_3= 3.5Ω. R_2 and R_4 are non-linear resistances of the same value and their V-I characteristic is shown on graph (9). The required solutions are the values of the voltages V_1 and V_2. The graph plotting procedure is as follows:-

1) The values of the co-contents of all the resistances are plotted as a function of the voltages from 0-10 volts

2) The co-contents of the resistors R_1 and R_4 are added together in such a way that the sum of the potential drops across R_1 and R_4 remains constant and equal to 10 volts. This total co-content is then plotted against the value of V_1, from 0 - 10 volts (graph 10)

$$\text{ie. } (J_1+J_4) = \frac{1}{2}\frac{(10-V_1)^2}{R_1} + \frac{1}{2}\frac{V_1^2}{R_4}$$

3) Similarly the sum of the co-contents of the resistors $R_2$and R_5,(J_2+J_5) is plotted against the value of V_2 from the range of 0-10 volts.(Graph 11).

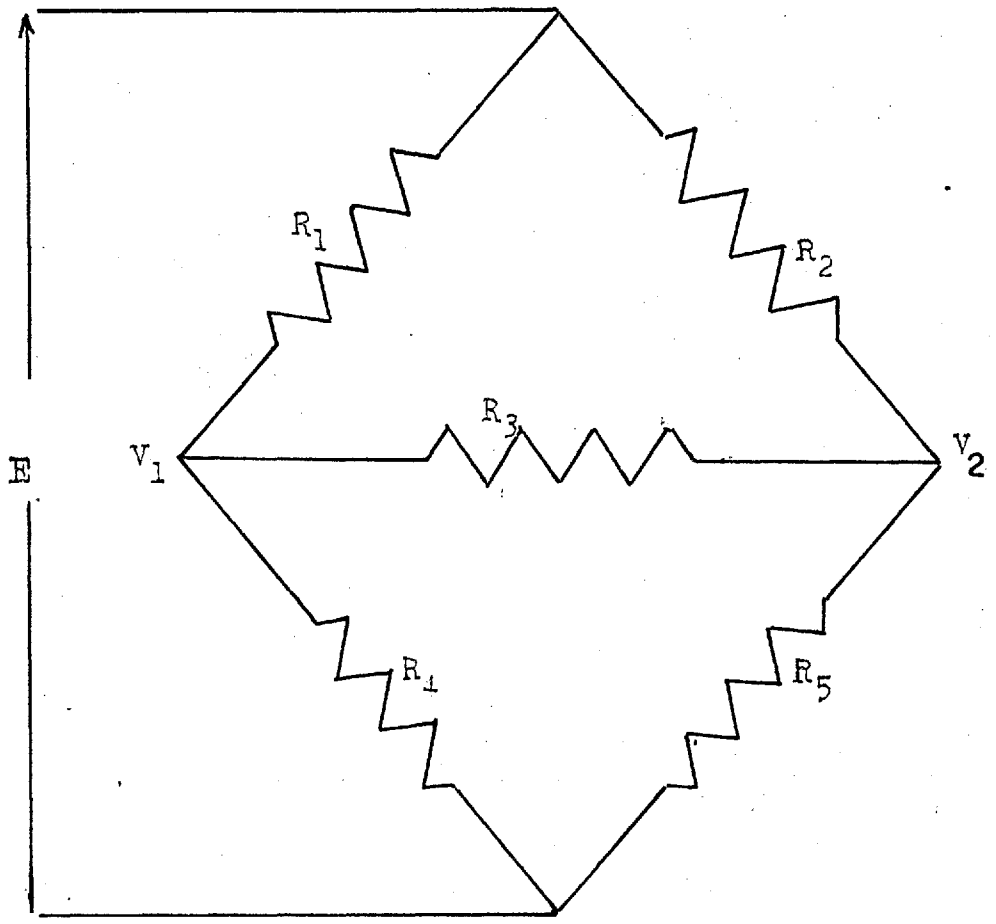


Fig. 2.9

R_2 and R_4 are non-linear resistors

4) The co-contents of all the resistors R_1 R_2 R_3 R_4 and R_5 are then added together in the following manner.

a) The value of the total co-contents of the resistors R_2 and R_5, (J_2+J_5) at V_2 equal to 1 volt, is added to the total co-contents of the resistors R_1 and R_4, (J_1+J_4) for all the values of V_1.ie. from 0 to 10 volts.

b) On the same graph the value of the co-contents of the resistors R_3, J_3, at the voltages $(V_1 \sim V_2)$ ie. V_1 -1 volts, is added, for all values of V_1 (graph 12)

This curve displays the total content of the whole network for all values of V_1, when V_2 is being kept constant at 1 volt . The minimum position of this curve gives the actual value of the voltage V_1 of the network, if the voltage V_2 is being kept at 1 volt.

5) Repeat 4) by increasing V_2 to 2, 3,....10 volts. This will give a set of curves as shown in graph 12, and each one will have a minimum value at a different value of V_1.

6) Plot all the minimum values of the total content as a function of V_1. This gives a curve shown in graph 13. The minimum position on graph 13 gives the

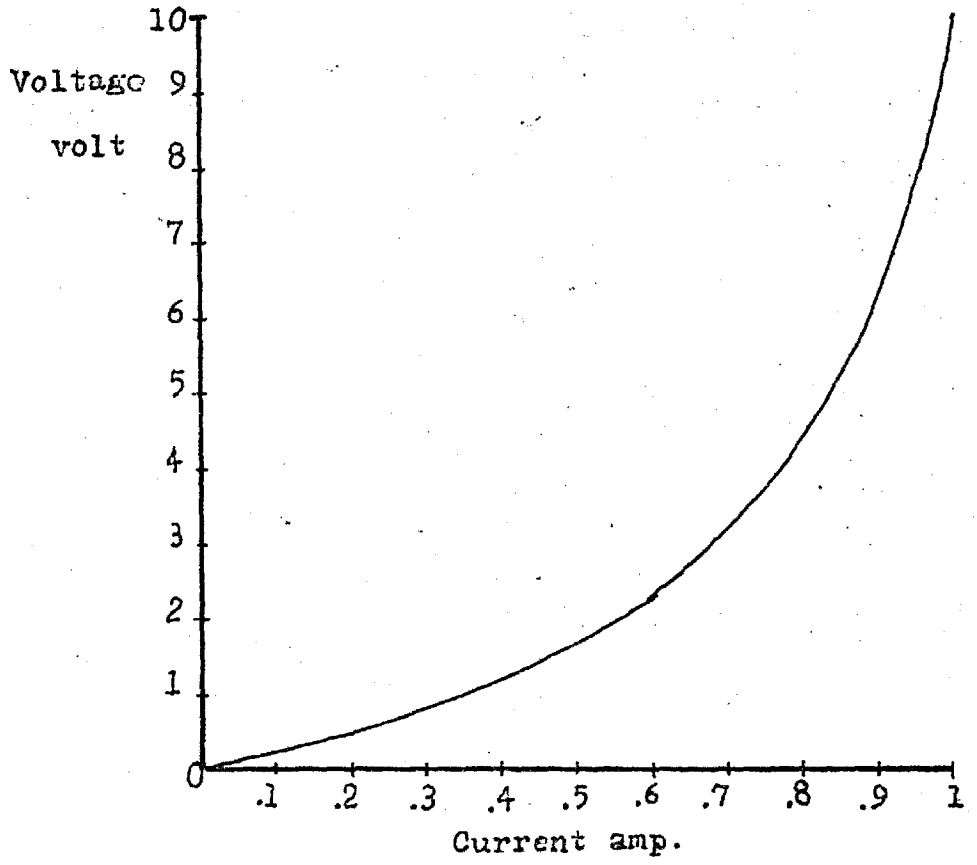
actual value of the voltage V_1 of the network. The actual value of V_2 can then be easily computed.

The result obtained gives V_1 and V_2 equal to 7.5 and 2.5 volts respectively, which compare very closely with the values obtained by the numerical method ie. V_1= 7.58 volts and V_2 2.4 volts.

Graph 9

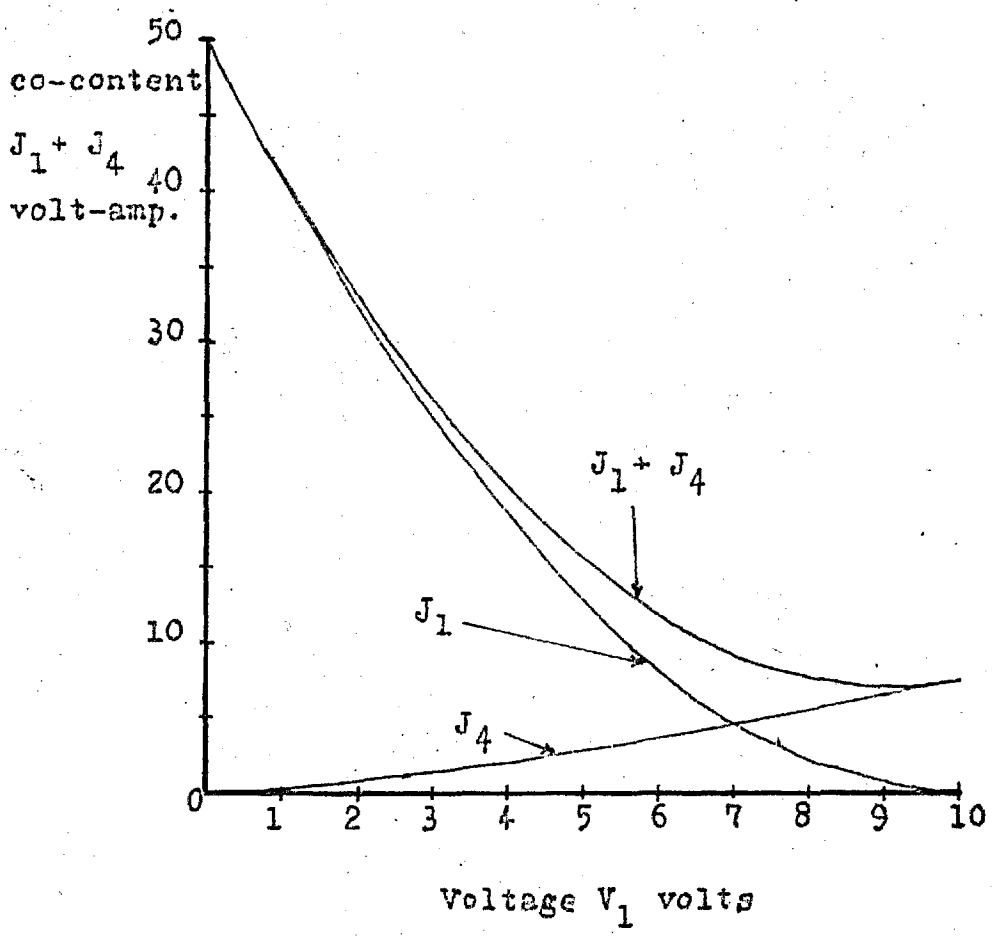
V-I Characteristic

of R_2 and R_4



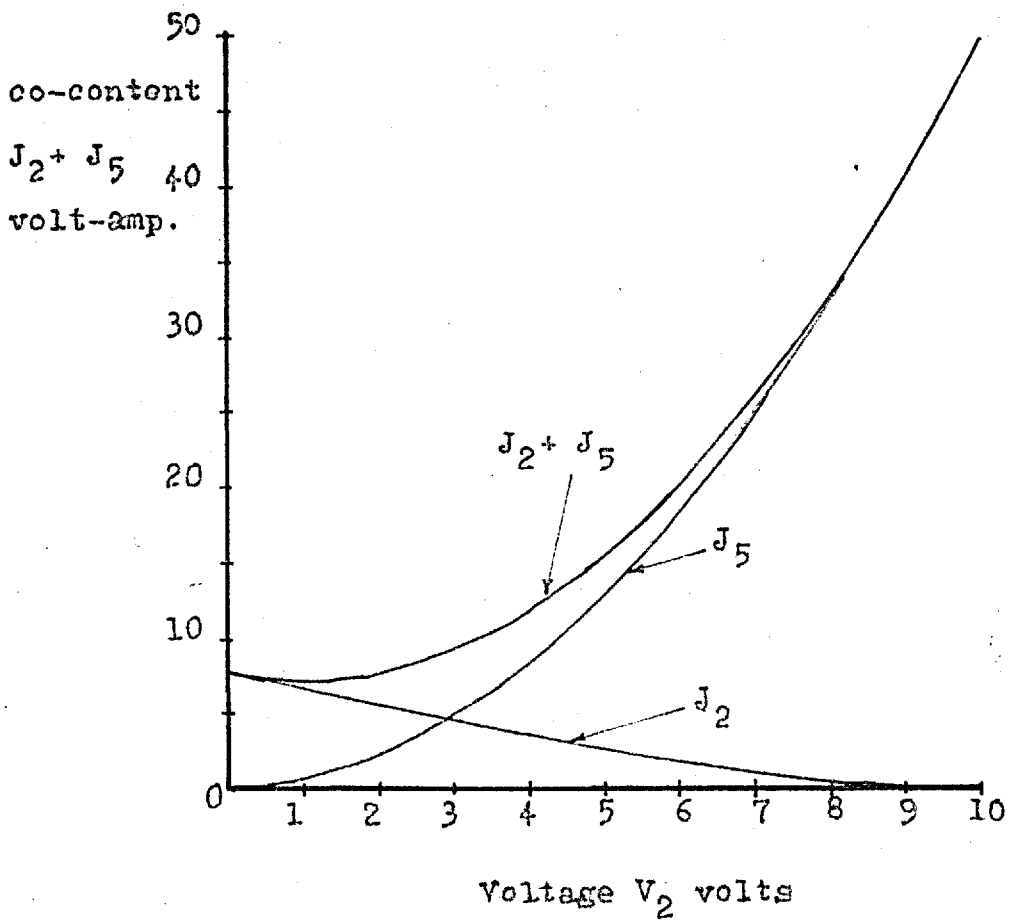
Graph 10

The Sum of the co-contents of R_1 and R_4.



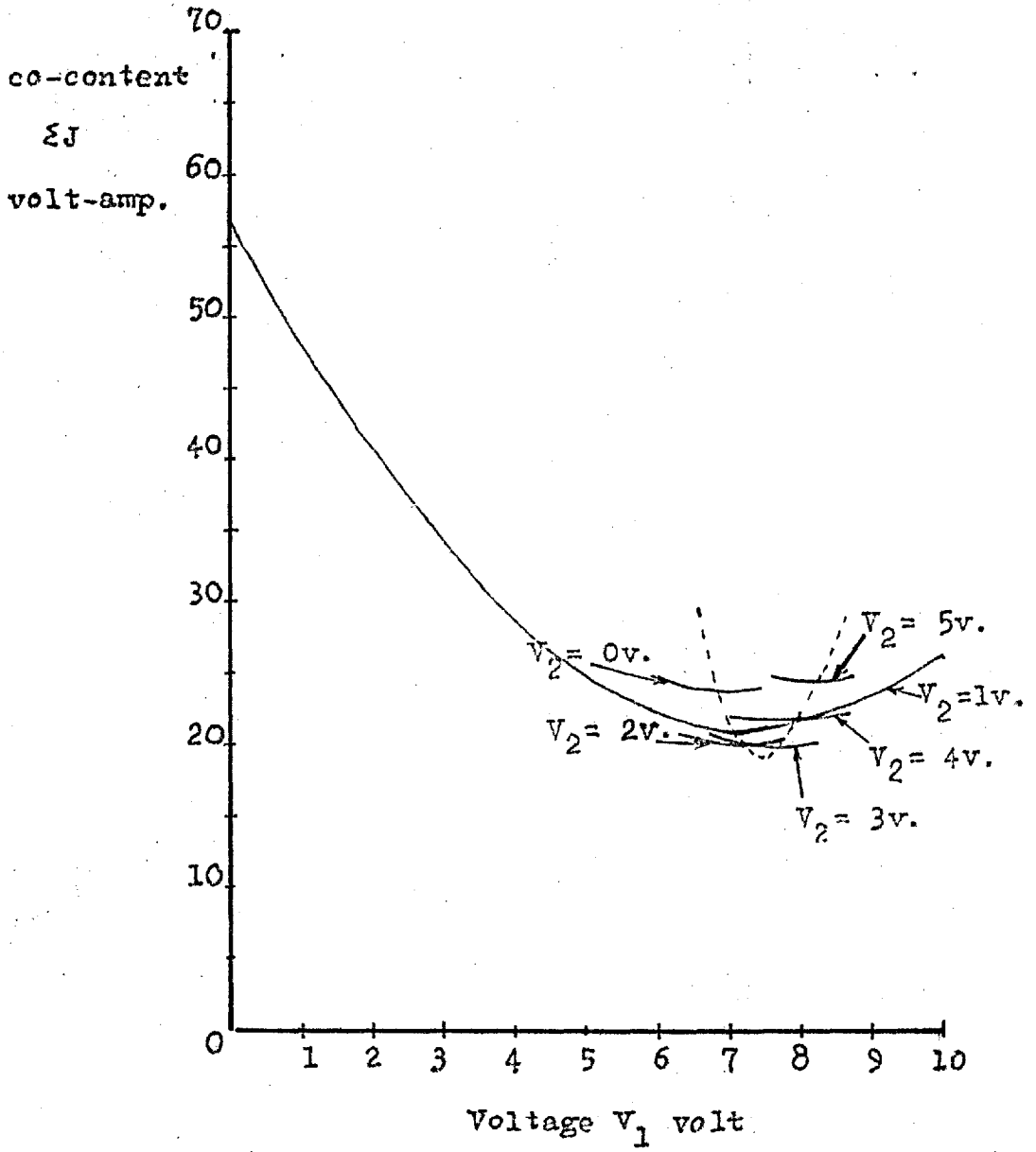
Graph 11

The Sum of the co-contents of R_2 and R_5



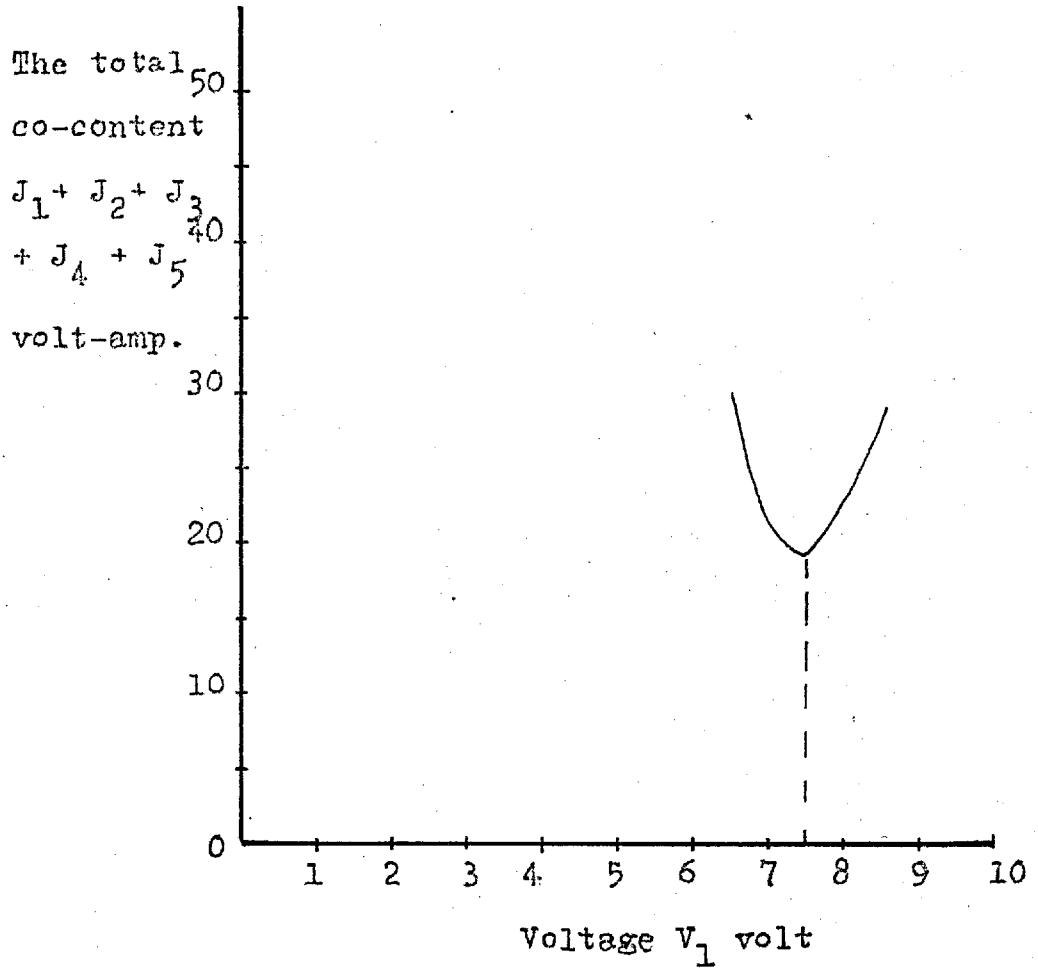
Graph 12

The total co-content of the network for each constant value of V_2.



Graph 13

The total Co-content of the network



2.5 The Application of Analogue Devices to the Graphical Method.

The graphical method discussed in both sections 2.3 and 2.4 is rather tedious, but it has been found possible to speed up the work by using function generators and adders. This method may now seem inferior to the new graphical method decribed in the latter part of this thesis, but nevertheless when the experiment was carried out, it provided a new method of solving finite difference equations, (Appendix I), in cases where the iterative method failed to converge.(4)

The function generators are tapped potentiometers fitted with ten variable-resistance shunts, so that the voltage on the slider can be made to represent the content G(i) or the co-content J(v) of a resistance, while the angular position of the slider represents the current i, for content, or the voltage v for co-content.(Appendix II)

When dealing with the ladder section of Fig.2.6 three function generators and an adder are used as shown in Fig.2.10

When the system of Fig.2.10 is set up it is first used to find the content looking towards the right at aa' on fig.2.5 as a function of i_1.

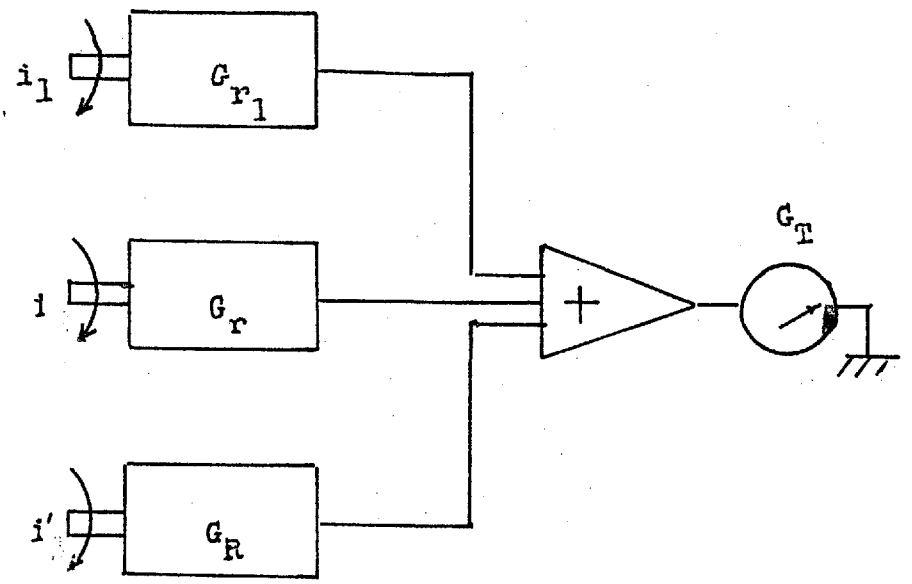


Fig.2.10

G_T indicates the minimum sum of the contents

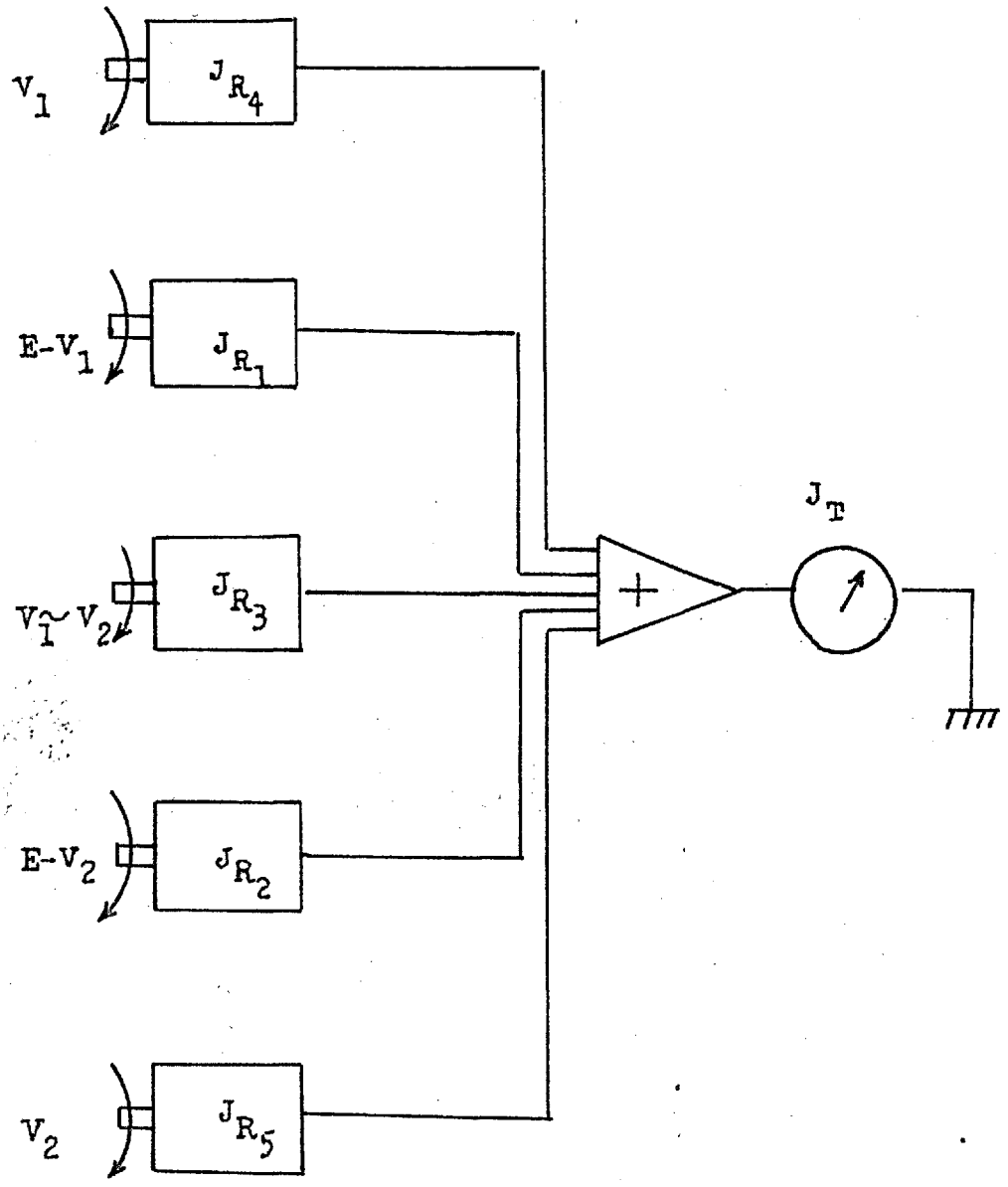


Fig. 2.11

J_T indicates the minimum sum of the co-contents

When the content of this section as a function of i_1 is obtained, it is then ready for the next similar stage of the process in which the content looking in at bb' will be found as a function of i_2.

When dealing with the Bridge circuit as in section 2.4(Fig.2.9) five function generators and an adder are used, and connected together as shown in Fig.2.11.

The voltage V_2 is set in small steps and V_1 is varied from 0 to 10 volts. The input voltage to the function generator JR_3 is adjusted to the difference between V_1 and V_2. The minimum output for each V_2 setting is then obtained, and the smallest value of the minimum output indicates the actual distribute voltage, V_1 and V_2.

2.6 The Application of the Non-linear Ladder Network

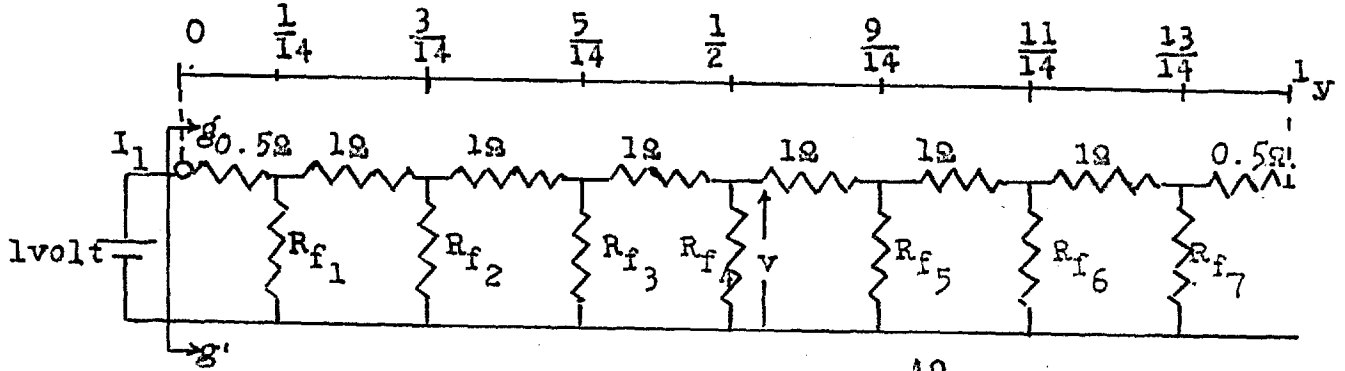
The non-linear ladder network has been used as an analogue of the finite-difference form ofaone-dimensional non-linear differential equation.(Appendix I). Its application in this case has assisted in scientific and engineering fields, and its usefulness has been proved by the evidence of papers published by many scientists. The solution of the non-linear ladder network by the method shown in section 2.3 and 2.5 was employed successfully even in cases where the iterative methods of Liebmann and Bailey (9)

Karplus[5] and Hutcheon[7] failed to converge, (These failures, which were analysed by Fisher[4], occur even in linear cases in certain circumstances, ie. the simple harmonic equation.(Illustrated in the section 2.3).

Since a new graphical method based on Cherry's diagram, has been discovered, and is discussed in the latter part of this thesis, and this is in many ways simpler and quicker than this graphical method, only one non-linear example(for the solution of a differential equation) will be given. This example can also be used for comparing this method with the graphical method described in the next chapter.(This is the same problem as that published by Kovattana and Barker. A reprint of the paper is given inside the back page of the thesis.)

Example This example is taken from Hutcheon and Spalding[7]'s paper on steady-state heat-flow, with convective losses, in uniform bars. The differential equation to be solved is $\frac{d^2\theta}{dy^2} = K\theta^{5/4}$ in which θ represents temperature at a distance y along the bar. The numerical value of K is 12.5 and the boundary values are $\theta_0=1$ at y=0 and $\frac{d\theta}{dy}=0$ at y=1 for the example presented here.

The ladder network for this solution is as shown in Fig.2.14. For the node potential V to be numerically equal to the temperatures θ in the bar at the values



For all shunt elements $R_{f_n} = \frac{49}{12.5 v_n^{1/4}}$ Ω

where v is in volts.

Fig. 2.14

of y indicated, the shunt resistance R_{f_n} will have the value $R_{f_n} = \frac{RV_n}{h^2 KV_n^{5/4}}$, for R=1Ω, h=7,

then $R_{f_n} = \frac{49}{12.5V^{1/4}}\Omega$

The process of computing is similar to that discussed in section 2.3 and 2.5 and the total content of the impedance of the network looking in at gg' is shown on graph 14 as a function of the current I_1. Graph 15 shows the total content of the whole circuit including the content of the voltage generator connected at the terminal gg! This gives a minimum value between the values of current I_1 equal to .40 and .50.

When compared with the new graphical method decribed in the latter part of the thesis, this method is long and laborious.

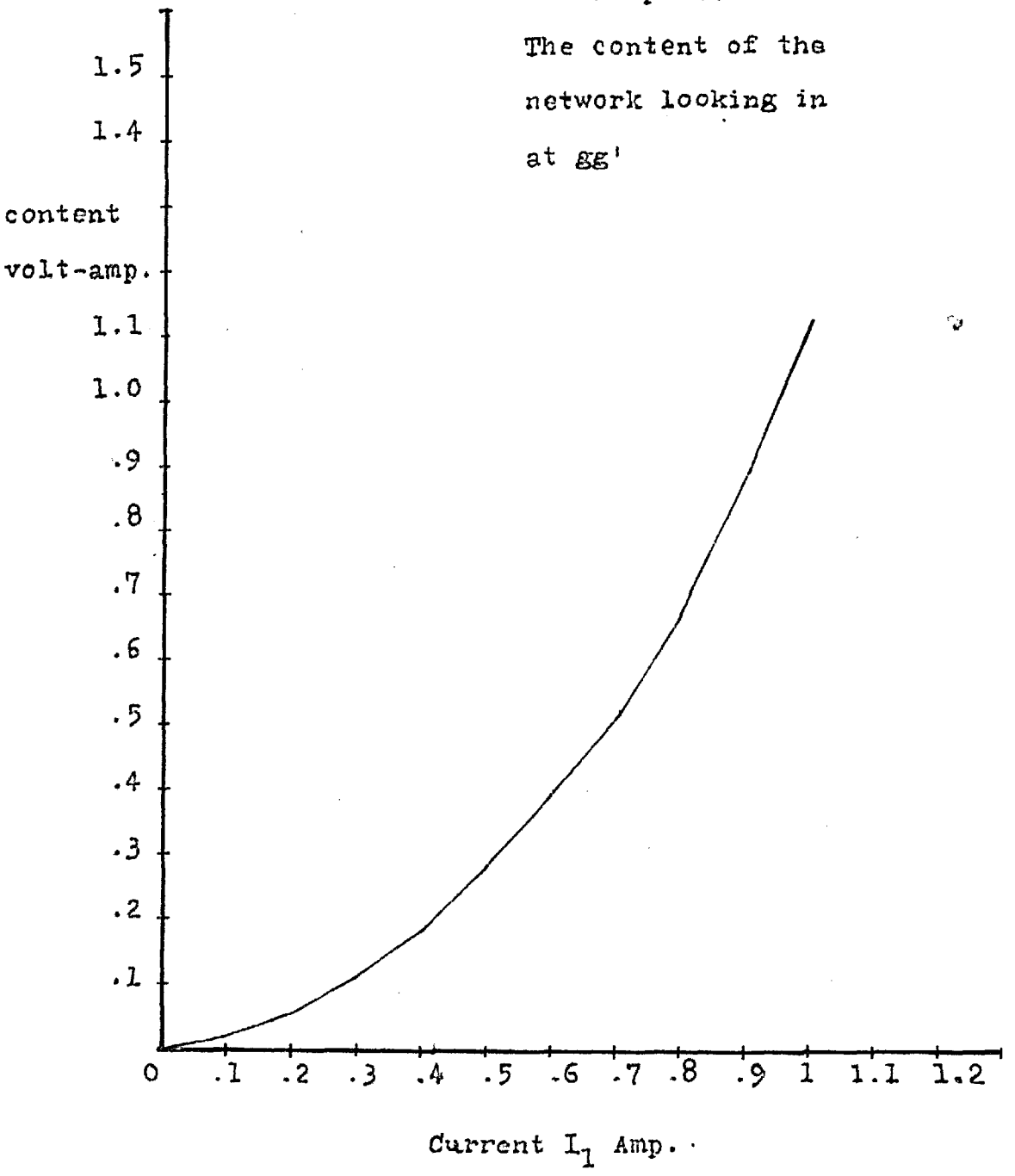
To obtain the final result of I_1, the numerical calculation is used. This is done very rapidly because of the good start given by the two values obtained. The final value of I_1 is .458 amp. The node voltages are then reading from left to right on Fig.2.14.

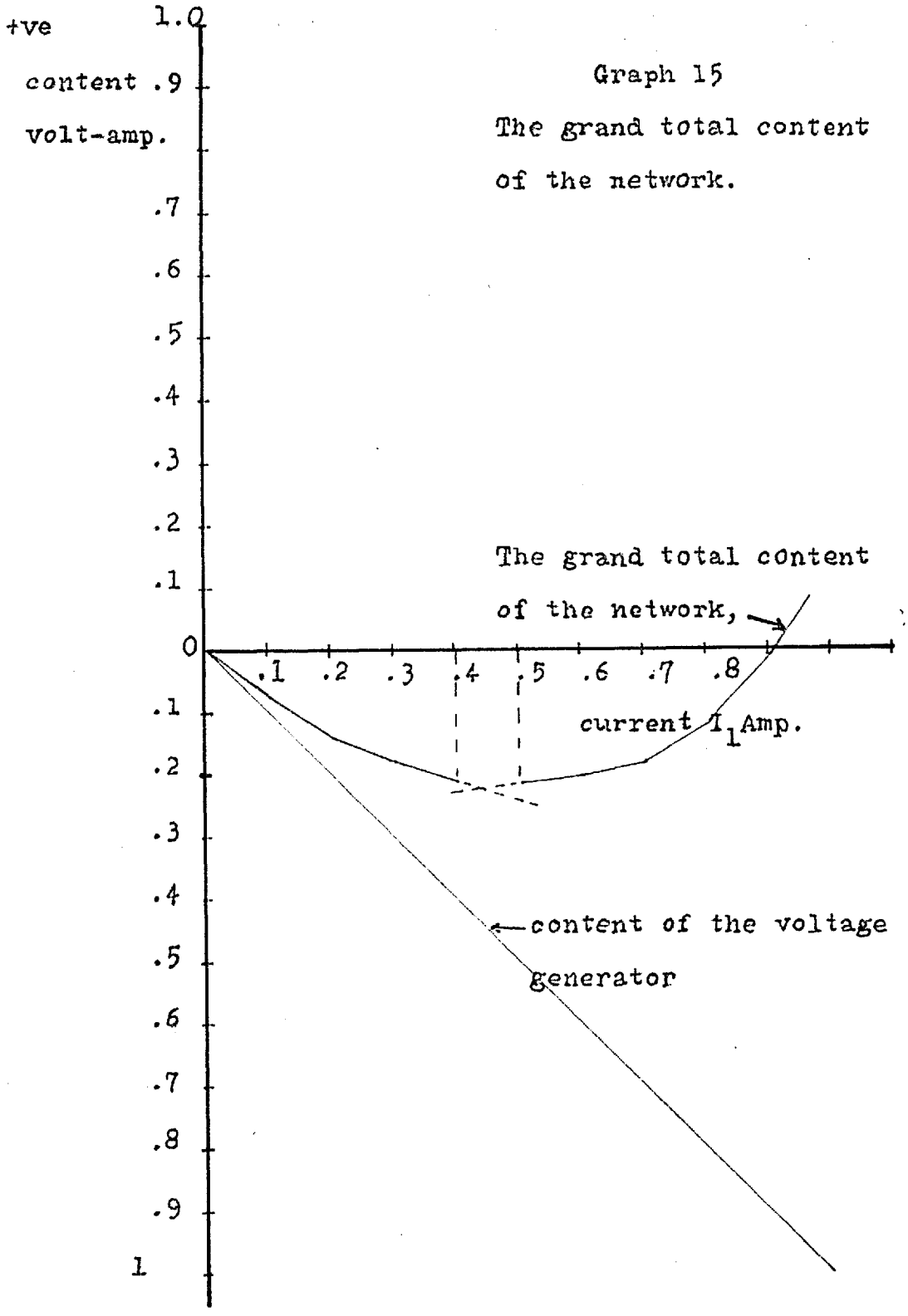
1.0, 0.77, 0.50, 0.33, 0.22, 0.15, 0.11, 0.09

which are similar to those obtained by Hutcheon and Spalding.(7)

Graph 14

The content of the network looking in at gg'





Graph 15

The grand total content of the network.

CHAPTER III.

The Solution of the Ladder Network by a New Visual Iterative Method based on graphs, and Its Application to solve Second-order Differential Equations.

In 1951 Cherry[1] showed that the network topology, the V-I characteristics of each resistance, Kirchhoff's circuit laws and the power relationships of a pure resistance network energised by a single power, can be illustrated in a rectangular diagram. I have found that in solving a non-linear resistance network expecially a ladder network, the solution can be obtained very rapidly by drawing the V-I characteristics of the constituent resistances on rectangular sheets of transparent glass and arranging them visually on top of one another to form Cherry' diagram. This idea is then developed to use templates instead of glass sheets to solve second-order differential equations. This diagram is also extended to cover the resistance network which includes negative non-linear resistances(ie. resistances with V-I characteristics with negative slope) and the circuits which have more than one energy source.

Since the ladder network as analogue of the differential equation has very wide and useful applications, this method of solution will be discussed and illustrated

mainly with reference to the ladder network and its application to various types of problems.

3.1 The Diagram for a Ladder Network of Positive Resistance.

To introduce the method, the ladder network of Fig. 3.1 is considered. In this network the series elements are linear resistors, the shunt resistors are non-linear and e.m.f's E_1 and E_2 ($E_1 > E_2$) are applied at the ends. The voltages at the two intermediate nodes have to be found. For the moment, however, suppose that the problem has already been solved so that the final diagram can be exhibited, as in Fig.3.2. Within an area defined by two rectangles of sides $E_1 \times I_1$ and $E_2 \times I_2$, the following appear:-

a) Portions of the V-I characteristic of the resistors 1,3 and 5 which have slopes R_1,R_3 and R_5 respectively;

b) Portions of the non-linear V-I characteristic of the shunt resistors 2 and 4, starting at zero potential and continuing up to their actual operating points.

c) All these V-I characteristics appear, diagonally inside minor rectangles which are fitted together in a way which satisfies both of Kirchhoff's laws. For example the sum of the voltage drops across 1,3 and 5,

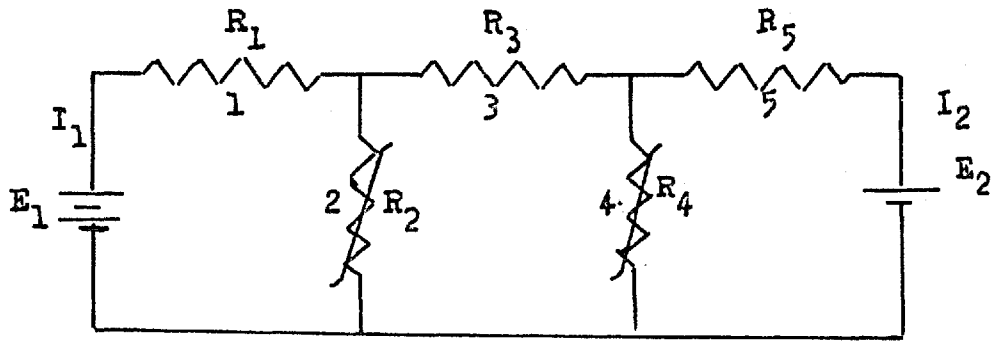

R1
R3
R5
I1
1
3
5
I2
E1
2
R2
4
R4
E2

Fig. 3.1

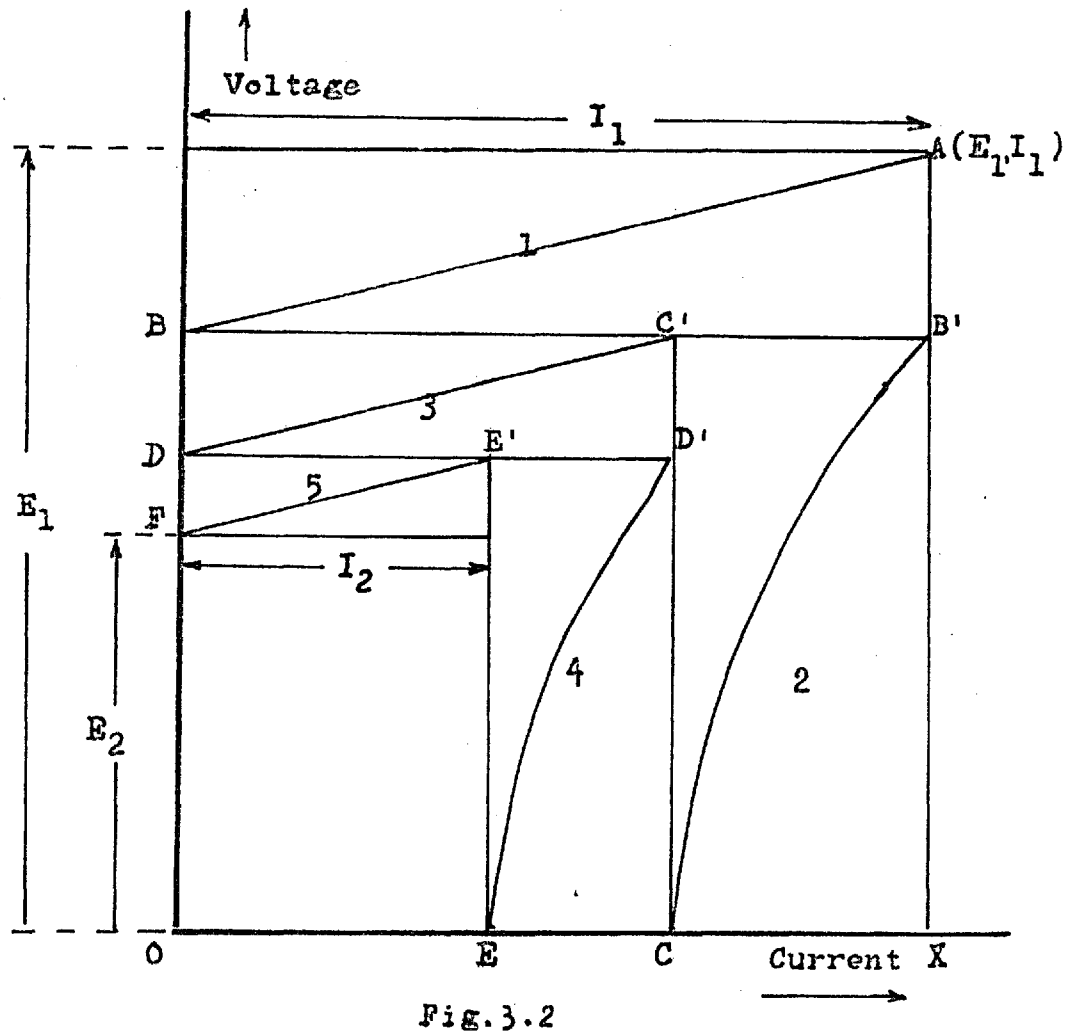

Voltage
I1
A(E1,I1)
1
B
C'
B'
3
D
E'
D'
5
E1
F
I2
4
2
E2
O
E
C
Current
X

Fig. 3.2

should be E_1-E_2, and this relationship is satisfied by the voltages represented along the left-hand edge of the main rectangle (on which all the node potentials appear). Next taking horizontal distances in Fig.3.2, the current in 1, which is I_1, can be seen to equal the sum of the currents in 2 and 3, similarly, the sum of the currents in 4 and 5 is the current in 3.

d) Finally, the area of each minor rectangle represents the power dissipated in a resistance, and the total area of these minor rectangles 1-5 is $E_1I_1-E_2I_2$, which is the power supplied by the two sources, so that the law of conservation of energy is satisfied.

The method of solving the problem when only E_1 and E_2 are given is very simple. After estimating I_1 an attempt is made to draw the diagram to scale using the V-I characteristics in the following order; 1,2,3,4 and 5. (Templates will be needed for the non-linear V-I characteristics, and these can be cut out of graph paper of the same kind as is used for the rectangle diagram). The drawing is made as follows:- On Fig.3.2 is the point (E_1,I_1); AB is drawn of slope R_1, whence AB' is I_1R_1, leaving voltage XB'= OB for the rest of the network. The current flowing through the non-linear resistor R_2 is found by sliding its V-I characteristic to

the position 2 (it passes through B' and has its origin at C) whence current CX flows through it, and OC flows into R_3. C'D of slope R_3 is drawn next, from C' which is vertically above C, and OD will be the next voltage across R_4. The current in R_4 is obtained by sliding its V-I characteristic to the position 4. E'F is drawn of slope R_5 and if I_1 were correctly chosen, OF would be equal to E_2. On completion of the drawing, the distance OF on Fig.3.2 is unlikely to be equal to E_2, but it will be obvious in what sense to adjust I_1 in order to improve it, and a second attempt can then be made. Because vision is incorporated in this trial and error process the solution can be rapidly obtained. If the ladder network is long and highly attenuating and $E_2 \rightarrow 0$, the voltages remote from the source E_1 cannot be found accurately unless a very large drawing is used. In this case, it is better to improve I_1, in the latter stages only, by numerical calculation. If OA is calculated for two values of I_1, preferably one larger and one smaller than the true value, improvement by interpolation is rapid because of the good start given by the graphical method. (The calculations are simple:- starting with E_1 and I_1, the voltage drops and currents can be found as the ladder is traversed from left to right.)

3.2 The Diagram for a Ladder-network containing Negative Shunt Resistances.

When the shunt resistors R_2 and R_4 in Fig.3.1 are negative, the diagram is drawn similarly to the above except that the currents C'B' and E'D' are added to the currents BB' and DD' (Fig.3.2) when the potentials OB and OD are positive. The final diagram is shown in Fig.3.3

To avoid possible confusion due to inexperience the following rule may be useful.

Rule.

a) The V-I characteristics are used in the order 1,2,3, 4,5 as labelled on Fig.3.1

b) Each V-I characteristic lies diagonally inside a rectangle.

c) Each successive V-I curve has its operating point at that corner of the previous rectangle which is not occupied by the previous V-I curve, and which is not on either the V or I axis of the diagram.

d) If a series element is being dealt with, the V-I curve has its origin on the V-axis and if a shunt element is concerned, it is on the I-axis.

Applying this to the network of Fig.3.1 R_4 and R_5 have negative non-linear characteristics. I_1 is

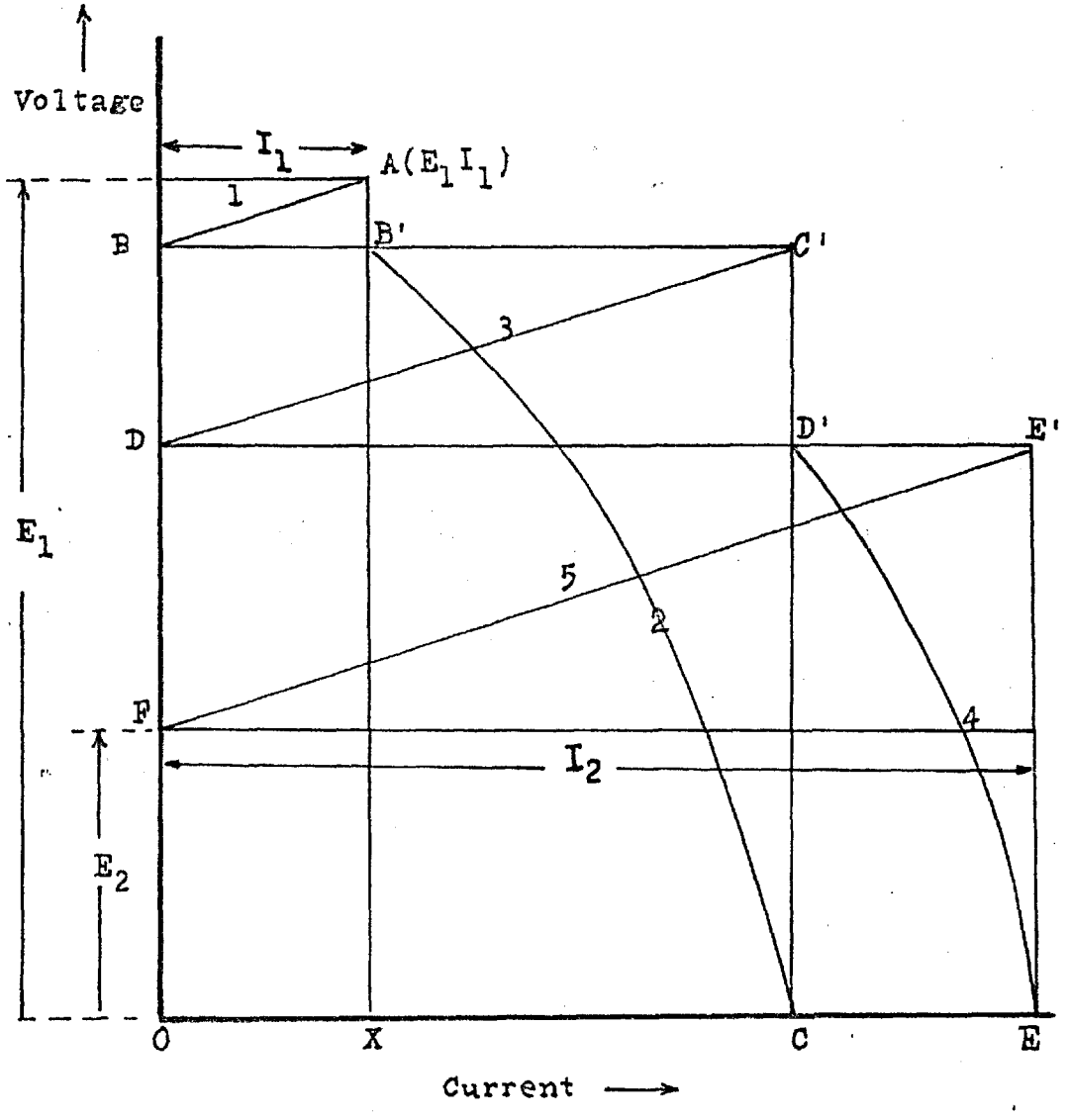


Fig.3.3 Rectangle diagram for network of Fig.3.1

R_2 and R_4 are negative non-linear resistances

drawn to locate the starting point A (where A is I_1, E_1) on the diagram shown in Fig.3.3. A line of slope R_1 is drawn from A to intersect the voltage axis at B. This defines the first rectangle, for resistor R_1, of which B' is the corner which satisfies the rules. The V-I characteristic of R_2 is drawn passing through B' and has its origin at C on the current axis. This curve is the diagonal of the second rectangle, for resistance 2, whose corner C' is the starting point for the next V-I line.

3.3 The finite difference solution of $\nabla^2\psi = f(\psi,x,K)$ by the graphical method.

When the ladder network as shown in Fig.3.1 is used as analogue of the finite-difference form of the equation $\frac{d^2\psi}{dx^2} = f(\psi,x,K)$, E_1 and E_2 will be the value of the boundary conditions, and the shunt resistors R_2 and R_4 will be the functions of $f(\psi,x,K)$.

In the actual computing process there will be more than three sections as in Fig.3.1. The shunt resistors R_2, R_4 are referred to as R_{f_n} where index n denotes the node position. In most cases the series resistors R_1 R_3 R_5.....have the same value and will be referred to as resistor 'R' unless otherwise stated.

Since the resistors R_{f_n} are presented purely for the purpose of helping to indicate the amount of current applied to the node n, then if this current can be found more quickly by another method, it is not necessary to draw the V-I characteristics of R_{f_n} as CB' and ED' in Fig.3.2 provided that B'C' and D'E' can be correctly located. Thus it follows that the templates may not have to be made in some equations.

3.4 Classification of the equation.

In some cases the number of templates for each equation to be solved can be reduced. To minimise the number of templates, some consideration of each equation must be made and the equation may be classified as follows:-

1) $\frac{d^2\psi}{dx^2} = f(x,K)$ in this case the function varies with the value of x only.

2) $\frac{d^2\psi}{dx^2} \equiv f(\psi,K)$ function varies with the value of ψ only.

3) $\frac{d^2\psi}{dx^2} = f(\psi,x,K)$ function varies with both the values of x and ψ

1) $\frac{d^2\psi}{dx^2} = f(x,K)$, when function f(x,K) varies with x only, such as $f(x,K) = K(ax^n+bx^{n-1}+cx^{n-2}.....,)$ the current I_{f_n} to be supplied to each node has a fixed value for each node and equals $\frac{h^2}{R} f(x,K)$. [see Appendix I and use

$x = hn$. Since the value of x is known for each node then the value of I_{f_n} can be calculated. Thus no template is required for this equation. The method of solving the problem is similar to that shown in Fig.3.2 except that CX and EC are always constant for the BB' and DD' node. C' and E' can be located by a pair of dividers.

2) $\frac{d^2\psi}{dx^2} = f(\psi,K)$ i.e. $f(\psi,K) = K(a\psi^n+b\psi^{n-1}+c\psi^{n-2}..)$

In this case the current I_{f_n} depends only on the value of ψ which is in fact the value of the voltage of node n. $(I_{f_n} = \frac{h^2}{R} f(\psi,K)_n)$. The relationship of I_{f_n} and ψ_n can be plotted and is referred to as the V-I characteristic of the shunt resistor R_{f_n}. It is obvious that the same curve is used for all nodes, and therefore only one template is required.

3) $\frac{d^2\psi}{dx^2} = f(\psi,x,K)$ This function can take many forms such as $f(\psi,x,K) = g(\psi) + G(x)$ or $f(\psi,x,K) = g(\psi) \cdot G(x)$, or a mixture of both forms. In the first case I_{f_n} is equal to $\frac{h^2}{R} g(\psi)_n + \frac{h^2}{R} G(x)_n$. Let $i_{gn} = \frac{h^2}{R} g(\psi)_n$ and $i_{Gn} = \frac{h^2}{R} G(x)_n$ then $I_{f_n} = i_{gn} + i_{Gn}$ i_{Gn} are calculated before-hand and their values are constant for each node. i_{gn} is obtained from the curve(template) i.e. the relationship of $i_g = \frac{h^2}{R} g(\psi)$

C'B' is obtained by drawing the curve cB' which gives c'B', and then C'c'(C'c = $\frac{h^2}{R}G(h.1)$) is added.

Fig.3.4

In the second case I_{f_n} is equal to $\frac{h^2}{R}g(\psi).G(x)$

There are two methods which may be used:-

a) Templates are made for the relation $\frac{h^2}{R}g(\psi).G(hn)$ for all values of n, and the diagram is drawn as in Fig.3.2

b) Only one template is made for the relation $\frac{h^2}{R}g(\psi)$. To obtain the current C'B'(Fig.3.5), the curve cB' is drawn and cX is then the value of $\frac{h^2}{R}g(\psi_1)$, this value is multiplied by G(h.1) which is then equal to $\frac{h^2}{R}g(\psi_1).G(h.1)$ = C'B'.

3.5 Example

Eg.1,

The equation is $\frac{d^2\varphi}{dx^2} = \frac{4}{9}\frac{1}{\sqrt{\varphi}}$, and the boundary condition is given as follows:-

1) when x = 0, φ = 0

2) when x = 10, φ = 21.5

The current $I_{f_n} = \frac{h^2}{R}\cdot\frac{4}{9}\cdot\frac{1}{\sqrt{\varphi}}$; a 10 sections network with h equal to 1 is taken. The value of R is arbitrarily chosen as 1Ω. Thus resistors R_{f_n} will have values equal to $\frac{9}{4}V_n^{3/2}$ where V_n is the potential between the n^{th} node and ground. The V-I characteristic of all the

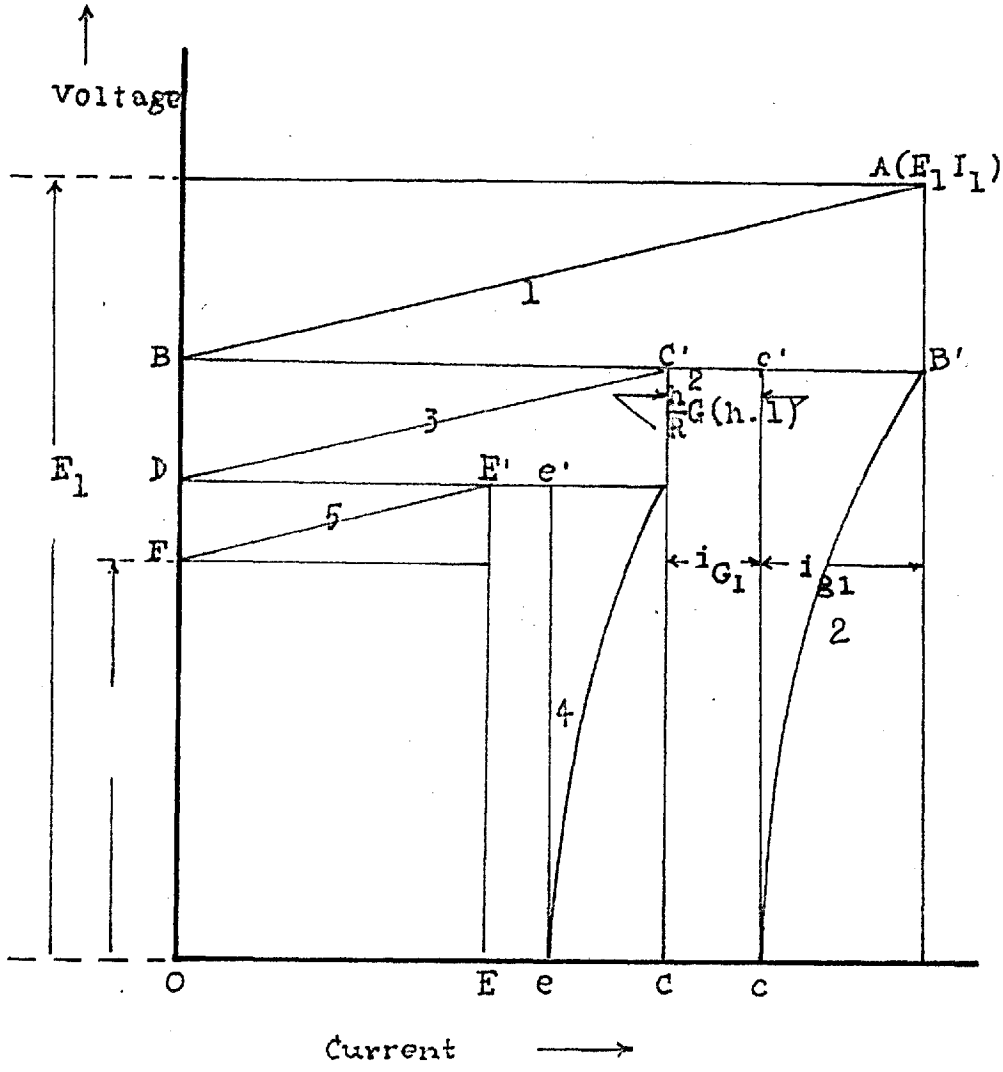


Fig.3.4 The values of $\frac{h^2}{R}G(h.n)$ are calculated before-hand and remain constant for each node

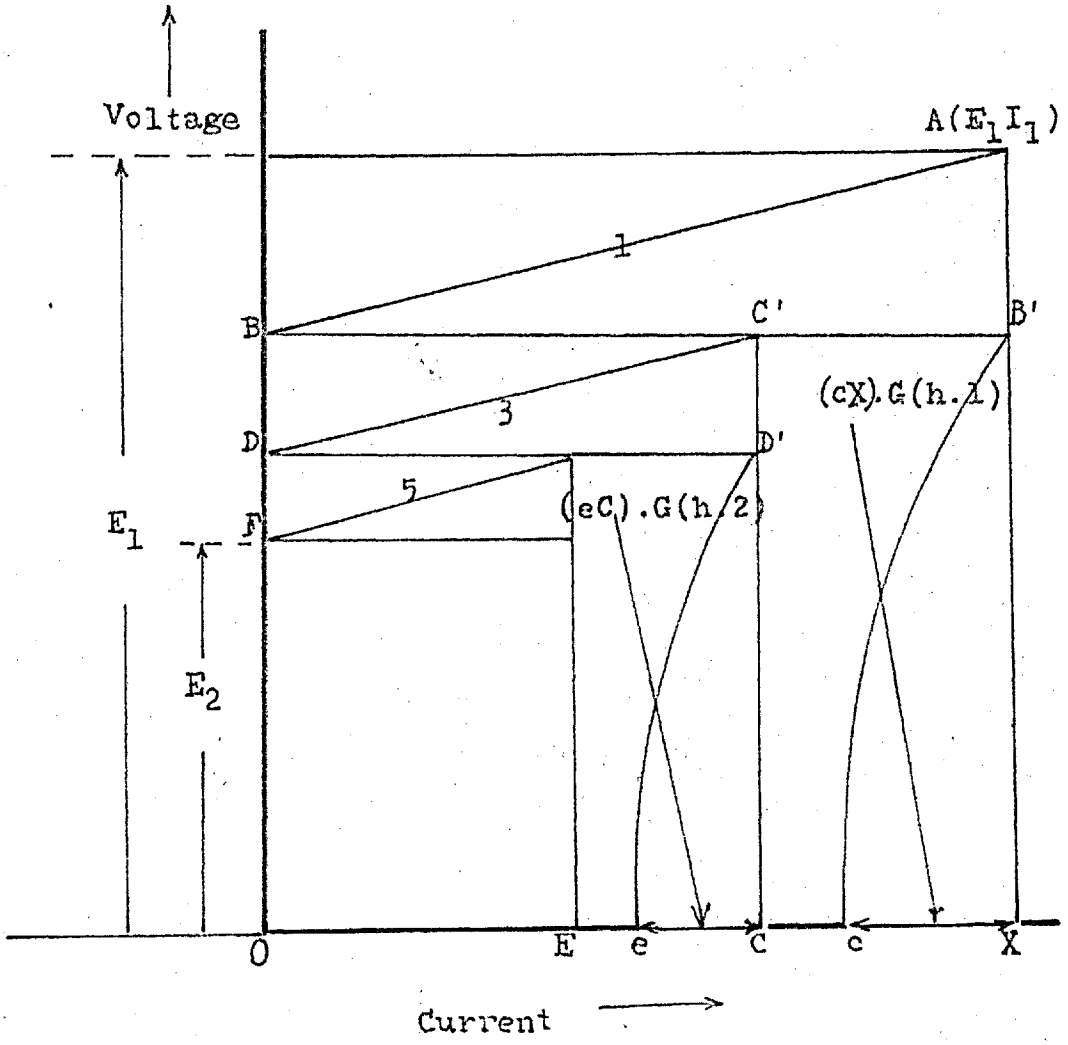


Fig.3.5

resistors R_f is as shown in Fig.3.6

To obtain the solution of this network, the right hand end section of the network is used as the starting point of the drawing as shown in Fig.3.7. The estimated current I_{10} and the boundary voltage 21.5 volts forms the starting point A. The slope of the series resistor R is drawn from point A to intersect the voltage axis at point B, and the rectangle AA'BB' is drawn. The template R_f is slid along the current axis until the curve coincides with point B'. C and C' are then marked on the drawing.

Fig.3.8 displays the voltage along the node of the network for various values of the estimated currents I_{10}. The two values of I_{10}, 2.5 amps. and 3 amps. form the two dotted lines which indicate that the correct solution lies in between. The correct value of the current I_{10} is 2.8 amps.

Results.

x	φ exact solns.[6] of differential eqn.	φ computed solutions
0	0	0
1	1	1
2	2.52	2.61
3	4.33	4.41

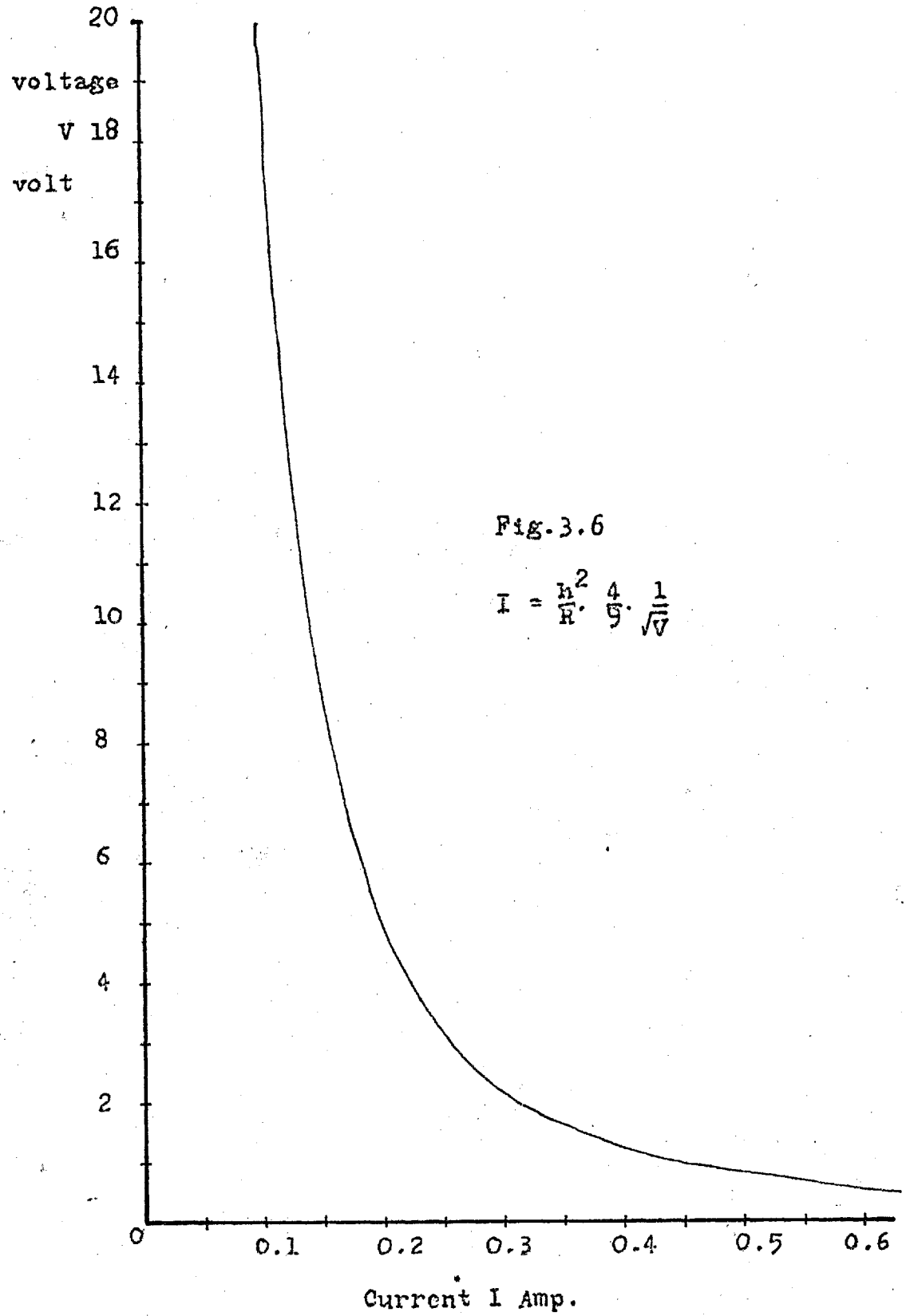
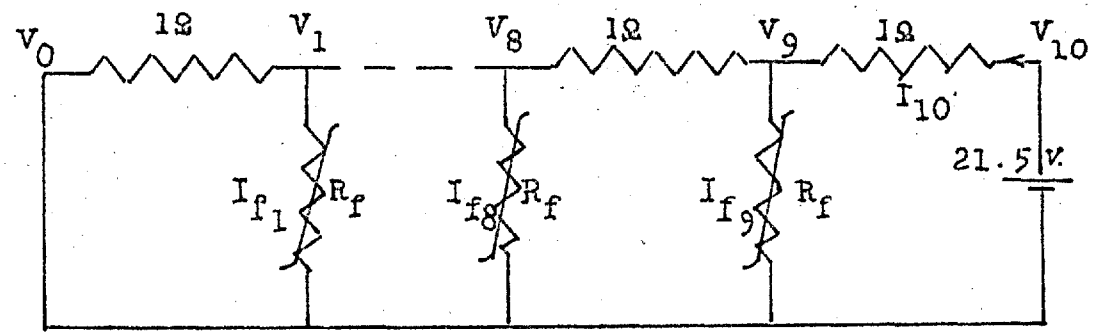


Fig.3.6

$$I = \frac{h^2}{R} \cdot \frac{4}{9} \cdot \frac{1}{\sqrt{V}}$$


V_0
1Ω
V_1
V_8
1Ω
V_9
1Ω
V_{10}
I_{10}
21.5 v.
I_{f_1} R_f
I_{f_8} R_f
I_{f_9} R_f

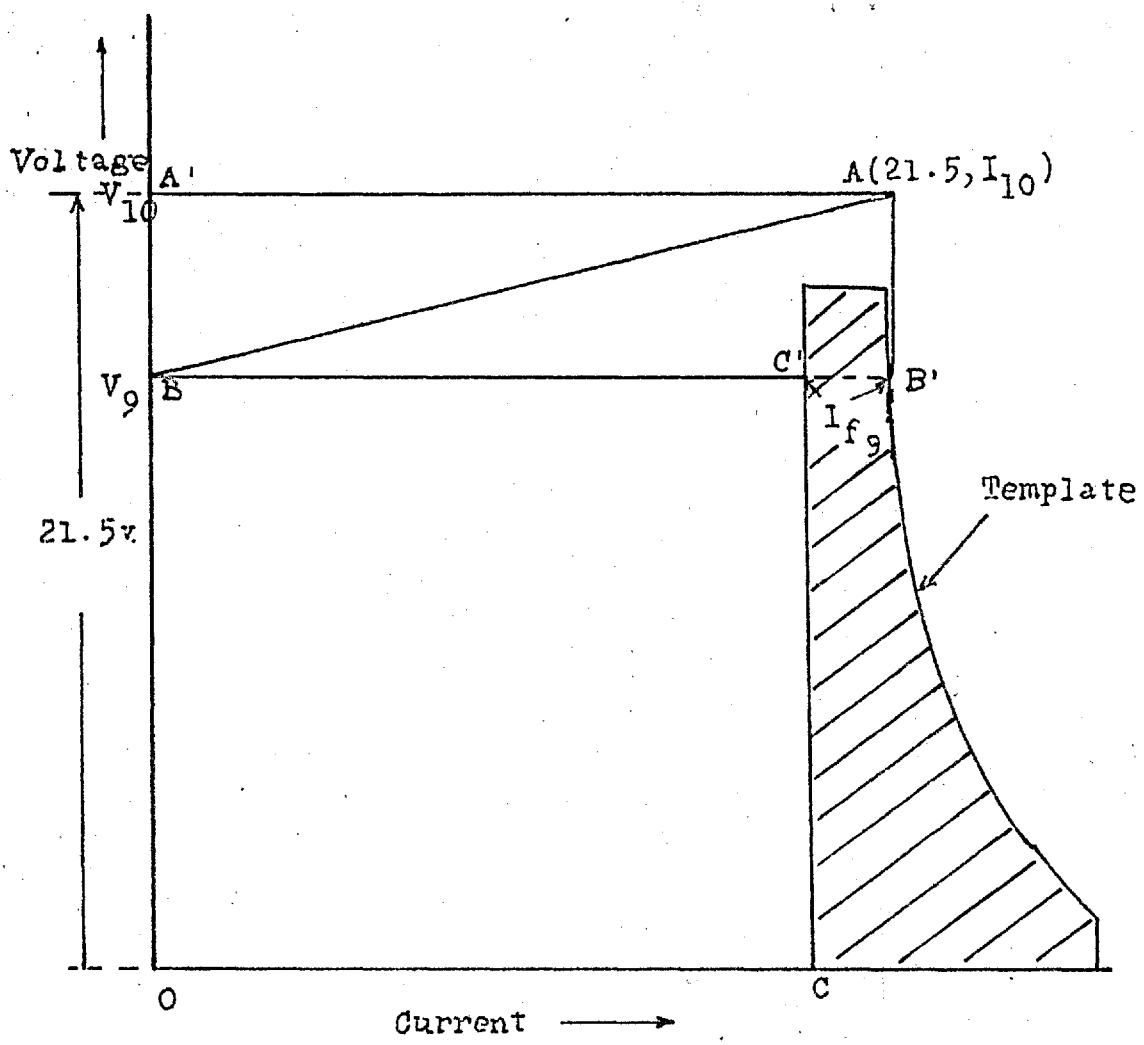

Voltage
V_{10}
A'
A(21.5, I_{10})
V_9
B
C'
B'
I_{f_9}
Template
21.5v.
O
C
Current

Fig. 3.7

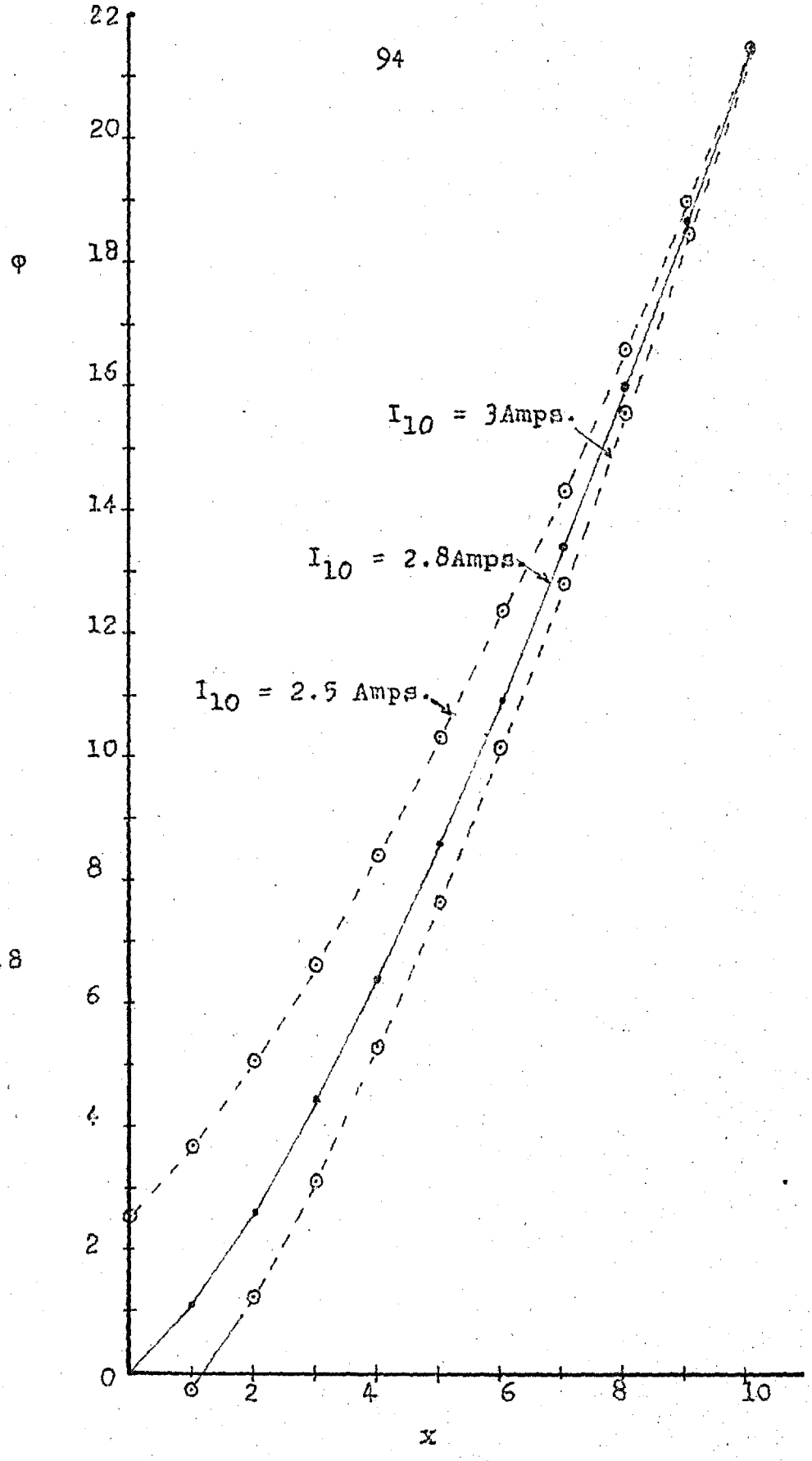

22
20
18
16
14
12
10
8
6
4
2
0
φ
2
4
6
8
10
x
I_{10} = 3Amps.
I_{10} = 2.8Amps.
I_{10} = 2.5 Amps.

Fig.3.8

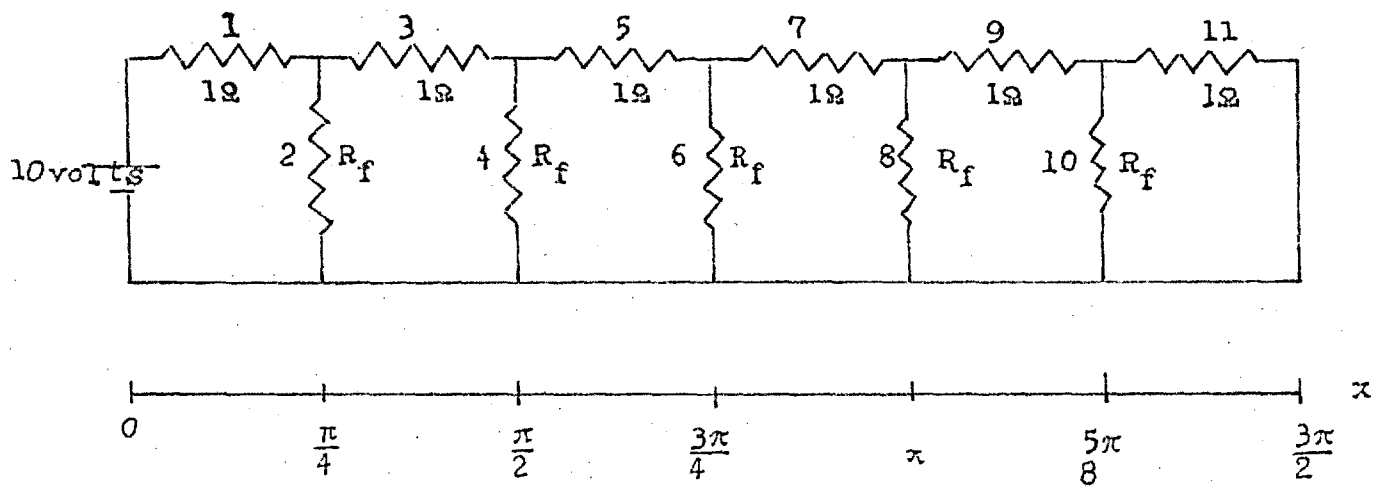
4	6.35	6.42
5	8.55	8.60
6	10.9	10.94
7	13.4	13.41
8	16.0	16.0
9	18.7	18.7
10	21.5	21.5

Eg.2,

A case for which the analogue computor will encounter convergence difficulties is taken as the next example. Consider :-

$$\frac{d^2\varphi}{dx^2} + K^2\varphi = 0$$

with $K^2 = 1$, $\varphi = 10$ at $x=0$ and $\varphi=0$ at $x=3\pi/2$, for which the exact solution is $\varphi = 10\cos.x$. The analogue network, which has six intervals is employed which gives the value of h, a finite difference interval, equal to $\pi/4$ and $R_f = -1.621$. The solutions obtained by graphical drawing depart slightly from the solutions obtained from 10cos.x, but the error is due to finite difference approximation and is not relevant to the solving of the network; errors in the network solution cannot be detected. The complete diagram is as shown in Fig.3.8b. The drawing is commenced in the order 1,2,3,-- of the resistors shown in Fig.3.8a



$R_f = -1.621\ \Omega$

Fig.3.8a Resistance network for simple harmonic motion

Rectangle diagram for network of Fig.3.8a

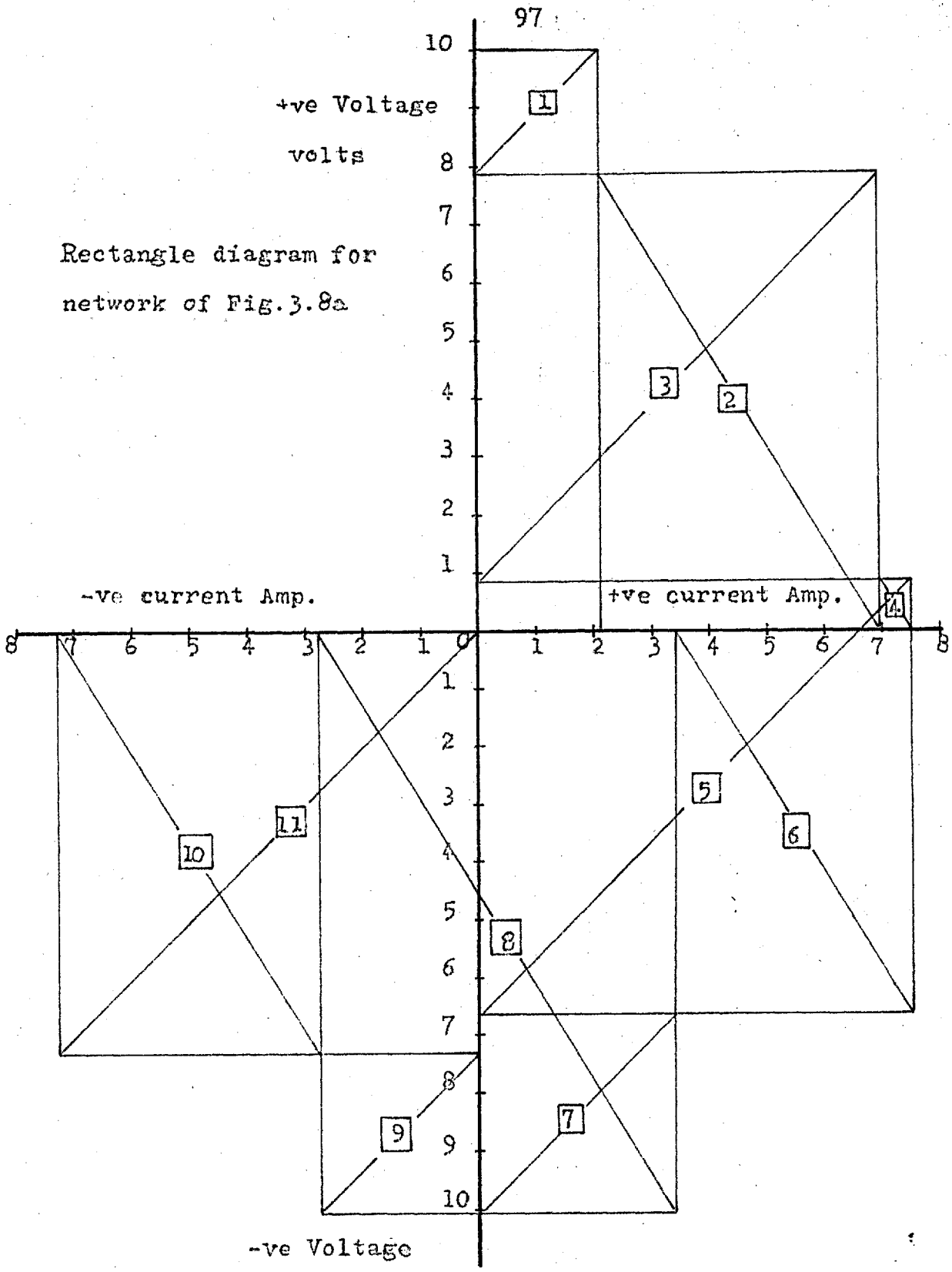


Fig 3.8b

3.6 The Solution of Ordinary Linear and Non-linear Differential Equations by the Graphical Method

The equation in the form of

$\frac{d^2\varphi}{dx^2} + f(x)\frac{d\varphi}{dx} - K\varphi + F(x,\varphi) = 0$ is to be considered.

The presence of a first derivative introduces an additional feature to the problem, but the procedure in applying graphical drawing follows the same pattern.

The analysis of the ordinary diffential equation in the finite-difference form, is given in Appendix III and every term used in this section refers to it.

The finite-difference form of the first derivative $(\frac{d\varphi}{dx})_n$ causes the values of the series resistors R_1 R_3 R_5..R_{n+1} (Fig.3.1) to be different and the calculation necessary to obtain these values is given in equation (A3.21) of Appendix III. If they have non-linear values then templates are made for each of them.

The value of the shunt resistors for each node is equal to the value of the resistor R'_n and R_{Fn} (Fig A3.1 p. 199) connected together in parallel, and a separate template is required for each node, having a V-I characteristic of the combined values of R'_n and R_{Fn} i.e. $R_{f_n} = \frac{R'_n \cdot R_{Fn}}{R_{Fn}+R'_n}$

An example of this type of equation is given in Eg.1 section 4.2

3.7 The Electrical Analogue of the Graphical Method.

To speed up the graphical drawing an electrical circuit which will perform in the same way as the graphical drawing, may be used. Consider the ladder network in Fig. 3.11.

To obtain the graphical solution of this network the current I_1 is estimated and the node voltages V_1 V_2 .....V_N are obtained from the drawing of the V--I characteristics of the resistances in the circuit.

Thus I_1 is the only independent variable. The variation of the value of I_1 will then cause the values of V_1 V_2 .....V_N to change accordingly. Thus it follows that V_N can have any value by suitably choosing the value of I_1. For the solutions of the ladder network shown in Fig. 3.11, the value of I_1 will have to be adjusted until the value of V_N is equal to the value of e_N. Then V_1 V_2 .....V_N will have values equal to the values of the node voltages e_1 e_2 .....e_N.

If a circuit as shown in Fig. 3.12 is used, similar behaviour to that of the graphical drawing can be obtained. (This circuit is by no means a topological analogue form of the ladder network in Fig. 3.11, but the series resistors R_1 R_2 .....R_N will have the same value as those of the ladder network). The blocks

in the diagram represent devices which will give voltage at the right-hand end without taking any current from the left-hand end, i.e. cathode followers.

To obtain a similar result to that of the graphical drawing the circuit is operated in the following way:-

1) The estimated value of the current I_1 is caused to flow through the resistor R_1, by adjusting the variable voltage source V_1. V_1 will be the first estimated voltage of the first node similar to that on the graphical drawing.

2) The current I_2 is obtained by adjusting the variable voltage source V_2 until the amount of current is equal to the sum of the estimated current I_1 and the current obtained from the V-I characteristic of the resistor R_{f_1}, for the corresponding voltage V_1.

3) The currents I_3 I_4 .....I_N are got in the same way as the current I_2, and the values of the node voltages V_3 V_4.....V_N corresponding to the estimated value I_1 are obtained.

The change in the value of the current I_1 will cause the values of the voltage V_N and the rest of the node voltages to alter accordingly.

This arrangement can obviously be made automatic

so that only I_1 requires adjustment and the rest of the currents will adjust themselves accordingly. Thus V_N will respond simultaneously with the value of I_1. It would then enable V_N to be speedily adjusted to the value of the boundary condition when a second-order differential equation is being solved.

The replacement of the V-I characteristic of the R_f resistor when the new equation is to be solved, can be made simple by installing a universal function generator, and in most cases only one type of function generator will be required.

This arrangement may not be appreciated on a one dimensional problem, but it will be of great value when a two dimensional problem is involved, and it will be more convenient if all the values at the boundary points can be displayed simultaneously on an oscilloscope.

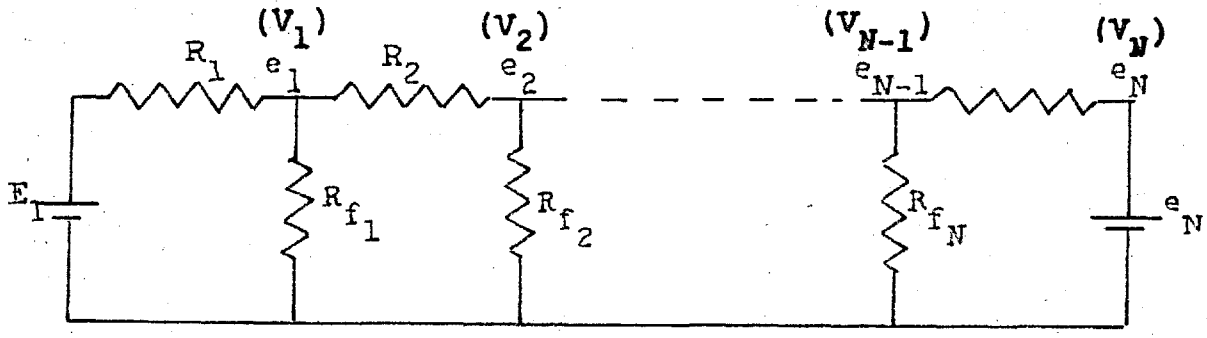


Fig.3.11

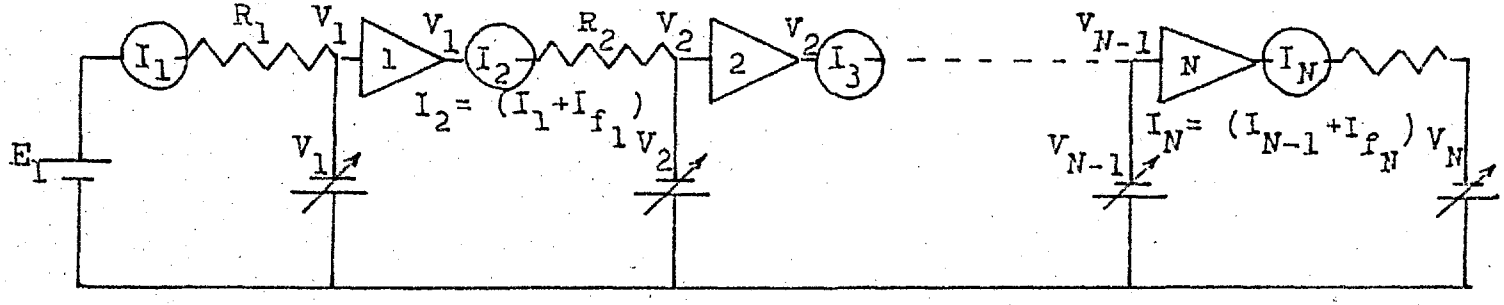


Fig.3.12

CHAPTER IV

The Solution of the Eigen Value Problem by the Graphical Drawing.

The equation $\nabla^2\varphi+K^2\varphi = 0$ is widely used to express vibration and oscillation problems. They are not easy to solve by conventional numerical methods, because the solutions of the mode distribution φ and the unknown eigen value parameter K have to be found. K is equal to $2\pi/\lambda$ where λ is the wave-length of the oscillation. It has been shown by M.E.Fisher[4] that when the value of K is greater than 1, the conventional iterative process on a resistance network analogue technique fails to converge. This however does not occur when the graphical method is applied, as already shown in the example 2 section 3.5 When this problem is to be solved, the given boundary conditions in most cases are the values of φ at the first node and the last node, together with the value of the slope at one point, (normally at the first node or the last node).

4.1 Applying the Graphical Method.

The finite difference solutions, and the eigen value K, of the above equation or other similar equations, can be obtained by the graphical method as illustrated

in section 3.3. The only additional procedure is that the correct values of the shunt resistors of the ladder network have to be found. Since the value of the current I_{f_n} is $-\frac{h^2}{R}K^2\varphi_n$ and the numerical value of φ_n is equal to the value of the voltage at node n, then the value of each of the shunt resistors is $-\frac{R}{h^2K^2}$. In this case, K is unknown, and therefore the values of the shunt resistors are also unknown. (R is the value of the series resistors of the ladder network, and h is the finite-difference interval). It must be noted that for any value of K the finite difference solutions of the above equation can be obtained if only two boundary conditions are to be satisfied. To satisfy also the third boundary condition the value of K becomes critical. To solve the above equation two values of K are first estimated and the finite-difference solutions for each value of K which satisfy two of the given boundary conditions, are obtained. The examination of these two sets of solutions and the third boundary condition, will show in what way the value of K has to be adjusted so that the third boundary condition will be satisfied.

When the values of the boundary conditions are given as the values of the solutions at both boundary

points and the value of the first derivative at one of the boundary points, then the correct finite difference solutions may be obtained in a shorter time, if the computing is commenced at the boundary point where two conditions are known.

This factor is illustrated below. Consider Fig.4 and assume that the finite difference solutions and the eigen value K of the equation $\frac{d^2\varphi}{dx^2} = -K^2\varphi$ are to be obtained. The following boundary conditions are given.

1) $\varphi = E_1$ when $x = 0$

2) $\varphi = E_2$ when $x = 1$

3) $\frac{d\varphi}{dx}_0 = 0$ when $x = 0$

It is also assumed that for $(\frac{d\varphi}{dx})_0 = 0$.

$4\varphi_1 - 3\varphi_0 - \varphi_2 = 0$ .....(Appendix IV) .....(4.1)

φ_0, φ_1, and φ_2 are equal to the value of the voltage of the ladder network at nodes 0,1 and 2 respectively. Let K_1 be the first estimated value of K, and two sets of solutions are obtained. The first set of solutions satisfies the boundary conditions 1 and 2 and the second set satisfies the 1st. and the 3rd. boundary conditions as shown in Fig. 4.1.

The first set of solutions is obtained by the method explained in section 3.1. All the values of the node voltages have to be found for every estimated

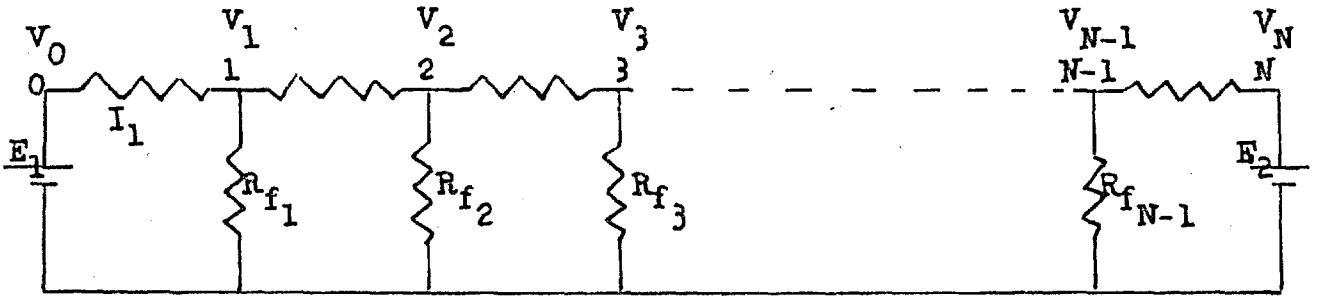
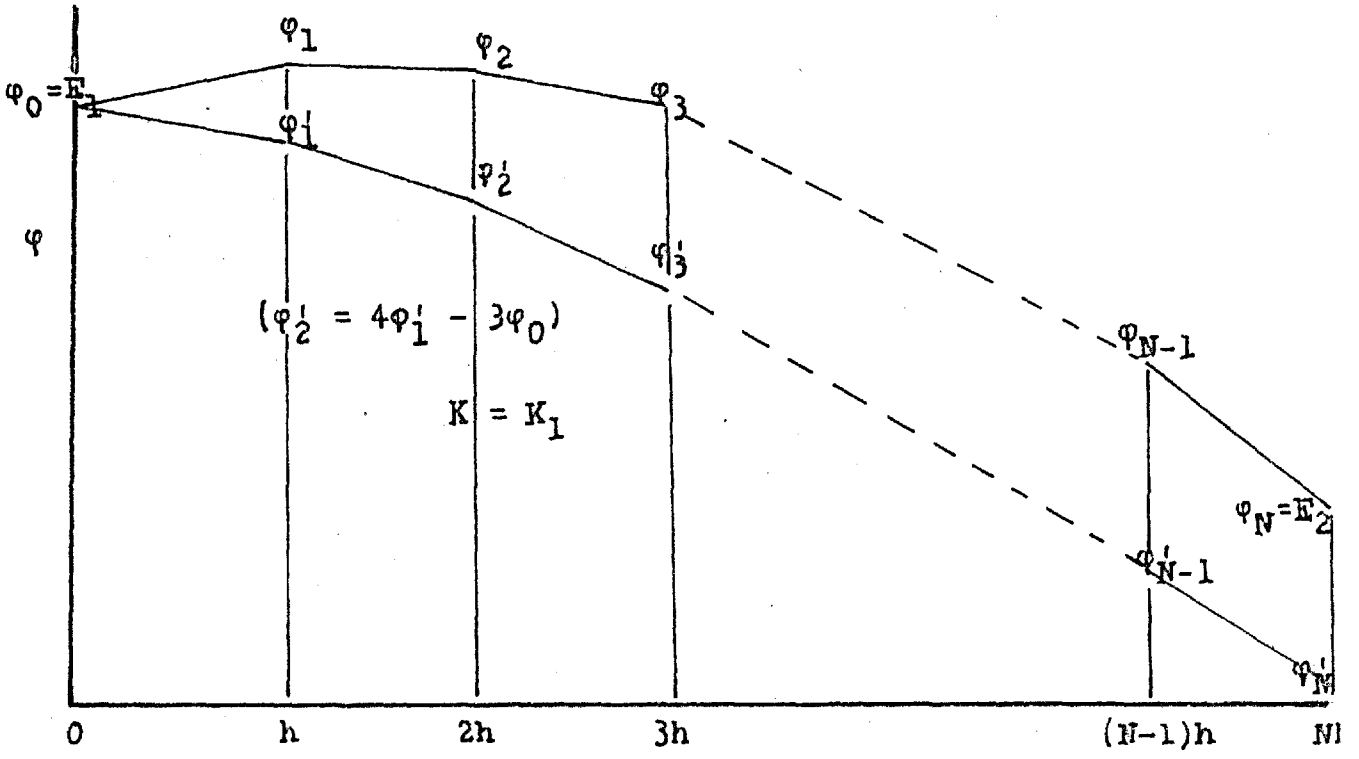


Fig.4



x = nh

Fig.4.1

value of I_1.

To obtain the second set of solutions, I_1 is estimated, and the values of the voltage of the 1st. and 2nd. nodes only are obtained. These values are then substituted in the equation (4.1). When the correct value of I_1 is found (when equation 1 is satisfied) then the rest of the node voltages are computed.

Since the values of the voltages of the first two nodes only are needed to be computed for each estimated value of I_1, then the second set of solutions can be obtained in a much shorter time than the first set.

4.2 Example.

Eg. 1. The equation $\frac{d^2\varphi}{d\rho^2} + \frac{1}{\rho}\frac{d\varphi}{d\rho} = -\beta^2(1-4\rho^2)\varphi$ ...(4.2)

has been investigated by many authors and the first eigen value β has been obtained. In this example an attempt is also made to obtain the second eigen value which could not be found by the other iterative processes using analogue computor techniques. The boundary conditions are given as follows:-

1) $\varphi = 1$ for $\rho = 0$

2) $\varphi = 0$ for $\rho = 0.5$

3) $\frac{d\varphi}{d\rho} = 0$ for $\rho = 0$

This condition is only possible for ~~two~~ specific values of β.

This example was also used by [illegible] author.

The ten sections network is employed, and therefore the value of h, the finite interval, is 0.05.

The values of the series resistors and the shunt resistors are computed from the equation given by the Appendix III, and the value of K_1 (also Appendix III) is given as 1.

The value of the series resistor R_n in this case may be calculated from the following formula :-

$$R_n = \frac{1}{400.(n-\frac{1}{2})} \quad \ldots\ldots\ldots(4.3)$$

and R_{f_n} the value of the shunt resistor is

$$R_{f_n} = -\frac{1}{\beta^2(1-4\rho^2)n} \quad \ldots.(\rho = 0.05n)..(4.4)$$

To determine the value of the slope at the point $\rho = 0$ the following equation is used.

$$\left(\frac{d\varphi}{d\rho}\right)_0 = \frac{1}{2h}\ (4\varphi_1 - 3\varphi_0 - \varphi_2) \ldots\ldots\ldots(4.5)$$

Fig. 4.2 shows the distribution of the voltages along the ladder network for three values of β. When β has the value 5.3 the voltage at the last node of the network is above the earth potential and does not meet the requirements of the boundary condition. For the value of $\beta = 5.5$, the voltage at the last node becomes negative. The two dotted lines show the distribution of the voltages when the values of β are

5.3 and 5.5. The solid line shows the distribution of the voltages when β has the value 5.413. For this value of β, the three boundary conditions are satisfied and therefore the values of the voltages at each node are numerically equal to the values of the required finite-difference solutions.

To obtain the distribution of the voltages along the ladder network for each value of β, the drawing commences from the node n=0, and the voltages of nodes 1,2,.....10 are obtained. At node n=0, two boundary conditions are given. i.e. 1)φ = 1 and 2) $(\frac{d\varphi}{d\rho})_0 = 0$. The first boundary condition is readily satisfied when the boundary voltage equal to 1 volt is used. The second boundary condition is obtained from the relationship given by equation (4.5). i.e. $4\varphi_1 - 3\varphi_0 - \varphi_2 = 0$ or

$$\varphi_2 = 4\varphi_1 - 3\varphi_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\varphi_n = V_n) \ldots\ldots (4.6)$$

The second boundary condition is met by obtaining the values of the node voltages V_1 and V_2 for various values of I_1, until the values of the voltages V_1 and V_2 satisfy equation (4.5). When both boundary conditions are satisfied, then the rest of the node voltages are obtained in the usual way. For both values of β ie. 5.3 and 5.5, the third boundary condition is not met, as is shown in Fig.4.2, but the correct value of β

is clearly indicated to have the value in between these two values.

The values of the resistors R_n and R_{f_n} for the value of ρ equal to 5.413, 5.3 and 5.5 are tabulated below:-

n	$\frac{1}{R_n}$	$-\frac{1}{R_{f_n}}$ (ρ= 5.413)	(ρ=5.3)	(ρ=5.5)
1	200	29	27.809	29.95
2	602	56.256	53.93	58.08
3	1000	79.789	76.68	82.58
4	1408	98.45	94.38	101.64
5	1818	109.875	105.34	113.44
6	2222	112.51	107.87	116.16
7	2631	108.10	104.21	112.23
8	3000	84.384	80.9	87.12
9	3887	50.103	48.03	51.72
10	3773	00.00	00.00	00.00

The second eigen value is obtained in the same way as the first eigen-value, except that the solutions display 3/4 of the wave length, as shown in Fig.4.3. The value of ρ which satisfies all three boundary conditions is 13.13. The equation(4.5) is also employed to indicate the zero slope at the point ρ = 0

It has been shown that there are other relationships which may be used to indicate the value of the slope at a point, apart from that of the equation (4.5) (see Appx. IV). The error introduced to the finite-difference solutions by all these relationships varies for each problem. To reduce the error introduced to the finite-difference solutions by all these relationships, the reduction in the value of h (the finite-difference interval), is commonly used. In obtaining the second eigen value it is obvious that more sections will be required for greater accuracy. Since this example is purely a demonstration of applying the graphical method, then the reduction of the value of the finite-difference interval h, or the employment of other relationships to indicate the value of the slope at a point, will not bring an appreciable advantage.

Results

$\beta = 5.413$

ρ	φ exact solutions of differential eqn.	φ computed solutions
0	1	1
0.05	0.98	0.98
0.1	0.93	0.928
0.15	0.85	0.845
0.2	0.74	0.738

0.25	0.62	0.615
0.3	0.48	0.485
0.35	0.35	0.355
0.4	0.22	0.228
0.45	0.11	0.11
0.5	0.00	0.00

$\beta = 13.13$

ρ	φ computed solutions
0	1
0.05	0.898
0.1	0.6
0.15	0.229
0.2	-0.11
0.25	-0.342
0.3	-0.43
0.35	-0.398
0.4	-0.28
0.45	-0.133
0.5	-0.00

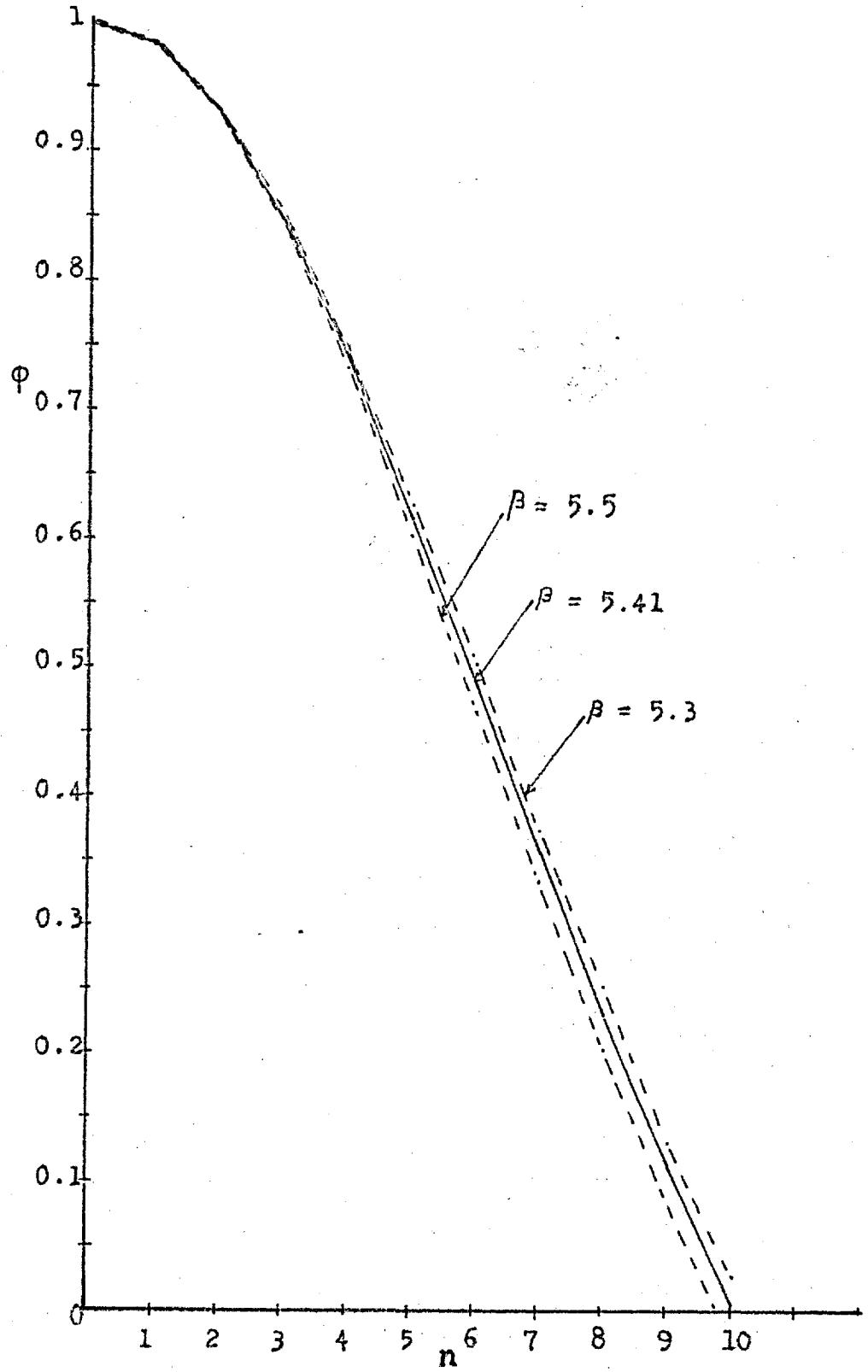


Fig. 4.2

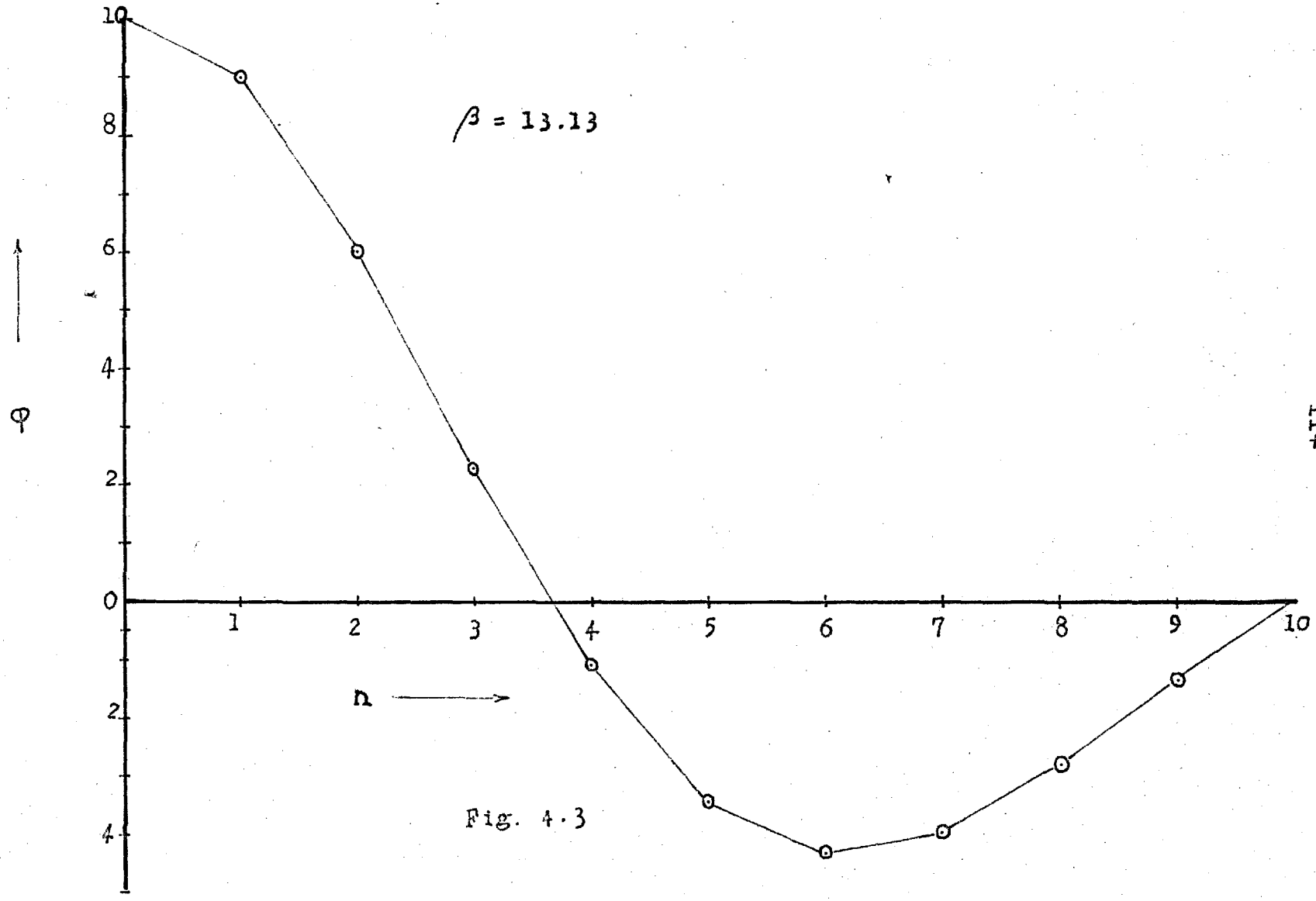


Fig. 4.3

Eg. 2.

The following differential equation is the expression of the propagation of a steady laminar spherical flame. The problem has been investigated by the Mechanical Eng. Dept., Imperial College, by means of a resistance network analogue computor.

The equation is

$$\frac{d^2\tau}{d\xi^2} = -\left(\frac{\ln\xi}{\ln\xi}\right)^4 \cdot \frac{1}{\xi^2}\left(30\ (1-\tau)\tau^4\right)$$

and the boundary conditions are given as follows:-

1) $\tau = 0$ when $\xi = 0$

2) $\tau = 1$ when $\xi = 1$

3) $\dfrac{1}{\left(\frac{d\tau}{d\xi}\right)_{\xi=0}} = 0.32$

It is beyond the scope of this thesis to examine this problem in detail and give all the definitions of all the notations, but nevertheless the above differential equation is of an interesting form, and the given information is sufficient for this equation to be solved by the graphical method, and for the unknown eigen value λ to be obtained.

The V-I characteristic of the shunt resistor R_{f_n} is obtained from the following equation.

$$I_{f_n} = -\frac{h^2}{R}\lambda\left(\frac{\ln\xi}{\ln\xi}\right)^4 \cdot\frac{1}{\xi^2}\left(30(1-\tau_n)\right)\tau_n^4$$

τ_n is equal to the value of the voltage at node n of the resistance network analogue. R = 1Ω, is the value of the series resistors, and h = 0.05, is the value of the finite difference interval. The V-I characteristics of each shunt resistor are not equal and take the form as shown in Fig. 4.4. The value of the current I_{f_n} becomes zero when τ has the value equal to 0 and 1 volt, and has its maximum value when is equal to 0.8 volts.

In order that the results can be compared with the result obtained by the resistance network analogue computor, the same number of finite intervals as that of the computor was employed in the graphical drawing. The first slope of the solution, i.e. $\frac{\tau_1-\tau_0}{h}$ is assumed to have the value equal to $\left(\frac{d\tau}{d\xi}\right)_{\xi=0} = \frac{1}{0.32}$ (This technique is also employed with the resistance network computor).

The unknown eigen value λ obtained by the graphical method is 0.623. This value is the same as that obtained by the resistance network analogue computor.

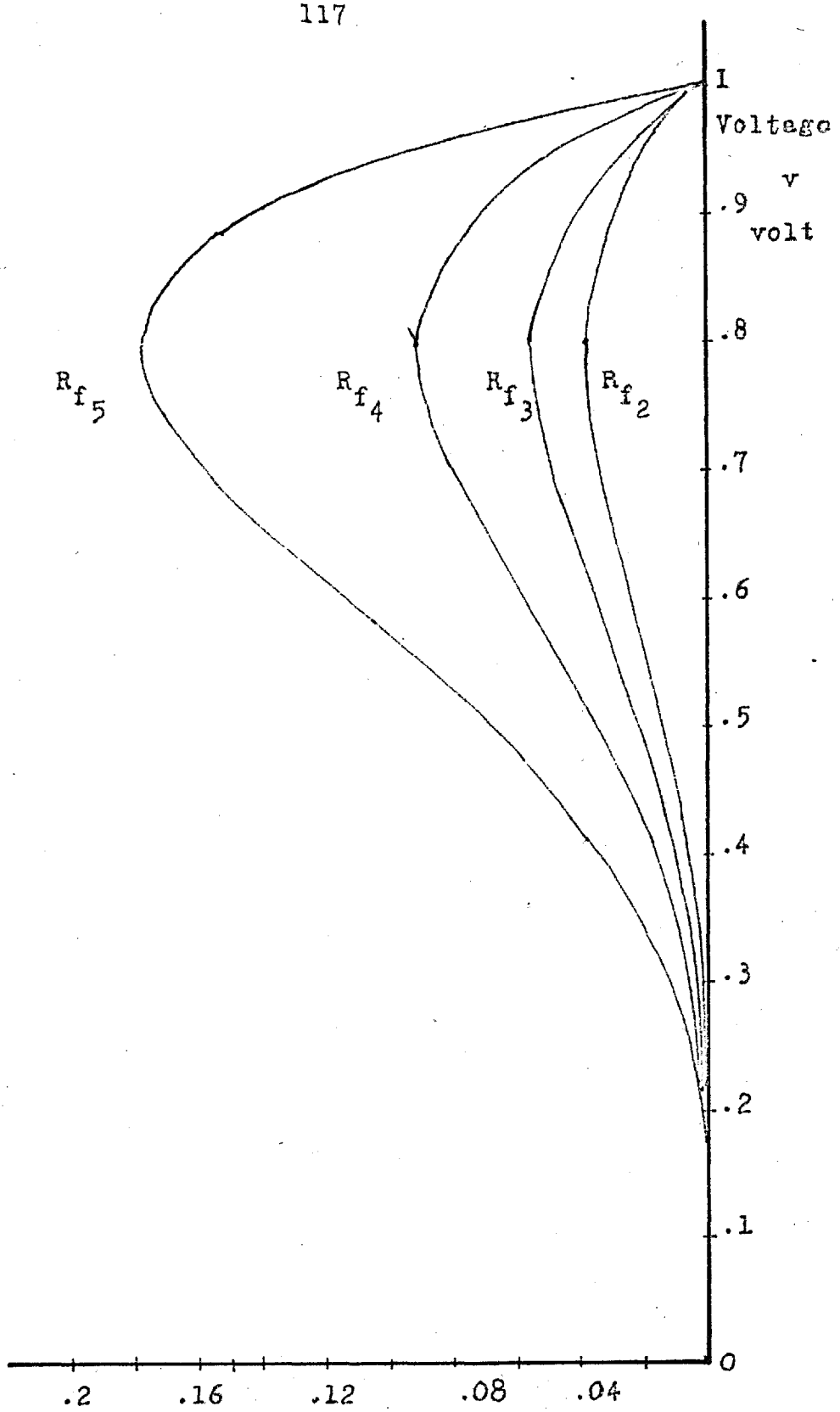

1
Voltage
v
volt
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
R_{f_5}
R_{f_4}
R_{f_3}
R_{f_2}
.2
.16
.12
.08
.04
-ve current I amp.

Fig4.4

CHAPTER V

The Solution of the Biharmonic Equation by the Visual Iterative Method based on Graphs.

The application of the graphical method to solve various second-order differential equations has been described in Chapter III, and in this section the fourth-order biharmonic equation will be brought within the scope of the method. This equation is of condiderable importance in the theory of deflection of beams on elastic foundations (linear or non-linear elastic foundations), and in cases of the vibration of beams. In the latter cases the solution of the eigen-value problem is important.

Many physical problems can be expressed in terms of a fourth-order differential equation $\nabla^4\psi = f(\psi,x,K)$ when $\nabla^4 = \nabla^2(\nabla^2) = \left(\frac{d^4}{dx^4}\right)$ and K is a constant which may be known (in deflection of beam problems) or unknown (in vibration problems, i.e. eigen-value problems).

Many authors have shown that it is possible to use two similar resistance networks in cascade to solve the fourth-order differential equation $\frac{d^4\psi}{dx^4} = f(\psi,x,K)$ Their arrangements are very similar in form as shown in Fig. 5.

The two resistance networks R and R' are coupled together at corresponding nodes by a high value resistor

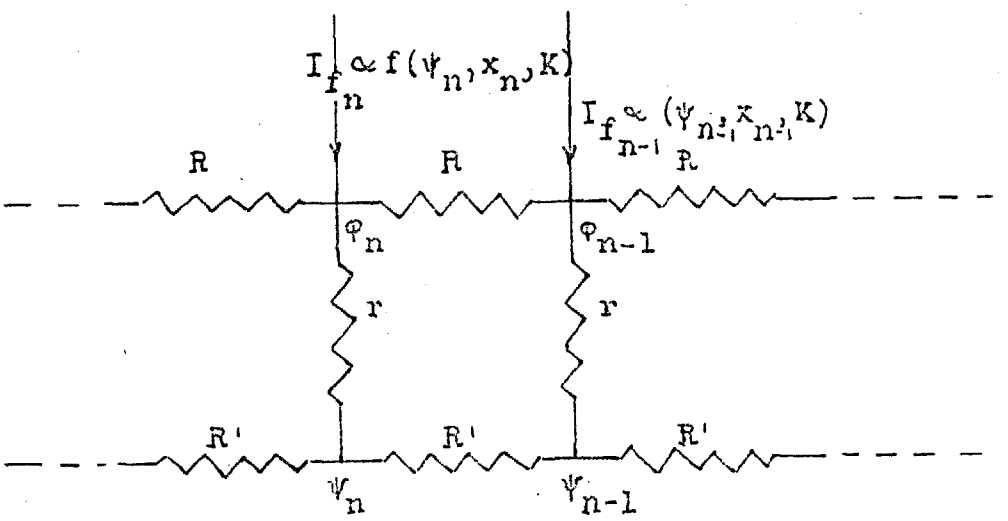


Fig.5. Two resistance networks coupled together

'r'. Currents proportional to the value of $f(\psi,x,K)$ are applied to the appropriate nodes of the R-networks. When the boundary conditions of both networks are adjusted to the specified values, the numerical value of the voltages at each node of the R'-network will be approximately equal to the numerical value of the finite-difference solution ψ of the equation. The corresponding values of the node voltages of the R-network are proportional to the numerical values of $\frac{d^2\psi}{dx^2}$ and referred to as φ.

The accuracy of the solution of this network depends on the ratio of r to R, and r to R', G.Liebmann[3] has shown that to obtain solutions accurate to within 1% of the true finite-difference solutions, r must be at least ten times greater than the value of R and R', and thus it follows that $\varphi \gg \psi$.

5.1 The Solution of Fourth-order differential equations by applying the graphical method to two resistance networks coupled together

Graphical methods can be applied to solve these networks in a similar way to those previously described, (Chapter III). The graphical representations of the relationships between voltages and currents of R-network and R'-network, are separately drawn. The networks and

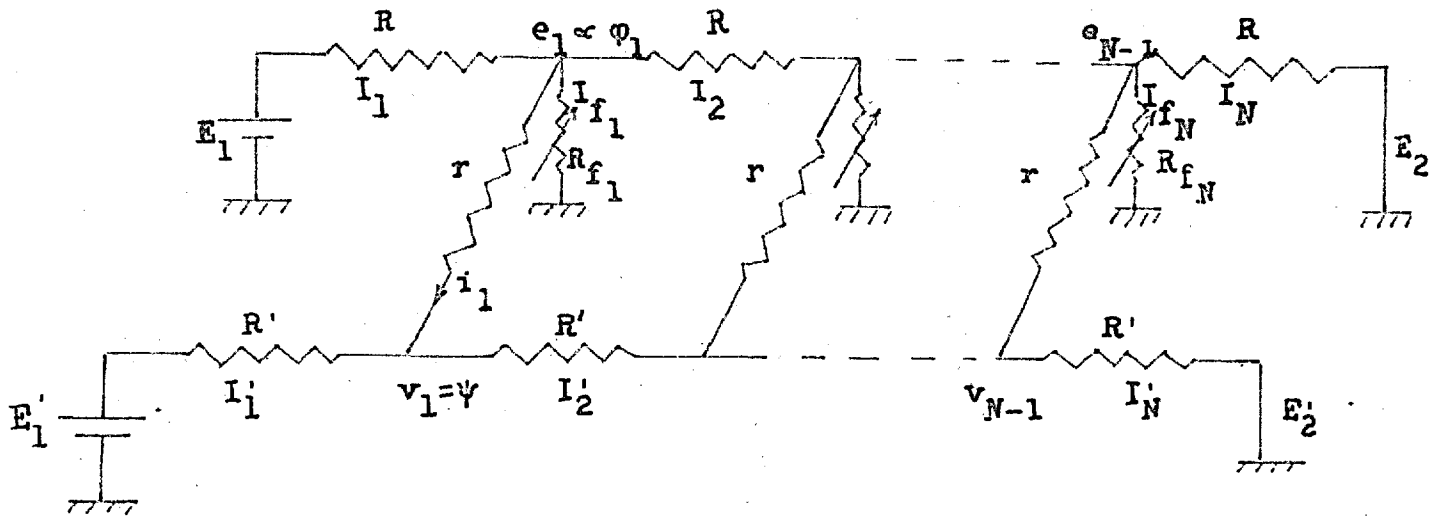
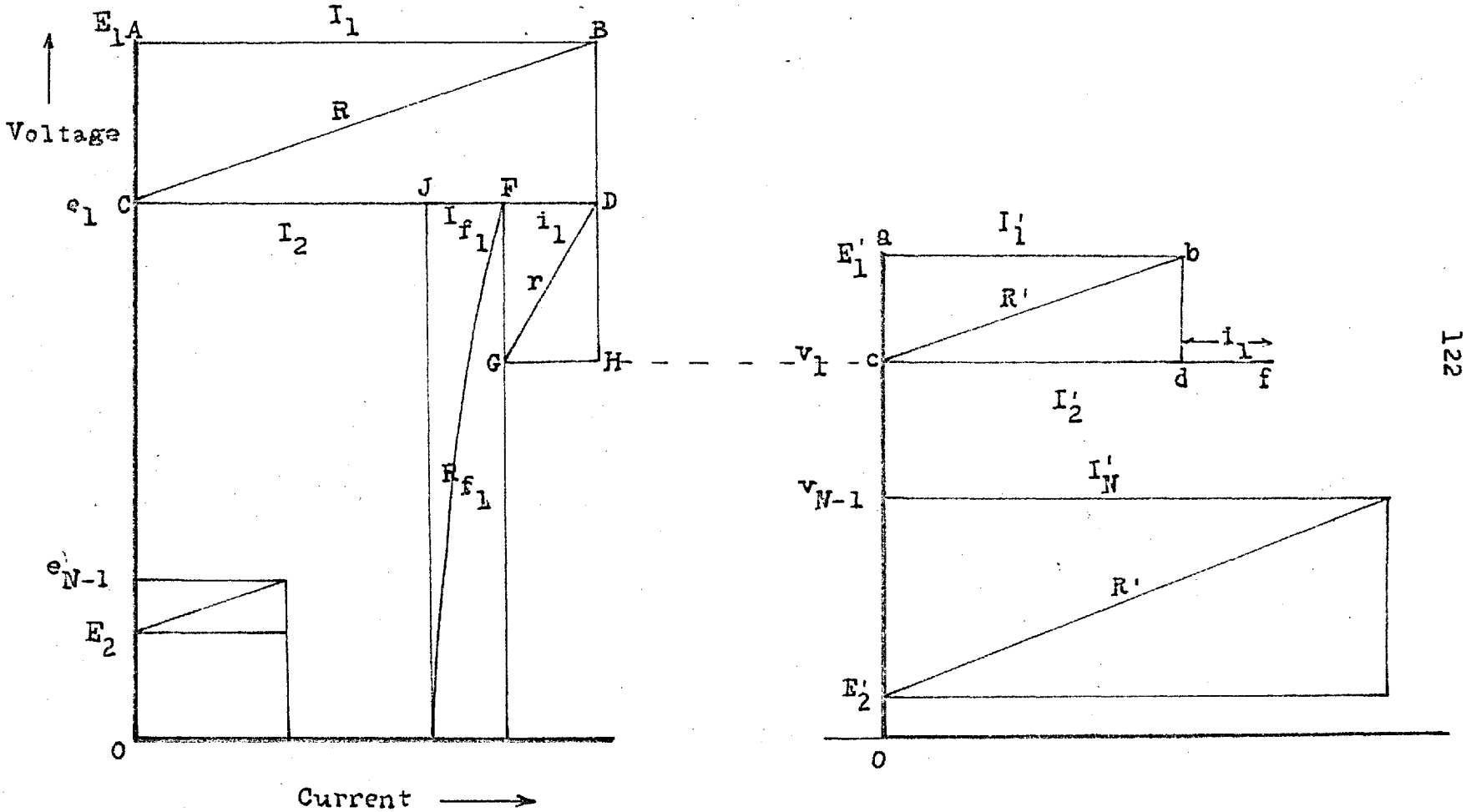

R
$e_1 \propto \varphi_1$
R
e_{N-1}
R
I_1
I_{f_1}
I_2
I_{f_N}
I_N
E_1
R_{f_1}
R_{f_N}
E_2
r
r
r
i_1
R'
R'
R'
E'_1
I'_1
$v_1=\psi$
I'_2
v_{N-1}
I'_N
E'_2

Fig. 5.1



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Fig. 5.2 Rectangle diagrams for network of fig. 5.1

graphical representations are shown together in Figs 5.1 and 5.2.

The value of the currents I_1 and I_1' are first estimated, E_1 and E_1' being the value of the left-hand boundary conditions. The vertical lines OA and oa represent the value of the voltages E_1 and E_1' and the horizontal lines AB and ab represent the estimated value of the currents I_1 and I_1' respectively.

The V-I characteristic of the resistor R is drawn, starting at C from the vertical line OA and passing through the point B. OC will then be the value of the voltage e_1 of the R-network. In a similar way the value of the voltage v_1 of the R'-network is obtained, and is represented by the line oc. Both networks are joined at this point by a resistor r, and therefore there must be a current flowing from one network into the other network. The value of this current is determined by the value of the voltage e_1 and v_1 and the value of the resistor r. Since in this case e_1 is greater than v_1, then the current i_1 will flow from the R-network into the R'-network. i.e. i_1 will be subtracted from I_1 and added to I_1'. The current i_1 is found by drawing the V-I characteristic of the resistance r from the level equal to v_1 passing through point D. DF or GH will then represent the value of

i_1. i_1 is added to the R'-network by extending cd to f. At the node e_1 the current I_{f_1} will be fed or drawn out of this node according to the sign of the function $f(\psi,x,K)$ of this node, and may be represented by a resistor R_{f_1}. (It can either have positive or negative value, and can also have a non-linear characteristic). The resistor R_{f_1} will be connected from this node to earth. Supposing it has a non-linear positive value, then the current will flow from e_1 to earth, and will be subtracted from I_1. This is done by drawing the V-I characteristic of the resistor R_{f_1} starting from earth potential and passing through point F. JF will then represent I_{f_1}. We are now left with the currents I_2 and I_2', for the R and R' networks respectively. These currents will flow into the adjoining resistors. The process is then repeated starting with voltages e_1 and v_1 and the currents I_2 and I_2', and is continued until the last resistor of the network R and R' is drawn, which will give the values of E_2 and E_2'. The values of E_2 and E_2' are not likely to be the same as the boundary values prescribed for the equation, but the values of E_2 and E_2' can be compared with the given boundary conditions and the results used to determine the new values for I_1 and I_1', so that after several attempts the required values of E_2 and E'_2 will

be obtained. Fig.5.3a shows a complete V-I drawing for the network shown in Fig.5.3.

It can be seen from Fig.5.3a that if the value of r is very large, the values of the voltages and currents of the R-network will be very much larger than those of the R'-network. If the drawing of the R'-network is very small, as is always the case, then the error in drawing will be appreciably high. This must be avoided by having a very large drawing for the R-network, so that the drawing of the R'-network remains a reasonable size. Alternatively different scales could be used for each network but this is not a practical proposition, as changing the scale can cause some confusion when both drawings are done simultaneously.

5.2 The Solution of Fourth-order differential equations by applying the graphical method to two separate resistance networks.

The fourth order differential equation $\nabla^4\psi = f(\psi,x,K)$, may be written as $\nabla^2(\nabla^2\psi)=f$, and if φ is equal to $\nabla^2\psi$, then $\nabla^2\varphi=f(\psi,x,K)$ or $\frac{d^2\varphi}{dx^2} = f(\psi.x,K)$. By using a network of resistances, R, and applying the currents $I_f = \frac{h^2}{R}f(\psi,x,K)$ into the appropriate nodes, the value of the voltage at each node will be equal to the value of the finite-dif-

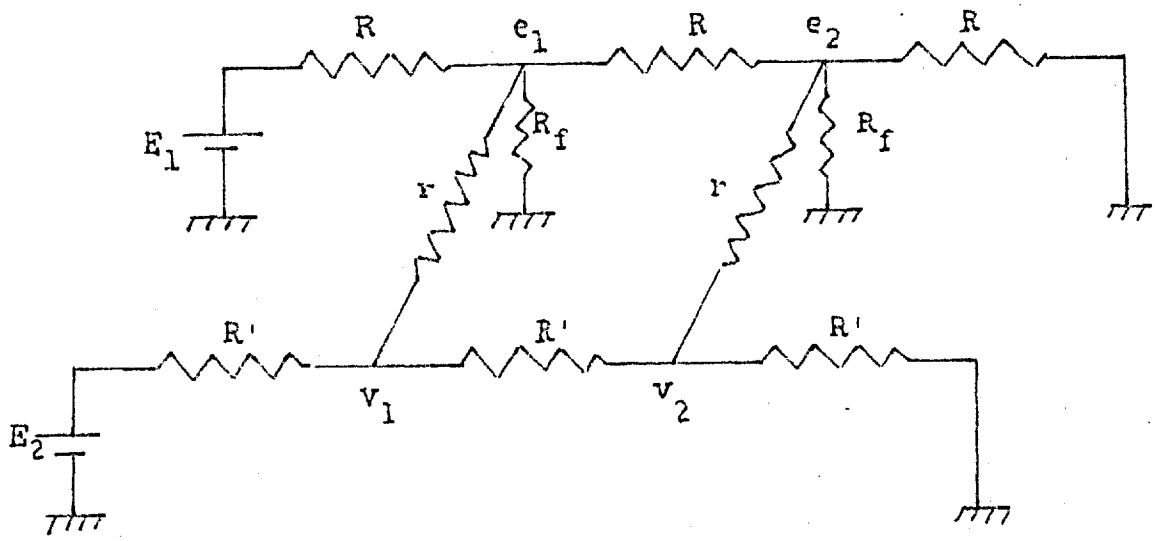


Fig. 5.3

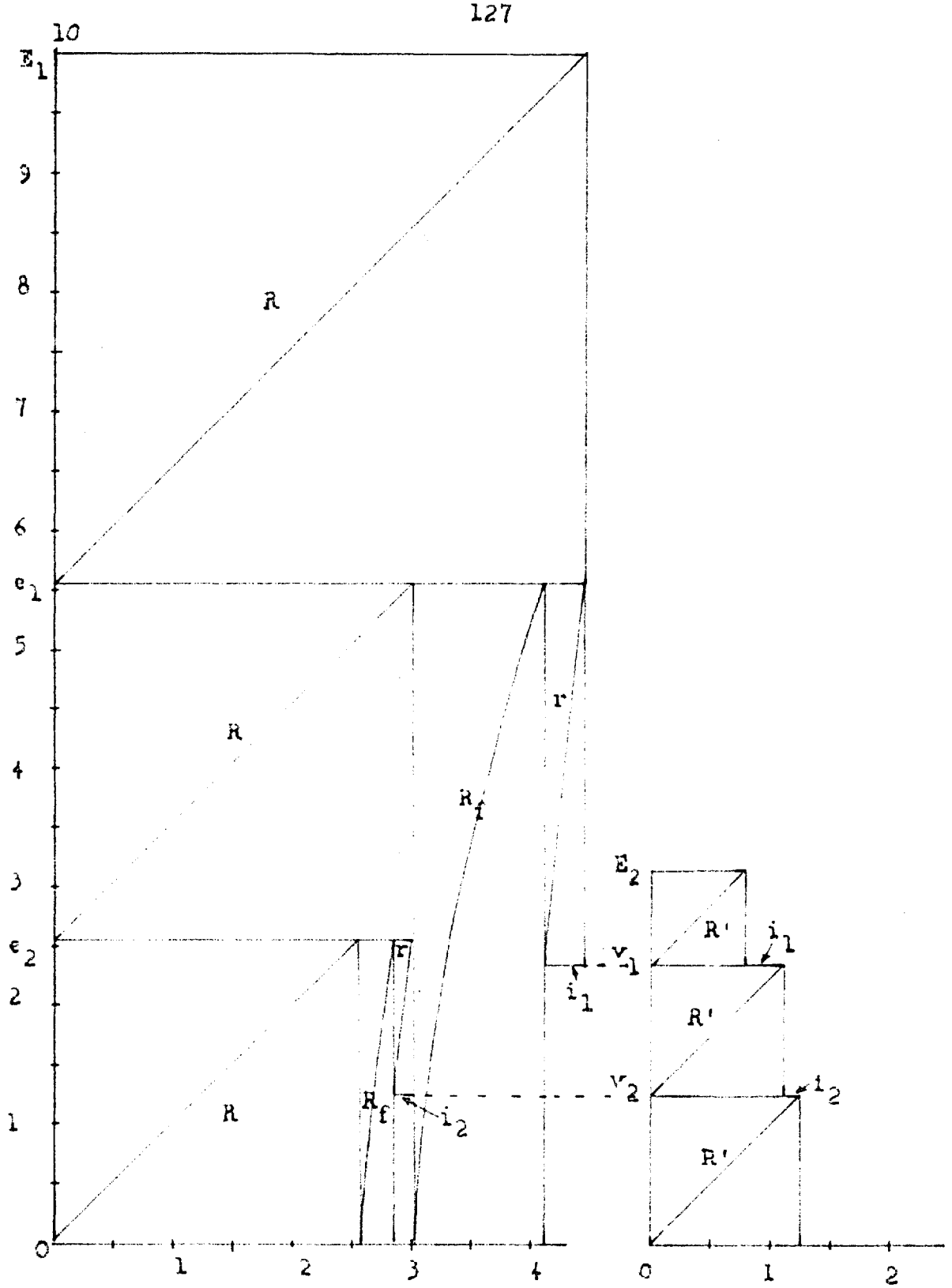

10
E1
9
8
R
7
6
e1
5
R
r
4
Rf
E2
3
R'
i1
e2
r
V1
2
i1
R'
1
R
Rf
i2
V2
i2
R'
0
1
2
3
4
0
1
2

Fig. 5.3a

ference solution φ at the corresponding distance.i.e. $\varphi_0, \varphi_1, \varphi_2 \ldots\ldots \varphi_N$. If another network of resistances R' is used, and has the same number of nodes, as the R-network and if currents $I'_f = \frac{h^2}{R'}\varphi_0, \frac{h^2}{R'}\varphi_1 \ldots\ldots \frac{h^2}{R'}\varphi_N$ are applied to the corresponding nodes, then the voltage at each node will have a value equal to the finite-difference solutions ψ, since $\frac{d^2\psi}{dx^2} = \varphi$.

By using two separate networks, the disadvantage mentioned above of the node voltages of one network being much greater than those of the other network, is overcome, unless the solution φ is greater than the solution ψ in the first case. In such cases the scaling method must be introduced. The resistors 'r' are no longer required and therefore more exact solutions are possible.

5.3 <u>The Classification of the Equations</u>

The fourth-order differential equation can be classified into three group as follows:-

1) $\frac{d^4\psi}{dx^4} = f(x,K)$ where $f(x,K)$ varies with the value of x only.

2) $\frac{d^4\psi}{dx^4} = f(\psi,K)$ when $f(\psi,K)$ varies with the value of ψ only.

3) $\frac{d^4\psi}{dx^4} = f(\psi,x,K)$

In all these cases K is a known constant. For the problem

where K is an unknown constant, there is a different approach to the problem which will be discussed later.

Solution of $\frac{d^4\psi}{dx^4} = f(x,K)$

This type of fourth-order differential equation is very common in the problem of beam bending with a uniform or non-uniform distributed load, or with a concentrated load at a known distance. The beam itself may have a non-uniform cross-section which is a function of distance.

To obtain the solution of this class of equation, the values of φ or $\frac{d^2\psi}{dx^2}$ are first obtained. The current I_{f_n} applied to the node n is equal to $\frac{h^2}{R}f(x,K)$. Since the values of x and K are independent of the value of φ or ψ, then R_{f_n} (Fig.5.4) will have the characteristic of a current generator. The graphical drawing of the V-I relationship of the network is as shown in Fig 5.4b.

OA represents the voltage of the boundary condition when x=0. AB or CD represent I_1, which is an estimated current. The V-I characteristic of R is drawn to obtain the value of the voltage of node 1(OC). I_{f_1} is added or subtracted from I_1 and is represented by the line DF. CF will then be the value of the current I_2 flowing into the next resistor R. The process is repeated until the last resistor of the network is drawn which will give the

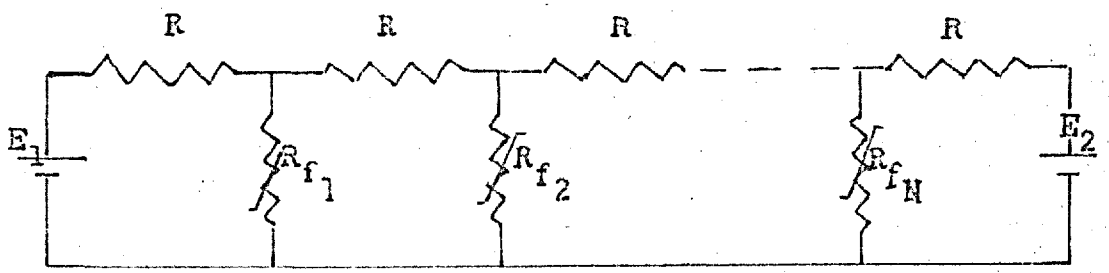


Fig. 5.4

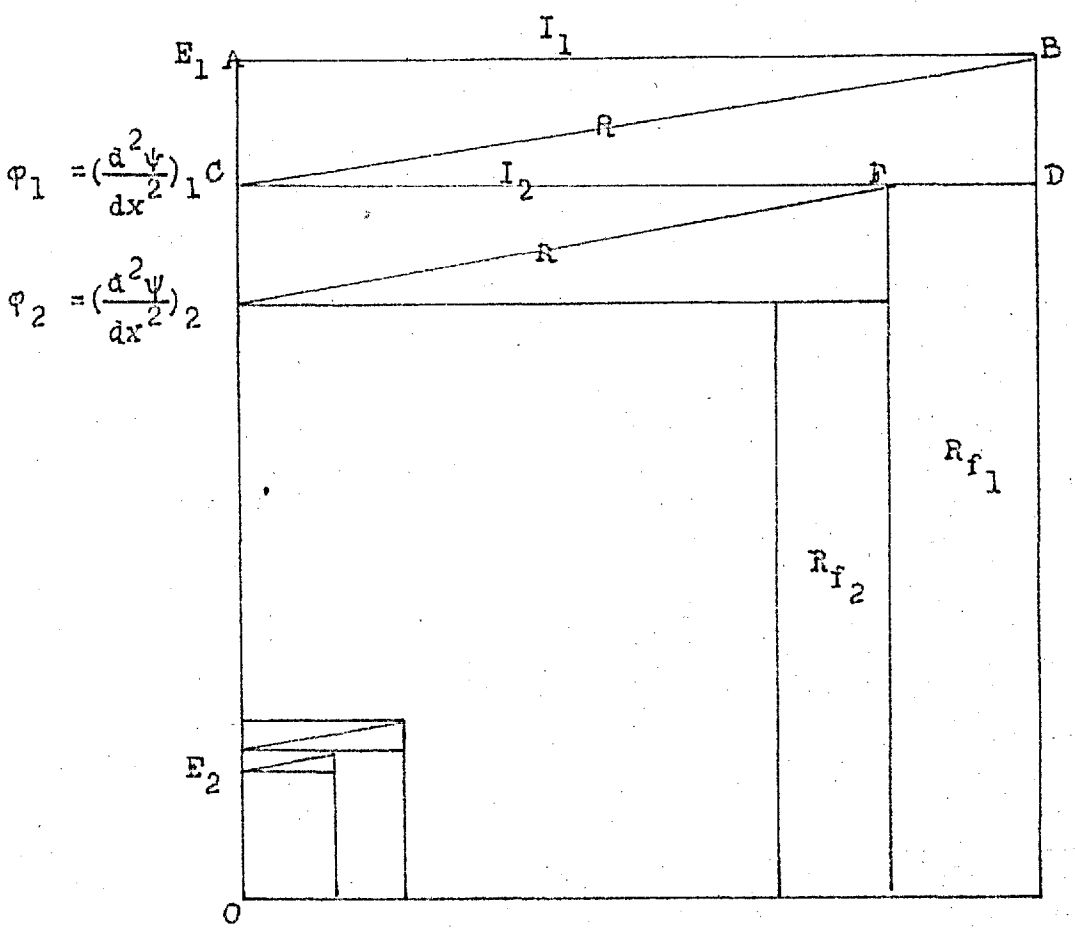


Fig. 5.4b

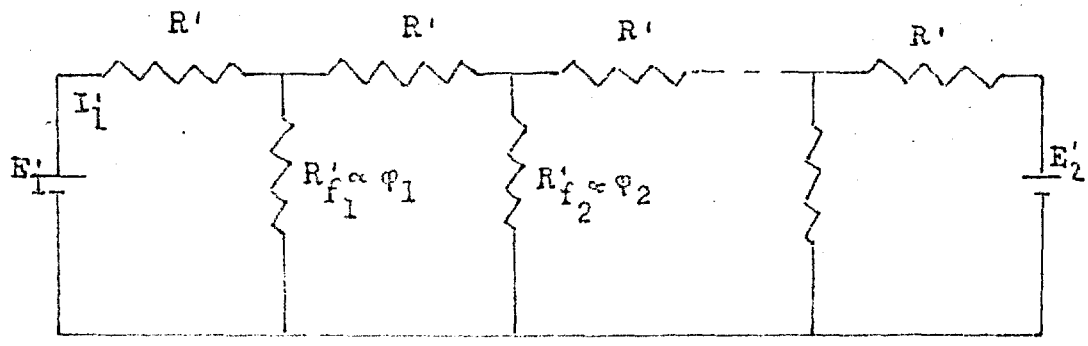


Fig. 5. 5

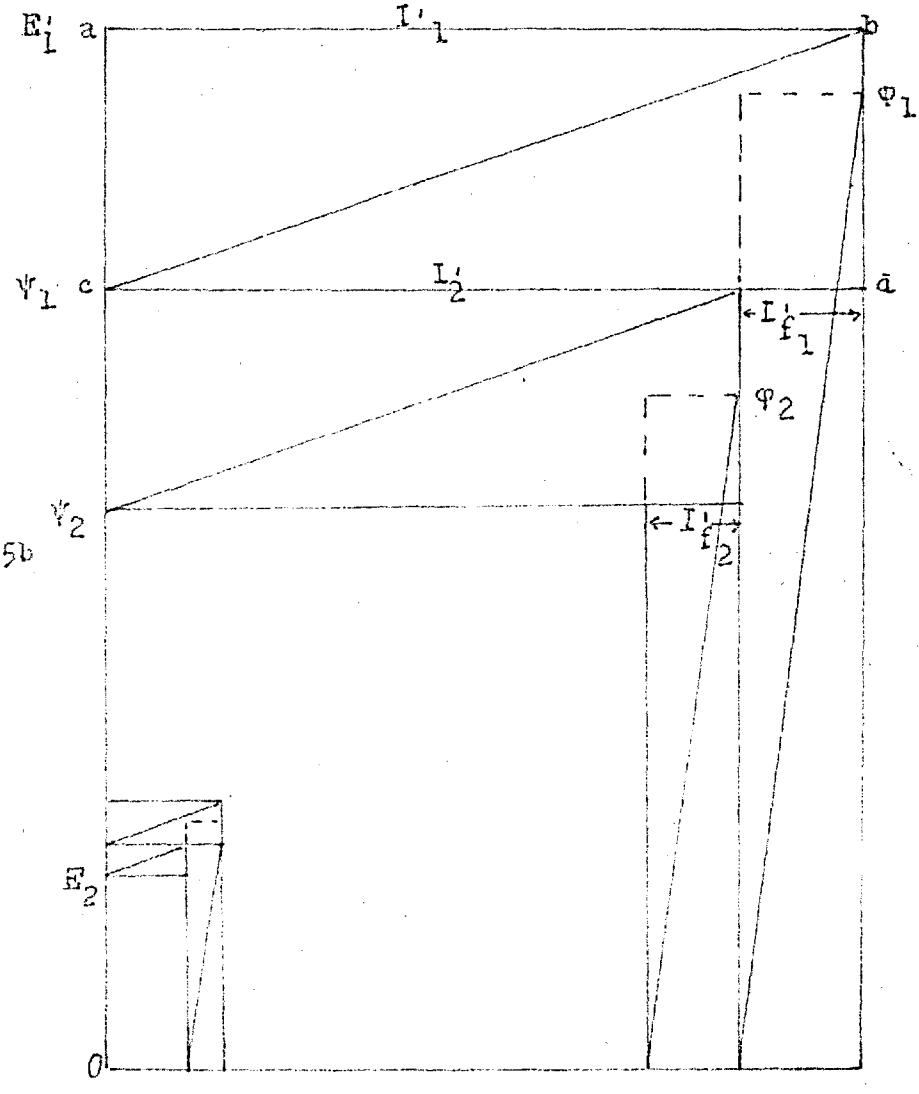


Fig. 5. 5b

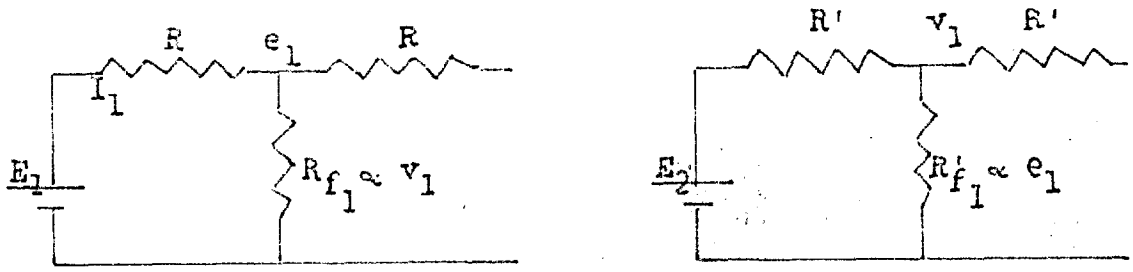
value of the voltage E_2. E_2 is unlikely to have the same value as the boundary condition of this node. By succesive comparisons of the value of E_2 with the value of the given boundary condition, however, a new value of I_1 will eventually be produced which will make the new value of E_2 equal to the value of the boundary condition. Thus it follows that the node voltages are the solution φ (or $\frac{d^2\psi}{dx^2}$). In the problem of beam bending, the values of φ are proportional to the values of the bending moment along the beam, and therefore the values of the boundary conditions are proportional to the values of the bending moment at both ends of the beam.

The second V-I relationship of the R'-network is drawn in a similar way. The current I'_f applied to each node is equal to $\frac{h^2}{R'}.\varphi_n$ or $\frac{h^2}{R'}.(\frac{d^2\psi}{dx^2})_n$. When all the boundary conditions are satisfied, the node voltages will then equal the numerical value of the solution ψ.Fig.5.5 and 5.5b. This process is very straight forward as the solution of the second network in no way modifies the solutions of the first one. The values of ψ in beam problem correspond to the values of the deflection along the beam, and the values of the boundary conditions are the values of deflection at both ends.

Solution of $\frac{d^4\psi}{dx^4} = f(\psi, K)$

This equation can be applied to the problem concerning vibration of beam, or any problem which may be expressed as $\frac{d^4\psi}{dx^4} = K(a\psi^N + b\psi^{N-1} \ldots\ldots \psi)$. To obtain the solution ψ, voltage and current relationships of both resistance networks, R and R', are drawn simultaneously. (Fig.5.6) The currents I_1, and I_1' are first estimated, and the value of the voltages at the first node of both networks is obtained in the usual way. The current I_{f_1} applied by R_{f_1} to the R-network is $\frac{h^2}{R} f(\psi, K)$. Since this current is a function of ψ, therefore it is also a function of the node voltage of the R'-network. Hence in this case the resistance R_{f_n} of the R-network has a V-I characteristic depending on the value of the node voltage of the R'-network and independent of the node voltage of the R-network. Similarly the resistance R'_{f_1} has a V-I characteristic depending on the node voltage of the R-network.

To obtain the current I_{f_1} the V-I characteristic of the resistor R_{f_1} is drawn on the diagram of the R-network, starting from earth potential and passing through the point D'. OD' is equal to oc, and the value of I_{f_1} is represented by the line FD. Similarly


R
e1
R
I1
E1
Rf1 ∝ v1
R'
v1
R'
E2
R'f1 ∝ e1

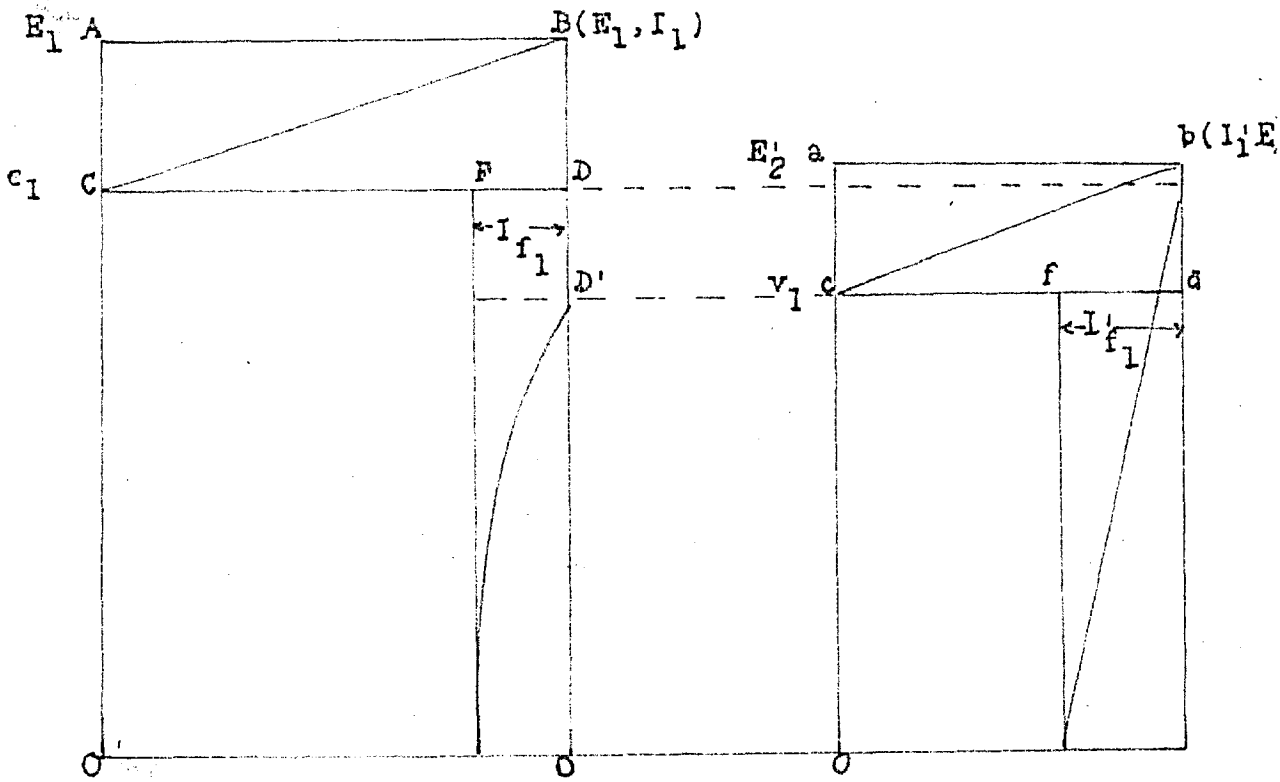

E1
A
B(E1, I1)
c1
C
F
D
-If1
D'
E'2
a
b(I1'E
v1
c
f
d
-I'f1
O
O
O

Fig. 5.6

I'_{f_i} is found on the drawing of the R'-network, and is represented by the line fd.

Solution of $\frac{d^4\psi}{dx^4} = f(\psi,x,K)$

This equation appears in most of the problems of beams resting on elastic foundations, such as a freely supported beam with a linear, non-linear or concentrated load, resting on a linear or non-linear spring. It also applies to the problem of a sheet pile wall with non-linear active and passive load on both sides.

This equation may be divided into three classes.

a) when the function $f(\psi,x,K)$ takes the form $f(\psi,x,K) = K\{G(x)+g(\psi)\}$, $G(x)$ is a function of x alone and $g(\psi)$ is a function of ψ alone.

b) when the function $f(\psi,x,K)$ takes the form $f(\psi,x,K) = K\{G(x).g(\psi)\}$

c) when the function $f(\psi,x,K)$ takes any other form which is not in either class a) or b).

In class a), the current I_{f_n} applied to each node of the R-network is equal to $\frac{h^2}{R}KG(x)+\frac{h^2}{R}Kg(\psi)$ Therefore the resistances R_{f_n} may be replaced by two resistances in parallel. One will have a V-I characteristic depending on the $\frac{h^2}{R}KG(x)$ term, and the other will depend on $\frac{h^2}{R}Kg(\psi)$. Since the value

of $\frac{h^2}{R}Kg(\psi)$ term depends on the value of ψ i.e. the voltage of the R'-network, then the V-I characteristic of this resistance is the same in all nodes, i.e. only one template is required. The value $\frac{h^2}{R}KG(x)$ depends on the value of x, i.e. the position of the node, and therefore each node will require a different constant current. This value can easily be calculated beforehand.

In drawing the V-I relationships of all the resistances of the R and R' network, the procedure is very similar to that in 2) section 5.3 except that on the drawing of the R-network the extra current in terms of $\frac{h^2}{R}KG(x)$ is added to it.Fig.5.7.

Class b) In this class the value of the currents I_{f_n} is equal to $\frac{h^2}{R}K\{G(x)\}.\{g(\psi)\}$,and therefore R_{f_n} will have the V-I characteristic depending of the value of ψ and x. To obtain the value of the current I_{f_n} the value of $\frac{h^2}{R}Kg(\psi)$ is first found by using a template, and then multiplying by the precalculated value G(nh). I_{f_n} can also be obtained by using a template which is a function of $\frac{h^2}{R}KG(nh).g(\psi)$ for each node.

Class c) From both class a) and b)it is obvious that if the current I_{f_n} has the value composed of the

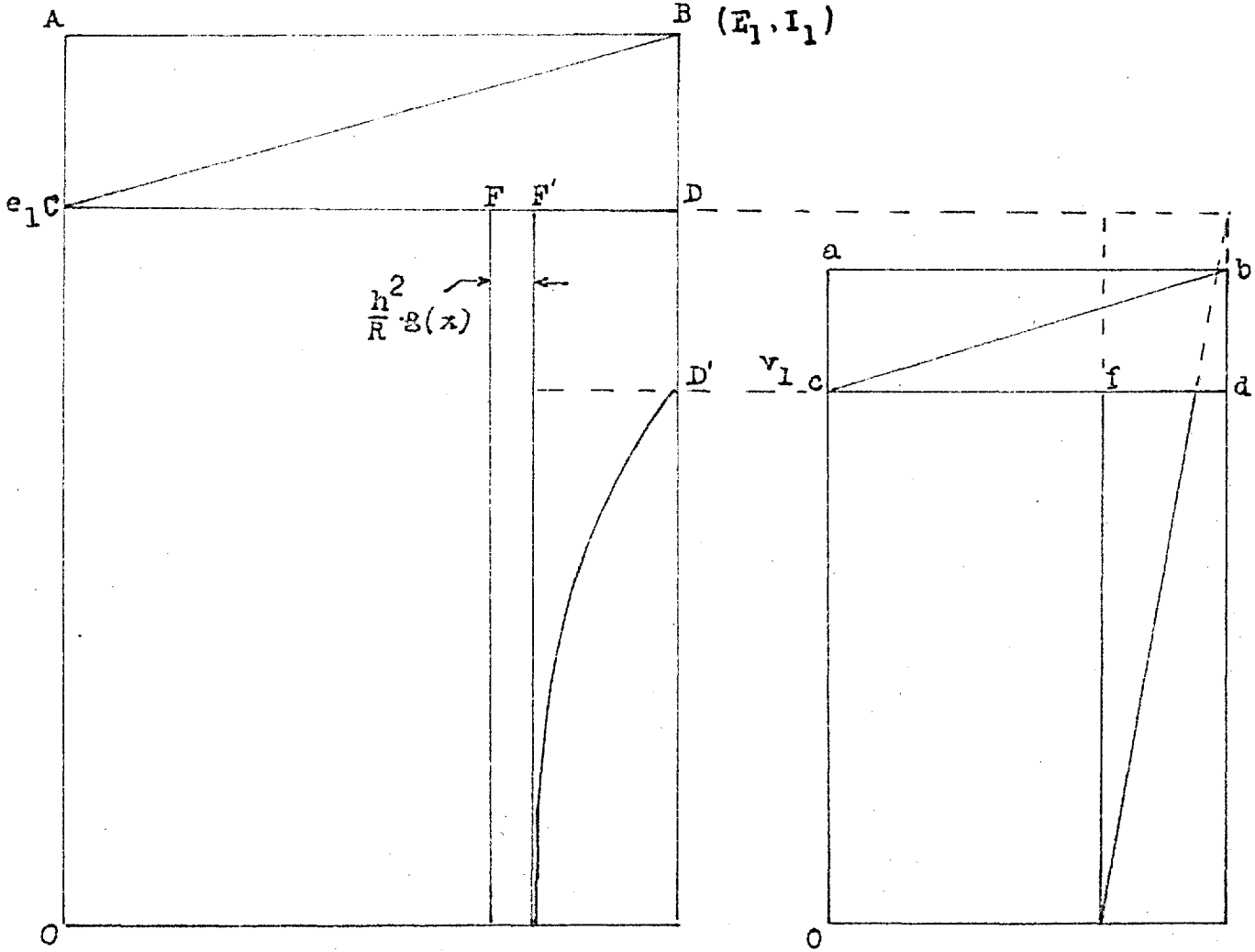


Fig. 5.7

terms mentioned in class a) and b), several templates have to be made for each node, and careful planning will help to minimise the number of templates.

5.4 The Boundary Conditions.

If the boundary values at both ends of the system are known for both networks i.e. for the network for the equation $\frac{d^2\varphi}{dx^2} = f(\psi,x,K)$, and for the network for equation $\frac{d^2\psi}{dx^2} = \varphi$, then the method of solving the equation is straight forward. In some cases the boundary conditions are given as the slope of the solution at one end or the other i.e. φ_O and $(\frac{d\varphi}{dx})_N$ or $(\frac{d\varphi}{dx})_O$ and φ_N, and similarly for the ψ-network. If the slope $\frac{d\varphi}{dx}$ (or $\frac{d\psi}{dx}$) is specified at a boundary then the problem remains as to how this can be set up. It so happens that the slopes can be expressed in terms of φ(or ψ), at points near the boundary, see appendix IV. But a new relation which is found to be more suitable to this graphical method will be derived:-

When solving fourth-order differential equations by employing two similar resistance networks, it is obvious that both networks are doing exactly the same thing, i.e. producing the solution of a second-order differential equation. Therefore both networks can

be dealt with by the same method, in which φ and ψ can both be represented by 'y'. Both equations will take the form

$$\frac{d^2y}{dx^2} = f(y,x,K)$$

It is required to find what relation $(\frac{dy}{dx})_n$ has with the neighboring value of y separated by the distance h as shown in Fig.5.8a and 5.8b. n denotes the node which the value of the slope is precribed.

For n = 0 Fig. 5.8a

By Taylor's expansion the following is obtained :-

$$y_2 = y_1 + h(\frac{dy}{dx})_1 + \frac{h^2}{2}(\frac{d^2y}{dx^2})_1 + \frac{h^3}{6}(\frac{d^3y}{dx^3})_1 + \frac{h^4}{24}(\frac{d^4y}{dx^4})_1 + \ldots$$

$$y_0 = y_1 - h(\frac{dy}{dx})_1 + \frac{h^2}{2}(\frac{d^2y}{dx^2})_1 - \frac{h^3}{6}(\frac{d^3y}{dx^3})_1 + \frac{h^4}{24}(\frac{d^4y}{dx^4})_1 + \ldots$$

$$y_1 + y_0 = 2y_1 + h^2(\frac{d^2y}{dx^2})_1 + \frac{h^4}{12}(\frac{d^4y}{dx^4}) + \ldots\ldots\ldots(5.1)$$

if h is small enough then the terms omitted will be very small, but in most cases the terms $\frac{d^4y}{dx^4}$, $\frac{d^5y}{dx^5}$ etc., are very small or equal to zero.

Similar to above y_1 expanded in terms of y_0 gives:-

$$y_1 = y_0 + h(\frac{dy}{dx})_0 + \frac{h^2}{2}(\frac{d^2y}{dx^2})_0 + \frac{h^3}{6}(\frac{d^3y}{dx^3})_0 + \frac{h^4}{24}(\frac{d^4y}{dx^4})_0 \ldots(5.2)$$

Substituting y_0 (5.2) in (5.1) gives

$$(\frac{dy}{dx})_0 = \frac{1}{h}\Big(y_1 - y_2 + \frac{h^2}{2}(\frac{d^2y}{dx^2})_0 + h^2(\frac{d^2y}{dx^2})_1 + \frac{h^3}{6}(\frac{d^3y}{dx^3})_0 + \frac{h^4}{24}(\frac{d^4y}{dx^4})_0 + \frac{h^4}{12}(\frac{d^4y}{dx^4})_1\Big) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.3)$$

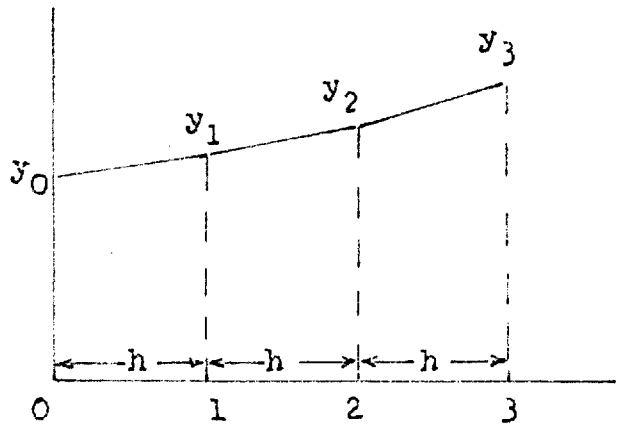


Fig. 5.8a

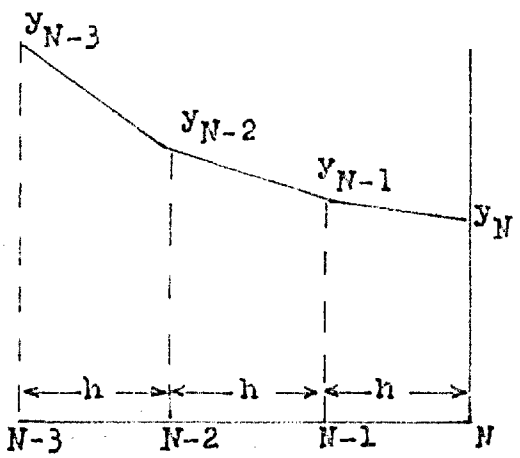


Fig. 5.8b

Similarly for n = N Fig.5.8b

$$\left(\frac{dy}{dx}\right)_N = \frac{1}{h}\Big(y_{N-1} - y_{N-2} + \frac{h^2}{2}\left(\frac{d^2y}{dx^2}\right)_N + h^2\left(\frac{d^2y}{dx^2}\right)_{N-1} - \frac{h^3}{6}\left(\frac{d^3y}{dx^3}\right)_N$$

$$+ \frac{h^4}{24}\left(\frac{d^4y}{dx^4}\right)_N + \frac{h^4}{12}\left(\frac{d^4y}{dx^4}\right)_{N-1} \cdots\cdots\cdots\cdots\cdots\cdots (5.4)$$

Since a second-order differential equation is to be solved, the known equation of $\left(\frac{d^2y}{dx^2}\right)$ will be furnished, and thus the value of $\left(\frac{d^2y}{dx^2}\right)_n, \left(\frac{d^2y}{dx^2}\right)_{n+1}, \left(\frac{d^2y}{dx^2}\right)_{n-1}$--- may be obtained from the solution y_n, y_{n+1} y_{n-1}. In some cases however, they are independent of the solutions and may be obtained before-hand. The equation can also be differentiated to find the value of $\left(\frac{d^3y}{dx^3}\right)$ and $\left(\frac{d^4y}{dx^4}\right)$ and so on.

W.E. Milne[33] has shown also that the slope $\left(\frac{dy}{dx}\right)$ can be expressed in terms of the solution y_n, y_{n+1}, y_{n+2}, ---- which is shown in appendix IV.

The relations shown in equation (5.3) and (5.4) are most suitable to be applied to the ladder network, because only the first two values of the solutions may be required to determine whether the boundary condition is satisfied before the rest of the solutions are computed.

5.5 Example.

In many non-linear cases, the exact finite-difference solutions cannot be obtained by the iterative methods

of other writers for these fail to converge. Therefore to illustrate the new methods, linear examples are taken, since only these can be solved by the numerical method. There would be no difficulty in providing non-linear examples, but there are no alternative solutions with which to compare them. (It is not the non-linearity which gives trouble, but wave-like properties)

Eg. 1

This example illustrates the equation which exhibits wave-like solutions, and in which the values of the boundary condition are known. The equation is $\frac{d^4\psi}{dx^4} = \psi$. The boundary condition is when x=0, ψ= -10 and $\frac{d^2\psi}{dx^2} = 10$, and when $x = \frac{3\pi}{2}$, $\frac{d^2\psi}{dx^2}$ and ψ are equal to zero. The current applied to the R-network is $I_{f_n} = \frac{h^2}{R} \psi_n$ and that to the R'-network is $I_{f'_n} = \frac{h^2}{R'} (\frac{d^2\psi}{dx^2})_n$ If the values of the currents I_{f_n} and I'_{f_n} are negative, the current will be fed into the node, and if the values are positive the current will be drawn out. The solution of the above equation is $\frac{d^2\psi}{dx^2} = 10\cos x$ and ψ= -10cos x. The value of R and R' is chosen to be 1Ω. h is equal to $\frac{\pi}{4}$. The graphical drawing is shown in Fig. 5.9a,b,&c and drawing procedure is as follows:-

1) Draw the V-I characteristics of the shunt resistors R_{f_n} and R'_{f_n} for both R and R'-networks.

In this case they both have the value 1.621Ω, ie. $\frac{R}{h^2}\Omega$ Fig.5.9a.

2) For R-network, the value of the boundary condition when x = 0 is 10 volts, and therefore on the vertical axis of Fig.5.9b the value of the boundary condition is marked at A. The next step is to estimate the value of current I_1 which will give the starting point B. From B the V-I characteristic of the resistance R is drawn, having its origin at point C on the vertical axis. Complete the rectangle ABCD.

3) Repeat the process for R' network, on Fig.5.9c and complete the rectangle abcd.

4) Obtain the current I_{f_1} from the V-I characteristic of R_{f_n}. The voltage corresponding to this current is oc on Fig.5.9c which thus gives a negative current. This current is therefore added to the current I_1 at D to E, and the next rectangle CEFG is drawn.

5) Repeat (4) for the current I'_{f_1} which has a positive value, and therefore is drawn out of the node 1 of the R-network ie. extend cd to e. (oc is negative). The rectangle cefg is drawn using the V-I characteristic of R' to obtain point g.

6) Repeat (2) to (5) until the last resistors of both R and R' networks are drawn which will give

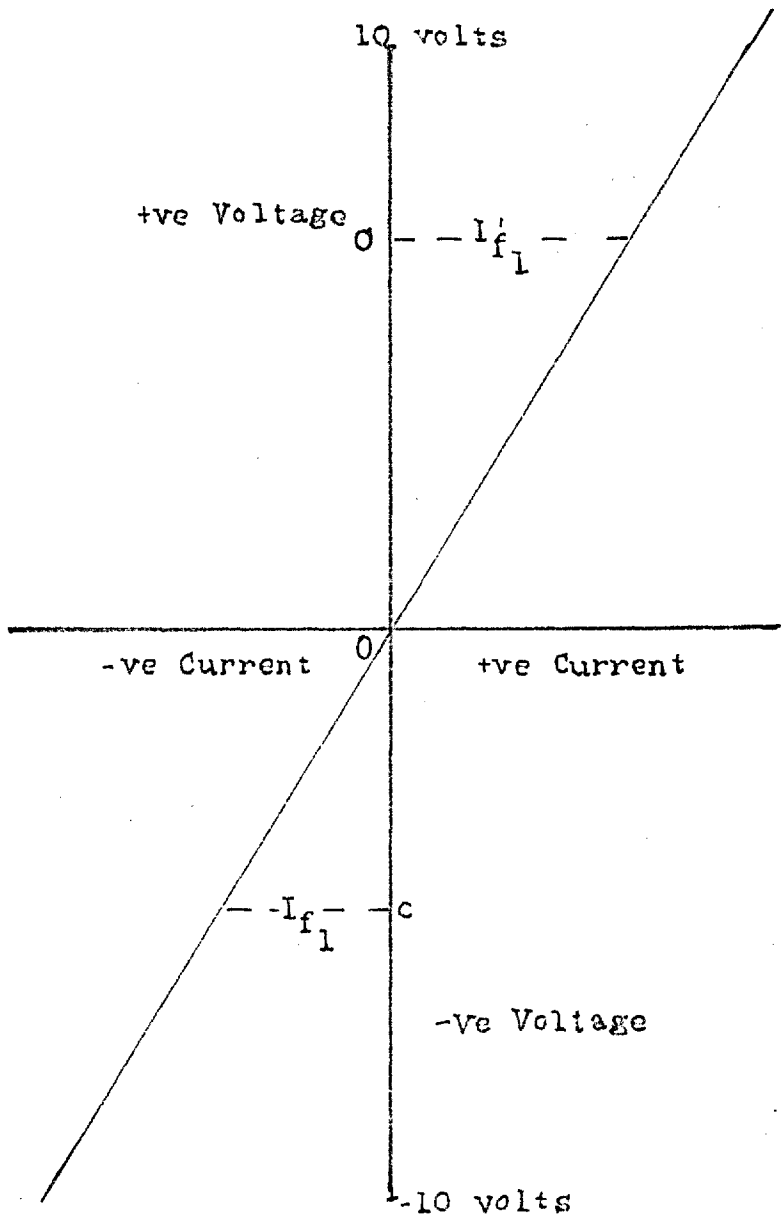


Fig.5.9a V-I Characteristic of the shunt resistors R_{f_n} and R'_{f_n} . I_{f_1} is the current correspond to the voltage oc in Fig.5.9c and I'_{f_1} is the current correspond to the voltage oc(5.9b)

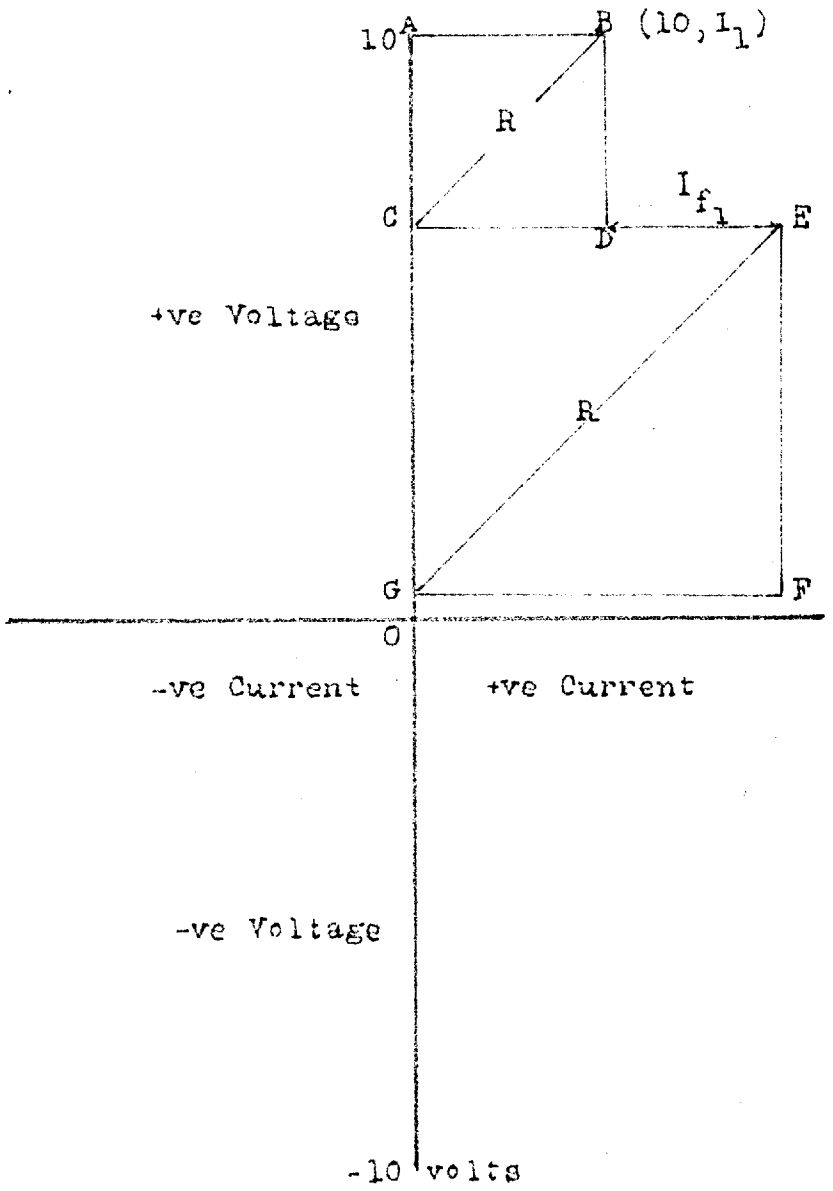


Fig. 5.9b V-I drawing for R-network

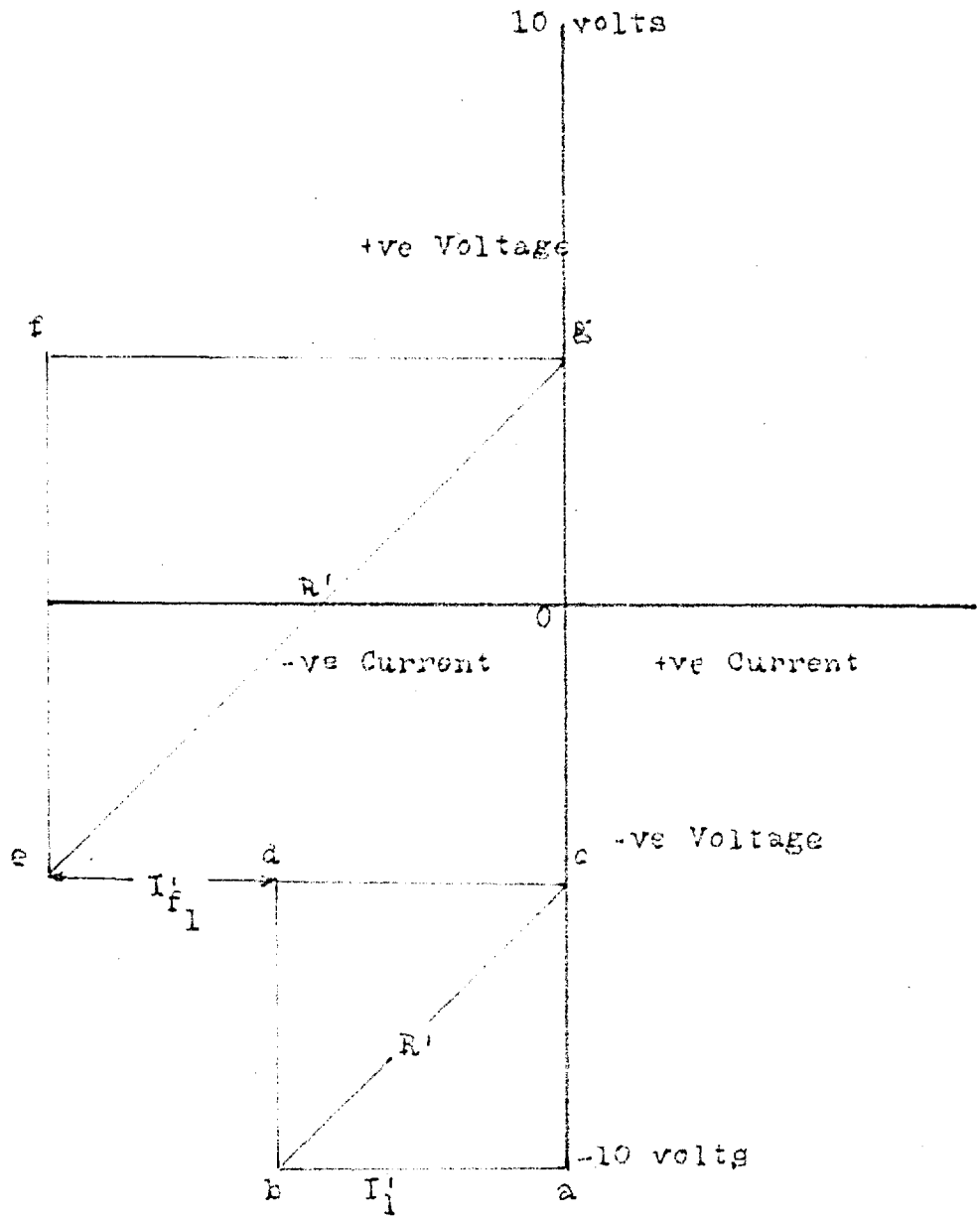


Fig.5.9c V-I drawing for R'-network

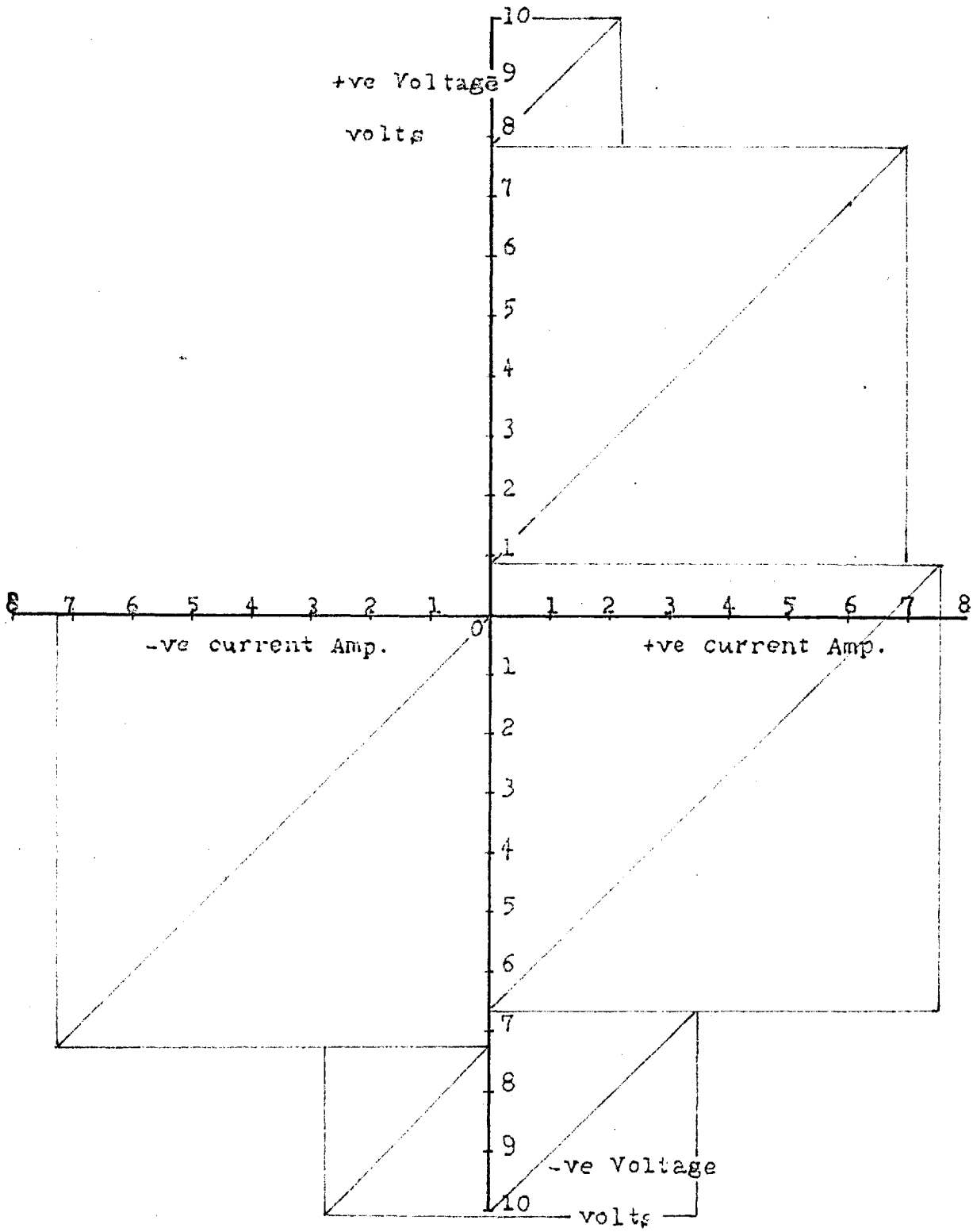


Fig5.10 10cos x

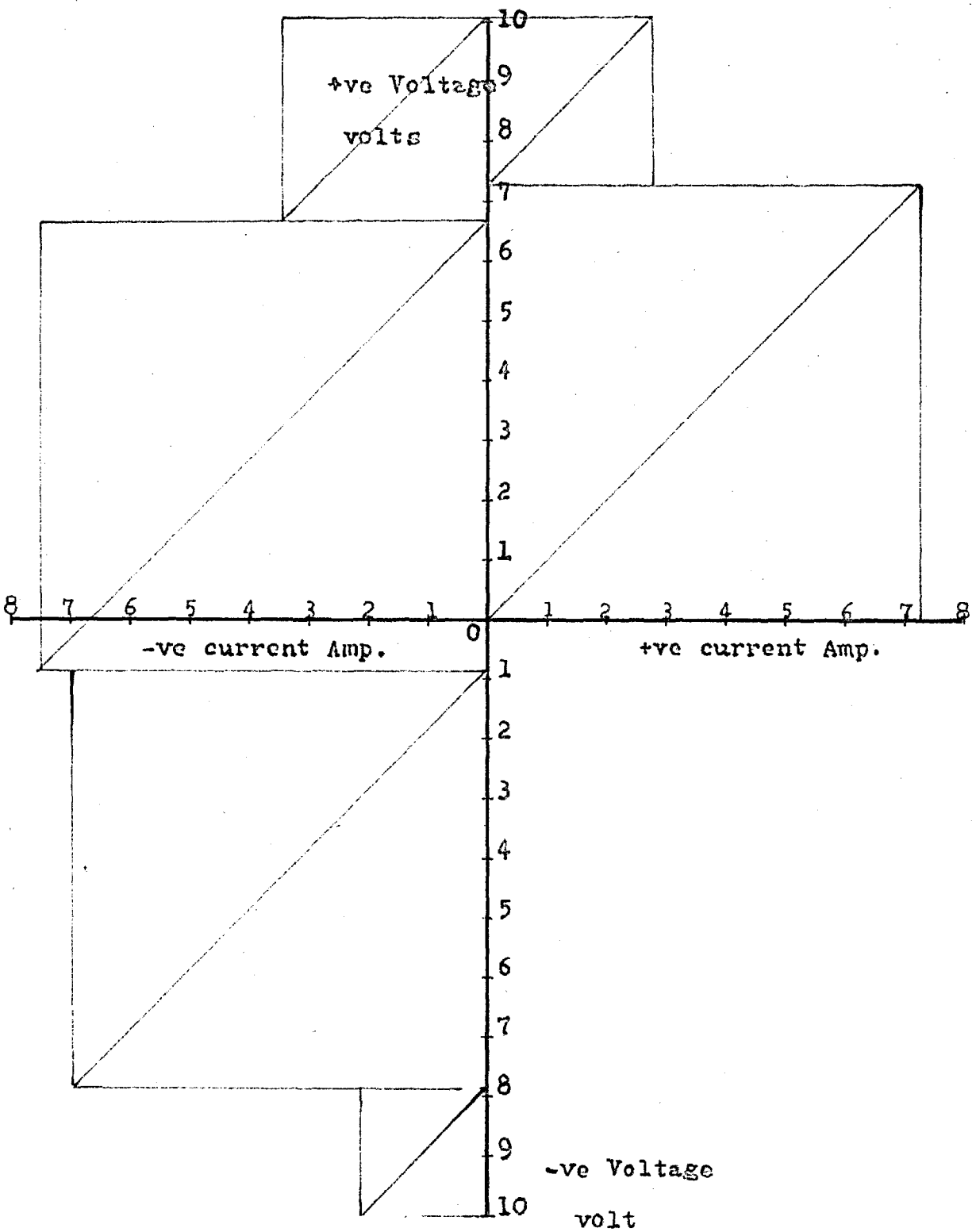
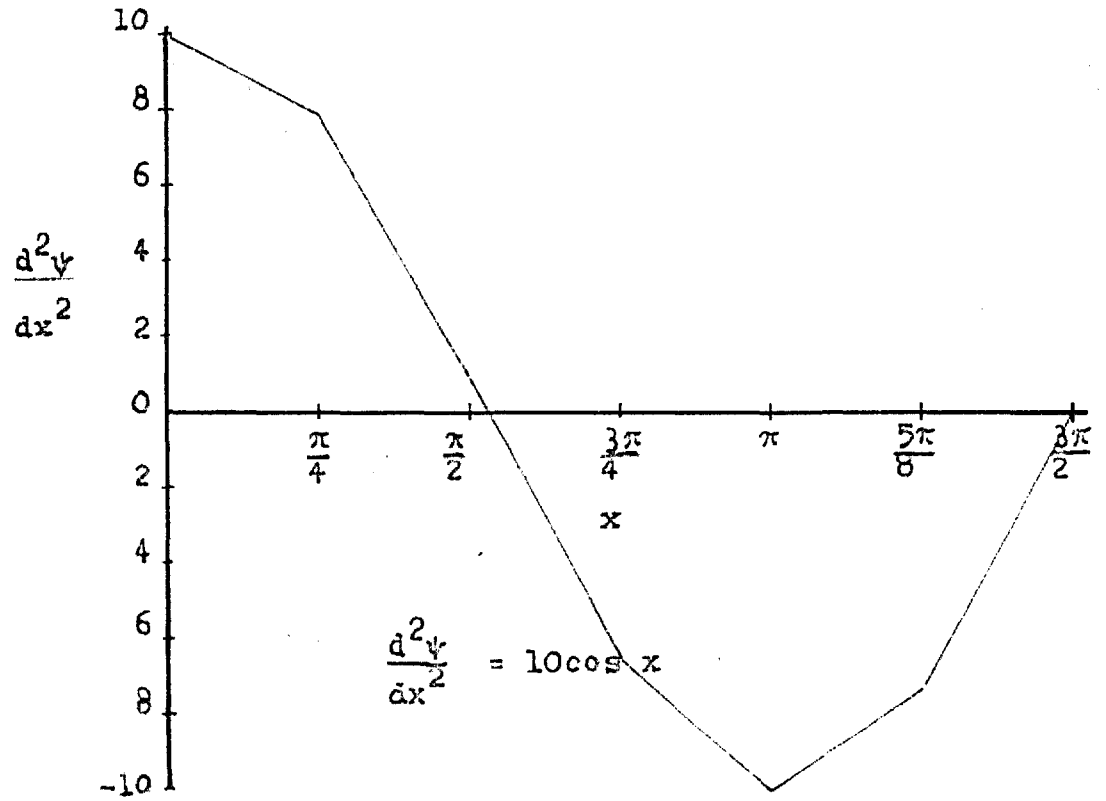
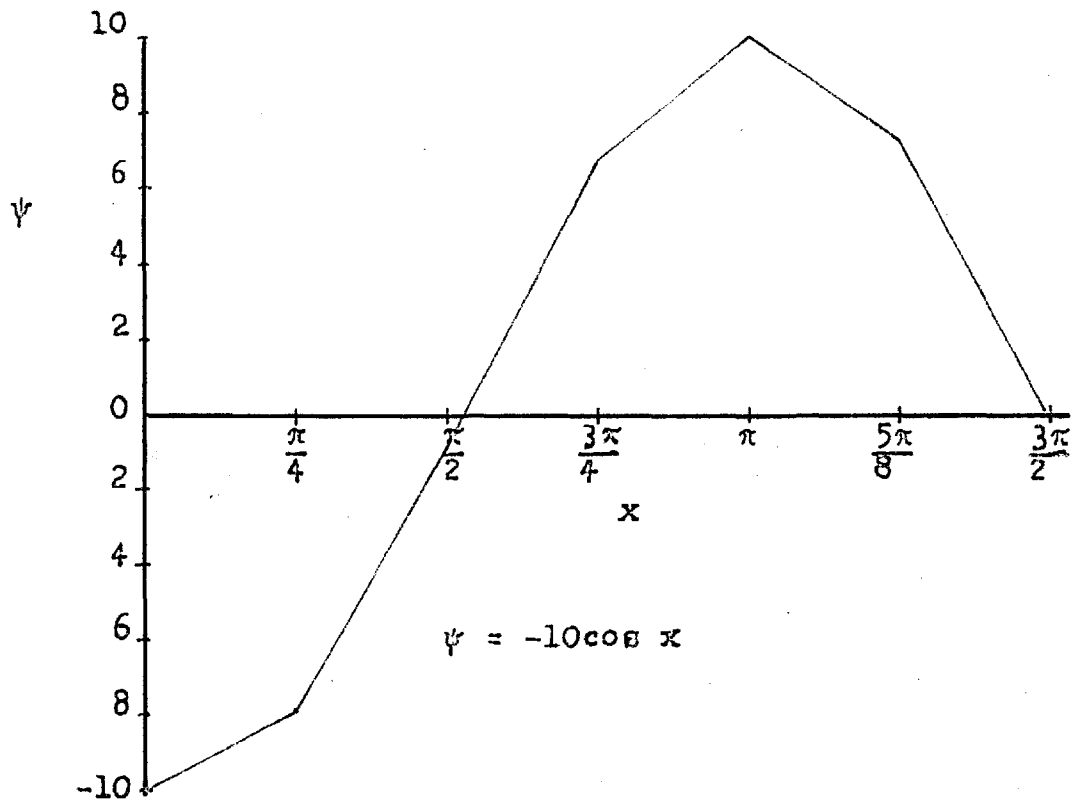


Fig5.10b -10cos x


10
8
6
4
2
0
2
4
6
8
-10
$\frac{d^2\psi}{dx^2}$
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
π
$\frac{5\pi}{8}$
$\frac{3\pi}{2}$
x
$\frac{d^2\psi}{dx^2} = 10\cos x$


10
8
6
4
2
0
2
4
6
8
-10
ψ
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{4}$
π
$\frac{5\pi}{8}$
$\frac{3\pi}{2}$
x
$\psi = -10\cos x$

the values of the voltages of the last nodes of both networks. The whole process is repeated until the correct values of the currents I_1 and I_1' are obtained which will then give the node voltages of both networks equal to the numerical values of $\frac{d^2\psi}{dx^2}$ and ψ. Fig.5.10.

Results

	Exact Finite-difference Soln.		Computed Soln.	
x	$\frac{d^2\psi}{dx^2}$	ψ	$\frac{d^2\psi}{dx^2}$	ψ
0	10.00	-10.00	10.00	-10.00
$\frac{\pi}{4}$	7.862	-7.862	7.865	-7.865
$\frac{\pi}{2}$	0.88	-0.88	0.878	-0.878
$\frac{3\pi}{4}$	-6.6439	6.6439	-6.6674	6.6674
π	-10.0665	10.0665	-10.0997	10.0997
$\frac{5\pi}{8}$	-7.278	7.278	-7.3025	7.3025
$\frac{3\pi}{2}$	0.00	0.00	0.00	0.00

Eg.2

This example illustrates the equation in which the boundary conditions are given as the values of the slopes of the equation i.e. $(\frac{d^3\psi}{dx^3})$ and $\frac{d\psi}{dx}$.
The equation is $\frac{d^4\psi}{dx^4} = K$, where $K = 20$.
The given boundary conditions are when $x = 0$ $\frac{d^3\psi}{dx^3}$ and $\frac{d^2\psi}{dx^2} = 0$, and when $x = 1$ $\frac{d\psi}{dx}$ and ψ are equal to zero.

The values R and R' of the network are equal to 1Ω

and $h = \frac{1}{10}$

The resistance networks representing this equation are as shown in the fig.5.13. The first network gives the value of $(\frac{d^2\psi}{dx^2})_n$, and the second network gives the value of ψ. The computing procedure is as discussed in (1) sec. 5.3 and the value of $I_{f_n} = \frac{h^2}{R}20$ and $I'_{f_n} = \frac{h^2}{R}(\frac{d^2\psi}{dx^2})_n$.

Applying the Boundary Condition.

Since $\frac{d^3\psi}{dx^3} = 0$ when $x = 0$ and if φ_n is equal to $(\frac{d^2\psi}{dx^2})_n$, then $(\frac{d\varphi}{dx})_0 = 0$. The condition when $(\frac{d\varphi}{dx})_0 = 0$ is that

$$\varphi_2 = \varphi_1 + \frac{h^2}{2}(\frac{d^2\varphi}{dx^2})_0 + h^2(\frac{d^2\varphi}{dx^2})_1 \text{ (from eqn.5.3)}$$

$(\frac{d^3\varphi}{dx^3})$, $(\frac{d^4\varphi}{dx^4})_0$ are equal to zero.

$$(\frac{d^2\varphi}{dx^2})_0 = (\frac{d^2\varphi}{dx^2})_1 = 20$$

The equation becomes

$$\varphi_2 = \varphi_1 + 0.3$$

Similarly for the network R'

n= 10 (eqn.5.4) $(\frac{d\psi}{dx})_{10} = 0$

then $\psi_8 = \psi_9 + \frac{h^2}{2}(\frac{d^2\psi}{dx^2})_{10} + h^2\psi_9 - \frac{h^3}{6}(\frac{d^3\psi}{dx^3})_{10} + \frac{h^4}{24}(\frac{d^4\psi}{dx^4})_{10}$

$$+ \frac{h^4}{12}(\frac{d^4\psi}{dx^4})_9$$

$$(\frac{d^4\psi}{dx^4}) = (\frac{d^4\psi}{dx^4})_9 = (\frac{d^2\varphi}{dx^2})_{10} = 20$$

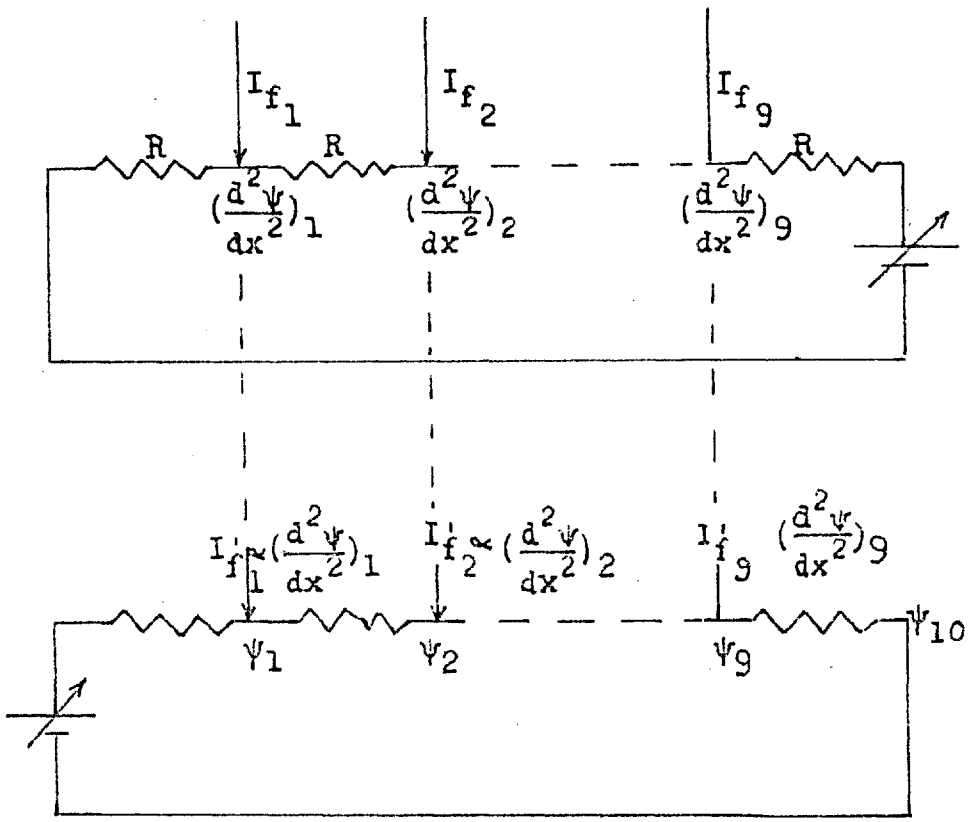

I_{f_1}
I_{f_2}
I_{f_9}
R
R
R
$(\frac{d^2\psi}{dx^2})_1$
$(\frac{d^2\psi}{dx^2})_2$
$(\frac{d^2\psi}{dx^2})_9$
$I'_{f_1} \propto (\frac{d^2\psi}{dx^2})_1$
$I'_{f_2} \propto (\frac{d^2\psi}{dx^2})_2$
$I'_{f_9} \propto (\frac{d^2\psi}{dx^2})_9$
ψ_1
ψ_2
ψ_9
ψ_{10}

Fig. 5.13

The values of $(\frac{d^2\psi}{dx^2})_{10}, (\frac{d^2\psi}{dx^2})_9, (\frac{d^3\psi}{dx^3})_{10}$ are obtained from the solutions of the R-network.

then $\psi_8 = \psi_9 + 0.12792$

Results

x	Exact Solutions of differential equation	Exact Solutions of finite-difference eqn.	Computed Solutions
0	2.5	2.4942	2.495
.1	2.1667	2.16228	2.162
.2	1.8347	1.83136	1.830
.3	1.5067	1.50444	1.504
.4	1.1880	1.18652	1.19
.5	0.8854	0.8846	0.885
.6	0.6086	0.60768	0.607
.7	0.3667	0.36676	0.367
.8	0.1747	0.17484	0.174
.9	0.047	0.04692	0.047
1	0.00	0.00	0.00

5.6 Eigen value Problems.

When the constant K of $f(\psi,x,K)$ is an unknown value, the equation becomes an eigen value problem. This does not however present any greater problem in finding both the solution ψ and the value of K.

The procedure is very similar to that discussed in Chapter IV. Firstly the value of K is estimated, and then the solution of the equation which satisfies two given boundary conditions is obtained for both R and R-1 networks. It will be evident that the third boundary conditions of both networks will not be satisfied unless K is correct. When the solution is obtained for the first value of K, the value of K is then either increased or reduced to obtain a new set of solutions. It will now be possible to see whether the third boundary conditions of both networks are approaching or departing from the given value. This will indicate whether the value of K should be increased or decreased. The value of K is then improved until all the boundary conditions are satisfied. After the first comparison of the two values of K, the next value of K can easily be estimated which will bring it very near to the correct solution. The example for obtaining the value of K is already given in detail in Chapter IV.

CHAPTER VI

The Application of the Visual Iterative Method based on graphs to the Design of the Valve Circuit.

In the orthodox way of designing valve amplifiers, either graphically or numerically, the valves are in most cases assumed to have linear characteristics. This is because the non-linearity of the valve characteristic is treated as a complex problem. In this section an attempt will be made to simplify the design of the valve circuit when a non-linear characteristic is included. Furthermore an attempt is also made to describe a valve amplifier as a circuit consisting only of linear and non-linear resistors which can be explained in terms of simple mathematics or a graph.

W.R. Evans and D.H. Mitchell[28] have illustrated the power and flexibility of solving non-linear circuits by a graphical method. The new graphical method discussed in Chapter III with its power and flexibility, can also be used and will cope with even more complex forms of non-linear circuits. When the number of resistors in the circuit increases and the circuit becomes more complex, the method of W.R. Evans and D.H. Mitchell may lead to confusion. This is because the V-I characteristics of each resistor are not positioned in the same way as the

resistors in the circuit, i.e. network topology does not appear on their diagrams. Furthermore the V-I characteristics are drawn on top of one another which makes the drawing more difficult to analyse. In this new graphical method, the topology of the network is clearly illustrated and the overlapping of the V-I characteristics in the diagram only occurs when negative resistors are present.

6.1 The Single-valve Amplifier Circuit.

The planar electrical diagram as described by Cherry is used to show the relationship of the voltages and currents of an amplifier circuit shown in Fig.6

The diagram (Fig.6a) is logically drawn so that the V-I characteristic of each component has a position corresponding to the position of each component in the circuit.

One of the V-I characteristics of a valve corresponding to its grid-cathode potential v_{g_n} is employed for each diagram. The only potential which cannot be correctly represented on this diagram is the input potential. This is already known or can be easily obtained from the given grid-cathode potential v_{g_n} for each V-I characteristic of the valve.

The potential at each point on the amplifier circuit is ~~customarily~~ shown on the vertical axis,

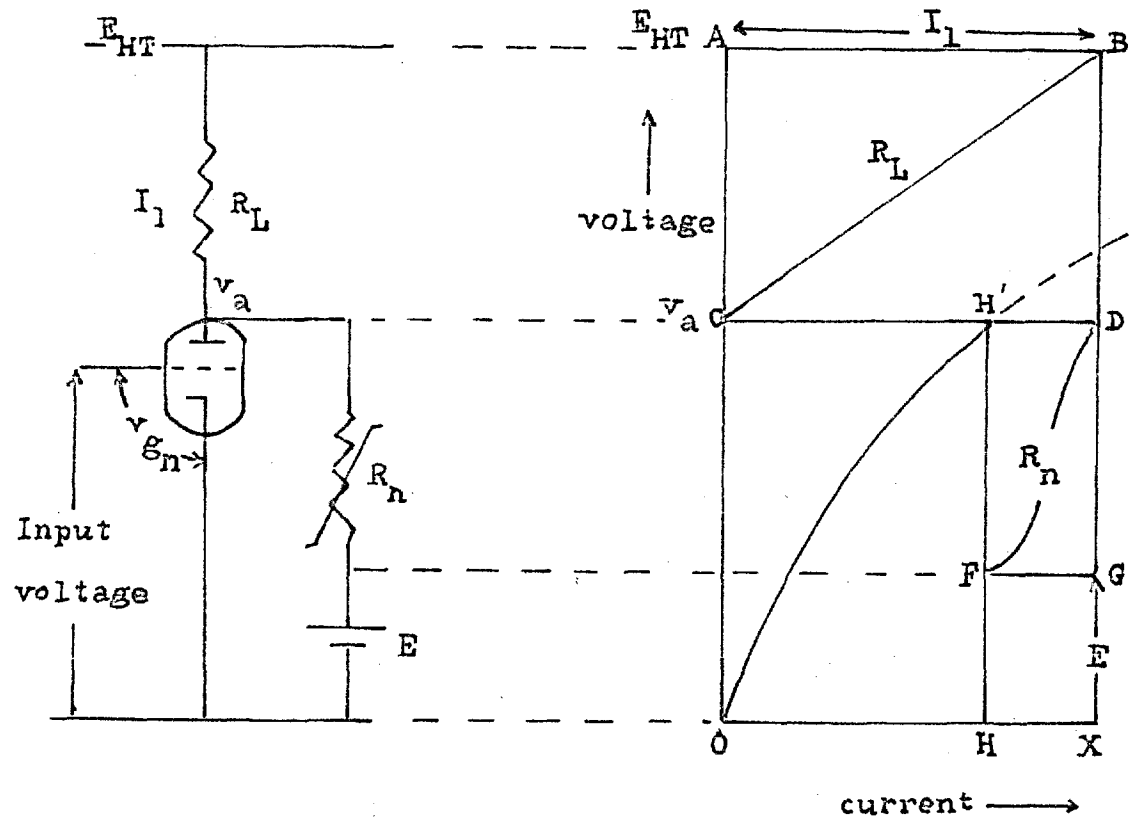


Fig.6

Fig.6a

V-I characteristic of the valve corresponding to the grid cathode v_{g_n}

and therefore the vertical axis of the graphical diagram will display the potential at each point. On the graphical diagram Kirchhoff's voltage law is readily satisfied, since the sum of the potential drop in the valve (OC), and the potential drop in the resistor R_L(CA), is always equal to the supply voltage E_{HT}(OA). It is also true that the sum of the potential drop in the resistor R_n(GD), the resistor R_L(DB) and the voltage source E(XG) is equal to the supply voltage E_{HT}(BX or AO). The current relationship is displayed by the horizontal axis and also satisfies Kirchhoff's current law because the current flowing in the resistor R_L, which is represented on the graphical diagram as the line CD, is equal to the sum of the current flowing into the valve (represented by the line CH') and the non-linear resistor R_n (represented by H'D).

The diagonal lines CB, OH' and FD are the V-I characteristics of the resistor R_L, the valve (which has a V-I characteristic similar to a non-linear resistor) and the non-linear resistor R_n respectively. Points C, O and F are their origins. Finally each minor rectangle represents the power dissipated in each component, and the total area of these minor rectangles

is equal to the power supplied by the two sources.

To obtain this relationship, the drawing ~~may be~~ is made on the V-I characteristic of the valve, which ~~has~~ may have its vertical line as a current axis, and the voltage axis is on the horizontal line, but in this paper the vertical line is the voltage axis. The drawing is commenced as follows:-

1) Mark the E_{HT} point on the voltage axis, and estimate the current I_1 which will give a starting point B. Fig.6a)

2) Draw the V-I characteristic of the resistor R_L starting from the voltage axis and passing through point B, and complete the rectangle ABCD.

3) Where the line CD intersects with the valve characteristic, the line H'H is drawn. CH' will represent the value of the current flowing into the valve.

4) Draw the V-I characteristic of the resistor R_n (using a template) starting from the line H'H and passing through the point D, and complete the rectangle H'DGF.

If the current I_1 is the correct value, GX will be equal to the voltage E, but if GX is not equal to the voltage E, It will be obvious in what way I_1 needs to be adjusted.

It was found that if an extra template of the V-I characteristic of the resistor R_L is also employed, this diagram can be drawn in a matter of minutes.

To obtain any value of the anode voltage v_a which corresponds to any value of the grid cathode voltage v_{g_n}, one of these diagrams is drawn. Each of these diagrams may be drawn on the top of one another so that all the anode voltages may be displayed simultaneously for each value of the grid cathode voltage, or input potential. (Fig.6.1). The relationship of the anode voltage and grid input potential is plotted in Fig.6.1a.

6.2. The Amplifier Circuit with Cathode Resistor.

The graphical drawing in Fig.6.2a shows the voltage and current relationship of the amplifier circuit shown in Fig.6.2.

The diagram displays the V-I characteristic of each component in the circuit. The cathode resistor R_c is connected in the cathode circuit of the amplifier, and therefore its V-I characteristic is drawn below the V-I characteristic of the valve V_1 in the diagram Fig.6.2a. Since the cathode resistor R_c takes the same amount of current as the valve it is drawn in the diagram to share the same amount of current. The net

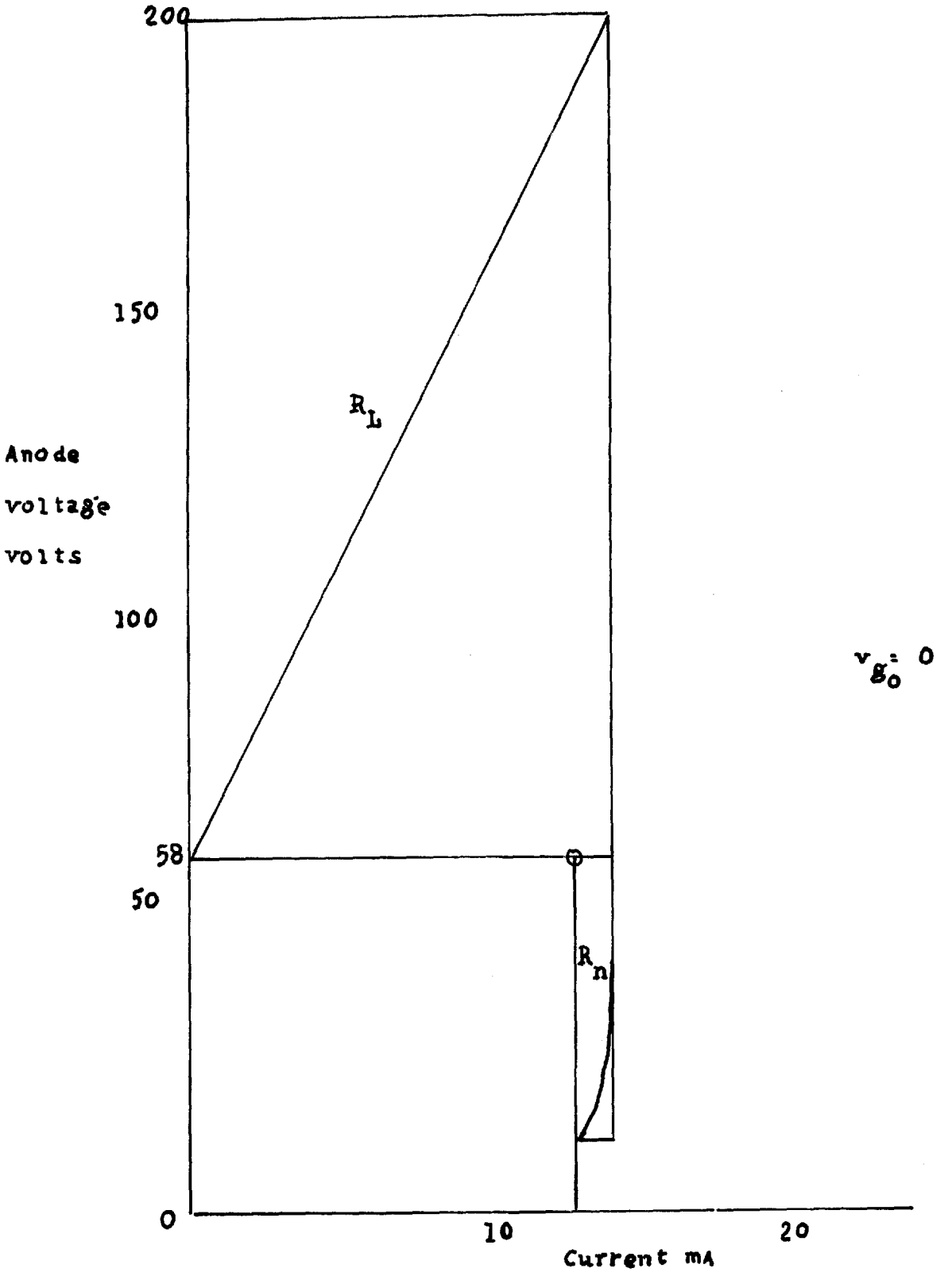
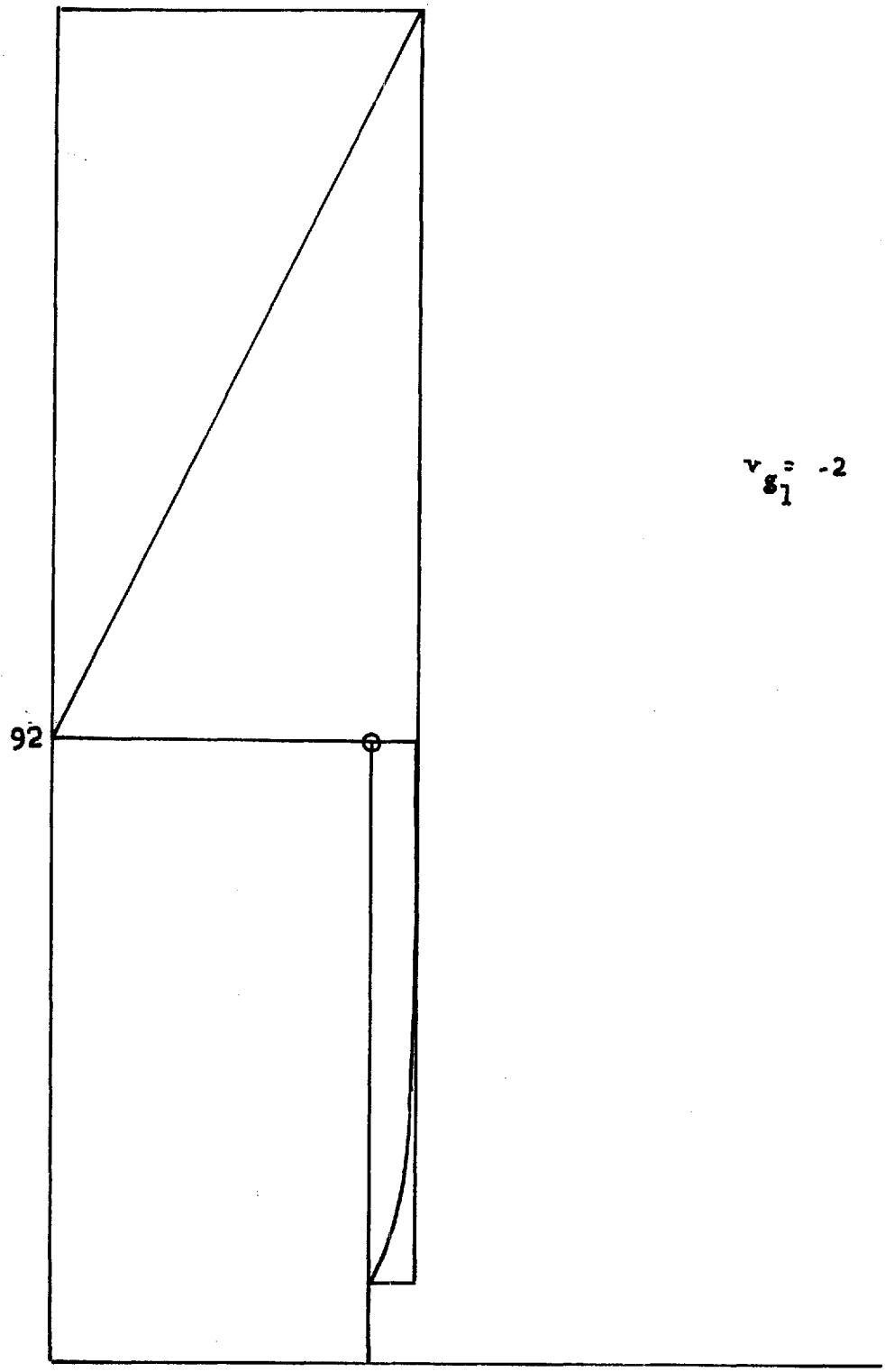
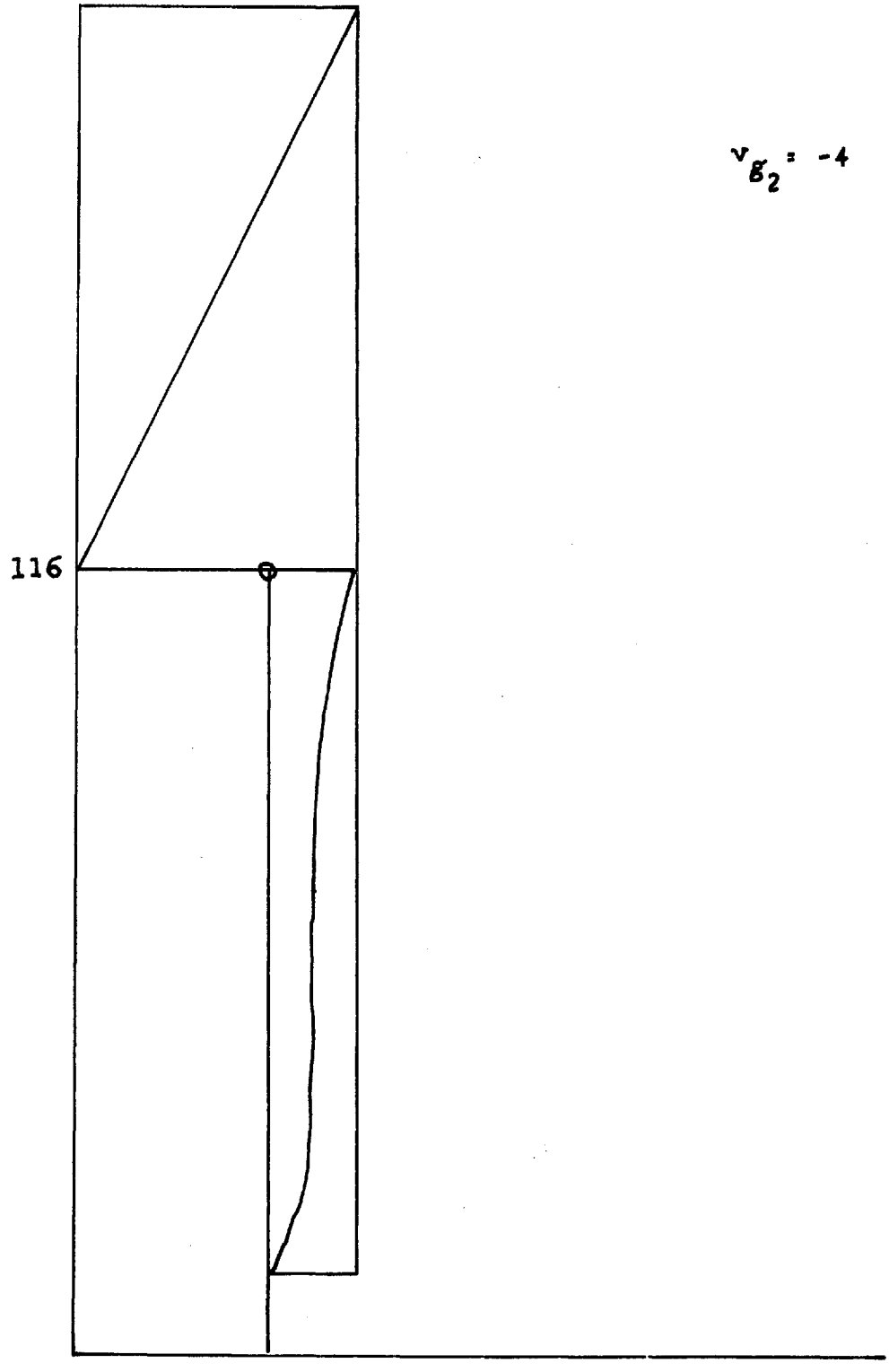

200
150
100
58
50
0
Anode
voltage
volts
R_L
R_n
$v_{g_o} = 0$
10
20
Current mA

Fig. 6.1



$v_{g_1} = -2$


116
$v_{g_2} = -4$

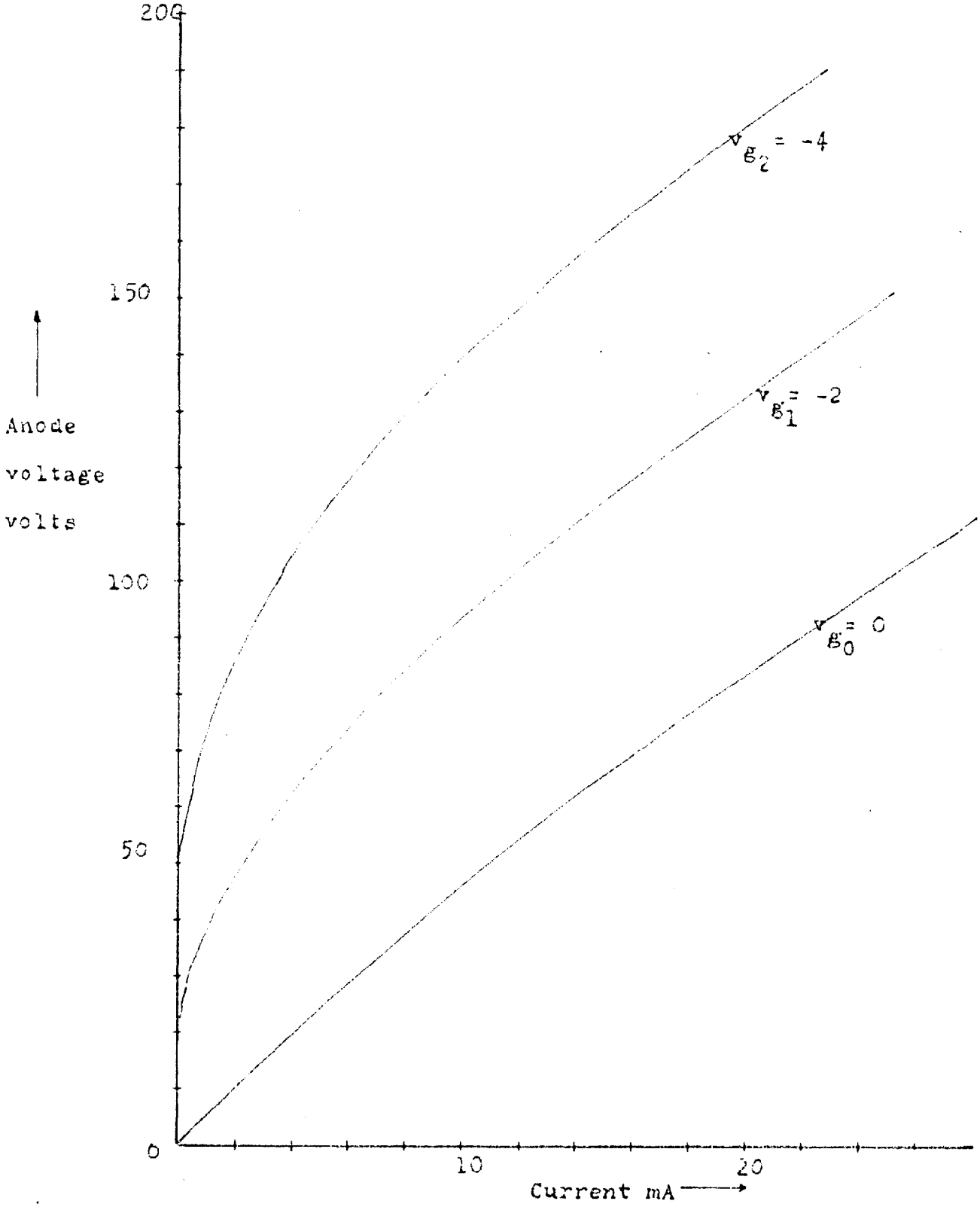

200
150
100
50
0
Anode
voltage
volts
$V_{g_2} = -4$
$V_{g_1} = -2$
$V_{g_0} = 0$
10
20
Current mA

Fig.6.1

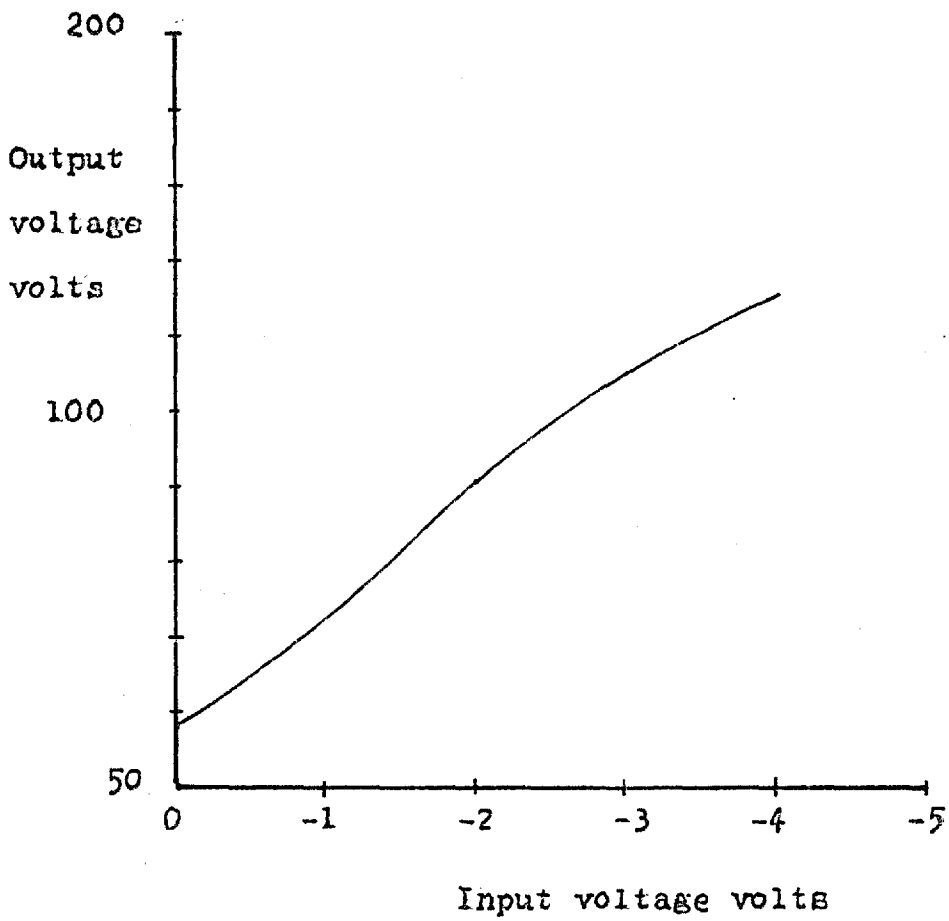


Fig.6.1a

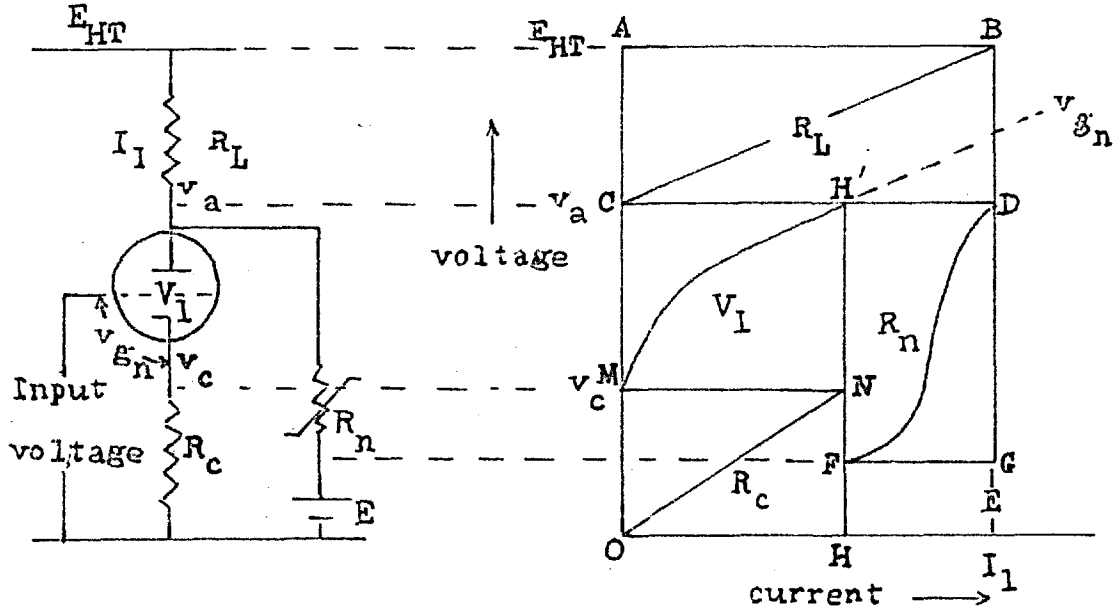


Fig.6.2

Fig.6.2a

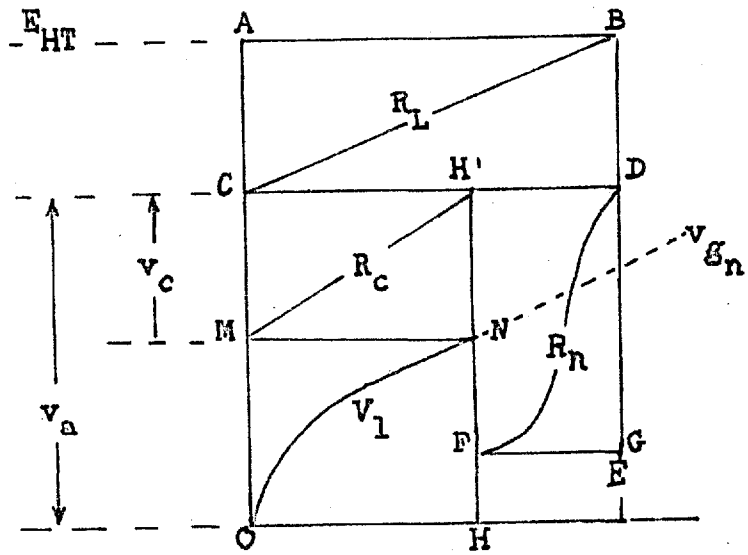
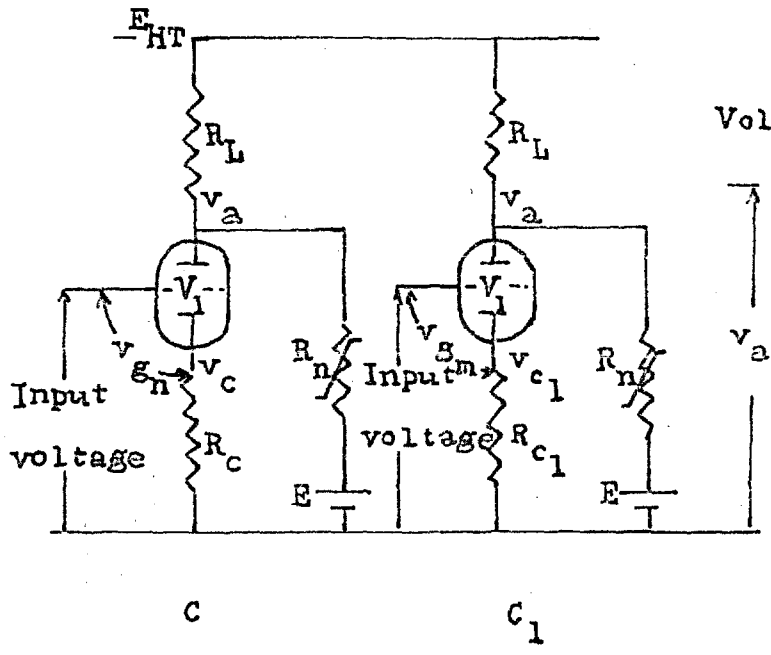


Fig. 6.3

value of the input voltage required to cause the valve to be at this operating point is $(v_c + v_{g_n})$.

In obtaining the solution of the above circuit it is found more convenient to rearrange the drawing so that the V-I characteristic of the cathode resistor R_c is drawn above the valve V-I characteristic, as shown in Fig.6.3.

It is also possible to illustrate the change in the gain of the amplifier caused by the alteration of the value of the cathode resistor R_c on the graphical diagram shown in Fig.6.4. Consider the circuits C and C_1, in Fig.6.4a. The corresponding values of the components of both circuit are equal except that the value of the cathode resistor R_{c_1} is greater than the value of the cathode resistor R_c. The graphical diagrams of both circuits are drawn on top of one another as shown in Fig.6.4. The anode voltages of both diagrams are arranged so that they both have the same value, and thus all the corresponding values of the currents of both diagrams will also be the same. The value of the cathode resistor of one diagram is greater than the other, ie. R_{c_1} R_c, and therefore the only difference between these two diagrams is that the rectangle CH'M'N' is greater than the rectangle CH'MN.



C C_1

Fig.6.4a $R_{c_1} > R_c$

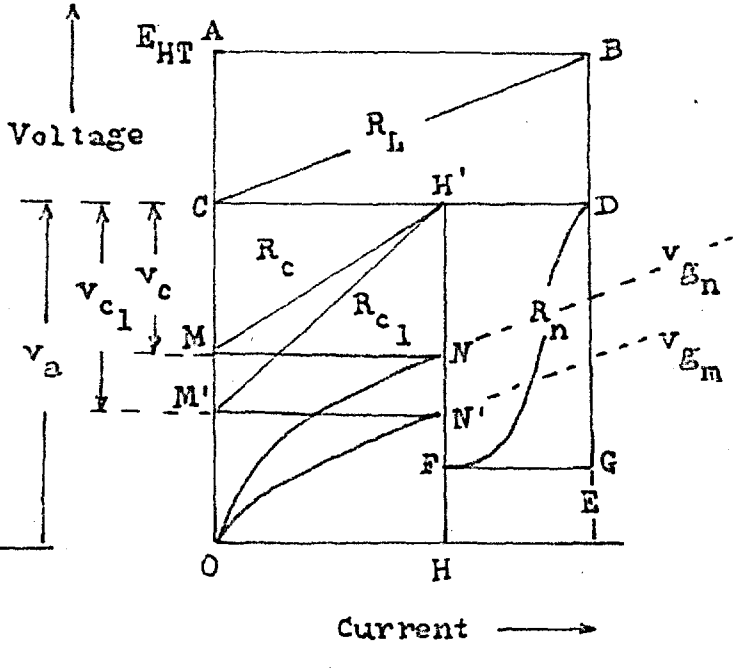


Fig.6.4 $v_{c_1} > v_c$

This causes the valve to operate on two different V-I characteristics, v_{g_n} and v_{g_m}.

For the circuit C to be at this operating point the required input potential is $v_c + v_{g_n}$. Similarly for the circuit C_1, the required input potential is $v_{c_1} + v_{g_m}$. Since in most cases v_c and v_{c_1} will be approximately equal to the value of the input voltages of the circuits C and C_1 respectively. v_{c_1} is greater than v_c as is shown on the diagram, and therefore the amplification of the circuit C is greater than that of C_1.

6:3 The Cathode-follower Circuit.

The solution of the cathode-follower circuit can be obtained by the graphical method in a similar way to that of the current feed-back circuit, section 6.2. The difference between these two circuits is that the output of the cathode-follower is taken from the cathode of the valve, and there is no load resistor R_L in the cathode-follower circuit. The circuit and the graphical diagram is as shown in Fig.6.5, and Fig.6.5a respectively.

In Fig.6.5a, the diagonal line CB is the V-I characteristic of the valve corresponding to its cathode-grid potential v_{g_n}. The lines OH' and FD are the V-I characteristics of the cathode resistor R_c and the load

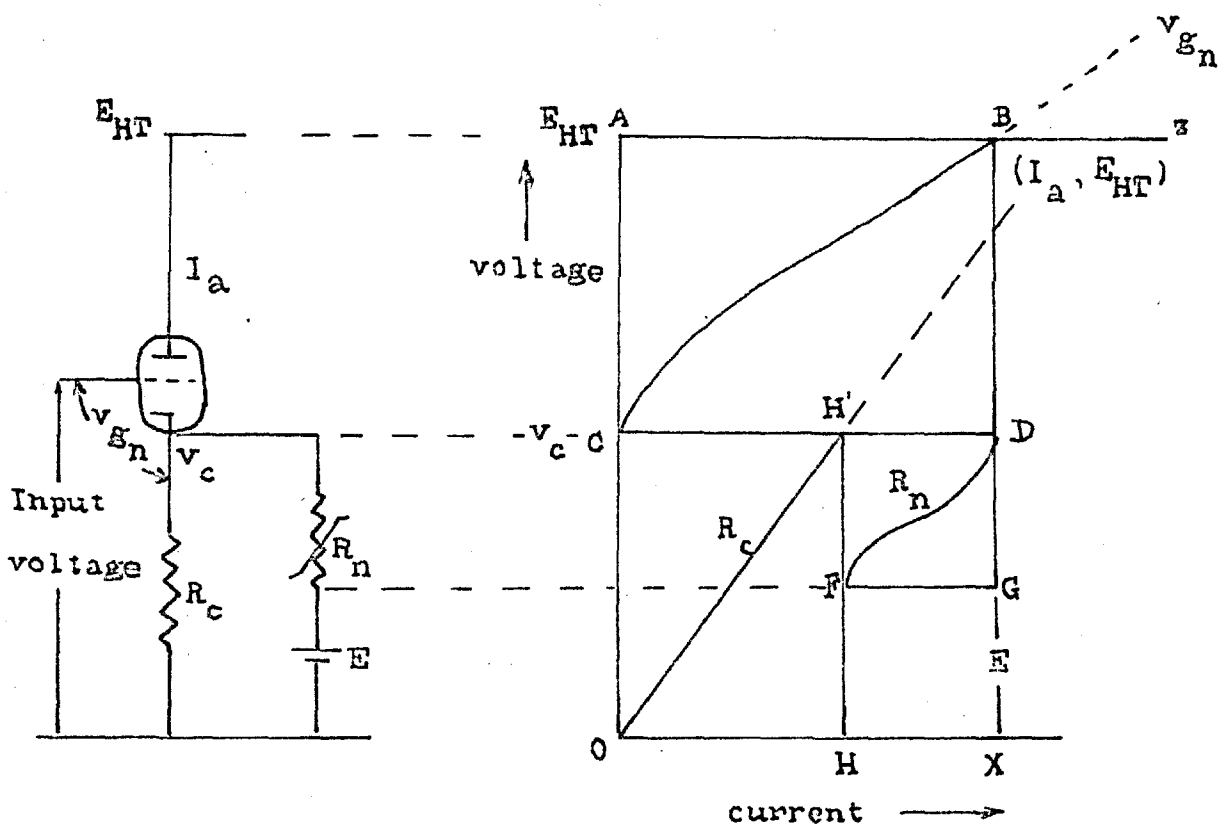


Fig.6.5

Fig.6.5a

resistor R_n respectively. The vertical line OA = E_{HT} is the supply voltage, and the vertical line HF = E is the potential to which the load resistor R_n is connected.

To obtain the solution of this circuit, templates are made for the V-I characteristics of the valve for each value of cathode-grid potential v_{g_n}, and also a template is made for the V-I characteristic of the load resistor R_n if it has a non-linear characteristic. The V-I characteristic of the cathode resistor R_c is plotted on the graph paper on which the diagram is to be drawn, with its origin at point O. Point A is marked on the voltage axis so that OA = E_{HT}, and a horizontal line AZ is drawn. The fitting of these characteristics together is then completed in the usual way.

After all the V-I characteristics of the valve corresponding to all the grid-cathode potentials v_{g_n} are used, the values of the cathode voltage v_c corresponding to the values of the input voltage $(v_c + v_{g_n})$ can be plotted to give the relationship between the output voltage and the input voltage.

It should be noted that the value of the input voltage is $v_c + v_{g_n}$ and the output voltage is v_c. Since v_c is very much greater than v_{g_n} then the input

voltage is approximately equal to the output voltage.

In order to avoid making templates for the V-I characteristics of the valve, the V-I characteristics of the valve may be plotted on the graph paper on which the diagram is to be drawn as shown in Fig.6.6 together with the V-I characteristic of the cathode resistor R_c. The origin of the valve V-I characteristic is placed at point A where $OA = E_{HT}$ and the values of the anode voltage are plotted in the reverse direction.

The solution is obtained by estimating the value of the anode current I_a which will give the starting point B. From B a vertical line is drawn to intersect the valve V-I characteristic at point D. BD will then be the value of the potential drop in the valve. From this point the procedure is similar to that of the drawing of the diagram shown in Fig.6.5a.

Fig.6.7 shows the graphical diagrams of a cathode-follower plotted on top of one another for three valve V-I characteristics corresponding to the grid-cathode voltage 0,-2, and-4 volts respectively. The potential to which one end of the resistor R_n is connected is 90 volts. For the cathode-grid potential equal to 0 volts, the cathode potential is above 90 volts, and therefore

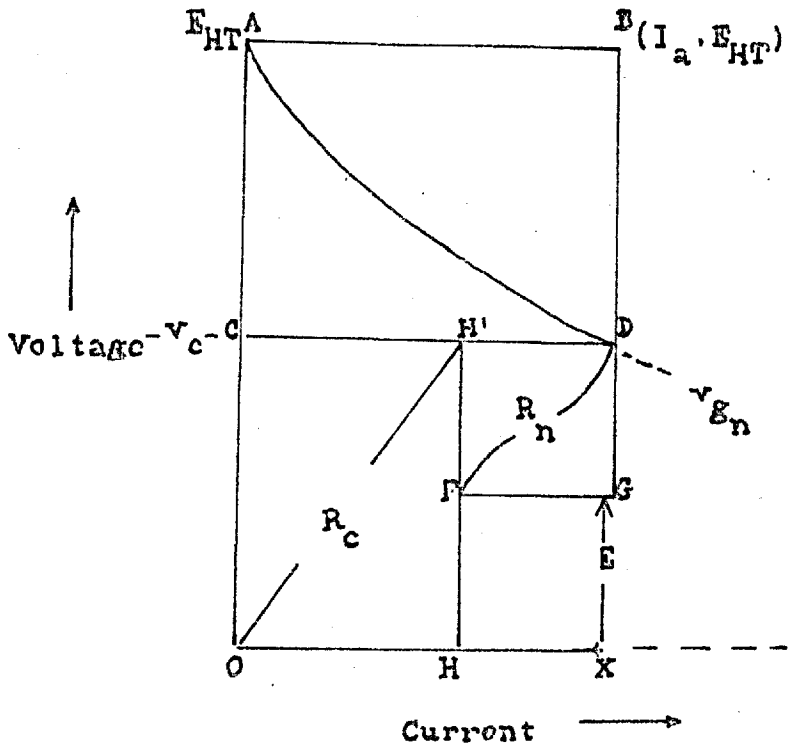


Fig.6.6

the current is flowing from the cathode potential through the resistor R_n to the 90 volts source. For the grid-cathode potential equal to -2 volts, the cathode potential is at 90 volts, therefore no current flows into the resistor R_n. Lastly for the grid-cathode potential equal to -4 volts, the cathode potential is 69 volts. Therefore the current flows from the 90 volts source through the resistor R_n to the cathode potential.

In the design of a cathode-follower, the following points must be observed:-

1) The maximum current through the valve and the maximum dissipation of the valve must not exceed the given limit.

2) The grid current must be kept at a minimum.

3) The valve must not be cut off.

All these points are already displayed on the graphical diagram shown in Fig.6.7.

When the grid-cathode potential of the valve increases beyond 0 volts, the valve will begin to draw grid current, and therefore the maximum value of the grid-cathode within which the valve will operate is 0 volts. This will also give the maximum value of the current through the valve. As the grid-cathode potential

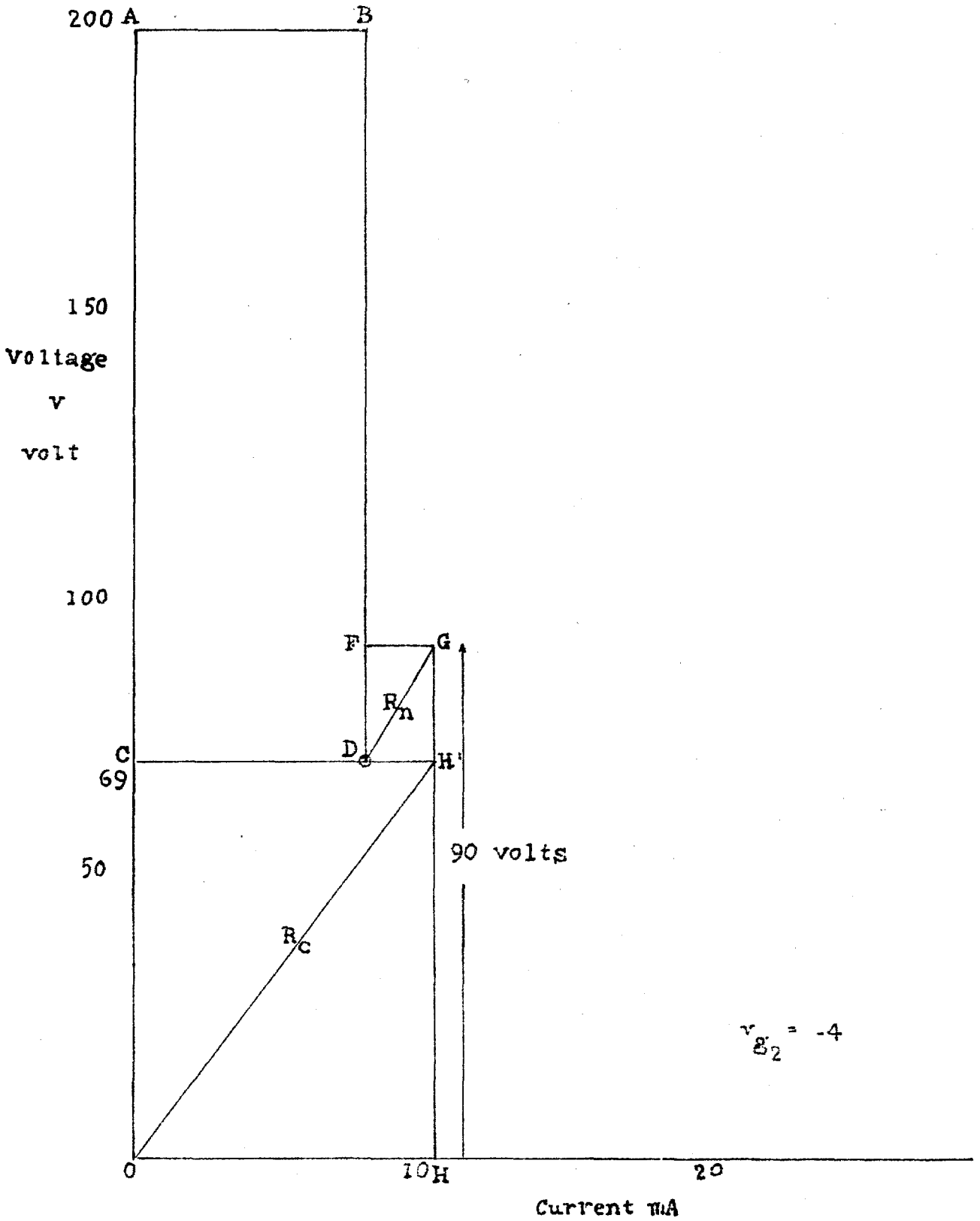
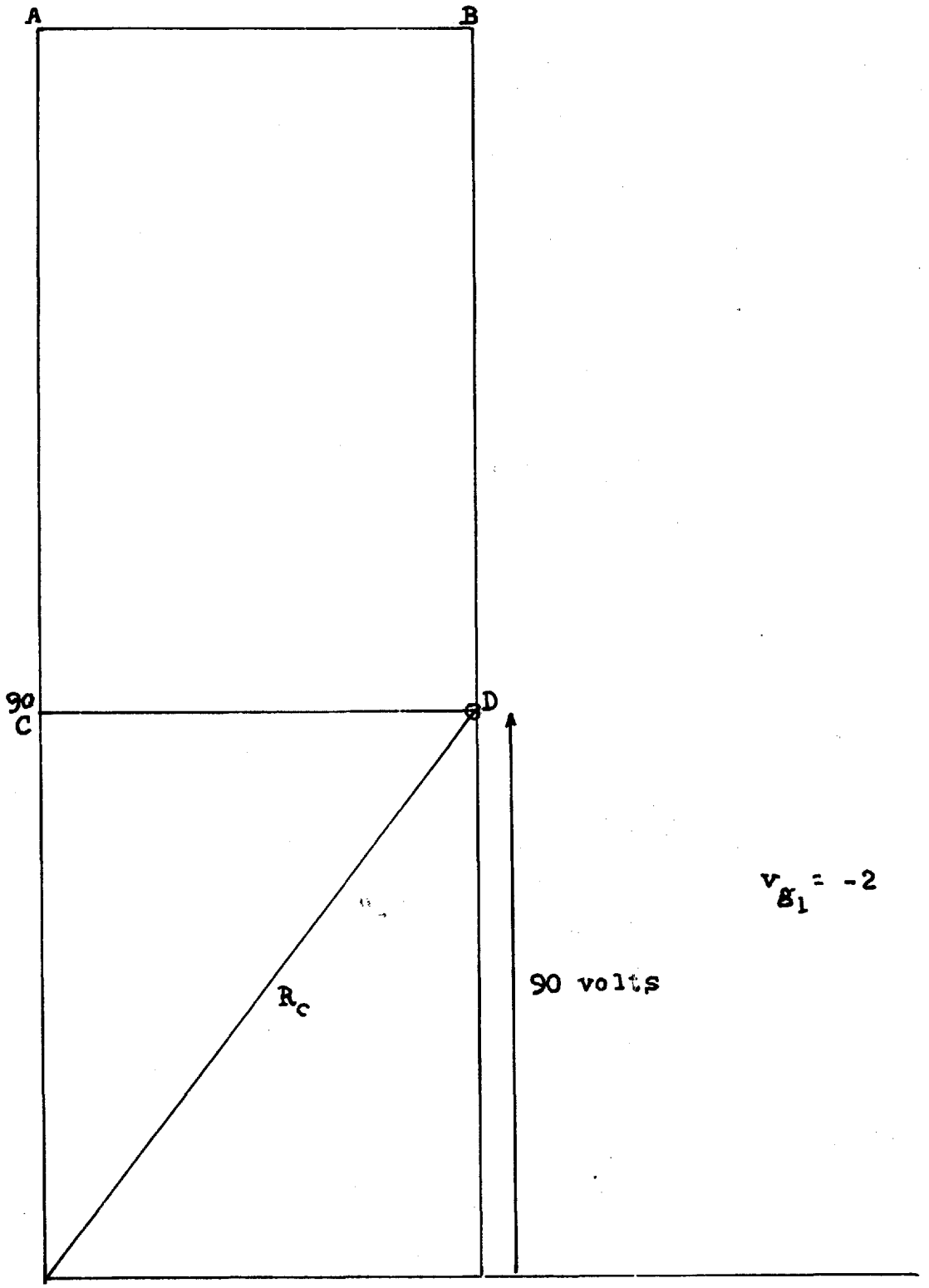
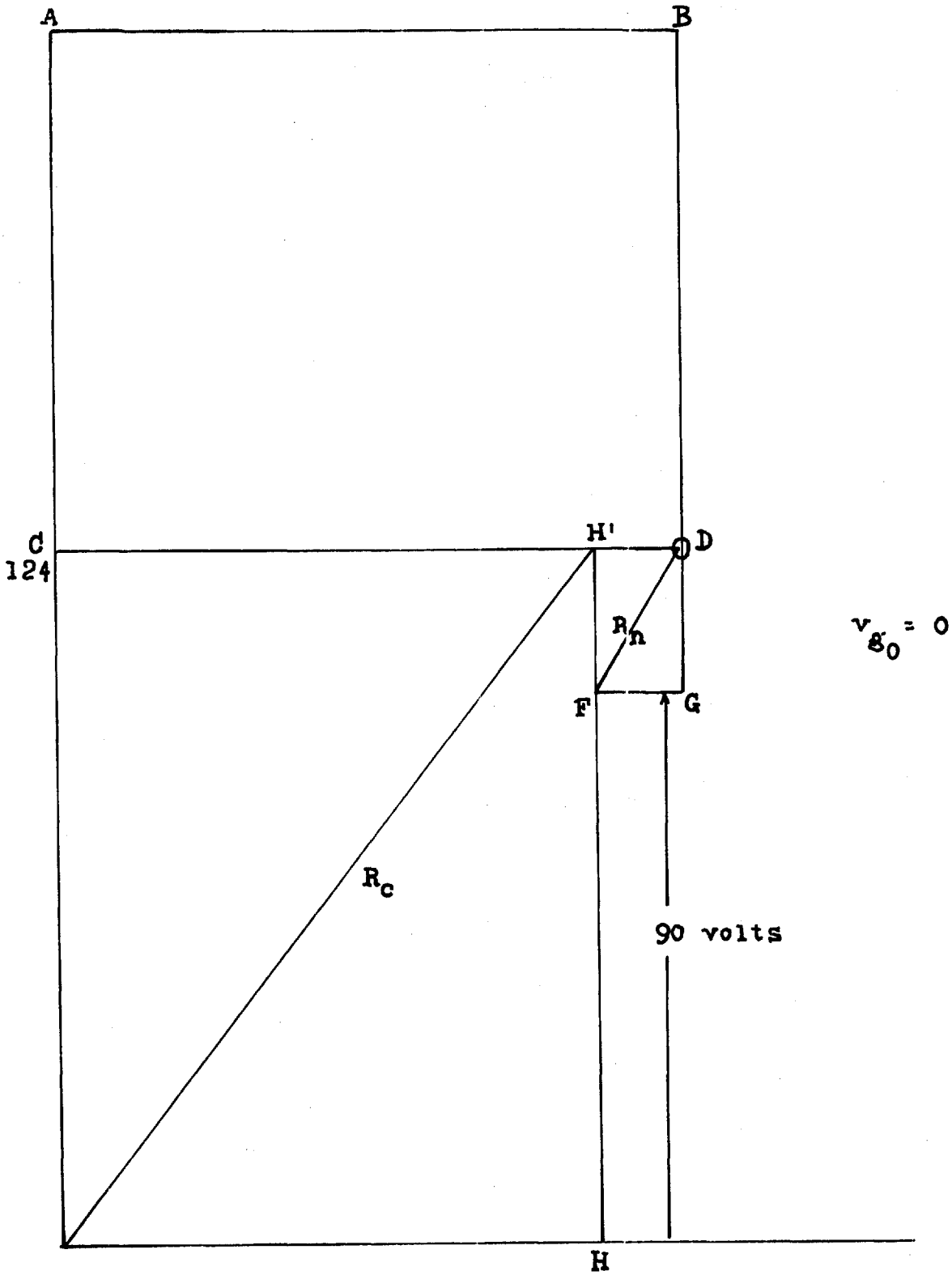

200
A
B
150
Voltage
V
volt
100
F
G
R_n
C
69
D
H'
50
90 volts
R_c
V_{g_2} = -4
0
10
H
20
Current mA

Fig. 6.7


A
B
90
C
D
R_c
90 volts
$v_{g_1} = -2$


A
B
C
124
H'
D
R_h
$v_{g_0} = 0$
F
G
R_c
90 volts
H

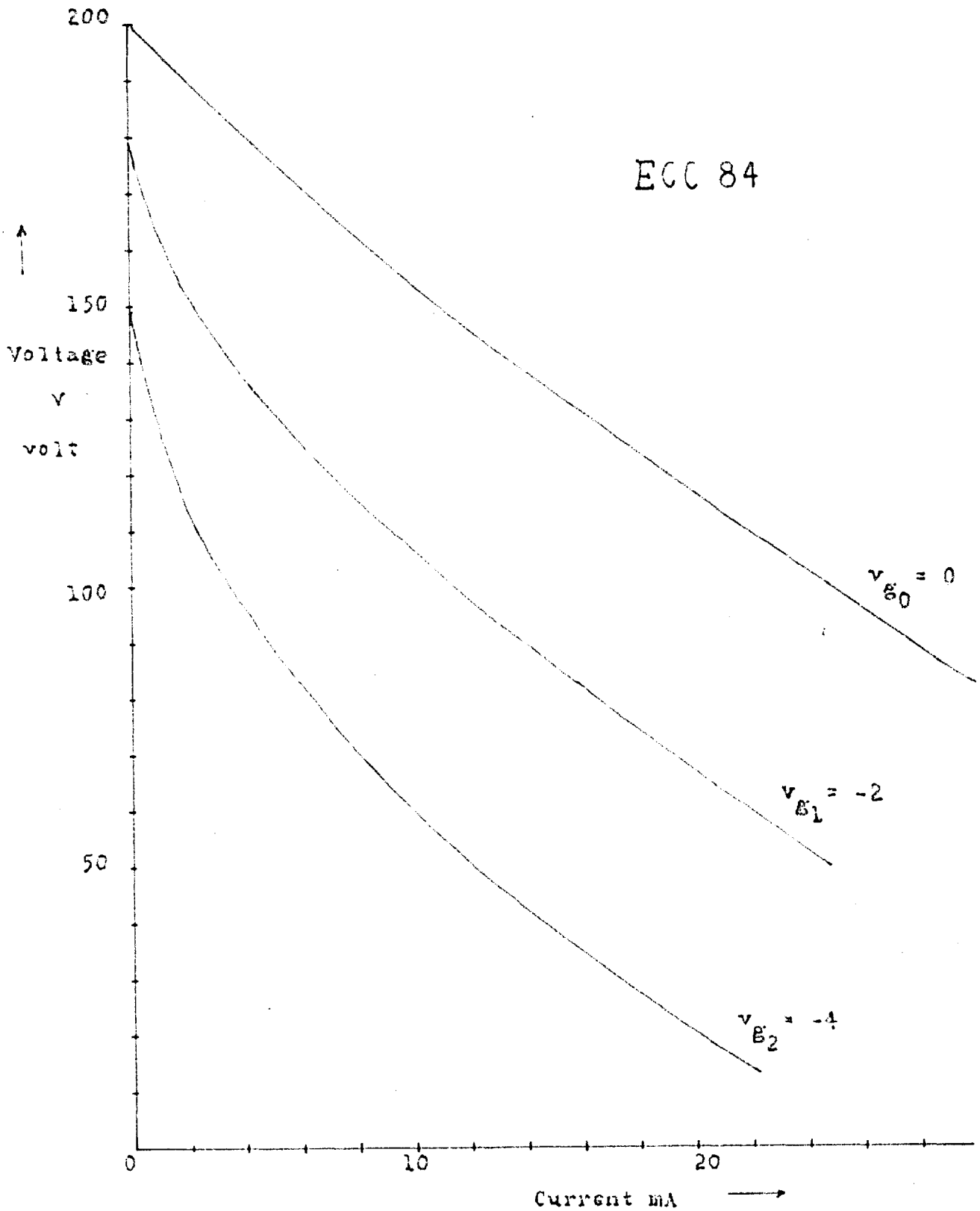

ECC 84
200
150
100
50
0
10
20
Voltage
V
volt
Current mA
$V_{g_0} = 0$
$V_{g_1} = -2$
$V_{g_2} = -4$

Fig. 6.7

of the valve decreases, it will cause the current through the valve to reduce, and therefore the length of the line CD on the diagram is shortened. When point D coincides with point C, the valve will cut off, and therefore this will be the other limit of grid-cathode potential within which the valve will operate. The anode dissipation is the area ABCD and therefore for the whole working range of the cathode-follower this area must not exceed the maximum rating of the valve.

6.4 The Cathode-coupled Amplifier.

The cathode-coupled amplifier, or as it is sometimes called, the "longtail pair", is widely used as the first stage of a D.C. amplifier. The method of obtaining the exact solution of this circuit by the numerical method is rather long and complicated. This process can be simplified and speeded up by the use of the graphical diagram. To illustrate the graphical method, the circuit is first divided into two parts, each part containing one valve. (V_1 and V_2). The diagram showing the relationship between the voltage and current of each part is also illustrated separately. Once the method is understood, it will be obvious how the graphical diagram may be drawn as one diagram.

Consider the circuit as shown in Fig.6.8. It is

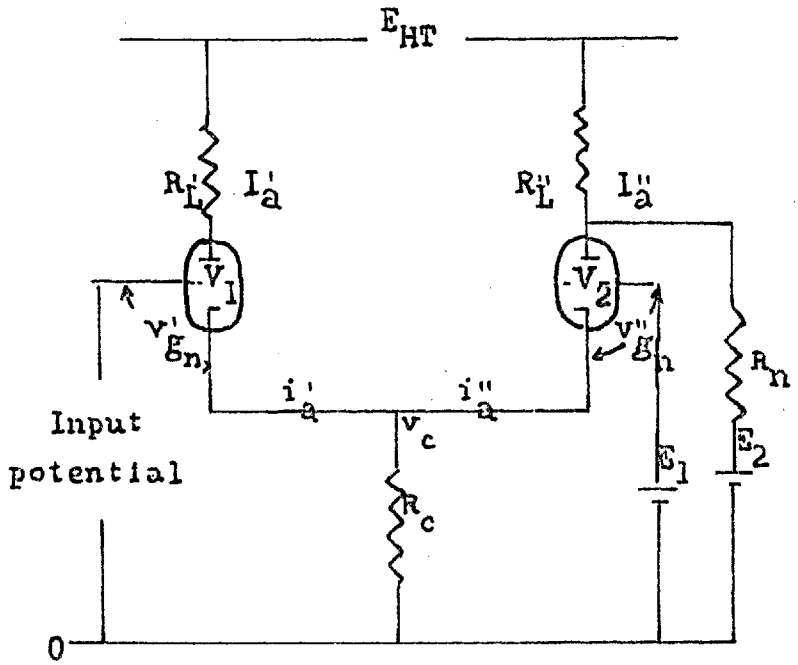

E_HT
R'_L
I'_a
R''_L
I''_a
V_1
V_2
v'_gn
v''_gn
i'_a
i''_a
R_n
Input
potential
v_c
E_1
E_2
R_c
0

Fig. 6.8

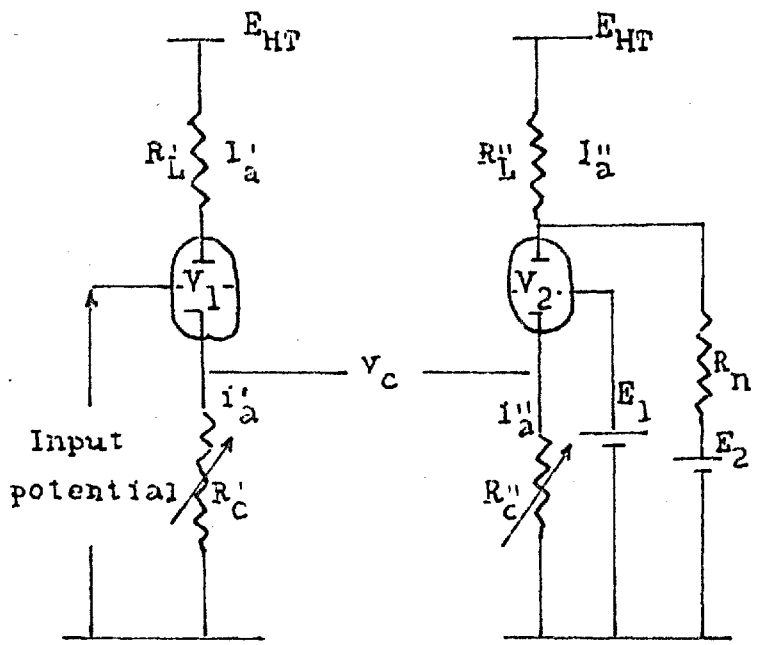

E_HT
R'_L
I'_a
V_1
v_c
i'_a
Input
potential
R'_c
E_HT
R''_L
I''_a
V_2
R_n
E_1
i''_a
E_2
R''_c
Input part
Output part

Fig. 6.9

assumed that both valves have similar characteristics. (When both valves have different characteristics, this graphical method is still applicable without any more labour). The circuit as shown in Fig.6.9 is similar to the circuit shown in Fig.6.8, provided that the cathode voltage remains the same throughout the operation, and the sum of the values of the resistors R_c' and R_c'' is equal to R_c. In Fig.6.9 the solution of the output part is first obtained by the graphical diagram. From this result the value of the current i_a' of the input part can be found. On the input part of the circuit, the anode current i_a' and the cathode voltage is known, and thus the potential drop in the valve can be assessed. From this, the required grid-cathode potential of the valve V_1 can be found and then the input potential. For each value of the output voltage, the input voltage is computed (graphically), and thus the relationship between the input and the output potential can be obtained.

In the graphical diagram of the output part shown in Fig.6.10, the resistor R_c'' is omitted because the cathode potential v_c is already known for each V-I characteristic of the valve. ie.$=v_{g_n}+v_c=E_1$. The value of the current i_a'' will be known when the diagram is completely drawn,

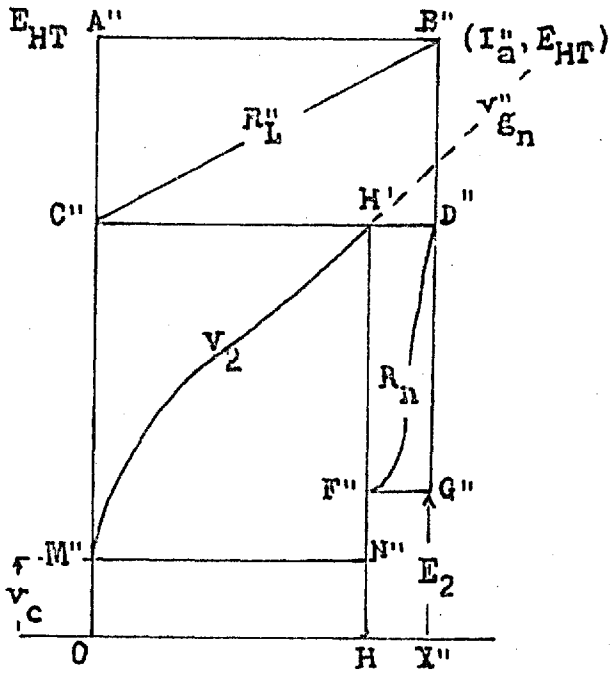


Fig.6.10 Output part diagram

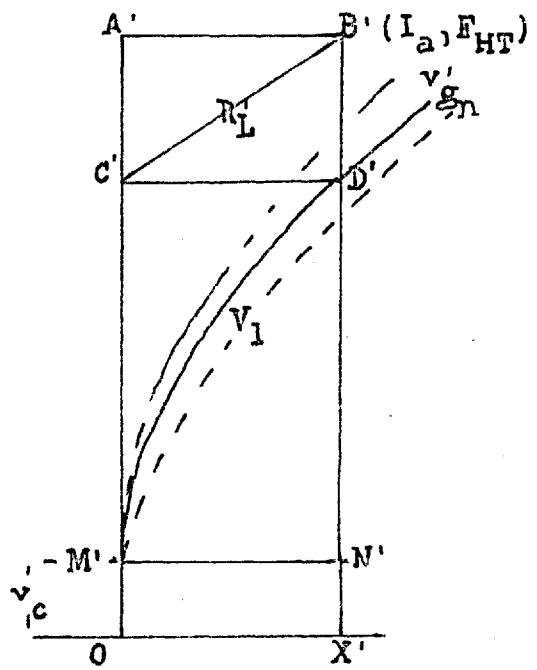


Fig.6.11 Input part diagram

as it has the same value as the current flowing in the valve.

To draw the diagram the following values are already known:-

1) The grid-cathode v''_{g_n} for the V-I characteristic of the valve.

2) The fixed grid-bias potential E_1.

3) The supply voltage E_{HT}.

The cathode potential v_c is then equal to $E_1 - v''_{g_n}$.

On the graph paper the values of the E_{HT} voltage and the cathode potential are located at the points A" and M" on the vertical axis. The current I''_a is estimated to form the starting point B", and the V-I characteristic of the resistor R''_L is drawn passing through it, with its origin at point C". Complete the rectangle A"B"C"D". Next the valve V-I characteristic is drawn starting from point M" and intersecting the line C"D" at H'. This will give the amount of current flowing in the load resistor R_n ie.H'D". If this current is of the correct value the potential F"H will be equal to E_2 when the V-I characteristic of the load resistor R_n is drawn.

The drawing of the input part is done with the

following information already known:-

1) The anode current $I_a' = i_a'$, i.e. it is equal to

$$\frac{v_c}{R_c} - i_a''$$

2) The cathode voltage v_c.

3) The supply voltage E_{HT}

The drawing is commenced as follows:- Mark the value of the E_{HT} voltage, the cathode potential v_c and the current I_a' at point A', M' and B' respectively.

The V-I characteristic of the resistor R_L' is drawn next and then the rectangle A'B'C'D' is completed. This will give the potential drop in the valve equal to M'C'. At point M' draw several V-I characteristics of the valve V_1 for various values of the grid cathode potential v'_{g_n} until the one which passes through point D is obtained. This will then give the value of the input potential corresponding to the value of the output potential in the output diagram.

These two diagrams can be combined together in many ways, the most effective way being to reverse the current direction of the input diagram and join it to the output diagram on the voltage axis shown in Fig. 6.12. This will give a distinction between all the components of each valve, except the cathode resistor.

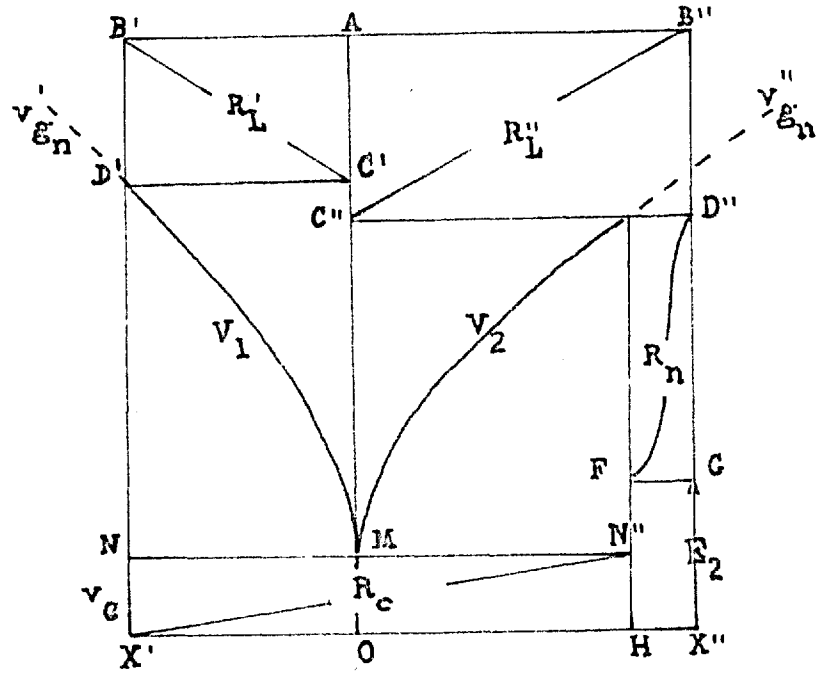


Fig. 6.12

When the output side is completely drawn, the V-I characteristic of the resistor R_c may be drawn starting at the current axis and passing through point N". OX' will then be the value of the current i_a'.

Since the cathode potential v_c (Fig. 6.12) does not remain constant over the operating range of the amplifier, the origin of the V-I characteristics of the valve for each diagram will not remain at the same point. It is therefore necessary for templates to be made for all the V-I characteristics of both valves. This can however be avoided, and all the V-I characteristics of the valve are plotted on the graph paper on which the diagram is going to be drawn having their origins at the same point. To do this it is convenient to rearrange the position of the V-I characteristics of all the elements even though the network topology is not then so clearly exhibited by the diagram. This rearranging is already shown in Fig. 6.3.

CONCLUSION.

The solutions of non-linear resistance networks have been discussed and one numerical method and two graphical methods have been demonstrated, each of them possessing different interesting properties. It is evident that to solve non-linear resistance networks without having to make any approximation of the non-linear properties, the graphical method explained in Chapters III IV V and VI is the best method to use. This method is also suitable for use in most non-linear problems where a fast calculating machine is not available.

The numerical method discussed in Chapter I is suitable for use with a digital computer. This is because the programme for solving linear simultaneous equations is already known, and only slight modifications in programming will be needed to cope with this numerical method. For a small number of unknown currents the calculation can be done by the use of a desk calculating machine or slide rule, but when the number of unknown currents becomes greater than five, it can lead to dire confusion in solving large linear simultaneous equations. This is a common problem

and to overcome this difficulty the digital computer may be employed. This numerical method is a form of the relaxation method, and therefore it is possible that the method may fail to converge. In my own experience of working out a large number of problems this has not occurred, but it cannot be assumed that the method will always converge. It would be most valuable if some conditions could be obtained under which the calculation will diverge, but this is mainly a mathematical problem.

The application of analogue computers to the numerical method has been considered and proved to be unsuitable, because the error in the results can be very great, due to the errors made by the analogue computer. Also it is only suitable for small problems and therefore there is no appreciable advantage over the desk calculating machine.

The graphical method shown in Chapter II is rather slow and tedious, but the purpose of looking into this problem was to investigate the theory of content graphically, and to extend it. From the graphic investigation of the theory of content, an interesting property is suggested. This is the property of instability. In many cases when the total content has more

than one stationary point, the system is unstable, and the circuit can exist in any one of those states. (Section 2.2). Slight external disturbance can easily cause the circuit to change its state. This property may well be used in the non-linear servo-system to indicate instability.

The graphical method discussed in Chapter III is most useful especially when it is used for the solution of second-order differential equations. When it is used for the solution of differential equations, the only error involved in the solution is caused by the use of a finite-difference to represent the differential equation. This error is common in all resistance network analogues and cannot be avoided. The error which is caused by the construction can easily be lessened by increasing the size of the graph paper used, and it is then very small compared with the error caused by finite-difference approximation. This method is economical and very simple to understand. The problem can be solved in a much shorter time by this method than by using analogue resistance network computers (as Hutcheon and Spalding did). The time taken in setting the non-linear function generator, and building a resistance board for each problem must be taken into account, not to mention the building of the whole computer, and in addition the

analogue computer cannot easily be used to solve problems which exhibit wave-like solutions. This graphical method can be expanded to cope with two dimensional problems of second-order partial differential equations. To do this it may be found necessary to use an automatic version of the electrical system explained in Section 3.7. The automatic electrical system will be more useful than the automatic analogue resistance computers built by Hutcheon and Spalding, J. Ph. Korthals Altes and W.J. Karplus, because of its convergence properties.

To solve the eigen-value problem by the method discussed in Chapter IV would take a slightly longer time than solving straight forward second-order differential equations. This is understandable because more solutions are required from the equation which gives less information than the straight forward second-order differential equation. The method explained by this section is entirely different from the method described by G. Liebmann.(3) His method however, leads to many calculations in order to obtain the eigen value and many repetitions of the process of setting up the analogue network. The new graphical method only requires a very few estimations of the value of K (Section 4.1). By using the equation 4.5 section 4.1

a more accurate value of K can be obtained. In the method described by G. Liebmann the eigen value can only be obtained if the solutions have zero slope at one boundary point. The graphical method discussed in this section overcomes this problem, and it also can be used to obtain higher eigen values which cannot be computed by other iterative processes.

The solution of fourth-order differential equations discussed in Chapter V is applied to many problems concerning structures and framework. When solving fourth-order differential equations by the graphical method, the use of two separate resistance networks is preferable as this reduces not only construction error but also gives a more accurate solution.

It has been brought to my attention that satisfactory solutions of the distribution of stress of a sheet pile wall with non-linear active and passive loads has not yet been obtained. The difficulty seems to be in finding a suitable method of solving non-linear fourth-order differential equations. The method discussed in Chapter V may well be suitable for solving such problems.

To solve two resistance networks simultaneously as in cases 2 and 3 in section 5.3, a longer time may

be needed to arrive at the solution, because there are two independent variables I_1 and I_1' and because I_{f_n} is a function of the R'-network. Therefore the use of the automatic electrical analogue of the graphical method (Section 3.7) will considerably speed up the computing process. This will also be necessary when the two dimensional fourth-order partial differential equations are explored.

The use of this new graphical method to assist in the design of valve amplifier circuits has the following advantages -

1) The maximum current through the valve is shown on the diagram.
2) The maximum dissipation of the valve is shown.
3) The precise relationship between the output and the input voltage is shown.
4) It is simple to understand and can be used in most types of valve circuits.

This method can also be used to assist in teaching, because of the clear relationship of voltage and current between each element in the circuit. In the case of the Cathode-follower, the diagram illustrates the reason why the output voltage approximately equals the input voltage. The diagram of the amplifier circuit

containing the cathode resistor illustrates the change of the over-all gain, for different values of the cathode resistor. Finally for the case of the "cathode coupled" amplifier, a more accurate relationship between the output and input voltage than in any existing analytical method can be obtained, and no difficulty arises in designing this circuit when both valves have different characteristics.

APPENDIX I

To illustrate the relation between the finite difference solution and the ladder resistance network, Figs. A1.1 and A1.2 are considered.

To solve a differential equation

$$\frac{d^2\varphi}{dx^2} = f(\varphi, x, K) \quad \text{.....(A1.1)}$$

with $\varphi = \varphi_0$ when $x = 0$

and $\varphi = \varphi_N$ when $x = x_N$

by means of the finite-difference approximation method, the first step is to replace the derivative by its finite-difference approximation so that

$$\frac{\varphi_{n+1} + \varphi_{n-1} - 2\varphi_n}{h^2} = f(\varphi_n, nh, K) \text{.....(A1.2)}$$

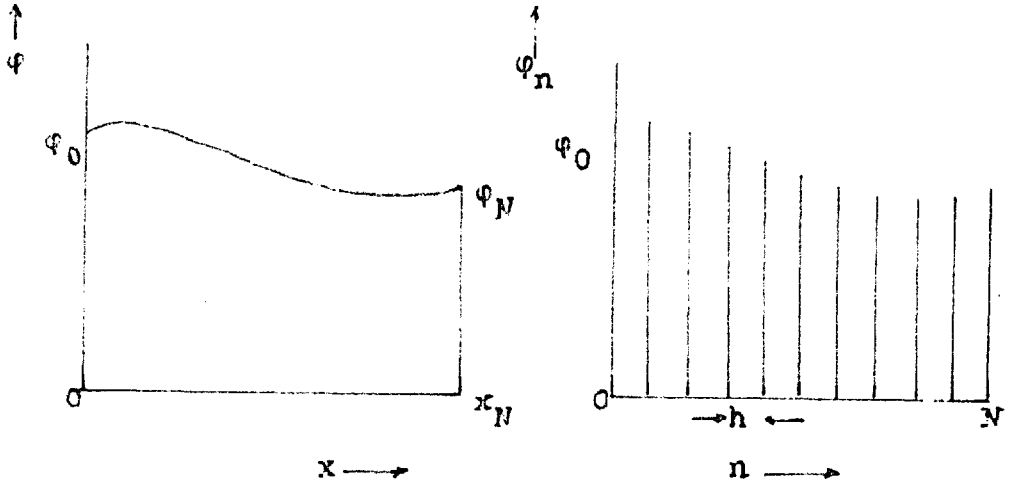
Consider next a ladder network of resistances as shown in Fig. A1.2, for which equating to zero the sum of the currents arriving at the n^{th} node give

$$\frac{V_{n-1} - V_n}{R} + \frac{V_{n+1} - V_n}{R} - \frac{V_n}{R_{f_n}} = 0$$

$$\frac{V_{n+1} + V_{n-1} - 2V_n}{R} = \frac{V_n}{R_{f_n}} \text{.....(A1.3)}$$

$$= I_{f_n} \text{.....(A1.4)}$$

This is similar in form to the finite-difference equation, (Eqn. A1.2) and this too can be used as the



Soln. of diff. Eqn(A1.1) Soln.of finite diff. Eqn.(A1.2)

Fig.A1.1

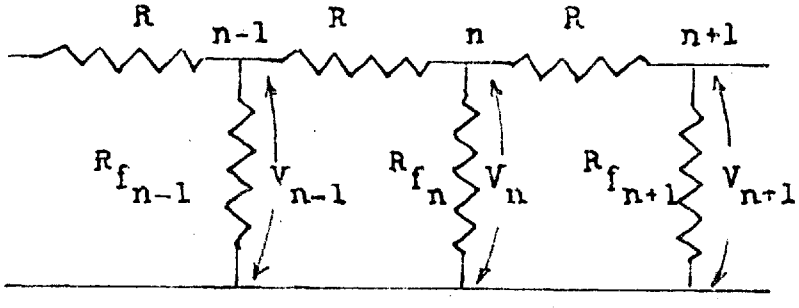


Fig.A1.2

basis of an analogue solution of Eqn. A1.2, and therefore of an approximate solution of the differential equation Eqn. A1.1.

For φ_n and V_n to have equal numerical values $\frac{h^2}{R} f(\varphi_n, nh, K)$ must be equal to V_n/R_{f_n} or I_{f_n}

$$\text{thus } R_{f_n} = \frac{RV_n}{h^2 f(\varphi_n, nh, K)}$$

If f is not proportional to φ, R_{fn} is then a non-linear resistance.

When $f(\varphi_n, nh, K)$ is negative, R_{f_n} is a negative resistance.

It is then obvious that if the circuit is connected up, with all the appropriate resistors in their places, and with correct boundary voltage applied at both ends of the network, then the voltage of node n will have a numerical value equal to the numerical value of φ_n.

The non-linear or negative resistor cannot be easily made, and therefore to solve this differential equation by the analogue method the current I_{f_n} has to be fed or extracted from the node n, depending on the sign of $f(\varphi_n, nh, K)$ i.e. negative = feeding, and positive = extracting.

APPENDIX II

The Design of the Universal Function Generator.

The main requirement of a universal function generator is that the instrument must be easily set to generate the required function with a high degree of accuracy. This requirement can be met by using a tapped potentiometer shunted between each pair of adjacent taps by a variable resistor, and this arrangement is powered by a current generator. The current generator consists of one pentode valve and a few components. It is powered by an external voltage stabilizer.

The non-linear function is built up by interpolation of linear function and in this case it is based on linear interpolation of voltages. The dependent variable is represented by the numerical value of the voltage V at the slider terminal of the tapped potentiometer. The independent variable is represented by the angular position of the potentiometer output arm.

The circuit of the universal function generator is shown in Fig.A1.1 It consists of a linear tapped potentiometer and variable resistors shunted

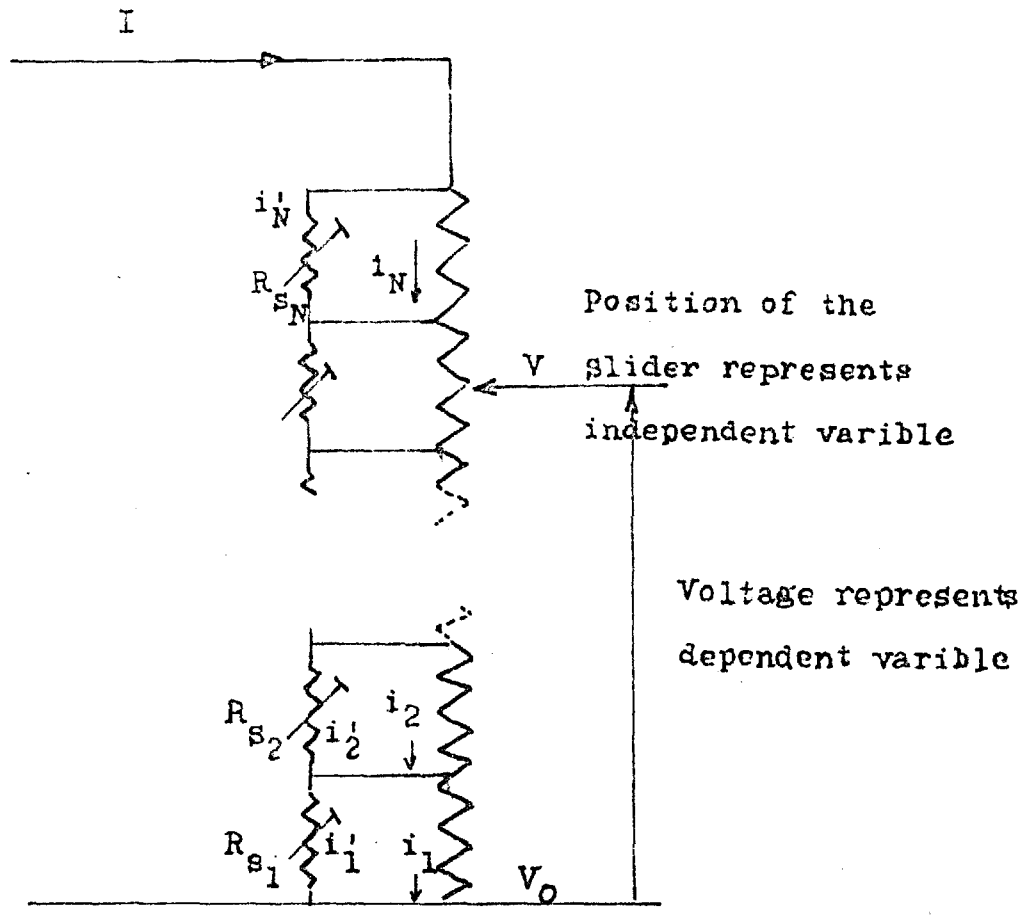

I
i'_N
R_{S_N}
i_N
V
Position of the slider represents independent varible
Voltage represents dependent varible
R_{S_2}
i'_2
i_2
R_{S_1}
i'_1
i_1
V_o

Fig.1

across each pair of taps. The eleven taps divide the potentiometer equally with regard to both the resistive value and the angle of rotation of the slider. The values of all the shunt resistors are also equal and have a maximum value of about ten times the value of the resistance between taps, and a minimum value of zero.

The difference between the potential of two adjacent taps is the value of the slope of the required function at the corresponding independent variables. To obtain the slope, the shunt resistor across that pair of taps is adjusted. Note that with this circuit arrangement the adjustment of the value of any shunt resistor does not alter the slope of the other pair of taps. This is because the total amount of current flowing between each pair of taps remains constant and each current is equal to current I of the current generator. i.e. $i_1+i_1' = i_2+ i_2' = \ldots\ldots\ldots = i_n+i_n' = I$. Therefore any alteration in the value of its shunt resistor does not alter the total value of the current flowing between that pair of taps. The value of current I also determines the value of the maximum slope between any pair of

taps, and therefore the maximum slope of the function to be generated determines the value of the current I.

The current generator is constructed from a pentode valve as shown in Fig. 2, and it is powered by an external stabilized H.T. supply. If the external supply is not connected to any reference potential, then the voltage at point b, corresponding to the voltage V_o of Fig. 1 can be connected to any desired reference voltage.

To set up the function generator to the required function as shown in Fig. 3, the X axis is first divided into ten equal intervals, each interval corresponding to a pair of taps on the potentiometer. The maximum slope is next located. On this occasion it is between 1 and 0 which corresponds to taps 1 and 0. The shunt resistor connected across these taps is adjusted to its maximum value so that most of the current flowing across these taps will flow in the resistance coil between these taps of the tapped potentiometer. Next a volt meter is connected between taps 1 and 0 and the value of the current I of the current generator is adjusted until the value of the required slope is reached. To adjust the rest

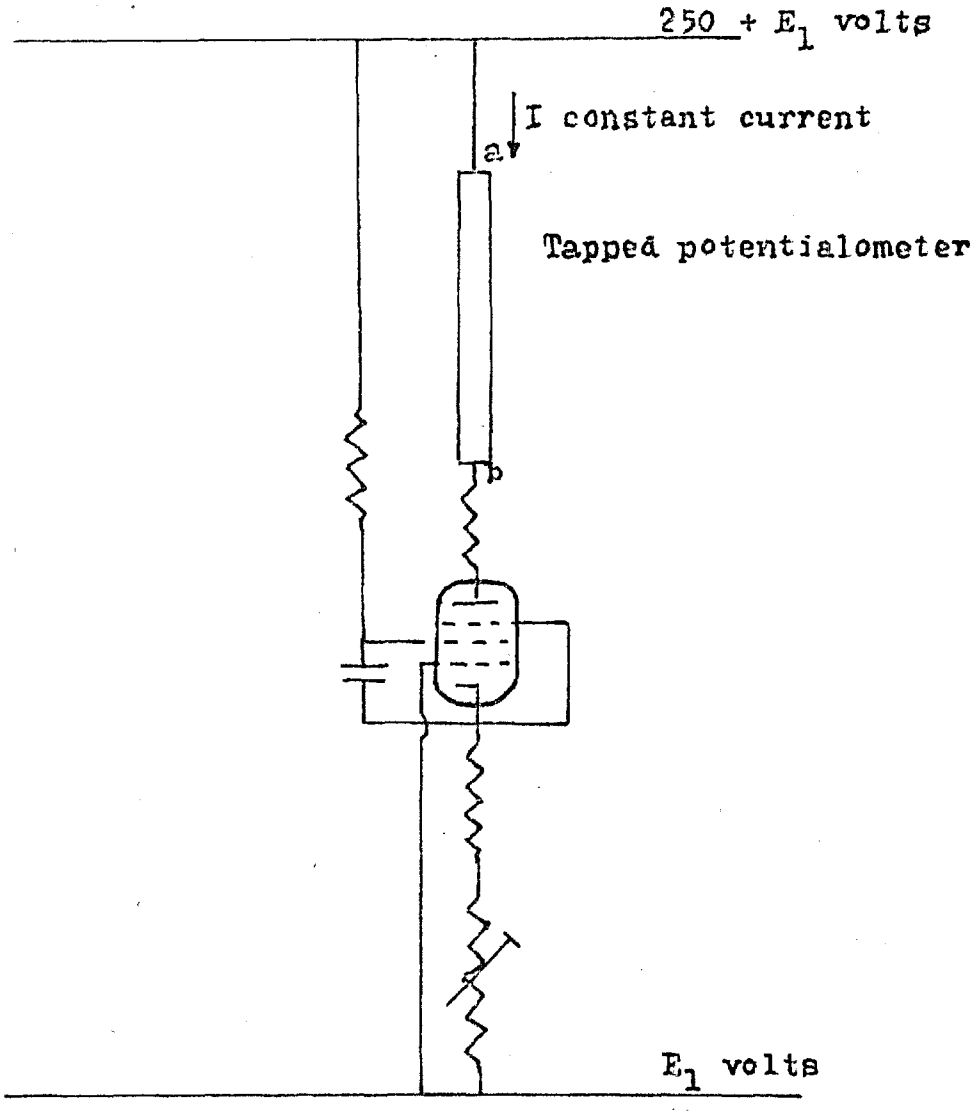

250 + E_1 volts
I constant current
a
Tapped potentialometer
b
E_1 volts

Fig. 2

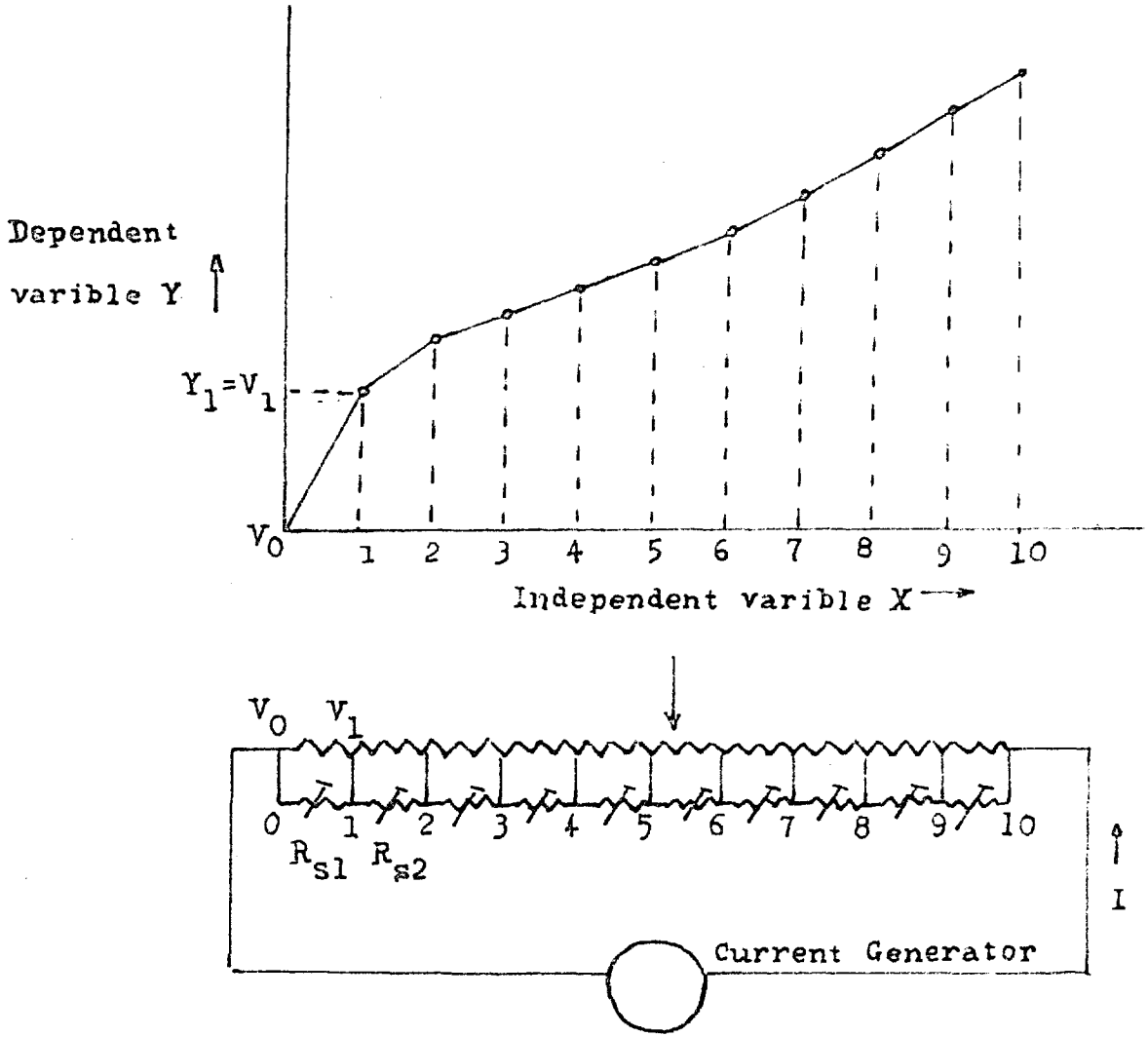

Dependent
varible Y
$Y_1=V_1$
V_0
1 2 3 4 5 6 7 8 9 10
Independent varible X
V_0 V_1
0 1 2 3 4 5 6 7 8 9 10
R_{s1} R_{s2}
I
Current Generator

Fig. 3

of the slopes is quite simple and is done by connecting a volt meter across the appropriate pair of taps and adjusting its shunt resistance until the required value of the slope is obtained. No re-checking is necessary if the current I remains constant.

APPENDIX III

To llustrate the relation between the finite-difference solution of Ordinary Linear and Non-linear Differential Equations and the ladder resistance network. The equation in the form of

$$\frac{d^2\varphi}{dx^2} + f(x)\frac{d\varphi}{dx} - K\varphi + F(x,\varphi) = 0 \quad \ldots\ldots(A3.1)$$

is considered.(Soroka[26]). The presence of a first derivative introduces an additional feature to the problem, i.e. causes the values of the series resistances $R_1, R_2 \ldots R_N$(Fig.A3.1) to be different and the calculation necessary to obtain these values is as follows:-

The finite-difference form of the first derivative $(\frac{d\varphi}{dx})_n$ is $\frac{\varphi_{n+1} - \varphi_{n-1}}{2h}$. Thus the finite difference form of the equation A3.1 becomes:-

$$\frac{\varphi_{n+1} + \varphi_{n-1} - 2\varphi_n}{h^2} + f_n(\frac{\varphi_{n+1} - \varphi_{n-1}}{2h}) - K\varphi_n + F(x_n,\varphi_n) = 0 \quad \ldots\ldots\ldots\ldots\ldots(A3.2)$$

If n is equal to 1, the equation becomes

$$\frac{\varphi_2+\varphi_0-2\varphi_1}{h^2} + \frac{f_1}{2h}(\varphi_2-\varphi_0) - K\varphi_1 + F(x_1,\varphi_1) = 0 \ldots\ldots(A3.3)$$

Collecting terms involving common solutions.

$$(\frac{1}{h^2} - \frac{f_1}{2h})\varphi_0 + (\frac{1}{h^2} + \frac{f_1}{2h})\varphi_2 - (K + \frac{2}{h^2})\varphi_1 + F(x_1,\varphi_1) = 0 \quad (A.3.4)$$

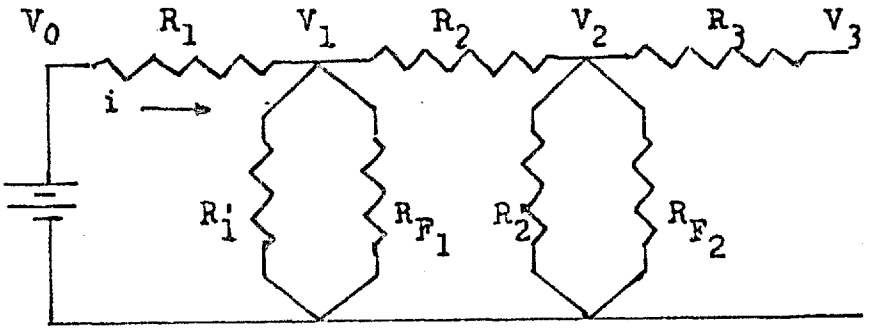


Fig. (A3.1)

Consider next a ladder network of resistances as shown in Fig.A3.1 for which, equating to zero the sum of the currents arriving at the first node, gives

$$\frac{V_0-V_1}{R_1} + \frac{V_2-V_1}{R_2} - \frac{V_1}{R_{F_1}} - \frac{V_1}{R_1'} = 0 \qquad \text{.....(A3.5)}$$

$$\frac{V_0}{R_1} + \frac{V_2}{R_2} - \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_1'}\right)V_1 - \frac{V_1}{R_{F_1}} \equiv 0 \qquad \text{.....(A3.6)}$$

This is similar in form to the differential equation, Eqn.(A3.4) and

$$\frac{1}{R_1} = \left(\frac{1}{h^2} - \frac{f_1}{2h}\right) \qquad \text{.....(A3.7)}$$

$$\frac{1}{R_2} = \left(\frac{1}{h^2} + \frac{f_1}{2h}\right) \qquad \text{.....(A3.8)}$$

$$\frac{1}{R'} = K \qquad \text{.....(A3.9)}$$

$$\frac{1}{R_{F_1}} = -F(x_1,\varphi_1)/V_1 = -F(x_1,V_1)/V_1 \qquad \text{.....(A3.10)}$$

The term $\frac{f_1}{2h}$ in Eqn.A3.7 and A3.8, is due to the presence of the first derivative in Eqn. A3.1. Since it is added to the eqn.A3.8 and subtracted in the eqn.A3.7 it means that the resistors connecting nodes have different values, depending on which end of the resistance is being considered as the node. When the 2nd node is considered the resistor R_2 has the value $\frac{1}{R_2} = \left(\frac{1}{h^2} - \frac{f_2}{2h}\right)$.

It is important to make the value of R_2 equal in both cases.

When the 2nd node is considered, the equations (A3.4) and (A3.6).become

$$(\frac{1}{h^2} - \frac{f_2}{2h})\varphi_1 + (\frac{1}{h^2} + \frac{f_2}{2h})\varphi_3 - (K + \frac{2}{h^2})\varphi_2 + F(x_2,\varphi_2) = 0 \quad \text{.....(A3.11)}$$

$$\frac{V_1}{R_2} + \frac{V_3}{R_3} - (\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_2'})V_2 - \frac{V_2}{R_{F_2}} = 0 \quad \text{.....(A3.12)}$$

If Eqns. (A3.6) and (A3.11) are multiplied by K_1 and K_2 respectively, they yield

for the 1st. node

$$\frac{1}{R_1} = (\frac{1}{h^2} - \frac{f_1}{2h})K_1 \quad \text{.....(A3.13)}$$

$$\frac{1}{R_2} = (\frac{1}{h^2} + \frac{f_1}{2h})K_1 \quad \text{.....(A3.14)}$$

$$\frac{1}{R_1'} = KK_1 \quad \text{.....(A3.15)}$$

$$\frac{1}{R_{F_1}} = -F(x_1,V_1)K_1/V_1 \quad \text{.....(A3.16)}$$

and for the 2nd. node

$$\frac{1}{R_2} = (\frac{1}{h^2} - \frac{f_2}{2h})K_2 \quad \text{.....(A3.17)}$$

$$\frac{1}{R_3} = (\frac{1}{h^2} + \frac{f_2}{2h})K_2 \quad \text{.....(A3.18)}$$

$$\frac{1}{R_2'} = KK_2 \quad \text{.....(A3.19)}$$

$$\frac{1}{R_{F_2}} = -F(x_2, V_2)K_2/V_2 \quad \text{.....(A3.20)}$$

Since R_2 for both nodes must have the same value then the Eqn. A3.14 must be equal to Eqn.A3.17. This gives

$$K_2 = \frac{(\frac{1}{h^2} + \frac{f_1}{2h})}{(\frac{1}{h^2} - \frac{f_2}{2h})} \cdot K_1$$

For node n then

$$K_n = \frac{\frac{1}{h^2} + \frac{f_{n-1}}{2h}}{\frac{1}{h^2} - \frac{f_n}{2h}} \cdot K_{n-1}$$

and $$\frac{1}{R_n} = (\frac{1}{h^2} - \frac{f_n}{2h})K_n \quad \text{.....(A3.21)}$$

APPENDIX IV

W.E.Milne("Numerical Calculus") has shown that the slope $(\frac{dy}{dx})_n$ can be expressed in terms of the solution y_n, y_{n+1}, y_{n+2}.....and a remainder term as follows:-

In the expression for the remainder term; the notation $y^{(n+1)}$ means the $(n+1)^{th}$ derivative of y taken at some point x = z lying between x_0 and x_N.

N = 2, three points

$$(\frac{dy}{dx})_0 = \frac{1}{2h}(-3y_0 + 4y_1 - y_2) + \frac{h^2}{3}.y^{(3)}$$

$$(\frac{dy}{dx})_2 = \frac{1}{2h}(y_0 - 4y_1 + 3y_2) + \frac{h^2}{3}.y^{(3)}$$

N = 3, four points

$$(\frac{dy}{dx})_0 = \frac{1}{6h}(-11y_0 + 18y_1 - 9y_2 + 2y_3) - \frac{h^3}{4}y^{(4)}$$

$$(\frac{dy}{dx})_3 = \frac{1}{6h}(-2y_0 + 9y_1 \cdot\cdot 18y_2 + 11y_3) + \frac{h^3}{4}.y^{(4)}$$

N = 4, five points

$$(\frac{dy}{dx})_0 = \frac{1}{12h}(-25y_0+48y_1-36y_2+16y_3-3y_4) + \frac{h^4}{5}.y^{(5)}$$

$$(\frac{dy}{dx})_4 = \frac{1}{12h}(3y_0-16y_1+36y_2-48y_3+25y_4) + \frac{h^4}{5}.y^{(5)}$$

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Solution of ladder networks containing non-linear resistances

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Abstract

A graphical method of solving network problems involving non-linear and/or negative resistances is described and fully developed for the case of ladder networks. Such networks have been used for the solution of non-linear differential equations, using an analogy based on finite differences, and the examples given in this paper are of this application.

Introduction

In 1951, Cherry[1] gave a description of a planar electrical resistance network which, in a single diagram, showed the network topology, the *V–I* characteristics of each resistance, Kirchhoff's circuit laws and the power relationships. The present paper uses Cherry's diagram to solve problems involving resistance networks and shows that it is particularly useful when some or all of the resistances are non-linear. It is also applied to networks containing negative resistances. A resistance network can be the analogue of the finite-difference form of a non-linear differential equation and, as many papers on this analogy have appeared in this Journal, our illustrative examples will be taken from this field. The new method of solution is of special interest because it can be used for those cases where the iterative analogue techniques described by Liebmann and Bailey,[2] Karplus[3] and Hutcheon[4] fail to converge.

Ladder network of positive resistances

To introduce the method, the ladder network of Fig. 1 is considered. In this, all the series elements are linear

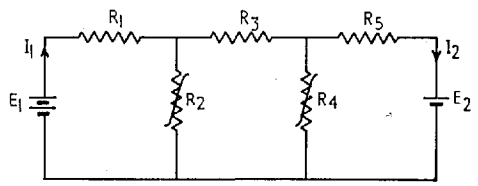


Fig. 1. Ladder network with non-linear shunt elements

resistors, the shunt resistors are non-linear resistors and e.m.f.'s E_1 and E_2 ($E_1 > E_2$) are applied at the ends. The voltages at the two intermediate nodes have to be found. However, for the moment, suppose that the problem has already been solved so that the final diagram can be exhibited, as in Fig. 2. Within an area defined by the two rectangles of sides $E_1 \times I_1$ and $E_2 \times I_2$ the following appear:

(*a*) Portions of the *V–I* characteristics of the resistors 1, 3 and 5 which have slopes R_1, R_3 and R_5 respectively;

(*b*) Portions of the non-linear *V–I* characteristics of the shunt resistors 2 and 4, starting at zero potential and continuing up to their actual operating points;

(*c*) All these *V–I* characteristics appear, diagonally, inside minor rectangles which are fitted together in a way which

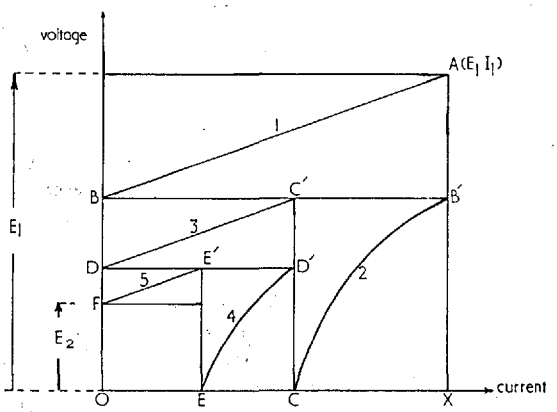


Fig. 2. Rectangle diagram for network of Fig. 1
$OX = I_1$, $OE = I_2$ and $XA = E_1$

satisfies both of Kirchhoff's Laws. For example, the sum of the voltage drops across 1, 3 and 5 should be $E_1 - E_2$, and this relationship is satisfied by the voltages represented along the left-hand edge of the main rectangle (on which all the node potentials appear). Next, taking horizontal distances in Fig. 2, the current in 1, which is I_1, can be seen to equal the sum of the currents in 2 and 3; similarly, the sum of the currents in 4 and 5 is the current in 3.

(*d*) Finally, the area of each minor rectangle represents the power dissipated in a resistance, and the total area of these minor rectangles 1–5 is $E_1I_1 - E_2I_2$, which is the power supplied by the two sources, so that the law of conservation of energy is satisfied.

The method of solving the problem when only E_1 and E_2 are given is very simple. After estimating I_1, an attempt is made to draw the diagram, to scale, using the *V–I* characteristics in the following order: 1, 2, 3, 4 and 5. (Templates will be needed for the non-linear *V–I* characteristics, and, if not many cases are to be solved, these can be cut out of graph paper of the same kind as is used for the rectangle diagram. If a lot of work is to be done, it is worth making metal templates for the characteristics.) The drawing is made as follows. On Fig. 2, A is the point (E_1, I_1); AB is drawn of slope R_1, whence AB' is I_1R_1, leaving voltage $XB' = OB$ for the rest of the network. The current traversing the non-linear resistor R_2' is found by sliding its *V–I* characteristic to the position 2 (it passes through B' and has its origin at C) whence current CX flows through it and OC remains to traverse R_3. $C'D$ of slope R_3 is drawn next, from C' which is vertically above C, and OD will be the

voltage across R_4. The current in R_4 is obtained by sliding its V–I characteristic to the position 4. $E'F$ is drawn of slope R_5 and, if I_1 were correctly chosen, OF would be equal to E_2. On completion of the drawing, the distance OF on Fig. 2 is unlikely to be equal to E_2, but it will be obvious in what sense to adjust I_1 in order to improve it, and a second attempt can then be made. We have been surprised by the speed with which this trial and error process leads to the correct solution, especially when a little intelligent interpolation is applied.

If the ladder network is long and highly attenuating and $E_2 = 0$, the voltages remote from the source E_1 cannot be found accurately unless a very large drawing is used. In this case, it is better to improve I_1, in the latter stages only, by numerical calculation. If OA is calculated for two values of I_1, preferably one larger and one smaller than the true value, improvement by interpolation is rapid because of the good start given by the graphical method. (The calculations are simple: starting with E_1 and I_1, the voltage drops and currents can be found as the ladder is traversed from left to right.)

Example 1

The first example is taken from Hutcheon and Spalding's paper[5] on steady-state heat flow, with convective losses, in uniform bars. The differential equation to be solved is:

$$\frac{d^2\theta}{dy^2} = k\theta^{5/4} \qquad (1)$$

in which θ represents temperature at a distance y along the bar. The numerical value of k is 12·5 and the boundary values are $\theta = 1$ at $y = 0$ and $d\theta/dy = 0$ at $y = 1$, for the example presented here.

It is not necessary in this Journal to repeat the derivation of the finite-difference analogue network. It is sufficient to say that the node potentials V on the ladder network of Fig. 3 should be numerically equal to the temperatures θ in the bar at the values of y indicated. (Any discrepancy between the values of θ satisfying equation (1) and the node potentials will be due to the coarseness of the finite-difference approximation used.)

For the network of Fig. 3, I_1 was estimated to be 0·40 A. Foolscap graph paper was used and a paper template cut out for the non-linear V–I characteristic. On successive drawings it was found that $I_1 = 0{\cdot}40$ A was too small, 0·50 A was too much, 0·45 A was too small, 0·46 A was too big and 0·458 A seemed very close. The node voltages were then, reading from left to right on Fig. 3:

1·0, 0·77, 0·50, 0·33, 0·22, 0·15, 0·11, 0·09 V.

Hutcheon and Spalding's values are obtained from their Fig. 7*a*, bottom graph, and are, of course, similar.

Ladder network containing negative resistances

Negative resistances appear in finite-difference analogue networks as shunt elements when, if

$$\nabla^2\phi = f(\phi, x, y, z) \qquad (2)$$

is being solved, $f(\phi, x, y, z)$ is negative. (In passing, it should be mentioned that constant-current sources are often used in place of the shunt resistors, in analogue equipment, particularly if the resistances are negative.)

If the solution ϕ is wave-like along the ladder, the normal iterative analogue computers fail to converge to a solution under conditions given by Fisher,[6] although Korthals Altes[7] has suggested a tedious practical method of double iteration which does give convergence. No such troubles beset the graphical method of this paper.

Example 2

A case for which the analogue computers will encounter convergence difficulties is taken as the next example. Consider:

$$\frac{d^2\phi}{dx^2} + k^2\phi = 0 \qquad (3$$

with $k^2 = 1$, $\phi = 10$ at $x = 0$ and $\phi = 0$ at $x = 3\pi/2$, for which the exact solution is $\phi = 10\cos x$. The analogue network is shown in Fig. 4 for a finite difference interval of $\pi/4$. Its solution by means of the rectangle diagram is carried out as already described but it is slightly complicated by the fact that the component rectangles will trespass outside the main rectangle $E_1 \times I_1$. There is however a simple rule to enable the diagrams to be drawn quickly, a rule which applies also to the ladder network of positive resistances.

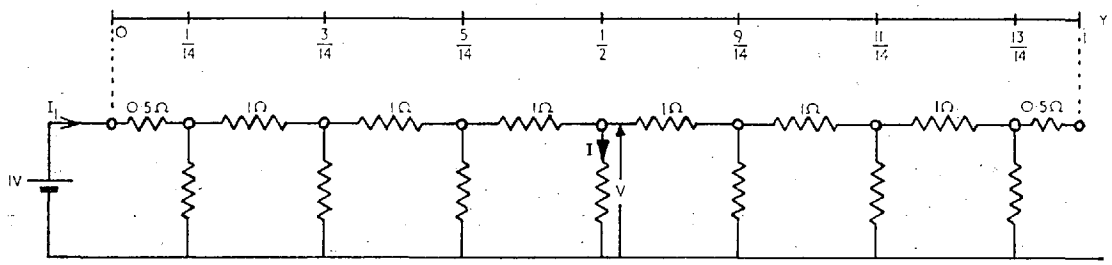


Fig. 3. Finite-difference analogue network for heat-flow problem

For all shunt elements:

$$I = \frac{12{\cdot}5}{49} V^{5/4},$$

where V is in volts and I in amperes

Rule

(*a*) The V–I characteristics are used in the order 1, 2, 3, 4, . . . as labelled on Fig. 4.

(*b*) Each V–I characteristic lies diagonally inside a rectangle.

(*c*) Each successive V–I curve has its operating point at that corner of the previous rectangle which is not occupied by the previous V–I curve and which is not on either the V- or I-axis of the diagram.

(*d*) If a series element is being dealt with, the V–I curve has its origin on the V-axis and if a shunt element, it is on the I-axis.

Applying this to the network of Fig. 4, an initial estimate of I_1 is made and this determines the starting point A

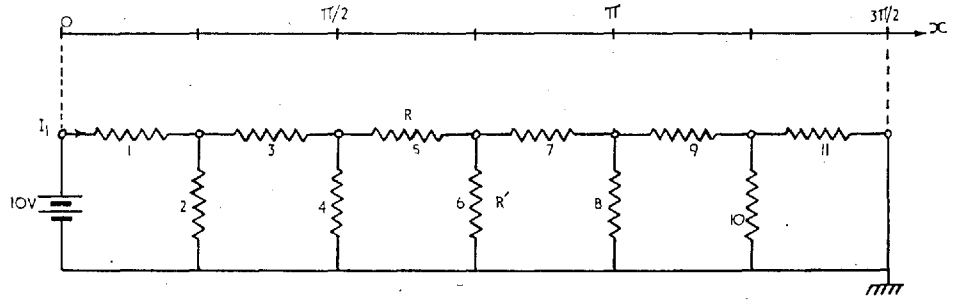


Fig. 4. Network for vibration problem.
All shunt elements $R = 1\ \Omega$
All shunt elements $R' = -1{\cdot}621\ \Omega$

(co-ordinates: 10V, I_1A) on the diagram shown in Fig. 5. A line of slope 1 Ω is drawn from A to intersect the voltage axis at B. This defines the first rectangle, for resistor 1, of which B' is the corner which satisfies the rule. From B', a line of slope $-1{\cdot}621\ \Omega$ is drawn to intersect the current axis at C. This line is the diagonal of the second rectangle,

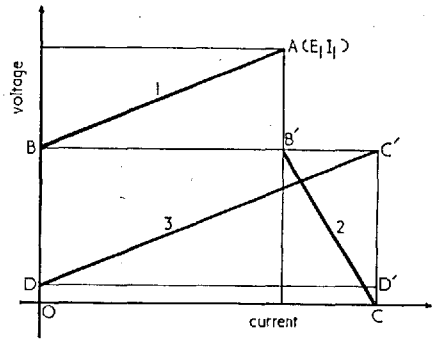


Fig. 5. V–I diagram for the first three elements of the network in Fig. 4
The node potentials are E_1, OB and OD

for resistance 2, whose corner C' is the starting point for the next V–I line. Close examination of Fig. 5 shows that Kirchhoff's laws will be satisfied on the diagram. When the diagram is completed, the value of I_1 can be assessed and another trial made. In the example, the correct value of I_1 is 2·13 A and Fig. 6 shows a graph of the node voltages along the network. These depart slightly from $10 \cos x$, but the error is due to finite differencing and is not relevant to the network-solving; errors in the network solution cannot be detected in Fig. 6.

Other applications

It seems likely that the solution of some two-dimensional problems involving the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(\phi, x, y),$$

and the solution of one-dimensional elastic problems, involving a fourth-order differential equation, can be carried out by extensions of the graphical method. We are already tackling *eigen*value problems. However, the application of non-linear resistance networks is not restricted to the single group of circuits considered so far. For example, for mine-ventilation networks[8] and gas and water-distribution systems, useful preliminary investigations might well be carried out graphically. Bridge networks containing tungsten filament

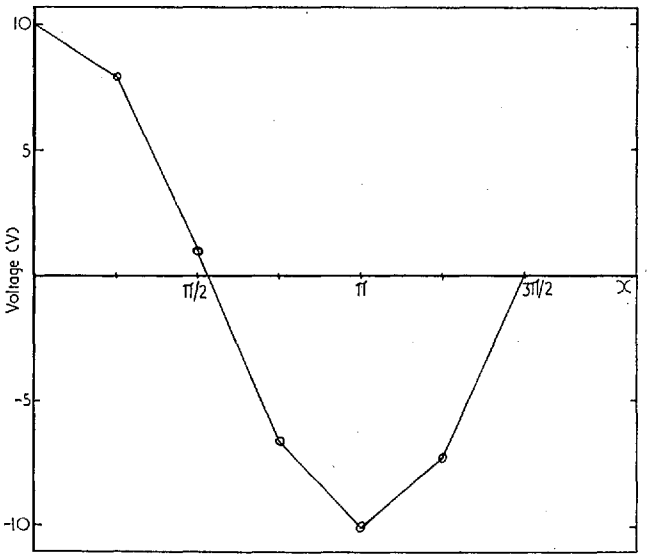


Fig. 6. Node potentials on network of Fig. 4 approximate to $10 \cos x$

lamps and d.c. amplifier networks might provide other useful applications.

This paper has described a method of solving non-linear resistance networks. It should, however, be possible to interpret it as a graphical method of solving the finite difference approximation to a differential equation without mentioning the resistance network analogy at all.

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