

PLANCK and WMAP constraints on generalised Hubble flow inflationary trajectories

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(Dated: July 2, 2018)

We use the Hamilton–Jacobi formalism to constrain the space of possible single field, inflationary Hubble flow trajectories when compared to the WMAP and PLANCK satellites Cosmic Microwave Background (CMB) results. This method yields posteriors on the space of Hubble Slow Roll (HSR) parameters that uniquely determine the history of the Hubble parameter during the inflating epoch. The trajectories are used to numerically determine the observable primordial power spectrum and bispectra that can then be compared to observations. Our analysis is used to infer the most likely shape of the inflaton potential $V(\phi)$ and also yields a prediction for, f_{NL} , the dimensionless amplitude of the non-Gaussian bispectrum.

I. INTRODUCTION

Recent PLANCK results [1] have confirmed, with the highest precision to date, the existence of a spectrum of primordial curvature perturbations on super-horizon scales with a power law with a spectral index close to but not equal to unity. This picture has now been verified over roughly three decades of scales probed by primary Cosmic Microwave Background (CMB) anisotropies that can be related to the primordial curvature perturbations on super-horizon scales via a well defined set of photon perturbation transfer functions. The quoted value for the scalar spectral index of $n_s = 0.9603 \pm 0.0073$ seems to be in good agreement with many models of cosmological inflation [2]. The fact that n_s is not compatible with unity is also interpreted by many to support the actual *existence* of an inflationary epoch in the very early universe.

The interpretation of the result, in the context of inflationary model selection, is complicated by the large number of inflationary models that are compatible with the CMB observations (see for example section 2 of [2] for a review of the landscape of models). The models range from the simplest chaotic model with a single scalar field to massively, multi-field models inspired by dimensional compactification in string theory. A typical discriminatory approach is to analyse the consistency of a particular model in the space of parameters such as n_s and tensor-to-scalar ratio r constrained directly by the data. However it becomes readily apparent that this combination does not refine the space of possible models to an extent at which conclusions about the fundamental nature of the inflaton can be made. Including higher-order parameters (in either slow-roll approximation or perturbation expansion) such as running of the spectral index with wavenumber k $dn_s/d \ln k$ or non-Gaussianity amplitude f_{NL} greatly enhances the ability to reject or falsify

models. However the data has not reached the sensitivity to detect the expected higher-order signals.

An alternative method adopted here is known as the Hubble flow equation method. This method [3–6] assumes inflation was driven by a single scalar field and employs the Hamilton-Jacobi framework [7] to define a hierarchy of differential equations that can be used to generate inflationary trajectories consistent with *any* inflationary potential up to a certain order in derivatives of the Hubble rate H with respect to the inflaton field value ϕ . Within this framework one can dispense with proposing a *single* model consisting of a parametrised potential and constrain directly the space of allowed inflationary trajectories described by the evolution of the Hubble parameter $H \equiv \dot{a}/a$.

This approach allows one to compare all possible inflationary trajectories with a given complexity with no loss of accuracy. This is because, for a given truncation of the hierarchy of differential equations, the value of the Hubble rate during the inflationary epoch can be evaluated to arbitrary precision. Once the history of H has been obtained it is then possible to calculate all observable quantities to the desired precision irrespective of whether the trajectory satisfies slow-roll conditions.

One can then use the hierarchy of Hubble flow parameters as the base parameters being constrained. This has two advantages. Firstly, the space of Hubble flow parameters explores the space of all inflationary potentials allowed at a certain order consistently. Secondly, Bayesian model comparison is simplified for a given Hubble flow order since there is a *single* model proposition which reduces the selection to a comparison of likelihood values for two different points in the Hubble parameter space with no need to calculate Bayesian evidence.

In the work reported here PLANCK total intensity results, together with WMAP polarisation results [8] are used to directly constrain the space of Hubble flow parameters with priors given by the set of Assumptions 1–4. The constrained space of Hubble flow parameters can then be related to “conventional” parameters including

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n_s , r , $dn_s/d\ln k$, and f_{NL} without the need to redefine the model from a Bayesian perspective. The definition of *detections* of the conventional parameters $n_s - 1$, r , $dn_s/d\ln k$, and f_{NL} has no meaning within this analysis and the constraints can be viewed as ranges *allowed* by the observations i.e. *predictions* given the underlying set of assumptions.

This *paper* is organised as follows. In section II we review the Hubble flow formalism and describe how to obtain observables to compare with data in section III. In section IV we show the results obtained by constraining Hubble flow trajectories using the latest CMB data. We also describe the derived constraints on primordial spectral parameters and on the inflaton potential. In section V we discuss our results and future extensions.

II. HUBBLE FLOW EQUATIONS

The Hamilton-Jacobi approach to analysing the dynamics of inflation consists of changing the independent variable in the Friedmann equations from cosmological time t to the value of the inflaton scalar field ϕ . The only assumption required for this change of variable is that ϕ is a monotonic function of t . The Friedmann equation and the inflaton's equation of motion then take on the following form

$$\dot{\phi} = -2M_{\text{pl}}^2 H'(\phi), \quad (1)$$

$$[H'(\phi)]^2 - \frac{3}{2M_{\text{pl}}^2} H(\phi)^2 = -\frac{1}{2M_{\text{pl}}^4} V(\phi), \quad (2)$$

where dot denotes a derivative with respect to t , prime denotes a derivative with respect to ϕ , $H \equiv \dot{a}/a$ is the Hubble rate for the FRW scale factor $a(t)$, M_{pl} is the Planck mass and $V(\phi)$ is the inflaton potential. One of the advantages of performing this change of variables is that one can merely pick a function $H(\phi)$ and this will correspond to an exact solution of a corresponding potential $V(\phi)$ in (2).

The system can be further simplified by introducing an infinite hierarchy of Hubble flow parameters ¹

$${}^\ell\lambda = (2M_{\text{pl}}^2)^\ell \frac{(H')^{\ell-1}}{H^\ell} \frac{d^{\ell+1}H}{d\phi^{\ell+1}}. \quad (3)$$

The first of these parameters, ${}^0\lambda \equiv \epsilon$, is a proxy for the acceleration of the scale factor and it is straightforward to verify that the relation

$$\epsilon = 2M_{\text{pl}}^2 \left(\frac{H'(\phi)}{H(\phi)} \right)^2 = \frac{-\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2M_{\text{pl}}^2 H^2} < 1, \quad (4)$$

is a necessary and sufficient condition for the universe to be undergoing inflation with $\ddot{a}/a > 0$ ². The $\ell = 1$ and 2 flow parameters can also be identified with the usual slow roll parameter $\eta = {}^1\lambda = -(\ddot{\phi}/H\dot{\phi})$ and $\xi = {}^3\lambda$.

A further change of variable can be introduced by using the relation between the rate of change in e -folds $N = \ln(a/a_i)$, where a_i is the value of the scale factor at the beginning of inflation, and cosmological time t with $dN/dt = H$. The entire system can then be re-cast as an infinite hierarchy of differential ‘‘Hubble flow’’ equations with N as the independent variable

$$\frac{dH}{dN} = -\epsilon H, \quad (5)$$

$$\frac{d\epsilon}{dN} = 2\epsilon(\epsilon - \eta), \quad (6)$$

$$\frac{d{}^\ell\lambda}{dN} = [\ell\epsilon - (\ell - 1)\eta] {}^\ell\lambda - {}^{\ell+1}\lambda, \quad (7)$$

with solutions $H(N)$ and ${}^\ell\lambda(N)$.

This is the most natural set of variables to use when constructing single field inflationary trajectories and the solution of the infinite system provides a *complete* set of exact solutions for the background evolution that are consistent with single field inflation and monotonic time evolution of ϕ . Truncating the hierarchy at ℓ_{max} provides an *incomplete* set of solutions that are nonetheless still *exact*.

In practice a set of solutions for a given ℓ_{max} is obtained by integrating the system with a set of random initial conditions for H and ${}^\ell\lambda$ for $\ell = 0, 1, \dots, \ell_{\text{max}}$. The system can be integrated forward or backwards to obtain an exact solution describing the dynamics of the background within a required window in e -foldings N .

A. Hubble flow measure

The Hubble flow method of generating random inflationary trajectories has a well known measure problem due to the seemingly arbitrary choice of proposal density and location for the initial conditions in H and ${}^\ell\lambda$. The existence of attractors in the phase space of the ${}^\ell\lambda$ complicates the interpretation of the imposed measure and the nature of trajectories obtained.

A number of choices have been made in the literature [3–6]. These include starting at arbitrary points and integrating forward or backwards to select trajectories with enough e -folds. Different choices have been made with regards to the encounters with fixed points in the HSR phase space where ϵ asymptotes to a constant and ${}^\lambda\ell \rightarrow 0$ for $\lambda > 1$. These can be interpreted as eternally inflating solutions that can be allowed or discarded if only trajectories where inflation ends are to be allowed. In all cases

¹ Sometimes called Hubble-Slow-Roll (HSR) parameters to contrast with the Potential-Slow-Roll (PSR) parameters.

² This is in contrast to the the PSR ϵ_V for which $\epsilon_V < 1$ is only an *approximate* condition for inflation.

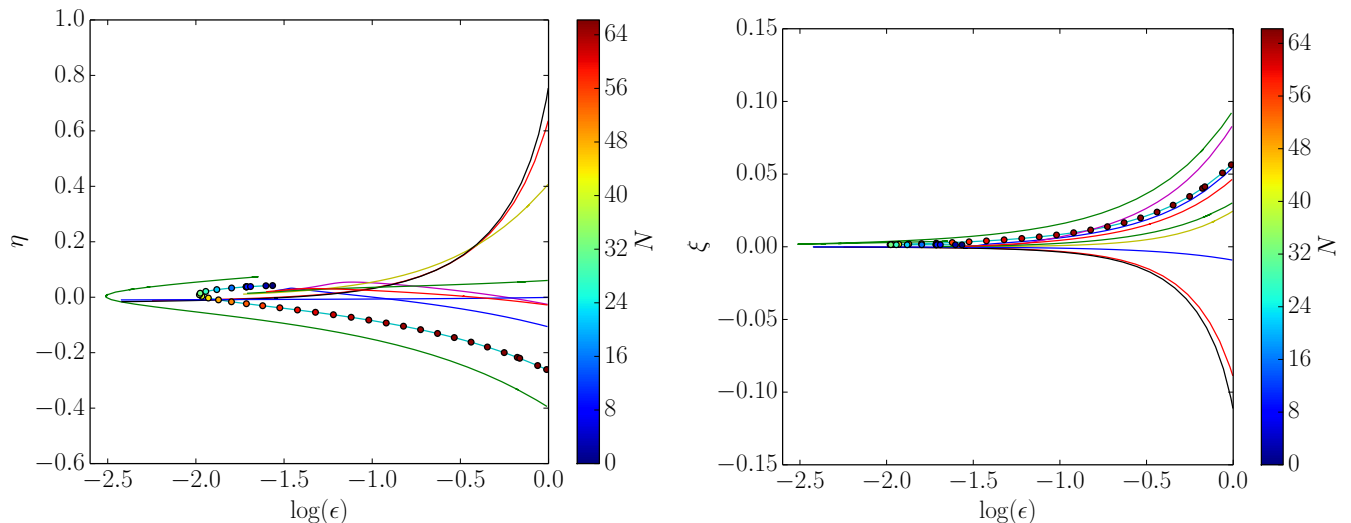


FIG. 1: Ten random trajectories drawn using the scheme described in (??). The evolution of η (left) and ξ (right) are plotted against $\log(\epsilon)$ from the end of inflation ($\epsilon = 1$) back to a time when the largest scale of interest k_* was a few order of magnitude smaller than the horizon scale. One of the trajectories also shows points colour coded by e -folding number N as points colour coded with respect to e -fold N . $N = 0$ corresponds a few e -folds before the k_* exits the horizon. All trajectories are evolving away (as N increases) from a “slow-roll” attractor with $\epsilon \ll 1$, $\eta \ll 1$, and $\xi \ll 1$. For most trajectories observable scales exit the horizon, when $N \sim \mathcal{O}(1) \rightarrow \mathcal{O}(10)$ and the flow parameters are well within the slow-roll limit. Trajectories with larger, negative final η values are ones where the trajectory is furthest from the slow-roll regime at early times.

the proposal densities for the HSR have been uniform and the random draws have been made *wherever* each trajectory’s integration was started.

In this work the simplest possible assumptions compatible with the data are made to define the choice of location for the initial conditions

Assumption 1 *A phase of accelerated expansion (inflation) with $\ddot{a} > 0$ occurred before the radiation dominated, decelerating phase of the standard big bang model.*

Assumption 2 *Inflation lasted a minimum number of e -folds such that all scales that are sub-horizon sized today were super-horizon by the end of inflation.*

Assumption 3 *Inflation ended when the universe stopped accelerating i.e. $\ddot{a}$ switched sign.*

Assumption 4 *Inflation was driven by a single scalar field ϕ .*

In line with these assumptions the initial conditions are drawn at the *end* of inflation i.e. a fixed point where $\epsilon = 1$. Only the remaining flow parameters for $\ell = 1, \dots, \ell_{\max}$ are then drawn from uniform distributions with fixed ranges. A value for N_0 is drawn from a uniform distribution and the system (5) is integrated *backwards* a total number of e -folds N_0 . The e -folding N_0 is interpreted as the e -folding where the largest mode k_0 in the observable window is sufficiently smaller than the horizon to allow normalisation using the Bunch-Davies adiabatic limit [9]. This ensures that the system is integrated

far back enough for the calculation of all observables required for comparison with data. In all cases considered in this work N_0 is drawn with a uniform distribution in the range $N_0 = [60, 70]$, this allows for the uncertainty in the total number of e -folds that occurred after the end of inflation due to the details of reheating. The uncertainty impacts our ability to connect a given scale exiting the horizon at a given time during inflation with a physically observable scale that subsequently re-entered the horizon during the decelerating epoch (see e.g. equation (24) of [2]).

B. Potential reconstruction

Each trajectory generated in this manner corresponds to a realisation of inflation with particular initial conditions and potential $V(\phi)$. Given a trajectory one can reconstruct the potential function probed during the evolution as the solution to the Hubble flow system is equivalent to selecting a solution by specifying a potential $V(\phi)$ and initial conditions for ϕ and $\dot{\phi}$.

For example if $\ell\lambda = 0$ for all $\ell > 0$ then the only remaining non-zero parameter is ϵ . This implies $H(\phi)$ is a linear function and hence $V(\phi)$ is quadratic. The solutions for $\epsilon(N)$ and therefore $H(N)$ and $\phi(N)$ can then be obtained easily. The potential is obtained by combining (2) and (4) to get

$$V[\phi(N)] = 3M_{pl}^2 H^2(N) \left[1 - \frac{\epsilon(N)}{3} \right]. \quad (8)$$

III. CALCULATION OF OBSERVABLES

A. Power spectrum

The evolution of background, homogeneous quantities during inflation is fully determined by the Hubble flow trajectory. Background also determines the evolution of the inflaton perturbations that end up as super-horizon primordial curvature perturbations that seed structure formation after inflation. The power spectrum of primordial curvature perturbations can be calculated numerically for any given Fourier wavenumber $k \equiv |\mathbf{k}|$. This is done by integrating the Mukhanov-Sasaki [13, 14] equation for the Fourier expanded comoving curvature perturbation $\zeta(\mathbf{k})$. The isotropic power spectrum is defined as the variance of the curvature perturbations as

$$\begin{aligned} \langle \zeta(\mathbf{k}) \zeta(\mathbf{k}') \rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') P_\zeta(k) \\ &\equiv (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') |\zeta(k)|_{k \ll aH}^2, \end{aligned} \quad (9)$$

and is evaluated at a time when the amplitude of the mode has converged on superhorizon scales ($k \ll aH$).

Expressed in terms of N the Mukhanov-Sasaki equation becomes

$$\frac{d^2 \zeta(k)}{dN^2} + (3 + \epsilon - 2\eta) \frac{d\zeta(k)}{dN} + \frac{k^2}{a^2 H^2} \zeta(k) = 0, \quad (10)$$

from which it can also be seen that the amplitude of $\zeta(k)$ is conserved on superhorizon scales.

The initial condition for integration of (10) is set when $k \gg aH$ for each mode being solved for in which case the adiabatic Bunch-Davies conditions can be assumed and the mode asymptotes to the form

$$\zeta(k) \rightarrow \frac{e^{-ik\tau}}{2a\sqrt{k\epsilon}}, \quad (11)$$

with τ the conformal time defined by $dN/d\tau = aH$. The phase of $\zeta(k)$ is irrelevant and only the rate of change for the initial condition on $d\zeta_k/dN$ is required such that the value of τ at when the mode is normalised need never be evaluated explicitly.

In the following (10) is integrated for a range of modes of interest for observational comparison; $10^{-5} < k < 10^{-1}$ in units of Mpc^{-1} . This is done for each flow trajectory drawn at random in order to compare the resulting power spectrum to observations via calculation of CMB angular power spectrum

$$C_L = \int k^2 dk P_\zeta(k) |\Delta_L(k)|^2, \quad (12)$$

where L here is the angular multipole and $\Delta_L(k, \eta_0)$ is the multipole expanded radiation transfer function for the mode k integrated to the present. The CMB angular power spectrum is evaluated using modified version of CAMB [15] where $P_\zeta(k|\lambda)$ is used as input to (12) instead of the conventional assumption

$$k^3 P_\zeta(k) = A_s \left(\frac{k}{k_\star} \right)^{n_s(k_\star) + \frac{1}{2} \frac{dn_s}{d \ln k} + \dots}, \quad (13)$$

i.e. a power law with amplitude A_s and spectral index given by n_s and higher derivative contributions. The power spectrum of tensor modes $P_h(k|\lambda)$ is calculated in a similar fashion for the same range of wavenumbers and the tensor contribution to the CMB angular power spectrum is also calculated. In this case the full functional form of $P_h(k|\lambda)$ replaces the parametrisation in terms of the tensor-to-scalar ratio r and tensor spectral index n_t .

It is important to note that the numerical integration of mode evolution provides *exact* solutions (within numerical tolerances) without use of any “slow-roll” assumptions. The results obtained are therefore valid also in the case when the flow parameters are not small as long as other necessary conditions of weak coupling and linearity are satisfied³.

The remaining stochastic parameter is the initial condition for H . This value only affects the overall amplitude of the perturbation spectra and does not modify the solution for the flow parameters. There is therefore more freedom in choosing where to impose a normalisation. For this work a value for $\ln(\tilde{H}_{\text{inf}})$ is drawn from a uniform distribution and used to normalise the Hubble rate of the trajectory at a time when a chosen pivot scale $k_\star$ has been outside the horizon for a few e -foldings i.e. when it's amplitude has converged as

$$H|_{k_\star \sim aH} = \frac{10^6}{4\pi\sqrt{2\pi}} \tilde{H}_{\text{inf}}. \quad (14)$$

The value of $\tilde{H}_{\text{inf}}$ is then related linearly to the final amplitude of the curvature power spectrum.

$$H(N_c) = 4\pi\sqrt{2\pi\epsilon(N_c)} M_{pl} A_s \quad (15)$$

B. Non-Gaussianity

The bispectrum is defined as

$$\langle \zeta(\mathbf{k}) \zeta(\mathbf{k}') \zeta(\mathbf{k}'') \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}' + \mathbf{k}'') B(k, k', k''), \quad (16)$$

where momentum conservation forces $\mathbf{k}, \mathbf{k}', \mathbf{k}''$ to form a closed triangle and isotropy implies $B(k, k', k'')$ only depends on their magnitudes. It is convenient to work with a dimensionless bispectrum, which is independent of the power spectrum amplitude, often denoted as

$$\begin{aligned} f_{\text{NL}}(k, k', k'') &\equiv \frac{5}{6} B(k, k', k'') / \\ &[P_\zeta(k) P_\zeta(k') + P_\zeta(k) P_\zeta(k'') + P_\zeta(k') P_\zeta(k'')] . \end{aligned} \quad (17)$$

There are many “type” of f_{NL} with different weightings to $P_\zeta(k)$ in the denominator while the above definition is frequently called $f_{\text{NL}}^{\text{local}}$. The calculation of the bispectrum relies on the “in-in” formalism to calculate

³ For further details of our numerical integration scheme see [12]

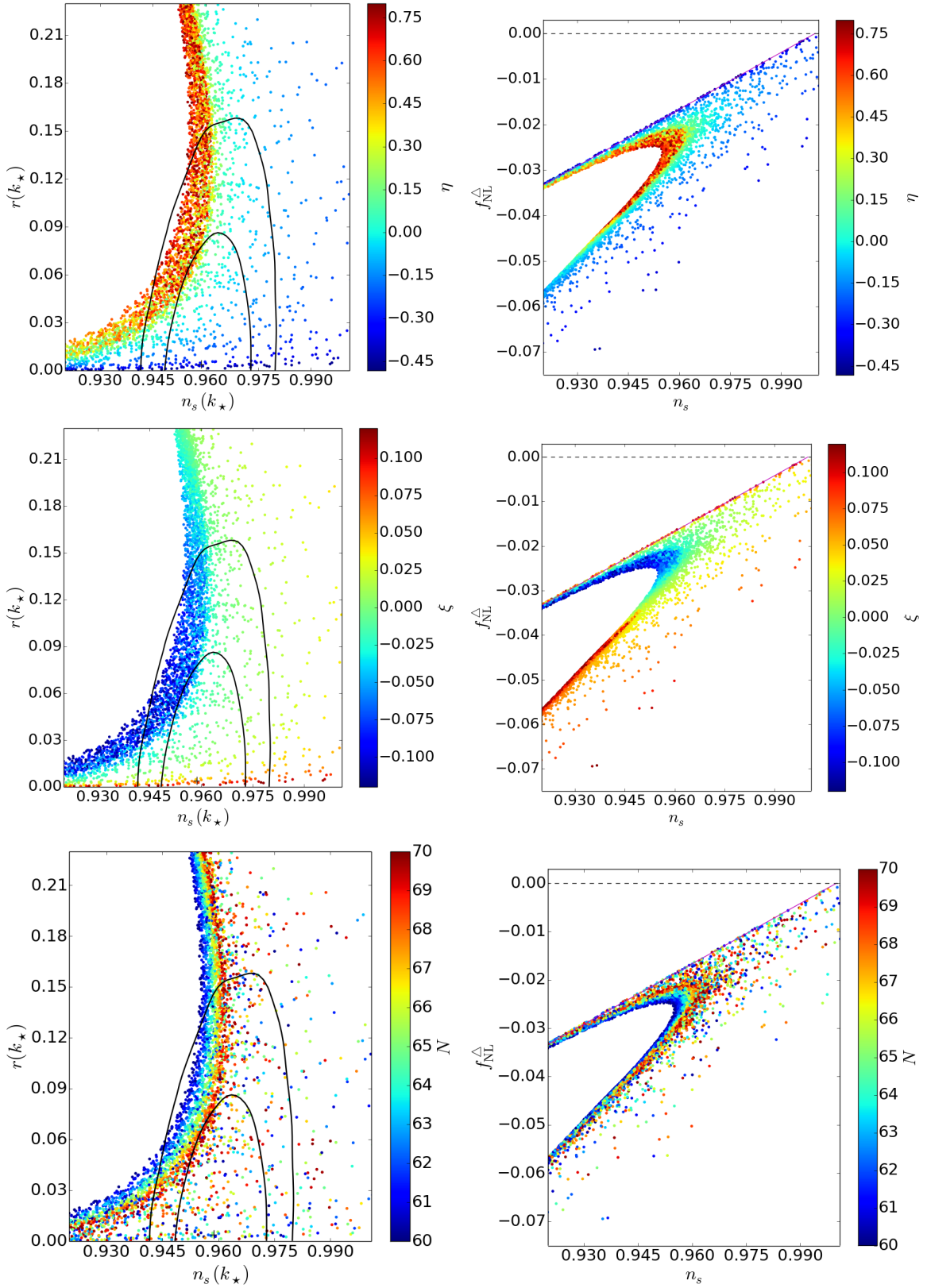


FIG. 2: Hubble flow proposal densities projected into the space of derived parameters n_s , r , and f_{NL}^{Δ} . The derived parameters are obtained from numerical calculation of scalar and tensor power spectra and f_{NL}^{Δ} and evaluated at the pivot scale $k_{\star} = 0.05 \text{ Mpc}^{-1}$. The contours indicate the 68% and 95% confidence regions in the n_s - r plane from the PLANCKr reference fits [10]. Each point represents the derived quantities at the pivot scale obtained for each of ~ 10000 random Hubble flow trajectories generated using the uniform sampling described in section II A. The points are colour coded according to the random values η , ξ , and N_0 used to generate the trajectory. The inflationary attractor and the level of its correlation to the underlying flow parameters is clearly visible in both n_s - r and n_s - f_{NL}^{Δ} planes. The f_{NL}^{Δ} attractor follows a consistency relation given by $f_{\text{NL}} \sim \frac{5}{12}(n_s - 1)$ [11, 12] shown as the solid (magenta) line.

TABLE I: Uniform MCMC priors for cosmological parameters and a short description of each parameter. PLANCK Nuisance parameters are not listed here but are included with the same prior settings as used in [10]. The second block are derived parameters that are not used to randomly sample trajectories.

Parameter	Prior range	Definition
$\omega_b \equiv \Omega_b h^2$	[0.005,0.1]	Baryon density today
$\omega_c \equiv \Omega_c h^2$	[0.001,0.99]	Cold dark matter density today
$100 \theta_{MC}$	[0.5,10.0]	$100 \times \text{CosmoMC}$ sound horizon to angular diameter distance ratio approximation
τ	[0.01,0.8]	Optical depth to reionisation
$\ln(\tilde{H}_{\text{inf}})$	[2.5,3.5]	Log of rescaled Hubble rate at time of Horizon exit of scale k_*
N_0	[60,70]	Number of e -folds for which trajectory is integrated back from end of inflation
$^0\lambda \equiv \epsilon$	1.0	Flow parameter value at end of inflation
$^1\lambda \equiv \eta$	[-1.0,1.0]	Flow parameter value at end of inflation
$^2\lambda \equiv \xi$	[-0.2,0.2]	Flow parameter value at end of inflation
$n_s(k_*)$	...	Scalar spectral index measured from trajectory spectrum at scale $k_* = 0.05 \text{ Mpc}^{-1}$
$r(k_*)$	...	Tensor-to-scalar ratio measured from trajectory spectra at scale $k_* = 0.05 \text{ Mpc}^{-1}$
$n_t(k_*)$	...	Tensor spectral index measured from trajectory spectrum at scale $k_* = 0.05 \text{ Mpc}^{-1}$
$f_{\text{NL}}^\Delta(k_*)$	...	Equilateral non-Gaussianity amplitude at scale $k_* = 0.05 \text{ Mpc}^{-1}$
$\epsilon(k_*)$	...	Flow parameter value shortly after mode k_* exits the horizon
$\eta(k_*)$	...	Flow parameter value shortly after mode k_* exits the horizon
$\xi(k_*)$	...	Flow parameter value shortly after mode k_* exits the horizon

correlation-functions in time-dependent backgrounds for interacting quantum fields.

$$\langle \zeta^3(t) \rangle = -i \int_{-\infty}^t dt' \langle [\zeta^3(t), H_{\text{int}}(t')] \rangle, \quad (18)$$

Just as in flat space when we express our fields as a sum of plane waves (solutions to Klein-Gordon equation, Dirac equation etc.), here we express ζ as a sum of solutions of (10).

$$\zeta(t, \mathbf{x}) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \left(\zeta_{\mathbf{p}}(t) a_{\mathbf{p}} + \zeta_{-\mathbf{p}}^*(t) a_{-\mathbf{p}}^\dagger \right) e^{i\mathbf{p} \cdot \mathbf{x}}. \quad (19)$$

$\zeta_{\mathbf{p}}(t)$ by definition satisfies equation (10) with initial condition (11). The interaction Hamiltonian $H_{\text{int}}(t')$ is obtained from expanding the action for ζ to third order which produces cubic interactions with time-dependent coupling constants [12, 16–19].

$$S_3 = \int d^4 x a^3 \epsilon \left[(2\eta - \epsilon) \zeta \dot{\zeta}^2 + \frac{1}{a^2} \epsilon \zeta (\partial \zeta)^2 - (\epsilon - \eta) \zeta^2 \partial^2 \zeta - 2\epsilon \left(1 - \frac{\epsilon}{4} \right) \dot{\zeta} \partial_i \zeta \partial_i \partial^{-2} \dot{\zeta} + \frac{\epsilon^2}{4} \partial^2 \zeta \partial_i \partial^{-2} \dot{\zeta} \partial_i \partial^{-2} \dot{\zeta} \right], \quad (20)$$

Using this expression for $H_{\text{int}}(t')$ in (18) produces the

following expression for f_{NL}^Δ

$$f_{\text{NL}} = \frac{1}{3|\zeta|^4} \times \mathcal{I} \left[\zeta^{*3} \int_{N_0}^{N_2} dN f_1 \zeta^3 + f_2 \zeta \zeta'^2 \right], \quad (21)$$

where $\zeta = \zeta_k, \zeta_\beta = \zeta_{\beta k}$ and $\zeta' = d\zeta/dN$. The functions f_i are given by

$$f_1 = \frac{5k^2 a \epsilon}{H} (2\eta - 3\epsilon), \quad f_2 = -5H a^3 \epsilon \left(4\eta - \frac{3}{4} \epsilon^2 \right), \quad (22)$$

N_0 and N_2 are e -folds when ζ_k is deep inside and far outside the horizon respectively. The subtleties involved for dealing with this integral numerically are fully explored in [12].

IV. CONSTRAINTS ON HUBBLE FLOW TRAJECTORIES

A. Base parameters

Having defined a measure for generating random Hubble flow trajectories one can now ask whether the resulting observables i.e. scalar and tensor power spectra are compatible with observations and/or gain constraints on

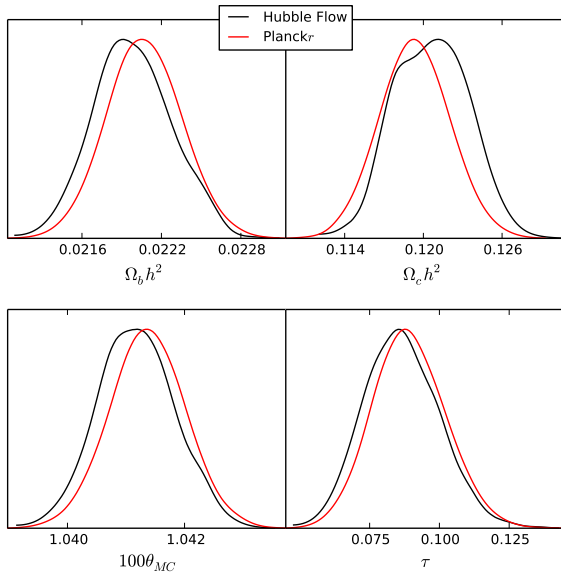


FIG. 3: Comparison of 1d marginalised posteriors in the overlapping, parameters between the reference PLANCKr run and the Hubble flow case with $\ell_{\max} = 2$. There is no significant change in the constraints as expected. The PLANCK nuisance parameters are not shown but also show no significant change in constraints.

the allowed space of flow parameters. To do this the set of parameters defining the random trajectory N_0 , H_{inf} , and ${}^1\lambda$, ${}^2\lambda$, ..., ${}^{\ell_{\max}}\lambda$ can be used as base parameters in an MCMC exploration of the likelihood of CMB observations. In this case the set of flow parameters replaces the conventional parametrisation of scalar and tensor primordial power i.e. A_s , n_s , $dn_s/d\ln k$, etc., and r , n_t , $dn_t/d\ln k$, etc.

Here, the CosmoMC [20] code is used, together with a modified version of CAMB, to explore the likelihood of the Hubble flow parameters with respect to CMB observations. The parameter set used in the exploration in this case is the combination of radiation transfer parameters ω_b , the physical density of baryons, ω_c , the physical density of cold dark matter, θ_{MC} , the angular diameter distance parameter used by CosmoMC [10], and τ , the optical depth parameter, and the set of flow parameters N_0 , H_{inf} , and ${}^0\lambda$, ${}^1\lambda$, ..., ${}^{\ell_{\max}}\lambda$. The flow parameters only affect the primordial scalar and tensor spectra and are therefore probed as *fast* parameters in CosmoMC runs. In practice at each step in the MCMC we compute the trajectory resulting from the set of proposed flow parameters and then numerically evaluate the corresponding scalar and tensor power spectra. We also evaluate numerically the value of f_{NL}^{Δ} , the dimensionless amplitude of the bispectrum in the equilateral configuration, for the pivot scale.

For CMB observations, the latest PLANCK temperature only results [21] are used together with WMAP polarisation measurements. In all the runs described in this work the PLANCK likelihood settings

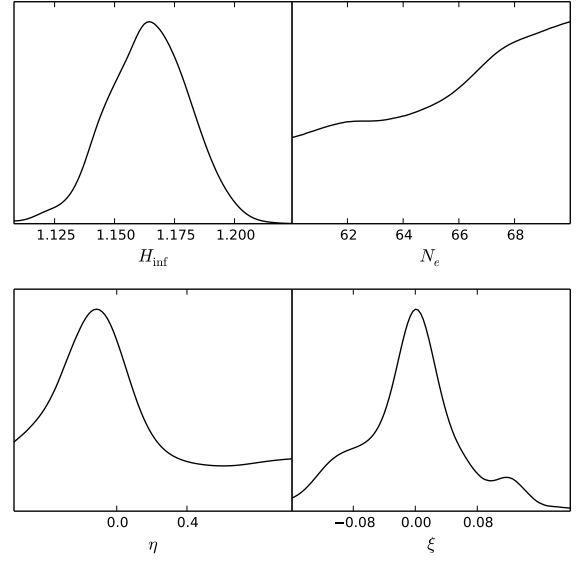


FIG. 4: 1d marginalised posteriors for the Hubble flow parameters. These parameters replace the conventional A_s , n_s , r , n_t , $dn_s/d\ln k$, etc. H_{inf} is equivalent to the scalar amplitude parameter A_s and is well constrained, as expected, whereas the e -folds parameter N_e is unconstrained. This is also expected since there is little sensitivity in the observable to the *total* duration of inflation and N_e can be regarded as an additional nuisance parameter. The flow parameters have posteriors peaked around 0.

and nuisance parameters are set as in the standard “PLANCK+WP” combination (see [10] for details). The PLANCK+WP “base r planck lowl lowLike” (abbreviated to PLANCKr in the following) MCMC chains [20] using the conventional parametrisation A_s , n_s for the primordial scalar spectrum with a tensor extension parametrised solely by r can be used as a reference run to compare with the results reported below⁴.

The conventional parameters A_s , n_s , r , etc., can be calculated directly from the power spectra obtained by numerical integration of the mode equations and can then be treated as *derived* parameters for each accepted flow trajectory in the MCMC chains. f_{NL}^{Δ} can also be treated as a *derived* parameter to gain insight into the level of non-Gaussianity preferred by the current data in the context of random Hubble flow proposal. It is instructive to visualise how the Hubble flow proposal density used here projects into the space of derived parameters. Figure 2 shows the scatter of trajectories in the n_s - r and n_s - f_{NL}^{Δ} planes. The derived quantities are evaluated from the numerically obtained spectra at a pivot scale $k_{\star} = 0.05 \text{ Mpc}^{-1}$. The points are also colour coded according to the random value of η , ξ , and N_0 used to generate the trajec-

⁴ For the tensor spectral index the inflationary consistency relation is used to treat it as a function of n_s .

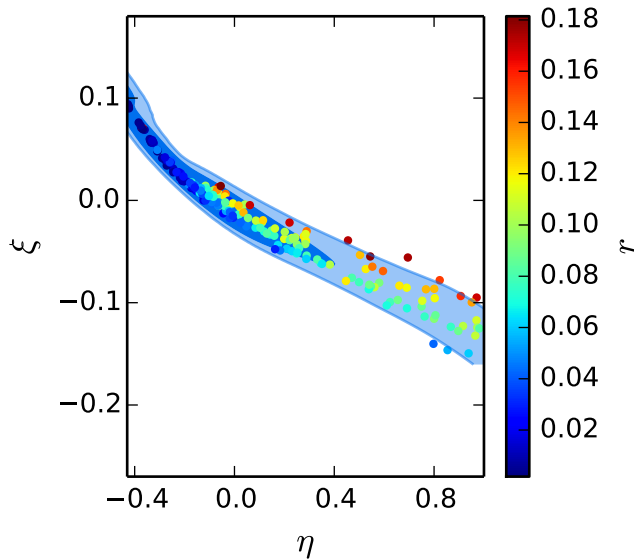


FIG. 5: The 2d marginalised posterior for ξ and η , the base flow parameters for the Hubble flow $\ell_{\max} = 2$ run. The contours denote the 68% and 95% significance levels. The coloured scatter plot indicates the value of $r \sim 16\epsilon(k_*)$ for each sample in the chain. The two base parameters are highly correlated and the unconstrained, large positive η tail is correlated with larger values of r .

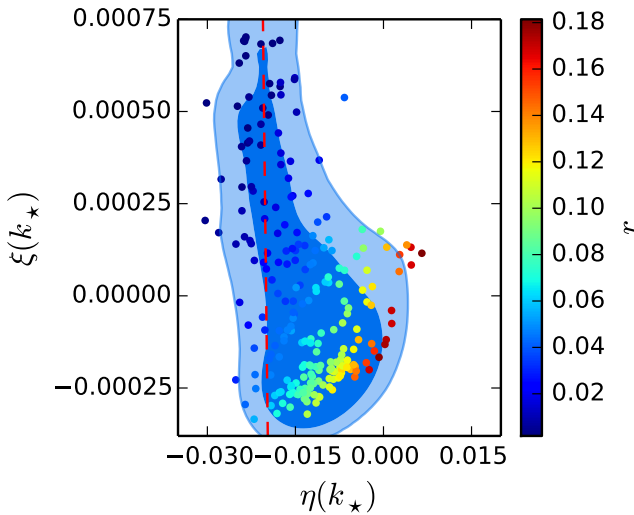


FIG. 6: Same as Figure 5 but for ξ and η values at the observationally relevant scale k_* . The red (dashed) line indicates the expected value of ξ as $r \rightarrow 0$ as given by the second order slow-roll approximation.

tory. The value of the random flow parameters is highly correlated with the resulting values of n_s , r , and f_{NL}^{Δ} and the scatter shows a strong “inflationary” attractor [3–6]. The attractor overlaps the PLANCK r constraints for the n_s - r combination in a corner of the region between the 68% and 95% contours.

TABLE II: Parameter constraints from the marginalised posteriors for both Hubble flow $\ell_{\max} = 2$ and PLANCK r runs. Parameters marked with † are derived ones in the Hubble flow run. Upper limits are 95% significance values.

	Hubble Flow	PLANCK r
$\Omega_b h^2$	$0.02198^{+0.00028}_{-0.00032}$	$0.02207^{+0.00028}_{-0.00028}$
$\Omega_c h^2$	$0.1206^{+0.0028}_{-0.0031}$	$0.1193^{+0.0026}_{-0.0026}$
$100\theta_{MC}$	$1.04117^{+0.00063}_{-0.00069}$	$1.04137^{+0.00063}_{-0.00063}$
τ	$0.087^{+0.013}_{-0.015}$	$0.089^{+0.012}_{-0.014}$
H_{inf}	$1.164^{+0.017}_{-0.017}$	-
$\log(10^{10} A_s)$	-	$3.09^{+0.024}_{-0.027}$
$^{\dagger}n_s$	$0.9579^{+0.0072}_{-0.0090}$	$0.9623^{+0.0075}_{-0.0075}$
$^{\dagger}r$	< 0.143	< 0.126
$^{\dagger}f_{\text{NL}}^{\Delta}$	$-0.0205^{+0.0037}_{-0.0057}$	-

We consider a Hubble flow system with $\ell_{\max} = 2$ (i.e. including ϵ , η , and ξ) for the MCMC exploration. This allows potentials that include up to order 6 polynomials in ϕ . The uniform priors chosen for this run are shown in table II together with a description of each base and derived parameter. The run uses seven base parameters which is the same number used for the conventional PLANCK r run. The PLANCK nuisance parameters are omitted for brevity.

The chains are run until the R^{-1} convergence parameter [20] falls below 0.1. Figure 3 shows the resulting 1-dimensional marginalised posterior distribution for the conventional parameters that determine the form of the radiation perturbation transfer functions. These shared by both PLANCK r and the Hubble flow runs. The marginalised posteriors are very similar between the two runs indicating that there is no *tension* in the transfer parameters with respect to how the primordial perturbation spectrum is sampled.

Figure 4 shows the marginalised posteriors for the flow parameters that do not have counterparts in the conventional runs. The overall amplitude is tightly constrained as expected - it takes the same role as the conventional amplitude A_s . The total number of e -folds is unconstrained and acts as an extra nuisance parameter which is marginalised in the given interval. The two flow parameters whose values are allowed to vary at the end of inflation, η and ξ have posteriors that are peaked around zero. The ξ parameter is also well constrained with respect to its uniform prior. Positive values of η are unconstrained and the posterior approaches a uniform distribution that extends to the $\eta = 1$ limit of the uniform prior.

The two Hubble flow parameters are highly correlated as seen in Figure 5. The large η , negative ξ tail however is correlated with larger values of $r \sim 16\epsilon(k_*)$ and therefore lower upper limits on the tensor-to-scalar ratio will help in eliminating the large η tail and break the degeneracy. The resulting 2d posterior for the hubble flow

parameters at observable scales can be seen in Figure 6 that also includes a line indicating the consistency of η and ξ with the slow-roll limit expression in the limit that $r \sim 16\epsilon(k_*) \rightarrow 0$ and $n_s - 1 \rightarrow -0.04$.

Trajectories with positive η values, and hence positive derivative in ϵ at the end of inflation approach the slow-roll limit very quickly as they are evolved backwards towards the observable window. These therefore almost always result in acceptable values for e.g. n_s, r , etc. Negative values of η are cutoff by the data at $\eta \sim -0.42$. The reason for this strong cutoff can be seen in Figure 1 where trajectories with larger negative values of η at the end of inflation approach the slow-roll limit much slower, giving values of $|n_s - 1|$ that are typically larger and therefore in disagreement with observations.

The best-fit sample in the chain for the $\ell_{\max} = 2$ Hubble flow run has a negative log-likelihood $-\ln L \equiv \mathcal{L} = 4903.2521$ compared to the PLANCKr one of $\mathcal{L} = 4904.3370$ giving a better fit by $\Delta\mathcal{L} = 1.085$. The two runs have a comparable number of degrees of freedom since the N_e can be considered as an additional nuisance parameter. The marginalised constraints on parameters in both Hubble flow and PLANCKr run are compared in Table II.

B. Derived parameters

It is useful to compare the marginalised posteriors in the derived r and n_s parameters between the Hubble flow run and the conventional PLANCKr case. Figure 7 shows the 2d marginalised constraints for this combination together with their respective best-fit sample location. The Hubble flow case prefers higher values of r due to the proposal density peaking at $r \sim 0.075$ for acceptable values of n_s . Constraints on n_s are similar in both cases although the Hubble flow constraints disfavour relatively large values of n_s compared to PLANCKr.

A novel feature of this method is that existing data *already* constrains the possible values of f_{NL}^{Δ} . This is simply due to the fact that each trajectory has a non-vanishing bispectrum and therefore the data will constrain this degree of freedom too. Figure 8 shows the 2d marginalised constraints in the n_s vs f_{NL}^{Δ} plane. Since most of the trajectories are in the slow-roll regime when the pivot scale k_* is leaving the horizon the posterior for f_{NL}^{Δ} agree well with the limiting consistency condition $f_{\text{NL}}^{\Delta} \approx 5(n_s - 1)/12$ [12, 16]. This result, of course, should not be interpreted as a detection of non-Gaussianity but rather as an indication of what amplitudes of the bispectrum are consistent with the general single field inflationary solutions for a Hubble flow system with $\ell_{\max}$. If measurements of primordial non-Gaussianity ever reach the sensitivity to constrain the level of $f_{\text{NL}}^{\Delta} \sim 10^{-2}$ then the measurement will provide a fundamental consistency check for single field inflation.

In Figure 9 we also show the approximately 200 best-fit

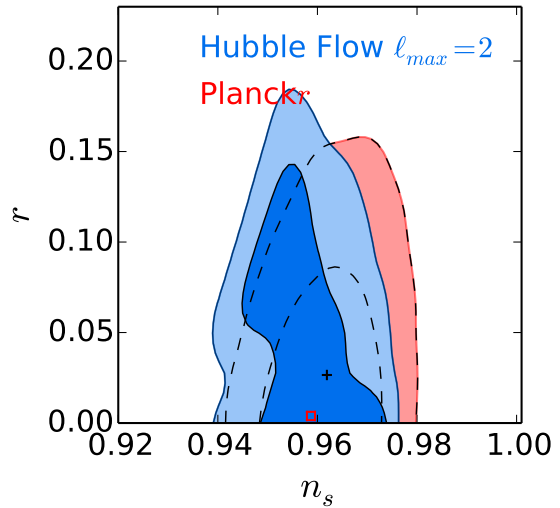


FIG. 7: The 2d marginalised posterior for n_s and r . These are derived at the pivot scale k_* in the Hubble flow case. The same contours for the PLANCKr run are shown for comparison. The cross and square indicate the position of the best-fit sample for the Hubble flow and PLANCKr run respectively. The Hubble flow case prefers higher values of r due to the proposal density peaking at $r \sim 0.075$ for acceptable values of n_s .

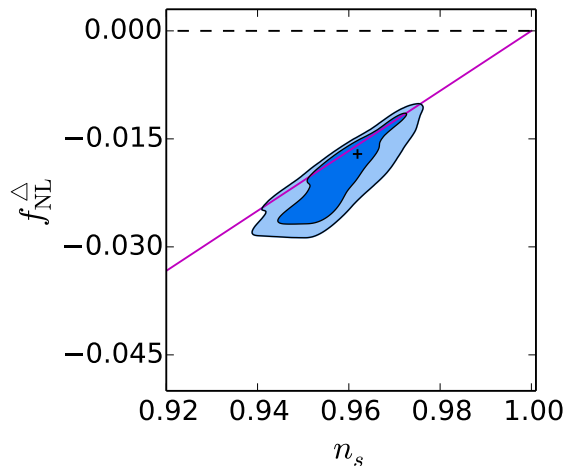


FIG. 8: The 2d marginalised posterior for n_s and f_{NL}^{Δ} at the pivot scale k_* . The cross indicates the location of f_{NL}^{Δ} of the best-fit sample. The line shows the slow-roll consistency condition $f_{\text{NL}}^{\Delta} \approx 5(n_s - 1)/12$.

power spectra in the chains. The spectra are colored and weighted by their $\Delta\mathcal{L}$ with respect to the best-fit sample to emphasise the best fitting curves. The best-fitting spectra are very close to power laws with respect to $\ln k$. The best fitting spectra have very similar normalisations at the pivot scale $k_* = 0.05h \text{ Mpc}^{-1}$ as the normalisation of the primordial spectrum is once of the best constrained parameters.

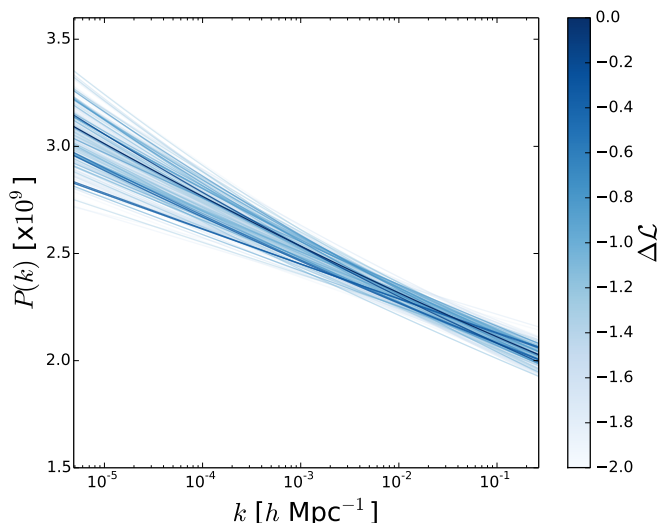


FIG. 9: The primordial curvature power spectra for all samples s within $\Delta\mathcal{L} = 2$ of the best-fit sample. This is an indication of all spectra allowed within the 95% significance level. Each spectrum is colored and weighted on a scale given by $\Delta\mathcal{L}$ and $1/(1 - \Delta\mathcal{L})$ respectively in order to emphasise the best-fitting spectra.

C. Inflaton potential

Each trajectory in the MCMC chain yields an individual potential and we can therefore translate directly the constraints on our base parameters into the space of allowed potentials using (8). For the $\ell_{\max} = 2$ Hubble flow run the best-fit potential is one given by $\eta \sim \xi \sim 0$ i.e. with small curvature. Figure 10 shows all the potentials in the MCMC chain that have $\Delta\mathcal{L} = 2$ with respect to the best-fit sample. There are some 200 samples within this range. Each potential is weighted by its $\Delta\mathcal{L}$ value so the darkest curves are the most likely.

The range in η probed by the sample is large and extends from $\eta \sim -0.4$ to $\eta \sim 1$. This translates to potentials that are both convex and concave, and those that include an inflection point. This is simply a feature of the degeneracy in the contribution from both ϵ and η to the scalar tilt n_s , the only shape spectral parameter, aside from amplitude, that has been constrained so far. If r were to be detected in future it would help to constrain the sign of the curvature of the potential in the observable regime ($\Delta\phi \sim 0$).

An n^{th} -order polynomial fit to the best-fit sample potential converges for $n = 5$ and gives a potential $V(\varphi)$

$$V(\varphi) = V_0 \left(1 + \sum_{n=1}^{n=4} \lambda_n \varphi^n \right), \quad (23)$$

with $\varphi = \Delta\phi$, and $V_0 = 1.50 \times 10^{-12}$, $\lambda_1 = 2.20$,

$\lambda_2 = 0.66$, $\lambda_3 = -6.00 \times 10^{-2}$, $\lambda_4 = 1.78 \times 10^{-3}$, and $\lambda_5 = -1.98 \times 10^{-5}$. The best-fit sample potential and the polynomial fit are shown in the left panel of Figure 10. The coefficients λ_n for $n \leq 4$ converge for higher order fits with $n > 4$ and the potential does not change appreciably in the interval $\varphi = 0 \rightarrow \mathcal{O}(10)$. Note that given (8) an $\ell_{\max} = 2$ flow system allows for potentials that include terms up to ϕ^6 .

V. DISCUSSION

We have obtained constraints on generalised, single field inflation trajectories using the Hamilton-Jacobi formalism. The Hubble flow system was used as base parameters in an MCMC exploration of the likelihood with respect to the latest CMB data. This allowed us to obtain marginalised posteriors on the flow parameters that define the evolution of the Hubble parameter $H(N)$ as a function of e -folds N during inflation. Alternatively, the constraints can be viewed as a selection in the space of inflaton potentials $V(\phi)$.

Our method also includes the numerical calculation of primordial bispectra and we obtained *predictions* based on current data of consistent bispectrum amplitude f_{NL} for the equilateral case.

Further exploration will be left for future work. In particular it will be of interest to extend the system to higher $\ell_{\max}$ to allow for more structure in the trajectories. This is currently limited by the fact that the highly correlated space of HSR parameters result in a very inefficient MCMC exploration. More work to explore the likelihood more efficiently or defining new sets of HSR parameters may help in extending this line of work to systems with higher $\ell_{\max}$.

Future data from CMB and also large scale structure will also provide deeper probes of non-Gaussianity which will provide tighter constraints in the space of trajectories. This will be particularly important if the discovery and characterisation of tensor modes will turn out to elude future CMB polarisation measurements due to foreground contamination. In that case non-Gaussianity measurements will possibly provide the only way to break shape degeneracies and reveal the precise form of the inflaton potential over the range of scales accessible to observations.

Acknowledgments

JSH is supported by a STFC studentship. CRC and JSH acknowledge the hospitality of the Perimeter Institute for Theoretical Physics and the Canadian Institute for Theoretical Astrophysics where some of this work was carried out.

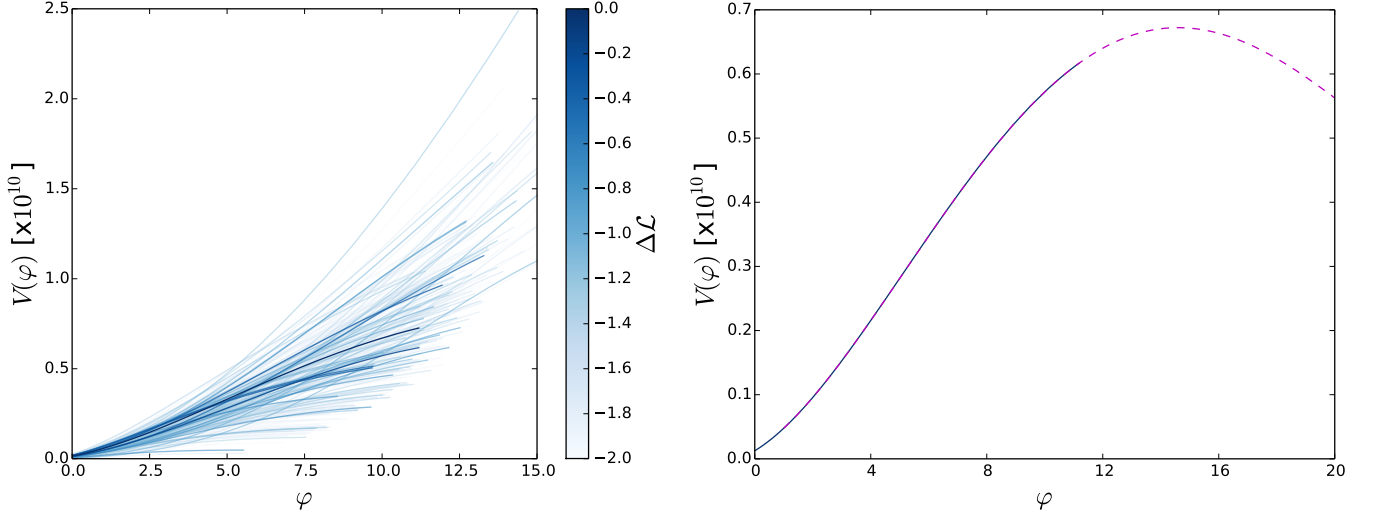


FIG. 10: *Left*: All sampled potentials within $\Delta\mathcal{L} = 2$ of the best-fit sample. This is an indication of all potential shapes and normalisations allowed with the 95% significance level. The x -axis shows the change in ϕ from the final value where inflation ends ($\phi \equiv \Delta\phi = 0$). Both ϕ and V are in units of $M_{\text{pl}} = 1$. The weighting of curves is the same as in Figure 9. *Right*: The best-fit sample potential (solid) and its 4th-order polynomial fit (dashed).

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