

Thesis
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Dedication

To

My Mother and My Father

ABSTRACT

The coupled-integral equations formalism of L. D. Faddeev is used in the low-energy three nucleon system for p-d scattering and the ppn bound state. The approximations involve the use of non-local separable potentials to describe the low energy nucleon-nucleon interactions. The effects of the nucleon-nucleon short-range interactions including angular-momentum, helicities and isospin are considered. The effect of the Coulomb potential is also completely accounted for by the use of the Coulomb modified functions g_c , rather than the original separable potential functions g .

The results are in good agreement with the experimental data and are compared with calculations of others.

PREFACE

The work described in this thesis was carried out under the supervision of Dr. S. Hochberg, between July 1967 and October 1969 in the Department of Mathematics, Imperial College of Science and Technology, London.

Except where otherwise stated, the material contained herein is original and has not been previously presented for a degree in this or any other university.

It is a pleasure for me to express deep gratitude to my supervisor, Dr. S. Hochberg, for suggesting this problem and for his constant encouragement and valuable guidance throughout this work. Our stimulating discussions and his suggestion have a great value to me.

I would like to thank my teachers Professor M. Dutta and Dr. P. K. Roy of Calcutta, for their constant encouragement.

The staffs of the Computer Centre of Imperial College and Mr. R. Sibbel have graciously advised me upon writing most of the computer programmes for the problems.

I thank all my friends who gave me financial help and encouragement when I was setting out for England.

Thanks are also due to Miss Cathy Smart for undertaking the difficult task of typing this thesis.

Without the help of my parents, my education would not have been possible. To them I therefore dedicate the first-fruits of their years of loving labour.

M. Dhara.

CONTENTS

	<u>Pages</u>
ABSTRACT	3
PREFACE	4
CONTENTS	6
CHAPTER 1: INTRODUCTION	7
CHAPTER 2: PRELIMINARIES	10
I. KINEMATICS AND NOTATIONS	10
II. THE TWO-PARTICLE SYSTEM	14
CHAPTER 3: THE THREE-PARTICLE SYSTEM	29
CHAPTER 4: THE SEPARABLE APPROXIMATIONS	53
CHAPTER 5: THE COULOMB INTERACTIONS	60
CHAPTER 6: THE MODIFIED FADDEEV EQUATIONS	69
CHAPTER 7: THE OBSERVABLES	78
CHAPTER 8: CONCLUSION	90
APPENDIX I	92
APPENDIX II	102
APPENDIX III	104
REFERENCES	105

CHAPTER 1
INTRODUCTION

Faddeev [1-4] who gave for the first time the correct formulation of non-relativistic three-particle scattering, has opened a way for calculations on three-body problems in their full complexity. The difficulty with using the usual Lippmann-Schwinger equations [5] to study the interaction of three or more particles is that the kernels are not compact because of the presence of disconnected processes in which only two of the particles interact. In contrast to equations of the Lippmann-Schwinger type the Faddeev equations have kernels that depend upon the solution of the three two-particle subsystems. Thus all the properties of the three-particle problem may be derived as long as one knows the exact (off the energy shell) two-body scattering amplitudes.

Nevertheless the three-body equations in their full generality are still very complicated and cannot easily be solved without any approximations. Such approximations

were made by Amado [6], Rosenberg [7], Lovelace [8,9] and others.

In the following, we further develop the work of Omnes [10] who extended Faddeev's theory to non-zero angular momentum. We also formulate the Coulomb Green's functions which are essential in discussing the He^3 bound state. Further the concepts of helicities and isospin are used in our formulation.

Chapter 2 is devoted to a discussion of some general aspects of two-particle scattering. The notations and the brief review of the results by Jacob and Wick [11] which are relevant for our purposes are also discussed. Following the work of Omnes [10], the Faddeev equations with angular momentum, helicities and isospins are described and the system of coupled integral equations in one variable is derived in Chapter 3. Chapter 4 contains the approximation of the separable interactions.

In Chapter 5, we briefly review the results by Harrington [16] and the Coulomb wave function is modified in obtaining g_c from g for the S-state. Chapter 6 contains the discussion of the bound state properties of

the three-nucleon system to obtain the He^3 wave function and p-d scattering amplitude. In Chapter 7 we discuss our numerical results. The conclusion and final remarks are in Chapter 8.

We slightly modify the special quadrature rules [39] which allow us to get accurate results with a small number of mesh points. This method is discussed in the appendix.

CHAPTER 2

PRELIMINARIES

In this Chapter, we shall first discuss the kinematics of a non-relativistic three-particle system and introduce our notations. In the second part of this Chapter, we shall briefly review the two-particle system using the operator techniques.

I. KINEMATICS AND NOTATIONS

Let us consider three particles of masses m_1, m_2, m_3 with momenta p_1, p_2, p_3 respectively. The usual relation between the kinetic energies (E_1, E_2, E_3), momenta and masses are

$$E_\alpha = \frac{p_\alpha^2}{2m_\alpha}, \quad \alpha = 1, 2, 3. \quad (2.1)$$

The total momentum $\underline{P}$ is given by

$$\underline{P} = p_1 + p_2 + p_3. \quad (2.2)$$

We introduce the momentum of particle 1 relative to particle 2 in their own centre-of-mass system as

$$\underline{k}_3 = \frac{m_2 \underline{p}_1 - m_1 \underline{p}_2}{m_1 + m_2} , \quad (2.3)$$

and their kinetic energy in this system is given by

$$E_{12} = \frac{k_3^2}{2\mu} , \quad (2.4)$$

where μ is the reduced mass of particles 1 and 2, namely

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} .$$

The angle between two momenta $\underline{p}_1$ and $\underline{p}_2$ in the centre-of-mass system is given by θ_{12} , where

$$\cos \theta_{12} = \frac{p_1^2 + p_2^2 - p_3^2}{2 p_1 p_2} , \quad (2.5)$$

and $p_1 = |\underline{p}_1|$, $p_2 = |\underline{p}_2|$.

This can also be expressed in terms of masses and kinetic energies, thus

$$\cos \theta_{12} = \frac{m_1 E_1 + m_2 E_2 - m_3 E_3}{2(m_1 m_2 E_1 E_2)^{1/2}} . \quad (2.6)$$

In the centre-of-mass system $\underline{p}_1, \underline{p}_2, \underline{p}_3$ add up to zero, i.e. $\underline{p}_1 + \underline{p}_2 + \underline{p}_3 = 0$ and symbolically form a triangle with sides equal to p_1, p_2, p_3 respectively in magnitudes and directions. This triangle is completely defined up to a translational displacement.

Similarly, a triangle can be formed by taking the sides equal in magnitude to the kinetic energies E_1, E_2, E_3 respectively.

Again we can easily obtain the cosine of the angle γ_3 between p_3 in the total centre-of-mass system and k_3 in the relative centre-of-mass system of particles 1 and 2; thus

$$\cos \gamma_3 = \frac{p_3 \cdot k_3}{p_3 k_3}, \quad (2.7)$$

where

$$\begin{aligned} p_3 \cdot k_3 &= p_3 \cdot (m_2 p_1 - m_1 p_2) (m_1 + m_2)^{-1} \\ &= (m_2 p_3 \cdot p_1 - m_1 p_3 \cdot p_2) (m_1 + m_2)^{-1} \\ &= (m_2 p_3 p_1 \cos \theta_{31} - m_1 p_3 p_2 \cos \theta_{23}) (m_1 + m_2)^{-1} \\ &= \left[m_2 (4m_1 m_3 E_1 E_3)^{1/2} \frac{(m_3 E_3 + m_1 E_1 - m_2 E_2)}{2(m_1 m_2 E_1 E_3)^{1/2}} \right. \\ &\quad \left. - m_1 (4m_2 m_3 E_2 E_3)^{1/2} \frac{(m_2 E_1 + m_3 E_3 - m_1 E_1)}{2(m_2 m_3 E_2 E_3)^{1/2}} \right] \times \\ &\quad (m_1 + m_2)^{-1} \\ &= [(m_1 + m_2)(m_1 E_1 - m_2 E_2) + (m_2 - m_1)m_3 E_3] (m_1 + m_2)^{-1}. \end{aligned} \quad (2.8)$$

The momentum of the particle 3 relative to the centre-of-mass of the subsystem of particles 1 and 2 can be written as

$$\underline{q}_3 = \frac{1}{M} [m_3(\underline{p}_1 + \underline{p}_2) - (m_1 + m_2)\underline{p}_3] , \quad (2.9)$$

where M stands for the total mass of the particles 1, 2, 3.

This relation can be written in general form

$$\underline{q}_\alpha = \frac{m_\alpha \underline{P}}{M} - \underline{p}_\alpha , \quad \alpha = 1, 2, 3 . \quad (2.10)$$

From these, we can write

$$\sum_{\alpha=1}^3 \underline{q}_\alpha = 0 . \quad (2.11)$$

Again the $\underline{p}_\alpha$ are expressed in terms of $\underline{P}$, $\underline{k}_\alpha$, $\underline{q}_\alpha$; thus

$$\begin{aligned} \underline{p}_1 &= \frac{m_1}{M} \underline{P} + \frac{m_1}{m_1 + m_2} \underline{q}_3 - \underline{k}_3 \\ \underline{p}_2 &= \frac{m_2 \underline{P}}{M} + \frac{m_1}{m_1 + m_2} \underline{q}_3 - \underline{k}_3 \\ \underline{p}_3 &= \frac{m_3 \underline{P}}{M} - \underline{q}_3 \end{aligned} \quad (2.12)$$

We can also write the $\underline{k}_\alpha$ in terms of the $\underline{q}_\alpha$, thus

$$\underline{k}_1 = - \frac{m_3}{m_2+m_3} \underline{q}_1 - \underline{q}_2 \quad (2.13)$$

$$\underline{k}_2 = \underline{q}_1 + \frac{m_1}{m_1+m_3} \underline{q}_2$$

etc.

It is convenient to use the momentum $\underline{K}_3$ (total subsystem momentum), when considering the subsystem of particles 1 and 2. Thus we take

$$\underline{K}_3 = \underline{k}_1 + \underline{k}_2 \quad (2.14)$$

giving

$$\underline{p}_1 = \frac{m_1}{m_1+m_2} \underline{K}_3 + \underline{k}_3 \quad (2.15)$$

$$\underline{p}_2 = \frac{m_1}{m_1+m_2} \underline{K}_3 - \underline{k}_3$$

Similar expressions can be written down for other pairs of particles.

II. TWO - PARTICLE SYSTEM

We are now going to review some results [1,8,11] in two-particle potential scattering. In order to distinguish

between similar operators in the two- and three-particle Hilbert space we will adopt the following notation.

Capital letters will refer to operators in the three-particle space while small letters will be reserved for operators in the two-particle Hilbert space.

Let the free Hamiltonian operator, h_0 , be defined in the centre-of-mass system. Let p_1, p_2 be the momenta of particles 1 and 2 respectively on the space of square integrable function of two real variables p_1, p_2 ^{11 rec} and p_2 ; i.e. on Hilbert space $L_2(p_1, p_2)$. We also define $p = p_1 + p_2$ and $p' = p_1' + p_2'$ for initial and final momenta respectively. Then we have

$$\langle p | h_0 | p' \rangle = p^2 \delta(p - p') . \quad (2.16)$$

The resolvent operator of h_0 is

$$g_0(z) = [h_0 - zI]^{-1} , \quad (2.17)$$

where I is the unit operator and z is a complex parameter. It can also be defined in the momentum space representation by

$$\langle \underline{p} | g_0(z) | \underline{p}' \rangle = [p^2 - z]^{-1} g_0(\underline{p} - \underline{p}') . \quad (2.18)$$

In this representation, if v is a local potential, then its locality requires that it depends only on $(\underline{p} - \underline{p}')$. We have

$$\langle \underline{p} | v | \underline{p}' \rangle = v(\underline{p} - \underline{p}') . \quad (2.19)$$

We assume $v(\underline{k})$ is square integrable. Secondly $v(\underline{k})$ is analytic in $\underline{k}$ for a domain containing the real axis. ~~For assuming this analyticity,~~ We assume v to be a generalized Yakawa potential of range μ .

$$v(\underline{k}) = \int_{\mu}^{\infty} \frac{g(\omega) d\omega}{k^2 + \omega^2} . \quad (2.20)$$

The weight function $g(\omega)$ can contain delta function, but its integral

$$I(\omega) = \int_{\mu}^{\omega} g(v) dv ,$$

must be of bounded variation on $(\mu, +\infty)$. Therefore, we can see that $v(\underline{k})$ will be analytic in $\underline{k}$ in a

domain including the real axis and $v(\underline{k}) = O(k^{-2})$, as $|\underline{k}| \rightarrow \infty$. Finally, time reversal invariance requires that $v(\underline{k})$ have the real valued property

$$v(-\underline{k}) = \overline{v(\underline{k})} . \quad (2.21)$$

The total two-particle Hamiltonian is given by

$$h = h_0 + v . \quad (2.22)$$

Its Green's function operator is

$$g(z) = [h - zI]^{-1} . \quad (2.23)$$

We note that $g(z)$ of course will not exist for all values of z . If z is equal to the energy of a bound state or lies on the real positive axis, then $g(z)$ ceases to exist.

The property of the resolvents g and g_0 are governed by two identities known as the first and second resolvent equations. The first resolvent equation for g may be obtained directly from (2.23)

$$g(z_1) - g(z_2) = g(z_1)(h - z_1)[g(z_1) - g(z_2)](h - z_2)g(z_2)$$

$$\begin{aligned}
&= g(z_1)[(h-z_2) - (h-z_1)]g(z_2) \\
&= (z_1-z_2)g(z_1)g(z_2) .
\end{aligned} \tag{2.24}$$

Similarly, the first resolvent equation for g_o is

$$g_o(z_1) - g_o(z_2) = (z_1-z_2)g_o(z_1)g_o(z_2) . \tag{2.25}$$

The resolvent operators $g(z)$ and $g_o(z)$ will satisfy an identity called the second resolvent equation. This result follows by using both (2.23) and (2.17).

$$\begin{aligned}
g(z) &= g_o(z)(h_o-z)g(z) \\
&= g_o(z)(h_o+v-z)g(z) - g_o(z)vg(z) \\
&= g_o(z) - g_o(z)vg(z) .
\end{aligned} \tag{2.26}$$

This second resolvent equation can be deduced in another form namely

$$g(z) = g_o(z) - g(z)vg_o(z) . \tag{2.27}$$

If this is written out in terms of matrix elements, it gives an integral equation for $\langle p | g(z) | p' \rangle$. The formal solution for $g(z)$ may be obtained from (2.26) in the following way.

$$g(z) = g_0(z)[1 - g_0(z)v]^{-1} \quad (2.28)$$

$$= g_0(z)[1 - vg_0(z)]^{-1} . \quad (2.29)$$

The kernel representation of $g_0(z)v$ is

$$\langle p | g_0(z)v | p' \rangle = v(p-p')[p^2-z]^{-1} . \quad (2.30)$$

Now we introduce the t -matrix or transition amplitude operator. The transition operator is defined to be the operator t such that

$$t(s+i\xi) = v - vg(s+i\xi)v . \quad (2.31)$$

The most general version of t is obtained by replacing $s + i\xi$ with z , where z is any point in the complex plane. That is,

$$t(z) = v - vg(z)v . \quad (2.32)$$

Its matrix elements $\langle p | t(z) | p' \rangle$ on the energy shell defined by

$$p^2 = p'^2 = s \quad (2.33)$$

give the physical transition amplitude from the initial

state with momentum p to the final state with momentum p' .

By multiplying Eq. (2.32) from the left and from the right by $g_0(z)$, we get with the help of the second resolvent Eq. (2.27)

$$\begin{aligned} t(z)g_0(z) &= [v - vg(z)v]g_0(z) \\ &= v[g_0(z) - g(z)vg_0(z)] \\ &= vg(z) , \end{aligned} \tag{2.34}$$

and

$$\begin{aligned} g_0(z)t(z) &= g_0(z)[v - vg(z)v] \\ &= [g_0(z) - g_0vg(z)]v \\ &= g(z)v \end{aligned} \tag{2.35}$$

where (2.26) has been used.

Then we can write Eq. (2.32) by using (2.34)

$$t(z) = v - t(z)g_0(z)v \tag{2.36}$$

and using (2.35)

$$t(z) = v - vg_0(z)t(z) . \tag{2.37}$$

Both (2.36) and (2.37) are operator version of the Lippmann-Schwinger equation for the t -matrix. We can express these operator equations as integral equations thus

$$\langle p | t(z) | p' \rangle = \langle p | v | p' \rangle - \int \frac{dp'' \langle p | v | p'' \rangle \langle p'' | t(z) | p' \rangle}{p''^2 - z} . \quad (2.38)$$

Since $g_0 v$ is analytic and compact the solution of (2.36) may be written

$$\begin{aligned} t(z) &= [1 + v g_0(z)]^{-1} v \\ &= v [1 + g_0(z) v]^{-1} . \end{aligned} \quad (2.39)$$

Then the solution of the integral equation (2.38) is unique except at poles of $[1 + g_0(z) v]^{-1}$.

We now discuss the scattering amplitude for non-zero angular momentum and helicities. Following Jacob and Wick [11], we use the component of spin along a fixed direction, say the z -axis, to classify the possible polarized state of a particle. The states are labelled

by the component of the spin along the direction of motion of the particle which we may call the helicity quantum number or briefly helicity.

Let $\lambda_1, \lambda_2, E(\lambda_1', \lambda_2', E')$ be the helicities and the total kinetic energy of particles 1 and 2 in the initial (final) state respectively. Let $J, M(= J_z)$ (J', M') be the angular momentum and its z-component for the initial (final) state. We define $|E, J, M, \lambda_1, \lambda_2\rangle$ and $|E', J', M', \lambda_1', \lambda_2'\rangle$ to be the initial and final states respectively in the centre-of-mass system. The S-matrix can be written in the form [11]

$$\begin{aligned} \langle E', J', M', \lambda_1', \lambda_2' | S | E, J, M, \lambda_1, \lambda_2 \rangle \\ = \delta(E'-E) \delta_{J', J} \delta_{M', M} \langle \lambda_1', \lambda_2' | S^J(E) | \lambda_1, \lambda_2 \rangle . \end{aligned} \quad (2.40)$$

The reduction of the S-matrix to submatrices $S^J(E)$, belonging to definite values of E and J , is, of course, an essential step in any general collision analysis. The diagonalization of S^J and S^{Jz} is usually achieved by algebraic addition of the spin vectors and relative orbital momentum vector. As a result an element of the submatrix $S^J(E)$ is labelled by

$$\langle L's' | S^J(E) | Ls \rangle ,$$

where $Ls(L's')$ are orbital and total spin quantum numbers for the initial (final) state.

If J_x, J_y, J_z are the components of angular momentum for the particle, a finite rotation with Eulerian angles α, β, γ is performed by the operator $R_{\alpha\beta\gamma}$ where

$$R_{\alpha\beta\gamma} = \exp(-i\alpha J_x) \exp(-i\beta J_y) \exp(-i\gamma J_z) . \quad (2.41)$$

It is easy to construct the states with zero linear momentum and definite total angular momentum J and its component $M(= J_z)$. Since the magnitude p of the relative momentum and the helicities λ_1 and λ_2 are invariant against rotations in (2.41), we can assign definite values to them, together with J and M .

Let $|p, J, M, \lambda_1, \lambda_2\rangle$ be the symbol for such a state.

We can define this state with a suitable normalization factor N as

$$|p, J, M, \lambda_1, \lambda_2\rangle = \frac{N}{2\pi} \int d\Omega D_{M\lambda}^{*J}(\alpha\beta\gamma) R_{\alpha\beta\gamma} \psi_{p\lambda_1\lambda_2} , \quad (2.42)$$

where $\lambda = \lambda_1 - \lambda_2$ and $\psi_{\lambda_1 \lambda_2}$ is the phase of the state and

$$du = d\alpha \, d \cos \beta \, d\gamma . \quad (2.43)$$

The integral extends over the domain $0 < \alpha < 2\pi$,
 $0 < \beta < \pi$, $0 < \gamma < 2\pi$ and the star means complex conjugation. Furthermore,

$$D_{MM'}^J(\alpha\beta\gamma) = \exp(-iM\alpha) d_{MM'}^J(\beta) \exp(-iM'\gamma) , \quad (2.44)$$

is the matrix corresponding to $R_{\alpha\beta\gamma}$ in the irreducible representation D^J .

If we choose the normalization factor

$$N = [(2J+1)/4\pi]^{1/2} \quad (2.45)$$

and use the orthogonality relation [12] for the d-function namely

$$\int_0^\pi d_{m\lambda}^j(\beta) d_{m\lambda}^{j'}(\beta) d(\cos \beta) = \delta_{jj'} 2[2j+1]^{-1} , \quad (2.46)$$

and

$$\frac{1}{2} \sum_j (2j+1) d_{m\lambda}^j(\beta) d_{m\lambda}^j(\beta') = \delta(\cos \beta - \cos \beta') . \quad (2.47)$$

where $\lambda = \lambda_1 - \lambda_2$ and $\psi_{\mathbb{P}\lambda_1\lambda_2}$ is the phase of the state and

$$du = d\alpha d \cos \beta d\gamma . \quad (2.43)$$

The integral extends over the domain $0 < \alpha < 2\pi$,
 $0 < \beta < \pi$, $0 < \gamma < 2\pi$ and the star means complex
 conjugation. Furthermore,

$$D_{MM'}^J(\alpha\beta\gamma) = \exp(-iM\alpha) d_{MM'}^J(\beta) \exp(-iM'\gamma) , \quad (2.44)$$

is the matrix corresponding to $R_{\alpha\beta\gamma}$ in the irreducible
 representation D^J .

If we choose the normalization factor

$$N^2 = [(2J+1)/4\pi]^{1/2} \quad (2.45)$$

and use the orthogonality relation [12] for the d-function
 namely

$$\int_0^\pi d_{m\lambda}^j(\beta) d_{m\lambda}^{j'}(\beta) d(\cos \beta) = \delta_{jj'} 2[2j+1]^{-1} , \quad (2.46)$$

and

$$\frac{1}{2} \sum_j (2j+1) d_{m\lambda}^j(\beta) d_{m\lambda}^j(\beta') = \delta(\cos \beta - \cos \beta') . \quad (2.47)$$

We easily find that the angular momentum-helicity states are normalized in a standard fashion, thus

$$\langle J'M' \lambda'_1 \lambda'_2 | JM \lambda_1 \lambda_2 \rangle = \delta_{J'J} \delta_{\lambda'_1 \lambda_1} \delta_{\lambda'_2 \lambda_2} \delta_{MM'} \quad (2.48)$$

Also the transformation matrix

$$\langle \theta \phi; \lambda_1 \lambda_2 | JM; \lambda'_1 \lambda'_2 \rangle = N \delta_{\lambda_1 \lambda'_1} \delta_{\lambda_2 \lambda'_2} D_{M\lambda}^{*J}(\phi, \theta, -\phi) \quad (2.49)$$

satisfies the unitary conditions

$$\begin{aligned} \int d\Omega \langle \theta \phi; \lambda_1 \lambda_2 | JM \lambda_1 \lambda_2 \rangle \langle \theta' \phi'; \lambda_1 \lambda_2 | J'M' \lambda_1 \lambda_2 \rangle^* \\ = \delta_{JJ'} \delta_{MM'} \end{aligned}$$

and

$$\begin{aligned} \sum_{JM} \langle \theta \phi; \lambda_1 \lambda_2 | JM \lambda_1 \lambda_2 \rangle \langle \theta' \phi'; \lambda_1 \lambda_2 | JM \lambda_1 \lambda_2 \rangle^* \\ = \mathcal{C}_2(\theta \phi; \theta' \phi') \end{aligned} \quad (2.50)$$

and $\mathcal{C}_2$, the two-dimensional Δ -function on the unit sphere, viz:

$$\mathcal{C}_2(\theta \phi; \theta' \phi') = \delta(\cos \theta - \cos \theta') \delta(\phi - \phi') .$$

These relations are useful in the following discussion of the S-matrix.

The differential cross-section can be written down

$$d\sigma = (2\pi p^{-1})^2 \left| \langle \theta, \phi; \lambda_1' \lambda_2' | T(E) | 0, 0; \lambda_1 \lambda_2 \rangle \right|^2 d\Omega, \quad (2.51)$$

where $d\Omega$ ($= \sin \theta d\theta d\phi$) is the final solid angle in the centre-of-mass system and the operator T is related to S by the usual equation

$$S - \mathbb{1} = iT \quad (2.52)$$

where $\mathbb{1}$ is the unit matrix. The cross-section refers to given values of the initial and final helicities.

By using equations (2.45) and (2.49), we can obtain the matrix elements S in the helicities representation and in the JM representation, Eq. (2.40), viz:

$$\begin{aligned} & \langle \theta, \phi; \lambda_1' \lambda_2' | S(E) | 0, 0; \lambda_1 \lambda_2 \rangle \\ &= \sum_{JM} \langle \theta, \phi; \lambda_1' \lambda_2' | JM; \lambda_1' \lambda_2' \rangle \langle JM; \lambda_1' \lambda_2' | \\ & \quad | S(E) | J'M'; \lambda_1 \lambda_2 \rangle \langle J'M'; \lambda_1 \lambda_2 | 0, 0; \lambda_1 \lambda_2 \rangle \\ &= (4\pi)^{-1} \sum_J (2J+1) \langle \lambda_1' \lambda_2' | S(E) | \lambda_1 \lambda_2 \rangle D_{\lambda\mu}^{*J}(\phi, \theta, -\phi), \end{aligned} \quad (2.53)$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_1' - \lambda_2'$.

Again with the help of equations (2.51), (2.52) and (2.53), the differential cross-section becomes

$$d\sigma = \left| f_{\lambda_1' \lambda_2'; \lambda_1 \lambda_2}(\theta, \phi) \right|^2 d\Omega, \quad (2.54)$$

where

$$f_{\lambda_1' \lambda_2'; \lambda_1 \lambda_2}(\theta, \phi) = p^{-1} \sum_J \left(J + \frac{1}{2} \right) \langle \lambda_1' \lambda_2' | T^J(E) | \lambda_1 \lambda_2 \rangle \exp[i(\lambda - \mu)\phi] d_{\lambda\mu}^J(\theta). \quad (2.55)$$

$T^J(E)$ is identical with $-iS^J(E)$ except in the case of elastic scattering. Then we can write Eq. (2.52) as

$$\begin{aligned} \langle \lambda_1' \lambda_2' | S^J(E) | \lambda_1 \lambda_2 \rangle &= \delta_{\lambda_1 \lambda_2} \delta_{\lambda_1' \lambda_2'} \\ &= i \langle \lambda_1' \lambda_2' | T^J(E) | \lambda_1 \lambda_2 \rangle. \end{aligned} \quad (2.56)$$

We summarize our two-body results; all the important equations governing two-body scattering have been expressed in operator form. We have deduced the first and second resolvent equations. Simple algebraic manipulations then lead to the Lippmann-Schwinger equations for the t -matrix. We have also been able, by using the spin components (helicity states) not only in the initial and final states, but also in the angular momentum states, to reduce the

S-matrix to a simpler form. The above approach to two-body scattering is applicable to the solution of the three-body scattering problem. By using the first and second resolvent identities together with our two-body results we will obtain Faddeev's equations.

CHAPTER 3

THE THREE - PARTICLE- SYSTEM

Here we shall summarise the derivation of the Faddeev equations which help one overcome certain defects of the Lippmann-Schwinger equations for the three-body problem. We also discuss the angular momentum analysis for the three-body problem based on Omnes [10]. Further, the concepts of spin and iso-spin are introduced.

Let us consider first three non-relativistic spinless particles of masses m_1, m_2, m_3 and their momenta p_1, p_2, p_3 respectively. We shall always assume that the total centre-of-mass momentum $\underline{P}(= p_1 + p_2 + p_3)$ is zero. This condition implies that only two of the three momentum vectors are independent. The momentum configuration of a system can be described by the pair of vectors $(\underline{k}_3, p_3)$ where $\underline{k}_3$ is defined in (2.3). The unperturbed three-particle Hamiltonian H_0 is given by

$$H_0 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} . \quad (3.1)$$

The appropriate three-particle Hilbert space for these variables is that of square integrable functions of six real variables on $L_2(\underline{k}, \underline{p})$. Now we define the form of a two-body interaction in the three-particle Hilbert space. A two-body interaction will not involve the third particle and thus the potential v_{12} may be written in the three-particle momentum space as

$$\begin{aligned} \langle p_1, p_2, p_3 | v_{12} | p'_1, p'_2, p'_3 \rangle &= \delta(p_3 - p'_3) \langle p_1 p_2 | v_{12} | p'_1 p'_2 \rangle \\ &= \delta(p_3 - p'_3) \langle \underline{k}_3 | v_{12} | \underline{k}'_3 \rangle . \end{aligned} \quad (3.2)$$

The Hamiltonian of the three-body system has the form
($\hbar = 1$)

$$H = H_0 + v \quad (3.3)$$

$$\text{where } v = \sum_{\alpha=0}^3 v_{\alpha} \text{ and } v_0 = 0 .$$

The three-body forces could be included, but we shall leave them out for the sake of simplicity. Then one defines the Green's function operators

$$G(z) = [H - ZI]^{-1} \quad (3.4)$$

and

$$G_0(z) = [H_0 - ZI]^{-1} \quad (3.5)$$

where Z is a complex parameter and I is the unit operator.

As before we start with the second resolvent equation relating $G_0(z)$ and $G(z)$. Algebraic manipulations identical to those that give us (2.26) and (2.27) show that

$$G(z) = G_0(z) - G_0(z)vG(z) \quad (3.6)$$

$$G(z) = G_0(z) - G(z)vG_0(z) \quad (3.7)$$

We can express the scattering amplitude operator $T(z)$ of the two-body system in the Hilbert space variable of the three-body system. Then $T(z)$ denotes a state in which one of the three-particles does not interact with the other two. Thus

$$T(z) = v - vG(z)v. \quad (3.8)$$

Combining (3.8) with (3.6) and (3.7) gives us

$$T(z) = v - v G_o(z)T(z) \quad (3.9)$$

$$= v - T(z)G_o(z)v, \quad (3.10)$$

where we have also used

$$G(z)v = G_o(z)T(z)$$

and

$$vG(z) = T(z)G_o(z) .$$

The formal solution of equation (3.9) is

$$T(z) = [1 + vG_o(z)]^{-1}v .$$

By expansion of $[1 + vG_o(z)]^{-1}$, we get

$$\begin{aligned} T(z) &= [1 - vG_o(z) + vG_o(z)vG_o(z) - \dots]v \\ &= v - vG_o(z)v + vG_o(z)vG_o(z)v - \dots . \end{aligned} \quad (3.11)$$

We can rearrange this series in the following way by

using $v = v_1 + v_2 + v_3$,

$$\begin{aligned} T(z) &= v_1 - v_1G_o(z)v_1 + v_1G_o(z)v_1G_o(z)v_1 \dots \\ &\quad + v_2 - v_2G_o(z)v_2 + v_2G_o(z)v_2G_o(z)v_2 \dots \\ &\quad + v_3 - v_3G_o(z)v_3 + v_3G_o(z)v_3G_o(z)v_3 \dots \end{aligned}$$

$$\begin{aligned}
& + (v_1 - v_1 G_0(z) v_1 + \dots) G_0(z) (v_2 - v_2 G_0(z) v_2 + \dots) \\
& + \dots
\end{aligned}
\tag{3.12}$$

Now $T_1(z)$ is denoted the sum of the chain

$$T_1(z) = v_1 - v_1 G_0(z) v_1 + \dots \tag{3.13}$$

Obviously, $T_1(z)$ is the solution of the equation

$$T_1(z) = v_1 - v_1 G_0(z) T_1(z), \tag{3.14}$$

which is obtained from (3.9) if the other two potentials v_2 and v_3 are equal to zero.

Here $T_1(z)$ is the amplitude for the process in which particles 2 and 3 scatter, while particle 1 goes straight through. It is not quite the same as the two-particle amplitude $t_1(z)$ for particles 2 and 3. The relation between them is given in the momentum space by

$$\langle p_1, \underline{k}_1 | T_1(z) | p'_1, \underline{k}'_1 \rangle = \delta(p_1 - p'_1) \delta(\underline{k}_1 - \underline{k}'_1) | t_1(z - p_1^2) | \underline{k}'_1 \rangle.
\tag{3.15}$$

Lovelace [8] has shown that an important consequence of (3.15) is that, if the $t_\alpha(z)$ have bound state poles at $z = -E_\alpha$,

then $T_\alpha(z)$ will have branch points there, with cuts going from $-E_\alpha$ to $+\infty$. These are the right hand cuts for the scattering of the bound state. We define the energy scale so that $z = 0$ corresponds to the three-particle threshold for a three-particle system and to the two-particle threshold for a two-particle system.

The amplitude $T(z)$ for a transition between an initial configuration of the three particles with momenta p_1, p_2, p_3 and a final configuration with momenta p'_1, p'_2, p'_3 is given by the matrix element of the scattering matrix

$$\lim_{z \rightarrow E + i0} \langle p'_1 p'_2 p'_3 | T(z) | p_1 p_2 p_3 \rangle . \quad (3.16)$$

Faddeev has shown that $T(z)$ can be written as a sum of the three contributions,

$$T(z) = T^{(1)}(z) + T^{(2)}(z) + T^{(3)}(z) \quad (3.17)$$

where $T^{(\alpha)}(z)$ are defined by

$$T^{(\alpha)}(z) = v_\alpha - v_\alpha G_0(z) T(z) , \quad \alpha = 1, 2, 3 . \quad (3.18)$$

With the help of Eqs. (3.14), (3.17) and (3.18), the

components of (3.17) form a set of equations which are conveniently written in matrix form

$$\begin{pmatrix} T^{(1)}(z) \\ T^{(2)}(z) \\ T^{(3)}(z) \end{pmatrix} = \begin{pmatrix} T_1(z) \\ T_2(z) \\ T_3(z) \end{pmatrix} - \begin{pmatrix} 0 & T_1(z) & T_1(z) \\ T_2(z) & 0 & T_2(z) \\ T_3(z) & T_3(z) & 0 \end{pmatrix} \begin{pmatrix} T^{(1)}(z) \\ T^{(2)}(z) \\ T^{(3)}(z) \end{pmatrix} \quad (3.19)$$

We can write Eq. (3.19) in the following form

$$T^{(\alpha)}(z) = T_\alpha(z) - T_\alpha(z)G_0(z)[T^{(\beta)}(z) + T^{(\gamma)}(z)] \quad (3.20)$$

The integral equation for $\langle \underline{p}, \underline{k} | T(z) | \underline{p}', \underline{k}' \rangle$ is simply derived from the operator Eq. (3.20) by taking the plane wave expectation values and using the completeness relation for the plane wave states. Thus we have

$$\begin{aligned} \langle \underline{p}, \underline{k} | T^{(\alpha)}(z) | \underline{p}', \underline{k}' \rangle &= \langle \underline{p}, \underline{k} | T_\alpha(z) | \underline{p}', \underline{k}' \rangle - \\ &\int \frac{d\underline{k}'' d\underline{p}'' \langle \underline{p}, \underline{k} | T_\alpha(z) | \underline{p}'', \underline{k}'' \rangle}{\frac{\underline{p}''^2}{2n} + \frac{\underline{k}''^2}{2\mu} - z} \langle \underline{p}'', \underline{k}'' | [T^{(\beta)}(z) + T^{(\gamma)}(z)] | \underline{p}', \underline{k}' \rangle \end{aligned} \quad (3.21)$$

where

$$\frac{1}{n} = \frac{1}{m_1} + \frac{1}{m_2 + m_3} . \quad (3.22)$$

We can write this more explicitly by using the relationship, (3.15), between two-body and three-body amplitudes.

Our plane wave equation becomes

$$\begin{aligned} \langle \underline{p}_\alpha \underline{k}_\alpha | T^{(\alpha)}(z) | \underline{p}_\alpha' \underline{k}_\alpha' \rangle &= \delta(\underline{p}_\alpha - \underline{p}_\alpha') \langle \underline{k}_\alpha | t_\alpha(z - \frac{p_\alpha^2}{2n_\alpha}) \\ &| \underline{k}_\alpha' \rangle - \int \frac{d\underline{p}_\alpha'' d\underline{k}_\alpha'' \delta(\underline{p}_\alpha - \underline{k}_\alpha'') \langle \underline{k}_\alpha | t_\alpha(z - \frac{p_\alpha^2}{2n_\alpha}) | \underline{k}_\alpha' \rangle}{\frac{p_\alpha''^2}{2n_\alpha} + \frac{k_\alpha''^2}{2\mu_\alpha} - z} \times \\ &\langle \underline{p}_\alpha'' \underline{k}_\alpha'' | [T^{(\beta)}(z) + T^{(\gamma)}(z)] | \underline{p}_\alpha' \underline{k}_\alpha' \rangle . \end{aligned} \quad (3.23)$$

This makes explicitly the fact that the principal ingredient contained in the kernel for the Faddeev's integral equation is the off-shell two-body transition amplitude $\langle \underline{k}_\alpha | t_\alpha(z - \frac{p_\alpha^2}{2n_\alpha}) | \underline{k}_\alpha'' \rangle$. When we have obtained this kernel and have solved Eq. (3.23), then we have solved all three-body problems for the three particles interacting through potentials v_α . The operator kernel $K(z)$ is also defined from Eq. (3.19) by

$$K(z) = \begin{pmatrix} 0 & T_1(z) & T_1(z) \\ T_2(z) & 0 & T_2(z) \\ T_3(z) & T_3(z) & 0 \end{pmatrix} G_0(z) . \quad (3.24)$$

This is called Faddeev's kernel whose matrix elements contain Δ -functions of the momenta. The important property of equations (3.19) is that the square of the kernel does not contain any Δ -function, owing to zeros on the main diagonal. Furthermore, Faddeev [4] has proved that $K^2(z)$ is completely continuous, i.e.

$$\text{Trace } K^2(z)K^2(z)^\dagger < \infty \quad (3.25)$$

apart from the values of z which are not equal to the energy of a physical state. Lovelace [8] has shown that the scattering matrix $T(z)$ is an analytic function of z except for these values of z which are equal to the energy of a physical state, e.g. when $K^2(z)$ denotes a bound state.

We denote the momentum state $|p_1 p_2 p_3\rangle$ by a single parameter $|p\rangle$. The normalization of these states is given by

$$\langle \underline{p}' | \underline{p} \rangle = \zeta_3(\underline{p}' - \underline{p}) . \quad (3.26)$$

We are considering that the states with momentum $\underline{p}'$ in an arbitrary direction specified by polar angles θ, ϕ may be defined by means of a suitable rotation applied to the states having momentum $\underline{p}$ in the positive z-direction. The rotation is conveniently defined to be a rotation through an angle θ in the positive direction about the axis $\underline{p} \times \underline{p}'$.

It is easy to construct the states with zero linear momentum and definite total angular momentum J and its third component $J_z (= M)$. Since the magnitude p of the relative momentum and the helicities λ are invariant under rotations, we can assign definite values to them, together with J and M . An eigenstate of this set of observables is denoted by $|\underline{p}, E, J, M, \lambda\rangle$ where E and λ stand for E_1, E_2, E_3 and $\lambda_1, \lambda_2, \lambda_3$ respectively. In the centre-of-mass system ($\underline{p} = 0$), we can denote the initial and final states by $|E, J, M, \lambda\rangle$ and $|E', J', M', \lambda'\rangle$ respectively. In order to evaluate the operator products we need to know how to transform in the

three-body Hilbert space from the basis $|p_1, p_2, p_3\rangle$ to the basis $|E_1, E_2, E_3, J, M, \lambda\rangle$. Since the rotation matrix element $D_{M\lambda}^J(\psi, \theta, \phi)$ is just the eigenfunction of J^2, M, λ on the angular space (ψ, θ, ϕ) we have

$$\langle p_1, p_2, p_3 | E_1, E_2, E_3, J, M, \lambda \rangle = N \delta_3 \left(\frac{p^2}{2m} - E \right) D_{M\lambda}^{J*}(\psi, \theta, \phi), \quad (3.27)$$

where N is some scalar to be determined. We can determine N by fixing our normalizations for the states $|p_1, p_2, p_3\rangle$ and $|E_1, E_2, E_3, J, M, \lambda\rangle$. The conventional normalization condition for $|E_1, E_2, E_3, J, M, \lambda\rangle$ is taken to be

$$\begin{aligned} \langle E_1, E_2, E_3, J, M, \lambda | E'_1, E'_2, E'_3, J', M', \lambda' \rangle \\ = \delta_3(E-E') \delta_{JJ'} \delta_{MM'} \delta_{\lambda\lambda'}. \end{aligned} \quad (3.28)$$

Thus Eq. (3.28) may be written by using the completeness of the momentum states and after integrating over p_3

$$\int dp_1 dp_2 \langle E_1, E_2, E_3, J, M, \lambda | p_1, p_2, p_3 \rangle \langle p_1, p_2, p_3 | E'_1, E'_2, E'_3, J', M', \lambda' \rangle =$$

$$= \int d\mathbf{p}_1 d\mathbf{p}_2 \delta_3(E - \frac{\mathbf{p}_1^2}{2m_1}) \delta_3(E' - \frac{\mathbf{p}_2^2}{2m_2}) |N|^2 D_{M\lambda}^J(\psi, \theta, \phi) D_{M\lambda}^{J*}(\psi, \theta, \phi) . \quad (3.29)$$

By changing the variables of integration to $\mathbf{p}_1^2/2m_1$, $\mathbf{p}_2^2/2m_2$, $\mathbf{p}_3^2/2m_3$, ψ , θ , ϕ , we can integrate easily. For $\mathbf{p}_1$ and $\mathbf{p}_2$ restricted to lie in the plane defined by ψ , θ , ϕ we can write

$$d\mathbf{p}_1 d\mathbf{p}_2 = m_1 m_2 m_3 d(\mathbf{p}_1^2/2m_1) d(\mathbf{p}_2^2/2m_2) d(\mathbf{p}_3^2/2m_3) dR , \quad (3.30)$$

where dR is the measure of the rotation group

$$dR = \sin \theta d\theta d\psi d\phi , \quad (3.31)$$

and

$$\begin{aligned} \int dR &= \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \\ &= 8\pi^2 . \end{aligned} \quad (3.32)$$

Thus Eq. (3.29) may be simplified to

$$|N|^2 m_1 m_2 m_3 \delta_3(E-E') \delta_{JJ'} \delta_{MM'} \delta_{\lambda\lambda'} \frac{8\pi^2}{(2J+1)} \quad (3.33)$$

by using the orthogonality relation for the D-functions. Comparing (3.33) with (3.28) gives us $|N|^2$

$$|N|^2 = \frac{2J+1}{8\pi^2 m_1 m_2 m_3} . \quad (3.34)$$

We have to find the Faddeev equations (3.20) when taking the matrix elements between the states $|E, J, M, \lambda\rangle$ and $|E', J, M, \lambda'\rangle$. Thus

$$\begin{aligned} \langle E', J, M, \lambda' | T^{(\alpha)}(z) | E, J, M, \lambda \rangle \\ = \langle E', J, M, \lambda' | T_{\alpha}(z) | E, J, M, \lambda \rangle \\ - \langle E', J, M, \lambda' | T_{\alpha}(z) G_0(z) [T^{(\beta)}(z) + T^{(\gamma)}(z)] \\ | E, J, M, \lambda \rangle . \end{aligned} \quad (3.35)$$

In the initial and final states, we have used the same J and M indices because J^2 and M both commute with the total Hamiltonian, H and from the definition of the three-body transition operator (3.8), it follows immediately that $T(z)$ is diagonal in J and M . The expectation value of $G_0(z)$ is very simple in the state $|E, J, M, \lambda\rangle$ because all J^2, M, λ commute with each other and H_0 so that

$$\begin{aligned}
\langle E', J', M', \lambda' | G_0(z) | E, J, M, \lambda \rangle = \\
= \delta_3(E-E') \delta_{JJ'} \delta_{MM'} \delta_{\lambda\lambda'} [E_1 + E_2 + E_3 - z]^{-1} . \quad (3.36)
\end{aligned}$$

Then Eq. (3.35) becomes

$$\begin{aligned}
\langle E', J, M, \lambda' | T^{(\alpha)}(z) | E, J, M, \lambda \rangle = \langle E', J, M, \lambda' | T_\alpha(z) \\
| E, J, M, \lambda \rangle - \sum_{\lambda''} \int \frac{dE'' \langle E', J, M, \lambda' | T_\alpha(z) | E'', J, M, \lambda'' \rangle}{E_1'' + E_2'' + E_3'' - z} \times \\
\langle E'', J, M, \lambda'' | [T^{(\beta)}(z) + T^{(\gamma)}(z)] | E, J, M, \lambda \rangle . \quad (3.37)
\end{aligned}$$

This is an integral equation which contains the kernel in terms of known two-body functions.

We can now evaluate the inhomogeneous terms in the Faddeev equations, namely, $\langle E', J, M, \lambda' | T_\alpha(z) | E, J, M, \lambda \rangle$ by extending the procedure of Omnes [10] to the angular momentum and to the helicities. For simplicity, let us consider $\alpha = 1$. The expectation value of $T_1(z)$ between the states $\langle E', J, M, \lambda' |$ and $| E, J, M, \lambda \rangle$ is

$$\begin{aligned}
\langle E', J, M, \lambda' | T_1(z) | E, J, M, \lambda \rangle \equiv \langle T_1 \rangle = \int \int dp_2 dp_2' dp_3 dp_3' \times \\
\langle E', J, M, \lambda' | p_2 p_3 \rangle \langle p_2 p_3 | T_1(z) | p_2' p_3' \rangle \langle p_2' p_3' | E, J, M, \lambda \rangle . \quad (3.38)
\end{aligned}$$

This can be rewritten, using the relation between T_1 and t_1 , as

$$\begin{aligned}
 \langle T_1 \rangle &= (m_1 m_2 m_3)^2 |N|^2 \int dR d(p_1^2/2m_1) d(p_2^2/2m_2) \\
 &\quad d(p_3^2/2m_3) \int dR' d(p_1'^2/2m_1) d(p_2'^2/2m_2) d(p_3'^2/2m_3) \\
 &\quad \zeta_3(E - \frac{p^2}{2m}) \zeta_3(E' - \frac{p'^2}{2m}) D_{M\lambda}^J(R) D_{M\lambda'}^{J*}(R') \\
 &\quad \zeta_{\lambda_1 \lambda_1'} \delta(p_1 - p_1') \langle \underline{k}_1 | t_1(z - E_1) | \underline{k}_1' \rangle, \quad (3.39)
 \end{aligned}$$

where $|N|^2$ is the normalization factor and $\wedge = \lambda_2 - \lambda_3$ and $\wedge' = \lambda_2' - \lambda_3'$.

Since p_1 and p_2 depend on the Euler angles in R and R' , the only complicated factor in this integral is $\delta(p_1 - p_1')$. Thus we may use the representation of the delta function in spherical coordinates [27] to get

$$\begin{aligned}
 \delta(p_1 - p_1') &= \frac{1}{p_1^2} \delta(p_1 - p_1') \delta(\cos \theta - \cos \theta') \delta(\psi - \psi') \\
 &= \frac{1}{m_1 p_1} \delta(E_1 - E_1') \delta(\cos \theta - \cos \theta') \delta(\psi - \psi'). \quad (3.40)
 \end{aligned}$$

Substituting Eq. (3.40) in Eq. (3.39), we get

$$\begin{aligned}
\langle E', J, M, \lambda' | T_1(z) | E, J, M, \lambda \rangle &= (m_1 m_2 m_3) \cdot \frac{(2j+1)}{8 \pi^2} \times \\
&\frac{1}{m_1 p_1} \delta(E_1 - E'_1) \delta_{\lambda_1 \lambda'_1} \int dR \int dR' D_{M\wedge}^J(R) D_{M\wedge}^{J*}(R') \times \\
&\langle \underline{k}_1 | t_1(z - E_1) | \underline{k}'_1 \rangle \delta(\cos \theta - \cos \theta') \delta(\psi - \psi') \quad (3.41)
\end{aligned}$$

Integrating over the delta-functions $\delta(\cos \theta - \cos \theta')$, $\delta(\psi - \psi')$ gives us

$$\begin{aligned}
\langle T_1(z) \rangle &= \frac{\delta(E_1 - E'_1)}{m_1 (2m_1 E_1)^{1/2}} \cdot \frac{(2j+1)}{8 \pi^2} (m_1 m_2 m_3) \times \\
&\int d\phi \, d\phi' \, d \cos \theta \, d\psi \, D_{M\wedge}^J(\phi, \theta, \psi) \\
&D_{M\wedge}^{J*}(\phi', \theta, \psi) \langle \underline{k}_1 | t_1(z - E_1) | \underline{k}'_1 \rangle \delta_{\lambda_1 \lambda'_1} \quad (3.42)
\end{aligned}$$

We choose the direction of the helicities to lie along the p_1 direction. This direction is uniquely determined by θ, ψ . In fact, we can deduce a relation between $D_{M\wedge}^{J*}(\phi', \theta, \psi)$ and $D_{M\wedge}^{J*}(\phi, \theta, \psi)$ as

$$\begin{aligned}
D_{M\wedge}^{J*}(\phi', \theta, \psi) &= \exp(-i \wedge \phi') d_{M\wedge}^J(\theta) \exp(-i M \psi) \\
&= \exp(i \wedge (\phi - \phi')) \exp(-i \wedge \phi) d_{M\wedge}^J(\theta) \exp(-i M \psi)
\end{aligned}$$

$$= \exp(i \wedge (\phi - \phi')) D_{M \wedge}^{J*}(\phi, \theta, \psi) . \quad (3.43)$$

We have $\lambda_1 = \lambda'_1$. Then Eq. (3.42) becomes, using
 $u = \phi - \phi'$

$$\begin{aligned} \langle T_1(z) \rangle &= \frac{m_1 m_2 m_3}{m_1 (2m_1 E_1)^{\frac{1}{2}}} \delta(E_1 - E'_1) \left[\frac{(2j+1)}{8\pi^2} \int dR D_{M \wedge}^J(R) \times \right. \\ &\quad \left. D_{M \wedge}^{J*}(R') \right] \int_0^{2\pi} du \exp(i \wedge u) \langle \underline{k}_1 | t_1(z - E_1) | \underline{k}'_1 \rangle . \end{aligned} \quad (3.44)$$

For the choice of the direction of the helicities, we have

$$\begin{aligned} \langle T_1(z) \rangle &= \frac{m_1 m_2 m_3}{m_1 (2m_1 E_1)^{\frac{1}{2}}} \delta(E_1 - E'_1) \delta_{\wedge \wedge'} \int_0^{2\pi} du \exp(i \wedge u) \times \\ &\quad \langle \underline{k}_1 | t_1(z - E_1) | \underline{k}'_1 \rangle . \end{aligned} \quad (3.45)$$

We may expand the two-body amplitude by using the partial wave projection [12] as

$$\begin{aligned} \langle \underline{k}_1 | t_1(z - E_1) | \underline{k}'_1 \rangle &= \frac{1}{4\pi^2 \mu_1} \sum_{\ell} (2\ell+1) t_{1\ell}(k_1, k'_1; z - E_1) \times \\ &\quad P_{\ell}(\cos \gamma_1 \cos \gamma'_1 + \sin \gamma_1 \sin \gamma'_1 \cos u) . \end{aligned} \quad (3.46)$$

The Legendre polynomial may be now expressed in terms of $Y_{\ell m}$'s

$$\begin{aligned}
 P_{\ell}(\cos \gamma_1 \cos \gamma_1' + \sin \gamma_1 \sin \gamma_1' \cos u) \\
 &= \frac{4\pi}{2\ell+1} \sum_m Y_{\ell m}^*(\gamma_1, \phi_1) Y_{\ell m}(\gamma_1', \phi_1') \\
 &= \frac{4\pi}{2\ell+1} \sum_m Y_{\ell m}(\gamma_1, 0) Y_{\ell m}(\gamma_1', 0) \exp(-imu) . \quad (3.47)
 \end{aligned}$$

Then Eq. (3.45) can be written with the help of Eqs. (3.46) and (3.47) as

$$\begin{aligned}
 \langle T_1(z) \rangle &= \frac{m_2 + m_3}{\pi(2m_1 E_1)^{\frac{1}{2}}} \delta(E_1 - E_1') \sum_{\ell} Y_{\ell \wedge_1}(\gamma_1, 0) Y_{\ell \wedge_1'}(\gamma_1', 0) \times \\
 &\quad \delta_{\wedge_1 \wedge_1'} t_{\ell}(k_1, k_1'; z - E_1) \left[\int_0^{2\pi} du \exp(iu(\wedge_1 - m)) \right] . \\
 &\hspace{25em} (3.48)
 \end{aligned}$$

The integral term in square bracket is zero unless $m = \wedge_1$.

We can now write down the most general form of the matrix element $\langle T_{\alpha} \rangle$ thus

$$\langle E', J, M, \lambda' | T_{\alpha}(z) | E, J, M, \lambda \rangle = \frac{m_{\beta} + m_{\gamma}}{\pi(2m_{\alpha} E_{\alpha})^{\frac{1}{2}}} \delta(E_{\alpha} - E_{\alpha}')$$

$$\delta_{\alpha} \delta_{\alpha'} \sum_{\ell} Y_{\ell}(\gamma_{\alpha}, 0) Y_{\ell}(\gamma_{\alpha'}, 0) t_{\alpha\ell}(k_{\alpha}, k_{\alpha'}; z - E_{\alpha}) . \quad (3.49)$$

Up to now we have assumed that the particles have no initial degrees of freedom. We now introduce isotopic spin. The three-particle states are represented by $|p_1, p_2, p_3, I_1 I_{1z}, I_2 I_{2z}, I_3 I_{3z}\rangle$ where $I_{\alpha}, I_{\alpha z}$ designate the total isotopic spin and z-component respectively of the α th particle. These states can be written in the form $|p_1, p_2, p_3, I_1(I_2 I_3) I_{23} I_{1z}\rangle$. In this wave function I_2 and I_3 are coupled to form I_{23} which is coupled in turn to I_1 to form the total isotopic spin I . We can write the inhomogeneous term including angular momentum, helicities and isotopic spin as

$$\langle E', J, M, \lambda', I'_{23} | T_1(z) | E, J, M, \lambda, I_{23} \rangle = \langle I_1(I_2 I_3) I_{23} I_{1z} | \\ | (I_3 I_1) I_{31} I_2 I_{2z} \rangle \langle E', J, M, \lambda' | T_1(z) | E, J, M, \lambda \rangle . \quad (3.50)$$

We use an additional subscript 23 to emphasize that the two-body scattering matrix also depends on I_{23} . The matrix elements $\langle I_1(I_2 I_3) I_{23} I_{1z} | (I_3 I_1) I_{31} I_2 I_{2z} \rangle$ are directly related to the 6-j symbols according to the formula

$$\begin{aligned}
& \langle I_1(I_2 I_3) I_{23} II_z \mid (I_3 I_1) I_{31} I_2 II_z \rangle \\
& = (-1)^{I_1+I_2+I_3+I} [(2I_{31}+1)(2I_{23}+1)]^{\frac{1}{2}} \times \\
& \quad \left\{ \begin{matrix} I_2 & I_3 & I_{23} \\ I_1 & I & I_{31} \end{matrix} \right\} \quad (3.51)
\end{aligned}$$

We have considered $I_{31} = I'_{23}$. The recoupling of three angular momenta and the 6-j symbols are discussed in appendix.

Finally one gets the inhomogeneous term $T_1(z)$ in the representation of angular momentum, helicities and isospin with the combination of Eqs. (3.48), (3.50) and (3.51)

$$\begin{aligned}
\langle E', J, M, \lambda', I'_{23} \mid T_1(z) \mid E, J, M, \lambda, I_{23} \rangle & = (-1)^{I_1+I_2+I_3+I} \times \\
& [(2I_{31}+1)(2I_{23}+1)]^{\frac{1}{2}} \left\{ \begin{matrix} I_2 & I_3 & I_{23} \\ I_1 & I & I_{31} \end{matrix} \right\} \frac{m_2+m_3}{\pi(2m_1 E_1)^{\frac{1}{2}}} \times \\
& \delta_{\lambda_1 \wedge_1, \lambda'_1} \delta_{(E'_1-E_1)} \sum_{\ell} Y_{\ell \wedge_1}(\gamma_1, 0) Y_{\ell \wedge'_1}(\gamma'_1, 0) \\
& t_{1\ell}(k_1, k'_1; z-E_1) . \quad (3.52)
\end{aligned}$$

We can write the kernel $K_\alpha(E', E'')$ in general form

$$K_\alpha(E', E'') = (-1)^{I_1 + I_2 + I_3 + I} [(2I_{31} + 1)(2I_{23} + 1)]^{\frac{1}{2}} \times$$

$$\left\{ \begin{matrix} I_2 & I_3 & I_{23} \\ I_1 & I & I_{31} \end{matrix} \right\} \frac{m_\beta + m_\gamma}{\pi (2m_\alpha E_\alpha)^{\frac{1}{2}}} \delta_\alpha \wedge_\alpha \delta_\alpha (E'_\alpha - E_\alpha) \times$$

$$(E_\alpha'' + E_\beta'' + E_\gamma'' - z)^{-1} \sum_k Y_{k\wedge_\alpha}(\gamma_\alpha, 0) Y_{k\wedge'_\alpha}(\gamma'_\alpha, 0)$$

$$t_{\alpha k}(k_\alpha, k'_\alpha; z - E_\alpha). \quad (3.53)$$

Basdevant [28] and Ahmadzadeh and Tjon [29] have shown that if the kernel $K(z)$ is separable the system of integral equations in three variables are reduced to a system of coupled integral equations in one variable only. We follow Basdevant's reduction method to obtain from three variables to one variable coupled integral equations. The kernel defined in Eq. (3.53) can be written in the form

$$K_\alpha(E', E'') = \delta(E'_\alpha - E''_\alpha) A_\alpha(E') B_\alpha(E'') \quad (3.54)$$

where

$$A_\alpha(E') = (-1)^{I_1 + I_2 + I_3 + I} [(2I_{31} + 1)(2I_{23} + 1)]^{\frac{1}{2}} \left\{ \begin{matrix} I_2 & I_3 & I_{23} \\ I_1 & I & I_{31} \end{matrix} \right\}$$

$$\times \delta_{\alpha} \wedge_{\alpha} \wedge_{\alpha} \frac{m_{\beta} + m_{\gamma}}{\pi(2m_{\alpha} E_{\alpha})^{\frac{1}{2}}} Y_{\ell \wedge_{\alpha}}(\gamma'_{\alpha}, 0) t_{\alpha \ell}(k_{\alpha}, k'_{\alpha}, z - E_{\alpha})$$

and

$$B_{\alpha}(E'') = \left(\sum_{\beta} E_{\beta}'' - z \right)^{-1} Y_{\ell \wedge_{\alpha}}(\gamma''_{\alpha}, 0) .$$

Then the Faddeev equations (3.37) become

$$T^{(\alpha)}(E', E) = T_{\alpha}(E', E) - A_{\alpha}(E') \int B_{\alpha}(E'') \zeta(E'_{\alpha} - E''_{\alpha}) \times \\ [T^{(\beta)}(E'', E) + T^{(\gamma)}(E'', E)] dE'' . \quad (3.55)$$

Its solution can be defined as

$$T^{(\alpha)}(E'_{\alpha}, E) = T_{\alpha}(E'_{\alpha}, E) - A_{\alpha}(E'_{\alpha}) X_{\alpha}(E'_{\alpha}, E) . \quad (3.56)$$

Substituting (3.56) in (3.55), we obtain a new set of integral equations, involving the function $X_{\alpha}(E'_{\alpha}, E)$

$$X_{\alpha}(E'_{\alpha}, E) = \int B_{\alpha}(E'') \zeta(E'_{\alpha} - E''_{\alpha}) [T_{\beta}(E''_{\alpha}, E) + T_{\gamma}(E''_{\alpha}, E)] dE'' \\ - \sum_{\beta} \int Z_{\alpha\beta}(E'_{\alpha}, E''_{\alpha}) X_{\beta}(E''_{\alpha}, E) dE''_{\alpha} . \quad (3.57)$$

Introducing the functions $\mathcal{Y}_{\alpha}(E'_{\alpha}, E)$ and $Z_{\alpha\beta}(E'_{\alpha}, E''_{\alpha})$ thus

$$\gamma_{\alpha}(E'_{\alpha}, E) = \int B_{\alpha}(E'') \delta(E'_{\alpha} - E''_{\alpha}) [T_{\beta}(E'', E) + T_{\gamma}(E'', E)] dE''$$

and

$$Z_{\alpha\beta}(E'_{\alpha}, E'_{\alpha}) = (1 - \delta_{\alpha\beta}) \int B_{\alpha}(E) A_{\beta}(E) \delta(E'_{\alpha} - E_{\alpha}) \delta(E''_{\beta} - E_{\beta}) dE ,$$

Eq. (3.57) becomes

$$X_{\alpha}(E'_{\alpha}, E) = \gamma_{\alpha}(E'_{\alpha}, E) - \sum_{\beta} \int Z_{\alpha\beta}(E'_{\alpha}, E'_{\alpha}) X_{\beta}(E'', E) dE'' \quad (3.58)$$

or written in matrix form

$$\begin{pmatrix} X_1(E'_{\alpha}, E) \\ X_2(E'_{\alpha}, E) \\ X_3(E'_{\alpha}, E) \end{pmatrix} = \begin{pmatrix} \gamma_1(E'_{\alpha}, E) \\ \gamma_2(E'_{\alpha}, E) \\ \gamma_3(E'_{\alpha}, E) \end{pmatrix} - \begin{pmatrix} 0 & Z_{12} & Z_{13} \\ Z_{21} & 0 & Z_{23} \\ Z_{31} & Z_{32} & 0 \end{pmatrix} \begin{pmatrix} X_1(E''_{\alpha}, E) \\ X_2(E''_{\alpha}, E) \\ X_3(E''_{\alpha}, E) \end{pmatrix} . \quad (3.59)$$

At this point let us assess what we have accomplished. The form of the equations we have obtained is quite analogous to that of the original Faddeev equations; what must be pointed out as extremely important from a practical

point of view is that we have now a problem involving a system of coupled integral equations in one variable only.

CHAPTER 4THE SEPARABLE APPROXIMATIONS

In the last section, we have discussed the formal aspects of three-particle scattering. All the properties of the three-particle system may be derived as long as one knows the exact scattering amplitudes (off-energy shell) for the two-particle system and information about the form of the three-body force.

Attempts to solve the three-body problems were made by several authors [6, 7, 9, 10]. Though starting with different formalisms Lovelace and Rosenberg in potential theory and Amado in field theory, the three-body equations of these theories are very similar. However, the most comprehensive calculations are due to Lovelace.

With regard to two-body interaction, it should be noted that the two-particle scattering data only determine the scattering amplitude on the energy shell. Information regarding the off-shell scattering amplitude can be obtained by the study of multiparticle systems. In particular, the Faddeev equations provide a basis for

the study of this off-shell behaviour and its effects on the three-particle system. However, this is not a very practical proposition since the number of degrees of freedom involved in the three-particle problem renders difficult attempts for numerical solution without approximation (S. Lee's Ph.D. thesis, University of London, 1969). One possible approximation is a separable two-body potential.

The Lovelace theory is applicable to a three-particle system in which each two-particle subsystem can form bound states or resonances. It exploits the fact that the energy and form factor of the bound and resonance states determine the behaviour of the two-particle amplitude (off-and on-the energy shell) in the vicinity of the poles associated with these states. The assumption of the separable potential leads to an approximation of the two-particle amplitude which is separable in the initial and final variable.

The separable approximation of the two nucleon system has been proposed by Yamaguchi [13] and has since been used extensively [14, 15]. We here discuss its

theoretical justification. This approximation gives rise to an adequate representation of the low energy two nucleon data. It would be desirable to include the effects of the nucleon-nucleon short-range and tensor forces. The inclusion of these phenomena would greatly increase the numerical difficulties involved in the solution of the three-nucleon problem. As we believe their effects will be small, we have chosen to represent them by a single adjustable parameter hoping that with a single choice of this constant a good representation of several three-nucleon observables can be obtained.

Let us consider a two-body system with Hamiltonian $h (= h_0 + v)$. The relation between the resolvent of the Hamiltonian h and the t -matrix is given by

$$t(z) = v - v g(z) v . \quad (4.1)$$

If h gives rise to a bound state function $\psi_n(\mathbf{p})$ with binding energy E_n , then as is well known $g(z)$ and hence $t(z)$ will have a pole at $z = -E_n$. It has been shown [8] that $g(z)$ has the following spectral representation

$$\langle p | g(z) | p' \rangle = - \sum_n \frac{\psi_n(p) \psi_n(p')}{z + E_n} + \frac{1}{\pi} \int_0^\infty \frac{dq q^2 \psi_\ell(p, q)}{q^2 - z - i\epsilon} \times \psi_\ell(p', q) \quad (4.2)$$

where $\psi_\ell(p, q)$ are the continuum wave functions.

The assumption that the bound states dominate the two-particle interaction leads to the approximation.

Thus

$$\langle p | g(z) | p' \rangle = - \sum_n \frac{\psi_n(p) \psi_n(p')}{z + E_n} . \quad (4.3)$$

If we have just one separable potential with partial wave

$$v_\ell(p, p') = \lambda_\ell g_\ell(p) g_\ell(p') , \quad (4.4)$$

then we can easily get a closed form for the off-shell t -matrix which has been given by many authors [14].

Thus

$$\langle p | t_\ell(z) | p' \rangle = \sum_\ell g_\ell(p) g_\ell(p') t_\ell(z) , \quad (4.5)$$

where

$$t_{\ell}(z) = [\lambda_{\ell}^{-1} - (2\pi^2)^{-1} \int_0^{\infty} \frac{q^2 dq |g_{\ell}(q)|^2}{q^2 - z + i\epsilon}]^{-1}. \quad (4.6)$$

The separable potential will have a bound state, if there is a point $z = -E_{\ell}$ for which

$$\lambda_{\ell}^{-1} = (2\pi^2)^{-1} \int_0^{\infty} \frac{q^2 dq |g_{\ell}(q)|^2}{q^2 + E_{\ell}}. \quad (4.7)$$

Then, Eq. (4.6) becomes

$$t_{\ell}(z) = + \frac{1}{z + E_{\ell}} \left[(2\pi^2)^{-1} \int_0^{\infty} \frac{q^2 dq |g_{\ell}(q)|^2}{(q^2 + E_{\ell})(q^2 - z + i\epsilon)} \right]^{-1} \dots \quad (4.8)$$

If we take the second factor constant, then the t -matrix will reduce to the bound state contribution. Thus the t -matrix becomes

$$\langle p | t_{\ell}(z) | p' \rangle = \sum_{\ell} g_{\ell}(p) g_{\ell}(p') (z + E_{\ell})^{-1}, \quad (4.9)$$

where $g_{\ell}(p)$ is the form factor of the bound state.

We see that the factors of the separable potential are just the bound state form factors provided the

coupling constants λ_ℓ of Eq. (4.4) are chosen to give these form factors the right normalization

$$(2\pi^2)^{-1} \int_0^\infty \frac{q^2 dq |g_\ell(q)|^2}{(q^2 + E_\ell)^2} = 1 . \quad (4.10)$$

It is instructive to consider the separable amplitude as an off-shell form of N/D equations. The amplitude (4.5) may be written in this way

$$\langle p | t_\ell(z) | p' \rangle = \frac{N_\ell(z)}{D_\ell(z)} , \quad (4.11)$$

where $D_\ell(z)$ has only the right hand cut and $N_\ell(z)$ only the left hand cut. Therefore, we have

$$t_\ell(z) = \frac{1}{D_\ell(z)} , \quad (4.12)$$

and

$$g_\ell(p) = [N_\ell(p^2)]^{\frac{1}{2}} . \quad (4.13)$$

The bound state form factors $g_\ell(p)$ satisfy [21] dispersion relations in p^2 with only a left hand cut and the off-shell t -matrix is given by

$$\langle p | t_{\chi}(z) | p' \rangle = \frac{[N_{\chi}(p^2)N_{\chi}(p'^2)]^{\frac{1}{2}}}{D_{\chi}(z)} \quad (4.14)$$

provided that the potential wave is determined by a bound state. The numerator $N_{\chi}(z)$ is determined by the form factor of the bound state and hence contains the left hand cut [22]. The form factors that will be considered correspond to pole or double pole approximations to the left hand cut.

CHAPTER 5

THE COULOMB INTERACTIONS

In this section we study the Coulomb interaction. Charged particles are much easier to study experimentally than neutral particles, but their Coulomb interactions complicate the theory considerably. The Coulomb potential is, of course, not separable and at low energies, with the particles held together in the bound or resonant states, can not be treated as a small perturbation. Another exact method for dealing with the Coulomb interactions is therefore required.

The three-particle equations require the exact two-particle t-matrix elements (off-energy shell) and these are at present not known precisely. A method for dealing with this difficulty will be described below. The behaviour of the two-particle interaction via a short-range separable potential together with a Coulomb potential was studied by Harrington [16]. He illustrated the nature of the Coulomb modification to the g-functions in a simple s-wave problem with a single term (central

force part) of separable potential of Yamaguchi type [13]. We shall apply the technique to calculating s-wave g-functions with the separable potentials of Yamaguchi [13].

We consider a Coulomb potential with cut-off. The potential v is expressed as $v = v_s + v_c$ which is the sum of a separable potential v_s and Coulomb potential v_c . The cut-off radius R of v_c is much larger than the range of v_s .

The three-particle system is described by the Hamiltonian H where

$$H = H_0 + v_s + v_c \quad (5.1)$$

and H_0 is the kinetic energy operator. The Coulomb Green's functions are given by [18]

$$G^c(z) = [H - z]^{-1} \quad (5.2)$$

and

$$G_0^c(z) = [H_0 + v_c - z]^{-1}. \quad (5.3)$$

Then we can write $G_0^c(z)$ in momentum space as

$$\langle \underline{p}'_1, \underline{k}'_1 | G_o^c(z) | \underline{p}_1, \underline{k}_1 \rangle = \delta(\underline{p}'_1 - \underline{p}_1) \langle \underline{k}'_1 | g_o^c(z - \frac{p_1^2}{2M}) | \underline{k}_1 \rangle. \quad (5.4)$$

Since $\underline{k}_1$ is the relative momentum of particles 2 and 3 then Eq. (5.4) can be written with the Coulomb function

$\langle \underline{k}_1 | \phi_q^c \rangle$ in momentum space, as

$$\langle \underline{p}'_1, \underline{k}'_1 | G_o^c(z) | \underline{p}_1, \underline{k}_1 \rangle = \delta(\underline{p}'_1 - \underline{p}_1) (2\pi^2)^{-1} \times \int_0^\infty \frac{q^2 dq \langle \underline{k}'_1 | \phi_q^c \rangle \langle \phi_q^c | \underline{k}_1 \rangle}{z - \frac{p_1^2}{2M} - \frac{q^2}{2\mu}}. \quad (5.5)$$

We assume the Coulomb function $\langle \underline{k} | \phi_q^c \rangle$ to be strongly peaked at $\underline{k} = \underline{q}$, both in magnitude and direction.

Then Eq. (5.5) becomes to a first approximation

$$\langle \underline{p}'_1, \underline{k}'_1 | G_o^c(z) | \underline{p}_1, \underline{k}_1 \rangle = \delta(\underline{p}'_1 - \underline{p}_1) \delta(\underline{k}'_1 - \underline{k}_1) (2\pi^2)^{-1} \times \int_0^\infty \frac{q^2 dq |\langle \underline{k}_1 | \phi_q^c \rangle|^2}{z - \frac{p_1^2}{2M} - \frac{q^2}{2\mu}}. \quad (5.6)$$

The short-range interacting separable potential

$v_{s,\ell}$ in its partial wave form is

$$v_{s,\ell}(q',q) = \lambda_\ell g_\ell(q')g_\ell(q) . \quad (5.7)$$

Harrington [16] has shown that when $v_{s,\ell}$ is separable the scattering potential $v_{sc,\ell}$ which includes the Coulomb potential $v_{c,\ell}$ is also separable. Thus

$$v_{sc,\ell}(q',q) = \exp(i\Delta_{c,\ell}(q'))\lambda_\ell g_{c,\ell}(q')g_{c,\ell}(q)\exp(i\Delta_{c,\ell}(q)), \quad (5.8)$$

where

$$g_{c,\ell}(q)\exp(i\Delta_{c,\ell}(q)) = (2\pi^2)^{-1} \int_0^\infty dq q^2 |\langle \underline{k} | \phi_{\underline{q}}^c \rangle| g_\ell(q), \quad (5.9)$$

$\Delta_{c,\ell}$ is the Coulomb phase-shift, and $v_{sc,\ell} = v_{s,\ell} + v_{c,\ell}$.

Since $v_{sc,\ell}$ is separable, the matrix amplitude (4.5) becomes

$$\begin{aligned} \langle \underline{q}' | t_{\ell}^c(z) | \underline{q} \rangle &= \sum_{\ell} \exp(i\Delta_{c,\ell}(q')) g_{c,\ell}(q') g_{c,\ell}(q) \\ &\quad t_{c,\ell}(z) \exp(i\Delta_{c,\ell}(q)) , \end{aligned} \quad (5.10)$$

where

$$t_{c,\ell}(z) = \left[\lambda_{c,\ell}^{-1} - (2\pi^2)^{-1} \int_0^\infty \frac{q^2 dq |g_{c,\ell}(q)|^2}{q^2 - z + i\epsilon} \right]^{-1} . \quad (5.11)$$

The asymptotic form of the radial wave function $W_\ell(q, r)$ in configuration-space for the cut-off Coulomb potential is given by

$$W_\ell(q, r) \sim \sin(qr - \frac{1}{2} \ell \pi + \Delta_{c, \ell}(q)) . \quad (5.12)$$

Yost, Wheeler and Breit [17] have originally shown that the radial Coulomb wave function $F_\ell(q, r)$ has the asymptotic form

$$F_\ell(q, r) \sim \sin(qr - x \log 2qr - \frac{1}{2} \ell \pi + \Delta_{c, \ell}(q)) , \quad (5.13)$$

where $\Delta_{c, \ell}$, the Coulomb phase-shift is given by

$$\Delta_{c, \ell}(q) = \arg \Gamma(\ell + ix) , \quad (5.14)$$

and

$$x = \mu e^2 q^{-1} . \quad (5.15)$$

The wave function $W_\ell(q, r)$ satisfies the Schrödinger equation with a cut-off [23], rather than exact Coulomb potential (for $qR \gg \ell(\ell+1) + x^2(q)$ and $r < R$).

This is required in order that $W_\ell(q, r)$ has its asymptotic form when $r = R$. Then we have for $r \geq R$

$$\begin{aligned}
W_\ell(q, r) &\equiv F_\ell(q, r) \\
&= (2i)^{-\ell-1} C_\ell(x) M_{ix, \ell+\frac{1}{2}}(2iqr) ,
\end{aligned} \tag{5.16}$$

where $M_{\mu, \nu}$ is the Whittaker function and $C_\ell(x)$, the barrier penetration factor [17]

$$\begin{aligned}
C_\ell^2(x) &= 2^{2\ell} ((2\ell+1)!)^{-2} (\ell^2 + x^2) [(\ell-1)^2 + x^2] \dots \\
&\quad (1+x^2) 2\pi x (\exp(2\pi x) - 1)^{-1} .
\end{aligned} \tag{5.17}$$

The function $g_{c, \ell}(q)$ is defined [16] in terms of $W_\ell(q, r)$ by

$$g_{c, \ell}(q) = 4\pi q^{-1} \int_0^\infty r \, dr \, G_\ell(r) W_\ell(q, r) , \tag{5.18}$$

where $G_\ell(r)$ is defined by

$$G_\ell(r) = (2\pi^2)^{-1} \int_0^\infty q^2 \, dq \, g_\ell(q) j_\ell(qr) \tag{5.19}$$

and j_ℓ is a spherical Bessel function.

With the help of (5.16), Eq. (5.18) becomes

$$g_{c, \ell}(q) = 4\pi q^{-1} \int_0^\infty r \, dr \, G_\ell(r) (2i)^{-\ell-1} C_\ell(x) M_{ix, \ell+\frac{1}{2}}(2iqr) . \tag{5.20}$$

We can now compute the functions $g_{c,\ell}(q)$ for the different angular momentum states.

We consider the non-local separable potential of Yamaguchi type [13] for the S-state in the form

$$v_{s,0}(q',q) = \lambda_0 g_0(q') g_0(q) \quad (5.21)$$

where

$$g_0(q) = (\beta_0^2 + q^2)^{-1} \quad (5.22)$$

and β_0 is a constant parameter.

So Eqn. (5.19) becomes

$$G_0(r) = (2\pi^2 r)^{-1} \int_0^\infty q (\beta_0^2 + q^2)^{-1} \sin(qr) dq .$$

Now we have the result [24]

$$\int_0^\infty \frac{x^{2m+1} \sin(ax) dx}{(z+x^2)^{n+1}} = \frac{(-1)^{m+n}}{n!} \frac{\pi}{2} \frac{d^n}{dz^n} [z^m \exp(-a\sqrt{z})] . \quad (5.23)$$

Then $G_0(r)$ becomes

$$G_0(r) = (4\pi r)^{-1} \exp(-\beta_0 r) . \quad (5.24)$$

We get $g_{c,o}(q)$ with the help of Eqns. (5.18) and (5.24)

$$g_{c,o}(q) = 4\pi q^{-1} \int_0^{\infty} r dr (4\pi r)^{-1} \exp(-\beta_o r) (2i)^{-1} c_o(x) \times \\ M_{ix, \frac{1}{2}}(2iqr)$$

We also have the result [25]

$$\int_0^{\infty} \exp(-st) t^{\mu-\frac{1}{2}} M_{\lambda, \mu}(at) dt \\ = a^{\mu+\frac{1}{2}} \left[(2\mu+1) (s-\frac{1}{2}a)^{\lambda-\mu-\frac{1}{2}} (s+\frac{1}{2}a)^{-\lambda-\mu-\frac{1}{2}} \right]. \quad (5.25)$$

Then the final form of $g_{c,o}(q)$ becomes

$$g_{c,o}(q) = (\beta_o^2 + q^2)^{-1} c_o(x) \exp[2x \tan^{-1}(q\beta_o^{-1})], \quad (5.26)$$

where

$$c_o^2(x) = 2\pi x [\exp(2\pi x) - 1]^{-1}. \quad (5.27)$$

All calculations of (5.26) are in appendix III.

This function g_c is useful in finding the correct separable-potential parameters for the scattering of

two charged particles, especially in those cases where there are no neutral counterparts. With this information the off-shell two particle T-matrix needed as input in the Faddeev equations of the three-body problem can be calculated. In the next section, we shall use the function g_c to modify the Faddeev equations of our p-p-n problem.

CHAPTER 6

THE MODIFIED FADDEEV EQUATIONS

The aim of this chapter is to study the three-body bound state problem with two of the particles charged. We shall use the approximate form of the Coulomb Green's functions in momentum space, suggested by Schulman [19] and Adya [20]. Both of them left out spin, isospin, angular momentum, tensor force and hard core saturation effects. They only considered the two-body short-range interaction with the central force term of the Yamaguchi potential [13]. Alessandrini et al [30] discussed the Coulomb energy Δ_c of He^3 and the probability $p_{3/2}$ of finding the He^3 in an $I = 3/2$ -state. They extended the formalism of Alt et al [31] to spin and isospin and also showed that the Coulomb interaction violates the isospin symmetry. We shall try to modify the Faddeev equations with the Coulomb potential including angular momentum, helicities and isospin.

We have already discussed the three-particle system of the uncharged particles and their Green's functions in

Chapter 3. The transition to the charged particle system is now straightforward. The total Hamiltonian for this system is defined in (5.1) and the Green's functions $G^c(z)$ and $G_o^c(z)$ are also defined in (5.2) and (5.3) respectively.

Then we can write

$$G^c(z) = G_o^c(z) - G_o^c(z)T(z)G_o^c(z) , \quad (6.1)$$

where

$$T(z) = \sum_{\alpha=1}^3 T^{(\alpha)}(z) . \quad (6.2)$$

The $T^{(\alpha)}(z)$ satisfy the coupled integral equations

$$T^{(\alpha)}(z) = T_{\alpha}^c(z) - T_{\alpha}^c(z)G_o^c(z) \sum_{\beta \neq \alpha} T^{(\beta)}(z) , \quad (6.3)$$

$(\alpha, \beta = 1, 2, 3)$.

The $T_{\alpha}^c(z)$ is operator in the three-body space for the charged particles. We can write $T_{\alpha}^c(z)$ in analogous to ^{3.52}~~(3.28)~~

$$\langle E', J, M, \lambda', I'_{23} | T_{\alpha}^c(z) | E, J, M, \lambda, I_{23} \rangle = \frac{m_{\beta} + m_{\gamma}}{m_1 m_2 m_3 \sqrt{2m_{\alpha} E_{\alpha}}}$$

$$\delta_{\alpha} (E_{\alpha}' - E_{\alpha}) (-1)^{I_1 + I_2 + I_3 + I} [(2I_{23}' + 1)(2I_{23} + 1)]^{\frac{1}{2}} \times$$

$$\left\{ \begin{matrix} I_2 & I_3 & I_{23} \\ I_1 & I & I_{31} \end{matrix} \right\} \delta_{\wedge \wedge'} \sum_{\ell} Y_{\ell \wedge}(\gamma, 0) Y_{\ell \wedge'}(\gamma', 0) \times$$

$$t_{\ell}^c(k_{\alpha}, k'_{\alpha}; z - E_{\alpha}) \quad (6.4)$$

where

$$t_{\ell}^c(k_{\alpha}, k'_{\alpha}; z) = g_{c, \ell}(k_{\alpha}) g_{c, \ell}(k'_{\alpha}) t_{c, \ell}(z - E_{\alpha}), \quad (6.5)$$

and $g_{c, \ell}(k)$ and $t_{c, \ell}(z)$ are given in (5.14) and (5.11) respectively.

Let us denote the factor $K_{\wedge \wedge'}^I(\gamma, \gamma')$ by

$$K_{\wedge \wedge'}^I(\gamma, \gamma') = (-1)^{I_1 + I_2 + I_3 + I} [(2I_{23}' + 1)(2I_{23} + 1)]^{\frac{1}{2}} \times$$

$$\left\{ \begin{matrix} I_2 & I_3 & I_{23} \\ I_3 & I & I_{31} \end{matrix} \right\} \delta_{\wedge \wedge'} \sum_{\ell} Y_{\ell \wedge}(\gamma, 0) Y_{\ell \wedge'}(\gamma', 0). \quad (6.6)$$

The operators $T_1(z)$ and $T_2(z)$ which we have already discussed in Chapter 3, are equal in our case (p-n subsystems). They are not affected by the Coulomb potential. Only the operator $T_3^c(z)$ will be affected

for the Coulomb potential. The integral equation, in its angular momentum, helicities and isospin decomposed form for the Coulomb effect, is obtained from (6.4) and (6.3).

Thus

$$\begin{aligned}
 \langle E', J, M, \lambda', I'_{23} | T^{(\alpha)}(z) | E, J, M, \lambda, I_{23} \rangle &= \frac{m_\beta + m_\gamma}{m_1 m_2 m_3 \sqrt{2m_\alpha E_\alpha}} \times \\
 &\left\{ \delta(E'_\alpha - E_\alpha) K^I_{\wedge' \wedge''}(\gamma, \gamma') t^c_\ell(k_\alpha, k'_\alpha; z - E_\alpha) \right. \\
 &- \int dE'' K^I_{\wedge' \wedge''}(\gamma', \gamma'')(E''_\alpha + E''_\beta + E''_\gamma - z - v_c)^{-1} \times \\
 &\delta(E''_\alpha - E'_\alpha) t^c_\ell(k'_\alpha, k''_\alpha; z - E''_\alpha) \langle E'', J, M, \lambda'', I''_{23} | \\
 &\left. (T^{(\beta)}(z) + T^{(\gamma)}(z)) | E', J, M, \lambda', I'_{23} \rangle \right\}. \quad (6.7)
 \end{aligned}$$

We can write the kernel in (3.29) including Coulomb potential

$$\begin{aligned}
 K_3(E', E'') &= K^I_{\wedge' \wedge''}(\gamma', \gamma'')(E''_1 + E''_2 + E''_3 - z - v_c)^{-1} \times \\
 &\delta(E''_3 - E'_3) t^c_\ell(k'_3, k''_3; z - E_3). \quad (6.8)
 \end{aligned}$$

Again this kernel can be written in analogous to (3.54)

$$K_3(E', E'') = \delta(E'_3 - E''_3) F_3(E'') A_3(E'), \quad (6.9)$$

where

$$F_3(E'') = (\sum E'' - z - v_c)^{-1} Y_{\ell \wedge 3}''(\gamma_3'', 0) g_{c, \ell}(k_3'') , \quad (6.10)$$

and

$$A_3(E') = (-1)^{I_1 + I_2 + I_3 + I} [(2I_{12}' + 1)(2I_{12} + 1)]^{\frac{1}{2}} \times$$

$$\left\{ \begin{matrix} I_1 & I_2 & I_{12} \\ I_3 & I & I_{23} \end{matrix} \right\} \delta_{\ell \wedge 3} \wedge_3' \frac{m_1 + m_2}{m_1 m_2 m_3 \sqrt{2m_3 E_3}} \times$$

$$Y_{\ell \wedge 3}''(\gamma', 0) g_{c, \ell}(k_3') t_{c, \ell}(z - E_3') . \quad (6.11)$$

Then Eq. (6.7) becomes

$$T^{(3)}(E', E) = T_3^c(E', E) - A_3(E') \int F_3(E'') \delta(E_3' - E_3'') \times$$

$$[T^{(1)}(E'', E) + T^{(2)}(E'', E)] dE'' . \quad (6.12)$$

According to ^{3.57}~~(3.34)~~, Eq. (6.12) can be written as

$$X_3(E_3', E) = \int F_3(E'') \delta(E_3' - E_3'') [T_1(E'', E) + T_2(E'', E)] \times$$

$$dE'' = \int \sum_{\beta} K_{3\beta}(E_3', E_3'') X_{\beta}(E_3'', E) dE_3'' , \quad (6.13)$$

$$(\beta = 1, 2)$$

where

$$K_{3\beta}(E'_3, E''_3) = (1 - \delta_{3\beta}) \int F_3(E) A_\beta(E) \delta(E'_3 - E_3) \delta(E''_3 - E_\beta) dE. \quad (6.14)$$

Let us denote $B_3(E', E)$ by

$$B_3(E'_3, E) = \int F_3(E'') \delta(E'_3 - E''_3) [T_1(E'', E) + T_2(E'', E)] dE''. \quad (6.15)$$

Then Eq. (6.13) becomes

$$X_3(E'_3, E) = B_3(E'_3, E) - \sum_{\beta} K_{3\beta}(E'_3, E'') X_{\beta}(E''_3, E) dE''_3. \quad (6.16)$$

Similarly, $X_{\alpha}(E'_{\alpha}, E)$ ($\alpha = 1, 2$) can be written as

$$X_{\alpha}(E'_{\alpha}, E) = B_{\alpha}(E'_{\alpha}, E) - \sum_{\alpha} K_{\alpha 3}(E'_{\alpha}, E''_{\alpha}) X_3(E''_{\alpha}, E) dE''_{\alpha}. \quad (6.17)$$

Combining (6.16) and (6.17), we get the equation in matrix form

$$\begin{pmatrix} X_1(E'_1, E) \\ X_2(E'_2, E) \\ X_3(E'_3, E) \end{pmatrix} = \begin{pmatrix} B_1(E'_1, E) \\ B_2(E'_2, E) \\ B_3(E'_3, E) \end{pmatrix} - \begin{pmatrix} 0 & K_{12}(E'_1, E''_2) & K_{13}(E'_1, E''_3) \\ K_{21}(E'_2, E''_1) & 0 & K_{23}(E'_2, E''_3) \\ K_{31}(E'_3, E''_1) & K_{32}(E'_3, E''_2) & 0 \end{pmatrix} \times \begin{pmatrix} X_1(E''_1, E) \\ X_2(E''_2, E) \\ X_3(E''_3, E) \end{pmatrix} \quad (6.18)$$

When particles 1 and 2 are the same, then equation (6.18) becomes

$$\begin{pmatrix} X_1(E'_1, E) \\ X_1(E'_1, E) \\ X_3(E'_1, E) \end{pmatrix} = \begin{pmatrix} B_1(E'_1, E) \\ B_1(E'_1, E) \\ B_3(E'_1, E) \end{pmatrix} - \begin{pmatrix} 0 & 0 & K_{13}(E'_1, E''_1) \\ 0 & 0 & K_{13}(E'_1, E''_1) \\ K_{31}(E'_1, E''_1) & K_{31}(E'_1, E''_1) & 0 \end{pmatrix} \times \begin{pmatrix} X_1(E''_1, E) \\ X_1(E''_1, E) \\ X_3(E''_1, E) \end{pmatrix} \quad (6.19)$$

Eqs. (6.19) are after one iteration, integral equations for $X_1(E'_1, E)$ and $X_3(E'_1, E)$, thus

$$X_1(E'_1, E) = B_1(E'_1, E) - K_{13}(E'_1, E)B_3(E'_1, E) + \int_0^\infty 2K_{13}(E'_1, E''_1) \times K_{31}(E'_1, E''_1)X_1(E''_1, E)dE''_1, \quad (6.20)$$

and

$$X_3(E'_1, E) = B_3(E'_1, E) - 2K_{31}(E'_1, E)B_1(E'_1, E) + \int_0^\infty 2K_{31}(E'_1, E''_1) \times K_{13}(E'_1, E''_1)X_3(E''_1, E)dE''_1. \quad (6.21)$$

Combining Eqs. (6.20) and (6.21), we can write the p-d scattering amplitude matrix equation as

$$\begin{aligned}
 X_{\alpha}(E'_{\alpha}, E) = & \sum_{\alpha} B_{\alpha}(E'_{\alpha}, E) - \sum_{\alpha \neq \beta} K_{\alpha\beta}(E'_{\alpha}, E) B_{\beta}(E'_{\beta}, E) + \\
 & 2 \sum_{\alpha \neq \beta} \int_0^{\infty} K_{\alpha\beta}(E'_{\alpha}, E''_{\alpha}) K_{\beta\alpha}(E'_{\beta}, E''_{\beta}) X_{\alpha}(E''_{\alpha}, E) dE''_{\alpha} , \\
 & (\alpha, \beta = 1, 1, 3) , \quad (6.22)
 \end{aligned}$$

when $K_{\alpha\beta} = 0$, if $\alpha = \beta$.

Eq. (6.22) forms the basis of many of our predictions for proton-deuteron scattering. At this stage, it is convenient to make a number of results.

The bound state wave function equation can be written from equation (3.20) as

$$T^{(\alpha)}(z) = - G_0(z) T_{\alpha}(z) \sum_{\beta \neq \alpha} T^{(\beta)}(z) , \quad (\alpha, \beta = 1, 2, 3) .$$

Then the He^3 wave function becomes (6.23)

$$\begin{aligned}
 Y_{\alpha}(E'_{\alpha}, E) = & 2 \sum_{\alpha \neq \beta} \int_0^{\infty} K_{\alpha\beta}(E'_{\alpha}, E''_{\alpha}) K_{\beta\alpha}(E'_{\beta}, E''_{\beta}) Y_{\alpha}(E''_{\alpha}, E) dE''_{\alpha} . \\
 & (6.24)
 \end{aligned}$$

Thus we have a set of homogeneous integral equations that has a solution only at an energy equal to the binding energy of the three-nucleon system.

For the treatment of isospin, we choose to work in a basis in which the total isospin I is diagonal. Since the Coulomb interaction violates I -conservation not I_z , the bound state scattering operators will be diagonal in I_z , but will have nonvanishing matrix elements between the states of different I . We can write the equations in a basis in the Hilbert space in which the $SU(2)$ symmetry breaking in the three-nucleon system is more transparent. When $I_z = -\frac{1}{2}$, they reduce to the well-known equation for n - d scattering [9], but for $I_z = \frac{1}{2}$, they show implicitly the Coulomb effects.

The eigenstates of I available for the three-nucleon system are $I = \frac{1}{2}$ and $I = \frac{3}{2}$. If the pairs of nucleons are in a deuteron-state, the total isospin can only be $I = \frac{1}{2}$. However, if the pairs of nucleons are in a nucleon-state, the total isospin can take both values $\frac{1}{2}$ or $\frac{3}{2}$. We shall discuss the results for the S -state in the next section.

CHAPTER 7

THE OBSERVABLES

Here we report on the results of our numerical calculations of the three-body bound state described in the last chapter. In the following we describe Faddeev's T-matrix for p-d (ppn) elastic collisions and the binding energy of He^3 as well as the Coulomb energy Δ_c of He^3 . We also compute the phase-shifts from the calculated T-matrix elements for the $\ell = 0$ partial wave.

Firstly we calculated the strength of the coupling constant λ . The value of the coupling constant is conveniently determined from the equation [32]

$$\lambda^{-1} = \pi^2 \beta^{-3} - 2\pi^2 a_s^{-1} \beta^{-4} \quad (7.1)$$

which relates it to the singlet scattering length a_s . Table 1* contains the strengths of the inverse of the coupling constants for n-p, p-p and n-n collisions with corresponding input parameters.

* See on page 86.

We also determined the strength of the Coulomb potential $v_c(r) = e^2 r^{-1}$. Taking the Fourier sine transform into momentum space viz. $v_c(p) = e^2 \pi/2$ we obtained for $v_c(p)$ a value of 0.011466 (using $e^2 = \frac{1}{137}$). Throughout our calculation we assumed the equal masses for n and p, thus $m = m_p = m_n = 4.76 \text{ F}^{-1}$ and reduced mass $\mu = \frac{1}{2} \times 4.76 \text{ F}^{-1}$.

We replaced the momentum variables $\underline{k}$ by the energy variables ω using the relations

$$|\underline{k}_1| = \frac{1}{2} \sqrt{\frac{m\omega_1}{2}} \quad \text{and} \quad |\underline{k}_3| = \sqrt{2m\omega_1}$$

respectively. Then $t_o(\underline{k}_1, \underline{k}'_1; z)$ in (4.5) and $t_o^c(\underline{k}_3, \underline{k}'_3; z)$ in (6.5) became

$$t_o(\underline{k}_1, \underline{k}'_1; z) \equiv t_o(\omega_1, \omega'_1; z) = \left(\beta^2 + \frac{m\omega_1}{2}\right)^{-1} \left(\beta^2 + \frac{m\omega'_1}{2}\right)^{-1} \times$$

$$\left[\lambda_{np}^{-1} - (2\pi^2)^{-1} \int_0^\infty \left(\frac{m^3 \omega'_1}{32}\right)^{\frac{1}{2}} \left(\beta^2 + \frac{m\omega'_1}{2}\right)^{-2} \left(z - \frac{m\omega'_1}{2}\right)^{-1} d\omega'_1 \right], \quad (7.2)$$

and

$$t_o^c(\underline{k}_3, \underline{k}'_3; z) \equiv t_o^c(\omega_1, \omega'_1; z) = (\beta^2 + 2m\omega_1)^{-1} (\beta^2 + 2m\omega'_1)^{-1} \times$$

$$(4\pi^2 \mu^2 e^4 / (4m^2 \omega_1 \omega'_1))^{\frac{1}{2}} (\exp(2\pi\mu e^2 / (2m\omega_1)^{\frac{1}{2}}) - 1)^{-\frac{1}{2}} \times$$

$$\begin{aligned}
& (\exp(2\pi\mu e^2/(2m\omega_1')^{\frac{1}{2}}) - 1)^{-\frac{1}{2}} \exp[(2\mu e^2/(2m\omega_1')^{\frac{1}{2}}) \cdot \chi \\
& \tan^{-1}((2m\omega_1')^{\frac{1}{2}} \beta^{-1})] \exp[(2\mu e^2/(2m\omega_1')^{\frac{1}{2}}) \cdot \chi \\
& \tan^{-1}((2m\omega_1')^{\frac{1}{2}} \beta^{-1})) [\lambda_{pp}^{-1} - (2\pi^2)^{-1} \int_0^{\infty} (2m^3 \omega_1')^{\frac{1}{2}} \cdot \chi \\
& [2\pi\mu e^2/(2m\omega_1')^{\frac{1}{2}}] \exp[(4\mu e^2/(2m\omega_1')^{\frac{1}{2}}) \tan^{-1}((2m\omega_1')^{\frac{1}{2}} \beta^{-1})] \times \\
& (\beta^2 + 2m\omega_1')^{-2} (z - 2m\omega_1')^{-1} [\exp(2\pi\mu e^2/(2m\omega_1')^{\frac{1}{2}}) - 1]^{-1} \times \\
& d\omega_1']^{-1} . \quad (7.3)
\end{aligned}$$

For solving integral equations (6.20) and (6.21) we used the quadrature rules described by Osborn [39] slightly modified. We transformed the integral equation to matrix form by approximating the integrations by finite sums. The infinite limits of the integrals were easily changed into finite ones by a conformal mapping. Gaussian quadrature was used to convert the integral into a sum. Our quadrature rules evaluated exactly all the rapidly varying terms in the integrals and allowed us to get accurate results with a small number of mesh points.

All calculations were carried out on an IBM 7094 II computer at the Computer Centre of Imperial College. We used five point Gaussian quadrature [33]. We also used the energy units of inverse Fermis instead of Mev. The relation between them was $139.4 \text{ Mev} = 1.415 \text{ F}^{-1}$. Faddeev's T-matrices were estimated at different (total kinetic) energy levels viz. 5 Mev, 8 Mev, 10 Mev and 15 Mev for p-d elastic scattering. The phase-shifts were determined from these matrices.

Lovelace [9] has shown that the T-matrix satisfies the principal value form of the Lippmann-Schwinger equation when $i\epsilon$ is dropped i.e. $\text{Im}(z) = 0$ and the integral becomes a principal valued integral. Let $A_{\ell}(p,q;k^2)$ be the amplitude which is the solution of the T-matrix equation. Lovelace derived the relation between A_{ℓ} and t_{ℓ} thus:

$$t_{\ell}(p,q; k^2+i\epsilon) = A_{\ell}(p,q; k^2) -$$

$$\frac{2\pi^2 ik A_{\ell}(p,k;k^2) A_{\ell}(k,q;k^2)}{1 + 2\pi^2 ik A_{\ell}(k,k;k^2)} . \quad (7.4)$$

Since A_ℓ is real and symmetric in p and q , Eq. (7.4) satisfies the off-shell unitarity relation. The on-shell T-matrix is related to the phase-shifts by

$$A_\ell(k, k; k^2) = - \frac{\tan \delta_\ell}{2\pi^2 k} . \quad (7.5)$$

Using this relation we estimated the phase-shifts for the S-state. Our results and those of other authors are shown in table 2*. BHM's theory [26] included $\ell = 0, 1, 2$ whereas ours only $\ell = 0$. This probably explains the small energy dependence of our $\ell = 0$ phase-shifts and the discrepancy at higher energies. CG determined their phase-shifts $\ell = 0$ from the experimental differential cross-sections and the phase-shifts for $\ell \geq 1$ calculated in Born approximation.

For the binding energy of He^3 , we determined the largest eigenvalue of the kernel of the homogeneous equations (6.24) for a definite energy. When the largest eigenvalue was equal to 1, then this energy was called the binding energy of He^3 . Weinberg [34]

* See page 87.

discussed the properties of the eigenvalues and eigenvectors of the kernel of the Lippmann-Schwinger equations for real potentials. Phillips [35] used the matrix equation of the form

$$K_{ij}(E_b) \mathcal{Y}_j(E_b) = k \mathcal{Y}_j(E_b) . \quad (7.6)$$

He employed an iterative method for finding the largest eigenvalue $k_{\max}$ of $K_{ij}(E_b)$. If E_b was the He^3 binding energy, $k_{\max} = 1$. We observed the energy E_b by searching for the zeros of the Fredholm determined of Eq. (6.24). The difference between the binding energies of H^3 and He^3 yielded the Coulomb energy Δ_c of He^3 .

Our model estimated the He^3 binding energy which was in good agreement with the experimental value. In table 3*, the binding energy and the Coulomb energy Δ_c of He^3 are shown together with those of Adya [20] and the experimental values. Adya took an approximation of the form factor $g_c(k) = c(x) \exp[x \tan^{-1}(k/\beta)]$ and calculated the Coulomb energy Δ_c of He^3 from his

* See page 88.

estimated triton binding energy of 8.3 Mev. In our case, no approximation to the form factor $g_c(k) = \frac{1}{\beta^2 + k^2} c(x) \times \exp[x \tan^{-1}(k/\beta)]$ was made and the Coulomb energy Δ_c of He^3 was determined from the experimental value of the triton binding energy of 8.49 Mev.

Many authors [37] have suggested that the Coulomb energy Δ_c of He^3 may be considerably smaller than the binding energy difference between He^3 and H^3 . Most recently Okamoto and Lucas [38] have shown that the effect of a soft repulsive core of the nuclear forces in the H^3 wave function indicates that this discrepancy may be of the order 18%. If this discrepancy exists, it implies a charge asymmetry of the nuclear forces, in particular, that the n-n interaction is stronger than the p-p interaction [38].

Most of the previous estimates of the Coulomb energy Δ_c of He^3 were obtained from the difference between the calculated binding energy of H^3 and the experimental value of the binding energy of He^3 . Our estimates of the binding energy of H^3 and the Coulomb energy of He^3

are shown in table 4* together with the results of other authors which were all obtained from the three-nucleon problems.

The Coulomb energy Δ_c of He^3 determined from the difference between our calculated binding energies of He^3 and H^3 is 1.96 Mev.

* See page 89.

TABLE 1

$$\beta = 1.36 \text{ F}^{-1}$$

Collisions	scattering lengths $a_s(\text{F})$	values of $\lambda^{-1}(\text{F}^3)$
n-p	- 23.7	4.167037
p-p	- 7.61	4.681787
n-n	- 20.0	4.212076

TABLE 2

Energy (Mev)	Phase-shifts (radian)		
	Ours	BHM[26]	CG*
5.18			- 1.38
4.20		- 1.15	
5.00	- 1.17		
9.66			- 1.77
7.40		- 1.52	
8.00	- 1.24		
10.00	- 1.31		
11.50		- 1.82	
15.00	- 1.35		

* R. S. Christian and J. L. Gammel, *Phy.Rev.*91,100(1953).

TABLE 3

Results	Binding Energy of He^3 (Mev)	Coulomb Energy Δ_c of He^3 (Mev)
Ours	8.03	0.46
Adya	7.0	1.3
Expt	7.718	0.764

TABLE 4

Authors	Potentials*	Binding energy of H^3 (Mev)	Coulomb Energy Δ_c of He^3 (Mev)
Ours	C_y	9.99	1.5
SK[40]	C_y	12.05	4.332
SK[41]	$C_y + T_y$	9.8	2.092
AAY[42]	$Z = 0$	11.01	3.292
	$Z \neq 0$	8.53	0.812
BM[43]	$C_y + T_y$	10.4	2.682
Phillips [35]	$V_4 = 0$	11.1	3.382
	$V_4 \neq 0$	9.1	1.382
MSB[32]	$C_y + T_y$	8.85	1.132
PSK[44]	$C_y + T_y$	8.59	0.872
Expt[36]		8.49	0.764

* C_y and T_y are the abbreviations of the central and tensor part of Yamaguchi potential.

CHAPTER 8CONCLUSION

Although a great deal of attention has been paid to the triton using exact three-body techniques surprisingly little work on He^3 has been done. Starting with the Faddeev coupled integral equations for a three-body system in momentum space, all the particles have a short-range force and two of them have a Coulomb interaction. The effect of the Coulomb potential is completely accounted for by the use of the Coulomb modified functions g_c , rather than the original separable potential functions g . These functions g_c are useful in finding the correct separable potential parameters for the scattering of two charged particles, especially in those cases where there are no neutral counterparts. With this information the off-shell two-particle T-matrix needed as input in the Faddeev equations of the three-body problem has been calculated.

Our approach is to approximate the two-nucleon interactions and to treat the three-nucleon interactions

exactly. The advantages of our technique are that we are approximating the best known part of the problem and having chosen the two nucleon scattering amplitudes we investigated the following three-nucleon problems:

- (a) the S-phase shifts for p-d scattering;
- (b) the binding energies of He^3 and H^3 and the structure of the three-nucleon bound state wave functions.

Our results are in agreement with those of others and the experimental data. It shows that our basic assumptions of the approximation scheme including angular momentum, helicities and isospin are justified. Finally, we conclude that the similar ideas are applicable to solve more than three-nucleon problems.

APPENDIX I
NUMERICAL METHOD

This section describes the techniques which allow us to approximate an integral equation by a finite linear system of algebraic equations which may be easily solved by a computer. The integrals are approximated by some finite sum rules. The general form of these rules follows. Let the set,

$$Y(N) = \left\{ y_i : y_i \in (a,b), i \leq N \right\}$$

be a discrete set of N points on the interval (a,b) of integration. We may write down the general form of a quadrature rule while considering the integrand as a product of two factors $w(x)$, $h(x)$. All the singularities and rapidly varying portions of the integrand are contained in $w(x)$, whereas $h(x)$ is a smooth, singularity-free function with a rapidly converging Taylor series. Then

$$\int_a^b w(x)h(x)dx = \sum_{i=1}^N h(x_i)w_i + \epsilon_N(h), \quad (I.1)$$

where ϵ_N is the error associated with this quadrature rule and the factors w_i and $w(x)$ are usually called weights and weight functions respectively. The weights $\{w_i\}$ are a set of N numbers and can be determined when ϵ_N is small. So the weights can be calculated when $\epsilon_N(x^k) = 0$ for $k = 0, 1, \dots, N-1$, i.e. we calculate

$$I_k = \int_a^b x^k w(x) dx$$

either by some numerical scheme for each integer value of k or analytically and then solve the linear $N \times N$ matrix problem given by

$$I_k = x_1^k w_1 + x_2^k w_2 + \dots + x_N^k w_N, \quad k = 0, 1, \dots, N-1, \quad (I.2)$$

This can be written in the matrix form

$$\begin{pmatrix} I_0 \\ I_1 \\ \vdots \\ I_{N-1} \end{pmatrix} = \begin{pmatrix} x_1^0 & x_2^0 & \dots & x_N^0 \\ x_1^1 & x_2^1 & \dots & x_N^1 \\ \vdots & \vdots & & \vdots \\ x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}$$

It is, therefore, possible to calculate the weights w_i when I_k and x^k are known. We expand $h(x)$ in a Taylor series. Thus

$$h(x) = \sum_{n=0}^{N-1} a_n x^n + R_N(x) , \quad (I.3)$$

where $R_N(x)$ is the remainder term. By definition, the error term of the weight is

$$\xi_N(h) = \int_a^b R_N(x) w(x) dx . \quad (I.4)$$

The first N terms in the power series for $h(x)$ are integrated exactly by the defined quadrature rule in (I.1). The quadrature rule is acceptable when

$\xi_N(k) < \delta$ where δ is some small positive number. We just increase N , for a given δ , until the above inequality is satisfied.

We consider the Fredholm integral equation

$$u(x) = v(x) + \int_a^b K(x,y) u(y) dy , \quad a \leq x \leq b , \quad (I.5)$$

where the kernel $K(x,y)$ and $v(x)$ are known functions. We assume that a unique solution, $u(x)$ exists. The existence of a solution can usually be examined either by Fredholm theory or by showing that the kernel K represents a compact operator. The first step in transforming Eq. (I.5) into a finite matrix form is to replace the integral by a sum. Thus we have

$$u(x) = v(x) + \sum_{i=1}^N K(x,y_i)u(y_i)w_i + \xi_N(K(x,y_i)u(y_i)) . \quad (I.6)$$

This equation is valid for all $x \in (a,b)$. If we consider the x given by the N point subset $Y(N)$ of (a,b) , then Eq. (I.6) becomes N equations for N unknown $u(y_i)$. $\xi_N(K(x,y_i)u(y_i))$ may be dropped from the right hand side of Eq. (I.6), for the accuracy of the quadrature rule when $\xi_N(K(x,y_i)u(y_i)) < \delta$ where δ is sufficiently small. The resulting $N \times N$ matrix equation then becomes

$$\bar{u}(x_i) = v(x_i) + \sum_{j=1}^N K(x_i,y_j)\bar{u}(y_j)w_j . \quad (I.7)$$

Eq. (I.7) is completely different from Eq. (I.6) because of neglecting $\xi_N(K(x, y_i)u(y_i))$. The equation is also satisfied by $u(x_i)$

$$u(x_i) = v(x_i) + \sum_{j=1}^N K(x_i, y_j)u(y_j)w_j + \xi_N(K(x_i, y_j)u(y_j)) . \quad (I.8)$$

Subtracting Eq. (I.7) from Eq. (I.8) and defining the error in $\bar{u}$ as

$$E(x_i) = u(x_i) - \bar{u}(x_i) ,$$

then $E(x_i)$ satisfies the integral equation

$$E(x_i) = \xi_N(K(x_i, y)u(y)) + \sum_{j=1}^N K(x_i, y_j)E(y_j)w_j . \quad (I.9)$$

Thus the error is determined by

$$E(x_i) = \sum_j (1 - K_{ij}w_j)^{-1} \xi_N(K(x_j, y)u(y)) , \quad (I.10)$$

where $(1 - K_{ij}w_j)^{-1}$ is the inverse of the matrix

$\delta_{ij} - K(x_i, y_j)w_j$. Hence, if $|\xi_N(K(x_i, y)u(y))| \ll u(x_i)$,

we can expect the error in replacing $u(x_i)$ by $\bar{u}(x_i)$

to be small. If the $u(x_i)$ have been determined once,

it is easy to find u for all x by using Eq. (I.6) and neglecting the error term. Caution is necessary when neglecting $\mathcal{E}_N(K(x,y)u(y))$. This technique of interpolating u is generally much more accurate than linear or Lagrangian methods. So when the integral equation is used to interpolate $u(x)$ the error may be considerably larger for a general point x than that for the points in $Y(N)$. In spite of this, the integral equation is its own best interpolator.

The quadrature rules like (I.1) are valid for infinite interval $[0, \infty)$. We obtain such rules by conformally mapping $[0, \infty)$ into a finite interval $[-1, 1)$ and then using Gaussian formulas for the finite interval.

Let us integrate $h(x)$ on $[0, \infty)$ according to the quadrature rule,

$$\int_0^{\infty} h(x) dx = \sum_{i=1}^N h(x_i) w_i . \quad (I.11)$$

The problem is to determine w_i and x_i so that the approximation is the best we can obtain for N points.

Let us consider the map $x \rightarrow y \in [-1, 1)$ given by

$$y = \frac{x-1}{x+1} , \quad x = \frac{1+y}{1-y} . \quad (\text{I.12})$$

The Jacobian is

$$\frac{dx}{dy} = \frac{2}{(1-y)^2} = \frac{(1+x)^2}{2} \geq 0 . \quad (\text{I.13})$$

Thus the integral becomes

$$\int_0^\infty h(x) dx = \int_{-1}^1 2h(x(y))(1-y)^{-2} dy . \quad (\text{I.14})$$

Now the integral on the right is easily solvable by Gauss' method. This method is constructed so that

$$\int_{-1}^{+1} u(y) dy = \sum_{i=1}^N u(y_i) \bar{w}_i \quad (\text{I.15})$$

is exact for u in the class of polynomials of order $2N-1$ or less. Thus if the $2N$ components $\left\{ y_1, y_2, \dots, y_N , \bar{w}_1, \bar{w}_2, \dots, \bar{w}_N \right\}$ are chosen so that for $u \equiv (1, y^1, y^2, \dots, y^{2N-1})$ then Eq. (I.14) is exact. The values of abscissa y_i and weights $\bar{w}_i$ have been tabulated [33] for many values of N (≤ 100) . By using the Gaussian abscissa and weights appropriate to Eq. (I.11), we obtain an

integration rule for $h(x)$

$$\int_0^{\infty} h(x) dx = \sum_{i=1}^N h(x_i) \frac{(1+x_i)^2}{2} \bar{w}_i, \quad (I.16)$$

where the x_i are determined by Eq. (I.12) from y_i .

The weights for the quadrature rule are obviously,

$$w_i = \frac{1}{2} (1+x_i)^2 \bar{w}_i. \quad (I.17)$$

So Eq. (I.16) becomes

$$\int_0^{\infty} h(x) dx = \sum_{i=1}^N h(x_i) w_i. \quad (I.18)$$

We illustrate our technique by the known integral

$$I(x) = \int_0^{\infty} e^{-x} dx.$$

We obtained the values with the percentage errors 0.26% and 0.008% for $N = 5, 10$ respectively.

For checking our computer programmes, we took the integral equation

$$f(x) = 1 + \int_0^{\infty} e^{-x-y} f(y) dy , \quad (I.19)$$

which can be solved analytically. Let

$$f(x) = 1 + Ae^{-x} \quad (I.20)$$

be the solution of Eq. (I.19). Substituting (I.20) in (I.19), we get

$$A = 2$$

so the analytic solution of Eq. (I.19) is

$$f(x) = 1 + 2e^{-x}$$

which depends on x .

The values of $f(x)$ determined analytically and by the computer are given in table 5*. Table 5 verifies the correctness of our computer programmes.

* See page 101.

TABLE 5

Values of x	Values of f(x)	
	Analytic	Computer
0.05	2.90	2.9083836
0.30	2.48	2.4851000
1.00	1.74	1.7374740
3.30	1.07	1.0715090
20.00	1.00	1.0000000

APPENDIX II

Here we have calculated the value of

$$(-1)^{i_1+i_2+i_3+I} [(2i_{12}+1)(2i_{23}+1)]^{\frac{1}{2}} \begin{Bmatrix} i_1 & i_2 & i_{12} \\ i_3 & I & i_{23} \end{Bmatrix}$$

occurring in Eq. (6.11). In our problem, we have

$i_1 = i_2 = i_3 = \frac{1}{2}$ while i_{12} and i_{23} are 1. The total isospin I be $\frac{3}{2}$. We have used the general expression for 6-j symbol described by Edmonds' Eqn. (6.3.7) as

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \Delta(a,b,c) \Delta(a, \overset{e}{\cancel{d}}, \overset{f}{\cancel{e}}) \Delta(d,b,f) \\ \times \Delta(d,e, \overset{c}{\cancel{f}}) W \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix},$$

where

$$\Delta(a,b,c) = \left[\frac{(a+b-c)! (a-b+c)! (-a+b+c)!}{(a+b+c+1)!} \right]^{\frac{1}{2}}$$

and

$$W \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \sum_z \frac{(-1)^z (z+1)!}{[(z-a-b-c)! (z-a-e-f)! (z-d-b-\cancel{a})! \times \underset{f}{}]} \times (z-d-e-c)! (a+b+d+e-z)! (b+c+e+f-z)! \times (c+a+f+d-z)!]$$

and where the sum is over all positive values of z such that no factorial in the denominator has negative argument.

Then the value of

$$(-1)^{i_1+i_2+i_3+I} [(2i_{12}+1)(2i_{23}+1)]^{\frac{1}{2}} \begin{Bmatrix} i_1 & i_2 & i_{12} \\ i_3 & I & i_{23} \end{Bmatrix}$$

is 1. This is necessarily so since the transformation matrix is here a 1 by 1 unitary matrix.

APPENDIX III

The calculation of the integral (5.26)

$$\begin{aligned}
 (2ik)^{-1} C_0(x) & \int_0^{\infty} \exp(-\beta_0 r) M_{ix, \frac{1}{2}}(2ikr) dr \\
 & = C_0(x) (2ik)^{-1} (2ik) \Gamma(2) (\beta_0 - ik)^{ix-1} (\beta_0 + ik)^{-ix-1}
 \end{aligned}$$

For simplification, we put

$$\beta_0 = r \cos \theta$$

and

$$k = r \sin \theta .$$

Then

$$r^2 = \beta_0^2 + k^2$$

and

$$\theta = \tan^{-1}(k/\beta_0) .$$

Therefore, the integral becomes

$$\begin{aligned}
 & = C_0(x) r^{ix-1} r^{-ix-1} (\cos \theta - i \sin \theta)^{-ix-1} \\
 & \quad \times (\cos \theta + i \sin \theta)^{-ix-1} \\
 & = C_0(x) r^{-2} \exp(2x\theta) \\
 & = C_0(x) (\beta_0^2 + k^2)^{-1} \exp[2x \tan^{-1}(k/\beta_0)] .
 \end{aligned}$$

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