"THE FORMULATION OF THE STATE VARIABLE EQUATIONS FOR THE ANALYSIS AND SYNTHESIS OF LINEAR ELECTRIC NETWORKS"

by

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ABSTRACT

In this thesis, the application of state space theory to the analysis and synthesis of electrical networks is considered.

The state equations consist of a set of first order linear differential equations of the form

$$\frac{d\underline{x}}{dt} = \underline{A}\underline{x} + \underline{B}\underline{u}$$

where u is the input vector and x the state vector.

The problem of setting up these equations for the case of RLC active networks, containing controlled generators, is examined. A topological approach is taken which gives a method for finding, by inspection, the maximum number of state equations, the maximum order of complexity, for a class of networks.

An algebraic approach is also taken. A method is suggested for forming the equations and a way of determining the order of complexity given. It is shown that, with very small restrictions the problem of finding the maximum order of complexity can be greatly simplified. Necessary and sufficient conditions are given for the network to possess a unique solution.

The relationship between the concepts of system theory and network synthesis are considered. In particular, the relationship between the order of complexity and the degree is considered. The problems associated with poles at infinity are considered and a more general transformation suggested for generating all equivalent irreducible

realizations for systems with poles at infinity.

The use of state space theory for network synthesis is reviewed and the problems of active network synthesis are considered.

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LIST OF PRINCIPAL SYMBOLS

Much of the work of this thesis involves the use of matrix algebra and it is necessary to distinguish between matrices, vectors and scalars. The general rule used here is that matrices are represented by upper case Roman letters, vectors by lower case Roman letters, and scalars by Greek letters. There are a number of exceptions, where this rule cannot be applied because of some well known convention, e.g. s for the Laplace operator. It is hoped that such cases will be obvious. Indices will also violate this rule.

Two other conventions will be used. These are that A^t is the transpose of A and $\{A\}_{ij}$ is the element of A lying in the ith row and the jth column.

The meaning of the remaining principal symbols is given below

Symbol	definition	first used on page
$\underline{A}$	state equation matrix	18
B	loop matrix	26
B_f	submatrix of fundamental loop matrix	28
$\underline{B}$	state equation matrix	18
C	capacitance matrix	30
$\underline{C}$	state equation matrix	108
$\mathcal{C}$	capacitive part of $\mathcal{Y}$	48
D	differential operator	80

e	voltage source vector	31
E_i	matrix block of canonic form	118
$F(\lambda)$	function matrix in λ	114
G	conductance matrix	48
$\mathcal{G}$	conductance part of $\mathcal{Y}$	48
H	hybrid matrix	80
$H_0(D)$	branch equation matrix	82
i	current vector	29
i_m	loop current vector	36
j	current source vector	31
J_i	Jordan block of order i	118
K	arbitrary real (n x n) matrix	149
L	inductance matrix	30
$\mathcal{L}$	inductive part of $\mathcal{Z}$	48
L_{ϵ_i}	matrix block of canonic form	200
L_{η_i}	" " " " "	200
M	matrix relating generators to replacement sources	109
$N(\lambda)$	Smith canonic form	147
$\underline{P}$	state equation matrix	18
Q	cut-set matrix	26
Q_f	submatrix of fundamental cut-set matrix	29
$\underline{Q}$	state equation matrix	18
R	resistance matrix	30

$\underline{R}$	state equation matrix	109
$\mathcal{R}$	resistive part of $\mathcal{Z}$	48
s	Laplace operator	31
S	arbitrary matrix in transformation	114
t	time	18
T	non-singular transformation matrix	138
u	input vector	18
u_1	replacement source vector	108
$\underline{u}_1$	generator vector	108
u_2	input source vector	108
U	unit matrix	28
v	voltage vector	28
v_n	node pair voltage vector	35
x	state vector	18
$\bar{x}$	complement of x	168
y	output vector	18
Y	admittance matrix	37
$\mathcal{Y}$	node admittance matrix	41
Z	impedance matrix	38
$\mathcal{Z}$	loop impedance matrix	41
β	number of branches in network	27
β_L	number of inductors in network	92
$\delta(A)$	degree of A	147

Δ_n	det $\mathcal{Y}$	85
Δ_m	det $\mathcal{Z}$	85
ξ_i	minimal indices	200
η_i	minimal indices	200
λ	general polynomial variable	82
μ	nullity of network	92
ν	number of nodes in a network	27
π	number of separate parts in a network	27
ρ	rank of network	92
$\rho(A)$	rank of A	48
σ	order of complexity	20
τ	number of trees in network	92

LIST OF PRINCIPAL SUBSCRIPTS

subscript	subscripts refer to	first used on page
b	tree branches	38
C	capacitors	30
e	voltage sources	31
g	conductances	50
j	current sources	31
ℓ	links	38
L	inductors	30
r	resistances	30

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Chapter 1

INTRODUCTION

1.1 BASIC PHILOSOPHY

The classical approaches to network theory using impedance and admittance analysis are by now well established.[61,67] Although they have proved adequate for most problems, there are still a number of unsolved problems outstanding for which a new approach may be useful. The approach which is considered here is based upon the methods developed in modern system theory.

The subject of system theory has emerged as a unifying approach to the analysis of various complex systems, including control and sequential switching systems. As the complexity of such systems increased so the need for a deeper mathematical understanding became more apparent. In the search for such mathematical machinery the similarities between these various fields appeared and so the unified approach of system theory was developed. Such a unification has resulted in a sharing of results and a clarification of the basic problems of all systems. It has not replaced research in more specific fields, since each subject has its own individual characteristics which the general theory cannot consider. In this work the application of such ideas to electric network theory and the resulting problems are considered.

An immediate problem in such a general subject is to find a characterization of the system which is sufficiently general to apply

to a large class of systems and yet particular enough to give useful results. The state space description has been found to be such a characterization. This uses the concept of the state of a system and describes the system by a set of first order differential equations in the continuous case or difference equations in the discrete case. Such a set of first order differential equations is the normal form in which most of the theory of differential equations has been written[22] and so it is possible to make use of many well established mathematical theories, such as Liapunov's stability theory.[43] Since the approach is in the time domain it is possible to consider both non-linear and time varying systems as well as the more usual linear time invariant systems.

Here the state-space approach is taken and the basic network problems involved in its application to the case of general linear, lumped, finite, time-invariant networks are considered. Thus the problem is a continuous one and the basic approach will involve differential equations.

1.2 THE STATE OF A SYSTEM AND THE STATE EQUATIONS

A rigorous definition of the state of a system has been given in the literature,[72] but for the purposes of the present work a much simpler, intuitive approach will be sufficient.

Consider a system, Fig. 1.1, with a set of inputs, denoted by the

vector u, and a set of outputs denoted by the vector y. The state variables of the system, denoted by the vector x, are a numerical set of independent variables having the two important properties:-

(a) If they are specified at one instant of time, then, for a given set of inputs u, they are determined for all time

$$\text{i.e.} \quad \frac{dx}{dt} = F_1(x,u) \tag{1.1}$$

(b) If they are specified, then for a given set of inputs, the outputs y are also specified uniquely

$$\text{i.e.} \quad y = F_2(x,u) \tag{1.2}$$

The state of the whole system is characterized at any time by the state vector, x and the behaviour of the system is described by the trajectory of the state vector in the state space. Once the input vector is fixed, a single point on the trajectory serves to determine uniquely the whole of the trajectory in later time. This point may thus be regarded as representing a minimal set of initial conditions.

In many cases, including the case considered here, equations (1.1) and (1.2) may be simplified to the simple linear time-invariant equations

$$\dot{x} = \underline{A}x + \underline{B}u \tag{1.3}$$

$$y = \underline{P}x + \underline{Q}u \tag{1.4}$$

where $\dot{x}$ represents $\frac{dx}{dt}$ and $\underline{A}$, $\underline{B}$, $\underline{P}$ and $\underline{Q}$ are matrices compatible with their corresponding vectors. There is not necessarily any relationship between the dimensions of the vectors x, u and y. Equations (1.3) and

(1.4) are often known as the state equations.

It is a well known fact from the theory of differential equations[22] that a set of linear inhomogeneous equations of the form shown in equation (1.3) always has a solution for a given initial x, provided the input vector u is continuous. Thus the existence of state equations represents a sufficient condition for the existence of a unique output solution, y, for a given continuous input u, and x represents the minimal set of initial conditions which can be arbitrarily specified.

1.3 HISTORICAL BACKGROUND IN NETWORK THEORY

Although many of the ideas used in state space theory have been known for some time to network theorists[32], it was not until 1957 that Bashkow[4] first pointed out the advantages to be obtained by expressing the network equations in the matrix form

$$\dot{x} = \underline{A}x + \underline{B}u$$

where x was some set of network variables and u a set of network inputs. He gave a method for obtaining a minimal set of such equations for the RLC case, by introducing extra reactive elements to simplify the elimination procedure.

In 1959 Bryant[16] presented the first really systematic study of the problem, giving an expression for the number of "natural frequencies"

of an RLC network. By giving a systematic procedure for forming the equations using the properties of the network topology he showed that, with his definition of natural frequency, the number of natural frequencies was the same as the number of equations in Bashkow's procedure. In a later letter[18] he gave an explicit normal form for these equations. Since Bashkow's equations are, in fact, the state equations of the network, the number of natural frequencies is the same as the dimension of the state space which Bryant termed the order of complexity, σ , of the network.

Although Bryant's results were originally obtained for the case of RLC networks, he was later able to extend them to the case of loosely coupled coils.[17] Since then a number of alternative solutions have been given to the problem of determining the state equations for the RLC case.[8,11,13,53]

More general networks containing transformers, gyrators, transistors etc have been considered, using controlled generators as the basic element. Since such an element is active, such general networks have often been termed "active networks". In these cases the term "active" means "containing an active element" and refers to the structure of the network. This differs from the definition used in n-port theory[48] which relates to the power entering the network. In fact such active networks are not active in the n-port sense since they can be used to model transformers and gyrators. "Active" in this topological sense is used as a general term meaning "not specifically RLC" rather than meaning "non passive". Although this

is rather inconsistent, it is by now in common usage in the literature[62] and will be used here.

The work on active networks is less complete. Dervisoglu[24] gave some necessary and sufficient conditions for the formation of the state equations for a very restricted class of networks. He later gave an algorithm[25] for forming a set of state equations from a general set of equations and this was programmed by Pottle.[52] Tow[65] considered the problems in the case when transformers are included. Roe[58], Declaris and Saeks[23] and Rane and Lang[56] all gave methods for forming the equations for active networks but all make assumptions on the topology of the network or its element values. Koenig and Tokad[40,41] have given some results on the interconnection of state models using a graph theory approach.

Thus it seems that the problem of the formulation of the state equations for a general active network has not been fully understood and no simple, general approach is available.

Recently the possibility of using state space theory for the synthesis as well as analysis of networks was suggested by Kalman.[37] Since then a number of papers have been published concerned with this problem in the case of networks with transformers[1,44,65,71] and those without.[26,68-70] In this field many problems are still unsolved.

1.4 AIMS AND LAYOUT OF THE THESIS

The main aim of this thesis is to consider state space theory as a means of studying electric networks. The problems of setting up the state space equations will be considered and a formulation presented which gives some insight into the processes involved. The analytic methods developed will be then used to look at the problem of synthesis and an approach to active network synthesis will be presented. The thesis will therefore be set out in the following manner.

Chapters 2 and 3 will contain mostly introductory work. Chapter 2 is an introduction to the important classical results of network theory used in the later chapters. Chapter 3 gives an introduction to the formation of the state equations for RLC networks, explaining the basic concepts and proving results which will form the basis for the later work. The extension to the case of negative resistors will also be considered here.

Chapters 4 and 5 are concerned with the problems of general active networks. In Chapter 4 the problem is introduced and a topological treatment given of the problem of determining the order of complexity. Chapter 5 is an algebraic treatment of the problem and contains solutions to the problems of forming the state equations, the existence of solutions of active networks and the determination of the order of complexity.

Chapters 6 and 7 are concerned with the use of state space theory for synthesis. Chapter 6 introduces the relevant system theory concepts and considers their application to the case of networks. The

relationship between these and classical network synthesis concepts are also considered. Chapter 7 gives a critical review of the present state space synthesis methods and uses the concepts developed so far for the synthesis of active networks.

Throughout the thesis an understanding of elementary matrix algebra[35] is assumed, but as the properties of polynomial matrices are important to the results of this thesis, a short appendix is included giving the most important results.

1.5 STATEMENT OF ORIGINALITY

Since the commencement of this work a number of papers have appeared considering similar problems to those of this thesis and a number of similar results have been given. Unless otherwise stated, the results and conclusions presented here were obtained independently by the author. In particular, all results stated in the form of theorems are to the author's knowledge original.

The original contributions in this thesis are

(a) The treatment of negative resistors;

(b) The approach to the formation of the state equations for an active network;

(c) The determination of and bounds on the order of complexity of an active network;

(d) The conditions for the existence of a unique solution for an active network;

(e) The treatment of poles at infinity and the generation of equivalent realizations;

(f) The relationship between the order of complexity and the degree;

(g) The approach to the synthesis of active networks.

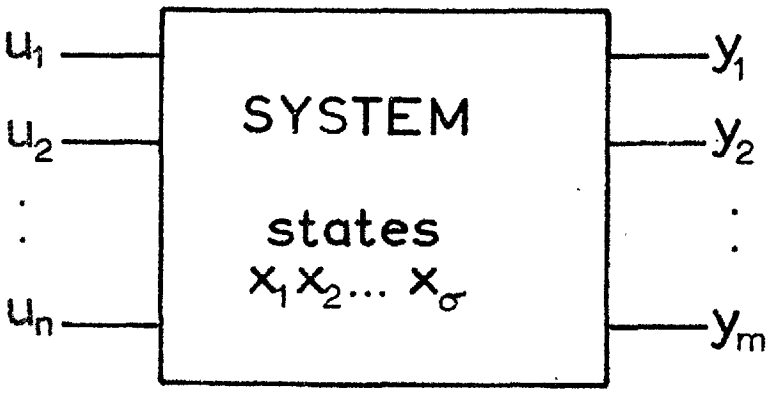


Fig.1.1. General system.

Chapter 2

BASIC NETWORK ANALYSIS

2.1 INTRODUCTION

Throughout this thesis, use will be made of the basic properties of electric networks, their graphs and the conventional methods of analysis. This chapter is intended to introduce the basic definitions and results on which all the later work is based.

2.2 NETWORK DESCRIPTION

2.2.1 Kirchhoff's laws and graph theory[62]

An electric network may be described by three sets of independent equations. These correspond to Kirchhoff's voltage and current laws and to the element relationships, which relate the branch voltages and currents. In order to express Kirchhoff's laws in a useful algebraic form it is necessary to use some matrices commonly used in graph theory. These are the loop and cut-set matrices, B and Q, of a graph.

Each column of a loop matrix corresponds to a branch of the graph and each row corresponds to a loop. A loop is simply a closed path in the graph, along which no node is touched by more than two branches in the loop. The value of each matrix element b_{ij} is

given by:-

b_{ij} = +1 if element j is in loop i and the orientations of the element and the loop agree.

b_{ij} = -1 if element j is in loop i and the orientations of the element and the loop do not agree.

b_{ij} = 0 if element j is not in loop i.

Thus the matrix contains only +1, -1 and 0 as its elements and its dimension depends on the number of loops in the network. In general the rows will not be independent and it can be shown that only $(\beta - \nu + \pi)$ rows are linearly independent where

β = number of branches in the network

ν = number of nodes in the network

π = number of separate parts in the network.

A tree of a connected graph is a subgraph which contains a minimum set of branches and connects all the nodes in the graph. It can be shown that such a subgraph contains no loops and $\nu - 1$ branches. The branches not contained in this tree subgraph are called chords or links.

If the graph is not connected, this concept can be generalised and is termed a forest. This is a disjoint set of trees, one for each separate connected subgraph. In this case a forest contains $(\nu - \pi)$ branches. It is usual to think of a network as being connected and to work in terms of trees rather than forests. The extension of any results to a forest is usually obvious, since each separate part can be treated as a graph in its own right.

In this thesis it will usually be assumed that the network considered is connected.

If a link is added to a given tree then a single loop is formed, consisting of the link and a unique path in the tree. Such a loop is called a fundamental loop of the tree. Since each tree has $\beta - \nu + 1$ links then there are $\beta - \nu + 1$ fundamental loops corresponding to each tree and it can be shown that these define a set of linearly independent rows in the loop matrix. Usually only a set of $\beta - \nu + 1$ independent rows are considered as the loop matrix, and if these are written on the basis of some fundamental loops the matrix takes the form

$$B = \left[U \; B_f \right]$$

where U is a unit matrix of order $\beta - \nu + 1$, whose columns correspond to the links of the tree.

Kirchhoff's law for a network can be expressed in terms of any set of $\beta - \nu + 1$ linearly independent rows of the loop matrix simply as

$$Bv = 0 \qquad (2.1)$$

where v is the vector composed of the branch voltages in the network i.e. there are $\beta - \nu + 1$ equations in β unknowns.

A cut-set is a minimal set of branches of a graph such that their removal increases the number of separate parts by one. The cut set matrix is a matrix whose columns correspond to the graph branches, whose rows correspond to the cut-sets and whose elements q_{ij} have the

following properties:-

q_{ij} = +1 if branch j is in cut-set i and the orientations of the branch and cut-set agree.

q_{ij} = -1 if branch j is in cut-set i and the orientations of the branch and the cut-set do not agree.

q_{ij} = 0 if branch j does not lie in cut-set i.

Again the dimension of the cut-set matrix is quite large, but normally only a matrix whose rows are linearly independent is considered. In this case such a matrix has $\nu - \pi$ rows. Again if the graph is assumed to be connected then π becomes 1.

If a tree is chosen then each branch of the tree defines a unique cut-set with some of the links. Such a cut-set is called a fundamental cut-set and for each tree there are $\nu - 1$ such cut-sets. These can be shown to correspond to a set of $\nu - 1$ linearly independent rows of the cut-set matrix, which can be written in the partitioned form

$$Q = \left[Q_f \; U \right]$$

where U is the unit matrix of order $\nu - 1$ and its columns correspond to the tree branches of the graph.

Kirchhoff's current law for the network may be expressed by the equation

$$Qi = 0 \qquad (2.2)$$

where i is a vector composed of the branch currents and Q is the cut-set matrix of $\nu - 1$ linearly independent rows. Hence there are $\nu - 1$

equations in β unknowns.

Thus Kirchhoff's laws give a total of β independent equations in 2β variables. The remaining β equations are supplied by the element equations.

Before these are considered it may be noted that the loop and cut-set matrices can be related by the equation

$$BQ^t = QB^t = 0 \qquad (2.3)$$

where Q^t is the transpose of Q.

In the case of fundamental loops and cut-sets,

$$\begin{bmatrix} U & B_f \end{bmatrix} \begin{bmatrix} Q_f^t \\ U \end{bmatrix} = 0$$

$$\text{i.e.} \quad B_f = -Q_f^t. \qquad (2.4)$$

2.2.2 Passive network elements

In the RLC case the element equations consist of the matrix equations

$$i_C = C \frac{dv_C}{dt}$$

$$v_r = R\, i_r$$

$$v_L = L \frac{di_L}{dt}$$

If Laplace transforms are taken of these equations they become

$$i_C = Cs\, v_C$$

$$v_r = R\, i_r$$

$$v_L = Ls\, i_L$$

where s is the Laplace operator.

The equations for uncontrolled driving sources are simply

$$v_e = e$$

and $$i_j = j$$

Here the sources are regarded as separate branches.

2.2.3 Active network elements

A very useful method of representing an active element in a network is the controlled generator. Such an element occurs in many equivalent circuits for transistors and valves. The element is useful because it gives not only activity, but also non-reciprocity, since it is unilateral.

The constraint introduced into the network by a controlled generator may be any of four different types. These are the four different types of generator shown in Fig. 2.1.

(a) current controlled current generators

(b) voltage controlled current generators

(c) voltage controlled voltage generators

(d) current controlled voltage generators.

Such elements are often referred to as controlled sources but for convenience of notation a distinction will be made here. The

term 'source' will always be taken as implying an uncontrolled source, whilst the term 'generator' will always imply a controlled generator.

It is important to distinguish between the four types of generator, mainly because of the dimension of their element values. Type (b) has a conductance form, type (d) a resistance form and (a) and (c) are dimensionless. Such distinctions are important when doing admittance analysis for example, when it is difficult to include voltage generators.

Here it is assumed that the generators are controlled by either a branch current or voltage. The branch may in general be a passive element, a source, or another generator. This is not an important restriction since in general any variable in the network can be represented as a linear combination of branch variables. In such a case the generator would become a combination of generators, one for each controlling branch variable. A simple illustration of this is shown in Fig. 2.2, where a current generator is controlled by the voltage across two resistors. This can be converted to 2 generators in parallel, each controlled by the voltage across a single resistor.

By using such a device, both currents through short circuits and voltages across open circuits can be considered.

It is also assumed that, as in the case of sources, the generators are regarded as separate network branches and are not necessarily associated with any other branches of the network. This gives more generality, but it means that it is not possible to use the well known "source conversion", illustrated in Fig. 2.3.

Guillemin[33] has shown that any source can always be associated with some appropriate adjacent passive element. For example, a voltage source can always be associated with some series impedance. This converts the two elements, the source and impedance into a single network branch. Since the two elements have been merged into one element, some variable, either the voltage or current in the passive element, has been lost as a branch variable. When only uncontrolled sources are considered this is not important, but with controlled generators in the network it is, since such a branch variable may well be used as a controlling variable of some other generator. Thus a certain amount of care must always be exercised when using this result in a network and it is not possible to assume it can always be used in the active case.

2.2.4 Removal of elements

Removing a branch from a network normally implies the open circuiting of the branch, i.e. removing the branch and leaving the nodes unaffected. A branch can, however, be removed by short circuiting it. i.e. removing the branch and coalescing the terminal nodes. Here in the case of passive elements removal will always mean open circuiting unless otherwise stated.

In the case of sources and generators a different convention will be used. Voltage generators and sources will always be removed by short circuiting and current generators and sources will always be

removed by open circuiting.

Similarly voltage generators and sources will be added to the network by splitting a node into two and inserting the generator or source. Current generators and sources will be added by connecting them between pairs of existing nodes of the network.

2.3 CONVENTIONAL ANALYSIS METHODS FOR PASSIVE NETWORKS[10,62]

2.3.1 Preliminaries

The choice of analysis methods for a network depends to a large extent upon the way in which the elements are characterized. If they are given as admittances then node analysis is used, if as impedances, then loop analysis is used. For most networks, if a Laplace transformation is taken, it is possible to convert from an admittance to an impedance characterization although in the case of active networks this may not always be possible. In fact if the network contains active terms having only impedance characterizations and others having only admittances characterizations it may not be possible to use either. Because of this a third, mixed method is also available[10,34] although it is little used because it is only needed in these special cases.

For the purposes of this thesis only the case of passive networks will be considered here, although all three approaches will be developed.

Seshu and Reed[62] have proved the following important result:

Lemma 2.1 If an RLC network has a unique solution, there exists a tree such that all voltage sources are tree branches and all current sources are links.

The proof of this result will not be given at this point since it can be found in the literature and also will be considered in greater detail when the case of active networks is considered in Chapters 4 and 5 where it will appear as a special case. It is important because it represents a necessary condition which must be used to set up the equations.

A second basic result[62] comes from equation (2.3). All possible voltage solutions of the network are described by the equations

$$Bv = 0 \tag{2.5}$$

where B is a $(\beta - \nu + 1) \times \beta$ matrix and v is a β vector. Since B is of rank $\beta - \nu + 1$ this implies that there are $\beta - (\beta - \nu + 1)$ independent solutions i.e. $\nu - 1$ solutions.

But the fact that

$$BQ^t = 0$$

implies that each column of Q^t or row of Q is such a solution. Since there are $\nu - 1$ independent rows in Q, these form such a set of solutions to equation (2.5) and all other solutions are linear combinations of these.

$$\text{i.e. } v = Q^t v_n \tag{2.6}$$

where v_n is an arbitrary $(\nu - 1)$ vector. Such a vector can be regarded as a set of variables representing the voltages between various nodes

in the network and the equation is called the node pair transformation.

If the independent rows of Q are written on the basis of a tree, then Q can be written as

$$Q = \left[U \; Q_f \right]$$

by suitable ordering of the branches.

$$\text{Thus } v = \begin{bmatrix} U \\ Q_f^t \end{bmatrix} v_n \tag{2.7}$$

and so v_n can be regarded as the voltages across a set of tree branches of the network.

In a dual way

$$Qi = 0$$

$$\text{implies that } i = B^t i_m \tag{2.8}$$

where i_m is a $(\beta - \nu + 1)$ vector and the transformation is called a loop transformation. In terms of trees and cotrees

$$i = \begin{bmatrix} B_f^t \\ U \end{bmatrix} i_m \tag{2.9}$$

so that i_m can be regarded as the set of currents flowing through any set of links in the network.

The starting point of all three methods of analysis is the selection of a tree, based upon Lemma 2.1. In general the network has four classes, voltage sources, elements with admittance representations, elements with impedance representations and current sources and these will be indicated by the subscripts 1,2,3 and 4. The tree is formed by first including all voltage sources, then as many admittances as

possible, then as many impedances as possible. The tree is then complete because all current sources can be taken as link by Lemma 2.1.

2.3.2 Node admittance analysis

If there are no impedance branches then node admittance analysis is appropriate and Kirchhoff's current law can be written as

$$\begin{bmatrix} U_1 & Q_{12} & Q_{14} \\ & Q_{22} & Q_{24} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ j_4 \end{bmatrix} = 0 \qquad (2.10)$$

where j_4 is the vector giving the values of the current sources. Note that because of the choice of the tree the first equations simply give the currents in all the voltage sources in terms of i_2 and j_4.

Now $i_2 = Y_2 v_2$

where Y_2 is the diagonal admittance matrix of all the passive branches.

Thus the second set of equations in equation (2.10) becomes

$$Q_{22}\, Y_2\, v_2 = - Q_{24} j_4$$

Using the node pair transformation, equation (2.6), in this case gives

$$\begin{bmatrix} e_1 \\ v_2 \\ v_4 \end{bmatrix} = \begin{bmatrix} U_1 & \\ Q_{12}^t & Q_{22}^t \\ Q_{14}^t & Q_{24}^t \end{bmatrix} \begin{bmatrix} v_{n1} \\ v_{n2} \end{bmatrix}$$

and it can be seen that $e_1 = v_{n1}$.

Then $Q_{22}\, Y_2\, Q_{22}^t\, v_{n2} + Q_{22}\, Y_2\, Q_{12}^t\, e_1 = - Q_{24} j_4$

or $$Q_{22}\, Y_2\, Q_{22}^t\, v_{n2} = - Q_{24} j_4 - Q_{22}\, Y_2\, Q_{12}^t\, e_1 \qquad (2.11)$$

This is the conventional form for the equations for node admittance analysis. v_{n2} determines all other network variables by the node pair transformation, element equations and Kirchhoff's current law. Thus the solution of equation (2.11) gives a complete solution to the network. v_{n2} is the node pair voltage vector and $Q_{22}\, Y_2\, Q_{22}^t$ is the node admittance matrix. If Q_{22} is partitioned into its tree branch and link parts, indicated by subscripts b and ℓ, then

$$Q_{22} = \left[U_b \; Q_{22\ell} \right]$$

and $$Q_{22}\, Y_2\, Q_{22}^t = Y_b + Q_{22\ell}\, Y_{2\ell}\, Q_{22\ell}^t$$

and v_{n2} becomes the tree branch voltages, v_{2b}.

Thus equation (2.11) becomes

$$\left[Y_{2b} + Q_{22\ell}\, Y_{2\ell}\, Q_{22\ell}^t \right] v_{2b} = - Q_{22\ell}\, Y_{2\ell}\, Q_{12\ell}^t\, e_1 - Q_{24} j_4 \qquad (2.12)$$

2.3.3 Loop impedance analysis

Similarly if the elements all have an impedance form loop analysis can be used. Kirchhoff's voltage law may be written as

$$\begin{bmatrix} B_{31} & B_{33} & \cdot \\ B_{41} & B_{43} & U_4 \end{bmatrix} \begin{bmatrix} e_1 \\ v_2 \\ v_4 \end{bmatrix} = 0$$

and the element equations as

$$v_3 = Z_3\, i_3$$

where Z_3 is the diagonal impedance matrix of the passive branches.

Following a dual procedure to that of the last section the

equations can be reduced to

$$B_{33}\, Z_3\, B_{33}^t\, i_{m3} = - B_{31}\, e_1 - B_{33}\, Z_3\, B_{43}^t\, j_4 \qquad (2.13)$$

where i_{m3} is the loop current vector and $B_{33}\, Z_3\, B_{33}^t$ is the loop impedance matrix. Again i_{m3} determines all other network variables.

If B_{33} is again partitioned into its tree and cotree parts, b and ℓ, then

$$B_{33}\, Z_3\, B^t_{33} = Z_{3\ell} + B_{33b}\, Z_{3b}\, B_{33b}^t$$

and i_{m3} is a set of link currents $i_{3\ell}$.

Then equation (2.13) becomes

$$\left[Z_{3\ell} + B_{33b}\, Z_{3b}\, B_{33b}^t \right] i_{3\ell} = - B_{33}\, Z_{3b}\, B_{43}^t\, j_4 - B_{31}\, e_1 \qquad (2.14)$$

2.3.4 Mixed analysis

The mixed method of analysis is a combination of the last two methods and is more complex. Kirchhoff's current law gives

$$\begin{bmatrix} U_1 & & & Q_{12\ell} & Q_{13\ell} & Q_{14} \\ & U_{2b} & & Q_{22\ell} & Q_{23\ell} & Q_{24} \\ & & U_{3b} & 0 & Q_{33\ell} & Q_{34} \end{bmatrix} \begin{bmatrix} i_1 \\ i_{2b} \\ i_{3b} \\ i_{2\ell} \\ i_{3\ell} \\ j_4 \end{bmatrix} = 0 \qquad (2.15)$$

where subscripts ℓ and b indicate links and tree branches respectively. Note that the extra zero in the third row of the matrix corresponding to $Q_{32\ell} = 0$, arises because of the order of selection of the tree branches.

Similarly Kirchhoff's voltage law gives

$$\begin{bmatrix} B_{21} & B_{22b} & & U_{2\ell} & \\ B_{31} & B_{32b} & B_{33b} & & U_{3\ell} \\ B_{41} & B_{42b} & B_{43b} & & U_4 \end{bmatrix} \begin{bmatrix} e_1 \\ v_{2b} \\ v_{3b} \\ v_{2\ell} \\ v_{3\ell} \\ v_4 \end{bmatrix} = 0 \qquad (2.16)$$

The element equations are

$$\begin{bmatrix} i_{2b} \\ i_{2\ell} \\ v_{3b} \\ v_{3\ell} \end{bmatrix} = \begin{bmatrix} Y_{2b} & & & \\ & Y_{2\ell} & & \\ & & Z_{3b} & \\ & & & Z_{3\ell} \end{bmatrix} \begin{bmatrix} v_{2b} \\ v_{2\ell} \\ i_{3b} \\ i_{3\ell} \end{bmatrix}$$

where now some elements of the diagonal matrix have admittance form and others impedance form.

The first set of equations (2.15) gives the currents through the voltage generators. The second set of equations gives

$$\begin{bmatrix} U_{2b} & Q_{22\ell} \end{bmatrix} \begin{bmatrix} i_{2b} \\ i_{2\ell} \end{bmatrix} + Q_{23\ell}\, i_{3\ell} = - Q_{24} j_4$$

or $$\begin{bmatrix} U_{2b} & Q_{22\ell} \end{bmatrix} \begin{bmatrix} Y_{2b} & \\ & Y_{2\ell} \end{bmatrix} \begin{bmatrix} v_{2b} \\ v_{2\ell} \end{bmatrix} + Q_{23\ell}\, i_{3\ell} = - Q_{24} j_4$$

Now use of the node pair transformation gives

$$\begin{bmatrix} U_{2b} & Q_{22\ell} \end{bmatrix} \begin{bmatrix} Y_{2b} & \\ & Y_{2\ell} \end{bmatrix} \begin{bmatrix} U_{2b} \\ Q_{22\ell}^t \end{bmatrix} v_{2b} + \begin{bmatrix} U_{2b} Q_{22\ell} \end{bmatrix} \begin{bmatrix} Y_{2b} & \\ & Y_{2\ell} \end{bmatrix} \begin{bmatrix} 0 \\ Q_{22\ell}^t \end{bmatrix} e_1 = -Q_{24} j_4 - Q_{23\ell}\, i_{3\ell}$$

$$\text{i.e.} \left[Y_{2b} + Q_{22\ell}\, Y_{2\ell}\, Q_{22\ell}^{t} \right] v_{2b} + Q_{23\ell}\, i_{3\ell} = - Q_{22\ell} Y_{2\ell} Q_{12\ell}^{t}\, e_1 - Q_{24} j_4 \tag{2.17}$$

Note the similarity between equations (2.17) and (2.12).

Using a similar approach, but starting from equations (2.16) it can be shown that

$$\left[Z_{3\ell} + B_{33b} Z_{3b} B_{33_b}^{t} \right] i_{3\ell} + B_{32b} v_{2b} = -B_{33} Z_{3b} B_{43b}^{t} j_4 - B_{31} e_1 \tag{2.18}$$

This equation is similar to equation (2.14).

The mixed equations can now be written together as

$$\begin{bmatrix} (Y_{2b} + Q_{22\ell} Y_{2\ell} Q_{22\ell}^{t}) & Q_{23\ell} \\ B_{32b} & (Z_{3\ell} + B_{33b} Z_{3b} B_{33b}^{t}) \end{bmatrix} \begin{bmatrix} v_{2b} \\ i_{3\ell} \end{bmatrix} = - \begin{bmatrix} Q_{22\ell} Y_{2\ell} Q_{12\ell}^{t} E_1 & Q_{24} \\ B_{31} & B_{33b} Z_{3b} B_{43b}^{t} \end{bmatrix} \begin{bmatrix} e_1 \\ j_4 \end{bmatrix} \tag{2.19}$$

Note that if all the impedance elements are open circuited equation (2.19) reduces to the simple node admittance analysis equation (2.12). Similarly if all admittance elements are short circuited the equations reduce to the loop impedance analysis equations (2.14). The left hand side of equation (2.19) contains the node admittance matrix Y of the admittance subnetwork and the loop impedance matrix Z of the impedance subnetwork and a coupling term $Q_{23\ell}$.

Note that from equation (2.4)

$$B_f = - Q_f^{t}$$

then $B_{32b} = - Q_{23\ell}^{t}$

The set of variables given by vector $\begin{bmatrix} v_{2b} \\ i_{3\ell} \end{bmatrix}$ is a mixed set containing

the node pair voltages of the admittance subnetwork and the loop currents of the impedance subnetwork.

The equations can thus be written in the form

$$\begin{bmatrix} \mathcal{Y} & -B_{32b}^{t} \\ B_{32b} & \mathcal{Z} \end{bmatrix} \begin{bmatrix} v_n \\ i_m \end{bmatrix} = \begin{bmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \end{bmatrix} \qquad (2.20)$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ represent driving terms.

2.3.5 The inclusion of coupling

In the last sections it was assumed that there is no coupling between elements. This assumption is implicit in the assumption that all admittance and impedance matrices are diagonal. In fact, if coupling only takes place between the elements of one class all the results still follow immediately.

In the admittance case, if

$$\begin{bmatrix} i_{2b} \\ i_{2\ell} \end{bmatrix} = \begin{bmatrix} Y_{2bb} & Y_{2b\ell} \\ Y_{2\ell b} & Y_{2\ell\ell} \end{bmatrix} \begin{bmatrix} v_{2b} \\ v_{2\ell} \end{bmatrix}$$

then the admittance matrix becomes

$$\begin{bmatrix} U_{2b} & Q_{22\ell} \end{bmatrix} \begin{bmatrix} Y_{2bb} & Y_{2b\ell} \\ Y_{2\ell b} & Y_{2\ell\ell} \end{bmatrix} \begin{bmatrix} U_{2b} \\ Q_{22\ell}^{t} \end{bmatrix}$$

The same approach can be applied to impedance analysis and to the mixed form of analysis, but this is only if no coupling occurs between impedance and admittance elements. Such cases cannot be treated at

all with admittance or impedance analysis, but can still be analysed using the mixed method. Unfortunately such coupling will affect the symmetry of equation (2.20) since terms will be added to the $Q_{23\ell}$ and B_{32b} terms.

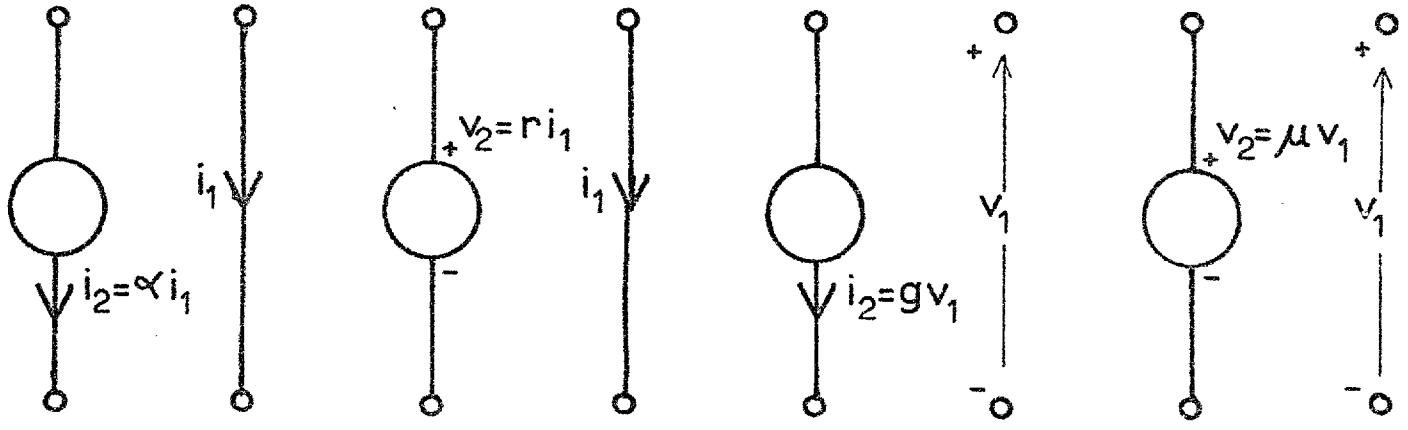


Fig..2.1 The four types of generator.

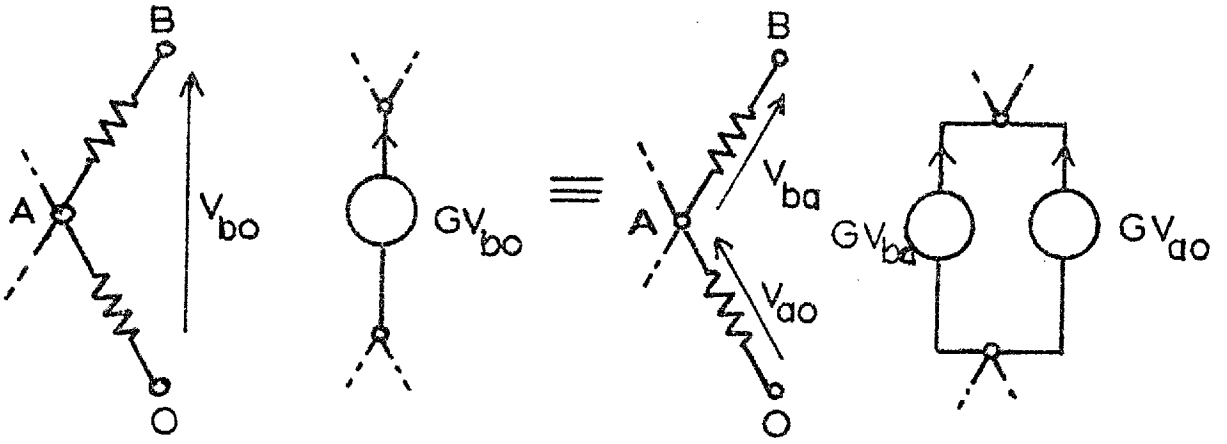


Fig.2.2. Illustration of generator equivalence.

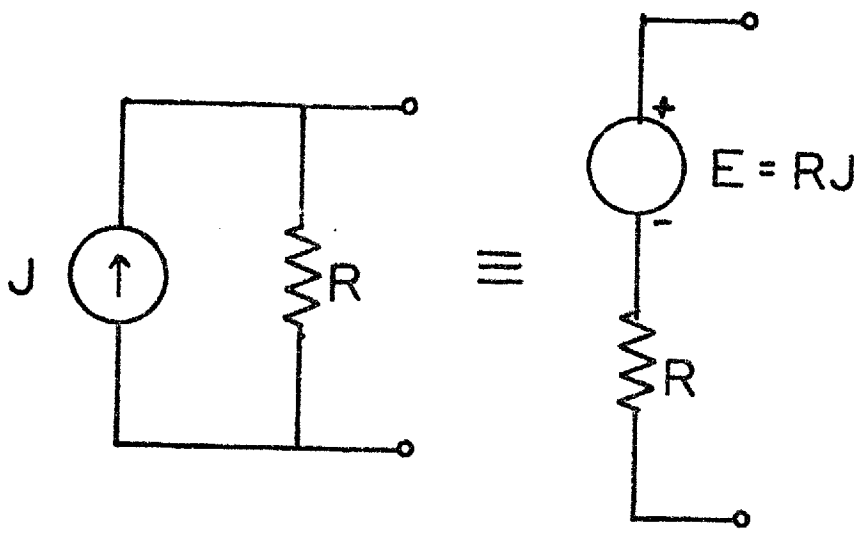


Fig.2.3 The source conversion.

Chapter 3

THE FORMATION OF THE STATE EQUATIONS FOR RLC NETWORKS

3.1 INTRODUCTION

As an introduction to the concepts involved and a foundation for later work, this chapter consists of a thorough presentation of the formation of the state equations for RLC networks. This problem was originally solved by Bryant.

In this chapter the number of state variables in the network and the way in which they can be chosen will be considered. Methods for forming the state equations will be compared and an explicit form for the equations given. The conditions for the existence of a unique solution to the network will be considered. The formation of the equations will also be extended to the case of negative resistors.

A number of alternative approaches[8,11,13,53] to the problem of obtaining the state equations for RLC networks have been presented since Bryant originally solved the problem. The approach taken here is chosen because it yields insight into the processes taking place and gives some interesting interpretations of the results which more direct approaches do not have.

The approach is based upon the mixed method of analysis described in the last Chapter.

3.2 THE FORMATION OF THE EQUATIONS

3.2.1 The setting up of the equations

The basic difference between the formation of the equations in classical terms and state variable terms is that the latter is concerned with first order differential equations in a time domain, while the former is algebraic and works with the Laplace transform or complex frequency variable, s. The problem of working in a time domain is that it is not possible to find inverses for any differential terms, since these would be integrations. This means, for example, that although the element equation for capacitors can be given in an admittance form as

$$i = C \frac{dv}{dt}$$

it cannot be given in an impedance form in the time domain. It is for this reason that the mixed form of analysis is important, since it allows the use of both admittance and impedance equations. Thus capacitors and conductances can be taken in admittance form and resistances and inductors in impedance form. For a linear passive network the distinction between conductances and resistances is purely arbitrary, but as it will be useful later it will be assumed at this point that some partitioning of the resistors has been made.

With such a partitioning, Y of equation (2.20) becomes the node admittance matrix of a capacitor and conductance subnetwork and Z becomes the loop impedance matrix of a resistance and inductor subnetwork.

$\mathcal{Y}$ may be written as

$$\mathcal{Y} = \begin{bmatrix} Q_C & Q_g \end{bmatrix} \begin{bmatrix} C\frac{d}{dt} & \\ & G \end{bmatrix} \begin{bmatrix} Q_C^t \\ Q_g^t \end{bmatrix}$$

$$= Q_C\, C\, Q_C^t\, \frac{d}{dt} + Q_g\, G\, Q_g^t$$

where $\begin{bmatrix} Q_C & Q_g \end{bmatrix}$ is the partitioned cut-set matrix of this subnetwork and C and G are diagonal capacitance and conductance matrices.

This can be written as

$$\mathcal{Y} = \mathcal{C}\frac{d}{dt} + \mathcal{G}$$

and similarly

$$\mathcal{Z} = \mathcal{R} + \mathcal{L}\frac{d}{dt}$$

Equation (2.20) can now be written as

$$\begin{bmatrix} \mathcal{C} & \\ & \mathcal{L} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} v_n \\ i_m \end{bmatrix} = - \begin{bmatrix} \mathcal{G} & Q_{32} \\ B_{23b} & \mathcal{R} \end{bmatrix} \begin{bmatrix} v_n \\ i_m \end{bmatrix} - \begin{bmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \end{bmatrix} \qquad (3.1)$$

The equations are now in first order differential equation form, but as yet the vector $\begin{bmatrix} v_n \\ i_m \end{bmatrix}$ is not the state vector because its components may not be dynamically independent. Since the number of dynamically independent variables cannot exceed the rank of the left hand matrix of equation (3.1),

then the number of dynamically independent variables

$$\leqslant \rho(\mathcal{C}) + \rho(\mathcal{L})$$

where $\rho(\mathcal{C})$ is the rank of matrix $\mathcal{C}$.

But $\mathcal{C} = Q_C \, C \, Q_C^t$

and so $\rho(\mathcal{C}) = \rho(Q_C \, C \, Q_C^t)$.

But it is a well known result in matrix algebra[35] that

$$\rho(AB) \leqslant \rho(A)$$
$$\leqslant \rho(B)$$

and if A is nonsingular, then

$$\rho(AB) = \rho(B)$$

Since C is nonsingular

$$\rho(\mathcal{C}) = \rho(Q_C)$$
$$= \text{the number of independent columns in } Q_C.$$

It is a well known result of graph theory that if a set of columns of a cut-set matrix are independent they must correspond to a set of elements forming a tree of the graph.[62]

Hence $\rho(Q_C)$ is equal to the maximum number of capacitors which can be included in a tree of the capacitor and conductance subgraph.

Thus $\rho(\mathcal{C})$ equals the maximum number of capacitors which can be included in a tree of the capacitor and conductance subgraph.

Similarly, $\rho(\mathcal{L})$ equals the maximum number of inductors which can be included in a cotree of the impedance subgraph.

Thus the number of dynamically independent variables $\leqslant$ the maximum number of capacitors which can be included in a tree plus the maximum number of inductors which can be included in the cotree. This follows from the fact that the admittance and impedance subgraphs are disjoint.

In order to show that this is the actual number of variables

it is necessary to show that the remaining variables can be eliminated. This can be shown by actually doing the elimination and forming the explicit form.

3.2.2 The elimination process

In order to eliminate the non-dynamically independent variables from equation (3.1) it is necessary to look into the form of the equations in more detail. The elimination can proceed if the rank of the left hand side matrix is exhibited by explicitly obtaining rows of zeros in the matrix. The results in the last section on the rank of this matrix indicate how this form can be obtained. If the tree used to write the equations is chosen to maximise the number of inductors in the co-tree, then the required all zero rows are produced.

Thus the tree used to set up the mixed equations should be chosen so that it contains the maximum number of voltage sources, capacitors, conductances, resistances and inductors in that order. For such a tree, Kirchhoff's current law can be written as

$$\begin{bmatrix} U_{be} & Q_{eC} & Q_{eg} & Q_{er} & Q_{eL} & Q_{ej} \\ U_{bc} & Q_{CC} & Q_{Cg} & Q_{Cr} & Q_{CL} & Q_{Cj} \\ U_{bg} & & Q_{gg} & Q_{gr} & Q_{gL} & Q_{gj} \\ U_{br} & & & Q_{rr} & Q_{rL} & Q_{rj} \\ U_{bL} & & & & Q_{LL} & Q_{Lj} \end{bmatrix} \begin{bmatrix} i_{be} \\ i_{bC} \\ i_{bg} \\ i_{br} \\ i_{bL} \\ i_{\ell C} \\ i_{\ell g} \\ i_{\ell r} \\ i_{\ell L} \\ i_{\ell j} \end{bmatrix} = 0 \qquad (3.2)$$

where the subscripts e, C, g, r, L, j, b and ℓ correspond to voltage sources, capacitors, conductances, resistances, inductors, current sources, tree branches and links.

Thus $i_{\ell j}$ = j, the current source vector.

Similarly, the equations for Kirchhoff's voltage law may be expressed as

$$\begin{bmatrix} B_{Ce} & B_{CC} & & & & U_{\ell C} & & & & \\ B_{ge} & B_{gC} & B_{gg} & & & & U_{\ell g} & & & \\ B_{re} & B_{rC} & B_{rg} & B_{rr} & & & & U_{\ell r} & & \\ B_{Le} & B_{LC} & B_{Lg} & B_{Lr} & B_{LL} & & & & U_{\ell L} & \\ B_{je} & B_{jC} & B_{jg} & B_{jr} & B_{jL} & & & & & U_{\ell j} \end{bmatrix} \begin{bmatrix} v_{be} \\ v_{bC} \\ v_{bg} \\ v_{br} \\ v_{bL} \\ v_{\ell C} \\ v_{\ell g} \\ v_{\ell r} \\ v_{\ell L} \\ v_{\ell j} \end{bmatrix} = 0 \qquad (3.3)$$

and again v_{be} = e, the voltage source vector.

The element equations may be written as

$$\begin{bmatrix} i_{bC} \\ i_{bg} \\ v_{br} \\ v_{bL} \\ i_{\ell C} \\ i_{\ell g} \\ v_{\ell r} \\ v_{\ell L} \end{bmatrix} = \begin{bmatrix} C_b \frac{d}{dt} & & & & & & & \\ & G_b & & & & & & \\ & & R_b & & & & & \\ & & & L_b \frac{d}{dt} & & & & \\ & & & & C_\ell \frac{d}{dt} & & & \\ & & & & & G_\ell & & \\ & & & & & & R_\ell & \\ & & & & & & & L_\ell \frac{d}{dt} \end{bmatrix} \begin{bmatrix} v_{bC} \\ v_{bg} \\ i_{br} \\ i_{bL} \\ v_{\ell C} \\ v_{\ell g} \\ v_{\ell r} \\ v_{\ell L} \end{bmatrix} \qquad (3.4)$$

With this partitioning the mixed equations become

$$\begin{bmatrix} (C_b + Q_{CC}\, C_\ell\, Q_{CC}^t) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & (L_\ell + B_{LL} L_b B_{LL}^t) \end{bmatrix} \begin{bmatrix} \dot{v}_{bC} \\ \dot{v}_{bg} \\ \dot{i}_{\ell r} \\ \dot{i}_{\ell L} \end{bmatrix}$$

$$+ \begin{bmatrix} Q_{Cg}\, G_i\, Q_{Cg}^t & Q_{Cg}\, G_\ell\, Q_{gg}^t & Q_{Cr} & Q_{CL} \\ Q_{gg}\, G_\ell\, Q_{Cg}^t & (G_b + Q_{gg} G_\ell\, Q_{gg}^t) & Q_{gr} & Q_{gL} \\ B_{rC} & B_{rg} & (R_\ell + B_{rr} R_b B_{rr}^t) & B_{rr} R_b B_{Lr}^t \\ B_{LC} & B_{Lg} & B_{Lr} R_b B_{rr}^t & B_{Lr} R_b B_{Lr}^t \end{bmatrix} \begin{bmatrix} V_{bC} \\ V_{bg} \\ i_{\ell r} \\ i_{\ell L} \end{bmatrix} =$$

$$- \begin{bmatrix} Q_{Cg}\, G_\ell\, Q_{eg}^t & Q_{Cj} \\ Q_{gg}\, G_\ell\, Q_{eg}^t & Q_{gj} \\ B_{re} & B_{rr} R_b\, B_{jr}^t \\ B_{Le} & B_{Lr} R_b\, B_{jr}^t \end{bmatrix} \begin{bmatrix} E \\ J \end{bmatrix} - \begin{bmatrix} Q_{CC}\, C_\ell\, Q_{eC}^t & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & B_{LL} L_b B_{jL}^t \end{bmatrix} \begin{bmatrix} \dot{E} \\ \dot{J} \end{bmatrix} \tag{3.5}$$

The second and third sets of rows of the equations contain no differential terms and so are only algebraic relationships among the variables. As no differential terms of v_{bg} and $i_{\ell r}$ occur at all the problem is to eliminate these variables using the algebraic equations. This is possible if the submatrix

$$\begin{bmatrix} (G_b + Q_{gg}\, G_\ell\, Q_{gg}^t) & Q_{gr} \\ B_{rg} & (R_\ell + B_{rr} R_b B_{rr}^t) \end{bmatrix} \tag{3.6}$$

possesses an inverse i.e. if it is non-singular. To show this it will

be first shown that the matrix is positive definite. This requires a number of well known results in matrix theory.[35]

The quadratic form of a real (n x n) matrix, A, is defined as the product

$$\varphi(x) = x^t Ax$$

where x is an arbitrary real n-vector.

The matrix is said to be positive definite if $\varphi(x) > 0$ for all non-zero vectors x.

and positive semidefinite if

$$\varphi(x) \geqslant 0 \text{ for all non-zero vectors } x.$$

A positive definite matrix is always nonsingular.

Lemma 3.1[62] If A is a real, (n x n) positive definite matrix and T is a real (r x n) matrix of rank r, ($r \leqslant n$), then $T A T^t$ is also positive definite.

Proof Let $y = T^t x$ where y is an n-vector and

x is an r-vector.

If T^t is of rank r, then if x is a non zero vector y is also non zero.

But $x^t T P T^t x = y^t P y > 0$

since P is a positive definite matrix.

Hence $T P T^t$ is also positive definite.

With the aid of these facts it is now possible to prove the

following lemma.

Lemma 3.2

$$\begin{bmatrix} (G_b + Q_{gg} G_\ell Q_{gg}^t) & Q_{gr} \\ B_{rg} & (R_\ell + B_{rr} R_b B_{rr}^t) \end{bmatrix}$$

is a positive definite matrix and therefore possesses an inverse.

Proof Consider the quadratic form

$$\psi(x) = \begin{bmatrix} x_1^t & x_2^t \end{bmatrix} \begin{bmatrix} (G_b + Q_{gg} G_\ell Q_{gg}^t) & Q_{gr} \\ B_{rg} & (R_\ell + B_{rr} R_b B_{rr}^t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1^t (G_b + Q_{gg} G_\ell Q_{gg}^t) x_1 + x_2^t (R_\ell + B_{rr} R_b B_{rr}^t) x_2$$

$$+ x_1^t Q_{gr} x_2 + x_2^t B_{rg} x_1$$

Now

$$(G_b + Q_{gg} G_\ell Q_{gg}^t) = \begin{bmatrix} U & Q_{gg} \end{bmatrix} \begin{bmatrix} G_b & . \\ . & G_\ell \end{bmatrix} \begin{bmatrix} U \\ Q_{gg}^t \end{bmatrix}$$

and the matrix $\begin{bmatrix} G_b & \\ & G_\ell \end{bmatrix}$ is positive definite since it contains only positive diagonal elements. Therefore by Lemma 3.1, $(G_b + Q_{gg} G_\ell Q_{gg}^t)$ is also positive definite. Similarly $(R_\ell + B_{rr} R_b B_{rr}^t)$ can also be shown to be positive definite.

Since $\psi(x)$ is a scalar,

$$x_1^t Q_{gr} x_2 = (x_1^t Q_{gr} x_2)^t = x_2^t Q_{gr}^t x_1$$

But $Q_{gr}^{t} = - B_{rg}$,

and so $x_1^t Q_{gr} x_2 = - x_2^t B_{rg} x_1$

Thus $\psi(x)$ is the sum of two positive definite quadratic forms and so is itself positive definite.

Thus the given matrix is positive definite and hence non-singular, possessing an inverse.

Thus it is possible to use the second and third sets of equations in equation (3.5) to solve for v_{bg} and $i_{\ell r}$ in terms of v_{bc}, $i_{\ell L}$, e, j, $\dot{e}$ and $\dot{j}$, and eliminate them from the remaining differential equations.

By the use of Lemma 3.1 in the manner of the proof of Lemma 3.2 it can be shown that matrices $(C_b + Q_{cc} C_\ell Q_{cc}^t)$ and $(L_\ell + B_{LL} L_b B_{LL}^t)$ are both positive definite and therefore possess inverses. Thus it is possible to bring the equations into the standard state equation form of equation (1.3), with v_{bC} and $i_{\ell L}$ as the state variables.

Hence the number of dynamically independent variables is equal to the maximum number of capacitors which can be included in the tree plus the maximum number of inductors which can be included in the cotree.

This can be restated as saying that the number of dynamically independent variables in the network is the number of capacitors plus the number of inductors less the number of all capacitor loops and all inductor cut-sets, after all sources have been removed.

These results can be extended to the case of mutual coupling between reactive elements of the same class. This is because the results concerning positive definiteness still apply, if tight coupling is excluded. e.g. In the case of mutual inductance the inductor

equations become

$$\begin{bmatrix} v_{\ell L} \\ v_{bL} \end{bmatrix} = \begin{bmatrix} L_{\ell\ell} & L_{\ell b} \\ L_{b\ell} & L_{bb} \end{bmatrix} \begin{bmatrix} \dot{i}_{\ell L} \\ \dot{i}_{bL} \end{bmatrix}$$

and the matrix to be inverted has the form

$$\begin{bmatrix} U & B_{LL} \end{bmatrix} \begin{bmatrix} L_{\ell\ell} & L_{\ell b} \\ L_{b\ell} & L_{bb} \end{bmatrix} \begin{bmatrix} U \\ B_{LL}^{t} \end{bmatrix}$$

Again using Lemma 3.1, this is positive definite and possesses an inverse if

$$\begin{bmatrix} L_{\ell\ell} & L_{\ell b} \\ L_{b\ell} & L_{bb} \end{bmatrix}$$

is positive definite. Such a condition is in fact true, provided the loosely coupled case only is considered. In the tightly coupled case the determinant of this coefficient matrix is zero.

Similar results will apply to the case of coupled capacitors.

3.2.3 The choice of state variables

In the last section the state variables were taken as the independent capacitor voltages and independent inductor currents in order to simplify the elimination procedure. This is not a necessary choice and is purely a matter of convenience. Other sets of variables are possible, but they have the common properties that they are components

of the vectors v_n and i_m for the admittance and impedance subnetworks.

As a simple example consider the network shown in Fig. 3.1. In this case there are 2 state variables. These are v_1 and v_2 using the approach of the last section. But v_1 and v_3 also form a suitable set. This fact is not surprising since

$$v_1 - v_2 - v_3 = 0$$

Hence either v_1 and v_2 fix v_3 or v_1 and v_3 fix v_2.

Another suitable set of variables would be v_1 and i_3, when the resistor is regarded as a resistance in the impedance subnetwork.

The only bound that can be put on the number of voltages and current variables is that there cannot be more voltage variables than in v_n. In fact this result means that the number of capacitors and inductors have no direct control over the number of voltage or current variables which can be taken. Only the total number of reactive elements, less redundancies, fixes the total number of state variables. This means that it is possible to take current variables in an RC network, as was illustrated in the discussion of Fig. 3.1. The number of current variables possible is limited by the number of current variables in i_m, which is governed by the number of resistors in the network. The total number of variables is still limited by the capacitor subgraph.

The dimensions of v_n and i_m are dependent on the choice of admittance and impedance elements, which in the case of linear resistors is quite arbitrary.

Consider as a second example the network shown in Fig. 3.2.

This example has 2 state variables and v_1 and i_3 form such a pair. If R_2 is taken as a conductance then the dimension of v_n is 2 and the dimension of i_m is 1. Thus it may be possible to take 2 voltages as variables, such as v_1 and v_2. Since the dimension of i_m cannot be greater than one, even with R_2 regarded as a resistance, it is never possible to have more than one current state variable.

Normally such unusual state variables are not used, and the state variables are generally associated with the appropriate capacitor voltages and inductor currents. This choice is simple, and as can be seen from the last section, always forms a suitable set of variables.

3.2.4 The classification of the resistor network

In section 3.2.2 the elimination procedure was given without any consideration of the way the resistor network was partitioned into resistances and conductances and the key result of Lemma 3.2 was proved for any partitioning. It is this flexibility in the partitioning which allows for the number of alternative approaches to the problem of finding the state equations which have been suggested in the literature. Branin[12] claimed that his method was the best because the order of the required inverse matrix was less than that of Bryant or Brown. Since most of the time needed for setting up the equations on a computer is used in finding this inverse it is of some practical importance. The time taken to invert a matrix, using conventional programmes, seems to depend upon the order of the matrix

and so the minimization of the order would be worthwhile.

Although not explicitly stated in his paper, Bryant's original approach[16] was to divide the resistors so that all conductances could be placed in the tree and all resistances could be placed in the cotree. This approach was taken because of symmetry. It implies that G_ℓ and $R_b = 0$, and so the matrix (3.6) becomes

$$\begin{bmatrix} G_b & Q_{gr} \\ -Q_{gr}^t & R_\ell \end{bmatrix}$$

Certain other terms of equation (3.5) also become zero, but these seem to give no advantage. Bryant must therefore invert a matrix whose order is equal to the total number of resistors in the network.

In fact Bryant gave an inverse for the matrix explicitly, which in fact does not require the inverse of such a large matrix, but rather inverts two smaller matrices, whose orders are the number of resistive links and the number of resistive tree branches. Basically the aim of this approach is to maintain a symmetry in the equations when obtaining an explicit form for the state equations.

An alternative choice is to take all the tree branches as resistances i.e. $G_b = 0$. In this case the second set of equations in equation (3.5) vanish completely, as do the columns corresponding to v_{bg}. Thus the matrix (3.6) becomes

$$\begin{bmatrix} R_\ell + B_{rr} R_b B_{rr}^t \end{bmatrix}$$

The order of this matrix is the number of resistive links. The procedure for selecting the tree restricts the way in which the

links may be split up into resistances and conductances. From the last result it is best to minimise the number of resistance links. But since the tree is formed by selecting the conductances and then the resistances, any conductance chords must have a capacitor and voltage source path between their nodes, so that they are contained in an otherwise all capacitor and voltage source loop. Hence in this case the minimum order for matrix (3.6) is given by the number of resistors in the cotree minus the number of resistors which close all capacitor and voltage source paths. Such an approach was taken by Price and Bers.[53]

A possible simplification is to assume that there are no conductance chords. This means that the order of the matrix is slightly increased.

Dually, it can be assumed that there are no resistive links i.e. $R_{\ell} = 0$. This means that the matrix (3.6) becomes

$$\left[G_b + Q_{gg} G_{\ell} Q_{gg}^t \right]$$

and the third set of equations in equation (3.5) along with the columns corresponding to $i_{\ell r}$ vanish. The choice of resistance tree branches is restricted in this case, to only those resistors which lie in an otherwise all inductor and current source cut-set. The minimum order of the matrix is then given by the number of resistors in the tree less the number of resistors in the tree which lie in such cut-sets.

Again, for simplicity, it is possible to assume that there are no resistive tree branches. This approach which produces a slightly larger matrix, was taken by Brown.[13]

An alternative approach is to work, not with separate capacitors and inductors, but to assume that each capacitor has a shunt conductance across it and each inductance has a series resistance. The conductances and resistances may have zero values, so this is just a device for fixing the partitioning of some of the resistors. After this has been done the remaining resistors must be treated in one of the above ways. This type of approach was given by Branin.[11]

In fact the assumption, that any resistor in shunt with a capacitor is a conductance, is a special case of the restriction that any resistor, having an all capacitor and voltage source path between its terminal nodes, is a conductance. So this assumption is a special case of Price and Bers' partitioning. Similarly the case of resistors in series with inductors is a special case of the dual approach to Price and Bers. Thus Branin's approach seems to offer no new advantages.

The best approach seems to be that of Price and Bers, or the dual of their approach, when either G_b or $R_c = 0$. In general it is not possible to fulfil both these conditions at the same time unless the network takes a very special form. The difference between the two methods is not definite, but depends on the particular example considered, in much the same way as the difference between loop and node analysis.

The algorithm for finding a tree with this partitioning is quite simple. In the resistance case i.e. when $G_b = 0$, the procedure

is:

(a) Include all voltage sources in the tree;

(b) Short circuit all voltage sources and open circuit all resistors, inductors and current sources. Find a tree in this capacitor subgraph;

(c) Short circuit all capacitors and add the resistors. All resistor self loops are regarded as conductances, the remaining are resistances. Find a tree in the resistance subgraph;

(d) Short circuit all the resistors and add the inductors. Find a tree in the inductor subgraph;

(e) All current sources will be in the cotree.

Because of solvability, all the voltage sources will be in the tree and all the current sources in the cotree. All conductance branches defined by step (c) will also lie in the cotree.

A dual algorithm is possible for the conductance case, when $R_\ell = 0$

(a) All current sources are in the cotree;

(b) Remove all current sources and short circuit all resistors, capacitors and voltage sources. Find a cotree in the inductance subgraph.

(c) Remove all inductors and replace all resistors. Any resistors which do not lie in a resistor loop are regarded as resistances, the remaining are conductances. Form a cotree in the conductance subgraph.

(d) Remove all resistors and add all capacitors. Form a cotree in the capacitor subgraph.

(e) All voltage sources are included in the tree.

Again the tree contains all voltage sources and no current sources on the grounds of solvability, but now all resistances are included in the tree.

Although this is the dual of the first algorithm it seems much more complicated to programme since most computer methods work with trees and not cotrees, and prefer to add branches by "solder" connection and not by "pliers" connection. The basic reason for this is purely a conceptual one in that node splitting seems a rather unnatural process in the normal network background.

3.2.5 The explicit form of the state equations

To obtain a general form for the state equations the explicit inverse of the matrix (3.6) is needed. The general form of this matrix does not give a simple result and so some simplification in the partitioning of the resistor subgraph must be made. If it is assumed that all resistor elements are resistances then the problems of notation are all set aside and the inverse is simply

$$(R_{\ell} + B_{rr}\, R_b\, B_{rr}^t)^{-1}$$

and there are no conductance terms.

With such an elimination the equations are reduced to

$$\begin{bmatrix} (C_b + Q_{CC} C_\ell Q_{CC}^t) & \\ & (L_\ell + B_{LL} L_b B_{LL}^t) \end{bmatrix} \begin{bmatrix} \dot{v}_{bC} \\ \dot{i}_L \end{bmatrix}$$

$$+ \begin{bmatrix} -Q_{Cr}(R_L + B_{rr} R_b B_{rr}^t)^{-1} B_{rC} & Q_{CL} - Q_{Cr}(R_L + B_{rr} R_b B_{rr}^t)^{-1} B_{rr} R_b B_{Lr}^t \\ B_{LC} - B_{Lr} R_b B_{rr}^t (R_L + B_{rr} R_b B_{rr}^t)^{-1} B_{rC} & B_{Lr} R_b B_{Lr}^t - B_{rr} R_b B_{Lr}^t (R_L + B_{rr} R_b B_{rr}^t)^{-1} B_{rr} R_b B_{Lr} \end{bmatrix} \begin{bmatrix} v_{bC} \\ i_{\ell r} \end{bmatrix}$$

$$+ \begin{bmatrix} -Q_{Cr}(R_\ell + B_{rr} R_b B_{rr}^t)^{-1} B_{re} & Q_{Cj} - Q_{Cr}(R_\ell + B_{rr} R_b B_{rr}^t)^{-1} B_{rr} R_b B_{jr}^t \\ B_{Le} - B_{Lr} R_b B_{rr}^t (R_\ell + B_{rr} R_b B_{rr}^t)^{-1} B_{re} & B_{Lr} R_b B_{jr}^t - B_{Lr} R_b B_{rr}^t (R_\ell + B_{rr} R_b B_{rr}^t)^{-1} B_{rr} R_b B_{jr}^t \end{bmatrix} \begin{bmatrix} e \\ j \end{bmatrix}$$

$$+ \begin{bmatrix} Q_{CC} C_\ell Q_{eC}^t & \cdot \\ \cdot & B_{LL} L_b B_{jL}^t \end{bmatrix} \begin{bmatrix} \dot{e} \\ \dot{j} \end{bmatrix} = 0 \qquad (3.7)$$

Because all the resistors have been taken as resistances some of the symmetry has been destroyed. But since

$$(B_{LC} - B_{Lr} R_b B_{rr}^t (R_\ell + B_{rr} R_b B_{rr}^t)^{-1} R_{rC})^t = -(Q_{CL} - Q_{Cr}(R_\ell + B_{rr} R_b B_{rr}^t)^{-1} B_{rr} R_b B_{Lr}^t)$$

the equations do show that the symmetry and skew symmetry of equation (3.1) has been preserved.

The coefficient matrix of the differential terms has not been reduced to a unit matrix in order to simplify the expressions. In section 3.2.2 it was shown that this coefficient matrix has an inverse.

The symmetry of the equations can be restored either by forming the lower equations by taking all resistors as conductances, or by using the matrix identities proved in the following lemmas.

Lemma 3.3[13] $(F + GHK)^{-1} = F^{-1} + F^{-1}G(H^{-1} + K F^{-1}G) KF^{-1}$

Proof $(F + GHK) = (U + GHKF^{-1})F$

Inserting an extra matrix and its inverse gives

$$(F + GHK) = \{U + GH(H^{-1} + KF^{-1}G) (H^{-1} + KF^{-1} G)^{-1}KF^{-1} \} F$$

$$= \{U + (G + GHKF^{-1}G) (H^{-1}+ KF^{-1}G)^{-1}KF^{-1} \} F$$

$$= \{U + (F + GHK)F^{-1}G (H^{-1}+ KF^{-1}G)^{-1} KF^{-1} \} F$$

$$\therefore (F + GHK) \{ F^{-1} + F^{-1}G(H^{-1} + KF^{-1}G)^{-1} KF^{-1} \} = U$$

$$\text{i.e. } (F + GHK)^{-1} = F^{-1} + F^{-1}G(H^{-1} + KF^{-1}G)^{-1} KF^{-1}.$$

Lemma 3.4[13] $(F + GHK)^{-1} GH = F^{-1}G(H^{-1} + KF^{-1}G)^{-1}$

Proof $(F + GHK)F^{-1}G = (G + GHKF^{-1}G)$

$$= GH(H^{-1} + KF^{-1}G)$$

i.e. $(F + GHK)^{-1}GH = F^{-1}G(H^{-1} + KF^{-1} G)^{-1}.$

With these lemmas, equation (3.7) can be written as

$$\begin{bmatrix} (C_b + Q_{CC}C_\ell Q_{CC}^t) & \\ & (L_\ell + B_{LL}L_b B_{LL}^t) \end{bmatrix} \begin{bmatrix} \dot{v}_{bC} \\ \dot{i}_{\ell L} \end{bmatrix} =$$

$$\begin{bmatrix} Q_{Cr}(R_\ell+B_{rr}R_b B_{rr}^t)^{-1}B_{rC} & Q_{Cr}(R_\ell+B_{rr}R_b B_{rr}^t)^{-1}B_{rr}R_b B_{Lr}^t - Q_{CL} \\ B_{Lr}(R_b^{-1}+Q_{rr}R_\ell^{-1}Q_{rr}^t)^{-1}Q_{rr}R^{-1}Q_{Cr}^t - B_{LC} & B_{Lr}(R_b^{-1}+Q_{rr}R_\ell^{-1}Q_{rr})^{-1}Q_{rL} \end{bmatrix} \begin{bmatrix} v_{bC} \\ i_{\ell L} \end{bmatrix}$$

$$+ \begin{bmatrix} Q_{Cr}(R_\ell+B_{rr}R_b B_{rr}^t)^{-1}B_{re} & Q_{Cr}(R_\ell+B_{rr}R_b B_{rr}^t)^{-1}B_{rr}R_b B_{jr}^t - Q_{cj} \\ B_{Lr}(R_b^{-1}+Q_{rr}R_\ell^{-1}Q_{rr}^t)^{-1}Q_{rr}R^{-1}Q_{er}^t - B_{Le} & B_{Lr}(R_b^{-1}+Q_{rr}R_\ell^{-1}Q_{rr})^{-1}Q_{rj} \end{bmatrix} \begin{bmatrix} e \\ j \end{bmatrix}$$

$$- \begin{bmatrix} Q_{CC}\, C_\ell\, Q_{eC}^t & \\ & B_{LL}L_b B_{jL}^t \end{bmatrix} \begin{bmatrix} \dot{e} \\ \dot{j} \end{bmatrix} \qquad (3.8)$$

This is the form which was originally given by Bryant.[18]

3.3 STATE VARIABLE DESCRIPTION OF A MULTIPORT NETWORK

In later work the network will be regarded as a "black box" driven by some voltage and current sources. This system will be characterized by the state equations and so it is important to know what outputs can be obtained from the system, i.e. what current passes through the voltage sources and what voltage is across the current sources. Because of the nature of the state equations these can be obtained as expressions involving only the state variables and the inputs. i.e. i_{be} and $v_{\ell j}$ can be obtained in terms of v_{bC}, $i_{\ell L}$, e and j.

In the case of the current i_{be}, this is given from equation (3.2) by the relationship.

$$i_{be} = - Q_{eC}\, i_{\ell C} - Q_{er}\, i_{\ell r} - Q_{eL}\, i_{\ell L} - Q_{ej}\, j$$

where all resistors are taken as resistances.

Then

$$i_{be} = - Q_{ej}\, j + Q_{eC}\, C_{\ell}\, B_{Ce}\, \dot{e} + Q_{eC}\, C_{\ell}\, B_{CC}\, \dot{v}_{bC} - Q_{er}\, i_{\ell r} - Q_{eL}\, i_{\ell L}$$

Similarly

$$v_{\ell j} = - B_{je}\, e + B_{jL}\, L_b\, Q_{Lj}\, \dot{j} + B_{jL}\, L_b\, Q_{LL}\, \dot{i}_{\ell L} - B_{jL}\, v_{br} - B_{jC}\, v_{bC}$$

Eliminating $i_{\ell r}$ and v_{br} in the same way as for the state equations themselves, produces the required equations. These can be written

along with equation (3.7) in the form

$$\begin{bmatrix} (C_b+Q_{CC}C_\ell Q_{CC}^t)^{-1} & \cdot & \cdot & \cdot \\ \cdot & (L_\ell+B_{LL}L_bB_{LL}^t)^{-1} & \cdot & \cdot \\ Q_{eC}C_\ell B_{CC}(C_b+Q_{CC}C_\ell Q_{CC}^t)^{-1} & \cdot & U & \cdot \\ \cdot & B_{jL}L_bQ_{LL}(L_\ell+B_{LL}L_bB_{LL}^t)^{-1} & \cdot & U \end{bmatrix}^{-1} \begin{bmatrix} \dot{v}_{bC} \\ \dot{i}_{\ell L} \\ i_{be} \\ v_{\ell j} \end{bmatrix}$$

$$= \begin{bmatrix} Q_{Cr}(R_\ell+B_{rr}R_bB_{rr}^t)^{-1}B_{rC} & Q_{Cr}(R_\ell+B_{rr}R_bB_{rr}^t)^{-1}B_{rr}R_bB_{Lr}^t-Q_{CL} \\ B_{Lr}(R_b^{-1}+Q_{rr}R_\ell^{-1}Q_{rr}^t)^{-1}Q_{rr}R_\ell^{-1}Q_{Cr}^t-B_{LC} & B_{Lr}(R_b^{-1}+Q_{rr}R_\ell^{-1}Q_{rr}^t)^{-1}Q_{rL} \\ Q_{er}(R_\ell+B_{rr}R_bB_{rr}^t)^{-1}B_{rC} & Q_{er}(R_\ell+B_{rr}R_bB_{rr}^t)^{-1}B_{rr}R_bB_{Lr}-Q_{eL} \\ B_{jr}(R_b^{-1}+Q_{rr}R_\ell^{-1}Q_{rr}^t)^{-1}Q_{rr}R_\ell Q_{Cr}^t-B_{jC} & B_{jr}(R_b^{-1}+Q_{rr}R_\ell^{-1}Q_{rr}^t)^{-1}Q_{rL} \end{bmatrix} \begin{bmatrix} v_{bC} \\ i_{\ell L} \end{bmatrix}$$

$$+ \begin{bmatrix} Q_{Cr}(R_\ell+B_{rr}R_bB_{rr}^t)^{-1}B_{re} & Q_{Cr}(R_\ell+B_{rr}R_bB_{rr}^t)^{-1}B_{rr}R_bB_{jr}-Q_{Cj} \\ B_{Lr}(R_b^{-1}+Q_{rr}R_\ell^{-1}Q_{rr}^t)^{-1}Q_{rr}R_\ell^{-1}Q_{er}^t-B_{Le} & B_{Lr}(R_b^{-1}+Q_{rr}R_\ell^{-1}B_{rr})^{-1}Q_{rj} \\ Q_{er}(R_\ell+B_{rr}R_bB_{rr}^t)^{-1}B_{re} & Q_{er}(R_\ell+B_{rr}R_bB_{rr}^t)^{-1}B_{rr}R_bB_{jr}-Q_{ej} \\ B_{jr}(R_b^{-1}+Q_{rr}R_\ell^{-1}Q_{rr}^t)^{-1}Q_{rr}R_\ell^{-1}Q_{er}^t-B_{je} & B_{jr}(R_b^{-1}+Q_{rr}R_\ell^{-1}Q_{rr})^{-1}Q_{rj} \end{bmatrix} \begin{bmatrix} e \\ j \end{bmatrix}$$

$$+ \begin{bmatrix} Q_{CC}\,R_\ell^{-1}\,B_{Ce} & \\ \cdot & B_{LL}\,L_b\,Q_{Lj} \\ Q_{eC}\,C_\ell\,B_{Ce} & \cdot \\ \cdot & B_{jL}\,L_b\,Q_{Lj} \end{bmatrix} \begin{bmatrix} \dot{e} \\ \cdot \\ j \end{bmatrix} \qquad (3.9)$$

Again the coefficient matrix of the differential terms and the outputs has been kept on the left hand side of the equation for clarity, although its inverse has been found.

Equation (3.9) is not quite in the standard multiport form. This is because of the signs of i_{be} and v_{ℓ_j}. Here they have been taken with the same convention as all other branches in the network. Usually in n-port theory these signs would be reversed, so that the current in a voltage source "flows out" of the source.

3.4 THE EXISTENCE OF A UNIQUE SOLUTION TO A NETWORK

In section 3.2 it has been shown that, if the necessary condition of Lemma 2.1 is fulfilled, then the equations can always be brought to the form

$$\dot{x} = \underline{A}x + \underline{B}u$$

where in this case $x = \begin{bmatrix} v_{bc} \\ i_{\ell_L} \end{bmatrix}$ and $u = \begin{bmatrix} e \\ j \\ \dot{e} \\ \dot{j} \end{bmatrix}$

It was pointed out in section 1.2 that the existence of such a set of state equations implies a unique solution to the system provided the inputs are continuous. Thus section 3.2 is a proof that Lemma 2.1 and the continuity of the inputs and their first derivations are necessary and sufficient conditions for the network to have a unique solution.

In special cases the differentials of the input vector do not occur in the equations and the conditions can be relaxed. Inspection of the explicit form, equation (3.8), shows that the first derivative

terms for a given voltage source only appear if it lies in a loop containing only voltage sources and capacitors and for a given current source if it lies in a cut-set of inductors and current sources. If this is not the case then the first derivative does not appear in the equations and the only requirement is that the driving function is continuous.

The initial conditions of the network can be any set of state variables.

These results are of some interest since as yet it has not been possible to show the existence of a unique solution using the conventional methods of Laplace transformation and loop or node analysis. Such methods require the proof that the resulting functions of the Laplace variable, s, have an inverse Laplace transformation as a time function, and then that these time functions satisfy the initial conditions. This is, as yet, an unsolved problem.[62]

3.5 NETWORKS CONTAINING NEGATIVE RESISTORS

3.5.1 The order of complexity

The most simple approach to active networks is to represent the activity by negative resistors. Such an element is a source of power in the network but does not affect its reciprocity. Since the elements are only 2-terminal the approach given in section 3.2 still applies, except that since the matrices are no longer automatically positive

definite it is not possible to say, in general, if the necessary inverses exist.

It is still possible to say that the order of complexity is less than $\rho(\mathcal{C}) + \rho(\mathcal{L})$, but the actual value is not clear. If the inverse of the matrix (3.6) exists then the elimination is the same as in the positive resistor case. If not, then the number of variables will be less than the passive counterpart. The problem of finding the actual order of complexity is algebraic in nature and is closely connected with the general problem of the existence of a state model for the network and the solvability of the network.

A necessary and sufficient condition for the order of complexity to be the same as for its passive counterpart is that the elimination proceeds as in the passive case, i.e. that the matrix (3.6) is non-singular.

If the resistor network is partitioned so that all tree branches are regarded as conductances, unless they lie across otherwise all inductor and current source cut-sets, and all links are regarded as resistors, unless they lie in a voltage source otherwise all capacitor loops, then,

$$Q_{gg} = 0$$

$$\text{and } Q_{rr} = 0$$

and the matrix (3.6) becomes

$$\begin{bmatrix} G_b & Q_{gr} \\ B_{rg} & R_\ell \end{bmatrix}$$

where the same notation as section 3.22 applies. This can be expressed

in the following theorem.

Theorem 3.1

The order of complexity of a network containing capacitors, inductors, uncontrolled sources and positive and negative resistors is the same as that for the network with all resistors positive if the matrix

$$\begin{bmatrix} G_b & Q_{gr} \\ B_{rg} & R_\ell \end{bmatrix} \tag{3.10}$$

is non-singular, with the above notation.

Since G_b and R_ℓ are both diagonal, their inverses exist i.e. $G_b^{-1} = R_b$ and $R_\ell^{-1} = G_\ell$.

Also $Q_{gr} = - B_{rg}^t$.

Thus $$\det \begin{bmatrix} G_b & Q_{gr} \\ B_{rg} & R_\ell \end{bmatrix} = \det G_b \cdot \det(R_\ell + B_{rg} R_b B_{rg}^t)$$

and $$= \det R_\ell \cdot \det(G_b + Q_{gr} G_\ell Q_{gr}^t)$$

Thus the condition that matrix (3.10) is non singular is equivalent to the conditions that either

$$(R_\ell + B_{rg} R_b B_{rg}^t) \quad \text{or} \quad (G_b + Q_{gr} G_\ell Q_{gr}^t)$$

is non-singular.

A physical interpretation can be given to this result. Consider the network obtained by short circuiting all voltage sources and capacitors, open circuiting all current sources and inductances, open circuiting all resistor self loops and short circuiting all resistors

that are in no loops. Then the cut-set matrix of the remaining network is

$$Q = \left[U \; Q_{gr} \right]$$

and the node admittance matrix is

$$= \left[U \; Q_{gr} \right] \begin{bmatrix} G_b & \\ & G_\ell \end{bmatrix} \begin{bmatrix} U \\ Q_{gr}^t \end{bmatrix}$$

$$= \left[G_b + Q_{gr} \, G_\ell \, Q_{gr}^t \right]$$

Similarly the loop impedance matrix of this network is

$$= \left[R_\ell + B_{rg} \, R_b \, B_{rg}^t \right]$$

Thus theorem 3.1 is equivalent to the condition that this resistive subnetwork has a unique solution.

3.5.2 An extension to non-linear resistors

Because of the way in which the resistor network was partitioned there are a certain number of resistors which are classed as resistance tree branches and conductance links. These branches do not effect the formation of the state equations since they are not included in the matrix (3.10). This is because the independent variables of their equations can be written directly in terms of the state variables using Kirchhoff's laws. For example, the voltage across a resistance tree branch is given directly as the sum of capacitor and source voltages along a path between its terminals.

This means that as long as these elements possess the correct characterization i.e. conductance or resistance, their exact nature is not important. They may be linear, non-linear, positive or negative. These results provide a simple demonstration of some well known results in the theory of non-linear networks.[20]

Quite often, in physical devices, negative resistance is associated with non-linearity, since the negative region extends only over a short region of its characteristic. Such characteristic curves are often single valued functions in only one variable and so possess only one characterization. Fig. 3.3 shows typical examples. In Fig. 3.3(a) the characteristic is "current controlled" and has only a resistance characterization. Similarly in Fig. 3.3(b) the characteristic is "voltage controlled" and so possesses only a conductance characterization.

The state equations can only be formed for a network containing such elements if their positions are chosen so that their inverses are not required. From the above reasoning this means that such networks have a solution only if each voltage controlled element has an all capacitor and voltage source path between its terminals and each current controlled element lies in an all inductor and current source cut-set.

3.5.3 Example

Consider the example shown in Fig. 3.4. Kirchhoff's laws

give

$$v_g + v_r + v_c + v_L = 0$$

and $$i_g = i_r = i_c = i_L$$

The equations can be reduced to

$$L \frac{di_L}{dt} = - i_L R - v_c - v_g$$

$$c \frac{dv_c}{dt} = i_L$$

$$i_L = g(v_g)$$

Since the non linear resistor is voltage controlled no inverse exists for the last equation and it is impossible to eliminate v_g. The network possesses no state model and no solution. In practice such problems are avoided by postulating a small 'stray' capacitance, C_s, across the negative resistor. This means that the voltage controlled element always has a capacitor path between its terminals and always has a solution.

In this case the last equation becomes

$$C_s \frac{dv_g}{dt} = i_L - g(v_g)$$

and the order of complexity becomes 3. Since the equations are for a non-linear system, they no longer have the simple constant coefficient form, and now functions are involved.

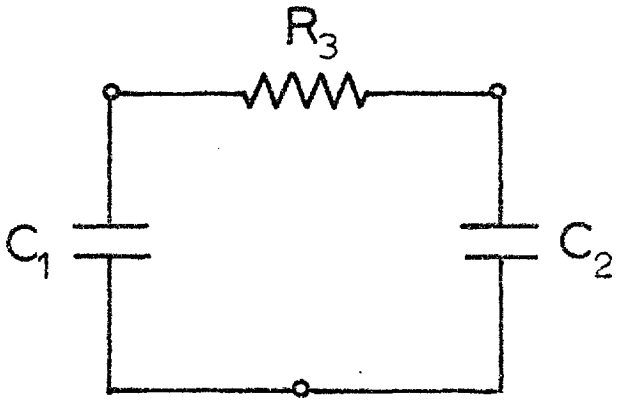


Fig.3.1. Example of section 3.2.3.

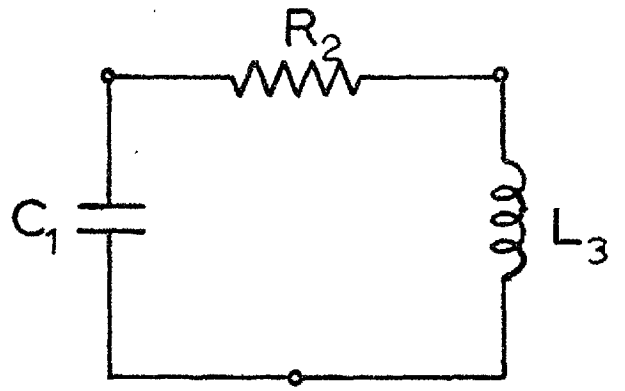


Fig.3.2. Example of section 3.2.3.

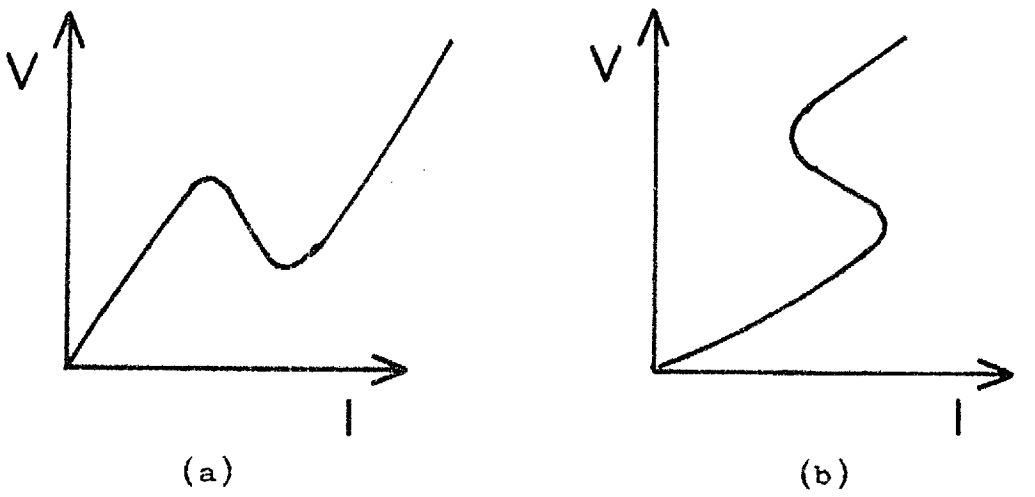


Fig.3.3. Non-linear characteristics.

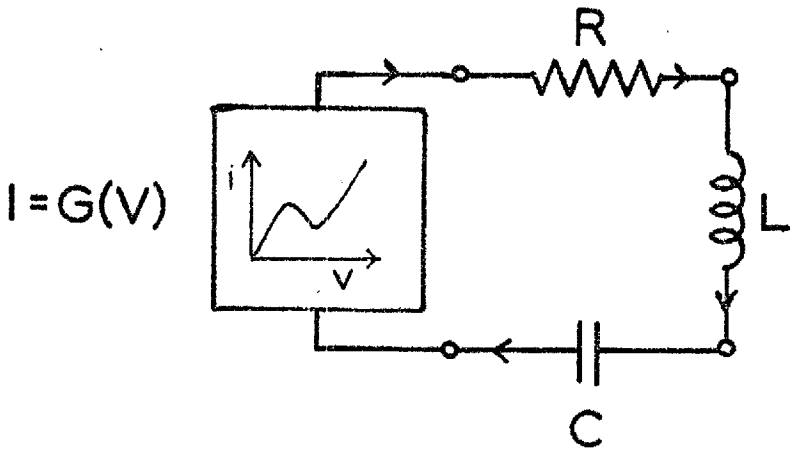


Fig.3.4. Example of section 3.5.2.

Chapter 4

THE ORDER OF COMPLEXITY OF AN ACTIVE NETWORK

4.1 INTRODUCTION

In this Chapter, and the next, the problems of active networks are considered. The term "active network" is used for a network which contains controlled generators of the type considered in section 2.2.3. Such elements can be used to model most linear devices.

The main difficulties with such devices are that they are basically 4-terminal and also active. Thus the problems are far greater than in RLC networks.

4.2 THE PROBLEMS WITH NETWORKS CONTAINING CONTROLLED GENERATORS

When controlled generators are included in the network the problems are both topological and algebraic. The primary problem is that of solvability. In the case of passive networks it was possible with the normal element relationships and Kirchhoff's laws to show that the network had a solution provided small assumptions were made about the positioning and nature of the driving sources. In the case of controlled generator networks this is not so, because of algebraic difficulties. It is possible to propose networks which do not have

solutions for certain element values. An example is given in Fig. 4.1, where the network does not have a solution if $g = (G_3 + G_2)$.

If the network possesses a solution then the network possesses a state model. But the order of complexity and the way to choose the actual state variables is not a simple matter. There seem two basic reasons for this;

(a) There is an algebraic one which is basically similar to the negative resistor case. The order of complexity depends upon the element values. An example of such a network is shown in Fig. 4.2. Here the order of complexity is 4 unless $g = G_3$ when it reduces to 3. Such a problem is due to a non-singularity arising during the reduction process.

(b) The second reason is topological. Since a controlled generator is not just a two terminal device, it provides a means for constraining variables from different branches apart from Kirchhoff's laws. Thus the arguments given for 2-terminal elements do not apply. It may not be possible to separate out the admittance and impedance branches in the manner used in the last chapter. e.g. A current generator which is normally associated with admittance analysis may be controlled by the voltage across an inductor, normally an impedance element. Because of such couplings it is possible to raise the order of complexity above $\rho(\mathcal{G}) + \rho(\mathcal{L})$, the order of complexity of the corresponding passive network formed by removing the generators. An example of this is given in Fig. 4.3, where the order of complexity is 4 compared with 3 for the passive network. (See 4.4.4)

These two effects cannot be completely isolated from one another, since the formation of the state equations will always involve some algebraic considerations. But it may be possible to limit the topology of the network in such a way that the possibility of the second effect can be eliminated. It is this problem which is considered for the remainder of this Chapter. The algebraic problem is eliminated by assuming that all elements have arbitrarily specifiable, non-zero, finite values. This assumption will mean it is possible to consider the solution of the topological problem for a class of networks.

The approach used for the study is an indirect one by way of the natural frequencies of the network, an approach used by Bryant in his early work for the RLC case.

4.3 THE EXISTENCE OF A UNIQUE SOLUTION

For any network there are three sets of equations. These are statements of Kirchhoff's current law,

$$Qi = 0 \qquad (4.1)$$

Kirchhoff's voltage law,

$$Bv = 0 \qquad (4.2)$$

and the element relationships.

In order to write the element equations for a network containing controlled elements the network must first be partitioned. The sets are

1) voltage sources whose currents do not control a generator;

2) voltage sources whose currents control a generator;

3) voltage generators whose currents do not control a generator;

4) voltage generators whose current control a generator;

5) passive elements;

6) current generators whose voltages control a generator;

7) current generators whose voltages do not control a generator;

8) current sources whose voltages control a generator;

9) current sources whose voltages do not control a generator.

The equation for the passive network can be written in the mixed form

$$Y_C(D)\, v_5 = Z_{LR}(D)\, i_5$$

where $Y_C(D)$ and $Z_{LR}(D)$ are diagonal, Y(D) containing all admittance and unit terms, and Z(D) contains all impedance and unit terms. D is the differential operator.

The sources are described by

$$\begin{aligned} v_1 &= e_1 & \qquad i_8 &= j_8 \\ v_2 &= e_2 & \qquad i_9 &= j_9 \end{aligned} \tag{4.3}$$

The equations for the controlled generators have the form

$$\begin{bmatrix} v_3 \\ v_4 \\ i_6 \\ i_7 \end{bmatrix} = \begin{bmatrix} H_{32} & H_{34} & H_{3i} & H_{3v} & H_{36} & H_{38} \\ H_{42} & H_{44} & H_{4i} & H_{4v} & H_{46} & H_{48} \\ H_{62} & H_{64} & H_{6i} & H_{6v} & H_{66} & H_{68} \\ H_{72} & H_{74} & H_{7i} & H_{7v} & H_{76} & H_{78} \end{bmatrix} \begin{bmatrix} i_2 \\ i_4 \\ i_5 \\ v_5 \\ v_6 \\ v_8 \end{bmatrix} \tag{4.4}$$

where the subscripts i and v have been introduced to distinguish

between columns corresponding to i_5 and v_5.

All the network equations can be written in the matrix form

$$\begin{bmatrix} B_3 & B_4 & B_5 & B_6 & B_7 & B_8 & B_9 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 & Q_7 \\ -U_3 & \cdot & H_{3v} & H_{36} & \cdot & H_{38} & \cdot & \cdot & H_{32} & \cdot & H_{34} & H_{3i} & \cdot & \cdot \\ \cdot & -U_4 & H_{4v} & H_{46} & \cdot & H_{48} & \cdot & \cdot & H_{42} & \cdot & H_{44} & H_{4i} & \cdot & \cdot \\ \cdot & \cdot & Y_C(D)_5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -Z_{LR}(D)_5 & \cdot & \cdot \\ \cdot & \cdot & H_{6v} & H_{66} & \cdot & H_{68} & \cdot & \cdot & H_{62} & \cdot & H_{64} & H_{6i} & -U_6 & \cdot \\ \cdot & \cdot & H_{7v} & H_{76} & \cdot & H_{78} & \cdot & \cdot & H_{72} & \cdot & H_{74} & H_{7i} & \cdot & -U_7 \end{bmatrix} \begin{bmatrix} v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix}$$

$$= \begin{bmatrix} B_1 & B_2 & \cdot & \cdot \\ \cdot & \cdot & Q_8 & Q_9 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ j_8 \\ j_9 \end{bmatrix} \qquad (4.5)$$

This is a set of non-homogeneous equations in all the unknown voltage and current variables. The network is said to possess a unique solution if it is possible to calculate all these variables in terms of the driving functions and the initial conditions. This definition differs from that of Doclaris and Saeks[23] who did not

consider it necessary to find the currents through voltage generators or voltages across current generators. This can best be seen in their example shown in Fig. 4.4, where the currents in the voltage generators are arbitrary. Here the controlled generators are regarded as actual network branches, and not simply constraints, and so a network is of no interest if it is possible to have an arbitrary amount of current flowing in any branch, as occurs in Declaris and Saeks' example.

Equation (4.5) is a set of first order differential equations. If the coefficient matrix of the left hand side is $H_0(D)$, then from the appendix it can be seen that the condition for these equations to have a unique solution is that det $H_0(\lambda)$ is non zero where λ is an arbitrary variable. This condition gives the following theorem.

Theorem 4.1. If the equations of an active network containing resistors, capacitors, inductors, generators and sources possess a unique solution, then

(a) there exists a tree such that all voltage sources are tree branches and all current sources are links;

and (b) there exists a tree such that all the voltage sources and generators, whose currents do not control any generators are tree branches and all the current sources and generators, whose voltages do not control generators are links.

Proof If a solution to equation (4.5) exists then the polynomial matrix $H_0(\lambda)$ must be non-singular. Hence all the rows are

linearly independent. In particular the first $\beta - \nu + 1$ rows, corresponding to the Kirchhoff's voltage law equations, are linearly independent. This means that the submatrix $[B_3B_4B_5B_6B_7B_8B_9]$ must contain a non-singular major. A well known property of a loop matrices[62] is that the columns of a non-singular major correspond to the elements of a cotree. Thus there is a cotree amongst the elements corresponding to the columns of this submatrix. Hence the elements corresponding to columns of $[B_1B_2]$ must lie in a tree. By applying similar arguments to the next $\nu - 1$ rows corresponding to the Kirchhoff's current law equations it can be proved that the elements corresponding to the columns of $[Q_8Q_9]$ lie in a cotree.

Thus the voltage sources lie in the tree and the current sources lie in a cotree. Seshu and Reed have proved the following lemma[62]

Lemma 4.1 Let G be a connected graph. Let S_1 and S_2 be the edge disjoint subsets of G, such that (a) there exists a tree T_1 such that the edges of S_1 are links of T_1 (not necessarily the whole link set of T_1) and (b) there exists a tree T_2 such that the edges of S_2 are branches of T_2 (again T_2 may have other branches besides the edges of S_2); then there exists a tree T for which the elements of S_2 are branches and the elements of S_1 are chords.

From this lemma, since the voltage and current sources are clearly disjoint subsets, then it is possible to find a tree such that all voltage sources are tree branches and all current sources are links. This proves the first statement of the theorem.

If $H_0(\lambda)$ is non singular all the columns must be linearly independent. In particular the columns corresponding to v_7 and v_9 must be linearly independent. Because of the zeros in these columns this implies that the columns of the submatrix $[B_7 B_9]$ are linearly independent, and so that the elements corresponding to the columns of $[B_7 B_9]$ form part of a cotree.

Similarly by examination of the columns corresponding to i_1 and i_3 it can be shown that the elements corresponding to the columns of $[Q_1 Q_3]$ form part of a tree. This tree may be taken as being the one forming the cotree and so the second half of the theorem is proved.

In the passive case there are no generators and so only the first half of the theorem applies. This was lemma 2.1, given by Seshu and Reed[62]. In the case when the generators are controlled by the passive elements only, the second result reduces to a generalization of the first. This is that there exists a tree such that all voltage sources and voltage generators are in the tree and all current sources and current generators are in the cotree. This result was given by Dervisoglu[25] and Purslow and Spence[54].

Theorem 4.1 represents the necessary topological restriction on the network. It is not a sufficient condition for the existence of a solution since such a result would involve algebraic considerations involving the element values.

4.4 THE ORDER OF COMPLEXITY OF AN ACTIVE NETWORK

Because of the algebraic restrictions in the network, which have been neglected at present, it is not possible to give a simple procedure for eliminating the unwanted variables. An alternative approach is to view the natural frequencies of the network. This does not involve the actual element values and is thus more suited to the present problem.

4.4.1 Natural frequencies and the order of complexity

The problem of determining the number of natural frequencies of a network has been a classical problem in network theory. The early approach was to consider non-zero natural frequencies, since these can usually be observed by driving point measurements at any point in the network. This assumes that no cancellations occur between the numerator and denominator of the driving point function, which implies complete controlability and observability in the sense of Gilbert[31] and Kalman.[38]

The first significant results were obtained by Reza[57] who used the order of complexity for the number of non zero natural frequencies. These were the zeros of the determinants of the node admittance and loop impedance matrices, Δ_n and Δ_m. Tsang[66] and Cederbaum[19] showed that these determinants were related, so that the number of zeros was the same. Bryant[15] and Bers[7] both eventually gave formulas in terms of the topology of the network for the RLC network case.

Bryant's later work[16] was slightly different since he defined natural frequencies as being "the roots of the determinant polynomial of the operator matrix of the network equations when these are formed as a set of first order differential equations for the branch voltages and currents". This means that the natural frequencies are the zeros of det $H_0(D)$. Such a definition differs only from previous definitions by considering zero frequencies. Under such a definition this number, the order of complexity, is the dimension of the state space.

This can easily be seen[27] by taking the Laplace transform of the state equations

$$sx = \underline{A}x + \underline{B}u$$

$$\text{or} \quad x = (sU - \underline{A})^{-1} \underline{B}u$$

Since all other variables are algebraically related to x and u, then the zeros of det $(sU - \underline{A})$ will give all the natural frequencies of the network. The number of zeros is equal to the order of $\underline{A}$, i.e. the dimension of the state space, which is called the order of complexity.

Bryant related the determinant det $H_0(D)$ to the determinants of the node admittance and loop impedance matrices in the case of RLC networks and applied the topological formulas for such determinants to give the order of complexity. It is this approach which is taken here.

4.4.2 The node admittance matrix

Although the basic problem is to expand det $H_0(D)$ this will be approached indirectly using the determinant of the node admittance matrix. In order to use this approach, a restriction must be made on the type of network considered. The networks considered will contain resistors, capacitors, inductors, sources and current generators controlled by variables associated with passive elements only. The restriction on the class of generators considered is not very severe since in many cases voltage generators can be exchanged for current generators, as was mentioned in section 2.2.3. Thus the results can be applied, in fact, to a much larger class of networks than the initial restriction would imply.

From theorem 4.1, equation (4.1) can be rewritten, with the notation of equation (4.5) on the basis of a tree which contains all the controlled current generators

$$\begin{bmatrix} U_1 & Q_{15} & Q_{17} \\ & Q_{5'5} & Q_{5'7} \end{bmatrix} \begin{bmatrix} i_1 \\ i_5 \\ i_7 \end{bmatrix} = \begin{bmatrix} -Q_{19} \\ -Q_{5'9} \end{bmatrix} i_9 \qquad (4.6)$$

where U_1 is a unit matrix and the subscript 5' indicates that the matrix is partitioned according to the fundamental cut-sets defined by the voltage sources. The remainder of the tree comes from the passive elements. The matrix $Q_{5'5}$ is not therefore a square matrix but $\begin{bmatrix} Q_{5'5} & Q_{5'7} \end{bmatrix}$ corresponds to the cut-set matrix of the network with all uncontrolled sources removed.

Hence $$i_1 = -\begin{bmatrix} Q_{15} & Q_{17} & Q_{19} \end{bmatrix} \begin{bmatrix} i_5 \\ i_7 \\ j_9 \end{bmatrix}$$

and $$\begin{bmatrix} Q_{5'5} & Q_{5'7} \end{bmatrix} \begin{bmatrix} i_5 \\ i_7 \end{bmatrix} = - Q_{5'9} \; j_9 \qquad (4.7)$$

The node pair transformation may be written as $v = Q^t v_n$

or $$\begin{bmatrix} v_1 \\ v_5 \\ v_7 \\ v_9 \end{bmatrix} = \begin{bmatrix} U_1 & . \\ Q_{15}^t & Q_{5'5}^t \\ Q_{17}^t & Q_{5'7}^t \\ Q_{19}^t & Q_{5'9}^t \end{bmatrix} \begin{bmatrix} v_{n1} \\ v_{n2} \end{bmatrix} \qquad (4.8)$$

The element equations now are simply

$$i_5 = Y_5 \, v_5$$

$$i_7 = \begin{bmatrix} H_{7v} & H_{7i} \end{bmatrix} \begin{bmatrix} v_5 \\ i_5 \end{bmatrix}$$

$$= \begin{bmatrix} H_{7v} + H_{7i} \, Y_5 \end{bmatrix} v_5$$

where Y_5 is a diagonal matrix.

Substituting into equation (4.7) gives

$$\begin{bmatrix} Q_{5'5} & Q_{5'7} \end{bmatrix} \begin{bmatrix} Y_5 \\ H_{7v} + H_{7i} \, Y_5 \end{bmatrix} v_5 = - Q_{5'9} \, j_9 \qquad (4.9)$$

Now $v_{n1} = v_1 = e_1$

Substituting for v_5 from equation (4.8) into equation (4.9) gives the

node admittance equations

$$\left[Q_{5'5}\ Q_{5'7}\right]\begin{bmatrix} Y_5 \\ H_{7v}+H_{7i}\,Y_5 \end{bmatrix} Q^{t}_{5'5}\, v_{n2} = -\,Q_{5'5}\, j_9 - Q_{5'5}Q_{5'7}\begin{bmatrix} Y_5 \\ H_{7v}+H_{7i}Y_5 \end{bmatrix} Q^{t}_{15}$$

and the node admittance matrix is

$$\left[Q_{5'5}\ Q_{5'7}\right]\begin{bmatrix} Y_5 \\ H_{7v}+H_{7i}\,Y_5 \end{bmatrix} Q^{t}_{5'5}$$

From the discussions of equation (4.6), $\left[Q_{5'5}\ Q_{5'7}\right]$ is the cut-set matrix of the graph containing the passive elements and the current generators, and $Q_{5'5}$ is the cut-set matrix of the graph containing only the passive elements. Also Y_5 is a diagonal submatrix corresponding to the passive branches and $\left[H_{7v}+H_{7i}\,Y_5\right]$ is a rectangular submatrix which has only one non-zero element in any row.

Since the properties of det $H_0(D)$ and the node admittance matrix are being investigated, the driving terms play no part. Thus they can be removed in the conventional way, and no consideration will be given to them. For the remainder of this chapter sources will be included in the general term RLC networks.

The properties of the determinant of this node admittance matrix may now be investigated using topological formulas in terms of some trees of the network. The work assumes for simplicity that the passive subnetwork formed by removing all generators is connected. All results can be still applied if the passive submatrix is disconnected by regarding the trees as a set of disconnected trees, or a forest.

In order to expand the determinant of the node admittance matrix it is necessary to introduce the concept of an active tree and a passive tree.

A passive tree is a complete tree of the .

passive subnetwork.

An active tree is a complete tree of the network containing some controlled generators, chosen so that the passive elements of the tree and the controlling elements of the generators also form a passive tree.

Lemma 4.2

The terms of the expansion of the determinant of the node admittance matrix are the passive and active tree admittance products of the network.

Proof

Using the Binet-Cauchy theorem[35], the determinant of the admittance matrix, Δ_n, can be written as

$$\Delta_n = \sum_{p,q} Q_p \, Q_q \, Y_{pq} \qquad (4.10)$$

where Q_p and Q_q are the majors of $\left[Q_{5'5} \; Q_{5'7}\right]$ and $Q_{5'5}$ respectively, corresponding to column sets p and q. Y_{pq} is a minor of the matrix

$$Y = \begin{bmatrix} Y_5 \\ H_{7v} + H_{7i} \, Y_5 \end{bmatrix}$$

contained in rows p and columns q.

Since Y_5 is diagonal, each row of Y has one non-zero entry. Hence any non-zero minor will have a value equal to the product of the admittance terms contained in a set of rows p. Since each column contains a single passive branch element and perhaps some active elements controlled by this branch, then for Y_{pq} to be a non-zero minor, set q must correspond identically to a set p if set p is completely passive, or must correspond to the set p when each active term is replaced by its controlling element.

But any major of a fundamental cut-set matrix is non-singular if, and only if its columns correspond to a tree, when its value is $\pm$ 1. Hence sets p and q must both correspond to trees, q being a passive tree and p being either the same passive tree or a corresponding active tree. Hence the expansion consists of all the passive and active tree admittance products.

This result is similar to those of Percival,[51] Mayeda[46] and Coates[21], but a non-symmetrical approach has been used to preserve a simple topology which is useful in the later work.

4.4.3 The order of complexity

To determine the order of complexity det $H_0(D)$ must be investigated. After Laplace transformation this becomes det $H_0(S)$. The relationship between det $H_0(S)$ and the determinant of the node admittance matrix, Δ_n, is given by the following lemma.

Lemma 4.3

For an RLC network containing current generators controlled by the variables associated with the passive elements,

$$\det H_0(s) = \pm \left(\begin{matrix}\text{product of} \\ \text{all inductors}\end{matrix}\right) \left(\begin{matrix}\text{product of} \\ \text{all resistors}\end{matrix}\right) s^{\beta_L} \Delta_n$$

where β_L is the number of inductors in the network.

Proof

Let ρ be the rank and μ the nullity of the network. i.e. $\rho = \beta - \nu + 1$ and $\mu = \nu - 1$. Hence $\rho + \mu$ is the number of elements, β. Let Q_p be a major of Q consisting of a set of ρ columns, p. Hence $Q_p \neq 0$, if p corresponds to a tree.

Let B_p be the major of B consisting of a set of μ columns formed by the exclusion of the set of columns p. Hence $B_p \neq 0$ if p is a tree and the μ columns are a cotree. It can be shown[16] that

$$\det \begin{bmatrix} Q \\ B \end{bmatrix} = \pm \tau$$

where τ = number of trees of the network graph. Expanding the determinant by a Laplace expansion yields

$$\sum_{\text{all trees}} Q_p B_p (-1)^{\frac{1}{2}\mu(2\rho+\mu+1)+\sum \gamma_p}$$

where $\sum \gamma_p$ is the sum of all the indices of the columns of the cotree corresponding to tree set p. Since Q_p & B_p have values of ± 1, each set p must contribute a ± 1 term to the expansion, and, since only τ are available and the sum is $\pm \tau$, there can be no cancellation.

$$\text{i.e.} \quad B_p Q_p (-1)^{\frac{1}{2}\mu(2\rho+\mu+1)+\sum \gamma_p} = \epsilon$$

when ϵ is a constant $= \pm 1$ for all p. But $\frac{1}{2}\mu(2\rho + \mu + 1)$ is also a constant for all p, hence

$$B_p = \epsilon Q_p (-1)^{\sum \gamma_p} \tag{4.11}$$

for all p in the network.

Now equation (4.10) shows that

$$\sum_{p,q} Q_p Q_q Y_{pq} = \Delta_n \tag{4.12}$$

where q is a set of columns of Q chosen only from the columns corresponding to the passive elements.

Now

$$H_0(s) = \begin{bmatrix} B_5 & B_7 & B_9 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & Q_1 & Q_5 & Q_7 \\ Y_{C5} & \cdot & \cdot & \cdot & -Z_{LR5} & \cdot \\ H_{7v} & \cdot & \cdot & \cdot & H_{7i} & -U_7 \end{bmatrix}$$

Using theorem 4.1 the rows due to the fundament loops defined by current sources and the rows due to the fundamental cut sets defined by voltage sources can be written explicitly.

$$\text{i.e.} \quad H_0(S) = \begin{bmatrix} B_{95} & B_{97} & U_9 & \cdot & \cdot & \cdot \\ B_{5'5} & B_{5'7} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & U_1 & Q_{15} & Q_{17} \\ \cdot & \cdot & \cdot & \cdot & Q_{5'5} & Q_{5'7} \\ Y_{C5} & \cdot & \cdot & \cdot & -Z_{LR5} & \cdot \\ H_{7v} & \cdot & \cdot & \cdot & H_{7i} & -U_7 \end{bmatrix}$$

thus $\det H_0(s) = \begin{bmatrix} B_{5'5} & B_{5'7} & \cdot & \cdot \\ \cdot & \cdot & Q_{5'5} & Q_{5'7} \\ Y_{C5} & \cdot & -Z_{LR5} & \cdot \\ H_{7v} & \cdot & H_{7i} & -U_7 \end{bmatrix}$

Since Z_{LR5} is a diagonal submatrix, the terms of Z_{LR} may be taken out of the matrix as factors giving after slight manipulation.

$$\det H_0(s) = \begin{pmatrix}\text{product} \\ \text{of all } L_i S\end{pmatrix} \begin{pmatrix}\text{product} \\ \text{of all } R_j\end{pmatrix} \begin{bmatrix} B_{5'5} & B_{5'7} & \cdot & \cdot \\ \cdot & \cdot & Q_{5'5} & Q_{5'7} \\ Y_5 & \cdot & -U_5 & \cdot \\ (H_{7v}+H_{7i}Y_5) & \cdot & \cdot & -U_7 \end{bmatrix}$$

The determinant may now be expanded by a Laplace expansion, taking care over the signs. The expansion consists of the product of the cofactors of the first two sets of rows and the corresponding minors of the second two sets of rows. Because all entries under $B_{5'7}$ are zero, all non vanishing terms must be such that $B_{5'7}$ is included in the cofactor of the top rows.

Hence this cofactor is

$$Q_p B_q \, (-1)^{\frac{1}{2}\beta_1(1+\beta_1) + \sum p + \mu\beta_1 + \sum \gamma_q}$$

where β_1 is the number of rows in the first two sets of rows. i.e. the number of elements in the network less the number of sources,

$\sum p$ is the sum of the indices of the set of columns p

and $\sum \gamma_q$ is the sum of the indices of the set of columns left

when q is removed. Since there is a diagonal matrix in the second β_1 rows, the minor corresponding to the above cofactor is

$$K Y_{pq} (-1)^{\mu + \frac{1}{2}\mu(\mu+1) + \sum \gamma_p}$$

where $K = \pm 1$, i.e. a constant for all p and q. Thus the term may be written from equation (4.11), as

$$\epsilon K Q_p Q_q Y_{pq} (-1)^{2\sum \gamma_q + \sum p + \sum \gamma_p}$$

But $\sum p + \sum \gamma_p = \frac{1}{2} e (1+e)$ which is a constant, and $2 \sum \gamma_q$ is even. Hence the term is

$$\epsilon' Q_p Q_q Y_{pq}$$

and the determinant is $\epsilon' \sum_{p,q} Q_p Q_q Y_{pq}$

Comparing this with equation (4.12), shows that

$$\det H_0(s) = \pm \left\{ \begin{matrix} \text{product of} \\ \text{all } L_i S \end{matrix} \right\} \left\{ \begin{matrix} \text{product of} \\ \text{all } R_j \end{matrix} \right\} \Delta_n$$

$$\text{or } \det H_0(s) = \pm \left(\begin{matrix} \text{product of} \\ \text{all inductances} \end{matrix} \right) \left(\begin{matrix} \text{product of} \\ \text{all resistances} \end{matrix} \right) s^{\beta_L} \Delta_n$$

With these results it is now possible to expand det $H_0(S)$ and obtain the order of complexity. To do this it is necessary to use the concept of a normal tree. This will be defined here as a tree, either passive or active, chosen so that the difference between the number of capacitors and the number of inductors contained in it is maximized.

This definition differs from that used by Bryant[16] since he was able to show that it was possible in the RLC case to maximise the number of capacitors and minimize the number of inductors independently. This is not possible in the case of active elements.

Theorem 4.2

The order of complexity of an RLC network containing current generators, controlled by variables associated with passive elements, is given by the number of capacitors contained in the normal tree plus the number of inductors contained in its cotree.

Proof

The order of complexity is the highest index of s in the polynomial $\det H_0(s)$. From lemma 4.3,

$$\det H_0(s) = \Delta_n \cdot s^{\beta_L} \cdot \text{constant}.$$

Therefore highest index = β_L + highest index of Δ_n.

Since each term of Δ_n corresponds to either a passive or an active tree, the normal tree will give the maximum index, assuming no cancellations. This will be the number of capacitors minus the number of inductors in the tree. Hence the order of complexity

σ = number of capacitors in normal tree - number of inductors in normal tree + number of inductors in network.

= number of capacitors in normal tree + number of inductors in the cotree of the normal tree.

The advantage of using topological formulas can now be seen. No account has been taken of the possibility of cancellation of terms

in the expansion, although some tree admittance products may have negative signs. Cancellations can only affect the degree of the polynomial if special values are taken and so are a representation of the algebraic problem in the elimination procedure. By ignoring these cancellations we consider only the effects which are truly topological.

Although theorem 4.2 is a solution to the problem it is rather unsatisfactory, because it leaves the problem of finding the normal tree unsolved. In general it has not been possible to find an algorithm for determining such a tree because of the difficulty of including the restrictions on the nature of active trees.

An alternative approach to the problem is to consider the case when the order of complexity differs from that of the passive case. If the current generators are open circuited the order of complexity of the passive network can be determined from the passive normal tree as indicated in Chapter 3. This tree is also a tree of the active network and so contributes to the expansion. If the active network is to have a different order of complexity there are two possibilities.

(a) The contribution of this passive tree admittance product may be cancelled out by the contribution of some active tree admittance products. Since this depends on algebraic values, it is not considered.

(b) The normal tree of the whole network may be an active tree. This represents the topological problem.

It is possible to set restrictions on the network so that it is

always possible to select a passive normal tree.

Theorem 4.3

The order of complexity of an RLC network containing current generators, controlled by variables associated with passive elements, can be found from the passive RLC network, obtained by removing the sources and generators, if the network contains no current generator controlled by the voltage across an inductor, which lies in a cut-set containing only inductors, at least one generator and possibly some current sources.

Proof

The proof depends on showing that with the above restrictions it is always possible to find a passive tree whose tree admittance product has an index at least as great as that of any active tree.

Consider an active tree, containing in general both types of current generator. Then from the definition of an active tree there must be some passive tree corresponding to this active tree, sharing the same passive elements but containing, instead of the generators, their controlling elements. Hence the passive elements common to both trees will have the same contribution to the indices in s, and so will be disregarded by short circuiting them. This simplifies the network to one which contains an all generator tree, the corresponding all controlling element tree and some other passive elements corresponding to the links common to both trees in the original network. Any other generators have been removed, along

with any sources, which play no part in these results.

To give a current controlled current generator the dimension of admittance it is necessary to convert the controlling current to a voltage using the element equation for the controlling element. Since the parameter of the generator is a constant, the contribution of such a generator to the active tree admittance product differs only by a constant from the controlling elements contribution to the passive tree.

A current generator controlled by the voltage across a resistor will have the dimension of a conductance and so will have the same contribution to the active tree as its controlling element has to the passive tree.

In the case of a current generator controlled by the voltage across a capacitor, the generator does not contribute anything to the index of the active tree admittance product, but the capacitor contributes an increase in the passive index of one. Hence the inclusion of such a generator reduces the index by one, relative to the passive tree.

In fact the only possibility of getting a higher index arises when current generators controlled by voltages across inductors are included. Each generator will increase the index by one. Thus it is possible that the active tree has a higher index than the corresponding passive tree. But if such a generator is controlled by an inductor which is not in an all inductor cut-set, then it is possible to modify the passive tree by exchanging some non-inductor link for

the inductor, forming another tree. If a resistor is now included the index is equal to that of the active tree and if a capacitor is included it is one greater. Thus in general a passive tree admittance product of at least equal index can always be obtained if there are no generators controlled by voltages across inductors in an all indicator cut-set in the passive subnetwork.

The condition that each active tree must have a corresponding passive tree means that each controlling element must lie in a fundamental cut-set of the active tree defined by a generator of some kind. Hence it follows that the restrictions can be extended to say that the cut-set must contain only inductors and at least one current generator and possibly some current sources.

The restrictions of this theorem can be stated simply by means of an algorithm, based on the proof of the theorem.

(a) Short circuit all capacitors and resistors and voltage sources;

(b) Open circuit all current sources and inductors whose voltages do not control a generator.

Then the restriction is that the remaining network must contain no cut-set containing both generators and inductors.

The dual result can also be obtained for the case of voltage generators.

Theorem 4.4

The order of complexity of an RLC network containing voltage generators, controlled by the variables associated with passive elements,

can be found from the passive RLC network obtained by removing all the generators and sources, if the network contains no voltage generator controlled by the currents through a capacitor which lies in a loop containing only capacitors, at least one generator and possibly some voltage sources.

Again a simple algorithm exists for checking the restriction.

(a) Open circuit all inductors, resistors and current sources;

(b) Short circuit all voltage sources and capacitors whose voltages do not control a generator.

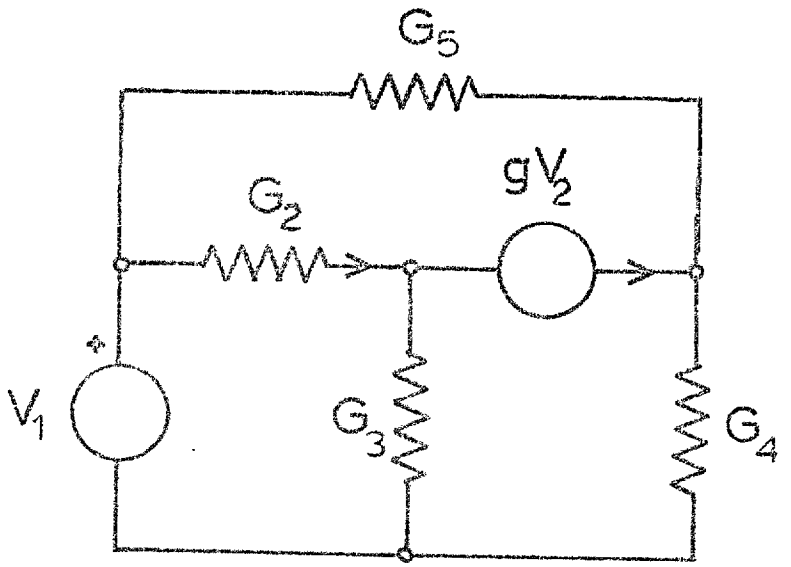
Then the restriction is that the remaining network must contain no loops containing both generators and capacitors.

4.4.4 Examples

(a) It can be seen that the example of Fig. 4.3 is a network which violates theorem 4.3 since there is a cut-set containing 2 inductors and a current generator $\{ L_4, L_5\ g \}$ and v_4 controls the current generator. This example is not as general as possible, since both the generator and L_4 lie in the same cut set. In fact the critical fact is that v_4 lies in an inductor and generator cut set.

(b) A second example is given in Fig. 4.5. Here the conditions of theorem 4.3 are violated by two generators, J_1 and J_3, since L_4 lies in cut set $\{ L_3\ L_4\ J_5 \}$ and L_2 lies in $\{ J_2\ L_2\ L_3 \}$.

The type of generator which lies in the cut-set is not important, in this case one is current controlled and the other is voltage controlled. The order of complexity can be found using theorem 4.2 and is 4. The order of complexity of the RLC subnetwork is 3. Although in this case the order of complexity is higher in the active network this is not necessarily so. Theorem 4.3 only states a sufficient condition for the orders of complexity to be equal, not a necessary one. In fact if generator J_5 is removed the cut set $\{J_2\ L_2\ L_3\}$ still remains and the conditions of theorem 4.3 are still violated, but the order of complexity of the active network falls to 3.



Fig,4.1. Example of section 4.2.

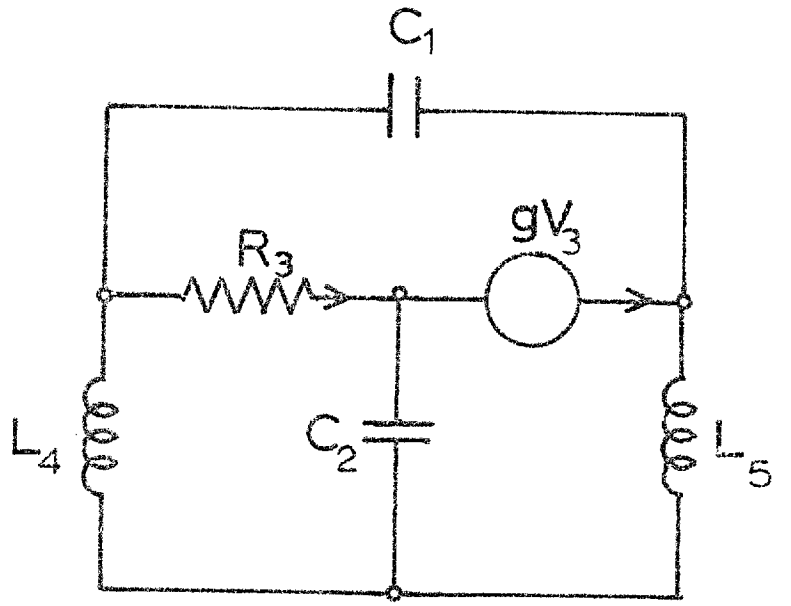


Fig.4.2. Example of section 4.2.

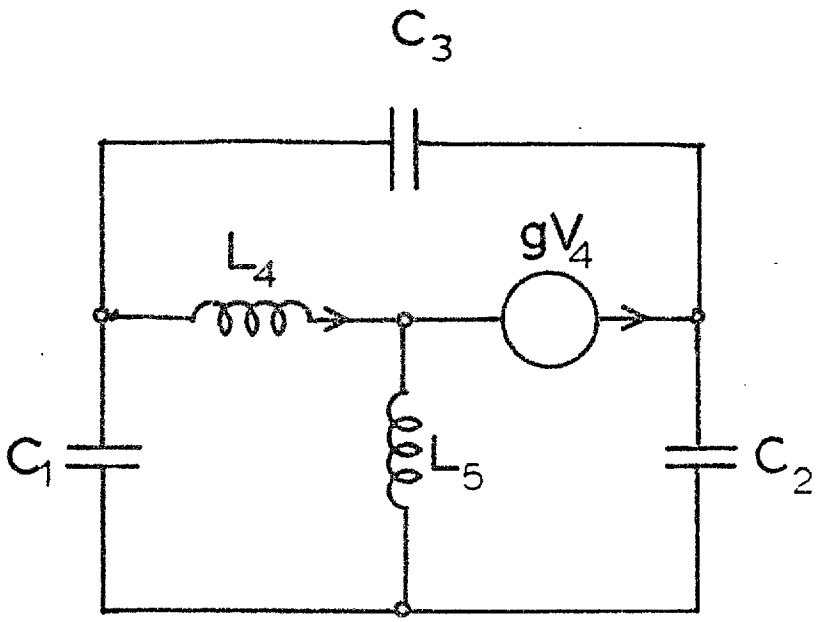


Fig.4.3. Example of section 4.2.

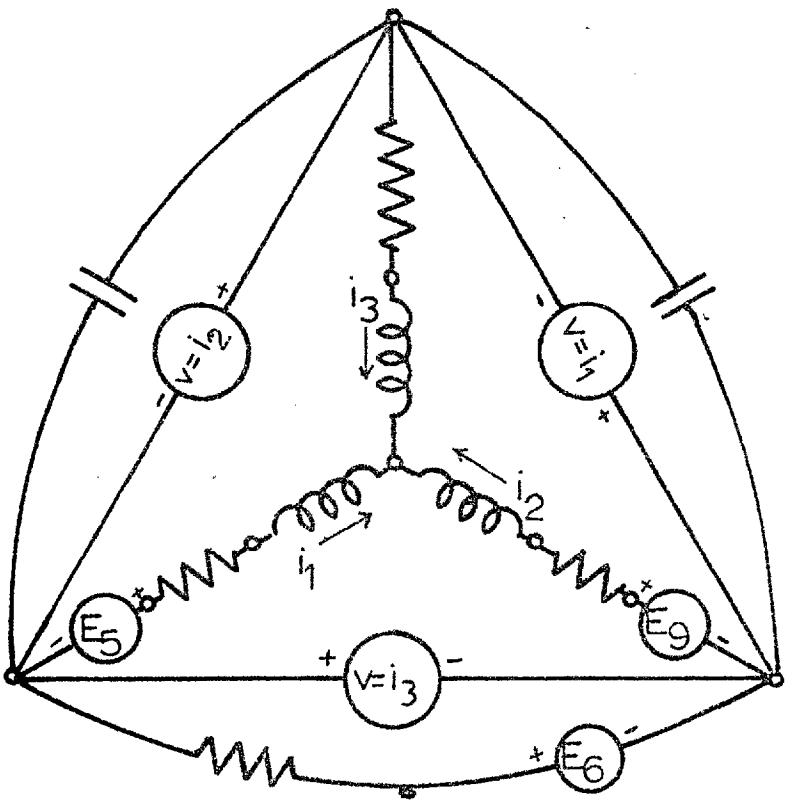
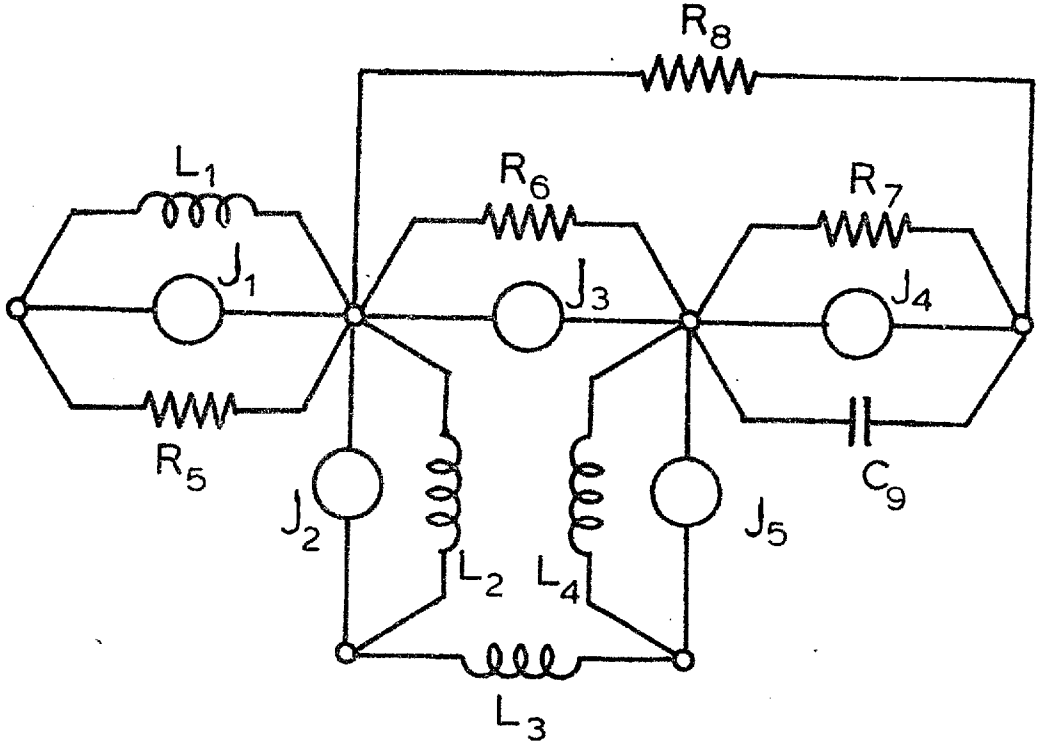


Fig.4.4. Declaris and Saeks' example.



$$J_1 = g_1 V_4$$

$$J_2 = g_2 V_9$$

$$J_3 = g_3 V_2$$

$$J_4 = g_4 V_1$$

$$J_5 = \alpha_5 I_6$$

Fig.4.5. Example of section 4.4.4.

Chapter 5

THE FORMATION OF THE STATE EQUATIONS FOR AN ACTIVE NETWORK

5.1 INTRODUCTION

As was pointed out in the last chapter the problem of forming the state equations for an active network is a complex one. It involves both topological and algebraic questions. The topological problems were considered in chapter 4, for a restricted class of networks. The algebraic problems involve not only the order of complexity of the network, but also the actual existence of the state equations, which is equivalent to the existence of a unique solution to the network.

A number of workers have considered these problems but their solutions have not been very general. Roe[58] gave an early formulation but did not consider the possibility that the order of complexity could be above that of the passive network, or that the various inverse matrices required might not exist. More recent contributions by Rane and Lang[56] and Declaris and Saeks[23] both make restrictions on the positioning of the controlled generators which are similar to those mentioned in Theorems 4.3 and 4.4 of the last Chapter. By making these restrictions the problem of the order of complexity rising above its passive counterpart appears to have been eliminated, but in the actual elimination procedure again the non-singularity of various matrices is assumed. Dervisoglu[25] has given the only

treatment which considers the problem with no algebraic assumptions and has given an algorithm for forming the equations for a quite general class of networks directly from the branch equations describing the network.

Here, an approach to the problem is proposed based upon the results given previously for passive networks, viewing the active network as an RLC multiport network terminated by active elements. The active elements are regarded as controlled generators. The most general definition of a controlled generator, given in section (2.23) is taken here.

Since the algebraic aspects of the problem are now considered Theorem 4.1 must be extended to cover the algebraic properties of the network. Some conditions must be placed upon the element values to ensure that a unique solution exists to the network.

5.2 THE EXISTENCE OF A UNIQUE SOLUTION TO THE NETWORK

5.2.1 Replacement sources

The approach taken here is to convert the active network to a passive one, by converting all the generators to uncontrolled sources. Following the example of Malik and Hale[45], these uncontrolled sources will be termed "replacement sources".

Theorem 4.1 has given a necessary condition for the network to have a solution. These conditions mean that there are no all voltage

source loops and no all current source cut-sets. It does not say that there are no all voltage source and generator loops or all current source and generator cut-sets, but if such do exist, then the current in an element of such a loop, or the voltage across an element of such a cut-set, controls some generator of the network. From these results it is clear that if, in an active network, all voltage generators are replaced by voltage sources and all current generators are replaced by current sources, then some all voltage source loops or all current cut sets may result. In the case of a voltage source loop, this may be avoided by changing one of the voltage replacement sources to a current replacement source. Similarly in the case of a current source cut-set, this may be avoided by changing one of the current replacement sources to a voltage replacement source. These sources will be called exchanged replacement sources when it is needed to distinguish them from the ordinary replacement sources.

By this process, the resulting network is an RLC network with current and voltage sources positioned in such a way that lemma 2.1 is satisfied. In Chapter 3, it was shown that such a network always has a state model, which may be written as

$$\dot{x} = \underline{A}x + \underline{B}u + \underline{C}\dot{u} \qquad (5.1)$$

where x is the state vector and u is the input vector, corresponding to u_1 the replacement source vector and u_2 the input source vector. u_1 is obtained from $\underline{u}_1$ the generator vector, by regarding the elements as sources and making the appropriate changes for the exchanged

replacement sources. It is a property of the state equations that all the other variables in the network are fixed by x, u and $\dot{u}$. This includes the variables, w, which control the generators by the generator equation

$$\underline{u}_1 = Hw \tag{5.2}$$

where H is the hybrid matrix representing the generator terms. Thus w can be written in the form

$$w = \underline{P}x + \underline{Q}u + \underline{R}\dot{u} \tag{5.3}$$

Now since the only difference between u_1 and $\underline{u}_1$ is in the cases of loops of voltages sources and generators, and in the cases of cut-sets of current sources and generators, they can be related by Kirchhoff's laws. For example, in the case of a loop of voltage sources and generators, one voltage generator is replaced by a current source, and its voltage is given by application of Kirchhoff's voltage law around the loop. Thus the voltage is a linear function of the remaining source voltages.

Similarly, if a replacement current source is changed to a voltage source because it is in a current source cut-set, then the current in it is given as a linear function of the other currents.

Thus if u_1 is ordered with all exchange replacement sources at the end of the vector, the generator vector can be related to the source and replacement source vector by

$$\underline{u}_1 = Mu = M_1u_1 + M_2u_2 \tag{5.4}$$

where M_1 has the form $\begin{bmatrix} U & 0 \\ M_{21} & 0 \end{bmatrix}$ and where U is a

unit matrix and M_{21} is the matrix corresponding to the constraints due to Kirchhoff's laws amongst the sources, replacement sources and exchanged replacement sources and

$$M_2 = \begin{bmatrix} 0 \\ M_{22} \end{bmatrix}$$

5.2.2 The conditions for the existence of a unique solution

The generator equation, (5.2) can now be written in the form,

$$Mu = H \left\{ \underline{P}x + \underline{Q}u + \underline{R}\dot{u} \right\}$$

or $$H\underline{R}_1\dot{u}_1 = - H\underline{P}x + (M_1 - H\underline{Q}_1)u_1 + (M_2 - H\underline{Q}_2)u_2 - H\underline{R}_2\dot{u}_2$$

For the network to possess a unique solution the system of equations

$$\dot{x} = \underline{A}x + \underline{B}_1u_1 + \underline{B}_2u_2 + \underline{C}_1\dot{u}_1 + \underline{C}_2\dot{u}_2$$

and $$H\underline{R}_1\dot{u}_1 = - H\underline{P}x + (M_1 - H\underline{Q}_1)u_1 + (M_2 - H\underline{Q}_2)u_2 - H\underline{R}_2\dot{u}_2$$

must have a unique solution.

These can be written as

$$\begin{bmatrix} U & -\underline{C}_1 \\ 0 & H\underline{R}_1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u}_1 \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B}_1 \\ -H\underline{P} & (M_1 - H\underline{Q}_1) \end{bmatrix} \begin{bmatrix} x_1 \\ u_1 \end{bmatrix} + \begin{bmatrix} \underline{B}_2 \\ (M_2 - H\underline{Q}_2) \end{bmatrix} u_2 + \begin{bmatrix} \underline{C}_2 \\ -H\underline{R}_2 \end{bmatrix} \dot{u}_2 \qquad (5.5)$$

In the appendix it is shown that such a set of first order equations possesses a unique solution if and only if the polynomial matrix

$$\begin{bmatrix} U & -\underline{C}_1 \\ 0 & H\underline{R}_1 \end{bmatrix} \lambda - \begin{bmatrix} \underline{A} & \underline{B}_1 \\ -H\underline{P} & (M_1 - H\underline{Q}_1) \end{bmatrix} \qquad (5.6)$$

is non-singular and u_2 and all higher order differentials are continuous.

This can be expressed in the following theorem.

Theorem 5.1

A RLC network containing voltage and current sources and voltage and current generators of all types possesses a unique solution if and only if

(a) there exists a tree such that all voltage sources may be regarded as tree branches and all current sources may be regarded as links;

(b) the matrix

$$\begin{bmatrix} U\lambda - \underline{A} & -(\underline{C}_1\lambda + \underline{B}_1) \\ -H\underline{P} & (H\underline{R}_1\lambda + H\underline{Q}_1 - M_1) \end{bmatrix} \qquad (5.7)$$

is non-singular.

So far in this section, only the first half of theorem 4.1 has been used. The question of which variables control which generators has not been considered. Because of this, theorem 5.1 does not include the second, necessary condition of theorem 4.1. It will now be shown that such a condition is implicit in the second condition of theorem 5.1.

In order to show this, consider an exchanged replacement current source in an otherwise all voltage source and replacement voltage source loop. An inspection of the explicit form for the state equations, equation (3.8) shows that such a source makes no contribution

to the differential equations and consequently there is a corresponding column of zeros in matrices $\underline{B}_1$ and $\underline{C}_1$ above. Inspection of equation (3.5) in the derivation of the state equations shows that such a source makes no contribution to either the resistor voltages and currents, the currents in the capacitors or voltages in inductors. The currents in inductors and voltages in capacitors are of course state variables or linear combinations of state variables, and sources, so again the exchanged replacement source makes no contribution. The equation for the multiport network, equation (3.9), shows that the only variables to which this source makes a contribution are the currents in the voltage sources lying in the voltage source loops with the source. This arises from the Q_{ej} term in equation (3.9).

Now consider the equation for the controlling variables, y. There is a zero column in $\underline{R}_1$ corresponding to this exchanged replacement source, and there will only be a contribution in $\underline{Q}_1$ if vector y contains a current in the exchanged replacement source, or another of the currents in the voltage sources and replacement sources contained in the loop with the source.

Since the source is an exchanged replacement source, M_1 has a column of zeros corresponding to it. Therefore there is a column of zeros in the matrix (5.7) corresponding to this exchanged replacement source unless the current in one of the branches of the loop containing the voltage sources and voltage generators is an element of y, and therefore controls some generator.

Similar reasoning shows that any cut set of all current sources and current generators also implies a column of zeros in matrix (5.7) unless the voltage across one of these elements controls a generator of the network. These results imply the existence of a tree as given in the second condition of theorem 4.1.

An example where the second half of theorem 4.1 is violated is Declaris and Saeks' example, shown in Fig. 4.4. In this case one of the generators must be regarded as a replacement current source, and since no currents in the loop of generators control any of the generators in the network, this results in a column of zeros in matrix (5.7).

In fact in this example there are 3 equations in the second set of equations of equation (5.5), with zeros in the columns corresponding to the exchanged replacement source. By correct choice of the element values it is possible to make these 3 equations linearly dependent and so they can be reduced to 2, ignoring the question of solving for the exchanged replacement source current. It turns out that the remaining equations can then be solved uniquely and Declaris and Saeks claim that the network is then solvable. As was explained in section 4.3 this differs from the definition of solvability used here, since the exchanged replacement source current cannot be calculated. This current is of course equivalent to the current flowing through one of the generators.

5.2.3 Physical interpretation

Some physical understanding can be given to the second condition of theorem 5.1. Consider the network formed by removing all the sources and replacing all the generators by the appropriate replacement sources. The equations (5.1) and (5.3) now become

$$\dot{x} = \underline{A}x + \underline{B}_1 u_1 + \underline{C}_1 \dot{u}_1$$

$$y = \underline{P}x + \underline{Q}_1 u_1 + \underline{R}_1 \dot{u}_1$$

If a Laplace transformation is taken the equations become

$$(sU-\underline{A})x = (\underline{B}_1 + \underline{C}_1 s)\, u_1$$

and

$$y = \underline{P}x + (\underline{Q}_1 + \underline{R}_1 s)u_1$$

The vector x can now be eliminated to give a relationship between the controlling vector y and the replacement source vector u_1

$$y = \left[\underline{P}(sU-\underline{A})^{-1} (\underline{B}_1 + \underline{C}_1 s) + (\underline{Q}_1 + \underline{R}_1 s) \right] u_1$$

where the transfer function matrix can be given as

$$F(s) = \underline{P}(sU-\underline{A})^{-1} (\underline{B}_1 + \underline{C}_1 s) + (\underline{Q}_1 + \underline{R}_1 s)$$

The determinant of matrix (5.7) is

$$\det \begin{bmatrix} (U\lambda-\underline{A}) & -(\underline{C}_1\lambda + \underline{B}_1) \\ H\underline{P} & (H\underline{R}_1\lambda + H\underline{Q}_1 - M_1) \end{bmatrix}$$

Since $(U\lambda-A)$ is non-singular then the determinant may be written as

$$\det (U\lambda-\underline{A}) \det (H\underline{P}(U\lambda-\underline{A})^{-1}(\underline{C}_1\lambda+\underline{B}_1) + H\underline{R}_1\lambda + H\underline{Q}_1-M_1)$$

$$= \det (U\lambda-\underline{A}) \det (HF(\lambda) - M_1)$$

The condition for there to exist a unique solution is that this determinant is non-zero.

i.e. $\det(HF(\lambda) - M_1) \neq 0$

Since H and M_1 do not depend on λ, this can be rewritten as

$$\det (HF(s) - M_1) \neq 0 \qquad (5.8)$$

Thus the solvability condition can be written in the form of the following theorem.

Theorem 5.2

An RLC network containing voltage and current sources and voltage and current generators of all types possesses a unique solution if, and only if

(a) there exists a tree such that the voltage sources may be regarded as the tree branches and the current sources as the links;

(b) $\det (HF(s) - M_1) \neq 0$

where F(s) is the transfer function relating the controlling variables to the replacement source variables, H is the generator matrix and M_1 is defined by equation (5.4).

This result can be simplified in the case when there are no voltage generator and source loops or current generator and source cut sets. In this case there are no exchanged replacement sources and M_1 becomes a unit matrix. This special case was recently given by Malik and Hale,[45] except that, as they themselves pointed out, their proof relies on Laplace transforms and it is not possible to prove the sufficiency of the conditions. In the case of theorems 5.1 and

5.2 it has been assumed here only that the input source functions and all their higher order differentials are continuous.

5.2.4 An example

Consider the example shown in Fig. 5.1. In this case there is a loop of voltage sources and generators. Thus generator 2 can be taken as a voltage source and generator 3 as a current source. Then the state equation is

$$\dot{v}_4 = - C_4^{-1} g_5 v_4 + C_4^{-1} g_5 v_2$$

and the controlling variables are given by

$$\begin{bmatrix} i_3 \\ i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} v_4 + \begin{bmatrix} 0 & 0 & 1 \\ g_6 & -g_6 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix}$$

But $$\begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_1 + \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ i_3 \end{bmatrix}$$

$$= \begin{bmatrix} r_2 & \\ & r_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ g_6 & -g_6 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 & -1 & r_2 \\ (r_3 g_6 - 1) & (1 - r_3 g_6) & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_3 \end{bmatrix}$$

The equations of the network are

$$\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_4 \\ \dot{v}_2 \\ \dot{i}_3 \end{bmatrix} = \begin{bmatrix} -C_4^{-1}g_5 & C_4^{-1}g_5 & 0 \\ 0 & -1 & r_2 \\ 0 & (1-r_3g_6) & 0 \end{bmatrix} \begin{bmatrix} v_4 \\ v_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ (r_3g_6-1) \end{bmatrix} v_1$$

Thus the network has a solution if

$$\det \begin{bmatrix} s + C_4^{-1}g_5 & -C_4^{-1}g_5 & 0 \\ 0 & +1 & -r_2 \\ 0 & -(1-r_3g_6) & 0 \end{bmatrix} \neq 0$$

i.e. if $r_2(1-r_3g_6)(s + C_4^{-1} g_5) \neq 0$

i.e. if $r_3 \neq \frac{1}{g_6}$

If $g_6 \neq \frac{1}{r_3}$ then the state equations become simply

$$\dot{v}_4 = - C_4^{-1}g_5v_4 + C_4^{-1} g_5v_1$$

N. B. The condition $r_3 \neq \frac{1}{g_6}$ implies that $v_3=0$

5.3 THE FORMATION OF THE STATE EQUATIONS

5.3.1 Classical method

It is shown in the appendix, that a set of equations of the form

$$Ax + B\dot{x} = f(t)$$

can be transformed into the canonic form

$$\begin{bmatrix} J & \\ & U_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} U_1 & \\ & E_2 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \quad (5.9)$$

by a non singular constant transformation, where J is the direct sum of Jordan blocks of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & \cdot & \cdot & 0 \\ \cdot & \lambda_i & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \lambda_i \end{bmatrix}$$

corresponding to the finite elementary divisors of $(A + B\lambda)$, $(\lambda-\lambda_1)^p$ and E is the direct sum of matrices of the form

$$E_i = \begin{bmatrix} 0 & 1 & \cdot & \cdot & 0 \\ \cdot & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

corresponding to the infinite elementary divisors of $(A + B\lambda)$ and U_1 and U_2 are unit matrices.

The second set of equations can be used to solve algebraically for y_2.

i.e. the equations from

$$U_i\, y_{(i)} + E_i \dot{y}_{(i)} = f_{(i)}(t)$$

are

$$y_{(i)_1} + \frac{d}{dt} y_{(i)_2} = f_{(i)_1}$$

$$y_{(i)_2} + \frac{d}{dt} y_{(i)_3} = f_{(i)_2}$$

$$\cdot \qquad\qquad \cdot$$

$$\cdot \qquad\qquad \cdot$$

$$y_{(i)_n} + \qquad 0 \qquad = f_{(i)_n}$$

$$\text{i.e.} \quad y_{(i)_n} = f_{(i)_n}$$

$$y_{(i)_{n-1}} = f_{(i)_{n-1}} - \frac{d}{dt} f_{(i)_n}$$

$$\text{and} \quad y_{(i)_j} = f_{(i)_j} + \sum_{k=1}^{n-j} (-1)^k \frac{d^k}{dt^k} f_{(i)_{k+j}}$$

The solutions for such a block involve in general n-1 differentials of the driving functions f(t), where n is the order of the block. The orders of these blocks are the indices of the infinite elementary divisors. Thus the number of differentials needed depends upon the highest index of the infinite elementary divisors.

In the case of equation (5.5), the driving terms are functions of u_2 and $\dot{u}_2$, thus in general the number of differentials of u_2 that are needed correspond to the highest index of the infinite elementary divisions of the pencil of matrices (5.6).

Thus in principle, the state equations can always be found by taking equation (5.5) to the canonic form shown in equation (5.9).

5.3.2 Dervisoglu's algorithm

Dervisoglu suggested an alternative approach to the problem[25], by using a straightforward elimination procedure in an iterative manner to give the equations without transforming the variables.

The procedure starts by finding the number of independent rows, r_1, of B. It is then possible to reduce the remaining $n-r_1$ rows of B to zero rows, giving $n-r_1$ algebraic equations from A, relating the variables of x. Since $\det(B\lambda+A)\neq 0$, then there must be a minor of full rank in these $n-r_1$ rows of A, and so $n-r_1$ variables can be solved for to give

$$x_1 = C\,x_2 + f_1(t)$$

These variables can be eliminated from the remaining differential equations using this equation and its differential,

$$\dot{x}_1 = C\dot{x}_2 + \dot{f}_1(t)$$

to give $A_2x_2 + B_2\dot{x}_2 = F\,f(t) + F_1\dot{f}(t)$

This process can be repeated each time reducing the size of the x vector until eventually $\det B_{(i)} \neq 0$ and so the equations

$$A_{(i)}\,x_{(i)} + B_{(i)}\dot{x}_{(i)} = Ff + F_1\dot{f} + F_2\ddot{f} + \cdots F_{(i-1)}f^{(i-1)}$$

and so $$\dot{x}_{(i)} = -B_{(i)}^{-1}\left[A_{(i)}x_{(i)} + Ff + F_1\dot{f} + F_2\ddot{f} + \cdots F_{(i-1)}f^{(i-1)}\right]$$

where $$f^{(i)} = \frac{d^i}{dt^i} f.$$

Since $\det (A+B\lambda)\neq 0$ the process will always terminate, either with $\det B \neq 0$ or $B \equiv 0$, in which case the equations are all algebraic. Thus the equations can always be reduced by this method.

In Chapter 3, it can be seen that the process used for reducing the equations (3.5) to the explicit form can be regarded as a simple

application of this algorithm, which terminates after one cycle. Similarly the condition of theorem 3.1 in the case of negative resistors is a condition that the elimination terminates after one cycle.

Dervisoglu suggested this method for forming the state equations starting with the set of all equations for the network, shown in equation (4.5). By doing this very little can be discovered about the nature of the solvability conditions and the order of complexity.[25]

5.4 THE ORDER OF COMPLEXITY OF AN ACTIVE NETWORK

5.4.1 General results

The number of independent first order differential equations in equation (5.5) is given by the number of finite elementary divisors of the pencil of matrices (5.6). From the appendix it is clear that the number of finite elementary divisors is equal to the degree of the determinant polynomial of the pencil of matrices. This can be expressed in the theorem.

Theorem 5.3

The order of complexity of an RLC active network containing all types of generators is given by the degree of the determinant polynomial of the matrix

$$\begin{bmatrix} U\lambda - \underline{A} & -(\underline{C}_1\lambda + \underline{B}_1) \\ H\underline{P} & (H\underline{R}_1\lambda + H\underline{Q}_1 - M_1) \end{bmatrix}$$

It does not seem possible to make a more positive statement about the order of complexity than this, but it is possible to give some bounds on it.

The order of complexity cannot be greater than the number of equations in equation (5.5), i.e the sum of the orders of the vectors x and u_1. In fact, by taking the determinant of the matrix (5.6), it can be seen that the order of complexity cannot be greater than the rank of the left hand matrix. This is given by the dimension of x plus $\rho(H\underline{R}_1)$.

The dimension of x is the order of complexity of the passive network obtained by replacing the generators by the appropriate replacement sources. Since the first step in obtaining the order of complexity is to remove all sources, and replacement sources, then the dimension of x is the same as the dimension of the network formed by removing all the sources and generators directly.

From the discussion following Theorem 5.1, the columns of $\underline{R}_1$ corresponding to exchanged replacement sources contain no non-zero elements. Thus the rank of $H\underline{R}_1$ is no greater than the number of non-exchanged replacement sources. Thus an upper bound on the order of complexity is the order of complexity of the network obtained by removing all sources and generators, plus the number of generators, less the number of all voltage generator and source loops less the number of all current generator and source cut-sets.

This bound is rather high in practice because it is unlikely that $\rho(H\underline{R}_1)$ will ever reach this number, since only in a few special

cases will there be any non-zero elements in $\underline{R}_1$. It can be seen directly from equation (4.5) that the highest possible order of complexity is equal to the number of reactive elements. Normally the number of degeneracies in the passive network will be small, so that the $\rho(H\underline{R}_1)$ must be very small.

5.4.2 The properties of matrix $\underline{R}_1$

The equations for the network can be written from equations (5.1) and (5.3) in the form

$$\dot{x} = \underline{A}x + \begin{bmatrix} \underline{B}_1 & \underline{B}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \underline{C}_1 & \underline{C}_2 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \qquad (5.10)$$

$$y = \underline{P}x + \begin{bmatrix} \underline{Q}_1 & \underline{Q}_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \underline{R}_1 & \underline{R}_2 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \qquad (5.11)$$

Now if an element of y is as an element of x, u_1 or u_2, then it will be directly related algebraically to x, u_1 u_2 and so gives no non-zero contribution to $\underline{R}_1$. If y is a capacitor voltage or an inductor current not in x, then it is related to them by Kirchhoff's laws and is an algebraic function of x, u_1 and u_2, again giving no non-zero contributions to $\underline{R}_1$.

Also $\underline{u}_1 = M_1 u_1 + M_2 u_2$

so that $\underline{u}_1$ also gives no contributions to $\underline{R}_1$.

From equation (3.5) it can be seen that v_{bg} and $i_{\ell r}$ can be

directly related to x, u_1 and u_2 and i_{bg} and $v_{\ell r}$ can be related through Ohm's law. $i_{\ell g}$ and v_{br} can be obtained from equations (3.2) and (3.3) and $v_{\ell g}$ and i_{br} from Ohms law. Thus all resistor currents and voltages can be obtained in terms of x, u_1 and u_2 and make no contributions to $\underline{R}_1$.

There are now only two cases, and their duals left to consider. These are the currents in capacitors (and the voltages in inductors) and the currents in voltage sources and replacement sources (and voltages in current sources and replacement sources).

In the case of currents in a capacitor defining a state variable, then the current can be obtained directly by

$$i_C = \text{capacitance} \times \dot{v}_C$$

where $\dot{v}_C$ is the differential of a state variable, given by equation (3.8) in terms of x, u_1, u_2, $\dot{u}_1$, and $\dot{u}_2$. Thus $\dot{u}_1$ terms may now be involved, giving a contribution to $\underline{R}_1$. From equation (3.8) it can be seen that

$$\left[\underline{C}_1 \; \underline{C}_2\right] = (C_b + Q_{CC} C_\ell Q_{CC}^t)^{-1} Q_{CC} C_\ell B_{Ce}$$

and so $\underline{C}_1$ will consist of a submatrix of this matrix corresponding to the division of the matrix B_{Ce} into u_1 and u_2. For convenience, the distinction between sources and replacement sources will be dropped.

If $\{A\}_{ij}$ is the ij th element of A_1 then,

Lemma 5.1 $\{ (C_b + Q_{CC} C_\ell Q_{CC}^t)^{-1} \}_{ij}$ is non-zero only if the tree branch capacitors i and j lie in an all capacitor and voltage source loop.

Proof This can be proved from the fact that $(C_b + Q_{CC} C_\ell Q_{CC}^t)$ is the node admittance matrix of the capacitor formed by short circuiting all voltage sources and open circuiting all other non-capacitor branches. Then

$\left\{(C_b + Q_{CC} C_\ell Q_{CC}^t)^{-1}\right\}_{ij}$ may be taken as Z_{ij}, the transfer impedance relating the voltage across tree branch capacitor C_i with the current from a current source across tree branch capacitor C_j.

Seshu and Reed[62] have shown that

$$Z_{ij} = \frac{\sum (T_{2_{12,1'2}} \text{ products}) - \sum (T_{2_{12',1'2}} \text{ products})}{\sum T_1 \text{ products}}$$

where T is a tree, T_2 a two tree and 1, 2, 1' and 2' the nodes shown in Fig. 5.2.

The existence of a 2-tree, either $T_{2_{12,1'2'}}$ or $T_{2_{12',1'2}}$ means that C_i and C_j lie in a loop since the 2-tree can be combined with C_i and C_j to form such a loop. Hence $\left\{(C_b + Q_{CC} C_\ell Q_{CC}^t)^{-1}\right\}_{ij} \neq 0$ means that tree branches C_i and C_j lie in an all capacitor loop of the capacitor subgraph. Hence the lemma follows.

By inspection, $\left\{ Q_{CC} C_\ell B_{Ce} \right\}_{jk} \neq 0$ if a capacitor tree branch j is in an all capacitor and voltage source loop with the source k.

Hence if $\left\{ (C_b + Q_{CC} C_\ell Q_{CC}^t)^{-1} Q_{CC} C B_{Ce} \right\}_{jk} \neq 0$ then a tree capacitor i lies in an all capacitor and source loop with tree capacitor j, which also lies in an all capacitor and source loop with source k. If capacitor j is common to two such loops, then

there is a similar loop containing both tree capacitor i and source k.

Thus if $\{[\underline{C}_1\ \underline{C}_2]\}_{ik} \neq 0$, then there is a loop containing only capacitors and sources but containing capacitor i and source k. If source k is a replacement source, then there is a contribution to $\underline{C}_1$. Thus it has been shown that the choice of tree capacitor current as a controlling variable cannot give a contribution to $\underline{R}_1$, unless there exists an all capacitor and voltage source loop, containing the capacitor and some replacement voltage source, (and therefore a voltage generator of the original network).

If the capacitor does not define a state variable voltage, then it must lie in a loop containing only capacitors and voltage sources. Thus differentiating the Kirchhoff's voltage law equation for the loop shows again that the current will only give a contribution to $\underline{R}_1$ if the capacitor lies in a loop containing only capacitors, perhaps voltage sources and at least one replacement source.

Similar reasoning shows that the voltage across an inductor only gives a contribution to $\underline{R}_1$ if the inductor lies in a cut-set containing only inductors, possibly current sources, and at least one replacement source.

In the case when a current through a voltage source, replacement or otherwise, is considered, the contribution can be seen from equation (3.9) to be

$$Q_{eC}C_{\ell}B_{Ce} + Q_{eC}C_{\ell}B_{CC}(C_b + Q_{CC}C_{\ell}Q_{CC}^{t})^{-1}Q_{CC}C_{\ell}B_{Ce}$$

If $\{Q_{eC}C_{\ell}B_{CC}\}_{ji} \neq 0$ then there exist an all capacitor and source loop containing tree capacitor i and source j.

If $\left\{ Q_{eC} C_{\ell} B_{CC} (C_b + Q_{CC} C_{\ell}^t Q_{CC}^t)^{-1} Q_{CC} C_{\ell} B_{Ce} \right\}_{ik} \neq 0$ then there exist an all capacitor and voltage source loop containing tree capacitor i and voltage source j and also a similar loop containing tree capacitor i and voltage source k. Thus there exists a capacitor and source loop containing both sources j and k a tree capacitor.

But if $\left\{ Q_{eC} C_{\ell} B_{Ce} \right\}_{jk} \neq 0$ then there exist a capacitor and voltage source loop containing sources j and k and a link capacitor. Hence no tree capacitors are required.

Thus, in general, a non-zero jk term arises if there is a loop containing only capacitors, perhaps some sources and the jth and kth voltage sources. A contribution to matrix $\underline{R}_1$ may thus occur if the current in a particular voltage source or replacement voltage source, if there exist a capacitor path and voltage source loop containing this source and a voltage replacement source. This source could, of course, be the voltage replacement source.

The dual applies to the case of current sources. These results can be expressed in the form of a lemma.

Lemma 5.2

If $\left\{ \underline{R}_1 \right\}_{ij} \neq 0$ in equation (5.3) then either

(a) if y_i is a current variable, it is associated with an element in a loop of the network containing only capacitors, perhaps some voltage sources and at least one replacement voltage source; or

(b) if y_i is a voltage variable, it is associated with an element in a cut-set of the network containing only inductors, perhaps some current sources, and at least one replacement current source.

It should be noted that the converse is not necessarily true, since the converse of lemma 5.1 is not true. This is because Z_{ij} can be zero by the cancellation of terms.

Lemma 5.2 gives necessary conditions which must occur before $\{ \underline{R}_1 \}_{ij} \neq 0$ and so before the $\rho(H\underline{R}_1) > 0$. Thus it gives necessary conditions for the order of complexity to rise above the passive order of complexity, the dimension of vector x.

It can be seen that the conditions of Theorems 4.3 and 4.4 are sufficient conditions to ensure that $\underline{R}_1 = 0$. The topological conditions of the last chapter amount to the condition that the elimination procedure starts with the rank of the left hand matrix of equation (5.5) equal to the passive order of complexity. Thus any algebraic considerations can only reduce the order of complexity below the passive one.

Lemma 5.2 applies to a completely general class of networks and so the results of Theorems 4.3 and 4.4 can now be generalized.

Theorem 5.4

The order of complexity of a RLC network containing sources and generators of all types is not greater than the order of complexity of the RLC network obtained by removing all sources and generators, provided the generators are not controlled by

(a) a current associated with an element contained in a loop of the network containing only capacitors, possibly some voltage sources and at least one voltage generator;

(b) a voltage associated with an element contained in a cut-set of

the network containing only inductors, possibly some current sources and at least one current generator.

5.5 AN EXAMPLE

Consider the circuit shown in Fig. 5.3. It is assumed for simplicity that all elements have unity value and the generators are replaced by the equations

$$v_1 = i_3$$
$$v_2 = i_5$$
$$i_9 = v_6$$

It can be seen that this network violates condition (a) of theorem 5.4 since loop $\{234\}$ contains only capacitors and voltage generators and current i_3 controls as generators. Thus the order of complexity may be greater than the order of complexity of the network, with all sources and generators removed, which is 2. Note also that the cut-set $\{7,8,9\}$ contains inductors and a current generator, but this does not violate the conditions of Theorem 5.4 since none of the voltages control a generator.

If the generators are now regarded as sources, there are no loops or cut-sets of sources so the state equations can be simply

written as

$$\begin{bmatrix} \dot{v}_3 \\ \dot{i}_8 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} v_3 \\ i_8 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \cdot & \frac{1}{2} & \cdot & \cdot \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_9 \\ i_{10} \end{bmatrix} + \begin{bmatrix} \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & -\frac{1}{2} & \cdot \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ i_9 \\ \dot{i}_{10} \end{bmatrix}$$

Now the controlling variables are given by

$$\begin{bmatrix} i_3 \\ i_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \cdot \\ 1 & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} v_3 \\ i_8 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -1 & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_9 \\ i_{10} \end{bmatrix} + \begin{bmatrix} \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ i_9 \\ \dot{i}_{10} \end{bmatrix}$$

$$= \begin{bmatrix} v_1 \\ v_2 \\ i_9 \end{bmatrix}$$

Combining these equations gives the equations of the network

$$\begin{bmatrix} 1 & \cdot & \cdot & -\frac{1}{2} & \cdot \\ \cdot & 1 & \cdot & \cdot & \frac{1}{2} \\ \cdot & \cdot & \cdot & \frac{1}{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \dot{v}_3 \\ \dot{v}_8 \\ \dot{v}_1 \\ \dot{v}_2 \\ \dot{i}_9 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \cdot & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \cdot & \cdot & \cdot & \frac{1}{2} & \cdot \\ \frac{1}{2} & \cdot & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -1 & \cdot & 1 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} v_3 \\ i_8 \\ v_1 \\ v_2 \\ i_9 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ \cdot \\ \frac{1}{2} \\ \cdot \\ \cdot \end{bmatrix} i_{10} + \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \dot{i}_{10}$$

The rank of the left hand side matrix is 3 and so the last 2 equations can be used to eliminate 2 variables. Since the column corresponding to $\dot{v}_1$ has no non-zero entries, eliminating v_1 is simpler

than $\dot{v}_2$, although the choice is unimportant rearranging the equations gives

$$\begin{bmatrix} 1 & . & -\frac{1}{2} & . & . \\ . & 1 & . & . & \frac{1}{2} \\ . & . & \frac{1}{2} & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix} \begin{bmatrix} \dot{v}_3 \\ \dot{i}_8 \\ \dot{v}_2 \\ \dot{v}_1 \\ \dot{i}_9 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & . & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ . & . & \frac{1}{2} & . & . \\ \frac{1}{2} & . & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & . & 2 & 1 & . \\ . & . & . & . & 1 \end{bmatrix} \begin{bmatrix} v_3 \\ i_8 \\ v_2 \\ v_1 \\ i_9 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ . \\ \frac{1}{2} \\ . \\ . \end{bmatrix} i_{10} + \begin{bmatrix} . \\ . \\ . \\ . \\ 1 \end{bmatrix} \dot{i}_{10}$$

Using the last 2 equations, v_1 and i_9 can be eliminated to give

$$\begin{bmatrix} 1 & . & -\frac{1}{2} \\ . & 1 & . \\ . & . & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \dot{v}_3 \\ \dot{i}_8 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} . & . & -\frac{1}{2} \\ . & . & \frac{1}{2} \\ 1 & . & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} v_3 \\ i_8 \\ v_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ . \\ \frac{1}{2} \end{bmatrix} i_{10} + \begin{bmatrix} \frac{1}{2} \\ . \\ -\frac{1}{2} \end{bmatrix} \dot{i}_{10} + \begin{bmatrix} . \\ \frac{1}{2} \\ . \end{bmatrix} \ddot{i}_{10}$$

$$\begin{bmatrix} \dot{v}_3 \\ \dot{i}_8 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & . & -2 \\ . & . & -\frac{1}{2} \\ 2 & . & -3 \end{bmatrix} \begin{bmatrix} v_3 \\ i_8 \\ v_2 \end{bmatrix} + \begin{bmatrix} . \\ . \\ 1 \end{bmatrix} i_{10} + \begin{bmatrix} . \\ . \\ -1 \end{bmatrix} \dot{i}_{10} + \begin{bmatrix} . \\ \frac{1}{2} \\ . \end{bmatrix} \ddot{i}_{10}$$

Thus the order of complexity is 3, the predicted maximum. Note also that second order derivatives occur.

5.6 OTHER ACTIVE ELEMENTS

As was mentioned in section 1.3 most network elements can be represented by controlled sources. For example the gyrator is given

by the equations

$$i_1 = g\, v_2$$

$$i_2 = -g\, v_2$$

This is simply two current generators whose voltages control one another, as shown in Fig. 5.4. From theorem 5.4 a sufficient condition for the order of complexity of a network containing such a gyrator to be the same as for the passive network obtained by open circuiting the ports is that no port of the gyrator lies across a cut set containing only inductors and possibly current sources and generators.

Alternatively, the gyrator can be modelled as two voltage sources in which case the conditions are the dual of the above.

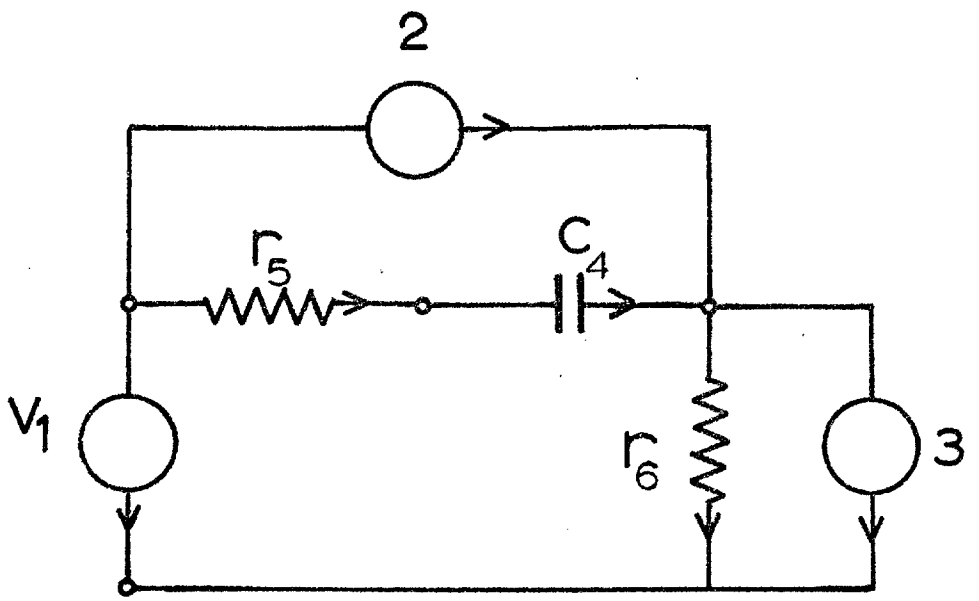
Ideal transformers can be treated in a similar manner.

Negative resistors can be simply modelled by a voltage generator controlled by the current flowing through it, as shown in Fig. 5.5.

Thus the order of complexity will be no greater than that of the passive network formed by short circuiting the negative resistor, if they lie in no loops containing only capacitors and voltage sources. It should be noted that this approach differs from that given in Chapter 3 for negative resistor networks. There the negative resistor was changed to a positive one and here, it is removed. This means that in Chapter 3 it was shown that if a negative resistor had a capacitor path between its terminals then the order of complexity was unchanged, but here the order of complexity

may increase. The apparent contradiction results from the fact that in the controlled generator point of view, the passive network may have a lower order of complexity than in the approach of Chapter 3.

Thus the controlled generator approach may not be the best for every situation, but it possesses the advantage that it is general.



$$v_2 = r_2 i_3 \qquad v_3 = r_3 i_6$$

Fig.5.1. Example of section 5.24

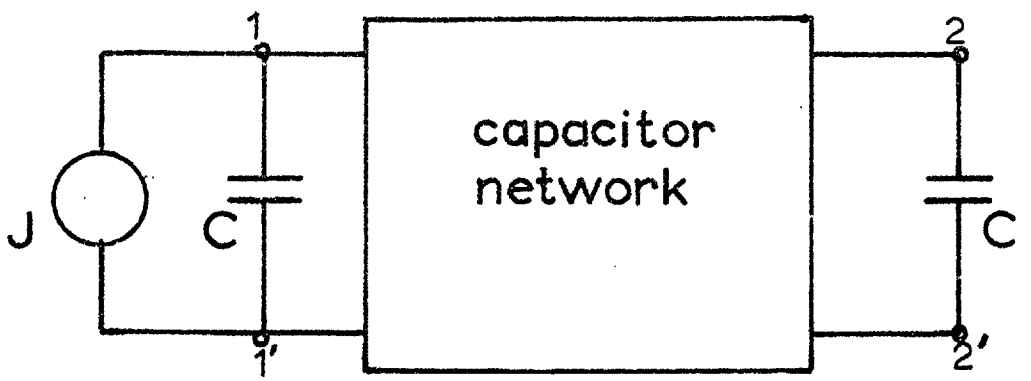
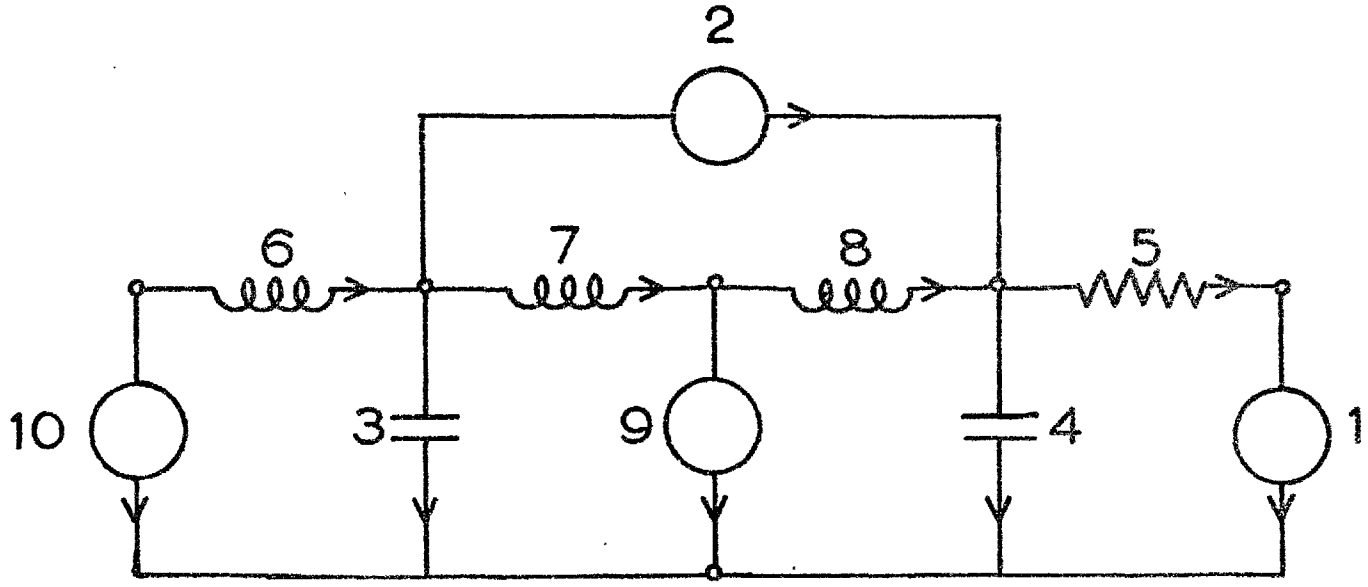


Fig.5.2. Illustration for lemma 5.1



$V_1 = I_3$; $V_2 = I_5$; $I_9 = V_6$; I_{10} = CONSTANT

Fig.5.3. Example of section 5.6.

All element values are unity.

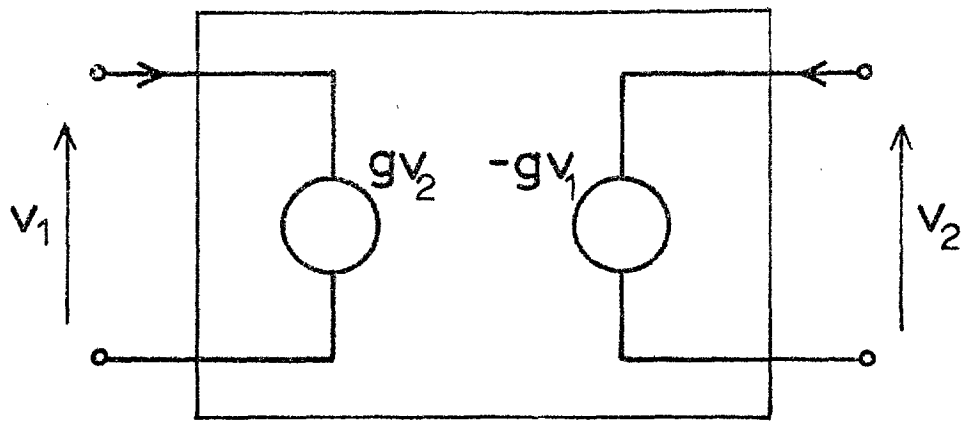


Fig.5.4. Equivalent circuit for gyrator.

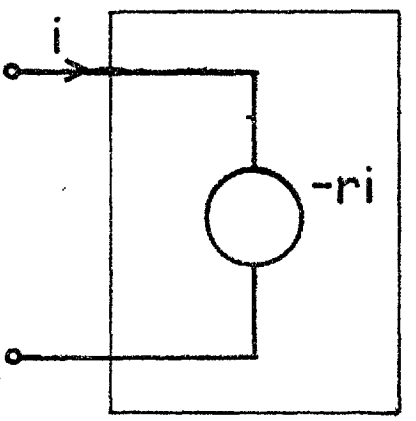


Fig.5.5. Equivalent circuit for negative resistor.

Chapter 6

THE APPLICATION OF SYSTEM THEORY TO NETWORK SYNTHESIS

6.1 INTRODUCTION

System theorists have been concerned with the basic engineering problem of finding a system which has a given behaviour. This, in network terms, is the basic problem of network synthesis. In this chapter the methods used in system theory are introduced and their relevance to electrical network synthesis is discussed. In particular the problems which arise from poles at infinity are considered and the relationship between the concepts of degree and order of complexity is examined.

6.2 THE THEORY OF IRREDUCIBLE REALIZATIONS.[31,38]

If a system is described by its state equations in the form

$$\begin{aligned} \dot{x} &= \underline{A}x + \underline{B}u \\ y &= \underline{P}x \end{aligned} \tag{6.1}$$

then its input output relationship is given by the equation

$$y = F(s)u = \underline{P}(sU - \underline{A})^{-1}\,\underline{B}u$$

where s is the Laplace variable

and $$F(s) = \underline{P}(sU - \underline{A})^{-1}\,\underline{B}$$

The triple of matrices $\{\underline{A},\underline{B},\underline{P}\}$ is said to be a realization of the transfer matrix F(s). It represents the state model of a system with this required input-output description, F(s). The dimension of the state vector is said to be the dimension of the realization.

It is always possible to transform the variable x in the form $x = Tx_1$, where T is a non-singular transformation and this gives

$$\dot{x}_1 = T^{-1}\underline{A}T\, x_1 + T^{-1}\,\underline{B}u$$
$$y = \underline{P}T\, x_1$$

This has the same input output relationship F(s). Thus all the triples of the form $\{T^{-1}\underline{A}\,T,\ T^{-1}\underline{B},\ \underline{P}T\}$ are realizations of the given F(s).

In general it is always possible to transform any matrix to its canonic Jordan form, as is pointed out in the appendix. In the case of a matrix with distinct eigen values it can be reduced to a diagonal matrix, with the diagonal elements being the eigen values of the matrix. For the sake of simplicity, if it is assumed that $\underline{A}$ has distinct eigen values, then a transformation T can be found which reduces equation (6.1) to the form

$$\dot{x} = \underline{A}_1 x + \underline{B}_1 u$$
$$y = \underline{P}_1 x \qquad (6.2)$$

where $\underline{A}_1$ is a diagonal matrix with the eigen values of $\underline{A}$, λ_i, the natural frequencies of the system, along the diagonal.

$$F(s) = \underline{P}_1(sU - \underline{A}_1)^{-1}\,\underline{B}_1$$

Thus the eigen values of $\underline{A}$ are the poles of $F(s)$

$$\text{and} \quad \left\{ F(s) \right\}_{ij} = \sum_{k=1}^{\delta} \frac{\left\{ \underline{P}_1 \right\}_{ik} \left\{ \underline{B}_1 \right\}_{kj}}{(s - \lambda_k)}$$

where δ is the order of A, the dimension of the realization.

Now if $\left\{ \underline{B}_1 \right\}_{kj} = 0$ for all j then the term $\frac{1}{(s-\lambda_k)}$ would not appear in $F(s)$. In such a case the natural frequency λ_k is said to be uncontrollable. Similarly if $\left\{ \underline{P}_1 \right\}_{ik} = 0$ for all i, then again λ_k makes no contribution, and is said to be unobservable. For a natural frequency of the system to appear in the transfer matrix $F(s)$, the natural frequency must be both controllable and observable. If it is not, it will still exist, but only internally in the system. In fact it corresponds to cancelling terms in the transfer matrix.

This concept can be easily generalized to the case where the eigen values of $\underline{A}$ are no longer necessarily distinct.[38]

As far as $F(s)$ is concerned, any uncontrollable or unobservable natural frequency is just redundant and can be removed. Thus in equation (6.2), if λ_i is either unobservable or uncontrollable, then the ith row of the first set of the equations and the ith column of $\underline{A}_1$ and $\underline{P}_1$ can be dropped to give a new system described by the equations

$$\dot{x}_2 = \underline{A}_2 x + \underline{B}_2 u$$

$$y = \underline{P}_2 x$$

and again $$F(s) = \underline{P}_2 (sU - \underline{A}_2^{-1}) \underline{B}_2$$

This new realization has dimension of $\delta - 1$. Thus the dimension of a realization can be reduced by the number of uncontrollable and unobservable natural frequencies. A realization possessing no such natural frequencies is said to be an irreducible realization, or a minimal realization. Kalman has shown that such an irreducible realization of a given F(s) is unique, except for a non singular transformation, T. It is this uniqueness which makes the concept of such importance, since the transfer matrix F(s) serves only to define the set of irreducible realizations. In fact methods are available for constructing an irreducible realization from F(s).[71] Any reducible realization contains some natural frequencies which are arbitrary and in no way connected with F(s).

The state description of the system given in equation (6.1), is rather restrictive since it implies that $F(\infty) = 0$. If $F(\infty)$ is a matrix of non-zero but finite elements, then the equations take the form

$$\dot{x} = \underline{A}x + \underline{B}u$$

$$y = \underline{P}x + \underline{Q}u$$

where $\underline{Q} = F(\infty)$ and is fixed by F(s). The realization is then taken as $\{ \underline{A}, \underline{B}, \underline{P}, \underline{Q} \}$.

6.3 NETWORK INTERPRETATION

The concepts of system theory can all be interpreted in terms of networks and can be used to clarify various problems in the subject

of network synthesis. In network theory the basic problem is to find a multiport network with a prescribed port behaviour, specified by a real rational matrix Y(s). This electric network will be termed a synthesis of Y(s) to avoid confusion with the term realization, used previously here in the system theory sense.

The concepts of uncontrollable and unobservable natural frequencies are very familiar in network theory since cancellations frequently occur between numerators and denominators in driving point functions of networks. An example of such a network is the constant resistance network pictured in Fig. 6.1, which has two natural frequencies, neither of which appear at the output.

Any network having uncontrollable or unobservable natural frequencies will be termed a 'reducible network'. This does not necessarily mean that an alternative controllable and observable network exists, with the same input-output behaviour, but rather, it means that the realization of the network is non-minimal. It may not be possible to synthesize such a minimal realization.

The number of reactive elements will not correspond directly to the number of state variables partly because some reactive elements are redundant and partly because some extra may be needed to include poles at infinity. In order to understand the relationship between the number of reactive elements and a realization, it is necessary to consider the inclusion of poles at infinity and also the concept of the degree of a rational matrix.

6.4 POLES AT INFINITY IN STATE SPACE THEORY[55]

The state space approach finds particular difficulty in the treatment of the behaviour of F(s) at $s = \infty$. Intrinsic in the standard definitions of the state equations

$$\dot{x} = \underline{A}x + \underline{B}u$$

$$y = \underline{P}x + \underline{Q}u$$

is the fact that the matrix F(s) has no poles at infinity. Finding a realization in this form for F(s) amounts to factorizing F(s) in the form

$$F(s) = \underline{P}(sU - \underline{A})^{-1}\underline{B} + \underline{Q}$$

which only applies if F(s) has no poles at infinity.

In general the factorization will take the form

$$F(s) = \underline{P}(sU-\underline{A})^{-1}\underline{B} + \underline{Q} + \underline{R}_1 s + \underline{R}_2 s^2 + \ldots\ldots \underline{R}_i s^i \quad (6.3)$$

and the state equations will take the form

$$\dot{x} = \underline{A}x + \underline{B}u$$

$$y = \underline{P}x + \underline{Q}u + \underline{R}_1\dot{u} + \underline{R}_2\ddot{u} + \ldots\ldots \underline{R}_i u^{(i)} \quad (6.4)$$

where $$u^{(i)} = \frac{d^i u}{dt^i}$$

Kalman's[39] approach to this problem was to follow McMillan[47] in his work on the degree of a rational matrix and to use a Möbius transformation

$$s = \frac{a\,\omega + b}{c\,\omega + d} \qquad ad \neq bc$$

This can be arranged to map all the poles in the s-plane, including those at infinity into finite points in the ω-plane. Thus

the matrix is changed to $F(\omega)$, which has no poles at infinity and can be given as a simple realization.

In electric networks, poles at infinity frequently occur and having to do frequency transformations in order to characterize the matrix seems to be rather unfortunate. In Chapter 3 it was shown that the state equations of an RLC network take the form

$$\begin{aligned} \dot{x} &= \underline{A}x + \underline{B}u + \underline{C}\dot{u} \\ y &= \underline{P}x + \underline{Q}u + \underline{R}\dot{u} \end{aligned} \tag{6.5}$$

and in Chapter 5 it was shown that higher order derivatives of the input vector u can also occur when the network is active.

It can be seen that the factorization shown in equation (6.3) can always be performed, since it is possible to identify $\underline{Q}$ and all the $\underline{R}_k$ terms directly from the matrix.

e.g. $R_i = \left[\frac{1}{s^i} \quad F(s) \right]_{s=\infty}$

If these terms are subtracted from $F(s)$ the remainder $F_1(s)$ can be factorized as

$$F_1(s) = \underline{P}(Us - \underline{A})^{-1}\underline{B} \quad ,$$

an irreducible realization, in the usual manner. Thus an irreducible realization of the form shown in equation (6.4) can always be found.

For the matrix of an RLC network such a realization will always take the form

$$\begin{aligned} \dot{x} &= \underline{A}x + \underline{B}u \\ y &= \underline{P}x + \underline{Q}u + R\dot{u} \end{aligned} \tag{6.6}$$

where the matrix $\underline{C} = 0$.

Kalman[38] has shown that all irreducible realizations $\{ \underline{A}, \underline{B}, \underline{P} \}$ are linked by the non-singular transformation T, so that

$$\begin{aligned} \dot{x} &= T^{-1}\underline{A}T\, x + T^{-1}\underline{B}u \\ y &= \underline{P}T\, x + \underline{Q}u + \underline{R}\dot{u} \end{aligned} \tag{6.7}$$

characterizes all alternative irreducible realizations of F(s), of this form. The fact that many electrical networks have a state representation in the form of equation (6.5) with a non-zero matrix $\underline{C}$ means that T no longer characterizes all the possible realizations of F(s) if F(s) has a pole at infinity.

A more general transformation which does generate all the equivalent irreducible realizations of a matrix F(s) with a pole at infinity is one of the form

$$x = T\omega + Su$$

where T is the usual non-singular square matrix and S is an arbitrary, compatible, constant matrix.

If this transformation is applied to equation (6.5), then

$$\begin{aligned} \dot{\omega} &= T^{-1}\underline{A}T\omega + T^{-1}(\underline{B} + \underline{A}S)u + T^{-1}(\underline{C}-S)\dot{u} \\ y &= \underline{P}T\omega + (\underline{Q} + \underline{P}S)u + \underline{R}\dot{u} \end{aligned} \tag{6.8}$$

Eliminating ω gives

$$y = \underline{P}T(Us - T^{-1}\underline{A}T)^{-1}T^{-1}(\underline{B}+\underline{A}S)u + \underline{P}T(Us-T^{-1}\underline{A}T)^{-1}T^{-1}(\underline{C}-S)us$$

$$+ (\underline{Q} + \underline{P}S)u + \underline{R}\, us$$

$$= \underline{P}(Us-\underline{A})^{-1}(\underline{B}+\underline{C}s)u + \underline{Q}u + \underline{R}su + \underline{P}(Us-\underline{A})^{-1}(\underline{A}S-Ss)u + \underline{P}Su$$

$$= \underline{P}(Us-\underline{A})^{-1}(\underline{B}+\underline{C}s)u + \underline{Q}u + \underline{R}su$$

$$= F(s)u$$

So equation (6.8) defines an equivalent irreducible realization of $F(s)$.

If $S = \underline{C}$, then equation (6.8) is reduced to the form of equation (6.6), Thus any two equivalent irreducible realizations can be brought by suitable choices of S to the form of equation (6.6). By Kalman's results, any two such equivalent realizations are linked by some non-singular transformation matrix and so the 2 original realizations can be linked by a transformation of the above form.

These results can be stated in the following theorem.

Theorem 6.1

If a realization of a given real rational matrix takes the form

$$\dot{x} = \underline{A}x + \underline{B}u + \underline{C}\dot{u}$$

$$y = \underline{P}x + \underline{Q}u + \underline{R}\dot{u}$$

then all other equivalent realizations, of the same dimension, are realized by the transformation

$$x = T x_1 + Su$$

where T is a non-singular matrix and S is arbitrary, but compatible.

The inclusion of the fact that the equivalent realizations

have the same dimension means that, although the realizations need not be irreducible, all equivalents of all dimensions cannot be generated.

This result is useful only for cases such as RLC networks when the equations take the form indicated in the theorem. For systems with higher order derivatives, such as active networks the equations have the form

$$\dot{x} = \underline{A}x + \underline{B}u + \underline{C}_1\dot{u} + \underline{C}_2\dot{u}^{\cdot} + \ldots. \underline{C}_i u^{(i)}$$
$$y = \underline{P}x + \underline{Q}u + \underline{R}_1\dot{u} + \underline{R}_2\dot{u}^{\cdot} + \ldots. \underline{R}_i u^{(i)}$$

In this case all equivalents can be generated by a generalization of the transformation of theorem 4.1, that is by

$$\omega = Tx + Su + S_1\dot{u} + \ldots.. S_{i-1}u^{(i-1)}$$

6.5 THE CONCEPT OF DEGREE AND ITS RELATIONSHIP TO THE ORDER OF COMPLEXITY

6.5.1 McMillan's degree[47]

The concept of degree was rigorously defined by McMillan, as a generalization of the degree of a rational function to matrices of rational functions. In order to do this he introduced a canonic form for a rational matrix F(s), which is known as the Smith-McMillan canonic form.

If $\phi(s)$ is the lowest common denominator of all the elements

of F(s), then the product $\phi(s)\ F(s)$ is a polynomial matrix in s and has a Smith Canonic form

$$\text{i.e.}\quad \phi(s)\ F(s) \;=\; P(s)\ N(s)\ Q(s)$$

where N(s) is a diagonal matrix with the invariant factors of $\phi(s)\ F(s)$ along the diagonal. Dividing through by polynomial $\phi(s)$ gives

$$F(s) \;=\; P(s)\ \phi^{-1}(s)\ N(s)\ Q(s)$$

and $\phi^{-1}(s)\ N(s)$ becomes a diagonal matrix of rational functions, when $\phi^{-1}(s)$ has been taken into N(s) and any cancellations performed.

$$\phi^{-1}(s)N(s) = \begin{bmatrix} \dfrac{\psi_1(s)}{\phi_1(s)} & & & & & \\ & \dfrac{\psi_2(s)}{\phi_2(s)} & & & & \\ & & \ddots & & & \\ & & & \dfrac{\psi_r(s)}{\phi_r(s)} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

McMillan defined the degree of the real rational matrix, F(s), as being the sum of the degrees of these rational functions

$$\text{i.e.}\quad \delta(F(s)) \;=\; \sum_{i=1}^{r} \delta(\phi_i(s))$$

where $\delta(A)$ is the degree of A.

The case of F(s) having a pole at infinity was treated by first using a Möbuis transformation to bring all points into the finite plane.

An alternative approach to the problem, still using Smith-McMillan canonic forms, is to first split F(s) up into partial fractions, in the form

$$F(s) = \frac{F_1(s)}{(s-\alpha_1)^{p_1}} + \frac{F_2(s)}{(s-\alpha_2)^{p_2}} + \cdots \frac{F_q(s)}{(s-\alpha_q)^{p_q}}$$

The degree is given by the sum of the degrees of each term of the expansion

$$\text{i.e.} \quad \delta\{F(s)\} = \sum_{i=1}^{q} \delta\left\{\frac{F_i(s)}{(s-\alpha_i)^{p_i}}\right\}$$

The Smith-McMillan forms of each term are simpler than in the general case because the lowest common denominator is simply $(s-\alpha_i)^{p_i}$ for each term.

This approach has the advantage that poles at infinity can be easily treated. A pole at infinity produces a term $s^{p_\infty} F_\infty(s)$ in the expansion. The degree of such a matrix can be found by replacing s by $\frac{1}{s}$ and finding the degree of this simple matrix.

6.5.2 The Duffin-Hazony degree[28]

Duffin and Hazony have suggested a definition of degree which is based upon the concept of the order of an n-port, introduced by Tellegen[64]. This is defined as "the order of the differential equation of the free oscillations of the system formed by connecting the ports of the n-port to the ports of an n-port containing no reactances."

The Duffin-Hazony definition of degree is in terms of the determinant

$$\det\,(F(s) + K) \;=\; \frac{\phi_N}{\phi_D}$$

where ϕ_N and ϕ_D are relatively prime polynomials and K a real square matrix. The degree is defined as the maximum degree of ϕ_N over all possible K.

6.5.3 Kalman's contribution[39]

Kalman made an important contribution to this theory by showing that McMillan's[1] degree and the Duffin-Hazony degree are both equal to the dimension of the irreducible realization of the matrix.

Consider a realization of F(s) of the form

$$\dot{x} = \underline{A}x + \underline{B}u$$

$$y = \underline{P}x + \underline{Q}u$$

If $y = -Ku$, then

$$\dot{x} = \underline{A}x + \underline{B}u$$

and $$0 = \underline{P}x + (\underline{Q} + K)u$$

Now taking the determinant

$$\phi(s) = \det \begin{bmatrix} (sU - \underline{A}) & -B \\ -\underline{P} & -(\underline{Q} + K) \end{bmatrix}$$

gives $$\phi(s) = \det\,(sU-\underline{A}).\det\,(\underline{P}(sU-\underline{A})^{-1}\underline{B} + \underline{Q} + K)$$

$$= \det\,(sU-\underline{A}).\det\,(F(s) + K).$$

$$\text{i.e.} \quad \det\,(F(s) + K) \;=\; \frac{\phi(s)}{\det(sU - \underline{A})}$$

Some cancellations may occur in this expression. If the realization is reducible, then some of the zeros of det $(sU - \underline{A})$ are arbitrary and cannot appear in det $(F(s) + K)$. Thus these will always cancel out with $\phi(s)$. Some other zeros may cancel, but Kalman has shown that a K can be found so that $\phi(s)$ contains no common zeros with the det $(sU - \underline{A})$ of an irreducible realization. Thus the Duffin-Hazony degree is equal to the dimension of the irreducible realization, since $\phi(s)$ and det $(sU - \underline{A})$ have the same degree.

This treatment assumes that F(s) has no poles at infinity. In such cases a Möbuis transformation is used as in McMillans method. It is more difficult to show the connection between McMillans degree and the dimension of the irreducible realization and as the proof is of no interest in this present work it will not be given here.

6.5.4 The order of complexity

The definitions of degree are purely properties of rational matrices and are not directly related to any physical problem. Kalman's work is slightly more physical because he introduces the concept of a realization which, although it considered abstractly, is connected with the physical interpretation of the matrix. Kalman has shown that the dimension of an irreducible realization of a real

rational matrix is equal to the degree of the matrix.

The use of the equation

$$y = -Ku$$

can be interpreted as the termination of the n-port network

$$y = F(s)u$$

by a non reactive, (or resistive), n-port.

This new network has some properties and it is these which will be considered here.

If $\dot{x} = \underline{A}x + \underline{B}u$

$$y = \underline{P}x + \underline{Q}u + \underline{R}_1\dot{u} + \underline{R}_2\ddot{u} + \dots \underline{R}_i u^{(i)}$$

is a general realization of $F(s)$, then terminating the n port in the resistive network given by

$$y = -Ku$$

gives $\dot{x} = \underline{A}x + \underline{B}u$

$$0 = \underline{P}x + (\underline{Q}+K)u + \underline{R}_1\dot{u} + \dots \underline{R}_i u^{(i)}.$$

as the new system. Because the system is more general than in Chapter 5, the result is not a set of first order differential equations in x and u and the problem of finding the new order of complexity is more difficult. In this case, extra state variables are needed besides vectors x and u. These will be $u_1, u_2 \dots u_{i-1}$.

where $u_1 = \dot{u}$

$$u_2 = \dot{u}_1 = \ddot{u}$$

$$u_i \quad \dot{u}_{i-1} = u^{(i)}$$

Thus the equations can be written in first order form as

$$\begin{bmatrix} U & & & & \\ & U & & & \\ & & U & & \\ & & & \ddots & \\ & & & & \underline{R}_i \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \\ \dot{u}_1 \\ \vdots \\ \dot{u}_{i-1} \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B} & & & \\ & U & & & \\ & & U & & \\ & & & \ddots & \\ \underline{P}(\underline{Q}+K) & \underline{R}_1 & \underline{R}_2 & \cdots & \underline{R}_{i-1} \end{bmatrix} \begin{bmatrix} x \\ u \\ u_1 \\ \vdots \\ u_{i-1} \end{bmatrix}$$

The order of complexity of the system is thus the degree of the polynomial

$$\det \begin{bmatrix} \lambda U-\underline{A} & -\underline{B} & & & & \\ & \lambda U & -U & & & \\ & & \lambda U & -U & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \\ \underline{P} & (\underline{Q}+K) & \underline{R}_1 & \underline{R}_2 & \cdots & \underline{R}_{i-1}+\lambda\underline{R}_i \end{bmatrix}$$

$$= \det \begin{bmatrix} (\lambda U-\underline{A}) & -\underline{B} \\ \underline{P} & (\underline{Q}+K) + \underline{R}_1\lambda + \underline{R}_2\lambda^2 \cdots \underline{R}_i\lambda_1^i \end{bmatrix}$$

$$= \det (\lambda U-\underline{A}) \det (\underline{P}(\lambda U-\underline{A})^{-1}\underline{B} + (\underline{Q}+K) + \underline{R}_1\lambda + \cdots \underline{R}_i\lambda_1^i)$$

$$= \det (\lambda U-\underline{A}) \det (F(\lambda) + K) = \phi(\lambda)$$

Thus the degree of $\phi(\lambda)$ is the order of complexity of the network.

$$\text{Now} \quad \det\,(F(\lambda) + K) \quad = \quad \frac{\phi(\lambda)}{\det(\lambda U - \underline{A})}$$

Kalman has shown that provided the realization is irreducible, K can be chosen so no cancellations occur, and so the degree will be the order of complexity with the terminating resistive n-port characterized by matrix K.

This illustrates the meaning of degree and its importance in network synthesis. If a matrix F(s) is synthesized as a network, then no matter how the network is synthesized it must be possible to terminate it with a resistive n-port and achieve an order of complexity equal to the degree. If the synthesis is reducible then it will be possible to go above this bound, but it can always be reached. This idea is similar to Tellegen's approach in defining the term order. It can be seen from his definition, given in section 6.5.2, that it represents the highest degree of $\phi(\lambda)$ after cancellation and so represents the highest order of complexity in the irreducible case.

The order of complexity cannot exceed the number of reactive elements in the network and so the order of complexity represents the minimum number of reactive elements needed, for a synthesis of that dimension. The degree represents the minimum possible dimension of the realization, or the number of dynamically independent variables needed to synthesize a given matrix. These 2 numbers indicate the two forms of redundancy which can occur in network synthesis.

Consider any n-port network which is a synthesis of a given real rational matrix F(s). If the n-port is terminated by a n-port

resistive network and the order of complexity maximized, then the difference between this order of complexity and the degree of F(s) represents the number of uncontrollable or unobservable natural frequencies present in the network. The difference between the number of reactive elements and the order of complexity represents the number of excess reactive elements. A network having more reactive elements than the order of complexity is termed a degenerate network.

The order of complexity is greater than or equal to the degree of the n-port and is less than or equal to the number of reactive elements. Thus the degree represents the minimum number of reactive elements needed to synthesize a given rational matrix. Such a synthesis would be irreducible and non-degenerate.

Since these definitions require that the maximum order of complexity be found, it would be useful to know the nature of the resistive network which would achieve this. In the RLC case such a network is particularly simple.

6.5.5 The relationship between the degree and the order of complexity in the RLC case

Lemma 6.1

If an RLC n-port network, given by the admittance matrix Y(s), has a realization

$$\dot{x} = \underline{A}x + \underline{B}u + \underline{C}\dot{u}$$

$$y = \underline{P}x + \underline{Q}u + \underline{R}\dot{u}$$

then the increase in the order of complexity of the network due to the replacement of the sources by resistors is given by the rank of matrix $\underline{R}$.

Proof The equations for the network, when it is terminated in a set of resistors given by the equation

$$y = -Gu,$$

is given by

$$\begin{bmatrix} U & -\underline{C} \\ & -\underline{R} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{P} & \underline{Q}+G \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

The order of complexity is given by the degree of the determinant

$$\phi(\lambda) = \det \begin{bmatrix} U\lambda - \underline{A} & -(\underline{C}\lambda + \underline{B}) \\ \underline{P} & -(\underline{R}\lambda + \underline{Q} + G) \end{bmatrix}$$

The degree of $\phi(\lambda)$ cannot be greater than the rank of the matrix

$$\begin{bmatrix} \lambda U & \lambda\underline{C} \\ & \lambda\underline{R} \end{bmatrix}$$

which is equal to $\rho(\underline{R}) + \sigma$

where $\rho(\underline{R})$ is the rank of $\underline{R}$ and σ is the order of complexity of the network driven by sources, i.e. the order of $\underline{A}$.

Hence if σ_1 is the new order of complexity

then $$\sigma_1 \leqslant \sigma + \rho(\underline{R}) \qquad (6.9)$$

Consider the matrix $\underline{R}$. It is given in equation (3.9) as

$$\underline{R} = Q_{eC}\, C_\ell B_{Ce} + Q_{eC}\, C_\ell B_{GC}(C_b + Q_{CC}C_\ell Q_{GC}^t)^{-1}Q_{CG}R_\ell^{-1}B_{Ce}$$

$$= Q_{eC}C_\ell\{U + B_{CG}(C_b + Q_{CC}C_\ell Q_{GC}^t)^{-1}Q_{CC}R^{-1}\}B_{Ce}$$

It is a well known result of matrix algebra which says that

if $A = B \cdot C$

$\rho(A) \leqslant \rho(B)$

and $\leqslant \rho(C)$

Hence $\rho(\underline{R}) \leqslant \rho(Q_{eG})$ (6.10)

Now consider the effects upon the network equations of exchanging the sources for resistors. When the normal tree was chosen to define the equations the voltage sources were taken first, then the capacitors and then the remaining elements. Thus the cut-set matrix has the form

$$\begin{bmatrix} U_{be} & & & Q_{eC} & Q_{ei} \\ & U_{bC} & & Q_{CC} & Q_{Ci} \\ & & U_i & Q_{iC} & Q_{ii} \end{bmatrix}$$

where the notation is the same as in Chapter 3 except that the subscript i represents the remaining elements, other than capacitors and voltage sources.

Now since Y(s) is an admittance matrix, vector u represents all voltage sources. If these are changed to resistors the choice of tree must be changed, and the capacitor tree chosen before them. This means that some of the capacitors corresponding to link capacitors in the fourth column of the cut set matrix now become tree capacitors.

The number of capacitors will depend upon the number of independent rows in Q_{eG}, and so will be $\rho(Q_{eC})$. Since each tree capacitor represents a state variable, the only effect of changing the voltage sources to resistors is to increase the order of complexity by $\rho(Q_{eC})$

$$\text{i.e.} \quad \sigma_1 = \sigma + \rho(Q_{eC}) \tag{6.11}$$

Comparing the inequalities (6.9) and (6.10) with equation (6.11) gives

$$\sigma_1 \leqslant \sigma + \rho(\underline{R})$$
$$\leqslant \sigma + \rho(Q_{eC}) = \sigma_1$$
$$\text{i.e.} \quad \rho(\underline{R}) = \rho(Q_{eC}) = \sigma_1 - \sigma$$

which proves the lemma.

Theorem 6.2

The order of complexity of the network, formed by terminating an irreducible RLC n-port network synthesis of Y(s) with resistors, is equal to the degree of Y(s).

Proof If Y(s) describes an RLC n-port network, by the results of section 6.4 its irreducible realization can be expressed in the form

$$\dot{x} = \underline{A}x + \underline{B}u$$
$$y = \underline{P}x + \underline{Q}u + \underline{R}\dot{u}$$
$$\text{or} \quad Y(s) = \underline{P}(sU - \underline{A})^{-1}\underline{B} + \underline{R}s + Q$$
$$= Y_1(s) + \underline{R}s$$

where $Y_1(s)$ is finite at $s = \infty$.

Thus the degree of Y(s) is given by

$$\delta\{Y(s)\} = \delta\{Y_1(s)\} + \delta\{\underline{R}s\}$$

using McMillans definition of degree and considering the pole at infinity separately.

Since $Y_1(s)$ has no poles at infinity,

$$\delta\{Y_1(s)\} = \sigma \text{ , the order of } \underline{A}.$$

Since $\underline{R}$ is a constant matrix, $\frac{1}{s}\underline{R}$ has invariant factors of unity. The number is $\rho(\underline{R})$.

Thus the Smith McMillan canonic form is

$$\begin{bmatrix} \frac{1}{s} & & & & & \\ & \frac{1}{s} & & & & \\ & & \ddots & & & \\ & & & \frac{1}{s} & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}$$

where there are $\rho(\underline{R})$ non-zero terms.

i.e. $\delta\{\underline{R}s\} = \rho(\underline{R})$

$\therefore \quad \delta\{Y(s)\} = \sigma + \rho(\underline{R})$

If now an RLC network is found to synthesis the n-port irreducibly, its order of complexity will be σ. If now resistors are added to the ports, by lemma 6.1 the order of complexity will rise to $\sigma + \rho(\underline{R})$, and is equal to $\delta\{Y(s)\}$.

These results mean that adding the resistors displays the effects of the poles at infinity, even if the synthesis is reducible.

Thus for the RLC case it is possible to check on the degeneracy of the network by simply adding resistors across the various ports.

These results can be similarly proved for the dual case of an impedance matrix Z(s), and since $\underline{R}$ is the direct sum of two matrices, one corresponding to the capacitors and voltage sources and the other the inductors and current sources, it could be extended to the case of a hybrid matrix H(s).

6.5.6 The relationship between the degree and the order of complexity in the general case

It has been shown that the degree of an n-port corresponds to the maximum order of complexity of an irreducible synthesis of the n-port embedded in a resistive n-port. In the last section it was shown that if the synthesis can be taken as an RLC network the embedding network could be taken simply as a set of terminating resistors for the ports. In the general case this is not true and the more general resistive network is needed.

This can be simply demonstrated by an example

$$Y(s) = \begin{bmatrix} \frac{s^2-2s+2}{s-1} & s^3-1 \\ & \frac{s-1}{s-2} \end{bmatrix}$$

$$= \frac{1}{(s-1)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{(s-2)} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} (s-1) & (s^3-1) \\ 0 & 1 \end{bmatrix}$$

The degree of Y(s) is therefore 5.

A minimal realization gives

$$\frac{dx}{dt} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} u + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{d^3u}{dt^3} .$$

Terminating this by resistors given by

$$y = \begin{bmatrix} -g_1 & \\ & -g_2 \end{bmatrix} u$$

$$\text{gives} \begin{bmatrix} (s-1) & . & 1 & . \\ . & (s-2) & . & 1 \\ 1 & . & (s-1)+g_1 & (s^3-1) \\ . & 1 & . & 1+g \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0$$

The order of complexity of the network is 3. If a general resistive embedding is taken

$$y = - \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} u$$

the order of complexity can be raised to 5.

In this example there is a $\frac{d^3u}{dt^3}$ term, but this is not significant.

Consider $Y(s) = \begin{bmatrix} 1 & (s-1) \\ 0 & 1 \end{bmatrix}$

which has a degree of 1, but the order of complexity is always zero unless the network is embedded in a general resistive 2-port. Y(s) may be synthesized as a 2-port shown in Fig. 6.2.

The reason for the simplicity of the RLC case is that it is purely topological in nature. The resistors simply move all poles at infinity to finite points which can be counted in the order of complexity. The occurrence of poles at infinity is again purely topological, being decided on the fact that in the admittance case, a capacitor path exists between a pair of port terminals. Thus the pole can be moved simply by inserting the appropriate resistor.

In the general case the problem is confused by the fact that poles at infinity can arise by algebraic constraints. Fig. 6.2 illustrates an example where the algebraic constraints effect the poles at infinity. Here some terms of the residue matrix of this pole are supressed.

Fig. 6.3 shows an example of a network which has a pole at infinity but no capacitor across the port terminals.

The driving point admittance of this one port is given by

$$Y(s) = \frac{s^2 + 1}{s}$$

In fact in this case the algebraic constraints just maintains points A and B at the same potential and so effectively brings the capacitor directly across the port.

It is these algebraic effects which cause the trouble, although they may be necessary to the synthesis as in the network shown in Fig. 6.2. The basic problem is that, unlike the RLC case, it is not

possible by inspection to say if there is any degeneracy in the network. As an example consider the network of Fig. 6.4. This network is an equivalent of the network of Fig. 6.3, yet it uses an extra capacitor to give the behaviour at infinity. This only apparent when the maximum order of complexity is evaluated and found to be less than the number of reactive elements.

Thus in the general case it is not possible to simplify the condition, and the maximum order of complexity is obtained by adding a resistive n-port. In such a case, care must be taken in choosing the n-port, because it is possible to connect n-ports illegally and violate the n-port constraints.[67]

6.6 REDUCIBILITY AND DEGENERACY IN CLASSICAL NETWORK SYNTHESIS

The concept of degree plays an important part in network synthesis since if the network is synthesized with this many elements, it is both irreducible and non-degenerate. It is known that if transformers and gyrators are allowed, then any positive real matrix can be synthesized in this form. For the one port case this was established by Brune[14] and in the n-port by McMillan[47] and Oono[50]. If transformers are not allowed, then the problems become more difficult, and many are as yet unsolved. The problem for one ports has been solved. For the 2 element kind network, the Foster and Cauer methods[3] both yield networks which are irreducible and non-

degenerate. In the RLC one port case Bott and Duffin[9] have shown a method of synthesis. Seshu[60] has shown that the Bott-Duffin network represents the smallest general network which will realize a biquadratic function. The Bott-Duffin cycle produces a network which is a reducible network, although it is not degenerate. The multiport network problem is, as yet, unsolved. The example of the biquadratic function indicates that, in fact, transformerless synthesis techniques will not result in irreducible networks. As yet there seems no advantage in allowing degeneracy.

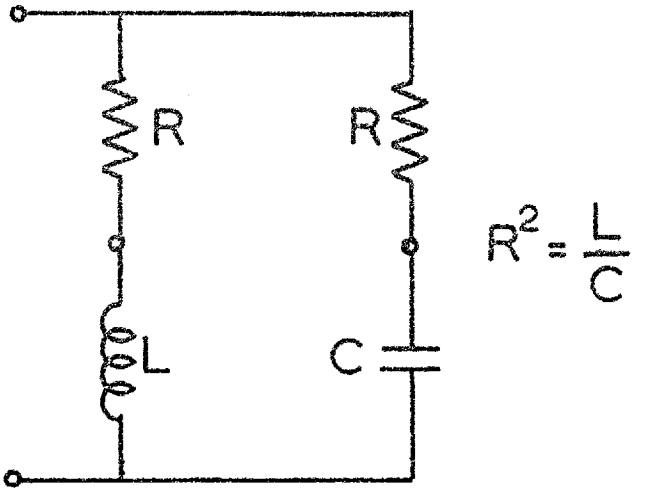


Fig.6.1. Constant resistance network.

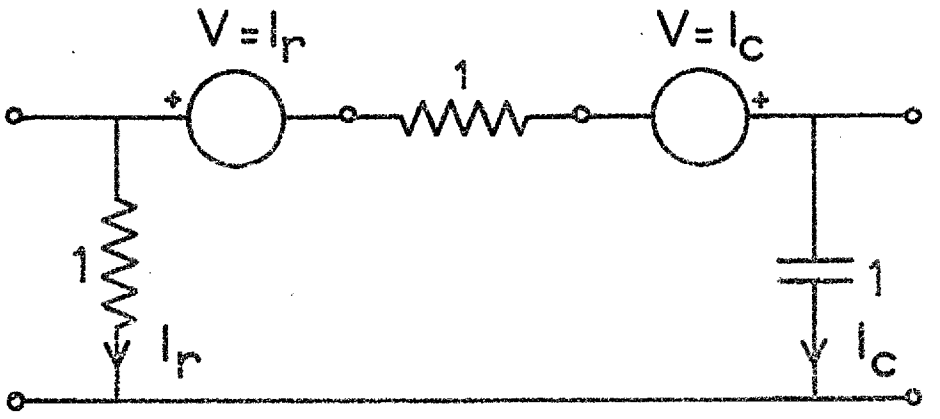


Fig.6.2. Example of section 6.5.6.

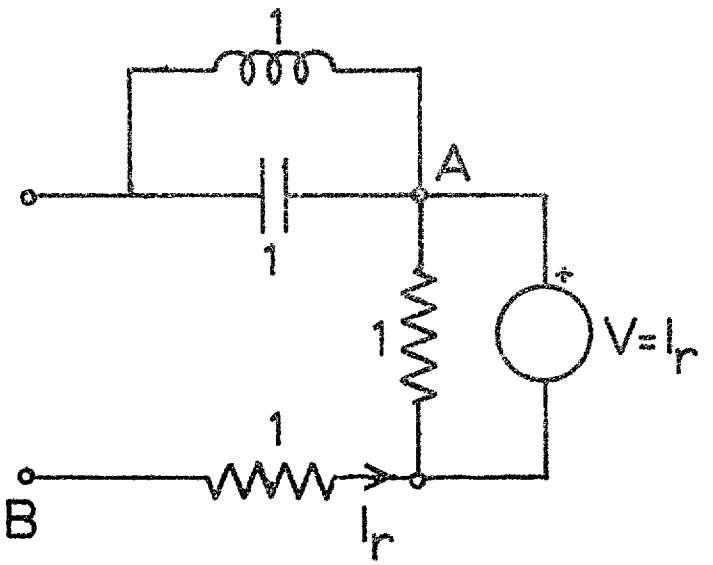


Fig.6.3. Example of section 6.5.6.

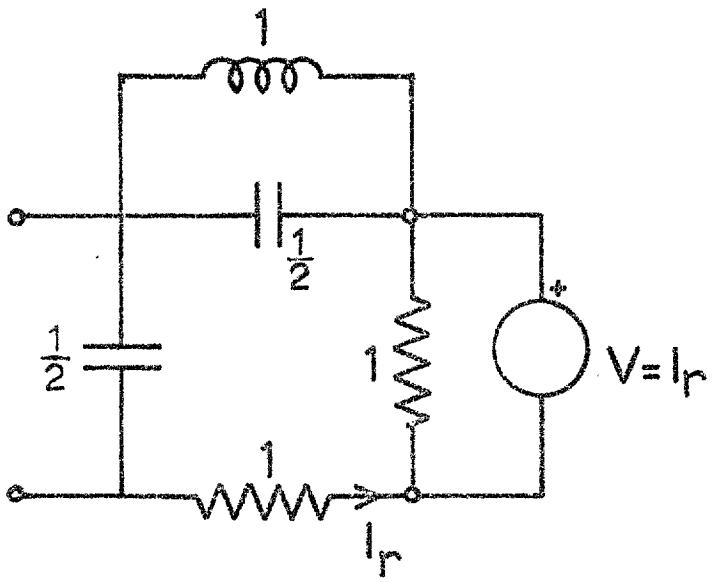


Fig.6.4. Equivalent network of Fig.6.3.

Chapter 7

THE SYNTHESIS OF ELECTRICAL NETWORKS USING STATE SPACE TECHNIQUES

7.1 INTRODUCTION

The relevance of the concepts of a realization, controllability, observability and order of complexity was shown in the last chapter. In this chapter the application of these ideas in the field of network synthesis is considered. This consists of a critical review of the methods now available and, based on the work of the last chapters, a new approach to active network synthesis is presented.

In this section the term active synthesis is used in the n-port sense[48] as the opposite of passive synthesis and not in the general sense used in the rest of the thesis. Here passive synthesis includes transformers and active synthesis involves some active elements such as amplifiers.

7.2 GENERAL REMARKS

The uniqueness of the irreducible networks realization makes it much easier to synthesize networks that are irreducible rather than reducible. A synthesis procedure allowing reducible networks allows the inclusion of arbitrary uncontrollable and unobservable poles. This gives a large degree of freedom in synthesis, but unless there is some guidance as

to how it can be used, it is useless. The Bott-Duffin[9] synthesis is the outstanding example where this has been done. Richards' theorem[3] indicates the way in which an extra cancelling factor can be introduced to aid physical realizability.

In the state space approach the way in which reducible realizations are formed is very clear,[2] but the problem of knowing which one to choose is little understood and normally the emphasis is towards irreducible synthesis.

The problems of synthesis fall roughly into two categories;

(i) Given a set of state equations, what are the conditions on the equations that a network can be found having these equations, and how can the network be found?

(ii) Given a real rational matrix Y(s), ways exist of constructing an irreducible realization. Normally such a realization cannot be synthesized directly, but use can be made of transformations to convert the equations into a synthesizable form. The problem is to find such a transformation. This approach is powerful because such transformations generate all equivalent irreducible realizations, and so covers all the realizations representing equivalent networks of Y(s). Thus the problem of network equivalence can be solved.

7.3 THE SYNTHESIS OF PASSIVE NETWORKS

7.3.1 Networks with transformers

The case of RLC networks with transformers has been considered by a number of authors.[1,44,71] Their approach has been to consider the network as a resistive multiport network, terminated in reactive elements, as shown in Fig. 7.1.

If $Z(s)$ is an $(n \times n)$ matrix of degree δ, then the multiport is an $(n + \delta)$ port network containing, resistors transformers, and perhaps gyrators. Since transformers are included, all reactive elements can have normalised values of 1 farad and 1 henry. If H is a hybrid matrix describing the network, then

$$\begin{bmatrix} \dot{\bar{x}} \\ i \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

where x is the voltages in the capacitors and current on the inductors, $\bar{x}$ is the compliment of x, and v and i are the port variables. Since all reactive elements have normalized values $\dot{\bar{x}} = -\dot{x}$.

So the system can be reduced to

$$\begin{aligned} i &= \left[H_{22} - H_{21} (sU + H_{11})^{-1} H_{12} \right] v \\ &= Z(s)\, v. \end{aligned}$$

This can be compared with the irreducible realization of $Z(s)$

$$Z(s) = \underline{P}(sU - \underline{A})^{-1} \underline{B} + \underline{Q}$$

It can be seen that the realization corresponds to possible submatrices of the hybrid matrix H. Thus part (i) of the synthesis problem is solved. The conditions on the coefficient matrices of the

state equations is that they correspond to a hybrid resistive matrix.

The problem is thus reduced to the problem of finding a transformation T, which for a given realization $\{ \underline{A}, \underline{B}, \underline{P}, \underline{Q} \}$ gives the matrix

$$M = \begin{bmatrix} -T^{-1}\underline{A}\,T & -T^{-1}\underline{B} \\ \underline{P}T & \underline{Q} \end{bmatrix}$$

as a realizable hybrid matrix. This approach was taken by Layton[44]. Because of the nature of the realization, Y(s) cannot have poles at infinity.

Anderson and Newcomb[1] have suggested an alternative approach. Since gyrators are allowed, all capacitors can be considered as gyrators terminated in inductors. Thus for the case of impedance matrix synthesis, the hybrid matrix M can be simplified to an impedance matrix. This approach is also suitable for RL networks.

Youla and Tissi[71] have another approach based on scattering matrices instead of impedance and admittance matrices. Because the multiport network is taken in its scattering description, networks with poles at infinity in their impedance and admittance matrices can be considered. With this general description it is possible to characterise all possible multiports containing resistors, transformers and gyrators. Since all such characterizations are obtained for one irreducible realization, it is possible to characterize all irreducible equivalent networks of the matrix.

Anderson and Newcomb did propose a way of overcoming the problem of poles at infinity. The solution presented was simply to

remove the pole at infinity and synthesize this as a separate network.

i.e. $Z(s) = Z_1(s) + Ls$

where L is the residue of the pole at infinity and $Z_1(s)$ is finite at infinity. L can be synthesized as a separate all reactive network, containing inductors, transformers and gyrators. The remaining matrix $Z_1(s)$ can be treated in the usual manner.

The limitation of this approach can best be seen by a simple example.[55] Anderson and Newcomb used impedance matrices, but the approach can be used for admittance matrices also.

Consider the driving point impedance

$$Y(s) = \frac{s(s+1)}{(2s+1)}$$

The pole at infinity can be extracted as a capacitor across the input terminals as

$$Y(s) = \frac{s}{2} + \frac{s}{2(1+2s)}$$

The realization of the remainder is

$$\left\{-\frac{1}{2}, -\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\}$$

The state equations may be written in the form

$$\dot{x} = -\frac{1}{2}x - \frac{1}{4}e$$

$$j = \frac{1}{2}x + \frac{1}{4}e + \frac{1}{2}\dot{e}$$

where e and j are the input voltages and currents at the port terminals.

The state variable x may be interpreted as the voltage across a capacitor, C as in Fig. 7.2.

The synthesis problem is thus reduced to the problem of synthesizing a resistive 2-port connecting the capacitor with the input.

Its conductance matrix is

$$G = \begin{bmatrix} \frac{1}{2}C & \frac{Ct}{4} \\ \frac{1}{2t} & \frac{1}{4} \end{bmatrix}$$

where t is the nonsingular transformation, in this case a number, and C is the value of the capacitor. Since the example is a simple one, C need not be taken as unity, but can be used to obtain a transformerless network. The condition that a 2-port can be synthesized as a resistor network is the dominance condition.[67] In this case it implies that

$$t \geqslant 2 \quad \text{and} \quad C = \frac{2}{t^2}$$

If $t = 2$, then $C = \frac{1}{2}$ and the matrix becomes

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

which is simply a single conductance of 4 ohms. The whole synthesis is given in Fig. 7.3.

This is the second Foster form of classical synthesis.[67] Other equivalents are obtained by varying C. But all these have the capacitor across the port terminals because of the way the pole at infinity was extracted.

A well known equivalent of this network which does not have such a capacitor is the first Foster canonic form,[67] shown in Fig. 7.4. The state equations of this network have the form

$$\dot{x} = -\frac{1}{2}x + \frac{1}{2}e + \frac{1}{2}\dot{e}$$

$$y = -\frac{1}{2}x + \frac{1}{2}e + \frac{1}{2}\dot{e}$$

The reason this cannot be reached is that the realizations are of the form given in equation (6.6) of the last chapter, when $\underline{C} = 0$. If a more general transformation of the type indicated in section 6.4 is used then this realization can be found by the transformation

$$x = -\frac{1}{2} x_1 + \frac{1}{2} e$$

From the results of this section it can be seen that in order to obtain all realizations this type of transformation must be used. The problem is that the synthesis of the resulting realizations is not known. The advantage of the scattering matrix approach is that such networks have no poles at infinity in the scattering matrix and can be treated in the usual manner. By using this approach it is possible to prove many of the facts about n-port synthesis.[71,48]

7.3.2 Networks without transformers

A more challenging field has been that of synthesis without transformers. In this field conventional methods have failed to produce any really complete synthesis theory. Youla and Tissi's approach is general and will, of course, apply to transformerless networks, but the problem of synthesizing a scattering matrix as a resistive network is an unsolved one. Layton's and Anderson and Newcomb's approaches are more hopeful since they involve impedance and admittance matrices which are at least partially understood.[67] Anderson and Newcomb's use of gyrators is of course not

possible and so the RLC problem is essentially the problem of synthesizing a hybrid matrix. In the RL or RC case of course the problem is simplified, as was illustrated in the example in the last section, but it also illustrated that in general all realizations cannot be confined to realizations of the form shown in equation (6.6). In the general case of transformerless synthesis this will be of even greater importance since the number of transformerless realizations will be quite small and so it is important to find all of them. The realizations produced will be of the form

$$\dot{x} = \underline{A}x + \underline{B}u + \underline{C}\dot{u}$$

$$y = \underline{P}x + \underline{Q}u + \underline{R}\dot{u}$$

and the first problem is to develop a synthesis theory for such equations.

This was first considered by Dervisoglu[26] when he considered the synthesis of the state equations for portless RLC networks. This means that the state equations were simply

$$\dot{x} = \underline{A}x$$

To perform the synthesis he made two assumptions;

a) the network was "half degenerate". By this he meant that although poles at the origin where allowed no degeneracy occurred due to all capacitor loops and all inductor cut sets. This was termed non-degenerate in the last chapter.

b) The resistive network was connected.

The principle of his approach was to factorize $\underline{A}$ into the form

$$\underline{A} = \begin{bmatrix} C^{-1} & \\ & L^{-1} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ -H_{12}^{t} & H_{22} \end{bmatrix}$$

where C and L were diagonal with positive elements along the diagonal

H_{11} and H_{22} were real and symmetric

and H_{12} was real.

C and L correspond to the capacitor and inductor values and H is a hybrid matrix describing the resistive subnetwork and the interconnections between the reactive and resistive subnetworks. Dervisoglu described a scheme for extracting these descriptions from H and reduced the problem to the synthesis of a hybrid resistor network and the connection matrices. The connectedness of the resistive network was essential to guarantee the hybrid matrix could be transformed into either an impedance or admittance matrix.

The assumption of no degeneracy was made to simplify the factorization of $\underline{A}$. Since C^{-1} and L^{-1} are diagonal the sign pattern of $\underline{A}$ is the same as H, and so can be used to determine the partitioning of the capacitor rows from the inductor rows. The extraction of C^{-1} and L^{-1} can then be performed, to give H its symmetric and skew symmetric form. This factorization is not unique since any constant multiplier can be extracted from H without ffecting the symmetry. Thus the synthesis procedure consists of taking a factorization and attempting to synthesize it. If it fails, due to some point in the procedure failing, then another factorization must be tried. The only way to show that such equations have no network is by trying all possibilities.

Although no synthesis procedure has ever needed to employ degeneracy, it appears that Dervisoglu's method can be extended to a larger class of networks if degeneracy is included. An example of a set of equations which cannot be synthesized with Dervisoglu's method is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ 0 & \frac{1}{3} & 0 \\ -\frac{1}{2} & -\frac{1}{6} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Because of the zero, placed non symmetrically off the main diagonal it is not possible to factorize the matrix. In fact it can be factorised on the degenerate form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & . & . \\ . & \frac{2}{3} & -\frac{1}{3} \\ . & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and so it can be synthesized in the form given in Fig. 7.5. The network is degenerate since it contains a capacitor loop.

The fact that degeneracy is needed here does not, of course, mean that the general problem of admittance and impedance synthesis will need degeneracy, since Dervisoglu's method is for portless networks and so does not allow for the use of any transformations to improve the form of the equations. It also assumes that the state variables are the voltages across capacitors and the current through inductors which is also not necessarily so.

A general approach has been taken by Yaralagadda and Tokad,

firstly for LC networks,[68,69] and more recently for RLC networks.[70] They considered the general case when the network has ports and degeneracy. The procedure is based on a comparison between the given equations and the general explicit form. By equating the appropriate terms it is possible to obtain linear equations in submatrices of the capacitor node admittance matrix, the inductor loop impedance matrix, and the resistor hybrid matrix. If the linear equations are not consistant no solution exists. The main problem appears to be the synthesis of the hybrid resistor network, which in general cases may neither possess an impedance or admittance equivalent.

The approach depends upon the fact that the division of the state variables between capacitor voltages and inductor currents is previously known. If not, then like Dervisoglu's method, many partitionings may be possible, which must be tried until the procedure works.

Thus, as yet, no satisfactory procedure exists to the problem of the synthesis of the state equations as RLC networks without transformers.

7.4 THE SYNTHESIS OF RC ACTIVE NETWORKS

7.4.1 Basic approach

With the advent of integrated circuits the synthesis of networks without inductors has become very important, because inductors cannot be made satisfactorily with these techniques. Most of the work done has been based upon special configurations[63], using a number of different active elements. At present the most common active element seems to be the high gain amplifier,[36] which of course can be modelled simply by a controlled generator.

Sandberg[59] has given a general treatment of networks with controlled generators and has shown that an n-port network can be synthesized using a minimum of n generators. This seems to be the only bound that has been placed upon controlled generator RC active networks.

The state space approach is particularly suited to the case of RC active networks because the state variables can be associated with capacitor voltages only. This removes the problem of distinguishing between the current and voltage variables, which occur in the RLC case.

7.4.2 The importance of irreducibility in active network synthesis

In the last chapter it was shown that there is a minimum dimension for the state vector of an n port network and this corresponds

to the maximum number of natural frequencies which can appear at the ports. The importance of this concept of irreducibility in passive network synthesis lies in the fact that an irreducible synthesis may contain a maximum number of reactive elements. It is therefore mainly of interest from the point of view of efficiency. In fact, as was pointed out in section 6.6 allowing reduceability broadens the class of matrices which can be synthesized.

In the case of active networks a second fact becomes important, that of stability. In passive networks all poles lie in the left half plane and so any uncontrollable or unobservable natural frequency will not affect the stability. But in an active network, the poles can appear anywhere and so it is possible to find examples of networks which have poles in the right half plane which are either uncontrollable or unobservable. This will mean that although the network has the required terminal behaviour, variables within the network will go off to infinity. Since in practice no system is completely linear these large voltages and currents will eventually affect the behaviour at the ports. A simple example is in an analogue computer circuit, where one of the operational amplifiers would overload. Thus, to study the stability of an active network, it is not sufficient to consider port behaviour only, and in synthesis it is important to know where all the natural frequencies are.

This fact has been noted in active network synthesis e.g. page 94 of reference 63, but it seems to have been little used. For example, Sandberg's result[59] on the minimum number of controlled

generators is achieved without any consideration of the stability of the whole network, although the number of reactive elements used will be much greater, in general, than the degree of the matrix being synthesized.

The simplest way to overcome this problem is to aim at irreducible synthesis. This has the advantages that it has no hidden instability and also uses the minimum number of reactive components.

7.4.3 Synthesis of active n-port networks

The starting point for a synthesis method giving irreducible networks will be an irreducible realization of the given real rational matrix. This can be obtained in any of the well known methods[39,71] and the problem is reduced to the synthesis of such a realization. A basic problem is that no explicit form for the equations exists because the form of the equations depends upon so many different variables.

An active network can be described by the equations

$$\begin{aligned} \dot{x} &= \underline{A}x + \underline{B}u + \underline{C}\dot{u} \\ w &= \underline{P}_1 x + \underline{Q}_1 u + \underline{R}_1 \dot{u} \\ y &= \underline{P}_2 x + \underline{Q}_2 u + \underline{R}_2 \dot{u} \end{aligned} \tag{7.1}$$

and $\quad \underline{u}_1 = Mu = Hw \quad ; \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

where the notation is the same as in Chapter 5 except that now the

equation of the outputs y has been added.

The system is described by the equation

$$y = Y(s)u_2$$

Equations (7.1) can be reduced to form a standard realization of Y(s), but the dimension of this realization depends upon many considerations and can vary from the order of $A + \rho(\underline{R}_1)$ to zero. Even if Y(s) has no pole at infinity, matrix $\underline{R}_1$ may exist and so it is still not possible in general to identify the various matrices.

An alternative approach will be taken here. This is to consider the network as a resistive multiport with some ports terminated in capacitors, as in Fig. 7.6. This is similar to the approaches of Layton, and Anderson and Newcomb in the case of passive networks.

It is assumed that Y(s) is a real rational matrix, possessing no poles at infinity. It therefore has a realization

$$\dot{x} = \underline{A}x + \underline{B}u$$

$$y = \underline{P}x + \underline{Q}u$$

The equations for the multiport network shown in Fig. 7.6 are given by

$$i_1 = Y_{11} v_1 + Y_{12} v_2$$

$$i_2 = Y_{21} v_1 + Y_{22} v_2$$

$$i_1 = -C \dot{v}_1$$

i.e.

$$\dot{v}_1 = -C^{-1} Y_{11} v_1 - C^{-1} Y_{12} v_2$$

$$i_2 = Y_{21} v_1 + Y_{22} v_2$$

By comparison $\underline{A} = -C^{-1}Y_{11}$, $\underline{B} = -C^{-1}Y_{12}$

$\underline{P} = Y_{21}$ and $\underline{Q} = Y_{22}$

If all capacitors are assumed to have unit value, then $C = U$ and the problem is reduced to the synthesis of a resistive $(n + \delta)$ port network given by the admittance matrix

$$Y_0 = \begin{bmatrix} -A & -B \\ P & Q \end{bmatrix}$$

This is a special case of Sandbergs' general RC result and so by Sandbergs' result, $n + \delta$ generators will in general be needed. A simpler method than Sandbergs' can be given.

Lemma 7.1 An $(m \times m)$ resistive matrix can be synthesized with m generators.

Proof

Consider a 2 m-port network of resistors terminated in m voltage controlled voltage sources.

The equation of the 2m-port can be written as

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^t & Y_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

If $v_2 = G\, v_1$, where G is a diagonal matrix representing the generator constraints, then

$$i_1 = Y_1 v_1 = (Y_{11} + Y_{12}G)v_1$$

i.e. $$Y_1 - Y_{11} = Y_{12}\, G.$$

When all the generator terms are assumed equal to g , this becomes

$$Y_1 - Y_{11} = g\, Y_{12}$$

A sufficient condition for a real symmetric matrix to represent a resistor network is that the matrix be dominant.[67] This means that

$$Y_{ii} \geqslant \sum_{j \neq i} |Y_{ij}|$$

Y_{11} may be chosen as a symmetric dominant matrix. g can then be chosen sufficiently large to satisfy the dominance condition in the submatrix $\left[Y_{11} Y_{12} \right]$. Y_{22} can also be chosen arbitrarily to be symmetrical and again ensure dominance. Thus it is possible to find a dominant (2m x 2m) matrix which can be synthesized as a balanced resistor network.

The matrix Y_0 can therefore be synthesized very simply as a resistor network with $n + \delta$ generators, controlled by the input ports. Thus the whole synthesis becomes a $2(n + \delta)$-port resistor network, with δ ports terminated in unit capacitors, $n + \delta$ ports terminated in voltage generators controlled by the voltages across the other $(n + \delta)$ ports, i.e. the inputs and state variable voltages.

This result can be summarized in the following theorem.

Theorem 7.1

Any real rational (n x n) admittance matrix Y(s) of degree δ, having no poles at infinity can be synthesized as an RC active network containing δ capacitors and $(n + \delta)$ generators.

This method does not use the minimum number of generators, which Sandberg has shown to be n. This is because the method

intrinsically involves the synthesis of an $(n + \delta)$ port resistive network which in general needs $(n + \delta)$ generators. Sandberg's method does not, in general, use the minimum number of capacitors, δ.

The method proposed here has the disadvantage of requiring a balanced resistor network which is not very practical. This may be overcome in some cases by a good choice of the realization $\{ \underline{A}, \underline{B}, \underline{C}, \underline{D} \}$ and also Y_{11}. The proof here has been given here for an arbitrary realization. The method of synthesizing Y_0 given here is simple but also has a rather restricted topology, and other methods, such as Sandberg's, could be used for this simple special case.

If poles at infinity exist, these will have to be synthesized separately as a second multiport network in a manner similar to Anderson and Newcomb's for the passive case. This will require a large number of extra generators.

Example Consider the one port network

$$i = \frac{1}{s} v$$

This is the equation of a unit inductor

$$Y(s) = \frac{1}{s}$$

and the realization is

$$\dot{x} = 0\,x + 1\,u$$

$$y = 1\,x + 0\,u$$

Thus the resistive network is a two port

$$Y_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This is of course the well known equations for a gyrator. It can be simply synthesized with two current generators as shown in Fig. 5.4. An alternative realization can be obtained using the method given in Lemma 7.1.

$$Y_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{Let} \quad Y_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{then} \quad Y_0 - Y_{11} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = g\, Y_{12}$$

$$\therefore \quad Y_{12} = \begin{bmatrix} -\frac{1}{g} & -\frac{1}{g} \\ \frac{1}{g} & -\frac{1}{g} \end{bmatrix}$$

If $g = 3$

$$\text{then} \quad Y_1 = \begin{bmatrix} 1 & . & -\frac{1}{3} & -\frac{1}{3} \\ . & 1 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & 1 & . \\ -\frac{1}{3} & -\frac{1}{3} & . & 1 \end{bmatrix}$$

where Y_{22} is taken as $Y_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

This can be synthesized as a 4-port resistor network. Thus the complete network is shown in Fig. 7.7.

7.4.4 Transfer function synthesis

The concepts of a realization and degree of a real rational matrix are not confined to n-ports or even to square matrices. Kalman's

treatment[38,39] is completely general. Thus the process of obtaining a realization can be used to find a realization of a transfer function. This is in practice often all that is needed in filter synthesis. Thus the state approach is particularly useful in the case of active filters. It is possible to synthesize transfer functions having the minimum number of reactive elements.

In the case of a single transfer function Kalman[38] has given a very simple way of form the realization. If the transfer function is given by

$$Y(s) = \frac{\sum_{i=0}^{n-1} b_i s^i}{\sum_{i=0}^{n} a_i s^i} \qquad \text{where } a_n = 1$$

then the realization is given by

$$\underline{A} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \cdot & \cdot & \\ & & & 0 & 1 \\ -a_0 & -a_1 & -a_2 & & -a_{n-1} \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

and $\underline{P} = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix}$

Since there is a zero at infinity, $Q = 0$.

If there is no zero at infinity, Q must be removed before $\underline{A}$, $\underline{B}$ and $\underline{P}$ are found.

Thus the realization of a single transfer function can be found by inspection and all others can be found by taking the appropriate

transformation.

The problem of synthesizing such a realization using operational amplifiers is quite simple. The first order linear differential equation has been synthesized on an analogue computer for many years and the methods are by now well known.[29] These can be used in a similar way to synthesis transfer functions.

As an example consider the biquadratic function,

$$y(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

This has a realization

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 \end{bmatrix} x$$

This can be synthesized using two integrators as shown in Fig. 7.8.

Such a 2 integrator loop was suggested by Girling[36] as a means of active network synthesis, purely on the basis of the similarity between operational amplifiers circuits and analogue computers. The flexibility introduced by being able to transform the realization may well be useful in obtaining some optimization, either of sensitivity or numbers of elements. The nature of the synthesis, which uses each capacitor with an amplifier as an integrator, means that at least δ amplifiers will be essential. But it may be possible in some cases

to avoid the use of others for summing and sign reversal.

Recently Newcomb and Anderson[49] suggested that state space techniques of this nature may well be useful in the synthesis of integrated circuits.

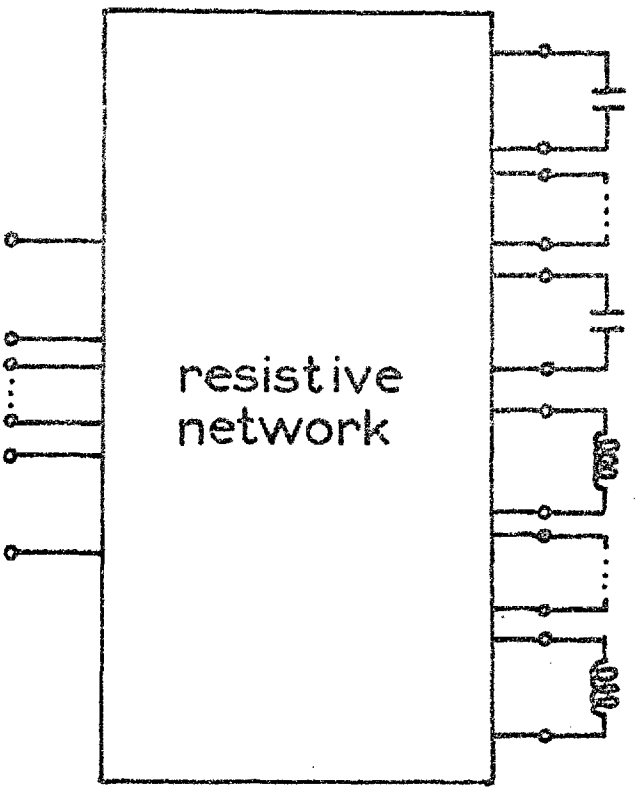


Fig.7.1. General form of network for synthesis.

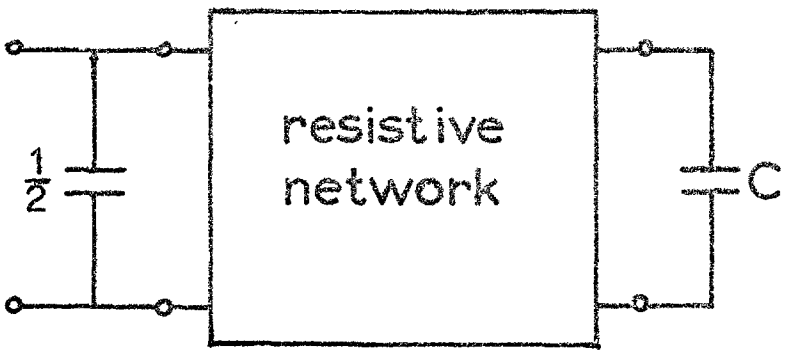


Fig.7.2. Form of networks synthesized in example.

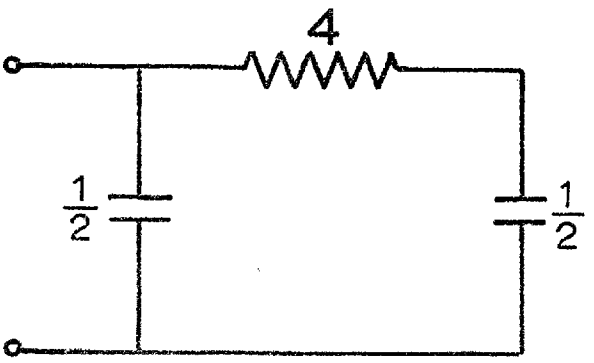


Fig.7.3. Second Foster form.

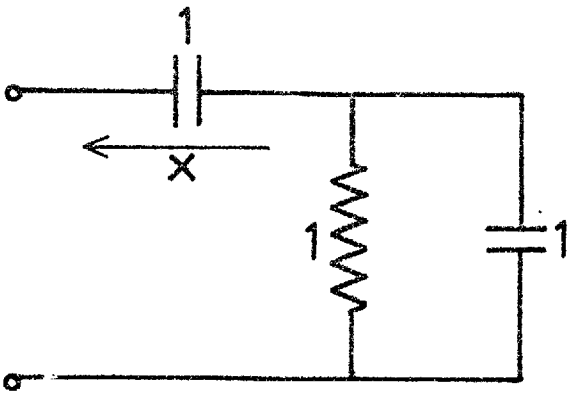


Fig.7.4. First Foster form.

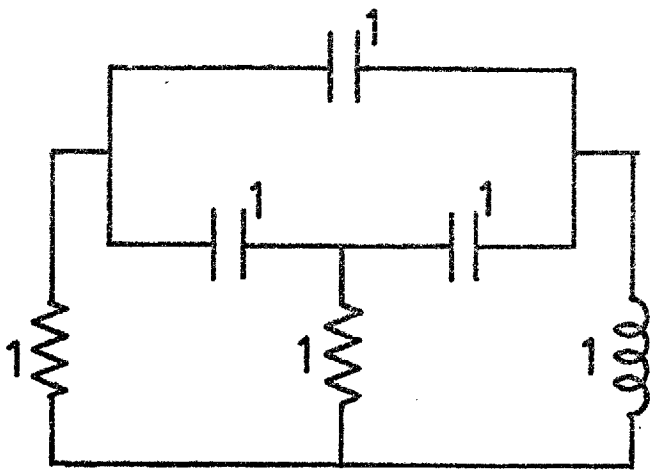


Fig.7.5. A degenerate network.

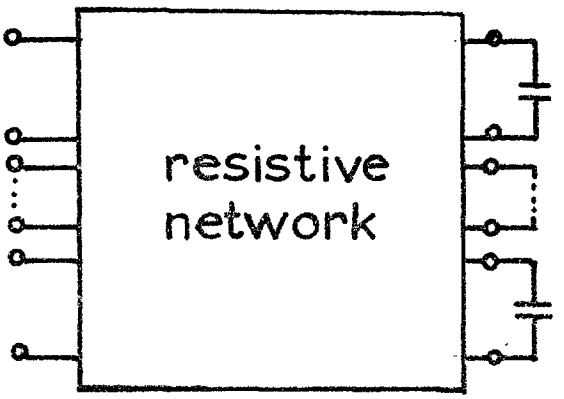


Fig.7.6. Form of active networks.

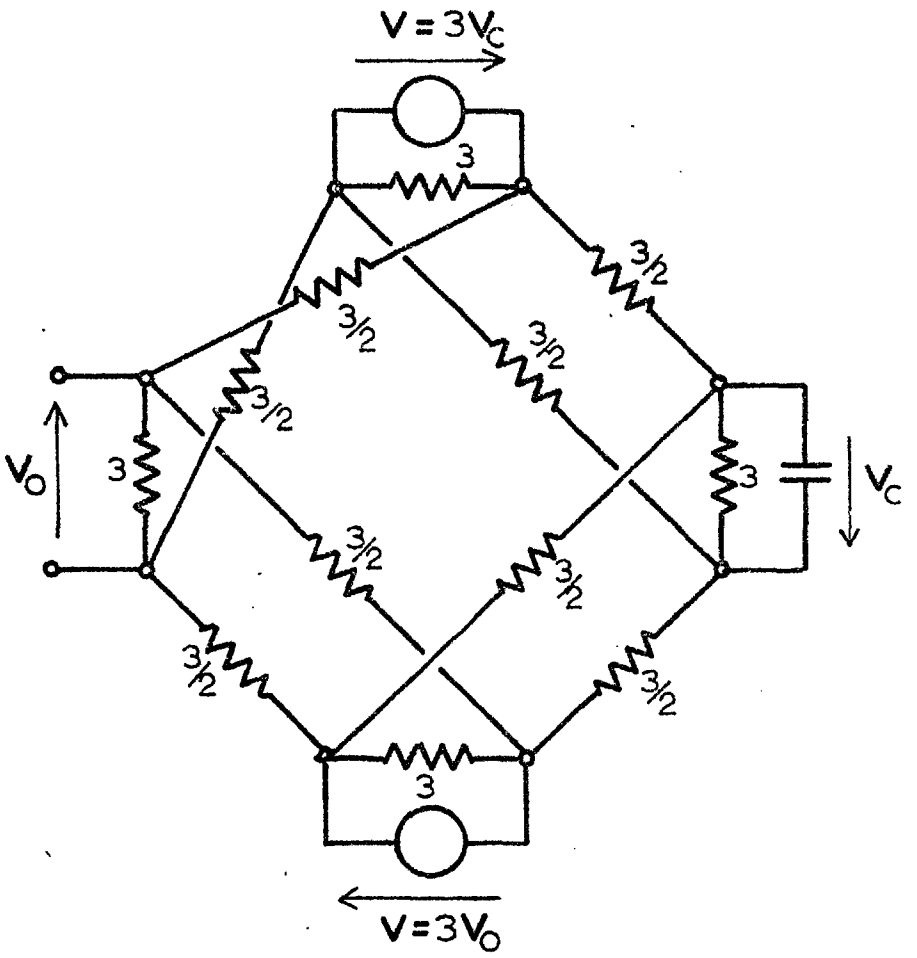


Fig.7.7. Example of section 7.4.3.

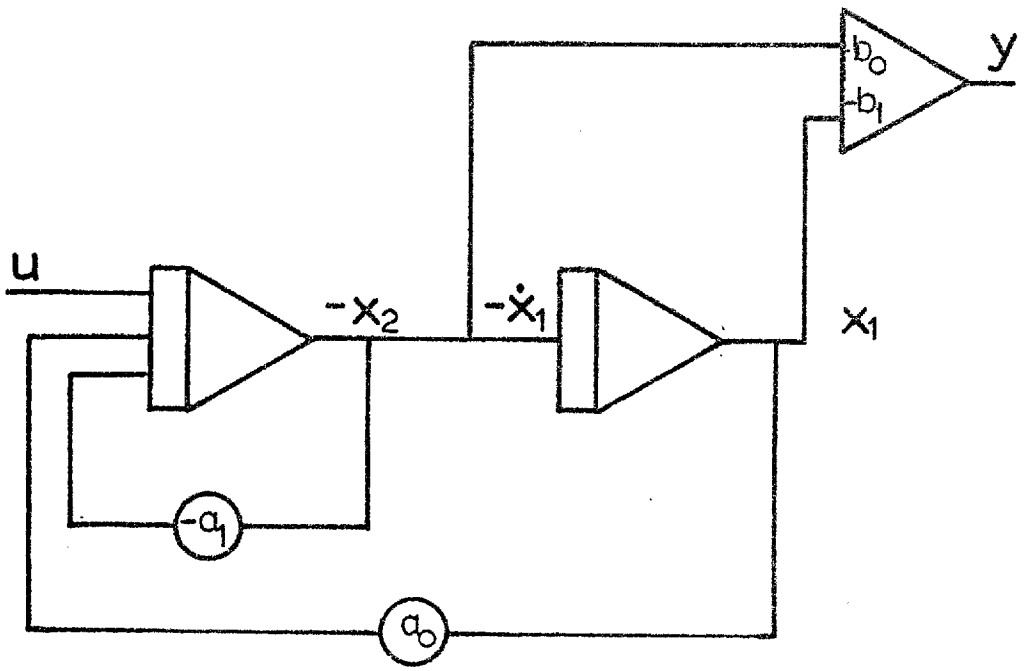


Fig.7.8. Synthesis of example in section 7.4.4.

Chapter 8

CONCLUSIONS

8.1 GENERAL CONCLUSIONS

An investigation of the formulation of the state equations for linear electric networks has been performed from the point of view of analysis and synthesis. The problem of forming the equations has been considered for the general case of RLC networks with all types of controlled generators. It was shown that, for this case, the problem was both topological and algebraic.

The topological problem can only give a maximum value for the order of complexity. A method has been given for finding this maximum order of complexity, by inspection of the network, for a restricted class of networks. It is possible, even in the general case, to give some small restrictions on the positioning of the controlling variables, so that the procedure is simplified and only the network with all generators removed need be considered.

An algebraic approach to the problem of setting up the equations has been suggested, based on extracting all the generators from the network. It was not possible to give an explicit form for the equations, but an algebraic method was given for finding the order of complexity. With these techniques, it was possible to give the necessary and sufficient conditions for the existence of a solution to an active network. These conditions are both topological and algebraic.

In the work on synthesis, a generalization of the present transformations was suggested which generates all equivalent irreducible realizations for systems with poles at infinity. This is important because electric networks represent an example of such systems. The problem of relating the order of complexity of a network to the degree was also considered, and it was shown that the problem is particularly simple in the case of n-ports which can be synthesized as RLC networks.

The methods of synthesis using state space techniques were considered. It was pointed out that although the problems of the synthesis of passive networks with transformers and gyrators are reasonably well understood, the case of RLC networks is not. The use of these methods for the synthesis of RC active networks was also considered, and it was shown that it is possible to synthesize a n-port with the minimum number of reactive elements, but only at the expense of extra active terms.

8.2 FUTURE RESEARCH

A number of points in this work could be extended for future research.

(a) The fact that the results on the order of complexity in Chapter 5 can be used to extend the results of Chapter 4 seems to imply that the class of networks considered in Chapter 4 was, perhaps, unnecessarily restrictive. The approach of Chapter 4 was

unsymmetrical, since the node admittance matrix was used. It could, perhaps, be possible to generalize Chapter 4 and simplify some of the proofs by expanding det $H_0(D)$ directly. This was, in fact, attempted but no method was found.

(b) The cases of networks containing transformers and gyrators are important ones. Although the synthesis of such networks can be achieved, this is done by an algebraic procedure which produces rather unsatisfactory results. It was pointed out in Chapter 5 that such networks can be included in the general procedure. They are, in fact, very interesting special cases, since it is a well known fact that such networks contain only simple poles at infinity. Thus the elimination procedure must have a special form which could be investigated. This could lead to some interesting results on the order of complexity of such networks.

(c) In Chapter 6, it was shown that the relationships between the degree and order of complexity was very simple for the case of RLC n-ports. Since the concept of degree is a property of the matrix only, this raises the question of what properties of the matrix give this simple result. Since the conditions for a matrix to be synthesizable as an RLC network are rather complex, it seems likely that these results could be extended to a larger class of networks, such as RLC networks with transformers and perhaps gyrators.

(d) The problem of transformerless RLC network synthesis is still of interest. Since it has been shown that the networks in the general case will be reducible, the problem of making use of the freedom of

reducible realizations is an important one. Also it has yet to be shown that any degenerate RLC n-port has an equivalent with no degeneracy. All procedures, except for Dervisoglu's, require no degeneracy, and this special case requires that the state variables are fixed.

(e) The procedure for RC active n-port synthesis has a number of disadvantages. The main one is that the method requires a balanced resistor network. From the practical point of view a more attractive network would be a grounded one. Attempts were made to find a method giving this type of network, but were unsuccessful and it still remains a problem. A second problem is the question of the number of active elements. It is not known if an n-port can be synthesized with the minimum number of reactive and active elements.

Finally, in Chapter 7 it was shown that any irreducible realization could be used in the synthesis procedure. It should be possible to make use of this to improve the sensitivity of the resulting networks.

APPENDIX. POLYNOMIAL MATRICES[30]

The Smith Canonic form

Let $A(\lambda)$ and $B(\lambda)$ be two rectangular matrices, whose elements are each polynomials in λ, then $A(\lambda)$ and $B(\lambda)$ are said to be equivalent if

$$B(\lambda) = P(\lambda)A(\lambda)Q(\lambda)$$

where $P(\lambda)$ and $Q(\lambda)$ are square polynomial matrices, with constant non-zero determinants. If $P(\lambda)$ and $Q(\lambda)$ are constant matrices independent of λ, then $A(\lambda)$ and $B(\lambda)$ are said to be strictly equivalent.

Any polynomial matrix $A(\lambda)$ is equivalent to a canonic form, known as the Smith normal form of $A(\lambda)$,

$$v(\lambda) = \begin{bmatrix} \phi_1(\lambda) & & & & & & \\ & \phi_2(\lambda) & & & & & \\ & & \ddots & & & & \\ & & & \phi_r(\lambda) & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

where $\phi_i(\lambda)$ are monic polynomials and are called the invariant factors of $N(\lambda)$. They have the following properties;

(a) $\phi_i(\lambda)$ divides $\phi_{i+1}(\lambda)$;

(b) the product $\phi_1 \ldots \phi_i(\lambda)$ equals the greatest common division of all minors of $A(\lambda)$ of order i, $\mathcal{F}_i(\lambda)$

i.e. $\mathcal{F}_i(\lambda) = \phi_1 \ldots \phi_i(\lambda)$

$$\text{and so } \phi_i(\lambda) = \frac{\chi_i(\lambda)}{\chi_{i-1}(\lambda)}$$

$\chi_i(\lambda)$ is also taken as monic.

(c) All $\phi_i(\lambda)$ are unique for a given $A(\lambda)$. Since $\phi_{r+1}(\lambda) = 0$ then $\chi_{r+1}(\lambda) = 0$ and so r is the rank of $A(\lambda)$.

In a given number field each invariant factor can be factorized as

$$\phi_i(\lambda) = \{\psi_1(\lambda)\}^{p_{i_1}} \{\psi_2(\lambda)\}^{p_{i_2}} \ldots\ldots \{\psi_s(\lambda)\}^{p_{i_s}}$$

These factors are called elementary divisors.

In general, 2 matrices are only equivalent if they have the same invariant factors and therefore the same elementary divisors.

The Jordan normal form

An important special case of a λ matrix is the linear binomial matrix

$$A(\lambda) = A_0 + A_1\lambda$$

where A_0 and A_1 are constant, square matrices.

For 2 such equivalent binomials,

$$A(\lambda) = A_0 + A_1\lambda$$
$$\text{and } B(\lambda) = B_0 + B_1\lambda$$

if $\det A_1 \neq 0$ and $\det B_1 \neq 0$, then they are strictly equivalent

$$\text{i.e. } (A_0 + A_1\lambda) = P(B_0 + B_1\lambda)Q$$

where P and Q are constant non singular matrices.

A special case of this is the characteristic matrix of a given

constant matrix A, which is defined as

$$\lambda U - A$$

$\det(\lambda U - A) = 0$ is called the characteristic polynomial and the zeros of this equation are known as the eigen values of A.

This characteristic matrix can be treated as a polynomial matrix and A is said to have elementary divisors and invariant factors.

A matrix A is said to be similar to a matrix B if

$$A = P^{-1} BP$$

for some non-singular matrix P.

It can be seen that 2 similar matrices have the same elementary divisors, since

$$\lambda U - A = P^{-1} (\lambda U - B)P$$

and so their characteristic matrices are similar.

If the field of complex numbers is used to fix the elementary divisors, they are simply terms of the form

$$\{ \psi_i(\lambda) \}^{p_{ij}} = (\lambda - \lambda_j)^{p_{ij}}$$

and in the case of A , λ_j will be an eigen value of A.

The $(p_{ij} \times p_{ij})$ matrix

$$J_{ij} = \begin{bmatrix} \lambda_j & 1 & . & 0 \\ 0 & \lambda_j & 1 & . \\ . & . & . & . \\ . & . & . & 1 \\ 0 & . & 0 & \lambda_j \end{bmatrix}$$

has only one elementary divisor $(\lambda - \lambda_i)^{p_{ij}}$ and is known as a Jordan block. It can be seen that a quasi-diagonal matrix, with Jordan

blocks on the diagonal corresponding to the elementary divisors of A, will have the same elementary divisors as A and will therefore be similar to A.

This matrix, called a Jordan form, constitutes a general normal form for a constant matrix. In the case of distinct eigen value's, all elementary divisors will have order 1 and so the Jordan normal form is simply a diagonal matrix of eigen values.

The canonic form of $A_0 + \lambda A_1$ in the general case

Now consider the matrix

$$A_0 + \lambda A_1$$

This is called a regular pencil of matrices if

1) A_0 and A_1 are square matrices;

2) $\det (A_0 + \lambda A_1)$ is not identically zero.

All other pencils of matrices are called singular.

If $\det A_1$ and $\det B_1 \neq 0$, then the pencils of matrices

$$(A_1\lambda + A_0) \quad \text{and} \quad (B_1\lambda + B_0)$$

are strictly equivalent if they have the same elementary divisors.

If a new parameter ξ is introduced the pencil becomes

$$(\xi A_0 + \lambda A_1)$$

Thus $\det (\xi A_0 + \lambda A_1) = f(\lambda, \xi)$, a function of λ and ξ and the elementary divisors are also functions of λ and ξ i.e. $e(\lambda, \xi)$, where $e(\lambda,1)$ would be the conventional elementary divisors. But now, if $\det A_1 = 0$, elementary divisors of the form ξ^p can occur. These will be termed infinite elementary divisors.

Two regular pencils of matrices are strictly equivalent if and only if they have the same elementary divisors, including the infinite ones.

The canonic form of such a pencil of matrices uses Jordan blocks. The finite elementary divisors are represented by blocks of the form $(U\lambda + J_i)$ and infinite elementary divisors of order p, by (p x p) blocks of the form $(U + E_p\lambda)$.

Where E_p is (p x p) matrix block of the form

$$\begin{bmatrix} 0 & 1 & . & 0 \\ . & 0 & 1 & . \\ & . & . & \\ & & . & 1 \\ 0 & . & . & 0 \end{bmatrix}$$

Thus the canonic form for a regular pencil of matrices is

$$\begin{bmatrix} U_1 & \\ & E_2 \end{bmatrix} \lambda + \begin{bmatrix} J_1 & \\ & U_2 \end{bmatrix}$$

where J_1 is a quasi-diagonal matrix consisting of Jordan blocks, corresponding to finite elementary divisors and E_2 is a quasi-diagonal matrix containing matrices E_p, corresponding to the infinite elementary divisors.

In the singular case

$$(A + \lambda B)\, x = 0$$

has solutions $x(\lambda)$ in polynomial form. Let ε_i be the order of $x_i(\lambda)$. The number of linearly independent solutions is finite, q_1, and so

these can be chosen to minimize their order. Thus the orders will be

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_{n1} < \varepsilon_{n_1+1} = \ldots = \varepsilon_{n2} < \varepsilon_{n_2+1} \ldots = \varepsilon_{q_1}$$

since they may not all be different. These are called the minimal indices.

Similarly, a set of minimal indices can be defined for $(A + \lambda B)^t x = 0$. These are $\eta_1, \eta_2 \ldots\ldots \eta_{q_2}$.

Some of these indices will be zero. Because of the ordering, these will be the first indices,

i.e. the first g of ε_i

and the first h of η_i

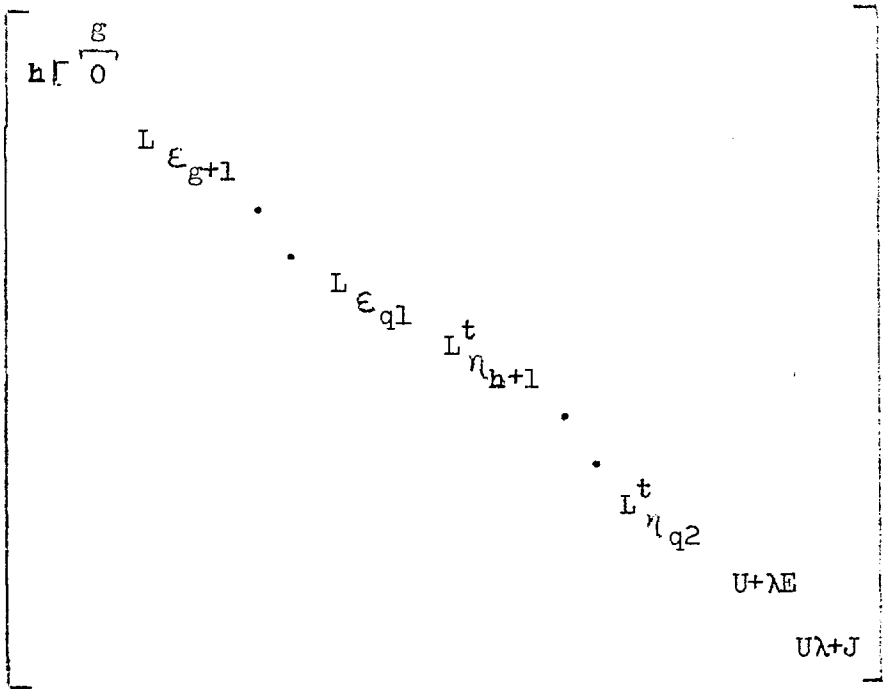
The canonic form for such a pencil of matrices can be written as a quasi-diagonal matrix. The first block is a (h x g) block of zeros corresponding to the zero minimal indices. Each non-zero ε_i gives rise to a (ε_i x ε_{i+1}) matrix of the form

$$L_{\varepsilon_i} = \begin{bmatrix} \lambda & 1 & . & 0 \\ . & \lambda & 1 & \\ . & . & . & . \\ 0 & & \lambda & 1 \end{bmatrix}$$

and each non-zero η_i gives rise to a ($\eta_i + 1$ x η_i) matrix of the form

$$L^t_{\eta_i} = \begin{bmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & 1 & . & \\ & & . & \lambda \\ 0 & & & 1 \end{bmatrix}$$

Thus the general canonic form of a pencil of matrices is a quasi diagonal matrix of the form



Application to differential equations

Consider the differential equation

$$Ax + B\frac{dx}{dt} = f(t).$$

By reducing the pencil of matrices $(A + \lambda B)$ to canonic form by a strictly equivalent transformation, a number of different types of equation result;

(a) The first block gives equations of the form

$$0\, x_i = f_i$$

which is only consistent if $f_i = 0$ and x can be arbitrary;

(b) The second block gives equations of the form

$$L_{\varepsilon_i}\, x = f(t)$$

$$\text{i.e.} \quad \begin{aligned} \frac{dx_1}{dt} + x_2 &= f_1(t) \\ \frac{dx_2}{dt} + x_3 &= f_2(t) \\ &\vdots \\ \frac{dx_{n-1}}{dt} + x_n &= f_{n-1}(t) \end{aligned}$$

These equations are consistent and have a solution, but x_1 is arbitrary.

(c) The next set of equations has the form

$$L^t_{\eta_1} x = f(t)$$

which means $\frac{dx_1}{dt} = f_1(t)$

$$\begin{aligned} \frac{dx_2}{dt} + x_1 &= f_2(t) \\ &\vdots \\ \frac{dx_n}{dt} + x_{n-1} &= f_n(t) \end{aligned}$$

which are only consistent if

$$f_1(t) - \frac{d}{dt} f_2(t) + \frac{d^2}{dt^2} f_3(t) + \ldots (-1)^n \frac{d^{n-1}}{dt} f_n(t) = 0$$

(d) The next has the form

$$U + E \frac{d}{dt} \quad x = f_1(t)$$

$$\text{i.e.} \quad \begin{aligned} x_1 + \frac{d}{dt} x_2 &= f_1(t) \\ x_2 + \frac{d}{dt} x_3 &= f_2(t) \\ &\vdots \\ x_n + 0 &= f_n(t) \end{aligned}$$

i.e.
$$x_n = f_n(t)$$
$$x_{n-1} = f_{n-1}(t) - \frac{d}{dt} f_n(t)$$
$$x_i = f_i(t) + \sum_{j=1}^{n-i} (-1)^j \frac{d^j}{dt^j} \cdot f_{i+j}(t).$$

Thus there is an algebraic solution to these equations.

(e) The last set of equations is of the form

$$U \frac{dx}{dt} + Jx = f(t)$$

which is the usual set of first order differential equations and always possesses a solution is f(t) is continuous.

Thus the only sets of equations which generally possess a unique solution are those in (d) and (e), the last two sets. Thus the condition for the differential equation to have a unique solution is that the pencil of matrices

$$A + \lambda B$$

be regular. The equations due to the infinite elementary divisors will be purely algebraic and will require higher order differentials of f(t) to solve them. The number of differentiations needed depends on the order of the infinite elementary divisors.

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