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On quasi-infinite divisibility, limit theorems and signatures

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Declaration of originality

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This space is dedicated to a quote, which is often intimately related to the author and usually reflects one of his/her life lesson. I do not have a quote, but I will try to summarise in a superficial and incomplete way one of my most important personal belief:

There is nothing more important than love and being loved. For love pushes us to understand and to not be afraid.

Abstract

In this thesis we investigate three main interconnected topics that lie in the interface between probability theory and mathematical statistics. They are quasi-infinite divisibility, limit theorems and signatures.

Concerning quasi-infinite divisibility, we present numerous results on quasi-infinitely divisible (QID) distributions, processes and random measures. Lindner, Pan and Sato in [LPS18] show that QID distributions are dense in the space of all probability distributions with respect to weak convergence. We start our presentation with the results on QID random measures. In particular, we prove spectral representations, we build stochastic integral using them, and show necessary and sufficient conditions for their integrability. Then, we present several results on QID stochastic processes, including various spectral representations. The results presented in this chapter, namely Chapter 2, have been subsequently published in the article [4] and constitute the fundamental basis for the ongoing work [5], where it is proved that QID random measures are dense in the space of completely random measures under convergence in distribution.

Concerning limit theorems, we present two main collection of results. In the first one, we derive some necessary and sufficient conditions for stationary ID multivariate random fields to be mixing. As an example we show that mixed moving average fields are mixing. In the second one, the main result is a central limit theorem for the multivariate Brownian semistationary (BSS) process. In particular, we study the joint asymptotic behaviour of its realised covariation using in-fill (*i.e.* high-frequency) asymptotics. For this central limit theorem we provide examples and feasible results, namely results that can be computed directly from data. These two collections of results will constitute the content of the works [6] and [7].

Finally, we focus on the signature of the fractional Brownian motion (fBm) with Hurst parameter $H > \frac{1}{2}$. The signature of a d-dimensional fBm is a sequence of iterated Stratonovich integrals along the paths of the fBm. We provide a bound for the expected signature of the fBm and show the rate of convergence of the expected signature of the linear piecewise approximation of the fBm to its exact value. Further, we present the general cubature method for the fBm with $H > \frac{1}{2}$ for small times. The cubature method is a numerical deterministic method based on signatures for the approximation of the expected solution of certain SDEs. These results will be published in the work [3].

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1 Introduction

This thesis explores three topics in probability theory and mathematical statistics, which are motivated by real world problems. As indicated in the title they are: quasi-infinite divisibility, limit theorems and signatures. In this introduction we give a general overview of these topics and describe the main results of the thesis.

We remark here that, in order to simplify the exposition of this thesis, we decided to put long proofs in the appendices at the end of each chapter. Further, we tried to be as consistent as possible with the notation throughout the thesis. However, in order to avoid any confusion we expose at the beginning of each chapter the notation used in that particular chapter.

The content of this thesis has been subsequently published in the articles [3, 4, 6, 7].

On quasi-infinite divisibility

Infinitely divisible (ID) distributions represent one of the main families of probability distributions. Their development goes back to the work of Lévy and De Finetti. Among the many properties they satisfy, one is particularly useful in practice: the characteristic function (c.f.), which is the Fourier transform of a probability distribution, of an ID distribution has a Lévy-Khintchine formulation. This formulation allows to write explicitly the c.f. in terms of three elements: the drift (which is a real constant), the Gaussian variance (which is a positive constant), and the Lévy measure (which is a measure on $\mathbb{R}$ that satisfies an integrability condition and has no mass at zero). Further, there is a one to one correspondence between an ID distribution and the triplet composed by these elements, which is called the characteristic triplet. Examples of ID distributions are the Normal and the Poisson distributions.

Recently, a series of works shed new light on a broader class of probability distributions called quasi-ID (QID) distributions, where informally the Lévy measure is now a signed measure (thus taking also negative values). Hence, every ID distribution is QID, but not every QID distribution is ID. A simple formal definition of QID random variable is the following: a random variable X

is QID if and only if there exists two ID random variables Y and Z s.t. $X + Y \stackrel{d}{=} Z$ and Y is independent of Z.

The recent work of Lindner, Pan and Sato [LPS18] represents one of the most important papers of this series. It shows that the set of QID distributions is dense in the set of all probability distributions w.r.t. weak convergence. In [Kh19], the authors complement this result by similar criteria of relative and stochastic compactness for quasi-infinitely divisible distributions with partial weak limits from this class. Moreover, in [LPS18] it is proved that a distribution concentrated on the integers is QID if and only if its characteristic function does not have zeroes. This last result is extended in [Ber19] to distributions which can be written as the sum of a measure concentrated on the integers and a measure which is absolutely continuous w.r.t. the Lebesgue measure. An interesting result shown in [Ber19] states that a distribution which has a Lévy measure with complex values cannot exist.

QID distributions have been shown to have links to the field of prime numbers as well. Indeed, in [Na15] it is shown that $\frac{\xi(\sigma-it)}{\xi(\sigma)}$, where ξ is the complete Riemann zeta function and $t, \sigma \in \mathbb{R}$, has a characteristic function which is QID, but not ID, for $\sigma > 1$. In addition, in [Na] it is shown that any zeta distribution defined by Dirichlet series $\sum_{n=1}^{\infty} a(n)n^{-s}$ with $a(1) > 0$, $a(n) \geq 0$ when $n \geq 2$ and $a(n) = O(n^{\varepsilon})$ for any $\varepsilon > 0$ is QID if $\Re(s) > 1$ is sufficiently large. Further, QID distributions turn out to be of importance in mathematical physics too, as seen in [CDM16] and [DM15].

The majority of our focus in Chapter 2 is devoted to the study of random measures. Random measures have a long history (think of the Poisson process, which is a point process) and many famous mathematicians have worked on them: Itô, Bochner, Renye, Jacod, Kallenberg, etc... The topic remains nowadays a very active areas of research as shown by the publications of several books on the subject (see [Ka17] and references therein). In Chapter 2 we provide spectral representations, mainly Lévy-Khintchine formulations, for QID random measures and using them we define QID stochastic integrals. Further, we provide sufficient and necessary conditions for the integrability of a measurable function with respect to QID random measures.

We point out that in ongoing work [5], which is a work that we do not present here, we prove that a certain class of QID random measures is dense under convergence in distribution in the space of all independently scattered random measures, also known as random measures with independent increments or as completely random measures. This result extends the result in [LPS18] for real valued random variables.

It is also possible to define ID processes, namely stochastic processes whose finite dimensional distributions are ID. One of the most pivotal works in the field of ID processes is given by Rajput and Rosinski's paper in 1989 [RR89]. In this chapter we provide new results on the spectral representation of discrete and centred continuous ID processes. Therefore, it appears natural to see to what extent the results in [RR89] extend to the QID case. This is part of the content of Chapter 2. Indeed, we extend the results of the celebrated 1989 paper by Rajput and Rosinski to the QID framework. However, we also investigate general questions like: What is the Lévy-Khintchine representation of QID processes? What is the Lévy-Khintchine representation when the QID process belongs to l_2 (the Hilbert space of square summable sequences)? What is the connection between QID processes and Lévy processes?

As expected, in this chapter we use many results from measure theory, but we also present new contributions. For example, in Theorem 4 in [Ho77], the author proves a sufficient and necessary condition for the extension of a signed bimeasure to a signed measure on the product σ-algebra. In Section 2.4, we extend this result and provide a unique integral representation for the Jordan decomposition of this signed measure. In addition, in Subsection 2.1.1, we provide a complete account on extension and on decomposition theorems for measures and signed measures on rings. We show that there exists a unique Jordan decomposition on δ-rings and that we can talk about extension theorems even for signed measures, among others.

On Limit Theorems

Mixing properties for ID random fields

In 1970 in his fundamental work [Mar70], Maruyama provided pivotal results for infinitely divisible (ID) processes. Among them, he proved that under certain conditions, known afterwards as Maruyama conditions, these processes are mixing (see Theorem 6 of [Mar70]). After him various authors contributed to this line of research, see for example Gross [Gr94] and Kokoszka and Taqqu [KT96]. In 1996 Rosinski and Zak [RZ96] extended Maruyama's results proving that a stationary ID process $(X_t)_{t\in\mathbb{R}}$ is mixing if and only if $\lim_{t\to\infty} \mathbb{E}\left[e^{i(X_t - X_0)}\right] = \mathbb{E}\left[e^{iX_0}\right]\mathbb{E}\left[e^{-iX_0}\right]$, provided the Lévy measure of X_0 has no atoms in $2\pi\mathbb{Z}$. More recently, Fuchs and Stelzer [FS13] extended some of the main results of Rosinski and Zak to the multivariate case. Parallel to this line of research, new developments have been obtained for ergodic and weak mixing properties of infinitely divisible *random fields*. In particular, see Roy [RoE07] and [RoE09] for Poissonian ID random fields and Roy [RoP10], Roy and Samorodnitsky [RS08] and [WRS13] for α-stable univariate random fields.

In Chapter 3, we fill an important gap by extending the results of Maruyama [Mar70], Rosinski and Zak [RZ96], [RZ97], and Fuchs & Stelzer [FS13] to the multivariate random field case. First, this is crucial for applications since many of them consider a multidimensional domain composed by both spatial and temporal components (and not just temporal ones). This is typically the case for many physical systems, like turbulences (*e.g.* [BS04] [BBV14]), and in econometrics (*e.g.* models based on panel data). Second, with the present work we also close the gap between the two lines of research presented above by focusing on the more general case of multivariate stationary ID random fields.

On the modelling/application level, we prove that multivariate mixed moving average fields are mixing. This is a relevant result since Lévy driven moving average fields are extensively used in many applications throughout various disciplines, such as brain imaging [JR13], tumour growth [BS07] and turbulence [BS04, BS07], among others.

Moreover, we discuss conditions which ensure that a multivariate random fields is weakly mixing. In particular, we show that the proofs of the results obtained for the mixing case can be slightly modified to obtain similar results for the weak mixing case. Finally, we prove that a multivariate stationary ID random field is weak mixing if and only if it is ergodic.

Multivariate BSS process

The second main topic on limit theorems concerns the Brownian semistationary (BSS) process. The univariate BSS process is a stochastic process of the following form:

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s dW_s + \int_{-\infty}^{t} q(t-s)a_s ds,$$

where μ is a constant, W is a Brownian measure on $\mathbb{R}$, g and q are non-negative deterministic functions on $\mathbb{R}$, with $g(t) = q(t) = 0$ for $t \leq 0$, and σ and a are càdlàg processes. The name Brownian semistationary process comes from the fact that when σ and a are stationary and independent of W then Y is stationary. These processes were firstly introduced in [BS09] and, since then, they have been extensively used in applications due to their flexibility and, thus, their capacity of modelling a variety of empirical phenomena. Two of the most notable fields of applications are turbulence and finance.

In the context of turbulence, where the process σ represents the intermittency of the dynamics, these processes are able to reproduce the key stylized features of turbulence data, such as homogeneity, stationarity, skewness, isotropy and certain scaling laws (see [BPS14, CHPP13] and discussion therein). In finance, the BSS process has been applied to the modelling of energy spot prices ([BBV13, Be17]) and of logarithmic volatility of futures ([BLP17b]), among others. Furthermore, fast and efficient simulation schemes for the univariate BSS are available ([BLP17a]).

One of the key aspects of the BSS process that has been analysed in great detail in the last decade is the asymptotic behaviour of its realised power variation. The realised power variation of a process Y_t is the sum of absolute powers of increments of a process, *i.e.*

$$\sum_{i=1}^{\lfloor nt \rfloor} |\Delta_i^n Y|^r, \quad \text{where} \quad \Delta_i^n Y := Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}.$$

For general semimartingales, the study of realised variation has a pivotal role in estimating the key aspects of the process under consideration, *e.g.* the integrated squared volatility given by $\int_0^t \sigma_s^2 ds$ (see [BCD11] for further discussions). This has led to the development of numerous works on this topic (see [Ja08] and the references therein). On the other hand, the BSS process is not in general a semimartingale and the theory of realised power variation for semimartingales does not apply in this case. New results based on different mathematical tools, mainly the ones presented in the works of Peccati, Nourdin and coauthors (see [NP12] and the references therein), have been obtained. In [BCD11] the multipower variation for BSS processes is presented, while in [GR19] Granelli and Veraart obtain the realised covariation for the bivariate BSS without drift. It is important to mention that, in the general multivariate setting, we have the work [BSh04] for the semimartingale case, but no works for the case of BSS processes outside the semimartingale framework.

In Chapter 4, we introduce the multivariate BSS process, study the joint asymptotic behaviour of its realised covariation, present feasible results and relevant examples. In particular,

we will study the asymptotic behaviour of

$$\sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(k)}}{\tau_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\tau_n^{(l)}} \right)_{k,l=1,...,p},$$

where $p \in \mathbb{N}$, $\tau_n^{(j)} > 0$ and $Y_t^{(j)}$ is the j-th component of the multivariate BSS process, for $j = 1, ..., p$. The work of this chapter is motivated by the manifold applications of the BSS process and it is not just an extension to the multivariate case of the results presented in [BCD11] and [GR19]. Indeed, in these previous works, the realised power variations and covariations were always scaled by a scaling factor (τ_n) restricted to a specific structure. We eliminate this restriction and this enables us to obtain all the feasible results presented in this chapter, which were not obtainable otherwise.
We remark that despite the more general theory developed here, no additional assumptions will be added other than the ones already introduced in [BCD11] and [GR19] (but used in a multivariate setting).

Due to the various potential applications of the multivariate BSS process, it appears natural to derive feasible results, namely results that can be computed directly from real data. We focus on two objects:

$$\left(\frac{\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)}}{\sqrt{\sum_{i=1}^{\lfloor nt \rfloor} \left(\Delta_i^n Y^{(k)} \right)^2} \sqrt{\sum_{i=1}^{\lfloor nt \rfloor} \left(\Delta_i^n Y^{(l)} \right)^2}} \right)_{k,l=1,...,p} \quad \text{and} \quad \left(\frac{\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)}}{\sum_{i=1}^{\lfloor nT \rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)}} \right)_{k,l=1,...,p}.$$

Both objects belong to the class of *realised covariation ratios*. Similar ratios tailored to the univariate case have been used in the literature to construct a consistent estimator of key parameters, *e.g.* the smoothness parameter α of the BSS process in [CHPP13]. The second object can be defined as the *relative covolatility* of the BSS process, since it represents the multivariate representation of the relative volatility concept introduced in [BPS14].

We refer to Section 4.5.1 for a presentation of the main results of this chapter (in a simplified framework).

On the signature of the fBm with $H > \frac{1}{2}$

The signature of a d-dimensional fractional Brownian motion (fBm) is a sequence of iterated Stratonovich integrals along the paths of the fBm. It is an object taking values in the tensor algebra over $\mathbb{R}^d$.

Signatures were firstly studied by K.T.-Chen in 1950's in a series of papers [Ch57], [Ch58] and [Ch77]. In the last twenty years the attention devoted to signatures has increased rapidly. This has been caused by the pivotal role they have in rough path theory, a field developed in the late nineties by Terry Lyons culminating in the paper [Te98], which is also at the base of the newly developed theory of regularity structures [Ha14]. The signature of a path summarises the essential properties of that path allowing the possibility to study SPDEs driven by that path.

We remark also the increasing importance of the signatures in the field of machine learning, see for example [NXL17] among many. In light of this increasing importance and of the fact that

rate of convergences are crucial to evaluate algorithmic efficiency, our results may have a *direct* impact to real world applications.

In 2015 Hao Ni and Weijun Xu [NX15] computed the sharp rate of convergence for expected Brownian signatures. They obtained a rate of convergence of 1; in formulas,

$$\left| \mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^I \right) - \mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{m,I} \right) \right| \leq Cm^{-1},$$

where $C > 0$, $\mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^I\right)$ is the $2k$-th level of the expected signature of the Brownian motion and $\mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{m,I}\right)$ is the $2k$-th level of the expected signature of the linear piecewise approximation of the Brownian motion with mesh size $\frac{1}{m}$. A more formal introduction is given in the next sections (see Chapter 5).

On the other hand, for the fBm no progress has been made in this direction. In particular, the rate of convergence for the expected signature of the fBm is not known for any value of the Hurst parameter $H \in (0,1)$. The first result of this chapter address this problem, obtaining the "sharp" rate of convergence for $H \in (\frac{1}{2}, 1)$. In formulas,

$$\left| \mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,I} \right) - \mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,m,I} \right) \right| \leq C' m^{-2H}, \tag{1.1}$$

where $C' > 0$, $\mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,I}\right)$ and $\mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,m,I}\right)$ are respectively the $2k$-th level of the expected signature of the fBm and of its linear piecewise approximation with mesh size $\frac{1}{m}$. To achieve this result, we used the results of Baudoin and Coutin in [BC07], who computed the expected signature for the fractional Brownian motion for $H > \frac{1}{2}$ and also for small times for $H \in (\frac{1}{3}, \frac{1}{2})$. We mention also the works [CQ02] and [LV07], where further properties of the signature of the fBm are studied.

In this chapter we focus on the weak rate of convergence and we refer to the work of Friz and Riedel [FR14] for the strong rate of convergence. It is possible to derive from their work a rate of H for $H > \frac{1}{2}$ (see also Deya, Neuenkirch and Tindel [DNT12] for similar results), while here we obtain a weak convergence rate of $2H$.

In the first result we also provide an explicit expression for the coefficient of the rate of convergence, namely C' in (1.1). We obtain an uniform bound for the coefficient for the $2k$-th term of the signature of

$$C' = \frac{\tilde{A}k(2k-1)}{(k-1)!2^k},$$

where $\tilde{A}$ is an explicit finite constant independent of k. This shows that for any fixed mesh size (*i.e.* m is fixed) the difference between the expected signature of the fBm and the expected signature of its linear piecewise approximation goes to zero (and it goes fast) as the number of iterated integrals increases (namely $k \to \infty$). We stress that for the Brownian motion case the uniform bound of the coefficient is not known (see [NX15]).

The first result holds for general mesh size, like the uniform mesh of size $\frac{1}{m}$ or the dyadic mesh (2^{-m}). In particular, the first result holds for any uniform mesh. For uniform mesh we mean the mesh in which the distances between any two consecutive grid points are all equal.

For the second result we move away from the linear piecewise approximation and focus just on the expected signature. In [CL16], Chevyrev and Lyons showed that the expected signature has infinite radius of convergence, but not a factorial decay. In this chapter we show that the expected signature has a factorial decay, indeed the sharp bound for the $2k$-th term of the signature is simply given by $\frac{1}{k!2^k}$ for all $H \in (\frac{1}{2}, 1)$.

In the $H > \frac{1}{2}$ case, our result gives an alternative proof, with sharper estimates, of the fact that the expected signature of the fractional Brownian motion has infinite radius of convergence, which by [CL16] implies that the expected signature determines the signature in distribution. Our estimate is also sharper than the ones obtained by Friz and Riedel [FR13], from which is possible to obtain a bound of $\frac{2^k C^{2k} (2k)^{k/2}}{(2k/2+)!}$ for some positive constant C independent of k, and by Neuenkirch, Nourdin, Rössler and Tindel [NNRT09], who showed a bound of $\frac{C^k}{\sqrt{(2k)!}}$ (with $C > 2$ and dependent on H). In formulas, our result is:

$$\mathbb{E}\left(\int_{\Delta^{2k}[s,t]} dB^{H,I}\right) \leq \frac{(t-s)^{2kH}}{k!2^k}.$$

In 2003 Lyons and Victoir [LV04] developed a numerical methods for approximating the solution of parabolic PDEs and general SDEs driven by Brownian motions called cubature method on Weiner space. In the cubature method the main first step is to obtain the cubature formula in which the signature of a path (in the case of [LV04] is the Brownian motion) is matched to a finite weighted sum of Dirac delta measures applied to the iterated integrals of the deterministic paths. In this chapter, we develop the framework of the cubature method for fBm, obtain the cubature formula for the fBm in a particular setting, and provide a bound for the approximation error for the general case. Further, we show a simple numerical example where these results apply.

We remark that a first version of this chapter was the content of the author's MRes project. However, the chapter was majorly during the PhD and the main results presented in this PhD thesis were not part of the MRes thesis.

On the connections between these topics

The main idea behind the structure of this thesis is the following. We start with the the QID framework, which is a very general and abstract setting, and present the relevant results. Then, in Chapter 3 we restrict our attention to the ID framework, in particular to the ID multivariate random fields, and investigate their long term behaviour. As it is possible to see, there are many connections between these two topics. A very clear link is provided by Section 3.2, where mixed moving average fields are analysed. The same definition of mixed moving averages requires the use of ID random measures, which is a subclass of QID random measures defined in Chapter 2. Moreover, the integrability conditions for mixed moving averages discussed in Section 3.2 are a specific example of the integrability conditions discussed in Section 2.4 of Chapter 2. Further, there are additional numerous connections due to the fact that both chapters have the work of Rajput and Rosinski [RR89] as starting point. Indeed, in Chapter 2 we extend it to the QID framework, while in Chapter 3 we analysed it from a multivariate random fields point of

view. Moreover, the extensive discussions on the Lévy measure in Chapter 2 facilitates the understanding of the proofs of Chapter 3, in which a considerable use of its properties is made.

Keeping the focus on limit theorems, we move in Chapter 4 to the analysis of BSS processes. Here we focus first on the multivariate Gaussian processes with stationary increments and then on the multivariate BSS processes. A first connection between this and the previous chapter is that the class of stationary multivariate Gaussian processes constitutes a major example for both the multivariate Gaussian processes with stationary increments and the multivariate ID random fields. Moreover, BSS processes are also called volatility modulated moving averages in the literature. This is because they are actually moving averages with an additional stochastic integrand, namely the volatility σ_s. Unfortunately, the introduction of σ generally destroys the infinitely divisibility of the moving average. Thus, we cannot in general hope to use the result in Chapter 3 for the BSS process. However, this is possible when $\sigma_s dW_s$ behaves as dL_s where L is a ID random measure (as defined in Chapter 2).

The Gaussian framework and the focus on the asymptotic behaviour, which are two fundamental elements of Chapter 4, constitutes the leitmotiv of Chapter 5. Here we restrict our attention on a specific Gaussian process, the fractional Brownian motion, and study the asymptotic behaviour of its signature. The asymptotic analysis takes here the form of a specific question: the rate of convergence of the expected signature of the linear piecewise approximation of the fBm to the expected signature of its exact value, as the mesh size goes to zero. The second main result in Chapter 5 is a bound for the truncated expected signature of the fBm, which shows its asymptotic behaviour with respect to the number of iterated integrals considered.

A more complete and connected collection of mathematical topics could have been achieved with the introduction of the ongoing works [1, 2, 5]. These are ongoing works that explore further aspects of the topics investigated in this thesis. Indeed, in [2] we analyse limit theorems for trawl processes, where a trawl process is ID random measure parametrised by time. In [1] we study feasible inference results for BSS processes, providing complementary results to the ones presented in Section 4.4. In [5], we continue the investigation of QID random measures, proving new results and extending some of the statements presented in Section 2.4.

We conclude this section by remarking that there are various and sometimes strong connections between the main topics of this thesis. However, this thesis also represents the author's mathematical journey of the last three years and his willingness to explore various and, sometimes, different areas of probability theory and mathematical statistics.

Outline of the thesis

Chapter 2 is structured as follows. In Section 2.1 we introduce the notation and preliminaries, including the introduction of QID random measures, and we provide some general topological and measure theoretical results.

In Section 2.2 we explore the connections between QID and ID random measures. We show that under certain conditions two ID random measure generates a unique QID random measure and vice versa. We present examples of QID random measures which are not necessarily ID. Then, QID stochastic integrals are defined. We extend the measure theoretical results at the core of Rajput and Rosinski's 1989 paper ([RR89]) to the signed measure framework, and prove that the

QID stochastic integrals are well defined. Further, we provide a Lévy-Khintchine representation and integrability conditions for QID stochastic integrals. Moreover, we explore the set of Λ-integrable functions and show that the linear stochastic integral mapping $f \to \int_S f d\Lambda$ from the space of Λ-integrable functions to the L_p space is continuous.

In Section 2.6 quasi-Lévy measures on l_2 and QID processes are defined. We prove that X is QID if there exist two ID processes Y and Z such that $X + Y \stackrel{d}{=} Z$ with Y independent of X. As mentioned above, the class of such QID processes is called *generated QID processes*. Further, we provide a Lévy-Khintchine representation for generated QID processes on l_2 and for generated QID processes of any arbitrary index set, and present a representation for discrete parameters generated QID processes *à la* Rajput and Rosinski. In the last subsection of Section 2.6, we show further results on general QID processes, provide some examples, and discuss the atomless condition of random measures in the QID framework.

Chapter 3 is structured as follows. In Section 3.1, we discuss some preliminaries on mixing and derive the mixing conditions for multivariate ID stationary random fields. In addition, we study some extensions and other related results. In Section 3.2, we prove that (sums of independent) mixed moving averages (MMA) are mixing, including MMA with an extended subordinated basis. In Section 3.3 we obtain weak mixing versions of the results obtained in Section 3.1 and we prove the equivalence between ergodicity and weak mixing for stationary ID random fields.

Chapter 4 is structured as follows. In Section 4.1, we introduce the multivariate BSS process, the general setting and the basic mathematical concepts of this chapter. It is usually the case that, when a univariate process is extended to the multivariate case, there is more than one way to do it. Hence, we present two possible multivariate extensions of the one-dimensional BSS process. Further, in the same chapter we introduce the Gaussian core, which is a key object for the mathematical understanding and estimation of the BSS process. In Sections 4.2 and 4.3, we present the joint central limit theorem (CLT) for general multivariate Gaussian processes with stationary increments and for the BSS processes, respectively. In particular, in these two chapters we are going to present different cases depending on which multivariate extension of the BSS process and values of the scaling factor τ_n are considered. In addition, in Section 4.3 we prove the weak law of large numbers (WLLN) for BSS processes. In Sections 4.4 we derive the feasible results and in Section 4.5 we present some examples.

We refer to Section 4.5.1 for a presentation of the main results of this chapter (in a simplified framework).

Finally, Chapter 5 is structured in the following way. In Section 5.1 we introduce some notations and preliminaries. In particular, we discuss the signature of the fractional Brownian motion in details and present relevant results by other authors. In Section 5.2 we state the main results of the chapter. In Section 5.3 we discuss the cubature method for the fBm and present a numerical example.

2

Quasi-infinite divisibility

2.1 Notations and preliminaries

In this section we introduce the notation and the preliminaries needed in this chapter. By a measure on a measurable space $(X, \mathcal{G})$ we always mean a positive measure on $(X, \mathcal{G})$, namely an $[0, \infty]$-valued σ-additive set function on $\mathcal{G}$ that assigns the value 0 to the empty set. Given a non-empty set X, the symbol $\mathcal{B}(X)$ stands for the Borel σ-algebra of X, unless stated differently. The law and the characteristic function of a random variable X will be denoted by $\mathcal{L}(X)$ and by $\hat{\mathcal{L}}(X)$, respectively. Given two measurable spaces $(X, \mathcal{G})$ and $(Y, \mathcal{F})$, we denote by $\mathcal{G} \times \mathcal{F}$ the Cartesian product and by $\mathcal{G} \otimes \mathcal{F}$ the product σ-algebra of $\mathcal{G}$ and $\mathcal{F}$. Finally, due to their frequent use we abbreviate the following words: random variable by r.v., random measure by r.m., characteristic function by c.f. and characteristic triplet by c.t..
Given the importance of signed measures in this chapter we recall now the definition and some properties.

Definition 2.1.1 (extended signed measure)**.** *Given a measurable space (X, Σ), that is, a set X with a σ-algebra Σ on it, an extended signed measure is a function $\mu : \Sigma \to \mathbb{R} \cup \{\infty, -\infty\}$ s.t. $\mu(\emptyset) = 0$ and μ is σ-additive, that is, it satisfies the equality $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ where the series on the right must converge in $\mathbb{R} \cup \{\infty, -\infty\}$ absolutely (namely the value of the series is independent of the order of its elements), for any sequence $A_1, A_2, \ldots$ of disjoint sets in Σ.*

As a consequence any extended signed measure can take plus or minus infinity as value, but not both. Throughout this chapter, the term 'signed measure' refers to 'extended signed measure'. Recall also that the *total variation* of a signed measure μ is defined as the measure $|\mu| : \Sigma \to [0, \infty]$ defined by

$$|\mu|(A) := \sup \sum_{j=1}^{\infty} |\mu(A_j)|, \tag{2.1}$$

where the supremum is taken over all the partitions $\{A_j\}$ of $A \in \Sigma$. The total variation $|\mu|$ is finite if and only if μ is finite. By the Hahn decomposition theorem, for any signed measure μ, there exist disjoint Borel sets C^+ and C^- with $C^+ \cup C^- = X$ and $C^+ \cap C^- = \emptyset$ s.t. $\mu(A \cap C^+) \geq 0$ and $\mu(A \cap C^-) \leq 0$ for every $A \in \Sigma$. In the following, we present the definition of mutually singular measures and the Jordan decomposition theorem.

Definition 2.1.2. *Two measures μ and ν on (X, Σ) are said to be* mutually singular *if there are disjoint sets $A, B \in \Sigma$ with $X = A \cup B$ and $\mu(A) = 0$ while $\nu(B) = 0$. In this case, we write $\mu \perp \nu$.*

Theorem 2.1.3 (Jordan Decomposition Theorem, see [Ka01])**.** *Let μ be a signed measure on (X, Σ). Then there exist two mutually singular positive measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$. Furthermore, if λ and ν are any two positive measures with $\mu = \lambda - \nu$, then for each $E \in \Sigma$ we have $\lambda(E) \geq \mu^+(E)$ and $\nu(E) \geq \mu^-(E)$. Finally, if $\lambda \perp \nu$, then $\lambda = \mu^+$ and $\nu = \mu^-$.*

Recall that while the Jordan decomposition is unique the Hahn decomposition is only essential unique, indeed μ-null sets can be transferred from C^+ to C^- and vice versa.

Definition 2.1.4 (Signed bimeasure)**.** *Let (X, Σ) and (Y, Γ) be two measurable spaces. A* signed bimeasure *is a function $M : \Sigma \times \Gamma \to [-\infty, \infty]$ such that:*
(i) *the function $A \to M(A, B)$ is a signed measure on Σ for every $B \in \Gamma$,*
(i) *the function $B \to M(A, B)$ is a signed measure on Γ for every $A \in \Sigma$.*

For a signed bimeasure M, we denote by M^+ and M^- the Jordan decomposition of $B \mapsto M(A, B)$ for fixed $A \in \Sigma$, and M_+ and M_- the Jordan decomposition of $A \mapsto M(A, B)$ for fixed $B \in \Gamma$.

Now, we introduce the concept of a quasi-Lévy type measure. Although it is called measure, it is not always a measure. This explains the need of the following definitions, which we recall from [LPS18]:

Definition 2.1.5. *Let $\mathcal{B}_r(\mathbb{R}) := \{B \in \mathcal{B}(\mathbb{R}) | B \cap (r, r) = \emptyset\}$ for $r > 0$ and $\mathcal{B}_0(\mathbb{R}) := \bigcup_{r>0} \mathcal{B}_r(\mathbb{R})$ be the class of all Borel sets that are bounded away from zero. Let $\nu : \mathcal{B}_0(\mathbb{R}) \to \mathbb{R}$ be a function such that $\nu_{|\mathcal{B}_r(\mathbb{R})}$ is a finite signed measure for each $r > 0$ and denote the total variation, positive and negative part of $\nu_{|\mathcal{B}_r(\mathbb{R})}$ by $|\nu_{|\mathcal{B}_r(\mathbb{R})}|$, $\nu^+_{|\mathcal{B}_r(\mathbb{R})}$ and $\nu^-_{|\mathcal{B}_r(\mathbb{R})}$ respectively. Then the* total variation *$|\nu|$, the* positive part *ν^+ and the* negative part *ν^- of ν are defined to be the unique measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying*

$$|\nu|(\{0\}) = \nu^+(\{0\}) = \nu^-(\{0\}) = 0$$

$$\text{and} \quad |\nu|(A) = |\nu_{|\mathcal{B}_r(\mathbb{R})}|,\ \nu^+(A) = \nu^+_{|\mathcal{B}_r(\mathbb{R})}(A),\ \nu^-(A) = \nu^-_{|\mathcal{B}_r(\mathbb{R})}(A),$$

for $A \in \mathcal{B}_r(\mathbb{R})$, for some $r > 0$.

As mentioned in [LPS18], ν is not a signed measure because it is defined on $\mathcal{B}_0(\mathbb{R})$, which is not a σ-algebra. In the case when it is possible to extend the definition of ν to $\mathcal{B}(\mathbb{R})$ such that ν is a signed measure, then we will identify ν with its extension to $\mathcal{B}(\mathbb{R})$ and speak of ν as a signed measure. Moreover, the uniqueness of $|\nu|$, ν^+ and ν^- is ensured by the Carathéodory's extension theorem (see Lemma 2.1.14).

Remark 2.1.6. For the sake of clarity, notice that $\mathcal{B}_0(\mathbb{R}) = \{B \in \mathcal{B}(\mathbb{R}) : 0 \notin \overline{B}\} \neq \{B \in \mathcal{B}(\mathbb{R}) : 0 \notin B\}$. Indeed, consider the set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ then A is a Borel set since it is a countable intersection of closed Borel sets. Notice that $A \in \{B \in \mathcal{B}(\mathbb{R}) : 0 \notin B\}$. Moreover, observe that for every $\epsilon > 0$ we can find an element of A that does not belong to $\mathcal{B}_\epsilon$. Hence, $A \notin \mathcal{B}_0(\mathbb{R})$.

Thus, note that saying that a property holds "for every $A \in \mathcal{B}_r(\mathbb{R})$ with $r > 0$" is equivalent to saying that it holds "for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$".

Throughout this chapter we define the centering function τ (on a general Hilbert space) as

$$\tau(x) := \begin{cases} x & \text{if} \quad \|x\| \leq 1, \\ \frac{x}{\|x\|} & \text{if} \quad \|x\| > 1. \end{cases}$$

where $\|\cdot\|$ is the norm of the Hilbert space considered. Notice that this centering function satisfies equation (1) in [LPS18]. The following definition has been introduced for the first time in [LPS18].

Definition 2.1.7 (quasi-Lévy type measure, quasi-Lévy measure, QID distribution)**.** *A quasi-Lévy type measure is a function $\nu : \mathcal{B}_0(\mathbb{R}) \to \mathbb{R}$ satisfying the condition in Definition 2.1.5 and such that its total variation $|\nu|$ satisfies $\int_{\mathbb{R}}(1 \wedge x^2)|\nu|(dx) < \infty$.*
Let μ be a probability distribution on $\mathbb{R}$. We say that μ is quasi-infinitely divisible if its characteristic function has a representation

$$\hat{\mu}(\theta) = \exp\left(i\theta\gamma - \frac{\theta^2}{2}a + \int_{\mathbb{R}}\left(e^{i\theta x} - 1 - i\theta\tau(x)\right)\nu(dx)\right)$$

where $a, \gamma \in \mathbb{R}$ and ν is a quasi-Lévy type measure. The characteristic triplet (γ, a, ν) of μ is unique (see [[Sa99], Exercise 12.2]), and a and γ are called the Gaussian variance *and the* drift *of μ, respectively.*
A quasi-Lévy type measure ν is called quasi-Lévy measure*, if additionally there exist a quasi-infinitely divisible distribution μ and some $a, \gamma \in \mathbb{R}$ such that (γ, a, ν) is the characteristic triplet of μ. We call ν the quasi-Lévy measure of μ.*

The above definition extends to the $\mathbb{R}^d$ case (for $d > 1$) as shown in Remark 2.4 in [LPS18]. Further, in this chapter we always keep the same order for the elements in the c.t.: the first element is the drift, the second one is the Gaussian variance, and the third one is the (quasi) Lévy measure.

A quasi-Lévy measure is always a quasi-Lévy type measure, while the converse is not true as pointed out in Example 2.9 of [LPS18]. Moreover, we say that a function f is *integrable with respect to quasi-Lévy type measure ν* if it is integrable with respect to $|\nu|$. Then, we define:

$$\int_B f d\nu := \int_B f d\nu^+ - \int_B f d\nu^-, \quad B \in \mathcal{B}(\mathbb{R}).$$

Given the above discussions, it appears clear why it is sometimes useful to work with the characteristic pair (ζ, γ) instead of (γ, a, ν) where ζ is a signed measure defined as:

$$\zeta(B) = a\delta_0(B) + \int_B (1 \wedge x^2)\nu(dx), \quad B \in \mathcal{B}(\mathbb{R}).$$

Then, the representation of c.f. of μ becomes:

$$\hat{\mu}(\theta) = \exp\left(i\theta\gamma + \int_{\mathbb{R}} g_\tau(x,\theta)\zeta(dx)\right),$$

where $g_\tau : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ defined by

$$g_\tau(x,\theta) = \begin{cases} (e^{i\theta x} - 1 - i\theta\tau(x))/(1\wedge x^2), & x \neq 0, \\ -\frac{\theta^2}{2}, & x = 0. \end{cases}$$

The function $g_\tau(\cdot,\theta)$ is bounded for each fixed $\theta\in\mathbb{R}$ and is continuous at zero (see [LPS18] for further details).
Let us now mention one of the few known explicit results on QID distributions.

Theorem 2.1.8 (Theorem 4.3.4 in [Cu75]). *Let $d\in\mathbb{N}$. The c.t. $(\gamma,0,\nu)$, where ν is a finite quasi-Lévy type measure, is the c.t. of a QID distribution on $\mathbb{R}^d$ if and only if $\exp(\nu) := \sum_{n=1}^\infty \frac{\nu^{*n}}{n!}$ is a measure. In that case, $\mu\sim(\gamma,0,\nu)$ is given by*

$$\mu = \frac{\delta_\gamma * \exp(\nu)}{\exp(\nu(\mathbb{R}^d))}.$$

We introduce now the framework adopted in this chapter when dealing with QID random measures. Throughout this chapter we denote by S an arbitrary non-empty set and by $\mathcal{S}$ a δ-ring with the additional condition that there exists an increasing sequence of sets $S_1, S_2, \cdots \in \mathcal{S}$ s.t. $\bigcup_{n\in\mathbb{N}} S_n = S$. In this framework S does not need to belong to $\mathcal{S}$ (thus $\mathcal{S}$ is not necessarily an algebra) and arbitrary subsets of S do not need to satisfy the condition $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{S}$ (thus $\mathcal{S}$ is not necessarily a σ-ring).

Definition 2.1.9 (Signed measure on a ring). *A set function $\mu(A)$ defined on the elements of a ring $\mathcal{R}$ with values in $[-\infty,\infty]$ will be called a signed measure, if $\mu(\emptyset)=0$ and if for every sequence $A_1, A_2, ...$ of disjoint sets of $\mathcal{R}$ for which $A=\bigcup_{k=1}^\infty A_k \in \mathcal{R}$ we have*

$$\mu(A) = \sum_{k=1}^\infty \mu(A_k) \tag{2.2}$$

and the relation (2.2) holds absolutely (namely independent of the order of its elements).

Similarly, it is possible to extend the definition of bimeasures on rings. Let us now introduce one of the central objects of this chapter.

Definition 2.1.10 (QID random measure). *Let $\Lambda = \{\Lambda(A) : A\in\mathcal{S}\}$ be a real valued stochastic process defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. We call Λ to be an independently scattered r.m., if, for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{S}$, the r.v. $\Lambda(A_n)$, $n=1,2,...$, are independent, and, if $\bigcup_{n=1}^\infty A_n \in \mathcal{S}$, then we have $\Lambda(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \Lambda(A_n)$ a.s. (where the series is assumed to converge almost surely). In addition, if $\Lambda(A)$ is a QID (ID) r.v., for every $A\in\mathcal{S}$, then we call Λ a QID (ID) r.m..*

In this chapter $\Lambda = \{\Lambda(A) : A\in\mathcal{S}\}$ will denote a QID r.m.. Since, for every $A\in\mathcal{S}$, $\Lambda(A)$ is a

QID r.v., its c.f. can be written in the Lévy-Khintchine form:

$$\hat{\mathcal{L}}(\Lambda(A))(\theta) := \mathbb{E}(e^{i\theta\Lambda(A)}) = \exp\left(i\theta\nu_0(A) - \frac{\theta^2}{2}\nu_1(A) + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)F_A(dx)\right), \quad (2.3)$$

where $-\infty < \nu_0(A) < \infty$, $0 \leq \nu_1(A) < \infty$ and F_A is a quasi-Lévy measure on $\mathbb{R}$, for $A \in \mathcal{S}$.

For the sake of completeness we report here Proposition 2.1 of [RR89].

Proposition 2.1.11 (Proposition 2.1 in [RR89]). (a) *Let Λ be an ID r.m. with the c.f. given by*

$$\mathbb{E}(e^{i\theta\Lambda(A)}) = \exp\left(i\theta\nu_0(A) - \frac{\theta^2}{2}\nu_1(A) + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)F_A(dx)\right). \quad (2.4)$$

Then $\nu_0 : \mathcal{S} \mapsto \mathbb{R}$ is a signed-measure, $\nu_1 : \mathcal{S} \mapsto [0,\infty)$ is a measure, F_A is a Lévy measure on $\mathbb{R}$, for every $A \in \mathcal{S}$, and $\mathcal{S} \ni A \mapsto F_A(B) \in [0,\infty)$ is a measure, for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$.
(b) *Let ν_0, ν_1 and F. satisfy the conditions given in* (a). *Then there exists a unique (in the sense of finite-dimensional distributions) ID r.m. Λ such that* (2.4) *holds.*
(c) *Let ν_0, ν_1 and F. as in* (a) *and define*

$$\lambda(A) = |\nu_0|(A) + \nu_1(A) + \int_{\mathbb{R}} (1 \wedge x^2)F_A(dx), \quad A \in \mathcal{S}.$$

Then $\lambda : \mathcal{S} \mapsto [0,\infty)$ is a measure such that $\lambda(A_n) \to 0$ implies $\Lambda(A_n) \xrightarrow{p} 0$ for every $\{A_n\} \subset \mathcal{S}$; further if $\Lambda(A'_n) \xrightarrow{p} 0$ for every $\{A'_n\} \subset \mathcal{S}$ s.t. $A'_n \subset A_n \in \mathcal{S}$, then $\lambda(A_n) \to 0$.

Moreover, we recall the following result from the work of Prékopa (Theorem 2.1 in [Pr56]). In doing this we also correct a typo in that statement.

Theorem 2.1.12 (Theorem 2.1 in [Pr56]). *In order that a finitely additive set function $\xi(A)$ defined on the elements of the ring $\mathcal{R}$ should be countably additive it is necessary and sufficient that, for every non-increasing sequence of sets $B_1, B_2, \ldots$ with $B_k \in \mathcal{R}$ $(k = 1, 2, \ldots)$ and $B_n \searrow \emptyset$, $\xi(B_n) \xrightarrow{p} 0$ as $n \to \infty$.*

We conclude with the following remark on the conditions under which a quasi-Lévy type measure is a signed measure. From the discussion done in this section we have not mentioned *when* a quasi-Lévy measure is a signed measure. The idea behind this is that ν is the difference of two Lévy measures μ_1 and μ_2 and when these measures are both infinite then we cannot find a signed measure such that $\nu = \mu_1 - \mu_2$. This is because we get the form $\infty - \infty$, which is not defined. However, a sufficient condition is that one of the two Lévy measures is finite because then $\nu = \mu_1 - \mu_2$ defines a signed measure. Further, we do not make any general assumption about it but we remark that in many interesting cases considered in the existing literature, mainly in [LPS18], quasi-Lévy measures are signed measures.

2.1.1 On extension and decomposition results

In this subsection we provide various known and new results regarding extension and decomposition theorems and make some topological remarks.

The first result is a well-known result: the Carathéodory's extension theorem.

Theorem 2.1.13 (Carathéodory's extension theorem, see Theorem 1.41 in [Kl14]). *Let μ be a pre-measure on a ring $\mathcal{R}$ of subsets of a space X (i.e. a countably additive set function, namely a set function such that $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} A_n$ for every disjoint $A_1, A_2, ... \in \mathcal{R}$ such that $\cup_{n=1}^{\infty} A_n \in \mathcal{R}$). Assume that μ is σ-finite (i.e. that there exists $S_1, S_2, ... \in \mathcal{R}$ such that $X = \cup_{n=1}^{\infty} S_n$ and that $\mu(S_n) < \infty$ for every $n \in \mathbb{N}$) then there exists a unique σ-finite measure $\bar{\mu}$ on $\sigma(\mathcal{R})$ such that $\mu(A) = \bar{\mu}(A)$ for all $A \in \mathcal{R}$.*

The next lemma justifies the well-posedness of the measures introduced in Definition 2.1.5. The following result and its proof are essential for the results of Section 2.6, where the l_2 and the uncountable settings are explored. It obviously extends to the $\mathbb{R}^d$ case.

Lemma 2.1.14. *A set function $\nu : \mathcal{B}_0(\mathbb{R}) \to \mathbb{R}$ is a quasi-Lévy type measure if and only if there exist two Lévy measures $\nu^{(1)}$ and $\nu^{(2)}$ s.t. $\nu_{|\mathcal{B}_r(\mathbb{R})}(A) = \nu^{(1)}_{|\mathcal{B}_r(\mathbb{R})}(A) - \nu^{(2)}_{|\mathcal{B}_r(\mathbb{R})}(A)$ for every $A \in \mathcal{B}_r(\mathbb{R})$ with $r > 0$. Moreover, this decomposition is unique.*

Proof. $\Rightarrow$: Let ν be a set function on $\mathcal{B}_0$ as in the Definition 2.1.5. We show that the positive and the negative parts and the total variation of ν are unique. We prove it for the positive part since the arguments for the others are the same. Let ν^+ be a set function on $\mathcal{B}_0 \cup \{0\}$ such that $\nu^+(A) = \nu^+_{|\mathcal{B}_r(\mathbb{R})}(A)$ for every $A \in \mathcal{B}_r(\mathbb{R})$, where $\nu^+_{|\mathcal{B}_r(\mathbb{R})}$ is the positive part of the Jordan decomposition of $\nu_{|\mathcal{B}_r(\mathbb{R})}$, that $\nu^+(\{0\}) = 0$ and that $\nu^+(\{0\} \cup A) = \nu^+(\{0\}) + \nu^+(A)$ for every $A \in \mathcal{B}_r(\mathbb{R})$, where $r > 0$. Observe that $\mathcal{B}_0 \cup \{0\}$ is a ring on $\mathbb{R}$. Then, it is possible to see that ν^+ is countably additive and since $\nu^+((-\infty, -\frac{1}{n}) \cup (\frac{1}{n}, \infty) \cup \{0\}) = \nu^+_{|\mathcal{B}_{\frac{1}{n}}(\mathbb{R})}((-\infty, -\frac{1}{n}) \cup (\frac{1}{n}, \infty)) < \infty$ and $\bigcup_{n=1}^{\infty} (-\infty, -\frac{1}{n}) \cup (\frac{1}{n}, \infty) \cup \{0\} = \mathbb{R}$, the set function is σ-finite. Then, by the Carathéodory's extension theorem we obtain that there exists a unique measure on $(\mathbb{R}, \sigma(\mathcal{B}_0 \cup \{0\}))$. Now, notice that any open interval on $\mathbb{R}$ is the countable union of elements of $\mathcal{B}_0 \cup \{0\}$, for example $(0, 1) = \bigcup_{n=1}^{\infty} (\frac{1}{n}, 1)$. Therefore, $\sigma(\mathcal{B}_0 \cup \{0\}) = \mathcal{B}$ and ν^+ is unique.

Now, if ν is a quasi-Lévy type measure then by definition $\int_{\mathbb{R}} (1 \wedge x^2) |\nu|(dx)$, $\int_{\mathbb{R}} (1 \wedge x^2) \nu^+(dx)$ and $\int_{\mathbb{R}} (1 \wedge x^2) \nu^-(dx)$ are all finite, hence ν^+, ν^- and $|\nu|$ are all Lévy measures.

$\Leftarrow$: Let $\nu^{(1)}$ and $\nu^{(2)}$ be two Lévy measures s.t. $\nu_{|\mathcal{B}_r(\mathbb{R})} = \nu^{(1)}_{|\mathcal{B}_r(\mathbb{R})} - \nu^{(2)}_{|\mathcal{B}_r(\mathbb{R})}$ for every $A \in \mathcal{B}_r(\mathbb{R})$ with $r > 0$. Since $\nu^{(1)}_{|\mathcal{B}_r(\mathbb{R})}$ and $\nu^{(2)}_{|\mathcal{B}_r(\mathbb{R})}$ are two finite measures then $\nu_{|\mathcal{B}_r(\mathbb{R})}$ is a finite signed measure. Thus, we denote by $\nu^+_{|\mathcal{B}_r(\mathbb{R})}$ and $\nu^-_{|\mathcal{B}_r(\mathbb{R})}$ its (unique) Jordan decomposition. Notice that $\nu^+_{|\mathcal{B}_r(\mathbb{R})}(A) = \nu^+_{|\mathcal{B}_s(\mathbb{R})}(A)$ for every $A \in \mathcal{B}_r(\mathbb{R})$ with $0 < s \leq r$. Moreover, let $\nu^+(\{0\}) = \nu^-(\{0\}) = 0$. Then, as in the previous part of the proof by Carathéodory's extension theorem we obtain the existence of two unique σ-finite measures ν^+ and ν^- on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. It is possible to see that they satisfy the conditions of Definition 2.1.5 and that

$$\int_{\mathbb{R}} (1 \wedge x^2) |\nu|(dx) \leq \int_{\mathbb{R}} (1 \wedge x^2) \nu^{(1)}(dx) - \int_{\mathbb{R}} (1 \wedge x^2) \nu^{(2)}(dx).$$

Therefore, ν is a (unique) quasi-Lévy type measure. □

Consider now the following general lemma.

Lemma 2.1.15. *Let X an arbitrary non-empty set and let $\mathcal{R}$ be a δ-ring. Then, for every $E \in \mathcal{R}$ we have that $\{E \cap B : B \in \mathcal{S}\}$ is a σ-algebra.*

Proof. Fix $E \in \mathcal{R}$. First, notice that any $A \in \{E \cap B : B \in \mathcal{R}\}$ is also an element of $\mathcal{R}$ since δ-rings are closed under (countable) intersections. Further, for every $A \in \{E \cap B : B \in \mathcal{R}\}$ we have that $E \setminus A \in \{E \cap B : B \in \mathcal{R}\}$, again by definition. Consider any $A_1, A_2, ... \in \{E \cap B : B \in \mathcal{R}\}$ then by De Morgan's law we have

$$\bigcup_{i=1}^{\infty} A_i = \left[\bigcap_{i=1}^{\infty} A_i^c\right]^c \left(= E \setminus \left[\bigcap_{i=1}^{\infty} [E \setminus A_i]\right]\right)$$

and since $\mathcal{R}$ (and so $\{E \cap B : B \in \mathcal{R}\}$) is closed under countable intersections, we have that $\bigcap_{i=1}^{\infty} A_i^c \in \{E \cap B : B \in \mathcal{R}\}$ and consequently that $\bigcup_{i=1}^{\infty} A_i \in \{E \cap B : B \in \mathcal{R}\}$. □

In the following lemma we extend the concept of Jordan decomposition to signed measures on δ-rings (see Definition 2.1.9).

Lemma 2.1.16. *Let X an arbitrary non-empty set and let $\mathcal{R}$ a δ-ring. Let μ be a (possibly infinite) signed measure on $\mathcal{R}$. Then, there exist two unique measures μ^+ and μ^- on $\mathcal{R}$ such that $\mu = \mu^+ - \mu^-$ and that on any $A \in \mathcal{R}$ they are mutually singular.*

Proof. For every $A \in \mathcal{R}$, let $\mu^+(A) := \sup_{B \in \mathcal{R}, B \subset A} \mu(A)$. Since by Lemma 2.1.15 we know that $\{B \in \mathcal{R}, B \subset A\}$ (namely $\{A \cap B : B \in \mathcal{R}\}$) is a σ-algebra, then $\mu^+(A)$ is the positive part of the Jordan decomposition of $\mu_{|\{B \in \mathcal{R}, B \subset A\}}(A)$. The same applies to $\mu^-(A) := -\inf_{B \in \mathcal{R}, B \subset A} \mu(A)$. Then, for $A \in \mathcal{R}$ we have $\mu(A) = \mu^+(A) - \mu^-(A)$ and both μ^+ and μ^- enjoy the properties of the Jordan decomposition. □

Remark 2.1.17. The above lemma also shows that there exists a Hahn decomposition for every $A \in \mathcal{R}$.

For the sake of completeness, we point out that a similar (but weaker) statement can be obtained for a real valued additive set functions on rings, see Section 3.10 (iv) in [Bog07].

In the following lemma we provide a version of the Carathéodory's extension theorem for signed measures.

Lemma 2.1.18. *Let X an arbitrary non-empty set and let $\mathcal{R}$ a δ-ring. Let μ be a σ-finite signed measure on $\mathcal{R}$ (namely there exists a sequence $S_1, S_2, ... \in \mathcal{R}$ s.t. $X = \cup_{n=1}^{\infty} S_n$ and that $-\infty < \mu(S_n) < \infty$ for every $n \in \mathbb{N}$). Then μ^+ and μ^- can be uniquely extended to two σ-finite measures on $(X, \sigma(\mathcal{R}))$.*

Proof. Since μ is σ-finite, then μ^+ and μ^- are also σ-finite. This is because if for any S_n we have that $\mu^+(S_n)$ is not finite then $\mu(S_n)$ cannot be finite, hence we have a contradiction. Then, by the Carathéodory's extension theorem we obtain the stated result. □

2.2 The connection between ID and QID random measures

In this section, we investigate the connection between ID and QID random measures. The first result is a lemma. It states that given a characteristic triplet (c.t.) satisfying certain conditions there exists a unique QID random measure with that c.t.. The techniques used to prove this

lemma are similar to the ones used in Proposition 2.1 of Rajput and Rosinski's work [RR89] and they are based on basic probabilistic results like the Chebychev's inequality. Building on these two results, we prove in Proposition 2.2.2 that given two ID random measures Λ_1 and Λ_2 there exists a unique QID random measure Λ such that, for every $A \in \mathcal{S}$, $\Lambda(A) + \Lambda_2(A) \stackrel{d}{=} \Lambda_1(A)$ with $\Lambda_2(A)$ is independent of $\Lambda(A)$.

In the second part of this section we go in the other direction. We start from a QID random measure and impose a minimal condition on its quasi-Lévy measure $F_A(B)$, where $A \in \mathcal{S}$ and $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$, to ensure that it behaves roughly as a bimeasure (which is similar to what happens in the ID setting). Then, given any such QID r.m. Lemma 2.2.4 states that we can find a unique corresponding c.t., while Proposition 2.2.7 shows that there exist two (not necessarily unique) ID random measures that satisfy the relation in the above paragraph (namely $\Lambda(A) + \Lambda_2(A) \stackrel{d}{=} \Lambda_1(A)$, for every $A \in \mathcal{S}$).

Let us now make a general remark. While we know that for any c.t. with a Lévy measure it corresponds an ID r.v., this is not necessarily true in the QID case. In other words, given any c.t. with a quasi-Lévy measure there is not always a corresponding distribution (however, if it does then the distribution is QID). Thus, the lack of one-to-one correspondence between c.t. and QID distributions force us to be careful when making the connection between QID r.m. and their c.t.. However, this affects more the statement of the results than the arguments to prove them.

In particular, we conclude this section with an example that shows why this minimal condition on $F_A(B)$ is needed. Indeed, we show an example in which under certain assumptions this condition fails, while under others it holds. In the latter, this example represents a general case for which all the results presented in this section hold.

We are now ready to state and prove our results. The following result represents a partial QID analogue of Proposition 2.1.11.

Lemma 2.2.1. *Let $\nu_0 : \mathcal{S} \mapsto \mathbb{R}$ be a signed measure, $\nu_1 : \mathcal{S} \mapsto [0, \infty)$ be a measure, G_A be a Lévy measure on $\mathbb{R}$ for every $A \in \mathcal{S}$ and $\mathcal{S} \ni A \mapsto G_A(B) \in [0, \infty)$ be a measure for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$. Let M be defined as G. Let $F_A(B) := G_A(B) - M_A(B)$ for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$ and $A \in \mathcal{S}$. Then, there is a unique quasi-Lévy type measure F_A for every $A \in \mathcal{S}$. Moreover, $\mathcal{S} \ni A \mapsto F_A(B) \in (-\infty, \infty)$ is a signed measure for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$.*
Further, if $(\nu_0(A), \nu_1(A), F_A)$ is the characteristic triplet of a QID random variable $\forall A \in \mathcal{S}$. Then there exists a unique (in the sense of finite-dimensional distributions) QID random measure Λ such that (2.3) *holds, and F_A is a quasi-Lévy measure for every $A \in \mathcal{S}$.*

Now, we present the first link between ID and QID random measures.

Proposition 2.2.2. *For every $A \in \mathcal{S}$, let $\Lambda(A)$ be a random variable and let two ID random measures Λ_1 and Λ_2 with $\Lambda(A) + \Lambda_2(A) \stackrel{d}{=} \Lambda_1(A)$ s.t. $\Lambda_2(A)$ is independent of $\Lambda(A)$. Then, $\Lambda = \{\Lambda(A) : A \in \mathcal{S}\}$ is a QID random measure and it is unique. In this case, we say that the r.m. Λ is* generated by two ID random measures.

Remark 2.2.3. The above result can be restated using instead of the condition that $\Lambda(A)$ is a random variable and there exist two ID random measures $\Lambda(A) + \Lambda_2(A) \stackrel{d}{=} \Lambda_1(A)$, such that

$\Lambda_2(A)$ is independent of $\Lambda(A)$, $\forall A \in \mathcal{S}$, the equivalent alternative conditions:
(i) $\forall A \in \mathcal{S}$ and $\theta \in \mathbb{R}$, $\mathcal{L}(\Lambda(A))$ is a distribution and

$$\hat{\mathcal{L}}(\Lambda(A))(\theta) = \frac{\hat{\mathcal{L}}(\Lambda_1(A))(\theta)}{\hat{\mathcal{L}}(\Lambda_2(A))(\theta)}, \tag{2.5}$$

(ii) $\forall A \in \mathcal{S}$ $\mathcal{L}(\Lambda(A))$ is a distribution and the c.t. of $\Lambda(A)$ can be rewritten as the difference of the c.t. of $\Lambda_1(A)$ and $\Lambda_2(A)$.

Proof. Let $\nu_0^{(j)}$, $\nu_1^{(j)}$ and $F_\cdot^{(j)}$ the measures corresponding to Λ_j, for $j = 1, 2$ (see Proposition 2.1.11). Let $\nu_0(A) := \nu_0^{(1)}(A) - \nu_0^{(2)}(A)$, $\nu_1(A) := \nu_1^{(1)}(A) - \nu_1^{(2)}(A)$ for all $A \in \mathcal{S}$. Moreover, let F_A be the unique quasi-Lévy measure defined by the difference between the Lévy measures $F_A^{(1)}$ and $F_A^{(2)}$, namely $F_A(B) := F_A^{(1)}(B) - F_A^{(2)}(B)$ for all $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$ and $|F_A|(\{0\}) = 0$, for all $A \in \mathcal{S}$. Then ν_0, ν_1 and $F_\cdot$ satisfies the conditions of Lemma 2.2.1.
Further, the condition that $\Lambda(A)$ is a random variable together with $\Lambda(A) + \Lambda_2(A) \overset{d}{=} \Lambda_1(A)$ implies that $\mathcal{L}(\Lambda(A))$ is QID with c.t. $(\nu_0(A), \nu_1(A), F_A)$, hence, by Lemma 2.7 in [LPS18] we have $\nu_1(A) \geq 0$, $\forall A \in \mathcal{S}$.
Then, by Lemma 2.2.1 we obtain the stated result (including the statement that Λ is uniquely determined). □

In the following we present two results, where the second is a partial "only if" result of Proposition 2.2.2.

Lemma 2.2.4. *Let Λ be a QID random measure and let $K_\cdot(B) : A \mapsto K_A(B)$ be a measure s.t. $K_A(B) \geq F_A(B)$, for every $A \in \mathcal{S}$ and $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$. Assume that $K_A(\cdot) : B \mapsto K_A(B)$ is Lévy measure for every $A \in \mathcal{S}$. Then $\nu_0 : \mathcal{S} \mapsto \mathbb{R}$ is a signed measure, $\nu_1 : \mathcal{S} \mapsto [0, \infty)$ is a measure, F_A is a quasi-Lévy measure on $\mathbb{R}$, for every $A \in \mathcal{S}$, and $\mathcal{S} \ni A \mapsto F_A(B) \in (-\infty, \infty)$ is a signed measure, for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$.*

Corollary 2.2.5. *Let Λ be a QID random measure with $F_\cdot^+(B) : A \mapsto F_A^+(B)$ a measure for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$. Then $\nu_0 : \mathcal{S} \mapsto \mathbb{R}$ is a signed measure, $\nu_1 : \mathcal{S} \mapsto [0, \infty)$ is a measure, F_A is a quasi-Lévy measure on $\mathbb{R}$, for every $A \in \mathcal{S}$, and $\mathcal{S} \ni A \mapsto F_A(B) \in (-\infty, \infty)$ is a signed measure, for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$. Moreover, we have that $\mathcal{S} \ni A \mapsto F_A^-(B) \in [0, \infty)$ is a measure, for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$.*

Proof. It follows from the same arguments of Lemma 2.2.4. □

Remark 2.2.6. Notice that we could use the assumption that $F_\cdot^-(\mathbb{R}) : A \mapsto F_A^-(\mathbb{R})$ is a measure, instead. Then, the arguments would be the same.

Proposition 2.2.7. *Let Λ be QID random measure and let $K_A(B)$ as in Lemma 2.2.4. Then, there exist two ID random measures Λ_1 and Λ_2 with $\Lambda(A) + \Lambda_2(A) \overset{d}{=} \Lambda_1(A)$ such that $\Lambda_2(A)$ is independent of $\Lambda(A)$ and has zero Gaussian part, $\forall A \in \mathcal{S}$.*

Proof. Let $F_A^{(1)} := K_A$, $F_A^{(2)}(B) := K_A(B) - F_A(B)$ for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$ and $F_A^{(2)}(\{0\}) = 0$. In addition, let $\nu_0^{(1)} := 2\nu_0$, $\nu_0^{(2)} := \nu_0$, $\nu_1^{(1)} := \nu_1$, and $\nu_1^{(2)} := 0$. By Lemma 2.2.4, we obtain that $F_A^{(j)}(\cdot) : \mathcal{B}(\mathbb{R}) \to [0, \infty)$, $F_\cdot^{(j)}(B) : \mathcal{S} \to [0, \infty)$ for any $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$,

$\nu_0^{(j)} : \mathcal{S} \to [0,\infty)$ and $\nu_0^{(j)} : \mathcal{S} \to [0,\infty)$ are all measures, for $j = 1, 2$. In addition, since F_A is a quasi-Lévy measure and K_A is a Lévy measure then $F_A^{(2)}$ is Lévy measure.
Then, by Proposition 2.1 point (ii) of [RR89] we know that there exist two unique ID random measures Λ_1 and Λ_2 such that, for every $A \in \mathcal{S}$, they have c.t. given by $(\nu_0^{(1)}(A), \nu_1^{(1)}(A), F_A^{(1)})$ and $(\nu_0^{(2)}(A), \nu_1^{(2)}(A), F_A^{(2)})$, respectively. Finally, it is possible to see that the following relation holds: for every $A \in \mathcal{S}$ and $\theta \in \mathbb{R}$, $\hat{\mathcal{L}}(\Lambda(A))(\theta) = \frac{\hat{\mathcal{L}}(\Lambda_1(A))(\theta)}{\hat{\mathcal{L}}(\Lambda_2(A))(\theta)}$. □

Remark 2.2.8. The reason why Λ_1 and Λ_2 are not determined uniquely by Λ is because given ν_0 we cannot exclude the possibility of finding two measures $\tilde{\nu}_0^{(1)}$ and $\tilde{\nu}_0^{(2)}$ such that $\nu_0(A) = \tilde{\nu}_0^{(1)}(A) - \tilde{\nu}_0^{(2)}(A)$, where $\tilde{\nu}_0^{(1)}(A) \neq 2\nu_0(A)$ and $\tilde{\nu}_0^{(2)}(A) \neq \nu_0(A)$. This leads to two measures $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ such that they satisfies the same conditions as Λ_1 and Λ_2. Moreover, the same reasoning applied to ν_0 may also applies to F_A.

In the following example we observe that the condition on $F_A^+(B)$ being a measure is not always satisfied even if we restrict to random measures Λ with $\mathcal{L}(\Lambda(A))$ concentrated on $\mathbb{Z}$, $\forall A \in \mathcal{S}$.

Example 2.2.9. Consider a quasi-Lévy measure F_A to be given by $F_A = \sum_{l\in\mathbb{Z}, l\neq 0} b_l^A \delta_l$, where b_l^A are real numbers, then $|F_A| = \sum_{l\in\mathbb{Z}, l\neq 0} |b_l^A| \delta_l$. Such a quasi-Lévy measure arises when we consider a distribution supported on the integers whose c.f. has no zeroes (see Theorem 8.1 in [LPS18]). While F_A^+ and F_A^- are Lévy measure for every fixed $A \in \mathcal{S}$ they are not always measures for fixed $B \in \mathcal{B}(\mathbb{R})$. Indeed, consider $A = A_1 \cup A_2$ with $A_1 \cap A_2 = \emptyset$. Since F_A is a signed measure, then $b_k^{A_1\cup A_2} = b_k^{A_1} + b_k^{A_2}$. However, it might happen that $b_k^{A_1}$ and $b_k^{A_2}$ have different signs. This implies that $F^+_{A_1\cup A_2}(\{k\}) \neq F^+_{A_1}(\{k\}) + F^+_{A_2}(\{k\})$ and $F^-_{A_1\cup A_2}(\{k\}) \neq F^-_{A_1}(\{k\}) + F^-_{A_2}(\{k\})$.

On the other hand, this example shows that if we consider $F_A = \sum_{l\in\mathbb{Z}, l\neq 0} b_l^A \delta_l$ such that b_l^A, with $l \in \mathbb{Z}$, do not change sign for all $A \in \mathcal{S}$, then this represents an example where the theory presented so far applies.

2.3 Examples of QID random measures

In the previous section we have seen that under certain conditions two ID random measures generate a QID random measure and vice versa. In this section we explore other cases and show necessary and sufficient conditions for the existence and uniqueness of QID r.m.. In particular, we show the existence of QID r.m. which are not ID r.m. and prove that there exists a corresponding c.t. to these QID r.m.. The techniques used to prove these results rely on the particular structure of these random measures, on some results of [LPS18], and on classical probabilistic results such as the Lévy continuity theorem and the Kolmogorov extension theorem.

2.3.1 The distribution concentrated on the integers

In this subsection we investigate the case where $\text{supp}(\mathcal{L}(\Lambda(A))) \subset \mathbb{Z}$. In other words, we consider $\mathcal{L}(\Lambda(A)) = \sum_{m\in\mathbb{Z}} a_m \delta_m$, for $a_m \in \mathbb{R}$. We start by showing the equivalent of Proposition 2.1 of [RR89].

Proposition 2.3.1. (i) *Let Λ be a QID random measure with* $\mathrm{supp}(\mathcal{L}(\Lambda(A)))\subset\mathbb{Z}$, $\forall A\in\mathcal{S}$. *Then for each $A\in\mathcal{S}$, $\nu_1(A)=0$, the drift of $\hat{\mathcal{L}}(\Lambda(A))\in\mathbb{Z}$ and F_A is a quasi-Lévy measure concentrated on $\mathbb{Z}\setminus\{0\}$ and finite. Moreover, $\nu_0:\mathcal{S}\mapsto\mathbb{R}$ is a signed measure, $\nu_1:\mathcal{S}\mapsto\mathbb{R}$ is the zero measure, F_A is a quasi-Lévy measure on $\mathbb{R}$, for every $A\in\mathcal{S}$, and $\mathcal{S}\ni A\mapsto F_A(B)\in(-\infty,\infty)$ is a signed measure, for every $B\in\mathcal{B}(\mathbb{R})$.*
(ii) *Let ν_0, ν_1 and $F_\cdot(B)$ be as in* (i) *and such that* (2.3) *is the c.f. of a distribution and has no zeros, for each $A\in\mathcal{S}$. Then there exists a unique (in the sense of finite-dimensional distributions) quasi-ID random measure Λ s.t.* (2.3) *holds.*

Proof. (i) The first assertion is the content of Theorem 8.1 of [LPS18]. For the finite additivity of ν_0, ν_1 and $F_\cdot(B)$ we follow the first part of the proof of Proposition 2.1 of [RR89]. Indeed, let $\{A_k\}_{k=1}^n$ be pairwise disjoint sets in $\mathcal{S}$. By the uniqueness of Lévy-Khintchine representation of the c.f. of a QID distribution and by $\hat{\mathcal{L}}\left(\Lambda\left(\bigcup_{k=1}^n A\right)\right)=\prod_{k=1}^n\hat{\mathcal{L}}(\Lambda(A_k))$, we obtain the finite additivity property. Now, let $\{A_n\}\subset\mathcal{S}$, $A_n\searrow\emptyset$. Since $\Lambda(A_n)\to 0$ then $\hat{\mathcal{L}}(\Lambda(A_n))\to\hat{\delta_0}$ and by Theorem 8.5 of [LPS18] we obtain that $\nu_0(A_n)\to 0$, $\nu_1(A_n)\to 0$ and $\sum_{l\in\mathbb{Z}}|F_{A_n}(\{l\})|\to 0$. Now, observe that since $F_{A_n}=\sum_{l\in\mathbb{Z},l\neq 0}b_{n,l}\delta_l$ for some $b_{n,l}\in\mathbb{R}$, then $|F_{A_n}|=\sum_{l\in\mathbb{Z},l\neq 0}|b_{n,l}|\delta_l$, and since $F_{A_n}(\{l\})=b_{n,l}$, we obtain that $|F_{A_n}|(\mathbb{R})\to 0$, because we have that $F_{A_n}(\{k\})\to 0$ $\forall k\in\mathbb{Z}$. Therefore, we deduce that $F_{A_n}(B)\to 0$ for every $B\in\mathcal{B}(\mathbb{R})$.

(ii) From Theorem 8.5 of [LPS18] we obtain that $(\nu_0(A),\nu_1(A),F_A)$ is the c.t. of a QID random variable $\forall A\in\mathcal{S}$. Further, the existence of a finitely additive independently scattered random measure $\Lambda=\{\lambda(A):A\in\mathcal{S}\}$ follows by the Kolmogorov extension theorem. To prove that Λ is countable additive let $A_n\searrow\emptyset$ with $\{A_n\}\subset\mathcal{S}$. Then, by definition of (signed) measures, $\nu_0(A_n)\to 0$, $\nu_1(A_n)\to 0$ and $F_{A_n}(B)\to 0$ for every $B\in\mathcal{B}(\mathbb{R})$ s.t. $0\notin\overline{B}$. From the last limit and since F_{A_n} is concentrated on the integers, we obtain that $|F_{A_n}|(\mathbb{R})\to 0$. Therefore, we have that $\Lambda(A_n)\xrightarrow{p}0$, hence Λ is countably additive. □

Open Question 2.3.2. Following Corollary 8.2 in [LPS18], it might be possible to extend the above result to the case of distributions concentrated on a lattice of the form $r+h\mathbb{Z}$ where $r\in\mathbb{R}$ and $h>0$. Is it really possible?

2.3.2 The distribution with an atom of mass greater than 1/2

In this subsection we explore the case where a random measure is such that $\mathcal{L}(\Lambda(A))(\{k_A\})>1/2$ for some $k_A\in\mathbb{R}$. We have the following result for $k_A=0$.

Proposition 2.3.3. (i) *Let Λ be an independent scattered r.m. with $p_A:=\mathcal{L}(\Lambda(A))(\{0\})>1/2$, $\forall A\in\mathcal{S}$. Then Λ is a QID random measure and $\nu_1(A)=0$, the drift of $\hat{\mathcal{L}}(\Lambda(A))$ equal zero, and F_A is a finite quasi-Lévy measure given by*

$$F_A=\left(\sum_{m=1}^{\infty}\frac{1}{m}(-1)^{m+1}\left(\frac{1-p_A}{p_A}\right)^m(\delta_0*\sigma_A)^{*m}\right)_{\mathbb{R}\setminus\{0\}},\tag{2.6}$$

where $\sigma_A=(1-p_A)^{-1}(\mathcal{L}(\Lambda(A))-p_A\delta_0)$, for every $A\in\mathcal{S}$.
Moreover, $\nu_0:\mathcal{S}\mapsto\mathbb{R}$ and $\nu_1:\mathcal{S}\mapsto\mathbb{R}$ are the zero measure, F_A is a quasi-Lévy measure on $\mathbb{R}$, for every $A\in\mathcal{S}$, and $\mathcal{S}\ni A\mapsto F_A(B)\in(-\infty,\infty)$ is a signed measure, for every $B\in\mathcal{B}(\mathbb{R})$.

(ii) *Let ν_0 and ν_1 be the zero measures, let μ_A be a distribution with c.t. $(0,0,F_A)$ where F_A is a quasi-Lévy measure on $\mathbb{R}$, for every $A \in \mathcal{S}$, and $\mathcal{S} \ni A \mapsto F_A(B) \in (-\infty,\infty)$ is a signed measure, for every $B \in \mathcal{B}(\mathbb{R})$. In particular, let*

$$F_A = \left(\sum_{m=1}^{\infty} \frac{1}{m} (-1)^{m+1} \left(\frac{1-p_A}{p_A} \right)^m (\delta_0 * \sigma_A)^{*m} \right)_{\mathbb{R}\setminus\{0\}},$$

where $p_A = \mu_A(\{0\}) > 1/2$ and $\sigma_A = (1-p_A)^{-1}(\mu_A - p_A\delta_0)$, $\forall A \in \mathcal{S}$. Then there exists a unique (in the sense of finite-dimensional distributions) quasi-ID random measure Λ s.t. $\mathcal{L}(\Lambda(A)) = \mu_A$, for every $A \in \mathcal{S}$.

Proof. (i) The first statement is the content of Theorem 4.3.7 in [Cu75] (see also Theorem 3.1 of [LPS18]). Now, the finite additivity follows from the same arguments used before (see the proof of Proposition 2.3.1). For countable additivity, let $\{A_n\} \subset \mathcal{S}$, $A_n \searrow \emptyset$. Since $\Lambda(A_n) \xrightarrow{p} 0$, then $\mathcal{L}(\Lambda(A_n)) \to \delta_0$, which implies that $p_{A_n} \to 1$. Then, from (2.6) and using the fact that F_{A_n} is finite we get $F_{A_n}(B) \to 0$ for every $B \in \mathcal{B}(\mathbb{R})$.

(ii) The existence and the finite additivity of Λ are straightforward. Concerning the countable additivity, notice that if $F_{A_n}(B) \to 0$ for every $B \in \mathcal{B}(\mathbb{R})$, then $p_{A_n} \to 1$, which implies that $\mathcal{L}(\Lambda(A_n)) \xrightarrow{d} \delta_0$. □

Under certain conditions, it is possible to have QID distributions with non-positive quasi-Lévy measure. An example of such conditions are presented in Lemma 2.8 in [LPS18]. Thus, we have the following remark.

Remark 2.3.4. All the results presented in Section II of [RR89] apply mutatis mutandis to the case of QID random measures with quasi-Lévy measure which take only non-positive values. Indeed, consider $F_A(\cdot)$ to be our non-positive quasi-Lévy measure, where $A \in \mathcal{S}$. Then all the results in Section II of [RR89] hold for the total variation $|F_A|(\cdot)$. But since $F_A = -|F_A|$ we obtain the statement.

2.4 QID Stochastic integrals

This is one of the main sections of this chapter. Here, we investigate various results concerning QID stochastic integrals. This section has three subsections. The presentation of the results in each of them is similar. However, each subsection has its own assumptions and results.

In the first subsection, similarly to the setting of Lemma 2.2.1, we assume that F is given by the difference between G and M. This gives us enough structure that allows us to use the contributions of Rajput and Rosinski in [RR89] as a starting point for the proof of our results, which are the following. First, to each QID random measure Λ we associate a measure λ on $\mathcal{S}$, which dominates Λ, namely if $\lambda(A_n) \to 0$ then $\Lambda(A_n) \xrightarrow{p} 0$. This λ can be extended to a measure on $\sigma(\mathcal{S})$, which we call the control measure of Λ and with an abuse of notation we denote it by λ. Second, we show that the (almost) bimeasure G can be extended to a measure on $\sigma(S) \otimes \mathcal{B}(\mathbb{R})$ and that it can be decomposed into ρ_G and λ, formally $\int_{S\times\mathbb{R}} h(s,x)G(ds,dx) = \int_S \int_{\mathbb{R}} h(s,x)\rho_G(s,dx)\lambda(ds)$, for every measurable function $h : S \times \mathbb{R} \mapsto [0,\infty]$, and where $\rho_G(s,\cdot)$ is a Lévy measure and $\rho_G(\cdot,B)$ is a Borel measurable function. The same holds for M. We use

these results to demonstrate that something similar holds for F. These results allow us to obtain a spectral representation for Λ in terms of λ.

At this point we introduce the stochastic integral $\int_S f d\Lambda$ and define what it means for a measurable function f to be Λ-integrable. Then, we show that this stochastic integral is well-defined. The just mentioned definition and result appear only in the first subsection because they do not depend on the particular QID random measure considered and, so, they hold for all the subsections without any modifications. Finally, we show under which conditions a measurable function f is Λ-integrable and demonstrate how this is related to being Λ_G and Λ_M integrable. In these last results we provide also a spectral representation for $\int_S f d\Lambda$.

In the other two subsections we investigate the same questions under different assumptions. In particular, in the second subsections we assume that $\mathcal{S}$ is a σ-algebra and that F is a bimeasure and satisfies a boundedness condition. In the last, subsection we assume again that $\mathcal{S}$ is a σ-algebra and in addition that F satisfies a weaker boundedness condition than the one of the second subsection. At the beginning of the third subsection, we show that when $\mathcal{S}$ is a σ-algebra, then the assumptions of the third subsections are the weakest ones in comparison with the ones of the second and of the third subsection. On the other hand, the results obtained in the second subsection are the most complete ones. For example, we are able to obtain an *if and only if* result for the integrability conditions of f w.r.t. Λ, while in the other two subsections we can only obtain an *if* result. [Note: in the first subsection we can also obtain an only if result but we need to assume that f is integrable w.r.t. both Λ_G and Λ_M.]

As mentioned in the introduction in [5], the authors show that a certain class of QID r.m. is dense in the class of all independently scattered r.m. with respect to convergence in distribution. We remark here that this class of QID r.m. satisfies the conditions of the first subsection. Further, in the case $\mathcal{S}$ is a σ-algebra, these QID r.m. satisfy the conditions of every subsection of this section (see [5] for details). This is a further motivation for investigating these three different settings in details. We conclude by remarking that this section represents the QID extension of Chapter II in [RR89], which is at the core of the theory of ID r.m. and processes. Indeed, we also extends the measure theoretical results on which the results in [RR89] are based.

2.4.1 The generating two ID random measure case

In this subsection we focus on the cases seen in Section 2.2. In particular, the setting is the one of Lemma 2.2.1 (or equivalently of Proposition 2.2.2), which is more general than the one of Lemma 2.2.4. Indeed, all the results presented in this section apply to the setting of Lemma 2.2.4 and they are obtained by simply substituting G by K and M by $K - F$.

The first result regards the construction of the control measure of Λ.

Proposition 2.4.1. *Let ν_0, ν_1 and F. be as in Lemma 2.2.1. Define*

$$\lambda(A) = |\nu_0|(A) + \nu_1(A) + \int_{\mathbb{R}} (1 \wedge x^2) G_A(dx) + \int_{\mathbb{R}} (1 \wedge x^2) M_A(dx), \quad A \in \mathcal{S}. \tag{2.7}$$

Then $\lambda : \mathcal{S} \mapsto [0, \infty)$ is a measure such that $\lambda(A_n) \to 0$ implies $\Lambda(A_n) \xrightarrow{p} 0$ for every $\{A_n\} \subset \mathcal{S}$.

Proof. First, notice that all the addends on the right hand side of (2.7) are measures. Hence, λ

is a measure. Further, let $A_n \searrow \emptyset$ with $\{A_n\} \subset \mathcal{S}$. Since $G_{A_1}(B) \geq G_{A_n}(B)$ for every $n \geq 1$ and $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$, we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}} (1 \wedge x^2) G_{A_n}(dx) = \lim_{n\to\infty} \int_{|x|<\epsilon} (1 \wedge x^2) G_{A_n}(dx) + \lim_{n\to\infty} \int_{|x|\geq\epsilon} (1 \wedge x^2) G_{A_n}(dx)$$

$$\leq \int_{|x|<\epsilon} (1 \wedge x^2) G_{A_1}(dx) + \lim_{n\to\infty} G_{A_n}(\{|x| \geq \epsilon\}).$$

The second addend goes directly to zero for every $\epsilon > 0$, and for the first we obtain the convergence to zero by letting $\epsilon \to 0$. The same holds for M. Hence, λ is countably additive. Moreover, using Theorem 4.3 point (a) in [LPS18] or the Lévy continuity theorem it is straightforward to see that if $\lambda(A_n) \to 0$ then $\Lambda(A_n) \xrightarrow{p} 0$. □

Definition 2.4.2. *Since $\lambda(S_n) < \infty$, $n = 1, 2, ...$ we extend λ to a σ-finite measure on $(S, \sigma(\mathcal{S}))$; we call λ the* control measure *of Λ.*

It is possible to use different control measures than the one mentioned. For example, we can consider $\tilde{\lambda} := \lambda_G + \lambda_M$, where λ_G and λ_M are the control measure of Λ_G and Λ_M. Observe that $\tilde{\lambda} \gg \lambda$ because the total variation of the drift of Λ is smaller than the sum of the total variations of the drift of Λ_G and Λ_M. This $\tilde{\lambda}$ has the additional property that if $\Lambda_G(A'_n) \to 0$ and $\Lambda_M(A'_n) \xrightarrow{p} 0$ for every $\{A'_n\} \subset \mathcal{S}$ s.t. $A'_n \subset A_n \in \mathcal{S}$, then (by Proposition 2.1.11) $\tilde{\lambda}(A_n) \to 0$ and, so, $\lambda(A_n) \to 0$.

The reason why we prefer λ is because it is potentially the smallest (up to a constant) measure such that $\lambda \gg \log \hat{\mathcal{L}}(\Lambda)(\theta)$ for every $\theta \in \mathbb{R}$. The reason why λ is only "potentially" the smallest has to do with an old and classical problem, the Maharam's Problem posed in 1947 by Maharam [Mah47], which has been solved in 2008 by Talagrand [Ta08]. In few words and applied to our case, the solution of this problem states that an exhaustive submeasure (think of $A \mapsto F_A^+(B)$ which we know it is not always a measure) is not always absolutely continuous w.r.t. a measure. In other words, given $A \mapsto F_A^+(B)$ is not always possible to find a measure $A \mapsto \hat{\lambda}(A)$ such that $\hat{\lambda}(A_n) \to 0$ implies $F_A^+(B) \to 0$. By considering that there exist two generating ID r.m. we impose that $G_A(B) \geq F_A^+(B)$ and $M_A(B) \geq F_A^-(B)$ (for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \bar{B}$), and thus solving the issue. We will see in the next two subsections how to obtain a control measure without this assumption (but with other assumptions).

Since in this subsection we are concerned with QID random measures then it is of secondary importance which specific couple of ID r.m. generate Λ. Indeed, given two ID r.m. Λ_G and Λ_M with c.f. $(\nu_0^G, \nu_1^G, G.)$ and $(\nu_0^M, \nu_1^M, M.)$ that generate the QID r.m. Λ, we can always find two different ID r.m. with (for example) c.t. $(0, \nu_1, G.)$ and $(-\nu_0, 0, M.)$ which also generate Λ.

Therefore, w.l.o.g. throughout this section we let $\Lambda_G \sim (0, \nu_1, G.)$ and $\Lambda_M \sim (-\nu_0, 0, M.)$. We denote by λ_G and λ_M the control measure of Λ_G and Λ_M, namely $\lambda_G(A) = \nu_1(A) + \int_{\mathbb{R}} (1 \wedge x^2) G_A(dx)$ and $\lambda_M(A) = \|\nu_0\|(A) + \int_{\mathbb{R}} (1 \wedge x^2) M_A(dx)$. Observe that in this setting we have that $\lambda = \tilde{\lambda}$.

The following result is a corollary of Lemma 2.3 in [RR89].

Corollary 2.4.3. *Let $F.$ be as in Lemma 2.2.4. Then there exist two unique σ-finite signed*

measures G and M on $\sigma(\mathcal{S}) \otimes \mathcal{B}(\mathbb{R})$ such that

$$G(A \times B) = G_A(B) \quad \textit{and} \quad M(A \times B) = M_A(B), \quad \textit{for all } A \in \mathcal{S},\ B \in \mathcal{B}(\mathbb{R}).$$

Moreover, there exist two functions $\rho_G, \rho_M : S \times \mathcal{B}(\mathbb{R}) \mapsto [0, \infty]$ such that
(i) *$\rho_G(s, \cdot), \rho_M(s, \cdot)$ are Lévy measures on $\mathcal{B}(\mathbb{R})$, for every $s \in S$,*
(ii) *$\rho_G(\cdot, B), \rho_M(\cdot, B)$ are Borel measurable function, for every $B \in \mathcal{B}(\mathbb{R})$,*
(iii) *$\int_{S\times\mathbb{R}} h(s,x)G(ds,dx) = \int_S \int_{\mathbb{R}} h(s,x)\rho_G(s,dx)\lambda(ds)$, for every $\sigma(\mathcal{S})\otimes\mathcal{B}(\mathbb{R})$-measurable function $h : S \times \mathbb{R} \mapsto [0, \infty]$, and the same holds for M. This equality can be extended to real and complex-valued functions h.*

Proof. The above statement is almost identical to Lemma 2.3 in [RR89]. The only differences are two: a different function, call it $\tilde{\rho}_G$, instead of ρ_G, and λ_G instead of λ. In the proof of Lemma 2.3 in [RR89] $\tilde{\rho}_G := \frac{d\lambda_0}{d\lambda_G}(s)(1 \wedge x^2)^{-1}q(s, dx)$ where $q(s, B)$ is such that $s \mapsto q(s, B)$ is Borel measurable and $B \mapsto q(s, B)$ is a probability measure. Moreover, it is shown that $G(A, B) = \int_A q(s, B)\lambda_0 ds$. Now, let

$$\rho_G(s, dx) = \frac{d\lambda_0}{d\lambda}(s)(1 \wedge x^2)^{-1}q(s, dx).$$

Since $s \mapsto \rho_G(s, B)$ is still measurable and since $\lambda_0(A) \leq \lambda_G(A) \leq \lambda(A)$ then $B \mapsto \rho_G(s, B)$ is still a Levy measure too. Finally, by the properties of the Radon-Nikodym derivative we get $\int_{S\times\mathbb{R}} h(s,x)G(ds,dx) = \int_S \int_{\mathbb{R}} h(s,x)\tilde{\rho}_G(s,dx)\lambda_G(ds) = \int_S \int_{\mathbb{R}} h(s,x)\rho_G(s,dx)\lambda(ds)$. □

In the following proposition the goal is to extend Corollary 2.4.3 to the QID setting. To achieve this, the proof of the next result follows the leitmotif of this chapter, namely it tries to obtain the equivalent QID result by considering the difference of two ID objects. The QID object in this case is $\rho(s, B)$ and is defined as the difference of $\rho_G(s, B)$ and $\rho_M(s, B)$. Then, we look at the Jordan decomposition of $\rho(s, \cdot)$, which we call $\rho_+(s, \cdot)$ and $\rho_-(s, \cdot)$, and check that they satisfy the same conditions of ρ_G and ρ_M.

Once we have done this, we can define a measure on $\sigma(\mathcal{S}) \otimes \mathcal{B}(\mathbb{R})$, which we call $\tilde{F}^+$ and which satisfies (2.8), see below. The reason why we adopt the notation $\tilde{F}^+(C)$ and not $F^+(C)$, where $C \in \sigma(\mathcal{S})\otimes\mathcal{B}(\mathbb{R})$ is because we cannot obtain a signed measure on this product σ-algebra. Briefly, this is because we are considering an extension of our space from $\mathcal{S}\times\mathcal{B}(\mathbb{R})$ to $\sigma(\mathcal{S})\otimes\mathcal{B}(\mathbb{R})$ and we know that extensions do not applies smoothly to signed measures (or to real valued bi-set functions).

In particular, if we were in the ID case, this $\tilde{F}^+$ should have been F, seen as the extension of the bi-set function $F(A, B) = F_A(B)$ from $\mathcal{S}\times\mathcal{B}(\mathbb{R})$ to a measure $F(C)$ on $\sigma(\mathcal{S})\otimes\mathcal{B}(\mathbb{R})$. However, in the QID case we cannot make this extension for F^+ and F^- (see next paragraph). To solve this problem, we could directly impose that F^+ and F^- extend to measures on $\sigma(\mathcal{S}) \otimes \mathcal{B}(\mathbb{R})$. However (and this is a crucial point), we would have obtained the same results as the ones presented in this subsection. But, in this subsection we assume something weaker, namely that F^+ and F^- are dominated by G and M, which can be extended to measures on $\sigma(\mathcal{S}) \otimes \mathcal{B}(\mathbb{R})$ (as shown in Corollary 2.4.3).

The reasons why we cannot extend $F_A^+(B)$ and $F_A^-(B)$ from $\mathcal{S}\times\mathcal{B}(\mathbb{R})$ to measures $F^+(C)$ and

$F^-(C)$ on $\sigma(\mathcal{S})\otimes\mathcal{B}(\mathbb{R})$ is the following: the bi-set functions $F_A^+(B)$ and $F_A^-(B)$ do not necessarily behave like bimeasures. Indeed, consider the simplified case in which $\mathcal{S}$ is a σ-algebra and that $F_A(B)$ is a bimeasure on $\mathcal{S}\times\mathcal{B}(\mathbb{R})$. We do not know whether $A\mapsto F_A^+(B)$ and $A\mapsto F_A^+(B)$ are measures or not. In particular, recall that $F_A^+(B)=\sup_{B_n\in\mathcal{B}(\mathbb{R}),B_n\subset B}\sum_{n=1}^\infty F_A(B_n)$; hence we know that $B\mapsto F_A^+(B)$ is a measure (by the Jordan decomposition), but we do not necessarily know that $A\mapsto F_A^+(B)$ is a measure. See Example 4.9, where an explicit case is provided.

Of course we could try to avoid extending $F_A^+(B)$ and $F_A^-(B)$ and just try to extend $F_A(B)$ – after all, the final objective is to get a representation like (2.9). However, this requires certain assumptions on F or on $\mathcal{S}$, which are explored in the next two subsections.

Proposition 2.4.4. *Let F. be as in Lemma 2.2.4. Then there exists a unique function $\rho: S\times\mathcal{B}_0(\mathbb{R})\mapsto[-\infty,\infty]$ s.t.*

$$\rho(s,B)=\rho_G(s,B)-\rho_M(s,B),\quad s\in S,\ B\in\mathcal{B}(\mathbb{R})\ s.t.\ 0\notin\overline{B}$$

The function $\rho(s,\cdot)$ is a quasi-Lévy type measure on $\mathcal{B}(\mathbb{R})$ for every $s\in S$. Moreover, its positive and negative part are two unique functions $\rho^+,\rho^-: S\times\mathcal{B}(\mathbb{R})\mapsto[0,\infty]$ such that
(i) *$\rho^+(s,\cdot),\rho^-(s,\cdot)$ are Lévy measures on $\mathcal{B}(\mathbb{R})$, for every $s\in S$,*
(ii) *$\rho^+(\cdot,B),\rho^-(\cdot,B)$ are Borel measurable function, for every $B\in\mathcal{B}(\mathbb{R})$,*
Moreover, there exist two unique σ-finite measures $\tilde{F}^+$ and $\tilde{F}^-$ on $\sigma(\mathcal{S})\otimes\mathcal{B}(\mathbb{R})$ such that

$$\int_{S\times\mathbb{R}}h(s,x)\tilde{F}^+(ds,dx)=\int_S\int_{\mathbb{R}}h(s,x)\rho^+(s,dx)\lambda(ds),\tag{2.8}$$

for every $\sigma(\mathcal{S})\otimes\mathcal{B}(\mathbb{R})$-measurable function $h: S\times\mathbb{R}\mapsto[0,\infty]$, and the same holds for F^-. This equality can be extended to real and complex-valued functions h. Finally, for every $A\in\mathcal{S}$ and for every real $\mathcal{B}(\mathbb{R})$-measurable function g s.t. $\int_A\int_{\mathbb{R}}g(x)|\rho|(s,dx)\lambda(ds)<\infty$, we have that

$$\int_{\mathbb{R}}g(x)F_A(dx)=\int_A\int_{\mathbb{R}}g(x)\rho(s,dx)\lambda(ds)\tag{2.9}$$

and for every $B\in\mathcal{B}(\mathbb{R})$ s.t. $0\notin\overline{B}$,

$$\tilde{F}^+(A,B)\geq F_A^+(B)\quad and\quad \tilde{F}^-(A,B)\geq F_A^-(B).$$

Remark 2.4.5. The reason why there is no a signed measure $\tilde{F}$ on $\sigma(\mathcal{S})\otimes\mathcal{B}(\mathbb{R})$ s.t. $\tilde{F}(C)=\tilde{F}^+(C)-\tilde{F}^-(C)$ is that both $\tilde{F}^+$ and $\tilde{F}^-$ might have infinite values.

We present now one of the key results of this section. It represents the first step for the construction of the c.f. of $\int_S f\Lambda$. We remark that the ID version of following results is frequently used in the field of (applied) probability, especially when the control measure is the Lebesgue measure.

Proposition 2.4.6. *The characteristic function of $\Lambda(A)$ can be written as:*

$$\mathbb{E}(e^{i\theta\Lambda(A)})=\exp\left(\int_A K(\theta,s)\lambda(ds)\right),\quad\theta\in\mathbb{R},A\in\mathcal{S},$$

where

$$K(\theta,s)=i\theta a(s)-\frac{\theta^2}{2}\sigma^2(s)+\int_{\mathbb{R}}e^{i\theta x}-1-i\theta\tau(x)\rho(s,dx),$$

$a(s)=\frac{d\nu_0}{d\lambda}(s)$, $\sigma^2(s)=\frac{d\nu_1}{d\lambda}(s)$ *and* ρ *has been introduced in Proposition 2.4.4, and* $\exp(K(\theta,s))$ *is the characteristic function of a QID random variable if it exists. Moreover, we have*

$$|a(s)|+\sigma^2(s)+\int_{\mathbb{R}}(1\wedge x^2)\rho_G(s,dx)+\int_{\mathbb{R}}(1\wedge x^2)\rho_M(s,dx)=1,\qquad \lambda\text{-}a.e..$$

Proof. The first statement follows from the Lévy-Khintchine formulation, Corollary 2.4.3 and Proposition 2.4.4. In particular, since $\int_A\int_{\mathbb{R}}(1\wedge x^2)|\rho|(s,dx)\lambda(ds)<\infty$ then

$$\int_{\mathbb{R}}e^{i\theta x}-1-i\theta\tau(x)F_A(dx)=\int_A\int_{\mathbb{R}}e^{i\theta x}-1-i\theta\tau(x)\rho(s,dx)\lambda(ds),$$

and so

$$\mathbb{E}(e^{i\theta\Lambda(A)})=\exp\left(i\theta\nu_0(A)-\frac{\theta^2}{2}\nu_1(A)+\int_{\mathbb{R}}e^{i\theta x}-1-i\theta\tau(x)F_A(dx)\right)$$

$$=\exp\left(\int_A i\theta a(s)-\frac{\theta^2}{2}\sigma^2(s)+\int_{\mathbb{R}}\left(e^{i\theta x}-1-i\theta\tau(x)\right)\rho(s,dx)\lambda(ds)\right).$$

The second statement follows from the fact that for every $A\in\mathcal{S}$, we have

$$\int_A\left(|a(s)|+\sigma^2(s)+\int_{\mathbb{R}}(1\wedge x^2)\rho_G(s,dx)+\int_{\mathbb{R}}(1\wedge x^2)\rho_M(s,dx)\right)\lambda(ds)$$

$$=|\nu_0|(A)+\nu_1(A)+\int_{\mathbb{R}}(1\wedge x^2)G_A(dx)+\int_{\mathbb{R}}(1\wedge x^2)M_A(dx)=\lambda(A)=\int_A\lambda(ds).$$

□

The following definition of the stochastic integral is the same as the one presented in [RR89] but extended to the framework of QID random measure. As mentioned in the introduction of this section, this definition and the next lemma are common to the three subsections since they consider general QID random measures with no assumptions. Hence, they only appear here.

Definition 2.4.7. *Let* $f(s)=\sum_{j=1}^n x_j\mathbf{1}_{A_j}(s)$ *be a real simple function on* S*, where* $A_j\in\mathcal{S}$ *are disjoint. Then, for every* $A\in\sigma(\mathcal{S})$*, we define*

$$\int_A f d\Lambda=\sum_{j=1}^n x_j\Lambda(A\cap A_j).$$

Further, a measurable function $f:(S,\sigma(\mathcal{S}))\to(\mathbb{R},\mathcal{B}(\mathbb{R}))$ *is said to be* Λ*-integrable if there exists a sequence* $\{f_n\}$ *of simple functions such that*

(i) $f_n\to f$, λ*-a.e.,*

(ii) *for every* $A\in\sigma(\mathcal{S})$*, the sequence* $\{\int_A f_n d\Lambda\}$ *converges in probability as* $n\to\infty$*.*

If f *is* Λ*-integrable, then we write*

$$\int_A f d\Lambda=\mathbb{P}-\lim_{n\to\infty}\int_A f_n d\Lambda,$$

where $\{f_n\}$ *satisfies* (i) *and* (ii).

Notice that for a simple function $f(s) = \sum_{j=1}^n x_j \mathbf{1}_{A_j}(s)$ we have the following representation of the c.f.:

$$\mathbb{E}\left(e^{i\theta \int_A f d\Lambda}\right) = \prod_{j=1}^n \mathbb{E}\left(e^{i\theta x_j \Lambda(A\cap A_j)}\right)$$

$$= \prod_{j=1}^n \exp\left(\int_{A\cap A_j} i\theta x_j a(s) - \frac{\theta^2 x_j^2}{2}\sigma^2(s) + \int_{\mathbb{R}} \left(e^{i\theta x_j x} - 1 - i\theta x_j \tau(x)\right)\rho(s,dx)\lambda(ds)\right)$$

$$= \exp\left(\int_A i\theta f(s) a(s) - \frac{\theta^2 f^2(s)}{2}\sigma^2(s) + \int_{\mathbb{R}} \left(e^{i\theta f(s) x} - 1 - i\theta f(s)\tau(x)\right)\rho(s,dx)\lambda(ds)\right).$$

In the following result, we prove that $\int_A f d\Lambda$ does not depend on the approximating sequence, hence it is well defined. This does not follow neither from [UW67] nor from [Ro87] since they focus on ID random variables, but nonetheless we use some of their arguments. In particular, in [UW67] (which is the work cited in [RR89]) the random measures considered are *atomless* (see Section 2.6.5 for the definition and for further details) while we (and [Ro87]) do not have such a restriction.

Lemma 2.4.8. *Let* Λ *be a QID random measure and let* f *be* Λ*-integrable then* $\int_A f d\Lambda$ *is well defined, for every* $A \in \sigma(\mathcal{S})$.

Proof. Let $\{f_n\}$ and $\{g_n\}$ be two real simple functions on S as in the Definition 2.4.7 and satisfying (i) and (ii) with the same limit. Let $h_n = f_n - g_n$ $(n = 1, 2, ...)$. Then, the sequence $\{h_n\} \to 0$ λ-a.e. and $\int_A h_n d\Lambda$ converges in probability for every $A \in \sigma(\mathcal{S})$. It remains to show that $\int_A h_n d\Lambda \xrightarrow{p} 0$ for every $A \in \sigma(\mathcal{S})$.
Let $N_n(A) = \int_A h_n d\Lambda$ then N_n is a measure on $L_0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$, which is the set of all measurable random elements defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}$. In particular, N_n is absolutely continuous w.r.t. λ and for every $A \in \sigma(\mathcal{S})$ we have that $N(A) := \lim_{n\to\infty} N_n(A)$ exists. In particular the existence is guaranteed by the convergence of $\int_A h_n d\Lambda$. Applying the Hahn-Saks-Vitali theorem we obtain that N is a measure on $L_0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ and $N \ll \lambda$.
Since $\{h_n\}$ converges λ-a.e., we can apply the Egorov (or more correctly Lusin) theorem and obtain that for every set A in $\sigma(\mathcal{S})$ we have $A = \bigcup_{k=0}^\infty A_k$ where A_k's are pairwise disjoint sets of finite λ-measure belonging to $\sigma(\mathcal{S})$ s.t. $\lambda(A_0) = 0$ and $h_n \to 0$ uniformly on every set A_k's for $k > 0$. Then, for every A_k we have

$$\hat{\mathcal{L}}\left(N(A_k)\right) = \lim_{n\to\infty} \hat{\mathcal{L}}\left(N_n(A_k)\right) = \lim_{n\to\infty} \exp\left(\int_{A_k} K(\theta h_n(s), s)\lambda(ds)\right)$$

$$= \lim_{n\to\infty} \exp\left(\int_{A_k} i\theta h_n(s) a(s) - \frac{\theta^2}{2} h_n^2(s)\sigma^2(s) + \int_{\mathbb{R}} \left(e^{i\theta h_n(s) x} - 1 - i\theta h_n(s)\tau(x)\right)\rho(s,dx)\lambda(ds)\right)$$

$$= 1,$$

using the dominated convergence theorem and that $\sup_{s\in A_k} h_n(s) \to 0$ as $n \to \infty$. Hence, $N(A_k) = 0$ in probability for every $k > 0$. Then, since $N(A)$ is a measure we have that $N(A) = \sum_{k=0}^\infty N(A_k) = 0$, for every $A \in \sigma(\mathcal{S})$. □

It is now possible to give a representation of the c.f. of $\int_S f d\Lambda$.

Proposition 2.4.9. *If f is Λ-integrable, then $\int_S |K(\theta f(s), s)|\lambda(ds) < \infty$, where K is given in Proposition 2.4.6, and*

$$\hat{\mathcal{L}}\left(\int_S f d\Lambda\right)(\theta) = \exp\left(\int_S K(\theta f(s), s)\lambda(ds)\right), \quad \theta \in \mathbb{R}.$$

Proof. The statement follows from the same arguments used in the proof of Proposition 2.6 of [RR89]. Let us sketch them. First, notice that the statement holds for simple function as in Definition 2.4.7. Let $\{f_n\}$ be a sequence of simple functions. Let $\mu_{\theta,n}(A) := \int_A K(\theta f_n(s), s)\lambda(ds)$ where $A \in \sigma(\mathcal{S})$ and observe that it is a complex measure for every $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \mu_{\theta,n}(A) = \lim_{n\to\infty} \log \hat{\mathcal{L}}\left(\int_A f_n d\Lambda\right)(\theta) = \log \hat{\mathcal{L}}\left(\int_S f d\Lambda\right)(\theta) = \mu_\theta(A),$$

for every $A \in \sigma(\mathcal{S})$ and $\theta \in \mathbb{R}$, we conclude that by the Hahn-Saks-Vitali theorem μ_θ is a countably additive complex measure. Further, since $\mu_\theta << \lambda$ then $\mu_\theta(A) = \int_A h_\theta(s)\lambda(ds)$. By the continuity of $K(\cdot, s)$ for each $s \in S$ we deduce that $K(\theta f_n(s), s) \to K(\theta f(s), s)$ a.e.-λ, and using Egorov (Lusin) theorem we have that $S = \bigcup_{j=0}^{\infty} A_j$, where $\lambda(A_0) = 0$ and $\lambda(A_j) < \infty$ and with $K(\theta f_n(s), s) \to K(\theta f(s), s)$ a.e.-λ uniformly in $s \in A_j$ for $j \in \mathbb{N}$. Thus, $h_\theta(s) = K(\theta f(s), s)$ a.e.-λ on A_j with $j \geq 1$ because for every $A \in \sigma(\mathcal{S})$

$$\int_{A\cap A_j} h_\theta(s)\lambda(ds) = \mu_\theta(A \cap A_j) = \lim_{n\to\infty} \int_{A\cap A_j} K(\theta f_n(s), s)\lambda(ds) = \int_{A\cap A_j} K(\theta f(s), s)\lambda(ds).$$

But since A_0 is a λ-null set then $h_\theta(s) = K(\theta f(s), s)$ a.e.-λ on S. □

We are now ready to present the main result of this subsection, which concerns the integrability conditions for $\int_S f d\Lambda$.

Theorem 2.4.10. *Let $f : S \to \mathbb{R}$ be a $\sigma(\mathcal{S})$-measurable function. Then f is Λ-integrable if the following three conditions hold:*
(i) $\int_S |U(f(s), s)|\lambda(ds) < \infty$,
(ii) $\int_S |f(s)|^2\sigma^2(s)\lambda(ds) < \infty$,
(iii) $\int_S V_0(f(s), s)\lambda(ds) < \infty$,

$$\text{where} \quad U(u, s) = ua(s) + \int_{\mathbb{R}} \tau(xu) - u\tau(x)\rho(s, dx), \quad V_0(u, s) = \int_{\mathbb{R}} (1 \wedge |xu|^2)|\rho|(s, dx).$$

Further, the c.f. of $\int_S f d\Lambda$ can be written as
(iv) $\hat{\mathcal{L}}\left(\int_S f d\Lambda\right)(\theta) = \exp\left(i\theta a_f - \frac{1}{2}\theta^2\sigma_f^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)F_f(dx)\right)$, *where*

$$a_f = \int_S U(f(s), s)\lambda(ds), \quad \sigma_f^2 = \int_S |f(s)|^2\sigma^2(s)\lambda(ds), \quad \textit{and}$$

F_f is the unique quasi-Lévy measure determined by the difference of the Lévy measures $\tilde{F}_f^+$ and

$\tilde{F}_f^-$, *which are defined as: for every* $B \in \mathcal{B}(\mathbb{R})$

$$\tilde{F}_f^+(B) = \tilde{F}^+(\{(s,x) \in S \times \mathbb{R} : \, f(s)x \in B \setminus \{0\}\})$$

$$\textit{and} \quad \tilde{F}_f^-(B) = \tilde{F}^-(\{(s,x) \in S \times \mathbb{R} : \, f(s)x \in B \setminus \{0\}\}).$$

Lemma 2.4.11. *For every* $u \in \mathbb{R}$, $s \in S$ *and* $d > 0$,

$$\sup\{|U(cu,s)| : |c| \leq d\} \leq d|U(u,s)| + (1+d)^3 V_0(u,s).$$

Proof. Let $|c| \leq d$. Define $R(c,u,s) := \int_{\mathbb{R}} \tau(xcu) - c\tau(ux)\rho(s,dx)$. We can rewrite $U(cu,s)$ as follows

$$U(cu,s) = cua(s) + \int_{\mathbb{R}} \tau(xcu) - cu\tau(x)\rho(s,dx)$$

$$= cua(s) + c\int_{\mathbb{R}} \tau(xu) - u\tau(x)\rho(s,dx) + \int_{\mathbb{R}} \tau(xcu) - c\tau(ux)\rho(s,dx) = cU(u,s) + R(c,u,s).$$

Observe that $\tau(xcu) - c\tau(ux) = 0$ if $|ux| \leq (1 \wedge |c|^{-1})$ and $|\tau(xcu) - c\tau(ux)| \leq 1 + d$ otherwise. Thus, we obtain, using Chebychev's inequality, that

$$|R(c,u,s)| \leq (1+d)\int_{|ux|>(1\wedge|c|^{-1})} |\rho|(s,dx) \leq (1+d)|\rho|(s,\{x : (1 \wedge |ux|) \geq (1 \wedge |c|^{-1})\})$$

$$\leq \frac{1+d}{(1 \wedge |c|^{-2})}\int_{\mathbb{R}} (1 \wedge |ux|^2)|\rho|(s,dx) \leq (1+d)^3 V_0(u,s).$$

□

The reason why it is not possible to get an "only if" result is because we have

$$\int_{\mathbb{R}} (1 \wedge x^2)|F_f|(dx) \leq \int_S \int_{\mathbb{R}} (1 \wedge |f(s)x|^2)|\rho|(s,dx)\lambda(ds),$$

where $|F_f|$ is the total variation of the quasi-Lévy measure F_f. Hence, we do not have an equality but only an inequality, which implies that, if $\int_{\mathbb{R}}(1 \wedge x^2)|F_f|(dx) < \infty$, then it is not necessarily true that $\int_S \int_{\mathbb{R}}(1 \wedge |f(s)x|^2)|\rho|(s,dx)\lambda(ds) < \infty$ (*i.e.* point (iii) in Theorem 2.4.10). This is because we have that

$$\int_{\mathbb{R}} (1\wedge x^2)|F_f|(dx) \leq \int_{\mathbb{R}} (1\wedge x^2)\tilde{F}_f^+(dx) + \int_{\mathbb{R}} (1\wedge x^2)\tilde{F}_f^-(dx) = \int_S \int_{\mathbb{R}} (1\wedge|f(s)x|^2)|\rho|(s,dx)\lambda(ds),$$

where the inequality comes from the following argument. Recall that $\tilde{F}_f^+(B) = \tilde{F}^+(\{(s,x) \in S \times \mathbb{R} : \, f(s)x \in B \setminus \{0\}\})$, for every $B \in \mathcal{B}(\mathbb{R})$. Now, let $C^+ \in \sigma(\mathcal{S}) \otimes \mathcal{B}(\mathbb{R})$ s.t. $\tilde{F}^+(A) > 0$ for every $A \subset C^+$ with $A \in \sigma(\mathcal{S}) \otimes \mathcal{B}(\mathbb{R})$ and similarly define C^- for $\tilde{F}^-$. In words, C^+ and C^- are the disjoint sets of $\sigma(\mathcal{S}) \otimes \mathcal{B}(\mathbb{R})$ for which $\tilde{F}^+$ and $\tilde{F}^-$ take positive values, respectively. [Note that these sets are disjoints because for every $s \in S$ $\rho^+(s,\cdot)$ and $\rho^-(s,\cdot)$ are orthogonal measures.]

Then, there might exists a non-empty set $B \in \mathcal{B}(\mathbb{R})$ (with $0 \notin B$ for simplicity) such that for every $y \in B$ we have that $y = f(s)x = f(s')x'$ for some $(s,x) \in C^+$ and $(s',x') \in C^-$. Notice that since $f(s)x$ is a $\sigma(\mathcal{S}) \otimes \mathcal{B}(\mathbb{R})$-measurable function from $S \times \mathbb{R} \to \mathbb{R}$ and that $B \in \mathcal{B}(\mathbb{R})$

then the sets of such (s,x) and (s',x') are measurables. In other words, there might exists a non-empty set $B \in \mathcal{B}(\mathbb{R})$ such that the sets $\{y \in B : y = f(s)x \text{ with } (s,x) \in C^+\}$ and $\{y \in B : y = f(s)x \text{ with } (s,x) \in C^-\}$ are equal. When this happens both $\tilde{F}_f^+(B)$ and $\tilde{F}_f^-(B)$ are strictly positive, thus explaining the inequality.

However, by adding a further assumption we have the following result, which can be seen as the "only if" part of Theorem 2.4.10.

Theorem 2.4.12. *Let $f : S \to \mathbb{R}$ be a $\sigma(\mathcal{S})$-measurable function. Then, if f is integrable with respect to Λ_G and Λ_M, the three conditions* (i), (ii) *and* (iii) *of Theorem 2.4.10 hold, namely f is Λ-integrable. Further, the c.f. of $\int_S f d\Lambda$ can be written as in point* (iv) *of Theorem 2.4.10.*

Proof. From the fact that $|\hat{\mathcal{L}}\left(\int_S f d\Lambda_G\right)|^2$ and $|\hat{\mathcal{L}}\left(\int_S f d\Lambda_M\right)|^2$ are the c.f. of ID-random variables it follows that

$$\left|\hat{\mathcal{L}}\left(\int_S f d\Lambda_G\right)(\theta)\right|^2 = \exp\left(-\theta^2\sigma_{G,f}^2 + 2\int_{\mathbb{R}} \cos(\theta x) - 1 G_f(dx)\right),$$

where $G_f(B) = G(\{(s,x) \in S \times \mathbb{R} : f(s)x \in B \setminus \{0\}\})$ and where $\sigma_{G,f}^2 = \int_S |f(s)|\sigma_G^2(s)\lambda_G(ds)$. By the properties of the Radon-Nikodym derivative we have that

$$\sigma_{G,f}^2 = \int_S |f(s)|\sigma_G^2(s)\lambda_G(ds) = \int_S |f(s)|\frac{d\nu^{(1)}}{d\lambda_G}(s)\lambda_G(ds) = \int_S |f(s)|\frac{d\nu^{(1)}}{d\lambda}(s)\lambda(ds) = \sigma_f^2.$$

Therefore we get that $\sigma_f^2 < \infty$ and that

$$\int_{\mathbb{R}} (1 \wedge y^2)\tilde{F}_f^+(dy) + \int_{\mathbb{R}} (1 \wedge y^2)\tilde{F}_f^-(dy) \leq \int_{\mathbb{R}} (1 \wedge y^2)G_f(dy) + \int_{\mathbb{R}} (1 \wedge y^2)M_f(dy) < \infty,$$

where M_f is defined similarly to G_f, and thus we obtain (ii) and (iii). Further, since $|\tau(x) - \sin(x)| \leq 2(1 \wedge x^2)$, (i) follows from noticing that

$$\begin{aligned} |U(u,s)| &\leq \left|ua(s) + \int_{\mathbb{R}} \sin(xu) - u\tau(x)\rho(s,dx)\right| + \left|\int_{\mathbb{R}} \tau(xu) - \sin(xu)|\rho|(s,dx)\right| \\ &\leq |\operatorname{Im} K(u,s)| + 2V_0(u,s), \end{aligned}$$

which is finite because of (iii) and of Proposition 2.4.9. □

We end this section with the following result, which links $\int_A f_n d\Lambda$ with $\int_A f_n d\Lambda_G$ and $\int_A f_n d\Lambda_M$.

Proposition 2.4.13. *Let f be integrable w.r.t. Λ_G and Λ_M. Then, for every $\theta \in \mathbb{R}$, we have*

$$\hat{\mathcal{L}}\left(\int_S f d\Lambda\right)(\theta) = \frac{\hat{\mathcal{L}}\left(\int_S f d\Lambda_G\right)(\theta)}{\hat{\mathcal{L}}\left(\int_S f d\Lambda_M\right)(\theta)}. \tag{2.10}$$

Proof. There are at least two ways to prove this result. The first is to use the properties of the Radon-Nikodym derivative and use point (iv) in Theorem 2.4.10. The second is consider the sequence of simple functions (f_n) of the proof of Theorem 2.4.10 together with Proposition 2.4.9 and Theorem 2.4.12. Let us show the latter in detail.

Consider f to be $\Lambda^{(1)}$ and $\Lambda^{(2)}$ integrable. Then we know that f is also Λ-integrable by Theorem 2.4.12. Now consider the sequence (f_n) as in the proof of Theorem 2.4.10, then we know that $\{\int_A f_n d\Lambda\}_{n=1}^{\infty}$, $\{\int_A f_n d\Lambda_G\}_{n=1}^{\infty}$ and $\{\int_A f_n d\Lambda_M\}_{n=1}^{\infty}$ converges in probability for every $A \in \sigma(\mathcal{S})$. Since for every $n \in \mathbb{N}$

$$\hat{\mathcal{L}}\left(\int_S f_n d\Lambda\right)(\theta) = \frac{\hat{\mathcal{L}}\left(\int_S f_n d\Lambda_G\right)(\theta)}{\hat{\mathcal{L}}\left(\int_S f_n d\Lambda_M\right)(\theta)},$$

then we have that

$$\hat{\mathcal{L}}\left(\int_S f d\Lambda\right)(\theta) = \lim_{n\to\infty} \hat{\mathcal{L}}\left(\int_S f_n d\Lambda\right)(\theta) = \lim_{n\to\infty} \frac{\hat{\mathcal{L}}\left(\int_S f_n d\Lambda_G\right)(\theta)}{\hat{\mathcal{L}}\left(\int_S f_n d\Lambda_M\right)(\theta)} = \frac{\hat{\mathcal{L}}\left(\int_S f d\Lambda_G\right)(\theta)}{\hat{\mathcal{L}}\left(\int_S f d\Lambda_M\right)(\theta)}.$$

□

As we mentioned at the beginning of this section, we have that $\Lambda_G \sim (0, \nu_1, G.)$ and $\Lambda_M \sim (-\nu_0, 0, M.)$, however all the results hold for more general ID r.m.. A sufficient condition for two generic ID r.m. (call them Λ_J and Λ_H) is that Λ_J and Λ_H generate Λ and that their control measure, call them λ_X and λ_Y, are s.t. $\lambda_X(A) \leq C\lambda(A)$ and $\lambda_X(A) \leq C'\lambda(A)$ where $C, C' \in (0, \infty)$ for every $A \in \sigma(\mathcal{S})$. For the sake of brevity we left to the curious reader to check this statement. However, it would interesting to have the weakest possible conditions that two generating ID r.m. must satisfy in order to guarantee the validity of all the results presented in this section, which in particular satisfy (2.10).

Open Question 2.4.14. Given a QID r.m., what is the largest class of generating ID r.m. that satisfy (2.10)?

The answer to this question is also linked with Theorem 2.6.22 and Open Question 2.6.23.

2.4.2 The bounded case

In this subsection we work with the following assumptions. We let $\mathcal{S}$ be a σ-algebra, $F(A, B)$ be a bimeasure on $\mathcal{S} \times \mathcal{B}(\mathbb{R})$ and

$$\sup_I \sum_{i\in I} |F_{A_i}(B_i)| < \infty, \tag{2.11}$$

where the supremum is taken over all the finite families of disjoints elements of $\mathcal{S} \times \mathcal{B}(\mathbb{R})$. In other words, the supremum is taken over all the finite families of the form $(A_i, B_i)_{i\in I}$, where I is finite, such that $A_i \in \mathcal{S}$ and $B_i \in \mathcal{B}(\mathbb{R})$ and that the rectangles $(A_i \times B_i) \cap (A_j \times B_j) = \emptyset$ for every $i, j \in I$ with $i \neq j$.

Let us introduce the set function $\nu(A) : \mathcal{S} \mapsto [0, \infty)$ such that

$$\nu(A) := \sup_{I_A} \sum_{i\in I_A} |F_{A_i}(B_i)|,$$

where I_A is defined as I but with the constraint that $A_i \subset A$. To have a better idea of what kind of object ν is, compare it with the definition of total variation of a signed measure (see (2.1)).

The results in this section are similar to the ones of the previous section. The main difference is given by Theorem 2.4.18 and Lemma 2.4.19. Indeed, these two results (and consequently all

the others in this subsection) do not rely on ID results. In particular, Theorem 2.4.18 is a general measure theoretical result which extends a classical result by Horowitz on signed bimeasures (see Theorem 4 in [Ho77], which we report here as Theorem 2.4.17). We start with the following proposition on the control measure.

Proposition 2.4.15. *Let Λ be a QID random measure. Let $\nu_0 : \mathcal{S} \mapsto \mathbb{R}$ be a signed measure, $\nu_1 : \mathcal{S} \mapsto \mathbb{R}$ be a measure, F_A be a quasi-Lévy measure on $\mathbb{R}$ for every $A \in \mathcal{S}$, $\mathcal{S} \ni A \mapsto F_A(B) \in (-\infty, \infty)$ be a signed measure for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $(\nu_0(A), \nu_1(A), F_A)$ is the characteristic triplet of $\Lambda(A)$, $\forall A \in \mathcal{S}$. Assume that F satisfies* (2.11). *Define*

$$\lambda(A) = |\nu_0|(A) + \nu_1(A) + \nu(A). \tag{2.12}$$

Then $\lambda : \mathcal{S} \mapsto [0, \infty)$ is a measure such that $\lambda(A_n) \to 0$ implies $\Lambda(A_n) \xrightarrow{p} 0$ for every $\{A_n\} \subset \mathcal{S}$.

Proof. From Theorem 4 in [Ho77] we have that $\nu(A)$ defines a measure. Then it is straightforward to see that $\lambda(A)$ is a measure. Now if $\nu(A_n) \to 0$ then $F^+_{A_n}(B) \to 0$ and $F^-_{A_n}(B) \to 0$ for every $B \in \mathcal{B}(\mathbb{R})$, hence by the Lévy continuity theorem we obtain the stated result. □

Before presenting one of the main results of this section, which consists of an extension of Proposition 2.4 in [RR89], we give a remark on its topology.

Remark 2.4.16. Proposition 2.4 in [RR89] is stated for a standard Borel space, where a measurable space (X, Γ) is said to be standard Borel space if there exists a metric on X which makes it a complete separable metric space in such a way that Γ is then the Borel σ-algebra $\mathcal{B}(X)$. However, the following result holds for more general topological spaces: the Lusin measurable spaces. Lusin measurable spaces are measurable spaces isomorphic to a measurable space $(H, \mathcal{B}(H))$, where H is homeomorphic to a Borel subset of a compact metrizable space. Notice that H is usually called Lusin space or more correctly Lusin metrizable space. In this remark, we do not provide an extensive discussion (indeed, see [DM82], [Ho77] and [Ho86] for further details), we just mention that H can be any Polish space (thus, including the real line).

The next result also extends a classical result in measure theory: Theorem 4 in [Ho77]. Let us recall it here.

Theorem 2.4.17 (Theorem 4 in [Ho77])**.** *Let $(X, \mathcal{B})$ be a Lusin measurable space and let $(T, \mathcal{A})$ be an arbitrary measurable space. Let μ is a signed bimeasure on $\mathcal{A} \times \mathcal{B}$. Then, there exists a finite signed measure M on $\mathcal{A} \otimes \mathcal{B}$ such that $\mu(A, B) = M(A \times B)$ if and only if $\sup_I \sum_{i \in I} |\mu(A_i, B_i)| < \infty$. In particular, $M = M^+ - M^-$ where*

$$M^+(C) = \int_T \int_X \mathbf{1}_C(x, y) H^+(t, dx) \nu(dt) \quad \text{and} \quad M^-(C) = \int_T \int_X \mathbf{1}_C(x, y) H^-(t, dx) \nu(dt),$$

where $C \in \mathcal{A} \otimes \mathcal{B}$, H^+ and H^- are two sub-Markovian kernels, and $\nu(A) := \sup_{I_A} \sum_{i \in I} |\mu(A_i, B_i)| < \infty$ is a finite measure on $(T, \mathcal{A})$.

Proof. See Theorem 4 in [Ho77]. □

Theorem 2.4.18. *Let $(X,\mathcal{B})$ be a Lusin measurable space and let $(T,\mathcal{A})$ be an arbitrary measurable space. Let $Q_0(A,B)$ be a (possibly negative) function of $A\in\mathcal{A}$, $B\in\mathcal{B}$, satisfying:*
(a) *for every $A\in\mathcal{A}$, $Q_0(A,\cdot)$ is a signed measure on $(X,\mathcal{B})$,*
(b) *for every $B\in\mathcal{B}$, $Q_0(\cdot,B)$ is a signed measure on $(T,\mathcal{A})$,*
(c) $\sup_I \sum_{i\in I}|Q_0(A_i,B_i)|<\infty$.
Let $\nu(A):=\sup_{I_A}\sum_{i\in I}|Q_0(A_i,B_i)|$. Then there exists a unique finite signed measure Q on the product σ-algebra $\mathcal{A}\otimes\mathcal{B}$ s.t.

$$Q(A\times B)=Q_0(A,B)=\int_A q(t,B)\nu(dt), \tag{2.13}$$

with Jordan decomposition

$$Q^+(C)=\int_T\int_X \mathbf{1}_C(x,t)q^+(t,dx)\nu(dt)\quad \text{and}\quad Q^-(C)=\int_T\int_X \mathbf{1}_C(x,t)q^-(t,dx)\nu(dt),$$

for every $A\in\mathcal{A}$, $B\in\mathcal{B}$, $C\in\mathcal{A}\otimes\mathcal{B}$ where $q:T\times\mathcal{B}\to[-1,1]$ and $q^+,q^-:T\times\mathcal{B}\to[0,1]$ fulfil the following conditions:
(d) *for every t, $q(t,\cdot)$ is a signed measure on $\mathcal{B}$,*
(e) *for every $B\in\mathcal{B}$, $q(\cdot,B)$ is $\mathcal{A}$-measurable,*
(d)′ *$q^+(t,\cdot)$ and $q^-(t,\cdot)$ are the Jordan decomposition of $q(t,\cdot)$,*
(e)′ *$q^+(\cdot,B)$ and $q^-(\cdot,B)$ are $\mathcal{A}$-measurable functions.*
Further, if $q_1(\cdot,\cdot)$ is some other function satisfying (2.13), (d) *and* (e), *then off a set of ν-measure zero, $q_1(t,\cdot)=q(t,\cdot)$.*

Lemma 2.4.19. *Let F. be as in Proposition 2.4.15. Then there exists a unique finite measure F on $\mathcal{S}\otimes\mathcal{B}(\mathbb{R})$ such that*

$$F(A\times B)=F_A(B),\quad \text{for all } A\in\mathcal{S},\ B\in\mathcal{B}(\mathbb{R}).$$

Moreover, there exists a function $\rho:S\times\mathcal{B}(\mathbb{R})\mapsto[-\infty,\infty]$ such that
(i) *$\rho(s,\cdot)$ is a quasi-Lévy type measure on $\mathcal{B}(\mathbb{R})$, for every $s\in S$,*
(ii) *$\rho(\cdot,B)$ is a Borel measurable function, for every $B\in\mathcal{B}(\mathbb{R})$,*
(iii) *$\int_{S\times\mathbb{R}}h(s,x)F(ds,dx)=\int_S\int_{\mathbb{R}}h(s,x)\rho(s,dx)\lambda(ds)$, for every $\mathcal{S}\otimes\mathcal{B}(\mathbb{R})$-measurable function $h:S\times\mathbb{R}\mapsto[0,\infty]$. This equality can be extended to real and complex-valued functions h.*

Proof. Using Theorem 2.4.18, we have that $F_A(B)=F(A\times B)=\int_A q(t,B)\nu(dt)$ where q satisfies point (d) and (e) in that theorem. Since $\lambda\ll\nu$, then defining

$$\rho^+(s,dx):=\frac{d\nu}{d\lambda}(s)q^+(s,dx),\quad \rho^-(s,dx):=\frac{d\nu}{d\lambda}(s)q^-(s,dx),$$

$$\text{and}\quad \rho(s,dx):=\rho^+(s,dx)-\rho^-(s,dx)$$

we have that (ii) is satisfied and that

$$\int_{\mathbb{R}}(1\wedge x^2)|\rho|(s,dx)=\frac{d\nu}{d\lambda}(s)\int_{\mathbb{R}}(1\wedge x^2)q^+(s,dx)+\frac{d\nu}{d\lambda}(s)\int_{\mathbb{R}}(1\wedge x^2)q^-(s,dx)$$

$$\leq \frac{d\nu}{d\lambda}(s)\int_{\mathbb{R}} q^+(s,dx) + \frac{d\nu}{d\lambda}(s)\int_{\mathbb{R}} q^-(s,dx) \leq \frac{d\nu}{d\lambda}(s) + \frac{d\nu}{d\lambda}(s) \leq 2,$$

where the last inequality comes from the fact that we can always assume that $\frac{d\nu}{d\lambda}(s) \leq 1$ for all s (and the same for ν) and from the properties of q^+ and q^-. This proves (i).
Therefore, we have immediately (iii) since

$$\int_S \int_{\mathbb{R}} \mathbf{1}_C(s,x)\rho(s,dx)\lambda(ds) = \int_S \int_{\mathbb{R}} \mathbf{1}_C(s,x)q(s,dx)\nu(ds) = F(C).$$

The extension to real and complex integrand follows by standard arguments (*e.g.* see [Ho77]). □

Corollary 2.4.20. *Under the setting of Proposition 2.4.15,* $F^+(C) = \int_S \int_{\mathbb{R}} \mathbf{1}_C(s,x)\rho^+(s,dx)\lambda(ds)$ *and* $F^-(C) = \int_S \int_{\mathbb{R}} \mathbf{1}_C(s,x)\rho^-(s,dx)\lambda(ds)$.

Proof. First, notice that $\int_S \int_{\mathbb{R}} \mathbf{1}_C(s,x)\rho^+(s,dx)\lambda(ds)$ and $\int_S \int_{\mathbb{R}} \mathbf{1}_C(s,x)\rho^-(s,dx)\lambda(ds)$ are measures on $\mathcal{S}\otimes\mathcal{B}(\mathbb{R})$ and that $F(C) = \int_S \int_{\mathbb{R}} \mathbf{1}_C(s,x)\rho^+(s,dx)\lambda(ds) - \int_S \int_{\mathbb{R}} \mathbf{1}_C(s,x)\rho^-(s,dx)\lambda(ds)$. Finally, let E_s^+ and E_s^+ the Hahn decomposition of $\rho(s,\cdot)$ (which is the same as the one of $q(s,\cdot)$). Define $C^+ := \{(s,x) \in S\times\mathbb{R} :\ x \in E_s^+\}$ and $C^- := \{(s,x) \in S\times\mathbb{R} :\ x \in E_s^-\}$. It is possible to see that C^+ and C^- form an essential decomposition of $S\times\mathbb{R}$ and so the measures $\int_S \int_{\mathbb{R}} \mathbf{1}_C(s,x)\rho^+(s,dx)\lambda(ds)$ and $\int_S \int_{\mathbb{R}} \mathbf{1}_C(s,x)\rho^-(s,dx)\lambda(ds)$ are singular, thus concluding the proof.
Notice that an alternative proof follows directly from Theorem 2.4.18 and Lemma 2.4.19. □

Under the same setting as in Proposition 2.4.15, we obtain the following proposition.

Proposition 2.4.21. *Under the setting of Proposition 2.4.15, the c.f. of* $\Lambda(A)$ *can be written in the form:*

$$\mathbb{E}(e^{i\theta\Lambda(A)}) = \exp\left(\int_A K(\theta,s)\lambda(ds)\right), \quad \theta\in\mathbb{R},\ A\in\mathcal{S},$$

where

$$K(\theta,s) = i\theta a(s) - \frac{\theta^2}{2}\sigma^2(s) + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)\rho(s,dx),$$

$a(s) = \frac{d\nu_0}{d\lambda}(s)$, $\sigma^2(s) = \frac{d\nu_1}{d\lambda}(s)$ *and* ρ *is given by Lemma 2.4.19, and* $\exp(K(\theta,s))$ *is the characteristic function of a QID random variable if it exists. Moreover, we have*

$$|a(s)| + \sigma^2(s) + \frac{d\nu}{d\lambda}(s) = 1 \quad \lambda\text{-}a.e..$$

Proof. The first statement follows from the Lévy-Khintchine formulation (2.3) and Lemma 2.4.19.

The second statement follows from the fact that for every $A\in\mathcal{S}$, we have

$$\int_A \left(|a(s)| + \sigma^2(s) + \frac{d\nu}{d\lambda}(s)\right)\lambda(ds) = |\nu_0|(A) + \nu_1(A) + \nu(A) = \lambda(A) = \int_A d\lambda(ds).$$

□

Proposition 2.4.22. *Under the setting of Proposition 2.4.15, if* f *is* Λ*-integrable, then we have*

that $\int_S |K(tf(s),s)|\lambda(ds) < \infty$, *where* K *is given in Proposition 2.4.21, and*

$$\hat{\mathcal{L}}\left(\int_S f d\Lambda\right)(\theta) = \exp\left(\int_S K(\theta f(s),s)\lambda(ds)\right), \quad \theta \in \mathbb{R}.$$

Proof. The statement follows from the same arguments used in the proof Proposition 2.4.9. □

Differently to Theorem 2.4.10 in the following result we have directly an *if and only if* result for the integrability conditions of $\int_S f d\Lambda$.

Theorem 2.4.23. *Let* $f : S \to \mathbb{R}$ *be a* $\mathcal{S}$*-measurable function and consider the same setting as in Proposition 2.4.15. Then* f *is* Λ*-integrable if and only if the following three conditions hold:*
(i) $\int_S |U(f(s),s)|\lambda(ds) < \infty$,
(ii) $\int_S |f(s)|^2\sigma^2(s)\lambda(ds) < \infty$,
(iii) $\int_S V_0(f(s),s)\lambda(ds) < \infty$,

$$\text{where} \quad U(u,s) = ua(s) + \int_{\mathbb{R}} \tau(xu) - u\tau(x)\rho(s,dx), \quad V_0(u,s) = \int_{\mathbb{R}} (1 \wedge |xu|^2)|\rho|(s,dx),$$

Further, if f *is* Λ*-integrable, then the c.f. of* $\int_S f d\Lambda$ *can be written as*
(iv) $\hat{\mathcal{L}}\left(\int_S f d\Lambda\right)(\theta) = \exp\left(i\theta a_f - \frac{1}{2}\theta^2\sigma_f^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)F_f(dx)\right)$, *where*

$$a_f = \int_S U(f(s),s)\lambda(ds), \quad \sigma_f^2 = \int_S |f(s)|^2\sigma^2(s)\lambda(ds), \quad \text{and}$$

$$F_f(B) = F(\{(s,x) \in S \times \mathbb{R} : \ f(s)x \in B \setminus \{0\}\}), \quad B \in \mathcal{B}(\mathbb{R}).$$

Proof. $\Leftarrow$: Under the framework of this section we have that F is a finite signed measure (see Lemma 2.4.19) and so $\int_{\mathbb{R}}(1 \wedge y^2)|\tilde{F}_f|(dy) < \infty$. Moreover, since F is a finite signed measure then by Corollary 2.4.20 we have

$$\int_S \int_{\mathbb{R}} (1 \wedge |f(s)x|^2)|\rho|(s,dx)\lambda(ds) \leq \int_S \int_{\mathbb{R}} |\rho|(s,dx)\lambda(ds) < \infty.$$

We conclude this part of the proof with the same arguments used in the proof of Theorem 2.4.12.
$\Rightarrow$: It follows from similar arguments as the ones used in the proof of Theorem 2.4.10. □

We conclude this subsection with a discussion on the symmetric case, namely the case where $\bar{\Lambda}(A) = \Lambda(A) - \Lambda'(A)$ and $\Lambda'(A)$ is an independent copy of $\Lambda(A)$. We start with the following general lemma.

Lemma 2.4.24. *Let* μ *be a QID distribution. Then, the symmetrisation and the dual of* μ *are QID distributions.*

Proof. The dual is straightforward. Regarding the symmetrisation, we have $|\mu|^2 = \mu * \tilde{\mu}$, where $\tilde{\mu}$ denotes the dual of μ. Since the class of QID distributions is closed under convolution (see Remark 2.6 of [LPS18]), then $|\mu|^2$ is QID. □

Then, we have the following result under the framework of this subsection.

Proposition 2.4.25. *Let $\bar{\Lambda}(A) = \Lambda(A) - \Lambda'(A)$ where $\Lambda'(A)$ is an independent copy of $\Lambda(A)$ for every $A \in \mathcal{S}$. Then, $\bar{\Lambda}$ is a QID random measure. Further, under the setting of Proposition 2.4.15, for an arbitrary function $f : S \mapsto \mathbb{R}$, f is $\bar{\Lambda}$-integrable if and only if it is Λ-integrable.*

Proof. By Lemma 2.4.24 we have that $\bar{\Lambda}(A)$ is a QID r.v. for every $A \in \mathcal{A}$. Moreover, using the notations used before and Proposition 2.5 in [Sa99], we have that

$$\hat{\mathcal{L}}\left(\bar{\Lambda}(A)\right)(\theta) = |\hat{\mathcal{L}}\left(\Lambda(A)\right)(\theta)|^2 = \exp\left(-\theta^2\nu_1(A) + 2\int_{\mathbb{R}} \cos(\theta x) - 1 F_A(dx)\right).$$

$$= \exp\left(\int_A \left(-\theta^2\sigma^2(s) + 2\int_{\mathbb{R}} \cos(\theta x) - 1\rho(s,dx)\right)\lambda(ds)\right).$$

Then, by applying Theorem 2.4.23 we obtain the stated result. □

Remark 2.4.26. An "if" result for the framework of Section 2.4.1 is also possible if f is Λ_G and Λ_M integrable, where Λ_G and Λ_M are the ID r.m. that generate Λ.

2.4.3 The general case

In this subsection we weaken the assumptions made in the previous subsection. We work with the assumption that $\mathcal{S}$ is a σ-algebra and that, as usual (see Section 2.2), $F(A,\cdot)$ is a signed measure for every $A \in \mathcal{S}$ and $F(\cdot,B)$ is a signed measure for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$. We define for every $A \in \mathcal{S}$ and $B \in \mathcal{B}(\mathbb{R})$

$$J(A,B) := \int_B (1 \wedge x^2) F_A(dx).$$

We assume that

$$J \text{ is a signed bimeasure on } \mathcal{S} \times \mathcal{B}(\mathbb{R}) \text{ and } \sup_I \sum_{i\in I} |J(A_i,B_i)| < \infty. \tag{2.14}$$

Informally, the above assumption is weaker than assuming the "total variation" of the signed bimeasure J to be finite.

Let us introduce the set function $\xi(A) : \mathcal{S} \mapsto [0,\infty)$ such that

$$\xi(A) := \sup_{I_A} \sum_{i\in I_A} |J(A_i,B_i)|.$$

Notice that ξ is indeed a measure (see Theorem 2.4.17). Further, let E_A^+ and E_A^- be the Hahn decomposition of $\mathbb{R}$ under the signed measure F_A. Observe that

$$\xi(A) = \sup_{I_A} \sum_{i\in I_A} |J(A_i,B_i)| \geq \int_{E_A^+} (1\wedge x^2) F_A(dx) - \int_{E_A^-} (1\wedge x^2) F_A(dx) = \int_{\mathbb{R}} (1\wedge x^2)|F_A|(dx). \tag{2.15}$$

Therefore, since $\xi(S)$ is finite by assumption we have that $\int_{\mathbb{R}} (1\wedge x^2)|F_A|(dx) < \infty$.

The name of this section comes from the fact that if $\mathcal{S}$ is a σ-algebra then the assumptions on F presented in the previous sections (Sections 2.4.1 and 2.4.2) are stricter than the ones of this section. Indeed, for the bounded case, notice that if $F(A,B)$ is a signed bimeasure on $\mathcal{S} \times \mathcal{B}(\mathbb{R})$

and $\sup_I \sum_{i\in I} |F_{A_i}(B_i)| < \infty$, then F immediately satisfies the assumptions of this section. For the case of the two generating ID r.m., we have the following.

Lemma 2.4.27. *Given the notation above, we have that*

$$\int_{\mathbb{R}} (1\wedge x^2) G_S(dx) + \int_{\mathbb{R}} (1\wedge x^2) M_S(dx) < \infty \quad \Rightarrow \quad \xi(S) < \infty.$$

Proof. Observe that

$$\int_B (1\wedge x^2) F_A(dx) = \int_B (1\wedge x^2) G_A(dx) - \int_B (1\wedge x^2) M_A(dx)$$

and so using that $S \in \mathcal{S}$ (since $\mathcal{S}$ is a σ-algebra) we have that

$$\xi(S) = \sup_I \sum_{i\in I} |J(A_i, B_i)| \leq \sup_I \sum_{i\in I} \int_{B_i} (1\wedge x^2) G_{A_i}(dx) + \int_{B_i} (1\wedge x^2) M_{A_i}(dx).$$

Now, we show that for every family $(A_i, B_i)_{i\in I}$ we have

$$\sum_{i\in I} \int_{B_i} (1\wedge x^2) G_{A_i}(dx) \leq \int_{\mathbb{R}} (1\wedge x^2) G_S(dx).$$

Consider the case where all the A_i are all disjoints. Then, the B_i's could take any values. In particular, by the (finite) additivity of G we would have that

$$\sum_{i\in I} \int_{B_i} (1\wedge x^2) G_{A_i}(dx) \leq \sum_{i\in I} \int_{\mathbb{R}} (1\wedge x^2) G_{A_i}(dx) = \int_{\mathbb{R}} (1\wedge x^2) G_{\cup_{i\in I} A_i}(dx) \leq \int_{\mathbb{R}} (1\wedge x^2) G_S(dx).$$

Hence, it remains to investigate the case where the A_i's have at least one intersection. Let $(A_i)_{i\in I}$ be any finite family of sets of $\mathcal{S}$. Then, it is possible to find a finite set of disjoints elements in $\mathcal{S}$, call it $(A'_i)_{i\in K}$, such that $\bigcup_{i\in I} A_i = \bigcup_{i\in J} A'_i$. Then, each A'_i is a subset of one or more of the A_i's. This implies that the corresponding B_i of the A_i's, whose intersection is A'_i, cannot have intersections, because the rectangles (A_i, B_i)'s must be disjoint. Therefore, for each A'_i the union of the corresponding B_i's is a subset of $\mathbb{R}$. Thus, in formulae we have

$$\sum_{i\in I} \int_{B_i} (1\wedge x^2) G_{A_i}(dx) \leq \sum_{i\in J} \int_{\mathbb{R}} (1\wedge x^2) G_{A'_i}(dx) = \int_{\mathbb{R}} (1\wedge x^2) G_{\cup_{i\in J} A'_i}(dx)$$

$$= \int_{\mathbb{R}} (1\wedge x^2) G_{\cup_{i\in I} A_i}(dx) \leq \int_{\mathbb{R}} (1\wedge x^2) G_S(dx).$$

Since the same holds for M, we obtain the result □

Given these preliminaries, we are now ready to present the results of this subsection, whose presentation follows the same structure as the one of the previous subsections.

Proposition 2.4.28. *Let Λ be a QID random measure. Let $\nu_0 : \mathcal{S} \mapsto \mathbb{R}$ be a signed measure, $\nu_1 : \mathcal{S} \mapsto \mathbb{R}$ be a measure, F_A be a quasi-Lévy measure on $\mathbb{R}$ for every $A \in \mathcal{S}$, $\mathcal{S} \ni A \mapsto F_A(B) \in (-\infty, \infty)$ be a signed measure for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$ and s.t. $(\nu_0(A), \nu_1(A), F_A)$ is the*

characteristic triplet of $\Lambda(A)$, $\forall A \in \mathcal{S}$. *Assume that* F *satisfies* (2.14). *Define*

$$\lambda(A) = |\nu_0|(A) + \nu_1(A) + \xi(A). \tag{2.16}$$

Then $\lambda : \mathcal{S} \mapsto [0, \infty)$ *is a measure s.t.* $\lambda(A_n) \to 0$ *implies* $\Lambda(A_n) \xrightarrow{p} 0$ *for every* $\{A_n\} \subset \mathcal{S}$.

Proof. Since by Theorem 4 in [Ho77] we know that ξ is a measure, it is straightforward to see that $\lambda(A)$ is a measure. Now, let $\lambda(A_n) \to 0$ then we have that $|\nu_0|$, ν_1 and ξ go to zero. Then since $\xi(A_n) \to 0$ implies that $\int_{\mathbb{R}}(1 \wedge x^2)|F_{A_n}|(dx) \to 0$, then we have that $\Lambda(A_n) \xrightarrow{p} 0$ for every $\{A_n\} \subset \mathcal{S}$. □

Lemma 2.4.29. *Let* F. *be as in Proposition 2.4.28. There exists a function* $\rho : S \times \mathcal{B}_0(\mathbb{R}) \mapsto \mathbb{R}$ *such that*
(i) $\rho(s, \cdot)$ *is a quasi-Lévy type measure on* $\mathcal{B}(\mathbb{R})$, *for every* $s \in S$, *with positive and negative part denoted by* $\rho^+(s, \cdot)$ *and* $\rho^-(s, \cdot)$,
(ii) $\rho^+(\cdot, B)$ *and* $\rho^-(\cdot, B)$, *are Borel measurable functions, for every* $B \in \mathcal{B}(\mathbb{R})$,
Furthermore, there exist two unique σ*-finite measures* $\tilde{F}^+$ *and* $\tilde{F}^-$ *on* $\mathcal{S} \otimes \mathcal{B}(\mathbb{R})$ *such that* $\int_{S\times\mathbb{R}} h(s,x)\tilde{F}^+(ds, dx) = \int_S \int_{\mathbb{R}} h(s,x)\rho^-(s, dx)\lambda(ds)$, *for every* $\mathcal{S} \otimes \mathcal{B}(\mathbb{R})$*-measurable function* $h : S \times \mathbb{R} \mapsto [0, \infty]$, *and the same holds for* $\tilde{F}^-$. *This equality can be extended to real and complex-valued functions* h. *Finally, for every* $A \in \mathcal{S}$ *and for every* $\mathcal{B}(\mathbb{R})$*-measurable real function* g *s.t.* $\int_A \int_{\mathbb{R}} |g(x)||\rho|(s, dx)\lambda(ds) < \infty$, *we have that*

$$\int_{\mathbb{R}} g(x)F_A(dx) = \int_A \int_{\mathbb{R}} g(x)\rho(s, dx)\lambda(ds),$$

and for every $B \in \mathcal{B}(\mathbb{R})$ *s.t.* $0 \notin \overline{B}$,

$$\tilde{F}^+(A, B) \geq F_A^+(B) \quad \textit{and} \quad \tilde{F}^-(A, B) \geq F_A^-(B).$$

Using the above results, we obtain the following proposition.

Proposition 2.4.30. *Under the setting of Proposition 2.4.28, the c.f. of* $\Lambda(A)$ *can be written in the form:*

$$\mathbb{E}(e^{i\theta\Lambda(A)}) = \exp\left(\int_A K(\theta, s)\lambda(ds)\right), \quad \theta \in \mathbb{R}, A \in \mathcal{S},$$

where

$$K(\theta, s) = i\theta a(s) - \frac{\theta^2}{2}\sigma^2(s) + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)\rho(s, dx),$$

$a(s) = \frac{d\nu_0}{d\lambda}(s)$, $\sigma^2(s) = \frac{d\nu_1}{d\lambda}(s)$ *and* ρ *is given by Lemma 2.4.29, and* $\exp(K(\theta, s))$ *is the characteristic function of a QID random variable if it exists. Moreover, we have*

$$|a(s)| + \sigma^2(s) + \frac{d\xi}{d\lambda}(s) = 1, \qquad \lambda\textit{-a.e.}.$$

Proof. The first statement follows from the Lévy-Khintchine formulation (2.3) and Lemma 2.4.29.

The second statement follows from the fact that for every $A \in \mathcal{S}$, we have

$$\int_A \left(|a(s)| + \sigma^2(s) + \frac{d\xi}{d\lambda}(s)\right)\lambda(ds) = |\nu_0|(A) + \nu_1(A) + \xi(A) = \lambda(A) = \int_A d\lambda(ds).$$

□

Proposition 2.4.31. *Under the setting of Proposition 2.4.28, if f is Λ-integrable, then we have that $\int_S |K(tf(s),s)|\lambda(ds) < \infty$, where K is given in Proposition 2.4.30, and*

$$\hat{\mathcal{L}}\left(\int_S f d\Lambda\right)(\theta) = \exp\left(\int_S K(\theta f(s), s)\lambda(ds)\right), \quad \theta \in \mathbb{R}.$$

Proof. The statement follows from the same arguments used in the proof of Proposition 2.4.9. □

We state now the main theorem of this subsection on the integrability conditions of $\int_S f d\Lambda$.

Theorem 2.4.32. *Let $f : S \to \mathbb{R}$ be a $\mathcal{S}$-measurable function and consider the setting of Proposition 2.4.28. Then f is Λ-integrable if the following three conditions hold:*
(i) $\int_S |U(f(s),s)|\lambda(ds) < \infty$,
(ii) $\int_S |f(s)|^2\sigma^2(s)\lambda(ds) < \infty$,
(iii) $\int_S V_0(f(s),s)\lambda(ds) < \infty$,

$$\text{where} \quad U(u,s) = ua(s) + \int_{\mathbb{R}} \tau(xu) - u\tau(x)\rho(s,dx), \quad V_0(u,s) = \int_{\mathbb{R}} (1 \wedge |xu|^2)|\rho|(s,dx).$$

Further, the c.f. of $\int_S f d\Lambda$ can be written as
(iv) $\hat{\mathcal{L}}\left(\int_S f d\Lambda\right)(\theta) = \exp\left(i\theta a_f - \frac{1}{2}\theta^2\sigma_f^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)F_f(dx)\right)$, *where*

$$a_f = \int_S U(f(s),s)\lambda(ds), \quad \sigma_f^2 = \int_S |f(s)|^2\sigma^2(s)\lambda(ds), \quad \textit{and}$$

$F_f(B)$ is the unique quasi-Lévy measure determined by the difference of the Lévy measures $\tilde{F}_f^+$ and $\tilde{F}_f^-$, which are defined as: for every $B \in \mathcal{B}(\mathbb{R})$

$$\tilde{F}_f^+(B) = \tilde{F}^+(\{(s,x) \in S \times \mathbb{R} : \, f(s)x \in B \setminus \{0\}\})$$

$$\textit{and} \quad \tilde{F}_f^-(B) = \tilde{F}^-(\{(s,x) \in S \times \mathbb{R} : \, f(s)x \in B \setminus \{0\}\}).$$

Proof. It follows from similar arguments as the ones used in the proof of Theorem 2.4.10. □

2.5 Continuity of the stochastic integral mapping

In this section we explore the set of Λ-integrable functions and show a continuity property of the linear stochastic integral mapping $f \to \int_S f d\Lambda$ from the space of Λ-integrable functions to the L_p space (more precisely to $L_p(\Omega, \mathbb{P})$). In particular, the space of integrable functions is a subset of the corresponding Musielak-Orlicz modular space defined in Chapter III of [RR89]. We begin with some preliminaries. Let $q \in [0, \infty)$. Consider the following condition:

$$\mathbb{E}[|\Lambda(A)|^q] < \infty, \qquad \text{for all } A \in \mathcal{S}.$$

Observe that for $q = 0$ every Λ satisfies this condition. Throughout this section, we shall assume that the above condition is satisfied. Further, we assume that for all $s \in S$

$$\int_{|x|>1} |x|^q \rho^+(s, dx) < \infty. \tag{2.17}$$

From the arguments in the previous section we have that

$$\int_A \int_{|x|>1} |x|^q \rho^+(s, dx)\lambda(ds) \geq \int_{|x|>1} |x|^q F_A^+(dx).$$

Thus, by Theorem 6.2 point (a) in [LPS18] the assumption (2.17) implies that

$$\int_{|x|>1} |x|^q F_A^-(dx) < \infty \quad \text{and} \quad \int_{\mathbb{R}} |x|^q \mathcal{L}(\Lambda(A))(dx) < \infty.$$

Observe that, while Theorem 6.2 point (a) in [LPS18] is stated for the centering function $\tau(x) = x\mathbf{1}_{|x|\leq 1}$, the result holds for any centering function. This is because the proof is based on results on ID distributions with no restrictions on the choice of the centering function (see Theorem 25.3 of [Sa99]).
Now, define, for $0 \leq p \leq q$, $u \in \mathbb{R}$ and $s \in S$,

$$\Phi_p(u, s) = U^*(u, s) + u^2\sigma^2(s) + V_p(u, s), \tag{2.18}$$

where

$$U^*(u, s) = \sup_{|c|\leq 1} |U(cu, s)| \quad \text{and} \quad V_p(u, s) = \int_{\mathbb{R}} |xu|^p \mathbf{1}_{|xu|>1}(x) + |xu|^2 \mathbf{1}_{|xu|\leq 1}(x)|\rho|(s, dx).$$

In the following lemma, which is an equivalent of Lemma 3.1 in [RR89] in our framework, we abuse the notation and consider λ to take all the different formulations it has taken in this chapter, namely (2.7), (2.12) and (2.16).

Lemma 2.5.1. *The following are satisfied:*
(i) *for every $s \in S$, $\Phi_p(\cdot, s)$ is a continuous non-decreasing function on $[0, \infty)$ with $\Phi_p(0, s) = 0$,*
(ii) $\lambda(\{s : \Phi_p(u, s) = 0 \text{ for some } u = u(s) \neq 0\}) = 0$,
(iii) *there exists a constant $C > 0$ such that $\Phi_p(2u, s) \leq C\Phi_p(u, s)$, for all $u \geq 0$ and $s \in S$.*

Proof. Points (i) and (iii) are proved adapting the same arguments of the proof of the points (i) and (iii) of Lemma 3.1 in [RR89] to our framework. We sketch them for the sake of completeness. First, it is easy to prove that, for any fixed s, $U(\cdot, s)$ is continuous and so that $U^*(\cdot, s)$ is continuous. Then by dominated convergence theorem, we obtain the continuity of $\Phi_p(\cdot, s)$. Further, notice that $U^*(\cdot, s)$ is non-decreasing and that

$$|xu|^p \mathbf{1}_{|xu|>1}(x) + |xu|^2 \mathbf{1}_{|xu|\leq 1}(x) = \begin{cases} (|xu|^p \wedge |xu|^2) & \text{if } 0 \leq p \leq 2, \\ (|xu|^p \vee |xu|^2) & \text{if } p > 2. \end{cases} \tag{2.19}$$

is increasing in $x \geq 0$. To prove (iii) observe that from (2.19) and Lemma 2.4.11 we have

$$\Phi_p(2u,s) \leq 2|U(u,s)| + 27V_0(u,s) + 4u^2\sigma^2(s) + (2^p+4)V_p(u,s) \leq (2^p+31)\Phi_p(u,s).$$

To prove point (ii) we proceed as follow. Recall that we have different formulations of λ. If $\Phi_p(u,s) = 0$ for some $u = u(s) \neq 0$ then $\sigma^2(s) = 0$, $|\rho|(s,\mathbb{R}) = 0$ and $U(u,s) = 0$. The last two equalities imply that $a(s) = 0$. Thus,

$$S_0 := \{s : \Phi_p(u,s) = 0 \text{ for some } u = u(s) \neq 0\} = \{s : a(s) = \sigma^2(s) = |\rho|(s,\mathbb{R}) = 0\},$$

which also shows that S_0 is a measurable set. Let A be any measurable set in S_0. Notice that $\nu_0(A) = \int_A a(s)\lambda(ds) = 0$, whence we obtain that $|\nu_0|(S_0) = 0$.

So, given the above arguments we have that

$$\int_{S_0}\int_{\mathbb{R}} (1 \wedge |x|^2)|\rho|(s,dx)\lambda(ds) = 0,$$

$$\nu(S_0) = \sup_{I_{S_0}} \sum_{i\in I_{S_0}} |F_{A_i}(B_i)| = \sup_{I_{S_0}} \sum_{i\in I_{S_0}} \left| \int_{A_i}\int_{B_i} \rho(s,dx)\lambda(ds)\right| = 0,$$

and

$$\xi(S_0) = \sup_{I_{S_0}} \sum_{i\in I_{S_0}} \left| \int_{B_i} (1\wedge x^2)F_{A_i}(dx)\right| = \sup_{I_{S_0}} \sum_{i\in I_{S_0}} \left| \int_{A_i}\int_{B_i} (1\wedge x^2)\rho(s,dx)\lambda(ds)\right| = 0.$$

Therefore, we get (in its different formulations) that $\lambda(S_0) = 0$. □

Lemma 2.5.2. *Let $\{\mu_n\}$ be a sequence of QID probability distributions on $\mathbb{R}$ with c.t. (a_n, σ_n^2, G_n). Let $a_n \to 0$, $\sigma_n^2 \to 0$ ad $\int_{\mathbb{R}}(1\wedge|x|^2)|G_n|(dx) \to 0$. Assume that $\int_{|x|>1}|x|^b|G_n|(dx) < \infty$ for all $n \in \mathbb{N}$. Then, for every $b > 0$,*

$$\int_{|x|>1} |x|^b|G_n|(dx) \to 0 \Rightarrow \int_{\mathbb{R}} |x|^b\mu_n(dx) \to 0.$$

Proof. For every $n \in \mathbb{N}$, let μ_n^+ and μ_n^- the ID distributions with c.t. (a_n, σ_n^2, G_n^+) and $(0,0,G_n^-)$ respectively. Then, from Lemma 3.2 in [RR89] we obtain that

$$\int_{|x|>1} |x|^b G_n^+(dx) \to 0 \Rightarrow \int_{\mathbb{R}} |x|^b\mu_n^+(dx) \to 0,$$

and

$$\int_{|x|>1} |x|^b G_n^-(dx) \to 0 \Rightarrow \int_{\mathbb{R}} |x|^b\mu_n^-(dx) \to 0.$$

Now, for every $n \in \mathbb{N}$, let X_n, X_n^+ and X_n^- be the real valued r.v. with distributions μ_n, μ_n^+ and μ_n^-, respectively; thus, $X_n + X_n^- \stackrel{d}{=} X_n^+$ with X_n and X_n^- independent. Notice that for $b \leq 1$ we have

$$\mathbb{E}[|X_n|^b] \leq \mathbb{E}[|X_n + X_n^-|^b] + \mathbb{E}[|X_n^-|^b] = \mathbb{E}[|X_n^+|^b] + \mathbb{E}[|X_n^-|^b],$$

while for $b > 1$ we have

$$\mathbb{E}[|X_n|^b] \leq \lceil b \rceil \left(\mathbb{E}[|X_n + X_n^-|^{\lceil b \rceil}] + \mathbb{E}[|X_n^-|^{\lceil b \rceil}] \right) = \lceil b \rceil \left(\mathbb{E}[|X_n^+|^{\lceil b \rceil}] + \mathbb{E}[|X_n^-|^{\lceil b \rceil}] \right),$$

where $\lceil b \rceil$stands for the lowest natural number greater than b. Finally, by sending $n \to \infty$ we obtain the stated result. □

Before presenting the main result of this section, we need some preliminaries. Define the Musielak-Orlicz space as in [RR89]:

$$L_{\Phi_p}(S;\lambda) = \left\{ f \in L_0(S;\lambda) : \int_S \Phi_p(|f(s)|, s)\lambda(ds) < \infty \right\}.$$

By Lemma 2.5.1 this space has the following properties (see Chapter III of [RR89]). The space $L_{\Phi_p}(S;\lambda)$ is a complete linear metric space with the F-norm defined by

$$\|f\|_{\Phi_p} = \inf_{c>0} \left\{ \int_S \Phi_p(c^{-1}|f(s)|, s)\lambda(ds) \leq c \right\}.$$

Simple functions are dense in $L_{\Phi_p}(S;\lambda)$ and $L_{\Phi_p}(S;\lambda) \hookrightarrow L_0(S;\lambda)$ is continuous, where in the present case $L_0(S;\lambda)$ is equipped with the topology of convergence in λ measure on every set of finite λ-measure. Moreover, $\|f_n\|_{\Phi_p} \to 0 \Leftrightarrow \int_S \Phi_p(|f(s)|, s)\lambda(ds) \to 0$.

We remark that the above definitions and results hold both in the case λ is atomless and when it is not (see Chapter III of [RR89] and Musielak's book [Mu83]).

Theorem 2.5.3. *Let $0 \leq p \leq q$ and Φ_p defined as in* (2.18)*. Then*

$$\left\{ f : f \text{ is } \Lambda\text{-integrable and } \mathbb{E}\left[\left| \int_S f d\Lambda \right|^p \right] < \infty \right\} \supset L_{\Phi_p}(S;\lambda)$$

and the linear mapping

$$L_{\Phi_p}(S;\lambda) \ni f \mapsto \int_S f d\Lambda \in L_p(\Omega;\mathbb{P})$$

is continuous.

Remark 2.5.4. When $p = 0$ the statement becomes $\{f : f \text{ is } \Lambda\text{-integrable}\} \supset L_{\Phi_0}(S;\lambda)$.

Proof. We partially follow the proof of Theorem 3.3 in [RR89]. Let $f \in L_{\Phi_p}(S;\lambda)$, which is equivalent to consider $\int_S \Phi_p(|f(s)|, s)\lambda(ds) < \infty$. Then, the conditions (i), (ii) and (iii) of Theorem 2.4.32 (and also Theorems 2.4.10, 2.4.12 and 2.4.23) are satisfied. Thus, f is Λ-integrable. Moreover, we have that

$$\int_{\{|u|>1\}} |u|^p |F_f|(du) \leq \int_S \int_{\{|f(s)x|>1\}} |f(s)x|^p |\rho|(s, dx)\lambda(ds)$$

$$\leq \int_S V_p(|f(s)|, s)\lambda(ds) \leq \int_S \Phi_p(|f(s)|, s)\lambda(ds) < \infty.$$

Hence, thanks to Theorem 6.2 point (a) in [LPS18] we have that $\mathbb{E}\left[\left| \int_S f d\Lambda \right|^p \right] < \infty$. Thus, every element of $L_{\Phi_p}(S;\lambda)$ is also an element of $\left\{ f : f \text{ is } \Lambda\text{-integrable and } \mathbb{E}\left[\left| \int_S f d\Lambda \right|^p \right] < \infty \right\}$.

Now, let $f_n \to 0$ in $L_{\Phi_p}(S;\lambda)$, which implies that $\int_S \Phi_p(|f_n(s)|,s)\lambda(ds) \to 0$ as $n \to \infty$. Further, let a_{f_n}, $\sigma^2_{f_n}$ and F_{f_n}, be, respectively, the drift, the variance and the quasi-Lévy measure of $\hat{\mathcal{L}}(\int_S f_n d\Lambda)$; see point (iv) Theorem 2.4.32 (and also Theorems 2.4.10, 2.4.12 and 2.4.23). Then, we have that $a_{f_n} \to 0$, $\sigma^2_{f_n} \to 0$, and

$$0 \leq \int_{\mathbb{R}} |u|^p \mathbf{1}_{|u|>1}(u) + |u|^2 \mathbf{1}_{|u|\leq 1}(u) |F_{f_n}|(du)$$

$$\leq \int_{\mathbb{R}} |xf_n(s)|^p \mathbf{1}_{|xf_n(s)|>1}(x) + |xf_n(s)|^2 \mathbf{1}_{|xf_n(s)|\leq 1}(x) |\rho|(s,dx)\lambda(ds) = \int_S V_p(|f_n(s)|,s)\lambda(ds) \to 0, \tag{2.20}$$

as $n \to \infty$. Hence, using Lemma 2.5.2 we conclude that $\mathbb{E}\left[\left|\int_S f_n d\Lambda\right|^p\right] \to 0$ as $n \to \infty$, for $p > 0$, and $\int_S f_n d\Lambda \xrightarrow{p} 0$ if $p = 0$. □

2.6 QID stochastic processes

One of the underlying fundamental problems of the work of this chapter is the mismatch between the ID and the QID case. This mismatch is caused by the greater generality of the QID framework in comparison with the ID one. Hence, one of the main objective of the work of this chapter is to overcome this mismatch in order to obtain the QID equivalent version of the ID result. Every section has its own difficulties, which sometimes force us to make additional assumptions in order to achieve the equivalent QID result.

For example, in Section 2.2 the main difficulty is that the Jordan decomposition of $F_A(\cdot)$ (namely $F_A^+(B)$ and $F_A^-(B)$) are not measures on $\mathcal{S}$ for fixed B. In Section 2.4, in addition to this problem we have that, informally, $|F_f|$ is smaller than $|\rho|\lambda$, which in some cases (Sections 2.4.1 and 2.4.3) prevent us to obtain an if and only if result on the integrability conditions of f w.r.t. Λ. In Section 2.5, this same last problem does not allow us to conclude that $\left\{f : f \text{ is } \Lambda\text{-integrable and } \mathbb{E}\left[\left|\int_S f d\Lambda\right|^p\right] < \infty\right\} = L_{\Phi_p}(S;\lambda)$.

In this section there are two novel difficulties: dimensionality and consistency. Let us explain in details. In this section we investigate QID processes both in the countable and in the uncountable index set case. While in the former case we follow classical arguments, for the latter we need to be extremely careful. Indeed, the uncountable case has a trouble history even in the ID framework. This is due to the fact that the product σ-algebra and the cylindrical σ-algebra do not coincide anymore (the former is larger than the latter). Thus, it makes a difference if we consider the Lévy measure on the σ-ring generated by cylindrical subsets of $\mathbb{R}^T \setminus \{0\}$ or on the measurable space $(\mathbb{R}^T, \mathcal{B}^T)$, where $\mathcal{B}^T$ is the cylindrical σ-algebra. Thus, which is the "right", or at least more natural, definition of Lévy measure?

After almost 50 years, started with Maruyama's 1970 pivotal work [Mar70] in which the first framework has been investigated, a complete satisfactory answer has been given only recently by Rosinski in [Ro18], in which the author investigates the second framework. In this section we follow the framework and results of [Ro18] and extend them to the QID case.

Concerning consistency, we make the following remark. One of the first properties we introduce in this section is the *consistency* of a family of finite dimensional Lévy measures. In words, it says that a family of finite dimensional Lévy measures is consistent if, given any two elements

in this family, the projection of the first measure to the dimension of the second one coincides with the second measure and this holds for any set which does not contain the origin (see below for a formal definition). This definition is essential in [Ro18] because it gives enough structure to properly define a Lévy measure in the (uncountable) infinite dimensional space $\mathbb{R}^T$. Even though this concept is not modified in our QID framework, it brings a new level of technicality which we need to take care of in our proofs.

In addition to this, there is another aspect of consistency which is more problematic for us. We know that for every QID random variable X there exist two ID random variables Y and Z such that $X + Y \stackrel{d}{=} Z$ with Y independent of X. However, does this holds for QID processes?

We do not give an answer to this question, which represents one of the major open questions of this work. The underlying problem is the following. Consider any finite dimensional index set $I \subset T$ and let $X_I := (X_t)_{t\in I}$. We know that there exist two random variables Y_I and Z_I such that $X_I + Y_I \stackrel{d}{=} Z_I$ with Y_I independent of X_I. Actually, we know that there exist an infinite number of couples of $\mathbb{R}^I$-valued random variables that satisfy the above condition, namely that for every couple $(\tilde{Y}_I, \tilde{Z}_I)$ we have $X_I + \tilde{Y}_I \stackrel{d}{=} \tilde{Z}_I$. Then, we would like to prove that there exist two processes Y and Z on $\mathbb{R}^T$ such that $Y_I = (Y_t)_{t\in I}$ and $Z_I = (Z_t)_{t\in I}$. Thanks to the Kolmogorov extension theorem this reduces to obtain a "consistent" collection of finite-dimensional distributions $(Y_I)_{I\subset T}$. As it is possible to see the problem is not the existence of the Y_I, indeed for every I there is an infinite number of them, but their consistency. This problem can be seen from the point of view of Lévy measures (thanks to Theorem 2.8 in [Ro18]) and translates into the problem of finding a consistent family of finite dimensional Lévy measures.

While we are able to show such consistency in the finite case, namely when T is finite, (see Proposition 2.6.2 and in particular equation (2.21)), we are not able to show it for the infinite (either countable or uncountable) case, namely for processes. We show a simple and general example where the positive and negative parts of a quasi-Lévy measure, which is the natural way of selecting a couple of Lévy measures from a quasi-Lévy measure, do not produce a consistent family.

Let us now briefly mention the content of this section. In this section we introduce QID stochastic processes and present various spectral representation for them.

In the first subsection we start by defining QID processes and by introducing a subclass called generated QID processes. Informally, a generated QID processes is a stochastic process to which it is possible to associate two ID processes. We conjecture that any QID process is a generated QID process. In this subsection we prove some simple but interesting results on general QID random variables and processes.

In the second subsection, we investigate the case of generated QID processes with values in l_2. The main result is Theorem 2.6.9, where we provide the Lévy-Khintchine formulation. The arguments used to prove the results exploit the structure of the generated QID process, namely the structure of the underlying generating ID processes, and the structure of the l_2 space.

In the third subsection, we investigate QID stochastic processes with uncountable index set, namely $(X_t)_{t\in[0,T]}$, and we prove the Lévy-Khintchine formulation. The arguments rely on novel techniques which build on the results of [Ro18]. As mentioned before, in this subsection we need to pay attention to the fact that we are in the uncountable setting. For example, this force us

to introduce a new definition of quasi-Lévy measure.

In the fourth subsection, we provide a spectral representation of discrete parameters QID processes. The main idea of the proof is to connect discrete processes with processes with values in l_2, for which we have a representation.

In the last subsection, we start by showing further results and examples of QID processes. In particular, we provide a very general parametrisation of QID random variables, which is useful for the construction of QID processes. We conclude by discussing the atomless condition for QID and ID random measures.

2.6.1 QID processes: definitions and general results

We start with the definition of QID processes.

Definition 2.6.1 (QID processes)**.** *Let T be an arbitrary index set. A stochastic process $X = \{X_t;\, t \in T\}$ is said to be a QID* process *if and only if for every finite set of indices $t_1, \dots, t_k$ in the index set T*

$$X_{t_1,\dots,t_k} := (X_{t_1}, ..., X_{t_k})$$

is a multivariate QID random variable.

The existence of QID processes is ensured by the Kolmogorov extension theorem. From Definition 2.6.1, it is clear that the class of QID processes is strictly larger than the class of ID processes.

Before presenting other results we introduce some preliminaries. We use the general framework introduced in [Ro18]. Let T be an arbitrary (possibly uncountable) index set, $\mathbb{R}^T$ denote the space of all functions $x : T \to \mathbb{R}$ and $\mathcal{B}^T$ be its cylindrical product σ-algebra. Given an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for a stochastic process $X = (X_t)_{t\in T}$ we define its law $\mathcal{L}(X)$ as a probability measure on $(\mathbb{R}^T, \mathcal{B}^T)$ such that

$$\mathcal{L}(X)(A) = \mathbb{P}(\{\omega \in \Omega : (X_t(\omega))_{t\in T} \in A\}), \quad A \in \mathcal{B}^T.$$

We denote by x_S the restriction of x to $S \subset T$ and by 0_S the origin of $\mathbb{R}^S$, which depending on the setting is a point or a one-point set. Further, we let $\hat{T}$ to be defined as $\hat{T} := \{I \subset T : 0 < \mathrm{Card}(I) < \infty\}$, let $\pi_S : \mathbb{R}^T \to \mathbb{R}^S$ be the projection from $\mathbb{R}^T$ onto $\mathbb{R}^S$ (namely $\pi_S(x) = x_S$) and let $\mathcal{B}^S_{00} := \{B \in \mathcal{B}^S : 0_S \notin B\}$. Finally, we use as a cutoff function the bounded measurable function χ defined as $\chi : \mathbb{R} \mapsto \mathbb{R}$ such that $\chi(v) = 1 + o(|v|)$ as $v \to 0$ and $\chi(v) = O(|v|^{-1})$ as $|v| \to \infty$. Then, the truncation of $v = (v_1, ..., v_n) \in \mathbb{R}^n$ is defined by $[[v]] = (v_1\chi(|v_1|), ..., v_n\chi(|v_n|))$ and of $x \in \mathbb{R}^S$ by $[[x]](s) = x(s)\chi(|x(s)|)$, $s \in S$.

Let $\{\nu_I : I \in \hat{T}\}$ be a family of finite dimensional Lévy measures on $(\mathbb{R}^J, \mathcal{B}^J)$ s.t. for every $I, J \in \hat{T}$ with $I \subset J$

$$\nu_J \circ \pi_{IJ}^{-1} = \nu_I \text{ on } \mathcal{B}^I_{00},$$

where $\pi_{IJ} : \mathbb{R}^J \to \mathbb{R}^I$ is the natural projection. Then we say that $\{\nu_I : I \in \hat{T}\}$ is *consistent* (see Definition 2.6 in [Ro18]). Moreover, we say that a family of probability measures is *projective* if the above condition is satisfied on the whole $\mathcal{B}^S$ (and not just on $\mathcal{B}^I_{00}$).

The first result of this section concerns a general property of QID distributions.

Proposition 2.6.2. *All the marginal distributions of a multivariate QID distribution are QID.*

Proof. Let $d \in \mathbb{N}$ and $X = (X_1, ..., X_d)$ be QID. Then there exists at least two ID r.v. Y and Z s.t. $X + Y \stackrel{d}{=} Z$. Consider any $I \subset \{1, ..., d\}$. Let $\theta = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$ be s.t. $\theta_i = 0$ for $i \notin I$ and let $\theta_I \in \mathbb{R}^I$ containing all the non-zero element of θ. Then

$$\hat{\mathcal{L}}(X_I)(\theta_I) = \hat{\mathcal{L}}(X)(\theta) = \frac{\hat{\mathcal{L}}(Z)(\theta)}{\hat{\mathcal{L}}(Y)(\theta)} = \frac{\hat{\mathcal{L}}(Z_I)(\theta_I)}{\hat{\mathcal{L}}(Y_I)(\theta_I)}. \tag{2.21}$$

Thus, we obtain the stated results by Theorem 3.1 in [HS78], namely by the fact that all the marginal distributions of a multivariate ID distribution are ID. □

The above result shows that in the finite dimensional case we can always find two consistent families of finite dimensional Lévy measures and that for each sub-dimension I the difference of the respective Lévy measure generates the quasi-Lévy measure of X_I.

Recall from Proposition 3.2 in [Ka01] that for two stochastic processes $X^{(1)}$ and $X^{(2)}$ we have that $X^{(1)} \stackrel{d}{=} X^{(2)}$ if and only if $(X^{(1)}_{t_1}, ..., X^{(1)}_{t_k}) \stackrel{d}{=} (X^{(2)}_{t_1}, ..., X^{(2)}_{t_k})$ where $t_1, ..., t_k \in T$ and $k \in \mathbb{N}$. We have the following simple result.

Theorem 2.6.3. *Let T be an arbitrary index set. A stochastic process $X = \{X_t; t \in T\}$ is QID if there exist two ID processes Y and Z such that $X + Y \stackrel{d}{=} Z$ with Y independent of X. In this case, we say that X is* generated *by Y and Z and that X is a* generated QID process.

Proof. Let X be a stochastic process and let Y and Z two ID processes s.t. $X + Y \stackrel{d}{=} Z$ with Y independent of X. Consider any finite set of indices $t_1, ..., t_k$ in the index set T. Since $(Y_{t_1}, ..., Y_{t_k})$ and $(Z_{t_1}, ..., Z_{t_k})$ are ID distributed, $(X_{t_1}, ..., X_{t_k})$ is QID distributed. Then by definition the process $X = \{X_t; t \in T\}$ is QID. □

We sometimes say that Y and Z are the *generating ID process of* X. Moreover, we remark that while for any given two ID processes there exists a unique QID process (when it exists), it might happen that for a given QID processes there exist more than just two generating ID processes. We present now the main question left open from this chapter.

Open Question 2.6.4. Is it true that any QID process is generated by two ID processes? In other words, is it true that any QID process is a generated QID process?

The answer of this question is not trivial. It lies on the capacity of building canonical ID distributions from QID distributions and it involves the use of the axiom of choice. In particular, the axiom of choice is used to show the existence of a family of ID probability measures (actually there exist uncountably many such families). Then, in order to use the Kolmogorov extension theorem we need to show that this family of probability measures is consistent. However, despite Proposition 2.6.2 and equation (2.21), which provide consistency for each finite set of indices, it is not clear how to obtain the result as shown in the following example.

Example 2.6.5. Let (a, σ^2, ν) be the c.t. of a QID r.v. $X = (X_1, X_2)$ on $\mathbb{R}^2$. Let (a_1, σ_1^2, ν_1) and (a_2, σ_2^2, ν_2) the c.t. of X_1 and X_2, respectively. From Proposition 2.6.2 we have that $\nu_1(A) = \nu((A, \mathbb{R}))$ and $\nu_2(B) = \nu((\mathbb{R}, B))$ for every $A, B \in \mathcal{B}_{00}(\mathbb{R})$. Assume that the quasi-Lévy measures

are signed measures and consider their Jordan decomposition. For example let E_1^+ and E_2^+ be the Hahn decomposition of $\mathbb{R}$ under the signed measure ν_1 and assume w.l.o.g. that $\{0\} \notin E_1^+$. Then, $0 \leq \nu_1^+(\mathbb{R}) = \nu_1(E_1^+) = \nu((E_1^+, \mathbb{R}))$. Applying the same argument to ν_2 we have that $0 \leq \nu_2^+(\mathbb{R}) = \nu_2(E_2^+) = \nu((\mathbb{R}, E_2^+))$. Then we have that $(E_1^+, \mathbb{R}) \cup (\mathbb{R}, E_2^+)$ is a subset of $\mathbb{R}^2$ where ν is positive. On the other hand by applying the same arguments we have that $(E_1^- \setminus \{0\}, \mathbb{R}) \cup (\mathbb{R}, E_2^+ \setminus \{0\})$ is a subset of $\mathbb{R}^2$ where ν is negative. But these two sets have intersections, which means that the marginals of the Jordan decomposition of ν (call them ν^+ and ν^- with marginals $\tilde{\nu}_1^+$, $\tilde{\nu}_2^+$, $\tilde{\nu}_1^-$ and $\tilde{\nu}_2^-$) are s.t. $\nu_1^+(\mathbb{R}) = \nu_1(E_1^+) = \nu((E_1^+, \mathbb{R})) \leq \nu^+((E_1^+, \mathbb{R})) \leq \nu^+((\mathbb{R}, \mathbb{R})) = \tilde{\nu}_1^+(\mathbb{R})$, where the last equality comes from the fact that ν^+ is a Lévy measure of a ID r.v. whose marginals exist and are ID thanks to Theorem 3.1 in [HS78]. A similar result hold for the other cases.

This implies that the projection of ν^+ is not always ν_1^+, and similarly for ν^-. In other words, the positive and negative part of a quasi-Lévy measure are always not consistent (and, hence, projective). Hence, given $X + Y \stackrel{d}{=} Z$ where $Z = (Z_1, Z_2)$ and $Y = (Y_1, Y_2)$ are two ID r.v. on $\mathbb{R}^2$ with quasi-Lévy measures ν^+ and ν^- respectively, it is not always true that Z_1 and Y_1 have quasi-Lévy measure given by the positive and negative part of ν_1, and similarly for Z_2 and Y_2. Therefore, a family of ID probability measures where each probability measure has the Lévy measure given by the positive (or negative) part of the quasi-Lévy measure of the respective QID probability measure is not always projective.

The above example takes place in $\mathbb{R}^2$, thus T contains only two elements and so it is finite. This shows that taking the positive and negative parts of a quasi-Lévy measure does not necessarily produce a consistent family of finite dimensional Lévy measures, even in the finite case. Of course, this does not implies that there are no consistent family of finite dimensional Lévy measures in this case. Indeed, Proposition 2.6.2 show their existence whenever T is finite.

2.6.2 Spectral representation of generated QID processes: the l_2 case

In this subsection we present a spectral representation of generated QID processes with values in l_2. This setting represents the most general setting under which the definitions and the arguments to prove the results are similar to ones of the finite dimensional case.

Let index set T be countable. Let $\mathcal{B}^{l_2}$ be the Borel σ-algebra on l_2. We note that since l_2 is separable the Borel and cylindrical σ-algebras coincide (see page 38 in [LT91]). Then, we have the following definition.

Definition 2.6.6 (quasi-Lévy type measure on $(l_2, \mathcal{B}^{l_2})$). *Let $\mathcal{B}_r^{l_2} := \{B \in \mathcal{B}^{l_2} | B \cap \{x \in l_2 : \|x\| < r\} = \emptyset\}$ and $\mathcal{B}_0^{l_2} := \bigcup_{r>0} \mathcal{B}_r^{l_2}$. Let $\nu : \mathcal{B}_0^{l_2} \to \mathbb{R}$ be a set function s.t. $\nu_{|\mathcal{B}_r^{l_2}}$ is a finite signed measure for each $r > 0$ and denote the total variation, positive and negative part of $\nu_{|\mathcal{B}_r^{l_2}}$ by $|\nu_{|\mathcal{B}_r^{l_2}}|$, $\nu^+_{|\mathcal{B}_r^{l_2}}$ and $\nu^-_{|\mathcal{B}_r^{l_2}}$ respectively. Then the* total variation $|\nu|$, *the* positive part ν^+ *and the* negative part ν^- *of ν are defined to be the unique measures on $(l_2, \mathcal{B}^{l_2})$ satisfying*

$$|\nu(\{0_T\})| = \nu^+(\{0_T\}) = \nu^-(\{0_T\}) = 0$$

$$\text{and} \quad |\nu|(A) = |\nu_{|\mathcal{B}_r^{l_2}}|, \ \nu^+(A) = \nu^+_{|\mathcal{B}_r^{l_2}}(A), \ \nu^-(A) = \nu^-_{|\mathcal{B}_r^{l_2}}(A),$$

for $A \in \mathcal{B}_r^{l_2}$, for some $r > 0$.

A quasi-Lévy type measure on $(l_2, \mathcal{B}^{l_2})$ *is a function $\nu : \mathcal{B}_0^{l_2} \to \mathbb{R}$ with the above properties and s.t. $\int_{l_2}(1 \wedge \|x\|^2)|\nu|(dx) < \infty$.*

Lemma 2.6.7. *$\mathcal{B}_r^{l_2}$ is a σ-algebra.*

Proof. Since the space $Y_r := \{x \in l_2 : \|x\| \geq r\}$ is a non-empty subspace of l_2 then we have $Z_r := \{B \subseteq Y_r | B \in \mathcal{B}^{l_2}\} = \mathcal{B}_r^{l_2}$. Observe now that Y_r is measurable (namely $Y_r \in \mathcal{B}^{l_2}$) because l^2 is a complete separable metric space and the norm is continuous, then Z_r is the restriction of $\mathcal{B}^{l_2}$ on Y_r. Hence, $\mathcal{B}_r^{l_2}$ is a σ-algebra and the argument applies to any $r > 0$. □

If $\nu : \mathcal{B}_0^{l_2} \to \mathbb{R}$ is s.t. $\nu_{|\mathcal{B}_r^{l_2}}$ is a finite signed measure for each $r > 0$, then we have that $|\nu_{|\mathcal{B}_r^{l_2}}|(A) = |\nu_{|\mathcal{B}_s^{l_2}}|(A)$ for every $A \in \mathcal{B}_r^{l_2}$ with $0 < s \leq r$ and the same applies to $\nu^+_{|\mathcal{B}_r^{l_2}}$ and $\nu^-_{|\mathcal{B}_r^{l_2}}$. Moreover, since we are in the countable framework (the index set considered T is countable since $l_2 \subset \mathbb{R}^{\mathbb{N}}$ hence $0_T \in \mathcal{B}^{l_2}$) then $|\nu|$, ν^+ and ν^- are well defined and the condition $|\nu(\{0\})| = \nu^+(\{0\}) = \nu^-(\{0\}) = 0$ together with the Carathéodory's extension theorem ensure their uniqueness. Notice that we have the equivalent of Lemma 2.1.14 in this framework.

Lemma 2.6.8. *A set function $\nu : \mathcal{B}_0^{l_2} \to \mathbb{R}$ is a quasi-Lévy type measure if and only if there exist two Lévy measures $\nu^{(1)}$ and $\nu^{(2)}$ s.t. $\nu_{|\mathcal{B}_r(\mathbb{R})}(A) = \nu^{(1)}_{|\mathcal{B}_r^{l_2}}(A) - \nu^{(2)}_{|\mathcal{B}_r^{l_2}}(A)$ for $A \in \mathcal{B}_r^{l_2}$, for some $r > 0$. Moreover, ν is unique.*

Proof. It follows from the same arguments used in the proof of Lemma 2.1.14. □

We have the following representation of a certain QID processes.

Theorem 2.6.9. *Let X be a discrete parameter QID process with values in l_2 s.t. there exist two generating ID processes with values in l_2. Then there exists an unique triplet $(z_0, \mathcal{K}, \nu)$ consisting of $z_0 \in l_2$, a non-negative definite function operator $\mathcal{K} : l_2 \to \mathbb{R}$ and a quasi-Lévy type measure ν on $(l_2, \mathcal{B}^{l_2})$ s.t.*

$$\hat{\mathcal{L}}(X)(y) = \exp\left(i\langle z_0, y\rangle - \frac{1}{2}\langle y, \mathcal{K}y\rangle + \int_{l_2}\left(e^{i\langle y,x\rangle} - 1 - i\langle \tau(x), y\rangle\right)\nu(dx)\right) \tag{2.22}$$

where $y \in l_2$.

Proof. Since X, Y and Z are l_2-valued stochastic processes, then they can be seen as three r.v. on $(l_2, \mathcal{B}^{l_2})$, with the property that $X + Y \overset{d}{=} Z$ with Y independent of X. Now, let $\nu^{(1)}$ and $\nu^{(2)}$ be the Lévy measures of the l_2 valued ID processes Z and Y, respectively. Then, we have

$$\hat{\mathcal{L}}(X)(y) = \hat{\mathcal{L}}(Z)(y)/\hat{\mathcal{L}}(Y)(y) = \exp\Bigg(i\langle z_0, y\rangle$$

$$-\frac{1}{2}\langle y, \mathcal{K}y\rangle + \int_{l_2}\left(e^{i\langle y,x\rangle} - 1 - i\langle \tau(x), y\rangle\right)\nu^{(1)}(dx) - \int_{l_2}\left(e^{i\langle y,x\rangle} - 1 - i\langle \tau(x), y\rangle\right)\nu^{(2)}(dx)\Bigg).$$

Let $\nu : \mathcal{B}_0^{l_2} \to \mathbb{R}$ be a set function s.t. $\nu_{|\mathcal{B}_r^{l_2}}(A) = \nu^{(1)}_{|\mathcal{B}_r^{l_2}}(A) - \nu^{(2)}_{|\mathcal{B}_r^{l_2}}(A)$ for every $A \in \mathcal{B}_r^{l_2}$ with $r > 0$. Then by Lemma 2.6.8, ν is uniquely determined by the difference $\nu^{(1)} - \nu^{(2)}$. Indeed the absence of uniqueness of $\nu^{(1)} - \nu^{(2)}$ will violate the condition $X + Y \overset{d}{=} Z$ due to the uniqueness of the

Lévy-Khintchine representation on l_2 (see Theorem 4.10 Chapter 4 in [Pa67]). Hence, we obtain the expression (2.22). Finally, it directly follows that z_0 and $\mathcal{K}$ are uniquely determined. □

Theorem 2.6.10. *Consider any $(z_0, \mathcal{K}, \nu)$ as in Theorem 2.6.9 such that the map*

$$l_2 \ni y \mapsto \exp\left(i\langle z_0, y\rangle - \frac{1}{2}\langle y, \mathcal{K}y\rangle + \int_{l_2} \left(e^{i\langle y,x\rangle} - 1 - i\langle \tau(x), y\rangle \right) \nu(dx) \right) \tag{2.23}$$

is the c.f. of a probability distribution. Then, there exists a QID process X with values in l_2 which is unique in distribution and such that (2.23) is the c.f. of X.

Proof. It is enough to see that the positive and negative part of ν, namely ν^+ and ν^-, are Lévy measures on $\mathcal{B}^{l_2}$. Hence, we can extract two ID processes with values in l_2. Then, these two processes generate a QID process with values in l_2. □

Open Question 2.6.11. In Theorem 2.6.9 we have the condition that discrete parameter QID process with values in l_2 must be generated by two generating ID processes with values in l_2. In order to get rid of this condition we need to prove that for every discrete parameter QID process with values in l_2 there exist two generating ID processes with values in l_2. Is this possible?

2.6.3 Spectral representation of generated QID processes: the uncountable case

In this subsection we investigate the representation of generated QID processes. For the first results we assume that the quasi-Lévy measure of QID processes, whose definition is provided below, is a signed measure. This situation happens when one of the two generating ID processes has a finite Lévy measure (see below). This is because in that case $\nu = \nu_1 - \nu_2$ is a signed measure. In the second part of this subsection, we do not consider such condition and we investigate general generated QID processes.

We rewrite here the definition of Lévy measure on $(\mathbb{R}^T, \mathcal{B}^T)$, firstly introduced in [Ro18], and introduce the definition of quasi-Lévy type measure in this framework.

Definition 2.6.12 (Definition 2.1 in [Ro18])**.** *A measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ is said to be a Lévy measure if the following two conditions hold*
(L1) *for every $t \in T$ we have $\int_{\mathbb{R}^T} |x(t)|^2 \wedge 1 \nu(dx) < \infty$,*
(L2) *for every $A \in \mathcal{B}^T$ we have $\nu(A) = \nu_*(A \setminus 0_T)$, where ν_* is the inner measure of ν.*

Definition 2.6.13. *A signed measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ is said to be a quasi-Lévy type measure if the following two conditions hold*
(QL1) *for every $t \in T$ we have $\int_{\mathbb{R}^T} |x(t)|^2 \wedge 1 |\nu|(dx) < \infty$,*
(QL2) *for every $A \in \mathcal{B}^T$ we have $|\nu(A)| = \nu_*^+(A \setminus 0_T) + \nu_*^-(A \setminus 0_T)$, where ν_*^+ and ν_*^- are the inner measures of the Jordan decomposition of ν.*

We present now the first result of this subsection, namely that for any two Lévy measures (one of which is finite) there exists a quasi-Lévy type measure.

Lemma 2.6.14. *Let $\nu^{(1)}$ and $\nu^{(2)}$ be two Lévy measures on $(\mathbb{R}^T, \mathcal{B}^T)$, with one of the two being finite. Let $\nu_* := \nu_*^{(1)} - \nu_*^{(2)}$ and let $\nu(A) := \nu_*(A)$ for every $A \in \mathcal{B}^T$. Then, ν is quasi-Lévy type measure on $(\mathbb{R}^T, \mathcal{B}^T)$.*

Proof. First, notice that $\nu = \nu^{(1)} - \nu^{(2)}$. Hence, ν is a signed measure on $(\mathbb{R}^T, \mathcal{B}^T)$. Let ν^+ and ν^- be the Jordan decomposition of ν and let ν_*^+ and ν_*^- be their inner measures. Further, let E^+ and E^- be the Hahn decomposition of $\mathbb{R}^T$ under ν. Then, for every $A \in \mathcal{B}^T$ we have

$$\nu_*^+(A \setminus 0_T) = \sup_{\substack{B \subset A \setminus 0_T \\ B \in \mathcal{B}^T}} \nu^+(B) = \sup_{\substack{B \subset A \setminus 0_T \\ B \in \mathcal{B}^T}} \nu(B \cap E^+) = \sup_{\substack{B \subset A \cap E^+ \setminus 0_T \\ B \in \mathcal{B}^T}} \nu(B)$$

$$= \sup_{\substack{B \subset A \cap E^+ \setminus 0_T \\ B \in \mathcal{B}^T}} \nu^{(1)}(B) - \nu^{(2)}(B) \geq \sup_{\substack{B \subset A \cap E^+ \setminus 0_T \\ B \in \mathcal{B}^T}} \nu^{(1)}(B) - \sup_{\substack{B \subset A \cap E^+ \setminus 0_T \\ B \in \mathcal{B}^T}} \nu^{(2)}(B)$$

$$= \nu_*^{(1)}(A \cap E^+ \setminus 0_T) - \nu_*^{(2)}(A \cap E^+ \setminus 0_T) = \nu^{(1)}(A \cap E^+) - \nu^{(2)}(A \cap E^+) = \nu(A \cap E^+) = \nu^+(A).$$

Since the other direction follows from definition, we have that $\nu_*^+(A \setminus 0_T) = \nu^+(A)$. Applying the same argument to ν^- we conclude that $|\nu(A)| = \nu_*^+(A \setminus 0_T) + \nu_*^-(A \setminus 0_T)$. Finally, observe that $\forall t \in T$ we have that $\int_{\mathbb{R}^T} |x(t)|^2 \wedge 1 |\nu|(dx) \leq \int_{\mathbb{R}^T} |x(t)|^2 \wedge 1 \nu^{(1)}(dx) + \int_{\mathbb{R}^T} |x(t)|^2 \wedge 1 \nu^{(2)}(dx) < \infty$. □

We can present now the Lévy-Khintchine formulation of generated QID processes with quasi-Lévy measure.

Theorem 2.6.15. *Let T be an arbitrary index set and let $X = (X_t)_{t \in T}$ be a QID process s.t. there exist two of generating ID process with one having finite Lévy measure. Then there exists a unique triplet (b, Σ, ν) where Σ is a non-negative definite function on $T \times T$, ν is a quasi-Lévy type measure on $(\mathbb{R}^T, \mathcal{B}^T)$ and a function $b \in \mathbb{R}^T$ s.t. for every $I \in \hat{T}$ and $\theta \in \mathbb{R}^I$*

$$\hat{\mathcal{L}}(X_I)(\theta) = \exp\left(i\langle \theta, b_I \rangle - \frac{1}{2}\langle \theta, \Sigma_I \theta \rangle + \int_{\mathbb{R}^T} (e^{i\langle \theta, x_I \rangle} - 1 - i\langle \theta, [[x_I]] \rangle) \nu(dx) \right), \tag{2.24}$$

where Σ_I is the restriction of Σ to $I \times I$. The triplet (b, Σ, ν) is called the generating triplet of X and ν the quasi-Lévy measure of X.

Theorem 2.6.16. *Consider any (b, Σ, ν) as in Theorem 2.6.15 such that, for every $I \in \hat{T}$, the map*

$$\mathbb{R}^I \ni \theta \mapsto \exp\left(i\langle \theta, b_I \rangle - \frac{1}{2}\langle \theta, \Sigma_I \theta \rangle + \int_{\mathbb{R}^T} (e^{i\langle \theta, x_I \rangle} - 1 - i\langle \theta, [[x_I]] \rangle) \nu(dx) \right) \tag{2.25}$$

is the c.f. of a probability distribution. Then, there exists a QID process $X = (X_t)_{t \in T}$ which is unique in distribution and such that (2.25) is the c.f. of X_I.

Proof. It is enough to see that the positive and negative part of ν, namely ν^+ and ν^-, are Lévy measures on $(\mathbb{R}^T, \mathcal{B}^T)$. Hence, we can extract two ID processes which in turn generate a QID process which is unique in distribution. □

From the arguments used in the previous result we have the following Lévy-Khintchine representation of general QID processes, namely the QID equivalent of Theorem 2.8 in [Ro18].

Theorem 2.6.17. *Let T be an arbitrary index set and let $X = (X_t)_{t \in T}$ be a generated QID process. Then there exists a triplet $(b, \Sigma, (\nu^+, \nu^-))$ where Σ and b are as in the Theorem 2.6.15*

and (ν^+, ν^-) *is a couple of Lévy measure on* $(\mathbb{R}^T, \mathcal{B}^T)$ *s.t. for every* $I \in \hat{T}$ *and* $\theta \in \mathbb{R}^I$

$$\hat{\mathcal{L}}(X_I) = \exp\Bigg(i\langle\theta, b_I\rangle - \frac{1}{2}\langle\theta, \Sigma_I\theta\rangle$$

$$+ \int_{\mathbb{R}^T} (e^{i\langle\theta, x_I\rangle} - 1 - i\langle\theta, [[x_I]]\rangle)\nu^+(dx) - \int_{\mathbb{R}^T} (e^{i\langle\theta, x_I\rangle} - 1 - i\langle\theta, [[x_I]]\rangle)\nu^-(dx)\Bigg).$$

$(b, \Sigma, (\nu^+, \nu^-))$ *is called the generating triplet of* X*, and it is unique up to other couples of Lévy measures* (ν^Y, ν^Z) *of ID processes that generate* X*.*

Conversely, consider any $(b, \Sigma, (\nu^+, \nu^-))$ *as above s.t., for every* $I \in \hat{T}$*, the map*

$$\mathbb{R}^I \ni \theta \mapsto \exp\Bigg(i\langle\theta, b_I\rangle - \frac{1}{2}\langle\theta, \Sigma_I\theta\rangle$$

$$+ \int_{\mathbb{R}^T} (e^{i\langle\theta, x_I\rangle} - 1 - i\langle\theta, [[x_I]]\rangle)\nu^+(dx) - \int_{\mathbb{R}^T} (e^{i\langle\theta, x_I\rangle} - 1 - i\langle\theta, [[x_I]]\rangle)\nu^-(dx)\Bigg) \tag{2.26}$$

is the c.f. of a probability distribution. Then, there exists a QID process $X = (X_t)_{t\in T}$ *which is unique in distribution and such that (2.26) is the c.f. of* X_I*.*

Proof. It follows from the same arguments used in the proof of Theorem 2.6.15. □

The above results have two important implications. First, if there are QID processes which are not generated QID processes then they have no c.t. associated to them. Second, generated QID processes can be defined as the class of processes for which it is possible to associate a c.t..

Remark 2.6.18. The reason why we have the uniqueness "up to other couples of Lévy measures (ν^Y, ν^Z) of ID processes that generate X" is because given two ID processes, say Y and Z, that generate X we cannot exclude that there exist two other ID processes, say $\bar{Y}$ and $\bar{Z}$ that generate. In other words, there might exist four (or more) Lévy measures call them $\nu^Y, \nu^Z, \nu^{\bar{Y}}$ and $\nu^{\bar{Z}}$ corresponding to $Y, Z, \bar{Y}$ and $\bar{Z}$, respectively, s.t. for every $I \in \hat{T}$ and $\theta \in \mathbb{R}^I$

$$\int_{\mathbb{R}^T} (e^{i\langle\theta, x_I\rangle} - 1 - i\langle\theta, [[x_I]]\rangle)\nu^Z(dx) - \int_{\mathbb{R}^T} (e^{i\langle\theta, x_I\rangle} - 1 - i\langle\theta, [[x_I]]\rangle)\nu^Y(dx)$$

$$= \int_{\mathbb{R}^T} (e^{i\langle\theta, x_I\rangle} - 1 - i\langle\theta, [[x_I]]\rangle)\bar{\nu}^Z(dx) - \int_{\mathbb{R}^T} (e^{i\langle\theta, x_I\rangle} - 1 - i\langle\theta, [[x_I]]\rangle)\bar{\nu}^Y(dx).$$

Notice that $\nu := \nu^+ - \nu^-$ might not be defined for two reasons. First, because we might have $\infty - \infty$. Second because we cannot apply the extension arguments used for the definitions of the quasi-Lévy measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ or on $(l_2, \mathcal{B}^{l_2})$ (see Definitions 2.1.5 and 2.6.6), because the restrictions do not form σ-algebras and $0_T \notin \mathcal{B}_T$.

Open Question 2.6.19. It is not possible to extend the arguments applied to the definition of quasi-Lévy type measure in $\mathbb{R}^d$ or in l_2. However, is it still possible to obtain a sensible definition of quasi-Lévy type measure on $(\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T))$ in the general case (namely without assuming it to be a signed measure)?

2.6.4 Spectral representation of generated QID processes: the discrete case

In this subsection we investigate the spectral representation of discrete generated QID processes, thus not necessarily l_2-valued. However, we use the results obtained for l_2-valued generated QID processes. In particular, we multiply our discrete generated QID processes by a sequence of constants in order to satisfy the conditions of Theorem 2.6.9 on l_2-valued processes. Then, thanks to Proposition 2.6.20 (see below) we uniquely link the c.t. of the l_2-valued generated QID processes to the c.t. of a QID random measure on ∂U, which is the boundary of the unit ball in l_2. Finally, in the main theorem of this subsection we give a representation of the original discrete generated QID processes in terms of an integral on ∂U whose measure the obtained QID random measure. Further, the integrand is just the projection from ∂U to l_2 times the inverse of the above mentioned constants.

Informally, we uniquely connect the general discrete framework to the l_2 one and subsequently to the ∂U one, so that finally we have a connection between the general discrete framework and the ∂U one. In this subsection we heavily rely on the notations and arguments of Section VI in [RR89].

Before presenting the result we need some preliminaries. Let $X = \{X_n;\, n = 1, 2, ...\}$ be a discrete parameters QID process with generating ID processes Y and Z; let $b_n > 0$ be s.t. $\{b_n X_n\}, \{b_n Y_n\}, \{b_n Z_n\} \in l_2$ almost surely. Let $\mathcal{L}(\{b_n X_n\}) \sim (z_0, \mathcal{K}, M)$, where $z_0 \in l_2$, $\mathcal{K}$ is the covariance operator and M is the quasi Lévy measure of μ. Notice that since $X + Y \stackrel{d}{=} Z$ then $\{b_n X_n\} + \{b_n Y_n\} \stackrel{d}{=} \{b_n Z_n\}$, which implies that $\{b_n Y_n\}$ with c.t. $(z_0^{(Y)}, \mathcal{K}^{(Y)}, M^{(Y)})$ and $\{b_n Z_n\}$ with c.t. $(z_0^{(Z)}, \mathcal{K}^{(Z)}, M^{(Z)})$ are such that $z_0 = z_0^{(Z)} - z_0^{(Y)}$, $\mathcal{K} = \mathcal{K}^{(Z)} - \mathcal{K}^{(Y)}$ and M is uniquely generated by $M^{(Z)}$ and $M^{(Y)}$.

However, notice that we can consider w.l.o.g. that $\{b_n Z_n\} \sim (0, \mathcal{K}, M^{(Z)})$ and $\{b_n Y_n\} \sim (-z_0, 0, M^{(Y)})$. This is because of the following argument, which we adopted multiple times in this chapter. Let Q and R be two l_2 ID process with c.t. $(0, \mathcal{K}, M^{(Z)})$ and $(-z_0, 0, M^{(Y)})$, thus, $\{b_n X_n\} - R \stackrel{d}{=} Q$. Then, we have that $X - \{b_n^{-1} R_n\} \stackrel{d}{=} \{b_n^{-1} Q_n\}$, and that $\{b_n^{-1} R_n\}$ and $\{b_n^{-1} Q_n\}$ are ID (where the latter can be seen from their finite dimensional distributions). Therefore, throughout this section for any discrete parameters QID process we consider such generating ID processes.

Let us continue with the preliminaries. For every $y \in l_2$, we have that

$$\mathcal{K}(y) = \sum_{j=1}^{\infty} \beta_j \langle e_j, y \rangle e_j,$$

where $\beta_j \geq 0$, $\sum_{j=1}^{\infty} \beta_j < \infty$ and $\{e_j\}$ is an orthonormal set in l_2. Moreover, let

$$\nu_0 = \begin{cases} \|z_0\| \delta_{(z_0/\|z_0\|)} & \text{if } z_0 \neq 0, \\ 0 & \text{if } z_0 = 0, \end{cases} \qquad \text{and} \qquad \nu_1 = \sum_{j=1}^{\infty} \beta_j \delta_{(e_j)}$$

be two finite measures on $\mathcal{B}(\partial U)$. Let us recall and reformulate Lemma 4.1 and Proposition 4.2 in [RR89], which concerns a "polar factorisation" of a Lévy measure on l_2.

Proposition 2.6.20 (Part of Lemma 4.1 and of Proposition 4.2 in [RR89]). *Let M be a Lévy*

measure on l_2, let ∂U be the boundary of the unit ball in l_2 and let Ψ be the map $\Psi : \partial U \times \mathbb{R}^+ \to l_2 \setminus \{0\}$ s.t. $\Psi(u,x) = xu$. Then F is a unique measure on $\mathcal{B}(\partial U \times \mathbb{R}^+)$ satisfying

$$M = F \circ \psi^{-1}.$$

Moreover, let $F_A(\cdot) = F(A \times \cdot)$. Then $F_A(\cdot)$ can be naturally extended to $\mathbb{R}$ and is a Lévy measure on $\mathbb{R}$.

Proof. See Lemma 4.1, Proposition 4.2 and the preliminaries of Section VI in [RR89]. □

From the above we let $F_A^{(Z)}$ and $F_A^{(Y)}$ be the unique Lévy measures associated with $M^{(Z)}$ and $M^{(Y)}$, for every $A \in \mathcal{B}(\partial U)$. Let F_A be the unique quasi-Lévy type measure generated by $F_A^{(Z)}$ and $F_A^{(Y)}$. Let Λ_Z and Λ_Y be the associate ID random measure of Z and Y, namely $\Lambda_Z \sim (0, \nu_1, F_\cdot^{(Z)})$ and $\Lambda_Y \sim (-\nu_0, 0, F_\cdot^{(Y)})$. Similarly we have the following definition.

Definition 2.6.21. *Let X, ν_0, ν_1 and $F_\cdot$ be as above, then the QID random measure on $\mathcal{B}(\partial U)$ with parameters $(\nu_0, \nu_1, F_\cdot)$ is called* the associated QID random measure of X, *if it exists. Let us denote it by Λ.*

Observe that the control measure λ of Λ is given by $\lambda(A) = \nu_0(A) + \nu_1(A) + \int_{\mathbb{R}} (1 \wedge x^2) F_A^{(Z)} + \int_{\mathbb{R}} (1 \wedge x^2) F_A^{(Y)}$ for every $A \in \mathcal{B}(\partial U)$. Moreover, denote by π_j the j-th coordinate projection in l_2. We can now finally present the main result of this section.

Theorem 2.6.22. *Let $X = \{X_n\}$ be a discrete parameters QID process and consider its two generating discrete parameters ID processes Y and Z as above. Let $f_n = b_n^{-1} \pi_n$. Then, f_n is integrable w.r.t. Λ_Y and Λ_Z, for every $n \in \mathbb{N}$. Further, assuming that $\left\{\int_{\partial U} f_n d\Lambda_Y\right\}_{n\in\mathbb{N}}$ is independent of X, we have*

$$\{X_n\}_{n\in\mathbb{N}} - \left\{\int_{\partial U} f_n d\Lambda_Y\right\}_{n\in\mathbb{N}} \stackrel{d}{=} \left\{\int_{\partial U} f_n d\Lambda_Z\right\}_{n\in\mathbb{N}}.$$

Finally, if the associated QID random measure of X exists then f_n's are Λ-integrable and

$$\{X_n\}_{n\in\mathbb{N}} \stackrel{d}{=} \left\{\int_{\partial U} f_n d\Lambda\right\}_{n\in\mathbb{N}}. \tag{2.27}$$

Proof. The first statement is proved in Theorem 4.9 in [RR89]. Now, observe that thanks to Theorem 4.9 in [RR89] for any fixed $k \in \mathbb{N}$ and $a_1, ..., a_k \in \mathbb{R}$ we have that

$$\hat{\mathcal{L}}\left(\sum_{j=1}^k a_j b_n X_n\right) = \frac{\hat{\mathcal{L}}\left(\sum_{j=1}^k a_j b_n Z_n\right)}{\hat{\mathcal{L}}\left(\sum_{j=1}^k a_j b_n Y_n\right)} = \frac{\hat{\mathcal{L}}\left(\sum_{j=1}^k a_j \int_{\partial U} f_j d\Lambda_Z\right)}{\hat{\mathcal{L}}\left(\sum_{j=1}^k a_j \int_{\partial U} f_j d\Lambda_Y\right)}.$$

Thus, we obtain second statement. For the last statement, we have that f_n's are integrable thanks to 2.4.12, where the ID r.m. Λ_G and Λ_M there are the ID r.m. Λ_Z and Λ_Y here. Moreover, thanks to Proposition 2.4.13 we have that for any fixed $k \in \mathbb{N}$ and $a_1, ..., a_k \in \mathbb{R}$ we have that

$$\frac{\hat{\mathcal{L}}\left(\sum_{j=1}^k a_j \int_{\partial U} f_j d\Lambda_Z\right)}{\hat{\mathcal{L}}\left(\sum_{j=1}^k a_j \int_{\partial U} f_j d\Lambda_Y\right)} = \hat{\mathcal{L}}\left(\sum_{j=1}^k a_j \int_{\partial U} f_j d\Lambda\right),$$

hence, obtaining the result. □

Open Question 2.6.23. Is it possible to show that for every discrete parameters QID process there exists a QID r.m. s.t. we have the representation (2.27)? In other words, is it true that for every X there exists a Λ s.t. (2.27) holds?

2.6.5 Further results and examples on QID processes and the atomless condition

In this subsection, we start by presenting additional results and examples on QID processes, and conclude with a discussion of the atomless condition of QID random measures. Recall that a process X is QID if $X+X^{(2)} \stackrel{d}{=} X^{(1)}$ where $X^{(2)}$ and $X^{(1)}$ are ID processes with $X^{(2)}$ independent of X. One of the first questions that might come to mind is: if $X^{(2)}$ and $X^{(1)}$ are Lévy processes, would X be a QID process without being an ID process? The answer is negative, as shown in the following result.

Proposition 2.6.24. *Let $X^{(1)}$ and $X^{(2)}$ be two Lévy processes with triplets $(a^{(1)}, \sigma^{(1)}, \nu^{(1)})$ and $(a^{(2)}, \sigma^{(2)}, \nu^{(2)})$. Let X be a stochastic process independent of $X^{(2)}$. The equality $X_t + X_t^{(2)} \stackrel{d}{=} X_t^{(1)}$ for every $t \geq 0$ holds only if $\sigma^{(1)} \geq \sigma^{(2)}$ and $\nu^{(1)} \geq \nu^{(2)}$. In this case X_t is an ID r.v. for every $t \geq 0$.*

Proof. If $\sigma^{(1)} < \sigma^{(2)}$ then X_1 will not be a random variable. If $\nu^{(1)} < \nu^{(2)}$ then X_t is a QID r.v. for every $t \geq 0$ without being ID. But then, for every n we would have that $X_{\frac{1}{n}}$ is a QID r.v. hence for every $n \in \mathbb{N}$ we would have that $\hat{\mathcal{L}}(X_1) = (\hat{\mathcal{L}}(X_{\frac{1}{n}}))^n$, which implies that X_1 is ID. Thus, we have a contradiction. However, if $\sigma^{(1)} \geq \sigma^{(2)}$ and $\nu^{(1)} \geq \nu^{(2)}$ then X_t is a ID r.v. for every $t \geq 0$, hence, X might potentially be a Lévy process. □

However, we have the following positive general result, which provides a general toolbox for the construction of QID processes. We use the representation function $c(\cdot)$ (see [LPS18]) which is a function $c : \mathbb{R} \to \mathbb{R}$ which is bounded, Borel measurable and satisfies $\lim_{x \to 0}(c(x) - x)/x^2 = 0$.

Proposition 2.6.25. *Let X_0 be any QID random variable and let (a, σ, ν) be its characteristic triplet. Then*

$$\hat{\mathcal{L}}(X_t) := \exp\left(i\theta h(t) + i\theta a_t f(t) - \frac{1}{2}\sigma_t f(t)^2\theta^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta c(x)\nu_t(f(t)^{-1}dx) \right)$$

is the characteristic function of a QID random variable for every $t \geq 0$, where $h(\cdot)$, $a.$, $f(\cdot)$ and $\sigma.$ are real valued functions with $f(t) \neq 0$ and $\sigma_t \geq \sigma \geq 0$ for all $t \geq 0$, $\nu_t(\cdot)$ is a set function such that $\nu_t(A) \geq \nu(A)$ for every $t \geq 0$ and $A \in \mathcal{B}_0(\mathbb{R})$, and $c(\cdot)$ is any representation function. Moreover, let $k \in \mathbb{N}$ and let $X_0^{(1)}, ..., X_0^{(k)}$ be QID random variables. Let $h^{(j)}(\cdot)$, $a.^{(j)}$, $f^{(j)}(\cdot)$, $\sigma.^{(j)}$ and $\nu_t^{(j)}(\cdot)$ satisfy the above respective properties and similarly define $\hat{\mathcal{L}}(X_t^{(j)})$. Then

$$\hat{\mathcal{L}}(Y) := \hat{\mathcal{L}}(X_{t_1}^{(1)}) \cdots \hat{\mathcal{L}}(X_{t_k}^{(k)})$$

is the characteristic function of a QID random variable for every $t_1, ..., t_k \geq 0$.

Proof. Let for the moment $f(t) = 1$ and $h(t) = 0$ for every $t \geq 0$. Since for any QID distribution μ with c.t. (b, γ, η) if there exists $\gamma' \geq \gamma$ and $\eta'(A) \geq \eta(A)$ for every $A \in \mathcal{B}_0(B)$ then the

c.t. $(b'-b, \gamma'-\gamma, \eta'-\eta)$, where $b' \in \mathbb{R}$, is the c.t. of a ID distribution, call it $\tilde{\mu}$, (see Remark 2.6 point (a) in [LPS18]) hence since the class of QID distributions is closed under convolution then we have that $\mu' = \mu * \tilde{\mu}$ is QID with c.t. (b', γ', η') (see Remark 2.6 point (c) in [LPS18]). Thus, we have that

$$\exp\left(a_t\theta - \frac{1}{2}\sigma_t\theta^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta c(x)\nu_t(dx)\right)$$

is the c.f. of a QID distribution. Now, since the class of QID distributions is closed under shifts and dilation (see Remark 2.6 point (b) in [LPS18]) then

$$\exp\left(i\theta\left(\tilde{h}(t) + \int_{\mathbb{R}} c(mx) - mc(x)\nu_t(dx) + a_t f(t)\right)\right.$$

$$\left.-\frac{1}{2}\sigma_t f(t)^2\theta^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta c(x)\nu_t(f(t)^{-1}dx)\right)$$

is the c.f. of QID distribution, where $\tilde{h}$ is any real valued function. Hence, by setting $\tilde{h}$ such that $h(t) = \tilde{h}(t) + \int_{\mathbb{R}} c(mx) - mc(x)\nu_t(dx)$ we have the first statement.

The second statement follows by the fact that the class of QID distributions is closed under convolution. □

Corollary 2.6.26. *Under the same conditions of Proposition 2.6.25 we have that for every $t \geq 0$ there exist two QID random variables Y_t and Z_t such that*

$$\hat{\mathcal{L}}(Y_t) = \exp\left(i\theta a_t - \frac{1}{2}\sigma_t\theta^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta c(x)\nu_t(dx)\right)$$

and

$$\hat{\mathcal{L}}(Z_t) = \exp\left(i\theta h(t) + i\theta a f(t) - \frac{1}{2}\sigma f(t)^2\theta^2 + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta c(x)\nu(f(t)^{-1}dx)\right).$$

Proof. It follows directly from Proposition 2.6.25. □

From the formulation $X + X^{(2)} \stackrel{d}{=} X^{(1)}$ and from the previous results many examples can be constructed. We do not deal with the existence of particular QID stochastic processes since it is a major topic and we leave it for further research. We only point out that most of the potential results seem to rely on the multidimensional extension of the results presented in [LPS18]. Hence, the following question.

Open Question 2.6.27. Is it possible to extend the results in [LPS18] and [Ber19] to the multidimensional case, namely to distributions on $\mathbb{R}^d$?

Further, we remark that a first example of QID process has been investigated in [ZLK17], although they mostly focus on the level of distributions and in particular on a subclass of QID distributions call discrete pseudo-compound Poisson distributions, which have applications in insurance mathematics.

We conclude this subsection with some remarks on the atomless condition for random measures which is in few occasions mentioned in the references used in this chapter and that characterise certain ID random measures.

One of the main properties that independently scattered random measure might satisfy is the atomless condition. Indeed, the work of Prékopa [Pr57] is centred on this condition and in Theorem 2.2 of [Pr57] he proves that if an independently scattered random measure satisfies the atomless condition then it is an ID random measure.

Definition 2.6.28 (atom, atomless)**.** *Let Λ be a completely additive set function defined on a δ-ring $\mathcal{S}$. A set $A \in \mathcal{S}$ is called an* atom *relative to the set function Λ if for every $C \subseteq A$ with $C \in \mathcal{S}$ we have either $\Lambda(C) \overset{a.s.}{=} 0$ or $\Lambda(C) \overset{a.s.}{=} \Lambda(A)$. Moreover, the completely additive set function Λ is called* atomless *if for every atom A we have $\Lambda(A) \overset{a.s.}{=} 0$.*

The set function Λ is atomless if and only if for every $A \in \mathcal{S}$ satisfying $\mathbb{P}(|\Lambda(A)| > 0) > 0$ there exist sets $A_1, A_2 \in \mathcal{S}$ with $A_1, A_2 \subset A$ with $A_1 \cap A_2 = \emptyset$ such that $\mathbb{P}(|\Lambda(A_1)| > 0) > 0$ and $\mathbb{P}(|\Lambda(A_2)| > 0) > 0$. This implies that, for a random measure ξ on a Polish space X, $x \in X$ is an atom of ξ if and only if $\mathbb{P}(|\xi(\{x\})| > 0) > 0$.

The atomless condition is for random measures what the continuity in probability is for continuous time stochastic processes. Notice that we have mentioned explicitly continuous time processes because for discrete time ones the condition is meaningless. Recall that every σ-ring is a δ-ring, but not every δ-ring is a σ-ring, and that every σ-algebra is a σ-ring, but not every σ-ring is a σ-algebra. Then, we have the following result.

Corollary 2.6.29. *It does not exist a QID r.m. on a σ-ring which is atomless and which is not ID.*

Proof. The result is straightforward since any atomless independently scattered r.m. is ID, see Theorem 2.2 in [Pr57].

□

In Section 2.2, we have shown the connections between QID and ID random measure. From this discussion it appears clear that, in case the ID random measures considered are atomless, the generated QID r.m. is indeed an ID r.m.. This leads us to the following question: is it true that ID r.m. are always atomless? The answer is negative. This is because if either the drift ν_0 or the Gaussian part ν_1 or the measure $F_\cdot(B)$ for some $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$ have at least one atom then the corresponding ID measure is not atomless. Therefore, the results shown in Section 2.2 applies to any possible ID and QID r.m.. We refer to Chapter 3.3 in [Ka17] for a lucid exposition of ID random measures.

Furthermore, we point out that the atomless condition is also not relevant for the spectral representation of the discrete parameter QID process presented in Section 2.6. The associated r.m. might potentially have atoms and this is also true in the ID framework.

2.7 Conclusion

In this chapter, we have investigated QID distributions, r.m. and processes, and explored their connections with the ID framework. First, we have shown the connections between QID and ID r.m. and proved an almost "if and only if" correspondence. In particular, we have shown

that given any two ID r.m. we can always find a QID r.m. that is generated by them, and that given a QID r.m. (and assuming a certain condition) we can always find two ID r.m.. Then, after presenting some examples of QID r.m. which are not ID, we developed the integration theory with respect to QID r.m. under different settings. We have defined QID integrals, proved spectral representations, and shown necessary and sufficient conditions for the integrability of deterministic measurable functions w.r.t. a QID r.m.. Further, we have shown that the integral mapping is continuous. In the last part of the chapter, we have investigated the results on QID processes. The main results concern Lévy-Khintchine representations, which are demonstrated for QID processes with both countable and uncountable index set. We concluded with some results on the construction of QID processes.

One of the main underlying issues of the present chapter has been the following. Given any two ID distributions it is always possible find a QID distribution that is generated by them. We have shown that this is also true for QID r.m. and processes. However, while for QID distributions the opposite is also true, namely that for any QID distribution there exist two ID measures, this is still an open question for QID r.m. and processes. In case this holds true also for QID r.m. and processes, our work provides several complete results in a very general setting for both QID r.m. and QID processes. In case this is false, which means that there exist some QID r.m. or processes which are not generated QID r.m. and processes, our work does not lose generality because our results hold in various general settings and not only for generated QID r.m. and processes.

2.8 Appendices

Appendix A: Proofs of Section 2.2

Proof of Lemma 2.2.1

Since G_A and M_A are Lévy measures and since $F_A(B) = G_A(B) - M_A(B)$ for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$ and $A \in \mathcal{S}$, then there is a unique quasi-Lévy type measure F_A for every $A \in \mathcal{S}$. Moreover, since $F_A(B) = G_A(B) - M_A(B)$, it follows immediately that $A \mapsto F_A(B)$ is a signed measure for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$.
If $(\nu_0(A), \nu_1(A), F_A)$ is the c.t. of a QID random variable then $B \mapsto F_A(B)$ is by definition a quasi-Lévy measure. Moreover, the existence of a finitely additive independently scattered random measure $\Lambda = \{\lambda(A) : A \in \mathcal{S}\}$ follows by a standard application of Kolmogorov extension theorem.

To prove that Λ is countable additive let $A_n \searrow \emptyset$ with $\{A_n\} \subset \mathcal{S}$. Then, by definition of (signed) measures, $\nu_0(A_n) \to 0$, $\nu_1(A_n) \to 0$, $G_{A_n}(B) \to 0$ and $M_{A_n}(B) \to 0$ for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$. In particular, since $A_n \searrow \emptyset$ (thus $A_1 \supset A_2 \supset ...$) then by assumption we have that $G_{A_1}(B) \geq G_{A_2}(B) \geq ...$ and $M_{A_1}(B) \geq M_{A_2}(B) \geq ...$ for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$. Notice that $G_A(B) \geq F_A^+(B)$ and $M_A(B) \geq F_A^-(B)$ for all $A \in \mathcal{S}$ and $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$. Then, we obtain that for every $n \in \mathbb{N}$

$$\int_{\mathbb{R}}(1\wedge x^2)|F_{A_n}|(dx) \leq \int_{\mathbb{R}}(1\wedge x^2)G_{A_n}(dx) + \int_{\mathbb{R}}(1\wedge x^2)M_{A_n}(dx)$$

$$\leq \int_{|x|<\epsilon}(1\wedge x^2)G_{A_1}(dx) + \int_{|x|<\epsilon}(1\wedge x^2)M_{A_1}(dx) + G_{A_n}(\{|x|\geq\epsilon\}) + M_{A_n}(\{|x|\geq\epsilon\}),$$

where $\epsilon > 0$. The last two terms go directly to zero as $n \to \infty$ for every $\epsilon > 0$ because $G_\cdot(B)$ and $M_\cdot(B)$ are measures. For the firsts two terms we obtain the convergence to zero as $n \to \infty$ by letting $\epsilon \to 0$. Hence, $\int_{\mathbb{R}}(1\wedge x^2)|F_{A_n}|(dx) \to 0$ as $n \to \infty$.

Now, by the Lévy continuity theorem we obtain that $\Lambda(A_n) \xrightarrow{d} 0$, which implies that $\Lambda(A_n) \xrightarrow{p} 0$ and, thus, by Theorem 2.1.12 the countable additivity of Λ. The uniqueness follows by the one to one relation between $(\nu_0(A), \nu_1(A), F_A)$ and $\Lambda(A)$, for every $A \in \mathcal{S}$.

Proof of Lemma 2.2.4

First, since Λ is a QID r.m. it follows that F_A is a quasi-Lévy measure on $\mathbb{R}$, for every $A \in \mathcal{S}$.

Now, let $\{A_k\}_{k=1}^n$ be pairwise disjoint sets in $\mathcal{S}$. By the uniqueness of the Lévy-Khintchine representation of a quasi-ID distribution, it follows, using $\hat{\mathcal{L}}(\Lambda(\bigcup_{k=1}^n A_k)) = \prod_{k=1}^n \hat{\mathcal{L}}(\Lambda(A_k))$, that all three set functions ν_0, ν_1 and $F_\cdot(B)$ (for every fixed $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$) are finitely additive.

Let now $\{A_n\} \subset \mathcal{S}$, $A_n \searrow \emptyset$. Then since $\Lambda(A_n) \xrightarrow{p} 0$ we have that $\nu_0(A_n) \to 0$, $\nu_1(A_n) \to 0$ and $\int_{\mathbb{R}}(1\wedge x^2)F_{A_n}(dx) \to 0$. Concerning $F_{A_n}(B)$, observe that $K_{A_1}(B) \geq K_{A_2}(B) \geq ...$ and that $K_{A_j}(B) \geq F^+_{A_j}(B)$, where $j \in \mathbb{N}$, for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$ by assumption. Then,

$$\lim_{n\to\infty}\int_{\mathbb{R}}(1\wedge x^2)F^+_{A_n}(dx) \leq \lim_{n\to\infty}\int_{\mathbb{R}}(1\wedge x^2)K_{A_n}(dx) \leq \int_{|x|<\epsilon}(1\wedge x^2)K_{A_1}(dx) + \lim_{n\to\infty}K_{A_n}(\{|x|\geq\epsilon\}).$$

The second addend goes directly to zero for every $\epsilon > 0$ because $K_\cdot(B)$ is a measure, while for the first we obtain the convergence to zero by letting $\epsilon \to 0$. Moreover, since

$$\int_{\mathbb{R}}(1\wedge x^2)F_{A_n}(dx) \to 0 \ \text{ as } n\to\infty, \text{ then } \ \int_{\mathbb{R}}(1\wedge x^2)F^-_{A_n}(dx) \to 0 \ \text{ as } n\to\infty.$$

By the Chebychev's inequality, for every $\epsilon > 0$ we have

$$F^+_{A_n}(\{|x|\geq\epsilon\}) \leq \epsilon^{-2}\int_{\mathbb{R}}(1\wedge x^2)F^+_{A_n}(dx),$$

hence, we obtain that $F^+_{A_n}(B) \to 0$ for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$; and the same holds for $F^-_{A_n}(B)$. Therefore, we have $|F_{A_n}(B)| \to 0$, and thus $F_{A_n}(B)$ is a signed measure, for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$.

Appendix B: Proofs of Section 2.4

Proof of Proposition 2.4.4

Since, for every $s \in S$, $\rho_G(s,B)$ and $\rho_M(s,B)$ are two Lévy measures, we obtain the first statement by the definition of quasi-Lévy type measure.

This implies that, for each $s \in S$, $\rho(s, B)$ has a positive and negative part defined as measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then, it is easy to obtain (i). Concerning (ii), we have the following. Let $f : \mathbb{R} \to \mathbb{R}$ be s.t. $\int_{\mathbb{R}} f(x)|\rho|(s, dx) < \infty$ for every $s \in S$ and let $f^+(x) := f(x) \vee 0$ and $f^-(x) := -f(x) \vee 0$. Observe that the following are all measures

$$B \mapsto \int_B f^+(x)\rho^+(s, dx), \quad B \mapsto \int_B f^+(x)\rho^-(s, dx),$$

$$B \mapsto \int_B f^+(x)\rho_G(s, dx), \quad \text{and} \quad B \mapsto \int_B f^+(x)\rho_M(s, dx).$$

Moreover, we have that

$$\int_B f^+(x)\rho^+(s, dx) - \int_B f^+(x)\rho^-(s, dx) = \int_B f^+(x)\rho(s, dx)$$

$$= \int_B f^+(x)\rho_G(s, dx) - \int_B f^+(x)\rho_M(s, dx).$$

Since $B \mapsto \int_B f^+(x)\rho^+(s, dx) - \int_B f^+(x)\rho^-(s, dx)$ is a finite signed measure, it has a Hahn decomposition. Denote it by E_s^+ and E_s^-. Further, since $\rho^+(s, B)$ and $\rho^-(s, B)$ are mutually singular we have that

$$\int_{E_s^+} f^+(x)\rho^+(s, dx) - \int_{E_s^+} f^+(x)\rho^-(s, dx) = \int_{E_s^+} f^+(x)\rho^+(s, dx)$$

and similarly for E_s^-. Therefore, for every $B \in \mathcal{B}(\mathbb{R})$ we deduce that

$$\int_B f^+(x)\rho^+(s, dx) = \int_{B \cap E_s^+} f^+(x)\rho^+(s, dx) = \sup_{A \subset B} \int_A f^+(x)\rho^+(s, dx)$$

$$= \sup_{A \subset B} \int_A f^+(x)\rho_G(s, dx) - \int_A f^+(x)\rho_M(s, dx).$$

Observe that the last two terms are Borel measurable functions because $\rho_G(s, B)$ and $\rho_M(s, B)$ are Borel measurable functions thanks to Corollary 2.4.3, the difference between two measurable functions is measurable, and the supremum over a countable set of a measurable function is measurable (the set is countable because $\mathcal{B}(\mathbb{R})$ is generated by a countable collection of sets, see the proof of Theorem 2.4.18). Since the difference of two measurable function is measurable we conclude that $\int_B f^+(x)\rho^+(s, dx)$ is measurable and this holds for every $B \in \mathcal{B}(\mathbb{R})$. The same holds for $\int_B f^+(x)\rho^-(s, dx)$ and for $\int_B f^-(x)\rho^+(s, dx)$ and $\int_B f^-(x)\rho^-(s, dx)$. Therefore, since

$$\int_B f(x)\rho^+(s, dx) = \int_B f^+(x)\rho^+(s, dx) - \int_B f^-(x)\rho^+(s, dx)$$

(and similarly for ρ^-) for every $B \in \mathcal{B}(\mathbb{R})$, we obtain that $\int_B f(x)\rho^+(s, dx)$ and so $f(x)\rho^+(s, dx)$ are Borel a measurable functions.

Let us now take $f(x) = 1 \wedge x^2$. It is possible to see that it is a valid choice since $\int_{\mathbb{R}}(1 \wedge x^2)|\rho|(s, dx) < \infty$. Then $s \mapsto (1 \wedge x^2)\rho^+(s, dx)$ is a measurable function and so (as done in the

proof of Lemma 2.3 in [RR89]) we have that

$$\rho^+(s,dx) = \left(1 \wedge x^2\right)^{-1} (1 \wedge x^2)\rho^+(s,dx)$$

is a Borel measurable function. The same applies to $\rho^-(s,dx)$ and, therefore, we obtain (ii).

Now, notice that the set function

$$\tilde{F}^+(C) := \int_S \int_{\mathbb{R}} \mathbf{1}_C(s,x)\rho^+(s,dx)\lambda(ds), \quad C \in \sigma(\mathcal{S}) \otimes \mathcal{B}(\mathbb{R})$$

is a well defined measure on $\sigma(\mathcal{S}) \otimes \mathcal{B}(\mathbb{R})$. Then, using standard measure theoretical arguments we get (2.8).

The last statement follows by simple computations. Indeed, let $g(x)$ be a non-negative function s.t. $\int_A \int_{\mathbb{R}} g(x)|\rho|(s,dx)\lambda(ds) < \infty$ (notice that this includes also $g(x) = \mathbf{1}_B(x)$ for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$). Then, for every $A \in \mathcal{S}$, $s \mapsto \mathbf{1}_A(s)g(x)\rho(s,dx)$ is a Borel measurable function and

$$\begin{aligned}\int_A \int_{\mathbb{R}} g(x)\rho(s,dx)\lambda(ds) &= \int_S \int_{\mathbb{R}} \mathbf{1}_A(s)g(x)\rho^+(s,dx)\lambda(ds) - \int_S \int_{\mathbb{R}} \mathbf{1}_A(s)g(x)\rho^-(s,dx)\lambda(ds) \\ &= \int_S \int_{\mathbb{R}} \mathbf{1}_A(s)g(x)\rho_G(s,dx)\lambda(ds) - \int_S \int_{\mathbb{R}} \mathbf{1}_A(s)g(x)\rho_M(s,dx)\lambda(ds) \\ &= \int_{\mathbb{R}} g(x)G_A(dx) - \int_{\mathbb{R}} g(x)M_A(dx) = \int_{\mathbb{R}} g(x)F_A(dx).\end{aligned}$$

Further, let $g(x) \geq 0$ (*e.g.* $\mathbf{1}_B(x)$) and let E_A^+ and E_A^- be the Hahn decomposition of $\mathbb{R}$ under $F_A(\cdot)$, then

$$\begin{aligned}\int_{\mathbb{R}} g(x)F_A^+(dx) &= \int_{E_A^+} g(x)F_A(dx) = \int_S \int_{E_A^+} \mathbf{1}_A(s)g(x)\rho(s,dx)\lambda(ds) \\ \leq \int_S \int_{E_A^+} \mathbf{1}_A(s)g(x)\rho^+(s,dx)\lambda(ds) &\leq \int_S \int_A g(x)\rho^+(s,dx)\lambda(ds) = \int_{\mathbb{R}} g(x)\tilde{F}^+(A,dx).\end{aligned}$$

Proof of Theorem 2.4.10

In this proof we follow some of the arguments used in the proof of Theorem 2.7 in [RR89]. Assume that (i), (ii), (iii) hold. Consider the sets $S_1, S_2, ... \in \mathcal{S}$ s.t. $\bigcup_{n\in\mathbb{N}} S_n = S$. Let $A_n = \{s : |f(s)| \leq n\} \cap S_n$. We have that $\{A_n\} \subset \mathcal{S}$ and $A_n \nearrow S$. Consider a sequence (f_n) of simple $\mathcal{S}$-measurable functions, s.t. $f_n(s) = 0$ if $s \notin A_n$, $|f_n(s) - f(s)| \leq \frac{1}{n}$ if $s \in A_n$, and $|f_n(s)| \leq |f(s)|$ for all $s \in S$. Then, $f_n \to f$ everywhere on S as $n \to \infty$. Further, notice that for every $A \in \sigma(\mathcal{S})$ and $n, m \geq 1$, $|\left(f_n(s) - f_m(s)\right)\mathbf{1}_A(s)| \leq 2|f(s)|$. Then by Lemma 2.4.11 (see the next result) we derive

$$|U\left(\left(f_n(s) - f_m(s)\right)\mathbf{1}_A(s), s\right)| \leq 2|U(f(s), s)| + 27V_0(f(s), s).$$

Thus, by the dominated convergence theorem, we obtain that for every $A \in \sigma(\mathcal{S})$,

$$\lim_{n,m\to\infty} \int_S U((f_n(s) - f_m(s)), \mathbf{1}_A(s), s)\lambda(ds) = 0, \tag{2.28}$$

$$\lim_{n,m\to\infty} \int_S (f_n(s) - f_m(s))^2 \mathbf{1}_A(s)\sigma^2(s)\lambda(ds) = 0, \tag{2.29}$$

$$\text{and} \quad \lim_{n,m\to\infty} \int_S V_0((f_n(s) - f_m(s))\mathbf{1}_A(s), s)\lambda(ds) = 0. \tag{2.30}$$

In addition, from (i), (ii), (iii) and Proposition 2.4.6 it is possible to obtain (iv) for the simple measurable functions $(f_n(s) - f_m(s))\mathbf{1}_A$. Indeed, let $g_{n,m,A}(s) := (f_n(s) - f_m(s))\mathbf{1}_A$, we have

$$\begin{aligned}
&\hat{\mathcal{L}}\left(\int_S g_{n,m,A} d\Lambda\right)(\theta) \\
&= \exp\left(i\theta a_{g_{n,m,A}} - \frac{1}{2}\theta^2\sigma^2_{g_{n,m,A}} + \int_S\int_{\mathbb{R}} \left(e^{i\theta g_{n,m,A}(s)x} - 1 - i\theta g_{n,m,A}(s)\tau(x)\right)\rho(s,dx)\lambda(ds)\right) \\
&= \exp\left(i\theta a_{g_{n,m,A}} - \frac{1}{2}\theta^2\sigma^2_{g_{n,m,A}} + \int_S\int_{\mathbb{R}} \left(e^{i\theta g_{n,m,A}(s)x} - 1 - i\theta g_{n,m,A}(s)\tau(x)\right)\rho^+(s,dx)\lambda(ds)\right. \\
&\qquad \left. - \int_S\int_{\mathbb{R}} \left(e^{i\theta g_{n,m,A}(s)x} - 1 - i\theta g_{n,m,A}(s)\tau(x)\right)\rho^-(s,dx)\lambda(ds)\right) \\
&= \exp\left(i\theta a_{g_{n,m,A}} - \frac{1}{2}\theta^2\sigma^2_{g_{n,m,A}} + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)\tilde{F}^+_{g_{n,m,A}}(dx)\right. \\
&\qquad \left. - \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)\tilde{F}^-_{g_{n,m,A}}(dx)\right) \\
&= \exp\left(i\theta a_{g_{n,m,A}} - \frac{1}{2}\theta^2\sigma^2_{g_{n,m,A}} + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta\tau(x)F_{g_{n,m,A}}(dx)\right)
\end{aligned}$$

Now, using (iv) for simple functions, and the equations (2.28), (2.29), and (2.30), we obtain that $\lim_{n,m\to\infty} \hat{\mathcal{L}}\left(\int_S (f_n(s) - f_m(s))\mathbf{1}_A d\Lambda\right)(\theta) = 1$ for every $\theta \in \mathbb{R}$ and $A \in \sigma(\mathcal{S})$. Therefore, the sequence $\{\int_A f_n d\Lambda\}_{n=1}^\infty$ converges in probability for every $A \in \sigma(\mathcal{S})$, namely f is Λ-integrable and, then, the c.f. of $\int_S f d\Lambda$ can be written as in (iv).

Proof of Theorem 2.4.18

From Theorem 2.4.17, we know that ν is a measure on $(T, \mathcal{A})$, that there exist two sub-Markovian kernels $\tilde{q}_+$ and $\tilde{q}_-$ such that

$$\tilde{Q}^+(C) = \int_T\int_X \mathbf{1}_C(x,t)\tilde{q}_+(t,dx)\nu(dt) \quad \text{and} \quad \tilde{Q}^-(C) = \int_T\int_X \mathbf{1}_C(x,t)\tilde{q}_-(t,dx)\nu(dt),$$

and that there exists a unique finite signed measure Q on the product σ-algebra $\mathcal{A}\otimes\mathcal{B}$ s.t.

$$Q(A\times B) = Q_0(A,B) = \int_A q(t,B)\nu(dt),$$

where $q(t,B) = \tilde{q}_+(t,B) - \tilde{q}_-(t,B)$. In particular, since $\tilde{q}_+$ and $\tilde{q}_-$ are two sub-Markovian kernels then q satisfies point (d) and (e) of the statement.

Hence, it remains to show the Jordan decomposition of $q(t,\cdot)$, for every $t \in T$, and $Q(\cdot)$. This simply reduces to find the Jordan decomposition of $q(t,\cdot)$, for every $t \in T$. One could think that they are given by $\tilde{q}_+(t,\cdot)$ and $\tilde{q}_-(t,\cdot)$, but we only know that they exist without knowing they are the Jordan decomposition of $q(t,\cdot)$, for every $t \in T$.

We know that $q(t,\cdot)$ is a finite (actually absolutely bounded by 1) signed measure on $\mathcal{B}$ for every $t \in T$. Hence, it has a Jordan decomposition: $q^+(t,\cdot)$ and $q^-(t,\cdot)$. So, it remains to show that $q^+(\cdot,B)$ and $q^-(\cdot,B)$ are $\mathcal{A}$-measurable functions, for every $B \in \mathcal{B}$. First, recall that any Lusin space, as any standard Borel space, is countably generated (indeed, they are both isomorphic to separable metric spaces). Now, the algebra generated by a countable set is countable, so we can assume w.l.o.g. that there is a countable algebra $\mathcal{E}$ s.t. $\sigma(\mathcal{E}) = \mathcal{B}$. Then, we have

$$q^+(t,B) = \sup_{A \in \mathcal{B}, A \subset B} q(t,A).$$

Since for every $\epsilon > 0$, $t \in T$ and $B \in \mathcal{B}$, there exists $A \in \mathcal{E}$ s.t. $q(t, A\Delta B) < \epsilon$, (where $A\Delta B := (A \setminus B) \cup (B \setminus A)$). Then, we have

$$q^+(t,B) = \sup_{A \in \mathcal{E}, A \subset B} q(t,A)$$

which is a supremum over a countable set. Further, since the supremum over a countable set of measurable functions is measurable we obtain the stated result.

Proof of Lemma 2.4.29

First, notice that $J(A,B)$ satisfies the assumptions of Theorem 2.4.18 with $(T,\mathcal{A}) = (S,\mathcal{S})$ and $(X,\mathcal{B}) = (\mathbb{R},\mathcal{B}(\mathbb{R}))$. Therefore, there exists a finite signed measure Q on the product σ-algebra $\mathcal{S} \otimes \mathcal{B}(\mathbb{R})$ such that

$$Q(A \times B) = J(A,B) = \int_A q(s,B)\xi(ds) = \int_A q^+(s,B)\xi(ds) - \int_A q^-(s,B)\xi(ds),$$

where q^+ and q^- satisfy $(d)'$ and $(e)'$, and q satisfies (d) and (e) of Theorem 2.4.18. Since $\lambda \gg \xi$, define

$$\rho^+(s,dx) := \frac{d\xi}{d\lambda}(s)(1 \wedge x^2)^{-1} q^+(s,dx), \quad \text{and} \quad \rho^-(s,dx) := \frac{d\xi}{d\lambda}(s)(1 \wedge x^2)^{-1} q^-(s,dx).$$

Notice that

$$\int_{\mathbb{R}} (1 \wedge x^2)\rho^+(s,dx) = \frac{d\xi}{d\lambda}(s) \int_{\mathbb{R}} q^+(s,dx) \leq 1,$$

where the last inequality comes from the fact that we can always assume that $\frac{d\xi}{d\lambda}(s) \leq 1$ for all $s \in S$. Hence, $\rho^+(s,\cdot)$ is a Lévy measure on $\mathbb{R}$ for all $s \in S$. The same is true for $\rho^-(s,\cdot)$. Further, let

$$\rho(s,B) := \rho^+(s,B) - \rho^-(s,B) \quad \text{for all } s \in S,\ B \in \mathcal{B}(\mathbb{R}) \text{ s.t. } 0 \notin \overline{B}.$$

Then $\rho(s,\cdot)$ is a quasi-Lévy type measure, thus obtaining (i). Using the fact that $\rho^+(s,B)$ and $\rho^-(s,B)$ are mutually singular for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $\overline{B}$, we obtain (ii). Now, let

$$\tilde{F}^+(C) = \int_S \int_{\mathbb{R}} \mathbf{1}_C(s,x)\rho^+(s,dx)\lambda(ds), \tag{2.31}$$

where $C \in \mathcal{S} \otimes \mathcal{B}(\mathbb{R})$, then $\tilde{F}^+$ is a well defined measure that satisfies, for every $A \in \mathcal{S}$ and $B \in \mathcal{B}(\mathbb{R})$,

$$\tilde{F}^+(A \times B) = \int_A \int_B \rho^+(s,dx)\lambda(ds) = \int_A \int_B (1 \wedge x^2)^{-1} q^+(s,dx)\xi(ds)$$

$$= \int_{A\times B} (1 \wedge x^2)^{-1} Q^+(ds,dx) \geq \int_B (1 \wedge x^2)^{-1} J^+(A,dx) = \int_B F_A^+(dx) = F_A^+(B),$$

where Q^+ is the positive Jordan decomposition of the measure Q on $\mathcal{S} \otimes \mathcal{B}(\mathbb{R})$. Concerning $J^+(A,dx)$, recall that the notation M^+ and M^- for a bimeasure M stands for the Jordan decomposition of $B \mapsto M(A,B)$ for fixed A. Finally, notice that for any $\mathcal{B}(\mathbb{R})$-measurable real function g s.t. $\int_A \int_{\mathbb{R}} |g(x)||\rho|(s,dx)\lambda(ds) < \infty$ we have

$$\int_A \int_{\mathbb{R}} g(x)\rho(s,dx)\lambda(ds)$$

$$= \int_A \int_{\mathbb{R}} g(x)\rho^+(s,dx)\lambda(ds) - \int_A \int_{\mathbb{R}} g(x)\rho^-(s,dx)\lambda(ds)$$

$$= \int_{A\times\mathbb{R}} g(x)(1 \wedge x^2)^{-1} Q(ds,dx) = \int_{\mathbb{R}} g(x)(1 \wedge x^2)^{-1} J(A,dx) = \int_{\mathbb{R}} g(x) F_A(dx).$$

Appendix C: Proofs of Section 2.6

Proof of Theorem 2.6.15

Let X^1 and X^2 the ID processes generating X, with one having finite Lévy measure, and denote by $\nu^{(1)}$ and $\nu^{(2)}$ the respective Lévy measures on $(\mathbb{R}^T, \mathcal{B}^T)$. Let $\nu_* := \nu_*^{(1)} - \nu_*^{(2)}$ and let $\nu(A) := \nu_*(A)$ for every $A \in \mathcal{B}^T$. Then, by Lemma 2.6.14 ν is quasi-Lévy type measure on $(\mathbb{R}^T, \mathcal{B}^T)$. Further, we have that, for every $I \in \hat{T}$ and $\theta \in \mathbb{R}^I$,

$$\hat{\mathcal{L}}(X_I)(\theta) = \frac{\hat{\mathcal{L}}(X_I^1)(\theta)}{\hat{\mathcal{L}}(X_I^2)(\theta)} = \exp\Bigg(i\langle\theta, b_I\rangle - \frac{1}{2}\langle\theta, \Sigma_I\theta\rangle$$

$$+ \int_{\mathbb{R}^T} (e^{i\langle\theta,x_I\rangle} - 1 - i\langle\theta, [[x_I]]\rangle)\nu^{(1)}(dx) - \int_{\mathbb{R}^T} (e^{i\langle\theta,x_I\rangle} - 1 - i\langle\theta, [[x_I]]\rangle)\nu^{(2)}(dx)\Bigg)$$

$$= \exp\Bigg(i\langle\theta, b_I\rangle - \frac{1}{2}\langle\theta, \Sigma_I\theta\rangle + \int_{\mathbb{R}^T} (e^{i\langle\theta,x_I\rangle} - 1 - i\langle\theta, [[x_I]]\rangle)\nu(dx)\Bigg),$$

which gives (2.24). From the above formula we also deduce that the $\nu \circ \pi_I^{-1} = \nu_I$ on $\mathcal{B}_{00}^I$, $I \in \hat{T}$, where ν_I is the quasi-Lévy measure of X_I.

Notice ν^+ and ν^- are Lévy measures on $(\mathbb{R}^T, \mathcal{B}^T)$. In particular, notice that $\nu_I^+ = \nu^+ \circ \pi_I^{-1}$

and $\nu_I^- = \nu^- \circ \pi_I^{-1}$ are Lévy measures such that $\nu_I(A) = \nu_I^+(A) - \nu_I^-(A)$ for every $A \in \mathcal{B}(\mathbb{R})$, for $I \in \hat{T}$ (although it is not necessarily true that ν_I^+ and ν_I^- are mutually singular on $(\mathbb{R}^I, \mathcal{B}(\mathbb{R}^I))$).

Let us prove uniqueness. Notice that the arguments used in the proof of Lemma 2.1.14, which were applicable for the $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and the $(l_2, \mathcal{B}^{l_2})$ cases, do not apply here due to the different topological framework.
Assume that there exists another quasi-Lévy type measure $\tilde{\nu}$ on $(\mathbb{R}^T, \mathcal{B}^T)$ (hence $\tilde{\nu}^+$ and $\tilde{\nu}^-$ are two Lévy measures on $(\mathbb{R}^T, \mathcal{B}^T)$) s.t., for every $I \in \hat{T}$ and $\theta \in \mathbb{R}^I$,

$$\hat{\mathcal{L}}(X_I)(\theta) = \exp\left(i\langle \theta, b_I \rangle - \frac{1}{2}\langle \theta, \Sigma_I \theta \rangle + \int_{\mathbb{R}^T} (e^{i\langle \theta, x_I \rangle} - 1 - i\langle \theta, [[x_I]] \rangle) \tilde{\nu}(dx) \right), \tag{2.32}$$

but with $\tilde{\nu} \neq \nu$. Notice that (2.32) together with the uniqueness of quasi-Lévy measures imply that $\nu_I = \tilde{\nu} \circ \pi_I^{-1} =: \tilde{\nu}_I$ on $\mathcal{B}_{00}^I$, $I \in \hat{T}$, and that there exists a QID process $\tilde{X}$ with c.t. $(b, \Sigma, \tilde{\nu})$ s.t. $\tilde{X} \overset{d}{=} X$.

However, observe that since $\tilde{\nu} \neq \nu$ then $\tilde{\nu}^+ - \tilde{\nu}^- \neq \nu^+ - \nu^-$. Recall that by Corollary 2.9 in [RR89] that for every c.t. (b, Σ, μ) where μ is a Lévy measures there exist a unique (in distribution) ID process. Let $b^{(1)}$ and $b^{(2)}$ two functions in $\mathbb{R}^T$ s.t. $b = b^{(1)} - b^{(2)}$ and let $(b^{(1)}, \Sigma, \nu^+)$, $(b^{(2)}, 0, \nu^-)$, $(b^{(1)}, \Sigma, \tilde{\nu}^+)$ and $(b^{(2)}, 0, \tilde{\nu}^-)$ the c.t. of the ID processes Z, Y, $\tilde{Z}$ and $\tilde{Y}$, respectively. Then observe that $X - Y \overset{d}{=} Z$ and $\tilde{X} - \tilde{Y} \overset{d}{=} \tilde{Z}$. Moreover, we have that, for every $I \in \hat{T}$ and $\theta \in \mathbb{R}^I$,

$$\frac{\hat{\mathcal{L}}(Z_I)(\theta)}{\hat{\mathcal{L}}(Y_I)(\theta)} = \frac{\hat{\mathcal{L}}(\tilde{Z}_I)(\theta)}{\hat{\mathcal{L}}(\tilde{Y}_I)(\theta)} \Rightarrow \hat{\mathcal{L}}(Z_I)(\theta)\hat{\mathcal{L}}(\tilde{Y}_I)(\theta) = \hat{\mathcal{L}}(\tilde{Z}_I)(\theta)\hat{\mathcal{L}}(Y_I)(\theta). \tag{2.33}$$

Now, let $\nu_I^{Z\tilde{Y}} := \nu_I^+ + \tilde{\nu}_I^-$ and $\nu_I^{\tilde{Z}Y} := \tilde{\nu}_I^+ + \nu_I^-$. It is possible to see that $\{\nu_I^{\tilde{Z}Y} : I \in \hat{T}\}$ and $\{\nu_I^{Z\tilde{Y}} : I \in \hat{T}\}$ are two consistent families of Lévy measures. Hence, they generate two unique Lévy measures $\nu^{\tilde{Z}Y}$ and $\nu^{Z\tilde{Y}}$ on $(\mathbb{R}^T, \mathcal{B}^T)$. Observe that $\nu^+ + \tilde{\nu}^-$ and $\tilde{\nu}^+ + \nu^-$ are two Lévy measure on $(\mathbb{R}^T, \mathcal{B}^T)$ because the sum of two Lévy measures on $(\mathbb{R}^T, \mathcal{B}^T)$ is still a Lévy measure. Since $\nu^{\tilde{Z}Y} \circ \pi_I^{-1} = \nu_I^{\tilde{Z}Y} = \tilde{\nu}_I^+ + \nu_I^- = (\nu^+ + \tilde{\nu}^-) \circ \pi_I^{-1}$ for every $I \in \hat{T}$ and by the uniqueness of Lévy measures on $(\mathbb{R}^T, \mathcal{B}^T)$, we conclude that $\nu^{\tilde{Z}Y} = \nu^+ + \tilde{\nu}^-$ and similarly that $\nu^{Z\tilde{Y}} = \tilde{\nu}^+ + \nu^-$. However, from (2.33) we have that $\nu_I^{Z\tilde{Y}} = \nu_I^{\tilde{Z}Y}$ which implies that $\nu^+ + \tilde{\nu}^- = \tilde{\nu}^+ + \nu^-$, hence a contradiction to $\tilde{\nu}^+ - \tilde{\nu}^- \neq \nu^+ - \nu^-$ and, so, to $\tilde{\nu} \neq \nu$.

3

Limit Theorems: Mixing properties of ID random fields

3.1 Preliminaries and Results on Mixing Conditions

In this section we analyse mixing conditions for stationary infinite divisible random fields. We work with the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra on the vector field $\mathbb{R}^d$. We write $\mathcal{L}(X_t)$ for the distribution, or law, of the random variable X_t. Now, let $(\theta_t)_{t\in\mathbb{R}^l}$ be a measure preserving $\mathbb{R}^l$ action on $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the random field $X_t(\omega) = X_0 \circ \theta_t(\omega)$, $t \in \mathbb{R}^l$. The random field $(X_t)_{t\in\mathbb{R}^l}$ defined in this way is stationary and, conversely, any stationary measurable random field can be expressed in this form. Further, we have, with a little bit of abuse of notation, $\theta_v(B) := \{\theta_v(\omega) \in \Omega : \omega \in B\} = \{\omega' \in \Omega : X_0(\omega') = X_v(\omega) \text{ for } \omega \in B\}$. We denote by $\|\cdot\|_\infty$ the supremum norm.
Then $(X_t)_{t\in\mathbb{R}^l}$ is mixing if and only if (see Wang, Roy and Stoev [WRS13] equation (4.4)):

$$\lim_{n\to\infty} \mathbb{P}(A \cap \theta_{t_n}(B)) = \mathbb{P}(A)\mathbb{P}(B), \tag{3.1}$$

for all $A, B \in \sigma_X$ and all $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$, where $\sigma_X := \sigma(\{X_t : t \in \mathbb{R}^l\})$ is the σ-algebra generated by $(X_t)_{t\in\mathbb{R}^l}$ and $\mathcal{T} := \left\{(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^l : \lim_{n\to\infty} \|t_n\|_\infty = \infty\right\}$.
The following definition is based on the characteristic function of $(X_t)_{t\in\mathbb{R}^l}$ (see [WRS13] equation (A.6)):

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\exp\left(i\sum_{j=1}^{r} \beta_j X_{s_j}\right) \exp\left(i\sum_{k=1}^{q} \gamma_k X_{p_k+t_n}\right)\right] \\
&= \mathbb{E}\left[\exp\left(i\sum_{j=1}^{r} \beta_j X_{s_j}\right)\right] \mathbb{E}\left[\exp\left(i\sum_{k=1}^{q} \gamma_k X_{p_k}\right)\right],
\end{aligned} \tag{3.2}$$

for all $r, q \in \mathbb{N}, \beta_j, \gamma_k \in \mathbb{R}, p_j, s_k \in \mathbb{R}^l$ and $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$. Further, for the multivariate (or $\mathbb{R}^d$-valued) random field we have the following definition based on the characteristic function of $(X_t)_{t\in\mathbb{R}^l}$.

Definition 3.1.1. *Let $(X_t)_{t\in\mathbb{R}^l}$ be an $\mathbb{R}^d$-valued stationary random field. Then $(X_t)_{t\in\mathbb{R}^l}$ is said to be mixing if for all $\lambda = (s_1, ..., s_m)', \mu = (p_1, ..., p_m)' \in \mathbb{R}^{ml}$ and $\theta_1, \theta_2 \in \mathbb{R}^{md}$*

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(i\langle\theta_1, X_\lambda\rangle + i\langle\theta_2, X_{\tilde{\mu}_n}\rangle\right)\right] = \mathbb{E}\left[\exp\left(i\langle\theta_1, X_\lambda\rangle\right)\right]\mathbb{E}\left[\exp\left(i\langle\theta_2, X_\mu\rangle\right)\right], \tag{3.3}$$

where $X_\lambda := (X'_{s_1}, ..., X'_{s_m})' \in \mathbb{R}^{md}$ and $\tilde{\mu}_n = (p_1 + t_n, ..., p_m + t_n)'$, where $(t_n)_{n\in\mathbb{N}}$ is any sequence in $\mathcal{T}$.

Further, we recall the definition of an infinitely divisible random field.

Definition 3.1.2. *An $\mathbb{R}^d$ valued random field $(X_t)_{t\in\mathbb{R}^l}$ (or its distribution) is said to be infinitely divisible if for every $(X_{t_1}, ..., X_{t_k})$, where $k \in \mathbb{N}$, and for every $n \in \mathbb{N}$ there exist i.i.d random vectors $Y_i^{(n,k)}, i = 1, ..., n$, in $\mathbb{R}^{d\times k}$ (possibly on a different probability space) such that $(X_{t_1}, ..., X_{t_k}) \stackrel{d}{=} Y_1^{(n,k)} + \cdots + Y_n^{(n,k)}$.*

It is straightforward to see that the above definition is equivalent to the following definition. An $\mathbb{R}^d$ valued random field $(X_t)_{t\in\mathbb{R}^l}$ (or its distribution) is said to be infinitely divisible if for every finite dimensional distribution of $(X_t)_{t\in\mathbb{R}^l}$, namely $F_{t_1,...,t_k}(x_1, ..., x_k) := \mathbb{P}(X_{t_1} < x_1, ..., X_{t_k} < x_k)$ where $k \in \mathbb{N}$, and for every $n \in \mathbb{N}$ there exists a probability measure $\mu_{n,k}$ on $\mathbb{R}^{d\times k}$ such that $F_{t_1,...,t_k}(x_1, ..., x_k) = \underbrace{\mu_{n,k} * \cdots * \mu_{n,k}}_{\text{(n times)}} = \mu_{n,k}^{*n}$.
We are now ready to state our first result.

Theorem 3.1.3. *Let $(X_t)_{t\in\mathbb{R}^l}$, with $l \in \mathbb{N}$, be an $\mathbb{R}^d$-valued strictly stationary infinite divisible random field such that Q_0, the Lévy measure of $\mathcal{L}(X_0)$, satisfies $Q_0(\{x = (x_1, ..., x_d)' \in \mathbb{R}^d : \exists j \in \{1, ..., d\}, x_j \in 2\pi\mathbb{Z}\}) = 0$. Then $(X_t)_{t\in\mathbb{R}^l}$ is mixing if and only if*

$$\lim_{n\to\infty} \mathbb{E}\left[e^{i(X_{t_n}^{(j)} - X_0^{(k)})}\right] = \mathbb{E}\left[e^{iX_0^{(j)}}\right] \cdot \mathbb{E}\left[e^{-iX_0^{(k)}}\right], \tag{3.4}$$

for any $j, k = 1, .., d$ and for any sequence $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$.

The above theorem relies on the following result, which is the multivariate random field extension of the Maruyama conditions (see Theorem 6 of [Mar70]).

Theorem 3.1.4. *Let $(X_t)_{t\in\mathbb{R}^l}$ be an $\mathbb{R}^d$-valued strictly stationary infinite divisible random field. Then $(X_t)_{t\in\mathbb{R}^l}$ is mixing if and only if*

$(MM1)$ the covariance matrix function $\Sigma(t_n)$ of the Gaussian part of $(X_{t_n})_{t_n\in\mathbb{R}^l}$ tends to 0, as $n \to \infty$,

$(MM2')$ $\lim_{n\to\infty} Q_{0t_n}(\|x\| \cdot \|y\| > \delta) = 0$ for any $\delta > 0$, where Q_{0t_n} is the Lévy measure of $\mathcal{L}(X_0, X_{t_n})$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$,

where $(t_n)_{n\in\mathbb{N}}$ is any sequence in $\mathcal{T}$.

Notice that the above conditions are fewer than the Maruyama conditions. This is because we used the following lemma, which is a multivariate random field extension of Lemma 1 of [Mag10] and Lemma 2.2 of [FS13].

Lemma 3.1.5. *Assume that* $\lim_{n\to\infty} Q_{0t_n}(\|x\|\cdot\|y\|>\delta)=0$ *holds for any* $\delta>0$*, where* Q_{0t_n} *is the Lévy measure of* $\mathcal{L}(X_0,X_{t_n})$ *on* $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$*, and* $(t_n)_{n\in\mathbb{N}}\in\mathbb{R}^l$*. Then one has*

$$\lim_{n\to\infty}\int_{0<\|x\|^2+\|y\|^2\le 1}\|x\|\cdot\|y\|Q_{0t_n}(dx,dy)=0.$$

3.1.1 Related results and extensions

In this section, we present various results which follow from, are related to or extend the theorems presented in the previous section.
The first result is a corollary which follows immediately from Theorem 3.1.3, and states that a multivariate random field is mixing if and only if its components are pairwise mixing.

Corollary 3.1.6. *An* $\mathbb{R}^d$*-valued strictly stationary i.d. random field* $X=(X_t)_{t\in\mathbb{R}^l}$ *with* $Q_0(\{x=(x_1,...,x_d)'\in\mathbb{R}^d:\exists j\in\{1,...,d\},x_j\in 2\pi\mathbb{Z}\})=0$ *is mixing if and only if the bivariate random fields* $(X^{(j)},X^{(k)})$*,* $j,k\in\{1,...,d\}$*,* $j<k$*, are all mixing.*

Proof. It follows immediately from Theorem 3.1.3. □

The following corollary is a generalization of Corollary 2.5 of [FS13].

Corollary 3.1.7. *Let* $(X_t)_{t\in\mathbb{R}^l}$ *be an* $\mathbb{R}^d$*-valued strictly stationary ID random field. Then with the previous notation,* $(X_t)_{t\in\mathbb{R}^l}$ *is mixing if and only if*

$$\lim_{n\to\infty}\left\{\|\Sigma(t_n)\|+\int_{\mathbb{R}^{2d}}\min(1,\|x\|\cdot\|y\|)Q_{0t_n}(dx,dy)\right\}=0, \tag{3.5}$$

for any $(t_n)_{n\in\mathbb{N}}\in\mathcal{T}$*.*

Proof. We can follow the argument by [FS13]. To this end, note that if we assume that (3.5) holds, then conditions $(MM1)$ and $(MM2')$ hold and, thus, Theorem 3.1.4 implies that $(X_t)_{t\in\mathbb{R}^l}$ is mixing.
For the other direction assume that $(X_t)_{t\in\mathbb{R}^l}$ is mixing then by Theorem 3.1.4 condition $(MM1)$ holds. Furthermore, for every $\delta>0$ with $Q_{jk}(\partial K_\delta)=0$, where $K_\delta:=\{(x,y)\in\mathbb{R}^2:x^2+y^2\le\delta^2\}$ and any $j,k=1,...,d$, $(cf.\ (3.21))$,

$$Q^{(jk)}_{0t_n}\Big|_{K_\delta^c}\rightharpoonup Q_{jk}\Big|_{K_\delta^c}\quad as\ n\to\infty. \tag{3.6}$$

for any $(t_n)_{n\in\mathbb{N}}\in\mathcal{T}$, where the symbol "$\rightharpoonup$" means convergence in the weak topology. In addition, we know that the Lévy measures Q_{jk} are concentrated on the axes of $\mathbb{R}^2$. Now consider a $\delta>0$ such that conditions (3.22) and (3.6) hold. Then we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^2}\min(1,|xy|)Q^{(jk)}_{0t_n}(dx,dy)\le\epsilon+\limsup_{n\to\infty}\int_{B_\delta^c}\min(1,|xy|)Q^{(jk)}_{0t_n}(dx,dy)=\epsilon,$$

where B_δ indicates the ball of radius δ centred at zero. By letting $\epsilon \searrow 0$ we obtain that $\limsup_{n\to\infty} \int_{\mathbb{R}^2} \min(1, |xy|) Q^{(jk)}_{0t_n}(dx, dy) = 0$ for any $j, k = 1, ..., d$. Finally,

$$\int_{\mathbb{R}^{2d}} \min\left(1, \sum_{k=1}^{d} |x_k| \cdot \sum_{j=1}^{d} |y_j|\right) Q_{0t_n}(dx, dy) \leq \sum_{j,k=1}^{d} \int_{\mathbb{R}^{2d}} \min(1, |x_k y_j|) Q_{0t_n}(dx, dy)$$

$$= \sum_{j,k=1}^{d} \int_{\mathbb{R}^2} \min(1, |xy|) Q^{(jk)}_{0t_n}(dx, dy) \to 0, \quad as \quad n \to 0.$$

Therefore, this implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2d}} \min(1, \|x\| \cdot \|y\|) Q_{0t_n}(dx, dy),$$

for any $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$, hence we obtain that condition (3.5) is satisfied. □

The next two results are a reformulation of Theorem 3.1.3. However, the first requires a short preliminary introduction, which will be useful for Section 4 as well. Recall that a *codifference* $\tau(X_1, X_2)$ of an ID real bivariate random vector (X_1, X_2) is defined as follows

$$\tau(X_1, X_2) := \log \mathbb{E}\left[e^{i(X_1 - X_2)}\right] - \log \mathbb{E}\left[e^{iX_1}\right] - \log \mathbb{E}\left[e^{-iX_2}\right],$$

where log is the distinguished logarithm as defined in [Sa99] p. 33. Following [FS13] we recall that the *autocodifference* function for an $\mathbb{R}^d$-valued strictly stationary ID process $(X_t)_{t\in\mathbb{R}}$ is defined as $\tau(t) = \left(\tau^{(jk)}(t)\right)_{j,k=1,...,d}$ with $\tau^{(jk)}(t) := \tau\left(X_0^{(k)}, X_t^{(j)}\right)$. For an $\mathbb{R}^d$-valued strictly stationary ID random field $(X_t)_{t\in\mathbb{R}^l}$ the *autocodifference field* $\tau(t)$ is defined as $\tau(t) = \left(\tau^{(jk)}(t)\right)_{j,k=1,...,d}$ with $\tau^{(jk)}(t) := \tau\left(X_0^{(k)}, X_t^{(j)}\right)$, where $t \in \mathbb{R}^l$.

Corollary 3.1.8. *Let $(X_t)_{t\in\mathbb{R}^l}$, with $l \in \mathbb{N}$, be an $\mathbb{R}^d$-valued strictly stationary infinite divisible random field such that Q_0, the Lévy measure of $\mathcal{L}(X_0)$, satisfies $Q_0(\{x = (x_1, ..., x_d)' \in \mathbb{R}^d : \exists j \in \{1, ..., d\}, x_j \in 2\pi\mathbb{Z}\}) = 0$. Then $(X_t)_{t\in\mathbb{R}^l}$ is mixing if and only if $\tau(t_n) \to 0$ as $n \to \infty$ for any sequence $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$.*

Proof. It follows immediately from Theorem 3.1.3. □

Corollary 3.1.9. *Let $(X_t)_{t\in\mathbb{R}^l}$, with $l \in \mathbb{N}$, be an $\mathbb{R}^d$-valued strictly stationary infinite divisible random field such that Q_0, the Lévy measure of $\mathcal{L}(X_0)$, satisfies $Q_0(\{x = (x_1, ..., x_d)' \in \mathbb{R}^d : \exists j \in \{1, ..., d\}, x_j \in 2\pi\mathbb{Z}\}) = 0$. Then $(X_t)_{t\in\mathbb{R}^l}$ is mixing if and only if*

$$\lim_{\|t\|\to\infty} \mathbb{E}\left[e^{i(X_t^{(j)} - X_0^{(k)})}\right] = \mathbb{E}\left[e^{iX_0^{(j)}}\right] \cdot \mathbb{E}\left[e^{-iX_0^{(k)}}\right], \tag{3.7}$$

for any $j, k = 1, .., d$, where $\|\cdot\|$ is any norm on $\mathbb{R}^l$ (e.g. the sup or the Euclidean norm) and $t \in \mathbb{R}^l$.

Proof. "⇒": Assume that $(X_t)_{t\in\mathbb{R}^l}$ is mixing. Then by Theorem 3.1.3 we know that

$$\lim_{n\to\infty} \mathbb{E}\left[e^{i(X_{t_n}^{(j)} - X_0^{(k)})}\right] = \mathbb{E}\left[e^{iX_0^{(j)}}\right] \cdot \mathbb{E}\left[e^{-iX_0^{(k)}}\right] \tag{3.8}$$

holds for any $j,k=1,..,d$ and for any sequence $(t_n)_{n\in\mathbb{N}}\in\mathcal{T}$. Now consider the following simple result.

Let $M_1=(A_1,d_1)$ and $M_2=(A_2,d_2)$ be two metric spaces. Let $S\subseteq A_1$ be an open set of M_1. Let f be a mapping defined on S. Then $\lim_{x\to c} f(x)=l$ iff for any sequence $(x_n)_{n\in\mathbb{N}}$ of points in S such that $\forall n\in\mathbb{N}: x_{n\neq c}$ and $\lim_{n\to\infty} x_n=c$ we have $\lim_{n\to\infty} f(x_n)=l$.

From this result and from the fact that we are considering any sequence such that $\lim_{n\to\infty}\|t_n\|_\infty=\infty$, we obtain equation (3.7).

"$\Leftarrow$": Assume that (3.7) holds. Then we have that (3.8) holds by the result stated above. Then by Theorem 3.1.3 we obtain that $(X_t)_{t\in\mathbb{R}^l}$ is mixing.

Now consider the set $\mathcal{E}:=\left\{(t_n)_{n\in\mathbb{N}}\subset\mathbb{R}^l:\lim_{n\to\infty}\|t_n\|=\infty, \text{where } \|\cdot\| \text{ is any norm on } \mathbb{R}^l\right\}$. Notice that any sequence $(t_n)_{n\in\mathbb{N}}\in\mathcal{T}$ belongs to $\mathcal{E}$ and vice versa, because on the finite dimensional vector space $\mathbb{R}^l$ any norm $\|\cdot\|_a$ is equivalent to any other norm $\|\cdot\|_b$. Hence, we obtain that equation (3.8) holds for any $(t_n)_{n\in\mathbb{N}}\in\mathcal{E}$, and by applying the argument above we obtain our result. □

Remark 3.1.10. It is possible to see that the extension that we have done in the above corollary, can be applied to all our results that holds for "any sequence $(t_n)_{n\in\mathbb{N}}\in\mathcal{T}$".

Recall that from Proposition 11.10 of [Sa99] projections of Lévy processes (measures) are again Lévy processes (measures).The next result is a multivariate and random field extension of Theorem 2 of Rosinski and Zak [RZ96] and it will help us to generalise Theorem 3.1.3.

Theorem 3.1.11. *Let $(X_t)_{t\in\mathbb{R}^l}$ be an $\mathbb{R}^d$-valued stationary ID random field such that Q_0, the Lévy measure of X_0, satisfies $Q_0(\{x=(x_1,...,x_d)'\in\mathbb{R}^d:\exists j\in\{1,...,d\},x_j\in 2\pi\mathbb{Z}\})\neq 0$. Let $Q_0^{(j)}$ be the one dimensional Lévy measure on the j-th coordinate of Q_0, let $\mathcal{Q}_j$ be the set of atoms of $Q_0^{(j)}$ and let*

$$Z_j=\{z\in\mathbb{R}: z=2\pi k/y,\ \ y\in\mathcal{Q}_j,\ \ k\in\mathbb{Z}\}$$

for $j=1,...,d$. Then $(X_t)_{t\in\mathbb{R}^l}$ is mixing if and only if for some $a\in\mathbb{R}^d$, with $a_j\notin Z_j\cup\{0\}$ for $j=1,..,d$, we have

$$\lim_{n\to\infty}\mathbb{E}\left[e^{i(a_jX_{t_n}^{(j)}-a_kX_0^{(k)})}\right]=\mathbb{E}\left[e^{ia_jX_0^{(j)}}\right]\cdot\mathbb{E}\left[e^{-ia_kX_0^{(k)}}\right],$$

for any $j,k=1,..,d$ and for any sequence $(t_n)_{n\in\mathbb{N}}\in\mathcal{T}$.

Proof. Consider an element $a\in\mathbb{R}^d$, with $a_j\notin Z_j\cup\{0\}$ for $j=1,..,d$. We know that the set of atoms of any σ-finite measure is a countable set (the proof is straightforward) and that any Lévy measure is σ-finite. Hence, our a_j exists, (as in one dimensional case, see Theorem 2 in [RZ96]), for every $j=1,..,d$, and thus our a exists. Now, let

$$M_a:=\begin{pmatrix}a_1 & 0 & \cdots & 0\\ 0 & a_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & a_d\end{pmatrix}.$$

Notice that M_a is an invertible $d\times d$ matrix and X_t is a d-dimensional column vector. We have that $(X_t)_{t\in\mathbb{R}^l}$ is mixing if and only if $(M_aX_t)_{t\in\mathbb{R}^l}$ is mixing. This is because by looking at the

Definition 3.1.1 it is enough to show that for every $m \in \mathbb{N}$, $\lambda = (s_1, ..., s_m)' \in \mathbb{R}^{ml}$ and $\theta = (\theta_1, ..., \theta_m)' \in \mathbb{R}^{md}$ we have $\langle \theta, M_a \star X_\lambda \rangle = \langle \tilde{\theta}, X_\lambda \rangle$, where $M_a \star X_\lambda := (M_a X_{s_1}, ..., M_a X_{s_m})'$ and $\tilde{\theta} \in \mathbb{R}^{md}$. Notice that for $m = 1$ we have $M_a \star X_t := M_a X_t$, $t \in \mathbb{R}^l$. Indeed, we have $\langle \theta, M_a \star X_\lambda \rangle = \sum_{j=1}^{d} \sum_{k=1}^{m} a_j X_{s_k}^{(j)} \theta_{jk} = \langle M_a \star \theta, X_\lambda \rangle = \langle \tilde{\theta}, X_\lambda \rangle$, where $M_a \star \theta := (M_a \theta_1, ..., M_a \theta_m)' \in \mathbb{R}^{md}$.
Now, the Lévy measure Q_0^a of $M_a X_0$ is given by $Q_0^a(\cdot) = Q_0(M_a^{-1}(\cdot))$ (see Proposition 11.10 of [Sa99]). Since a is such that $a_j \notin Z_j \cup \{0\}$ for $j = 1, .., d$, Q_0^a has no atoms in the set $\{x = (x_1, ..., x_d)' \in \mathbb{R}^d : \exists j \in \{1, ..., d\}, x_j \in 2\pi\mathbb{Z}\}$. This is because

$$Q_0^a(\{x = (x_1, ..., x_d)' \in \mathbb{R}^d : \exists j \in \{1, ..., d\}, x_j \in 2\pi\mathbb{Z}\})$$

$$= Q_0(\{x = (x_1, ..., x_d)' \in \mathbb{R}^d : \exists j \in \{1, ..., d\}, x_j \in 2\pi\mathbb{Z}/a_j\}),$$

$$\leq Q_0(2\pi\mathbb{Z}/a_1 \times \mathbb{R} \times \cdots \times \mathbb{R}) + Q_0(\mathbb{R} \times 2\pi\mathbb{Z}/a_2 \times \mathbb{R} \times \cdots \times \mathbb{R})$$

$$+ \cdots + Q_0(\mathbb{R} \times \cdots \times \mathbb{R} \times 2\pi\mathbb{Z}/a_d) = \sum_{j=1}^{d} Q_0^{(j)}(2\pi\mathbb{Z}/a_j) = 0,$$

where we used the fact that since $a_j \notin Z_j \cup \{0\}$ then $2\pi k/a_j \neq y$, for any $k \in \mathbb{Z}$ and $y \in \mathcal{Q}_j$, for every $j = 1, ..., d$. Finally, by using Theorem 3.1.3 the proof is complete. □

From this result we have the following generalisation of Theorem 3.1.3.

Corollary 3.1.12. *Let $(X_t)_{t\in\mathbb{R}^l}$ be an $\mathbb{R}^q$-valued stationary ID random field. Then $(X_t)_{t\in\mathbb{R}^l}$ is mixing if and only if $\mathcal{L}(X_{t_n} - X_0) \overset{n\to\infty}{\to} \mathcal{L}(X_0 - X_0')$ for any $j, k = 1, .., d$ and for any sequence $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$, where X_0' is an independent copy of X_0.*

Proof. This result is an immediate consequence of Theorem 3.1.3, Corollary 3.1.6 and Theorem 3.1.11. □

We end this section with a simple general result which will also be useful for the next section.

Proposition 3.1.13. *Let $(X_t)_{t\in\mathbb{R}^l}$ be a linear combination of independent, stationary, ID and mixing random fields. In other words, let $r \in \mathbb{N}$ and let $(X_t)_{t\in\mathbb{R}^l} \overset{d}{=} (\sum_{k=1}^{r} Y_t^k)_{t\in\mathbb{R}^l}$, where $(Y_t^k)_{t\in\mathbb{R}^l}$, $k = 1, ..., r$, are independent $\mathbb{R}^q$-valued stationary, ID and mixing random fields. Then $(X_t)_{t\in\mathbb{R}^l}$ is stationary, ID and mixing.*

3.2 Mixed moving average field

In this section we will focus on a specific random field: the mixed moving average (MMA) random field. Before introducing this random field we need to recall the definition of an $\mathbb{R}^d$-valued Lévy basis and the related integration theory. They represents specific examples of the definitions and the settings presented in Chapter 2. Indeed, Lévy basis are also called infinitely divisible independently scattered random measures in the literature.

In the following let S be a non-empty topological space, $\mathcal{B}(S)$ be the Borel-σ-field on S and π be some probability measure on $(S, \mathcal{B}(S))$. We denote by $\mathcal{B}_0(S \times \mathbb{R}^l)$ the collection of all Borel sets in $S \times \mathbb{R}^l$ with finite $\pi \otimes \lambda^l$-measure, where λ^l denotes the l-dimensional Lebesgue measure.

Definition 3.2.1. *A d-dimensional Lévy basis on $S \times \mathbb{R}^l$ is an $\mathbb{R}^d$-valued random measure $\Lambda = \{\Lambda(B) : B \in \mathcal{B}_0(S \times \mathbb{R}^l)\}$ satisfying:*
(i) the distribution of $\Lambda(B)$ is infinitely divisible for all $B \in \mathcal{B}_0(S \times \mathbb{R}^l)$,
(ii) for an arbitrary $n \in \mathbb{N}$ and pairwise disjoint sets $B_1, ..., B_n \in \mathcal{B}_0(S \times \mathbb{R}^l)$ the random variables $\Lambda(B_1), .., \Lambda(B_n)$ are independent,
(iii) for any pairwise disjoint sets $B_1, B_2, ... \in \mathcal{B}_0(S \times \mathbb{R}^l)$ with $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}_0(S \times \mathbb{R}^l)$ we have, almost surely, $\Lambda(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} \Lambda(B_n)$.

Throughout this section we shall restrict ourselves to time-homogeneous and factorisable Lévy bases, i.e. Lévy bases with characteristic function given by

$$\mathbb{E}\left[e^{i\langle\theta,\Lambda(B)\rangle}\right] = e^{\psi(\theta)\Pi(B)}, \tag{3.9}$$

for all $\theta \in \mathbb{R}^d$ and $B \in \mathcal{B}_0(S \times \mathbb{R}^l)$, where $\Pi = \pi \otimes \lambda^l$ is the product measure of the probability measure π on S and the Lebesgue measure λ^l on $\mathbb{R}^l$ and

$$\psi(\theta) = i\langle\gamma,\theta\rangle - \frac{1}{2}\langle\theta,\Sigma\theta\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle\theta,x\rangle} - 1 - i\langle\theta,x\rangle \mathbf{1}_{[0,1]}(\|x\|)\right) Q(dx)$$

is the cumulant transform of an ID distribution with characteristic triplet (γ, Σ, Q). We note that the quadruple (γ, Σ, Q, π) determines the distribution of the Lévy basis completely and therefore it is called the *generating quadruple*. Now, we provide an extension of Theorem 3.2 of [FS13], which does not need a proof since it is a combination of Theorem 3.2 of [FS13] and Theorem 2.7 of [RR89]. It concerns the existence of integrals with respect to a Lévy basis.

Remark 3.2.2. In this section we are considering a q-valued random field, since the d is used for the $\mathbb{R}^d$-valued Lévy basis, and we denote by $M_{q\times d}(\mathbb{R})$ the collection of $q \times d$ matrices over the field $\mathbb{R}$.

Theorem 3.2.3. *Let Λ be an $\mathbb{R}^d$-valued Lévy basis with characteristic function of the form (3.9) and let $f : S \times \mathbb{R}^l \to M_{q\times d}(\mathbb{R})$ be a measurable function. Then f is Λ-integrable as a limit in probability in the sense of Rajput and Rosinski [RR89] (see also Section 2.4), if and only if*

$$\int_S \int_{\mathbb{R}^l} \left\| f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x(\mathbf{1}_{[0,1]}(\|f(a,s)x\|) - \mathbf{1}_{[0,1]}(\|x\|))Q(dx)\right\| ds\pi(dA) < \infty,$$

$$\int_S \int_{\mathbb{R}^l} \|f(A,s)\Sigma f(A,s)'\| ds\pi(dA) < \infty, \qquad \textit{and}$$

$$\int_S \int_{\mathbb{R}^l} \int_{\mathbb{R}^d} \min(1, \|f(A,s)x\|^2) Q(dx) ds\pi(dA) < \infty.$$

If f is Λ-integrable, the distribution of $\int_S \int_{\mathbb{R}^l} f(A,s)\Lambda(dA,ds)$ is infinitely divisible with characteristic triplet $(\gamma_{int}, \Sigma_{int}, v_{int})$ given by

$$\gamma_{int} = \int_S \int_{\mathbb{R}^l} f(A,s)\gamma + \int_{\mathbb{R}^d} f(A,s)x(\mathbf{1}_{[0,1]}(\|f(a,s)x\|) - \mathbf{1}_{[0,1]}(\|x\|))v(dx)ds\pi(dA),$$

$$\Sigma_{int} = \int_S \int_{\mathbb{R}^l} f(A,s)\Sigma f(A,s)' ds\pi(dA), \qquad \textit{and}$$

$$v_{int}(B) = \int_S \int_{\mathbb{R}^l} \int_{\mathbb{R}^d} \mathbf{1}_B(f(A,s)x)Q(dx)ds\pi(dA),$$

for all Borel sets $B \subseteq \mathbb{R}^q \setminus \{0\}$.

Proof. This theorem is a specific representation of Theorem 3.2 of [FS13] and Theorem 2.7 of [RR89]. □

Let us now introduce the main object of interest of this section: the mixed moving average random field.

Definition 3.2.4. *(Mixed Moving Average Random Field). Let* Λ *be an* $\mathbb{R}^d$*-valued Lévy basis on* $S \times \mathbb{R}^l$ *and let* $f : S \times \mathbb{R}^l \to M_{q\times d}(\mathbb{R})$ *be a measurable function. If the random field*

$$X_t := \int_S \int_{\mathbb{R}^l} f(A, t-s)\Lambda(dA, ds)$$

exists in the sense of Theorem 3.2.3 for all $t \in \mathbb{R}^l$*, it is called an* q*-dimensional mixed moving average random field (MMA random field for short). The function* f *is said to be its kernel function.*

MMA random field have been discussed in Surgailis *et al.* [SRMC93] and Veraart [Ve15]. Note that an MMA random field is an ID and strictly stationary random field. The following lemma is a direct application of Corollary 3.1.7 to our MMA random field case.

Lemma 3.2.5. *Let* $(X_t)_{t\in\mathbb{R}^l} \overset{d}{=} (\int_S \int_{\mathbb{R}^l} f(A, t-s)\Lambda(dA, ds))_{t\in\mathbb{R}^l}$ *be an MMA random field where* Λ *is an* $\mathbb{R}^d$*-valued Lévy basis on* $S \times \mathbb{R}^l$ *with generating quadruple* (γ, Σ, Q, π) *and* $f : S \times \mathbb{R}^l \to M_{q\times d}(\mathbb{R})$ *is a measurable function. Then* $(X_t)_{t\in\mathbb{R}^l}$ *is mixing if and only if*

$$\lim_{n\to\infty} \left\{ \left\| \int_S \int_{\mathbb{R}^l} f(A, -s)\Sigma f(A, t_n - s)' ds\pi(dA) \right\| \right.$$

$$\left. + \int_S \int_{\mathbb{R}^l} \int_{\mathbb{R}^d} \min(1, \|f(A, -s)x\| \cdot \|f(A, t_n - s)x\|)Q(dx)ds\pi(dA) \right\} = 0,$$

for any $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$.

The following theorem is the main result of this section, while the next corollary is an extension of it.

Theorem 3.2.6. *Let* $(X_t)_{t\in\mathbb{R}^l} \overset{d}{=} (\int_S \int_{\mathbb{R}^l} f(A, t-s)\Lambda(dA, ds))_{t\in\mathbb{R}^l}$ *be an MMA random field where* Λ *is an* $\mathbb{R}^d$ *valued Lévy basis on* $S \times \mathbb{R}^l$ *with generating quadruple* (γ, Σ, Q, π) *and* $f : S \times \mathbb{R}^l \to M_{q\times d}(\mathbb{R})$ *is a measurable function. Then* $(X_t)_{t\in\mathbb{R}^l}$ *is mixing.*

From the above result and Proposition 3.1.13, we have this corollary.

Corollary 3.2.7. *Sums of independent MMA random fields are stationary, ID and mixing random fields.*

Proof. This corollary is an immediate consequence of Theorem 3.2.6 and Proposition 3.1.13. □

Remark 3.2.8. The above corollary holds for any linear combination of independent MMA, including MMA with different Lévy basis and different parameter space S.

3.2.1 Meta-times and subordination

In this section we give a brief introduction of the concepts of meta-times and subordination, and present a result which is a corollary of Theorem 3.2.6.

First, we recall the definition of an homogeneous Lévy sheet (see [BP12] definition 2.1). Let $\triangle_a^b F$ indicate the increments of a function F over an interval $(a,b] \subset \mathbb{R}_+^k$ and let $a \leq b$ indicate $a^i \leq b^i$ for $i = 1, ..., k$ (see [BP12]), where $\mathbb{R}_+^m = \{x \in \mathbb{R}^m : x^i \geq 0, i = 1, ..., m\}$ and $m \in \mathbb{N}$. Let $k, l \in \mathbb{N}$.

Definition 3.2.9. *Let $X = \{X_t : t \in \mathbb{R}_+^k\}$ be a family of random vectors in $\mathbb{R}^d$. We say that X is an homogeneous Lévy sheet on $\mathbb{R}_+^k$ if $X_t = 0$ for all $t \in \{t \in \mathbb{R}_+^k : t^j = 0 \text{ for some } j \in \{1, ..., k\}\}$ a.s., $\triangle_{a_1}^{b_1} X, ..., \triangle_{a_n}^{b_n} X$ are independent whenever $n \geq 2$ and $(a_1, b_1], ..., (a_n, b_n] \subset \mathbb{R}_+^k$ are disjoint, X is continuous in probability, $\triangle_{a+t}^{b+t} X \overset{d}{=} \triangle_a^b X$ for all $a, b, t \in \mathbb{R}^k$ with $a \leq b$, and all sample paths of X are* lamp.

The concept of *lamp* (i.e. *limits along monotone paths*) is the analogue of *càdlàg*, but in the multiparameter setting. If $X = \{X_t : t \in \mathbb{R}_+^k\}$ is a homogeneous Lévy sheet then $\mathcal{L}(\triangle_a^b X) \in ID(\mathbb{R}^d)$.

Now let $X = \{X_t : t \in \mathbb{R}_+^k\}$ be an $\mathbb{R}^d$-valued homogeneous Lévy sheet on $\mathbb{R}_+^k$ and $\Lambda_X = \{\Lambda_X(A) : A \in \mathcal{B}(\mathbb{R}_+^k)\}$ be the homogeneous Lévy basis induced by X, namely $\Lambda_X([0,t]) = X_t$ a.s. for all $t \in \mathbb{R}_+^k$. Let $T = \{T_t : t \in \mathbb{R}_+^k\}$ be an $\mathbb{R}_+$-valued homogeneous Lévy sheet and $\Lambda_T = \{\Lambda_T(A) : A \in \mathcal{B}(\mathbb{R}_+^k)\}$ be the nonnegative homogeneous Lévy basis induced by T. Define $\mathcal{F}^T = \sigma(\Lambda_T(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k))$ to be the σ-field generated by Λ_T. Then there exists a $(\mathcal{F}^T, \mathcal{B}(\mathbb{R}_+^k), \mathcal{B}(\mathbb{R}^k))$-measurable mapping $\phi_T : \Omega \times \mathbb{R}_+^k \to \mathbb{R}^k$ such that for all $\omega \in \Omega$ and $A \in \mathcal{B}_b(\mathbb{R}_+^k)$, the set $\mathbf{T}(A)(\omega)$, given by $\mathbf{T}(A)(\omega) = \{x \in \mathbb{R}_+^k : \phi_T(\omega, x) \in A\}$ is bounded and

$$\Lambda_T(A)(\omega) = Leb(\mathbf{T}(A)(\omega)).$$

For each ω, $\mathbf{T}(\cdot)(\omega)$ is called a *meta-time associated with* $\Lambda_T(\cdot)(\omega)$. Let $M = \{M(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$ be defined as

$$M(A)(\omega) = \Lambda_X(\mathbf{T}(A))(\omega)$$

for all $A \in \mathcal{B}(\mathbb{R}_+^k)$. We say that M appears by *extended subordination of* Λ_X *by* Λ_T (or *of* X *by* T). Then by Theorem 5.1 of [BP12] we have that M ia a homogeneous Lévy basis.
Therefore, we have the following corollary of Theorem 3.2.6.

Corollary 3.2.10. *Let $X = \{X_t : t \in \mathbb{R}^k\}$ be an $\mathbb{R}^d$-valued homogeneous Lévy sheet on $\mathbb{R}^k$ and $\Lambda_X = \{\Lambda_X(A) : A \in \mathcal{B}(\mathbb{R}^k)\}$ be the homogeneous Lévy basis induced by X. Let $T = \{T_t : t \in \mathbb{R}^k\}$ be an $\mathbb{R}_+$-valued homogeneous Lévy sheet and $\Lambda_T = \{\Lambda_T(A) : A \in \mathcal{B}(\mathbb{R}^k)\}$ be the nonnegative homogeneous Lévy basis induced by T. Let $M = \{M(A) : A \in \mathcal{B}_b(\mathbb{R}_+^k)\}$ be an extended subordination of Λ_X by Λ_T. Let $(Y_t)_{t \in \mathbb{R}^l} \overset{d}{=} (\int_{\mathbb{R}^{k-l}} \int_{\mathbb{R}^l} f(B, t-s) M(dB, ds))_{t \in \mathbb{R}^l}$, where $f : \mathbb{R}^{k-l} \times \mathbb{R}^l \to M_{q \times d}(\mathbb{R})$ is a measurable function. Then $(Y_t)_{t \in \mathbb{R}^l}$ is mixing.*

Proof. It is sufficient to notice that the framework introduced above holds for the case $\mathbb{R}^k$ and not just for $\mathbb{R}_+^k$ (see [BP12]) and that M is an $\mathbb{R}^d$-valued homogeneous Lévy basis on $\mathbb{R}^k$. Then by using Theorem 3.2.6 we obtain the result. □

3.3 Weak mixing and ergodicity

In this section we will first show how to modify our results to obtain weak mixing version of the results presented before and then prove that for stationary ID random fields ergodicity and weak mixing are equivalent. We start with a definition of a density one set and of weak mixing for stationary random fields.

Definition 3.3.1. *A set $E \subset \mathbb{R}^l$ is said to have density zero in $\mathbb{R}^l$ with respect to the Lebesgue measure λ if*

$$\lim_{T\to\infty} \frac{1}{(2T)^l} \int_{(-T,T]^l} \mathbf{1}_E(x)\lambda(dx) = 0.$$

A set $D \subset \mathbb{R}^l$ is said to have density one in $\mathbb{R}^l$ if $\mathbb{R}^l \setminus D$ has density zero in $\mathbb{R}^l$.

The class of all sequences on D that converge to infinity will be denoted by

$$\mathcal{T}_D := \left\{(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}^l \cap D : \lim_{n\to\infty} \|t_n\|_\infty = \infty\right\}.$$

Definition 3.3.2. *Consider the random field $X_t(\omega) = X_0 \circ \theta_t(\omega)$, $t \in \mathbb{R}^l$, where $\{\theta_t\}_{t\in\mathbb{R}^l}$ is a measure preserving $\mathbb{R}^l$-action. Let σ_X be the σ-algebra generated by the field $(X_t)_{t\in\mathbb{R}^l}$. We say that $(X_t)_{t\in\mathbb{R}^l}$ is weakly mixing if there exists a density one set D such that*

$$\lim_{n\to\infty} \mathbb{P}(A \cap \theta_{t_n}(B)) = \mathbb{P}(A)\mathbb{P}(B),$$

for all $A, B \in \sigma_X$ and all $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}_D$.

We are now ready to state the weak mixing version of Theorem 3.1.3.

Theorem 3.3.3. *Let $(X_t)_{t\in\mathbb{R}^l}$, with $l \in \mathbb{N}$, be an $\mathbb{R}^d$-valued strictly stationary infinite divisible random field such that Q_0, the Lévy measure of $\mathcal{L}(X_0)$, satisfies $Q_0(\{x = (x_1, ..., x_d)' \in \mathbb{R}^d : \exists j \in \{1, ..., d\}, x_j \in 2\pi\mathbb{Z}\}) = 0$. Then $(X_t)_{t\in\mathbb{R}^l}$ is weakly mixing if and only if there exists a density one set $D \subset \mathbb{R}^l$ such that*

$$\lim_{n\to\infty} \mathbb{E}\left[e^{i(X^{(j)}_{t_n} - X^{(k)}_0)}\right] = \mathbb{E}\left[e^{iX^{(j)}_0}\right] \cdot \mathbb{E}\left[e^{-iX^{(k)}_0}\right],$$

for any $j, k = 1, .., d$ and for any sequence $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}_D$.

Proof. It is possible to see that the argument used in the first part of the proof of Theorem 3.1.3 applies and holds also for the case $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}_D$. Moreover, using Theorem 3.3.4 the proof is complete. □

Theorem 3.3.4. *Let $(X_t)_{t\in\mathbb{R}^l}$ be an $\mathbb{R}^d$-valued strictly stationary infinite divisible random field. Then $(X_t)_{t\in\mathbb{R}^l}$ is weakly mixing if and only if there exists a density one set $D \subset \mathbb{R}^l$ such that*

$(MM1)$ the covariance matrix function $\Sigma(t_n)$ of the Gaussian part of $(X_{t_n})_{t_n\in\mathbb{R}^l}$ tends to 0, as $n \to \infty$,

$(MM2')$ $\lim_{n\to\infty} Q_{0t_n}(\|x\| \cdot \|y\| > \delta) = 0$ for any $\delta > 0$, where Q_{0t_n} is the Lévy measure of

$\mathcal{L}(X_0, X_{t_n})$ *on* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$,

where $(t_n)_{n\in\mathbb{N}}$ *is any sequence in* $\mathcal{T}_D$.

Proof. It is possible to see that the arguments used in the proof of Theorem 3.1.4 go through for the case $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}_D$. □

Remark 3.3.5. It is possible to obtain weak mixing version also for the following results previously stated: Corollary 3.1.6, Corollary 3.1.7, Corollary 3.1.8, Corollary 3.1.9, Theorem 3.1.11, Corollary 3.1.12 and Proposition 3.1.13. The proofs of these results are omitted because they follow exactly the same arguments used in the proofs of their respective results. The only difference is that we have $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}_D$ and not $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$ but this does not trigger any change in the arguments used in the proofs of these results.

Among these results, we have the following corollary, which is the weak mixing version of Corollary 3.1.7 and it will be useful for our next result: the equivalence between weak mixing and ergodicity for stationary ID random fields.

Corollary 3.3.6. *Let* $(X_t)_{t\in\mathbb{R}^l}$ *be an* $\mathbb{R}^d$*-valued strictly stationary ID random field. Then with the previous notation,* $(X_t)_{t\in\mathbb{R}^l}$ *is weakly mixing if and only if there exists a density one set* $D \subset \mathbb{R}^l$ *such that*

$$\lim_{n\to\infty} \left\{ \|\Sigma(t_n)\| + \int_{\mathbb{R}^{2d}} \min(1, \|x\| \cdot \|y\|) Q_{0t_n}(dx, dy) \right\} = 0$$

for any $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}_D$.

Proof. See discussion in Remark 3.3.5. □

In order to prove the equivalence between ergodicity and weak mixing for ID stationary random fields we will need various preliminary results, some of which have already been proven in the literature. We start with two known results.

Lemma 3.3.7. [Lemma 4.3 of the ArXiv last version (i.e. v3) of [WRS13]]. *Let* $f : \mathbb{R}^l \to \mathbb{R}$ *be nonnegative and bounded. A necessary and sufficient condition for*

$$\lim_{T\to\infty} \frac{1}{(2T)^l} \int_{(-T,T]^l} f(t)dt = 0$$

is that there exists a subset D *of density one in* $\mathbb{R}^l$ *such that*

$$\lim_{n\to\infty} f(t_n) = 0, \qquad \text{for any} \quad (t_n)_{n\in\mathbb{N}} \in \mathcal{T}_D.$$

Lemma 3.3.8. [Theorem 2.3.2 of [Cu75]]. *Let* μ *be a finite measure on* $\mathbb{R}^l$. *Then*

$$\lim_{T\to\infty} \frac{1}{(2T)^l} \int_{(-T,T]^l} \hat{\mu}(t)dt = \mu(\{0\}),$$

where $\hat{\mu}$ *denotes the Fourier transform of* μ.

The following lemma is an adaptation to our framework of Lemma 3 of [RZ97].

Lemma 3.3.9. *Let $(X_t)_{t\in\mathbb{R}^l}$ be an $\mathbb{R}^d$-valued stationary ID random field and Q_{0t}^{jk} be the Lévy measure of $\mathcal{L}(X_0^{(j)}, X_t^{(k)})$. Then for every $\delta > 0$ and $j,k = 1,...,d$, the family of finite measures of $(Q_{0t}^{jk}|_{K_\delta^c})_{t\in\mathbb{R}^l}$ is weakly relatively compact and*

$$\lim_{\delta\to 0}\sup_{t\in\mathbb{R}^l}\int_{K_\delta}|xy|Q_{0t}^{jk}(dx,dy) = 0, \tag{3.10}$$

where $K_\delta = \{(x,y) : x^2+y^2 \le \delta^2\}$.

Proof. This result comes directly from the proof of Lemma 3 of [RZ97]. □

Now we will investigate the auto-codifference matrix of the $\mathbb{R}^d$-valued stationary ID random field $(X_t)_{t\in\mathbb{R}^l}$, which was already introduced in Section 2.1. Consider

$$\tau(t_n) = \left(\tau^{(jk)}(t_n)\right)_{j,k=1,...,d},$$

with

$$\tau^{(jk)}(t_n) := \tau\left(X_0^{(k)}, X_{t_n}^{(j)}\right) = \log\mathbb{E}\left[e^{i(X_0^{(k)}-X_{t_n}^{(j)})}\right] - \log\mathbb{E}\left[e^{iX_0^{(k)}}\right] - \log\mathbb{E}\left[e^{-iX_{t_n}^{(j)}}\right] \tag{3.11}$$

$$= \sigma_{t_n}^{jk} + \int_{\mathbb{R}^2}(e^{ix}-1)\overline{(e^{iy}-1)}Q_{0t_n}^{jk}(dx,dy),$$

where $\Gamma_{t_n}^{jk}$ is the covariance function of the Gaussian part of $(X_0^{(k)}, X_{t_n}^{(j)})$ and it is given by

$$\Gamma_{t_n}^{jk} = \begin{pmatrix}\sigma_0^{jk} & \sigma_{t_n}^{jk}\\ \sigma_{t_n}^{jk} & \sigma_0^{jk}\end{pmatrix}.$$

Proposition 3.3.10. *Let $(X_t)_{t\in\mathbb{R}^l}$ be an $\mathbb{R}^d$-valued ID random field (not necessarily stationary). Then for any $j,k = 1,...,d$ the function*

$$\mathbb{R}^l\times\mathbb{R}^l \ni (s,t) \to \tau^{(jk)}(X_s^{(j)}, X_t^{(k)}) \in \mathbb{C}$$

is non-negative definite.

Proof. We argue as in Proposition 2 of [RZ97]. Without loss of generality let $t \ge s$ with $t,s\in\mathbb{R}^l$. As seen above, we have

$$\tau\left(X_s^{(k)}, X_t^{(j)}\right) = \sigma_{t-s}^{jk} + \int_{\mathbb{R}^2}(e^{ix}-1)\overline{(e^{iy}-1)}Q_{st}^{jk}(dx,dy). \tag{3.12}$$

Since $(s,t)\to\sigma_{t-s}^{jk}$ is nonnegative definite because it is a covariance function, it just remains to show that the second element on the RHS of (3.12) is non-negative definite. However, this is a consequence of Lemma 4 in [RZ97]. □

We can now state and later prove (see Appendix 3) the second main theorem of this section, which states the equivalence between ergodicity and weak mixing for ID stationary random fields.

Theorem 3.3.11. *Let $l, d \in \mathbb{N}$. Let $(X_t)_{t\in\mathbb{R}^l}$ be an $\mathbb{R}^d$-valued stationary ID random field. Then $(X_t)_{t\in\mathbb{R}^l}$ is ergodic if and only if it is weakly mixing.*

3.4 Conclusion

In this chapter, we have focused on stationary ID random fields and their long time behaviour. First, we have shown that under certain conditions, which constitute an extension of the celebrated Maruyama's conditions, these random fields are mixing. Then, by a simple modification of the arguments, we have extended these results to the weak mixing case. Further, we have demonstrated that for these random fields weak mixing and ergodicity are equivalent. These results both unify and generalise the results of the existing literature. As a prime example of the relevance of the theoretical results presented, we have shown that mixed moving averages (MMA) are mixing. For the definition of the MMA we used the concept of ID random measures, which have been extensively investigated in Chapter 2. The connections between the present chapter and the previous one are so relevant that one might wonder whether the results presented in this chapter hold also for QID random measures and processes. We do not provide an answer to this question and leave this question open for future investigations. After all, the QID world has only marginally been explored and several questions remain unanswered.

3.5 Appendices

Appendix A: Proofs of Section 3.1

Proof of Theorem 3.1.3

This proof is an extension to the random field case of the proof of Theorem 2.1 of [FS13].

"$\Rightarrow$": Assume $(X_t)_{t\in\mathbb{R}^l}$ to be mixing. This implies

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(i\langle\theta_1, X_0\rangle + i\langle\theta_2, X_{t_n}\rangle\right)\right] = \mathbb{E}\left[\exp\left(i\langle\theta_1, X_0\rangle\right)\right]\mathbb{E}\left[\exp\left(i\langle\theta_2, X_0\rangle\right)\right],$$

for any $\theta_1, \theta_2 \in \mathbb{R}^d$. Now, setting $(\theta_1, \theta_2) = (-e_k, e_j)$, $j, k = 1, ..., d$ with e_j the unit j-th vector in $\mathbb{R}^d$, equation (3.4) is satisfied.

"$\Leftarrow$": Assume that equation (3.4) holds for every $j, k = 1, .., d$, then we have

$$\lim_{n\to\infty} \mathbb{E}\left[e^{i\left(X^{(j)}_{t_n} + X^{(k)}_0\right)}\right] = \mathbb{E}\left[e^{iX^{(j)}_0}\right]\mathbb{E}\left[e^{iX^{(k)}_0}\right], \tag{3.13}$$

for every $j, k = 1, .., d$. This is true because of the following reasoning. In particular, we extend the proof of Rosinski and Zak [RZ96], Theorem 1, to the multivariate random field case. Assume that equation (3.4) holds. We will initially prove the following. For every $Y \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ (complex valued)

$$\lim_{n\to\infty} \mathbb{E}\left[e^{iX^{(j)}_{t_n}}\bar{Y}\right] = \mathbb{E}\left[e^{iX^{(j)}_0}\right]\mathbb{E}\left[\bar{Y}\right], \tag{3.14}$$

for $j = 1, .., d$. It is possible to see that equation (3.14) holds for $Y \in H_0 := lin\{1, e^{iX^{(j)}_t}, t \in$

$\mathbb{R}^l\} := \{Z \in L^2(\Omega,\mathcal{F},\mathbb{P}) : Z = a_0 1 + \sum_{i=1}^n a_i e^{iX_{t_i}^{(j)}}, t_i \in \mathbb{R}^l, n \in \mathbb{N}\}$. Consider now the L^2-closure of H_0 and call it H. Then, by standard density argument equation (3.14) is true for any $Y \in H$.

Now, consider any $Y \in L^2(\Omega,\mathcal{F},\mathbb{P})$ (complex valued). We can write $Y = Y_1 + Y_2$, where $Y_1 \in H$ and $Y_2 \in H^\perp := \{W \in L^2(\Omega,\mathcal{F},\mathbb{P}) : \mathbb{E}[Z\bar{W}] = 0 \ \forall Z \in H\}$, where $\mathbb{E}[\cdot\,\bar{\cdot}]$ denotes the inner product (usually written as $<\cdot,\cdot>$) from $L^2(\Omega,\mathcal{F},\mathbb{P}) \times L^2(\Omega,\mathcal{F},\mathbb{P})$ to $\mathbb{R}$. Notice that we are using the conjugate (i.e. in symbol " $\bar{}$ ") because this is how the inner product space over a complex space is defined. Further, notice that we can write $Y = Y_1 + Y_2$ since the L^2-space endowed with that inner product is a Hilbert space.

Since $\mathbb{E}[e^{iX_{t_n}}\bar{Y}_2] = 0$ for every $t_n \in \mathbb{R}^l$ and $\mathbb{E}[\bar{Y}_2] = 0$ by definition of $H^\perp$, we get that

$$\lim_{n\to\infty} \mathbb{E}\left[e^{iX_{t_n}}\bar{Y}\right] = \lim_{n\to\infty} \mathbb{E}\left[e^{iX_{t_n}}\bar{Y}_1\right] = \mathbb{E}\left[e^{iX_0}\right]\mathbb{E}\left[\bar{Y}_1\right] = \mathbb{E}\left[e^{iX_0}\right]\mathbb{E}\left[\bar{Y}\right].$$

Hence we have equation (3.14). Putting now $Y = e^{-iX_0^{(k)}}$ in (3.14), for $k = 1,..,d$, we obtain

$$\lim_{n\to\infty} \mathbb{E}\left[e^{i(X_{tn}^{(j)} + X_0^{(k)})}\right] = \mathbb{E}\left[e^{iX_0^{(j)}}\right]\mathbb{E}\left[e^{iX_0^{(k)}}\right], \tag{3.15}$$

which is eq. (3.13).

We now prove that equations (3.4) and (3.13) imply the multidimensional Maruyama conditions:

$(MM1)$ the covariance matrix function $\Sigma(t_n)$ of the Gaussian part of $(X_{t_n})_{t_n\in\mathbb{R}^l}$ tends to 0, as $n \to \infty$, where $(t_n)_{n\in\mathbb{N}}$ is any sequence in $\mathcal{T}$.

$(MM2)$ $\lim_{n\to\infty} Q_{0t_n}(\|x\| \cdot \|y\| > \delta) = 0$ and $\lim_{n\to\infty} \int_{0<\|x\|^2+\|y\|^2\le 1} \|x\| \cdot \|y\| Q_{0t_n}(dx,dy) = 0$ for any $\delta > 0$, where Q_{0t_n} is the Lévy measure of $\mathcal{L}(X_0, X_{t_n})$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

Actually, we will not prove $(MM2)$ but we will prove instead the following condition:

$(MM2')$ $\lim_{n\to\infty} Q_{0t_n}(\|x\| \cdot \|y\| > \delta) = 0$ for any $\delta > 0$, where Q_{0t_n} is the Lévy measure of $\mathcal{L}(X_0, X_{t_n})$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

This is because in Lemma 3.1.5 we will prove that $(MM2')$ implies $(MM2)$.

Regarding $(MM1)$, we have the following. Since (X_0, X_{t_n}) has a $2d$-dimensional ID distribution, its characteristic function can be written, using the Lévy-Khintchine formulation for every $\theta = (\theta_1,\theta_2)' \in \mathbb{R}^d \times \mathbb{R}^d$, as (see Theorem 8.1 of [Sa99] and the proof of Theorem 2.1 of [FS13])

$$\mathbb{E}\left[e^{i\langle\theta_1,X_0\rangle + i\langle\theta_2,X_{t_n}\rangle}\right] = \exp\left\{ i\left\langle \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \begin{pmatrix}\alpha^1\\ \alpha^2\end{pmatrix}\right\rangle - \frac{1}{2}\left\langle \begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \Sigma(t_n)\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\right\rangle \right.$$

$$\left. + \int_{\mathbb{R}^{2d}} \left(e^{i<\theta_1,x>+i<\theta_2,y>} - 1 - i<\theta_1,x>\mathbf{1}_{[0,1]}(\|x\|) - i<\theta_2,y>\mathbf{1}_{[0,1]}(\|y\|)\right) Q_{0t_n}(dx,dy)\right\}. \tag{3.16}$$

By substituting $(-e_k, e_j)$, $(0, e_j)$ and $(-e_k, 0)$, $j,k = 1,...,d$ for (θ_1,θ_2) in (3.16) we get the description of equation (3.4) in terms of the covariance matrix function of the Gaussian part of

$(X_{t_n}^{(j)}, X_0^{(k)})$ and the Lévy measure Q_{0t_n}, namely

$$\lim_{n\to\infty} \mathbb{E}\left[e^{i\left(X_{t_n}^{(j)} - X_0^{(k)}\right)}\right] \left(\mathbb{E}\left[e^{iX_0^{(j)}}\right] \mathbb{E}\left[e^{-iX_0^{(k)}}\right]\right)^{-1}$$

$$= \lim_{n\to\infty} \exp\left\{\sigma_{jk}(t_n) + \int_{\mathbb{R}^{2d}} \left(e^{i\left(y^{(j)} - x^{(k)}\right)} - e^{iy^{(j)}} - e^{-ix^{(k)}} + 1\right) Q_{0t_n}(dx, dy)\right\} = 1,$$

for arbitrary $j, k = 1, ..., d$, where $\sigma_{jk}(t_n)$ is the (k, j)-th element of $\Sigma(t_n)$. By using the identity $\text{Real}(e^{i(-x+y)} - e^{-ix} - e^{iy} + 1) = (\cos x - 1)(\cos y - 1) + \sin x \sin y$ and taking the logarithm on both sides, we get

$$\lim_{n\to\infty} \left[\sigma_{jk}(t_n) + \int_{\mathbb{R}^{2d}} \left((\cos x^{(k)} - 1)(\cos y^{(j)} - 1) + \sin x^{(k)} \sin y^{(j)}\right) Q_{0t_n}(dx, dy)\right] = 0, \quad (3.17)$$

for any $j, k = 1, .., d$. Applying the same argument to $\mathbb{E}\left[e^{i\left(X_{t_n}^{(j)} + X_0^{(k)}\right)}\right] \left(\mathbb{E}\left[e^{iX_0^{(j)}}\right] \mathbb{E}\left[e^{iX_0^{(k)}}\right]\right)^{-1}$ we obtain

$$\lim_{n\to\infty} \left[-\sigma_{jk}(t_n) + \int_{\mathbb{R}^{2d}} \left((\cos x^{(k)} - 1)(\cos y^{(j)} - 1) - \sin x^{(k)} \sin y^{(j)}\right) Q_{0t_n}(dx, dy)\right] = 0, \quad (3.18)$$

for any $j, k = 1, .., d$. Putting together equations (3.17) and (3.18), due to the consistency of the Lévy measures (see Proposition 11.10 of [Sa99] and the proof of Theorem 2.1 of [FS13]),

$$\lim_{n\to\infty} \int_{\mathbb{R}^{2d}} \left(\cos x^{(k)} - 1)(\cos y^{(j)} - 1\right) Q_{0t_n}(dx, dy) = \lim_{n\to\infty} \int_{\mathbb{R}^2} (\cos x - 1)(\cos y - 1) Q_{0t_n}^{(jk)}(dx, dy) = 0, \quad (3.19)$$

for any $j, k = 1, .., d$, where $Q_{0t_n}^{(jk)}$ denotes the Lévy measure of $\mathcal{L}(X_0^{(k)}, X_{t_n}^{(j)})$ on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$.

Now, for fixed $j, k \in \{1, ..., d\}$ the family of measures $\{\mathcal{L}(X_0^{(k)}, X_{t_n}^{(j)})\}_{n\in\mathbb{N}}$ is tight. To see this, note that by letting $K_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\}$ and using the stationarity of $(X_t)_{t\in\mathbb{R}^l}$ we have

$$\mathbb{P}((X_0, X_{t_n}) \notin K_r) \leq \mathbb{P}(\|X_0\| > 2^{-\frac{1}{2}} r) + \mathbb{P}(\|X_{t_n}\| > 2^{-\frac{1}{2}} r) = 2\mathbb{P}(\|X_0\| > 2^{-\frac{1}{2}} r),$$

hence $\lim_{r\to\infty} \sup_{t_n\in\mathbb{R}^l} \mathbb{P}((X_0, X_{t_n}) \notin K_r) = 0$. Since $\{\mathcal{L}(X_0, X_{t_n})\}_{n\in\mathbb{N}}$ is tight, then $\{\mathcal{L}(X_0^{(k)}, X_{t_n}^{(j)})\}_{n\in\mathbb{N}}$ is tight. Notice that we are proving more than necessary because it is sufficient to prove the above limit for $\sup_{n\in\mathbb{N}}$. Thus, by Prokhorov's theorem we have that the family $\{\mathcal{L}(X_0^{(k)}, X_{t_n}^{(j)})\}_{n\in\mathbb{N}}$ is sequentially compact (in the topology of weak convergence). Choose any sequence $(\tau_n)_{n\in\mathbb{N}} \in \mathcal{T}$ and let F_{jk} be a cluster (or accumulation) point of the family $\{\mathcal{L}(X_0^{(k)}, X_{\tau_n}^{(j)})\}_{n\in\mathbb{N}}$. Then, using Lemma 7.8 of page 34 of Sato's book [Sa99], we have that F_{jk} is an ID distribution on $\mathbb{R}^2$ with some Lévy measure Q_{jk}.

Moreover, let $(t_m)_{m\in\mathbb{N}}$ be a subsequence of $(\tau_n)_{n\in\mathbb{N}}$ such that

$$\mathcal{L}(X_0^{(k)}, X_{t_m}^{(j)}) \rightharpoonup F_{jk}, \quad as \ m \to \infty. \quad (3.20)$$

Notice that $(t_m)_{m\in\mathbb{N}} \in \mathcal{T}$ as well. We know that F_{jk} exists by Prokhorov theorem on Euclidean spaces. Then, for every $\delta > 0$ with $Q_{jk}(\partial K_\delta) = 0$,

$$Q^{(jk)}_{0t_m}\Big|_{K^c_\delta} \rightharpoonup Q_{jk}\Big|_{K^c_\delta}, \quad as \ m \to \infty. \tag{3.21}$$

Since $(\cos x - 1)(\cos y - 1) \geq 0$ and using equations (3.19) and (3.21), we deduce that

$$0 \leq \int_{K^c_\delta} (\cos x - 1)(\cos y - 1) Q_{jk}(dx, dy) = \lim_{n\to\infty} \int_{K^c_\delta} (\cos x - 1)(\cos y - 1) Q^{(jk)}_{0t_m}(dx, dy)$$

$$\leq \lim_{n\to\infty} \int_{\mathbb{R}^2} (\cos x - 1)(\cos y - 1) Q^{(jk)}_{0t_n}(dx, dy) = 0.$$

Every Lévy measure Q_{jk} is concentrated on the set of straight lines $\{(x, y) \in \mathbb{R}^2 : x \in 2\pi\mathbb{Z} \ or \ y \in 2\pi\mathbb{Z}\}$. This is because only on these lines the integrand $(\cos x - 1)(\cos y - 1)$ is zero, otherwise it is positive, and because δ can be taken arbitrarily small.

By the stationarity of the process and (3.20), the projection of Q_{jk} onto the first and second axis coincides with $Q^{(k)}_0$ and $Q^{(j)}_0$, respectively, on the complement of every neighbourhood of zero (because (3.21) holds on any complement of every neighbourhood of zero). Recall the assumption on Q_0, i.e. $Q_0(\{x = (x_1, ..., x_d)' \in \mathbb{R}^d : \exists j \in \{1, ..., d\}, x_j \in 2\pi\mathbb{Z}\}) = 0$. Hence, we have for every $a \in \mathbb{Z}$, $a \neq 0$,

$$Q_{jk}(\{2\pi a\} \times \mathbb{R}) = Q^{(k)}_0(\{2\pi a\}) = Q_0(\mathbb{R} \times ... \times \mathbb{R} \times \{2\pi a\} \times \mathbb{R} \times ... \times \mathbb{R})$$

$$\leq Q_0(\{x \in \mathbb{R}^d : \exists l \in \{1, ..., d\}, x_l \in 2\pi\mathbb{Z}\}) = 0,$$

and similarly $Q_{jk}(\mathbb{R} \times \{2\pi a\}) = 0$. Therefore, Q_{jk}, $j, k = 1, ..., d$ is concentrated on the axes of $\mathbb{R}^2$ and on each of them coincides with $Q^{(k)}_0$ and $Q^{(j)}_0$.

It is important to stress the main ideas of the above argument. First, we showed that Q_{jk} is concentrated only on $\{(x, y) : x \in 2\pi\mathbb{Z} \text{ or } y \in 2\pi\mathbb{Z}\}$ and then using the assumption of our theorem we showed that only when $x = 0$ or $y = 0$ the measure Q_{jk} is non-zero. Further, the stationarity of the process allows the fact that when we project Q_{jk} on the axes the projections coincide with $Q^{(k)}_0$ and $Q^{(j)}_0$, avoiding the case of having $Q^{(k)}_0$ on one axis and $Q^{(j)}_{t_m}$ on the other.

Now observe that, for every $t \in \mathbb{R}^l$, using the consistency of the Lévy measure

$$\int_{K_\delta} |xy| Q^{(jk)}_{0t}(dx, dy) \leq \frac{1}{2} \int_{K_\delta} (x^2 + y^2) Q^{(jk)}_{0t}(dx, dy)$$

$$\leq \frac{1}{2} \int_{|x|\leq\delta} x^2 Q^{(jk)}_{0t}(dx, dy) + \frac{1}{2} \int_{|y|\leq\delta} y^2 Q^{(jk)}_{0t}(dx, dy)$$

$$= \frac{1}{2} \int_{|x|\leq\delta} x^2 Q^{(k)}_0(dx) + \frac{1}{2} \int_{|y|\leq\delta} y^2 Q^{(j)}_0(dy) < \epsilon, \tag{3.22}$$

for any positive ϵ and any $j, k = 1, ..., d$, if only δ is small enough. This implies that, by (3.22) for every $j, k = 1, ..., d$ we have for all $m \in \mathbb{N}$

$$\int_{K_\delta} |\sin x \sin y| Q^{(jk)}_{0t_m}(dx, dy) \leq \int_{K_\delta} |xy| Q^{(jk)}_{0t_m}(dx, dy) < \epsilon,$$

for sufficiently small $\delta > 0$. Since Q_{jk} is concentrated on the axes of $\mathbb{R}^2$ then we have that

$\int_{K_\delta^c} \sin x \sin y Q_{jk}(dx,dy) = 0$ and using (3.21) we get $\lim_{m\to\infty} \int_{K_\delta^c} \sin x \sin y Q_{0t_m}^{(jk)}(dx,dy) = 0$. Thus,

$$\lim_{m\to\infty} \int_{\mathbb{R}^2} \sin x \sin y Q_{0t_m}^{(jk)}(dx,dy) = 0, \tag{3.23}$$

for every $j,k = 1,...,d$. Combining equations (3.17), (3.19) and (3.23) we obtain that $\sigma_{jk}(t_m) \to 0$ as $m \to \infty$ for all $j,k = 1,...,d$. Since $(t_m)_{m\in\mathbb{N}}$ is a subsequence of an arbitrary sequence $\tau_n \in \mathcal{T}$, it follows that $\sigma_{jk}(t_n) \to 0$ as $n \to \infty$ and thus $\Sigma(t_n) \to 0$ as $n \to \infty$, for any $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$. This is because we have used the fact that if a sequence has the property that any subsequence has a further subsequence that converges to the same limit, then the sequence converges to that limit. Hence, condition $(MM1)$ follows.

To prove $(MM2')$, observe that, for any $m \in \mathbb{N}$,

$$Q_{0t_m}\left(\|x\|^2 \cdot \|y\|^2 > \delta^2\right) = Q_{0t_m}\left(\sum_{j,k=1}^{d} \left(x^{(k)}y^{(j)}\right)^2 > \delta^2\right) \leq \sum_{j,k=1}^{d} Q_{0t_m}^{(jk)}\left(\left|x^{(k)}y^{(j)}\right| \geq \frac{\delta}{d}\right).$$

In view of (3.21) we also get

$$\limsup_{m\to\infty} Q_{0t_m}^{(jk)}\left(\left|x^{(k)}y^{(j)}\right| \geq \frac{\delta}{d}\right) \leq Q_{jk}\left(\left|x^{(k)}y^{(j)}\right| \geq \frac{\delta}{d}\right) = 0,$$

for any $\delta > 0$ and $j,k = 1,...,d$. Hence, $\lim_{m\to\infty} Q_{0t_m}(\|x\| \cdot \|y\| > \delta) = 0$ for any $\delta > 0$ and together with the fact that $(t_k)_{k\in\mathbb{N}}$ is a subsequence of an arbitrary sequence $\tau_n \in \mathcal{T}$ we obtain condition $(MM2')$.

Now, by Theorem 3.1.4 the proof is complete.

Proof of Theorem 3.1.4

This proof is an extension to the random field case of the proof of Theorem 2.3 of [FS13].

"$\Rightarrow$": We have shown in the proof of Theorem 3.1.3 that mixing implies conditions $(MM1)$ and $(MM2')$. In particular, mixing implies formula (3.4), which implies $(MM1)$ and $(MM2')$.

"$\Leftarrow$": For the other direction we have the following. Assume that $(MM1)$ and $(MM2')$ hold. Then by Lemma 3.1.5 condition $(MM2)$ holds. We need to prove that the process is mixing, i.e. for all $m \in \mathbb{N}, \lambda = (s_1,...,s_m)', \mu = (p_1,...,p_m)' \in \mathbb{R}^{ml}$ and $\theta_1, \theta_2 \in \mathbb{R}^{md}$

$$\lim_{n\to\infty} \mathbb{E}\left[\exp\left(i\langle\theta_1, X_\lambda\rangle + i\langle\theta_2, X_{\tilde{\mu}_n}\rangle\right)\right] = \mathbb{E}\left[\exp\left(i\langle\theta_1, X_\lambda\rangle\right)\right] \mathbb{E}\left[\exp\left(i\langle\theta_2, X_\mu\rangle\right)\right], \tag{3.24}$$

where $X_\lambda := (X'_{s_1},...,X'_{s_m})' \in \mathbb{R}^{md}$ and $\tilde{\mu}_n = (p_1+t_n,...,p_m+t_n)'$, where $(t_n)_{n\in\mathbb{N}}$ is any sequence in $\mathcal{T}$.

The family of $\mathbb{R}^{2md}$-valued distributions of the ID random fields $(X_\lambda, X_{\tilde{\mu}_n})$, denoted by $\{\mathcal{L}(X_\lambda, X_{\tilde{\mu}_n})\}_{n\in\mathbb{N}}$ is tight. Indeed, let K be the $2md$-dimensional ball around the origin with radius of length $\sqrt{2m}a$, then by stationarity of the process $(X_t)_{t\in\mathbb{R}^l}$ we have

$$\mathbb{P}((X_\lambda, X_{\tilde{\mu}_n}) \notin K) \leq \sum_{j=1}^{m} \mathbb{P}(\|X_{s_j}\| \geq a) + \sum_{j=1}^{m} \mathbb{P}(\|X_{p_j+t_n}\| \geq a) = 2m \sum_{j=1}^{m} \mathbb{P}(\|X_0\| \geq a),$$

hence $\lim_{a\to\infty} \sup_{\lambda,\mu,t_n} \mathbb{P}((X_\lambda, X_{\tilde{\mu}_n}) \notin K) = 0$. Again notice that we are proving more than necessary because it is sufficient to prove the above limit for $\sup_{\lambda,\mu,n}$. Let $(\alpha^1, \Sigma^1, Q^1)$ and $(\alpha^2, \Sigma^2, Q^2)$ be the characteristic triplets of $\mathcal{L}(X_\lambda)$ and $\mathcal{L}(X_\mu)$ respectively.

Suppose F with characteristic triplet (α, R, Q) is a cluster point of the distributions of $(X_\lambda, X_{\tilde{\mu}_n})$. As in the proof of the previous theorem, we have that F is the limit as $r \to \infty$ of the distributions of $(X_\lambda, X_{\tilde{\mu}_r})$, where $\tilde{\mu}_r = (p_1 + t_r, ..., p_m + t_r)'$ with t_r is a subsequence of t_n. Let $(\alpha_r, \Sigma_r, Q_r)$ be the characteristic triplets of $(X_\lambda, X_{\tilde{\mu}_r})$. By Lemma 7.8 of [Sa99] F is an ID distribution on $\mathbb{R}^{2md}$. We denote by $\Phi_r(\theta_1, \theta_2)$ the characteristic function of $\mathcal{L}(X_\lambda, X_{\tilde{\mu}_r})$ at the point $(\theta_1, \theta_2) \in \mathbb{R}^{md} \times \mathbb{R}^{md}$. The logarithm of $\Phi_r(\theta_1, \theta_2)$ can be written (see proof of Theorem 3.1.3 and Theorem 2.3.of [FS13]) as

$$\log \Phi_r(\theta_1, \theta_2) = i \left\langle \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \begin{pmatrix} \alpha_r^1 \\ \alpha_r^2 \end{pmatrix} \right\rangle - \frac{1}{2} \left\langle \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \Sigma_r \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \right\rangle$$

$$+ \int_{\{\|x\|<\delta, \|y\|<\delta\}} \left(e^{i\langle\theta_1,x\rangle + i\langle\theta_2,y\rangle} - 1 - i\langle\theta_1, x\rangle \mathbf{1}_{[0,1]}(\|x\|) - i\langle\theta_2, y\rangle \mathbf{1}_{[0,1]}(\|y\|) \right) Q_r(dx, dy)$$

$$+ \int_{\{\|x\|\geq\delta \text{ or } \|y\|\geq\delta\}} \left(e^{i\langle\theta_1,x\rangle + i\langle\theta_2,y\rangle} - 1 - i\langle\theta_1, x\rangle \mathbf{1}_{[0,1]}(\|x\|) - i\langle\theta_2, y\rangle \mathbf{1}_{[0,1]}(\|y\|) \right) Q_r(dx, dy)$$

$$:= I_1 + I_2 + I_3 + I_4.$$

We need to prove that $\log \Phi_r(\theta_1, \theta_2) \to \log \Phi_1(\theta_1) + \log \Phi_2(\theta_2)$ as $r \to \infty$ for all $\theta_1, \theta_2 \in \mathbb{R}^{md}$ where Φ_1 and Φ_2 are the characteristic functions of X_λ and X_μ, respectively.

It is possible to see immediately that $I_1 = i\langle\alpha^1, \theta_1\rangle + i\langle\alpha^2, \theta_2\rangle$, just by setting $\alpha_r = (\alpha_r^1, \alpha_r^2)' = (\alpha^1, \alpha^2)'$. Further, by condition $(MM1)$ I_2 converges to $-\frac{1}{2}\langle\Sigma^1\theta_1, \theta_1\rangle - \frac{1}{2}\langle\Sigma^2\theta^2, \theta^2\rangle$ as $r \to \infty$, where Σ^1 and Σ^2 are the md-dimensional covariance matrix of (X_λ) and (X_μ) respectively.

For I_4, we have

$$\lim_{r\to\infty} I_4 = \int_{\{\|x\|\geq\delta \text{ or } \|y\|\geq\delta\}} \left(e^{i\langle\theta_1,x\rangle + i\langle\theta_2,y\rangle} - 1 - i\langle\theta_1, x\rangle \mathbf{1}_{[0,1]}(\|x\|) - i\langle\theta_2, y\rangle \mathbf{1}_{[0,1]}(\|y\|) \right) Q(dx, dy)$$

$$= \int_{\{\|x\|\geq\delta\}} \left(e^{i\langle\theta_1,x\rangle} - 1 - i\langle\theta_1, x\rangle \mathbf{1}_{[0,1]}(\|x\|) \right) Q^1(dx)$$

$$+ \int_{\{\|y\|\geq\delta\}} \left(e^{i<\theta_2,y>} - 1 - i < \theta_2, y > \mathbf{1}_{[0,1]}(\|y\|) \right) Q^2(dy).$$

This is because of the identity

$$\exp\left(\sum_{i=1}^m a_i + \sum_{i=1}^m b_i\right) - 1 = \exp\left(\sum_{i=1}^m a_i\right) - 1 + \exp\left(\sum_{i=1}^m b_i\right) - 1,$$

which is valid when $a_i b_j = 0$ for any $1 \leq i, j \leq m$, and because, letting $x = \left(x^{(1)'}, ..., x^{(m)'}\right) \in (\mathbb{R}^d)^m$ and $y = \left(y^{(1)'}, ..., y^{(m)'}\right) \in (\mathbb{R}^d)^m$, we have

$$Q(\|x\| \cdot \|y\| > \delta) \leq \liminf_{r\to\infty} Q_r(\|x\| \cdot \|y\| > \delta)$$

$$\leq \liminf_{r\to\infty} \sum_{j,i=1}^{m} Q_{0,t_r+p_i-s_j}\left(|\|x^{(j)}\|\cdot\|y^{(i)}\|>\frac{\delta}{m}\right) \overset{(MM2')}{=} 0,$$

for any $\delta>0$, which shows in particular that $Q(\|x\|\cdot\|y\|>0)=0$.

Analogously to x and y we denote by $\theta_1^{(j)}$ and $\theta_2^{(j)}$ the j-th $\mathbb{R}^d$-component of θ_1 and θ_2, respectively. Concerning I_3, consider the multivariate Taylor expansion of

$$e^{i\langle\theta_1,x\rangle+i\langle\theta_2,y\rangle}-1-i\langle\theta_1,x\rangle\mathbf{1}_{[0,1]}(\|x\|)-i\langle\theta_2,y\rangle\mathbf{1}_{[0,1]}(\|y\|)=:g(x,y)$$

with respect to the variable $(x,y)'$ at the point $(x_0,y_0)'\equiv 0$. For any $\delta>0$ small enough, we obtain

$$I_3=-\frac{1}{2}\Bigg[\int_{\{\|x\|<\delta,\|y\|<\delta\}}\left(\sum_{j=1}^{m}\left\langle\theta_1^{(j)},x^{(j)}\right\rangle\right)^2+\left(\sum_{j=1}^{m}\left\langle\theta_2^{(j)},y^{(j)}\right\rangle\right)^2 Q_r(dx,dy)$$

$$+2\int_{\{\|x\|<\delta,\|y\|<\delta\}}\left(\sum_{j,k=1}^{m}\left\langle\theta_1^{(j)},x^{(j)}\right\rangle\left\langle\theta_2^{(k)},y^{(k)}\right\rangle\right)Q_r(dx,dy)\Bigg]+R,$$

where R is the reminder and in the integral form it is given by

$$R=\int_{\{\|x\|<\delta,\|y\|<\delta\}}\frac{1}{6}\int_0^1(1-t)^2D^3_{x,y}(g(tx,ty))dtQ_k(du),$$

and so

$$6|R|\leq\int_{\{\|x\|<\delta,\|y\|<\delta\}}\int_0^1|\langle\theta_1,tx\rangle+\langle\theta_2,ty\rangle|^3\left|e^{i\langle\theta_1,tx\rangle+i\langle\theta_2,ty\rangle}\right|dtQ_r(dx,dy)$$

$$=\int_{\{\|x\|<\delta,\|y\|<\delta\}}|\langle\theta_1,x\rangle+\langle\theta_2,y\rangle|^3Q_r(dx,dy)\int_0^1t^3dt=\frac{1}{4}\int_{\{\|x\|<\delta,\|y\|<\delta\}}|\langle\theta_1,x\rangle+\langle\theta_2,y\rangle|^3Q_r(dx,dy)$$

$$\leq\frac{1}{4}\left\|\begin{pmatrix}\theta_1\\\theta_2\end{pmatrix}\right\|^3\int_{\{\|x\|<\delta,\|y\|<\delta\}}\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|^3Q_r(dx,dy)$$

$$\leq\frac{1}{4}\left\|\begin{pmatrix}\theta_1\\\theta_2\end{pmatrix}\right\|^3 2\delta\left(\int_{\{0<\|x\|<\delta\}}\|x\|^2Q^1(dx)+\int_{\{0<\|y\|<\delta\}}\|y\|^2Q^2(dx)\right)$$

and thus $6|R|<\epsilon$ for any positive ϵ if only δ is sufficiently small. Notice that our estimates are sharper than the ones of [FS13] because we work with the explicit integral form of the remainder. Moreover, we obtain for every $j,k=1,...,m$ and any δ small enough that

$$\int_{\{\|x\|<\delta,\|y\|<\delta\}}\left|\left\langle\theta_1^{(j)},x^{(j)}\right\rangle\left\langle\theta_2^{(k)},y^{(k)}\right\rangle\right|Q_r(dx,dy)$$

$$\leq\left\|\theta_1^{(j)}\right\|\cdot\left\|\theta_2^{(k)}\right\|\int_{\{\|x\|<\delta,\|y\|<\delta\}}\left\|x^{(j)}\right\|\cdot\left\|y^{(k)}\right\|Q_r(dx,dy)$$

$$\leq\left\|\theta_1^{(j)}\right\|\cdot\left\|\theta_2^{(k)}\right\|\int_{\{0<\|x^{(j)}\|^2+\|y^{(k)}\|^2\leq 2\delta^2\}}\left\|x^{(j)}\right\|\cdot\left\|y^{(k)}\right\|Q_{0,t_r+p_k-s_j}\left(dx^{(j)},dy^{(k)}\right)$$

$$\leq\left\|\theta_1^{(j)}\right\|\cdot\left\|\theta_2^{(k)}\right\|\int_{\{0<\|x^{(j)}\|^2+\|y^{(k)}\|^2\leq 1\}}\left\|x^{(j)}\right\|\cdot\left\|y^{(k)}\right\|Q_{0,t_r+p_k-s_j}\left(dx^{(j)},dy^{(k)}\right)\overset{r\to\infty}{\to}0,$$

by using condition $(MM2)$. Finally, we have

$$\left|\frac{1}{2}\int_{\{\|x\|<\delta,\|y\|<\delta\}}\langle\theta_1,x\rangle^2 Q_r(dx,dy)+\int_{\{0<\|x\|<\delta\}} e^{i\langle\theta_1,x\rangle}-1-i\langle\theta_1,x\rangle\mathbf{1}_{[0,1]}(\|x\|)Q^1(dx)\right|\leq J_1+J_2.$$

For J_1, we have

$$J_1=\left|\frac{1}{2}\int_{\{0<\|x\|<\delta,\|y\|<\delta\}}\langle\theta_1,x\rangle^2 Q_r(dx,dy)-\frac{1}{2}\int_{\{0<\|x\|<\delta\}}\langle\theta_1,x\rangle^2 Q_r(dx,dy)\right|$$

$$\leq\int_{\{0<\|x\|<\delta\}}\langle\theta_1,x\rangle^2 Q_r(dx,dy)\leq\|\theta_1\|^2\int_{\{0<\|x\|<\delta\}}\|x\|^2 Q^1(dx).$$

For J_2, we have

$$J_2=\left|\int_{\{0<\|x\|<\delta\}}\frac{1}{2}\langle\theta_1,x\rangle^2+e^{i\langle\theta_1,x\rangle}-1-i\langle\theta_1,x\rangle\mathbf{1}_{[0,1]}(\|x\|)Q^1(dx)\right|,$$

and, by using the multivariate Taylor expansion and noticing that in this case the expansion is only for the variable x, we obtain

$$J_2\leq\|\theta_1\|^3\delta\int_{\{0<\|x\|<\delta\}}\|x\|^2 Q^1(dx).$$

Similar arguments apply to the second addend of the first term of I_3.

Combining all the different results, we get

$$\lim_{r\to\infty}\log\Phi_r(\theta_1,\theta_2)=\log\Phi_1(\theta_1)+\log\Phi_2(\theta_2),\quad\forall\theta_1,\theta_2\in\mathbb{R}^{md},$$

and consequently we obtain the desired result in (3.24), which concludes the proof.

Proof of Lemma 3.1.5

Fix $\epsilon>0$, put $B_\delta=\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d:\|x\|^2+\|y\|^2\leq\delta^2\}$ and $R_\delta=\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d:\delta^2<\|x\|^2+\|y\|^2\leq 1\}$. Then, we get

$$\int_{\{0<\|x\|^2+\|y\|^2\leq 1\}}\|x\|\cdot\|y\|Q_{0t_n}(dx,dy)$$

$$=\int_{B_\delta}\|x\|\cdot\|y\|Q_{0t_n}(dx,dy)+\int_{R_\delta}\|x\|\cdot\|y\|Q_{0t_n}(dx,dy):=P_1+P_2.$$

We will estimate the terms P_1 and P_2 separately. Using the stationarity of Q_{0t_n} (due to the stationarity of $(X_{t_n})_{t_n\in\mathbb{R}^l}$) and the consistency of the Lévy measure, we get

$$|P_1|\leq\frac{1}{2}\int_{B_\delta}\|x\|^2 Q_{0t_n}(dx,dy)+\frac{1}{2}\int_{B_\delta}\|y\|^2 Q_{0t_n}(dx,dy)$$

$$\leq\frac{1}{2}\int_{\{\|x\|^2\leq\delta^2,y\in\mathbb{R}^d\}}\|x\|^2 Q_{0t_n}(dx,dy)+\frac{1}{2}\int_{\{\|y\|^2\leq\delta^2,x\in\mathbb{R}^d\}}\|y\|^2 Q_{0t_n}(dx,dy)=\int_{\|x\|\leq\delta}\|x\|^2 Q_0(dx).$$

Here Q_0 is the Lévy measure of X_0. Thus, for some appropriately small δ_0 we have

$$|P_1| = \int_{B_{\delta_0}} \|x\| \cdot \|y\| Q_{0t_n}(dx, dy) \leq \frac{\epsilon}{2}. \tag{3.25}$$

For the second term, set $c_0 = \min\left\{\delta_0, \frac{\epsilon}{8q}\right\}$ with $q = Q_0\left(\|x\|^2 > \frac{\delta_0^2}{2}\right) < \infty$. Then, for $C = R_{\delta_0} \cap \{\|x\| \cdot \|y\| > c_0\}$ we obtain

$$|P_2| = \int_C \|x\|\cdot\|y\| Q_{0t_n}(dx, dy) + \int_{R_{\delta_0}\setminus C} \|x\|\cdot\|y\| Q_{0t_n}(dx, dy) \leq \frac{1}{2} Q_{0t_n}(C) + \int_{R_{\delta_0}\setminus C} \frac{\epsilon}{8q} Q_{0t_n}(dx, dy)$$

$$\leq \frac{1}{2} Q_{0t_n}(\|x\| \cdot \|y\| > c_0) + \frac{\epsilon}{8q} Q_{0t_n}(R_{\delta_0} \setminus C)$$

$$\leq \frac{1}{2} Q_{0t_n}(\|x\| \cdot \|y\| > c_0) + \frac{\epsilon}{8q} Q_{0t_n}\left(\left\{(x,y) : \|x\|^2 > \frac{\delta_0^2}{2}\right\} \cup \left\{(x,y) : \|y\|^2 > \frac{\delta_0^2}{2}\right\}\right)$$

$$\leq \frac{1}{2} Q_{0t_n}(\|x\| \cdot \|y\| > c_0) + \frac{\epsilon}{4q} Q_{0t_n}\left(\|x\|^2 > \frac{\delta_0^2}{2}\right) = \frac{1}{2} Q_{0t_n}(\|x\| \cdot \|y\| > c_0) + \frac{\epsilon}{4}.$$

For n large enough we have $Q_{0t_n}(\|x\| \cdot \|y\| > c_0) < \frac{\epsilon}{2}$ and therefore

$$|P_2| = \int_{R_\delta} \|x\| \cdot \|y\| Q_{0t_n}(dx, dy) < \frac{\epsilon}{2}. \tag{3.26}$$

Finally, combining (3.25) and (3.26), and letting $\epsilon \to 0$, we obtain the result of the lemma.

Proof of Proposition 3.1.13

By independence of the random fields $(Y_t^k)_{t\in\mathbb{R}^l}$, $k = 1, ..., r$, we have that the Lévy-Khintchine representation of $(X_t)_{t\in\mathbb{R}^l}$ can be written as the product of the Lévy-Khintchine representation of the $(Y_t^k)_{t\in\mathbb{R}^l}$, $k = 1, ..., r$. In other words, for any $n \in \mathbb{N}$ and $\theta_j \in \mathbb{R}^d$, $j = 1, ..., n$, we have

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^n \langle \theta_j, X_{t_j}\rangle\right)\right] = \mathbb{E}\left[\exp\left(i\sum_{j=1}^n \langle \theta_j, \sum_{k=1}^r Y_{t_j}^k\rangle\right)\right] = \prod_{k=1}^r \mathbb{E}\left[\exp\left(i\sum_{j=1}^n \langle \theta_j, Y_{t_j}^k\rangle\right)\right].$$

Now $(X_t)_{t\in\mathbb{R}^l}$ is stationary since for any $h \in \mathbb{R}^l$

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^n \langle \theta_j, Y_{t_j}^k\rangle\right)\right] = \mathbb{E}\left[\exp\left(i\sum_{j=1}^n \langle \theta_j, Y_{t_j+h}^k\rangle\right)\right]$$

because $(Y_t^k)_{t\in\mathbb{R}^l}$ is stationary, for any $k = 1, .., r$. Moreover, $(X_t)_{t\in\mathbb{R}^l}$ is ID since it is a sum of independent ID random fields.

To show that $(X_t)_{t\in\mathbb{R}^l}$ is mixing we proceed as follows. Consider any sequence $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$ and consider the joint random field (X_0, X_{t_n}). First, notice that the covariance function of the Gaussian part of $(X_{t_n})_{t_n\in\mathbb{R}^l}$, call it $\Sigma_X(t_n)$, is given by the sum of the covariance functions of the $(Y_t^k)_{t\in\mathbb{R}^l}$, call them $\Sigma_{Y^k}(t_n)$, $k = 1, ..., r$. Moreover, notice also that the Lévy measure of the Lévy-Khintchine formula of the law $\mathcal{L}(X_{t_n}, X_0)$, call it $Q_{X,0t_n}$, is given by the sum of the Lévy

measures of the Lévy-Khintchine formula of the laws $\mathcal{L}(Y^k_{t_n}, Y^k_0)$, call them $Q_{Y^k,0t_n}$, $k = 1, ..., r$. It is possible to see this in formulae.

$$\mathbb{E}\left[\exp\left(i\langle\theta_1, X_{t_n}\rangle + i\langle\theta_2, X_0\rangle\right)\right] = \mathbb{E}\left[\exp\left(i\langle\theta_1, \sum_{k=1}^r Y^k_{t_n}\rangle + i\langle\theta_2, \sum_{k=1}^r Y^k_0\rangle\right)\right]$$

$$= \prod_{k=1}^r \mathbb{E}\left[\exp\left(i\langle\theta_1, Y^k_{t_n}\rangle + i\langle\theta_2, Y^k_0\rangle\right)\right] = \prod_{k=1}^r \exp\left\{i\left\langle\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \begin{pmatrix}\alpha^k_1\\ \alpha^k_2\end{pmatrix}\right\rangle - \frac{1}{2}\left\langle\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \Gamma^k(t_n)\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\right\rangle\right.$$

$$\left. + \int_{\mathbb{R}^{2d}}\left(e^{i<\theta_1,x>+i<\theta_2,y>} - 1 - i<\theta_1, x>\mathbf{1}_{[0,1]}(\|x\|) - i<\theta_2, y>\mathbf{1}_{[0,1]}(\|y\|)\right) Q_{Y^k,0t_n}(dx, dy)\right\}$$

$$= \exp\left\{i\left\langle\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \begin{pmatrix}\sum_{k=1}^r \alpha^k_1\\ \sum_{k=1}^r \alpha^k_2\end{pmatrix}\right\rangle - \frac{1}{2}\left\langle\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \sum_{k=1}^r \Gamma^k(t_n)\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\right\rangle\right.$$

$$\left. + \int_{\mathbb{R}^{2d}}\left(e^{i<\theta_1,x>+i<\theta_2,y>} - 1 - i<\theta_1, x>\mathbf{1}_{[0,1]}(\|x\|) - i<\theta_2, y>\mathbf{1}_{[0,1]}(\|y\|)\right) \sum_{k=1}^r Q_{Y^k,0t_n}(dx, dy)\right\}$$

$$= \exp\left\{i\left\langle\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \begin{pmatrix}\alpha^1_X\\ \alpha^2_X\end{pmatrix}\right\rangle - \frac{1}{2}\left\langle\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \Gamma_X(t_n)\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}\right\rangle\right.$$

$$\left. + \int_{\mathbb{R}^{2d}}\left(e^{i<\theta_1,x>+i<\theta_2,y>} - 1 - i<\theta_1, x>\mathbf{1}_{[0,1]}(\|x\|) - i<\theta_2, y>\mathbf{1}_{[0,1]}(\|y\|)\right) Q_{X,0t_n}(dx, dy)\right\}$$

where

$$\Gamma^k(t_n) = \begin{pmatrix}\Sigma(0) & \Sigma_{Y^k}(t_n)\\ \Sigma_{Y^k}(t_n) & \Sigma(0)\end{pmatrix}.$$

In order to prove that $(X_t)_{t\in\mathbb{R}^l}$ is mixing we need to show that conditions $(MM1)$ and $(MM2')$ hold. However, these conditions hold for each $\Sigma_{Y^k}(t_n)$ and $Q_{Y^k,0t_n}$, where $k = 1, ..., r$. Moreover, since both the covariance matrix function $\Sigma_X(t_n)$ of the Gaussian part of $(X_t)_{t\in\mathbb{R}^l}$ and its Lévy measure $Q_{X,0t_n}$ are sums of $\Sigma_{Y^k}(t_n)$ and $Q_{Y^k,0t_n}$ respectively, for $k = 1, ..., r$, then we have that these conditions hold also for them. Hence, $(X_t)_{t\in\mathbb{R}^l}$ is mixing.

Appendix B: Proofs of Section 3.2

Proof of Lemma 3.2.5

The argument of the proof of Lemma 3.4 of [FS13] can be extended to our setting. To this end, notice that we can write

$$\begin{pmatrix}X_0\\ X_{t_n}\end{pmatrix} = \int_S\int_{\mathbb{R}^l}\begin{pmatrix}f(A, -s)\\ f(A, t_n - s)\end{pmatrix}\Lambda(dA, ds), \qquad t_n \in \mathbb{R}^l.$$

From this and from Theorem 3.2.3 it is possible to compute the covariance matrix function of the Gaussian part of $(X_t)_{t\in\mathbb{R}^l}$ by

$$\Gamma(t_n) = \begin{pmatrix} \Sigma(0) & \Sigma(t_n) \\ \Sigma(t_n) & \Sigma(0) \end{pmatrix},$$

where

$$\Sigma(t_n) = \int_S \int_{\mathbb{R}^l} f(A,-s)\Sigma f(A,t_n-s)' ds\pi(dA), \qquad t_n \in \mathbb{R}^l.$$

The Lévy measure Q_{0t_n} of $\mathcal{L}(X_0, X_{t_n})$ is given by

$$Q_{0t_n}(B) = \int_S \int_{\mathbb{R}^l} \int_{\mathbb{R}^d} \mathbf{1}_B(f(A,-s)x, f(A,t_n-s)x) Q(dx) ds\pi(dA),$$

for all Borel sets $B \subseteq \mathbb{R}^{2q} \setminus \{0\}$, using again Theorem 3.2.3. Therefore, given this explicit representation of Q_{0t_n} we have that

$$\int_{\mathbb{R}^{2q}} \min(1, \|x\|\cdot\|y\|) Q_{0t_n}(dx,dy) = \int_S \int_{\mathbb{R}^l} \int_{\mathbb{R}^d} \min(1, \|f(A,-s)x\|\cdot\|f(A,t_n-s)x\|) Q(dx) ds\pi(dA).$$

Now, by using Corollary 3.1.7, we complete the proof.

Proof of Theorem 3.2.6

We generalise the arguments given in the proof of Theorem 3.5 of [FS13]. Following Lemma 3.2.5, in order to prove this theorem it is sufficient to show that

$$\|\Sigma(t_n)\| = \left\| \int_S \int_{\mathbb{R}^l} f(A,-s)\Sigma f(A,t_n-s)' ds\pi(dA) \right\| \overset{n\to\infty}{\to} 0$$

and

$$\int_{\mathbb{R}^{2d}} \min(1, \|x\|\cdot\|y\|) Q_{0t_n}(dx,dy)$$

$$= \int_S \int_{\mathbb{R}^l} \int_{\mathbb{R}^d} \min(1, \|f(A,-s)x\|\cdot\|f(A,t_n-s)x\|) Q(dx) ds\pi(dA) \overset{n\to\infty}{\to} 0,$$

for any $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$.

First, we concentrate on proving that $\|\Sigma(t_n)\| \overset{n\to\infty}{\to} 0$. Consider that by the existence of the MMA field we have (see Theorem 3.2.3)

$$\int_S \int_{\mathbb{R}^l} \left\| f(A,t-s)\Sigma^{\frac{1}{2}} \right\|^2 ds\pi(dA) < \infty,$$

for any $t \in \mathbb{R}^l$, where $\Sigma^{\frac{1}{2}}$ denotes the unique square root of Σ. Therefore for any $t \in \mathbb{R}^l$, the function $g_t : S\times\mathbb{R}^l \to \mathbb{R}, (A,s) \mapsto \|f(A,t-s)\Sigma^{\frac{1}{2}}\|$ is an element of $L^2(S\times\mathbb{R}^l, \mathcal{B}(S\times\mathbb{R}^l), \pi\otimes\lambda^l; \mathbb{R})$. The fact that the measure $\pi\otimes\lambda^l$ is σ-finite implies that every L^2-function can be approximated (in the L^2-norm) by an elementary function in

$$\mathcal{E} := \Bigg\{ f \in L^2(S\times\mathbb{R}^l, \mathcal{B}(S\times\mathbb{R}^l), \pi\otimes\lambda^l; \mathbb{R}) : f = \sum_{i=1}^k c_i \mathbf{1}_{D_i\times R_i}, k \in \mathbb{N},$$

$$c_i \in \mathbb{R}, D_i \in \mathcal{B}(S), R_i \in \mathcal{B}(\mathbb{R}^l), i = 1,...,k \Bigg\}.$$

Let us now fix an arbitrary $\epsilon > 0$ and choose an elementary function $\tilde{g} \in \mathcal{E}$ such that

$$\|g_0 - \tilde{g}\|_{L^2} = \left(\int_S \int_{\mathbb{R}^l} |g_0(A,s) - \tilde{g}(A,s)|^2 ds\pi(dA) \right)^{\frac{1}{2}} < \epsilon.$$

We have that for any $t_n \in \mathbb{R}^l$

$$\|\Sigma(t_n)\| = \left\| \int_S \int_{\mathbb{R}^l} f(A,-s)\Sigma^{\frac{1}{2}} \left(f(A,t_n - s)\Sigma^{\frac{1}{2}} \right)' ds\pi(dA) \right\|$$

$$\leq \int_S \int_{\mathbb{R}^l} g_0(A,s) \cdot g_t(A,s) ds\pi(dA)$$

$$= \int_S \int_{\mathbb{R}^l} (g_0(A,s) - \tilde{g}(A,s)) \cdot g_t(A,s) + \tilde{g}(A,s)(g_t(A,s) - \tilde{g}(A,s-t_n)) + \tilde{g}(A,s)\tilde{g}(A,s-t_n) ds\pi(dA).$$

By the Cauchy-Schwarz inequality we obtain

$$\|\Sigma(t_n)\| \leq \epsilon \cdot \|g_{t_n}\|_{L^2} + \|\tilde{g}\|_{L^2} \cdot \left(\int_S \int_{\mathbb{R}^l} |g_{t_n}(A,s) - \tilde{g}(A,s-t_n)|^2 ds\pi(dA) \right)^{\frac{1}{2}}$$

$$+ \int_S \int_{\mathbb{R}^l} |\tilde{g}(A,s)\tilde{g}(A,s-t_n)|^2 ds\pi(dA).$$

Now notice that

$$\left(\int_S \int_{\mathbb{R}^l} |g_{t_n}(A,s) - \tilde{g}(A,s-t_n)|^2 ds\pi(dA) \right)^{\frac{1}{2}}$$

$$= \left(\int_S \int_{\mathbb{R}^l} |g_0(A,s-t_n) - \tilde{g}(A,s-t_n)|^2 ds\pi(dA) \right)^{\frac{1}{2}} < \epsilon$$

and that $\|g_{t_n}\|_{L^2} = \|g_0\|_{L^2}$, by a simple change of variables. In addition, $\|\tilde{g}\|_{L^2} \leq \|g_0\|_{L^2} + \|\tilde{g} - g_0\|_{L^2} < \|g_0\|_{L^2} + \epsilon$.

Finally, it is possible to see that

$$\int_S \int_{\mathbb{R}^l} |\tilde{g}(A,s)\tilde{g}(A,s-t_n)|^2 ds\pi(dA) = 0,$$

for sufficiently large t_n (or equivalently for sufficiently large n, where t_n is an element of $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$). This is because $\tilde{g} \in \mathcal{E}$. In particular, given $\tilde{g} \in \mathcal{E}$ then $\tilde{g}(A,s) = \sum_{i=1}^k c_i \mathbf{1}_{D_i \times R_i}(A,s), k \in \mathbb{N}$, hence $\tilde{g}(A,s-t_n) = 0$ for t_n sufficiently large using the fact that the rectangles R_i cannot cover the whole $\mathbb{R}^l$ since $\int_S \int_{\mathbb{R}^l} \|\tilde{g}\|^2 ds\pi(dA) < \infty$, for any $i = 1, ..., k$. Therefore, we have

$$\|\Sigma(t_n)\| < \epsilon \cdot \|g_0\|_{L^2} + (\epsilon + \|g_0\|_{L^2}) \cdot \epsilon,$$

for sufficiently large n. This yields $\|\Sigma(t_n)\| \to 0$ as $n \to \infty$, for any $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$.

We now move to the second objective of the proof. Indeed, we now prove that

$$\int_{\mathbb{R}^{2q}} \min(1, \|x\| \cdot \|y\|) Q_{0t_n}(dx,dy) \overset{n\to\infty}{\to} 0,$$

for any $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$.

Consider an arbitrary $\epsilon > 0$ and set $B_r := \{(x, y) \in \mathbb{R}^q \times \mathbb{R}^q : \|x\|^2 + \|y\|^2 \leq r^2\}$. Recall now the argument used to prove (3.21). In that argument we did not assume that the random field was mixing, but only that was stationary and ID. Thus, we have that for any $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$ the following holds

$$\sup_{n\geq n_0} Q_{0t_n}(\mathbb{R}^{2q} \setminus B_R) \leq \epsilon,$$

for some $R > 1$ and some $n_0 > 0$. Therefore, for all $n \geq n_0$, we obtain that

$$\int_{\mathbb{R}^{2q}} \min(1, \|x\| \cdot \|y\|) Q_{0t_n}(dx, dy) \leq \int_{B_R} \min(1, \|x\| \cdot \|y\|) Q_{0t_n}(dx, dy) + \epsilon.$$

Now notice that when $\max\{\|u\|, \|v\|\} \leq R$ we have

$$\min\{\|u\| \cdot \|v\|, 1\} \leq R \cdot \min\{\|u\|, 1\} \cdot \min\{\|v\|, 1\}.$$

Hence, we have

$$\int_{B_R} \min(1, \|x\| \cdot \|y\|) Q_{0t_n}(dx, dy) \leq R \cdot \int_{B_R} \min(1, \|x\|) \cdot \min(1, \|y\|) Q_{0t_n}(dx, dy)$$

$$\leq R \cdot \int_S \int_{\mathbb{R}^l} \int_{\mathbb{R}^d} \min(1, \|f(A, -s)x\|) \cdot \min(1, \|f(A, t_n - s)x\|) Q(dx) ds \pi(dA).$$

By the existence of the MMA random field, we have that for any $t_n \in \mathbb{R}^l$ the function $h_{t_n} : S \times \mathbb{R}^l \times \mathbb{R}^d \to \mathbb{R}$, $h_{t_n}(A, s, x) := \min(1, \|f(A, t_n - s)x\|)$ is an element of $L^2(S \times \mathbb{R}^l \times \mathbb{R}^d, \mathcal{B}(S \times \mathbb{R}^l \times \mathbb{R}^d), \pi \otimes \lambda^l \otimes Q; \mathbb{R})$. Also, the fact that every Lévy measure is σ-finite implies that the product measure $\pi \otimes \lambda^l \otimes Q$ is σ-finite as well and therefore it is possible to use the same approximation argument used above in the first part of this proof to show that

$$\int_S \int_{\mathbb{R}^l} \int_{\mathbb{R}^d} \min(1, \|f(A, -s)x\|) \cdot \min(1, \|f(A, t_n - s)x\|) Q(dx) ds \pi(dA) \overset{n\to\infty}{\to} 0,$$

for any $(t_n)_{n\in\mathbb{N}} \in \mathcal{T}$, which completes the proof.

Appendix C: Proofs of Section 3.3

Proof of Theorem 3.3.11

This proof is a multivariate and a random field extension of the proof of Theorem 1 of [RZ97].

"$\Rightarrow$": It is well known that any weakly mixing random field is ergodic.

"$\Leftarrow$": For the other direction we argue as follows. Let $(X_t)_{t\in\mathbb{R}^l}$ be an ergodic $\mathbb{R}^d$-valued stationary ID random field. In this proof we will work with $j, k = 1, ..., d$ and we will not repeat it every time. We showed before that

$$\tau(t) = \left(\tau^{(jk)}(t)\right)_{j,k=1,...,d}$$

with

$$\tau^{(jk)}(t) := \tau\left(X_0^{(k)}, X_t^{(j)}\right) = \sigma_t^{jk} + \int_{\mathbb{R}^2} (e^{ix} - 1)\overline{(e^{iy} - 1)} Q_{0t}^{jk}(dx, dy) \tag{3.27}$$

is the autocodifference field of $(X_t)_{t\in\mathbb{R}^l}$. Since $(X_t)_{t\in\mathbb{R}^l}$ is an $\mathbb{R}^d$-valued ID and stationary random field, we have that

$$\tau\left(X_s^{(k)}, X_t^{(j)}\right) = \tau\left(X_0^{(k)}, X_{t-s}^{(j)}\right).$$

Hence, as shown before in Proposition 3.3.10, the function

$$\tau^{(jk)}(t) = \tau\left(X_0^{(k)}, X_t^{(j)}\right)$$

is nonnegative definite and $\tau^{(jk)}(0) = -\log\left(\mathbb{E}[e^{iX_0^{(j)}}]\cdot\mathbb{E}[e^{iX_0^{(k)}}]\right)$ which is a constant. Hence, we can use Bochner's theorem, which implies that there exists a finite Borel measure v on $\mathbb{R}^l$ such that

$$\tau^{(jk)}(t) = \int_{\mathbb{R}^l} e^{i\langle t,\lambda\rangle} v(d\lambda).$$

Thus,

$$\mathbb{E}[e^{i(X_0^{(j)}-X_t^{(k)})}]\left(\mathbb{E}[e^{iX_0^{(j)}}]\cdot\mathbb{E}[e^{iX_0^{(k)}}]\right)^{-1} = e^{\tau^{(jk)}(t)} = e^{\hat{v}(t)} = \widehat{\exp(v)}(t),$$

where $\exp(v) = \sum_{n=o}^{\infty}(v^{*n}/n!), v^{*0} = \delta_0$ and the symbol " $\hat{}$ " denotes the Fourier transform. Notice that the last equality comes from the convolution theorem. Hence, $\tau^{(jk)} = \hat{v}$. Moreover, since both terms on the RHS of (3.27) are non-negative definite thanks to Proposition 3.3.10, again by Bochner's theorem there exists finite Borel measures v_G and v_P on $\mathbb{R}^l$ such that

$$\tau^{(jk)}(t) = \hat{v}_G(t) + \hat{v}_P(t). \tag{3.28}$$

Thus $v = v_G + v_P$. The ergodicity of the process implies that

$$\frac{1}{(2T)^l}\int_{(-T,T]^l}\mathbb{E}\left[e^{i(X_0^{(j)}-X_t^{(k)})}\right]dt \overset{T\to\infty}{\to} \mathbb{E}\left[e^{iX_0^{(j)}}\right]\mathbb{E}\left[e^{-iX_0^{(k)}}\right].$$

Hence, combining this result with Lemma 3.3.8, we obtain

$$\frac{1}{(2T)^l}\int_{(-T,T]^l}\widehat{\exp(v)}(t)dt \overset{T\to\infty}{\longrightarrow} \exp(v_G+v_P)(\{0\}) = 1,$$

which implies that

$$v_G(\{0\}) + v_P(\{0\}) = 0 \quad \text{and} \quad (v_G+v_P)*(v_G+v_P)(\{0\}) = 0, \tag{3.29}$$

and so $v_G * v_G(\{0\}) = 0$. Therefore, we have that

$$\frac{1}{(2T)^l}\int_{(-T,T]^l}(\sigma_t^{jk})^2dt = \frac{1}{(2T)^l}\int_{(-T,T]^l}(\hat{v}_G(t))^2dt \overset{T\to\infty}{\longrightarrow} v_G * v_G(\{0\}) = 0.$$

Since σ_t^{jk} is real, we deduce from Lemma 3.3.7 that there exists a set D of density one in $\mathbb{R}^l$, such that

$$\lim_{n\to\infty}\sigma_{t_n}^{jk} = 0, \qquad \text{for any} \quad (t_n)_{n\in\mathbb{N}} \in \mathcal{T}_D. \tag{3.30}$$

Now we would like to prove a similar result for the Lévy measure of $\mathcal{L}(X_0^{(j)}, X_{t_n}^{(k)})$, so that we can then apply Corollary 3.3.6, which will give us weak mixing of our random field $(X_t)_{t\in\mathbb{R}^l}$.

By equations (3.27), (3.28), (3.29) and Lemma 3.3.8 we have that

$$\frac{1}{(2T)^l}\int_{(-T,T]^l}\int_{\mathbb{R}^2}(e^{ix}-1)\overline{(e^{iy}-1)}Q_{0t}^{jk}(dx,dy)dt = \frac{1}{(2T)^l}\int_{(-T,T]^l}\hat{v}_P(t)dt \overset{T\to\infty}{\longrightarrow} v_P(\{0\}) = 0.$$

When taking the real part of $\frac{1}{(2T)^l}\int_{(-T,T]^l}\int_{\mathbb{R}^2}(e^{ix}-1)\overline{(e^{iy}-1)}Q_{0t}^{jk}(dx,dy)dt$ we get that

$$\frac{1}{(2T)^l}\int_{(-T,T]^l}\int_{\mathbb{R}^2}[(\cos x-1)(\cos y-1)+\sin x\sin y]Q_{0t}^{jk}(dx,dy)dt \overset{T\to\infty}{\longrightarrow} 0. \tag{3.31}$$

Now consider the two integrands of eq. (3.31). By stationarity and Proposition 3.3.10, the functions

$$t\to\int_{\mathbb{R}^2}(\cos x-1)(\cos y-1)Q_{0t}^{jk}(dx,dy), \quad \text{and} \quad t\to\int_{\mathbb{R}^2}\sin x\sin yQ_{0t}^{jk}(dx,dy)$$

are non-negative definite. By Bochner's theorem, this implies that there exist finite Borel measures λ_1 and λ_2 such that equation (3.31) can be written as

$$\frac{1}{(2T)^l}\int_{(-T,T]^l}\hat{\lambda}_1dt+\frac{1}{(2T)^l}\int_{(-T,T]^l}\hat{\lambda}_2dt \overset{T\to\infty}{\longrightarrow} 0,$$

and using Lemma 3.3.8 we obtain $\lambda_1(\{0\})=\lambda_2(\{0\})=0$. Focusing on the first one, we conclude that

$$\frac{1}{(2T)^l}\int_{(-T,T]^l}\int_{\mathbb{R}^2}(\cos x-1)(\cos y-1)Q_{0t}^{jk}(dx,dy)dt \overset{T\to\infty}{\longrightarrow} \lambda_1(\{0\})=0. \tag{3.32}$$

Define $R_T(dx,dy)=\frac{1}{(2T)^l}\int_{(-T,T]^l}Q_{0t}^{jk}(dx,dy)dt$. Then we have

$$\int_{\mathbb{R}^2}(\cos x-1)(\cos y-1)R_T(dx,dy) \overset{T\to\infty}{\longrightarrow} 0. \tag{3.33}$$

It is possible to notice that the family of finite measures $(R_T|_{K_\delta^c})_{T>0}$ is weakly relatively compact for every $\delta>0$. This is because by Lemma 3.3.9 $(Q_{0t}^{jk}|_{K_\delta^c})_{t\in\mathbb{R}^l}$ is weakly relatively compact for every $\delta>0$. The goal is now to show that

$$\int_{\mathbb{R}^2}\min(|xy|,1)R_T(dx,dy) \overset{T\to\infty}{\longrightarrow} 0. \tag{3.34}$$

So, let $T_n\to\infty$, $T_n\in\mathbb{R}$. Using the diagonalization procedure we can find a subsequence (T_n') of (T_n) and a measure R on $\mathbb{R}^2\setminus\{0\}$ such that $R_{T_n'}|_{K_\delta^c}\Rightarrow R|_{K_\delta^c}$ as $n\to\infty$ for every $\delta>0$. Now, notice that $(\cos x-1)(\cos y-1)\geq 0$ and $(\cos x-1)(\cos y-1)=0$ if $x=2\pi k$ or $y=2\pi k$, for $k\in\mathbb{N}$. Moreover, by eq. (3.33) we have that

$$\int_{K_\delta^c}(\cos x-1)(\cos y-1)R(dx,dy)=0,$$

for every $\delta>0$. Therefore, the measure R is concentrated on the set of lines $\{(x,y): x\in 2\pi\mathbb{Z} \text{ or } y\in 2\pi\mathbb{Z}\}$. Since the random field is stationary, the projections of Q_{0t} (and of Q_{0t}^{jk}) onto the first and second axis (excluding zero) are equal to Q_0 (and to Q_0^j for the first axis and to Q_0^k

for the second). The same holds for R_T and for R.

Suppose for now that $Q_0(\{x = (x_1, ..., x_d)' \in \mathbb{R}^d : \exists j \in \{1, ..., d\}, x_j \in 2\pi\mathbb{Z}\}) = 0$, which implies that for $\alpha \in \mathbb{Z} \setminus \{0\}$ we have $Q_0^{jk}(2\pi\alpha \times \mathbb{R}) = Q_0^{jk}(\mathbb{R} \times 2\pi\alpha) = 0$. Then R must be concentrated on the axes of $\mathbb{R}^2$. Hence, for every $\delta > 0$ such that $R(\{(x, y) : x^2 + y^2 = \delta^2\}) = 0$,

$$\limsup_{n\to\infty} \int_{\mathbb{R}^2} \min(|xy|, 1) R_{T_n'}(dx, dy)$$

$$\leq \limsup_{n\to\infty} \int_{K_\delta^c} \min(|xy|, 1) R_{T_n'}(dx, dy) + \sup_{T>0} \frac{1}{(2T)^l} \int_{(-T,T]^l} \int_{K_\delta} |xy| Q_{0t}^{jk}(dx, dy)dt.$$

Eq. (3.10) implies that the last quantity can be made arbitrarily small. Hence, (3.34) follows. From eq. (3.34) and Lemma 3.3.7 we obtain that there exists a set D' of density one in $\mathbb{R}^l$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \min(|xy|, 1) Q_{0t_n}^{jk}(dx, dy) = 0, \qquad \text{for any} \quad (t_n)_{n\in\mathbb{N}} \in \mathcal{T}_{D'}. \tag{3.35}$$

In case D of eq. (3.30) and D' of eq. (3.35) are different this is not a problem because the intersection of two (or a finite number) of density one sets is again a density one set.

Hence, following the proof of Corollary 3.1.7 (and using its weak mixing version, namely Corollary 3.3.6) we obtain that $(X_t)_{t\in\mathbb{R}^l}$ is weakly mixing with the additional assumption that $Q_0(\{x = (x_1, ..., x_d)' \in \mathbb{R}^d : \exists j \in \{1, ..., d\}, x_j \in 2\pi\mathbb{Z}\}) = 0$. However, this assumption can be eliminated by first using the fact that if $(X_t)_{t\in\mathbb{R}^l}$ is ergodic then $(M_a X_t)_{t\in\mathbb{R}^l}$ is ergodic too, and then by following the arguments of Theorem 3.1.11. In particular, let $Q_0^{(j)}$ be the one dimensional Lévy measure on the j-th coordinate of Q_0, let $\mathcal{Q}_j$ be the set of atoms of $Q_0^{(j)}$ and let $Z_j = \{z \in \mathbb{R} : z = 2\pi k/y,\ y \in \mathcal{Q}_j,\ k \in \mathbb{Z}\}$ for $j = 1, ..., d$. Since each Z_j, $j = 1, ..., d$, is countable, there exists $a \in \mathbb{R}^d$, with $a_j \notin Z_j \cup \{0\}$. Consider the random field $(M_a X_t)_{t\in\mathbb{R}^l}$ and let Q_0^a be the Lévy measure of $M_a X_0$. Then Q_0^a has no atoms in the set $(\{x = (x_1, ..., x_d)' \in \mathbb{R}^d : \exists j \in \{1, ..., d\}, x_j \in 2\pi\mathbb{Z}\})$; since $(M_a X_t)_{t\in\mathbb{R}^l}$ is also ergodic, it is weakly mixing by the arguments of this proof. This implies that $(X_t)_{t\in\mathbb{R}^l}$ is weakly mixing as well and the proof is complete.

4

Limit Theorems: The multivariate BSS process

4.1 Notations and Preliminaries

In this section we are going to explore the setting and some of the basic mathematical tools used throughout this chapter.
Let $T > 0$ denote a finite time horizon and let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered complete probability space. In the following we always assume that $p, n \in \mathbb{N}$ and that $\mathcal{B}(\mathbb{R})$ denotes the class of Borel sets of $\mathbb{R}$. We recall the definition of a Brownian measure.

Definition 4.1.1. *An $\mathcal{F}_t$-adapted Brownian measure $W : \Omega \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ is a Gaussian stochastic measure such that, if $A \in \mathcal{B}(\mathbb{R})$ with $\mathbb{E}[(W(A))^2] < \infty$, then $W(A) \sim N(0, Leb(A))$, where Leb is the Lebesgue measure. Moreover, if $A \subseteq [t, \infty)$, then $W(A)$ is independent of $\mathcal{F}_t$.*

We will assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ supports p independent $\mathcal{F}_t$-Brownian measures on $\mathbb{R}$. Consider the stochastic process $\{\mathbf{G}_t\}_{t\in[0,T]}$ defined as

$$\mathbf{G}_t := \begin{pmatrix} G_t^{(1)} \\ \vdots \\ G_t^{(p)} \end{pmatrix} = \int_{-\infty}^{t} \begin{pmatrix} g^{(1,1)}(t-s) & \dots & g^{(1,p)}(t-s) \\ \vdots & \ddots & \vdots \\ g^{(p,1)}(t-s) & \dots & g^{(p,p)}(t-s) \end{pmatrix} \begin{pmatrix} dW_s^{(1)} \\ \vdots \\ dW_s^{(p)} \end{pmatrix},$$

where the integral has to be considered componentwise, for $i, j = 1, ..., p$, $g^{(i,j)} \in L^2((0, \infty))$ are deterministic functions and continuous on $\mathbb{R} \setminus \{0\}$, and $(W^{(1)}, ..., W^{(p)})$ are jointly Gaussian $\mathcal{F}_t$-Brownian measures on $\mathbb{R}$. Thus, we have $G_t^{(i)} = \sum_{j=1}^{p} \int_{-\infty}^{t} g^{(i,j)}(t-s) dW_s^{(j)}$. We call the process $\{\mathbf{G}_t\}_{t\in[0,T]}$ the multivariate Gaussian core, and it is possible to see that it is a stationary Gaussian process and it has stationary increments. The Gaussian core will play a crucial role in the limit theorems for the BSS process.

Remark 4.1.2. Notice that we do not assume independence of the Brownian measures. The only requirement is that they are jointly Gaussian so that the process $\{\mathbf{G}_t\}_{t\in[0,T]}$ is Gaussian. This level of generality is needed to prove the central limit theorem (CLT) for the BSS process. In fact, as we will later see, proving a CLT for the Gaussian core driven by independent Brownian measures is not sufficient for proving the CLT for the BSS process.

For $j \in \{1,...,p\}$ and $l \in \{1,...,n\}$, let $\tau_n^{(j)}$ be a (scaling) constant depending on $G^{(j)}$ and n whose explicit form will be introduced later on, and let $\Delta_l^n G^{(j)} := G^{(j)}_{\frac{l}{n}} - G^{(j)}_{\frac{l-1}{n}}$. Since $\{\mathbf{G}_t\}_{t\in[0,T]}$ is a Gaussian process, we can use the machinery of Malliavin calculus. In particular, let $\mathcal{H}$ be the Hilbert space generated by the random variables given by:

$$\left(\frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}\right)_{n\geq 1, 1\leq l\leq \lfloor nt\rfloor, j\in\{1,...,p\}} \tag{4.1}$$

equipped with the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ induced by $L^2(\Omega,\mathcal{F},\mathbb{P})$, *i.e.* for $X, Y \in \mathcal{H}$ we have $\langle X, Y\rangle_{\mathcal{H}} = \mathbb{E}[XY]$. Notice that $\mathcal{H}$ is a closed subset of $L^2(\Omega,\mathcal{F},\mathbb{P})$ composed by L^2-Gaussian random variables generated by (4.1). In particular, we have an isonormal Gaussian process because (4.1) are jointly Gaussian random variables since they are (rescaled) increments of the Gaussian process $\{\mathbf{G}_t\}_{t\in[0,T]}$. Following the setting of [NP12], we assume that $\mathcal{F}$ is generated by $\mathcal{H}$. Finally, recall that any element of $L^2(\Omega,\mathcal{F},\mathbb{P})$ has a unique decomposition in terms of the Wiener chaos expansion of $\mathcal{H}$ (see [NP12]).

Next, we present and define the multivariate BSS process. Since there are several ways to generalise a univariate BSS process to a multivariate one, we will present two particularly relevant multivariate extensions.

Definition 4.1.3. *Consider p jointly Brownian measures $W^{(1)},...,W^{(p)}$. Further consider p^2 non-negative deterministic functions $g^{(1,1)},...,g^{(p,p)} \in L^2((0,\infty))$ which are continuous on $\mathbb{R}\setminus\{0\}$ and such that $g^{(i,j)}(t) = 0$ for $t \leq 0$ and for $i,j = 1,...,p$. Let $\sigma^{(1,1)},...,\sigma^{(p,p)}$ be càdlàg, $\mathcal{F}_t$-adapted stochastic processes and assume for all $t \in [0,T]$ and $i,j,k = 1,...,p$ that $\int_{-\infty}^t (g^{(i,j)}(t-s)\sigma_s^{(j,k)})^2 ds < \infty$ for every $t \geq 0$. Let $\{\boldsymbol{U}_t\}_{t\in[0,T]} = \{(U_t^{(1)},...,U_t^{(p)})\}_{t\in[0,T]}$ be a stochastic process in the nature of a drift term. Let $\boldsymbol{Y}_t = (Y_t^{(1)},...,Y_t^{(p)})$ be given by*

$$\boldsymbol{Y}_t := \int_{-\infty}^t \begin{pmatrix} g^{(1,1)}(t-s) & \cdots & g^{(1,p)}(t-s) \\ \vdots & \ddots & \vdots \\ g^{(p,1)}(t-s) & \cdots & g^{(p,p)}(t-s) \end{pmatrix} \begin{pmatrix} \sigma_s^{(1,1)} & \cdots & \sigma_s^{(1,p)} \\ \vdots & \ddots & \vdots \\ \sigma_s^{(p,1)} & \cdots & \sigma_s^{(p,p)} \end{pmatrix} \begin{pmatrix} dW_s^{(1)} \\ \vdots \\ dW_s^{(p)} \end{pmatrix} + \begin{pmatrix} U_t^{(1)} \\ \vdots \\ U_t^{(p)} \end{pmatrix}$$

and

$$\boldsymbol{X}_t := \begin{pmatrix} X_t^{(1)} \\ \vdots \\ X_t^{(p)} \end{pmatrix} = \int_{-\infty}^t \begin{pmatrix} g^{(1,1)}(t-s)\sigma_s^{(1,1)} & \cdots & g^{(1,p)}(t-s)\sigma_s^{(1,p)} \\ \vdots & \ddots & \vdots \\ g^{(p,1)}(t-s)\sigma_s^{(p,1)} & \cdots & g^{(p,p)}(t-s)\sigma_s^{(p,p)} \end{pmatrix} \begin{pmatrix} dW_s^{(1)} \\ \vdots \\ dW_s^{(p)} \end{pmatrix} + \begin{pmatrix} U_t^{(1)} \\ \vdots \\ U_t^{(p)} \end{pmatrix}.$$

Then the vector valued processes $\{\boldsymbol{Y}_t\}_{t\in[0,T]}$ and $\{\boldsymbol{X}_t\}_{t\in[0,T]}$ are both called multivariate Brownian semistationary processes.

Remark 4.1.4. Note that in the above definition we consider the case where the matrices are

of dimension $p \times p$. However, the above definition can be extended straightforwardly to include general rectangular matrices.

Now, we will discuss properties of the tensor product in Hilbert spaces. Consider two real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ endowed with the inner products $\langle \cdot, \cdot \rangle_{\mathcal{H}_1}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$ respectively. Given $f, x \in \mathcal{H}_1$ and $g, y \in \mathcal{H}_2$, we denote by $[f \otimes g](x, y) := \langle x, f \rangle_{\mathcal{H}_1} \langle y, g \rangle_{\mathcal{H}_2}$ the bilinear form $f \otimes g : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{R}$. Let $\mathcal{K}$ be the set of all finite linear combinations of such bilinear forms, namely $\mathcal{K} := span(f \otimes g : f \in \mathcal{H}_1, g \in \mathcal{H}_2)$. We are going to present a result on the inner product for this space.

Lemma 4.1.5. *The bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ on $\mathcal{K}$ defined by $\langle\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle\rangle := \langle f_1, f_2 \rangle_{\mathcal{H}_1} \langle g_1, g_2 \rangle_{\mathcal{H}_2}$ is symmetric, well defined and positive definite, and thus defines a scalar product on $\mathcal{K}$.*

Proof. This is a well known result, for details see Reed and Simon's book [RS75]. □

Observe that $\mathcal{K}$ endowed with $\langle\langle \cdot, \cdot \rangle\rangle$ is not complete. In the next three definitions we introduce the notion of a tensor product between Hilbert spaces, and the symmetrisation and contraction of a tensor product.

Definition 4.1.6. *The tensor product of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ defined to be the completion of $\mathcal{K}$ under the scalar product $\langle\langle \cdot, \cdot \rangle\rangle$. Further, we denote by $\mathcal{H}_1^{\otimes n}$ the n-fold tensor product between $\mathcal{H}_1$ and itself.*

Definition 4.1.7. *If $f \in \mathcal{H}^{\otimes n}$ is of the form $f = h_1 \otimes \cdots h_n$ for $h_1, ..., h_n \in \mathcal{H}$, then the symmetrisation of f, denoted by $\tilde{f}$, is defined by $\tilde{f} := \frac{1}{n!} \sum_\sigma h_{\sigma(1)} \otimes \cdots h_{\sigma(n)}$, where the sum is taken over all permutations of $\{1, ..., n\}$. The closed subspace of $\mathcal{H}^{\otimes n}$ generated by the elements of the form $\tilde{f}$ is called the n-fold symmetric tensor product of $\mathcal{H}$, and is denoted by $\mathcal{H}^{\odot n}$.*

Definition 4.1.8. *Let $g = g_1 \otimes \cdots \otimes g_n \in \mathcal{H}^{\otimes n}$ and $h = h_1 \otimes \cdots \otimes h_n \in \mathcal{H}^{\otimes m}$. For any $0 \leq p \leq n \wedge m$, we define the p-th contraction of $g \otimes h$ as the following element of $\mathcal{H}^{\otimes m+n-p}$: $g \otimes_p h := \langle g_1, h_1 \rangle_{\mathcal{H}} \cdots \langle g_p, h_p \rangle_{\mathcal{H}} g_{p+1} \otimes \cdots \otimes g_n \otimes h_{p+1} \otimes \cdots \otimes h_m$. Note that, even if g and h are symmetric, their p-th contraction is not, in general, a symmetric tensor. We therefore denote by $g \tilde{\otimes}_p h$ its symmetrisation.*

Let us now move to the discussion of multiple integrals in the Malliavin calculus setting (see section 2.7 of [NP12]). We denote by $I_p : \mathcal{H}^{\odot p} \to \mathcal{W}_p$ the isometry from the symmetric tensor product $\mathcal{H}^{\odot p}$, equipped with the norm $\sqrt{p!} \| \cdot \|_{\mathcal{H}^{\otimes p}}$, onto the p-th Wiener chaos $\mathcal{W}_p$. In other words, the image of a p-th multiple integral lies in the p-th Wiener chaos. The first property that we are going to present is the *isometry property of integrals.*

Proposition 4.1.9. *Fix integers $1 \leq q \leq p$, as well as $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$. We have*

$$\mathbb{E}[I_p(f) I_q(g)] = \begin{cases} p! \langle f, g \rangle_{\mathcal{H}^{\otimes p}}, & \text{if } p = q, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. See Proposition 2.7.5 in [NP12]. □

Moreover, we have the following product formula for multiple integrals.

Theorem 4.1.10. *Let $p, q \geq 1$. If $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$, then*

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} I_{p+q-2r}(f \tilde{\otimes}_r g).$$

Proof. See Theorem 2.7.10 in [NP12]. □

Similarly to [GR19] we apply the product formula for multiple integrals to conclude that for $i, j = 1, ..., p$

$$\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} = I_1\left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}}\right) I_1\left(\frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}\right) = \sum_{r=0}^{1} r! \binom{1}{r}\binom{1}{r} I_{2-2r}\left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \tilde{\otimes}_r \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}\right)$$

$$= I_2\left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \tilde{\otimes} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}\right) + \mathbb{E}\left[\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}\right]$$

$$\Rightarrow I_2\left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \tilde{\otimes} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}\right) = \frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} - \mathbb{E}\left[\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}\right].$$

Now, we introduce the space $\mathcal{D}([0,T], \mathbb{R}^n)$. This space is the set of all càdlàg functions from $[0,T]$ to $\mathbb{R}^n$ and it is called the Skorokhod space. The norm in this space is defined as $\|f\|_{\mathcal{D}([0,T],\mathbb{R}^n)} := \sup_{t\in[0,T]} \|f\|_{\mathbb{R}^n}$ where $f \in \mathcal{D}([0,T], \mathbb{R}^n)$ and $\|\cdot\|_{\mathbb{R}^n}$ is any norm on $\mathbb{R}^n$ (it is a finite dimensional vector space, thus all the norms are equivalent). This metric works fine for $C([0,T], \mathbb{R}^n)$ (the space of continuous functions from $[0,T]$ to $\mathbb{R}^n$), but it is stronger than the usual Skorokhod metric J_1 (or M_1). However, in this chapter the functions to which our random elements (*i.e.* random variables and stochastic processes) convergence are continuous, and in this case these metrics are all equivalent.

Let us recall some results on stable convergence. We use the notations $\overset{u.c.p.}{\to}$, $\overset{P}{\to}$, $\overset{st}{\to}$, $\overset{d}{\to}$ for convergence uniformly on compacts in probability, convergence in probability, stable convergence, and convergence in distribution, respectively. Recall that a sequence of stochastic processes (X_n) converges to the limit X uniformly on compacts in probability if $\mathbb{P}\Big(\sup_{s\leq t} |X_s^n - X_s| > \epsilon\Big) \to 0$ as $n \to \infty$ for each $t, \epsilon > 0$. In the case of the Skorokhod space with uniform metric, suppose X_n, X are $\mathcal{D}([0,T], \mathbb{R}^d)$-valued stochastic processes defined on the same filtered probability space, then we have that $X_n \overset{P}{\to} X$ if and only if $X_n \overset{u.c.p.}{\to} X$, since they are both equal to $\lim_{n\to\infty} \mathbb{P}(\sup_{t\in[0,T]} \|X_{n,t} - X_t\|_{\mathbb{R}^d})$.

Remark 4.1.11. To make the notation less cumbersome and when it does not create confusion, we avoid to write "as $n \to \infty$". For example, we write $X_n \overset{p}{\to} X$ for $X_n \overset{p}{\to} X$ as $n \to \infty$.

Theorem 4.1.12 (Continuous mapping theorem)**.** *Let (S, m) be a metric space and let $(S_n, m) \subset (S, m)$ be arbitrary subsets and $g_n : (S_n, m) \mapsto (E, \mu)$ be arbitrary maps ($n \in \mathbb{N} \cup \{0\}$) such that, for every sequence $x_n \in (S_n, m)$, if $x_{n'} \to x$ along a subsequence and $x \in (S_0, m)$ then $g_{n'}(x_{n'}) \to g_0(x)$. Then, for arbitrary maps $X_n : \Omega_n \mapsto (S_n, m)$ and every random element X with values in (S_0, m) such that $g_0(X)$ is a random element in (E, μ): if $X_n \overset{d}{\to} X$, then $g_n(X_n) \overset{d}{\to} g_0(X)$; if $X_n \overset{P}{\to} X$, then $g_n(X_n) \overset{P}{\to} g_0(X)$; and if $X_n \overset{a.s.}{\to} X$, then $g_n(X_n) \overset{a.s.}{\to} g_0(X)$.*

Proof. See Theorem 18.11 of [Va98]. □

Notice that (S, m) might be a function space like the Skorokhod space endowed with the uniform metric. For stable convergence we have the following theorem, see Theorem 1 in [AE78].

Theorem 4.1.13. *Let X_n be random elements defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $X_n \xrightarrow{st} X$, that σ is any fixed $\mathcal{F}$-measurable random variable and that $g(x, y)$ is a continuous function of two variables. Then, $g(X_n, \sigma) \xrightarrow{st} g(X, \sigma)$.*

Proof. It follows from Theorem 4.1.12 and the definition of stable convergence. □

Proposition 4.1.14. *Let X_n, Y_n and Y be random elements defined on the same probability space and assume that $Y_n \xrightarrow{P} Y$ and $X_n \xrightarrow{st} X$. Then, $(X_n, Y_n) \xrightarrow{st} (X, Y)$.*

Proof. See Section 2 of [Ja97]. □

We end this section with some asymptotic results. We start by reporting a simplified version of Theorem 6.2.3 in [NP12].

Theorem 4.1.15. *Let $b \geq 2$ and consider*

$$\boldsymbol{F}_n := (I_2(f_{1,n}), ..., I_2(f_{b,n})).$$

Let $\boldsymbol{C} \in \mathcal{M}^{b\times b}(\mathbb{R})$ be a symmetric, nonnegative definite matrix and let $\boldsymbol{N} \sim \mathcal{N}_b(0, \boldsymbol{C})$. Assume that for any $r, s = 1, ..., b$

$$\lim_{n\to\infty} \mathbb{E}[I_2(f_{s,n})I_2(f_{r,n})] = (\boldsymbol{C})_{s,r} \quad \text{and} \quad I_2(f_{s,n}) \xrightarrow{d} \mathcal{N}(0, (\mathbf{C})_{s,s}).$$

Then, $\boldsymbol{F}_n \xrightarrow{d} \boldsymbol{N}$ as $n \to \infty$.

Proof. See Theorem 6.2.3 in [NP12]. □

We have the following simple corollary of Theorem 5.2.7 and Theorem 6.2.3 in [NP12].

Corollary 4.1.16. *Let $\boldsymbol{F}_n$ and $\boldsymbol{C}$ as in Theorem 4.1.15. Assume that for any $r, s = 1, ..., b$*

$$\lim_{n\to\infty} \mathbb{E}[I_2(f_{s,n})I_2(f_{r,n})] = (\boldsymbol{C})_{s,r} \quad \text{and} \quad \lim_{n\to\infty} \|f_{s,n} \otimes_1 f_{s,n}\|_{\mathcal{H}^{\otimes 2}} = 0.$$

Then, $\boldsymbol{F}_n \xrightarrow{d} \boldsymbol{N}$ as $n \to \infty$.

Moreover, we report here the part of Proposition 2 in [AE78] needed in this chapter. This result concerns *mixing* limits, which are defined as follows. Let Y_n be a sequence of r.v.s in $(\Omega, \mathcal{F}, \mathbb{P})$ converging stably to a r.v. Y. If Y can be taken to be independent of $\mathcal{F}$ then the limit is said to be mixing. We denote it by $Y_n \xrightarrow{mixing} Y$.

Proposition 4.1.17. *Suppose that $Y_n \xrightarrow{d} Y$. Then the following are equivalent:*
(i) $Y_n \xrightarrow{mixing} Y$,
(ii) *for all fixed $k \in \mathbb{N}$ and $B \in \sigma(Y_1, ..., Y_k)$, $\mathbb{P}(B) > 0$,*

$$\lim_{n\to\infty} \mathbb{P}(Y_n \leq x|B) = F_Y(x),$$

where $F_Y(x)$ is the distribution function of Y.

Proof. See Proposition 2 in [AE78]. □

4.2 Joint CLT for multivariate Gaussian processes with stationary increments

As mentioned in the introduction, one of the differences from previous works on limit theorems for BSS processes is that we use a different scaling factor τ. In this section we are going to present two cases. For the first case we use the same τ used in the literature, while in the second we use a new formulation. The differences between the two approaches will be pointed out subsequently.

Remark 4.2.1. Throughout this section $\{\mathbf{G}_t\}_{t\in[0,T]}$ is a general multivariate Gaussian process with stationary increments. Thus, it is not necessarily the Gaussian core.

4.2.1 Case I

For $i,j=1,...,p$ and $k\in\mathbb{N}$, let us define the scaling factor by

$$\tau_n^{(j)} := \sqrt{\mathbb{E}\left[\left(\Delta_1^n G^{(j)}\right)^2\right]}, \tag{4.2}$$

and the multivariate process $\{\mathbf{Z}_t^n\}_{t\in[0,T]} = \{(Z^n_{(1,1),t},...,Z^n_{(p,p),t})^\top\}_{t\in[0,T]}$ as

$$Z^n_{(i,j),t} := \frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor} I_2\left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}}\tilde{\otimes}\frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}\right), \quad \text{and} \quad r_{i,j}^{(n)}(k) := \mathbb{E}\left[\frac{\Delta_1^n G^{(i)}}{\tau_n^{(i)}}\frac{\Delta_{1+k}^n G^{(j)}}{\tau_n^{(j)}}\right].$$

Thanks to Theorem 2.7.7 of [NP12] (reported in this work as Theorem 4.1.10), $Z^n_{(i,j),t}$ belongs to $\mathcal{W}_2$, namely the second Wiener chaos. In addition, we will consider the following assumption on the correlation. It states that, uniformly in n, the squared autocorrelations $(r_{i,j}^{(n)}(k))^2$ are summable, which means $r_{i,j}^{(n)}(k)$ goes to zero sufficiently fast as $k\to\infty$.

Assumption 4.2.2. *Let the limit $\lim_{n\to\infty} r_{i,j}^{(n)}(k)$ exists for any $k\in\mathbb{N}$ and let $(\xi(k))_{k\in\mathbb{N}}$ be a sequence such that, for any $k,n\in\mathbb{N}$, $\left(r_{i,j}^{(n)}(k)\right)^2 \le \xi(k)$ and $\sum_{k=1}^\infty \xi(k)<\infty$, for $i,j=1,...,p$.*

Remark 4.2.3. Let $\lim_{n\to\infty}\sum_{k=1}^\infty \left(r_{i,j}^{(n)}(k)\right)^2<\infty$, for $i,j=1,...,p$. For $x,y,z,w=1,...,p$, the following limit holds

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} k\left(r_{x,z}^{(n)}(k)r_{y,w}^{(n)}(k)+r_{y,z}^{(n)}(k)r_{x,w}^{(n)}(k)\right)+\frac{1}{n}\sum_{k=n+1}^{2n-1}(2n-k)\left(r_{x,z}^{(n)}(k)r_{y,w}^{(n)}(k)+r_{y,z}^{(n)}(k)r_{x,w}^{(n)}(k)\right)$$

$$=0$$

This is because given two sequences of real numbers $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ then if $\sum_{k=1}^\infty (a_k)^2<\infty$ and $\sum_{k=1}^\infty (b_k)^2<\infty$ then $\lim_{l\to\infty}\frac{1}{l}\sum_{k=1}^l k\,(a_k)^2=0$ and $\lim_{l\to\infty}\frac{1}{l}\sum_{k=1}^l k\,(b_k)^2=0$ which imply that $\lim_{l\to\infty}\frac{1}{l}\sum_{k=1}^l k a_k b_k=0$ (see the use of condition (3.9) in [BCD11]).

We now present two propositions regarding the process $\{\mathbf{Z}_t^n\}_{t\in[0,T]}$ and they will lead us to the main theorem of this section. The first one is on the convergence of the finite dimensional distributions, while the second one is on the tightness of the law of the process.

Proposition 4.2.4. *Let $d \in \mathbb{N}$ and let $(a_l, b_l]$ be pairwise disjoint intervals in $[0,T]$ where $l = 1, .., d$. Consider*

$$\boldsymbol{Z}_{b_l}^n - \boldsymbol{Z}_{a_l}^n = (Z_{(1,1),b_l}^n - Z_{(1,1),a_l}^n, ..., Z_{(1,p),b_l}^n - Z_{(1,p),a_l}^n, Z_{(2,1),b_l}^n - Z_{(2,1),a_l}^n, ..., Z_{(p,p),b_l}^n - Z_{(p,p),a_l}^n).$$

Then, under the Assumption 4.2.2, $(\boldsymbol{Z}_{b_l}^n - \boldsymbol{Z}_{a_l}^n)_{1\le l\le d} \xrightarrow{d} (\boldsymbol{D}^{\frac{1}{2}}(B_{b_l} - B_{a_l}))_{1\le l\le d}$ as $n \to \infty$, where $\boldsymbol{D} \in \mathcal{M}^{p^2\times p^2}(\mathbb{R})$ and B_t is a p^2-dimensional Brownian motion. In particular, associating for each combination (i,j) a combination $((x,y),(z,w))$, where $x,y,z,w = 1,...,p$, using the formula $(i,j) \leftrightarrow ((\lfloor\frac{i-1}{p}\rfloor + 1, i - p\lfloor\frac{i-1}{p}\rfloor), (\lfloor\frac{j-1}{p}\rfloor + 1, j - p\lfloor\frac{j-1}{p}\rfloor))$, we have

$$(\boldsymbol{D})_{ij} = (\boldsymbol{D})_{(x,y),(z,w)}$$

$$= \frac{1}{n}\sum_{k=1}^{n-1}(n-k)\left(r_{x,z}^{(n)}(k)r_{y,w}^{(n)}(k) + r_{y,z}^{(n)}(k)r_{x,w}^{(n)}(k) + r_{z,x}^{(n)}(k)r_{w,y}^{(n)}(k) + r_{z,y}^{(n)}(k)r_{w,x}^{(n)}(k)\right)$$

$$+ \left(r_{x,z}^{(n)}(0)r_{y,w}^{(n)}(0) + r_{y,z}^{(n)}(0)r_{x,w}^{(n)}(0)\right).$$

Proposition 4.2.5. *Under the Assumption 4.2.2, let $\mathbf{P}^n$ be the law of the process $\{\boldsymbol{Z}_t^n\}_{t\in[0,T]}$ on the Skorokhod space $\mathcal{D}([0,T],\mathbb{R}^{p^2})$. Then, the sequence $\{\mathbf{P}^n\}_{n\in\mathbb{N}}$ is tight.*

Theorem 4.2.6. *Let the Assumption 4.2.2 hold. Then we have*

$$\left\{\frac{1}{\sqrt{n}}\sum_{l=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}}\frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} - \mathbb{E}\left[\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}}\frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}\right]\right)_{i,j=1,...,p}\right\}_{t\in[0,T]} \xrightarrow{st} \left\{\boldsymbol{D}^{\frac{1}{2}}B_t\right\}_{t\in[0,T]},$$

where $\boldsymbol{D}$ and B_t are given in Proposition 4.2.4. In particular, B_t is independent of $G^{(1)}, ..., G^{(p)}$ and the convergence is in $\mathcal{D}([0,T],\mathbb{R}^{p^2})$, namely the Skorokhod space equipped with the uniform metric.

4.2.2 Case II

In this subsection we will show that the results presented in the previous section hold for different choices of τ_n, *i.e.* the scaling factor. In particular, the scaling factor which we will use in this section will have an order equal to or greater than the order of the previous scaling factor as n goes to infinity. We will denote this new scaling factor by $\tau_n^{(\beta(j))}$ for $j = 1, ..., p$ and maintain the notation of $\tau_n^{(j)}$ for the scaling factor introduced in the previous section.
Indeed, let $\tau_n^{(j)} = O\left(\tau_n^{(\beta(j))}\right)$ for $j = 1, ..., p$ (*e.g.* consider $\tau_n^{(\beta(j))} = \max_{j=1,...,p}\left(\tau_n^{(j)}\right)$ for some js and $\tau_n^{(\beta(j))} = \tau_n^{(j)}$ for others, or see also Example 4.2.10). Let $i,j = 1, ..., p$. In this section we will work with the Hilbert space generated by the Gaussian random variables

$$\left(\frac{\Delta_l^n G^{(j)}}{\tau_n^{(\beta(j))}}\right)_{n\ge1,1\le l\le\lfloor nt\rfloor,j\in\{1,...,p\}} \quad \text{and let} \quad r_{i,j}^{(\beta),(n)}(k) := \mathbb{E}\left[\frac{\Delta_1^n G^{(i)}}{\tau_n^{(\beta(i))}}\frac{\Delta_{1+k}^n G^{(j)}}{\tau_n^{(\beta(j))}}\right].$$

Moreover, the following assumption is the analogue of Assumption 4.2.2.

Assumption 4.2.7. *Let the limit* $\lim_{n\to\infty} r_{i,j}^{(\beta),(n)}(k)$ *exists for any* $k \in \mathbb{N}$ *and let* $(\xi(k))_{k\in\mathbb{N}}$ *be a sequence such that, for any* $k, n \in \mathbb{N}$, $\left(r_{i,j}^{(\beta),(n)}(k)\right)^2 \leq \xi(k)$ *and* $\sum_{k=1}^{\infty}\xi(k) < \infty$, *for* $i, j = 1, ..., p$.

We can now present and prove a modification of the main result of the previous section.

Theorem 4.2.8. *Let the Assumption 4.2.7 hold. Let* $\alpha, \beta = 1, 2$. *Then we have*

$$\left\{\frac{1}{\sqrt{n}}\sum_{l=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(\beta(i))}}\frac{\Delta_l^n G^{(j)}}{\tau_n^{(\beta(j))}} - \mathbb{E}\left[\frac{\Delta_l^n G^{(i)}}{\tau_n^{(\beta(i))}}\frac{\Delta_l^n G^{(j)}}{\tau_n^{(\beta(j))}}\right]\right)_{i,j=1,...,p}\right\}_{t\in[0,T]} \xrightarrow{st} \left\{\boldsymbol{D}^{(\beta)\frac{1}{2}}B_t\right\}_{t\in[0,T]}.$$

In particular, associating for each combination (i, j) *a combination* $((x, y), (z, w))$ *where* $x, y, z, w = 1, ..., p$ *using the formula* $(i, j) \leftrightarrow ((\lfloor\frac{i-1}{p}\rfloor + 1, i - p\lfloor\frac{i-1}{p}\rfloor), (\lfloor\frac{j-1}{p}\rfloor + 1, j - p\lfloor\frac{j-1}{p}\rfloor))$, *we have*

$$(\boldsymbol{D})_{ij}^{(\beta)} = (\boldsymbol{D})_{(x,y),(z,w)}^{(\beta)} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}(n-k)\Big(r_{x,z}^{(\beta),(n)}(k)r_{y,w}^{(\beta),(n)}(k) + r_{y,z}^{(\beta),(n)}(k)r_{x,w}^{(\beta),(n)}(k)$$

$$+r_{z,x}^{(\beta),(n)}(k)r_{w,y}^{(\beta),(n)}(k) + r_{z,y}^{(\beta),(n)}(k)r_{w,x}^{(\beta),(n)}(k)\Big) + \Big(r_{x,z}^{(\beta),(n)}(0)r_{y,w}^{(\beta),(n)}(0) + r_{y,z}^{(\beta),(n)}(0)r_{x,w}^{(\beta),(n)}(0)\Big).$$

Further, B_t *is a* p^2*-dimensional Brownian motion independent of* $G^{(1)}, ..., G^{(p)}$ *and the convergence is in* $\mathcal{D}([0, T], \mathbb{R}^{p^2})$, *namely the Skorokhod space equipped with the uniform metric.*

Proof. It follows from the same arguments as in the proofs of Proposition 4.2.4, Proposition 4.2.5, and Theorem 4.2.6. This is because the only difference is that now we have a greater denominator than before (*i.e.* $\tau_n^{(\beta(j))} \geq \tau_n^{(j)}$) which changes neither the logic of the arguments nor the computations. This is because in our framework we do not need that $\mathbb{E}\left[\left(\frac{\Delta_i^n G^{(k)}}{\tau_n^{(k)}}\right)^2\right] = 1$. This was different for the previous literature where the equality was needed in order to use Theorem 2.7.7 in combination with Theorem 2.7.8 of [NP12]. □

Remark 4.2.9. Notice that while the larger value of the $\tau_n^{(\beta(j))}$ does not trigger any modification in the proof of the results, it may reduce some of the components of $\mathbf{D}^{(\beta)}$ to zero. For example if we assume that $\mathbb{E}\left[\left(\frac{\Delta_i^n G^{(j)}}{\tau_n^{(j)}}\right)^2\right] < C$, where $C > 0$, and that $\lim_{n\to\infty}\frac{\tau_n^{(j)}}{\tau_n^{(\beta(j))}} = 0$, for every $j = 1, ..., p$, then all the components of $\mathbf{D}^{(\beta)}$ reduce to zero. This is because by stationarity

$$r_{i,j}^{(\beta),(n)}(k) = \mathbb{E}\left[\frac{\Delta_1^n G^{(i)}}{\tau_n^{(\beta(i))}}\frac{\Delta_{1+k}^n G^{(j)}}{\tau_n^{(\beta(j))}}\right] \leq \mathbb{E}\left[\left(\frac{\Delta_1^n G^{(i)}}{\tau_n^{(\beta(i))}}\right)^2 + \left(\frac{\Delta_{1+k}^n G^{(j)}}{\tau_n^{(\beta(j))}}\right)^2\right] = \frac{\tau_n^{(i)}}{\tau_n^{(\beta(i))}} + \frac{\tau_n^{(j)}}{\tau_n^{(\beta(j))}} \to 0.$$

Example 4.2.10. In this example, let $\{\mathbf{G}_t\}_{t\in[0,T]}$ be a Gaussian core. Consider a partition of the set $\{1, ..., p\}$ and call its elements $I_{\alpha_1}, ..., I_{\alpha_v}$ for some $v \in \mathbb{N}$. Hence, $I_{\alpha_h} \subset \{1, ..., p\}$ for $h = 1, ..., v$, and $I_{\alpha_h} \cap I_{\alpha_l} = \emptyset$ for $h, l = 1, ..., v$ with $h \neq l$. For $h = 1, .., v$, define

$$\tau_n^{(\alpha_h)} := \sqrt{\mathbb{E}\left[\left(\sum_{i\in I_{\alpha_h}}\Delta_1^n G^{(i)}\right)^2\right]}, \tag{4.3}$$

and consider $\frac{\Delta_l^n G^{(j)}}{\tau_n^{(\beta(j))}}$ where $\beta(j) := \alpha_h$ when $j \in I_{\alpha_h}$. In addition, assume that, for $h = 1, ..., v$, the $\mathcal{F}_t$-Brownian measures $W^{(i)}$ are *independent* for $i \in I_{\alpha_h}$. This means that $\mathbb{E}[W^{(i)}W^{(j)}] = 0$ if $i, j \in I_{\alpha_h}$ for some $h = 1, ..., v$. Then $\tau_n^{(\beta(j))} \geq \tau_n^{(j)}$ for $j = 1, .., p$, where $\tau_n^{(j)}$ has been defined in (4.2) and, thus, we can apply Theorem 4.2.8.

4.3 Joint CLT for the multivariate BSS process

In this section we will present and prove our main results consisting of the joint central limit theorem for the two types of multivariate BSS processes. First, we will present the CLT for the scaling factor τ used in the literature (*i.e* Case I) and then the CLT for the new formulation (*i.e.* Case II). For Case II we have two scenarios depending on which multivariate extension of the univariate BSS process we consider (see Definition 4.1.3). In this and in the next two sections we will use a multivariate version of the continuous mapping theorem applied to stable convergence (for reference see [AE78]). Moreover, we will adopt the following three assumptions. These are the analogues of Assumption (CLT) and conditions in Section 4.1 in [BCD11], and Assumptions 2.1, 2.2, 4.1 and 4.2 in [GR19]. The only difference is that they also focus on the convergence of the autocorrelations $r^{(n)}$, which for the sake of brevity and clarity of exposition we decided not to focus on in this work. Let $\Delta_n := 1/n$.

Assumption 4.3.1. *For $k, l = 1, ..., p$, let $g^{(k,l)}$ be differentiable on $(0, \infty)$ with derivative $(g^{(k,l)})' \in L^2((\epsilon, \infty))$ for all $\epsilon > 0$ and $((g^{(k,l)})')^2$ be non-increasing in $[b^{(k,l)}, \infty)$ for some $b^{(k,l)} > 0$. Let $\sigma^{(k,l)}$ have $\alpha^{(k,l)}$-Hölder continuous sample paths, for $\alpha^{(k,l)} \in \left(\frac{1}{2}, 1\right)$. Define*

$$\pi_n^{(k,l)}(A) := \frac{\int_A \left(g^{(k,l)}(s + \Delta_n) - g^{(k,l)}(s)\right)^2 ds}{\int_0^\infty \left(g^{(k,l)}(s + \Delta_n) - g^{(k,l)}(s)\right)^2 ds}, \quad \text{where} \quad A \in \mathcal{B}(\mathbb{R}).$$

We impose that there exists a constant $\lambda^{(k,l)} < -1$ such that for any $\epsilon_n = O\left(n^{-\kappa}\right)$, $\kappa \in (0, 1)$, we have

$$\pi_n^{(k,l)}((\epsilon_n, \infty)) = O(n^{\lambda^{(k,l)}(1-\kappa)}).$$

In the next sections we are going to present different cases. Each case will have particular τ_n and $r^{(n)}$, but the underlying assumptions will have the same structure. Hence, for the last two assumptions we are going to use the variables τ_n and $r^{(n)}$, whose specific form will not be introduced here, but instead it will be specified in the context where the assumptions are used. In other words, we prefer to have a general form for these two assumptions (with an unspecified τ_n and $r^{(n)}$) in order to avoid repeating the same assumptions with different τ_n and $r^{(n)}$ for each case.

Assumption 4.3.2. *Let the limit $\lim_{n\to\infty} r^{(n)}(k)$ exist for any $k \in \mathbb{N}$ and let $(\xi(k))_{k\in\mathbb{N}}$ be a sequence such that, for any $k, n \in \mathbb{N}$, $\left(r^{(n)}(k)\right)^2 \leq \xi(k)$ and $\sum_{k=1}^\infty \xi(k) < \infty$.*

Assumption 4.3.3. *Let $k, l, m = 1, ..., p$. Let*

$$\sup_{s\in(-\infty,T]} \mathbb{E}\left[\left(\sigma_s^{(l,m)}\right)^2\right] < \infty.$$

Moreover, for any $t > 0$ let

$$\int_1^\infty \left((g^{(k,l)})'(s)\right)^2 \left(\sigma_{t-s}^{(l,m)}\right)^2 ds < \infty.$$

Let us discuss the intuition behind these assumptions. First of all Assumption 4.3.2 is the analogue of Assumption 4.2.2 and Assumption 4.2.7 of the previous section and it concerns the summability of the autocorrelations of the Gaussian core under consideration.
Assumption 4.3.1 regards the behaviour of the deterministic kernel g. In the first sentence we are not imposing that $(g^{(m,l)})' \in L^2((0,\infty))$ in order to allow the theory to be applicable also outside the semimartingale case. Moreover, the intuition behind the conditions on $\pi_n^{(k,l)}$ is that without them the increments $\Delta_i^n Y^{(k)}$ might contain substantial information about the volatility far outside of the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ $\left(i.e.\ [(i-1)\Delta_n, i\Delta_n]\right)$ potentially leading to different stochastic limit. In other words, the mass of $\pi_n^{(k,l)}$ is more and more concentrated toward 0 as n increases.

Further, it is shown in Section 4.3 in [BCD11] in the univariate case that Assumptions 4.3.1 and 4.3.2 are for example satisfied in two cases. These cases can be easily extended to our multivariate framework since the Assumptions 4.3.1 and 4.3.2 apply to each $g^{(k,l)}$ where k and l are fixed. The first case consist of the function $g^{(k,l)}(x) = x^\delta \mathbf{1}_{(0,1]}(x)$, $x > 0$, for $\delta \in \left(-\frac{1}{2}, 0\right)$. The second one consist of the function $g^{(k,l)}(x) = x^\delta L_g(x)$ with $(g^{(k,l)})'(x) = x^{\delta-1} L_{g'}(x)$, where L_g and $L_{g'}$ are two continuous functions on $(0,\infty)$ and slowly varying at 0, under the assumption that $g^{(k,l)} \in L^2((0,\infty))$, $(g^{(k,l)})' \in L^2((\epsilon,\infty))$ for any $\epsilon > 0$, $(g^{(k,l)})$ is non-increasing on (b,∞) for some $b > 0$, and $\delta \in \left(-\frac{1}{2}, 0\right)$.

In many situations the condition of Assumption 4.3.3 is satisfied, for example when the stochastic volatilities are second order stationary or more generally when $\sup_{s\in(-\infty,T]} \mathbb{E}\left[\left(\sigma_s^{(l,m)}\right)^2\right] < \infty$. Concerning the second condition, it is satisfied when we assume a fast decay to zero of the derivative of the deterministic kernel $(g^{(k,l)})'(s)$ when s goes to infinity.

In Section 4.5 we will show that these assumptions are going to be satisfied for an important and practical case which is found in many real world applications: the gamma kernel. Finally, since the assumptions of this section and of [BCD11] and [GR19] are similar, it is suggested to see Section 4.3 of [BCD11] and Section 2.1 in [GR19] for further discussions on the assumptions.

4.3.1 Case I

We will start with some preliminaries, where we focus on the bivariate case in order to simplify the exposition. The notations for higher dimensions are analogously the same. Consider the stochastic process $\{\mathbf{Y}_t\}_{t\in[0,T]}$ defined as

$$\mathbf{Y}_t := \begin{pmatrix} Y_t^{(1)} \\ Y_t^{(2)} \end{pmatrix} = \int_{-\infty}^t \begin{pmatrix} g^{(1,1)}(t-s) & g^{(1,2)}(t-s) \\ g^{(2,1)}(t-s) & g^{(2,2)}(t-s) \end{pmatrix} \begin{pmatrix} \sigma_s^{(1,1)} & \sigma_s^{(1,2)} \\ \sigma_s^{(2,1)} & \sigma_s^{(2,2)} \end{pmatrix} \begin{pmatrix} dW_s^{(1)} \\ dW_s^{(2)} \end{pmatrix} + \begin{pmatrix} U_t^{(1)} \\ U_t^{(2)} \end{pmatrix},$$

where $g^{(i,j)}(\cdot), i,j = 1,2$ are deterministic functions and $W^{(1)}, W^{(2)}$ are two (possibly dependent) $\mathcal{F}_t$-adapted jointly Brownian measures on $\mathbb{R}$. From a modelling point of view the dependence of the Brownian measures is not very important since it is always possible to shift it from the

Brownian measure to the stochastic volatilities by just rewriting the latter. We have

$$Y_t^{(1)} = \int_{-\infty}^{t} g^{(1,1)}(t-s)\sigma_s^{(1,1)} + g^{(1,2)}(t-s)\sigma_s^{(2,1)} dW_s^{(1)}$$

$$+ \int_{-\infty}^{t} g^{(1,1)}(t-s)\sigma_s^{(1,2)} + g^{(1,2)}(t-s)\sigma_s^{(2,2)} dW_s^{(2)} + U_t^{(1)}, \quad \text{and}$$

$$Y_t^{(2)} = \int_{-\infty}^{t} g^{(2,1)}(t-s)\sigma_s^{(1,1)} + g^{(2,2)}(t-s)\sigma_s^{(2,1)} dW_s^{(1)}$$

$$+ \int_{-\infty}^{t} g^{(2,1)}(t-s)\sigma_s^{(1,2)} + g^{(2,2)}(t-s)\sigma_s^{(2,2)} dW_s^{(2)} + U_t^{(2)}.$$

Let us define, for $k, r, m = 1, 2$,

$$\Delta_i^n Z^{(k,r,m)} := \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(k,r)}(i\Delta_n - s)\sigma_s^{(r,m)} dW_s^{(m)} + \int_{-\infty}^{(i-1)\Delta_n} \Delta_i^n g^{(k,r)}(s)\sigma_s^{(r,m)} dW_s^{(m)},$$

where we have that $\Delta_i^n g^{(k,l)}(s) := g^{(k,l)}(i\Delta_n - s) - g^{(k,l)}((i-1)\Delta_n - s)$. Then, we have

$$\Delta_i^n Y^{(k)} = \Delta_i^n Z^{(k,1,1)} + \Delta_i^n Z^{(k,2,1)} + \Delta_i^n Z^{(k,1,2)} + \Delta_i^n Z^{(k,2,2)} + \Delta_i^n U^{(k)}.$$

For $m, r, k = 1, 2$, let $G_t^{(k,r;m)} := \int_{-\infty}^{t} g^{(k,r)}(t-s) dW_s^{(m)}$. Hence,

$$\Delta_i^n G^{(k,r;m)} = \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(k,r)}(i\Delta_n - s) dW_s^{(m)} + \int_{-\infty}^{(i-1)\Delta_n} \Delta_i^n g^{(k,r)}(s) dW_s^{(m)}.$$

Further, let $\tau_n^{(k,r)} := \sqrt{\mathbb{E}\left[\left(\Delta_1^n G^{(k,r;m)}\right)^2\right]}$ and $r_{k,r,m;l,q,w}^{(n)}(h) := \mathbb{E}\left[\frac{\Delta_1^n G^{(k,r;m)}}{\tau_n^{(k,r)}} \frac{\Delta_{1+h}^n G^{(l,q;w)}}{\tau_n^{(l,q)}}\right]$.

The Gaussian core $\{\mathbf{G}_t\}_{t\in[0,T]}$ is here given by $\mathbf{G}_t = \left(G_t^{(k,r;m)}\right)_{m,k,r=1,2}$, and it is a Gaussian process since $W^{(1)}$ and $W^{(2)}$ are jointly Gaussian. We work with the separable Hilbert space $\mathcal{H}$ generated by the jointly Gaussian random variables $\left(\frac{\Delta_1^n G^{(k,r;m)}}{\tau_n^{(k,r)}}\right)_{n\geq 1, 1\leq l\leq \lfloor nt\rfloor, k,r,m\in\{1,2\}}$. Further, observe that $\mathbb{E}\left[\frac{\Delta_1^n G^{(k,r;m)}}{\tau_n^{(k,r)}} \frac{\Delta_{1+h}^n G^{(l,q;w)}}{\tau_n^{(l,q)}}\right] = \mathbb{E}\left[\frac{\Delta_i^n G^{(k,r;m)}}{\tau_n^{(k,r)}} \frac{\Delta_{i+h}^n G^{(l,q;w)}}{\tau_n^{(l,q)}}\right]$ for any $i = 1, ..., \lfloor nt\rfloor$. Before presenting the CLT, we introduce an assumption, similar to assumption (4.10) in [BCD11], that controls the asymptotic behaviour of the drift process. In particular, it states that the drift process has to be sufficiently smooth so that the sum of its increments does not grow too fast as $n \to \infty$.

Assumption 4.3.4. *Let* $\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt\rfloor} \frac{\Delta_i^n Z^{(k,r,m)}}{\tau_n^{(k,r)}} \Delta_i^n U^{(l)} \overset{u.c.p.}{\to} 0$ *and* $\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt\rfloor} \Delta_i^n U^{(k)} \Delta_i^n U^{(l)} \overset{u.c.p.}{\to} 0$ *for any* $k, l, r, m = 1, ..., p$.

Theorem 4.3.5. *Under Assumptions 4.3.1, 4.3.2 and 4.3.3 applied to* $\tau_n^{(l,q)}, r_{k,r;l,q}^{(n)}$ *for* $l, q, k, r = 1, ..., p$, *and Assumption 4.3.4, we have the following stable convergence.*

$$\left\{\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\left(\sum_{r,m=1}^{p} \frac{\Delta_i^n Z^{(k,r,m)}}{\tau_n^{(k,r)}} + \Delta_i^n U^{(k)}\right)\left(\sum_{q,w=1}^{p} \frac{\Delta_i^n Z^{(l,q,w)}}{\tau_n^{(l,q)}} + \Delta_i^n U^{(l)}\right)\right.\right.$$

$$- \sum_{r,m,q,w=1}^{p} \mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\tau_n^{(k,r)}} \frac{\Delta_1^n G^{(l,q,w)}}{\tau_n^{(l,q)}}\right] \int_0^t \sigma_s^{(r,m)} \sigma_s^{(q,w)} ds \Bigg]_{k,l=1,...,p} \Bigg\}_{t\in[0,T]} \xrightarrow{st} \left\{ \int_0^t V_s \boldsymbol{D}^{1/2} dB_s \right\}_{t\in[0,T]}$$

in $\mathcal{D}([0,T], \mathbb{R}^{p^2})$*, where* $\boldsymbol{D}$ *and* V_s *are introduced in Appendix A, and* B_s *is a* p^6*-dimensional Brownian motion.*

Remark 4.3.6. As all the other CLTs and WLLN in this section, this result is not feasible because the scaling factor $\tau_n^{(k,r)}$, for $k, r = 1, ..., p$, depends on the Gaussian core $\mathbf{G}_t$ which is not observable, and, so, this result cannot be computed directly from the data. Note also that in Theorem 4.3.5 and Corollary 4.3.7 the increments $\Delta_i^n Z^{(k,r,m)}$ appear, which are not observable either.

It is possible to obtain a *vech* formulation of our results, thus reducing their dimensions without losing any information. This is possible because of the symmetry of our object of study, that is there is no difference between $\frac{\Delta_i^n G^{(k,r,m)}}{\tau_n^{(k,r)}} \frac{\Delta_i^n G^{(l,q,w)}}{\tau_n^{(l,q)}}$ and $\frac{\Delta_i^n G^{(l,q,w)}}{\tau_n^{(l,q)}} \frac{\Delta_i^n G^{(k,r,m)}}{\tau_n^{(k,r)}}$ and between $\frac{\Delta_i^n Z^{(k,r,m)}}{\tau_n^{(k,r)}} \frac{\Delta_i^n Z^{(l,q,w)}}{\tau_n^{(l,q)}}$ and $\frac{\Delta_i^n Z^{(l,q,w)}}{\tau_n^{(l,q)}} \frac{\Delta_i^n Z^{(k,r,m)}}{\tau_n^{(k,r)}}$. Hence, we have the following formulation of our results.

Corollary 4.3.7. *Under Assumptions 4.3.1, 4.3.2 and 4.3.3 applied to* $\tau_n^{(l,q)}, r_{k,r;l,q}^{(n)}$ *for* $l, q, k, r = 1, ..., p$*, and Assumption 4.3.4, we have the following stable convergence.*

$$\left\{ \sqrt{n} \left[\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\sum_{r,m=1}^{p} \frac{\Delta_i^n Z^{(k,r,m)}}{\tau_n^{(k,r)}} + \Delta_i^n U^{(k)} \right) \left(\sum_{q,w=1}^{p} \frac{\Delta_i^n Z^{(l,q,w)}}{\tau_n^{(l,q)}} + \Delta_i^n U^{(l)} \right) \right.$$

$$\left. - \sum_{r,m,q,w=1}^{p} \mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\tau_n^{(k,r)}} \frac{\Delta_1^n G^{(l,q,w)}}{\tau_n^{(l,q)}}\right] \int_0^t \sigma_s^{(r,m)} \sigma_s^{(q,w)} ds \Bigg]_{k=1,...,p; l\leq k} \Bigg\}_{t\in[0,T]} \xrightarrow{st} \left\{ \int_0^t V_s \boldsymbol{D}^{1/2} dB_s \right\}_{t\in[0,T]}$$

in $\mathcal{D}([0,T], \mathbb{R}^{\frac{p}{2}(p+1)})$*, where* $\boldsymbol{D}$ *and* V_s *are introduced in Appendix A, and* B_s *is a* $\frac{p^5}{2}(p+1)$*-dimensional Brownian motion.*

Proof. It follows from the same arguments as the ones used in the proof of Theorem 4.3.5. Indeed, in Theorem 4.3.5 we had $k, l = 1, ..., p$, but now we have that $k = 1, .., p$ and $l \leq k$. As before, this does change the dimension of the objects considered (in particular see the matrix $\boldsymbol{D}$ and V_s in Appendix A), but not the logic of the arguments. □

Remark 4.3.8. Notice that $\frac{p^5}{2}(p+1)$ comes from $p^4(p^2 - \sum_{j=1}^{p-1} j)$ where $\sum_{j=1}^{p-1} j$ indicates that we are not considering the strictly upper (or lower) triangular elements of the $p \times p$ matrix. Moreover, for the remaining sections and subsections we will always adopt the vech version of our results.

4.3.2 Case II: first scenario

Despite the process $\mathbf{Y}_t$ being the same as in the previous section, we introduce a new formulation for the τ. This formulation is in line with the one presented in Section 4.2.2. Furthermore, in this section we present and prove the results for one of the two versions of the multivariate BSS process introduced in Definition 4.1.3. In the next section we will do the same for the other version.

Consider the stochastic process $\{\mathbf{Y}_t\}_{t\in[0,T]} = \{(Y_t^{(1)},...,Y_t^{(p)})\}_{t\in[0,T]}$ given by

$$\mathbf{Y}_t = \int_{-\infty}^{t} \begin{pmatrix} g^{(1,1)}(t-s) & \cdots & g^{(1,p)}(t-s) \\ \vdots & \ddots & \vdots \\ g^{(p,1)}(t-s) & \cdots & g^{(p,p)}(t-s) \end{pmatrix} \begin{pmatrix} \sigma_s^{(1,1)} & \cdots & \sigma_s^{(1,p)} \\ \vdots & \ddots & \vdots \\ \sigma_s^{(p,1)} & \cdots & \sigma_s^{(p,p)} \end{pmatrix} \begin{pmatrix} dW_s^{(1)} \\ \vdots \\ dW_s^{(p)} \end{pmatrix} + \begin{pmatrix} U_t^{(1)} \\ \vdots \\ U_t^{(p)} \end{pmatrix}. \quad (4.4)$$

Assume that the $\mathcal{F}_t$-Brownian measures are all *independent* of each other. The Gaussian core is given by $\{\mathbf{G}_t\}_{t\in[0,T]}$ with

$$\mathbf{G}_t = \left(G_t^{(k,r,m)}\right)_{k,r,m=1,...,p} = \left(\int_0^t g^{(k,r)}(t-s)dW_s^{(m)}\right)_{k,r,m=1,...,p}.$$

Let us define, for $k = 1,...,p$, the sequence $(\bar{\tau}_n^{(k)})_{n\in\mathbb{N}}$ such that, for each $n \in \mathbb{N}$, $\max_{r=1,...,p} \tau_n^{(k,r)} = O\left(\bar{\tau}_n^{(k)}\right)$. For example, we may assume that $\bar{\tau}_n^{(k)} := \sqrt{\mathbb{E}\left[\left(\sum_{r=1}^p \Delta_1^n G_t^{(k,r,r)}\right)^2\right]}$ or that $\bar{\tau}_n^{(k)} := \max_{r=1,...,p}\sqrt{\mathbb{E}\left[\left(\Delta_1^n G_t^{(k,r,r)}\right)^2\right]}$. Notice that in the previous expressions we focused on (k,r,r) instead of (k,r,m) this is because by the independence of the Brownian measures we obtain that $\mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_{1+h}^n G^{(l,q,w)}}{\bar{\tau}_n^{(l)}}\right] = 0$ whenever $m \neq w$, hence it is sufficient to define the scaling factors just in terms of (k,r,r). Moreover, let for $h \in \mathbb{N}$, $\bar{r}_{k,r,m;l,q,w}^{(n)}(h) := \mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_{1+h}^n G^{(l,q,w)}}{\bar{\tau}_n^{(l)}}\right]$, and observe that if $m \neq w$ then $\bar{r}_{k,r,m;l,q,w}^{(n)}(h) = 0$.

Assumption 4.3.9. *Let* $\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n(Y^{(k)}-U^{(k)})}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n U^{(l)}}{\bar{\tau}_n^{(l)}} \xrightarrow{u.c.p.} 0$ *and* $\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n U^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n U^{(l)}}{\bar{\tau}_n^{(l)}} \xrightarrow{u.c.p.} 0$ *for any* $k,l = 1,...,p$.

Observe that the above assumption is similar to Assumption 4.3.4. The only difference is that now the increments of the drift are divided by the scaling factor due to the different formulation of the theorems. Indeed, now the process Y_t contains the drift U_t.

Theorem 4.3.10. *Under Assumptions 4.3.1, 4.3.2 and 4.3.3 applied to* $\bar{\tau}_n^{(l)}, \bar{r}_{k,r,m;l,q,m}^{(n)}$ *for* $l,q,k,r,m = 1,...,p$, *and Assumption 4.3.9, we have the following stable convergence.*

$$\left\{\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} - \sum_{r,m,q=1}^{p}\mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_1^n G^{(l,q,m)}}{\bar{\tau}_n^{(l)}}\right]\int_0^t \sigma_s^{(r,m)}\sigma_s^{(q,m)}ds\right]_{k=1,...,p;l\leq k}\right\}_{t\in[0,T]}$$

$$\xrightarrow{st}\left\{\int_0^t V_s \boldsymbol{D}^{1/2}dB_s\right\}_{t\in[0,T]}$$

in $\mathcal{D}([0,T],\mathbb{R}^{\frac{p}{2}(p+1)})$, *where* $\boldsymbol{D}$ *and* V_s *are introduced in Appendix A, and* B_s *is a* $\frac{p^5}{2}(p+1)$*-dimensional Brownian motion.*

Proof. It follows from the same arguments as the ones used in the proof of Theorem 4.3.5, the only difference is the use of Theorem 4.2.8 instead of Theorem 4.2.6. In particular, the elements that converge to zero (which we called A_n in the proof of Theorem 4.3.5) still converge to zero

since we are using a larger denominator. For the other elements (which we called C_n) we do have the stated convergence due to Theorem 4.2.8. □

4.3.3 Case II: second scenario

In this section we present and prove the results for the other version of the multivariate BSS process introduced in Definition 4.1.3. In addition, at the price of a simple assumption on the stochastic volatilities σs, this new form allows for a definition of τ_n in terms of the BSS process $\{\mathbf{X}_t\}_{t\in[0,T]}$ (see below) and not in terms of Gaussian process $\{\mathbf{G}_t\}_{t\in[0,T]}$.

Consider the stochastic process $\{\mathbf{X}_t\}_{t\in[0,T]} = \{(X_t^{(1)},...,X_t^{(p)})\}_{t\in[0,T]}$ given by

$$\mathbf{X}_t = \int_{-\infty}^{t} \begin{pmatrix} g^{(1,1)}(t-s)\sigma_s^{(1,1)} & \cdots & g^{(1,p)}(t-s)\sigma_s^{(1,p)} \\ \vdots & \ddots & \vdots \\ g^{(p,1)}(t-s)\sigma_s^{(p,1)} & \cdots & g^{(p,p)}(t-s)\sigma_s^{(p,p)} \end{pmatrix} \begin{pmatrix} dW_t^{(1)} \\ \vdots \\ dW_t^{(p)} \end{pmatrix} + \begin{pmatrix} U_t^{(1)} \\ \vdots \\ U_t^{(p)} \end{pmatrix}. \tag{4.5}$$

Assume that the $\mathcal{F}_t$-Brownian measures are all *independent* of each other and of the drift process, and that the volatilities, $\sigma_s^{(k,m)}$ for any $k,m=1,...,p$, are second order stationary. Let us define, for $k=1,...,p$, $\tilde{\tau}_n^{(k)} := \sqrt{\mathbb{E}\left[\left(\Delta_1^n X^{(k)}\right)^2\right]}$. Following Example 4.2.10, it is possible to observe that $\tilde{\tau}_n^{(k)}$ is similar to $\tau_n^{(\alpha_h)}$ introduced in (4.3) (except for the drift component). The Gaussian core is given by $\{\mathbf{G}_t\}_{t\in[0,T]}$ with $\mathbf{G}_t = \left(G_t^{(k,m)}\right)_{k,m=1,...,p} = \left(\int_0^t g^{(k,m)}(t-s)dW_s^{(m)}\right)_{k,m=1,...,p}$ and the partition is in the k variables, namely we split $\{1,...,p\}^2$ into $\{1\}\times\{1,...,p\},...,\{p\}\times\{1,...,p\}$. Notice that for each element of the partition the associated Brownian measures are independent of each other. Further, we define $\tilde{r}_{k,m;l,w}^{(n)}(h) := \mathbb{E}\left[\frac{\Delta_1^n G^{(k,m)}}{\tilde{\tau}_n^{(k)}}\frac{\Delta_{1+h}^n G^{(l,w)}}{\tilde{\tau}_n^{(l)}}\right]$; observe that if $m\neq w$ then $\tilde{r}_{k,m;l,w}^{(n)}(h)=0$ and that the rate of $\tilde{\tau}_n^{(k)}$ is greater than or equal to the order of $\tau_n^{(k,r)}$ $n\to\infty$ for any $r=1,..,p$.

Theorem 4.3.11. *Under Assumptions 4.3.1, 4.3.2 and 4.3.3 applied to $\tilde{\tau}_n^{(l)}, \tilde{r}_{k,m;l,m}^{(n)}$ and Assumption 4.3.9 applied to $X^{(k)}$ for $l,k,m=1,...,p$, we have the following stable convergence.*

$$\left\{\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n X^{(k)}}{\tilde{\tau}_n^{(k)}}\frac{\Delta_i^n X^{(l)}}{\tilde{\tau}_n^{(l)}} - \sum_{m=1}^{p}\mathbb{E}\left[\frac{\Delta_1^n G^{(k,m)}}{\tilde{\tau}_n^{(k)}}\frac{\Delta_1^n G^{(l,m)}}{\tilde{\tau}_n^{(l)}}\right]\int_0^t \sigma_s^{(k,m)}\sigma_s^{(l,m)}ds\right]_{k=1,...,p;l\leq k}\right\}_{t\in[0,T]}$$

$$\xrightarrow{st}\left\{\int_0^t V_s \boldsymbol{D}^{1/2}dB_s\right\}_{t\in[0,T]}$$

in $\mathcal{D}([0,T],\mathbb{R}^{\frac{p}{2}(p+1)})$, where $\boldsymbol{D}$ and V_s are introduced in Appendix A and B_s is a $\frac{p^3}{2}(p+1)$-dimensional Brownian motion.

Proof. It follows from the arguments of Theorem 4.3.5 and the results of Theorem 4.2.8. In particular, in the present case we have that the converging element is given by

$$C_n = \frac{1}{\sqrt{n}}\left(\sum_{j=1}^{\lfloor ht\rfloor}\sum_{i\in I_{h,n}(j)}\sum_{m=1}^{p}\left(\frac{\Delta_i^n G^{(k,m)}}{\tilde{\tau}_n^{(k)}}\frac{\Delta_i^n G^{(l,m)}}{\tilde{\tau}_n^{(l)}}\right.\right.$$

$$-\mathbb{E}\left[\frac{\Delta_i^n G^{(k,m)}}{\tilde{\tau}_n^{(k)}}\frac{\Delta_i^n G^{(l,m)}}{\tilde{\tau}_n^{(l)}}\right]\Big)\sigma^{(r,m)}_{(j-1)\Delta_h}\sigma^{(q,m)}_{(j-1)\Delta_h}\Bigg)_{k,l=1,...,p;l\leq k}.$$

□

Remark 4.3.12. It is possible to obtain a similar result to Theorem 4.3.11 using a formulation for the τs which is not written in terms of the σs, (hence without the assumption of the independence of the Brownian measures and on the expected squared value of the σs required for Theorem 4.3.11). This could be obtained by proceeding as we have done in the previous section (*i.e.* Case II first scenario). So why did we introduce this formulation of the τs? The reason comes from the novel possibility, provided by the multidimensional structure of the BSS process presented in this section (see (4.5)), to write τ directly in terms of X. The benefit of doing this is that if we have an estimate of $\mathbb{E}\left[\left(\Delta_1^n X^{(k)}\right)^2\right]$ then Theorem 4.3.11 becomes a feasible central limit theorem.

4.3.4 Weak laws of large numbers

From the central limit theorems proved in the previous sections it is possible to derive the weak law of large numbers (WLLN). First, we present the following lemma which follows from the definition of uniform convergence in probability.

Lemma 4.3.13. *Consider p real valued stochastic processes $\{H_t^{(1)}\}_{t\in[0,T]},...,\{H_t^{(p)}\}_{t\in[0,T]}$. Further, consider sequences of random variables $\{H_t^{(i),n}\}_{t\in[0,T]}$ such that $H_t^{(i),n} \overset{u.c.p.}{\to} H_t^{(i)}$, for $i = 1,...,p$ and for any $t\in[0,T]$, then $(H_t^{(1),n},...,H_t^{(p),n}) \overset{u.c.p.}{\to} (H_t^{(1)},...,H_t^{(p)})$.*

Proof. Consider the case $p=2$, since for $p>2$ the proof uses exactly the same arguments. Let $t\in[0,T]$ and let $|\cdot|$ denotes the Euclidean norm, we have

$$\mathbb{P}\left(\sup_{s\in[0,t]}|(H_s^{(1),n},H_s^{(2),n})-(H_s^{(1)},H_s^{(2)})|>\epsilon\right)\leq\mathbb{P}\left(\sup_{s\in[0,t]}|H_s^{(1),n}-H_s^{(1)}|+|H_s^{(2),n}-H_s^{(2)}|>\epsilon\right)$$

$$\leq\mathbb{P}\left(\sup_{s\in[0,t]}|H_s^{(1),n}-H_s^{(1)}|>\frac{\epsilon}{2}\right)+\mathbb{P}\left(\sup_{s\in[0,t]}|H_s^{(2),n}-H_s^{(2)}|>\frac{\epsilon}{2}\right)\to 0,\quad \text{as } n\to\infty.$$

□

We will derive the WLLN for the first scenario of Case II and point out that using the same arguments it is possible to obtain similar results for all the CLTs presented in this chapter.
Let $\bar{r}_{k,r,m;l,q,w}(h) := \lim_{n\to\infty}\bar{r}^{(n)}_{k,r,m;l,q,w}(h)$ and $\tilde{r}_{k,m;l,w}(h) := \lim_{n\to\infty}\tilde{r}^{(n)}_{k,m;l,w}(h)$, for $k,r,m,l,q,w = 1,...,p$.

Theorem 4.3.14. *Let the assumptions of Theorem 4.3.10 hold. Then we have*

$$\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}\right)_{k=1,...,p;l\leq k}\overset{u.c.p.}{\to}\left(\sum_{r,m,q=1}^{p}\bar{r}_{k,r,m;l,q,m}(0)\int_0^t\sigma_s^{(r,m)}\sigma_s^{(q,m)}ds\right)_{k=1,...,p;l\leq k}.$$

Proof. Fix $l, k \in \{1, ..., p\}$. From Theorem 4.3.10 we have that

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\tau_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\tau_n^{(l)}} - \sum_{r,m,q=1}^{p}\mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_1^n G^{(l,q,m)}}{\bar{\tau}_n^{(l)}}\right]\int_0^t \sigma_s^{(r,m)}\sigma_s^{(q,m)}ds\right]$$

converges in distribution. Now, by Slutsky's theorem we have that

$$\left[\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\tau_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\tau_n^{(l)}} - \sum_{r,m,q=1}^{p}\mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_1^n G^{(l,q,m)}}{\bar{\tau}_n^{(l)}}\right]\int_0^t \sigma_s^{(r,m)}\sigma_s^{(q,m)}ds\right] \xrightarrow{d} 0,$$

$$\Rightarrow \left[\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\tau_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\tau_n^{(l)}} - \sum_{r,m,q=1}^{p}\mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_1^n G^{(l,q,m)}}{\bar{\tau}_n^{(l)}}\right]\int_0^t \sigma_s^{(r,m)}\sigma_s^{(q,m)}ds\right] \xrightarrow{P} 0.$$

Then, by the triangular inequality it follows that

$$\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\tau_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\tau_n^{(l)}} \xrightarrow{P} \sum_{r,m,q=1}^{p}\bar{r}_{k,r,m;l,q,m}(0)\int_0^t \sigma_s^{(r,m)}\sigma_s^{(q,m)}ds.$$

The uniform convergence in probability comes from Remark 4.25 of [Gr17], while the joint uniform convergence follows from Lemma 4.3.13. □

Remark 4.3.15. Similar weak laws of large numbers corresponding to all the others central limit theorems presented in this chapter, including the ones for the multivariate Gaussian processes with stationary increments, can be derived using the same arguments as above.

4.4 Feasible results

In all the limit theorems presented above, we considered scaling of the increments of the corresponding processes X, Y or G by τ_n. However, in empirical applications τ_n would not be known, which makes the limit theorems infeasible in the sense that they are not computable from empirical data. We now want to move on to derive related feasible results which can be implemented in empirical applications. To this end we somehow need to get rid off the τ_ns. A natural way of doing this is to consider suitable ratios of the statistics considered above so that the scaling parameters cancel out.

In this section we will focus on two kinds of feasible results, which differ from each other for the type of ratio considered. They are the *correlation ratio* and the *relative covolatility*. For the latter see [BPS14]. We will show feasible results for both scenarios of Case II. Moreover, we will present the results using the *vech* formulation, however similar results hold true for the general formulation. The reason why we focus only on Case II is because for Case I it is not possible to get rid off the scaling factor τ (unless in trivial cases) and, hence, to get feasible limit theorems. This is one of the main benefit of the introduction of Case II.

Remark 4.4.1. From the feasible results developed in this section it is possible to obtain estimates for the mean of our process. Thus, we have "first order" feasible results. It is an open question whether it is possible to obtain "second order" feasible results, namely estimates for the

asymptotic covariance. For the univariate BSS process, this question has been solved for the power variation case in [BPS14], but it still remains open for the multipower variation case.

In this section we will make considerable use of certain random variables, which can be considered as certain types of covariation measures, and in order to simplify the exposition we decided to use the following formulation. For any $l, k = 1, ..., p$, we define

$$\bar{R}_{t,n}^{(k,l)} := \sum_{r,m,q=1}^{p} \mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_1^n G^{(l,q,m)}}{\bar{\tau}_n^{(l)}}\right] \int_0^t \sigma_s^{(r,m)} \sigma_s^{(q,m)} ds,$$

$$\bar{R}_t^{(k,l)} := \sum_{r,m,q=1}^{p} \bar{r}_{k,r,m;l,q,m}(0) \int_0^t \sigma_s^{(r,m)} \sigma_s^{(q,m)} ds,$$

$$\tilde{R}_{t,n}^{(k,l)} := \sum_{m=1}^{p} \mathbb{E}\left[\frac{\Delta_1^n G^{(k,m)}}{\tilde{\tau}_n^{(k)}} \frac{\Delta_1^n G^{(l,m)}}{\tilde{\tau}_n^{(l)}}\right] \int_0^t \sigma_s^{(k,m)} \sigma_s^{(l,m)} ds,$$

$$\text{and} \quad \tilde{R}_t^{(k,l)} := \sum_{m=1}^{p} \tilde{r}_{k,m;l,m}(0) \int_0^t \sigma_s^{(k,m)} \sigma_s^{(l,m)} ds.$$

4.4.1 Correlation ratio

In this section, by a slight abuse of notation, we will consider $\bar{\tau}_n^{(k)} = \sqrt{\mathbb{E}\left[\left(\Delta_1^n Y^{(k)}\right)^2\right]}$, $\bar{r}_{k,m;l,w}^{(n)}(h) = \mathbb{E}\left[\frac{\Delta_1^n G^{(k,m)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_{1+h}^n G^{(l,w)}}{\bar{\tau}_n^{(l)}}\right]$ and $\bar{r}_{k,m;l,w}(h) = \lim_{n\to\infty} \bar{r}_{k,m;l,w}^{(n)}(h)$, for $k, m, l, w = 1, ..., p$. Moreover, observe that, for $k, m = 1, ..., p$,

$$\sum_{r,q=1}^{p} \mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_1^n G^{(k,q,m)}}{\bar{\tau}_n^{(k)}}\right] \int_0^t \sigma_s^{(r,m)} \sigma_s^{(q,m)} ds \geq 0,$$

since it is a quadratic form. Therefore, for $k = 1, ..., p$ we have that $\bar{R}_{t,n}^{(k,k)} \geq 0$ and $\bar{R}_t^{(k,k)} \geq 0$ (and $= 0$ only in the trivial case). The same applies to $\tilde{R}_{t,n}^{(k,k)}$ and $\tilde{R}_t^{(k,k)}$.
Before presenting the main results of this section, we introduce the following lemma, which is a generalisation of the functional delta method, see Chapter 3.9 of [VW96], and of Proposition 2 of [PV10].

Lemma 4.4.2. *Let $\mathbb{D}$ and $\mathbb{E}$ be metrizable topological vector spaces and r_n constants such that $r_n \to \infty$, as $n \to \infty$. Let $\phi : \mathbb{D}_\phi \subset \mathbb{D} \to \mathbb{E}$ be continuous and satisfy*

$$r_n(\phi(\theta_n + r_n^{-1} h_n) - \phi(\theta_n)) \to \phi'(\theta, h), \tag{4.6}$$

for all converging sequences θ_n, h_n with $h_n \to h \in \mathbb{D}_0 \subset \mathbb{D}$, $\theta_n \to \theta \in \mathbb{D}_\phi$ and with $\theta_n + r_n^{-1} h_n \in \mathbb{D}_\phi$ for all n, and some arbitrary map $\phi'(\theta, h)$ from $\mathbb{D}_\phi \times \mathbb{D}_0$ to $\mathbb{E}$. If $Y_n, \bar{Y}_n : \Omega_n \to \mathbb{D}_\phi$ are maps with $Y_n \xrightarrow{P} Y$ and $r_n(Y_n - \bar{Y}_n) \xrightarrow{st} X$, where X is separable and takes its values in $\mathbb{D}_0$, then $r_n\left(\phi(Y_n) - \phi(\bar{Y}_n)\right) \xrightarrow{st} \phi'(Y, X)$.

Proof. Since $\sqrt{n}(Y_n - \bar{Y}_n) \xrightarrow{st} X$ then $|Y_n - \bar{Y}_n| \xrightarrow{P} 0$ and, given that $Y_n \xrightarrow{P} Y$, applying triangular

inequality we can deduce that $\bar{Y}_n \xrightarrow{P} Y$. Hence, we have $(Y_n, \bar{Y}_n) \xrightarrow{P} (Y, Y)$, and using the properties of the stable convergence we have $(\bar{Y}_n, r_n(Y_n - \bar{Y}_n)) \xrightarrow{st} (Y, X)$.
Now, define for each n a map $g_n(\theta_n, h_n) := r_n(\phi(\theta_n + r_n^{-1}h_n) - \phi(\theta_n))$ on $\{(h_n, \theta_n) : \theta_n + r_n^{-1}h_n \in \mathbb{D}_\phi\}$. These maps are continuous in $\mathbb{E}$ since ϕ is continuous and by (4.6) they converge to $\phi'(\theta, h)$. Applying the continuous mapping theorem, we obtain $g_n(\bar{Y}_n, r_n(Y_n - \bar{Y}_n)) \xrightarrow{st} \phi'(Y, X)$, which is our result. □

For the next results, we will use the following lemma.

Lemma 4.4.3. *Let $I \subset \mathbb{R}$ be a compact interval and consider $\mathcal{D}(I, (0, \infty))$, namely the space of càdlàg functions from I to $(0, \infty)$. Then, for any $f \in \mathcal{D}(I, (0, \infty))$ we have that $\inf_{t\in I} f(t) > 0$. Moreover, let $f_n \in \mathcal{D}(I, (0, \infty))$, $n \in \mathbb{N}$, such that $f_n \to f$ in $\mathcal{D}(I, (0, \infty))$, namely $\sup_{t\in I} |f_n(t) - f(t)| \to 0$ as $n \to \infty$. Then, $\inf_{n\in\mathbb{N}} \inf_{t\in I} f_n(t) > 0$.*

Proof. First, recall that for a càdlàg function the left limits must exist, that is for every $t \in I$ we have that $\lim_{s\uparrow t} f(s)$ exists, which means that $\lim_{s\uparrow t} f(s) \in (0, \infty)$. Assume that $\inf_{t\in I} f(t) = 0$, then there exists a sequence $(t_n)_{n\in\mathbb{N}} \subset I$ such that $\lim_{n\to\infty} f(t_n) = 0$ and by compactness there exists a subsequence $(t_{n_k})_{n_k\in\mathbb{N}}$ such that $t_{n_k} \to t \in I$ as $n_k \to \infty$ and so $\lim_{t_{n_k}\to t} f(t_{n_k}) = 0$. Hence, there exists either a subsequence $(t_{n_{k_l}})_{n_{k_l}\in\mathbb{N}}$ of $(t_{n_k})_{n_k\in\mathbb{N}}$ which converges to the left to t and so $\lim_{t_{n_{k_l}}\uparrow t} f(t_{n_{k_l}}) = 0$, or a subsequence $(t_{n_{k_j}})_{n_{k_j}\in\mathbb{N}}$ of $(t_{n_k})_{n_k\in\mathbb{N}}$ which converges to the right to t and so $\lim_{t_{n_{k_j}}\downarrow t} f(t_{n_{k_j}}) = 0$, or both. In all the three cases we have a contradiction. This proves the first statement.

Now, consider the second statement. Let $\inf_{t\in I} f(t) = 2\varepsilon$, for some $\varepsilon > 0$. Since $f_n \to f$ in the uniform metric then there exists a $\tilde{n} \in \mathbb{N}$ s.t. $\forall n \geq \tilde{n}$ we have that $\sup_{t\in I} |f_n(t) - f(t)| < \varepsilon$. This implies that, $\forall n \geq \tilde{n}$, $f_n(t) > f(t) - \varepsilon$ for every $t \in I$ and so that $\inf_{t\in I} f_n(t) \geq \inf_{t\in I} f(t) - \varepsilon = \varepsilon$. Furthermore, $\forall n < \tilde{n}$ we have that $\inf_{t\in I} f_n(t) > 0$ and since $\tilde{n}$ is finite we get that $\inf_{n\in\mathbb{N}} \inf_{t\in I} f_n(t) \geq \min\left(\min_{n<\tilde{n}} \inf_{t\in I} f_n(t), \varepsilon\right) > 0$. □

We can now present the main results of this section: the feasible WLLN and CLT.

Proposition 4.4.4. *Let $\epsilon > 0$. Under the assumptions of Theorem 4.3.10 and for any interval $[\epsilon, T]$,*

$$\left(\frac{\sum_{i=1}^{\lfloor nt\rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)}}{\sqrt{\sum_{i=1}^{\lfloor nt\rfloor} \left(\Delta_i^n Y^{(k)}\right)^2} \sqrt{\sum_{i=1}^{\lfloor nt\rfloor} \left(\Delta_i^n Y^{(l)}\right)^2}}\right)_{k=1,\dots,p; l\leq k} \xrightarrow{u.c.p.} \left(\frac{\bar{R}_t^{(k,l)}}{\sqrt{\bar{R}_t^{(k,k)} \bar{R}_t^{(l,l)}}}\right)_{k=1,\dots,p; l\leq k}. \quad (4.7)$$

Proof. Fix $l, k \in \{1, ..., p\}$ such that $l \leq k$, we have for $n \geq 1/\epsilon$ (the case $n < 1/\epsilon$ is a trivial one and moreover we are concerned with the behaviour as $n \to \infty$)

$$\left(\sum_{i=1}^{\lfloor nt\rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)}\right) \left(\sum_{i=1}^{\lfloor nt\rfloor} \left(\Delta_i^n Y^{(k)}\right)^2 \sum_{i=1}^{\lfloor nt\rfloor} \left(\Delta_i^n Y^{(l)}\right)^2\right)^{-1/2} - \frac{\bar{R}_t^{(k,l)}}{\sqrt{\bar{R}_t^{(k,k)} \bar{R}_t^{(l,l)}}}$$

$$= \left[\left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right) - \bar{R}_t^{(k,l)} \right] \left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \right)^2 \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right)^2 \right)^{-1/2}$$
$$+ \frac{\bar{R}_t^{(k,l)} \left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \right)^2 \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right)^2 \right)^{-1/2}}{\sqrt{\bar{R}_t^{(k,k)} \bar{R}_t^{(l,l)}}}$$
$$\left[\sqrt{\bar{R}_t^{(k,k)} \bar{R}_t^{(l,l)}} - \left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \right)^2 \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right)^2 \right)^{1/2} \right] \xrightarrow{u.c.p.} 0,$$

where for the term in the first square bracket the u.c.p. convergence to zero comes from the fact that we have LLN results (see Theorem 4.3.14). For the second square bracket we have the following. First, we use the continuous mapping theorem knowing the joint convergence in probability and using the continuous function $g(x,y) = \sqrt{xy}$ (notice that in our case x, y are positive). Then, we pass from the convergence in probability to the uniform convergence using the fact that the paths are non-decreasing in time and the paths of the limiting process are continuous almost surely. Concerning the elements outside the square brackets, they do not interfere with the uniform convergence since their suprema are bounded for any $t \in [\epsilon, T]$ (and that is why we have considered $\epsilon > 0$).
Finally, the joint convergence follows from Lemma 4.3.13. □

Proposition 4.4.5. *Under the assumptions of Theorem 4.3.10, we have for any $\epsilon > 0$*

$$\left\{ \sqrt{n} \left(\frac{\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)}}{\sqrt{\sum_{i=1}^{\lfloor nt \rfloor} \left(\Delta_i^n Y^{(k)} \right)^2} \sqrt{\sum_{i=1}^{\lfloor nt \rfloor} \left(\Delta_i^n Y^{(l)} \right)^2}} - \frac{\bar{R}_{t,n}^{(k,l)}}{\sqrt{\bar{R}_{t,n}^{(k,k)} \bar{R}_{t,n}^{(l,l)}}} \right)_{k=1,\dots,p; l \leq k} \right\}_{t \in [\epsilon, T]}$$
$$\xrightarrow{st} \left\{ \left(\frac{1}{\sqrt{\bar{R}_t^{(k,k)} \bar{R}_t^{(l,l)}}} \left(\int_0^t (V_s \boldsymbol{D}^{1/2})_{(k,l)} dB_s^{(k,l)} - \frac{1}{2} \frac{\bar{R}_t^{(k,l)}}{\bar{R}_t^{(k,k)}} \int_0^t (V_s \boldsymbol{D}^{1/2})_{(k,k)} dB_s^{(k,k)} \right. \right.$$
$$\left. \left. - \frac{1}{2} \frac{\bar{R}_t^{(k,l)}}{\bar{R}_t^{(l,l)}} \int_0^t (V_s \boldsymbol{D}^{1/2})_{(l,l)} dB_s^{(l,l)} \right) \right)_{k=1,\dots,p; l \leq k} \Bigg\}_{t \in [\epsilon, T]}$$

in $\mathcal{D}([\epsilon, T], \mathbb{R}^{\frac{p}{2}(p+1)})$, where $(V_s \boldsymbol{D}^{1/2})_{(k,l)}$ indicates that we are considering only the (k,l) row of the matrix $(V_s \boldsymbol{D}^{1/2})$ (see Appendix A) and $B_s^{(k,l)}$ are one-dimensional independent Brownian motions.

Similar results can be obtained for the second scenario of Case II.

Proposition 4.4.6. *Let $\epsilon > 0$. Under the assumptions of Theorem 4.3.11 and for any interval $[\epsilon, T]$,*

$$\left(\frac{\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n X^{(k)} \Delta_i^n X^{(l)}}{\sqrt{\sum_{i=1}^{\lfloor nt \rfloor} \left(\Delta_i^n X^{(k)} \right)^2} \sqrt{\sum_{i=1}^{\lfloor nt \rfloor} \left(\Delta_i^n X^{(l)} \right)^2}} \right)_{k=1,\dots,p; l \leq k} \xrightarrow{u.c.p.} \left(\frac{\tilde{R}_t^{(k,l)}}{\sqrt{\tilde{R}_t^{(k,k)} \tilde{R}_t^{(l,l)}}} \right)_{k=1,\dots,p; l \leq k}.$$

Proof. It follows from the same arguments as the ones used in the proof of Proposition 4.4.4. □

Proposition 4.4.7. *Under the assumptions of Theorem 4.3.11, we have for any $\epsilon > 0$*

$$\left\{ \sqrt{n} \left(\frac{\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n X^{(k)} \Delta_i^n X^{(l)}}{\sqrt{\sum_{i=1}^{\lfloor nt \rfloor} \left(\Delta_i^n X^{(k)}\right)^2} \sqrt{\sum_{i=1}^{\lfloor nt \rfloor} \left(\Delta_i^n X^{(l)}\right)^2}} - \frac{\tilde{R}_{t,n}^{(k,l)}}{\sqrt{\tilde{R}_{t,n}^{(k,k)} \tilde{R}_{t,n}^{(l,l)}}} \right)_{k=1,\dots,p; l \leq k} \right\}_{t \in [\epsilon, T]}$$

$$\xrightarrow{st} \left\{ \left(\frac{1}{\sqrt{\tilde{R}_t^{(k,k)} \tilde{R}_t^{(l,l)}}} \left(\int_0^t (V_s \boldsymbol{D}^{1/2})_{(k,l)} dB_s^{(k,l)} - \frac{1}{2} \frac{\tilde{R}_t^{(k,l)}}{\tilde{R}_t^{(k,k)}} \int_0^t (V_s \boldsymbol{D}^{1/2})_{(k,k)} dB_s^{(k,k)} \right. \right. \right.$$

$$\left. \left. \left. - \frac{1}{2} \frac{\tilde{R}_t^{(k,l)}}{\tilde{R}_t^{(l,l)}} \int_0^t (V_s \boldsymbol{D}^{1/2})_{(l,l)} dB_s^{(l,l)} \right) \right)_{k=1,\dots,p; l \leq k} \right\}_{t \in [\epsilon, T]}$$

in $\mathcal{D}([\epsilon, T], \mathbb{R}^{\frac{p}{2}(p+1)})$, where $(V_s \boldsymbol{D}^{1/2})_{(k,l)}$ indicates that we are considering only the (k,l) row of the matrix $(V_s \boldsymbol{D}^{1/2})$ (see Appendix A) and $B_s^{(k,l)}$ are one-dimensional independent Brownian motions.

Proof. It follows from the same arguments of Proposition 4.4.5. □

4.4.2 Relative covolatility

In this section we are looking at the relative volatility case (see [BPS14]). Similarly to the previous section, we present first the results for the first scenario and then for the second one of Case II.

Proposition 4.4.8. *Assume that $\forall n \in \mathbb{N}, \sum_{i=1}^{\lfloor nT \rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)} \neq 0$ a.s. for $k = 1..., p$ and $l \leq k$, then under the assumptions of Theorem 4.3.10 we have*

$$\left(\frac{\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)}}{\sum_{i=1}^{\lfloor nT \rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)}} \right)_{k,l=1,\dots,p; l \leq k} \xrightarrow{u.c.p.} \left(\frac{\bar{R}_t^{(k,l)}}{\bar{R}_T^{(k,l)}} \right)_{k=1,\dots,p; l \leq k}.$$

Proof. Fix k, l. We have

$$\left(\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)} \right) \left(\sum_{i=1}^{\lfloor nT \rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)} \right)^{-1} - \frac{\bar{R}_t^{(k,l)}}{\bar{R}_T^{(k,l)}}$$

$$= \left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} - \bar{R}_t^{(k,l)} \right) \left(\frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} \frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right)^{-1}$$

$$+ \frac{\bar{R}_t^{(k,l)}}{\bar{R}_T^{(k,l)}} \left(\frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} \frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right)^{-1} \left(\bar{R}_T^{(k,l)} - \frac{1}{n} \sum_{i=1}^{\lfloor nT \rfloor} \frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right) \xrightarrow{u.c.p.} 0,$$

by Theorem 4.3.14. Notice that the supremum of $\bar{R}_t^{(k,l)}$ over $t \in [0, T]$ is bounded since the σs are compact on bounded intervals. Finally, the joint convergence follows from Lemma 4.3.13. □

Proposition 4.4.9. *Assume that* $\forall n \in \mathbb{N}, \sum_{i=1}^{\lfloor nT \rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)} \neq 0$ *a.s. for* $k = 1..., p$ *and* $l \leq k$*, then under the assumptions of Theorem 4.3.10 we have*

$$\left\{ \sqrt{n} \left(\frac{\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)}}{\sum_{i=1}^{\lfloor nT \rfloor} \Delta_i^n Y^{(k)} \Delta_i^n Y^{(l)}} - \frac{\bar{R}_{t,n}^{(k,l)}}{\bar{R}_{T,n}^{(k,l)}} \right)_{k=1,...,p; l \leq k} \right\}_{t \in [0,T]}$$

$$\xrightarrow{st} \left\{ \left(\frac{1}{\bar{R}_T^{(k,l)}} \int_0^t (V_s \boldsymbol{D}^{1/2})_{(k,l)} dB_s^{(k,l)} - \frac{\bar{R}_t^{(k,l)}}{\left(\bar{R}_T^{(k,l)}\right)^2} \int_0^T (V_s \boldsymbol{D}^{1/2})_{(k,l)} dB_s^{(k,l)} \right)_{k=1,...,p; l \leq k} \right\}_{t \in [0,T]}$$

in $\mathcal{D}([0,T], \mathbb{R}^{\frac{p}{2}(p+1)})$*, where* $(V_s \boldsymbol{D}^{1/2})_{(k,l)}$ *indicates that we are considering only the* (k,l) *row of the matrix* $(V_s \boldsymbol{D}^{1/2})$ *(see Appendix A) and* $B_s^{(k,l)}$ *are one-dimensional independent Brownian motions.*

Similar results can be obtained for the second scenario of Case II.

Proposition 4.4.10. *Assume that* $\forall n \in \mathbb{N}, \sum_{i=1}^{\lfloor nT \rfloor} \Delta_i^n X^{(k)} \Delta_i^n X^{(l)} \neq 0$ *a.s. for any* $k = 1, ..., p$ *and* $l \leq k$*, then under the assumptions of Theorem 4.3.11 we have*

$$\left(\frac{\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n X^{(k)} \Delta_i^n X^{(l)}}{\sum_{i=1}^{\lfloor nT \rfloor} \Delta_i^n X^{(k)} \Delta_i^n X^{(l)}} \right)_{k=1,...,p; l \leq k} \xrightarrow{u.c.p.} \left(\frac{\tilde{R}_t^{(k,l)}}{\tilde{R}_T^{(k,l)}} \right)_{k=1,...,p; l \leq k}.$$

Proof. It follows from the same arguments as the ones used in the proof of Proposition 4.4.8. □

Proposition 4.4.11. *Assume that* $\forall n \in \mathbb{N}, \sum_{i=1}^{\lfloor nT \rfloor} \Delta_i^n X^{(k)} \Delta_i^n X^{(l)} \neq 0$ *a.s. for* $k = 1..., p$ *and* $l \leq k$*, then under the assumptions of Theorem 4.3.11 we have*

$$\left\{ \sqrt{n} \left(\frac{\sum_{i=1}^{\lfloor nt \rfloor} \Delta_i^n X^{(k)} \Delta_i^n X^{(l)}}{\sum_{i=1}^{\lfloor nT \rfloor} \Delta_i^n X^{(k)} \Delta_i^n X^{(l)}} - \frac{\tilde{R}_{t,n}^{(k,l)}}{\tilde{R}_{T,n}^{(k,l)}} \right)_{k=1,...,p; l \leq k} \right\}_{t \in [0,T]}$$

$$\xrightarrow{st} \left\{ \left(\frac{1}{\tilde{R}_T^{(k,l)}} \int_0^t (V_s \boldsymbol{D}^{1/2})_{(k,l)} dB_s^{(k,l)} - \frac{\tilde{R}_t^{(k,l)}}{\left(\tilde{R}_T^{(k,l)}\right)^2} \int_0^T (V_s \boldsymbol{D}^{1/2})_{(k,l)} dB_s^{(k,l)} \right)_{k=1,...,p; l \leq k} \right\}_{t \in [0,T]},$$

in $\mathcal{D}([0,T], \mathbb{R}^{\frac{p}{2}(p+1)})$*, where* $(V_s \boldsymbol{D}^{1/2})_{(k,l)}$ *indicates that we are considering only the* (k,l) *row of the matrix* $(V_s \boldsymbol{D}^{1/2})$ *(see Appendix A) and* $B_s^{(k,l)}$ *are one-dimensional independent Brownian motions.*

Proof. It follows from the same arguments of Proposition 4.4.9. □

Remark 4.4.12. Similar feasible results for general multivariate Gaussian processes with stationary increments can be derived by just setting all the σs to be equal to 1.

4.5 Examples

4.5.1 The diagonal case

Due to the complexity of the presentation of the results, mainly due to their generality and their multidimensional nature, we now present a setting under which the results of this chapter simplify considerably. Nonetheless the results of this section are already a theoretical and feasible extension of the existing literature due to their multidimensional, joint and feasible nature (indeed the work [GR19] provides neither joint nor feasible results while the work [BCD11] only covers the one-dimensional case).

Let $p \in \mathbb{N}$. Consider the stochastic process $\{\mathbf{Y}_t\}_{t\in[0,T]} = \{(Y_t^{(1)}, ..., Y_t^{(p)})\}_{t\in[0,T]}$ given by

$$\mathbf{Y}_t = \int_{-\infty}^{t} \begin{pmatrix} g^{(1)}(t-s)\sigma_s^{(1)} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & g^{(p)}(t-s)\sigma_s^{(p)} \end{pmatrix} \begin{pmatrix} dW_s^{(1)} \\ \vdots \\ dW_s^{(p)} \end{pmatrix} + \begin{pmatrix} U_t^{(1)} \\ \vdots \\ U_t^{(p)} \end{pmatrix}. \tag{4.8}$$

Assume that the $\mathcal{F}_t$-Brownian measures are jointly Gaussian (hence we allow for dependency). For $k = 1, ..., p$, let $G_t^{(k)} := \int_0^t g^{(k)}(t-s)dW_s^{(k)}$. Further, let us define the scaling factor as in Section 4.2.1. For $k = 1, ..., p$, let $\tau_n^{(k)} := \sqrt{\mathbb{E}\left[\left(\Delta_1^n G^{(k)}\right)^2\right]}$ and $r_{k,l}^{(n)}(h) := \mathbb{E}\left[\frac{\Delta_1^n G^{(k)}}{\tau_n^{(k)}} \frac{\Delta_{1+h}^n G^{(l)}}{\tau_n^{(l)}}\right]$. It is possible to see that in this setting Case I and the first scenario of Case II coincide. Moreover, if we additionally assume that the volatilities are second order stationary with variance normalised to 1 (*i.e* $\mathbb{E}\left[(\sigma_s^{(k)})^2\right] = 1$, for any $k = 1, ..., p$ and $s \in (-\infty, T]$), then $\mathbb{E}\left[\left(\Delta_1^n G^{(k)}\right)^2\right] = \mathbb{E}\left[\left(\Delta_1^n Y^{(k)}\right)^2\right]$. Hence, by this additional assumption all the cases presented in this work coincide.

Then, we have the following central limit theorem, which follows directly from Theorem 4.3.5.

Corollary 4.5.1. *Under Assumptions 4.3.1, 4.3.2, 4.3.3 and 4.3.9 applied to $\tau_n^{(l)}, r_{k,l}^{(n)}$ for $l, k = 1, ..., p$, we have the following stable convergence:*

$$\left\{\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor} \frac{\Delta_i^n Y^{(k)}}{\tau_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\tau_n^{(l)}} - \mathbb{E}\left[\frac{\Delta_1^n G^{(k)}}{\tau_n^{(k)}} \frac{\Delta_1^n G^{(l)}}{\tau_n^{(l)}}\right] \int_0^t \sigma_s^{(k)}\sigma_s^{(l)} ds\right]_{k,l=1,...,p}\right\}_{t\in[0,T]}$$

$$\xrightarrow{st} \boldsymbol{D}^{1/2}\left\{\left(\int_0^t \sigma_s^{(k)}\sigma_s^{(l)} dB_s^{(k,l)}\right)_{k,l=1,...,p}\right\}_{t\in[0,T]}$$

in $\mathcal{D}([0,T], \mathbb{R}^{p^2})$, where $B_s^{(k,l)}$, for $l, k = 1, ..., p$, are independent one-dimensional Brownian motions and $\boldsymbol{D}$ is a $p^2 \times p^2$ matrix defined in Proposition 4.2.4.

Remark 4.5.2. It is possible to see that in this framework we have a clear separation between the deterministic and the stochastic part of the limiting process. Indeed, the deterministic kernel functions $g^{(1)}, ..., g^{(p)}$ constitute just a matrix of constants multiplying the limiting stochastic process.

Let $r_{k,l}(0) := \lim_{n\to\infty} r_{k,l}^{(n)}(0)$. The previous result leads to the following weak law of large numbers.

Corollary 4.5.3. *Let the assumptions of Corollary 4.5.1 hold. Then we have*

$$\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\tau_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\tau_n^{(l)}}\right)_{k,l=1,\dots,p} \xrightarrow{u.c.p.} \left(r_{k,l}(0)\int_0^t \sigma_s^{(k)}\sigma_s^{(l)}ds\right)_{k,l=1,\dots,p}.$$

Now, recall that $\bar{R}_{t,n}^{(k,l)} := \mathbb{E}\left[\frac{\Delta_1^n G^{(k)}}{\tau_n^{(k)}}\frac{\Delta_1^n G^{(l)}}{\tau_n^{(l)}}\right]\int_0^t \sigma_s^{(k)}\sigma_s^{(l)}ds$ and that $\bar{R}_t^{(k,l)} := r_{k,l}(0)\int_0^t \sigma_s^{(k)}\sigma_s^{(l)}ds$. Further let, $\mathcal{R}_t^{(k,l)} := \sqrt{\bar{R}_{t,n}^{(k,k)}\bar{R}_{t,n}^{(l,l)}} = \sqrt{\int_0^t(\sigma_s^{(k)})^2ds\int_0^t(\sigma_s^{(l)})^2ds}$. Therefore, we have that

$$\frac{\bar{R}_t^{(k,l)}}{\mathcal{R}_t^{(k,l)}} = r_{k,l}(0)\frac{\int_0^t \sigma_s^{(k)}\sigma_s^{(l)}ds}{\sqrt{\int_0^t(\sigma_s^{(k)})^2ds\int_0^t(\sigma_s^{(l)})^2ds}}.$$

We have the following feasible results on the asymptotic behaviour of the correlation ratio and of the relative covolatility. They follow directly from the results presented in Chapter 4.4.

Corollary 4.5.4. *Under the assumptions of Corollary 4.5.1, we have that for any $\epsilon > 0$*

$$\left(\frac{\sum_{i=1}^{\lfloor nt\rfloor}\Delta_i^n Y^{(k)}\Delta_i^n Y^{(l)}}{\sqrt{\sum_{i=1}^{\lfloor nt\rfloor}\left(\Delta_i^n Y^{(k)}\right)^2}\sqrt{\sum_{i=1}^{\lfloor nt\rfloor}\left(\Delta_i^n Y^{(l)}\right)^2}}\right)_{k,l=1,\dots,p} \xrightarrow{u.c.p.} \left(\frac{\bar{R}_t^{(k,l)}}{\mathcal{R}_t^{(k,l)}}\right)_{k,l=1,\dots,p},$$

and

$$\left\{\sqrt{n}\left(\frac{\sum_{i=1}^{\lfloor nt\rfloor}\Delta_i^n Y^{(k)}\Delta_i^n Y^{(l)}}{\sqrt{\sum_{i=1}^{\lfloor nt\rfloor}\left(\Delta_i^n Y^{(k)}\right)^2}\sqrt{\sum_{i=1}^{\lfloor nt\rfloor}\left(\Delta_i^n Y^{(l)}\right)^2}} - \frac{\bar{R}_{t,n}^{(k,l)}}{\mathcal{R}_t^{(k,l)}}\right)_{k,l=1,\dots,p}\right\}_{t\in[\epsilon,T]} \xrightarrow{st} \boldsymbol{D}^{1/2}$$

$$\left\{\left(\frac{1}{\mathcal{R}_t^{(k,l)}}\left(\int_0^t \sigma_s^{(k)}\sigma_s^{(l)}dB_s^{(k,l)} - \frac{1}{2}\bar{R}_t^{(k,l)}\left(\frac{\int_0^t(\sigma_s^{(k)})^2dB_s^{(k,k)}}{\int_0^t(\sigma_s^{(k)})^2ds} + \frac{\int_0^t(\sigma_s^{(l)})^2dB_s^{(l,l)}}{\int_0^t(\sigma_s^{(l)})^2ds}\right)\right)\right)_{k,l=1,\dots,p}\right\}_{t\in[\epsilon,T]}$$

in $\mathcal{D}([\epsilon,T],\mathbb{R}^{p^2})$, where $B_s^{(k,l)}$ are one-dimensional independent Brownian motions and $\boldsymbol{D}$ is a $p^2\times p^2$ matrix defined in Proposition 4.2.4.

Corollary 4.5.5. *Assume that $\forall n\in\mathbb{N}, \sum_{i=1}^{\lfloor nT\rfloor}\Delta_i^n Y^{(k)}\Delta_i^n Y^{(l)} \neq 0$ a.s. for $k,l = 1...,p$, then under the assumptions of Corollary 4.5.1 we have*

$$\left(\frac{\sum_{i=1}^{\lfloor nt\rfloor}\Delta_i^n Y^{(k)}\Delta_i^n Y^{(l)}}{\sum_{i=1}^{\lfloor nT\rfloor}\Delta_i^n Y^{(k)}\Delta_i^n Y^{(l)}}\right)_{k,l=1,\dots,p} \xrightarrow{u.c.p.} \left(\frac{\bar{R}_t^{(k,l)}}{\bar{R}_T^{(k,l)}}\right)_{k,l=1,\dots,p}.$$

and

$$\left\{\sqrt{n}\left(\frac{\sum_{i=1}^{\lfloor nt\rfloor}\Delta_i^n Y^{(k)}\Delta_i^n Y^{(l)}}{\sum_{i=1}^{\lfloor nT\rfloor}\Delta_i^n Y^{(k)}\Delta_i^n Y^{(l)}} - \frac{\bar{R}_{t,n}^{(k,l)}}{\bar{R}_{T,n}^{(k,l)}}\right)_{k,l=1,\dots,p}\right\}_{t\in[0,T]}$$

$$\xrightarrow{st} \boldsymbol{D}^{1/2}\left\{ \left(\frac{1}{\bar{R}_T^{(k,l)}} \int_0^t \sigma_s^{(k)}\sigma_s^{(l)} dB_s^{(k,l)} - \frac{\bar{R}_t^{(k,l)}}{\left(\bar{R}_T^{(k,l)}\right)^2} \int_0^T \sigma_s^{(k)}\sigma_s^{(l)} dB_s^{(k,l)} \right)_{k,l=1,\dots,p} \right\}_{t\in[0,T]}$$

in $\mathcal{D}([0,T],\mathbb{R}^{p^2})$*, where* $B_s^{(k,l)}$ *are one-dimensional independent Brownian motions and* $\boldsymbol{D}$ *is a* $p^2 \times p^2$ *matrix defined in Proposition 4.2.4.*

4.5.2 The gamma kernel

Here we explore an example of the multivariate process $\{\mathbf{G}_t\}_{t\in[0,T]}$ which satisfies the assumptions presented in Chapter 4.3. We will focus on the gamma kernel (see [BCD11]) because it plays a central role in the modelling of (atmospheric) turbulences, which is one of the main objectives of the development of BSS processes. We will show that the process obtained from using the gamma kernel in the kernel matrix satisfies our assumptions. Consider the stochastic process $\{\mathbf{G}_t\}_{t\in[0,T]}$ defined as

$$\mathbf{G}_t := \begin{pmatrix} G_t^{(1)} \\ \vdots \\ G_t^{(p)} \end{pmatrix} = \int_{-\infty}^t \begin{pmatrix} g^{(1,1)}(t-s) & \dots & g^{(1,p)}(t-s) \\ \vdots & \ddots & \vdots \\ g^{(p,1)}(t-s) & \dots & g^{(p,p)}(t-s) \end{pmatrix} \begin{pmatrix} dW_s^{(1)} \\ \vdots \\ dW_s^{(p)} \end{pmatrix},$$

where $g^{(i,j)}(t) = t^{\delta^{(i,j)}} e^{-\lambda^{(i,j)}t}\mathbf{1}_{[0,\infty)}(t)$ and $W^{(i)}$ are independent Gaussian $\mathcal{F}_t$-Brownian measures on $\mathbb{R}$, for $i,j = 1,...,p$. We consider the case of independent Brownian measures for the sake of clarity and remark that the extension to the dependent case is immediate from our computations. Thus, we have $G_t^{(i)} = \sum_{j=1}^p \int_{-\infty}^t g^{(i,j)}(t-s)dW_s^{(j)}$ and

$$\mathbb{E}\left[G_{t+h}^{(i)} G_t^{(j)}\right] = \sum_{l=1}^p \int_{-\infty}^t g^{(i,l)}(t+h-s)g^{(j,l)}(t-s)ds$$

$$= \sum_{l=1}^p \int_0^\infty g^{(i,l)}(x+h)g^{(j,l)}(x)dx = \sum_{l=1}^p e^{-\lambda^{(i,l)}h} \int_0^\infty (x+h)^{\delta^{(i,l)}} x^{\delta^{(j,l)}} e^{-(\lambda^{(i,l)}+\lambda^{(j,l)})x} dx.$$

It is important to notice that if $\delta^{(i,j)} \in (-\frac{1}{2},0)\cup(0,\frac{1}{2})$, then $\int_{-\infty}^t g^{(i,j)}(t-s)dW_s^{(j)}$ is not a semimartingale (see [BCD11] and [GR19]). We will first investigate Assumption 4.3.2 and then Assumption 4.3.1. Observe that by stationarity

$$r_{i,j}^{(n)}(k) := \mathbb{E}\left[\frac{\Delta_1^n G^{(j)}}{\tau_n^{(j)}} \frac{\Delta_{1+k}^n G^{(i)}}{\tau_n^{(i)}}\right]$$

$$= \frac{2\mathbb{E}\left[G_{\frac{1}{n}}^{(j)} G_{\frac{1+k}{n}}^{(i)}\right] - \mathbb{E}\left[G_{\frac{1}{n}}^{(j)} G_{\frac{k}{n}}^{(i)}\right] - \mathbb{E}\left[G_0^{(j)} G_{\frac{1+k}{n}}^{(i)}\right]}{\left(2\mathbb{E}\left[(G_0^{(j)})^2\right] - 2\mathbb{E}\left[G_{\frac{1}{n}}^{(j)} G_0^{(j)}\right]\right)^{\frac{1}{2}} \left(2\mathbb{E}\left[(G_0^{(i)})^2\right] - 2\mathbb{E}\left[G_{\frac{1}{n}}^{(i)} G_0^{(i)}\right]\right)^{\frac{1}{2}}}$$

$$= \frac{\sum_{l=1}^p \int_0^\infty 2g^{(i,l)}(x+\frac{k}{n})g^{(j,l)}(x) - g^{(i,l)}(x+\frac{k-1}{n})g^{(j,l)}(x) - g^{(i,l)}(x+\frac{k+1}{n})g^{(j,l)}(x)dx}{2\left(\sum_{l=1}^p \int_0^\infty \left(g^{(i,l)}(x)\right)^2 - g^{(i,l)}(x+\frac{1}{n})g^{(i,l)}(x)dx\right)^{\frac{1}{2}}}$$

$$\cdot \frac{1}{\left(\sum_{l=1}^{p}\int_0^\infty \left(g^{(j,l)}(x)\right)^2 - g^{(j,l)}(x+\frac{1}{n})g^{(j,l)}(x)dx\right)^{\frac{1}{2}}}. \tag{4.9}$$

Remark 4.5.6. Since $\int_0^\infty \left[\left(g^{(i,l)}(x)\right)^2 - g^{(i,l)}(x+\frac{1}{n})g^{(i,l)}(x)\right]dx > 0$, for any $i,l = 1,...,p$, we have that (4.9) is bounded by

$$\leq \sum_{l=1}^{p}\left|\frac{\int_0^\infty 2g^{(i,l)}(x+\frac{k}{n})g^{(j,l)}(x) - g^{(i,l)}(x+\frac{k-1}{n})g^{(j,l)}(x) - g^{(i,l)}(x+\frac{k+1}{n})g^{(j,l)}(x)dx}{2\left(\int_0^\infty \left(g^{(i,l)}(x)\right)^2 - g^{(i,l)}(x+\frac{1}{n})g^{(i,l)}(x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty \left(g^{(j,l)}(x)\right)^2 - g^{(j,l)}(x+\frac{1}{n})g^{(j,l)}(x)dx\right)^{\frac{1}{2}}}\right|. \tag{4.10}$$

From here it is possible to see that the results and examples of [GR19] directly apply to our framework because each summand in (4.10) is the correlation coefficient "$r_{i,j}^{(n)}(k)$" in their work.

Using the results in the supplementary material of [GR19] (see also eq. (12) on page 234 in [Ba54]) we have that the numerator in (4.9) is given by

$$\sum_{l=1}^{p} K_1^{(i,j),(l)} e^{-\lambda^{(i,l)}\frac{k}{n}} \sum_{r=0}^{\infty} \frac{(1+\delta^{(j,l)})_r}{(\delta^{(i,l)}+\delta^{(j,l)}+2)_r}\frac{1}{r!}\left(\lambda^{(i,l)}+\lambda^{(j,l)}\right)^r \left(2\left(\frac{k}{n}\right)^{r+\delta^{(i,l)}+\delta^{(j,l)}+1}\right.$$

$$\left. - \left(\frac{k-1}{n}\right)^{r+\delta^{(i,l)}+\delta^{(j,l)}+1} e^{\lambda^{(i,l)}\frac{1}{n}} - \left(\frac{k+1}{n}\right)^{r+\delta^{(i,l)}+\delta^{(j,l)}+1} e^{-\lambda^{(i,l)}\frac{1}{n}}\right)$$

$$+ \sum_{l=1}^{p} K_2^{(i,j),(l)} e^{-\lambda^{(i,l)}\frac{k}{n}} \sum_{r=0}^{\infty} \frac{(\delta^{(i,l)})_r}{(\delta^{(i,l)}+\delta^{(j,l)})_r}\frac{1}{r!}\left(\lambda^{(i,l)}+\lambda^{(j,l)}\right)^r \left(2\left(\frac{k}{n}\right)^{r}\right.$$

$$\left. - \left(\frac{k-1}{n}\right)^{r} e^{\lambda^{(i,l)}\frac{1}{n}} - \left(\frac{k+1}{n}\right)^{r} e^{-\lambda^{(i,l)}\frac{1}{n}}\right),$$

where $K_1^{(i,j),(l)} := \frac{\Gamma\left(\delta^{(j,l)}+1\right)\Gamma\left(-1-\delta^{(j,l)}-\delta^{(i,l)}\right)}{\Gamma\left(-\delta^{(i,l)}\right)}$, $K_2^{(i,j),(l)} := \frac{\Gamma\left(\delta^{(j,l)}+\delta^{(i,l)}+1\right)}{\left(\lambda^{(i,l)}+\lambda^{(j,l)}\right)^{\delta^{(j,l)}+\delta^{(i,l)}+1}}$,

and where $(a)_n := a(a+1)\cdots(a+n-1) = \prod_{q=0}^{n-1}(a+q) = \frac{\Gamma(a+n)}{\Gamma(a)}$, with $(a)_0 := 1$. Further, let $\bar{\delta} := \min_{l=1,...,p} \delta^{(j,l)} + \delta^{(i,l)}$. It is possible to see that, as $n \to \infty$, the numerator is of order $\left(\frac{1}{n}\right)^{1+\bar{\delta}}$, because $\delta^{(i,l)} < 1/2$, for every $i,l = 1,...,p$, in order to be in the non-semimartingale case.

Moreover, regarding the denominator we observe that, for every $l = 1,...,p$,

$$\int_0^\infty \left[\left(g^{(i,l)}(x)\right)^2 - g^{(i,l)}\left(x+\frac{1}{n}\right)g^{(i,l)}(x)\right]dx = \frac{\Gamma(2\delta^{(i,l)}+1)}{\left(2\lambda^{(i,l)}\right)^{2\delta^{(i,l)}+1}}$$

$$-K_1^{(i,i),(l)} e^{-\lambda^{(i,l)}\frac{1}{n}} \sum_{r=0}^{\infty} \frac{(1+\delta^{(i,l)})_r}{(2\delta^{(i,l)}+2)_r}\frac{1}{r!}\left(2\lambda^{(i,l)}\right)^r \left(\frac{1}{n}\right)^{r+2\delta^{(i,l)}+1}$$

$$-K_2^{(i,i),(l)} e^{-\lambda^{(i,l)}\frac{1}{n}} \sum_{r=0}^{\infty} \frac{(\delta^{(i,l)})_r}{(2\delta^{(i,l)})_r}\frac{1}{r!}\left(2\lambda^{(i,l)}\right)^r \left(\frac{1}{n}\right)^{r},$$

and using that $e^{-\lambda^{(i,l)}x} = 1 - \lambda^{(i,l)}x + O(x^2)$ and that $K_2^{(i,i),(l)} = \frac{\Gamma(2\delta^{(i,l)}+1)}{\left(2\lambda^{(i,l)}\right)^{2\delta^{(i,l)}+1}}$ we have that

$$= -K_1^{(i,i),(l)} e^{-\lambda^{(i,l)}\frac{1}{n}} \sum_{r=0}^{\infty} \frac{(1+\delta^{(i,l)})_r}{(2\delta^{(i,l)}+2)_r} \frac{1}{r!} \left(2\lambda^{(i,l)}\right)^r \left(\frac{1}{n}\right)^{r+2\delta^{(i,l)}+1}$$

$$- \left(1 - \lambda^{(i,l)}\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) K_2^{(i,i),(l)} \sum_{r=1}^{\infty} \frac{(\delta^{(i,l)})_r}{(2\delta^{(i,l)})_r} \frac{1}{r!} \left(2\lambda^{(i,l)}\right)^r \left(\frac{1}{n}\right)^r$$

$$- \left(-\lambda^{(i,l)}\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) K_2^{(i,i),(l)}$$

$$= -K_1^{(i,i),(l)} \left(1 - \lambda^{(i,l)}\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) \sum_{r=0}^{\infty} \frac{(1+\delta^{(i,l)})_r}{(2\delta^{(i,l)}+2)_r} \frac{1}{r!} \left(2\lambda^{(i,l)}\right)^r \left(\frac{1}{n}\right)^{r+2\delta^{(i,l)}+1}$$

$$- \left(1 - \lambda^{(i,l)}\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) K_2^{(i,i),(l)} \sum_{r=2}^{\infty} \frac{(\delta^{(i,l)})_r}{(2\delta^{(i,l)})_r} \frac{1}{r!} \left(2\lambda^{(i,l)}\right)^r \left(\frac{1}{n}\right)^r$$

$$- \left(1 - \lambda^{(i,l)}\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) K_2^{(i,i),(l)} \lambda^{(i,l)} \left(\frac{1}{n}\right) - \left(-\lambda^{(i,l)}\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) K_2^{(i,i),(l)}$$

$$= -K_1^{(i,i),(l)} \left(1 - \lambda^{(i,l)}\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) \sum_{r=0}^{\infty} \frac{(1+\delta^{(i,l)})_r}{(2\delta^{(i,l)}+2)_r} \frac{1}{r!} \left(2\lambda^{(i,l)}\right)^r \left(\frac{1}{n}\right)^{r+2\delta^{(i,l)}+1}$$

$$- \left(1 - \lambda^{(i,l)}\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) K_2^{(i,i),(l)} \sum_{r=2}^{\infty} \frac{(\delta^{(i,l)})_r}{(2\delta^{(i,l)})_r} \frac{1}{r!} \left(2\lambda^{(i,l)}\right)^r \left(\frac{1}{n}\right)^r + O\left(\frac{1}{n^2}\right).$$

Now, let $\hat{\delta} := \min_{l=1,\dots,p} \delta^{(i,l)} + \min_{l=1,\dots,p} \delta^{(j,l)}$. From the above computations it is possible to see that, as $n \to \infty$, the denominator in (4.9) is of order $\left(\frac{1}{n}\right)^{1+\hat{\delta}}$.

Therefore, looking at the order as $n \to \infty$ we have that (4.9) reduces to $\left(\frac{1}{n}\right)^{\bar{\delta}-\hat{\delta}}$. Observe that $\bar{\delta} - \hat{\delta} \geq 0$ by definition. Hence, if $\bar{\delta} > \hat{\delta}$ then $\lim_{n\to\infty} \left(r_{i,j}^{(n)}(k)\right)^2 = 0$. While if $\bar{\delta} = \hat{\delta}$, we have that, for any $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \left(r_{i,j}^{(n)}(k)\right)^2 = C\left(2k^{\bar{\delta}+1} - (k-1)^{\bar{\delta}+1} - (k+1)^{\bar{\delta}+1}\right)^2 < \infty, \tag{4.11}$$

where $C \in \mathbb{R}$ is a finite constant independent of k. Hence, the first part of Assumption 4.3.2 is satisfied.

Let $\bar{R}^{(i,j),(l)}(t) := \int_0^\infty \left[\left(g^{(i,l)}(x)\right)^2 + \left(g^{(j,l)}(x)\right)^2 - 2g^{(i,l)}(x+t)\, g^{(j,l)}(x)\right] dx$, thus we have that $\mathbb{E}\left[(G_{s+t}^{(i)} - G_s^{(j)})^2\right] = \sum_{l=1}^p \bar{R}^{(i,j),(l)}(t)$. Recall that a function $L : (0,\infty) \to \mathbb{R}$ is called slowly varying at 0 when the identity $\lim_{t\to 0^+} \frac{L(\lambda t)}{L(t)} = 1$ holds, for any fixed $\lambda > 0$. If L is continuous on $(0,\infty)$, we have that

$$|L(t)| \leq Ct^{-\alpha}, \quad t \in (0, \lambda], \tag{4.12}$$

for any $\alpha > 0$ and any $\lambda > 0$ (where the constant $C > 0$ depends on α and λ, see [BCD09a]). Now, consider the following conditions.

(A1) $\bar{R}^{(i,i),(l)}(t) = t^{1+2\delta^{(i,l)}} L_0^{(i,i,l)}(t)$ *for some* $\delta^{(i,l)} \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ *and some positive slowly varying at 0 function* $L_0^{(i,i,l)}$*, which is continuous on (0, ∞), for every* $l = 1, ..., p$.

(A2) $(\bar{R}^{(i,j),(l)})''(t) = t^{\delta^{(i,l)}+\delta^{(j,l)}-1} L_2^{(i,j,l)}(t)$ *for some slowly varying at 0 function* $L_2^{(i,j,l)}$*, which is continuous on (0, ∞), for every* $l = 1, ..., p$.

(A3) *Denote* $\tilde{L}_0^{(i,j,l)}(t) := \sqrt{L_0^{(i,i,l)}(t)L_0^{(j,j,l)}(t)}$. *There exists* $d \in (0,1)$ *s.t., for every* $l = 1, ..., p$,

$$\limsup_{t\to 0^+} \sup_{y\in(t,t^d)} \frac{L_2^{(i,j,l)}(y)}{\tilde{L}_0^{(i,j,l)}(t)} < \infty.$$

The following result is the multivariate version of Lemma 1 in [BCD09a].

Lemma 4.5.7. *Suppose that conditions (A1)-(A3) hold. Let* $\beta := \max_{l=1,\dots,p} \delta^{(i,l)} + \delta^{(i,l)} + 1$ *and let* $\epsilon > 0$ *with* $\epsilon < 2-\beta$. *Define the sequence* $r(k)$ *by* $r(k) = (k-1)^{\beta+\epsilon-2}$, $k \geq 2$, *and* $r(0), r(1) \geq C$, *with* $C > r_{i,j}^{(n)}(0), r_{i,j}^{(n)}(1)$. *Then, there exists a natural number* $n_0(\epsilon)$ *such that* $|r_{i,j}^{(n)}(k)| \leq Cr(k)$, $k \geq 0$, *for all* $n > n_0(\epsilon)$. *Further, let* $\beta + \epsilon - 2 < -1/2$ *then* $\sum_{k=1}^{\infty} r^2(k) < \infty$.

Proof. Observe that

$$r_{i,j}^{(n)}(k) = \sum_{l=1}^{p} \frac{\bar{R}^{(i,j,l)}\left(\frac{k-1}{n}\right) - 2\bar{R}^{(i,j,l)}\left(\frac{k}{n}\right) + \bar{R}^{(i,j,l)}\left(\frac{k+1}{n}\right)}{2\sqrt{\left(\sum_{l=1}^{p} \bar{R}^{(i,i,l)}\left(\frac{1}{n}\right)\right)\left(\sum_{l=1}^{p} \bar{R}^{(j,j,l)}\left(\frac{1}{n}\right)\right)}}.$$

Under conditions (A1)-(A2), for $2 \leq k \leq n$ by (4.10) and by the mean value theorem (applied twice) we have that

$$|r_{i,j}^{(n)}(k)| \leq \sum_{l=1}^{p} \left| \frac{1}{2}\left(\frac{1}{n}\right)^{1-\delta^{(i,l)}-\delta^{(j,l)}} \frac{(\bar{R}^{(i,j),(l)})''\left(\frac{k+\theta_k^n}{n}\right)}{\tilde{L}_0^{(i,j,l)}\left(\frac{1}{n}\right)} \right|$$

$$= \sum_{l=1}^{p} \left| \frac{1}{2}\left(\frac{1}{n}\right)^{1-\delta^{(i,l)}-\delta^{(j,l)}} \left(\frac{k+\theta_k^n}{n}\right)^{\delta^{(i,l)}+\delta^{(j,l)}-1} \frac{L_2^{(i,j,l)}\left(\frac{k+\theta_k^n}{n}\right)}{\tilde{L}_0^{(i,j,l)}\left(\frac{1}{n}\right)} \right|, \tag{4.13}$$

where $|\theta_k^n| < 1$. In particular, for $2 \leq k \leq \lfloor n^{1-d} \rfloor$, by the condition (A3), we obtain

$$|r_{i,j}^{(n)}(k)| \leq C'(k-1)^{\delta^{(i,l)}+\delta^{(j,l)}-1} \leq C'(k-1)^{\beta-2},$$

where C' is a positive constant. Moreover, for $\lfloor n^{1-d} \rfloor \leq k \leq n$, using (4.12) we have that

$$|r_{i,j}^{(n)}(k+1)| \leq \sum_{l=1}^{p} \left| \frac{1}{2} k^{\delta^{(i,l)}+\delta^{(j,l)}-1} \frac{L_2^{(i,j,l)}\left(\frac{k+\theta_k^n}{n}\right)}{\tilde{L}_0^{(i,j,l)}\left(\frac{1}{n}\right)} \right| \leq \sum_{l=1}^{p} \left| \frac{1}{2} k^{\delta^{(i,l)}+\delta^{(j,l)}-1+\epsilon} n^{(d-1)\epsilon} \frac{L_2^{(i,j,l)}\left(\frac{k+\theta_k^n}{n}\right)}{\tilde{L}_0^{(i,j,l)}\left(\frac{1}{n}\right)} \right|$$

$$\leq C'' k^{\delta^{(i,l)}+\delta^{(j,l)}-1+\epsilon} \leq C'' k^{\beta+\epsilon-2},$$

where C'' is a positive constant. The last statement is an immediate consequence. □

Remark 4.5.8. Our proof uses the same arguments as the ones in the proof of Lemma 1 in [BCD09a]. We have an additional factor $\left(\frac{1}{n}\right)^{1-\delta^{(i,l)}-\delta^{(j,l)}}$ in (4.13), which appears to have been forgotten on one occasion in the proof of Lemma 1 in [BCD09a].

Now, we need to check that in our example conditions (A1)-(A3) are satisfied and that $r_{i,j}^{(n)}(0)$ and $r_{i,j}^{(n)}(1)$ are uniformly bounded. First, it is easy to see that $|r_{i,j}^{(n)}(0)| \leq 1$ since it is the correlation function of $\Delta_1^n G^{(i)}$ and $\Delta_1^n G^{(j)}$. Further, from (4.11), we have that $r_{i,j}^{(n)}(k)$ is

a converging sequence, hence bounded, for any $k \in \mathbb{N}$. Thus, $|r_{i,j}^{(n)}(1)|, |r_{i,j}^{(n)}(1)| < C$, where C is a positive constant independent of n. Condition (A1) is the same as condition (A1) for the univariate case in [BCD11] (see [BCD09b] for more details), thus it is satisfied for the Gamma kernel example. For (A2) we have that

$$\int_0^\infty g^{(i,l)}(x+t)g^{(j,l)}(x)dx = K_1^{(i,j),(l)} e^{-\lambda^{(i,l)}t} \sum_{r=0}^{\infty} \frac{(1+\delta^{(j,l)})_r}{(\delta^{(i,l)}+\delta^{(j,l)}+2)_r} \frac{1}{r!} \left(\lambda^{(i,l)}+\lambda^{(j,l)}\right)^r t^{r+\delta^{(i,l)}+\delta^{(j,l)}+1}$$

$$+K_2^{(i,j),(l)} e^{-\lambda^{(i,l)}t} \sum_{r=0}^{\infty} \frac{(\delta^{(i,l)})_r}{(\delta^{(i,l)}+\delta^{(j,l)})_r} \frac{1}{r!} \left(\lambda^{(i,l)}+\lambda^{(j,l)}\right)^r t^r,$$

for any $l = 1, ..., p$, and taking the second derivative we get

$$(\bar{R}^{(i,j),(l)})''(t) = t^{\delta^{(i,l)}+\delta^{(j,l)}-1}\left[-2(\delta^{(i,l)}+\delta^{(j,l)}+1)(\delta^{(i,l)}+\delta^{(j,l)})K_1^{(i,j),(l)}+O\left(\min(t^{1-\delta^{(i,l)}-\delta^{(j,l)}}, t)\right)\right].$$

Hence, (A2) is satisfied.

Finally, from [BCD11] (see also the supplementary material of [GR19]) we have $\lim_{t\to 0^+} \tilde{L}_0^{(i,i,l)} = 2^{-1-2\delta^{(i,l)}} \frac{\Gamma(\frac{1}{2}-\delta^{(i,l)})}{\Gamma(\frac{3}{2}+\delta^{(i,l)})}$ and so $\lim_{t\to 0^+} \tilde{L}_0^{(i,j,l)} = 2^{-1-\delta^{(i,l)}-\delta^{(j,l)}} \sqrt{\frac{\Gamma(\frac{1}{2}-\delta^{(i,l)})\Gamma(\frac{1}{2}-\delta^{(j,l)})}{\Gamma(\frac{3}{2}+\delta^{(i,l)})\Gamma(\frac{3}{2}+\delta^{(j,l)})}} =: K_0^{(i,j),(l)}$. Thus, we have

$$\limsup_{t\to 0^+} \sup_{y\in(t,t^d)} \frac{L_2^{(i,j,l)}(y)}{\tilde{L}_0^{(i,j,l)}(t)}$$

$$\leq \limsup_{t\to 0^+} \frac{-2(\delta^{(i,l)}+\delta^{(j,l)}+1)(\delta^{(i,l)}+\delta^{(j,l)})K_1^{(i,j),(l)} + C\left(\min(t^{d(1-\delta^{(i,l)}-\delta^{(j,l)})}, t^d)\right)}{\tilde{L}_0^{(i,j,l)}(t)}$$

$$= \frac{-2(\delta^{(i,l)}+\delta^{(j,l)}+1)(\delta^{(i,l)}+\delta^{(j,l)})K_1^{(i,j),(l)}}{K_0^{(i,j),(l)}} < \infty,$$

for some $C > 0$ and for every $l = 1, ..., p$. Therefore, (A3) and, so, Assumption 4.3.2 are satisfied.

Given the fact that our $\pi_n^{(m,l)}$ has the same structure as π_n in [BCD11], then the same arguments used in [BCD11] hold here and we can conclude that if $\delta^{(m,l)} \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, for every $m, l = 1, .., p$, then Assumption 4.3.1 is satisfied.

Combining the ranges obtained, we conclude that when $\delta^{(i,j)} \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ with $\max_{l=1,...,p} \delta^{(i,l)} + \delta^{(j,l)} < 1/2$ for every $i, j = 1, ..., p$, then all the results presented in this chapter apply to our example.

4.6 Conclusion

In this chapter, we have investigated the joint asymptotic behaviour of the realised covariation of multivariate Gaussian processes with stationary increments and of the multivariate BSS process. We have shown central limit theorems for the realised covariation of these processes. For the BSS process, we have provided several results according to the setting adopted; the objective was to cover all the possible interesting cases. Then, we have shown how to make these results feasible, namely directly computable from data. These feasible results are based on ratios that

eliminate the infeasible scaling factor τ_n. Again for the sake of completeness we have presented different approaches. All the results presented in this chapter extend the current literature on the volatility estimation for stochastic processes outside the semimartingale framework, which is a relevant topic for empirical applications, but of which little is known. A main example has also been provided: the Gamma kernel. This represents a pivotal example for the study and modelling of turbulences and it constitutes one of the main motivation for the study of BSS process. This work triggers various interesting questions. For example, the feasible results presented use ratios to avoid the computation of the infeasible scaling factor τ_n. But is it possible to estimate τ_n directly from data? The ongoing work [1] goes in this direction.

4.7 Appendices

Appendix A: the matrices D and V for the BSS process

In this appendix we are going to specify the explicit structure and value of the matrices $\mathbf{D}$ and $\mathbf{V}_s$. We present them in the appendix because to present them we need combinatorial arguments which are easy but tedious, and they are similar for the different cases presented in Chapter 4.3.

Case I

Let $\boldsymbol{D}^{1/2} \in \mathcal{M}^{p^6 \times p^6}(\mathbb{R})$ be defined as

$$(\boldsymbol{D})_{z,y} := \lim_{n\to\infty} \frac{1}{n} \sum_{h=1}^{n-1} (n-h)$$

$$\begin{aligned}
&\Big(r^{(n)}_{k_z,r_z,m_z;k_y,r_y,m_y}(h) r^{(n)}_{l_z,q_z,w_z;l_y,q_y,w_y}(h) + r^{(n)}_{l_z,q_z,w_z;k_y,r_y,m_y}(h) r^{(n)}_{k_z,r_z,m_z;l_y,q_y,w_y}(h) \\
&+ r^{(n)}_{k_y,r_y,m_y;k_z,r_z,m_z}(h) r^{(n)}_{l_y,q_y,w_y;l_z,q_z,w_z}(h) + r^{(n)}_{k_y,r_y,m_y;l_z,q_z,w_z}(h) r^{(n)}_{l_y,q_y,w_y;k_z,r_z,m_z}(h) \Big) \\
&+ \Big(r^{(n)}_{k_z,r_z,m_z;k_y,r_y,m_y}(0) r^{(n)}_{l_z,q_z,w_z;l_y,q_y,w_y}(0) + r^{(n)}_{l_z,q_z,w_z;k_y,r_y,m_y}(0) r^{(n)}_{k_z,r_z,m_z;l_y,q_y,w_y}(0) \Big),
\end{aligned}$$

where for each of the $p^6 \times p^6$ combinations of (z,y) there exists a unique combination of $((r_z,m_z,q_z,w_z,k_z,l_z),(r_y,m_y,q_y,w_y,k_y,l_y))$ where each of these elements takes value in $\{1,...,p\}$. Let $\nu(r,m,q,w)$ be any permutation of the set of the different combinations of $r,m,q,w \in \{1,...,p\}$ and let $\nu_s(r,m,q,w)$ determine the s-th element of $\nu(r,m,q,w)$. Notice that $\nu(r,m,q,w)$ has certain order for its element, which is not relevant for us since we only care about the consistent use of the order adopted. Recall that by the notation $(\cdot)_{\nu(r,m,q,w)}$ we mean that the s-th component of the vector is $\nu_s(r,m,q,w)$. Then, the association is given by

$$(z,y) \leftrightarrow \Bigg(\Big(\nu_{z-\lfloor \frac{z-1}{p^4} \rfloor p^4}(r,m,q,w), \lfloor \frac{\lfloor \frac{z-1}{p^4} \rfloor}{p} \rfloor + 1, \lfloor \frac{z-1}{p^4} \rfloor + 1 - p \lfloor \frac{\lfloor \frac{z-1}{p^4} \rfloor}{p} \rfloor \Big),$$

$$\Big(\nu_{y-\lfloor \frac{y-1}{p^4} \rfloor p^4}(r,m,q,w), \lfloor \frac{\lfloor \frac{y-1}{p^4} \rfloor}{p} \rfloor + 1, \lfloor \frac{y-1}{p^4} \rfloor + 1 - p \lfloor \frac{\lfloor \frac{y-1}{p^4} \rfloor}{p} \rfloor \Big) \Bigg).$$

For a proof of this statement for the case $p = 2$ see the proof of Theorem 4.3.5; the extension to the case $p > 2$ is trivial. Moreover, define for $s \in [0, T]$ the $p^2 \times p^6$ matrix

$$V_s := \begin{pmatrix} \sigma_s & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \sigma_s \end{pmatrix}, \tag{4.14}$$

where $\sigma_s := \left(\sigma_s^{(r,m)}\sigma_s^{(q,w)}\right)^{\top}_{\nu(r,m,q,w)}$, so it is a row vector of p^4 elements (here the consistency of the order of the elements of $\nu(r, m, q, w)$ is fundamental), and $\mathbf{0}$ is a row vector of p^4 elements containing only zeros. Hence, σ_s and $\mathbf{0}$ contain the same number of elements.

In case of the *vech* notation, the association for $\boldsymbol{D}$ is the following. First, let us define for $i \in \mathbb{N}$

$$\chi(i) := \left\lfloor \sqrt{2i} + \frac{1}{2} \right\rfloor, \quad \xi(i) := i - \frac{1}{2}\left\lfloor \frac{\sqrt{8i-7}-1}{2} \right\rfloor \left(\left\lfloor \frac{\sqrt{8i-7}-1}{2} \right\rfloor + 1 \right),$$

then we have

$$(z, y) \leftrightarrow \Bigg(\left(\nu_{z - \lfloor \frac{z-1}{p^4} \rfloor p^4}(r, m, q, w), \chi\left(\left\lfloor \frac{z-1}{p^4} \right\rfloor \right), \xi\left(\left\lfloor \frac{z-1}{p^4} \right\rfloor \right) \right),$$

$$\left(\nu_{y - \lfloor \frac{y-1}{p^4} \rfloor p^4}(r, m, q, w), \chi\left(\left\lfloor \frac{y-1}{p^4} \right\rfloor \right), \xi\left(\left\lfloor \frac{y-1}{p^4} \right\rfloor \right) \right) \Bigg) \tag{4.15}$$

The association (4.15) comes from the fact that we have $k = 1, ..., p$ with $k \leq l$. The couple (k, l) has the following sequence $(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), (5,1), ...$ which is a fractal sequence of form given by $(\xi(i), \chi(i))$ where i is the i-th term of the sequence. Concerning the matrix V_s, we have the same structure as (4.14). However, the dimension is now $\frac{p}{2}(p+1) \times \frac{p^5}{2}(p+1)$. The dimension and the value of the σ_s and of the $\mathbf{0}$ is the same as before and, hence, we are considering fewer of them (*e.g.* there are $\frac{p}{2}(p+1)$ number of σ_s while in (4.14) we had p^2 number of them).

Case II first and second scenario

For Case II first scenario, we have the same as in Case I. The only difference is the use of $\bar{r}^{(n)}$ instead of $r^{(n)}$.

For Case II first scenario, we have the following. For this case we have something similar to the previous sections but simpler since there are not any more the variables r and q. Let $\boldsymbol{D}^{1/2} \in \mathcal{M}^{p^4 \times p^4}(\mathbb{R})$ be defined as

$$(\boldsymbol{D})_{z,y} := \lim_{n\to\infty} \frac{1}{n} \sum_{h=1}^{n-1} (n-h) \Big(\tilde{r}^{(n)}_{k_z,m_z;k_y,m_y}(h)\tilde{r}^{(n)}_{l_z,w_z;l_y,w_y}(h) + \tilde{r}^{(n)}_{l_z,w_z;k_y,m_y}(h)\tilde{r}^{(n)}_{k_z,m_z;l_y,w_y}(h)$$

$$\tilde{r}^{(n)}_{k_y,m_y;k_z,m_z}(h)\tilde{r}^{(n)}_{l_y,w_y;l_z,w_z}(h) + \tilde{r}^{(n)}_{k_y,m_y;l_z,w_z}(h)\tilde{r}^{(n)}_{l_y,w_y;k_z,m_z}(h) \Big)$$

$$+ \left(\tilde{r}^{(n)}_{k_z,m_z;k_y,m_y}(0)\tilde{r}^{(n)}_{l_z,w_z;l_y,w_y}(0) + \tilde{r}^{(n)}_{l_z,w_z;k_y,m_y}(0)\tilde{r}^{(n)}_{k_z,m_z;l_y,w_y}(0) \right),$$

where for each of the $p^4 \times p^4$ combinations of (z, y) there is a unique combination of $((m_z, w_z, k_z, l_z), (m_y, w_y, k_y, l_y))$. The association is given by

$$(z,y) \leftrightarrow \Bigg(\Big(\mu_{z-\lfloor \frac{z-1}{p^2} \rfloor p^2}(m,w), \lfloor \frac{\lfloor \frac{z-1}{p^2} \rfloor}{p} \rfloor + 1, \lfloor \frac{z-1}{p^2} \rfloor + 1 - p \lfloor \frac{\lfloor \frac{z-1}{p^2} \rfloor}{p} \rfloor \Big),$$

$$\Big(\mu_{y-\lfloor \frac{y-1}{p^2} \rfloor p^2}(m,w), \lfloor \frac{\lfloor \frac{y-1}{p^2} \rfloor}{p} \rfloor + 1, \lfloor \frac{y-1}{p^2} \rfloor + 1 - p \lfloor \frac{\lfloor \frac{y-1}{p^2} \rfloor}{p} \rfloor \Big) \Bigg),$$

where $\mu(m,w)$ is any permutation of the set of all the possible combinations of $m, w \in \{1, ..., p\}$ (*i.e.* any permutation of the set $((1,1),(1,2)..., (1,p),(2,1),(2,2),...,(2,p),...,(p,p))$) and $\mu_s(m,w)$ is the s-th element of $\mu(m,w)$. Moreover, define for $s \in [0,T]$ the $p^2 \times p^4$ matrix

$$V_s := \begin{pmatrix} \sigma_{(1,1),s} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \sigma_{(p,p),s} \end{pmatrix}, \tag{4.16}$$

where $\sigma_{(k,l),s} := \left(\sigma_s^{(k,m)} \sigma_s^{(l,w)} \right)^\top_{\mu(m,w)}$ and $\mathbf{0}$ is a row vector of p^2 elements containing only zeros.

For the *vech* notation we have that the structure of the matrices $\boldsymbol{D}$ and V remain the same but their dimension reduce. Similarly to the previous cases, the association is given by

$$(z,y) \leftrightarrow \Bigg(\Big(\mu_{z-\lfloor \frac{z-1}{p^2} \rfloor p^2}(m,w), \chi\left(\lfloor \frac{z-1}{p^4} \rfloor \right), \xi\left(\lfloor \frac{z-1}{p^4} \rfloor \right) \Big),$$

$$\Big(\mu_{y-\lfloor \frac{y-1}{p^2} \rfloor p^2}(m,w), \chi\left(\lfloor \frac{y-1}{p^4} \rfloor \right), \xi\left(\lfloor \frac{y-1}{p^4} \rfloor \right) \Big) \Bigg).$$

Concerning the matrix V_s, we have the same structure as (4.16). However, the dimension is now $\frac{p}{2}(p+1) \times \frac{p^3}{2}(p+1)$. The value of the σ_s and of the $\mathbf{0}$ is the same as before and, hence, we are considering fewer of them (*e.g.* there are $\frac{p}{2}(p+1)$ number of σ_s while in (4.16) we had p^2 number of them).

Appendix B: Proofs of Section 4.2

Proof of Proposition 4.2.4

In order to prove this result we need to use Corollary 4.1.16. Notice that now $I_2(f_{s,n})$ takes the form of $Z^n_{(x,y),b_l} - Z^n_{(z,w),a_l}$ and $b = p^2 d$.

First, we compute the covariances. Notice that it is sufficient to focus on the case $l = 2$ and $a_1 = 0, b_1 = a_2 = 1, b_2 = 2$. Recall the isometry property of integrals (*i.e.* Proposition 4.1.9) and that for $f_1, g_1, f_2, g_2 \in \mathcal{H}$ we have $\langle f_1 \otimes g_1, f_2 \otimes g_2 \rangle_{\mathcal{H}^{\otimes 2}} := \langle f_1, f_2 \rangle_{\mathcal{H}} \langle g_1, g_2 \rangle_{\mathcal{H}}$. Then, for

$x, y, z, w = 1, ..., p$, we have

$$\mathbb{E}[(Z^n_{(x,y),1} - Z^n_{(x,y),0})(Z^n_{(z,w),2} - Z^n_{(z,w),1})] = \frac{2}{n}\left\langle \sum_{i=1}^{n} \frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \tilde{\otimes} \frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}}, \sum_{j=n+1}^{2n} \frac{\Delta_j^n G^{(z)}}{\tau_n^{(z)}} \tilde{\otimes} \frac{\Delta_j^n G^{(w)}}{\tau_n^{(w)}} \right\rangle_{\mathcal{H}^{\otimes 2}}$$

$$= \frac{1}{2n} \sum_{i=1}^{n} \sum_{j=n+1}^{2n} \left\langle \frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \otimes \frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}}, \frac{\Delta_j^n G^{(z)}}{\tau_n^{(z)}} \otimes \frac{\Delta_j^n G^{(w)}}{\tau_n^{(w)}} \right\rangle_{\mathcal{H}^{\otimes 2}}$$

$$+ \left\langle \frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}} \otimes \frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}}, \frac{\Delta_j^n G^{(z)}}{\tau_n^{(z)}} \otimes \frac{\Delta_j^n G^{(w)}}{\tau_n^{(w)}} \right\rangle_{\mathcal{H}^{\otimes 2}} + \left\langle \frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \otimes \frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}}, \frac{\Delta_j^n G^{(w)}}{\tau_n^{(w)}} \otimes \frac{\Delta_j^n G^{(z)}}{\tau_n^{(z)}} \right\rangle_{\mathcal{H}^{\otimes 2}}$$

$$+ \left\langle \frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}} \otimes \frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}}, \frac{\Delta_j^n G^{(w)}}{\tau_n^{(w)}} \otimes \frac{\Delta_j^n G^{(z)}}{\tau_n^{(z)}} \right\rangle_{\mathcal{H}^{\otimes 2}}$$

$$= \frac{1}{2n} \sum_{i=1}^{n} \sum_{j=n+1}^{2n} \mathbb{E}\left[\frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \frac{\Delta_j^n G^{(z)}}{\tau_n^{(z)}}\right] \mathbb{E}\left[\frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}} \frac{\Delta_j^n G^{(w)}}{\tau_n^{(w)}}\right]$$

$$+ \mathbb{E}\left[\frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}} \frac{\Delta_j^n G^{(z)}}{\tau_n^{(z)}}\right] \mathbb{E}\left[\frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \frac{\Delta_j^n G^{(w)}}{\tau_n^{(w)}}\right]$$

$$+ \mathbb{E}\left[\frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \frac{\Delta_j^n G^{(w)}}{\tau_n^{(w)}}\right] \mathbb{E}\left[\frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}} \frac{\Delta_j^n G^{(z)}}{\tau_n^{(z)}}\right] + \mathbb{E}\left[\frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}} \frac{\Delta_j^n G^{(w)}}{\tau_n^{(w)}}\right] \mathbb{E}\left[\frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \frac{\Delta_j^n G^{(z)}}{\tau_n^{(z)}}\right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=n+1}^{2n} \mathbb{E}\left[\frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \frac{\Delta_j^n G^{(z)}}{\tau_n^{(z)}}\right] \mathbb{E}\left[\frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}} \frac{\Delta_j^n G^{(w)}}{\tau_n^{(w)}}\right]$$

$$+ \mathbb{E}\left[\frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}} \frac{\Delta_j^n G^{(z)}}{\tau_n^{(z)}}\right] \mathbb{E}\left[\frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \frac{\Delta_j^n G^{(w)}}{\tau_n^{(w)}}\right]$$

$$= \frac{1}{n} \sum_{k=1}^{n} k \left(r^{(n)}_{x,z}(k) r^{(n)}_{y,w}(k) + r^{(n)}_{y,z}(k) r^{(n)}_{x,w}(k) \right) + \frac{1}{n} \sum_{k=n+1}^{2n-1} (2n-k) \left(r^{(n)}_{x,z}(k) r^{(n)}_{y,w}(k) + r^{(n)}_{y,z}(k) r^{(n)}_{x,w}(k) \right).$$

Hence, we have that, for $x, y, z, w = 1, ..., p$,

$$\mathbb{E}[(Z^n_{(x,y),1} - Z^n_{(x,y),0})(Z^n_{(z,w),2} - Z^n_{(z,w),1})]$$

$$= \frac{1}{n} \sum_{k=1}^{n} k \left(r^{(n)}_{x,z}(k) r^{(n)}_{y,w}(k) + r^{(n)}_{y,z}(k) r^{(n)}_{x,w}(k) \right) + \frac{1}{n} \sum_{k=n+1}^{2n-1} (2n-k) \left(r^{(n)}_{x,z}(k) r^{(n)}_{y,w}(k) + r^{(n)}_{y,z}(k) r^{(n)}_{x,w}(k) \right).$$

Under Assumption 4.2.2 we have that $\lim_{n\to\infty} \mathbb{E}[(Z^n_{(x,y),1} - Z^n_{(x,y),0})(Z^n_{(z,w),2} - Z^n_{(z,w),1})] = 0$.
For the variances, following similar computations as above and using that $r^{(n)}_{x,z}(-k) = r^{(n)}_{z,x}(k)$, we have

$$\mathbb{E}[(Z^n_{(x,y),1} - Z^n_{(x,y),0})(Z^n_{(z,w),1} - Z^n_{(z,w),0})] = \frac{2}{n}\left\langle \sum_{i=1}^{n} \frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \tilde{\otimes} \frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}}, \sum_{j=1}^{n} \frac{\Delta_j^n G^{(z)}}{\tau_n^{(z)}} \tilde{\otimes} \frac{\Delta_j^n G^{(w)}}{\tau_n^{(w)}} \right\rangle_{\mathcal{H}^{\otimes 2}}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} r_{x,z}^{(n)}(i-j)r_{y,w}^{(n)}(i-j) + r_{y,z}^{(n)}(i-j)r_{x,w}^{(n)}(i-j)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{i} r_{x,z}^{(n)}(i-j)r_{y,w}^{(n)}(i-j) + r_{y,z}^{(n)}(i-j)r_{x,w}^{(n)}(i-j)$$

$$+\frac{1}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n} r_{z,x}^{(n)}(j-i)r_{w,y}^{(n)}(j-i) + r_{z,y}^{(n)}(j-i)r_{w,x}^{(n)}(j-i)$$

$$= \frac{1}{n}\sum_{k=1}^{n-1}(n-k)\left(r_{x,z}^{(n)}(k)r_{y,w}^{(n)}(k) + r_{y,z}^{(n)}(k)r_{x,w}^{(n)}(k) + r_{z,x}^{(n)}(k)r_{w,y}^{(n)}(k) + r_{z,y}^{(n)}(k)r_{w,x}^{(n)}(k)\right)$$

$$+\left(r_{x,z}^{(n)}(0)r_{y,w}^{(n)}(0) + r_{y,z}^{(n)}(0)r_{x,w}^{(n)}(0)\right). \tag{4.17}$$

Under Assumption 4.2.2 we have that $\lim_{n\to\infty}\mathbb{E}[(Z^n_{(x,y),1} - Z^n_{(x,y),0})(Z^n_{(z,w),1} - Z^n_{(z,w),0})] < \infty$, by using that for $a, b \in \mathbb{R}$ we have $|ab| \leq \frac{1}{2}(a^2 + b^2)$.
Further, since we need to use a matrix formulation for our result, we need to associate an element $((x,y),(z,w)))$ with $x, y, z, w = 1, ..., p$ to (i,j) with $i, j = 1, ..., p^2$. Recall that we have the vector

$$(Z^n_{(1,1),b_l} - Z^n_{(1,1),a_l}, ..., Z^n_{(1,p),b_l} - Z^n_{(1,p),a_l}, Z^n_{(2,1),b_l} - Z^n_{(2,1),a_l}, ..., Z^n_{(p,p),b_l} - Z^n_{(p,p),a_l}).$$

and we can rename it as

$$(Z^n_{(1),b_l} - Z^n_{(1),a_l}, ..., Z^n_{(p),b_l} - Z^n_{(p),a_l}, Z^n_{(p+1),b_l} - Z^n_{(p+1),a_l}, ..., Z^n_{(p^2),b_l} - Z^n_{(p^2),a_l}).$$

In this way we are making the following association between (x,y) and i: $i = 1 \leftrightarrow (1,1)$; $i = 2 \leftrightarrow (1,2)$;...; $i = p \leftrightarrow (1,p)$; $i = p+1 \leftrightarrow (2,1)$;...; $i = 2p \leftrightarrow (2,p)$; $i = 2p+1 \leftrightarrow (3,1)$;...; $i = p^2 \leftrightarrow (p,p)$. By symmetry, the same association applies to (z,w) and j. Thus, this can be written compactly as $(i,j) \leftrightarrow ((\lfloor\frac{i-1}{p}\rfloor + 1, i - p\lfloor\frac{i-1}{p}\rfloor), (\lfloor\frac{j-1}{p}\rfloor + 1, j - p\lfloor\frac{j-1}{p}\rfloor))$. Therefore, we have

$$(\boldsymbol{D})_{i,j} = (\boldsymbol{D})_{((\lfloor\frac{i-1}{p}\rfloor+1, i-p\lfloor\frac{i-1}{p}\rfloor),(\lfloor\frac{j-1}{p}\rfloor+1, j-p\lfloor\frac{j-1}{p}\rfloor))},$$

where $(\boldsymbol{D})_{((x,y),(z,w))} = \lim_{n\to\infty}\mathbb{E}[(Z^n_{(x,y),1} - Z^n_{(x,y),0})(Z^n_{(z,w),1} - Z^n_{(z,w),0})]$ equals to (4.17).
Now we have shown the convergence of the variances and covariances for intervals $[a_l, b_l]$ of length 1, however it is straightforward to change the summation indices from $\sum_{i=1}^{n}$ to $\sum_{i=\lfloor na_l\rfloor+1}^{\lfloor nb_l\rfloor}$, and in addition observe that

$$\lim_{n\to\infty}\frac{\lfloor nb_l\rfloor - \lfloor na_l\rfloor}{n} = \lim_{n\to\infty}\frac{nb_l - na_l}{n} + o\left(\frac{1}{n}\right) = b_l - a_l.$$

The last step is to show that for $l = 1, .., d$ and $x, y = 1, ..., p$,

$$\left\| \left(\frac{1}{\sqrt{n}}\sum_{i=\lfloor na_l\rfloor+1}^{\lfloor nb_l\rfloor} \frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \tilde{\otimes} \frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}}\right) \otimes_1 \left(\frac{1}{\sqrt{n}}\sum_{i=\lfloor na_l\rfloor+1}^{\lfloor nb_l\rfloor} \frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \tilde{\otimes} \frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}}\right) \right\|_{\mathcal{H}^{\otimes 2}} \to 0. \tag{4.18}$$

However, this is true under the Assumption 4.2.2 by using the same computations done in the proof of Theorem 3.2 of [GR19]. Let us sketch them. Without loss of generality we can focus on

$d = 1$ and $a_1 = 0$ and $b_1 = 1$. By a simple computations it is possible to show that

$$\left(\frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \tilde{\otimes} \frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}}\right) \otimes_1 \left(\frac{\Delta_j^n G^{(x)}}{\tau_n^{(x)}} \tilde{\otimes} \frac{\Delta_j^n G^{(y)}}{\tau_n^{(y)}}\right) = \frac{1}{4} \sum_{\substack{\{a,a'\}=\{x,y\} \\ \{b,b'\}=\{x,y\}}} r_{a,b}^{(n)}(j-i) \frac{\Delta_i^n G^{(a')}}{\tau_n^{(a')}} \otimes \frac{\Delta_j^n G^{(b')}}{\tau_n^{(b')}}$$

and from this we obtain that

$$\frac{1}{n^2} \left\| \sum_{i,j=1}^{n} \left(\frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \tilde{\otimes} \frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}}\right) \otimes_1 \left(\frac{\Delta_i^n G^{(x)}}{\tau_n^{(x)}} \tilde{\otimes} \frac{\Delta_i^n G^{(y)}}{\tau_n^{(y)}}\right) \right\|_{\mathcal{H}^{\otimes 2}}^2$$

$$= \frac{1}{16n^2} \sum_{\substack{\{a,a'\}=\{x,y\} \\ \{b,b'\}=\{x,y\} \\ \{\alpha,\alpha'\}=\{x,y\} \\ \{\beta,\beta'\}=\{x,y\}}} \sum_{i,j,i',j'=1}^{n} r_{a,b}^{(n)}(j-i) r_{\alpha,\beta}^{(n)}(j'-i') r_{a',\alpha'}^{(n)}(i'-i) r_{b',\beta'}^{(n)}(j'-j). \tag{4.19}$$

The last step is to show that (4.19) goes to zero as $n \to \infty$. By Hölder inequality and some computations, we get that

$$(4.19) \leq \frac{1}{8} \sum_{\substack{\{a,a'\}=\{x,y\} \\ \{b,b'\}=\{x,y\} \\ \{\alpha,\alpha'\}=\{x,y\} \\ \{\beta,\beta'\}=\{x,y\}}} \sum_{k\in\mathbb{Z}} \left[\left(r_{\alpha,\beta}^{(n)}(k)\right)^2 + \left(r_{b',\beta'}^{(n)}(k)\right)^2\right] \left(\frac{1}{\sqrt{n}} \sum_{|i|<n} \left|r_{a,b}^{(n)}(i)\right|\right) \left(\frac{1}{\sqrt{n}} \sum_{|j|<n} \left|r_{a',\alpha'}^{(n)}(j)\right|\right).$$

Then, by Assumption 4.2.2 we have that

$$\sum_{k\in\mathbb{Z}} \left[\left(r_{\alpha,\beta}^{(n)}(k)\right)^2 + \left(r_{b',\beta'}^{(n)}(k)\right)^2\right] < \infty$$

and that

$$\frac{1}{\sqrt{n}} \sum_{|i|<n} \left|r_{a,b}^{(n)}(i)\right| \to 0, \quad \frac{1}{\sqrt{n}} \sum_{|j|<n} \left|r_{a',\alpha'}^{(n)}(j)\right| \to 0,$$

as $n \to \infty$, and therefore we obtain the desired convergence (4.18).

Finally, observe that the matrix denoted by $\mathbf{C}$ in Corollary 4.1.16 is here a $dp^2 \times dp^2$ matrix given by

$$\mathbf{C} = \begin{pmatrix} (b_1 - a_1)\boldsymbol{D} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & (b_d - a_d)\boldsymbol{D} \end{pmatrix},$$

where the $(b_l - a_l)\boldsymbol{D}$ are $p^2 \times p^2$ matrices and the $\mathbf{0}$ are also $p^2 \times p^2$ matrices but only composed by zeros. Thus, we obtain the representation of the statement.

Proof of Proposition 4.2.5

It follows from the tightness of the components of the vector $\mathbf{Z}_t^n$ which is proved following the same arguments as in the proof of Theorem 4.3 of [GR19] or of Theorem 7 of [Co12].

In particular, using the computations done in the proof of Proposition 4.2.4 for the variances, we have

$$\mathbb{E}\left[(Z^n_{(x,y),t} - Z^n_{(x,y),s})^2\right] = \frac{1}{n}\mathbb{E}\left[\left(\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} I_2\left(\frac{\Delta_l^n G^{(x)}}{\tau_n^{(x)}}\tilde{\otimes}\frac{\Delta_l^n G^{(y)}}{\tau_n^{(y)}}\right)\right)^2\right]$$

$$= \frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{n}\frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor}\mathbb{E}\left[\left(\sum_{i=1}^{\lfloor nt\rfloor-\lfloor ns\rfloor} I_2\left(\frac{\Delta_l^n G^{(x)}}{\tau_n^{(x)}}\tilde{\otimes}\frac{\Delta_l^n G^{(y)}}{\tau_n^{(y)}}\right)\right)^2\right]$$

$$= \frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{n}\left[\frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor}\left(\frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor-\lfloor ns\rfloor-1}(n-k)\Big(r^{(n)}_{x,z}(k)r^{(n)}_{y,w}(k) + r^{(n)}_{y,z}(k)r^{(n)}_{x,w}(k)\right.\right.$$

$$\left.\left.+r^{(n)}_{z,x}(k)r^{(n)}_{w,y}(k) + r^{(n)}_{z,y}(k)r^{(n)}_{w,x}(k)\Big) + \Big(r^{(n)}_{x,z}(0)r^{(n)}_{y,w}(0) + r^{(n)}_{y,z}(0)r^{(n)}_{x,w}(0)\Big)\right)\right] \leq C\frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{n},$$

for a constant $C > 0$, where we used the fact that the object inside the square brackets converges and hence it is bounded. Then, by the equivalence of the L^p norms for $1 < p < \infty$ on a fixed (sum of) Wiener chaos (see Theorem 2.7.2 in [NP12]), we obtain that

$$\mathbb{E}\left[(Z^n_{(x,y),t} - Z^n_{(x,y),s})^4\right]^{1/2} \leq C\frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{n}$$

Then, by the Cauchy-Schwarz inequality we have for any $t \geq r \geq s$ and $\lambda > 0$

$$\mathbb{P}\left(|Z^n_{(x,y),t} - Z^n_{(x,y),r}| \geq \lambda, |Z^n_{(x,y),r} - Z^n_{(x,y),s}| \geq \lambda\right)$$

$$\leq \frac{C}{\lambda^4}\frac{\lfloor nt\rfloor - \lfloor nr\rfloor}{n}\frac{\lfloor nr\rfloor - \lfloor ns\rfloor}{n} \leq C\frac{(t-s)^2}{\lambda^4}.$$

Finally, we obtain tightness by using Theorem 13.5 in [Bi99].

Proof of Theorem 4.2.6

First, notice that

$$\left\{\frac{1}{\sqrt{n}}\sum_{l=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}}\frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} - \mathbb{E}\left[\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}}\frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}\right]\right)_{i,j=1,\dots,p}\right\}_{t\in[0,T]} \xrightarrow{d} \left\{\mathbf{D}^{\frac{1}{2}}B_t\right\}_{t\in[0,T]},$$

follows from Theorem 13.1 in [Bi99] by using the finite dimensional distribution convergence proved in Proposition 4.2.4 and by the tightness proved in Proposition 4.2.5.

The independence of B_t from $G_t^{(1)}, \dots, G_t^{(p)}$ is given by the fact that $G_t^{(i)}, \dots, G_t^{(p)}$ belong to the first Wiener chaos, while B_t is the limiting process of objects belonging to the second Wiener chaos. Moreover, we have

$$\left\{\left(G_t^{(i)}\right)_{i=1,\dots,p}, \frac{1}{\sqrt{n}}\sum_{l=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}}\frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} - \mathbb{E}\left[\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}}\frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}\right]\right)_{i,j=1,\dots,p}\right\}_{t\in[0,T]}$$

$$\xrightarrow{d} \left\{ \left(G_t^{(i)} \right)_{i=1,\dots,p}, \mathbf{D}^{\frac{1}{2}} B_t \right\}_{t\in[0,T]}, \tag{4.20}$$

in the space $\mathcal{D}([0,T], \mathbb{R}^p \times \mathbb{R}^{p^2})$, namely the Skorokhod space equipped with the uniform metric. Notice that this result comes from the convergence of the finite dimensional distributions of (4.20), which follows from the arguments at the beginning of this proof together with the orthogonality of different Wiener chaos, and from the tightness of the law of (4.20), which follows from the tightness of each component of the vector proved in Proposition 4.2.5.

Concerning the convergence of the finite dimensional distributions, notice that for any $t \in [0,T]$ each element of $\left(G_t^{(i)}\right)_{i=1,\dots,p}$ and of $\frac{1}{\sqrt{n}} \sum_{l=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} - \mathbb{E}\left[\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} \right] \right)_{i,j=1,\dots,p}$ belong to the first and second Wiener chaos, respectively; hence, for any $i,j,k = 1,...,p$ and $s,t \in [0,T]$, we have

$$\mathbb{E}\left[G_s^{(k)} \frac{1}{\sqrt{n}} \sum_{l=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} - \mathbb{E}\left[\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} \right] \right) \right] = 0.$$

Then, from this argument we have that

$$\lim_{n\to\infty} \mathbb{E}\left[G_s^{(k)} \frac{1}{\sqrt{n}} \sum_{l=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} - \mathbb{E}\left[\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} \right] \right) \right] = 0,$$

and notice that each component of $\frac{1}{\sqrt{n}} \sum_{l=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} - \mathbb{E}\left[\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} \right] \right)_{i,j=1,\dots,p}$ and of $\left(G_t^{(i)}\right)_{i=1,\dots,p}$ converges to a normal distribution. Then, by Theorem 4.1.15 we obtain the convergence of the finite dimensional distributions, that is for any $M \in \mathbb{N}$ and disjoint intervals $[a_m, b_m]$ with $m = 1,...,M$

$$\left(\left(G_{b_m}^{(i)} - G_{a_m}^{(i)} \right)_{i=1,\dots,p}, \frac{1}{\sqrt{n}} \sum_{l=a_m}^{\lfloor nb_m \rfloor} \left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} - \mathbb{E}\left[\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}} \frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}} \right] \right)_{i,j=1,\dots,p} \right)_{m=1,\dots,M}$$

$$\xrightarrow{d} \left(\left(G_{b_m}^{(i)} - G_{a_m}^{(i)} \right)_{i=1,\dots,p}, \boldsymbol{D}^{1/2} (B_{b_m} - B_{a_m}) \right)_{m=1,\dots,M}.$$

In order to obtain the stable convergence it is sufficient to use Proposition 4.1.17. Observe that condition (ii) in that proposition is implied by the convergence in (4.20) using Bayes' theorem and independence of the limiting process $\{B_t\}_{t\in[0,T]}$ from $\left(\{G_t^{(i)}\}_{t\in[0,T]}\right)_{i=1,\dots,p}$. In particular, notice that by Bayes' theorem: (4.20) $\Rightarrow$ for all fixed $n_1,...,n_k \in \mathbb{N}$ and for all

$$A \in \sigma\left(\frac{1}{\sqrt{n_1}} \sum_{l=1}^{\lfloor n_1 t \rfloor} \left(\frac{\Delta_l^{n_1} G^{(i)}}{\tau_{n_1}^{(i)}} \frac{\Delta_l^{n_1} G^{(j)}}{\tau_{n_1}^{(j)}} - \mathbb{E}\left[\frac{\Delta_l^{n_1} G^{(i)}}{\tau_{n_1}^{(i)}} \frac{\Delta_l^{n_1} G^{(j)}}{\tau_{n_1}^{(j)}} \right] \right)_{i,j=1,\dots,p}, \dots, \right.$$

$$\left. \frac{1}{\sqrt{n_k}} \sum_{l=1}^{\lfloor n_k t \rfloor} \left(\frac{\Delta_l^{n_k} G^{(i)}}{\tau_{n_k}^{(i)}} \frac{\Delta_l^{n_k} G^{(j)}}{\tau_{n_k}^{(j)}} - \mathbb{E}\left[\frac{\Delta_l^{n_k} G^{(i)}}{\tau_{n_k}^{(i)}} \frac{\Delta_l^{n_k} G^{(j)}}{\tau_{n_k}^{(j)}} \right] \right)_{i,j=1,\dots,p} \right),$$

with $\mathbb{P}(A) > 0$, we have that

$$\lim_{n\to\infty}\mathbb{P}\left(\left(\frac{1}{\sqrt{n}}\sum_{l=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}}\frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}-\mathbb{E}\left[\frac{\Delta_l^n G^{(i)}}{\tau_n^{(i)}}\frac{\Delta_l^n G^{(j)}}{\tau_n^{(j)}}\right]\right)<x^{(i,j)}\right)_{i,j=1,\dots,p}\middle|A\right)$$

$$=\mathbb{P}\left(\left(\left(\boldsymbol{D}^{1/2}B_t\right)_{i,j}<x^{(i,j)}\right)_{i,j=1,\dots,p}\middle|A\right).$$

In addition, by the independence of the limiting process $\{B_t\}_{t\in[0,T]}$ from $\left(\{G_t^{(i)}\}_{t\in[0,T]}\right)_{i=1,\dots,p}$ we get

$$\mathbb{P}\left(\left(\left(\boldsymbol{D}^{1/2}B_t\right)_{i,j}<x^{(i,j)}\right)_{i,j=1,\dots,p}\middle|A\right)=\mathbb{P}\left(\left(\left(\boldsymbol{D}^{1/2}B_t\right)_{i,j}<x^{(i,j)}\right)_{i,j=1,\dots,p}\right).$$

Following the same computations, it is possible to show the result for any set of points $\{t_1,....,t_a\}\in[0,T]^a$ for $a\in\mathbb{N}$ and not just one $t\in[0,T]$. Therefore, we obtain mixing convergence and hence stable convergence in $\mathcal{D}([0,T],\mathbb{R}^{p^2})$.

Appendix C: Proofs of Section 4.3

Proof of Theorem 4.3.5

Let us assume for now that $p=2$ and that the drift process $\mathbf{U}_t=0$ for any $t\in[0,T]$. We can split our formulation into two components A_n+C_n, where A_n contains the elements that go to zero, while C_n contains the ones that do not converge to zero. In this proof we will use the so-called *blocking technique*, see [BCD11], [CHPP13] and [GR19] for details. Let $1\le h\le n$ and let us first focus on C_n, which is defined as

$$C_n:=\frac{1}{\sqrt{n}}\Bigg(\sum_{j=1}^{\lfloor ht\rfloor}\sum_{i\in I_{h,n}(j)}\sum_{r,m,q,w=1}^{2}\Bigg(\frac{\Delta_i^n G^{(k,r;m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q;w)}}{\tau_n^{(l,q)}}$$

$$-\mathbb{E}\left[\frac{\Delta_i^n G^{(k,r;m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q;w)}}{\tau_n^{(l,q)}}\right]\Bigg)\sigma^{(r,m)}_{(j-1)\Delta_h}\sigma^{(q,w)}_{(j-1)\Delta_h}\Bigg)_{k,l=1,2}, \tag{4.21}$$

where $I_{h,n}(j)=\left\{i\middle|\frac{i}{n}\in\left(\frac{j-1}{h},\frac{j}{h}\right]\right\}$, which can be rewritten as

$$=\frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor ht\rfloor}\sum_{i\in I_{h,n}(j)}V_{(j-1)\Delta_h}\left(\frac{\Delta_i^n G^{(k,r;m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q;w)}}{\tau_n^{(l,q)}}-\mathbb{E}\left[\frac{\Delta_i^n G^{(k,r;m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q;w)}}{\tau_n^{(l,q)}}\right]\right)_{r,m,q,w,k,l=1,2}$$

where

$$V_{(j-1)\Delta_h}=\begin{pmatrix}\sigma_{(j-1)\Delta_h} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \sigma_{(j-1)\Delta_h} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \sigma_{(j-1)\Delta_h} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \sigma_{(j-1)\Delta_h}\end{pmatrix}.$$

Here, $\sigma_{(j-1)\Delta_h} = \left(\sigma^{(r,m)}_{(j-1)\Delta_h}\sigma^{(q,w)}_{(j-1)\Delta_h}\right)^\top_{r,m,q,w=1,2}$ (hence it is a row vector of 16 elements), and $\mathbf{0}$ is a row vector of 16 elements containing only zeros. Hence, $V_{(j-1)\Delta_h}$ is a 4×64 matrix. Now, by Theorem 4.2.6, we have that

$$\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_i^n G^{(k,r;m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q;w)}}{\tau_n^{(l,q)}} - \mathbb{E}\left[\frac{\Delta_i^n G^{(k,r;m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q;w)}}{\tau_n^{(l,q)}}\right]\right)_{r,m,q,w,k,l=1,2}\right\}_{t\in[0,T]}$$

$$\xrightarrow{st}\left\{\boldsymbol{D}^{1/2}B_t\right\}_{t\in[0,T]}, \quad \text{as} \quad n\to\infty,$$

in $\mathcal{D}([0,T],\mathbb{R}^{64})$, where the symmetric matrix $\boldsymbol{D}^{1/2}$ is a 64×64 matrix and B_t is a 64-dimensional Brownian motion. In particular, in order to define the elements of the matrix $\boldsymbol{D}$ we proceed with the following association of (z,y) to $((r_z,m_z,q_z,w_z,k_z,l_z),(r_y,m_y,q_y,w_y,k_y,l_y))$. This part of the proof is similar to the analogue one in the proof of Proposition 4.2.4, but there are some differences. First, in the latter we had the association of fewer elements (namely $(i,j)\leftrightarrow((x,y),(z,w))$). Second, as it is possible to see from (4.21) in the present case there is a summation over r,m,q,w and hence there is no a specific ordering of r,m,q,w when we consider it as a vector (given that we modify the ordering of $\sigma_{(j-1)\Delta_h}$ accordingly). Indeed, let $\nu(r,m,q,w)$ be any permutation of the set $\{(1,1,1,1),(1,1,1,2),(1,1,2,1),(1,1,2,2),...,(2,2,2,2)\}$, namely the set of all the possible combinations of $r,m,q,w\in\{1,2\}$, and let $\nu_s(r,m,q,w)$ determine the s-th element of $\nu(r,m,q,w)$. It is possible to see that $\nu(r,m,q,w)$ contains 2^4 elements, hence $s\in\{1,...,2^4\}$. Then, we have

$$\begin{aligned}
&C_n\\
&= \frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor ht\rfloor}\sum_{i\in I_{h,n}(j)} V_{(j-1)\Delta_h}\left(\frac{\Delta_i^n G^{(k,r;m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q;w)}}{\tau_n^{(l,q)}} - \mathbb{E}\left[\frac{\Delta_i^n G^{(k,r;m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q;w)}}{\tau_n^{(l,q)}}\right]\right)_{r,m,q,w,k,l=1,2}\\
&= \frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor ht\rfloor}\sum_{i\in I_{h,n}(j)} V^\nu_{(j-1)\Delta_h}\left(\frac{\Delta_i^n G^{(k,r;m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q;w)}}{\tau_n^{(l,q)}} - \mathbb{E}\left[\frac{\Delta_i^n G^{(k,r;m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q;w)}}{\tau_n^{(l,q)}}\right]\right)_{\nu(r,m,q,w);k,l=1,2}
\end{aligned}$$

where $V^\nu_{(j-1)\Delta_h}$ has the same form of $V_{(j-1)\Delta_h}$, but with $\sigma^\nu_{(j-1)\Delta_h} = \left(\sigma^{(r,m)}_{(j-1)\Delta_h}\sigma^{(q,w)}_{(j-1)\Delta_h}\right)^\top_{\nu(r,m,q,w)}$ instead of $\sigma_{(j-1)\Delta_h}$. In particular, given a vector $\mathbf{X}=(X_1,...,X_{2^4})$, by the notation $(\mathbf{X})_{\nu(r,m,q,w)}$ we mean that the sth component of the vector is $\nu_s(r,m,q,w)$, that is

$$(\mathbf{X})_{\nu(r,m,q,w)} = (X_{\nu_1(r,m,q,w)}, X_{\nu_2(r,m,q,w)},..., X_{\nu_{2^4}(r,m,q,w)}).$$

We can now write the relation between z and $(r_z,m_z,q_z,w_z,k_z,l_z)$: for $z=s\leftrightarrow(\nu_s(r,m,q,w),1,1)$; $z=2^4+1\leftrightarrow(\nu_1(r,m,q,w),1,2)$;...; $z=2^4+s\leftrightarrow(\nu_s(r,m,q,w),1,2)$;...; $z=2^5+1\leftrightarrow(\nu_1(r,m,q,w),2,1)$;...; $z=2^5+s\leftrightarrow(\nu_s(r,m,q,w),2,1)$;...; $z=2^5+2^4+1\leftrightarrow(\nu_1(r,m,q,w),2,2)$;...; $z=2^5+2^4+s\leftrightarrow(\nu_s(r,m,q,w),2,2)$;...; $z=2^6\leftrightarrow(\nu_{2^4}(r,m,q,w),2,2)$. By symmetry the same relation applies to y and $(r_y,m_y,q_y,w_y,k_y,l_y)$. This can be written compactly as

$$(z,y)\leftrightarrow\left(\left(\nu_{z-\lfloor\frac{z-1}{2^4}\rfloor 2^4}(r,m,q,w), \lfloor\frac{\lfloor\frac{z-1}{2^4}\rfloor}{2}\rfloor+1, \lfloor\frac{z-1}{2^4}\rfloor+1-2\lfloor\frac{\lfloor\frac{z-1}{2^4}\rfloor}{2}\rfloor\right)\right.$$

$$, \left(\nu_{y-\lfloor \frac{y-1}{2^4} \rfloor 2^4}(r,m,q,w), \lfloor \frac{\lfloor \frac{y-1}{2^4} \rfloor}{2} \rfloor + 1, \lfloor \frac{y-1}{2^4} \rfloor + 1 - 2\lfloor \frac{\lfloor \frac{y-1}{2^4} \rfloor}{2} \rfloor \right)\Bigg).$$

Moreover, here it becomes evident why we need Theorem 4.2.6 to hold for jointly Gaussian Brownian measures and not just for independent ones: this is because the vector of Brownian measures is not composed of independent Brownian measures but rather of the same Brownian measures recurring repeatedly.

Continuing with the proof, we observe that by using the properties of the stable convergence and the assumption on the $\mathcal{F}$-measurability of the σs, we obtain the following stale convergence for C_n when h is fixed:

$$C_n \xrightarrow{st} \left\{ \sum_{j=1}^{\lfloor ht \rfloor} V_{(j-1)\Delta_h} \boldsymbol{D}^{1/2} \left(B_{j\Delta_h} - B_{(j-1)\Delta_h} \right) \right\}_{t\in[0,T]}, \quad \text{as} \quad n \to \infty$$

in $\mathcal{D}([0,T],\mathbb{R}^4)$ where the dimensions are 4×64, 64×64 and 64×1, respectively. The convergence in this space is implied by the convergence in $\mathcal{D}([0,T],\mathbb{R}^{64})$. In addition, since the stochastic volatilities are càdlàg we have

$$\sum_{j=1}^{\lfloor ht \rfloor} V_{(j-1)\Delta_h} \boldsymbol{D}^{1/2} \left(B_{j\Delta_h} - B_{(j-1)\Delta_h} \right) \xrightarrow{P} \int_0^t V_s \boldsymbol{D}^{1/2} dB_s,$$

as $h \to \infty$. From this we obtain the stable convergence of C_n.

Concerning the elements in A_n, we can use the same arguments adopted in Propositions 7.1, 7.2 and 7.5 in [GR19] and Theorem 4 of [BCD11]. This is because we can focus on the single elements

$$\sqrt{n}\left[\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Z^{(k,r,m)}}{\tau_n^{(k,r)}} \frac{\Delta_i^n Z^{(l,q,w)}}{\tau_n^{(l,q)}} - \mathbb{E}\left[\frac{\Delta_i^n G^{(k,r,m)}}{\tau_n^{(k,r)}} \frac{\Delta_i^n G^{(l,q,w)}}{\tau_n^{(l,q)}} \right] \int_0^t \sigma_s^{(r,m)} \sigma_s^{(q,w)} ds \right] \tag{4.22}$$

and directly apply their arguments using the assumptions of this theorem. This is because for each (k,r,m,l,q,w) we have that $A_{(k,r,m,l,q,w),n}$ converges to zero in distribution in $\mathcal{D}([0,T],\mathbb{R}^4)$, which implies that they converge jointly to zero stably in distribution.

For the sake of completeness, let us sketch their arguments and show how they apply to our case. First notice that (4.22) can be split into elements that converge to zero and elements that do not. The second ones belong to C_n and so they have already been treated in the previous part of this proof. The first ones are given by:

$$A_{(k,r,m,l,q,w),n} := \underbrace{\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Z^{(k,r,m)}}{\tau_n^{(k,r)}} \frac{\Delta_i^n Z^{(l,q,w)}}{\tau_n^{(l,1)}} - \sigma_{(i-1)\Delta_n}^{(r,m)} \sigma_{(i-1)\Delta_n}^{(q,w)} \frac{\Delta_i^n G^{(k,r,m)}}{\tau_n^{(k,r)}} \frac{\Delta_i^n G^{(l,q,w)}}{\tau_n^{(l,q)}} \right)}_{A_t^n}$$

$$+ \underbrace{\frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \sigma_{(i-1)\Delta_n}^{(r,m)} \sigma_{(i-1)\Delta_n}^{(q,w)} \frac{\Delta_i^n G^{(k,r,m)}}{\tau_n^{(k,r)}} \frac{\Delta_i^n G^{(l,q,w)}}{\tau_n^{(l,q)}}}_{A_t'^{n,h\,(1)}}$$

$$\underbrace{-\frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor ht\rfloor}\sigma^{(r,m)}_{(j-1)\Delta_h}\sigma^{(q,w)}_{(j-1)\Delta_h}\sum_{i\in I_{h,n}(j)}\frac{\Delta_i^n G^{(k,r,m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q,w)}}{\tau_n^{(l,q)}}}_{A_t'^{n,h\,(2)}}$$

$$+\underbrace{\mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\tau_n^{(k,r)}}\frac{\Delta_1^n G^{(l,q,w)}}{\tau_n^{(l,q)}}\right]\left(\frac{\sqrt{n}}{h}\sum_{j=1}^{\lfloor ht\rfloor}\sigma^{(r,m)}_{(j-1)\Delta_h}\sigma^{(q,w)}_{(j-1)\Delta_h}-\frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor nt\rfloor}\sigma^{(r,m)}_{(j-1)\Delta_n}\sigma^{(q,w)}_{(j-1)\Delta_n}\right)}_{A_t''^{n,h}}$$

$$+\underbrace{\mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\tau_n^{(k,r)}}\frac{\Delta_1^n G^{(l,q,w)}}{\tau_n^{(l,q)}}\right]\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor nt\rfloor}\sigma^{(r,m)}_{(j-1)\Delta_n}\sigma^{(q,w)}_{(j-1)\Delta_n}-\sqrt{n}\int_0^t\sigma_s^{(r,m)}\sigma_s^{(q,w)}ds\right)}_{D_t^n}.$$

Further, let $A_t'^{n,h} := A_t'^{n,h\,(1)} - A_t'^{n,h\,(2)}$. Notice that $\#I_{h,n}(j) \in \{\lfloor \frac{n}{h}\rfloor, \lfloor \frac{n}{h}\rfloor + 1\}$, which implies that $\#I_{h,n}(j) = \frac{n}{h} + e_{h,n}(j)$ with $e_{h,n}(j) \in (-1, 1]$, for every $1 \le h \le n$ and $j \ge 1$. Then, we have

$$\tilde{C}_{(k,r,m,l,q,w),n} = \frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor ht\rfloor}\sigma^{(r,m)}_{(j-1)\Delta_h}\sigma^{(q,w)}_{(j-1)\Delta_h}\sum_{i\in I_{h,n}(j)}\frac{\Delta_i^n G^{(k,r,m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q,w)}}{\tau_n^{(l,q)}}$$

$$-\mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\tau_n^{(k,r)}}\frac{\Delta_1^n G^{(l,q,w)}}{\tau_n^{(l,q)}}\right]\frac{\sqrt{n}}{h}\sum_{j=1}^{\lfloor ht\rfloor}\sigma^{(r,m)}_{(j-1)\Delta_h}\sigma^{(q,w)}_{(j-1)\Delta_h}$$

$$=\underbrace{\frac{1}{\sqrt{n}}\sum_{j=1}^{\lfloor ht\rfloor}\sigma^{(r,m)}_{(j-1)\Delta_h}\sigma^{(q,w)}_{(j-1)\Delta_h}\sum_{i\in I_{h,n}(j)}\left(\frac{\Delta_i^n G^{(k,r,m)}}{\tau_n^{(k,r)}}\frac{\Delta_i^n G^{(l,q,w)}}{\tau_n^{(l,q)}}-\mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\tau_n^{(k,r)}}\frac{\Delta_1^n G^{(l,q,w)}}{\tau_n^{(l,q)}}\right]\right)}_{C_{(k,r,m,l,q,w),n}}$$

$$+\sum_{j=1}^{\lfloor ht\rfloor}\sigma^{(r,m)}_{(j-1)\Delta_h}\sigma^{(q,w)}_{(j-1)\Delta_h}\left(\frac{1}{\sqrt{n}}\#I_{h,n}(j)-\frac{\sqrt{n}}{h}\right)\mathbb{E}\left[\frac{\Delta_1^n G^{(k,r,m)}}{\tau_n^{(k,r)}}\frac{\Delta_1^n G^{(l,q,w)}}{\tau_n^{(l,q)}}\right].$$

Since $\frac{1}{\sqrt{n}}\#I_{h,n}(j)-\frac{\sqrt{n}}{h} = \frac{e_{h,n}(j)}{\sqrt{n}}$, the second addendum in the above equation (*i.e.* $\tilde{C}_{(k,r,m,l,q,w),n} - C_{(k,r,m,l,q,w),n}$) goes to zero almost surely as $n \to \infty$. We remark that in this proof the order is always $n \to \infty$ first and $h \to \infty$ afterwards. Notice that $A_{(k,r,m,l,q,w),n} + \tilde{C}_{(k,r,m,l,q,w),n}$ give us (4.22), and that $C_{(k,r,m,l,q,w),n}$ is the (k, r, m, l, q, w)-th element of the vector C_n.

Let $A_t'''^{n,h} := A_t'^{n,h} + A_t''^{n,h}$. Observe that our assumptions are sufficient to use Propositions 7.1, 7.2 and 7.5 in [GR19], which would allow us to prove the convergence to zero of A_t^n, of $A_t'''^{n,h}$ and of D_t^n, respectively. In particular, Assumptions 2.1, 2.2, 4.1 and 4.2 in [GR19] and Assumption (CLT) in [BCD11] are the analogue of Assumptions 4.3.1, 4.3.2 and 4.3.3 together with Definition 4.1.3 (see also the discussion after Assumption 4.3.3). The only main difference between our and their assumptions is that they additionally focus on the limiting object of the convergence of the correlations $r^{(n)}$, which is denoted by $\rho_\vartheta^{i,j}(k)$ in [GR19] and by $\rho(k)$ in [BCD11]. However, for the sake of brevity and clarity of exposition we decided not to focus on this issue in our work. This allows us to get rid of some assumptions, in particular some part of Assumption 2.2 in [GR19] and some part of Assumption (CLT) in [BCD11]. Therefore, by our assumptions and by Propositions 7.1, 7.2 and 7.5 in [GR19], we obtain the convergence to zero

of A_t^n, of $A_t^{\prime\prime\prime n,h}$ and of D_t^n, respectively.

Since C_n converges stably and A_n converges stably to zero, we have that they jointly converge stably. Notice also that $\tilde{C}_n - C_n$ converges to zero almost surely and so it does not affect the limit. This concludes the proof for the case $\mathbf{U}_t = \mathbf{0}$, where $\mathbf{0}$ is a vector of zeros.

Now consider $\mathbf{U}_t \neq \mathbf{0}$. In order to get the stated result we need to prove that the following elements converge uniformly on compacts in probability (u.c.p.) to zero, so that the stable convergence obtained so far in this proof remains the same. These elements are

$$\sqrt{n}\Bigg[\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_i^n Z^{(k,1,1)}}{\tau_n^{(k,1)}}+\frac{\Delta_i^n Z^{(k,2,1)}}{\tau_n^{(k,2)}}+\frac{\Delta_i^n Z^{(k,1,2)}}{\tau_n^{(k,1)}}+\frac{\Delta_i^n Z^{(k,2,2)}}{\tau_n^{(k,2)}}\right)\Delta_i^n U^{(l)}$$

$$+\left(\frac{\Delta_i^n Z^{(l,1,1)}}{\tau_n^{(l,1)}}+\frac{\Delta_i^n Z^{(l,2,1)}}{\tau_n^{(l,2)}}+\frac{\Delta_i^n Z^{(l,1,2)}}{\tau_n^{(l,1)}}+\frac{\Delta_i^n Z^{(l,2,2)}}{\tau_n^{(l,2)}}\right)\Delta_i^n U^{(k)}+\Delta_i^n U^{(k)}\Delta_i^n U^{(l)}\Bigg]_{k,l=1,2}.$$

Thanks to Assumption 4.3.4 (with $p=2$) they go to zero u.c.p. componentwise (*i.e.* for fixed k,l) and, hence, jointly. Thus, using the properties of the stable convergence, the proof for the case $p=2$ is complete.

Now, let $p \geq 2$. The proof for this case follows from the same arguments as the ones just presented for the case $p=2$. Indeed, having $p \geq 2$ does change the dimension of the objects considered (in particular see the matrix $\boldsymbol{D}$ and V_s in the Appendix), but it does not affect in any way the logic of the arguments.

Appendix D: Proofs of Section 4.4

Proof of Proposition 4.4.5

First we prove the statement for fixed k and l. As in the previous proof we concentrate on the case $n \geq 1/\epsilon$. We have

$$\sqrt{n}\left[\left(\sum_{i=1}^{\lfloor nt\rfloor}\Delta_i^n Y^{(k)}\Delta_i^n Y^{(l)}\right)\left(\sum_{i=1}^{\lfloor nt\rfloor}\left(\Delta_i^n Y^{(k)}\right)^2\sum_{i=1}^{\lfloor nt\rfloor}\left(\Delta_i^n Y^{(l)}\right)^2\right)^{-1/2}-\frac{\bar{R}_{t,n}^{(k,l)}}{\sqrt{\bar{R}_{t,n}^{(k,k)}\bar{R}_{t,n}^{(l,l)}}}\right]$$

$$=\frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}-\bar{R}_{t,n}^{(k,l)}\right)}{\sqrt{\bar{R}_{t,n}^{(k,k)}\bar{R}_{t,n}^{(l,l)}}}-\frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}\right)}{\sqrt{\bar{R}_{t,n}^{(k,k)}\bar{R}_{t,n}^{(l,l)}\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\right)^2\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}\right)^2\right)}}$$

$$\cdot\left(\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\right)^2\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}\right)^2\right)^{1/2}-\sqrt{\bar{R}_{t,n}^{(k,k)}\bar{R}_{t,n}^{(l,l)}}\right)$$

$$=\frac{\sqrt{n}}{\sqrt{\bar{R}_{t,n}^{(k,k)}\bar{R}_{t,n}^{(l,l)}}}\left(1,-\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}\right)\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\right)^2\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}\right)^2\right)^{-1/2}\right)$$

$$\left(\left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right) - \bar{R}_{t,n}^{(k,l)}, \right.$$

$$\left. \left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \right)^2 \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right)^2 \right)^{1/2} - \sqrt{\bar{R}_{t,n}^{(k,k)} \bar{R}_{t,n}^{(l,l)}} \right)^\top.$$

Notice that

$$Z_{1,n}^{(k,l)} := \frac{1}{\sqrt{\bar{R}_{t,n}^{(k,k)} \bar{R}_{t,n}^{(l,l)}}}$$

$$\left(1, - \left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right) \left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \right)^2 \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right)^2 \right)^{-1/2} \right)$$

$$\overset{u.c.p.}{\to} \frac{1}{\sqrt{\bar{R}_t^{(k,k)} \bar{R}_t^{(l,l)}}} \left(1, - \frac{\bar{R}_t^{(k,l)}}{\sqrt{\bar{R}_t^{(k,k)} \bar{R}_t^{(l,l)}}} \right) =: Z_1^{(k,l)},$$

by Proposition 4.4.4 and by noticing that for any $\delta > 0$

$$\mathbb{P}\left(\sup_{t \in [\epsilon, T]} \left(\sqrt{\bar{R}_t^{(k,k)} \bar{R}_t^{(l,l)}} \right)^{-1} > \delta \right) = \mathbb{P}\left(\left(\sqrt{\bar{R}_\epsilon^{(k,k)} \bar{R}_\epsilon^{(l,l)}} \right)^{-1} > \delta \right).$$

For the other term, by Theorem 4.3.11 and Lemma 4.4.2 applied to $\mathbf{g} : \mathcal{D}([\epsilon, T], \mathbb{R} \times (0, \infty)^2) \to \mathcal{D}([\epsilon, T], \mathbb{R} \times (0, \infty))$ (where both Skorokhod spaces as well as the Euclidean spaces are equipped with the uniform metric), where $\mathbf{g}$ is defined as $\mathbf{g}(\{\mathbf{x}_t\}_{t \in [\epsilon, T]}) = \{g(x_{1,t}, x_{2,t}, x_{3,t})\}_{t \in [\epsilon, T]} = \{(x_{1,t}, \sqrt{x_{2,t} x_{3,t}})\}_{t \in [\epsilon, T]}$ we have that

$$Z_{2,n}^{(k,l)} := \left\{ \sqrt{n} \left(\left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right) - \bar{R}_{t,n}^{(k,l)}, \right. \right.$$

$$\left. \left. \left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \right)^2 \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right)^2 \right)^{1/2} - \sqrt{\bar{R}_{t,n}^{(k,k)} \bar{R}_{t,n}^{(l,l)}} \right)^\top \right\}_{t \in [\epsilon, T]}$$

$$\overset{st}{\to} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} \sqrt{\frac{\bar{R}_t^{(l,l)}}{\bar{R}_t^{(k,k)}}} & \frac{1}{2} \sqrt{\frac{\bar{R}_t^{(k,k)}}{\bar{R}_t^{(l,l)}}} \end{pmatrix} \right.$$

$$\left. \left(\int_0^t (V_s \boldsymbol{D}^{1/2})_{(k,l)} dB_s^{(k,l)}, \int_0^t (V_s \boldsymbol{D}^{1/2})_{(k,k)} dB_s^{(k,k)}, \int_0^t (V_s \boldsymbol{D}^{1/2})_{(l,l)} dB_s^{(l,l)} \right)^\top \right\}_{t \in [\epsilon, T]}$$

$$= \left\{ \begin{pmatrix} \int_0^t (V_s \boldsymbol{D}^{1/2})_{(k,l)} dB_s^{(k,l)} \\ \frac{1}{2} \sqrt{\frac{\bar{R}_t^{(l,l)}}{\bar{R}_t^{(k,k)}}} \int_0^t (V_s \boldsymbol{D}^{1/2})_{(k,k)} dB_s^{(k,k)} + \frac{1}{2} \sqrt{\frac{\bar{R}_t^{(k,k)}}{\bar{R}_t^{(l,l)}}} \int_0^t (V_s \boldsymbol{D}^{1/2})_{(l,l)} dB_s^{(l,l)} \end{pmatrix} \right\}_{t \in [\epsilon, T]} =: Z_2^{(k,l)},$$

where $B_s^{(k,l)}, B_s^{(k,k)}$ and $B_s^{(l,l)}$ are three independent Brownian motions. Notice that we have not yet investigated whether such functional $\mathbf{g}$ satisfies the conditions of Lemma 4.4.2, but we will do it now. First, we check that $\mathbf{g}$ is continuous and then, using the notations of Lemma 4.4.2, that $\mathbf{g}'(\{\theta\}_{t\in[\epsilon,T]}, \{h\}_{t\in[\epsilon,T]}) = \{\nabla g(\theta_t)h_t\}_{t\in[\epsilon,T]}$, where ∇g is the Jacobian matrix of g. Comparing the notations of Lemma 4.4.2 to the present framework we notice that $\mathcal{D}([\epsilon,T],\mathbb{R}\times(0,\infty)^2) = \mathbb{D}_\phi$, $\mathcal{D}([\epsilon,T],\mathbb{R}^3) = \mathbb{D}_0 = \mathbb{D}$, $\mathcal{D}([\epsilon,T],\mathbb{R}^2) = \mathbb{E}$, $\mathbf{g} = \phi$, and $\mathbf{g}' = \phi'$. Hence, $\theta, \theta_n \in \mathcal{D}([\epsilon,T],\mathbb{R}\times(0,\infty)^2)$ and $h_n, h \in \mathcal{D}([\epsilon,T],\mathbb{R}^3)$ with $\theta_n + r_n^{-1}h_n \in \mathcal{D}([\epsilon,T],\mathbb{R}\times(0,\infty)^2)$, for every $n\in\mathbb{N}$.

To prove the continuity of $\mathbf{g}$, we need to show that for every $(\{(x_{n,t}^{(1)}, x_{n,t}^{(2)}, x_{n,t}^{(3)})\}_{t\in[\epsilon,T]})_{n\in\mathbb{N}} \to \{(x_t^{(1)}, x_t^{(2)}, x_t^{(3)})\}_{t\in[\epsilon,T]}$ in $\mathcal{D}([\epsilon,T],\mathbb{R}\times(0,\infty)^2)$ we have

$$\lim_{n\to\infty}\sup_{t\in[\epsilon,T]}\left\|\left(x_{n,t}^{(1)} - x_t^{(1)}, \sqrt{x_{n,t}^{(2)}x_{n,t}^{(3)}} - \sqrt{x_t^{(2)}x_t^{(3)}}\right)\right\|_\infty = 0.$$

For the first component it is straightforward, while for the second we have

$$\sup_{t\in[\epsilon,T]}\left|\sqrt{x_{n,t}^{(2)}x_{n,t}^{(3)}} - \sqrt{x_t^{(2)}x_t^{(3)}}\right| \le \sup_{t\in[\epsilon,T]}\left|\sqrt{x_{n,t}^{(2)}}\left(\sqrt{x_{n,t}^{(3)}} - \sqrt{x_t^{(3)}}\right)\right| + \left|\sqrt{x_t^{(3)}}\left(\sqrt{x_{n,t}^{(2)}} - \sqrt{x_t^{(2)}}\right)\right|$$

$$\le \sup_{t\in[\epsilon,T]}\left|\sqrt{x_{n,t}^{(2)}}\right|\sup_{t\in[\epsilon,T]}\left|\sqrt{x_{n,t}^{(3)}} - \sqrt{x_t^{(3)}}\right| + \sup_{t\in[\epsilon,T]}\left|\sqrt{x_t^{(3)}}\right|\sup_{t\in[\epsilon,T]}\left|\sqrt{x_{n,t}^{(2)}} - \sqrt{x_t^{(2)}}\right|.$$

Notice that $x_{n,t}^{(3)} - x_t^{(3)} = \left(\sqrt{x_{n,t}^{(3)}} - \sqrt{x_t^{(3)}}\right)\left(\sqrt{x_{n,t}^{(3)}} + \sqrt{x_t^{(3)}}\right)$. Further, by Lemma 4.4.3 we have that $\sup_{t\in[\epsilon,T]}\frac{1}{\sqrt{x_t^{(3)}}} < \infty$ because $x_t^{(3)} : [\epsilon,T] \to (0,\infty)$ and it is a càdlàg function. The same applies to $x_t^{(2)}$. Hence, we have that

$$\sup_{t\in[\epsilon,T]}\left|\sqrt{x_{n,t}^{(3)}} - \sqrt{x_t^{(3)}}\right| \le \sup_{t\in[\epsilon,T]}\frac{1}{\left(\sqrt{x_t^{(3)}} + \sqrt{x_{n,t}^{(3)}}\right)}\sup_{t\in[\epsilon,T]}\left|x_{n,t}^{(3)} - x_t^{(3)}\right|$$

$$\le \sup_{t\in[\epsilon,T]}\frac{1}{\sqrt{x_t^{(3)}}}\sup_{t\in[\epsilon,T]}\left|x_{n,t}^{(3)} - x_t^{(3)}\right| \to 0 \Rightarrow \sup_{t\in[\epsilon,T]}\left|\sqrt{x_{n,t}^{(2)}x_{n,t}^{(3)}} - \sqrt{x_t^{(2)}x_t^{(3)}}\right| \to 0, \quad \text{as } n\to\infty.$$

Regarding the map $\mathbf{g}'$, we need to show that

$$\lim_{n\to\infty}\sup_{t\in[\epsilon,T]}\left\|\left(r_n\left(\theta_{n,t}^{(1)} + r_n^{-1}h_{n,t}^{(1)} - \theta_{n,t}^{(1)}\right) - h_t^{(1)},\right.\right.$$

$$\left.\left. r_n\sqrt{(\theta_{n,t}^{(2)} + r_n^{-1}h_{n,t}^{(2)})(\theta_{n,t}^{(3)} + r_n^{-1}h_{n,t}^{(3)})} - r_n\sqrt{\theta_{n,t}^{(2)}\theta_{n,t}^{(3)}} - \frac{h_t^{(2)}}{2}\sqrt{\frac{\theta_t^{(3)}}{\theta_t^{(2)}}} - \frac{h_t^{(3)}}{2}\sqrt{\frac{\theta_t^{(2)}}{\theta_t^{(3)}}}\right)\right\|_\infty = 0. \quad (4.23)$$

For the first component it is straightforward since $h^{(n)} \to h$, while for the second using the Taylor series for the bivariate function $f(x,y) = r_n\sqrt{(x + r_n^{-1}h_{n,t}^{(2)})(y + r_n^{-1}h_{n,t}^{(3)})}$ we have

$$\sup_{t\in[\epsilon,T]}\left|r_n\sqrt{(\theta_{n,t}^{(2)} + r_n^{-1}h_{n,t}^{(2)})(\theta_{n,t}^{(3)} + r_n^{-1}h_{n,t}^{(3)})} - r_n\sqrt{\theta_{n,t}^{(2)}\theta_{n,t}^{(3)}} - \frac{h_t^{(2)}}{2}\sqrt{\frac{\theta_t^{(3)}}{\theta_t^{(2)}}} - \frac{h_t^{(3)}}{2}\sqrt{\frac{\theta_t^{(2)}}{\theta_t^{(3)}}}\right|$$

$$\leq \sup_{t\in[\epsilon,T]}\left|\frac{h^{(2)}_{n,t}}{2}\sqrt{\frac{\theta^{(3)}_{n,t}}{\theta^{(2)}_{n,t}}}+\frac{h^{(3)}_{n,t}}{2}\sqrt{\frac{\theta^{(2)}_{n,t}}{\theta^{(3)}_{n,t}}}-\frac{h^{(2)}_{t}}{2}\sqrt{\frac{\theta^{(3)}_{t}}{\theta^{(2)}_{t}}}-\frac{h^{(3)}_{t}}{2}\sqrt{\frac{\theta^{(2)}_{t}}{\theta^{(3)}_{t}}}\right|$$

$$+\frac{r_n^{-1}}{2}\sup_{t\in[\epsilon,T]}\left|\frac{h^{(2)}_{n,t}h^{(3)}_{n,t}}{\sqrt{\theta^{(2)}_{n,t}\theta^{(3)}_{n,t}}}-\frac{1}{4}\left(h^{(2)}_{n,t}\right)^2\sqrt{\frac{\theta^{(3)}_{n,t}}{\left(\theta^{(2)}_{n,t}\right)^3}}-\frac{1}{4}\left(h^{(3)}_{n,t}\right)^2\sqrt{\frac{\theta^{(2)}_{n,t}}{\left(\theta^{(3)}_{n,t}\right)^3}}\right|+o(r_n^{-1}). \tag{4.24}$$

Since $\theta^{(2)}_{n,t}$ and $\theta^{(3)}_{n,t}$ take values in $(0,\infty)$ and are càdlàg functions, then by Lemma 4.4.3 we have that, for each $n\in\mathbb{N}$, $\sup_{t\in[\epsilon,T]}\left|\frac{1}{\sqrt{\theta^{(j)}_{n,t}}}\right|<\infty$ for $j=1,2$. Moreover, $\theta^{(2)}_{n,t}$, $\theta^{(3)}_{n,t}$, $h^{(2)}_{n,t}$ and $h^{(3)}_{n,t}$ converge in the uniform metric to $\theta^{(2)}_t$, $\theta^{(3)}_t$, $h^{(2)}_t$ and $h^{(3)}_t$, respectively, while $r_n^{-1}\to 0$. Therefore,

$$\lim_{n\to\infty}\frac{r_n^{-1}}{2}\sup_{t\in[\epsilon,T]}\left|\frac{h^{(2)}_{n,t}h^{(3)}_{n,t}}{\sqrt{\theta^{(2)}_{n,t}\theta^{(3)}_{n,t}}}-\frac{1}{4}\left(h^{(2)}_{n,t}\right)^2\sqrt{\frac{\theta^{(3)}_{n,t}}{\left(\theta^{(2)}_{n,t}\right)^3}}-\frac{1}{4}\left(h^{(3)}_{n,t}\right)^2\sqrt{\frac{\theta^{(2)}_{n,t}}{\left(\theta^{(3)}_{n,t}\right)^3}}\right|=0.$$

Finally, the $o(r_n^{-1})$ term in (4.24) comes from the fact that the remaining terms of the Taylor expansion have a further multiplying factor of r_n^{-1}. Now, observe that

$$\left|\frac{h^{(2)}_{n,t}}{2}\sqrt{\frac{\theta^{(3)}_{n,t}}{\theta^{(2)}_{n,t}}}-\frac{h^{(2)}_{t}}{2}\sqrt{\frac{\theta^{(3)}_{t}}{\theta^{(2)}_{t}}}\right|\leq\left|\frac{h^{(2)}_{n,t}}{2}\left(\sqrt{\frac{\theta^{(3)}_{n,t}}{\theta^{(2)}_{n,t}}}-\sqrt{\frac{\theta^{(3)}_{t}}{\theta^{(2)}_{t}}}\right)\right|+\left|\sqrt{\frac{\theta^{(3)}_{t}}{\theta^{(2)}_{t}}}\left(\frac{h^{(2)}_{n,t}}{2}-\frac{h^{(2)}_{t}}{2}\right)\right|,$$

$$\sqrt{\frac{\theta^{(3)}_{n,t}}{\theta^{(2)}_{n,t}}}-\sqrt{\frac{\theta^{(3)}_{t}}{\theta^{(2)}_{t}}}=\frac{1}{\sqrt{\theta^{(2)}_t}}\left(\sqrt{\theta^{(3)}_{n,t}}-\sqrt{\theta^{(3)}_t}\right)+\frac{1}{\sqrt{\theta^{(2)}_t}}\sqrt{\frac{\theta^{(3)}_{n,t}}{\theta^{(2)}_{n,t}}}\left(\sqrt{\theta^{(2)}_t}-\sqrt{\theta^{(2)}_{n,t}}\right),$$

and that $\sup_{t\in[\epsilon,T]}\left|\frac{1}{\sqrt{\theta^{(2)}_t}}\right|<\infty$ by Lemma 4.4.3 because $\theta^{(2)}_t$ is a càdlàg function with values in $(0,\infty)$ (and the same holds for the other θs). Then, taking the limit as $n\to\infty$ in (4.24), we obtain the desired result (4.23).

Further, since $Z^{(k,l)}_{1,n}\xrightarrow{P}Z^{(k,l)}_1$ and $Z^{(k,l)}_{2,n}\xrightarrow{st}Z^{(k,l)}_2$ we get that $(Z^{(k,l)}_{1,n},Z^{(k,l)}_{2,n})\xrightarrow{st}(Z^{(k,l)}_1,Z^{(k,l)}_2)$. Finally, by applying the continuous mapping theorem for the stable convergence using the continuous function $f(Z^{(k,l)}_{1,n},Z^{(k,l)}_{2,n})=\{Z^{(k,l)}_{1,n}(t)Z^{(k,l)}_{2,n}(t)\}_{t\in[\epsilon,T]}$, we obtain our result for fixed k and l.

For the joint stable convergence we proceed similarly thanks to the uniform metric. Let

$$\Theta_n:=\left\{\begin{pmatrix}Z^{(1,1)}_{1,n}(t) & 0 & \cdots & 0\\ 0 & \ddots & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & Z^{(p,p)}_{1,n}(t)\end{pmatrix}\right\}_{t\in[\epsilon,T]}.$$

Then,

$$\left\{\sqrt{n}\left(\frac{\sum_{i=1}^{\lfloor nt\rfloor}\Delta^n_iY^{(k)}\Delta^n_iY^{(l)}}{\sqrt{\sum_{i=1}^{\lfloor nt\rfloor}\left(\Delta^n_iY^{(k)}\right)^2}\sqrt{\sum_{i=1}^{\lfloor nt\rfloor}\left(\Delta^n_iY^{(l)}\right)^2}}-\frac{\bar{R}^{(k,l)}_{t,n}}{\sqrt{\bar{R}^{(k,k)}_{t,n}\bar{R}^{(l,l)}_{t,n}}}\right)_{k=1,\dots,p;l\leq k}\right\}_{t\in[\epsilon,T]}$$

$$= \left\{ \Theta_n(t) \left(Z_{2,n}^{(1,1)}(t), ..., Z_{2,n}^{(p,p)}(t) \right)^\top \right\}_{t\in[\epsilon,T]}.$$

Using Lemma 4.3.13 and the arguments used before for fixed k and l we obtain the u.c.p. convergence of Θ_n. Now, we would like to prove the stable convergence for $\left\{ \left(Z_{2,n}^{(1,1)}(t), ..., Z_{2,n}^{(p,p)}(t) \right)^\top \right\}_{t\in[\epsilon,T]}$. Define the function $\tilde{\mathbf{g}} : \mathcal{D}([\epsilon,T],(0,\infty)^p \times \mathbb{R}^{\frac{p}{2}(p-1)}) \to \mathcal{D}([\epsilon,T],(0,\infty)^{\frac{p}{2}(p+1)} \times \mathbb{R}^{\frac{p}{2}(p+1)})$ as (using our variables)

$$\tilde{\mathbf{g}} \left(\left\{ \left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right)_{k=1,...,p; l\leq k} \right\}_{t\in[\epsilon,T]} \right)$$

$$= \left\{ \left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(1)}}{\bar{\tau}_n^{(1)}} \right)^2, \sqrt{\left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(1)}}{\bar{\tau}_n^{(1)}} \right)^2 \right)^2}, \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(2)}}{\bar{\tau}_n^{(2)}} \frac{\Delta_i^n Y^{(1)}}{\bar{\tau}_n^{(1)}}, \right.$$

$$\sqrt{\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(2)}}{\bar{\tau}_n^{(2)}} \right)^2 \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(1)}}{\bar{\tau}_n^{(1)}} \right)^2}, \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(2)}}{\bar{\tau}_n^{(2)}} \right)^2, \sqrt{\left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(2)}}{\bar{\tau}_n^{(2)}} \right)^2 \right)^2},$$

$$\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(3)}}{\bar{\tau}_n^{(3)}} \frac{\Delta_i^n Y^{(1)}}{\bar{\tau}_n^{(1)}}, \sqrt{\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(3)}}{\bar{\tau}_n^{(3)}} \right)^2 \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(1)}}{\bar{\tau}_n^{(1)}} \right)^2}, \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(3)}}{\bar{\tau}_n^{(3)}} \frac{\Delta_i^n Y^{(2)}}{\bar{\tau}_n^{(2)}},$$

$$\sqrt{\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(3)}}{\bar{\tau}_n^{(3)}} \right)^2 \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(2)}}{\bar{\tau}_n^{(2)}} \right)^2},$$

$$\left. ..., \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(p)}}{\bar{\tau}_n^{(p)}} \right)^2, \sqrt{\left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(p)}}{\bar{\tau}_n^{(p)}} \right)^2 \right)^2} \right)^\top \Bigg\}_{t\in[\epsilon,T]}.$$

Notice that the above formulation is just a multidimensional extension of the formulation of $\mathbf{g}$ given in the first part of this proof. In particular, the function $\tilde{\mathbf{g}}$ can be seen as associating to any three dimensional vector of the form

$$\left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}, \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \right)^2, \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right)^2 \right)$$

a two dimensional vector of the form

$$\left(\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}, \sqrt{\frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}} \right)^2 \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} \right)^2} \right).$$

for any $k = 1, .., p$ and $l \leq k$.

Then, as in the first part of the proof thanks to Theorem 4.3.11 and Lemma 4.4.2 applied to $\tilde{\mathbf{g}}$, we obtain the stable convergence as in the statement. Moreover, we use the same arguments as the ones used for fixed k and l to prove that $\tilde{\mathbf{g}}$ satisfies the required conditions of Lemma 4.4.2 thanks to the properties of the uniform metric.

Proof of Proposition 4.4.9

Fix k, l. We have

$$\sqrt{n}\left[\frac{\sum_{i=1}^{\lfloor nt\rfloor}\Delta_i^n Y^{(k)}\Delta_i^n Y^{(l)}}{\sum_{i=1}^{\lfloor nT\rfloor}\Delta_i^n Y^{(k)}\Delta_i^n Y^{(l)}} - \frac{\bar{R}_{t,n}^{(k,l)}}{\bar{R}_{T,n}^{(k,l)}}\right] = \frac{\sqrt{n}}{\bar{R}_{T,n}^{(k,l)}}\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} - \bar{R}_{t,n}^{(k,l)}\right)$$

$$-\frac{\sqrt{n}}{\bar{R}_{T,n}^{(k,l)}}\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}\right)\left(\frac{1}{n}\sum_{i=1}^{\lfloor nT\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{\lfloor nT\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} - \bar{R}_{T,n}^{(k,l)}\right),$$

which can be rewritten in vector notation as

$$= \frac{\sqrt{n}}{\bar{R}_{T,n}^{(k,l)}}\left(1, -\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}\right)\left(\frac{1}{n}\sum_{i=1}^{\lfloor nT\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}\right)^{-1}\right)$$

$$\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} - \bar{R}_{t,n}^{(k,l)}, \frac{1}{n}\sum_{i=1}^{\lfloor nT\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} - \bar{R}_{T,n}^{(k,l)}\right)^{\top}.$$

Notice that

$$\frac{1}{\bar{R}_{T,n}^{(k,l)}}\left(1, -\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}\right)\left(\frac{1}{n}\sum_{i=1}^{\lfloor nT\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}}\right)^{-1}\right) \overset{u.c.p}{\to} \frac{1}{\bar{R}_T^{(k,l)}}\left(1, -\frac{\bar{R}_t^{(k,l)}}{\bar{R}_T^{(k,l)}}\right)$$

using Proposition 4.4.8, and that

$$\left\{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} - \bar{R}_{t,n}^{(k,l)}, \frac{1}{n}\sum_{i=1}^{\lfloor nT\rfloor}\frac{\Delta_i^n Y^{(k)}}{\bar{\tau}_n^{(k)}}\frac{\Delta_i^n Y^{(l)}}{\bar{\tau}_n^{(l)}} - \bar{R}_{T,n}^{(k,l)}\right)^{\top}\right\}_{t\in[0,T]}$$

$$\overset{st}{\to}\left\{\left(\int_0^t (V_s\boldsymbol{D}^{1/2})_{(k,l)}dB_s^{(k,l)}, \int_0^T (V_s\boldsymbol{D}^{1/2})_{(k,l)}dB_s^{(k,l)}\right)^{\top}\right\}_{t\in[0,T]}, \quad \text{in} \quad \mathcal{D}([0,T],\mathbb{R}^2),$$

by Theorem 4.3.10, where $(V_s\boldsymbol{D}^{1/2})_{(k,l)}$ indicates that we are considering only the (k,l) row of the matrix $(V_s\boldsymbol{D}^{1/2})$. Then using the properties of the stable convergence and the continuous mapping theorem we conclude the proof for fixed k, l.

For the joint case we proceed as we have done in the proof of Proposition 4.4.5. In particular, we have an analogue of Θ_n which converges u.c.p. since its elements do. Moreover, we have an analogue of $\left\{\left(Z_{2,n}^{(1,1)}(t), ..., Z_{2,n}^{(p,p)}(t)\right)^{\top}\right\}_{t\in[\epsilon,T]}$ whose stable convergence in the Skorokhod space is guaranteed by Theorem 4.3.11 and by the continuous mapping theorem using the function given by $\mathbf{g}(\{x_1(t), ..., x_{\frac{p}{2}(p+1)}(t)\}_{t\in[0,T]}) = \{(x_1(t), x_1(T), ..., x_{\frac{p}{2}(p+1)}(t), x_{\frac{p}{2}(p+1)}(T))\}_{t\in[0,T]}$. Finally, using the properties of the stable convergence we obtain the stated result.

5

On the signature of the fBm

In this chapter we turn our attention to the expected signature of the fractional Brownian motion with Hurst parameter $H > 1/2$. The expected signature is of interest since it is one of the main protagonists of rough path theory and of its applications to real worlds problems, especially on machine learning. We will now introduce the notation and some preliminaries for this chapter and then present our main results.

5.1 Notation and Preliminaries

The notation used in this chapter is in line with the one used in the papers of Baudoin and Coutin [BC07], Lyons [Te98], and Lyons and Victoir [LV04] and in the book by Lyons, Caruana, and Lévy [LCL07].

The fractional Brownian motion is defined as follows.

Definition 5.1.1. *Let H be a constant belonging to $(0,1)$. A fractional Brownian motion (fBm) $(B^H(t))_{t0}$ of Hurst index H is a continuous and centered Gaussian process with covariance function*

$$\mathbb{E}\left[B^H(t)B^H(s)\right] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

For $H = \frac{1}{2}$ then the fBm is a Bm. Further, the multi-dimensional fBm has coordinate components that are independent and identically distributed copies of one dimensional fBm. Moreover, recall that the fBm has the following two properties:
(i) (scaling) for any $c > 0$, $c^H B^H_{\cdot/c}$ is a fBm,
(ii) (stationary increments) for any $h > 0$, $B^H_{\cdot+h} - B^H_h$ is a fBm.

Now, we define the simplex $\Delta^k[s,t]$, where $T \in [0,\infty)$ and $s,t \in [0,T]$,

$$\Delta^k[s,t] := \{(t_1,...,t_k) \in [s,t]^k : t_1 < ... < t_k\}.$$

Further, we define the following iterated integrals. Let $I = (i_1,...,i_k) \in \{1,..,d\}^k$ be a word with

length k then

$$\int_{\Delta^k[s,t]} dB^{H,I} := \int_{s\leq t_1<\ldots<t_k\leq t} dB_{t_1}^{H,i_1}\cdots dB_{t_k}^{H,i_k}$$

and

$$\int_{\Delta^k[s,t]} dB^{H,m,I} := \int_{s\leq t_1<\ldots<t_k\leq t} dB_{t_1}^{H,m,i_1}\cdots dB_{t_k}^{H,m,i_k},$$

where B^H is the fractional Brownian motion with Hurst parameter H and $B^{H,m}$ is its linear piecewise approximation. In addition, $B^{H,i}$ is the i-th coordinate component of the fBm B^H and the iterated integrals can be defined in the sense of Young [Yo36].

In the following, we provide the definition of the linear piecewise approximation $B^{H,m}$. The only requirement we need is that the linear approximation comes from a uniform grid. For the sake of clarity, we choose to present the setting using the uniform mesh of size m^{-1}, namely where the distances between any two consecutive grid points are all equal to m^{-1} (*i.e.* $q_m = m^{-1}$).

Let $s,u\in[0,\infty)$ and consider an interval of time $[s,u]$. Let $t_i^{[s,u]} := \frac{i}{m}(u-s)+s$ for $i=0,\ldots,m$. If $t\in[t_i^{[s,u]},t_{i+1}^{[s,u]}]$ then

$$B^{H,m}_{[u,s],t} := B^{H,m}_{t_i^{[s,u]}} + \frac{m}{u-s}(t-t_i^{[s,u]})(B^{H,m}_{t_{i+1}^{[s,u]}} - B^{H,m}_{t_i^{[s,u]}}).$$

Observe that the definition of the linear piecewise approximation depends on the extremes of the interval considered, in this case u and s. Notice also that B_t^m satisfies the scaling and stationary increments property, once the interval is properly modified. Indeed, we have that for the stationary increments property the interval to be taken is, for any $h>0$, $[s+h,u+h]$ and we have

$$B^{H,m}_{[s+h,u+h],t+h} = B^{H,m}_{t_i^{[s+h,u+h]}} + \frac{m}{u-s}(t+h-t_i^{[s+h,u+h]})(B^{H,m}_{t_{i+1}^{[s+h,u+h]}} - B^{H,m}_{t_i^{[s+h,u+h]}})$$

$$\overset{Law}{=} B^{H,m}_{t_i^{[u,s]}} + \frac{m}{u-s}(t-t_i^{[u,s]})(B^{H,m}_{t_{i+1}^{[u,s]}} - B^{H,m}_{t_i^{[u,s]}}) = B^{H,m}_{[u,s],t}$$

and for the scaling property the interval to be taken, for any $c>0$, is $[s/c,u/c]$ and we have

$$c^H B^{H,m}_{[u/c,s/c],t/c} = c^H B^{H,m}_{t_i^{[s/c,u/c]}} + c^H\frac{cm}{u-s}(t/c-t_i^{[s/c,u/c]})(B^{H,m}_{t_{i+1}^{[s/c,u/c]}} - B^{H,m}_{t_i^{[s/c,u/c]}})$$

$$= c^H B^{H,m}_{t_i^{[s,u]}/c} + c^H\frac{cm}{u-s}(t/c-t_i^{[s,u]}/c)(B^{H,m}_{t_{i+1}^{[s,u]}/c} - B^{H,m}_{t_i^{[s,u]}/c})$$

$$\overset{Law}{=} B^{H,m}_{t_i^{[u,s]}} + \frac{m}{u-s}(t-t_i^{[u,s]})(B^{H,m}_{t_{i+1}^{[u,s]}} - B^{H,m}_{t_i^{[u,s]}}) = B^{H,m}_{[u,s],t}.$$

Further, from now on we will not write explicitly the interval of the linear approximation, unless it is necessary in order to avoid confusion. Hence, we denote $t_i := t_i^{[s,u]}$ and $B_t^{H,m} := B^{H,m}_{[u,s],t}$.

One of the main concepts of this chapter is the signature of the fractional Brownian motion. In order to give a definition of signature we need to introduce first the following concept: the *p-variation* of a path.

Definition 5.1.2. *Let $p\geq 1$ be a real number. Let $X: J\to E$ be continuous path, where J is a compact interval and E is a finite dimensional Banach space. The p-variation of X on the*

interval J is defined by

$$\|X\|_{p,J} = \left[\sup_{\mathcal{D}\subset J}\sum_{i=0}^{r-1}|X_{t_i} - X_{t_{i+1}}|^p\right]^{\frac{1}{p}},$$

where the supremum is taken over all the partitions of J.

A path X is said to be of finite p-variation over the interval J if $\|X\|_{p,J} < \infty$. Further, for the signature of a continuous path we mean the following.

Definition 5.1.3. *Let E be a finite dimensional Banach space. Let $X : [0,T] \to E$ be a continuous path with finite p-variation for some $p < 2$. The signature of X is the element $S(X)$ of $T(E) := \{\mathbf{a} = (a_0, a_1, ...)|\forall n \geq 0, a_n \in E^{\otimes n}\}$ defined as follows*

$$S(X_{[0,T]}) = (1, X^1_{[0,T]}, X^2_{[0,T]}, ...),$$

where, for each $n \geq 1$,

$$X^n_{[0,T]} = \int_{0<u_1<...<u_n<T} dX_{u_1} \otimes ... \otimes dX_{u_n}.$$

The integration in the previous definition is defined in the sense of Young [Yo36].

The fractional Brownian motion is a p-variation path for all $p > \frac{1}{H}$. Hence, for $H \in (\frac{1}{2}, 1)$ we have $p < 2$. Notice that for I, a word composed by the letters $(i_1, ..., i_{2k})$ where each $i_l \in 1, ..., d$, we have that $dB^{H,I} := dB^{H,i_1} \cdots dB^{H,i_{2k}}$. Therefore, we have that the $2k$-th element of the expected signature of the fBm and its piecewise approximation are respectively given by

$$\mathbb{E}\left(\int_{0<u_1<...<u_{2k}<T} dB^H_{u_1} \otimes \cdots \otimes dB^H_{u_{2k}}\right) = \sum_{i_1=1}^{d}\cdots\sum_{i_{2k}=1}^{d}\mathbb{E}\left(\int_{\Delta^{2k}[0,T]} dB^{H,I}\right) e_{i_1}\otimes\cdots\otimes e_{i_{2k}}, \quad \text{and} \tag{5.1}$$

$$\mathbb{E}\left(\int_{0<u_1<...<u_{2k}<T} dB^{H,m}_{u_1} \otimes \cdots \otimes dB^{H,m}_{u_{2k}}\right) = \sum_{i_1=1}^{d}\cdots\sum_{i_{2k}=1}^{d}\mathbb{E}\left(\int_{\Delta^{2k}[0,T]} dB^{H,m,I}\right) e_{i_1}\otimes\cdots\otimes e_{i_{2k}}, \tag{5.2}$$

where $(e_1, ..., e_d)$ is the basis of $\mathbb{R}^d$. Notice that we will consider only the $2k$ case since for odd values the expected value of the iterated integrals is zero due to the symmetry of the fBm.

We recall two results which can be found for example in Baudoin and Coutin's paper [BC07]. First, we present a reformulation of the Isserlis' (or Wick's) theorem.

Lemma 5.1.4. *Let $G = (G_1, ..., G_{2k})$ be a centered Gaussian vector. We have*

$$\mathbb{E}(G_1 \cdots G_{2k}) = \frac{1}{k!2^k}\sum_{\sigma\in\mathcal{G}_{2k}}\prod_{l=1}^{k}\mathbb{E}(G_{\sigma(2l-1)}G_{\sigma(2l)}), \tag{5.3}$$

where $\mathcal{G}_{2k}$ is the group of the permutations of the set $\{1, ..., 2k\}$.

Therefore, from the previous lemma we have

$$\mathbb{E}\left(\int_{\Delta^{2k}[s,t]} dB^{H,m,I}\right) = \frac{1}{k!2^k}\sum_{\sigma\in\mathcal{G}_{2k}}\int_{\Delta^{2k}[s,t]}\prod_{l=1}^{k}\mathbb{E}\left(\frac{dB^{H,m,i_{\sigma(2l-1)}}}{dt_{\sigma(2l-1)}}\frac{dB^{H,m,i_{\sigma(2l)}}}{dt_{\sigma(2l)}}\right)dt_1\cdots dt_{2k}. \tag{5.4}$$

Notice that if $t_{\sigma(2l-1)} \in [t_j, t_{j+1}]$ and $t_{\sigma(2l)} \in [t_i, t_{i+1}]$ then, in the case of the uniform mesh of size m^{-1} (*i.e.* $q_m = m^{-1}$)

$$\mathbb{E}\left(\frac{dB^{H,m,i_{\sigma(2l-1)}}}{dt_{\sigma(2l-1)}}\frac{dB^{H,m,i_{\sigma(2l)}}}{dt_{\sigma(2l)}}\right) = \delta_{i_{\sigma(2l)},i_{\sigma(2l-1)}} H(2H-1)m^2 \int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} |x-y|^{2H-2}dxdy. \quad (5.5)$$

Further, recall Theorem 31 of [BC07].

Theorem 5.1.5. *Assume $H > \frac{1}{2}$. Letting $I = (i_1, ..., i_{2k})$ be a word, then*

$$\mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,I}\right) = \frac{H^k(2H-1)^k}{k!2^k}\sum_{\sigma\in\mathcal{G}_{2k}}\int_{\Delta^{2k}[0,1]}\prod_{l=1}^{k}\delta_{i_{\sigma(2l)},i_{\sigma(2l-1)}}|t_{\sigma(2l)}-t_{\sigma(2l-1)}|^{2H-2}dt_1\cdots dt_{2k}. \quad (5.6)$$

We conclude this section with the following remark.

Remark 5.1.6. By using the scaling and the stationary increments property of the fBm, we have that

$$\mathbb{E}\left(\int_{\Delta^{2k}[s,t]} dB^{H,I}\right) = \mathbb{E}\left(\int_{\Delta^{2k}[0,t-s]} dB^{H,I}\right) = (t-s)^{2kH}\mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,I}\right), \quad \text{and}$$

$$\mathbb{E}\left(\int_{\Delta^{2k}[s,t]} dB^{H,m,I}\right) = \mathbb{E}\left(\int_{\Delta^{2k}[0,t-s]} dB^{H,m,I}\right) = (t-s)^{2kH}\mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,m,I}\right).$$

Hence, it is sufficient to focus on the case $\Delta^{2k}[0,1]$ in the proofs of our results.

5.2 Main Results

The first result shows the rate of convergence of the expected signature of the linear piecewise approximation of the fBm to its exact value. In this result, we also give an explicit expression for the coefficient. Moreover, we state the results in its generality, meaning that we use the expression q_m to stands for the size of the uniform mesh considered. In formulas, $q_m := t_{i+1} - t_i$. Examples are $q_m = m^{-1}$ and $q_m = 2^{-m}$.

Theorem 5.2.1. *Let $H > \frac{1}{2}$, $T \in [0, \infty)$ and $0 \le s < t \le T$. Let $I = (i_1, ..., i_{2k})$ be a word where $i_l \in \{1, ..., d\}$ for $l = 1, ..., 2k$, then for all m*

$$\left|\mathbb{E}\left(\int_{\Delta^{2k}[s,t]} dB^{H,I}\right) - \mathbb{E}\left(\int_{\Delta^{2k}[s,t]} dB^{H,m,I}\right)\right| \le Cq_m^{2H}(t-s)^{2kH},$$

where q_m is the size of the uniform mesh considered, $C = \frac{\tilde{A}k(2k-1)}{(k-1)!2^k}$ and $\tilde{A}$ is a finite constant depending only on H.

It is important to stress that for the Brownian motion case, namely $H = \frac{1}{2}$, Ni and Xu in [NX15] proved that the sharp rate of convergence is given by 1, which is in line with the result presented here (*i.e.* $q_m^1 = q_m^{2\frac{1}{2}}$).

The following theorem provides a sharp bound for the value for the expected signature of the fBm. In the $H > \frac{1}{2}$ case, this result in particular implies Chevyrev-Lyons' result that the expected signature of fBm has infinite radius of convergence. This in turns implies that the expected signature determines the signature in distribution.

Theorem 5.2.2. *Let $H > \frac{1}{2}$, $T \in [0, \infty)$ and $0 \leq s < t \leq T$. Let $I = (i_1, ..., i_{2k})$ be a word where $i_l \in \{1, ..., d\}$ for $l = 1, ..., 2k$, then*

$$\mathbb{E}\left(\int_{\Delta^{2k}[s,t]} dB^{H,I}\right) \leq \frac{(t-s)^{2kH}}{k!2^k}.$$

In order to present the results on the cubature method we require an extended formal introduction. For this reason we present the results in all their completeness and formality in Section 6. Here we provide a high-level description of the main results. The first theorem can be described as follow:

Theorem 5.3.2 (High-level description). We provide an explicit bound for the approximation error of the cubature method for the fBm with $H > 1/2$.

In other words, we provide the bound of the difference between the expected solution of an SDE driven by a fBm with $H > 1/2$ and the deterministic solution obtained by the cubature method. In particular, with this result we extend Proposition 2.1, Lemma 3.1 and Proposition 3.2 of [LV04] to the fBm case for $H > 1/2$.

The second main result can be summarised as follow:

Theorem 5.3.5 (High-level description). We provide the cubature formula for the one dimensional fBm with $H > 1/2$ up to degree 5.

It is important to remark that for $H = \frac{1}{2}$, *i.e.* when the fBm is just a Brownian motion, the results obtained in this theorem perfectly coincide with the results of Lyons and Victoir obtained in [LV04].

5.3 The cubature method for the fBm

In this section we present the results on cubature method for the fBm in details. We start this section with a brief introduction to the cubature method for the fractional Brownian motion.

The cubature method is a numerical method used to obtain approximate solutions to SDEs and parabolic PDEs. The first main step is to obtain the cubature formula, which we define below for the d-dimensional fBm (for $d \in \mathbb{N}$). The setting is the same as the one presented for the Bm case in [LV04]. So, the probability space we are working on is $(C_0^0([0,T], \mathbb{R}^d), \mathcal{F}, \mathbb{P})$ where $C_0^0([0,T], \mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued continuous functions defined in $[0,T]$ which starts at zero (*i.e.* the Wiener space), $\mathcal{F}$ is its Borel σ-field and $\mathbb{P}$ is the Wiener measure. As in [LV04], let $B_t : C_0^0([0,T], \mathbb{R}^d) \to \mathbb{R}^d$ such that $B_t^i(\omega) = \omega^i(t)$ for $t \in [0,T]$ and $i = 1, .., d$,

then $\{B_t\}_{t\in[0,T]} = \{B_t^1, ..., B_t^d\}_{t\in[0,T]}$ is a $\mathbb{R}^d$-valued Brownian motion on the probability space $(C_0^0([0,T],\mathbb{R}^d), \mathcal{F}, \mathbb{P})$.

Now, recall that it is possible to write the fBm in terms of a Bm, as done in [BPQ]. In particular we follow the formulation in [BPQ] and consider the stochastic process $B_t^{H,i}(\omega) = \int_0^t K(t,s)dB_s^i(\omega) \left(= \int_0^t K(t,s)d\omega^i(s)\right)$ for $t \in [0,T]$ and $i = 1,..,d$, where $H \geq 1/2$. Then, the process $\{B_t^H\}_{t\in[0,T]} = \{B_t^{H,1}, ..., B_t^{H,d}\}_{t\in[0,T]}$ is a $\mathbb{R}^d$-valued fractional Brownian motion on the probability space $(C_0^0([0,T],\mathbb{R}^d), \mathcal{F}, \mathbb{P})$.
As in [LV04], the process $\{\xi_{t,x}\}_{t\in[0,T]}$ is a stochastic process like our $\{B_t\}_{t\in[0,T]}$, that is $\xi_{t,x} : C_0^0([0,T],\mathbb{R}^d) \to \mathbb{R}^N$ (for $N \in \mathbb{N}$), hence $\xi_{t,x} : \omega \mapsto \xi_{t,x}(\omega)$. Further, we will consider the process $\{\hat{B}_t^H\}_{t\in[0,T]} = \{t, B_t^{H,1}, ..., B_t^{H,d}\}_{t\in[0,T]}$.

Moreover, let $C_{0,bv}^0([0,T],\mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued continuous paths of bounded variation defined in $[0,T]$ and which starts at zero.

We present the definition of the cubature formula for the fBm for $H \geq 1/2$ on the Wiener space.

Definition 5.3.1. *Let $m \in \mathbb{N}$ and $H \geq \frac{1}{2}$. Let define $\mathcal{A}_m := \{(i_1, ..., i_k) \in \{0, ..., d\}^k, 2Hk + (2-2H) \times card\{l, i_l = 0\} \leq m$ and $k \in \mathbb{N}\}$. We say that the paths*

$$\bar{\omega}_1, ..., \bar{\omega}_n \in C_{0,bv}^0([0,T],\mathbb{R}^d)$$

and the positive weights $\lambda_1, ..., \lambda_n$ define a cubature formula of degree m at time T, if and only if, for all $(i_1, ..., i_k) \in \mathcal{A}_m$,

$$\mathbb{E}\left[\int_{0<t_1<,...,<t_k<T} d\hat{B}_{t_1}^{H,i_1} \cdots d\hat{B}_{t_k}^{H,i_k}\right] = \sum_{j=1}^n \lambda_j \int_{0<t_1<,...,<t_k<T} d\hat{\omega}_j^{i_1}(t_1) \cdots d\hat{\omega}_j^{i_k}(t_k).$$

where, for $l = 1, ..., k$,

$$\hat{B}_{t_l}^{H,i_l} := \begin{cases} t_l, & if \quad i_l = 0, \\ B_{t_l}^{i_l H} & if \quad i_l \neq 0, \end{cases} \qquad and \quad \hat{\omega}_{t_l}^{i_l} := \begin{cases} t_l, & if \quad i_l = 0, \\ \bar{\omega}_{t_l}^{i_l} & if \quad i_l \neq 0. \end{cases}$$

Notice that the formulation of $\mathcal{A}_m$ comes from the fact that for a word I, which is a set $(i_1, ..., i_k) \in \{0, ..., d\}^k$ for some $k \in \mathbb{N}$, we have that

$$\mathbb{E}\left(\int_{\Delta^k[s,t]} d\hat{B}^{H,I}\right) = \mathbb{E}\left(\int_{\Delta^k[0,t-s]} d\hat{B}^{H,I}\right) = (t-s)^{kH+(1-H)\times card\{j,i_j=0\}} \mathbb{E}\left(\int_{\Delta^k[0,1]} d\hat{B}^{H,I}\right)$$

(see [NNRT09] Proposition 4.8 for similar discussions) and the reason why we multiply by 2 in $\mathcal{A}_m$ is to be consistent with the formulation provided by [LV04] for the Bm case.

Once the cubature formula is obtained, it is possible to derive approximate solutions of SDEs driven by fBm. Indeed, let $C_b^\infty(\mathbb{R}^N, \mathbb{R}^N)$ be the space of $\mathbb{R}^N$-valued smooth functions defined on $\mathbb{R}^N$ whose derivatives of any order are bounded. We regard elements of $C_b^\infty(\mathbb{R}^N, \mathbb{R}^N)$ as vector fields on $\mathbb{R}^N$. In other words, for $V \in C_b^\infty(\mathbb{R}^N, \mathbb{R}^N)$ with $V = (V^{(1)}, ..., V^{(N)})$ and for $f \in C_b^\infty(\mathbb{R}^N)$, we let $Vf = \sum_{j=1}^N V^{(j)}(x) \frac{\partial}{\partial x^{(j)}} f(x)$. Consider $V_0, ..., V_d$ to be such vector fields. Let $\xi_{t,x}$, $t \in [0,T], x \in \mathbb{R}^N$, be the solution of the SDE:

$$\mathrm{d}\xi_{t,x} = \sum_{i=0}^{d} V_i(\xi_{t,x})\mathrm{d}\hat{B}_t^{H,i}, \quad \text{with} \quad \xi_{0,x} = x. \tag{5.7}$$

Moreover, let $\Phi_{T,x}(\omega_j^*)$, where $\omega_j^* \in C_0^0([0,T],\mathbb{R}^d)$, be the solution at time T of the ODE:

$$\mathrm{d}y_{t,x,j} = \sum_{i=0}^{d} V_i(y_{t,x})\mathrm{d}\hat{\omega}_j^i(t), \quad \text{with} \quad y_{0,x} = x. \tag{5.8}$$

The core message of the cubature method is that the weighted sum of the solutions of the ODEs (5.8) for $j = 1, ..., n$, where the weights are given by $\lambda_1, ..., \lambda_n$ of the cubature formula, approximates the expected solution of the SDE (5.7).

In the following theorem, we provide an explicit bound for the approximation error of the cubature method for the fBm with $H > 1/2$. Hence, we provide the theoretical foundations of the cubature method for the fBm, extending Proposition 2.1, Lemma 3.1 and Proposition 3.2 of [LV04]. Before proving the theorem let us introduce some notation. Let $\|\cdot\|_{\mathbb{R}^N}$ denote the Euclidean norm in $\mathbb{R}^N$ and let $|I|$ indicate the length of the word I, where $I \in \{0, ..., d\}^k$ for some $k \in \mathbb{N}$. We will also use the supremum norm $\|\cdot\|_\infty$, *e.g.* $\|V_{i_1} \cdots V_{i_k} f\|_\infty = \sup_{x\in\mathbb{R}^N} |(V_{i_1} \cdots V_{i_k} f)(x)|$.

Theorem 5.3.2. *Consider any function $f : \mathbb{R}^N \to \mathbb{R}$, whose derivatives of any order exist and are bounded, and vector fields $V_0, ..., V_d \in C_b^\infty(\mathbb{R}^N, \mathbb{R}^N)$. Let $T \in [0, \infty)$ and assume that there exists $M > 0$ and $0 \le \gamma < 1/2$ such that, for every word I, $\|(V_{i_i} \cdots V_{i_k} f)\|_\infty \le M^{|I|}(|I|!)^\gamma$. Then, we have*

$$\sup_{x\in\mathbb{R}^N} \left|\mathbb{E}\left(f(\xi_{T,x})\right) - \sum_{j=1}^{n} \lambda_j f(\Phi_{T,x}(\omega_j))\right| \le \begin{cases} L_1(T) & \text{if } T \ge 1, \\ L_2(T) & \text{if } T < 1, \end{cases}$$

where

$$L_1(T) := CT^{(m+2)/2}\left(1 + M^{(m+2)/2}\sum_{k=0}^{\infty}\frac{(dMKT)^k}{(k!)^{1/2-\gamma}}\right),$$

$$L_2(T) := C'T^{2H} + C''T^{H(m+2)/2}M^{(m+2)/2}\sum_{k=0}^{\infty}\frac{(dMKT^H)^k}{(k!)^{1/2-\gamma}},$$

and where $C, C', C'' > 0$ are constants independent of T and $K = \sqrt{\frac{2}{H(2H-1)}}$.

Remark 5.3.3. Notice that our result differs from Proposition 3.2 of [LV04] for two reasons. First, in that proposition lack the consideration of the case $T < 1$. Second, we do not have the possibility to use the Itô isometry of Proposition 2.1 in [LV04], but we have to rely on Theorem 3.4 in [BZ12]. We also point out that in case sharper estimates for the remainder of the stochastic Taylor expansion for the fBm are found, these will directly improve our result.

From the scaling properties of the fBm, we obtain the following simple proposition, which is an equivalent of Proposition 2.5 in [LV04] for the fBm.

Proposition 5.3.4. *Assume that $\bar{\omega}_1, ..., \bar{\omega}_n \in C_{0,bv}^0([0,1],\mathbb{R}^d)$ and the weights $\lambda_1, ..., \lambda_n$ define a cubature formula for the fBm on the Wiener space of degree m for time 1. Define, for $j = 1, ..., n$, the paths $\bar{\omega}_j^{(T)} \in C_{0,bv}^0([0,T],\mathbb{R}^d)$ by $\bar{\omega}_j^{(T),i}(t) = T^H\bar{\omega}_j^i(t/T)$ for $i = 1, ..., d$. The paths $\bar{\omega}_j^{(T)}$ and*

the weights λ_j , $j = 1, ..., n$, then define a cubature formula for the fBm on the Wiener space of degree m at time T.

Proof. It follows from the scaling property of the fractional Brownian motion. □

The last step of the cubature method is to extend it from small times to any times. This step is usually called concatenation step. Indeed, for the Bm case it is sufficient to divide the time interval in smaller subintervals and then concatenate the solutions obtained in each subinterval (see [LV04] for details).

In this chapter, we do not deal with the concatenation step. This is because for the fBm we do not have the independence of the increment property (*i.e.* the Markov semigroup property) that we have for the Brownian motion case. Hence, the problem of the concatenation is open and non-trivial. On the other hand, as pointed out in the Introduction in [BFL13] "in many practical applications K (the number of concatenation steps) remains small". Therefore, despite we do not provide the concatenation step our method can be nonetheless potentially applied to many empirical applications which consider any interval of time.

The following result is the second main result of this section. In this result we present the cubature formula for the one dimensional fBm up to degree 5. As mentioned above, the cubature formula allows to put into practice the cubature method.

It is important to remark that for $H = \frac{1}{2}$ the results obtained in this theorem perfectly coincide with the results of Lyons and Victoir obtained in [LV04]. Further, due to the fact that the proof is tedious and mainly based on linear algebra computations we decided to only sketch it here (see the supplementary material of [3] for further details).

Theorem 5.3.5. *Let $H \geq \frac{1}{2}$. Define $\hat{B}$ and $\hat{\omega}$ to be*

$$\hat{B}_{t_l}^{H,i_l} = \begin{cases} t_l, & if \quad i_l = 0, \\ B_{t_l}^H & if \quad i_l = 1, \end{cases} \qquad and \qquad \hat{\omega}_{t_l,j}^{i_l} = \begin{cases} t_l, & if \quad i_l = 0, \\ \omega_{t_l,j} & if \quad i_l = 1, \end{cases}$$

for $l = 1, ..., k$ and $j = 1, ..., n$ and where $\omega_j \in C_{0,bv}^0([0,T], \mathbb{R}^d)$. The weights $\{\lambda_1, \lambda_2, \lambda_3\}$ and the paths $\{\omega_{t,1}, \omega_{t,2}, \omega_{t,3}\}$ will satisfy the cubature formula

$$\mathbb{E}\left[\int_{0<t_1<,...,<t_k<T} d\hat{B}_{t_1}^{H,i_1} \cdots d\hat{B}_{t_k}^{H,i_k}\right] = \sum_{j=1}^{3} \lambda_j \int_{0<t_1<,...,<t_k<T} d\hat{\omega}_{t_1,j}^{i_1} \cdots d\hat{\omega}_{t_1,j}^{i_k},$$

for the following degree

$$Degree = \begin{cases} 5 & for \quad \frac{1}{2} \leq H < \frac{2}{3}, \\ 4 & for \quad \frac{2}{3} \leq H < 1, \end{cases}$$

if $\lambda_1 = \lambda_2 = \frac{1}{6}$ and $\lambda_3 = \frac{2}{3}$, and

$$\omega_{t,1} = \begin{cases} (2\sqrt{3} - \beta)t, & t \in [0, \frac{1}{3}], \\ (\sqrt{3} - \beta) + (2\beta - \sqrt{3})t, & t \in [\frac{1}{3}, \frac{2}{3}], \\ (\beta - \sqrt{3}) + (2\sqrt{3} - \beta)t, & t \in [\frac{2}{3}, 1], \end{cases}$$

where $\beta := \dfrac{\sqrt{-96H^2+66H+57}}{2H+1}$, *and* $\omega_{t,2} = -\omega_{t,1}$ *and* $\omega_{t,3} = 0$ *for* $t \in [0,1]$.

Sketch of the proof. The first step is to compute the expected iterated integrals for different combinations of words, that is for different combinations of time path t and fBm B_t^H. For example, for the degree $2H+2$ we need to compute:

$$\mathbb{E}\left(\int_0^1 \int_0^{u_2} dB_{u_1}^H du_2\right) = \int_0^1 \mathbb{E}\left(B_{u_2}^H\right) du_2 = 0$$

and

$$\mathbb{E}\left(\int_0^1 \int_0^{u_2} du_1 dB_{u_2}^H\right) = \mathbb{E}\left(\int_0^1 u_2 dB_{u_2}^H\right) = 0.$$

The second step is to obtain the respective iterated integrals of the ω_i weighted by the λ_i. Hence, for degree $2H+2$ we have:

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_2} d\omega_{i,u_1} du_2 \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \omega_{i,u_2} du_2 = 0$$

and

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_2} du_1 d\omega_{i,u_2} \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 u_2 d\omega_{i,u_2} = 0.$$

Then we have a system of equations where the unknowns are the ω_i, the λ_i and n. Solving this system give us our result. □

We conclude with a final remark on the solution(s) obtained for the cubature formula.

Remark 5.3.6. From the proof of this theorem we obtain two solutions of our system of equations, which determine the slope of our paths ωs. The reason why we focused only on one solution is because Lyons and Victoir focused on that particular solution in their paper. However, both solutions are feasible. Hence, we have two valid solutions for the cubature formula of the fractional Brownian motion for this kind of structure (*i.e.* piecewise linear paths with change of slopes at $t = \frac{1}{3}$ and $t = \frac{2}{3}$).
Indeed, the structure adopted is already arbitrary and hence it is important to have a solution and not to have a particular or unique solution.

In the next section we are going to explore a concrete application of the theoretical results presented in this section.

5.3.1 Example

In this section we present a practical example of the cubature method for the fBm with $H > 1/2$. We consider the geometric fractional Brownian motion (gfBm):

$$\xi_{t,x} = x + \int_0^t b\xi_{s,x} \mathrm{d}s + \int_0^t a\xi_{s,x} \mathrm{d}B_s^H, \tag{5.9}$$

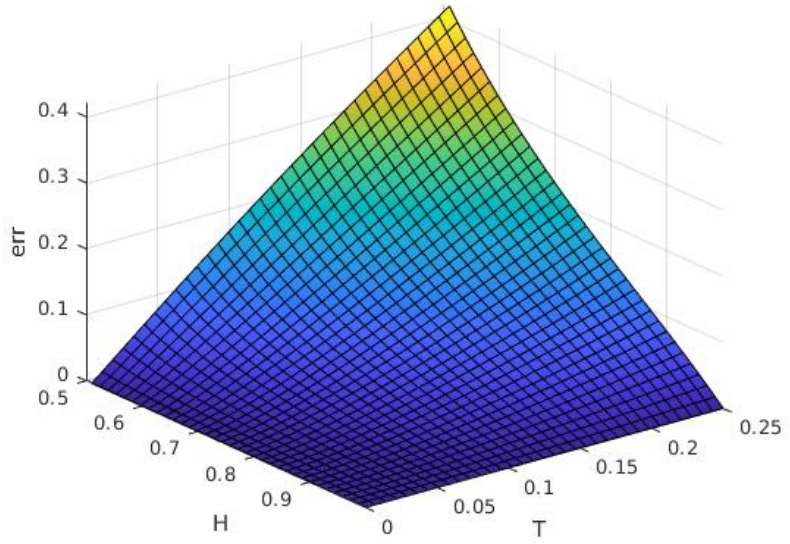


Figure 5.1: Error function for the cubature method of the gfBm with $a = 2$, $b = -3/2$, $x = 1$, $H \in (1/2, 1)$ and $T \in [0, 1/4]$.

where $x, a, b \in \mathbb{R}$. Using Theorem 4.3.1 in [Zha98], it is possible to prove that (5.9) has a unique solution, which is given by

$$\xi_{t,x} = x \exp\left(bt + aB_t^H\right) \quad \Rightarrow \quad \mathbb{E}[\xi_{t,x}] = x \exp\left(bt + \frac{a^2 t^{2H}}{2}\right).$$

The ODEs that we need to solve to implement the cubature method are

$$y_{t,x,j} = x + \int_0^t b y_{s,x,j} \mathrm{d}s + \int_0^t a y_{s,x,j} \mathrm{d}\omega_{s,j} \quad \Rightarrow \quad y_{t,x,j} = x \exp\left(bt + a\omega_{t,j}\right),$$

where $(t, \omega_{t,j})$ are the cubature paths, for $j = 1, 2, 3$. The reason why we decided to focus on the SDE (5.9) is because it has an explicit solution allowing the approximation error to measure uniquely the error coming from the cubature method. The objective is to compute the following error function:

$$err(T, H) = \left| \mathbb{E}\left(\xi_{T,x}\right) - \sum_{j=1}^{3} \lambda_j y_{T,x,j} \right|. \tag{5.10}$$

In terms of the notations of the previous section, we have that: $d = 2$, $N = 1$, $V_0(x) = bx$, $V_1(x) = ax$, and $f(x) = x$. Observe that this setting does not fit into the framework of the previous section since V_0, V_1 and f are not bounded. However, this is typical of the literature on the cubature method (see the Introduction in [BFL13] for a discussion).

In Fig. 5.1 we plot the error function for $a = 2$, $b = -3/2$ and $x = 1$ in the intervals $H \in (1/2, 1)$ and $T \in [0, 1/4]$. It is possible to see that the error function becomes smaller as T decreases and as H increases. This is in accordance with the theory. In particular, when T decreases the cubature method becomes more accurate since it is based on the (stochastic) Taylor expansion around zero. Moreover, as H increases the fBm has smoother paths which are easier to be matched by the cubature paths and, so, by the cubature formula.

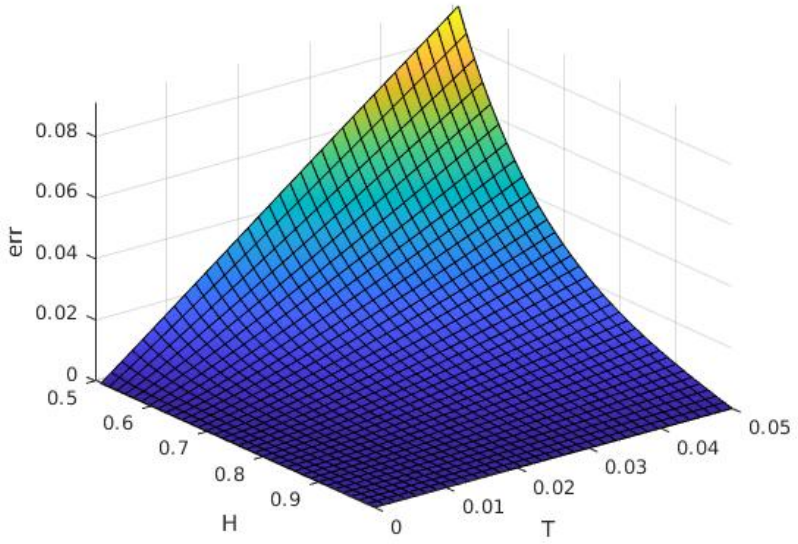


Figure 5.2: Error function for the cubature method of the gfBm with $a = 2$, $b = -3/2$, $x = 1$, $H \in (1/2, 1)$ and $T \in [0, 1/20]$.

In Fig. 5.2 we plot the error function for $a = 2$, $b = -3/2$ and $x = 1$ in the intervals $H \in (1/2, 1)$ and $T \in [0, 1/20]$. It is possible to see that the error becomes very small. This is primarily caused by the higher precision of the cubature method around $T = 0$. This is supported theoretically by the fact that the cubature method is based on the (stochastic) Taylor expansion around zero. Moreover, one might think that when T is small the absolute value of the solution of (5.9) is small hence the error is small. However, this phenomenon does not depend on the absolute value of the solution as supported by the following additional empirical results.

An alternative error function is given by

$$err^{new}(T, H) = \left| 1 - \frac{\sum_{j=1}^{3} \lambda_j y_{T,j}}{\mathbb{E}\left(\xi_{T,x}\right)} \right|$$

The advantage of this new error function is that it is not influenced by the absolute value of the solution of the SDE considered. When we focus on our example, we found that this new error function shows the same behaviour as the previous error function (5.10). The only difference is the scale of the error which is now consistently (*i.e.* for any value of H and T) smaller than the previous one.

Although the new numerical results have a smaller error then the one computed with the previous error function, we decided to not present them since they show the same behaviour as the ones already shown (see Fig. 5.1 and Fig. 5.2).

5.4 Conclusion

In this chapter, we have investigated various properties of the expected signature of the fBm for $H > 1/2$. We have shown the sharp rate of convergence of the truncated expected signature of the linear piecewise approximation of the fBm to its exact value, namely the truncated expected

signature of the fBm. Further, we have provided a bound for the truncated expected signature of the fBm, demonstrating that the expected signature of the fBm goes to zero fast as the number of iterated integrals increases. Apart from the theoretical interest of our results, in applied machine learning the (truncated) expected signature of a path is used as a summary of the behaviour of the path. Our results, which describes various properties of the expected signature, might be used to have bounds for convergence for the models developed. In addition, to this we have developed the cubature method for the fBm for $H > 1/2$. By doing this, we have developed a complete numerical deterministic method that provides approximations of the expected solution of certain SDEs driven by the fBm. We have shown a practical example of this method using the geometric fractional Brownian motion. The main limitation is the lack of the concatenation step which forces our method to be applicable only for small-time solutions.

5.5 Appendices

Appendix A: Proof of Theorem 5.2.1

In this section we prove Theorem 5.2.1. Hence, we show the rate of convergence of the expected signature of the piecewise approximation of the fBm to its exact value and its coefficient. The sharpness of the rate of convergence is a delicate matter and it will be discussed at the end of this section.

As mentioned before, this theorem covers the weak convergence rate of the signature of the fBm. A previous result on the strong convergence rate was obtained by Friz and Riedel in [FR14].

Remark 5.5.1. For the sake of clarity, we prove Theorem 5.2.1 and all the results in this section for the case of the uniform mesh of size m^{-1}, namely $q_m = m^{-1}$. However, all the results holds for any uniform mesh.

Recall from Theorem 5.1.5 that we have

$$\left|\mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,I}\right) - \mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,m,I}\right)\right|$$

$$= \left| \frac{H^k(2H-1)^k}{k!2^k} \sum_{\sigma\in\mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \prod_{l=1}^{k} \delta_{i_{\sigma(2l)},i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} \right.$$

$$\left. - \prod_{l=1}^{k} \delta_{i_{\sigma(2l)},i_{\sigma(2l-1)}} \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t_{\sigma(2l)}, t_{\sigma(2l-1)}) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy dt_1 \cdots dt_{2k} \right|.$$

In order to understand this as a whole we need to first understand the single parts of it. In other words let us focus on

$$\int_{u_2}^{v_2} \int_{u_1}^{v_1} \Big(|t-s|^{2H-2} - \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \Big) ds dt,$$

where $u_1, u_2, v_1, v_2, t, s \in \{t_1, ..., t_{2k}\}$. There are two possible cases. The first one is when t and

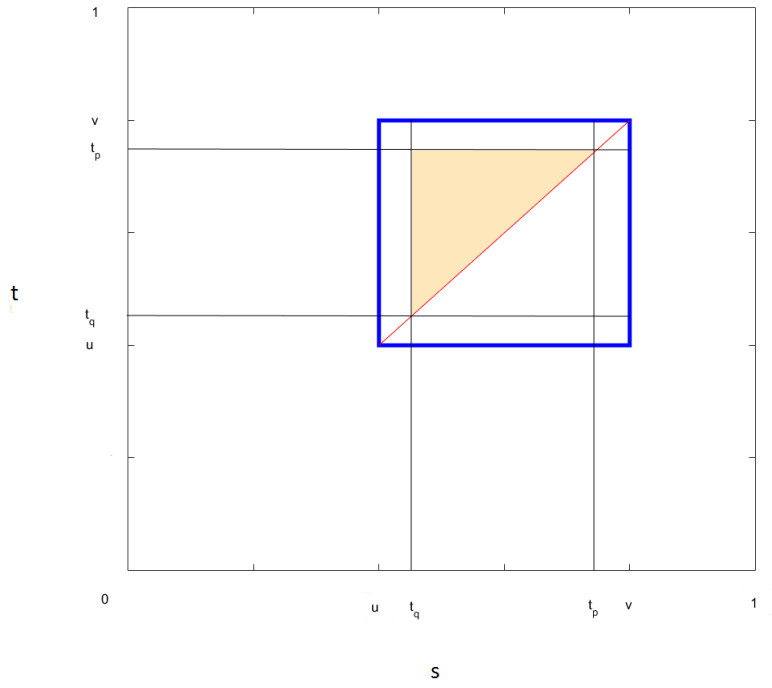


Figure 5.3: Representation of the area of the double integral (5.13).

s are next to each other. In this case, we have

$$\int_u^v \int_u^t \Big(|t-s|^{2H-2} - \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dxdy \Big) dsdt \quad (5.11)$$

with $u, v, t, s \in \{t_1, ..., t_{2k}\}$ with $u \neq v \neq t \neq s$. The second one is that s and t are not next to each other. Hence, we have

$$\int_{u_2}^{v_2} \int_{u_1}^{v_1} \Big(|t-s|^{2H-2} - \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dxdy \Big) dsdt \quad (5.12)$$

with $u_1 \neq u_2 \neq v_1 \neq v_2 \neq t \neq s$.

We will focus first on the first case (5.11), which is more delicate. We are going to split this double integral into different parts according to our piecewise approximation. In particular, let t_q be the lowest grid time above u and t_p the highest grid time below v, as shown in Figure 5.3. Then we have that our double integral can be rewritten in the following way:

$$\int_{u<s<t<v} = \int_u^v \int_u^t = \int_u^{t_q} \int_u^t + \int_{t_q}^{t_p} \int_u^{t_q} + \int_{t_p}^{v} \int_u^{t_q} + \int_{t_p}^{v} \int_{t_q}^{t_p} + \int_{t_p}^{v} \int_{t_p}^{t} + \int_{t_q}^{t_p} \int_{t_q}^{t}. \quad (5.13)$$

In order to better understand this decomposition it is helpful to look at the upper triangle in Figure 5.3 (*i.e.* the triangle from the red diagonal to the upper and left blue edges). The first integral is the small triangle in the lower left. From here you first move up and then move right. The last integral is the central yellow triangle in Figure 5.3. For this integral the piecewise approximation and the exact value are equal, hence their difference is zero. This is because the

piecewise approximation is equal to the exact value if the points in time considered coincide with the grid points.

What we want to show now is the speed at which the difference between the piecewise approximation and the exact value for these integrals is going to zero as the grid size goes to zero (*i.e.* $m \to \infty$). In order to facilitate the reading and understanding of the content of this section, we decided to split the proof of Theorem 5.2.1 in different small propositions.

The following propositions regards the rate of convergence of each double integral of the right hand side of equation (5.13). First, let us focus on the integrals: $\int_u^{t_q}\int_u^t$ and $\int_{t_p}^v\int_{t_p}^t$. We have the following proposition.

Proposition 5.5.2. *The differences*

$$\left|\int_u^{t_q}\int_u^t\Big(|t-s|^{2H-2}-\sum_{i,j=1}^m \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s)m^2\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|x-y|^{2H-2}dxdy\Big)dsdt\right|$$

and

$$\left|\int_{t_p}^v\int_{t_p}^t\Big(|t-s|^{2H-2}-\sum_{i,j=1}^m \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s)m^2\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|x-y|^{2H-2}dxdy\Big)dsdt\right|$$

are bounded by

$$\frac{m^{-2H}}{H(2H-1)}.$$

Proof.

$$\begin{aligned}
&\int_u^{t_q}\int_u^t\Big(|t-s|^{2H-2}-\sum_{i,j=1}^m \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s)m^2\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|x-y|^{2H-2}dxdy\Big)dsdt\\
&\quad=\int_u^{t_q}\int_u^t\Big(|t-s|^{2H-2}-m^2\int_{t_{q-1}}^{t_q}\int_{t_{q-1}}^{t_q}|x-y|^{2H-2}dxdy\Big)dsdt\\
&\quad=\frac{(t_q-u)^{2H}}{2H(2H-1)}-m^2\int_u^{t_q}\int_u^t\frac{(t_q-t_{q-1})^{2H}}{H(2H-1)}dsdt\\
&\quad=\frac{(t_q-u)^{2H}}{2H(2H-1)}-m^{2-2H}\frac{(t_q-u)^2}{2H(2H-1)}.
\end{aligned}$$

Notice that

$$\begin{aligned}
\left|\frac{(t_q-u)^{2H}}{2H(2H-1)}-m^{2-2H}\frac{(t_q-u)^2}{2H(2H-1)}\right| &\le \frac{(t_q-u)^{2H}}{2H(2H-1)}+m^{2-2H}\frac{(t_q-u)^2}{2H(2H-1)}\\
&\le \frac{m^{-2H}}{2H(2H-1)}+m^{2-2H}\frac{m^{-2}}{2H(2H-1)}=\frac{m^{-2H}}{H(2H-1)}.
\end{aligned}$$

The same reasoning done for this integral applies *mutatis mutandis* to the integral $\int_{t_p}^v\int_{t_p}^t$. □

Let us focus on the integrals: $\int_{t_q}^{t_p}\int_u^{t_q}$ and $\int_{t_p}^v\int_{t_q}^{t_p}$. We have the following proposition.

Proposition 5.5.3. *The differences*

$$\left| \int_{t_q}^{t_p} \int_{u}^{t_q} \Big(|t-s|^{2H-2} - \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dxdy \Big) dsdt \right|$$

and

$$\left| \int_{t_p}^{v} \int_{t_q}^{t_p} \Big(|t-s|^{2H-2} - \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dxdy \Big) dsdt \right|$$

are bounded by

$$\left(\frac{2^{2H}+2}{H(2H-1)} + (4-4H) \sum_{i=1}^{\infty} i^{2H-3} \right) m^{-2H}.$$

Proof. We have

$$\int_{t_q}^{t_p} \int_{u}^{t_q} \Big(|t-s|^{2H-2} - \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s) m^m \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dxdy \Big) dsdt$$

$$= \int_{t_q}^{t_p} \int_{u}^{t_q} \Big(|t-s|^{2H-2} - \sum_{t_i=t_q}^{t_{p-1}} \mathbf{1}_{[t_i,t_{i+1}]}(t) m^2 \int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} |x-y|^{2H-2} dxdy \Big) dsdt$$

$$= \sum_{t_i=t_q}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_{u}^{t_q} \Big(|t-s|^{2H-2} - m^2 \int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} |x-y|^{2H-2} dxdy \Big) dsdt.$$

Here we need to split the sum in two parts. The first part is composed by the first element of the sum, while the second is composed by the whole sum without the first element. Let us first consider the first part.

$$\int_{t_q}^{t_{q+1}} \int_{u}^{t_q} \Big(|t-s|^{2H-2} - m^2 \int_{t_q}^{t_{q+1}} \int_{t_{q-1}}^{t_q} |x-y|^{2H-2} dxdy \Big) dsdt$$

$$= \frac{(t_{q+1}-u)^{2H} - (t_q-u)^{2H} - (t_{q+1}-t_q)^{2H}}{2H(2H-1)}$$

$$-m(t_{q+1}-t_q)(t_q-u) \frac{(t_{q+1}-t_{q-1})^{2H} - (t_q-t_{q-1})^{2H} - (t_{q+1}-t_q)^{2H}}{2H(2H-1)}.$$

Notice that every element in the numerators is smaller or equal than m^{-2H} except for $(t_{q+1}-u)^{2H}$ and $(t_{q+1}-t_{q-1})^{2H}$, for which the following relation holds $(t_{q+1}-u)^{2H} \leq (t_{q+1}-t_{q-1})^{2H} = 2^{2H} m^{-2H}$. Therefore, taking absolute value of each element we have

$$\leq \frac{2^{2H}+2}{H(2H-1)} m^{-2H}.$$

Now let's focus on the second part, that is

$$\sum_{t_i=t_{q+1}}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_{u}^{t_q} \Big(|t-s|^{2H-2} - m^2 \int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} |x-y|^{2H-2} dydx \Big) dsdt$$

$$= \sum_{t_i=t_{q+1}}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_{u}^{t_q} \Big((t-s)^{2H-2} - m^2 \int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} (x-y)^{2H-2} dydx \Big) dsdt$$

$$= \sum_{t_i=t_{q+1}}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_{u}^{t_q} m^2 \Big(\int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} (t-s)^{2H-2} - (x-y)^{2H-2} dydx \Big) dsdt.$$

By applying the Mean Value Theorem we have

$$\leq \sum_{t_i=t_{q+1}}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_{u}^{t_q} m^2 \Big(\int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} \sup_{\xi \in [(t-s),(x-y)]} (2-2H)\xi^{2H-3} |t-s-(x-y)| dydx \Big) dsdt$$

$$= \sum_{t_i=t_{q+1}}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_{u}^{t_q} m^2 \Big(\int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} (2-2H)(t_i - t_q)^{2H-3} |t-x+y-s| dydx \Big) dsdt.$$

Observe that $|t-x| \leq m^{-1}$ and $|y-s| \leq m^{-1}$, hence $|t-x+y-s| \leq 2m^{-1}$. Thus, we have

$$\leq \sum_{t_i=t_{q+1}}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_{u}^{t_q} m^2 \Big(\int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} (4-4H)(t_i - t_q)^{2H-3} m^{-1} dydx \Big) dsdt$$

$$\leq (4-4H) m^{-3} \sum_{t_i=t_{q+1}}^{t_{p-1}} (t_i - t_q)^{2H-3}.$$

Moreover, it is possible to notice that

$$\leq (4-4H) m^{-3} \sum_{i=1}^{m} \left(\frac{i}{m} \right)^{2H-3} = (4-4H) m^{-2H} \sum_{i=1}^{m} i^{2H-3} \leq (4-4H) m^{-2H} \sum_{i=1}^{\infty} i^{2H-3}$$

$$\leq K m^{-2H},$$

where K is a finite constant independent of m, and that $\sum_{i=1}^{\infty} i^{2H-3} < \infty$ for $H < 1$.
A similar reasoning applies to the integral $\int_{t_p}^{v} \int_{t_q}^{t_p}$. □

The last integral to be analysed is $\int_{t_p}^{v} \int_{u}^{t_q}$.

Proposition 5.5.4. *The difference*

$$\left| \int_{t_p}^{v} \int_{u}^{t_q} \Big(|t-s|^{2H-2} - \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dxdy \Big) dsdt \right|$$

is bounded by

$$\frac{3^{2H} + 10(2^{2H}) + 2}{2H(2H-1)} m^{-2H}.$$

Proof. Here we need to distinguish two cases. One in which $|u-v| \leq \frac{2}{m}$ and the other in which $|u-v| > \frac{2}{m}$. Let's focus first on the case $|u-v| > \frac{2}{m}$.

$$\int_{t_p}^{v} \int_{u}^{t_q} \Big(|t-s|^{2H-2} - \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dxdy \Big) dsdt$$

$$= \int_{t_p}^{v}\int_{u}^{t_q}\Big(|t-s|^{2H-2} - m^2\int_{t_p}^{t_{p+1}}\int_{t_{q-1}}^{t_q}|x-y|^{2H-2}dxdy\Big)dsdt$$

$$= \int_{t_p}^{v}\int_{u}^{t_q} m^2\Big(\int_{t_p}^{t_{p+1}}\int_{t_{q-1}}^{t_q}|t-s|^{2H-2} - |x-y|^{2H-2}dxdy\Big)dsdt.$$

By Mean Value Theorem we have

$$\left|\int_{t_p}^{v}\int_{u}^{t_q} m^2\Big(\int_{t_p}^{t_{p+1}}\int_{t_{q-1}}^{t_q}|t-s|^{2H-2} - |x-y|^{2H-2}dxdy\Big)dsdt\right|$$

$$\leq \int_{t_p}^{v}\int_{u}^{t_q} m^2\Big(\int_{t_p}^{t_{p+1}}\int_{t_{q-1}}^{t_q}\sup_{\xi\in[(t-s),(x-y)]}(2-2H)\xi^{2H-3}|t-s-(x-y)|dxdy\Big)dsdt$$

$$= \int_{t_p}^{v}\int_{u}^{t_q} m^2\Big(\int_{t_p}^{t_{p+1}}\int_{t_{q-1}}^{t_q}(2-2H)(t_p-t_q)^{2H-3}|t-x+y-s|dxdy\Big)dsdt.$$

Since $|u-v| > \frac{2}{m}$ then $t_p - t_q \geq m^{-1}$.

$$\leq \int_{t_p}^{v}\int_{u}^{t_q} m^2\Big(\int_{t_p}^{t_{p+1}}\int_{t_{q-1}}^{t_q}(4-4H)m^{-2H+3}m^{-1}dxdy\Big)dsdt$$

$$\leq (4-4H)m^{-2H}.$$

Finally, consider the second case, *i.e.* $|u-v| \leq \frac{2}{m}$.

$$\int_{t_p}^{v}\int_{u}^{t_q}\Big(|t-s|^{2H-2} - m^2\int_{t_p}^{t_{p+1}}\int_{t_{q-1}}^{t_q}|x-y|^{2H-2}dxdy\Big)dsdt$$

$$= \frac{(v-u)^{2H} - (t_p-u)^{2H} - (v-t_q)^{2H} + (t_p-t_q)^{2H}}{2H(2H-1)}$$

$$-m^2(v-t_p)(t_q-u)\frac{(t_{p+1}-t_{q-1})^{2H} - (t_p-t_{q-1})^{2H} - (t_{p+1}-t_q)^{2H} + (t_p-t_q)^{2H}}{2H(2H-1)}.$$

Now, since $|u-v| \leq \frac{2}{m}$ then $t_p - t_q \leq \frac{1}{m}$ so $t_{p+1} - t_{q-1} \leq \frac{3}{m}$. Therefore, we have

$$\leq \left(\frac{3^{2H} + 10(2^{2H}) + 2}{2H(2H-1)}\right)m^{-2H}.$$

Moreover, since

$$(4-4H) < \frac{3^{2H} + 10(2^{2H}) + 2}{2H(2H-1)},$$

then we can use the right hand side of the above inequality as the coefficient for the rate of convergence for the integral $\int_{t_p}^{v}\int_{u}^{t_q}$ independently of the value of u and v. □

Therefore, we have the following proposition.

Proposition 5.5.5. *The difference*

$$\left|\int_{u<s<t<v}\Big(|t-s|^{2H-2} - \sum_{i,j=1}^{m}\mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s)m^2\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|x-y|^{2H-2}dxdy\Big)dsdt\right|$$

is bounded by

$$Am^{-2H},$$

where A is given by

$$A = 2\left(\frac{1}{H(2H-1)} + \frac{2^{2H}+2}{H(2H-1)} + (4-4H)\sum_{i=1}^{\infty} i^{2H-3}\right) + \frac{3^{2H}+10(2^{2H})+2}{2H(2H-1)}.$$

Proof. It is immediate from the previous propositions, from the fact that for the double integral $\int_{t_q}^{t_p}\int_{t_q}^{t}$ the difference is zero as discussed before and by considering equation (5.13). Indeed, the difference for the double integral $\int_{t_q}^{t_p}\int_{t_q}^{t}$ is given by

$$\int_{t_q}^{t_p}\int_{t_q}^{t}\Big(|t-s|^{2H-2} - \sum_{i,j=1}^{m}\mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s)m^2\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|x-y|^{2H-2}dxdy\Big)dsdt.$$

Notice that this difference can be expressed as the sum of two classes of integrals. The first class contains the off-diagonal terms, which can be expressed as

$$\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}\Big(|t-s|^{2H-2} - m^2\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|x-y|^{2H-2}dxdy\Big)dsdt,$$

where $t_i, t_j \in \{t_q, t_{q+1}, ..., t_{p-1}, t_p\}$. It is possible to see that this difference is immediately zero.

The second case is

$$\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}\Big(|t-s|^{2H-2} - m^2\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}|x-y|^{2H-2}dxdy\Big)dsdt,$$

where $t_i \in \{t_q, t_{q+1}, ..., t_{p-1}\}$. This difference is equal to

$$\begin{aligned}
&= \frac{(t_{i+1}-t_i)^{2H}}{2H(2H-1)} - \int_{t_i}^{t_{i+1}}\int_{t_i}^{t} m^2\frac{(t_{i+1}-t_i)^{2H}}{H(2H-1)}dsdt \\
&= \frac{m^{-2H}}{2H(2H-1)} - \frac{m^{2-2H}}{H(2H-1)}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t} dsdt \\
&= \frac{m^{-2H}}{2H(2H-1)} - \frac{m^{2-2H}}{H(2H-1)}\frac{m^{-2}}{2} = 0.
\end{aligned}$$

This result could also be directly seen from the symmetric (with respect to the diagonal) property of the function $|t-s|^{2H-2}$. □

Now, recall the second case (*i.e.* equation (5.12)), which we rewrite here:

$$\int_{u_2}^{v_2}\int_{u_1}^{v_1}\Big(|t-s|^{2H-2} - \sum_{i,j=1}^{m}\mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s)m^2\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|x-y|^{2H-2}dxdy\Big)dsdt$$

with $u_1 \neq u_2 \neq v_1 \neq v_2 \neq t \neq s$.

Concerning this case we have not any more a triangle but a rectangle. First, observe that these rectangles never lie on the diagonal. This is because we have $v_2 > u_2 > v_1 > u_1$. The value

of the double integral for this rectangle can be obtained by the difference of double integrals of properly chosen triangles. That is

$$\int_{u_2}^{v_2}\int_{u_1}^{v_1} = \int_{u_1}^{v_2}\int_{u_1}^{t} - \int_{u_1}^{u_2}\int_{u_1}^{t} - \int_{v_1}^{v_2}\int_{v_1}^{t} + \int_{v_1}^{u_2}\int_{v_1}^{t}.$$

Therefore, we have the following proposition.

Proposition 5.5.6. *The difference*

$$\left|\int_{u_2}^{v_2}\int_{u_1}^{v_1}\Big(|t-s|^{2H-2} - \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s)m^2\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|x-y|^{2H-2}dxdy\Big)dsdt\right|$$

is bounded by

$$4Am^{-2H},$$

where A is given in Proposition 5.5.5.

Proof. Let

$$f(t,s) := |t-s|^{2H-2} - \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t,s)m^2\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|x-y|^{2H-2}dxdy.$$

Then from the discussion in the above paragraph we have that

$$\int_{u_2}^{v_2}\int_{u_1}^{v_1} f(t,s)dsdt$$

$$= \int_{u_1}^{v_2}\int_{u_1}^{t} f(t,s)dsdt - \int_{u_1}^{u_2}\int_{u_1}^{t} f(t,s)dsdt - \int_{v_1}^{v_2}\int_{v_1}^{t} f(t,s)dsdt + \int_{v_1}^{u_2}\int_{v_1}^{t} f(t,s)dsdt$$

$$\leq 4Am^{-2H},$$

where the last inequality follows from Proposition 5.5.5. □

We present now a general result on combinatorial properties of the iterated integrals.

Proposition 5.5.7. *Let $k \in \mathbb{N}$ and let $h : \mathbb{R}^{2k-2} \to \mathbb{R}$ be a $2k-2$-dimensional integrable function. Then,*

$$\sum_{\sigma\in\mathcal{G}_{2k}} \int_{0<t_1<\ldots<t_{\sigma(2j-1)-1}<t_{\sigma(2j-1)+1}<\ldots<t_{\sigma(2j)-1}<t_{\sigma(2j)+1}<\ldots<t_{2k}<1}$$

$$h(t_{\sigma(1)}, t_{\sigma(2)}, ..., t_{\sigma(2j)-1}, t_{\sigma(2j)+1}, ..., t_{\sigma(2j-1)-1}, t_{\sigma(2j-1)+1}, ..., t_{\sigma(2k)})$$

$$dt_1\cdots dt_{\sigma(2j)-1}dt_{\sigma(2j)+1}\cdots dt_{\sigma(2j-1)-1}dt_{\sigma(2j-1)+1}\cdots dt_{2k}$$

$$= 2k(2k-1)\sum_{\tau\in\mathcal{G}_{2k-2}}\int_{\Delta^{2k-2}[0,1]} h(s_{\tau(1)}, s_{\tau(2)}, ..., s_{\tau(2k-2)})ds_1\cdots ds_{2k-2}, \tag{5.14}$$

where $\mathcal{G}_{2k-2}$ is the set of all permutations of the set $\{1, ..., 2k-2\}$ and $j = 1, ..., k$.

Proof. For the moment, fix $\sigma \in \mathcal{G}_{2k}$ and consider the case $j = 1$. By a simple change of notation we have

$$\int_{0<t_1<\ldots<t_{\sigma(1)-1}<t_{\sigma(1)+1}<\ldots<t_{\sigma(2)-1}<t_{\sigma(2)+1}<\ldots<t_{2k}<1} h(t_{\sigma(3)},t_{\sigma(4)},\ldots,t_{\sigma(2k)})$$
$$dt_1\cdots dt_{\sigma(2)-1}dt_{\sigma(2)+1}\cdots dt_{\sigma(1)-1}dt_{\sigma(1)+1}\cdots dt_{2k}$$
$$=\int_{\Delta^{2k-2}[0,1]} h(s_{\tau(1)},s_{\tau(2)},\ldots,s_{\tau(2k-2)})\mathrm{d}s_1\cdots\mathrm{d}s_{2k-2},$$

where $\tau\in\mathcal{G}_{2k-2}$. Now, given a permutation $\tau\in\mathcal{G}_{2k-2}$ there are different σs, say $\sigma_1,\sigma_2,\ldots\in\mathcal{G}_{2k}$ such that they satisfy the same equality as above. In particular, for a given τ there are precisely $2k(2k-1)$ of these σs that satisfy the equality. That is

$$\sum_{j=1}^{2k(2k-1)}\int_{0<t_1<\ldots<t_{\sigma_j(1)-1}<t_{\sigma_j(1)+1}<\ldots<t_{\sigma_j(2)-1}<t_{\sigma_j(2)+1}<\ldots<1} h(t_{\sigma_j(3)},t_{\sigma_j(4)},\ldots,t_{\sigma_j(2k)})$$
$$dt_1\cdots dt_{\sigma(2)-1}dt_{\sigma(2)+1}\cdots dt_{\sigma(1)-1}dt_{\sigma(1)+1}\cdots dt_{2k}$$
$$=2k(2k-1)\int_{\Delta^{2k-2}[0,1]} h(s_{\tau(1)},s_{\tau(2)},\ldots,s_{\tau(2k-2)})\mathrm{d}s_1\cdots\mathrm{d}s_{2k-2}.$$

The $2k(2k-1)$ factor comes from the possible values that $\{\sigma(1),\sigma(2)\}$ may assume in the set $\{1,\ldots,2k\}$ leaving the *order* of the remaining $2k-2$ elements, which are the elements $\sigma(3),\ldots,\sigma(2k)$, unchanged. That is the $2k(2k-1)$ comes from the 2-permutation of $2k$.

The next step is to see that it is possible to reformulate all the permutations $\sigma\in\mathcal{G}_{2k}$ in terms of the permutations $\tau\in\mathcal{G}_{2k-2}$ times $2k(2k-1)$. In particular, we have the following equality

$$\sum_{\sigma\in\mathcal{G}_{2k}}\int_{0<t_1<\ldots<t_{\sigma(1)-1}<t_{\sigma(1)+1}<\ldots<t_{\sigma(2)-1}<t_{\sigma(2)+1}<\ldots<1} h(t_{\sigma(3)},t_{\sigma(4)},\ldots,t_{\sigma(2k)})$$
$$dt_1\cdots dt_{\sigma(2)-1}dt_{\sigma(2)+1}\cdots dt_{\sigma(1)-1}dt_{\sigma(1)+1}\cdots dt_{2k}$$
$$=2k(2k-1)\sum_{\tau\in\mathcal{G}_{2k-2}}\int_{\Delta^{2k-2}[0,1]} h(s_{\tau(1)},s_{\tau(2)},\ldots,s_{\tau(2k-2)})\mathrm{d}s_1\cdots\mathrm{d}s_{2k-2},$$

which is our desired equality (5.14). The reason why this holds is that it is possible to decompose the permutations $\mathcal{G}_{2k}$ (the σs) into the 2-permutations of $2k$ (which are $2k(2k-1)$ permutations σ for *each* permutation τ) times the permutations $\mathcal{G}_{2k-2}$ (the number of different τs). Indeed, for $\mathcal{G}_{2k}$ we have $(2k)!$ permutations (and of course they are all different from each other) and for the other we have $(2k-2)!\cdot 2k(2k-1)=(2k)!$ permutations (and they are also all different from each other).

It is possible to see that the same arguments apply to the case $j=1,\ldots,k$. For example, take $j=k$ then it is easy to see that we have

$$\sum_{\sigma\in\mathcal{G}_{2k}}\int_{0<t_1<\ldots<t_{\sigma(2k)-1}<t_{\sigma(2k)+1}<\ldots<t_{\sigma(2k-1)-1}<t_{\sigma(2k-1)+1}<\ldots<t_{2k}<1} h(t_{\sigma(1)},t_{\sigma(2)},\ldots,t_{\sigma(2k-2)})$$
$$dt_1\cdots dt_{\sigma(2k)-1}dt_{\sigma(2k)+1}\cdots dt_{\sigma(2k-1)-1}dt_{\sigma(2k-1)+1}\cdots dt_{2k}$$
$$=2k(2k-1)\sum_{\tau\in\mathcal{G}_{2k-2}}\int_{\Delta^{2k-2}[0,1]} h(s_{\tau(1)},s_{\tau(2)},\ldots,s_{\tau(2k-2)})\mathrm{d}s_1\cdots\mathrm{d}s_{2k-2}.$$

□

Now we are ready to prove Theorem 5.2.1.

Proof (Theorem 5.2.1).

$$\left|\mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,I}\right) - \mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,m,I}\right)\right| =$$

$$= \left| \frac{H^k(2H-1)^k}{k!2^k} \sum_{\sigma\in\mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \prod_{l=1}^{k} \delta_{i_{\sigma(2l)},i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} \right.$$

$$\left. - \prod_{l=1}^{k} \delta_{i_{\sigma(2l)},i_{\sigma(2l-1)}} \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t_{\sigma(2l)}, t_{\sigma(2l-1)}) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dxdydt_1 \cdots dt_{2k} \right|.$$

Now, by using the relation

$$\prod_{l=1}^{k} a_l - \prod_{l=1}^{k} b_l = (a_1 - b_1)\prod_{l=2}^{k} b_i + a_1(a_2 - b_2)\prod_{l=3}^{k} b_i + \cdots + \prod_{l=1}^{k-1} a_l(a_k - b_k),$$

where $a_i, b_i \in \mathbb{R}$ for $i = 1, ..., k$, we have

$$= \left| \frac{H^k(2H-1)^k}{k!2^k} \sum_{\sigma\in\mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \delta_{i_{\sigma(2)},i_{\sigma(1)}} \Big(|t_{\sigma(2)} - t_{\sigma(1)}|^{2H-2} \right.$$

$$\left. - \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t_{\sigma(2)}, t_{\sigma(1)}) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dxdy \Big) \right.$$

$$\prod_{l=2}^{k} \delta_{i_{\sigma(2l)},i_{\sigma(2l-1)}} \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t_{\sigma(2l)}, t_{\sigma(2l-1)}) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dxdy$$

$$+ ... + \prod_{l=1}^{k-1} \delta_{i_{\sigma(2l)},i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} \delta_{i_{\sigma(2k)},i_{\sigma(2k-1)}} \Big(|t_{\sigma(2k)} - t_{\sigma(2k-1)}|^{2H-2} \right.$$

$$\left. - \sum_{i,j=1}^{m} \mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t_{\sigma(2k)}, t_{\sigma(2k-1)}) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dxdy \Big) dt_1 \cdots dt_{2k} \right|. \quad (5.15)$$

By Fubini's theorem, given an integrable function $g : \mathbb{R}^{2k} \to \mathbb{R}$ and denoting $\mathbf{1}_{\{s<t<r\}} := \mathbf{1}_{[s,r]}(t)$ we have that

$$\int_{\Delta^{2k}[0,1]} g dt_1 \cdots dt_{2k} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g \mathbf{1}_{\{0<t_1<t_2\}} \cdots \mathbf{1}_{\{t_{2k-1}<t_{2k}<1\}} dt_1 \cdots dt_{2k}$$

$$\overset{Fubini}{=} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g \mathbf{1}_{\{0<t_1<t_2\}} \cdots \mathbf{1}_{\{t_{2k-1}<t_{2k}<1\}}$$

$$dt_{\sigma(2l)} dt_{\sigma(2l-1)} dt_1 \cdots dt_{\sigma(2l)-1} dt_{\sigma(2l)+1} \cdots dt_{\sigma(2l-1)-1} dt_{\sigma(2l-1)+1} \cdots dt_{2k}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g \mathbf{1}_{\{t_{\sigma(2l)-1}<t_{\sigma(2l)}<t_{\sigma(2l)+1}\}} \mathbf{1}_{\{t_{\sigma(2l-1)-1}<t_{\sigma(2l-1)}<t_{\sigma(2l-1)+1}\}} dt_{\sigma(2l)} dt_{\sigma(2l-1)}$$

$$\mathbf{1}_{\{0<t_1<t_2\}}\cdots\mathbf{1}_{\{t_{\sigma(2l)-2}<t_{\sigma(2l)-1}<t_{\sigma(2l)+1}\}}\mathbf{1}_{\{t_{\sigma(2l)-1}<t_{\sigma(2l)+1}<t_{\sigma(2l)+2}\}}$$

$$\cdots\mathbf{1}_{\{t_{\sigma(2l-1)-2}<t_{\sigma(2l-1)-1}<t_{\sigma(2l-1)+1}\}}\mathbf{1}_{\{t_{\sigma(2l-1)-1}<t_{\sigma(2l-1)+1}<t_{\sigma(2l-1)+2}\}}\cdots\mathbf{1}_{\{t_{2k-1}<t_{2k}<1\}}$$

$$dt_1\cdots dt_{\sigma(2l)-1}dt_{\sigma(2l)+1}\cdots dt_{\sigma(2l-1)-1}dt_{\sigma(2l-1)+1}\cdots dt_{2k}$$

$$=\int_{0<t_1<\ldots<t_{\sigma(2l)-1}<t_{\sigma(2l)+1}<\ldots<t_{\sigma(2l-1)-1}<t_{\sigma(2l-1)+1}<\ldots<t_{2k}<1}\int_{t_{\sigma(2l-1)-1}}^{t_{\sigma(2l-1)+1}}\int_{t_{\sigma(2l)-1}}^{t_{\sigma(2l)+1}}g$$

$$dt_{\sigma(2l)}dt_{\sigma(2l-1)}dt_1\cdots dt_{\sigma(2l)-1}dt_{\sigma(2l)+1}\cdots dt_{\sigma(2l-1)-1}dt_{\sigma(2l-1)+1}\cdots dt_{2k},$$

where we assumed without loss of generality that $t_{\sigma(2l)} < t_{\sigma(2l-1)}$. Notice that, for example, $\mathbf{1}_{\{0<t_1<t_2\}}$ is a function of t_1 and that $\mathbf{1}_{\{0<t_1<t_2\}}\mathbf{1}_{\{t_1<t_2<t_3\}} = \mathbf{1}_{\{0<t_1<t_3\}}\mathbf{1}_{\{t_1<t_2<t_3\}} = \mathbf{1}_{\{0<t_1<t_2\}}\mathbf{1}_{\{0<t_2<t_3\}}$. In our case the function g is given by the integrand of (5.15), which is integrable since $H > \frac{1}{2}$.

Now, by using this fact together with Proposition 5.5.5 and Proposition 5.5.6, we have that (5.15) is bounded by

$$\leq \frac{m^{-2H}4AH^k(2H-1)^k}{k!2^k}\sum_{\sigma\in\mathcal{G}_{2k}}\Bigg[\int_{0<t_1<\ldots<t_{\sigma(2)-1}<t_{\sigma(2)+1}<\ldots<t_{\sigma(1)-1}<t_{\sigma(1)+1}<\ldots t_{2k}<1}$$

$$\prod_{l=2}^{k}\delta_{i_{\sigma(2l)},i_{\sigma(2l-1)}}\sum_{i,j=1}^{m}\mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t_{\sigma(2l)},t_{\sigma(2l-1)})m^2\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|x-y|^{2H-2}dxdy$$

$$dt_1\cdots dt_{\sigma(2)-1}dt_{\sigma(2)+1}\cdots dt_{\sigma(1)-1}dt_{\sigma(1)+1}\cdots dt_{2k}$$

$$+\ldots+\int_{0<t_1<\ldots<t_{\sigma(2k)-1}<t_{\sigma(2k)+1}<\ldots<t_{\sigma(2k-1)-1}<t_{\sigma(2k-1)+1}<\ldots t_{2k}<1}$$

$$\prod_{l=1}^{k-1}\delta_{i_{\sigma(2l)},i_{\sigma(2l-1)}}|t_{\sigma(2l)}-t_{\sigma(2l-1)}|^{2H-2}dt_1\cdots dt_{\sigma(2k)-1}dt_{\sigma(2k)+1}\cdots dt_{\sigma(2k-1)-1}dt_{\sigma(2k-1)+1}\cdots dt_{2k}\Bigg]. \tag{5.16}$$

From Proposition 5.5.7 we have

$$=\frac{m^{-2H}4AH^k(2H-1)^k}{k!2^k}2k(2k-1)\sum_{\tau\in\mathcal{G}_{2k-2}}\Bigg[\int_{\Delta^{2k-2}[0,1]}\prod_{l=1}^{k-1}\sum_{i,j=1}^{m}\mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(s_{\tau(2l)},s_{\tau(2l-1)})$$

$$m^2\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|x-y|^{2H-2}\mathrm{d}x\mathrm{d}y\mathrm{d}s_1\cdots\mathrm{d}s_{2k-2}$$

$$+\ldots+\int_{\Delta^{2k-2}[0,1]}\prod_{l=1}^{k-1}\delta_{i_{\tau(2l)},i_{\tau(2l-1)}}|s_{\tau(2l)}-s_{\tau(2l-1)}|^{2H-2}\mathrm{d}s_1\cdots\mathrm{d}s_{2k-2}. \tag{5.17}$$

In particular, in terms of the notation of Proposition 5.5.7 we have used the functions

$$h_1(t_{\sigma(3)},t_{\sigma(4)},\ldots,t_{\sigma(2k)})=\prod_{l=2}^{k}\delta_{i_{\sigma(2l)},i_{\sigma(2l-1)}}\sum_{i,j=1}^{m}\mathbf{1}_{[t_i,t_{i+1}]\times[t_j,t_{j+1}]}(t_{\sigma(2l)},t_{\sigma(2l-1)})$$

$$m^2\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|x-y|^{2H-2}dxdy$$

$$\vdots$$

$$h_k(t_{\sigma(1)}, t_{\sigma(2)}, ..., t_{\sigma(2k-2)}) = \prod_{l=1}^{k-1} \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2}$$

Once we do not consider the delta functions (namely we assume they are all equal to 1), it is possible to observe that the functions

$$f_1(s_1, ..., s_{2k-2}) := \sum_{\tau \in \mathcal{G}_{2k-2}} \prod_{l=1}^{k-1} \sum_{i,j=1}^{m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(s_{\tau(2l)}, s_{\tau(2l-1)}) m^2 \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} \mathrm{d}x\mathrm{d}y$$

$$\vdots$$

$$f_k(s_1, ..., s_{2k-2}) := \sum_{\tau \in \mathcal{G}_{2k-2}} \prod_{l=1}^{k-1} |s_{\tau(2l)} - s_{\tau(2l-1)}|^{2H-2}$$

are symmetric with respect to the diagonal of the $(2k-2)$-dimensional hypercube. This holds because we are considering all possible permutations of the set $(1, ..., 2k-2)$ (notice that $\sum_{\tau \in \mathcal{G}_{2k-2}}$ is part of the functions and that, thanks to the assumption on the delta functions, all the possible permutations are considered), which means that we are considering all the possible orders of the elements $s_1, ..., s_{2k-2}$. This implies that if we change the order of the integrals the value of the iterated integral will not change. Therefore, we have symmetry and we obtain that, for $j = 1, ..., k$,

$$\int_{\Delta^{2k-2}[0,1]} f_j(s_1, ..., s_{2k-2}) \mathrm{d}s_1 \cdots \mathrm{d}s_{2k-2} = \frac{1}{(2k-2)!} \int_{[0,1]^{2k-2}} f_j(s_1, ..., s_{2k-2}) \mathrm{d}s_1 \cdots \mathrm{d}s_{2k-2}.$$

We remark that without the assumption that all the delta functions were equal to one we could not have symmetry since we could not have all the possible permutations (hence orders) of the set $(1, ..., 2k-2)$. Therefore, in conclusion we have that the formula (5.17) is bounded by

$$m^{-2H} \frac{4AH^k(2H-1)^k}{k!2^k} \frac{2k(2k-1)k}{(2k-2)!} \sum_{\tau \in \mathcal{G}_{2k-2}} \int_{[0,1]^{2k-2}} \prod_{l=1}^{k-1} |s_{\tau(2l)} - s_{\tau(2l-1)}|^{2H-2} \mathrm{d}s_1 \cdots \mathrm{d}s_{2k-2}$$

$$\leq m^{-2H} \frac{4AH^k(2H-1)^k}{k!2^k} \frac{2k(2k-1)k}{(2k-2)!} \frac{(2k-2)!}{H^{k-1}(2H-1)^{k-1}} = m^{-2H} \frac{8AH(2H-1)}{(k-1)!2^k} k(2k-1), \tag{5.18}$$

where we have used the fact that when the extremes of the integrals are grid points (in this case they are 0 and 1) the piecewise approximation and the exact value coincide. Recall that A is given by

$$A = 2\left(\frac{1}{H(2H-1)} + \frac{2^{2H}+2}{H(2H-1)} + (4-4H) \sum_{i=1}^{\infty} i^{2H-3} \right) + \frac{3^{2H} + 10(2^{2H}) + 2}{2H(2H-1)}$$

and let $\tilde{A}$ to be defined as

$$\tilde{A} := 8AH(2H-1) = 16\left(3 + 2^{2H} + H(2H-1)(4-4H)\sum_{i=1}^{\infty} i^{2H-3}\right) + 4(3^{2H} + 10(2^{2H}) + 2)$$

$$= 56(1+2^{2H}) + 4(3^{2H}) + 16H(2H-1)(4-4H)\sum_{i=1}^{\infty} i^{2H-3}.$$

Thus, equation (5.18) is equal to $m^{-2H}\frac{\tilde{A}k(2k-1)}{(k-1)!2^k}$, which concludes the proof. □

Notice that

$$m^{-2H}\frac{\tilde{A}k(2k-1)}{(k-1)!2^k} \to 0 \qquad as \qquad k \to \infty.$$

Thus, for any fixed m, the coefficient of the rate of convergence goes to zero as the number of iterated integrals increases and it goes very fast.

The sharpness of the bound is a delicate matter. It has been proved by [NX15] that for the Brownian motion the sharp rate of convergence is 1, which motivates us to believe in the sharpness of our result. In order to be completely sure about it we need to prove that

$$\limsup_{m\to\infty} m^{2H}\left|\mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,I}\right) - \mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,m,I}\right)\right| = 0.$$

However, the piecewise approximation makes this idea extremely difficult to put in practice. This is because, when we focus on the double integral $\int_{u<s<t<v}$, the piecewise approximation and the exact value are equal when the times u and v correspond to grid points t_q and t_p respectively, but they are not equal otherwise. Hence, as $m \to \infty$ the difference between the exact value and the piecewise approximation goes to zero not linearly. This makes the problem of the sharpness of the rate of convergence to appear impossible.

A possible solution is to consider the iterated integral as a whole without focusing on the general double integral. But this seems a difficult way of dealing with the problem. Indeed, we proceeded on with this way at the beginning of this chapter obtaining a bound with a rate of convergence of $2H$ but with a non integrable coefficient. For these reasons we are confident that the rate of convergence of $2H$ is sharp.

As a final remark, notice that $m^{-2H} < m^{-1}$, which means that the difference goes to zero faster for the fBm than the Bm. This is an expected result since the fBm with $H > \frac{1}{2}$ is a smoother path than the Bm. This higher smoothness is related to the positive correlation of the increments of the fBm with $H > \frac{1}{2}$. Indeed, we can see that the rate of convergence becomes smaller (hence the convergence faster) as the positive correlation (represented by H) increases.

Appendix B: Proof of Theorem 5.2.2

In this section we focus solely on the expected signature of the fractional Brownian motion, for which we prove a simple but sharp estimate for it: Theorem 5.2.2. The proof of this theorem is very short and the arguments are similar to the ones used at the end of the proof of Theorem 5.2.1.

Proof of Theorem 5.2.2. Consider the $2k$-term of the expected signature of the fractional Brownian motion for $H > \frac{1}{2}$, we have

$$\mathbb{E}\left(\int_{\Delta^{2k}[0,1]} dB^{H,I}\right) = \frac{H^k(2H-1)^k}{k!2^k} \sum_{\sigma\in\mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \prod_{l=1}^{k} \delta_{i_{\sigma(2l)},i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} dt_1 \cdots dt_{2k}$$

$$\leq \frac{H^k(2H-1)^k}{k!2^k} \sum_{\sigma\in\mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \prod_{l=1}^{k} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} dt_1 \cdots dt_{2k}$$

$$= \frac{H^k(2H-1)^k}{k!2^k} \frac{1}{(2k)!} \sum_{\sigma\in\mathcal{G}_{2k}} \int_{[0,1]^{2k}} \prod_{l=1}^{k} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} dt_1 \cdots dt_{2k}$$

$$= \frac{H^k(2H-1)^k}{k!2^k} \frac{1}{(2k)!} \sum_{\sigma\in\mathcal{G}_{2k}} \prod_{l=1}^{k} \int_0^1 \int_0^1 |t-s|^{2H-2} ds dt = \frac{1}{k!2^k},$$

by using only the fact that the function

$$\sum_{\sigma\in\mathcal{G}_{2k}} \prod_{l=1}^{k} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2}$$

is symmetric with respect to the diagonal of the $2k$-dimensional hypercube. □

A similar but less sharp estimate was obtained in Proposition 4.8 of [NNRT09]. Indeed, using the explicit formulation of this proposition reported in Proposition 3.3 of [BZ12], it is possible to see that they obtained a uniform bound for the $2k$-th term of the expected signature of

$$\frac{2^k}{H^k(2H-1)^k\sqrt{(2k)!}}.$$

Observe that

$$\frac{2^k}{H^k(2H-1)^k\sqrt{(2k)!}} > \frac{2^k}{\sqrt{(2k)!}} > \frac{1}{k!2^k}.$$

The advantage of our result is that the estimate is sharp, it is independent of H and the proof is very short.

Moreover, our bound is sharper than the bound of $\frac{2^k C^{2k} (2k)^{k/2}}{(2k/2+)!}$ in [FR13], where C is some positive constant independent of k. Their bound is obtained from Remark 1 point (2) in the second ArXiv version of their work, for the case $\rho = q = 1$ and $p = 2+$ (see also Remark 1 point (3)). In particular, we have that $\rho = 1$ because the covariance of the fBm with $H > 1/2$ has finite ($\rho =$)1-variation. Looking at the leading order terms, it is easy to see that our bound is sharper than their bound.

Appendix C: Proof of Theorem 5.3.2

First of all, notice that we have the following stochastic Taylor expansion (see Proposition 2.1 of [LV04] together with [BZ12]). For any f which is smooth (*i.e.* infinitely differentiable) and whose derivatives of any order are bounded, we have

$$f(\xi_{T,x}) = \sum_{(i_1,\dots,i_k)\in\mathcal{A}_m} V_{i_1}\cdots V_{i_k} f(x) \int_{0<t_1<,\dots,<t_k<T} d\hat{B}^{H,i_1}_{t_1}\cdots d\hat{B}^{H,i_k}_{t_k} + R_m(T,x,f), \quad \text{where}$$

$$R_m(T,x,f) = \sum_{(i_1,\dots,i_k)\in\mathcal{A}_m,(i_0,\dots,i_k)\notin\mathcal{A}_m} \int_{0<t_0<,\dots,<t_k<T} V_{i_0}\cdots V_{i_k} f(\xi_{t_0,x}) d\hat{B}^{H,i_0}_{t_0}\cdots d\hat{B}^{H,i_k}_{t_k}.$$

Assume that the paths $\bar{\omega}_1,\dots,\bar{\omega}_n \in C^0_{0,bv}([0,T],\mathbb{R}^d)$ and the weights $\lambda_1,\dots,\lambda_n$ define a cubature formula for the fBm of degree m for time T. Now notice that by the cubature formula, we have

$$\mathbb{E}\left[\int_{0<t_1<,\dots,<t_k<T} d\hat{B}^{H,i_1}_{t_1}\cdots d\hat{B}^{H,i_k}_{t_k}\right] = \sum_{j=1}^{n}\lambda_j \int_{0<t_1<,\dots,<t_k<T} d\hat{\omega}^{i_1}_j(t_1)\cdots d\hat{\omega}^{i_k}_j(t_k), \qquad (5.19)$$

where $\hat{\omega}(t) = (t,\bar{\omega}(t))$. Observe that from our setting, we have

$$\mathbb{E}\left[\int_{0<t_1<,\dots,<t_k<T} d\hat{B}^{H,i_1}_{t_1}\cdots d\hat{B}^{H,i_k}_{t_k}\right] = \int_{C^0_0([0,T],\mathbb{R}^d)}\int_{0<t_1<,\dots,<t_k<T} d\hat{B}^{H,i_1}_{t_1}(\omega)\cdots d\hat{B}^{H,i_k}_{t_k}(\omega)\mathbb{P}(d\omega),$$

so eq. (5.19) can be rewritten as

$$\int_{C^0_0([0,T],\mathbb{R}^d)}\int_{0<t_1<,\dots,<t_k<T} d\hat{B}^{H,i_1}_{t_1}(\omega)\cdots d\hat{B}^{H,i_k}_{t_k}(\omega)\mathbb{P}(d\omega)$$

$$= \sum_{j=1}^{n}\lambda_j \int_{0<t_1<,\dots,<t_k<T} d\hat{\omega}^{i_1}_j(t_1)\cdots d\hat{\omega}^{i_k}_j(t_k).$$

From this formula we can obtain the probability measure $\mathbb{Q}_T = \sum_{j=1}^n \lambda_j\delta_{\omega^\star_j}$ on $(C^0_0([0,T],\mathbb{R}^d),\mathcal{F})$, namely on the Wiener space with its Borel σ-field. In particular, let

$$Z(\omega) := \int_{0<t_1<,\dots,<t_k<T} d\hat{B}^{H,i_1}_{t_1}(\omega)\cdots d\hat{B}^{H,i_k}_{t_k}(\omega),$$

and so

$$\int_{C^0_0([0,T],\mathbb{R}^d)}\int_{0<t_1<,\dots,<t_k<T} d\hat{B}^{H,i_1}_{t_1}(\omega)\cdots d\hat{B}^{H,i_k}_{t_k}(\omega)\mathbb{P}(d\omega) = \int_{C^0_0([0,T],\mathbb{R}^d)} Z(\omega)\mathbb{P}(d\omega),$$

and by applying the measure $\mathbb{Q}_T$ we have

$$\mathbb{E}_{\mathbb{Q}_T}[Z] = \sum_{j=1}^{n}\lambda_j \int_{C^0_0([0,T],\mathbb{R}^d)} Z(\omega)\delta_{\omega^\star_j}d\omega = \sum_{j=1}^{n}\lambda_j Z(\omega^\star_j)$$

$$= \sum_{j=1}^{n}\lambda_j \int_{0<t_1<,\dots,<t_k<T} d\hat{B}^{H,i_1}_{t_1}(\omega^\star_j)\cdots d\hat{B}^{H,i_k}_{t_k}(\omega^\star_j),$$

where from the definition of the fBm we have, for $i_l \neq 0$, $\hat{B}^{H,i_l}_{t_l}(\omega^\star_j) = \int_0^{t_l} K(t_l,s)d\omega^{\star,i_l}_j(s)$, while, for $i_l = 0$, $\hat{B}^{H,i_l}_{t_l}(\omega^\star_j) = t_l$.

It is possible to observe that we have not mentioned how we select these $\omega^\star_j \in C^0_0([0,T],\mathbb{R}^d)$.

We do it now. Choose $\omega_j^\star \in C_0^0([0,T],\mathbb{R}^d)$ such that $\hat{\omega}_j^{i_l}(t_l) = \int_0^{t_l} K(t_l,s)d\omega_j^{\star,i_l}(s)$ for $i_l \neq 0$. We know that such $\omega_j^\star$ exists since $\bar{\omega} \in C^0_{\mathrm{bv},0}([0,T],\mathbb{R}^d)$, where we had $\hat{\omega}(t) = (t,\bar{\omega}(t))$. Thus, $\mathbb{Q}_T = \sum_{j=1}^n \lambda_j \delta_{\omega_j^\star}$ is a probability measure on $(C_0^0([0,T],\mathbb{R}^d),\mathcal{F})$. Hence, we have $\hat{B}_{t_l}^{H,i_l}(\omega_j^\star) = \hat{\omega}_j^{i_l}(t_l)$ and, consequently,

$$\sum_{j=1}^n \lambda_j \int_{0<t_1<,\dots,<t_k<T} d\hat{B}_{t_1}^{H,i_1}(\omega_j^\star)\cdots d\hat{B}_{t_k}^{H,i_k}(\omega_j^\star) = \sum_{j=1}^n \lambda_j \int_{0<t_1<,\dots,<t_k<T} d\hat{\omega}_j^{i_1}(t_1)\cdots d\hat{\omega}_j^{i_k}(t_k),$$

that is

$$\mathbb{E}_{\mathbb{Q}_T}\left[\int_{0<t_1<,\dots,<t_k<T} d\hat{B}_{t_1}^{H,i_1}\cdots d\hat{B}_{t_k}^{H,i_k}\right] = \sum_{j=1}^n \lambda_j \int_{0<t_1<,\dots,<t_k<T} d\hat{\omega}_j^{i_1}(t_1)\cdots d\hat{\omega}_j^{i_k}(t_k).$$

Therefore, from eq. (5.19) we obtain the following formulation:

$$\mathbb{E}\left[\int_{0<t_1<,\dots,<t_k<T} d\hat{B}_{t_1}^{H,i_1}\cdots d\hat{B}_{t_k}^{H,i_k}\right] = \mathbb{E}_{\mathbb{Q}_T}\left[\int_{0<t_1<,\dots,<t_k<T} d\hat{B}_{t_1}^{H,i_1}\cdots d\hat{B}_{t_k}^{H,i_k}\right]. \tag{5.20}$$

It is very important to notice that while $B_{t_l}^{H,i_l}(\omega)$ is a fBm under the Wiener measure, it is not any more under $\mathbb{Q}_T$, and that the equality (5.20) holds only for all $(i_1,...,i_k) \in \mathcal{A}_m$ by definition of the cubature formula. Moreover, observe that

$$\mathbb{E}_{\mathbb{Q}_T}[f(\xi_{T,x})] = \sum_{j=1}^n \lambda_j \int_{C_0^0([0,T],\mathbb{R}^d)} f(\xi_{T,x}(\omega_j))\delta_{\omega_j^*} d\omega = \sum_{j=1}^n \lambda_j f(\xi_{T,x}(\omega_j^*))$$

and that $\xi_{T,x}(\omega_j^*)$ is the solution of

$$\mathrm{d}\xi_{t,x}(\omega_j^*) = \sum_{i=0}^d V_i(\xi_{t,x}(\omega_j^*))\mathrm{d}\hat{B}_t^{H,i}(\omega_j^*), \quad \text{with} \quad \xi_{0,x}(\omega_j^*) = x,$$

or, equivalently, the solution of

$$\mathrm{d}y_{t,x} = \sum_{i=0}^d V_i(y_{t,x})\mathrm{d}\hat{\omega}_j^i(t), \quad \text{with} \quad y_{0,x} = x,$$

which we defined before to be $\Phi_{t,x}(\omega_j^*)$. Therefore,

$$\sum_{j=1}^n \lambda_j f(\Phi_{T,x}(\omega_j^*)) = \mathbb{E}_{\mathbb{Q}_T}[f(\xi_{T,x})].$$

Let us now adapt the stochastic Taylor formula to the probability measure $\mathbb{Q}_T$. In particular, using the scaling property of the fBm, which is inherited by the respective cubature paths we have that

$$|\mathbb{E}_{\mathbb{Q}_T}[R_m(T,x,f)]|$$

$$= \left|\mathbb{E}_{\mathbb{Q}_T}\left[\sum_{(i_1,\dots,i_k)\in\mathcal{A}_m,(i_0,\dots,i_k)\notin\mathcal{A}_m} \int_{0<t_0<,\dots,<t_k<T} V_{i_0}\cdots V_{i_k} f(\xi_{t_0,x}) d\hat{B}_{t_0}^{H,i_0}\cdots d\hat{B}_{H,t_k}^{i_k}\right]\right|$$

$$\leq \sum_{j=1}^{n} \lambda_j \sum_{(i_1,\dots,i_k)\in\mathcal{A}_m,(i_0,\dots,i_k)\notin\mathcal{A}_m} \left| \int_{0<t_0<,\dots,<t_k<T} V_{i_0}\cdots V_{i_k} f(\xi_{t_0,x}(\omega_j^\star)) d\hat{\omega}_j^{i_1}(t_1)\cdots d\hat{\omega}_j^{i_k}(t_k) \right| \tag{5.21}$$

$$\leq \sum_{i=j}^{n} \lambda_j \sum_{(i_1,\dots,i_k)\in\mathcal{A}_m,(i_0,\dots,i_k)\notin\mathcal{A}_m} \|V_{i_0}\cdots V_{i_k} f\|_\infty$$

$$T^{1+kH+(1-H)\times card\{l,i_l=0\}} \int_{0<t_0<,\dots,<t_k<1} \left|d\hat{\omega}_j^{i_0}(t_0)\right|\cdots\left|d\hat{\omega}_j^{i_k}(t_k)\right|$$

$$\leq C' T^{(m+2)/2} \sup_{(i_1,\dots,i_k)\in\mathcal{A}_m,(i_0,\dots,i_k)\notin\mathcal{A}_m} \|V_{i_0}\cdots V_{i_k} f\|_\infty, \quad \text{if } T\geq 1, \text{ and}$$

$$(5.21) \leq \sum_{i=j}^{n} \lambda_j \sum_{(i_1,\dots,i_k)\in\mathcal{A}_m,(i_0,\dots,i_k)\notin\mathcal{A}_m} \|V_{i_0}\cdots V_{i_k} f\|_\infty$$

$$T^{H+kH+(1-H)\times card\{l,i_l=0\}} \int_{0<t_0<,\dots,<t_k<1} \left|d\hat{\omega}_j^{i_0}(t_0)\right|\cdots\left|d\hat{\omega}_j^{i_k}(t_k)\right|$$

$$\leq C'' T^{2H} \sup_{(i_1,\dots,i_k)\in\mathcal{A}_m,(i_0,\dots,i_k)\notin\mathcal{A}_m} \|V_{i_0}\cdots V_{i_k} f\|_\infty, \quad \text{if } T<1,$$

where $C', C'' > 0$ and does not depend on T. Notice that the 1 in $1+kH+(1-H)\times card\{l, i_l = 0\}$ and the H in $H + kH + (1-H)\times card\{l, i_l = 0\}$ comes from the fact that we might have $i_0 = 0$ (hence we will have the scaling T) or $i_0 = 1$ (hence we will have the scaling T^H), since $T \geq T^H$ for $T \geq 1$ we take T to get the upper bound for $T \geq 1$ and T^H otherwise for $T < 1$. Moreover, for the last inequality in both cases we used the fact that for any word in $\mathcal{A}_m$ we have by its definition that $H \leq kH + (1-H)\times card\{l, i_l = 0\} \leq m/2$.

We are not able to get a similar bound for $|\mathbb{E}[R_m(T, x, f)]|$. Indeed, [BZ12] is partially devoted to the study of this quantity. In [BZ12] a different form of the remainder is considered, but the two can be linked together. In particular, from Theorem 3.4 in [BZ12] we have

$$|\mathbb{E}[R_m(T,x,f)]| \leq C_\gamma \frac{(dMKT)^{(m+2)/2}}{(((m+2)/2)!)^{1/2-\gamma}} \sum_{k=0}^{\infty} \frac{(dMKT)^k}{(k!)^{1/2-\gamma}}, \quad \text{if } T\geq 1 \text{ and}$$

$$|\mathbb{E}[R_m(T,x,f)]| \leq C_\gamma \frac{(dMKT^H)^{(m+2)/2}}{(((m+2)/2)!)^{1/2-\gamma}} \sum_{k=0}^{\infty} \frac{(dMKT^H)^k}{(k!)^{1/2-\gamma}}, \quad \text{if } T<1,$$

where M and K are defined in the statement of the theorem. We used $(m+2)/2$ for the "$N+1$" in the statement of Theorem 3.4 in [BZ12] because it is the minimum number of iterated integrals of $\mathcal{A}_{m+2}$. Further, we used T instead of T^H because we want a bound which is independent of the exact composition of the word I:

$$\mathbb{E}\left(\left|\int_{\Delta^k[0,T]} d\hat{B}^{H,I}\right|^2\right)^{1/2} \leq T^{kH+(1-H)\times card\{j,i_j=0\}} \frac{K^k}{\sqrt{k!}} \leq \begin{cases} T^k \dfrac{K^k}{\sqrt{k!}} & \text{if } T\geq 1, \\ T^{kH} \dfrac{K^k}{\sqrt{k!}} & \text{if } T<1. \end{cases}$$

By doing this we are able to consider the drift (namely $V_0 \neq 0$), which is not considered explicitly in Theorem 3.4 in [BZ12]. Further, we have not mentioned that the vector fields V_is are analytic since being infinitely differentiable and having bounded derivatives imply that they are global

analytic.

Finally, observe that

$$|\mathbb{E}[f(\xi_{T,x})] - \mathbb{E}_{\mathbb{Q}_T}[f(\xi_{T,x})]| \leq |\mathbb{E}[R_m(T,x,f)]| + |\mathbb{E}_{\mathbb{Q}_T}[R_m(T,x,f)]|$$

$$+\left|(\mathbb{E}-\mathbb{E}_{\mathbb{Q}_T})\left[\sum_{(i_1,\dots,i_k)\in\mathcal{A}_m} V_{i_1}\cdots V_{i_k}f(x)\int_{0<t_1<,\dots,<t_k<T} d\hat{B}_{t_1}^{H,i_1}\cdots d\hat{B}_{t_k}^{H,i_k}\right]\right|.$$

For the first two terms we have proved their upper bounds, while the last term is zero by definition of $\mathbb{Q}_T$, namely by the cubature formula. Then, we have

$$\sup_{x\in\mathbb{R}^N}\left|\mathbb{E}\left(f(\xi_{T,x})\right) - \sum_{j=1}^{n}\lambda_j f(\Phi_{T,x}(\omega_j))\right| \leq \begin{cases} Z_1(T)+Z_3(T) & \text{if } T\geq 1,\\ Z_2(T)+Z_4(T) & \text{if } T<1.\end{cases}$$

where

$$Z_1(T) := C'T^{(m+2)/2}\sup_{(i_1,\dots,i_k)\in\mathcal{A}_m,(i_0,\dots,i_k)\notin\mathcal{A}_m}\|V_{i_1}\cdots V_{i_k}f\|_\infty,$$

$$Z_2(T) := C''T^{2H}\sup_{(i_1,\dots,i_k)\in\mathcal{A}_m,(i_0,\dots,i_k)\notin\mathcal{A}_m}\|V_{i_1}\cdots V_{i_k}f\|_\infty,$$

$$Z_3(T) := C_\gamma T^{(m+2)/2}\frac{(dK)^{(m+2)/2}}{(((m+2)/2)!)^{1/2-\gamma}}\sup_{x\in\mathbb{R}^N}M_x^{(m+2)/2}\sum_{k=0}^{\infty}\frac{(dM_xKT)^k}{(k!)^{1/2-\gamma}},$$

$$Z_4(T) := C_\gamma T^{H(m+2)/2}\frac{(dK)^{(m+2)/2}}{(((m+2)/2)!)^{1/2-\gamma}}\sup_{x\in\mathbb{R}^N}M_x^{(m+2)/2}\sum_{k=0}^{\infty}\frac{(dM_xKT^H)^k}{(k!)^{1/2-\gamma}},$$

where $C', C'', C_\gamma > 0$ are constants independent of T.

Finally, notice that $\sup_{x\in\mathbb{R}^N} M_x = M$ because from the statement of the theorem M is such that

$$\sup_{x\in\mathbb{R}^N}|(V_{i_i}\cdots V_{i_k}f)(x)| = \|(V_{i_i}\cdots V_{i_k}f)\|_\infty \leq M^{|I|}(|I|!)^\gamma$$

and thus M does not depend on x. Therefore, we have

$$Z_3(T) = C_\gamma T^{(m+2)/2}\frac{(dK)^{(m+2)/2}}{(((m+2)/2)!)^{1/2-\gamma}}M^{(m+2)/2}\sum_{k=0}^{\infty}\frac{(dMKT)^k}{(k!)^{1/2-\gamma}},$$

$$Z_4(T) = C_\gamma T^{H(m+2)/2}\frac{(dK)^{(m+2)/2}}{(((m+2)/2)!)^{1/2-\gamma}}M^{(m+2)/2}\sum_{k=0}^{\infty}\frac{(dMKT^H)^k}{(k!)^{1/2-\gamma}}.$$

6

Conclusion and future directions

In this section we recall the main highlights of this thesis and indicate future potential research directions.

In Chapter 2 we have extended the theory of QID distributions to random measures and stochastic processes. In particular, we studied the existence and uniqueness of QID random measures and how they are related to ID random measures. We have seen how to define stochastic integrals w.r.t. QID random measures and show for which conditions on a functions f we have that $\int f d\Lambda$ is defined. Moreover, we have investigated various representations of a QID random measure and of a QID stochastic process.

There are several and pivotal questions that are left open. First, we know that QID distributions are dense in the space of all probability distributions under the weak convergence. In [5], it is shown that a similar result holds for completely random measures. Can we extend this to processes? In other words, can we prove that for any stochastic process there is a sequence of QID processes that converge to it in distribution? Or at least in finite dimensional distribution, namely without tightness?
Second, what properties does a QID process satisfy? Is there a similar Lévy-Itô decomposition? Third, it has been seen that QID distributions are related to the Riemann zeta function. Which insights on this function can we extract from QID random measures and stochastic processes?

In Chapter 3 we derived various results concerning ergodicity and mixing properties of multivariate stationary infinitely divisible random fields. A possible future direction consists of the investigation of statistical properties of the results presented in this chapter. For example for multivariate stochastic processes, showing that mixed moving average (MMA) processes are mixing implies that the corresponding moment based estimator (like the generalised method of moments (GMM)) are consistent (see [FS13]). However, it is not clear that a similar result holds for the random fields case. Other possible directions might be the extension of the present results to the case of random fields on manifolds or on infinite dimensional vector spaces. However, the literature is not as developed as for the $\mathbb{R}^l$-case and for these settings further work is required.

In Chapter 4 we introduced the multivariate BSS process and studied the joint asymptotic behaviour of its realised covariation, presenting limit theorems, feasible results and an explicit example. We also provided central limit theorems and weak laws of large numbers for general multivariate Gaussian processes with stationary increments. There are at least two directions which will be worth exploring in more detail in the future:

First, is it possible to find feasible estimates for the asymptotic variance of the multivariate BSS processes? *I.e.* can "second order" feasible results be obtained in addition to the "first order" feasible results we already presented?

Second, we considered the asymptotic theory for BSS processes also outside the semimartingale setting. In doing so, we concentrate on a particular scenario (as described by the assumptions on the deterministic function g in Assumption 4.3.1). However, one can imagine other scenarios which lead to BSS processes (or other volatility modulated Gaussian processes) beyond the semimartingale framework. Can similar asymptotic results for the (scaled) realised covariation be obtained in such settings?

Finally, in Chapter 5 we present a bound for the expected signature of the fBm, provide the rate of convergence of the expected signature of the linear piecewise approximation of the fBm to its exact value, and develop the general cubature method for the fBm with $H > \frac{1}{2}$ for small times. Possible future directions are inextricably linked with the development of explicit formulas for the expected signature of the process at hand. A major open problem is to find an explicit formulation for the expected signature of all Gaussian processes with covariance with bounded ρ-variation, with $\rho \in [1, 2)$. A possible direction would be to explore the cubature method for Lévy rough paths, for which the explicit formula for their expected signature is provided (see [FrS17]).

Further future directions of the topics discussed in this thesis are also explored in the ongoing works [1, 2, 5]. In [1], feasible inference results for BSS processes are explored. It is shown that using both in-fill and long-term asymptotics a feasible CLT for BSS processes can be obtained. In [2], limit theorems for trawl processes are achieved. For example, it is shown that under certain assumption trawl processes, which are time parametrisations of ID random measures, converge in distribution to Gaussian moving averages. Finally, in [5], QID random measures are further explored. It is shown that a family of quasi-infinitely divisible (QID) random measures is dense in the class of all independently scattered random measures with respect to convergence in distribution. Moreover, an extension of Theorem 2.4.18 is provided, allowing a generalisation and unification of some of the results on QID random measures of Chapter 2.

Bibliography

[AE78] Aldous, D. J. and Eagleson, G. K. (1978). On mixing and stability of limit theorems, *The Annals of Probability* **6**, 325-331.

[BBV13] Barndorff-Nielsen, O. E., Benth, F. E. and Veraart, A. E. D. (2013). Modelling energy spot prices by volatility modulated Lévy-driven Volterra processes. *Bernoulli* **19** (3), 803-845.

[BBV14] Barndorff-Nielsen, O. E., Benth, F. E. and Veraart, A. E. D. (2014) Recent advances in ambit stochastics with a view towards tempo-spatial stochastic volatility/intermittency, *Banach C. Publ.*, 104, 25-60.

[BCD09a] Barndorff-Nielsen, O. E., Corcuera, J. M. and Podolskij, M. (2009). Power variation for Gaussian processes with stationary increments. *Stochastic Process. Appl.* **119**, 1845-1865.

[BCD11] Barndorff-Nielsen, O. E., Corcuera, J. M. and Podolskij, M. (2011). Multipower variation for Brownian semistationary processes. *Bernoulli* **17**(4), 1159-1194.

[BCD09b] Barndorff-Nielsen, O. E., Corcuera, J. M. and Podolskij, M. (2009). Multipower variation for Brownian semistationary processes (full version). *CREATES research paper 2009-21*, Aarhus Univ.

[BPS14] Barndorff-Nielsen, O. E., Pakkanen, M. S. and Schmiegel, J. (2014). Assessing relative volatility/ intermittency/energy dissipation. *Electron. J. Statist.* **8**, no. 2, 1996-2021.

[BP12] Barndorff-Nielsen O. E., and Pedersen J., (2012) Meta times and extended subordination, *Theory of Probability and its Applications*, Vol. 56 No. 2, 319-327.

[BS04] Barndorff-Nielsen, O. E., and Schmiegel, J. (2004) Lévy-based Tempo-Spatial Modelling; with Applications to Turbulence, *Uspekhi Matematicheskikh Nauk*, 159, 63-90.

[BS07] Barndorff-Nielsen, O. E., and Schmiegel, J. (2007) Ambit processes: with applications to turbulence and tumour growth, *Stochastic Analysis and Applications: Abel Symposium 2005*, 93-124.

[BS09] Barndorff-Nielsen O. E. and Schmiegel J. (2009). Brownian semistationary processes and volatility/intermittency, in: *H. Albrecher, et al. (Eds.), Advanced Financial Modelling,* pp. 1-26.

[BSh04] Barndorff-Nielsen, O. E. and Shephard, N. (2004). Econometric Analysis of Realized Covariation: High Frequency Based Covariance, Regression, and Correlation in Financial Economics. *Econometrica*, **72**, 885-925.

[Ba54] Bateman, H. (1954). *Tables of integral transforms, California Institute of Technology Bateman Manuscript Project*, New York: McGraw-Hill, edited by Erdélyi Arthur.

[BC07] Baudoin, F., and Coutin, L. (2007) Operators associated with a stochastic differential equation driven by fractional Brownian motions. *Stoch. Proc. Appl.* 117, 5, 550-574.

[BZ12] Baudoin, F., and Zhang, X., (2012) Taylor expansion for the solution of a stochastic differential equation driven by fractional Brownian motions. *Electron. J. Probab.*, 51, 1-21.

[BFL13] Bayer, C., Friz P., and Loeffen, R., (2013) Semi-closed form cubature and applications to financial diffusion models. *Quantitative Finance*, 13, 5, 769-782.

[Be17] Bennedsen, M. (2017). A rough multi-factor model of electricity spot prices.*Energ. Econ.* **63**, 301-313.

[BLP17a] Bennedsen M., Lunde, A. and Pakkanen, M. (2017). Hybrid scheme for Brownian semistationary processes *Finance and Stochastics* **21**(4), 931-965.

[BLP17b] Bennedsen M., Lunde, A. and Pakkanen M. (2017), Decoupling the short- and long-term behavior of stochastic volatility. e-print: arXiv:1610.00332.

[Ber19] Berger, D. (2019) On Quasi-Infinitely Divisible Distributions with a Point Mass. To appear in *Mathematische Nachrichten.*

[Bi99] Billingsley, P. (1999). *Convergence of probability measures.* Second Edition. Wiley series in Probability and Statistics.

[BPQ] Boedihardjo, H., Papavasiliou, A., Qian, Z. (2013) Expected signature of Gaussian processes with strictly regular kernels. *ArXiv:1304.4930.*

[Bog07] Bogachev, V. I. (2007) *Measure theory. Vol I.* Springer-Verlag, Berlin.

[CDM16] Chaiba H., Demni, N.,and Mouayn, Z. (2016) Analysis of generalized negative binomial distributions attached to hyperbolic Landau levels. *Journal of Mathematical Physics* 57, 072-103.

[Ch57] Chen, K. (1957) Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula, *Ann. Math.*, 65, 1, 163-178.

[Ch58] Chen, K. (1958) Integration of paths âĂŞ a faithful representation of paths by non-commutative formal power series. *Trans. Am. Math. Soc.*, 89, 395-407.

[Ch77] Chen, K. (1977) Iterated path integrals, *Bull. Am. Math. Soc.*, 83, 5, 831-879.

[CL16] Chevyrev, I., and Lyons, T. (2016) Characteristic functions of measures on geometric rough paths. *Ann. Probab.* 44, 6, 4049-4082.

[Co12] Corcuera, J. M. (2012). New Central Limit Theorems for Functionals of Gaussian Processes and their Applications. In: *Methodology and Computing in Applied Probability* **14.3**, pp. 477-500

[CHPP13] Corcuera, J. M., Hedevang, E., Podolskij, M. and Pakkanen, M. (2013). Asymptotic theory for Brownian semistationary processes with application to turbulence. *Stoch. Proc. Appl.* **123**(7), 2552-2574

[CQ02] Coutin, L., and Qian, Z. (2002) Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theor. Relat. Fields*, 122(1):108-140.

[Cu75] Cuppens, R. (1975) *Decomposition of Multivariate Probabilities.* Academic Press, New York.

[DM82] Dellacherie, C., and Meyer, P. A. (1982) *Probabilities and Potential Vol. A.,* North-Holland Mathematics.

[DM15] Demni, N., and Mouayn, Z. (2015) Analysis of generalized Poisson distributions associated with higher Landau levels. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 18(04).

[DNT12] Deya, A., Neuenkirch A., and Tindel, S. (2012) A Milstein-type scheme without Levy area terms for SDEs driven by fractional Brownian motion, *Ann. Inst. H. Poincaré (B)*, 48, 2, 518-550.

[FS13] Fuchs, F., and Stelzer, R., (2013) Mixing conditions for multivariate infinitely divisible processes with an application to mixed moving averages and the supOU stochastic volatility model, *ESAIM: PS*, 17, 455-471.

[FR13] Friz, P., and Riedel, S. (2013) Integrability of (non-)linear rough differential equations and integrals. *Stoch. Anal. Appl.*, 31, 2, 336-358.

[FR14] Friz, P., and Riedel, S. (2014) Convergence rates for the full Gaussian rough paths. *Ann. Inst. H. Poincaré (B)*, 50, 1, 154-194.

[FrS17] Friz, P., and Shekhar, A. (2017). General rough integration, Lévy rough paths and a Lévy-Kintchine-type formula. *Ann. Probab.* 45, no. 4, 2707-2765.

[Gr17] Granelli, A. (2017). Limit theorems and stochastic models for dependence and contagion in financial markets. PhD thesis, Imperial College London.

[GR19] Granelli, A. and Veraart, A. E. D. (2019). A central limit theorem for the realised covariation of a bivariate Brownian semistationary process. To appear in *Bernoulli*.

[Gr94] Gross A., (1994) Some mixing conditions for stationary symmetric stable stochastic processes, *Stoch. Proc. Appl.* 51, 277-295.

[Ha14] Hairer, M. (2014) A theory of regularity structures. *Inventiones mathematicae*, 198, 2, 269-504..

[HS78] Horn R.A, and Steutel F.W. (1978) Multivariate infinitely divisible distributions. *Stoch. Proc. Appl.*, 6, 139-151.

[Ho77] Horowitz J. (1977) Une remarque sur les bimesures. *Séminaire de probabilités*, 11 , p. 59-64.

[Ho86] Horowitz J. (1986) Gaussian random measures. *Stoch. Proc. Appl.* 22, 129-133.

[Ja97] Jacod J. (1997). On continuous conditional Gaussian martingales and stable convergence in law. *Séminaire de Probabilités* XXXI, 232-246.

[Ja08] Jacod J. (2008), Asymptotic properties of realized power variations and related functionals of semimartingales. *Stoch. Proc. Appl.*, **118**, pp. 517-559.

[JR13] Jónsdóttir K., Rønn-Nielsen A.,Mouridsen K, Vedel Jensen E., (2013) Lévy-based modelling in brain imaging, *Scand. J. Stat.*, 40, 3, 511-529.

[Ka01] Kallenberg O. (2001) Foundations of modern probability. Second Edition. *Springer*.

[Ka17] Kallenberg O. (2017) Random Measures, Theory and Applications. *Springer*.

[Kh19] Khartov A. A. (2019) Compactness criteria for quasi-infinitely divisible distributions on the integers. *Statistics and Probability Letters*, 153, 1-6.

[Kl14] Klenke A. (2014) *Probability Theory: A Comprehensive Course*. Second Edition, Springer.

[KT96] Kokoszka P. and Taqqu M.S., (1996) A characterization of mixing processes of type G, *J. Theor. Probab.*, 9, 3-17.

[LT91] Ledoux, M., and Talagrand, M. (1991) *Probability in Banach spaces: isoperimetry and processes*, Springer-Verlag.

[LPS18] Lindner A., Pan L., and Sato K. (2018) On quasi-infinitely divisible distribution, *Trans. Amer. Math. Soc.*, **370**, 8483-8520

[Te98] Lyons, T., (1998) Differential equations driven by rough signals. *Rev. Mat. Iberoam.* 14, 2, 215-310.

[LCL07] Lyons, T., Caruana, M., and Lévy, T. (2007) *Differential equations driven by rough paths*, vol. 1908 of Lecture Notes in Mathematics, Springer.

[LV04] Lyons, T., and Victoir, T. (2004) Cubature on Wiener space. *Proc. R. Soc. Lond. Ser. A* 460, 2041, 169-198.

[LV07] Lyons, T., and Victoir, T. (2007) An extension theorem to rough paths. *Ann. Inst. H. Poincare Anal. Non Lineaire* 24, 5, 835-847.

[Mag10] Magdziarz, M., A note on Maruyama's mixing theorem. *Theor. Probab. Appl.*, 54, 322-324 (2010).

[Mah47] Maharam D., (1947) An algebraic characterization of measure algebras, *Ann. of Math.*, 48, 154-167.

[Mar70] Maruyama, G., (1970) Infinitely divisible processes. *Theor. Probab. Appl.*, 15, 1-22.

[Mu83] Musielak, J. (1983) *Orlicz spaces and modular spaces.* Lect. Notes Math., Vol. 1034. Springer.

[Na15] Nakamura, T. A complete Riemann zeta distribution and the Riemann hypothesis. *Bernoulli*, 21(1), 604-617, (2015).

[Na] Nakamura, T. Zeta distributions generated by Dirichlet series and their (quasi) infinitely divisibility, *private communication.*

[NNRT09] Neuenkirch, A., Nourdin, I., Rössler, A., Tindel, S. (2009) Trees and asymptotic expansions for fractional stochastic differential equations. *Ann. Inst. Poincaré (B)* 45, 1, 157-174.

[NXL17] Ni, H., Xie, Z., Lyons, T., Jin L., and Sun, Z. (2017) Learning Spatial-Semantic Context with Fully Convolutional Recurrent Network for Online Handwritten Chinese Text Recognition. *IEEE Transactions on Pattern Analysis and Machine Intelligence.*

[NX15] Ni, H., and Xu, W. (2015) Concentration and exact convergence rates for expected Brownian signatures, *Electronic Communications in Probability*, 20, 8.

[NP12] Nourdin, I. and Peccati G. (2012). *Normal approximations with Malliavin calculus: from Stein's method to universality.* Vol. 192. Cambridge University Press.

[1] Otryakhin, D., Pakkanen, M., Passeggeri, R., and Veraart, A. E. D. (2019+). Feasible inference for BSS processes. *Working paper.*

[2] Pakkanen, M., Passeggeri, R. Sauri, O. and Veraart, A. E. D. (2019+). Limit theorems for trawl processes. *Working paper.*

[3] Passeggeri, R. (2019). On the signature and cubature of the fractional Brownian motion for $H > \frac{1}{2}$. In press in *Stochastic Processes and their Applications.* https://doi.org/10.1016/j.spa.2019.04.013.

[4] Passeggeri, R. (2019). Spectral representations of quasi-infinitely divisible processes. In press in *Stochastic Processes and their Applications.* https://doi.org/10.1016/j.spa.2019.05.014

[5] Passeggeri, R. (2019). On quasi-infinitely divisible random measures. *ArXiv preprint*:1906.06736

[6] Passeggeri, R. and Veraart, A. E. D. (2019). Mixing properties of multivariate infinitely divisible random fields. *J. Theor. Probab.*, 32: 1845-1879 https://doi.org/10.1007/s10959-018-0864-7

[7] Passeggeri, R. and Veraart, A. E. D. (2019). Mixing properties of multivariate infinitely divisible random fields. *Advances in Applied Probability*, 51(3), 667-716. doi:10.1017/apr.2019.30

[Pa67] Parthasarathy, K. R. Probability Measures on Metric Spaces. *Probability and Mathematical Statistics: A Series of Monographs and Textbooks*, Academic Press, (1967).

[PV10] Podolskij, M. and Vetter, M. (2010). Understanding limit theorems for semimartingales: a short survey. *Statistica Neerlandica* Vol. 64, nr. 3, pp. 329-351

[Pr56] Prékopa, A. (1956) On stochastic set functions I. *Acta Math. Acad. Sci. Hung.* 7, 215-263.

[Pr57] Prékopa, A. (1957) On stochastic set functions III. *Acta Math. Acad. Sci. Hung.* 8, 337-400.

[RR89] Rajput, B. S. and Rosinski, J. (1989) Spectral representations of infinitely divisible processes. *Probab. Theory Relat. Fields*, 82, 451-487.

[RS75] Reed, M. and Simon, B. (1975). *Methods of modern mathematical physics*, Vol. 2, Academic press.

[Ro87] Rosinski, J. (1987) Bilinear random integrals, *Dissertationes Mathematicae CCLIX*, Warsaw.

[Ro18] Rosinski, J. (2018) Representations and isomorphism identities for infinitely divisible processes. *Ann. Probab.*, 46 (6), 3229-3274.

[RZ96] Rosinski, J., and Zak, T., (1996) Simple conditions for mixing of infinitely divisible processes. *Stoch. Proc. Appl.*, 61, 277-288.

[RZ97] Rosinski, J., and Zak, T., (1997) The equivalence of ergodicity and weak mixing for infinitely divisible processes. *J. Theor. Probab.*, 10, 1.

[RoE07] Roy E., (2007) Ergodic properties of Poissonian ID processes. *Ann. Probab.*, 35, 551-576.

[RoE09] Roy E., (2009) Poisson suspensions and infinite ergodic theory. *Ergod. Theor. Dyn. Syst*, 29, 667-683.

[RoP10] Roy P., (2010) Nonsingular group actions and stationary SαS random fields. *Proc. Am. Math. Soc.*, 138, 2195-2202.

[RS08] Roy P., and Samorodnitsky, G., (2008) Stationary symmetric α-stable discrete parameter random fields. *J. Theor. Probab.*, 21, 212-233.

[Sa99] Sato, K., (1999) *Lévy Processes and Infinitely Divisible Distributions,* vol. 68 of Cambridge Studies in Advanced Mathematics, Cambridge University Press.

[SRMC93] Surgailis, D., Rosinski, J., Mandrekar, V., and Cambanis, S., (1993) Stable mixed moving averages, *Probab. Theor. Relat. Fields* 97(4), 543-558.

[Ta08] Talagrand, M. (2008) Maharam's problem. *Ann. Math.* 168 , 981-1009.

[UW67] Urbanik, K., and Woyczynski, W. A. (1967) Random integrals and Orlicz spaces. *Bull. Acad. Polon. Sci.* 15, 161-169.

[Va98] Van der Vaart, A. W. (1998). *Asymptotic statistics.* New York: Cambridge University Press.

[VW96] Van der Vaart, A. W. and Wellner, J. A. (1996). *Weak Convergence and empirical processes: with applications to statistics.* Springer Series in Statistics.

[Ve15] Veraart, A. E. D., (2015) Stationary and multi-self-similar random fields with stochastic volatility, *Stochastics*, 87:5, 848-870.

[WRS13] Wang, Y., Roy, P. and Stoev, S. A. (2013) Ergodic properties of sum- and max-stable stationary random fields via null and positive group actions. *Ann. Probab.* 41, 1, 206-228.

[Yo36] Young, L. C. (1936) An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.* 67, 1, 251-282.

[Zha98] Zhäle, M. Integration with respect to fractal functions and stochastic calculus. I. *Probab. Theor. Relat. Fields* 111, 333-374.

[ZLK17] Zhang, H., Li, B., and Kerns, G. J. (2017) A characterization of signed discrete infinitely divisible distributions. *Stud. Sci. Math. Hung.*, 54(4), 446-470.