

Schurly You're Joking - Quantum-Classical Correspondence in Phase Space for Non-Hermitian Systems

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Abstract

In closed quantum systems described by Hermitian Hamiltonians the Husimi distributions of stationary states are closely related to classical phase space structures. To each state there is assigned a unique area of the Planck cell partitioned phase space. The localisation of the stationary states upon specific regions of phase space can be related to the classical dynamics and energies. In systems described by non-normal operators, and in particular non-Hermitian Hamiltonians, the general non-orthogonality of the stationary states stands in the way of the application of many standard ideas of quantum-classical correspondence. Many of these can be recovered if the Schur vectors are used instead of the eigenvectors. For specific classes of non-Hermitian systems, with simple loss profiles, this correspondence between the Schur vectors and features of the classical phase space are well investigated. For general non-Hermitian systems with more complex gain-loss profiles, however, there is a lack of methods and understanding. In this thesis we will address this problem by associating to each Schur vector an area of phase space described by the classical analogue of the quantum norm, which we describe as classical norm maps. An algorithm will be introduced that constructs from the norm maps a classical density that shows remarkably close correspondence to the Husimi distributions of the Schur vectors. We will demonstrate this correspondence in both regular and chaotic non-Hermitian systems.

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Dedication

This thesis is dedicated to my Father. I love you with all of my heart. I wish you had been able to read this.

I also dedicate this work to Jessica Brady. Jess you are, and always will be, our alpha.

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Chapter 1

Introduction

It has been nearly a century since the phrase *quantum mechanics* was first introduced by Born and Jordan in their 1925 paper “Zur Quantenmechanik”. For as long as there has been quantum there has been a concerted effort to understand the connection between the worlds of quantum and classical mechanics. In this thesis we contribute further to the investigation of this correspondence, in the context of open quantum systems described by non-Hermitian Hamiltonians. In this first chapter we provide a brief summary of some of the important concepts relevant for the thesis, leading up to the formulation of our objectives.

1.1 Quantum and Classical Mechanics

Of the two theories, quantum mechanics can be regarded as the more fundamental theory of nature. Traditionally it describes systems arising from the quantisation of conservative dynamical systems - systems whose phase space volume is preserved. The correspondence principle says that conservative classical mechanics arises as a (highly non-trivial) limit of quantum mechanics. One concept that is ubiquitous in both the quantum and classical description of reality is that of energy. It is the concept of energy that can form a bridge between quantum and classical descriptions of physical systems. The connection between the two theories is clearest when using a Hamiltonian formulation of classical mechanics.

The total energy of a classical system is represented by the Hamiltonian H which can be written in terms of the canonical variables q and p (generalised position and momentum) that span the phase space. The dynamics are

governed by Hamilton's equations of motion,

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}. \quad (1.1)$$

For a system with n degrees of freedom the phase space is $2n$ -dimensional. The observables of classical systems, the quantities we can measure in the system, are real valued smooth functions on the phase space. Sets of classical observables can be, theoretically, measured simultaneously with any desired precision.

In classical mechanics q and p play dual roles, describing both the states of a system and observables. This is different in quantum mechanics. Quantum systems are described in Hilbert space and the states $|\psi\rangle$ of the quantum system are vectors belonging to this space. The observables of classical mechanics become linear Hermitian operators acting on the Hilbert space. In standard quantum mechanics observables have real eigenvalues (corresponding to potential measurement outcomes) and orthogonal eigenvectors, which provide a basis for the relevant Hilbert space.

Conservative quantum systems are described by a Hermitian Hamiltonian $\hat{H}$ - the observable representing the total energy of the system. The Hamiltonian is the generator of the unitary time evolution in the system. The dynamics of the quantum state obey the Schrödinger equation

$$|\dot{\psi}(t)\rangle = -\frac{i}{\hbar} \hat{H}(t) |\psi(t)\rangle. \quad (1.2)$$

When the Hamiltonian is time-independent the solution to the Schrödinger equation (1.2) can be expressed in terms of the eigenstates $|\phi_n\rangle$ of the Hamiltonian, fulfilling

$$\hat{H} |\phi_n\rangle = E_n |\phi_n\rangle. \quad (1.3)$$

The solution of the Schrödinger equation for $|\psi(0)\rangle = |\phi_n\rangle$ is simply given by

$$|\psi_n(t)\rangle = e^{-\frac{i}{\hbar} E_n t} |\phi_n\rangle. \quad (1.4)$$

That is, an eigenstate only evolves with a complex phase factor. Hence, these states are stationary. They are fixed points of the quantum dynamics in the Hilbert space. An arbitrary initial state $|\psi(0)\rangle$ can be expanded into the basis of eigenstates as

$$|\psi(0)\rangle = \sum_n c_n |\phi_n\rangle, \quad (1.5)$$

and its time evolution is then given simply by

$$|\psi(t)\rangle = \sum_n c_n e^{-\frac{iE_n t}{\hbar}} |\phi_n\rangle. \quad (1.6)$$

Thus, though stationary states do not have an obvious classical counterpart, they play an often central role in the study of quantum systems.

In contrast to the classical case, two quantum observables can only be measured simultaneously if they commute. The position operator and momentum operators $\hat{q}$, $\hat{p}$ do not commute, but instead fulfil the fundamental commutation relation $[\hat{q}, \hat{p}] = i\hbar$. From this follows Heisenberg's famous uncertainty principle, stating that the product of the uncertainties of the position and momentum is bounded from below by Planck's constant

$$\Delta q \Delta p \geq \frac{\hbar}{2}. \quad (1.7)$$

This rules out the existence of a quantum phase-space point in the classical sense.

From a geometric perspective it is natural to view the Hilbert space as the phase space of a quantum system and the Schrödinger equation as a special case of Hamilton's equations. To compare quantum and classical dynamics, however, it is more useful to consider the space of observables, specifically $\hat{p}$ and $\hat{q}$. The uncertainty principle places fundamental restrictions on how localised a state can be in position and momentum. The best quantum analogue to a classical phase-space point is a minimum uncertainty state. These states are known as coherent states. We will now introduce the coherent states in their natural setting, the Harmonic oscillator, and then review how they can be used to define a quantum quasi-probability distribution.

1.2 Coherent States and the Husimi Distribution

The Hamiltonian of the quantum harmonic oscillator is given by

$$\hat{H} = \hbar\omega \left(\frac{1}{2} + \hat{a}^\dagger \hat{a} \right), \quad (1.8)$$

with the annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$ defined as

$$\hat{a} = \frac{1}{\sqrt{2\hbar}} (\hat{q} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}} (\hat{q} - i\hat{p}). \quad (1.9)$$

Here $\hat{q}$ and $\hat{p}$ are the canonical position and momentum operators fulfilling $[\hat{q}, \hat{p}] = i\hbar\hat{\mathbb{I}}$. The number operator $\hat{N}$ is defined as $\hat{N} = \hat{a}^\dagger \hat{a}$, and has eigenvectors $|n\rangle$ with integer eigenvalues $n \in \mathbb{Z}_0^+$. The Hamiltonian (1.8) is diagonal in the basis $\{|n\rangle\}$ and can be expressed as

$$\hat{H} = \sum_{n=0}^{\infty} \hbar\omega \left(n + \frac{1}{2}\right) |n\rangle \langle n|. \quad (1.10)$$

The coherent states $|z\rangle$ can be defined as eigenvectors of the annihilation operator

$$\hat{a} |z\rangle = z |z\rangle, \quad (1.11)$$

where $z = \sqrt{\frac{\hbar}{2}}(q + ip) \in \mathbb{C}$, corresponding to the expectation values of $\hat{p}$

$$\langle z | \hat{p} | z \rangle = \sqrt{2\hbar} \text{Im}(z) = q, \quad (1.12)$$

and $\hat{q}$

$$\langle z | \hat{q} | z \rangle = \sqrt{2\hbar} \text{Re}(z) = p. \quad (1.13)$$

In the harmonic oscillator basis $\{|n\rangle\}$, the coherent states are expanded as

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle. \quad (1.14)$$

The position representation of a coherent state is a Gaussian

$$\phi_z(x) = \langle x | z \rangle = (\pi\hbar)^{-\frac{1}{4}} e^{-\frac{1}{2\hbar}(x-q)^2 + \frac{i}{\hbar}p(x-q)}. \quad (1.15)$$

A key property of the coherent states is that they saturate the lower bound of the uncertainty relation (1.7). That is, the uncertainties of $\hat{q}$ and $\hat{p}$ evaluated in coherent states satisfy

$$\langle \Delta q \rangle_z \langle \Delta p \rangle_z = \frac{\hbar}{2}. \quad (1.16)$$

In this way, the coherent states occupy the minimal quantum mechanical area in phase space. This is as close as Heisenberg will let us get to the concept of a phase-space point of the classical system. The coherent states afford a one-to-one mapping between the points in the phase space and a unique vector in the Hilbert space of the quantised system [1].

The coherent states form an overcomplete basis for the Hilbert space with

$$\frac{1}{\pi} \int |z\rangle \langle z| d^2z = \hat{\mathbb{I}}. \quad (1.17)$$

They play a significant role in the quantum-classical correspondence by facilitating the construction of a quasi-probability distribution in phase space, known as the Husimi distribution. The Husimi distribution can be defined as

$$H_\phi(q, p) = \frac{1}{2\pi\hbar} |\langle z|\phi \rangle|^2. \quad (1.18)$$

That is, we project our wavefunction of interest $|\phi\rangle$ onto the coherent state $|z\rangle$ centred at some (q_0, p_0) in phase space. This is the probability of finding the system described by the state $|\phi\rangle$ in a coherent state $|z\rangle$. If $\hbar$ is small, the coherent state is well localised in both position and momentum, and the probability intuitively corresponds to the probability of finding the system in the respective phase-space point. The term quasi-probability as applied to the Husimi distribution refers to it satisfying some but not all of Kolmogorov's axioms of probability. In particular the Husimi distribution does not produce the correct marginals for the canonical variables q, p , however it satisfies the remaining axioms of positivity and normalisability [2]. The Husimi function is normalised according to

$$\int_q \int_p dq dp H_\phi(q, p) = 1. \quad (1.19)$$

The Husimi distributions are a powerful tool in the study of quantum classical correspondence and will feature heavily throughout this thesis.

1.3 Expectation Value Dynamics

Let us now consider how the quantum dynamics of localised, and in particular coherent, states is related to the classical dynamics. Starting from the Schrödinger equation, the time-evolution of the expectation value

$$\langle \hat{A} \rangle = \frac{\langle \psi | \hat{A} | \psi \rangle}{\langle \psi | \psi \rangle}, \quad (1.20)$$

of an observable $\hat{A}$ is found as

$$\begin{aligned} \frac{d}{dt} \langle \hat{A} \rangle &= \frac{d}{dt} \frac{\langle \psi | \hat{A} | \psi \rangle}{\langle \psi | \psi \rangle} \\ &= \frac{\langle \dot{\psi} | \hat{A} | \psi \rangle + \langle \psi | \partial_t \hat{A} | \psi \rangle + \langle \psi | \hat{A} | \dot{\psi} \rangle}{\langle \psi | \psi \rangle} \\ &= -\frac{i}{\hbar} \langle [\hat{A}, \hat{H}] \rangle + \left\langle \frac{\partial \hat{A}}{\partial t} \right\rangle. \end{aligned} \quad (1.21)$$

When the operator $\hat{A}$ is time-independent (1.21) reduces to

$$\frac{d}{dt} \langle \hat{A} \rangle = -\frac{i}{\hbar} \langle [\hat{A}, \hat{H}] \rangle. \quad (1.22)$$

The expectation values in (1.22) refer to generic quantum states. The magic happens when the evaluation takes place in coherent states because then

$$\langle \hat{A} \rangle_z = A(z) + \mathcal{O}(\hbar), \quad (1.23)$$

which in the classical limit $\hbar \rightarrow 0$ converges to the classical phase space function $A(z)$ at the phase-space point $z = q_0 + ip_0$. Using (1.22) the position and momentum expectation values follow the dynamical equations

$$\begin{aligned} \frac{d}{dt} \langle \hat{q} \rangle &= -\frac{i}{\hbar} \langle [\hat{q}, \hat{H}] \rangle \\ \frac{d}{dt} \langle \hat{p} \rangle &= -\frac{i}{\hbar} \langle [\hat{p}, \hat{H}] \rangle. \end{aligned} \quad (1.24)$$

For Hamiltonians that are quantisations of analytic functions of p and q , the commutators can be rewritten such that

$$\begin{aligned} \frac{d}{dt} \langle \hat{q} \rangle &= \left\langle \frac{\partial \hat{H}(q, p)}{\partial \hat{p}} \right\rangle \\ \frac{d}{dt} \langle \hat{p} \rangle &= \left\langle -\frac{\partial \hat{H}(q, p)}{\partial \hat{q}} \right\rangle. \end{aligned} \quad (1.25)$$

The similarity between equations (1.25) and the classical equations of motion is often referred to as the Ehrenfest theorem. Despite their similarity to Hamilton's equations the quantum dynamics only follow the classical dynamics when

$$\left\langle \frac{\partial \hat{H}}{\partial \hat{p}} \right\rangle \approx \frac{\partial H(\langle \hat{p} \rangle, \langle \hat{q} \rangle)}{\partial \langle \hat{p} \rangle}, \quad (1.26)$$

and

$$\left\langle \frac{\partial \hat{H}}{\partial \hat{q}} \right\rangle \approx \frac{\partial H(\langle \hat{p} \rangle, \langle \hat{q} \rangle)}{\partial \langle \hat{q} \rangle}. \quad (1.27)$$

When the Hamiltonian is at most quadratic in p and q equation (1.26) and (1.27) hold for arbitrarily long times. For more general Hamiltonians we can Taylor expand the expectation values $\left\langle \frac{\partial \hat{H}}{\partial \hat{p}} \right\rangle$ and $\left\langle \frac{\partial \hat{H}}{\partial \hat{q}} \right\rangle$ around the expectation values of $\hat{p}$ and $\hat{q}$, to see that as long as the covariances in $\hat{p}$ and $\hat{q}$ are small, the expectation value dynamics follows approximately the classical

dynamics. In fact, the classical dynamics can be viewed as the approximate quantum dynamics under the assumption that an initially coherent state remains coherent for all times. For Hamiltonians that are higher than quadratic order, an initially coherent state will remain well localised for a while, but will eventually spread out, leading to a breakdown of the approximations (1.26) and (1.27). The associated time-scale, measured as the time at which the covariances grow beyond a certain threshold, is known as the Ehrenfest time τ_E .

When the underlying classical system is regular, the Ehrenfest time in general scales as $\tau_E \propto \frac{1}{\sqrt{\hbar}}$. However, even simple classical dynamical systems can have a range of Ehrenfest times that can differ quite dramatically across the phase space. This is demonstrated in Figure 1.1 where in the left column we show a phase-space portrait of a double-well system described by the Hamiltonian $H = \frac{1}{2}(p^2 - q^2) + \frac{1}{4}q^4$. A selected classical trajectory is highlighted in red and the quantum expectation value dynamics (blue) for an initially coherent state centred at the same initial condition. We follow the evolution of the classical and quantum systems up to a final time $t = 8\pi$. For the initial condition in the top row the quantum expectation value dynamics of the wave packet stay close to the classical trajectory over the evolution period. We see this in Figure 1.1b where we plot the classical position versus the quantum expectation value of the position. Despite the regularity of the dynamics, a shift in the initial position from $q_0 = 0.8$ to $q_0 = 0.2$ results in a much faster divergence between the quantum and classical expectation dynamics. This can be seen clearly in the phase space plot Figure (1.1c) and Figure(1.1d) and is related to the proximity of an unstable classical fixed point on the maximum of the potential barrier between the two wells.

The difference between quantum and classical mechanics is most pronounced in the presence of chaos in the classical dynamics. When the underlying classical system dynamics are chaotic the Ehrenfest time scales as $\tau_E \approx \frac{1}{\lambda} \ln(\hbar)$, where λ is the classical Lyapunov exponent [3]. This is due to the famous exponential divergence of nearby orbits of classically chaotic systems. This discrepancy between the two descriptions makes classically chaotic systems a particularly interesting subject in the study of quantum-classical correspondence. In the next section we will briefly summarise some background material on chaotic systems that will be useful for this thesis.

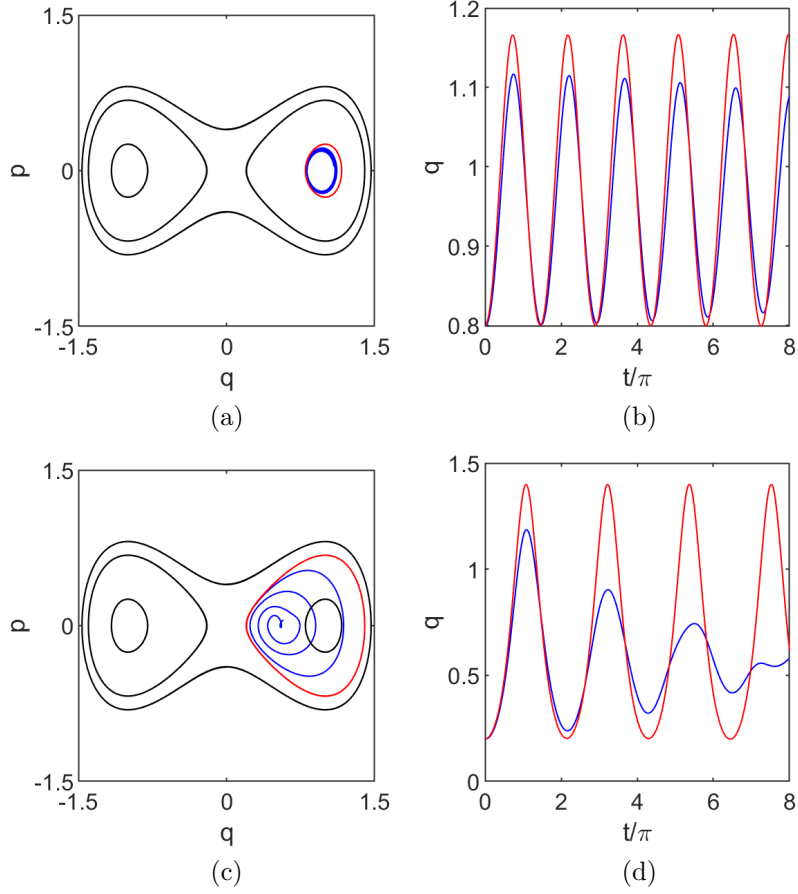


Figure 1.1: The left column depicts phase-space trajectories in a double-well system. The classical trajectory (red) and the quantum trajectory (blue) have initial conditions $(q_0, p_0) = (0.8, 0)$ (top) and $(q_0, p_0) = (0.2, 0)$ (bottom). In both the quantum and classical cases the trajectory is followed up to a final time $t = 8\pi$. The quantum system has a matrix dimension of $N = 150$. The right column depicts the quantum expectation value of position (blue) and the classical position (red).

1.4 Chaos in Hamiltonian Systems

1.4.1 Classical Hamiltonian Chaos

Classically chaotic systems are characterised most famously (excluding Butterflies) by an exponential sensitivity to initial conditions [3]. The emergence of chaos is a commonplace phenomenon, famously arising in the solution to

the three-body problem [4], and as the bane of the general public in weather forecasts [5].

Within the literature the definition of chaos can be varied [3, 6, 7]. Here we follow [7]: Let $f : \Omega \rightarrow \Omega$ be a map on a set Ω , f is chaotic on Ω if

1. Ω has *dense periodic orbits*. This means that arbitrarily close to any point in Ω there exists a periodic orbit. It should be noted that these include stable and unstable periodic orbits. This property implies an underlying structure to the seemingly random dynamics.
2. f is *topologically transitive*, i.e., for any two open subsets $\omega_1, \omega_2 \subset \Omega$ $\exists k > 0$ such that $f^k(\omega_1) \cap \omega_2 \neq \emptyset$. Intuitively a topologically transitive map f has points that will move from an arbitrarily small neighbourhood into another.
3. f has a *sensitivity to initial conditions*, i.e., $\exists \delta > 0$ such that, for any $z \in \Omega$ and small neighbourhood N_z $\exists z' \in N_z$ and $k > 0$ such that $|f^k(z_1) - f^k(z_2)| > \delta$.

In some cases the sensitivity to initial conditions is implied by the other two conditions [8]. Nevertheless, the condition of exponential sensitivity has become the hallmark of chaotic motion. It can be characterised through the Lyapunov exponent, σ which measures the rate of divergence of nearby trajectories in the phase space. This is defined as

$$\sigma(z) = \lim_{t \rightarrow \infty} \lim_{\delta \rightarrow 0} \frac{1}{t} \ln \left(\frac{|\delta z(t)|}{|\delta z_0|} \right). \quad (1.28)$$

For $\sigma > 0$ the trajectory is classified as chaotic [9] [3].

The study of Hamiltonian chaos has been facilitated by “toy models” that illustrate the main features of chaotic systems. A particularly well-studied class of Hamiltonians are simple delta-kicked Hamiltonians of the form

$$H = \frac{p^2}{2} + V(q) \sum_{n=-\infty}^{\infty} \delta(t - n\tau). \quad (1.29)$$

While Hamiltonians of the type (1.29) do not conserve energy, they nevertheless conserve phase space volume. The global dynamics of periodic time-dependent systems are conveniently analysed by considering stroboscopic snapshots. That is, instead of the continuous dynamics one considers a map, or Poincaré section. Consider an n -dimensional phase space with an evolution law $\dot{z} = f(z)$. Let us denote by S an $(n - 1)$ dimensional hyper-surface. The

Poincaré section is a mapping from $S \rightarrow S$ which is obtained by capturing the intersections of the trajectories with the hypersurface S , as illustrated in Figure 1.2. For periodic systems it is natural to place the Poincaré section such that it coincides with the driving period τ the system.

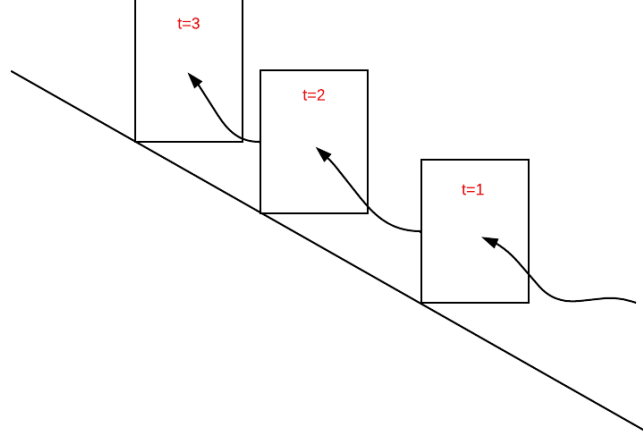


Figure 1.2: Schematic illustration of a Poincaré section showing a trajectory in the phase space intersecting the hypersurface S at times $t \in \mathbb{N}$.

For a delta-kicked system of the type (1.29), integration of Hamilton's equations from just after the n -th kick at, $t = n\tau + \epsilon$, to just after the next kick, $t = (n + 1)\tau + \epsilon$ yields

$$\begin{aligned} q_{n+1} &= q_n + p_n \\ p_{n+1} &= p_n - V'(q_{n+1}). \end{aligned} \tag{1.30}$$

Periodic orbits with period $m\tau$ will then intersect the Poincaré section at m different points and those with period τ appear as fixed points. Regular (non-chaotic) trajectories undergo periodic or quasiperiodic motion and present clear structures in the Poincaré sections. Chaotic trajectories present a seemingly stochastic motion and the only “structure” is the area of the Poincaré section such trajectories explore.

Figure 1.3a shows an example of the Poincaré section of a kicked rotor, a standard example in chaos, which will play a prominent role in this thesis. The dynamics depicted are mixed, that is there are regions of regular motion (the central region) and regions of chaos. When surrounded by chaotic motion these regular regions are sometimes described as (regular) islands in a chaotic sea.

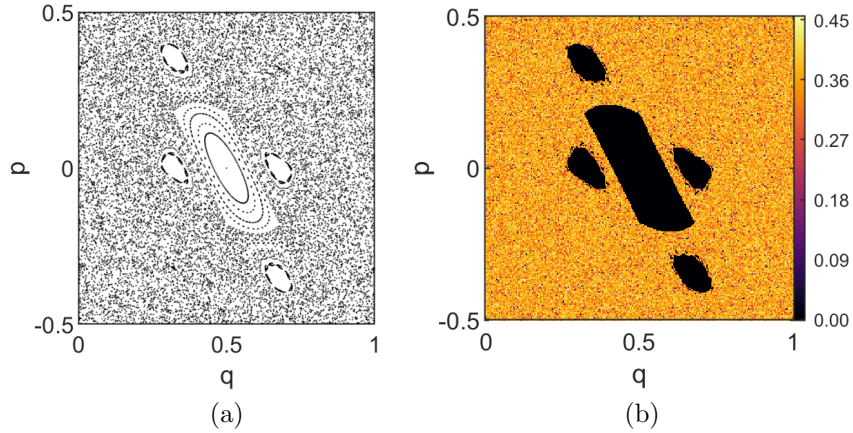


Figure 1.3: Example of a Poincaré section (for a kicked rotor that will be discussed in detail later) with mixed dynamics (left), and heat map of the corresponding Lyapunov exponents (right). The Lyapunov heat maps are constructed from an initial distribution of $N^2 = (250)^2$ points uniformly distributed across the phase space. The Lyapunov exponents are averaged for $t = 2000$ iterations after an initial propagation of $t_0 = 1000$ steps.

In Figure 1.3b we depict the Lyapunov exponent associated to various initial conditions in the phase space as a heat map. These heat maps are constructed by choosing a uniform distribution of N^2 initial conditions on the phase space. The average Lyapunov exponent is then calculated through the Benettin algorithm [10, 11] and via equation (1.28). Each initial condition is coloured by the value of the average Lyapunov exponent at a (large) final time. The calculation of the average Lyapunov exponent is typically conducted after some initial time t_0 . That is we first evolve the system up to a time t_0 , then calculate the Lyapunov exponent from t_0 up to some large final time. The purpose of t_0 is to remove any transient behaviour in trajectories. The Lyapunov exponent is positive when the trajectories are chaotic. When the trajectories are regular the Lyapunov exponent is zero.

1.4.2 Quantum Maps

The quantum analogue of a classical phase-space map is an iterative map between quantum states. For conservative (phase-space conserving) classical systems the corresponding quantum map is represented by a unitary operator, the time-evolution operator $\hat{U}(t, t_0)$, that maps a state at time t_0 to a state at time t as

$$|\psi(t)\rangle = \hat{U}(t, t_0) |\psi(t_0)\rangle, \quad (1.31)$$

For systems with an underlying Hamiltonian dynamics, the time-evolution operator $\hat{U}(t, t_0)$ is itself a solution to the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \hat{U}(t) = \hat{H}(t) \hat{U}(t), \quad (1.32)$$

with $\hat{U}(t_0, t_0) = \hat{\mathbb{I}}$, which is solved by

$$\hat{U}(t, t_0) = \hat{T} \exp \left(-\frac{i}{\hbar} \int_{t_0}^t \hat{H}(t') dt' \right), \quad (1.33)$$

Here $\hat{T}$ denotes the time-ordering operator that accounts for the possible non-commutativity of $\hat{H}(t)$ at different times [12].

In general the solution to the Schrödinger equation for an explicitly time-dependent Hamiltonian $\hat{H}(t)$ cannot be written as the product of a time-dependent phase factor and stationary state. However, a special case exists when the Hamiltonian is time periodic $\hat{H}(t + \tau) = \hat{H}(t)$. A special role is played by the one-period evolution operator

$$\hat{F}_0 := \hat{U}(\tau, 0) = \hat{T} \exp \left(-\frac{i}{\hbar} \int_0^\tau \hat{H}(t) dt \right), \quad (1.34)$$

for which, according to the Floquet theorem [13,14], there exists the following spectral representation

$$\hat{F}_0 := \sum_n e^{-\frac{i\epsilon_n \tau}{\hbar}} |\phi_n\rangle \langle \phi_n|. \quad (1.35)$$

To understand the significance of the operator $\hat{F}_0$ we consider the propagator over n -periods

$$\begin{aligned} \hat{F}_n &= \hat{T} \exp \left(-\frac{i}{\hbar} \int_0^{n\tau} \hat{H}(t) dt \right) \\ &= \hat{T} \exp \left(-\frac{i}{\hbar} \sum_{m=1}^n \int_{(m-1)\tau}^{m\tau} \hat{H}(t) dt \right). \end{aligned} \quad (1.36)$$

Now since $\hat{H}(t)$ is periodic

$$\begin{aligned} \hat{F}_n &= \hat{T} \exp \left(-\frac{i}{\hbar} \sum_{m=1}^n \int_{(m-1)\tau}^{m\tau} \hat{H}(t) dt \right) \\ &= \hat{T} \prod_{m=1}^n \exp \left(-\frac{i}{\hbar} \int_0^\tau \hat{H}(t) dt \right), \end{aligned} \quad (1.37)$$

and because terms at multiples of the period commute

$$\begin{aligned}\hat{F}_n &= \prod_{m=1}^n \hat{T} \exp\left(-\frac{i}{\hbar} \int_0^\tau \hat{H}(t) dt\right) \\ &= \hat{F}_0^n.\end{aligned}\tag{1.38}$$

Thus knowledge of the one-period evolution operator $\hat{F}_0$ permits the stroboscopic evolution of arbitrary quantum states [14] over multiples of the period τ . From the spectral representation (1.35) we can write

$$\hat{F} |\phi_n\rangle = e^{-\frac{i\tau\epsilon_n}{\hbar}} |\phi_n\rangle.\tag{1.39}$$

The Floquet modes $\{|\phi_n\rangle\}$ provide a complete basis for the Hilbert space and play an analogous role to the eigenstates of $\hat{H}$ in time-independent systems. The associated ϵ_n are known as the *quasienergies*, and are determined modulo $\frac{2\pi\hbar}{\tau}$.

1.5 Quantum-Classical Correspondence in Phase Space

In quantum mechanics we are generally interested in the spectral features of operators and their associated eigenvectors, while in classical systems they have no obvious analogy. Thus it is natural to ask how the eigenvalues and eigenvectors of the quantum system are related to the classical system.

The Husimi distribution allows us to form a map between the quantum Hilbert space and the classical phase space. The localisation of the Husimi distributions can be understood through *Weyls law* which associates a phase-space area of h to each eigenstate in phase space. For finite-dimensional Hilbert spaces associated to compact phase spaces, the totality of all N states will occupy the entire phase space and a single state will occupy a phase space area of $h = 1/N$.

Husimi distributions of individual eigenstates localise on classical phase space structures [15]. This is illustrated for two examples in Figure 1.4 and 1.5. Figure 1.4 shows Husimi distributions of eigenstates of a harmonic oscillator. The left and central figures show the ground state and 9th excited state, respectively. It is clearly visible how these are organised according to the contour lines of the classical Hamiltonian with the ground state localising around the classical fixed point. Figure 1.4c illustrates how we can build up the phase space by summing over the first 10 eigenfunctions with energy $E < E_{10}$. We see that the Husimi distribution is effectively uniform on the area associated to the 10 lowest-energy eigenfunctions.

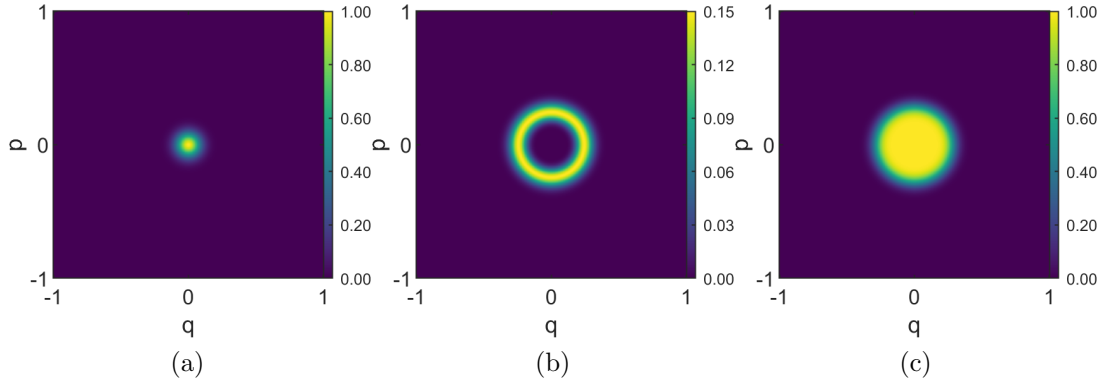


Figure 1.4: Husimi-functions of the harmonic oscillator. From left to right: Husimi function of the ground state, 10th eigenstate, first 10 eigenstates. The quantum system has a matrix dimension of $N = 100$.

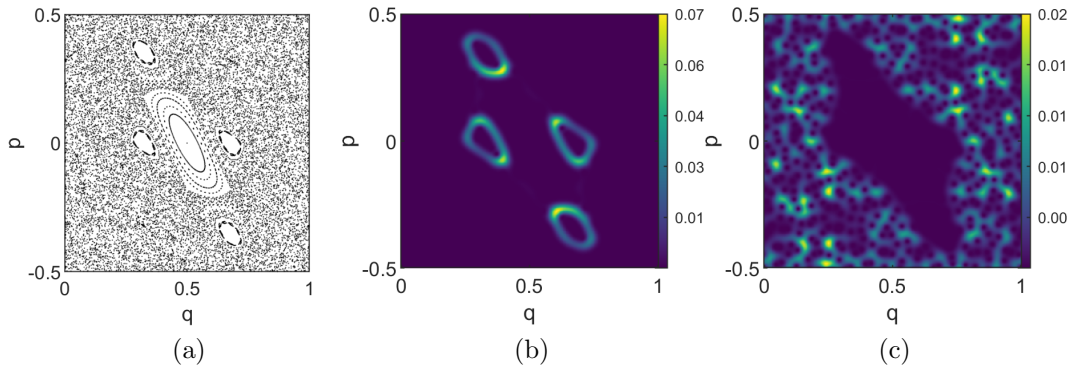


Figure 1.5: The classical Poincaré section of the kicked rotor, Husimi distribution of a regular eigenstate, Husimi distribution of a chaotic eigenstate. The quantum system has a matrix dimension of $N = 1001$.

In chaotic systems what is known as the *chaotic eigenfunction hypothesis* manifests in a quasi-random distribution of the Husimi functions of eigenstates over the relevant classically chaotic region, with the exception of scar states, that localise on unstable classical periodic orbits [16]. In mixed regular-chaotic systems one can distinguish between quantum states that localise on regular islands and those that localise in the classically chaotic sea. This classification of the eigenstates is known as the *semiclassical eigenfunction hypothesis* [17]. In Figure 1.5 we show the Husimi distributions of selected eigenstates of the Floquet operator of the kicked rotor for parameter values that lead to mixed regular-chaotic dynamics. The corresponding

classical Poincaré section is depicted in the left panel. As predicted by the semi-classical eigenfunction hypothesis the eigenstates either localise upon regular structures, or are delocalised over the chaotic sea.

1.6 Non-Hermitian Quantum Theories

So far we have discussed idealised quantum and classical systems in which overall probabilities are conserved. In recent years, the question on how quantum-classical correspondence extend to systems experiencing loss has come into focus [18, 19, 20, 21, 22, 23, 24, 25]. Most realistic physical systems experience loss and dissipation in some form. Indeed, in experiments this can be brought on by the process of measurement itself [26]. Quantum mechanically, a common approach is to model loss through non-Hermitian Hamiltonians [27, 28, 29]. In this section we provide some background on the most relevant concepts of this research area.

1.6.1 Non-Hermitian Hamiltonians

To see why non-Hermitian Hamiltonians are related to loss, let us consider a time-independent non-Hermitian Hamiltonian $\hat{\mathcal{H}}$, divided into Hermitian and anti-Hermitian part

$$\hat{\mathcal{H}} = \hat{H} - i\hat{\Gamma}, \quad (1.40)$$

with $\hat{H} = \hat{H}^\dagger$ and $\Gamma = \Gamma^\dagger$. The anti-Hermitian term $-i\hat{\Gamma}$ describes the loss profile of the system. The loss profile manifests in the spectrum of $\hat{\mathcal{H}}$, which in general has complex eigenvalues. For simplicity let us assume that $\hat{\mathcal{H}}$ has a discrete spectrum. Then, writing a complex eigenenergy as $\lambda_n = \epsilon_n - i\gamma_n$ where $\epsilon_n \in \mathbb{R}$ and $\gamma_n \in \mathbb{R}^+$, the corresponding eigenstate $|\phi_n\rangle$ evolves in time as

$$\begin{aligned} \hat{U} |\phi_n\rangle &= e^{-i\hat{\mathcal{H}}t} |\phi_n\rangle \\ &= e^{-i(\epsilon_n - i\gamma_n)t} |\phi_n\rangle. \end{aligned} \quad (1.41)$$

That is, the norm $w_n = \langle \phi_n | \phi_n \rangle$ of the state is not constant in time but changes as

$$w_n(t) = e^{-2\gamma_n t} w_n(0). \quad (1.42)$$

Interpreting the norm as an overall probability, an eigenvalue with negative imaginary part can be used to describe an exponential decay. Complex energies were used in this context as early as 1928 by Gamow for the description of nuclear decay [29].

One way to think of a non-Hermitian Hamiltonian is as modelling a subsystem of some larger quantum system. A particle initially in this subsystem need not stay there for all time. Thus the probability within the subsystem decays with time. Examples of systems modelled by lossy non-Hermitian Hamiltonians are those involving decay [28, 29] and scattering phenomenon [19, 27]. This non-conservation of the probability may also arise through a gain of wave amplitude associated to complex energies with positive imaginary parts. An example of this is the the amplification of light in a classical optical system [27].

Consider an initial state $|\psi(t=0)\rangle$ that is a superposition of eigenstates of the Hamiltonian (1.40) such that

$$|\psi(t=0)\rangle = \sum_{n=1}^N \psi_n |\phi_n\rangle. \quad (1.43)$$

The time evolution of this state is then given by

$$|\psi(t)\rangle = \sum_{n=1}^N e^{-\frac{i}{\hbar}(\epsilon_n - i\gamma_n)t} \psi_n |\phi_n\rangle, \quad (1.44)$$

similar to the standard Hermitian case, with the difference of relative exponential growth/decay of the coefficients of the different eigenstates. If we order the $|\phi_n\rangle$ such that

$$\gamma_1 < \gamma_2 < \dots < \gamma_{N-1} < \gamma_N, \quad (1.45)$$

then for large times we find

$$|\psi(t \rightarrow \infty)\rangle \rightarrow e^{-\frac{i}{\hbar}(\epsilon_1 - i\gamma_1)t} \psi_1 |\phi_1\rangle, \quad (1.46)$$

if $\psi_1 \neq 0$. For a linear superposition of eigenstates with $\psi_n \neq 0$, the long term dynamics is dominated by the state with the smallest imaginary part of the eigenvalue and non-zero initial coefficient. On intermediate timescales an initial state that is a superposition of a number of eigenstates will dynamically *decay* into the subspace spanned by the $|\phi_n\rangle$ with smaller γ_n . That is, there is a natural structure of invariant subspaces of the Hilbert space arising from the hierarchy of eigenvectors.

1.6.2 $\mathcal{PT}$ -Symmetry

In recent years non-Hermitian Hamiltonians with a balanced gain and loss structure have become increasingly popular. This special class of non-Hermitian

Hamiltonians are known as PT-symmetric Hamiltonians. These first began to attract attention due to the possibility of having a purely real spectrum, despite being non-Hermitian [30]. There are different definitions of PT-symmetry in the literature. Here we use the following convention: A Hamiltonian $\hat{H}$ is said to be PT-symmetric if $[\hat{H}, \hat{\mathcal{P}}\hat{\mathcal{T}}] = 0$. The parity operator $\hat{\mathcal{P}}$ is an unitary operator with $\hat{\mathcal{P}}^2 = \mathbb{I}$. The time-reversal operator $\hat{\mathcal{T}}$ is an anti-unitary operator with $\hat{\mathcal{T}}^2 = \pm\mathbb{I}$. It follows that $\hat{\mathcal{P}}\hat{\mathcal{T}}$ is an anti-unitary operator. The condition for the famous reality of an eigenvalue of a PT-symmetric non-Hermitian Hamiltonian is that the energy eigenstates of $\hat{H}$ are also eigenfunctions of the PT-operator [31]. To see this consider, an energy eigenstate of the PT-symmetric Hamiltonian $\hat{H}|\phi\rangle = E|\phi\rangle$. We have $[\hat{H}, \mathcal{PT}] = 0$, that is

$$\mathcal{PT}\hat{H}|\phi\rangle = E^*\mathcal{PT}|\phi\rangle = \hat{H}\mathcal{PT}|\phi\rangle = E\mathcal{PT}|\phi\rangle. \quad (1.47)$$

We have $(\mathcal{PT})^2 = \mathbb{I}$, and thus the eigenvalues of the $\mathcal{PT}$ are unimodular. If $|\phi\rangle$ is also an eigenvector of $\mathcal{PT}$ we have $\mathcal{PT}|\phi\rangle = |\phi\rangle$ (up to a phase which can be removed by an appropriate gauge transformation), then $E = E^*$ and the eigenvalues are real. A PT-symmetric Hamiltonian with a purely real spectrum is referred to in the literature as having *unbroken* PT-symmetry. When the eigenfunctions of the Hamiltonian are no longer eigenfunctions of the $\mathcal{PT}$ operator the corresponding eigenvalues come in complex conjugate pairs. This case is often referred to as “broken PT-symmetry”. Consider now a time-independent PT-symmetric Hamiltonian with discrete spectrum in the regime of “broken” PT-symmetry. The norm evolves according to

$$w(t) = e^{-2\gamma_n t} \langle \phi_n | \phi_n \rangle. \quad (1.48)$$

Each state $|\phi_n\rangle$ with $\gamma_n > 0$ will have a partner $|\phi_{n'}\rangle$ with $\gamma_{n'} < 0$ and $\gamma_n = -\gamma_{n'}$. We see how this can be interpreted as representing an open system with balanced gain and loss [30, 32, 33, 34].

In this thesis we will specifically consider PT-symmetric quantum maps. Following the definition in [35, 36], a quantum map, described by a Floquet operator $\hat{F}$, is PT-symmetric if

$$(\mathcal{PT})\hat{F}(\mathcal{PT})^{-1} = \hat{F}^{-1}. \quad (1.49)$$

For a PT-symmetric Floquet Matrix $\hat{F}$, Floquet states $|\phi_n\rangle$, fulfilling

$$\hat{F}|\phi_n\rangle = \lambda_n |\phi_n\rangle, \quad (1.50)$$

are non-orthogonal.

In the “unbroken” regime the eigenvalues λ are unimodular as in the closed case and lie on the unit circle. In the case of “broken” PT-symmetry, the

eigenvalues will either lie on the unit circle, or there will exist $\lambda_n, \lambda_{n'}$ such that $\lambda_n = \lambda_{n'}^*$. Writing $\lambda_n = r_n e^{-i\theta_n}$, for $\text{Re}(\lambda_n) \in (1, \infty]$, then $\lambda_{n'} = \frac{1}{r_n} e^{i\theta_n}$, for $\text{Re}(\lambda_{n'}) \in [0, 1)$. In terms of the quasienergies $\epsilon_n = i \ln(\lambda_n)$ this presents as complex conjugate pairs or real quasienergies.

1.6.3 Non-Normal Operators and Biorthogonality

In addition to giving rise to complex eigenvalues, non-Hermitian operators are in general non-normal. A matrix $\hat{\mathcal{A}}$ is normal if it commutes with its conjugate transpose $\hat{\mathcal{A}}\hat{\mathcal{A}}^\dagger = \hat{\mathcal{A}}^\dagger\hat{\mathcal{A}}$. Normality is a necessary and sufficient condition for unitary diagonalizability. In other words, non-Hermitian Hamiltonians do not generally possess an orthogonal set of eigenvectors. For special cases the set of eigenvectors may not even be complete. This happens at what are known as exceptional points in parameter space. While these have interesting properties they are of measure zero for generic non-Hermitian and PT-symmetric systems, and we shall not consider them in this thesis. Generally, the eigenvectors of a non-normal operator $\hat{\mathcal{A}}$ form a non-orthogonal basis of the Hilbert space. Nevertheless, the notion of orthogonality that is lost can be replaced with the notion of biorthogonality. Here we follow the introduction to biorthogonality in [37]. Consider a non-Hermitian time-independent Hamiltonian $\hat{\mathcal{H}} = \hat{H} - i\hat{\Gamma}$, with non-degenerate eigenvalues and right eigenstates $|\phi_n\rangle$ such that

$$\hat{\mathcal{H}}|\phi_n\rangle = \lambda_n|\phi_n\rangle, \quad \langle\phi_n|\hat{\mathcal{H}}^\dagger = \lambda_n^*\langle\phi_n|. \quad (1.51)$$

Similarly for the right eigenstates $|\chi_n\rangle$ of the Hermitian adjoint $\hat{\mathcal{H}}^\dagger$ we have

$$\hat{\mathcal{H}}^\dagger|\chi_n\rangle = \chi_n|\chi_n\rangle, \quad \langle\chi_n|\hat{\mathcal{H}} = \chi_n^*\langle\chi_n|. \quad (1.52)$$

Although $\{|\phi_n\rangle\}$ are non-orthogonal they are, nevertheless, linearly independent and thus span the Hilbert space. This allows us to uniquely pair the $|\phi_n\rangle$ of (1.51) to the $|\chi_n\rangle$ of (1.52) and identify $\lambda_n = \chi_n^*$. From this identification follows the normalisation condition

$$\langle\chi_n|\phi_m\rangle = \delta_{nm} \langle\chi_n|\phi_n\rangle. \quad (1.53)$$

Where appropriate normalisation allows the freedom to choose $\langle\chi_n|\phi_n\rangle = 1$, a convention frequently used in quantum chemistry [37]. This construction is known as biorthogonality. It allows us to recover for example the completeness relation

$$\mathbb{I} = \sum_n \frac{|\phi_n\rangle\langle\chi_n|}{\langle\chi_n|\phi_n\rangle}. \quad (1.54)$$

1.6.4 The Schur Decomposition

While non-normal operators cannot be diagonalised by unitary matrices they can be brought to another useful form by unitary transformations. The *Schur* decomposition [19,24,38] exists for any matrix $\hat{\mathcal{A}}$, and it takes the form

$$\hat{\mathcal{A}} = \hat{Q} \hat{R} \hat{Q}^\dagger \quad (1.55)$$

where $\hat{Q}$ is a unitary matrix and $\hat{R}$ is an upper triangular matrix with the eigenvalues of $\hat{\mathcal{A}}$ on the diagonal. The columns of $\hat{Q}$ are the Schur vectors which form an orthonormal basis $\{|\nu_j\rangle\}$. This Schur basis can be constructed from the non-orthogonal set of eigenstates $\{|\phi_n\rangle\}$, by a Gram-Schmidt orthogonalisation scheme. The first Schur vector $|\nu_1\rangle$ is also an eigenvector $|\phi_1\rangle$ of $\hat{\mathcal{A}}$. The second Schur vector is the normalised component of $|\phi_2\rangle$ orthogonal to $|\nu_1\rangle$, that is

$$|\nu_2\rangle \propto |\phi_2\rangle - \langle\phi_2|\nu_1\rangle |\nu_1\rangle. \quad (1.56)$$

The remaining Schur vectors are constructed accordingly, such that we have

$$\begin{aligned} |\nu_1\rangle &= |\phi_1\rangle \\ |\nu_2\rangle &\propto |\phi_2\rangle - \langle\phi_2|\nu_1\rangle |\nu_1\rangle \\ &\vdots \\ |\nu_n\rangle &\propto |\phi_n\rangle - \sum_{j=1}^{n-1} \frac{\langle\nu_j|\phi_n\rangle}{\langle\nu_j|\nu_j\rangle} |\nu_j\rangle. \end{aligned} \quad (1.57)$$

Depending on the ordering of the original eigenvectors, there are $N!$ different sets of Schur vectors. One natural ordering for the eigenvectors $\{|\phi_n\rangle\}$ is through their associated eigenvalues $\{\lambda_n\}$ with the ordering $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$. This leads to a natural structure of invariant subspaces spanned by the orthogonal basis of Schur vectors. That is, if the initial state is a linear combination of the first n Schur vectors, it remains within the space spanned by these for all times, and the first n Schur vectors provide an orthogonal basis for this subspace. This subspace is identical to that spanned by the first n eigenvectors, however, the Schur vectors provide an orthogonal basis. Throughout this thesis the Schur vectors of a quantum system will become our main objects of interest.

1.6.5 Classical Dynamics

While for Hermitian quantum systems the classical analogue has been studied long before quantum theory was even conceived, for non-Hermitian systems

it is not clear a priori what their classical counterpart is. In this thesis we use “classical” phase-space dynamics that arise from a coherent state approximation of the quantum dynamics of expectation values, in the same way in which Hamilton’s equations arise for the Hermitian case. These have been introduced in [39] and we will summarise them here.

The time evolution of an expectation value $\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle / \langle \psi | \psi \rangle$ of a time-independent generator $\hat{A}$ under a non-Hermitian Hamiltonian $\hat{\mathcal{H}} = \hat{H} - i\hat{\Gamma}$ evolves according to

$$i\hbar \frac{d}{dt} \langle \hat{A} \rangle = \langle [\hat{A}, \hat{H}] \rangle - 2i\Delta_{A\Gamma}^2, \quad (1.58)$$

where

$$\Delta_{AB} = \frac{1}{2} \langle [A, B]_+ \rangle - \langle A \rangle \langle B \rangle \quad (1.59)$$

denotes the covariance.

In the Hermitian case $\Gamma = 0$ the classical limit arises through the association between the commutator $\langle [\hat{A}, \hat{H}] \rangle$ and the Poisson bracket $\{A, H\}$. We will now explore how to interpret the additional anticommutator term in Equation (1.58) and its effect on the classical dynamics. Following [39] the ladder operators have the following commutation relations with an operator $\hat{F}$

$$[\hat{a}, \hat{F}] = \frac{\partial \hat{F}}{\partial \hat{a}^\dagger}, \quad [\hat{a}^\dagger, \hat{F}] = -\frac{\partial \hat{F}}{\partial \hat{a}}. \quad (1.60)$$

The anticommutator in (1.58) can be rewritten in terms of the relation (1.60) as

$$[\hat{a}, \hat{F}]_+ = 2\hat{F}\hat{a} + \frac{\partial \hat{F}}{\partial \hat{a}^\dagger}, \quad [\hat{a}^\dagger, \hat{F}]_+ = 2\hat{a}^\dagger \hat{F} + \frac{\partial \hat{F}}{\partial \hat{a}}. \quad (1.61)$$

We interpret the expectation value dynamics in the semiclassical limit as the classical dynamics. Writing (1.58) in terms of (1.61) and evaluating in coherent states yields

$$i\dot{z} = \frac{\partial H}{\partial z^*} - i\frac{\partial \Gamma}{\partial z^*} = \frac{\partial \mathcal{H}}{\partial z^*}, \quad i\dot{z}^* = -\frac{\partial H}{\partial z} - i\frac{\partial \Gamma}{\partial z} = -\frac{\partial \mathcal{H}}{\partial z}, \quad (1.62)$$

where z represents the complex phase space coordinate. Translated into p and q the coherent state approximation yields the equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} - \frac{\partial \Gamma}{\partial q}, \quad \dot{p} = -\frac{\partial H}{\partial q} - \frac{\partial \Gamma}{\partial p}. \quad (1.63)$$

In the Hermitian limit, $\Gamma = 0$, these reduce to the familiar Hamilton’s equations (1.1). In general the non-Hermiticity adds the gradient of the non-Hermitian term $i\Gamma$ to the equations of motion. This results in a modified

classical dynamics whenever the non-Hermitian term has a non-trivial form i.e, whenever $\nabla\Gamma \neq 0$. Such dynamics typically lead to dissipative dynamics, where neither energy H nor the phase space volume are conserved. In the special case of PT-symmetric quantum systems the resulting dynamics (1.63) are an example of what is known as reversible dynamical systems [40], which, despite not having a Hamiltonian structure can still show characteristics typically associated to conservative systems.

In addition to the modified dynamics there is a further degree of freedom that survives the semiclassical limit, the *classical norm* $w(t)$ [25, 41]. In the presence of a non-Hermitian Hamiltonian $\hat{\mathcal{H}} = \hat{H} - i\hat{\Gamma}$ the norm of a quantum state is no longer dynamically conserved and evolves according to the equation

$$\frac{d}{dt}w = -\frac{2}{\hbar} \langle \psi | \hat{\Gamma} | \psi \rangle = -\frac{2}{\hbar} \langle \Gamma \rangle w. \quad (1.64)$$

In the semiclassical limit the norm of the quantum state is approximated by the classical variable w , which follows the equation of motion

$$\dot{w}(t) = -\frac{2}{\hbar} \Gamma(p, q) w(t). \quad (1.65)$$

In the Hermitian limit, the classical norm will reduce to unity for all classical trajectories.

1.7 Non-Hermitian Quantum-Classical Correspondence

The understanding of quantum-classical correspondence in non-Hermitian systems is still rudimentary. Some insights have been gained through the study of a particular class of non-normal operators, describing systems with escape through an opening. The majority of research thus far has focused on the opening of paradigmatic chaotic systems such as the bakers map and kicked rotor via an opening in the phase space [42]. These models are conceptually very simple: take your favourite classically chaotic system and place an opening (hole) in the phase space. When trajectories enter the opening they will leave the system forever, or equivalently are trapped in the opening for all time. These systems are effectively non-Hermitian, however the construction of the relevant operator, the quantum propagator, is usually performed without reference to such an underlying non-Hermitian Hamiltonian. The main question that has been addressed in this context concerns the phase space localisation of stationary (or Floquet) states of the quantum system

on classical structures. The loss induces a concept of a lifetime associated to each trajectory, and the characteristic structures are determined around classical structures not seen in conservative chaotic systems, namely chaotic saddles. When the corresponding classical system is chaotic, the phase space support of the eigenfunctions with the longest lifetimes is given by fractal sets in the phase space which are intimately related to the chaotic saddle of the classical system. This is in stark contrast to closed quantum chaotic systems where chaotic eigenstates are generally delocalised over large regions of the phase space.

In closed systems a standard idea of quantum-classical correspondence is the partitioning of the classical phase space into Planck cells [43]. This is facilitated through a basis of semiclassically localised states, which is precisely the role played by the coherent states [19]. This readily allows an association between the phase space support of the eigenfunctions and the relevant classical structures. The Husimi distribution of the totality of the eigenfunctions resolves the identity on the phase space [43,44], becoming the uniform density. More generally, if the relevant quantum operator is normal, the eigenfunctions will form an orthonormal basis for the Hilbert space, and the same results apply. However when the system is described by a non-normal operator, e.g those generated by non-Hermitian Hamiltonians, a simple Planck cell construction of the phase space is no longer guaranteed [38]. The non-orthogonality of the eigenfunctions leads to the identity on the phase space no longer being satisfied by the Husimi distributions. This can be interpreted as a large number of states occupying the same Planck cells. In [19] it has been demonstrated that analysing the phase space distributions of Schur vectors instead of eigenvectors is key for understanding the quantum classical correspondence in systems with openings.

In general, however it is not clear how the results from systems with escape through an opening extend to more general non-Hermitian systems. In particular, the understanding is usually centred around properties of the chaotic saddle, a feature which is not generic to non-Hermitian systems. The underlying classical dynamics of systems with an opening remains unchanged from the classical case, apart from the additional lifetime concept. This again limits the applicability to systems that have a more general gain-loss profile where the classical dynamics changes.

One system with such a gain-loss profile that has been investigated recently is a PT-symmetric kicked top [24]. Here a correspondence was found between the classical norm and the phase-space area associated (though the Husimi distribution) to sets of Schur vectors following [19]. In this thesis we

will investigate in detail how the quantum-classical correspondence occurs in systems whose gain-loss profile leads to a modified classical dynamics, such as in [24]. As in [19, 24] we will consider the Schur vectors to be the relevant quantum states to build this correspondence. In particular, we will propose a classical density that accurately reproduces the prominent features of the quantum Schur vectors in the phase space. We will investigate the resulting quantum-classical correspondence in a variety of non-Hermitian systems, with a focus on a PT-symmetric kicked rotor, and a non-Hermitian implementation of the Duffing oscillator. We now give an outline of the content in the remainder of this thesis.

1.8 Outline

We begin with a review of many standard results concerning the kicked rotor in chapter 2. We will explore the classical dynamics and introduce the torus quantisation of the model which will carry forward to later examples. Following this, we will take an in depth look at systems with escape through an opening. We will reproduce many of the main results from the literature, in the context of the kicked rotor model introduced previously.

In chapter 3 we will introduce one of the main systems of interest in this thesis, a PT-symmetric kicked rotor. We will study the classical features of this system and introduce the concept of norm maps. We will then proceed to the quantum analysis of the model and study the relation between the Husimi distributions of the Schur vectors and these norm maps. These form the basis of the quantum classical correspondence to come.

In chapter 4 we will introduce the full quantum classical correspondence through the introduction of a classical density that reproduces the features of the quantum Husimi distributions. The classical density is obtained numerically and we will begin by introducing the relevant algorithm. We will then use the algorithm to construct the classical density and compare it to the quantum distributions of Schur vectors in a number of non-Hermitian systems. In particular, we will consider the PT-symmetric kicked rotor, introduced in the previous chapter, and a novel implementation of the Duffing oscillator. We will also demonstrate that our method of quantum classical correspondence works for the systems with escape through an opening introduced in the previous chapter.

In chapter 5 we will provide a summary of the research covered within the thesis and consider the open questions that remain.

All results presented in Chapter 3 and 4 are original and currently being prepared for publication

Chapter 2

A Tale of 2 Kicked Rotors

In this chapter we give a brief summary of the traditional (closed) kicked rotor, with a particular focus on the correspondence between the features of the classical phase space and the quantum Husimi distributions. We will begin on the classical side and review the Poincaré dynamics on the torus, as well as the various chaotic regimes in relation to the kicking parameter. We will also introduce some standard tools of Hamiltonian dynamical systems, namely the stability of fixed points and their related unstable and stable manifolds. We will discuss the standard quantisation of the quantum map on the torus as well as the representation of the Husimi distribution in the torus coherent states.

We will then introduce a kicked rotor with escape through an opening, various versions of which have been investigated in the literature before. In the context of these systems we will reproduce various results from quantum systems with escape through an opening, some of which have originally been studied in other systems. These will include the characterisation of the transient chaotic systems through their chaotic saddles and their associated stable and unstable manifolds, the localisation of quantum eigenfunctions with long lifetimes upon these structures and the distributions of eigenstates in various energy regimes as described by the fractal-Weyl Law. Finally, we will discuss the correspondence of the Schur vectors to the classical structures in these systems.

2.1 The Traditional Kicked Rotor

The kicked rotor, first introduced by Chirikov [45,46] serves as a simple model to illustrate the transition from regular to chaotic dynamics. It is a prototypical system in chaos; an originally integrable system, whose integrability is destroyed by the non-linearity induced by the kicking term - a periodic driving applied for an infinitesimal time [3, 45, 47]. While the system had long been an important theoretical example, it has been experimentally realised in the context of cold atoms [48], with subsequent realisations providing insights into fundamental quantum effects such as decoherence [49] and noise [50].

2.1.1 The Classical Kicked Rotor

The system is described by the time-periodic Hamiltonian

$$H = \frac{p^2}{2} + k \cos(q) \sum_{n=-\infty}^{\infty} \delta(t - n\tau), \quad (2.1)$$

where k denotes the kicking strength.

Depending on the physical properties one wishes to study, it is customary to consider the phase-space geometry as a cylinder or torus [45, 51, 52, 53, 54]. Here we consider a toroidal phase space, that is, $(q, p) \in \mathcal{S}^1 \times \mathcal{S}^1$. The fact that the phase space is compact aides in the study of quantum-classical correspondence. Throughout the thesis we use a rescaled map on the torus which maps the unit cell $(q, p) \in [0, 1) \times [-\frac{1}{2}, \frac{1}{2})$ to itself. Consequently the Hamiltonian (2.1) takes the form

$$H = \frac{p^2}{2} + \frac{k}{4\pi^2} \cos(2\pi q) \sum_{n=-\infty}^{\infty} \delta(t - n), \quad (2.2)$$

where we have set the kicking period to one, and will do so throughout the thesis unless otherwise stated.

The classical map $\mathcal{M}$, obtained via integration of Hamilton's equations from just after a kick to just after the next one is given by

$$\begin{aligned} q_{n+1} &= \text{mod } (q_n + p_n)_1 \\ p_{n+1} &= \text{mod } \left(p_n + \frac{k}{2\pi} \sin(2\pi q_{n+1}) + \frac{1}{2} \right)_1 - \frac{1}{2}. \end{aligned} \quad (2.3)$$

The fixed points of the map (2.3), which we will denote $\bar{z}_j = (\bar{q}_j, \bar{p}_j)$, are located at $\bar{z}_1 = (0, 0)$, $\bar{z}_2 = (0.5, 0)$. Fixed points are classified by their linear

stability - the behaviour of trajectories in a small neighbourhood around them. The linearisation, about any point of the map, is determined by the Jacobian

$$\mathcal{J}_{\mathcal{M}} = \begin{bmatrix} \partial_q q_{n+1} & \partial_p q_{n+1} \\ \partial_q p_{n+1} & \partial_p p_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ k \cos(2\pi q_{n+1}) & 1 + k \cos(2\pi q_{n+1}) \end{bmatrix}. \quad (2.4)$$

The stability of the fixed points is classified by the eigenvalues of the Jacobian evaluated at the fixed points, $\mathcal{J}_{\mathcal{M}}(\bar{z})$. For closed systems, the dynamics are area-preserving which means that $|\det \mathcal{J}_{\mathcal{M}}| = 1$. This results in three classifications for the fixed points: *Elliptic Fixed Point*: $|\text{tr}(\mathcal{J}_{\mathcal{M}})| < 2$; $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\lambda_1 = \lambda_2^*$ and $|\lambda|^2 = 1$; *Hyperbolic Fixed Point*: $|\text{tr}(\mathcal{J}_{\mathcal{M}})| > 2$, $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 = \frac{1}{\lambda_2}$; *Parabolic Fixed Point* $|\text{tr}(\mathcal{J}_{\mathcal{M}})| = 2$, $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 = \lambda_2 = \pm 1$.

Fixed points still play an important role in the analysis of the global dynamics. For the kicked rotor, the fixed point $\bar{z}_2 = (0.5, 0)$ is an elliptic fixed point for $k < 4$, and hyperbolic for $k > 4$ [3]. To see this, consider the Jacobian of the fixed point $\bar{z}_2$

$$\mathcal{J}_{\mathcal{M}}(\bar{z}_2) = \begin{bmatrix} 1 & 1 \\ -k & 1 - k \end{bmatrix} \quad (2.5)$$

The matrix has $\text{tr}(\mathcal{J}_{\mathcal{M}}) = 2 - k$, and thus $|\text{tr}(\mathcal{J}_{\mathcal{M}})| < 2$ for $k < 4$. The fixed point at $\bar{z}_1 = (0, 0)$ is a hyperbolic fixed point for all values of the kicking strength. Hyperbolic fixed points have an associated stable and unstable manifold. For a hyperbolic fixed point $\bar{z}$ the *stable manifold* is defined as [55]

$$W^s(\bar{z}) = \{z_j \in \Omega : \lim_{n \rightarrow \infty} \mathcal{M}^n z_j = \bar{z}\}. \quad (2.6)$$

That is, the stable manifold is the set of points that approach $\bar{z}$ under forward iterations of the map. Similarly the *unstable manifold* is defined as

$$W^u(\bar{z}) = \{z_j \in \Omega : \lim_{n \rightarrow \infty} \mathcal{M}^{(-n)} z_j = \bar{z}\}, \quad (2.7)$$

and is the set of points that approach $\bar{z}$ under backward iterations of the map. For an invertible map the stable and unstable manifolds cannot self intersect [3, 7]. While the stable and unstable manifolds cannot self intersect they may, and often do, intersect one another. The set of points where stable and unstable manifolds of the same fixed point intersect are known as *homoclinic points*. *Heteroclinic points* are the set of intersections where the stable manifold belongs to one fixed point and the unstable to another, or vica-versa. It is known that chaos first arises near these homoclinic and heteroclinic intersections [56].

Let us now visualise these concepts for the kicked rotor. In Figure 2.1a we show the Poincaré section of the kicked rotor for $k = 0.5$. The majority of trajectories are regular, moving on or close to the invariant tori of the integrable system. Figure 2.1b shows a heat map of the initial conditions of trajectories coloured by their Lyapunov exponent. Chaotic trajectories have a positive Lyapunov exponent, and we see that this occurs in the vicinity of the unstable fixed point $\bar{z}_1 = (0, 0)$, and defines a bounded curve. Figure 2.1c shows the sections of the unstable and stable manifold in the vicinity of the unstable fixed point where we can clearly see the homoclinic intersections.

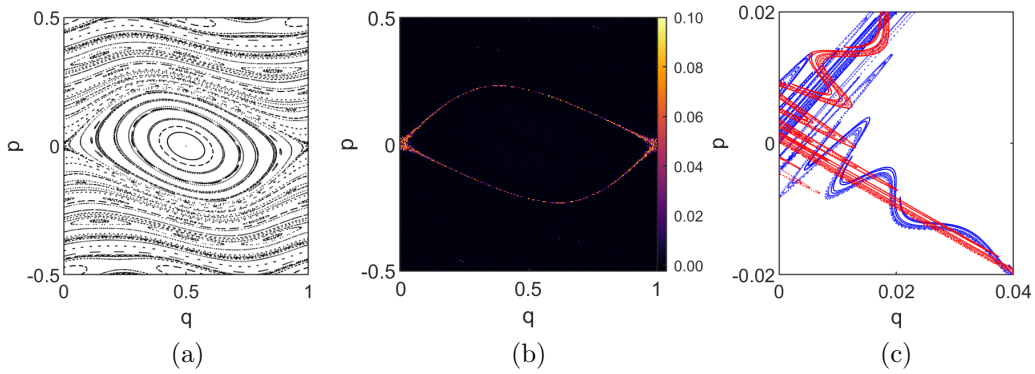


Figure 2.1: Poincaré section (left), heat map of the Lyapunov exponents (middle) and homoclinic intersection (right) of the classical kicked rotor for $k = 0.5$. The homoclinic intersection is shown in the neighbourhood of the unstable fixed point $\bar{z}_1 = (0, 0)$ where the stable manifold $W^s(\bar{z}_1)$ is depicted in blue and the unstable manifold $W^u(\bar{z}_1)$ in red.

As we increase the kicking the effects of the non-linearity leads to the destruction of the invariant tori. The last invariant torus is destroyed at the critical value $k_c \approx 0.97$. Figure 2.2a shows the dynamics for $k = 1.1$ which results in a mixed phase space. The regular dynamics occur in a large central island, as well as in resonance chains in the upper and lower momentum plane. Surrounding these is the chaotic sea. Figure 2.2b shows the dynamics for $k = 2.5$. The elliptic fixed point still remains as the centre of a regular island. The island is surrounded by four smaller regular islands whose trajectories visit each island with a period of 4. The chaotic sea covers a large proportion of the space for this value of k . Figure 2.2c shows the dynamics for $k = 5$. The elliptic fixed point is replaced by a hyperbolic fixed point and the central island has split up into two small regular islands that are part of a 2-cycle, surrounded by a large chaotic sea.

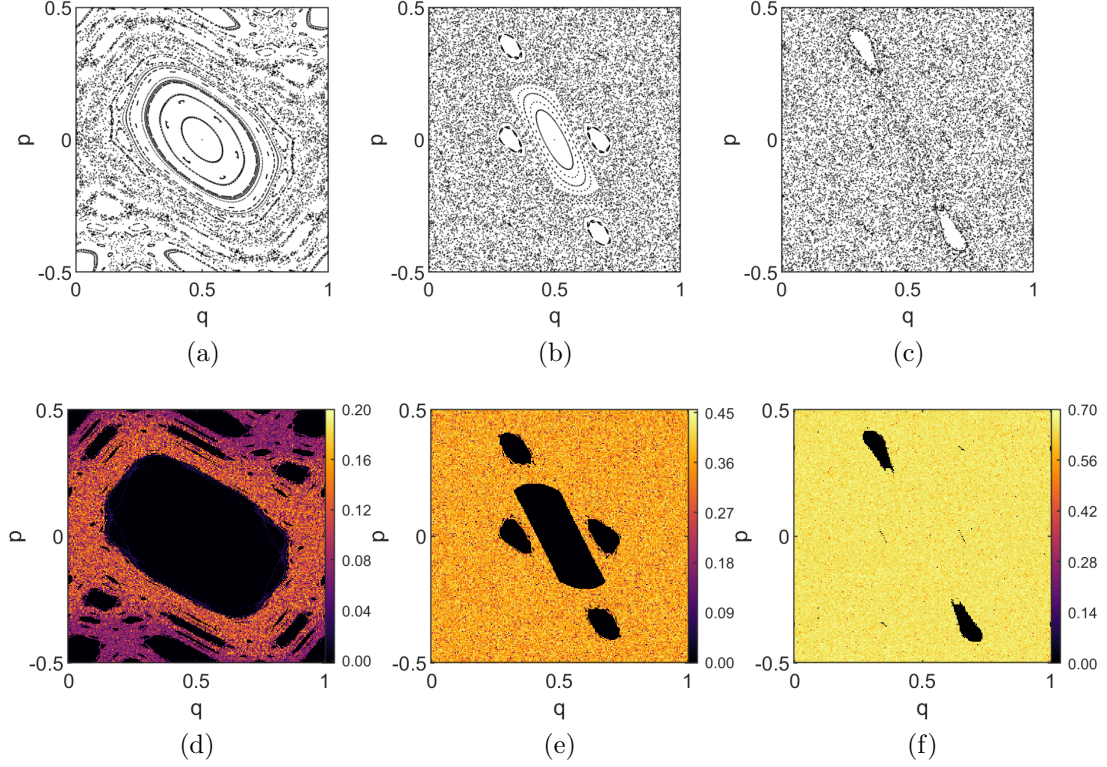


Figure 2.2: Poincaré sections and Lyapunov heat maps of the kicked rotor for different value of k : $k = 1.1$ (left), $k = 2.5$ (middle) and $k = 5$ (right). The Lyapunov heat maps are constructed from an initial distribution of $N^2 = (250)^2$ points uniformly distributed across the phase space. The Lyapunov exponents are averaged for $t = 2000$ iterations after an initial propagation of $t_0 = 1000$ steps.

The heat maps of the Lyapunov exponents are shown in Figure 2.2d ($k = 1.1$), Figure 2.2e ($k = 2.5$), Figure 2.2f ($k = 5$), and show more clearly the regular vs chaotic regions of the dynamics. The system is globally chaotic for $k \geq 7$ [19]. Figure 2.3a shows the Poincaré section for $k = 10$, indeed there are no regular regions visible. Figures 2.3b and 2.3c shows the stable and unstable manifold for the fixed point $\bar{z}_2 = (0.5, 0)$, respectively. We observe that underlying the chaotic dynamics there is still a high degree of structure.

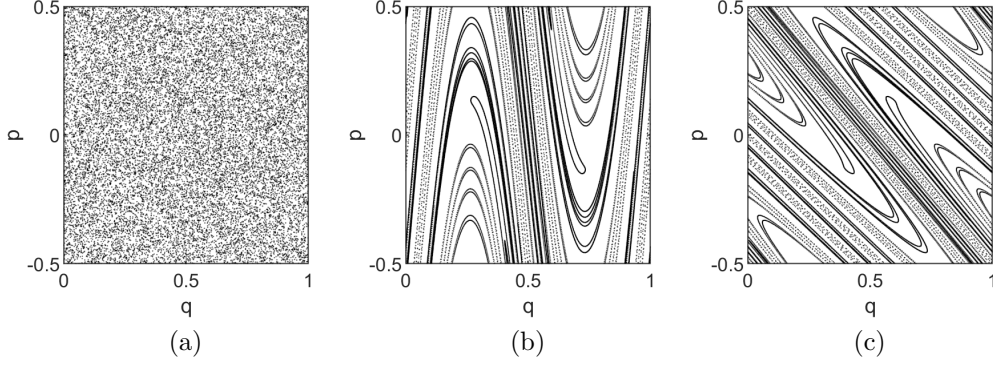


Figure 2.3: The Poincaré section and stable and unstable manifolds of the kicked rotor for $k = 10$. The stable (middle) and unstable (right) manifolds are associated to the hyperbolic fixed point $\bar{z}_2 = (0.5, 0)$. The Poincaré section is shown in the left plot.

2.1.2 Torus Quantisation of the Kicked Rotor

Here we use the quantisation of the kicked rotor on the torus following the pioneering work of Izrailev [51, 52]. In what follows, we will present a slightly longer derivation of the matrix elements of the Floquet operator $\hat{F}$.

Since we are on the torus we will use the explicit form of the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{k}{4\pi^2} \cos(2\pi\hat{q}) \sum_{n=-\infty}^{\infty} \delta(t - n). \quad (2.8)$$

Classically the map (2.3) has boundary conditions that identify $q = q + m$, $p = p + m$ for integer shift $m \in \mathbb{Z}$. Quantum mechanically this translational invariance manifests itself through

$$\begin{aligned} |q_{j+N}\rangle &= |q_j\rangle \\ |p_{j+N}\rangle &= |p_j\rangle, \end{aligned} \quad (2.9)$$

where $\{|q_j\rangle\}$, $\{|p_j\rangle\}$, are the discrete position and momentum eigenstates in the *finite-dimensional Hilbert space* $\mathcal{H}_N$.

The discrete position states $|q_j\rangle \in \mathcal{H}_N$ are eigenstates of the *discrete position operator*

$$\hat{q} = \sum_{l=1}^N q_l |q_l\rangle \langle q_l|, \quad (2.10)$$

with

$$q_l = \frac{1}{N}l, \quad (2.11)$$

where the index l runs from 1, N where N is the dimension of the discretised Hilbert space $\mathcal{H}_N$. Similarly, the discrete momentum states $|p_k\rangle \in \mathcal{H}_N$ are eigenstates of the *discrete momentum operator*

$$\hat{p} = \sum_{m=1}^N p_m |p_m\rangle \langle p_m|, \quad (2.12)$$

with

$$p_m = \frac{1}{N}m, \quad (2.13)$$

for $m \in (1, 2, \dots, N)$. Under this construction $\hbar$ cannot take arbitrary values, and instead must take the form

$$\hbar = \frac{1}{2\pi N}. \quad (2.14)$$

which is sometimes referred to as the quantum resonance condition [47,57,58]. This also follows from semiclassical arguments; since each state occupies an area h in the phase space, and there are N states in our unit phase space, we have $h = \frac{1}{N}$. The overlap between the discrete position and momentum basis vectors in $\mathcal{H}_N$ is given by

$$\begin{aligned} \langle q_l | p_m \rangle &= \frac{1}{\sqrt{N}} e^{\frac{i}{\hbar} q_l p_m} \\ &= \frac{1}{\sqrt{N}} e^{\frac{2i\pi}{N} l m}, \end{aligned} \quad (2.15)$$

Using Floquet theory we can construct a quantum map for the kicked rotor (2.8). The one-period evolution operator $\hat{F}$ factorises into the product

$$\begin{aligned} \hat{F} &= \hat{T} \exp \left(-\frac{i}{\hbar} \int_0^\tau dt' \hat{H}(t') \right) \\ &= \lim_{\delta \rightarrow 0} \hat{T} \exp \left(\frac{i}{\hbar} \int_{\tau-\delta}^{\tau+\delta} dt' \hat{H}(t') \right) \hat{T} \exp \left(\frac{i}{\hbar} \int_{0+\delta}^{\tau-\delta} dt' \hat{H}(t') \right) \\ &= \lim_{\delta \rightarrow 0} \hat{T} \exp \left(\frac{i}{\hbar} \delta \hat{p}^2 - \frac{ik}{4\pi^2 \hbar} \cos(\hat{q}) \right) \hat{T} \exp \left(\frac{i}{2\hbar} (\tau - 2\delta) \delta \hat{p}^2 \right) \\ &= \exp \left(-\frac{ik}{4\pi^2 \hbar} \cos(\hat{q}) \right) \exp \left(-\frac{i\tau}{2\hbar} \hat{p}^2 \right). \end{aligned} \quad (2.16)$$

We will denote these two operators by the subscript “ k ” and “ nk ” which denote the “kicked” and “non-kicked” (free) motion. We then write the map $\hat{F}$ as

$$\hat{F} = \hat{U}_k \hat{U}_{nk} = e^{-\frac{ik}{4\pi^2\hbar} \cos(\hat{q})} e^{-\frac{i}{2\hbar} \hat{p}^2}. \quad (2.17)$$

The quantum map (2.17) lends itself to a nice numerical treatment since the $\hat{U}_{nk}$ is diagonal in the momentum representation, likewise the $\hat{U}_k$ is diagonal in the position representation.

We will now show how to obtain the matrix elements for $\hat{F}$ in the discrete position representation under the torus conditions in $\mathcal{H}_N$. The evolution operator of the free motion acts on the discrete momentum basis as

$$\begin{aligned} \hat{U}_{nk} |p_m\rangle &= e^{-\frac{i}{2\hbar} \hat{p}^2} |p_m\rangle \\ &= e^{-\frac{i}{2\hbar} p_m^2} |p_m\rangle \\ &= e^{-\frac{i\pi}{N} m^2} |p_m\rangle. \end{aligned} \quad (2.18)$$

Similarly the quantisation of the evolution operator of the kick acts on the position basis as

$$\begin{aligned} \hat{U}_k |q_l\rangle &= e^{-\frac{ik}{4\pi^2\hbar} \cos(\hat{q})} |q_l\rangle \\ &= e^{-\frac{ik}{4\pi^2\hbar} \cos(\hat{q}_l)} |q_l\rangle \\ &= e^{-\frac{kN}{2\pi} \cos(\frac{2\pi}{N} l)} |q_l\rangle. \end{aligned} \quad (2.19)$$

The matrix elements of $\hat{F}$, in the position basis $|q_l\rangle, |q_{l'}\rangle$, are thus found as

$$\begin{aligned} F_{ll'} &= \langle q_l | \hat{F} | q_{l'} \rangle \\ &= \frac{1}{N} e^{-\frac{ikN}{2\pi} \cos(\frac{2\pi}{N} l)} \left(\sum_m e^{-\frac{iN}{4\pi} m^2 + \frac{2\pi i}{N} m(l-l')} \right), \end{aligned} \quad (2.20)$$

where k is the classical kicking strength and N is the matrix dimension. The semiclassical limit corresponds to $N \rightarrow \infty$ for fixed classical kicking strength k . Note that some authors write (2.20) in terms of $T = \frac{2\pi}{N}$ and $\tilde{k} = k/T$ (see for example [51, 52, 59, 60]) for which the semiclassical limit corresponds to $\tilde{k} \rightarrow \infty$, $T \rightarrow 0$, with k fixed.

2.1.3 Quantum-Classical Correspondence

Due to the toroidal geometry the standard coherent states (1.15) need to be adjusted to respect the underlying topology of the phase space. Here we will

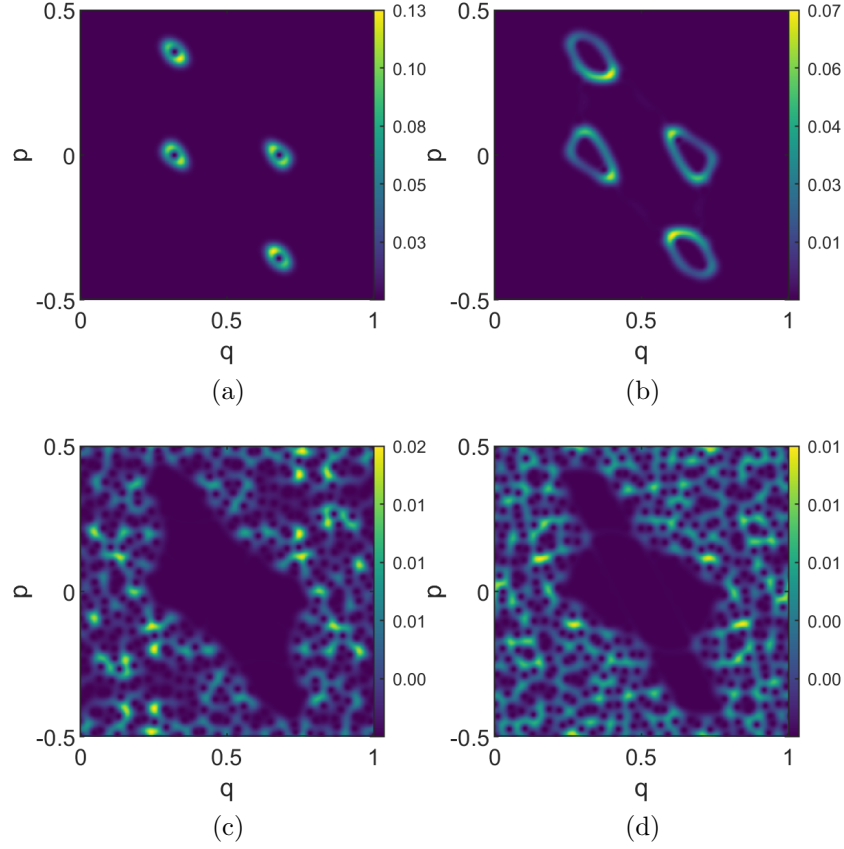


Figure 2.4: Husimi distributions of selected eigenfunctions of the kicked rotor for kicking strength $k = 2.5$ and matrix dimension $N = 1001$. Regular eigenfunctions are shown in the top row and chaotic eigenfunctions in the bottom row.

use torus coherent states as defined in [57]. In the discrete position basis $\{|q_l\rangle\}$ these can be written as

$$|z\rangle = N_z \sum_l \sum_{\mu=-\infty}^{\infty} e^{\frac{-\pi N}{2}(|z|^2 - z^2) - \frac{\pi}{N}(q_l - z + \mu)^2 + i\pi\mu} |q_l\rangle. \quad (2.21)$$

These can be used analogously to the standard coherent states to define the Husimi distribution. We use these to look at an example of the quantum classical correspondence in the kicked rotor for the mixed phase space $k = 2.5$. In Figure 2.4 we see the Husimi distributions of selected eigenstates which, as predicted by the semi-classical eigenfunction hypothesis, localise upon regular structures, or are delocalised over the chaotic sea.

2.2 A Kicked Rotor with Escape

Here we will review the literature on open models of chaotic systems with escape through an opening. To aid in the illustration of the concepts we will reproduce the main results in an open version of the kicked rotor.

2.2.1 The Classical Model

Systems with escape through an opening are one of the most obvious ways to introduce loss into a system. The opening is defined as a region of phase space which acts as a hole. Trajectories that enter the opening will not exit it again. Consider the kicked rotor phase space with an opening. For simplicity we consider the open region to be a strip in phase space defined by $q \in [q_L, q_R]$. The strip is shown schematically Figure 2.5a where the opening is represented by the black region. Outside of the opening the dynamics evolve under the map of the closed system

$$\begin{aligned} q_{n+1} &= \text{mod}_1(q_n + p_n) \\ p_{n+1} &= \text{mod}_1(p_n + \frac{k}{2\pi} \sin(2\pi q_{n+1}) + 1/2) - 1/2. \end{aligned} \quad (2.22)$$

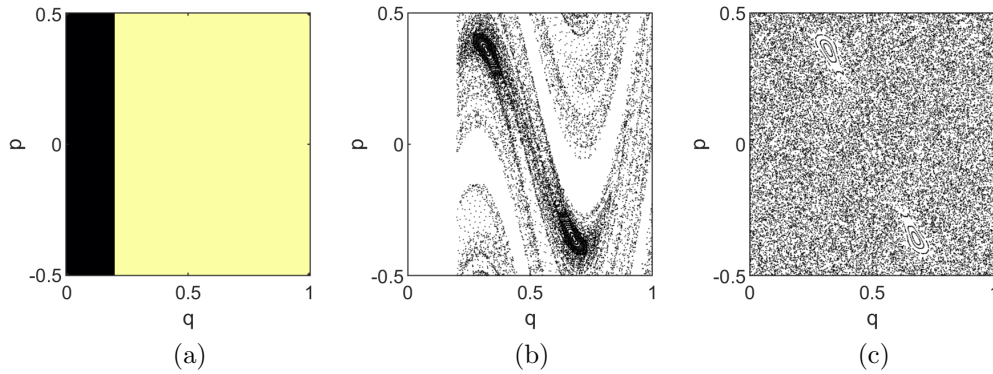


Figure 2.5: The left column depicts the opening (black) within the Poincaré section with endpoints $q_L = 0$, $q_R = 0.2$. In the middle column the Poincaré section of the open kicked rotor is shown for kicking strength $k = 5$ and is generated by iterating $N^2 = (30)^2$ uniformly distributed initial conditions for $t = 50$ iterations. In the right column the corresponding Poincaré section of the closed system is shown for the same parameters.

In Figure 2.5c we show the Poincaré dynamics without the opening for $k = 5$, where we have a largely chaotic phase space with two small regular islands.

In Figure 2.5b the dynamics in the open system are shown. The regular islands are dynamically connected to each other. Since they do not contain the opening, trajectories in these islands survive for all time. The chaotic sea is a different story, chaotic trajectories explore most of the available phase space. Due to this, the chaotic trajectories have a finite lifetime. To understand the structures that arise in connection to this, we turn to the theory of transient chaos.

2.2.2 Transient Chaos

Transient chaos is the study of chaotic dynamics that take place on finite-time scales. The need for such a theory arises naturally in the presence of an opening. A chaotic trajectory will almost certainly enter the opening after some finite time as it jumps around the phase space. When the dynamics of the closed system are globally chaotic almost every trajectory in the phase space of the open system acquires a finite lifetime. The survival probability, $p(t)$, for the chaotic trajectories that can visit the opening is given by

$$p(t) \sim e^{-\Lambda t}, \quad (2.23)$$

where Λ denotes the decay rate and is related to the average lifetime τ by $\tau \approx \Lambda^{-1}$. Despite almost all trajectories obeying (2.23), there is a set of measure zero that does not leave the system under forward or backwards iteration. The dynamics in transient fully chaotic systems are structured around these sets, which are known as chaotic saddles [61].

A *chaotic saddle* is a dynamical invariant and possesses a stable and unstable manifold defined similarly as for hyperbolic fixed points. The stable manifold is defined as the set of all trajectories that approach the saddle as $t \rightarrow \infty$. The unstable manifold is defined as the set of all trajectories that approach the saddle as $t \rightarrow -\infty$. The chaotic saddle as well as its stable and unstable manifolds are fractals. In the literature, the stable manifold, which we denote as K_+ is often referred to as the *forward trapped set* and the unstable manifold, we denote as K_- , the *backward trapped set* [20, 21]. For the globally chaotic case $k = 10$, Figure 2.6 shows the chaotic saddle together with its stable and unstable manifolds for an opening with endpoints $(q_L, q_R) = (0, 0.2)$ in the first row, and $(q_L, q_R) = (0.4, 0.6)$ in the second. These were obtained using the sprinkler method. (For an outline of the method see for example [61]).

Visually these structures can be understood by colouring the trajectories by their lifetime. In Figure 2.7a we show the lifetime plot of the classical trajectories under the backward dynamics for an opening at $(q_L, q_R) = (0, 0.2)$.

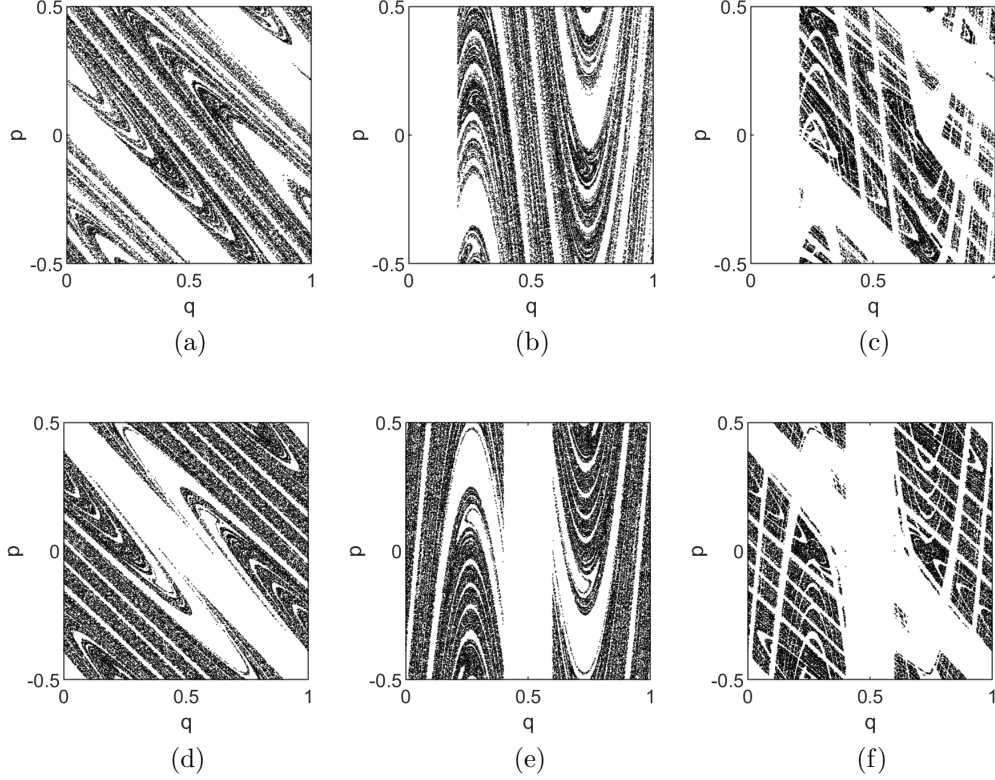


Figure 2.6: From left to right: The stable manifold (Forward trapped set K_+), the unstable manifold (backward trapped set K_-), and the chaotic saddle ($K_+ \cap K_-$) of the open kicked rotor with kicking strength $k = 10$. In the top row the opening is defined by $(q_L, q_R) = (0, 0.2)$ and in the bottom row $(q_L, q_R) = (0.4, 0.6)$. The plots are obtained from the sprinkler method with $t^* = 10$ and $N^2 = (500)^2$ uniformly distributed initial conditions.

There is a clear correspondence between the regions of escape and the empty regions of the unstable manifold 2.6b. We see the same correspondence between Figure 2.7b and Figure 2.6e, where the opening is shifted to the position $(q_L, q_R) = (0.4, 0.6)$. This can also be applied to the phase space under the forward dynamics. This is shown in Figure 2.7c which corresponds to the stable manifold of Figure 2.6a with opening $(q_L, q_R) = (0, 0.2)$. Similarly for an opening at $(q_L, q_R) = (0.4, 0.6)$, Figure 2.7d corresponds to Figure 2.6d.

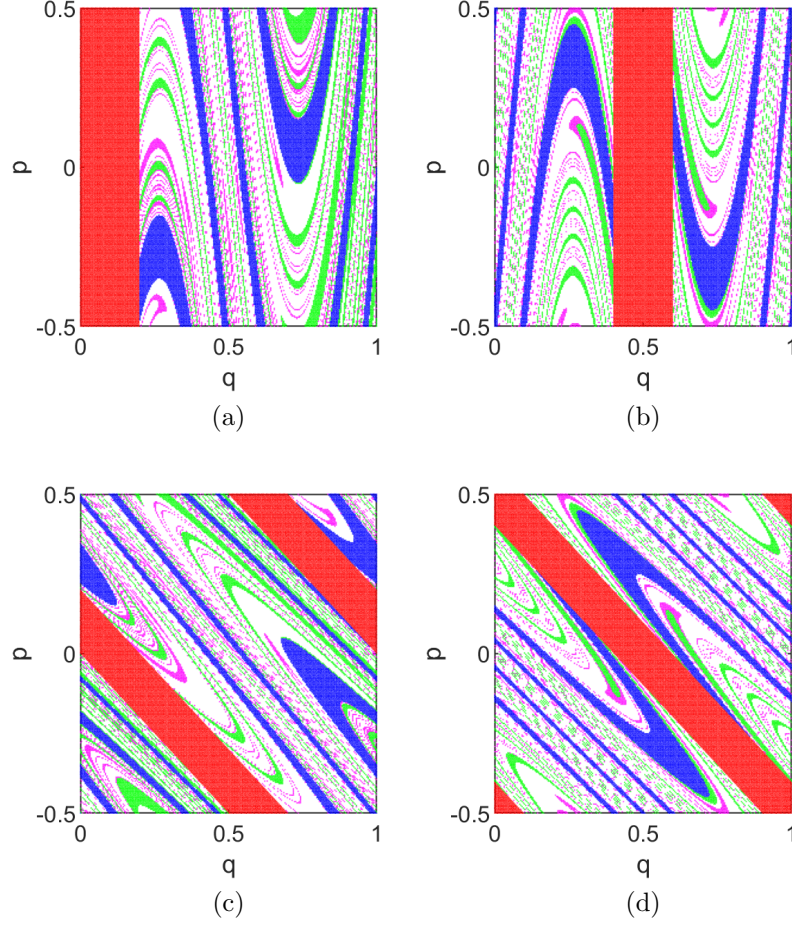


Figure 2.7: Lifetime of trajectories in the kicked rotor with an opening for a kicking strength $k = 10$. The colours denote the number of iterations before escape and correspond to: 0 (red), 1 (blue), 2 (green), and 3 (magenta). The left column corresponds to an opening $(q_L, q_R) = (0, 0.2)$ and the right column to an opening $(q_L, q_R) = (0.4, 0.6)$. The top row corresponds to the backwards dynamics and the bottom row to the forward dynamics. Each plot is generated from an initial distribution of $N^2 = (250)^2$ uniformly distributed initial conditions.

2.2.3 The Quantum Model

Quantum mechanically systems with escape through an opening can be modelled as truncations of unitary propagators. The quantum model is simply described by the projection of the Floquet operator onto the complement of

the opening [20, 38]. The resulting Floquet operator is given by

$$\hat{\mathcal{F}} = \hat{F}\hat{\Pi}, \quad (2.24)$$

where $\hat{\Pi}$ is the projector onto the compliment of the opening. In the position representation, with an opening that spans the entire p-range, as considered here, this essentially amounts to setting M of the N columns of $\hat{F}$ to zero. The resulting matrix $\hat{\mathcal{F}}$ is non-normal and thus its eigenstates $\{|\phi_n\rangle\}$, the Floquet states of $\hat{\mathcal{F}}$, are non-orthogonal. The decaying lifetimes attached to quantum states through the norm mirrors that of the trajectories in the classical system. Due to the non-orthogonality of the eigenstates it is necessary to distinguish between the right and left eigenstates of $\hat{\mathcal{F}}$ [20]. The right eigenstates satisfy

$$\hat{\mathcal{F}}|\phi_n\rangle = \lambda_n|\phi_n\rangle, \quad (2.25)$$

while the left eigenstates $|\chi_n\rangle$ are defined via the relation

$$\langle\chi_n|\hat{\mathcal{F}} = \langle\chi_n|\lambda_n. \quad (2.26)$$

2.2.4 Husimi Distributions of the Eigenstates

In a study of a purely chaotic Baker map, it has been observed that Husimi distributions of the right eigenstates with the longest lifetimes localise on the unstable manifold of the saddle, K_- , while the left eigenstates with the longest lifetimes localise on the stable manifold of the chaotic saddle K_+ [20]. The classification of a “long lifetime” is with respect to the *Ehrenfest time* τ_E [20, 38, 62]. The Ehrenfest time in this context is defined as the time it takes an minimally localised state to have a width greater than the opening, thus experiencing a reflection when leaving the system [20, 38, 63, 64, 65]. Thus, the Ehrenfest time defines a natural timescale for quantum classical correspondence in this system [20]. For our model setting $k = 10$ ensures global chaos of the closed classical system. Figure 2.8b shows the Husimi distribution of the right eigenstates with longest lifetimes. We see a clear correspondence to the classical unstable manifold of the chaotic saddle, Figure 2.8a. Similarly, in Figure 2.8d we show the summed Husimi distribution of the left eigenstates with longest lifetimes. We observe a clear correspondence to the stable manifold of the classically chaotic saddle, Figure 2.8c. Thus, unlike the closed case where the chaotic eigenstates are delocalised over the chaotic sea, the chaotic eigenstates in these systems are highly structured.

In closed systems, the fraction of states with energies $E_n < E$ is independent of Plancks constant, h , and associated to the classical phase space area belonging to energies $H < E$. In open chaotic systems the fractal nature of the

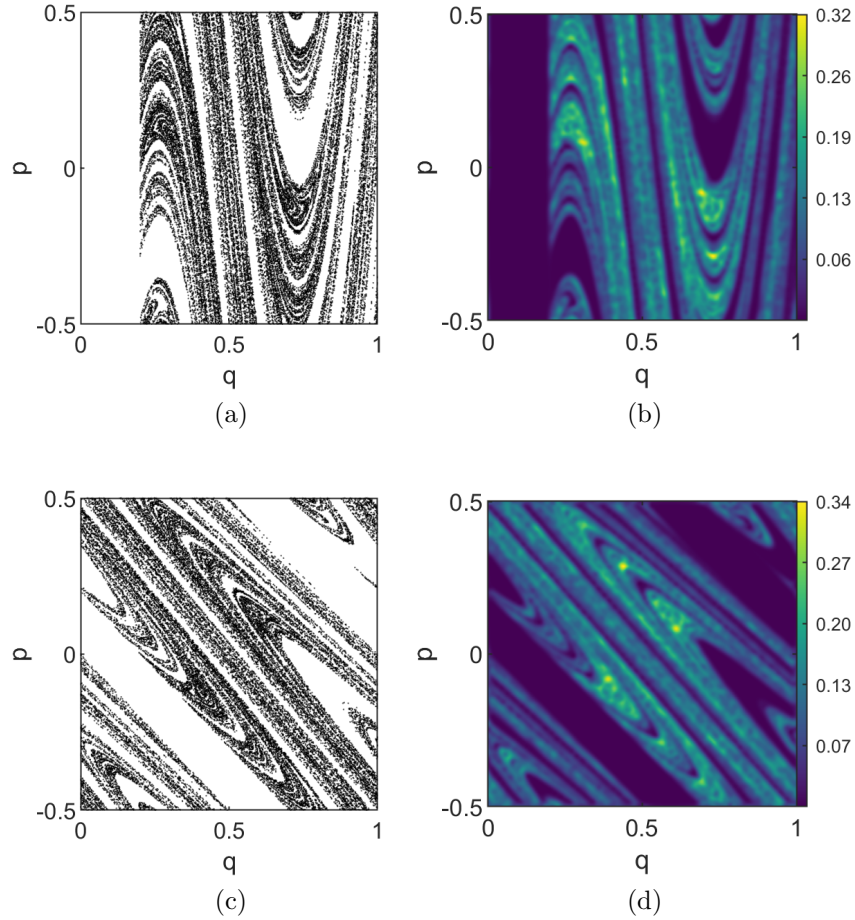


Figure 2.8: Husimi distributions of the longest lived right (left) eigenstates and the corresponding (un)stable manifold of the chaotic saddle of the open kicked rotor. In both the quantum and classical system the kicking strength is $k = 10$ with opening $(q_L, q_R) = (0, 0.2)$. The Husimi of longest lived eigenstates correspond to a quantum matrix dimension of $N = 1001$. The top row depicts the unstable manifold of the chaotic saddle K_- (left) and the Husimi distribution of the right eigenstates with longest lifetimes (right). The bottom row depicts the stable manifold of the chaotic saddle K_+ (left) and the Husimi distribution of the left eigenstates with longest lifetimes (right).

sets on which the states localise has consequences for the spectral counting function, and results in a non-trivial scaling with $\hbar$. This is often referred to as a fractal Weyl law. To illustrate this, following the notation introduced in [19], Figure 2.9 shows the fraction $p(\mu > \mu_n)$ of eigenvalues with absolute value $|\mu_n| > \mu$. States that localise on the structures with a fractal nature

see the non-trivial geometric scaling and more states are needed to resolve this regions of the phase space. Since the number of states is fixed, this results in different energy bands of the spectrum having a greater “weight”. We observe this in Figure 2.9a through the flattening of the bands of states with intermediate lifetimes as the matrix dimension is increased ($\hbar$ is getting smaller).

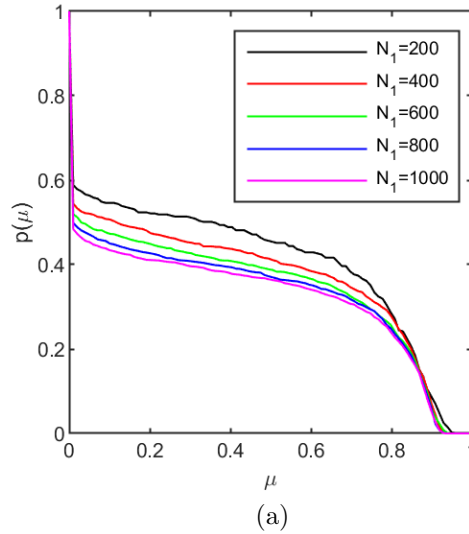


Figure 2.9: The distribution of eigenvalue lifetimes $\mu_n = |\lambda_n|$ as a function of μ for various matrix dimensions in the quantum open kicked rotor. In the quantum model the kicking strength is $k = 10$ and the opening is defined by $(q_L, q_R) = (0, 0.2)$. The matrix dimension $N = 2N_1 + 1$ is varied over $N_1 \in \{200, 400, 600, 800, 1000\}$.

Returning to the phase space localisation of the eigenstates the association between states with long lifetimes and classical structures necessarily leaves out a number of eigenstates from the quantum-classical correspondence. The association between states of intermediate lifetimes and classical structures does not possess a simple correspondence to the manifolds $K_{\pm}$ of the chaotic saddles. It has been proposed that the Husimi distributions of sets with intermediate lifetimes can be related to classical structures not corresponding to $\hat{K}_{\pm}$ through conditionally invariant measures [21, 22, 23].

To avoid complications arising from the non-orthogonality it has been proposed in [19, 38] to use the Schur vectors. The Schur vectors provide an orthonormal basis and allow us to recover traditional ideas of quantum-classical correspondence such as the partitioning of phase space into Planck cells.

2.2.5 Husimi Distributions of the Schur states

The Schur decomposition has been used in previous investigations into a kicked rotor with an opening [62]. There the Schur states have been used to demonstrate a correspondence in both the fully chaotic and mixed regimes [19]. It was found that the Schur vectors with short lifetimes corresponded to those trajectories that escape from the system the fastest. As an example in Figure 2.10a we show the Husimi distribution of the Schur states with shortest lifetimes selected using the fractal Weyl law. In Figure 2.10b we show the classical phase space where the points are coloured according to after how many iterations the corresponding trajectories escape, a classical lifetime. This shows a clear correspondence between the Husimi distributions of the Schur vectors and the lifetimes of classical trajectories. Finally, observe that the Husimi distribution of all the Schur states, the union of Figure 2.10a and Figure 2.10c, is the uniform density on the phase space.

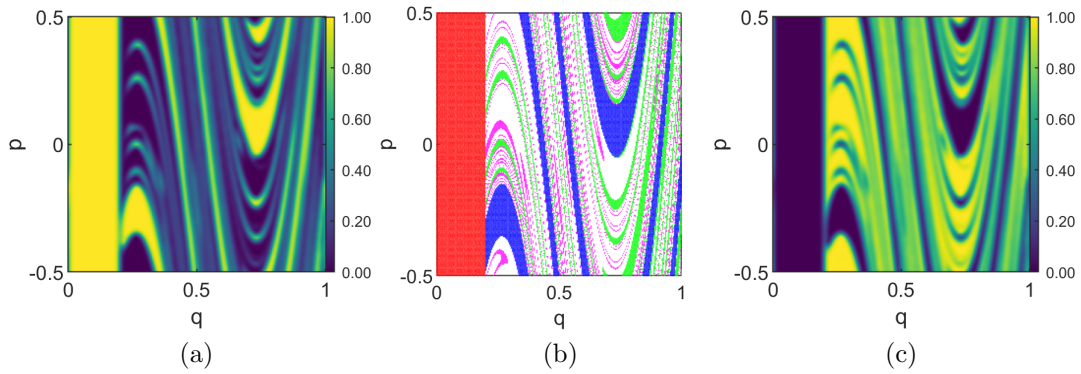


Figure 2.10: Husimi distribution of sets of states with short lifetimes and those with long lifetimes together with the classical lifetime map of the backwards trapped set for the open kicked rotor for a kicking strength $k = 10$ and opening $(q_L, q_R) = (0, 0.2)$. The Husimi distributions correspond to a matrix dimension $N = 1001$ and display the set of states with the shortest lifetimes (left) and the longest lifetimes (right) as predicted by the fractal Weyl law. The middle plot displays lifetime map of the backwards trapped set with colours denoting escape after the following number of iterations: 0 (red), 1 (blue), 2 (green), and 3 (magenta).

Chapter 3

A $\mathcal{PT}$ -Symmetric Kicked Rotor

The work in this chapter is in the process of being submitted for publication.

3.1 A $\mathcal{PT}$ -Symmetric Kicked Rotor

PT-symmetric extensions of kicked rotors with complex kicking terms have been discussed in the literature before with a focus on complexified classical dynamics and momentum diffusion [66, 67, 68]. Here we introduce a simple PT-symmetric generalisation of a kicked rotor

$$\hat{H} = \frac{\hat{p}^2}{2} + i\frac{\gamma}{2\pi}\hat{p} + \frac{k}{4\pi^2}\cos(2\pi\hat{q})\sum_{n=-\infty}^{\infty}\delta(t-n), \quad (3.1)$$

with real kicking strength $k \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ describing the strength of the gain-loss profile. The Hamiltonian (3.1) is PT-symmetric in the sense that it is invariant with respect to the parity operator $\hat{\mathcal{P}} : \hat{q} \rightarrow -\hat{q}, \hat{p} \rightarrow -\hat{p}$ and the time-reversal operator $\hat{\mathcal{T}} : i \rightarrow -i, \hat{q} \rightarrow -\hat{q}$.

In this section we will form a correspondence between the Husimi distributions of the Schur vectors and an object defined by the classical norm, which we will refer to as *norm maps*. We will find that even in the presence of chaos these norm maps present a high degree of structure. In particular, the norm maps develop fractal-like structures when the corresponding dynamics are largely or fully chaotic. We will find that Husimi distributions of sets of Schur vectors will have a clear correspondence to the norm maps generated by the forwards or backwards dynamics depending on the ordering of the Schur vectors. Finally we will propose an analytic argument for why the Schur

vectors correspond to the classical norm maps. We begin by exploring the classical dynamics generated by the PT-symmetric Hamiltonian (3.1)

3.1.1 The Classical Map

For the PT-symmetric kicked rotor Hamiltonian (3.1), the coherent state approximation results in a simple modification of the equations of motion

$$\begin{aligned}\dot{q} &= p \\ \dot{p} &= \frac{\gamma}{2\pi} + \frac{k}{2\pi} \sin(2\pi q) \delta(t - n).\end{aligned}\tag{3.2}$$

The integration of (3.2) is performed from just after a kick to just after the next as in the closed case discussed before. This yields the classical map

$$\begin{aligned}q_{n+1} &= \text{mod}_1 \left(q_n + p_n + \frac{\gamma}{2\pi} \right) \\ p_{n+1} &= \text{mod}_1 \left(p_n + \frac{k}{2\pi} \sin(2\pi q_{n+1}) + \frac{\gamma}{2\pi} + \frac{1}{2} \right) - \frac{1}{2},\end{aligned}\tag{3.3}$$

where we have again used the toroidal geometry, represented on the unit cell $(q, p) \in [0, 1) \times [-0.5, 0.5)$. The non-Hermitian term simply induces an additional constant shift in q and p in each iteration. The time-reversed form of the kicked rotor map (3.3) is given by

$$\begin{aligned}p_{n+1} &= \text{mod}_1 \left(p_n - \frac{k}{2\pi} \sin(2\pi q_{n+1}) - \frac{\gamma}{2\pi} + \frac{1}{2} \right) - \frac{1}{2} \\ q_{n+1} &= \text{mod}_1 \left(q_n - p_n - \frac{\gamma}{2\pi} \right).\end{aligned}\tag{3.4}$$

Calculating the Jacobian of the PT-symmetric dynamics (3.3) we find

$$\mathcal{J}_{\mathcal{M}} = \begin{bmatrix} \partial_q q_{n+1} & \partial_p q_{n+1} \\ \partial_q p_{n+1} & \partial_p p_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ k \cos(2\pi q_{n+1}) & 1 + k \cos(2\pi q_{n+1}) \end{bmatrix}\tag{3.5}$$

which is the same as in the closed system with $\det(\mathcal{J}_{\mathcal{M}}) = +1$. This means that the modified map is still area and orientation preserving, as in the closed system, a feature that is possible due to the PT-symmetry of the system, but not generic for PT-symmetric systems. Despite the open nature of the classical system, there will not arise the more exotic features of typical dissipative classical models such as sources, sinks, strange attractors.

The fixed point structure of the map is modified in the PT-symmetric system from that of the closed system. The unstable fixed point of the closed system

at $(0,0)$ is destroyed by the shift of q, p by γ . The remaining fixed point at $(0.5, 0)$ is shifted to the new location

$$\begin{aligned}\bar{p}_\gamma &= -\frac{\gamma}{2\pi} \\ \bar{q}_\gamma &= 0.5 - \frac{1}{2\pi} \sin^{-1} \left(\frac{\gamma}{k} \right).\end{aligned}\tag{3.6}$$

The shift of the fixed point can be seen for example in Figure 3.1c. When $|\gamma| > k$ the fixed point vanishes completely. When $|\gamma| < k$ the fixed point, in a similar manner to the closed system, exhibits a transition from elliptic to hyperbolic. From (3.5) and the hyperbolic fixed point condition $|\text{tr}(\mathcal{J}_\mathcal{M})| > 2$, the condition for the fixed point to become hyperbolic is $k > \sqrt{16 + \gamma^2}$.

Together with the map (3.3), we have from equation (1.65) the classical norm, the evolution of which here takes the simple form

$$w_{n+1} = e^{2\gamma p_n} w_n.\tag{3.7}$$

We also consider the norm map of the backwards dynamics which gives the time-reversed form of the norm

$$w_{n+1} = e^{-2\gamma p_{n+1}} w_n,\tag{3.8}$$

where the sign of γ is flipped because time-reversal exchanges regions of gain and loss.

3.1.2 Phase-space Dynamics

To understand the classical features of the model we begin by comparing the Poincaré sections generated by the map (3.3) to those of the closed case. Here we will consider two values of γ : $\gamma = 0.001$, which will have significance when we analyse the quantum system and $\gamma = 0.3$ which will bring out more clearly the dynamical differences between the closed and the PT-symmetric classical systems. Quantum mechanically we will be restricted in our choice of γ by the numerical instabilities, hence our choice of including such a small value.

In Figure 3.1 the dynamics of the closed system (left) are shown for two relatively small values of k ($k = 0.5$ on the top and $k = 1.1$ on the bottom). The dynamics of the PT-symmetric system are shown in the middle and right column for $\gamma = 0.001$, $\gamma = 0.3$, respectively. The translation of the elliptic fixed point to the new coordinates given by (3.6) is most evident for $\gamma = 0.3$.

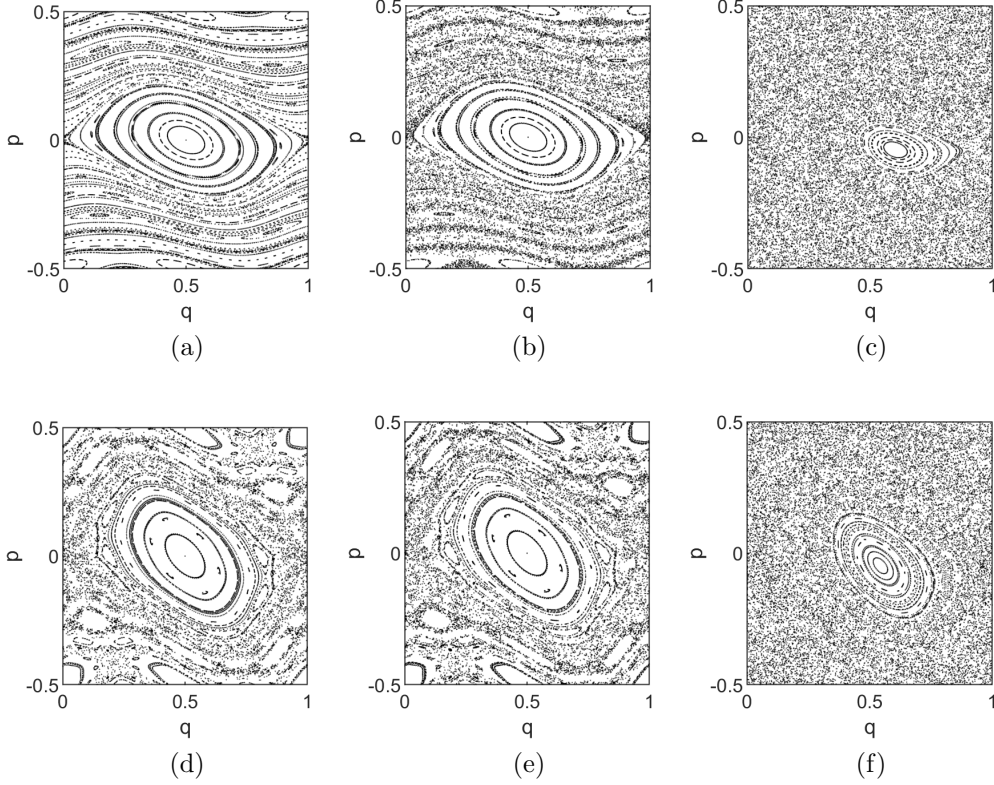


Figure 3.1: Poincaré sections of the PT-symmetric kicked rotor for different values of non-Hermitian term γ and kicking strength k . From left to right $\gamma = 0$, $\gamma = 0.001$ and $\gamma = 0.3$. In the top row the kicking strength is $k = 0.5$ and in the bottom row $k = 1.1$. Each Poincaré section is generated by iterating $N^2 = (15)^2$ initial conditions under $t = 50$ of the forwards map (3.3).

Further we observe that the non-Hermiticity seems to increase the regions of chaos.

To quantify this effect we turn to the Lyapunov exponents of the system. In Figure 3.2 the heat map of the Lyapunov exponents is shown for the same parameter values as in Figure 3.1. In the closed case these small kicking strengths are characteristic of the transition from almost fully regular dynamics for $k = 0.5$, to mixed dynamics for $k = 1.1$. We observe that the effect of γ is to introduce chaos into a much larger phase-space area when compared to that of the closed system. When the kicking strength is at $k = 1.1$ the effect of $\gamma = 0.001$ is less dramatic, but we still observe an in-

crease in chaos in the values of the Lyapunov exponent. For $\gamma = 0.3$ we find widespread chaos except for regions close to the elliptic fixed point. Interestingly, the chaotic sea for $k = 1.1$ in this case is somewhat smaller than that for $k = 0.5$.

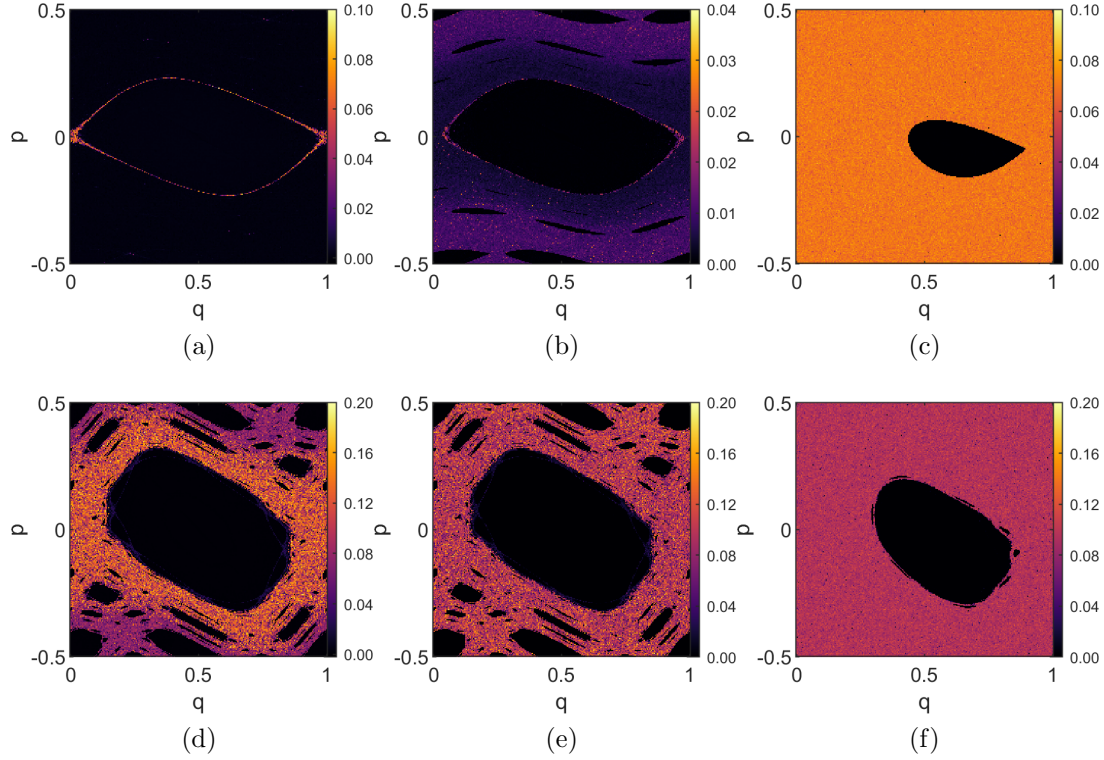


Figure 3.2: Heat maps of the Lyapunov exponent for the PT-kicked rotor for different values of γ and k . From left to right $\gamma = 0$, $\gamma = 0.001$ and $\gamma = 0.3$. In the top row the kicking strength is $k = 0.5$ and in the bottom row $k = 1.1$. The Lyapunov heat maps are constructed from an initial distribution of $N^2 = (250)^2$ points uniformly distributed across the phase space. The Lyapunov exponents are averaged for $t = 2000$ iterations of the forwards map (3.3) after an initial propagation of $t_0 = 1000$ steps.

In Figure 3.3 we show the Poincaré sections for the same two γ values further into the chaotic regime for $k = 2.5$ and $k = 5$. The effect of γ here is less noticeable between the closed case, $\gamma = 0$, and the case $\gamma = 0.001$. Again we observe that when $\gamma = 0.3$ the chaotic phase space area is enlarged for $k = 2.5$. In Figure 3.3f we have $k = 5$ and the fixed point has become hyperbolic. There is nothing left to protect the regular islands and the dynamics becomes fully chaotic.

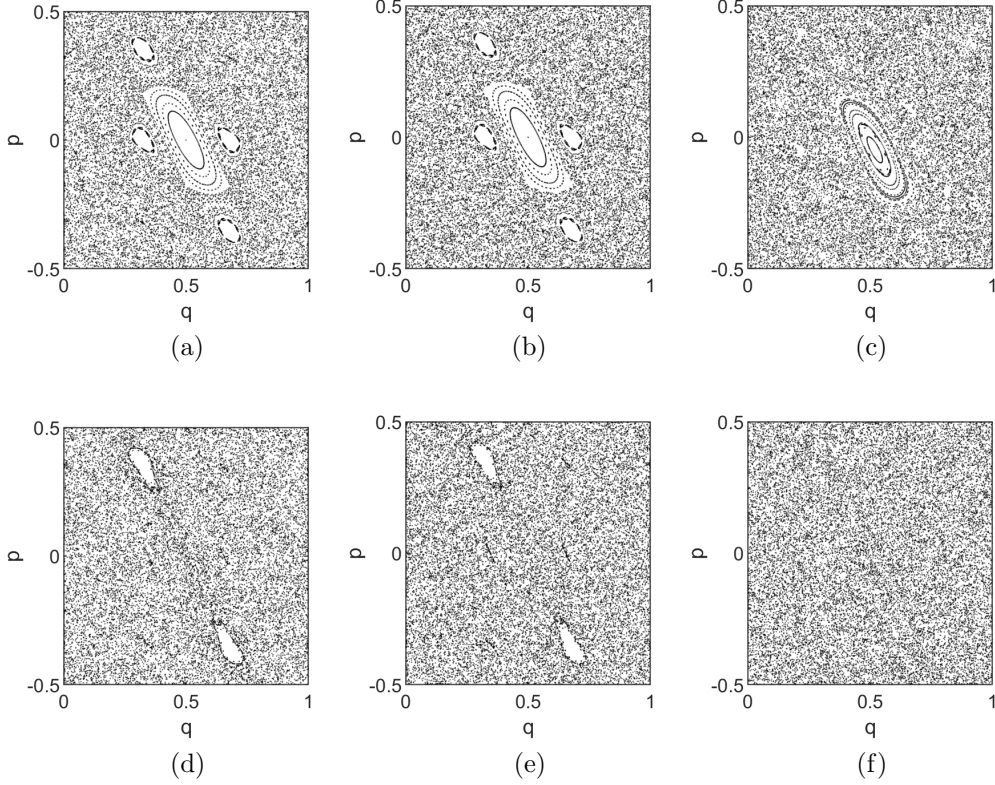


Figure 3.3: Poincaré sections of the PT-symmetric kicked rotor for different values of non-Hermitian term γ and kicking strength k . From left to right $\gamma = 0$, $\gamma = 0.001$ and $\gamma = 0.3$. In the top row the kicking strength is $k = 2.5$ and in the bottom row $k = 5$. Each Poincaré section is generated by iterating $N^2 = (15)^2$ initial conditions under $t = 50$ of the forwards map (3.3).

We visualise the classical norm through heatmaps which we will refer to as norm maps. These are constructed by colour coding the initial conditions of trajectories according to their averaged norm at some final time t_f . To make the distinction clear between the norm maps generated by the forwards (3.7) or backwards (3.8) map the norm maps will be prefaced with forward or backward to indicate the direction of time. Figure 3.4a shows the forward norm map for $k = 1.1$ and $\gamma = 0.001$. The corresponding Poincaré section is shown in Figure 3.4b. Due to the balance of gain and loss we can distinguish between regions in phase space of average norm growth, stability, and decay. By norm stability we are referring to trajectories whose norm on average stays close to unity. In Figure 3.4a these are the reddish regions. We will describe the sets of initial conditions corresponding to these regions as gain

sets, stable sets and loss sets. In Figure 3.4c we show the backward norm map.

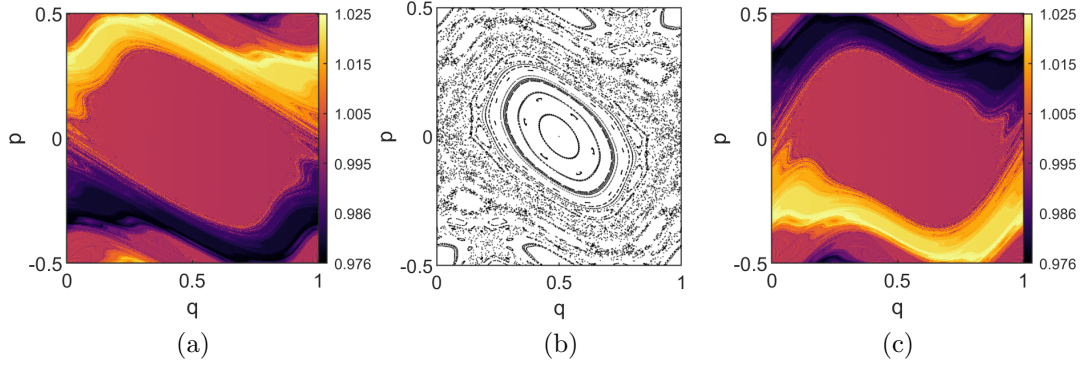


Figure 3.4: Norm maps and Poincaré section of the PT-kicked rotor for kicking strength $k = 1.1$ and $\gamma = 0.001$. The central plot shows the Poincaré section which is the same as in Figure 3.1e. The forwards norm map is shown in the left plot and the backwards norm map in the right. The norm maps are formed from $N^2 = (500)^2$ initial conditions and are calculated with a final time $t_f = 60$.

The distinction of gain and loss sets is independent of the chaotic or regular character of the corresponding trajectory. To illustrate this, consider the backwards norm map in Figure 3.4c. Figure 3.5a shows the initial conditions associated to the regular trajectories of the time-reversed map. These are isolated using the Lyapunov exponent. In Figure 3.5b we show the same set, but remove those points whose norm would place them in the gain and loss sets. Figure 3.5a and Figure 3.5b differ, thus the gain and loss sets contain both regular and chaotic trajectories. Yet, in this particular example there is some connection between regularity and norm behaviour of the trajectories.

Those regular trajectories in Figure 3.5a that belong to the gain and loss sets belong to regular island chains which restricts their motion to the positive or negative momentum regions respectively. This is due to the form of gain loss profile which leads to a monotonic increase or decrease of the norm for trajectories bounded to the upper or lower momentum plane. The trajectories that are both regular and belong to the stable set, the initial conditions of which are depicted in Figure 3.5b, on average spend an equal time in gain and loss regions. The same applies to chaotic trajectories with stable norm behaviour.

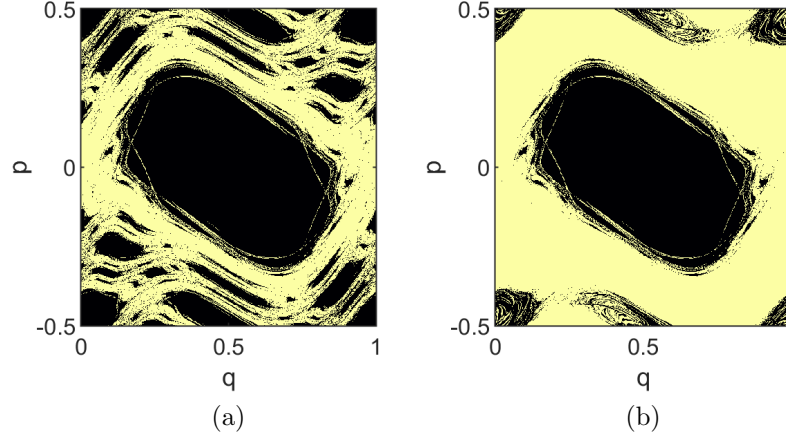


Figure 3.5: Regular trajectories (black) of the PT-symmetric kicked rotor for kicking strength $k = 1.1$ and $\gamma = 0.001$ and their correspondence to the stable set of the backwards norm map. The left plot shows the subset of $N^2 = 500^2$ initial conditions associated to regular trajectories. These are calculated from the Lyapunov heat map with $t_0 = 1000$ and $t = 2000$. The right plot shows the subset of these that are also in the stable set of the backwards norm map.

In Figure 3.6 we show the forward (left) and backwards (right) norm maps for two values of the kicking strength $k = 5$ (top) and $k = 10$ (bottom). For the case $k = 5$ the Poincaré section Figure 3.6b is mixed and we observe little structure, while for the case $k = 10$ the Poincaré section 3.6e is dominated by a single featureless chaotic sea. In stark contrast to this, both the forward and backwards norm maps show a rich structure within the phase-space area associated to the chaotic sea. We note that the structures visible in the forward and backward norm maps show a strong similarity to the stable and unstable manifolds of the hyperbolic fixed points we encountered in the closed kicked rotor - a comparison we can make only because of the size of γ .

Let us now look at what is hidden beneath the chaos in the norm maps for the case $\gamma = 0.3$. Figure 3.7 shows the backward norm maps corresponding to the kicking strengths $k = 0.5$, $k = 1.1$, $k = 5$ and $k = 10$. For $k = 0.5$ (Figure 3.7a) and $k = 1.1$ (Figure 3.7b) we observe a rich and seemingly fractal, structure in the chaotic seas surrounding the regular islands. It is interesting to note that the underlying structures emerging in the backward norm map Figure 3.7c for $k = 5$ and 3.7d for $k = 10$ are quite similar to those seen in the case of $\gamma = 0.001$, as in this case the kicking term dominates the

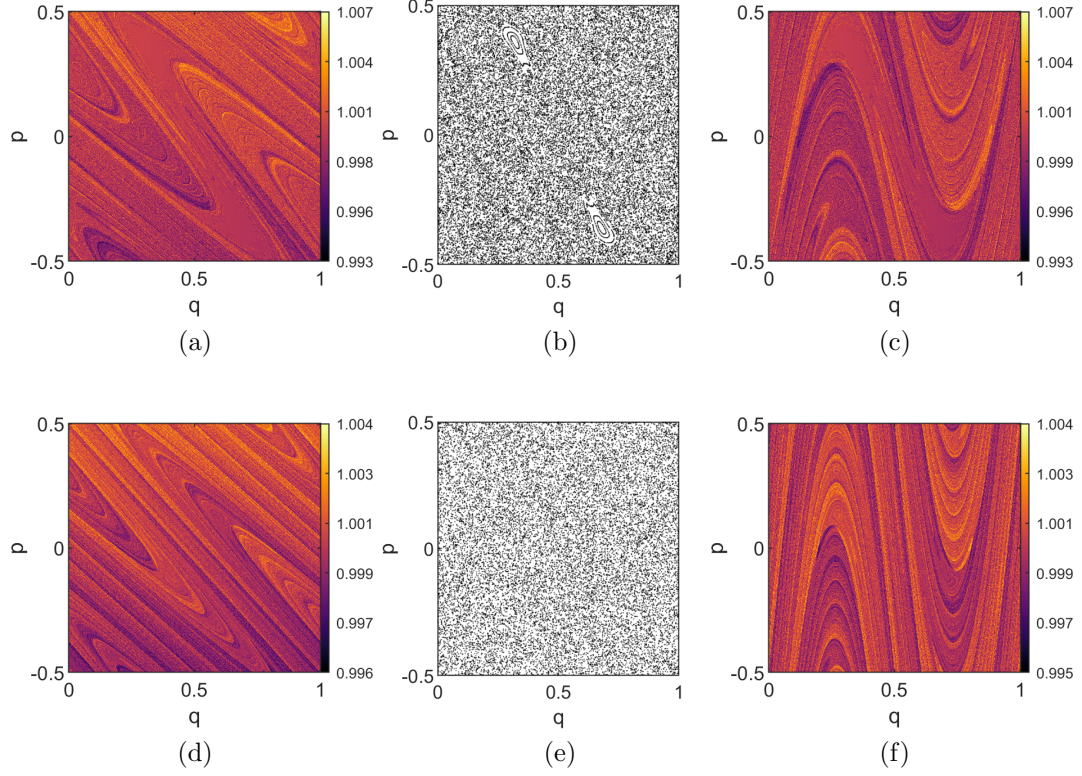


Figure 3.6: Norm maps and Poincaré section of the PT-kicked rotor with $\gamma = 0.001$ and two values of the kicking strength k . The top row corresponds to the kicking strength $k = 5$ and the bottom row to $k = 10$. The forwards norm map is shown on the left and the backwards norm map on the right with each having $N^2 = (500)^2$ initial conditions and a final time $t_f = 22$. The central plot in each case shows the Poincaré section generated from $N^2 = 25$ initial conditions and $t = 50$ forward iterations of the PT-kicked rotor map.

dynamics and both values of γ can be viewed as small perturbations.

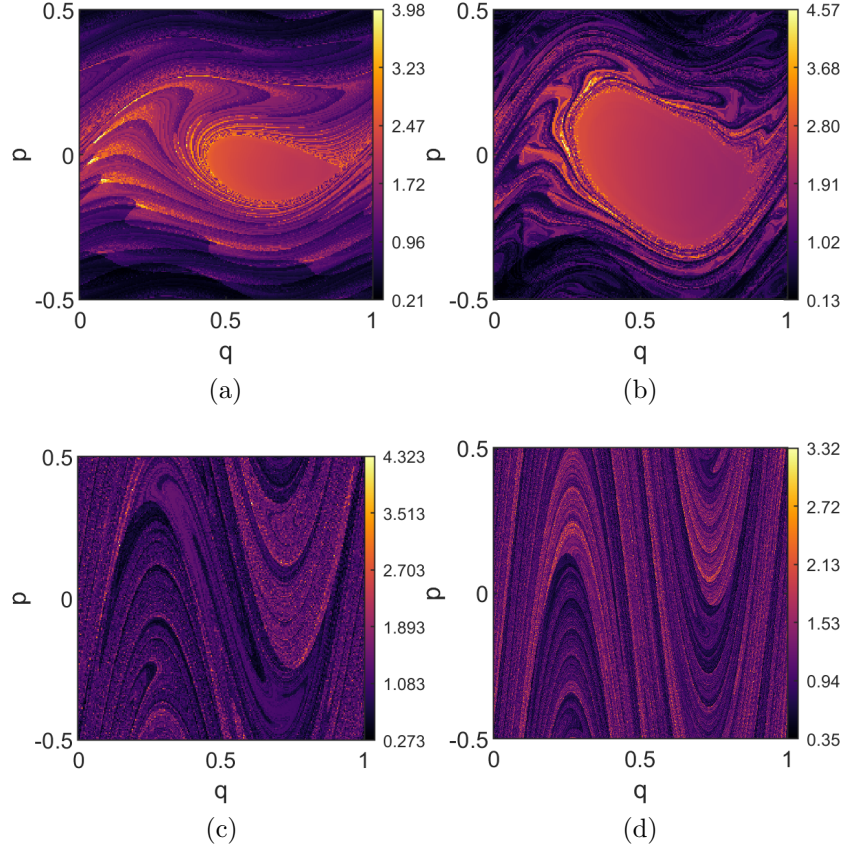


Figure 3.7: Backward norm maps of the PT-kicked rotor for $\gamma = 0.3$ and various values of the kicking strength k . From left to right the kicking strength is $k = 0.5$, $k = 1.1$, $k = 5$ and $k = 10$. The backwards norm maps are generated from $N^2 = (500)^2$ initial conditions and have final times $t_f = 50$ (top left), $t_f = 50$ (top right), $t_f = 15$ (bottom left) and $t_f = 10$ (bottom right).

3.1.3 The Quantum Model

Our PT-symmetric Hamiltonian (3.1) is still time periodic and we again study the quantum map $\hat{F}$ describing the free-evolution followed by a kick

$$\hat{F} = \hat{U}_k \hat{U}_{nk}. \quad (3.9)$$

The operator for the free motion $\hat{U}_{nk}$ is now given by

$$\hat{U}_{nk} = e^{-\frac{i}{2\hbar}\hat{p}^2 + \frac{\gamma}{2\pi\hbar}\hat{p}}, \quad (3.10)$$

while the kicking part $\hat{U}_k$

$$\hat{U}_k = e^{-\frac{ik}{4\pi^2\hbar} \cos(2\pi\hat{q})}, \quad (3.11)$$

remains unchanged by the gain and loss. Due to our asymmetric choice of time frame, the map $\hat{F}$ is not PT-symmetric in the strict sense, but is a similarity transform of a PT-symmetric map $\tilde{F}$ given by

$$\tilde{F} = U_{nk}^{1/2} U_k U_{nk}^{1/2}. \quad (3.12)$$

We have

$$\begin{aligned} \hat{F} &= U_k U_{nk} \\ &= U_{nk}^{-\frac{1}{2}} U_{nk}^{\frac{1}{2}} U_k U_{nk}^{\frac{1}{2}} U_{nk}^{\frac{1}{2}} \\ &= U_{nk}^{-\frac{1}{2}} \tilde{F} U_{nk}^{\frac{1}{2}}, \end{aligned} \quad (3.13)$$

thus, $\hat{F}$ shares the same spectrum as the PT-symmetric map $\tilde{F}$. This procedure of identifying PT-symmetry in quantum maps with asymmetric time frames is related to the concept of *symmetry time frames* introduced in [69].

The fact that the non-Hermitian term is a function of $\hat{p}$ in our PT-symmetric Hamiltonian (3.1) has the advantage of $\hat{U}_k$ ($\hat{U}_{nk}$) again being diagonal in the position (momentum) basis. The procedure of quantisation of the Floquet operator is thus analogous to the closed case described in section 2.1.2. The matrix elements in the discrete position basis $\{|q_l\rangle\}$ are now given by

$$F_{ll'} = \frac{1}{N} e^{-\frac{ikN}{2\pi} \cos(\frac{2\pi l}{N})} \left(\sum_m e^{-\frac{iN}{4\pi} m^2 + \gamma m + \frac{2\pi i}{N} m(l-l')} \right), \quad (3.14)$$

where k is the classical kicking strength and $N = 2N_1 + 1$, with $N_1 \in \mathbb{N}$, is the dimension of the Hilbert space $\mathcal{H}_N$. The Floquet states $|\phi_n\rangle$ satisfy

$$\hat{F} |\phi_n\rangle = \lambda_n |\phi_n\rangle, \quad (3.15)$$

where the quasienergies ϵ_n can be obtained by taking the logarithm of the eigenvalues $\lambda_n = e^{-i\epsilon_n}$. The transition from “unbroken” to “broken” PT-symmetry manifests through the quasienergies. Writing the quasienergies as $\epsilon_n = \theta_n + i\mu_n$, the “unbroken” regime $\epsilon_n \in \mathbb{R}$ corresponds to $\theta_n \in \mathbb{R}$ and $\mu_n = 0$ for each ϵ_n . In the “broken” regime either: $\epsilon_n \in \mathbb{R}$ and $\mu_n = 0$ or $\epsilon_n \in \mathbb{C}$ and $\exists \epsilon_{n'} \in \mathbb{C}$ such that $\epsilon_n^* = \epsilon_{n'}$ i.e., $\mu_n = -\mu_{n'}$.

3.1.4 Spectral Properties

In practice, the value of γ is restricted by the numerical sensitivity of the quantum systems to non-Hermitian perturbations [24, 70]. If γ is too large, the spectrum obtained by numerical diagonalization will be “far away” from the true spectrum. This process is known as *spectral pollution* [71]. To avoid this problem we restrict the choice of γ to be $\gamma \leq 0.003$ throughout our discussion of the quantum PT-symmetric kicked rotor. We will mainly focus on the case of $\gamma = 0.001$. As we will see this is large enough for some interesting physics, but also keeps us safe from spectral instabilities.

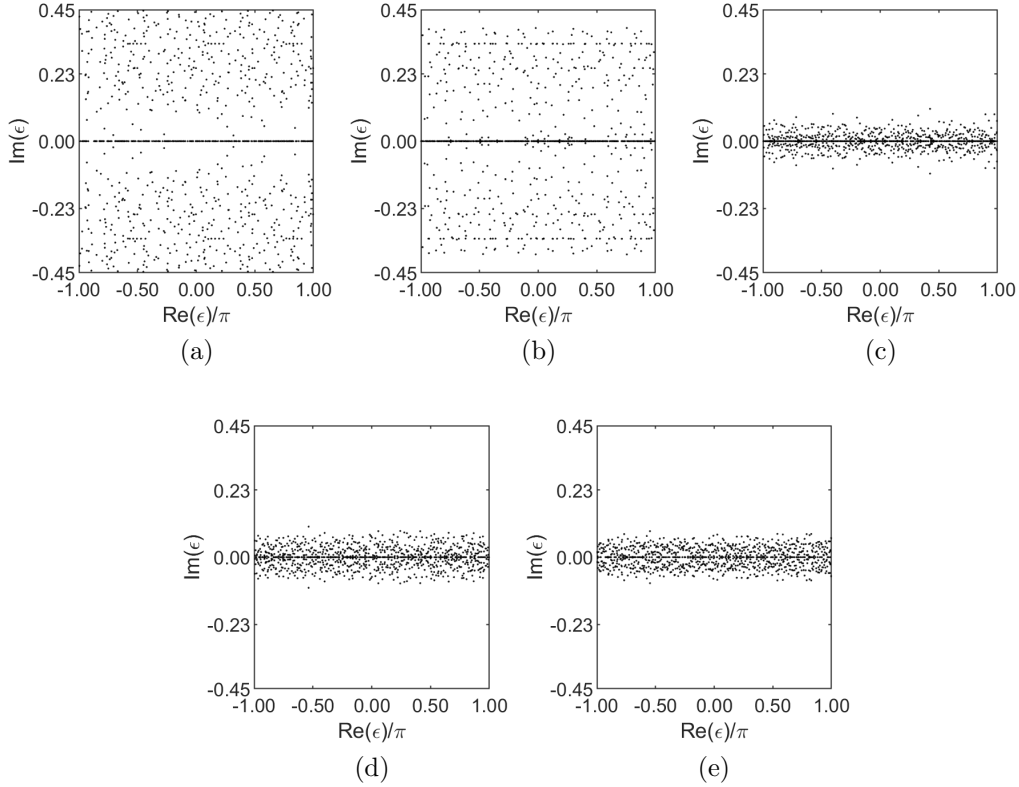


Figure 3.8: Quasienergy components of the PT-kicked rotor for different values of the kicking strength k and $\gamma = 0.001$. From top left to bottom right $k = 0.5$, $k = 1.1$, $k = 2.5$, $k = 5$ and $k = 10$. For each kicking strength the matrix dimension of the quantum system is $N = 1001$.

We will now form our first link between the quantum system and the norm maps by showing that the scaling of the eigenvalues with largest imaginary parts as a function of the kicking strength follows the same trend as the

largest value of the norm. Figure 3.8 shows the typical behaviour of the quasienergy spectrum for $\gamma = 0.001$, matrix dimension $N = 1001$, and different values of the kicking strength corresponding to the different chaotic regimes. The quasienergies are depicted in the complex plane. We see that even for this small value of γ , which classically had a relatively little effect on the dynamics, the PT-symmetry is in the broken regime for all values of the kicking strength. For larger values of the kicking $k = 2.5$, $k = 5$ and $k = 10$, there is a noticeable decrease in the largest imaginary part.

We investigate this further in Figure 3.9a, where the largest imaginary part μ_+ is plotted for $k \in [0, 10]$. The main feature of the observed dependence of the largest imaginary part on the kicking strength can be understood by considering the classical norm dynamics. For this purpose we consider the backwards norm map for $k = 1.1$, displayed in Figure 3.4c. In the corresponding Poincaré dynamics we observed that there were islands of regular trajectories confined to the upper and lower momentum plane. Accordingly this confinement means that the classical norm, $w_{n+1} = e^{2\gamma p_{n+1}} w_n$, will grow/decay monotonically in these regions. For larger values of the kicking strength the classical phase space becomes more chaotic and these regular regions disappear. Thus, trajectories will no longer be confined to these regions of large $|p|$ and the relative rate of maximum norm increase is slower.

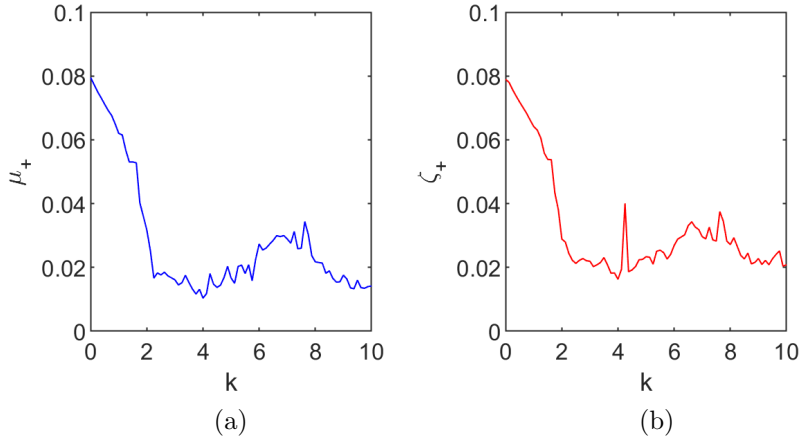


Figure 3.9: The maximum value of the imaginary part of the quasienergy (left) and the largest value of the logarithm of the classical norm (right) of the PT-kicked rotor as a function of the kicking strength k . In each plot the kicking strength $k \in [0, 10]$ is sampled uniformly at intervals of $\delta k = 0.125$. The quantum system has a matrix dimension $N = 1001$. The classical norm is calculated from $N^2 = 500$ initial conditions with a final time $t_f = 80$.

We argue that the quantum norm, which grows or decays with a rate determined by $\mu_n = \text{Im}(\epsilon_n)$, reflects this classical norm rate $\zeta := \ln(w_{n+1}) = 2\gamma\Delta$ where $\Delta \in \mathbb{R}$. We demonstrate this in Figure 3.9b where we plot the largest value of the logarithm of the classical norm ζ_+ at a final time $t_f = 80$, chosen such that trajectories will likely be following their long term norm behaviour. We observe that the general shape of the curve follows the quantum curve (Figure 3.9a), and that both curves have approximately the same minima around $k = 4$. In Figure 3.10a we again plot the value of the classical norm but now with different final times for each kicking strength. For each kicking strength the final time, displayed in Figure 3.10b, is chosen such that it minimises the difference between the quantum μ_+ and the classical ζ_+ . We will revisit this association between different kicking strengths and final times in the next chapter.

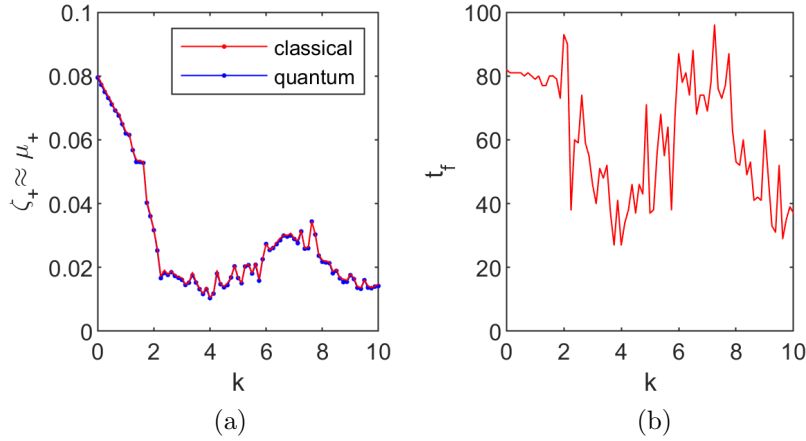


Figure 3.10: The classical norm parameter ζ fitted to the quantum data displayed in Figure 3.9a (left) and the corresponding final times (right) for each value of the kicking strength k .

In a similar manner to our considerations of the classical norm behaviour we can form three natural sets of states based on the growth/decay rates of the quantum norm $w(t)$

$$\lim_{t \rightarrow \infty} w(t) = \begin{cases} \infty & : \mu_n = \text{Im}(\epsilon_n) \in \mathbb{R}^+ \\ 1 & : \mu_n = \text{Im}(\epsilon_n) = 0 \\ 0 & : \mu_n = \text{Im}(\epsilon_n) \in \mathbb{R}^- \end{cases} \quad (3.16)$$

In general we will refer to states as *gain states* if $\mu_n > 0$, *stable states* if $\mu_n = 0$, and *loss states* if $\mu_n < 0$. Since this ordering is based on the

eigenvalues is also applies to the Schur vectors whose construction preserves the structure of the eigenvalues. When dividing the states in gain, stable, and loss sets, we need to be mindful of numerical error. The condition for the stable set, $\mu_n = 0$, has to be related to $\mu_n = \pm\delta$, for a small parameter δ to account for numerical inaccuracies. In general we use a cut-off parameter of the order 10^{-6} . We have verified that these results are robust in a large range $10^{-4} \geq \delta \geq 10^{-12}$.

We have established a connection to the norm maps through the scaling of the eigenvalue with largest imaginary part. Further, we have made an argument about the region of the norm map this Schur vector is associated too. As we will see later in this section this correspondence is a general one and the norm maps have already shown us the underlying structures the Schur vectors will localise on. We will now demonstrate how this information can be obtained from the scaling of the number of Schur vectors associated to the gain, stable, and loss sets.

In the top row of Figure 3.11 we depict the fraction of quasienergies, $p(\mu)$, whose imaginary part $\mu_n = \text{Im}(\epsilon_n)$ is greater than μ . The various curves depict differing matrix dimension $N = 2N_1 + 1$ where $N_1 = 200$ (black), $N_1 = 400$ (red), $N_1 = 600$ (green), $N_1 = 800$ (blue), $N_1 = 1000$ (magenta). The imaginary parts μ_n scale with the matrix dimension, to account for this we rescale the eigenvalues in the spectrum by their respective largest imaginary part. This results in imaginary parts falling into the interval $\mu_n \in (-1, 1)$. The three sets of interest will correspond to the following sub intervals: loss states $\mu \in [1, -\delta)$, stable states $\mu \in [\delta, \delta]$ and gain states $\mu \in (\delta, 1]$, where δ denotes the numerical cut-off parameter. In the bottom row we depict the fraction of states belonging to the gain set as a function of the matrix dimension.

In Figure 3.11 the left column corresponds to a kicking strength $k = 1.1$. Classically this corresponds to the mixed phase space and forwards and backwards norm maps depicted in Figure 3.4. We observe that the relative fraction of states as a function of μ changes little with N . Also the fraction of states in the gain state varies relatively little between $N_1 = 500$, and $N_1 = 1000$. This suggests that for $k = 1.1$, the bulk of the spectrum follows the standard Weyls law. From this we may infer that the underlying phase space area associated to the sets of gain, stable, and loss states will have on average a non-fractal nature.

In the middle column the kicking strength is $k = 5$, and in the right column $k = 10$. The corresponding classical dynamics are largely chaotic for $k = 5$, and fully chaotic for $k = 10$. The corresponding Poincaré sections together

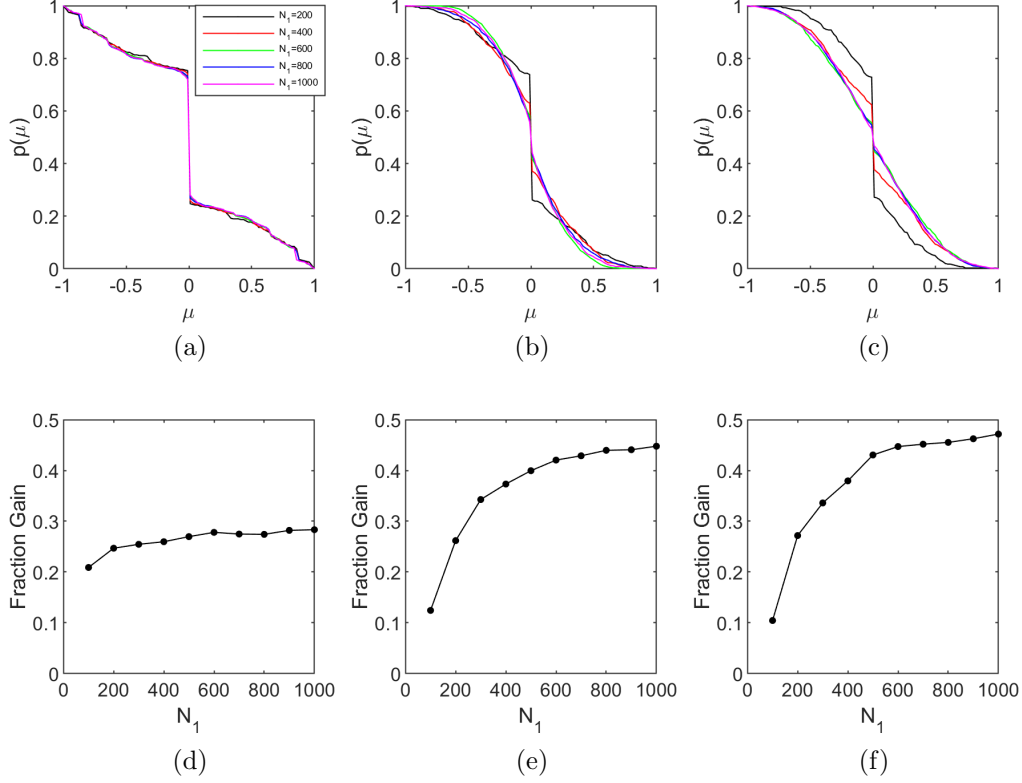


Figure 3.11: Scaling of the gain, stable, and loss sets as a function of matrix dimension. In the top row we show the Fraction of Schur states with $\mu > \mu_n$ for matrix dimension $N = 2N_1 + 1$ with: $N_1 = 200$ (black), $N_1 = 400$ (red), $N_1 = 600$ (green), $N_1 = 800$ (blue) and $N_1 = 1000$ (magenta). Bottom: Fraction of total states in the gain set for various N_1 . From left to right: $k = 1.1$, $k = 5$, $k = 10$ $\gamma = 0.001$.

with the associated norm maps are depicted in Figure 3.6 and Figure 3.6e, respectively. In both cases we observe that the relative fraction of states belonging to the stable set has a sensitive dependence on the matrix dimension. This is reflected in the fraction of states associated to the gain sets. From this behaviour we infer that the phase space area associated to the gain states is increasing as the matrix dimension increases ($\hbar$ is getting smaller). This suggests that geometrically the dimension of the underlying structure is not scaling as an integer, i.e their underlying structure is a fractal. This observation links into what appears to be fractal structures in the underlying classical norm maps.

3.1.5 The Husimi Distributions of the Schur Vectors

We make a distinction between the orderings of the Schur vectors based on their leading eigenfunction $|\phi_{\pm}\rangle$. Recall that we define these orderings as follows

$$|\phi_{\pm}\rangle \begin{cases} |\phi_{+}\rangle & : |\lambda_1|^2 > |\lambda_2|^2 > \dots > |\lambda_N|^2 \\ |\phi_{-}\rangle & : |\lambda_1|^2 < |\lambda_2|^2 < \dots < |\lambda_N|^2 \end{cases} \quad (3.17)$$

Of the 2 orderings, we consider the more fundamental to be the one with leading eigenfunction $|\phi_{+}\rangle$. Under this ordering, the orthogonalisation procedure builds up a series of invariant subspaces spanned by the longest lived states. An arbitrary quantum state expanded in the Schur basis will thus decay into these invariant subspaces. To view the phase-space localisation of the gain, stable, and loss sets we plot the summed Husimi distribution of all the Schur vectors associated to a given set. Since the Schur vectors are an orthonormal set the summed Husimi distribution of all of the Schur vectors would be the uniform density on the whole phase space.

We begin with the case $k = 1.1, \gamma = 0.001$ the classical Poincaré section of which was depicted in Figure 3.1b, and showed a mixed phase space with a large regular region. In Figure 3.12 we show the Husimi distribution of the Schur vectors with leading eigenfunction $|\phi_{+}\rangle$ belonging to the gain set in Figure 3.12a, the stable set in Figure 3.12b, and loss set Figure 3.12c. There is a clear association between the three sets and the regions of norm growth, stability, and loss in the norm map Figure 3.4c. It is interesting to note that these Husimi distributions correspond to the backwards norm map, formed from the time-reversed classical map, with the gain states corresponding to the regions of backwards decaying norm, and the loss set to the region of backwards growing norm.

In Figure 3.13 we depict the Husimi distributions of the three sets of Schur vectors, when sorted according to increasing imaginary part, i.e with leading eigenfunction $|\phi_{-}\rangle$. Again we observe a clear correspondence to the norm map, but this time to the forwards propagation (as depicted in Figure 3.4a). This correspondence to the structures arising from forward and backwards dynamics is similar, but distinct, from the correspondence to the stable and unstable manifolds of the chaotic saddle in the kicked rotor with an opening.

Let us further consider the more chaotic case $k = 5, \gamma = 0.001$ where the Poincaré section and the norm maps, depicted in Figure 3.6, showed different structures. The classical Poincaré section (Figure 3.6b) has two regular islands surrounded by a large chaotic sea. In Figure 3.14 we show the Husimi

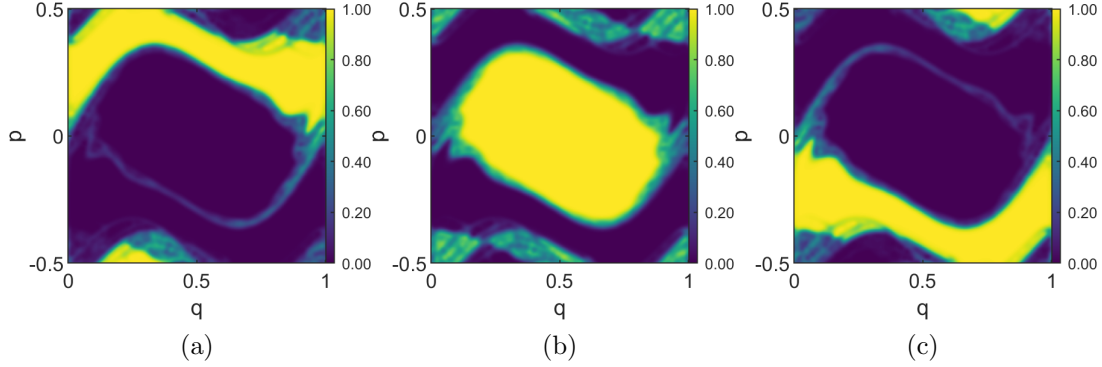


Figure 3.12: Husimi distributions of the Schur vectors of the quantum PT-symmetric kicked rotor with $\gamma = 0.001$ and $k = 1.1$ and matrix dimension $N = 1001$. The Husimi distributions are over the quantum gain (left), stable (middle), and loss (right) sets with leading eigenfunction $|\phi_+\rangle$.

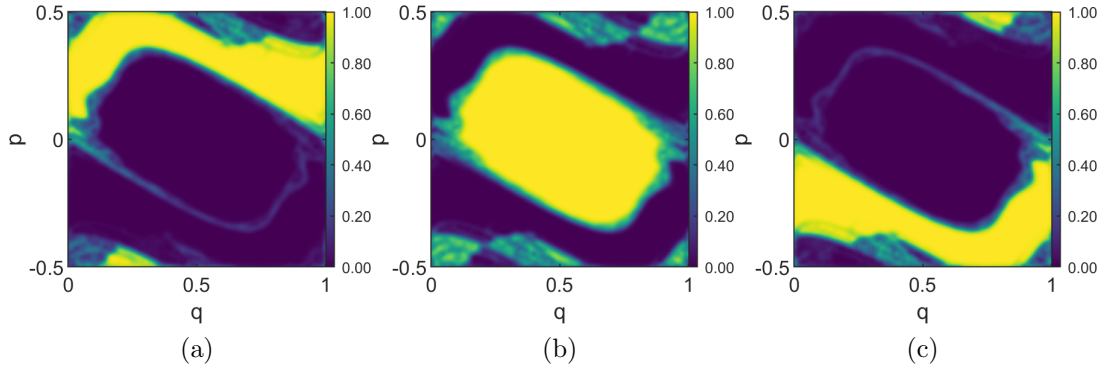


Figure 3.13: Husimi distributions of the Schur vectors of the quantum PT-symmetric kicked rotor with $\gamma = 0.001$ and $k = 1.1$ and matrix dimension $N = 1001$. The Husimi distributions are over the quantum gain (left), stable (middle), and loss (right) sets with leading eigenfunction $|\phi_-\rangle$.

distribution of the Schur vectors with leading eigenfunction $|\phi_+\rangle$ belonging to the gain set (Figure 3.14a), the stable set (Figure 3.14b), and loss set (Figure 3.14c). We again find a clear to association to the backwards norm map of the subsystem.

Similarly, in Figure 3.15, where the leading eigenfunction is $|\phi_-\rangle$ we find the same connection to the classical forwards norm map in Figure 3.6a. Thus despite the largely chaotic dynamics in the phase space the Schur vectors associated to these regions are highly structured. This is in contrast to the

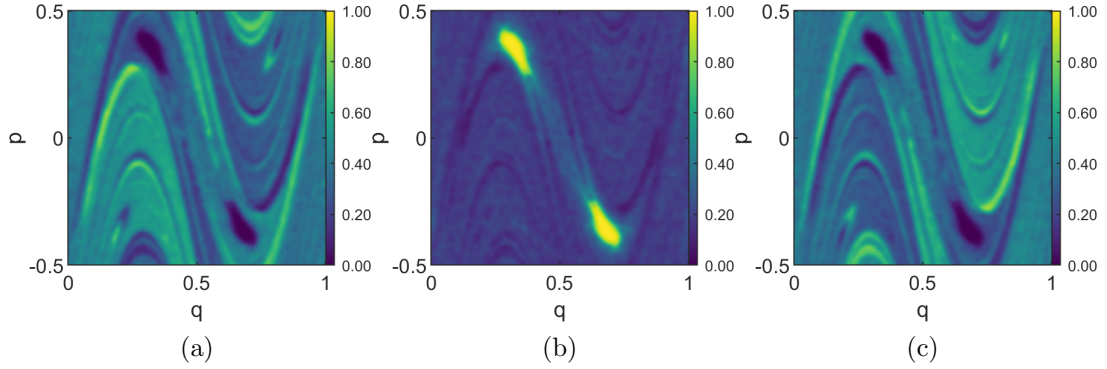


Figure 3.14: Husimi distributions of the Schur vectors of the quantum PT-symmetric kicked rotor with $\gamma = 0.001$ and $k = 5$ and matrix dimension $N = 1001$. The Husimi distributions are over the quantum gain (left), stable (middle), and loss (right) sets with leading eigenfunction $|\phi_+\rangle$.

closed case where the states supported on the area associated to chaotic trajectories were delocalised over the chaotic sea.

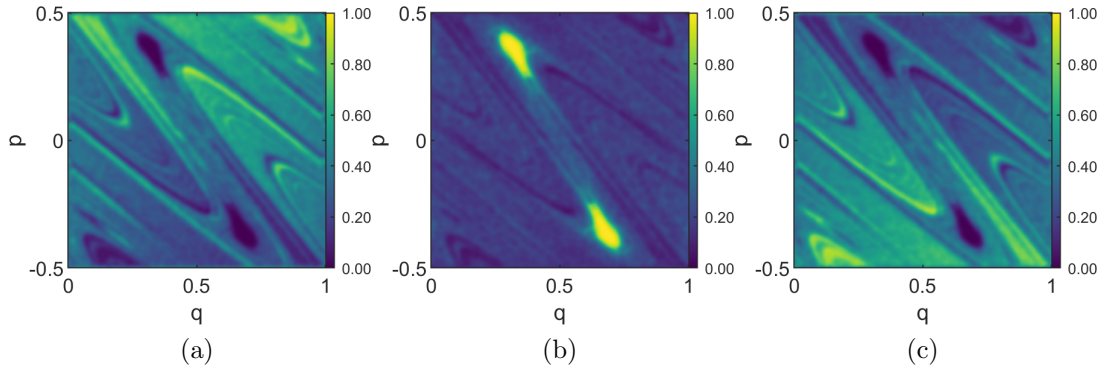


Figure 3.15: Husimi distributions of the Schur vectors of the quantum PT-symmetric kicked rotor with $\gamma = 0.001$ and $k = 5$ and matrix dimension $N = 1001$. The Husimi distributions are over the quantum gain (left), stable (middle), and loss (right) sets with leading eigenfunction $|\phi_-\rangle$.

The localisation of the Schur vectors on the fractal structures associated to the norm maps for $k = 5$ is linked to the scaling behaviour of the spectral counting functions in Figure 3.11. The reduction in the fraction of states belonging to the stable set as a function of the matrix dimension reflects the fractal nature of the support of the gain and loss sets. That the fraction of gain states Figure 3.11e saturates below $f = 0.5$, and thus the fraction

of stable states saturates at a non-zero value, can be understood through the support of the stable states in the Husimi distributions 3.14b and 3.15b. We see that in these cases the regions of highest localisation in the Husimi distributions corresponds to those regions of phase space associated to regular islands in the classical dynamics. The states that are supported on these regions cannot vanish and thus their relative fraction will follow a standard Weyl scaling for large matrix dimension.

We conjecture that the correspondence between the backward (forward) norm maps and the Schur vectors with leading eigenfunction $|\phi_+\rangle$ ($|\phi_-\rangle$) is a general relation. In the coming chapter we will provide in the coming chapter evidence for this from multiple examples. Let us finish this chapter providing some analytic arguments that might support this conjecture.

3.2 Schur Vectors and the Norm

We see a clear identification between the Schur vectors and the norm maps. When the leading eigenfunction is $|\phi_-\rangle$ this is to the forward norm map. When the leading eigenfunction is $|\phi_+\rangle$, the backwards norm map. Here we provide a sketch of an analytical argument that will demonstrate the association between the classical backwards norm map and the quantum Schur vectors. We will argue that for eigenvalues with non-degenerate imaginary parts the Schur vectors sorted according to decreasing imaginary parts are eigenvectors of the operator $\hat{W} = \hat{U}(t)\hat{U}^\dagger(t)$ in the limit of $t \rightarrow \infty$. This operator, which correspond to the singular-value decomposition, is the quantum analogue of the classical backwards norm map.

Let us denote the right eigenvectors of $\hat{U}(t)$ by $|\phi_n\rangle$ and the corresponding eigenvalues by $e^{-i\lambda_n t}$. Note that the $|\phi_n\rangle$ do not depend on time. We denote the corresponding left eigenvectors by $|\chi_n\rangle$. Using the normalisation convention $\langle\phi_n|\chi_m\rangle = \delta_{nm}$, we can write $\hat{U}$ and $\hat{U}^\dagger$ in a non-normal version of the spectral decomposition as

$$\hat{U}(t) = \sum_m e^{-i\lambda_m t} |\phi_m\rangle \langle\chi_m|, \quad (3.18)$$

and

$$\hat{U}^\dagger(t) = \sum_n e^{i\lambda_n^* t} |\chi_n\rangle \langle\phi_n|. \quad (3.19)$$

Thus, we have

$$\hat{W}(t) = \hat{U}\hat{U}^\dagger = \sum_{m,n} e^{-i(\lambda_m - \lambda_n^*)t} |\phi_m\rangle \langle\chi_m|\chi_n\rangle \langle\phi_n|. \quad (3.20)$$

We will now argue that the naturally ordered Schur vectors of the time evolution operator $\hat{U}$ become the eigenvectors of the operator $\hat{W}(t)$ in the limit $t \rightarrow \infty$. For this purpose, we will first express the Schur vectors $|\nu_j\rangle$ in the non-orthogonal basis of left eigenvectors $|\chi_j\rangle$ of $\hat{U}$, to make use of the biorthogonality when evaluating matrix elements of $\hat{W}$. We start with an ordered set of eigenvectors, where the ordering is through the eigenvalues $\lambda_n = \epsilon_n + i\mu_n$ such that $|\mu_1| > |\mu_2| > \dots > |\mu_N|$ is fulfilled. The first Schur vector $|\nu_1\rangle$ is then chosen as the longest lived eigenvector $|\phi_1\rangle$, which, depending on the system can be a superposition of all left eigenvectors that is

$$|\nu_1\rangle = |\phi_1\rangle = \sum_{n=1}^N a_n^{(1)} |\chi_n\rangle. \quad (3.21)$$

The second Schur vector is the normalised component of $|\phi_2\rangle$ orthogonal to $|\nu_1\rangle$. That means it does not have a contribution from $|\chi_1\rangle$, which is the only one of the left eigenvectors that is not orthogonal to $|\phi_1\rangle$, and we have

$$|\nu_2\rangle = \sum_{n=2}^N a_n^{(1)} |\chi_n\rangle. \quad (3.22)$$

Constructing the third Schur vector as the component of $|\phi_3\rangle$ that is orthogonal to the first two, and so on, we find that the j^{th} Schur vector is a linear superposition of the left eigenvectors $|\chi_n\rangle$ with $n \geq j$, and the first $j - 1$ left eigenvectors have no contribution,

$$|\nu_j\rangle = \sum_{n=j}^N a_n^{(j)} |\chi_n\rangle. \quad (3.23)$$

Applying $\hat{W}$ to a Schur vector $|\nu_k\rangle$ we find

$$\hat{W}|\nu_k\rangle = \sum_{l=1}^N \sum_{l'=k}^N e^{(\mu_l + \mu_{l'})t} a_{l'}^{(k)} e^{-i(\epsilon_l - \epsilon_{l'}^*)t} \langle \chi_l | \chi_{l'} \rangle |\phi_m\rangle. \quad (3.24)$$

Projecting onto another Schur vector we find the matrix elements of the operator $\hat{W}$ in the Schur basis

$$\langle \nu_j | \hat{W} | \nu_k \rangle = \sum_{l=j}^N \sum_{l'=k}^N e^{(\mu_l + \mu_{l'})t} a_l^{(j)*} a_{l'}^{(k)} e^{-i(\epsilon_l - \epsilon_{l'}^*)t} \langle \chi_{l'} | \chi_l \rangle. \quad (3.25)$$

For large values of t each of these is dominated by a single term and the matrix elements of $\hat{W}$ have the asymptotic form

$$\langle \nu_j | \hat{W} | \nu_k \rangle \sim c_{jk} e^{(\mu_j + \mu_k)t}, \quad (3.26)$$

where the c_{jk} are complex numbers with $c_{jk} = c_{kj}^*$ that do not depend on time. While this does not appear diagonal, we can argue how the $|\nu_k\rangle$ emerge as eigenvectors in an asymptotic way. Let us divide by the dominant term $e^{2\mu_1 t}$. Then all elements of the matrix, apart from the first diagonal element exponentially approach zero. Thus, clearly the vector $|\nu_1\rangle$ is an eigenvector for large times. Other than that, the fastest decaying matrix elements are those of the last row and column. Thus, for sufficiently large t the vector $|\nu_N\rangle$ approximately becomes an eigenvector with eigenvalue zero. Constraining to the space orthogonal to the vector $|\nu_N\rangle$, the fastest decaying elements are now those of the second to last row and column, and we can argue that the vector $|\nu_{N-1}\rangle$ appears as another approximate eigenvector with eigenvalues zero.

That is, in the limit $t \rightarrow \infty$ the operator $\hat{W}(t) = \hat{U}(t)\hat{U}^\dagger(t)$ has eigenvectors $|\nu\rangle$. Since this operator is Hermitian we can evoke standard ideas of quantum classical correspondence. I.e. the eigenvectors of the operator $\hat{W}$ are organised according to the features of the classical phase space function associated to $\hat{W}$. We define the classical phase-space function as $\langle z | \hat{W} | z \rangle$ in leading order, where $|z\rangle$ denotes a coherent state. This corresponds to the norm of a backwards evolved state in a semiclassical limit. Thus under this ordering the correspondence of the Schur vectors is to the classical norm evolved *backwards in time*.

Chapter 4

Classical Density for Non-Hermitian Systems

In this chapter we will build on the conjectured correspondence between the Husimi distributions of Schur vectors and the norm maps introduced in the previous chapter. We will introduce an algorithm that generates a classical density that closely resembles the quantum Husimi distributions. We will demonstrate its validity for a number of example systems.

We will begin with a general outline of the algorithm which is rooted in a quantum state counting argument. We will then apply the algorithm to a simple non-Hermitian Hamiltonian, the harmonic oscillator with complex frequency. This Hamiltonian is a normal operator, its eigenvectors and Schur vectors are the eigenvectors of the standard harmonic oscillator. The Husimi distribution of the harmonic oscillator eigenstates presents a simple geometry in phase space and will serve as a simple illustration of the state counting argument intrinsic to the density calculation. While the classical density does not exactly reproduce the quantum Husimi distributions, it accurately captures the main features in this simple case.

We will then return to the PT-symmetric kicked rotor, where we will focus mostly on the classical density associated to the Husimi distributions of the gain, stable, and loss sets introduced in the previous chapter. This will be done for a range of kicking strengths corresponding to various degrees of chaos in the classical phase space. We will close this section by demonstrating that the correspondence also applies to subsets of the gain and loss sets. Following the PT-symmetric kicked rotor we will return to the example of the kicked rotor with escape through an opening discussed previously. This will serve

as another benchmark for the classical density.

As a final example system we will introduce a novel approach to the classical Duffing oscillator - a coherent state approximation of a non-Hermitian Hamiltonian. The Duffing oscillator is a dissipative chaotic system. The dissipation enters in the model in the form of friction, which we will model with the anti-Hermitian term of our non-Hermitian Hamiltonian. We will explore the quantum-classical correspondence in the Duffing oscillator and produce one of its famous strange attractors in our Husimi distributions and classical densities.

4.1 The Classical Density Algorithm

4.1.1 The Idea

In the previous chapter we conjectured that the association between the classical norm maps and the Husimi distributions of the Schur vectors was due to the Schur vectors quantising the operator associated to the backward (forwards) norm $\hat{W}$ ($\tilde{W} = \hat{U}^\dagger \hat{U}$). We observed that the phase-space area associated to the gain, stable, and loss sets of the Schur vectors with leading eigenfunction $|\phi_+\rangle$ ($|\phi_-\rangle$) are encoded in the backward (forward) norm maps. Building on this, we view the phase-space area associated to a Husimi distribution of m ordered Schur states as a subset $\mathcal{S}_m$ of the norm map. Consider the set $\mathcal{S}_m$ of initial conditions in the phase space that satisfy

$$\mathcal{S}_m = \left\{ z : \alpha_-^{(m)} \leq \langle w_{tf} \rangle \leq \alpha_+^{(m)} \right\}, \quad (4.1)$$

where $\alpha_-^{(m)}, \alpha_+^{(m)} \in \mathbb{R}$, and $\langle w_{tf} \rangle$ is the averaged norm evaluated at a final time $t = t_f$. A priori the values $\alpha_-^{(m)}, \alpha_+^{(m)}$ are unknown, and will be determined self-consistently. For the moment, let us imagine that we have made an ansatz for the $\alpha_\pm^{(m)}$ and have picked a suitable value of t_f . We will now construct a classical density $\mathcal{D}_m$ on this set.

To mimic the intrinsic Gaussian nature of the Husimi distribution, we construct the classical density $\mathcal{D}_m(q, p)$ by the convolution of a uniform distribution on the set $\mathcal{S}_m$ with a 2D Gaussian smoothing kernel. The standard deviation σ of the smoothing kernel is chosen to match the minimum quantum uncertainty of the coherent states $\sigma^2 = \frac{\hbar}{2}$. This process is illustrated in a 1-D example in Figure 4.1, where a uniform distribution, displayed in Figure 4.1a, is smoothed by convolution with a Gaussian smoothing kernel, resulting in the distribution shown in Figure 4.1b. In this example we choose $N = 101$

initial conditions $\{q_j\}$ over the interval $q \in [0, 1]$ and a uniform distribution non-zero for $0.4 \leq q \leq 0.6$. For the Gaussian smoothing we have that $\sigma^2 = \frac{\hbar}{2}$, where $N = 101$. In practical implementations N is the dimension of the quantum Hilbert space and the $\{q_j\}$ can be thought of as the centre of the corresponding coherent states used in the Husimi distribution.

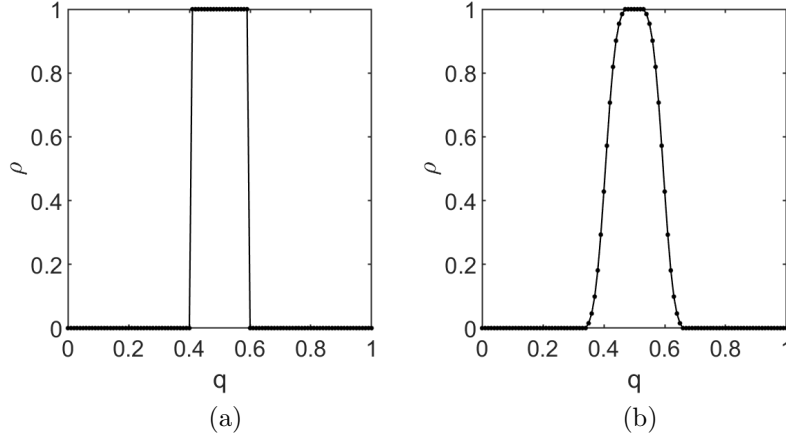


Figure 4.1: Gaussian smoothing of a uniform density. The initial uniform distribution $\rho(x)$ (left) the smoothed distribution $\tilde{\rho}(x)$ (right).

Following the Gaussian smoothing of $\mathcal{S}_m$, which is determined by our ansatz for $\alpha_{\pm}^{(m)}$, we have the classical density $\mathcal{D}_m$. Since our goal is to construct a $\mathcal{D}_m$ that closely resembles the quantum Husimi distribution, let us now explore a simple argument with which we can condition the $\alpha_{\pm}^{(m)}$. More appropriately we will condition the $\mathcal{D}_m$ which will, in turn, condition the $\alpha_{\pm}$.

A standard semiclassical argument is that there is one state per area h in the phase space. For a unit phase space with N orthogonal states this implies that $h = \frac{1}{N}$. This state counting is trivially fulfilled for the quantum Husimi distribution. Since the Husimi distribution is normalised

$$\int \int H_{\psi_n}(q, p) dq dp = 1, \quad (4.2)$$

we can count m states, of a total N states, as the fraction

$$f_Q^{(m)} = \frac{1}{N} \sum_{n=1}^m \int \int H_{\psi_n}(q, p) dq dp, \quad (4.3)$$

which is just the number $f_Q^{(m)} = m/N$. We define the classical analogous

fraction $f_C^{(m)}$ as

$$f_C^{(m)} = \frac{1}{N} \int \int \mathcal{D}(q, p) dq dp. \quad (4.4)$$

We then condition the classical density, using the ansatz of a Gaussian smoothed uniform distribution on the set $\mathcal{S}_n$, based on this state counting, by requiring that $f_Q^{(m)} = f_C^{(m)}$. The full algorithm is summarised in Figure 4.2. Before moving to illustrative physical examples, let us explain the main components of the algorithm.

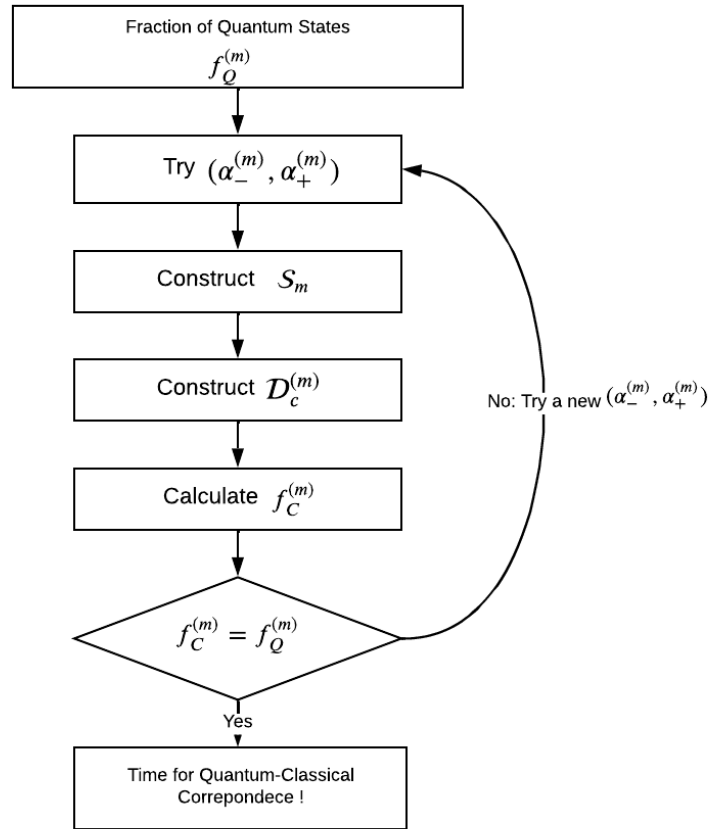


Figure 4.2: The Algorithm for the Classical Density.

4.1.2 Algorithm Outline

We will now introduce the general outline for implementing the classical density algorithm summarised in Figure 4.2. For the purpose of illustration, we consider a classical discrete-time dynamical system evolving under a map

$\mathcal{M}$. The map $\mathcal{M}$ is generated by a one-dimensional kicked non-Hermitian Hamiltonian $\mathcal{H}(t) = H(t) - i\Gamma$ where the Hermitian part $H(t)$ has the form $H(t) = \frac{p^2}{2} + \sum_n V(q)\delta(t - n)$ and the anti-Hermitian term $\Gamma = \gamma f(q, p)$ has $f(q, p)$ and γ , such that it induces a strict loss profile across the two-dimensional phase space. The quantum counterpart is described by Floquet theory and characterised by N ordered Schur states. To each Schur state $|\nu_n\rangle$ there is an associated eigenvalue λ_n with $|\lambda_n| \leq 1$. If we choose the leading eigenfunction to be $|\phi_- \rangle$ i.e., the ordering in the Schur decomposition is $|\lambda_1| < |\lambda_2| < \dots < |\lambda_N|$, the set of states $\{|\nu_N\rangle, |\nu_{N-1}\rangle, \dots, |\nu_{N-m}\rangle\}$ are the m states with longest lifetimes and we wish to construct the associated classical density $\mathcal{D}_m$ for these $m < N$ states. In essence this task can be broken down into three parts: the construction of the forwards norm map (due to the choice of leading eigenfunction), the partitioning of the norm map into the set $\mathcal{S}_m$ through the partitioning parameters $\alpha_{\pm}^{(m)}$, and finally the calculation of the classical density $\mathcal{D}_m$ on the set $\mathcal{S}_m$.

We begin by introducing the ubiquitous physical quantities that will enter into the algorithm, these are: the fraction of quantum states $f_Q^{(m)}$, the partitioning parameters $\alpha_{\pm}$, the reduced Planck's constant $\hbar$, and the final time t_f . The final time t_f determines how long we will evolve the dynamics when generating the norm map. Because the classical system we consider here is a discrete-time dynamical system, we necessarily have $t_f \in \mathbb{Z}$. The partitioning parameters $\alpha_{\pm}^{(m)}$ determine the set $\mathcal{S}_m$ and are initialised with an ansatz. The quantum fraction $f_Q^{(m)} = m/N$ provides our termination condition for the algorithm and is the means by which we condition the classical density $\mathcal{D}_m$. The reduced Planck's constant $\hbar$ defines the width of the Gaussian smoothing kernel for which we have $\sigma^2 = \frac{\hbar}{2}$. Due to the assumption of the unit area of the phase space in the present example we have $\hbar = \frac{1}{2\pi N}$.

It is clear that the algorithm has a natural dependence on the dimension of the quantum system N . In the calculation of the Husimi distributions on the quantum side we use a grid of N^2 coherent states uniformly distributed over the phase space. The classical q, p intervals are sampled at the centre of the coherent states, resulting in a grid of N^2 initial conditions. Using these initial conditions we can now construct the norm map.

The construction of the norm map needs to be performed only once for a given final time t_f . For each of the N^2 initial conditions the classical norm w is calculated up to a final time t_f . The average of the norm is taken and its value at t_f associated to its given initial condition. The values are stored in an $N \times N$ array. This procedure is outlined in Algorithm 1.

Algorithm 1 The Norm Map

```

1: for Initial condition do
2:   for  $t = 1 : t_f$  do
3:     Evolve initial condition under  $\mathcal{M}^t$  and calculate norm
4:   end for
5:   Calculate averaged norm for initial condition
6: end for

```

For a classical dynamical system that is not described by a map the procedure is analogous to that outlined in Algorithm 1 above. In this case the loop in lines 2-5 can be replaced by numerical integration over the interval $t \in [0, t_f]$. The bottleneck in execution time of the norm map comes from the calculation of the classical evolution. When the classical dynamics are described by a map the execution speed can be greatly improved upon by application of the map to the whole array of initial conditions.

Following the construction of the norm map, the ansatz of $\alpha_{\pm}^{(m)}$ can be used to partition the norm map into the set $\mathcal{S}_m$. From this set we can then construct a classical density $\mathcal{D}_m$ which has the associated classical counting number $f_C^{(m)}$ defined by equation (4.4). Generally the ansatz for $\alpha_{\pm}^{(m)}$ will not be correct and we will condition the $\alpha_{\pm}^{(m)}$ such that classical density fulfils the condition $f_Q^{(m)} = f_C^{(m)}$. Before discussing the details of this conditioning, we will first outline the construction of the classical density for the ansatz $\alpha_{\pm}^{(m)}$.

Our choice of the ansatz $\alpha_{\pm}^{(m)}$ is informed by our search for the phase-space area associated to the m longest lived states. The loss encoded in the system manifests through the classical norm $w_{n+1} = e^{-2\gamma f(q_n, p_n)} w_n$. Since the system is lossy, it follows that $\forall t_f \in \mathbb{N} : w_{t_f} \leq 1$. If there exist initial conditions whose norm $w_{t_f} = 1$ these would naturally correspond to the phase space area of the states with longest lifetimes. For this reason we can set $\alpha_+^{(m)} = 1$. Even if no such initial conditions exist, setting $\alpha_+^{(m)} = 1$ is still a sufficient upper bound. The choice of $\alpha_-^{(m)}$ need simply be picked so at least one initial condition is present in the set $\mathcal{S}_m$ (since we will condition the $\alpha_-^{(m)}$ later in the algorithm). The construction of $\mathcal{S}_m$ is shown in lines 1:5 of Algorithm 2.

The set $\mathcal{S}_m$ is represented numerically as an $N \times N$ array consisting of ones at those initial conditions that belong to $\mathcal{S}_m$ and zero otherwise. This representation reflects the assumption that the density on the set $\mathcal{S}_m$ is the

uniform density. To construct the classical density $\mathcal{D}_m$ we apply a 2d Gaussian smoothing kernel to the set $\mathcal{S}_m$. For the examples we will explore in this thesis we use the function “imgaussfilt()” from the MATLAB image processing toolbox [72] for this purpose. We note that the process of Gaussian smoothing is a commonplace technique and equivalent prebuilt functions are readily available in other programming languages. The Gaussian smoothing is represented in line 6 of Algorithm 2, and takes as an input the set $\mathcal{S}_m$ and standard deviation $\sigma = \sqrt{\frac{h}{2}}$. Finally in line 7 of Algorithm 2 we calculate the classical counting number $f_C^{(m)}$. We terminate the algorithm if $f_C^{(m)} = f_Q^{(m)}$, however since in this first pass we used our ansatz for $\alpha_{\pm}$ this condition is unlikely to be met. We will now finish the overview of the classical density algorithm by outlining how we self consistently condition the $\alpha_{\pm}^{(m)}$ and the classical density $\mathcal{D}_m$.

Algorithm 2 Constructing the Classical Density

```

1: for Initial condition do
2:   if  $\alpha_- \leq \text{Norm at Initial condition} \leq \alpha_+$  then
3:     Initial condition  $\in \mathcal{S}_m$ 
4:   end if
5: end for
6:  $\mathcal{D}_m = \text{GaussianSmoothing}(\mathcal{S}_m, \sigma)$ 
7:  $f_C^{(m)} = \frac{1}{N^2} \sum_i \sum_j \mathcal{D}_m(i, j)$ 

```

As discussed previously we can set $\alpha_+^{(m)} = 1$ and the conditioning requires us to optimise the value of $\alpha_-^{(m)}$. Practically, the termination condition $f_C^{(m)} = f_Q^{(m)}$ is implemented numerically as $|f_C^{(m)} - f_Q^{(m)}| < \delta$ where $\delta \in \mathbb{R}$ is a small positive number and $\delta < h = \frac{1}{N}$. Minimising the value of $|f_C^{(m)} - f_Q^{(m)}|$ can be implemented in a variety of ways. In this thesis we implement the minimisation of $|f_C^{(m)} - f_Q^{(m)}|$ through a simple bracketing method. For the example lossy system we consider here the procedure is outlined in Algorithm 3. Writing α_- in the form of the norm $\alpha_-^{(m)} = e^{-\gamma \Delta_-^{(m)}}$ we then choose two values $\Delta_{upper}, \Delta_{lower}$ that result in $f_C^{(m)} > f_Q^{(m)}$ for Δ_{upper} and $f_C^{(m)} < f_Q^{(m)}$ for Δ_{lower} . We then define a new value $\Delta_{new} = \Delta_{lower} + \frac{1}{2}|\Delta_{lower} - \Delta_{upper}|$. Calculating the classical counting number $f_C^{(m)}$ with the $\alpha_-^{(m)}$ from Δ_{new} either $|f_C^{(m)} - f_Q^{(m)}| < \delta$, in which case the Algorithm 3 terminates, or $|f_C^{(m)} - f_Q^{(m)}| > \delta$. In the latter case we readjust the boundaries of the interval as $\Delta_{upper} = \Delta_{new}$ if $f_C^{(m)} > f_Q^{(m)} + \delta$, or $\Delta_{lower} = \Delta_{new}$ otherwise. This process repeats until the termination condition $|f_C^{(m)} - f_Q^{(m)}| < \delta$ is met.

Algorithm 3 Conditioning the Classical Density

```

1: while  $|f_C^{(m)} - f_Q^{(m)}| > \delta$  do
2:    $\Delta_{new} = \Delta_{new} + \frac{1}{2}|\Delta_{lower} - \Delta_{upper}|$ 
3:   Construct Classical Density  $\mathcal{D}_m$  with  $\alpha_-$ ,  $\Delta_{new}$ 
4:   if  $|f_C^{(m)} - f_Q^{(m)}| < \delta$  then
5:     Exit Loop
6:   else if  $|f_C^{(m)} - f_Q^{(m)}| > \delta$  then
7:     if  $f_C^{(m)} > f_Q^{(m)} + \delta$  then
8:        $\Delta_{upper} = \Delta_{new}$ 
9:        $\Delta_{lower} = \Delta_{lower}$ 
10:    else
11:       $\Delta_{lower} = \Delta_{new}$ 
12:       $\Delta_{upper} = \Delta_{upper}$ 
13:    end if
14:  end if
15: end while

```

As a first illustrative example for the construction of the density, we will consider the complex harmonic oscillator.

4.2 A Complex Harmonic Oscillator

Non-Hermitian systems generally have two properties that require our attention, the non-orthogonality of the eigenfunctions, and modification of the norm through the gain-loss profile of the system. There is a special class of non-Hermitian Hamiltonians which remain normal. These arise through the multiplication of a Hermitian Hamiltonian $\hat{H}$ with a complex factor $k \in \mathbb{C}$ giving a non-Hermitian Hamiltonian $\hat{\mathcal{H}} = k\hat{H}$. A Hamiltonian of this type that has been studied in the literature is the complex harmonic oscillator [39, 73, 74].

Here we consider the complex harmonic oscillator described by the non-Hermitian Hamiltonian

$$\hat{\mathcal{H}} = \frac{\hbar\tilde{\omega}}{2}(\hat{p}^2 + \hat{q}^2) \quad : \quad \tilde{\omega} \in \mathbb{C}. \quad (4.5)$$

Writing $\tilde{\omega} = \omega - i\gamma$, we assume that $\gamma > 0$. This allows us to interpret (4.5) as a damped harmonic oscillator.

4.2.1 Classical Dynamics

The dynamics arising from the coherent state approximation is governed by the evolution equation

$$\begin{aligned}\dot{q} &= p - \gamma q \\ \dot{p} &= -q - \gamma p.\end{aligned}\tag{4.6}$$

These can be solved directly as

$$\begin{aligned}q(t) &= e^{-\gamma t} (q_0 \cos(\omega t) + p_0 \sin(\omega t)) \\ p(t) &= e^{-\gamma t} (p_0 \cos(\omega t) - q_0 \sin(\omega t)).\end{aligned}\tag{4.7}$$

A typical example of a trajectory is depicted in the left panel of Figure 4.3, with initial condition $(q_0, p_0) = (0.5, 0.5)$, damping value of $\gamma = 0.1$, up to a final time $t_f = 8\pi$. We observe the trajectory spirals towards the classical fixed point $(0, 0)$. This spiralling inwards is a sign of dissipation in the system. The point $(0, 0)$ is an attracting fixed point known as a focus. Every trajectory asymptotically approaches this fixed point. This type of fixed point is not possible in Hamiltonian systems as the phase space volume must be conserved.

The classical norm evolves according to

$$\dot{w} = -\gamma(q^2 + p^2)w.\tag{4.8}$$

The solution to (4.8), can be found exactly as

$$w(t) = e^{-(q_0^2 + p_0^2)(1 - e^{-2\gamma t})}\tag{4.9}$$

4.2.2 Quantum Dynamics and Husimi Distributions

Writing the Hamiltonian (4.5) in terms of the number operator $\hat{N}$ yields

$$\hat{\mathcal{H}} = \tilde{\omega} \left(\hat{N} + \frac{1}{2} \right),\tag{4.10}$$

from which we see that like its closed counterpart, the Hamiltonian is linear in the generators of the harmonic oscillator algebra. Thus, the expectation value dynamics, when evaluated with coherent states, will follow the classical equations of motion exactly. Note however, that unlike in the Hermitian version, this is only true for coherent states.

The eigenvectors of (4.5) are the same as those of the Hermitian harmonic oscillator. Figure 4.3b depicts the Husimi distribution for the ground state,

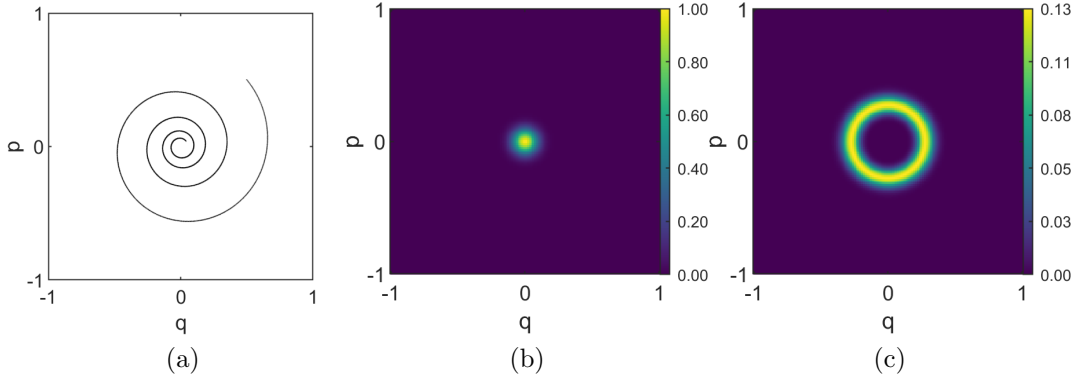


Figure 4.3: Classical and quantum expectation value dynamics and Husimi distributions for the complex harmonic oscillator. The right and central plots depict the Husimi distributions of the ground state (middle) and 9th excited state (right) for a matrix dimension $N = 150$. The left plot depicts the expectation dynamics of a coherent state with initial condition $(q_0, p_0) = (0.5, 0.5)$.

and Figure 4.3c the 9th excited state. The correspondence, or lack thereof, between the Husimi distribution and the classical trajectories has rather surprisingly not been addressed in the literature [39, 73, 74]. We will see that again, the Schur vectors (or the eigenvectors here) correspond to regions of the classical norm map. The Schur decomposition coincides with the diagonalization here, and the Schur vectors are the same as the eigenvectors regardless of their respective ordering. On the classical side this manifests in the backwards and forward norm map coinciding. However, there is still a natural ordering for the eigenvectors $\{|\phi_n\rangle\}$ through the eigenvalues $\lambda_n = \hbar\tilde{\omega}(n + 1/2)$.

4.2.3 Classical Density

Let us now demonstrate the classical density algorithm to study quantum-classical correspondence of the complex harmonic oscillator. The left and middle plots of Figure 4.4 depict the Husimi distributions we will compare against when constructing the classical density. These are the Husimi distribution of the ground state (left) and the set of the first ten states (middle). The right plot (Figure 4.4c) depicts the norm map for $\gamma = 0.1$, and $t_f = 10$, from which we will construct the set $\mathcal{S}_n$. Note that the results in the simple case do not depend sensitively on t_f .

Let us begin by considering the phase-space area associated to the ground

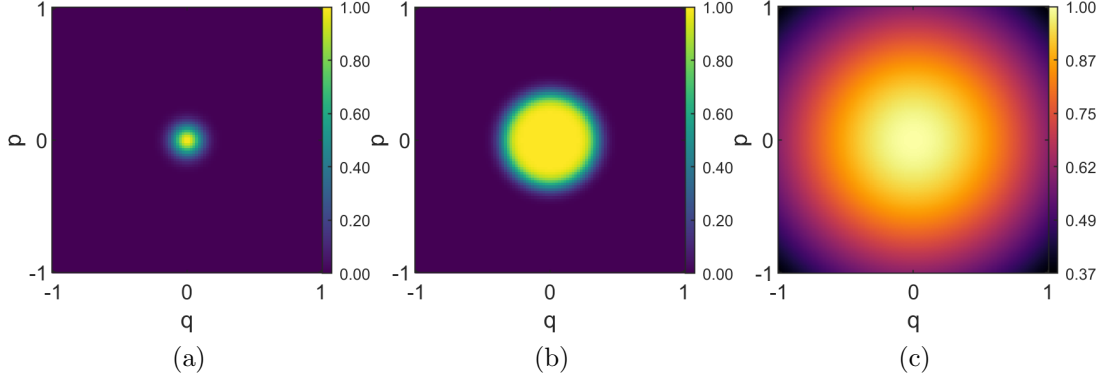


Figure 4.4: Quantum Husimi distributions and classical norm of the complex harmonic oscillator for $\gamma = 0.1$. The Husimi distribution depicted are those of the ground state (middle) and the summed distribution over the first 10 states (right) with a matrix dimension $N = 150$. The left plot depicts the classical norm map with final time $t_f = 10$.

state $|\phi_1\rangle$, which is the set $\mathcal{S}_1$ of points in the norm map

$$\mathcal{S}_1 = \left\{ z : \alpha_-^{(1)} \leq w_{t_f} \leq \alpha_+^{(1)} \right\}. \quad (4.11)$$

We pick trial values of $\alpha_{\pm}^{(1)}$, then construct the classical density $\mathcal{D}_1$, on the set $\mathcal{S}_1$. For one state, using the counting argument (4.3), this correspond to an area of $1/N$, that is $f_Q^{(1)} = 1/N$. Now if $\alpha_{\pm}$ are picked “correctly” then $f_C^{(1)} = \frac{1}{N}$, if not we change $\alpha_{\pm}$ until we have $f_C^{(1)} = \frac{1}{N}$. This condition gives us, in principle, both a classical density and the phase-space area associated to the states from the norm map.

In Figure 4.5a we show the Husimi distribution of the ground state, and in Figure 4.5b the constructed classical density. We see that the density localises on the correct phase-space area but some discrepancy between the densities at the centre of the distribution. This is displayed in Figure 4.5c which shows a cut of the Husimi and classical density along the line $p = 0$. The quantum density is shown in blue, and the classical in red.

The procedure for the classical analogue of the Husimi distribution over the first m states is analogous, where we define the set $\mathcal{S}_m = \bigcup_{n=1}^m \mathcal{S}_n$. In Figure 4.6a we show the Husimi distribution of the first 10 eigenstates. The classical density is constructed using the condition $f_C^{(10)} = 10/N$. 4.6b shows the classical density and we a good agreement between the two plots. In Figure 4.6c the cut along the density at $p = 0$ is shown. Having demonstrated the

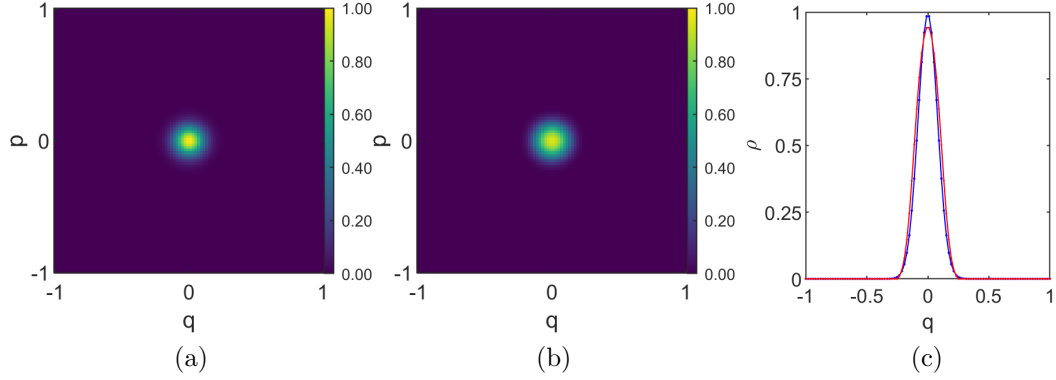


Figure 4.5: Husimi distribution and classical density for the complex harmonic oscillator for $\gamma = 0.1$. The Husimi distributions shown in the left plot is of the ground state of the quantum system with dimension $N = 150$, the middle column shows the classical density associated to the Husimi distributions and the column on the right compares the Husimi density (blue) with the classical density (red) through the line $p = 0$.

algorithm for the simple complex harmonic oscillator let us now turn to our main system of interest, the PT-symmetric kicked rotor.

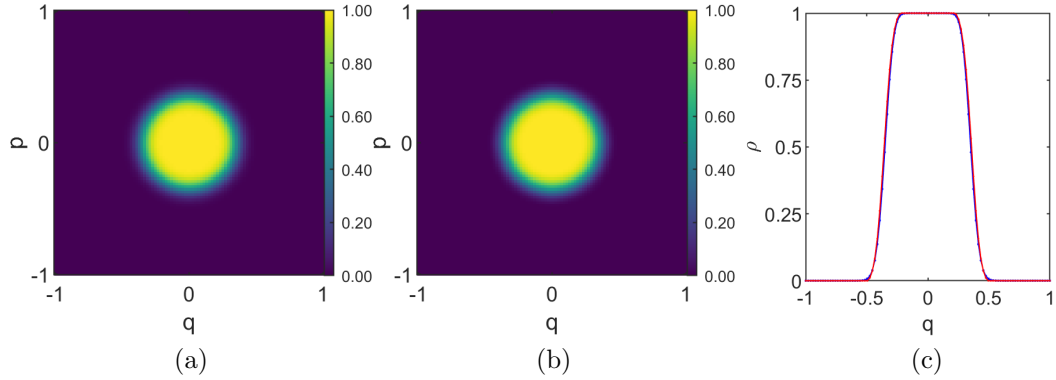


Figure 4.6: Husimi distribution and classical density for the complex harmonic oscillator for $\gamma = 0.1$. The Husimi distributions shown in the left plot is summed over the first 10 states, the associated classical density is shown in the middle plot and the column on the right compares the Husimi density (blue) with the classical density (red) through the line $p = 0$. The quantum system has a matrix dimension $N = 150$.

4.3 The PT-Symmetric Kicked Rotor

Here we form three classical support sets, $\mathcal{S}_+$ for the support on the gain set, $\mathcal{S}_0$ for the support on the stable set and $\mathcal{S}_-$ for the support on the loss set. When the leading eigenfunction is $|\phi_+\rangle$, these sets are constructed as

$$\begin{aligned}\mathcal{S}_+ &= \{z : \langle w_{t_f} \rangle < e^{2\gamma\Delta_+}\} \\ \mathcal{S}_- &= \{z : \langle w_{t_f} \rangle > e^{2\gamma\Delta_-}\} \\ \mathcal{S}_0 &= \{z : e^{2\gamma\Delta_+} \leq \langle w_{t_f} \rangle \leq e^{2\gamma\Delta_-}\}.\end{aligned}\tag{4.12}$$

The form $\alpha_\pm = e^{2\gamma\Delta_\pm}$ of the partitioning parameter is chosen to reflect the functional form of the classical norm. Because the dynamics are reversed with this leading eigenfunction the gain and loss sets correspond to decaying ($\Delta_+ \in \mathbb{R}^-$) and growing ($\Delta_- \in \mathbb{R}^+$) norm, respectively. When the leading eigenfunction is $|\phi_-\rangle$, and the norm follows the forward dynamics, we have

$$\begin{aligned}\mathcal{S}_+ &= \{z : \langle w_{t_f} \rangle > e^{2\gamma\tilde{\Delta}_+}\} \\ \mathcal{S}_- &= \{z : \langle w_{t_f} \rangle < e^{2\gamma\tilde{\Delta}_-}\} \\ \mathcal{S}_0 &= \{z : e^{2\gamma\tilde{\Delta}_-} \leq \langle w_{t_f} \rangle \leq e^{2\gamma\tilde{\Delta}_+}\}.\end{aligned}\tag{4.13}$$

Here the set $\mathcal{S}_+$ corresponds to regions of growing ($\tilde{\Delta}_+ \in \mathbb{R}^+$) classical norm and the set $\mathcal{S}_-$ to regions of decaying ($\tilde{\Delta}_- \in \mathbb{R}^-$) classical norm. A simple identification can be made between the $\tilde{\Delta}_\pm$ in the sets (4.13) and the $\Delta_\pm$ appearing in (4.12), namely $\tilde{\Delta}_+ = \Delta_-$ and $\tilde{\Delta}_- = \Delta_+$. Furthermore, owing to the PT-symmetry of the system we have $e^{\Delta_+} = e^{-\Delta_-}$ and $e^{\tilde{\Delta}_+} = e^{-\tilde{\Delta}_-}$.

Using the state counting argument the fraction of gain states will be $f_Q^{(+)}$, the fraction of stable states $f_Q^{(0)}$, and the fraction of loss states $f_Q^{(-)}$. We then condition the classical densities by picking $\Delta_\pm$ such that

$$\begin{aligned}f_C^{(+)} &= \frac{1}{N} \int \int \mathcal{D}_+(q, p) dq dp = f_Q^{(+)} \\ f_C^{(0)} &= \frac{1}{N} \int \int \mathcal{D}_0(q, p) dq dp = f_Q^{(0)} \\ f_C^{(-)} &= \frac{1}{N} \int \int \mathcal{D}_-(q, p) dq dp = f_Q^{(-)}.\end{aligned}\tag{4.14}$$

We will now explore how the conditioning of the density using (4.14) also determines the final time t_f . In practice, the conditioning will provide a lower bound on t_f , however taking t_f too large is not only computationally impractical, but can also lead to the accumulation of numerical errors.

4.3.1 Determining t_f

The determination of t_f is a question of resolving the structures that arise in the Husimi distributions, while also satisfying the requirements 4.14. We will focus on the case $k = 5$ and $\gamma = 0.001$, with leading eigenfunction $|\phi_+\rangle$, and matrix dimension $N = 1001$, which has Husimi distributions shown in Figure 4.7a. For the classical density $\mathcal{D}_+$, associated to the gain set $\mathcal{S}_+$, we condition the partitioning parameter $\alpha_+ = e^{\Delta_+}$ such that $f_C^{(+)} = f_Q^{(+)}$. The set $\mathcal{S}_+$ has an implicit dependence on the final time t_f , since it arises through the partitioning of the norm map which is propagated up to a final time t_f . However, if the final time t_f is too short then the norm map will not contain a sufficient number of initial conditions associated to the gain set $\mathcal{S}_+$ to satisfy $f_C^{(+)} = f_Q^{(+)}$. In this case, the only way for the condition $f_C^{(+)} = f_Q^{(+)}$ to be satisfied will be for α_+ to be chosen such that the set $\mathcal{S}_+$ contains initial conditions that truly belong to the stable set $\mathcal{S}_0$. This situation is depicted in Figure 4.7 where the final time is $t_f = 40$ and we have, from the quantum spectrum, that $f_Q^{(+)} = f_Q^{(-)} = 0.3966$, $f_Q^{(0)} = 1 - f_Q^{(+)}$ (since $f_Q^{(+)} + f_Q^{(0)} + f_Q^{(-)} = 1$). In this case the condition $f_C^+ = f_Q^+$ results in a value of $\Delta_+ = -0.2020$. The classical density is depicted in Figure 4.7b where we observe a clear overestimation in the support when compared to the quantum Husimi distribution Figure 4.7a.

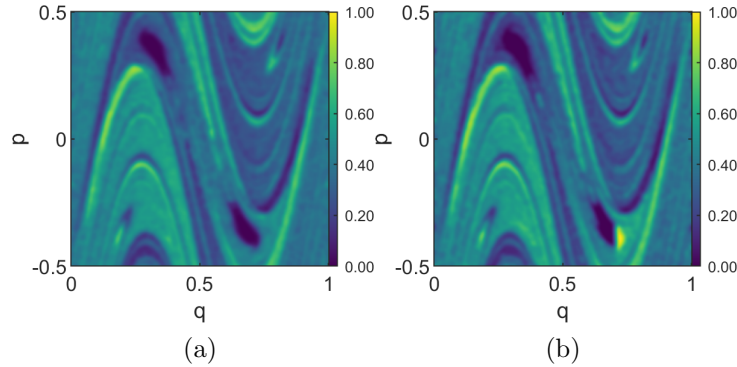


Figure 4.7: Husimi distribution of the gain states of the PT-symmetric kicked rotor and classical density for $k = 5$, $\gamma = 0.001$. The Husimi distribution (left) is summed over the gain states, with leading eigenfunction $|\phi_+\rangle$, for a matrix dimension $N = 1001$. The classical density (right) $\mathcal{D}_+$ for a final time $t_f = 42$ and $\sigma^2 = \frac{1}{2}\hbar$ with $\hbar = (2\pi N)^{-1}$.

It is clear that the discrepancies between the two arise in the neighbourhood of the classical regular islands. One possible mechanism for this can be the

stickiness of trajectories. For chaotic trajectories, trajectories asymptotically in the chaotic sea, the stickyness can confine them close to regular islands for intermediate times [56]. Effectively, on these intermediate timescales the norm maps cannot yet distinguish the boundary of the regular regions. Thus, we should view the classical trajectories as possessing a characteristic transient time scale before they join their asymptotic gain, stable, or loss set. Picking a sufficiently large t_f that resolves the boundary of the regular regions ensures that enough of these times are below t_f .

The minimum such final time that satisfies $f_C^{(+)} = f_Q^{(+)}$ and gives the correct regions of support is $t_f = 58$. The classical support of the set $\mathcal{S}_+$ is depicted in Figure 4.8a. The quantum Husimi distribution of the gain states is shown in Figure 4.8b. Finally, the classical density is shown in Figure 4.8c which bears a striking resemblance to the Husimi distribution of the gain states.

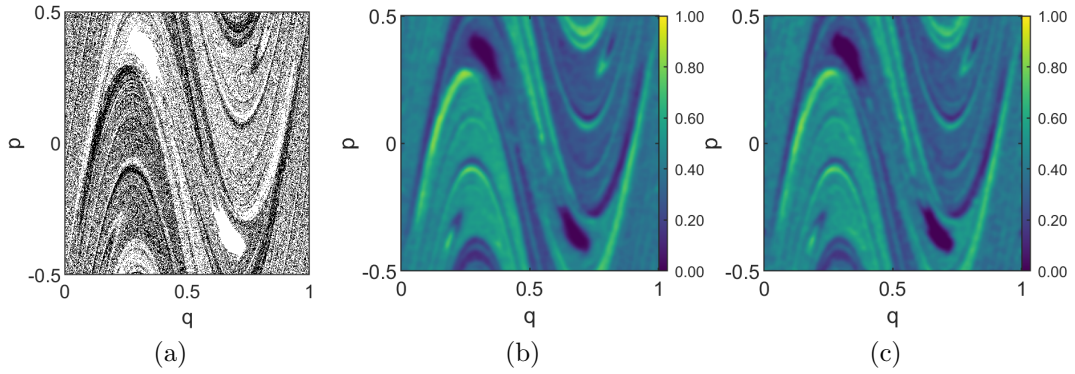


Figure 4.8: The classical support of the gain set (left), Quantum Husimi distribution (middle) and associated classical density $\mathcal{D}_+$ of the gain states of the PT-symmetric kicked rotor for $k = 5$, $\gamma = 0.001$, $t_f = 58$, $\Delta_+ = -0.2423$ and leading eigenfunction $|\phi_+\rangle$.

The same analysis can be carried out for the leading eigenfunction $|\phi_-\rangle$. All the parameters, are the same as in the previous example. In Figure 4.9a we show the support of the gain states in the phase space with leading eigenfunction $|\phi_-\rangle$. We see that the support agrees with the phase space area related to the Husimi distribution of the gain states 4.9b. Finally in Figure 4.9c we show the classical density. We see a good agreement between the constructed classical density and the Husimi distribution of the gain states.

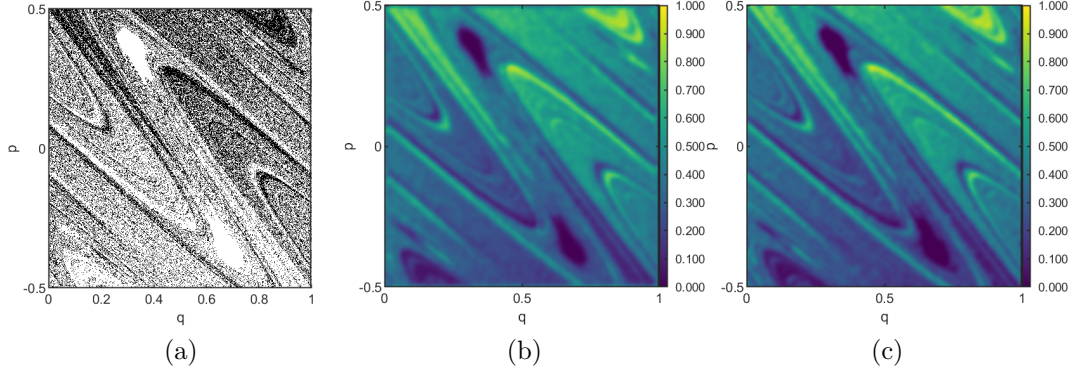


Figure 4.9: The classical support of the gain set (left), Quantum Husimi distribution (middle) and associated classical density $\mathcal{D}_+$ of the gain states of the PT-symmetric kicked rotor for $k = 5$, $\gamma = 0.001$, $t_f = 58$, $\tilde{\Delta}_+ = 0.2423$ and leading eigenfunction $|\phi_-\rangle$.

4.3.2 Quantum-Classical Correspondence in the PTKR

For brevity, in this section we will restrict the analysis to the case of $|\phi_+\rangle$. Furthermore, we will show only the gain and stable sets. The loss sets that we omit are simply the complement of the union of the stable and gain distributions. In all examples in this section we use the value $\gamma = 0.001$. In Figures 4.10 and 4.11 we show the classical phase space area associated to the Husimi distribution of the gain and stable set, the Husimi distribution of the Schur vectors belonging to the same set and the corresponding classical densities for four different values of k ranging from 1.1 to 10. For a given kicking strength, the first row corresponds to gain and the one below to the stable set. In Figure 4.10 the first row corresponds to the gain set for $k = 1.1$ and the third to the gain set for $k = 2.5$. In Figure 4.11 the first row corresponds to the gain set for $k = 5$ and the third to the gain set for $k = 10$.

In Figure 4.10 and 4.11 we see a good agreement between the quantum Husimi distributions of the gain and stable sets and their associated classical densities. The weakest of the results is in the bottom half of Figure 4.10 where $k = 2.5$. In this case, we observe that both the underlying fractal structure associated to the gain set and the region associated to stable occupy large areas of the phase space. The discrepancy between the Husimi distribution and the classical density may be due to this region being more sensitive to numerical instabilities.

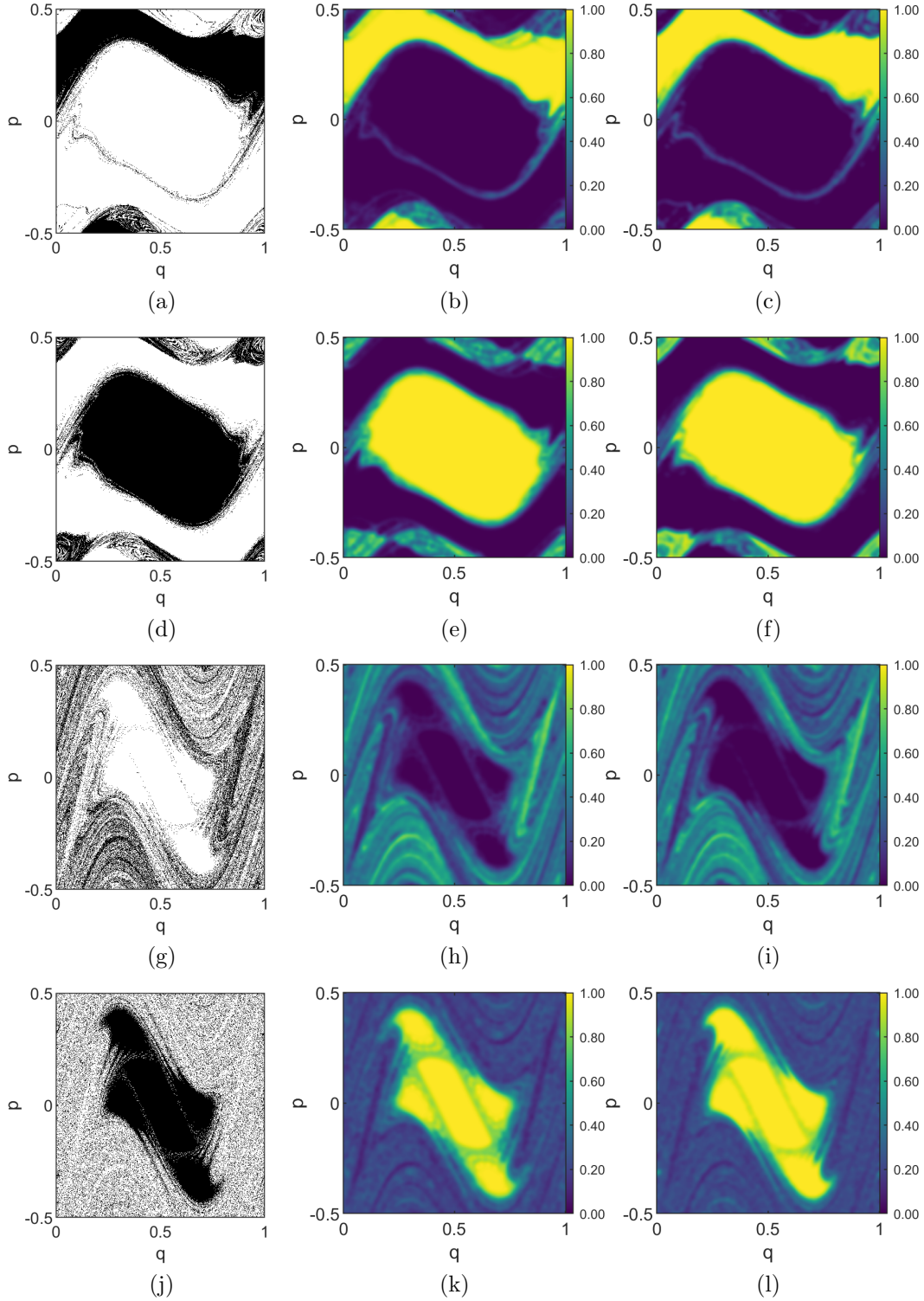


Figure 4.10: The classical support (left), quantum Husimi distribution (middle) and classical density of the gain states (row 1 and 3) and stable states (row 2 and 4). The kicking strength is $k = 1.1$ (rows 1 and 2), $k = 2.5$ (rows 3 and 4). For $k = 1.1$: $t_f = 80$, $\Delta_+ = -0.8444$ and $f_Q^{(+)} = 0.2697$ and for $k = 2.5$: $t_f = 80$, $\Delta_+ = -0.2833$ and $f_Q^{(+)} = 0.30759$. We have $\sigma^2 = \frac{1}{2}\hbar$ with $\hbar = (2\pi N)^{-1}$ and $N = 1001$.

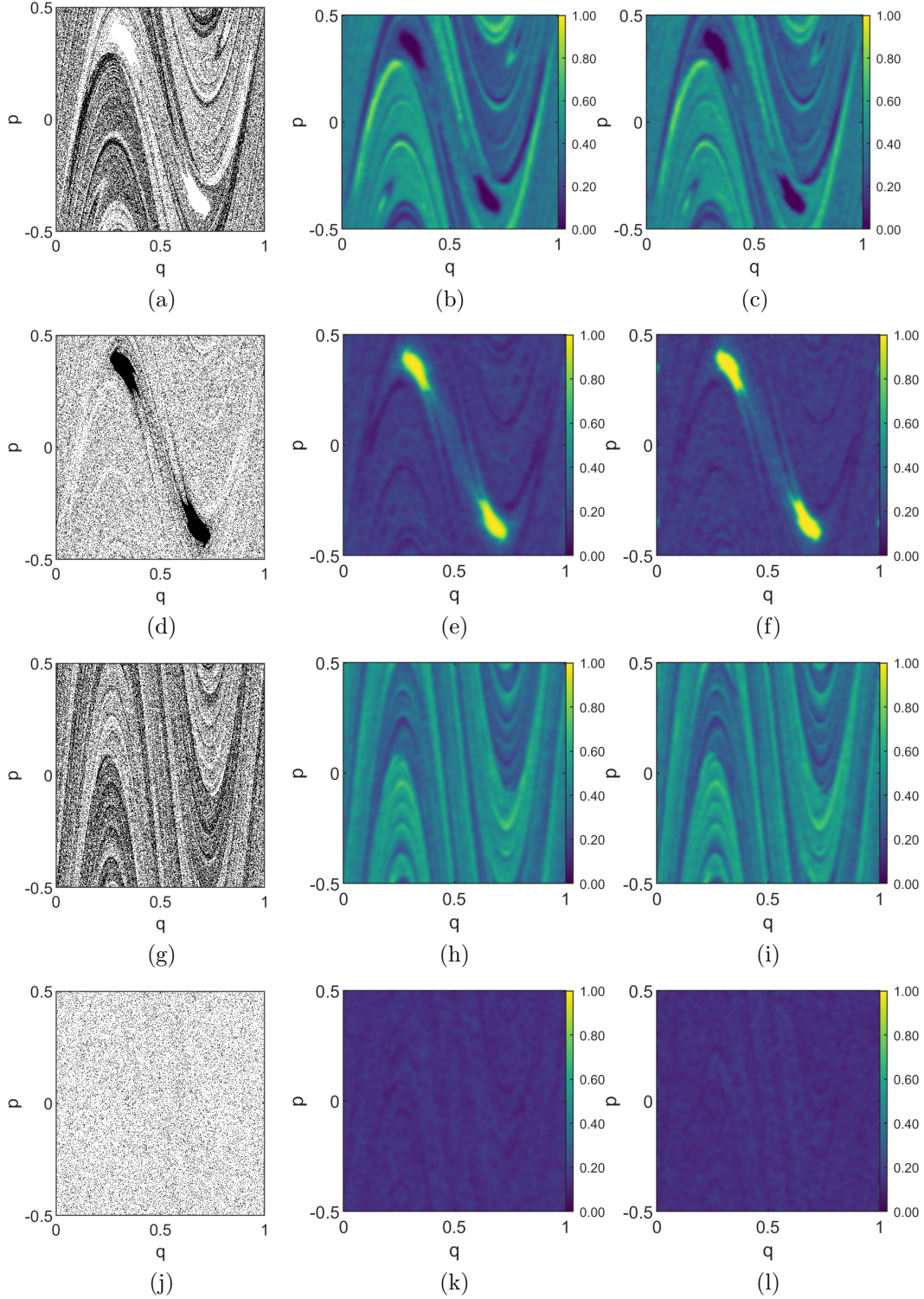


Figure 4.11: The classical support (left), quantum Husimi distribution (middle) and classical density of the gain states (row 1 and 3) and stable states (row 2 and 4). The kicking strength is $k = 5$ (rows 1 and 2), $k = 10$ (rows 3 and 4). For $k = 5$: $t_f = 58$, $\Delta_+ = -0.2423$ and $f_Q^{(+)} = 0.3988$ and for $k = 10$: $t_f = 40$, $\Delta_+ = -0.1589$, $f_Q^{(+)} = 0.4286$. We have $\sigma^2 = \frac{1}{2}\hbar$ with $\hbar = (2\pi N)^{-1}$ and $N = 1001$.

We can also extend the analysis to subsets of the gain, loss sets. An example of this is shown in Figure 4.12 where we depict the Husimi distribution of the 20 longest lived Schur states in the globally chaotic regime $k = 7.7$. This kicking strength corresponds to the maximum value of the imaginary part of the quasienergy in the chaotic regime $5 < k < 9$. This is depicted in Figure 4.12a. The corresponding Husimi distribution is shown in Figure 4.12b. The classical density, Figure 4.12c is conditioned with the quantum state counting number $f_Q^{(+)} = 20/1001$. We see a good agreement between the quantum Husimi distribution and the associated classical density.

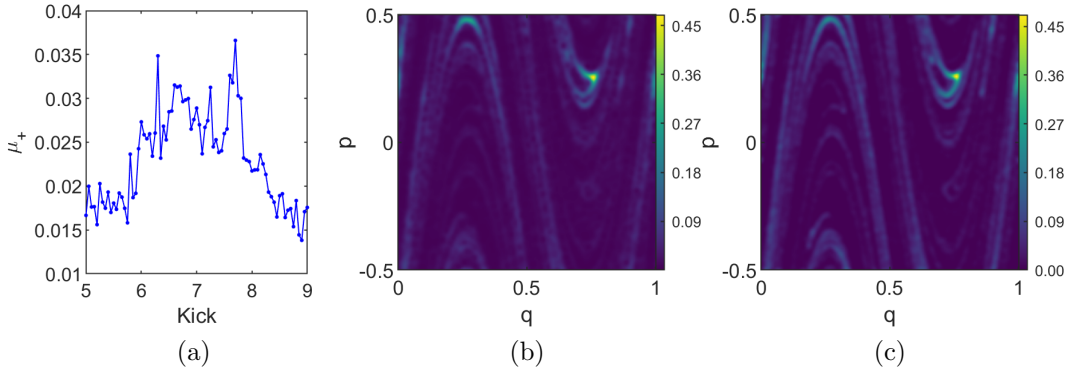


Figure 4.12: In the left plot we show the largest absolute imaginary part as a function of kicking strength in the range $5 \leq k \leq 10$. In the remaining plots we show the Husimi distribution (middle) and associated classical density (right) corresponding to the 20 longest lived eigenstates for $k = 7.7$ and $\gamma = 0.001$ and $\sigma^2 = \frac{1}{2}\hbar$ with $\hbar = (2\pi N)^{-1}$ and $N = 1001$.

Having convinced ourselves of the applicability of our classical density as an approximation for the Husimi distribution of Schur vectors for the PT-kicked rotor, let us return briefly to the kicked rotor with escape through an opening.

4.4 The Kicked Rotor with an opening

The kicked rotor with escape through an opening can be viewed as the limit of the following non-Hermitian Hamiltonian

$$\mathcal{H} = H - i\gamma\mathbb{I}_O, \quad (4.15)$$

where H is the Hamiltonian of the closed system, $\gamma > 0$ is the loss strength and $\mathbb{I}_O$ is a characteristic function of the opening i.e., the identity on the

set of points on the opening. For an opening that is defined across the entire momentum range the characteristic function is defined by its end points q_L, q_R such that

$$\mathbb{I}_O(q) = \begin{cases} 1 & : q \in [q_L, q_R] \\ 0 & : \text{otherwise} \end{cases} \quad (4.16)$$

From the coherent state approximation the classical dynamics remain unchanged from the Hermitian dynamics whenever the gradient of the imaginary part of the non-Hermitian Hamiltonian vanishes. The information about the loss in the system manifests through the classical norm. The classical norm evolves as

$$w_{n+1} = \begin{cases} e^{-2\gamma} w_n & : q_n \in [q_L, q_R] \\ w_n & : \text{otherwise} \end{cases} \quad (4.17)$$

The Hamiltonian (4.15) and norm (4.17) define a system with partial escape through an opening. For the system with full escape through an opening we take $\gamma \rightarrow \infty$. The norm at a given time is then

$$w_{n+1} = (1 - \mathbb{I}_O) w_n \quad (4.18)$$

where we have used the definition of $\mathbb{I}_O$ from equation (4.16). It is clear from (4.18) that the total escape from the opening is represented by a norm value of zero for all subsequent times. From this we can see that a hierarchy arises amongst the classical trajectories based on the number of steps before they reach the opening. As discussed in Chapter 2, the Husimi distributions of the Schur vectors can be related to the lifetimes of trajectories, i.e the number of steps a trajectory is separated in time from the opening [19]. This result is precisely what our classical norm reduces to.

We will now construct the classical density on these sets. We form the support set $\mathcal{S}_+$

$$\mathcal{S}_+ = \{z : \alpha_- \leq \langle w_{tf} \rangle\}, \quad (4.19)$$

for the Schur vectors with the longest lifetimes. Similarly we form the support set $\mathcal{S}_-$

$$\mathcal{S}_- = \{z : \langle w_{tf} \rangle \geq \alpha_-\}, \quad (4.20)$$

for the Schur vectors with the shortest lifetimes. Now using the state counting argument, we say the fraction of the total states of long-lived states is $f_Q^{(+)}$ and that of the short-lived states is $f_Q^{(-)}$. Then we condition the classical density, by choosing the parameter α_- such that the classical density $\mathcal{D}_\pm$ satisfies

$$f_C^{(+)} = \frac{1}{N} \int \int \mathcal{D}_+(q, p) dq dp = f_Q^{(+)}, \quad (4.21)$$

$$f_C^{(-)} = \frac{1}{N} \int \int \mathcal{D}_-(q, p) dq dp = f_Q^{(-)}. \quad (4.22)$$

In Figure 4.13b we plot the Husimi distribution of the 400 Schur states with the shortest lifetimes for the globally chaotic case $k = 10$. In Figure 4.13c we show the classical density and again see a good agreement.

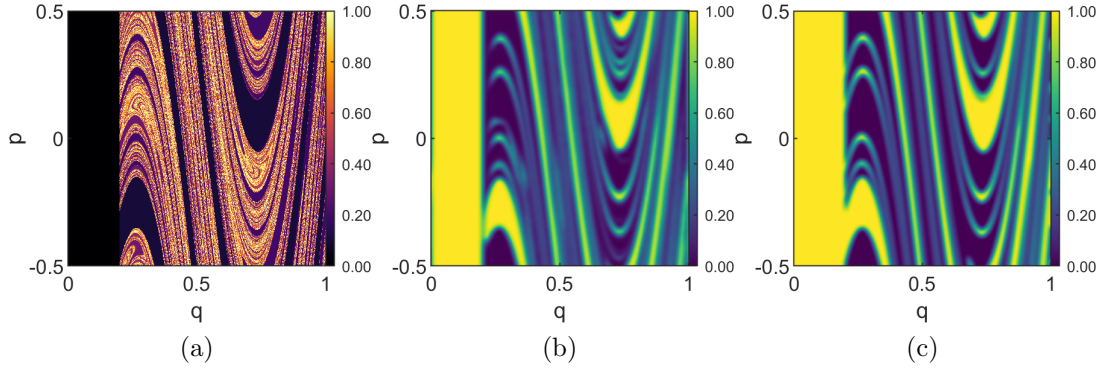


Figure 4.13: The classical norm map (left), Husimi distribution (middle) and associated classical density of the open kicked rotor with $k = 10$, $t_f = 10$, and $f_Q^{(-)} = 0.4825$ and $\sigma^2 = \frac{1}{2}\hbar$ with $\hbar = (2\pi N)^{-1}$ and $N = 1001$.

Having satisfied ourselves that the density indeed works for the kicked rotor with an opening we now proceed to our final example, the Duffing oscillator.

4.5 A Non-Hermitian Duffing Oscillator

The Duffing oscillator is a famous example of classical dissipative chaos. It is a periodically forced oscillator subject to dissipation and commonly described by the equation

$$\ddot{q} + \gamma\dot{q} + \beta q + \alpha q^3 = F \cos(\omega t), \quad (4.23)$$

where $\alpha, \beta, \gamma, F \in \mathbb{R}$. Adjusting the values of the constants α, β, γ, F allows us to probe various physical situations. When the system is unforced, $F = 0$, and there is no damping, $\gamma = 0$, the energy is conserved. For negative values of β this describes a particle in a double well potential. The interesting physics happens when the system is forced and damped. In the presence of forcing $F > 0$, the system can become chaotic. For $\gamma \neq 0$ the system is dissipative and the characteristic signatures of chaos are different to those seen in Hamiltonian chaotic systems. In particular strange attractors

can emerge. Strange attractors are attracting sets of a complicated (usually fractal) type in the phase space, upon which the trajectories are chaotic. The Duffing oscillator has well known strange attractors. The attractor which we will be interested in is shown in Figure 4.14a with parameters $F = 0.3, \gamma = 0.2, \alpha = 1, \beta = -0.75$.

The general approach to studying quantum analogues of the Duffing equation has been either through a Linblad approach or through the use of stochastic Schrödinger equations [75,76]. We propose here that the Duffing equation can be viewed as the coherent state approximation of the following non-Hermitian quantum Hamiltonian

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2} - \frac{i\gamma}{2}\hat{p}^2 + \frac{\beta}{2}\hat{q}^2 + \frac{\alpha}{4}\hat{q}^4 - F\hat{q}\cos(\omega t), \quad (4.24)$$

where F is the forcing term, β the strength of the quadratic potential, α the strength of the quartic potential, and the anti-Hermitian part of the Hamiltonian $-\frac{i\gamma}{2}\hat{p}^2$ plays the role of friction. The main advantage in adopting a non-Hermitian Hamiltonian approach is the simple numerical implementation and analysis that it affords.

We will now show that the coherent state approximation applied to the Hamiltonian (4.24) yields the Duffing equation (4.23). The classical equations of motion arising from the coherent state approximation are

$$\dot{q} = p, \quad \dot{p} = -\beta q - \alpha q^3 + F \cos(\omega t) - \gamma p. \quad (4.25)$$

Taking the time-derivative of the expression for $\dot{q}$ in (4.25) yields

$$\ddot{q} = -\beta q - \alpha q^3 + F \cos(\omega t) - \gamma p, \quad (4.26)$$

or

$$\ddot{q} + \gamma \dot{q} + \beta q + \alpha q^3 = F \cos(\omega t), \quad (4.27)$$

which is the desired Duffing equation.

Alongside the equations of motion we have the classical norm which follows the equation of motion

$$\dot{w} = -\gamma p^2 w. \quad (4.28)$$

The classical norm map for a relatively short final time $t_f = 4$ corresponding to the strange attractor in Figure 4.14a is depicted in Figure 4.14b. The structure of the strange attractor is clearly visible in the norm map.

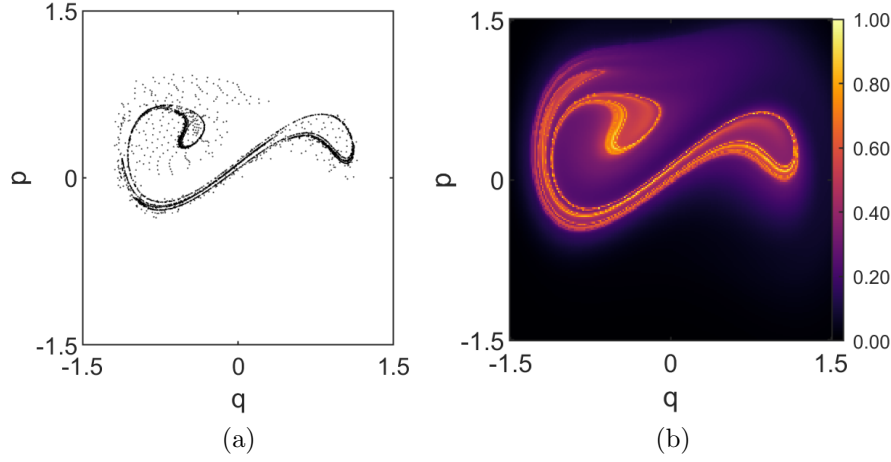


Figure 4.14: Classical Poincaré section and norm map of the Duffing oscillator. In the left plot we show a strange attractor of the Duffing oscillator and in the right the norm map for $F = 0.3, \gamma = 0.2, \alpha = 1, \beta = -0.75$, and $t_f = 4$.

4.5.1 The Quantum System

We now detail the construction of the Floquet operator for the non-Hermitian Hamiltonian (4.24). For the quantum system, the periodicity of the Hamiltonian allows us to construct the Floquet operator $\hat{U}$ over a single period. However, unlike the kicked Hamiltonian encountered previously, the driving term $Fq \cos(\omega t)$ is continuous in time. As a result, the time evolution operator has to be evaluated as

$$\hat{U}(t, t_0) = \hat{T} e^{-i \int_{t_0}^t dt' H(t')}, \quad (4.29)$$

and does not simplify into two factors as for kicked systems. We must take into account the non-commutativity of the Hamiltonian $[H(t), H(t')]$. For a sufficiently small time interval δt the Floquet operator can be approximated by a split step method.

$$\hat{U}(t, t_0) \approx \prod_j e^{-i \hat{H}(t_j) \frac{\delta t}{\hbar}} \quad (4.30)$$

We will now study the quantum-classical correspondence through the classical density introduced previously. First we will consider the special case of a driven quartic oscillator with damping, arising when $\beta = 0$.

4.5.2 The Driven Dissipative Quartic Oscillator

For $\beta = 0$, the Duffing oscillator reduces to a dissipative driven quartic oscillator, described by the non-Hermitian Hamiltonian

$$\hat{\mathcal{H}} = \frac{p^2}{2} - \frac{i\gamma}{2}p^2 + \frac{\alpha}{4}q^4 - Fq \cos(\omega t). \quad (4.31)$$

The Hermitian driven quartic oscillator ($\gamma = 0$) classically possesses a mixed regular-chaotic phase space, which we depict in the classical Poincaré section Figure 4.15a and the classical Lyapunov heat map Figure 4.15b, for parameters $F = 0.3$ and $\alpha = 1$. We observe a bounded chaotic sea surrounding a regular island. The chaotic sea itself is surrounded by regular tori that are only slightly perturbed by the effect of the driving.

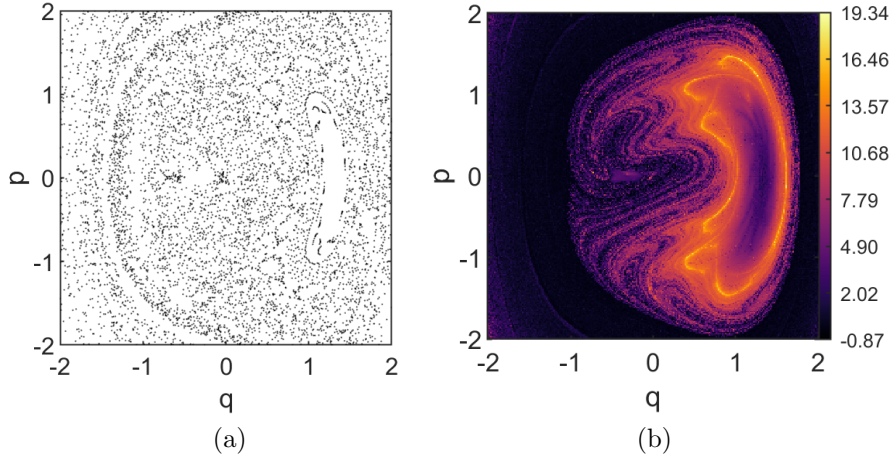


Figure 4.15: Classical Poincaré section (left) and Lyapunov heat map (right) of the driven quartic oscillator with $F = 0.3$ and $\alpha = 1$. The Lyapunov heat map is formed of $N^2 = 250$ uniformly distributed initial conditions with $t_0 = 20$ and final time $t = 80$.

Let us shown now the effect of the damping on the classical system. We use the same parameters as in the Hermitian case above, but include the damping with parameter $\gamma = 0.01$. The Lyapunov heatmap of the dissipative system is shown in Figure 4.16a. We observe that the boundary that surrounded the regular island, and had the largest Lyapunov exponent in the closed case (Figure 4.15b), is deformed by the damping. The chaotic sea now encloses a new structure in the phase space which is displayed in Figure 4.16b. The Poincaré dynamics associated to this region are shown in Figure 4.16c where we observe a clear divide at the boundary between the chaotic sea and this new structure.

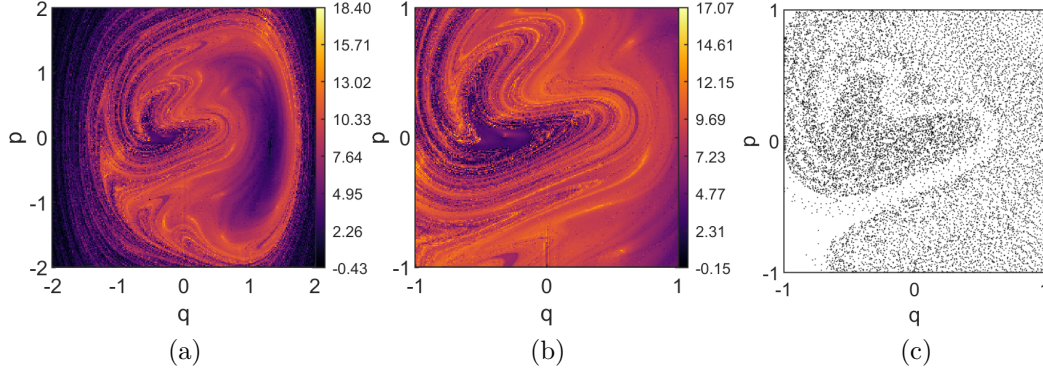


Figure 4.16: Lyapunov heat maps and Poincaré section of the dissipative driven quartic oscillator. The Lyapunov heat maps show the general behaviour across the phase space (left) and around the structure that is surrounded by the chaotic boundary. The Poincaré associated to this region of interest is displayed in the right plot. Each plot has associated parameters $F = 0.3$, $\alpha = 1$ and $\gamma = 0.01$

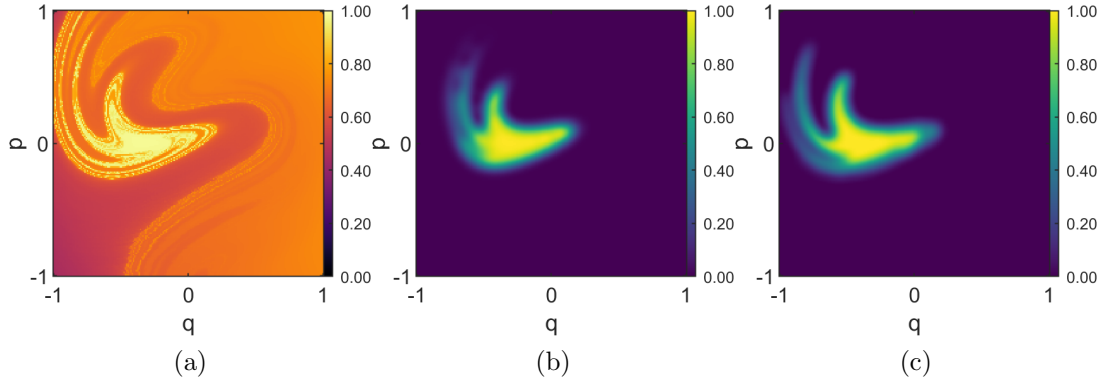


Figure 4.17: classical norm map (left), Quantum Husimi distribution (middle), and classical density (right) for the dissipative driven quartic oscillator. $F = 0.2$, $\gamma = 0.01$, $t_f = 5$, $f_Q^{(LL)} = 0.06$.

The quantum phase space area associated to the 8 Schur vectors with longest lifetimes, for a matrix dimension $N = 200$, is displayed in Figure 4.17b. The Husimi distribution of the 12 Schur vectors, with leading eigenfunction $|\phi_+\rangle$, with longest lifetimes is shown in Figure 4.17b. We associate the phase space area of the Schur vectors to the backwards norm map, displayed in Figure 4.17a for $t_f = 3$, and observe a clear correspondence between the two. It is interesting to note that we also see a close resemblance to the Poincaré

dynamics and the phase space area associated to the Schur states. We now complete the full quantum-classical correspondence by constructing the classical density $\mathcal{D}_{LL}$ associated to these longest lived states (where the superscript “LL” denotes “longest-lived”). The quantum counting parameter which conditions the classical density is $f_Q^{(LL)} = 12/200 = 0.06$. The conditioned density is displayed in Figure 4.17c and we see a good agreement.

4.5.3 The Duffing Oscillator

We now construct the classical density for the Duffing oscillator with classical parameters $F = 0.3$, $\alpha = 1$, $\beta = -0.75$, $\gamma = 0.05$. This corresponding backwards norm map is shown in Figure 4.18a for final time $t_f = 4$. Because of numerical instabilities, the convergence in the Harmonic oscillator basis occurs for only the 8 Schur states with longest lifetimes when using a matrix dimension of $N = 175$ (For a useful review on the numerical convergence of the Harmonic oscillator basis in both Hermitian and non-Hermitian systems see [77]). The Husimi distribution of these 8 Schur vectors with longest lifetimes is shown in Figure 4.18b. This gives the quantum state counting parameter $f_Q^{(LL)} = 8/175 = 0.045$. The classical density $\mathcal{D}_{LL}$ is displayed in Figure 4.18c where we see a good agreement with the Quantum Husimi distribution and a beautiful representation of the strange attractor in this non-Hermitian quantum system and its classical density.

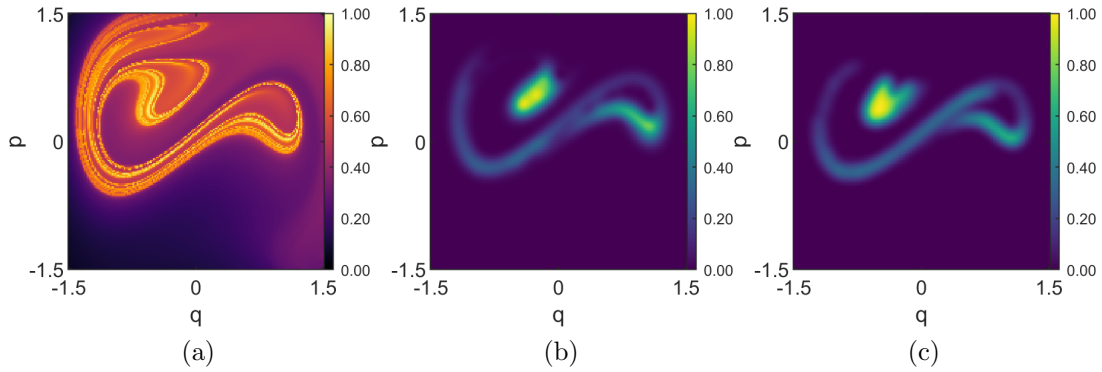


Figure 4.18: Norm map, Quantum Husimi distribution and associated classical density depicting the strange attractor of the of the Duffing oscillator for parameters $F = 0.3$, $\gamma = 0.2$, $\alpha = 1$, $\beta = -0.75$. The Husimi distribution corresponds to the 8 Longest lived states.

Chapter 5

Summary and Further Work

5.1 Summary

In this thesis we have investigated quantum classical correspondence in a variety of non-Hermitian systems. We have developed a correspondence between the classical norm maps and the quantum Husimi distributions of the Schur vectors. We have introduced an algorithm that provides a classical density that closely resembles the quantum Husimi distributions. We have demonstrated effectiveness of the density in a variety of non-Hermitian systems including a PT-symmetric kicked rotor, a non-Hermitian Duffing oscillator, and a complex harmonic oscillator. We have seen a good agreement between the classical density in all of the systems we have compared it against. Surprisingly the correspondence seems to work regardless of the chaos in the classical phase space. Thus, the algorithm for the classical density provides a general tool for performing quantum-classical correspondence in non-Hermitian quantum systems.

5.2 Further Work

5.2.1 Analytic Arguments for Quantum-Classical Correspondence

In Chapter 3 of this thesis we introduced a qualitative argument concerning the connection between the classical norm maps and quantum Schur vectors. This argument sought to clarify the relation between the leading eigenfunction in the Schur decomposition $|\phi_+\rangle$ ($|\phi_-\rangle$) and the observed cor-

correspondence between quantum Husimi distributions and backward (forward) norm map. We argued that in the long time limit, the Schur vectors with leading eigenfunction $|\phi_+\rangle$ become the eigenvectors of the quantum operator $\hat{W} = \hat{U}(t)\hat{U}^\dagger(t)$. Since this operator is Hermitian we can evoke standard ideas of quantum classical correspondence. I.e. the eigenvectors of the operator $\hat{W}$ are organised according to the features of the classical phase space function associated to $\hat{W}$. We define the classical phase-space function as $\langle z | \hat{W} | z \rangle$ in leading order, where $|z\rangle$ denotes a coherent state. This corresponds to the norm of a backwards evolved state in a semiclassical limit. In the future we hope to provide a rigorous proof of this statement. The analytic argument also assumed that all the eigenvectors had complex eigenvalues. It will be interesting to see how the argument is modified when a mix of complex and real eigenvalues are considered, which is clearly important for PT-symmetric systems in the regime of broken PT-symmetry.

It may also be interesting to investigate the quantum mechanical interpretation of the Schur vectors by viewing the Schur vectors themselves as eigenvectors (or Floquet modes) of some, possibly fictitious, Hermitian matrix (unitary matrix generated by a fictitious Hamiltonian). Similar interpretations have already been made between PT-symmetric Hamiltonians and Hermitian Hamiltonians [78].

5.2.2 Husimi distributions of eigenvectors

It would be interesting to see if the norm can be used to predict the Husimi distributions of eigenvectors. Figure 5.1 shows the Husimi distributions of the eigenvectors of the PT-kicked rotor for $k = 5$ and $\gamma = 0.001$. We observe localisation on the structures of both the forward and backwards classical norm maps.

We have made some progress towards the classical distributions associated to the Husimi distributions using what we refer to as “privileged trajectories”. The “privileged trajectories” refer to the sets of trajectories with overall norm growth, stability, or decay. For a given set of these trajectories we then relate the classical phase space area associated to the Husimi distributions to subsets of the phase space visited by these trajectories. Examples of the distributions we obtain are shown in Figure 5.2. The correspondence is not exact but we see similarities between these and the Husimi distributions of gain and loss in Figure 5.1. In particular we note that the classical distributions exhibit the asymmetry seen in the Husimi distributions in the upper and lower p -plane.

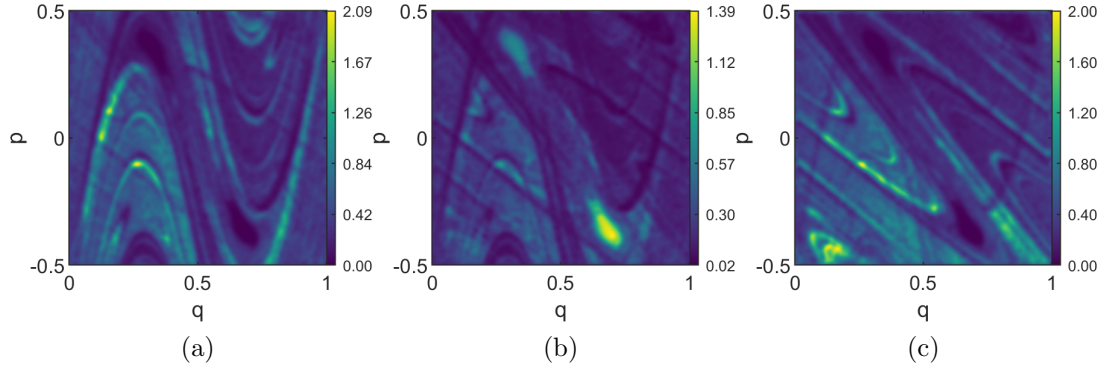


Figure 5.1: The Husimi distributions of the eigenvectors corresponding to the gain set (left), stable set (middle), and loss set (right) of the PT-symmetric kicked rotor for kicking strength $k = 5$ and Matrix dimension $N = 1001$.

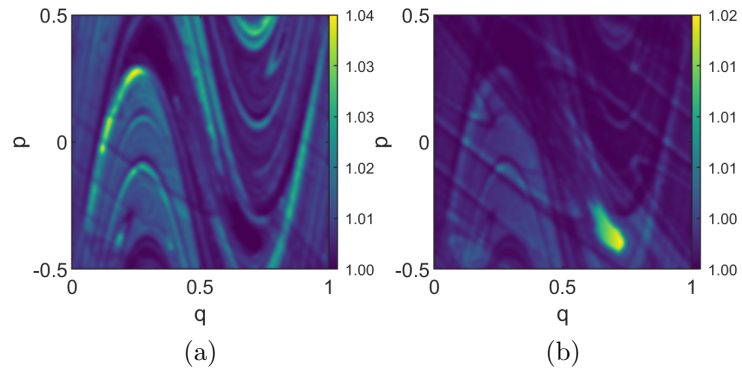


Figure 5.2: Two examples of Privileged trajectories which show a resemblance to the Husimi distributions of the gain states (left) and stable states (right).

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