SCATTERING OF SOUND BY FINITE THIN ELASTIC PLATES AND CAVITIES

by

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May 1982

Submitted for the degree of Doctor of Philosophy of the University of London and for the Diploma of Imperial College

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Acknowledgements

I would like to thank Dr. F.G. Leppington for all the support he has given whilst supervising this project and for his real enthusiasm when discussing all aspects of the work. My admiration and appreciation cannot be stressed highly enough. I am also extremely grateful to the S.E.R.C. and A.U.W.E. Portland for the financial support of a C.A.S.E. studentship.

Abstract

A theoretical investigation is made of the acoustic field scattered by a finite thin elastic plate set in an infinite hard screen. The problem is examined when forcing is supplied by a monopole (or quadrupole) source in the near field and this is contrasted with results found with incident plane wave forcing. It is also desired to calculate the interior fields of cavities and ducts (or any other geometry) which may be appended to one side of the screen.

In general, no exact solutions can be found and so asymptotic methods are employed by allowing physical parameters to be large or small. Suitable non-dimensional variables include a fluid loading parameter (a measure of the influence of the fluid on plate deflections) and the plate length measured with respect to an acoustic wavelength.

It is found that leading order approximations in all limits break down near to a plate resonance, which is defined to occur when large amplitude standing waves form on the plate. In the heavy fluid loading limit the leading order estimate is simply the field generated in the absence of the plate, but by using the method of matched asymptotic expansions it is found that, at a resonance, eigensolutions and travelling plate waves must be added to the acoustic potential. Similarly, in the low loading limit, correction potentials are derived by the use of Green's functions and physical argument, and these validate the solution at both plate and cavity resonances.

In the low frequency (or small plate) model the field is easily found but in the alternate large plate limit the problem is more difficult, and so a new method is presented to simplify the analysis of this and similar problems.

The effects of plate curvature and flow are also considered in this work.

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Chapter 1

Introduction

(i) Brief History

Before stating the problems under analysis in this thesis it is useful to briefly review the history of acoustics. It is hoped that this introduction will illuminate the context in which the work is presented and thus help to justify the specialised nature of both the problem and the analysis. A short survey of the literature may also allow the reader a glimpse of the beauty of the subject (both for its elegant mathematics and powerful physical arguments) and highlight the work performed by distinguished individuals in the field.

The first analysis of sound was made by the Greek philosophers, including Pythagoras (6th century B.C.), who were aware (though in the vaguest of terms) of the propagation of sound through the air. Unfortunately the subject did not develop quickly but lay dormant until mechanics and other related fields had expanded enough to aid theoretical analysis. Up to the beginning of this century acoustics had been studied in two discrete sections. Firstly sound generation by plucked strings, vibrating plates, etc., and secondly the propagation of sound, including the phenomenon of pipe resonance and ideas of diffraction etc.

In the field of sound production, Galileo Galilei (1564-1642) published a work in 1638 discussing the relationship between frequency of vibration and pitch, and it was also clear that he had an understanding of the dependence of the frequency of a stretched string on its length, tension and density. Further work on strings and organ pipes was made by Sauver (1653-1716) who, incidentally, named the study of sound "acoustics", and Taylor (1685-1731) was the first to find a dynamical solution of the vibrating string. Sauver also suggested the notions of

the fundamental frequency and higher harmonics and was the first to note the principle of superposition which is attributed to Daniel Bernoulli (1700-1782). D'Alembert and Euler did much work in the subject in the mid 18th century, followed by Fourier and Lagrange a few years later. Lagrange's study included predictions on the harmonic frequencies of closed and open pipes. During the 19th century the vibrations of plates, membranes and bars were studied by many notable mathematicians including of course Rayleigh (1842-1919).

Returning to the propagation of sound it is interesting to note that even during the Galilean period there was still much confusion as to whether sound needed a medium through which to propagate. Indeed it was not until 1687 that Newton made the first attempt to postulate a wave theory of sound. His arguments included some specific and arbitrary assumptions but led to an estimate of the wave speed which was in reasonable agreement with experimental evidence of the day. Lagrange followed with a more rigorous general derivation which was found to predict the same result as that of Newton. The matter was finally cleared by Laplace (1749-1827) who had the insight to use the adiabatic law of elasticity in his calculation.

By the end of the 18th century the use of the wave equation with boundary conditions had become familiar practice and in the next century much was accomplished by Poisson (1781-1840), Helmholtz (1821-1894) and Rayleigh. Reflection and refraction of sound in a problem with two fluids was studied by Green (1793-1841) and this work highlighted the similarity between the diffraction and radiation of sound and light. The work on the theory of light by the early pioneers, including Huyghens, is therefore of great importance in the later study of acoustics.

A significant piece of analysis in optics is that by Sommerfeld (1896) on diffraction of a plane-wave by a semi-infinite screen. He derived a formally exact solution (in terms of Fresnel integrals) but the problem of diffraction by a slit in an infinite hard screen (Schwarzschild (1902)) does not have an exact solution. It will become clear to the reader that any diffraction problem, other than those with the most simple boundary conditions have only approximate solutions.

The beginning of the 20th century saw massive expansion within the world of science, as well as industrialisation of western society. The field of fluid mechanics blossomed, and within it the subjects of hydrodynamic stability, aerodynamics, water waves and vortex mechanics were developing rapidly. All these subjects overlap with acoustics, as will be discussed later, and had arisen because of technological advancements such as the airplane.

With the advent of the wireless there became a need for a better understanding of diffraction of electromagnetic radiation. Problems with complicated boundaries were now studied with approximations made in both the long wave limit (tending to the equivalent electrostatics problem), and in optics (ray theory incorporated). Difficulty was encountered with short wave radio where no approximations could be made.

New mathematical techniques were becoming available to the physicist, and allowed a better understanding of the problem studied, enabling generalisations to be made. Many new problems could be studied using the techniques of complex variable theory (with generalised functions), integral transforms, modal expansions etc. Asymptotic analysis (including the method of steepest descents) allowed a more rigorous method of approximation to be made and Levine and Schwinger (1948, 1949) produced an approximate technique involving a variational principle

(Baker and Copson (1950)). Wiener and Hopf (1931) presented a method of solution of a class of integral equations which led to a great advance in diffraction theory, being of use in problems with semi-infinite geometries.

The following three decades saw the continued development of compressible fluid mechanics with the rise of shock wave theory. There was renewed interest in acoustics through the use of sonar in military applications and from industry in loudspeaker design and architectural acoustics. Many of the problems already analysed in electromagnetic theory had direct application to acoustics, and the same approximations could be made in both subjects. For example geometrical ray theory is used in the study of room acoustics, and the low frequency approximation is also of use in a wide variety of applications. During this period great advances were made in the numerical solution of scattering problems and experimental methods also improved.

The field of acoustics had, by this time, grown so diverse that in this synopsis it is necessary to restrict attention to advances in the field of diffraction, concerning the theoretical investigation of complicated boundary value problems. Jones (1952A) greatly simplified problems with semi-infinite geometries by showing a method of obtaining the Wiener-Hopf functional equation without first deriving the equivalent integral equation form. D.S. Jones (1952B) also looked at the problem of diffraction by a finite wave-guide and derived the potential as the solution of a pair of coupled integral equations. These equations are analogous to those derived by Schwarzschild (1902) in the finite gap problem but Jones used a modified form of the Wiener-Hopf technique. Noble (1958) published the definitive text on the Wiener-Hopf technique applying the method to a wide range of diffraction problems. The later

chapters in the book show how a modified form of the Wiener-Hopf technique may be applied to problems with finite geometries. As with the finite wave-guide problem, the modified technique always derives the potential as the solution of a pair of coupled integral equations. The integral equations can be solved in various asymptotic limits, the usual being the high frequency limit. It is very informative to study the bibliography in Noble's book as it clearly illustrates the range of problems of interest to engineers and applied mathematicians at the time of publication. Examples include work in both the field of water waves and electromagnetic or acoustic wave theory by such authors as Carlson and Heins (1946), Heins (1948), Jones (1953) and Williams (1954).

As a direct result of aircraft instability the study of subsonic flutter has been popular since the 1930's, although much of the later work has been concerned with high Mach number flows (Shen (1952)), Fung (1954)). The instability of a finite elastic plate set in an infinite baffle has been studied using a modal expansion for the deflection (Dowell (1966), Ellen (1972)). The solution then reduces to finding the coefficients in the expansion and approximate results of the eigenvalue problem are found by use of the Galerkin method. The same boundary value problem (but with acoustic forcing) is analysed in §4, using an entirely different approach, and it is therefore useful to compare results obtained by both methods.

The next decade saw the method of matched asymptotic expansions become widely used in the fields of aerodynamics, water waves and acoustics. A useful book describing the application of the method to a wide range of problems was written by Van Dyke (1964). A modified form of Van Dyke's matching principle is given in a paper by Leppington (1972) who analyses the radiation and scattering of surface water waves by

partially immersed objects, this scheme being used in §3. Although the method is not justifiable in a rigorous mathematical sense it does highlight the physical features of the problem and therefore is also of use in avoiding difficulties in numerical computation. At the same time as "matching" was becoming an important technique in applied mathematics the field of aerodynamic sound generation was also blossoming. As suggested by Lighthill (1952, 1954) it was shown that the acoustic field generated by compact turbulence is identical to that induced by a distribution of quadrupole sources. It was later shown that the far field generated by the turbulence could be greatly enhanced if surfaces or edges were present in the problem (Curle (1955), Ffowcs Williams and Hall (1970), Crighton and Leppington (1970)). The rapid growth in the use of the jet engine in commercial aeroplanes prompted much of the work in this field as noise "pollution" was becoming a significant political issue. The method of matched asymptotic expansions was a valuable tool in the analysis of many problems in this field, examples of which include work by Crow (1970), Cannell (1974).

Since the beginning of 1960 the only major new development in acoustics has been the analysis of problems involving non-linearities in the boundary conditions or governing equations. Studies range from the non-linear effects of tubes near resonance (Keller (1977)), to propagation of acoustic waves satisfying the Burgers' equation (Blackstock (1972)). Most problems of direct interest to industry and the military are, however, still linear, and in the last decade have included studies on the sound generated by infinite and semi-infinite vortex sheets (Jones and Morgan (1972), Jones (1973), Crighton and Leppington (1974)) as well as scattering by compliant surfaces or shear layers (Leppington (1977), Leppington and Levine (1973), Jones (1977)).

In recent years the scattering of sound by elastic plates or membranes has become important in both aerodynamic noise theory and underwater acoustics. First analysis included infinite plates (Junger and Feit (1972), Crighton (1971)) followed by semi-infinite flexible surfaces (Cannell (1975), (1976)) and surfaces with periodically arranged struts (Leppington 1978). This thesis attempts the analysis of scattering by finite elastic plates which are more physically realistic than infinite models and the work was initiated because of its direct application to problems in underwater acoustics.

(ii) The problem under investigation

The major thesis parts (2, 3, 4 and 6) are presented as self contained papers each having its own introduction and discussion. The study examines the scattering of sound by finite plates and membranes in the presence of external geometry (e.g. ducts or cavities). No exact solutions to the problems can be derived and so approximate estimates have been found in various asymptotic limits. Having established the asymptotic form of the scattered field the concluding discussion (§8) shows that generalisations can be made to give order of magnitude estimates over all values of the physical parameters in the problem.

As suggested previously the problem with no external geometry could be analysed using the modified Wiener-Hopf technique but the potential is given as the solution of a pair of coupled integral equations. Asymptotic solutions of the equations could be found in various limits but more direct methods are employed in the following analysis which give a better understanding of the physical nature of the problem.

Turning now to look at the asymptotic limits chosen in the problem, the elastic plate equation (§3 equation (1.3)) can be seen to relate the surface deflection to the acoustic pressure of the fluid around the plate. It is therefore convenient to define a fluid loading parameter (§3 eqn. (1.10)) which, roughly speaking, is a measure of the influence of the sound field on the plate deflection. The limits of large and small fluid loading, for a range of frequencies of the acoustic source, correspond physically to the plate being immersed in water and air respectively. The ratio of the plate length to wavelength of the

forcing potential is another parameter which can be allowed to become very large or small in order to simplify the problem. Chapter 2 takes the small fluid loading limit and uses a method similar to that of Leppington (1976). In this limit the leading order solution to the scattered field is that generated by the plate if it were vibrating in a vacuum. This solution breaks down near a plate resonance and Leppington derived a corrected approximation (valid at a resonance) by a simple physical argument. In Chapter 2 the problems of plates and membranes backing ducts or cavities is tackled in the same way, whereby the leading order solution predicts infinite behaviour at both plate and cavity (or duct) resonances. A method is presented which modifies the leading order result and predicts a large but finite potential at a cavity resonance.

In Chapter 3 the same model is analysed in the alternate heavy loading limit. To leading order the acoustic potential behaves as that induced by the geometry in the absence of the plate but this result is shown to be invalid close to the plate. This is due to the presence of travelling waves on the elastic surface. Asymptotic matching is performed between "inner" regions close to the plate edges and the "outer" field, and the scheme shows that non-attenuating plate waves are launched from each edge. For specific values of the plate length, standing plate waves are therefore formed, and this is identified as a plate resonance. To accommodate the change in order of magnitude of the acoustic field near a resonance, the matching scheme is suitably modified (similar to a scheme by Alker (1978)); this yields the result that near resonance, eigensolutions of the outer problem, with singularities at the plate edges, also become present at leading order.

The problem is modified in §4 to include flow and this is analysed for both moderate and small Mach number. The matching scheme in the previous chapter is adopted for this model, and for a large subsonic flow velocity it is found that the inner lengthscale is different from that in the zero flow case, resulting in a modified resonance condition. As the Mach number is reduced to zero the result in §3 is not recovered hence suggesting a separate analysis for small flow velocities. This is performed and the results match smoothly onto those predicted by the studies for zero and large flow velocities.

An important feature of the problems discussed in sections 3 and 4 was the presence of plate resonances. This occurred because waves generated at each edge travelled along the plate without attenuation. Many problems in underwater acoustics involve curved plates and so it was thought necessary to examine the effect of slight curvature on the attenuation of the travelling plate waves. This is discussed in §5 and results reveal that the energy loss of plate waves, as they travel from one edge to the other, is exponentially small so plate resonances are not significantly suppressed.

Returning to the work in section 3 it is found that when performing matching the "outer" boundary value problems cannot, in general, be solved explicitly. The problems are well behaved (i.e. singularities are only present at the plate edges or infinity) and so are open to numerical computation. If, however, an analytical solution is required then further asymptotic approximations must be made. This is performed in §3 with the "small gap" limit and a solution is found by conformal transformation. Of more interest is the "small gap" problem with a cavity on one side. This is tackled in §8 by use of Green's functions

and results show that cavity resonances severely alter the characteristics of the acoustic field. The other obvious limit, which possibly has greater application in underwater acoustics, is that of a gap which is large compared to an acoustic wavelength. As stated previously this can be analysed by the method of Schwarzschild or the modified Wiener-Hopf technique. Both methods, however, are extremely complicated and so it was thought useful to develop a new approach with the aim of simplifying the algebra. Chapter 6 is devoted to this method, which reduces the analysis by applying the large gap approximation at an early stage in the procedure. It is also shown that more complicated geometries, including problems with resonance, can also be tackled in a straightforward manner.

The last chapter of the main contents of this thesis (§7) shows how the problem of a lightly loaded finite elastic plate near resonance may be tackled by the method of modal expansions. The problem reduces to determining the coefficients of the expansion for the plate deflection and the results are analogous to those found by Leppington (1976). The ease of use of the method is apparent and this chapter also tackles the three-dimensional problem of flow over a rectangular flexible plate. The condition for divergence instability is sought and the result is found to agree with that predicted by Ellen (1972). If one of the dimensions of the plate was allowed to tend to infinity (thus reducing the problem to flow over an infinite strip) then this problem could be tackled using alternative techniques (as performed in §4). However, although the method used is not mathematically rigorous, no other procedures can be used for fully three-dimensional problems.

Finally, having established the asymptotic forms of the acoustic field in preceding chapters, the concluding discussion (§8) draws generalisations about its behaviour over all values of the physical parameters. Comment is also made about possible applications of the methods developed and further work, relevant to the physical problem, is suggested.

Chapter 2

Scattering of sound by finite elastic surfaces bounding ducts or cavities near resonance

An offprint from

QUARTERLY JOURNAL OF MECHANICS AND APPLIED MATHEMATICS

Volume XXXV, Part 1 February 1982

CLARENDON PRESS • OXFORD

SCATTERING OF SOUND BY FINITE ELASTIC SURFACES BOUNDING DUCTS OR CAVITIES NEAR RESONANCE

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[Received 19 November 1980]

SUMMARY

A finite membrane is set in one of the infinite plane rigid walls of a two-dimensional duct and the whole is immersed in a stationary acoustic medium. A plane wave is incident onto the elastic surface and the sound field is examined in the limit of small fluid loading. Leading-order asymptotic solutions for the potential in the duct break down near both duct and membrane resonances, but improved estimates are derived which include the effect of the small, but finite, radiation damping in the system. The method is valid for a wide range of geometries and is performed on the problem with an elastic plate bounding one side of a cavity.

1. Introduction and governing equations

MANY problems occur in aerodynamic noise theory where the sound field interacts with compliant surfaces. Work has been carried out, in general on models with infinite or semi-infinite flexible surfaces (**1, 2, 3**), or with line constraints (**4**). Difficulty arises with finite elastic surfaces, which are more physically realistic, due to the increased complexity of the boundary geometry and so most solutions involve asymptotic or other approximate techniques which assume the fluid loading parameter, α, to be large or small. The fluid-loading parameter, α, is a measure of the mass of the fluid (in a fluid wavelength) to the mass of the elastic plate and conveniently, for a certain acoustic frequency range, the limit α small corresponds to the plate being immersed in air.

Another phenomenon involved with finite flexible surfaces is the possibility of resonance which has been considered by Leppington (**5**), and this paper extends that work by looking at duct and cavity geometries which create further resonant conditions.

All problems investigated are two-dimensional and assuming simple harmonic time dependence, with angular frequency ω, the velocity potential can be expressed as

$$\mathrm{Re}\{\Phi(x, y)\exp(-i\omega t)\}, \tag{1.1}$$

and the time factor, $\exp(-i\omega t)$, will henceforth be suppressed. Cartesian coordinates (x, y) are chosen so that the membrane (or plate) lies on the plane $y = 0$ and $|x| < a$, and the whole is immersed in an inviscid, stationary,

compressible fluid, whose velocity potential therefore satisfies

$$(\partial^2/\partial x^2+\partial^2/\partial y^2+k^2)\Phi=0, \tag{1.2}$$

where $k=\omega/c$ is the fluid wave number and c is the sound speed in the fluid.

The pressure fluctuations $p(x, y)$ and membrane (or plate) deflection $\eta(x)$ can be expressed in terms of Φ by

$$p(x, y)=\rho_0 i\omega\Phi(x, y), \tag{1.3}$$

$$\eta(x)=(i/\omega)\Phi_y(x, 0), \tag{1.4}$$

where ρ_0 is the mean fluid density and the suffix refers to partial differentiation. Using the linearised membrane equation, which relates pressure to deflection, and using (1.3) and (1.4) yields

$$(\partial^2/\partial x^2+\mu^2)\Phi_y-\alpha\Phi|_-^+=0, \qquad |x|<a, \qquad y=0, \tag{1.5}$$

where

$$\mu=\omega(m/T)^{\frac{1}{2}} \tag{1.6}$$

is the wave number for a membrane in vacuo, T is the surface tension, and m is the mass per unit length of the membrane. Also $\Phi|_-^+$ signifies the discontinuity in the velocity potential across the membrane and $\Phi|_-^+=\Phi_+-\Phi_-$, where Φ_+ denotes the potential just above the surface, and Φ_- just below. The fluid loading parameter α is

$$\alpha=\rho_0\mu^2/m, \tag{1.7}$$

and is assumed small in this paper. Similarly for the elastic plate it is found that

$$(\partial^4/\partial x^4-\mu_1^4)\Phi_y+\alpha_1\Phi|_-^+=0, \qquad |x|<a, \qquad y=0, \tag{1.8}$$

where

$$\mu_1^4=mh\omega^2/D, \qquad \alpha_1=\rho_0\omega^2/D, \tag{1.9}$$

and D is the bending stiffness, h is the plate half-thickness and the suffixes attached to α_1, μ_1 simply distinguish the plate parameters from the membrane parameters.

All the models discussed in this paper need a radiation condition at infinity to ensure outgoing waves and also edge conditions. For the membrane $\eta=0$ is taken as the condition at either edge, and $\eta=\eta_x=0$ for the plate problem.

2. Leading-order approximation as $\alpha\to 0$

Specifications for the duct problem are

$$\left.\begin{aligned}(\nabla^2+k^2)\Phi&=0, \quad \text{all} \quad x, y>-h,\\ (\partial^2/\partial x^2+\mu^2)\Phi_y-\alpha\Phi|_-^+&=0, \quad y=0, \quad |x|<a,\\ \Phi_y&=0, \quad y=0, \quad |x|\geqslant a,\\ \Phi_y&=0, \quad y=-h, \quad -\infty<x<\infty.\end{aligned}\right\} \tag{2.1}$$

Suppose the problem is forced by incident waves of the form

$$\phi_i(x, y) = \tfrac{1}{2}e^{-iky\cos\theta}(e^{ikx\sin\theta} + e^{-ikx\sin\theta}), \tag{2.2}$$

which helps to simplify the algebra by preserving symmetry in x, and θ is the angle between the incident direction and the y-axis. It is convenient to express the total potential as

$$\Phi = \begin{cases} \phi_i + \phi_r + \phi, & y > 0, \\ \phi, & y < 0, \end{cases} \tag{2.3}$$

where ϕ_r is the reflected wave and ϕ is the 'diffracted' potential. The problem for ϕ is now

$$\left.\begin{aligned} (\nabla^2 + k^2)\phi = 0, &\qquad \text{all } x, y, \\ (\partial^2/\partial x^2 + \mu^2)\phi_y - \alpha\phi|_-^+ = 2\alpha\cos k_2 x, &\qquad y = 0, \quad |x| < a, \\ \phi_y = 0, &\qquad y = 0, \quad |x| > a, \\ \phi_y = 0, &\qquad y = -h, \quad -\infty < x < \infty, \end{aligned}\right\} \tag{2.4}$$

where $k_2 = k\sin\theta$. Also ϕ must satisfy a radiation condition at infinity to ensure only outgoing waves.

Away from a resonance ϕ and ϕ_y are of order α and so the membrane equation in (2.4) can be approximated to leading order, as $\alpha \to 0$, by

$$(\partial^2/\partial x^2 + \mu^2)\phi_y = 2\alpha\cos k_2 x, \qquad y = 0, \qquad |x| < a, \tag{2.5}$$

which immediately gives a solution for ϕ_y on $y = 0$,

$$\phi_y(x, 0) = \begin{cases} \dfrac{2\alpha}{(\mu^2 - k_2^2)}\left(\cos k_2 x - \dfrac{\cos k_2 a}{\cos \mu a}\cos \mu x\right), & |x| < a, \\ 0, & |x| \geqslant a. \end{cases} \tag{2.6}$$

A solution for $\phi(x, y)$ is now obtainable using Fourier transforms. Defining

$$\psi(s, y) = \int_{-\infty}^{\infty} e^{isx}\phi(x, y)\, dx, \tag{2.7}$$

and

$$\phi(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\psi(s, y)\, ds, \tag{2.8}$$

then the transformed boundary conditions become

$$\psi_y(s, -h) = 0, \tag{2.9}$$

$$\psi_y(s, 0) = \frac{2\alpha}{(\mu^2 - k_2^2)}\left[\frac{\sin a(s + k_2)}{s + k_2} + \frac{\sin a(s - k_2)}{s - k_2} - \frac{\cos k_2 a}{\cos \mu a}\left\{\frac{\sin a(s + \mu)}{s + \mu} + \frac{\sin a(s - \mu)}{s - \mu}\right\}\right]. \tag{2.10}$$

Taking the transform of the Helmholtz equation gives a solution

$$\psi = Ae^{\gamma y} + Be^{-\gamma y} \quad \text{for} \quad s > k, \tag{2.11}$$

where $\gamma = (s^2 - k^2)^{\frac{1}{2}}$. Now γ may be regarded as a complex function of s with branch cuts from $s = \pm k$ to $\pm\infty$, with $\gamma = -ik$ when $s = 0$, and the integration path of (2.8) passes above the negative real axis and below the positive axis. Applying the boundary conditions in (2.9), (2.10) gives

$$\psi(s, y) = \psi_y(s, 0) \frac{\cosh \gamma(y+h)}{\gamma \sinh \gamma h}, \qquad -h < y < 0. \tag{2.12}$$

Thus

$$\phi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_y(s, 0) \frac{\cosh \gamma(y+h)}{\gamma \sinh \gamma h} e^{-isx}\, ds, \qquad -h < y < 0, \tag{2.13}$$

where the integrand has simple poles at

$$\gamma = \pm n\pi i/h, \qquad n = 0, 1, 2, \ldots, \tag{2.14}$$

and so the solution can be found by using the method of residues. Closing the path of integration in the lower half-s-plane picks up the contribution from the poles at

$$s = \begin{cases} -k, \\ -(k^2 - (n\pi/h)^2)^{\frac{1}{2}}, & \text{where} \quad k > n\pi/h, \\ -i((n\pi/h)^2 - k^2)^{\frac{1}{2}}, & \text{where} \quad n\pi/h > k. \end{cases} \tag{2.15}$$

For $x > 0$ the contribution from the arc in the lower half-plane goes to zero as the radius becomes large, and so

$$\phi(x, y) = -2\pi i \sum_{n=0}^{\infty} b_1^{(n)}, \qquad -h < y < 0, \tag{2.16}$$

where $b_1^{(n)}$ is the nth residue. Calculating the residues using an expansion of $\sinh \gamma h$ about $-n\pi i$ gives, for the pole at

$$s = -i((n\pi/h)^2 - k^2)^{\frac{1}{2}},$$

$$b_1^{(n)} = \frac{i}{2\pi} \psi_y^{(n)} \frac{\cos(n\pi y/h)}{((n\pi)^2 - (kh)^2)^{\frac{1}{2}}} \exp\{-((n\pi/h)^2 - k^2)^{\frac{1}{2}} x\}, \qquad n\pi/h < k, \tag{2.17}$$

where

$$\psi_y^{(n)} = \psi_y(s, 0), \text{ evaluated at } s = -i((n\pi/h)^2 - k^2)^{\frac{1}{2}},$$

and similarly for $s = -(k^2 - (n\pi/h)^2)^{\frac{1}{2}}$, $n \neq 0$,

$$b_1^{(n)} = -\frac{1}{2\pi} \psi_y^{(n)} \frac{\cos(n\pi y/h)}{((kh)^2 - (n\pi)^2)^{\frac{1}{2}}} \exp\{i(k^2 - (n\pi/h)^2)^{\frac{1}{2}} x\}. \tag{2.18}$$

When $n=0$ the pole is at $x=-k$ and

$$b_1^{(0)} = -\psi_y^{(0)} \frac{e^{ikx}}{4\pi kh}. \tag{2.19}$$

By inspection of (2.17), (2.18) and (2.19) it is clear that when k approaches $n\pi/h$ for $n=0, 1, 2, 3...$, then $b_1^{(n)}$ becomes infinite and so this approximate solution becomes invalid near a duct resonance.

3. Solution near a duct resonance

It can be seen that the solution for $\phi(x, y)$, $-h<y<0$, away from a resonance behaves as

$$\phi(x, y) = \sum_{n=0}^{\infty} \frac{i\psi_y^{(n)}}{hC_n} \cos(n\pi y/h) \exp(iC_n|x|), \tag{3.1}$$

where

$$C_n = \begin{cases} 2k, & n=0, \\ (k^2-(n\pi/h)^2)^{\frac{1}{2}}, & 0<n<kh/\pi, \\ i((n\pi/h)^2-k^2)^{\frac{1}{2}}, & n>kh/\pi, \end{cases} \tag{3.2}$$

and $\psi_y^{(n)} = \psi_y(C_n, 0)$. The potential above the plate can be shown to be

$$\phi(x, y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_y(s, 0) \frac{e^{-isx-\gamma y}}{\gamma} ds, \qquad y>0, \tag{3.3}$$

where the integration path is the same as in (2.8).

Near the pth resonance, i.e. $C_p \to 0$, it is expected that $\phi(x, y)$ behaves as

$$\phi(x, y) \sim K \cos(p\pi y/h) + \sum_{\substack{n=0 \\ n\neq p}}^{\infty} \frac{i\psi_y^{(n)}}{hC_n} \cos(n\pi y/h) \exp(iC_n|x|) - \psi_y^{(p)}|x| \cos(p\pi y/h)/h, \qquad -h<y<0, \tag{3.4}$$

where

$$K = i\psi_y^{(p)}/(hC_p), \tag{3.5}$$

and will be large but finite at resonance (assumed of order unity or larger). The solution is therefore known when K is determined. It is convenient to write

$$\phi_y(x, 0) \equiv v_1(x) + v_2(x), \tag{3.6}$$

and the corresponding potential and its transform are

$$\left.\begin{aligned} \phi(x, y) &= \phi_1(x, y) + \phi_2(x, y), \\ \psi(s, y) &= \psi_1(s, y) + \psi_2(s, y), \end{aligned}\right\} \tag{3.7}$$

with $\phi_{1y}(x, 0) \equiv v_1(x)$ etc.

The plate equation in (2.4), using (3.3) and (3.4), now becomes

$$\left(\frac{d^2}{dx^2}+\mu^2\right)v_1(x)=2\alpha\cos k_2x-\alpha K, \tag{3.8}$$

and

$$\left(\frac{d^2}{dx^2}+\mu^2\right)v_2(x)=-\frac{\alpha}{2\pi}\int_{-\infty}^{\infty}\psi_y(s,0)\frac{e^{-isx}}{\gamma}\,ds$$
$$-\alpha\sum_{\substack{n=0\\n\neq p}}^{\infty}\frac{i\psi_y^{(n)}}{hC_n}\exp(iC_n|x|)+\alpha\psi_y^{(p)}|x|/h, \tag{3.9}$$

on $y=0$ and $|x|<a$. By hypothesis and by inspection of (3.8), (3.9), v_2 is taken to be an order of magnitude smaller than v_1, as $\alpha\to 0$, and hence (3.9) can be approximated by

$$\left(\frac{d^2}{dx^2}+\mu^2\right)v_2(x)=-\frac{\alpha}{2\pi}\int_{-\infty}^{\infty}\psi_{1y}(s,0)\frac{e^{-isx}}{\gamma}\,ds$$
$$-\alpha\sum_{\substack{n=0\\n\neq p}}^{\infty}\frac{i\psi_{1y}^{(n)}}{hC_n}\exp(iC_n|x|)+\alpha\psi_{1y}^{(p)}|x|/h. \tag{3.10}$$

Both v_1 and v_2 can now be determined and to satisfy the zero-displacement end conditions it is found that

$$v_1(x)=\frac{2\alpha}{\mu^2-k_2^2}\left(\cos k_2x-\frac{\cos k_2a}{\cos\mu a}\cos\mu x\right)$$
$$-\frac{\alpha K}{\mu^2}\left(1-\frac{\cos\mu x}{\cos\mu a}\right),\qquad |x|<a, \tag{3.11}$$

and

$$v_2(x)=-\frac{i\alpha^2K}{\pi\mu^2}\int_{-k}^{k}r(t)\left(\cos tx-\frac{\cos ta}{\cos\mu a}\cos\mu x\right)\frac{dt}{(k^2-t^2)^{\frac{1}{2}}(t^2-\mu^2)}+$$
$$+\frac{i\alpha^2K}{\mu^2}\sum_{n=0}^{n\leq p}\frac{r(C_n)}{hC_n(\mu^2-C_n^2)}\left(\cos C_nx-\frac{\cos C_na}{\cos\mu a}\cos\mu x\right)+$$
$$+iO(\alpha^2)+O(\alpha^2K), \tag{3.12}$$

where

$$r(s)=\frac{\sin sa}{s}-\frac{(s\sin sa\cos\mu a-\mu\cos sa\sin\mu a)}{(s^2-\mu^2)\cos\mu a}.$$

It will be seen that v_2 will be needed only to calculate K and that v_1 will be a valid approximation to $\phi_y(x,0)$ holding for all values of the wave number except at a plate resonance.

To finally evaluate K, equation (3.5), is used with (3.6) to give

$$K=\frac{i}{hC_p}\psi_{1y}(C_p,0)+\psi_{2y}(C_p,0) \tag{3.13}$$

as $C_p \to 0$, and after simplifying it is found that

$$K = \frac{2iq(k_2)}{(\mu^2 - k_2^2)}\{\alpha + O(\alpha^2)\}\left[hC_p + \frac{2\alpha i r(0)}{\mu^2} + iO(\alpha^2) + \right.$$
$$\left. + \frac{\alpha^2}{\mu^2}\left\{\sum_{n=0}^{n<p} \frac{r(C_n)q(C_n)}{hC_n(\mu^2 - C_n^2)} - \frac{1}{\pi}\int_{-k}^{k} \frac{r(t)q(t)}{(k^2-t^2)^{\frac{1}{2}}(t^2-\mu^2)}\,dt\right\} + O(\alpha^3)\right]^{-1}, \quad (3.14)$$

where $r(s)$ is given in (3.12), and $q(t)$ by

$$q(t) = 2\left(\frac{\sin ta}{t} - \cos ta \frac{\tan \mu a}{\mu}\right). \quad (3.15)$$

Thus the approximation to $\phi_y(x, 0)$, as $\alpha \to 0$, which holds for all k except at a plate resonance is given by $v_1(x)$ in (3.11), and can be used in (3.1) and (3.2) to determine the improved $\phi(x, y)$ which is now valid at a duct resonance.

Consistent with the initial assumption for the order in α of K, it is found that

$$K = \begin{cases} O(\alpha^0), & \text{when} \quad k = p\pi/h + \epsilon, \\ O(\alpha^{-1}), & \text{when} \quad k = p\pi/h - \epsilon, \end{cases} \quad (3.16)$$

for $\epsilon > 0$ as $\epsilon \to 0$. This is seen by examining the denominator of (3.14) which tends to $O(\alpha)$ if $C_p \to +\epsilon$ and $O(\alpha^2)$ if $C_p \to i\epsilon$ (when $r(0) < 0$). Note that if $r(0) > 0$ then $K = O(\alpha^0)$ and if $r(0) = 0$ then $K = O(\alpha^{-1})$ for both $k = p\pi/h \pm \epsilon$. The case $r(0) = 0$ corresponds to the plate length satisfying

$$\mu a = \tan \mu a. \quad (3.17)$$

It is therefore an interesting conclusion that the scattered potential ϕ is of order α, or α^0 near a duct resonance depending on the wave number being slightly greater than, or less than, a specific value.

4. Solution near a membrane resonance

Resonance of the membrane can occur in the model, as well as the duct resonance, and following a method by Leppington (**5**) an asymptotic solution is found which is valid for all k. A membrane resonance occurs when

$$\mu = \mu_n = (n + \tfrac{1}{2})\pi/a, \qquad n = 0, 1, 2, \ldots, \quad (4.1)$$

which corresponds to $\cos \mu_n a \equiv 0$. Thus, from (2.6), $\phi_y(x, 0)$ becomes undefined at these resonant values, as the second term becomes infinite. To obtain a valid asymptotic solution at resonance it is convenient to expect that the second term in (2.6) will become large (but finite) and so (2.6) is replaced by

$$\phi_y(x, 0) \sim v_1(x) = \begin{cases} \dfrac{2\alpha}{(\mu^2 - k_2^2)}(\cos k_2 x - K \cos k_2 a \cos \mu x), & |x| < a, \\ 0, & |x| > a. \end{cases} \quad (4.2)$$

Following the method derived by Leppington (**5**), it is found that

$$\frac{1}{K}=\cos\mu a+\frac{\alpha}{2\pi}\int_{-\infty}^{\infty}\frac{b(s)(1+\coth\gamma h)}{\gamma(s^2-\mu^2)}\cos sa\,ds, \tag{4.3}$$

where

$$b(s)=\frac{\sin(s+\mu)a}{s+\mu}+\frac{\sin(s-\mu)a}{s-\mu}.$$

The solution for $\phi(x, y)$ has now been found and is

$$\phi(x, y)\sim-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\bar{v}_1(s)}{\gamma}e^{-isx-\gamma y}\,ds, \qquad y>0, \tag{4.4}$$

$$\phi(x, y)\sim\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\bar{v}_1(s)\cosh\gamma(y+h)}{\gamma\sinh\gamma h}e^{-isx}\,ds, \qquad -h<y<0, \tag{4.5}$$

where $\bar{v}_1(s)$ is the Fourier transform of $v_1(x)$.

5. Approximate solution near a cavity resonance

The same technique described in sections 3 and 4 can be used to find an approximate solution to the sound field in the model problem with a cavity behind the elastic plate or membrane. As an example take the boundary-value problem defined by

$$(\nabla^2+k^2)\Phi=0, \quad \text{all } x, y, \tag{5.1}$$

$$(\partial^4/\partial x^4-\mu^4)\Phi_y+\alpha\Phi|_-^+=0, \qquad |x|<l, \qquad y=0, \tag{5.2}$$

$$\Phi_y=0, \qquad |x|>l, \qquad y=0, \tag{5.3}$$

$$\Phi_y=0, \qquad |x|<a, \qquad y=-h, \tag{5.4}$$

$$\Phi_x=0, \qquad x=\pm a, -h<y<0, \tag{5.5}$$

$$\Phi_y=\Phi_{yx}=0, \qquad x=\pm l, \qquad y=0. \tag{5.6}$$

This is the two-dimensional problem of an elastic plate of length $2l$ set in a rigid baffle, with a cavity attached, of length $2a$ and height h. Suppose the incident forcing is as previously defined, i.e.

$$\phi_i(x, y)=\tfrac{1}{2}e^{-iky\cos\theta}(e^{ikx\sin\theta}+e^{-ikx\sin\theta}), \tag{5.7}$$

and again express the total potential as

$$\Phi=\begin{cases}\phi_i+\phi_r+\phi, & y>0,\\ \phi, & y<0.\end{cases} \tag{5.8}$$

By using a Green's function approach it can be shown that (5.2) becomes

$$(\partial^4/\partial x^4-\mu^4)\phi_y(x, 0)=-\alpha(\phi_i+\phi_r)$$
$$-\alpha\int_{-L}^{L}\{G_+(x, 0; x_1, 0)+G_-(x, 0; x_1, 0)\}\phi_y(x_1, 0)\,dx_1, \tag{5.9}$$

where G_+ is a Green's function for the half-plane above the plate and G_- is a Green's function for the cavity, given by

$$G_-(x, y; x_1, y_1) = \sum_{m,n} \frac{2\cos\{(m\pi/2a)(x_1+a)\}\cos\{(m\pi/2a)(x+a)\}}{ah\{k^2-(n\pi/h)^2-(m\pi/2a)^2\}} \times \\ \times \cos(n\pi y/h)\cos(n\pi y_1/h). \quad (5.10)$$

G_+ and G_- satisfy the condition that their derivatives in the direction normal to all surfaces are zero.

By inspection it is seen that ignoring terms of order α^2 in (5.9) gives

$$(\partial^4/\partial x^4 - \mu^4)\phi_y(x, 0) = -2a\cos k_2 x. \quad (5.11)$$

If, however, a cavity resonance is approached then

$$k_{pq} = \{k^2-(q\pi/h)^2-(p\pi/2a)^2\} \to 0,$$

and assuming $k_{pq} \sim \alpha$, the leading-order terms in (5.9) now become

$$(\partial^4/\partial x^4 - \mu^4)\phi_{1y}(x, 0) = -2\alpha\cos k_2 x - \alpha K\cos\{p\pi(x+a)/(2a)\}, \quad (5.12)$$

and

$$(\partial^4/\partial x^4 - \mu^4)\phi_{2y}(x, 0) = \frac{\alpha i}{2\pi}\int_{-k}^{k} \Phi_{1y}(s, 0)\frac{\cos sx}{(k^2-s^2)^{\frac{1}{2}}}\,ds + O(\alpha^2 K),$$

where

$$\phi_y(x, 0) = \phi_{1y}(x, 0) + \phi_{2y}(x, 0), \qquad \phi_{1y} \gg \phi_{2y},$$

and

$$K = \frac{2}{k_{pq}ah}\int_{-L}^{L} \cos\{p\pi(x_1+a)/(2a)\}\phi_y(x_1, 0)\,dx_1. \quad (5.13)$$

The solution to (5.12) which satisfies the end conditions (5.6) is

$$\phi_y(x, 0) = -\alpha\{p(x) + f(x)/D\}, \quad (5.14)$$

where

$$p(x) = \frac{2\cos k_2 x}{(k_2^4-\mu^4)} + \frac{K\cos\{p\pi(x+a)/(2a)\}}{\{(p\pi/2a)^4-\mu^4\}} + \frac{\alpha iK}{2\pi}\int_{-k}^{k} \frac{E(s)\cos sx}{(k^2-s^2)^{\frac{1}{2}}(s^4-\mu^4)}\,ds, \quad (5.15)$$

$$f(x) = -\{\mu p(l)\sin\mu l + p'(l)\cos\mu l\}\cosh\mu x + \\ + \{-\mu p(l)\sinh\mu l + p'(l)\cosh\mu l\}\cos\mu x, \quad (5.16)$$

$$D = \mu(\sinh\mu l\cos\mu l + \cosh\mu l\sin\mu l), \quad (5.17)$$

and

$$E(s) = \frac{2\cos(p\pi/2)\{s\sin sl\cosh(p\pi l/2a) - (p\pi/2a)\cos sl\sin(p\pi l/2a)\}}{\{(p\pi/2a)^4-\mu^4\}\{s^2-(p\pi/2a)^2\}} \\ -\frac{2z}{s^2+\mu^2}(s\sin sl\cosh\mu l + \mu\cos sl\sinh\mu l) + \\ + \frac{2t}{s^2-\mu^2}(s\sin sl\cos\mu l - \mu\cos sl\sin\mu l),$$

where t and z are given in (5.21), (5.22).

Substituting (5.14) into (5.13) and performing the integration finally yields

$$K = -\frac{\alpha}{B}\cos(p\pi/2)\left[\frac{2}{(k_2^4-\mu^4)}\left\{\frac{\sin\{((p\pi/2a)+k_2)l\}}{(p\pi/2a)+k_2}+\frac{\sin\{((p\pi/2a)-k_2)l\}}{(p\pi/2a)-k_2}\right\}+\right.$$
$$+r\left\{\frac{\sin\{((p\pi/2a)+\mu)l\}}{(p\pi/2a)+\mu}+\frac{\sin\{((p\pi/2a)-\mu)l\}}{(p\pi/2a)-\mu}\right\} \tag{5.18}$$
$$\left.-\frac{2u}{(p\pi/2a)^2+\mu^2}\{(p\pi/2a)\sin(p\pi l/2a)\cosh\mu l+\mu\cos(p\pi l/2a)\sinh\mu l\}\right],$$

where

$$B = \tfrac{1}{2}a\,hk_{pq}+\frac{i\alpha^2}{2\pi}\int_{-k}^{k}\frac{\cos(p\pi/2)E(s)}{(k^2-s^2)^{\frac{1}{2}}(s^4-\mu^4)}\left[\frac{\sin\{((p\pi/2a)+s)l\}}{(p\pi/2a)+s}+\right.$$
$$+\frac{\sin\{((p\pi/2a)-s)l\}}{(p\pi/2a)-s}+r_1\left\{\frac{\sin\{((p\pi/2a)+\mu)l\}}{(p\pi/2a)+\mu}+\frac{\sin\{((p\pi/2a)-\mu)l\}}{(p\pi/2a)-\mu}\right\}$$
$$\left.-\frac{2u_1}{(p\pi/2a)^2+\mu^2}\{(p\pi/2a)\sin(p\pi l/2a)\cosh\mu l+\mu\cos(p\pi l/2a)\sinh\mu l\}\right]ds, \tag{5.19}$$

$$u = \{\mu g(l)\sin\mu l + g'(l)\cos\mu l\}/D, \tag{5.20}$$

$$u_1 = \{\mu g_1(l)\sin\mu l + g_1'(l)\cos\mu l\}/D, \tag{5.20a}$$

$$r = \{-\mu g(l)\sinh\mu l + g'(l)\cosh\mu l\}/D, \tag{5.21}$$

$$r_1 = \{-\mu g_1(l)\sinh\mu l + g_1'(l)\cosh\mu l\}/D, \tag{5.21a}$$

$$z = \{\mu j(l)\sin\mu l + j'(l)\cos\mu l\}/D, \tag{5.22}$$

$$t = \{-\mu j(l)\sinh\mu l + j'(l)\cosh\mu l\}/D, \tag{5.23}$$

$$\left.\begin{aligned} g(x) &= 2\cos k_2 x/(k_2^4-\mu^4), \qquad j(x) = \frac{\cos\{(p\pi/2a)(x+a)\}}{\{(p\pi/2a)^4-\mu^4\}}, \\ g_1(x) &= \cos sx, \end{aligned}\right\} \tag{5.24}$$

and $g'(l)$ denotes $dg/dx|_{x=l}$.

The approximate potential which is valid near a cavity resonance is now known and is

$$\phi(x, y) = -\int_{-L}^{L} G_-(x, y; x_1, 0)\phi_{1y}(x_1, 0)\,dx_1, \qquad -h<y<0, \qquad |x|<a, \tag{5.25}$$

which near a resonance becomes, to leading order,

$$\phi(x, y)\sim -K\cos\{(p\pi/2a)(x+a)\}\cos(q\pi y/h),$$
$$-h<y<0, \qquad |x|<a, \quad \text{as} \quad k_{pq}\to 0. \tag{5.26}$$

6. Conclusions

Section 4 dealt with the duct problem near a membrane resonance and the asymptotic solution for the velocity potential, as $\alpha \to 0$, was found to go

from $O(\alpha)$ away from a resonance to order unity near a resonance. A totally rigorous argument to justify the method used would require a knowledge of the magnitude of the discarded terms in the asymptotic expansions. Leppington (**5**), however, shows agreement between this method and other rigorous methods applied to different boundary-value problems and hence gives an empirical justification for the method.

Section 3 showed that the velocity potential for the duct problem near a duct resonance also varied from $O(\alpha)$ away from resonance to a higher order. The technique used for deriving the asymptotic solution near a duct resonance is also difficult to justify rigorously but it is a plausible method that has the advantage that it can be used for a wide variety of problems (as shown in section 5 for the elastic plate in a cavity problem).

Acknowledgements

I wish to thank Dr. F. G. Leppington for many helpful discussions, and advice, on this work, and the Science Research Council and A.U.W.E. Portland for the financial support of a C.A.S.E. studentship.

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Chapter 3

Scattering of sound by a heavily loaded finite elastic plate

Proc. R. Soc. Lond. A 378, 89–117 (1981)
Printed in Great Britain

Scattering of sound by a heavily loaded finite elastic plate

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(*Communicated by J. T. Stuart, F.R.S. – Received* 8 *October* 1980 – *Revised* 27 *February* 1981)

A plane wave is incident onto a finite plate set in a rigid baffle, and the scattered field is examined in the limit when fluid–plate coupling effects are large. An asymptotic solution is obtained, matching an outer region with inner regions at either edge of the plate. Waves are found to be present on the flexible surface, and resonance is shown to occur for particular values of the plate half-length, a. Away from a resonance, the leading term in the expansion of the outer potential is the solution of the boundary value problem in the absence of the plate. As a resonance is approached, however, eigensolutions, with singularities at the plate edges, also become present at this order.

1. Introduction

(*a*) *Previous work*

Many problems occur in underwater acoustics and aerodynamic noise theory where sound waves interact with flexible surfaces. Previous work has mainly centred on exact analyses for infinite or semi-infinite membranes or elastic plates (Junger & Feit 1972; Crighton 1971; Cannell 1975, 1976). The problems with finite elastic plates, which are more physically realistic, have in general been tackled by approximate methods (Leppington 1976; Handscomb 1977) although the problem of scattering by plates with line constraints (Leppington 1978) has a formally exact solution.

The problem of a finite plate near resonance, set in an infinite baffle, has been looked at by Leppington (1976), and an asymptotic solution was found in the low-fluid-loading limit (i.e. when the fluid is relatively light compared with the elastic plate). This class of problem has not previously been tackled in the high-fluid-loading limit, and this paper presents a solution by the method of matched asymptotic expansions. The possibility of resonance of the finite elastic plate has been taken into consideration in the solution.

(*b*) *Governing equations*

Cartesian coordinates (x', y') are chosen so that the flexible surface has the equilibrium position $y' = 0$, $|x'| < a$. The two-dimensional model has a plane finite elastic plate, length $2a$, set in a rigid baffle that lies on the plane $y' = 0$. The plate is fastened to the baffle so that the deflexion and its gradient at each edge are

always zero. The whole is immersed in an inviscid, compressible, stationary fluid, which satisfies the wave equation; and a fluid loading parameter α, which gives a measure of the fluid mass, is assumed large.

Any external geometry or forcing can be present in the problem (for example, cavities, point sources) as long as the sound field can be found for the same geometry and forcing but without the elastic plate. For a general geometry the analogous problem without the elastic plate is not trivial and so it is very convenient that a numerical solution can be used to calculate the coefficients needed in the matching scheme.

For two-dimensional motions with simple-harmonic time dependence, frequency ω, the total velocity potential can be written as

$$\mathrm{Re}\{\phi'(x', y')\exp(-\mathrm{i}\omega t)\}; \tag{1.1}$$

the time factor is henceforth suppressed.

The pressure fluctuation $p(x', y')$ and the surface deflexion of the plate $\eta(x')$ are given in terms of ϕ' by

$$p = \rho_0 \mathrm{i}\omega\phi'(x', y'), \quad \eta = (\mathrm{i}/\omega)\,\phi'_{y'}(x', 0), \tag{1.2}$$

where the suffix y denotes partial differentiation and ρ_0 is the mean fluid density.

The displacement η satisfies the linearized thin elastic plate equation

$$D\,\partial^4\eta/\partial x'^4 - 2mh\omega^2\eta = -[p]^{\pm}_{\mp} \quad (y' = 0,\ |x'| < a), \tag{1.3}$$

where D is the bending stiffness, h the plate half-thickness, m the plate density, and $[p]^{\pm}_{\mp}$ the pressure difference across the plate. It is finally assumed that the deflexion and its gradient are zero at each end of the plate, which implies that

$$\phi'_{y'} = \phi'_{y'x'} = 0 \quad (y' = 0,\ x' = \pm a). \tag{1.4}$$

The general boundary value problem can now be written down for $\phi'(x', y')$:

$$(\partial^2/\partial x'^2 + \partial^2/\partial y'^2 + k^2)\,\phi' = 0 \quad (\text{all } x', y'), \tag{1.5}$$

$$\phi'_{y'} = 0 \quad (y' = 0,\ |x'| > a), \tag{1.6}$$

$$\phi'_{y'} = \phi'_{y'x'} = 0 \quad (y' = 0,\ x' = \pm a), \tag{1.7}$$

$$(\partial^4/\partial x'^4 - \mu^4)\,\phi'_{y'} + \tfrac{1}{2}\alpha\phi'|^{\pm}_{\mp} = 0 \quad (y' = 0,\ |x'| < a), \tag{1.8}$$

where

$$\mu^4 = 2mh\omega^2/D, \tag{1.9}$$

$$\alpha = 2\rho\omega^2/D, \tag{1.10}$$

and μ is assumed of order k.

(c) *Matching regions*

By examining the plate equation (1.8) it is clear that in the heavy-fluid-loading limit ($\alpha \to \infty$) the first approximation to the acoustic field would be the potential if the elastic plate were not present, i.e. equation (1.8) becomes $\phi'|^{\pm}_{\mp} = 0$. This solution is expected to be valid at most points, except near the edges where the derivative

terms in the plate equation cannot be neglected. Thus this approximation is expected to break down when

$$\frac{\partial}{\partial y'}\left(\frac{\partial^4\phi'}{\partial x'^4}\right) = O(\tfrac{1}{2}\alpha\phi'|^{+}_{-}),$$

which, crudely, occurs at a distance l from an edge, where

$$1/l^5 = O(\alpha) \quad \text{or} \quad l = O(1/\alpha^{\frac{1}{5}}).$$

Matching can therefore be expected to be performed between an outer region based on a convenient outer length scale $1/k$ say, where k is the fluid wavenumber, and an inner region defined by an inner length-scale $1/\alpha^{\frac{1}{5}}$. The small parameter of the problem is therefore $\epsilon = k/\alpha^{\frac{1}{5}}$.

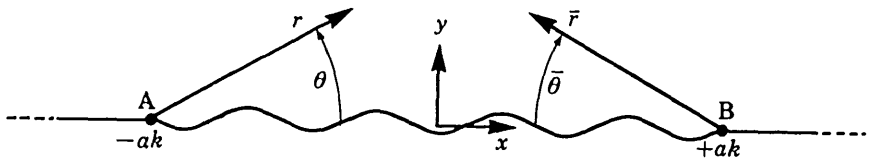


FIGURE 1. Outer-region coordinates.

Defining the non-dimensional coordinates as

$$x = x'k, \quad y = y'k \tag{1.11}$$

in the outer region, and

$$X = (a+x')\alpha^{\frac{1}{5}}, \quad Y = y'\alpha^{\frac{1}{5}}, \tag{1.12}$$

$$\bar{X} = (a-x')\alpha^{\frac{1}{5}}, \quad \bar{Y} = y'\alpha^{\frac{1}{5}} \tag{1.13}$$

in the left-hand-edge and right-hand-edge inner regions respectively and substituting into the general problem (equations (1.5)–(1.8)) gives the inner and outer boundary problems.

The boundary value problem for the outer region is thus

$$\left.\begin{aligned} (\nabla^2+1)\phi &= 0 \quad (\text{all } x,y),\\ \phi_y &= 0 \quad (y=0,\ |x|>ak),\\ \epsilon^5\{\phi_{yxxxx}-(\mu/k)^4\phi_y\}+\tfrac{1}{2}\phi|^{+}_{-} &= 0 \quad (y=0,\ |x|<ak), \end{aligned}\right\} \tag{1.14}$$

where

$$\epsilon = k/\alpha^{\frac{1}{5}}; \tag{1.15}$$

the geometry is shown in figure 1.

It is convenient to define polar coordinates r, θ, $\bar{r}$, $\bar{\theta}$ which will be used in the matching procedure. Thus, from figure 1,

$$r = \{(x+ak)^2+y^2\}^{\frac{1}{2}}, \quad \theta = \arctan\{y/(x+ak)\}, \tag{1.16}$$

$$\bar{r} = \{(x-ak)^2+y^2\}^{\frac{1}{2}}, \quad \bar{\theta} = \arctan\{y/(ak-x)\}. \tag{1.17}$$

The left inner problem is, after the substitution of inner coordinates (1.12),

$$\left.\begin{aligned}(\partial^2/\partial X^2+\partial^2/\partial Y^2+\epsilon^2)\Phi &= 0 \quad (\text{all } X, Y),\\ \Phi_Y &= 0 \quad (Y=0,\ X<0),\\ \{\partial^4/\partial X^4-(\mu\epsilon/k)^4\}\Phi_Y+\tfrac{1}{2}\Phi|^+_- &= 0 \quad (Y=0,\ X>0),\\ \Phi_Y=\Phi_{YX} &= 0 \quad (Y=0,\ X=0),\end{aligned}\right\} \tag{1.18}$$

with similar equations for the right-hand region.

The polar coordinates for the left- and right-hand regions are

$$R=(X^2+Y^2)^{\frac{1}{2}},\quad \theta=\arctan(Y/X), \tag{1.19}$$

$$\bar{R}=(\bar{X}^2+\bar{Y}^2)^{\frac{1}{2}},\quad \bar{\theta}=\arctan(\bar{Y}/\bar{X}). \tag{1.20}$$

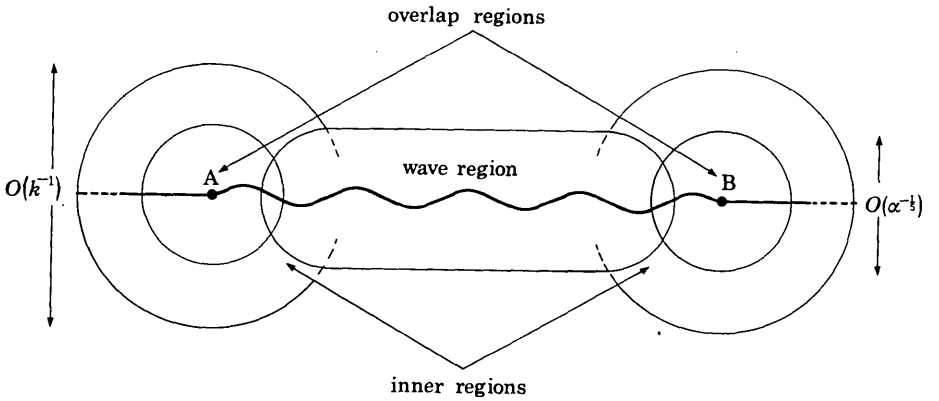


FIGURE 2. The matching regions.

(*d*) *Wave region*

There is also another régime where the outer field solution is not expected to be valid. This is the region close to the surface of the plate, where the potential includes not only the outer solution but also travelling wave solutions that can be supported on the elastic plate. These waves, which are shown in later analysis not to decay with distance along the plate, are generated in the inner regions at each edge of the plate, the energy being supplied by the incident acoustic wave. For convenience the travelling waves will be referred to as the 'wavy' part of the potential, and the remainder of the acoustic field will be called the 'non-wavy' part (although this potential includes acoustic wave terms). Because the travelling waves are generated in the inner region the thickness of this 'wave' region will be of order $(1/\alpha^{\frac{1}{2}})$.

(*e*) *Resonance*

Because the waves on the elastic plate are non-attenuating, and as will be shown the reflected wave at an edge has the same magnitude as the incoming one, a

situation could arise when, to first order, standing waves occur on the plate. This 'resonant' condition is shown in the wave-matching section (§2*h*) to occur when

$$\alpha^{\frac{1}{2}}a = \tfrac{1}{8}\pi + \tfrac{1}{2}n\pi, \quad (n = 0, 1, 2, 3, 4, \ldots).$$

As a resonance is approached it can be expected that the magnitude of the travelling waves will increase. Thus the order in ϵ of the waves will change as $\alpha^{\frac{1}{2}}a$ varies and this suggests that the waves are multiplied by functions of ϵ that are not necessarily power series. Let the function multiplying the wave travelling in the negative x-direction be $A(\epsilon)$ say, and let $B(\epsilon)$ multiply the positive x-direction wave, where $A(\epsilon)$ and $B(\epsilon)$ have to be found.

Alker (1978) found a similar type of resonant behaviour for surface waves between two partially immersed cylinders in water, and found that the non-wavy part of the potential also contained terms that depended on the non-power-series functions of ϵ. Thus it can be expected that the non-wavy potential in this problem also has eigensolutions multiplied by $A(\epsilon)$ and $B(\epsilon)$, and this is taken into consideration in the matching procedure in the following section.

2. Matching procedure

(*a*) *Notation*

All coefficients in an expansion of a potential that is known in principle (as the solution of an uniquely posed boundary value problem for that potential) are written as letters from the English alphabet. Capital letters are used for inner-potential coefficients, small letters for outer-region coefficients. Unknown coefficients are denoted by Greek letters, and bars over coefficients denote right-hand edge-matching.

(*b*) *Matching principle*

Matching is performed by using a modified form of Van Dyke's (1964) principle. This states that

$$\phi^{(s,t)} \equiv \Phi^{(t,s)} \quad \text{for any } s, t.$$

The function $\phi^{(s)}$ denotes the outer expansion, truncated after terms of order ϵ^s. Letting $r = \epsilon R$ and expanding as $\epsilon \to 0$ (R fixed, $r \to 0$) gives the expansion to order ϵ^t, denoted by $\phi^{(s,t)}$. Similarly replacing R by r/ϵ in $\Phi^{(t)}$, expanding and truncating after ϵ^s, gives $\Phi^{(t,s)}$.

(*c*) *Procedure*

In the Introduction it was indicated that the whole acoustic field is affected crucially by the product $\alpha^{\frac{1}{2}}a$, which value determines whether the plate is near or away from a resonance. It is useful, therefore, to treat the total potential as three separate parts, one having a fixed order in ϵ, and the other two multiplied by functions of ϵ, $A(\epsilon)$ and $B(\epsilon)$ say, that cannot be represented by ordinary power series and whose magnitudes change as resonance is approached. Thus let

$$\phi(x, y; \epsilon) = \phi_0(x, y; \epsilon) + A(\epsilon)\phi_1(x, y; \epsilon) + B(\epsilon)\phi_2(x, y; \epsilon) \qquad (2.1)$$

in the outer region. The functions $A(\epsilon)$ and $B(\epsilon)$ are $A(\epsilon:\beta)$, $B(\epsilon;\beta)$, where $\beta = ka$, and $A(\epsilon)$ and $B(\epsilon)$ will be shown to vary from $O(\epsilon^{\frac{1}{2}})$ away from a plate resonance to $O(\epsilon^{-\frac{1}{2}})$ at resonance. It is now convenient to treat ϕ_0, ϕ_1, ϕ_2 independently and match with three separate inner potentials, say Φ_0, Φ_1, Φ_2, where the total inner potential has the corresponding form

$$\Phi = \Phi_0 + A(\epsilon)\Phi_1 + B(\epsilon)\Phi_2. \tag{2.2}$$

Equation (1.14) defines the boundary value problem for all potentials ϕ_0, ϕ_1, and ϕ_2.

(*d*) *Matching* ϕ_0, Φ_0

The potential ϕ_0 is expressed as an expansion in the small parameter ϵ as follows:

$$\phi_0 \sim \phi_{01} + g_1(\epsilon)\phi_{02} + g_2(\epsilon)\phi_{03} + \ldots, \tag{2.3}$$

with $1 \gg g_1 \gg g_2$ as $\epsilon \to 0$. If $g_2(\epsilon) > \epsilon^5$ then ϕ_{01}, ϕ_{02}, ϕ_{03} satisfy the boundary conditions

$$\phi|^{+}_{-} = 0 \quad (y = 0,\ |x| < ka), \tag{2.4}$$

$$\phi_y = 0 \quad (y = 0,\ |x| > ka), \tag{2.5}$$

and the governing equation

$$(\nabla^2 + 1)\phi = 0 \quad (\text{all } x, y). \tag{2.6}$$

It is assumed that the forcing for this problem is a plane wave incident on the plate and baffle. This type of problem gives a potential (ϕ_{01}) which, as $r \to 0$ (r defined in equation (1.16)), behaves as

$$\underset{r\to 0}{\phi_{01}} \sim a_{01} + b_{01} r^{\frac{1}{2}} \sin \tfrac{1}{2}\theta + c_{01} r \cos\theta + d_{01} r^{\frac{3}{2}} \sin \tfrac{3}{2}\theta + \ldots. \tag{2.7}$$

The constants a_{01}, b_{01}, etc., are determined by the geometry of the model and also by the angle of incidence and magnitude of the incoming wave, and can be considered as known in principle for a particular problem.

Before proceeding with matching, note that all terms in the expansions of the individual potentials (e.g. equation (2.7)) are either even or odd in y. The even terms (of the form $r^{\pm n}\cos n\theta$) automatically satisfy $\phi|^{+}_{-} = 0$ and $\phi_{yxxx} = 0$ on $y = 0$ and therefore do not require an inner and outer matching, as the even solution is valid for the whole region. Thus to simplify matching all even parts of the incident potential are subtracted off, to be included again in equation (2.44) and thereafter.

By using the matching principle of § 2*b*, equation (2.7) becomes

$$\underset{r\to 0}{\phi_0^{(0)}} = \underset{r\to 0}{\phi_{01}} = b_{01} r^{\frac{1}{2}} \sin \tfrac{1}{2}\theta + d_{01} r^{\frac{3}{2}} \sin \tfrac{3}{2}\theta + \ldots, \tag{2.8}$$

and rewriting in terms of R and expanding to order $\epsilon^{\frac{3}{2}}$ (R fixed) give

$$\phi_0^{(0,\frac{3}{2})} = \epsilon^{\frac{1}{2}} b_{01} R^{\frac{1}{2}} \sin \tfrac{1}{2}\theta + d_{01}\epsilon^{\frac{3}{2}} R^{\frac{3}{2}} \sin \tfrac{3}{2}\theta. \tag{2.9}$$

The boundary value problem for Φ_0, Φ_1, Φ_2 is defined by (1.18) and expanding Φ_0 gives

$$\Phi_0 \sim G_1(\epsilon)\Phi_{01} + G_2(\epsilon)\Phi_{02} + G_3(\epsilon)\Phi_{03}. \tag{2.10}$$

It is known that the forcing in the inner problem is due to singular behaviour as $R\to\infty$ and also possibly to an incoming wave along the plate, of magnitude $A(\epsilon)$ into inner region A, and $B(\epsilon)$ into inner region B (see figure 3). It will be shown that to the order of matching considered in this paper all the inner potentials take one of three forms:

$\Phi_{\frac{1}{2}}$ the potential with $R^{\frac{1}{2}}$ behaviour as $R\to\infty$ (equation (3.33)),

$\Phi_{\frac{3}{2}}$ the potential with $R^{\frac{3}{2}}$ behaviour as $R\to\infty$ (equation (3.42)),

Φ_{w} the potential with $R^{-\frac{1}{2}}$ behaviour as $R\to\infty$ (equation (3.28)).

Note that Φ_{w} is forced by an incoming wave. The three potentials are analysed in the Wiener–Hopf discussion (§3) and they satisfy the boundary problem

$$\left.\begin{aligned} \nabla^2\Phi &= 0 \quad (\text{all } Y, X),\\ \Phi_Y &= 0 \quad (Y=0,\ X<0),\\ \Phi_{YXXXX}+\tfrac{1}{2}\Phi\big|_-^+ &= 0 \quad (Y=0,\ X>0),\\ \Phi_Y=\Phi_{YX} &= 0 \quad (Y=X=0). \end{aligned}\right\} \tag{2.11}$$

Returning to the problem for Φ_0, equation (2.9) suggests that

$$G_1(\epsilon)=\epsilon^{\frac{1}{2}},\quad G_2(\epsilon)=\epsilon^{\frac{3}{2}},$$

and it can be shown that $G_3(\epsilon)=\epsilon^{\frac{5}{2}}$. Also by inspection of equation (2.9) it is clear that to match correctly Φ_{01} behaves as $R^{\frac{1}{2}}$, and Φ_{02} as $R^{\frac{3}{2}}$, as $R\to\infty$. The Wiener–Hopf analysis in §3 shows the forms of $\Phi_{\frac{1}{2}}$, $\Phi_{\frac{3}{2}}$, and so

$$\begin{aligned} \Phi_{01} &\underset{R\to\infty}{=} \alpha_{01}\,\Phi_{\frac{1}{2}}\\ &\sim \alpha_{01}(R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta+A_{\frac{1}{2}}R^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta+B_{\frac{1}{2}}\sin\tfrac{3}{2}\theta\,R^{-\frac{3}{2}}+\ldots), \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \Phi_{02} &\underset{R\to\infty}{=} \alpha^1_{02}\,\Phi_{\frac{3}{2}}+\alpha^2_{02}\,\Phi_{\frac{1}{2}}\\ &\sim \alpha^1_{02}(R^{\frac{3}{2}}\sin\tfrac{3}{2}\theta+A_{\frac{3}{2}}R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta+B_{\frac{3}{2}}R^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta+\ldots)\\ &\quad+\alpha^2_{02}(R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta+A_{\frac{1}{2}}R^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta+B_{\frac{1}{2}}R^{-\frac{3}{2}}\sin\tfrac{3}{2}\theta+\ldots). \end{aligned} \tag{2.13}$$

Therefore

$$\Phi_0^{(\frac{3}{2})}=\phi^{\frac{1}{2}}\alpha_{01}\Phi_{\frac{1}{2}}+\epsilon^{\frac{3}{2}}(\alpha^1_{02}\Phi_{\frac{3}{2}}+\alpha^2_{02}\Phi_{\frac{1}{2}}). \tag{2.14}$$

The matching principle $\Phi^{(\frac{3}{2},0)}\equiv\phi^{(0,\frac{3}{2})}$ equates the coefficients α^1_{02}, α_{01} in equations (2.12) and (2.13) to the known coefficients in equation (2.9), yielding

$$\alpha_{01}=b_{01}\quad\text{and}\quad\alpha^1_{02}=d_{01}.$$

Higher-order matching is now possible and from equation (2.14)

$$\begin{aligned} \Phi_0^{(\frac{3}{2},2)} &= \alpha_{01}r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta+\alpha^1_{02}r^{\frac{3}{2}}\sin\tfrac{3}{2}\theta\\ &\quad+\epsilon(\alpha_{01}A_{\frac{1}{2}}r^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta+(\alpha^1_{02}A_{\frac{3}{2}}+\alpha^2_{02})\,r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta)\\ &\quad+\epsilon^2(\alpha_{01}B_{\frac{1}{2}}r^{-\frac{3}{2}}\sin\tfrac{3}{2}\theta+(\alpha^1_{02}B_{\frac{3}{2}}+\alpha^2_{02}A_{\frac{1}{2}})\,r^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta). \end{aligned} \tag{2.15}$$

Thus in equation (2.3) $g_1(\epsilon) = \epsilon$, $g_2(\epsilon) = \epsilon^2$ and ϕ_{02}, ϕ_{03} are eigensolutions of the boundary value problem defined by equations (2.4)–(2.6), with singularities at each edge of the plate. From the terms in (2.15) it can be shown that ϕ_{02}, ϕ_{03} have singular behaviour of the form

$$\phi_{02} = \Gamma^1_{02}\phi_{\frac{1}{2}\mathrm{A}} + \Gamma^2_{02}\phi_{\frac{1}{2}\mathrm{B}} \tag{2.16}$$

and

$$\phi_{03} = \Gamma^1_{03}\phi_{\frac{1}{2}\mathrm{A}} + \Gamma^2_{03}\phi_{\frac{1}{2}\mathrm{B}} + \Gamma^3_{03}\phi_{\frac{3}{2}\mathrm{B}} + \Gamma^4_{03}\phi_{\frac{3}{2}\mathrm{A}}, \tag{2.17}$$

where $\phi_{\frac{1}{2}\mathrm{A}}$ is the eigensolution with $r^{-\frac{1}{2}}$ behaviour at edge A,

$$\phi_{\frac{1}{2}\mathrm{A}} \underset{r\to 0}{\sim} r^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta + a_{\frac{1}{2}\mathrm{A}} r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta + b_{\frac{1}{2}\mathrm{A}} r^{\frac{3}{2}}\sin\tfrac{3}{2}\theta + \ldots; \tag{2.18}$$

similarly

$$\phi_{\frac{1}{2}\mathrm{B}} \underset{r\to 0}{\sim} a_{\frac{1}{2}\mathrm{B}} r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta + b_{\frac{1}{2}\mathrm{B}} r^{\frac{3}{2}}\sin\tfrac{3}{2}\theta + \ldots, \tag{2.19}$$

$$\phi_{\frac{3}{2}\mathrm{B}} \underset{r\to 0}{\sim} a_{\frac{3}{2}\mathrm{B}} r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta + b_{\frac{3}{2}\mathrm{B}} r^{\frac{3}{2}}\sin\tfrac{3}{2}\theta + \ldots, \tag{2.20}$$

$$\phi_{\frac{3}{2}\mathrm{A}} \sim r^{-\frac{3}{2}}\sin\tfrac{3}{2}\theta + a_{\frac{3}{2}\mathrm{A}} r^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta + b_{\frac{3}{2}\mathrm{A}} r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta + \ldots, \tag{2.21}$$

and the Γ are to be determined by matching. At the right-hand edge there are equations equivalent to (2.18)–(2.21):

$$\phi_{\frac{1}{2}\mathrm{A}} \underset{\bar{r}\to 0}{\sim} \bar{a}_{\frac{1}{2}\mathrm{A}} \bar{r}^{\frac{1}{2}}\sin\tfrac{1}{2}\bar{\theta} + \bar{b}_{\frac{1}{2}\mathrm{B}} \bar{r}^{\frac{3}{2}}\sin\tfrac{3}{2}\bar{\theta} + \ldots. \tag{2.22}$$

The potential ϕ_{02} is that due to an $r^{-\frac{1}{2}}$ singularity at the left-hand edge (of magnitude Γ^1_{02}) plus that due to an $r^{-\frac{1}{2}}$ singularity at B of magnitude Γ^2_{02}. Similarly ϕ_{03} is the superposition of four potentials due to singularities (of order $r^{-\frac{1}{2}}$ and $r^{-\frac{3}{2}}$) at each edge of the plate. Thus, with the matching terminology,

$$\begin{aligned}\phi_0^{(2,\frac{3}{2})} = {} & b_{01} r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta + d_{01} r^{\frac{3}{2}}\sin\tfrac{3}{2}\theta \\ & + \epsilon(\Gamma^1_{02}\sin\tfrac{1}{2}\theta r^{-\frac{1}{2}} + (\Gamma^1_{02} a_{\frac{1}{2}\mathrm{A}} + \Gamma^2_{02} a_{\frac{1}{2}\mathrm{B}}) r^{\frac{1}{2}}\sin\tfrac{1}{2}\theta) \\ & + \epsilon^2(\Gamma^4_{03}\sin\tfrac{3}{2}\theta r^{-\frac{3}{2}} + (\Gamma^1_{03} + \Gamma^4_{03} a_{\frac{3}{2}\mathrm{A}}) r^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta).\end{aligned} \tag{2.23}$$

Using $\phi_0^{(2,\frac{3}{2})} \equiv \Phi_0^{(\frac{3}{2},2)}$ gives

$$\left.\begin{aligned} \alpha_{01} = b_{01}, \quad \alpha^1_{02} = d_{01}, \quad \Gamma^1_{02} = A_{\frac{1}{2}} b_{01}, \quad \Gamma^4_{03} = B_{\frac{1}{2}} b_{01}, \\ d_{01} A_{\frac{3}{2}} + \alpha^2_{02} = A_{\frac{1}{2}} b_{01} a_{\frac{1}{2}\mathrm{A}} + \Gamma^2_{02} a_{\frac{1}{2}\mathrm{B}}, \\ d_{01} B_{\frac{3}{2}} + a^2_{02} A_{\frac{1}{2}} = \Gamma^1_{03} + B_{\frac{1}{2}} b_{01} a_{\frac{3}{2}\mathrm{A}}. \end{aligned}\right\} \tag{2.24}$$

Notice that there is one equation too few for the determination of all the unknown coefficients. This equation is found from relating the two matching problems (left-hand and right-hand) and will be discussed in §2*f*.

(*e*) *Matching* ϕ_1, Φ_1

With Φ_1 expanded as

$$\Phi_1 \sim \Phi_{10} + h_1(\epsilon)\Phi_{11} + h_2(\epsilon)\Phi_{13}, \quad \text{say,} \tag{2.25}$$

it is found that Φ_{10} satisfies the problem defined by equations (2.11). From arguments discussed in the Introduction it is clear that Φ_{10} must be the solution that is forced by an incoming wave, i.e.

$$\Phi_{10} = \Phi_{\mathrm{w}}. \tag{2.26}$$

Equation (3.28) (§3*b*) gives the Wiener–Hopf solution for this problem which, as $R \to \infty$, results in the non-wavy part of the potential behaving as

$$\underset{R\to\infty}{\Phi_{10}} \sim A_{\mathrm{w}}(R^{-\frac{1}{2}} \sin \tfrac{1}{2}\theta + B_{\mathrm{w}} R^{-\frac{3}{2}} \sin \tfrac{3}{2}\theta + C_{\mathrm{w}} R^{-\frac{5}{2}} \sin \tfrac{5}{2}\theta + \ldots) \tag{2.27}$$

and so

$$\Phi_1^{(0,\frac{1}{2})} = \epsilon^{\frac{1}{2}} A_{\mathrm{w}} r^{-\frac{1}{2}} \sin \tfrac{1}{2}\theta + A_{\mathrm{w}} B_{\mathrm{w}} \epsilon^{\frac{3}{2}} r^{-\frac{3}{2}} \sin \tfrac{3}{2}\theta. \tag{2.28}$$

The form of ϕ_1 can now be expected to be

$$\phi_1 \sim \epsilon^{\frac{1}{2}} \phi_{10} + \epsilon^{\frac{3}{2}} \phi_{11} + \epsilon^{\frac{5}{2}} \phi_{12}. \tag{2.29}$$

Since ϕ_{10} and ϕ_{11} satisfy the problem given by (2.4)–(2.6) these eigensolutions must be of the form

$$\phi_{10} = \Gamma^1_{10} \phi_{\frac{1}{2}\mathrm{A}} + \Gamma^2_{10} \phi_{\frac{1}{2}\mathrm{B}}, \tag{2.30}$$

$$\phi_{11} = \Gamma^1_{11} \phi_{\frac{1}{2}\mathrm{A}} + \Gamma^2_{11} \phi_{\frac{1}{2}\mathrm{B}} + \Gamma^3_{11} \phi_{\frac{3}{2}\mathrm{B}} + \Gamma^4_{11} \phi_{\frac{3}{2}\mathrm{A}}, \tag{2.31}$$

where $\phi_{\frac{1}{2}\mathrm{A}}$, etc., are defined in equations (2.18)–(2.21). The Γ are determined by matching, and

$$\phi_1^{(\frac{1}{2},0)} = \Gamma^1_{10} R^{-\frac{1}{2}} \sin \tfrac{1}{2}\theta + \Gamma^4_{11} R^{-\frac{3}{2}} \sin \tfrac{3}{2}\theta \tag{2.32}$$

is equated to (2.28) to give

$$\Gamma^1_{10} = A_{\mathrm{w}}, \quad \Gamma^4_{11} = A_{\mathrm{w}} B_{\mathrm{w}}. \tag{2.33}$$

Higher-order matching is again possible:

$$\begin{aligned} \phi_1^{(\frac{1}{2},1)} = {} & \Gamma^1_{10} R^{-\frac{1}{2}} \sin \tfrac{1}{2}\theta + \Gamma^4_{11} R^{-\frac{3}{2}} \sin \tfrac{3}{2}\theta \\ & + \epsilon\{(\Gamma^1_{10} a_{\frac{1}{2}\mathrm{A}} + \Gamma^2_{10} a_{\frac{1}{2}\mathrm{B}}) R^{\frac{1}{2}} \sin \tfrac{1}{2}\theta + (\Gamma^1_{11} + \Gamma^4_{11} a_{\frac{3}{2}\mathrm{A}}) R^{-\frac{1}{2}} \sin \tfrac{1}{2}\theta\}. \end{aligned} \tag{2.34}$$

Now $h_1(\epsilon)$ in equation (2.25) must be equal to ϵ to match with the outer region and so Φ_{11} is an eigensolution of the boundary value problem (2.11) that can be shown to have $R^{\frac{1}{2}}$ behaviour as $R \to \infty$, i.e. $\Phi_{11} = \alpha_{11} \Phi_{\frac{1}{2}}$. Thus equating $\Phi_1^{(1,\frac{1}{2})}$ with equation (2.34), by using the matching principle in §2*b*, finally determines the coefficients:

$$\left.\begin{aligned} & \Gamma^1_{10} = A_{\mathrm{w}}, \quad \Gamma^4_{11} = A_{\mathrm{w}} B_{\mathrm{w}}, \\ & A_{\mathrm{w}} a_{\frac{1}{2}\mathrm{A}} + \Gamma^2_{10} a_{\frac{1}{2}\mathrm{B}} = \alpha_{11}, \quad \Gamma^1_{11} + A_{\mathrm{w}} B_{\mathrm{w}} a_{\frac{3}{2}\mathrm{A}} = \alpha_{11} A_{\frac{1}{2}}. \end{aligned}\right\} \tag{2.35}$$

(*f*) *Relating the matching at* A *and* B

There are more unknown coefficients than equations to solve for both ϕ_0 and ϕ_1. The extra conditions necessary to determine all the unknown coefficients are found by comparing the outer field for both edges A and B.

Consider the ϕ_0 potential first: it can be seen from (2.7) that

$$\underset{\bar{r}\to 0}{\phi_{01}} \sim \bar{a}_{01} + \bar{b}_{01} \bar{r}^{\frac{1}{2}} \sin \tfrac{1}{2}\bar{\theta} + \bar{c}_{01} \bar{r} \cos \bar{\theta} + \ldots, \tag{2.36}$$

where $\bar{r}$, $\bar{\theta}$ are shown in figure 1, and $\bar{a}_{01}$, $\bar{b}_{01}$ etc. are right-hand coefficients (thus denoted by a bar) that are in principle known. Thus by analogy with (2.24) it is found that

$$\left.\begin{gathered}\bar{\alpha}_{01} = \bar{b}_{01}, \quad \bar{\alpha}^1_{02} = \bar{d}_{01}, \quad \Gamma^2_{02} = A_{\frac{1}{2}}\bar{b}_{01}, \\ \Gamma^3_{03} = B_{\frac{1}{2}}\bar{b}_{01}, \quad \bar{d}_{01}A_{\frac{3}{2}} + \bar{\alpha}^2_{02} = A_{\frac{1}{2}}\bar{b}_{01}\bar{a}_{\frac{1}{2}\mathrm{B}} + \Gamma^1_{02}\bar{a}_{\frac{1}{2}\mathrm{A}}, \\ \bar{d}_{01}B_{\frac{3}{2}} + \bar{\alpha}^2_{02}A_{\frac{1}{2}} = \Gamma^2_{03} + B_{\frac{1}{2}}\bar{b}_{01}\bar{a}_{\frac{3}{2}\mathrm{B}},\end{gathered}\right\} \tag{2.37}$$

where $\bar{a}_{\frac{1}{2}\mathrm{A}}$ is defined by

$$\phi_{\substack{\frac{1}{2}\mathrm{A} \\ \bar{r}\to 0}} \sim \bar{a}_{\frac{1}{2}\mathrm{A}}\bar{r}^{\frac{1}{2}}\sin\tfrac{1}{2}\bar{\theta}. \tag{2.38}$$

There are now just enough equations to solve for all the unknowns, $\bar{\alpha}_{01}$, Γ^2_{02}, etc.

Return to the ϕ_1 potential: the unknown coefficients can be determined by matching at the right-hand edge. From the matching principle at edge B, $\bar{\phi}_1^{(\frac{3}{2},1)}$, by analogy with (2.34), is given by

$$\begin{aligned}\bar{\phi}_1^{(\frac{3}{2},1)} = {} & \Gamma^2_{10}\bar{R}^{-\frac{1}{2}}\sin\tfrac{1}{2}\bar{\theta} + \Gamma^3_{11}\bar{R}^{-\frac{3}{2}}\sin\tfrac{3}{2}\bar{\theta} \\ & + \epsilon\{(\Gamma^2_{10}\bar{a}_{\frac{1}{2}\mathrm{B}} + \Gamma^1_{10}\bar{a}_{\frac{1}{2}\mathrm{A}})\bar{R}^{\frac{1}{2}}\sin\tfrac{1}{2}\bar{\theta} + (\Gamma^2_{11} + \Gamma^3_{11}\bar{a}_{\frac{3}{2}\mathrm{B}})\bar{R}^{-\frac{1}{2}}\sin\tfrac{1}{2}\bar{\theta}\}.\end{aligned} \tag{2.39}$$

The inner potential for the right-hand edge has a term $\bar{\Phi}_1$ multiplied by $A(\epsilon)$, which must match with ϕ_1 as $\bar{r}\to 0$. From equation (2.39) the form of $\bar{\Phi}_1$ is

$$\bar{\Phi}_1 \sim \bar{\Phi}_{11} + \epsilon\bar{\Phi}_{12}, \tag{2.40}$$

where $\bar{\Phi}_{11}$, $\bar{\Phi}_{12}$ satisfy the problem defined in (2.11). The potential $\bar{\Phi}_{11}$ has no forcing and so must be identically zero, while $\bar{\Phi}_{12}$ can be seen to behave as

$$\bar{\Phi}_{12} = \bar{\alpha}_{12}\bar{\Phi}_{\frac{1}{2}}. \tag{2.41}$$

By using the equality $\qquad \bar{\phi}_1^{(1,\frac{3}{2})} \equiv \bar{\phi}_1^{(\frac{3}{2},1)}$

and the equations in (2.35), the coefficients can be calculated:

$$\left.\begin{gathered}\Gamma^1_{10} = A_{\mathrm{w}}, \quad \Gamma^2_{10} = 0, \quad \Gamma^1_{11} = A_{\mathrm{w}}(a_{\frac{1}{2}\mathrm{A}}A_{\frac{1}{2}} - a_{\frac{3}{2}\mathrm{A}}B_{\mathrm{w}}), \\ \Gamma^2_{11} = A_{\mathrm{w}}A_{\frac{1}{2}}\bar{a}_{\frac{1}{2}\mathrm{A}}, \quad \Gamma^3_{11} = 0, \quad \Gamma^4_{11} = A_{\mathrm{w}}B_{\mathrm{w}}, \\ \alpha_{11} = A_{\mathrm{w}}a_{\frac{1}{2}\mathrm{A}}.\end{gathered}\right\} \tag{2.42}$$

The potential ϕ_2 is due to an incoming wave to the right-hand inner region of magnitude $B(\epsilon)$ and is analogous with ϕ_1. Thus by inspection it is found that

$$\left.\begin{gathered}\Gamma^1_{20} = 0 \quad \Gamma^4_{21} = 0, \quad \Gamma^2_{20} = A_{\mathrm{w}}, \quad \Gamma^3_{21} = A_{\mathrm{w}}B_{\mathrm{w}}, \\ \Gamma^1_{21} = A_{\mathrm{w}}A_{\frac{1}{2}}a_{\frac{1}{2}\mathrm{B}}, \quad \Gamma^2_{21} = A_{\mathrm{w}}(\bar{a}_{\frac{1}{2}\mathrm{B}}A_{\frac{1}{2}} - \bar{a}_{\frac{3}{2}\mathrm{B}}B_{\mathrm{w}}), \\ \alpha_{22} = A_{\mathrm{w}}a_{\frac{1}{2}\mathrm{B}}.\end{gathered}\right\} \tag{2.43}$$

(*g*) *Asymptotic solution*

The matching is now complete and so equations (2.1) and (2.2) yield the inner and outer potentials

$$\begin{aligned}\phi \sim{}& \phi_{01}+\epsilon A_{\frac12}(b_{01}\phi_{\frac12\mathrm{A}}+\bar b_{01}\phi_{\frac12\mathrm{B}})\\ &+\epsilon^2[\{d_{01}B_{\frac12}-B_{\frac12}b_{01}a_{\frac32\mathrm{A}}+A_{\frac12}(A_{\frac12}b_{01}a_{\frac12\mathrm{A}}+A_{\frac12}\bar b_{01}a_{\frac12\mathrm{B}}-d_{01}A_{\frac32})\}\phi_{\frac12\mathrm{A}}\\ &+B_{\frac12}(\bar b_{01}\phi_{\frac32\mathrm{B}}+b_{01}\phi_{\frac32\mathrm{A}})\\ &+\{\bar d_{01}B_{\frac32}-B_{\frac12}\bar b_{01}\bar a_{\frac32\mathrm{B}}+A_{\frac12}(A_{\frac12}\bar b_{01}\bar a_{\frac12\mathrm{B}}+A_{\frac12}b_{01}\bar a_{\frac12\mathrm{A}}-\bar d_{01}A_{\frac32})\}\phi_{\frac12\mathrm{B}}]\\ &+A(\epsilon)\,\epsilon^{\frac12}[A_{\mathrm w}\phi_{\frac12\mathrm{A}}\\ &+\epsilon\{A_{\mathrm w}(a_{\frac12\mathrm{A}}A_{\frac12}-a_{\frac32\mathrm{A}}B_{\mathrm w})\,\phi_{\frac12\mathrm{A}}+A_{\mathrm w}A_{\frac12}\bar a_{\frac12\mathrm{A}}\phi_{\frac12\mathrm{B}}+A_{\mathrm w}B_{\mathrm w}\phi_{\frac32\mathrm{A}}\}]\\ &+B(\epsilon)\,\epsilon^{\frac12}[A_{\mathrm w}\phi_{\frac12\mathrm{B}}\\ &+\epsilon\{A_{\mathrm w}(\bar a_{\frac12\mathrm{B}}A_{\frac12}-\bar a_{\frac32\mathrm{B}}B_{\mathrm w})\,\phi_{\frac12\mathrm{B}}+A_{\mathrm w}A_{\frac12}a_{\frac12\mathrm{B}}\phi_{\frac12\mathrm{A}}+A_{\mathrm w}B_{\mathrm w}\phi_{\frac32\mathrm{B}}\}],\end{aligned}\tag{2.44}$$

$$\begin{aligned}\Phi ={}& \Phi_{\mathrm{left}}\\ \sim{}& \epsilon^{\frac12}b_{01}\Phi_{\frac12}+\epsilon^{\frac32}\{d_{01}\Phi_{\frac32}+(A_{\frac12}b_{01}a_{\frac12\mathrm{A}}+A_{\frac12}\bar b_{01}a_{\frac12\mathrm{B}}-d_{01}A_{\frac32})\Phi_{\frac12}\}\\ &+A(\epsilon)\{\Phi_{\mathrm w}+\epsilon A_{\mathrm w}a_{\frac12\mathrm{A}}\Phi_{\frac12}+O(\epsilon^2)\}+\epsilon B(\epsilon)\{A_{\mathrm w}a_{\frac12\mathrm{B}}\Phi_{\frac12}+O(\epsilon)\},\end{aligned}\tag{2.45}$$

$$\begin{aligned}\bar\Phi ={}& \Phi_{\mathrm{right}}\\ \sim{}& \epsilon^{\frac12}\bar b_{01}\bar\Phi_{\frac12}+\epsilon^{\frac32}\{\bar d_{01}\bar\Phi_{\frac32}+(A_{\frac12}\bar b_{01}\bar a_{\frac12\mathrm{B}}+A_{\frac12}b_{01}\bar a_{\frac12\mathrm{A}}-\bar d_{01}A_{\frac32})\bar\Phi_{\frac12}\}\\ &+B(\epsilon)\{\bar\Phi_{\mathrm w}+\epsilon A_{\mathrm w}\bar a_{\frac12\mathrm{B}}\bar\Phi_{\frac12}+O(\epsilon^2)\}+\epsilon A(\epsilon)\{A_{\mathrm w}\bar a_{\frac12\mathrm{A}}\bar\Phi_{\frac12}+O(\epsilon)\}.\end{aligned}\tag{2.46}$$

(*h*) *Wave matching*

Having performed matching on the non-wavy part of the potential, the values of $A(\epsilon)$ and $B(\epsilon)$ can be determined by using equations (2.45) and (2.46), by matching the waves from the left- and right-hand edges.

The only coupled waves that can be supported on the elastic plate, in the outer region, are of the form

$$\phi_{\mathrm c}=\text{constant}\times\exp\{-(s^{*2}-k^2)^{\frac12}\,|y/k|\pm\mathrm{i}s^*(x/k)\},\tag{2.47}$$

where the coupled wavenumber s^* must satisfy

$$K(s^*)=\alpha-(s^{*2}-k^2)^{\frac12}(s^{*4}-\mu^4)=0.\tag{2.48}$$

The only waves of interest are those that have $\mathrm{Im}\,(s^*)=0$ as these do not decay with distance along the plate. The s^* that satisfies this condition is

$$s^*=\alpha^{\frac15}\{1+\tfrac{1}{10}\epsilon^2+O(\epsilon^4)\}\quad\text{for small }\epsilon.$$

Thus
$$\phi_{\mathrm c}=\text{constant}\times\{1+O(\epsilon^2)\}\exp(\pm\mathrm{i}x/\epsilon-|y|/\epsilon)\tag{2.49}$$

Figure 3 shows the reflected waves $\phi_{\mathrm A}$, $\phi_{\mathrm B}$, say, from edges A and B respectively. For continuity of waves along the plate it is clear that

$$A(\epsilon)\{1+O(\epsilon^2)\}\exp(-\mathrm{i}X_1/\epsilon-|y|/\epsilon)=\phi_{\mathrm B},\tag{2.50}$$

$$B(\epsilon)\{1+O(\epsilon^2)\}\exp(-\mathrm{i}X_2/\epsilon-|y|/\epsilon)=\phi_{\mathrm A},\tag{2.51}$$

where $X_1=x+ka$, $X_2=ka-x$.

From the previous matching the inner non-wavy potentials were determined ((2.45) and (2.46)) which give the form of the reflected waves ϕ_A, ϕ_B. The Wiener–Hopf analysis (§ 3*b*, equation (3.26)) shows that for an incident wave of magnitude 1, the reflected wave has magnitude 1 also, but a phase change of $e^{3i\pi/4}$. Thus from (2.45) ϕ_A will be of the form

$$\phi_A = [K_1\epsilon^{\frac{1}{2}} + O(\epsilon^{\frac{3}{2}}) + A(\epsilon)\{e^{3i\pi/4} + K_2\epsilon + O(\epsilon^2)\} + B(\epsilon)\,\epsilon\{K_3 + O(\epsilon)\}]\exp(iX_1/\epsilon - |y|/\epsilon), \quad (2.52)$$

and similarly from (2.46)

$$\phi_B = [\bar{K}_1\epsilon^{\frac{1}{2}} + O(\epsilon^{\frac{3}{2}}) + A(\epsilon)\,\epsilon\{\bar{K}_3 + O(\epsilon)\} + B(\epsilon)\{e^{3i\pi/4} + \bar{K}_2\epsilon + O(\epsilon^2)\}]\exp(+iX_2/\epsilon - |y|/\epsilon). \quad (2.53)$$

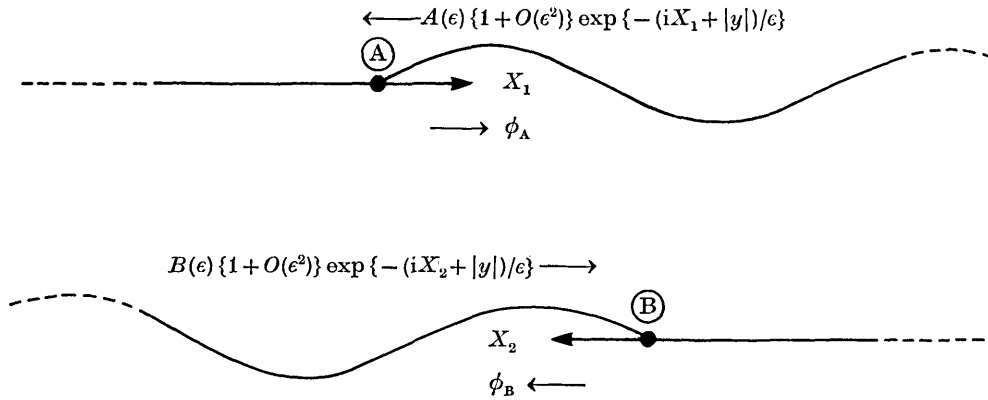


FIGURE 3. Wave matching.

Note that K_1, K_2, K_3, $\bar{K}_1$, $\bar{K}_2$, $\bar{K}_3$ are determined in the Wiener–Hopf analysis of the inner region:

$$\left.\begin{aligned} K_1 &= (\pi/10)^{\frac{1}{2}} b_{01}\, e^{7\pi i/8}, & \bar{K}_1 &= (\pi/10)^{\frac{1}{2}} \bar{b}_{01}\, e^{7\pi i/8}, \\ K_2 &= a_{\frac{1}{2}A}\, e^{5\pi i/4}, & \bar{K}_2 &= \bar{a}_{\frac{1}{2}B}\, e^{5\pi i/4}, \\ K_3 &= a_{\frac{1}{2}B}\, e^{5\pi i/4}, & \bar{K}_3 &= \bar{a}_{\frac{1}{2}A}\, e^{5\pi i/4}. \end{aligned}\right\} \quad (2.54)$$

Substituting (2.52) and (2.53) into the simultaneous equations (2.50) and (2.51) determines $A(\epsilon)$, $B(\epsilon)$ after changing from local coordinates X_1, X_2. Thus finally it is found that

$$\left.\begin{aligned} A(\epsilon) &= \{(\bar{K}_1 e^{-2i\beta/\epsilon} + K_1 e^{3i\pi/4})\,\epsilon^{\frac{1}{2}} + O(\epsilon^{\frac{3}{2}})\}/D(\epsilon), \\ B(\epsilon) &= \{(K_1 e^{-2i\beta/\epsilon} + \bar{K}_1 e^{3i\pi/4})\,\epsilon^{\frac{1}{2}} + O(\epsilon^{\frac{3}{2}})\}/D(\epsilon), \end{aligned}\right\} \quad (2.55)$$

where $\beta = ka$, $\epsilon = k/\alpha^{\frac{1}{2}}$ and

$$D(\epsilon) = (e^{-4i\beta/\epsilon} - e^{3\pi i/2}) - \epsilon\{e^{3\pi i/4}(K_2 + \bar{K}_2) + e^{-2i\beta/\epsilon}(K_3 + \bar{K}_3)\} + O(\epsilon^2). \quad (2.56)$$

Resonance is said to occur whenever the $O(1)$ term in $D(\epsilon)$ vanishes, i.e. whenever

$$e^{-4i\beta/\epsilon} - e^{3i\pi/2} = 0, \quad \text{or equivalently} \quad (2.57)$$

$$\alpha^{\frac{1}{2}}a = \tfrac{1}{8}\pi + \tfrac{1}{2}\pi n \quad (n = 0, 1, 2, 3, \ldots). \quad (2.58)$$

The initial assumption that $A(\epsilon)$, $B(\epsilon)$ change their order in ϵ, as a resonance is approached, was correct, and by inspection of (2.55) it is seen that $A(\epsilon)$, $B(\epsilon)$ are $O(\epsilon^{\frac{1}{2}})$ away from a resonance and $O(\epsilon^{-\frac{1}{2}})$ at a resonance.

3. Inner problem: the Wiener–Hopf solution

(a) *Boundary value problem*

By substituting the expansions for the inner potentials, i.e. Φ_0 given by (2.10), into the boundary problem (1.18), and collecting terms of like order in ϵ, the problems for the potentials Φ_{01}, Φ_{11} etc. are defined. On the assumption that $\mu/k = O(1)$, it is found that to the order reached in this paper all the potentials satisfy the boundary value problem

$$\left.\begin{aligned} \nabla^2\Phi &= 0 \quad (\text{all } X, Y), \\ \Phi_Y &= 0 \quad (Y = 0,\ X < 0), \\ \Phi_{YXXXX} + \tfrac{1}{2}\Phi\big|_{-}^{+} &= 0 \quad (Y = 0,\ X > 0), \\ \Phi_Y = \Phi_{YX} &= 0 \quad (Y = X = 0). \end{aligned}\right\} \tag{3.1}$$

The potentials are forced by either an incoming wave (at large R) on the elastic plate, or singular behaviour as $R \to \infty$ of $R^{\frac{1}{2}}$ or $R^{\frac{3}{2}}$-form. The form of the forcing has been examined in § 2 for each potential, but all the potentials have a solution that includes an outgoing wave along the plate. By solving the inner problem by means of the Wiener–Hopf technique the magnitude and phase angle of the outgoing wave can be determined in terms of the magnitude and phase of the forcing. Although the detailed structure of the inner problem is known (in terms of an integral) it is not of any interest to the outer problem and so will not be examined.

(b) *Forcing by an incoming wave; Φ_{w} potential*

It is convenient to split $\Phi = \Phi_{\mathrm{w}}$, say, into the incoming wave term plus a scattered term ϕ (which includes the outgoing wave):

$$\Phi_{\mathrm{w}} = \phi_{\mathrm{i.w.}} + \phi.$$

It is assumed that the magnitude of the incoming wave is 1, and in inner coordinates,

$$\left.\begin{aligned} \phi_{\mathrm{i.w.}} &= \exp(-Y - \mathrm{i}X) \quad (Y > 0), \\ &= -\exp(Y - \mathrm{i}X) \quad (Y < 0). \end{aligned}\right\} \tag{3.2}$$

In terms of ϕ, the problem now becomes

$$\left.\begin{aligned} \nabla^2\phi &= 0 && (\text{all } X, Y), \\ \phi_{YXXXX} + \phi &= 0 && (Y = 0,\ X > 0), \\ \phi_Y &= \exp(-\mathrm{i}X) && (Y = 0,\ X < 0), \\ \phi_Y &= 1, \quad \phi_{YX} = -\mathrm{i} && (X = Y = 0), \end{aligned}\right\} \tag{3.3}$$

it being noted that ϕ is odd in y to produce waves on the plate.

Let $\Theta(s, Y)$ represent the full range Fourier transform of $\phi(X, Y)$ with respect to X:

$$\Theta(s, Y) = \int_{-\infty}^{+\infty} e^{isX} \phi(X, Y)\, dX, \tag{3.4}$$

where s is complex. To accommodate the two-part boundary conditions on $y = 0$ it is convenient to split Θ into half-range transforms defined by

$$\Theta_+(s, Y) = \int_0^{\infty} e^{isX} \phi(X, Y)\, dX, \tag{3.5}$$

$$\Theta_-(s, Y) = \int_{-\infty}^{0} e^{isX} \phi(X, Y)\, dX. \tag{3.6}$$

It can be shown that Θ_+ is regular in the upper half s-plane, U_+ say, shown in figure 4, while Θ_- is regular in U_-. The two half-planes do not overlap, but touch along XX (figure 4) which is a line lying just above the negative real axis and just below the positive real axis. This is sufficient to enable a solution to be found for all s by analytic continuation arguments.

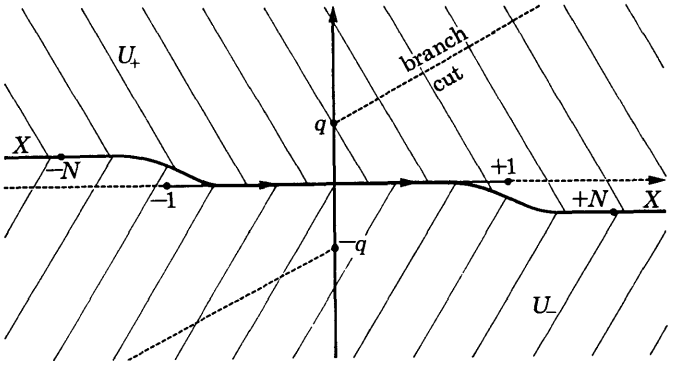


FIGURE 4. Complex s- or ζ-plane.

Note that the term $|s|$, s complex, is defined as

$$|s| = \lim_{q \to 0} \{(s + iq)(s - iq)\}^{\frac{1}{2}}, \tag{3.7}$$

where q is a fictitious dissipation term, and the line XX (in figure 4) passes between $s = iq$ and $s = -iq$. The branch of $|s|$ is chosen so that

$$\left.\begin{aligned} |s| &= s \quad \text{when } \mathrm{Re}\,(s) > 0, \\ |s| &= -s \quad \text{when } \mathrm{Re}\,(s) < 0, \end{aligned}\right\} \tag{3.8}$$

and $s_\pm = \lim_{q \to 0} (s \pm iq)^{\frac{1}{2}}$.

Taking the Fourier transform of the governing equation and using the fact that the transform will be odd in y gives

$$\left.\begin{aligned}\Theta(s, Y) &= F(s)\,\mathrm{e}^{-|s|Y} \quad (Y > 0),\\ &= -F(s)\,\mathrm{e}^{|s|Y} \quad (Y < 0).\end{aligned}\right\} \tag{3.9}$$

By restricting attention to the half-plane $Y > 0$, and using the abbreviated notation Θ_+ for $\Theta_+(s, 0)$, etc., the transformed boundary conditions become

$$\Theta'_- = \partial\{\Theta_-(s, Y)\}/\partial Y|_{Y=0} = -\mathrm{i}/(s-1), \tag{3.10}$$

$$N(s) + s^4\Theta'_+ + \Theta_+ = 0, \tag{3.11}$$

where $N(s)$ is regular over all s and

$$N(s) = -\phi^0_{YXXX} + \mathrm{i}s\phi^0_{XYX} + s^2\phi^0_{YX} - \mathrm{i}s^3\phi^0_Y,$$

the superscript 0 denoting quantities evaluated at $X = Y = 0$. From (3.9) it is known that

$$\Theta_+ + \Theta_- = F(s) = -(\Theta'_+ + \Theta'_-)/|s|, \tag{3.12}$$

which, on rearranging and using (3.10), (3.11), gives the Wiener–Hopf functional equation

$$\Theta'_+(1/|s| - s^4) - \mathrm{i}/|s|(s-1) = N(s) - \Theta_-. \tag{3.13}$$

The Wiener–Hopf kernel $K(s)$ is given by

$$K(s) = (|s|)^{-1} - s^4, \tag{3.14}$$

which can be written as

$$K(s) = K_+(s)/K_-(s), \tag{3.15}$$

where $K_+(s)$ is regular and non-zero in U_+, and $K_-(s)$ is regular and non-zero in U_-. Since $K(s)$ is an even function, the quotient factors can be normalized by insisting that

$$K_+(-s) = 1/K_-(s). \tag{3.16}$$

By rearranging (3.13), with the use of (3.15), it is found that

$$\begin{aligned}K_+(s)\,\Theta'_+ - \mathrm{i}\{K_+(s) - K_+(1)\}/(s-1) &= N(s)\,K_-(s) - \Theta_-K_-(s)\\ &\quad + \mathrm{i}\{K_+(1) + s^4K_-(s)\}/(s-1),\end{aligned} \tag{3.17}$$

where the left-hand side is regular in U_+, and the right-hand side is regular in U_-.

Let $J(s)$ define a function equal to both sides of (3.17) along the line of common analyticity XX. Thus the definition of $J(s)$ can be extended throughout the complex s plane by analytic continuation. The form of $J(s)$ is found by examining the asymptotic behaviour of the terms in (3.17) as $s \to \infty$. Finite pressure at $X = Y = 0$ requires that

$$\underset{X\to 0+}{\phi_Y} \sim \phi^0_Y + X\phi^0_{YX} + (X^2/2!)\phi^0_{YXX} + O(X^3), \tag{3.18}$$

and so

$$\underset{s\to\infty \text{ in } U_+}{\Theta'_+} \sim \mathrm{i}\phi^0_Y/s - \phi^0_{YX}/s^2 - \mathrm{i}\phi^0_{YXX}/s^3 + O(s^{-5}). \tag{3.19}$$

Appendix A(*c*), equation (A 25), shows that

$$K_+(s) = O(s^2) \quad \text{as} \quad s \to \infty \quad \text{in } U_+. \tag{3.20}$$

Putting the values of ϕ^0_Y, ϕ^0_{YX} (from (3.3)) into (3.19) and then expanding the left-hand side of (3.17) for large s gives

$$K_+(s)\left[\frac{\mathrm{i}}{s}+\frac{\mathrm{i}}{s^2}+O(s^{-3})-\frac{\mathrm{i}}{s}\left\{1+\frac{1}{s}+O(s^{-2})\right\}\right]+\frac{\mathrm{i}K_+(1)}{s-1}, \tag{3.21}$$

which has a leading term of $O(s^{-1})$. Repeating for the right-hand side of (3.17) finally shows that $J(s) \to 0$ as $s \to \infty$; hence $J(s) \equiv 0$ by Liouville's theorem. Thus

$$\Theta'_+ = \{\mathrm{i}/(s-1)\}\{1-K_+(1)/K_+(s)\}, \tag{3.22}$$

or from (3.12)

$$\Theta(s, Y) = \mathrm{i}K_+(1)\,\mathrm{e}^{-|s|\,Y}/|s|(s-1)\,K_+(s) \quad (Y > 0). \tag{3.23}$$

Thus the solution of the inner problem is

$$\Phi_{\mathrm{w}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{i}K_+(1)\exp(-\mathrm{i}sX-|s|\,Y)\,\mathrm{d}s}{|s|\,(s-1)\,K_+(s)} + \exp(-\mathrm{i}X-Y) \quad (Y > 0), \tag{3.24}$$

where the path of integration is along the line XX, and there are branch cuts as shown in figure 4.

By closing the path of integration in the lower half s-plane, the magnitude of the reflected wave can be found by using the residue theorem. By inspection it is clear that the pole of interest is at $s = -1$, which gives an unattenuated outgoing wave and so

$$\begin{aligned}\phi_{\mathrm{r.w.}} &= -2\pi\mathrm{i} \times (\text{residue at } s = -1)\\ &= \lim_{s\to -1}\{K_+(1)\,(s+1)\exp(-\mathrm{i}sX-|s|\,Y)/|s|\,(s-1)\,K_+(s)\}.\end{aligned} \tag{3.25}$$

From Appendix A(*b*), equation (A 17), it is found, finally, that

$$\phi_{\mathrm{r.w.}} = \mathrm{e}^{3\pi\mathrm{i}/4}\,\mathrm{e}^{\mathrm{i}X-Y}. \tag{3.26}$$

The form of the non-wavy part of the potential as $R \to \infty$ is required in the matching procedure. The behaviour of Φ_{w} as $Y = 0, X \to -\infty$ is found by expanding the kernel of the integral in (3.24) as $s \to O_+$ and then integrating term by term. Appendix A(*d*), equation (A 28), shows that

$$K_+(s) \sim \mathrm{e}^{\mathrm{i}\pi/4}s^{-\frac{1}{2}}\{1+(\mathrm{e}^{-3\pi\mathrm{i}/10}/\sin\tfrac{1}{5}\pi)\,s+O(s^2)\}, \tag{3.27}$$

and the far-field form of Φ_{w} is, from Appendix B(*a*), (B 5),

$$\Phi_{\mathrm{w}} \sim \mathrm{e}^{3\pi\mathrm{i}/8}\left(\frac{10}{\pi}\right)^{\frac{1}{2}}\{\sin(\tfrac{1}{2}\theta)\,R^{-\frac{1}{2}}+\tfrac{1}{2}\cot\tfrac{1}{5}\pi\sin(\tfrac{3}{2}\theta)\,R^{-\frac{3}{2}}+O(R^{-\frac{5}{2}})\}, \tag{3.28}$$

and so

$$A_{\mathrm{w}} = \mathrm{e}^{3\pi\mathrm{i}/8}(10/\pi)^{\frac{1}{2}}, \quad B_{\mathrm{w}} = \tfrac{1}{2}\cot\tfrac{1}{5}\pi.$$

(c) *Forcing by $R^{\frac{1}{2}}$-behaviour as $R\to\infty$; $\Phi_{\frac{1}{2}}$ potential*

If $\Phi_{\frac{1}{2}} \sim R^{\frac{1}{2}}\sin\frac{1}{2}\theta$ as $R\to\infty$, it does not have a Fourier transform in the normal sense, but by inspection $\psi = \Phi_{\frac{1}{2}X}$ is transformable as

$$\psi \sim -\sin\tfrac{1}{2}\theta/2R^{\frac{1}{2}} \quad \text{as} \quad R\to\infty.$$

Now $\psi(X, Y)$ satisfies the boundary value problem in (3.1), not (3.3), and so the analogous Wiener–Hopf functional equation to (3.17) is

$$K_+(s)\,\Psi'_+ = M(s)\,K_-(s) - \Psi_-K_-(s), \tag{3.29}$$

where K_+, K_- are defined by (3.14), (3.15),

$$\left.\begin{aligned} \Psi_+ &= \int_0^\infty \psi(X,0)\,\mathrm{e}^{\mathrm{i}sX}\,\mathrm{d}X, \quad \Psi_- = \int_{-\infty}^0 \psi(X,0)\,\mathrm{e}^{\mathrm{i}sX}\,\mathrm{d}X, \\ \text{and} \qquad M(s) &= \{-\phi^0_{YXXXX} + \mathrm{i}s\phi^0_{YXXX} + s^2\phi^0_{YXX} - \mathrm{i}s^3\phi^0_{YX}\}. \end{aligned}\right\} \tag{3.30}$$

$J(s)$ is again defined to equal both sides of (3.29) and is determined by asymptotic values of the terms as $s\to\infty$. Now

$$\underset{s\to\infty \text{ in } U_+}{\Psi'_+} \sim \frac{\mathrm{i}\phi^0_{YX}}{s} - \frac{\phi^0_{YXX}}{s^2} + O(s^{-3}) \tag{3.31}$$

through finite pressure arguments as $X\to O_+$, and the given condition that $\phi^0_{YX} = O$ results in (3.31) behaving as $O(s^{-2})$ as $s\to\infty$. Since $K_+(s) = O(s^2)$ as $s\to\infty$ in U_+, the left-hand side of (3.29) behaves as $\underset{s\to\infty \text{ in } U_+}{K_+(s)\,\Psi'_+} = O(1)$, and it can be shown that the right-hand side goes as $O(s)$. $J(s)$ must therefore be a constant, p say, and so

$$\Psi(s, Y) = p\,\mathrm{e}^{-|s|Y}/|s|\,K_+(s) \quad (Y > 0). \tag{3.32}$$

The constant, p, is determined in Appendix B(b), equation (B 10), from the fact that $\Phi_{\frac{1}{2}} \sim R^{\frac{1}{2}}\sin\frac{1}{2}\theta$ as $R\to\infty$, and it is found that

$$\left.\begin{aligned} p &= \tfrac{1}{2}\pi^{\frac{1}{2}}, \\ \underset{R\to\infty}{\Phi_{\frac{1}{2}}} &\sim R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta + \{\mathrm{i}\,\mathrm{e}^{-3\pi\mathrm{i}/10}/2\sin(\pi/5)\}\sin\tfrac{1}{2}\theta R^{-\frac{1}{2}} + O(R^{-\frac{3}{2}}) \\ \text{or} \qquad A_{\frac{1}{2}} &= \mathrm{i}\,\mathrm{e}^{-3\pi\mathrm{i}/10}/2\sin(\pi/5). \end{aligned}\right\} \tag{3.33}$$

The outgoing wave is determined in Appendix B(b):

$$\phi_{\text{o.w.}} = -(\pi/10)^{\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}\pi/8}\,\mathrm{e}^{\mathrm{i}X-Y}. \tag{3.34}$$

Thus in equations (2.52) and (2.53), from (2.12), (2.42),

$$\left.\begin{aligned} K_1 &= -(\pi/10)^{\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}\pi/8}\,b_{01}, \quad K_2 = \mathrm{e}^{5\pi\mathrm{i}/4}\,a_{\frac{1}{2}\mathrm{A}}, \\ \bar{K}_1 &= -(\pi/10)^{\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}\pi/8}\,\bar{b}_{01}, \quad \bar{K}_2 = \mathrm{e}^{5\pi\mathrm{i}/4}\,\bar{a}_{\frac{1}{2}\mathrm{B}}, \text{ etc.} \end{aligned}\right\} \tag{3.35}$$

(d) Forcing by $R^{\frac{3}{2}}$-behaviour as $R\to\infty$; $\Phi_{\frac{3}{2}}$ potential

Repeating the procedure in §3(*c*) by defining a new variable τ, say,

$$\tau = \Phi_{\frac{3}{2}XX}, \tag{3.36}$$

which is Fourier transformable as $\Phi_{\frac{3}{2}} \sim R^{\frac{3}{2}}\sin\frac{3}{2}\theta$ as $R\to\infty$. Thus the analogue equation to (3.29) becomes

$$K_+(s)\,T'_+ = P(s)\,K_-(s) - T_-K_-(s), \tag{3.37}$$

where
$$T_+(s) = \int_0^\infty \tau(X,0)\,\mathrm{e}^{\mathrm{i}sX}\,\mathrm{d}X, \quad T_-(s) = \int_{-\infty}^0 \tau(X,0)\,\mathrm{e}^{\mathrm{i}sX}\,\mathrm{d}X,$$
and
$$P(s) = (-\phi^0_{YXXXXX} + \mathrm{i}s\phi^0_{YXXXX} + s^2\phi^0_{YXXX} - \mathrm{i}s^3\phi^0_{YXX}). \tag{3.38}$$

To determine $J(s)$ again, the form of T'_+ as $s\to\infty$ is found to be

$$\underset{s\to\infty \text{ in } U_+}{T'_+} \sim \mathrm{i}\phi^0_{YXX}/s + O(s^{-2}) \tag{3.39}$$

and so the left-hand side of (3.37) is $O(s)$ as $s\to\infty$ in U_+. Thus $J(s)$ has the general form $J(s) = ms + l$, but this gives a behaviour, as $R\to\infty$, for $\Phi_{\frac{3}{2}}$ as

$$\underset{R\to\infty}{\Phi_{\frac{3}{2}}} \sim C_1(R^{\frac{3}{2}}\sin\tfrac{3}{2}\theta + \ldots) + C_2(R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta + \ldots). \tag{3.40}$$

It is required that $C_2 \equiv 0$ and $C_1 = 1$ which determines m and l. By inspection, if $C_2 = 0$, then $m = 0$, and so it is found that

$$T(s, Y) = l\,\mathrm{e}^{-|s|Y}/|s|K_+(s) \quad (Y > 0). \tag{3.41}$$

The constant l is determined in Appendix B(*c*) from the condition that $C_1 = 1$ in (3.40). It is finally found that

$$l = 3\pi^{\frac{1}{2}}/4, \quad A_{\frac{3}{2}} = -3\mathrm{i}\,\mathrm{e}^{-3\mathrm{i}\pi/20}/\sin\tfrac{1}{5}\pi,$$
$$\underset{R\to\infty}{\Phi_{\frac{3}{2}}} \sim R^{\frac{3}{2}}\sin\tfrac{3}{2}\theta - \{3\mathrm{i}\,\mathrm{e}^{-3\pi\mathrm{i}/10}/\sin\tfrac{1}{5}\pi\}\sin(\tfrac{1}{2}\theta)\,R^{\frac{1}{2}} + \ldots. \tag{3.42}$$

The outgoing wave is also determined in Appendix B(*c*):

$$\phi_{\text{o.w.}} = \tfrac{3}{2}(\pi/10)^{\frac{1}{2}}\,\mathrm{e}^{3\pi\mathrm{i}/8}\,\mathrm{e}^{\mathrm{i}X-Y} \quad (Y > 0). \tag{3.43}$$

4. Prototype model problem

The matching procedure discussed in this paper is now applied to a particular geometry, to show a method of approach to calculate the magnitudes of the eigen-solutions and wave solutions.

The prototype problem consists of a finite elastic plate, length $2a$, set in an

infinite baffle lying on $y' = 0$, and the limit $ka \to 0$ is taken. The boundary value problem is defined, for $\phi'(x', y')$, by

$$\left.\begin{aligned} (\nabla'^2 + k^2)\phi' &= 0 \quad (\text{all } x', y'), \\ \phi'_{y'} &= 0 \quad (y' = 0, \infty > |x'| > a), \\ \phi'_{y'} = \phi'_{y'x'} &= 0 \quad (y' = 0,\ x' = \pm a), \\ (\partial^4/\partial x'^4 - \mu^4)\,\phi'_{y'} + (\tfrac{1}{2}\alpha)\,\phi'|^{\pm} &= 0 \quad (y' = 0,\ |x'| < a), \end{aligned}\right\} \tag{4.1}$$

and

$$ka \to 0. \tag{4.2}$$

The incident wave has potential $e^{-ik y'}$, corresponding to normal incidence.

By making the outer-region transformation and taking the expansion of ϕ_0, the first term ϕ_{01} satisfies equations (2.4)–(2.6). If ϕ_{01} is split so that

$$\left.\begin{aligned} \phi_{01} &= \phi_{\text{inc}} + \phi_{\text{ref}} + \phi_s \quad (y > 0), \\ &= \phi_s \quad (y < 0), \end{aligned}\right\} \tag{4.3}$$

where $\phi_{\text{inc}} = e^{-iky}$, $\phi_{\text{ref}} = e^{iky}$, then ϕ_s is an odd function in y and its boundary problem is

$$(\nabla^2 + 1)\,\phi_s = 0 \quad (\text{all } x, y), \tag{4.4}$$

$$\phi_s|^{\pm} = -2 \quad (y = 0, |x| < ka), \tag{4.5}$$

$$\phi_{sy} = 0 \quad (y = 0, \infty > |x| > ka). \tag{4.6}$$

The final condition is that ϕ_s must only represent an outward travelling wave as

$$r = (x^2 + y^2)^{\frac{1}{2}} \to \infty.$$

In the limit as $ka \to 0$ the gap AB (see figure 1) goes to a point and the potential ϕ_s becomes simply the Green function solution of a source at $y = O_+$, $x = 0$, and a sink at $y = O_-$, $x = 0$. Thus the solution is

$$\phi_s(x, y) \sim (m/2i)\,H_0^{(1)}(r) \quad (y > 0), \tag{4.7}$$

where $H_0^{(1)}(r)$ is the Hankel function of the first kind, and m is the magnitude of the source at the origin and must be determined from examining the region $r \to 0$.

It is clear that because of this new small parameter, $ka = \beta$ say, the outer region discussed in the matching section consists of an inner and an outer region. The parameter ϕ_s is the scattered potential of the outer–outer region and the inner–outer region is defined by the transformation

$$x = \beta\tilde{x}, \quad y = \beta\tilde{y}, \tag{4.8}$$

where $\beta = ka$, and $\beta \gg \epsilon$.

Thus $\tilde{\phi}$, the inner–outer scattered potential, satisfies

$$\left.\begin{aligned} (\tilde{\nabla}^2 + \beta^2)\tilde{\phi} &= 0 \quad (\text{all } \tilde{x}, \tilde{y}), \\ \tilde{\phi} &= -1 \quad (\tilde{y} = O_+, |\tilde{x}| < 1), \\ \tilde{\phi}_{\tilde{y}} &= 0 \quad (\tilde{y} = 0, |\tilde{x}| > 1), \\ \tilde{\phi} &\text{ is odd in } y. \end{aligned}\right\} \tag{4.9}$$

It is convenient to calculate m by using a Green function approach for ϕ_s and a matching-type approach for the eigensolutions.

From Green's theorem it is easily proven that

$$\phi_s(x,y) = \int_{-\beta}^{+\beta} \{G(x_0,y_0;x,y)\,\phi_{sy_0}(x_0)\}|_{y_0=0}\,dx_0, \tag{4.10}$$

where

$$G(x_0,y_0;x,y) = (\tfrac{1}{4}i)\langle H_0^{(1)}[\{(x-x_0)^2+(y-y_0)^2\}^{\frac{1}{2}}] + H_0^{(1)}[\{(x-x_0)^2+(y+y_0)^2\}^{\frac{1}{2}}]\rangle. \tag{4.11}$$

Thus
$$\phi_s(x,0) = \frac{1}{2i}\int_{-\beta}^{\beta} H_0^{(1)}(|x-x_0|)\,\phi_{sy_0}|_{y_0=0}\,dx_0, \tag{4.12}$$

but from (4.9) $\phi_s(x,0) = -1$ on the gap, which gives

$$-2i = \int_{-\beta}^{\beta} H_0^{(1)}(|x-x_0|)\,\phi_{sy_0}|_{y_0=0}\,dx_0. \tag{4.13}$$

In the limit as $\beta \to 0$ the argument of the Hankel function is at most 2β and so it can be approximated by

$$H_0^{(1)}(|x-x_0|) \sim (2i/\pi)\ln(|x-x_0|) - (2i/\pi)(\ln 2 - \gamma + i\pi/2), \tag{4.14}$$

where γ is the Euler constant, $\gamma = 0.5772....$ Putting (4.14) into (4.13) and differentiating under the integral with respect to x gives

$$\int_{-\beta}^{\beta} \frac{\phi_{sy_0}(x_0)}{(x-x_0)}\bigg|_{y_0=0} dx_0 = 0, \tag{4.15}$$

which is a standard integral with a solution

$$\phi_{sy_0}(x)|_{y_0=0} = Q/(\beta^2-x^2)^{\frac{1}{2}} \quad (|x| < \beta). \tag{4.16}$$

Q is determined by substituting the solution back into (4.13), to get

$$-\pi = Q\int_{-\beta}^{\beta} \frac{\ln(|x-x_0|)}{(\beta^2-x_0^2)^{\frac{1}{2}}}\,dx_0 - (\ln 2 - \gamma + i\pi/2)\,Q\int_{-\beta}^{\beta} \frac{dx_0}{(\beta^2-x_0^2)^{\frac{1}{2}}}, \tag{4.17}$$

which gives
$$Q = -1/(\ln\beta - 2\ln 2 + \gamma - i\pi/2). \tag{4.18}$$

Substituting into (4.10) and again taking the limit as $\beta \to 0$ gives

$$\phi_s(x,y) = \frac{Q}{2i}\lim_{\beta\to 0}\int_{-\beta}^{+\beta} \frac{H_0^{(1)}[\{(x-x_0)^2+y^2\}^{\frac{1}{2}}]}{(\beta^2-x_0^2)^{\frac{1}{2}}}\,dx_0, \tag{4.19}$$

which finally becomes

$$\phi_s(x,y) \sim -\pi H_0^{(1)}\{(x^2+y^2)^{\frac{1}{2}}\}/2i\,(\ln\beta - 2\ln 2 + \gamma - i\pi/2) \quad (y > 0). \tag{4.20}$$

The constant m in (4.7) is now known to be

$$m = -\pi/(\ln\beta - 2\ln 2 + \gamma - i\pi/2). \tag{4.21}$$

By using (4.16) the form of ϕ_{01} can be found. From equation (2.7)

$$\phi_{01} \sim a_{01} + b_{01} r^{\frac{1}{2}} \sin \tfrac{1}{2}\theta + c_{01} r \cos\theta + d_{01} r^{\frac{3}{2}} \sin \tfrac{3}{2}\theta + \ldots,$$

where $$r = \{(x+\beta)^2 + y^2\}^{\frac{1}{2}}, \quad \theta = \arctan\{y/(x+\beta)\}.$$

For this model problem the potential on the plate is (from (4.3) and (4.9))

$$\phi_{01} = 1 \quad (y = 0, \ |x| < \beta), \tag{4.22}$$

and so $a_{01} = 1$, $c_{01} = c_{01} = 0$, etc. Now

$$\begin{aligned} \phi_{01y}|_{y=0} = \phi_{sy}|_{y=0} &= (1/r)\,\partial\phi_{01}/\partial\theta|_{\theta=0} \\ &= \tfrac{1}{2} b_{01}(x+\beta)^{-\frac{1}{2}} + \tfrac{3}{2} d_{01}(x+\beta)^{\frac{1}{2}} + \tfrac{5}{2} f_{01}(x+\beta)^{\frac{3}{2}} + \ldots. \end{aligned} \tag{4.23}$$

Expanding (4.16) for $x+\beta \to 0$ gives

$$\phi_{sy}|_{y=0} = \frac{Q}{(2\beta)^{\frac{1}{2}}} (x+\beta)^{-\frac{1}{2}} \left\{ 1 + \frac{(x+\beta)}{4\beta} + \frac{3(x+\beta)^2}{32\beta^2} + \ldots \right\}. \tag{4.24}$$

Therefore equating coefficients finally gives

$$\underset{r \to 0}{\phi_{01}} \sim 1 + Q(2r/\beta)^{\frac{1}{2}} \sin \tfrac{1}{2}\theta + \tfrac{1}{3} Q (r/2\beta)^{\frac{3}{2}} \sin \tfrac{3}{2}\theta + \ldots, \tag{4.25}$$

where $r \ll \beta$ and Q is defined by (4.18).

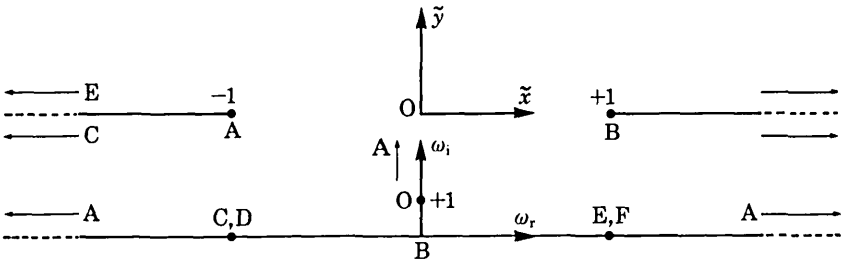


FIGURE 5. Complex ω- and z-planes.

The eigensolution potentials, ϕ_{02}, ϕ_{10} etc., to leading order in β are only influenced by the inner–outer field (defined in (4.8)), and so the boundary value problem becomes incompressible, therefore yielding to solution by a conformal transformation which is shown in figure 5.

Letting the inner–outer region eigensolution $\tilde{\phi}_e = \mathrm{Re}\{\lambda(z)\}$, where $z = \tilde{x} + i\tilde{y}$, and posing the transformation

$$w = w_R + i w_I = \{(z-1)/(z+1)\}^{\frac{1}{2}} \tag{4.26}$$

maps the whole z-plane in figure 5 onto the top half of the w-plane.

By inspection of the w-plane it is clear that a class of solutions is

$$\lambda(w) = \text{constant} \times w^{\pm n}, \quad \text{where } n = 1, 3, 5, 7, \ldots, \tag{4.27}$$

or

$$\lambda(z) = \{(z-1)/(z+1)\}^{\pm n/2} \times \text{constant}. \tag{4.28}$$

These solutions are the eigensolutions exhibiting singular behaviour at either edge A or B, the singularity being of order $\frac{1}{2}n$. Thus

$$\tilde{\phi}_{\mathrm{e}}(\tilde{x},\tilde{y}) = \mathrm{Re}\,[\{(z-1)/(z+1)\}^{\pm n/2}] \times \text{constant}. \tag{4.29}$$

Defining the branch of $(z^2-1)^{\pm n/2}$ by

$$(z^2-1)^{\pm n/2} = \mathrm{e}^{\pm n\pi \mathrm{i}/2} \quad \text{at} \quad z = 0 \tag{4.30}$$

and letting $u = z+1$ gives

$$\tilde{\phi}_{\mathrm{e}} = \mathrm{Re}\,\{(2-u)^{\pm n/2}\,\mathrm{e}^{\pm \mathrm{i}n\pi/2}\,u^{\mp n/2}\} \times \text{constant}. \tag{4.31}$$

If $|u| \to 0$ then the eigensolution can be expanded to show its behaviour near A. Thus

$$\underset{|u|\to 0}{\tilde{\phi}_{\mathrm{e}}} \sim \text{constant} \times \mathrm{Re}\left[\left(\frac{2\mathrm{e}^{\mathrm{i}\pi}}{u}\right)^{\pm n/2}\left\{1 \mp \frac{nu}{4} + \frac{u^2 n}{32}(n \mp 2)\ldots\right\}\right]. \tag{4.32}$$

This general solution can now give us the particular eigensolutions of interest. Expanding (4.32) for the particular n of interest, adjusting the constant to give the correct magnitude, and changing to outer coordinate r, where $r = \beta|u|$, finally gives

$$\underset{r\to 0}{\phi_{\frac{1}{2}\mathrm{A}}} \sim r^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta + \frac{r^{\frac{1}{2}}}{\beta}\sin\tfrac{1}{2}\theta + \frac{r^{\frac{3}{2}}}{32\beta^2}\sin\tfrac{3}{2}\theta + \ldots, \tag{4.33}$$

$$\underset{r\to 0}{\phi_{\frac{3}{2}\mathrm{A}}} \sim r^{-\frac{3}{2}}\sin\tfrac{3}{2}\theta - \frac{3r^{-\frac{1}{2}}}{4\beta}\sin\tfrac{1}{2}\theta - \frac{3r^{\frac{1}{2}}}{32\beta^2}\sin\tfrac{1}{2}\theta + \ldots, \tag{4.34}$$

$$\underset{r\to 0}{\phi_{\frac{1}{2}\mathrm{B}}} \sim \frac{r^{\frac{1}{2}}}{2\beta}\sin\tfrac{1}{2}\theta + \frac{r^{\frac{3}{2}}}{8\beta^2}\sin\tfrac{3}{2}\theta + \frac{3r^{\frac{5}{2}}}{64\beta^3}\sin\tfrac{5}{2}\theta + \ldots, \tag{4.35}$$

$$\underset{r\to 0}{\phi_{\frac{3}{2}\mathrm{B}}} \sim \frac{r^{\frac{3}{2}}}{8\beta^3}\sin\tfrac{3}{2}\theta + \frac{3r^{\frac{5}{2}}}{32\beta^4}\sin\tfrac{5}{2}\theta + \frac{15r^{\frac{7}{2}}}{256\beta^5}\sin\tfrac{7}{2}\theta + \ldots, \tag{4.36}$$

where $r \ll \beta$. Comparing (4.33)–(4.36) with equations (2.18)–(2.21) determines the coefficients $a_{\frac{1}{2}\mathrm{A}}$, $b_{\frac{1}{2}\mathrm{A}}$, etc.

It is also worth noting that in the inner–outer region the ϕ_{01} potential has an analytic solution of the form

$$\begin{aligned}\tilde{\phi}_{01}(\tilde{x},\tilde{y}) &\sim 1 - Q\,\mathrm{Re}\left[\ln\left\{\frac{(z-1)^{\frac{1}{2}} - (z+1)^{\frac{1}{2}}}{(z+1)^{\frac{1}{2}} + (z+1)^{\frac{1}{2}}}\right\}\right] \quad (\tilde{y} > 0)\\ &\sim 1 + Q\,\mathrm{Re}\left[\ln\left\{\frac{(z+1)^{\frac{1}{2}} - (z+1)^{\frac{1}{2}}}{(z-1)^{\frac{1}{2}} + (z+1)^{\frac{1}{2}}}\right\}\right] \quad (\tilde{y} < 0).\end{aligned} \tag{4.37}$$

Because of the symmetry of the geometry, and forcing, the problem is even in x and so all the right-hand coefficients are equal to the left-hand ones, i.e. $K_1 = \bar{K}_1$. Thus from (2.44), after substituting the particular values of b_{01}, $a_{\frac{1}{2}\mathrm{A}}$, etc., it is found that in the outer region

$$\phi \sim \phi_{01} + \{A(\epsilon)\,\epsilon^{\frac{1}{2}}\phi_{\frac{1}{2}\mathrm{A}} + B(\epsilon)\,\epsilon^{\frac{1}{2}}\phi_{\frac{1}{2}\mathrm{B}}\}\,(10/\pi)^{\frac{1}{2}}\,\mathrm{e}^{3\pi\mathrm{i}/8}, \tag{4.38}$$

where

$$A(\epsilon) = B(\epsilon) \sim \frac{Q\mathrm{e}^{7\pi\mathrm{i}/8}\,(\pi/5\beta)^{\frac{1}{2}}\,(\mathrm{e}^{-2\mathrm{i}\beta/\epsilon} + \mathrm{e}^{3\mathrm{i}\pi/4})\,\epsilon^{\frac{1}{2}} + O(\epsilon^{\frac{3}{2}})}{(\mathrm{e}^{-4\mathrm{i}\beta/\epsilon} - \mathrm{e}^{3\pi\mathrm{i}/2}) - (\epsilon/\beta)\,\mathrm{e}^{5\pi\mathrm{i}/4}\,(\mathrm{e}^{-2\mathrm{i}\beta/\epsilon} + 2\mathrm{e}^{3\pi\mathrm{i}/4}) + O(\epsilon^2)}, \tag{4.39}$$

and ϕ_{01}, $\phi_{\frac{1}{2}A}$, $\phi_{\frac{1}{2}B}$ are known explicitly in the inner–outer region. Equation (4.38) could have been expanded to $O(\epsilon^2)$, $O\{A(\epsilon)\,\epsilon^{\frac{3}{2}}\}$ as all the coefficients are known to this order.

5. Discussion

(*a*) *Resonance*

The fundamental difference between the finite and infinite elastic plate is the ability of the finite plate to support standing waves. The effect of this resonant situation is clearly seen in the analysis, even though the heavy-fluid-loading limit might seem to reduce the effect of the plate on the acoustic far field. It is seen from (2.44) that in the outer region

$$\begin{aligned} \phi &\sim \phi_{01} + O(\epsilon) \quad \text{away from resonance,} \\ &\sim \phi_{01} + O(1) \quad \text{at a resonance,} \end{aligned}$$

which shows that the eigensolution potentials, with singularities at the plate edges, are of the same order as ϕ_{01} at a resonance. Thus the initial tentative idea that, to leading order, the potential in the outer region behaves as if the plate were not present, is incorrect at resonance. Also drawn from the analysis is the fact that in the wave region (see figure 2) at a resonance, the leading terms are the coupled waves of order $\epsilon^{-\frac{1}{2}}$.

(*b*) *Accuracy*

To calculate $A(\epsilon)$, $B(\epsilon)$ required expanding the potentials ϕ_0, ϕ_1, ϕ_2 to second order (i.e. $O(\epsilon)$). At a resonance $A(\epsilon)$, $B(\epsilon)$ become dependent on this second-order term and so inaccuracy can be expected as both viscous and mechanical damping effects were neglected.

(*c*) *Geometry*

To use the method outlined in this paper for a specific geometry requires the solution of the ϕ_{01} and eigensolution potentials for the analogue problem without the presence of the elastic plate. As has already been discussed this will not be trivial for most geometries and so could be solved by approximate methods, i.e. numerical or asymptotic.

I wish to thank Dr F. G. Leppington for many helpful discussions, and advice, on this work, and the Science Research Council and A.U.W.E. Portland for the financial support of a C.A.S.E. studentship.

Appendix A. Factorization of the Wiener–Hopf kernel

(a) *Determining* $\ln(M_+(s))$

From (3.15) $$K_+(s)/K_-(s) = 1/|s| - s^4.$$

If a new function $M(s)$ is defined by

$$M(s) = M_+(s)/M_-(s) = 1 - |s|s^4, \tag{A 1}$$

then $$K_+(s) = s_+^{-\frac{1}{2}} M_+(s)\, e^{i\pi/4}, \tag{A 2}$$

and $$K_-(s) = s_-^{-\frac{1}{2}} M_-(s)\, e^{i\pi/4}, \tag{A 3}$$

whence $$K_+(s) = \{K_-(-s)\}^{-1},$$

where $M_+(s)$, $M_-(s)$ are regular and non-zero in U_+, U_- respectively. It can be shown that

$$\ln\{M_+(s)\} = (2\pi i)^{-1} \lim_{N\to\infty} \int_{-N}^{N} \frac{\ln(1-|\zeta|\zeta^4)}{\zeta - s}\, d\zeta, \tag{A 4}$$

where N, $-N$ lie on XX, the path along the real axis in the complex ζ plane, from $-\infty$ to $+\infty$. Path XX lies above the branch cut $-\infty < \zeta < -1$ and below the branch cut $1 < \zeta < \infty$, as in figure 4, and s must be a point above XX in the ζ-plane. Now

$$\ln\{M_+(s)\} = (2\pi i)^{-1} \lim_{N\to\infty} \left\{ \int_{XX_1}^{0} \frac{\ln(1+\zeta^5)}{\zeta - s}\, d\zeta + \int_{XX_2}^{N} \frac{\ln(1-\zeta^5)}{\zeta - s}\, d\zeta \right\}, \tag{A 5}$$

where XX_1 is the path along XX from $-N$ to 0 and XX_2 is the path on XX from 0 to N. By making the substitution $\zeta = -\zeta'$ in the first integral, and formally taking the limit, (A 5) becomes

$$\ln\{M_+(s)\} = \frac{s}{\pi i} \int_{XX_2}^{\infty} \frac{\ln(1-\zeta^5)}{\zeta^2 - s^2}\, d\zeta = \frac{I(s)}{2\pi i}, \tag{A 6}$$

say. Integration by parts yields,

$$\begin{aligned} I(s) &= \left[\ln(1-\zeta^5) \ln\left(\frac{\zeta - s}{\zeta + s}\right) \right]_0^{\infty} + 5 \int_0^{\infty} \frac{\zeta^4}{1-\zeta^5} \ln\left(\frac{\zeta - s}{\zeta + s}\right) d\zeta \\ &= 5 \int_0^{\infty} \frac{\zeta^4}{1-\zeta^5} \ln\left(\frac{\zeta - s}{\zeta + s}\right) d\zeta, \end{aligned} \tag{A 7}$$

which can be differentiated, i.e.

$$\frac{dI}{ds} = -5 \int_0^{\infty} \frac{\zeta^4}{1-\zeta^5} \left(\frac{1}{\zeta - s} + \frac{1}{\zeta + s} \right) d\zeta. \tag{A 8}$$

Now, it is known that $$\frac{5\zeta^4}{1-\zeta^5} = \sum_{i=-2}^{+2} (\zeta_i - \zeta)^{-1}, \tag{A 9}$$

where ζ_i are the roots of $1 - \zeta^5 = 0$, and so

$$\begin{aligned} \frac{dI}{ds} &= \sum_{i=-2}^{+2} \int_0^{\infty} \left(\frac{1}{\zeta - s} - \frac{1}{\zeta - \zeta_i} \right) \frac{1}{s - \zeta_i} + \left(\frac{1}{\zeta - \zeta_i} - \frac{1}{\zeta + s} \right) \frac{1}{s + \zeta_i} d\zeta \\ &= \left[\sum_{i=-2}^{+2} \left\{ \ln\left(\frac{\zeta - s}{\zeta - \zeta_i}\right) \frac{1}{s - \zeta_i} - \ln\left(\frac{\zeta + s}{\zeta - \zeta_i}\right) \frac{1}{s + \zeta_i} \right\} \right]_0^{\infty}. \end{aligned} \tag{A 10}$$

Care being taken to evaluate angles correctly in (A 10), after the five branch cuts in the ζ-plane have been defined, it is found that

$$\frac{dI}{ds} = -\sum_{i=-2}^{+2} \frac{\ln s - i\pi}{s-\zeta_i} + \sum_{i=-2}^{+2} \frac{\ln s}{s+\zeta_i} - \frac{i\pi}{5}\left(\frac{1}{s-\zeta_2} - \frac{1}{s+\zeta_2}\right) - \frac{3i\pi}{5}\left(\frac{1}{s-\zeta_1} - \frac{1}{s+\zeta_1}\right)$$
$$-i\pi\left(\frac{1}{s-1} - \frac{1}{s+1}\right) + \frac{3i\pi}{5}\left(\frac{1}{s-\zeta_{-1}}\right) + \frac{i\pi}{5}\left(\frac{1}{s-\zeta_{-2}} - \frac{1}{s+\zeta_{-2}}\right), \tag{A 11}$$

where $\zeta_1 = e^{2\pi i/5}$, $\zeta_2 = e^{4\pi i/5}$, $\zeta_{-1} = e^{-2i\pi/5}$, $\zeta_{-2} = e^{-4i\pi/5}$. Integrating (A 11) with respect to s, and noting that $I(0) = 0$, gives

$$I(s) = -\frac{2}{5}\int_0^{s^5} \frac{\ln u}{(u^2-1)}\,du + i\pi \ln(s^4+s^3+s^2+s+1) - (i\pi)^2\tfrac{8}{5}$$
$$+ i\pi\left[\frac{1}{5}\ln\left\{\frac{(s+\zeta_2)(s-\zeta_{-2})}{(s-\zeta_2)(s+\zeta_{-2})}\right\} + \tfrac{3}{5}\ln\left\{\frac{(s+\zeta_1)(s-\zeta_{-1})}{(s-\zeta_1)(s+\zeta_{-1})}\right\} + \ln(s+1)\right]. \tag{A 12}$$

(*b*) *Evaluation of* $K_+(s)$ *when* $s = 1, -1$

Taking (A 12) when $s = 1$ gives

$$I(1) = -\frac{2}{5}\int_0^1 \frac{\ln u}{u^2-1}\,du + i\pi \ln 5 - (i\pi)^2\tfrac{8}{5}$$
$$+ i\pi\left[\tfrac{1}{5}\ln\left\{\frac{(1+\zeta_2)(1-\zeta_{-2})}{(1-\zeta_2)(1+\zeta_{-2})}\right\} + \tfrac{3}{5}\ln\left\{\frac{(1+\zeta_1)(1-\zeta_{-1})}{(1-\zeta_1)(1+\zeta_{-1})}\right\} + \ln 2\right]$$
$$= -\tfrac{1}{20}\pi^2 + i\pi \ln(10) - \tfrac{4}{5}(i\pi)^2. \tag{A 13}$$

From (A 2) it is now found that

$$K_+(1) = M_+(1)\,e^{i\pi/4} = \exp\{I(1)/2\pi i\}\,e^{i\pi/4}$$
$$= 10^{\frac{1}{2}}\,e^{-i\pi/8}. \tag{A 14}$$

Evaluating $\lim_{s\to-1}\{K_+(s)\}$ by rearranging (3.14) and (3.15) gives

$$K_+(s) = (1-s^{10})\{|s|(1+|s|s^4)K_+(-s)\}^{-1}, \tag{A 15}$$

which, by expanding $(1-s^{10})$, gives

$$\lim_{s\to-1}\{K_+(s)/(s+1)\} = 5/K_+(1) = (\tfrac{5}{2})^{\frac{1}{2}}\,e^{i\pi/8}. \tag{A 16}$$

Finally, equation (3.25) can be evaluated:

$$\phi_{\text{r.w.}} = \lim_{s\to-1}\left\{\frac{s+1}{K_+(s)}\right\}\frac{K_+(1)}{(-2)}\,e^{iX-Y}$$
$$= e^{3\pi i/4}\,e^{iX-Y}. \tag{A 17}$$

(c) *Expansion of $K_+(s)$ for large s*

Rearranging $\mathrm{d}I/\mathrm{d}s$ in the form

$$\frac{\mathrm{d}I}{\mathrm{d}s} = -\frac{10s^4 \ln s}{s^{10}-1} + \frac{5\mathrm{i}\pi s^4}{s^5-1} + \frac{\mathrm{i}\pi}{5}\left(\frac{2\zeta_{-2}}{s^2-\zeta_{-2}^2} - \frac{2\zeta_2}{s^2-\zeta_2^2}\right) + \frac{3\mathrm{i}\pi}{5}\left(\frac{2\zeta_{-1}}{s^2-\zeta_{-1}^2} - \frac{2\zeta_1}{s^2-\zeta_1^2}\right) - \frac{2\mathrm{i}\pi}{s^2-1}, \tag{A 18}$$

and expanding in powers of s^{-1} as $s \to \infty$, gives

$$\frac{\mathrm{d}I}{\mathrm{d}s}_{s\to\infty} = -\frac{10\ln s}{s^6}\{1+O(s^{-10})\} + \frac{5\mathrm{i}\pi}{s}\{1+O(s^{-5})\} + \frac{\mathrm{i}\pi}{5s^2}\{2\zeta_{-2}-2\zeta_2+O(s^{-2})\} + \frac{3\mathrm{i}\pi}{5s^2}\{2\zeta_{-1}-2\zeta_1+O(s^{-2})\} - \frac{2\mathrm{i}\pi}{s^2}\{1+O(s^{-2})\}. \tag{A 19}$$

Therefore, grouping terms of like power yields

$$\frac{\mathrm{d}}{\mathrm{d}s}_{s\to\infty}[\ln\{M_+(s)\}] = \frac{5}{2s} - \frac{1}{s^2}[1+\tfrac{2}{5}\mathrm{i}\{\sin\tfrac{4}{5}\pi + 3\sin\tfrac{2}{5}\pi\}] + O(s^{-4}). \tag{A 20}$$

Integrating (A 20) and expanding in $1/s$ gives

$$\ln\{M_+(s)\}_{s\to\infty} = \ln G + \tfrac{5}{2}\ln s + \frac{\mathrm{e}^{3\mathrm{i}\pi/10}}{\sin\frac{1}{5}\pi} + O(s^{-3}), \tag{A 21}$$

or

$$M_+(s)_{s\to\infty} = Gs^{\frac{5}{2}}\left\{1 + \frac{\mathrm{e}^{3\mathrm{i}\pi/10}}{\sin\frac{1}{5}\pi}s^{-1} + O(s^{-2})\right\}, \tag{A 22}$$

where G is the constant of integration. G is determined by examining (A 12) and (A 21) as s formally goes to ∞. Thus

$$\lim_{s\to\infty}[\ln\{M_+(s)\} - \tfrac{5}{2}\ln s] = \ln G = \frac{\mathrm{i}}{5\pi}\int_0^\infty \frac{\ln u\,\mathrm{d}u}{u^2-1} - \frac{4\pi\mathrm{i}}{5}, \tag{A 23}$$

which gives

$$G = \mathrm{e}^{-3\pi\mathrm{i}/4}. \tag{A 24}$$

Finally, substituting the value of $M_+(s)$ from (A 22) into (A 2) gives

$$K_+(s)_{s\to\infty} = \mathrm{e}^{-\mathrm{i}\pi/2}s^2\left\{1 + \frac{\mathrm{e}^{3\mathrm{i}\pi/10}}{\sin\frac{1}{5}\pi}s^{-1} + O(s^2)\right\}. \tag{A 25}$$

(d) *Expansion of $K_+(s)$ for small s*

Taking (A 11) and expanding in powers of s gives, after rearranging,

$$\frac{\mathrm{d}I}{\mathrm{d}s}_{s\to 0} = \frac{2\mathrm{i}\pi}{5}\left\{\frac{1}{\zeta_2} - \frac{1}{\zeta_{-2}} + 3\left(\frac{1}{\zeta_1} - \frac{1}{\zeta_{-1}}\right) + 5\right\} + 2s^2\frac{\mathrm{i}\pi}{5}\left\{\frac{1}{\zeta_2^3} - \frac{1}{\zeta_{-2}^3} + 3\left(\frac{1}{\zeta_1^3} - \frac{1}{\zeta_{-1}^3}\right) + 5\right\} + O(s^4\ln s). \tag{A 26}$$

Integrating (A 26) with respect to s, since $I(0) = 0$, yields

$$\ln\{M_+(s)\} \underset{s\to 0}{=} I(s)/2\pi\mathrm{i} \underset{s\to 0}{=} \{\mathrm{e}^{-3\mathrm{i}\pi/10}/\sin\tfrac{1}{5}\pi\}\,s + O(s^3). \tag{A 27}$$

Thus from (A 2) it is found that

$$M_+(s)\,\mathrm{e}^{\mathrm{i}\pi/4}s^{-\frac{1}{2}} \underset{s\to 0}{=} K_+(s) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{s^{\frac{1}{2}}}\left\{1 + \frac{\mathrm{e}^{-3\mathrm{i}\pi/10}}{\sin\frac{1}{5}\pi}s + O(s^2)\right\} \tag{A 28}$$

Appendix B

(*a*) *Far field due to forcing by an incoming wave*

From (3.24)
$$\underset{Y=0,\,X<0}{\Phi_{\mathrm{w}}} = \frac{\mathrm{i}}{2\pi}K_+(1)\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}s|X|}\,\mathrm{d}s}{|s|\,(s-1)\,K_+(s)}, \tag{B 1}$$

where the path of integration is along XX in figure 4. By closing the contour in the upper half s-plane the non-wavy part of the potential can be determined by integrating around the upper branch cut (see figure 4). Letting the branch cut go from $\mathrm{i}k$ to $\mathrm{i}\infty$ it is found that

$$\begin{aligned}\underset{Y=0,\,X\to-\infty}{\Phi_{\mathrm{w}}} &= \lim_{q\to 0}\left\{\frac{\mathrm{i}K_+(1)}{\pi}\int_{1\infty}^{\mathrm{i}q}\frac{\mathrm{e}^{\mathrm{i}s|X|}\,\mathrm{d}s\,\mathrm{e}^{-\mathrm{i}\pi/4}}{(s-\mathrm{i}q)^{\frac{1}{2}}\,(s-1)\,M_+(s)}\right\}\\ &= \frac{\mathrm{i}K_+(1)}{\pi}\int_0^{\infty}\frac{\mathrm{e}^{-s|X|}\,\mathrm{d}x}{s^{\frac{1}{2}}(1-\mathrm{i}s)\,M_+(\mathrm{i}s)}.\end{aligned} \tag{B 2}$$

The integral is now in a form that enables us to obtain an asymptotic solution by using Watson's lemma. Thus expanding $\{(1-\mathrm{i}s)\,M_+(\mathrm{i}s)\}^{-1}$ for small s gives, from (A 28),

$$\{(1-\mathrm{i}s)\,M_+(\mathrm{i}s)\}^{-1} \sim 1 - \cot\tfrac{1}{5}\pi s + O(s^2), \tag{B 3}$$

and inserting into (B 2) gives

$$\begin{aligned}\underset{Y=0,\,X\to-\infty}{\Phi_{\mathrm{w}}} &\sim \frac{\mathrm{i}K_+(1)}{\pi}\left\{\int_0^{\infty}\frac{\mathrm{e}^{-s|X|}}{s^{\frac{1}{2}}}\,\mathrm{d}s - \cot\tfrac{1}{5}\pi\int_0^{\infty}\mathrm{e}^{-s|X|}s^{\frac{1}{2}}\,\mathrm{d}s + O(|X|^{-\frac{5}{2}})\right\}\\ &= \mathrm{e}^{3\mathrm{i}\pi/8}\frac{(10)^{\frac{1}{2}}}{\pi}\left\{\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-u^2}\,\mathrm{d}u}{|X|^{\frac{1}{2}}} - \frac{\cot\frac{1}{5}\pi}{2|X|^{\frac{3}{2}}}\int_{-\infty}^{\infty}\mathrm{e}^{-u^2}\,\mathrm{d}u + O(|X|^{-\frac{5}{2}})\right\}.\end{aligned} \tag{B 4}$$

Thus for general R, θ, it is found that

$$\underset{R\to\infty}{\Phi_{\mathrm{w}}}(R,\theta) \sim \left(\frac{10}{\pi}\right)^{\frac{1}{2}}\mathrm{e}^{3\pi\mathrm{i}/8}\left\{\frac{\sin\frac{1}{2}\theta}{R^{\frac{1}{2}}} + \frac{\cot\frac{1}{5}\pi}{2R^{\frac{3}{2}}}\sin\tfrac{3}{2}\theta + O(R^{-\frac{5}{2}})\right\}. \tag{B 5}$$

(*b*) *Far field with $R^{\frac{1}{2}}$ behaviour*

From (3.32) the X-derivative of the potential on $Y = 0$, $X < 0$ is given by

$$\Phi_{\frac{1}{2}X} = \psi(X,0) = \frac{p}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}s|X|}\,\mathrm{d}s}{|s|\,K_+(s)}, \tag{B 6}$$

where p is a constant to be determined and the path of integration is XX (figure 4).

Repeating the previous procedure in Appendix B(a) of estimating the integral around the branch cut for large $|X|$ by using Watson's lemma, it is found that

$$\underset{Y=0,\,X\to-\infty}{\Phi_{\frac{1}{2}X}} = -\frac{p}{\pi^{\frac{1}{2}}}\left\{\frac{1}{|X|^{\frac{1}{2}}} - \frac{\mathrm{i}\,\mathrm{e}^{-3\mathrm{i}\pi/10}}{2\sin(\frac{1}{5}\pi)\,|X|^{\frac{3}{2}}} + O(|X|^{-\frac{5}{2}})\right\}. \tag{B 7}$$

The form of $\Phi_{\frac{1}{2}}$, as $R\to\infty$, is

$$\underset{R\to\infty}{\Phi_{\frac{1}{2}}} \sim R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta + A_{\frac{1}{2}}R^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta + O(R^{-\frac{3}{2}}), \tag{B 8}$$

and differentiation gives, on $Y = 0$,

$$\underset{Y=0,\,X\to-\infty}{\Phi_{\frac{1}{2}X}} = -\frac{1}{2}\left\{\frac{1}{|X|^{\frac{1}{2}}} - \frac{A_{\frac{1}{2}}}{|X|^{\frac{3}{2}}} + O(|X|^{-\frac{5}{2}})\right\}. \tag{B 9}$$

Thus, equating with (B 7) evaluates the constants

$$p = \tfrac{1}{2}\pi^{\frac{1}{2}}, \tag{B 10}$$

$$A_{\frac{1}{2}} = \mathrm{i}\,\mathrm{e}^{-3\mathrm{i}\pi/10}/2\sin\tfrac{1}{5}\pi. \tag{B 11}$$

By using this value of p, and equation (3.32), the outgoing wave can be determined from the residue theorem. Thus

$$\psi_{\text{o.w.}} = -\frac{2\pi\mathrm{i}}{4\pi^{\frac{1}{2}}}\lim_{s\to-1}\left\{\frac{s+1}{K_+(s)}\right\}\mathrm{e}^{\mathrm{i}X-Y} \quad (Y>0), \tag{B 12}$$

and (A 16) gives

$$\psi_{\text{o.w.}} = -\mathrm{i}(\pi/10)^{\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}\pi/8}\,\mathrm{e}^{\mathrm{i}X-Y}. \tag{B 13}$$

Finally it is known that $\psi = \phi_X$, and

$$\phi_{\text{o.w.}} = -\mathrm{i}\psi_{\text{o.w.}}.$$

Thus

$$\phi_{\text{o.w.}} = -(\pi/10)^{\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}\pi/8}\,\mathrm{e}^{\mathrm{i}X-Y}. \tag{B 14}$$

(c) *Far field with $R^{\frac{3}{2}}$ behaviour*

From (3.41) and (B 7) the form of the second derivative is known, i.e.

$$\underset{Y=0,\,X\to-\infty}{\Phi_{\frac{3}{2}XX}} = -\frac{l}{\pi^{\frac{1}{2}}}\left\{|X|^{-\frac{1}{2}} - \frac{\mathrm{i}\,\mathrm{e}^{-3\mathrm{i}\pi/10}}{\sin\frac{1}{5}\pi}\,|X|^{-\frac{3}{2}} + O(|X|^{-\frac{5}{2}})\right\}. \tag{B 15}$$

$\Phi_{\frac{3}{2}}$ is known to be, as $R\to\infty$,

$$\Phi^{\frac{3}{2}} \sim R^{\frac{3}{2}}\sin\tfrac{3}{2}\theta + A_{\frac{3}{2}}R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta + O(R^{-\frac{1}{2}}), \tag{B 16}$$

and so

$$\underset{Y=0,\,X\to-\infty}{\Phi_{\frac{3}{2}XX}} = -\tfrac{3}{4}|X|^{-\frac{1}{2}} - \tfrac{1}{4}A_{\frac{3}{2}}|X|^{-\frac{3}{2}} + O(|X|^{-\frac{5}{2}}). \tag{B 17}$$

Thus from (B 15) and (B 17) it is found that

$$l = \tfrac{3}{4}\pi^{\frac{1}{2}}, \quad A_{\frac{3}{2}} = -3\mathrm{i}\,\mathrm{e}^{-3\mathrm{i}\pi/10}/\sin\tfrac{1}{5}\pi. \tag{B 18}$$

Again from the residue theorem, the outgoing wave is calculated, by using the fact that

$$\phi_{\text{o.w.}} = -\phi_{XX\,\text{o.w.}},$$

and

$$\phi_{XX\,\text{o.w.}} = -il \lim_{s \to -1} \left\{ \frac{s+1}{K_+(s)} \right\} e^{lX-Y} \quad (Y > 0). \tag{B 19}$$

From (3.41). Thus finally the outgoing wave along the plate, generated by an $R^{\frac{3}{2}}$-far-field behaviour, is

$$\phi_{\text{o.w.}} = \tfrac{3}{2}(\pi/10)^{\frac{1}{2}} e^{l3\pi/8} e^{lX-Y} \quad (Y > 0). \tag{B 20}$$

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Chapter 4

Scattering of sound by an elastic plate with flow

Summary

An elastic plate, set in an infinite baffle and immersed in a fluid moving with a uniform subsonic velocity, is excited by an acoustic source. The scattered sound field is analysed when fluid-plate coupling is large, and a solution is found by the use of matched asymptotic expansions.

The far field is found to approximate to the solution as if the elastic plate were absent. At a plate resonance, however, the outer field must include eigensolutions with singularities at the plate edges, and close to the plate the dominant terms are travelling plate waves. These plate waves are found to have a wavelength independent of the frequency of the source. It is also shown that a plate resonance corresponds to a divergence instability of aerodynamic flutter theory and that the stability results found in this paper are in agreement with those obtained by using modal expansions.

The limit as the Mach number goes to zero is found to be singular, suggesting an analysis of the model for small flow velocity. This calculation is performed and the results match smoothly onto both the solutions for a stationary fluid and large subsonic flow.

1. Introduction

There has been much attention in recent years on acoustic problems with finite or infinite flexible surfaces. With finite geometries asymptotic or other approximate methods must be employed to obtain an estimate of the sound field (Leppington 1976, Handscomb 1977). The problem of a thin elastic plate set in an infinite baffle has been analysed in the limits of both small and large fluid loading (Leppington 1976, Abrahams 1981 respectively) and this paper extends the latter work by introducing a uniform subsonic flow to the fluid enveloping the plate. The method of matched asymptotic expansions is employed with modification to allow for the presence of plate resonances. It is also expected, due to the addition of flow to the problem, that flutter or divergence instability of the elastic plate may occur. This problem has been analysed by use of modal expansions to calculate the onset of instability (Weaver and Unny 1970, Ellen 1972) and the results will be shown to be in agreement with those found in this paper.

In section 2 the problem is analysed for a Mach number of order unity ($M < 1$) and, as will be shown, the results do not reduce to that given by Abrahams (1981) when the flow velocity goes to zero (i.e. $M \to 0$ is a singular limit). It is therefore necessary to analyse separately the problem when the Mach number is small (Chapter 4) as this result matches onto both the $M = 0$ and $M = O(1)$ results.

Cartesian coordinates (x,y) are chosen in the two-dimensional problem, and taking simple harmonic time dependence, with angular frequency ω, the velocity potential can be defined as

$$\mathrm{Re}\{\phi(x,y)\exp(-i\omega t)\}. \qquad (1.1)$$

Note that the time factor will henceforth be suppressed for brevity. An elastic plate (lying on $y = 0$, $|x| < a$) is immersed in a compressible

inviscid fluid and assuming a uniform flow in the x direction the potential satisfies the corresponding convected wave equation

$$\phi_{xx}(1-M^2) + \phi_{yy} + k^2\phi + 2ikM\phi_x = 0, \tag{1.2}$$

where M (=U/c) is the Mach number, U the flow velocity, c the sound speed in the fluid, and k (=ω/c) the acoustic wavenumber.

For a thin elastic plate, of length 2a, the equation governing small flexural vibrations is

$$(D\frac{\partial^4}{\partial x^4} - M_p\omega^2)\eta(x) = -p(x,0)\Big|_-^+, \quad |x| < a, \; y = 0, \tag{1.3}$$

where D is the plate bending stiffness, M_p its mass per unit area and η,p denote the plate deflection and fluid pressure respectively. Note that $p(x,0)\Big|_-^+$ denotes the discontinuity in pressure across the plate. The deflection η is related to the potential by

$$\phi_y = \begin{cases} -i\omega\eta + U\eta_x, & |x| < a, \; y = 0, \\ 0, & |x| > a, \; y = 0, \end{cases} \tag{1.4}$$

where the subscripts denote partial differentiation, and similarly

$$p(x,0) = -\rho(U\phi_x - i\omega\phi), \quad |x| < a, \; y = 0, \tag{1.5}$$

where ρ is the mean fluid density. Boundary conditions are required at the plate ends, and these are taken to be

$$\eta = \eta_x = 0 \qquad x = \pm a, \; y = 0, \tag{1.6}$$

which corresponds to clamped edges. Equation (1.3) can be rearranged to give

$$(\frac{\partial^4}{\partial x^4} - \mu^4)\eta(x) + \frac{i\alpha}{2\omega}(\phi + iM\phi_x/k)\Big|_-^+ = 0, \quad |x| < a, \; y = 0, \tag{1.7}$$

where $\alpha = 2\rho\omega^2/D$, $\mu^4 = M_p\omega^2/D$, and in this paper the heavy loading limit is taken, corresponding to $\alpha \to \infty$.

2. Uniform Subsonic Flow (M = O(1))

(a) Inner and outer region lengthscales

It is expected that the basic structure of the acoustic potential will be similar to that in the problem with no flow and so the matching scheme in Abrahams (1981) will be used. It was shown that for large fluid loading ($\alpha \to \infty$) the outer potential (away from the plate) is to leading order the solution of the boundary value problem with the elastic plate removed. This "outer" approximation becomes invalid near to the plate and it was noted that standing waves are present on the plate (resulting in plate resonances) for particular lengths of the flexible surface. When a resonance occurs, eigensolutions, with singular behaviour at the plate edges, become of the same order as the outer solution. To perform asymptotic matching between the outer and inner regions it is convenient to split the potential into the form

$$\phi = \phi_0 + A(\varepsilon)\,\phi_1 + B(\varepsilon)\phi_2 \tag{2.1}$$

in the outer region,

$$\Phi = \Phi_0 + A(\varepsilon)\,\Phi_1 + B(\varepsilon)\Phi_2, \tag{2.2}$$

in the inner region, where $A(\varepsilon)$, $B(\varepsilon)$ are functions of the small parameter, ε, given in (2.7), and change their order in ε as a resonance is approached. Matching is then performed separately between ϕ_0 and Φ_0, ϕ_1 and Φ_1 etc.

Away from the plate the coordinates can be non-dimensionalised by using the suitable lengthscale k^{-1}, hence

$$kx = X, \quad ky = Y, \tag{2.3}$$

where (X,Y) are the new outer region coordinates. An inner region lengthscale can be found by examining when the outer region approximation fails. This occurs (from 1.7) when

$$\frac{\partial^4 \eta}{\partial x^4} \sim \frac{\alpha U \phi_x}{\omega^2} \quad \text{and} \quad \phi_y \sim U\eta_x, \tag{2.4}$$

or at a distance ℓ from an edge, where the inner lengthscale, ℓ, is given by

$$\ell = (k^2/\alpha)^{1/3} \text{ assuming } \phi_x, \phi_y \sim \phi/\ell, \quad M = O(1). \tag{2.5}$$

Hence $(\bar{x},\bar{y}),(\bar{x}',\bar{y})$, the left and right hand inner region coordinates are given by

$$\ell^{-1}(a+x) = \bar{x}, \quad \ell^{-1}y = \bar{y}, \tag{2.6}$$
$$\ell^{-1}(a-x) = \bar{x}'.$$

The matching parameter ε, first written in (2.1), is the ratio of inner to outer lengthscales,

$$\varepsilon = k\ell = (k^5/\alpha)^{1/3} << 1, \tag{2.7}$$

and in this paper $\ell << a$.

(b) Matching ϕ_0, Φ_0

The outer potential ϕ_0 can be shown to be expanded in the form

$$\phi_0 \sim \phi_{01} + \varepsilon\phi_{02} + \varepsilon^2\phi_{03} + \ldots, \tag{2.8}$$

where ϕ_{01} is the potential of the problem in the absence of the plate and with external forcing. The potentials ϕ_{02} and ϕ_{03} satisfy the same boundary value problem and are eigensolutions with singular behaviour at the edges of the plate. Scaling on the acoustic wavelength (2.3) and replacing ϕ_0 by the expansion in (2.8) gives the boundary value problem

satisfied by $\phi_{01},\phi_{02},\phi_{03}$; this is defined by

$$(1-M^2)\,\phi_{XX} + \phi_{YY} + \phi + 2iM\phi_X = 0 \qquad \text{all } X,Y, \tag{2.9}$$

$$\phi_Y = 0 \qquad |X| > ka,\ Y = 0, \tag{2.10}$$

$$(\phi + iM\phi_X)\big|_-^+ = 0 \qquad |X| < ka,\ Y = 0. \tag{2.11}$$

This problem cannot be solved exactly but an asymptotic estimate can be found for small or large values of ka (Abrahams 1981, 1982). Note that the outer problem can have more complex geometries as long as these boundaries are of the order of one acoustic wavelength from the plate.

The outer potential near to an edge can therefore be shown to exhibit the form

$$\phi_{01} \underset{R\to 0}{\sim} a_{01} + b_{01}R^{\frac{1}{2}} \sin\tfrac{1}{2}\theta + c_{01}R\cos\theta + d_{01}R^{3/2}\sin\tfrac{3}{2}\theta + \ldots, \tag{2.12}$$

$$\text{where } R^2 = (1-M^2)^{-1}(X+ka)^2 + Y^2\ , \qquad \theta = \tan^{-1}\{(1-M^2)^{\frac{1}{2}}Y/(X+ka)\} \tag{2.13}$$

near to the left hand edge, and near to the right edge

$$\bar{R}^2 = (1-M^2)^{-1}(X-ka)^2 + Y^2\ , \tag{2.14}$$

$$\bar{\theta} = \tan^{-1}\{(1-M^2)^{\frac{1}{2}}Y/(-X+ka)\}.$$

The coefficients a_{01},b_{01}, etc. can be taken to be known in principle (found by numerical solution of the outer problem or otherwise) and are dependent on the external geometry and forcing (incident wave or monopole source for instance). The expansions of the potentials (i.e. (2.12)) can be split into two parts, one being even in y, the other odd in y. The even function automatically satisfies the plate boundary

condition and therefore does not require an inner and outer matching as it is valid for the whole region. Thus it is convenient to subtract off this part of the potential to be included again after the matching has been performed, and then attention is restricted to analysing the region $y > 0$. The inner problems are again found by substituting the inner coordinates into (1.2)-(1.7), thus the left hand inner problem becomes

$$\Phi_{xx}^{--}(1-M^2) + \Phi_{yy}^{--} + \varepsilon^2\Phi + 2i\varepsilon M\Phi_x^- = 0 \qquad \text{all } \bar{x},\bar{y}, \tag{2.15}$$

$$\Phi_y^- = 0 \qquad \bar{x} < 0,\ \bar{y} = 0, \tag{2.16}$$

$$\bar{\eta} = \bar{\eta}_x^- = 0 \qquad \bar{x} = 0,\ \bar{y} = 0, \tag{2.17}$$

$$\left.\begin{aligned} &2c(\partial^4/\partial\bar{x}^4 - (\varepsilon\mu/k)^4)\bar{\eta} + i(\varepsilon\Phi + iM\Phi_x^-)\big|_-^+ = 0 \\ &\Phi_y^- = -ic\varepsilon\bar{\eta} + U\bar{\eta}_x^- \end{aligned}\right\} \quad \bar{x} > 0,\ \bar{y} = 0 \tag{2.18}$$

and similarly for the right hand region. From (2.8) it can be expected that Φ_0 can be expanded as

$$\Phi_0 = g_1(\varepsilon)(\Phi_{01} + \varepsilon\Phi_{02} + \varepsilon^2\Phi_{03} + \ldots.), \tag{2.19}$$

which after substituting into (2.15)-(2.18) defines the problem for Φ_{01}, written as

$$\Phi_{xx}^{--}(1-M^2) + \Phi_{yy}^{--} = 0 \qquad \text{all } \bar{x},\bar{y} > 0, \tag{2.20}$$

$$\Phi_y^- = 0 \qquad \bar{x} \leq 0,\ \bar{y} = 0, \tag{2.21}$$

$$\Phi_{xxxy}^{----} - M^2\Phi_x^- = 0 \qquad \bar{x} > 0,\ \bar{y} = 0. \tag{2.22}$$

Changing now to inner coordinates, the leading order term in ϕ_0 (from (2.12)) has the form

$$\phi_0 \underset{r\to\infty}{\sim} \varepsilon^{1/2} b_{01} r^{1/2} \sin\tfrac{1}{2}\theta\ , \tag{2.23}$$

where

$$r^2 = (1-M^2)^{-1}\bar{x}^2 + \bar{y}^2, \quad \theta = \tan^{-1}\{\bar{y}(1-M^2)^{\frac{1}{2}}/\bar{x}\} , \tag{2.24}$$

and this must match with the inner potential Φ_0 giving $g_1(\varepsilon) = \varepsilon^{1/2}$. Thus the inner potential Φ_{01} is forced by singular behaviour ($r^{\frac{1}{2}}$) as $r \to \infty$, and similarly Φ_{02} behaves as

$$\Phi_{02} \sim d_{01} r^{3/2} \sin \tfrac{3}{2}\theta .$$

The form, as $r \to \infty$, of the potential Φ_0 can be written as

$$\underset{r\to\infty}{\Phi_{01}} \sim b_{01}(r^{\frac{1}{2}} \sin \tfrac{1}{2}\theta + A_{01} r^{-\frac{1}{2}} \sin \tfrac{1}{2}\theta + B_{01} r^{-3/2} \sin \tfrac{3}{2}\theta + \ldots.), \tag{2.25}$$

where A_{01}, B_{01} are found by solution of the Φ_{01} problem. It is therefore found from continued matching that

$$\underset{R\to 0}{\phi_{02}} \sim A_{01} b_{01} R^{-\frac{1}{2}} \sin \tfrac{1}{2}\theta, \quad \underset{R\to 0}{\phi_{03}} \sim B_{01} b_{01} R^{-3/2} \sin \tfrac{3}{2}\theta. \tag{2.26}$$

Because of the symmetry of the geometry it is clear that the eigenpotentials ϕ_{02}, ϕ_{03} must have the form

$$\begin{aligned} \phi_{02} &= A_{01} b_{01} \phi_{\frac{1}{2}L} + A_{01} \bar{b}_{01} \phi_{\bar{\frac{1}{2}}L} \\ \phi_{03} &= B_{01} b_{01} \phi_{\frac{3}{2}L} + B_{01} \bar{b}_{01} \phi_{\bar{\frac{3}{2}}L} , \end{aligned} \tag{2.27}$$

where $\phi_{\frac{1}{2}L}$ denotes a potential satisfying (2.9)-(2.11) with a singularity of $O(R^{-1/2})$ at the left hand edge etc., and the bar denotes a right hand coefficient (i.e. $\bar{b}_{01}$).

The problem for Φ_{01} ((2.20)-(2.22)) can be solved using the Wiener-Hopf technique (see chapter 3) and the solution is composed of a scattered field plus an outgoing wave on the elastic plate of the form (3.23)

$$\Phi_{01\omega} = \alpha_{01} b_{01} e^{ib\bar{x} - (1-M^2)^{\frac{1}{2}} b\bar{y}}, \bar{x} > 0, \ \bar{y} > 0, \tag{2.28}$$

where α_{01} is a known complex constant, and

$$b = (M^2/(1-M^2)^{1/2})^{1/3} \quad . \tag{2.29}$$

This non-attenuating wave must obviously match onto an outgoing wave (travelling on the plate but far from either edge) which satisfies the full plate and governing equations ((1.2) and (1.7)). Taking this wave to be

$$\phi_{wave} = e^{is(x+a)-\gamma y} \tag{2.30}$$

then to satisfy all conditions

$$\gamma = (1-M^2)^{1/2}(s-k/(1+M))^{1/2}(s+k/(1-M))^{1/2}, \text{ and}$$

$$\gamma(s^4-\mu^4) - \alpha(1-Ms/k)^2 = 0. \tag{2.32}$$

But if $\alpha \to \infty$ s can be expanded in powers of ε to give from (2.30) after replacing x by $\bar{x}'$ etc.,

$$\phi_{wave} = e^{-ib\bar{x}'-(1-M^2)^{\frac{1}{2}}b\bar{y}}(1 + \varepsilon\{(2-M^2)i\bar{x}'-2(1-M^2)^{\frac{1}{2}}(2M^2-1)\bar{y}\}/(3M(1-M^2)) + O(\varepsilon^2)), \tag{2.33}$$

$$b = (M^2/(1-M^2)^{\frac{1}{2}})^{1/3}.$$

Similarly for a wave in the opposite direction

$$\bar{\Phi}_{wave} = e^{-is_1(x+a)-\gamma_1 y} \quad , \tag{2.34}$$

$$\gamma_1 = (1-M^2)^{\frac{1}{2}}(s_1+k/(1+M))^{\frac{1}{2}}(s_1-k/(1-M))^{\frac{1}{2}} \tag{2.35}$$

$$(s_1^4-\mu^4)\gamma_1 - \alpha(1 + Ms_1/k)^2 = 0, \tag{2.36}$$

which becomes

$$\bar{\Phi}_{wave} = e^{-ib\bar{x}-(1-M^2)^{\frac{1}{2}}b\bar{y}}(1 -$$

$$\varepsilon\{(2-M^2)i\bar{x}-2(1-M^2)^{\frac{1}{2}}(2M^2-1)\bar{y}\}/(3M(1-M^2)) + O(\varepsilon^2)). \qquad (2.37)$$

Thus, for consistent matching, part of the outgoing wave term in Φ_{02} must add on to $\Phi_{01\omega}$ (2.28) to merge into the wave in (2.33). This will be shown when solving the inner problems in Chapter 3.

(c) Matching ϕ_1,Φ_1

The potentials ϕ_1,ϕ_2 have been introduced into the problem to allow for possible plate resonances (by changing the order in ε of $A(\varepsilon)$, $B(\varepsilon)$). The forcing for the ϕ_1,Φ_1 matching can therefore be taken as an un-attenuated plate wave travelling into the left hand edge, having initially been launched from the right edge. The potential Φ_1 can be expanded as

$$\Phi_1 \sim \Phi_{11} + \varepsilon\Phi_{12} + \ldots , \qquad (2.38)$$

where Φ_{11} satisfies the boundary conditions in (2.20)-(2.22) with forcing by an incoming wave along the plate of the form

$$\Phi_{11i} = \exp(-ib\bar{x} - (1-M^2)^{\frac{1}{2}}b\bar{y}) .$$

The form of Φ_{11} as $r \to \infty$ can now be written as

$$\Phi_{11} \underset{r\to\infty}{\sim} A_{11}r^{-1/2} \sin\tfrac{1}{2}\theta + B_{11}r^{-3/2} \sin\tfrac{3}{2}\theta + \ldots\ldots, \qquad (2.39)$$

as there is no forcing from the outer scattered potential. Thus, using the expansion

$$\phi_1 \sim \varepsilon^{1/2}\phi_{11} + \varepsilon^{3/2}\phi_{12} + \varepsilon^{5/2}\phi_{13} , \qquad (2.40)$$

where ϕ_{11}, ϕ_{12} satisfy (2.9)-(2.11), it is clear that

$$\phi_{11} \underset{R\to 0}{\sim} a_{11}(R^{-1/2}\sin\tfrac{1}{2}\theta + a_{\frac{1}{2}}R^{1/2}\sin\tfrac{1}{2}\theta + \ldots.), \tag{2.41}$$

and so $a_{11} = A_{11}$. Continued matching also gives

$$\Phi_{12} \underset{r\to\infty}{\sim} A_{11}a_{\frac{1}{2}}r^{1/2}\sin\tfrac{1}{2}\theta \tag{2.42}$$

where Φ_{12} satisfies the problem given by

$$\Phi_{12\bar{x}\bar{x}}(1-M^2) + \Phi_{12\bar{y}\bar{y}} = -2iM\Phi_{11\bar{x}}, \text{ all } \bar{x},\bar{y} > 0, \tag{2.43}$$

$$\Phi_{12\bar{y}} = 0 \qquad , \bar{x} \leq 0, \bar{y} = 0, \tag{2.44}$$

$$\Phi_{12\bar{y}\bar{x}\bar{x}\bar{x}} - M^2\Phi_{12\bar{x}} = -i(\Phi_{11\bar{y}\bar{x}\bar{x}} + M^2\Phi_{11})/M, \ \bar{x} > 0, \ \bar{y} = 0. \tag{2.45}$$

Note that Φ_{12} is forced by (i) $r^{1/2}$ behaviour at infinity, (ii) terms in both (2.43) and (2.45), and (iii) Φ_{12} must also have an incoming wave of the form

$$\Phi_{12i} = \frac{e^{ib\bar{x}-(1-M^2)^{\frac{1}{2}}b\bar{y}}}{3M(1-M^2)}(2(1-M^2)^{1/2}(2M^2-1)\bar{y} - i(2-M^2)\bar{x}) \tag{2.46}$$

to match with the incoming wave in the Φ_{11} problem (equation (2.37)). As a final point the leading order outer potential for ϕ_1 can be re-written as

$$\phi_{11} = A_{11}\phi_{\frac{1}{2}L}, \tag{2.47}$$

where $\phi_{\frac{1}{2}L}$ is given in (2.27).

(d) Matching ϕ_2, Φ_2

The matching procedure for ϕ_2, Φ_2 is analogous to that with the previous potentials (ϕ_1, Φ_1) where Φ_2 is the right hand inner region potential and the forcing is by a right travelling wave along the plate. Again expanding the potentials

$$\Phi_2 \sim \Phi_{21} + \varepsilon\Phi_{22} + \ldots\ldots, \tag{2.48}$$

it is found that Φ_{21} satisfies the boundary value problem defined in (2.20)-(2.22) when $\bar{x}$ is replaced by $\bar{x}'$ (2.6) with an incident plate wave providing the forcing. Thus it is clear that

$$\Phi_{21} = \Phi_{11}(\bar{x}', \bar{y}) , \tag{2.49}$$

and so

$$\underset{\bar{r}\to\infty}{\Phi_{21}} \sim A_{11}\bar{r}^{-1/2}\sin\frac{1}{2}\bar{\theta} + B_{11}\bar{r}^{-3/2}\sin\frac{3}{2}\bar{\theta} , \tag{2.50}$$

where $\bar{r}^2 = (1-M^2)^{-1}\bar{x}'^2 + \bar{y}^2$, $\bar{\theta} = \tan^{-1}(\bar{y}(1-M^2)/\bar{x}')$. (2.51)

Matching to leading order gives

$$\phi_2 \sim \phi_{21} = \varepsilon^{1/2} A_{11}\phi_{\frac{1}{2}R} + O(\varepsilon^{3/2}), \tag{2.52}$$

where $\phi_{\frac{1}{2}R}$ is an eigensolution of the problem defined by equations (2.9)-(2.11) and has the form

$$\underset{\bar{R}\to 0}{\phi_{\frac{1}{2}R}} \sim \bar{R}^{-1/2}\sin\frac{1}{2}\bar{\theta} + \bar{a}_{\frac{1}{2}}\bar{R}^{1/2}\sin\frac{1}{2}\bar{\theta} , \tag{2.53}$$

$$\underset{R\to 0}{\phi_{\frac{1}{2}R}} \sim b_{\frac{1}{2}}R^{1/2}\sin\frac{1}{2}\theta, \quad \bar{R}^2 = (1-M^2)^{-1}(X-ka)^2 + Y^2.$$

The constants $a_{\frac{1}{2}}$, $\bar{b}_{\frac{1}{2}}$ are found by solving the outer problems for $\phi_{\frac{1}{2}L}$, $\phi_{\frac{1}{2}R}$, and are again assumed known in principle. The boundary value problem for ϕ_{22}, analogous with ϕ_{12}, is found to be

$$\Phi_{22\bar{x}'\bar{x}'}(1-M^2) + \Phi_{22\bar{y}\bar{y}} = 2iM\Phi_{21\bar{x}'} \quad , \text{all } \bar{x}', \bar{y} > 0, \tag{2.54}$$

$$\Phi_{22\bar{y}} = 0 \quad , \bar{x}' \leq 0, \ \bar{y} = 0, \tag{2.55}$$

$$\Phi_{22\bar{y}\bar{y}\bar{x}'\bar{x}'} - M^2\Phi_{22\bar{x}'} = i(\Phi_{21\bar{y}\bar{y}\bar{x}'} + M^2\Phi_{21})/M, \ \bar{x}' > 0, \ \bar{y} = 0. \tag{2.55}$$

It can also be shown that Φ_{22} must behave as

$$\Phi_{22} \underset{\bar{r}\to\infty}{\sim} A_{11}\bar{a}_{\frac{1}{2}}\bar{r}^{-1/2} \sin\frac{1}{2}\bar{\theta} \quad , \tag{2.56}$$

and has an incident plate wave providing the forcing, of the form (from (2.33))

$$\Phi_{22i} = \frac{e^{-ib\bar{x}'-(1-M^2)^{\frac{1}{2}}b\bar{y}}}{3M(1-M^2)}\left((2-M^2)i\bar{x}' - 2(1-M^2)^{\frac{1}{2}}(2M^2-1)\bar{y}\right). \tag{2.57}$$

As a final observation note that the potential ϕ_2, as $R \to 0$, will behave as

$$\phi_2 \underset{R\to 0}{\sim} \varepsilon^{1/2}A_{11}\phi_{\frac{1}{2}R} \underset{R\to 0}{=} \varepsilon^{1/2}A_{11}b_{\frac{1}{2}}R^{1/2} \sin\frac{1}{2}\theta + O(\varepsilon^{3/2}) \tag{2.58}$$

which will therefore match with part of the inner left hand potential, Φ_{2L} say, giving

$$\Phi_{2L} \sim \frac{\varepsilon A_{11}b_{\frac{1}{2}}}{b_{01}}\ \Phi_{01} + O(\varepsilon^2). \tag{2.59}$$

This can be repeated at the right hand edge to match the ϕ_1 potential.

(e) Asymptotic solution

Matching has been accomplished to leading order and the solution may be written as

$$\phi_{total} = \phi_{01} + \varepsilon(A_{01}b_{01}\phi_{\frac{1}{2}L} + A_{01}\bar{b}_{01}\phi_{\frac{1}{2}R}) + O(\varepsilon^2)$$
$$+ A(\varepsilon)(\varepsilon^{1/2}A_{11}\phi_{\frac{1}{2}L} + O(\varepsilon^{3/2}))$$
$$+ B(\varepsilon)(\varepsilon^{1/2}A_{11}\phi_{\frac{1}{2}R} + O(\varepsilon^{3/2})) , \qquad (2.60)$$

with inner approximations

$$\Phi_{left} = \varepsilon^{1/2}\Phi_{01} + O(\varepsilon^{3/2}) + A(\varepsilon)(\Phi_{11} + \varepsilon\Phi_{12} + O(\varepsilon^2))$$
$$+ B(\varepsilon)(\varepsilon A_{11}b_{\frac{1}{2}}\Phi_{01}/b_{01} + O(\varepsilon^2)) \qquad (2.61)$$

$$\Phi_{right} = \varepsilon^{1/2}\Phi_{01}\bar{b}_{01}/b_{01} + O(\varepsilon^{3/2}) + A(\varepsilon)(\varepsilon A_{11}\bar{b}_{\frac{1}{2}}\Phi_{01}/b_{01} + O(\varepsilon^2))$$
$$+ B(\varepsilon)(\Phi_{21} + \varepsilon\Phi_{22} + O(\varepsilon^2)). \qquad (2.62)$$

Near the plate, the total potential is the sum of the travelling waves on the plate and the potential in (2.60), i.e.

$$\phi_{total} + A(\varepsilon)\exp(-is_1(x+a)-\gamma_1 y) + B(\varepsilon)\exp(is(x-a) - \gamma y), \qquad (2.63)$$

where γs etc. satisfy equations (2.32), (2.36). It now remains to solve the inner problems Φ_{01}, Φ_{11} etc. so that the coefficients A_{01}, A_{11} etc., can be found. The functions $A(\varepsilon)$, $B(\varepsilon)$ are determined by equating the incoming and outgoing waves at the left hand edge with those at the right edge and this will give the condition for plate resonance (3.62)-(3.65).

3. The Inner Problem

(i) The Φ_{11} potential

Taking the inner problem Φ_{11} it will be shown that a solution can be found by use of the Wiener-Hopf technique. The potential Φ_{11} satisfies the boundary conditions

$$\phi_{xx}(1-M^2) + \phi_{yy} = 0 \qquad \text{all } x,y > 0, \tag{3.1}$$

$$\phi_y = 0 \qquad x \leq 0,\ y = 0, \tag{3.2}$$

$$\phi_{xxxy} - M^2\phi_x = 0 \qquad x > 0,\ y = 0, \tag{3.3}$$

where for convenience $\bar{x},\bar{y}$ is written as x,y and forcing is supplied by an incoming plate wave

$$\phi_i = \exp(-ibx - (1-M^2)^{\frac{1}{2}}by),\ b^3 = M^2/(1-M^2)^{\frac{1}{2}} . \tag{3.4}$$

Note that the plate equation in (3.3) can be integrated to give

$$\phi_{xxy} - M^2\phi = \text{constant}, \qquad x > 0,\ y = 0, \tag{3.5}$$

but from matching considerations it can be shown that this constant must be zero. Subtracting off the incident wave by introducing a new potential ψ, where

$\psi = \phi - \phi_i$, the problem becomes

$$\psi_{xx}(1 - M^2) + \psi_{yy} = 0 \qquad \text{all } x,y > 0 \tag{3.6}$$

$$\psi_y = (1 - M^2)^{\frac{1}{2}}be^{-ibx} \qquad x \leq 0,\ y = 0, \tag{3.7}$$

$$\psi_{xxy} - M^2\psi = 0 \qquad x > 0,\ y = 0. \tag{3.8}$$

The half range Fourier transforms are defined as

$\Psi(x,y) = \Psi_+(x,y) + \Psi_-(x,y)$, where

$$\Psi_+(s,y) = \int_0^\infty e^{isx}\psi(x,y)dx, \tag{3.9}$$

$$\Psi_-(s,y) = \int_{-\infty}^0 e^{isx}\psi(x,y)dx, \tag{3.10}$$

noting that Ψ_+ is analytic in an upper region of the complex s plane and similarly Ψ_- is analytic in the lower half plane. Both Ψ_+, Ψ_- are assumed to have algebraic growth as $|s| \to \infty$ in their respective half planes. Transforming the governing equation (3.6) gives the result

$$\Psi(s,y) = A(s)\exp(-y|s|(1-M^2)^{\frac{1}{2}}), \tag{3.11}$$

which allows only outgoing wave solutions. Note that

$$|s| = \lim_{q\to 0}\{(s + iq)^{\frac{1}{2}}(s-iq)^{\frac{1}{2}}\}, \tag{3.12}$$

where q is a fictitious dissipation term included for mathematical convenience, and

$$|s| = \begin{cases} s & \mathrm{Re}(s) > 0 \\ -s & \mathrm{Re}(s) < 0. \end{cases} \tag{3.13}$$

For simplicity the abbreviated notation $\Psi_+ = \Psi_+(s,0)$, etc., will be used and the transformed boundary conditions become

$$\Psi_-' = \partial\{\Psi_-(s,y)\}/\partial y\big|_{y=0} = -(1-M^2)^{\frac{1}{2}}bi/(s-b), \tag{3.14}$$

$$N(s) - s^2\Psi_+' - M^2\Psi_+ = 0, \tag{3.15}$$

where $N(s) = -\psi_{xy}(0,0) + is\psi_y(0,0)$ and is regular over all s. From (3.13) it is known that

$$\Psi_+ + \Psi_- = A(s) = -(\Psi_+' + \Psi_-')/(|s|(1-M^2)^{1/2}), \tag{3.16}$$

which, on rearranging and using (3.14), (3.15), gives the Wiener-Hopf functional equation

$$\frac{N(s)}{M^2} K_-(s) + \Psi_- K_-(s) - \frac{(1-M^2)^{\frac{1}{2}} bi}{(s-b)} (s^2 K_-(s) + K_+(b)) \tag{3.17}$$

$$= -\Psi_+' K_+(s) + \frac{(1-M^2)^{\frac{1}{2}} bi}{(s-b)} (K_+(s) - K_+(b)) .$$

The Wiener-Hopf kernel K(s) is given by

$$K(s) = \frac{(1-M^2)^{-\frac{1}{2}}}{|s|} - \frac{s^2}{M^2} , \tag{3.18}$$

or $K(s) = K_+(s)/K_-(s)$, $K_+(s) = 1/K_-(-s)$,

where $K_+(s)$, $K_-(s)$ are regular and non-zero in their respective upper and lower half s-planes. The left hand side of (3.17) is therefore analytic in the lower half plane (the pole at s = b lying just above the line joining the two half planes), and similarly the right hand side is regular in the region Im(s) > 0. Note that a regular function J(s), say, can equal both sides of (3.17) for values of s lying on the line of common analyticity. By analytic continuation J(s) can be extended throughout the whole s plane and the form of J(s) is found by examining the behaviour of (3.17) as $|s| \to \infty$. By using the fact (from A9) that

$$K_+(s) \sim O(s) \qquad \text{as } s \to \infty \text{ , etc.,} \tag{3.19}$$

together with the edge condition $\psi_y(0,0) = b(1-M^2)^{1/2}$, J(s) behaves as $J(s) \to 0$ as $|s| \to \infty$ and so by Liouville's theorem $J(s) \equiv 0$. Thus Ψ_+' is now found from (3.17), and (3.11) can be written as

$$\Psi(s,y) = \frac{biK_+(b)}{|s|(s-b)K_+(s)} \exp(-|s|(1-M^2)^{1/2} y), \tag{3.20}$$

or finally the solution is

$$\Phi_{11}(x,y) = \frac{ib}{2\pi}\int_{-\infty}^{\infty} \frac{K_+(b)\exp(-is\bar{x}-|s|(1-M^2)^{\frac{1}{2}}\bar{y})ds}{K_+(s)|s|(s-b)}$$
$$y>0$$
$$+ e^{-ib\bar{x}-(1-M^2)^{\frac{1}{2}}b\bar{y}} \quad . \tag{3.21}$$

The integral can be estimated for large r hence giving the coefficients A_{11}, B_{11} (2.39) and the pole in the integrand at s = -b (in $K_+(s)$) gives the outgoing plate wave. After much algebra the outgoing wave (from A15) is determined

$$\Phi_{11\omega} = e^{5i\pi/4}\ e^{ib\bar{x}-\bar{y}b(1-M^2)^{\frac{1}{2}}} \quad . \tag{3.22}$$

The above procedure could be repeated for Φ_{01} which would determine the coefficients A_{01}, B_{01} (2.25), and the coefficient α_{01} multiplying the outgoing wave, i.e. the outgoing plate wave term in Φ_{01} is (from 2.28)

$$\Phi_{01\omega} = \alpha_{01}b_{01}\exp(ib\bar{x} - (1-M^2)^{\frac{1}{2}}b\bar{y}), \tag{3.23}$$

where α_{01} is assumed known in this analysis.

(ii) <u>The Φ_{12} potential</u>

The boundary value problem for Φ_{12} is written in (2.43)-(2.45) and forcing is supplied by an incoming wave along the plate (2.46). As in section 3i a new potential, ϕ, can be defined which is given (from 2.46) by

$$\phi = \Phi_{12} - \Phi_{12i} \tag{3.24}$$

and, after substitution into the relevant equations, ϕ satisfies the following problem

$$\phi_{xx}(1-M^2) + \phi_{yy} + 2iM\psi_x = 0, \quad \text{all } x,y > 0, \tag{3.25}$$

$$\phi_y = -e^{-ibx}(Ax+b), \quad x \leq 0,\ y = 0, \tag{3.26}$$

$$\phi_{yxxx} - M^2\phi_x + \frac{i}{M}(\psi_{yxx} + M^2\psi) = 0, \qquad x > 0,\ y = 0. \tag{3.27}$$

The potential ψ is the inverse Fourier transform of equation (3.11), and

$$A = \frac{(2-M^2)ib}{3M(1-M^2)^{\frac{1}{2}}}, \qquad B = \frac{2(2M^2-1)}{3M(1-M^2)^{\frac{1}{2}}}. \tag{3.28}$$

Transforming the governing equation gives

$$\Phi(s,y) = D(s)e^{-|s|y(1-M^2)^{\frac{1}{2}}} + \frac{yMibK_+(b)e^{-|s|y(1-M^2)^{\frac{1}{2}}}}{s(1-M^2)^{\frac{1}{2}}(s-b)K_+(s)}, \tag{3.29}$$

where $\Phi(s,y)$ is the Fourier transform of $\phi(x,y)$, and Φ_+, Φ_-' etc., are half range transforms as previously defined in (3.9), (3.10) and can be shown to have the same regions of analyticity as for the Ψ_+, Ψ_- functions. The problem now reduces to determining D(s) for the given boundary conditions. The plate equation transforms to give

$$P(s) + is^3\Phi_+' + iM^2s\Phi_+ + \frac{i}{M}(-s^2\Psi_+' + M^2\Psi_+) = 0, \tag{3.30}$$

where

$$P(s) = \{-\phi^0_{yxx} + is\phi^0_{yx} + s^2\phi^0_y + M^2\phi^0 - \frac{i}{M}\psi^0_{yx} - \frac{s}{M}\psi^0_y\}, \tag{3.31}$$

the superscript denoting x = y = 0, and

$$\phi^0_y = -B, \quad \psi^0_y = b(1-M^2)^{1/2}.$$

The identities,

$$\Phi_+' + \Phi_-' = -|s|(1-M^2)^{\frac{1}{2}}D(s) - \frac{M}{s(1-M^2)}(\Psi_+' + \Psi_-'), \tag{3.32}$$

$$\Phi_+ + \Phi_- = D(s) \tag{3.33}$$

can be used, together with the left hand boundary condition, to give the Wiener-Hopf equation. After much algebra this equation is found to be

$$P(s)K_-(s) + is^3K_-(s)\left(\frac{A}{(s-b)^2} - \frac{Bi}{(s-b)}\right) - iM^2s\Phi_-K_-(s)$$

$$+ iM^2\left(\frac{A}{(s-b)^2}(sK_+(b) + b(s-b)K_+'(b)) - \frac{Bib}{(s-b)}K_+(b)\right)$$

$$- \frac{M^3bK_+(b)}{(s-b)(1-M^2)^{\frac{1}{2}}}\left(1 + \frac{R_-(s)}{M^2} + \frac{R_+(b)}{M^2}\right) + \frac{2(1-M^2)^{\frac{1}{2}}b}{M(s-b)}(s^2K_-(s)-K_+(b)\{R_-(s)+R_+(b)\})$$

$$= isM^2K_+(s)\Phi_+' - iM^2\left(\frac{A}{(s-b)^2}(sK_+(s)-sK_+(b) - b(s-b)K_+'(b)\right)$$

$$- \frac{Bi}{(s-b)}(sK_+(s)-bK_+(b)) + \frac{MbK_+(b)}{(s-b)(1-M^2)^{\frac{1}{2}}}(R_+(s) - R_+(b))$$

$$+ \frac{2b(1-M^2)^{\frac{1}{2}}K_+(b)}{M(s-b)}(R_+(s) - R_+(b)), \qquad (3.34)$$

where $K_+(s)$, $K_-(s)$ are defined in (3.18),

$$K'(b) = dK(s)/ds\,|_{s=b}, \quad \text{and}$$

$$R_+(s) + R_-(s) = s^2/K(s), \qquad (3.35)$$

$R_+(s)$, $R_-(s)$ being regular in the upper and lower half planes respectively. The left hand side of (3.34) is regular in the lower half plane, and the right hand side is regular in the upper half plane. By allowing both sides of (3.34) to be equal to a function regular over all s, and examining the behaviour of each side as $|s| \to \infty$, the extended form of Liouville's theorem states that this function must be a constant, C say. Note that C is determined by matching onto the outer region and therefore from (2.42) must be proportional to $A_{11}a_{\frac{1}{2}}$. It is now assumed to be a known constant and so the function D(s) can be written as

$$D(s) = \frac{1}{|s|(1-M^2)^{\frac{1}{2}}}\{\frac{AK_+(b)}{(s-b)^2K_+(s)} + \frac{AbK_+'(b)}{(s-b)sK_+(s)} - \frac{BibK_+(b)}{(s-b)sK_+(s)}$$

$$- \frac{ib(2-M^2)K_+(b)}{M^3(1-M^2)^{\frac{1}{2}}(s-b)sK_+(s)}(R_+(s)-R_+(b)) + \frac{iC}{M^2sK_+(s)}\} + \frac{MibK_+(b)}{|s|s(1-M^2)(s-b)K_+(s)} \quad (3.36)$$

The solution can finally be written as

$$\Phi_{12}(\bar{x},\bar{y}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{D(s) + \frac{\bar{y}MibK_+(b)}{s(s-b)(1-M^2)^{\frac{1}{2}}K_+(s)}\}e^{-|s|\bar{y}(1-M^2)^{\frac{1}{2}}-is\bar{x}}\,ds$$

$$+ \Phi_{12i} \quad . \quad (3.37)$$

The outgoing wave can again be found by picking up the pole at $s = -b$, and from Appendix B, (B10), after considerable algebra

$$\Phi_{12\omega} = \{e^{i\pi/4}\,(\frac{(2-M^2)i\bar{x}}{3M(1-M^2)} + \frac{2(1-M^2)^{\frac{1}{2}}(2M^2-1)\bar{y}}{3M(1-M^2)}$$

$$- \frac{Ce^{i\pi/8}b(2/3)^{\frac{1}{2}}}{M^3}\}\,e^{ib\bar{x}-b(1-M^2)^{\frac{1}{2}}\bar{y}} \quad . \quad (3.38)$$

The analogous outgoing wave $\Phi_{22\omega}$ can be written down after noting that the forcing in (2.54), (2.53), (2.57) is the negative of that in the Φ_{12} problem. Thus it is found that

$$\Phi_{22} = \{e^{5i\pi/4}\,(\frac{(2-M^2)i\bar{x}'}{3M(1-M^2)} + \frac{2(1-M^2)^{\frac{1}{2}}(2M^2-1)\bar{y}}{3M(1-M^2)}$$

$$- \frac{De^{i\pi/8}b(2/3)^{\frac{1}{2}}}{M^3}\}e^{ib\bar{x}'-b(1-M^2)^{\frac{1}{2}}\bar{y}} \quad , \quad (3.39)$$

where D is found from matching with the outer field and can be considered to be a known constant.

(iii) Determining $A(\varepsilon)$, $B(\varepsilon)$

It is clear that the outgoing wave from the left hand edge must be equal to the incoming wave into the right hand inner region as the

waves do not decay with distance along the plate. This must also be true for waves emanating from the right hand edge and travelling into the left hand edge.

The outgoing wave from the left hand edge is the sum of all waves generated by the potentials in (2.61). Thus, using (3.22), (3.23), (3.38), (2.59) the total outgoing wave is

$$\Phi_{\text{left}\omega} = [\varepsilon^{\frac{1}{2}}\alpha_{01}b_{01} + O(\varepsilon^{3/2}) + B(\varepsilon)(\varepsilon A_{11}b_{\frac{1}{2}}\alpha_{01} + O(\varepsilon^2))$$
$$+ A(\varepsilon)(e^{5i\pi/4}\{1-\varepsilon(2-M^2)i\bar{x} + 2(1-M^2)^{\frac{1}{2}}(2M^2-1)\bar{y})/(3M(1-M^2))\} \qquad (3.40)$$
$$- \varepsilon M^{-2}Ce^{i\pi/8}b(2/3)^{\frac{1}{2}} + O(\varepsilon^2)]\exp(ib\bar{x} - b(1-M^2)^{\frac{1}{2}}\bar{y}),$$

or in outer coordinates

$$\Phi_{\text{left}\omega} = [\varepsilon^{\frac{1}{2}}\alpha_{01}b_{01} + O(\varepsilon^{3/2}) + B(\varepsilon)(\varepsilon A_{11}b_{\frac{1}{2}}\alpha_{01} + O(\varepsilon^2))$$
$$+ A(\varepsilon)(e^{5i\pi/4} - \varepsilon M^{-2}Ce^{i\pi/8}b(2/3)^{\frac{1}{2}} + O(\varepsilon^2))]e^{is(x+a)-\gamma y}, \qquad (3.41)$$

where γ,s satisfy equations (2.31), (2.32). This wave must be equal to the incident wave into the right edge which, from (2.63), gives

$$\Phi_{\text{left}\omega} = B(\varepsilon)e^{is(x-a)-\gamma y}. \qquad (3.42)$$

Similarly the left going wave condition gives

$$\Phi_{\text{right}\omega} = [\varepsilon^{\frac{1}{2}}\alpha_{01}\bar{b}_{01} + O(\varepsilon^{3/2}) + A(\varepsilon)(\varepsilon A_{11}\bar{b}_{\frac{1}{2}}\alpha_{01} + O(\varepsilon^2))$$
$$+ B(\varepsilon)(e^{5i\pi/4} - \varepsilon M^{-2}De^{i\pi/8}b(2/3)^{\frac{1}{2}} + O(\varepsilon^2))]e^{-is_1(x-a)-\gamma_1 y}$$
$$= A(\varepsilon)e^{-is_1(x+a)-\gamma_1 y}, \qquad (3.43)$$

γ_1, s_1 satisfy (2.35), (2.36).

Solving the above simultaneous equations finally gives the results

$$A(\varepsilon) = \varepsilon^{\frac{1}{2}}\alpha_{01}\{b_{01}e^{5i\pi/4} + \bar{b}_{01}\exp(-2ika[b/\varepsilon-(2-M^2)/(3M(1-M^2))]) + O(\varepsilon)\}/E(\varepsilon) \tag{3.44}$$

$$B(\varepsilon) = \varepsilon^{\frac{1}{2}}\alpha_{01}\{b_{01}\exp(-2ika[b/\varepsilon+(2-M^2)/(3M(1-M^2))]) + \bar{b}_{01}e^{5i\pi/4} + O(\varepsilon)\}/E(\varepsilon), \tag{3.45}$$

where

$$E(\varepsilon) = \exp(-4iakb/\varepsilon) - \exp(5i\pi/2) + \varepsilon\{\frac{2iak^2}{M^2(1-M^2)^2}[14M^4 - 49M^2 + 38]\exp(-4iakb/\varepsilon) - A_{11}\alpha_{01}e^{-2iakb/\varepsilon}[b_{\frac{1}{2}}\exp(-2ika(2-M^2)/(3M(1-M^2))) + \bar{b}_{\frac{1}{2}}\exp(2ika(2-M^2)/(3M(1-M^2)))] + (D+C)\frac{b}{M^2}(2/3)^{\frac{1}{2}}\} + O(\varepsilon^2). \tag{3.46}$$

A resonance is defined to occur when $A(\varepsilon)$, $B(\varepsilon)$ change from $O(\varepsilon^{\frac{1}{2}})$ to $O(\varepsilon^{-\frac{1}{2}})$, and this condition occurs when

$$e^{-4iakb/\varepsilon} = e^{5i\pi/2} \qquad \text{or} \tag{3.47}$$

$$\frac{2\rho U^2 a^3}{(1-M^2)^{\frac{1}{2}}D} = (\frac{3\pi}{8} + \frac{n\pi}{2})^3 \qquad n = 0,1,2,\ldots . \tag{3.48}$$

This resonance condition can be altered by changing the plate edge conditions. For instance, if the plate were simply supported at the edges, i.e.

$$\eta = \eta_{xx} = 0 \qquad x = \pm a,\ y = 0, \tag{3.49}$$

then repeating the previous analysis would yield the resonance condition

$$\frac{2\rho U^2 a^3}{(1-M^2)^{\frac{1}{2}} D} = \left(\frac{\pi}{24} + \frac{n\pi}{2}\right)^3 \qquad n = 0,1,2,\ldots \; . \tag{3.50}$$

Note that if fluid was on only one side of the plate (with a vacuum on the other) then the resonance condition (3.50) would become

$$\frac{\rho U^2 a^3}{(1-M^2)^{\frac{1}{2}} D} = \left(\frac{\pi}{24} + \frac{n\pi}{2}\right)^3 \qquad n = 0,1,2,\ldots \; . \tag{3.51}$$

4. Low Mach number flow

The resonance condition obtained in §3 is valid for a Mach number of order unity, and does not reduce to the zero flow resonance condition (Abrahams 1981)

$$\alpha^{1/5} a = \frac{1}{8}\pi + \frac{1}{2} n\pi, \qquad n = 0,1,2,\ldots, \tag{4.1}$$

as the Mach number is reduced. The reason for this is a breakdown of the leading order expansion of the equations (2.15)-(2.19) written in (2.20)-(2.22). This is therefore a singular limit suggesting that an examination be made of the problem when the Mach number is of the order of the other small parameter in the problem. As suggested in the zero flow problem the small parameter is taken as

$$\varepsilon = k/\alpha^{1/5}, \tag{4.2}$$

and so the Mach number is defined as

$$M = m\varepsilon \text{ , where } m = O(1). \tag{4.3}$$

The inner coordinates can now be written as

$$\begin{aligned} \bar{x} &= \alpha^{1/5}(a + x), \quad \bar{y} = \alpha^{1/5} y \\ \bar{x}' &= \alpha^{1/5}(a - x), \end{aligned} \tag{4.4}$$

for the left and right inner regions respectively, and the outer coordinates scale on the acoustic wavelength, thus

$$X = kx, \qquad Y = ky. \tag{4.5}$$

The leading order outer problem is found after rescaling on the outer coordinates, and is defined by

$$\phi_{XX} + \phi_{YY} + \phi = 0, \qquad \text{all } X,Y, \tag{4.6}$$

$$\phi_Y = 0 \qquad |X| > ka, \tag{4.7}$$

$$\phi\,|_-^+ = 0 \qquad |X| < ka. \tag{4.8}$$

Similarly the inner problems can be written as

$$\Phi_{\bar{x}\bar{x}} + \Phi_{\bar{y}\bar{y}} + \varepsilon^2(\Phi + 2im\Phi_{\bar{x}} - m^2\Phi_{\bar{x}\bar{x}}) = 0 \qquad \text{all } \bar{x},\bar{y}\,, \tag{4.9}$$

$$\Phi_{\bar{y}} = 0 \qquad , \quad \bar{x} < 0, \tag{4.10}$$

$$\bar{\eta} = \bar{\eta}_{\bar{x}} = 0 \qquad , \quad \bar{x} = 0, \tag{4.11}$$

$$\varepsilon(\partial^4/\partial\bar{x}^4 + \mu^4\varepsilon^4)\bar{\eta} + (i\Phi - m\Phi_{\bar{x}})/2c\,|_-^+ = 0, \quad y = 0,\ \bar{x} > 0,$$

$$\Phi_{\bar{y}} = -\varepsilon c(i\bar{\eta} - m\bar{\eta}_{\bar{x}})\,, \qquad y = 0,\ \bar{x} > 0. \tag{4.13}$$

The matching scheme for this low Mach number limit is no different from that described in chapter 2, and will therefore not be analysed. To calculate the condition for resonance it is only necessary to obtain the reflection coefficient of a plate wave travelling into an edge. Thus the leading order inner problem is found to be (after noting that Φ is odd in y)

$$\Phi_{0\bar{x}\bar{x}} + \Phi_{0\bar{y}\bar{y}} = 0 \qquad \text{all } \bar{x},\bar{y} > 0, \tag{4.14}$$

$$\Phi_{0\bar{y}} = 0 \qquad , \quad \bar{x} < 0,\ \bar{y} = 0, \tag{4.15}$$

$$\bar{\eta}_0 = \bar{\eta}_{0\bar{x}} = 0 \qquad , \quad \bar{x} = 0,\ \bar{y} = 0,$$

$$\partial^4\bar{\eta}_0/\partial\bar{x}^4 + \frac{1}{c}(i\Phi_0 - m\Phi_{0x}) = 0, \quad \bar{x} > 0,\ \bar{y} = 0, \tag{4.16}$$

$$\Phi_{0\bar{y}} = -c(i\bar{\eta}_0 - m\bar{\eta}_{0\bar{x}}) \qquad , \quad \bar{x} > 0,\ \bar{y} = 0, \tag{4.17}$$

where $\bar{\eta} = \eta_0 + \varepsilon\eta_1 + \ldots,$ (4.18)

$$\Phi = \varepsilon\Phi_0 + \varepsilon^2\Phi_1 + \ldots \quad . \tag{4.19}$$

Forcing is due to a travelling plate wave into the inner region and will have the form

$$e^{-id\bar{x}-d\bar{y}} \tag{4.20}$$

where d satisfies the plate equation and kinematic boundary condition in the inner region. Thus d must be a solution of

$$d^5 = (1 + md)^2. \tag{4.21}$$

The inner problem (4.14)-(4.17) can be solved using the Wiener-Hopf technique which yields

$$\Phi_0(\bar{x},\bar{y}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{\frac{diK_+(d)}{(s-d)K_+(s)} - \frac{f}{K_+(s)}\} \frac{e^{-is\bar{x}-|s|\bar{y}}}{|s|}\, ds, \tag{4.22}$$

where

$$K(s) = \frac{K_+(s)}{K_-(s)} = (1 + sm)^2/|s| - s^4, \tag{4.23}$$

and f is a constant found by matching or by edge conditions.

The non-attenuating outgoing plate wave can be found in limit $m \to 0$, using the results

$$d = 1 + O(m), \quad f = 0 + O(m), \tag{4.24}$$

thus

$$\Phi_{Owave} = e^{3i\pi/4}\, e^{i\bar{x}-\bar{y}} + O(m). \tag{4.25}$$

Equating the waves at either edge of the plate gives the following resonance condition

$$\alpha^{1/5}a = \frac{1}{8}\pi + \frac{1}{2}n\pi, \qquad n = 0,1,2,\ldots, \tag{4.26}$$

which is identical to that in the zero flow problem. Alternatively when the factor m in (4.27) becomes very large it is found that, from (4.21),

$$d = m^{2/3} \quad \text{and}$$

$$K(s) \sim (m^2/|s| - s^2)s^2 + O(m^{-1}) . \tag{4.27}$$

The constant, f, is found to be

$$f = -iK_+(d), \tag{4.28}$$

which gives the required behaviour at infinity, and so the outgoing wave is found, by picking up the pole at s = -d, to be

$$\Phi_{0\text{wave}} = e^{5i\pi/4}e^{id\bar{x}-d\bar{y}} + O(m^{-1}). \tag{4.29}$$

The resonance condition therefore becomes

$$d\alpha^{1/5}a = \frac{3\pi}{8} + \frac{n\pi}{2}, \quad n = 0,1,2,\ldots., \tag{4.30}$$

or, using (4.3) to give $d = (M/\varepsilon)^{2/3}$,

$$\frac{2\rho U^2 a^3}{D} = (\frac{3\pi}{8} + \frac{n\pi}{2})^3, \qquad n = 0,1,2,\ldots . \tag{4.31}$$

This result must be equivalent to the resonance condition found in chapter 3 in the limit $M \to 0$, and by expanding (3.48) in powers of M this is found to be correct.

It has now been shown that the potential of the problem with small Mach number flow smoothly matches the zero flow solution to that of the M = O(1) problem.

5. Discussion

It has been shown that in the heavy loading limit the outer potential behaves to leading order as if the plate was absent. Near a resonance, however, eigensolutions of the outer boundary value problem with singularities at the plate edges also become present at this leading order. Close to the plate it is found that near resonance the leading order terms are standing plate waves (of order $\varepsilon^{-1/2}$) and these have a wavelength of the order of the size of the inner region. These results are similar in form to those in the zero flow problem except that for $M = O(1)$ the wavelength of the plate waves is given by (2.5),

$$\ell = (k^2/\alpha)^{1/3} = (D/\rho c^2)^{1/3} . \tag{5.1}$$

This result is interesting because it shows that the wavelength (or frequency) of the un-attenuated waves on the flexible plate is independent of the frequency of the acoustic source supplying the forcing. Thus for any frequency of forcing it can be expected that the frequency of plate waves will be a constant dependent only on D,ρ,c,U, subject of course to the heavy loading approximation

$$\varepsilon = k\ell << 1. \tag{5.2}$$

As was discussed briefly in the introduction, it was hoped that comparison could be made between this analysis and flutter analysis using modal expansions. Much work has been performed on calculating the onset of flutter of finite or infinite elastic plates (Weaver and Unny 1970, Dowell 1966, Bohon and Dixon 1964). The usual method for calculating instability is to replace the unknown plate deflection by a modal expansion (where each term satisfies the plate edge conditions) and the problem is solved using a Galerkin approximation. The validity

of Galerkin's method is difficult to justify in a rigorous sense and it is therefore helpful to solve the problem by an alternative technique (in this case by matched asymptotic expansions) to compare the predictions made. The two-dimensional problem of flow of a compressible fluid over one side of a simply supported elastic plate was analysed by Ellen (1972) (again using a truncated modal expansion) and his results predicted divergence instability (when the complex frequency crosses the real axis at $\omega = 0$) when

$$\frac{\rho U^2 a_E^2}{D(1-M^2)^{\frac{1}{2}}} = \frac{\pi^5 n^4}{2\{n\pi Si(n\pi)-1+(-1)^n\}}, \quad n = 1,2,\ldots, \tag{5.3}$$

$$a_E = 2a, \quad Si(z) = \int_0^z \frac{\sin x \, dx}{x}, \text{ subject to the restriction } ka_E << 1. \tag{5.4}$$

In the analysis presented in this paper a time harmonic solution was assumed. If, instead, a Fourier transform in time had been taken, i.e.

$$\Phi(\underline{x},\omega) = \int_{-\infty}^{\infty} \phi(\underline{x},t)e^{i\omega t} \, dt, \tag{5.5}$$

where $\phi(\underline{x},t) = 0$ for $t < 0$, then

$$\phi(\underline{x},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\underline{x},\omega)e^{-i\omega t} d\omega, \tag{5.6}$$

where the poles of $\Phi(\underline{x},\omega)$ must lie below the real axis. Thus to satisfy causality the result in (2.60) must have poles (of complex ω) lying in the lower half plane. This condition leads to the result (from (3.63)) that the frequency vanishes (i.e. a divergence instability occurs) at a plate resonance. This paper therefore predicts divergence instability for a simply supported plate with flow over one side when (from (3.51))

$$\frac{\rho U^2 a_E^2}{D(1-M^2)^{\frac{1}{2}}} = (\frac{\pi}{12} + n\pi)^3 \qquad n = 0,1,2,\ldots, \tag{5.7}$$

with the conditions

$$k\ell << 1, \qquad \ell/a << 1. \tag{5.8}$$

Note that the latter constraint implies that (5.7) is strictly valid only for large n. It can be seen that the values of the dimensionless parameter

$$\rho U^2 a_E{}^2 / [D(1-M^2)^{\frac{1}{2}}]$$

predicted by both methods are within 2% of each other for $n \geq 1$ and both tend to the same value $(n\pi^3)$ as $n \to \infty$. The $n = 0$ term in (5.7) violates the latter constraint in (5.8) and does not satisfy the condition that $ka_E << 1$ in (5.4) and can therefore be rejected; it is reassuring, however, to find that the results are in very good agreement, even for low n (when neither approximation is strictly valid). As a final point it can be noted that an advantage of the present method over that of Ellen is the ability to predict resonance for any plate edge condition whereas the modal expansion technique has only been applied to simply supported panels.

The author would like to thank Dr. F.G. Leppington for much advice and assistance, and the S.E.R.C. and A.U.W.E. Portland for the financial support of a C.A.S.E. studentship.

Appendix A

Factorization of the Wiener-Hopf Kernel

From (3.18)

$$K(s) = (|s|(1-M^2)^{\frac{1}{2}})^{-1} - s^2/M^2, \tag{A1}$$

which can be rewritten as

$$K_+(s) = \frac{e^{i\pi/4}}{MS_+^{\frac{1}{2}}} Q_+(s), \quad K_-(s) = MS_-^{\frac{1}{2}} e^{i\pi/4} Q_-(s), \tag{A2}$$

where $S_+^{\frac{1}{2}} = \lim_{q\to 0} (s + iq)^{\frac{1}{2}}$ etc., and

$$\frac{Q_+(s)}{Q_-(s)} = (b^3 - s^2|s|), \quad Q_+(s) = Q_-(-s) \ . \tag{A3}$$

Q_+, Q_- are regular and non-zero in the same regions as K_+, K_- respectively, and it can be shown that

$$\ln[Q_+(s)] = (2\pi i)^{-1} \lim_{N\to\infty} \int_{-N}^{N} \frac{\ln\,(b^3 - \zeta^2|\zeta|)d\zeta}{(\zeta - s)} \ , \tag{A4}$$

where $I_m s > I_m \zeta$.

Note that the integration path in (A4) passes below the point $\zeta = b$ and so

$$\ln[Q_+(s)] = \frac{I(s)}{2\pi i} \text{ say } = \frac{2s}{2\pi i} \int_0^{\infty} \frac{\ln(b^3 - \zeta^3)d\zeta}{(\zeta^2 - s^2)} \ . \tag{A5}$$

After much algebra, with careful interpretation of the branch of the logarithm chosen, it is found that

$$\frac{dI}{ds} = \sum_{n=-1}^{+1} \{ \frac{1}{(b_n - s)} \ln(\frac{s}{b}) + \frac{1}{(s+b_n)} \ln(\frac{s}{b})\} - \frac{2\pi i}{3} (\frac{1}{(b_1 - s)} + \frac{2}{(b_{-1} - s)})$$

$$+ \pi i \{\frac{1}{(s+b_0)} + \frac{1}{3(s+b_1)} - \frac{1}{3(s+b_{-1})} \} \ , \tag{A6}$$

where $b_n = be^{2n\pi i/3}$. To estimate the order of $K_+(s)$, as $s \to \infty$, the expansion of (A6) in powers of 1/s is

$$\frac{dI}{ds} \sim \frac{3\pi i}{s} + O(\frac{1}{s^2}), \qquad s \to \infty, \tag{A7}$$

which, after integrating, gives

$$Q_+(s) = Ps^{3/2} + O(s^{1/2}), \quad s \to \infty, \tag{A8}$$

where P is a constant, and so from (A2)

$$K_+(s) = O(s) \text{ when } s \to \infty. \tag{A9}$$

It is now required to find $K_+(b)$ so that the coefficient of the outgoing wave in equation (3.21) can be found. Firstly, by integrating by parts, (A5) can be rewritten as

$$\frac{I(s)}{2\pi i} = \frac{3}{2} \ln b + \frac{3}{2\pi i} \int_0^\infty \frac{\zeta^2}{(b^3-\zeta^3)} \ln (\frac{\zeta-s}{\zeta+s})\, d\zeta , \tag{A10}$$

and so $$I(0) = 3\pi i \ln b, \tag{A11}$$

as the integral in (A10) is zero when s = 0. The expression in (A6) can now be integrated and, letting s = b, the equation simplifies to

$$\frac{I(b)-I(0)}{2\pi i} = \ln\{\frac{Q_+(b)}{b^{3/2}}\} = -\frac{1}{3\pi i} \int_0^\infty \frac{\ln v\, dv}{(v^2 - 1)} + \frac{1}{2} \ln 6 - \frac{i\pi}{6} . \tag{A12}$$

The integral can be shown to have the value $\pi^2/8$ and so substituting (A12) into (A2) finally gives

$$K_+(b) = be^{i\pi/8}\sqrt{6}/M. \tag{A13}$$

The pole contribution at s = -b in (3.21) gives the outgoing wave for Φ_{11}, and can be written as

$$\Phi_{11\omega} = b \lim_{s \to -b} \frac{K_+(b) e^{-is\bar{x} - |s|\bar{y}(1-M^2)^{\frac{1}{2}}}}{K_+(s)|s|(s-b)}$$

$$= -\frac{K_+^2(b)}{2b} \lim_{s \to -b} \left\{ \frac{s+b}{K(s)} \right\} e^{ib\bar{x} - b\bar{y}(1-M^2)^{\frac{1}{2}}} . \qquad \text{(A14)}$$

The coefficient in front of the exponential in (A14) is found to be

$$-K_+^2(b)M^2/6b^2,$$

and using (A13) the solution becomes

$$\Phi_{11\omega} = e^{5i\pi/4} e^{ib\bar{x} - b\bar{y}(1-M^2)^{\frac{1}{2}}} . \qquad \text{(A15)}$$

Appendix B

Outgoing wave $\Phi_{12\omega}$

The expression for Φ_{12} is written in (3.37) and the outgoing wave is again found by closing the integral in the lower half plane thus picking up the pole at s = -b. Noting that the term in D(s), (3.36), containing $R_+(s)$ has a pole of order two at s = -b, the residue of the pole can be written as

Residue (at s = -b) =

$$\frac{1}{2\pi b^2(1-M^2)^{\frac{1}{2}}}\{\frac{AK_+(b)}{4b}+\frac{AK_+'(b)}{2}-\frac{BibK_+(b)}{2b}+\frac{iK_+(b)(2-M^2)R_+(b)}{2M^3(1-M^2)^{\frac{1}{2}}}$$

$$-\frac{iC}{M^2}+\frac{MiK_+(b)}{2(1-M^2)^{\frac{1}{2}}}+\frac{\bar{y}MiK_+(b)b}{2}\}e^{ib\bar{x}-b\bar{y}(1-M^2)^{\frac{1}{2}}}\lim_{s\to -b}\{\frac{s+b}{K_+(s)}\}$$

$$-\frac{ibK_+(b)(2-M^2)}{2\pi M^3(1-M^2)}\lim_{s\to -b}\frac{d}{ds}\{\frac{R_+(s)(s+b)^2e^{is\bar{x}-|s|\bar{y}(1-M^2)^{\frac{1}{2}}}}{sK_+(s)|s|(s-b)}\}\quad . \qquad (B1)$$

Using the identities

$$R_+(s) = s^2/K(s) - R_-(s), \qquad (B2)$$

$$R_-(-b) = R_+(b), \quad \text{and} \qquad (B3)$$

$$\lim_{s\to -b}\{\frac{s+b}{K_+(s)}\} = K_+(b)\lim_{s\to -b}\{\frac{s+b}{K(s)}\}\ , \qquad (B4)$$

the last term in (B1) can be written as

$$- \frac{i(2-M^2)K_+(b)e^{ib\bar{x}-b\bar{y}(1-M^2)^{\frac{1}{2}}}}{4\pi M^3(1-M^2)} \left[\{(\bar{y}(1-M^2)^{\frac{1}{2}} - i\bar{x})K_+(b)\right.$$

$$- K_+'(b) + \frac{K_+(b)}{2b}\} \lim_{s\to -b}\{\frac{s+b}{K(s)}\}^2 + K_+(b) \lim_{s\to -b}\{\frac{d}{ds}(\frac{s+b}{K(s)})^2\}$$

$$\left. - \frac{K_+(b)}{b^2} R_+(b) \lim_{s\to -b}\{\frac{d}{ds}\ \frac{(s+b)^2}{K(s)}\}\right] . \qquad \text{(B5)}$$

It is easily shown that

$$\lim_{s\to -b}\{\frac{s+b}{K(s)}\} = \frac{M^2}{3b} \quad , \qquad \text{(B6)}$$

$$\lim_{s\to -b}\{\frac{d}{ds}\ \frac{(s+b)^2}{K(s)}\} = \frac{M^2}{3b} \quad , \quad \text{and} \qquad \text{(B7)}$$

$$\lim_{s\to -b}\{\frac{d}{ds}(\frac{s+b}{K(s)})^2\} = 0, \qquad \text{(B8)}$$

and these values are substituted into (B1) and (B6). Using (3.28), (B5) and (B1) the outgoing wave of Φ_{12}, after considerable cancellation, is determined; thus

$$\Phi_{12\omega} = -2\pi i\{\text{residue at } s = -b\} =$$

$$\{\frac{(2M^2-1)\bar{y}K_+^{\ 2}(b)b}{9M} + \frac{i(2-M^2)K_+^{\ 2}(b)b\bar{x}}{18M(1-M^2)^{\frac{1}{2}}} - \frac{CK_+(b)}{3M^2}\} e^{ib\bar{x}-b\bar{y}(1-M^2)^{\frac{1}{2}}} , \qquad \text{(B9)}$$

or, using (A13),

$$\Phi_{12\omega} = \{-e^{5i\pi/4}\ (\frac{(2-M^2)i\bar{x}}{3M(1-M^2)} + \frac{2(1-M^2)^{\frac{1}{2}}(2M^2-1)\bar{y}}{3M(1-M^2)})$$

$$- \frac{ce^{i\pi/8}b(2/3)^{\frac{1}{2}}}{M^3}\}e^{ib\bar{x}-b\bar{y}(1-M^2)^{\frac{1}{2}}} . \qquad \text{(B10)}$$

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Chapter 5

The attenuation of travelling waves on slightly curved elastic plates

In the two preceding thesis chapters it was noted that plate waves (of small wavelength) can travel along a plane elastic plate without attenuation. Thus, for specific values of the plate length, standing waves are formed, and this is identified as a plate resonance. Many applications in underwater acoustics include curved surfaces and it is therefore useful to examine the attenuation of travelling plate waves due to curvature. It is expected that energy is shed from the wave as it propagates along the plate, and if this loss is large then no standing waves can form, therefore suppressing any resonances.

Purely for convenience this analysis will study cylindrical plates of circular cross-section. Cylindrical coordinates r,θ,z are adopted and only two-dimensional motions are assumed (with no z dependence). From Timoshenko (1959, p.513), the equation for an elastic plate of circular cross-section, after adding inertia terms, is found to be

$$\frac{\partial^2 v}{\partial\theta^2} - \frac{\partial\eta}{\partial\theta} + \frac{h^2}{12a^2}\left(\frac{\partial^3\eta}{\partial\theta^3} + \frac{\partial^2 v}{\partial\theta^2}\right) = \frac{ma^2}{E}(1-\nu^2)\frac{\partial^2 v}{\partial t^2}, \qquad (1)$$

$$\frac{\partial v}{\partial\theta} - \eta - \frac{h^2}{12a^2}\left(\frac{\partial^4\eta}{\partial\theta^4} + \frac{\partial^3 v}{\partial\theta^3}\right) = \frac{ma^2}{E}(1-\nu^2)\frac{\partial^2\eta}{\partial t^2} + \frac{a^2 p}{Eh}(1-\nu^2), \qquad (2)$$

where η,v are the radial and circumferential deflections of the plate respectively, and E and ν have their usual meanings. The undisturbed plate lies on the circle of radius $r = a$, and p,m,h respectively denote the pressure discontinuity across the plate, the mass per unit volume of the plate, and the plate thickness. It is now assumed, for simplicity, that no fluid lies inside the circle and that the fluid outside satisfies the two dimensional wave equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0, \tag{3}$$

where ϕ is the velocity potential and c is the fluid propagation speed. The potential $\phi(r,\theta)$ is related to η,p by

$$p = -\rho\phi_t \qquad r > a, \tag{4}$$

$$\eta_t = \phi_r(a,\theta) \qquad r = a, \tag{5}$$

(the subscripts denoting partial differentiation), and if assuming time harmonic motion, the travelling plate wave solution may, by separation of variables, be written as

$$\phi(r,\theta) = Ae^{i\theta\mu}H_\mu^{(1)}(kr)e^{-i\omega t}. \tag{6}$$

Note that ω is the angular frequency, $k = \omega/c$, ρ the fluid density, A is an arbitrary amplitude, and $H_\mu^{(1)}$ is a Hankel function of the first kind of order μ,($H_\mu^{(1)} = J_\mu + iY_\mu$).

Assuming that the circumferential displacement of the plate is of the form

$$v = v_0 e^{i\theta\mu} e^{-i\omega t}, \tag{7}$$

the travelling wave solution (6) may be substituted into the plate equations (1), (2), using (4) and (5) to derive coupled equations in the unknowns v_0 and μ. These are given by

$$-\mu^2 v_0 + \frac{\mu}{\omega} AkH_\mu^{(1)}(ka) + \frac{h^2}{12a^2}\left(Ak\omega^{-1}\mu^3 H_\mu'^{(1)}(ka) - v_0\mu^2\right)$$

$$= -ma^2 G v_0 \omega^2, \tag{8}$$

$$\mu v_0 - \frac{k}{\omega} AH_\mu'^{(1)}(ka) - \frac{h^2}{12a^2}\left(Ak\omega^{-1}\mu^4 H_\mu'^{(1)}(ka) - v_0\mu^3\right)$$

$$= -\frac{m}{\omega} a^2 G\omega^2 kAH_\mu'^{(1)}(ka) + \frac{a^2}{h} G\rho\omega AH_\mu^{(1)}(ka) \,, \tag{9}$$

where $G = (1-\nu^2)/E$ and $H_\mu'^{(1)}(ka) = \frac{\partial}{\partial(kr)} H_\mu^{(1)}(kr)\Big|_{r=a}$.

Rearranging, to eliminate v_0, yields

$$\mu^2(1 + h^2\mu^2/(12a^2))^2[\mu^2(1 + h^2/(12a^2)) - ma^2G\omega^2]^{-1} \tag{10}$$

$$= 1-ma^2G\omega^2 + a^2G\rho\omega^2 H_\mu^{(1)}(ka)/(H_\mu'^{(1)}(ka)kh) + h^2\mu^4/(12a^2).$$

To compare the results for circular and flat plates it is necessary to let

$$\theta = x/a, \tag{11}$$

where x is now the distance along the plate and thus the wavenumber is

$$s = \mu/a. \tag{12}$$

As the radius, a, tends to infinity, the leading order result for s should reduce to that given in part 3. Substituting (12) into (10) and letting $a \to \infty$ gives, to leading order,

$$-mG\omega^2 + \frac{G\rho\omega^2}{kh}\left(H_\mu^{(1)}(ka)/H_\mu'^{(1)}(ka)\right) + \frac{h^2s^4}{12} = 0. \tag{13}$$

It is assumed that the travelling waves are of very small wavelength ($s >> k$), and so, using the relevant asymptotic expansions for the Hankel function and its derivative, it is found that

$$\frac{H_\mu^{(1)}(ka)}{H_\mu'^{(1)}(ka)} \sim \frac{-k}{(s^2-k^2)^{\frac{1}{2}}}\left(1 + ie^{2a(s^2-k^2)^{\frac{1}{2}}}[(s-(s^2-k^2)^{\frac{1}{2}})/k]^{2sa}\right), \tag{14}$$

$$\mu >> ka, \ \mu \to \infty \ .$$

The attenuation of the wave is given by the imaginary part of s, and so it is convenient to substitute

$$s = s_0 + is_1, \tag{15}$$

into (13), after using (14), and then collect real and imaginary terms. After careful expansion of terms, it is finally found that

$$S_0 \sim (12\rho\omega^2 G/h^3)^{1/5} \ , \tag{16}$$

and $$s_1 \sim (S_0/5)\exp(-2as_0[\ln(2S_0/k) - 1]). \tag{17}$$

The travelling wave solution (6), after using the expansion for the Hankel function, can now be expressed, for large a, in the form

$$\phi(x,y) \sim B \exp(is_0x - s_0y - i\omega t)e^{-s_1x} \ , \tag{18}$$

where B is a constant, and $\exp(-s_1x)$ is the attenuation factor. It is seen that if the radius, a, formally tends to infinity, then the previous travelling wave solution is recovered, with wavenumber, s_0, the same as that in §3. For large but finite a, the attenuation, from (17), is clearly exponentially small when $as_0 >> 1$ and can therefore be neglected. It is therefore assumed that plate resonances will not be suppressed if the elastic surface has slight curvature (in the sense that $as_0 >> 1$).

As a final point it is useful to calculate the energy flux from a section of the plate (between x_1 and x_2, say) into the fluid. The power radiated is given by

$$\text{Power} = -\rho\omega \int_{x_1}^{x_2} \text{Im}(\phi^* \phi_r) dx \tag{19}$$

where * denotes the complex conjugate and the subscript r means partial differentiation in the radial direction. Substituting the expression for $\phi(r,\theta)$ given by (6) into equation (19), and expanding for large plate radius, a, allows the integral to be performed. Thus, it is finally found that the energy flux from the plate is given by

$$\text{Power} = \frac{8A^2\rho\omega}{5\pi a}(x_2 - x_1) \ . \tag{20}$$

Chapter 6

Scattering of sound by large finite geometries

Summary

Diffraction problems with finite geometries do not usually have exact solutions and so asymptotic methods, which require a large or small parameter, are employed to obtain approximate results. This paper presents a formal method for approximating the acoustic potential when the finite length in the geometry is large compared to an acoustic wavelength. The method presented is exactly analogous to other approaches, including the modified Wiener-Hopf technique, but is advantageous because of its relative ease of use and applicability to many problems. This is shown by using the method on problems with resonances and complicated geometries.

1. Introduction

The problem of diffraction by slits (in infinite baffles) or finite screens has been well studied in the past. All the analytic work has required a small or large parameter to obtain approximate solutions. This paper presents a new method of obtaining an asymptotic solution when the ratio of the screen (or slit) length to acoustic wavelength is large, and is presented because of its simplicity of use and applicability to a wide range of problems.

Schwarzschild (1902) used a Green's function method on the slit problem and obtained a pair of coupled integral equations. He proposed an iterative method of solution by taking the electromagnetic (or acoustic) potential due to two Sommerfeld half-plane problems (one for each end and ignoring the interaction terms) as the first approximation, and substituted these into the equations to find the second approximation which contained an interaction term (i.e. the effect on one edge due to the presence of the other). This series was shown by Schwarzschild to converge for all slit widths but proved extremely difficult to use analytically, and the convergence was too slow for numerical use. An account of this can be found in Baker & Copson (1950). Fox (1949) used essentially the same technique for diffraction of a pulse through a slit and it is plausible that this system of iteration would be useful (i.e. converge rapidly) when the slit width is large, as then the interaction between edges would necessarily be small.

This led Karp & Russek (1956) to obtain an approximate (but not asymptotic) solution when the ratio of slit width to wavelength was large. They simplified the mathematics by using an essentially physical argument whereby they assumed that the effect of one of the half planes on the other could be replaced by the potential due to a line source at

the edge of the half plane. A few years later Noble (1958) attacked the same problem using a modified form of the Wiener-Hopf technique which involved a great deal of algebra. This method led to a pair of coupled integral equations which were analogous to those of Schwarzschild, and these were then approximated in the same large gap limit.

The present method uses ideas proposed in all the work described above but is somewhat similar to a technique used by Leppington (1968) in a finite dock water wave problem. Each problem is organised so that the potential can be split into four (or more) parts; two half plane problems plus two interaction potentials. These interaction potentials are then approximated in a rigorous asymptotic sense rather than by the physical argument of Karp and Russek.

In section 2 the problem of a slit in a soft screen is examined but to show the direct analogy between this method and that of Noble the approximations to the interaction potentials are made later in the analysis than necessary. This also helps to show that the interaction potentials are replaced by the leading order terms in their asymptotic expansions. In section 3 the problem of a slit in a hard screen is examined. The ease of use of the method is shown when the large gap approximation is made at an early stage in the analysis.

The next section (4) shows how the method can be applied to a wide range of problems with more complicated geometries; in this case the problem of diffraction by two large gaps (far apart) in a hard baffle. Finally the technique is extended to the problem of scattering by a large finite membrane which highlights the way of extending the method to include resonances.

2. Scattering by a finite screen

Assuming simple harmonic time dependence with angular frequency ω, the velocity potential can be written as

$$\mathrm{Re}\{\phi(x,y)\exp(-i\omega t)\} \tag{2.1}$$

where ϕ is the acoustic potential, (x,y) are cartesian coordinates of the two-dimensional problem, and the time factor is henceforth omitted for brevity. To show the relation between the methods of this paper and the modified Wiener-Hopf technique it is convenient to examine the prototype problem of diffraction by a finite slit, length 2a, in an infinite baffle on which $\phi = 0$. The boundary value problem for the total potential, ϕ_t, is

$$(\nabla^2 + k^2)\phi_t = 0, \qquad \text{all } x,y, \tag{2.2}$$

$$\phi_t = 0 \quad , \qquad |x| > a,\ y = 0, \tag{2.3}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $k = \omega/c$ the acoustic wavenumber, and c the sound speed in the fluid. To ensure a solution of the boundary value problem which does not allow energy to radiate from the edges, it must be insisted that

$$r|\nabla\phi| \to 0 \text{ as } \quad r = ((x \pm a)^2 + y^2)^{\frac{1}{2}} \to 0. \tag{2.3a}$$

This condition must also be satisfied by each of the semi-infinite potentials at their respective edges.

Assuming that the baffle is irradiated from above by incident waves of the form

$$\phi_i = \exp(-ikx\cos\theta - iky\sin\theta), \tag{2.4}$$

where θ is the angle between incident waves and the x axis, it is then most convenient to split the potential ϕ_t into the form

$$\phi_t = \begin{cases} \phi_i + \phi, & y > 0 \\ \phi_i - \phi, & y < 0, \end{cases} \tag{2.5}$$

where ϕ, the diffracted potential, is an odd function of y. Using the fact that there is no discontinuity in velocity across the slit, and dealing with $y > 0$ only, the problem becomes

$$(\nabla^2 + k^2)\phi = 0 \qquad \text{all } x,y > 0,$$

$$\phi = -\exp(-ikx\cos\Theta) \qquad |x| > a,\ y = 0, \tag{2.6}$$

$$\phi_y = 0 \qquad |x| < a,\ y = 0,$$

plus a radiation condition which allows only outgoing waves at infinity.

It was suggested in the introduction that as $ka \to \infty$ the interaction between the edges becomes small and so ϕ can be approximated to leading order by the two semi-infinite edge problems, ϕ_1 and ϕ_2 say, where ϕ_n is the solution of

$$(\nabla^2 + k^2)\phi_n = 0$$

$$\phi_n = -\exp(-ikx\cos\Theta), \qquad x_n \begin{cases} < 0, & n = 1,\ y = 0, \\ > 0, & n = 2, \end{cases}$$

$$\phi_{ny} = 0 \quad , \qquad x_n \begin{cases} > 0, & n = 1,\ y = 0, \\ < 0, & n = 2 \end{cases} \tag{2.7}$$

and $n = 1,2$ with $x_1 = x+a$, $x_2 = x-a$. The edge coordinates have been chosen in this section to compare the results with those by Noble and in later sections x_2 will be of opposite sign. The potential ϕ also needs correction potentials which will be shown to be small in some sense (because of the way the total potential was split (2.5)), and these potentials g_1, g_2 can easily be shown to satisfy

$$(\nabla^2 + k^2)g_n = 0, \qquad \text{all } x_n, y,$$

$$g_n = -\phi_n - g_m, \qquad x_n \begin{cases} < 0, & n=1,\ y = 0 \\ > 0, & n = 2 \end{cases}$$

$$g_{ny} = 0 \quad , \qquad x_n \begin{cases} > 0, & n = 1,\ y = 0 \\ < 0, & n = 2 \end{cases} \tag{2.8}$$

where n,m = 1 or 2 and n ≠ m. Justification for splitting the potential into this form

$$\phi = \phi_1 + \phi_2 + g_1 + g_2, \tag{2.9}$$

can be found in Leppington (1968); ϕ_1, ϕ_2 are solved by the Wiener-Hopf technique, and g_1, g_2 are governed by coupled problems which, it will be shown, can be easily uncoupled when ka becomes large.

(i) Equivalence with Modified Wiener Hopf technique

Defining the half range Fourier transforms as

$$\Phi_{n+}(s,y) = \int_0^\infty \phi_n(x_n,y)e^{isx_n}\,dx_n, \tag{2.10}$$

analytic in an upper half of the complex s plane ($s = \sigma + it$), and

$$\Phi_{n-}(s,y) = \int_{-\infty}^0 \phi_n(x_n,y)e^{isx_n}\,dx_n, \tag{2.11}$$

similarly analytic in the lower half s plane, the inverse transform is

$$\phi_n(x_n,y) = \frac{1}{2\pi}\int_{-\infty}^\infty \Phi_n(s,y)e^{-isx_n}\,ds, \tag{2.12}$$

with $\Phi_n = \Phi_{n+} + \Phi_{n-}$.

Note that the transforms are assumed to be analytic and of algebraic growth in their respective half planes as $|s| \to \infty$. A transform with a bar on top denotes that the alternate x variable was used, i.e.

$$\bar{\Phi}_{1+}(s,y) = \int_0^\infty \phi_1(x_2,y)e^{isx_2}\,dx_2, \tag{2.13}$$

$$\bar{G}_{2-}(s,y) = \int_{-\infty}^0 g_2(x_1,y)e^{isx_1}\,dx_1, \tag{2.14}$$

etc., and it is found that

$$\bar{\Phi}_1(s,y) = e^{-2ias}\Phi_1(s,y), \tag{2.15}$$

$$\bar{G}_2(s,y) = e^{2ias}G_2(s,y). \tag{2.16}$$

Transforming the governing equations gives the result

$$\Phi(s,y) = A(s)e^{-\gamma y} \tag{2.17}$$

where A(s) is to be found and

$$\gamma = (s^2-k^2)^{\frac{1}{2}} = -i\,(k^2-s^2)^{\frac{1}{2}} \quad . \tag{2.18}$$

It is convenient to allow k to be slightly complex during the analysis, thus

$$k = k_1 + ik_2, \qquad k_2 > 0 \text{ and } k_2 << 1, \tag{2.19}$$

and due to the branch of γ chosen in (2.18) the path of the inversion integral passes just above the branch cut $-\infty < s < -k$, and below the cut $k < s < \infty$.

From (2.17) it follows that

$$A(s) = \Phi_+ + \Phi_- = -\frac{1}{\gamma}(\Phi_+' + \Phi_-') \tag{2.20}$$

where $\Phi_+ = \Phi_+(s) \doteq \Phi_+(s,0)$ etc. and

$$\Phi_+' + \Phi_-' = \frac{\partial\Phi}{\partial y}(s,y)\bigg|_{y=0} \quad . \tag{2.21}$$

Returning to the ϕ_1 problem, the boundary conditions defined in (2.7) are transformed, i.e.

$$\Phi_{1-} = \frac{ie^{ika\cos\Theta}}{(s-k\cos\Theta)} \quad , \qquad \Phi_{1+}' = 0, \tag{2.22}$$

where Φ_{1-} is analytic in the region $\tau = \mathrm{Im}(s) < k_2 \cos\theta$. Substituting into (2.20) and rearranging gives the Wiener-Hopf functional equation, which is defined in the strip $-k_2 < \tau < k_2 \cos\theta$,

$$\frac{\Phi_{1+}(s)}{K_+(s)} + \frac{ie^{ika\cos\theta}}{(s-k\cos\theta)}\left\{\frac{1}{K_+(s)} - \frac{1}{K_+(k\cos\theta)}\right\} = -\frac{\Phi'_{1-}(s)}{K_-(s)} - \frac{ie^{ika\cos\theta}}{(s-k\cos\theta)K_+(k\cos\theta)}, \tag{2.23}$$

where $K_-(s)/K_+(s) = \gamma$, $K_+(s) = 1/K_-(-s) = e^{i\pi/4}/(s+k)^{\frac{1}{2}}$, (2.24)

and $K_+(s)$, $K_-(s)$ are analytic in $\tau > -k_2$, $\tau < k_2$ respectively. The left hand side of (2.23) is analytic in the region $\tau > -k_2$, the right in $\tau < k_2$, and therefore by analytic continuation arguments both sides must be equal to a function analytic in the whole s plane. The edge condition (2.3a) specifies the form of the potential for large $|s|$ as

$$\Phi_{1+}(s) = O(|s|^{-1}), \quad \Phi'_{1-}(s) = O(|s|^{-\varepsilon}), \quad \varepsilon > 0,$$

(ε can in fact be shown to equal $\varepsilon = \frac{1}{2}$) and so by examining the behaviour of both sides of the equation as $|s| \to \infty$, Liouville's theorem shows that this function must be identically zero; hence

$$\Phi'_{1-}(s) = -\frac{iK_-(s)e^{ika\cos\theta}}{(s-k\cos\theta)K_+(k\cos\theta)}, \quad \text{and} \tag{2.25}$$

$$\Phi_1(s,y) = \frac{ie^{ika\cos\theta}K_+(s)e^{-\gamma y}}{(s-k\cos\theta)K_+(k\cos\theta)}. \tag{2.26}$$

Repeating for Φ_2 using

$$\Phi_{2+} = -\frac{ie^{-ika\cos\theta}}{(s-k\cos\theta)}, \text{ analytic in } \tau > k_2\cos\theta, \tag{2.27}$$

$$\Phi'_{2-} = 0, \quad \text{gives} \tag{2.28}$$

$$- \frac{ie^{-ika\cos\Theta}}{(s-k\cos\Theta)}(K_-(s) - K_-(k\cos\Theta)) + \Phi_{2-}K_-(s)$$

$$= -K_+(s)\Phi'_{2+}(s) + \frac{ie^{-ika\cos\Theta}}{(s-k\cos\Theta)} K_-(k\cos\Theta) \qquad (2.29)$$

$$\equiv J(s),$$

defined in the strip $k_2 > \tau > k_2\cos\Theta$, and again using Liouville's theorem it is found that $J(s) \equiv 0$ for all s. Thus the solution is

$$\Phi'_{2+}(s) = \frac{ie^{-ika\cos\Theta}K_-(k\cos\Theta)}{(s-k\cos\Theta)\,K_+(s)}, \quad \text{or} \qquad (2.30)$$

$$\Phi_2(s,y) = -\frac{ie^{-ika\cos\Theta}K_-(k\cos\Theta)e^{-\gamma y}}{(s-k\cos\Theta)K_-(s)}. \qquad (2.31)$$

To calculate the interaction potentials the boundary conditions defined in (2.8) transform to give

$$G_{1-} = -\bar{\Phi}_{2-} - \bar{G}_{2-}, \quad G'_{1+} = 0, \qquad (2.32)$$

$$G_{2+} = -\bar{\Phi}_{1+} - \bar{G}_{1+}, \quad G'_{2-} = 0, \qquad (2.33)$$

which are analytic in the same upper and lower regions as before. To obtain the Wiener-Hopf equations for G_1, G_2 the following sum split is used

$$f(s) = \frac{1}{2\pi i}\int_{-\infty+ic}^{+\infty+ic}\frac{f(\zeta)}{(\zeta-s)}\,d\zeta - \frac{1}{2\pi i}\int_{-\infty+id}^{+\infty+id}\frac{f(\zeta)d\zeta}{(\zeta-s)} \qquad (2.34)$$

$$= f_+(s) + f_-(s) \quad \text{respectively,}$$

where f(s) is regular in some strip $\tau_- < \tau < \tau_+$ (with suitable behaviour as $|\sigma| \to \infty$), $\tau_- < c < \tau < d < \tau_+$, and $f_+(s)$ is analytic in $\tau > \tau_-$, $f_-(s)$ in $\tau < \tau_+$.

The Wiener-Hopf equation for $g_1(x,y)$ can be written (from 2.32) as

$$R_+(s)+P_+(s)+\frac{G_{1+}}{K_+(s)} = \frac{G'_{1-}}{K_-(s)} - R_-(s)-P_-(s), \qquad (2.34a)$$

which is valid in the strip $k_2 > \mathrm{Im}(s) > k_2 \cos\theta$, and the functions $R_+(s)$, $P_-(s)$ etc. are given by (2.34), (2.39), (2.41). Each side of this equation is equal to a function regular in this strip, and by analytic continuation arguments is valid over all s. Taking the right hand side first, it is easily shown that the imposed edge condition for g_1 ensures, as $|s| \to \infty$ in the lower half plane,

$$G'_{1-}(s) = O(|s|^{-\varepsilon}), \quad \text{for some } \varepsilon > 0.$$

Similarly the expression for $R_-(s)$ in (2.39) shows that

$$R_-(s) = O(|s|^{-1}) \text{ as } |s| \to \infty.$$

The edge condition for $g_2(x,y)$ is used to give

$$G'_{2+}(s) = O(|s|^{-\varepsilon}), \ \varepsilon > 0, \ |s| \to \infty,$$

and without specifying more about this function (as yet unknown) the integral expression (2.41) reveals

$$P_-(s) \to 0$$

as $s \to \infty$ in the lower half plane. Hence the right hand side of (2.34a) tends to zero for large $|s|$ in the lower half plane. Similarly it can be shown that the left hand side also tends towards zero as $s \to \infty$ in the upper plane and so by Liouville's theorem either side must be identically zero.

Thus employing (2.32), (2.34) in (2.20) gives

$$G'_{1-} = (R_-(s) + P_-(s))K_-(s), \tag{2.35}$$

or

$$G_1(s,y) = -K_+(s)(R_-(s) + P_-(s))e^{-\gamma y},$$

where it is assumed that both sides of the Wiener-Hopf equation are identically zero, and

$$R_-(s) = -\frac{1}{2\pi i}\int_{-\infty+id}^{+\infty+id} \frac{\bar{\Phi}_{2-}(\zeta)d\zeta}{K_+(\zeta)(\zeta-s)}, \quad k_2 > d > k_2 \cos\Theta, \; d > \tau, \tag{2.36}$$

$$P_-(s) = -\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia} \frac{\bar{G}_{2-}(\zeta)d\zeta}{K_+(\zeta)(\zeta-s)}, \quad k_2 > a > -k_2, \; a > \tau. \tag{2.37}$$

Again using the expression in (2.34) with (2.31) gives $R_-(s)$ in the form

$$R_-(s) = -\frac{ie^{-ika\cos\Theta}}{(2\pi i)^2}\int_{-\infty+id}^{\infty+id} \frac{d\zeta}{K_+(\zeta)(\zeta-s)}\int_{-\infty+ib}^{\infty+ib} \frac{e^{2ita}K_-(k\cos\Theta)dt}{(t-\zeta)(t-k\cos\Theta)K_-(t)}, \tag{2.38}$$

$d > \tau$, $k_2 > b > d > k_2 \cos\Theta$, and reversing the order of integration allows the ζ integration to be performed, thus

$$R_-(s) = \frac{ie^{-ika\cos\Theta}}{2\pi i}K_-(k\cos\Theta)\int_{-\infty+id}^{\infty+id} \frac{e^{2ita}dt}{K_+(t)K_-(t)(t-k\cos\Theta)(t-s)} \tag{2.39}$$

$$= \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{e^{2ita}\Phi'_{2+}(t)dt}{K_-(t)(t-s)}.$$

An alternative form for $R_-(s)$ can be found by allowing d to pass into the region $k_2 \cos\Theta > d > -k_2$, which will pick up the pole contribution at $t = k\cos\Theta$ resulting in

$$R_-(s) = \frac{ie^{ika\cos\Theta}}{(s-k\cos\Theta)K_+(k\cos\Theta)} + \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{e^{2ita}\Phi_{2+}'^{*}(t)dt}{K_-(t)(t-s)} \quad , \tag{2.40}$$

where * denotes that $\Phi_{2+}^{*}(t)$ is analytic in $I_m t > d$ except for the singularity at $t = k\cos\Theta$. $P_-(s)$ can similarly be expressed as

$$P_-(s) = -\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{e^{2ita}G_2(t)dt}{K_+(t)(t-s)} = \frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{e^{2ita}G_{2+}'(t)dt}{K_-(t)(t-s)} \tag{2.41}$$

where $k_2 > d > -k_2$, $\tau < d$, which finally gives $G_{1-}'(s)$, $G_1(s,y)$ after the integral representations of $P_-(s)$, $R_-(s)$ are substituted into (2.35).

Repeating the above procedure for the G_2 potential yields

$$\begin{aligned} G_{2+}'(s) &= \frac{1}{K_+(s)}\left(Q_+(s) + T_+(s)\right), \\ G_2(s,y) &= -\frac{1}{K_-(s)}\left(Q_+(s) + T_+(s)\right)e^{-\gamma y} \quad , \end{aligned} \tag{2.42}$$

where

$$Q_+(s) = -\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{e^{-2ita}K_+(t)\Phi_{1-}'(t)dt}{(t-s)} \quad , \tag{2.43}$$

$$T_+(s) = -\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{e^{2ita}K_+(t)G_{1-}'(t)dt}{(t-s)K_-(t)} \quad , \tag{2.44}$$

and $k_2\cos\Theta > c > -k_2$, $c < \tau$.

Defining the two functions Ψ_+^*, Ψ_- as

$$\Psi_+(s) = \bar{\Phi}'_{1+} + \Phi'^{*}_{2+} + \bar{G}'_{1+} + G'_{2+} ,$$

$$\Psi_-(s) = \Phi'_{1-} + \bar{\Phi}'_{2-} + G'_{1-} + \bar{G}'_{2-} , \qquad (2.45)$$

noting that from the boundary conditions,

$$\bar{\Phi}'_{1+} = \bar{G}'_{1+} = \bar{\Phi}'_{2-} = \bar{G}'_{2-} = 0,$$

the problem reduces to the solution of the following pair of coupled integral equations

$$\Psi_+^*(s) = \frac{iK_-(k\cos\Theta)e^{-ika\cos\Theta}}{(s-k\cos\Theta)K_+(s)} - \frac{1}{2\pi iK_+(s)} \int_{-\infty+ic}^{\infty+ic} \frac{e^{-2ita}K_+(t)\Psi_-(t)dt}{t-s} , \qquad (2.46)$$

$$\Psi_-(s) = \frac{K_-(s)}{2\pi i} \int_{-\infty+id}^{\infty+id} \frac{e^{2ita}\Psi_+^*(t)dt}{K_-(t)(t-s)} . \qquad (2.47)$$

In the first equation $\tau > c$, $k_2\cos\Theta > c > -k_2$, in the second $\tau < d$, $k_2\cos\Theta > d > -k_2$ and the asterisk again denotes that $\Psi_+^*(s)$ has a pole at $s = k\cos\Theta$. This is identical to the equation quoted in Noble (1958) §5.5,6 where in his analysis $\gamma = K_+(s)K_-(s)$ and $A = -k\sin\Theta$. Noble then approximates the integrals in the limit $ka \to \infty$ which involves a complicated asymptotic analysis using Whittaker functions (later simplified by Jones (1964)). The following method uncouples the interaction potentials at the beginning therefore reducing algebra to a minimum.

(ii) Approximation in the limit ka $\to\infty$

An assumption implicit in this and Noble's analysis is that $\Psi_-(t)$, $\Psi_+^*(t)$ scale on the length 1/k, and not on the gap length a, which then allows an asymptotic estimate of the integrals to be found by use of Watson's lemma. It is therefore apparent that in this large gap case the leading order terms for the potential are simply the two semi-infinite problem potentials (ϕ_1,ϕ_2). The correction potentials g_1, g_2 are of the same order as $R_-(s)$, $Q_+(s)$, which will be shown to be $O(\frac{1}{(ka)^{1/2}})$, and consequently the terms $P_-(s)$, $T_+(s)$, which couple the equations, are of order $O((ka)^{-1})$. Thus to obtain an asymptotic solution, neglecting terms of O(1/ka), it is possible to uncouple the problem at the start by assuming that one of the boundary conditions in (2.8) is

$$g_n = -\phi_n + O(\frac{1}{ka}). \qquad x_n \begin{cases} < 0, & n=1 \\ > 0, & n = 2,\ y = 0\ . \end{cases} \tag{2.48}$$

The problem now reduces to estimating the integral representations of $R_-(s)$, $Q_+(s)$ when ka $\to \infty$. For convenience, and with no loss of generality let the branch cuts in the s plane be rotated so that they lie along s = -k to -k - i∞, and s = k to k + i∞. Thus, deforming the path of integration in (2.39) around the branch cut in the upper half plane from k to k + i∞ gives

$$R_-(s) = \frac{2ie^{-ika\cos\Theta}}{2\pi i} K_-(k\cos\Theta) \int_k^{k+i\infty} \frac{e^{2ita-i\pi/2}dt}{(t-k)^{\frac{1}{2}}(t+k)^{-\frac{1}{2}}(t-k\cos\Theta)(t-s)} \tag{2.49}$$

which, after making the substitution t = k + ivk, becomes

$$R_-(s) = \frac{\sqrt{2}e^{2ika-ika\cos\Theta}}{\pi} K_-(k\cos\Theta) \int_0^{\infty} \frac{e^{-2vka-\pi i/4}dv(1+iv/2)^{\frac{1}{2}}}{v^{\frac{1}{2}}(1-\cos\Theta+iv)(k-s+ivk)}\ . \tag{2.50}$$

Letting $ka \to \infty$ implies that the maximum contribution from the integrand is near $v = 0$ as long as $\Theta \neq 0, \pi$ and the path of s can be deformed away from the branch point k. Accepting these restrictions finally yields

$$R_{-}(s) \sim - \frac{ik^{\frac{1}{2}} e^{-ika(\cos\Theta - 2)}}{\sqrt{\pi ka}(k-s)(1-\cos\Theta)^{\frac{1}{2}}} + O\left(\frac{1}{(ka)^{3/2}}\right). \tag{2.51}$$

It is now possible to write ϕ_1, ϕ_2 in terms of integrals (which can also be rearranged as Fresnel integrals) and a far field estimate of g_1, g_2 will be derived. Firstly, from (2.26), (2.31),

$$\phi_1 = \frac{ik \sin\Theta}{2\pi(k-k\cos\Theta)^{\frac{1}{2}}} \int_{ia-\infty}^{ia+\infty} \frac{e^{-isa+ika\cos\Theta} e^{-isx-\gamma y}}{(s-k\cos\Theta)(k+s)^{\frac{1}{2}}} ds, \tag{2.52}$$

$$-k_2 < a < k_2 \cos\Theta,$$

$$\phi_2 = - \frac{ie^{-ika\cos\Theta} k\sin\Theta}{2(k+k\cos\Theta)^{\frac{1}{2}}} \int_{ib-\infty}^{ib+\infty} \frac{e^{isa-isx-\gamma y} ds}{(s-k\cos\Theta)(k-s)^{\frac{1}{2}}} \tag{2.53}$$

$$= - \frac{ie^{-ika\cos\Theta} k\sin\Theta}{2\pi(k+k\cos\Theta)^{\frac{1}{2}}} \int_{ia-\infty}^{ia+\infty} \frac{e^{isa-isx-\gamma y} ds}{(s-k\cos\Theta)(k-s)^{\frac{1}{2}}} - e^{-ikx\cos\Theta + iky\sin\Theta},$$

$k_2 \cos\Theta < b < k_2$, $-k_2 < a < k_2 \cos\Theta$,

and g_1, the interaction potential, is

$$g_1(x,y) = \frac{ik^{\frac{1}{2}} e^{ika(2-\cos\Theta)+i\pi/4}}{\sqrt{\pi ka}(1-\cos\Theta)^{\frac{1}{2}} 2\pi} \int_{-\infty+ic}^{\infty+ic} \frac{e^{-isa-isx-\gamma y} ds}{(k-s)(s+k)^{\frac{1}{2}}} \tag{2.54}$$

$$+ O\left(\frac{1}{ka}\right), \qquad -k_2 < c < k_2,$$

$$\text{or } g_1(x,y) \sim \frac{ik^{\frac{1}{2}}e^{ika(2-\cos\Theta)+i\pi/4}}{\sqrt{\pi ka}(1-\cos\Theta)^{\frac{1}{2}}2\pi} \; I \quad \text{say.}$$

To estimate the integral I as x,y become large it is convenient to convert to polar coordinates and make the transformation

$$s = -k\cos(\theta_1 + it), \qquad (-\infty < t < \infty), \tag{2.55}$$

where $x_1 = r_1\cos\theta_1$, $y = r_1\sin\theta_1$, $r_1 = \sqrt{x_1^2+y^2}$, and so

$$\gamma = -i(k^2-s^2)^{\frac{1}{2}} = -ik\sin(\theta_1+it). \tag{2.56}$$

Thus

$$I = -ik\int_{-\infty}^{\infty} \frac{\sin(\theta_1+it)e^{ikr_1\cosh t}}{(k+k\cos(\theta_1+it))(k-k\cos(\theta_1+it))^{\frac{1}{2}}}\,dt, \tag{2.57}$$

and by standard stationary phase arguments, when θ_1 is not near $0,\pi$, I approximates to

$$I \sim -\frac{i\sin\theta_1 e^{ikr_1+\pi i/4}}{k^{\frac{1}{2}}(1+\cos\theta_1)(1-\cos\theta_1)^{\frac{1}{2}}}\sqrt{\frac{2\pi}{kr_1}} \tag{2.58}$$

as kr_1 becomes large. If it is also assumed that $r >> a$ where $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(\frac{y}{x})$, then

$$r_1 = r + a\cos\theta + O(a/r), \quad \theta_1 = \theta + O(a/r), \tag{2.59}$$

and finally

$$g_1(x,y) \sim \frac{ie^{2ika}e^{ikr}(\sin\frac{1}{2}\Theta\cos\frac{1}{2}\theta)^{-1}e^{ika(\cos\theta-\cos\Theta)}}{2\pi\sqrt{2ka}\;\sqrt{kr}} + O(\frac{1}{ka}). \tag{2.60}$$

Repeating the above procedure gives

$$g_2(x,y) \sim \frac{ie^{2ika}e^{ikr}}{2\pi\sqrt{2ka}\,\sqrt{kr}} (\cos \tfrac{1}{2}\Theta \sin \tfrac{1}{2}\theta)^{-1} e^{-ika(\cos\theta - \cos\Theta)} + O(\frac{1}{ka}), \quad (2.61)$$

$ka \to \infty$, $kr \to \infty$, $r >> a$.

These results again agree with the calculations by Noble and Karp & Russek but note that the interaction potential $g_1 + g_2$ in this paper is a rigorous estimate to $O(\frac{1}{ka})$ whereas the previous work must be expanded and then truncated after $O(\frac{1}{ka})$ terms to yield a true asymptotic approximation. As a final note if Θ or θ are close to $0,\pi$ then the expressions (2.51) and (2.58) are incorrect and the integrals in (2.50), (2.57) must be reinterpreted.

3. Large Gap Problem

To show the relative simplicity of the method consider the problem of a large finite gap in an infinite hard screen irradiated by plane incident waves normal to the screen. Thus the total potential, ϕ_t, satisfies

$$\begin{aligned} (\nabla^2 + k^2)\phi_t &= 0 && \text{all } x,y, \\ \phi_{ty} &= 0 && |x| > a,\ y = 0, \\ \phi_t\big|_-^+ &= 0 && |x| < a,\ y = 0, \end{aligned} \quad (3.1)$$

with the edge condition given in (2.3a), and note $\phi_t\big|_-^+ = 0$ denotes that the potential across the gap is continuous.

Again choosing a form for the scattered potential such that the interaction potential will be small, take

$$\phi_t = e^{-iky} + \phi(x,y), \quad (3.2)$$

where $\phi(x,y) = -\phi(x,-y)$, and so the boundary value problem for ϕ is

$$(\nabla^2 + k^2)\phi = 0 \qquad \text{all } x,y,$$

$$\phi_y = ik \qquad |x| > a,\ y = 0, \tag{3.3}$$

$$\phi = 0 \qquad |x| < a,\ y = 0,$$

plus a radiation condition at infinity. As in §2 the potential is split into four parts, i.e.

$$\phi = \phi_1 + \phi_2 + g_1 + g_2$$

where all parts satisfy the Helmholtz equation and

$$\phi_{ny} = ik \qquad x_n < 0,\ y = 0, \tag{3.4}$$

$$\phi_n = 0 \qquad x_n > 0,\ y = 0, \tag{3.5}$$

$$g_{ny} = -\phi_{my} - g_{my} \qquad x_n < 0,\ y = 0, \tag{3.6}$$

$$g_n = 0 \qquad x_n > 0,\ y = 0, \tag{3.7}$$

$n, m = 1,2,\ n \neq m,\ x_1 = x+a,\ x_2 = a-x.$

Taking transforms as before and rearranging into the Wiener-Hopf equation it is found that

$$\Phi_1(s,y) = -\frac{K_+(0)ke^{-\gamma y}}{K_-(s)s} \tag{3.8}$$

$$\text{and}\quad \phi_1(x,y) = -\frac{k}{2\pi}\int_{-\infty}^{\infty} \frac{K_+(0)e^{-isx-\gamma y-isa}}{sK_-(s)}\,ds, \tag{3.9}$$

where $K_+(s)$, $K_-(s)$ were defined in (2.24) and the path of integration passes above the branch point at $s = -k$ but below both the pole at $s = 0$ and point $s = k$. Note that, by enforcing the edge condition (2.3a) both sides of the Wiener- Hopf equation can be shown to be identically zero.

Due to the symmetry of forcing and geometry it is easy to see that $\phi_1(x_1,y)$ must be

$$\phi_1(x_1,y) = \phi_2(x_2,y) \quad \text{and}$$
$$g_1(x_1,y) = g_2(x_2,y), \tag{3.10}$$

therefore

$$\phi_2(x,y) = -\frac{kK_+(0)}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-isa+isx-\gamma y}}{sK_-(s)}\,ds, \tag{3.11}$$

and the path of integration lies below s = 0.

As ka $\to \infty$, the equation in (3.6) can be assumed to reduce to

$$g_{ny} = -\phi_{my} \qquad x_n < 0, \quad y = 0, \tag{3.12}$$

(where the order of the discarded term will be found later), and when n = 1 this transforms to

$$G'_{1-} = -\bar{\Phi}'_{2-} \quad . \tag{3.13}$$

From (3.11), after changing s to -s, it is found that

$$\bar{\Phi}'_{2-}(s) = \frac{kK_+(0)}{2\pi i}\int_{-\infty}^{\infty} \frac{K_-(\zeta)e^{2i\zeta a}}{\zeta(\zeta-s)}\,d\zeta \tag{3.14}$$

where the ζ path lies above $\zeta = 0$ and s and below $\zeta = k$. The semi-infinite Wiener-Hopf problem for g_1 can now be solved giving

$$G_1(s,y) = \frac{kK_+(0)e^{-\gamma y}}{2\pi iK_-(s)}\int_{-\infty}^{\infty} \frac{K_+(\zeta)K_-(\zeta)e^{2i\zeta a}d\zeta}{(\zeta-s)\zeta}\,, \tag{3.15}$$

where the path of integration again passes above $\zeta = -k,0,s$ and below $\zeta = k$. The integral can now be approximated for large ka and so the interaction potentials are

$$g_1(x,y) = \frac{k^{\frac{1}{2}} e^{2ika-\pi i/4}}{16(\pi ka)^{3/2}} \int_{-\infty}^{\infty} \frac{e^{-isx-isa-\gamma y}}{(s-k)^{3/2}}\, ds + O(\frac{1}{(ka)^{5/2}}), \qquad (3.16)$$

and, from (3.10),

$$g_2(x,y) = \frac{k^{\frac{1}{2}} e^{2ika-\pi i/4}}{16(\pi ka)^{3/2}} \int_{-\infty}^{\infty} \frac{e^{isx-isa-\gamma y}}{(s-k)^{3/2}}\, ds + O(\frac{1}{(ka)^{5/2}}), \qquad (3.17)$$

where the path for both integrals lies above $s = -k$, and below $s = k$.

It is interesting to note that the leading order term in the interaction potential in this problem is $O((ka)^{-3/2})$ whilst in the previous case (which has the analogous problem of scattering by a finite hard strip) this potential is much larger $O((ka)^{-1/2})$. By inspection it is also seen that the discarded coupling term in (3.6) is of order $(ka)^{-3}$.

4. Double Gap Problem

The method can be extended to more complicated geometries and this section examines the problem of scattering by two gaps (each length 2a, separated by a distance 2a) in a hard baffle irradiated by plane waves incident normally onto the baffle. Thus the problem for the total potential ϕ_t can be defined by

$$\begin{aligned} &(\nabla^2 + k^2)\phi_t = 0 && \text{all } x,y, \\ &\phi_{ty} = 0 && 0 < |x| < a,\ 3a < |x|,\ y = 0, \\ &\phi_t\Big|_-^+ = 0 && a < |x| < 3a,\ y = 0. \end{aligned} \qquad (4.1)$$

To ensure uniqueness the usual condition, that no energy radiates from the edges, is employed as in previous sections (see (2.3a)).

In the previous two sections the scattered potential was chosen (by subtracting off suitable parts of the total potential) so that the forcing for the g potentials was small. This required in §2 that between the edges the boundary condition was $\phi_y = 0$, and in §3, $\phi = 0$. In this problem the same must hold between each of the four edges and so for convenience the scattered potential, ϕ, is defined by

$$\phi_t = \begin{cases} \phi + e^{-iky} + e^{+iky}, & 0 < |x| < a, \quad 3a < |x|, \quad y > 0, \\ \phi, & 0 < |x| < a, \quad 3a < |x|, \quad y < 0, \\ \phi + e^{-iky}, & a < |x| < 3a, \quad \text{all } y, \end{cases} \tag{4.2}$$

where e^{-iky} is the incident wave and ϕ is odd in y. The potential, ϕ, can now be split into the four half-plane problems plus the interaction terms, i.e.

$$\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4 + g_1 + g_2 + g_3 + g_4 , \tag{4.3}$$

where all terms satisfy the Helmholtz equation, and on $y = 0$

$$\phi_{ny} = 0, \quad x_n < 0; \quad \phi_n = 0, \quad x_n > 0, \tag{4.4}$$

$$g_{1y} = -\phi_{4y} - g_{4y}, \quad x_1 < 0; \quad g_1 = 0, \quad x_1 > 0, \tag{4.5}$$

$$g_{2y} = -\phi_{3y} - g_{3y}, \quad x_2 < 0; \quad g_2 = 0, \quad x_2 > 0, \tag{4.6}$$

$$g_{3y} = -\phi_{2y} - g_{2y}, \quad x_3 < 0; \quad g_3 = -\phi_4 - g_4, \quad x_3 > 0, \tag{4.7}$$

$$g_{4y} = -\phi_{1y} - g_{1y}, \quad x_4 < 0; \quad g_4 = -\phi_3 - g_3, \quad x_4 > 0, \tag{4.8}$$

where $x_1 = 3a+x, \quad x_2 = 3a-x, \quad x_3 = x-a, \quad x_4 = -x-a.$ (4.9)

Note that for ϕ_t to be continuous each ϕ_n must be discontinuous and it is this discontinuity which provides the forcing. The g_n problems, however, are continuous, and when $ka \to \infty$ it is again possible to simplify them.

Looking at the forcing terms (ϕ_{4y},ϕ_{3y}) for g_1,g_2 in (4.5), (4.6) it is known from §3 that these are $O((ka)^{-3/2})$ whilst g_3,g_4 have forcing terms (ϕ_4,ϕ_3) of $O((ka)^{-\frac{1}{2}})$. Therefore to leading order $O((ka)^{-\frac{1}{2}})$ g_1,g_2 can be neglected and the boundary conditions for g_3,g_4 become

$$\begin{aligned} g_{3y} &= 0 && x_3 < 0, \\ g_3 &= -\phi_4 && x_3 > 0, \\ g_{4y} &= 0 && x_4 < 0, \\ g_4 &= -\phi_3 && x_4 > 0. \end{aligned} \tag{4.10}$$

Note that each of the four leading order potentials and the two interaction potentials satisfies the edge condition

$$r|\nabla\phi_n| \longrightarrow 0 \text{ as } r = (x_n^2+y^2)^{\frac{1}{2}} \longrightarrow 0 \tag{4.10a}$$

(i) Problem for ϕ_1

From the symmetry of forcing and geometry $\phi_1(x_1,y) = \phi_n(x_n,y)$ and so only the ϕ_1 problem will be examined. Using the same notation as in previous sections the transformed boundary conditions (from (4.4)) give

$$\Phi_1' = 0, \qquad \Phi_{1+} = 0, \tag{4.11}$$

and the governing equation yields

$$\Phi(s,y) = A(s)e^{-\gamma y} + \frac{ie^{iky}}{s}, \tag{4.12}$$

where the last term is included because of the discontinuity in potential and the pole ($s = 0$) lies slightly above the real axis. The Wiener-Hopf equation becomes

$$K_-(s)\Phi_{1-}(s) - \frac{iK_-(s)}{s} + \frac{kK_+(0)}{s} = -K_+(s)\Phi_{1+}' - \frac{k}{s}(K_+(s)-K_+(0)) \equiv J(s) \tag{4.13}$$

and by examining the behaviour of both sides as $|s| \to \infty$ in the respective half planes it is found that $J(s) \equiv 0$.

Thus substituting $\Phi_1(s,0)$ into (4.12) and taking the inverse transform gives

$$\phi_n(x_n,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\frac{ie^{iky}}{s} - \frac{kK_+(0)e^{-\gamma y}}{sK_-(s)}\}\ e^{-isx_n}\,ds\ , \qquad (4.14)$$

where the integration path lies above the branch cut from s = -k to $-\infty$, and below s = 0 and the cut from s = k to $+\infty$.

(ii) Problem for g_3

As before the symmetry of the problem implies that $g_3(x_3,y) = g_4(x_4,y)$ and also it is clear that

$$\Phi_4(-s,y) = e^{2isa}\bar{\Phi}_4(s,y) \qquad (4.15)$$

where $\bar{\Phi}_4(s,y) = \int_{-\infty}^{\infty}\phi_4(x_3,y)e^{isx_3}dx_3$ etc.

Transforming the boundary conditions in (4.10) gives

$$G'_{3-} = 0,\quad G_{3+} = -\bar{\Phi}_{4+} = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\{\frac{i}{\zeta} - \frac{kK_+(0)K_+(\zeta)}{\zeta}\}\frac{e^{-2i\zeta a}d\zeta}{(\zeta-s)}\ , \qquad (4.16)$$

where ζ passes above $\zeta = -k$ and below $\zeta = s,k$. Again forming the Wiener-Hopf equation, and by analytic continuation arguments showing that each side is identically zero (using the edge condition (4.10a)), the G'_{3+} potential is

$$G'_{3+}(s) = \frac{1}{K_+(s)2\pi i}\int_{-\infty}^{\infty}\frac{K_-(t)\bar{\Phi}_{4+}(t)dt}{(t-s)} \qquad (4.17)$$

$$= -\frac{1}{2\pi iK_+(s)}\int_{-\infty}^{\infty}\{\frac{i}{\zeta} - \frac{kK_+(0)K_+(\zeta)}{\zeta}\}\frac{K_-(\zeta)e^{-2i\zeta a}}{(\zeta-s)}\ d\zeta\ ,$$

the integration path being the same as in (4.16).

For large ka this becomes

$$G'_{3+}(s) = \frac{e^{\frac{i\pi}{4}+2ika}}{2\sqrt{\pi ka}\ k^{\frac{1}{2}}(s+k)^{\frac{1}{2}}} + O(\frac{1}{(ka)^{3/2}}) \qquad (4.18)$$

and so

$$g_n(s,y) = -\frac{e^{\frac{i\pi}{4}+2ika}}{4\pi\sqrt{\pi ka}\,k^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{e^{-isx_n-\gamma y}}{(s+k)(s-k)^{\frac{1}{2}}}\,ds + O(\frac{1}{(ka)^{3/2}}). \tag{4.19}$$

Thus the problem has now been solved in the large gap limit with terms of order $(ka)^{-1}$ being neglected, i.e.

$$\phi(x,y) = \sum_{n=1}^{4} \phi_n(x,y) + g_3 + g_4 + O(\frac{1}{ka}) \ . \tag{4.20}$$

5. Resonance Problem

As an example involving resonance take the problem of a membrane of length 2a, set in a hard baffle, with fluid in the region $y \geq 0$, and irradiated by incident forcing (chosen to preserve symmetry in x) of the form

$$\phi_i = e^{-iky\,\sin\theta}\cos(kx\cos\theta), \tag{5.1}$$

where θ is the angle between the incident wave and the line $y = 0$. The problem is defined as

$$(\nabla^2 + k^2)\phi_t = 0 \qquad \text{all } x,y > 0, \tag{5.2}$$

$$\phi_{ty} = 0 \qquad |x| \geq a,\ y = 0, \tag{5.3}$$

$$(\partial^2/\partial x^2 + \mu^2)\phi_{ty} - \alpha\phi_t = 0 \qquad |x| < a,\ y = 0, \tag{5.4}$$

where μ is the membrane wavenumber in a vacuum, and α is a fluid loading parameter, assumed small in this case, and the derivation of (5.4) can be found in Leppington (1976).

Now choose ϕ so that there is no forcing term in the membrane equation, i.e.

$$\phi_t = \phi_i + Ae^{iky\,\sin\theta}\cos(kx\cos\theta) + \phi \ , \tag{5.5}$$

where $$A = \frac{ik \sin\theta(k^2\cos^2\theta - \mu^2) - \alpha}{ik \sin\theta(k^2\cos^2\theta - \mu^2) + \alpha} \tag{5.6}$$

which give the boundary conditions for ϕ

$$\phi_y \sim \frac{2\alpha}{(k_0^2-\mu^2)}\cos(k_0 x), \qquad \alpha \to 0, \quad |x| > a, \; y = 0, \tag{5.7}$$

$$(\partial^2/\partial x^2 + \mu^2)\phi_y - \alpha\phi = 0, \quad |x| < a, \quad y = 0, \tag{5.8}$$

where $k_0 = k\cos\theta$. Also a radiation condition is required of ϕ such that there are only outgoing waves at infinity. Repeating the procedure of defining half plane potentials when

$$\phi = \phi_1 + \phi_2 + g_1 + g_2,$$

the boundary problems become

$$\phi_{ny} \sim \frac{2\alpha}{(k_0^2-\mu^2)}\cos(k_0(x_n-a)) \qquad x_n < 0, \; y = 0, \tag{5.9}$$

$$(\partial^2/\partial x^2 + \mu^2)\phi_{ny} - \alpha\phi_n = 0 \qquad x_n > 0, \; y = 0, \tag{5.10}$$

$$g_{ny} = -\phi_{my} - g_{my} \qquad x_n < 0, \; y = 0, \tag{5.11}$$

$$(\partial^2/\partial x^2 + \mu^2)g_{ny} - \alpha g_n = 0 \qquad x_n > 0, \; y = 0, \tag{5.12}$$

where $n,m = 1,2, \quad n \neq m, \; x_1 = a + x, \; x_2 = a - x.$

The difficulty with plate or membrane problems is that the surfaces can support travelling waves and so in (5.11) ϕ_{my} is not necessarily small as previously found but, as will be shown in this section, has an O(1) wave term. Thus allow

$$\phi_n = \begin{cases} \psi_n + Pe^{itx_n - \gamma(t)y} & x_n > 0, \\ \psi_n & x_n < 0, \end{cases} \tag{5.13}$$

$$g_n = \begin{cases} h_n + pe^{itx_n - \gamma(t)y} & x_n > 0, \\ h_n & x_n < 0, \end{cases} \tag{5.14}$$

where t satisfies $\gamma(t^2-\mu^2) = \alpha$, $\gamma = \sqrt{t^2-k^2}$. P must be the same amplitude coefficient for both ϕ_1, ϕ_2 because of the symmetry of the geometry and forcing and together with p must be determined. If, as $ka \to \infty$, only a leading order estimate is required for the potential then ψ_n, h_n can be neglected when (5.13), (5.14) are substituted into (5.11) as it is known that they will be smaller than O(1). Thus (5.11) approximates to

$$g_{ny} = \gamma(t)(P + p)e^{2ita - itx_n}, \qquad x_n < 0, \quad y = 0; \tag{5.15}$$

hence if the g_n potential is solved, the outgoing wave will be of the form

$$C(P + p)e^{itx_n - \gamma y} \tag{5.16}$$

and so from (5.14)

$$C(P + p) = p, \text{ giving}$$

$$P + p = \frac{P}{1-C}, \tag{5.17}$$

where P and C are determined by solving the ϕ,g problems. Note that the denominator of (5.17) can be small if $C \simeq 1$ and this is the definition of resonance for the membrane.

Performing the Wiener-Hopf analysis it is found that

$$\phi_n(x_n,y) = \frac{-\alpha}{2\pi i(k_0^2-\mu^2)} \int_{-\infty}^{\infty} \{\frac{e^{-ik_0a}(\mu-k_0)}{(s+k_0)} + \frac{e^{ik_0a}(\mu+k_0)}{(s-k_0)}\} \frac{e^{-isx_n-\gamma y}}{\gamma(s+\mu)}\, ds \tag{5.18}$$

where the integration path lies below $s = -k_0$, k_0, k and above $s = -k, -\mu$. From (5.18) the coefficient multiplying the outgoing wave, P, can be determined, thus

$$P = \frac{2\alpha \cos k_0 a}{(\mu^2-k_0^2)(\mu^2-k^2)^{1/2}} + O(\alpha^2). \tag{5.19}$$

Finally, the g potentials can be written as

$$g_n(x_n,y) = -\frac{(\mu^2-k^2)^{\frac{1}{2}}}{\pi i}\mu(P+p)e^{2i\mu a} \int_{-\infty}^{\infty} \frac{e^{-isx_n-\gamma y}\, ds}{(s^2-\mu^2)\sqrt{s^2-k^2}} + O(\alpha), \tag{5.20}$$

where the path is as in (5.18), and, from the reflected wave, after considerable algebra the constant P+p is found to be

$$P + p = -\frac{Pe^{-ia\mu}}{2(-\cos \mu a + \alpha I)} + O(\alpha^2), \tag{5.21}$$

where $(2\pi \operatorname{cosec} \mu a)I = \frac{2\mu^2-k^2}{\mu^2(\mu^2-k^2)^{3/2}} \cosh^{-1}(\frac{\mu}{k})$

$$- \frac{1}{\mu(\mu^2-k^2)} + \frac{\pi i(2\mu^2-k^2)}{2\mu^2(\mu^2-k^2)^{3/2}} + \frac{\pi a}{\mu\sqrt{\mu^2-k^2}} \quad . \tag{5.22}$$

The $O(\alpha)$ term in the denominator of (5.21) has been kept because at a resonance $\cos \mu a \equiv 0$ and so

$$P + p \sim -\frac{Pe^{-ia\mu}}{2\alpha I} \quad , \tag{5.23}$$

which is an order of magnitude in α larger than away from resonance. Note that implicit in this analysis is the requirement that $\alpha >> (ka)^{-3/2}$ although both α,ka are assumed small.

By deforming the path of integration in (5.18), (5.20) into the hyperbolic path

$$s = -k \cos(\theta + i\tau) \qquad -\infty < \tau < \infty$$

a far field estimate can be obtained in the form

$$\phi \sim -(2\pi kr)^{-1/2} F(\theta) e^{ikr+\pi i/4}, \qquad r \to \infty, \quad r >> a, \tag{5.24}$$

where $$\phi(x,y) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{v(s)e^{-\gamma y - isx}}{\gamma}\, ds, \tag{5.25}$$

$$F(\theta) = v(-k \cos\theta) = v(-k_1), \tag{5.26}$$

and $$r = (x^2+y^2)^{1/2}, \quad \theta = \tan^{-1}(y/x). \tag{5.27}$$

The total far field amplitude function $F(\theta)$ is given by

$$F(\theta) = F_\phi(\theta) + F_g(\theta) \tag{5.28}$$

where, from (5.18)-(5.21), after rearranging,

$$F_\phi(\theta) = \frac{4i\alpha}{(\mu^2-k_0^2)(\mu^2-k_1^2)} \{ \frac{i}{(k_1^2-k_0^2)} (\sin(k_0 a) \cos(k_1 a)(\mu^2-k_1^2)k_0$$

$$-\cos(k_0 a) \sin(k_1 a)(\mu^2-k_0^2)k_1) - \mu \cos(k_0 a) \cos(k_1 a)\} + O(\alpha^2), \tag{5.29}$$

$$F_g(\theta) = -\frac{4\mu i\alpha \cos(k_0 a) e^{i\mu a} \cos(k_1 a)}{(\mu^2-k_1^2)(\mu^2-k_0^2)(-\cos(\mu a) + \alpha I)} + O(\alpha^2). \tag{5.30}$$

Near a resonance the dominant contribution to $F(\theta)$ will come from (5.30) alone, but if μ is near k_1 or k_0 then part of (5.29) must be included to cancel the zero in the denominator of F_g. The far field amplitude function valid near a resonance, and for all μ, is therefore found, after discarding the smaller terms, to be

$$F(\theta) \sim \frac{4\alpha\mu \sin(\mu a)}{(-\cos(\mu a)+\alpha I)} \frac{(\cos(\mu a)-\cos(k_1 a))}{(\mu^2 - k_1^2)} \frac{(\cos(\mu a)-\cos(k_0 a))}{(\mu^2 - k_0^2)} \tag{5.31}$$

where I is given by (5.22). This result agrees with Leppington (1976) who uses an entirely different method which is more direct, but does not have the advantage of applicability to such a wide range of problems. An example of this is the same problem of a finite membrane (or plate) in the limit $\alpha \to \infty$ which has been solved using matched asymptotic expansions (Abrahams (1981a)). The present method could easily be applied to this problem (in the case ka large) whilst the method by Leppington could not be used.

6. Concluding Remarks

(i) Higher Order Terms

Further terms in the asymptotic expansion for the interaction potentials could be made by splitting this potential into several parts. Take, for instance, the problem in §2 and let $g_n = j_n + q_n$ where from (2.8) choose

$$j_n = -\phi_n \qquad x_n \begin{cases} <0, & n = 1, \\ >0, & n = 2, \end{cases} \qquad y = 0, \tag{6.1}$$

and

$$q_n = -j_m - q_m \qquad x \begin{cases} <0, & n = 1, \\ >0, & n = 2, \end{cases} \qquad y = 0 \ . \tag{6.2}$$

From the analysis it was shown that

$$j_n = \frac{1}{(ka)^{1/2}} \; j_n^{(0)} + O(\frac{1}{(ka)^{3/2}}) \tag{6.3}$$

where $j_n^{(0)}$ is an O(1) function of x,y, and it follows that solving the q_n problem (after discarding the coupling term, q_m) gives

$$q_n = \frac{1}{ka} \, q_n^{(0)} + O(\frac{1}{(ka)^{3/2}}) \quad . \tag{6.4}$$

The interaction potential, g_n, is determined to $O((ka)^{-1})$ once $q_n^{(0)}$ has been found, thus

$$g_n = \frac{j_n^{(0)}}{(ka)^{1/2}} + \frac{q_n^{(0)}}{ka} + O(\frac{1}{(ka)^{3/2}}) \; , \tag{6.5}$$

whereas before g_n was only determined to $O((ka)^{-1/2})$. This procedure could be repeated to determine g_n to any order in its asymptotic expansion in powers of $(ka)^{-1/2}$.

(ii) Further Studies

This paper has presented a technique which is useful in problems where a length scale of the geometry is large. Each section has shown how the method can be applied to a particular class of problems. Further studies using an analysis as in §§3,4 could include problems with arrays of slits (finite or infinite) or semi-infinite screens arranged at arbitrary angles to each other.

Resonance problems with such geometries as a finite wave-guide (Jones (1952)), or elastic plate set in a duct (Abrahams (1981b)), could all be solved in the same way as §5 in the appropriate short wave limit.

The author wishes to thank Dr. F.G. Leppington for much advice and assistance, and the Science Research Council and A.U.W.E. Portland for the financial support of a C.A.S.E. studentship.

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Chapter 7

The Modal Method

This chapter shows how the method of modal expansions can be used in the solution of boundary value problems. Two examples are presented in which the plate deflection is written as an infinite sine series, and the problems reduce to determining the coefficients in this series.

Firstly the problem of a finite thin elastic plate set in an infinite hard screen is analysed in the light loading limit and the results obtained are analogous to those found by Leppington (1976); although the modal method offers a saving in algebra. Few methods are available to solve three dimensional problems and so the second model in this chapter is presented to show how the modal method is used to determine divergence instability of flow over a rectangular elastic plate.

(i) Two-dimensional problem

Returning to the first model, and using the derivation in the introduction of chapter 3, the problem can be written as

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2 + k^2)\phi = 0 \qquad y > 0, \qquad (1)$$

$$\phi_y = 0 \qquad y = 0, \ |x| > a, \qquad (2)$$

$$\phi_y = \phi_{yxx} = 0 \qquad y = 0, \ x = \pm a, \qquad (3)$$

$$(\partial^4/\partial x^4 - \mu^4)\phi_y + \alpha\phi = 0 \qquad y = 0, \ |x| < a, \qquad (4)$$

$$\mu^4 = mh\omega^2/D, \quad \alpha = \rho\omega^2/D. \qquad (5)$$

Note that the plate edges are simply supported and there is no fluid below the plate.

Assuming forcing by an incident plane wave, and, after subtracting off the incoming and reflected waves (ϕ_{inc} and ϕ_{ref}), the scattered potential now has an inhomogeneous plate equation of the form

$$(\partial^4/\partial x^4 - \mu^4)\phi_y + \alpha\phi = A(x) \qquad y = 0, \quad |x| < a, \tag{6}$$

where $A(x) = -\alpha(\phi_{inc} + \phi_{ref})|_{y=0}$. It is easily shown that the potential is related to the plate deflection by

$$\phi(x,y) = \int_{-a}^{a} G(x,y;x_1,0)\phi_y(x_1,0)dx_1, \tag{7}$$

where G is a Green's function for the problem with zero normal derivative on y = 0. By taking a Fourier transform in the x direction it is found that G, on y = 0, is given by

$$G(x,0;x_1,0) = \frac{-i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{is(x_1-x)}}{(k^2-s^2)^{\frac{1}{2}}}\, ds. \tag{8}$$

Note that the integration path passes just above the cut (in the complex s plane) from s = -k to $-\infty$, and below the branch cut s = k to ∞. The branch of the square root chosen is such that, when s = 0,

$$(s^2-k^2)^{\frac{1}{2}} = -ik. \tag{9}$$

Substituting the integral form of the potential (7) into equation (6) results in an integro-differential equation in $\phi_y(x,0)$ which must be solved. It is useful to pose an expansion for the deflection such that each term satisfies the edge conditions (3), and one such expansion is

$$\phi_y(x,0) = \sum_{n=1}^{\infty} b_n \sin[n\pi(x+a)/2a], \qquad |x| < a. \tag{10}$$

The problem now reduces to calculating the coefficients, b_n, from the following equation

$$\sum_{n=1}^{\infty} b_n\{\sin[n\pi(x+a)/2a]((n\pi/2a)^4 - \mu^4)$$

$$- \frac{\alpha i}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-isx}}{(k^2-s^2)^{\frac{1}{2}}}\int_{-a}^{a} e^{isx_1}\sin[n\pi(x_1+a)/2a]dx_1 ds\} \tag{11}$$

$$= A(x), \qquad |x| < a.$$

In the light loading limit, $\alpha \to 0$, the second term on the left hand side of equation (11) can be neglected. Therefore, by multiplying both sides by a sine and integrating over the range $-a < x < a$, the coefficients are simply given by

$$b_n = \int_{-a}^{a} A(x)\sin(n\pi(x+a)/2a)dx a^{-1}((n\pi/2a)^4-\mu^4)^{-1} + O(\alpha). \tag{12}$$

This is a valid asymptotic estimate away from a plate resonance, but near a resonance, i.e. when

$$(n\pi/2a)^4 - \mu^4 \to 0, \tag{13}$$

one of the coefficients in (12) becomes infinite. Thus part of the discarded term in (11) must be kept to allow the resonant coefficient to remain finite. The approximate form of equation (11), valid near to the pth resonance is therefore assumed to be

$$\sum_{n=1}^{\infty} b_n\{((n\pi/2a)^4 - \mu^4)\sin(n\pi(x+a)/2a)\}$$

$$- \frac{\alpha i b_p}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-isx}}{(k^2-s^2)^{\frac{1}{2}}}\int_{-a}^{a} e^{isx_1}\sin(p\pi(x_1+a)/2a)dx_1 ds = A(x). \tag{14}$$

This leads to modified forms for the coefficients, b_n, valid near to the pth resonance, and written as

$$b_p = \int_{-a}^{a} A(x)\sin(p\pi(x+a)/2a)dx/\lambda \tag{15}$$

where

$$\lambda = a((p\pi/2a)^4-\mu^4) - \frac{2\alpha i(p\pi)^2}{\pi(2a)^2}\int_{-\infty}^{\infty} ds\ \frac{[\sin^4(p\pi/2)\cos^2 sa+\cos^4(p\pi/2)\sin^2 sa]}{(k^2-s^2)^{\frac{1}{2}}(s^2-(n\pi/2a)^2)^2}$$

and

$$b_{\substack{n\\ n\neq p}} = \{\int_{-a}^{a} A(x)\sin(n\pi(x+a)/2a)dx \tag{16}$$

$$-\frac{b_p\alpha inp\pi}{2a^2}\int_{-\infty}^{\infty}\frac{(\sin^2(p\pi/2)\cos sa-i\cos^2(p\pi/2)\sin sa)(\sin^2(n\pi/2)\cos sa+i\cos^2(n\pi/2)\sin sa)}{(k^2-s^2)^{\frac{1}{2}}(s^2-(p\pi/2a)^2)(s^2-(n\pi/2a)^2)}$$

$$ds\} \times a^{-1}((n\pi/2a)^4 - \mu^4)^{-1}.$$

Note that the denominator in (15) cannot become zero and so at a resonance b_p tends to order α^{-1}, which is large but finite as expected. Near a resonance all other coefficients in the expansion are modified but remain of order unity. The potential can easily be found by substituting the expansion for the plate deflection (10), with coefficients given in (15), (16), into equation (7).

(ii) Three-dimensional problem

The next example using the modal method seeks to find the condition for divergence instability of flow over a rectangular plate. This work is presented to show how the technique may be applied to three dimensional problems, although Galerkin's approximation is necessary in the analysis.

Assuming uniform subsonic flow, with velocity U, in the x direction, the problem (from Ellen (1972)) may be written as

$$((1-M^2)\partial^2/\partial^2 x + \partial^2/\partial y^2 + \partial^2/\partial z^2 + 2ikM\partial/\partial x + k^2)\phi = 0, \quad y > 0, \tag{17}$$

$$\eta = 0, \qquad y = 0, \text{ all } x,z \text{ except } |x| < a, \quad |z| < b, \tag{18}$$

$$\eta = \eta_{xx} = 0, \qquad y = 0, \; x = \pm a, \; |z| < b; \quad z = \pm b, \; |x| < a, \tag{19}$$

$$(\partial^4/\partial z^4 + 2\partial^4/\partial x^2\partial z^2 + \partial^4/\partial x^4 - \mu^4)\eta - \frac{\alpha i}{\omega}\left(1 + \frac{iM}{k}\,\partial/\partial x\right))\phi = 0, \tag{20}$$

$$y = 0, \; |x| < a, \quad |z| < b.$$

The parameters k, M, α, ω etc. are defined in §4 and the deflection η is now related to the acoustic potential by the kinematic condition

$$\phi_y = -i\omega\eta + U\eta_x \qquad y = 0, \; |x| < a, \quad |z| < b. \tag{21}$$

The problem has now been completely defined and eigensolutions are sought which, as discussed in the final part of thesis chapter 4, correspond to conditions of plate resonance.

As in the previous example, the deflection is represented by an infinite series, each term of which satisfies the pin-jointed edge conditions. Hence

$$\eta(x,z) = \sum_{n,m=1}^{\infty} d_{nm} \sin(n\pi(x+a)/2a)\sin(m\pi(z+b)/2b) \tag{22}$$

for $|x| < a$, $|z| < b$ and Σ represents a double sum. In the previous example a Green's function approach was used to relate the acoustic potential to its derivative on the plate. It is more convenient in this problem to use Fourier transforms in both x and z and this leads to the condition

$$\phi(x,z,y) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Phi_y(s,t,0)}{\gamma}\exp(-isx-itz-\gamma_y)dsdt, \qquad (23)$$

where $\gamma(s,t) = (s^2(1-M^2)+t^2-2Msk-k^2)^{\frac{1}{2}}$ and the branch of γ is chosen such that $\gamma(0,0) = -ik$. The transform variable is, as usual, defined as

$$\Phi_y(s,t,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_y(x,z,y)\exp(isx+itz)dxdz\ , \qquad (24)$$

and this can be expressed as an infinite sum of modes by substituting equation (22) into (21) and then performing the double integration. Assuming that differentiation under the integral can be performed, and changing order of the summation and integration, the plate equation (20) is written in modal form as

$$\sum_{n,m=1}^{\infty} d_{nm}\{p_{nm}\sin(n\pi(x+a)/2a)\sin(m\pi(z+b)/2b) \qquad (25)$$
$$-\frac{\alpha_{mn}}{4ab}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(1+ms/k)^2 r(n;sa)r(m;tb)\exp(-isx-itz)dsdt}{\gamma(s^2-(n\pi/2a)^2)(t^2-(m\pi/2b)^2)}\} = 0\ ,$$

where

$$p_{nm} = (n\pi/2a)^4 + (m\pi/2b)^4 + 2(nm\pi^2/(4ab))^2 - \mu^4,$$

and

$$r(\eta;\zeta) = i\cos^2(\eta\pi/2)\sin\zeta - \sin^2(\eta\pi/2)\cos\zeta\ .$$

It is clear that all coefficients, d_{nm}, must be identically zero unless, for a particular mode, the expression in brackets in (2.5) is zero. The Galerkin approximation is now employed to obtain a resonance condition which is independent of the x,z variables. This is achieved by multiplying through by the product

$$\sin(p\pi(x+a)/2a)\sin(q\pi(z+b)/2b), \tag{26}$$

and then integrating over the surface of the plate. This leads to the condition for divergence instability of the n,m mode

$$(nA + m^2/(nA))^2$$

$$- \frac{\rho U^2 m^2 b^3}{\pi^2 D} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{s^2(1-(-1)^n\cos 2s)(1-(-1)^m\cos 2t)\,ds\,dt}{((1-m^2)A^2s^2+t^2)^{\frac{1}{2}}(s^2-(n\pi/2)^2)^2(t^2-(m\pi/2)^2)^2} \tag{27}$$

$$= 0,$$

where A = b/a, ρ, D given in chapter 4, and for n,m = 1 this gives the maximum flow velocity for which the elastic plate is stable. By using the identity

$$K_0(pv) = \int_0^{\infty} dt\ \cos(pvt)/(t^2+1)^{\frac{1}{2}}\ , \tag{28}$$

where K_0 is a modified Bessell function of order zero, this expression can be rewritten as

$$(nA + m^2/(nA))^2$$

$$- \frac{16\rho U^2 b^3}{\pi^3 B} \int_0^1 K_0(bvn\pi)((1-v)\cos(vm\pi) + \pi^{-1}\sin(vm\pi))dv = 0. \tag{29}$$

This is identical to the result predicted by Ellen (1972) also using a Galerkin approximation but it would obviously be useful in further studies to compare this result with that found by other methods.

8. Closing Chapter

(i) Comparison of Results

In the first part of this final thesis chapter the previous results are used to estimate the sound field inside a cavity. An elastic plate is set in one side of the cavity and the other side of the plate lies on an infinite baffle (figure 1). Order of magnitude estimates, in various limits, are derived for the acoustic potential when forcing is supplied by a source near to the plate, and this is then repeated for an incident plane wave.

To reduce algebra the reciprocity principle is employed which states that irrespective of the boundary geometry in the problem the sound field at any point ((1) say) due to a source at another position (2) is identical to the field at (2) when the source is at (1). Hence a monopole source is introduced inside the cavity (of O(1)) and the order of the acoustic potential at a position near the plate and in the far field is determined. This corresponds directly to an estimate of the sound field inside the cavity due to both forcing by an O(1) source near the plate and in the far field.

Assuming a compressible, inviscid fluid and simple harmonic time dependence, a velocity potential can be defined as

$$\mathrm{Re}\{\phi(x,y)e^{-i\omega t}\} \tag{1.1}$$

where (x,y) are cartesian coordinates of the two dimensional problem, ϕ satisfies the helmholtz equation, and ω is the angular frequency of the source. The time factor $\exp\{-\omega t\}$ will henceforth be omitted for brevity. The acoustic potential is related to the pressure by

$$p = i\omega\rho_0\phi(x,y) , \tag{1.2}$$

and the deflection of the plate is

$$\eta = (\tfrac{i}{\omega})\,\phi_y(x,0), \qquad y = 0, \tag{1.3}$$

where ρ_0 is the mean density of the fluid. From Leppington (1[illegible] the equation for an elastic plate can be written as

$$(\partial^4/\partial x^4 - \mu^4)\phi_y + \alpha\phi\big|_-^+ = 0, \qquad |x| < a, \quad y = 0, \tag{1.4}$$

where $\mu^4 = mh\omega^2/D$, the fluid loading parameter is $\alpha = \rho_0\omega^2/D$, D the bending stiffness, h the plate thickness and m the mass per unit volume. The term $\phi\big|_-^+$ is the discontinuity in the potential across the plate, and 2a is the length of the plate. Boundary conditions need to be imposed at the ends of the plate to specify the method of clamping. In this paper the plate is assumed to be rigidly held, i.e. $\phi_y = \phi_{yx} = 0$ are the relevant conditions.

The boundary value problem can now be expressed in terms of ϕ, thus (from Fig. 1)

$$(\nabla^2 + k^2)\phi = 0 \qquad \text{all } x,y, \tag{1.5}$$

$$(\partial^4/\partial x^4 - \mu^4)\phi_y + \alpha\phi\big|_-^+ = 0 \qquad |x| < a, \ y = 0, \tag{1.6}$$

$$\phi_y = 0 \qquad |x| > a, \ y = 0, \tag{1.7}$$

$$\phi_y = 0 \qquad |x| < \ell, \ y = -h, \tag{1.8}$$

$$\phi_x = 0 \qquad x = \pm\ell, \ 0 > y > -h, \tag{1.9}$$

$$\phi_y = \phi_{yx} = 0 \qquad x = \pm a, \ y = 0 \ . \tag{1.10}$$

2. Large, heavily loaded plate problem

An analysis has been made for the finite plate problem with general external geometry in the heavy loading limit ( ). A matching technique was used whereby it was shown that away from the elastic plate (and also away from a plate resonance) the leading order potential satisfied the condition

$$\phi\big|_-^+ = 0 , \qquad |x| < a, \quad y = 0, \tag{2.1}$$

which replaces the plate equation. Thus this "outer" problem is now significantly simpler to analyse. Near a resonance it was also shown that in the far field, eigensolutions, with singularities at the plate edges, change their order of magnitude and become the same order as the outer potential, i.e. resonance occurs when

$$2^{1/5}\,\alpha^{1/5}a = \frac{1}{8}\pi + \frac{1}{2}n\pi, \qquad (n = 0,1,2,3,\ldots) \qquad (2.2)$$

and

$$\text{eigenpotential} = \begin{cases} O(1) & \text{near a resonance} \\ O(\varepsilon) & \text{away from a resonance,} \end{cases} \qquad (2.3)$$

where $\varepsilon = k/\alpha^{1/5}$ and $\varepsilon \to 0$ in the heavy loading limit. Similarly, close to the plate the total potential is made up of the outer solution eigensolutions, and also travelling waves present upon the plate, where these wave solutions are shown in reference (2) to be

$$\text{travelling waves} = \begin{cases} O(\varepsilon^{1/2}) & \text{away from resonance,} \\ O(\varepsilon^{-1/2}) & \text{at a resonance.} \end{cases} \qquad (2.4)$$

Therefore, near a resonance, the dominant contribution to the sound field near the plate is due to the travelling wave solutions.

Returning to the outer problem, where the plate has been removed, it is now interesting to examine the sound fields at points (1) and (2) (far field), in figure 1, due to a source at the back of the cavity and away from the edges of the plate (3). The assumption is now made that the effect of image sources is small as the cavity side walls are very far apart and therefore in this large gap limit there will be no cavity resonances. This would appear to be valid from physical argument that the large gap allows too much acoustic energy to escape from the cavity

for the potential inside to be large near a resonance. Thus an O(1) source at (3) will produce an O(1) outer-potential field at (1) because the cavity depth is h (order unity). In the far field, (2), however, the potential will be the leading order term in the expansion for the field generated by a two-dimensional source in free space, i.e.

$$\lim_{kr\to\infty} H_0^{(1)}(kr) \sim \frac{e^{ikr}}{(kr)^{1/2}} F(\theta) = O(\frac{1}{(kr)^{1/2}}) \ , \tag{2.5}$$

where $H_0^{(1)}$ is the Hankel function of the first kind, of order zero. Note that r is the distance between the point (2) and the centre of the plate, $r >> a$, and $F(\theta)$ is an amplitude coefficient dependent on $\theta = \tan^{-1}(y/x)$. Similarly, as the distance between the source at (3) and a plate edge is large, it is again possible to assume, neglecting wall effects, that the sound field at the edge is

$$O(\frac{1}{(ka)^{1/2}}) \ ,$$

and hence will be the order of the forcing for the eigenpotential and travelling wave terms. It can now be observed that

at (1)

$$\phi_{total} = O(1) + O(f(\varepsilon)(ka)^{-1/2}) \ , \tag{2.6}$$

at (2)

$$\phi_{total} = O((kr)^{-1/2}) + O(\varepsilon^{1/2} f(\varepsilon)(ka)^{-1/2}(kr)^{-1/2}), \tag{2.7}$$

where

$$f(\varepsilon) = \begin{cases} O(\varepsilon^{1/2}) & \text{away from a plate resonance,} \\ O(\varepsilon^{-1/2}) & \text{at a plate resonance.} \end{cases} \tag{2.8}$$

It can be seen that if $\varepsilon << (ka)^{-1}$ then near a resonance the second term in (2.6) is dominant and so the potential at (1) is $O(\varepsilon^{-1/2})$ larger than at (2).

By reciprocity it is now clear that an O(1) monopole source at (1) and (2) will generate at (3) the sound fields given by (2.6) and (2.7) respectively.

3. Small, heavily loaded plate

The procedure in chapter 2 is now repeated in the limit $ka = \beta \to 0$, and it is assumed that $\varepsilon << \beta << 1$. In this problem the presence of cavity resonances does alter the sound field and therefore must be included in the analysis. As before, the total potential is the sum of the outer potential plus the contributions from the travelling waves on the plate and also the eigensolutions with singular behaviour at the plate edges. From the analysis in Appendix 1, and using the reciprocity principle it can be stated that away from a cavity resonance the potential inside the cavity due to a monopole source on the plate and in the far field is given respectively by

$$\phi_{total} = O(1) + O\left(\frac{f(\varepsilon)}{\ln \beta \beta^{1/2}}\right), \tag{3.1}$$

$$\phi_{total} = O\left(\frac{(kr)^{-1/2}}{\ln \beta}\right) + O\left(\frac{\varepsilon^{1/2} f(\varepsilon)}{\ln \beta (\beta kr)^{1/2}}\right), \tag{3.2}$$

where $f(\varepsilon)$, r were defined in the previous chapter. Near a cavity resonance the analogous results to (3.1), (3.2) become (from A19, A18)

$$\phi_{total} = O(\ln \beta) + O(f(\varepsilon)\beta^{-1/2}) \ , \tag{3.3}$$

when the source is on the plate, and

$$\phi_{total} = O((kr)^{-1/2}) + O(\frac{\varepsilon^{1/2} f(\varepsilon)}{\beta^{1/2}(kr)^{1/2}}) \ , \tag{3.4}$$

if the source is in the far field.

4. Light Loading Limit ($\alpha \to 0$)

In the previous two chapters the difficulty in the heavy loading limit was that the approximation made in the far field broke down near a plate resonance and was also invalid near the plate. In the light loading limit the problem becomes easier to analyse as to leading order the plate acts as if it were in a vacuum and therefore the plate equation can be uncoupled. This approximation breaks down at both plate and cavity resonances and a better estimate must be made which can include the finite radiation damping in the system. An analysis has been made of this type of problem ([illegible]) and the method is worked through briefly in Appendix 2 when cavity resonances (i.e. $R_{pq} = k^2-(p\pi/2\ell)^2 - (q\pi/h)^2 \Rightarrow 0$) are present. Summarising the results after using the reciprocity principle, it is found that (from B9, B10)

$$\phi_{total} = \begin{cases} O(1) & \text{away from a cavity resonance,} \\ O(k^5/\alpha) & \text{at a cavity resonance,} \end{cases} \tag{4.1}$$

for the total potential at (3) (fig. 1) due to a source (O(1)) at (1); similarly if the source were at (2) then the potential at (3) would be (from B11, B12)

$$\phi_{total} = \begin{cases} O(\alpha/(k^5(kr)^{1/2})) & \text{away from a cavity resonance,} \\ O((kr)^{-1/2}) & \text{at a cavity resonance.} \end{cases} \tag{4.2}$$

A similar result has been found by Leppington (1) in the light loading limit when a plate resonance is approached. For an elastic plate, a resonance is defined to occur when the condition

$$\sin \mu a \cosh \mu a + \cos \mu a \sinh \mu a = 0 \ . \tag{4.3}$$

This corresponds to

$$\mu a = \frac{3\pi}{4} + \eta(n) + n\pi, \qquad n = 0,1,2,\ldots ,$$

where $0 < \eta < \frac{\pi}{4}$ and $\eta(n) \to 0$ as $n \to \infty$. Again assuming a monopole source insice the cavity and subtracting off the Green's function for the cavity, the scattered field can be shown to be $O(\alpha/k^5)$ away from a resonance. Near a plate resonance however the plate deflection becomes large and it is found that consequently the scattered potential changes from $O(\alpha/k^5)$ to $O(1)$. Thus the total potential inside the cavity (3) is

$$\phi_{total} = \begin{cases} O(1) & \text{away from a plate resonance,} \\ O(1) & \text{at a plate resonance,} \end{cases} \tag{4.5}$$

due to a source at (1), and

$$\phi_{total} = \begin{cases} O(\alpha/(k^5(kr)^{1/2})) & \text{away from a plate resonance,} \\ O((kr)^{-1/2}) & \text{at a plate resonance,} \end{cases} \tag{4.6}$$

at (3) due to a source in the far field.

5. Discussion

A summary of the results noted in the previous chapters can be found in Table 1. To show how this information can be interpreted compare the sound field due to an O(1) quadrupole source (due to a turbulent boundary layer, say) near the plate with an incident plane wave of order unity. The incident wave can be replaced by a monopole source at a large distance, r, from the plate and must be of strength $O((kr)^{1/2})$ to generate a plane wave (O(1)) at the plate. The parameters of the plate/cavity geometry can be altered in order to obtain the largest ratio of

$$\frac{\text{sound field generated at (3) by a source at (2)}}{\text{sound field generated at (3) by a source at (1)}} \quad . \tag{5.1}$$

Firstly note that a quadrupole source generates a sound field which behaves as a second derivative of the potential due to a monopole source. It is clear that except for the small gap limit

$$\frac{\partial^2\phi}{\partial n^2} \sim \phi k^2 \quad , \qquad \text{(n is any direction)} \tag{5.2}$$

which, if k = O(1), results in

$$O(\text{quadrupole sound field}) = O(\text{monopole field}). \tag{5.3}$$

However, if $ka = \beta \to 0$, then .

$$\frac{\partial^2\phi}{\partial x^2} \sim \frac{\phi}{a^2} \tag{5.4}$$

and so

$$O(\text{quadrupole sound field}) = O(\text{monopole field}/\beta^2) \ . \tag{5.5}$$

Thus the ratio in (5.1) will be very small in the limits $\beta \to 0$, $\alpha \to \infty$ and by physical reasoning it can be deduced that this is the case for

all α. Similarly it can be generalised that as $\beta \to \infty$ there will be no change in order of magnitude of the sound field at a cavity resonance for all values of the fluid loading parameter α.

Turning to the limit $\alpha \to 0$ it is seen from Table 1 that the ratio in (5.1) changes from $O(\alpha/k^5)$ to $O(1)$ as a plate resonance is approached and if $(\varepsilon ka)^{1/2} \geq O(1)$ then this ratio is also $O(1)$ at a plate resonance when $\alpha \to \infty$. Thus to obtain the optimum ratio in (5.1) (which is at best only $O(1)$) it appears that for a given α the parameters of the geometry should be chosen so that a plate resonance occurs (i.e. standing waves are present on the surface) and it is also helpful if ka is large.

Table 1

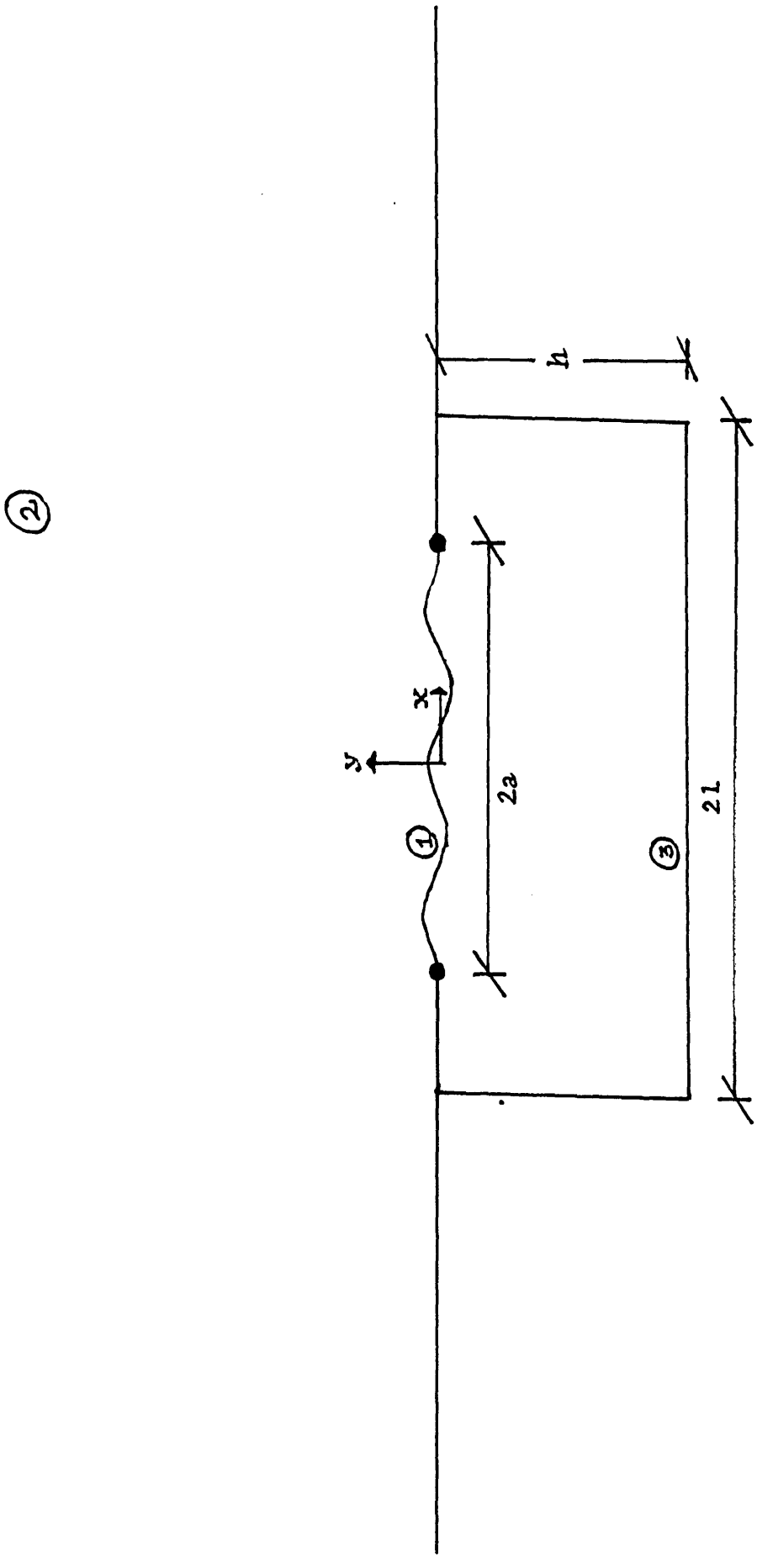
Sound field at (3)

Source at (1) of order unity			
parameters	sound field	cavity resonance	plate resonance
$\alpha \to \infty$, $\beta \to \infty$	$O(1)$	N/A	$O(1)$, $O((\varepsilon ka)^{-1/2})$ whichever greater
$\alpha \to \infty$, $\beta \to 0$	$O(1)$	$O(1n\ \beta)$ [$O((\varepsilon\beta)^{-\frac{1}{2}})$ at both resonances]	$O(1/((\varepsilon\beta)^{\frac{1}{2}} 1n\ \beta))$
$\alpha \to 0$	$O(1)$	$O(k^5/\alpha)$	$O(1)$

Source at (2) of order $(kr)^{\frac{1}{2}}$

parameters	sound field	cavity resonance	plate resonance
$\alpha \to \infty$, $\beta \to \infty$	$O(1)$	N/A	$O(1)$
$\alpha \to \infty$, $\beta \to 0$	$O(1/1n\ \beta)$	$O(1)$ [$O(\beta^{-\frac{1}{2}})$ at both resonances]	$O(1/(\beta^{\frac{1}{2}} 1n\ \beta))$
$\alpha \to 0$	$O(\alpha/k^5)$	$O(1)$	$O(1)$

Figure 1



Appendix 1

Taking the outer problem defined in (1.5), (1.7), (1.8), (1.9) and (2.1), and using the non-dimensional lengthscales

$$x = k\underline{x}, \qquad y = k\underline{y} \ , \tag{A1}$$

where $\underline{x},\underline{y}$ are the dimensional coordinates used in chapter 1, the problem becomes

$$\begin{aligned}
&(\nabla^2 + 1)\Phi = 0 && \text{all } x,y,\\
&\Phi\big|_{-}^{+} = 0 && |x| < \beta,\ y = 0,\\
&\Phi_y = 0 && |x| > \beta,\ y = 0\ , \\
&\Phi_y = 0 && |x| < k\ell,\ y = -kh,\\
&\Phi_x = 0 && x = \pm k\ell,\ 0 > y > -kh.
\end{aligned} \tag{A2}$$

In the limit $\beta \to 0$ the problem defined by (A2) can be solved, and the forcing is a monopole source inside the cavity. The Green's function for the cavity, with derivatives normal to cavity walls zero, is well known and can be written as

$$\phi_i = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \cos\left[\frac{m\pi}{2\ell k}(x'+\ell k)\right] \cos\left[\frac{m\pi}{2\ell k}(x+\ell k)\right] \cos\left(\frac{n\pi y'}{hk}\right)\cos\left(\frac{n\pi y}{hk}\right)$$

$$\times \frac{2}{h\ell\left[k^2-(m\pi/2\ell)^2-(n\pi/h)^2\right]} \ , \tag{A3}$$

where the right hand side is multiplied by $\frac{1}{2}$ when n or m = 0, (x', y') are coordinates of the source and $|x| < \ell k$, $|x'| < \ell k$, $-kh < y < 0$, $-kh < y' < 0$. It is convenient to assume the forcing potential to be that defined in (A3) and subtracting this off the total potential gives

$$\phi = \begin{cases} \phi & y > 0, \\ \phi_i + \phi & y < 0, \end{cases} \tag{A4}$$

thus the scattered potential ϕ satisfies (A2) except that the forcing is now due to a discontinuity across the gap

$$\phi\big|_-^+ = \phi_i \qquad |x| < \beta,\ y = 0. \tag{A5}$$

The forcing potential in (A5) can be approximated to

$$\phi_i(x,0;\ x',\ y') = \sum_{m,n=0}^{\infty} \cos\left(\frac{m\pi}{2}\right)\cos\left(\frac{m\pi}{2\ell k}(x'+\ell k)\right) \cos\left(\frac{n\pi y'}{hk}\right)$$

$$\times \frac{2}{h\ell[k^2-(m\pi/2\ell)^2-(n\pi/h)^2]} + O(\beta), \qquad |x| < \beta, \tag{A6}$$

if it is assumed that when $m\beta = O(1)$ the denominator in (A6) is large enough to allow this and further terms to be neglected. Alternatively the potential ϕ_i could have been re-written in terms of an array of image sources and then approximated in a more rigorous sense as $\beta \to 0$. Thus if the source is away from the gap in the cavity wall ϕ_i can be written

$$\phi_i(x,0;\ x',y') = K + O(\beta), \tag{A7}$$

$|x| < \beta$, $(x'^2+y'^2)^{\frac{1}{2}} >> \beta$, K independent of x. If, however, the source is on the gap then the alternate form for the potential approximates to

$$\phi_i(x,0;x',0) = \frac{H_0^{(1)}}{4i}(|x-x'|) + \sum_{n=1}^{\infty} \frac{H_0^{(1)}}{2i}(2nh)$$

$$+ \sum_{\substack{m,n=-\infty \\ m\neq 0}}^{\infty} \frac{H_0^{(1)}}{4i}((2nh)^2+(2\ell m)^2)^{\frac{1}{2}} + O(\beta), \tag{A8}$$

or, as $|x-x'| \leq 2\beta$,

$$\phi_i(x,0;\, x',0) = \frac{1}{2\pi}\left(\ln(|x-x'|)-(\ln 2-\gamma+\frac{i\pi}{2}) + C + O(\beta)\right), \tag{A9}$$

where C is a known constant which behaves as

$$C \sim \frac{C_1}{R_{pq}} \quad , \quad C_1 = O(1)$$

as $R_{pq} = k^2 - (\frac{p\pi}{2\ell})^2 - (\frac{q\pi}{h})^2 \to 0$. Using Green's theorem it is easily shown that

$$\underset{y<0}{\phi(x,y)} = -\int_{-\beta}^{\beta} \phi_i(x',0;\, x,y)\phi_y(x',0)dx', \tag{A10}$$

$$\underset{y>0}{\phi(x,y)} = \int_{-\beta}^{\beta} G_+(x',0;\, x,y)\phi_y(x',0)dx',$$

where $G_+(x',y';\, x,y)$ is the Green's function for the half space above the gap,

$$G_+(x',y;\, x,y) = \frac{H_0^{(1)}((x-x')+(y-y')^2)^{\frac{1}{2}}+H_0^{(1)}((x-x')^2+(y+y')^2)^{\frac{1}{2}}}{4i}. \tag{A11}$$

Using the equations (A10) and (A5), (A7) gives, after differentiating, the integral equation

$$\int_{-\beta}^{\beta} \frac{\phi_y(x',0)dx'}{(x-x')} = 0, \tag{A12}$$

which finally gives the result

$$\phi_y(x,0) = \frac{2K(\beta^2-x^2)^{-\frac{1}{2}}}{3(\ln\beta - (2\ln 2 - \gamma + \frac{i\pi}{2}) + 2\pi C/3)}, \tag{A13}$$

where K is given in (A7).

From this the magnitude (in β) of the travelling waves and eigenpotentials due to the presence of the elastic plate is found to be $O((\beta \ln \beta)^{-\frac{1}{2}})$. The total potential is also found from (A10) to be

$$\Phi(x,y) \sim - \frac{iK\pi H_0^{(1)}(x^2+y^2)^{\frac{1}{2}}}{3(\ln \beta - (2 \ln 2 - \gamma + i\pi/2) + 2\pi C/3)} , \qquad (A14)$$

when $y > 0$, $(x^2+y^2)^{\frac{1}{2}} >> \beta$, and when $y < 0$

$$\Phi(x,y) = \frac{-4 K\pi}{3(\ln\beta-(2\ln 2-\gamma+i\pi/2)+2\pi C/3)} \times \sum_{m,n=0}^{\infty} \frac{\cos(\frac{m\pi}{2})\cos(\frac{n\pi y}{hk})\cos\frac{m\pi(x+\ell k)}{2\ell k}}{h\ell(k^2-(\frac{m\pi}{2\ell})^2 - (\frac{n\pi}{h})^2)}$$

$$+ \phi_i(x,y; x',y') . \qquad (A15)$$

If the parameter $k^2 - (\frac{m\pi}{2\ell})^2 - (\frac{n\pi}{h})^2$ is not small for any m,n then (A14) and (A15) show that

$$\Phi = O(\frac{1}{\ln \beta}) , \qquad y > 0, \qquad (A16)$$

and $$\Phi \sim \phi_i = O(1), \quad y < 0. \qquad (A17)$$

When a cavity resonance is approached (i.e. $R_{pq} = k^2 - (p\pi/2\ell)^2-(q\pi/h)^2 \to 0$) both K and C become large and it is found that

$$\Phi(x\ y) \sim - \frac{i \cos(q\pi y/hk)}{\cos(p\pi/2)} \cos(p\pi(x'+\ell k)/2\ell k)H_0^{(1)}(x^2+y^2)^{\frac{1}{2}}$$

$$= O(1), \qquad y > 0, \qquad \text{and} \qquad (A18)$$

$$\Phi(x,y) \sim \frac{3 \ln \beta}{2\pi\cos^2(p\pi/2)} \cos[\frac{p\pi}{2\ell k}(x+\ell k)]\cos[\frac{p\pi}{2\ell k}(x'+\ell k)]\cos(\frac{q\pi y}{hk})\cos(\frac{q\pi y'}{hk})$$

$$= O(\ln \beta), \quad y \leq 0. \qquad (A19)$$

Appendix 2

Assuming monopole source forcing at (x', y') (again using dimensionless coordinates in A1) inside the cavity, it is convenient to allow the potential to be split as,

$$\Phi = \begin{cases} \phi & y > 0, \\ \phi + \phi_i & y < 0, \end{cases} \tag{B1}$$

where $\phi_i(x,y; x',y')$ is defined in (A3) and is the Green's function for the cavity. By using the scheme in reference (3) the scattered potential ϕ is taken to be the sum of two potentials ϕ_1, ϕ_2 ($\phi_1 >> \phi_2$), given by the plate equations

$$k(k^4 \frac{\partial^4}{\partial x^4} - \mu^4)\phi_{1y}(x,0) = \alpha\phi_i(x,0;x',y') - \alpha \int_{-ka}^{ka} \phi_i(x,0;x_1,0)\phi_y(x_1,0)dx_1 , \tag{B2}$$

$$k(k^4 \frac{\partial^4}{\partial x^4} - \mu^4)\phi_{2y}(x,0) = -\alpha \int_{-ka}^{ka} G_+(x,0;x_1,0)\phi_y(x_1,0)dx_1, \tag{B3}$$

$|x| < ka$, and $G_+(x,y; x',y')$ is written in (A11). In chapter 5 of Abrahams (3) this problem is examined when forcing is by an incident wave and so, with adjustment for a source inside the cavity, the plate equations can be approximated as $\alpha \to 0$ and $R_{pq} = \{k^2-(p\pi/2\ell)^2-(q\pi/h)^2\} \to 0$ by

$$k(k^4 \frac{\partial^4}{\partial x^4} - \mu^4)\phi_{1y} = \alpha \sum_{\substack{m,n=0 \\ (m,n\neq p,q)}}^{\infty} \cos[\frac{m\pi}{2\ell k}(x+\ell k)]\cos[\frac{m\pi}{2\ell k}(x'+\ell k)]\cos(\frac{n\pi y'}{hk}) \times \frac{2}{h\ell R_{mn}}$$

$$- \alpha K \cos[\frac{p\pi}{2ka}(x+ka)], \tag{B4}$$

$$k(k^4 \frac{\partial^4}{\partial x^4} - \mu^4)\phi_{2y} = -\alpha \int_{-ka}^{ka} G_+(x,0;\ x_1,0)\phi_{1y}(x_1,0)dx_1, \tag{B5}$$

$$\text{where } K = \frac{2}{R_{pq} h\ell} \{ \int_{-ka}^{ka} \cos[\frac{p\pi(x_1+ka)}{2ka}]\phi_y(x_1,0)dx_1 - \cos[\frac{p\pi}{2\ell k}(x'+\ell k)]\cos(\frac{q\pi y'}{hk}) \} \tag{B6}$$

and is large, but finite, near resonance. The constant K can be determined once ϕ_{1y}, ϕ_{2y} are found from (B4), (B5) and these values are then substituted into (B6). The order of K is dependent upon the parameter R_{pq} and is found to be of the form

$$K = \frac{O(\alpha/k^5)}{\{\ell h R_{pq} + iO(\alpha^2/k^{10})\}} , \tag{B7}$$

or

$$K = \begin{cases} O(\alpha/k^5) & \text{away from a cavity resonance,} \\ O(k^5/\alpha) & \text{near a cavity resonance.} \end{cases} \tag{B8}$$

The order of magnitude of the potential can now be determined from

$$\Phi(x,y) = \phi_i(x,y;\ x',y') - \int_{-ka}^{ka} \phi_i(x,y;\ x_1,0)\phi_y(x_1,0)dx_1, \tag{B9}$$

which, near a cavity resonance becomes

$$\Phi(x,y) \sim -K \cos\{p\pi(x+\ell k)/2\ell k\}\cos(q\pi y/hk), \tag{B10}$$

$0 > y > -kh$, $|x| < \ell k$, as $R_{pq} \to 0$. Similarly for $y > 0$ and $r = (x^2+y^2)^{\frac{1}{2}} \to \infty$

$$\Phi(x,y) \sim \frac{H_0^{(1)}(r)}{2i} \int_{-\beta}^{\beta} \phi_y(x_1,0)dx_1, \tag{B11}$$

$$\text{where} \qquad \phi_y = O(\frac{\alpha}{k^5}) + O(\frac{\alpha K}{k^5}) \quad . \tag{B12}$$

(ii) Concluding Remarks

Many problems with finite plates and complicated geometries occur in aerodynamic sound theory and underwater acoustics. It is therefore hoped that the methods of analysis presented in this thesis can be of direct application to some of these models.

In the light loading limit a method was presented (§2) which was applicable to plate problems in the presence of cavities and ducts. The technique is simple to apply to a wide range of models, but the algebra does increase with increased complexity of the geometry.

In chapter 3 the "outer" problem of the matching scheme was not specified since no analytic solutions can be found in general. The only cases where analytic work can be performed is in the limits of large and small gap, but as suggested, "outer" problems with complicated geometries could be solved by simple numerical methods. The matching scheme derived in §3 can be used with many problems involving resonances and this was shown by performing this analysis in the flow problem of section 4.

Several boundary value problems with large finite geometries were analysed in chapter 6. The method was shown to be applicable to a wide range of models, including those with resonances.

Perhaps the most important extension to the work in this thesis would be to scattering by rectangular or circular plates. As suggested in section 7 few methods are available to tackle fully three dimensional problems. Asymptotic matching may be of use if the plate is large (compared to an acoustic wavelength) and the heavy loading limit is taken. Difficulty is expected with the rectangular plate as the solution of the "inner" problem will include determining the sound field generated by a

quarter plane elastic plate (Radlow (1961)). The circular plate will be easier as the inner region will consist of an elastic plate, infinite in one direction, and semi-infinite in the other (compare this with the problem by Jones (1950)).

Finally, as well as extending the particular area of acoustics under examination, it is hoped that the methods developed in this thesis will aid analysis of other problems in fluid mechanics.

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